Spectral characterization of quaternionic positive definite functions on the real line

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Abstract

This paper is concerned with the spectral characteristics of quaternionic positive definite functions on the real line. We generalize the Stone’s theorem to the case of a right quaternionic linear one-parameter unitary group via two different types of functional calculus. From the generalized Stone’s theorems we obtain a correspondence between continuous quaternionic positive definite functions and spectral systems, i.e., unions of a spectral measure and a unitary anti-self-adjoint operator that commute with each other; and then deduce that the Fourier transform of a continuous quaternionic positive definite function is an unusual type of quaternion-valued measure which can be described equivalently in two different ways. One is related to spectral systems on $\mathbb{R}^+$ (induced by the first generalized Stone’s theorem), the other is related to non-negative finite Borel measures on $\mathbb{R}^3$ (induced by the second generalized Stone’s theorem). An application to weakly stationary quaternionic random processes is also presented.

Keywords: Quaternionic positive definite function, Spectral characterization, Stone’s theorem, Unitary group, Random process

2010 MSC: 43A35, 47D03, 47A60, 47S10, 42A38

1. Introduction

The notion of positive definiteness is an important and basic notion in mathematics, which occurs in a variety of algebraic settings. At present there exists a rather satisfactory theory of complex-valued positive definite functions on abelian semigroups. The further development splits mainly into two orientations: the theory of positive definite functions on non-abelian semigroups, and that of positive definite functions taking values in non-commutative algebras. This paper is concerned with the latter. Our interest lies especially in the spectral properties of quaternionic positive definite functions.

There are two major difficulties that we encounter:
Firstly, the conceptual framework in the quaternionic case is incomplete. Many vital concepts in the classical theory of positive definite functions have no proper counterparts in the quaternionic setting. For instance, in the complex case, for every locally compact group $G$, its Pontryagin dual, which is composed of continuous group homomorphisms from $G$ to the unit circle in the complex plane, has a natural group structure given by the ordinary function multiplication, thus it is also called the dual group of $G$; whereas, the analog composed of continuous group homomorphisms from $G$ to the unit sphere in the real quaternion algebra, possesses no natural group structure due to the non-commutative nature of quaternions. As of now, a proper definition for the dual group, along with other vital concepts, still has not been found in the quaternionic case.

Secondly, as is well known, several branches of the functional analysis play crucial roles in the theory of positive definite functions. In contrast to the complex case, the functional analysis in quaternionic vector spaces still remains to be perfected; even how to define the spectrum of a quaternionic linear operator is a controversial issue. It causes the absence of some important analytical tools.

It seems impossible to overcome all the mentioned difficulties and establish a complete theory for quaternionic positive definite functions in a short period of time. So we start with the quaternionic positive definite functions on $\mathbb{R}$, one of the simplest locally compact abelian groups. Based on the recent contributions (see, e.g., [1, 5, 6, 10]) on the normal operators in quaternionic Hilbert spaces, we are able to give a spectral characterization of quaternionic positive definite functions on the real line.

Our strategy is as follows:

First we establish two generalized Stone theorems for right quaternionic linear one-parameter unitary groups via two different types of functional calculus in quaternionic Hilbert spaces. Explicitly speaking, the generalized Stone theorems state that every strongly continuous right linear one-parameter unitary group $U(t)$ can be expressed as

\[ U(t) = e^{tA}|_S, \]

and

\[ U(t) = e^{tA}|_L, \]

for all $t \in \mathbb{R}$ with $A$ being a normal operator. Here $e^{tA}|_S$ is defined as the functional calculus for the function $e^{tx}$ on the $S$-spectrum of $A$, and $e^{tA}|_L$ is defined as the functional calculus on the left spectrum.

