THE ENUMERATION OF MAXIMALLY CLUSTERED PERMUATIONS

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ABSTRACT. The maximally clustered permutations are characterized by avoiding the classical permutation patterns \{3421, 4312, 4321\}. This class contains the freely braided permutations and the fully commutative permutations. In this work, we show that the generating functions for certain fully commutative pattern classes can be transformed to give generating functions for the corresponding freely braided and maximally clustered pattern classes. Moreover, this transformation of generating functions is rational. As a result, we obtain enumerative formulas for the pattern classes mentioned above as well as the corresponding hexagon-avoiding pattern classes where the hexagon-avoiding permutations are characterized by avoiding \{46718235, 46781235, 56718234, 56781234\}.

1. INTRODUCTION

The maximally clustered permutations introduced in [Los07] are a generalization of the freely braided permutations developed in [GL02] and [GL04], and these in turn include the fully commutative permutations studied in [Ste96] as a subset. In [Jon07], an explicit formula was obtained for the Kazhdan–Lusztig polynomials of maximally-clustered hexagon-avoiding permutations, generalizing an earlier result of [BW01] that identified the 321-hexagon avoiding permutations. The enumeration of the 321-hexagon avoiding permutations was first given by [SW04] who showed that these elements satisfy a linear constant-coefficient recurrence with 7 terms.

Theorem 1.1. [SW04] The number \(c_n\) of 321-hexagon-avoiding permutations in \(S_n\) satisfies the recurrence
\[
c_{n+1} = 6c_n - 11c_{n-1} + 9c_{n-2} - 4c_{n-3} - 4c_{n-4} + c_{n-5}
\]
for all \(n \geq 8\) with initial conditions given in Figure 1.

Theorem 1.1 was extended in [MS03] and also proved using an enumeration scheme as described in [Vat08]. Figure 1 shows the number \(b_n\) and \(m_n\) of freely-braided hexagon-avoiding and maximally-clustered hexagon-avoiding permutations respectively, in \(S_n\) for \(n \leq 15\). It can be shown that each of these pattern classes have a rational generating function by Proposition 13 and Corollary 10 of [ALR05]. In this paper, we develop a method for determining the generating function precisely.

| \(n\) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|----|----|----|----|----|----|
| # 321-hexagon-avoiding | 5 | 14 | 42 | 132 | 429 | 1426 | 4806 | 16329 | 55740 | 190787 | 654044 | 2244135 | 7904047 |
| # freely-braided hexagon-avoiding | 6 | 20 | 71 | 280 | 971 | 3670 | 13986 | 53369 | 204352 | 783408 | 3005284 | 11533014 | 44267854 |
| # maximally-clustered hexagon-avoiding | 6 | 21 | 78 | 298 | 1157 | 4535 | 17872 | 70644 | 279708 | 1108462 | 4395045 | 17431206 | 69144643 |

FIGURE 1. Enumeration of hexagon-avoiding classes

One feature of our main results, Theorem 2.4 and Theorem 3.6 below, is that they provide many examples of classes characterized by permutation pattern avoidance that have rational generating functions.

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We also apply Theorem 2.4 to enumerate the maximally-clustered permutations introduced in [Los07]. The freely braided permutations have previously been enumerated in [Man04] and we recover this result from Theorem 2.4 as well.

Section 1 describes the combinatorial model used in our enumeration. In Section 2 we give a result that enables us to find enumerative formulas for the pattern classes mentioned above in a unified way. In Section 3 we show how the methods of proof from Section 2 can be applied in the fully commutative case to recover the main result of [SW04] as well as some new generating functions.

1.1. **Background.** We view the symmetric group $S_n$ as the Coxeter group of type $A$ with generating set $S = \{s_1, \ldots, s_{n-1}\}$ and relations of the form $(s_i s_{i+1})^3 = 1$ together with $(s_i s_j)^2 = 1$ for $|i - j| \geq 2$ and $s_i^2 = 1$. We also refer to elements in the symmetric group by the $1$-line notation $w = [w_1 w_2 \ldots w_n]$ where $w$ is the bijection mapping $i$ to $w_i$. Then the generators $s_i$ are the adjacent transpositions interchanging the entries $i$ and $i + 1$ in the $1$-line notation. Suppose $w = [w_1 \ldots w_n]$, then $p = [p_1 \ldots p_k]$ is another permutation in $S_k$ for $k \leq n$. We say $w$ contains the permutation pattern $p$ or $w$ contains $p$ as a $1$-line pattern whenever there exists a subsequence $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that

$$w_{i_a} < w_{i_b} \text{ if and only if } p_a < p_b$$

for all $1 \leq a < b \leq k$. We call $(i_1, i_2, \ldots, i_k)$ the pattern instance. For example, $[53241]$ contains the pattern $[321]$ in several ways, including the underlined subsequence. If $w$ does not contain the pattern $p$, we say that $w$ avoids $p$. A pattern class is a set of permutations characterized by avoiding a set of permutation patterns. For example, the maximally clustered permutations are characterized by avoiding the permutation patterns

(1.1) $[3421], [4312], \text{ and } [4321]$ by [Los07, Proposition 3.7], while the freely braided permutations are characterized by avoiding

(1.2) $[4231], [3421], [4312], \text{ and } [4321]$ as permutation patterns by [GL02, Proposition 5.1.1].

Recall that the products of generators from $S$ with a minimal number of factors are called reduced expressions, and $l(w)$ is the length of such an expression for $w \in S_n$. Given $w \in S_n$, we represent reduced expressions for $w$ in sans serif font, say $w = w_1 w_2 \cdots w_p$, where each $w_i \in S$. We call any expression of the form $s_i s_{i+1} s_i$ a short-braid. There is a well-known theorem of Matsumoto [Mat64] and Tits [Tit69], which states that any reduced expression for $w$ can be transformed into any other by applying a sequence of relations of the form $(s_i s_{i+1})^3 = 1$ together with $(s_i s_j)^2 = 1$ for $|i - j| > 1$. The theorem implies that the set of all generators appearing in any reduced expression for $w$ is well-defined. We call this set of generators the support of $w$ and denote it by $\text{supp}(w)$. We say that the permutation $w$ is connected if the subscripts of the generators appearing in $\text{supp}(w)$ form a nonempty interval in $\{1, 2, \ldots, n - 1\}$.

As in [Ste96], we define an equivalence relation on the set of reduced expressions for a permutation by saying that two reduced expressions are in the same commutativity class if one can be obtained from the other by a sequence of commuting moves of the form $s_i s_j \leftrightarrow s_j s_i$, where $|i - j| \geq 2$. If the reduced expressions for a permutation $w$ form a single commutativity class, then we say $w$ is fully commutative.

