Symmetric pairs and pseudosymmetry of $\Theta$-Yetter-Drinfeld categories for Hom-Hopf algebras

Abstract: In this paper, we investigate a more general category of $\Theta$-Yetter-Drinfeld modules ($\Theta \in \text{Aut}(H(H))$) over a Hom-Hopf algebra, which unifies two different definitions of Hom-Yetter-Drinfeld category introduced by Makhlouf and Panaite, Li and Ma, respectively. We show that the category of $\Theta$-Yetter-Drinfeld modules with a bijective antipode $S$ is a braided tensor category and some solutions of the Hom-Yang-Baxter equation and the Yang-Baxter equation can be constructed by this category. Also by the method of symmetric pairs, we prove that if a $\Theta$-Yetter-Drinfeld category over a Hom-Hopf algebra $H$ is symmetric, then $H$ is trivial. Finally, we find a sufficient and necessary condition for a $\Theta$-Yetter-Drinfeld category to be pseudosymmetric.

Keywords: Hom-Hopf algebra, $\Theta$-Yetter-Drinfeld module, symmetric pair

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1 Introduction

The genesis of Hom-structures may be found in the physics literature from the 1990, concerning quantum deformations of algebras of vector fields, especially Witt and Virasoro algebras. These classes of examples led to the development of Hom-Lie algebras; see [1–4]. Recently, various Hom-Lie structures have been studied further by many scholars; see [5–12]. The idea of tailoring associativity-like conditions by linear maps was migrated to other algebraic structures. The concepts of Hom-algebras, Hom-coalgebras, Hom-Hopf algebras and quasi-triangular Hom-bialgebras were introduced and further developed; see [13–22].

Yetter-Drinfeld modules are known to be at the origin of a very vast family of solutions to the Yang-Baxter equation. Therefore, it is very natural to extend the notion of Yetter-Drinfeld module into the Hom-setting. The main purpose to study Yetter-Drinfeld module in the Hom-setting is to prove that the category of Yetter-Drinfeld modules over a Hom-Hopf algebra with a bijective antipode is also a braided tensor category. Furthermore, we can construct some solutions of the classical Yang-Baxter equation and the Hom-Yang-Baxter equation introduced by Yau [12]. Two directions of the study were developed, one considering the category of Yetter-Drinfeld modules under monoidal Hom-Hopf algebras introduced by Caenepeel and Goyvaerts [23]; see also [24–26] and others, discussing the category of Yetter-Drinfeld modules under Hom-Hopf algebras in [14,27]. For a monoidal Hom-Hopf algebra, Caenepeel and Goyvaerts explained it from a categorical point of view. Therefore, we can write the corresponding conditions by the categorical method. Different from the monoidal Hom-Hopf algebra, the conditions in Hom-Hopf algebra

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are obtained more unpredictable than the monoidal version. For a Hom-Hopf algebra, there exists two
definitions of Yetter-Drinfeld modules in the studies of Makhlouf and Panaite [14] and Li and Ma [27],
respectively. It is obvious that these two definitions of Yetter-Drinfeld modules are not equivalent. Surpris-
ingly, such two categories of Yetter-Drinfeld modules are both braided tensor categories and provide
solutions of the Yang-Baxter equation and the Hom-Yang-Baxter equation. The motivation of the definition
of Yetter-Drinfeld modules by Makhlouf and Panaite relies on the main tool called “twisting principle,”
which was introduced by Yau [11]. But for the definition by Li and Ma, they gave it by massive calculations
to satisfy the corresponding conditions. Naturally inspired by the above two definitions, the authors in [28]
gave the definition of \( \nu\)-Yetter-Drinfeld modules, \( \nu \in \mathbb{Z} \); see (2.15). And they showed that every category of
\( \nu\)-Yetter-Drinfeld modules is also a braided tensor category and provides some solutions of the Yang-Baxter
equation and the Hom-Yang-Baxter equation. Therefore, the \( \nu\)-Yetter-Drinfeld module unifies the above two
definitions of Yetter-Drinfeld modules. Observing the compatibility of \( \nu\)-Yetter-Drinfeld modules, we find
that the condition (2.16) is equivalent to the condition (2.15). In other words, we have proven that if
\( p = 2\nu - 2 \) is even, the definition of \( \nu\)-Yetter-Drinfeld is favorable. One natural question to ask is whether
the definition of \( \nu \) is favorable or not if \( p \) is odd. Furthermore, there exist more general favorable definitions
of Yetter-Drinfeld module over a Hom-Hopf algebra. In this paper, we give the definition of \( \Theta\)-Yetter
Drinfeld modules, \( \Theta \in \text{Aut } H(H) \) where we denote the group of all Hom-Hopf algebra automorphisms of
\( H \) by \( \text{Aut } H(H) \) and prove that even if \( p \) is odd, the definition of Yetter-Drinfeld is in the same way favorable.
Meanwhile, all the definitions of Yetter-Drinfeld modules over a Hom-Hopf algebra that appeared in
[14,27,28] can be regarded as special cases of the \( \Theta\)-Yetter-Drinfeld module.

This paper is organized as follows. In Section 1, we recall some definitions and results which will be
used later. In Section 2, after introducing the concept of \( \Theta\)-Yetter-Drinfeld category \( ^H_HYM \Theta \), we prove that
every \( \Theta\)-Yetter Drinfeld category over a Hom-Hopf algebra with a bijective antipode is a braided tensor
category and that every \( \Theta\)-Yetter Drinfeld category over a Hom-Hopf algebra provides a solution of the Hom-
Yang-Baxter equation and the Yang-Baxter equation. In Section 3, by the method of symmetric pairs,
we show that symmetric \( \Theta\)-Yetter-Drinfeld categories \( ^H_HYM \Theta \) over a Hom-Hopf algebra are all trivial.
The results obtained in this section generalize the corresponding results in [28–31]. In Section 4, we find
a necessary and sufficient condition for a \( \Theta\)-Yetter-Drinfeld category \( ^H_HYM \Theta \) over a Hom-Hopf algebra \( H \)
to be pseudosymmetric.

2 Preliminaries

We work over a base field \( k \). All algebras, linear spaces etc. will be over \( k \), and unadorned \( \otimes \) means \( \otimes_k \).
For a comultiplication \( \Delta : C \rightarrow C \otimes C \) on a vector space \( C \), we use a Sweedler-type notation
\( \Delta(c) = c_1 \otimes c_2 \) for all \( c \in C \). In what follows, we assume that \( k\)-linear maps \( a \) are bijective, although some notions are
not supposed to be bijective.

**Definition 2.1.** [14,32] A Hom-associative algebra is a quadruple \( (A, \mu, 1_A, a) \) (abbr. \( (A, a) \)) where \( A \) is
a \( k\)-linear space, \( \mu : A \otimes A \rightarrow A \) and \( a : A \rightarrow A \) are \( k\)-linear maps such that
\[
\alpha(\alpha') = \alpha(\alpha a'), \quad a(1_A) = 1_A, \tag{2.1}
\]
\[
\alpha(a)(a' a'') = (aa')(\alpha a''), \quad 1_A a = a(a) \tag{2.2}
\]
are satisfied for \( a, a', a'' \in A \). Here we use the notation \( \mu(a \otimes a') = aa' \).

A morphism \( f : (A, \mu, 1_A, a) \rightarrow (B, \mu_B, 1_B, a_B) \) of Hom-associative algebra is a linear map \( f : A \rightarrow B \) such that
\( a_B \circ f = f \circ a_A, f(1_A) = 1_B \) and \( f \circ \mu_A = \mu_B \circ (f \otimes f) \).

Let \( (A, a) \) and \( (B, b) \) be two Hom-algebras. Then \( (A \otimes B, a \otimes b) \) in a Hom-associative algebra (called
a tensor product Hom-associative algebra) with the multiplication \((a \otimes b)(a' \otimes b') = aa' \otimes bb' \) and its unit
\( 1_A \otimes 1_B \).
Definition 2.2. [14,18,33] A Hom-coassociative coalgebra is a quadruple \((C, \Delta, \varepsilon_C, \beta)\) (abbr. \((C, \beta)\)), where \(C\) is a \(k\-linear space, \Delta : C \to C \otimes C, \varepsilon_C : C \to k\) and \(\beta : C \to C\) are \(k\-linear maps such that
\[
\beta(c_1) \otimes \beta(c_2) = \beta(c_1) \otimes \beta(c_2), \quad \varepsilon_C \circ \beta = \varepsilon_C,
\]
\[(2.3)\]
\[
\beta(c_1) \otimes c_2 \otimes c_2 = c_1 \otimes c_2 \otimes \beta(c_2), \quad \varepsilon_C(c_1)c_2 = c_1\varepsilon_C(c_2) = \beta(c).
\]
\[(2.4)\]
are satisfied for \(c \in C\).

A morphism \(g : (C, \Delta_C, \varepsilon_C) \to (D, \Delta_D, \varepsilon_D)\) of Hom-coassociative coalgebras is a linear map \(g : C \to D\) such that \(\alpha_D \circ g = g \circ \alpha_C, \varepsilon_C = \varepsilon_D \circ g\) and \((g \otimes g) \circ \Delta_C = \Delta_D \circ g\).

