INFINITELY MANY BUBBLING SOLUTIONS AND NON-DEGENERACY RESULTS TO FRACTIONAL PRESCRIBED CURVATURE PROBLEMS

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ABSTRACT. We consider the following fractional prescribed curvature problem
\[
(−Δ)^s u = K(y)u^{2^*_s-1}, \quad u > 0, \quad y ∈ \mathbb{R}^N,
\]
where \( s \in (0, \frac{1}{2}) \) for \( N = 3 \), \( s \in (0, 1) \) for \( N ⩾ 4 \) and \( 2^*_s = \frac{2N}{N-2s} \) is the fractional critical Sobolev exponent, \( K(y) \) has a local maximum point in \( r ∈ (r_0 - δ, r_0 + δ) \).

First, for any sufficient large \( k \), we construct a \( 2k \)-bubbling solution to (0.1) of some new type, which concentrate on an upper and lower surfaces of an oblate cylinder through the Lyapunov-Schmidt reduction method. Furthermore, a non-degeneracy result of the multi-bubbling solutions is proved by use of various Pohozaev identities, which is new in the study of the fractional problems.

Keywords: Fractional critical elliptic equation, Infinitely many solutions, Lyapunov-Schmidt Reduction method, Non-degeneracy, Pohozaev identities.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the following fractional prescribed curvature equation
\[
(−Δ)^s u = K(y)u^{2^*_s-1}, \quad u > 0, \quad y ∈ \mathbb{R}^N,
\]
where \( s \in (0, \frac{1}{2}) \) for \( N = 3 \), \( s \in (0, 1) \) for \( N ⩾ 4 \) and \( 2^*_s = \frac{2N}{N-2s} \) is the fractional critical Sobolev exponent. \( (−Δ)^s \) is the nonlocal operator defined as
\[
(−Δ)^s u = c(N, s)P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,
\]
where \( P.V. \) is the principal value and \( c(N, s) = \pi^{−(2s+\frac{N}{2})} \Gamma(s+\frac{N}{2})/\Gamma(−s) \). For more details on the fractional Laplace operator, one can refer to [6, 7].

Recently, problems with fractional Laplacian have been extensively studied, see for example [1–5, 9–12, 17–20, 22] and references therein. Particularly, various critical problems for the fractional Laplacian were considered in [2, 5, 14, 15] and references therein.

When \( s = 1 \), Wei and Yan in [21] consider the prescribed scalar curvature problem
\[
−Δu = K(|y|)u^{2^*-1}, \quad u > 0, \quad y ∈ \mathbb{R}^N,
\]
and they proved the existence of infinitely many solutions concentrated on a circle. Later, Guo, Musso, Peng and Yan in [14] studied the non-degeneracy of this \( k \)-bubbling solution, by which, they use the gluing method to construct a new solutions. Precisely, with \( n \gg k \), they glue the \( k \)-bubbling solution, which concentrates at the vertices of the regular \( k \)-polygon in the \((y_1, y_2)\)-plane, with a \( n \)-spike solution, whose centers lie in another circle in the \((y_3, y_4)\)-plane. In [8], the existence of this solution \( 2k \) Aubin-Talenti bubbles are centred at points lying on the top and the bottom circles of a
cylinder is studied, whose energy can be made arbitrarily. In this paper, we extend the results to the fractional case nontrivially and study the corresponding non-degeneracy results.

To illustrate our results, we first give some denotations and definitions. Define $\dot{H}^s(\mathbb{R}^N)$ space as the closure of the set $C_c^\infty(\mathbb{R}^N)$ of compact supported smooth functions under the norm

$$||u||_{\dot{H}^s(\mathbb{R}^N)} = ||(-\Delta)^{\frac{s}{2}} u||_{L^2(\mathbb{R}^N)} = ||\xi^s \mathcal{F}(u)(\xi)||_{L^2(\mathbb{R}^N)},$$

where $S$ is the Schwartz space of rapidly decaying $C^\infty$ functions on $\mathbb{R}^N$, $\mathcal{F}$ is the Fourier transformation of $\phi$ by:

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i\xi x} \phi(x) dx, \ \forall \phi \in S.$$ 

Another equivalent space is defined by

$$D^s(\mathbb{R}^N) := \{u \in L^\infty(\mathbb{R}^N) : ||(-\Delta)^{\frac{s}{2}} u||_{L^2(\mathbb{R}^N)} < \infty\}.$$

Through the extension formulation of $(-\Delta)^s$ introduced in [4], the equation (1.1) is equivalent to a degenerate elliptic equation with a Neumann boundary problem, which is fundamentally different from the classical elliptic equation and defined on the upper half-space $\mathbb{R}^+_N = \{(y, t) : y \in \mathbb{R}^N, t > 0\}$. For $\forall u \in \dot{H}^s(\mathbb{R}^N)$, we set

$$\tilde{u}(y, t) = P_s[u] := \int_{\mathbb{R}^N} P_s(y - \xi, t) u(\xi) d\xi, \ (y, t) \in \mathbb{R}_+^{N+1} := \mathbb{R}^N \times [0, +\infty),$$

where

$$P_s(x, t) = \beta(N, s) \frac{t^2 s}{(|x|^2 + t^2)^{\frac{N+2s}{2}}},$$

with a constant $\beta(N, s)$ such that $\int_{\mathbb{R}^N} P_s(x, 1) dx = 1$. Thus, $\tilde{u} \in L^2(t^{1-2s}, \mathbb{R}_+^{N+1})$ and $\tilde{u} \in C^\infty(\mathbb{R}_+^{N+1})$.

It is well known that for any $x \in \mathbb{R}^N$ and $\Lambda > 0$ the functions

$$U_{x, \Lambda}(y) = (4 \pi \gamma)^{\frac{N-2s}{2}} \frac{\Lambda}{1 + \Lambda^2 |y - x|^2}^{\frac{N-2s}{2}},$$

with $\gamma = \frac{\pi^{\frac{N+2s}{2}}}{\Gamma(\frac{N+2s}{2})}$, are the only solutions to the problem (see [16])

$$(-\Delta)^s u = u^{2_s^* - 1}, \ u > 0 \ in \mathbb{R}^N. \ \ \ (1.2)$$

Moreover, the functions

$$Z_0(y) = \frac{\partial U}{\partial \Lambda}|_{\Lambda=1}, \ Z_i(y) = \frac{\partial U}{\partial y_i}(y), \ i = 1, 2, \cdots, N$$

solving the linearized problem

$$(-\Delta)^s \phi = (2_s^* - 1) U^{2_s^*-2} \phi, \ u > 0 \ in \mathbb{R}^N,$$

are the kernels of the linearized operator associated with (1.2).

Throughout this paper, we assume that $K(r)$ is a bounded radial potential function, and has a local maximum at $r_0$ satisfying

$$K(r) = K(r_0) - c_0 |r - r_0|^m + O(|r - r_0|^{m+\delta}), \ r \in (r_0 - \delta, r_0 + \delta), \ \ \ \ (1.3)$$

where $c_0 > 0, \delta > 0$ are some constants, and $m \in \left(\frac{N-2s}{2}, N - 2s\right)$. Without loss of generality, we may assume that $K(r_0) = 1$. For any integer $k > 0$, by use of the transformation $u(y) = u^{\frac{N+2s}{2}} u^{\frac{N+2s}{m}}$, the
Under the assumptions of Theorem 1.1, there exists an integer \( k \) where

\[
\theta \quad \text{such that for all} \quad k \geq k_0, \text{the problem (1.4) has a solution } u_k \text{ of the following form:}
\]

\[
 u_k = W_{r_k, h_k, \Lambda_k}(y) + \omega_k(y),
\]

where \( \omega_k \in H_s, (r_k, h_k, \Lambda_k) \in D \) and \( \omega_k \) satisfies \( ||\omega_k||_{L^\infty} \to 0 \) as \( k \to \infty \).

Denote

\[
 L_k \eta = (-\Delta)^s \eta - (2^*_s - 1)K\left(\frac{\mu}{s}\right)\eta^{2^*_s - 2} \eta,
\]

The non-degeneracy of \( L_k \) makes it possible to glue two multiple bubbling solutions concentrated in two orthogonal subspaces together to construct infinitely many solutions of new type. More precisely, we have the following result.
Theorem 1.3. Assume $N \geq 3$, and $s \in (0,1)$, for $N \geq 4$, $s \in (0,\frac{1}{2})$ for $N = 3$. Suppose that $K(y)$ satisfies (1.3). Let $\xi \in H_s$ be a solution of $L_k \xi = 0$. Then $\xi = 0$.

Remark 1.4. First of all, compared with the previous known results about the construction of the $k$-bubbling solution of the fractional equation, which concentrates on a circle, our solution obtained in this paper concentrates both on the upper and lower sides of an oblate cylinder. In this process, there is one more variable parameter; that is, the height $h$ of the cylinder, which makes the proof of the construction of the solution become more subtle. In addition, we study the non-degeneracy of this type of bubbling solutions, which makes it possible to continue the gluing process to construct other new solutions centered symmetrically at arranged infinitely many points. Moreover, in the process of using two kinds of Pohozaev identities to prove the non-degeneracy, the analysis of the main terms further shows great difference relative to the case without the parameter $h$.

Compared with the Laplacian, the nonlocal operators make it rather difficult to apply local Pohozaev identities, which is of great importance in the study of non-degeneracy and other properties of the solutions. By the aid of harmonic extension, we put two kinds of equivalent problems together, i.e., local and nonlocal ones, and delicately handle some integral terms induced by the extensions. The algebra decay at infinity of the approximate solutions, in stark contrast to the classical Laplacian, makes it crucial to give accurate and precise estimates of the solutions.

This paper is organized as follows. In Section 2, we use the Lyapunov-Schmidt reduction procedure to get a finite dimensional setting. In section 3, the main results of the corresponding finite dimensional problems are obtained. Then in section 4, the non-degeneracy result for the positive multi-bubbling solutions constructed in Theorem 1.2 is proved by use of the local Pohozaev identities. In the Appendix, we give energy expressions and some useful tools and estimates.

2. Preliminaries and the reduction framework

Let

$$||u||_s = \sup_{y \in \mathbb{R}^n} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2s}{2} + \tau}} + \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2s}{2} + \tau}} \right)^{-1} |u(y)|,$$

and

$$||f||_{s*} = \sup_{y \in \mathbb{R}^n} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N+2s}{2} + \tau}} + \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N+2s}{2} + \tau}} \right)^{-1} |f(y)|,$$

where $\frac{N-2s-m}{N-2s} < \tau < \min\{\frac{N+2s}{N-2s}, 1 + \epsilon\}$ and $\epsilon > 0$ is a small constant.

Denote

$$\mathcal{Z}_{1,j} = \frac{\partial U_{s,j}}{\partial r}, \quad \mathcal{Z}_{2,j} = \frac{\partial U_{s,j}}{\partial h} \quad \text{for} \ j = 1, \ldots, k,$$

$$\mathcal{Z}_{3,j} = \frac{\partial U_{s,j}}{\partial \lambda}, \quad \mathcal{Z}_{3,j} = \frac{\partial U_{s,j}}{\partial \lambda} \quad \text{for} \ j = 1, \ldots, k.$$

Define

$$\mathcal{E} = \left\{ v : v \in H_s, \int_{\mathbb{R}^n} U_{s,j}^{2l-2} \mathcal{Z}_{l,j} v = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} U_{s,j}^{2l-2} \mathcal{Z}_{l,j} v = 0, \ j = 1, \ldots, k, \ l = 1, 2, 3 \right\}. \quad (2.1)$$
Moreover, denote the splitting domains $\Omega_j$ for $j = 1, \ldots, k$ as

$$
\Omega_j := \left\{ y = (y_1, y_2, y_3, y''') \in \mathbb{R}^3 \times \mathbb{R}^{N-3} : \left( \frac{(y_1, y_2)}{|(y_1, y_2)|}, \left( \cos \frac{2(j-1)\pi}{k}, \sin \frac{2(j-1)\pi}{k} \right) \in \mathbb{R}^2 \right) \right\},
$$

and we divide the $\Omega_j$ into upper and lower regions:

$$
\Omega_j^+ = \left\{ y = (y_1, y_2, y_3, y''') \in \Omega_j, y_3 \geq 0 \right\},
$$

$$
\Omega_j^- = \left\{ y = (y_1, y_2, y_3, y''') \in \Omega_j, y_3 < 0 \right\}.
$$

Obviously,

$$
\mathbb{R}^N = \bigcup_{j=1}^k \Omega_j, \quad \Omega_j^+ \cup \Omega_j^- 
$$

and

$$
\Omega_j \cap \Omega_i = \emptyset, \quad \Omega_j^- \cap \Omega_j^- = \emptyset, \quad \text{if } i \neq j.
$$

Consider the following linearized problem:

$$
\begin{cases}
(\Delta^2 \phi_k - (2s^2 - 1)K(\frac{y}{\mu})W_{r, h, \Lambda}^{2s^2 - 2}\phi_k = f_k + \sum_{i=1}^3 c_i \left( \sum_{j=1}^k U_{\xi_j, \Lambda}^{2s^2 - 2i, j} + \sum_{j=1}^k U_{\xi_j, \Lambda}^{2s^2 - 2i, j} \right), \\
\phi_k \in \mathcal{E},
\end{cases}
$$

with some constants $c_i$.

**Lemma 2.1.** Assume that $\phi_k$ solves problem (2.2) for $f = f_k$. If $\|f_k\|_\infty$ goes to zero as $k$ goes to infinity, so does $\|\phi_k\|_\infty$.

**Proof.** We prove it by contradiction. Assume that there exist $f_k$ with $\|f_k\|_\infty \to 0$ as $k \to \infty$ $(r_k, h_k, \Lambda_k)$ satisfies (1.5) and $\phi_k$ solves problem (2.2) for $f = f_k, \Lambda = \Lambda_k, r = r_k, h = h_k$ with $\|\phi_k\|_\infty \geq c > 0$. Without loss of generality, we always assume that $\|\phi_k\|_\infty \equiv 1$. For the sake of convenience, we drop the subscript $k$.

