Non-orthogonal bases for quantum metrology

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Abstract
Many quantum statistical models are most conveniently formulated in terms of non-orthogonal bases. This is the case, for example, when mixtures and superpositions of coherent states are involved. In these instances, we show that the analytical evaluation of the quantum Fisher information matrix may be greatly simplified by avoiding both the diagonalization of the density matrix and the orthogonalization of the basis. The key ingredient in our method is the Gramian matrix (i.e. the matrix of scalar products between basis elements), which may be interpreted as a metric tensor for index contraction. As an application, we derive novel analytical results for several estimation problems involving noisy Schrödinger cat states.

Keywords: quantum metrology, nonorthogonal bases, Schrödinger cat states

1. Introduction
The quantum Fisher information (QFI) matrix, and the associated symmetric logarithmic derivatives (SLDs) are central theoretical tools in quantum estimation theory and quantum metrology [1–4]. Under certain regularity assumptions, the QFI matrix provides the ultimate precision bounds on the estimation of unknown parameters encoded in a density matrix: the quantum Cramér–Rao bounds. More in detail, the covariance matrix of any unbiased estimator is, up to a constant, bounded from below by the inverse of the QFI matrix. The SLDs are

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instead self-adjoint operators that serve a double purpose: while they provide a possible route to calculate the QFI matrix elements, their commutators also determine whether the quantum Cramér–Rao bounds may be saturated by physically realisable measurements [5, 6].

The associated applications are plenty, including phase and frequency estimation [4, 7–17], estimation of noise parameters [18–23], joint estimation of unitary and/or noise parameters [24–31], sub-wavelength resolution of optical sources [32–38], nano-scale thermometry [39–45], and estimation of Hamiltonian parameters in the presence of phase-transitions [46–48]. The most common approach for calculating the QFI matrix involves the diagonalisation of the density matrix \( \rho \) [3]. This method has the tendency to yield tedious calculations, often preventing analytical treatment even in apparently simple quantum statistical models. In continuous-variable systems the calculation can be greatly simplified for Gaussian states, in which case all the quantities of interest are univocally determined by the first and second moments of an appropriate set of quadrature (or canonical) operators [49–56]. For finite-dimensional systems, Šafraňek has recently pointed out that the diagonalization of \( \rho \) is unnecessary [57], and one may build a formal solution for the SLDs via vectorization techniques. However, this approach still assumes the availability of an orthonormal basis spanning the support of \( \rho \) and its derivatives. As can be appreciated from several recent works on sub-wavelength resolution, however, the construction of such orthonormal bases can become extremely tedious even for rank-2 states, whenever additional basis elements are needed to span the support of the SLDs—for example see [32, 37] among many others. [58, 59], instead, employed non-orthogonal states to expand the eigenvectors of \( \rho \), but still relied on lengthy diagonalization methods to calculate the QFI.

The purpose of this work is to show that, in fact, both the diagonalization of the density matrix and the construction of an orthonormal basis may be safely bypassed in quantum metrology calculations, by introducing a nontrivial metric tensor for index contraction (the Gramian matrix of the chosen basis). We note that a problem-specific version of the theory presented here, which however was not formulated in terms of a modified metric, was already employed by one of the authors in a recent paper [38]. Here, we expand on these ideas to show a more general and elegant procedure that will streamline the calculation of the SLDs and QFI matrix in many cases of interest. Our method is applicable to any finite-rank quantum statistical model involving non-orthogonal quantum states. To demonstrate the power of the approach, we provide novel analytical results for several quantum estimation problems involving noisy superpositions of coherent states (or ‘noisy cat states’). As a useful sanity check for the reliability of our methods, we also show how known results on phase estimation with entangled coherent states [58, 59] can be recovered by our procedure with very little effort.

The paper is organised as follows: in section 2 we briefly review the quantum Cramér–Rao bound and outline the central mathematical problem of quantum estimation theory, i.e. the calculation of the SLDs; in section 3 we explain the core of our calculation method relying on non-orthogonal bases; in section 4 we show examples of novel analytical results that can be obtained with our method, while in section 5 we draw our conclusions.

2. Quantum Cramér–Rao bounds: brief review

Following standard local quantum estimation theory, we consider a quantum statistical model of the form

\[
\rho \equiv \rho(\lambda),
\]

that is, a family of quantum states depending on a set of parameters \( \lambda = (\lambda_1, ..., \lambda_N) \). It is assumed that the functional form of \( \rho \) is known, but that the exact values of the parameters
\(\lambda\) are not. The aim of local estimation theory is to set the ultimate bounds on how precisely these parameters can be estimated, given \(M \gg 1\) copies of the state \(\varrho\), and assuming the ability to perform any measurement allowed by quantum mechanics. For any unbiased estimator \(\hat{\lambda}\), i.e. a function of the measured data providing the correct guess on average, \(E[\hat{\lambda}] = \lambda\), the quantum Cramér-Rao bound (QCRB) holds

\[
V \geq \frac{1}{M} H^{-1},
\]

where \(V\) is the covariance matrix of the estimator, with elements \(V_{\mu \nu} = E[\hat{\lambda}_\mu \hat{\lambda}_\nu] - E[\hat{\lambda}_\mu]E[\hat{\lambda}_\nu]\), while \(H\) is the QFI matrix whose elements are

\[
H_{\mu \nu} = \frac{1}{2} \text{Tr}[\varrho[L_\mu, L_\nu]].
\]

