Measure of tripartite quantum entanglement in multi-particle system of two-level atoms

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Abstract

We propose a direct measure of tripartite quantum entanglement in an arbitrary symmetric pure state of \(N\) correlated two-level atoms (qubits). We compute the third order moments of the collective pseudo-spin operators in terms of the individual atomic operators of the atoms in the assembly and find that all the bipartite quantum correlation terms among the atoms cancel out leaving only the all possible tripartite quantum correlation terms among them. We observe that the third order moments are made up of solely the tripartite quantum correlations among the atoms. This helps to extract out only the tripartite quantum correlations among the \(N\) atoms and propose a measure and quantification of these correlations. We also propose the necessary and sufficient condition for the presence of tripartite quantum entanglement in such multi-atomic systems. We conjecture the way of determining the third order moments of the pseudo-spin operators of the two-level atoms in the assembly, experimentally.

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I. INTRODUCTION

Quantum entanglement in multi-particle systems is an important topic of research over the last two decades. A lot of work has been done in this direction [1-25]. The works of Wootters [27, 28], Peres [30] and Horodecki [29], and several other researchers have given a strong understanding of quantum entanglement in bipartite states [26]. But, quantum entanglement in multi-particle systems is yet to be completely understood.

In this paper we consider an arbitrary symmetric pure state of \(N\) correlated two-level atoms (qubits) and calculate the third order moments of the collective pseudo-spin operators in mutually perpendicular directions in a plane perpendicular to the collective mean pseudo-spin vector. We find that all possible bipartite correlation terms among
the atoms cancel out. We are left with only the tripartite correlation terms. This helps to extract out the tripartite correlation terms among the \( N \) atoms and propose a measure of tripartite quantum correlations among the atoms.

The computation of higher order moments and study of higher order squeezing have already been discussed in the context of degenerate parametric down-conversion, harmonic generation and resonance fluorescence from an atom in Ref. \([31]\). In the above mentioned three situations the higher order moments have been computed for the electric field operators of the radiation field. In this paper we calculate third order moments of the spin operators for the atoms and use it to quantify tripartite entanglement. The motivation for the calculation of third order moments also lies on the fact that, third order moments solely reveal the tripartite quantum correlations among the atoms.

The organization of the paper is as follows. In Sec. II we briefly describe a two-level atom, pseudo-spin operators and moments of the pseudo-spin operators in mutually perpendicular directions in a plane perpendicular to the the mean pseudo-spin vector. In Sec. III we present the detail computation of these moments for three two-level atoms and quantify the amount of tripartite quantum entanglement present in such system. In Sec. IV we extend these ideas in the context of \( N \) two-level atoms. In Sec. V we conjecture the way of determining the third order moments experimentally. In Sec. VI we present the summary and conclusion.

II. TWO-LEVEL ATOM, PSEUDO-SPIN OPERATORS AND THIRD ORDER MOMENTS

An atom has many electronic energy levels, but when it interacts with an external monochromatic electromagnetic field, the atom makes a transition from one of its energy level to the other. In this case, we mainly concentrate on those two energy levels and hence the atom is called as a two-level atom. We consider a system of \( N \) such two-level atoms. Now, if among the assembly of \( N \) such two-level atoms, the \( n \)-th atom has the upper and lower energy levels, denoted as \( |u_n\rangle \) and \( |l_n\rangle \), respectively, then, we can construct the pseudo-spin operators (with \( \hbar = 1 \)),

\[
\hat{J}_{nx} = (1/2)(|u_n\rangle\langle l_n| + |l_n\rangle\langle u_n|),
\]

\[
\hat{J}_{ny} = (-i/2)(|u_n\rangle\langle l_n| - |l_n\rangle\langle u_n|),
\]

\[
\hat{J}_{nz} = (1/2)(|u_n\rangle\langle u_n| - |l_n\rangle\langle l_n|),
\]

such that,

\[
[\hat{J}_{nx}, \hat{J}_{ny}] = i\hat{J}_{nz},
\]

and two more relations with cyclic changes in \( x, y \) and \( z \) \([34]\). For the entire system of \( N \)
two-level atoms, we have collective pseudo-spin operators,

\[ \hat{J}_x = \sum_{i=1}^{N} \hat{J}_{ix}, \quad \hat{J}_y = \sum_{i=1}^{N} \hat{J}_{iy}, \quad \hat{J}_z = \sum_{i=1}^{N} \hat{J}_{iz}. \]  

(5)

The individual atomic operators satisfy

\[ [\hat{J}_{1x}, \hat{J}_{2y}] = 0, \quad [\hat{J}_{1x}, \hat{J}_{1y}] = i\hat{J}_{1z}, \]

\[ [\hat{J}_{2x}, \hat{J}_{2y}] = i\hat{J}_{2z}, \ldots \]  

(6)

As a consequence of these commutation relations, the collective pseudo-spin operators \( \hat{J}_x, \hat{J}_y \) and \( \hat{J}_z \) satisfy,

\[ [\hat{J}_x, \hat{J}_y] = i\hat{J}_z \]

(7)

and two more relations with cyclic changes in \( x, y \) and \( z \).

