Persistence probabilities of mixed FBM and other mixed processes

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Abstract
We consider the sum of two self-similar centred Gaussian processes with different self-similarity indices. Under the assumption of non-negative correlations and some further minor conditions, we show that the asymptotic behaviour of the persistence probability of the sum is the same as for the process with the greater self-similarity index. In particular, this covers the mixed fractional Brownian motion introduced in (Cheridito 2001 Bernoulli 7 913–34) and shows that the corresponding persistence probability decays asymptotically polynomially with persistence exponent \(1 - \max(1/2, H)\), where \(H\) is the Hurst parameter of the underlying fractional Brownian motion.

Keywords: fractional Brownian motion, Gaussian process, mixed processes, persistence, reproducing kernel Hilbert space, self-similarity, stationary process

1. Introduction

Studying the so-called persistence or survival probabilities of stochastic processes is a classical issue in probability theory. For real-valued processes, persistence in particular concerns the event of staying on a half-line for an untypically long time, and one is interested in the asymptotic behaviour of the probability of this event. For many processes \(X = (X_t)_{t \geq 0}\) of interest, the persistence probability decays asymptotically polynomially to zero, i.e.

\[
P \left( \sup_{t \in [0,T]} X_t \leq 1 \right) = T^{-\theta + o(1)}, \quad T \to \infty,
\]

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for some constant $\theta \in (0, \infty)$, and the aim is to determine the so-called persistence exponent $\theta$ of $X$. This persistence exponent is of special interest in physics as it is an important characteristic of many complex systems in nature, related to how fast the system returns from a disordered initial condition to its stationary state. The analysis of persistence probabilities of various processes has received considerable attention in recent years, both in theoretical physics and mathematics. For an overview of relevant processes and known results from a physics point of view, we recommend the surveys [14, 33, 49] as well as the monographs [35, 47], while we refer to [9] for a survey of the mathematics literature in this context.

Recall that fractional Brownian motion (FBM) $B^H$ with Hurst parameter $H \in (0,1)$ is the unique normalized centred Gaussian process which is $H$-self-similar and has stationary increments. Molchan has shown already in 1999 that $B^H$ satisfies (1) for the persistence exponent $\theta = 1 - H$, see [39], but still up to now, persistence probabilities of FBM and related processes have been studied extensively in theoretical physics and mathematics. For instance, we refer to [41, 42] where the Hausdorff dimension of Lagrangian regular points for the inviscid Burgers equation with FBM initial velocity is related to the two-sided persistence probabilities of integrated FBM, motivated by [52, 54].

FBM has many desirable properties and is therefore subject of applications in a variety of fields with both short-range and long-range dependent phenomena. In particular, it has received large attention in the physics literature, where it is typically called an anomalous diffusion. Such diffusions exhibiting the same properties as FBM have been observed to occur in various systems. For example, diffusions with short-range properties are observed in the motion of charge carriers with certain semiconductors [45, 51], in the spreading of tracer molecules in subsurfaces [50], on random site percolation clusters [28], whereas diffusions with long-range properties are observed in active motion within biological cells [16], spreading in layered velocity fields [34] and in mediated surface exchange [57], see also [55] for further references and applications in physics. Additional mentionable applications in which FBM naturally arises are the modelling of polymers, see [13, 27], where such models appear under the name Edwards model, and the modelling of fluctuating interfaces, where FBM turned out to be an equivalent for the Edwards–Wilkinson model in the case of an initial condition with fully-developed steady-state roughness, see [30].

Several of such diffusions may have two sources of randomness, one with memory and another one either with short or without memory. Mixed processes, in particular mixed FBM as introduced in [17], qualify for such modeling purposes, as the component without memory can be captured by standard Brownian motion and the component with long range effects can be captured by FBM. A concrete example of diffusions with such properties is given by anomalous diffusions with both accelerating and retarding behaviour, see [21] for more detailed discussions, so that it is clear that the above mentioned mixed processes are relevant in physical applications. Moreover, a particular type of such mixed processes has been considered in theoretical physics [60] with the focus on fractal dimensions, see also [48, 56] for further mentionable references. It naturally becomes of both theoretical and practical interest to understand the persistence behaviour of such processes.

The main purpose of this paper is to show that the persistence probability of mixed FBM asymptotically behaves indeed as in (1) and to determine the corresponding persistence exponent. Mixed FBM $M_t^{H, \alpha}$ is defined as

$$M_t^{H, \alpha} := B_t^{1/2} + \alpha B^H_t, \quad t \geq 0,$$

where $\alpha \in \mathbb{R} \setminus \{0\}$, $H \in (0,1)$, $B^H$ is an FBM with Hurst parameter $H$ and $B^{1/2}$ is an independent Brownian motion. Note that this process still has stationary increments, but is not self-similar itself.
Infact, we will derivethedesired persistenceresult for a more general class of sumsof self-similar centred Gaussian processes with different self-similarity indices, covering not only the mixed FBM $M^{H,K}$, but also e.g. the case of completely correlated mixed FBM (ccmFBM) introduced in [20]. Note that the latter process neither is self-similar nor has stationary increments. Thus, our result contributes to the amount of rather rare persistence results for stochastic processes violating both the properties of self-similarity and stationary increments. Self-similarity is a valuable property in the context of persistence as in this case, one is able to apply the so-called Lamperti transformation to get a stationary process, and concerning persistence, many powerful tools are available for the class of stationary centred Gaussian processes, see [8, 11, 18, 19, 23–25]. In particular, combined with non-negative (and non-degenerate) covariances, self-similarity always guarantees the existence of the persistence exponent. In the case that self-similarity is not available, the property of stationary increments turned out to be appropriate as another property that can be used to prove the existence of the persistence exponent, see [5]. Besides, one could derive persistence results even outside of the Gaussian setting if one assumes both self-similarity and stationary increments, see [7, 38].

The outline of the paper is as follows. In section 2, we will introduce the class of mixed self-similar processes that are suitable for our purposes and present our main result that for these processes, the persistence probability decays asymptotically polynomially with the persistence exponent of the self-similar process with the greater self-similarity index. In section 3, we will then use this result to derive persistence results for the (completely correlated) mixed FBM and other explicit mixed processes of interest. Subsequently, in section 4, we will prove the main result. Finally, in section 5, we will give a conclusion of our results.

2. Main result

Recall that for $H > 0$, a stochastic process $(X_t)_{t \geq 0}$ is called $H$-self-similar if $(X_{ct}) \overset{d}{=} (c^H X_t)$. We consider the sum of two self-similar centred Gaussian processes with different self-similarity indices, i.e. $X^H + Y^K$, where $X^H$ is an $H$-self-similar centred Gaussian process, $Y^K$ is a $K$-self-similar centred Gaussian process and $K < H$. The main result of this paper, which is given in the following theorem, states that under the assumption that $X^H$ and $X^H + Y^K$ have non-negative covariance functions, respectively, and that some minor conditions hold as specified below, the persistence probability of $X^H + Y^K$ has—up to terms of lower order—the same asymptotic behaviour as the persistence probability of $X^H$.

