Tests for Comparing Weighted Histograms. Review and Improvements

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Abstract

Histograms with weighted entries are used to estimate probability density functions. Computer simulation is the main application of this type of histograms. A review on chi-square tests for comparing weighted histograms is presented in this paper. Improvements to these tests that have a size closer to its nominal value are proposed. Numerical examples are presented for evaluation and demonstration of various applications of the tests.

Key words: homogeneity test, random sum of random variables, fit weighted histogram, Monte-Carlo simulation.
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1. Introduction

A histogram with $m$ bins for a given probability density function (PDF) $p(x)$ is used to estimate the probabilities $p_i$ that a random event belongs to bin $i$:

$$p_i = \int_{S_i} p(x) dx, \quad i = 1, \ldots, m.$$  

Integration in (1) is carried out over the bin $S_i$ and $\sum_1^m p_i = 1$. A histogram can be obtained as a result of a random experiment with the PDF $p(x)$.

A frequently used technique in data analysis is to compare two distributions through comparison of histograms. The hypothesis of homogeneity states that two histograms represent random values with identical distributions [1]. It is equivalent to the existing $m$ constants $p_1, \ldots, p_m$, such that

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\[ \sum_{i=1}^{m} p_i = 1, \text{ and the probability of belonging to the } i^{\text{th}} \text{ bin for some measured value in both experiments is equal to } p_i. \]

Let us denote the numbers of random events belonging to the \( i^{\text{th}} \) bin of the first and second histogram as \( n_{1i} \) and \( n_{2i} \), respectively. The total number of events in the histograms is equal to \( n_j = \sum_{i=1}^{m} n_{ji} \), where \( j = 1, 2 \).

It has been shown by Pearson [2] that the goodness of fit test statistic

\[ \sum_{i=1}^{m} \frac{(n_{ji} - n_j p_i)^2}{n_j p_i} \]

has approximately a \( \chi^2_{m-1} \) distribution. For two statistically independent histograms with probabilities \( p_1, ..., p_m \), the statistic

\[ \sum_{j=1}^{2} \sum_{i=1}^{m} \frac{(n_{ji} - n_j p_i)^2}{n_j p_i} \]

has approximately a \( \chi^2_{2m-2} \) distribution. If the probabilities \( p_1, ..., p_m \) are not known, the estimation of \( p_i \) is carried out by the following expression:

\[ \hat{p}_i = \frac{n_{1i} + n_{2i}}{n_1 + n_2}, \]

as shown in [1]. By substituting expression (4) in (3), the statistic

\[ X^2 = \sum_{j=1}^{2} \sum_{i=1}^{m} \frac{(n_{ji} - n_j \hat{p}_i)^2}{n_j \hat{p}_i} = \frac{1}{n_1 n_2} \sum_{i=1}^{m} \frac{(n_2 n_{1i} - n_1 n_{2i})^2}{n_{1i} + n_{2i}} \]

is obtained. This statistic has approximately a \( \chi^2_{m-1} \) distribution because \( m - 1 \) parameters are estimated [1]. The statistic (5) was first proposed in [3] and is widely used to test the hypothesis of homogeneity.

A weighted histogram or a histogram with weighted events [4–6] is used to estimate the probabilities \( p_i \) (1) as well. The sum of weights of events for the bin \( i \) is defined as:

\[ W_i = \sum_{k=1}^{n_i} w_i(k), \]

where \( n_i \) is the number of events in the bin \( i \) and \( w_i(k) \) is the weight of the \( k^{\text{th}} \) event in the \( i^{\text{th}} \) bin. The statistic

\[ \hat{p}_i = W_i / n \]

(7)
is used to estimate $p_i$, where $n = \sum_{i=1}^{m} n_i$ is a total number of events for the histogram with $m$ bins. Weights of events are chosen in such a way that the estimate (7) is unbiased:

$$E[\hat{p}_i] = p_i.$$  

(8)

Because of the condition $\sum_i p_i = 1$, we will further call the above defined weights “normalized” as opposed to the unnormalized weights $\tilde{w}_i(k)$ which are $\tilde{w}_i(k) = \text{const} \cdot w_i(k)$.

Comparison of two weighted histograms and comparison of weighted and unweighted histograms as well as fitting weights of simulated random events to an experimental histogram are all important parts of data analysis. Tests for comparing weighted histograms have been developed in [7, 8] while tests for Poisson weighted histograms have been proposed in [9].

This paper is organized as follows. In Section 2 generalization of the chi-square homogeneity test is discussed and improvements for the test are proposed. A test for histograms with unnormalized weights as well as improvements of that test are discussed in Section 3. Tests for comparison of two Poisson weighted histograms are discussed in Section 4. Restrictions for chi-square test application are discussed in Section 5. Applications and verification of the tests are demonstrated using numerical examples in Section 6.

2. Homogeneity test for comparison two histograms with normalized weights

Let us consider two histograms with normalized weights, and the subindex $j$ will be used to differentiate them. A total sum of weights of events $W_{ji}$ in the $i$th bin of the $j$th histogram $j = 1, 2; i = 1, \ldots, m$ can be considered as a sum of random variables

$$W_{ji} = \sum_{k=1}^{n_{ji}} w_{ji}(k),$$  

(9)

where the number of events $n_{ji}$ is also a random value and the weights $w_{ji}(k), k = 1, \ldots, n_{ji}$ are independent random variables with the same PDF for a given bin [4, 6]. Let us introduce a variable

$$r_{ji} = E[w_{ji}] / E[w_{ji}^2],$$  

(10)
which is the ratio of the first moment to the second moment of the distribution of weights in the bin $i$. Let us estimate $r_{ji}$ using

$$
\hat{r}_{ji} = \frac{\sum_{k=1}^{n_{ji}} w_{jki}(k)}{\sum_{k=1}^{n_{ji}} w_{jki}^2(k)}.
$$

(11)

As shown in [4] the statistic

$$
\frac{1}{n_j} \sum_{i \neq k} \frac{\hat{r}_{ji} W_{jki}^2}{p_i} + \frac{1}{n_j} \frac{(n_j - \sum_{i \neq k} \hat{r}_{ji} W_{jki})^2}{1 - \sum_{i \neq k} \hat{r}_{ji} p_i} - n_j,
$$

(12)

where sums extend over all the bins $i$, except for the bin $k$, which has approximately a $\chi^2_{m-1}$ distribution and is a generalization of the Pearson’s statistic [2] [4, 6, 10]. It should be noted that it is only valid for the case when $1 - \sum_{i \neq k} \hat{r}_{ji} p_i > 0$. The last inequality means that estimation of a covariance matrix for variables $W_{j1}, ..., W_{jk-1}, W_{jk+1}, ..., W_{jm}$ is positive definite.