The simultaneous eigenvectors of \( \hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \) and \( \hat{J}_z \) are denoted as \( |j, m\rangle \)

where

\[ \hat{J}^2 |j, m\rangle = j(j + 1) |j, m\rangle \]  

(8)

and

\[ \hat{J}_z |j, m\rangle = m |j, m\rangle. \]  

(9)

The quantum number \( j \) is related to the number of atoms \( N \) as \( j = N/2 \) and \( m = -j, -j+1, ..., (j-1), j \). The collective quantum state vector for a system of \( N \) two-level atoms can be expressed as a linear superposition of \( |j, m\rangle \) as

\[ |\psi_j\rangle = \sum_{m=-j}^{j} c_m |j, m\rangle. \]  

(10)

In this paper we calculate the third order moments of the collective pseudo-spin operators in two mutually perpendicular directions in a plane perpendicular to the mean collective pseudo-spin vector. The motivation of doing so lies on the fact that while investigating spin squeezing in such systems, we calculate the second order moments

\[ \Delta J_{x,y}^2 = \langle \psi_j | \hat{J}_{x,y}^2 | \psi_j \rangle - \langle \psi_j | \hat{J}_{x,y} | \psi_j \rangle^2 \]  

(11)

in a plane perpendicular to the mean spin vector

\[ \langle \hat{J} \rangle = \langle \hat{J}_x \rangle \hat{i} + \langle \hat{J}_y \rangle \hat{j} + \langle \hat{J}_z \rangle \hat{k}, \]  

(12)

where \( \hat{i}, \hat{j} \) and \( \hat{k} \) are the unit vectors along positive \( x, y \) and \( z \) axes respectively. This is because only in that plane the quantum fluctuations in Eqs. (11) bring out the original quantum correlations among the atoms.

Now, the mean pseudo-spin vector \( \langle \hat{J} \rangle \) points in an arbitrary direction in space. Therefore, we conventionally rotate the co-ordinate system \( \{x, y, z\} \) to \( \{x', y', z'\} \), such that \( \langle \hat{J} \rangle \) points along the \( z' \) axis and calculate the second order moments in \( \hat{J}_{x'} \) and \( \hat{J}_{y'} \) for the state \( |\psi_j\rangle \). These moments are

\[ \Delta J_{x',y'}^2 = \langle \psi_j | \hat{J}_{x',y'}^2 | \psi_j \rangle - \langle \psi_j | \hat{J}_{x',y'} | \psi_j \rangle^2. \]  

(13)

Now, in the \( x' - y' \) plane

\[ \langle \psi_j | \hat{J}_{x'} | \psi_j \rangle = \langle \psi_j | \hat{J}_{y'} | \psi_j \rangle = 0, \]  

(14)

and hence, these moments reduce to

\[ \Delta J_{x',y'}^2 = \langle \psi_j | \hat{J}_{x',y'}^2 | \psi_j \rangle. \]  

(15)
The third order moments are defined as

$$\Delta J_{x',y'}^3 = \langle \psi_j | (\hat{J}_{x',y'} - \langle \hat{J}_{x',y'} \rangle)^3 | \psi_j \rangle,$$  

which due to Eq. (14) reduces to

$$\Delta J_{x',y'}^3 = \langle \psi_j | \hat{J}_{x',y'}^3 | \psi_j \rangle.$$  

A collective state vector $|\alpha\rangle$ for a system of $N$ atoms is said to be quantum mechanically correlated if $|\alpha\rangle$ cannot be expressed as a product of the individual atomic state vectors, i.e.,

$$|\alpha\rangle \neq |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \ldots |\alpha_N\rangle,$$  

where $|\alpha_1\rangle$, $|\alpha_2\rangle$,...,$|\alpha_N\rangle$ are the state vectors of the $N$ individual atoms.

We, now, proceed to calculate the third order moments $\Delta J_{x',y'}^3$ for a system of three two-level atoms and define a parameter to quantify the amount of tripartite quantum entanglement in such system.

The collective pseudo-spin operators, from Eqs. (5), are

$$\hat{J}_{x',y',z'} = \hat{J}_{1x,y,z} + \hat{J}_{2x,y,z} + \hat{J}_{3x,y,z}.$$  

Now, the mean spin vector $\langle \hat{J} \rangle$ points in an arbitrary direction in space. So, assuming that it lies in the first octant of the coordinate system, we perform a rotation of the coordinate system from $\{x, y, z\}$ to $\{x', y', z'\}$, such that $\langle \hat{J} \rangle$ points along the $z'$ axis. The operators $\{\hat{J}_{x'}, \hat{J}_{y'}, \hat{J}_{z'}\}$ in the rotated frame $\{x', y', z'\}$ are related to $\{\hat{J}_x, \hat{J}_y, \hat{J}_z\}$ in the unrotated frame $\{x, y, z\}$ as

$$\hat{J}_{x'} = \hat{J}_x \cos \theta \cos \phi + \hat{J}_y \cos \theta \sin \phi$$

$$- \hat{J}_z \sin \theta,$$  

$$\hat{J}_{y'} = - \hat{J}_x \sin \phi + \hat{J}_y \cos \phi,$$  

$$\hat{J}_{z'} = \hat{J}_x \sin \theta \cos \phi + \hat{J}_y \sin \theta \sin \phi$$

$$+ \hat{J}_z \cos \theta,$$  

III. TRIPARTITE QUANTUM EN-TANGLEMENT IN A SYSTEM OF THREE TWO-LEVEL ATOMS

We consider a system of three two-level atoms. A symmetric quantum state vector for such a system in the $\{m_1, m_2, m_3\}$ representation can be written as

$$|\psi_3\rangle = C_1 \frac{1}{\sqrt{3}} \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right] + \frac{1}{\sqrt{3}} \left[ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right]$$

$$+ \left[ -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right] + C_4 \left[ -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right].$$  

(19)
where,

\[ \cos \theta = \frac{\langle \hat{J}_z \rangle}{|\langle \mathbf{J} \rangle|} \]  

\[ \cos \phi = \frac{\langle \hat{J}_x \rangle}{\sqrt{\langle \hat{J}_x \rangle^2 + \langle \hat{J}_y \rangle^2}} \]  

\[ \cos \theta \text{ and } \cos \phi, \text{ we have } \langle \hat{J}_{x'} \rangle = 0, \langle \hat{J}_{y'} \rangle = 0, \]

and the mean spin vector points along the \( z' \) axis.

