Review of Neutrino Oscillations With Sterile and Active Neutrinos

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Abstract

Recently neutrino oscillation experiments have shown that it is very likely that there are one or two sterile neutrinos. In this review neutrino oscillations with one, two, three sterile and three active neutrinos, and parameters that are consistent with experiments, are reviewed.

1 Introduction

This is a review of the method introduced by Sato and collaborators for three active neutrinos[1][2] extended to three active neutrinos plus one, two, or three sterile neutrinos. The transition probability for a neutrino of flavor $f_1$ to oscillate to a neutrino of flavor $f_2$, $P(\nu_{f_1} \to \nu_{f_2})$, is derived using S-Matrix theory, which is discussed in the next section with $f_1, f_2 \to$ muon, electron neutrinos.

In the following three sections the derivation of $P(\nu_\mu \to \nu_e)$ is described for one, two, three sterile neutrinos, with predictions using parameters of four recent neutrino oscillation experiments. In all three sections a U-matrix approach is used, introduced with a 3x3 U-matrix[1], and extended to a 4x4 U-martix with three active and one sterile neutrino a 5x5 U-martix with three active and two sterile neutrinos and a 6x6 U-marxix with three active and three sterile neutrinos.

From these sections the dependence of the $P(\nu_\mu \to \nu_e)$ neutrino oscillation probability on the number of sterile neutrinos and oscillation parameters will be shown.
2 \( \mathcal{P}(\nu_\mu \rightarrow \nu_e) \) for Active Neutrinos Derived Using Improved S-Matrix Theory

Neutrinos are produced as \( \nu_f \), with \( f = \text{flavor} = e, \mu, \tau \). They do not have definite mass, which is the cause of neutrino oscillations, which we now discuss. Active neutrinos with flavors \( \nu_e, \nu_\mu, \nu_\tau \) are related to neutrinos with definite mass \( \nu_m \), \( m=1,2,3 \) by the \( 3 \times 3 \) unitary matrix, \( U \),

\[
\nu_f = U \nu_m ,
\]

where \( \nu_f, \nu_m \) are \( 3 \times 1 \) column vectors and \( U \) a \( 3 \times 3 \) matrix. Therefore the electron state produced at time \( t=0 \) is

\[
|\nu_e > = \sum_{i=1}^{3} U_{1i} |\nu_{m_j} > .
\]

Making use of quantum theory, a state with energy \( E \) satisfies

\[
id/dt|E(t) > = E|E(t) > \quad \text{giving} \quad |E(t) > = e^{-iEt} |E(0) > ,
\]

or the electron neutrino state of Eq(2) at time \( t \) for the neutrino at rest \( (e=m, \text{ with } c \equiv 1) \) is

\[
|\nu_e, t > = \sum_{i=1}^{3} U_{1i} e^{-im_i t} |m_i, 0 > \quad \text{or}
\]

\[
|\nu_e, t > = \sum_{i=1}^{3} c_f |\nu_f > ,
\]

which shows that an electron neutrino produced at time \( t=0 \) oscillates to neutrinos of different flavors at time \( t \). Therefore an electron neutrino produced at \( t=0 \) when it travels a distance \( L \simeq t \) (as the velocity of the very low mass neutrinos is almost the speed of light) oscillates into \( e, \mu, \tau \) neutrinos. By placing detectors at a distance \( L \) this oscillation has been measured. The \( \nu_\mu \) neutrino has a similar relationship, and can also oscillate to a sterile neutrino as we discuss in sections below.

The 3x3 active neutrino U-matrix is \( (\sin \theta_{ij} \equiv s_{ij}, \text{ etc}) \).

\[
U = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\
 s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13}
\end{pmatrix}
\]

where \( c_{12} = .83 \), \( s_{12} = .56 \), \( s_{23} = c_{23} = .7071 \), \( s_{13} = 0.19 \), and \( \delta_{CP}=0 \) are used.

Given the Hamiltonian, \( H(t) \), for neutrinos, the neutrino state at time \( t = t_0 \) is obtained from the state at time \( t = t_0 \) from the S-matrix, \( S(t,t_0) \), by

\[
|\nu(t) > = S(t,t_0)|\nu(t_0) >
\]

\[
i \frac{d}{dt} S(t,t_0) = H(t) S(t,t_0) .
\]
In the vacuum the S-matrix is obtained from
\[ S_{ab}(t, t_0) = \sum_{j=1}^{3} U_{aj} e^{iE_j(t-t_0)} U_{bj}^*, \] (7)
while for neutrinos travelling through the earth the potential \( V = 1.13 \times 10^{-13} \) eV is included.

The transition probability \( \mathcal{P}(\nu_\mu \rightarrow \nu_e) \) is obtained from the S-Matrix element \( S_{12} \):
\[ \mathcal{P}(\nu_\mu \rightarrow \nu_e) = (\text{Re}[S_{12}])^2 + (\text{Im}[S_{12}])^2. \] (8)

From Ref\[4\]
\[ \text{Re}[S_{12}] = s_{23} a [\cos(\Delta L) \text{Im}[I_{\alpha^*}] - \sin(\Delta L) \text{Re}[I_{\alpha^*}]], \]
\[ \text{Im}[S_{12}] = -c_{23} \sin 2\theta \sin \omega L - s_{23} a [\cos(\Delta L) \text{Re}[I_{\alpha^*}]] + \sin(\Delta L) \text{Im}[I_{\alpha^*}], \] (9)

with \( \Delta L = -(V+\delta)/2, \Delta = \delta m_{13}^2/(2E), \delta = \delta m_{12}^2/(2E) \), where the neutrino mass differences are \( \delta m_{12}^2 = 7.6 \times 10^{-5} (eV)^2 \), \( \delta m_{13}^2 = 2.4 \times 10^{-3} (eV)^2 \), \( \sin 2\theta = s_{12} c_{12} \), \( a = s_{13}(\Delta - s_{12}^2 \delta) \), and \( E \) is the neutrino energy. Note that \( t \rightarrow L \), where \( L \) is the baseline, for \( v_\nu \approx c \). The neutrino-matter potential \( V = 1.13 \times 10^{-13} \) eV.

