Two-dimensional particle motion in a random potential under ac bias

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(March 22, 2022)

We study the Brownian motion of a single particle coupled to an external ac field in a two-dimensional random potential. We find that for small fields a large-scale vorticity pattern of the steady-state net currents emerges, a consequence of local symmetry breaking. In this regime the net currents are highly correlated, the spatial correlation function follows a logarithmic dependence, and the correlation length is of the order of the system size. For large external fields correlations disappear and only random net currents are observed. Numerical analysis indicates that the correlation length scales as a power law with both the size of the system and the amplitude of the ac field.

PACS numbers: 79.20.Rf, 64.60.Ht, 68.35.Rh

Driven diffusion in random media is a much studied problem impacting on a number of various fields. By now, it is well established that the interplay between the annealed randomness of the diffusion process and the quenched randomness of the media gives rise to unexpected scaling phenomena that have been extensively studied in recent decades. The range of applicability of the problem of driven diffusive motion in random media covers various fields, with relaxation phenomena in spin glasses, dislocation motion in disordered crystals, transport in porous media, and turbulent diffusion being but a few typical examples. In the presence of an external ac field, the motion of a particle is the result of the combined effect of thermally activated diffusivity and periodic motion, induced by the coupling to the applied field. At time scales much larger than the period of the driving force, one would expect that the influence of the field is negligible and the dynamics of the system can be described by the classical Brownian motion. Indeed, while the particle moves back and forth along the direction of the ac field, a stroboscopic view of the system, obtained by taking pictures only at times that are integer multiples of the external field period, would still show a randomly diffusing particle, almost as if the external field was absent in the system. In contrast to this intuitive picture, we demonstrate that the presence of an external ac field can fundamentally change the nature of the dynamics, when quenched randomness is present in the system.

An indication, that an ac field bias may drastically influence the nature of diffusive motion in a random potential comes from the recent advances in the field of thermal ratchets dealing with one-dimensional (1D) systems, where the $x \to -x$ symmetry is broken. In equilibrium, a particle moving in a periodic asymmetric potential displays a simple diffusive behavior. However, if the particle is also driven by an ac field, it drifts in the direction defined by the asymmetry of the potential, on time scales exceeding the period of the ac field. The appearance of this non-equilibrium steady-state net current is called the ratchet effect, and the average drift velocity of the particle is often referred to as the ratchet velocity. It must be noted, however, that the ratchet velocity is expected to vanish when the particle is moving in a quenched random potential, since such a potential obeys inversion symmetry (in a statistical sense). In particular, the ratchet velocity in a system of identical energy wells separated by high enough energy barriers is always zero, irrespective of the distribution of the barrier heights.

On the other hand, in two-dimensional (2D) systems any finite region of a random potential exhibits some degree of broken symmetry, and the particle can follow different trajectories during the two half periods of the ac driving. In this Letter, we show that the interplay between the local symmetry breaking in a random 2D potential landscape and the motion of the particle biased by an external ac field leads to the appearance of a highly correlated steady-state net current field, which is characterized by a large-scale vorticity. We also investigate the scaling properties of these fields.

As a model system, we investigate the driven dynamics of a single particle moving in a random uncorrelated potential on a 2D square Euclidean lattices with varied number of identical sites $N = L \times L$, where each site is connected to its nearest neighbors by bonds of unit length. Random potential barriers are assigned to each bond and periodic boundary

![FIG. 1: Schematic illustrating the system under consideration. A particle on the site $(i, j)$ hops to its neighboring sites, with hopping rates, $k_{ij}^{\pm}$, defined by both the potential barriers between these sites and by the ac field.](image-url)
conditions are assumed. A schematic illustration of the system's geometry is shown in Fig. 1. The particle, diffusing by an external ac field, applied along the x-direction. We define the local probability currents \( J^x_{i,j} \) [\( J^y_{i,j} \)] on site \((i,j)\) as the currents flowing over the potential barriers between the lattice sites \((i,j)\) and \((i+1,j)\) \([\{(i,j + 1)\}]\). As shown in Fig. 1 a particle, located at site \((i,j)\) at an arbitrary moment of time \(t\), can hop to any of the four nearest neighbor lattice sites by overcoming the potential barriers, \( E_{ij} \), assigned to each bond. To simplify the notation, we denote by \( E^r_{i,j}, E^l_{i,j}, E^u_{i,j}\) and \( E^d_{i,j}\), the local potential barriers associated with the right \( E^r_{i,j} = E_{(i,j)} \rightarrow (i+1,j) \), left \( E^l_{i,j} = E_{(i,j)} \rightarrow (i-1,j) \), up \( E^u_{i,j} = E_{(i,j)} \rightarrow (i,j+1)\), and down \( E^d_{i,j} = E_{(i,j)} \rightarrow (i,j-1) \) jumps, such that \( E^r_{i,j} = E^l_{i+1,j} \) and \( E^d_{i,j} = E^u_{i,j+1}\). The barrier heights are chosen to be random and quenched, with the spatial distribution given by \( E^r_{i,j} = E_0 + U \eta^r_{i,j}\), where \( \gamma \) stands for \( r, l, u, \) and \( d; \) \( E_0 \) is a constant; and \( \eta^r_{i,j} \equiv \eta^r_{i+1,j} \) and \( \eta^d_{i,j} \equiv \eta^d_{i,j+1}\) are uncorrelated random numbers uniformly distributed between 0 and 1. Throughout the paper we measure the energies in units of \( k_B T\), where \( k_B \) stands for the Boltzmann constant, and \( T \) is the absolute temperature.

