Interleaved sequences of geometric sequences binarized with Legendre symbol of two types

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Abstract

A pseudorandom number generator is widely used in cryptography. A cryptographic pseudorandom number generator is required to generate pseudorandom numbers which have good statistical properties as well as unpredictability. An m-sequence is a linear feedback shift register sequence with maximal period over a finite field. M-sequences have good statistical properties, however we must nonlinearize m-sequences for cryptographic purposes. A geometric sequence is a binary sequence given by applying a nonlinear feedforward function to an m-sequence. Nogami, Tada and Uehara proposed a geometric sequence whose nonlinear feedforward function is given by the Legendre symbol. They showed the geometric sequences have good properties for the period, periodic autocorrelation and linear complexity. However, the geometric sequences do not have the balance property. In this paper, we introduce geometric sequences of two types and show some properties of interleaved sequences of the geometric sequences of two types. These interleaved sequences have the balance property and double the period of the geometric sequences by the interleaved structure. Moreover, we show correlation properties and linear complexity of the interleaved sequences. A key of our observation is that the second type geometric sequence is the complement of the left shift of the first type geometric sequence by half-period positions.

1 Introduction

In cryptography, a pseudorandom number generator is used to generate a private key, a public key, a session key, a keystream and so on. A cryptographic pseudorandom number generator is required to generate pseudorandom numbers which have good statistical properties as well as unpredictability.

A linear feedback shift register (LFSR) [5, 10] is a fast pseudorandom number generator which generates well-distributed sequences with long period. In particular, an m-sequence is an LFSR sequence with maximal period over a finite field. However, we must nonlinearize m-sequences for cryptographic purposes. A geometric sequence [2] is a binary sequence given by applying a nonlinear feedforward function to an m-sequence. As geometric sequences, GMW sequences [9, 20], cascaded GMW sequences [13, 7] and so on have been well known.

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Nogami, Tada and Uehara [17] proposed a geometric sequence whose non-linear feedforward function is given by the Legendre symbol. They showed the geometric sequences have good properties for period, periodic autocorrelation and linear complexity. Moreover, the studies on generalization and extension of the sequences [12, 14, 1, 19, 18] have made progress in recent years. On the other hand, the sequences do not have the balance property. It may become an obstruction for the fast implementation proposed by Nogami, Tada and Uehara [16].

The purpose of this paper is to construct sequences from the geometric sequences, which have the balance property. Since the geometric sequences are described by the Legendre symbol, the geometric sequences have a property similar to that of Legendre sequences [25]. Legendre sequences of two types were defined and properties of the interleaved sequences of Legendre sequences of two types were studied by Tang and Gong [22] and some other researchers. In this paper, we introduce geometric sequences of two types and show properties of interleaved sequences of the geometric sequences of two types. More concretely, we show the period, correlation properties and linear complexity, and show that the sequences have the balance property. A key of our observation is that the second type geometric sequence is the complement of the left shift of the first type geometric sequence by half-period positions.

This paper is organized as follows: In Sect. 2, we recall the definition of periodic autocorrelation, periodic cross-correlation and linear complexity. In Sect. 3, we introduce the geometric sequences of two types and show some properties of the sequences. In Sect. 4, we propose interleaved sequences of the geometric sequences of two types and show the period, correlation properties and linear complexity of the interleaved sequences. Finally, we describe some conclusions in Sect. 5.

We give some notations. For a prime power $q$, $\mathbb{F}_q$ denotes the finite field with $q$ elements. For a polynomial $f(x)$ over a field $\mathbb{F}$, $\deg f$ denotes the degree of $f(x)$. For a prime number $p$ and an integer $a$, $(a/p)$ denotes the Legendre symbol. For a finite field $\mathbb{F}_p$ and the extension field $\mathbb{F}_{p^m}$ of degree $m$ of $\mathbb{F}_p$, $\text{Tr}_{\mathbb{F}_{p^m}/\mathbb{F}_p} : \mathbb{F}_{p^m} \to \mathbb{F}_p$ denotes the trace map, namely, $\text{Tr}_{\mathbb{F}_{p^m}/\mathbb{F}_p}(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{m-1}} \in \mathbb{F}_p$ for $\alpha \in \mathbb{F}_{p^m}$. For a finite set $A$, $\# A$ denotes the number of elements in $A$. For an integer $n > 1$ and an integer $a$, $a \mod n$ denotes the integer $u \in \{1, 2, \ldots, n\}$ such that $u \equiv a \mod n$.