1.2. **Heaps.** If $w = w_1 \cdots w_k$ is a reduced expression, then as in [Ste96] we define a partial ordering on the indices $\{1, \ldots, k\}$ by the transitive closure of the relation $i < j$ if $i < j$ and $w_i$ does not commute with $w_j$. We label each element $i$ of the poset by the corresponding generator $w_i$. It follows from the definition that if $w$ and $w'$ are two reduced expressions for a permutation $w$ that are in the same commutativity class, then the labeled posets of $w$ and $w'$ are isomorphic. This isomorphism class of labeled posets is called the heap of $w$, where $w$ is a reduced expression representative for a commutativity
class of \( w \). In particular, if \( w \) is fully commutative then it has a single commutativity class, and so there is a unique heap of \( w \).

As in [BW01], we will represent a heap as a set of lattice points embedded in \( \mathbb{N}^2 \). To do this, we assign coordinates \( (x, y) \in \mathbb{N}^2 \) to each entry of the labeled Hasse diagram for the heap of \( w \) in such a way that:

1. An entry represented by \( (x, y) \) is labeled \( s_i \) in the heap if and only if \( x = i \), and
2. If an entry represented by \( (x, y) \) is greater than an entry represented by \( (x', y') \) in the heap, then \( y > y' \).

Since the Coxeter graph of type \( A \) is a path, it follows from the definition that \( (x, y) \) covers \( (x', y') \) in the heap if and only if \( x = x' + 1 \), \( y > y' \), and there are no entries \( (x'', y'') \) such that \( x'' \in \{x, x'\} \) and \( y'' < y' < y \). Hence, we can completely reconstruct the edges of the Hasse diagram and the corresponding heap poset from a lattice point representation. This representation will enable us to make arguments “by picture” that would otherwise be difficult to formulate. Although there are many coordinate assignments for any particular heap, the \( x \) coordinates of each entry are fixed for all of them, and the coordinate assignments of any two entries only differs in the amount of vertical space between them.

**Example 1.2.** One lattice point representation of the heap of \( w = s_2 s_3 s_1 s_2 s_4 \) is shown below, together with the labeled Hasse diagram for the unique heap poset of \( w \).

Suppose \( x \) and \( y \) are a pair of entries in the heap of \( w \) that correspond to the same generator \( s_i \), so that they lie in the same column \( i \) of the heap. Assume that \( x \) and \( y \) are a minimal pair in the sense that there is no other entry between them in column \( i \). Then, for \( w \) to be reduced, there must exist at least one non-commuting generator between \( x \) and \( y \), and if \( w \) is short-braid avoiding, there must actually be two non-commuting labeled heap entries that lie strictly between \( x \) and \( y \) in the heap. We call these two non-commuting labeled heap entries a resolution of the pair \( x, y \). If the generators lie in distinct columns, we call the resolution a distinct resolution. The Lateral Convexity Lemma of [BW01] characterizes fully commutative permutations \( w \) as those for which every minimal pair in the heap of \( w \) has a distinct resolution.

**Definition 1.3.** If \( s_i \in \text{supp}(w) \), we say that \( s_i \) supports \( w \). We also say that column \( i \) supports \( w \). If \( w \) is fully commutative and connected then \( \text{supp}(w) = \{s_i, s_{i+1}, \ldots, s_j\} \) for some \( i, j \), and since every minimal pair of entries in the heap of \( w \) has a distinct resolution we must have exactly one entry in columns \( i \) and \( j \) of the heap. In this situation, we call columns \( i + 1, i + 2, \ldots, j - 1 \) the internal columns of the heap of \( w \) and we call columns \( i \) and \( j \) the extremal columns of the heap of \( w \).

We now describe a notion of containment for heaps. Recall from [BJ07] that an orientation preserving Coxeter embedding \( f : \{s_1, \ldots, s_{k-1}\} \to \{s_1, \ldots, s_{n-1}\} \) is an injective map of Coxeter generators such that for each \( m \in \{2, 3\} \), we have

\[
(s_i s_j)^m = 1 \text{ if and only if } (f(s_i) f(s_j))^m = 1
\]

and the subscript of \( f(s_i) \) is less than the subscript of \( f(s_j) \) whenever \( i < j \). We view this as a map of permutations, which we also denote \( f : S_k \to S_n \), by extending it to a word homomorphism which can then be applied to any reduced expression in \( S_k \).

Recall that a subposet \( Q \) of \( P \) is called convex if \( y \in Q \) whenever \( x < y < z \) in \( P \) and \( x, z \in Q \). Suppose that \( w \) and \( h \) are permutations. We say that \( w \) heap-contains \( h \) if there exist commutativity
classes represented by \( w \) and \( h \), together with an orientation preserving Coxeter embedding \( f \) such that the heap of \( f(h) \) is contained as a convex labeled subposet of the heap of \( w \). If \( w \) does not heap-contain \( h \), we say that \( w \) heap-avoids \( h \). To illustrate, \( w = s_2 s_3 s_1 s_2 s_4 \) from Example 1.2 heap-contains \( s_1 s_2 s_3 \) under the Coxeter embedding that sends \( s_i \mapsto s_{i+1} \), but \( w \) heap-avoids \( s_1 s_2 s_1 \).

In type \( A \), the heap construction can be combined with another combinatorial model for permutations in which the entries from the 1-line notation are represented by strings. The points at which two strings cross can be viewed as adjacent transpositions of the 1-line notation. Hence, we may overlay strings on top of a heap diagram to recover the 1-line notation for the permutation, by drawing the strings from bottom to top so that they cross at each entry in the heap where they meet and bounce at each lattice point not in the heap. Conversely, each permutation string diagram corresponds with a heap by taking all of the points where the strings cross as the entries of the heap.

For example, we can overlay strings on the two heaps of \([3214]\). Note that the labels in the picture below refer to the strings, not the generators.

![Heap Diagrams](image)

For a more leisurely introduction to heaps and string diagrams, as well as generalizations to Coxeter types \( B \) and \( D \), see [BJ07]. Cartier and Foata [CF69] were among the first to study heaps of dimers, which were generalized to other settings by Viennot [Vie89]. Stembridge has studied enumerative aspects of heaps [Ste96, Ste98] in the context of fully commutative elements. Green has also considered heaps of pieces with applications to Coxeter groups in [Gre03, Gre04a, Gre04b].

### 1.3. Maximally clustered elements.

In [Los07], Losonczy introduced the maximally clustered elements of simply laced Coxeter groups.

**Definition 1.4.** [Los07] A braid cluster is an expression of the form

\[
s_{i_1} s_{i_2} \cdots s_{i_k} s_{i_k+1} s_{i_k} \cdots s_{i_2} s_{i_1}
\]

where each \( s_{i_p} \) for \( 1 \leq p \leq k \) has a unique \( s_{i_q} \) with \( p < q \leq k + 1 \) such that \( |i_p - i_q| = 1 \).