An important quantity is \( I_{\alpha^*} \)
\[ I_{\alpha^*} = \int_0^t dt' \alpha^*(t') e^{-i\Delta t'}, \] (10)
with \( \alpha(t) = \cos(\omega t) - i \cos 2\theta \sin(\omega t) \), \( \omega = \sqrt{\delta^2 + V^2 - 2\delta V \cos(2\theta_{12})}/2 \). In Ref\[4\], as in Ref\[5\], one used \( \delta, \omega \ll \Delta \) to obtain
\[ \text{Re}[I_{\alpha^*}] \approx \sin \Delta L/\Delta, \]
\[ \text{Im}[I_{\alpha^*}] \approx (1 - \cos \Delta L)/\Delta. \] (11)

In an improved theory\[6\] it was shown that
\[ \text{Re}[I_{\alpha^*}] = \left[(\omega - \Delta \cos 2\theta) \cos \Delta L \sin \omega L - (\Delta - \omega \cos 2\theta) \sin \Delta L \cos \omega L\right]/(\omega^2 - \Delta^2), \]
\[ \text{Im}[I_{\alpha^*}] = \left[\Delta + \omega \cos 2\theta - (\Delta + \omega \cos 2\theta) \cos \Delta L \cos \omega L - (\omega + \Delta \cos 2\theta) \sin \Delta L \sin \omega L\right]/(\omega^2 - \Delta^2). \] (12)

From Eqs\[9,12\] \( \mathcal{P}(\nu_\mu \rightarrow \nu_e) = (\text{Re}[S_{12}])^2 + (\text{Im}[S_{12}])^2 \) is obtained, giving the results shown in Figure 1.
$\mathcal{P}(\nu_\mu \rightarrow \nu_e)$ with old and precise $I_{\alpha^*}(E, L)$.

Figure 1: $\mathcal{P}(\nu_\mu \rightarrow \nu_e)$ for MINOS ($L=735$ km), MiniBooNE ($L=500$ m), JHF-Kamioka ($L=295$ km), and CHOOZ ($L=1.03$ km) using the improved $3\times3$ mixing matrix. Solid curve for precise $I_{\alpha^*}(E, L)$ and dashed curve for approximate $I_{\alpha^*}(E, L)$, $s_{13}=0.19$. 
3 $\mathcal{P}(\nu_\mu \to \nu_e)$ With Three Active and One Sterile Neutrino

Active neutrinos have only weak and gravitational interactions, and are therefore difficult to detect in neutrino oscillation experiments. Sterile neutrinos have no interaction except gravity and therefore cannot be detected via the apparatus used in neutrino oscillation or other experiments. For an overview of sterile neutrinos and neutrino oscillations see Ref[8] in which sterile neutrino states are investigated using neutrino oscillation data. These authors, J. Koppe et. al., considered one and two sterile neutrinos and discussed both experimental and theoretical publications.

In the present section we discuss neutrino oscillations with one sterile neutrino, while in the next section two sterile neutrinos are discussed.

Motivated by an experiment measuring neutrino oscillations[9], which suggested the existence of at least one sterile neutrino and estimated the mass differences and mixing angles with active neutrinos, estimates of $\mathcal{P}(\nu_\mu \to \nu_e)$ were made[10]. We now review this article.

This is an extension of the method introduced by Sato and collaborators for three active neutrino oscillations[1, 2] to three active neutrinos plus one sterile neutrino. Active neutrinos with flavors $\nu_e, \nu_\mu, \nu_\tau$ and a sterile neutrino $\nu_s$ are related to neutrinos with definite mass by

$$\nu_f = U \nu_m,$$

where $U$ is a 4x4 matrix and $\nu_f, \nu_m$ are 4x1 column vectors.

$$U = O^{23} \phi O^{13} O^{12} O^{24} O^{34}$$

with

$$O^{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{23} & s_{23} & 0 \\ 0 & -s_{23} & c_{23} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad O^{13} = \begin{pmatrix} c_{13} & 0 & s_{13} & 0 \\ 0 & 1 & 0 & 0 \\ -s_{13} & 0 & c_{13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$O^{12} = \begin{pmatrix} c_{12} & s_{12} & 0 & 0 \\ -s_{12} & c_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad O^{14} = \begin{pmatrix} c_\alpha & 0 & 0 & s_\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_\alpha & 0 & 0 & c_\alpha \end{pmatrix},$$

$$O^{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & 0 & s_\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -s_\alpha & 0 & c_\alpha \end{pmatrix}, \quad O^{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_\alpha & s_\alpha \\ 0 & 0 & -s_\alpha & c_\alpha \end{pmatrix},$$

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\delta_{CP}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
with $c_{12} = .83$, $s_{12} = .56$, $s_{23} = c_{23} = .7071$. We use $s_{13} = .15$ from the Daya Bay Collaboration[7]. In our present work we assume the angles $\theta_{ij} \equiv \alpha$ for all three $j$, and $s_{\alpha}, c_{\alpha} = \sin \alpha, \cos \alpha$. An important aspect of this work was to find the dependence of neutrino oscillation probabilities on $s_{\alpha}, c_{\alpha}$.

