Quantum spin pump on a finite antiferromagnetic chain through bulk states

Nan-Hong Kuo\textsuperscript{1}, Sujit Sarkar\textsuperscript{2}, C. D. Hu\textsuperscript{1,3}\textsuperscript{*}

\textsuperscript{1}Department of Physics, National Taiwan University, Taipei, Taiwan, R.O.C.
\textsuperscript{2}PoornaPraijna Institute of Scientific Research, 4 Sadashivanagar, Bangalore-5600 80, India
\textsuperscript{3}Center for Theoretical Sciences, National Taiwan University, Taipei, Taiwan, R.O.C.

Abstract

We studied the possibility of the spin pump in a $S=1/2$ antiferromagnetic chain. The spin chain is mapped into a fermion system and bosonization is utilized to transform the equation of motion to a sine-Gordon equation. The sine-Gordon equation on a finite chain with different boundary conditions is solved. Among numerous solutions, the static soliton is compatible with the original physical system. By varying adiabatically a angle $\phi$ in the phase space composed of applied electric and magnetic fields, the spin states change between the Néel state and dimer state and a quantized spin $S = 1$ is transported by the bulk state from one end of the system to the other.

PACS: 75.10.Pq, 75.10.Jm, 03.65.Vf
Keywords: spin chain, sine-Gordon equation, quantum spin transport
1. Introduction: An adiabatic quantum pump is a device that generates a dc current by a cyclic variation of some system parameters, the variation being slow enough so that the system remains close to the ground state throughout the pumping cycle. After the pioneering work of Thouless[1] and Niu and Thouless[2], the quantum adiabatic pumping physics gets more attention. It is applied to the systems like open quantum dots[3-5], superconducting quantum wires[6,7], the Luttinger quantum wire[8], the interacting quantum wire[9] and of course the spin systems.

In recent years, spintronics become an exciting new field new field of research. Various proposals of generating spin current have been studied. Among them, an adiabatic spin pumping process is most interesting. Quantum spin pumping physics probably has inspired by the phenomenal work of Thouless[1], which is clearly related to the topological explanation of quantum hall effect by Thouless et.al.[10]. However Halperin[11] pointed out before that the quasi-one dimensional edge states played an important role in quantum hall effect. Hatsugai[12,13] showed that the edge states indeed have topological meaning and thus confirm their importance.

Shindou[14] has shown that the origin of spin transport is due to the edge state of the system. Fu, Kane and Mele[15] and Fu and Kane[16] studied the similar problem. Among their contribution, they found that the edge state crossing (Kramers degeneracy) is essential for spin pumping. So the possibility of spin transport through the bulk states of the system is not reveal from these studies and leave this bulk state spin transport as a open problem.

Here we mention very briefly the basic theme of spin transport in adiabatic process. Suppose we consider a spin chain and constructed a parameter space with \((h_{st}, \Delta) = R(\cos \phi, \sin \phi)\) where \(h_{st}\) is the applied magnetic field \(\Delta\) the dimer states bond strength. Fixing \(R\) and varying \(\phi\) adiabatically in time, one can argue that a line integral of \(A_n(K) = (i/2\pi) < n(K)|\nabla_K|n(K) >\), where \(n(K)\) is the Bloch function for the \(n\)-th band, and \(K = (k, \Delta, h_{st})\), on a closed loop yields exactly \(\pm 1\) due to the singularity at the origin. In other words, \(A\) is related to a fictitious magnetic field \(B_n(K) = \nabla_K \times A_n(K)\). One with the Stokes’ theorem, can express the line integral in terms of surface integral \((\int B \cdot dS)\) where the integration is on two dimensional closed surface enclosing the origin. This is exactly the quantization of particle transport proposed by Thouless[1].

It is well known to us that one can express spin chain problem into a spin less fermion problem with Jordan-Wigner transformations. One can use this kind of adiabatic variation of parameters as a tool of quantized spin transport. Shindou considered the spin polarization \(P_s = \frac{1}{N} \sum_{j=1}^{N} jS_j^z\) and divided it into two parts, bulk state part and edge state part and he concluded that edge state part of spin polarization contribute to spin transport. Fu and Kane[16] considered a similar system with an additional interaction of spin-orbit coupling. They calculated the energy bands of the bulk states and end (edge) states and were able to show clearly that whenever there is Kramers degeneracy of end states, there is spin transport and it has a \(Z_2\) symmetry.
We have already mentioned that in all previous studies of spin transport, the contribution is coming from the edge states. Here, most probably first in the literature, we raise the question, whether the edge states are indispensable in spin transport? We do the rigorous analytical exercises to complete the search of this question. One can see during our analytical derivation that spin transport is nothing but the transport of soliton in the system. The plan of our paper is the following: In section (2), we present model Hamiltonian and continuum field theoretical studies. Section (3) is for analysis of sine-Gordon equation on a finite chain for different boundary conditions. Section (4) is for the detail analysis for static soliton solution. Section (5) is reserved for results and discussions.

