Protecting qudit operations from noise by continuous dynamical decoupling

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We have developed a procedure of generalized continuous dynamical decoupling (GCDD) for an arbitrary d-level system (qudit). We show that using this GCDD we can protect an arbitrary qudit gate from general noise. We illustrate a practical implementation for the qutrit case during the action of a Hadamard gate, using the three magnetic hyperfine states of the ground energy level of $^{87}$Rb and laser beams whose intensities and phases are modulated according to our prescription.

The development and implementation of effective quantum computers is of great interest to the scientific community as well as to the world economy as a whole [1]. Indeed, quantum computers promise to revolutionize many important tasks, even with a reduced number of algorithms known to be more efficient than their classical analogues [2]. In this sense, reducing errors while keeping quantum algorithms simple is an important aspect to be addressed. Several strategies have been proposed to contrast decoherence effects, such as reservoir engineering methods [3,7], optimal quantum control protocols [8,9], measurement-based control [10,12], and pulsed dynamical decoupling of qubits [13,16]. Some of us worked at continuous dynamical decoupling strategies, applied to single- and two-qubit systems [17,20].

Continuous dynamical decoupling techniques have been theoretically investigated and experimentally implemented in several contexts. For example, these kinds of techniques have been applied in the case of nitrogen-vacancy centres to separate a single nuclear spin signal from the bath noise [21] to extend the coherence time for the electron spin [22,23] while protecting quantum gates [23], to provide single-molecule magnetic resonance spectroscopy [24], and in the context of sensing high frequency fields [25]. Furthermore, they can be used to create a dephasing-insensitive quantum computation scheme in an all-in-all connected superconducting circuit [26], and for engineering an optical clock transition in trapped ions, robust against external field fluctuations [27].

Although qubits are virtually ubiquitous in the development of quantum algorithms [2], d-level systems (qudits) seem to be potentially more powerful for information processing [28,40]. Indeed, the use of higher-dimensional quantum systems brings significant advantages, allowing for information coding with increased density and thus reducing the number of multi-particle interactions. Specifically, the use of qudits brings improvements in the building of quantum logic gates and the simplification of the design of circuits [28,31], in the security of quantum key distribution protocols [32,36], in performing quantum computation [37,41], as well as in the realization of fundamental tests of quantum mechanics [42]. In particular, powerful error correction procedures have been proposed for qudits [43,46]. We remark that some of the above advantages have been pointed out already for the case of qutrits, see e.g. [29,30,32,33]. Several setups have been considered to experimentally implement qudits, including optical systems [47,49], superconductors [31,50] and atomic spins [51,52].

In this letter, we present a complete theoretical prescription for a generalized continuous dynamical decoupling (GCDD) of an arbitrary qudit from environmental noise. We consider the situation in which an arbitrary quantum operation is applied to an unknown qudit state while errors, due to environmental effects, might occur, reducing the fidelity of the operation. By fidelity we mean a measure of how close is the obtained state to the target state which would have been obtained if the ideal noise-free version of the intended operation had taken place. Our prescription consists of a continuously-varying control Hamiltonian and a modification of the intended quantum operation that, together, improve the fidelity of the process.

The GCDD procedure.— Let $H_G$ be the Hamiltonian generating the intended evolution of an arbitrary input state of the qudit, in the ideal, noise-free case. That is, after a gate operation time $\tau$, the desired evolution operator acting on an initial qudit state is given by $U_G = \exp(-iH_G\tau/\hbar)$. However, in the presence of environmental effects the evolution of the same initial qudit state is not going to be given by $U_G$, because of the perturbations introduced by the noise during $\tau$. Our aim is, thus, instead of using $H_G$, to use external fields whose interaction with the qudit is described by the non-autonomous Hamiltonian $H_{\text{lab}}(t)$, acting continuously during the time interval $\tau$. We choose $H_{\text{lab}}(t)$ such that, despite the presence of noise, it generates an effective evolution of the qudit that, at least up to a high enough fidelity, is the same as the one in the ideal noiseless case, which is provided by $U_G$. It is important to remark that $H_{\text{lab}}(t)$ acts only on the qudit Hilbert space.

To describe $H_{\text{lab}}(t)$, we first split this control field Hamiltonian into two terms: $H_{\text{gate}}(t)$, which will provide the modified gate Hamiltonian that in the end will
effectively reproduce the action generated by the ideal \( H_G \), and \( H_c(t) \), which is the control Hamiltonian that will continuously decouple the qudit evolution from the interference of the environment. Associated with \( H_c(t) \) there is a unitary operator \( U_c(t) \) that we require to be periodic with a period \( t_0 \) and to satisfy the dynamical-decoupling condition [33]

\[
\int_0^{t_0} dt \left[ U_c^\dagger(t) \otimes I_E \right] H_{int} \left[ U_c(t) \otimes I_E \right] = 0,
\]

where \( I_E \) is the identity operator of the environmental Hilbert space and \( H_{int} \) is the Hamiltonian interaction term coupling the qudit with its environment. We choose \( \tau \) to be an integer multiple of \( t_0 \), as we explain below.

The total Hamiltonian of the system and the environment is then written as:

\[
H_{tot}(t) = \left[ H_{gate}(t) + H_c(t) \right] \otimes I_E + I_d \otimes H_E + H_{int},
\]

where \( H_E \) is the free Hamiltonian of the environment and \( I_d \) is the identity operator of the qudit Hilbert space of dimension \( d \). In the picture obtained by unitarily transforming Eq. (2) using \( U_c(t) \), we obtain the Hamiltonian in what we henceforth call the control picture:

\[
H(t) \equiv \left[ U_c^\dagger(t) \otimes I_E \right] H_{tot}(t) \left[ U_c(t) \otimes I_E \right]
\]

\[
+ i\hbar \frac{d}{dt} U_c(t) \otimes I_E = H_G \otimes I_E + I_d \otimes H_E + H_{int},
\]

where, as we explain shortly, we have chosen \( H_{gate}(t) \) as

\[
H_{gate}(t) \equiv U_c(t) H_G U_c^\dagger(t).
\]

Of course, \( H_E \) is invariant under this unitary transformation because the environment operators commute with the ones acting on the qudit states. It is not difficult to see that the total evolution operator \( U(t) \), in the control picture, associated with \( \hat{H}(t) \), is also periodic with a period \( t_0 \), so that \( U(\tau) = I_d \otimes I_E \). Thus, at the end of the modified gate operation, the qudit state in the control picture coincides with the one in the original picture at time \( \tau \), explaining why we have chosen Eq. (4) and \( \tau \) as an integer multiple of \( t_0 \). Since \( H_{gate}(t) \) is used in the presence of continuous dynamical decoupling, the evolution proceeds, effectively, as if only \( U_G \) described the qudit evolution in the control picture. At time \( \tau \), even in the original picture the qudit state is the one that the ideal evolution would produce, up to a high-enough fidelity.

The dissipative dynamics is assumed to be resulting from a perturbing interaction between the qudit and its environment described by the very general Hamiltonian:

\[
H_{int} = \sum_{r,s=0}^{d-1} \sum_{r'=s=0}^{d-1} \langle r | s \rangle \otimes B_{r,s},
\]

where \( B_{r,s} \), for \( r,s = 0,1,2,\ldots,d-1 \) are operators that act on the environmental states and \( |k\rangle \), for \( k = 0,1,2,\ldots,d-1 \), are the \( d \) states comprising the qudit. For convenience, without loss of generality, we can always define \( H_{int} \) and \( H_E \) in such a way that

\[
\sum_{r=0}^{d-1} B_{r,r} = 0.
\]

**Our prescription.**— Now, we prescribe how to construct the required \( U_c(t) \). Let us consider a basis set for a Hilbert space of dimension \( d \), henceforth called the qudit space, whose normalized state vectors are labeled \( |k\rangle \), for \( k = 0,1,\ldots,d-1 \). This is also to be considered the logical basis. We define \( H_L \) as the Hermitian operator whose action on the logical basis states gives

\[
H_L |k\rangle \equiv k \hbar \omega_d |k\rangle,
\]

for \( k = 0,1,\ldots,d-1 \), where

\[
\omega_d \equiv d \omega_0
\]

and \( \omega_0 \) is the control frequency corresponding to the dynamical-decoupling period \( \tau_0 \):

\[
\omega_0 = \frac{2\pi}{t_0}
\]

The quantum-Fourier transform of the logical basis is given by [2]

\[
|\psi_n\rangle \equiv \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp \left( \frac{2\pi i n j}{d} \right) |j\rangle,
\]

for \( n = 0,1,\ldots,d-1 \). We define the Hermitian operator \( H_F \) by its action on the quantum-Fourier transformed basis, that is,

\[
H_F |\psi_n\rangle \equiv n \hbar \omega_0 |\psi_n\rangle,
\]

for \( n = 0,1,\ldots,d-1 \). The required control unitary transformation is given by

\[
U_c(t) \equiv \exp (-i \omega_r t) \exp \left( -i \frac{H_L}{\hbar} t \right) \exp \left( -i \frac{H_F}{\hbar} t \right),
\]

where we have defined a real constant \( \omega_r \) as

\[
\omega_r \equiv -\frac{\text{Tr} \{H_L\} + \text{Tr} \{H_F\}}{\hbar d}.
\]

