A Constant-Factor Approximation Algorithm for the
Asymmetric Traveling Salesman Problem

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Abstract

We give a constant-factor approximation algorithm for the asymmetric traveling salesman problem. Our approximation guarantee is analyzed with respect to the standard LP relaxation, and thus our result confirms the conjectured constant integrality gap of that relaxation.

Our techniques build upon the constant-factor approximation algorithm for the special case of node-weighted metrics. Specifically, we give a generic reduction to structured instances that resemble but are more general than those arising from node-weighted metrics. For those instances, we then solve Local-Connectivity ATSP, a problem known to be equivalent (in terms of constant-factor approximation) to the asymmetric traveling salesman problem.

Keywords: approximation algorithms, asymmetric traveling salesman problem, combinatorial optimization, linear programming

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*This is the full version of the paper. There is also a 10-page extended abstract available from the authors’ homepages, which gives an overview of the main ideas and techniques.

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1 Introduction

The traveling salesman problem — to find the shortest tour visiting \( n \) given cities — is one of the best-known NP-hard optimization problems.

Without any assumptions on the distances, a simple reduction from the problem of deciding whether a graph is Hamiltonian shows that it is NP-hard to approximate the shortest tour to within any factor. Therefore it is common to relax the problem by allowing the tour to visit cities more than once. This is equivalent to assuming that the distances satisfy the triangle inequality: the distance from city \( i \) to \( k \) is no larger than the distance from \( i \) to \( j \) plus the distance from \( j \) to \( k \). All results mentioned and proved in this paper refer to this setting.

If we further assume the distances to be symmetric, then Christofides’ classic algorithm from 1976 [Chr76] is guaranteed to find a tour of length at most \( \frac{3}{2} \) times the optimum. Improving this approximation guarantee is a notorious open question in approximation algorithms. There has been a flurry of recent progress in the special case when distances are given as unweighted shortest path metrics [GSS11, MS16, Muc12, SV14]. However, even though the standard linear programming (LP) relaxation is conjectured to approximate the optimum within a factor of \( \frac{4}{3} \), it remains an elusive problem to improve upon Christofides’ algorithm.

If we do not restrict ourselves to symmetric distances, we obtain the more general asymmetric traveling salesman problem (ATSP). Compared to the symmetric setting, the gap in our understanding is much larger, and the current algorithmic techniques have failed to give any constant approximation guarantee. This is intriguing especially since the standard LP relaxation, also known as the Held-Karp lower bound, is conjectured to approximate the optimum to within a small constant. In fact, it is only known that its integrality gap\(^1\) is at least 2 [CGK06].

The first approximation algorithm for ATSP was given by Frieze, Galbiati and Maffioli [FGM82], achieving an approximation guarantee of \( \log_2(n) \). Their elegant “repeated cycle cover” approach was refined in several papers [Blä08, KLSS05, FS07], but there was no asymptotic improvement in the approximation guarantee until the more recent \( O(\log n / \log \log n) \)-approximation algorithm by Asadpour et al. [AGM+10]. They introduced a new and influential approach to ATSP based on a connection to the graph-theoretic concept of thin spanning trees. This has further led to improved algorithms for special cases of ATSP, such as graphs of bounded genus [GS11]. Moreover, Anari and Oveis Gharan recently exploited this connection to significantly improve the best known upper bound on the integrality gap of the standard LP relaxation to \( O(\text{poly} \log \log n) \) [AG15]. This implies an efficient algorithm for estimating the optimal value of a tour within a factor \( O(\text{poly} \log \log n) \) but, as their arguments are non-constructive, no approximation algorithm for finding a tour of matching guarantee.

Around the same time, an alternative approach was introduced by Svensson [Sve15]. It reduces the task of approximating ATSP to a seemingly easier problem called Local-Connectivity ATSP. The paper [Sve15] also gave an algorithm for Local-

\(^{1}\)Recall that the integrality gap is defined as the maximum ratio between the optimum values of the exact (integer) formulation and of its relaxation.
Connectivity ATSP restricted to the special case of node-weighted metrics, implying a constant-factor approximation algorithm for that special case. We have generalized this to graphs with at most two different edge weights in subsequent work [STV16]. In this paper, we build upon and generalize both of these results to give a constant-factor approximation algorithm for all metrics.

**Theorem 1.1.** There is a polynomial-time algorithm for ATSP that returns a tour of value at most a constant times the Held-Karp lower bound.

We remark that we have not optimized the constant of the approximation guarantee, instead favoring simplicity. However, we believe that further developments are needed to get close to the lower bound of 2 on the integrality gap [CGK06] and the inapproximability of 75/74 [KLS13]. See the conclusions (Section 9) for further discussion and open problems.

**Brief overview of approach and outline of paper.** The paper [Sve15] has introduced the problem Local-Connectivity ATSP and showed that it is equivalent (in terms of constant-factor approximation) to the asymmetric traveling salesman problem. Further, it gave an efficient solution to Local-Connectivity ATSP for node-weighted graphs. In [STV16] we gave a solution for graphs with two different edge weights. This, however, turned out to be technically challenging. In fact, it is unclear if the same approach can be extended even to a fixed number of different edge weights.

In the current paper we take a different route. Instead of trying to directly tackle Local-Connectivity ATSP in arbitrary weighted graphs, the first part of our argument uses a sequence of natural reductions to reduce the problem of approximating ATSP in general to that of approximating ATSP on special, structured instances called vertebrate pairs. These instances enjoy properties that make them amenable for Local-Connectivity ATSP.

The reduction to structured instances is itself done in several steps that we now briefly explain. It will be convenient to define ATSP in terms of its graphic formulation:

**Definition 1.2.** The input for ATSP is a pair \((G, w)\), where \(G\) is a strongly connected directed graph (digraph) and \(w\) is a nonnegative weight function defined on the edges. The objective is to find a closed walk of minimum weight that visits every vertex at least once.

Without loss of generality, one could assume that \(G\) is a complete digraph. However, for our reductions, it will be important that \(G\) may not be complete. We also remark that a closed walk that visits every vertex at least once is equivalent to an Eulerian multiset of edges that connects the graph. (An edge set of a digraph is Eulerian if the in-degree of each vertex equals its out-degree.)

In the preliminaries (Section 2.2) we exploit the extreme point structure of the LP relaxation and of its dual to show that we can focus on laminarily-weighted ATSP instances: there is a laminar family \(\mathcal{L}\) of vertex sets and a non-negative vector \((y_S)_{S \in \mathcal{L}}\) such that any edge \(e\) has \(w(e) = \sum_{S \in \mathcal{L}, e \in \delta(S)} y_S\), where \(\delta(S)\) denotes the edges in the cut defined by \(S\). See the left part of Figure 1 for an example. This is a generalization of
node-weighted metrics, which correspond to the case when the laminar family only contains singleton sets.

We further explore the structure of sets in $\mathcal{L}$: in Section 3 we study paths inside sets $S \in \mathcal{L}$ and, in Section 4, we introduce analogs of the classic graph-theoretic operations of contracting and inducing on such a set. These operations naturally give rise to a recursive algorithm that, intuitively, works as long as the contraction of some set $S \in \mathcal{L}$ results in a significant decrease in the value of the LP relaxation. In Section 5 we formally analyze this recursive algorithm and reduce the task of approximating ATSP to that of approximating irreducible instances: those where no set $S \in \mathcal{L}$ brings about a significant decrease of the LP value if contracted.

Informally, every set $S \in \mathcal{L}$ in an irreducible instance has two vertices $u, v \in S$ such that the shortest path from $u$ to $v$ crosses a large (weighted) fraction of the sets $R \in \mathcal{L}$: $R \subseteq S$ (otherwise contracting $S$ into a single vertex, endowed with a node-weight equal to the weight of the shortest path, would lead to a decrease in the LP value). This insight, together with the approximation algorithm for node-weighted metrics in [Sve15], allows us to construct a low-weight subtour $B$ that does not necessarily visit every vertex but crosses every non-singleton set of $\mathcal{L}$. See the right part of Figure 1 for an example. We refer to $B$ as a backbone, and to the ATSP instance and the backbone together as a vertebrate pair. This reduction allows us to further assume that our input is such a vertebrate pair; it is presented in Section 6.

In each of the above stages, we prove a theorem of the form: if there is a constant-factor approximation for ATSP on more structured instances, then there is a constant-factor approximation for ATSP on less structured instances. For instance, an algorithm for irreducible instances implies an algorithm for laminarily-weighted instances. One can also think of making a stronger and stronger assumption on the instance without loss of generality, making it increasingly resemble a node-weighted metric.

Having reduced the original problem to vertebrate pairs, in Section 7 we give an algorithm for Local-Connectivity ATSP for such instances. The main technical ingredient of the argument is Lemma 7.3, where we use the idea of finding a minimum-weight circulation in an auxiliary “split graph”. This is inspired by the paper [STV16] on ATSP with two edge-weights, although the concepts are not identical, and (perhaps surprisingly) the proof for general metrics is simpler. By our reductions and the result in [Sve15] this then implies the main result, Theorem 1.1, as explained in Section 8.

2 Preliminaries

We let $\mathbb{R}_+$ denote the set of nonnegative real numbers. The support of a function $f : X \to \mathbb{R}_+$ is the subset $\{x \in X : f(x) > 0\}$. For a subset $Y \subseteq X$, we also use $f(Y) = \sum_{x \in Y} f(x)$.

For a digraph $G$, we let $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively. We simply use $V$ and $E$ whenever the graph is clear from the context. By an edge set $F \subseteq E$, we always mean an edge multiset: the same edge can be present in multiple copies. We will refer to the union of two edge (multi)sets as a multiset (adding up the multiplicities of every edge).

A closed walk is a subtour, and it is a tour if it visits every vertex at least once. As
Figure 1: On the left we give an example of a laminarly-weighted ATSP instance. The sets of the laminar family are shown in gray. We depict a single edge $e$ that crosses three sets in the laminar family, say $S_1$, $S_2$, $S_3$, and so $w(e) = y_{S_1} + y_{S_2} + y_{S_3}$. On the right, we give an example of a vertebrate pair. Notice that the backbone (depicted as the cycle) crosses all non-singleton sets of the laminar family, though it may not visit all the vertices.

mentioned in the introduction, a tour is equivalent to an Eulerian multiset of edges that connects the graph. Similarly, a subtour is equivalent to an Eulerian multiset of edges that form a single component. In the sequel we use this viewpoint.

For a vertex set $U \subseteq V$, we let $G[U]$ denote the subgraph induced by $U$. That is, $G[U]$ is the subgraph of $G$ whose vertex set is $U$ and whose edge set consists of all edges in $E(G)$ with both endpoints in $U$. We also let $G/U$ denote the graph obtained by contracting the vertex set $U$, i.e., by replacing the vertices in $U$ by a single new vertex $u$ and redirecting every edge with one endpoint in $U$ to the new vertex $u$. This may create parallel edges in $G/U$. We keep all parallel copies; thus, every edge in $G/U$ will have a unique preimage in $G$.

For vertex sets $S, T \subseteq V$ we let $\delta(S, T) = \{(u, v) \in E : u \in S \setminus T, v \in T \setminus S\}$. For a set $S \subseteq V$ we let $\delta^+(S) = \delta(S, V \setminus S)$ denote the set of outgoing edges, and we let $\delta^-(S) = \delta(V \setminus S, S)$ denote the set of incoming edges. Further, let $\delta(S) = \delta^-(S) \cup \delta^+(S)$. For a vertex $v \in V$ we let $\delta^+(v) = \delta^+([v])$ and $\delta^-(v) = \delta^-([v])$. For an edge (multi)set $F \subseteq E$, we use $\delta_F(S, T) = \delta(S, T) \cap F$, $\delta_F^+(S) = \delta^+(S) \cap F$, etc. Further, let $1_F$ denote the indicator vector of $F$, which has a coordinate for each edge $e$ with value equal to the multiplicity of $e$ in $F$.

For a set $S \subseteq V$ we let $S_{\text{in}}$ and $S_{\text{out}}$ be those vertices of $S$ that have an incoming edge from outside of $S$ and those that have an outgoing edge to outside of $S$, respectively. That is,

$$S_{\text{in}} = \{v \in S : \delta^-(S) \cap \delta^-(v) \neq \emptyset\}, \quad \text{and} \quad S_{\text{out}} = \{v \in S : \delta^+(S) \cap \delta^+(v) \neq \emptyset\}.$$

2.1 Held-Karp Relaxation

Given an edge-weighted digraph $(G, w)$, the Held-Karp relaxation has a variable $x(e) \geq 0$ for every edge $e \in E$. The intended solution is that $x(e)$ should equal the number of times $e$ is used in the solution. The linear programming relaxation $\text{LP}(G, w)$ is now
defined as follows:

\[
\begin{align*}
\text{minimize} & \quad \sum_{e \in E} w(e)x(e) \\
\text{subject to} & \quad x(\delta^+(v)) = x(\delta^-(v)) \quad \text{for } v \in V, \\
& \quad x(\delta(S)) \geq 2 \quad \text{for } \emptyset \neq S \subseteq V, \\
& \quad x \geq 0.
\end{align*}
\]

(LP(G, w))

The optimum value of this LP is called the Held-Karp lower bound. The first set of constraints says that the in-degree should equal the out-degree for each vertex, i.e., the solution should be Eulerian. We call a vector \(x\) satisfying these constraints an Eulerian vector. The second set of constraints enforces that the solution is connected. These are sometimes referred to as subtour elimination constraints. Notice that the Eulerian property implies \(x(\delta^- (S)) = x(\delta^+(S)) = 1\) for every set \(S \subseteq V\), and therefore these constraints are equivalent to \(x(\delta^+(S)) \geq 1\) for all \(\emptyset \neq S \subseteq V\), which appear more frequently in the literature. We use the above formulation as it enables some simplifications in the presentation.

We say that a set \(S \subseteq V\) is tight with respect to a solution \(x\) to LP(G, w) if \(x(\delta^- (S)) = 2\), that is, \(x(\delta^- (S)) = x(\delta^+(S)) = 1\).

Let us now formulate the dual linear program DUAL(G, w). We associate variables \((\alpha_v)_{v \in V}\) and \((y_S)_{\emptyset \neq S \subseteq V}\) with the first and second set of constraints of LP(G, w), respectively.

\[
\begin{align*}
\text{maximize} & \quad \sum_{\emptyset \neq S \subseteq V} 2 \cdot y_S \\
\text{subject to} & \quad \sum_{S : (u, v) \in \delta(S)} y_S + \alpha_u - \alpha_v \leq w(u, v) \quad \text{for } (u, v) \in E, \\
& \quad y \geq 0.
\end{align*}
\]

(DUAL(G, w))

For singleton sets \{u\}, we will also use the notation \(y_u = y_{\{u\}}\). We remark that although the Held-Karp relaxation has exponentially many constraints, it is well-known that we can solve it in polynomial time either using the ellipsoid method with a separation oracle, or by formulating an equivalent compact (polynomial-size) linear program.

An optimal solution to DUAL(G, w) can also be found in polynomial time. To see this, note that when solving LP(G, w), the ellipsoid method returns a polynomial number of constraints used as cutting planes. These provide a polynomial-size family \(\mathcal{S} \subseteq 2^V\) such that DUAL(G, w) has an optimal solution with support contained in \(\mathcal{S}\). Moreover, we show that one can efficiently obtain a dual optimal solution with a “nice” support, as stated below. Recall that a family \(\mathcal{L} \subseteq 2^V\) of vertex subsets is laminar if for any \(A, B \in \mathcal{L}\) we have either \(A \subseteq B\) or \(B \subseteq A\) or \(A \cap B = \emptyset\).

**Lemma 2.1.** For every edge-weighted digraph \((G, w)\) there exists an optimal solution \((\alpha, y)\) to DUAL(G, w) such that the support of \(y\) is a laminar family of vertex subsets. Moreover, such a solution can be computed in polynomial time.
Proof. We start by showing the existence of a laminar optimal solution using a standard uncrossing argument (see e.g. [CFN85] for an early application of this technique to the Held-Karp relaxation of the symmetric traveling salesman problem). Select \((\alpha, y)\) to be an optimal solution to DUAL\((G, w)\) minimizing \(\sum S |y_S|\). That is, among all dual solutions that maximize the dual objective \(2 \sum S y_S\), we select one that minimizes \(\sum S |S| y_S\). We claim that the support \(L = \{ S : y_S > 0 \}\) is a laminar family. Suppose not, i.e., that there are sets \(A, B \in L\) such that \(A \cap B, A \setminus B, B \setminus A \neq \emptyset\). Then we can obtain a new dual solution \((\alpha, \hat{y})\), where \(\hat{y}\) is defined, for \(\varepsilon = \min(y_A, y_B) > 0\), as
\[
\hat{y}_S = \begin{cases} 
  y_S - \varepsilon & \text{if } S = A \text{ or } S = B, \\
  y_S + \varepsilon & \text{if } S = A \setminus B \text{ or } S = B \setminus A, \\
  y_S & \text{otherwise.}
\end{cases}
\]
That \((\alpha, \hat{y})\) remains a feasible solution follows since \(\hat{y}\) remains non-negative (by the selection of \(\varepsilon\)) and since for any edge \(e\) we have
\[
1_{e \in \delta(A)} + 1_{e \in \delta(B)} \geq 1_{e \in \delta(A \setminus B)} + 1_{e \in \delta(B \setminus A)}.
\]
Therefore \(\sum_{S: e \in \delta(S)} \hat{y}_S \leq \sum_{S: e \in \delta(S)} y_S\) and so the constraint corresponding to edge \(e\) remains satisfied. Further, we clearly have \(2 \sum S \hat{y}_S = 2 \sum_S y_S\). In other words, \((\alpha, \hat{y})\) is an optimal dual solution. However,
\[
\sum S |S| (y_S - \hat{y}_S) = (|A| + |B| - |A \setminus B| - |B \setminus A|) \varepsilon > 0,
\]
which contradicts that \((\alpha, y)\) was selected to be an optimal dual solution minimizing \(\sum S |S| y_S\). Therefore, there can be no such sets \(A\) and \(B\) in \(L\), and so it is a laminar family.