**Theorem 1.** For $0 < K < H$, let $X^H$ and $Y^K$ be self-similar centred Gaussian processes with a.s. càdlàg sample paths and self-similarity indices $H$ and $K$, respectively. Let us assume that the covariance functions of the processes $X^H$ and $X^H + Y^K$ are non-negative, respectively, and that the autocovariance function $\tau \mapsto \text{cov}(X^H_t, e^{-\tau H} X^H_1)$ of the Lamperti transform of $X^H$ is continuous, integrable and not the zero function. Further let

$$
\theta_X := - \lim_{T \to \infty} \frac{\log P(\sup_{t \leq T} X^H_t \leq 1)}{\log T}
$$

be the persistence exponent of $X^H$. Then

$$
P \left( \sup_{t \in [0,T]} X^H_t + Y^K_t \leq 1 \right) = T^{-\theta_X + o(1)}, \quad T \to \infty.
$$
The result remains true if one replaces \(Y^K\) by a finite sum of self-similar centred Gaussian processes \(Y_{Ki}\) with self-similarity indices \(K_i < H\), as our proof in section 4 can be easily adapted to this setting. As already mentioned in the introduction, the assumption of non-negative covariances of \(X^H\) (together with the self-similarity and the assumption on the Lamperti transform) guarantees the existence of \(\theta_X\). Note that for the mixed process \(X^H + Y^K\) on the contrary, the condition of non-negative covariances does not yield the existence of a persistence exponent \(\text{a priori}\), as the mixed process is not self-similar anymore. Further note that we do not need any direct assumption on the covariance function of \(Y^K\) nor on the correlation of \(X^H\) and \(Y^K\). Thus, in particular, \(X^H\) and \(Y^K\) do not need to be independent and a persistence exponent of \(Y^K\) does not necessarily have to exist.

3. Mixed FBM and further corollaries

Mixed FBM. Let us now come back to the case of mixed FBM, which we defined in (2). Note that this is a special case of the so-called fractional mixed FBM, which covers all linear combinations of independent FBM with different Hurst parameters, see [22, 36]. Recall that FBM \(B^H\) has the covariance function \((s, t) \mapsto \frac{1}{2}([t^H + s^H - |t - s|^H])\), which is non-negative. Due to the independence of the underlying FBM, this directly implies also the non-negativity of the covariance function of the (fractional) mixed FBM. Note that the continuous and integrable function \(\tau \mapsto \frac{1}{2}(e^{H\tau} + e^{-H\tau} - |e^{\tau/2} - e^{-\tau/2}|^H)\) is the autocovariance function of the Lamperti transform of \(B^H\), given by \((e^{-rH}B^H r)_{r \in \mathbb{R}}\). Regarding persistence, it was shown in [39] that

\[
\mathbb{P}
\left(\sup_{t \in [0, T]} B^H_t \leq 1\right) = T^{-(1-H)+o(1)}, \quad T \to \infty.
\]

This yields the following corollary of theorem 1 for the (fractional) mixed FBM.

Corollary 2. For \(0 < K < H < 1\), let \(B^H\) and \(B^K\) be independent FBM with Hurst parameters \(H\) and \(K\), respectively, and \(a, b \in \mathbb{R}\) with \(ab \neq 0\). Then

\[
\mathbb{P}
\left(\sup_{t \in [0, T]} aB^H_t + bB^K_t \leq 1\right) = T^{-(1-H)+o(1)}, \quad T \to \infty.
\]

In particular, for the mixed FBM as defined in (2), we have

\[
\mathbb{P}
\left(\sup_{t \in [0, T]} M^H_{t, a} \leq 1\right) = T^{-(1-\max\{\frac{1}{2}, H\})+o(1)}, \quad T \to \infty.
\]

Note that the local behaviour of fractional mixed FBM is completely different: in [59], it was shown that \(aB^H + bB^K\) is locally equivalent to \(bB^K\) if and only if \(H - K > 1/4\).

Completely correlated mixed FBM. Recall that corollary 2 assumes the independence of \(B^H\) and \(B^K\). As mentioned in the introduction, theorem 1 also covers the case of ccmFBM. Under this term, it was introduced recently in [20], while the process itself had already been studied as the driving process of an SDE in [37, section 3.2.3]. The definition is as follows. Let \(B^H\) be an FBM with Hurst parameter \(H \in (0, 1)\). Then, there exists a Brownian motion \(W\) such that

\[
B_t^H = \int_0^t k_H(t, s)dW_s, \quad t \geq 0,
\]
where \( k_H \) is the so-called Molchan–Golosov kernel, see [43, section 5.1.3] and (7) (8) below. The ccmFBM \( X_{t}^{H,a,b} \) is given by
\[
X_{t}^{H,a,b} := aW_{t} + bB_t^H, \quad t \geq 0,
\]
where \( a, b \in \mathbb{R} \) with \( ab \neq 0 \) and \( W \) is the same Brownian motion as in (4). Similarly to the fractional mixed FBM, as \( k_{1/2} \equiv 1 \) (see (8)), one can generalize \( X_{t}^{H,a,b} \) to linear combinations \( aB_t^H + bB_t^K \) of FBMs generated by the same Brownian motion \( W \) via the Molchan–Golosov kernels \( k_H \) and \( k_K \) with different Hurst parameters \( H \) and \( K \), which were discussed recently in [44] and which we want to call fractional ccmFBM. Using the Itô-isometry, the fractional ccmFBM has the covariance function
\[
(s, t) \mapsto a^2\mathbb{E}[B^H_s B^H_t] + b^2\mathbb{E}[B^K_s B^K_t] + ab\int_0^\infty \mathbb{E}[(k_H(t, u)k_K(s, u) + k_H(s, u)k_K(t, u))]du.
\]
Set \( C(H) := \sqrt{\frac{2H(\frac{1}{2}-H)\Gamma(H+\frac{1}{2})}{12-24H}} \). Then, for \( H > 1/2 \) and \( 0 < s < t \), we have
\[
k_H(t,s) = \frac{C(H)}{(1-H)^{\frac{1}{2}}} t^{-\frac{1}{2}} + \int_s^t u^{-H-\frac{1}{2}}(u-s)^{H-\frac{1}{2}}du \geq 0,
\]
whereas for \( H \leq 1/2 \) and \( 0 < s < t \), it holds
\[
k_H(t,s) = \frac{C(H)}{(1-H)^{\frac{1}{2}}} t^{-\frac{1}{2}} + \left( \frac{1}{2} - H \right) \int_s^t u^{-H-\frac{1}{2}}(u-s)^{H-\frac{1}{2}}du \geq 0.
\]
Thus, the covariance function of the (fractional) ccmFBM is non-negative, if \( ab > 0 \), and theorem 1 together with (3) gives the following corollary.

**Corollary 3.** For \( 0 < K < H < 1 \) and a Brownian motion \( W \), define \( B^H_t := \int_0^t k_H(t, s)dW_s \) and \( B^K_t := \int_0^t k_K(t, s)dW_s \). Further let \( a, b \in \mathbb{R} \) with \( ab > 0 \). Then
\[
\mathbb{P}\left( \sup_{t \in [0,T]} aB^H_t + bB^K_t \leq 1 \right) = T^{-(1-H)+o(1)}, \quad T \to \infty.
\]
In particular, for the ccmFBM as defined in (5), we have
\[
\mathbb{P}\left( \sup_{t \in [0,T]} X_{t}^{H,a,b} \leq 1 \right) = T^{-(1-\max\{\frac{1}{2},H\})+o(1)}, \quad T \to \infty.
\]

Further important self-similar centred Gaussian processes are integrated FBM and fractionally integrated Brownian motion, also called Riemann–Liouville process. Similarly to mixed FBM, one can define mixed integrated FBM and mixed Riemann–Liouville processes.