From Eq(14), the 4x4 $U$ matrix is

$$
\begin{pmatrix}
    c_{12}c_{13}c_{\alpha} & c_{13}(s_{12}c_{\alpha} - c_{12}s_{\alpha}^2) & -c_{13}s_{\alpha}(c_{12}c_{\alpha} + s_{12}) & c_{13}s_{\alpha}c_{\alpha}(c_{12}c_{\alpha} + s_{12}) + s_{13}s_{\alpha} \\
    Ac_{\alpha} & -As_{\alpha}^2 + Bc_{\alpha} & -As_{\alpha}^2c_{\alpha} - Bs_{\alpha}^2 & c_{13}s_{23}e^{i\delta_{CP}}c_{\alpha} \\
    Cc_{\alpha} & -Cs_{\alpha}^2 + Dc_{\alpha} & -Cs_{\alpha}^2c_{\alpha} - Ds_{\alpha}^2 & c_{13}c_{23}e^{i\delta_{CP}}c_{\alpha} \\
    -s_{\alpha} & -s_{\alpha}c_{\alpha} & -s_{\alpha}^2 & c_{\alpha}
\end{pmatrix}
$$

with

$$
A = -(c_{23}s_{12} + c_{12}s_{13}s_{23}e^{i\delta_{CP}}) \\
B = (c_{23}c_{12} - s_{12}s_{13}s_{23}e^{i\delta_{CP}}) \\
C = (s_{23}s_{12} - c_{12}s_{13}c_{23}e^{i\delta_{CP}}) \\
D = -(s_{23}c_{12} + s_{12}s_{13}s_{23}e^{i\delta_{CP}}).
$$

(15)

Using the formalism of Refs.1,2 extended to four neutrinos, the transition probability $\mathcal{P}(\nu_\mu \rightarrow \nu_\tau)$ is obtained from the 4x4 $U$ matrix and the neutrino mass differences $\delta m_{ij}^2 = m_i^2 - m_j^2$ for a neutrino beam with energy $E$ and baseline $L$ by [1]

$$
\mathcal{P}(\nu_\mu \rightarrow \nu_\tau) = \sum_{i=1}^{4} \sum_{j=1}^{4} U_{1i}U_{1j}^* U_{2i}U_{2j}^* e^{-i(\delta m_{ij}^2/E)L},
$$

(16)

or, with $\delta_{CP} = 0$ $U_{ij}^* = U_{ij}$,

$$
\mathcal{P}(\nu_\mu \rightarrow \nu_\tau) = U_{11}U_{21}[U_{11}U_{21} + U_{12}U_{22}e^{-i\Delta L} + U_{13}U_{23}e^{-i\Delta L} + U_{14}U_{24}e^{-i\gamma L}] + U_{12}U_{22}[U_{12}U_{22} - U_{13}U_{23}e^{-i\Delta L} + U_{14}U_{24}e^{-i\gamma L}] + U_{13}U_{23}[U_{13}U_{23} - U_{12}U_{22}e^{-i\Delta L} + U_{14}U_{24}e^{-i\gamma L}] + U_{14}U_{24}[U_{14}U_{24} - U_{12}U_{22} + U_{13}U_{23}]e^{-i\gamma L} + U_{14}U_{24},
$$

(17)

with $\delta = \delta m_{12}^2/2E$, $\Delta = \delta m_{13}^2/2E$, $\gamma = \delta m_{14}^2/2E$ (j=1,2,3). The neutrino mass differences are $\delta m_{12}^2 = 7.6 \times 10^{-5}(eV)^2$, $\delta m_{13}^2 = 2.4 \times 10^{-3}(eV)^2$; and we use both $\delta m_{14}^2 = 0.9(eV)^2$ and $\delta m_{24}^2 = 0.043(eV)^2$, since $\delta m_{14}^2 = 0.043(eV)^2$ was the best fit parameter found via the 2013 MiniBooNE analysis, while $\delta m_{24}^2 = 0.9(eV)^2$ is the best fit using the 2013 MiniBooNE data and previous experimental fits[9].

Note that in Refs.3,4 $\mathcal{P}(\nu_\mu \rightarrow \nu_\tau) = |S_{12}|^2$, with $S_{12}$ obtained from the 3x3 $U$-matrix and the $\delta m_{ij}$ parameters. Therefore our formalism, given by Eq(17), is quite different, and as will be shown for the same $L, E$ the magnitude of $\mathcal{P}(\nu_\mu \rightarrow \nu_\tau)$ is also different. Since the S-matrix formalism was not used in Refs.1,2 for the 3x3 study, $\mathcal{P}(\nu_\mu \rightarrow \nu_\tau)$ was quite different from Refs.3,4.
From Eq(17),

\[
\mathcal{P}(\nu_\mu \rightarrow \nu_e) = U_{11}^2 + U_{12}^2 + U_{13}^2 + U_{14}^2 + 2U_{11}U_{21}U_{12}U_{22}\cos\delta + 2(1 - U_{12}^2 - U_{13}^2)\cos\Delta + 2U_{14}U_{24}(1 - U_{12}^2 - U_{13}^2)\cos\gamma.
\]  

(18)

Using the parameters given above,

\[
\begin{align*}
U_{11} &= 0.822 c_\alpha & U_{12} &= -0.554 s_\alpha + 0.084 c_\alpha \\
U_{13} &= -0.822 s_\alpha c_\alpha - 0.554 c_\alpha + 1.5 c_\alpha & U_{14} &= 0.822 c_\alpha s_\alpha + 0.554 s_\alpha c_\alpha + 1.5 s_\alpha \\
U_{21} &= -0.484 c_\alpha & U_{22} &= 0.484 c_\alpha + 0.527 c_\alpha \\
U_{23} &= 0.484 c_\alpha - 0.527 s_\alpha + 0.7 c_\alpha & U_{24} &= -0.484 c_\alpha s_\alpha + 0.527 s_\alpha c_\alpha + 0.7 s_\alpha.
\end{align*}
\]  

(19)

With the addition of a sterile neutrino, the 4th neutrino, there are three new angles, \(\theta_{14}, \theta_{24}, \) and \(\theta_{34}\). The main assumption is that these three angles are the same, \(\theta_{ji} = \alpha\). The angle \(\alpha\) is the main parameter that is being studied.

Two values for the sterile-active mass differences are used. The most widely accepted value for \(m_4^2 - m_1^2\) is 0.9(\(eV\))^2 [9], but we also use \(m_4^2 - m_1^2 = 0.043(\text{eV})^2\) from the 2013 MiniBooNE result to test the sensitivity of \(\mathcal{P}(\nu_\mu \rightarrow \nu_e)\) to the sterile neutrino-active neutrinos mass differences. Since \(m_4^2 - m_1^2 >> m_j^2 - m_i^2\) for \((i,j)=1,2,3\), we assume that \(m_4^2 - m_j^2 = m_i^2 - m_1^2\).