Let \( w \) be a permutation and let \( N(w) \) denote the number of \([321]\) pattern instances in \( w \). We say \( w \) is **maximally clustered** if there is a reduced expression for \( w \) of the form

\[
a_0 c_1 a_1 c_2 a_2 \cdots c_M a_M
\]

where each \( a_i \) is a reduced expression, each \( c_i \) is a braid cluster with length \( 2n_i + 1 \) and \( N(w) = \sum_{i=1}^{M} n_i \). Such an expression is called **contracted**. In particular, \( w \) is **freely braided** if there is a reduced expression for \( w \) with \( N(w) \) disjoint short-braids.

This is not the original definition for the maximally clustered elements; however it is equivalent. The remarks in Section 5 of [GL02] show that the number of \([321]\) pattern instances in \( w \) equals the number of contractible triples of roots in the inversion set of \( w \). Corollary 4.11(ii) and Corollary 4.13 of [Los07] prove that \( w \) is a contracted reduced expression for a maximally clustered element if and only if it has the form given in Definition 1.4. Moreover, it follows from the proof of [Los07, Corollary 4.11(ii)] that the \( a_i \) in Definition 1.4 are fully commutative.

Recall that [BJS93] showed that \( w \) is fully commutative whenever \( N(w) = 0 \). In this work we will frequently use the fact that any braid cluster has the canonical form of Lemma 1.5.
Lemma 1.5. Suppose \( x = s_{i_1}s_{i_2} \ldots s_{i_k}s_{i_1} \) is a braid cluster of length \( 2k + 1 \) in type \( A \). Then, \( x = s_{m+1}s_{m+2} \ldots s_{m+k}s_{m+k+1} \ldots s_{m+2s_{m+1}} \) for some \( m \).

Proof. Since \( s_{i_{k+1}} \) is a transposition and conjugation preserves cycle type, \( x \) is a transposition and we may write \( x = (m+1 \ m+k' + 2) \) in cycle notation for some \( k', m \geq 0 \). This transposition is given by the expression \( x = s_{m+1}s_{m+2} \ldots s_{m+k'}s_{m+k'+1} \ldots s_{m+2s_{m+1}} \), which is a reduced expression for \( x \) by [Los07, Lemma 4.3]. Since the length of reduced expressions for \( x \) is an invariant of \( x, k' = k \) and the result follows. \( \square \)

Recall the following structural lemma about contracted reduced expressions.

Lemma 1.6. [Jon07, Lemma 2.3] Let \( w \) be a contracted reduced expression for a maximally clustered permutation, so \( w \) has the form

\[
a_0c_1a_1 \ldots c_Ma_M
\]

where each \( c_j \) is a braid cluster, and the \( a_j \) are short-braid avoiding. Then, any generator \( s_i \) that appears in any of the braid clusters \( c_j \) does not appear anywhere else in \( w \).

Lemma 1.7. Let \( w \) be a contracted reduced expression for a maximally clustered permutation. If the generator \( s_i \) supports a braid cluster in \( w \) then \( s_i \) supports a braid cluster in every contracted reduced expression for \( w \).

Proof. By [Los07, Lemma 4.11(i)], the contracted reduced expressions form a complete set of representatives for the commutativity classes of \( w \). Suppose the generator \( s_i \) supports a braid cluster in \( w \) in the sense of Definition 1.3 but there exists a contracted reduced expression \( w' = a_0c_1a_1 \ldots c_Ma_M \) for \( w \) in which \( s_i \) does not support a braid cluster. By the theorem of Matsumoto [Mat64] and Tits [Tit69], it suffices to consider a pair of heaps represented by \( w \) and \( w' \) that are related by a single short-braid move.

Observe that each short-braid move on \( w' \) can change the length of at most one braid cluster. This is clear if the short-braid move involves two entries from a single braid cluster, or one entry from some braid cluster \( c_i \) together with some entry of \( a_j \) for \( j \in \{i-1, i\} \). In the case that some \( a_i \) is the identity, note that there are no short-braid moves involving an entry from \( c_i \) and an entry from \( c_{i+1} \) by Lemma 1.6.

Therefore, by the equation \( N(w) = \sum_{i=1}^{M} n_i \) from Definition 1.4, we actually have that no short-braid move on \( w' \) can change the length of any of the braid clusters because \( N(w) \) does not depend on the reduced expression for \( w \). Thus we have shown that the length of each braid cluster remains the same over all contracted reduced expressions for \( w \). Finally, since no single short-braid move can change the support of a braid-cluster without changing its length, we have shown that the support of each braid cluster also remains the same over all contracted reduced expressions for \( w \). \( \square \)

Putting these lemmas together, we can show that there is a canonical heap associated to any maximally clustered permutation. See Example 1.10 for an illustration.

Definition 1.8. Let \( \mathcal{C} \) be a commutativity class for a permutation \( w \). Suppose that the set of columns \( \{1, \ldots, n\} \) of the heap of \( \mathcal{C} \) can be partitioned into intervals

\[
\tilde{C}_0 = [1, p_1 - 1], \tilde{B}_1 = [p_1, q_1], \tilde{C}_1 = [q_1 + 1, p_2 - 1], \ldots, \tilde{B}_k = [p_k, q_k], \tilde{C}_k = [q_k + 1, n]
\]

satisfying:

(1) For each \( i \), the heap of \( \mathcal{C} \) restricted to \( \tilde{B}_i \) is a braid cluster in the canonical form of Lemma 1.5.

(2) For each \( i \), every minimal pair of entries in any column from \( \tilde{C}_i \) has a distinct resolution, possibly using entries from columns \( q_i \) or \( p_{i+1} \).

Then, we say that the heap of \( \mathcal{C} \) has a braid cluster column decomposition given by \( \tilde{C}_0, \tilde{B}_1, \tilde{C}_1, \ldots, \tilde{B}_k, \tilde{C}_k \).
Proposition 1.9. Every maximally clustered permutation \( w \) has a unique commutativity class \( C \) such that the heap of \( C \) has a braid cluster column decomposition. Conversely, every heap having a braid cluster column decomposition corresponds to a maximally clustered permutation.

Proof. Let \( w \) be a maximally clustered permutation. By [Los07, Lemma 4.11(i)], the contracted reduced expressions form a complete set of representatives for the commutativity classes of \( w \). By Lemma 1.7, each generator either supports a unique braid cluster in some contracted reduced expression, in which case the generator must also support the braid cluster in every contracted reduced expression, or the generator is not used in any braid cluster. Hence, the set of braid clusters appearing in any contracted reduced expression for \( w \) is independent of the choice of reduced expression. Therefore, specifying a commutativity class of \( w \) depends only on a choice of commutativity class for each braid cluster. We choose each braid cluster to have the form given in Lemma 1.5 and declare this to be our canonical commutativity class \( C \) for \( w \).

Consider the heap associated to \( C \). We can partition the support of \( w \) into segments \( B_1, B_2, \ldots, B_k \) corresponding to the generators that support braid clusters. The complementary columns \( \{1, \ldots, n\} \setminus \bigcup_{i=1}^{k} B_i \) form intervals that we denote \( C_0, C_1, \ldots, C_k \).