The operators \(L_\mu\) appearing above are known as symmetric logarithmic derivatives (SLDs), and can be found by solving the operator Lyapunov equation \([1–3]\)

\[
2\partial_\mu \varrho = \varrho L_\mu + L_\mu \varrho,
\]

where the shorthand \(\partial_\mu \equiv \frac{\partial}{\partial \lambda_\mu}\) shall be used throughout the manuscript. In the multi-parameter case, the matrix inequality \((2)\) describing the QCRB is often rewritten in terms of the scalar QCRB

\[
\text{Tr}[GV] \geq \frac{1}{M} \text{Tr}[GH^{-1}],
\]

where \(G\) is a generic positive definite matrix (or weight matrix). It is known that the scalar QCRB may be saturated by a physical measurement iff the following matrix elements vanish for all \(\mu \neq \nu\):

\[
\Gamma_{\mu \nu} = \frac{1}{2i} \text{Tr}[\varrho[L_\mu, L_\nu]].
\]

It is important to note that, in general, saturating the QCRB requires global measurements over many copies of the state \(\varrho\) \([5]\). However, if the stronger compatibility condition \([L_{\bar{\mu}}, L_{\bar{\nu}}] = 0\) holds for a given pair of indices \(\bar{\mu}, \bar{\nu}\), the common eigenbasis of \(L_{\bar{\mu}}\) and \(L_{\bar{\nu}}\) provides a projective measurement that can be implemented on a single copy of \(\varrho\) and is optimal for both \(\lambda_{\bar{\mu}}\) and \(\lambda_{\bar{\nu}}\). It is worth pointing out that equation \((4)\) does not have a unique solution when \(\varrho\) is not full rank; however, the quantities in equations \((3)\) and \((6)\) are uniquely defined (see the appendix for a general proof, or \([60]\) for the pure state case).

To conclude this section, we emphasize that solving equation \((4)\) embodies the central mathematical problem of quantum estimation theory. In what follows, we outline how equation \((4)\) may be converted into a Lyapunov matrix equation by selecting a suitable (not necessarily orthonormal) set of quantum states spanning the support of \(\varrho\) and its derivatives.

### 3. Expanding operators on a non-orthogonal basis

The observation at the core of our work is that the density matrix \(\varrho\) may be expanded as

\[
\varrho = \sum_{i} R_{ij} |\psi_i\rangle \langle \psi_j|,
\]

where it is not necessary to assume that the set \(\{|\psi_i\rangle\}\) is orthonormal—the only requirement is that it should be linearly independent and contain enough states to expand \(\varrho\) and its derivatives.
derivatives. Note that both the coefficients $R_{ij}$ and the states $|\psi_i\rangle$ may depend on the unknown parameters $\lambda$. For brevity we shall refer to the set $\mathcal{B} \equiv \{|\psi_i\rangle\}$, as a basis, keeping in mind that in general it may span only a subspace of the full Hilbert space: the joint support of $\rho$ and its derivatives. While the quantum states $|\psi_i\rangle$ may be chosen in whichever way simplifies the problem at hand, it is worth mentioning at least one explicit method to construct such a set. This is comprised of the following steps:

1. Pick a convenient, linearly independent set of quantum states $\mathcal{B}_\rho$, spanning the support of $\rho$.
2. Compute the set of all the nonzero derivatives $\mathcal{B}' \equiv \{\partial_\mu |\psi\rangle |\psi\rangle \in \mathcal{B}_\rho, \mu \text{ s.t. } \partial_\mu |\psi\rangle \neq 0\}$.
3. As a starting point, set $\mathcal{B} = \mathcal{B}_\rho$.
4. For each $|\psi\rangle \in \mathcal{B}'$, check whether $\mathcal{B} \cup \{|\psi\rangle\}$ is a linearly independent set. If so, enlarge the set $\mathcal{B}$ by including the state $|\psi\rangle$; if not, discard $|\psi\rangle$.

We note that the above method returns a basis $\mathcal{B}$ that spans the joint support of $\rho$ and all the SLDs. In section 3.2, we discuss how the calculation of each $L_\mu$ may be simplified even further by employing a smaller subset $\mathcal{B}^{(\mu)} \subseteq \mathcal{B}$, spanning the support of $\rho$ and $L_\mu$ only. Explicit examples will be given in section 4. In the next subsection we explain how to convert equation (4) into a set of matrix Lyapunov equations by introducing an appropriate metric tensor characterizing the set $\mathcal{B}$.

### 3.1. Setting up the SLD equations

We shall adopt the following convention: a matrix with elements $A_{ij}$ will be denoted as $\mathbf{A}$. For example, according to equation (7) the state $\rho$ is represented by the matrix $\mathbf{R}$. Our first step towards solving equation (4) is to expand the derivatives of $\rho$ as per

$$\partial_\mu \rho = \sum_{ij} D_{ij}^{(\mu)} |\psi_i\rangle \langle \psi_j|,$$

which defines the matrices $\mathbf{D}^{(\mu)}$, and we do the same for the (yet unknown) SLDs,

$$L_\mu = \sum_{ij} L_{ij}^{(\mu)} |\psi_i\rangle \langle \psi_j|,$$

implicitly defining the matrices $\mathbf{L}^{(\mu)}$. The next ingredient we need is the matrix $\mathbf{S}$ of scalar products between the chosen basis vectors, also known as Gramian matrix. Its elements are

$$S_{ij} \equiv \langle \psi_i | \psi_j \rangle.$$

In passing, we remark that the set $\mathcal{B}$ is linearly independent iff $\det[\mathbf{S}] \neq 0$; this can provide a useful sanity check before the start of the procedures outlined below. It can be checked by direct calculation that the matrix $\mathbf{S}$ acts as a metric tensor for contracting indices. E.g.:

$$\rho L_\mu = \sum_{ij} R_{ij} |\psi_i\rangle \langle \psi_j| \sum_{kl} L_{kl}^{(\mu)} |\psi_k\rangle \langle \psi_l|$$

$$= \sum_{ij} \left( \sum_{jk} R_{ij} L_{jk}^{(\mu)} \frac{\langle \psi_j | \psi_k \rangle}{S_{jk}} \right) |\psi_i\rangle \langle \psi_j|$$

$$= \sum_{ij} \left( \sum_{jk} R_{ij} S_{jk} L_{kl}^{(\mu)} \right) |\psi_i\rangle \langle \psi_j|.$$
showing that $gL_\mu$ corresponds to the matrix $RSL(\mu)$. With a similar calculation, one can show that for example $\text{Tr}[gL_\mu] = \text{Tr}[RSL(\mu)]$, where one of the $S$ matrices comes from the operator multiplication, the other from the index contraction implicit in the trace. For completeness we also note that the physical requirements $\text{Tr}[\rho] = 1$ and $\rho \geq 0$ translate into the conditions $\text{Tr}[RS] = 1$ and $R \geq 0$ (equivalently, one may notice that the nonzero eigenvalues of $\rho$ coincide with the nonzero eigenvalues of the matrix $RS$).

