Abstract: In this paper, we use the mean value theorem of Dirichlet $L$-functions, the properties of Gauss sums and Dedekind sums to study the hybrid mean value problem involving Dedekind sums and the two-term exponential sums, and give an interesting identity and asymptotic formula for it.

Keywords: Dedekind sums, The two-term exponential sums, Hybrid mean value, Identity, Asymptotic formula

MSC: 11L03, 11F20

1 Introduction

Let $q$ be a natural number and $h$ an integer prime to $q$. The classical Dedekind sums

$$S(h, q) = \sum_{a=1}^{q} \left( \frac{a}{q} \right) \left( \frac{ah}{q} \right),$$

where

$$\left( \frac{x}{y} \right) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer}, \end{cases}$$

describes the behaviour of the logarithm of the eta-function (see [1, 2]) under modular transformations. The various arithmetical properties of $S(h, q)$ were investigated by many authors, who obtained a series of results, see [3–10]. For example, W. P. Zhang and Y. N. Liu [10] studied the hybrid mean value problem of Dedekind sums and Kloosterman sums

$$K(m, n; q) = \sum_{a=1}^{q} e \left( \frac{ma + n\overline{a}}{q} \right),$$

where $q \geq 3$ is an integer, $\sum_{a=1}^{q}$ denotes the summation over all $1 \leq a \leq q$ with $(a, q) = 1$, $e(y) = e^{2\pi i y}$, and $\overline{a}$ denotes the multiplicative inverse of $a \mod q$. They proved the following results:

**Theorem A.** Let $p$ be an odd prime, then one has the identity

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1; p)|^2 \cdot |K(b, 1; p)|^2 \cdot S(a \cdot \overline{b}, p) = \begin{cases} \frac{1}{12} \cdot p^2 \cdot ((p - 1)(p - 2) - 12 \cdot h_p^2), & \text{if } p \equiv 3 \mod 4; \\ \frac{1}{12} \cdot p^2 \cdot ((p - 1)(p - 2), & \text{if } p \equiv 1 \mod 4, \end{cases}$$

where $h_p$ denotes the class number of the quadratic field $\mathbb{Q}(\sqrt{-p})$. 

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Theorem B. Let $p$ be an odd prime, then one has the asymptotic formula
\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1; p)|^2 \cdot |K(b, 1; p)|^2 \cdot S^2(a \cdot \overline{b}, q) = \frac{1}{24} p^5 + O \left( \frac{p^4 \cdot \exp \left( \frac{3 \ln \ln p}{\ln p} \right)}{p} \right),
\]
where \( \exp(y) = e^y \).

On the other hand, W. P. Zhang and D. Han [11] studied the sixth power mean of the two-term exponential sums, and proved that for any prime \( p > 3 \) with \( (3, p - 1) = 1 \), one has the identity
\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} e \left( \frac{n^3 + an}{p} \right)^6 = 5p^4 - 8p^3 - p^2.
\]
It is natural that one will ask, for the two-term exponential sums
\[
E(r, s) = \sum_{n=0}^{p-1} e \left( \frac{rn^3 + sn}{p} \right),
\]
whether there exists an identity (or asymptotic formula) similar to Theorem A (or Theorem B). The answer is yes.

The main purpose of this paper is to show this point. That is, we shall use the mean value theorem of Dirichlet \( L \)-functions, the properties of Gauss sums and Dedekind sums to prove the following similar conclusions:

Theorem 1.1. Let \( p > 3 \) be an odd prime with \( (3, p - 1) = 1 \), then we have the identity
\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |E(a, 1)|^2 \cdot |E(b, 1)|^2 \cdot S \left( a \cdot \overline{b}, p \right) = \begin{cases} 
\frac{1}{12} (p - 2)(p - 1)p^2, & \text{if } p \equiv 1 \mod 4; \\
\frac{1}{12} (p - 2)(p - 1)p^2 - p^2 h_p^2, & \text{if } p \equiv 3 \mod 4,
\end{cases}
\]
where \( h_p \) denotes the class number of the quadratic field \( \mathbb{Q}(\sqrt{-p}) \).