By definition, this column partition satisfies property (1) of Definition 1.8. Any minimal pair of entries from some \( C_i \) without a distinct resolution would either form a short-braid or form an extension of an existing braid-cluster, contradicting the definition of the column partition we have given. Hence, the column partition satisfies property (2) of Definition 1.8 as well.

To prove the converse statement, fix a heap with a column partition satisfying properties (1) and (2) and let \( w \) be a corresponding expression. We show that \( w \) is reduced and maximally clustered.

If \( w \) is not reduced then there exists a sequence of braid moves bringing \( w \) to an expression whose heap has two entries in the same column with no entries between them. But the available braid moves in any expression related to \( w \) only involve entries from some braid cluster supported on columns \( B_i \) because the entries in the complementary columns \( \bigcup_{i=0}^{k} C_i \) all have distinct resolutions by property (2) and this remains true after performing arbitrarily many braid moves in \( \bigcup_{i=1}^{k} B_i \). Since the braid clusters supported on each \( B_i \) are reduced, we never have any opportunity for cancellation. Hence, \( w \) is reduced.

Moreover, we can read a contracted reduced expression for \( w \) from the given heap. We begin by linearly ordering the braid clusters \( B_1, \ldots, B_k \) in a way which is compatible with the order in which they appear in the heap poset. First read all of the entries in columns \( C = \bigcup_{i=0}^{k} C_i \) that lie below the entries of \( B_1 \) in the heap poset. Then for each \( i < k \) we sequentially read the entries of \( B_i \) followed by the entries from columns \( C \) appearing between \( B_i \) and \( B_{i+1} \) in the heap poset. Finally, we read the entries from columns \( C \) that appear above \( B_k \) in the heap poset.

Since the braid clusters have the canonical form of Lemma 1.5, we can verify that each braid cluster \( B_i \) supported on \( n_i + 1 \) columns has \( n_i \) [321]-instances using a string diagram overlaid on the heap, as illustrated in Figure 2. By Lemma 1.6 each braid cluster \( B_i \) takes exactly one string lying next to a column in \( C_i \) and crosses it with exactly one string lying next to a column in \( C_{i+1} \). Since each minimal pair of entries from any column of \( C_i \) has a distinct resolution by property (2), there are no other [321]-instances in \( w \). Therefore, the number of [321]-instances in \( w \) is exactly \( \sum_{i=0}^{k-1} n_i \), and each braid cluster \( B_i \) has length \( 2n_i + 1 \). Hence, \( w \) is maximally clustered by Definition 1.4.

Example 1.10. Suppose \( w \) is given by the contracted reduced expression

\[(s_5)(s_1s_2s_3s_4s_5s_2s_1)(s_6s_5s_9)(s_7s_8s_7)(s_6).\]
Then the heap of $w$ is drawn below, together with its braid cluster column decomposition. The braid clusters are shown in grey.

2. Enumeration

Let $S^P = \bigcup_{n \geq 1} S^P_n$ denote the permutations characterized by avoiding a set of 1-line patterns $P$. The most important pattern classes for this work are the maximally clustered permutations and the freely braided permutations, characterized by avoiding the patterns from (1.1) and (1.2), respectively. Given a finite set $H$ of permutations, let $S^P(H)$ be the subset of $S^P$ consisting of those permutations that heap-avoid the patterns in $H$.

In [Jon07, Section 4] we described how to find a set of 1-line patterns $Q$ such that $S^P(\{h\}) = SQ$, when possible. Example 11.1 of [BJ07] shows that some heap patterns $h$ have no such set $Q$.

Let $r(h)$ denote the rank of the symmetric group containing $h$. Define $U^P(h)$ to be the set of all elements in $S^P_{r(h)}$ that heap-contain $h$. This is a finite set because the rank is fixed.

**Definition 2.1.** [Jon07, Definition 4.2] Let $p \in S^P$. Then, we say that $p$ is an ideal pattern in $S^P$ if for every $q \in S^P_{r(p)+1}$ containing $p$ as a 1-line pattern, we have that $q$ heap-contains $p$.

For example, [Ten06, Theorem 3.8] implies that $p$ is an ideal pattern in $S^0$ if $p$ avoids $[2143]$. Definition 2.1 describes a finite test that extends to permutations of all ranks according to the following result.
Theorem 2.2. [Jon07, Theorem 4.4] Suppose $S^P(H)$ is the subset of permutations characterized by avoiding a finite set $P$ of 1-line patterns and heap-avoiding a finite set $H$ of permutations. If each of the elements in $P' = \bigcup_{h \in H} U^P(h)$ is an ideal pattern, then $S^P(H) = S^P \cup P'$, so is characterized by avoiding the permutations in $P \cup P'$ as 1-line patterns.

We can apply Theorem 2.2 to study certain classical permutation pattern classes using heap-avoidance. For example, it is straightforward to verify that

$$S^{[321]}(\{s_5s_6s_7s_8s_4s_5s_6s_2s_3s_4s_5s_1s_2s_3\}) = S^{[321]}, [46718235],[46781235],[56718234],[56781234]$$

using Theorem 2.2. Similar statements [Jon07, Corollary 4.5] hold for the freely braided and maximally clustered permutations. The permutations that heap-avoid

$[46718235] = s_5s_6s_7s_8s_4s_5s_6s_2s_3s_4s_5s_1s_2s_3$

are called hexagon-avoiding after [BW01].

Let $H$ be a finite set of connected fully commutative permutations each of whose heaps contains at least two entries in each internal column in the sense of Definition 1.3 or let $H = \emptyset$. Suppose $F_n$ is the subset of the fully commutative permutations on $n$ generators that are characterized by heap-avoiding the set of patterns from $H$, so $F_n = S^{[321]}(H) \subset c^{[321]}_n$. We define $|F_0|$ to be 1, corresponding to the empty heap. Our main result in this section is that we can transform the generating function $F(x) = \sum_{n \geq 0} |F_n| x^n$ to obtain generating functions for the corresponding freely braided and maximally clustered pattern classes. Moreover, the transformation is a rational function of $F(x)$, so it preserves this important property of the generating function.

We first consider the number of permutations in $F_n$ that have one or both of the extremal generators $\{s_1, s_n\}$ present in the heap. Recall that every minimal pair of entries in a fully commutative heap must have a distinct resolution, so in particular any fully commutative heap has at most a single entry in its leftmost and rightmost columns. Let $L_n$ be the permutations from $F_n$ that have $s_n$ present in their heap, and $R_n$ be the set of permutations from $F_n$ having $s_1$ present in the heap. By the bijection that reverses the ordering of the subscripts of the generators, we find that $|L_n| = |R_n|$. Let $M_n$ denote the set of permutations in $F_n$ with both $s_1$ and $s_n$ present in the heap. Then, we have the following enumerative result.

