Research Article

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Carathéodory solutions of Sturm-Liouville dynamic equation with a measure of noncompactness in Banach spaces

Abstract: In this paper, we present the existence result for Carathéodory type solutions for the nonlinear Sturm-Liouville boundary value problem (SLBVP) in Banach spaces on an arbitrary time scale. For this purpose, we introduce an equivalent integral operator to the SLBVP by means of Green’s function on an appropriate set. By imposing the regularity conditions expressed in terms of Kuratowski measure of noncompactness, we prove the existence of the fixed points of the equivalent integral operator. Mönch’s fixed point theorem is used to prove the main result. Finally, we also remark that it is straightforward to guarantee the existence of Carathéodory solutions for the SLBVP if Kuratowski measure of noncompactness is replaced by any axiomatic measure of noncompactness.

Keywords: Sturm-Liouville equation, Banach space, Measure of noncompactness, Carathéodory solutions, Time scale

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1 Introduction

The measure of noncompactness which is one of the fundamental tools in the theory of nonlinear analysis was initiated by the pioneering articles of Kuratowski [24], Darbo [15] and was developed by Banaś and Goebel [7] and many researchers in the literature. The applications of the measure of noncompactness (for the weak case, the measure of weak noncompactness developed by DeBlasi [16]) can be seen in the wide range of applied mathematics: theory of differential equations (see [3, 11, 26, 32] and references therein), difference equations [1, 17], integral equations [5], differential inclusions (multi-valued functional differential equations) [10, 18, 27] and inevitably in the theory of dynamic equations [12–14, 33].

After Hilger initiated the concept of time scale (or measure chain) [20, 21], the time scale unified the theory that can be described various kinds of discrete ($h$-difference and $q$-difference) and continuous models. The intrinsic advantage of time scale is that it provides not only a unified approach to study the discrete intervals with uniform step size (i.e., lattice $h\mathbb{Z}$), continuous intervals and discrete intervals with nonuniform step size (for instance $q$-numbers), but also, more interestingly, it gives an opportunity to extend the approach to study the combination of continuous and discrete intervals. Therefore, the concept of time scales can build bridges between the continuous, discrete and $q$-discrete analysis. Nevertheless, the theory of dynamic equations in Banach spaces is not sufficiently developed.

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A first result on this research area belongs to Hilger (Theorem 5.7 of [21]). Recently, there have been research activities for the existence of weak and strong type solutions of dynamic equations in Banach spaces [12–14, 23, 29, 30, 33]. However the existence of the Carathéodory type solutions of dynamic equations in Banach spaces is still quite a new research area.

In the literature, many authors endeavoured to improve both differential and difference equations in Banach spaces and their applications in the last century. Since the dynamic equations on time scale unify the differential and difference equations, we evolve the results to time scale which is a more unifying general framework.

In [31], Topal, et al. present the existence of the solution of the equation

\[(p(t)x^\Delta(t))^{\nabla} + \lambda \phi(t)f(t, x(t)) = 0, \quad 0 < t < \infty,\]

for an unbounded time scale. Comparing with [31], the primary contributions of this work are the followings: First, we prove the existence of the solutions in Banach spaces rather than in \(\mathbb{R}\). Consequently, we deal with a Carathéodory nonlinear term instead of a continuous one.

In this paper, we prove the existence of a Carathéodory solution of the boundary value problem

\[
(p(t)x^\Delta(t))^{\nabla} + f(t, x(t), x^\Delta(t), x^{\nabla}(t)) = 0, \quad t \in J = [0, \infty) \cap \mathbb{T},
\]

\[
\alpha_1 x(0) - \beta_1 \lim_{t \to 0^+} p(t)x^\Delta(t) = 0,
\]

\[
\alpha_2 \lim_{t \to \infty} x(t) + \beta_2 \lim_{t \to \infty} p(t)x^\Delta(t) = 0,
\]

in Banach spaces. The existence result for the classical solutions of the same problem is given by Yantir, et al. [33]. We follow the following procedure for our main result. First, we express the SLBVP (1)-(2) as an integral equation by the virtue of Green’s function. Consequently we construct an integral operator associated to this integral equation. For the compactness condition, we use Kuratowski measure of noncompactness. Finally, we apply Mönch fixed point theorem [25] for the existence result.