Theorem 1.2. Let \( p > 3 \) be a prime with \( (3, p - 1) = 1 \), then we have the asymptotic formula
\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |E(a, 1)|^2 \cdot |E(b, 1)|^2 \cdot S^2 \left( a \cdot \overline{b}, p \right) = \frac{1}{24} p^5 + O \left( \frac{p^4 \cdot \exp \left( \frac{3 \ln \ln p}{\ln p} \right)}{p} \right).
\]
It is very interesting that the results in our paper are exactly the same as in reference [10]. This means that there is close relationship between Kloosterman sums and two-term exponential sums. In fact, some close relationships can be found in W. Duke and H. Iwaniec [12].

2 Several lemmas

To complete the proof of our theorems, we need to prove several lemmas. Hereinafter, we shall use some properties of characters mod \( q \) and Dirichlet \( L \)-functions, all of these can be found in reference [13], so they will not be repeated here.

Lemma 2.1. Let \( p \) be an odd prime, \( a \) be any integer with \( (a, p) = 1 \). For any non-principal character \( \chi \mod p \), we have the identity
\[
\sum_{n=0}^{p-1} \chi \left( an^2 + bn + c \right) = \frac{\chi(4) \tau(\chi_2) \tau(\chi \overline{\chi}_2)}{\tau(\chi)} \chi \left( \frac{4ac - b^2}{p} \right) \left( \frac{4ac - b^2}{p} \right). 
\]
where \( \chi_2 = \left( \frac{2}{p} \right) \) denotes the Legendre symbol mod \( p \).
Proof. From the definition and properties of Gauss sums we have
\[
\sum_{n=0}^{p-1} \chi(an^2 + bn + c) = \frac{1}{\tau(\chi)} \sum_{r=1}^{p-1} \tau(r) \sum_{n=0}^{p-1} e\left(\frac{ran^2 + bn + c}{p}\right).
\]
\[
= \frac{\tau(\chi)}{\tau(\chi)} \sum_{r=1}^{p-1} \tau(r) e\left(\frac{4rc - r^2b^2}{p}\right) \sum_{n=0}^{p-1} e\left(\frac{ra(2n + b\bar{a})^2}{p}\right).
\]
\[
= \frac{\tau(\chi)}{\tau(\chi)} \sum_{r=1}^{p-1} \tau(r) e\left(\frac{r(4c - b^2\bar{a})}{p}\right) \sum_{n=0}^{p-1} e\left(\frac{ran^2}{p}\right).
\]
(1)

For any integer \(a\) with \((a, p) = 1\), from Theorem 7.5.4 of [14] we know that
\[
\sum_{n=0}^{p-1} e\left(\frac{an^2}{p}\right) = \left(\frac{a}{p}\right) \cdot \tau(\chi_2).
\]
(2)

Combining (1) and (2) we have the identity
\[
\sum_{n=0}^{p-1} \chi(an^2 + bn + c) = \frac{\tau(\chi_2) \tau(\chi_2)}{\tau(\chi)} \left(\frac{a}{p}\right) \sum_{r=1}^{p-1} \tau(r) \left(\frac{r}{p}\right) e\left(\frac{r(4c - b^2\bar{a})}{p}\right).
\]
This proves Lemma 2.1.

Lemma 2.2. Let \(p\) be an odd prime. Then for any non-principal character \(\chi\) mod \(p\) with \(\chi^3 \neq \chi_0\) (the principal character mod \(p\)), we have the identity
\[
\sum_{a=1}^{p-1} \chi(a) \left| \sum_{n=0}^{p-1} e\left(\frac{an^3 + n}{p}\right) \right|^2 = \chi(4)\chi_2(3)\tau(\chi_2)\tau(\chi_2)\tau(\chi^3).
\]