Lemma 2.3. Let $H$ be a finite set of fully commutative permutations or let $H = \emptyset$. Suppose

$$F(x) = \sum_{n \geq 0} |S^{[321]}(H)| x^n.$$

Then,

$$L(x) = \sum_{n \geq 0} |L_n| x^n = F(x) - xF(x) - 1$$

and

$$M(x) = \sum_{n \geq 0} |M_n| x^n = F(x) - 2xF(x) + x^2F(x) - 1.$$

Proof. This follows using inclusion-exclusion together with the observation that $|F_0| = 1$ and $|F_1| = 2$ so $|L_0| = |M_0| = |M_1| = 0$, which is required by definition. $\square$

We are now in a position to prove our main enumerative theorem.

Theorem 2.4. Let $H$ be a finite set of connected fully commutative permutations each of whose heaps contains at least two entries in each internal column or let $H = \emptyset$. Suppose

$$F(x) = \sum_{n \geq 0} |S^{[321]}(H)| x^n,$$
\[ L(x) = F(x) -xF(x) - 1, \text{ and } M(x) = F(x) -2xF(x) + x^2F(x) - 1. \]

Then, we have
\[
\sum_{n \geq 0} |S_{n+1}^{\{[3421],[4231],[4312],[4321]\}}(H)\| x^n = F(x) + \frac{L(x)^2}{1-M(x)}
\]

and
\[
\sum_{n \geq 0} |S_{n+1}^{\{[3421],[4312],[4321]\}}(H)\| x^n = F(x) + \frac{L(x)^2}{1-x-M(x)}.
\]

**Proof.** Suppose \( w \) is an element of \( S_{n+1}^{\{[3421],[4312],[4321]\}}(H) \). By Proposition 1.9, we may choose a commutativity class \( C \) of \( w \) so that the heap of \( C \) has a specific form where each column either supports a braid cluster, in which case there are no other generators in that column, or else the column is not used in any braid cluster.

Let \( k \) be the number of braid clusters in \( w \). We partition the columns \( \{1, \ldots, n\} \) of the heap of \( C \) into intervals \( C_0, B_1, C_1, B_2, \ldots, B_k, C_k \) based on the location of the braid clusters in \( w \). We define the \( r \)th interval \( C_r \) to be \([p, q]\) where \( p \) is the rightmost column of the \( r \)th braid cluster and \( q \) is the leftmost column of the \((r+1)\)st braid cluster, setting \( p = 1 \) for \( r = 0 \) and \( q = n \) for \( r = k \). The \( B_i \) then consist of the internal columns of the \( i \)th braid cluster. In particular, if \( w \) is freely braided then every \( B_i = \emptyset \). If \( w \) is fully commutative, then \( C_0 = \{1, \ldots, n\} \).

For example, suppose \( w \) is given by the contracted reduced expression
\[ (s_5)(s_1s_2s_3s_4s_3s_2s_1)(s_6s_5s_9)(s_7s_8s_7)(s_6). \]

Then the heap of \( w \) is drawn below, together with its column partition. The braid clusters are shown in grey.

Observe that the entries appearing in columns \( B_i \) are completely determined by the positions of the entries from the rightmost column in \( C_{i-1} \) and the leftmost column in \( C_i \), as these are the ends of the braid cluster supported by \( B_i \).

Next, we define a map \( \pi : \{C_0, C_1, \ldots, C_k\} \to S_{n+1} \) to project each interval of columns to a permutation. If \( k = 0 \), then \( \pi(C_0) = C_0 \). Otherwise, observe that the leftmost column in each \( C_i \) for \( i > 0 \) is the rightmost column of some braid cluster, so it consists of a single entry. The rightmost column \( q \) in each \( C_i \) for \( i < k \) is the leftmost column of some braid cluster, so it consists of two entries, and there can be no entries between them in column \( q - 1 \) by Definition 1.4. Hence, we may define a permutation \( \pi(C_i) \), whose heap is obtained from the heap of \( w \) restricted to the columns of \( C_i \), by collapsing the two entries in column \( q \) to a single entry.
In the example above, \( \pi(C_1) \) has the heap

```
• • • •
• •
[C_1]
```

Next, we claim that for each \( i \),

\[
\pi(C_i) \in \begin{cases} 
F_{|C_0|} & \text{if } k = 0, \\
L_{|C_0|} & \text{if } k > 0 \text{ and } i = 0, \\
M_{|C_1|} & \text{if } k > 0 \text{ and } 1 \leq i < k, \\
R_{|C_k|} & \text{if } k > 0 \text{ and } i = k.
\end{cases}
\]

In particular, we show that each \( \pi(C_i) \) corresponds to a fully commutative permutation that heap-avoids the patterns from \( H \).

The claim is clear if \( k = 0 \), so suppose \( k > 0 \), and let \( C_i = [p, q] \) with \( 0 < i < k \). Then \( \pi(C_i) \) is fully commutative because every minimal pair of entries in \( \pi(C_i) \) has a distinct resolution by property (2) in Definition 1.8 from Proposition 1.9.

Next, suppose there exists an instance of a heap-pattern \( h \in H \) among the entries of \( \pi(C_i) \). Then we must have such a heap-instance in \( w \) which is a contradiction. This follows because the heap of \( w \) with respect to the commutativity class where the \( (i + 1) \)st braid cluster is of the form

\[
s_{m+k+1}s_m+1 \ldots s_{m+1}s_m s_{m+1} \ldots s_{m+k}s_{m+k+1}
\]

has precisely the entries of \( \pi(C_i) \) in columns \( C_i \), and so \( w \) heap-contains \( h \). As this cannot occur, each \( \pi(C_i) \) heap-avoids the patterns from \( H \).

It follows from the construction that if \( k > 0 \) then \( \pi(C_0) \) has an entry in its rightmost column, \( \pi(C_k) \) has an entry in its leftmost column, and every other \( \pi(C_i) \) has entries in both the leftmost and rightmost columns. Otherwise, \( k = 0 \), and \( \pi(C_0) = C_0 \in F_{|C_0|} \). Thus, we have proved (2.1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Internal columns of a braid cluster}
\end{figure}