Based on the first generalized Stone theorem we construct a correspondence between continuous quaternionic positive definite functions and spectral systems, namely, unions of a spectral measure and a unitary anti-self-adjoint operator that commute with each other (for more details, one may refer to Theorem 4.7). It leads to the conclusion that the Fourier transform of a continuous quaternionic positive definite function is a slice-condensed measure, which is an unusual type of quaternionic valued measure related to spectral systems (see Definition 4.3). More precisely, if $\varphi$ is a continuous quaternionic positive definite function on $\mathbb{R}$, then there exists a unique slice-condensed measure $\mu$ such
that

$$\varphi(t) = \int_{\mathbb{R}^+} \cos(tx)d\text{Re}\mu(x) + \int_{\mathbb{R}^+} \sin(tx)d(\mu - \text{Re}\mu)(x),$$

and vice versa. After that we apply the second generalized Stone theorem to show the concept of slice-condensed measure can be defined equivalently (as Definition 4.2) in a more concrete way: A quaternionic regular Borel measure $\mu$ on $\mathbb{R}^+$ is slice-condensed if and only if there exists a non-negative finite regular Borel measure $\Gamma$ on $H_I$, namely the 3-dimensional real vector space of pure imaginary quaternions, s.t.,

$$\mu = \rho_* (\Gamma + \frac{x}{|x|} \Gamma),$$

where $\rho_*$ is the push-forward mapping induced by the function $\rho : x \mapsto |x|$, $x \in H_I$.

The present paper is organized as follows: Some preliminaries are given in Section 2. Two generalized Stone’s theorems for right quaternionic linear unitary groups are established in Section 3. We devote Section 4 to the spectral characteristics of quaternionic positive definite functions on the real line; especially a generalized Bochner’s theorem is established in this section. An application to weakly stationary quaternionic random processes is presented in Section 5. Section 6 is the final remark.

2. Preliminaries

We would like to introduce some basic information about two types of functional calculus in quaternionic Hilbert spaces.

Let $H$ denote the real quaternion algebra

$$\{q = q_0i_0 + q_1i_1 + q_2i_2 + q_3i_3, q_i \in \mathbb{R} \ (i = 0, 1, 2, 3)\},$$

where $i_0 = 1$, $i_3 = i_1i_2$, and $i_1, i_2$ are the generators of $H$, subject to the following identities:

$$i_1^2 = i_2^2 = -1, \quad i_1i_2 = -i_2i_1.$$

For all $q \in H$, its conjugate is defined as $\overline{q} := q_0i_0 - q_1i_1 - q_2i_2 - q_3i_3$, and its norm given by $|q| := \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$. $S$ will denote the set of all imaginary units, namely,

$$S := \{q = q_0i_0 + q_1i_1 + q_2i_2 + q_3i_3 \in H : q_0 = 0 \text{ and } |q| = 1\}.$$

Consider the subalgebra $C_j$ generated by a imaginary unit $j \in S$. It can be easily seen that $C_j$ in fact is a complex field since $j^2 = -1$. Let $C_j^+$ denote the set of all $p \in C_j$ with $\text{Im}p \geq 0$, i.e.,

$$C_j^+ := \{q = q_0i_0 + q_jj \in C_j : q_0 \in \mathbb{R}, q_j \geq 0\}.$$
Let $V$ be a right vector space over $\mathbb{H}$. An inner product on $V$ is a map $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{H}$ with the following properties:

\[
\langle x, y \rangle = \overline{\langle y, x \rangle}, \\
\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \\
\langle xp, y \rangle = \langle x, y \rangle p, \\
\langle x, yp \rangle = \overline{p} \langle x, y \rangle,
\]

and

\[
\langle x, x \rangle \geq 0, = 0 \text{ if and only if } x = 0,
\]

for all $x, y, z \in V$ and $q \in \mathbb{H}$. If $\langle \cdot, \cdot \rangle$ is an inner product, then $\|x\| = \sqrt{\langle x, x \rangle}$ is norm on $V$. A right vector space $V$ over $\mathbb{H}$ endowed with an inner product which makes $V$ a complete norm space is called a quaternionic Hilbert space (see, e.g., [1, 6]).

Let $\mathcal{H}$ be a quaternionic Hilbert space. The set of all right linear bounded operators on $\mathcal{H}$ will be denoted by $B(\mathcal{H})$, and the set of all right linear operators on the subspaces of $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$. For any operator $T$, the definition domain, the range and the kernel will be denoted by $D(T)$, $R(T)$ and $\text{Ker}(T)$ respectively. The concepts of unitary, normal, self-adjoint and anti-self-adjoint operators are defined in the same way as the case that $\mathcal{H}$ is a real or complex Hilbert space.