Figure 2 shows the results for \(\mathcal{P}(\nu_\mu \rightarrow \nu_e)\) for the four experiments with \(m_4^2 - m_1^2 = 0.9(\text{eV})^2\) and \(\alpha = 45^\circ, 60^\circ, 30^\circ\). As one can see, \(\mathcal{P}(\nu_\mu \rightarrow \nu_e)\) is very strongly dependent on \(\alpha\).

Next \(m_4^2 - m_1^2 = 0.043(\text{eV})^2\) was used, as found in the recent MiniBooNE experiment, to study the effects of \(m_4^2 - m_1^2\) on \(\mathcal{P}(\nu_\mu \rightarrow \nu_e)\), with \(\alpha = 45^\circ, 30^\circ, 60^\circ\), as shown in Figure 3.

Note for \(\alpha = 0\) (no sterile-active mixing) \(U_{14} = 0\). Therefore, \(\mathcal{P}(\nu_\mu \rightarrow \nu_e)\) is a 3x3 theory; however, we find that \(\mathcal{P}(\nu_\mu \rightarrow \nu_e)\) is different with the model of Refs. [11] [2].

An article Neutrino Oscillations With Recently Measured Sterile-Active Neutrino Mixing Angle \(\sin(\alpha) = 0.16\) was recently published [11], with estimates of \(\mathcal{P}(\nu_\mu \rightarrow \nu_e)\), is shown in Figure 4.
Figure 2: The ordinate is $P(\nu_\mu \rightarrow \nu_e)$ for MINOS (L=735 km), MiniBooNE (L=500 m), JHF-Kamioka (L=295 km), and CHOOZ (L=1.03 km) using the 4x4 $U$ matrix with $\delta m^2_{24} = 0.9(eV)^2$, $s_{13} = 0.15$, and (a),(b),(c) for $\alpha = 45^\circ, 60^\circ, 30^\circ$. The dashed curves are for $\alpha = 0 (3x3)$. 
Figure 3: The ordinate is $P(\nu_\mu \to \nu_e)$ for MINOS (L=735 km), MiniBooNE (L=500m), JHF-Kamioka (L=295 km), and CHOOZ (L=1.03 km) using the 4x4 U matrix with $\delta m^2_{4j} = 0.043(\text{eV})^2$, $s_{13} = 0.15$, and (a),(b),(c) for $\alpha = 45^\circ$, $60^\circ$, $30^\circ$. The dashed curves are for $\alpha = 0$ (3x3).
Figure 4: The ordinate is $P(\nu_{\mu} \rightarrow \nu_{e})$ for MINOS (L=735 km), MiniBooNE (L=500 m), JHF-Kamioka (L=295 km), and CHOOZ (L=1.03 km) using the 4x4 U matrix with $\delta m^{2}_{4j} = 0.9 (eV)^{2}$ and $\sin(\alpha) \simeq 0.16$. The dashed curves are for $\alpha = 0$ (3x3).
4 \( P(\nu_\mu \rightarrow \nu_e) \) With Three Active and Two Sterile Neutrinos

Recent reviews of experimental data on neutrino oscillations\cite{8, 12} find that there probably are two sterile neutrino with the mass and mixing angles used in the present review, which is based on Ref\cite{13}.

With three active and two sterile neutrinos one uses a 5x5 U-matrix to estimate the transition porbability for a muon neutrino to oscillate to an electron neutrino \( P(\nu_\mu \rightarrow \nu_e) \). The U-matrix that relates neutrinos with definite mass \( m \) to neutrinos with definite flavor \( f \), Eq(13), with the sterile-active mixing angles \( s_\alpha = \sin(\alpha), c_\alpha = \cos(\alpha), s_\beta = \sin(\beta), c_\beta = \cos(\beta) \), where \( \alpha = \theta_{i4}, \beta = \theta_{i5} \) are sterile-active neutrino mixing angles, with \( i=1,2,3 \), and \( \delta_{CP}=0 \), is

\[
U = O^{23}O^{13}O^{12}O^{14}O^{24}O^{34}O^{15}O^{25}O^{35}O^{45},
\]

where \( (O^{45}, \text{giving sterile-sterile neutrino mixing, is not shown}) \)

\[
O^{23} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & c_{23} & s_{23} & 0 & 0 \\
0 & -s_{23} & c_{23} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad O^{13} = \begin{pmatrix}
c_{13} & 0 & s_{13} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-s_{13} & 0 & c_{13} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
O^{12} = \begin{pmatrix}
c_{12} & s_{12} & 0 & 0 & 0 \\
-s_{12} & c_{12} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad O^{14} = \begin{pmatrix}
c_{\alpha} & 0 & 0 & s_{\alpha} & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-s_{\alpha} & 0 & 0 & c_{\alpha} & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
O^{24} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & c_{\alpha} & 0 & s_{\alpha} & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -s_{\alpha} & 0 & c_{\alpha} & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad O^{34} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & c_{\alpha} & s_{\alpha} & 0 \\
0 & 0 & -s_{\alpha} & c_{\alpha} & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\phi = \begin{pmatrix}
c_{\beta} & 0 & 0 & 0 & s_{\beta} \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-s_{\beta} & 0 & 0 & 0 & c_{\beta} \\
\end{pmatrix}, \quad O^{25} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & c_{\beta} & 0 & 0 & s_{\beta} \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & -s_{\beta} & 0 & 0 & c_{\beta} \\
\end{pmatrix}
\]
The active neutrino mass differences are $m_{ij} = m_i^2 - m_j^2$ for a neutrino beam with energy $E$ and baseline $L$ by

$$\mathcal{P}(\nu_\mu \to \nu_e) = \text{Re} \left[ \sum_{i=1}^{5} \sum_{j=1}^{5} U_{1i}^* U_{2j}^* U_{3j} U_{2i} e^{-i(\delta m_{ij}^2/E)L} \right].$$

(21)

From Eq.(20), multiplying the nine 5x5 $O$ matrices, one obtains the matrix $U$. With $\delta_{CP}=0$, $U_{ii}^\ast = U_{ij}$, so we only need $U_{1j}, U_{2j}$. The active neutrino mixing parameters[3] are $c_{23} = s_{23} = 0.7071, c_{13} = 0.989, s_{13} = 0.15, c_{12} = 0.83, s_{12} = 0.56$.