Since (2.2), we get

$$
\phi(y) \leq C \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2s}} W_{r,h,\Lambda}^{2s^2 - 2} \phi(z) dz + \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2s}} f(z) dz
$$

$$
+ \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2s}} \sum_{i=1}^3 c_i \left( \sum_{j=1}^k U_{\xi_j, \Lambda}^{2s^2 - 2i,j} + \sum_{j=1}^k U_{\xi_j, \Lambda}^{2s^2 - 2i,j} \right) dz.
$$

For the first term, using Lemma [A.3] we have

$$
\left| \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2s}} K(\frac{|z|}{\mu}) W_{r,h,\Lambda}^{2s^2 - 2} \phi(z) dz \right| \leq \|f\|_\infty \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2s}} W_{r,h,\Lambda}^{2s^2 - 2} \sum_{j=1}^k \left( \frac{1}{(1 + |z - \xi_j|)^{\frac{N-2s}{2} + r \theta}} + \frac{1}{(1 + |z - \xi_j|)^{\frac{N-2s}{2} + r \theta}} \right) dz
$$

$$
\leq \|f\|_\infty \sum_{j=1}^k \left( \frac{1}{(1 + |y - \xi_j|)^{\frac{N-2s}{2} + r \theta}} + \frac{1}{(1 + |y - \xi_j|)^{\frac{N-2s}{2} + r \theta}} \right),
$$

and for the second term, from Lemma [A.2], it holds that

$$
\left| \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2s}} f(z) dz \right|
$$
Next, we estimate
\[
\|f\|_{\infty} \leq \frac{1}{|z - y|^{N-2s}} \sum_{j=1}^{k} \left( \frac{1}{(1 + |z - x_j|)^{\frac{N+2s}{2}}} + \frac{1}{(1 + |y - x_j|)^{\frac{N+2s}{2}}} \right) dz.
\]

Since
\[
|\overline{Z}_{1,j}| \leq \frac{C}{(1 + |y - x_j|)^{N-2s}}, \quad |\overline{Z}_{2,j}| \leq \frac{Cr}{(1 + |y - x_j|)^{N-2s}}, \quad |\overline{Z}_{3,j}| \leq \frac{C}{(1 + |y - x_j|)^{N-2s}},
\]

(2.3)

Combining (2.3), we obtain,
\[
\left| \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2s}} \left( \sum_{j=1}^{k} U_{\delta_{2},\mu}^{2s-2} \overline{Z}_{1,j} + \sum_{j=1}^{k} U_{\delta_{2},\mu}^{2s-2} \overline{Z}_{2,j} \right) dz \right|
\leq C \sum_{j=1}^{k} \left( \frac{1 + r\delta_{2}}{(1 + |y - x_j|)^{\frac{N+2s}{2}}} + \frac{1 + r\delta_{2}}{(1 + |y - x_j|)^{\frac{N+2s}{2}}} \right),
\]

where
\[
\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}
\]

Next we estimate \(c_i, i = 1, 2, 3\). Multiplying (2.2) by \( \overline{Z}_{i,1} \) and integrating, it shows that \(c_i\) satisfies
\[
\left\langle (-\Delta)^s \phi - (2^*_s - 1)K \left( \frac{|y|}{\mu} \right) W_{r,h,\Lambda}^{2s-2} \overline{Z}_{i,1}, \phi \right\rangle - \left\langle t, \overline{Z}_{i,1} \right\rangle
\]
\[
= \left\langle \sum_{j=1}^{k} c_i \left( \sum_{j=1}^{k} U_{\delta_{2},\mu}^{2s-2} \overline{Z}_{1,j} + \sum_{j=1}^{k} U_{\delta_{2},\mu}^{2s-2} \overline{Z}_{2,j} \right), \overline{Z}_{i,1} \right\rangle.
\]

(2.4)

where \(\langle x, y \rangle = \int_{\mathbb{R}^N} x y dz\). By using Lemma A.1, we have
\[
\left| \left\langle f, \overline{Z}_{i,1} \right\rangle \right| \leq \|f\|_{\infty} \int_{\mathbb{R}^N} \frac{1 + r\delta_{2}}{(1 + |z - x|)^{N-2s}} \sum_{j=1}^{k} \left( \frac{1}{(1 + |y - x_j|)^{\frac{N+2s}{2}}} + \frac{1}{(1 + |y - x_j|)^{\frac{N+2s}{2}}} \right) dz.
\]

Next, we calculate
\[
\left\langle (-\Delta)^s \phi - (2^*_s - 1)K \left( \frac{|y|}{\mu} \right) W_{r,h,\Lambda}^{2s-2} \overline{Z}_{i,1}, \phi \right\rangle
\]
\[
= \left\langle (-\Delta)^s \overline{Z}_{i,1} - (2^*_s - 1)K \left( \frac{|y|}{\mu} \right) W_{r,h,\Lambda}^{2s-2} \overline{Z}_{i,1}, \phi \right\rangle
\]
\[
= (2^*_s - 1) \left\langle \left( U_{\delta_{2},\mu}^{2s-2} - W_{r,h,\Lambda}^{2s-2} \overline{Z}_{i,1}, \phi \right) - (2^*_s - 1) \left[ K \left( \frac{|y|}{\mu} \right) - 1 \right] W_{r,h,\Lambda}^{2s-2} \overline{Z}_{i,1}, \phi \right\rangle,
\]
\[
:= H_1 - H_2.
\]
For $H_1$, it holds that

$$
H_1 \leq C\|\phi\|_s \int_{\mathbb{R}^N} (U_{\phi, r, h, \lambda}^{2k-2} - W_{r, h, \lambda}^{2k-2}) \left[ 1 + \frac{r \delta_2}{\mu^r} \right] d\tau
$$

\begin{align*}
&= \frac{1}{(1 + |y - x_j|)^{\frac{\alpha}{2} + \theta}} + \frac{1}{(1 + |y - x_j|)^{\frac{\alpha}{2} + \theta}} d\tau

&\leq C\|\phi\|_s \left[ 1 + \frac{r \delta_2}{\mu^r} \right].
\end{align*}

Using the condition of $K(|y|)$, we obtain $H_2$,

\begin{align*}
H_2 &\leq C\|\phi\|_s \int_{\mathbb{R}^N} \left[ K\left( \frac{|y|}{\mu} \right) - 1 \right] W_{r, h, \lambda}^{2k-2} \sum_{j=1}^{k} \left[ 1 + \frac{r \delta_2}{\mu^r} \right] d\tau

&= C\|\phi\|_s \int_{\|y\| > \sqrt{p}} \left[ K\left( \frac{|y|}{\mu} \right) - 1 \right] W_{r, h, \lambda}^{2k-2} \sum_{j=1}^{k} \left[ 1 + \frac{r \delta_2}{\mu^r} \right] d\tau

&= C\|\phi\|_s \int_{\|y\| > \sqrt{p}} \left[ K\left( \frac{|y|}{\mu} \right) - 1 \right] W_{r, h, \lambda}^{2k-2} \sum_{j=1}^{k} \left[ 1 + \frac{r \delta_2}{\mu^r} \right] d\tau

&= C\|\phi\|_s \int_{\|y\| > \sqrt{p}} \left[ K\left( \frac{|y|}{\mu} \right) - 1 \right] W_{r, h, \lambda}^{2k-2} \sum_{j=1}^{k} \left[ 1 + \frac{r \delta_2}{\mu^r} \right] d\tau

&= C\|\phi\|_s \int_{\|y\| > \sqrt{p}} \left[ K\left( \frac{|y|}{\mu} \right) - 1 \right] W_{r, h, \lambda}^{2k-2} \sum_{j=1}^{k} \left[ 1 + \frac{r \delta_2}{\mu^r} \right] d\tau

&= C\|\phi\|_s \left[ 1 + \frac{r \delta_2}{\mu^r} \right].
\end{align*}

By using (2.6) and (2.7), we get

$$
\langle (-\Delta)^s \psi - (2_s^* - 1)K\left( \frac{|y|}{\mu} \right) W_{r, h, \lambda}^{2k-2}, \phi, \psi_{i,1} \rangle = C\|\phi\|_s \left[ 1 + \frac{r \delta_2}{\mu^r} \right].
$$

However, there is a constant $c > 0$ satisfies

$$
\langle \int_{\mathbb{R}^N} \left[ \sum_{j=1}^{k} U_{\phi, r, h, \lambda}^{2k-2} z_{i,j} + \sum_{j=1}^{k} U_{\phi, r, h, \lambda}^{2k-2} z^*_{i,j} \right] \rangle = C_i \delta_2 (1 + \delta_2 r^2),
$$

where $C_i$ is a constant. So, we find that by substituting (2.5), (2.8) and (2.9) into equation (2.4),

$$
c_i = \frac{1 + \frac{r \delta_2}{\mu^r}}{1 + r^2 \delta_2} O\left( \frac{1}{\mu^r} \right). $$

Thus,

$$
\|\phi\|_s \leq \|f\|_{s*} + \frac{C_i}{(1 + |y - x_j|)^{\frac{\alpha}{2} + \theta}} + \frac{C_i}{(1 + |y - x_j|)^{\frac{\alpha}{2} + \theta}}.
$$

\[ \sum_{j=1}^{k} \left[ (1 + |y - x_j|)^{\frac{\alpha}{2} + \theta} + (1 + |y - x_j|)^{\frac{\alpha}{2} + \theta} \right].
\)
Since $\|\phi\|_* = 1$, we find that there is $R > 0$ from (2.10), such that
\[ \|\phi\|_{L^\infty(B_R(x_i))} \geq a > 0, \]  
(2.11)
for some $x_i$. But $\tilde{\phi}(y) = \phi(y - x_i)$ converges uniformly in any compact set to a solution $u$ of
\[ (-\Delta)^s u - (2^*_s - 1)U_{0\Lambda}^{2^*_s - 2} u = 0, \text{ in } \mathbb{R}^N, \]  
(2.12)
for some $\Lambda \in [L_1, L_2]$, $u$ must be a linear combination of the functions
\[ \frac{\partial U_{0\Lambda}}{\partial \Lambda} \bigg|_{\Lambda = 1} = \frac{\partial U_{01}}{\partial y_1}, \quad \frac{\partial U_{01}}{\partial y_3}. \]
However $u$ is perpendicular to the kernel of (2.12). So, $u = 0$, which is in conflict with (2.11).

From Lemma 2.1 using the same argument as the proof for Proposition 2.2 in [8], we prove the following result:

**Lemma 2.2.** There exist $k_0 > 0$ and a constant $C > 0$, independent of $k$, such that for all $k \geq k_0$ and all $f \in L^\infty(\mathbb{R}^N)$, problem (2.2) has a unique solution $\phi \equiv L_\lambda(f)$. Furthermore,
\[ \|L_\lambda(f)\|_* \leq C\|f\|_{**}, \quad |c_i| \leq \frac{C}{1 + r\delta_2}\|f\|_{**}, \quad i = 1, 2, 3. \]  
(2.13)

Now, we consider the following problem (2.14),
\[ \begin{aligned}
(-\Delta)^s (W_{r,h,\Lambda} + \phi) &= K\left(\frac{|y|}{\mu}\right)(W_{r,h,\Lambda} + \phi)^{2^*_s - 1} + \sum_{i=1}^3 c_i \left( \sum_{j=1}^k U_{x_{ij},\Lambda}^{2^*_s - 2} Z_{i,j} + \sum_{j=1}^k U_{x_{ij},\Lambda}^{2^*_s - 2} Z_{i,j} \right), \\
\phi &\in \mathcal{E}.
\end{aligned} \]  
(2.14)
where $\mathcal{E}$ is as (2.1).

Rewrite above equation (2.14) as
\[ \begin{aligned}
(-\Delta)^s \phi - (2^*_s - 1)K\left(\frac{|y|}{\mu}\right)W_{r,h,\Lambda}^{2^*_s - 2} \phi &= N(\phi) + l_k + \sum_{i=1}^3 c_i \left( \sum_{j=1}^k U_{x_{ij},\Lambda}^{2^*_s - 2} Z_{i,j} + \sum_{j=1}^k U_{x_{ij},\Lambda}^{2^*_s - 2} Z_{i,j} \right), \\
\phi &\in \mathcal{E}.
\end{aligned} \]  
(2.15)
where
\[ N(\phi) = K\left(\frac{|y|}{\mu}\right)((W_{r,h,\Lambda} + \phi)^{2^*_s - 1} - W_{r,h,\Lambda}^{2^*_s - 1} - (2^*_s - 1)W_{r,h,\Lambda}^{2^*_s - 2} \phi), \]
and
\[ l_k = K\left(\frac{|y|}{\mu}\right)W_{r,h,\Lambda}^{2^*_s - 1} - \left( \sum_{j=1}^k U_{x_{ij},\Lambda}^{2^*_s - 1} + \sum_{j=1}^k U_{x_{ij},\Lambda}^{2^*_s - 1} \right). \]  
(2.16)

In order to prove that (2.15) is uniquely solvable using the contraction mapping Theorem, we need to estimate $N(\phi)$ and $l_k$.

**Lemma 2.3.** If $s \in (0, 1)$ for $N \geq 4$ and $s \in (0, \frac{1}{2})$ for $N = 3$, and $\|\phi\|_* \leq 1$, then
\[ \|N(\phi)\|_{**} \leq C\|\phi\|_{**}^{2^*_s - 1,2}. \]

**Proof.** Since the proof is similar to the [12], we omit it here.

Next, we estimate $l_k$. 

\[ \Box \]
Lemma 2.4. If \( s \in (0, 1) \) for \( N > 4 \) and \( s \in (0, \frac{1}{2}) \) for \( N = 3 \), then there is a small \( \epsilon > 0 \), such that

\[
\|l_k\|_{**} \leq C \left( \frac{1}{\mu} \right)^{\frac{N}{2} + \epsilon}.
\]

Proof. First we rewrite equation (2.16) as

\[
l_k = K \left( \frac{|y|}{\mu} \right) \left( \sum_{j=1}^{k} U_{j, \Lambda}^{2s} - \sum_{j=1}^{k} U_{\Lambda, \alpha}^{2s} \right) + \sum_{j=1}^{k} \left( U_{j, \Lambda}^{2s} - U_{\Lambda, \alpha}^{2s} \right) \left( \left| \frac{|y|}{\mu} \right| - 1 \right) = J_1 + J_2.
\]

By symmetry, supposing \( y \in \Omega_i \), we obtain

\[
|y - x_i| \leq |y - x_1| \quad \text{and} \quad |y - x_i| \leq |y - x_j| \leq |y - x_k|, \quad j = 2, 3, \ldots, k.
\]

So, it holds that

\[
|J_1| = K \left( \frac{|y|}{\mu} \right) \left( \sum_{j=1}^{k} U_{j, \Lambda}^{2s} - \sum_{j=1}^{k} U_{\Lambda, \alpha}^{2s} \right) \leq C \left( U_{j, \Lambda}^{2s} \right) \left( \sum_{j=2}^{k} \left( 1 + |y - x_j|^2 \right)^{N-2s} \right) \leq C \left( \frac{1}{(1 + |y - x_i|^2)^{N-2s}} \right)^{2s-1}
\]