Proof. From Lemma 2.1, the definition and properties of Gauss sums we have
\[
\sum_{a=1}^{p-1} \chi(a) \left\| \sum_{n=0}^{p-1} e\left(\frac{an^3 + n}{p}\right) \right\|^2
\]
\[
= \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \chi(a) e\left(\frac{a(m^3 - n^3) + m - n}{p}\right)
\]
\[
= \tau(\chi) \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \tau(m^3 - n^3) e\left(\frac{m - n}{p}\right)
\]
\[
= \tau(\chi) \sum_{m=0}^{p-1} \tau^3(m) e\left(\frac{m}{p}\right) + \tau(\chi) \sum_{m=0}^{p-1} \sum_{n=1}^{p-1} \tau(m^3 - n^3) e\left(\frac{m - n}{p}\right)
\]
\[
= \tau(\chi) \sum_{m=0}^{p-1} \tau^3(m) e\left(\frac{m}{p}\right) + \tau(\chi) \sum_{m=0}^{p-1} \tau(m^3 - 1) \sum_{n=1}^{p-1} \tau^3(n) e\left(\frac{n(m - 1)}{p}\right)
\]
\[
= \tau(\chi) \tau(\chi^3) + \tau(\chi) \tau(\chi^3) \sum_{m=0}^{p-1} \chi((m - 1)^3) \tau(m^3 - 1)
\]
\[
= 2\tau(\chi) \tau(\chi^3) + \tau(\chi) \tau(\chi^3) \sum_{m=2}^{p-1} \chi((m - 1)^3) \tau(m^3 - 1)
\]
\[
= 2\tau(\chi) \tau(\chi^3) + \tau(\chi) \tau(\chi^3) \sum_{m=1}^{p-2} \chi(m^3) \tau(m^3 + 3m^2 + 3m)
\]
\[ = 2\tau(\chi)\tau(\chi^3) + \tau(\chi)\tau(\chi^3) \sum_{m=1}^{p-2} \chi(1 + 3m^2 + 3m) \]

\[ = 2\tau(\chi)\tau(\chi^3) + \tau(\chi)\tau(\chi^3) \sum_{m=1}^{p-2} \chi(3m^2 + 3m + 1) \]

\[ = \tau(\chi)\tau(\chi^3) \sum_{m=0}^{p-1} \chi(3m^2 + 3m + 1) \]

\[ = \chi(4)\chi_Z(3)\tau(\chi_Z)\tau(\chi^3). \]

This proves Lemma 2.2. \qed

**Lemma 2.3.** Let \( q > 2 \) be an integer. Then for any integer \( a \) with \( (a, q) = 1 \), we have the identity

\[ S(a, q) = \frac{1}{\pi^2 q} \sum_{d \mid q} d^2 \phi(d) \sum_{\chi \bmod d \atop \chi(-1)=-1} \chi(a)|L(1, \chi)|^2, \]

where \( L(1, \chi) \) denotes the Dirichlet \( L \)-function corresponding to character \( \chi \bmod d \).

**Proof.** See Lemma 2 of [9]. \qed

**Lemma 2.4.** For any odd prime \( p \), we have the asymptotic formula

\[ \sum_{\chi \bmod p \atop \chi(-1)=-1} |L(1, \chi)|^4 = \frac{5}{144} \pi^4 \cdot p + O\left( \exp\left( \frac{3\ln p}{\ln p} \right) \right). \]

**Proof.** See Lemma 6 of [15]. \qed

### 3 Proof of the theorems

In this section, we shall complete the proof of our theorems. First we prove Theorem 1.1. From Lemma 2.3 with \( q = p \) (an odd prime) we have

\[ S(a, p) = \frac{1}{\pi^2 p} \cdot \frac{p}{p-1} \cdot \sum_{\chi \bmod p \atop \chi(-1)=-1} \chi(a)|L(1, \chi)|^2 \quad \text{(3)} \]

and

\[ \sum_{\chi \bmod p \atop \chi(-1)=-1} |L(1, \chi)|^2 = \frac{\pi^2(p-1)}{p} \sum_{a=1}^{p-1} \left( \frac{a}{p} - \frac{1}{2} \right)^2 = \frac{\pi^2}{12} \cdot \frac{(p-1)^2(p-2)}{p^2}. \quad \text{(4)} \]

