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Corresponding Author: Prof. Francisco Marques, PH.D.

Corresponding Author's Institution: UPC Barcelona Tech

First Author: Francisco Marques, PH.D.

Order of Authors: Francisco Marques, PH.D.; Alvaro Meseguer, Ph.D.; Juan M Lopez, Ph.D.; Rafael Pacheco, Ph.D.
Please find attached a paper entitled “Hopf bifurcation with zero frequency and imperfect $SO(2)$ symmetry,” by Marques, Meseguer, Lopez and Pacheco, for consideration as a regular article in Physica D.

This work analyzes the breaking of the $SO(2)$ symmetry in a dynamical system close to a Hopf bifurcation whose frequency changes sign along a curve in parameter space. The complete unfolding is very complex, and we have derived and analyzed five different normal forms corresponding to the different symmetry breaking terms up to order two included. In all cases the behavior when the symmetry breaking is weak shows the presence of a pinning region limited by infinite period bifurcations. The motivation stems from experimental and numerical studies in fluid dynamics showing the pinning of rotating waves to defects. The normal form results agree with the observed and computed results.
Rotating waves are periodic solutions in $SO(2)$ equivariant dynamical systems. Their precession frequency changes with parameters, and it may change sign, passing through zero. When this happens, the dynamical system is very sensitive to imperfections that break the $SO(2)$ symmetry, and the waves may become trapped by the imperfections, resulting in steady solutions that exist in a finite region in parameter space, the so-called pinning phenomenon. In this study we analyze the breaking of the $SO(2)$ symmetry in a dynamical system close to a Hopf bifurcation whose frequency changes sign along a curve in parameter space. The problem is more complex than expected, and the complete unfolding is of codimension six. A detailed analysis of different types of imperfections indicates that a pinning region surrounded by infinite-period bifurcation curves appears in all cases. Complex bifurcational processes, strongly dependent on the specifics of how the symmetry is broken, appear very close to the intersection of the Hopf bifurcation and the pinning region. Scaling laws of the pinning region width, and partial breaking of $SO(2)$ to $Z_m$, are also considered. Previous experimental and numerical studies of pinned rotating waves are reviewed in light of the new theoretical results.
Hopf bifurcation with zero frequency and imperfect $SO(2)$ symmetry

F. Marques$^{a,*}$, A. Meseguer$^a$, J. M. Lopez$^b$, J. R. Pacheco$^{b,c}$

$^a$Departament de Física Aplicada, Universitat Politècnica de Catalunya, Grivona Salgado s/n, Mòdul B4 Campus Nord, 08034 Barcelona, Spain
$^b$School of Mathematical and Statistical Sciences, Arizona State University, Tempe AZ 85287, USA
$^c$Environmental Fluid Dynamics Laboratories, Department of Civil Engineering and Geological Sciences, University of Notre Dame, Notre Dame, Indiana 46556, USA

Abstract

Rotating waves are periodic solutions in $SO(2)$ equivariant dynamical systems. Their precession frequency changes with parameters, and it may change sign, passing through zero. When this happens, the dynamical system is very sensitive to imperfections that break the $SO(2)$ symmetry, and the waves may become trapped by the imperfections, resulting in steady solutions that exist in a finite region in parameter space, the so-called pinning phenomenon. In this study we analyze the breaking of the $SO(2)$ symmetry in a dynamical system close to a Hopf bifurcation whose frequency changes sign along a curve in parameter space. The problem is more complex than expected, and the complete unfolding is of codimension six. A detailed analysis of different types of imperfections indicates that a pinning region surrounded by infinite-period bifurcation curves appears in all cases. Complex bifurcational processes, strongly dependent on the specifics of how the symmetry is broken, appear very close to the intersection of the Hopf bifurcation and the pinning region. Scaling laws of the pinning region width, and partial breaking of $SO(2)$ to $Z_m$, are also considered. Previous experimental and numerical studies of pinned rotating waves are reviewed in light of the new theoretical results.

Keywords: Bifurcations with symmetry; imperfections; pinning; homoclinic and heteroclinic dynamics.

1. Introduction

Dynamical systems theory plays an important role in many areas of mathematics and physics because it provides the building blocks that allow us to understand the changes many physical systems experience in their dynamics.

*Corresponding author
Email address: marques@fa.upc.edu (F. Marques)
when parameters are varied. These building blocks are the generic bifurcations (saddle-node, Hopf, etc.) that any arbitrary physical system must experience under parameter variation, regardless of the physical mechanisms underlying the dynamics. When one single parameter of the system under consideration is varied, codimension-one bifurcations are expected. If the system depends on more parameters, higher codimension bifurcations are going to appear, and they act as organizing centers of the dynamics.

The presence of symmetries changes the nature and type of bifurcations that a dynamical system may undergo. Symmetries play an important role in many idealized situations, where simplifying assumptions and the consideration of simple geometries result in dynamical systems equivariant under a certain symmetry group. Bifurcations with symmetry have been widely studied [13, 15, 6, 14, 7, 9]. However, in any real system, the symmetries are only approximately fulfilled, and the breaking of the symmetries, due to the presence of noise, imperfections and/or other phenomena, is always present. There are numerous studies of how imperfect symmetries lead to dynamics that are unexpected in the symmetric problem, e.g. [17, 5, 19, 16, 10, 21]. However, a complete theory is currently unavailable.

One observed consequence of imperfections in systems that support propagating waves is that the waves may become trapped by the imperfections [e.g., see 17, 32, 29, 30]. In these various examples, the propagation direction is typically biased. However, a more recent problem has considered a case where a rotating wave whose sense of precession changes sign is pinned by symmetry-breaking imperfections [1]. We are unaware of any systematic analysis of the associated normal form dynamics for such a problem and this motivates the present study.

When a system is invariant to rotations around an axis (invariance under the $SO(2)$ symmetry group), Hopf bifurcations result in rotating waves, consisting of a pattern that rotates around the symmetry axis at a given precession frequency without changing shape. This frequency is parameter dependent, and in many problems, when parameters are varied, the precession frequency changes sign along a curve in parameter space. What has been observed in different systems is that in the presence of imperfections, the curve of zero frequency becomes a band in parameter space. Within this band, the rotating wave becomes a steady solution. This is the so-called pinning phenomenon. It can be understood as the attachment of the rotating pattern to some stationary imperfection of the system, so that the pattern becomes steady, as long as its frequency is small enough so that the imperfection is able to stop the rotation. This pinning phenomenon bears some resemblance to the frequency locking phenomena, although in the frequency locking case we are dealing with a system with two non-zero frequencies and their ratio becomes constant in a region of parameter space (a resonance horn).

In the present paper, we analyze the breaking of $SO(2)$ symmetry in a dynamical system close to a Hopf bifurcation whose frequency changes sign along a curve in parameter space. The analysis shows that breaking $SO(2)$ symmetry is much more complex than expected, resulting in a bifurcation of codimension
six. Although it is not possible to analyze in detail such a complex and high-codimension bifurcation, we present here the analysis of five different ways to break $SO(2)$ symmetry. This is done by introducing into the normal form all the possible terms, up to and including second order, that break the symmetry. In all cases we find that a band of pinning solutions appears around the zero frequency curve from the symmetric case, and that the band is delimited by curves of infinite-period bifurcations. The details of what happens when the infinite-period bifurcation curves approach the Hopf bifurcation curve are different in the five cases, and involve complicated dynamics with several codimension-two bifurcations occurring in a small region of parameter space as well as several global bifurcations.

Interest in the present analysis is twofold. First of all, although the details of the bifurcational process close to the zero-frequency Hopf point are very complicated and differ from case to case, for all cases we observe the appearance of a pinning band delimited by infinite-period bifurcations of homoclinic type. Secondly, some of the scenarios analyzed are important *per se* because they correspond to the generic analysis of a partial breaking of the $SO(2)$ symmetry, so that after the introduction of perturbations, the system still retains a discrete symmetry (the $Z_2$ case is analyzed in detail).

The paper is organized as follows. In section §2 the properties of a Hopf bifurcation with $SO(2)$ symmetry with the precession frequency crossing through zero are summarized, and the general unfolding of the $SO(2)$ symmetry breaking process is discussed. The next sections explore the particulars of breaking the symmetry at order zero (§4), one (§3) and two (§5). Section §3 is particularly interesting because it corresponds to the symmetry-breaking process $SO(2) \rightarrow Z_2$ that can be realized experimentally. Some considerations of the $SO(2) \rightarrow Z_3$ symmetry-breaking are presented in §5.3. Section §6 analyzes the remaining partial symmetry breakings, $SO(2) \rightarrow Z_m$ for $m \geq 4$. Section §7 extracts the general features of the pinning problem from the analysis of specific cases carried out in the earlier sections. In particular, §7.1 presents comparisons with experiments and numerical computations in a real problem in fluid dynamics, illustrating the application of the general theory developed in the present study. Finally, §7.2 summarizes the codimension-two global bifurcations of limit cycles obtained in the present study.

### 2. Hopf bifurcation with $SO(2)$ symmetry and zero frequency

The normal form for a Hopf bifurcation is

$$
\dot{z} = z(\mu + i\omega - c|z|^2),
$$

where $\omega$ and $c$ are functions of parameters, but generically at the bifurcation point ($\mu = 0$) both are different from zero. It is the non-zero character of $\omega$ that allows one to eliminate the quadratic terms in $z$ in the normal form. This is because the normal form $\dot{z} = P(z, \bar{z})$ satisfies

$$
P(e^{-i\omega t}z, e^{i\omega t}\bar{z}) = e^{-i\omega t}P(z, \bar{z}).
$$

3
If $\omega = 0$ this equation becomes an identity and $P$ cannot be simplified. The case $\omega = 0$ is a complicated bifurcation and depends on details of the double-zero eigenvalue linear part $L$. If $L$ is not completely degenerate, i.e.,

$$L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ (3)

then we have the well-studied Takens–Bogdanov bifurcation, whereas the completely degenerate case,

$$L = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ (4)

is a high-codimension bifurcation that has not been completely analyzed.

If the system has $SO(2)$ symmetry, it must also satisfy

$$P(e^{im\theta} z, e^{-im\theta} \bar{z}) = e^{im\theta} P(z, \bar{z}),$$ (5)

where $Z_m$ is the discrete symmetry retained by the bifurcated solution; when the group $Z_m$ is generated by rotations of angle $2\pi/m$ around an axis of $m$-fold symmetry, as is usually the case with $SO(2)$, then the group is also called $C_m$.

Equations (2) and (5) are completely equivalent, and have the same implications for the normal form structure. Advancing in time is the same as rotating the solution by a certain angle ($\omega t = m\theta$); the bifurcated solution is a rotating wave. Therefore, if $\omega$ becomes zero by varying a second parameter, we still have the same normal form (1), due to (5), with $\omega$ replaced by a small parameter $\nu$:

$$\dot{z} = z(\mu + i\nu - c|z|^2).$$ (6)

The Hopf bifurcation with $SO(2)$ symmetry and zero frequency is, in this sense, trivial. Introducing the modulus and phase of the complex amplitude $z = re^{i\phi}$, the normal form becomes

$$\dot{r} = r(\mu - ar^2),$$  

$$\dot{\phi} = \nu - br^2,$$ (7)

where $c = a + ib$. We assume $c \neq 0$, and in fact, by scaling $z$, we will consider $|c| = 1$, i.e.

$$c = a + ib = ie^{-i\alpha_0} = \sin \alpha_0 + i \cos \alpha_0, \quad b + ia = e^{i\alpha_0},$$ (8)

which helps simplify subsequent expressions. We will assume for definitiveness that $a$ and $b$ are both positive, which corresponds to the fluid dynamics problem that motivated the present analysis (see [1, 23] and §7.1); for other signs of $a$ and $b$, analogous conclusions can be drawn.