### 2.1. Functional calculus based on the S-spectrum

For a densely defined operator $T \in \mathcal{L}(\mathcal{H})$, its S-spectrum (see Definition 2.12. of [1]) is defined as

\[
\sigma_S(T) := \mathbb{H} \setminus \rho_S(T),
\]

where $\rho_S(T)$ is the S-resolution set of $T$ given by

\[
\rho_S(T) := \{ q \in \mathbb{H} : \text{Ker}(\mathcal{R}_q(T)) = 0, \ R(\mathcal{R}_q(T)) \text{ is dense in } \mathcal{H} \}
\]

with $\mathcal{R}_q(T) := T^2 - 2\text{Re}(q)T + |q|^2I$.

A resolution of the identity in a quaternionic Hilbert space is defined as follows.

**Definition 2.1.** Let $\mathcal{M}$ be the $\sigma$-algebra of all Borel sets on a locally compact Hausdorff space $\Omega$, and $\mathcal{H}$ be a quaternionic Hilbert space. A resolution of the identity on $\Omega$ is a mapping

\[
E : \mathcal{M} \mapsto B(\mathcal{H})
\]

with the following properties:

1. $E(\emptyset) = 0, \ E(\Omega) = I$.
2. $E(\omega)$ is a right linear self-adjoint projection for all $\omega \in \mathcal{M}$.
3. $E(\omega' \cap \omega'') = E(\omega')E(\omega'')$ holds for all $\omega', \ \omega'' \in \mathcal{M}$.
4. If $\omega' \cap \omega'' = \emptyset$, then $E(\omega' \cup \omega'') = E(\omega') + E(\omega'')$. 

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5. For every \(x, y \in \mathcal{H}\), the set function \(E_{x,y}\) defined by
\[
E_{x,y}(\omega) = \langle E(\omega)(x), y \rangle
\]
is a quaternion-valued regular Borel measure on \(\Omega\).

**Theorem 2.2.** Let \(T\) be a right linear normal operator on a quaternionic Hilbert space \(\mathcal{H}\) and \(j\) be an imaginary unit in \(S\). There exists a uniquely determined resolution of the identity \(E_j\) on \(\sigma_S(T) \cap \mathbb{C}_j^+\), such that
\[
\langle Tx, y \rangle = \int_{\sigma_S(A) \cap \mathbb{C}_j^+} \text{Re}(p)d\langle E_j x, y \rangle(p) + \int_{\sigma_S(A) \cap \mathbb{C}_j^+} \text{Im}(p)d\langle JE_j x, y \rangle(p),
\]
for all \(x \in D(T)\) and all \(y \in \mathcal{H}\), where \(J \in \mathcal{B}(\mathcal{H})\) is a unitary anti-self-adjoint operator associated with \(T\).

\(E_j\) is called the spectral measure of \(T\). Note that \(E_j\) and \(J\) commute with each other.

**Definition 2.3.** Let \(\mathcal{H}\) be a quaternionic Hilbert space. If a resolution of the identity \(E\) on a locally compact Hausdorff space \(\Omega\) commutes with a unitary anti-self-adjoint operator \(J \in \mathcal{B}(\mathcal{H})\), then \((E, J)\) is called a spectral system on \(\sigma_S(T) \cap \mathbb{C}_j^+\).

From this point of view, \((E_j, J)\) in Theorem 2.2 (Theorem 6.2 of [1]) is a spectral system on \(\sigma_S(A) \cap \mathbb{C}_j^+\).

**Definition 2.4.** A subset \(\Omega\) of \(\mathbb{H}\) is said to be axially symmetric if it satisfies the following property: For an arbitrary element \(p_0 + p_1 j\) \((p_0, p_1 \in \mathbb{R}, j \in S)\) in \(\Omega\), \(p_0 + p_1 j'\) also belongs to \(\Omega\) for all \(j' \in S\).

**Definition 2.5.** Let \(\Omega\) be an axially symmetric subset of \(\mathbb{H}\). Set
\[
D := \{(u, v) \in \mathbb{R}^2 : u + vj \in \Omega \text{ for some } j \in S\}.
\]
A function \(f : \Omega \mapsto \mathbb{H}\) is called an intrinsic slice function if it can be composed as
\[
f(u + vj) = f_0(u, v) + f_1(u, v)j, \quad \forall \ u, v \in D, \ \forall \ j \in S
\]
where \(f_0\) and \(f_1\) are both real-valued functions defined on \(D\).