We can observe that for the above choice of

\[ \hat{J}_x^3 = \hat{J}_x^3 \cos^3 \theta \cos^3 \phi + \hat{J}_x^2 \hat{J}_y \cos^3 \theta \sin \phi \cos^2 \phi - \hat{J}_x^2 \hat{J}_z \sin \theta \cos^2 \theta \cos^2 \phi + \]

\[ \hat{J}_y \hat{J}_x \cos^2 \theta \sin^2 \phi \cos \phi + \hat{J}_x^3 \cos^3 \phi - \hat{J}_y \hat{J}_x \hat{J}_z \sin \theta \cos^2 \theta \cos \phi + \]

\[ \hat{J}_z \hat{J}_y \hat{J}_x \sin \theta \cos \phi - \hat{J}_z \hat{J}_y \hat{J}_z \sin \theta \cos \phi \]

We, now, express the above expression in terms of the individual atomic operators \( \hat{J}_{x,y,z} \), \( \hat{J}_{2x,y,z} \), and \( \hat{J}_{3x,y,z} \). The motivation behind this is to find out how the individual atoms are correlated with each other. For that we use the following results.

\[ \hat{J}_{nx,y,z}^2 = 1/4, \quad \hat{J}_{n+1x,y,z}^3 = (1/4) \hat{J}_{n+1x,y,z} \]  

\[ \hat{J}_{nx} \hat{J}_{ny} = i/2 \hat{J}_{nz}, \quad \hat{J}_{ny} \hat{J}_{nz} = i/2 \hat{J}_{nx} \]  

\[ \hat{J}_{nx} \hat{J}_{nx} = i/2 \hat{J}_{ny} \]  

Using Eqs.(26), (27), (28) and the fact that \( \hat{J}_{nx}, \hat{J}_{ny}, \text{ and } \hat{J}_{nz} \) anticommute with each other, we evaluate all the operator parts of Eq. (25a) separately and the results are as shown below.

\[ \hat{J}_x^3 = \frac{7}{4} \left( \hat{J}_{1x} + \hat{J}_{2x} + \hat{J}_{3x} \right) + 6 \hat{J}_{1x} \hat{J}_{2x} \hat{J}_{3x} \]  

\[ \hat{J}_x^2 \hat{J}_y = \frac{3}{4} \left( \hat{J}_{1y} + \hat{J}_{2y} + \hat{J}_{3y} \right) + i \hat{J}_{1x} \hat{J}_{2x} + i \hat{J}_{1x} \hat{J}_{2x} + \]

\[ + i \hat{J}_{1x} \hat{J}_{3x} + i \hat{J}_{1x} \hat{J}_{3x} + i \hat{J}_{2x} \hat{J}_{3x} + i \hat{J}_{2x} \hat{J}_{3x} + \]

\[ 2 \hat{J}_{1x} \hat{J}_{2x} \hat{J}_{3x} + 2 \hat{J}_{1x} \hat{J}_{2y} \hat{J}_{3x} + 2 \hat{J}_{1y} \hat{J}_{2x} \hat{J}_{3x} \]  

\[ \hat{J}_x^2 \hat{J}_z = \frac{3}{4} \left( \hat{J}_{1x} + \hat{J}_{2x} + \hat{J}_{3x} \right) - i \hat{J}_{1y} \hat{J}_{2x} - i \hat{J}_{1x} \hat{J}_{2y} - \]

\[ - i \hat{J}_{1x} \hat{J}_{3x} - i \hat{J}_{1x} \hat{J}_{3y} - i \hat{J}_{2y} \hat{J}_{3x} - i \hat{J}_{2x} \hat{J}_{3y} + \]

\[ 2 \hat{J}_{1x} \hat{J}_{2x} \hat{J}_{3x} + 2 \hat{J}_{1x} \hat{J}_{2x} \hat{J}_{3x} + 2 \hat{J}_{1x} \hat{J}_{2x} \hat{J}_{3x}, \]  