By Lemma A.1 since \( \left( \frac{N + 2s - 2 - \tau}{N - 2s} \right) \frac{m}{N - 2s} > \frac{m}{2} + \epsilon \), from (1.6) we get

\[
\frac{1}{(1 + |y - x_i|^2)^{N-2s}} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|^2)^{N-2s}} \leq \frac{1}{\left( \frac{N + 2s - 2 - \tau}{N - 2s} \right) \left( \frac{m}{2} + \epsilon \right)} \sum_{j=1}^{k} \frac{1}{\left| x_j - x_i \right|^{\frac{N + 2s - 2 - \tau}{2}}}
\]

and

\[
\frac{1}{(1 + |y - x_i|^2)^{N-2s}} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|^2)^{N-2s}} \leq \frac{1}{\left( \frac{N + 2s - 2 - \tau}{N - 2s} \right) \left( \frac{m}{2} + \epsilon \right)} \sum_{j=1}^{k} \frac{1}{\left| x_j - x_i \right|^{\frac{N + 2s - 2 - \tau}{2}}}
\]

By Lemma A.1 since \( \left( \frac{N + 2s - 2 - \tau}{N - 2s} \right) \frac{m}{N - 2s} > \frac{m}{2} + \epsilon \), from (1.6) we get

\[
\frac{1}{(1 + |y - x_i|^2)^{N-2s}} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|^2)^{N-2s}} \leq \frac{1}{\left( \frac{N + 2s - 2 - \tau}{N - 2s} \right) \left( \frac{m}{2} + \epsilon \right)} \sum_{j=1}^{k} \frac{1}{\left| x_j - x_i \right|^{\frac{N + 2s - 2 - \tau}{2}}}
\]

and

\[
\frac{1}{(1 + |y - x_i|^2)^{N-2s}} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|^2)^{N-2s}} \leq \frac{1}{\left( \frac{N + 2s - 2 - \tau}{N - 2s} \right) \left( \frac{m}{2} + \epsilon \right)} \sum_{j=1}^{k} \frac{1}{\left| x_j - x_i \right|^{\frac{N + 2s - 2 - \tau}{2}}}
\]
By Hölder inequalities, for \(\frac{N+2s}{2s}(\frac{N-2s}{N+2s}) > 1\), there holds that
\[
\left(\sum_{j=2}^{k} \frac{1}{(1 + |y - \vec{x}_j|)^{N-2s}}\right)^{2s-1} \leq \left(\sum_{j=2}^{k} \frac{1}{(1 + |y - \vec{x}_j|)^{\frac{N+2s}{2s}}} \right)^{\frac{N}{N+2s}} \leq \frac{1}{(1 + |y - \vec{x}_1|)^{1 + \tau}} \left(\frac{1}{\mu}\right)^{\frac{N}{N+2s}},
\]
\[\tag{2.19}\]
Combining (2.17), (2.18), (2.19), we obtain
\[
\|J_1\|_{**} \leq \left(\frac{1}{\mu}\right)^{\frac{N}{N+2s}}.
\]

Next, we consider \(J_2\) to estimate
\[
J_2 \leq 2\left(K\left(\frac{|y|}{\mu}\right) - 1\right)\left(U_{2s-1}^{2s-1} - 1\right) \sum_{j=2}^{k} U_{2s}^{2s-1} - 1\right) \sum_{j=2}^{k} U_{2s}^{2s-1} - 1\right) (1 + \|y - \vec{x}_1\|)^{\frac{N+2s}{2s} + \tau} \left(\frac{1}{\mu}\right)^{\frac{N}{N+2s}} \leq \frac{1}{(1 + |y - \vec{x}_1|)^{\frac{N+2s}{2s} + \tau}} \left(\frac{1}{\mu}\right)^{\frac{N}{N+2s}},
\]
\[\tag{2.20}\]
where the last inequalities is due to \(m \in (\frac{N-2s}{2s}, N-2s), \frac{N}{2} + \epsilon < \frac{N+2s}{2s} - \tau\).

When \(y \in \Omega_1^n\) and \(||y| - \mu r_0| < \delta\), we have \(\|y - \vec{x}_1\| \leq |\|y\| - \mu r_0| + |\vec{x}_1| - \mu r_0| \leq 2\delta\). From the condition of \(K(|y|)\), we get
\[
\left(K\left(\frac{|y|}{\mu}\right) - 1\right) \leq \frac{C}{\mu''(\|y\| - |\vec{x}_1|)^{\frac{N}{2} + \epsilon}} + \frac{1}{\mu''(\|y\| - |\vec{x}_1|)^{\frac{N}{2} + \epsilon}}.
\]
Thus,
\[
\left(K\left(\frac{|y|}{\mu}\right) - 1\right) U_{2s}^{2s-1} - 1\right) \sum_{j=2}^{k} U_{2s}^{2s-1} - 1\right) (1 + |y - \vec{x}_1|)^{\frac{N+2s}{2s} + \tau} \left(\frac{1}{\mu}\right)^{\frac{N}{N+2s}} \leq \frac{C}{\mu''(\|y\| - |\vec{x}_1|)^{\frac{N}{2} + \epsilon}} - \frac{C}{\mu''(\|y\| - |\vec{x}_1|)^{\frac{N}{2} + \epsilon}}.
\]
\[\tag{2.21}\]
For the second term, we get
\[
\left(K\left(\frac{|y|}{\mu}\right) - 1\right) \sum_{j=2}^{k} U_{2s}^{2s-1} - 1\right) \sum_{j=2}^{k} U_{2s}^{2s-1} - 1\right) (1 + |y - \vec{x}_1|)^{\frac{N+2s}{2s} + \tau} \left(\frac{1}{\mu}\right)^{\frac{N}{N+2s}} \leq \frac{C}{(1 + |y - \vec{x}_1|)^{\frac{N+2s}{2s} + \tau}} \left(\frac{1}{\mu}\right)^{\frac{N}{N+2s}}.
\]
\[\tag{2.22}\]
Combing (2.20), (2.21), (2.22), we obtain

$$\|J_2\|_\ast \leq C \left( \frac{1}{\mu} \right)^{\frac{p}{2} + \epsilon}. $$

Next, we prove the following result through the contraction mapping Theorem.

**Lemma 2.5.** There exist $k_0 > 0$, such that for all $k \geq k_0, (r, h, \Lambda) \in \mathbb{D}$. The problem (2.14) has a unique solution $\phi = \phi(r, h, \Lambda)$, satisfying

$$\|\phi\|_\ast \leq C \left( \frac{1}{\mu} \right)^{\frac{p}{2} + \epsilon}$$  (2.23)

and

$$|c_i| \leq \frac{C}{1 + \delta r^2} \left( \frac{1}{\mu} \right)^{\frac{p}{2} + \epsilon}, $$  (2.24)

where $\epsilon > 0$ is a small constant.

**Proof.** Define

$$\tilde{S} := \left\{ \phi : \phi \in \mathbb{E} \cap C_0^\infty (\mathbb{R}^N), \|\phi\|_\ast \leq C \left( \frac{1}{\mu} \right)^{\frac{p}{2} + \epsilon} \right\}. $$

By Lemma 2.2, $L$ is invertible, so (2.15) is equivalent to

$$\phi = A(\phi) := L_k(N(\phi)) + L_k(l_k). $$

We will conclude that $A$ is a contraction mapping from $\tilde{S}$ to $\tilde{S}$. In fact, by Lemmas 2.3 and Lemma 2.4, we get

$$\|A(\phi)\|_\ast \leq C\|N(\phi)\|_\ast + C\|l_k\|_\ast \leq C \left( \frac{1}{\mu} \right)^{\frac{p}{2} + \epsilon} + C \left( \frac{1}{\mu} \right)^{\frac{p}{2} + \epsilon}, $$

So, $A$ maps $\tilde{S}$ to $\tilde{S}$. And if $p \leq 3$, we have $|N'(t)| \leq |t|^{p-2}$; if $p > 3$, we have $|N'(t)| \leq C(W_{r,h,\Lambda}^{2-3} |t| + |t|^{p-2})$. Thus if $p \leq 3$, for all $\phi_1, \phi_2 \in \tilde{S}$, it holds that

$$\|A(\phi_1) - A(\phi_2)\|_\ast = \|L_k(N(\phi_1)) - L_k(N(\phi_2))\|_\ast \leq \frac{1}{2}\|\phi_1 - \phi_2\|_\ast. $$

Therefore, $A$ is a contracting mapping. The case $p > 3$ can be obtained similarly. According to the contraction mapping Theorem there is a unique $\phi \in \mathbb{E}$, such that

$$\phi = A(\phi). $$

Besides, by Lemma 2.2 we conclude (2.23) and (2.24). \qed

3. **Finite dimensional problem**

Define

$$F(r, h, \Lambda) = I(W_{r,h,\Lambda} + \phi), \quad \forall (r, h, \Lambda) \in \mathbb{D}, $$

where $r = |x_1|$ and $\phi$ is the function gained in Lemma 2.5.

In this section, we will give the energy expansions for $F(r, h, \Lambda), \frac{\partial F(r, h, \Lambda)}{\partial \Lambda}$ and $\frac{\partial F(r, h, \Lambda)}{\partial h}$. 
Proposition 3.1. We have

\[
F(r, h, \Lambda) = k \left( A + \frac{A_1}{\Lambda^m \mu^m} + \frac{A_2}{\Lambda^{m-2} \mu^m} (\mu r_0 - r)^2 - \frac{B_1 k^{N-2s}}{\Lambda^{N-2s} (r \sqrt{1 - h^2})^{N-2s}} - \frac{B_2}{\Lambda^{N-2s} (r h)^{N-2s}} \right) + \frac{C}{\mu^{m+\theta}} + O \left( \frac{1}{\mu^{N-2s}} \right),
\]

where \(A, A_1, A_2, B_1\) and \(B_2\) are some positive constants.

Proof. Since

\[
\langle f'(W_{r,h,\Lambda} + \phi), \phi \rangle = 0, \quad \forall \phi \in \tilde{S},
\]

there is a constant \(\theta \in (0, 1)\) such that

\[
F(r, h, \Lambda) = I(W_{r,h,\Lambda}) + \frac{1}{2} D^2 I(W_{r,h,\Lambda} + \theta \phi)(\phi, \phi)
\]

\[
= I(W_{r,h,\Lambda}) - \frac{2s - 1}{2} \int_{\mathbb{R}^N} \frac{1}{\mu} \left( (W_{r,h,\Lambda} + \theta \phi)^{2s} - W_{r,h,\Lambda}^{2s} \right) \phi^2 + \frac{1}{2} \int_{\mathbb{R}^N} (N(\phi) + l_k) \phi
\]

\[
= I(W_{r,h,\Lambda}) + O \left( \int_{\mathbb{R}^N} (|\phi|^{2s} + |N(\phi)| |\phi|) \right).
\]

Since

\[
\int_{\mathbb{R}^N} (|N(\phi)| |\phi| + |l_k| |\phi|) \leq C (|N(\phi)|_{*+} + |l_k|_{*+}) |\phi|_{*}
\]

\[
\times \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} \frac{1}{1 + |y - x_j|^{\frac{N}{2s} + \tau}} + \sum_{j=1}^{k} \frac{1}{1 + |y - x_j|^{\frac{N}{2s} + \tau}} \right)
\]

then by Lemma [A.1] we obtain

\[
\sum_{j=1}^{k} \frac{1}{1 + |y - x_j|^{\frac{N}{2s} + \tau}} \sum_{i=1}^{k} \frac{1}{1 + |y - x_i|^{\frac{N}{2s} + \tau}} \leq C \sum_{j=1}^{k} \frac{1}{1 + |y - x_j|^{N+\tau}}.
\]

Thus

\[
\int_{\mathbb{R}^N} (|N(\phi)| |\phi| + |l_k| |\phi|) \leq C k (|N(\phi)|_{*+} + |l_k|_{*+}) |\phi|_{*}
\]

\[
\leq \frac{C k}{\mu^{m+\theta}}.
\]

On the other hand, we have

\[
\int_{\mathbb{R}^N} |\phi|^{2s} \leq C |\phi|^{2s} \int_{\mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{1}{1 + |y - x_j|^{\frac{N}{2s} + \tau}} + \sum_{j=1}^{k} \frac{1}{1 + |y - x_j|^{\frac{N}{2s} + \tau}} \right)^{2s}.
\]

By using Lemma [A.1] if \(y \in \Omega^*_r\), we can find a small constant \(\alpha > 0\) such that

\[
\sum_{j=2}^{k} \frac{1}{1 + |y - x_j|^{\frac{N}{2s} + \tau}} + \sum_{j=1}^{k} \frac{1}{1 + |y - x_j|^{\frac{N}{2s} + \tau}}
\]
\[ \leq C \sum_{j=1}^{k} \frac{1}{(1 + |y - \bar{x}_j|)^{N-2s+\frac{\mu+\theta}{2}+\frac{\theta}{s}}} \left( \frac{1}{|x_j - \bar{x}_1|^{\frac{N-2s+\mu+\theta}{s}}} + \frac{1}{|x_j - \bar{x}_1|^{\frac{N-2s+\mu+\theta}{s}}} \right) \]

and so

\[ \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - \bar{x}_j|)^{N-2s+\frac{\mu+\theta}{2}+\frac{\theta}{s}}} + \sum_{j=1}^{k} \frac{1}{(1 + |y - \bar{x}_j|)^{N-2s+\frac{\mu+\theta}{2}+\frac{\theta}{s}}} \right)^2 \leq C \]

Since

\[ \int_{\mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - \bar{x}_j|)^{N-2s+\frac{\mu+\theta}{2}+\frac{\theta}{s}}} + \sum_{j=1}^{k} \frac{1}{(1 + |y - \bar{x}_j|)^{N-2s+\frac{\mu+\theta}{2}+\frac{\theta}{s}}} \right)^2 \leq C, \]

we have

\[ \int_{\mathbb{R}^N} |\phi|^2 \leq Ck||\phi||^2_s \leq \frac{Ck}{\mu^{\mu+\theta}}. \]

**Proposition 3.2.**

\[ \frac{\partial F(r, h, \Lambda)}{\partial h} = k \left( \frac{B_2(N - 2s - 1)k}{\Lambda^N - 2sN - 2s} \frac{\Lambda^{N - 2s + N - 2s - 1}}{(1 - \mu^2)^{N - 2s + 2}} \right) \]

\[ + O \left( \frac{1}{\mu^{N - 2s + N - 2s - 1/2}} \right) \]

where \( B_1 \) and \( B_2 \) are some positive constants.