2. Hamiltonian and Continuum field theoretical studies

We consider a spin 1/2 chain of finite length, described by the Heisenberg Hamiltonian similar to that of Shindou[14]. A controlled dimerization amplitude and applied magnetic field are also present. The total Hamiltonian has three parts:

\[ H(t) = H_0 + H_{\text{dim}} + H_{\text{st}} \]  

where

\[ H_0 = J \sum_{i=1}^{N} S_i \cdot S_{i+1}, \]  

\[ H_{\text{dim}} = \frac{\Delta(t)}{2} \sum_{i=1}^{N} \left( -1 \right)^i (S^+_i S^-_{i+1} + S^-_i S^+_i S^+_i S^-_{i+1}), \]  

and

\[ H_{\text{st}} = h_{\text{st}}(t) \sum_i (-1)^i S^z_i. \]

\[ H_{\text{dim}} \] is the bond alternation term which can be induced by applying an electric field to the spin chain to alter the exchange interaction. It introduces into the system the strength of dimerization \( \Delta(t) \). \( H_{\text{st}} \) is the coupling of the system to a staggered external field \( h_{\text{st}}(t) \). The time-dependent bond strength \( \Delta(t) \) and staggered field \( h_{\text{st}}(t) \) can be varied adiabatically so as to create a parameter space for Berry phase. We write \( \Delta \) and \( h_{\text{st}} \) as \( (h_{\text{st}}, \Delta) = R(\cos \phi, \sin \phi) \), with \( R \) fixed. Varying \( \phi \) adiabatically, we expect spins to be transported. We shall also argue that going through one cycle along the loop, there will be quantized spin component transported from one end to the other.

The method of bosonization[17-20] has been used successfully to treat various one-dimensional systems, including the spin chains. It is suitable for the system we are considering. To this end, we first make the Jordon-Winger transformation to represent spins by fermion field \( f_i \) and \( f_i^+ \). Then, the bosonizations of \( f_i \) and \( f_i^+ \) will performed.

\[ f_j = \exp(i\pi \sum_{k=1}^{j-1} S^+_k S^-_k) S^z_j, \]
\[ f_j^\dagger = S_j^+ \exp(-i\pi \sum_{i}^j S_i^+ S_i^-), \]  
\( j \)

and

\[ f_j \cong R(x_j)e^{ikFx_j} + L(x_j)e^{-ikFx_j}, \]

\( f_j^\dagger \cong R^\dagger(x_j)e^{-ikFx_j} + L^\dagger(x_j)e^{ikFx_j}, \]

where

\[ R(x) = \frac{1}{\sqrt{2\pi\alpha}} \eta_1 \exp[i(\theta_+(x) + \theta_-(x))/2] \]

and

\[ L(x) = \frac{1}{\sqrt{2\pi\alpha}} \eta_2 \exp[-i(\theta_+(x) + \theta_-(x))/2] \]

are the slowly varying fields, and \( \eta_1 \) and \( \eta_2 \) are the Klein factors. Here, \( k_F \) is the Fermi wave vector and \( \alpha \) is the lattice constant. For a half filled system we have \( k_F = \pi/2\alpha \). In the following derivations, we left out the details because they can be found in many textbooks[17-20].

\[ S_j^z = f_j^\dagger f_j - \frac{1}{2} \]

\[ = \frac{\partial_x \theta_+(x_j)}{2\pi} - (-1)^j \frac{1}{\pi\alpha} \sin \theta_+(x_j) \]

and

\[ S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ = f_j^\dagger f_{j+1} + f_{j+1}^\dagger f_j \]

\[ = -\alpha [4\pi\tilde{\Pi}^2 + \frac{1}{4\pi} (\partial_x \tilde{\theta}_+)^2] - (-1)^j \frac{1}{\pi\alpha} \cos \tilde{\theta}_+(x_j). \]

Here \( \theta_+ = \) is bosonization phase and \( \tilde{\Pi}(x) = -(1/4\pi)\partial_x \theta_-(x) \) is the conjugate momentum of \( \theta_+(x) \). Substituting eqs. (11) and (12) into eqs. (1-4), and dropping the rapidly varying components such as \( \sum_j (-1)^j \cos \tilde{\theta}_+(x_j) \), we obtained

\[ H = \int dx \{ v[\pi\eta \tilde{\Pi}^2 + \frac{1}{4\pi\eta} (\partial_x \tilde{\theta}_+)^2] - \frac{R}{\pi\alpha^2} \sin(\tilde{\theta}_+ + \phi) + \frac{J}{2\pi^2\alpha^3} \cos 2\tilde{\theta}_+ \} \]

where the velocity

\[ v = J \sqrt{1 + \frac{2}{\pi}} \]

and the quantum parameter

\[ \eta = \frac{J}{v} \]

were discussed in ref. 17-20. Thus, we have the equation of motion

\[ \partial_t^2 \tilde{\theta}_+ = v^2 \partial_x^2 \tilde{\theta}_+ + \frac{2J R}{\alpha^2} \cos(\tilde{\theta}_+ + \phi) + \frac{2J^2}{\pi\alpha^3} \sin 2\tilde{\theta}_+ \]
The term of \( \sin 2\tilde{\theta}_+ \) is irrelevant in the sense of renormalization group analysis, so we consider only the part
\[
\partial^2_t \tilde{\theta}_+ = \partial^2_z \tilde{\theta}_+ + \cos(\tilde{\theta}_+ + \varphi).
\] (17)

where we have change variables: \( z = \sqrt{2JRx/\nu\alpha} \) and \( \tau = \sqrt{2Jt/\alpha} \). It is similar to the standard sine-Gordon equation
\[
\partial^2_t \tilde{\theta}_+ - \partial^2_z \tilde{\theta}_+ + \sin \tilde{\theta}_+ = 0
\] (18)

which has been well-studied. However, for our purpose which is to study the spin transport, we will solve it on a chain of finite length on which the phase \( \phi \) is no longer a trivial constant but introduces new meaning to the solution. This way, one can recognize the motion of spins from one end to the other.