It is now very straightforward to show that

\[
\int_0^{t_0} dt U_c^\dagger(t) AU_c(t) = \frac{t_0}{d} \hbar \text{Tr} \{A\},
\]

where \( A \) is any operator acting on the qudit space. Therefore, it follows from Eqs. [5], [6], and [11] that Eq. (1) is satisfied. We have then established what we named the GCDD procedure.
To implement the GCDD, we need a prescription for $H_{c}(t)$ and $H_{\text{gate}}(t)$. Using Eq. (12), we can calculate the control Hamiltonian, $H_{c}(t)$, as:

$$H_{c}(t) = i\hbar \frac{dU_{c}(t)}{dt} U_{c}^\dagger(t)$$

where, for simplicity, we have defined

$$U_{L}(t) \equiv \exp \left(-i\frac{H_{L}t}{\hbar}\right).$$

Equation (4) gives $H_{\text{gate}}(t)$ in terms of $U_{c}(t)$ and $H_{G}$. Hence, in the laboratory we need to generate external fields such that they interact with the qudit according to the following Hamiltonian:

$$H_{\text{lab}}(t) \equiv U_{c}(t) H_{G} U_{c}^\dagger(t) + H_{c}(t)$$

$$= U_{c}(t) H_{G} U_{c}^\dagger(t) + \hbar \omega_{r} I_{d} + H_{L} + U_{L}(t) H_{F} U_{L}^\dagger(t).$$

The term proportional to the unit matrix is immaterial to the dynamics, since it only involves a shift of the origin of our scale of energy values and, thus, gives rise to a global phase factor multiplying the evolved state vector.

**Application of the GCDD.**— To illustrate the GCDD method, we describe a possible implementation of a particular qutrit quantum gate, exploiting the three magnetic hyperfine states of the ground energy level of $^{87}$Rb. Figure 1 shows the relevant D$_2$-line hyperfine states of $^{87}$Rb. As shown in the figure, our qutrit space comprises the subspace spanned by the three magnetic states of the F = 1 ground level of $^{87}$Rb, with magnetic quantum numbers $m_{F} = -1, 0, 1$. We represent these degenerate states by the kets $|m\rangle$, for $m = -1, 0, 1$, respectively. We use two-photon transitions to couple these states among themselves in a controlled way, but we also need three independent detunings ($\Delta_{s}$, with $s = 1, 2, 3$) from the first-excited hyperfine state in the 5$P_{3/2}$ state manifold, namely, the one with total-angular-momentum quantum number $F' = 0$ and respective projection $m_{F}' = 0$, represented here by the ket $|e\rangle$ shown in the figure. Each laser beam is red-detuned from $|e\rangle$. Therefore, for each of the three different laser colors we can use the linear polarization and both circular polarizations, thus obtaining a total of nine independent Rabi frequencies. These independent control parameters are enough to emulate the action of any $3 \times 3$ Hermitian matrix used to represent a generic single-qutrit quantum gate, $H_{G}$, together with the control fields described by $H_{c}(t)$, that are required for the generalized continuous dynamical decoupling. We refer the reader to Appendices A and B where we present all the details and numbers that we can use, in principle, to implement such a qutrit and effective control Hamiltonian. Here, it suffices to say that the effective two-photon interaction we get, using the rotating-wave approximation and adiabatic elimination of $|e\rangle$ [90], is

$$H_{\text{eff}}(t) = \hbar \sum_{m=-1}^{1} \sum_{s=1}^{3} \frac{\Omega_{s,m}(t) \Omega_{s,-m}(t)}{\Delta_{s}} |m\rangle \langle n|,$$  

(18)

where $\Omega_{s,m}(t)$, for $s = 1, 2, 3$ and $m = -1, 0, 1$, are adiabatically time-varying Rabi frequencies allowing one to emulate the time dependent control Hamiltonian of Eq. (17) up the immaterial term proportional to $I_{3}$. In Appendix B, Secs. 1 and 2, we describe in detail the connection between the experimentally controlled Hamiltonian of Eq. (18) and the prescribed one of Eq. (17).

Next, we illustrate the GCDD method for the single-qutrit gate Hamiltonian (in the basis $\{|-1\rangle, |0\rangle, |1\rangle\}$)

$$H_{G} = \frac{\pi \hbar}{4\sqrt{3} \tau} \begin{pmatrix} 4\sqrt{3} - 2 & -2 & -2 \\ -2 & 2\sqrt{3} + 1 & 2\sqrt{3} + 1 \\ -2 & 2\sqrt{3} + 1 & 2\sqrt{3} + 1 \end{pmatrix},$$

(19)

which is known as the Hadamard gate Hamiltonian for a qutrit, where $\tau$ is the characteristic gate time (see Appendix C, Sec. 3). For example, starting from the state $|0\rangle$, after the action of the Hadamard operation, the output state after a time $\tau$ becomes

$$|\psi\rangle = -\frac{i}{\sqrt{3}} |\langle -1| + \exp(i\varphi) |0\rangle + \exp(-i\varphi) |1\rangle,$$  

(20)

where $\varphi = 2\pi/3$. In our simulations we consider two paradigmatic noises due to baths of thermal bosons, chosen to disturb our intended gate operation (see Appendix C). The amplitude damping noise simulates thermal dissipation involving, respectively, $|-1\rangle$ and $|0\rangle$, and
Figure 2. Numerical solutions for the GCDD to overcome amplitude damping and dephasing during the action of the Hadamard gate (the time \( t \) in units of the gate time \( \tau \)). Here, the bath correlation time is \( \tau_c = \tau/4 \), and various coupling constants are all equal leading to an effective coupling parameter \( \lambda = 0.1 \) (see Appendix C for details). The solid line represents the fidelity with no protection, while the dotted, dot-dashed, and dashed ones refer to the protective scheme with \( \omega_0/(2\pi) = 2/\tau_c \), \( 4/\tau_c \), and \( 16/\tau_c \), respectively. In the inset, we represent the gate fidelity (fidelity at time \( \tau \)) as a function of \( n = \omega_0\tau_c/(2\pi) \) (the interpolated curve just guides the reading).

\[ \rho_{\text{ff}}(t) = \frac{1}{2} \left( \rho + \sigma \right) \]

where \( \rho \) and \( \sigma \) are the initial state and the control Hamiltonian, respectively. The fidelity is defined for two arbitrary states \( \rho \) and \( \sigma \) as

\[ \text{FID} = \frac{\text{Tr}\{\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}\}}{\text{Tr}\{\sqrt{\rho}\}}. \]

The master equation governing the dissipative dynamics is

\[ \frac{d\rho}{dt} = -i[H_G, \rho] + \sum_{n} \Gamma_n \left( \rho_n - \rho \right) \]

where \( H_G \) is the interaction Hamiltonian with the environment. The fidelity reported in Fig. 2 shows that if we do not use the GCDD method during the time \( \tau \) in which the Hadamard gate operates and let the noise affect the dynamics, starting from the state \( |0\rangle \), the fidelity rapidly decreases. If we simulate the same gate operation under the same noise, starting from the same initial state, but now with the GCDD method turned on, we obtain better results by increasing the control frequency \( \omega_0 \). The inset shows the gate fidelity, i.e., the fidelity at time \( \tau \), as a function of \( n = \omega_0\tau_c/(2\pi) \). This implies \( t_0 = \tau_c/n \), according to Eq. (9). The figure shows that the application of the GCDD procedure allows to obtain very high values for the fidelity, thus overcoming the negative influence of the environment. The protective scheme improves when \( \omega_0 \) increases, with the final gate fidelity moving towards one. In particular, the smaller is \( t_0 \) with respect to the bath correlation time \( \tau_c \), the more effective is the decoupling procedure. We stress out that, by construction, the time at which to look for a state close to the original target is exactly the time \( \tau \) at which the original gate would have produced that state in the absence of the environment and of the control fields. The actual value of the gate time \( \tau \) is not specified in these simulations, the other quantities being given in units of it. Its value must just be such that the derivation of the effective Hamiltonian of Eq. (18), done in Appendix C Sec. 2, can be coherently performed. We also observe that the results shown in Fig. 2 have been obtained when the various coupling constants involved in the interaction Hamiltonian with the environment are all equal leading to an effective coupling parameter \( \lambda = 0.1 \) (see Appendix C for details). We have also tested some configurations with the various coupling constants not all equal, finding similar results.

The illustration of the GCDD method shown in Fig. 2 for our qutrit model using \(^{87}\text{Rb}\) and a modulated set of laser beams, can, in principle, be realistically implemented in the laboratory. Even if only the qutrit case has been considered, our results show that quantum computation could be implemented using laser light and atomic systems, which are available in setups with trapped ions, for example. This kind of implementation is attractive because it already presents long coherence times, implying high efficiency of our procedure. We remark that the implementation of our procedure can, in principle, be extended to the case of a qudit with more levels.

In conclusion, here we have presented a GCDD procedure to decouple an arbitrary qudit from any possible noise and still apply a given quantum gate on it. We expect that if the GCDD method is applied to a system of \( N \) qubits, it can decouple it from noise even if the errors introduced are on more than one qubit at a time. Our method is particularly relevant since there are not schemes of continuous dynamical decoupling for qudits, let alone in the case of generalized noise.