In addition to the thermally activated diffusion, the particle is also driven by an external ac force, \( 2 \phi(t) \), applied along the x-direction (see Fig. 1). Since the energy barriers are placed halfway between adjacent sites, the ac force modulates their height by \( -\phi(t) \) in the positive x-direction and by \( +\phi(t) \) in the negative one. Thus, for large enough barriers, the hopping rates of the particle occupying site \((i,j)\) can be written as:

\[
k^r_{i,j} = \nu \exp\{-[U \eta^r_{i,j} - \phi(t)]\},
\]

\[
k^l_{i,j} = \nu \exp\{-[U \eta^l_{i,j} + \phi(t)]\},
\]

\[
k^u_{i,j} = \nu \exp\{-[U \eta^u_{i,j}]\},
\]

\[
k^d_{i,j} = \nu \exp\{-[U \eta^d_{i,j}]\},
\]

where \( \nu = \nu_0 \exp\{-E_0\} \), and the attempt frequency \( \nu_0 \) is also a constant.

Next, we turn to an ensemble description, in which the probability of the particle occupying site \((i,j)\) is denoted by \( P_{i,j} \), and the time evolution of this probability distribution is described by the following set of master equations:

\[
\frac{\partial P_{i,j}}{\partial t} = -J^x_{i,j} + J^x_{i-1,j} - J^y_{i,j} + J^y_{i,j-1},
\]

where

\[
J^x_{i,j} = k^r_{i,j} P_{i,j} - k^l_{i+1,j} P_{i+1,j},
\]

\[
J^y_{i,j} = k^u_{i,j} P_{i,j} - k^d_{i,j+1} P_{i,j+1}
\]

are the \( x \) and \( y \) components of the local probability currents \( \vec{J}_{i,j} \), as defined above.

For simplicity, in the following we restrict ourselves to the consideration of symmetric square wave external fields, i.e., when the field \( \phi(t) \) alternates between \( +F \) and \( -F \) at constant time intervals \( \tau_F \). We also assume that \( \tau_F \) is much larger than the relaxation time of the entire system, estimated as \( \tau_{relax} \approx L^2 \max(1/k^2_{ij}) \). In this case, for each half period of the ac driving the probability distribution \( P_{i,j} \) and currents \( \vec{J}_{i,j} \) relax to their steady state values: \( P_{i,j} (+F) \), \( \vec{J}_{i,j} (-F) \), and \( P_{i,j} (-F), \vec{J}_{i,j} (-F) \), which can be determined from the stationary solution of the master equation for \( \phi(t) = +F \) and \( \phi(t) = -F \), respectively. From direct analogy with the ratchet velocities, we can then define the net currents in the system as:

\[
\vec{J}_{i,j} = \frac{1}{2}[\vec{J}_{i,j} (+F) + \vec{J}_{i,j} (-F)].
\]

The appearance of non-zero net currents is a characteristics of systems with locally broken symmetry, and their magnitudes are given by higher than linear terms in the response functions (i.e., by \( F^2 \) and higher order terms).

We solved the system of Eqs. (1)-(4) numerically by using the conjugate gradient method, modified for sparse matrices. In Figs. 2(a)-(d), we show the steady state local net current patterns obtained for a system with linear size \( L = 50 \), randomness parameter \( U = 0.5 \), and external field amplitudes \( F = 0.01, 0.05, 0.1, \) and 0.9. These figures offer a visual proof for the emergence of highly non-trivial steady-state net current fields, characterized by long-range correlations and large-scale vorticity. The vorticity structures are most apparent for small external field amplitudes \( (F \ll U) \). As \( F \) increases, however, the correlations gradually disappear and the net current field converges to a nearly random structure.

It must be emphasized that numerical solution of Eqs. (1)-(4) for arbitrarily small values of \( F \) and/or for arbitrarily large system sizes is not possible. This is due to the slow convergence of the conjugate gradient method in these limits and, most importantly, due to the growing impact of the randomization, induced by the numerical errors.

To understand the nature of the observed correlations and investigate the scaling properties of the system, we compute the ensemble-averaged current-current correlation function, defined in the real space as:

\[
C(r) = \left\langle \frac{1}{\sum_{(i,j)} 2N_r} \sum_{i,j} \sum_{i',j'} (\vec{J}(r_{i,j}) - \vec{J}(r_{i',j'}))^2 \right\rangle.
\]

Here summation over all the lattice site indexes \((i,j)\) is implied, while summation over \((i',j')\) indexes is performed only for lattice site pairs such that \(|r_{i,j} - r_{i',j'}| = r, N_r\) being the number of such pairs. The averaging \( \langle ... \rangle \) was taken for various different realizations of the disorder \( \{\eta^r_{i,j}\} \). The correlation function defined this way is bounded from below by 0 (in case of perfect correlation), from above by 4 (perfect anti-correlation), and takes the value of 2 for vanishing correlation.