Let \((C, \alpha)\) and \((D, \beta)\) be two Hom-coassociative coalgebras. Then \((C \otimes D, \alpha \otimes \beta)\) is a Hom-coassociative coalgebra (called a tensor product Hom-coassociative coalgebra) with the comultiplication \(\Delta(c \otimes d) = c_1 \otimes d_1 \otimes c_2 \otimes d_2\) and its counit \(\varepsilon_C \otimes \varepsilon_D\).

Definition 2.3. [18,19] A Hom-bialgebra is a sextuple \((H, \mu, 1_H, \Delta, \varepsilon, \alpha)\) (abbr. \((H, \alpha)\)), where \((H, \mu, 1_H, \alpha)\) is a Hom-associative algebra and \((H, \Delta, \varepsilon, \alpha)\) is a Hom-coassociative coalgebra such that \(\Delta\) and \(\varepsilon\) are morphisms of Hom-algebras, i.e.,
\[
\Delta(hh') = \Delta(h)\Delta(h'), \quad \Delta(1_H) = 1_H \otimes 1_H,
\]
\[(2.5)\]
\[
\varepsilon(hh') = \varepsilon(h)\varepsilon(h'), \quad \varepsilon(1_H) = 1.
\]
\[(2.6)\]
Furthermore, if there exists a linear map \(S : H \to H\) such that
\[
S(h_1)h_2 = h_1S(h_2) = \varepsilon(h)1_H, \quad S(\alpha(h)) = \alpha(S(h)),
\]
\[(2.7)\]
then we call \((H, \mu, 1_H, \Delta, \varepsilon, \alpha, S)\) (abbr. \((H, \alpha, S)\)) a Hom-Hopf algebra.

Let \((H, a, S)\) and \((H', a', S')\) be two Hom-Hopf algebras. The linear map \(f : H \to H'\) is called a Hom-Hopf algebra map if \(f \circ \alpha = a' \circ f, f \circ S = S' \circ f\) and \(f\) is a Hom-algebra map and a Hom-coalgebra map.

Definition 2.4. [18,32] Let \((A, \mu, 1_A, \alpha_d)\) be a Hom-associative algebra, \(M\) a linear space and \(\alpha_M : M \to M\) a linear map. A left \(A\)-Hom-module \((M, \alpha_M)\) consists of a linear map \(A \otimes M \to M, a \otimes m \mapsto a \cdot m\) satisfying the following conditions:
\[
\alpha_M(a \cdot m) = \alpha_d(a) \cdot \alpha_M(m);
\]
\[(2.8)\]
\[
\alpha_d(a) \cdot (a' \cdot m) = (a \cdot a') \cdot \alpha_M(m), \quad 1_H \cdot m = \alpha_M(m)
\]
\[(2.9)\]
for all \(a, a' \in A\) and \(m \in M\). If \((M, \alpha_M)\) and \((N, \alpha_N)\) are left \(A\)-Hom-modules (both are \(A\)-Hom-actions, denoted by \(\cdot\)), a morphism of left \(A\)-Hom-modules \(f : M \to N\) is a linear map satisfying the following conditions:
\[
a_N \circ f = f \circ \alpha_M \quad \text{and} \quad f(a \cdot m) = a \cdot f(m), \quad a \in A, m \in M.
\]

Definition 2.5. [18,32] Let \((C, \Delta_C, \varepsilon, \alpha_C)\) be a Hom-coassociative coalgebra, \(M\) a linear space and \(\alpha_M : M \to M\) a linear map. A left \(C\)-Hom-comodule \((M, \alpha_M)\) consists of a linear map \(\lambda : M \to C \otimes M\) (usually denoted by \(\lambda(m) = m(-1) \otimes m(0)\)) satisfying the following conditions:
\[
(\alpha_C \otimes \alpha_M) \cdot \lambda = \lambda \cdot \alpha_M;
\]
\[(2.10)\]
\[
(\Delta_C \otimes \alpha_M) \cdot \lambda = (\alpha_C \otimes \lambda) \cdot \lambda, \quad (\varepsilon \otimes id) \cdot \lambda = \alpha_M.
\]
\[(2.11)\]
If \((M, \alpha_M)\) and \((N, \alpha_N)\) are left \(C\)-Hom-comodules, with Hom-coactions \(\lambda_M : M \to C \otimes M\) and \(\lambda_N : N \to C \otimes N\), a morphism of left \(C\)-Hom-comodules \(g : M \to N\) is a linear map satisfying the conditions \(\alpha_N \circ g = g \circ \alpha_M\) and \((id_C \otimes g) \cdot \lambda_M = \lambda_N \cdot g\).

Definition 2.6. [14] Let \((H, \mu_H, 1_H, \Delta_{H}, \varepsilon_H, \alpha_{Hj})\) be a Hom-bialgebra, \(M\) a linear space and \(\alpha_M : M \to M\) a linear map such that \((M, \alpha_M)\) is a left \(H\)-Hom-module with Hom-action \(H \otimes M \to M, h \otimes m \mapsto h \cdot m\) and
a left $H$-Hom-comodule with Hom-coaction $M \to H \otimes M$, $m \mapsto m_{(-1)} \otimes m_{(0)}$. Then $(M, a_M)$ is called a left-left Yetter-Drinfel module over $H$ if, for all $h \in H$ and $m \in M$, there holds the following identity:

$$
(h_1 \cdot m)_{(-1)} a_M^2(h_2) \otimes (h_1 \cdot m)_{(0)} = a_M^2(h_1) a_M(m_{(-1)}) \otimes a_M(h_2) \cdot m_{(0)}.
$$

(2.12)

Notice that the choice of the compatibility condition (2.12) due to [14] was motivated by the twisting principle, but the compatibility condition of Yetter-Drinfel module over a Hom-bialgebra is not the only choice as in (2.12). The following definition of Yetter-Drinfel module was due to Li and Ma [27].

**Definition 2.7.** [27] Let $(H, \mu_H, 1_H, \Delta_H, \varepsilon_H, a_H)$ be a Hom-bialgebra, $(M, a_M)$ a left $H$-Hom-module with Hom-action $H \otimes M \to M$ denoted by $h \otimes m \mapsto h \cdot m$ and $(M, a_M)$ a left $H$-Hom-comodule with Hom-coaction $M \to H \otimes M$ denoted by $m \mapsto m_{(-1)} \otimes m_{(0)}$. Then we call $(M, a_M)$ a left-left Yetter-Drinfel module over a Hom-bialgebra $(H, a_H)$ if, for all $h \in H$ and $m \in M$, there holds the following condition:

$$
h_H a_H(m_{(-1)}) \otimes a_H^1(h_2) \cdot m_{(0)} = (a_H^2(h_1) \cdot m)_{(-1)} h_H^2 \otimes (a_H^2(h_1) \cdot m)_{(0)},
$$

(2.13)

**Remark 2.8.** Obviously, the compatibility condition (2.12) is different from the condition (2.13). It is not hard to show that the definition of Yetter-Drinfel module in Definition 2.7 does not satisfy the twisting principle. Indeed, by a direct computation, we have

$$(a_H^Q(h_{(1)}) \cdot m_{(-1)}) \cdot h_{(2)} \otimes (a_H^Q(h_{(1)}) \cdot m_{(0)}) \neq h_{(1)} \cdot a_H^Q(m_{(-1)}) \otimes a_H^Q(h_{(2)}) \cdot m_{(0)},$$

where the multiplication and comultiplication of $H_{a_H}$ were denoted by $h \cdot h' = a_H(h h')$ and $h_{(1)} \otimes h_{(2)} = a_H(h_1) \otimes a_H(h_2)$, respectively.

**Remark 2.9.** Since $a$ is bijective, it is not hard to obtain that the condition (2.13) is equivalent to

$$
h_1 \cdot m_{(-1)} a_H^2(h_2) \otimes (h_1 \cdot m)_{(0)} = a_H^2(h_1) a_H(m_{(-1)}) \otimes a_H(h_2) \cdot m_{(0)},
$$

(2.14)

In [28], the authors gave a more general compatibility condition including the definitions in [14,27] as the special cases.