To find a laminar optimal solution in polynomial time, we start with an arbitrary dual optimal solution. As noted above, one can be computed in polynomial time. Now we apply the above uncrossing operation to obtain a laminar optimal solution. A result by Karzanov [Kar96, Theorem 2] shows that if we carefully select the sequence of pairs \(A, B\) to uncross, this can be performed in polynomial time (although for an arbitrary sequence, the number of uncrossing steps may not be polynomially bounded). \(\square\)

### 2.2 Node-Weighted and Laminarily-Weighted ATSP

In [Sve15] a polynomial-time constant-factor approximation algorithm was given for node-weighted ATSP. A weighted digraph \((G, w)\) is called node-weighted if there exists a function \(h : V \to \mathbb{R}_+\) such that \(w(u, v) = h(u) + h(v)\) for every \((u, v) \in E\).\(^2\) The main result of [Sve15] is the following:

**Theorem 2.2.** For every \(\varepsilon > 0\) there is a polynomial-time algorithm that, for any node-weighted digraph \((G, w)\), returns a tour of weight at most \((2\varepsilon + 1)\times\text{the optimum value of } \text{LP}(G, w)\).

\(^2\) In [Sve15], the definition is slightly different: \(w(u, v) = f(u)\) for every \((u, v) \in E\) for a function \(f : V \to \mathbb{R}_+\). The two definitions are equivalent: by assigning \(f(u) = 2h(u)\), the weight of any tour is equal for the weights \(w(u, v) = f(u)\) and for the weights \(w(u, v) = h(u) + h(v)\).
We use $\alpha_{nw} = 27 + \varepsilon$ to refer to the approximation guarantee of this algorithm in order to exhibit the dependence of our results on this bound.

We now introduce a more complex class of cost functions, which takes solutions to LP($G, w$) and DUAL($G, w$) into account.

**Definition 2.3.** A tuple $I = (G, L, x, y)$ is called a laminarly-weighted ATSP instance if $G$ is a strongly connected digraph, $L$ is a laminar family of vertex subsets, $x$ is a feasible solution to LP($G, 0$), and $y : L \to \mathbb{R}_+$. We further require that $x_e > 0$ for every $e \in E$ and that every set $S \in L$ be tight with respect to $x$, i.e., that $x(\delta^+(S)) = x(\delta^-(S)) = 1$. We define the induced weight function $w_I : E \to \mathbb{R}_+$ as

$$w_I(e) = \sum_{S \in L : e \in \delta(S)} y_S \quad \text{for every } e \in E.$$

Given an instance $I$ as in the definition, the vectors $x$ and $y$ have the following important property. Define a dual solution $(\bar{\alpha}, \bar{y})$ by setting $\bar{\alpha}_u = 0$ for all $u \in V$, and $\bar{y}_S = y_S$ if $S \in L$ and $\bar{y}_S = 0$ otherwise. Then complementary slackness implies that for the induced weight function $w_I$, the vector $x$ is an optimal solution to LP($G, w_I$) and $(\bar{\alpha}, \bar{y})$ is an optimal solution to DUAL($G, w_I$).

Our first main insight is that ATSP with arbitrary weights can be reduced to the laminarly-weighted ATSP problem.

**Theorem 2.4.** Assume we have a polynomial-time algorithm that finds a solution of weight at most $\alpha$ times the Held-Karp lower bound for every laminarly-weighted ATSP instance. Then there is a polynomial-time algorithm for the general ATSP problem that finds a solution of weight at most $\alpha$ times the Held-Karp lower bound.

**Proof.** Consider an arbitrary edge-weighted strongly connected digraph $(G, w)$. Let $x$ be an optimal solution to LP($G, w$) and let $(\alpha, y)$ be an optimal solution to DUAL($G, w$) as guaranteed by Lemma 2.1, that is, $y$ has a laminar support $L$. We now define a pair $(G', w')$ as

$$V(G') = V(G), \quad E(G') = \{e \in E(G) : x(e) > 0\}, \quad \text{and} \quad w'(u, v) = w(u, v) - \alpha_u + \alpha_v.$$

We claim that $I = (G', L, x, y)$ is a laminarly-weighted ATSP instance whose induced weight function $w_I$ equals $w'$. To see this, recall that $x$ is a primal optimal solution and that $(\alpha, y)$ is a dual optimal solution (for $(G, w)$). Therefore complementary slackness implies that every set in $L$ is tight with respect to $x$ and that for every edge $(u, v) \in E(G')$, the weight $w'(u, v) = w(u, v) - \alpha_u + \alpha_v$ equals the sum of $y_S$-values for the sets $S$ crossed by $(u, v)$. Finally, we have $x_e > 0$ for every $e \in E(G')$ by definition. So $I$ satisfies all the properties of Definition 2.3, i.e., it is a laminarly-weighted instance.

We now argue that an $\alpha$-approximate solution for $I$ with respect to the Held-Karp relaxation LP($G', w'$) implies an $\alpha$-approximate solution for the original instance $(G, w)$ with respect to LP($G, w$). To this end, we make the following observation:

**Claim.** For any Eulerian vector $x \in \mathbb{R}_+^{E(G')}$, we have $\sum_{e \in E(G')} w'(e)x(e) = \sum_{e \in E(G')} w'(e)x(e)$.

Therefore the Held-Karp lower bound is the same for $(G, w)$ and for $(G', w')$, and any solution for $(G', w')$ is a solution of the same weight for $(G, w)$.

$\square$
Figure 2: (a) The structure of a tight set $S$ with strongly connected components $S_1, \ldots, S_\ell$. Every path traversing $S$ enters at a vertex in $S_{in} \subseteq S_1$, then visits all strongly connected components, which form a “path” structure, before it exits from a vertex in $S_{out} \subseteq S_\ell$.

(b) The structure of the path $P$ from $u \in S_{in}$ to $v \in S_{out}$ for a tight set $S$ as given by Lemma 3.2. The path crosses the set that contains $u$ but not $v$ once and it crosses the sets of $\mathcal{L}$ that are disjoint from $\{u, v\}$ at most twice.

In the rest of the paper we work exclusively with laminarly-weighted ATSP instances $I = (G, \mathcal{L}, x, y)$. We will refer to these as simply instances. Recall that $w_I(F)$ is the induced weight of an edge multiset $F \subseteq E$ in the instance $I$. We will omit the subscript and use simply $w(F)$ whenever $I$ is clear from the context. For a set $S \subseteq V$ we define

$$\text{value}_I(S) = 2 \cdot \sum_{R \in \mathcal{L} : R \subseteq S} y_R$$

to be the fractional dual value associated with the sets strictly inside $S$. Again, we will omit the subscript whenever clear from the context. We also use $\text{value}(I) = \text{value}_I(V)$; note that this equals the Held-Karp lower bound of the instance. Indeed, as noted above, $y$ can be extended to an optimal dual solution to $\text{DUAL}(G, w)$, and hence the optimum value for $\text{DUAL}(G, w)$ equals $2 \cdot \sum_{S \in \mathcal{L}} y_S$, which is equal to the primal optimum value $\sum_{e \in E} w(e)x(e)$ for $\text{LP}(G, w)$ by strong duality.

3 Paths in Tight Sets

An instance $I = (G, \mathcal{L}, x, y)$ will be fixed throughout this section. We say that a path $P$ traverses a set $S$ if both endpoints of $P$ are in $V \setminus S$ and $P$ contains at least one vertex in $S$. We now exhibit properties of paths traversing tight sets. In particular, we show that the strongly connected components of a tight set $S$ enjoy a nice path-like structure as depicted in Figure 2.

Lemma 3.1. For a tight set $S \subseteq V$ we have the following properties:

(a) Every path from a vertex $u \in S_{in}$ to a vertex $v \in S_{out}$ (and thus every path traversing $S$) visits every strongly connected component of $S$.

(b) For every $u \in S_{in}$ and $v \in S$ there is a path from $u$ to $v$ inside $S$. The same holds for every $u \in S$ and $v \in S_{out}$.
Proof. We remark that (a) can also be seen to follow from the “τ-narrow cut” structure as introduced in [AKS15] by setting τ = 0. We give a different proof that we find simpler for our setting.

Let \( S_1, S_2, \ldots, S_\ell \) be the vertex sets of the strongly connected components of \( S \), indexed using a topological ordering. We thus have that each subgraph \( G[S_i] \) is strongly connected and that there is no edge from a vertex in \( S_i \) to a vertex in \( S_j \) if \( i > j \).

By the above, we must have \( \delta^-(S_1) \subseteq \delta^-(S) \). Moreover, since \( x(\delta^-(S_1)) \geq 1 = x(\delta^-(S)) \), we can further conclude that \( \delta^-(S_1) = \delta^-(S) \) (recall that all edges have positive x-value) and \( x(\delta^-(S_1)) = x(\delta^+(S_1)) = 1 \) (i.e., \( S_1 \) is a tight set).

Similarly, we can show by induction on \( k \geq 2 \) that

\[
\delta^-(S_k) = \delta^+(S_{k-1}) \quad \text{and} \quad x(\delta^-(S_k)) = x(\delta^+(S_k)) = 1.
\]

To see this, note that \( \delta^-(S_k) \subseteq \delta^-(S) \cup \bigcup_{i<k} \delta^+(S_i) \). However, \( \delta^-(S) = \delta^-(S_1) \) (which is disjoint from \( \delta^-(S_k) \)), and for \( i < k - 1 \), the induction hypothesis gives that \( \delta^+(S_i) = \delta^+(S_{i+1}) \) (which is also disjoint from \( \delta^-(S_k) \)). The only term left in the union is \( i = k - 1 \) and so \( \delta^-(S_k) \subseteq \delta^+(S_{k-1}) \). Moreover, \( 1 \leq x(\delta^-(S_k)) \leq x(\delta^+(S_{k-1})) = 1 \), which implies the statement for \( k \).

Finally, we have that \( \delta^+(S_\ell) = \delta^+(S) \). To recap, all incoming edges of \( S \) are into \( S_1 \), the set of outgoing edges of every component is the set of incoming edges of the next one, and all outgoing edges of \( S \) are from \( S_\ell \). This shows (a), i.e., that every path traversing \( S \) needs to enter through \( S_1 \), exit through \( S_\ell \), and pass through every component on the way.

Finally, (b) follows because \( S_{\infty} \subseteq S_1 \) (similarly \( S_{\text{out}} \subseteq S_\ell \)), each two consecutive components are connected by an edge, and each component is strongly connected. □

Lemma 3.2. Let \( S \subseteq V \) be a nonempty set such that \( \mathcal{L} \cup \{S\} \) is a laminar family. Suppose \( u, v \in S \) are two vertices such that there is a path from \( u \) to \( v \) inside \( S \). Then we can in polynomial time find a path \( P \) from \( u \) to \( v \) inside \( S \) that crosses every set in \( \mathcal{L} \) at most twice. Thus, the path satisfies \( w(P) \leq \sum_{R \in \mathcal{L}, R \subseteq S} 2 \cdot y_R = \text{value}(S) \).

In addition, if either \( u \in S_{\text{in}} \) or \( v \in S_{\text{out}} \), then \( P \) crosses every tight set \( R \in \mathcal{L} \) at most \( 2 - |R \cap (u, v)| \) times. Thus, it satisfies \( w(P) \leq \sum_{R \in \mathcal{L}, R \subseteq S} (2 - |R \cap (u, v)|) \cdot y_R \).

Proof. Since \( \mathcal{L} \cup \{S\} \) is a laminar family, any path inside \( S \) only crosses those sets \( R \in \mathcal{L} \) that have \( R \subseteq S \). Now, to prove both statements, it is enough to find, in polynomial time, a path \( P \) inside \( S \) that for each \( R \in \mathcal{L} \) with \( R \subseteq S \) satisfies

\[
|P \cap \delta(R)| \leq \begin{cases} 2 & \text{if } |R \cap (u, v)| = 0, \\ 1 & \text{if } |R \cap (u, v)| = 1, \\ 2 & \text{if } |R \cap (u, v)| = 2, \\ 0 & \text{if } |R \cap (u, v)| = 2 \text{ and either } u \in S_{\text{in}} \text{ or } v \in S_{\text{out}}. \end{cases}
\]

The algorithm for finding \( P \) starts with any path \( P \) from \( u \) to \( v \) inside \( S \). Such a path is guaranteed to exist by the assumptions of the lemma and can be easily found in polynomial time. Now, while \( P \) does not satisfy the above conditions, select a set \( R \in \mathcal{L} \) of maximum cardinality that violates one of the above conditions. We remark that the selected set \( R \) is tight since \( R \in \mathcal{L} \). Therefore Lemma 3.1(b) implies that there is
a path from any \( u' \in R \) to any \( v' \in R \) inside \( R \) if either \( u' \in R_{in} \) or \( v' \in R_{out} \). Using this, the algorithm now modifies \( P \) depending on which of the above conditions is violated:

**Case 1:** \(|R \cap \{u, v\}| = 0\). Let \( u' \) be the first vertex visited by \( P \) in \( R \) and let \( v' \) be the last. Then \( u' \in R_{in} \) and \( v' \in R_{out} \), which implies by Lemma 3.1(b) that there is a path \( Q \) from \( u' \) to \( v' \) inside \( R \). We update \( P \) by letting \( Q \) replace the segment of \( P \) from \( u' \) to \( v' \). This ensures that the set \( R \) is no longer violated, since the path \( P \) now only enters and exits \( R \) once.

**Case 2:** \(|R \cap \{u, v\}| = 1\). This case is similar to the previous one. Suppose that \( u \in R \) and \( v \notin R \) (the other case is analogous). Let \( v' \) be the last vertex visited by \( P \) in \( R \). Then \( v' \in R_{out} \), and again by Lemma 3.1(b) there is a path \( Q \) from \( u \) to \( v' \) inside \( R \). We update \( P \) by letting \( Q \) replace the segment of \( P \) from \( u \) to \( v' \). This ensures that the set \( R \) is no longer violated, since the path \( P \) now only exits \( R \) once.

**Case 3:** \(|R \cap \{u, v\}| = 2\). Let \( u' \) be the first vertex visited by \( P \) in \( R_{in} \). By Lemma 3.1(b), there is a path \( Q \) from \( u' \) to \( v \) inside \( R \). We modify \( P \) by letting \( Q \) replace the segment of \( P \) from \( u' \) to \( v \). This ensures that the set \( R \) is no longer violated, since the path \( P \) now only enters and exits \( R \) at most once.

**Case 4:** \(|R \cap \{u, v\}| = 2\) and either \( u \in S_{in} \) or \( v \in S_{out} \). Suppose that \( u \in S_{in} \) (the case \( v \in S_{out} \) is analogous). Then, as \( R \subseteq S \), \( R \cap S_{in} \subseteq R_{in} \). So, by Lemma 3.1(b), there is a path \( Q \) from \( u \) to \( v \) inside \( R \). We replace \( P \) by \( Q \) and the set \( R \) is no longer violated.

At termination, the above algorithm returns a path satisfying all the desired conditions and thus the lemma. It remains to argue that the algorithm terminates in polynomial time. A laminar family contains at most \( 2n - 1 \) sets, so it is easy to efficiently identify a violated set \( R \) of maximum cardinality. The algorithm then, in polynomial time, modifies \( P \) by simple path computations so that the set \( R \) is no longer violated. Moreover, since the modifications are such that new edges are only added within the set \( R \), they may only introduce new violations to sets contained in \( R \) – sets of smaller cardinality. It follows, since we always select a violated set of maximum cardinality, that any set \( R \) in \( L \) is selected in at most one iteration. Hence, the algorithm runs for at most \(|L| \leq 2n - 1\) iterations (and so it terminates in polynomial time). \( \square \)

### 4 Contracting and Inducing on a Tight Set

In this section we generalize two natural graph-theoretic constructions that allow one to decompose the problem of finding a tour with respect to a vertex set \( S \). The first relies on contracting \( S \) (see Definition 4.2 in Section 4.1) and the second relies on inducing on \( S \) (see Definition 4.6 in Section 4.2).