Putting the above steps together, equation (4) can be recast as the matrix Lyapunov equation

$$2D(\mu) = L(\mu)SR + RSL(\mu),$$

which may be solved with standard linear algebraic methods to provide the matrices $L(\mu)$ and hence the SLDs. This can be achieved either by ‘brute force’, i.e. interpreting equation (12) as a linear system of equations for the variables $L(\mu)$, or by more sophisticated methods such as matrix vectorization [57]. Once the SLDs are known, the QFI matrix $H_{\mu\nu}$, as well as the matrix of averaged commutators $\Gamma_{\mu\nu}$ (both real matrices by construction), may be calculated via the relation

$$H_{\mu\nu} + i\Gamma_{\mu\nu} = \text{Tr}[RSL(\mu)SL(\nu)S],$$

which follows directly from equations (3) and (6). The calculation methods embodied by equations (12) and (13) constitute the main result of our work, and to the best of our knowledge have not appeared before in the quantum metrology literature. To conclude this section, we explain how the matrices of interest can be in general expressed in terms of standard quantum brakets and the $S$ matrix. Given a generic operator $A = \sum_\mu(A)_{\mu}\langle\psi\rangle$, we indicate as $\tilde{A}$ the matrix with elements $(\tilde{A})_{\mu\nu} = \langle\psi|\langle\psi|\langle\psi|A|\psi\rangle|\psi\rangle$. With straightforward algebra we get

$$\tilde{A} = SAS,$$

hence

$$A = S^{-1}\tilde{A}S^{-1}.$$ (15)

### 3.2. Constructing different bases for different parameters

So far we have assumed that the full set $\mathcal{B}$ was employed to calculate each SLD. In practice, to find a particular SLD $L_\mu$, it is sufficient (and advisable!) to include in equation (7), and in the subsequent derivations, a subset $B(\mu) \subseteq \mathcal{B}$ spanning the support of $\varrho$ and $\partial_\mu \varrho$ only. This in turn defines a reduced Gramian matrix which we will denote as $S(\mu)$, while the state $\varrho$ will be represented by the matrix $R(\mu)$. To avoid making our notation too cumbersome the matrix representation of the SLDs will be indicated by the symbol $L(\mu)$ also in this case. Furthermore, in many cases it may not be convenient to construct the set $\mathcal{B}$ by following the procedure outlined at the start of section 3, and one may instead begin by constructing each set $B(\mu)$ separately. In the simplest scenario, the union $\bigcup_\mu B(\mu)$ will be linearly independent and can be identified with the set $\mathcal{B}$—in passing we recall that the validity of such an assumption can be checked $a$ posteriori from the condition $\text{det}(S) \neq 0$. If so, the matrix representations of $\varrho$ and $L_\mu$, with respect to the full basis set $\mathcal{B}$, can be easily found by adding rows and columns of zeros wherever appropriate—see section 4 for an explicit example. In the most general scenario, however, some additional manipulations may be needed in order to construct the set $\mathcal{B}$ by discarding enough redundant (linearly dependent) vectors from the union set $\bigcup_\mu B(\mu)$. As a consequence, some additional (but straightforward) algebra will be required to find the matrix representations of the SLDs $L_\mu$ with respect to the full set $\mathcal{B}$. 

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In the next section, we shall apply the developed machinery to estimation problems involving coherent states. The latter embody one of the most common and useful examples of a non-orthogonal set of states.

4. Application: noisy Schrödinger cat states and entangled coherent states

4.1. Estimation of parameters characterizing a noisy cat state

A Schrödinger cat state is typically defined as the superposition of two coherent states of a single Bosonic mode (or a single quantum harmonic oscillator), that is

$$|\psi_{\text{cat}}\rangle = \frac{|\alpha\rangle + |\alpha\rangle}{\sqrt{2(1 + s)}}$$

where $s = (|\alpha| - |\alpha|) = e^{-2|\alpha|^2}$. The coherent states $|\alpha\rangle$ are defined as the eigenstates of the annihilation operator $a$, i.e. $a|\alpha\rangle = \alpha|\alpha\rangle$, where the Bosonic commutation relation $[a, a^\dagger] = 1$ is assumed. Since coherent states are traditionally seen as classical states in quantum optics, the superposition of two coherent states with opposite amplitude is often considered an example of a Schrödinger cat state (i.e. the superposition of two distinct classical configurations). Besides being interesting from a fundamental point of view, Schrödinger cat states are considered a resource for quantum information processing [61–66] (for example we will discuss in the next subsection how their non-classicality allows to obtain a quantum enhanced precision in the estimation of a displacement in phase space).