The bifurcation frequency in (7) is now the small parameter $\nu$. The bifurcated solution $RW_m$ exists only for $\mu > 0$, and has amplitude $r = \sqrt{\mu/a}$ and frequency $\omega = \nu - b\mu/a$. The limit cycle $RW_m$ becomes an invariant set of steady solutions along the straight line $\mu = a\nu/b$ (labeled L in figure 1) where the frequency of $RW_m$ goes to zero: the angle between L and the horizontal ($\mu = 0$) axis is $\alpha_0$. The bifurcation diagram, and a schematic of the bifurcations along a one-dimensional path, is shown in figure 1.
$H - H + \mu \nu \mu = a \nu / b$

$A C - C + P_0 A H - H + L A - - ++ - - P_0 (a) (b) \alpha_0$

Figure 1: Hopf bifurcation with $SO(2)$ symmetry and zero frequency. (a) Bifurcation diagram; thick lines are bifurcation curves. (b) Bifurcations along the path $A$ shown in (a). The fixed point curve is labeled with the signs of its eigenvalues. $C_\pm$ are the limit cycles born at the Hopf bifurcations $H_-^-$ and $H_+^+$, that rotate in opposite senses. $L$ is the line where the limit cycle becomes an invariant curve of fixed points.

2.1. Unfolding the Hopf bifurcation with zero frequency

If the $SO(2)$ symmetry in the normal form (6) is completely broken, and no symmetry remains, then the restrictions imposed on the normal form by (5) disappear completely and all the terms in $z$ and $\bar{z}$ missing from (6) will reappear multiplied by small parameters. This means that the normal form will be

$$\dot{z} = z(\mu + i\nu - c|z|^2) + \epsilon_1 + \epsilon_2 \bar{z} + \epsilon_3 z^2 + \epsilon_4 z \bar{z} + \epsilon_5 \bar{z}^2,$$

where additional cubic terms have been neglected because we assume $c \neq 0$ and that $cz|z|^2$ will be dominant. As the $\epsilon_i$ are complex, we have a problem with 12 parameters. Additional simplifications can be made in order to obtain the so-called hypernormal form; this method is extensively used by Kuznetsov [20], for example. These simplifications include an infinitesimal translation of $z$ (two parameters), a time reparametrization depending on the quadratic terms in $z$ (three parameters), and an arbitrary shift in the phase of $z$ (one parameter). Using these transformations the twelve parameters can be reduced to six. A complete analysis of a normal form depending on six parameters, i.e., a codimension-six bifurcation, is completely beyond the scope of the present paper. In the literature, only codimension-one bifurcations have been completely analyzed. Most of the codimension-two bifurcations for ODE and maps have also been analyzed, except for a few bifurcations for maps that remain outstanding [20]. A few codimension-three and very few codimension-four bifurcations have also been analyzed [8, 12]. To our knowledge, there is no systematic analysis of bifurcations of codimension greater than two.

In the following sections, we consider the five cases, $\epsilon_1$ to $\epsilon_5$, separately. A combination of analytical and numerical tools allows for a detailed analysis of these bifurcations. We extract the common features of the different cases when $\epsilon_i \ll \sqrt{\mu^2 + \nu^2}$, which captures the relevant behavior for weakly breaking $SO(2)$ symmetry. In particular, the $\epsilon_2$ case exhibits very interesting and rich dynamics that may be present in some practical cases, when the $SO(2)$ symmetry group
is not completely broken, but a $Z_2$ symmetry group, generated by the half-turn, remains.

Some general comments can be made here about these five cases, which are of the form

$$\dot{z} = z(\mu + i\nu - c|z|^2) + \epsilon z^q \bar{z}^{p-q},$$

(10)

for integers $0 \leq q \leq p \leq 2$, excluding the case $p = q = 1$ which is $SO(2)$ equivariant. By changing the origin of the phase of $z$, we can modify the phase of $\epsilon$ so that it becomes real and positive. Then, by re-scaling $z$, the time $t$, and the parameters $\mu$ and $\nu$ as

$$(z, t, \mu, \nu) \rightarrow (\epsilon^\delta z, \epsilon^{-2\delta} t, \epsilon^{2\delta} \mu, \epsilon^{2\delta} \nu), \quad \delta = \frac{1}{3-p},$$

(11)

we obtain (10) with $\epsilon = 1$, effectively leading to codimension-two bifurcations in all five cases. We expect complex behavior for $\mu^2 + \nu^2 \ll \epsilon^2$, when the three parameters are of comparable size, while the effects of small imperfections breaking $SO(2)$ will correspond to $\mu^2 + \nu^2 \gg \epsilon^2$. From now on $\epsilon = 1$ will be assumed, and we can restore the explicit $\epsilon$ dependence by reversing the transformation (11).

3. Symmetry breaking of $SO(2)$ to $Z_2$, with an $\epsilon \bar{z}$ term

The $\epsilon \bar{z}$ term in (9) corresponds to breaking $SO(2)$ symmetry in a way that leaves a system with $Z_2$ symmetry, corresponding to invariance under a half turn. The normal form to be analyzed is (10) with $p = 1$, $q = 0$ and $\epsilon = 1$:

$$\dot{z} = z(\mu + i\nu - c|z|^2) + \bar{z}.$$  

(12)

The new normal form (12) is still invariant to $z \rightarrow -z$, or equivalently, the half-turn $\phi \rightarrow \phi + \pi$. This is all that remains of the $SO(2)$ symmetry group, which is reduced to $Z_2$, generated by the half-turn. In fact, the $Z_2$ symmetry implies that $P(z, \bar{z})$ in $\dot{z} = P(z, \bar{z})$ must be odd: $P(-z, -\bar{z}) = -P(z, \bar{z})$, which is (5) for $\theta = \pi$, the half turn. Therefore, (12) is the unfolding corresponding to the symmetry breaking of $SO(2)$ to $Z_2$.

Writing the normal form (12) in terms of the modulus and phase of $z = re^{i\phi}$ gives

$$\dot{r} = r(\mu - ar^2) + r \cos 2\phi,$$

$$\dot{\phi} = \nu - br^2 - \sin 2\phi.$$  

(13)

3.1. Fixed points and their bifurcations

The normal form (12), or (13), admits up to five fixed points. One is the origin $r = 0$, the trivial solution $P_0$. The other fixed points come in two pairs of $Z_2$-symmetric points: one is the pair $P_+ = r_+ e^{i\phi_+}$ and $P_*^+ = -r_+ e^{i\phi_+}$, and
the other pair is $P_\pm = r_\pm e^{i\phi_\pm}$ and $P^*_\pm = -r_\pm e^{i\phi_\pm}$. Coefficients $r_\pm$ and $\phi_\pm$ are given by

$$r_\pm^2 = a\mu + b\nu \pm \Delta, \quad \phi_\pm = (\alpha_0 \pm \alpha_1)/2,$$

(14)

$$\Delta^2 = 1 - (a\nu - b\mu)^2, \quad e^{i\alpha_1} = av - b\mu - i\Delta,$$

(15)

The details of the computations are given in Appendix A. There are three different regions in the $(\mu, \nu)$-parameter plane:

Region II: $\mu^2 + \nu^2 < 1$,
Region III: $\mu^2 + \nu^2 > 1$, and $|a\nu - b\mu| < 1$, and $a\mu + b\nu > 0$,  

(16)

with Region I being the remaining parameter space. These three regions are separated by four curves along which steady bifurcations between the different fixed points take place, as shown in figure 2.

In region I only $P_0$ exists, in region II three fixed points $P_0, P_+, P^*_+$ exist, and in region III all five points $P_0, P_+, P^*_+, P_-, P^*_-$ exist. Along the semicircle

PF$_+$: $\mu^2 + \nu^2 = 1$ and $a\mu + b\nu < 0$,  

(17)

the two symmetrically-related solutions $P_+$ and $P^*_+$ are born in a pitchfork bifurcation of the trivial branch $P_0$. Along the semicircle

PF$_-$: $\mu^2 + \nu^2 = 1$ and $a\mu + b\nu > 0$,  

(18)

the two symmetrically-related solutions $P_-$ and $P^*_-$ are born in a pitchfork bifurcation of the trivial branch $P_0$. Along the two half-lines

SN: $\mu = (a\nu - 1)/b$, \quad SN': $\mu = (a\nu + 1)/b$, both with $a\mu + b\nu > 0$,  

(19)

a saddle-node bifurcation takes place. It is a double saddle-node, due to the $Z_2$ symmetry; we have one saddle-node involving $P_+$ and $P_-$, and the $Z_2$-symmetric saddle-node between $P^*_+$ and $P^*_-$.

Bifurcation diagrams along the paths A and B in figure 2(a) are shown in parts (b) and (c) of the same figure.
We can compare with the original problem with $SO(2)$ symmetry, corresponding to $\epsilon = 0$. In order to do that, the $\epsilon$ dependence will be restored in this paragraph. The single line $L$ ($\mu = \nu \tan \alpha_0$) where $\omega = 0$ and nontrivial fixed points exist in the perfect problem, becomes a region of width $2\epsilon$ in the imperfect problem, where up to four fixed points exist, in addition to the base state $P_0$; they are the remnants of the circle of fixed points in the original problem. Solutions with $\omega = 0$, that existed only along a single line in the absence of imperfections, now exist in a region bounded by the semicircle $F_+$ and the half-lines $av - b\mu = \pm \epsilon$; this region will be termed the pinning region. It bears some relationship with the frequency-locking regions appearing in Neimark-Sacker bifurcations, in the sense that here we also have frequency locking, but with $\omega = 0$. The width of the pinning region is proportional to $\epsilon$, a measure of the breaking of $SO(2)$ symmetry due to imperfections.

3.2. Hopf bifurcations of the fixed points

In the absence of imperfections ($\epsilon = 0$) the $P_0$ branch looses stability to a Hopf bifurcation along the curve $\mu = 0$. Let us analyze the stability of $P_0$ in the imperfect problem. Using Cartesian coordinates $z = x + iy$ in (12) we obtain

$$
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
\mu + 1 & -\nu \\
\nu & \mu - 1
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} - (x^2 + y^2) \begin{bmatrix}
ax - by \\
bx + ay
\end{bmatrix}.
$$

The eigenvalues of $P_0$ are the eigenvalues of the linear part of (20), $\lambda_{\pm} = \mu \pm \sqrt{1 - \nu^2}$. There is a Hopf bifurcation ($\Im \lambda_{\pm} \neq 0$) when $\mu = 0$ and $|\nu| > 1$, i.e. on the line $\mu = 0$ outside region II; the Hopf frequency is $\omega = \text{sign}(\nu) \sqrt{\nu^2 - 1}$. The sign of $\omega$ is the same as the sign of $\nu$, from the $\dot{\phi}$ equation in (13). Therefore, we have a Hopf bifurcation with positive frequency along $H_+$ ($\mu = 0$ and $\nu > 1$) and a Hopf bifurcation with negative frequency along $H_-$ ($\mu = 0$ and $\nu < 1$). The bifurcated periodic solutions are stable limit cycles $C_+$ and $C_-$. The Hopf bifurcations of the $P_{\pm}$ and $P^\pm$ points can be studied analogously. The eigenvalues of the Jacobian of the right-hand side of (20) at a fixed point characterize the different bifurcations that the fixed point can undergo. Let $T$ and $D$ be the trace and determinant of $J$. The eigenvalues are given by

$$
\lambda^2 - T\lambda + D = 0 \Rightarrow \lambda = \frac{1}{2}(T \pm \sqrt{Q}), \quad Q = T^2 - 4D.
$$

A Hopf bifurcation takes place for $T = 0$ and $Q < 0$. The equation $T = 0$ at the four points $P_{\pm}$ results in the ellipse $\mu^2 - 4ab\nu^2 + 4a^2\nu^2 = 4a^2$ (see Appendix A for details). This ellipse is tangent to $SN'$ at the point $(\mu, \nu) = (2b, (b^2 - a^2)/a)$. The condition $Q < 0$ is only satisfied by $P_+$ and $P^+_0$ on the elliptic arc $H_0$ from $(\mu, \nu) = (0, -1)$ to $(2b, (b^2 - a^2)/a)$:

$$
\mu = 2ab\nu + 2a\sqrt{1 - a^2\nu^2}, \quad \nu \in [-1, (b^2 - a^2)/a].
$$

The elliptic arc $H_0$ is shown in figure 3. Along this arc a pair of unstable limit cycles $C_0$ and $C^*_0$ are born around the fixed points $P_+$ and $P^+_0$, respectively.
3.3. Codimension-two points

The local codimension-one bifurcations of the fixed points are now completely characterized. There are two curves of saddle-node bifurcations, two curves of pitchfork bifurcations and three Hopf bifurcation curves. These curves meet at five codimension-two points. The analysis of the eigenvalues at these points, and of the symmetry of the bifurcating points \((P_0, P_+ \text{ and } P_-)\), characterizes these points as two degenerate pitchforks \(DP_+ \text{ and } DP_-\), two Takens–Bogdanov bifurcations with \(Z_2\) symmetry, \(TB_+\) and \(TB_-\), and a double Takens–Bogdanov bifurcation \(TB\), as shown in figure 3.