Conversely, suppose there exists a fixed a column partition \( \{C_0, B_1, C_1, \ldots, B_k, C_k\} \) as in the beginning of the proof and we are given either a single heap \( c^{(0)} \in F_{|C_0|} \) if \( k = 0 \) or a sequence of heaps

\[
c^{(0)}, b^{(1)}, c^{(1)}, b^{(2)}, \ldots, b^{(k)}, c^{(k)}
\]

where \( k > 0 \), \( c^{(0)} \in L_{|C_0|} \), \( c^{(k)} \in R_{|C_k|} \), every other \( c^{(i)} \in M_{|C_i|} \) and each \( b^{(i)} \) is a heap fragment on \( |B_i| \) columns with the canonical form shown in Figure 3. Then, we may form a maximally clustered permutation \( w \) with \( k \) braid clusters as follows. If \( k = 0 \), take \( w \) to be the permutation whose heap is \( c^{(0)} \). Otherwise, for each \( i < k \) we expand the single entry in the rightmost column of \( c^{(i)} \) to a pair of entries. This has the effect of reversing the \( \pi \) map on each \( c^{(i)} \), which we denote by \( \pi^{-1}(c^{(i)}) \). Then, glue
the columns together in order to form a single heap, so that the rightmost pair of entries in each \( \pi^{-1}(c^{(i)}) \) surrounds the leftmost pair of entries in \( b^{(i+1)} \) if any, or the unique leftmost entry in \( \pi^{-1}(c^{(i+1)}) \) if \( b^{(i+1)} \) is empty. Also, each \( b^{(i)} \) is glued to \( \pi^{-1}(c^{(i)}) \) so that the pair of entries in the rightmost column of \( b^{(i)} \) surrounds the unique entry in the leftmost column of \( \pi^{-1}(c^{(i)}) \). We call the permutation to which this heap corresponds \( w \).

Every minimal pair of entries \( x \) and \( y \) from an internal column of \( \pi^{-1}(c^{(i)}) \) must have a distinct resolution because \( c^{(i)} \) is fully commutative. Hence, we observe that the heap we constructed is reduced and maximally clustered because it has the form given in Proposition 1.9.

Next, observe that if \( w \) heap-contains a pattern \( h \in H \) then the pattern instance must lie in one of the \( c^{(i)} \). To see this, let \( x \) and \( y \) be a minimal pair of entries lying in column \( i \in [2, n - 1] \) of \( h \). Because \( h \) is fully commutative, \( x \) and \( y \) have a distinct resolution, so there exist entries from columns \( i + 1 \) and \( i - 1 \) lying strictly between \( x \) and \( y \) in the heap poset. However, no column in the heap of any commutativity class for a braid cluster has this property by the uniqueness statement in Definition 1.4, since each commutativity class for a braid cluster has a reduced expression representative that is itself a braid cluster. Thus, no heap-instance of \( h \) uses the internal columns of a braid cluster. Since we are assuming that each \( c^{(i)} \) heap-avoids the patterns from \( H \), we have that \( w \) heap-avoids the patterns from \( H \).

Finally, note that if we compose the constructions we have given above in either order, then we obtain the identity. Hence, we have shown a bijection and the enumerative formulas follow. \( F(x) \) contributes terms of the form where \( k = 0 \). Using Lemma 2.3, we obtain the generating function that counts sequences of the form given in (2.2) as the product

\[
L(x) \cdot \frac{1}{1 - x} \cdot \frac{1}{1 - M(x)} \cdot \frac{1}{1 - x} \cdot L(x)
\]

because each term of the sequence in (2.2) is independent. Multiplying the numerator and denominator of the third factor by \((1 - x)\), we obtain the second result. In the case where \( w \) is freely braided, each \( b^{(i)} \) term is \( \emptyset \), so the corresponding generating function is simply

\[
L(x) \cdot \frac{1}{1 - M(x)} \cdot L(x)
\]

which gives the first result. \( \square \)

**Corollary 2.5.** Let \( H \) be a finite set of connected fully commutative permutations each of whose heaps contains at least two entries in each internal column. If \( F(x) = \sum_{n \geq 0} |S_{n+1}^{[321]}(H)| x^n \) is a rational (respectively, algebraic) generating function, then

\[
\sum_{n \geq 0} |S_{n+1}^{[3421],[4231],[4312],[4321]}(H)| x^n \quad \text{and} \quad \sum_{n \geq 0} |S_{n+1}^{[3421],[4312],[4321]}(H)| x^n.
\]

are also rational (respectively, algebraic).

Using Theorems 2.2 and 2.4, we can enumerate several interesting classes characterized by permutation pattern avoidance. Note that we index the coefficients in the generating functions by the rank of the Coxeter group rather than by the number of entries appearing in the 1-line notation.
| [321]-avoiding | Generating function | Initial sequence |
|---------------|---------------------|-----------------|
| \( L(x) \) for \( H = \emptyset \) | \( \frac{1-2x-\sqrt{1-4x}}{2x^2} \) | \( 1 + 2x + 5x^2 + 14x^3 + 42x^4 + 132x^5 + 429x^6 + 1430x^7 + \ldots \) |
| \( M(x) \) for \( H = \emptyset \) | \( \frac{1-4x+3x^2-2x^3-(x-1)^2\sqrt{1-4x}}{2x^2} \) | \( x + 3x^2 + 9x^3 + 28x^4 + 90x^5 + 297x^6 + 1001x^7 + \ldots \) |
| Freely-braided | \( \frac{2x-2x^2-2x\sqrt{1-4x}}{-1+4x-2x^2+2x^3+(x-1)^2\sqrt{1-4x}} \) | \( 2x^2 + 6x^3 + 19x^4 + 62x^5 + 207x^6 + 704x^7 + \ldots \) |
| Maximally-clustered | \( \frac{2x}{-1+4x-2x^2+\sqrt{1-4x}} \) | \( 1 + 2x + 6x^2 + 21x^3 + 78x^4 + 298x^5 + 1157x^6 + 4539x^7 + \ldots \) |
| [321]-hexagon avoiding | \( \frac{-x^5+x^4+3x^3-4x^2+4x-1}{x^6-4x^2-4x^3+9x^4-11x^5+6x^6-1} \) | \( 1 + 2x + 5x^2 + 14x^3 + 42x^4 + 132x^5 + 429x^6 + 1426x^7 + \ldots \) |
| \( L(x) \) for \( H = \{[46718235]\} \) | \( \frac{2x^3+2x^2-2x^3+3x^2-x}{x^6-4x^2-4x^3+9x^4-11x^5+6x^6-1} \) | \( x + 3x^2 + 9x^3 + 28x^4 + 90x^5 + 297x^6 + 997x^7 + \ldots \) |
| \( M(x) \) for \( H = \{[46718235]\} \) | \( \frac{-x^5+2x^3+4x^2+4x^3+6x^3^2-2x^2}{x^6-4x^2-4x^3+9x^4-11x^5+6x^6-1} \) | \( 2x^2 + 6x^3 + 19x^4 + 62x^5 + 207x^6 + 700x^7 + \ldots \) |
| Freely-braided hexagon-avoiding | \( \frac{-x^5-2x^3+2x^3^2+4x^3+4x^3^2}{x^6-8x^2+4x^4+3x^3+6x-1} \) | \( 1 + 2x + 6x^2 + 20x^3 + 71x^4 + 260x^5 + 971x^6 + 3670x^7 + \ldots \) |
| Maximally-clustered hexagon-avoiding | \( \frac{3x^3+x^2-5x^2+7x^2-5x+1}{-3x^6+4x^3+8x^4-14x^3+15x^2-7x+1} \) | \( 1 + 2x + 6x^2 + 21x^3 + 78x^4 + 298x^5 + 1157x^6 + 4535x^7 + \ldots \) |

**Remark 2.6.** The first line is the Catalan generating function which appears in this context by [SS85], while the sixth line is Theorem 1.1 due to [SW04]. The freely braided permutations have previously been enumerated in [Man04], while the other formulas seem to be new. They follow from Theorem 2.2 together with Lemma 2.3 and Theorem 2.4 by taking \( H = \emptyset \) and \( H = \{[46718235]\} \), respectively.