2 Preliminaries

In this section, we recall the definition of periodic autocorrelation, periodic cross-correlation and linear complexity. For details, see [15, 10.3, 10.4].

2.1 Periodic autocorrelation and periodic cross-correlation

Let $S^{(1)} = (s^{(1)}_n)_{n \geq 0}$ and $S^{(2)} = (s^{(2)}_n)_{n \geq 0}$ be $N$-periodic sequences over $\mathbb{F}_2$. For $0 \leq \tau \leq N-1, \tau \in \mathbb{Z}$, $R_{S^{(1)}, S^{(2)}}(\tau)$ is defined as

$$R_{S^{(1)}, S^{(2)}}(\tau) = \sum_{i=0}^{N-1} (-1)^{s^{(1)}_i + s^{(2)}_{i+\tau}}.$$
Note that $s_1^{(1)} + s_2^{(2)}$ means an addition in $\mathbb{F}_2$. If $S^{(1)} \neq S^{(2)}$, then $R_{S^{(1)},S^{(2)}}(\tau)$ is called the periodic cross-correlation of $S^{(1)}$ and $S^{(2)}$. If $S := S^{(1)} = S^{(2)}$, then $R_S(\tau) := R_{S,S}(\tau)$ is called the periodic autocorrelation of $S$. By the definition, $R_S(0) = N$.

### 2.2 Linear complexity

Let $q$ be a prime power. Let $S = (s_n)_{n \geq 0}$ be a sequence over $\mathbb{F}_q$. The linear complexity $L(S)$ of $S$ is the length $L$ of the shortest linear recurrence relation

$$s_{n+L} = a_{L-1}s_{n+L-1} + \cdots + a_0s_n, \quad n \geq 0$$

for some $a_0, \ldots, a_{L-1} \in \mathbb{F}_q$, and the minimal polynomial $m(x)$ of $S$ is the polynomial $m(x) = x^L - a_{L-1}x^{L-1} - \cdots - a_0 \in \mathbb{F}_q[x]$.

Now, assume that $S$ is an $N$-periodic sequence. $S(x)$ denotes the polynomial $s_0 + s_1x + \cdots + s_{N-1}x^{N-1} \in \mathbb{F}_q[x]$. Then $m(x) = (x^N - 1)/\text{GCD}(x^N - 1, S(x))$ and $L(S) = N - \deg \text{GCD}(x^N - 1, S(x))$.

### 3 Geometric sequences binarized with Legendre symbol

Nogami, Tada and Uehara [17] proposed a geometric sequence whose nonlinear feedforward function is given by the Legendre symbol. In this section, we survey properties of the geometric sequences and define the geometric sequences of two types as the Legendre sequences (See [22, III, B]). Let $p > 2$ be a prime number and $m > 1$ an integer. Let $\omega \in \mathbb{F}_{p^m}$ be a primitive element in $\mathbb{F}_{p^m}$.

#### 3.1 The geometric sequences of the first type

The geometric sequence $T^{(1)} = (t_n^{(1)})_{n \geq 0}$ of the first type is defined as

$$t_n^{(1)} = \begin{cases} 
0 & \text{if } \frac{\text{Tr}_{p^m/p}(\omega^n)}{p} = 1 \\
1 & \text{if } \frac{\text{Tr}_{p^m/p}(\omega^n)}{p} = -1, \quad n \geq 0, \\
0 & \text{if } \frac{\text{Tr}_{p^m/p}(\omega^n)}{p} = 0
\end{cases}$$

$T^{(1)}$ is a periodic sequence of period $N := 2(p^m - 1)/(p - 1)$. Note that $N$ is an even integer. For $0 \leq \tau \leq N - 1, \tau \in \mathbb{Z}$, the autocorrelation of $T^{(1)}$ is

$$R_{T^{(1)}}(\tau) = \begin{cases} 
N = \frac{2(p^{m-1})}{p-1} & \text{if } \tau = 0 \\
N_1 := -2p^{m-1} + \frac{2(p^{m-1}-1)}{p-1} & \text{if } \tau = \frac{N}{2} \\
N_2 := \frac{2(p^{m-2}-1)}{p-1} & \text{otherwise},
\end{cases}$$