It is clear that if \( (3, p-1) = 1 \), then for any odd character \( \chi \bmod p \), we have \( \chi^3 \neq \chi_0 \), the principal character \( \bmod p \). Note that \( |\tau(\chi)| = \sqrt{p} \), if \( \chi \neq \chi_0 \). So from (3) and Lemma 2.2 we have

\[ \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |E(a, 1)|^2 \cdot |E(b, 1)|^2 \cdot S \left( a \cdot \overline{b}, p \right) \]

\[ = \frac{1}{\pi^2} \cdot \frac{p}{p-1} \cdot \sum_{\chi \bmod p \atop \chi(-1)=-1} \left| \chi(4)\chi_Z(3)\tau(\chi_Z)\tau(\chi^3) \right|^2 \cdot |L(1, \chi)|^2 \]
\[ |\tau(\chi \chi_2)|^2 \cdot |L(1, \chi)|^2. \]  

(5)

If \( p \equiv 1 \mod 4 \), then \( \chi \chi_2 \neq \chi_0 \) for all odd character \( \chi \mod p \). This time, from (4) and (5) we have

\[
\begin{align*}
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |E(a, 1)|^2 \cdot |E(b, 1)|^2 \cdot S \left( a \cdot \mathfrak{b}, p \right)
&= \frac{1}{\pi^2} \cdot \frac{p^4}{p-1} \cdot \sum_{\chi \mod p} \sum_{\chi(-1)\equiv -1} |L(1, \chi)|^2 \\
&= \frac{(p-2)(p-1)p^2}{12} - \frac{p^3}{\pi^2} |L(1, \chi_2)|^2 \\
&= \frac{(p-2)(p-1)p^2}{12} - p^2 \cdot h^2_p.
\end{align*}
\]

(6)

If \( p \equiv 3 \mod 4 \), then \( \chi_2(-1) = -1 \). This time we have \( |\tau(\chi_2 \chi_2)| = 1 \) and \( |\tau(\chi \chi_2)| = \sqrt{p} \), \( \chi \neq \chi_2 \). Combining (5) and (6) we obtain

\[
\begin{align*}
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |E(a, 1)|^2 \cdot |E(b, 1)|^2 \cdot S \left( a \cdot \mathfrak{b}, p \right)
&= \frac{1}{\pi^2} \cdot \frac{p^4}{p-1} \cdot \sum_{\chi \mod p} \sum_{\chi(-1)\equiv -1} |L(1, \chi)|^2 + \frac{1}{\pi^2} \cdot \frac{p^3}{p-1} |L(1, \chi_2)|^2 - \frac{1}{\pi^2} \cdot \frac{p^4}{p-1} |L(1, \chi)|^2 \\
&= \frac{(p-2)(p-1)p^2}{12} - \frac{p^3}{\pi^2} |L(1, \chi_2)|^2 \\
&= \frac{(p-2)(p-1)p^2}{12} - p^2 \cdot h^2_p.
\end{align*}
\]

(7)

where we have used the identity \( L(1, \chi_2) = \pi h_p / \sqrt{p} \).

Combining (6) and (7) we may immediately deduce Theorem 1.1.