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Hamiltonian whose action can be described by a time-dependent gate to prescribe how to generate the fields in the laboratory against the noise described by Eqs. (5) and (6). We wish on the qudit state, prescribing how to protect its action the Institut UTINAM for its hospitality and financial support during his visit in Besançon. B.B. acknowledges support by the French “Investissements d’Avenir” program, project ISITE-BFC (contract ANR-15-IDEX-03).

Appendix A: The laboratory Hamiltonian

Here, we address an arbitrary quantum gate operating on the qudit state, prescribing how to protect its action against the noise described by Eqs. (5) and (6). We wish to prescribe how to generate the fields in the laboratory whose action can be described by a time-dependent gate Hamiltonian $H_{\text{gate}}(t)$ and a control Hamiltonian $H_c(t)$, both generating the evolution of the qudit state driven by such external fields. Let us start by considering the gate Hamiltonian,

$$H_{\text{gate}}(t) = U_c(t) H_G U_c^\dagger(t), \quad (A1)$$

appearing in Eq. (17). The gate Hamiltonian $H_G$ is what we want as effectively performing, after the time interval $\tau$, the gate action. It can be expanded in the computational basis by

$$H_G = \hbar \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} g_{r,s} |r\rangle \langle s|, \quad (A2)$$

where $g_{r,s}^* = g_{s,r}$ because $H_G$ is Hermitian.

In the particular implementation we choose in the next section to illustrate the present GCDD prescription in the case of a qutrit, an effective Hamiltonian emulating $H_{\text{lab}}(t)$ up to an immaterial term proportional to $I_3$ [see Eq. (17)] is obtained as a non-positive operator. This is not a problem, since for whatever $H_G$ we intend to use, we can always proceed as follows. If some of the eigenvalues of $H_G$ are positive, let’s choose, of these, the one with the highest absolute value, say, $g_0$, with $g_0 \geq 0$, where the equal sign is chosen if there are no positive eigenvalues of $H_G$. Then, let’s define the Hermitian operator $G$ such that

$$H_G = \hbar g_0 I_d - \hbar G. \quad (A3)$$

Therefore, $-G$ does not have positive eigenvalues and, thus, it’s a non-positive operator. Notice that $G$, however, is a non-negative operator (we have introduced the minus sign appearing in Eq. (A3) just for convenience).

Now, let us take a look at $H_c(t)$. From Eqs. (15) and (16) we obtain

$$H_c(t) = \hbar \omega_r I_d + H_L + \exp \left(-i \frac{H_L}{\hbar} t \right) H_F \exp \left(i \frac{H_L}{\hbar} t \right). \quad (A4)$$

It is easy to see, from Eqs. (7) and (11), that if we define Hermitian operators $H_L'$ and $H_F'$ by

$$H_L' = \hbar (d-1) \omega_d I_d - H_L \quad (A5)$$

and

$$H_F' = \hbar (d-1) \omega_0 I_d - H_F, \quad (A6)$$

respectively, then $H_L'$ and $H_F'$ are both non-negative operators and now, using Eqs. (A5) and (A6), we obtain, from Eq. (17) and Eqs. (A1), (A3), and (A4), that

$$H_{\text{lab}}(t) = \hbar \left[g_0 + \omega_r + (d^2 - 1) \omega_0 \right] I_d - \left[H_L' \right.$$

$$\left. + \exp \left(-i \frac{H_L}{\hbar} t \right) H_F' \exp \left(i \frac{H_L}{\hbar} t \right) + \hbar U_c(t) G U_c^\dagger(t) \right], \quad (A7)$$

where we have used Eq. (8). Since a unitary transformation of a non-negative operator is still a non-negative operator and $H_L'$, $H_F'$, and $G$, as we have defined them, are all non-negative, it follows that the last term within square brackets on the right-hand side of Eq. (A7) is a non-negative operator. Then, we can define

$$\Upsilon(t) \equiv \sqrt{V}, \quad (A8)$$

where

$$V = H_L' + \exp \left(-i \frac{H_L}{\hbar} t \right) H_F' \exp \left(i \frac{H_L}{\hbar} t \right)$$

$$+ \hbar U_c(t) G U_c^\dagger(t), \quad (A9)$$

and rewrite Eq. (A7) as

$$H_{\text{lab}}(t) = \hbar \omega_g I_d - \hbar \Upsilon(t) \Upsilon(t), \quad (A10)$$

where

$$\omega_g \equiv g_0 + \omega_r + (d^2 - 1) \omega_0. \quad (A11)$$

In the next section, we apply the GCDD method to the case of a qutrit, but, in principle, we could use the same two-photon atomic transitions involving any number of Zeeman hyperfine states. Thus, although the model implementation we use is limited to the simple case of the ground state hyperfine states of the $^{87}$Rb atom, other atomic systems could be used to obtain control over systems of qudits of dimension $d > 3$. Let us, then, proceed with the illustration of the GCDD method.
The $^{87}\text{Rb}$ atom, in the absence of external magnetic fields, has a ground-state manifold of three degenerate magnetic states. This is so because $^{87}\text{Rb}$ has a nuclear spin equal to $3/2$ and a fundamental electronic manifold of states with symmetry $5^2S_{1/2}$. This amounts to a hyperfine ground state with total angular momentum $F = 1$, so that there are three magnetic states whose projections along the quantization axis have quantum numbers $m_F = -1, 0, 1$. These three ground states are degenerate in the absence of magnetic fields and we denote them by $|m\rangle$, for $m = -1, 0, 1$ (the notations $1$ and $+1$ are both used in the following). The $5^2S_{1/2}$ ground manifold of states (including also the five magnetic states of the $F = 2$ ground level, besides the already-mentioned three $F = 1$ states) can be excited to states of the $5^2P_{3/2}$ excited manifold by absorbing photons with wavelengths of about 780 nm (called the $D_2$ spectral line of $^{87}\text{Rb}$). The lowest-energy magnetic hyperfine state of the $5^2P_{3/2}$ manifold is not degenerate and has a total-angular-momentum quantum number $F' = 0$, whose projection is $m'_{F'} = 0$. We denote this state by $|e\rangle$. If we use only a frequency corresponding to a virtual transition with wavelength greater than the optical 780 nm, that is, if we use only photons that are red-detuned with detunings $\Delta \nu < 0$ from the $F = 1 \leftrightarrow F' = 0$ transition, then we can approximate the relevant set of atomic states to be the one involving only the ground states $|m\rangle$, for $m = -1, 0, 1$, and the excited state $|e\rangle$. Thus, for this restricted Hilbert space, denoted by $\mathcal{H}_3$, we have the identity operator

$$I_4 = \sum_{m=-1}^{1} |m\rangle \langle m| + |e\rangle \langle e|. \quad (B1)$$

If the photons are detuned far enough to the red of the transitions $|m\rangle \leftrightarrow |e\rangle$, for $m = -1, 0, 1$, then the excited state is no longer going to be effectively populated, avoiding spurious transitions to the $F = 2$ ground states through spontaneous emission from $|e\rangle$. The effective qutrit, therefore, as we describe below, consists of the states $|m\rangle$, with $m = -1, 0, 1$, whose Hilbert space we denote by $\mathcal{H}_3$. The control over the states in $\mathcal{H}_3$ using the GCDD method is accomplished through two-photon transitions as we explain in the following.

1. The interaction Hamiltonian between the atom and the laser beams

Now, we introduce nine laser beams, whose electric-field vectors, each being the resultant with a different polarization, can be written as

$$E_{\pm 1} (t) = \sum_{s=1}^{3} |\delta_{s, \pm 1} (t) \hat{\xi}_{\pm 1} \exp (-i \omega_s t) + \delta_{s, \pm 1}^* (t) \hat{\xi}_{\pm 1}^* \exp (i \omega_s t)| \quad (B2)$$

and

$$E_0 (t) = \sum_{s=1}^{3} |\delta_{s, 0} (t) \exp (-i \omega_s t) + \delta_{s, 0}^* (t) \exp (i \omega_s t)|, \quad (B3)$$

where the polarization vectors are chosen, in terms of a space-fixed system of Cartesian coordinates, as

$$\hat{\xi}_{\pm 1} \equiv \begin{pmatrix} \hat{x} \pm i \hat{y} \\ \sqrt{2} \end{pmatrix}, \quad (B4)$$

representing, respectively, the $\sigma^\pm$ polarizations, and

$$\hat{\xi}_0 \equiv \hat{z}, \quad (B5)$$

representing the $\pi$ polarization. Here, the $z$-axis of this system is chosen to represent the quantization axis. It is noteworthy that in Eqs. (B2) and (B3), for each polarization, there are three different superposed amplitudes, $\delta_{s, \pm 1} (t)$ and $\delta_{s, 0} (t)$, each corresponding to a different polarization-independent frequency, $\omega_s$, for $s = 1, 2, 3$. Figure 1 shows the scheme we are describing. The amplitudes $\delta_{s, \pm 1} (t)$ and $\delta_{s, 0} (t)$, as we discuss below, must follow a prescribed relatively slow time-dependent modulation. It is worth mentioning that we treat the driving electric fields of Eqs. (B2) and (B3) as classical, intense laser fields. We are justified to use such a semiclassical approach because of the relatively high intensities and detuning magnitudes used, so that quantum fluctuations of the number of photons is completely negligible in the regime we consider here.