**Mixed integrated FBM.** Let us first consider the case of integrated FBM. For \( H \in (0,1) \), let \( B^H_t \) be an FBM. Integrated FBM \( B^H_t \) is then defined as
\[
B^H_t := \int_0^t B^H_s ds, \quad t \geq 0.
\]
As $B^H$ is $H$-self-similar and has a non-negative covariance function, the so-defined $t^H$ is $(1 + H)$-self-similar and has again a non-negative covariance function. The autocovariance function of the Lamperti transform of $t^H$, given by $(e^{-\tau(1+H)}t^H)_\tau \in \mathbb{R}$, can be found in [40, lemma 2], and one easily sees that this is indeed a continuous and integrable function. As already mentioned in the context of theorem 1, this guarantees the asymptotic behaviour of the persistence probability of $t^H$ as in (1) with some persistence exponent $\theta(H) \in (0, \infty)$. However, the value $\theta(H)$ is unknown unless $H = 1/2$: for integrated Brownian motion $I^{1/2}$, one could show using Markov techniques that $\theta(I^{1/2}) = 1/4$ (see [26, 29, 53]). For the general case, Molchan and Khoklov stated in [42] the conjecture that $\theta(H) = H(1 - H)$. In [6], it was shown that $\theta_I$ is a continuous function and asymptotically equivalent to the conjectured $H(1 - H)$ for $H \to 0$ and $H \to 1$.

Therefore, theorem 1 yields the following corollary for mixed integrated FBM.

**Corollary 4.** For $0 < K < H < 1$, let $B^H$ and $B^K$ be independent FBM with Hurst parameters $H$ and $K$, respectively, and $a, b \in \mathbb{R}$ with $ab \neq 0$. Let $t_I^H = \int_0^t B^H_i ds$ and $t_K^K = \int_0^t B^K_i ds$. Further let $\theta_I : (0, 1) \to (0, \infty)$ denote the persistence exponent of integrated FBM depending on the Hurst parameter. Then

$\mathbb{P} \left( \sup_{t \in [0, T]} a t_I^H + b t_K^K \leq 1 \right) = T^{-\theta_I(H)+\alpha(1)}, \quad T \to \infty.$

Of course, the same result also holds for the integral of (fractional) cmnFBM, again in the case $ab > 0$, as the only difference in verifying the assumptions of theorem 1 is that the covariance function of the mixed process has additional summands. But these are given as the double integral of the additional summands in (6), which is again non-negative if $ab > 0$.

**Mixed Riemann–Liouville processes.** As a last example, we want to consider mixed Riemann–Liouville processes, which were introduced in [15, section 8]. For a Brownian motion $W$ and $H > 0$, define

$$R^H_t := \int_0^t (t-s)^{H-\frac{1}{2}} dW_s, \quad t \geq 0,$$

to be the Riemann–Liouville fractional integral of $W$. Note that $R^{n+1/2}/n!$ for $n \in \mathbb{N}$ is simply the $n$-times integrated Brownian motion $W$. Thus, the Riemann–Liouville process $R^H$ for $H > 0$ is a fractionally integrated Brownian motion. Further note that for $H \in (0, 1)$, the process $R^H$ is closely related to FBM $B^H$ via the Mandelbrot–van Ness integral representation, which states that

$$R^H_t + \int_{-\infty}^t (t-s)^{H-\frac{3}{2}} dW_s = (t-s)^{H-\frac{1}{2}} dW_s, \quad t \geq 0,$$

is an independent decomposition of FBM (with a non-normalized variance), see e.g. [37, theorem 1.3.1]. The Riemann–Liouville process $R^H$ is an $H$-self-similar centred Gaussian process with a non-negative covariance function. Thus, the persistence probability of $R^H$ decays asymptotically polynomially with some persistence exponent $\theta_R : (0, \infty) \to (0, \infty)$ depending on $H$. However, similarly to $t^H$, the exact value is unknown except for the Brownian cases $\theta_R(1/2) = 1/2$ (Brownian motion) and $\theta_R(3/2) = 1/4$ (integrated Brownian motion). In [4], it was shown that $\theta_R$ is non-increasing, while in [6], it was proven that $\theta_R$ is continuous, tends to $\infty$ and is in the range $H^{-1}$ to $H^{-2}$ for $H \to 0$. 
The autocovariance function of the Lamperti transform of $\mathcal{R}^H$, given by $(e^{-\tau H} \mathcal{R}^H)_\tau \in \mathbb{R}$, can be found in [32, equation (12)], and one assures oneself that this is again a continuous and integrable function. Thus, theorem 1 yields the following corollary.

**Corollary 5.** For $0 < K < H$ and independent Brownian motions $W^{(1)}$ and $W^{(2)}$, define $\mathcal{R}^H := \int_0^T (t-s)^{\frac{1}{2}} dW_s^{(1)}$ and $\mathcal{R}^K := \int_0^T (t-s)^{K-\frac{1}{2}} dW_s^{(2)}$. Let $a, b \in \mathbb{R}$ with $ab \neq 0$ and $\theta_R : (0, \infty) \rightarrow (0, \infty)$ denote the persistence exponent of the Riemann–Liouville process depending on the Hurst parameter. Then

$$
\mathbb{P}
\left( \sup_{c \in [0,T]} a\mathcal{R}^H_c + b\mathcal{R}^K_c \leq 1 \right) = T^{-\theta_R(\delta)+o(1)}, \quad T \rightarrow \infty.
$$

Again, in the case $ab > 0$, the same result also holds for the completely correlated mixed Riemann–Liouville process, where $\mathcal{R}^H$ and $\mathcal{R}^K$ are generated by the same Brownian motion (instead of two independent Brownian motions), as the covariance function of the mixed process gets additional summands which are non-negative.

### 4. Proof of the main result

In this section, we give the proof of theorem 1. The main idea is as follows. We restrict the interval $[0, T]$ of persistence to an interval $[a(T), T]$, where $a(T)$ has to be small enough such that the asymptotic order of the persistence probability does not change and large enough such that we are able to control the range of the process $Y^K$ on the interval $[a(T), T]$. It turns out that $a(T) := (\log T)^p$ for $p$ large enough is a suitable choice. The following lemma shows that the probability that $Y^K$ exceeds $\tau$ for $\gamma > K$ on the interval $[a(T), T]$ is of negligible order.

**Lemma 6.** Let $Y^K$ be as in theorem 1, $\theta \geq 0$, $\gamma > K$ and $\delta > 0$. Then there is a $p \geq e^2$ such that for $T$ large enough, it holds

$$
\mathbb{P}(\exists \tau \in [(\log T)^p, T] : |Y^K| > \tau) \leq T^{-\theta - \delta}.
$$

**Proof.** We estimate

$$
\mathbb{P}(\exists \tau \in [(\log T)^p, T] : |Y^K| > \tau) \leq \sum_{s = [(\log T)^p]}^{[T]} \mathbb{P}(\exists \tau \in [s, s+1] : |Y^K| > \tau)
\leq \sum_{s = [(\log T)^p]}^{[T]} \mathbb{P}\left( \sup_{c \in [s,s+1]} |Y^K| > s^\gamma \right).
$$

For $s = [(\log T)^p], \ldots, [T]$ and $\sigma^K_s := \mathbb{V}[Y^K] \vee \sup_{c \in [1,2]} \mathbb{V}[Y^K - Y^K_1]$, we may further estimate

$$
\mathbb{P}\left( \sup_{c \in [s,s+1]} |Y^K| > s^\gamma \right) \leq \mathbb{P}\left( |Y^K| > \frac{s^\gamma}{2} \right) + \mathbb{P}\left( \sup_{c \in [s,s+1]} |Y^K| > s^\gamma, |Y^K| \leq \frac{s^\gamma}{2} \right)
\leq \mathbb{P}\left( |\mathcal{N}(0,1)| > \frac{g^\gamma - \gamma}{2\sigma^K} \right) + \mathbb{P}\left( \sup_{c \in [s,s+1]} |Y^K| - |Y^K_1| > \frac{s^\gamma}{2} \right)
\leq c_1 e^{-2(\gamma-K)/(8\sigma^K)} + \mathbb{P}\left( \sup_{c \in [s,s+1]} |Y^K - Y^K_1| > \frac{s^\gamma}{2} \right)
= c_1 e^{-2(\gamma-K)/(8\sigma^K)} + \mathbb{P}\left( \sup_{c \in [1,1+1]} |Y^K - Y^K_1| > \frac{g^\gamma - \gamma}{2} \right)
$$

(10)
Thus, we have to estimate $t$ belonging to the reproducing kernel Hilbert space (RKHS) of the process, see [4, proposition 1.6].