The better power of test, as was shown in [6], was achieved for $k_j$, where

$$
k_j = \arg\min_{i} \frac{\hat{p}_i}{\hat{r}_{ji}}.
$$

(13)

2.1. Median test statistic for comparison of weighted histograms with normalized weights

Following [2], for two statistically independent histograms with probabilities $p_1, ..., p_m$ the statistic has approximately a $\chi^2_{m-1}$ distribution:

$$
\hat{X}_k^2 = 2 \sum_{j=1}^{2} \frac{1}{n_j} \sum_{i \neq k} \frac{\hat{r}_{ji} W_{jki}^2}{\hat{p}_i} + 2 \sum_{j=1}^{n_j} \frac{1}{1 - \sum_{i \neq k} \hat{r}_{ji} \hat{p}_i} - \sum_{j=1}^{2} n_j.
$$

(14)

The probabilities $p_i$ are not known and estimators $\hat{p}_1, \ldots, \hat{p}_{k-1}, \hat{p}_{k+1}, \ldots, \hat{p}_m$ can be determined by minimizing (14) under the following constraints:

$$
\hat{p}_i > 0, 1 - \sum_{i \neq k} \hat{p}_i > 0, 1 - \sum_{i \neq k} \hat{r}_{1i} \hat{p}_i > 0, \text{ and } 1 - \sum_{i \neq k} \hat{r}_{2i} \hat{p}_i > 0.
$$

(15)

The problem to determine the estimators of the probabilities $\hat{p}_i$ by minimizing (14) has been solved numerically by coordinate-wise optimization in [7, 8]. For every step, the minimum for one probability with others fixed ones can be found using the Brent algorithm [11].
A test statistic obtained as a median value of the formula (14) for a different choice of the excluded bin

$$\hat{X}_{Med} = \text{Med} \{ \hat{X}_1^2, \hat{X}_2^2, \ldots, \hat{X}_m^2 \}$$

(16)

was proposed in [7, 8] and has approximately a $\chi^2_{m-1}$ distribution if the hypothesis of homogeneity is valid.

The median is calculated for the set of statistically dependent random variables $\hat{X}_i^2$, with each variable having approximately $\chi^2_{m-1}$ distribution [6, 10]. The median statistic (16) coincides with the statistic (5) in case of two histograms with unweighted entries. Numerical investigations of the median tests (see Section 6.1 and Ref. [4]) show that the size of the test (16) exceeds slightly its nominal value making it the main disadvantage of this approach. The question, what deviation from the nominal size is acceptable for chi-square methods, has different answers.

In the classical work dedicated to chi-square tests [17] disturbance is regarded as unimportant when the nominal size of a test is 5%, with the exact size lying between 4% and 6%, and when the nominal size of a test is 1%, with the exact size lying between 0.7% and 1.5%. According to this criteria the disturbance of the median test can be considered unimportant.

However, according to [4], the disturbance of the median statistics is important. The authors of [9] have proposed tests for comparison of histograms of an equivalent number of unweighted events with false interpretation of these tests as tests for histograms with weighted entries. The methods from [9] are discussed in section 4 with numeric evaluation shown in subsection 6.2.1.

2.2. New test statistic for comparison of weighted histograms with normalized weights

The median test (16) can be improved by using the results for goodness of fit test for weighted histograms [6].

The new test statistic is

$$\hat{X}^2 = \sum_{j=1}^{2} \frac{1}{n_j} \sum_{i \neq k_j} \frac{\hat{r}_{ji} W_{ji}^2}{\hat{p}_i} + \sum_{j=1}^{2} \frac{1}{n_j} \left( \frac{n_j - \sum_{i \neq k_j} \hat{r}_{ji} W_{ji})^2}{1 - \sum_{i \neq k_j} \hat{r}_{ji} \hat{p}_i} - 1 \right) n_j. \quad (17)$$

The estimation of the probabilities $\hat{p}_1, \ldots, \hat{p}_m$ is determined by minimizing (17) under the following constraints:

$$\hat{p}_i > 0, \sum_i \hat{p}_i = 1, 1 - \sum_{i \neq k_1} \hat{r}_{1i} \hat{p}_i > 0, \text{ and } 1 - \sum_{i \neq k_2} \hat{r}_{2i} \hat{p}_i > 0, \quad (18)$$
where \( k_j \) is defined as
\[
\begin{align*}
    k_j &= \arg\min_i \frac{\hat{r}_{ji}}{\hat{p}_i}.
\end{align*}
\] (19)

The test statistic asymptotically has a \( \chi^2_{m-1} \) distribution and a size closer to its nominal value than the test (16) if the hypothesis of homogeneity is valid.

The bin \( k_j \) with the lowest information content is excluded to get the robust statistic \( \hat{X}^2 \) and it is plausible that the test (17) has higher power than the median test (16). Detail explanation of this choice is presented in Subsection 2.3 of [6].

3. Homogeneity test for histograms with unnormalized weights

In practice one is often confronted with cases when a histogram is defined up to an unknown normalization constant. Let us denote bin content of histograms with unnormalized weights as \( \bar{W}_{ji} \), then \( W_{ji} = \bar{W}_{ji} C_j \), and the test statistic (12) can be written as
\[
\begin{align*}
    \frac{C_j}{n_j} \sum_{i \neq k} \hat{r}_{ji} \bar{W}_{ji}^2 \frac{p_i}{p_i} + \frac{1}{n_j} \frac{(n_j - \sum_{i \neq k} \hat{r}_{ji} \bar{W}_{ji})^2}{1 - C_j^{-1} \sum_{i \neq k} \hat{r}_{ji} p_i} - n_j,
\end{align*}
\] (20)

with \( \hat{r}_{ji} = C_j \bar{r}_{ji} \). An estimator \( \hat{C}_{jk} \) for the constant \( C_j \) is found in [4] by minimizing (20) and is equal to
\[
\begin{align*}
    \hat{C}_{jk} &= \sum_{i \neq k} \hat{r}_{ji} p_i + \sqrt{\frac{\sum_{i \neq k} \hat{r}_{ji} p_i}{\sum_{i \neq k} \hat{r}_{ji} \bar{W}_{ji}^2 / p_i}} (n_j - \sum_{i \neq k} \hat{r}_{ji} \bar{W}_{ji}).
\end{align*}
\] (21)

Substituting (21) for (20) and replacing \( \hat{r}_{ji} \) with the estimate \( \hat{\bar{r}}_{ji} \) we get the test statistic
\[
\begin{align*}
    \frac{\hat{C}_{jk}}{n_j} \sum_{i \neq k} \hat{\bar{r}}_{ji} \bar{W}_{ji}^2 \frac{p_i}{p_i} + \frac{1}{n_j} \frac{(n_j - \sum_{i \neq k} \hat{\bar{r}}_{ji} \bar{W}_{ji})^2}{1 - C_j^{-1} \sum_{i \neq k} \hat{\bar{r}}_{ji} p_i} - n_j,
\end{align*}
\] (22)

The estimate \( \hat{\bar{r}}_{ji} \) in (22) is calculated in the same way as the estimate \( \hat{r}_{ji} \) in (11).