The laser beams of Eqs. (B2) and (B3) interact with the atom according to the Hamiltonian

$$H_{\text{int}} (t) = -d \cdot [E_{-1} (t) + E_0 (t) + E_{+1} (t)]$$

$$= -d \cdot \sum_{q=-1}^{+1} E_q (t), \quad (B6)$$

since we have the three resultant laser fields continuous and simultaneously present, each one with a different polarization. Here, $d$ is the atomic electric-dipole operator, which is Hermitian. In Cartesian coordinates, we write

$$d = d_x \hat{x} + d_y \hat{y} + d_z \hat{z}, \quad (B7)$$

and, using Eq. (B1) in $d = I_4 d I_4$, we obtain

$$d = \sum_{m=-1}^{1} |m\rangle \langle e| d |m\rangle \langle m| + \sum_{m=-1}^{1} |m\rangle \langle d |e\rangle \langle m|$$

$$= \sum_{m=-1}^{1} |e\rangle \langle m| (d |e\rangle)^* \langle m| + \sum_{m=-1}^{1} |m\rangle \langle m| d |e\rangle \langle e|, \quad (B8)$$
where we have used the fact that the electronic excited state has a parity that is opposite to the parity of the ground states, that is,

\[ \langle m | d | m' \rangle = 0 \quad \text{and} \quad \langle e | d | e \rangle = 0, \quad \text{(B9)} \]

for \( m, m' = -1, 0, 1 \). Now, we can write the operator \( d \) in terms of its spherical-tensor components:

\[
d = d_x \hat{x} + d_y \hat{y} + d_z \hat{z}
\]

\[
= -\frac{d_x + id_y}{\sqrt{2}} \hat{e}_{+1} + \frac{d_x - id_y}{\sqrt{2}} \hat{e}_{-1} + d_0 \hat{e}_0,
\]

\[ \text{(B10)} \]

that is,

\[
d = \sum_{q=-1}^{+1} d_q \hat{e}_q^*, \quad \text{(B11)}
\]

where we have used Eqs. (B4) and (B5) and defined its spherical components as usual:

\[
d_{\pm 1} \equiv \mp \frac{d_x \pm id_y}{\sqrt{2}} \quad \text{(B12)}
\]

and

\[
d_0 \equiv d_z. \quad \text{(B13)}
\]

Because \( |e \rangle \) has zero angular momentum, from Eq. (B11) it follows that

\[
\langle m | d | e \rangle \equiv \sum_{q=-1}^{+1} \langle m | d_q | e \rangle \hat{e}_q^* = \langle m | d_m | e \rangle \hat{e}_m^*, \quad \text{(B14)}
\]

since total angular momentum is conserved. From the Wigner-Eckart theorem \[51\], we have

\[
\langle m | d_q | e \rangle = \delta_{s,m} D,
\]

\[ \text{(B15)} \]

where \( D \) is a reduced matrix element of the dipole operator and is, thus, independent of \( m \) or \( q \). Hence, we rewrite Eq. (B14), using Eq. (B15), as:

\[
\langle m | d | e \rangle = D \hat{e}_m^* \quad \text{(B16)}
\]

Substituting Eq. (B16) into Eq. (B8), we obtain

\[
d = \sum_{m=-1}^{+1} \langle | e \rangle (D \hat{e}_m^*)^* \langle m | + \sum_{m=-1}^{+1} | m \rangle D \hat{e}_m^* \langle e | \]

\[ \text{Equation (B17)} \]

Substituting Eqs. (B2), (B3) and (B17) into Eq. (B6) gives

\[
H_{\text{int}}(t) = -\sum_{q=-1}^{+1} \sum_{s=1}^{3} (-1)^q D^* \delta_{s,q}^* (t) \exp(-i\omega_s t) |e \rangle \langle -q |
\]

\[
-\sum_{q=-1}^{+1} \sum_{s=1}^{3} (-1)^q D \delta_{s,q} (t) \exp(i\omega_s t) | -q \rangle \langle e |,
\]

\[ \text{(B18)} \]

which we have used the rotating-wave approximation \[52\], which we justify below. We have also used Eqs. (B4) and (B5) to calculate the scalar products between polarization vectors. Now, we introduce the Rabi frequencies:

\[
h \Omega_{s,q}(t) \equiv (-1)^q D^* \delta_{s,q}^* (t), \quad \text{(B19)}
\]

for \( s = 1, 2, 3 \) and \( q = -1, 0, +1 \). Rabi frequencies, \( \Omega_{s,q}(t)/2\pi \), of the order of a few MHz, let us say, roughly

\[
\frac{\Omega_{s,q}(t)}{2\pi} \sim 1 \text{ MHz}, \quad \text{(B20)}
\]

are routinely obtained in the context of optical manipulation of rubidium \[57, 58\]. These independent control parameters are enough to emulate the action of any \( 3 \times 3 \) Hermitian matrix used to represent a generic single-qutrit quantum gate together with the control fields required for the continuous dynamical decoupling, as we explain below. Given Eq. (B19), we can rewrite Eq. (B18) as

\[
H_{\text{int}}(t) = -\hbar \sum_{q=-1}^{+1} \sum_{s=1}^{3} \Omega_{s,q}(t) \exp(-i\omega_s t) |e \rangle \langle -q |
\]

\[
-\hbar \sum_{q=-1}^{+1} \sum_{s=1}^{3} \Omega_{s,q}(t) \exp(i\omega_s t) | -q \rangle \langle e |. \quad \text{(B21)}
\]

This is the interaction Hamiltonian whose effective version, for large detunings to the red of the D2 line, allows us to realize in the laboratory the GCDD Hamiltonian of Eq. (A10). In the following we show how to do this through adiabatic elimination of the excited state \( |e \rangle \).

\[ \text{2. Effective implementation of the GCDD Hamiltonian for the atomic qutrit} \]

The unperturbed atomic Hamiltonian is written as

\[
H_{\text{atom}} = \hbar \omega_g \sum_{m=-1}^{1} | m \rangle \langle m | + \hbar \omega_e | e \rangle \langle e |, \quad \text{(B22)}
\]

where we have taken the ground-state energy as \( \hbar \omega_g \) given by Eq. (A11). In Eq. (B22), \( \hbar (\omega_e - \omega_g) \) is equal to the energy corresponding to the D2 line, with wavelength given by 780.241 nm, which corresponds to a frequency of the order of 384.230 THz:

\[
\frac{\omega_e - \omega_g}{2\pi} \approx 384.230 \text{ THz}. \quad \text{(B23)}
\]

Using Eq. (B22) as the unperturbed Hamiltonian in the usual interaction picture with the interaction Hamilto-
nian of Eq. (B21), we have
\[
H_I(t) = \exp \left( i \frac{H_{\text{atom}}}{\hbar} t \right) H_{\text{int}}(t) \exp \left( -i \frac{H_{\text{atom}}}{\hbar} t \right)
\]
\[
= -\hbar \sum_{q=-1}^{+1} \sum_{s=1}^{3} \Omega_{s,q}(t) \exp \left( -i \Delta_s t \right) |e\rangle \langle -q|
\]
\[
- \hbar \sum_{q=-1}^{+1} \sum_{s=1}^{3} \Omega_{s,q}^*(t) \exp \left( i \Delta_s t \right) |-q\rangle \langle e|, \quad (B24)
\]
where the detuning is defined by
\[
\Delta_s = \omega_s - \omega_e + \omega_g. \quad (B25)
\]
The coherence times involved in superpositions of atomic quantum states are of the order of a second or longer [54, 58], thus, because the quantum-gate operation \( \tau \) is an integer multiple of \( t_0 \) and should be shorter than these typical coherence times, we can take, roughly,
\[
t_0 \sim 0.1 \text{ s}. \quad (B26)
\]
Hence, because we take Eq. (B26) as valid, we see that
\[
\frac{\omega_0}{2\pi} = \frac{1}{t_0} \sim 10 \text{ Hz} \quad (B27)
\]
is the corresponding rough estimate we can take for \( \omega_0 \) [cf. Eq. (9)]. As we show below, about ten Hz for \( \omega_0/(2\pi) \) are enough for the GCDD method to work. That is, about ten Hz corresponds to the order of magnitude of the Hamiltonians we need to emulate the effective two-photon Hamiltonian [see also Eqs. (1), (3), and (11)].