The degenerate pitchforks, \(DP_+ \text{ and } DP_-\), correspond to the transition between supercritical and subcritical pitchfork bifurcations. At these points, a saddle-node curve is born, and only fixed points are involved in the neighboring dynamics. The only difference between \(DP_+\) and \(DP_-\) is the stability of the base state \(P_0\); it is stable outside the circle \(\mu^2 + \nu^2 = 1\) at \(DP_+\) and unstable at \(DP_-\). Schematics of the bifurcations along a one-dimensional path in parameter space around the \(DP_+\) and \(DP_-\) points are illustrated in figure 4. The main difference, apart from the different stability properties of \(P_0, P_+ \text{ and } P_-\), is the existence of the limit cycle \(C_-\) surrounding the three fixed points in case \((b)\), \(DP_-\).

The Takens–Bogdanov bifurcation with \(Z_2\) symmetry has two different scenarios [8], and they differ in whether one or two Hopf curves emerge from the bifurcation point. In our problem, bifurcation point \(TB_+\) has a single Hopf curve, \(H_+\), while the \(TB_-\) point has two Hopf curves, \(H_-\) and \(H_0\), emerging from the bifurcation point. The scenario \(TB_+\) is depicted in figure 5, showing the bifurcation of fixed points in the symmetry breaking of \(SO(2)\) to \(Z_2\) case. Codimension-one bifurcation curves: Hopf \(H_+\), \(H_0\) and \(H_-\), pitchfork \(PF_+ \text{ and } PF_-\), saddle-node \(SN\) and \(SN'\). Codimension-two bifurcation points: degenerate pitchfork \(DP_+ \text{ and } DP_-\), Takens–Bogdanov \(TB_+ \text{ and } TB_-\). \(L\) is the zero-frequency curve in the \(SO(2)\) symmetric case.

Figure 3: Local bifurcations of fixed points in the symmetry breaking of \(SO(2)\) to \(Z_2\) case. Codimension-one bifurcation curves: Hopf \(H_+\), \(H_0\) and \(H_-\), pitchfork \(PF_+ \text{ and } PF_-\), saddle-node \(SN\) and \(SN'\). Codimension-two bifurcation points: degenerate pitchfork \(DP_+ \text{ and } DP_-\), Takens–Bogdanov \(TB_+ \text{ and } TB_-\). \(L\) is the zero-frequency curve in the \(SO(2)\) symmetric case.
Figure 4: Schematics of the degenerate pitchfork bifurcations \((a)\) \(DP_+\) and \((b)\) \(DP_−\). On the left, bifurcation curves emanating from \(DP_±\) in parameter space are shown, along with a closed one-dimensional path (dashed); shown on the right are schematics of the bifurcations along the closed path, starting and ending at \(A\) (\(DP_+\)) and \(B\) (\(DP_−\)). The fixed point curves are labeled with the signs of their eigenvalues. \(C_−\) is the periodic solution born at the curve \(H_−\).
Figure 5: Takens–Bogdanov bifurcation with $Z_2$ symmetry $TB_+$. The top left shows bifurcation curves emanating from $TB_+$ in parameter space, with a closed one-dimensional path. The top right shows a schematic of the bifurcations along the closed path, starting and ending at $A$. The fixed point curves are labeled with the signs of their eigenvalues. $C_+$ is the periodic solution born at the curve $H_+$. The region inside the dashed rectangle on the right contains the states locally connected with the bifurcation $TB_+$. The bottom panels show four numerically computed phase portraits, at points labeled $a$, $b$, $c$ and $d$, for the specified parameter values.
Figure 6: Takens–Bogdanov bifurcation with $Z_2$ symmetry, $\text{TB}_-$. The top left shows bifurcation curves emanating from $\text{TB}_-\text{in parameter space, with a closed one-dimensional path. The top right shows a schematic of the bifurcations along the closed path, starting and ending at A. The fixed point curves are labeled with the signs of their eigenvalues.} C_-\text{ is the periodic solution born at the curve H}_-; C_0^+$ and $C'_0^+$\text{ are the unstable cycles born simultaneously at the Hopf bifurcation H}_0\text{ around the fixed points } P_+\text{ and } P_+^*;\text{ and } C_0\text{ is the cycle around both fixed points that remains after the gluing bifurcation. The bottom panels show three numerically computed phase portraits, at points labeled a, b and c, for } \mu = 0.5 \text{ and } \nu \text{ as indicated, illustrating the gluing and cyclic fold bifurcations.}

The scenario $\text{TB}_-$, is depicted in figure 6, showing the bifurcation diagram as well as the bifurcations along a closed one-dimensional path around the codimension-two point. We have also included the $P_+$ and $P_+^*$ solutions that merge with the $P_-$ and $P_-^*$ fixed points along the saddle-node bifurcation curve SN, although they are not locally connected to the codimension-two point, in order to show all the fixed points in the phase space of (12). A curve of global bifurcations, a heteroclinic cycle Het connecting $P_-$ and $P_-^*$, is born at $\text{TB}_+$. The heteroclinic cycle is formed when the limit cycle $C_+$ simultaneously collides with the saddles $P_-$ and $P_-^*$.
ing the three fixed points $P_0$, $P_+$ and $P^*_+$. The second global bifurcation curve corresponds to a saddle-node of cycles, where $C_0$ and $C_-$ collide and disappear.

A generic Takens–Bogdanov bifurcation (without symmetry) takes place at the TB point on the SN' curve. At the same point in parameter space, but separate in phase space, two $Z_2$ symmetrically related Takens–Bogdanov bifurcations take place, with $P_+$ and $P^*_+$ being the bifurcating states. A schematic of the bifurcations along a one-dimensional path in parameter space around the TB point is shown in figure 7. Apart from the states locally connected to both TB bifurcation, there also exist the base state $P_0$ and the limit cycle $C_-\ldots$

3.4. Global bifurcations

In the analysis of the local bifurcations of fixed points we have found three curves of global bifurcations, a gluing curve Glu and a saddle-node of cycles emerging from TB$_-$ (CF, cyclic-fold), a heteroclinic loop born at TB$_+$ (Het), and a homoclinic loop emerging from TB (Hom). One wonders about the fate of these global bifurcation curves, and about possible additional global bifurcations. Numerical simulations of the solutions of the normal form ODE system (12), or equivalently (20), together with dynamical systems theory considerations have been used to answer these questions, and a schematic of all local and global bifurcation curves is shown in figure 8.

The gluing bifurcation Glu born at TB$_-$ and the two homoclinic loops emerging from the two Takens–Bogdanov bifurcations TB (bifurcations of the symmetric fixed points $P_+$ and $P^*_+$) meet at the point PGl on the circle $P^*_-$ (Het), where the base state $P_0$ undergoes a pitchfork bifurcation (see figure 9a). At that point, the two homoclinic loops of the gluing bifurcation, both homoclinic at the same point on the stable $P_0$ branch, split when the two fixed points $P_-$ and $P^*_-$ bifurcate from $P_0$ (see figure 9b). The two homoclinic loops are then

Figure 7: Double Takens–Bogdanov bifurcation TB: left, bifurcation curves emanating from TB in parameter space, with a closed one-dimensional path; right, schematics of the bifurcations along the closed path, starting and ending at $A$. 

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attached to the bifurcated points and separate along the curve Hom. The large unstable limit cycle $C_0$, after the pitchfork bifurcation $PF_-$, collides simultaneously with both $P_-$ and $P^*_-$, forming a heteroclinic loop along the curve Het$_0$ (see figure 9c and d2). Both curves Hom and Het$_0$ are born at PGl and separate, leaving a region in between where none of the cycles $C_{0^+}$, $C_{0^+}^*$ and $C_0$ exist. The unstable periodic solution $C_0$ merges with the stable periodic solution $C_-$ that was born in $H_-$ and existed in region I, resulting in a cyclic-fold bifurcation of periodic solutions CF. Phase portraits around PGl are shown in figure 9(d).

The curve Hom born at PGl ends at the double Takens–Bogdanov point TB. Locally, around both Takens–Bogdanov bifurcations at TB, after crossing the homoclinic curve the limit cycles $C_{0^+}$ and $C_{0^+}^*$ disappear, and no cycles remain. The formation of a large cycle $C_0$ at Het$_0$ surrounding both fixed points $P_+$ and $P^*_+$ is a global bifurcation involving simultaneously both $P_-$ and $P^*_-$ unstable points. It is the re-injection induced by the presence of the $Z_2$ symmetry that is responsible for this global phenomenon [2, 3, 28, 20]. The two global bifurcation curves Het$_0$ and CF become very close when leaving the PGl neighborhood, and merge at some point in a CFH (Cyclic-Fold–Heteroclinic collision) codimension-two global bifurcation. After CFH, the stable limit cycle $C_-$, instead of undergoing a cyclic-fold bifurcation, directly collides with the saddle points $P_-$ and $P^*_-$ along the Het’ bifurcation curve (see figure 8). In fact Het$_0$ and Het’ are collisions of a limit cycle with $P_-$ and $P^*_-$, but the limit cycle....
Figure 9: Pitchfork-gluing bifurcation PGl. Top left, bifurcation curves emanating from PGl in parameter space; top right, schematics of the bifurcations along the line Glu–Hom. The bottom row shows four numerically computed phase portraits, at points labeled 1, 2, 3 and 4, for the specified parameter values.
(a) $\mu = 1.6, \nu = 0.19$

(b) $\mu = 1.6, \nu = 0.213$

(c) $\mu = 3.0, \nu = 0.19$

(d) $\mu = 3.0, \nu = 1.5858$

Figure 10: Numerically computed phase portraits in the $\epsilon \bar{z}$ case, for $\alpha_0 = 45^\circ$ and $\mu$ and $\nu$ as indicated; cases (a) and (b) are below the SnicHet' point and cases (c) and (d) are above the SnicHet' point.

is on a different branch of the saddle-node of cycles CF on each side of CFH. The two limit cycles born at CF are extremelly close together in the neighborhood of CFH, and it is impossible to see them in a phase portrait, except with a very large zoom around $P_-$ or $P^*_-$.

When increasing $\mu^2 + \nu^2$, the heteroclinic loop Het' born at CFH intersects the SN' curve at a codimension-two global bifurcation point SnicHet'. When they intersect, the saddle-node appears precisely on the limit cycle, resulting in a SNIC bifurcation (a saddle-node on an invariant circle bifurcation). At the SnicHom point, the saddle-node and homoclinic bifurcation curves become tangents, and the saddle-node curve becomes a SNIC bifurcation curve for larger values of $\mu^2 + \nu^2$. Figure 10 shows numerically computed phase portraits for $\alpha_0 = 45^\circ$, below and above the SnicHet', located at $\nu \approx 0.1290$, $\mu \approx 1.505$. In the first case (figure 10a) the saddle-node bifurcation SN' takes place in the interior of $C_-$, while in the second case (figure 10b) SN' happens precisely on top of $C_-$, resulting in a SNIC bifurcation.