Now we prove Theorem 1.2. From (3) we have

\[
\begin{align*}
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |E(a, 1)|^2 \cdot |E(b, 1)|^2 \cdot S^2 \left( a \cdot \mathfrak{b}, p \right)
&= \frac{1}{\pi^4} \cdot \frac{p^2}{(p-1)^2} \cdot \sum_{\chi \mod p} \sum_{\eta \mod p} \left( \sum_{a=1}^{p-1} \chi(a) \eta(a) |E(a, 1)|^2 \right)^2 \\
&\times \left( \sum_{h=1}^{p-1} \overline{\mathfrak{b}(h)|E(h, 1)|^2} \right) |L(1, \chi)|^2 \cdot |L(1, \eta)|^2 \\
&= \frac{1}{\pi^4} \cdot \frac{p^2}{(p-1)^2} \cdot \sum_{\chi \mod p} \sum_{\eta \mod p} \left( \sum_{a=1}^{p-1} \chi(a) \eta(a) |E(a, 1)|^2 \right)^2 |L(1, \chi)|^2 \cdot |L(1, \eta)|^2 \\
&+ \frac{1}{\pi^4} \cdot \frac{p^2}{(p-1)^2} \cdot \left( \sum_{a=1}^{p-1} |E(a, 1)|^2 \right)^2 \sum_{\chi \mod p} \sum_{\chi(-1)\equiv -1} |L(1, \chi)|^4 \\
&= R_1 + R_2.
\end{align*}
\]

(8)

Now we estimate \( R_1 \) and \( R_2 \) in (8) respectively. Note that the identity

\[
\sum_{a=1}^{p-1} |E(a, 1)|^2 = p^2.
\]
from Lemma 2.4 we have the asymptotic formula

$$R_2 = \frac{1}{\pi^2} \cdot \frac{p^6}{(p-1)^2} \left( \frac{5\pi^4}{144} \cdot p + O \left( \exp \left( \frac{3 \ln \ln p}{\ln p} \right) \right) \right) = \frac{5}{144} \cdot p^5 + O \left( p^4 \cdot \exp \left( \frac{3 \ln \ln p}{\ln p} \right) \right). \tag{9}$$

If \( p \equiv 1 \mod 4 \), then there exist two odd characters \( \chi \) and \( \eta \) such that \( \chi \eta \chi_2 = \chi_0 \). This time, we have the estimate

$$\sum_{\chi \mod p} \sum_{\eta \mod p} |L(1, \chi)|^2 \cdot |L(1, \eta)|^2 = \sum_{\chi \mod p} |L(1, \chi)|^2 \cdot |L(1, \chi \chi_2)|^2 = O(p).$$

So for any prime \( p > 3 \), from (4), Lemma 2.2 and Lemma 2.4 we also have the asymptotic formula

$$R_1 = \frac{1}{\pi^2} \cdot \frac{p^5}{(p-1)^2} \sum_{\chi \mod p} \sum_{\eta \mod p} |L(1, \chi)|^2 \cdot |L(1, \eta)|^2$$

$$+ \frac{1}{\pi^4} \cdot \frac{p^4}{(p-1)^2} \sum_{\chi \mod p} \sum_{\eta \mod p} |L(1, \chi)|^2 \cdot |L(1, \eta)|^2$$

$$- \frac{1}{\pi^5} \cdot \frac{p^5}{(p-1)^2} \sum_{\chi \mod p} \sum_{\eta \mod p} |L(1, \chi)|^2 \cdot |L(1, \eta)|^2$$

$$= \frac{1}{\pi^4} \cdot \frac{p^5}{(p-1)^2} \left( \sum_{\chi \mod p} |L(1, \chi)|^2 \right)^2$$

$$+ O \left( p^4 \right)$$

$$= \frac{1}{\pi^4} \cdot \frac{p^5}{(p-1)^2} \cdot \frac{\pi^4 \cdot (p-1)^2}{144} \cdot \frac{p^2}{p^2} + O \left( p^4 \right)$$

$$= \frac{1}{144} \cdot p^5 + O \left( p^4 \right). \tag{10}$$

Combining (8), (9) and (10) we have the asymptotic formula

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |E(a, 1)|^2 \cdot |E(b, 1)|^2 \cdot S^2 \left( a \cdot b, p \right)$$

$$= \frac{1}{144} \cdot p^5 + \frac{5}{144} \cdot p^5 + O \left( p^4 \cdot \exp \left( \frac{3 \ln \ln p}{\ln p} \right) \right)$$

$$= \frac{1}{24} \cdot p^5 + O \left( p^4 \cdot \exp \left( \frac{3 \ln \ln p}{\ln p} \right) \right).$$

This completes the proofs of our results.

**Competing interests**

The authors declare that they have no competing interests.

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