To implement the GCDD method in the context of laser control of an atomic qutrit, here we show how to use two-photon transitions. We need detunings that are much greater in magnitude than the typical few MHz of the Rabi frequencies [cf. Eq. (B20)], so that we can use the rotating-wave approximation [59] and adiabatic elimination of the state \( |e\rangle \) [56]. One can easily implement this, given the relatively large difference in energies in the transitions indicated in Fig. 1. As we can see in this figure, the detunings can be as large as a few GHz, and still the states of the ground \( F = 2 \) level do not get involved in the transitions (they are at the energy corresponding to 6.8 GHz above the \( F = 1 \) states we use). Since the detunings we use are negative, meaning that the photons excite a virtual level well below the \( |e\rangle \) excited state, the higher excited states are not going to interfere with our transition scheme. Moreover, the D2-line natural line-width for \(^{87}\text{Rb}\) is of the order of 6 MHz, so that laser photons detuned to the red of the D2 transition frequency by a few GHz will not populate the excited state \( |e\rangle \). As we show in detail below, the magnitudes involved in the effective two-photon Hamiltonian are proportional to the square of Rabi frequencies divided by the detuning, which can be substantially higher than the few tens of MHz (at least) required for an efficient GCDD implementation. Typically, if we use, roughly,
\[
\frac{\Delta_s}{2\pi} \sim 1 \text{ GHz}, \quad (B28)
\]
using Eq. (B20) we find that
\[
\frac{\Omega^2}{2\pi |\Delta_s|} \sim 1 \text{ kHz}. \quad (B29)
\]
Hence, using large detunings as in Eq. (B28), we end up with an effective Hamiltonian (explained below) that can have its magnitude as in Eq. (B29), flexibly above the minimal requirement of Eq. (B27) for the GCDD method to work, as we have discussed above.

Now, let us write the interaction-picture state as
\[
|\psi_I(t)\rangle = \sum_{m=-1}^{1} C_m(t) |m\rangle + C_e(t) |e\rangle, \quad (B30)
\]
since \( |\psi_I(t)\rangle \in \mathcal{H}_3 \). We can introduce the following projection operators:
\[
P_g \equiv \sum_{m=-1}^{1} |m\rangle \langle m| \quad (B31)
\]
and
\[
P_e \equiv |e\rangle \langle e|. \quad (B32)
\]
We immediately see that
\[
|\psi_I(t)\rangle = (P_g + P_e) |\psi_I(t)\rangle = P_g |\psi_I(t)\rangle + P_e |\psi_I(t)\rangle. \quad (B33)
\]
From the interaction-picture Schrödinger equation and Eq. (B33), we obtain
\[
i\hbar \frac{d}{dt} |\psi_I(t)\rangle = H_I(t) |\psi_I(t)\rangle
\]
\[
= H_I(t) [P_g |\psi_I(t)\rangle + P_e |\psi_I(t)\rangle]
\]
\[
= H_I(t) P_g |\psi_I(t)\rangle + H_I(t) P_e |\psi_I(t)\rangle. \quad (B34)
\]
Therefore, by applying the projectors of Eqs. (B31) and (B32) to both sides of Eq. (B34), we obtain the coupled Schrödinger equations:
\[
i\hbar \frac{d}{dt} P_g |\psi_I(t)\rangle = P_g H_I(t) P_g |\psi_I(t)\rangle
\]
\[
+ P_g H_I(t) P_e |\psi_I(t)\rangle \quad (B35)
\]
and
\[
i\hbar \frac{d}{dt} P_e |\psi_I(t)\rangle = P_e H_I(t) P_g |\psi_I(t)\rangle
\]
\[
+ P_e H_I(t) P_e |\psi_I(t)\rangle. \quad (B36)
\]
From Eq. (B24) it is evident that
\[ P_g H_I (t) P_g = 0 \quad \text{and} \quad P_e H_I (t) P_e = 0. \] (B37)

Thus, Eqs. (B35) and (B36) become
\[ i \hbar \frac{d}{dt} P_g |\psi_I (t)\rangle = P_g H_I (t) P_e |\psi_I (t)\rangle \] (B38)
and
\[ i \hbar \frac{d}{dt} P_e |\psi_I (t)\rangle = P_e H_I (t) P_g |\psi_I (t)\rangle. \] (B39)

By formally integrating Eq. (B39) we obtain
\[ P_e |\psi_I (t)\rangle = P_e |\psi_I (0)\rangle - \frac{i}{\hbar} \int_0^t dt' P_e H_I (t') P_g |\psi_I (t')\rangle. \] (B40)

Our intention is to start with the atom in the ground-state subspace, that is, the population of the excited state is initially zero. Thus, using this fact, that is,
\[ P_e |\psi_I (0)\rangle = 0, \] (B41)
in Eq. (B40), we obtain
\[ P_e |\psi_I (t)\rangle = - \frac{i}{\hbar} \int_0^t dt' P_e H_I (t') P_g |\psi_I (t')\rangle. \] (B42)

Substitution of Eq. (B42) into Eq. (B38) gives
\[ i \hbar \frac{d}{dt} P_g |\psi_I (t)\rangle \]
\[ = - \frac{i}{\hbar} P_g H_I (t) P_e \int_0^t dt' P_e H_I (t') P_g |\psi_I (t')\rangle, \] (B43)
where we have used the fact that \( P_e \) is a projector operator and, therefore, \( P_e^2 = P_e \). From Eqs. (B24), (B30), and (B31), and (B32), we see that
\[ P_g H_I (t) P_e = - \hbar \sum_{q=-1}^{+1} \sum_{s=1}^{3} \Omega_{s,q}^* (t) \exp (i \Delta_q t) |q\rangle \langle e| \] (B44)
and
\[ \int_0^t dt' P_e H_I (t') P_g |\psi_I (t')\rangle \]
\[ = - |e\rangle \hbar \sum_{q=-1}^{+1} \sum_{s=1}^{3} \int_0^t dt' \Omega_{s,q} (t') \exp (- i \Delta_q t') C_{-q} (t'). \] (B45)

From Eqs. (B30), (B31), (B43), (B44) and (B45) we obtain
\[ \frac{d}{dt} C_m (t) = - \sum_{q=-1}^{+1} \int_0^t dt' K_{m,-q} (t, t') C_{-q} (t'), \] (B46)
for \( m = -1, 0, 1 \), where we have defined the kernel function:
\[ K_{m,-q} (t, t') \equiv \sum_{s=1}^{3} \sum_{s'=-1}^{3} \exp (i \Delta_s t - i \Delta_s t') \]
\[ \times \Omega_{s,-m}^* (t) \Omega_{s',q} (t'). \] (B47)

We can also arrange Eqs. (B46) and (B47) in matrix format:
\[ \frac{d}{dt} C (t) = - \int_0^t dt' K (t, t') C (t'), \] (B48)
where we have defined
\[ C (t) = \begin{bmatrix} C_{-1} (t) \\ C_0 (t) \\ C_1 (t) \end{bmatrix}, \] (B49)
Iteration of Eq. (B48) yields:
\[ C (t') = C (t) - \int_t^{t'} dt_1 \int_0^{t_1} dt_2 K (t_1, t_2) C (t) + (-1)^2 \int_t^{t'} dt_1 \int_0^{t_1} dt_2 K (t_1, t_2) \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 K (t_3, t_4) C (t) + \ldots \] (B51)

Let us calculate a generic element of the first kernel integral in Eq. (B51):
\[ \int_t^{t'} dt_1 \int_0^{t_1} dt_2 K_{m,-q} (t_1, t_2) \equiv \sum_{s=1}^{3} \sum_{s'=-1}^{3} \int_t^{t'} dt_1 \int_0^{t_1} dt_2 \exp (i \Delta_s t_1 - i \Delta_s t_2) \Omega_{s,-m}^* (t_1) \Omega_{s',q} (t_2) \]
\[ = \sum_{s=1}^{3} \sum_{s'=-1}^{3} \int_t^{t'} dt_1 \exp (i \Delta_s t_1) \Omega_{s,-m}^* (t_1) \int_0^{t_1} dt_2 \exp (- i \Delta_s t_2) \Omega_{s',q} (t_2). \] (B52)
Using Eq. (B54), we obtain whose inverse is given by

\[ G_{s',q'}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau \exp (i\omega \tau) \Omega_{s',q'}(\tau) , \quad (B53) \]

whose inverse is given by

\[ \Omega_{s',q'}(t_2) \equiv \int_{-\infty}^{+\infty} d\omega \exp (-i\omega t_2) G_{s',q'}(\omega) . \quad (B54) \]

Using Eq. (B54), we obtain

\[
\int_0^{t_1} dt_2 \exp (-i\Delta s t_2) \Omega_{s',q'}(t_2) \\
= \int_0^{t_1} dt_2 \exp (-i\Delta s t_2) \int_{-\infty}^{+\infty} d\omega \exp (-i\omega t_2) G_{s',q'}(\omega) \\
= \int_{-\infty}^{+\infty} d\omega G_{s',q'}(\omega) \int_0^{t_1} dt_2 \exp [-i(\Delta s' + \omega) t_2] , \quad (B55)
\]

where we have changed the order of the integrals. Whatever form \( G_{s',q'}(\omega) \) might have, it is assumed to be centered about a value \( \omega \approx \omega_0 > 0 \) with an absolute value much smaller than \( |\Delta_s| \), so that we can write

\[
\int_0^{t_1} dt_2 \exp [-i(\Delta s' + \omega) t_2] = \frac{\exp [-i(\Delta s' + \omega) t_1] - 1}{-i(\Delta s' + \omega)} \\
\approx \frac{\exp [-i(\Delta s' + \omega) t_1] - 1}{-i\Delta s'} . \quad (B56)
\]