$$U_{11} = 0.821ca cb$$
$$U_{12} = (0.554ca - 0.821sa^2)cb - 0.821ca sb^2$$
$$U_{13} = (0.15ca - 0.554sa^2 - 0.821ca sa^2)cb - (0.554ca - 0.821sa^2)sb^2$$
$$\quad + 0.821ca cb sb^2$$
$$U_{14} = cb(0.15sa + 0.554ca sa + 0.821ca^2 sa) - 0.821ca cb^2 sb^2$$
$$\quad - (0.554ca - 0.821sa^2)cb sb^2 - (0.15ca - 0.554sa^2 - 0.821casb^2)sb^2$$
$$U_{15} = 0.821ca sb cb^3 + (0.15ca + 0.554ca sa + 0.821ca^2 sa)sb$$
$$\quad + (0.554ca - 0.821sa^2)cb^2 sb + (0.15ca - 0.554sa^2 - 0.821ca sa^2)cb sb$$
$$U_{21} = -0.484ca cb$$
$$U_{22} = (0.527ca + 0.484sa^2)cb + 0.484ca sb^2$$
$$U_{23} = (0.699ca - 0.527sa^2 + 0.484ca sa^2)cb - (0.527ca + 0.484sa^2)sb^2 + 0.484ca cb sb^2$$
$$U_{24} = cb(0.699sa + 0.527ca sa - 0.484ca^2 sa) + 0.484ca cb^2 sb^2$$
$$\quad - (0.527ca + 0.484sa^2)cb sb^2 - (0.699ca - 0.527sa^2 + 0.484ca sa^2)sb^2$$
$$U_{25} = -0.484ca sb cb^3 + (0.699sa + 0.527ca sa - 0.484ca^2 sa)sb$$
$$\quad + (0.527ca + 0.484sa^2)cb^2 sb + (0.699ca - 0.527sa^2 + 0.484ca sa^2)cb sb$$

The active neutrino mass differences are $\delta m_{12}^2 = m_2^2 - m_1^2 = 7.6 \times 10^{-5}(eV)^2, \delta m_{13}^2 = m_3^2 - m_1^2 \simeq 2.4 \times 10^{-3}(eV)^2$. From Ref[3] the two sterile-active mass differences are $\delta m_{4i}^2 = m_i^2 - m_4^2 \simeq 0.5 (eV)^2, \delta m_{5i}^2 = m_i^2 - m_5^2 \simeq 0.9 (eV)^2$, with i=1,2,3 for active neutrinos, and $\delta m_{54}^2 = m_6^2 - m_4^2 \simeq 0.4 (eV)^2$.  

12
For $\mu$ neutrino disappearance the sterile-active neutrino mixing angle $\theta_{\mu\mu}$ is given by\cite{8} $\sin 2\theta_{\mu\mu} = 2|U_{\mu 4}|\sqrt{1 - U_{\mu 4}^2}$, with a similar form for e-neutrino disappearance. From Ref\cite{8} $U_{\mu 4} \simeq U_{e 4} \simeq U_{\mu 5} \simeq U_{e 5} = 0.13 - 0.17$. Therefore $\alpha \simeq \beta \simeq 7.5^\circ$ to $10^\circ$, with $9.2^\circ$ the sterile-active mixing angle used in Refs\cite{10, 6}.

With the mass differences $\delta m_{12}^2$, $\delta m_{13}^2$, $\delta m_{23}^2$, $\delta m_{4i}^2$, $\delta m_{54}^2$ given above, and the definitions $\delta = \delta m_{12}^2/2E$, $\Delta = \delta m_{13}^2/2E$, $\gamma = \delta m_{4i}^2/2E$, $\lambda = \delta m_{5i}^2/2E$, $\kappa = \delta m_{54}^2/2E$,

\[
\mathcal{P}(\nu_e \rightarrow \nu_e) = Re[\nu_{11}U_{21}(U_{11}U_{21} + U_{12}U_{22}e^{-i\theta L} + U_{13}U_{23}e^{-i\Delta L} + \\
U_{14}U_{24}e^{-i\lambda L} + U_{15}U_{25}e^{-i\kappa L}) + \\
U_{12}U_{22}(U_{11}U_{21}e^{-i\delta L} + U_{12}U_{22} + U_{13}U_{23}e^{-i\Delta L} + \\
U_{14}U_{24}e^{-i\lambda L} + U_{15}U_{25}e^{-i\kappa L}) + U_{13}U_{23}(U_{11}U_{21}e^{-i\Delta L} + U_{12}U_{22}e^{-i\Delta L} \\
+ U_{13}U_{23} + U_{14}U_{24}e^{-i\lambda L} + U_{15}U_{25}e^{-i\kappa L}) + U_{14}U_{24}(U_{11}U_{21} + U_{12}U_{22} \\
+ U_{13}U_{23} + U_{14}U_{24} + U_{15}U_{25}e^{-i\kappa L}) \\
+ U_{15}U_{25}((U_{11}U_{21} + U_{12}U_{22} + U_{13}U_{23})e^{-i\lambda L} + U_{14}U_{24}e^{-i\kappa L} + U_{15}U_{25})]
\]

(23)

From Eq(23)

\[
\mathcal{P}(\nu_e \rightarrow \nu_e) = U_{11}^2U_{21}^2 + U_{12}^2U_{22}^2 + U_{13}^2U_{23}^2 + U_{14}^2U_{24}^2 + U_{15}^2U_{25}^2 + \\
2U_{11}U_{21}U_{12}U_{22}\cos\delta L + \\
2(U_{11}U_{21}U_{13}U_{23} + U_{12}U_{22}U_{13}U_{23})\cos\Delta L + \\
2U_{14}U_{24}(U_{11}U_{21} + U_{12}U_{22} + U_{13}U_{23})\cos\gamma L + \\
2U_{15}U_{25}(U_{11}U_{21} + U_{12}U_{22} + U_{13}U_{23})\cos\lambda L + \\
2U_{14}U_{24}U_{15}U_{25}\cos\kappa L.
\]

(24)

From Eq(17) and the discussion below that equation, $\alpha \simeq \beta \simeq 7.5^\circ$, with $sa = sb \simeq 0.131$ and $ca = cb \simeq 0.991$, and $\alpha \simeq \beta \simeq 10^\circ$, with $sa = sb \simeq 0.174$ and $ca = cb \simeq 0.985$, which are used to determine $U_{13}$, $U_{23}$ in Eq(22).