**Definition 2.10.** [28] Let $(H, a_H)$ be a Hom-bialgebra, $(M, a_M)$ a left $H$-Hom-module with Hom-action $H \otimes M \to M$, $h \otimes m \mapsto h \cdot m$ and $(M, a_M)$ a left $H$-Hom-comodule with Hom-coaction $M \to H \otimes M$, $m \mapsto m_{(-1)} \otimes m_{(0)}$. Then we call $(M, a_M)$ a left-left $v$-Yetter-Drinfel module over a Hom-bialgebra $(H, a_H)$ if, for all $h \in H$, $m \in M$ and $v \in \mathbb{Z}$, there holds the following condition:

$$
(a_H^{v-2}(h_1) \cdot m)_{(-1)} a_H^{v-2}(h_2) \otimes (a_H^{v-2}(h_1) \cdot m)_{(0)} = a_H^{v-2}(h_1) a_H(m_{(-1)}) \otimes a_H^{v-2}(h_2) \cdot m_{(0)}.
$$

(2.15)

**Remark 2.11.** Being similar to Remark 2.9, we readily see that the condition (2.15) is equivalent to

$$
h_1 \cdot m_{(-1)} a_H^{v-2}(h_2) \otimes (h_1 \cdot m)_{(0)} = a_H^{v-2}(h_1) a_H(m_{(-1)}) \otimes a_H(h_2) \cdot m_{(0)}, \quad v \in \mathbb{Z}.
$$

(2.16)

### 3 Θ-Yetter-Drinfel category

In this section, we introduce a more general Θ-Yetter-Drinfel category $(\Theta \in \text{Aut } H(H))$, which unifies two different definitions of Hom-Yetter-Drinfel category of Makhlouf and Panaïte [14] and Li and Ma [27], respectively. We show that the category of Θ-Yetter-Drinfel modules with a bijective antipode is a braided tensor category and some solutions of the Hom-Yang-Baxter equation and the Yang-Baxter equation can be constructed by this category.
In case that \((H, \mu, \Delta, \alpha, S)\) is a Hom-Hopf algebra, we denote the group of all Hom-Hopf algebra automorphisms of \(H\) by \(\text{Aut}_H(H)\). Inspired by the above three definitions of Hom-Yetter-Drinfeld modules, we can give a more general definition of Hom-Yetter-Drinfeld modules as follows.

**Definition 3.1.** Let \((H, \alpha_H)\) be a Hom-bialgebra and \(\Theta \in \text{Aut}_H(H)\), \((M, \alpha_M)\) a left \(H\)-Hom-module with Hom-action \(H \otimes M \to M\), \(h \otimes m \mapsto h \cdot m\) and \((M, \alpha_M)\) a left \(H\)-Hom-comodule with Hom-coaction \(M \to H \otimes M\), \(m \mapsto m_{(-1)} \otimes m_{(0)}\). Then we call \((M, \alpha_M)\) a left-left \(\Theta\)-Yetter-Drinfeld module over a Hom-bialgebra \((H, \alpha_H)\) if, for all \(h \in H\) and \(m \in M\), there holds the following condition:

\[
(h_1 \cdot m)_{(-1)} \Theta(h_2) \otimes (h_1 \cdot m)_{(0)} = \Theta(h_1) \alpha_H(m_{(-1)}) \otimes \alpha_H(h_2) \cdot m_{(0)},
\]

(3.1)

**Remark 3.2.** We find an interesting fact that every definition that appeared in references can be regarded as the special cases of \(\Theta\)-Yetter-Drinfeld module.

1. If \(\Theta = \alpha^{2v-2}\), \(v \in \mathbb{Z}\), the definition of \(\Theta\)-Yetter-Drinfeld module is the same as Definition 2.1 in [28].
2. If \(\Theta = \alpha^{-2}\), the definition of \(\Theta\)-Yetter-Drinfeld module is the same as Definition 2.7;
3. If \(\Theta = \alpha^2\), the definition of \(\Theta\)-Yetter-Drinfeld module is just Definition 2.6;
4. If \(\alpha_H = \text{id}, \alpha_M = \text{id}\) and \(\Theta = \text{id}\), the definition of \(\Theta\)-Yetter-Drinfeld module is exactly the usual Yetter-Drinfeld module;
5. We should note that it is impossible to get the definition only by the twisting principle.

**Example 3.3.** Let \((H, S, \alpha_H)\) be a Hom-Hopf algebra and \(\Theta \in \text{Aut}_H(H)\). Then \((H, \alpha_H)\) is a \(\Theta\)-Yetter-Drinfeld module in Definition 3.1 with a left \(H\)-Hom-action

\[
h \triangleright g = (\Theta \alpha_H^{-1}(h_1) \alpha_H^{-1}(g)) \Theta \alpha_H^{-1}(h_2)
\]

and a left \(H\)-Hom-coaction by the Hom-comultiplication \(\Delta\). Denote it by \(H_\Theta = (H, \triangleright, \Delta, \alpha_H)\). Dually, \((H, \alpha)\) is a \(\Theta\)-Yetter-Drinfeld module in Definition 3.1 with a left \(H\)-Hom-action by the Hom-multiplication \(\mu\) and a left \(H\)-Hom-coaction

\[
\lambda(h) = \Theta \alpha_H^{-1}(h_1) \alpha_H^{-1} \Theta \alpha_H^{-1}(h_2) \otimes \alpha_H^{-1}(h_{12});
\]

it will be denoted by \(H_\alpha = (H, \mu, \lambda, \alpha_H)\).

**Proof.** We only establish the first part of the above example here. First, it is easy to see that \((H_\Theta, \Delta, \alpha)\) is a left \((H, \alpha)\)-comodule. Second, we show that \((H_\Theta, \triangleright, \alpha)\) is a left \((H, \alpha)\)-module. In fact, for any \(h, g, l \in H_\Theta\), we have

\[
\alpha(h) \triangleright \alpha(g) = (\Theta \alpha^{-1}(h_1) \alpha^{-1}(g)) \Theta \alpha^{-1}(h_2) = \alpha(h \triangleright g),
\]

which proves the equality (2.8). For proving (2.9), we observe that

\[
I_H \triangleright g = (\Theta \alpha^{-1}(1) \alpha^{-1}(g)) \Theta \alpha^{-1}(1) = (1 \cdot g) \cdot 1 = \alpha(g)
\]

and

\[
\alpha(h) \triangleright (g \triangleright l) = \alpha(h) \triangleright ((\Theta \alpha^{-1}(g_2) \alpha^{-1}(l)) \Theta \alpha^{-1}(g_1))
\]

\[
= (\Theta \alpha^{-1}(h_1) \alpha^{-1}(\Theta \alpha^{-1}(g_2) \alpha^{-1}(l)) \Theta \alpha^{-1}(g_1)) \Theta \alpha^{-1}(h_2)
\]

\[
= ((\Theta \alpha^{-1}(h_1) \alpha^{-1}(g_2) \alpha^{-1}(l)) \Theta \alpha^{-1}(g_1)) \Theta \alpha^{-1}(h_2)
\]

Finally, we show the compatible condition (3.1) of \(\Theta\)-Yetter-Drinfeld module. For this end, the left-hand side of (3.1) can be expressed as

\[
\Theta(h_1) \alpha(l_1) \otimes \alpha(h_2) \cdot l_2 = \Theta(h_1) \alpha(l_1) \otimes (\Theta \alpha^{-1}(h_2) \alpha^{-1}(l_2)) \Theta \alpha^{-1}(h_{12}),
\]
and also its right-hand side can be given by
\[
(h_1 \cdot l_{(-1)}(l)) (h_2) \otimes (h_1 \cdot l_{(0)})
\]
\[
\tag{25}(2.3) \Rightarrow \tag{2.5}
((\Theta \alpha^{-\delta}(h_{11}) \alpha^{-\delta}(h_{12})) S(\Theta \alpha^{-\delta}(h_{222}))) \Theta(h_2) \otimes \Theta(\alpha^{-\delta}(h_{11}) \alpha^{-\delta}(h_{12})) S(\Theta \alpha^{-\delta}(h_{22}))
\]
\[
\tag{2.4}(2.2) = \tag{2.4}
(\Theta \alpha^{-\delta}(h_{11}) \alpha^{-\delta}(h_{12})) S(\Theta \alpha^{-\delta}(h_{222})) \otimes (\Theta \alpha^{-\delta}(h_{11}) \alpha^{-\delta}(h_{12})) S(\Theta \alpha^{-\delta}(h_{22}))
\]
\[
\tag{2.7}(2.3)(2.4) \Rightarrow \tag{2.7}
\Theta^{-\delta}(h_{11}) \Theta^{-\delta}(h_{12}) \otimes (\Theta \alpha^{-\delta}(h_{11}) \alpha^{-\delta}(h_{12})) S(\Theta \alpha^{-\delta}(h_{22}))
\]
\[
\tag{2.4} \Rightarrow \tag{2.4}
\Theta(h_1) \Theta^{-\delta}(h_{11}) \Theta^{-\delta}(h_{12}) \otimes (\Theta \alpha^{-\delta}(h_{11}) \alpha^{-\delta}(h_{12})) S(\Theta \alpha^{-\delta}(h_{22}))
\]
yielding the compatible condition (3.1) of Θ-Yetter-Drinfeld module and the proof is complete.

**Remark 3.4.** If we take Θ = a^{2v-2}, v ∈ Z in Example 3.3, we obtain Example 2.4 in [28]. Also, if we take Θ = a^{-2}, Example 3.3 is just Lemmas 3.3 and 3.4 in [31].

**Definition 3.5.** Let \((H, a_H)\) be a Hom-bialgebra and Θ ∈ Aut H(H). We denote by \(H^H \cap Y \mathcal{D}^\Theta\) the category whose objects are Θ-Yetter-Drinfeld modules \((M, a_M)\) over H, and the morphisms in the category are morphisms of left H-Hom-modules and left H-Hom-comodules.