We will start by applying the methods discussed in the previous section to an ‘imperfect’ realisation of equation (16), i.e. we shall consider states of the form

$$\rho_{\text{cat}} = \mathcal{N} \left[ |\alpha\rangle\langle\alpha| + |\alpha\rangle\langle-\alpha| + c \left( |\alpha\rangle\langle-\alpha| + |\alpha\rangle\langle\alpha| \right) \right]$$

where we have introduced an additional parameter $0 \leq c \leq 1$, which loosely speaking quantifies the coherence of the superposition, and we have defined a normalization constant $\mathcal{N} = 1/(2(1 + sc))$. By fixing $c = 1$ one obtains the pure Schrödinger cat state previously introduced, while for $0 \leq c < 1$, the quantum states are mixed and can be interpreted as ‘noisy’ (or more precisely partially decohered) Schrödinger cat states. For example, if one considers an initial pure cat state with coherent state amplitude $\alpha_0$, evolving in time according to an amplitude damping master equation

$$\dot{\rho} = -\gamma a a^\dagger \rho + \gamma a^\dagger a \rho / 2$$

one obtains that the evolved quantum state at time $t$ has the form given in equation (17) with parameters [67]

$$\alpha = \alpha_0 e^{-\gamma t / 2}, \quad c = \exp \left\{-2|\alpha_0|^2 \left( 1 - e^{-\gamma t} \right) \right\}.$$

In the following we will first discuss the estimation of the parameters $\lambda = \{c, \alpha\}$ characterizing the quantum state $\rho_{\text{cat}}$, and via a simple change of variables, the parameters $\tilde{\lambda} = \{\tilde{\gamma} = \gamma t, \alpha_0\}$, characterizing the evolution of a pure Schrödinger cat state in a lossy channel. Next, we will consider this class of states as probes for the estimation of a unitary displacement in phase...
space, discussing in detail the robustness of the estimation precision against decoherence, and comparing the results with the ones obtainable with lossy squeezed states as inputs.

In what follows, for the sake of simplicity, we will consider \( \alpha \in \mathbb{R} \) and \( \alpha_0 \in \mathbb{R} \). While keeping \( \alpha \) and \( \alpha_0 \) generic could be an interesting generalisation, it would effectively introduce a new parameter to be estimated (the complex phase of \( \alpha \)). The resulting estimation problem is still tractable with our methods, but it yields lengthier calculations and additional technical details that go beyond the scopes of this manuscript. Following the methods in section 3, we will initially discuss the estimation of the two parameters \( \lambda = \{ c, \alpha \} \) separately, i.e. we shall calculate the diagonal QFI matrix elements \( H_{cc} \) and \( H_{c \alpha} \). Then, we will show how to calculate the remaining element \( H_{c \alpha} \) and hence the full quantum Fisher information matrix for the two parameters, as well as the quantity \( \Gamma_{c \alpha} \) characterizing the joint estimation properties. Finally, we shall extend the discussion to the parameters \( \tilde{\lambda} = \{ \tilde{\gamma} = \gamma t, \alpha_0 \} \).

For the coherence parameter \( c \) it is straightforward to observe that we can describe all the operators in terms of the two-dimensional basis \( \mathcal{B}^{(c)} = \{ |\alpha\rangle, | - \alpha\rangle \} \), which spans the support of \( \varrho \). We can immediately identify the matrices of interest:

\[
    R^{(c)} = \mathcal{N} \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix},
\]

\[
    S^{(c)} = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix},
\]

\[
    D^{(c)} = \mathcal{N} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - (2s\mathcal{N}) R^{(c)},
\]

where the last expression follows from

\[
    \partial_c \varrho = \mathcal{N} \left( |\alpha\rangle\langle -\alpha| + | -\alpha\rangle\langle \alpha| \right) - (2s\mathcal{N}) \varrho.
\]

The Lyapunov equation \( 2D^{(c)} = L^{(c)}S^{(c)}R^{(c)} + R^{(c)}S^{(c)}L^{(c)} \) can be solved analytically, yielding

\[
    L^{(c)} = \mathcal{N} \begin{pmatrix} -\frac{2 + \alpha^2 - 2c^{2} + 4c}{(1-c)^2(1+c)} & 2 \frac{2 + \alpha^2 + 2c}{(1-c)^2(1+c)} \\ 2 \frac{2 + \alpha^2 + 2c}{(1-c)^2(1+c)} & -\frac{2 + \alpha^2 - 2c^{2} + 4c}{(1-c)^2(1+c)} \end{pmatrix}.
\]

We can then calculate the QFI via equation (13), obtaining

\[
    H_{cc} = \text{Tr}[R^{(c)}S^{(c)}L^{(c)}S^{(c)}L^{(c)}S^{(c)}] = \frac{1 - e^{-4\alpha^2}}{(1 - e^2)(e^{-2\alpha^2}c + 1)^2}.
\]

The QFI is monotonically increasing with the amplitude \( |\alpha| \), and in particular in the limit of large coherent state amplitudes \( |\alpha| \to \infty \) we have \( H_{cc} \to (1 - c^2)^{-1} \) (note that the QFI presents a divergence at \( c = 1 \), where the state changes rank and the quantum Cramér-Rao bound does not hold [68, 69]).

If we now focus on the estimation of the parameter \( \alpha \), we find that we can expand the operators of interest in the extended basis \( \mathcal{B}^{(\alpha)} = \{ |\alpha\rangle, | -\alpha\rangle, \partial_\alpha |\alpha\rangle, \partial_\alpha | -\alpha\rangle \} \), which spans the joint support of \( \varrho \) and \( L_\alpha \). By observing that \( \partial_\alpha |\alpha\rangle = (a^\dagger - a)|\alpha\rangle \), one can evaluate the scalar product matrix as
\[
S^{(\alpha)} = \begin{pmatrix}
1 & s & 0 & -2s\alpha \\
s & 1 & -2s\alpha & 0 \\
0 & -2s\alpha & 1 & s(4\alpha^2 - 1) \\
-2s\alpha & 0 & s(4\alpha^2 - 1) & 1
\end{pmatrix}.
\] (26)

As anticipated in section 3, the state matrix associated with the new basis can be trivially found by adding appropriate rows and columns of zeros, i.e.