**Corollary 2.7.** The number \( b_n \) of freely-braided hexagon-avoiding permutations in \( S_n \) satisfies the recurrence
\[
b_{n+1} = 6b_n - 9b_{n-1} + 3b_{n-2} + b_{n-3} - 8b_{n-4} - b_{n-5} + b_{n-6}
\]
and the number \( m_n \) of maximally-clustered hexagon-avoiding permutations in \( S_n \) satisfies the recurrence
\[
m_{n+1} = 7m_n - 15m_{n-1} + 14m_{n-2} - 8m_{n-3} - 4m_{n-4} + 3m_{n-5}
\]
for all \( n \geq 9 \), with initial conditions given by Figure 1.

3. DIAMOND REDUCTIONS

In this section, we restrict to considering fully commutative permutations.

**Definition 3.1.** Suppose that \( h \) is a connected, fully commutative permutation whose heap contains at least two entries in each internal column. We say that every minimal pair of entries together with the entries of their distinct resolution forms a minimal diamond inside the heap of \( h \). Form a new heap whose entries correspond to minimal diamonds. The poset structure of the new heap is inherited from the poset structure on the minimal diamonds in the heap of \( h \) by taking the transitive closure of the relation that two minimal diamonds are related if they share an edge. The heap obtained in this way from \( h \) is called the diamond reduction of \( h \).
Pictorially, the heap of the diamond reduction of $h$ is obtained from the heap of $h$ by adding entries at the centers of all minimal diamonds and then erasing the heap of $h$.

**Example 3.2.** The diamond reduction of the hexagon $s_5s_6s_7s_8s_4s_5s_6s_2s_3s_4s_5s_1s_2s_3$ is the 3-hexagon $s_4s_5s_2s_3s_4s_1s_2$.

**Proposition 3.3.** The diamond reduction is a bijection from the set of connected fully commutative heaps on $n$ columns with at least two entries in each internal column to the set of connected fully commutative heaps on $n-2$ columns.

**Proof.** Let $h$ be a connected fully commutative heap on $n$ columns with at least two entries in each internal column. The diamond reduction $g$ of $h$ gives a reduced heap because we can represent the minimal diamonds by their maximal entries (rather than the centers) and the result is a convex subposet of the heap of $h$. The heap of $g$ is also connected since every internal column of the heap of $h$ has a minimal diamond. The diamond reduction can be reversed by forming a heap with minimal diamonds centered at the heap entries of $g$. Hence, the diamond reduction of $h$ is fully commutative because any short-braid instance in the heap of $g$ would imply the existence of a short-braid in $h$ when we consider the minimal diamonds centered at the entries of the heap of $g$. We also reduce the number of columns by 2 since only the internal columns of the heap of $h$ support minimal diamonds that become entries in the heap of $g$. □

Our main goal in this section is to describe how the generating functions for $\{|S_{n+1}^{[321]}(h)|\}$ and $\{|S_{n+1}^{[321]}(g)|\}$ are related when $g$ is the diamond reduction of $h$.

**Lemma 3.4.** Let $H$ be a set of connected, fully commutative permutations. Suppose

$$F(x) = \sum_{n \geq 0} |S_{n+1}^{[321]}(H)| x^n$$

and

$$F_c(x) = \sum_{n \geq 0} \{|p \in S_{n+1}^{[321]}(H) : p \text{ is connected}\} x^n,$$

with $F(0) = 1$ and $F_c(0) = 0$. Then the generating functions are rationally related according to

$$F(x) = \frac{1 + F_c(x)}{1 - x - xF_c(x)} \quad \text{and} \quad F_c(x) = \frac{F(x) - xF(x) - 1}{1 + xF(x)}.$$

**Proof.** To see this, observe that every not-necessarily-connected $H$-avoiding heap on $n$ columns decomposes uniquely into connected components, each of which is counted by $F_c(x)$. Conversely, if we have an ordered collection $C$ of connected permutations that all heap-avoid the patterns from $H$, then we can place them together from left to right, to obtain a heap which has the heaps from $C$ as its connected components.

The generating function identity that we obtain from this observation is

$$F(x) = \frac{1}{1-x} + \frac{1}{1-x} \cdot F_c(x) \cdot \frac{1}{1 - \frac{x}{1-x} F_c(x)} \cdot \frac{1}{1-x}.$$
One fundamental subclass of the fully commutative permutations are those with no minimal diamonds at all. It is straightforward to see that this corresponds to having at most one entry in each column, or equivalently to heap-avoiding $s_2 s_1 s_3 s_2$. This pattern class was previously enumerated in [Fan98] and [Wes96]. Also, [Ten07] enumerates these permutations using the statistic of Coxeter length.

**Lemma 3.5.** We have

$$\sum_{n \geq 0} |S_{n+1}^{[321],[3412]}| \ x^n = \sum_{n \geq 0} |S_{n+1}^{[321]}(s_2 s_1 s_3 s_2)| \ x^n = \frac{1 - x}{1 - 3x + x^2}.$$  

**Proof.** The second equality follows from Theorem 2.2. Let $w$ be a connected element of $S_{n+1}^{[321]}(s_2 s_1 s_3 s_2)$. Then the heap of $w$ is a lattice path consisting of $n - 1$ steps that are either ”up” or “down.” As each step is independent, the generating function for these is $\frac{x^n}{n!}$. Applying Lemma 3.4 gives the result.

We are now in a position to state and prove our main result in this section.

**Theorem 3.6.** Suppose that $h$ is a connected, fully commutative permutation whose heap contains at least two entries in each internal column and let $g$ be the diamond reduction of $h$. If

$$G(x) = \sum_{n \geq 0} |S_{n+1}^{[321]}(g)| \ x^n, \quad G_c(x) = \sum_{n \geq 0} |\{p \in S_{n+1}^{[321]}(g) : p \text{ is connected}\}| \ x^n,$$

and $F(x) = \sum_{n \geq 0} |S_{n+1}^{[321]}(h)| \ x^n$, with $G(0) = 1 = F(0)$ and $G_c(0) = 0$, then the generating functions are rationally related according to

$$F(x) = \frac{1 - x - xG_c(x)}{1 - 3x + x^2 + (x^2 - x)G_c(x)} = \frac{1}{1 - 2x - x^2G(x)}.$$

**Proof.** Suppose $w \in S_{n+1}^{[321]}(h)$. We assign each of the columns $\{1, \ldots, n\}$ of the heap of $w$ to intervals $E_0, D_1, E_1, \ldots, D_k, E_k$ according to whether the column supports more than one entry. The intervals $D_i$ are defined to be precisely those that support maximal connected fully commutative subheaps of the heap of $w$ such that the internal columns of $D_i$ each support at least two entries. The remaining intervals $E_i$ consist of columns that support at most one entry. In particular, the $E_i$ may include empty columns.