### 3. Analysis of sine-Gordon equation on a finite chain

We shall analyze eq. (18) first. The result can be applied to eq. (17). Equation (18) has many kinds of solutions. The traveling-wave solutions, such as \( \arctan[\exp(\gamma(z - v\tau))] \) is not suitable for our purpose because they cannot meet fixed boundary conditions. For the finite-length systems, we consider the so called “separable solutions”\[21-23\].

\[
\phi(z, \tau) = 4 \arctan(A \frac{f(\beta z)}{g(\Omega \tau)}).
\] (19)

and \( f(\beta z) \) and \( g(\Omega \tau) \) must satisfy the following equations:
\[
(\partial_z f)^2 = \left( \frac{1}{\beta^2} \right)[-\kappa A^2 f^4 + \mu f^2 + (\lambda A^2)]
\] (20)

and
\[
(\partial_\tau g)^2 = \left( \frac{1}{\Omega^2} \right)[-\lambda g^4 + (\mu - 1)g^2 + \kappa].
\] (21)

with the requirements \( \mu^2 + 4\kappa \lambda \geq 0 \) and \( (\mu - 1)^2 + 4\kappa \lambda \geq 0 \). \( A, \mu, \kappa, \lambda, \beta \) and \( \Omega \) are mutually related constants. We will show how they are determined in a while.

First, we would like to put forward the observation that \( f(\beta z) \) and \( g(\Omega \tau) \) satisfying eqs. (20) and (21) are Jacobi elliptic functions (JEF)\[24\]. Jacobi elliptic functions are defined as the following:
\[
u = \int_0^{sn(u)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.
\] (22)

where \( sn(u) \) is one of the JEFs and \( k \) is a constant in the range [0,1]. The second JEF is \( cn(u) \) where \( sn^2(u) + cn^2(u) = 1 \). There are more JEFs. They can be found in Appendix A. The ones we are going to encounter are \( sc(u) = sn(u)/cn(u) \) and \( dn(u) = 1 - k^2 sn^2(u) \). Both \( f \) and \( g \) are JEFs and their constants are denoted by \( k_f \) and \( k_g \). \( \mu, \kappa, \lambda \) are constants determined by \( k_f \) and \( k_g \). The relations are different for different Jacobi elliptic functions.
Here we give an example of \( f(\beta z) = cn(\beta(z - z_0)) \) and \( g(\Omega \tau) = cn(\Omega \tau) \) where \( z_0 \) is a constant. With the equation for \( cn(u) \) (see Table I in Appendix A)

\[
(\partial_u cn(u))^2 = (1 - u^2)(1 - k^2 + k^2 u^2),
\]

we found from comparison with eq. (20) that \( \kappa A^2 = k_f^2 \), \( \mu = 2k_f^2 - 1 \) and \( \lambda/A^2 = 1 - k_f^2 \). As a result, we get

\[
k_f = \frac{A^2}{1 + A^2} + \frac{A^2}{\beta^2(1 + A^2)^2}. \tag{24}
\]

and,

\[
k_g = \frac{A^2}{1 + A^2} - \frac{A^2}{\Omega^2(1 + A^2)^2}. \tag{25}
\]

where

\[
\Omega^2 = \beta^2 + \frac{1 - A^2}{1 + A^2}. \tag{26}
\]

If we choose the fixed boundary condition \( \theta_+(z = 0) = \theta_+(z = L) = 0 \) with \( L \) being the length of the system, then we will have

\[
\beta L = 4lK(k_f) \tag{27}
\]

with

\[
K = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \tag{28}
\]

being the complete elliptic integral of the first kind, \( z_0 = L/4l \) and \( l \) is an integer.

Not all the combinations of JEFs can satisfy the sine-Gordon equation. A table of the differential equations for all the JEFs is given in Appendix A. We will discuss the solutions of the sine-Gordon equation in eq. (17) on a finite system under various boundary conditions. Although finite-length solutions are well-known, different boundary conditions and the presence of \( \phi \) will impose restrictions on the solutions and infuse physical meaning to the wave forms.

**Case 1: Periodic boundary condition** \( \hat{\theta}_+(z, \tau) = \hat{\theta}_+(z + L, \tau) \) The first boundary condition coming to mind is the periodic boundary condition. There are many combinations of JEFs that can satisfy the periodic boundary condition. Here are two examples.

1a:

\[
\hat{\theta}_+(z, \tau) = \frac{\pi}{2} - \varphi + 4 \arctan \{ Acn[\beta(z - z_0); k_f]cn[\Omega \tau; k_g] \} \tag{29}
\]

where \( \beta L = 4lK(k_f) \) and

1b:

\[
\hat{\theta}_+(z, \tau) = \frac{\pi}{2} - \varphi + 4 \arctan \{ Asc[\beta(z - z_0); k_f]dn[\Omega \tau; k_g] \} \tag{30}
\]
where $\beta L = 2lK(k_f)$ and $z_0$ is an arbitrary constant. For this boundary condition, there is no spin transport if one varies the parameter $\phi$ adiabatically. The reason is quite simple. Increasing $\phi$ only gives a constant change to $\theta(z)$ everywhere. Thus the fermion field operators on every site from Jordan-Wigner transformation acquire a constant phase and the spins remain the same.

**Case 2: Fixed boundary condition**

It seems that we can have the solutions like

$$\hat{\theta}^+(z, \tau) = \frac{\pi}{2} - \varphi + 4 \arctan\{\text{ncn}[\beta(z - z_0); k_f] \text{cn}[\Omega \tau; k_g]\}$$  

(31)

where $\beta L = 2lK(k_f)$. However, the presence of the adiabatic change term $\pi/2 - \varphi$ in front requires that $\text{cn}[\beta(z - z_0); k_f]$ to be finite. Then the function $\text{cn}[\Omega \tau; k_g]$ makes the inverse tangent function varying with time and hence, the forms in solution (32) cannot satisfy the fixed boundary condition.