\[
R^{(\alpha)} = N \begin{pmatrix}
1 & c & 0 \\
c & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\] (27)

while differentiating \( \varrho_{\text{cat}} \) with respect to the parameter \( \alpha \), one obtains

\[
D^{(\alpha)} = N \begin{pmatrix}
0 & 1 & c & 0 \\
1 & c & 1 & 0 \\
c & 1 & 0 & 0
\end{pmatrix} + \frac{4\alpha c}{1 + 4\alpha} R^{(\alpha)}.
\] (28)

Also in this case the Lyapunov equation for \( L^{(\alpha)} \) can be solved analytically; we do not report the full solution here as the formula is cumbersome and does not give any particular insight. However, we remark that in this case the SLD \( L^{(\alpha)} \) is not unique, since the support of \( \varrho_{\text{cat}} \) is smaller than the support of \( \partial_{\alpha} \varrho_{\text{cat}} \). As a consequence, the matrix \( L^{(\alpha)} \) features some free parameters. Yet, as explained in the appendix, the corresponding QFI and commutator matrices remain uniquely defined. For the QFI of the parameter \( \alpha \), in particular, we obtain the expression

\[
H_{\alpha,\alpha} = \frac{\text{Tr}[R^{(\alpha)}S^{(\alpha)}L^{(\alpha)}S^{(\alpha)}L^{(\alpha)}S^{(\alpha)}]}{4\left(1 - c^2e^{-4\alpha^2} + 4c\alpha^2e^{-2\alpha^2}\right)} \frac{1}{\left(1 + c e^{-2\alpha^2}\right)^2}.
\] (29)

By inspecting this formula one observes a non-trivial behaviour both in terms of the coherent states amplitude \( \alpha \) and of the coherence parameter \( c \).

In the case we are considering here, the basis \( B^{(c)} \) is a strict subset of the basis \( B^{(\alpha)} \). As \( \det[S^{(\alpha)}] \neq 0 \), one can simply extend the matrix \( L^{(c)} \) with block matrices filled with zeros, and directly use the matrices \( R = R^{(\alpha)} \) and \( S = S^{(\alpha)} \) for all the calculations. In particular one obtains a non-zero off-diagonal element by using the formula

\[
H_{c,\alpha} = \frac{1}{2}\left(\text{Tr}[R^{(\alpha)}S^{(\alpha)}L^{(\alpha)}S^{(\alpha)}L^{(\alpha)}S^{(\alpha)}] + \text{Tr}[R^{(\alpha)}S^{(\alpha)}L^{(\alpha)}S^{(\alpha)}S^{(\alpha)}]\right)
\frac{1}{\left(1 + c e^{-2\alpha^2}\right)^2}.
\] (30)

showing that the estimation of the two parameters is ultimately correlated.

The same matrices can be exploited to investigate the joint estimation properties of the two parameters. In particular one can calculate the average value of the commutator between the SLD operators,
\[ \text{Tr}[\varrho_{\text{cat}}[L_\gamma, L_{\alpha}]] = \text{Tr}[\mathcal{RSL}^{(c)}\mathcal{S}L^{(\alpha)}\mathcal{S}] - \text{Tr}[\mathcal{RSL}^{(\alpha)}\mathcal{S}L^{(c)}\mathcal{S}] = 0. \] (31)

This weak commutativity condition ensures that the scalar bound on the variances of the two parameters can be achieved, and in principle this will require a joint measurement on an asymptotically large number of copies of the quantum state \( \varrho_{\text{cat}} \) [5].

One can apply the same machinery to analyse the estimation of the parameters \( \hat{\lambda} = \{ \hat{\gamma} = \gamma t, \alpha_0 \} \) characterizing respectively the loss parameter of the master equation and the initial amplitude of the pure cat state. In fact, by a simple change of variables one can obtain the corresponding SLD operators and QFI matrix [3] as in the equations

\[
\tilde{\mathbf{H}} = \mathbf{BHB}^T, \\
\mathbf{L}^{(\gamma)} = B_{31}\mathbf{L}^{(c)} + B_{13}\mathbf{L}^{(\alpha)}, \\
\mathbf{L}^{(\alpha)} = B_{32}\mathbf{L}^{(c)} + B_{23}\mathbf{L}^{(\alpha)}
\]

where \( \mathbf{H} \) and \( \tilde{\mathbf{H}} \) denote respectively the QFI matrix for the old and new parameters, and where we have introduced the Jacobian matrix with elements \( B_{\mu\nu} = \partial \lambda_\mu / \partial \lambda_\nu \). As the analytical formulas are quite cumbersome and do not give any particular insight we decided not to report them here. By using the new SLD operators we can however prove that the weak-commutativity condition holds also for the new couple of parameters, i.e. \( \text{Tr}[\varrho_{\text{cat}}[L_\gamma, L_{\alpha}]] = 0 \), showing how \( \hat{\gamma} \) and \( \alpha_0 \) can in principle be jointly estimated with optimal precision.

4.2. Displacement estimation with noisy Schrödinger cat states

We now consider the (single-parameter) estimation of displacement in phase-space: an initial probe state \( \varrho_0 \) undergoes the unitary transformation

\[
\varrho_0 \rightarrow \varrho_\epsilon = D(\epsilon)\varrho_0D(\epsilon)^\dagger,
\]

where \( D(\beta) = \exp\{\beta a^\dagger - \beta^* a\} \) denotes the displacement operation in phase space, that, in the case of a purely imaginary parameter \( \beta = i\epsilon \), describes a displacement along the \( p = i(a^\dagger - a) / \sqrt{2} \) quadrature. In the case of a pure coherent state as probe, \( \varrho_0 = |\alpha\rangle\langle\alpha| \) the QFI is constant, equal to \( H^{(\alpha)} = 4 \). A non-classical scaling in terms of the average photon number \( \bar{n} = \text{Tr}[\varrho_0a^\dagger a] \) can on the other hand be achieved by employing pure non-classical states such as Schrödinger cat states as the one in equation (16) [61, 63]. In particular, in this case one gets (\( \alpha \in \mathbb{R} \))

\[
H^{(\alpha)}_{\text{cat}} = \frac{4(4\alpha^2 + 1 + e^{-2\alpha^2})}{1 + e^{-2\alpha^2}} \gtrsim 16\bar{n} + 4,
\]

(36)

where the average photon number reads \( \bar{n} = \alpha^2(1 - e^{-2\alpha^2})/(1 + e^{-2\alpha^2}) \). A similar linear scaling for large average photon numbers is achieved for input squeezed states [61, 70].