The statistic (22) has approximately a \( \chi^2_{m-2} \) distribution.
3.1. Median test statistic for comparison of weighted histograms with unnormalized weights

Following [7], for two statistically independent histograms with probabilities \( p_1, \ldots, p_m \), the statistic

\[
\hat{X}^2_k = \sum_{j=1}^{2} \frac{\hat{C}_{jk}}{n_j} \sum_{i \neq k} \frac{\hat{r}_{ji} \hat{W}_{ji}^2}{\hat{p}_i} + \frac{1}{n_j} \frac{(n_j - \sum_{i \neq k} \hat{r}_{ji} \hat{W}_{ji})^2}{1 - \hat{C}_{jk}^{-1} \sum_{i \neq k} \hat{r}_{ji} \hat{p}_i} - n_j,
\]

has approximately a \( \chi^2\) distribution. An estimation of the probabilities \( \hat{p}_1, \ldots, \hat{p}_{k-1}, \hat{p}_{k+1}, \ldots, \hat{p}_m \) can be found by minimizing (23) under the following constraints:

\[
\hat{p}_i > 0, \quad 1 - \sum_{i \neq k} \hat{p}_i > 0, \quad 1 - \hat{C}_{1k}^{-1} \sum_{i \neq k} \hat{r}_{1i} \hat{p}_i > 0, \quad \text{and} \quad 1 - \hat{C}_{2k}^{-1} \sum_{i \neq k} \hat{r}_{2i} \hat{p}_i > 0.
\]

The probabilities \( \hat{p}_i \) can be calculated numerically in the same way as described in Section 2. A test statistic that is “invariant” to the choice of the excluded bin can be obtained again as a median value of (25) for all possible choices of the excluded bin

\[
\hat{X}^2_{Med} = \text{Med} \{ \hat{X}^2_1, \hat{X}^2_2, \ldots, \hat{X}^2_m \}.
\]

The statistic \( \hat{X}^2_{Med} \) for the case of comparing two histograms with normalized and unnormalized weights can be given by the same formulas (23–25) with \( C_{1k} \equiv 1 \).

Both statistics \( \hat{X}^2_{Med} \) and \( \hat{X}^2_{Med} \) have approximately a \( \chi^2\) distribution if the hypothesis of homogeneity is valid.

3.2. New test statistic for comparison of weighted histograms with unnormalized weights

The median test (25) can be improved by using the results for goodness of fit test for weighted histograms [4].

A new test statistic is

\[
\hat{X}^2 = \sum_{j=1}^{2} \frac{\hat{C}_{jk}}{n_j} \sum_{i \neq k} \frac{\hat{r}_{ji} \hat{W}_{ji}^2}{\hat{p}_i} + \frac{1}{n_j} \frac{(n_j - \sum_{i \neq k} \hat{r}_{ji} \hat{W}_{ji})^2}{1 - \hat{C}_{jk}^{-1} \sum_{i \neq k} \hat{r}_{ji} \hat{p}_i} - n_j.
\]
Estimation of the probabilities $\hat{p}_i$ can be determined by minimizing (26) under the following constraints:

$$\hat{p}_i > 0, \sum_i \hat{p}_i = 1, 1 - \hat{C}_{1k_1}^{-1} \sum_{i\neq k_1} \hat{r}_{1i} \hat{p}_i > 0, \text{ and } 1 - \hat{C}_{2k_2}^{-1} \sum_{i\neq k_2} \hat{r}_{2i} \hat{p}_i > 0,$$

(27)

where $k_j$ is defined as

$$k_j = \arg\min_i \frac{\hat{p}_i}{\hat{r}_{ji}}.$$

(28)

The test statistic asymptotically has a $\chi^2_{m-2}$ distribution and a size closer to its nominal value. It is plausible that the test (26) has higher power than the test (25).

The statistic $\hat{X}^2$ for the case of comparing two histograms with normalized and unnormalized weights can be given by the same formulas (26–28) with $C_{1k_1} \equiv 1$.

Both statistics $\hat{X}^2$ and $\hat{X}^2$ have approximately a $\chi^2_{m-2}$ distribution if the hypothesis of homogeneity is valid.

4. Test for comparison of weighted Poisson histograms

A Poisson histogram [12, 13] is defined as a histogram with multi-Poisson distributions of a number of events for bins:

$$P(n_1, \ldots, n_m) = \prod_{i=1}^{m} e^{-n_0p_i}(n_0p_i)^{n_i}/n_i!,$$

(29)

where $n_0$ is a free parameter.

The probability distribution function (29) can be represented as a product of two probability functions: a Poisson probability distribution function for a number of events $n$ with the parameter $n_0$ and a multinomial probability distribution function of a number of events for bins of the histogram, with a total number of events equal to $n$ [12, 13]:

$$P(n_1, \ldots, n_m) = e^{-n_0}(n_0)^n/n! \times \frac{n!}{n_1!n_2!\ldots n_m!} p_1^{n_1} \ldots p_m^{n_m}.$$

(30)

A Poisson histogram can be obtained as a result of two random experiments, namely when a first experiment with a Poisson probability distribution function gives us a total number of events in the histogram $n$ and then
a histogram is obtained as a result of a random experiment with a PDF \( p(x) \) and with a total number of events equal to \( n \).

The concept of an equivalent number of unweighted events has been introduced in [9]. An equivalent number of unweighted events for \( i^{th} \) bin of weighted histogram is \( W_ir_i \). The authors proposed two test statistics for comparison of histograms with equivalent number of unweighted events contents of bins. These statistics were interpreted in [9] as statistics for comparison of original Poisson weighted histograms.

4.1. First statistic for comparing Poisson weighted histograms

The first statistic \( X_{p1}^2 \), in our notation, can be written as

\[
X_{p1}^2 = C^{-1} \sum_{i=1}^{m} \frac{(W_{1i} - CW_{2i})^2}{W_{1i}r_{2i}^{-1} + W_{2i}r_{1i}^{-1}}. \tag{31}
\]

The parameter \( C \) [9] is taken equal to

\[
C = \frac{\sum W_{1i}}{\sum W_{2i}}. \tag{32}
\]

The statistic (31) according to [9] has a \( \chi^2_m \) distribution if the hypothesis of homogeneity is valid.

4.2. Second statistic for comparing Poisson weighted histograms

The parameter \( C \) can also be estimated [9]. Here an estimator \( \hat{C} \) was found by minimizing (31) and is equal to

\[
\hat{C} = \sqrt{\sum \frac{W_{1i}^2}{W_{1i}r_{2i}^{-1} + W_{2i}r_{1i}^{-1}} \left( \sum \frac{W_{2i}^2}{W_{1i}r_{2i}^{-1} + W_{2i}r_{1i}^{-1}} \right)^{-1}}. \tag{33}
\]

The second statistic

\[
X_{p2}^2 = \hat{C}^{-1} \sum_{i=1}^{m} \frac{(W_{1i} - \hat{CW}_{2i})^2}{W_{1i}r_{2i}^{-1} + W_{2i}r_{1i}^{-1}}. \tag{34}
\]

has a \( \chi^2_{m-1} \) distribution if the hypothesis of homogeneity is valid [9].
5. Restrictions of chi-square test applications

The use of the chi-square test $X^2$ for the histograms with unweighted entries is inappropriate if any expected frequency $n_1\hat{p}_i$ or $n_2\hat{p}_i < 1$ or if the total number of bins with the expected frequency $n_1\hat{p}_i$ or $n_2\hat{p}_i < 5$ exceeds 20% of the total number of bins [14, 17].

Restrictions for weighted histograms can be obtained by replacing the above mentioned expected frequencies with equivalent frequencies of the equivalent number of unweighted events. For the test $\hat{X}^2$ they must be replaced with $n_1\hat{p}_i\hat{r}_{1i}$ and $n_2\hat{p}_i\hat{r}_{2i}$, while for the test $\hat{\check{X}}^2$ with $n_1\hat{p}_i\hat{r}_{1i}/C_{1k_1}$ and $n_2\hat{p}_i\hat{r}_{2i}/C_{2k_2}$.