2 Basic notions

After the concept of time scale was introduced, the calculus on time scales was fastly developed by many authors and presented in the articles unifying the discrete and continuous analysis, mostly in the theory of differential equations (see [2, 8, 9, 20–22] and references therein). The detailed information of the general theory of dynamic equations on time scale can be found in the books [8, 9, 22].

Let \((E, \| \cdot \|)\) be a Banach space and \(\mathbb{T}\) be a time scale (nonempty closed subset of real numbers \(\mathbb{R}\)). Throughout this paper, by an interval \([a, b]\) we mean \(\{t \in \mathbb{T} : a \leq t \leq b\}\). In particular, \(J = [0, \infty) = \{t \in \mathbb{T} : 0 \leq t < \infty\}\). Other types of intervals are assumed similarly. By a subinterval \(I\) of \(J\) we mean the time scale subinterval.

We utilize the (Kuratowski) measure of noncompactness [7] as the principle tool for the compactness condition in the main theorem. For any bounded subset \(A\) of \(E\), the Kuratowski measure of noncompactness of \(A\), denoted by \(\sigma(A)\), is defined by the infimum of all \(\varepsilon > 0\) such that there exists a finite covering of \(A\) by sets of diameter smaller than \(\varepsilon\). The properties, required for the rest of the article, of the measure of noncompactness \(\sigma\) are listed below:

(i) If \(A \subset B\) then \(\sigma(A) \leq \sigma(B)\).
(ii) \(\sigma(A) = \sigma(\bar{A})\), where \(\bar{A}\) denotes the closure of \(A\).
(iii) \(\sigma(A) = 0\) if and only if \(A\) is relatively compact,
(iv) \(\sigma(A \cup B) = \max\{\sigma(A), \sigma(B)\}\),
(v) \(\sigma(\lambda A) = |\lambda|\sigma(A) (\lambda \in \mathbb{R})\),
(vi) \(\sigma(A + B) \leq \sigma(A) + \sigma(B)\),
(vii) \(\sigma(\text{conv } A) = \sigma(A)\), where \(\text{conv } A\) denotes the convex extension of \(A\).

Next we generalize the result of Ambrosetti [4] for the space of functions which has continuous \(\Delta\) and \(\nabla\)-derivatives which is denoted by \(C^\Delta_1(\mathbb{T}, E)\) throughout the paper. The adaptation of Ambrosetti Lemma is one the essential tool in the proof of the main result. The proof of the Ambrosetti Lemma for \(C^\Delta_1(\mathbb{T}, E)\) can be found in [33].
Lemma 2.1 (Ambrosetti Lemma [33]). Let $A \in C^\Delta V(\mathbb{T}, E)$ be bounded. Also assume that $A^\Delta$ and $A^\nabla$ defined by \{ $g^\Delta : g \in A$ \} and \{ $g^\nabla : g \in A$ \} respectively, be bounded and equicontinuous. Then the measure of noncompactness in $C^\Delta V(\mathbb{T}, E)$ is given by

$$
\alpha(A) = \max \left\{ \sup_{t \in \mathbb{T}} \alpha(A(t)), \sup_{t \in \mathbb{T}} \alpha(A^\Delta(t)), \sup_{t \in \mathbb{T}} \alpha(A^\nabla(t)) \right\}.
$$

In the proof of the main theorem, the following fixed point result of Mönch will be used for the existence of fixed point of the integral operator which corresponds to the SLBVP (1)-(2).

Theorem 2.2 (Mönch Fixed Point Theorem [25]). Let $D$ be a closed convex subset of $E$, and let $F$ be a continuous map from $D$ into itself. If for some $x \in D$ the implication

$$
\tilde{V} = \overline{\text{conv}}(\{x \cup F(V)\}) \Rightarrow V \text{ is relatively compact}
$$

holds for every countable subset $V$ of $D$, then $F$ has a fixed point.