#### 4.1 Contracting a Tight Set

Consider an instance \( I = (G, L, x, y) \). Before defining the contraction of a set \( S \in L \), we need to define the “distance” functions \( d_S \) and \( D_S \). For \( S \in L \) and \( u, v \in S \), define
we have weight increase incurred by this operation. For example, in Figure 3, which equals $d(u, s)$.

An example of the contraction of a tight set $S$ and the lift of a tour. Only $y$-values of the sets $R \in \mathcal{L}: R \subseteq S$ are depicted. On the left, only edges that have one endpoint in $S$ are shown. These are exactly the edges that are incident to $s$ in the contracted instance. In the center, a tour of $I/S$ is illustrated, and on the right we depict the lift of that tour.

$D_S(u, v)$ to be the minimum weight of a path inside $S$ from $u$ to $v$ (if no such path exists, $D_S(u, v) = \infty$). We also let

$$D_S(u, v) = \sum_{R \in \mathcal{L}: u \in R \subseteq S} y_R + d_S(u, v) + \sum_{R \in \mathcal{L}: v \in R \subseteq S} y_R,$$

which equals $d_S(u, v) + \sum_{R \in \mathcal{L}: R \subseteq S} |R \cap \{u, v\}| \cdot y_R$. We remark that $D_S(u, u)$ might be strictly positive.

The intuition of the definition of $D_S$ is as follows. After contracting $S$, all sets of the laminar family are still present in the contracted instance, except for the sets strictly contained in $S$. Now, after finding a tour in the contracted instance, we need to lift it back to a subtour in the original instance. This is done as depicted in Figure 3: for each visit of the tour to $s$ (the vertex corresponding to the contraction of $S$) on the edges $(u^i_{in}, s), (s, v^i_{out})$, we obtain a subtour of the original instance by replacing $(u^i_{in}, s), (s, v^i_{out})$ by the corresponding edges (i.e., by their preimages) $(u^i_{in}, v^i_{in}), (u^i_{out}, v^i_{out})$ of $G$ together with the minimum-weight path inside $S$ from $v^i_{in}$ to $u^i_{out}$. The value $D_S(v^i_{in}, u^i_{out})$ is the weight increase incurred by this operation. For example, in Figure 3 we have

$$D_S(v^1_{in}, u^1_{out}) = \sum_{R \in \mathcal{L}: v^1_{in} \in R \subseteq S} y_R + d_S(v^1_{in}, u^1_{out}) + \sum_{R \in \mathcal{L}: u^1_{out} \in R \subseteq S} y_R.$$

Before formally defining the notions of contraction and lift, we state the following useful bound on $D_S(u, v)$.

**Fact 4.1.** For any $u, v \in S$ with $u \in S_{in}$ or $v \in S_{out}$ we have

$$D_S(u, v) \leq \text{value}(S).$$
Proof. Lemma 3.1(b) says that there is a path from \( u \) to \( v \) inside \( S \). Select \( P \) to be the path from \( u \) to \( v \) as guaranteed by Lemma 3.2. Since \( u \in S_{\text{in}} \) or \( v \in S_{\text{out}} \), we have

\[
d_S(u, v) \leq w(P) \leq \sum_{R \in L : R \subseteq S} (2 - |R \cap \{u, v\}|) \cdot y_R
\]

and thus

\[
D_S(u, v) = d_S(u, v) + \sum_{R \in L : R \subseteq S} |R \cap \{u, v\}| \cdot y_R \leq \sum_{R \in L : R \subseteq S} 2 \cdot y_R = \text{value}(S).
\]

\( \square \)

We now define the notion of contracting a tight set for an ATSP instance. In short, the contraction is the instance obtained by performing the classic graph contraction of \( S \), modifying \( L \) to remove the sets contained in \( S \), and increasing the \( y \)-value of the new singleton \( \{s\} \) corresponding to \( S \) so as to become \( y_S + \frac{1}{2} \max_{u \in S_{\text{in}}, v \in S_{\text{out}}} D_S(u, v) \). This increase is done in order to pay for the maximum possible weight increase incurred when lifting a tour in the contraction back to a subtour in the original instance (as depicted in Figure 3, defined in Definition 4.4, and analyzed in Lemma 4.5).

**Definition 4.2** (Contracting a tight set). The instance \((G', L', x', y')\) obtained from \( I = (G, L, x, y)\) by contracting \( S \in L \), denoted by \( I/S \), is defined as follows:

- The graph \( G' \) equals \( G/S \), i.e., the graph obtained from \( G \) by contracting \( S \). Let \( s \) denote the new vertex of \( G' \) that corresponds to the set \( S \).

- For each edge \( e' \in E(G') \), \( x'(e') \) equals \( x(e) \), where \( e \in E(G) \) is the preimage of \( e' \) in \( G \).

- The laminar family \( L' \) contains all remaining sets of \( L \):

\[
L' = \{ R \setminus S \cup \{s\} : R \in L, S \subseteq R \} \cup \{ R : R \in L, S \cap R = \emptyset \}.
\]

- The vector \( y' \) equals \( y \) (via the natural mapping) on all sets but \( \{s\} \). For \( \{s\} \) we define

\[
y'_S = y_S + \frac{1}{2} \max_{u \in S_{\text{in}}, v \in S_{\text{out}}} D_S(u, v).
\]

We remark that \( I/S \) as defined above is indeed an instance: \( L' \) is a laminar family of tight sets (since for each \( R \in L' \) we have \( x'(\delta(R')) = x(\delta(R)) \), where \( R \) is the preimage of \( R' \) in \( L \) via the natural mapping), \( y'_R > 0 \) is defined only for \( R \in L' \), and \( x' \) is a feasible solution to LP\((G', 0)\) that is strictly positive on all edges.

The way we defined the new dual weight \( y'_S \) implies the natural property that the value of the linear programming solution does not increase after contracting a tight set:

\(^3\)Recall that for notational convenience we allow parallel edges in \( G/S \) and therefore the preimage is uniquely defined.
Fact 4.3. \( \text{value}(I/S) = \text{value}(I) - \left( \text{value}_f(S) - \max_{u \in S_{in}, v \in S_{out}} D_S(u, v) \right) \leq \text{value}(I) \).

Proof. By definition,
\[
\begin{align*}
\text{value}(I/S) &= 2 \cdot \sum_{R \in L_1} y_R' = 2 \cdot y_s' + 2 \cdot \sum_{R \in L_1: R \subseteq S} y_R
\\ &= \max_{u \in S_{in}, v \in S_{out}} D_S(u, v) + 2 \cdot y_s + 2 \cdot \sum_{R \in L_1: R \subseteq S} y_R
\\ &= \max_{u \in S_{in}, v \in S_{out}} D_S(u, v) + \max_{u \in S_{in}, v \in S_{out}} D_S(u, v) = \text{value}(I) - \text{value}_f(S)
\end{align*}
\]
and so the equality of the statement holds. Finally, the inequality of the statement follows from Fact 4.1, which implies that \( \max_{u \in S_{in}, v \in S_{out}} D_S(u, v) \leq \text{value}_f(S) \). \( \square \)

Having defined the contraction of a tight set \( S \in L_1 \), we define the aforementioned operation of lifting a tour of the contracted instance \( I/S \) to a subtour in the original instance \( I \). When considering a tour (or a subtour), we order the edges according to an arbitrary but fixed Eulerian walk. This allows us to talk about consecutive edges.

Definition 4.4. For a tour \( T \) of \( I/S \), we define its lift to be the subtour of \( I \) obtained from \( T \) by replacing each consecutive pair \((u_{in}, s), (s, v_{out})\) of incoming and outgoing edges incident to \( s \) by their preimages \((u_{in}, v_{in})\) and \((u_{out}, v_{out})\) in \( G \), together with a minimum-weight path from \( v_{in} \) to \( u_{out} \) inside \( S \).\(^4\)

See Figure 3 for an illustration. It follows that the lift is a subtour (i.e., an Eulerian multiset of edges that forms a single component), because we added paths between consecutive edges in the tour of \( I/S \). However, the lift is usually not a tour of the instance \( I \), as it is not guaranteed to visit all the vertices in \( S \). To extend the lift to a tour, we use the concept of inducing on the tight set \( S \), which we introduce in Section 4.2.

We complete this section by bounding the weight of the lift of \( T \).

Lemma 4.5. Let \( T \) be a tour of the instance \( I/S \). Then the lift \( F \) of \( T \) satisfies \( w(F) \leq w_{I/S}(T) \).

Proof. Consider the tour \( T \) and let \((u_{in}^{(i)}, s), (s, v_{out}^{(i)})\), \((u_{in}^{(k)}, s), (s, v_{out}^{(k)})\) be the edges that \( T \) uses to visit the vertex \( s \) (which corresponds to the contracted set \( S \)). That is, \((u_{in}^{(i)}, s)\) and \((s, v_{out}^{(i)})\) are the incoming and outgoing edge of the \( i \)-th visit of \( T \) to \( s \). By the definition of contraction, we can write the weight of \( T \) as
\[
\begin{align*}
w_{I/S}(T) &= \sum_{R \in L_1: R \subseteq S} \alpha_R y_R + 2k \cdot y_s'
\\ &= \sum_{R \in L_1: R \subseteq S} \alpha_R y_R + k \cdot \left( 2 \cdot y_s + \max_{u \in S_{in}, v \in S_{out}} D_S(u, v) \right),
\end{align*}
\]

\(^4\) We remark that it is not really important that the minimum-weight path from \( v_{in} \) to \( u_{out} \) is selected to be inside \( S \); a minimum-weight path without this restriction would also work. We have chosen this definition as we find it more intuitive and it simplifies some arguments.
where \( \alpha_R = |\delta(R) \cap T| \). We now compare this weight to that of the lift \( F \). Let \((u^{(i)}_{\text{in}}, v^{(i)}_{\text{in}})\) and \((u^{(i)}_{\text{out}}, v^{(i)}_{\text{out}})\) be the edges of \( G \) that are the preimages of \((u^{(i)}_{\text{in}}, s)\) and \((s, v^{(i)}_{\text{out}})\). The lift \( F \) is obtained from \( T \) by replacing \((u^{(i)}_{\text{in}}, s), (s, v^{(i)}_{\text{out}})\) by \((u^{(i)}_{\text{in}}, v^{(i)}_{\text{in}}), (u^{(i)}_{\text{out}}, v^{(i)}_{\text{out}})\) and adding a minimum-weight path inside \( S \) from \( v^{(i)}_{\text{in}} \) to \( u^{(i)}_{\text{out}} \). So \( F \) crosses every \( R \in \mathcal{L} : R \not\subseteq S \) the same number of times \( \alpha_R \) as \( T \). To bound the weight incurred by crossing the tight sets “inside” \( S \), note that the \( i \)-th visit to the set \( S \) incurs a weight from crossing sets \( R \subset \subset S \) that equals

\[
2y_S + \sum_{R \in \mathcal{L} : v^{(i)}_{\text{in}} \in R \subseteq S} y_R + d_S(v^{(i)}_{\text{in}}, u^{(i)}_{\text{out}}) + \sum_{R \in \mathcal{L} : u^{(i)}_{\text{out}} \in R \subseteq S} y_R = 2y_S + D_S(v^{(i)}_{\text{in}}, u^{(i)}_{\text{out}}).
\]

Hence

\[
w'_T(F) = \sum_{R \in \mathcal{L} : R \subseteq S} \alpha_R y_R + \sum_{i=1}^{k} \left( 2 \cdot y_S + D_S(v^{(i)}_{\text{in}}, u^{(i)}_{\text{out}}) \right)
\]

\[
\leq \sum_{R \in \mathcal{L} : R \subseteq S} \alpha_R y_R + \sum_{i=1}^{k} \left( 2 \cdot y_S + \max_{u \in S_{\text{in}}, v \in S_{\text{out}}} D_S(u, v) \right)
\]

\[
= w'_{T[S]}(T).
\]

\( \square \)

### 4.2 Inducing on a Tight Set

In this section we introduce our notion of induced instances. This concept will be used for completing a lift of a tour of a contracted instance into a tour of the original instance (see Definition 4.8 of “contractible” below). Inducing on a tight set \( S \) is similar to contracting its complement \( V \setminus S \) into a single vertex \( \tilde{s} \) (see Definition 4.2), though the resulting laminar family and dual values are somewhat different: namely, we let \( y'_S = \text{value}(S)/2 \) and we remove \( S \) (as well as all supersets of \( S \)) from \( \mathcal{L}' \). The intuitive reason for the setting of \( y'_S \) is that each visit to \( \tilde{s} \) should pay for the most expensive shortest paths in the strongly connected components of \( S \) (see Figure 4 and the proof of Lemma 4.9).

We remark that the notion of inducing on \( S \) for ATSP instances differs compared to the graph obtained by inducing on \( S \) (in the usual graph-theoretic sense), as here we also have the vertex \( \tilde{s} \) corresponding to the contraction of the vertices not in \( S \). This is needed to make sure that we obtain an ATSP instance (in particular, that we obtain a feasible solution \( x' \) to the linear programming relaxation).

**Definition 4.6.** The instance \((G', \mathcal{L}', x', y')\) obtained from \( I = (G, \mathcal{L}, x, y) \) by inducing on a tight set \( S \in \mathcal{L} \), denoted by \( I[S] \), is defined as follows:

- The graph \( G' \) equals \( G/\tilde{S} \), i.e., the graph obtained from \( G \) by contracting \( \tilde{S} = V \setminus S \). Let \( \tilde{s} \) denote the new vertex of \( G' \) that corresponds to the set \( \tilde{S} \).
Figure 4: In the left figure we depict a tight set $S \in \mathcal{L}$ with two strongly connected components $S_1$ and $S_2$. The induced instance (right figure) is obtained by contracting $\bar{S} = V \setminus S$ into a vertex $\bar{s}$ and removing the tight set $S$ from $\mathcal{L}$. The solid edges are paths and edges of a tour in the induced instance. In Lemma 4.9 we obtain an Eulerian set of edges in the original instance (left figure) by adding the dashed paths, resulting in a tour of each strongly connected component.

- For each edge $e' \in E(G')$, $x'(e')$ equals $x(e)$, where $e \in E(G)$ is the preimage of $e'$ in $G$.

- The laminar family $\mathcal{L}'$ contains $[\bar{s}]$ and all sets that are strict subsets of $S$: $$\mathcal{L}' = \{R \in \mathcal{L} : R \subsetneq S\} \cup \{[\bar{s}]\}.$$  

We remark that $I[S]$ in an instance: $\mathcal{L}'$ is a laminar family of tight sets, $y'_R \geq 0$ is defined only for $R \in \mathcal{L}'$, and $x'$ is a feasible solution to LP($G'$, 0) that is strictly positive on all edges.

As for the value of $I[S]$, it is comprised of the $y$-values of sets strictly inside $S$, which contribute $\text{value}(S)$, and that of $[\bar{s}]$, which also contributes $2y'_{[\bar{s}]} = \text{value}(S)$. Thus we have

**Fact 4.7.** $\text{value}(I[S]) = 2 \text{value}(S)$.

As alluded to above, we will use the instance $I[S]$ to find an Eulerian multiset $F$ of edges of the original instance $I$ such that $F$ plus a lift of a tour in $I/S$ form a tour of the instance $I$. We say that such a set $F$ makes $S$ **contractible**:

**Definition 4.8.** We say that $S \in \mathcal{L}$ is contractible with respect to an Eulerian multiset $F \subseteq E$ of edges if the lift of any tour of $I/S$ plus the edge set $F$ is a tour of $I$.

As an example, if $F$ were a subtour visiting every vertex of $S$, then $S$ would be contractible with respect to $F$. (Of course, such a subtour $F$ can only exist if $S$ is strongly connected.) The following lemma shows that, in general, it is sufficient to find a tour of $I[S]$ in order to make $S$ contractible (see also Figure 4).

---

$^5$We again recall that parallel edges are allowed in $G/\bar{S}$, and thus the preimage of an edge is uniquely defined.
Lemma 4.9. Given a tour $T$ of $I[S]$, we can in polynomial time find an Eulerian multiset of edges $F \subseteq E$ such that $S$ is contractible with respect to $F$ and $w_I(F) \leq w_{I[S]}(T)$.

Proof. Let $S_1, \ldots, S_\ell$ be the strongly connected components of $S$ indexed using a topological ordering. We will use $T$ to obtain a low-weight tour $F_i$ inside each $S_i$, and define $F$ to be the union of these tours. Then $S$ is contractible with respect to $F$. Indeed, the lift of any tour of $I/S$ must contain a path traversing $S$, and any such path visits every connected component by Lemma 3.1(a).

Let us fix one component $S_i$. We obtain the tour $F_i$ of $S_i$ by reproducing the movements of $T$ inside $S_i$ (recall that we think of $T$ as an ordered Eulerian walk). More precisely, we retain those edges of $T$ that are inside $S_i$ and, every time $T$ exits $S_i$ on an edge $(u_{\text{out}}, v_{\text{out}}) \in \delta^+(S_i)$ and then returns to $S_i$ on an edge $(u_{\text{in}}, v_{\text{in}}) \in \delta^-(S_i)$, we also insert a minimum-weight path from $u_{\text{out}}$ to $v_{\text{in}}$ inside $S_i$. Such a path exists because $S_i$ is strongly connected. (This step corresponds to adding the dashed paths in Figure 4.) Then we set $F = F_1 \cup \ldots \cup F_\ell$.