In Fig. 5 the results of the two sterile neutrinos on $\mathcal{P}(\nu_e \rightarrow \nu_e)$ using Eq(24) and the parameters obtained from Refs\cite{8, 12} are shown for four experimental neutrino oscillation experiments.

The figure also shows $\mathcal{P}(\nu_e \rightarrow \nu_e)$ with $\alpha = \beta = 0^\circ$, giving the results of a recent 3x3 S-matrix calculation\cite{6} to compare to the results with two sterile neutrinos.
Using Eq(24), one finds $\mathcal{P}(\nu_\mu \rightarrow \nu_e)$

![Figure 5: $\mathcal{P}(\nu_\mu \rightarrow \nu_e)$ for MINOS(L=735 km), MiniBooNE(L=500 m), JHF-Kamioka(L=295 km), and CHOOZ(L=1.03 km). (a) solid $\alpha = \beta = 10.0^\circ$; (b) dash-dotted for $\alpha = \beta = 7.5^\circ$; (c) dashed curve for $\alpha = \beta = \gamma = 0^\circ$ giving the 3x3 result.](image-url)
This review of neutrino oscillations with three active and three sterile neutrinos is based on Ref[14]. Active neutrinos with flavors $\nu_e, \nu_\mu, \nu_\tau$ and three sterile neutrinos, $\nu_{s_1}, \nu_{s_2}, \nu_{s_3}$ are related to neutrinos with definite mass by

$$\nu_f = U \nu_m, \quad (25)$$

where $U$ is a 6x6 matrix and $\nu_f, \nu_m$ are 6x1 column vectors. We use the notation $s_{ij}, c_{ij} = \sin \theta_{ij}, \cos \theta_{ij}$, with $\theta_{12}, \theta_{23}, \theta_{13}$ the mixing angles for active neutrinos; and $s_\alpha = \sin(\alpha), c_\alpha = \cos(\alpha), s_\beta = \sin(\beta)$, etc, where $\alpha, \beta, \gamma$ are sterile-active neutrino mixing angles.

$$U = O^{23}O^{12}O^{14}O^{24}O^{34}O^{15}O^{25}O^{35}O^{45}O^{16}O^{26}O^{36}O^{46}O^{56} \quad (26)$$

with ($O^{45}, O^{46}, O^{56}$, giving sterile-sterile neutrino mixing, are not shown)

$$O^{23} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{23} & s_{23} & 0 & 0 & 0 \\
0 & -s_{23} & c_{23} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad O^{13} = \begin{pmatrix}
c_{13} & 0 & s_{13} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-s_{13} & 0 & c_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$O^{12} = \begin{pmatrix}
c_{12} & s_{12} & 0 & 0 & 0 & 0 \\
-s_{12} & c_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad O^{14} = \begin{pmatrix}
c_{\alpha} & 0 & 0 & s_{\alpha} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-s_{\alpha} & 0 & 0 & c_{\alpha} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

$$O^{24} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{\alpha} & 0 & s_{\alpha} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -s_{\alpha} & 0 & c_{\alpha} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad O^{34} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{\alpha} & s_{\alpha} & 0 & 0 \\
0 & 0 & -s_{\alpha} & c_{\alpha} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$
\[
\begin{align*}
O^{15} &= \begin{pmatrix}
  c_{\beta} & 0 & 0 & 0 & s_{\beta} & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
 -s_{\beta} & 0 & 0 & 0 & c_{\beta} & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} & \quad O^{25} &= \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & c_{\beta} & 0 & 0 & s_{\beta} & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
 0 & -s_{\beta} & 0 & 0 & c_{\beta} & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
O^{35} &= \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & c_{\beta} & 0 & s_{\beta} & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & -s_{\beta} & 0 & c_{\beta} & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} & \quad O^{16} &= \begin{pmatrix}
  c_{\gamma} & 0 & 0 & 0 & 0 & s_{\gamma} \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
 0 & -s_{\gamma} & 0 & 0 & c_{\gamma} & 0 \\
  0 & 0 & -s_{\gamma} & 0 & 0 & c_{\gamma} \\
\end{pmatrix} \\
O^{26} &= \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & c_{\gamma} & 0 & 0 & 0 & s_{\gamma} \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & -s_{\gamma} & 0 & 0 & c_{\gamma} & 0 \\
\end{pmatrix} & \quad O^{36} &= \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & -s_{\gamma} & 0 & 0 & c_{\gamma} \\
  0 & 0 & -s_{\gamma} & 0 & 0 & c_{\gamma} \\
\end{pmatrix}
\end{align*}
\]

\(P(\nu_\mu \rightarrow \nu_e)\) is obtained from the 6x6 U matrix and the neutrino mass differences \(\delta m_{ij}^2 = m_i^2 - m_j^2\) for a neutrino beam with energy \(E\) and baseline \(L\) by

\[
P(\nu_\mu \rightarrow \nu_e) = \text{Re}\left[\sum_{i=1}^{6} \sum_{j=1}^{6} U_{1i}^\ast U_{1j} U_{2i}^\ast U_{2j} e^{-i(\delta m_{ij}^2/E)L}\right],
\]

an extension of the 4x4\cite{10,6} theory with one sterile neutrino, which used the 3x3 formalism of Ref\cite{11}, to a 6x6 matrix formalism\cite{15}. From Eq\cite{26}, multiplying the 12 6x6 \(O\) matrices, we obtain the matrix \(U\). With \(\delta_{CP}=0\), \(U_{ij}^\ast = U_{ij}\), so we only need \(U_{1j}, U_{2j}\), with the numbers in Eqs\cite{28,29} given in Ref\cite{11}.