Now, we estimate the probability in (10) as follows:

$$
\mathbb{P} \left( \sup_{r \in [1,1+\tau^{-1}]} |Y^K_r - Y^K_t| > \frac{s^{\gamma-K}}{2} \right) \leq \mathbb{P} \left( \sup_{r \in [1,2]} |Y^K_r - Y^K_t| > \frac{s^{\gamma-K}}{2} \right)
$$

$$
\leq \mathbb{P} \left( \sup_{r \in [1,2]} (Y^K_r - Y^K_t) > \frac{s^{\gamma-K}}{2} \right) + \mathbb{P} \left( \sup_{r \in [1,2]} (Y^K_t - Y^K_r) > \frac{s^{\gamma-K}}{2} \right)
$$

$$
= 2\mathbb{P} \left( \sup_{r \in [1,2]} (Y^K_r - Y^K_t) > \frac{s^{\gamma-K}}{2} \right).
$$

The last probability is a probability of large deviation of a bounded Gaussian random function and can therefore be estimated by the tail of a one-dimensional Gaussian distribution.

More precisely, by e.g. [31, theorem 12.1], there exist constants $c_2 > 0$ and $d \in \mathbb{R}$ such that

$$
\mathbb{P} \left( \sup_{r \in [1,2]} (Y^K_r - Y^K_t) > \frac{s^{\gamma-K}}{2} \right) \leq c_2 e^{-\gamma/K - 2 - (\gamma - K)^2 + d^2/(2\sigma^2)}.
$$

Together with (11) and (10), this yields for $s = [(\log T)^p], \ldots, [T]$:

$$
\mathbb{P} \left( \sup_{t \in [s,s+1]} |Y^K_t| > s^\gamma \right) \leq e^{-2(\gamma - K)/\sigma^2 - 3e^{\gamma - K}}
$$

$$
\leq e^{-(\log T)^{2(\gamma - K)}/(2\sigma^2) + c_0(\log T)^{(\gamma - K)p}}
$$

for constants $c_3, c_0 > 0$. Combining this with (9), we get

$$
\mathbb{P}(\exists t \in [(\log T)^p, T] : |Y^K_t| > r) \leq (T - (\log T)^p + 2)e^{-(\log T)^{2(\gamma - K)}/(2\sigma^2) + c_0(\log T)^{(\gamma - K)p}}.
$$

Taking e.g. $p = \max \{1/(\gamma - K), e^2\}$, the right-hand-side decays faster than any polynomial, which shows the assertion. $
$

Thus, we can estimate the persistence probability of $X^H + Y^K$ on $[a(T), T]$ by the persistence probability of $X^H$ shifted by $t \mapsto t'$ on $[a(T), T]$. Shifting a Gaussian process by a deterministic function does not change the asymptotic order of the persistence probability if the function belongs to the reproducing kernel Hilbert space (RKHS) of the process, see [4, proposition 1.6]. Thus, we have to estimate $t \mapsto t'$ on $[a(T), T]$ by a function in the RKHS of $X^H$, which works for $\gamma < H$, as we will see in the following. The RKHS of a centred Gaussian process $X = (X_t)_{t \in \mathbb{T}}$, defined on some index set $\mathbb{T} \neq \emptyset$, is a Hilbert space of functions on $\mathbb{T}$ which plays an important role for $X$. More precisely, the RKHS of $X$ is given by

$$
\mathcal{H}_X := \left\{ t \mapsto \mathbb{E}[\xi X_t] \in \mathbb{H}_X := \text{span}\{X_t : t \in \mathbb{T}\} \right\},
$$

(12)

where the closure is in $L^2$, with inner product $\langle h_1, h_2 \rangle_{\mathcal{H}_X} := \mathbb{E}[\xi_1 \xi_2]$ for $h_1 = \mathbb{E}[\xi_1 X_t]$, $h_2 = \mathbb{E}[\xi_2 X] \in \mathcal{H}_X$.

For Brownian motion, it is well-known that its RKHS consists of the functions which are differentiable a.e., which start in the origin and whose derivative is square-integrable, see
e.g. [58, lemma 10.1]. Thus, there are functions in the RKHS of Brownian motion growing faster than \( t^r \) for \( t \to \infty \) if and only if \( \gamma < H = 1/2 \). This is also known for further specific \( H \)-self-similar Gaussian processes where explicit representations of the RKHS are available, such as for FBM (cf [10, corollary 4.2] for \( H > 1/2 \) and [46, proposition 6.1] for \( H < 1/2 \)).

To show this for general \( H \)-self-similar Gaussian processes \( X^H \) fulfilling the additional assumptions of theorem 1, we will go over to the Lamperti transform \( Z^H := e^{-H}X^H \), \( \tau \in \mathbb{R} \), of \( X^H \) and use an alternative representation of the RKHS of Gaussian stationary processes (GSPs) with a continuous autocovariance function via the spectral measure.

Recall that for a GSP \( Z = (Z_r)_{r \in \mathbb{R}} \) with a continuous autocovariance function \( r(\tau) := \text{cov}(Z_0, Z_\tau) \), \( \tau \in \mathbb{R} \), Bochner’s theorem provides a unique finite measure \( \mu \) on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\), satisfying

\[
r(\tau) = \int_{\mathbb{R}} e^{i\tau x} \, d\mu(x), \quad \tau \in \mathbb{R},
\]

which is called spectral measure of \( Z \), see e.g. [2, theorem 1.2.7]. The RKHS of \( Z \) can then be written in the form

\[
\mathcal{H}_Z = \left\{ \tau \mapsto \int_{\mathbb{R}} \varphi(x)e^{-\tau x} \, d\mu(x) \middle| \varphi \in L^2(\mu) \right\},
\]

see e.g. [2, comment 2.2.2(c)]. Using this representation, we first get the following result for the RKHS of a GSP \( Z \).

**Lemma 7.** Let \( Z = (Z_r)_{r \in \mathbb{R}} \) be a real-valued GSP with a continuous autocovariance function. If the spectral measure of \( Z \) is absolutely continuous w.r.t. the Lebesgue measure in some neighbourhood of the origin and the corresponding spectral density is bounded away from zero near zero, then for every \( \alpha \in (0, 1/2) \), there exists \( h \in \mathcal{H}_Z \) satisfying \( h(\tau) \sim c\tau^{-\alpha-1} \) for \( \tau \to \infty \) and some \( c > 0 \) as well as \( h(\tau) > 0 \) for all \( \tau \geq 0 \).