In order to fulfill the assumptions in the main result, we use the following mean value theorem for $V$-integrals which is proved in [33]. The reader can reach the definition an the basic properties of Lebesgue $V(\Delta)$-measure and Lebesgue $V(\Delta)$-integral in [9, 19].

Theorem 2.3 (Mean Value Theorem for $V$-integrals). If $u : J \rightarrow E$ is a $V$-integrable function, then

$$
\int_I u(t) Vt \in \mu_V(I) \cdot \overline{\text{conv}}(I),
$$

where $I$ is an arbitrary subinterval of $J$ and $\mu_V(I)$ is the Lebesgue $\nabla$-measure of $I$.

## 3 Main results

We recall that a function $f : \mathbb{T} \times E^3 \rightarrow E$ is a Carathéodory function if the following statements are satisfied:

(a) $f(t, x, x_1, x_2)$ is bounded,
(b) $f(t, x, x_1, x_2)$ is measurable with respect to $t \in \mathbb{T}$ for all $(x, x_1, x_2) \in E^3$
(c) $f(t, x, x_1, x_2)$ is continuous with respect to $x, x_1, x_2$ for $\mu_V$ almost all $t \in \mathbb{T}$.

A function $x : \mathbb{T} \rightarrow E$ is said to be a Carathéodory solution of the SLBVP (1)-(2) provided that $x$ is $\Delta$-differentiable on $\mathbb{T}^k, x^\Delta : \mathbb{T} \rightarrow E$ is $V$-differentiable on $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$ and $x^\Delta : \mathbb{T}^* \rightarrow E$ is continuous and $x(t)$ satisfies (1) $\mu_V$ a.e. in $\mathbb{T}^*$ and (2) is satisfied. Here $\mathbb{T}^k$ and $\mathbb{T}_k$ represent the region of $\Delta$ and $V$-differentiability respectively.

Throughout the paper, we assume the following conditions on $p, f, \alpha_1, \alpha_2, \beta_1, \beta_2$:

(A1) $p : \mathbb{T} \rightarrow \mathbb{R}$ is a $V$-differentiable function on $\mathbb{T}_k = [0, \infty)_k, p(t) \neq 0$, $\forall t \in \mathbb{T}$ and $p^\nabla : \mathbb{T}_k \rightarrow \mathbb{R}$ is continuous, and $\int_0^\infty \frac{\Delta s}{p(s)} < \infty$.

(A2) $f : \mathbb{T} \times E^3 \rightarrow E$ is a Carathéodory function.

(A3) $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers such that $|\alpha_1| + |\beta_1| \neq 0, |\alpha_2| + |\beta_2| \neq 0$ with

$$
\alpha_2 \beta_1 + \alpha_1 \beta_2 + \alpha_1 \alpha_2 \int_0^\infty \frac{\Delta s}{p(s)} \neq 0.
$$

The linear case of the equation (1) on a finite interval is introduced by Atici and Guseinov [6]. Up to this article, the second order dynamic equations with $\Delta^\Delta$-derivatives are studied. The problem of obtaining the symmetric Green’s function over a time scale interval is solved by studying $\Delta^\nabla$-derivatives in this article. Then Topal, et al. study the existence of positive solutions for the SLBVP (1)-(2) for real valued function $f$, for an unbounded time scale [31]. While constructing the integral equation which is equivalent to the SLBVP (1)-(2), we employ the techniques that are used in [31, 33].
In order to find the equivalent integral equation corresponding to the SLBVP (1)-(2), we first consider the homogenous equation

\[-(p(t)x^\Delta(t))^{\nabla} = 0 \tag{3}\]

together with the boundary conditions

\[x(0) = \beta_1, \quad \lim_{t \to 0^+} x[\Delta](t) = \alpha_1. \tag{4}\]

\[\lim_{t \to -\infty} x(t) = \beta_2, \quad \lim_{t \to -\infty} x[\Delta](t) = -\alpha_2. \tag{5}\]

respectively. Apparently, the solutions of (3)-(4) and (3)-(5) are given by

\[u_1(t) = \beta_1 + \alpha_1 \int_0^t \frac{\Delta s}{p(s)} \quad \text{and} \quad u_2(t) = \beta_2 + \alpha_2 \int_t^{\infty} \frac{\Delta s}{p(s)}. \tag{6}\]