The linear complexity of $T^{(1)}$ is $L(T^{(1)}) = N = 2(p^m - 1)/(p - 1)$ (See [2] for arbitrary $m$). Hence $L(T^{(1)})$ attains a maximal value. On the other hand, $T^{(1)}$ does not have the balance property. In fact, the number of zeros in one period of $T^{(1)}$ is $p^{m-1} + 2(p^{m-1}-1)/(p - 1)$, and the number of ones in one period of $T^{(1)}$ is $p^{m-1}$.
3.2 The geometric sequences of the second type

We define a geometric sequence \( T^{(2)} = (t_n^{(2)})_{n \geq 0} \) of the second type as

\[
t_n^{(2)} = \begin{cases} 
0 & \text{if } \left( \frac{\text{Tr}_{F_p, \mathbb{F}_p}(\omega^n)}{p} \right) = 1 \\
1 & \text{if } \left( \frac{\text{Tr}_{F_p, \mathbb{F}_p}(\omega^n)}{p} \right) = -1, \ n \geq 0. \\
1 & \text{if } \left( \frac{\text{Tr}_{F_p, \mathbb{F}_p}(\omega^n)}{p} \right) = 0 
\end{cases}
\]  

(3)

Lemma 1. For any \( n \geq 0 \in \mathbb{Z} \), \( t_n^{(2)} = t_{n+N/2}^{(1)} + 1 \in \mathbb{F}_2 \).

Proof. Since \( \omega^{N/2} \in \mathbb{F}_p \) is a primitive element in \( \mathbb{F}_p \),

\[
\left( \frac{\text{Tr}_{F_p, \mathbb{F}_p}(\omega^{n+N/2})}{p} \right) = \left( \frac{\omega^{N/2} \cdot \text{Tr}_{F_p, \mathbb{F}_p}(\omega^n)}{p} \right) = - \left( \frac{\text{Tr}_{F_p, \mathbb{F}_p}(\omega^n)}{p} \right).
\]

Hence \( t_n^{(1)} = t_{n+N/2}^{(1)} + 1 \) for any \( n \geq 0 \) such that \( \text{Tr}_{F_p, \mathbb{F}_p}(\omega^n) \neq 0 \). Noting that \( \text{Tr}_{F_p, \mathbb{F}_p}(\omega^{n+N/2}) = 0 \) if and only if \( \text{Tr}_{F_p, \mathbb{F}_p}(\omega^n) = 0 \), it follows that

\[
t_n^{(2)} = \begin{cases} 
t_n^{(1)} & \text{if } \text{Tr}_{F_p, \mathbb{F}_p}(\omega^n) \neq 0 \\
t_n^{(1)} + 1 & \text{if } \text{Tr}_{F_p, \mathbb{F}_p}(\omega^n) = 0 
\end{cases}
\]

Corollary 2. The period, periodic autocorrelation and linear complexity of \( T^{(2)} \) are the same as those of \( T^{(1)} \), respectively.

Proposition 3. For \( 0 \leq \tau \leq N - 1, \tau \in \mathbb{Z} \), \( R_{T^{(1)}, T^{(2)}}(\tau) = - R_{T^{(1)}}(\tau + N/2). \)

Proof. By Lemma 1,

\[
R_{T^{(1)}, T^{(2)}}(\tau) = \sum_{i=0}^{N-1} (-1)^{t_i^{(1)} + t_i^{(2)} + \tau} = \sum_{i=0}^{N-1} (-1)^{t_i^{(1)} + t_i^{(1)} + N/2} + 1
\]

\[
\begin{align*}
\quad &= \# \left\{ i \mid t_i^{(1)} = t_i^{(1)} + N/2 + 1, 0 \leq i \leq N - 1 \right\} \\
&\quad - \# \left\{ i \mid t_i^{(1)} \neq t_i^{(1)} + N/2 + 1, 0 \leq i \leq N - 1 \right\} \\
\quad &= \# \left\{ i \mid t_i^{(1)} \neq t_i^{(1)} + N/2, 0 \leq i \leq N - 1 \right\} \\
&\quad - \# \left\{ i \mid t_i^{(1)} = t_i^{(1)} + N/2, 0 \leq i \leq N - 1 \right\} \\
\quad &= - R_{T^{(1)}}(\tau + N/2).
\end{align*}
\]
4 Interleaved sequences

In order to construct a sequence that has the balance property, we introduce an interleaved sequence. Gong [6] introduced the concept of interleaved sequences that were obtained by merging sequences, and sequence interleaving was widely used to improve the balance property of existing sequences such as in [21, 11, 8, 4, 3, 24, 23].