\[
\begin{align*}
U_{11} &= .821ca \ cb \ cg \\
U_{12} &= cg((.554ca -.821sa^2)cb - .821ca \ sb^2) -.821ca \ cb \ sg^2 \\
U_{13} &= cg((.15ca -.554sa^2 -.821ca \ sa^2)cb - (.554ca -.821sa^2)sb^2 \\
& - .821ca \ cb \ sb^2) -.821ca \ cb \ cg \ sg^2 - ((.554ca -.821sa^2)cb - .821ca \ sb^2)sg^2 \\
U_{14} &= cg(cb(.15sa + .554ca \ sa + .821ca^2 \ sa) - .821ca \ cb \ sb^2 \\
& - (.554ca -.821sa^2)cb \ sb^2 - (.15ca -.554sa^2 -.821cas^2)sb^2) - .821ca \ cb \ sg^2cg^2 \\
& - cg((.554ca -.821sa^2)cb - .821ca \ sb^2)sg^2 - (cb(.15ca -.554sa^2 -.821cas^2) \\
& - .821ca \ cb \ sb^2 - (.554ca -.821sa^2)sb^2)sg^2
\end{align*}
\]
\[ U_{15} = \begin{align*} & cg(.821ca \ sb \ cb^3 + (.15sa + .554ca \ sa + .821ca^2 \ sa)\ sb \\ & + (.554ca - .821sa^2) \ cb \ sb + (.15ca - .554sa^2 - .821ca \ sa^2) \ cb \ sb) \\ & -.821ca \ cb \ cg^3 \ sg^2 - cg^2(cb(.554ca - .821sa^2) - .821sb^2) sg^2 \\ & - cg(cb(.15ca - .554sa^2 - .821ca \ sa^2) - .821ca \ cb \ sb^2 \\ & -(554ca - .821sa^2) sb^2 sg^2 - (cb(.15ca + .554ca \ sa + .821ca^2 \ sa)) - .821ca \ cb^2 \ sb^2 \\ & - cb(.554ca - .821sa^2) sb^2 + (.15ca - .554sa^2 - .821ca \ sa^2) sb^2) sg^2 \end{align*} \]

\[ U_{16} = \begin{align*} & -.821ca \ cb \ cg \ cg^4 + (.821ca \ cb^3 \ sb + (.15ca + .554ca \ sa + .821ca^2 \ sa)) \ sb \\ & + cb(554ca - .821sa^2) sb + cb(.15ca - .554sa^2 - .821ca \ sa^2) sb) sg \\ & + c g^3((554ca - .821sa^2) cb - .821ca \ sb^2) sg + \\ & cg^2(cb(.15ca - .554sa^2 - .821ca \ sa^2) - .821ca \ cb \ sb^2 \\ & -(554ca - .821sa^2) sb^2) sg \\ & + cg(cb(.15sa + .554ca \ sa + .821ca^2 \ sa)) - .821ca \ cb^2 \ sb^2 \\ & - cb(.554ca - .821sa^2) sb^2 - (15ca - .554sa^2 - .821ca \ sa^2) sb^2) sg, \end{align*} \] (28)

\[ U_{21} = -.484ca \ cb \ cg \]
\[ U_{22} = \begin{align*} & cg(.527ca + .484sa^2) cb - .821ca \ sb^2) + .484ca \ cb \ sg^2 \end{align*} \]
\[ U_{23} = \begin{align*} & cg((.699ca - .527sa^2 + .484ca \ sa^2) cb - (.527ca + .484sa^2) sb^2 + .484ca \ cb \ sb^2) \\ & + .484ca \ cb \ cg \ sg^2 - ((.527ca + .484sa^2) cb + .484ca \ sb^2) \ * \ sg^2 \end{align*} \]

\[ U_{24} = \begin{align*} & cg(cb(.699sa + .527ca \ sa - .484ca \ sa^2) + .484ca \ cb \ sb^2 \\ & -(527ca + .484sa^2) \ cb \ sb^2 - (.699ca - .527sa^2 + .484ca \ sa^2) sb^2) + .484ca \ cb \ sg^2 \ cg^2 \\ & - cg((.527ca + .484sa^2) cb + .484ca \ sb^2) sg^2 - (cb(.69ca - .527sa^2 + .484ca \ sa^2) + \\ & .484ca \ cb \ sb^2 - (.527ca + .484sa^2) \ sb^2) sg^2 - (cb(.699sa + .527ca \ sa - .484ca \ ca^2) sa) + \\ & .484ca \ cb^2 \ sb - cb(.527ca + .484sa^2) \ sb^2 + (.699ca - .527sa^2 + .484ca \ sa^2) \ sb^2) sg^2 \end{align*} \]

\[ U_{25} = \begin{align*} & cg(-.484ca \ sb \ cb^3 + (.699sa + .527ca \ sa - .484ca \ sa^2) \ sb \\ & + (.527ca + .484sa^2) \ cb \ sb + (.699ca - .527sa^2 + .484ca \ sa^2) \ cb \ sb) + .484ca \ cb \ cg^3 \ sg^2 \\ & - cg^2(cb(.527ca + .484sa^2) + .484ca \ sb^2) sg^2 - cg(cb(.699ca - .527sa^2 + .484ca \ sa^2) + \\ & .484ca \ cb \ sb^2 - (.527ca + .484sa^2) \ sb^2) sg^2 - (cb(.699sa + .527ca \ sa - .484ca \ ca^2) sa) + \\ & .484ca \ cb^2 \ sb - cb(.527ca + .484sa^2) \ sb^2 + (.699ca - .527sa^2 + .484ca \ sa^2) \ sb^2) sg^2 \end{align*} \]

\[ U_{26} = \begin{align*} & -.484ca \ cb \ sg \ cg^4 + (-.484ca \ cb^3 \ sb + (.699sa + .527cas \ sa - 484ca \ sa) \ sb \\ & + cb^2(.527ca + .484sa^2) \ sb + cb(.699ca - .527sa^2 + .484ca \ sa^2) \ sb) sg \\ & + cg^3((.527ca + .484sa^2) cb + .484ca \ sb^2) sg + \\ & cg^2(cb(.699ca - .527sa^2 + .484ca \ sa^2) + .484ca \ cb \ sb^2 - (.527ca + .484sa^2) \ sb^2) sg \\ & + cg(cb(.699sa + .527ca \ sa - .484ca^2 \ sa) + .484ca \ cb \ sb^2 - cb(.527ca + .484sa^2) \ sb^2 \\ & - (.699ca - .527sa^2 + .484ca \ sa^2) \ sb^2) sg, \end{align*} \] (29)
6 $\mathcal{P}(\nu_\mu \to \nu_e)$ For equal sterile neutrino masses