**Proof.** We have

\[ \frac{\partial F(r, h, \Lambda)}{\partial h} = \left( \frac{\partial F(r, h, \Lambda)}{\partial h} \right) + \left( \frac{\partial W_{r,h,\Lambda}}{\partial h} \right) \left( \frac{\partial \phi}{\partial h} \right) \]

\[ = \left( \frac{\partial F(r, h, \Lambda)}{\partial h} \right) + \left( \frac{\partial W_{r,h,\Lambda}}{\partial h} \right) \left( \frac{\partial \phi}{\partial h} \right) \]

\[ = \left( \frac{\partial F(r, h, \Lambda)}{\partial h} \right) + \left( \frac{\partial W_{r,h,\Lambda}}{\partial h} \right) \left( \frac{\partial \phi}{\partial h} \right) \]

\[ + \left( \frac{\partial W_{r,h,\Lambda}}{\partial h} \right) \left( \frac{\partial \phi}{\partial h} \right) \]

In views of the orthogonality, we get

\[ \left( \left( \frac{\partial W_{r,h,\Lambda}}{\partial h} \right) \left( \frac{\partial \phi}{\partial h} \right) \right) = -\left( \frac{\partial (\partial W_{r,h,\Lambda})}{\partial h} \left( \frac{\partial \phi}{\partial h} \right) \right) \]

Through Lemma 2.5, it holds that

\[ \left| \sum_{j=1}^{k} c_i \left( \frac{\partial (\partial W_{r,h,\Lambda})}{\partial h} \left( \frac{\partial \phi}{\partial h} \right) \right) \right| \]
\[C \leq C(c_1) \| \phi \| \cdot \int_{\mathbb{R}^N} \left( \frac{\partial_U U_{\tau,\Lambda}^{2\tau - 2}}{\partial h} + \frac{\partial_U U_{\Lambda,\Lambda}^{2\tau - 2}}{\partial h} \right) \]
\[\times \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{2\tau + \tau}} + \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{2\tau + \tau}} \right) \]
\[\leq C(c_1) \| \phi \| \cdot \int_{\mathbb{R}^N} \frac{\mu(1 + r \delta_2)}{(1 + |y - x_j|)^{N + 2\tau + \tau}} \]
\[\times \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{2\tau + \tau}} + \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{2\tau + \tau}} \right) \]
\[\leq C \left( \frac{1}{\mu} \right)^{m-1+2\varepsilon} . \]

We have
\[\int_{\mathbb{R}^N} K(\frac{|y|}{\mu})(W_{r,\Lambda} + \phi)^{2\tau - 1} \frac{\partial W_{r,\Lambda}}{\partial h} = \int_{\mathbb{R}^N} K(\frac{|y|}{\mu})W_{r,\Lambda}^{2\tau - 2} \frac{\partial W_{r,\Lambda}}{\partial h} + (2\tau - 1) \int_{\mathbb{R}^N} K(\frac{|y|}{\mu})W_{r,\Lambda}^{2\tau - 2} \frac{\partial W_{r,\Lambda}}{\partial h} \phi + O(\int_{\mathbb{R}^N} |\phi|^{2\tau}). \]

Furthermore, for every \( \phi \in \mathbb{B} \), we obtain
\[\int_{\mathbb{R}^N} K(\frac{|y|}{\mu})W_{r,\Lambda}^{2\tau - 2} \frac{\partial W_{r,\Lambda}}{\partial h} \phi = \int_{\mathbb{R}^N} K(\frac{|y|}{\mu})(W_{r,\Lambda}^{2\tau - 2} \frac{\partial W_{r,\Lambda}}{\partial h} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{2\tau - 2} \frac{\partial U_{x_j,\Lambda}}{\partial h} - \sum_{j=1}^{k} U_{\Lambda,\Lambda}^{2\tau - 2} \frac{\partial U_{\Lambda,\Lambda}}{\partial h}) \phi \]
\[+ \int_{\mathbb{R}^N} (K(\frac{|y|}{\mu}) - 1)(\sum_{j=1}^{k} U_{x_j,\Lambda}^{2\tau - 2} \frac{\partial U_{x_j,\Lambda}}{\partial h} + \sum_{j=1}^{k} U_{\Lambda,\Lambda}^{2\tau - 2} \frac{\partial U_{\Lambda,\Lambda}}{\partial h}) \phi \]
\[= 2k \int_{\Omega} K(\frac{|y|}{\mu})(W_{r,\Lambda}^{2\tau - 2} \frac{\partial W_{r,\Lambda}}{\partial h} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{2\tau - 2} \frac{\partial U_{x_j,\Lambda}}{\partial h} - \sum_{j=1}^{k} U_{\Lambda,\Lambda}^{2\tau - 2} \frac{\partial U_{\Lambda,\Lambda}}{\partial h}) \phi \]
\[+ 2k \int_{\Omega} (K(\frac{|y|}{\mu}) - 1)(U_{x_j,\Lambda}^{2\tau - 2} \frac{\partial U_{x_j,\Lambda}}{\partial h} + U_{\Lambda,\Lambda}^{2\tau - 2} \frac{\partial U_{\Lambda,\Lambda}}{\partial h}) \phi . \]

Since
\[| \int_{\Omega} K(\frac{|y|}{\mu})(W_{r,\Lambda}^{2\tau - 2} \frac{\partial W_{r,\Lambda}}{\partial h} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{2\tau - 2} \frac{\partial U_{x_j,\Lambda}}{\partial h} - \sum_{j=1}^{k} U_{\Lambda,\Lambda}^{2\tau - 2} \frac{\partial U_{\Lambda,\Lambda}}{\partial h}) \phi | \leq C \left( \frac{1}{\mu} \right)^{m-1+2\varepsilon} \]
and
\[| \int_{\Omega} (K(\frac{|y|}{\mu}) - 1)(U_{x_j,\Lambda}^{2\tau - 2} \frac{\partial U_{x_j,\Lambda}}{\partial h} + U_{\Lambda,\Lambda}^{2\tau - 2} \frac{\partial U_{\Lambda,\Lambda}}{\partial h}) \phi | \]
\[\leq \int_{|y| - \mu_0 < \sqrt{\mu}} (K(\frac{|y|}{\mu}) - 1)(U_{x_j,\Lambda}^{2\tau - 2} \frac{\partial U_{x_j,\Lambda}}{\partial h} + U_{\Lambda,\Lambda}^{2\tau - 2} \frac{\partial U_{\Lambda,\Lambda}}{\partial h}) \phi | \]
\[+ \int_{|y| - \mu_0 > \sqrt{\mu}} (K(\frac{|y|}{\mu}) - 1)(U_{x_j,\Lambda}^{2\tau - 2} \frac{\partial U_{x_j,\Lambda}}{\partial h} + U_{\Lambda,\Lambda}^{2\tau - 2} \frac{\partial U_{\Lambda,\Lambda}}{\partial h}) \phi | \]
Let \( B \) be the solution of
\[
\partial F(r, h, \Lambda) = k\left( -\frac{mA_1}{\Lambda^{m+1}r^m} - \frac{A_2(m-2)}{\Lambda^{m-1}r^m} (\mu r_0 - r)^2 + \frac{B_1(N-2s)k^{N-2s}}{\Lambda^{N-2s+1}(r\sqrt{1-h^2})^{N-2s}} \right) + \frac{B_2(N-2s)}{\Lambda^{N-2s+1}(r\sqrt{1-h^2})^{N-2s}} + O\left( \frac{1}{\mu^{N-2s+1}(r\sqrt{1-h^2})^{N-2s+1}} \right),
\]
(3.4)

where \( A_1, B_1 \) and \( B_2 \) are some positive constants.

**Proof.** The proof of this Lemma is similar to Proposition 3.2 and we omit it here.

Now we analyze \( F(r, h, \Lambda) \), \( \frac{\partial F(r, h, \Lambda)}{\partial n} \) and \( \frac{\partial F(r, h, \Lambda)}{\partial \Lambda} \).

Let \( h_0 \) be the solution of
\[
\frac{B_2(N-2s-1)k}{h^{N-2s}} \cdot \left( \frac{1}{\sqrt{1-h^2}} \right)^{N-2s+2} = 0.
\]

It holds that
\[
h_0 = \frac{1}{\left( 1 + \left( \frac{B_1(N-2s)k^{N-2s}}{B_2} \right)^{N-2s+1} \right)^{\frac{1}{2}}} = O\left( \frac{1}{k^{N-2s+1}} \right).
\]

Let \( \Lambda_0 \) be the solution of
\[
-\frac{mA_1}{\Lambda^{m+1}r^m} + \frac{B_1(N-2s)k^{N-2s}}{\Lambda^{N-2s+1}(r\sqrt{1-h^2})^{N-2s}} + \frac{B_2(N-2s)}{\Lambda^{N-2s+1}(r\sqrt{1-h^2})^{N-2s}} = 0.
\]

We get
\[
\Lambda_0 = \frac{1}{\mu(mA_1)^{N-2s-1}} \left( \frac{B_1(N-2s)k^{N-2s}}{h_0^{N-2s-1}} \cdot \left( \frac{1}{\sqrt{1-h^2}} \right)^{N-2s+1} + \frac{B_2(N-2s)k}{h_0^{N-2s-1}} \right)^{\frac{1}{N-2s-1}}.
\]

Since
\[
r = r_0 + O\left( \frac{1}{\mu^{1+\epsilon}} \right), \quad h = h_0 + o\left( \frac{1}{\mu^{\epsilon}} \right)
\]
and for some positive constants \( B_3, B_4, B_5 \) and \( B_6 \), by (3.1) we get

\[
\left( \frac{B_1k^{N-2s}}{(r\sqrt{1-h^2})^{N-2s}} + \frac{B_2}{(r\sqrt{1-h^2})^{N-2s}} \right) + B_5 \left( \frac{h}{h_0} \right)^{\epsilon} + B_6 \left( \frac{h}{h_0} \right)^2 = \frac{B_3}{\mu^{N-2s}} + \frac{B_4}{\mu^{N-2s}} - \frac{N-2s}{N-2s} \cdot \left( \frac{B_2}{B_1k^{N-2s}} \right)^{N-2s} \cdot \left( \frac{1 - \frac{h}{h_0}}{1 - \frac{h}{h_0}} \right)^3.
\]

(3.5)
Rewrite (3.1), (3.4) and (3.5) respectively as
\[
F(r, h, \Lambda) = k\left(\frac{A_1}{\Lambda^m} + \frac{A_2}{\Lambda^{m-2}} (\mu r_0 - r)^2 - \frac{B_3}{\Lambda^{N-2s}} (\mu r_0 - r)^{2+\theta} - \frac{B_4}{\Lambda^{N-2s}} \right) + \frac{B_5}{\Lambda^{N-2s}} \left(1 - \frac{h}{h_0}\right)^2
\]
\[+ O\left(\frac{1}{\mu^{N-2s}} \frac{h_0}{(N-2s)(N-2s-1^{2})} \left(1 - \frac{h}{h_0}\right)^3 + \frac{1}{\mu^{N-2s}} \frac{h_0}{(N-2s)(N-2s-1^{2})} \right)\right) + \left(3.6\right)
\]
\[\frac{\partial F(r, h, \Lambda)}{\partial \Lambda} = k\left(-\frac{m A_1}{\Lambda^{m+1}} - \frac{A_2}{\Lambda^{m-1}} (\mu r_0 - r)^2 + \frac{B_3}{\Lambda^{N-2s+1}} \right) + O\left(\frac{1}{\mu^{N-2s}} \frac{h_0}{(N-2s)(N-2s-1^{2})} \right), \quad \left(3.7\right)
\]
\[\frac{\partial F(r, h, \Lambda)}{\partial h} = k\left(-\frac{m A_1}{\Lambda^{m+1}} - \frac{A_2}{\Lambda^{m-1}} (\mu r_0 - r)^2 + \frac{B_3}{\Lambda^{N-2s+1}} \right) + O\left(\frac{1}{\mu^{N-2s}} \frac{h_0}{(N-2s)(N-2s-1^{2})} \right), \quad \left(3.8\right)
\]
where \(A, A_1, A_2, B_3, B_4, B_5, B_6, B_7\) are positive constants defined in Lemma B.1 and (3.5).

Define
\[F(r, h, \Lambda) = -F(r, h, \Lambda), \quad (r, h, \Lambda) \in \mathbb{D}.
\]
Let
\[\alpha_2 = k(-A + \beta),
\]
and
\[\alpha_1 = k\left(-A - \frac{A_1}{\Lambda^m} - \frac{B_3}{\Lambda^{N-2s}} \right) - \frac{1}{\mu^{m+2\theta}},
\]
where \(\theta, \beta > 0\) is a small constant.

Denote the energy level set
\[\tilde{F} = \{(r, h, \Lambda)| (r, h, \Lambda) \in \mathbb{D}, \tilde{F}(r, h, \Lambda) \leq \alpha\}.
\]
Consider
\[\left\{\frac{dr}{dt} = -D_r \tilde{F}, t > 0; \quad \frac{dh}{dt} = -D_h \tilde{F}, t > 0; \quad \frac{d\Lambda}{dt} = -D_{\Lambda} \tilde{F}, t > 0; \quad (r, h, \Lambda) \in \tilde{F}_{\alpha_2}.\right\}
\]

Proposition 3.4. The flow \((r(t), h(t), \Lambda(t))\) does not leave \(\mathbb{D}\) before it reaches \(\tilde{F}_{\alpha_2}\).

Proof. If \(\Lambda = \Lambda_0 + \frac{1}{\mu^2}\), such that \(|r - \mu r_0| \leq \frac{1}{\mu^2}\), and \(|h - h_0| \leq \frac{1}{\mu^2}\), we gain from (3.7) that
\[\frac{\partial \tilde{F}(W_{r,h}, \Lambda)}{\partial \Lambda} = k\left(\frac{c}{\mu^{m+2\theta}} + O\left(\frac{1}{\mu^{m+2\theta}}\right)\right) > 0,
\]
where \(c\) is a positive constant. Thus, the flow does not leave \(\mathbb{D}\).
Similarly, if \( \Lambda = \Lambda_0 - \frac{1}{\mu^{2\theta}} \), such that \( |r - \mu r_0| \leq \frac{1}{\mu^{\theta}} \), and \( |h - h_0| \leq \frac{1}{\mu^{\theta}} \), we gain from (3.7) that

\[
\frac{\partial \tilde{F}(W_{r,h,\Lambda})}{\partial \Lambda} = k\left( - \frac{c}{\mu^{m+2\theta}} + O\left( \frac{1}{\mu^{m+2\theta}} \right) \right) < 0.
\]

where \( c \) is a positive constant. Thus, the flow does not leave \( D \).

If \( h = h_0 + \frac{1}{\mu^{\theta}} \), such that \( |r - \mu r_0| \leq \frac{1}{\mu^{\theta}} \), and \( |\Lambda - \Lambda_0| \leq \frac{1}{\mu^{\theta}} \), we gain from (3.8) that

\[
\frac{\partial \tilde{F}(W_{r,h,\Lambda})}{\partial h} = - \frac{k}{\Lambda^{N-2s}} \left( \mu^{N-2s} - \frac{B_7}{N-2s} \right) \left( \mu^{N-2s} - \frac{1}{N-2s+1} \right) + O\left( \frac{1}{\mu^{N-2s}} \right) < 0.
\]

Thus, the flow does not leave \( D \).