Furthermore \(u_1(t)\) and \(u_2(t)\) satisfy (2). If we set

\[D = -W_t(u_1, u_2) = u_1[\Delta](t)u_2(t) - u_1(t)u_2[\Delta](t). \tag{7}\]

\(D\) is independent of \(t\) and hence

\[D = \alpha_2 \beta_1 + \alpha_1 \beta_2 + \alpha_1 \alpha_2 \int_0^{\infty} \frac{\Delta s}{p(s)} \neq 0. \tag{8}\]

Therefore for any linear \(\nabla\)-integrable function \(h : \mathbb{T} \to E\) and under the condition \(\int_0^{\infty} \frac{\Delta s}{p(s)} < \infty\), the linear equation

\[(p(t)x^\Delta(t))^{\nabla} + h(t) = 0, \tag{9}\]

with the boundary conditions (2) has a unique solution

\[x(t) = \int_0^{\infty} G(t, s)h(s)\nabla s, \tag{10}\]

where \(h : \mathbb{T} \to E\) is any \(\nabla\)-integrable function and

\[G(t, s) = \frac{1}{D} \begin{cases} u_1(t)u_2(s), & 0 \leq t < s < \infty, \\ u_1(s)u_2(t), & 0 \leq s < t < \infty. \end{cases} \tag{11}\]

From the results above, for Carathéodory type solutions, the SLBVP (1)-(2) is equivalent to the integral equation

\[x(t) = \int_0^{\infty} G(t, s)f(s, x(s), x^\Delta(s), x^\nabla(s))\nabla s. \tag{12}\]

To verify the equivalence (10), we assume that \(x(t)\) is the Carathéodory solution of the SLBVP (1)-(2) and

\[A = \{s \in [0, t] : (p(s)x^\Delta(s))^\nabla \neq -f(s, x(s), x^\Delta(s), x^\nabla(s))\}. \tag{13}\]

Since (1) holds \(\mu_\nabla\) a.e. \(t \in J\), we have \(\mu_\nabla(A) = 0\). Hence

\[\int_A G(t, s)f(s, x(s), x^\Delta(s), x^\nabla(s))\nabla s = 0. \tag{14}\]

Similarly, for the set

\[B = \{s \in [t, \infty) : (p(s)x^\Delta(s))^\nabla \neq -f(s, x(s), x^\Delta(s), x^\nabla(s))\}, \tag{15}\]
we have
\[ \int_B G(t, s) f(s, x(s), x^A(s), x^\nabla(s)) \, ds = 0. \] (12)

Therefore the equations (11) and (12) yield
\[
\int_0^\infty G(t, s) f(s, x(s), x^A(s), x^\nabla(s)) \, ds = \int_0^t G(t, s) f(s, x(s), x^A(s), x^\nabla(s)) \, ds \\
+ \int_t^\infty G(t, s) f(s, x(s), x^A(s), x^\nabla(s)) \, ds \\
= \int_{[0,t] \setminus A} G(t, s)[p(s)x^A(s)]^{\nabla} \, ds + \int_{(t, \infty) \setminus B} G(t, s)[p(s)x^A(s)]^{\nabla} \, ds.
\]
Using the integration by parts formula (Theorem 1.77 of [8]) and Theorem 2.10 of [6], we lead
\[
\int_0^\infty G(t, s) f(s, x(s), x^A(s), x^\nabla(s)) \, ds = x(t) \mu_a.e. t \in J,
\]
which implies \( x(t) \) is the Carathéodory solution of problem (10).

Conversely, let the function \( x \) be a Carathéodory solution of (10). One can easily show that \( x(t) \) satisfies (2) and
\[
(p(t)x^A(t))^{\nabla} = -f(t, x(t), x^A(t), x^\nabla(t)) \mu_a.e. t \in J.
\]
Hence \( x \) is the Carathéodory solution of the SLBVP (1)-(2).