In this section, we propose interleaved sequences of geometric sequences of two types, which were introduced in the previous section. From Lemma 1, the interleaved sequence has the balance property by the interleaved structure. Furthermore, we show correlation properties and linear complexity of the interleaved sequences.

4.1 Interleaved sequences and left cyclic shift sequences

For a family $\mathcal{S} = \{S^{(i)} = (s^{(i)}_n)_{n \geq 0} | 0 \leq i \leq T - 1\}$ of $N$-periodic sequences, the sequence $U = (u_n)_{n \geq 0}$ defined as

$$u_n = s^{(i)}_j \text{ if } n = j \times T + i, \ 0 \leq i \leq T - 1, \ 0 \leq j$$

is called an interleaved sequence of $\mathcal{S}$.

For an $N$-periodic sequence $S = (s_n)_{n \geq 0}$ and $e \in \{0, 1, \ldots, N - 1\}$, the sequence $L^e(S) = (L^e(s)_n)_{n \geq 0}$ defined as

$$L^e(s)_n = s_{n+e}, \ n \geq 0$$

is called a left cyclic shift sequence.

4.2 Interleaved sequences of the geometric sequences of two types

Let $p > 2$ be a prime number and $m > 1$ an integer. Let $\omega \in \mathbb{F}_{p^m}$ be a primitive element in $\mathbb{F}_{p^m}$. Let $T^{(1)}$ and $T^{(2)}$ be the sequences defined as (1) and (3) with respect to $p, m$ and $\omega$, respectively. Then $N := 2(p^m - 1)/(p - 1)$ is the period of $T^{(1)}$ and $T^{(2)}$.

For $e \in \{0, 1, \ldots, N - 1\}$, we define a sequence $S^e = (s^e_n)_{n \geq 0}$ as the interleaved sequence of $(T^{(1)}, L^e(T^{(2)}))$.

**Example 4.** Let $p = 5$ and $m = 2$, so that $N = 12$. Let $\alpha$ be a root of the irreducible polynomial $x^2 + 2x + 3$, and $\omega = 4\alpha \in \mathbb{F}_p(\alpha)$. Then $T^{(1)}$ and $T^{(2)}$ are given as

$$T^{(1)} = (1, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, \ldots),$$
$$T^{(2)} = (1, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1, \ldots),$$

respectively. Put $e = 4$. Then $S^e$ is given as

$$S^4 = (1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, \ldots).$$

**Example 5.** Let $p = 3$ and $m = 3$, so that $N = 26$. Let $\alpha$ be a root of the irreducible polynomial $x^3 + 2x^2 + 1$, and $\omega = 2\alpha^2 \in \mathbb{F}_p(\alpha)$. Then $T^{(1)}$ and $T^{(2)}$
Lemma 6. If \( \tau = 2\tau_0, 0 \leq \tau_0 \leq N - 1, \tau_0 \in \mathbb{Z} \), then

\[ R_{S^{1b},S^{2c}}(\tau) = R_{T^{(1)}}(\tau_0) + R_{T^{(2)}}(e_2 - e_1 + \tau_0). \]

If \( \tau = 2\tau_0 + 1, 0 \leq \tau_0 \leq N - 1, \tau_0 \in \mathbb{Z} \), then

\[ R_{S^{1b},S^{2c}}(\tau) = -R_{T^{(1)}}\left( e_2 + \tau_0 + \frac{N}{2} \right) - R_{T^{(2)}}\left( e_1 - \tau_0 - 1 + \frac{N}{2} \right). \]

Proof. Assume that \( \tau = 2\tau_0, 0 \leq \tau_0 \leq N - 1, \tau_0 \in \mathbb{Z} \). By Corollary 2,

\[ R_{S^{1b},S^{2c}}(\tau) = R_{T^{(1)}}(\tau_0) + R_{T^{(2)}}(e_2 - e_1 + \tau_0) = R_{T^{(1)}}(\tau_0) + R_{T^{(1)}}(e_2 - e_1 + \tau_0). \]