6. Evaluation of the tests’ sizes and power

The hypothesis of homogeneity $H_0$ is rejected if the value of the test statistic $\hat{X}^2$ is above a given threshold. The threshold $k_\alpha$ for a given nominal size of the test $\alpha$ can be defined from the equation

$$\alpha = P(\chi^2_l > k_\alpha) = \int_{k_\alpha}^{+\infty} \frac{x^{l/2-1}e^{-x/2}}{2^{l/2}\Gamma(l/2)} dx, \quad (35)$$

where $l = m - 1$.

Let us define the test size $\alpha_s$ for a given nominal size of the test $\alpha$ as the probability

$$\alpha_s = P(\hat{X}^2 > k_\alpha|H_0), \quad (36)$$

i.e. the probability that the hypothesis $H_0$ will be rejected if the distribution of the weights $W_{ji}$, $j = 1, 2$; $i = 1, \ldots, m$, for the bins of the histograms satisfies the hypothesis $H_0$. The deviation of a test size from its nominal value is an important test characteristic.

A second important characteristic of the test is its power $\beta$

$$\beta = P(\hat{X}^2 > k_\alpha|H_a), \quad (37)$$

i.e. the probability that the hypothesis of homogeneity $H_0$ will be rejected if the distributions of the weights $W_{ji}$, $j = 1, 2$; $i = 1, \ldots, m$ of the compared histograms do not satisfy the hypothesis $H_0$.

The same definitions with $l = m - 2$ in the formula (35) can be used for the test statistic $\hat{\check{X}}^2$ [26].
Let us consider an example of a weighted histogram for estimation of the probability $p_i$ (1) for a given PDF $p(x)$ in the form

$$p_i = \int_{S_i} p(x) dx = \int_{S_i} w(x)g(x)dx,$$

where

$$w(x) = \frac{p(x)}{g(x)}$$

is a weight function and $g(x)$ is some other PDF. The function $g(x)$ must be $>0$ for the points $x$, where $p(x) \neq 0$. The weight is equal to 0 if $p(x) = 0$.

A weighted histogram is a histogram obtained from a random experiment with the PDF $g(x)$, and the weights of the events are calculated according to (39).

To evaluate a size and power of the tests let us take the distribution

$$p(x) \propto \frac{2}{(x-10)^2 + 1} + \frac{1}{(x-14)^2 + 1}$$

defined on the interval $\{4, 16\}$ and represented by two Breit-Wigner peaks [15]. Three cases of the PDF $g(x)$ can be considered (Fig. 1):

$$g_1(x) = p(x)$$

$$g_2(x) = 1/12$$

$$g_3(x) \propto \frac{2}{(x-9)^2 + 1} + \frac{2}{(x-15)^2 + 1}$$

The distribution $g_1(x)$ (41) results in a histogram with unweighted entries, while the distribution $g_2(x)$ (42) is a uniform distribution on the interval $\{4, 16\}$. The distribution $g_3(x)$ (43) has the same form of parametrization as $p(x)$ (40), but with different values of the parameters.

Sizes of the tests for histograms with a number of bins equal to 5 and different weighted functions were calculated for the nominal size $\alpha$ equal to 0.05.

Calculations of the test sizes $\alpha_s$ were carried out using the Monte Carlo method with 10,000 runs, therefore it is reasonable to test the hypothesis
\( H_0^{(1)} : \alpha_s = 0.05 \) against the alternative \( H_a^{(1)} : \alpha_s \neq 0.05 \). For this purpose z statistics can be used \[16\]

\[
z = \frac{(\hat{\alpha}_s - 0.05)\sqrt{0.05 \times (1 - 0.05)}}{10000}, \tag{44}
\]

where \( \hat{\alpha}_s \) is an estimated value of \( \alpha_s \). If the null hypothesis is true then this test statistic has a standard normal distribution. For the standard normal distribution, 2.5% of the values lie below the critical value of \(-1.959964\), and 2.5% lie above 1.959964. Therefore, if a 2-sided hypothesis test is conducted with a significance level equal to 0.05, \( H_0^{(1)} \) is accepted when \(|z| \leq 1.959964\) or \(0.045728 \leq \hat{\alpha}_s \leq 0.054272\).

The results of calculation for a pair of histograms with either normalized weights or unnormalized weights as well as for two histograms with normalized and unnormalized weights are presented in Tables 1-6 for different weight functions, and different total number of events. To calculate sizes of tests two statistically independent weighted histograms were simulated. The distribution \( p(x) \) \[40\] was used for simulation of the first weighted histogram and the same distribution \( p(x) \) for simulation of the second one.

Weights \( p(x)/g_i(x) \) where used for histograms with normalized weighted entries as well as weights \( 2p(x)/g_i(x) \) and \( 3p(x)/g_j(x) \) for histograms with unnormalized weighted entries.

Powers of the tests were investigated for slightly different values of the amplitude of the second peak of the specified probability distribution function (Fig. 2):

\[
p_0(x) \propto \frac{2}{(x - 10)^2 + 1} + \frac{1.15}{(x - 14)^2 + 1}. \tag{45}\]

6.1. Tests for histograms with a multinomial distribution of events

A size of the tests was calculated for a different total number of events \( n_1 \) and \( n_2 \) in five bin histograms. In the following, numerical examples demonstrate applications of:

- The median test \( X_{Med}^2 \) \[16\] and the new test \( \hat{X}^2 \) \[17\] for comparison of weighted histograms with normalized weights (Table 1);

- The median test \( \hat{X}_{Med}^2 \) \[25\] and the new test \( \hat{X}^2 \) \[26\] for comparison of weighted histograms with unnormalized weights (Table 2);
The median test $\hat{X}^2_{Med}$ and the new test $\hat{X}^2_{Med'}$ for comparison of a weighted histogram with normalized weights and a histogram with unnormalized weights (Table 3).

Figure 1: Probability density functions $g_1(x) = p(x)$, $g_2(x) = 1/12$ and $g_3(x)$.

Figure 2: Probability density functions $p(x)$ (solid line) and $p_0(x)$ (dashed line).
Table 1: Sizes $\hat{\alpha}_s$ of the test $X^2_{Med}$ (16) for comparison of two histograms with normalized weighted entries (left panel) and sizes of the new test $\tilde{X}^2$ (17) (right panel) for different pairs of weights (last column) and numbers of events $n_1, n_2$. Sizes of the tests that do not satisfy the hypothesis $\alpha_s = 5\%$ with a significance level equal to 0.05 ($\hat{\alpha}_s > 5.4\%$ or $\tilde{\alpha}_s < 4.6\%$) are highlighted with gray.