(31)
\[
\begin{align*}
\hat{J}_x^2 \hat{J}_x &= \frac{3}{4} \left( \hat{J}_{1x} + \hat{J}_{2x} + \hat{J}_{3x} \right) - i \hat{J}_{1y} \hat{J}_{2y} \\
&- i \hat{J}_{1y} \hat{J}_{2y} - i \hat{J}_{1z} \hat{J}_{3y} - i \hat{J}_{1y} \hat{J}_{3z} \\
&- i \hat{J}_{2x} \hat{J}_{3y} - i \hat{J}_{2y} \hat{J}_{3x} + 2 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3x} \\
&+ 2 \hat{J}_{1y} \hat{J}_{2x} \hat{J}_{3y} + 2 \hat{J}_{1z} \hat{J}_{2y} \hat{J}_{3y}, \\
\hat{J}_x^3 &= \frac{7}{4} \left( \hat{J}_{1y} + \hat{J}_{2y} + \hat{J}_{3y} \right) + 6 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3y}, \\
\hat{J}_y^2 \hat{J}_z &= \frac{3}{4} \left( \hat{J}_{1y} + \hat{J}_{2y} + \hat{J}_{3y} \right) + i \hat{J}_{1y} \hat{J}_{2y} \\
&+ i \hat{J}_{1y} \hat{J}_{2y} + i \hat{J}_{1z} \hat{J}_{3y} + i \hat{J}_{1y} \hat{J}_{3z} \\
&+ i \hat{J}_{2y} \hat{J}_{3y} + i \hat{J}_{2y} \hat{J}_{3z} + 2 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3z} \\
&+ 2 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3z} + 2 \hat{J}_{1z} \hat{J}_{2y} \hat{J}_{3z}, \\
\hat{J}_z^2 \hat{J}_z &= \frac{3}{4} \left( \hat{J}_{1z} + \hat{J}_{2z} + \hat{J}_{3z} \right) + i \hat{J}_{1z} \hat{J}_{2z} \\
&+ i \hat{J}_{1z} \hat{J}_{2z} + i \hat{J}_{1z} \hat{J}_{3z} + i \hat{J}_{1z} \hat{J}_{2z} \\
&+ i \hat{J}_{2z} \hat{J}_{3z} + i \hat{J}_{2z} \hat{J}_{3z} + 2 \hat{J}_{1z} \hat{J}_{2z} \hat{J}_{3z} \\
&+ 2 \hat{J}_{1z} \hat{J}_{2z} \hat{J}_{3z} + 2 \hat{J}_{1z} \hat{J}_{2z} \hat{J}_{3z}, \\
\hat{J}_y^3 &= \frac{7}{4} \left( \hat{J}_{1y} + \hat{J}_{2y} + \hat{J}_{3y} \right) + 6 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3y},
\end{align*}
\]
\[
\begin{align*}
\hat{J}_y \hat{J}_x \hat{J}_z &= -\frac{3}{8} - \hat{J}_1 \hat{J}_2 - \hat{J}_1 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \\
&\quad - i \hat{J}_1 \hat{J}_2 - i \hat{J}_1 \hat{J}_3 - i \hat{J}_1 \hat{J}_2 \\
&\quad - i \hat{J}_2 \hat{J}_3 - i \hat{J}_2 \hat{J}_3 - i \hat{J}_1 \hat{J}_2 \\
&\quad + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 \\
&\quad + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3,
\end{align*}
\]

(43)

\[
\begin{align*}
\hat{J}_x \hat{J}_z \hat{J}_x &= \frac{1}{4} \left( \hat{J}_1 + \hat{J}_2 + \hat{J}_3 \right) + 2 \hat{J}_1 \hat{J}_2 \hat{J}_3 \\
&\quad + 2 \hat{J}_1 \hat{J}_2 \hat{J}_3 + 2 \hat{J}_1 \hat{J}_2 \hat{J}_3
\end{align*}
\]

(44)

\[
\begin{align*}
\hat{J}_x \hat{J}_x \hat{J}_x &= \frac{3}{4} \left( \hat{J}_1 + \hat{J}_2 + \hat{J}_3 \right) + i \hat{J}_1 \hat{J}_2 \\
&\quad + i \hat{J}_1 \hat{J}_3 + i \hat{J}_1 \hat{J}_3 + i \hat{J}_2 \hat{J}_3 \\
&\quad + i \hat{J}_1 \hat{J}_3 + i \hat{J}_2 \hat{J}_3 + 2 \hat{J}_1 \hat{J}_2 \hat{J}_3 \\
&\quad + 2 \hat{J}_1 \hat{J}_2 \hat{J}_3 + 2 \hat{J}_1 \hat{J}_2 \hat{J}_3
\end{align*}
\]

(45)

\[
\begin{align*}
\hat{J}_y \hat{J}_x \hat{J}_y &= -i \frac{3}{8} - i \hat{J}_1 \hat{J}_2 - i \hat{J}_1 \hat{J}_3 + i \hat{J}_1 \hat{J}_2 \\
&\quad - i \hat{J}_1 \hat{J}_2 + i \hat{J}_1 \hat{J}_3 - i \hat{J}_1 \hat{J}_2 \\
&\quad - i \hat{J}_2 \hat{J}_3 + i \hat{J}_2 \hat{J}_3 - i \hat{J}_1 \hat{J}_2 \\
&\quad + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 \\
&\quad + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3,
\end{align*}
\]

(46)

\[
\begin{align*}
\hat{J}_x \hat{J}_x \hat{J}_y &= \frac{3}{8} + i \hat{J}_1 \hat{J}_2 + i \hat{J}_1 \hat{J}_3 - i \hat{J}_1 \hat{J}_2 \\
&\quad + i \hat{J}_1 \hat{J}_2 - i \hat{J}_1 \hat{J}_3 + i \hat{J}_1 \hat{J}_2 \\
&\quad + i \hat{J}_2 \hat{J}_3 - i \hat{J}_2 \hat{J}_3 + i \hat{J}_1 \hat{J}_2 \\
&\quad + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 \\
&\quad + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3
\end{align*}
\]

(51)

\[
\begin{align*}
\hat{J}_y \hat{J}_x \hat{J}_y &= \frac{3}{8} + i \hat{J}_1 \hat{J}_2 + i \hat{J}_1 \hat{J}_3 - i \hat{J}_1 \hat{J}_2 \\
&\quad + i \hat{J}_1 \hat{J}_2 - i \hat{J}_1 \hat{J}_3 + i \hat{J}_1 \hat{J}_2 \\
&\quad + i \hat{J}_2 \hat{J}_3 - i \hat{J}_2 \hat{J}_3 + i \hat{J}_1 \hat{J}_2 \\
&\quad + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 \\
&\quad + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3 + \hat{J}_1 \hat{J}_2 \hat{J}_3
\end{align*}
\]