**Case 3: Free end boundary condition**

The solution is

$$\hat{\theta}^+(z, \tau) = \frac{\pi}{2} - \varphi + 4 \arctan\{\text{dn}[\beta(z - z_0); k_f] \text{sn}[\Omega \tau; k_g]\}$$  

(32)

where $\beta L = 2K_f$ and $\beta z_0 = K_f$. The energy is equal to $16\beta E(K)$ where

$$E(K) = \int_0^{\pi^2} \frac{\sqrt{1 - k^2t^2}}{\sqrt{1 - t^2}} \, dt$$  

(33)

is the complete elliptic integral of the second kind. This solution cannot provide the system with spin transport for the same reason as that in case 1.

4. **Detailed analysis of the static soliton case**

It is most interesting to study the solution of the static soliton of eq. (17). Let us first consider the boundary condition

$$\begin{align*}
\theta^+(z = 0) &= 0, \\
\theta^+(z = L) &= 2\pi.
\end{align*}$$  

(34)

The phase difference $2\pi$ implies that the fermion field or the spins have same boundary conditions at both ends and hence, a common case for a finite spin chain. On the other hand, it is a fixed boundary condition of $\hat{\theta}^+$. Therefore, different values of $\phi$ will induce distinct solutions. The soliton has the form

$$\hat{\theta}^+(z, \tau) = \frac{\pi}{2} - \varphi + 4 \arctan\{\text{scn}[\beta(z - z_0); k_f] \text{dcn}[\Omega \tau; k_g]\}$$  

(35)

where

$$k_f^2 = 1 - A^2 + \frac{A^2}{\beta^2(1 - A^2)},$$  

(36)
\[ k_g^2 = 1 - \frac{1}{A^2} + \frac{1}{\Omega^2(1 - A^2)}, \quad (37) \]
\[ \Omega = A\beta, \quad (38) \]
and
\[ \beta L = K(k_f). \quad (39) \]
Equations (36-38) can be derived by substituting eq. (35) into eq. (17) and eq. (39) comes from the boundary conditions in eq. (34). We seek the static solution because it can always satisfy above boundary conditions. In this case, we require \( k_g = 0 \), \( d\nu(\Omega, k_g = 0) = 1 \) and \( A \) takes a special value \( A_{th} \). In view of eqs. (36) and (37), the static soliton is
\[ \bar{\theta}_+(z, \tau) = \frac{\pi}{2} - \varphi + 4 \arctan\{A_{th} \text{sc}[\beta(z - z_0); k_f]\} \quad (40) \]
with
\[ \beta = 1/(1 - A_{th}^2) \quad (41) \]
and
\[ k_f = \sqrt{1 - A_{th}^4} \quad (42) \]
The boundary condition at \( z = 0 \) requires that
\[ \tan\left(\frac{\phi}{4} - \frac{\pi}{8}\right) = A_{th} \text{sc}(\beta z_0) \]
which determines \( z_0 \). For the boundary condition at \( z = L \), we have derived the following lemma:

For the solution in eq. (40) with
\[ \beta L = K(k_f = \sqrt{1 - A_{th}^4}) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)[1 - (1 - A_{th}^4)t^2]}} \quad (44) \]
the difference of \( \bar{\theta}_+(z = 0) \) and \( \bar{\theta}_+(z = L) \) is always equal to \( 2\pi \).

The derivation is given in Appendix B. Eqs. (40) and (44) are the main result of this paper. Equation (44) can be generalized as
\[ \beta L = lK \quad (45) \]
where \( l \) is any nonzero integer. As the larger the magnitude of \( l \), the higher the energy.

In Fig. 1 \( A_{th} \) evaluated with eq. (44) is plotted. It shows that \( A_{th} \) decreases rapidly with increasing \( L \). The magnitude of \( A_{th} \) is closely related to the wave form of \( \bar{\theta}_+ \). A small \( A_{th} \) results in a steep change in \( \bar{\theta}_+ \), or a sharp domain wall. As it will be shown later, it related to the quantum spin transport.

The energy of the static soliton can be calculated with
\[ \mathcal{E} = \int_0^L dz\left[\frac{1}{2} \left(\frac{\partial \bar{\theta}_+}{\partial \tau}\right)^2 + \frac{1}{2} \left(\frac{\partial \bar{\theta}_+}{\partial z}\right)^2 - \sin(\bar{\theta}_+ + \phi)\right]. \quad (46) \]
It can be shown that
\[
\frac{1}{2} \left( \frac{\partial \theta_+}{\partial z} \right)^2 + \sin(\theta_+ + \phi) - \frac{1}{2} \left( \frac{\partial \theta_+}{\partial \tau} \right)^2 |_{z=z_0} - \sin(\theta_+ + \phi) |_{z=z_0}
\]  

\[
= \frac{1}{2} \left( \frac{\partial \theta_+}{\partial z} \right)^2 + \sin(\theta_+ + \phi) - \frac{8A_{th}^2}{(1 - A_{th}^2)^2} - 1 = 0
\]

Hence, eq. (45) becomes
\[
\mathcal{E} = \int_0^L dz \left[ 1 + \frac{8A_{th}^2}{(1 - A_{th}^2)^2} - 2 \sin(\theta_+ + \phi) \right]
\]  

where the term of the time derivative is dropped for we are considering the static case. We can change the variable of integration and get
\[
\mathcal{E} = \sqrt{2} \int_{\tilde{\theta}_{+,1}}^{\tilde{\theta}_{+,2}} d\tilde{\theta}_+ \left[ 1 + \frac{8A_{th}^2}{(1 - A_{th}^2)^2} - \sin(\theta_+ + \phi) \right]^{1/2} - L \left[ 1 + \frac{8A_{th}^2}{(1 - A_{th}^2)^2} \right]
\]

where \( \tilde{\theta}_{+,1} = \pi/2 - \varphi + 4 \arctan(A_{th} \text{sc}(-\beta z_0)) \) and \( \tilde{\theta}_{+,2} = \pi/2 - \varphi + 4 \arctan(A_{th} \text{sc}(\beta L - \beta z_0)) \). It has been shown in Appendix B that \( \tilde{\theta}_{+,2} - \tilde{\theta}_{+,1} = 2\pi \). Therefore the total energy \( \mathcal{E} \) is independent of \( z_0 \) and \( \phi \) because the integration is over an entire period. \( \mathcal{E} \) depends on only one parameter, \( \beta \), for static soliton because \( A = A_{th} \) and \( A_{th} \) is also determined by \( \beta \). The spectrum is plotted in Fig. 2 with eq. (45). It is very similar to that of standing wave with \( \beta \) being the wave vector.