| $n_1$ | 200 | 400 | 800 | 1600 | 3200 | 6400 | 200 | 400 | 800 | 1600 | 3200 | 6400 | $w(z)$ |
|-------|-----|-----|-----|------|------|------|-----|-----|-----|------|------|------|--------|
| 200   | 4.9 | 4.5 | 4.7 | 4.6 | 5.2 | 4.9 | 4.2 | 4.7 | 4.6 | 4.9 | 4.6 | 4.8 | $1$    |
| 400   | 4.7 | 4.7 | 4.8 | 4.7 | 5.1 | 5.0 | 4.4 | 4.7 | 4.4 | 4.8 | 5.2 | 5.2 | $1$    |
| 800   | 4.9 | 4.5 | 4.7 | 4.9 | 5.3 | 5.4 | 4.9 | 4.7 | 4.5 | 4.9 | 5.3 | 4.9 | $1$    |
| 1600  | 5.2 | 4.7 | 5.3 | 4.7 | 5.5 | 5.1 | 4.4 | 4.9 | 5.0 | 4.9 | 4.9 | 5.0 | $1$    |
| 3200  | 4.9 | 5.0 | 5.0 | 4.7 | 4.8 | 4.9 | 5.2 | 5.3 | 5.0 | 4.9 | 4.9 | 4.8 | $1$    |
| 6400  | 5.2 | 5.3 | 5.0 | 5.0 | 5.2 | 5.3 | 5.2 | 5.1 | 4.8 | 5.3 | 4.8 | 5.0 | $1$    |

Table 2: Sizes $\hat{\alpha}_s$ of the test $\tilde{X}^2_{Med}$ (26) for comparison of two histograms with unnormalized weighted entries (left panel) and sizes of the new test $\tilde{X}^2$ (20) (right panel) for different pairs of weights (last column) and numbers of events $n_1, n_2$. Sizes of the tests that do not satisfy the hypothesis $\alpha_s = 5\%$ with a significance level equal to 0.05 ($\hat{\alpha}_s > 5.4\%$ or $\tilde{\alpha}_s < 4.6\%$) are highlighted with gray.

| $n_1$ | 200 | 400 | 800 | 1600 | 3200 | 6400 | $w(z)$ |
|-------|-----|-----|-----|------|------|------|--------|
| 200   | 5.5 | 5.7 | 5.1 | 5.0 | 5.1 | 5.1 | $1$    |
| 400   | 5.5 | 5.7 | 5.4 | 5.2 | 5.2 | 5.2 | $1$    |
| 800   | 5.7 | 5.5 | 5.8 | 4.8 | 5.0 | 5.1 | 5.1 | $1$    |
| 1600  | 5.3 | 5.5 | 5.7 | 5.2 | 4.9 | 4.9 | $1$    |
| 3200  | 5.4 | 5.3 | 5.4 | 5.6 | 5.3 | 5.4 | $1$    |
| 6400  | 5.6 | 5.5 | 5.4 | 5.3 | 5.3 | 5.5 | $1$    |

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Table 3: Sizes $\hat{\alpha}$ of the test $1 \hat{X}^2_Med$ (25) for comparison of two histograms with normalized and unnormalized weighted entries (left panel) and sizes of the new test $1 \hat{X}^2$ (26) (right panel) for different pairs of weights (last column) and numbers of events $n_1$, $n_2$. Sizes of the tests that do not satisfy the hypothesis $\alpha_s = 5\%$ ($\hat{\alpha}_s > 5.4\%$ or $\hat{\alpha}_s < 4.6\%$) with a significance level equal to 0.05 are highlighted with gray.

| $n_1$ | 200 | 400 | 800 | 1600 | 3200 | 6400 | 200 | 400 | 800 | 1600 | 3200 | 6400 | $w(x)$ |
|-------|-----|-----|-----|------|------|------|-----|-----|-----|------|------|------|--------|
| 200   | 5.4  | 5.6  | 5.2  | 5.3  | 5.3  | 5.3  | 4.9  | 4.8  | 4.9  | 5.1  | 4.8  | 4.7  | 1      |
| 400   | 5.7  | 5.8  | 5.4  | 5.3  | 5.3  | 5.3  | 5.3  | 5.0  | 5.0  | 4.8  | 5.1  | 5.0  | 1      |
| 800   | 5.6  | 6.0  | 5.8  | 5.2  | 5.2  | 5.5  | 5.1  | 4.9  | 4.6  | 5.2  | 5.1  | 5.1  | &p;     |
| 1600  | 5.4  | 5.7  | 6.0  | 5.4  | 5.7  | 5.4  | 5.3  | 5.3  | 4.7  | 5.1  | 4.9  | 5.0  | &p;     |
| 3200  | 5.7  | 5.7  | 6.0  | 5.5  | 5.5  | 5.5  | 5.3  | 5.0  | 4.9  | 5.2  | 4.8  | 5.0  | &p;     |
| 6400  | 5.7  | 5.8  | 5.7  | 5.5  | 5.6  | 5.6  | 5.0  | 5.1  | 4.6  | 4.5  | 5.2  | 5.2  | &p;     |

Distributions of p-value were studied by simulating 100 000 runs. In each run 3 200 events were simulated for one histogram and 6 400 events for another one. Distributions were calculated for:

- The median statistic $\hat{X}^2_{Med}$ (16) and the new statistic $\hat{X}^2$ (17) used for comparison of two histograms with normalized weights. The first histogram represents the PDF $p(x)$ with weights of events $p(x)$ and the second histogram represents the PDF $p(x)$ with weights of events $\frac{p(x)}{g_2(x)}$;

- The median statistic $\hat{X}^2_{Med}$ (25) and the new statistic $\hat{X}^2$ (26) used for comparison of two histograms with unnormalized weights. The first histogram represents the PDF $p(x)$ with weights $2p(x)$ and the second histogram represents the PDF $p(x)$ with weights of events $\frac{3p(x)}{g_2(x)}$. 


Figure 3: Distributions of p-value for the median statistics $\hat{X}^2_{Med}$ [16] and the new statistic $\hat{X}^2$ [17] used for comparison of two weighted histograms with normalized weights.
Figure 4: Distributions of p-value for the median statistics $\hat{X}^2_{Med}$ (25) and the new statistic $\hat{X}^2$ (26) used for comparison of two weighted histograms with unnormalized weights.
Conclusions to subsection 6.1

- Tables 1-3
  The sizes $\hat{\alpha}$s of the new tests $\hat{X}^2$ (17), $\hat{\mathcal{X}}^2$, $\hat{\mathcal{X}}^2 (26)$ are closer to a nominal value of a test size equal to 5%, than the sizes of the median statistics $\hat{X}^2_{Med}$ (16), $\hat{\mathcal{X}}^2_{Med}$, $\hat{\mathcal{X}}^2_{Med}$ (25).

- Figure 3
  The distribution of the new statistic $\hat{X}^2$ (17) for comparison of weighted histograms with normalized weights is closer to a $\chi^2_{m-1}$ distribution, than the distribution of the median statistic $\hat{X}^2_{Med}$ (16).

- Figure 4
  The distribution of the new statistic $\hat{\mathcal{X}}^2$ (26) for comparison of weighted histograms with unnormalized weights is closer to a $\chi^2_{m-2}$ distribution, than the distribution of the median statistics $\hat{\mathcal{X}}^2_{Med}$ (25).

6.2. Tests for histograms with Poisson distribution of events

The size and power of the tests were calculated for a different number of events defined by parameters $n_{01}$ and $n_{02}$ of a Poisson distribution in five bin histograms. In the following, numerical examples demonstrate application of:

- The test $X^2_{p2}$ (34) and the new test $\hat{X}^2$ (17) for comparison of weighted histograms with normalized weights (Tables 4 and 7);

- The test $X^2_{p2}$ (34) and the new test $\hat{\mathcal{X}}^2$ (26) for comparison of weighted histograms with unnormalized weights (Tables 5 and 8);

- The test $X^2_{p2}$ (34) and the new test $\hat{\mathcal{X}}^2 (26)$ for comparison of weighted histogram with normalized weights and the histogram with unnormalized weights (Tables 6 and 9).
### 6.2.1. Size of tests for the Poisson weighted histograms

A total number of events for histograms was simulated according to a Poisson distribution with parameters $n_{01}$ and $n_{02}$.