Assuming that all three sterile neutrinos have the same mass, sterile-active neutrino mass differences are $\delta m_{ij}^2 = m_i^2 - m_j^2 \simeq 0.9(eV)^2$, with $\delta m_{ij}^2$ taken from the best fit to neutrino oscillation data\[9\] (see Ref\[9\] for references to earlier experiments), from Eq\[27\] $\mathcal{P}(\nu_\mu \to \nu_e)$ is

\[
\mathcal{P}(\nu_\mu \to \nu_e) = \text{Re}[U_{11}U_{21}[U_{11}U_{21} + U_{12}U_{22}e^{-i\delta L} + U_{13}U_{23}e^{-i\Delta L} + (U_{14}U_{24} + U_{15}U_{25} + U_{16}U_{26})e^{-i\gamma L}] + \\
U_{12}U_{22}[U_{11}U_{21} + U_{12}U_{22} + U_{13}U_{23}e^{-i\Delta L} + (U_{14}U_{24} + U_{15}U_{25} + U_{16}U_{26})e^{-i\gamma L}] + \\
U_{13}U_{23}[U_{11}U_{21}e^{-i\Delta L} + U_{12}U_{22}e^{-i\gamma L} + U_{13}U_{23}e^{-i\gamma L} + (U_{14}U_{24} + U_{15}U_{25} + U_{16}U_{26})e^{-i\gamma L}] + \\
(U_{14}U_{24} + U_{15}U_{25} + U_{16}U_{26})[\{(U_{11}U_{21} + U_{12}U_{22} \\
+ U_{13}U_{23})e^{-i\gamma L} + U_{14}U_{24} + U_{15}U_{25} + U_{16}U_{26}\}], \tag{30}
\]

with $\delta = \delta m_{12}^2/2E$, $\Delta = \delta m_{13}^2/2E$, $\gamma = \delta m_{j_k}^2/2E$ (j=1,2,3;k=4,5,6). The neutrino mass differences are $\delta m_{12}^2 = 7.6 \times 10^{-5}(eV)^2$, $\delta m_{13}^2 = 2.4 \times 10^{-3}(eV)^2$; and $\delta m_{j_k}^2(j = 1, 2, 3; k = 4, 5, 6) = 0.9(eV)^2[9]$.

From Eq\[30\]

\[
\mathcal{P}(\nu_\mu \to \nu_e) = U_{11}^2U_{21}^2 + U_{12}^2U_{22}^2 + U_{13}^2U_{23}^2 + \\
(U_{14}U_{24} + U_{15}U_{25} + U_{16}U_{26})^2 + \\
2U_{11}U_{21}U_{12}U_{22}\cos\delta L + \\
2(U_{11}U_{21}U_{13}U_{23} + U_{12}U_{22}U_{13}U_{23})\cos\Delta L + \\
2(U_{14}U_{24} + U_{15}U_{25} + U_{16}U_{26}) \\
(U_{11}U_{21} + U_{12}U_{22} + U_{13}U_{23})\cos\gamma L . \tag{31}
\]

Note that $\alpha \simeq 9.2^\circ$ from a recent analysis of MiniBooNE data, which was used in a recent study of $\mathcal{P}(\nu_\mu \to \nu_e)$ with one sterile neutrino[10, 4]. The figure below shows $\mathcal{P}(\nu_\mu \to \nu_e)$ with $\alpha = \beta = \gamma = 0^\circ$, giving the results of a recent 3x3 S-matrix calculation[6]. In Fig. 6, for the other curves, the sterile-active mixing angle $\alpha = 9.2^\circ$, while $\beta$ and $\gamma$ are chosen to be $9.2^\circ$ and $20^\circ$ to compare the 6x6 to the previous 3x3 results.
Using Eq(31), one finds $\mathcal{P}(\nu_\mu \rightarrow \nu_e)$ for the 6x6 vs 3x3 theories:

Figure 6: $\mathcal{P}(\nu_\mu \rightarrow \nu_e)$ for MINOS(L=735 km), MiniBooNE(L=500 m), JHF-Kamioka(L=295 km), and CHOOZ(L=1.03 km). (a) solid, for $\alpha = \beta = \gamma = 9.2^\circ$; (b) dashed, for $\alpha, \beta, \gamma = 9.2^\circ, 20^\circ, 20^\circ$; (c) dash-dotted curve for $\alpha = \beta = \gamma = 0^\circ$ giving the 3x3 result.
7 Conclusions

In this review with sterile and active neutrinos the $\nu_\mu \rightarrow \nu_e$ transition probability $\mathcal{P}(\nu_\mu \rightarrow \nu_e)$ for only active and one, two, three sterile neutrinos, with a variety of parameters associated with the transition probability, were discussed. There is now evidence for the existence of two sterile neutrinos, but this is still uncertain. One motivation for this review is to help extract parameters from future neutrino oscillation experiments.

At the present time there is no experimental evidence for three sterile neutrinos. From studies of the Cosmic Microwave Background Radiation, e.g. WMAP\[^{[16]}\], one knows that about 23 percent of matter in the universe is Dark Matter, which consists of particles that have no interaction except gravity. Since sterile neutrinos also only have a gravitational interaction, if the particles of Dark Matter are Fermions (quantum spin 1/2) they might be massive sterile neutrinos. Thus a third sterile neutrino would exist.

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