For the existence of the Carathéodory solution for the SLBVP (1)-(2), we seek for the fixed point of the operator
\[
F(x)(t) = \int_0^\infty G(t, s) f(s, x(s), x^A(s), x^\nabla(s)) \, ds
\]
corresponding to integral equation (10). For \( x \in C_1^{\Delta\nabla}(\mathbb{T}, E) \), we define the norm of \( x \) by \( ||x|| = \{||x(t)|| : t \in \mathbb{T}\} \).

For the proof of the main result we define the followings:
\[
M_1 = \frac{1}{D} \left( \beta_1 + \alpha_1 \int_0^{\Delta s} p(s) \right) \left( \beta_2 + \alpha_2 \int_0^{\Delta s} p(s) \right),
\]
\[
M_2 = \sup_{t \in J} \left| \frac{1}{p(t)} \right|,
\]
\[
M_3 = \sup_{t \in J} \left| \frac{1}{p(p(t))} \right|,
\]
\[
M_0 = M_1 \int_0^\infty m(s) \, ds,
\]
\[
\tilde{G}(t, \tau) = \int_0^\infty |G(t, s) - G(\tau, s)| \cdot m(s) \, ds.
\]

We also define a closed, bounded subset \( \tilde{B} \) of \( E \) in the following way:
\[
\tilde{B} = \{x \in C(J, E) : ||x|| \leq M_0, ||x(t) - x(\tau)|| \leq \tilde{G}(t, \tau) \}.
\]
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\[
\| x^\Delta(t) - x^\Delta(\tau) \| \leq \left| \frac{1}{p(\tau)} - \frac{1}{p(t)} \right| \int_0^\infty m(s) \nabla s + \left| \frac{1}{p(\tau)} \right| \int_t^\tau m(s) \nabla s,
\]

\[
\| x^\nabla(t) - x^\nabla(\tau) \| \leq \left| \frac{1}{p(\rho(\tau))} - \frac{1}{p(\rho(t))} \right| \int_0^\infty m(s) \nabla s + \left| \frac{1}{p(\rho(\tau))} \right| \int_t^\tau m(s) \nabla s.
\]

**Theorem 3.1.** Assume that \( f \) is a Carathéodory function and there exist Lebesgue \( \nabla \)-integrable functions \( m, k : [0, \infty) \to \mathbb{R}^+ \) such that

\[
\| f(t, x, x_1, x_2) \| \leq m(t) \quad \text{for } t \in J, \ x, x_1, x_2 \in E,
\]

\[
\alpha(f(I \times A \times B \times C)) \leq k(t) \max\{\alpha(A), \alpha(B), \alpha(C)\},
\]

and

\[
0 < M_i \int_0^\infty k(s) \nabla s < 1, \ M_i > 0, \ \forall i = 1, 2, 3.
\]

for \( I \subset J \) and for any bounded subsets \( A, B, C \) of \( E \). Then the SLBVP (1)-(2) has a solution.

**Proof.** In order to fulfill the conditions of Mönch fixed point theorem (Theorem 2.2), we first show that the operator \( F \) defined by (13) maps \( \tilde{B} \) into \( \bar{B} \).

\[
\| F(x)(t) \| = \int_0^\infty G(t, s) f(s, x, x^\Delta, x^\nabla) \nabla s \leq \int_0^\infty |G(t, s)| \cdot \| f(s, x, x^\Delta, x^\nabla) \| \nabla s
\]

As

\[
G(t, s) \leq \frac{1}{D} u_1(\infty) u_2(0) = \frac{1}{D} \left( \beta_1 + \alpha_1 \int_0^{\Delta s} \frac{\Delta s}{p(s)} \right) \left( \beta_2 + \alpha_2 \int_0^{\Delta s} \frac{\Delta s}{p(s)} \right) = M_1,
\]

we get

\[
\| F(x)(t) \| \leq \int_0^\infty M_1 \| f(s, x(s), x^\Delta(s), x^\nabla(s)) \| \nabla s \leq M_1 \int_0^\infty m(s) \nabla s = M_0.
\]

Consequently, we show that the sets \( F(\tilde{B}), F^\Delta(\tilde{B}), F^\nabla(\tilde{B}) \) are equicontinuous.