**Remark.** Note that, by (14), if the spectral density \( p \) exists on whole \( \mathbb{R} \), any function \( h \in \mathcal{H}_Z \) in the RKHS of \( Z \) is the Fourier transform of a function of the form \( \varphi \cdot p \), where \( \varphi \in L^2(\mu) \). In view of Tauberian theorems for Fourier transforms, see e.g. [12, theorem 4.10.3], one would expect that \( h(\tau) \sim c\tau^{-\alpha-1} \) holds for \( \tau \to \infty \) and some \( \alpha \in (0, 1) \) if and only if \( \varphi(x) \sim c' x^{-\alpha} \) for \( x \to 0 \), which is in accordance with the condition \( \alpha \in (0, 1/2) \).

**Proof.** Let \( \alpha \in (0, 1/2) \). Similarly to the proof of [3, proposition 5], we will first construct a function \( h_1 \in \mathcal{H}_Z \) with the desired asymptotic behaviour, which unfortunately may attain non-positive values up to some \( \tau_0 > 0 \). Afterwards, we will show the existence of another function \( h_2 \in \mathcal{H}_Z \) which is non-negative on \([0, \infty)\), even positive on \([0, \tau_0]\), and decays faster than \( h_1 \). Then, the function \( h := h_1 + 2 \max_{\tau \in [0, \tau_0]} |h_1(\tau)|/\min_{\tau \in [0, \tau_0]} h_2(\tau) \cdot h_2 \) yields the assertion.

By assumption, there exists a spectral density \( p : (-x_0, x_0) \to [c_0, \infty) \) of \( \mu|_{(-x_0, x_0)} \) for some \( c_0, x_0 > 0 \), where \( \mu \) denotes the spectral measure of \( Z \).

**Construction of \( h_1 \):** we set \( \varphi_1(x) := 1_{|x| < x_0} |x|^{-\alpha}/p(x) \). Then, we have

\[
\int_{\mathbb{R}} \varphi_1^2(x) \, d\mu(x) \leq 2 \int_0^{x_0} x^{-2\alpha}/p(x) \, dx \leq 2c_0^{-1} \int_0^{x_0} x^{-2\alpha} \, dx \leq \infty
\]

as \( \alpha < 1/2 \). Thus \( \varphi_1 \in L^2(\mu) \). For the \( h_1 \) corresponding to \( \varphi_1 \) (as in (14)), we get
bounded, and Dirichlet’s test, the integral in the definition of $h$ satisfying $h$.

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$\mu$ the inversion theorem for characteristic functions yields $d$.

Let $X^H$ be as in theorem 1. Then, for every $\alpha \in (0, 1/2)$, there exists $h \in \mathcal{H}^H$ satisfying $h(t) \sim ct^\alpha(\log t)^{\alpha - 1}$ for $t \to \infty$ and some $c > 0$ as well as $h(t) \geq 1$ for all $t \geq 1$.

Proof. Let $\alpha \in (0, 1/2)$ and $Z^H := e^{-r_HX^H}$, $r \in \mathbb{R}$, be the Lamperti transform of $X^H$. Note that, since $r(\tau) := \mathbb{E}[Z^H \tau]$, $\tau \in \mathbb{R}$, is assumed to be integrable and—by definition of the spectral measure $\mu$ of $Z^H$ (cf. (13))—represents the characteristic function of the finite measure $\mu$, the inversion theorem for characteristic functions yields $d\mu(x) = \mu(x)dx$ on $\mathbb{R}$ with the density

\[ p(x) = \frac{1}{2\pi} \int e^{-irx} r(\tau) d\tau \to \frac{1}{2\pi} \int r(\tau) d\tau \in (0, \infty), \quad x \to 0, \]

Applying the last lemma to the Lamperti transform of $X^H$ gives the following corollary.

Corollary 8. Let $X^H$ be as in theorem 1. Then, for every $\alpha \in (0, 1/2)$, there exists $h \in \mathcal{H}^H$ satisfying $h(t) \sim ct^\alpha(\log t)^{\alpha - 1}$ for $t \to \infty$ and some $c > 0$ as well as $h(t) \geq 1$ for all $t \geq 1$. 

Proof. Let $\alpha \in (0, 1/2)$ and $Z^H := e^{-r_HX^H}$, $r \in \mathbb{R}$, be the Lamperti transform of $X^H$. Note that, since $r(\tau) := \mathbb{E}[Z^H \tau]$, $\tau \in \mathbb{R}$, is assumed to be integrable and—by definition of the spectral measure $\mu$ of $Z^H$ (cf. (13))—represents the characteristic function of the finite measure $\mu$, the inversion theorem for characteristic functions yields $d\mu(x) = \mu(x)dx$ on $\mathbb{R}$ with the density

\[ p(x) = \frac{1}{2\pi} \int e^{-irx} r(\tau) d\tau \to \frac{1}{2\pi} \int r(\tau) d\tau \in (0, \infty), \quad x \to 0, \]

\[ h_1(\tau) = \int \varphi_1(x)e^{-irx} d\mu(x) = \int \varphi_1(x) \cos(\tau x) d\mu(x) \]

\[ = 2 \int_0^{x_0} x^{-\alpha} \cos(\tau x) dx = 2\tau^{\alpha - 1} \int_0^{\tau x_0} y^{-\alpha} \cos(y) dy \sim c\tau^{\alpha - 1} \]

for $c := 2 \int_0^{\infty} x^{-\alpha} \cos(y) dy$ and $\tau \to \infty$. Note that due to $\alpha < 1$, the fact that $\tau^{-\alpha}$ is decreasing and fulfills $\lim_{\tau \to \infty} \tau^{-\alpha} = 0$, the fact that the integrals of $\cos(\cdot)$ over any interval are uniformly bounded, and Dirichlet’s test, the integral in the definition of $c$ exists and is positive. Further note that this fails for $\alpha \leq 0$.

Construction of $h_2$: due to the asymptotic behaviour of $h_1$, there exists $\tau_0 > \pi/x_0$ such that $h_1(\tau) > 0$ for $\tau \geq \tau_0$. Let $g : \mathbb{R} \to [0, \infty)$ be a smooth even function with $g(x) > 0$ for $|x| < \pi/(2\tau_0)$ and $g(x) = 0$ otherwise.

We set $f := g \ast g$ and $\varphi_2(x) := \frac{1}{\tau \pi} f(x)/\mu(x)$. Then $\varphi_2 \in L^2(\mu)$ as

\[ \int \varphi_2^2(x) d\mu(x) = \int_{-\pi/\tau}^{\pi/\tau} f^2(x)/\mu(x) dx \leq \frac{2\pi}{\tau \pi} \max_{x \in [-\pi/\tau, \pi/\tau]} f^2(x) \]

where we used that $\pi/\tau_0 < x_0$. Note that by definition of $f$ and $g$, we have $f(x) = 0$ for $|x| \geq \pi/\tau_0$. Thus, the $h_2$ corresponding to $\varphi_2$ fulfills

\[ h_2(\tau) = \int \varphi_2(x)e^{-irx} d\mu(x) = \int f(x)e^{-irx} dx \]

where we used in the second line that the Fourier transform of a convolution is given by the product of the Fourier transforms of the convoluted functions, as well as that $g$ vanishes outside of $(-\pi/(2\tau_0), \pi/(2\tau_0))$ by definition. Furthermore, by integration by parts, we have

\[ h_2(\tau) = \int f(x)e^{-irx} dx = \frac{1}{i\tau^2} \int f''(x)e^{-irx} dx \]

\[ \leq \frac{2\pi}{\tau \pi} \max_{x \in [-\pi/\tau, \pi/\tau]} |f''(x)| \cdot \tau^{-2}. \]

\[ \square \]
so that the conditions of lemma 7 are fulfilled. Hence, lemma 7 yields the existence of a function $\tilde{h} \in H_{2H}$ and $c_0 > 0$ such that $\tilde{h}(\tau) \sim c_0 \tau^{\alpha-1}$ for $\tau \to \infty$ and $\tilde{h}(\tau) > 0$ for all $\tau > 0$.