From a practical point of view, close to but before the SnicHet' point, the saddle-node bifurcation SN' is very closely followed by the heteroclinic collision.
of the limit cycle $C_-$ with the saddles $P_-$ and $P^*$, and it becomes almost undistinguishable from the SNIC' bifurcation. The scaling laws of the periods when approaching a heteroclinic or a SNIC bifurcation are different, having logarithmic or square root profiles:

$$T_{\text{Het}} = \frac{1}{\lambda} \ln \frac{1}{\mu - \mu_c} + O(1), \quad T_{\text{SNIC}} = \frac{k}{\sqrt{\mu - \mu_c}} + O(1), \quad (23)$$

where $\lambda$ is the positive eigenvalue of the saddle, and $k$ a constant. We have numerically computed the period of $C_-$ at $\nu = 0.6$ for decreasing $\mu$ values approaching SN' in the range $\mu \in [2.0135, 2.2]$, for $\alpha_0 = 45^\circ$. Figure 11(a) shows both fits using the values of the period over the whole computed range. The log fit overestimates the period while the square-root fit underestimates it, and this underestimate gets larger as the heteroclinic collision is approached. Figure 11(b) again shows both fits, but now using values close to the collision for the log fit and values far away from the collision for the square-root fit. Both fits are now very good approximations of the period in their corresponding intervals, and together cover all the values numerically computed. When the interval between the SN' bifurcation and the heteroclinic collision (in figure 11, $\mu_{\text{SN}} = 2.01420$ and $\mu_c = 2.01336$, respectively) is very small, it cannot be resolved experimentally (or even numerically in an extended systems with millions of degrees of freedom, as is the case in fluid dynamics governed by the three-dimensional Navier–Stokes equations). In such a situation the square-root fit looks good enough, because away from the SN' point, the dynamical system just feels the ghost of the about to be formed saddle-node pair and does not distinguish between whether the saddle-node appears on the limit cycle or very close to it. However, if we are able to resolve the very narrow parameter range between the saddle-node formation and the subsequent collision with the saddle, then the log fit matches the period in this narrow interval much better.
Due to the presence of two very close bifurcations (Het’ and SN’), the scaling laws become cross-contaminated, and from a practical point of view the only way to distinguish between a SNIC and a Homoclinic collision is by computing or measuring periods extremely close to the infinite-period bifurcation point. We can also see this from the log fit equation in (23); when both bifurcations are very close, \( \lambda \), the positive eigenvalue of the saddle, goes to zero (it is exactly zero at the saddle-node point), so the log fit becomes useless, except when the periods are very large.

There remains a global bifurcation curve to be analyzed, the heteroclinic loop Het born at TB+. As shown in figure 8, the curve Het intersects the SN curve tangentially at a codimension-two global bifurcation point SnicHet. Beyond this point, the SN curve becomes a line of SNIC bifurcations, where the double saddle-node bifurcations appear on the stable limit cycle \( C^+ \), which disappears on entering the pinning region III, exactly in the same way as has been discussed for the SnicHet’ bifurcation.

We can estimate the width of the pinning region as a function of the magnitude of the imperfection \( \epsilon \) and the distance \( d \) to the bifurcation point, \( w(d, \epsilon) \). The distance \( d \) will be measured along the line \( L \), and the width \( w(d) \) will be the width of the pinning region measured transversally to \( L \) at a distance \( d \) from the origin. It is convenient to introduce coordinates \((u, v)\) in parameter space, rotated an angle \( \alpha_0 \) with respect to \((\mu, \nu)\), so that the parameter \( u \) along \( L \) is precisely the distance \( d \):

\[
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} a\mu + b\nu \\ a\nu - b\mu \end{pmatrix}, \quad \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{24}
\]

where \( a = \sin \alpha_0 \), \( b = \cos \alpha_0 \); in the present problem, \( d = \sqrt{\mu^2 + \nu^2} \) and along \( L \) (of equation \( v = 0 \)), \( d = u \). Figure 12 shows the relationship between the two coordinate systems. If the pinning region is delimited by a curve of equation \( v = \pm h(u) \), then \( w = 2h(u) = 2h(d) \). With an imperfection of the form \( \epsilon \tilde{z} \), the case analyzed in this section, the pinning region is of constant width \( w = 2 \). By restoring the dependence on \( \epsilon \), we obtain a width of value \( w(d, \epsilon) = 2\epsilon \), independent of the distance to the bifurcation point; the width of the pinning region is proportional to \( \epsilon \).
4. Symmetry breaking of $SO(2)$ with an $\epsilon$ term

The normal form to be analyzed is (10) with $p = q = 0$ and $\epsilon = 1$:

$$\dot{z} = z(\mu + i\nu - c|z|^2) + 1.$$  \hfill (25)

The normal form (25), in terms of the modulus and phase $z = re^{i\phi}$, is

$$\dot{r} = r(\mu - ar^2) + \cos \phi,$$

$$\dot{\phi} = \nu - br^2 - \frac{1}{r} \sin \phi.$$  \hfill (26)

4.1. Fixed points and their steady bifurcations

The fixed points of (26) are given by a cubic equation in $r^2$, and so we do not have convenient closed forms for the corresponding roots (the Tartaglia explicit solution is extremely involved). The parameter space is divided into two regions I and II, with one and three fixed points respectively, separated by a saddle-node curve given by (see Appendix B)

$$(u, v) = \left(\frac{3 + 3s^2, 2\sqrt{3}s}{(2 + 6s^2)^{2/3}}, s \in (-\infty, +\infty)\right),$$  \hfill (27)

in $(u, v)$ coordinates (24), and shown in figure 13. The saddle-node curve is divided into three different arcs SN, SN’ and SN₀ by two codimension-two cusp bifurcation points, Cusp±. SN’ corresponds to values $s \in (-\infty, 1)$, SN₀ to $s \in (-1, +1)$ and SN to $s \in (1, +\infty)$. The cusp points Cusp± have values $s = \pm 1$. The curves SN and SN’ are asymptotic to the line L, and region II is the pinning region in this case. The two fixed points that merge on the saddle-node curve have phase space coordinates $z_2 = r_2e^{i\phi_2}$, and the third fixed point is $z_0 = r_0e^{i\phi_0}$, where

$$r_0 = \left(\frac{4}{1 + 3s^2}\right)^{1/3}, \quad r_2 = \left(\frac{1 + 3s^2}{4}\right)^{1/6},$$  \hfill (28)
and the phases are obtained from \( \sin \phi = r(\nu - br^2) \), \( \cos \phi = r(\alpha r^2 - \mu) \).

**Other bifurcations of the fixed points**

The Hopf bifurcations of the fixed points can be obtained by imposing the conditions \( T = 0 \) and \( D > 0 \), where \( T \) and \( D \) are the trace and determinant of the Jacobian of (25). These conditions result in two curves of Hopf bifurcations (see Appendix B for details):

\[
\begin{align*}
(\mu, \nu) &= a^{1/3}(1 - s^2)^{1/3} \left( 2, \frac{b}{a} \frac{s}{\sqrt{1 - s^2}} \right), \\
H_- : s &\in \left( -1, -\sqrt{(1 - b)/2} \right), \\
H_+ : s &\in \left( \sqrt{(1 + b)/2}, 1 \right).
\end{align*}
\]

For \( s \to \pm 1 \) both curves are asymptotic to the \( \mu = 0 \) axis (\( \nu \to \pm \infty \)), the Hopf curve for \( \epsilon = 0 \); the stable limit cycles born at these curves are termed \( C_- \) and \( C_+ \) respectively. The other ends of the \( H_\pm \) curves are on the saddle-node curves of fixed points previously obtained, and at these points \( T = D = 0 \), so they are Takens–Bogdanov points \( TB_\pm \), as shown in figure 14. The \( TB_- \) and \( Cusp_- \) codimension-two bifurcation points are very close, as shown in the zoomed-in figure 14(b). In fact, depending on the angle \( \alpha_0 \), the Hopf curve \( H_- \) is tangent to, and ends at, either \( SN' \) or \( SN_0 \). For \( \alpha_0 > 60^\circ \), \( H_- \) ends at \( SN_0 \), and for \( \alpha_0 = 60^\circ \) \( Cusp_- \) and \( TB_- \) coincide, and \( H_- \) ends at the cusp point, a very degenerate case.

From the Takens–Bogdanov points, two curves of homoclinic bifurcations emerge, resulting in global bifurcations around these points. There are also some additional global bifurcations around these points that we have explored numerically and using dynamical systems theory; these are summarized in figure 15. There are nine codimension-two points organizing the dynamics of the normal form (25). Apart from the cusp and Takens–Bogdanov points already
found, there are five new points: $Ba$, $CfHom$ and three different $Snic-Homoclinic$ bifurcations, $SnicHom_0$, $SnicHom$ and $SnicHom'$. On $H_-$, before crossing the $SN_0$ curve, the Hopf bifurcation becomes subcritical at the Bautin point $Ba$, and from this point a curve of cyclic-folds $CF$ appears. This curve is the limit of the subcritical region where two periodic solutions, $C_-$ and $C_0$, exist and they merge on $CF$. $C_0$ is the unstable limit cycle born in the branch of $H_-$ between $Cusp_-$ and $TB_-$, from now on termed the Hopf curve $H_0$. Inside the pinning region these two periodic solutions disappear when they collide with a saddle fixed point along the curves $Hom_0$ ($C_0$ collision) and $Hom'$ ($C_-$ collision), in the neighborhood of the Takens–Bogdanov point $TB_-$, where the homoclinic curve $Hom_0$ is born. Away from the $TB_-$, for increasing values of $\mu$ and $\nu$, the curve $Hom'$ becomes tangent to and collides with the $SN'$ curve, in a $SnicHom'$ bifurcation, analogous to the $SnicHet'$ bifurcation point found in the $\epsilon \bar{z}$ case, but without the $Z_2$ symmetry. Very close to $SnicHom'$, $SN'$ closely followed by $Hom'$ become indistinguishable from the $SNIC'$ bifurcation, in the same sense as discussed in section §3.4.

The two curves $Hom'$ and $Hom_0$ on approaching $SN_0$, result in a couple of codimension-two bifurcation points, $SnicHom_0$ and $CfHom$. The arc of the curve $SN_0$ between the two new points $SnicHom_0$ and $CfHom$, is a curve of saddle-node bifurcations taking place on the limit cycle $C_0$, resulting in the $SNIC_0$ bifurcation curve, as shown in figure 15, and in more detail in the numerically computed inset figure 16(a). $SnicHom_0$ is exactly the same bifurcation as $SnicHom$ and
Figure 16: (a) Zoom in parameter space around the codimension-two global bifurcation points SnicHom and CfHom. (b) Phase portraits around CfHom; plots 1 to 4 at $\nu = 0.473$, plots 5 to 8 at $\nu = 0.475$, for $\mu$ as specified.
SnicHom’.

The cyclic fold bifurcation curve CF intersects the SNIC\_0 bifurcation curve past the SnicHom\_0 point, i.e. when the SN\_0 curve is a line of SNIC bifurcations, at the point C\_Hom. On the SNIC curve, one of the limit cycles born at CF undergoes a SNIC bifurcation. At the point C\_Hom, the SNIC bifurcation happens precisely when both limit cycles are born at CF: it is saddle-node bifurcation of fixed points taking place on a saddle-node bifurcation of limit circles. After the C\_Hom point, the SNIC curve becomes an ordinary saddle-node bifurcation curve, and there is an additional homoclinic bifurcation curve emerging from this point C\_Hom, Hom’ in figure 15. The two limit cycles born at CF exist only on one side of the CF line, so when following a closed path around the C\_Hom point they must disappear. One of them undergoes a SNIC bifurcation on the SNIC curve and disappears. The other limit cycle collides with the saddle point born at the saddle point curve and disappears at the homoclinic collision Hom’.

Numerically computed phase portraits illustrating these processes around the C\_Hom point are shown in figure 16.

The other Takens–Bogdanov point TB\_+ does not present any additional complications. The homoclinic curve Hom emerging from it approaches and intersect the SN curve in a SnicHom codimension two point, as shown in figure 15, exactly as in the $\epsilon z$ case discussed in §3.

The width of the pinning $w(d, \epsilon)$ region away from the origin (large $s$) is easy to compute from (27):

$$d = u \sim (3s^2/4)^{1/3}, \quad w = 2v \sim (16/\sqrt{3}s)^{1/3} \Rightarrow w = 2/\sqrt{d}.$$  \hspace{1cm} (30)

Restoring the $\epsilon$ dependence, we obtain $w(d, \epsilon) = 2\epsilon/\sqrt{d}$. The pinning region becomes narrower away from the bifurcation point, and the width is proportional to $\epsilon$, the size of the imperfection.