Assume that \( \tau = 2\tau_0 + 1, 0 \leq \tau_0 \leq N - 1, \tau_0 \in \mathbb{Z} \). By Proposition 3,

\[ R_{S^{1b},S^{2c}}(\tau) = R_{T^{(1)},T^{(2)}}(e_2 + \tau_0) + R_{T^{(1)},T^{(1)}}(\tau_0 + 1 - e_1) = R_{T^{(1)},T^{(2)}}(e_2 + \tau_0) + R_{T^{(1)},T^{(2)}}(e_1 - \tau_0 - 1) = -R_{T^{(1)}}\left( e_2 + \tau_0 + \frac{N}{2} \right) - R_{T^{(2)}}\left( e_1 - \tau_0 - 1 + \frac{N}{2} \right). \]

The periodic autocorrelation of the proposed sequence is given as follows:

**Theorem 7.** Let \( p > 2 \) be a prime number and \( m > 1 \) an integer. Let \( \omega \in \mathbb{F}_{p^m} \) be a primitive element in \( \mathbb{F}_{p^m} \). Let \( T^{(1)} \) and \( T^{(2)} \) be the sequences defined as (1) and (3) with respect to \( p, m \) and \( \omega \), respectively. For \( e \in \{0, 1, \ldots, N - 1\} \), let \( S^e \) be the interleaved sequence of \( \{T^{(1)}, L^e(T^{(2)})\} \). Put \( N = 2(p^m - 1)/(p - 1) \). Then, for \( \tau = 2\tau_0, 0 \leq \tau_0 \leq N - 1, \tau_0 \in \mathbb{Z} \), the autocorrelation \( R_{S^e}(\tau) \) of \( S^e \) is given as

\[
R_{S^e}(\tau) = \begin{cases} 
2N & \text{if } \tau_0 = 0 \\
2N_1 & \text{if } \tau_0 = \frac{N}{2} \\
2N_2 & \text{otherwise.}
\end{cases}
\]
For \( \tau = 2\tau_0 + 1, 0 \leq \tau_0 \leq N - 1, \tau_0 \in \mathbb{Z} \), the autocorrelation \( R_S(\tau) \) of \( S^e \) is given as

\[
\begin{cases}
-N - N_2 & \text{if } \tau_0 \equiv -e - \frac{N}{2}, e - 1 + \frac{N}{2} \mod N \\
-N_1 - N_2 & \text{if } \tau_0 \equiv -e, e - 1 \mod N \\
-2N_2 & \text{otherwise}
\end{cases}
\]

(5)

if \( 2e \not\equiv 1 - N/2 \mod N \), and

\[
\begin{cases}
-N - N_1 & \text{if } \tau_0 \equiv -e, e - 1 \mod N \\
-2N_2 & \text{otherwise}
\end{cases}
\]

(6)

if \( 2e \equiv 1 - N/2 \mod N \). Here \( N_1 \) and \( N_2 \) are defined as in (2).

**Proof.** Assume that \( \tau = 2\tau_0, 0 \leq \tau_0 \leq N - 1, \tau_0 \in \mathbb{Z} \). Since \( R_S(\tau) = 2R_T(\tau_0) \) by Lemma 6, (4) is satisfied.

Assume that \( \tau = 2\tau_0 + 1, 0 \leq \tau_0 \leq N - 1, \tau_0 \in \mathbb{Z} \). Then we have \( R_S(\tau) = -R_T(1 + \tau_0 + N/2) - R_T(1 + \tau_0 - 1 + N/2) \) by Lemma 6. On the other hand,

\[
R_T(1 + \tau_0 + N/2) = \begin{cases} N & \text{if } \tau_0 \equiv -e - \frac{N}{2} \mod N \\ N_1 & \text{if } \tau_0 \equiv -e \mod N \\ N_2 & \text{otherwise} \end{cases}
\]

and

\[
R_T(1 + \tau_0 - 1 + N/2) = \begin{cases} N & \text{if } \tau_0 \equiv 1 + \frac{N}{2} \mod N \\ N_1 & \text{if } \tau_0 \equiv -e - 1 \mod N \\ N_2 & \text{otherwise}. \end{cases}
\]

If \( 2e \not\equiv 1 - N/2 \mod N \), then \(-e - N/2, -e, e - 1 + N/2, e - 1\) are incongruent modulo \( N \). Hence we have (5). If \( 2e \equiv 1 - N/2 \mod N \), then \(-e - N/2 \equiv e - 1 \mod N \) and \(-e \equiv e - 1 + N/2 \mod N \). Hence we have (6).