Substituting Eq. (B56) into Eq. (B55) we obtain

\[
\int_0^{t_1} dt_2 \exp (-i\Delta s t_2) \Omega_{s',q'}(t_2) \\
\approx \int_{-\infty}^{+\infty} d\omega G_{s',q'}(\omega) \frac{\exp [-i(\Delta s' + \omega) t_1] - 1}{\Delta s'} \\
= \frac{i\Omega_{s',q'}(t_1)}{\Delta s'} \exp (-i\Delta s t_1) - \frac{i\Omega_{s',q'}(0)}{\Delta s'} , \quad (B57)
\]

where we have used Eq. (B54). With the result of Eq. (B57) we can now tackle Eq. (B52):

\[
\int_t^{t'} dt_1 \int_0^{t_1} dt_2 K_{s,-q'}(t_1, t_2) = \sum_{s=1}^{3} \sum_{s'=1}^{3} \frac{i}{\Delta s'} \int_t^{t'} dt_1 \\
\exp [i(\Delta s - \Delta s') t_1] \Omega_{s,-m}^*(t_1) \Omega_{s',q'}(t_1) - \sum_{s=1}^{3} \sum_{s'=1}^{3} \\
\frac{\Omega_{s',q'}(0)}{\Delta s' \Delta s} \left[ \Omega_{s,-m}^*(t') \exp (i\Delta s t') - \Omega_{s,-m}^*(t) \exp (i\Delta s t) \right] . \quad (B58)
\]

We already see that all the terms in the double sum on the right-hand side of Eq. (B58) are of second order in the quotient between the order of magnitude of the Rabi frequencies [see Eq. (B20)] and the order of magnitude of the detunings. Let us define this order of magnitude more rigorously by assuming that, from all \( t \in [0, t_0] \), \( s = 1, 2, 3 \), and \( q = -1, 0, 1 \), we define \( \eta \) as the maximum absolute value of \( \Omega_{s,\pm}(t)/\Delta s \), that is,

\[
\eta \equiv \max \left\{ \left| \Omega_{s,\pm}(t) / \Delta s \right| : t \in [0, t_0], s \in \{1, 2, 3\}, \pm \in \{-1, 0, 1\} \right\} . \quad (B59)
\]

Using the rough estimates of Eqs. (B20) and (B28) we see that \( \eta \) can be even less than \( 10^{-3} \). Using Eq. (B54), we can now calculate the following integral:

\[
\int_t^{t'} dt_1 \exp [i(\Delta s - \Delta s') t_1] \Omega_{s,-m}^*(t_1) \Omega_{s',q'}(t_1) \\
= \int_{-\infty}^{+\infty} d\omega_2 \int_{-\infty}^{+\infty} d\omega_1 G_{s,-m}^*(\omega_2) G_{s',q'}(\omega_1) \\
\times \int_t^{t'} dt_1 \exp [i(\Delta s - \Delta s' + \omega_2 - \omega_1) t_1] . \quad (B60)
\]

Here, we have two situations: \( s \neq s' \) and \( s = s' \). Hence, taking these two cases into account, we obtain

\[
\int_t^{t'} dt_1 \exp [i(\Delta s - \Delta s' + \omega_2 - \omega_1) t_1] \\
= \delta_{s,s'} \int_t^{t'} dt_1 \exp [i(\omega_2 - \omega_1) t_1] \\
+ (1 - \delta_{s,s'}) \int_t^{t'} dt_1 \exp [i(\Delta s - \Delta s' + \omega_2 - \omega_1) t_1] \\
- (1 - \delta_{s,s'}) \int_t^{t'} dt_1 \exp [i(\Delta s - \Delta s' + \omega_2 - \omega_1) t] \quad (B61)
\]

We now substitute Eq. (B61) back into Eq. (B60). We have assumed that the functions \( G_{s,-m}^*(\omega_2) \) and \( G_{s',q'}(\omega_1) \) only contribute in the frequency region where \( \omega_1 \approx \omega_2 \approx \omega_0 \). Thus, if we choose the detunings such that, for \( s \neq s' \), the absolute difference \( |\Delta s - \Delta s'| \) is of the same order of magnitude of the max\( \{|\Delta s| \}_s \in \{1, 2, 3\} \), that is,

\[
|\Delta s - \Delta s'| \approx \max \{|\Delta s| \}_s \in \{1, 2, 3\} . \quad (B62)
\]
which we estimate as about a few GHz [see Eq. (B28)], then we can assume \( \Delta_\omega - \Delta_{\omega'} + \omega_1 \approx \Delta_s - \Delta_{s'} \) in the denominators of Eq. (B61) when this equation is negligible when compared with the terms with \( s = s' \). We then can assume

\[
\int \frac{dt' dt}{t} \exp [i (\Delta_s - \Delta_{s'}) t_1] \Omega_{s, -m}^*(t_1) \Omega_{s', q'} (t_1)
\]

\[
= \delta_{s, s'} \int \frac{dt' dt}{t} \Omega_{s, -m}^*(t_1) \Omega_{s', q'} (t_1)
\]

\[
+ (1 - \delta_{s, s'}) \frac{\Omega_{s, -m}^*(t') \Omega_{s', q'} (t') \exp [i (\Delta_s - \Delta_{s'}) t']}{i (\Delta_s - \Delta_{s'})}
\]

\[
- (1 - \delta_{s, s'}) \frac{\Omega_{s, -m}^*(t) \Omega_{s', q'} (t) \exp [i (\Delta_s - \Delta_{s'}) t]}{i (\Delta_s - \Delta_{s'})}.
\]

(B63)

After substituting Eq. (B63) into Eq. (B58) we end up with

\[
\int \frac{dt' dt}{t} \frac{dt_1 dt_2 K_{m, -q'} (t_1, t_2)}{t} = \sum_{s=1}^{3} \frac{\delta_{s, s'} \int dt_1}{t}
\]

\[
\Omega_{s, -m}^*(t_1) \Omega_{s', q'} (t_1) + \sum_{s=1}^{3} \sum_{s'=1}^{3} (1 - \delta_{s, s'}) \Omega_{s, -m}^*(t') \exp [i (\Delta_s - \Delta_{s'}) t']
\]

\[
- \frac{\Omega_{s', -m}^*(t) \Omega_{s', q'} (t) \exp [i (\Delta_s - \Delta_{s'}) t]}{i (\Delta_s - \Delta_{s'})}
\]

\[
\frac{\Omega_{s', -m}^*(t') \exp [i (\Delta_s - \Delta_{s'}) t']}{i (\Delta_s - \Delta_{s'})}.
\]

(B64)

We conclude, therefore, that defining the kernel matrix including terms with \( s \neq s' \), as in Eq. (B50), amounts to producing contributions of second order in \( \eta \) that are negligible when compared with the terms with \( s = s' \), which are of first order in \( \eta \), as we can directly verify in Eq. (B64) [cf. Eqs. (B59) and (B62)]. Therefore, in the present problem, we keep only the \( s = s' \) terms in Eq. (B64) and neglect any other terms of second order in \( \eta \):

\[
\int \frac{dt' dt}{t} \int \frac{dt_1 dt_2 K_{m, -q'} (t_1, t_2)}{t} \approx \sum_{s=1}^{3} \frac{i}{\Delta_s} \int \frac{dt_1 \Omega_{s, -m}^*(t_1) \Omega_{s, q'} (t_1)}{t}.
\]

Now we can differentiate Eq. (B65) with respect to \( t \) and get

\[
\int \frac{dt}{t} \frac{dt_1 dt_2 K_{m, -q'} (t_1, t_2)}{t} \approx \sum_{s=1}^{3} \frac{i}{\Delta_s} \Omega_{s, -m}^*(t) \Omega_{s, q'} (t).
\]

We see from the above discussion and Eqs. (B48) and (B51) that a time-local approximation of Eq. (B46) is of first order in \( \eta \) [see Eq. (B59)], and, using Eq. (B66), we can write it as

\[
\frac{dt}{dt} C_m (t) = \sum_{s=1}^{3} \frac{i}{\Delta_s} \Omega_{s, -m}^*(t) \Omega_{s, q'} (t) C_{-q'} (t) + \mathcal{O} (\eta^2)
\]

\[
= \frac{1}{\Delta_s} \sum_{s=1}^{3} \frac{i}{\Delta_s} \Omega_{s, -m}^*(t) \Omega_{s, q'} (t) C_{-q'} (t) + \mathcal{O} (\eta^2).
\]

Incidentally, the above discussion is also the explanation of our use of the rotating-wave approximation to obtain Eq. (B13), where we assume that \( |\Delta_s| = |\omega_s - \omega_c + \omega_g| > \|D_s \delta_s (t)\| = |\Omega_{s, q} (t)| \).