5. Symmetry breaking of $SO(2)$ with quadratics terms

5.1. The $\epsilon z \bar{z}$ case

The normal form to be analyzed in this case is

$$\dot{z} = z(\mu + i\nu - c|z|^2) + z\bar{z},$$  \hspace{1cm} (31)

or in terms of the modulus and phase $z = re^{i\phi}$,

$$\dot{r} = r(\mu - ar^2) + r^2 \cos \phi,$$

$$\dot{\phi} = \nu - br^2 - r \sin \phi.$$  \hspace{1cm} (32)

There are three fixed points: the origin $P_0 \ (r = 0)$ and the two solutions $P_\pm$ of the biquadratic equation $r^4 - 2(a\mu + b\nu + 1/2)r^2 + \mu^2 + \nu^2 = 0$, given by

$$r^2 = a\mu + b\nu + \frac{1}{2} \pm \sqrt{a\mu + b\nu + \frac{1}{4} - (a\nu - b\mu)^2}. \hspace{1cm} (33)$$
These solutions are born at the parabola \( a\mu + b\nu + 1/4 = (a\nu - b\mu)^2 \), and exist only in its interior, which is the pinning region III in figure 17. The parabola is a curve of saddle-node bifurcations. It can be seen (details in Appendix C) that \( P_+ \) is stable while \( P_- \) is a saddle in the whole of region III, so there are no additional bifurcations of fixed points in the \( z\bar{z} \) case, in contrast to the \( z^2 \) case.

As the perturbation is of second order, the Jacobian at \( P_0 \) is the same as in the unperturbed case, and the Hopf bifurcations of \( P_0 \) take place along the horizontal axis \( \mu = 0 \). As in the unperturbed case, the Hopf frequency is negative for \( \nu < 0 \) (\( H^- \)), it is zero at the origin, and becomes positive for \( \nu > 0 \) (\( H^+ \)). The Hopf curves \( H^- \) and \( H^+ \) extend in this case up to the origin, in contrast with the zero and first order cases examined previously, where the Hopf curves ended in Takens–Bogdanov bifurcations without reaching the origin.

The stable limit cycles \( C_- \) and \( C_+ \), born at the Hopf bifurcations \( H_- \) and \( H_+ \), upon entering region III collide with the saddle \( P_+ \) and disappear along two homoclinic bifurcation curves \( \text{Hom and Hom'} \) for small values of \( \mu \). For larger values of \( \mu \), the curves \( \text{Hom and Hom'} \) collide with the parabola at the codimension-two bifurcation points \( \text{SnicHom and SnicHom'} \), and for larger values of \( \mu \), the saddle-node bifurcations take place on the parabola and the limit cycles \( C_- \) and \( C_+ \) undergo SNIC bifurcations, as in the previous \( \epsilon \) and \( \epsilon\bar{z} \) cases. Figure 17(a) summarizes all the local and global bifurcation curves in the \( z\bar{z} \) case, and shows numerically computed phase portraits around the \( \text{SnicHom'} \) point. Figure 17(b) shows the SN and \( \text{Hom'} \) bifurcations before \( \text{SnicHom'} \) (at \( \mu = 0.03 \)), and figure 17(c) illustrates the SNIC' bifurcation after \( \text{SnicHom'} \) (at \( \mu = 0.033 \)).

5.2. The \( \epsilon z^2 \) case

The normal form to be analyzed in this case is (10) with \( p = q = 2 \) and \( \epsilon = 1 \):

\[
\dot{z} = z(\mu + i\nu - c|z|^2) + z^2. \tag{34}
\]

The normal form (34), in terms of the modulus and phase \( z = r e^{i\phi} \), reads

\[
\dot{r} = r(\mu - ar^2) + r^2 \cos \phi,
\]
\[
\dot{\phi} = \nu - br^2 + r \sin \phi. \tag{35}
\]

There are three fixed points: the origin \( P_0 \) \((r = 0)\) and the two solutions \( P_{\pm} \) of the biquadratic equation \( r^4 - 2(a\mu + b\nu + 1/2)r^2 + \mu^2 + \nu^2 = 0 \), which are the same as in the \( \epsilon z\bar{z} \) case (§5.1). In fact, the fixed points in these two cases have the same modulus \( r \) and their phases have opposite sign; changing \( \phi \rightarrow -\phi \) in (32) results in (35). Therefore, the bifurcation curves of the fixed points (excluding Hopf bifurcations of \( P_{\pm} \)) in this case are also given by figure 17(a).

It can be seen (details in Appendix C) that \( P_+ \) is a saddle in the whole of region III, while \( P_- \) is stable for \( \mu > 0 \), unstable for \( \mu < 0 \), and undergoes a Hopf bifurcation \( H_0 \) along the segment of \( \mu = 0 \) delimited by the parabola of saddle-node bifurcations. The points \( \text{TB}_{\pm} \) where \( H_0 \) meets the parabola are
Figure 17: (a) Schematic of bifurcation curves corresponding to the normal form with quadratic terms in the $\epsilon z \ddot{z}$ case. Phase portraits (b) crossing the $\text{Hom}'$ curve at $\mu = 0.03$, for $\nu$ values as specified, and (c) crossing the $\text{SNIC}'$ curve at $\mu = 0.033$. Thick grey lines correspond to the periodic orbit and the homoclinic and heteroclinic loops.
Takens–Bogdanov codimension-two bifurcations. Figure 18 summarize all the local and global bifurcation curves just described.

In the present case, the two Takens–Bogdanov bifurcations and the Hopf bifurcations along $H_0$ are degenerate, as shown in Appendix C. Detailed analysis and numerical simulations show that the Hopf and homoclinic bifurcation curves emerging from the Takens–Bogdanov point are both coincident with the $H_0$ curve previously mentioned. Moreover, the interior of the homoclinic loop is filled with periodic orbits, and no limit cycle exists on either side of $H_0$. This situation is illustrated in the phase portrait in figure 18(b). This highly degenerate situation will be broken by the presence of additional terms in the normal form, and of the continuous family of periodic orbits, only a few will remain. Dumortier et al. [11], who have analyzed in detail the unfolding of such a degenerate case, find that at most two of the periodic orbits survive.

Finally, and exactly in the same way as in the $z\ddot{z}$ case examined in the previous subsection, the stable limit cycles $C_-$ and $C_+$ born respectively at the Hopf bifurcations $H_-$ and $H_+$ and existing in regions $I_-$ and $I_+$ (figure 18a), on entering region III collide with the saddle $P_+$ and disappear along two homoclinic bifurcation curves $\text{Hom}$ and $\text{Hom}'$ for small values of $\mu$. These curves collide with the parabola at the codimension-two bifurcation points $\text{SnHom}_\pm$ and for larger values of $\mu$, the limit cycles $C_-$ and $C_+$ undergo SNIC bifurcations on the parabola. The two curves $\text{Hom}$ and $\text{Hom}'$ emerge from the origin, as in the previous case.
5.3. The $\epsilon \bar{z}^2$ case

The normal form to be analyzed in this case is

$$\dot{z} = z(\mu + i\nu - c|z|^2) + \bar{z}^2,$$

or in terms of the modulus and phase $z = re^{i\phi}$,

$$\dot{r} = r(\mu - ar^2) + r^2 \cos 3\phi,$$
$$\dot{\phi} = \nu - br^2 - r \sin 3\phi.$$ (37)

The fixed points are the origin $P_0$ ($r = 0$) and the solutions of the same bi-quadratic equation as in the two previous cases. However, there is an important difference: due to the factor 3 inside the trigonometric functions in (37), the $P_i^\pm$ points come in triplets ($i = 1, 2, 3$), each triplet has the same radius $r$ but their phases differ by 120°. This is a consequence of the invariance of the governing equation (36) to the $Z_3$ symmetry group generated by rotations of 120° around the origin. This invariance was not present in the two previous cases. The bifurcation curves of the fixed points (excluding the Hopf bifurcations of $P_i^\pm$) are still given by figure 18(a), but now a triplet of symmetric saddle-node bifurcations take place simultaneously on the SN and SN' curves.

It can be seen (details in Appendix C) that $P_i^\pm$ are saddles in the whole of region III, while $P_i^\|$ are stable, except for small angles $\alpha_0 < \pi/6$ in a narrow region close to SN' where they are unstable. For $\alpha_0 > \pi/6$, the bifurcation diagram is exactly the same as in the $z \bar{z}$ case (figure 17a), except that the homoclinic curves are now heteroclinic cycles between the triplets of saddles $P_i^\|$; this case is illustrated in figure 19(a). For $\alpha_0 < \pi/6$, two Takens–Bogdanov bifurcation points appear at the tangency points between the SN' curve and the

Figure 19: Bifurcation curves corresponding to the normal forms with quadratic terms (36) in the $\epsilon \bar{z}^2$ case. (a) $\alpha_0 > 30^\circ$, (b) $\alpha_0 < 30^\circ$. $H_0$ is tangent to the parabola at the Takens bogdanov points $TB_{\pm}$, and $H_0$, $Hom_0$ almost coincide with SN'; in the figure the distances have been exaggerated for clarity.
arc $H_0$ of the ellipse $(b\mu - 2a\nu)^2 + (a\mu - 1)^2 = 1$, as shown in figure 19(b). The arc $H_0$ is a Hopf bifurcation curve of $P_+^i$: three unstable limit cycles $C_0^i$ are born when the triplet $P_+^i$ becomes stable. These unstable limit cycles disappear upon colliding with the saddles $P_0^i$ on a curve of homoclinic collisions, Hom$_0$, that ends at the two Takens–Bogdanov points TB$_+$ and TB$_-$. This situation is very similar to what happens in the $\varepsilon\bar{z}$ case analyzed in section §3, where a Hopf curve $H_0$ appeared close to SN' joining two Takens–Bogdanov points. In both cases, the $SO(2)$ symmetry is not completely broken, but a $Z_m$ symmetry remains. For $\alpha_0 = \pi/6$, the ellipse $H_0$ becomes tangent to SN' and the two Takens–Bogdanov points coalesce, disappearing for $\alpha_0 > \pi/6$.

Finally, the stable limit cycles $C_-$ and $C_+$, born at the Hopf bifurcations $H_-$ and $H_+$, upon entering region III collide simultaneously with the saddles $P_i^i$, $i = 1, 2$ and 3, and disappear along two heteroclinic bifurcation curves Het and Het’ for small values of $\mu$. These curves collide with the parabola at the codimension-two bifurcation points SnicHet and SnicHet’. For larger values of $\mu$ the limit cycles $C_-$ and $C_+$ undergo SNIC bifurcations on the parabola. The two curves Hom and Hom’ emerge from the origin, as in the previous cases.

In the three quadratic cases, the pinning region is delimited by the same parabola $u = v^2 - 1/4$, using the $(u, v)$ coordinates introduced in (24) (see also figure 12). The width of the pinning region is easy to compute, and is given by $w = 2v = 2\sqrt{u+1/4} \sim 2\sqrt{d}$. By using (11), the dependence on $\epsilon$ is restored, resulting in $w(d, \epsilon) = 2\epsilon\sqrt{d}$. The width of the pinning region increases with the distance $d$ to the bifurcation point, and it is proportional to the amplitude of the imperfection $\epsilon$.