It remains to show that $F$ has low weight, i.e., that $w_I(F) \leq w_{I[S]}(T)$. For this, let $k$ be the number of times the tour $T$ visits the auxiliary vertex $s$. The weight incurred by every such visit is at least $2y'_s = \text{value}(S)$ (since the set $\{s\}$ is crossed twice in each visit). Thus we have

$$w_{I[S]}(T) \geq k \cdot \text{value}(S) + \sum_{i=1}^\ell w_{I[S]}(T \cap E(S_i)) = k \cdot \text{value}(S) + \sum_{i=1}^\ell w_I(T \cap E(S_i)).$$

On the other hand, each tour $F_i$ consists of all those edges of $T$ that are inside $S_i$, as well as $k$ shortest paths between some pairs of vertices in $S_i$. Indeed, if $T$ makes $k$ visits to the auxiliary vertex $s$, then we add exactly $k$ paths inside $S_i$ due to the pathlike structure of the strongly connected components of a tight set $S$ (Lemma 3.1). We now have the following claim, which allows us to bound the length of these paths by applying Lemma 3.2.

Claim. For each $i = 1, \ldots, \ell$, $\mathcal{L} \cup \{S_i\}$ is a laminar family.

Proof. Suppose toward a contradiction that $\mathcal{L} \cup \{S_i\}$ is not a laminar family. Then there must be a set $R \in \mathcal{L}$ such that $R \setminus S_i, S_i \setminus R$, and $S \cap R$ are all non-empty. Furthermore, since $\mathcal{L}$ is a laminar family and $S_i \subset S \in \mathcal{L}$, we must have $R \subset S$. We can thus partition $R$ into the three sets

$$R_{\leq i} = R \cap (S_1 \cup \cdots \cup S_{i-1}), \quad R_i = R \cap S_i, \quad R_{>i} = R \cap (S_{i+1} \cup \cdots \cup S_\ell).$$

In words, $R_{\leq i}$ is the part of $R$ that intersects vertices of the strongly connected components that are ordered topologically before $S_i$. Similarly, $R_{>i}$ is the part of $R$ that intersects vertices of the strongly connected components that are ordered topologically after $S_i$. Note that since $R$ is not contained in $S_i$, we have that either $R_{\leq i}$ or $R_{>i}$ is non-empty. We suppose $R_{\leq i} \neq \emptyset$ (the case $R_{>i} \neq \emptyset$ is analogous).

As $x$ is a feasible solution to LP$(G, 0)$, we have $x(\delta^+(R_{\leq i})) \geq 1$. Moreover, since $\delta(R_i \cup R_{>i}, R_{\leq i}) = \emptyset$ due to the topological ordering, we have $\delta^-(R_{\leq i}) \subset \delta^-(R)$ and thus

$$1 = x(\delta^-(R)) \geq x(\delta^-(R_{\leq i})) + x(\delta(S_i \setminus R_i, R_{\leq i})) \geq 1 + x(\delta(S_i \setminus R_i, R_i)), $$

where $x$ is a feasible solution to LP$(G, 0)$.
where the first equality follows since $R \in \mathcal{L}$ is a tight set. However, this is a contradiction because $x(\delta(S_i \setminus R_i, R_i)) > 0$; that holds since $S_i \setminus R_i = S_i \setminus R \neq \emptyset$ and $R_i \neq \emptyset$, $S_i$ is a strongly connected component, and $G$ only contains edges with strictly positive $x$-value.

By the above claim, we can apply Lemma 3.2 to obtain that a shortest path between two vertices inside $S_i$ has weight at most $\text{value}(S_i)$. Recall that $F_i$ consists of all those edges of $T$ that are inside $S_i$, as well as $k$ shortest paths between some pairs of vertices in $S_i$. Therefore, the weight of $F$ is

$$w_f(F) = \sum_{i=1}^{\ell} w_f(F_i)$$

$$\leq \sum_{i=1}^{\ell} [k \cdot \text{value}(S_i) + w_f(T \cap E(S_i))]$$

$$\leq k \cdot \text{value}(S) + \sum_{i=1}^{\ell} w_f(T \cap E(S_i))$$

$$\leq w_f[I[S]](T),$$

as required.

\end{proof}

5 Reduction to Irreducible Instances

In this section we reduce the problem of approximating ATSP on general (laminarily-weighted) instances to that of approximating ATSP on irreducible instances. Specifically, Theorem 5.3 says that any approximation algorithm for irreducible instances can be turned into an algorithm for general instances while losing only a constant factor in the approximation guarantee.

We now define the notions of reducible sets and irreducible instances. The intuition behind them is as follows. The operations of contracting and inducing on a tight set $S$ introduced in the last section naturally lead to the following recursive algorithm:

1. Select a tight set $S \in \mathcal{L}$.
2. Find a tour $T_S$ in the induced instance $I[S]$. Via Lemma 4.9, $T_S$ yields a set $F_S$ that makes $S$ contractible.
3. Recursively find a tour $T$ in the contraction $I/S$.
4. Output $F_S$ plus the lift of $T$.

For this scheme to yield a good approximation guarantee, we need to ensure that we can find a good approximate tour $T_S$ in $I[S]$ and that contracting the set $S$ results in a “significant” decrease in the value of the LP solution. If it does, we refer to the set $S$ as reducible:
Definition 5.1. We say that a set \( S \in \mathcal{L} \) is reducible if

\[
\max_{u \in S_{\text{in}}, v \in S_{\text{out}}} D_S(u, v) < \delta \cdot \text{value}(S),
\]

otherwise we say that \( S \) is irreducible. We also say that the instance \( I \) is irreducible if no set \( S \in \mathcal{L} \) is reducible.

We will use the value \( \delta = 0.75 \); however, we keep it as a parameter to exhibit the dependence of the approximation ratio on this value.

Note that singleton sets are never reducible. Moreover, we have the following observation:

Fact 5.2. Consider an instance \( I = (G, \mathcal{L}, x, y) \) and a set \( S \in \mathcal{L} \). If every set \( R \in \mathcal{L} : R \subseteq S \) is irreducible, then \( I[S] \) is irreducible. In particular, if \( I \) is an irreducible instance, then \( I[S] \) is irreducible for every \( S \in \mathcal{L} \).

Proof. Let \( I[S] = (G', \mathcal{L}', x', y') \). By definition, \( \mathcal{L}' = \{ R \in \mathcal{L} : R \subseteq S \} \cup \{ [s] \} \). Clearly, the singleton set \([s]\) is irreducible. Now consider a set \( R \in \mathcal{L} : R \subseteq S \); we need to show that \( R \) is irreducible in \( I[S] \). Note that \( R \) is also present in \( I \) and that the sets \( \{ Q \in \mathcal{L} : Q \subseteq R \} \) and \( \{ Q \in \mathcal{L}' : Q \subseteq R \} \) are identical. This implies that the distance function \( D_R \) is identical in the instances \( I \) and \( I[S] \). Moreover, the sets \( R_{\text{in}} \) and \( R_{\text{out}} \) are also the same in the two instances. Therefore, as \( R \) is irreducible in \( I \) by assumption, we have that \( R \) is also irreducible in \( I[S] \). \( \square \)

The above fact implies that if we select \( S \in \mathcal{L} \) to be a minimal reducible set, then the instance \( I[S] \) is irreducible. Hence, we only need to be able to find an approximate tour \( T_S \) for irreducible instances (in Step 2 of the above recursive algorithm). This is the idea behind the following theorem, and its proof is based on formally analyzing the aforementioned approach.

Theorem 5.3. Let \( \mathcal{A} \) be a polynomial-time \( \rho \)-approximation algorithm for irreducible instances. Then there is a polynomial-time \( \frac{2 \rho}{1-\rho} \)-approximation algorithm for general instances.

Proof. Consider a general instance \( I = (G, \mathcal{L}, x, y) \). If it is irreducible, we can simply return the result of a single call to \( \mathcal{A} \). So assume that \( I \) is not irreducible, i.e., that \( \mathcal{L} \) contains a reducible set. Let \( S \in \mathcal{L} \) be a minimal (inclusion-wise) reducible set, i.e., one such that all subsets \( R \in \mathcal{L} : R \subseteq S \) are irreducible.

We will work with the induced instance \( I[S] \). Recall that \( \text{value}(I[S]) = 2 \cdot \text{value}(S) \) (Fact 4.7). Moreover, \( I[S] \) is irreducible by Fact 5.2. We can therefore use \( \mathcal{A} \) to find a tour \( T_S \) of \( I[S] \). Since \( \mathcal{A} \) is a \( \rho \)-approximation algorithm, we have

\[
\text{w}_{I[S]}(T_S) \leq \rho \cdot \text{value}(I[S]) = 2\rho \cdot \text{value}(S).
\]

Next, we invoke the algorithm of Lemma 4.9 to obtain an Eulerian multiset of edges \( F_S \subseteq E \) such that \( S \) is contractible with respect to \( F_S \) and

\[
\text{w}_I(F_S) \leq \text{w}_{I[S]}(T_S) \leq 2\rho \cdot \text{value}(S). \tag{5.1}
\]

Now we recursively solve the contraction \( I/S \). (This is a smaller instance than \( I \), because \(|I/S| = |I| - |S| + 1\) and \(|S| \geq 2\) since a singleton set \( S \) would not have been
reducible.) Let $T$ be the tour obtained from the recursive call, and let $F$ be the lift of $T$ to $I$. We finally return $F_S \cup F$. This is a tour of $I$, as $S$ is contractible with respect to $F_S$.

The running time of this algorithm is polynomial since each recursive call consists at most of: one call to $\mathcal{A}$, the algorithm of Lemma 4.9, simple graph operations and one recursive call (for a smaller instance).

Finally, let us show that this is a $2\rho_1 - \delta$-approximation algorithm by induction on the instance size. We have

$$w_I(F \cup F_S) = w_I(F) + w_I(F_S) \leq w_{I/S}(T) + w_I(F_S) \leq \frac{2\rho}{1-\delta} \text{value}(I/S) + 2\rho \text{value}(S) < \frac{2\rho}{1-\delta} [\text{value}(I) - (1-\delta) \text{value}(S)] + 2\rho \text{value}(S) = \frac{2\rho}{1-\delta} \text{value}(I),$$

where the first inequality is by Lemma 4.5, the second follows since $T$ is a $\frac{2\rho}{1-\delta}$-approximate solution for $I/S$ and by (5.1), and the strict inequality is by Fact 4.3 and the reducibility of $S$. This shows that $F \cup F_S$ is a $\frac{2\rho}{1-\delta}$-approximate solution for $I$. \quad \square

6 Backbones and Reduction to Vertebrate Pairs

In this section we further reduce the task of approximating ATSP to that of finding a tour in instances with a backbone. For an example of such an instance see the right part of Figure 1 in the introduction.

Definition 6.1. We say that an instance $I = (G, \mathcal{L}, x, y)$ and a subtour $B$ form a vertebrate pair if every $S \in \mathcal{L}$ with $|S| \geq 2$ is crossed by $B$, i.e., $\delta(S) \cap B \neq \emptyset$. The set $B$ is referred to as the backbone of the instance.

Specifically, the main result of this section, Theorem 6.4, says that an algorithm for vertebrate pairs can be turned into an approximation algorithm for irreducible instances while losing only a constant factor in the guarantee. Combining this with Theorem 5.3 allows us to reduce the problem of approximating ATSP on general instances to that of approximating ATSP on vertebrate pairs.

The proof of Theorem 6.4 is done in two steps. First, in Section 6.1, we give an efficient algorithm for finding a quasi-backbone $B$ of an irreducible instance – a subtour that crosses a large (weighted) fraction of the sets in $\mathcal{L}$. We use the term quasi-backbone as $B$ might not cross all non-singleton sets, as would be required for a backbone. Then, in Section 6.2, we give the reduction to vertebrate pairs via a recursive algorithm (similar to the proof of Theorem 5.3 in the previous section).
The rerouting inside $S$ when obtaining the quasi-backbone $B$ from a lift $B'$ of a tour $T$ of the instance $I'$ obtained by contracting maximal sets in $L$.

The lift $F$ of the tour $T'$ found in the vertebrate pair $(I', B)$, where $I'$ was obtained by contracting $R_1$ and $R_2$.

The final tour $T$ obtained by adding results of recursive calls on $R_1$ and $R_2$ (i.e., on $I[R_1]$ and $I[R_2]$).

Figure 5: An illustration of the steps in the proofs of Lemma 6.3 (left) and Theorem 6.4 (center and right). Only one maximal set $S \in L$ is shown.

### 6.1 Finding a Quasi-Backbone

We give an efficient algorithm for calculating a low-weight quasi-backbone of an irreducible instance.

**Definition 6.2.** For an instance $I = (G, L, x, y)$, we call a subtour $B$ a quasi-backbone if

$$2 \sum_{S \in L} y_S \leq (1 - \delta) \text{value}(I),$$

where $L' = \{S \in L : \delta(S) \cap B = \emptyset\}$ contains those laminar sets that $B$ does not cross.

Recall that $\delta = 0.75$ is the parameter in Definition 5.1 (of irreducible instances). Also note that a backbone is not necessarily a quasi-backbone, as it may not satisfy the above inequality if a lot of $y$-value is on singleton sets. Although the backbones we construct do satisfy the inequality, we do not require this in the definition of the backbone to simplify the presentation. Recall that $\alpha_{NW} = 27 + \varepsilon$ denotes the approximation guarantee for node-weighted instances given by Theorem 2.2.

**Lemma 6.3.** There is a polynomial-time algorithm that, given an irreducible instance $I = (G, L, x, y)$, constructs a quasi-backbone $B$ such that $w(B) \leq \alpha_{NW} + 3$ value($I$) and $B \cap \delta(S) \neq \emptyset$ for every maximal non-singleton set $S \in L$.

**Proof.** Let $L_{\text{max}}$ be the family of all maximal sets in $L$. We define $I'$ to be the instance obtained from $I$ by contracting all sets in $L_{\text{max}}$. By Fact 4.3, the LP value does not increase, i.e., $\text{value}(I') \leq \text{value}(I)$. In $I'$, all laminar tight sets are singletons, therefore the new instance is node-weighted and we can use the $\alpha_{NW}$-approximation algorithm (Theorem 2.2) to find a tour $T$ in $I'$ with $w_T(T) \leq \alpha_{NW} \text{value}(I') \leq \alpha_{NW} \text{value}(I)$.

Now, to obtain a subtour $B$ of the original instance $I$, we consider the lift $B'$ of $T$ back to $I$ (see Definition 4.4). The lift $B'$ is a subtour of low weight. Indeed, $w_I(B') \leq w_T(T) \leq \alpha_{NW} \text{value}(I)$ by Lemma 4.5. It also crosses every maximal set $S \in L_{\text{max}}$. However, it might not yet satisfy the inequality of Definition 6.2. We therefore slightly modify the subtour $B'$ to obtain $B$ as follows. For each set $S \in L_{\text{max}}$: 

2
1. Suppose the first visit to $S$ in the subtour $B'$ arrives at a vertex $u^S \in S_{\text{in}}$ and departs from a vertex $v^S \in S_{\text{out}}$.

2. Replace the segment of $B'$ from $u^S$ to $v^S$ by the union of:
   - a shortest path from $u^S$ to $u^S_{\text{max}}$,
   - a path from $u^S_{\text{max}}$ to $v^S_{\text{max}}$ inside $S$ as given by Lemma 3.2,
   - and a shortest path from $v^S_{\text{max}}$ to $v^S$,

   where $u^S_{\text{max}} \in S_{\text{in}}$ and $v^S_{\text{max}} \in S_{\text{out}}$ are selected to maximize $D_S(u^S_{\text{max}}, v^S_{\text{max}})$.

See the left part of Figure 5 for an illustration. The existence of the second path above is guaranteed by Lemma 3.1(b) since $u^S_{\text{max}} \in S_{\text{in}}$. It is clear that the obtained multiset $B$ is a subtour (since $B'$ is a subtour), that it crosses every set in $L_{\text{max}}$ and that the algorithm for finding $B$ runs in polynomial time. It remains to bound the weight of $B$ and to show that $B$ satisfies the property of a quasi-backbone, i.e., the inequality of Definition 6.2.

For the former, note that the weight of $B$ is at most the weight of the lift $B'$ plus the weight of the three paths added for each set $S \in L_{\text{max}}$. For such a set $S \in L_{\text{max}}$, the weight of the path from $u^S$ to $u^S_{\text{max}}$ is at most value($S$) since there is a path from $u^S \in S_{\text{in}}$ to $u^S_{\text{max}}$ inside $S$ by Lemma 3.1(b) and such a path can be selected to have weight at most value($S$) by Lemma 3.2. By the same argument, we have that the weight of the path from $v^S_{\text{max}}$ to $v^S \in S_{\text{out}}$ is at most value($S$). Finally, by applying Lemma 3.2 again, we have that the path added from $u^S_{\text{max}}$ to $v^S_{\text{max}}$ is also bounded by value($S$). It follows that

$$w(B) \leq w(B') + 3 \cdot \sum_{S \in L_{\text{max}}} \text{value}(S) \leq w(B') + 3 \cdot \text{value}(I) \leq (\alpha_{NW} + 3) \cdot \text{value}(I),$$

as required. (In the second inequality we used that the sets $S \in L_{\text{max}}$ are disjoint.)