Similarly, if \( h = h_0 - \frac{1}{\mu^{\theta}} \), such that \( |r - \mu r_0| \leq \frac{1}{\mu^{\theta}} \), and \( |\Lambda - \Lambda_0| \leq \frac{1}{\mu^{\theta}} \), we gain from (3.8) that

\[
\frac{\partial \tilde{F}(W_{r,h,\Lambda})}{\partial h} = \frac{k}{\Lambda^{N-2s}} \left( \mu^{N-2s} - \frac{B_7}{N-2s} \right) \left( \mu^{N-2s} - \frac{1}{N-2s+1} \right) + O\left( \frac{1}{\mu^{N-2s}} \right) > 0.
\]

Thus, the flow does not leave \( D \).

Assuming \( |r - \mu r_0| = \frac{1}{\mu^{\theta}} \), since \( |\Lambda - \Lambda_0| \leq \frac{1}{\mu^{\theta}} |h - h_0| \leq \frac{1}{\mu^{\theta}} \), we get

\[
\frac{B_3}{\Lambda^{N-2s}} - \frac{A_1}{\Lambda^m} = \frac{B_3}{\Lambda_0^{N-2s}} - \frac{A_1}{\Lambda_0^m} + O\left( |\Lambda - \Lambda_0|^2 \right)
= \left( \frac{B_3}{\Lambda_0^{N-2s}} - \frac{A_1}{\Lambda_0^m} \right) + O\left( \frac{1}{\mu^{2\theta}} \right). \tag{3.9}
\]

So, using (3.4), (3.9), we obtain

\[
\tilde{F}(r, h, \Lambda) = k \left( - A + \left( \frac{B_3}{\Lambda_0^{N-2s}} - \frac{A_1}{\Lambda_0^m} \right) \frac{1}{\mu^m} - \frac{A_2}{\Lambda_0^{m-2} \mu^m + 2\theta} + O\left( \frac{1}{\mu^{m+2\theta}} \right) \right) < \alpha_1. \tag{3.10}
\]

Proof of Theorem 1.2 — Define

\[
\Gamma = \left\{ f : f(r, h, \Lambda) = (f_1(r, h, \Lambda), f_2(r, h, \Lambda), f_3(r, h, \Lambda)) \in \mathbb{D}, (r, h, \Lambda) \in \mathbb{D}, \right. \left. f(r, h, \Lambda) = (r, h, \Lambda), \text{if } |r - \mu r_0| = \frac{1}{\mu^{\theta}} \right\}.
\]

Let

\[
c = \inf_{f \in \Gamma} \sup_{(r, h, \Lambda) \in \mathbb{D}} \tilde{F}(f(r, h, \Lambda)).
\]

In fact we can follow [21] to prove

(i) \( \alpha_1 < c < \alpha_2 \);

(ii) \( \sup_{|r - \mu r_0| = \frac{1}{\mu^{\theta}}} \tilde{F}(f(r, h, \Lambda)) < \alpha_1, \forall f \in \Gamma. \)

Thus \( c \) is a critical value of \( \tilde{F} \).
4. The non-degeneracy of the solutions

In this section, we prove the non-degeneracy of the 2k-bubbling solutions constructed by Theorem 1.2. Recall

\[
(-\Delta)^s u = K(\frac{|y|}{\mu})u^{2^*_s-1},
\]

and the solution \( u_k \) for (4.1), satisfying

\[ u_k = W_{r,h,A} + \omega_k, \]

where \( W_{r,h,A} = \sum_{i=1}^{k} U_{r,i,A} + \sum_{i=1}^{k} U_{\lambda,i}. \)

In order to apply local Pohozaev identities, we quote the extension of \( \tilde{u} \) and \( \tilde{\xi} \) to have

\[
\begin{cases}
\text{div}(t^{1-2s}\nabla \tilde{u}) = 0, & x \in \mathbb{R}^{N+1}, \\
-\lim_{t \to 0} t^{1-2s} \partial_t \tilde{u}(y, t) = K(\frac{|y|}{\mu})u^{2^*_s-1} & x \in \mathbb{R}^N,
\end{cases}
\]

and

\[
\begin{cases}
\text{div}(t^{1-2s} \nabla \tilde{\xi}) = 0, & x \in \mathbb{R}^{N+1}, \\
-\lim_{t \to 0} t^{1-2s} \partial_t \tilde{\xi}(y, t) = (2^*_s - 1)K(\frac{|y|}{\mu})u^{2^*_s-2} & x \in \mathbb{R}^N.
\end{cases}
\]

For any smooth domain \( \Omega \) in \( \mathbb{R}^N \), we set

\[
\Omega^+ = \{(y, t), y \in \Omega, t > 0 \} \subseteq \mathbb{R}^{N+1},
\]

and \( \partial'' \Omega^+ = \{(y, t), y \in \partial \Omega, t > 0 \} \subseteq \mathbb{R}^{N+1}. \)

There hold the following Pohozaev identities.

**Lemma 4.1.** There hold that

\[
- \int_{\partial'' \Omega^+} t^{1-2s} \partial_t \tilde{u} \partial_t \tilde{\xi} = \int_{\partial'' \Omega^+} t^{1-2s} \partial_t \tilde{u} \partial_t \tilde{\xi} + \int_{\partial'' \Omega^+} t^{1-2s}(\nabla \tilde{u}, \nabla \tilde{\xi}) \nu_i
\]

and

\[
\int_{\Omega} u^{2^*_s-1} \xi (\nabla K(\frac{|y|}{\mu}), y - x_0)
\]

\[
= \int_{\partial \Omega} K(\frac{|y|}{\mu})u^{2^*_s-1} \xi (y, y - x_0) + \int_{\partial'' \Omega^+} t^{1-2s} \partial_t \tilde{u} (\nabla \tilde{\xi}, Y - X_0) + \int_{\partial'' \Omega^+} t^{1-2s} \partial_t \tilde{\xi} (\nabla \tilde{u}, Y - X_0)
\]

\[
- \int_{\partial'' \Omega^+} t^{1-2s}(\nabla \tilde{u}, \nabla \tilde{\xi})(y, Y - X_0) + \frac{N - 2s}{2} \int_{\partial'' \Omega^+} t^{1-2s} \partial_t \tilde{u} \partial_t \tilde{\xi}
\]

where \( Y = (y, t), X_0 = (x_0, 0) \in \Omega^+. \)

**Proof.** For a similar proof, we can refer to [13]. \( \square \)
To obtain the non-degeneracy result, we should improve the estimates of the $2k$-bubbling solution of (4.1) obtained in Theorem 1.2. Precisely, we have the following result.

**Lemma 4.2.** There holds that

$$|u_k(y)| \leq C \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s}} + C \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s}}, \quad \text{for all } y \in \mathbb{R}^N.$$  

**Proof.** Since (4.1), for some constant $\sigma_{N,s}$, we have

$$u_k = \sigma_{N,s} \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2s}} K \left( \frac{|z|}{\mu} \right) u_k^{2s-1}(z) dz.$$  

We estimate by Lemma 4.3 that

$$|u_k| \leq C \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2s}} u_k^{2s-1}(z) dz \leq \frac{1}{|z - y|^{N-2s}} \left( \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{N-2s}} \right)^{2s-1} dz.$$  

Let $\tau_1$ and $\tau$ be such that

$$N - 2s \leq \tau_1 < \tau.$$  

Since it holds that

$$\frac{N - 2s}{2} \tau_1 + \frac{(N + 2s)(\tau - \tau_1)}{N - 2s} = \frac{N - 2s}{2} + \tau + \frac{4(\tau - \tau_1)}{N - 2s} > \frac{N - 2s}{2} + \tau,$$

we continue this process to obtain the result. \qed

Recall the linear operator

$$L_k \xi = (-\Delta)^s \xi - (2s-1) K \left( \frac{|y|}{\mu} \right) u_k^{2s-2} \xi.$$  

We prove Theorem 1.3 by contradiction. Suppose that there exists some $k \to +\infty$ such that $||\xi_n||_s = 1$ and $L_k \xi_n = 0$. Let

$$\bar{\xi}_n(y) = \xi_n(y + x_1).$$  

**Lemma 4.3.** There exist some constants $b_0$, $b_1$ and $b_3$ such that

$$\bar{\xi}_n \to b_0 \psi_0 + b_1 \psi_1 + b_3 \psi_3,$$

uniformly in $C^1(B_R(0))$ for any $R > 0$,

$$\psi_0 = \frac{\partial U_{0,1}}{\partial \Lambda} |_{\Lambda=1}, \quad \psi_i = \frac{\partial U_{0,1}}{\partial y_i}, \quad i = 1, 3.$$
Proof. Since by assumption, $|\bar{\xi}| \leq C$, we assume that $\bar{\xi} \to \xi$ in $C_{loc}(\mathbb{R}^N)$. It is easy to find that $\xi$ satisfies
\[ (-\Delta)^s \xi = (2^*_s - 1)U^{2^*_s - 2} \xi, \quad \text{in } \mathbb{R}^N, \tag{4.4} \]
which gives that
\[ \xi = \sum_{i=0}^N b_i \psi_i. \]
Since $\xi_n$ is even in $y_i$, $i = 2, 4, \ldots, N$, we obtain that $b_i = 0$, $i = 2, 4, \ldots, N$. \hfill \Box

Similar to Lemma 4.2, we show the following result.

Lemma 4.4. There holds that
\[ |\xi_k(y)| \leq C \sum_{j=1}^k \frac{1}{(1 + |y - \bar{x}_j|)^{N-2s}} + C \sum_{j=1}^k \frac{1}{(1 + |y - \bar{x}_j|)^{N-2s}}, \quad \text{for all } y \in \mathbb{R}^N. \]

From Lemma 4.3 we know that $b_0$, $b_1$ and $b_3$ are both bounded. Now we are ready to prove the following result.

Lemma 4.5. It holds $b_0 = b_1 = b_3 = 0$.

Proof. **Step 1.** We apply the Pohozaev identities (4.2) in the domain $\Omega = B_{\delta}(\bar{x}_1)$ and obtain that
\[ - \int_{\partial^* B_{\delta}(\bar{x}_1)} t^{1-2s} \frac{\partial \tilde{\xi}_k}{\partial y_1} - \int_{\partial^* B_{\delta}(\bar{x}_1)} t^{1-2s} \frac{\partial \tilde{\xi}_k}{\partial y_1} + \int_{\partial^* B_{\delta}(\bar{x}_1)} t^{1-2s} \langle \nabla \tilde{u}_k, \nabla \tilde{\xi}_k \rangle y_1 \]
\[ = - \int_{\partial B_{\delta}(\bar{x}_1)} \partial K \left( \frac{|y|}{\mu} \right) u^{2^*_s - 1}_k \tilde{\xi}_k + \int_{\partial B_{\delta}(\bar{x}_1)} K \left( \frac{|y|}{\mu} \right) u^{2^*_s - 1}_k \tilde{\xi}_k y_1. \tag{4.5} \]
It is easy to observe that
\[ \int_{\partial B_{\delta}(\bar{x}_1)} K \left( \frac{|y|}{\mu} \right) u^{2^*_s - 1}_k \tilde{\xi}_k y_1 = O \left( \frac{1}{t^{2s+\sigma}} \right). \]

By Lemma A.6, we get
\[ E_1 := - \int_{\partial^* B_{\delta}(\bar{x}_1)} t^{1-2s} \frac{\partial \tilde{\xi}_k}{\partial y_1} - \int_{\partial^* B_{\delta}(\bar{x}_1)} t^{1-2s} \frac{\partial \tilde{\xi}_k}{\partial y_1} + \int_{\partial^* B_{\delta}(\bar{x}_1)} t^{1-2s} \langle \nabla \tilde{u}_k, \nabla \tilde{\xi}_k \rangle y_1 \]
\[ = - \int_{\partial^* B_{\delta}(\bar{x}_1)} t^{1-2s} \frac{\partial \sum_{i=1}^k \tilde{U}_{\pi,\lambda} + \sum_{i=1}^k \tilde{U}_{\pi,\lambda}}{\partial y_1} - \int_{\partial^* B_{\delta}(\bar{x}_1)} t^{1-2s} \frac{\partial \sum_{i=1}^k \tilde{U}_{\pi,\lambda} + \sum_{i=1}^k \tilde{U}_{\pi,\lambda}}{\partial y_1} \]
\[ + \int_{\partial^* B_{\delta}(\bar{x}_1)} t^{1-2s} \langle \nabla \sum_{i=1}^k \tilde{U}_{\pi,\lambda} + \sum_{i=1}^k \tilde{U}_{\pi,\lambda}, \nabla \tilde{\xi}_k \rangle y_1 \]
\[ \leq C b_0 \int_{\partial^* B_{\delta}(\bar{x}_1)} t^{1-2s} \frac{1}{|y - \bar{x}_1|^{N-2s+1}} \sum_{i=1}^k \frac{(\bar{x}_j - \bar{x}_1)}{|\bar{x}_j - \bar{x}_1|^{N-2s+2}} \]
\[ + C b_0 \int_{\partial^* B_{\delta}(\bar{x}_1)} t^{1-2s} \frac{1}{|y - \bar{x}_1|^{N-2s+1}} \sum_{i=1}^k \frac{(\bar{x}_j - \bar{x}_1)}{|\bar{x}_j - \bar{x}_1|^{N-2s+2}} + O \left( \frac{1}{t^{2s+\sigma}} \right) \]
\[ \leq Cb_0 \sum_{j=2}^{k} \frac{(\overline{x}_j - \overline{x}_1)_1}{|\overline{x}_j - \overline{x}_1|^{N-2s+2}} + Cb_0 \sum_{j=1}^{k} \frac{(\overline{x}_j - \overline{x}_1)_1}{|\overline{x}_j - \overline{x}_1|^{N-2s+2}} + O\left(\frac{1}{\mu^{m+\sigma}}\right). \]

Since
\[ |\overline{x}_j - \overline{x}_1| = 2r \sqrt{1 - h^2} \sin \frac{j\pi}{k} \]
and
\[ |\overline{x}_1 - \overline{x}_j| = 2r \sqrt{(1 - h^2) \sin^2 \left(\frac{(j-1)\pi}{k}\right)} + h^2, \quad j = 2, 3, \ldots, k, \]
we have
\[ (\overline{x}_j - \overline{x}_1)_1 = (\overline{x}_j - \overline{x}_1)_1 = r \sqrt{1 - h^2} \cos \frac{2(j-1)\pi}{k} - r \sqrt{1 - h^2} = -\frac{|\overline{x}_j - \overline{x}_1|^2}{2r \sqrt{1 - h^2}} \]
and
\[ (\overline{x}_1 - \overline{x}_1)_1 = 0. \]
Thus
\[ E_1 \leq \frac{Cb_0}{r \sqrt{1 - h^2}} \left( \sum_{j=2}^{k} \frac{1}{|\overline{x}_j - \overline{x}_1|^{N-2s}} + \sum_{j=2}^{k} \frac{|\overline{x}_j - \overline{x}_1|^2}{|\overline{x}_j - \overline{x}_1|^{N-2s}} \right) + O\left(\frac{1}{\mu^{m+\sigma}}\right) \]
\[ = b_0O\left(\frac{1}{(r \sqrt{1 - h^2})^{N-2s}}\right) = b_0O\left(\frac{1}{(\sqrt{1 - h^2})^{N-2s}}\right). \quad (4.6) \]