6. Symmetry breaking $SO(2) \to Z_m, m \geq 4$

For completeness, and also for intrinsic interest, we will explore the breaking of the $SO(2)$ symmetry to $Z_m$, so that the imperfections added to the normal form (6) preserve the $Z_m$ subgroup of $SO(2)$ generated by rotations of $2\pi/m$ (also called $C_m$). The lowest order monomial in $(z, \bar{z})$ not of the form $z|z|^2$ and equivariant under $Z_m$, is $\varepsilon \bar{z}^{m-1}$, resulting in the normal form

$$\dot{z} = z(\mu + i\nu - c|z|^2) + \epsilon \bar{z}^{m-1}. \quad (38)$$

In terms of the modulus and phase of the complex amplitude $z = re^{i\phi}$, the normal form becomes

$$\dot{r} = r(\mu - ar^2) + \epsilon r^{m-1} \cos m\phi,$$
$$\dot{\phi} = \nu - br^2 - \epsilon r^{m-2} \sin m\phi. \quad (39)$$

The cases $m = 2$ and $m = 3$ have already been examined in §3 and §5.3 respectively. When $m \geq 4$, the term $\epsilon \bar{z}^{m-1}$ is smaller than the remaining terms in (38), so the effect of the symmetry breaking is going to be small compared with the other cases analyzed in this paper. The fixed point solutions of (39), apart from the trivial solution $P_0$ ($r = 0$), are very close the zero-frequency line
Figure 20: Bifurcation curves corresponding to the normal forms retaining a $Z_m$ symmetry (39); (a) corresponds to the $m = 4$ case and (b) corresponds to the $m > 4$ cases.

Figure 21: Crossing the horn for $m = 5$. (a) Fixed point solutions of (39) at the five points in figure 20(b); grey points in 2 and 4 are the saddle-node points, that split in a stable point (black) and saddle (white). (b) Phase portraits corresponding to the five cases in (a).

L in the perfect system. Using the coordinates $(u, v)$ along and orthogonal to L (24), the nontrivial fixed points of (39) satisfy
\[(r^2 - u)^2 = \epsilon^2 r^{2m-4} - v^2.\] (40)

On L, $r^2 = u$; close to L, the fixed points $P_{\pm}$ are given by $r^2 \sim u \pm \sqrt{\epsilon^2 u^{m-2} - v^2}$. The pinning region, at dominant order in $\epsilon$, is $v = \pm \epsilon u^{(m-2)/2}$. This gives a wedge-shaped region around L for $m = 4$ and a horn for $m > 4$, as illustrated in figure 20. On the boundaries of the pinning region, the fixed points merge in saddle-node bifurcations that take place on the limit cycles $C_{\pm}$. These are curves of SNIC bifurcations, exactly the same phenomena that is observed in Neimark-Sacker bifurcations [4], and that we have encountered also in the previous cases analyzed in the present study. Due to the symmetry $Z_m$, from (39) we see that at the boundaries of the wedge, $m$ simultaneous saddle-node bifurcations take place. Figure 21 shows how the fixed points appear and
disappear in saddle-node bifurcations on the limit cycle $C_{\pm}$ when crossing the horn for the $m = 5$ case, at the five points in parameter space indicated in figure 20(b). The nontrivial fixed points, from (39) and $r^2 \sim \mu/a \sim \nu/b$, satisfy
\begin{align*}
r^2 &= \mu/a + \epsilon (\mu/a)^{(m-1)/2} \cos m\phi, \\
r^2 &= \nu/b - \epsilon (\nu/b)^{(m-1)/2} \sin m\phi, \tag{41}
\end{align*}
and the intersection of these circles modulated by the $\sin m\phi$ and $\cos m\phi$ terms is illustrated in figure 21(a). Phase portraits corresponding to the five points in parameter space are also schematically shown in figure 21(b).

The width of the pinning region in the symmetry breaking $SO(2) \to Z_m$ case is obtained from the shape of the horn region and is given by $w(d,\epsilon) = 2\nu = 2\epsilon d^{(m-2)/2}$. Again, as in all the preceding cases, the width of the pinning region is proportional to the amplitude of the imperfection $\epsilon$.

7. Summary and Conclusions

Here we summarize the features that are common to the different perturbations analyzed in the previous sections. The most important feature is that the curve of zero frequency splits into two curves with a region of zero-frequency solutions appearing in between (the so-called pinning region). Of the infinite number of steady solutions that exist along the zero-frequency curve in the perfect system with $SO(2)$ symmetry, only a small finite number remain. These steady solutions correspond to the pinned solutions observed in experiments and in numerical simulations, like the ones to be described in §7.1. The number of remaining steady solutions depends on the details of the symmetry-breaking imperfections, but when $SO(2)$ is completely broken and no discrete symmetries remain, there are three steady solutions in the pinning region III (see figure 22a). One corresponds to the base state, now unstable with eigenvalues $(+,+)$. The other two are born on the SNIC curves delimiting region III away from the origin. Of these two solutions, one is stable (the only observable state in region III) and the other is a saddle (see figure 22b and c). There are also the two Hopf bifurcation curves $H_{-}$ and $H_{+}$. The regions where the Hopf bifurcations meet the infinite-period bifurcations cannot be described in general, and as has been shown in the examples in the previous sections, will depend on the specifics of how the $SO(2)$ symmetry is broken, i.e. on the specifics of the imperfections present in the problem considered. These regions contain complex bifurcational processes, and are represented as grey disks in figure 22(a). The stable limit cycle existing outside III, in regions $I_{\pm}$, undergoes a SNIC bifurcation and disappears upon entering region III (see figure 22b). When the SNIC bifurcation curves approach the Hopf bifurcation curves (i.e. enter the grey disks regions), the saddle-node bifurcations do not occur on the stable limit cycle but very close, and the limit cycle disappears in a saddle-loop homoclinic collision that occurs very close to the saddle-node bifurcations. These homoclinic collisions behave like a SNIC bifurcation, except in a very narrow region in parameter space around the saddle-node curves, as has been discussed in §3.4.
Figure 22: Imperfect Hopf under general perturbations: (a) regions in parameter space; (b) and (c), bifurcation diagrams along the two one-dimensional paths, (1) and (2), respectively. The signs (++, −−, ...) indicate the sign of the real part of the two eigenvalues of the solution branch considered.
In all cases considered, the width of the pinning region scales linearly with the strength of the symmetry breaking $\epsilon$. In all cases, we have found $w(d, \epsilon) = 2\epsilon \Omega d^{(p-1)/2}$, where $p$ is the order of the symmetry breaking considered. For lower order terms, the width decreases ($\epsilon$ case, order zero) or remains constant ($\epsilon \bar{\varepsilon}$ case, order one) with increasing distance from the bifurcation point. For quadratic and higher order terms, the width increases with the distance. When arbitrary perturbations are included, we expect a behavior of the form $w(d, \epsilon) = \epsilon f(d)$, where the function $f$ will depend on the details of the symmetry-breaking terms involved. The size of the regions containing complex bifurcational processes (the grey disks in figure 22a) is of order $\epsilon$ or smaller, as we have seen in all cases considered. Therefore these regions are comparable in size or smaller than the width of the pinning region.

7.1. Comparison with a pinning case from fluid dynamics

Experiments in small aspect-ratio Taylor–Couette flows have reported the presence of a band in parameter space where rotating waves become steady non-axisymmetric solutions (a pinning effect) via infinite-period bifurcations [26]. Previous numerical simulations, assuming $SO(2)$ symmetry of the apparatus, were unable to reproduce these observations [22, 1]. Recent additional experiments suggest that the pinning effect is not intrinsic to the dynamics of the problem, but rather is an extrinsic response induced by the presence of imperfections that break the $SO(2)$ symmetry of the ideal problem. Additional controlled symmetry-breaking perturbations were introduced into the experiment by tilting one of the endwalls [1]. In a very recent paper, Pacheco et al. [23] conducted direct numerical simulations of the Navier–Stokes equations including the tilt of one endwall by a very small angle. Those simulations agree very well with the experiments and the normal form theory developed in this paper. A brief summary of those results follows.

Taylor–Couette flow consists of a fluid confined in an annular region with inner radius $r_i$ and outer radius $r_o$, capped by endwalls a distance $h$ apart. The endwalls and the outer cylinder are stationary, and the flow is driven by the rotation of the inner cylinder at constant angular speed $\Omega$. The system is governed by three parameters: the Reynolds number $Re = \Omega r d/\nu$, the aspect ratio $\Gamma = h/d$, and the radius ratio $\eta = r_i/r_o$ where $\nu$ is the kinematic viscosity of the fluid. Both in the experiments and the numerical simulations, the radius ratio was kept fixed at $\eta = 0.5$. $Re$ and $\Gamma$ were varied, and these correspond to the parameters $\mu$ and $\nu$ in the normal forms studied.

In the parameter region $(Re, \Gamma) \in (300, 860) \times (0.7, 1.6)$, there exists a steady axisymmetric one-vortex state that has a jet of angular momentum emerging from the inner cylinder boundary layer near one of the endwalls. This state, on increasing $Re$, suffers a Hopf bifurcation that breaks the $SO(2)$ axisymmetry and a rotating wave state emerges with azimuthal wave number $m = 2$. For slight variations in aspect ratio, the rotating wave may precess either prograde or retrograde with the inner cylinder. Various different experiments in this regime have been conducted in the nominally perfect system, i.e. with the $SO(2)$ symmetry to within the tolerances in building the apparatus, as well as with a small
Figure 23: Bifurcation diagrams for the one-cell state from (a) the experimental results of Abshagen et al. [1] with the natural imperfections of their system, and (b) the numerical results of Pacheco et al. [23] with a tilt of 0.1° on the upper lid. The dotted curve in both is the numerically determined Hopf curve with zero tilt.

tilt of an endwall [25, 26, 24, 1]. Figure 23(a) shows a bifurcation diagram from the laboratory experiments of Abshagen et al. [1]; these experiments show that without an imposed tilt, the natural imperfections of the system produce a remarkable pinning region, and that the additional tilting of one endwall increases the pinning region. Tilts of the order of 0.1° are necessary in order for the tilt to dominate over the natural imperfections. Figure 23(b) shows a bifurcation diagram from the numerical results of Pacheco et al. [23] in the same problem, showing a remarkable agreement with the experimental results. The effects of imperfections are seen to be only important in the parameter range where the Hopf frequency is close to zero. In this case, a pinning region appears, bounded by infinite period bifurcations of limit cycles. The correspondence of these results with the normal form theory described in the present study is excellent, strongly suggesting that the general remarks on pinning extracted from the study of the five particular cases analyzed are indeed realized both experimentally and numerically. These two studies are the only cases we know where quantitative data about the pinning region are available. Yet, even in these cases the dynamics close to the intersection of the Hopf curve with the pinning region, that according to our analysis should include complicated bifurcational processes, has not been explored, neither numerically nor experimentally. This is a very interesting problem that deserves further exploration.

7.2. Codimension-two bifurcations of limit cycles

The bifurcations that a limit cycle can undergo have been an active subject of research since dynamical systems theory was born. Even in the case of iso-
lated codimension-one bifurcations, a complete classification was not completed until fifteen years ago [31], when the blue-sky catastrophe was found. The seven possible bifurcations are: the Hopf bifurcation, where a limit cycle shrinks to a fixed point, and the length of the limit cycle reduces to zero. Three bifurcations where both the length and period of the limit cycle remain finite: the saddle node of cycles (or cyclic fold), the period doubling and the Neimark-Sacker bifurcations. Two bifurcations where the length remains finite but the period goes to infinity: the collision of the limit cycle with an external saddle forming a homoclinic loop, and the appearance of a saddle-node of fixed points on the limit cycle (the SNIC bifurcation). Finally, there is the blue-sky bifurcation where both the length and period go to infinity, corresponding to the appearance of a saddle-node of limit cycles transverse to the given limit cycle. The seven bifurcations are described in many books on dynamical systems, e.g. Shil’nikov et al. [27], Kuznetsov [20]; they are also described on the web page http://www.scholarpedia.org/article/Blue-sky_catastrophe maintained by A. Shil’nikov and D. Turaev.

Of the seven bifurcations, only four (Hopf, cyclic fold, homoclinic collision and SNIC) are possible in planar systems, as is the case in the present study, and we have found the four of them in the different scenarios explored. We have also found a number of codimension-two bifurcations of limit cycles. For these bifurcations a complete classification is still lacking, and it is interesting to list them because some of the bifurcations obtained are not very common. The codimension-two bifurcations of limit cycles associated to codimension-two bifurcations of fixed points can be found in many dynamical systems textbooks, and include Takens–Bogdanov bifurcations (present in almost all cases considered here) and the Bautin bifurcation (in the $\epsilon$ case). Codimension-two bifurcations of limit cycles associated only to global bifurcations are not so common. We have obtained five of them, that we briefly summarize here.