In the limit \( L \to \infty \), \( A_{th} \) becomes vanishingly small as it can be seen from eqs. (44) and (41). We thus have \( \beta, k_f \to 1 \) and \( K(k_f) \simeq \ln(4/A_{th}^2) \). With eq. (41) we get
\[
A_{th} \simeq 2 \exp(-L/2).
\]

The magnitude of \( A_{th} \) is small even for a modest length of \( L \). For example, when \( L = 24 \), \( A_{th} \simeq 2/e^{12} \simeq 1.23 \times 10^{-5} \). Since \( A_{th} \) can be viewed as the amplitude of the nonlinear wave, the wave form on a long spin chain becomes flat everywhere except for narrow regions \( cu(z - z_0) \sim 0 \). Therefore, one can expect a sudden change of \( \theta_+ \) or a sharp domain wall.

5. Results and discussion
In this section, we present the results of our calculation above. First of all, we plotted \( z_0 \) versus \( \phi \) in Fig. 3. \( z_0 \) can be viewed as a reference point of the solution of the sine-Gordon equation. Its movement is a clear indication that the solitons is set in motion by \( \phi \). Its motion is not smooth as one can easily see that there is an abrupt change near \( \phi = \pi/2 \), a manifestation of the nonlinearity of the solution. It decreases by a distance \( L \) when \( \phi \) increases by \( 2\pi \). To see more clearly how the soliton moves, we plotted in Figs. 4 \( \theta_+ \) versus lattice sites for different values of \( \phi \) at \( L = 24 \). We can go back to the original spin system to see how spins are transported by utilizing eqs. (11) and (12). Hence, \( S_{j}^{z} \) and \( S_{j+1}^{+}S_{j}^{-} + S_{j+1}^{-}S_{j}^{+} \) are also plotted.
In view of eq. (11), the "domain wall" or the region where there is a jump of $\hat{\theta}_+$ is the place where $\langle S^z \rangle$ is large. Hence, in Figs. 4 the jump of $\theta_+$ and the peak of $\langle S^z \rangle$ move together with varying $\phi$ and spins moves from right to left. These figures also show that the static soliton solution really is a Néel state in the spin chain except in the neighborhood of $\phi \simeq \pi/2$. In this range, the Néel state becomes unstable due to the dimer coupling $\Delta(t)$ we added. This is manifest in Figs. 4(b-d) where $S^+_j S^-_{j+1} + S^-_j S^+_j$ which is proportional to the dimer state amplitude is large. Recall that $\phi$ is defined in $(h_{st}, \Delta) = R(\cos \phi, \sin \phi)$. The dimer strength $\Delta$ is the largest when $\phi \simeq \pi/2$. It is when the Néel state becomes unstable and the dimer state amplitude becomes significant that the transport of spin becomes possible. Not coincidentally, one can find in Fig. 3 that $z_0$ changes abruptly in this range.

In Fig. 5, $\hat{\theta}_+, S^z_j$ and $S^+_j S^-_{j+1} + S^-_j S^+_j$ versus lattice sites for $\phi = \pi/2$ at a shorter length of $L$ ($L = 14$) is plotted for comparison. For smaller $L$, the curve of $\hat{\theta}_+$ is smoother or the domain wall is not as sharp. On the other hand, the edge (end) effect is more important. The directions of spins are less ordered for a shorter spin chain because the edge effect penetrates deeper into the "bulk".

We will elaborate more on how the spins are transported. This can be done with eq. (11). The spin polarization is

$$P_{S^z} = \frac{1}{L} \int_0^L z S^z(z) dz.$$ (51)

By integration by parts, we found

$$P_{S^z} = \int_0^L S^z(z') dz' - \frac{1}{L} \int_0^L \int_0^z S^z(z') dz' dz$$ (52)

To find the variation of $P_{S^z}$ due to $\phi$ we note that the first term remain constant as $\phi$ varies. This can be seen by substituting eq. (11) into the integration. The contribution of the oscillatory term vanishes as $L \to \infty$ and the term of the derivative gives unity due to our boundary condition, no matter what the value of $\phi$ is. Thus, denoting the variation of $P_{S^z}$ due to the adiabatic change of $\phi$ by $\delta P_{S^z}$, we have

$$\delta P_{S^z} = \frac{1}{L} \left\{ \left[ \int_0^L S^z(z') dz' \right]_{\phi = \phi_2} - \left[ \int_0^L S^z(z') dz' \right]_{\phi = \phi_1} \right\}$$

$$\simeq -\frac{1}{2\pi L} \left\{ \left[ \hat{\theta}_+(z) dz \right]_{\phi_2} - \left[ \hat{\theta}_+(z) dz \right]_{\phi_1} \right\}$$ (53)

where the second step can be reached by using the approximation

$$S^z \simeq (\hat{\theta}_+/\partial z)/2\pi$$ (54)
in integration for large $L$. In view of Figs. 4(b-d) where $\phi$ increases exceeding $\pi/2$, we found that $\hat{\theta}_+$ increases in the entire length of the system by approximately $2\pi$ and hence, $\delta P_{S_z} \simeq -1$ around $\varphi = \pi/2$ and a spin 1 is moved from right to left around $\varphi = \pi/2$. In Figs. 4(e-h), where $\hat{\theta}_+$ is almost constant away from ends, we did not see any spin movement in the bulk but rather, there are changes of spins at both ends.