#### Table 4: Sizes $\hat{\alpha}_s$ of the test $X^2_{p_2}$ [34] for comparison of two Poisson histograms with normalized weights (left panel) and sizes of the new test $\hat{X}^2$ [17] (right panel) for different pairs of weights (last column) and parameters $n_{01}$, $n_{02}$. Sizes of the tests that do not satisfy the hypothesis $\alpha_s = 5\%$ with a significance level equal to $0.05$ ($\hat{\alpha}_s > 5.4\%$ or $\hat{\alpha}_s < 4.6\%$) are highlighted with gray.

| $n_{02}$ | 200 | 400 | 800 | 1600 | 3200 | 6400 | $u(x)$ |
|---------|-----|-----|-----|------|------|------|--------|
| $n_{01}$ |     |     |     |      |      |      |        |
| 200     | 4.8 | 5.2 | 5.2 | 4.8  | 4.9  | 4.8  | 1.0    |
| 400     | 4.7 | 4.7 | 4.8 | 5.1  | 4.8  | 4.8  | 2.2    |
| 800     | 4.6 | 4.8 | 4.5 | 5.2  | 5.2  | 4.9  | 4.7    |
| 1600    | 5.1 | 4.9 | 5.1 | 4.4  | 4.6  | 4.9  |        |
| 3200    | 4.5 | 4.8 | 5.0 | 6.5  | 5.0  | 4.8  |        |
| 6400    | 4.9 | 4.5 | 5.3 | 5.2  | 4.9  | 4.7  | 1.0    |

| $n_{02}$ | 200 | 400 | 800 | 1600 | 3200 | 6400 | $u(x)$ |
|---------|-----|-----|-----|------|------|------|--------|
| $n_{01}$ |     |     |     |      |      |      |        |
| 200     | 4.7 | 5.2 | 4.4 | 4.7  | 4.8  | 4.8  | 4.1    |
| 400     | 5.1 | 5.2 | 4.9 | 4.7  | 4.9  | 5.2  |        |
| 800     | 5.2 | 5.2 | 5.0 | 5.0  | 4.6  | 4.9  |        |
| 1600    | 4.9 | 5.3 | 4.7 | 5.1  | 4.7  | 4.7  |        |
| 3200    | 5.2 | 4.9 | 5.0 | 4.7  | 4.9  | 5.2  |        |
| 6400    | 4.9 | 5.2 | 4.4 | 4.7  | 5.0  | 5.0  |        |

#### Table 5: Sizes $\hat{\alpha}_s$ of the test $X^2_{p_2}$ [31] for comparison of two Poisson histograms with unnormalized weighted entries (left panel) and sizes of the new test $\hat{X}^2$ [20] (right panel) for different pairs of weights (last column) and parameters $n_{01}$, $n_{02}$. Sizes of the tests that do not satisfy the hypothesis $\alpha_s = 5\%$ with a significance level equal to $0.05$ ($\hat{\alpha}_s > 5.4\%$ or $\hat{\alpha}_s < 4.6\%$) are highlighted with gray.

| $n_{02}$ | 200 | 400 | 800 | 1600 | 3200 | 6400 | $u(x)$ |
|---------|-----|-----|-----|------|------|------|--------|
| $n_{01}$ |     |     |     |      |      |      |        |
| 200     | 4.8 | 5.0 | 5.2 | 5.4  | 4.8  | 4.8  | 4.2    |
| 400     | 4.7 | 4.7 | 4.8 | 5.1  | 4.8  | 4.8  | 4.2    |
| 800     | 4.6 | 4.8 | 4.5 | 5.2  | 5.2  | 4.9  | 4.7    |
| 1600    | 5.1 | 4.9 | 5.1 | 4.4  | 4.6  | 4.9  | 4.7    |
| 3200    | 4.5 | 4.8 | 5.0 | 6.5  | 5.0  | 4.8  |        |
| 6400    | 4.9 | 4.5 | 5.3 | 5.2  | 4.9  | 4.7  | 1.0    |

| $n_{02}$ | 200 | 400 | 800 | 1600 | 3200 | 6400 | $u(x)$ |
|---------|-----|-----|-----|------|------|------|--------|
| $n_{01}$ |     |     |     |      |      |      |        |
| 200     | 4.7 | 5.2 | 4.4 | 4.7  | 4.8  | 4.8  | 4.1    |
| 400     | 5.1 | 5.2 | 4.9 | 4.7  | 4.9  | 5.2  |        |
| 800     | 5.2 | 5.2 | 5.0 | 5.0  | 4.6  | 4.9  |        |
| 1600    | 4.9 | 5.3 | 4.7 | 5.1  | 4.7  | 4.7  |        |
| 3200    | 5.2 | 4.9 | 5.0 | 4.7  | 4.9  | 5.2  |        |
| 6400    | 4.9 | 5.2 | 4.4 | 4.7  | 5.0  | 5.0  |        |

| $n_{02}$ | 200 | 400 | 800 | 1600 | 3200 | 6400 | $u(x)$ |
|---------|-----|-----|-----|------|------|------|--------|
| $n_{01}$ |     |     |     |      |      |      |        |
| 200     | 4.7 | 4.7 | 4.8 | 4.9  | 4.8  | 4.8  | 1.0    |
| 400     | 5.1 | 4.9 | 5.1 | 4.4  | 4.6  | 4.9  | 4.7    |
| 800     | 5.2 | 5.2 | 5.0 | 5.0  | 4.6  | 4.9  |        |
| 1600    | 4.9 | 5.3 | 4.7 | 5.1  | 4.7  | 4.7  |        |
| 3200    | 5.2 | 4.9 | 5.0 | 4.7  | 4.9  | 5.2  |        |
| 6400    | 4.9 | 5.2 | 4.4 | 4.7  | 5.0  | 5.0  |        |
Table 6: Sizes $\hat{\alpha}_s$ of the test $X^2_{p2}$ \cite{31} for comparison of two Poisson weighted histograms with normalized and unnormalized weighted entries (left panel) and sizes of the new test $\tilde{X}^2_{p2}$ \cite{20} (right panel) for different pairs of weights (last column) and parameters $n_{01}, n_{02}$. Sizes of the tests that do not satisfy the hypothesis $\alpha_s = 5\%$ with a significance level equal to $0.05 (\hat{\alpha}_s > 5.4\%$ or $\hat{\alpha}_s < 4.6\%)$ are highlighted with gray.