**Corollary 8.** \( S^e \) has period \( 2N = 4(p^m - 1)/(p - 1) \).

By the interleaved structure, \( S^e \) has the balance property, namely the number of zeros and ones in one period of \( S^e \) are \( N \).

**Remark 9.** If \( m \) is even, then \( 2e \not\equiv 1 - N/2 \mod N \).

**Example 10.** Let \( p = 11 \) and \( m = 2 \), so that \( N = 24 \). Let \( \alpha \) be a root of the irreducible polynomial \( x^3 + 7x + 2 \), and \( \omega = 9 + 2\alpha \in \mathbb{F}_p(\alpha) \). Figure 1 describes the graph of the periodic autocorrelation of \( S^{17} \).

**Example 11.** Let \( p = 5 \) and \( m = 3 \), so that \( N = 62 \). Let \( \alpha \) be a root of the irreducible polynomial \( x^3 + 3x^2 + 2x + 3 \), and \( \omega = 1 + \alpha + 2\alpha^2 \in \mathbb{F}_p(\alpha) \). Figure 2 and Fig. 3 describe the graph of the periodic autocorrelation of \( S^{25} \) and \( S^{16} \), respectively.

Now we show the cross-correlation distribution of two proposed sequences that have different shift widths.

**Theorem 12.** Let \( p > 2 \) be a prime number and \( m > 1 \) an integer. Let \( \omega \in \mathbb{F}_p^m \) be a primitive element in \( \mathbb{F}_p^m \). Let \( T^{(1)} \) and \( T^{(2)} \) be the sequences defined as (1) and (3) with respect to \( p, m, \) and \( \omega \), respectively. For \( e_1, e_2 \in \{0, 1, \ldots, N - 1\} \), \( e_1 < e_2 \), let \( S^{e_1} \) and \( S^{e_2} \) be the interleaved sequences of \( \{T^{(1)}, L^{e_1}(T^{(2)})\} \) and \( \{T^{(1)}, L^{e_2}(T^{(2)})\} \), respectively. Put \( N = 2(p^m - 1)/(p - 1) \). Then, for
Figure 1: The graph of the periodic autocorrelation of $S^{17}$ in the case of $p = 11, m = 2$.

Figure 2: The graph of the periodic autocorrelation of $S^{25}$ in the case of $p = 5, m = 3$.

Figure 3: The graph of the periodic autocorrelation of $S^{16}$ in the case of $p = 5, m = 3$. 
describes the graph of the periodic cross-correlation of \( S \) given as follows:

The minimal polynomial and linear complexity of the proposed sequence are

4.4 Linear complexity of the proposed sequences

\( \tau = 2\tau_0, 0 \leq \tau_0 \leq N - 1, \tau_0 \in \mathbb{Z} \), the cross-correlation \( R_{S^{e_1},S^{e_2}}(\tau) \) of \( S^{e_1} \) and \( S^{e_2} \) is given as

\[
\begin{cases}
N + N_2 & \text{if } \tau_0 \equiv 0, e_1 - e_2 \mod N \\
N_1 + N_2 & \text{if } \tau_0 \equiv \frac{N}{2}, e_1 - e_2 + \frac{N}{2} \mod N \\
2N_2 & \text{otherwise}
\end{cases}
\]

if \( e_2 - e_1 \not\equiv N/2 \mod N \), and

\[
\begin{cases}
N + N_1 & \text{if } \tau_0 = 0, \frac{N}{2} \\
2N_2 & \text{otherwise}
\end{cases}
\]

if \( e_2 - e_1 \equiv N/2 \mod N \). For \( \tau = 2\tau_0 + 1, 0 \leq \tau_0 \leq N - 1, \tau_0 \in \mathbb{Z} \), the cross-correlation \( R_{S^{e_1},S^{e_2}}(\tau) \) of \( S^{e_1} \) and \( S^{e_2} \) is given as