The functional calculus based on the S-spectrum, also called S-functional calculus, is defined as follows.

**Definition 2.6.** Let \(T\) be a right linear normal operator on a quaternionic Hilbert space \(\mathcal{H}\), and \((E_j, J)\) be the spectral system that arises in Theorem 2.2. For any intrinsic slice function \(f : \sigma_S(T) \mapsto \mathbb{H}\) with the real component \(\text{Re}(f)\) and the imaginary component \(\text{Im}(f)\) both bounded and Borel measurable, the S-functional calculus for \(f\) is defined by
\[
\langle f(T)x, y \rangle = \int_{\sigma_S(A) \cap \mathbb{C}_j^+} \text{Re}(f(p))d\langle E_j x, y \rangle(p) + \int_{\sigma_S(A) \cap \mathbb{C}_j^+} \text{Im}(f(p))d\langle JE_j x, y \rangle(p)
\]
for all \( x, y \in \mathbb{H} \)

**Remark 2.7.** The S-functional calculus retains almost all the important properties of the classical functional calculus in complex Hilbert spaces. There are two types of functional calculus in quaternionic Hilbert space (see \([2, 13]\)) quite similar with the S-functional calculus \([2]\). The functional calculus via intertwining quaternionic PVMs \([2]\) and the S-functional calculus \([1]\) are both based on the continuous functional calculus introduced by R. Ghiloni, V. Moretti, and A. Perotti \([6]\). The functional calculus via spectral systems \([13]\) has been established decades earlier than the other two, but no proper notion of quaternionic spectrum appears in \([13]\). So from our point of view, it may remain to be perfected further.

### 2.2. Functional calculus based on the left spectrum

In order to give a spectral characterization for quaternionic positive definite functions, we need apply two types of functional calculus, namely, the functional calculus based on the S-spectrum \([1]\) and the functional calculus based on the left spectrum \([10, 11]\), to certain quaternionic Hilbert spaces.

Let \( \mathcal{H} \) be a quaternionic Hilbert space, i.e., a right \( \mathbb{H} \)-vector space with an inner product \( \langle \cdot, \cdot \rangle \). Then the functional calculus based on the S-spectrum can be established on \( \mathcal{H} \) as shown in \([1]\). However, we can’t apply the functional calculus based on the left spectrum directly to \( \mathcal{H} \), since the basic setting in \([1]\) is different from that in \([10, 11]\). So we must make some adjustment as follows:

1. Introduce a new inner product \( \langle \cdot | \cdot \rangle \), given by
   \[
   \langle x | y \rangle = \langle x, y \rangle, \quad \forall x, y \in \mathcal{H}.
   \]

2. Construct a left \( \mathbb{H} \)-linear structure on \( \tilde{\mathcal{H}} \): Let \( \{x_a\}_{a \in \Sigma} \) be an orthonormal basis, the left scalar multiplication is defined as
   \[
   qx = \sum_{a \in \Sigma} x_a q \langle x, x_a \rangle, \quad \forall q \in \mathbb{H}, x \in \mathcal{H}.
   \]

Then the \( \mathbb{H} \)-vector space \( \tilde{\mathcal{H}} \) endowed with the inner product \( \langle \cdot | \cdot \rangle \) can be treated as a quaternionic Hilbert space defined in \([10, 11]\). We would like to emphasize that this very type of quaternionic Hilbert space in \([10, 11]\) will be called a bilateral quaternionic Hilbert space in our article to distinguish it from the other type of quaternionic Hilbert space introduced in the preceding subsection.