Eq. (B67) can be arranged in a matrix representation:

\[
\int \frac{dt}{t} C (t) \approx \sum_{s=1}^{3} H_{1, s} (t) C (t),
\]

where, for \( s = 1, 2, 3 \), we define

\[
H_{1, s} (t) = \frac{\hbar}{\Delta_s} \begin{bmatrix}
\Omega_{s, 1}^*(t) \Omega_{s, 1} (t) & \Omega_{s, 1}^*(t) \Omega_{s, 0} (t) & \Omega_{s, 1}^*(t) \Omega_{s, -1} (t) \\
\Omega_{s, 0}^*(t) \Omega_{s, 1} (t) & \Omega_{s, 0}^*(t) \Omega_{s, 0} (t) & \Omega_{s, 0}^*(t) \Omega_{s, -1} (t) \\
\Omega_{s, -1}^*(t) \Omega_{s, 1} (t) & \Omega_{s, -1}^*(t) \Omega_{s, 0} (t) & \Omega_{s, -1}^*(t) \Omega_{s, -1} (t)
\end{bmatrix}.
\]

We see, in Eq. (B69), that, effectively, we have found a Hamiltonian in the interaction picture given by \( \sum_{s=1}^{3} H_{1, s} (t) \). Now,

\[
C (t) = \sum_{m=-1}^{1} C_m (t) |m\rangle = P_g \langle \psi_I (t) | \rangle
\]

\[
= P_g \exp \left( i \frac{H_{\text{atom}} t}{\hbar} \right) \psi_S (t),
\]

since \( P_g \) is proportional to \( I_3 \), the identity operator acting on \( \mathbb{H}_3 \). The above equation simply means that once computed the evolution of the state \( C (t) \), the corresponding
state in Schrödinger picture state is obtained by multiplying for the immaterial global phase factor \( \exp(i\omega_g t) \). It follows that up to this global phase factor the dynamics in the qutrit subspace \( \mathcal{H}_3 \) is governed by the effective Hamiltonian

\[
H_{\text{eff}} (t) = \sum_{s=1}^{3} H_{I,s} (t) .
\]  

(B72)

We notice that the global phase factor disappears if we perform a shift in the energy scale, so that the qutrit states all have zero energy in the new scale.

To make this scheme work, we know that \( \Delta_s \in \mathbb{R} \) and \( \Delta_s < 0 \). Therefore, we adopt the convention that

\[
\sqrt{-\Delta} = i \sqrt{-\Delta} ,
\]

so that

\[
\left( \sqrt{-\Delta} \right)^* = -i \sqrt{-\Delta} = -\sqrt{-\Delta} ,
\]

(B74)

where \( \sqrt{-\Delta} \in \mathbb{R} \) and \( \sqrt{-\Delta} > 0 \). Thus, Eq. (B69) becomes

\[
H_{I,s} (t) = -\hbar \left[ \begin{array}{c}
\frac{\Omega_{1,s} (t)}{\sqrt{\Delta}} \\
\frac{\Omega_{0,s} (t)}{\sqrt{\Delta}} \\
\frac{\Omega_{2,s} (t)}{\sqrt{\Delta}} \\
\end{array} \right] \left[ \begin{array}{ccc}
\Omega_{1,s} (t) & \Omega_{0,s} (t) & \Omega_{2,s} (t) \\
\sqrt{\Delta} & \sqrt{\Delta} & \sqrt{\Delta} \\
\end{array} \right] ,
\]

(B75)

since, from Eq. (B74), it follows that

\[
\left( \sqrt{-\Delta} \right)^* \sqrt{-\Delta} = -\sqrt{-\Delta} \sqrt{-\Delta} = -\Delta_s .
\]

(B76)

Based on Eqs. (B72) and (B75) we can now express \( H_{\text{eff}} (t) \) in a way that is analogous to Eq. (A10), allowing us to connect the Rabi frequencies of this section with the elements of the operator \( \check{Y} \) of Sec. \( \Delta \).

\[
H_{\text{eff}} (t) = -\hbar \sum_{s=1}^{3} \left[ \begin{array}{c}
\frac{\Omega_{1,s} (t)}{\sqrt{\Delta}} \\
\frac{\Omega_{0,s} (t)}{\sqrt{\Delta}} \\
\frac{\Omega_{2,s} (t)}{\sqrt{\Delta}} \\
\end{array} \right] \left[ \begin{array}{ccc}
\Omega_{1,s} (t) & \Omega_{0,s} (t) & \Omega_{2,s} (t) \\
\sqrt{\Delta} & \sqrt{\Delta} & \sqrt{\Delta} \\
\end{array} \right] ,
\]

(B77)

where we have defined

\[
\check{Y} (t) \equiv \left[ \begin{array}{ccc}
\Omega_{1,s} (t) & \Omega_{0,s} (t) & \Omega_{2,s} (t) \\
\sqrt{\Delta} & \sqrt{\Delta} & \sqrt{\Delta} \\
\end{array} \right] .
\]

(B78)

But then, up to the immaterial term proportional to \( \mathbb{I} \) in Eq. (A10) (applied to a qutrit), if we choose the Rabi frequencies and detunings appearing in Eq. (B78) so that \( \check{Y} (t) \) is Hermitian, we can identify it with \( \check{Y} (t) \) and this is how we can implement the GCDD method for an atomic qutrit manipulated using two-photon transitions. Accordingly, thus, we choose, along the diagonal of Eq. (B78):

\[
\begin{align*}
\Omega_{1,1} (t) &= \Omega_{0,1} (t) \\
\Omega_{2,0} (t) &= \Omega_{2,0} (t) \\
\Omega_{3,-1} (t) &= \Omega_{3,-1} (t) \\
\end{align*}
\]

(B79)

Using Eqs. (B73) and (B74), we then obtain:

\[
\begin{align*}
\Omega_{1,1}^* (t) &= -\Omega_{1,1} (t) \\
\Omega_{2,0}^* (t) &= -\Omega_{2,0} (t) \\
\Omega_{3,-1}^* (t) &= -\Omega_{3,-1} (t) .
\end{align*}
\]

(B80)

For the off-diagonal elements of Eq. (B78), we choose:

\[
\begin{align*}
\Omega_{2,1} (t) &= \Omega_{1,0} (t) \\
\Omega_{3,1} (t) &= \Omega_{1,-1} (t) \\
\Omega_{3,0} (t) &= \Omega_{2,-1} (t) \\
\end{align*}
\]

(B81)

Now, we are able to identify the remaining independent elements of \( \check{Y} (t) \) with those of \( \Theta (t) \). By imposing Eqs. (B80) and (B81), and that \( \check{Y} (t) = \check{Y} (t) \), we obtain:

\[
\begin{align*}
\Omega_{1,1} (t) &= \sqrt{\Delta_1} \check{Y}_{0,0} (t) \\
\Omega_{1,0} (t) &= \sqrt{\Delta_1} \check{Y}_{0,1} (t) \\
\Omega_{1,-1} (t) &= \sqrt{\Delta_1} \check{Y}_{0,-1} (t) \\
\Omega_{2,0} (t) &= \sqrt{\Delta_2} \check{Y}_{1,0} (t) \\
\Omega_{2,-1} (t) &= \sqrt{\Delta_2} \check{Y}_{1,-1} (t) \\
\Omega_{3,-1} (t) &= \sqrt{\Delta_3} \check{Y}_{2,0} (t) .
\end{align*}
\]

(B82)

We observe that the above derivation leading to the effective Hamiltonian of Eq. (B77) could be coherently obtained also for values of \( t_0 \) different from the one chosen in Eq. (B26) but satisfying the conditions required for performing the various approximations involved in the derivation. In this sense, we do not fix a specific value for \( t_0 \) in the numerical simulations based on Appendix C and depicted in Fig. 2. Consequently, the value of the gate time \( \tau \) is not specified in these simulations and the other quantities are given in units of it.

3. The Hadamard Hamiltonian for a qutrit

As quantum gate to implement in our qutrit model we choose the Hadamard one. Starting from the ideal
definition of the Hadamard unitary quantum gate \( \tilde{U}_G \) in the ground-state subspace basis \( \{ \{ -1 \}, \{ 0 \}, \{ 1 \} \} \), namely,
\[
U_G = \frac{1}{i\sqrt{3}} \begin{bmatrix}
1 & 1 & 1 \\
1 & \exp(2\pi i/3) & \exp(4\pi i/3) \\
1 & \exp(4\pi i/3) & \exp(2\pi i/3)
\end{bmatrix}, \tag{B83}
\]
we can invert the equation
\[
U_G = \exp \left( -i \frac{H_G}{\hbar} \right), \tag{B84}
\]
to obtain the gate Hamiltonian:
\[
H_G = \frac{\pi \hbar}{4\sqrt{3} \tau} \begin{bmatrix}
4\sqrt{3} - 2 & -2 & -2 \\
-2 & 2\sqrt{3} + 1 & 2\sqrt{3} + 1 \\
-2 & 2\sqrt{3} + 1 & 2\sqrt{3} + 1
\end{bmatrix}. \tag{B85}
\]