\[
\begin{cases}
-N - N_2 & \text{if } \tau_0 \equiv -e_2 - \frac{N}{2}, e_1 - 1 + \frac{N}{2} \mod N \\
-N_1 - N_2 & \text{if } \tau_0 \equiv -e_2, e_1 - 1 \mod N \\
-2N_2 & \text{otherwise}
\end{cases}
\]

if \( e_1 + e_2 \not\equiv 1 \mod N \) and \( e_1 + e_2 \not\equiv 1 - N/2 \mod N \),

\[
\begin{cases}
-2N & \text{if } \tau_0 \equiv -e_2 - \frac{N}{2} \mod N \\
-2N_1 & \text{if } \tau_0 \equiv -e_2 \mod N \\
-2N_2 & \text{otherwise}
\end{cases}
\]

if \( e_1 + e_2 \equiv 1 \mod N \), and

\[
\begin{cases}
-N - N_1 & \text{if } \tau_0 \equiv -e_2 - \frac{N}{2}, -e_2 \mod N \\
-2N_2 & \text{otherwise}
\end{cases}
\]

if \( e_1 + e_2 \equiv 1 - N/2 \mod N \). Here \( N_1 \) and \( N_2 \) are defined as in (2).

Proof. Exactly like the proof of Theorem 7.

Example 13. Let \( p = 11 \) and \( m = 2 \), so that \( N = 24 \). Let \( \alpha \) be a root of the irreducible polynomial \( x^2 + 7x + 2 \), and \( \omega = 9 + 2\alpha \in \mathbb{F}_p(\alpha) \). Figure 4 describes the graph of the periodic cross-correlation of \( S^9 \) and \( S^{11} \). Figure 5 describes the graph of the periodic cross-correlation of \( S^8 \) and \( S^{18} \). Figure 6 describes the graph of the periodic cross-correlation of \( S^{11} \) and \( S^{14} \). Figure 7 describes the graph of the periodic cross-correlation of \( S^{2} \) and \( S^{11} \).

Example 14. Let \( p = 5 \) and \( m = 3 \), so that \( N = 62 \). Let \( \alpha \) be a root of the irreducible polynomial \( x^3 + 3x^2 + 2x + 3 \), and \( \omega = 1 + \alpha + 2\alpha^2 \in \mathbb{F}_p(\alpha) \). Figure 8 describes the graph of the periodic cross-correlation of \( S^{25} \) and \( S^{11} \). Figure 9 describes the graph of the periodic cross-correlation of \( S^{25} \) and \( S^{15} \). Figure 10 describes the graph of the periodic cross-correlation of \( S^{26} \) and \( S^{37} \). Figure 11 describes the graph of the periodic cross-correlation of \( S^{45} \) and \( S^{49} \).

4.4 Linear complexity of the proposed sequences

The minimal polynomial and linear complexity of the proposed sequence are given as follows:
Figure 4: The graph of the periodic cross-correlation of $S^9$ and $S^{11}$ in the case of $p = 11, m = 2$.

Figure 5: The graph of the periodic cross-correlation of $S^6$ and $S^{18}$ in the case of $p = 11, m = 2$.

Figure 6: The graph of the periodic cross-correlation of $S^{11}$ and $S^{14}$ in the case of $p = 11, m = 2$. 
Figure 7: The graph of the periodic cross-correlation of $S^2$ and $S^{11}$ in the case of $p = 11, m = 2$.

Figure 8: The graph of the periodic cross-correlation of $S^{25}$ and $S^{41}$ in the case of $p = 5, m = 3$.

Figure 9: The graph of the periodic cross-correlation of $S^4$ and $S^{35}$ in the case of $p = 5, m = 3$. 
Figure 10: The graph of the periodic cross-correlation of $S^{26}$ and $S^{37}$ in the case of $p = 5, m = 3$.

Figure 11: The graph of the periodic cross-correlation of $S^{45}$ and $S^{49}$ in the case of $p = 5, m = 3$. 
Theorem 15. Let $p > 2$ be a prime number and $m > 1$ an integer. Let $\omega \in \mathbb{F}_p^m$ be a primitive element in $\mathbb{F}_p^m$. Let $T^{(1)}$ and $T^{(2)}$ be the sequences defined as (1) and (3) with respect to $p, m$ and $\omega$, respectively. For $e \in \{0, 1, \ldots, N - 1\}$, let $S^e$ be the interleaved sequence of $\{T^{(1)}, L^e(T^{(2)})\}$. Put $N = 2(p^m - 1)/(p - 1)$. Then the minimal polynomial $m(x) \in \mathbb{F}_2[x]$ of $S^e$ is given as