We here want to exploit the method discussed in section 3, in order to calculate the QFI for the lossy Schrödinger cat state in equation (17) as input \( \varrho_0 \). Here one can employ the following basis for the joint support of \( \rho_\epsilon \) and \( L_\epsilon \):

\[
\mathcal{B}^{(c)} = \{ D(\epsilon)|\alpha\rangle, D(\epsilon)|-\alpha\rangle, \partial_\epsilon D(\epsilon)|\alpha\rangle, \partial_\epsilon D(\epsilon)|-\alpha\rangle \},
\]
leading to the following matrices

\[
S^{(\epsilon)} = \begin{pmatrix}
1 & 2\alpha & 0 \\
\epsilon & 1 & 0 \\
-2\alpha & 0 & 1 + 4\alpha^2 \\
0 & 2\alpha & 1 + 4\alpha^2
\end{pmatrix},
\]

(37)

\[
R^{(\epsilon)} = \mathcal{N} \begin{pmatrix}
1 & 0 \\
c & 1 \\
0 & 0
\end{pmatrix},
\]

(38)

\[
D^{(\epsilon)} = \mathcal{N} \begin{pmatrix}
0 & 1 & c \\
1 & c & 1 \\
c & 0 & 0
\end{pmatrix},
\]

(39)

Remarkably, an analytical solution of the Lyapunov equation can be obtained for the SLD operator, in terms of the matrix

\[
L^{(\epsilon)} = \begin{pmatrix}
0 & -\frac{4\alpha}{1 + \gamma c} & 0 & \frac{2}{\gamma} \\
\frac{4\alpha}{1 + \gamma c} & 0 & \frac{2}{\gamma} & 0 \\
0 & \frac{2}{\gamma} & 0 & \frac{2}{\gamma} \\
\frac{2}{\gamma} & 0 & -\frac{1}{\gamma^2} & 0
\end{pmatrix},
\]

(40)

yielding a QFI

\[
H_{\text{\bar{n}}} = \frac{4\left[4\alpha^2 \left(\frac{c^2}{1 + \gamma e^{-2\alpha^2}}\right) + \left(1 + \gamma e^{-2\alpha^2}\right)\right]}{(1 + \gamma e^{-2\alpha^2})^2}.
\]

(41)

As the average photon number of a lossy cat state reads

\[
\bar{n} = \alpha^2 \left(1 - \gamma e^{-2\alpha^2}\right),
\]

(42)

by fixing the noise constant \(c\) and varying the average photon number \(\bar{n}\) with the coherent amplitude \(\alpha\), one gets

\[
H_{\text{\bar{n}}} \approx 16\gamma^2 \bar{n} + 4,
\]

(43)

that is a linear scaling in \(\bar{n}\) is observed for any value of \(c > 0\). A different behaviour is however observed if we substitute the parameters as in equation (19), and look at the physically more relevant parameters \(\alpha_0\) and \(\gamma\), corresponding to a pure Schrödinger cat state evolving in a lossy channel. In fact in this case, by fixing \(\gamma\) and varying the photon number with the initial amplitude \(\alpha_0\) one obtains the plot in figure 1: the QFI is not even monotonically increasing with \(\bar{n}\) as soon as \(\gamma > 0\), and a maximum \(n_{\text{max}}\) is observed, whose value monotonically decreases with \(\gamma\).

In order to investigate the robustness of the estimation properties of cat states, we have also looked at how the QFI varies, at fixed \(\gamma\), varying the average photon number of the initial pure cat state, characterized by the coherent states amplitude \(\alpha_0\), that enters in the
lossy channel. We compare it with the QFI corresponding to a initial pure squeezed state $|\psi\rangle_{sq} = S(r)|0\rangle$, where $S(r) = e^{2r(a^\dagger a - a^2)}/\sinh r$. The corresponding QFI can be indeed evaluated analytically, by exploiting formulas for single-mode Gaussian states [49], obtaining

$$H_{\rho_{\text{sq}}} = \frac{4e^{2r}}{e^{2r}(1 - e^{-\gamma t}) + e^{-\gamma t}}.$$ \hspace{1cm} (44)

In figure 2, we plot both QFIs for a fixed value of $\bar{\gamma}$ and as a function of the average photon number $\bar{n} = \text{Tr}[\rho_{\text{cat}} a^\dagger a]$. From top to bottom: $\bar{\gamma} = \{0, 0.1, 0.2, 0.3, 0.5\}$.