(50)
\[ \hat{J}_y \hat{J}_z \hat{J}_y = \frac{1}{4} \left( \hat{J}_{1x} + \hat{J}_{2x} + \hat{J}_{3x} \right) + 2 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3y} \]
\[ + 2 \hat{J}_{1x} \hat{J}_{2x} \hat{J}_{3y} + 2 \hat{J}_{1x} \hat{J}_{2y} \hat{J}_{3y} \]
\[ + \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3y} \]  \hspace{1cm} (52)

\[ \hat{J}_z \hat{J}_y^2 = \frac{3}{4} \left( \hat{J}_{1x} + \hat{J}_{2x} + \hat{J}_{3x} \right) - i \hat{J}_{1x} \hat{J}_{2y} \]
\[ - i \hat{J}_{1x} \hat{J}_{3y} - i \hat{J}_{1y} \hat{J}_{2x} - i \hat{J}_{2x} \hat{J}_{3y} \]
\[ - i \hat{J}_{1y} \hat{J}_{3x} + i \hat{J}_{2y} \hat{J}_{3x} + 2 \hat{J}_{1x} \hat{J}_{2y} \hat{J}_{3y} \]
\[ + 2 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3y} + 2 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3y} \]
\[ + 2 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3y} \]  \hspace{1cm} (53)

\[ \hat{J}_y \hat{J}_z^2 = \frac{3}{4} \left( \hat{J}_{1x} + \hat{J}_{2x} + \hat{J}_{3x} \right) + i \hat{J}_{1x} \hat{J}_{2z} \]
\[ + i \hat{J}_{1x} \hat{J}_{3z} + i \hat{J}_{1y} \hat{J}_{2x} + i \hat{J}_{2x} \hat{J}_{3z} \]
\[ + i \hat{J}_{1y} \hat{J}_{3x} + i \hat{J}_{2y} \hat{J}_{3x} + 2 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3z} \]
\[ + 2 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3z} + 2 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3z} \]
\[ + 2 \hat{J}_{1y} \hat{J}_{2y} \hat{J}_{3z} \]  \hspace{1cm} (54)

We see from the above results that, the operators in Eq. (25a) are made up of individual atomic operators, bipartite correlation terms and tripartite correlation terms. Using Eqs. (29-53) in Eq. (25a), we see that all the bipartite correlation terms cancel and we are left with only the individual atomic operators and tripartite correlation terms. The final expression of Eq. (25a) reduces to

\[ \hat{J}_z^3 = \frac{i}{4} \left( \hat{J}_{1x} + \hat{J}_{2x} + \hat{J}_{3x} \right) + \sum_{p \neq q \neq r}^3 \left( \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \cos^3 \theta \cos^3 \phi + \hat{J}_{p} \hat{J}_{q} \hat{J}_{r} \cos^3 \theta \sin^3 \phi - \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \sin^3 \theta - 6 \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \sin \theta \cos^2 \theta \sin \phi \cos \phi + 3 \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \sin \phi \cos \phi \right) \]
\[ + 3 \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \sin \theta \cos^2 \phi + 3 \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \cos^3 \theta \sin^2 \phi \cos \phi \]
\[ + 3 \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \sin^2 \theta \cos \phi \cos \phi + 3 \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \sin^2 \theta \cos \phi \cos \phi \]  \hspace{1cm} (55a)

The first term on the right hand side is \( \frac{\hat{J}_z^3}{3} \) whose expectation value over \( |\psi_3\rangle \) in this coordinate frame, that is, \((x', y', z')\) is zero. Therefore, we write

\[ \Delta \hat{J}_x^3 = \langle \hat{J}_x^3 \rangle \]
\[ = \sum_{p \neq q \neq r}^3 \left( \langle \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \rangle \cos^3 \theta \cos^3 \phi + \langle \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \rangle \sin^3 \phi - 6 \langle \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \rangle \sin \theta \cos^2 \theta \sin \phi \cos \phi + 3 \langle \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \rangle \sin \phi \cos \phi \right) \]
\[ + 3 \langle \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \rangle \sin \theta \cos^2 \phi + 3 \langle \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \rangle \cos^3 \theta \sin^2 \phi \cos \phi \]
\[ + 3 \langle \hat{J}_{pz} \hat{J}_{qz} \hat{J}_{rz} \rangle \sin^2 \theta \cos \phi \cos \phi \]
Now, using the expressions of $\cos \theta$, $\cos \phi$ and $\sin \theta$ and $\sin \phi$ obtained from Eqs. (24) and (25), Eq. (55b) takes the form

$$3\langle \hat{J}_{p_z}\hat{J}_{q_x}\hat{J}_{r_y}\rangle \cos^3 \theta \sin \phi \cos^2 \phi - 3\langle \hat{J}_{p_x}\hat{J}_{q_y}\hat{J}_{r_z}\rangle \sin \theta \cos^2 \theta \cos^2 \phi + 3\langle \hat{J}_{p_x}\hat{J}_{q_x}\hat{J}_{r_z}\rangle \sin^2 \theta \cos \phi + 3\langle \hat{J}_{p_y}\hat{J}_{q_x}\hat{J}_{r_z}\rangle \sin^2 \theta \cos \theta \sin \phi\right).$$

(55b)

The terms $\langle \hat{J}_{p_z}\hat{J}_{q_x}\hat{J}_{r_y}\rangle$, $\langle \hat{J}_{p_y}\hat{J}_{q_y}\hat{J}_{r_z}\rangle$, $\ldots$, $\langle \hat{J}_{p_y}\hat{J}_{q_x}\hat{J}_{r_z}\rangle$ in the above equation are the tripartite correlation terms.