**CFH** A cyclic-fold and a homoclinic (or heteroclinic) collision occurring simultaneously; see §3.

**PGl** A gluing bifurcation with the saddle point undergoing a pitchfork bifurcation; it may happen in systems with $Z_2$ symmetry; see §3.

**CfHom** A cyclic-fold and a saddle-node occurring simultaneously, with the saddle-node appearing on the limit cycle when it is born at the cyclic-fold bifurcation; see §4.

**SnicHom** A SNIC bifurcation and a homoclinic collision occurring simultaneously; see §5.

**SnicHet** A double SnicHom bifurcation occurring simultaneously in $Z_2$ symmetric systems; see §3.

The SnicHom bifurcation is particularly important in our problem because it separates the two possible scenarios upon entering the pinning region: the stable limit cycle outside may disappear in a homoclinic collision or a SNIC bifurcation.
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Appendix A. Symmetry breaking of \( SO(2) \) to \( Z_2 \): computations

Fixed points

The fixed points of the normal form (12) are given by \( \dot{r} = \dot{\phi} = 0 \). One solution is \( r = 0 \) (named \( P_0 \)). The other fixed points are the solutions of

\[
\begin{align*}
\cos 2\phi &= ar^2 - \mu, \\
\sin 2\phi &= \nu - br^2,
\end{align*}
\]

resulting in the bi-quadratic equation

\[
r^4 - 2(a\mu + b\nu)r^2 + \mu^2 + \nu^2 - 1 = 0,
\]

whose solutions are

\[
r^2 = \frac{a\mu + b\nu \pm \Delta}{2},
\]

\[
e^{2i\phi} = (a\nu - b\mu \mp i\Delta)e^{i\alpha_0},
\]

and every \( \phi_\pm \) admits two solutions, differing by \( \pi \) (they are related by the symmetry \( Z_2, z \to -z \) discussed above). Introducing a new phase \( \alpha_1 \),

\[
a\nu - b\mu - i\Delta = e^{i\alpha_1},
\]

where \( \alpha_1 \in [-\pi, 0] \) because \( \Delta > 0 \), we immediately obtain:

\[
e^{2i\phi_\pm} = e^{i(\alpha_0 \pm \alpha_1)} \Rightarrow \phi_\pm = (\alpha_0 \pm \alpha_1)/2,
\]

with the other solution being \( (\alpha_0 \pm \alpha_1)/2 + \pi \); \( \alpha_1 \) is a function of \( (\mu, \nu) \) while \( \alpha_0 \) is a fixed constant. We have obtained two pairs of \( Z_2 \) symmetric points, \( P_+ = r_+e^{i\phi_+} \) and \( P^*_+ = -r_+e^{i\phi_+} \), and \( P_- = r_-e^{i\phi_-} \) and \( P^*_- = -r_-e^{i\phi_-} \).

Hopf bifurcations of fixed points

The Jacobian of the right-hand side of (20) is given by

\[
J = \begin{pmatrix}
\mu + 1 - 3ax^2 - ay^2 + 2bxy \\
\nu - 3bx^2 - by^2 - 2axy
\end{pmatrix}
\begin{pmatrix}
-\nu + bx^2 + 3by^2 - 2axy \\
\mu - 1 - ax^2 - 3ay^2 - 2bxy
\end{pmatrix}.
\]

The invariants of the Jacobian are the trace \( T \), the determinant \( D \) and the discriminant \( Q = T^2 - 4D \). They are given by

\[
T = 2(\mu - 2ar^2),
\]

\[
D = \mu^2 + \nu^2 - 1 - 4(a\mu + b\nu)r^2 - 3r^4 + 2(a(x^2 - y^2) - 2bxy),
\]

\[
Q = 4\left(1 - \nu^2 + 4b\nu r^2 + (1 - 4b^2) r^4 - 2(a(x^2 - y^2) - 2bxy)\right).
\]
The eigenvalues of the Jacobian matrix (A.6) in terms of the invariants are
\[ \lambda \pm = \frac{1}{2}(T \pm \sqrt{Q}). \]
For example, a Hopf bifurcation takes place iff \( T = 0 \) and \( Q < 0 \). For the fixed points \( P_s \) and \( P^*_s \), where \( s = \pm \), we obtain
\[
T(P_s) = T(P^*_s) = 2((b^2 - a^2)\mu - 2abv - 2as\Delta),
\]
\[
D(P_s) = D(P^*_s) = 4s\Delta r^2,
\]
\[
Q(P_s) = Q(P^*_s) = 4((μ - 2ar^2)^2 - 4s\Delta r^2).
\]
As a result, \( Q(P) = Q(P^*_s) > 0 \) and \( P, P^*_s \) never experience a Hopf bifurcation. After some computations, \( T(P_\pm^+) = 0 \) results in the ellipse
\[
\mu^2 - 4ab\mu v + 4a^2v^2 = 4a^2,
\]
centered at the origin, contained between the straight lines SN and SN’ and passing through the points \((\mu, v) = (0, \pm 1)\), the ends of the horizontal diameter of the circle \( \mu^2 + v^2 = 1 \). This ellipse is tangent to SN and SN’ at the points \((\mu, v) = (\pm 2b, (b^2 - a^2)/a)\). The condition \( Q < 0 \) is only satisfied on the elliptic arc from \((\mu, v) = (0, -1)\) to \((2b, (b^2 - a^2)/a)\) with \( \mu > 0 \); along this arc \( P_\pm^+ \) and \( P^*_\pm \) undergo a Hopf bifurcation. We have assumed that \( a \) and \( b \) are both positive. The properties of the ellipse are:
- major semi-axis \((1 - \ell_\pm)\mu = 2abv\), length \(2a/\sqrt{\ell_\pm}\),
- minor semi-axis \((1 - \ell_\pm)\nu = -2ab\mu\), length \(2a/\sqrt{\ell_\pm}\),
where \( 2\ell_\pm = 1 + 4a^2 \pm \sqrt{1 + 8a^2} \). The eccentricity \( e \) is given by
\[
\frac{2}{e^2} = 1 + \frac{1 + 4a^2}{\sqrt{1 + 8a^2}}.
\]

**Codimension-two bifurcations of fixed points**

The Jacobian evaluated at the three points TB+, TB− and TB is:
\[
J(TB+) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},
\]
\[
J(TB-) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},
\]
\[
J(TB) = \begin{pmatrix} 1 & (1 + b)/a \\ (b - 1)/a & -1 \end{pmatrix}.
\]
The three matrices have double-zero eigenvalues and are of rank one, so the three of them correspond to Takens–Bogdanov bifurcations. The state that bifurcates at the TB point is \( P_+ \), without any symmetry, so that it is an ordinary Takens–Bogdanov bifurcation, although the \( Z_2 \) symmetric state \( P^*_+ \) also bifurcates at the same point in parameter space (but removed in phase space) at another ordinary Takens–Bogdanov bifurcation. The state that bifurcates at the TB±
points is $P_0$. This state is $Z_2$ symmetric, and so these are Takens–Bogdanov bifurcations with $Z_2$ symmetry.

The Jacobian evaluated at the two points $DP_+$ and $DP_-$ is

$$J(DP_+) = \begin{pmatrix} 1 - b & -a \\ a & -1 - b \end{pmatrix}, \quad J(DP_-) = \begin{pmatrix} b + 1 & a \\ -a & b - 1 \end{pmatrix}. \quad (A.20)$$

The corresponding eigenvalues are $\lambda_+ = -2b$ and $\lambda_- = 0$ for $DP_+$, and $\lambda_+ = 2b$ and $\lambda_- = 0$ for $DP_-$. Both points are pitchfork bifurcations, and in order to determine if they are degenerate, their normal form needs to be computed in order to verify that the cubic term is zero. However, since in a degenerate pitchfork bifurcation a curve of saddle-node bifurcations emerges that is tangent to the pitchfork bifurcation curve, from figure 3 it is immediate apparent that both $DP_+$ and $DP_-$ are degenerate pitchfork bifurcations.

**Appendix B. Symmetry breaking of $SO(2)$ with an $\epsilon$ term: computations**

**Fixed points**

The fixed points of the normal form (25) are given by $\dot{r} = \dot{\phi} = 0$, i.e.

$$\begin{cases} 
\cos \phi = r(ar^2 - \mu), \\
\sin \phi = r(\nu - br^2),
\end{cases} \quad \Rightarrow \quad r^2[(\mu - ar^2)^2 + (\nu - br^2)^2] = 1, \quad (B.1)$$

resulting in the cubic equation $f(\rho) = \rho^3 - 2(a\mu + b\nu)\rho^2 + (\mu^2 + \nu^2)\rho - 1 = 0$, where $\rho = r^2$. This equation always has a real solution with $\rho > 0$, and in some regions in parameter space may have three solutions. The curve separating these behaviors is a curve of saddle-node bifurcations, where a couple of additional fixed points are born. This saddle-node curve is given by $f(\rho) = f'(\rho) = 0$; from these equations we can obtain $(\mu, \nu)$ as a function of $\rho$. In order to describe the curve it is better to use the rotated reference frame $(u, v)$, where the $u$-axis coincides with the line $L$, introduced in (24) (see also figure 12). The saddle-node curve is given by

$$(u, v) = \frac{1}{2\rho_2^2} \left( 1 + 2\rho_2^3, \pm \sqrt{4\rho_2^3 - 1} \right), \quad \rho_2 \in (2^{-2/3}, +\infty), \quad (B.2)$$

where $\rho_2$ is the double root of the cubic equation $f(\rho) = 0$. The third root $\rho_0$ is given by $\rho_0 \rho_2^2 = 1$. If there is any point where the three roots coincide (i.e. a cusp bifurcation point, where two saddle-node curves meet), it must satisfy $f(\rho) = f'(\rho) = f''(\rho) = 0$. There are two such points $\text{Cusp}_\pm$, given by $\rho_2 = 1$ and $(u, v)_{\text{Cusp}_\pm} = (3/2, \pm \sqrt{3}/2)$, dividing the saddle-node curve into three branches: $\text{SN}_0$, joining $\text{Cusp}_+$ and $\text{Cusp}_-$, and unbounded branches $\text{SN}$ and $\text{SN'}$ starting at $\text{Cusp}_+$ and $\text{Cusp}_-$ respectively, and becoming asymptotic to the line $L$. Along $\text{SN}_0$, $\rho_2 < 1 < \rho_0$, while along $\text{SN}$ and $\text{SN'}$, $\rho_0 < 1 < \rho_2$. At the cusp points, the three roots coincide and their common value is +1.
A better parametrization of the saddle-node curve is obtained by introducing
\[ s = \pm \sqrt{4(\rho_1^2 - 1)/3}, \]
so that now \( s \in (-\infty, +\infty), \) Cusp\(_{\pm}\) corresponds to \( s = \pm 1 \) and

\[ (u, v) = \left( \frac{3(1 + s^2), 2\sqrt{3}s}{2(1 + 3s^2)} \right)^{2/3}, \quad \rho_0 = \left( \frac{4}{1 + 3s^2} \right)^{2/3}, \quad \rho_2 = \left( \frac{1 + 3s^2}{4} \right)^{1/3}. \] (B.3)

**Hopf bifurcations of the fixed points**

Using Cartesian coordinates \( z = x + iy \) in (25) we obtain

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix} 1 & \mu - \nu \\ \nu & -\mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - (x^2 + y^2) \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix},
\] (B.4)

where we have set \( \epsilon = 1 \). The Jacobian of the right-hand side of (B.4) is given by

\[ J = \begin{pmatrix} \mu - 3ax^2 - ay^2 + 2bxy & -\nu + bx^2 + 3by^2 - 2axy \\ \nu - 3bx^2 - by^2 - 2axy & \mu - ax^2 - 3ay^2 - 2bxy \end{pmatrix}. \] (B.5)

The invariants of the Jacobian are given by

\[ T = 2(\mu - 2\rho^2), \quad D = \mu^2 + \nu^2 - 4(\mu + \nu)\rho^2 + 3\rho^4. \] (B.6)