**Proposition 3.6.** Let \((H, S, a_H)\) be a Hom-Hopf algebra and Θ ∈ Aut H(H). Let \((M, a_M)\) be both a left H-Hom-module and a left H-Hom-comodule with notations as in Definition 3.1. Then (3.1) in Definition 3.1 is equivalent to
\[
(\Theta a_H^{-\delta}(h_{11}) \alpha a_H^{-\delta}(h_{12})) \Theta^{-\delta}(S(h_2)) \otimes a_H^{-\delta}(h_{12}) m_{(-1)} \otimes (h \cdot m)_{(0)} = (h \cdot m)_{(-1)} \otimes (h \cdot m)_{(0)} \tag{3.2}
\]
for all \(h \in H\) and \(m \in M\).

**Proof.** (3.1) ⇒ (3.2). For all \(h \in H\) and \(m \in M\), we compute
\[
\tag{2.1}(2.5)(2.10) \Rightarrow \tag{2.1}
(\Theta a_H^{-\delta}(h_{11}) \alpha a_H^{-\delta}(h_{12})) \Theta^{-\delta}(S(h_2)) \otimes a_H^{-\delta}(h_{12}) \cdot (a_H^{-\delta}(m))_{(0)} \Rightarrow \Theta a_H^{-\delta}(h_{11}) \alpha a_H^{-\delta}(h_{12}) \Theta^{-\delta}(S(h_2)) \otimes a_H^{-\delta}(h_{12}) \cdot (a_H^{-\delta}(m))_{(0)}
\]
\[
\tag{2.3}(2.2)(2.4) = \tag{2.3}
\Theta a_H^{-\delta}(h_{11}) \alpha a_H^{-\delta}(h_{12}) \Theta^{-\delta}(S(h_2)) \otimes a_H^{-\delta}(h_{12}) \cdot (a_H^{-\delta}(m))_{(-1)} \otimes (h \cdot m)_{(0)}
\]
\[
\tag{2.7}(2.3)(2.4) = \tag{2.7}
(\Theta a_H^{-\delta}(h_{11}) \alpha a_H^{-\delta}(h_{12}) \Theta^{-\delta}(S(h_2)) \otimes a_H^{-\delta}(h_{12}) \cdot (a_H^{-\delta}(m))_{(-1)} \otimes (h \cdot m)_{(0)}
\]
(3.2) ⇒ (3.1). For all \(h \in H\) and \(m \in M\), we have
\[
(h_1 \otimes (h_1 \cdot m)) \Theta^{-\delta}(h_2) \otimes (h_1 \cdot m)_{(0)} \Rightarrow \tag{3.2}
\tag{3.2}
(\Theta a_H^{-\delta}(h_{11}) \alpha a_H^{-\delta}(h_{12})) \Theta^{-\delta}(S(h_2)) \otimes a_H^{-\delta}(h_{12}) \cdot (a_H^{-\delta}(m))_{(-1)} \otimes (h \cdot m)_{(0)}
\]
\[
\tag{2.2}(2.4) = \tag{2.2}
(\Theta a_H^{-\delta}(h_{11}) \alpha a_H^{-\delta}(h_{12})) \Theta^{-\delta}(S(h_2)) \otimes (h_2 \cdot m)_{(0)}
\]
\[
\tag{2.7}(2.3)(2.2) = \tag{2.7}
\Theta(h_1) \alpha a_H^{-\delta}(h_{12}) \Theta^{-\delta}(S(h_2)) \otimes (h_2 \cdot m)_{(0)}
\]
This completes the proof. □

In the following, we give solutions of the Hom-Yang-Baxter introduced and studied by Yau [12].

**Proposition 3.7.** Let \((H, a_H)\) be a Hom-bialgebra, Θ ∈ Aut H(H) and \((M, a_M), (N, a_N) \in H^H \cap Y \mathcal{D}^\Theta\). Define the linear map
\[
\tau_{M,N} : M \otimes N \rightarrow N \otimes M, \quad m \otimes n \mapsto \Theta a_H^{-\delta}(m_{(-1)}) \cdot n \otimes m_{(0)}, \tag{3.3}
\]
where \( m \in M \) and \( n \in N \). Then,

(i) \( \tau_{M,N} \circ (a_M \otimes a_N) = (a_N \otimes a_M) \circ \tau_{M,N} \), and

(ii) if \((P, a_p) \in \mathcal{H}(\mathcal{Y}, \mathcal{D})^\theta \), the maps \( \tau_{\ldots} \) satisfy the Hom-Yang-Baxter equation:

\[
(a_p \otimes \tau_{M,N}) \circ (\tau_{M,P} \otimes a_N) \circ (a_M \otimes \tau_{N,P}) = (\tau_{N,P} \otimes a_M) \circ (a_N \otimes \tau_{M,P}) \circ (\tau_{M,N} \otimes a_P).
\]

**Proof.** The property (i) can be easily proven. Now we claim the property (ii): given \( m \in M, n \in N \) and \( p \in P \), using the conditions (2.10), (2.1) and (2.2), we obtain

\[
(a_p \otimes \tau_{M,N}) \circ (\tau_{M,P} \otimes a_N) \circ (a_M \otimes \tau_{N,P})(m \otimes n \otimes p)
\]

\[
= \Theta^{-1}a_H^2(m_{(-1)}n_{(-1)}) \cdot a_P^2(p) \otimes \Theta^{-1}a_H^2(m_{(0)X_{(-1)}}) \cdot a_N(n_{(0)})
\]

\[
\circ a_M(m_{(0)X_{(0)}})(\tau_{N,P} \otimes a_M)(m \otimes n \otimes p)
\]

\[
= \Theta^{-1}a_H^2(a_H^{-1}(m_{(-1)}) \cdot a_N(n))_{(-1)} \cdot (\Theta^{-1}a_H^2(m_{(0)X_{(-1)}}) \cdot a_P(p))
\]

\[
\circ (a_M \circ \Theta^{-1}(m_{(-1)}) \cdot a_N(n))_{(0)} \circ a_M(m_{(0)X_{(0)}})
\]

\[
(\Theta^{-1} \circ (\Theta^{-1}a_H^2(m_{(-1)}) \cdot a_N(n))_{(-1)} \cdot \Theta^{-1}a_H^2(m_{(-1)}) \cdot a_P(p))
\]

\[
\circ (\Theta^{-1} \circ (\Theta^{-1}a_H^2(m_{(-1)}) \cdot a_N(n))_{(0)}) \circ a_M(m_{(0)X_{(0)}})
\]

\[
(\Theta^{-1} \circ (\Theta^{-1}a_H^2(m_{(-1)}) \cdot a_N(n))_{(-1)}) \cdot a_P(p)
\]

\[
\circ a_H^2(m_{(-1)}n_{(-1)}) \cdot a_N(n)_{(0)} \circ a_M(m_{(0)X_{(0)}})
\]

and the proof is complete. \( \square \)

Now we will state the main result in this section.

**Theorem 3.8.** Let \((H, a_H)\) be a Hopf-Hopf algebra with a bijective antipode \( S \) and a bijective \( a_H, \Theta \in \text{Aut} H(H) \).

Then the \( \Theta \)-Yetter-Drinfeld category \( \mathcal{H}(\mathcal{Y}, \mathcal{D})^\theta \) is a braided tensor category, with tensor product \( \otimes \), associativity constraints \( a_{\ldots, \ldots} \), braiding \( c_{\ldots, \ldots} \) and the unit defined as follows: for all \((M, a_M), (N, a_N), (P, a_P) \in \mathcal{H}(\mathcal{Y}, \mathcal{D})^\theta \),

(i) The tensor product \( \otimes = \otimes \), and the left \( H \)-Hom-module \( H \otimes (M \otimes N) \rightarrow M \otimes N \) and the left \( H \)-comodule \( M \otimes N \rightarrow H \otimes (M \otimes N) \) are defined, respectively, by

\[
h \otimes (m \otimes n) \mapsto h_1 \cdot m \otimes h_2 \cdot n,
\]

\[
p(m \otimes n) \mapsto a_H^2(m_{(-1)}n_{(-1)}) \otimes m_{(0)} \otimes n_{(0)};
\]

(ii) \( a_{M,N,P} : (M \otimes N) \otimes P \rightarrow M \otimes (N \otimes P) \) is given by

\[
(m \otimes n) \otimes p \mapsto a_H^2(m) \otimes (n \otimes a_P(p));
\]

(iii) \( c_{M,N} : M \otimes N \rightarrow N \otimes M \) is defined by

\[
m \otimes n \mapsto \Theta^{-1}(m_{(-1)}) \cdot a_N^3(n) \otimes a_M^3(m_{(0)});
\]

(iv) \( c_{M,N}^{-1} : N \otimes M \rightarrow M \otimes N \) is given by

\[
n \otimes m \mapsto a_M^3(m_{(0)}) \otimes S^{-1}(\Theta^{-1}(m_{(-1)})) \cdot a_N^3(n);
\]

(v) the unit is defined by \( I = (k, id_k) \).