Similarly, we can calculate the third order moment of $J'_{y'\prime}$. Using Eq. (22), (24), (25), (26), (27), (28) and the fact that $\hat{J}_{n_x}$, $\hat{J}_{n_y}$, and $\hat{J}_{n_z}$ anticommute with each other, we obtain the third order moment in $J'_{y'\prime}$ as

$$\Delta J^3_{x'\prime} = \frac{1}{[\langle \hat{J}_x \rangle^2 + \langle \hat{J}_y \rangle^2]^{3/2}} \sum_{p,q,r=1}^{3} \frac{3}{p \neq q \neq r} \left( \langle \hat{J}_{p_x}\hat{J}_{q_x}\hat{J}_{r_y}\rangle \langle \hat{J}_x \rangle^3 \langle \hat{J}_y \rangle^3 + \langle \hat{J}_{p_y}\hat{J}_{q_y}\hat{J}_{r_z}\rangle \langle \hat{J}_z \rangle^3 \langle \hat{J}_y \rangle^3 - \langle \hat{J}_{p_x}\hat{J}_{q_y}\hat{J}_{r_z}\rangle \langle \hat{J}_x \rangle^6 + \langle \hat{J}_y \rangle^6 + 3\langle \hat{J}_x \rangle^4 \langle \hat{J}_y \rangle^2 + 3\langle \hat{J}_x \rangle^2 \langle \hat{J}_y \rangle^4 \right) - 6\langle \hat{J}_{p_x}\hat{J}_{q_y}\hat{J}_{r_z}\rangle \langle \hat{J}_x \rangle^2 \langle \hat{J}_y \rangle^2 - 3\langle \hat{J}_{p_x}\hat{J}_{q_y}\hat{J}_{r_z}\rangle \langle \hat{J}_x \rangle^2 \langle \hat{J}_y \rangle^2 + \langle \hat{J}_x \rangle^2 \langle \hat{J}_z \rangle^2 \langle \hat{J}_y \rangle^2 + 3\langle \hat{J}_{p_y}\hat{J}_{q_y}\hat{J}_{r_z}\rangle \langle \hat{J}_x \rangle \langle \hat{J}_z \rangle \langle \hat{J}_y \rangle^3 - 3\langle \hat{J}_{p_y}\hat{J}_{q_y}\hat{J}_{r_z}\rangle \langle \hat{J}_x \rangle \langle \hat{J}_z \rangle \langle \hat{J}_y \rangle^3 + \langle \hat{J}_x \rangle^4 + \langle \hat{J}_y \rangle^4 + 2\langle \hat{J}_x \rangle^2 \langle \hat{J}_y \rangle^2 + 3\langle \hat{J}_{p_y}\hat{J}_{q_y}\hat{J}_{r_z}\rangle \langle \hat{J}_y \rangle^4 \right) + 2\langle \hat{J}_x \rangle^2 \langle \hat{J}_y \rangle^2\right).$$

(55c)

While calculating $\Delta J^3_{x'\prime}$, we find that all the bipartite correlation terms cancel out and it is solely made up of tripartite correlation terms like the previous case.

Since the quantum state vector in Eq. (19) is a symmetric state where all the atoms have been treated on equal footing, then, we have

$$\langle \hat{J}_{1_x} \rangle = \langle \hat{J}_{2_x} \rangle = \langle \hat{J}_{3_x} \rangle \quad (57)$$
$$\langle \hat{J}_{1_y} \rangle = \langle \hat{J}_{2_y} \rangle = \langle \hat{J}_{3_y} \rangle \quad (58)$$
$$\langle \hat{J}_{1_z} \rangle = \langle \hat{J}_{2_z} \rangle = \langle \hat{J}_{3_z} \rangle \quad (59)$$

In this case, using Eqs. (57), (58) and (59) in Eq. (55c) and (56), the third order moments of $\hat{J}_{x'\prime}$ and $\hat{J}_{y'\prime}$, reduce to
The relation terms on the right hand side of Eqs. (59a) and (60) vanish. This confirms that the third order moments of \( \hat{J}_{x'} \) and \( \hat{J}_{y'} \) are the direct measures of tripartite correlations among the atoms.

So, now, we construct a parameter that quantifies the amount of tripartite correlations in this system of three two-level atoms. Since \( \Delta J^3_{x'} \) and \( \Delta J^3_{y'} \) may be both positive or negative, and, to treat the tripartite correlations in both \( x' \) and \( y' \) quadratures on equal footing, we define the tripartite quantum entanglement parameter as the root mean squared value of \( \Delta J^3_{x'} \) and \( \Delta J^3_{y'} \) as

\[
S = \frac{1}{2} \left[ (\Delta J^3_{x'})^2 + (\Delta J^3_{y'})^2 \right]^{1/2}. \tag{71}
\]

Since, \( \Delta J^3_{x'} \) and \( \Delta J^3_{y'} \) are solely made up of tripartite quantum correlations only, we may take the numerical value of \( S \) as a measure of the amount of tripartite quantum entanglement present in the system. Whenever

\[
S = 0, \tag{72}
\]

Using Eqs. (61-70) and Eqs. (57-59) in Eqs. (59a) and (60), we observe that all the correlation terms on the right hand side of Eqs. (59a) and (60) vanish. This confirms that the third order moments of \( \hat{J}_{x'} \) and \( \hat{J}_{y'} \) are the direct measures of tripartite correlations among the atoms.
in the system.

The necessary and sufficient condition for the presence of tripartite quantum entanglement in the system is that,

\[ S > 0. \tag{73} \]

We prove it in this way. Whenever there is tripartite quantum entanglement in the system, the conditions in Eqs. \((61)\) to \((70)\) are not satisfied. This makes \(\Delta J^3_{x'}\) and \(\Delta J^3_{y'}\) non-zero, implying that \(S > 0\). So, \(S > 0\) is the necessary condition for the presence of tripartite quantum entanglement in the system. The condition \(S > 0\) is sufficient also for the presence of tripartite quantum entanglement in the system, because, if \(S > 0\), either \(\Delta J^3_{x'}\) or \(\Delta J^3_{y'}\) or both of them are non-zero. In this case all the conditions in Eqs. \((61)\) to \((70)\) are not satisfied and hence, there is tripartite quantum entanglement in the system.