To see the quantum spin transport (a spin of unity being transported) more clearly, we consider the limit $L \to \infty$ which can be simulate very closely by the case $L = 24$. The following equation will be very useful for our purpose

$$\frac{\partial \hat{\theta}_+}{\partial z} = \frac{4A_{th}\beta dn(\beta(z - z_0))}{cn^2(\beta(z - z_0)) + A^2_{th} sn^2(\beta(z - z_0))},$$

(55)

since it is the dominant contribution to $S^z$. From eqs. (41), (50), (A-11), (A-17) and (A-18), we find that

$$\frac{\partial \hat{\theta}_+}{\partial z} \simeq \frac{8e^{-L/2}cosh(z - z_0)}{1 + 4e^{-L} sinh^2(z - z_0)}.$$  

(56)

The peak of $\partial \hat{\theta}_+ / \partial z$ or $S^z$ is at

$$z - z_0 \simeq L/2.$$  

(57)

The larger $L$, the narrower the peak. On the other hand, $z_0$ is determined by the boundary condition. When $\phi = \pi/2 - \delta$ where $\delta$ is a small and positive number, we have

$$\tan(-\frac{\delta}{4}) = A_{th} sc(-\beta z_0)$$  

(58)

and we find that $z_0 \simeq L/2$ as long as $\delta$ remains finite (see eq. (50)). The resulting $S^z$ due to eqs. (11) and (55) has a peak at the right end and vanishes everywhere else. When $\phi = \pi/2$, eq. (57) gives $z_0 = 0$ and the peak of $S^z$ moves to the center of the spin chain. If $\phi = \pi/2 + \delta$, then $z_0 \simeq -L/2$, and the peak moves to the left end.

As for the quantity of spin transported, we can analyze the variation of $\hat{\theta}_+$. In view of eq. (40), as $\phi = \pi/2 - \delta$, $\hat{\theta}_+$ almost vanishes for the entire chain except for the right end. As $\phi$ increases to $\pi/2 + \delta$, $\hat{\theta}_+ \simeq 2\pi$ for the entire chain except for the left end where it drops to zero sharply. Hence, according to eq. (53), during the interval $\phi = \pi/2 - \delta$ to $\phi = \pi/2 + \delta$, a spin of unity is transported from the right end to the left end. As $\phi$ increases onward from $\pi/2 + \delta$, there is no spin transport in the bulk. Nevertheless, the plateau of $\hat{\theta}_+$ is lowered (see Figs. 4(e-h)) and as a result, $S^z$ at the left end decreases and a peak of $S^z$ at the right end start to grow. This kind of change continues until $\phi$ reaches $5\pi/2 - \delta$. At this stage the state of soliton returns to that of $\phi = \pi/2 - \delta$.

We conclude the analysis of our result by the following summary: There is a swift spin transport in the bulk during the short interval between $\phi = \pi/2 - \delta$ and $\phi = \pi/2 + \delta$ where $\delta$ can be made arbitrarily small if $L \to \infty$. The net spin
transported is unity. Beyond this interval, the spins at both ends vary with \( \phi \) but there is no spin change in the bulk.

The question will inevitably be raised: Can spin be transported? We have seen that the soliton returns to the starting state if \( \phi \) increases by \( 2\pi \). Thus there is no net spin transported in a cycle. However, in a realistic system, two ends of the spin chain must be connected to leads. The leads ought to serve as a spin source and a spin drain. Thus it is reasonable to envisage the following picture: At \( \phi = \pi/2 + \delta \) the left end can dump spin into a spin drain and the right end can extract spin from the source. When \( \phi = 5\pi/2 - \delta \) which is equivalent to \( \phi = \pi/2 - \delta \), the dumping of spin at the left end is complete and the peak of spin at the right end has grown into saturation. Then an unit of spin is transported from the right end to the left end when \( \phi \) increases from \( \phi = \pi/2 - \delta \) to \( \phi = \pi/2 + \delta \). This is in all intent and purpose, same as a physical system of spin transport. On this point, our system is same as the \( Z_2 \) spin pump proposed by Fu and Kane[16]. However, there is an important difference. Our spin transport is through a bulk state. This is completely different from Shindou’s[14] and Fu and Kane’s[15,16] pictures in which the level crossing of the end (edge) states is essential. Consequently, our spin transport is quantized because it is through a bulk state. Connecting to spin reservoir cannot destroy the quantization as it will do the transport due to end states.

This work is supported in part by NSC of Taiwan, ROC under the contract number NSC 95-2112-M-002-048-MY3.

Appendix A: In this appendix we listed some properties of the Jacobian elliptic functions. See eqs. (24-25) for the definition.

\[
\begin{align*}
    sn^2(u) + cn^2(u) &= 1, \quad (A-1) \\
    dn^2(u) + k^2 sn^2(u) &= 1, \quad (A-2) \\
    sn(u + K) &= cn(u)/dn(u) \quad (A-3) \\
    cn(u + K) &= -sn(u)/dn(u) \quad (A-4)
\end{align*}
\]

where \( k^2 = 1 - k^2 \). The derivatives of Jacobian elliptic functions are

\[
\begin{align*}
    \partial_u sn(u) &= cn(u)dn(u), \quad (A-5) \\
    \partial_u cn(u) &= -sn(u)dn(u), \quad (A-6) \\
    \partial_u dn(u) &= -k^2 sn(u)cn(u). \quad (A-7)
\end{align*}
\]
Using above equations we found the differential equations to be satisfied by the Jacobian elliptic functions and listed them in Table 1 where

\begin{align}
ns(u) &= 1/sn(u), & (A-8) \\
nc(u) &= 1/cn(u), & (A-9) \\
nd(u) &= 1/dn(u), & (A-10) \\
sc(u) &= sn(u)/cn(u), & (A-11) \\
sd(u) &= sn(u)/dn(u), & (A-12) \\
cd(u) &= cn(u)/dn(u), & (A-13) \\
cs(u) &= 1/sc(u), & (A-14) \\
ds(u) &= 1/sd(u), & (A-15) \\
dc(u) &= 1/cd(u). & (A-16)
\end{align}