**Proof.** (Sketch of the proof) By the definition of a braided tensor category, we need to investigate many conditions by a large number of computations. In fact, we have to prove that \( M \otimes N \) is also a \( \Theta \)-Yetter-Drinfeld module, \( a_{M,N,P} \) is an isomorphism of left \( H \)-Hom-modules and left \( H \)-comodules, \( c_{M,N} \) is
a morphism of left $H$-Hom-modules and left $H$-Hom-comodules, $c_{M,N}^2 \ast c_{M,N} = id_{M \otimes N}$, the pentagon axiom and hexagonal relation are satisfied, and so on. We think that it is a good exercise to a reader. For example, now we will prove the $\Theta$-Yetter-Drinfeld compatibility condition (3.1) for $M \otimes N$. Indeed, given $m \in M$ and $n \in N$, simple computations yield

\[
(h_1 \cdot (m \otimes n)) \Theta(h_2) \otimes (h_1 \cdot (m \otimes n)) = a_M^2((h_1 \cdot m)_{-1}(h_1 \cdot n)_{-1} \Theta(h_2)) \otimes (h_1 \cdot m)_{(0)} \otimes (h_1 \cdot n)_{(0)}
\]

\[
= a_M^2((a_M(h_1) \cdot m_{(1)}a_M(h_2)) \otimes (a_M(h_1) \cdot m)_{(0)} \otimes a_M(h_2 \cdot n_{(0)})
\]

\[
= a_M^2((a_M(h_1) \cdot m_{(1)} \Theta(h_2)) \otimes (h_1 \cdot m)_{(0)} \otimes a_M(h_2 \cdot n)_{(0)}
\]

\[
= a_M^2((a_M(h_1) \cdot m_{(1)} \Theta(h_2)) \otimes (h_1 \cdot m)_{(0)} \otimes a_M(h_2 \cdot n)_{(0)}
\]

\[
= (h_1 \cdot (m \otimes n)) \Theta(h_2) \otimes (h_1 \cdot (m \otimes n))
\]

Let us choose another part of the proof: the $H$-linearity of $c_{M,N}$ can be obtained by

\[
c_{M,N}(h \cdot (m \otimes n)) = \Theta^{-1}((h_1 \cdot m)_{-1}) \cdot a_M(h_2 \cdot n) \otimes a_M^{-1}((h_1 \cdot m)_{(0)})
\]

\[
= \Theta^{-1}((h_1 \cdot m)_{-1}) \cdot a_M(h_2 \cdot n) \otimes a_M^{-1}((h_1 \cdot m)_{(0)})
\]

\[
= \Theta^{-1}((h_1 \cdot m)_{-1}) \cdot a_M(h_2 \cdot n) \otimes a_M^{-1}((h_1 \cdot m)_{(0)})
\]

\[
= h \cdot (\Theta^{-1}((m \otimes n) \cdot a_M^{-1}(n))) \otimes (h_2 \cdot a_M^{-1}(m_{(0)})) = h \cdot c_{M,N}(m \otimes n),
\]

for all $h \in H$, $m \in M$ and $n \in N$. \hfill $\Box$

**Remark 3.9.** Theorem 4.4 in [14] can be regarded as a special case of Theorem 3.8 if we take $\Theta = a^2$. Also, if we take $\Theta = a^{-2}$, then Theorem 4.7 in [27] is just a special case of this theorem. Furthermore, if we take $\Theta = a^{2v-2}$, $v \in \mathbb{Z}$, Theorem 3.8 is the same as Theorem 2.12 in [28].

By Theorem 3.8, we know that the $\Theta$-Yetter-Drinfeld category $\mathcal{D}^\Theta$ is a braided tensor category. Therefore, we can obtain the following corollary.

**Corollary 3.10.** Let $(H, a_H)$ be a Hom-Hopf algebra with a bijective antipode $S$. Assume $(M, a_M), (N, a_N)$ and $(P, a_P) \in \mathcal{D}$. Then the braiding $\mathcal{C}$ satisfies the braid relation

\[
(id_P \otimes c_{M,N}) \ast (c_{M,P} \otimes id_N) \ast (id_M \otimes c_{N,P}) = (c_{N,P} \otimes id_M) \ast (id_N \otimes c_{M,P}) \ast (id_P \otimes c_{M,N}).
\]

**Remark 3.11.** This implies that $c_{M,N}$ is a solution of the Yang-Baxter equation for any object $M$ of a $\Theta$-Yetter-Drinfeld category $\mathcal{D}^\Theta$. By Proposition 3.6 and Theorem 3.8, we find an interesting fact that every $\Theta$-Yetter-Drinfeld category $\mathcal{D}^\Theta$ can give not only a solution of the Hom-Yang-Baxter equation but also a solution of the classical Yang-Baxter equation.

### 4 Symmetric pairs of $\Theta$-Yetter-Drinfeld categories over a Hom-Hopf algebra

By [30], we know that the Yetter-Drinfeld category over a Hopf algebra $H$ is symmetric if and only if $H = k$. In this section, we extend some interesting results to $\Theta$-Yetter-Drinfeld categories $\mathcal{D}^\Theta$, e.g., symmetric
\(\Theta\)-Yetter-Drinfeld categories \(H^H_HYD^\Theta\) over a Hom-Hopf algebra are all trivial. The results obtained in this section generalize the corresponding results in [28–31].

**Definition 4.1.** Let \((M, \alpha_M)\) and \((N, \alpha_N)\) be two objects in a \(\Theta\)-Yetter-Drinfeld category \(H^H_HYD^\Theta\). The pair \((M, N)\) is called a symmetric pair if \(c_{N,M}c_{M,N} = \text{id}_{M\otimes N}\). Particularly, the category \(H^H_HYD^\Theta\) is called symmetric whenever any pair of objects \(M, N \in H^H_HYD^\Theta\) is a symmetric pair.

**Theorem 4.2.** Let \((H, a)\) be a Hom-Hopf algebra, then \((H_2, H_1)\) is a symmetric pair if and only if \(H = k\).

**Proof.** By Example 3.3, \(H_1 = (H, \triangleright, \Delta, a_H)\) and \(H_2 = (H, \mu, \lambda, a_H)\) are two v-Yetter Drinfeld modules. On setting \(h \in H_2 \otimes H_1\), we have

\[
c_{H_1,H_2}(1_H \otimes h) = \Theta^{-1}(1_{H(-1)}) \triangleright a_H^{-1}(h) \otimes a_H^{-1}(1_{H(0)}) = 1_H \triangleright a_H^{-1}(h) \otimes 1_H \overset{(2.9)}{=} h \otimes 1_H.
\]

Furthermore, we have

\[
c_{H_1,H_2}(1_H \otimes h) = c_{H_1,H_2}(h \otimes 1_H)
= \Theta^{-1}(h_{(-1)}) \cdot a_H^{-1}(1_H) \otimes a_H^{-1}(h_{(0)})
= \Theta^{-1}(h) \cdot a_H^{-1}(1_H) \otimes a_H^{-1}(h_2)
= \Theta^{-1}(a_H(h_1) \otimes a_H^{-1}(h_2)).
\]

Since \((H_1, H_2)\) is a symmetric pair, i.e., \(c_{H_1,H_2}c_{H_2,H_1} = \text{id}_{H_2 \otimes H_1}\), it follows that \(1_H \otimes h = \Theta^{-1}a_H(h_1) \otimes a_H^{-1}(h_2)\). Applying \(id \otimes \epsilon\) to the above equality gives

\[
1_H \otimes \epsilon(h) = 1_H \epsilon(h) = \Theta^{-1}a_H(h_1) \otimes \epsilon(a_H^{-1}(h_2)) \overset{(2.3)}{=} \Theta^{-1}a_H(h_1) \otimes \epsilon(h_2) \overset{(2.4)}{=} \Theta^{-1}a_H(h)
\]

and it then follows that \(h = \Theta a_H^{-2}(\epsilon(h)1_H) = \epsilon(h)1_H\). This means \(H = k\), as desired. The converse is straightforward. \(\Box\)

The following corollary is easily obtained from the above theorem.

**Corollary 4.3.** Let \((H, a)\) be a Hom-Hopf algebra such that the \(\Theta\)-Yetter-Drinfeld category \(H^H_HYD^\Theta\) is symmetric. Then \(H = k\).

**Remark 4.4.**

1. If \(a = \text{id}\) and \(\Theta = \text{id}\), Theorem 4.2 is exactly the famous conclusion in [30], namely, the symmetric Yetter-Drinfeld category \(H^H_HYD\) over a Hopf algebra is trivial.

2. If \(\Theta = a^{-2}\), Theorem 4.2 is just Theorem 3.5 in [31]. Also, if \(\Theta = a^{2v-2}, v \in \mathbb{Z}\), Theorem 4.2 is just Theorem 3.2 in [28].