\[
\| F(x)(t) - F(x)(\tau) \| = \int_0^\infty |G(t, s) - G(\tau, s)| \cdot \| f(s, x(s), x^\Delta(s), x^\nabla(s)) \| \nabla s
\]

\[
\leq \int_0^\infty |G(t, s) - G(\tau, s)| \cdot \| f(s, x(s), x^\Delta(s), x^\nabla(s)) \| \nabla s
\]

\[
\leq \int_0^\infty |G(t, s) - G(\tau, s)| \cdot m(s) \nabla s = \tilde{G}(t, \tau)
\]

Hence \( F(\tilde{B}) \) is equicontinuous since \( t \to \tau \) implies \( \tilde{G}(t, \tau) \to 0 \).

For the equicontinuity of \( F^\Delta(\tilde{B}) \), we consider

\[
F(x)^\Delta(t) = \left( \int_0^\infty G(t, s) f(s, x(s), x^\Delta(s), x^\nabla(s)) \nabla s \right)^\Delta
\]
Therefore for all \( t, \tau \in J \), we have
\[
\|F(x^\Delta(t)) - F(x^\Delta(\tau))\| = \frac{1}{D} \left| \int_0^t \frac{-\alpha_2}{p(t)} u_1(s, x, x^\Delta, x^\nabla) \nabla s + \int_\tau^\infty \frac{\alpha_1}{p(t)} u_2(s, x, x^\Delta, x^\nabla) \nabla s \right|
\]

\[
+ \frac{1}{D} \left| \int_0^t \frac{\alpha_2}{p(t)} u_1(s, x, x^\Delta, x^\nabla) \nabla s - \int_\tau^\infty \frac{\alpha_1}{p(t)} u_2(s, x, x^\Delta, x^\nabla) \nabla s \right|
\]

\[
\leq \frac{1}{D} \left| \int_0^t \frac{\alpha_2}{p(t)} u_1(s) \nabla s + \int_\tau^\infty \frac{\alpha_1}{p(t)} u_2(s) \nabla s \right|
\]

\[
+ \frac{1}{D} \left| \int_0^t \frac{\alpha_2}{p(t)} u_1(s) \nabla s - \int_\tau^\infty \frac{\alpha_1}{p(t)} u_2(s) \nabla s \right|
\]

\[
\leq \frac{1}{D} \left| \int_0^t \frac{\alpha_2}{p(t)} u_1(s) \nabla s + \int_\tau^\infty \frac{\alpha_1}{p(t)} u_2(s) \nabla s \right|
\]

As \( t \to \tau \) then \( \|F(x^\Delta(t)) - F(x^\Delta(\tau))\| \to 0 \). Hence \( F^\Delta(\bar{B}) \) is equicontinuous.

The equicontinuity of \( F(\bar{B}) \) follows from the \( \nabla \)-derivative under the integral sign formula and the similar calculations as in the equicontinuity of \( F^\Delta(\bar{B}) \).

Now, we show the continuity of \( F \). Let \( x_n \to x \) in \( \bar{B} \). Then we have
\[
\|F(x_n)(t) - F(x)(t)\| = \left| \int_0^\infty G(t, s) \left| f(s, x_n, x_n^\Delta, x_n^\nabla) - f(s, x, x^\Delta, x^\nabla) \right| \nabla s \right|
\]

\[
\leq \int_0^\infty \| G(t, s) \| \left| f(s, x_n, x_n^\Delta, x_n^\nabla) - f(s, x, x^\Delta, x^\nabla) \right| \nabla s
\]

\[
\leq M \int_0^\infty \left| f(s, x_n, x_n^\Delta, x_n^\nabla) - f(s, x, x^\Delta, x^\nabla) \right| \nabla s.
\]

Since \( f \) is a Carathéodory function, by the virtue of Lebesgue dominated convergence theorem for the \( \nabla \)-integral (see [9]), we deduce that \( \|F(x_n)(t) - F(x)(t)\| \to 0 \). Thus the operator \( F \) is continuous.