By representation (12), this implies that there exists a random variable $\xi \in H_{2H}$ such that $h(\tau) = \mathbb{E}[\xi^{2H}]$, $\tau \in \mathbb{R}$. Plugging in the definition of $Z^H$, this gives $e^{-\alpha h(\tau)} = \mathbb{E}[\xi^{2H}]$ for $\tau \in \mathbb{R}$ and $h_0(t) := h(\log t) = \mathbb{E}[\xi^{2H}]$ for $t > 0$. Since span$\{Z^H : \tau \in \mathbb{R}\} = \text{span} \{X^H : t > 0\}$ and thus $H_{2H} = H_{xH}$, we get $h_0 \in H_{xH}$, by using again (12). Further, $h_0$ is a continuous function (by the continuity of the covariance function) and satisfies $h_0(t) \sim c_0 h_0(\log t)^{\alpha-1}$ for $t \to \infty$ as well as $h_0(t) > 0$ for all $t \geq 1$. In particular, we have $h_0(t) \to \infty$ for $t \to \infty$. Thus, there exists $t_0 > 1$ such that $h_0(t) \geq 1$ for $t \geq t_0$. Setting $h := h_0/(\min_{t \in [1,t_0]} h_0(t) \wedge 1)$ yields the assertion for $c := c_0/(\min_{t \in [1,t_0]} h_0(t) \wedge 1)$.

Now, we are ready to give the proof of theorem 1.

**Proof of theorem 1.** First observe that the persistence probability of $X^H$ behaves as in (1) with some persistence exponent $\theta_H \in [0, \infty)$. Indeed, as $X^H$ is a self-similar centred Gaussian process with a non-negative covariance function by assumption, its Lamperti transform $Z^H := e^{-\alpha t}X^H$, $\tau \in \mathbb{R}$, is a centred GSP with a non-negative correlation function. By subadditivity and Slepian’s lemma, this yields

$$
\mathbb{P}\left(\sup_{T \in [1,T]} X^H_\tau \leq 0\right) = \mathbb{P}\left(\sup_{T \in [1,T]} Z^H_\tau \leq 0\right) = T^{-\theta_H + o(1)}, \quad T \to \infty,
$$

for some $\theta_H \in [0, \infty)$. Consider the function $h$ in the RKHS of $X^H$ given by corollary 8 for e.g. $\alpha := 1/4$. On the one hand, we know by Slepian’s lemma that

$$
\mathbb{P}\left(\sup_{T \in [0,1]} X^H_\tau \leq 1\right) \geq \mathbb{P}\left(\sup_{T \in [0,1]} X^H_\tau \leq 1\right) \cdot \mathbb{P}\left(\sup_{T \in [1,T]} X^H_\tau \leq 1\right)
\geq \mathbb{P}\left(\sup_{T \in [0,1]} X^H_\tau \leq 1\right) \cdot \mathbb{P}\left(\sup_{T \in [1,T]} X^H_\tau \leq 0\right),
$$

while on the other hand, the fact that $h(t) \geq 1$ for $t \geq 1$ together with [4, proposition 1.6] yields that

$$
\mathbb{P}\left(\sup_{T \in [0,1]} X^H_\tau \leq 1\right) \leq \mathbb{P}\left(\sup_{T \in [1,T]} X^H_\tau \leq 1\right) \leq \mathbb{P}\left(\sup_{T \in [1,T]} X^H_\tau + h(t) \leq 1\right) T^{o(1)}
\leq \mathbb{P}\left(\sup_{T \in [1,T]} X^H_\tau \leq 0\right) T^{o(1)},
$$

showing that also $\mathbb{P}\left(\sup_{T \in [0,1]} X^H_\tau \leq 1\right) = T^{-\theta_H + o(1)}$ for $T \to \infty$.

Now, note that, by [4, proposition 1.6], we have

$$
\mathbb{P}\left(\sup_{T \in (\log T)^p, T} X^H_\tau \pm h(t) \leq 1\right) = T^{o(1)}, \quad T \to \infty.
$$

Fix $\gamma$ with $K < \gamma < H$. 

**Upper bound:**

\[ P\left( \max_{t \in [0,T]} X_t^H + Y_t^K \leq 1 \right) \]

\[ \leq P\left( \exists t \in [(\log T)^p, T] : |Y_t^K| > h(t) \right) + P\left( \max_{t \in [(\log T)^p, T]} X_t^H - h(t) \leq 1 \right) \]

\[ \leq P\left( \exists t \in [(\log T)^p, T] : |Y_t^K| > r^* \right) + P\left( \max_{t \in [(\log T)^p, T]} X_t^H \leq 1 \right) T^{o(1)} \]

\[ \leq T^{-\theta_X - \delta} + \frac{P\left( \max_{t \in [0,T]} X_t^H \leq 1 \right)}{P\left( \max_{t \in [(\log T)^p, T]} X_t^H \leq 1 \right)} T^{o(1)} \]

\[ \leq T^{-\theta_X - \delta} + T^{-\theta_X + o(1)} (\log T)^{\theta_X + o(1)} T^{o(1)} \]

\[ \leq T^{-\theta_X + o(1)} \]

for \( T \) large enough and \( \delta > 0 \), where \( p \) is chosen according to lemma 6 for \( \theta := \theta_X \). Here, the second inequality uses (15) and the property of \( h \) that \( h(t) \sim c t^H (\log t)^{-3/4} > r^* \) for \( t \) large enough, while the third inequality is lemma 6 together with Slepian’s lemma.

**Lower bound:** the opposite reasoning gives

\[ T^{-\theta_X + o(1)} = P\left( \max_{t \in [0,T]} X_t^H \leq 1 \right) \leq P\left( \max_{t \in [(\log T)^p, T]} X_t^H + h(t) \leq 1 \right) T^{o(1)} \]

\[ \leq P\left( \exists t \in [(\log T)^p, T] : |Y_t^K| > h(t) \right) T^{o(1)} \]

\[ + P\left( \max_{t \in [(\log T)^p, T]} X_t^H + Y_t^K \leq 1 \right) T^{o(1)} \]

\[ \leq P\left( \exists t \in [(\log T)^p, T] : |Y_t^K| > r^* \right) T^{o(1)} \]

\[ + P\left( \max_{t \in [(\log T)^p, T]} X_t^H + Y_t^K \leq 1 \right) T^{o(1)} \]

\[ \leq T^{-\theta_X - \delta} + \frac{P\left( \max_{t \in [0,T]} X_t^H + Y_t^K \leq 1 \right)}{P\left( \max_{t \in [(\log T)^p, T]} X_t^H + Y_t^K \leq 1 \right)} T^{o(1)}, \]

where we used the definition of \( \theta_X \) in the first, (15) in the second and lemma 6 as well as Slepian’s lemma in the fifth step. Precisely here, we use the assumption of non-negative covariances of \( X_t^H + Y_t^K \). So we have

\[ P\left( \max_{t \in [0,T]} X_t^H + Y_t^K \leq 1 \right) \geq T^{-\theta_X + o(1)} P\left( \max_{t \in [(\log T)^p]} X_t^H + Y_t^K \leq 1 \right). \]