3. The special case of Theorem 4.2 has been considered in the setting of monoidal Hom-Hopf algebras; see [26].

**Theorem 4.5.** Let \((H, a)\) be a Hom-Hopf algebra. Then \((H_2, H_1)\) is a symmetric pair if and only if \(H = k\).

**Proof.** Let \(h \otimes 1_H \in H_1 \otimes H_2\). Then we obtain

\[
c_{H_2,H_1}(h \otimes 1_H) = \Theta^{-1}(h_{(-1)}) \cdot a_H^{-1}(1_H) \otimes a_H^{-1}(h_{(0)})
= \Theta^{-1}(h_1) \cdot 1_H \otimes a_H^{-1}(h_2)
= \Theta^{-1}a_H(h_1) \otimes a_H^{-1}(h_2)
\]

and also...
where the sixth equality uses (2.3) and the property of the antipode S. Since \((H_1, H_2)\) is a symmetric pair, i.e., \(\psi_{H_1 H_2} = id_{H_1 H_2}\), it follows that \(H = k\) is equals to

\[
\Theta(H_1 H_2) \Theta^{-1}(H_1 H_2)(\Theta^{-1} a_{H_2}^2(h_{12})) = (\Theta a_{H_2}^2((\Theta^{-1} a_{H_2}^2(h_{12}))) \Theta^{-1} a_{H_2}^2(h_{12})) \Theta^{-1} a_{H_2}^2(h_{12})
\]

Applying \(\varepsilon \otimes id\) to the above equality yields \(\varepsilon(h_{12}) = \Theta^{-1} a_{H_2}^2(h_{12})\). It then follows that \(n = \Theta a_{H_2}^2(\varepsilon(h_{12})_{12}) = \varepsilon(h_{12}).\) This means \(H = k\), as required. The converse is obvious. \(\blacksquare\)

**Remark 4.6.** Note that Corollary 4.3 can also be obtained by Theorem 4.5.

## 5 Pseudosymmetry of \(\Theta\)-Yetter-Drinfeld categories over a Hom-Hopf algebra

In this section, we will find a necessary and sufficient condition for a \(\Theta\)-Yetter-Drinfeld category \(H_{\mathcal{Y} D \Theta}\) over a Hom-Hopf algebra \((H, S, a)\) to be pseudosymmetric. The main theorem obtained in this section generalizes the main conclusions in [28,30,31].

**Definition 5.1.** [34,35] Let \(C\) be a tensor category and \(c\) a braiding on \(C\). The braiding \(c\) is called a **pseudosymmetry** if, for all \(M, N, P \in C\), there holds the following condition:

\[
(id_M \otimes c_{M,N})(c_{P,M} \otimes id_N)(id_M \otimes c_{N,P}) = (c_{N,P} \otimes id_M)(id_N \otimes c_{P,M})(c_{M,N} \otimes id_P). \tag{5.1}
\]

In this case, \(C\) is called a **pseudosymmetric braided tensor category**.

Obviously, a symmetric braided tensor category must be a pseudosymmetric braided tensor category.

**Lemma 5.2.** Let \((H, S, a)\) be a cocommutative Hom-Hopf algebra. Then, the canonical braiding of the \(\Theta\)-Yetter-Drinfeld category \(c_{H_{\mathcal{Y}D \Theta}}\) is the usual flip map.

**Proof.** For all \(x \in H_2\) and \(y \in H_1\), we have

\[
c_{H_{\mathcal{Y}D \Theta}}(x \otimes y) = \Theta(x_{(-1)}) \triangleright a_{H_2}^2(x_{(0)})
\]

\[
= \Theta^{-1}(\Theta^{-1}(a_{H_2}^2)(x_{(2)}) \triangleright a_{H_2}^2(x_{(1)}) \otimes a_{H_2}^2(x_{(0)})
\]

\[
= \Theta^{-1}(\Theta a_{H_2}^2(x_{(1)}) \otimes a_{H_2}^2(x_{(2)}))
\]

\[
= (a_{H_2}^2(x_{(1)}) \otimes a_{H_2}^2(x_{(2)})) \cdot a_{H_2}^2(x_{(1)}) \otimes a_{H_2}^2(x_{(2)})
\]

\[
\overset{(2.4)}{=} (a_{H_2}^2(x_{(1)}) \otimes a_{H_2}^2(x_{(2)})) \cdot a_{H_2}^2(x_{(1)}) \otimes a_{H_2}^2(x_{(2)})
\]

\[
\overset{(2.7)}{=} 1 \cdot a_{H_2}^2(x_{(1)}) \otimes a_{H_2}^2(x_{(2)})
\]

where the fourth equality uses cocommutativity. \(\blacksquare\)
Lemma 5.3. Let \((H, S, \alpha)\) be a Hom-Hopf algebra. Then, the canonical braiding of \(\Theta\)-Yetter-Drinfeld category satisfies \((\varepsilon \otimes \text{id})_{c_{H, H}}(x \otimes y) = \varepsilon(y)x\) for all \(x \in H_2\) and \(y \in H_1\).

**Proof.** For all \(x \in H_2\), \(y \in H_1\), we calculate

\[
(\varepsilon \otimes \text{id})_{c_{H, H}}(x \otimes y) = (\varepsilon \otimes \text{id})(a_H^{-2\varepsilon+3}(x_{-1}) \cdot a_H^{\varepsilon-1}(y) \otimes a_H^{\varepsilon-1}(x_{(0)}))
\]

\[
= \Theta^{-1}(\Theta a_H^{-3}(x_1) \Theta a_H^{3}(S(x_2))) \cdot a_H^{\varepsilon-1}(y) \otimes a_H^{\varepsilon-1}(x_2)
\]

\[
= (\varepsilon \otimes \text{id})(a_H^{-3}(x_1) \cdot a_H^{3}(S(x_2))) \cdot a_H^{\varepsilon-1}(y) \otimes a_H^{\varepsilon-1}(x_2)
\]

\[
= (\varepsilon \otimes \text{id})(\Theta a_H^{3}(x_1) \Theta a_H^{-3}(S(x_2)))a_H^{\varepsilon-1}(y))
\]

\[
\times S(\Theta a_H^{-3}(x_1) \Theta a_H^{3}(S(x_2))) \cdot a_H^{\varepsilon-1}(x_2)
\]

\[
= \varepsilon(x_1) \varepsilon(x_2) \varepsilon(y) a_H^{-2\varepsilon+3}(x_{-1}) \otimes a_H^{\varepsilon-1}(x_2) = \varepsilon(y)x,
\]

completing the proof. \(\square\)

**Theorem 5.4.** Let \((H, a_H)\) be a Hom-Hopf algebra with a bijective antipode \(S\). Then, the canonical braiding of the \(\Theta\)-Yetter-Drinfeld category \(\mathcal{H}_{H}^{\Theta} Y \mathcal{D}^{\Theta}\) is pseudosymmetric if and only if \((H, a_H)\) is commutative and cocommutative.

**Proof.** \((\Leftarrow)\) Assume that \((H, a_H)\) is commutative and cocommutative. We first prove that the compatibility (3.2) becomes the equality

\[
(h \cdot m)_{(1-1)} \otimes (h \cdot m)_{(0)} = a_H(m_{(1)}) \otimes a_H(h) \cdot m_{(0)}
\]

for all \(h \in H\) and \(m \in M\). Indeed, routine manipulations give

\[
(h \cdot m)_{(1-1)} \otimes (h \cdot m)_{(0)} = ((\Theta a_H^{-3}(h_1) a_H^{3}(m_{(1)}))) a_H^{3}(S(h_2)) \otimes a_H^{\varepsilon-1}(h_2) m_{(0)}
\]

\[
\overset{(2,6)}{=} ((\Theta a_H^{-3}(h_1) a_H^{3}(m_{(1)}))) a_H^{3}(S(h_2)) \otimes a_H^{\varepsilon-1}(h_2) m_{(0)}
\]

\[
= a_H^{-3}(m_{(1)}) (\Theta a_H^{3}(h_1)) a_H^{3}(S(h_2)) \otimes a_H^{\varepsilon-1}(h_2) m_{(0)}
\]

\[
\overset{(2,3)}{=} m_{(1)} ((\Theta a_H^{3}(h_1)) a_H^{3}(S(h_2)) \otimes a_H^{\varepsilon-1}(h_2) m_{(0)}
\]

\[
\overset{(6,4)}{=} m_{(1)} ((\Theta a_H^{3}(h_1)) a_H^{3}(S(h_2)) \otimes h_2 m_{(0)}
\]

\[
\overset{(2,7)}{=} m_{(1)} \cdot a_H^{-3}(1_H) \varepsilon(h_1) \otimes h_2 \cdot m_{(0)}
\]

\[
= a_H(m_{(1)}) \otimes a_H(h) \cdot m_{(0)},
\]

where we used commutativity and cocommutativity of \(H\) in the third equality.