For convenience, let \( \hat{\mathcal{H}} \) denote the bilateral quaternionic Hilbert space transformed from a quaternionic Hilbert space \( \mathcal{H} \). An operator \( T \) on \( \hat{\mathcal{H}} \) is said to be quasi-linear if it is additive and \( \mathbb{R} \)-homogeneous, i.e.,

\[
T(x + y) = T(x) + T(y), \quad \forall x, y \in D(T),
\]
\[
T(qx) = T(xq) = qT(x), \quad \forall x \in D(T), q \in \mathbb{R}.
\]

The Banach space consisting of all bounded quasi-linear operators is denoted by \( \mathcal{B}_q(\hat{\mathcal{H}}) \). Note that every bounded right linear operator on \( \mathcal{H} \) is a bounded quasi-linear operator on \( \hat{\mathcal{H}} \), which is to say

\[
\mathcal{B}(\mathcal{H}) \subset \mathcal{B}_q(\hat{\mathcal{H}}).
\]
Definition 2.8. \([10]\) The Left spectrum, denoted by \(\sigma_L(T)\), of a closed densely defined quasi-linear operator \(T\) is the set of all \(q \in \mathbb{H}\) such that \(T - qI\) is not bijective from the definition domain \(D(T)\) onto the whole space \(\tilde{\mathcal{H}}\).

As shown in [10], for any (not necessarily bounded) normal operator \(T\), there exits a unique smallest quasi-commutative von Neumann algebra \(A \subset B_q(\tilde{\mathcal{H}})\) that \(T\) is affiliated with; moreover any quasi-commutative von Neumann algebra is \(*\)-isomorphic to \(C(\Lambda, \mathbb{H})\) for some compact Hausdorff space \(\Lambda\) via a generalized Gelfand transform. Then the \(*\)-isomorphism from \(C(\Lambda, \mathbb{H})\) to \(A\) induces an involution preserving bounded \(\mathcal{H}\)-algebra homomorphism \(\phi: B(\sigma_L(T), \mathbb{H}) \to B_q(\tilde{\mathcal{H}})\),

where \(B(\sigma_L(T), \mathbb{H})\) denotes the algebra consisting of all \(\mathbb{H}\)-valued bounded Borel measurable functions defined on \(\sigma_L(T)\). Then the functional calculus on the left spectrum is naturally defined as follows.

Definition 2.9. \([10]\) The functional calculus on the left spectrum is given by

\[ f(T) = \phi(f), \quad \forall f \in B(\sigma_L(T), \mathbb{H}). \]

Moreover, there exist regular \(\mathbb{R}\)-valued Borel measures \(\mu_{v,i,l}[x, y] (v, l = 0, 1, 2, 3; x, y \in \tilde{\mathcal{H}})\) such that

\[ (f(T)x|y) = \sum_{v,l=0}^{3} \int_{\sigma_L(T)} f_v i_l \ d\mu_{v,i,l}[x, y], \]

where \(f = \sum_{v=0}^{3} f_v i_v\) and \(f_v\) is \(\mathbb{R}\)-valued.

Remark 2.10. Another generalized Gelfand transform in quaternionic Hilbert spaces has been investigated by S. H. Kulkarni \([8, 9]\) quite earlier. By contrast, the theory established by S. V. Ludkovsky has a higher degree of completion (one may refer to \([10, 11]\) for more details).

To avoid misunderstanding, we would like to mention the following facts:

1. Whether \(V\) is a quaternionic Hilbert space or a bilateral quaternionic Hilbert space, the real part of its inner product \((\cdot|\cdot)\) is a real inner product. For a densely defined operator \(T\) on \(V\), its adjoint \(T^*\) in \((V, (\cdot|\cdot))\) is identical with its adjoint in the real Hilbert space \((V, \text{Re}(\cdot|\cdot))\). Furthermore,

\[ (Tx|y) = (x|T^*y), \quad x \in D(T), y \in D(T^*) \]

holds when \(T\) is right linear. But this equality may fail when \(T\) is quasi-linear.

For example, let’s observe the operators \(L_q\) and \(R_q\), i.e., the left and right scalar multiplication by \(p \in \mathbb{H}\), on a bilateral quaternionic Hilbert space \(V\). It can be easily verified that \(L_q^* = L^{\pi}, R_q^* = R^{\pi}\), and \((L_qx|y) = (x|L^{\pi}y)\) holds for all \(x, y \in V\). In contrast, \((R_qx|y) = (x|R^{\pi}y)\) is generally not valid. The key difference between \(L_q\) and \(R_q\) is that the former is right linear, while the later is not.
2. The left scalar multiplication in a (bilateral) quaternionic Hilbert space is often uncertain. It may cause some problems since the left spectrum depends on the left scalar multiplication.