In figure 2, we plot both QFIs for a fixed value of $\bar{\gamma}$ and as a function of the initial photon number $\bar{n}_0$. In both cases the linear scaling is lost in the presence of noise, but while also in this case, as soon as $\gamma > 0$, the cat state’s QFI is not monotonous and presents a maximum in $\bar{n}_0$, the squeezed state QFI is always monotonically increasing and reaches a limiting value. In more detail one finds that, for fixed $\bar{\gamma} > 0$, in the limit of respectively large $\alpha$ and large $r$,

$$\lim_{\alpha \to \infty} H_{\rho_{\text{cat}}} = 4,$$

$$\lim_{r \to \infty} H_{\rho_{\text{sq}}} = \frac{4}{1 - e^{-\gamma t}}.$$ \hspace{1cm} (45)

that is, while the cat states eventually yield the coherent state limit, a constant enhancement can be obtained if we exploit input squeezed states. This result indeed confirms the intuition that Schrödinger cat states are very fragile under the loss master equation: as one can observe from equation (19), the coherence parameter $c$ goes exponentially to zero, with an exponent that increases with $|\alpha_0|^2$, i.e. the larger is the energy of the cat state, the faster is the decay of the non-classical coherence terms.

4.3. Phase estimation with noisy entangled coherent states

An entangled coherent state (ECS) is a superposition of coherent states of two Bosonic modes, as per [71, 72]
$|\psi_{\text{ecs}}\rangle = \sqrt{K} \left( |\alpha\rangle|0\rangle + |0\rangle|\alpha\rangle \right)$, \hspace{1cm} (47)

where $K = \left[ 2(1 + e^{-|\alpha|^2}) \right]^{-1}$. The ECS has a structure similar to the N00N state, $|\psi_{\text{N00N}}\rangle = (|N,0\rangle + |0,N\rangle)/\sqrt{2}$, i.e. the superposition of Fock states with exactly $N$ photons either in the first or second mode. For this reason its usefulness in phase-estimation has been investigated, by considering the effect of a phase-rotation $U_\phi = e^{i\alpha j_2 a_j}$ on the second mode [58, 59, 73, 74]:

$|\psi_{\text{ecs}}(\phi)\rangle = U_\phi |\psi_{\text{ecs}}\rangle = \sqrt{K} \left( |\alpha\rangle|0\rangle + |0\rangle|\alpha e^{i\phi}\rangle \right)$, \hspace{1cm} (48)

where $a_j$ indicates the annihilation operator of mode $j$. In the above noiseless scenario the QFI of the phase parameter $\phi$ can be easily calculated and it was shown that the performance of an ECS is comparable to that of a N00N state: the QFI of both classes of states follows the Heisenberg scaling $H \approx \bar{n}$ in terms of the average photon number $\bar{n}$, where $\bar{n} = 2K|\alpha|^2$ for an ECS, and $\bar{n} = N$ for a NOON state. Furthermore, it was found that the ECS outperforms the NOON state at low photon numbers [73, 74].

If we assume that the amplitude damping channel (photon loss) corresponding to the master equation (18) affects both modes of the ECS (after the phase rotation), one can derive the mixed output state [58, 59, 73]

$\rho_{\text{ecs}}(\phi) = K \left[ |\tilde{\alpha}\rangle_1 |\tilde{\alpha}\rangle + |e^{i\phi} \tilde{\alpha}\rangle_2 |e^{i\phi} \tilde{\alpha}\rangle | + \eta (|\tilde{\alpha}\rangle_1 |e^{i\phi} \tilde{\alpha}\rangle_1 + |e^{i\phi} \tilde{\alpha}\rangle_2 |\tilde{\alpha}\rangle_2) \right]$, \hspace{1cm} (49)

where we have introduced the notation $|\beta\rangle_1 = |\beta\rangle|0\rangle$, $|\beta\rangle_2 = |0\rangle|\beta\rangle$, and the lossy parameters [67]

$\tilde{\alpha} = e^{-\gamma t/2} \alpha$, \hspace{1cm} (50)
\[ \eta = \exp\{-(1 - e^{-\gamma t})|\alpha|^2\}. \]  

An analytical formula of the QFI for this state has been already derived in the literature, by means of a lengthy diagonalization procedure of the density operator \([58, 59]\), and it was shown how the ECS outperforms the N00N state in the presence of loss, despite losing the Heisenberg scaling behaviour for large \(\bar{n}\).

In the following we will show how, thanks to our method, we can streamline the derivation of the same analytical formula for the QFI, avoiding the diagonalization procedure. We start by identifying the basis of the subspace spanned by \(\rho_{\text{ecs}}(\phi)\) and its derivative \(\partial_{\phi}\rho_{\text{ecs}}(\phi)\), as

\[
B(\phi) = \{ |\tilde{\alpha}\rangle_1, e^{i\phi}|\tilde{\alpha}\rangle_2, \partial_{\phi}|e^{i\phi}\tilde{\alpha}\rangle_2 \}.
\]

With straightforward algebra we can then write the matrices

\[
S(\phi) = \begin{pmatrix} 1 & e^{-|\tilde{\alpha}|^2} & 0 \\
\bar{\alpha} & 1 & i|\tilde{\alpha}|^2 \\
0 & -i|\tilde{\alpha}|^2 & |	ilde{\alpha}|^2 + |\tilde{\alpha}|^4 \end{pmatrix},
\]

\[
R(\phi) = K' \begin{pmatrix} 1 & 0 \\
\eta & 1 & 0 \\
0 & 0 & 1 \end{pmatrix},
\]

\[
D(\phi) = K' \begin{pmatrix} 0 & 0 & \eta \\
0 & 0 & 1 \\
\eta & 1 & 0 \end{pmatrix},
\]

where, in order to derive the elements of the Gramian matrix we have used the relationship

\[
\partial_{\phi}|e^{i\phi}\tilde{\alpha}\rangle_2 = i|\tilde{\alpha}|^2 e^{i\phi}\tilde{\alpha}\rangle_2.
\]