We estimate the right hand side of (4.5),
\[ \int_{B_r(x_1)} \frac{\partial K\left(\frac{|y|}{\mu}\right)}{\partial y_1} u_k^{2_s-1} \xi_k \]
\[ = \int_{B_r(x_1)} \left( \frac{\partial K\left(\frac{|y|}{\mu}\right)}{\partial y_1} - \frac{\partial K\left(\frac{|x_1|}{\mu}\right)}{\partial y_1} \right) u_k^{2_s-1} \xi_k + O\left(\frac{1}{\mu^{m+\sigma}}\right) \]
\[ = \int_{B_r(x_1)} \left( \nabla \frac{K\left(\frac{|y|}{\mu}\right)}{\partial y_1}, y - \overline{x}_1 \right) u_k^{2_s-1} \xi_k + O\left(\frac{1}{\mu^{m+\sigma}}\right) \]
\[ = \int_{\mathbb{R}^N} \left( \nabla \frac{K\left(\frac{|y|}{\mu}\right)}{\partial y_1}, \frac{y}{\mu} \right) U^{2_s-1}(b_0\psi_0 + b_1\psi_1 + b_3\psi_3) + O\left(\frac{1}{\mu^{m+\sigma}}\right) \]
\[ = \frac{K''\left(\frac{|\overline{x}_1|}{\mu}\right)b_1}{\mu \Lambda} \int_{\mathbb{R}^N} U^{2_s-1}\psi_1 + O\left(\frac{1}{\mu^{m+\sigma}}\right). \quad (4.7) \]

Combining (4.6) and (4.7), we obtain
\[ \frac{K''\left(\frac{|\overline{x}_1|}{\mu}\right)b_1}{\mu \Lambda} \int_{\mathbb{R}^N} U^{2_s-1}\psi_1 + O\left(\frac{1}{\mu^{m+\sigma}}\right) \leq b_0O\left(\frac{1}{\mu^{m+\sigma}}\right), \]
which gives \( b_1 = 0. \)

**Step 2.** We apply the Pohozaev identities in \( y_3 \) to get
\[ - \int_{\partial^* B^c_\epsilon(x_1)} t^{1-2s} \frac{\partial u_k}{\partial y_3} \frac{\partial \overline{\xi}_k}{\partial y_3} - \int_{\partial^* B^c_\epsilon(x_1)} t^{1-2s} \frac{\partial \overline{\xi}_k}{\partial y_3} \frac{\partial u_k}{\partial y_3} + \int_{\partial^* B^c_\epsilon(x_1)} t^{1-2s} \langle \nabla u_k, \nabla \overline{\xi}_k \rangle y_3 \]
\[ = -\int_{B_0(\frac{1}{\mu})} \partial K(\frac{\nu}{\mu}) \frac{\partial \tilde{\xi}_k}{\partial \nu} \sum_{i=1}^{N-2s} |\bar{x}_j - \bar{x}_i|^3 |\bar{x}_j - \bar{x}_i|^{N-2s+2} + \int_{\partial B_0(\frac{1}{\mu})} K(\frac{\nu}{\mu}) \frac{\partial \tilde{\xi}_k}{\partial \nu} \sum_{i=1}^{N-2s} \langle \nabla u_k, \nabla \tilde{\xi}_k \rangle y_3. \]

Again by Lemma 4.4, we have

\[ \int_{\partial B_0(\frac{1}{\mu})} K(\frac{\nu}{\mu}) \frac{\partial \tilde{\xi}_k}{\partial \nu} \sum_{i=1}^{N-2s} \langle \nabla u_k, \nabla \tilde{\xi}_k \rangle y_3 = O(1). \]

Let

\[ E_3 := -\int_{\partial^s B_0(\frac{1}{\mu})} |y - \bar{x}|^{N-2s+1} \sum_{i=1}^{N-2s} \langle \tilde{\xi}_k, \nabla \tilde{\xi}_k \rangle y_3 + \int_{\partial^s B_0(\frac{1}{\mu})} \langle \tilde{\xi}_k, \nabla \tilde{\xi}_k \rangle y_3 \]

\[ \leq C_0 \int_{\partial^s B_0(\frac{1}{\mu})} \langle \tilde{\xi}_k, \nabla \tilde{\xi}_k \rangle y_3 + O(1) \]

\[ \leq C_0 \sum_{j=1}^{N-2s+2} \langle \tilde{\xi}_k, \nabla \tilde{\xi}_k \rangle y_3 + O(1) \]

\[ = b_0 O(1) \]

Since \((x_j - x_i)_3 = |x_j - x_i| = -2rh\) and \((x_j - x_i)_3 = 0\), then

\[ E_3 \leq b_0 O(1) \]

(4.8)

Similar to (4.7), we have

\[ \int_{B_0(\frac{1}{\mu})} \partial K(\frac{\nu}{\mu}) \frac{\partial \tilde{\xi}_k}{\partial \nu} \sum_{i=1}^{N-2s} |\bar{x}_j - \bar{x}_i|^3 |\bar{x}_j - \bar{x}_i|^{N-2s+2} + \int_{B_0(\frac{1}{\mu})} K(\frac{\nu}{\mu}) \frac{\partial \tilde{\xi}_k}{\partial \nu} \sum_{i=1}^{N-2s} \langle \nabla u_k, \nabla \tilde{\xi}_k \rangle y_3. \]

Combining (4.8) and (4.9), we obtain

\[ \frac{K''(\frac{\nu}{\mu}) b_3}{\mu \Lambda} \int_{\mathbb{R}^N} U^{2s-1} \psi y_3 + O(1) \leq O(1) \]

Since

\[ \frac{(N - 2s + 1)^2 - (N - 2s + 1)(N - 2s - m)}{(N - 2s + 1)} > 1, \]

thus \(b_3 = 0\).
Step 3. We use (4.3) for Ω = ℜ^N, from Lemma A.7 we obtain
\[ \int_{\mathbb{R}^N} u_k^{2^* - 1} \xi_k \langle \nabla K \left( \frac{|y|}{\mu} \right), y \rangle = 0, \]
which gives
\[ \int_{\Omega^*_1} u_k^{2^* - 1} \xi_k \langle \nabla K \left( \frac{|y|}{\mu} \right), y \rangle = 0. \]

On the other hand, we have
\[ \int_{\Omega^*_1} u_k^{2^* - 1} \xi_k \langle \nabla K \left( \frac{|y|}{\mu} \right), y \rangle = \int_{B_o(\overline{x}_1)} u_k^{2^* - 1} \xi_k \langle \nabla K \left( \frac{|y|}{\mu} \right), y \rangle + O \left( \frac{1}{\mu^{m+\sigma}} \right) \]
\[ = \int_{B_o(\overline{x}_1)} u_k^{2^* - 1} \xi_k \langle \nabla K \left( \frac{|y|}{\mu} \right), y - \overline{x}_1 \rangle + r \int_{B_o(\overline{x}_1)} u_k^{2^* - 1} \xi_k \frac{\partial K \left( \frac{|y|}{\mu} \right)}{\partial y_1} + O \left( \frac{1}{\mu^{m+\sigma}} \right), \]
which gives
\[ -r \int_{B_o(\overline{x}_1)} u_k^{2^* - 1} \xi_k \frac{\partial K \left( \frac{|y|}{\mu} \right)}{\partial y_1} = \int_{B_o(\overline{x}_1)} u_k^{2^* - 1} \xi_k \langle \nabla K \left( \frac{|y|}{\mu} \right), y - \overline{x}_1 \rangle + O \left( \frac{1}{\mu^{m+\sigma}} \right). \]

However, we have
\[ \int_{B_o(\overline{x}_1)} u_k^{2^* - 1} \xi_k \langle \nabla K \left( \frac{|y|}{\mu} \right), y - \overline{x}_1 \rangle \]
\[ = \int_{B_o(\overline{x}_1)} u_k^{2^* - 1} \xi_k \langle \nabla K \left( \frac{|y|}{\mu} \right) - \nabla K \left( \frac{|\overline{x}_1|}{\mu} \right), y - \overline{x}_1 \rangle + O \left( \frac{1}{\mu^{m+\sigma}} \right) \]
\[ = \int_{B_o(\overline{x}_1)} u_k^{2^* - 1} \xi_k \langle \nabla^2 K \left( \frac{|\overline{x}_1|}{\mu} \right), y - \overline{x}_1 \rangle + O \left( \frac{1}{\mu^{m+\sigma}} \right) \]
\[ = \frac{b_0 \Delta K \left( \frac{|\overline{x}_1|}{\mu} \right)}{N\mu} \int_{\mathbb{R}^N} U^{2^* - 1} \psi_0 |y|^2 + o \left( \frac{1}{\mu^m} \right). \]

Thus we obtain
\[ \int_{B_o(\overline{x}_1)} u_k^{2^* - 1} \xi_k \frac{\partial K \left( \frac{|y|}{\mu} \right)}{\partial y_1} = -\frac{b_0 \Delta K \left( \frac{|\overline{x}_1|}{\mu} \right)}{N\mu^2} \int_{\mathbb{R}^N} U^{2^* - 1} \psi_0 |y|^2 + o \left( \frac{1}{\mu^m} \right). \]

Combining (4.7) and (4.11), we have
\[ -\frac{b_0 \Delta K \left( \frac{|\overline{x}_1|}{\mu} \right)}{N\mu^2} \int_{\mathbb{R}^N} U^{2^* - 1} \psi_0 |y|^2 + o \left( \frac{1}{\mu^m} \right) \leq b_0 O \left( \frac{1}{\mu^m (\sqrt{1 - h^2})^{N-2s}} \right). \]
Thus \( b_0 = 0. \) □
Appendix

A. Basic Estimates

In this section, we give some basic estimates. For each fixed $i$ and $j, i \neq j$, consider the following function

\[ g_{ij}(y) = \frac{1}{(1 + |x - x_i|)^\alpha (1 + |x - x_j|)^\beta} \]

where $\alpha \geq 1$ and $\beta \geq 1$ are two constants.

**Lemma A.1** (c.f. [21]). For any constant $0 < \sigma \leq \min(\alpha, \beta)$, there is a constant $C > 0$, such that

\[ g_{ij}(y) \leq \frac{C}{|x_i - x_j|^\tau} \left( \frac{1}{(1 + |x - x_i|)^{\alpha - \sigma}} + \frac{1}{(1 + |x - x_j|)^{\beta - \sigma}} \right) \]

**Lemma A.2** (c.f. [12]). For any constant $0 < \sigma \leq N - 2s$, there is a constant $C > 0$, such that

\[ \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N - 2s}} \frac{1}{(1 + |z - x_i|)^{s - \sigma}} dz \leq C \left( \frac{1}{(1 + |y|)^\sigma} \right) \]

**Lemma A.3.** Suppose $\tau \in (0, \frac{N - 2s}{2})$, $y = (y_1, y_2, \ldots, y_N)$. Then there is a small $\theta > 0$, such that when $y_3 \geq 0$,

\[ \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N - 2s}} W_{r, \hat{h}, \Lambda} \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{s - \sigma}{2} + \tau}} dz \leq C \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{s - \sigma}{2} + \tau + \theta}} \]

and when $y_3 < 0$,

\[ \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N - 2s}} W_{r, \hat{h}, \Lambda} \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{s - \sigma}{2} + \tau}} dz \leq C \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{s - \sigma}{2} + \tau + \theta}} \]

**Proof.** The proof of Lemma A.3 is similar to [12], we omit it here. \( \square \)

**Lemma A.4.** For any $\tau > 0$, there is a constant $C > 0$, such that

\[ \sum_{i=1}^{k} \frac{1}{|x_i - \bar{x}|^\tau} \leq \frac{Ck^\tau}{(r \sqrt{1 - h^2})^\tau} \sum_{i=2}^{k} \frac{1}{i^\tau} = \begin{cases} \frac{Ck^\tau (1 + O(\frac{1}{k})),} {O((r \sqrt{1 - h^2})^\tau (1 + O(\frac{\ln k}{k})))}, & \text{if } \tau > 1, \\ \frac{Ck^\tau (1 + O(\frac{\ln k}{k})))}{O((r \sqrt{1 - h^2})^\tau)}, & \text{if } \tau \leq 1, \end{cases} \tag{A.1} \]

and

\[ \sum_{i=1}^{k} \frac{1}{|x_i - \bar{x}|^\tau} = \frac{1}{(rh)^\tau} \sqrt{1 - h^2} (1 + o(\frac{1}{hk})) + \begin{cases} O(\frac{Ck^\tau (1 + O(\frac{\ln k}{k})))}{O((r \sqrt{1 - h^2})^\tau)}), & \text{if } \tau > 1, \\ O(\frac{Ck^\tau (1 + O(\frac{\ln k}{k})))}{O((r \sqrt{1 - h^2})^\tau)}), & \text{if } \tau \leq 1, \end{cases} \tag{A.2} \]

where $D_1 = \int_0^{+\infty} \frac{1}{(x^2 + 1)^\tau} dx$.

**Proof.** Recall

\[ \bar{x}_1 = r(\sqrt{1 - h^2}, 0, h, 0), \quad \bar{x}_i = r(\sqrt{1 - h^2} \cos \frac{2(i - 1)\pi}{k}, \sqrt{1 - h^2} \sin \frac{2(i - 1)\pi}{k}, h, 0), \]

\[ \bar{x}_i - \bar{x}_1 = r(\sqrt{1 - h^2} \cos \frac{2(i - 1)\pi}{k} - \sqrt{1 - h^2}, \sqrt{1 - h^2} \sin \frac{2(i - 1)\pi}{k}, 0, 0). \]
We obtain $|\bar{x}_i - \bar{x}| = 2r \sqrt{1 - h^2} \sin \left(\frac{(i-1)\pi}{k}\right)$. For any $\tau > 0$, it holds

$$
\sum_{i=2}^{k} \frac{1}{|\bar{x}_i - \bar{x}|^r} = \frac{1}{(2r \sqrt{1 - h^2})^r} \sum_{i=2}^{k} \frac{1}{(\sin \left(\frac{(i-1)\pi}{k}\right))^r}
$$
\[
= \begin{cases} 
\frac{1}{(2r \sqrt{1 - h^2})^r} \sum_{i=2}^{k} \frac{1}{(\sin \left(\frac{(i-1)\pi}{k}\right))^r} + \frac{1}{(2r \sqrt{1 - h^2})^r}, & \text{if } k \text{ is even}, \\
\frac{1}{(2r \sqrt{1 - h^2})^r} \sum_{i=2}^{k} \frac{1}{(\sin \left(\frac{(i-1)\pi}{k}\right))^r}, & \text{if } k \text{ is odd}.
\end{cases}
\]

Note that there exist two constants $C_1, C_2 > 0$, such that

$$
0 < C_1 \leq \frac{\sin \left(\frac{(i-1)\pi}{k}\right)}{\left(\frac{1}{k}\right)} \leq C_2, \quad i = 2, \ldots, \left(\frac{k}{2}\right) + 1.
$$

Thus for any $\tau > \frac{N-2k-m}{N-2k}$, we have (A.1).