We now proceed to extend these ideas for a system of \(N\) two-level atoms.

**IV. TRIPARTITE QUANTUM ENTANGLEMENT IN A SYSTEM OF \(N\) TWO-LEVEL ATOMS**

We consider \(N\) two-level atoms in an arbitrary symmetric pure state. The quantum state vector for this system in \(m_1, m_2, m_3, \ldots, m_N\) representation is

\[
|\Psi_N\rangle = G_1 \left| \frac{1}{2}, \frac{1}{2}, \ldots \right\rangle + \frac{G_2}{\sqrt{N C_t}} \left[ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots \right\rangle + \frac{1}{2}, \frac{1}{2}, \ldots \right\rangle + \ldots \left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots \right\rangle \right]
\]

where \(G_1, G_2, \ldots, G_{N+1}\) are constants and \(N C_t\) is given as

\[
N C_t = \frac{N!}{r!(N-r)!}. \tag{75}
\]

The third order moments in \(\hat{J}_{x'}\) and \(\hat{J}_{y'}\) for this state can be written in analogy to Eqs. \((55c)\) and \((56)\) as

\[
\Delta J^3_{x'} = \frac{1}{|\langle J | (\hat{J}_x)^2 + (\hat{J}_y)^2)^{3/2} | \sum_{p \neq q \neq r} \left( \langle \hat{J}_{p_x} \hat{J}_{q_x} \hat{J}_{r_x} \rangle \langle \hat{J}_x \rangle^3 \langle \hat{J}_x \rangle^3 + \langle \hat{J}_{p_y} \hat{J}_{q_y} \hat{J}_{r_y} \rangle \langle \hat{J}_y \rangle^3 \langle \hat{J}_y \rangle^3 - \langle \hat{J}_{p_x} \hat{J}_{q_x} \hat{J}_{r_x} \rangle \langle \hat{J}_x \rangle^3 \langle \hat{J}_x \rangle^3 \langle \hat{J}_y \rangle^3 \right) \right|}
\]
\begin{equation}
\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_x} \rangle \left[ (\langle \hat{J}_x \rangle)^6 + 6(\langle \hat{J}_x \rangle)^4(\langle \hat{J}_y \rangle)^2 + 3(\langle \hat{J}_x \rangle)^2(\langle \hat{J}_y \rangle)^4 - 6(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_x} \rangle)^2 + (\langle \hat{J}_y \rangle)^2 \right] = 6(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_x} \rangle)^2 + (\langle \hat{J}_y \rangle)^2 \langle \hat{J}_x \rangle^2 + 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle ^2) \langle \hat{J}_x \rangle^2(\langle \hat{J}_y \rangle)^2 - 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_x} \rangle)^2 + (\langle \hat{J}_y \rangle)^2 \langle \hat{J}_x \rangle^2 + 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle ^2) \langle \hat{J}_x \rangle^2(\langle \hat{J}_y \rangle)^2 - 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_x} \rangle)^2 + (\langle \hat{J}_y \rangle)^2 \langle \hat{J}_x \rangle^2 + 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle ^2) \langle \hat{J}_x \rangle^2(\langle \hat{J}_y \rangle)^2 - 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle ^2) \langle \hat{J}_x \rangle^2(\langle \hat{J}_y \rangle)^2 \right)
\end{equation}

for the state \(|\Psi_N\rangle\),

\[
\Delta J_{y}^3 = \frac{1}{[(\langle \hat{J}_x \rangle)^2 + (\langle \hat{J}_y \rangle)^2]^{3/2}} \sum_{p,q,r=1}^{N} \left( -\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_x} \rangle \langle \hat{J}_y \rangle \right)^3 + (\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle \langle \hat{J}_y \rangle)^3 \times (\langle \hat{J}_x \rangle)^2 + 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle \langle \hat{J}_y \rangle)^2 - 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_x} \rangle)^2 \langle \hat{J}_x \rangle^2(\langle \hat{J}_y \rangle)^2 + 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle \langle \hat{J}_y \rangle)^2 \langle \hat{J}_x \rangle^2(\langle \hat{J}_y \rangle)^2 - 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle \langle \hat{J}_y \rangle)^2 \langle \hat{J}_x \rangle^2(\langle \hat{J}_y \rangle)^2 \right)
\]

respectively, where the upper index 2 in the summations in Eqs. (55c) and (55) has been replaced by \(N\).

Therefore, using Eqs. (77), (78) and (79), we get

\[
\Delta J_{x'}^3 = \frac{1}{[(\langle \hat{J}_x \rangle)^2 + (\langle \hat{J}_y \rangle)^2]^{3/2}} \sum_{p,q,r=1}^{N} \left( -\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_x} \rangle \langle \hat{J}_y \rangle \right)^3 + (\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle \langle \hat{J}_y \rangle)^3 \times (\langle \hat{J}_x \rangle)^2 + 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle \langle \hat{J}_y \rangle)^2 - 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_x} \rangle)^2 \langle \hat{J}_x \rangle^2(\langle \hat{J}_y \rangle)^2 + 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle \langle \hat{J}_y \rangle)^2 \langle \hat{J}_x \rangle^2(\langle \hat{J}_y \rangle)^2 - 3(\langle \hat{J}_p, \hat{J}_q, \hat{J}_{r_y} \rangle \langle \hat{J}_y \rangle)^2 \langle \hat{J}_x \rangle^2(\langle \hat{J}_y \rangle)^2 \right)
\]

and, Now, if all the \(N\) atoms are uncorrelated, then, the conditions of Eqs. (61) to (70) are
satisfied, and, we get
\[ \Delta J_{x'}^3 = \Delta J_{y'}^3 = 0. \] (81)

Thus, for the system of \( N \) two-level atoms (qubits), we observe that the third order moments of \( \hat{J}_{x'} \) and \( \hat{J}_{y'} \) are directly related to the all possible tripartite correlations among the atoms.