A Hopf bifurcation takes place iff \( T = 0 \) and \( D > 0 \). When \( T = 0, 4\rho^2D = 4(\alpha\nu - b\mu)^2 - \mu^2 \). The fixed points satisfying \( T = 0 \) are given by \( f(\rho) = 0 \) and \( \mu = 2\rho\), resulting in the curve T in parameter space

\[ 4\alpha\nu(\alpha\nu - b\mu) = 8\alpha^3 - \mu^3, \] (B.7)

that can be parametrized as

\[ (\mu, \nu) = a^{1/3}(1 - s^2)^{1/3} \left( 2, \frac{b}{a} + \frac{s}{\sqrt{1 - s^2}} \right), \quad s \in (-1, +1). \] (B.8)

For \( s = \pm 1, \mu = 0 \) and \( \nu = \pm \infty \), and the curve is asymptotic to the \( \mu = 0 \) axis, the Hopf curve for \( \epsilon = 0 \). Along the T curve, the determinant \( D \) is given by

\[ D = a^{2/3}(1 - s^2)^{-1/3} \left( 2s^2 - 1 - 2\frac{b}{a}s\sqrt{1 - s^2} \right), \] (B.9)

resulting in two Hopf bifurcation curves (when \( D > 0 \)):

\[ H_- : \quad s \in \left(-1, -\sqrt{1-b}/2 \right), \quad H_+ : \quad s \in \left( \sqrt{(1+b)/2}, +1 \right). \] (B.10)

The end points of these curves have \( T = D = 0 \), and are Takens–Bogdanov bifurcation points TB\(_{\pm}\). They are precisely on the saddle-node curve (B.3), where both curves are tangent. The coordinates of the four points Cusp\(_{\pm}\) and TB\(_{\pm}\) are:

\[ (\mu, \nu)_{\text{Cusp}_+} = \frac{3}{2} \left( a - \frac{b}{\sqrt{3}}, b + \frac{a}{\sqrt{3}} \right), \quad (\mu, \nu)_{\text{Cusp}_-} = \frac{3}{2} \left( a + \frac{b}{\sqrt{3}}, b - \frac{a}{\sqrt{3}} \right), \] (B.11)

\[ (\mu, \nu)_{\text{TB}_+} = \frac{(2a, 2b + 1)}{(2(1 + b))^{1/3}}, \quad (\mu, \nu)_{\text{TB}_-} = \frac{(2a, 2b - 1)}{(2(1 - b))^{1/3}}. \] (B.12)
As $TB_\pm$ are on the saddle-node curve, its $\tilde{s}_\pm$ parameter, according to (B.3), can be computed. The result is $\tilde{s}_\delta = \delta \sqrt{(1 - \delta b)/3(1 + \delta b)}$, with $\delta = \pm \epsilon'$. Therefore $TB_+ \in SN_0$, closer to Cusp$_+$ than to Cusp$_-$. $TB_- \in SN'$ if $\alpha_0 < 60^\circ$, $TB_- \in SN_0$ when $\alpha_0 > 60^\circ$ and $TB_- = \text{Cusp}_-$ for $\alpha_0 = 60^\circ$.

Appendix C. Symmetry breaking of $SO(2)$ with quadratic terms: computations

The three cases (32), (35) and (37) can be dealt with by considering the normal form
\[
\dot{r} = r(\mu - ar^2) + r^2 \cos m\phi, \\
\dot{\phi} = \nu - br^2 + r \sin m\phi, \\
\]
where $m = 1$ for the $\epsilon z^2$ case (§5.2), $m = -1$ for the $\epsilon z \bar{z}$ case (§5.1) and $m = -3$ for the $\epsilon^2 z$ case (§5.3). The fixed points, other than the trivial solution $P_0$ ($r = 0$), in the three cases are given by the biquadratic equation $r^4 - 2(a\mu + b\nu + 1/2)r^2 + \mu^2 + \nu^2 = 0$, with solutions $P_\pm$
\[
r^2_\pm = a\mu + b\nu + 1/2 \pm (a\mu + b\nu + 1/4 - (a\nu - b\mu)^2)^{1/2}. \\
\]
The phases $\phi$ of the $P_\pm$ fixed points can be recovered from
\[
\cos m\phi = ar - \mu/r, \quad \sin m\phi = br - \nu/r. \\
\]
For $m = \pm 1$ the solution is unique; for $m = 3$ the solutions come in triples, differing by $2\pi/m$. It is convenient to use the phase space coordinates adapted to line $L$, introduced in (24) (see also figure 12). In terms of these coordinates, $r^2_\pm = u + 1/2 \pm \sqrt{u + 1/4 - v^2}$, and the fixed points $P_\pm$ exist only in the interior of the parabola $u = v^2 - 1/4$, whose axis is the line $L$. On the parabola, these points are born in saddle-node bifurcations. In order to explore additional bifurcations of these points, we compute the Jacobian matrix of the normal form (C.1),
\[
J = \begin{pmatrix}
\mu - 3ar^2 + 2r \cos m\phi & -mr^2 \sin m\phi \\
-2br + \sin m\phi & mr \cos m\phi
\end{pmatrix}, \\
\]
whose trace and determinant, for the $P_\pm$ points, are easily computed:
\[
T(P_\pm) = (m - 1)ar^2_\pm - (m + 1)\mu, \\
D(P_\pm) = -2m\sqrt{u + 1/4 - v^2} \left(\sqrt{(u + 1/2)^2 - u^2 - v^2} \pm (u + 1/2)\right). \\
\]
Therefore $\text{sign} D(P_\pm) = \mp \text{sign} m$, and for $m > 0$ ($m < 0$) only $P_+$ ($P_-$) may undergo a Hopf bifurcation.

The $\epsilon z \bar{z}$ case (§5.1). Here $m = -1$ and $T(P_\pm) = -2ar^2 < 0$, so there are no Hopf bifurcations. Moreover, $D(P_+) > 0$, so it is always stable and $D(P_-) < 0$, so it is a saddle. The only exception is when $r = 0$, and this only happens at $\mu = \nu = 0$, the degenerate high-codimension point at the origin.
The $\epsilon z^2$ case ($\S$5.2). Here $m = 1$, and $T = -2\mu$ is zero on the line $\mu = 0$ inside the parabola. On this line $H_0$, $P_+$ undergoes a Hopf bifurcation, and the points of contact with the parabola have $D = T = 0$ so they are Takens–Bogdanov bifurcations (see figure 18b). The Hopf and Takens–Bogdanov bifurcations are degenerate, as will be discussed in $\S$Appendix C.1.

The $\epsilon z^2$ case ($\S$5.3). Here $m = -3$ and $T = 2\mu - 4ar^2$. $P_-$ is a saddle, but $P_+$ undergoes a Hopf bifurcation when $T = 0$. The condition $T = 0$ for $P_+$ gives

$$\sqrt{a\mu + b\nu + 1/4 - (a\nu - b\mu)^2} = -\frac{1}{2a} \left((a^2 - b^2)\mu + 2ab\mu + a\right) > 0. \quad (C.6)$$

By squaring and simplifying, we obtain the ellipse $(b\mu - 2a\nu)^2 + (a\mu - 1)^2 = 1$ which is tangent to the line $\mu = 0$ at the origin, with its center at $(\mu, \nu) = (2a, b)/(2a^2)$, and whose elements are:

- major semiaxis parallel to $(1 - \ell_-)\mu = 2ab\nu$, length $1/\sqrt{\ell_-}$, \quad (C.7)
- minor semiaxis parallel to $(1 - \ell_-)\nu = -2ab\mu$, length $1/\sqrt{\ell_+}$, \quad (C.8)

where $2\ell_{\pm} = 1 + 4a^2 \pm \sqrt{1 + 8a^2}$. This ellipse has much in common with the one found in the $\epsilon z$ case, and the eccentricity $e$ is given by the same expression (A.16). For $\alpha_0 > \pi/6$, the ellipse is located in the interior of the parabola of saddle-nodes, for $\alpha_0 = \pi/6$ it becomes tangent to the parabola at a single point, and for $\alpha_0 < \pi/6$ it becomes tangent at the two points

$$\mu = \frac{1}{a} \left(1 - 2a^2 - sb\sqrt{1 - 4a^2}\right), \quad (C.9)$$
$$\nu = \frac{2}{a^2} \sqrt{1 - 4a^2} \left(b\sqrt{1 - 4a^2} - s(1 - 2a^2)\right), \quad s = \pm 1.$$ 

These are the points TB in figure 19(b). Only the points on the elliptic arc $H_0$ joining these two points satisfy (C.6), and along this arc $P_+$ undergoes a Hopf bifurcation.

Appendix C.1. A degenerate Takens-Bogdanov bifurcation

In the $\epsilon z^2$ case, numerical simulations of the normal form (34) show that the Hopf bifurcation $H_0$ and the Takens–Bogdanov points $TB_{\pm}$ are degenerate. This can also be found by direct computation. Let us work out the details for the $TB_-$ point.

The coordinates of $TB_-$ in parameter space are $(\mu, \nu) = (0, -0.5/(1 + b)) = (0, 1/4\cos^2(\alpha/2))$, where $P_\pm$ are born in a saddle-node bifurcation, and the fixed points are given by

$$r^2_{\pm} = \frac{1}{2(1 + b)} = \frac{1}{4\cos^2(\alpha/2)}, \quad z_\pm = \frac{c + i}{2(1 + b)} = \frac{1e^{-i\alpha/2}}{4\cos^2(\alpha/2)}. \quad (C.10)$$

40
In order to obtain the normal form corresponding to the Takens–Bogdanov point, a translation of the origin plus a convenient rescaling of \( z \) and time is made:

\[
t = 4\tau \cos(\alpha/2), \quad \zeta = 2(z - z_\pm) \cos(\alpha/2), \quad \bar{\mu} + i\bar{\nu} = 4(\mu + i\nu) \cos^2(\alpha/2). \quad \text{(C.11)}
\]

Substituting in (34) results in

\[
\dot{\zeta} = i\zeta + ie^{-2i\alpha} \zeta + e^{i\alpha/2} \zeta^2 + 2e^{-3i\alpha/2} |\zeta|^2 - ie^{-i\alpha} |\zeta|^2. \quad \text{(C.12)}
\]

In order to obtain the normal form, we introduce the real variables \((x_1, y_1)\)

\[
\zeta = (y_1 + 2ix_1)e^{-i\alpha}, \quad \text{(C.13)}
\]

so that the linear part of the ODE is transformed into Jordan form, and we obtain

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{y}_1
\end{pmatrix} = \begin{pmatrix}
y_1 \\
0
\end{pmatrix} + \cos(\alpha/2) \begin{pmatrix}
2x_1y_1 \\
4x_1^2 + 3y_1^2
\end{pmatrix} + \sin(\alpha/2) \begin{pmatrix}
-2x_1^2 - 3y_1^2/2 \\
4x_1y_1
\end{pmatrix}

- (4x_1^2 + y_1^2) \begin{pmatrix}
x \sin \alpha + (y/2) \cos \alpha \\
y \sin \alpha - 2x \cos \alpha
\end{pmatrix}. \quad \text{(C.14)}
\]

Now we can reduce the quadratic and cubic terms to normal form by an appropriate near-identity quadratic transformation \((x_1, y_1) \rightarrow (x_2, y_2)\). Knobloch [18] gives explicitly the normal form coefficients up to and including third order, in terms of the coefficients of the original ODE (in the form C.14); a nice summary is also given in Wiggins [33], §19.9. Using this explicit transformation, we obtain

\[
\begin{align*}
\dot{x}_2 &= y_2 \\
\dot{y}_2 &= 4x_2^2 \cos(\alpha/2) + 16x_2^3 \cos^2(\alpha/2) + O(4)
\end{align*} \quad \text{(C.15)}
\]

and the \(x_2y_2\) term in the normal form of the Takens–Bogdanov bifurcation is missing, resulting in a degenerate case, the so-called cusp case, of codimension three. The unfolding of this degenerate case has been analyzed in detail in Dumortier et al. [11]. Note that the ODE (C.15) is Hamiltonian at least up to order three, which helps to explain the continuous family of periodic orbits obtained in the interior of the homoclinic loop in figure 18(b).

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