$$m(x) = \frac{x^{2N} - 1}{x^{G(N, e)} - 1},$$

where $G(N, e) = \gcd(N/2^{\nu_2(N)}, -2e + 1 \mod N/2^{\nu_2(N)})$ and $\nu_2(N)$ is the exponent of the largest power of 2 which divides $N$. Therefore the linear complexity $L(S^e)$ of $S^e$ is given as

$$L(S^e) = 2N - G(N, e).$$

Proof. Since

$$L^e(T^{(2)})(x) = x^{N - e}T^{(2)}(x) \mod x^N - 1 \equiv x^{N - e} \left( x^N + \frac{x^N - 1}{x - 1} \right) \mod x^N - 1$$

we have

$$S^e(x) = T^{(1)}(x^2) + xL^e(T^{(2)})(x^2) \equiv T^{(1)}(x^2) + x \left( x^{3N - 2e}T^{(1)}(x^2) + x^{2N - 2e} \cdot \frac{x^{2N} - 1}{x^2 - 1} \right) \mod x^{2N} - 1 \equiv (x^{3N - 2e + 1} + 1)T^{(1)}(x^2) + x^{2N - 2e + 1} \cdot \frac{x^{2N} - 1}{x^2 - 1} \mod x^{2N} - 1.$$
Corollary 16. The upper and lower bounds on the linear complexity are obtained as follows:

1. \( L(S_e) \leq 2N - 1 \). The equality holds if and only if \( G(N, e) = 1 \).

2. \( L(S_e) \geq 2N - N/2^{\nu_2(N)} \). The equality holds if and only if \(-2e + 1 \equiv 0 \mod N/2^{\nu_2(N)}\).

Example 17. Let \( p = 11 \) and \( m = 2 \), so that \( N = 24 \). Let \( \alpha \) be a root of the irreducible polynomial \( x^2 + 7x + 2 \), and \( \omega = 9 + 2\alpha \in \mathbb{F}_p(\alpha) \). Then the linear complexity \( L(S_e) \) of \( S_e \) is given as
\[
L(S_e) = \begin{cases} 
47 & \text{if } e \equiv 0, 1 \mod 3 \\
45 & \text{if } e \equiv 2 \mod 3.
\end{cases}
\]
Note that \( N = 24 = 2^3 \cdot 3 \).

Example 18. Let \( p = 5 \) and \( m = 3 \), so that \( N = 62 \). Let \( \alpha \) be a root of the irreducible polynomial \( x^3 + 3x^2 + 2x + 3 \), and \( \omega = 1 + \alpha + 2\alpha^2 \in \mathbb{F}_p(\alpha) \). Then the linear complexity \( L(S_e) \) of \( S_e \) is given as
\[
L(S_e) = \begin{cases} 
93 & \text{if } e = 16, 47 \\
123 & \text{otherwise}.
\end{cases}
\]
Note that \( N = 62 = 2 \cdot 31 \) and \(-2 \cdot 16 + 1 \equiv -2 \cdot 47 + 1 \equiv 0 \mod 31 \).

5 Conclusion

In this paper, we propose interleaved sequences of geometric sequences of two types, which were introduced by Nogami, Tada and Uehara. Furthermore, we show correlation properties and linear complexity of the interleaved sequences. The proposed sequences have the balance property and double the period of the geometric sequences by the interleaved structure. The autocorrelation distributions of the proposed sequences have peaks and troughs at six shift values (four shift values for special cases). The cross-correlation distributions of the two proposed sequences that have different shift widths also have a few peaks and troughs. The question is whether a security is affected by these distributions. We obtain the linear complexity of the proposed sequences. The formula induces the upper and lower bounds on the linear complexity, and the conditions to attain the upper and lower bounds, respectively. Therefore, although the linear complexity do not attain the maximal value, one can choose a shift width with which a sequence has a large linear complexity, especially such as the period minus one. Thus the proposed sequences have good cryptographic properties for linear complexity. We only consider the frequency for a single symbol, however the proposed sequences do not have the balance property for block of length larger than one. As a future work, we should give a transformation, by which transformed sequences have the balance property for block of a certain length and keep some good properties of the proposed sequences.

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