In this case also, we define the measure of tripartite quantum entanglement among the atoms as the root mean squared value of \( \Delta J_{x'}^3 \) and \( \Delta J_{y'}^3 \) as
\[ S = \frac{1}{2} \left[ (\Delta J_{x'}^3)^2 + (\Delta J_{y'}^3)^2 \right]^{1/2}. \] (82)

Like our previous case the necessary and sufficient condition for the presence of tripartite quantum entanglement in this system of \( N \) two-level atoms is
\[ S > 0. \] (83)

V. CONJECTURE FOR DETERMINING THIRD ORDER MOMENTS BY EXPERIMENTAL TECHNIQUE

Computation of higher order moments and study of higher order squeezing has been described in Ref. [31] in the context of degenerate parametric down-conversion, harmonic generation and resonance fluorescence from an atom. In the above mentioned three situations the higher order moments have been computed for the electric field operators of the radiation field. Now, as far as we know, moments higher than second order of angular momentum or pseudo-spin operators for two-level atoms have not been determined experimentally yet. Below we describe the area of experiment where the third order moments of the angular momentum or pseudo-spin operators of two-level atoms can be determined.

In Ref. [32], Wineland et. al described population spectroscopy of \( N \) two-level atoms. They localize an ensemble of \( N \) identical two-level atoms in a trap. They denote the upper and lower energy levels of the atom as \( | + \frac{1}{2} \rangle \) and \( | - \frac{1}{2} \rangle \) respectively. They apply a classical radiation called the clock radiation of frequency \( \omega \) to the atoms. As a result an atom acquires a coherent superposition state \( C_1| + \frac{1}{2} \rangle + C_2| - \frac{1}{2} \rangle \). Then they detect the number of atoms in \( | + \frac{1}{2} \rangle \) state, which they denote as \( N_+ \). In the detection process each atom is projected either in state \( | + \frac{1}{2} \rangle \) or \( | - \frac{1}{2} \rangle \). So, there occurs a projection noise of the atoms. Due to the analogy between an individual two-level system interacting with radiation with that of the dynamics of a spin-half particle in a magnetic field [35], the quantum projection noise is proportional to \( \Delta J_z = \sqrt{\langle \hat{J}_z^2 \rangle - \langle \hat{J}_z \rangle^2} \), where \( \hat{J}_z \) is the collective angular momentum operator of \( N \) atoms, which we call as the pseudo-spin operator in Eq. (5) in our paper. By making measurements of the population in \( | + \frac{1}{2} \rangle \)
state $M$ times the average $(N_+)_M$ is calculated for various values of $\omega$. Consequently, a resonance curve is obtained as a function of $\omega$. For a particular value of $\omega$, the deviation of the apparent position of the curve from the true curve $\langle N_+ \rangle$ is given by
\[
\delta \omega_M = [(N_+)_M - \langle N_+ \rangle]/(\partial \langle N_+ \rangle/\partial \omega). \tag{84}
\]
The magnitude of the rms fluctuation of $\delta \omega$ for repeated measurements of $N_+$ at a particular value of $\omega$ is given by
\[
|\Delta \omega| = \Delta N_z(t_f)/|\partial \langle N_+ \rangle/\partial \omega|
= \Delta J_z(t_f)/|\partial \langle J_z(t_f) \partial \omega|, \tag{85}
\]
where $t_f$ is the "final" time corresponding to the time just after the clock radiation is applied. Then they describe population spectroscopy using Ramsey method. In Sec. V of that paper, where they describe spectroscopy of correlated particles, they discuss spin squeezing, thereby calculating the variance $\Delta J_\perp$ in a plane perpendicular to the mean pseudo-spin vector operator $\langle \hat{J} \rangle$. So they have determined the second order moments of the angular momentum or pseudo-spin operators $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ experimentally. Now, to calculate higher order squeezing one needs to calculate higher order moments. Also to calculate the skewness of the resonance curve of Sec. V of that paper one needs to calculate the third order moments of the angular momentum or pseudo-spin operators. Here we conjecture that with the same experimental technique as described in the paper of Wineland et. al the third order moments in the angular momentum or pseudo-spin operators can be determined, and as proposed by us we can calculate the amount of tripartite quantum entanglement present in the system.

VI. SUMMARY AND CONCLUSION

We proposed a measure of tripartite quantum entanglement present in an arbitrary symmetric pure state of $N$ two-level atoms. We calculated the third order moments of the collective pseudo-spin operators in mutually orthogonal directions in a plane perpendicular to the mean pseudo-spin vector operator $\langle \hat{J} \rangle$. We calculated the third order moments in terms of the individual atomic operators. We found that in the expressions of third order moments all the bipartite correlation terms cancel out. These moments depend solely on the tripartite quantum correlation terms among the atoms. This helped to construct a measure of tripartite quantum correlations present among the atoms. Since, the two third order moments $\Delta J^3_x$ and $\Delta J^3_y$ may be positive or negative, and, also to treat the two moments on equal footing, we defined the measure of tripartite quantum entanglement as the root mean squared value of the two moments. We also established the neces-
sary and sufficient condition for the presence of tripartite quantum entanglement in such multi-atomic systems. We conjecture the experiment where the third order moments of the angular momentum or pseudo-spin operators can be determined.

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