Appendix C: Simulating noise with thermal baths of bosons

Here, we explain how we use two baths of thermal bosons to simulate the perturbations caused by a noisy environment (see also some previous works of some of us on continuous dynamical decoupling of qubit systems \cite{17,20}). In the picture obtained by unitarily transforming Eq. (3) using \( U_c(t) \), we obtain the Hamiltonian in the control picture [cf. Eq. (3)]:
\[
H(t) = H_G \otimes \mathbb{I}_E + U_G \otimes H_E + \left[ U_c(t) \otimes \mathbb{I}_E \right] H_{\text{int}} \times \left[ U_c(t) \otimes \mathbb{I}_E \right]. \tag{C1}
\]
As explained after Eq. (29), in our example we consider a qutrit subject to independent amplitude damping and dephasing noises. We divide the interaction Hamiltonian and the environment Hamiltonian in two parts, i.e., \( H_{\text{int}} = H_{\text{int}}^{(1)} + H_{\text{int}}^{(2)} \) and \( H_E = H_E^{(1)} + H_E^{(2)} \), where the superscripts 1 and 2 refer, respectively, to the amplitude damping and dephasing interactions and the boson Hamiltonians, which, by themselves, give the unpe rurbed evolutions of the bosons. The first interaction-Hamiltonian term (which introduces the damping error) is given by:
\[
H_{\text{int}}^{(1)} = \lambda_{0,-1}^{(1)} \left( \langle 0 \rangle \langle -1 \rangle \otimes B^{(1)} + \langle -1 \rangle \langle 0 \rangle \otimes B^{(1)} \right) + \lambda_{0,1}^{(1)} \left( \langle 0 \rangle \langle 1 \rangle \otimes B^{(1)} + \langle 1 \rangle \langle 0 \rangle \otimes B^{(1)} \right),
\]
which is
\[
= \lambda_{0,1}^{(1)} \otimes B^{(1)} + \lambda_{1}^{(1)} \otimes B^{(1)}, \tag{C2}
\]
where \( B^{(1)} = \sum_k \hbar \omega_k a_k^{(1)} \) and \( \lambda^{(1)} = \lambda_{0,1}^{(1)} [\langle 0 \rangle \langle -1 \rangle + \langle 1 \rangle \langle 0 \rangle] \) and the bath Hamiltonian associated to this class of error is given by \( H_E^{(1)} = \sum_k \hbar \omega_k a_k^{(1)} a_k^{(1)} \). In a similar way, the second interaction-Hamiltonian term (which introduces the dephasing error) is given by
\[
H_{\text{int}}^{(2)} = \lambda_{0,-1}^{(2)} \left( \langle -1 \rangle \langle -1 \rangle \otimes B^{(2)} + \langle -1 \rangle \langle 0 \rangle \otimes B^{(2)} \right) + \lambda_{0,1}^{(2)} \left( \langle 1 \rangle \langle 1 \rangle \otimes B^{(2)} + \langle 1 \rangle \langle 0 \rangle \otimes B^{(2)} \right),
\]
which is
\[
= \lambda^{(2)} \otimes B^{(2)} + \lambda^{(2)} \otimes B^{(2)}, \tag{C3}
\]
where \( B^{(2)} = \sum_k \hbar g_k a_k^{(2)} \) and \( \lambda^{(2)} = \lambda_{0,-1}^{(2)} [\langle -1 \rangle \langle -1 \rangle + \langle -1 \rangle \langle 0 \rangle + \langle 0 \rangle \langle 0 \rangle] + \lambda_{0,1}^{(2)} [\langle 1 \rangle \langle 1 \rangle + \langle 1 \rangle \langle 0 \rangle + \langle 0 \rangle \langle 0 \rangle] \), and the bath Hamiltonian associated to this class of error is given by \( H_E^{(2)} = \sum_k \hbar \omega_k a_k^{(2)} a_k^{(2)} \) (here we suppose identical baths, besides being independent).

We notice that in this model Eq. (4) is always satisfied.

To obtain a numerical solution to the three-level system dynamics, we transform the total Hamiltonian to the well known interaction picture. It is written as:
\[
\tilde{H}_I(t) = \tilde{H}_I^{(1)}(t) + \tilde{H}_I^{(2)}(t), \tag{C4}
\]
where
\[
\tilde{H}_I^{(s)}(t) = \tilde{\Lambda}^{(s)}(t) \otimes \tilde{B}^{(s)}(t) + \tilde{\Lambda}^{(s)}(t) \otimes \tilde{B}^{(s)}(t), \tag{C5}
\]
for \( s = 1, 2 \), with \( \tilde{B}^{(s)}(t) = U_E^{(s)\dagger}(t)B^{(s)}U_E^{(s)}(t) \) and \( \tilde{\Lambda}^{(s)}(t) = U_G(t)U_c(t)\Lambda^{(s)}U_c(t)U_G(t) \), where \( U_G(t) = \exp(-iH_Gt/\hbar) \), and \( U_E^{(s)}(t) = \exp(-iH_E^{(s)}t/\hbar) \). With this transformation, the master equation is written as:
\[
\frac{d\tilde{\rho}_S(t)}{dt} = -\frac{i}{\hbar^2} \sum_{s=1}^{2} \int_0^t \text{Tr}_E \left\{ \left[ \tilde{H}_I^{(s)}(t'), \tilde{\Lambda}^{(s)}(t), \tilde{\rho}_S(t) \right] \right\} dt', \tag{C6}
\]
where \( \tilde{\rho}_S(t) = U_G^{(s)}(t)\tilde{\rho}_S(t)U_G^{(s)}(t)U_G(t) \) and \( \rho_E = \text{environment density matrix, here given by a thermal state, that is} \rho_E = \frac{1}{Z} \exp(-\beta \sum_{s=1}^{2} H_E^{(s)}), \) where \( Z = \text{partition function}, Z = Tr_E(\exp(-\beta \sum_{s=1}^{2} H_E^{(s)})) \), \( \beta = 1/(k_B T) \), \( k_B \) is the Boltzmann constant, and \( T \) is the absolute temperature of the baths.

Now, substituting \( \tilde{H}_I(t) \) into the master equation, we finally obtain:
\[
\frac{d\tilde{\rho}_S(t)}{dt} = -\frac{i}{\hbar^2} \sum_{s=1}^{2} \int_0^t \left\{ \left[ \tilde{\rho}_S(t) \Lambda^{(s)}(t'), \Lambda^{(s)}(t) \right] \tilde{G}_1(t, t') \right. \left. - \left[ \Lambda^{(s)}(t'), \tilde{\rho}_S(t), \Lambda^{(s)}(t) \right] \tilde{G}_1(t, t') \right. \left. + \left[ \tilde{\rho}_S(t) \Lambda^{(s)}(t'), \Lambda^{(s)}(t) \right] \tilde{G}_2(t, t') \right. \left. - \left[ \Lambda^{(s)}(t'), \tilde{\rho}_S(t), \Lambda^{(s)}(t) \right] \tilde{G}_2(t, t') dt'. \right. \tag{C7}
\]
where the correlation functions are written as
\[
\tilde{G}_1(t, t') = \frac{1}{\hbar^2} \text{Tr}_E \left\{ \tilde{B}^{(s)}(t)\rho_E \tilde{B}^{(s)}(t') \right\},
\]
\[
\tilde{G}_2(t, t') = \frac{1}{\hbar^2} \text{Tr}_E \left\{ \tilde{B}^{(s)}(t)\rho_E \tilde{B}^{(s)}(t') \right\}. \tag{C8}
\]
It is important to emphasize that, since we suppose identical baths, the correlation functions are equivalent for \( \tilde{B}^{(1)} \) and \( \tilde{B}^{(2)} \). Thus, the expressions for \( \tilde{G}_1(t, t') \) and \( \tilde{G}_2(t, t') \) are given by
\[
\tilde{G}_1(t, t') = \sum_k |g_k|^2 n_k \exp(-i\omega_k(t - t')),
\]
\[
\tilde{G}_2(t, t') = \sum_k |g_k|^2 (1 + n_k) \exp[i\omega_k(t - t')]. \tag{C9}
\]
where $n_k = 1/\left[ \exp(\beta \hbar \omega_k) - 1 \right]$ is the average number of photons in a mode with frequency $\omega_k$. Finally, in the continuum limit, the sums become integrals and we obtain, using $s = t - r$,

$$
G_1(t, t') = \int_0^\infty d\omega J(\omega) n(\omega) \exp \left[ -i \omega (t - t') \right],
$$

$$
G_2(t, t') = \int_0^\infty d\omega J(\omega) [1 + n(\omega)] \exp \left[ i \omega (t - t') \right], \tag{C10}
$$

where we have exploited the fact that the correlation functions are homogeneous in time, $n(\omega)$ is the continuous frequency version of $n_k$, namely, $n(\omega) = 1/\left[ \exp(\beta \hbar \omega) - 1 \right]$, and $J(\omega)$ is the spectral density.

In the numerical simulations of Fig. 2 we apply the GCDD to the qutrit considered in Sec. B and we choose the spectral density $J(\omega) = \alpha^2 \omega \exp(-\omega/\omega_c)$, where $\alpha$ is a dimensionless constant prefactor and $\omega_c$ is the angular cut-off frequency. We have also set equal all the $\lambda$ coupling constants, introducing as effective coupling parameter $\lambda = \alpha \lambda^{(s)}_{0,1} = \alpha \lambda^{(s)}_{0,1} = 0.1$ for $s = 1, 2$, and chosen $\omega_c = 4 \omega_{\text{gate}}$, being $\omega_{\text{gate}} = 2\pi/\tau$ where $\tau$ is the gate time, and $\hbar \omega_c/(k_B T) = 1$. As explained at the end of Sec. 32 we do not consider a specific value for the gate time $\tau$. The other quantities are then given in units of it.

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