We proceed to prove that $B$ satisfies the inequality of Definition 6.2. Recall that $L^* = \{S \in L : \delta(S) \cap B = \emptyset\}$ contains those laminar sets that $B$ does not cross. As $B$ crosses every $S \in L_{\text{max}}$ (i.e., $L_{\text{max}} \cap L^* = \emptyset$), it is enough to show the following:

**Claim.** For every $S \in L_{\text{max}}$ we have

$$\sum_{R \in L^* : R \subseteq S} 2y_R \leq (1 - \delta) \cdot \text{value}(S).$$

Once we have this claim, the property of a quasi-backbone indeed follows:

$$2 \sum_{R \in L^*} y_R = \sum_{S \in L_{\text{max}}} \sum_{R \in L^* : R \subseteq S} 2y_R \leq \sum_{S \in L_{\text{max}}} (1 - \delta) \cdot \text{value}(S) \leq (1 - \delta) \cdot \text{value}(I).$$

**Proof of Claim.** The intuition behind the claim is that, when forming $B$, we have added a path $P$ from $u^S_{\text{max}}$ to $v^S_{\text{max}}$. Since $S$ is irreducible, this path $P$ has a large weight. However, it is chosen so that it crosses each set in $L$ at most twice. Thus it must cross most (weighted by value) sets of $L$ contained in $S$.

---

*Recall that the edges of the subtour $B'$ are ordered by an Eulerian walk.*
Now we proceed with the formal proof. As \( u^S_{\text{max}} \in S_{\text{in}} \), the path \( P \) inside \( S \) from \( u^S_{\text{max}} \) to \( v^S_{\text{max}} \) that we have obtained from Lemma 3.2 crosses every tight set \( R \in \mathcal{L} \) at most \( 2 - |R \cap \{ u^S_{\text{max}}, v^S_{\text{max}} \}| \) times. Moreover, \( P \) (a subset of \( B \)) does not cross any set \( R \in \mathcal{L}^* \). Therefore

\[
d_S(u^S_{\text{max}}, v^S_{\text{max}}) \leq w(P) \leq \sum_{R \in \mathcal{L} \setminus \mathcal{L}^*: R \subseteq S} (2 - |R \cap \{ u^S_{\text{max}}, v^S_{\text{max}} \}|) \cdot y_R.
\]

Furthermore, we have that the quasi-backbone \( B \) crosses all sets in \( \mathcal{L}_{\text{max}} \) and visits both vertices \( u^S_{\text{max}} \) and \( v^S_{\text{max}} \). Therefore it must cross all sets \( R \in \mathcal{L} \) for which \( R \cap \{ u^S_{\text{max}}, v^S_{\text{max}} \} \) is non-empty; i.e., for all \( R \in \mathcal{L}^* \) we have \( |R \cap \{ u^S_{\text{max}}, v^S_{\text{max}} \}| = 0 \). It follows that that the quasi-backbone crosses most (weighted by value) laminar sets:

\[
\sum_{R \in \mathcal{L} \setminus \mathcal{L}^*: R \subseteq S} 2y_R = \sum_{R \in \mathcal{L} \setminus \mathcal{L}^*: R \subseteq S} (2 - |R \cap \{ u^S_{\text{max}}, v^S_{\text{max}} \}|) \cdot y_R + \sum_{R \in \mathcal{L} \setminus \mathcal{L}^*: R \subseteq S} |R \cap \{ u^S_{\text{max}}, v^S_{\text{max}} \}| \cdot y_R \\
\geq d_S(u^S_{\text{max}}, v^S_{\text{max}}) + \sum_{R \in \mathcal{L} \setminus \mathcal{L}^*: R \subseteq S} |R \cap \{ u^S_{\text{max}}, v^S_{\text{max}} \}| \cdot y_R \\
= D_S(u^S_{\text{max}}, v^S_{\text{max}}) \\
\geq \delta \text{ value}(S),
\]

where the last inequality is by the choice of \( u^S_{\text{max}}, v^S_{\text{max}} \) and by the irreducibility of \( S \). The claim now follows:

\[
\sum_{R \in \mathcal{L}^*: R \subseteq S} 2y_R = \text{value}(S) - \sum_{R \in \mathcal{L} \setminus \mathcal{L}^*: R \subseteq S} 2y_R \leq \text{value}(S) - \delta \text{ value}(S) = (1 - \delta) \text{ value}(S).
\]

\( \diamond \)

The proof of the above claim completes the proof of Lemma 6.3. \( \square \)

### 6.2 Obtaining a Vertebrate Pair via Recursive Calls

We now prove the main result of Section 6.

**Theorem 6.4.** Let \( \mathcal{A} \) be a polynomial-time algorithm that, given a vertebrate pair \((I', B)\), returns a tour of \( I' \) with weight at most \( \kappa \text{ value}(I') + \eta \omega_F(B) \) (for some \( \kappa, \eta \geq 0 \)). Then there is a polynomial-time \( \rho \)-approximation algorithm for irreducible instances, where \( \rho = \frac{\kappa + \eta (\alpha + 2)}{1 - 2(1-\delta)} \).

The proof of this theorem is somewhat similar to that of Theorem 5.3, in that the algorithm presented here will call itself recursively on smaller instances, as well as invoking the black-box algorithm \( \mathcal{A} \) (once per recursive call).

**Proof.** We briefly discuss the intuition first. Consider an irreducible instance \( I = (G, \mathcal{L}, x, y) \). By Lemma 6.3, we can find a quasi-backbone \( B \) – a subtour such that \( \sum_{S \in \mathcal{L}} 2y_S \leq (1 - \delta) \text{ value}(I) \), where as before \( \mathcal{L}^* = \{ S \in \mathcal{L} : \delta(S) \cap B = \emptyset \} \) contains the laminar sets that the quasi-backbone does not cross. This is a small fraction of the entire optimum value \( \text{value}(I) \), so we can afford to run an expensive procedure (say,
a $2\rho$-approximation) on the unvisited sets (using recursive calls) so as to make them contractible. Once we contract all these sets, $B$ will become a backbone in the contracted instance and we will have thus obtained a vertebrate pair, on which the algorithm $\mathcal{A}$ can be applied.\footnote{Note that we never actually find a backbone of the original, uncontracted instance.} See Figure 5 for an illustration.

We now formally describe the $\rho$-approximation algorithm $\mathcal{A}_{\text{irr}}$ for irreducible instances. Given an irreducible instance $I = (G, L, x, y)$, it proceeds as follows:

1. Invoke the algorithm of Lemma 6.3 to obtain a quasi-backbone $B$ with $w_I(B) \leq (\alpha_{\text{nw}} + 3) \cdot \text{value}(I)$. Denote by $L_{\text{max}}$ the family of maximal (inclusion-wise) non-singleton sets in $L^* = \{S \in L : \delta(S) \cap B = \emptyset\}$. (For example, in Figure 5, $R_1$ and $R_2$ are two such sets.)

2. For each $S \in L_{\text{max}}$, recursively call $\mathcal{A}_{\text{irr}}$ to find a tour $T_S$ in the instance $I[S]$ (which is irreducible by Fact 5.2). Then use $T_S$ and the algorithm of Lemma 4.9 to find an Eulerian multiset of edges $F_S$ such that $S$ is contractible with respect to $F_S$ and $w_I(F_S) \leq w_I[I[S](T_S)]$.

3. Let $I'$ be the instance obtained from $I$ by contracting all the maximal sets $S \in L_{\text{max}}$. We have that $(I', B)$ is a vertebrate pair by construction: we have contracted all tight sets that were not crossed by $B$ into single vertices, and so $B$ is a backbone of $I'$. We can therefore invoke the algorithm $\mathcal{A}$ on $(I', B)$. By assumption, this returns a tour $T'$ of $I'$ such that $w(T') \leq \kappa \cdot \text{value}(I') + \eta \cdot w_I(B)$.

4. Finally, return the tour $T$ consisting of the lift $F$ of $T'$ to $I$ together with $\bigcup_{S \in L_{\text{max}}} F_S$.

(See the center and right parts of Figure 5 for an illustration.)

We remark that $T$ is indeed a tour of $I$ since all sets $S \in L_{\text{max}}$ are contractible with respect to $\bigcup_{S \in L_{\text{max}}} F_S$.

Having described the algorithm, it remains to show that $\mathcal{A}_{\text{irr}}$ runs in polynomial time and that it has an approximation guarantee of $\rho$.

For the former, we bound the total number of recursive calls that $\mathcal{A}_{\text{irr}}$ makes. We claim that the total number of recursive calls on input $I = (G, L, x, y)$ is at most the cardinality of $L_{\geq 2} = \{S \in L : |S| \geq 2\}$. The proof is by induction on $|L_{\geq 2}|$. For the base case, i.e., when $|L_{\geq 2}| = 0$, there are no recursive calls since there are no non-singleton sets in $L^* \subseteq L$ and so $L_{\text{max}}^* = \emptyset$. For the inductive step, suppose that $L_{\text{max}}^* = \{S_1, S_2, \ldots, S_\ell\} \subseteq L^*$ and so there are $\ell$ recursive calls in this iteration – on the instances $I[S_1], I[S_2], \ldots, I[S_\ell]$. If we let $L_{i_{\geq 2}}^i$ denote the non-singleton laminar sets of $I[S_i]$ then, by the definition of inducing on a tight set, for every $R \in L_{i_{\geq 2}}$ we have $R \subseteq S_i$ and $R \in L_{i_{\geq 2}}$. It follows by the induction hypothesis that the total number of recursive calls that $\mathcal{A}_{\text{irr}}$ makes is

$$\ell + \sum_{i=1}^\ell |L_{i_{\geq 2}}^i| \leq \ell + |L_{\geq 2}| - \ell = |L_{\geq 2}|,$$

where the inequality holds because the sets $L_{i_{\geq 2}}^i$ are disjoint and $L_{i_{\geq 2}}^1 \cup L_{i_{\geq 2}}^2 \cup \cdots \cup L_{i_{\geq 2}}^\ell \subseteq L_{\geq 2} \setminus \{S_1, S_2, \ldots, S_\ell\}$. Hence, the total number of recursive calls $\mathcal{A}_{\text{irr}}$ makes is $|L_{\geq 2}| \leq |L|$.\footnote{Note that we never actually find a backbone of the original, uncontracted instance.}
which is at most linear in \(|V|\). The fact that \(A_{\text{irr}}\) runs in polynomial time now follows because each call runs in polynomial time. Indeed, the algorithm of Lemma 6.3, the algorithm of Lemma 4.9, and \(A\) all run in polynomial time.

We now complete the proof of the theorem by showing that \(A_{\text{irr}}\) is a \(\rho\)-approximation algorithm. We have by the assumptions on \(A\) and by Lemma 4.5 that the weight \(w_I(F)\) of the lift \(F\) of \(I\) is at most

\[
w_F(T') \leq \kappa \text{ value}(I') + \eta w_I(B) \leq \kappa \text{ value}(I) + \eta (\alpha_{\text{NW}} + 3) \text{ value}(I) = (\kappa + \eta (\alpha_{\text{NW}} + 3)) \text{ value}(I),
\]

where the second inequality follows by Fact 4.3 and since \(w_I(B) = w_I(T')(I')\) arises by contracting only sets not visited by \(B\), which preserves the weight of \(B\) and \(w_I(B) \leq (\alpha_{\text{NW}} + 3) \text{ value}(I)\).

Now, to show that \(w(T) = w(F) + w(\bigcup_{S \in \mathcal{L}_\text{max}^*} F_S) \leq \rho \text{ value}(I)\), we proceed by induction on the total number of recursive calls. In the base case, when no recursive calls are made, we have \(w(T) = w(F) \leq w_I(T') \leq (\kappa + \eta (\alpha_{\text{NW}} + 3)) \text{ value}(I) \leq \rho \text{ value}(I)\). For the inductive step, the induction hypothesis yields that for each \(S \in \mathcal{L}_\text{max}^\ast\) we have

\[
w(F_S) \leq w_{I[S]}(T_S) \leq \rho \text{ value}(I[S]) = 2\rho \text{ value}(S),
\]

where the equality is by Fact 4.7. Hence

\[
w\left(\bigcup_{S \in \mathcal{L}_\text{max}^*} F_S\right) = \sum_{S \in \mathcal{L}_\text{max}^*} w(F_S) \leq \sum_{S \in \mathcal{L}_\text{max}^*} 2\rho \text{ value}(S)
\]

\[
= \sum_{S \in \mathcal{L}_\text{max}^*} 2\rho \sum_{R \subseteq S} 2y_R \leq 2\rho \sum_{R \subseteq \mathcal{L}^*} 2y_R \leq 2\rho (1 - \delta) \text{ value}(I).
\]

The second equality holds because Lemma 6.3 guarantees that \(B\) crosses each maximal set of \(\mathcal{L}\), which implies that every \(R \in \mathcal{L}\) that is a subset of a set \(S \in \mathcal{L}_\text{max}^\ast\) is not crossed by \(B\), i.e., \(R \in \mathcal{L}^\ast\): thus we have \(\sum_{R \subseteq \mathcal{L}^\ast} 2y_R = \sum_{R \subseteq \mathcal{L}^\ast} 2y_R = \sum_{S \in \mathcal{L}_\text{max}^\ast} 2y_R\) for all \(S \in \mathcal{L}_\text{max}^\ast\). The last inequality holds because \(B\) is a quasi-backbone of \(I\) (see Definition 6.2). Summing up the weight of the lift \(F\) of \(T'\) and of \(\bigcup_{S \in \mathcal{L}_\text{max}^*} F_S\) we get

\[
w(T) \leq (\kappa + \eta (\alpha_{\text{NW}} + 3) + 2\rho (1 - \delta)) \text{ value}(I) = \rho \text{ value}(I),
\]

by the selection of \(\rho\) to equal \(\frac{\kappa + \eta (\alpha_{\text{NW}} + 3)}{1 - 2(1 - \delta)}\). This concludes the inductive step and the proof of the theorem. \(\square\)

7 Algorithm for Vertebrate Pairs

In this section we consider a vertebrate pair \((I, B)\) and prove the following theorem. This provides the algorithm required in Theorem 6.4.

**Theorem 7.1.** Let \((I, B)\) be a vertebrate pair\(^8\). For every \(\epsilon > 0\) there is a polynomial-time algorithm that returns a tour \(T\) of \(I\) with \(w(T) \leq 8(9 + \epsilon) \cdot \text{ value}(I) + (9 + \epsilon) \cdot w(B)\).

\(^8\) Throughout this section we will assume that \(B \neq \emptyset\). In the special case when \(B = \emptyset\), it must be the case that \(\mathcal{L}_\text{me} = \emptyset\) and thus the instance is node-weighted; we can simply apply the \(\alpha_{\text{NW}}\)-approximation algorithm of Theorem 2.2.
Recall that value(I) equals the Held-Karp lower bound. This theorem, together with Theorem 6.4, yields a constant-factor approximation algorithm for any irreducible instance. Combined with the reductions of the previous sections, this implies a constant-factor approximation algorithm for arbitrary instances. We provide an overview of the entire sequence of reductions in Section 8.

Our algorithm for vertebrate pairs uses the approach introduced in [Sve15] that reduces the problem of finding a tour in I to that of solving the Local-Connectivity ATSP problem on I. We first define the necessary concepts from [Sve15] in Section 7.1.

### 7.1 Local-Connectivity ATSP

The Local-Connectivity ATSP problem consists in finding “local” subtours that are only required to cross the sets of a given partition \( V = V_1 \cup \cdots \cup V_k \) of the vertices instead of connecting the entire graph (as in the standard ATSP). A “good” solution to Local-Connectivity ATSP has a local requirement: each subtour should not be much more expensive than the lower bound on the cost of visiting the vertices in the subtour (see the definition of lightness below).

That lower bound on the cost of visiting vertices is defined in terms of a “lower bound” function \( \text{lb} : V \rightarrow \mathbb{R}_+ \). The only requirement besides non-negativity is that the \( \text{lb} \) function needs to be fixed by our algorithm (as a function of the instance \( I \) and the backbone \( B \)) before it is allowed to access the given partition. In order to emphasize the relationship between the solution quality (see the definition of \( \alpha \)-lightness below) of our algorithm for Local-Connectivity ATSP and the final approximation guarantee for ATSP, we additionally assume the normalization \( \text{lb}(V) \leq \text{value}(I) \) (see [Sve15], Remark 3.1).