In Fig. 3 we show the normalized correlation function, \( C(r)/C(1) \), computed for different system sizes, and fixed external field amplitude, \( F = 0.01 \). As Fig. 3 demonstrates, the correlation function follows closely a logarithmic dependence on \( r \), for small radial distances, and deviates from that
as \( r \) grows beyond a certain value, with a saturation threshold determined by the system size, \( L \). This allows us to introduce a correlation length \( \xi \) in the system, which we define as a characteristic length at which the correlation vanishes: \( C(r) = 2 \), that is, \( 1 - 0.5 C(r) \sim \langle \hat{J}(r_{i,j})\hat{J}(r'_{i',j'}) \rangle \) turns zero.

On the basis of the above observations, we conclude that, in the most general form, the current-current correlation function is described well by the relation

\[
C(r) \sim \log(r) f \left( \frac{r}{\xi} \right),
\]

where \( f(x \leq 1) \sim \text{const} \). Note that the correlation length in Eq. (6) is dependent on both the amplitude of the external field and the system size; i.e., \( \xi = \xi(F, L) \). Moreover, as our results demonstrate, the correlation length, \( \xi \), scales with both of these quantities following power laws. Thus, for small amplitudes of the external field, the correlation length depends strongly on the system size, approximately following \( \xi(L) \sim L^\alpha \) relation, where the exponent \( \alpha = 0.983 \pm 0.008 \) (i.e., \( \xi \) grows nearly linearly with \( L \)). On the other hand, for intermediate external field amplitudes, \( \xi \) is found to be weakly dependent on \( L \) and to monotonically decrease with \( F \), following closely the inverse power law, \( \xi \sim F^{-\beta} \), where the exponent is \( \beta \simeq 0.78 \pm 0.01 \). The transition between the two regimes occurs at a certain field amplitude, \( F_1^*(L) \), dependent on \( L \). The above scaling laws can be collapsed into a single scaling relation:

\[
\xi \sim F^{-\beta} g(FL^{\alpha/\beta}),
\]

where \( g(x) \sim x \) for \( F \leq F_1^*(L) \) and \( g(x) \sim \text{const} \) for \( F_1^*(L) \leq F \leq F_2^*(L) \). The significance of \( F_2^*(L) \) will be discussed later. In Fig. 4 we show the numerical data collapse, performed according to Eq. (7). As one can observe, it provides satisfactory results for small and intermediate \( FL^{\alpha/\beta} \).

FIG. 2: Snapshots of the steady-state net current fields obtained for model systems with \( L = 50, U = 0.5 \), and different external field amplitudes: (a) \( F = 0.01 \); (b) \( F = 0.05 \); (c) \( F = 0.10 \); (d) \( F = 0.90 \).

FIG. 3: Normal-log plot of the current-current correlation function \( C(r)/C(1) \), computed for \( U = 0.5 \) and \( F = 0.01 \) for different system sizes.
In general, three regimes of the scaling behavior can be discriminated. For small $F$, the behavior is defined by the system size scaling and extends over the region of $F$ values up to $F_1^c(L)$. In the second regime, the scaling is dominated by the external field amplitude. This regime corresponds to the intermediate values of $F_1^c(L) \leq F \leq F_2^c(L)$. Note that substantial deviations from the scaling behavior predicted by Eq. (7) are observed for large values of $F$ (i.e., for $F \geq F_2^c(L)$). This behavior is an artifact of the discrete nature of the model system under consideration: since the particle is on the lattice, we cannot measure $\xi$ smaller than 1. As a result, in the limit of large $F$, the correlation length does not follow Eq. (7) down to arbitrary small distances, but rather saturates at $\xi \approx 1$.

In summary, we have studied the Brownian motion of a single particle on 2D Euclidean lattices of different sizes with quenched random potential, coupled to an external ac field. We have found that the interplay between the quenched randomness of the potential landscape and the external ac field bias leads to the emergence of large-scale vorticity patterns in the net steady-state currents. These currents are found to be strongly correlated, with the two-point correlation function following a logarithmic dependence. The properties of the current fields are observed to be largely independent of the particular realization of the disorder. The correlation length has been found to scale nearly linearly with the system size and display an inverse power law behavior with the external field amplitude.

An intuitive explanation for the appearance of system-wide structures is that in 2D the particle has the opportunity to "choose" globally different paths when driven in the two opposing directions. The superposition of these globally different flow fields results in large-scale flow patterns. This picture is also consistent with a slow (logarithmic) decay of the correlations and a correlation length being in the order of the system size. It is important to note that the above phenomenon cannot occur in 1D, where all the energy barriers are in series, and no alternative trajectories are possible.

Research at Notre Dame was supported by NSF-PHYS.