| $n_{01}$ | 200 | 400 | 800 | $n_{02}$ | 6400 | 1600 | 3200 | 6400 | $w(x)$ |
|----------|-----|-----|-----|----------|------|------|------|------|-------|
| 200      | 5.1 | 4.5 | 4.8 | 4.5 | 5.2 | 4.4 | 5.0 | 5.1 | 4.7 | 4.9 | 5.4 | 4.5 |
| 400      | 5.2 | 4.8 | 5.1 | 5.3 | 4.7 | 4.7 | 5.1 | 5.1 | 5.2 | 5.1 | 4.5 | 5.4 |
| 800      | 5.1 | 4.9 | 5.1 | 4.6 | 4.9 | 5.1 | 5.1 | 5.2 | 5.1 | 4.6 | 4.7 | 4.8 |
| 1600     | 4.8 | 4.7 | 5.0 | 4.8 | 4.9 | 4.9 | 5.1 | 5.2 | 5.0 | 5.1 | 4.9 | 5.0 |
| 3200     | 4.9 | 4.8 | 5.1 | 4.9 | 4.9 | 4.9 | 5.4 | 4.8 | 4.9 | 4.8 | 5.0 | 4.9 |
| 6400     | 5.2 | 5.1 | 4.9 | 5.0 | 4.8 | 5.0 | 5.0 | 5.1 | 4.9 | 5.1 | 4.7 | 4.7 |
| 200      | 5.1 | 4.9 | 4.8 | 4.6 | 4.8 | 4.8 | 4.9 | 4.8 | 4.8 | 4.9 | 4.7 | 5.0 |
| 400      | 4.7 | 4.8 | 4.4 | 5.2 | 4.7 | 4.9 | 4.8 | 4.6 | 4.9 | 4.5 | 5.3 | 5.1 |
| 800      | 5.2 | 4.9 | 5.2 | 4.9 | 5.2 | 5.2 | 5.3 | 5.0 | 5.4 | 4.8 | 5.2 | 4.8 |
| 1600     | 4.9 | 5.2 | 5.0 | 4.7 | 5.2 | 5.0 | 5.2 | 5.1 | 5.0 | 5.2 | 5.0 | 5.0 |
| 3200     | 5.2 | 4.8 | 4.6 | 4.9 | 5.3 | 4.6 | 5.2 | 4.7 | 4.9 | 5.0 | 4.6 | 4.6 |
| 6400     | 5.0 | 5.1 | 5.0 | 5.1 | 4.8 | 5.0 | 5.0 | 4.9 | 4.9 | 5.1 | 4.7 | 4.7 |
| 200      | 5.2 | 5.1 | 5.5 | 4.7 | 5.2 | 5.2 | 4.8 | 4.7 | 5.1 | 5.3 | 5.0 | 5.4 |
| 400      | 4.9 | 5.4 | 4.9 | 5.2 | 4.9 | 5.3 | 5.2 | 5.1 | 4.8 | 5.2 | 4.9 | 4.7 |
| 800      | 5.2 | 5.2 | 4.9 | 5.3 | 5.1 | 5.1 | 4.7 | 5.3 | 5.4 | 5.2 | 4.5 | 5.3 |
| 1600     | 4.5 | 5.3 | 5.1 | 4.9 | 5.2 | 5.0 | 5.4 | 4.9 | 5.2 | 5.3 | 4.7 | 5.1 |
| 3200     | 5.1 | 4.9 | 4.9 | 4.7 | 4.9 | 4.7 | 5.4 | 5.3 | 4.5 | 5.0 | 5.0 | 5.2 |
| 6400     | 4.7 | 5.2 | 5.1 | 5.1 | 5.4 | 5.4 | 4.9 | 5.4 | 5.2 | 4.8 | 4.9 | 5.1 |
| 200      | 5.3 | 4.7 | 5.2 | 5.1 | 5.1 | 4.9 | 5.5 | 5.7 | 5.4 | 5.2 | 5.0 | 5.7 |
| 400      | 5.0 | 4.9 | 4.9 | 5.0 | 5.0 | 5.0 | 5.1 | 5.4 | 5.3 | 4.9 | 5.0 | 5.1 |
| 800      | 4.9 | 5.2 | 4.8 | 5.0 | 4.6 | 5.3 | 5.2 | 5.5 | 5.1 | 4.8 | 4.6 | 4.9 |
| 1600     | 5.1 | 4.7 | 4.7 | 5.2 | 5.2 | 4.7 | 5.4 | 4.8 | 5.1 | 4.9 | 5.4 | 4.8 |
| 3200     | 4.6 | 4.8 | 5.8 | 5.1 | 5.1 | 5.1 | 5.0 | 5.3 | 5.1 | 5.2 | 4.9 | 4.9 |
| 6400     | 5.3 | 4.9 | 4.9 | 5.2 | 4.9 | 5.2 | 4.8 | 5.4 | 5.0 | 5.0 | 5.2 | 4.9 |
| 200      | 5.1 | 5.1 | 4.7 | 4.8 | 5.0 | 4.9 | 5.4 | 5.9 | 5.1 | 5.4 | 5.4 | 5.4 |
| 400      | 5.0 | 5.2 | 4.9 | 5.3 | 4.6 | 4.9 | 5.0 | 5.2 | 5.4 | 4.8 | 5.0 | 4.9 |
| 800      | 5.2 | 4.8 | 4.8 | 5.1 | 5.3 | 5.1 | 5.1 | 5.3 | 5.0 | 5.4 | 5.1 | 5.3 |
| 1600     | 4.6 | 4.9 | 4.8 | 5.3 | 4.9 | 4.9 | 4.7 | 4.8 | 4.9 | 4.6 | 5.2 | 5.0 |
| 3200     | 5.1 | 5.1 | 5.0 | 5.4 | 4.8 | 4.9 | 5.4 | 4.7 | 5.3 | 5.2 | 4.9 | 4.8 |
| 6400     | 4.8 | 4.9 | 4.9 | 5.1 | 5.3 | 5.1 | 5.2 | 5.3 | 5.6 | 5.0 | 4.7 | 4.6 |
| 200      | 5.1 | 4.9 | 4.9 | 4.8 | 5.1 | 4.9 | 5.3 | 5.4 | 5.7 | 6.0 | 5.6 | 5.0 |
| 400      | 5.0 | 5.2 | 4.9 | 5.3 | 4.6 | 4.9 | 5.0 | 5.2 | 5.3 | 5.0 | 5.4 | 4.7 |
| 800      | 4.7 | 4.9 | 5.1 | 5.2 | 5.3 | 4.9 | 5.4 | 5.1 | 5.0 | 5.4 | 4.8 | 4.9 |
| 1600     | 4.6 | 5.2 | 5.3 | 5.0 | 5.2 | 5.0 | 5.0 | 5.1 | 4.5 | 4.7 | 4.9 | 5.1 |
| 3200     | 5.0 | 4.8 | 4.7 | 4.8 | 5.1 | 5.0 | 5.1 | 4.9 | 5.5 | 5.1 | 5.0 | 5.1 |
| 6400     | 5.0 | 4.9 | 4.9 | 5.0 | 4.9 | 4.8 | 5.4 | 4.8 | 5.1 | 5.2 | 4.9 | 5.1 |

Distribution of p-value was studied by simulating 100 000 runs. In each run a number of events was simulated according to a Poisson distribution with the parameter $n_{01} = 3200$ for the first histogram and $n_{02} = 6400$ for the second one. The first histogram represents the $p(x)$ distribution with weights $\frac{2p(x)}{g_{21}(x)}$ and the second histogram represents the $p(x)$ distribution of event with weights $\frac{3p(x)}{g_{32}(x)}$. To compare two Poisson weighted histograms with unnormalized weights the new statistic $\tilde{X}^2_{p2}$ \cite{20}, the first statistic $X^2_{p1}$ \cite{31} and the second statistic $X^2_{p2}$ \cite{31} were used.
Figure 5: Distributions of p-value for the new statistic $\hat{X}^2$ (26), the first statistic $X^2_{p1}$ (31) and the second statistic $X^2_{p2}$ (34) used for comparison of two Poisson weighted histograms with unnormalized weights.