Now, we prove that a fixed point of operator \( F \) exists by using Mönch fixed point theorem. For this purpose, we let \( V \) be a countable subset of \( \bar{B} \) satisfying the condition
\[
\bar{V} = \text{conv}(\{ x \mid F(V) \}) \text{ for some } x \in \bar{B}.
\]

We define \( V(t) = \{ v(t) \in E : v \in V, t \in J \} \). Since \( V \) is equicontinuous, by Lemma 2.1, the function \( t \mapsto v(t) = \alpha(V(t)) \) is continuous on \( J \). For any \( \epsilon > 0 \), we separate the interval \( J = [0, K] \cup [K, \infty) \) into two parts such that
\[
\int_0^K G(t, s) f(s, V(s), V^\Delta(s), V^\nabla(s)) \nabla s < \frac{\epsilon}{2}.
\]

Lusin’s theorem guarantees that for any \( K > 0 \) and \( \delta > 0 \), there exists a closed subset \( \Omega \subset [0, K] \) such that \( k(t) \) is continuous on \( \Omega \) and \( \mu(V([0, K] \setminus \Omega)) < \delta \) and
\[
\int_{[0, K] \setminus \Omega} G(t, s) f(s, V(s), V^\Delta(s), V^\nabla(s)) \nabla s < \frac{\epsilon}{2}.
\]
We divide the closed subset $\Omega$ into $m$ parts: $0 = t_0 < t_1 < \cdots < t_m = \max(\Omega)$. For $i = 0, 1, 2, \ldots, m - 1$ we denote $T_i = [t_i, t_{i+1}]$. The mean value theorem for $\nu-$ integrals (Theorem 2.3) implies the following embedding:

$$\int_\Omega G(t, s) f(s, V, V^\Delta, V^\nabla) \nu_s = \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} G(t, s) f(s, V, V^\Delta, V^\nabla) \nu_s \subset \sum_{i=0}^{m-1} \mu_{\nu}(T_i) \left( G(t, T_i) f(T_i, V(T_i), V^\Delta(T_i), V^\nabla(T_i)) \right).$$

The definition of the operator $F$, properties of Kuratowski measure of noncompactness, Lebesgue $\nu$-measure, Lemma 2.1 and the assumption (14) allow us to have

$$\alpha(F(V)(t)) \leq \sum_{i=0}^{m-1} \mu_{\nu}(T_i) \alpha \left( G(t, T_i) f(T_i, V(T_i), V^\Delta(T_i), V^\nabla(T_i)) \right) \leq \sum_{i=0}^{m-1} \mu_{\nu}(T_i) \sup_{s \in T_i} G(t, s) \alpha \left( f(T_i, V(T_i), V^\Delta(T_i), V^\nabla(T_i)) \right) \leq M_1 \sum_{i=0}^{m-1} \mu_{\nu}(T_i) \sup_{s \in T_i} k(s) \max\{\alpha(V(T_i)), \alpha(V^\Delta(T_i)), \alpha(V^\nabla(T_i))\} \leq M_1 \alpha(V(t)) \sum_{i=0}^{m-1} \mu_{\nu}(T_i) k(q_i)$$

for all $t \in \Omega$ and for some $q_i \in T_i$. Therefore as

$$F(V)(t) = \int_{\Omega} G(t, s) f(s, V, V^\Delta, V^\nabla) \nu_s + \int_{[0, K] \setminus \Omega} G(t, s) f(s, V, V^\Delta, V^\nabla) \nu_s + \int_{(0, K]} G(t, s) f(s, V, V^\Delta, V^\nabla) \nu_s,$$

we have $\alpha(F(V)(t)) \leq M_1 \alpha(V(\Omega)) \int_0^\infty k(s) \nu s + \epsilon$. Since $\epsilon$ is arbitrary

$$\alpha(F(V)(t)) \leq M_1 \alpha(V(\Omega)) \int_0^\infty k(s) \nu s. \quad (22)$$