Intuitively, \( \text{lb}(v) \) encodes how much we are willing to pay to visit vertex \( v \). For a vertebrate pair \((I, B)\), we select \( \text{lb} \) as follows. Let \( V(B) \) denote the vertices that are incident to an edge in \( B \), and define

\[
\text{lb}(v) = \text{lb}(v) \cdot \frac{\text{value}(I)}{\text{lb}(V)}, \quad \text{where} \quad \text{lb}(v) = \begin{cases} \frac{1}{\sqrt{\text{value}(I)}} \cdot (\text{value}(I) + w(B)/4) & \text{if } v \in V(B), \\ 2y_v & \text{otherwise.} \end{cases}
\]

Note that \( \text{lb} \) is merely a version of \( \text{lb} \) scaled down so that \( \text{lb}(V) = \text{value}(I) \). For notational convenience we extend the vector \( y \) to all singletons so that \( y_v = 0 \) if \( \{v\} \notin \mathcal{L} \).

We also note that the reason for our particular choice of \( \text{lb} \) is to simplify the calculations in the proof of Lemma 7.6.

For an edge multiset \( F \) let \( \mathcal{C}(F) \) denote the set of strongly connected components of \((V, F)\). Note that if \( F \) is Eulerian, then every component of \((V, F)\) that is connected in the undirected sense is also strongly connected, and the edge set of every component forms a (possibly empty) subtour.

We now formally define the Local-Connectivity ATSP problem, having fixed the lower bound function \( \text{lb} \) as e.g. above. The input is a partition \( V = V_1 \cup V_2 \cup \cdots \cup V_k \) of the vertices \((k \geq 2)\), and the output an Eulerian multisubset \( F^* \) of \( E \) such that

\[
|\delta_{F^*}(V_i)| \geq 1 \quad \text{for } i = 1, 2, \ldots, k.
\]
(Recall that $\delta_{-}^{-1}(V_i) = \delta_{-}^{-1}(V_i) \cap F^*$. We say that an algorithm for Local-Connectivity ATSP is \emph{$\alpha$-light} on $I$ (with respect to lb) if, for any input $V_1 \cup \cdots \cup V_k$, the algorithm finds an Eulerian edge set $F^*$ such that
\[ w(\tilde{G}) \leq \alpha \cdot \text{lb}(\tilde{G}) \quad \text{for every component } \tilde{G} \in C(F^*). \]

Here we use the notation that for a connected component $\tilde{G}$ of $(V, F^*)$, $w(\tilde{G}) = \sum_{e \in E(\tilde{G})} w(e)$ (summation over the edges) and $\text{lb}(\tilde{G}) = \sum_{v \in V(\tilde{G})} \text{lb}(v)$ (summation over the vertices).

The main technical theorem of [Sve15] reduces the problem of finding a tour to that of solving Local-Connectivity ATSP, up to a constant factor.

\textbf{Theorem 7.2 ([Sve15, Theorem 5.1])}. Suppose that there is an algorithm $\mathcal{A}$ for Local-Connectivity ATSP that is $\alpha$-light on $I$. Then there exists a tour of $I$ of weight at most $5\alpha \text{value}(I)$. Moreover, for any $\varepsilon > 0$ a tour of weight at most $(9 + \varepsilon)\alpha \text{value}(I)$ can be found in time polynomial in the number $n = |V|$ of vertices, in $1/\varepsilon$, and in the running time of $\mathcal{A}$.

To show Theorem 7.1, it thus suffices to devise a polynomial-time algorithm for Local-Connectivity ATSP that is $\alpha$-light on $I$, where $\alpha = 8 + \text{w}(B)/\text{value}(I)$.

\section{The Algorithm for Vertebrate Pairs}

We now formulate our main technical lemma. Let $\mathcal{L}_{\geq 2}$ denote the family of non-singleton sets in $\mathcal{L}$. Recall that for an Eulerian multiset $F \subseteq E$, the edge set of each component in $C(F)$ is a subtour. We refer to them as the subtours in $F$.

\textbf{Lemma 7.3}. \textit{There is a polynomial-time algorithm that solves the following problem. Let $(I, B)$ be a vertebrate pair, and let $U_1, \ldots, U_\ell \subseteq V \setminus V(B)$ be disjoint non-empty vertex sets such that the subgraphs $G[U_1], \ldots, G[U_\ell]$ are strongly connected and for every $S \in \mathcal{L}_{\geq 2}$ and $i = 1, \ldots, \ell$ we have either $U_i \cap S = \emptyset$ or $U_i \subseteq S$. Then the algorithm finds an Eulerian multiset $F \subseteq E$ of edges such that:}

\begin{itemize}
  \item[(a)] $w(F) \leq 3 \cdot \text{value}(I)$,
  \item[(b)] $|\delta_{-}^{-1}(U_i)| \geq 1$ for every $i = 1, \ldots, \ell$,
  \item[(c)] $|\delta_{-}^{-1}(v)| \leq 4$ for every $v \in V$ with $x(\delta_{-}^{-1}(v)) = 1$,
  \item[(d)] any subtour in $F$ that crosses a tight set in $\mathcal{L}_{\geq 2}$ visits a vertex of the backbone.
\end{itemize}

The proof will be given in Section 7.3. It uses an auxiliary \emph{split graph}: a concept inspired by the authors’ previous work on ATSP on graphs with two different edge weights [STV16]. Let us give an informal explanation of the statement before we use it for Local-Connectivity ATSP. The requirements on the disjoint sets $U_1, \ldots, U_\ell$ are that they do not contain any vertex of the backbone, they are strongly connected, and $\mathcal{L}_{\geq 2} \cup \{U_1, \ldots, U_\ell\}$ is a laminar family in which the sets $U_1, \ldots, U_\ell$ are minimal

\footnote{To be precise, for a polynomial running time we should also be able to evaluate the lb function in polynomial time, which clearly is the case here.}
Figure 6: On the left, the “dotted” sets $U_1, \ldots, U_\ell$ of Lemma 7.3 are depicted. On the right, we show how the set $U_i$ is obtained by the algorithm for Local-Connectivity ATSP in Section 7.2: $V_i$ is intersected with a minimal non-singleton set $S$ to obtain $V'_i$ (the striped area). Then $U_i$ is a source component in the decomposition of $V'_i$ into strongly connected components. This implies that there are no edges from $V'_i \setminus U_i$ to $U_i$ and so any edge in $\delta(V'_i \setminus U_i, U_i)$ must come from outside of $V'_i$ and thus cross the tight set $S$.

(see the left part of Figure 6). Now, given such sets, the lemma says that there is a polynomial-time algorithm which outputs an Eulerian multiset $F$ of edges that (a) is of low weight, (b) crosses all the sets $U_1, \ldots, U_\ell$ (similarly to the requirement for a solution to Local-Connectivity ATSP), and (c) visits every tight vertex at most four times. Finally, we also have that (d) any subtour in $F$ that crosses a non-singleton tight set intersects the backbone. This property will be important for analyzing the lightness of our tour. Indeed, it will imply that any connected component of our solution $F^*$ to Local-Connectivity ATSP that is disjoint from the backbone does not cross a set in $L_{\geq 2}$. So any edge $(u, v)$ in such a component will have weight equal to $y_u + y_v$. Intuitively, this (almost) reduces the problem to the node-weighted case.

Description of algorithm. Equipped with Lemma 7.3, we are ready to describe our algorithm for Local-Connectivity ATSP. Let $V = V_1 \cup \cdots \cup V_k$ be the input, i.e., a partition of $V$ into $k \geq 2$ sets. Our output $F^* = B \cup P \cup F$ will consist of three Eulerian multisets of edges. The first set is the backbone $B$ itself. The set $P$ is selected so as to handle the special case of the backbone being fully inside a single partition class: in that case, $P$ will cross that partition class. Finally, $F$ is selected so as to cross any partition class whose vertices are disjoint from the vertices of the backbone. They are obtained as follows:

(P) If there is no partition class that contains all the vertices of the backbone, then we set $P = \emptyset$. Otherwise, let $V_i$ be the partition class with $V(B) \subseteq V_i$. We select arbitrary vertices $u \in V(B)$ and $v \in V \setminus V_i$, and we let $P$ be the edges of a minimum-weight path from $u$ to $v$ plus the edges of a minimum weight path from $v$ to $u$.

(F) For convenience, index the partition classes so that $V_1, \ldots, V_\ell$ are those sets that are disjoint from the vertices of the backbone (we have $0 \leq \ell < k$). For $i = 1, \ldots, \ell$ let $V'_i$ be the intersection of $V_i$ with a minimal set $S \in L_{\geq 2} \cup \{V\}$ with $S \cap V_i \neq \emptyset$. Then consider a decomposition of $V'_i$ into strongly connected components (with
remark and proceed to its analysis. This completes the description of our algorithm for Local-Connectivity ATSP. We proceed to its

Remark 7.4. One can improve the guarantee of the above algorithm by using a slightly more complex procedure and modifying the definition of $\overline{\alpha}$. Namely, one can show that either of the two sets $B$ and $P$ is always sufficient.

7.2.1 Analysis

It is clear that the algorithm runs in polynomial time: $B$ is provided in the input, $P$ is either the empty set or the union of two minimum-weight paths, and $F$ is obtained via the polynomial-time algorithm guaranteed by Lemma 7.3. In addition, the solution $F^* = B \cup P \cup F$ is Eulerian, since $B$, $P$, and $F$ are all Eulerian. It remains to show that $F^*$ crosses every set of the input (Lemma 7.5) and that it is $\alpha$-light (Lemma 7.6 and Corollary 7.7).

Lemma 7.5. We have $|\delta^{-}_F(V_i)| \geq 1$ for $i = 1, 2, \ldots, k$.

Proof. We first consider a partition $V_i$ that intersects the backbone, i.e., $V_i \cap V(B) \neq \emptyset$. If $V(B) \subseteq V_i$, then $B$ (and thus $F^*$) crosses $V_i$. Otherwise, in the special case when $V(B) \subseteq V_i$, $P$ is defined so that it enters $V_i$.

Let us now consider the more interesting case of $V_i$ being disjoint from the vertices of the backbone, i.e., $V(B) \cap V_i = \emptyset$ (and $i \leq t$). By property $(b)$ of Lemma 7.3, there exists an edge $e \in \delta^{-}_F(V_i)$. Then either $e \in \delta^{-}_F(V_i)$ (in which case we are done since $F \subseteq F^*$), or $e \in \delta_F(V_i \setminus U_i, U_i)$. Assume the latter case.

Using that $U_i$ was a source component in the decomposition of $V'_i$ into strongly connected components, $e$ must enter a set in $\mathcal{L}_{\alpha}$. Indeed, recall that $U_i$ was selected so that there is no edge from $V'_i \setminus U_i$ to $U_i$. Since $e \in \delta_F(V_i \setminus U_i, U_i)$ and $\delta(V'_i \setminus U_i, U_i) = \emptyset$, we must have $e \in \delta(V_i \setminus V'_i, U_i) \subseteq \delta(V_i \setminus V'_i, V_i')$. However, $V'_i$ was obtained by intersecting $V_i$ with a minimal set $S \in \mathcal{L}_{\alpha} \cup \{V_i \}$ with $S \cap V_i \neq \emptyset$. Thus we must have $e \in \delta^{-}_F(S)$ (and $S \neq V$). Now, property $(d)$ of Lemma 7.3 guarantees that the connected component (i.e., the subtour) of $F$ containing $e$ must visit $V(B)$. This subtour thus visits both $V_i$ (the head of $e$ is in $U_i \subseteq V_i$) and $V(B)$, which is disjoint from $V_i$. As such, the subtour must cross $V_i$, i.e., we have $|\delta^{-}_F(V_i)| \geq |\delta^{-}_F(V_i)| \geq 1$ as required. □

Lemma 7.6. For every connected component $\hat{G}$ of $(V, F^*)$, i.e., $\hat{G} \in \mathcal{C}(F^*)$, we have

$$w(\hat{G}) \leq 4 \overline{\alpha}(\hat{G}).$$
Proof. Recall that the backbone $B$ is a subtour, which means that it forms a single component. Therefore, since $B \subseteq F^*$, there is one component of $(V, F^*)$ that contains the backbone and the others are disjoint from the backbone.

First suppose that $\tilde{G}$ is the component that contains the backbone. Then

$$\overline{\text{lb}}(\tilde{G}) \geq \overline{\text{lb}}(V(B)) = \frac{1}{4} \cdot w(B) + \text{value}(I),$$

whereas

$$w(\tilde{G}) \leq w(F^*) = w(B) + w(P) + w(F) \leq w(B) + w(P) + 3 \cdot \text{value}(I),$$

where we used that $w(F) \leq 3 \cdot \text{value}(I)$ by property (a) of Lemma 7.3. Hence $w(\tilde{G}) \leq 4 \overline{\text{lb}}(\tilde{G})$ will follow once we show that $w(P) \leq \text{value}(I)$. This is trivially true if $P = \emptyset$. Otherwise, i.e., if $P$ is selected as the union of two minimum-weight paths, we have the following claim.

Claim. For any two vertices $u, v \in V$, the total weight of the minimum-weight $u-v$ path and the minimum-weight $v-u$ path is at most $\text{value}(I)$.

Proof of Claim. We will show that $x$ simultaneously supports one unit of flow from $u$ to $v$ and one unit of flow from $v$ to $u$. The subtour elimination constraints in LP($G, w$) guarantee the existence of a unit flow $f$ from $u$ to $v$ (by the min-cut max-flow theorem). Consider the flow $x' = x - f$; we claim that $x'(\delta^-(S)) \geq 1$ holds for every $S \subseteq V \setminus \{v\}$ with $u \in S$, and therefore we can find another unit flow $g$ from $v$ to $u$ such that $f + g \leq x$. Indeed, $x(\delta^+(S)) = x(\delta^{-}(S))$ since $x$ is Eulerian, and $f(\delta^+(S)) = f(\delta^{-}(S)) + 1$. Hence $x'(\delta^{-}(S)) = x'(\delta^+(S)) + 1 \geq 1$.

Now the claim follows since the weight of a minimum-weight $u-v$ path is at most the weight of $f$, and similarly for the $v-u$ path and $g$. Since $f + g \leq x$, their total weight is at most that of $x$, i.e., $\text{value}(I)$.

Now consider the case when $\tilde{G}$ is disjoint from the backbone. For the lower bound, we have

$$\overline{\text{lb}}(\tilde{G}) = \sum_{v \in V(\tilde{G})} \overline{\text{lb}}(v) = 2 \sum_{v \in V(\tilde{G})} y_v.$$

To bound the weight of $\tilde{G}$, note that $\tilde{G}$ does not contain any edges of $B$ or $P$ (recall that $P$, if non-empty, has an edge incident to a vertex $u \in V(B)$). Moreover, by property (d) of Lemma 7.3, the edges of $\tilde{G}$ (which is a subtour in $F$) do not cross any tight set in $L_{\geq 2}$. Therefore any edge $(u, v) \in E(\tilde{G})$ has weight $y_u + y_v$ and so

$$w(\tilde{G}) = \sum_{e \in E(\tilde{G})} w(e) = \sum_{v \in V(\tilde{G})} |\delta^-_G(v)| y_v \leq 8 \sum_{v \in V(\tilde{G})} y_v = 4 \overline{\text{lb}}(\tilde{G}),$$

where for the inequality we used that $y_v$ is only strictly positive if $x(\delta^-(v)) = 1$ (see Definition 2.3), in which case $|\delta^-_G(v)| = |\delta^-_F(v)| = 2|\delta^-_F(v)| \leq 8$ using property (c) of Lemma 7.3.

Corollary 7.7. The algorithm is $(8 + w(B)/\text{value}(I))$-light (with respect to $\text{lb}$).
Proof. Lemma 7.6 implies

\[ w(\tilde{G}) \leq 4 \overline{\text{lb}}(\tilde{G}) = 4 \cdot \frac{\overline{\text{lb}}(V)}{\text{value}(I)} \cdot \text{lb}(\tilde{G}). \]

Furthermore,\(^{10}\)

\[ \overline{\text{lb}}(V) = \text{value}(I) + \frac{w(B)}{4} + \sum_{v \in V \setminus V(B)} 2y_v \leq 2 \text{value}(I) + \frac{w(B)}{4}. \]

\[ \square \]

We have thus given a polynomial-time algorithm for Local-Connectivity ATSP for vertebrate pairs \((I, B)\) that is \((8 + \frac{w(B)}{\text{value}(I)})\)-light. Together with Theorem 7.2 this implies Theorem 7.1, the main result of this section.

The only missing ingredient now is a proof of the main technical Lemma 7.3. We devote Section 7.3 to it.

7.3 The Split Graph

The main step in the proof of Lemma 7.3 is to solve a circulation problem in an auxiliary graph that we construct in two steps. First, for the given instance \(I\), we define the split graph. In the second step, we further modify the split graph, taking the input \(U_1, U_2, \ldots, U_r\) of Lemma 7.3 into account.

Recall that \(\mathcal{L}_{\geq 2}\) denotes the family of non-singleton sets in \(\mathcal{L}\). Let us use an indexing \(\mathcal{L}_{\geq 2} \cup \{V\} = \{S_1, S_2, \ldots, S_\ell\}\) such that \(2 \leq |S_1| \leq |S_2| \leq \cdots \leq |S_\ell| = |V|\). For a vertex \(v \in V\) let

\[ \text{level}(v) = \min\{i : v \in S_i\} \]

be the index of the first (smallest) set that contains \(v\). We use these levels to define a partial order \(<\) on the vertices: let \(v < v'\) if \(\text{level}(v) < \text{level}(v')\). This partial order is used to classify the edges as follows. An edge \((u, v) \in E\) is a

- forward edge if \(v < u\),
- backward edge if \(u < v\),

and otherwise it is a neutral edge. Let \(E_f, E_b\) and \(E_n\) denote the sets of forward, backward, and neutral edges respectively.