Conclusions to subsection 6.2.1

- Tables 4-6
The sizes $\hat{\alpha}_s$ of the new tests $\hat{X}^2$ (17), $\hat{X}^2$, $X^2_{p1}$ and the second statistic $X^2_{p2}$ (34) used for comparison of Poisson weighted histograms are close to a nominal value of a test size equal to 5% as well as the sizes of test $X^2_{p2}$ (34) [9].
Figure 5

The distribution of the new statistic \( \hat{X}^2 \) (26) for comparison of weighted histograms with unnormalized weights is close to a \( \chi^2_{m-2} \) distribution while the distribution of the statistic \( X^2_{p2} \) (34) [9] is close to a \( \chi^2_{m-1} \) distribution.

Assumption that the statistic \( X^2_{p1} \) (31) [9] has a \( \chi^2_m \) distribution is wrong and the statistic \( X^2_{p1} \) (31) [9] cannot be recommended for use in data analysis.

6.2.2. Power of tests for comparison of Poisson weighted histograms

Calculation of power was performed for the specified probability distribution function \( p_0(x) \) (45).

Table 7: Power \( \beta \) of the new test \( \hat{X}^2 \) (17) used for comparison of two Poisson histograms with normalized weighted entries (right panel) and the exceedance of power of the test \( \hat{X}^2 \) (17) over the power of the test \( X^2_{p2} \) (34) (left panel) for different pairs of weights (last column) and parameters \( n_{01}, n_{02} \). Cases when the power of the test \( X^2_{p2} \) (34) exceeds the power of the new test \( \hat{X}^2 \) (17) are highlighted with gray.

| \( n_{01} \) | 200 | 400 | 800 | 1600 | 3200 | 6400 | 200 | 400 | 800 | 1600 | 3200 | 6400 | \( w(x) \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 200 | 0.4 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 400 | 0.2 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 800 | -0.1 | -0.01 | -0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 1600 | 0.1 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 3200 | 0.1 | -0.01 | -0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 6400 | 0.2 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 200 | 0.4 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 400 | 0.2 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 800 | -0.1 | -0.01 | -0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 1600 | 0.1 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 3200 | 0.1 | -0.01 | -0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 6400 | 0.2 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 200 | 0.4 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 400 | 0.2 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 800 | -0.1 | -0.01 | -0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 1600 | 0.1 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 3200 | 0.1 | -0.01 | -0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
| 6400 | 0.2 | 0.01 | 0.01 | 0.2 | 0.4 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 |
Table 8: Power $\beta$ of the new test $\hat{X}^2$ (26) used for comparison of two Poisson histograms with unnormalized weighted entries (right panel) and the exceedance of the power of the test $\hat{X}^2$ (20) over the power of the test $X^2_{p2}$ (31) (left panel) for different pairs of weights (last column) and parameters $n_{01}$, $n_{02}$. Cases when the power of the test $X^2_{p2}$ (31) exceeds the power of the new test $\hat{X}^2$ (20) are highlighted with gray.

| $n_{01}$ | 200 | 400 | 800 | 1600 | 3200 | 6400 | 200 | 400 | 800 | 1600 | 3200 | 6400 | $w(x)$ |
|--------|-----|-----|-----|------|------|------|-----|-----|-----|------|------|------|------|
| 200    | 0.6 | 0.6 | 0.6 | 0.6  | 0.6  | 0.6  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |
| 400    | 0.1 | 0.1 | 0.1 | 0.1  | 0.1  | 0.1  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |
| 800    | 0.1 | 0.1 | 0.1 | 0.1  | 0.1  | 0.1  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |
| 1600   | 0.1 | 0.1 | 0.1 | 0.1  | 0.1  | 0.1  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |
| 3200   | 0.1 | 0.1 | 0.1 | 0.1  | 0.1  | 0.1  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |
| 6400   | 0.1 | 0.1 | 0.1 | 0.1  | 0.1  | 0.1  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |

Table 9: Power $\beta$ of the new test $\hat{X}^2$ (20) used for comparison of two Poisson histograms with normalized and unnormalized weighted entries (right panel) and the exceedance of the power of the test $\hat{X}^2$ (20) over the power of the test $X^2_{p2}$ (31) (left panel) for different pairs of weights (last column) and parameters $n_{01}$, $n_{02}$. Cases when the power of the test $X^2_{p2}$ (31) exceeds the power of the new test $\hat{X}^2$ (20) are highlighted with gray.

| $n_{01}$ | 200 | 400 | 800 | 1600 | 3200 | 6400 | 200 | 400 | 800 | 1600 | 3200 | 6400 | $w(x)$ |
|--------|-----|-----|-----|------|------|------|-----|-----|-----|------|------|------|------|
| 200    | 0.1 | 0.1 | 0.1 | 0.1  | 0.1  | 0.1  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |
| 400    | 0.1 | 0.1 | 0.1 | 0.1  | 0.1  | 0.1  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |
| 800    | 0.1 | 0.1 | 0.1 | 0.1  | 0.1  | 0.1  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |
| 1600   | 0.1 | 0.1 | 0.1 | 0.1  | 0.1  | 0.1  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |
| 3200   | 0.1 | 0.1 | 0.1 | 0.1  | 0.1  | 0.1  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |
| 6400   | 0.1 | 0.1 | 0.1 | 0.1  | 0.1  | 0.1  | 0.0 | 0.0 | 0.0 | 0.0  | 0.0  | 0.0  | 2m(x) |

23
Conclusions to subsection 6.2.2

• Tables 7-9

In general, the powers of the new tests $\hat{X}^2$, $\hat{\hat{X}}^2$, and $\hat{X}_p^2$ are greater than the power of the test $X^2_{p2}$ developed for Poisson histograms.

• $X^2_{p2}$ is a test for comparing equivalent number of unweighted events histograms and it cannot be directly interpreted as a test for comparison of original weighted histograms.

As a summary, the numerical examples demonstrate superiority of the new tests for comparison of weighted histograms under existing tests including test applications for Poisson weighted histograms.

7. Conclusions

A review of the chi-square homogeneity tests for comparison of weighted histograms is presented in this work. Bin content of a weighted histogram is considered as a random sum of random variables that permit generalization of the classical homogeneity chi-square test for histograms with weighted entries. Improvements of the chi-square tests with better statistical properties are proposed.

Evaluation of the size and power of tests is done numerically for different types of weighted histograms with a different number of events and different weight functions. In general, the size of the new tests is closer to its nominal value and it is plausible that the power is greater than their power of currently available tests. The presented numerical examples demonstrate the superiority of the new tests over the previously proposed tests for Poisson weighted histograms.

The proposed tests can be used to fit Monte Carlo data to experimental data, to compare experimental data with Monte Carlo data and to compare two Monte Carlo data sets as well as to solve the unfolding problem by reweighting the events.
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