For example, suppose

$$w = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_5 s_6 s_7 s_8 s_{12} s_8 s_{10}.$$ 

Then the heap of $w$ is drawn below, together with its column assignments.

```
   . . . . . . . . .
   . . . . . . . . .
   . . . . . . . . .
   . . . . . . . . .
   . . . . . . . . .
   . . . . . . . . .
   \ E_0   \  [E_1]  \  [E_2]  \  \\
   \ |     |  ||    |  ||
   \ D_1   || D_2  
```

Let $w|_{D_i}$ denote the restriction of the heap of $w$ to the columns in the interval $D_i$. By the hypotheses given, each $w|_{D_i}$ must heap-avoid $h$. Moreover, we can form the diamond reduction of $w|_{D_i}$, whose heap must be connected, heap-avoids $g$, and has 2 fewer columns than $w|_{D_i}$. Each $w|_{E_i}$ must heap-avoid $s_2 s_1 s_3 s_2$ and contains entries in the extremal columns of $E_i$ that are shared with $D_i$ or $D_{i+1}$. 

Conversely, suppose there exists a fixed column assignment \( E_0, D_1, E_1, \ldots, D_k, E_k \) and we are given a sequence of heaps \( e_0, d_1, e_1, \ldots, d_k, e_k \) such that \( e_0 \in \mathcal{S}_{[E_0]+1}(s_2 s_1 s_3 s_2) \) with one entry in the rightmost column of \( E_0 \), \( d_i \in \mathcal{S}_{[D_i]-1}(g) \) and \( d_i \) is connected for \( 1 \leq i \leq k \), \( e_i \in \mathcal{S}_{[E_i]+1}(s_2 s_1 s_3 s_2) \) for \( 1 \leq i \leq k \) with one entry in each of the extremal columns of \( E_i \) and \( E_k \). Then we can apply Proposition 3.3 to form the reverse diamond reduction \( \tilde{d}_i \) of \( d_i \) and glue these heaps together to obtain an element of \( \mathcal{S}_{[n+1]}(h) \). Specifically, we identify the unique entry in the rightmost column of \( \tilde{d}_i \) for each \( 0 \leq i \leq k - 1 \), and we also identify the unique entry in the leftmost column of \( e_i \) with the unique entry in the rightmost column of \( \tilde{d}_i \) for each \( 1 \leq i \leq k \).

If we compose the constructions we have given in either order, we obtain the identity. Hence, we have shown a bijection.

Let \( E(x) \) be the generating function appearing in Lemma 3.5. Define \( E_{LR}(x) = E(x) - xE(x) - 1 \) and \( E_M(x) = E(x) - 2xE(x) + x^2E(x) - 1 + x \). It is straightforward to verify using inclusion-exclusion that \( E_{LR}(x) \) counts the number of \( s_2 s_1 s_3 s_2 \)-avoiding heaps with one entry in at least one of the two extremal columns, and \( E_M(x) \) counts the number of \( s_2 s_1 s_3 s_2 \)-avoiding heaps with one entry in both of the extremal columns.

Putting all of these observations together, we have the generating function identity

\[
(3.2) \quad F(x) = E(x) + E_{LR}(x) \frac{1}{1 - G_c(x)E_M(x)G_c(x)E_{LR}(x)}.
\]

Here, \( E(x) \) contributes terms of the form where \( k = 0 \). This formula simplifies to the first equality given in the statement. The second equality follows from Lemma 3.4.

**Remark 3.7.** In the nicest cases, this result allows us to reduce the problem of enumerating permutations that avoid a given heap pattern to counting lattice paths that avoid a certain consecutive subpath. This problem is well known to have a rational generating function and the transfer matrix method can be used to find the generating function explicitly. See [Sta97, Example 4.7.5] for details.

**Example 3.8.**

\[
\sum_{n \geq 0} |\mathcal{S}_{n+1}(s_4 s_5 s_2 s_3 s_4 s_5 s_3)| \quad x^n = \sum_{n \geq 0} |\mathcal{S}_{n+1}(s_4 s_5 s_2 s_3 s_4 s_3 s_2)| \quad x^n = \frac{1 - 3x + 2x^2 - x^3}{1 - 5x + 7x^2 - 4x^3 + x^4}.
\]

The first equality follows from Theorem 2.2. The heap pattern \( p = s_4 s_5 s_2 s_3 s_4 s_1 s_2 s_3 \) has a diamond reduction to \( q = s_1 s_3 s_2 \).

The number of connected permutations that heap-avoid \( q \) is the same as the number of lattice paths that never contain a consecutive up-down subpath. It is straightforward to see that there are \( n \) such lattice paths that use \( n \) nodes. The corresponding generating function is \( G_c(x) = \frac{1}{(1-x)^2} \) and substituting this into Equation (3.1) yields the result.

**Example 3.9.**

\[
\sum_{n \geq 0} |\mathcal{S}_{n+1}(s_5 s_6 s_7 s_8 s_9 s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_1 s_2 s_3)| \quad x^n = \sum_{n \geq 0} |\mathcal{S}_{n+1}(s_5 s_6 s_7 s_8 s_9 s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_1 s_2 s_3)| \quad x^n
\]
recovering Theorem 1.1

The first equality follows from Theorem 2.2. The hexagon pattern has a diamond reduction to $q = s_1 s_3 s_1 s_3 s_1 s_3$ as shown in Example 3.2. However, the diamond reduction of $q$ is $s_1 s_3$ which is disconnected. Hence, we must enumerate the permutations heap-avoiding $q$ directly. We use the same column assignment as in the proof of Theorem 3.6. In order to heap-avoid $q$, the intervals of columns supporting diamonds must restrict to fully commutative permutations whose diamond reduction is a connected monotonic lattice path. Also, we can no longer glue two diamond-containing regions together along a trivial lattice path with one column. Hence, we must modify the $E_M(x)$ term of Equation (3.2) by subtracting $x$.

The monotonic lattice paths are counted by $G_c(x) = \frac{2x}{1-x} - x$. Substituting this into Equation (3.2) along with $E_M(x) = E(x) - 2x E(x) + x^2 E(x) - 1$ gives the generating function for all $q$-heap-avoiding permutations. Applying Theorem 3.6 then yields the result.

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