On the other hand, we get

$$
|\bar{x}_1 - \bar{x}| = 2r \sqrt{1 - h^2} \sin \left(\frac{(i-1)\pi}{k}\right) + h^2.
$$

Similarly, we have

$$
\sum_{i=1}^{k} \frac{1}{|\bar{x}_1 - \bar{x}|^r} = \sum_{i=1}^{k} \frac{1}{(2r \sqrt{1 - h^2})^r} \left(\sin \left(\frac{(i-1)\pi}{k}\right) + h^2\right)^r
$$
\[
= \frac{2}{(2rh)^r} \sum_{i=1}^{\frac{k+1}{2}} \frac{1}{\left(1 + \frac{1-h^2 (i-1)^2 \pi^2}{k^2}\right)^{\frac{r}{2}}} + \begin{cases} 
O\left(\frac{Ck^r}{(r \sqrt{1 - h^2})^r}\right), & \text{if } \tau > 1, \\
O\left(\frac{Ck \ln k}{(r \sqrt{1 - h^2})^r}\right), & \text{if } \tau \leq 1.
\end{cases}
\]

We obtain

$$
\sum_{i=1}^{\frac{k+1}{2}} \frac{1}{\left(1 + \frac{1-h^2 (i-1)^2 \pi^2}{k^2}\right)^{\frac{r}{2}}} = \int_{0}^{\frac{k+1}{2}} \frac{1}{\left(1 + \frac{1-h^2 (i-1)^2 \pi^2}{k^2}\right)^{\frac{r}{2}}} dx + o(1)
$$
\[
= \frac{hk}{\sqrt{1 - h^2 \pi}} \int_{0}^{\frac{\pi^2 (i-1)^2 \pi^2}{h^2}} \frac{1}{(x^2 + 1)^{\frac{r}{2}}} dx + o(1)
\]
\[
= \frac{hk}{\sqrt{1 - h^2 \pi}} \int_{0}^{+\infty} \frac{1}{(x^2 + 1)^{\frac{r}{2}}} dx \left(1 + o\left(\frac{1}{hk}\right)\right).
\]

Thus, it holds (A.2). \qed

**Lemma A.5** (c.f. [13]). Let $\theta > 0$ is a constant. Suppose that $(y - x)^2 + r^2 > \delta^2$, $t > 0$ and $\alpha > N$. Then, when $0 < \beta < N$, we have

$$
\int_{\mathbb{R}^N} \frac{1}{(t + |z|^p)^{\frac{1}{2}}} \frac{1}{|y - z - x|^\theta} dz \leq C \left(\frac{1}{(1 + |y - x|^\theta)^{\alpha-N}} + \frac{1}{(1 + |y - x|)^{\alpha+\beta-N}}\right).
$$
Lemma A.6. Suppose that \((y - x)^2 + t^2 = \delta^2, \delta > 0\), then there is a constant \(C > 0\), such that
\[
\left| \nabla \tilde{U}_{x,A} \right| \leq \frac{C}{(1 + |y - x|^{N-2s+1})}.
\] (A.3)
and
\[
\left| \nabla \tilde{\psi} \right| \leq \frac{C}{(1 + |y - x|^{N-2s+1})}, \quad i = 0, 1, 2, \cdots, N. \tag{A.4}
\]

Proof. Since \(U_{x,A} = C(N, s)\frac{\Lambda^{\frac{N-2s}{2}}}{(1 + \Lambda^2 |y - x|^2)^{\frac{N-2s}{2}}},\) we have
\[
\tilde{U}_{x,A}(y, t) = \beta(N, s) \int_{\mathbb{R}^N} \frac{t^{2s}}{(y - \xi)^2 + t^2} \frac{\Lambda^{\frac{N-2s}{2}}}{(1 + \Lambda^2 |\xi - x|^2)^{\frac{N-2s}{2}}} \, d\xi.
\]
Note that for \(i = 1, \cdots, N,\)
\[
\frac{\partial}{\partial y_i} \int_{\mathbb{R}^N} \frac{t^{2s}}{(y - \xi)^2 + t^2} \frac{\Lambda^{\frac{N-2s}{2}}}{(1 + \Lambda^2 |\xi - x|^2)^{\frac{N-2s}{2}}} \, d\xi
\]
\[
= \frac{\partial}{\partial y_i} \int_{\mathbb{R}^N} \frac{1}{(1 + |\xi|^2)^{\frac{N-2s}{2}}} \frac{\Lambda^{\frac{N-2s}{2}}}{(1 + \Lambda^2 |y - \xi|^2)^{\frac{N-2s}{2}}} \, d\xi
\]
\[
\leq C \int_{\mathbb{R}^N} \frac{1}{(1 + |\xi|^2)^{\frac{N-2s}{2}}} \frac{\partial}{\partial y_i} (1 + |y - \xi|^2)^{\frac{N-2s}{2}} \, d\xi
\]
\[
\leq C \int_{\mathbb{R}^N} \frac{1}{(1 + |\xi|^2)^{\frac{N-2s}{2}}} (1 + |y - \xi|^2)^{\frac{N-2s}{2}} \, d\xi
\]
and
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^N} \frac{t^{2s}}{(y - \xi)^2 + t^2} \frac{\Lambda^{\frac{N-2s}{2}}}{(1 + \Lambda^2 |\xi - x|^2)^{\frac{N-2s}{2}}} \, d\xi
\]
\[
\leq C \int_{\mathbb{R}^N} \frac{1}{(1 + |\xi|^2)^{\frac{N-2s}{2}}} \sum_{i=1}^{N} (y - \xi^i)^2 (1 + |y - \xi|^2)^{\frac{N-2s}{2} + 1} \, d\xi.
\]
By Lemma A.5 we get
\[
\left| \nabla \tilde{U}_{x,A} \right| \leq C \int_{\mathbb{R}^N} \frac{1}{(1 + |\xi|^2)^{\frac{N+2s-1}{2}}} (1 + |y - \xi|^2)^{\frac{N-2s+1}{2}} \, d\xi
\]
\[
\leq C \int_{\mathbb{R}^N} \frac{1}{t^{2s-1}} \frac{1}{(1 + |z|^2)^{\frac{N+2s-1}{2}}} (1 + |y - z|^2)^{\frac{N-2s+1}{2}} \, d\xi
\]
\[
\leq C \int_{\mathbb{R}^N} \frac{1}{t^{2s-1}} \frac{1}{(1 + |y - x|^2)^{\frac{N-2s+1}{2}}} (1 + |y - x|^2)^{\frac{N-2s+1}{2}} \, d\xi
\]
\[
\leq \frac{C}{(1 + |y - x|^{N-2s+1})}.
\]
The proof of (A.4) is similar to (A.3), so we omit it here. \(\square\)

In order to apply the Pohozaev identities in the unbounded domain in \(\mathbb{R}^{N+1}_+\), it is necessary to verify
the following result.
Lemma A.7. Suppose $\Omega$ is an unbounded domain, we have that the following integrals are finite, i.e.,

$$
\left| \int_{\Omega(x,M,\infty)} t^{1-2s} \frac{\partial \tilde{u}}{\partial v} \frac{\partial \tilde{v}}{\partial v} \right| + t^{1-2s} \frac{\partial \tilde{v}}{\partial v} \partial \nu, \quad \left| \int_{\Omega(x,M,\infty)} t^{1-2s} |\nabla \tilde{u}| |\nabla \tilde{v}|, \quad \left| \int_{\Omega(x,M,\infty)} t^{1-2s} \left( \nabla \tilde{u}, \nabla \tilde{v} \right) (Y - (X, 0), v) \right|,$$

$$
\left| \int_{\Omega(x,M,\infty)} t^{1-2s} \frac{\partial \tilde{u}}{\partial v} \frac{\partial \tilde{v}}{\partial v} \right| + t^{1-2s} \frac{\partial \tilde{v}}{\partial v} \partial \nu, \quad \left| \int_{\Omega(x,M,\infty)} t^{1-2s} |\nabla \tilde{v} - (X, 0) + t^{1-2s} \frac{\partial \tilde{v}}{\partial v} \partial \nu \right| < +\infty.
$$

Proof. It suffices to prove that

$$
\int_{\Omega(x,M,\infty)} t^{1-2s} |\nabla \tilde{u}| |\nabla \tilde{v}| < +\infty, \quad \int_{\Omega(x,M,\infty)} t^{1-2s} |\nabla \tilde{u}| |\nabla \tilde{v}| |y - X_0 + t| < +\infty,
$$

$$
\int_{\Omega(x,M,\infty)} t^{1-2s} |\nabla \tilde{u}| |\tilde{v}| < +\infty, \quad \int_{\Omega(x,M,\infty)} t^{1-2s} |\nabla \tilde{u}| |\tilde{v}| < +\infty.
$$

We only give the proof for $s \leq \frac{1}{2}$, the case $s > \frac{1}{2}$ can be similarly estimated. From definition, we have

$$
\int_{\Omega(x,M,\infty)} t^{1-2s} |\nabla \tilde{u}| |\nabla \tilde{v}| [y - X_0] + t dt dy
$$

$$
\leq \int_{\Omega} \int_{M} t^{1-2s} |y - X_0| + t \left( \int_{R^N} \left( \frac{t^{2s-1} |u(x)|}{(y - x)^2 + t^2 \frac{2s-1}{2}} \right) dx \right) \left( \int_{R^N} \frac{t^{2s-1} |\tilde{v}(x)|}{(y - x)^2 + t^2 \frac{2s-1}{2}} dx \right) dt dy
$$

$$
= \int_{\Omega} \int_{M} \left( \int_{R^N} \frac{t^{2s-1} |y - X_0| + t}{(y - x)^2 + t^{2N+2s}} dx \right) \left( \int_{R^N} \frac{t^{2s-1} |\tilde{v}(x)|}{(y - x)^2 + t^{2N+2s}} dx \right) dt dy
$$

$$
\leq \int_{\Omega} \int_{R^N} \left( \int_{M} \left( \frac{t^{2s-1} |y - X_0| + t}{(y - x)^2 + t^{2N+4s}} dx \right) \right) \left( \int_{R^N} \frac{1}{(y - x)^2 + t^{2N+4s}} dx \right) \frac{1}{(1 + |x|)^{N-2s} dx} dy
$$

$$
\leq \int_{\Omega} \left( \int_{R^N} \left( \frac{1}{(y - x)^2 + M^{N+s}} \right) dx \right) \frac{1}{(1 + |x|)^{N-2s}} \frac{1}{(1 + |x|)^N dx} dy
$$

$$
\leq \left\{ \int_{\Omega} \frac{1}{(y + 1)^{2N-4s}} + \frac{(\ln |y|)^2}{(y + 1)^{2N-4s}} dy, \quad \text{if } s = \frac{1}{2}, \right.
$$

$$
\left. \int_{\Omega} \frac{1}{(y + 1)^{2N-4s}} + \frac{1}{(y + 1)^{2N-4s}} dy, \quad \text{if } s < \frac{1}{2}, \right\}
$$

$$
\leq \int_{\Omega} \frac{1}{(y + 1)^{2N-4s-1}} dy < +\infty.
$$

In fact, the last but one inequality for $s = \frac{1}{2}$ holds as follows. It suffices to obtain the estimate for $|y| \geq 2$. Let $d = \frac{1}{2}|y|$, there holds that

$$
\int_{B_d(0)} \frac{1}{(1 + |y - z|)^{N-2s}} \frac{1}{(1 + |z|)^N} dz \leq \int_{B_d(0)} \frac{1}{(1 + |y - z|)^{N-2s}} \frac{1}{(1 + |z|)^N} dz \leq \frac{\ln |y|}{(1 + |y|)^{N-2s}},
$$

and

$$
\int_{B_d(y)} \frac{1}{(1 + |y - z|)^{N-2s}} \frac{1}{(1 + |z|)^N} dz \leq \int_{B_d(y)} \frac{1}{(1 + |y - z|)^{N-2s}} \frac{1}{(1 + |z|)^N} dz \leq \frac{\ln |y|}{(1 + |y|)^{N-2s}}.
$$
When \( z \in \mathbb{R}^n \setminus (B_d(0) \cup B_d(y)) \), we have \(|z - y| \geq \frac{1}{2}|y|\) and \(|z| \geq \frac{1}{2}|y|\).

If \(|z| \geq 2|y|\), we get \(|z - y| \geq |z| - |y| \geq \frac{1}{2}|z|\) and

\[
\frac{1}{1 + |y - z|^{N-2s}} \leq \frac{1}{1 + |z|^{N-2s}} \leq \frac{1}{1 + |y|^{N-2s}}.
\]

If \(|z| \leq 2|y|\), then

\[
\frac{1}{1 + |y - z|^{N-2s}} \leq \frac{1}{1 + |z|^{N-2s}} \leq \frac{1}{1 + |y|^{N-2s}} \leq \frac{1}{1 + |z|^{N-2s}}.
\]

As a result,

\[
\int_{\mathbb{R}^n \setminus (B_d(0) \cup B_d(y))} \frac{1}{1 + |y - z|^{N-2s}} \leq \int_{\mathbb{R}^n \setminus (B_d(0) \cup B_d(y))} \frac{1}{1 + |z|^{N-2s}} \leq \int_{\mathbb{R}^n \setminus (B_d(0) \cup B_d(y))} \frac{1}{1 + |y|^{N-2s}}.
\]

Other terms can be handled similarly and we omit it here.

\[\square\]

**B. Energy expansion**

**Lemma B.1.** There is a small \( \varepsilon > 0 \), such that

\[
I(W_{r,h,\Lambda}) = k(A + \frac{A_1}{N^m \mu^m} + \frac{A_2}{N^{m-2} \mu^m} (\mu r_0 - r)^2 - \frac{B_1 k^{N-2s}}{\Lambda N^{2s} (r \sqrt{1 - h^2})^{N-2s}} - \frac{B_2 \mu}{\Lambda N^{2s} (r \sqrt{1 - h^2})^{N-2s}} - \frac{h k}{\sqrt{1 - h^2}})
\]

where \( A, A_1, A_2, B_1 \) and \( B_2 \) are some positive constants.