We then further estimate the right-hand side of (16) by replacing \( T \) in (16) by \( (\log T)^p \) and get
\[ P \left( \sup_{t \in [0,T]} X^H_t + Y^K_t \leq 1 \right) \geq T^{-\theta_x + o(1)} (\log T)^{-p\theta_x + o(1)} \]

\[ \times P \left( \sup_{t \in [0,(p \log \log T)^{\frac{1}{p}}]} X^H_t + Y^K_t \leq 1 \right). \]  

(17)

We set \( f_0(T) := \log \log T \) and \( f_N(T) := \log p + \log f_{N-1}(T) \) for \( N \geq 1 \). Using (16) iteratively then gives

\[ P \left( \sup_{t \in [0,T]} X^H_t + Y^K_t \leq 1 \right) \geq T^{-\theta_x + o(1)} (\log T)^{-p\theta_x + o(1)} (N+1) \]

\[ \times P \left( \sup_{t \in [0,((p \log \log T)^{\frac{1}{p}})^{\frac{1}{p}}]} X^H_t + Y^K_t \leq 1 \right), \]  

(18)

for \( N \in \mathbb{N} \). This can be seen by induction: the induction base is (17), while for the induction step, one has to note that

\[ (\log((pf_N(T))^p) = (p\log p + \log f_{N-1}(T))^p = (pf_N(T))^p. \]

Now we consider the function

\[ \varphi_p(x) := \log p + \log x, \quad x \in [2, \infty). \]

This is a contraction with Lipschitz constant 1/2. The Lipschitz constant can be computed by the fact that \( \varphi'_p(x) = 1/x \leq 1/2 \) for \( x \geq 2 \), while the self-map property of \( \varphi_p \) is deduced from the fact that \( \log p \geq 2 \) holds by lemma 6. Thus, the Banach fixed-point theorem yields a unique fixed-point \( a_p \geq 2 \) of \( \varphi_p \), which does not depend on \( T \). Further, as \( f_N(T) = \varphi_p(f_{N-1}(T)) \), we can estimate

\[ |f_N(T) - a_p| \leq 2^{1-N} |f_1(T) - f_0(T)| \]

\[ = 2^{1-N} |\log p + \log \log T - \log \log T| \leq 2^{1-N} \times 3 \log \log T \]

for \( N \in \mathbb{N} \) and \( T \) large enough, see e.g. [1, theorem 1.1(c)]. For \( N_T := \lceil \log \log T + \log 6 \rceil \), this implies

\[ |f_{N_T}(T) - a_p| \leq 2^{1-N_T} \times 3 \log T \leq 1. \]

Considering (18) for \( N := N_T \) consequently yields
\[ P \left( \sup_{t \in [0,T]} X_t^H + Y_t^K \leq 1 \right) \geq T^{-\theta + o(1)} P \left( \sup_{t \in [0,(pN_f T)^p]^p} X_t^H + Y_t^K \leq 1 \right) \]
\[
\geq T^{-\theta + o(1)} P \left( \sup_{t \in [0,(1+a)p]^p} X_t^H + Y_t^K \leq 1 \right) 
\]
\[
= T^{-\theta + o(1)},
\]
which finishes the proof. \(\square\)

5. Conclusion

In this paper, we have studied sums of self-similar centred Gaussian processes with different self-similarity indices and obtained rigorous results for the persistence probabilities of such mixed processes. This covers processes which are neither self-similar nor have stationary increments, and thus, our results are among the rare occasions where one is able to obtain persistence results without relying on one of these two powerful properties.

The central process in this setting is mixed FBM, which is the sum of an FBM and an independent Brownian motion. This process appears in several applications where one needs to consider two sources of randomness, one with long memory and one either with short or without memory. We have seen that the persistence probability of mixed FBM decays asymptotically polynomially with persistence exponent \(1 - \max(1/2,H)\), where \(H\) is the Hurst parameter of the underlying FBM. Thus, regarding persistence, the component with long memory dominates the component without memory.

This result has been deduced as a corollary of our main result, stating that for a more general class of mixed processes, the persistence probability of the mixed process decays asymptotically polynomially with the persistence exponent of the process with the greater self-similarity index. As further corollaries, we obtained rigorous results for the persistence exponents of the ccmFBM, the mixed integrated FBM and the mixed Riemann–Liouville process.

Note that our main result also covers sums of self-similar processes of different type if the self-similarity indices do not coincide, such as the sum of an FBM with Hurst parameter \(H\) and a Riemann–Liouville process with parameter \(K \neq H\). It would be an interesting open problem to determine the persistence exponent of sums of self-similar processes of different type, but with coinciding self-similarity indices (e.g. FBM and Riemann–Liouville process with the same Hurst parameter). More generally, it would be very useful to develop a proof technique which is able to handle processes (not necessarily self-similar) with coinciding orders of fluctuation, since relevant processes such as the fractional Ornstein–Uhlenbeck process can be written as mixed processes of this kind.

Further note that, as already mentioned in the context of theorem 1, the proof of theorem 1 can easily be adapted to the setting of a finite sum of self-similar processes with different self-similarity indices, with the result that the process with the largest self-similarity index dominates. From a mathematical point of view, it would be an interesting open problem to study the persistence probability when \(Y^K_i\) is replaced by a (well-defined) infinite sum of the form \(\sum_{i=1}^{\infty} \alpha_i Y^K_i\), where e.g. \(\sup_i K_i = H\), as in this case, our proof technique does not work anymore.
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Data availability statement

No new data were created or analysed in this study.