The idea behind the split graph and the three edge types is the following. We would like to ensure the crucial property \((d)\) of Lemma 7.3, which is roughly that, in the solution \(F\), any cycle that crosses a set in \(\mathcal{L}_{\geq 2}\) must then also visit the backbone. Note that such a cycle must contain both a forward and a backward edge. For consider the cyclic sequence of levels of vertices in the cycle: it is not constant, therefore it must both increase and decrease at some points. Now, we will guarantee property \((d)\)

\(^{10}\)We note that a tighter bound could be given if we used that the backbone \(B\) had been obtained via a quasi-backbone (see Section 6), in which case one could conclude that \(2 \sum_{v \in V(B)} y_v \leq (1 - \delta) \text{value}(I)\). We keep the weaker bound for simplicity.
Figure 7: An example of the construction of $G_{sp}$ from $G$, $L_{\geq 2}$, and the backbone $B$. The vertices of the backbone are depicted in black. On the left, the forward edges are straight, the backward edges are swirly, and the neutral edges are dashed. On the right, the edges of $G_{sp}$ without a preimage in $G$ are dotted.

by splitting each vertex $v$ into two copies $v^0$ and $v^1$ and forcing forward edges to go between the 1-vertices and backward edges to go between the 0-vertices. Since our cycle contains both types, its version in the split graph will visit both 0-vertices and 1-vertices. Clearly, it will need to also contain an edge from a 1-vertex to a 0-vertex. But, crucially, in the split graph the only such edges will be of the form $(v^1, v^0)$ for backbone vertices $v \in V(B)$. This will guarantee that the cycle visits the backbone. (See Fact 7.9 for a formal statement and proof of this claim.)

Now we define the split graph (see Figure 7 for an example):

**Definition 7.8.** The split graph $G_{sp}$ is defined as follows. For every $v \in V$ we create two copies $v^0$ and $v^1$ in $V(G_{sp})$. The edge set $E(G_{sp})$ contains the following edges:

- For every $v \in V \setminus V(B)$ we create an edge $(v^0, v^1)$ of weight 0.
- For every $v \in V(B)$ we create edges $(v^0, v^1)$ and $(v^1, v^0)$ of weight 0.
- For every forward edge $(u, v) \in E_f$ we create an edge $(u^1, v^1)$ of weight $w(u, v)$.
- For every backward edge $(u, v) \in E_b$ we create an edge $(u^0, v^0)$ of weight $w(u, v)$.
- For every neutral edge $(u, v) \in E_n$ we create edges $(u^0, v^0)$ and $(u^1, v^1)$ of weight $w(u, v)$.

We denote the weight function on the edges of the split graph by $w_{sp}$. Vertices $v^0$ will be called 0-vertices, and vertices $v^1$ will be called 1-vertices.

We can naturally map $G_{sp}$ to $G$ by mapping the two vertices $v^0$ and $v^1$ to $v$ for every $v \in V$. We call the edges $(u^p, v^q) \in E(G_{sp})$ the preimages of the edge $(u, v) \in E$ for all $p, q \in \{0, 1\}$ for which such an edge is present in $E(G_{sp})$.

Now we can state and prove the fundamental consequence of our construction.

**Fact 7.9.** Consider a cycle $C_{sp}$ in $G_{sp}$. If the image of $C_{sp}$ in $G$ (obtained by contracting every pair $v^0, v^1$ of vertices into a single vertex $v$) crosses a tight set in $L_{\geq 2}$, then it visits a vertex of the backbone.
We remark that the image of a cycle in \( G_{sp} \) does not necessarily form a cycle in \( G \) (as some vertices may be visited twice), but it is a subtour.

**Proof.** Consider a cycle \( C_{sp} \) in \( G_{sp} \) whose image in \( G \) crosses a tight set in \( L_{\geq 2} \). In other words, there is a set \( S_l \in L_{\geq 2} \) such that \( C_{sp} \cap \delta([v^0, v^1 : v \in S_l]) \neq \emptyset \). Let \( i^* \) be the smallest index of such a set. Then, by the choice of \( i^* \), an edge \((u_{in}, v_{in})\) in the image of \( C_{sp} \) that enters \( S_l \) is a forward edge, and an edge \((u_{out}, v_{out})\) in the image of \( C_{sp} \) that exits \( S_l \) is a backward edge. By the construction of \( G_{sp} \), the single preimage of \((u_{in}, v_{in})\) is \((u_{in}^1, v_{in}^1) \in E(G_{sp})\), and the single preimage of \((u_{out}, v_{out})\) is \((u_{out}^0, v_{out}^0) \in E(G_{sp})\). Thus \( C_{sp} \) must contain both \((u_{in}^1, v_{in}^1)\) and \((u_{out}^0, v_{out}^0)\). The statement now follows since the only edges in \( G_{sp} \) from a 1-vertex to a 0-vertex are of the type \((u^1, u^0)\), where \( u \in V(B) \) is a vertex of the backbone. \( \square \)

In order to work on the split graph, we will need to transform \( x \) into a circulation in the split graph with similar guarantees as \( x \). To do this, we take advantage of the fact that every set \( S \in L_{\geq 2} \) has a non-empty intersection with the vertices of the backbone.

**Lemma 7.10.** There is a polynomial-time algorithm that finds an Eulerian vector \( x_{sp} \in R^{|E(G_{sp})|}_+ \) such that the image of \( x_{sp} \) in \( G \) is \( x \), the optimal solution to LP\((G, w)\). In particular, \( \sum_{e \in E(G_{sp})} w(e)x_{sp}(e) = \sum_{e \in E(G)} w(e)x(e) = \text{value}(f) \), and \( x_{sp}(\delta([v^0, v^1 : v \in U])) = x(\delta(U)) \geq 2 \) for each proper subset \( \emptyset \neq U \subseteq V \).

**Proof.** The main ingredient of the proof is the following claim.

**Claim 7.11.** In polynomial time, we can find a non-negative vector \( f \in R^E_+ \) satisfying:

(a) \( f \leq x \),
(b) \( f(\delta^+(v)) \geq f(\delta^-(v)) \) for every \( v \in V \setminus V(B) \),
(c) \( f(e) = 0 \) for each backward edge \( e \in E_b \),
(d) \( f(e) = x(e) \) for each forward edge \( e \in E_f \).

Intuitively, \( f \) resembles a flow supported on \( x \) that saturates all forward edges, does not use backward edges, and has the backbone as a sink. Before we give the proof, let us first motivate the existence of such an \( f \) in a simple example scenario where there is only one non-singleton set \( S \in L_{\geq 2} \). Then we have \( E_f = \delta^-(S) \) and \( E_b = \delta^+(S) \), i.e., the forward/backward edges are exactly the incoming/outgoing edges of \( S \). The subtour elimination constraints imply (via the min-cut max-flow theorem) that \( x \) supports a unit flow between any pair of vertices. Let \( f \) be such a flow from any vertex outside \( S \) to a vertex \( v \in S \cap V(B) \) (such a \( v \) exists by the backbone property). It is easy to see that \( f \) satisfies the conditions of the claim. Indeed, since \( S \) is a tight set, \( f \) saturates all incoming (forward) edges. It also does not leave \( S \), i.e., use any backward edges.

Now we proceed to give the full proof, which uses a more general argument based on LP duality to argue the existence of \( f \).
Proof of Claim 7.11. We find $f \in \mathbb{R}_+^E$ in polynomial time by solving the following linear program:

\[
\begin{align*}
\text{maximize} & \quad \sum_{e \in E_f} f(e) \\
\text{subject to} & \quad f(\delta^+(v)) \geq f(\delta^-(v)) \quad \text{for } v \in V \setminus V(B), \\
& \quad f(e) = 0 \quad \text{for } e \in E_b, \\
& \quad 0 \leq f \leq x.
\end{align*}
\]

By the constraints of the linear program, we have that $f$ satisfies (a), (b), and (c). It remains to verify (d), or equivalently, to show that the optimum value of this program equals $x(E_f)$.

This will be shown using the dual linear program. The variables $(\pi_v)_{v \in V}$ correspond to the first set of constraints, and $(z(e))_{e \in E_f \cup E_n}$ to the capacity constraints on forward and neutral edges. No such variables are needed for backward edges. For notational simplicity, we introduce $\pi_v$ also for $v \in V(B)$, and set $\pi_v = 0$ in this case. The dual program can be written as follows.

\[
\begin{align*}
\text{minimize} & \quad \sum_{e \in E_f \cup E_n} x(e)z(e) \\
\text{subject to} & \quad \pi_v - \pi_u + z(u, v) \geq 1 \quad \text{for } (u, v) \in E_f, \\
& \quad \pi_v - \pi_u + z(u, v) \geq 0 \quad \text{for } (u, v) \in E_n, \\
& \quad \pi_v = 0 \quad \text{for } v \in V(B), \\
& \quad \pi, z \geq 0.
\end{align*}
\]

Note that setting $\pi = 0$, $z(e) = 1$ if $e \in E_f$, and $z(e) = 0$ if $e \in E_n$ is a feasible solution with objective value $x(E_f)$. We complete the proof by showing that this is an optimal dual solution.

Let us select a dual optimal solution $(\pi, z)$ which minimizes $\pi(V)$. We show that this minimum value is 0, that is, there exists a dual optimal solution with $\pi = 0$. This immediately implies that the above solution is a dual optimal one, because given $\pi = 0$ we get a constraint $z(u, v) \geq 1$ for all $(u, v) \in E_f$, making the objective value at least $x(E_f)$.

Towards a contradiction, assume that $\pi$ is not everywhere zero. Let us select a vertex $t \in V$ such that $\pi_t > 0$, and level($t$) = $i^*$ is minimal among all such vertices. Let $S = S_{i^*}$, and define $T = \{u \in S : \pi_u > 0\}$. Since $S \cap V(B) \neq \emptyset$, and $\pi_u = 0$ for all $u \in V(B)$, we see that $T$ is a proper subset of $S$. Let $F = \delta(V \setminus S, T)$ and $F' = \delta(T, S \setminus T)$. A depiction of the sets is as follows:
Let us show that the edges between $T$ and $S \setminus T$ can be only of certain types.

Claim. $F' = \delta(T, S \setminus T) \subseteq E_f \cup E_n$ and $\delta(S \setminus T, T) \subseteq E_b \cup E_n$.

Proof of Claim. By the choice of $i'$, for any $S_i \subseteq S$ we have that $\pi_v = 0$ for all $v \in S_i$; thus $S_i \cap T = \emptyset$, and so for every $u \in T$ we have $\text{level}(u) = i'$. Therefore, for every $(u, v) \in F'$, we must have $\text{level}(u) = i' \geq \text{level}(v)$, and for every $(u, v) \in \delta(S \setminus T, T)$, level$(v) = i' \geq \text{level}(u)$.

Let us construct another dual solution $(\pi', z')$ as follows. Let $\epsilon = \min\{\pi_u : u \in T\}$, and let

$$
\pi'_u = \begin{cases} 
\pi_u - \epsilon & \text{if } u \in T, \\
\pi_u & \text{otherwise,}
\end{cases} \quad \text{and} \quad z'(e) = \begin{cases} 
z(e) + \epsilon & \text{if } e \in F \cap (E_f \cup E_n), \\
z(e) - \epsilon & \text{if } e \in F', \\
z(e) & \text{otherwise.}
\end{cases}
$$

We show that $(\pi', z')$ is another optimal solution. Then, $\pi'(V) < \pi(V)$ gives a contradiction to the choice of $(\pi, z)$. The proof proceeds in two steps: first we show feasibility and then optimality.

Feasibility. We have $\pi' \geq 0$ by the choice of $\epsilon$. To show $z' \geq 0$, note that for $e = (u, v) \in F'$, by the Claim and the dual constraints for $e \in E_f \cup E_n$ we have $0 \leq \pi_v - \pi_u + z(u, v) \leq z(u, v) - \epsilon$, where for the second inequality we used that $\pi_u \geq \epsilon$ and $\pi_v = 0$. Thus, $z(e) \geq \epsilon$ and $z'(e) = z(e) - \epsilon \geq 0$ for every $e \in F'$. For an edge $e \notin F'$, we have $z'(e) \geq z(e) \geq 0$ and so we can conclude that $z' \geq 0$. We also have $\pi'_v = 0$ for $v \in V(B)$ since $T \cap V(B) = \emptyset$.

It remains to verify the constraints for $(u, v) \in (F \cup F') \cap (E_f \cup E_n)$, as well as for edges not in $\delta(T)$, we have $\pi'_v - \pi'_u + z'(u, v) = \pi_v - \pi_u + z(u, v)$, and therefore the constraint remains valid. Thus we may have $\pi'_v - \pi'_u + z'(u, v) \neq \pi_v - \pi_u + z(u, v) + \epsilon$ in only two cases: either (i) if $(u, v) \in \delta(T, V \setminus S)$, or (ii) if $(u, v) \in \delta(S \setminus T, T)$.

In case (i), the constraint on $(u, v)$ remains valid since $\pi'_v - \pi'_u + z'(u, v) = \pi_v - \pi_u + z(u, v) + \epsilon$. In case (ii), $\pi'_u = 0$, $\pi'_v \geq 0$, and $z'(u, v) \geq 0$. Further, we have shown in the above Claim that $(u, v) \in E_b \cup E_n$. There is no constraint for $(u, v) \in E_b$, and $\pi'_v - \pi'_u + z'(u, v) \geq 0$ holds if $(u, v) \in E_n$.

Optimality. When changing $(\pi, z)$ to $(\pi', z')$, the objective value increases by $\epsilon(x(F \cap (E_f \cup E_n)) - x(F' \cap (E_f \cup E_n))) = \epsilon(x(F \setminus E_b) - x(F'))$; we will show that this is non-positive. Recall that $S$ is either a tight set or $V$, so that $x(\delta^-(S)) \leq 1$. Since $T \subseteq S$, we have $1 \leq x(\delta^-(S \setminus T)) = x(\delta^-(S)) - x(F) + x(F') \leq 1 - x(F) + x(F')$, and therefore $x(F') \geq x(F) \geq x(F \setminus E_b)$. Thus, the objective value does not increase, therefore $(\pi', z')$ must be optimal.

The existence of the optimal solution $(\pi', z')$ with $\pi'(V) < \pi(V)$ contradicts the choice of $(\pi, z)$, which completes the proof of Claim 7.11.

Claim 7.11 implies Lemma 7.10 by defining $x_{sp}$ as follows:

- for every edge $(u^0, v^0) \in E(G_{sp})$, set $x_{sp}(u^0, v^0) = x(u, v) - f(u, v)$,
- for every edge $(u^1, v^1) \in E(G_{sp})$, set $x_{sp}(u^1, v^1) = f(u, v)$,
- for every $v \in V$, set $x_{sp}(v^0, v^1) = \max\{0, f(\delta^+(v)) - f(\delta^-(v))\},$

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We are ready to prove Lemma 7.3. We restate it here for convenience.

**Lemma (restatement of 7.3).** There is a polynomial-time algorithm that solves the following problem. Let \((I, B)\) be a vertebrate pair, and let \(U_1, \ldots, U_\ell \subseteq V \setminus V(B)\) be disjoint non-empty vertex sets such that the subgraphs \(G[U_1], \ldots, G[U_\ell]\) are strongly connected and for every \(S \in L_{\geq 2}\) and \(i = 1, \ldots, \ell\) we have either \(U_i \cap S = \emptyset\) or \(U_i \subseteq S\). Then the algorithm finds an Eulerian multiset \(F \subseteq E\) of edges such that:

(a) \(w(F) \leq 3 \cdot \text{value}(I)\).

(b) \(|\delta_F(U_i)| \geq 1\) for every \(i = 1, \ldots, \ell\).

(c) \(|\delta_F(v)| \leq 4\) for every \(v \in V\) with \(x(\delta^-(v)) = 1\).

(d) any subtour in \(F\) that crosses a tight set in \(L_{\geq 2}\) visits a vertex of the backbone.

The proof proceeds in two steps. In the first step, we further modify \(G_{sp}\) and \(x_{sp}\) to \(G'_{sp}\) and \(x'_{sp}\), by taking the sets \(U_i\) into account. In the second step, we obtain the set \(F\) by solving a minimum-weight integral circulation problem on \(G'_{sp}\).