**Proof.** Recall

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{1}{2} \int_{\mathbb{R}^N} K(\frac{|y|}{\mu}) |u|^2.
\]

We should calculate

\[
I(W_{r,h,\Lambda}) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} W_{r,h,\Lambda}|^2 - \frac{1}{2} \int_{\mathbb{R}^N} K(\frac{|y|}{\mu}) |W_{r,h,\Lambda}|^2
\]

\[:= I_1 + I_2.\]

First, by symmetry, we obtain

\[
\frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} W_{r,h,\Lambda}|^2 = \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k \int_{\mathbb{R}^N} (U_{i,h,\Lambda}^{2s-1} + U_{j,h,\Lambda}^{2s-1})(U_{i,h,\Lambda} + U_{j,h,\Lambda})
\]

\[
= \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k \int_{\mathbb{R}^N} (U_{i,h,\Lambda}^{2s-1} U_{i,h,\Lambda} + U_{j,h,\Lambda}^{2s-1} U_{j,h,\Lambda} + U_{i,h,\Lambda}^{2s-1} U_{j,h,\Lambda} + U_{j,h,\Lambda}^{2s-1} U_{i,h,\Lambda})
\]
Next, we consider $I_2$. By Lemma [A.1]

\[ \int_{\mathbb{R}^N} K(\frac{|y|}{\mu})|W_{r_i}\Lambda|^{2^*_i} = 2k \int_{\Omega_1^+} K(\frac{|y|}{\mu})(U_{\tau_i,\Lambda} + U_{21,\Lambda} + \sum_{j=2}^{k} U_{\tau_j,\Lambda} + \sum_{j=2}^{k} U_{\tau_j,\Lambda})^{2^*_i} \]

\[ = 2k \left( \int_{\Omega_1^+} K(\frac{|y|}{\mu})U_{\tau_i,\Lambda}^{2^*_i} + 2^*_i K(\frac{|y|}{\mu})U_{\tau_i,\Lambda}^{2^*_i - 1} \sum_{j=2}^{k} U_{\tau_j,\Lambda} + \sum_{j=2}^{k} U_{\tau_j,\Lambda} \right) + O(U_{\tau_i,\Lambda}^{2^*_i/2} + \sum_{j=2}^{k} U_{\tau_j,\Lambda}^{2^*_i/2}) \]

\[ := 2k(I_21 + I_22 + I_23). \]

For $I_21$, we divide the space $\Omega_1^+$ into two parts, namely,

\[ K_1 := \{ y \in \Omega_1^+ : ||y| - \mu r_0| \geq \delta \mu \} \]

and

\[ K_2 := \{ y \in \Omega_1^+ : ||y| - \mu r_0| < \delta \mu \}, \]

where $\delta$ is the constant in $K(|y|)$. For the space $K_1$, we have

\[ |y - \bar{x}_1| \geq ||y| - \bar{x}_1|| \geq ||y| - \mu r_0| - ||\bar{x}_1| - \mu r_0|| \geq \frac{1}{2} \delta \mu. \]

So

\[ \int_{K_1} \left( K(\frac{|y|}{\mu}) - 1 \right) U_{\tau_i,\Lambda}^{2^*_i} \leq C \int_{K_1} \left( \frac{1}{1 + |y - \bar{x}_1|} \right)^{2N} \]

\[ \leq \frac{C}{\mu^{N-r}} \int_{K_1} \left( \frac{1}{1 + |y - \bar{x}_1|} \right)^{N+r} \]

\[ = O(C(\mu^{N-r})). \]

For the space $K_2$, by the property of $K(|y|)$, we obtain that

\[ \int_{K_2} \left( K(\frac{|y|}{\mu}) - 1 \right) U_{\tau_i,\Lambda}^{2^*_i} \leq -\frac{c_0}{\mu^m} \int_{K_2} ||y| - \mu r_0|^m U_{\tau_i,\Lambda}^{2^*_i} + O\left( \frac{1}{\mu^{m+\theta}} \int_{K_2} ||y| - \mu r_0|^m U_{\tau_i,\Lambda}^{2^*_i} \right) \]

\[ = -\frac{c_0}{\mu^m} \int_{\mathbb{R}^N} ||y - \bar{x}_1| - \mu r_0|^m U_{0,\Lambda}^{2^*_i} + O\left( \frac{1}{\mu^{m+\theta}} \right) \]

\[ = -\frac{c_0}{\mu^m} \int_{\mathbb{R}^N \setminus B_{\frac{\mu}{2}}(0)} ||y - \bar{x}_1| - \mu r_0|^m U_{0,\Lambda}^{2^*_i} - \frac{c_0}{\mu^m} \int_{B_{\frac{\mu}{2}}(0)} ||y - \bar{x}_1| - \mu r_0|^m U_{0,\Lambda}^{2^*_i} \]
However, we have
\[ \frac{1}{\mu^m} \int_{\mathbb{R}^N \setminus B_{\frac{3}{2}r}(0)} |y - \overline{x}_1| - \mu r_0^m U_{0, \Lambda}^{2^*_i} \leq C \int_{\mathbb{R}^N \setminus B_{\frac{3}{2}r}(0)} \frac{|y - \overline{x}_1|}{\mu^m} + r_0^m U_{0, \Lambda}^{2^*_i} \leq \frac{C}{\mu^{N-r}}. \]

If \( y \in B_{\frac{3}{2}r}(0) \), since \( |\overline{x}_1| = r \), by symmetry property, we have
\[ \int_{B_{\frac{3}{2}r}(0)} |y - \overline{x}_1| - \mu r_0^m U_{0, \Lambda}^{2^*_i} = \int_{B_{\frac{3}{2}r}(0)} |y - re_1| - \mu r_0^m U_{0, \Lambda}^{2^*_i}, \]
where \( e_1 = (1, 0, \cdots, 0) \). Thus, we obtain
\[ \int_{B_{\frac{3}{2}r}(0)} |y - \overline{x}_1| - \mu r_0^m U_{0, \Lambda}^{2^*_i} = \int_{B_{\frac{3}{2}r}(0)} |y_1| - \mu r_0^m U_{0, \Lambda}^{2^*_i} + \frac{1}{2} m(m - 1) \int_{B_{\frac{3}{2}r}(0)} |y_1| - \mu r_0^m U_{0, \Lambda}^{2^*_i} - |y - r_0^m U_{0, \Lambda}^{2^*_i}|. \]

Hence, it holds that
\[ I_{21} = \int_{\Omega^*_1} K\left(\frac{y}{\mu}\right) U_{\pi_1, \Lambda}^{2^*_i - 1} \left( \sum_{j=2}^{k} U_{\pi_j, \Lambda} + \sum_{j=1}^{k} U_{\pi_j, \Lambda} \right) \]
\[ = \int_{\Omega^*_1} U_{\pi_1, \Lambda}^{2^*_i - 1} \sum_{j=2}^{k} U_{\pi_j, \Lambda} + \sum_{j=1}^{k} U_{\pi_j, \Lambda} + \int_{\Omega^*_1} \left( K\left(\frac{y}{\mu}\right) - 1\right) U_{\pi_1, \Lambda}^{2^*_i - 1} \sum_{j=2}^{k} U_{\pi_j, \Lambda} + \sum_{j=1}^{k} U_{\pi_j, \Lambda} \]
\[ = \int_{\mathbb{R}^N} U_{\pi_1, \Lambda}^{2^*_i - 1} \sum_{j=2}^{k} U_{\pi_j, \Lambda} + \sum_{j=1}^{k} U_{\pi_j, \Lambda} + \int_{\mathbb{R}^N} \left( K\left(\frac{y}{\mu}\right) - 1\right) U_{\pi_1, \Lambda}^{2^*_i - 1} \sum_{j=2}^{k} U_{\pi_j, \Lambda} + \sum_{j=1}^{k} U_{\pi_j, \Lambda} \]
\[ + \int_{\Omega^*_1} \left( K\left(\frac{y}{\mu}\right) - 1\right) U_{\pi_1, \Lambda}^{2^*_i - 1} \left( \sum_{j=2}^{k} U_{\pi_j, \Lambda} + \sum_{j=1}^{k} U_{\pi_j, \Lambda} \right) \]
\[ := \int_{\mathbb{R}^N} U_{\pi_1, \Lambda}^{2^*_i - 1} \left( \sum_{j=2}^{k} U_{\pi_j, \Lambda} + \sum_{j=1}^{k} U_{\pi_j, \Lambda} \right) + I_{221} + I_{222}. \]

For \( I_{221} \), we calculate
\[ \sum_{j=2}^{k} \int_{\mathbb{R}^N} U_{\pi_1, \Lambda}^{2^*_i - 1} U_{\pi_j, \Lambda} \]

We consider the first term. For any \( y \in \Omega_1^+ \), we have
\[
|y - \overline{x}_j| \geq |\overline{x}_1 - \overline{x}_j| - |y - \overline{x}_1| \geq \frac{1}{2} |\overline{x}_1 - \overline{x}_j|,
\]
if \( |y - \overline{x}_1| \leq \frac{1}{4} |\overline{x}_1 - \overline{x}_j| \)
and
\[
|y - \overline{x}_j| \geq |y - \overline{x}_1| \geq \frac{1}{4} |\overline{x}_1 - \overline{x}_j|,
\]
if \( |y - \overline{x}_1| > \frac{1}{4} |\overline{x}_1 - \overline{x}_j| \).

Similarly, we have
\[
|y - \overline{x}_j| \geq \frac{1}{4} |\overline{x}_1 - \overline{x}_j|.
\]
Thus, it holds that
\[
\left( \sum_{j=2}^{k} U_{\overline{x}_j, \Lambda} + \sum_{j=1}^{k} U_{\overline{x}_j, \Lambda} \right) \leq \frac{C}{1 + |y - \overline{x}_1|^{N-2x}} \left( \sum_{j=2}^{k} \frac{1}{1 + |y - \overline{x}_1|^\beta} + \sum_{j=1}^{k} \frac{1}{1 + |y - \overline{x}_j|^\beta} \right)
\]
Combing (B.1), (B.2), (B.3), (B.4) and (B.5), we obtain

$$I_{222} = O\left(\frac{k^{N-\epsilon}}{(r \sqrt{1 - h^2})^{N-\epsilon}}\right) + O\left(\frac{1}{(rh)^{N-\epsilon}}\right).$$

(B.4)

At last, we easily calculate $I_{23}$,

$$I_{23} = O\left(\int_{\Omega_1'} K(\frac{|y|}{\mu}) U^{2^*/2}_{x, A} \sum_{j=2}^{k} U_{x_j, A} + \sum_{j=1}^{k} U_{x_j, A}^{2^*/2}\right)
\leq O\left(\frac{k^{N-\epsilon}}{(r \sqrt{1 - h^2})^{N-\epsilon}}\right) + O\left(\frac{1}{(rh)^{N-\epsilon}}\right).$$

(B.5)

Combing (B.1), (B.2), (B.3), (B.4) and (B.5), we obtain

$$I(W_{r,h,A}) = k\left(1 - \frac{2}{2^*}\right) \int_{R^N} U^{2^*/2} + 2c_0 \int_{R^N} |y|^m U^{2^*/2} + c_0 m(m-1) \int_{R^N} |y|^{m-2} U^{2^*/2}
\leq \frac{B_1 k^{N-2s}}{\Lambda^{N-2s}(r \sqrt{1 - h^2})^{N-2s}} + \frac{B_2 h k}{\Lambda^{N-2s}(r \sqrt{1 - h^2})^{N-2s}} + O\left(\frac{1}{(r \sqrt{1 - h^2})^{N-2s}}\right),$$

where obviously, $\frac{1}{\mu^{N-2s}} \leq \frac{1}{\mu^m}$. □

Meanwhile, we also make the following expansions for $\frac{\partial I(W_{r,h,A})}{\partial \Lambda}$ and $\frac{\partial I(W_{r,h,A})}{\partial h}$.

**Lemma B.2.** We have

$$\frac{\partial I(W_{r,h,A})}{\partial \Lambda} = k\left(1 - \frac{m A_1}{\Lambda^{m+1} \mu^m} - \frac{A_2 (m - 2)}{\Lambda^{m-1} \mu^m} (\mu r_0 - r)^2 + \frac{B_1 (N-2s) k^{N-2s}}{\Lambda^{N-2s+1} (r \sqrt{1 - h^2})^{N-2s}} \right)
\leq \frac{B_2 (N-2s) h k}{\Lambda^{N-2s+1} (r \sqrt{1 - h^2})^{N-2s}} + O\left(\frac{1}{\mu^{N-2s}}\right),$$
where $A_1, A_2, B_1$ and $B_2$ are defined in Lemma [B.1].

**Proof.** The proof of this Lemma is similar to Lemma [B.1] so we omit it here. □

**Lemma B.3.** We have

$$\frac{\partial I(W_{r,h,\Lambda})}{\partial h} = k\frac{B_2(N - 2s - 1)k}{\Lambda N - 2s} - k\frac{B_1(N - 2s)k^{N - 2s}h}{\Lambda N - 2s + 2}$$

$$+ O\left(\frac{1}{\mu(N - 2s - \frac{N - 2s}{N - 2s + 2} (\frac{N - 2s + 2}{\Lambda N - 2s} + \theta))}\right),$$

where $A_1, A_2, B_1$ and $B_2$ are defined in Lemma [B.1].

**Proof.** First,

$$\frac{\partial I(W_{r,h,\Lambda})}{\partial h} = k\frac{1}{2h}\left(\int_{\mathbb{R}^N}(-(\Delta)^{s}W_{r,h,\Lambda})^2 - \frac{1}{2} \int_{\mathbb{R}^N} K\left(\frac{|y|}{\mu}\right)W_{r,h,\Lambda}^{2s+1}\right)$$

$$= k\frac{1}{2h}\int_{\mathbb{R}^N} U_{T_1,\Lambda}^{2s+2} - \sum_{j=1}^{k} U_{T_{j,\Lambda}} - \sum_{j=1}^{k} U_{T_{j,\Lambda}} - \int_{\mathbb{R}^N} K\left(\frac{|y|}{\mu}\right)W_{r,h,\Lambda}^{2s+1}\frac{\partial W_{r,h,\Lambda}}{\partial h}.$$

Similar to the proof of Lemma [B.1], we get,

$$\frac{\partial I(W_{r,h,\Lambda})}{\partial h} = k\frac{B_2(N - 2s - 1)k}{\Lambda N - 2s} - k\frac{B_2(N - 2s)k^{N - 2s}h}{\Lambda N - 2s + 2}$$

$$+ O\left(\frac{1}{\mu(N - 2s - \frac{N - 2s}{N - 2s + 2} (\frac{N - 2s + 2}{\Lambda N - 2s} + \theta))}\right).$$

In fact, we know

$$\frac{B_2k}{\Lambda N - 2s + 2 (\frac{N - 2s}{\Lambda N - 2s} + \theta)} \leq O\left(\frac{1}{\mu(N - 2s - \frac{N - 2s}{N - 2s + 2} (\frac{N - 2s + 2}{\Lambda N - 2s} + \theta))}\right).$$

□

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