Having checked those equations in Table I, one can see that there are many combinations of JEF that can satisfy eqs. (20) and (21) where the discriminant is \( \mu^2 + 4\kappa\lambda \) for \( k_f \) and \((\mu + 1)^2 + 4\kappa\lambda \) for \( k_g \).

For \( L \rightarrow \infty \), we find from eq. (44) that \( k \rightarrow 1 \) and

\begin{align}
ns(u) &\simeq \tanh(u), & (A-17) \\
nc(u) &\simeq dn(u) \simeq \sec h(u). & (A-18)
\end{align}

**Appendix B** In this Appendix, we prove the following lemma.

**Lemma 1** If \( \theta(z) = \pi/2 + \varphi + 4\arctan[A_{th}sc(\beta(L - z_0); k_f)] \) and \( A = A_{th}, \) then \( \hat{\theta}_+(z = L) - \hat{\theta}_+(z = 0) = 2\pi \) where \( l \) is a natural number defined by

\begin{equation}
\beta L = lK(k_f = \sqrt{1 - A_{th}^4}) = n \int_0^1 \frac{dt}{\sqrt{(1 - t^2)[1 - (1 - A_{th}^4)t^2]}} \tag{B-1}
\end{equation}

with \( \beta = 1/(1 - A_{th}^2). \)

Proof:

The boundary condition requires that at \( z = 0 \)

\[ \frac{\phi}{4} = -\arctan A_{th}sn(-\beta z_0) \tag{B-2} \]

and at \( z = L \)

\[ \frac{n\pi}{2} - \frac{\phi}{4} = \arctan A_{th}sc(\beta(L - z_0)) = \arctan A_{th}sc(lK - \beta z_0) \tag{B-3} \]

Note that the period of \( sc(u) \) is \( 2K. \) Hence, when \( z \) increases from 0 to \( L, l/2 \) periods will pass. Since the period of \( \arctan \) function is \( \pi, \) the term \( l\pi/2 \) has to be added on the left hand side of eq. (C-3). To see it more explicitly, define

\[ \alpha = \arctan(A_{th}sc(\beta(L - z_0))) = \arctan(A_{th}sc(lK - \beta z_0)), \tag{B-4} \]
and
\[ \varsigma = \arctan(A_{th} \text{sc}(-\beta z_0)), \] (B-5)
then we have
\[ \tan(\alpha - \varsigma) = \frac{\tan \alpha - \tan \varsigma}{1 + \tan \alpha \tan \varsigma} = A_{th} \frac{\text{sc}(lK - \beta z_0) - \text{sc}(-\beta z_0)}{1 + A_{th}^2 \text{sc}(lK - \beta z_0) \text{sc}(-\beta z_0)}. \] (B-6)

With eqs. (A-3) and (A-4), we have
\[ \text{sc}(\beta(nK - z_0)) = \frac{\text{sn}(nK - \beta z_0)}{\text{cn}(nK - \beta z_0)} = \frac{\text{cn}(nK - K - \beta z_0)}{-k' \text{sn}(nK - K - \beta z_0)}. \] (B-7)
where \( k_f^2 + k_f'^2 = 1 \). Since \( k_f^2 = 1 - A_{th}^2 \), we find that \( k_f' = A_{th}^2 \). Eq. (C-6) becomes
\[ \tan(\alpha - \varsigma) = -A_{th} \frac{\text{cs}(lK - K - \beta z_0)/k_f' - \text{sc}(-\beta z_0)}{1 - \text{sc}(-\beta z_0) \text{cs}(lK - K - \beta z_0)} \] (B-8)
where \( \text{cs}(u) = 1/\text{sc}(u) \). So the denominator of \( \tan(\alpha - \varsigma) \) vanishes as \( n = 1 \) and the numerator is finite. Thus \( \tan(\alpha - \varsigma) = \pm \infty \) and \( \alpha - \varsigma = \pm \pi/2 \). The sign is determined by the fact that as \( z \) increases, \( \hat{\theta}_+ \) also increases, thus the positive sign should be chosen and \( \hat{\theta}_+(z = L) - \hat{\theta}_+(z = 0) = 2\pi \). If \( l = 2 \), we can use eqs. (A-3) and (A-4) again or simply use the fact that the period of \( \text{sc}(u) \) is \( 2K \) to find out that the numerator vanishes. Thus, \( \tan(\alpha - \varsigma) = \pi \) and \( \hat{\theta}_+(z = L) - \hat{\theta}_+(z = 0) = 4\pi \). Hence, we conclude that \( \hat{\theta}_+(z = L) - \hat{\theta}_+(z = 0) = 4(\alpha - \varsigma) = 2l\pi \).

End of Proof.