An analytical solution of the Lyapunov equation for the SLD matrix can then be obtained, as

\[
L(\phi) = \frac{1}{\sinh(T|\alpha|^2)} \begin{pmatrix} 0 & -i\chi & -1 \\
i\chi & 0 & e^{T|\alpha|^2} \\
-1 & e^{T|\alpha|^2} & 0 \end{pmatrix},
\]

where we have defined the function

\[
\chi = KT|\alpha|^2 \left(1 + e^{(2T-1)|\alpha|^2}\right),
\]

and we have introduced the ‘transmissivity’ parameter \(T = e^{-\gamma}\), such that \(\tilde{\alpha} = \sqrt{T}\alpha\). This leads to the following analytical formula for the QFI

\[
H_{\rho_{\text{ecs}}} = 2K' e^{-T|\alpha|^2} \frac{\chi}{\sinh(T|\alpha|^2)} \times \left[ \chi^2/K + T|\alpha|^2 (e^{2T|\alpha|^2} - 2T|\alpha|^2 - 1) \right]\]

that, after some manipulation, can be shown to be equivalent to the formula

\[
H_{\rho_{\text{ecs}}} = 4KT|\alpha|^2 \left[ 1 + T|\alpha|^2 - K'T|\alpha|^2 \left(1 + \frac{1 - e^{-2(1-T)|\alpha|^2}}{1 - e^{-2T|\alpha|^2}}\right) \right],
\]

that was derived, via direct diagonalization of the density operator, in \([58, 59]\).
5. Conclusions

In summary, we presented a general method to perform quantum metrology calculations when \( \rho \) and its derivatives are initially expanded on a non-orthogonal basis. By avoiding two significant computational hurdles, i.e. the diagonalization of the density matrix and the orthogonalization of the basis, our method has the potential to provide novel analytical results in a variety of quantum estimations problems. While the method is naturally suited for quantum statistical models involving a finite set of coherent states, one can seek further extensions of our techniques to deal with problems involving the full (infinite dimensional) set of coherent states [75]. We are also confident that the ideas presented here will prove useful for a much wider class of quantum estimation problems, for example involving superpositions and mixtures of general Gaussian states or spin coherent states among many others. While it is well known that non-orthogonal states play a prominent role in several areas of quantum mechanics, such as the theories of entanglement [76] and Glauber non-classicality [77], our work shows that they can have useful applications also in quantum metrology and estimation theory.

Note added: During the revision of our manuscript we became aware of [58, 59], which exploited an expansion of \( \rho \) in terms of coherent states to derive the results of section 4.3. Yet, these works rely by construction on the diagonalization of the density matrix to calculate the QFI, one of the cumbersome steps that our method aims to avoid. To the best of our knowledge, our work remains the first to develop a fully general method for metrology calculations via the Gramian matrix and not resorting to the diagonalization of \( \rho \), as summarised by our main result in equations (12) and (13).

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Appendix. Uniqueness of the QFI matrix

In this appendix we prove that, if \( \rho \) is not full rank, then the SLDs are not unique, yet the quantity \( H_{\mu \nu} + \mathbf{i} \Gamma_{\mu \nu} = \text{Tr} [\rho L_{\mu} L_{\nu}] \) is still unambiguously defined. To do so we decompose the Hilbert space as \( \mathcal{H} = \mathcal{H}_\rho \oplus \mathcal{H}_\perp \), where \( \mathcal{H}_\rho \) is the support of \( \rho \) and \( \mathcal{H}_\perp \) its orthogonal complement (in what follows, the latter will not be empty by hypothesis). Correspondingly we may represent the operators appearing in equation (4) in block form, as per

\[
\rho = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix},
\]

(A.1)

\[
L_\mu = \begin{pmatrix} A_\mu & B_\mu \\ B_\mu^* & C_\mu \end{pmatrix},
\]

(A.2)

\[
\partial_\mu \rho = \begin{pmatrix} F_\mu & G_\mu \\ G_\mu^* & 0 \end{pmatrix}.
\]

(A.3)

By straightforward algebra, equation (4) becomes equivalent to the following pair of equations for the unknown operators \( A_\mu, B_\mu, C_\mu \):
\begin{align}
A_\mu P + PA_\mu &= 2F_\mu, \quad (A.4) \\
PB_\mu &= G_\mu. \quad (A.5)
\end{align}

From the above, it is readily apparent that there are no constraints on \(C_\mu\), beyond the requirement \(C_\mu = C_\mu^\dagger\) (recall that \(L_\mu\) must be Hermitian). On the other hand, we shall see in what follows that \(A_\mu\) and \(B_\mu\) must be unique.

Since \(P > 0\) by construction, it is also invertible, hence equation (A.5) is uniquely solved by \(B_\mu = P^{-1}G_\mu\). To deal with the Lyapunov equation (A.4), we instead have to prove that the linear superoperator \(P(\bullet, \bullet) = P \bullet + \bullet P\) is invertible. To do this, we label by \(|\lambda_i\rangle\langle\lambda_j|\) the eigenstates of \(P\), with corresponding eigenvalues \(\lambda_i > 0\). It is easy to check that \(P\) is diagonalised by the operators \(|\lambda_i\rangle\langle\lambda_j|\), with eigenvalues \(\lambda_i + \lambda_j > 0\). This proves that \(P\) is positive definite, hence invertible. The unique solution for \(A_\mu\) can then be formally written as \(A_\mu = 2P^{-1}F_\mu\).

To conclude our discussion, we shall see that the freedom in the choice of \(C_\mu\) does not affect the calculation of the QFI matrix \(H\) or the commutator matrix \(\Gamma\). By direct calculation we get

\[
\text{Tr} \left[ gL_\mu L_\nu \right] = \text{Tr} \left[ P(A_\mu A_\nu + B_\mu B_\nu^\dagger) \right],
\]

which only involves operators that are uniquely defined. Hence, in the above formulation (which relied on decomposing the Hilbert space into two orthogonal subspaces), it is evident that the free parameters responsible for the non-uniqueness of the SLDs are all contained in the operators \(C_\mu\). Instead, when dealing with non-orthogonal bases as in our main text, the free parameters may feature in any element of the matrices \(L^{(\mu)}\).

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