For instance, assume that we are discussing two closed densely defined operators \( A \) and \( B \); in one situation we may discover that there exists \( q \in \mathbb{H} \) so that \( A = qB \), then it follows naturally that \( \sigma_L(A) \) is identical with \( q\sigma_L(B) \); however, in another situation, if the left scalar multiplication changes, the equality \( \sigma_L(A) = q\sigma_L(B) \) may no longer hold true.

The uncertainty of left scalar multiplication has been mentioned in Section 1 of [5], and also can be observed in Lemma 3.5 and Theorem 3.6 in [1].

3. Stone’s theorems in quaternionic Hilbert spaces

In this section, we are going to apply both the functional calculuses based on the S-spectrum and the left spectrum to establish two generalized Stone’s theorems for one-parameter unitary groups in quaternionic Hilbert spaces. For precision, if \( f(T) \) is given by the functional calculus based on the S-spectrum, we denote it by \( f(T)|_S \); if it is given by the functional calculus based on the left spectrum, then denote it by \( f(T)|_L \). Only when there is no ambiguity, we will just write it as \( f(T) \).

**Definition 3.1.** A one-parameter unitary group on a quaternionic Hilbert space \( \mathcal{H} \) is a family \( U(t), t \in \mathbb{R}, \) of right linear unitary operators on \( \mathcal{H} \) with the following properties:

\[
U(0) = I, \quad U(s + t) = U(s)U(t) \quad \text{for all } s, t \in \mathbb{R}.
\]

A one-parameter unitary group is said to be strongly continuous if

\[
\lim_{s \to t} \| U(s)(x) - U(t)(x) \| = 0 \quad (3.1)
\]

for all \( t \in \mathbb{R} \) and all \( x \in \mathcal{H} \).

**Definition 3.2.** Let \( U(t) \) be a strongly continuous one-parameter unitary group on \( \mathcal{H} \). The infinitesimal generator of \( U(t) \) is the operator \( A \) defined by

\[
A(x) := \lim_{t \to 0} \frac{U(t)(x) - x}{t}, \quad (3.2)
\]

with its domain \( D(A) \) consisting of all \( x \in \mathcal{H} \) for which the limit exists in the norm topology on \( \mathcal{H} \).

This type of infinitesimal generator has been investigated in the study of semigroups over real algebras (see, e.g., [3, 4]). They are anti-self-adjoint in contrast to the classical infinitesimal generators.
Theorem 3.3. Suppose $U(t)$ is a strongly continuous one-parameter unitary group on $\mathcal{H}$. Then the infinitesimal generator $A$ is a right linear anti-self-adjoint operator, and

$$U(t) = e^{tA}|_S$$ for all $t \in \mathbb{R}$.

Here $e^{tA}|_S$ is defined as the functional calculus for the intrinsic slice function $e^{tx}$ on the S-spectrum of $A$. To be precise, Definition 2.6 says

$$\langle e^{tA}|_S x, y \rangle = \int_{\sigma_S(A) \cap \mathbb{C}_j^+} \text{Re}(e^{tp})d\langle E_j x, y \rangle(p) + \int_{\sigma_S(A) \cap \mathbb{C}_j^+} \text{Im}(e^{tp})d\langle JE_j x, y \rangle(p),$$

where $(E_j, J)$ is the spectral system associated with $A$. Moreover, we have

$$\text{Re}(e^{tp}) = \cos(t|p|) \quad \text{and} \quad \text{Im}(e^{tp}) = \sin(t|p|),$$

since $\sigma_S(A)$ is a subset of $\mathbb{H}_I$, namely, the 3-dimensional real vector space of pure imaginary quaternions.

Theorem 3.4. Suppose $U(t)$ is a strongly continuous one-parameter unitary group on $\mathcal{H}$. Then

$$U(t) = e^{tA}|_L$$ for all $t \in \mathbb{R},$$

where $A$ is the infinitesimal generator of $U(t)$.