References:
[1]. D. J. Thouless, Phy. Rev. B27, 6083 (1983).
[2]. Q. Niu and D. J. Thouless, J. Phys. A, 17, 2453 (1984).
[3]. P. W. Brouwer, Phy. Rev. B58, 10135 (1998).
[4]. T. A. Shutenko, I. L. Aleiner and B. L. Altshuler, Phy. Rev. B61, 10366 (2000).
[5]. Y. Levinson, Entin-O. Wohlm and P. Wolfe, Physica A302, 335 (2001); Wohlan-O. Entin and A. Afaron, Phy. Rev. B66, 35239 (2002).
[6]. M. Blaanboor, Phy. Rev. B65, 235318 (2002).
[7]. J. Wang and B. Wang, Phy. Rev. B65, 153311 (2002); B. Wang and J. Wang, Phy. Rev. B66, 201305 (2002).
[8]. P. Sharma and C. Chamon, Phy. Rev. Lett. 87, 96401 (2001) and cond-mat/0209291.
[9]. R. Citro, N. Anderi and Q. Niu, cond-mat/0306181
[10]. D. J. Thouless, M. Kohmoto, M. P. Nightingale and M. den Nijs, Phy. Rev. Lett. 49, 405 (1982).
[11]. B. I. Halperin, Phy. Rev. B25, 2185 (1982).
[12]. Y. Hatsugai, Phy. Rev. Lett. 71, 3697 (1993).
[13]. Y. Hatsugai, Phy. Rev. B48, 1185 (1993).
[14]. R. Shindou, J. Phy. Soc.Jpn. 74, 1214 (2005).
[15]. C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005).
[16]. L. Fu and C. L. Kane, Phys. Rev. B74, 195312 (2006).
[17]. E. Fradkin, 'Field Theories of Condensed Matter Systems', Addison-Wesley, Redwood City, CA (1991).
[18]. N. Nagaosa, 'Quantum Field Theory in Strongly Correlated Electronic Systems', Springer-Verlag, Berlin (1999).
[19]. A. O. Gogolin, A. A. Nersesyan and A. M. Tsvelik, 'Bosonization and Strongly Correlated Systems', Cambridge University Press (1998).
[20]. T. Giamarchi, 'Quantum Physics in One Dimension', Oxford University Press (2004).
[21]. G. Costabile et. al., App. Phys. Lett. 32, 587 (1978).
[22]. R. M. DeLeonardis et. al., J. App. Phys. 51, 1211(1982).
[23]. R. M. DeLeonardis et. al., J. App. Phys. 53, 699 (1982).
[24]. Derek. F. Lawden, 'Elliptic Function and Application', Springer-Verlag NY (1989).
Fig captions:

Fig. 1 Threshold amplitude ($A_{th}$) versus system length $L$ for the static soliton solution.

Fig. 2 Energy spectrum of the static soliton versus $n = \beta/L$ with the boundary condition $\theta_+(z = 0) = 0$ and $\theta_+(z = L) = 2n\pi$ when $L = 24$.

Fig. 3 $z_0$ versus adiabatical parameter $\varphi$ with $L = 24$ for static soliton and boundary conditions $\theta_+(z = 0) = 0$ and $\theta_+(z = L) = 2\pi$.

Fig. 4 $\tilde{\theta}_+$, $S_z$ and dimer state amplitude $S^+S^- + S^-S^+$ versus lattice sites with $L = 24$ and different values of $\varphi$ (a) $\varphi = 0$, (b) $\varphi = \pi/4$, (c) $\varphi = \pi/2$, (d) $\varphi = 3\pi/4$, (e) $\varphi = \pi$, (f) $\varphi = 5\pi/4$, (g) $\varphi = 3\pi/2$, (h) $\varphi = 3\pi/2$.

Fig. 5 $\tilde{\theta}_+$, $S_z$ and dimer state amplitude $S^+S^- + S^-S^+$ versus lattice sites with $L = 14$ and $\varphi = \pi/2$. 
Fig. 1
Fig. 2
Spectrum levels: \( I = \beta L/K \)
Fig. 3
Fig. 4(a)
Fig. 4(b)
Fig. 4(c)
Fig. 4(e)
Fig. 4(f)
Fig. 4(h)
Fig. 5
Table 1
Differential equations satisfied by Jacobian elliptic functions. See Appendix A for the definitions of Jacobian elliptic functions.

| JEF  | its equation | JEF  | its equation |
|------|--------------|------|--------------|
| \( y = \text{sn}(u) \) | \( (\partial_u y)^2 = (1 - y^2)(1 - k^2 y^2) \) | \( y = \text{cn}(u) \) | \( (\partial_u y)^2 = (1 - y^2)(1 - k^2 + k^2 y^2) \) |
| \( y = \text{dn}(u) \) | \( (\partial_u y)^2 = (y^2 - 1)(1 - k^2 - y^2) \) | \( y = \text{ns}(u) \) | \( (\partial_u y)^2 = (y^2 - 1)(y^2 - k^2) \) |
| \( y = \text{nc}(u) \) | \( (\partial_u y)^2 = (y^2 - 1)(1 - k^2) y^2 + k^2 \) | \( y = \text{nd}(u) \) | \( (\partial_u y)^2 = (1 - y^2)(1 - k^2) y^2 - 1 \) |
| \( y = \text{sc}(u) \) | \( (\partial_u y)^2 = (y^2 + 1)(1 + k^2 y^2) \) | \( y = \text{cd}(u) \) | \( (\partial_u y)^2 = (y^2 - 1)(k^2 y^2 - 1) \) |
| \( y = \text{sd}(u) \) | \( (\partial_u y)^2 = (1 - k^2 y^2)(1 + k^2 y^2) \) | \( y = \text{cs}(u) \) | \( (\partial_u y)^2 = (1 + y^2)(k^2 y^2 + y^2) \) |
| \( y = \text{dc}(u) \) | \( (\partial_u y)^2 = (k^2 - y^2)(1 - y^2) \) | \( y = \text{ds}(u) \) | \( (\partial_u y)^2 = (y^2 - k^2)(y^2 + k^2) \) |