Now we claim the equality (5.1). For this end, let \((M, a_M), (N, a_N)\) and \((P, a_P) \in \mathcal{H}_{H}^{\Theta} Y \mathcal{D}^{\Theta}\). Then, for all \(m \in M, n \in N, p \in P,\) we obtain that

\[
(id_P \otimes c_{M,N})(c_{P,M} \otimes id_N)(id_M \otimes c_{N,P})(m \otimes n \otimes p)
\]

\[
= (id_P \otimes c_{M,N})(c_{P,M} \otimes id_N)(m \otimes \Theta^{-1}(n_{-1}) \cdot a_P^{-1}(p) \otimes a_N^{-1}(n_{(0)}))
\]

\[
= (id_P \otimes c_{M,N})(c_{P,M} \otimes id_N)(m \otimes \Theta^{-1}(n_{-1}) \cdot a_P^{-1}(p) \otimes a_N^{-1}(n_{(0)}))
\]

\[
\overset{(5,2)}{=} (id_P \otimes c_{M,N})(c_{P,M} \otimes id_N)(m \otimes \Theta^{-1}(a_P(a_P^{-1}(p)))_{-1}) \cdot a_P^{-1}(m) \otimes a_N^{-1}(n_{(0)}))
\]

\[
= \Theta^{-1}(n_{-1}) \cdot a_P^{-2}(p_{(0)}) \otimes \Theta^{-1}(S^{-1} \Theta^{-1}(p_{-1}) a_M^{-2}(m))_{(1)} \cdot a_N^{-2}(n_{(0)}) \otimes a_M^{-2}(a_M^{\varepsilon-1}(m))_{(0)}
\]

\[
\overset{(2,8)}{=} \Theta^{-1}(n_{-1}) \cdot a_P^{-2}(p_{(0)}) \otimes \Theta^{-1}(a_P(a_P^{-1}(p)))_{-1} \cdot a_N^{-2}(n_{(0)}) \otimes \Theta^{-1}(S^{-1} \Theta^{-1}(p_{-1}) a_M^{-2}(m))_{(0)}
\]

\[
\overset{(10,2)}{=} \Theta^{-1}(n_{-1}) \cdot a_P^{-2}(p_{(0)}) \otimes \Theta^{-1}(a_P(a_P^{-1}(p)))_{-1} \cdot a_N^{-2}(n_{(0)}) \otimes \Theta^{-1}(S^{-1} \Theta^{-1}(p_{-1}) a_M^{-2}(m))_{(0)}
\]

and also
(c_{N,p} \otimes id_M)(id_N \otimes c_{N,M}^{-1})(c_{M,N} \otimes id_p)(m \otimes n \otimes p) \\
= (c_{N,p} \otimes id_M)(id_N \otimes c_{N,M}^{-1})(\Theta^{-1}(m_{-1}) \cdot a_M^{-1}(n) \otimes a_M^{-1}(m_{0}) \otimes p) \\
= (c_{N,p} \otimes id_M)(\Theta^{-1}(m_{-1}) \cdot a_M^{-1}(n) \otimes a_M^{-1}(p_{0}) \otimes S^{-1}\Theta^{-1}(p_{-1}) \cdot a_M^{-1}(m_{0})) \\
= \Theta^{-1}(1(\Theta^{-1}(m_{-1}) \cdot a_M^{-1}(n))_{-1} \cdot a_M^{-1}(p_{0}) \otimes a_M^{-1}((\Theta^{-1}(m_{-1}) \cdot a_M^{-1}(n))_{0}) \otimes S^{-1}\Theta^{-1}(p_{-1}) \cdot a_M^{-1}(m_{0})) \\
=(\Theta^{-1}(a_M^{-1}(n))_{(-1)}) \cdot a_M^{-1}(p_{0}) \otimes a_M^{-1}(a_M^{-1}(m_{-1}) \cdot a_M^{-1}(n))_{0}) \otimes S^{-1}\Theta^{-1}(p_{-1}) \cdot a_M^{-1}(m_{0})) \\
= \Theta^{-1}(n_{-1}) \cdot a_M^{-1}(p_{0}) \otimes \Theta^{-1}(m_{-1}) \cdot a_M^{-1}(n_{0}) \otimes S^{-1}\Theta^{-1}(p_{-1}) \cdot a_M^{-1}(m_{0}), \\
\text{as required.}

(\Rightarrow) \text{ Assume that the canonical braiding } c \text{ is pseudosymmetric. We will first show that } (H, a) \text{ is commutative. For this end, let } 1 \otimes \Theta a_H^{-2}(x) \otimes 1 \in H_2 \otimes H_1 \otimes H_2. \text{ Then we obtain}

(id \otimes \varepsilon \otimes id)(c_{H,H} \otimes id_H)(id_{H} \otimes c_{H,H}^{-1})(c_{H,H} \otimes id_H)(1 \otimes \Theta a_H^{-2}(x) \otimes 1) \\
= (id \otimes \varepsilon \otimes id)(c_{H,H} \otimes id_H)(id_{H} \otimes c_{H,H}^{-1})(\Theta a_H^{-2}(x) \otimes 1 \otimes 1) \\
= (id \otimes \varepsilon \otimes id)(c_{H,H} \otimes id_H)(\Theta a_H^{-2}(x) \otimes 1 \otimes 1) \\
= (id \otimes \varepsilon \otimes id)(a_H^{-1}(x) \otimes \Theta a_H^{-2}(x) \otimes 1) \\
\text{by 2.2 and 2.3. On the other hand, we have}

(id \otimes \varepsilon \otimes id)(c_{H,H} \otimes id_H)(c_{H,H}^{-1} \otimes id_H)(id_{H} \otimes c_{H,H})(1 \otimes \Theta a_H^{-2}(x) \otimes 1) \\
= (id \otimes \varepsilon \otimes id)(id_{H} \otimes c_{H,H})(\Theta a_H^{-2}(x) \otimes 1 \otimes 1) \\
= (id \otimes \varepsilon \otimes id)(a_H^{-1}(x) \otimes \Theta a_H^{-2}(x) \otimes 1) \\
\text{where the fourth equality follows by Lemma 5.3. So, we get}

x \otimes 1 = a_H^{-2}(x_{12}) \otimes a_H^{-2}(x_{0}) \otimes S^{-1}(a_H^{-2}(x_{11})) \\
\text{Furthermore, we have}

x_{2} \otimes x_{1} = x_{2} \otimes a_H^{-2}(1 \cdot x_{1}) \\
= a_H^{-2}(x_{12}) \otimes a_H^{-2}((a_H^{-2}(x_{2}) \otimes S^{-1}(a_H^{-2}(x_{11})))x_{1}) \\
\text{by 2.4 and 2.3.}

= a_H^{-2}(x_{12}) \otimes a_H^{-2}(a_H^{-2}(x_{2}) \otimes S^{-1}(a_H^{-2}(x_{11})))a_H^{-2}(x_{11})) \\
\text{Thus,}

= a_H^{-2}(x_{12}) \otimes S^{-1}(a_H^{-2}(x_{2}) \otimes S^{-1}(a_H^{-2}(x_{11})))a_H^{-2}(x_{11})) \\
\text{as required. Next we claim that } (H, a) \text{ is commutative. Indeed, given } 1 \otimes \Theta a_H^{-2}(x) \otimes \Theta a_H^{-2}(y) \in H_2 \otimes H_1 \otimes H_2, \text{ on one hand, we have}

(c_{H,H} \otimes id_H)(id_H \otimes c_{H,H}^{-1})(c_{H,H} \otimes id_H)(1 \otimes \Theta a_H^{-2}(x) \otimes \Theta a_H^{-2}(y)) \\
= (c_{H,H} \otimes id_H)(id_H \otimes c_{H,H}^{-1})(\Theta a_H^{-2}(x) \otimes 1 \otimes \Theta a_H^{-2}(y)) \\
= (c_{H,H} \otimes id_H)(\Theta a_H^{-2}(x) \otimes \Theta a_H^{-2}(y) \otimes S^{-1}(a_H^{-2}(y))) \\
= \Theta^{-1}(\Theta a_H^{-2}(x) \otimes \Theta a_H^{-2}(y)) \otimes \Theta a_H^{-2}(x) \otimes S^{-1}(a_H^{-2}(y)) \otimes S^{-1}(a_H^{-2}(y)) \\
= x \otimes x_{1} \otimes 1 \otimes 1 \otimes x \otimes x_{1} \otimes 1 \otimes 1 \otimes x \otimes x_{1},
= α_H^2(x_2) \cdot Θα_H^{-1}(y_2) \otimes Θα_H^{-1}(y_3) \otimes S^{-1}(α_H^{-1}(y_1))

= (Θα_H^2(α_H^{-1}(x_1))α_H^i(y_2))S(Θα_H^{-1}(α_H^{-1}(y_3))) \otimes Θα_H^{-1}(x_2) \otimes S^{-1}(α_H^{-1}(y_1))

= (Θα_H^{-1}(x_2)Θα_H^{-1}(y_3))S(Θα_H^{-1}(x_3)) \otimes Θα_H^{-1}(x_2) \otimes S^{-1}(α_H^{-1}(y_1))

where the first equality is followed by Lemma 5.2. On the other hand, we compute

\[
(id_H \otimes c_{H,H})(c_{H,H} \otimes id_H)(id_H \otimes c_{H,H})(id_H \otimes c_{H,H})(1 \otimes Θα_H^{-1}(x) \otimes Θα_H^{-1}(y))
\]

= (id_H \otimes c_{H,H})(c_{H,H} \otimes id_H)(1 \otimes α_H^{-1}(x) \cdot Θα_H^{-1}(y) \otimes Θα_H^{-1}(x))

= (id_H \otimes c_{H,H})(c_{H,H} \otimes id_H)(1 \otimes (Θα_H^{-1}(x_1)Θα_H^{-1}(y))S(Θα_H^{-1}(x_2))) \otimes Θα_H^{-1}(x_2))

= (id_H \otimes c_{H,H})((id_H \otimes c_{H,H})(Θα_H^{-1}(x_2))Θα_H^{-1}(y_2))S(Θα_H^{-1}(x_2)) \otimes Θα_H^{-1}(x_2) \otimes S^{-1}(α_H^{-1}(y_1)).