Note that $e^{tA}|_L$ is given as the functional calculus for the function $e^{tx}$ on the left spectrum of $A$ in the bilateral Hilbert space $\tilde{\mathcal{H}}$ transformed from $\mathcal{H}$. Moreover, according to Definition 2.9, we have

$$\langle e^{tA}|_L x, y \rangle = \sum_{l=0}^{3} \int_{\sigma_L(T)} \cos(|p|)i_l \, d\mu_{\nu_1, i_l}[x, y](p) + \sum_{v=1}^{3} \sum_{l=0}^{3} \int_{\sigma_L(T)} \frac{p_v}{|p|} \sin(|p|)i_l \, d\mu_{\nu_1, i_l}[x, y](p),$$

(3.3)

where $p \in \sigma_L(A)$ is composed as $p = p_1i_1 + p_2i_2 + p_3i_3$ with $p_v \in \mathbb{R}$, and $\mu_{\nu_1, i_l}[x, y]$ are the regular Borel measures on $\sigma_L(A)$ uniquely determined by $A$ and $x, y \in \tilde{\mathcal{H}}$.

A generalization of Stone’s theorem to the case of a one-parameter unitary group in a quaternionic Hilbert space has been established by S. V. Ludkovsky (see Theorem 2.33 in [12]). However, this version of Stone’s theorem does not fit in with our aims, because the infinitesimal generator defined in [12] is self-adjoint.

We would like to emphasize that the major difference between the earlier version of Stone’s theorem and our versions is that the spectrum of a self-adjoint generator is included in the real line, while that of an anti-self-adjoint generator is included in the 3-dimensional real vector space consisting of all pure imaginary quaternions.
Remark 3.5. The generalized Stone’s theorems (Theorems 3.3 and 3.4) can be proved in almost the same way as the case when \( H \) is a complex Hilbert space (see, e.g., Chap. 10 in [7]).

In fact, the following relation between a semigroup \( U(t) \) on a quaternionic Hilbert space and its infinitesimal generator \( A \):

\[
U(t) = e^{tA}
\]

has been verified in several settings (see, e.g., Theorem 3.2 in [3] and Theorem 6.3 in [4]). Compared with the previous contributions made by others, we have to admit Theorems 3.3 and 3.4 do not count as great progress. What really matters is that they are vital for us to achieve our purpose. Based on these two generalized Stone’s Theorems, we are able to reveal the spectral characteristics of quaternionic positive definite functions.

Lemma 3.6. Suppose \( A \) is a right linear anti-self-adjoint operator on \( \mathcal{H} \), and \( U(t) \) is a family of operators defined as

\[
U(t) = e^{tA}|_{\mathcal{S}}, \quad t \in \mathbb{R},
\]

then the following results hold true:

1. \( U(t) \) is a strongly continuous one-parameter unitary group.
2. For any \( x \in D(A) \),

\[
A(x) = \lim_{t \to 0} \frac{U(t)(x) - x}{t},
\]

where the limit is in the norm topology on \( \mathcal{H} \).
3. For any \( x \in \mathcal{H} \), if

\[
\lim_{t \to 0} \frac{U(t)(x) - x}{t}
\]

exists in the norm topology on \( \mathcal{H} \), then \( x \in D(A) \).

Proof. Since \( \sigma_S(A) \) contains only pure imaginary quaternions, the function \( f_t(p) := e^{tp} \) is a bounded continuous intrinsic slice function on \( \sigma_S(A) \). More precisely,

\[
f_t(jv) = \cos(tv) + j\sin(tv),
\]

holds for any \( j \in \mathbb{S} \) and any \( v \in \mathbb{R} \) with \( jv \in \sigma_S(A) \). Hence, for different \( j, j' \in \mathbb{S} \), \( f_t(jv) \) and \( f_t(j'v) \) share the same real and imaginary components that are both bounded and continuous. Therefore the functional calculus for \( f(p) \) on the S-spectrum of \( A \) is well defined as shown in Definition 2.6:

\[
\langle f_t(A)x, y \rangle = \int_{\sigma_S(A) \cap \mathbb{C}_+^j} \cos(t|p|)d\langle E_j(p)x, y \rangle + \\
\int_{\sigma_S(A) \cap \mathbb{C}_+^j} \sin(t|p|)d\langle JE_j(p)x, y \rangle,
\]