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References

[1] Agarwal P, Jleli M and Samet B 2018 Fixed Point Theory in Metric Spaces: Recent Advances and Applications (Berlin: Springer)
[2] Ash R B and Gardner M F 1975 Topics in stochastic processes Probability and Mathematical Statistics (New York: Academic)
[3] Aurzada F and Buck M 2018 Persistence probabilities of two-sided (integrated) sums of correlated stationary Gaussian sequences J. Stat. Phys. 170 784–99
[4] Aurzada F and Dereich S 2013 Universality of the asymptotics of the one-sided exit problem for integrated processes Ann. Inst. Henri Poincare B 49 236–51
[5] Aurzada F, Guillotin-Plantard N and Pène F 2018 Persistence probabilities for stationary increment processes Stoch. Process. Appl. 128 1750–71
[6] Aurzada F and Kilian M 2022 Asymptotics of the persistence exponent of integrated fractional Brownian motion and fractionally integrated Brownian motion Theory Probab. Appl. 67 77–88
[7] Aurzada F and Mönch C 2019 Persistence probabilities and a decorrelation inequality for the Rosenblatt process and Hermite processes Theory Probab. Appl. 63 664–70
[8] Aurzada F and Mukherjee S 2020 Persistence probabilities of weighted sums of stationary Gaussian sequences (arXiv:2003.01192)
[9] Aurzada F and Simon T 2015 Persistence probabilities and exponents Lévy Matters V (Lecture Notes in Mathematics vol 2149) (Berlin: Springer) pp 183–224
[10] Barton R J and Poor H V 1988 Signal detection in fractional Gaussian noise IEEE Trans. Inf. Theory 34 943–59
[11] Basu R, Dembo A, Feldheim N and Zeitouni O 2020 Exponential concentration for zeroes of stationary Gaussian processes Int. Math. Res. Not. 2020 9769–96
[12] Bingham N H, Goldie C M and Teugels J L 1987 Regular variation Encyclopedia of Mathematics and its Applications (Cambridge: Cambridge University Press)
[13] Bornales J, Oliveira M J and Streit L 2013 Self-repelling fractional Brownian motion—a generalized Edwards model for chain polymers Quantum Bio-Informatics V (Singapore: World Scientific) pp 389–401
[14] Bray A J, Majumdar S N and Schehr G 2013 Persistence and first-passage properties in nonequilibrium systems Adv. Phys. 62 225–361
[15] Cai C, Chigansky P and Kleptsyna M 2016 Mixed Gaussian processes: a filtering approach Ann. Probab. 44 3032–75
[16] Caspi A, Granek R and Elbaum M 2002 Diffusion and directed motion in cellular transport Phys. Rev. E 66 011916
[17] Cheridito P 2001 Mixed fractional Brownian motion Bernoulli 7 913–34
[18] Dembo A and Mukherjee S 2015 No zero-crossings for random polynomials and the heat equation Ann. Probab. 43 85–118
[19] Dembo A and Mukherjee S 2017 Persistence of Gaussian processes: non-summable correlations Probab. Theory Relat. Fields 169 1007–39
[20] Dufifinema J, Shokrollahi F, Sottinen T and Viitassari L 2021 Long-range dependent completely correlated mixed fractional Brownian motion (arXiv:2104.04992)
[21] Eab C H and Lim S C 2012 Accelerating and retarding anomalous diffusion J. Phys. A: Math. Theor. 45 145001
[22] El-Nouty C 2003 The fractional mixed fractional Brownian motion Stat. Probab. Lett. 65 111–20
[23] Feldheim N and Feldheim O 2015 Long gaps between sign-changes of Gaussian stationary processes Int. Math. Res. Not. 2015 3021–34
[24] Feldheim N, Feldheim O and Mukherjee S 2021 Persistence and Ball exponents for Gaussian stationary processes (arXiv:2112.04820)
[25] Feldheim N, Feldheim O and Nitzan S 2021 Persistence of Gaussian stationary processes: a spectral perspective Ann. Probab. 49 1067–96
[26] Goldman M 1971 On the first passage of the integrated Wiener process Ann. Math. Stat. 42 2150–5
[27] Grothaus M, Oliveira M J, da Silva J L and Streit L 2011 Self-avoiding fractional Brownian motion-the Edwards model J. Stat. Phys. 145 1513–23
[28] Havlin S and Ben-Avraham D 1987 Diffusion in disordered media Adv. Phys. 36 695–798
[29] Isozaki Y and Watanabe S 1994 An asymptotic formula for the Kolmogorov diffusion and a refinement of Sinai’s estimates for the integral of Brownian motion Proc. Japan Acad. A 70 271–6
[30] Krug J, Kallabis H, Majumdar S N, Cornell S J, Bray A J and Sire C 1997 Persistence exponents for fluctuating interfaces Phys. Rev. E 56 2702–12
[31] Lifshits M A 1995 Gaussian Random Functions (Dordrecht: Kluwer)
[32] Lim S C and Eab C H 2015 Some fractional and multifractional Gaussian processes: a brief introduction Int. J. Mod. Phys. Conf. Ser. 36 1560001
[33] Majumdar S N 1999 Persistence in nonequilibrium systemsCurr. Sci. 77 370–5 https://www.jstor.org/stable/24102955
[34] Matheron G and De Marsily G 1980 Is transport in porous media always diffusive? A counterexample Water Resour. Res. 16 901–17
[35] Metzler R, Oshanin G and Redner S 2014 First-Passage Phenomena and Their Applications (Singapore: World Scientific)
[36] Miao Y, Ren W and Ren Z 2008 On the fractional mixed fractional Brownian motion Appl. Math. Sci. 2 1729–38
[37] Mishura Y 2008 Stochastic Calculus for Fractional Brownian Motion and Related Processes (Lecture Notes in Mathematics vol 1929) (Berlin: Springer)
[38] Mönch C 2021 Universality for persistence exponents of local times of self-similar processes with stationary increments J. Theor. Probab.
[39] Molchan G M 1999 Maximum of a fractional Brownian motion: probabilities of small values Commun. Math. Phys. 205 97–111
[40] Molchan G 2008 Unilateral small deviations of processes related to the fractional Brownian motion Stoch. Process. Appl. 118 2085–97
[41] Molchan G 2017 The inviscid Burgers equation with fractional Brownian initial data: the dimension of regular Lagrangian points J. Stat. Phys. 167 1546–54
[42] Molchan G and Khokhlov A 2004 Small values of the maximum for the integral of fractional Brownian motion J. Stat. Phys. 114 923–46
[43] Nualart D 2006 The Malliavin calculus and related topics Probability and its Applications 2nd edn (Berlin: Springer)
[44] Nualart D and Sönmez E 2022 Regularization of differential equations by two fractional noises Stoch. Dyn. 102 103–16
[45] Pfister G and Scher H 1978 Dispersive (non-Gaussian) transient transport in disordered solids Adv. Phys. 27 747–98
[46] Pipiras V and Taqqu M S 2000 Integration questions related to fractional Brownian motion Probab. Theory Relat. Fields 118 251–91
[47] Redner S 2001 A Guide to First-Passage Processes (Cambridge: Cambridge University Press)
[48] Ruan D and Luo J 2018 The existence, uniqueness, and controllability of neutral stochastic delay partial differential equations driven by standard Brownian motion and fractional Brownian motion Discrete Dyn Nat. Soc. 2018 7502514
[49] Salcedo-Sanz S, Casillas-Pérez D, Del Ser J, Casanova-Mateo C, Cuadra L, Piles M and Camps-Valls G 2022 Persistence in complex systems Phys. Rep. 957 1–73
[50] Scher H, Margolin G, Metzler R, Klafte J and Berkowitz B 2002 The dynamical foundation of fractal stream chemistry: the origin of extremely long retention times Geophys. Res. Lett. 29 1–4
[51] Scher H and Montroll E W 1975 Anomalous transit-time dispersion in amorphous solids Phys. Rev. B 12 2455
[52] She Z-S, Aurell E and Frisch U 1992 The inviscid Burgers equation with initial data of Brownian type Commun. Math. Phys. 148 623–41
[53] Sinai Y G 1992 Distribution of some functionals of the integral of a random walk Theor. Math. Phys. 90 219–41
[54] Sinai Y G 1992 Statistics of shocks in solutions of inviscid Burgers equation Commun. Math. Phys. 148 601–21
[55] Sliusarenko O Y, Gonchar V Y, Chechkin A V, Sokolov I M and Metzler R 2010 Kramers-like escape driven by fractional Gaussian noise Phys. Rev. E 81 041119
[56] Stanislavsky A and Weron A 2019 Control of the transient subdiffusion exponent at short and long times Phys. Rev. Res. 1 023006
[57] Stapf S, Kimmich R and Seitter R-O 1995 Proton and deuteron field-cycling NMR relaxometry of liquids in porous glasses: evidence for Lévy-walk statistics Phys. Rev. Lett. 75 2855
[58] van der Vaart A W and van Zanten J H 2008 Reproducing Kernel Hilbert Spaces of Gaussian Priors vol 3 (Institute of Mathematical Statistics Collections) (Beachwood: Institute of Mathematical Statistics) pp 200–22
[59] van Zanten H 2007 When is a linear combination of independent fBm’s equivalent to a single fBm? Stoch. Process. Appl. 117 57–70
[60] Wu D 2011 Mixed fractional Brownian sheets and their applications J. Math. Phys. 52 063510