Analogously by the use of (15),

$$\alpha(F(V^\Delta)(t)) \leq \alpha \left( \int_{\Omega} G^\Delta(t, s) f(s, V(s), V^\Delta(s), V^\nabla(s)) \nu_s \right) \leq \sum_{i=0}^{m-1} \mu_{\nu}(T_i) \alpha \left( G^\Delta(t, T_i) f(T_i, V(T_i), V^\Delta(T_i), V^\nabla(T_i)) \right) \leq M_2 \sum_{i=0}^{m-1} \mu_{\nu}(T_i) \sup_{s \in T_i} k(s) \cdot \max\{\alpha(V(T_i)), \alpha(V^\Delta(T_i)), \alpha(V^\nabla(T_i))\} \leq M_2 \cdot \alpha(V(t)) \sum_{i=0}^{m-1} \mu_{\nu}(T_i) k(q_i)$$

for all $t \in \Omega$ and for some $q_i \in T_i$. Thus

$$\alpha(F(V^\Delta)(t)) \leq M_2 \alpha(V(\Omega)) \int_0^\infty k(s) \nu s. \quad (23)$$
Moreover
\[
\alpha(F(V)\overline{\mathbf{v}}(t)) \leq M_3\alpha(V(\Omega)) \int_0^\infty k(s)\overline{v}s
\] (24)
can be obtained in a similar way. Hence Lemma 2.1 and equations (22), (23), (24) enable us to have
\[
\alpha(F(V)) \leq \max_{i=1,2,3}\left\{ M_i \int_0^\infty k(s)\overline{v}s \right\} \cdot \alpha(V) < \alpha(V).
\] (25)

On the other hand since \( V = \overline{\text{conv}}(\{x\} \cup F(V)) \), by the properties of measure of noncompactness \( \alpha \) we obtain \( \alpha(V) \leq \alpha(F(V)) \). The formula (25) leads us to have \( \alpha(V) = 0 \). The Arzela-Ascoli theorem implies \( V \) is relatively compact. As a result, all the assumption hypothesis of Mönch fixed point theorem are fulfilled and we conclude that \( F \) has a fixed point which is the solution of the SLBVP (1)-(2).

Remark 3.2. Theorem 3.1 also holds if the measure of noncompactness \( \alpha \) is replaced by any axiomatic measure of noncompactness possessing the listed properties.

We demonstrate our main result with the following example:

Example 3.3. Consider the boundary value problem
\[
\left( \frac{1}{2^\gamma} x^\Delta(t) \right)^\gamma + \frac{1 + \sin(x(t)x^\Delta(t))}{3^\gamma} = 0, \quad t \in T = \{ \frac{n}{2} : n \in \mathbb{N}_0 \},
\] (26)
\[
\frac{1}{2} x(0) - \frac{1}{2} \lim_{t \to 0^+} p(t)x^\Delta(t) = 0,
\]
\[
\frac{1}{2} \lim_{t \to -\infty} x(t) + \frac{1}{2} \lim_{t \to -\infty} p(t)x^\Delta(t) = 0,
\] (27)
Clearly the conditions (A1)-(A3) hold. By the definition of \( D \), one can find
\[
D = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \int_0^\infty \frac{\Delta t}{2^\gamma} = \frac{5 + \sqrt{2}}{8} \approx 0.8 \neq 0.
\]
Using the definitions of \( M_1 \), \( M_2 \) and \( M_3 \), we find \( M_1 \approx 1.376 \), \( M_2 = 0.5 \) and \( M_3 = 0.5 \). If we choose \( m(t) = \frac{2}{3^\gamma} \), clearly the condition \( ||f(t, x(t), x^\Delta(t))|| \leq m(t) \) of Theorem 3.1 holds. Also
\[
M_0 = M_1 \int_0^\infty m(t)\Delta t \approx 1.376 \cdot 1.366 \approx 1.879.
\]
If we pick \( k(t) = 4^{-t} \), since \( \int_0^\infty k(s)\overline{v}s = 0.5 \) the condition (21) of Theorem 3.1 are fulfilled. Because of the character of the function \( f(t, x(t), x^\Delta(t)) = \frac{1 + \sin(x(t)x^\Delta(t))}{3^\gamma} \) the condition (20) of Theorem 3.1 is satisfied. As a result, the BVP (26)-(27) has a solution.

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