**Modifying the split graph.** The graph \(G'_{sp}\) is constructed as follows (see also Figure 8). For \(i = 1, \ldots, \ell\) let \(U_{sp}^i = \{v^0, v^1 : v \in U_i\} \subseteq V(G_{sp})\) be the set of vertices in the split graph corresponding to \(U_i\). We select edge sets \(X^-_i, X^+_i\), and a flow \(g_i\) as follows:

- Select a subset of edges \(X^-_i \subseteq \delta^-(U_{sp}^i)\) with \(x_{sp}(X^-_i) = 1/2\) such that either all edges in \(X^-_i\) enter 1-vertices, or all of them enter 0-vertices. This is possible since \(x_{sp}(\delta^-(U_{sp}^i)) \geq 1\) by Lemma 7.10.\(^{11}\)

- Define the set of outgoing edges \(X^+_i \subseteq \delta^+(U_{sp}^i)\) to be, intuitively, the edges over which the flow that entered \(U_{sp}^i\) by \(X^-_i\) exits \(U_{sp}^i\). That is, consider an arbitrary cycle decomposition of the circulation \(x_{sp}\), and look at the set of cycles containing the edges in \(X^-_i\). We define \(X^+_i\) as the set of edges on these cycles that first leave \(U_{sp}^i\) after entering \(U_{sp}^i\) on an edge in \(X^-_i\); clearly, \(x_{sp}(X^+_i) = 1/2\).\(^{12}\)

- Let \(g_i\) denote the flow on these cycle fragments connecting the heads of edges in \(X^-_i\) and the tails of edges in \(X^+_i\).

\(^{11}\)To obtain exactly 1/2, we might need to break an edge up into two copies, dividing its \(x_{sp}\)-value between them appropriately, and include one copy in \(X^-_i\) but not the other; we omit this for simplicity of notation, and assume there is such an edge set with exactly \(x_{sp}(X^-_i) = 1/2\).

\(^{12}\)Again, we might need to split some edges into two copies.
We define the weights \( w \) of the graph. The weight may only decrease:

Claim. One can find, in polynomial time, an integral circulation (i.e., an Eulerian vector) \( x \) solving the circulation problem. Having defined the modified split graph \( \mathcal{G}_i^\prime \), we set \( w_{\mathcal{G}_i} = w \), \( x_{\mathcal{G}_i} = x \), and those in \( \mathcal{G}_i^\prime \) to point to \( a_i \), and those in \( \mathcal{G}_n \) to point from \( a_i \). We further subtract the flow \( g_i \) inside \( \mathcal{G}_i^\prime \), hence the resulting vector \( x_{\mathcal{G}_i} \) will be Eulerian in \( \mathcal{G}_i^\prime \), with \( x_{\mathcal{G}_i}(\delta^+(a)) = x_{\mathcal{G}_i}(\delta^- (a)) = 1/2 \). We define the weights \( w_{\mathcal{G}_i} = w_{\mathcal{G}}(e) \) if \( e \) was not modified, and for every redirected edge \( e \) we set \( w_{\mathcal{G}_i} = \) as the weight of the edge it was redirected from. Thus, the total weight may only decrease: \( \sum_{e \in E(\mathcal{G}_i^\prime)} w_{\mathcal{G}_i} e x_{\mathcal{G}_i}(e) \leq \sum_{e \in E(\mathcal{G}_i)} w_{\mathcal{G}} e x_{\mathcal{G}}(e) = \text{value}(I) \) (it decreases if the flows \( g_i \) are nonzero on edges with positive weight).

Solving the circulation problem. Having defined the modified split graph \( \mathcal{G}_i^\prime \) and the circulation \( x_{\mathcal{G}_i} \), we now describe how to use them to find \( F \).

Claim. One can find, in polynomial time, an integral circulation (i.e., an Eulerian vector) \( z_{\mathcal{G}_i} \) in \( \mathcal{G}_i^\prime \) satisfying:

\[
- z_{\mathcal{G}_i}(\delta^-(v^0) \setminus \{(v^1, v^0)\}) \leq \left| 2x_{\mathcal{G}_i} \left(\delta^- (v^0) \setminus \{(v^1, v^0)\}\right) \right| \text{ for each } v^0 \in V(\mathcal{G}_i^\prime),
- z_{\mathcal{G}_i}(\delta^-(v^1) \setminus \{(v^0, v^1)\}) \leq \left| 2x_{\mathcal{G}_i} \left(\delta^- (v^1) \setminus \{(v^0, v^1)\}\right) \right| \text{ for each } v^1 \in V(\mathcal{G}_i),
- z_{\mathcal{G}_i}(\delta^+(a)) = 1 \text{ for each } i = 1, \ldots, \ell ,
- \sum_{e \in E(\mathcal{G}_i^\prime)} w_{\mathcal{G}_i} e z_{\mathcal{G}_i}(e) \leq 2 \cdot \text{value}(I).
\]
**Proof of Claim.** The well-known integrality of the circulation polyhedron asserts that, given integer upper and lower capacity bounds on the edges, if there is a feasible circulation, then for any weight function there is a minimum-weight integer circulation (see e.g. Corollary 12.2a in [Sch03]). We wish to use this theorem by transforming the above conditions into edge capacity bounds. We do so by introducing further auxiliary edges.

The reduction is as follows. For the first set of constraints, we split \( v^0 \) further into \( v^0 \) and \( \bar{v}^0 \), add an edge \((v^0, \bar{v}^0)\), and replace all incoming edges \((z, v^0) \in E(G'_{sp})\) for \( z \neq v^1 \) by \((z, \bar{v}^0)\). In that way, constraints from the first set become capacity constraints on the new edges \((\bar{v}^0, v^0)\) (which are integral because we take the ceiling). A similar construction is used for the second set of constraints. Finally, for the node capacity constraint on \( a_i \), we add a new \( \bar{a}_i \) with an edge \((\bar{a}_i, a_i)\), and redirect all incoming edges in \( \delta^-(a_i) \) to point to \( \bar{a}_i \). We add upper and lower bounds on these new edges as required.

Note that the fractional circulation \( 2x'_{sp} \) satisfies all the conditions in the claim; its natural image under these transformations will satisfy all edge capacity constraints. Hence we can find a minimum-weight integer circulation in the modified graph, which naturally maps back to an integral circulation \( z'_{sp} \) in \( G'_{sp} \) whose weight is at most that of \( 2x'_{sp} \), i.e., at most 2 \( \text{value}(I) \).

We then let \( F_{sp} \subseteq E(G_{sp}) \) be the multiset of edges of the (unmodified) split graph obtained by, for each \( e \in E(G'_{sp}) \), taking \( z'_{sp}(e) \) copies of the corresponding edge in \( G_{sp} \), i.e., if \( e \) is incident to an auxiliary vertex \( a_i \), then reverse the redirection of this edge so as to obtain its preimage in \( G_{sp} \). We thus have \( w_{sp}(F_{sp}) = \sum_{e \in E(G_{sp})} w'_{sp}(e)z'_{sp}(e) \leq 2 \cdot \text{value}(I) \).

We remark that \( F_{sp} \) may not be Eulerian. Specifically, since the in- and out-degree of \( a_i \) were exactly 1 in \( z'_{sp} \), in each component \( U_i \), there is a pair of vertices \( u_i^p, v_i^p \) which are the head and tail, respectively, of the mapped-back edges adjacent to \( a_i \). These are the only vertices whose in-degree may differ from their out-degree. (They differ unless \( u_i^p = v_i^p \).) As the third step, to repair this, for each \( i = 1, \ldots, \ell \) we route a walk \( P_i \) from \( u_i^p \) to \( v_i^p \) in \( G_{sp} \). The walk \( P_i \) is selected as follows:

- Let \( u_i, w_1, \ldots, w_i, v_i \) be a shortest path in \( G \) from \( u_i \) to \( v_i \) inside \( U_i \), which exists since \( G[U_i] \) is assumed to be strongly connected.
- By the assumption on \( U_i \) (that it is either contained in or disjoint from any set in \( L_{2,2} \)), all edges in \( G[U_i] \) cross no set in \( L_{2,2} \), i.e., are neutral edges. Hence the path \( \bar{P}_i = u_i^p, w_1^p, \ldots, w_i^p, v_i^p \) is present in \( G_{sp} \).
- If \( p = q \), then we define \( P_i = \bar{P}_i \). If \( p = 0 \) and \( q = 1 \), then we define \( P_i \) by appending the edge \((v_i^p, v_i^1)\) to \( \bar{P}_i \); note that this edge is always included in \( E(G_{sp}) \). Finally, the next claim shows that \( p = 1, q = 0 \) is not possible.

**Claim.** If the head of an edge in \( X_i^- \) is a 1-vertex, then the tail of every edge in \( X_i^+ \) is a 1-vertex.

**Proof of Claim.** By the way that \( X_i^- \) was defined, all edges in \( X_i^- \) enter 1-vertices if at least one does. Further, the only edges from 1-vertices to 0-vertices in \( G_{sp} \) are of the
type \((v^1, v^0)\) where \(v \in V(B)\), and it is assumed that \(U_i \cap V(B) = \emptyset\). Therefore, any flow that enters \(U_i^{sp} \cap 1\)-vertex only visits 1-vertices until it exits \(U_i^{sp}\).  

The Eulerian multiset \(F \subseteq E\) of edges in \(G\) is now obtained from the Eulerian set \(\bigcup_{i=1}^\ell P_i \cup F_{sp}\) of edges in \(G_{sp}\) by contracting every pair \(v^0, v^1\) into a single vertex \(v\). This completes the description of the algorithm for finding \(F\).

The computation of \(F\) can be done in polynomial time as it is based on solving a minimum-weight circulation problem on \(G_{sp}'\), together with basic graph operations. It remains to verify properties \((a)-(d)\) of the lemma.

For \((a)\), we use that the path \(P_i\) has the same weight as a shortest path in \(G[U_i]\) from a vertex \(u_i \in U_i\) to a vertex \(v_i \in U_i\). Since \(U_i\) is either disjoint from or contained in any tight set in \(\mathcal{L}_{\geq 2}\), every edge \((u, v)\) inside \(U_i\) has weight \(y_u + y_v\), and thus we have \(w(P_i) \leq 2 \cdot \sum_{v \in U_i} y_v \leq \text{value}(U_i)\). The overall weight of \(F\) is the weight of the edges corresponding to \(z_{sp}'\) plus the weight of the paths. Hence

\[
w(F) \leq 2 \cdot \text{value}(I) + \sum_{i=1}^\ell \text{value}(U_i) \leq 3 \cdot \text{value}(I)
\]

by disjointness of the sets \(U_i\).

Now consider property \((b)\). For each \(i = 1, \ldots, \ell\) there is an edge \(e \in F\) corresponding to \((i.e., the preimage of)\) the incoming edge \(e'\) of the auxiliary vertex \(a_i\) with \(z_{sp}'(e') = 1\). By construction, the edge \(e'\) corresponds to an edge \(e''\) in \(G_{sp}\) such that \(e'' \in X_i^{-} \subseteq \delta^{-}(U_i^{sp})\) and thus \(e \in \delta^{-}(U_i)\). So \((b)\) holds.

For property \((c)\), first notice that path \(P_i\) was formed from a shortest path that was completely contained in \(U_i\), and since the sets \(U_i\) (and thus \(U_i^{sp}\)) are disjoint, the degree in \(F\) of any vertex \(v\) is at most one more than the degree of \(\{v^0, v^1\}\) in the circulation \(z_{sp}'\). Thus

\[
|\delta_F^{-}(v)| \leq z_{sp}'(\delta^{-}(\{v^0, v^1\})) + 1 = z_{sp}'(\delta^{-}(v^0) \cup (v^1, v^0)) + z_{sp}'(\delta^{-}(v^1) \cup (v^0, v^1)) + 1 \\
\leq \left[2x_{sp}'(\delta^{-}(v^0) \cup (v^1, v^0)) + 2x_{sp}'(\delta^{-}(v^1) \cup (v^0, v^1))\right] + 1.
\]

Assume now that \(x(\delta^{-}(v)) = 1\). We have

\[
x(\delta^{-}(v)) = x_{sp}(\delta^{-}(\{v^0, v^1\})) \\
\geq x_{sp}'(\delta^{-}(\{v^0, v^1\})) \\
= x_{sp}'(\delta^{-}(v^0) \cup (v^1, v^0)) + x_{sp}'(\delta^{-}(v^1) \cup (v^0, v^1)),
\]

where the first equality is by Lemma 7.10. Thus we have to bound \([2p] + [2q] + 1\) subject to \(p + q < 1\). Clearly, the maximum is 4, and \((c)\) follows.

Finally, property \((d)\) immediately follows from Fact 7.9. Indeed, \(F\) was obtained from an Eulerian edge set in the split graph \(G_{sp}\) by contracting every pair \(v^0, v^1\) into a single vertex \(v\). In other words, \(F\) is the image of an Eulerian multiset of edges in the split graph \(G_{sp}\). As any Eulerian multiset can be be decomposed into cycles, this is equivalent to saying that \(F\) is the union of images of cycles in \(G_{sp}\). Fact 7.9 says that any such image that crosses a tight set in \(\mathcal{L}_{\geq 2}\) must visit a vertex in the backbone.

This concludes the proof of Lemma 7.3.
8 Completing the Puzzle: Proof of Theorem 1.1

We now combine the techniques and algorithms of the previous sections to obtain a constant-factor approximation algorithm for ATSP. In multiple steps, we have reduced ATSP to finding tours for vertebrate pairs. Every reduction step was polynomial-time and increased the approximation ratio by a constant factor. Hence, together they give a constant-factor approximation algorithm for ATSP.

We now give an overview of these reductions and set the parameters. Throughout, $\varepsilon > 0$ will be a fixed small value. We set $\delta = 0.75$. All approximation guarantees are with respect to the optimum value of the Held-Karp relaxation LP($G, w$). The reduction proceeds using the following algorithmic subroutines.

- Algorithm $\mathcal{A}_{NW}$ from [Sve15], which is a polynomial time $\alpha_{NW}$-approximation for node-weighted ATSP, with $\alpha_{NW} = 27 + \varepsilon$ (Theorem 2.2). This will be used to find a (quasi-)backbone for irreducible instances (Lemma 6.3).

- Algorithm $\mathcal{A}_{ver}$, which, for a vertebrate pair ($I, B$), finds a tour of cost $\kappa \text{value}(I) + \eta w(B)$, where $\kappa = 8(9 + \varepsilon)$ and $\eta = (9 + \varepsilon)$ (Theorem 7.1). Algorithm $\mathcal{A}_{ver}$ uses the reduction Theorem 7.2 from Local-Connectivity ATSP to ATSP (see [Sve15]). For Local-Connectivity ATSP, we use the $8 + w(B)/\text{value}(I)$-light algorithm of Section 7.2.

- Algorithm $\mathcal{A}_{irr}$ which, provided $\mathcal{A}_{ver}$ as above, obtains a polynomial-time $\rho$-approximation algorithm for irreducible instances, where $\rho = (\kappa + \eta(\alpha_{NW} + 3))/(1 - 2(1 - \delta)) = 684 + \varepsilon'$ (Theorem 6.4).

- Algorithm $\mathcal{A}_{lam}$, which converts the $\rho$-approximation algorithm $\mathcal{A}_{irr}$ to a $2\rho/(1 - \delta)$-approximation algorithm for an arbitrary laminarily-weighted instance $I$. Here, $2\rho/(1 - \delta) = 5472 + \varepsilon'' < 5500$ (Theorem 5.3).

- Our final Algorithm $\mathcal{A}_{ATSP}$, which reduces an arbitrary input weighted digraph $(G, w)$ to a laminarily-weighted instance, keeping the same approximation ratio (Theorem 2.4).

All in all we have thus obtained a polynomial-time algorithm for ATSP that returns a tour of value at most a constant (5500) times the Held-Karp lower bound.

9 Conclusion

In this paper we gave the first constant-factor approximation algorithm for ATSP. The result was obtained in two steps. First, we gave a generic reduction to ATSP instances with a backbone, i.e., vertebrate pairs. These instances were then solved using the connection to Local-Connectivity ATSP introduced in [Sve15]. We believe that, by specializing and optimizing the techniques of [Sve15] to the setting of this paper, the integrality gap of the LP relaxation can be upper-bounded by the hundreds. However, achieving an upper bound on the integrality gap that is close to the current lower bound of 2, even say an upper bound of 50, seems to require substantial progress. We raise this as an important open problem.
Open Question 9.1. Is the integrality gap of the standard LP relaxation upper-bounded by 2?

As mentioned in the introduction, Asadpour et al. [AGM+10] introduced a different approach for ATSP based on so-called thin spanning trees. Our algorithm does not imply a better construction of such trees and the $O(\text{poly log log } n)$-thin trees of [AG15] remain the best such (non-constructive) result. Whether trees of better thinness exist is an interesting question. Also, as shown in [AKS10], the construction of $O(1)$-thin trees would lead to a constant-factor approximation algorithm for the bottleneck ATSP problem. There, we are given a complete digraph with edge weights satisfying the triangle inequality, and we wish to find a Hamiltonian cycle that minimizes the maximum edge weight. A tight $2$-approximation algorithm for bottleneck symmetric TSP was given already in [Fle74, Lau81, PR84], but no constant-factor approximation is known for bottleneck ATSP.

Open Question 9.2. Is there a $O(1)$-approximation algorithm for bottleneck ATSP?

We believe that this is an interesting open question in itself, and progress on it may shed light on the existence of $O(1)$-thin trees.

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