Now applying ε \otimes Θ^{-1}α_H^2 \otimes S to the above equality yields

\[
x \otimes y = α_H^{-1}(x_2) \otimes (α_H^{-1}(x_1)α_H^{-1}(y))Sα_H^{-1}(y_2).
\]

and then we obtain

\[
yx = ((α_H^{-1}(x_1)α_H^{-1}(y))Sα_H^{-1}(y_2))α_H^{-1}(y_2)
\]

\[
= (α_H^{-1}(x_1)α_H^{-1}(y))(Sα_H^{-1}(y_2)α_H^{-1}(x_2)) \quad (2.2)
\]

\[
= (α_H^{-1}(y)Sα_H^{-1}(y_2)α_H^{-1}(x_2)) \quad (2.4)
\]

\[
= (α_H^{-1}(x_2)α_H^{-1}(y))(Sα_H^{-1}(x_1)α_H^{-1}(x_2)) \quad (2.7)
\]

Therefore H is commutative, as desired. The proof is complete. ■

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References

[1] J. T. Hartwig, D. Larsson, and S. D. Silvestrov, Deformation of Lie algebras using α-derivations, J. Algebra 295 (2006), no. 2, 314–361, DOI: https://doi.org/10.1016/j.jalgebra.2005.07.036.
[2] D. Larsson and S. D. Silvestrov, Quasi-Hom-Lie algebras, central extensions and 2-cocycle-like identities, J. Algebra 288 (2005), no. 2, 321–344, DOI: https://doi.org/10.1016/j.jalgebra.2005.02.032.
[3] D. Larsson and S. D. Silvestrov, Quasi-Lie algebras, in: J. Fuchs et al. (eds.), Noncommutative Geometry and Representation Theory in Mathematical Physics, American Mathematical Society, Contemporary Mathematics, 2005, Vol. 391, pp. 241–248.
[4] D. Larsson and S. D. Silvestrov, Quasi-deformations of sl(f) using twisted derivations, Commun. Algebra 35 (2007), no. 12, 4303–4318, DOI: https://doi.org/10.1080/00927870701545127.
[5] F. Ammar and A. Makhlouf, Hom-Lie superalgebras and Hom-Lie admissible superalgebras, J. Algebra 324 (2010), no. 7, 1513–1528, DOI: https://doi.org/10.1016/j.jalgebra.2010.06.014.
[6] J. Arnlind, A. Makhlouf, and S. Silvestrov, Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebra, J. Math. Phys. 51 (2010), 043515, DOI: https://doi.org/10.1063/1.3359004.
[7] Y. Y. Chen, Y. Wang, and L. Y. Zhang, The construction of Hom-Lie bialgebras, J. Lie Theory 20 (2010), 1075–1089.
[8] Y. Y. Chen, Z. W. Wang, and L. Y. Zhang, Quasi-triangular Hom-Lie bialgebras, J. Lie Theory 22 (2012), no. 4, 767–783, DOI: https://doi.org/10.1111/j.1937-5956.2012.01393.x.
[9] Q. Q. Jin and X. C. Li, Hom-Lie algebra structures on semi-simple Lie algebras, J. Algebra 319 (2008), 1398–1408, DOI: https://doi.org/10.1016/j.jalgebra.2007.12.005.
[10] G. Sigurdsson and S. D. Silvestrov, Graded quasi-Lie algebras of Witt type, Czech. J. Phys. 56 (2006), 1287–1291, DOI: https://doi.org/10.1007/s10582-006-0439-1.
[11] D. Yau, Enveloping algebra of Hom-Lie algebras, J. Gen. Lie Theory Appl. 2 (2008), 95–108, DOI: https://doi.org/10.4303/jglta/S070209.
[12] D. Yau, The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras, J. Phys. A 42 (2009), 165202, DOI: https://doi.org/10.1088/1751-8113/42/16/165202.
[13] A. Gohr, On Hom-algebras with surjective twisting, J. Algebra 324 (2010), 1483–1491, DOI: https://doi.org/10.1016/j.jalgebra.2010.05.003.
[14] A. Makhlouf and F. Panaite, Yetter-Drinfeld modules for Hom-bialgebras, J. Math. Phys. 55 (2014), 013501, DOI: https://doi.org/10.1063/1.4858875.
[15] A. Makhlouf and F. Panaite, Hom-L-R-smash products, Hom-diagonal crossed products and the Drinfeld double of a Hom-Hopf algebra, J. Algebra 441 (2015), 314–343, DOI: https://doi.org/10.1016/j.jalgebra.2015.05.032.
[16] A. Makhlouf and F. Panaite, Twisting operators, twisted tensor products and smash products for Hom-associative algebras, J. Math. Glasgow 58 (2016), 513–538, DOI: https://doi.org/10.1017/S0017089515000294.
[17] A. Makhlouf and S. D. Silvestrov, Hom-algebras structures, J. Gen. Lie Theory Appl. 2 (2008), 51–64, DOI: https://doi.org/10.4303/jglta/S070206.
[18] A. Makhlouf and S. D. Silvestrov, Hom-algebras and Hom-coalgebras, J. Algebra Appl. 9 (2010), no. 4, 553–589, DOI: https://doi.org/10.1142/S0219464910004117.
[19] A. Makhlouf and S. D. Silvestrov, Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras, in: S. Silvestrov et al. (eds.), Generalized Lie Theory in Mathematics, Physics and Beyond, Springer-Verlag, Berlin, 2009.
[20] D. Yau, Hom-algebras and homology, J. Lie Theory 19 (2009), 409–421.
[21] D. Yau, Hom-quantum groups III: Representations and module Hom-algebras, 2009, arXiv:0911.5402.
[22] D. Yau, Hom-quantum group I; Quasi-triangular Hom-bialgebras, J. Phys. A: Math. Theor. 45 (2012), no. 6, 065203, DOI: https://doi.org/10.1088/1751-8113/45/6/065203.
[23] S. Caenepeel and I. Goyvaerts, Monoidal Hom-Hopf algebras, Commun. Algebra 39 (2011), no. 6, 2216–2240, DOI: https://doi.org/10.1080/00927872.2010.490800.
[24] Y. Y. Chen and L. Y. Zhang, The category of Yetter-Drinfel'd Hom-modules and the quantum Hom-Yang-Baxter equation, J. Math. Phys. 55 (2014), 031702, DOI: https://doi.org/10.1063/1.4868964.
[25] L. Liu and B. L. Sheng, Radford's biproducts and Yetter-Drinfeld modules for monoidal Hom-Hopf algebras, J. Math. Phys. 55 (2014), 031701, DOI: https://doi.org/10.1063/1.4866760.
[26] S. Wang and S. Guo, Symmetries and the u-condition in Hom-Yetter-Drinfeld categories, J. Math. Phys. 55 (2014), 081708, DOI: https://doi.org/10.1063/1.4892081.
[27] H. Li and T. Ma, A construction of the Hom-Yetter-Drinfeld category, Colloquium Mathematicum 137 (2014), 43–65, DOI: https://doi.org/10.4064/cm137-1-4.
[28] X. L. Fang, T. H. Kim, and X. H. Zhang, Symmetry and pseudosymmetry of v-Yetter-Drinfeld categories for Hom-Hopf algebras, Int. J. Geom. Methods Mod. Phys. 9 (2017), 1750129, DOI: https://doi.org/10.1142/S0219887417501298.
[29] M. Cohen and S. Westreich, Determinants and symmetries in Yetter-Drinfeld categories, Appl. Categ. Structures 6 (1998), 267–289, DOI: https://doi.org/10.1023/A:1008668314522.
[30] B. Pareigis, Symmetric Yetter-Drinfeld categories are trivial, J. Pure Appl. Algebra 155 (2001), no. 1, 91–91, DOI: https://doi.org/10.1016/S0022-4049(99)00089-4.
[31] S. Wang and S. Guo, Symmetric pairs and pseudosymmetries in Hom-Yetter-Drinfeld categories, J. Algebra Appl. 16 (2017), no. 7, 1750125, DOI: https://doi.org/10.1142/S0219498817501250.
[32] D. Yau, Module Hom-algebras, 2008, arXiv:0812.4695.
[33] D. Yau, Hom-bialgebras and comodule Hom-algebras, Int. Electron. J. Algebra 8 (2010), 45–64.
[34] A. Joyal and R. Street, Braided tensor categories, Adv. Math. 102 (1993), no. 1, 20–78, DOI: https://doi.org/10.1006/aima.1993.1055.
[35] F. Panaite, M. D. Staic, and F. V. Oystaeyen, Pseudosymmetric braiding, twines and twisted algebras, J. Pure Appl. Algebra 214 (2010), no. 6, 867–884, DOI: https://doi.org/10.1016/j.jpaa.2009.08.008.