where \( (E_j, J) \) is the spectral system associated with \( A \).
The functional calculus on the S-spectrum is also a \( * \)-homomorphism of real (not quaternionic) Banach \( C^* \)-algebras like the classical functional calculus \cite{1}. It indicates that
\[
U(t)U(t)^* = f_t(A)\overline{f_t}(A) = (f_t\overline{f_t})(A) = 1(A) = I,
\]
\[
U(t)^*U(t) = \overline{f_t}(A)f_t(A) = (\overline{f_t}f_t)(A) = 1(A) = I,
\]
\[
U(s)U(t) = f_s(A)f_t(A) = (f_s f_t)(A) = f_{s+t}(A) = U(s+t);
\]
which is to say \( U(t) \) is a one-parameter unitary group. Moreover, for any \( x \in \mathcal{H} \) and \( s, t \in \mathbb{R} \), we have
\[
\|U(s)(x) - U(t)(x)\|^2 = \langle (f_s(A) - f_t(A))^*(f_s(A) - f_t(A))x, x \rangle
\]
\[
= \langle f_s - f_t \rangle^2(A)x, x \rangle.
\]
Definition 2.6 yields
\[
\langle \|f_s - f_t \|^2(A)x, x \rangle = \int_{\sigma_s(A) \cap \mathbb{C}^+_j} (\cos((s-t)|p|) - 1)^2 + (\sin((s-t)|p|))^2 d\mu_{j,x}(p)
\]
with \( \mu_{j,x} = \langle E_j(p)x, x \rangle \) being a finite Borel measure. The integral on the right side tends to zero as \( s \) approaches \( t \), by dominated convergence. Thus we reach the first conclusion:

1. \( U(t) \) is a strongly continuous one-parameter unitary group.

To see the second conclusion, first notice that Corollary 6.5 in \cite{1} indicates:
\[
\left\| \frac{U(t)(x) - x}{t} - A(x) \right\|^2 = \int_{\sigma_s(A) \cap \mathbb{C}^+_j} \left| \frac{e^{tp} - 1}{t} - p \right|^2 d\mu_{j,x}(p)
\]
\[
= \int_{\sigma_s(A) \cap \mathbb{C}^+_j} \left( \left| \cos(tp) - 1\right|^2 + \left| \sin(tp) \right|^2 \right) |p|^2 d\mu_{j,x}(p)
\]
is valid for all \( x \in \text{D}(A) \) and all \( t \in \mathbb{R} \). Then we apply the dominated convergence theorem again with \( 5|p|^2 \) as the dominating function to achieve the desired result:

2. For any \( x \in \text{D}(A) \),
\[
A(x) = \lim_{t \to 0} \frac{U(t)(x) - x}{t},
\]
where the limit is in the norm topology of \( \mathcal{H} \).

For the third conclusion, let \( A' \) be the infinitesimal generator of \( U(t) \). For any \( x, y \in \text{D}(A') \), one can easily see
\[
\langle A'(x), y \rangle = \lim_{t \to 0} \left\langle \frac{U(t)(x) - x}{t}, y \right\rangle
\]
\[
= \lim_{t \to 0} \left\langle x, \frac{U(-t)(y) - y}{t} \right\rangle
\]
\[
= \left\langle x, -A'(y) \right\rangle
\]
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Hence, $A'$ is anti-symmetric. Combining with the second conclusion we have:

1. $A'$ is an extension of the anti-self-adjoint operator $A$,
2. $\text{D}(A') \subseteq \text{D}((A')^*)$, 

and consequently $\text{D}(A)$ is identical with $\text{D}(A')$, which indicates the third conclusion:

3. For any $x \in \mathcal{H}$, if 
   \[ \lim_{t \to 0} \frac{U(t)(x) - x}{t} \]
   exists in the norm topology of $\mathcal{H}$, then $x \in \text{D}(A)$.

\[ \square \]

**Lemma 3.7.** For any strongly continuous one-parameter unitary group $U(t)$ on $\mathcal{H}$, its infinitesimal generator $A$ is anti-self-adjoint.

**Proof.** Set $\langle x, y \rangle_{j} := \frac{1}{2} \left( \langle x, y \rangle - j \langle x, y \rangle_{j} \right)$ with $j$ being an arbitrary imaginary unit. It can be easily seen that $\langle \cdot, \cdot \rangle_{j}$ is a $\mathbb{C}_{j}$-linear inner product, and the norm induced by $\langle \cdot, \cdot \rangle_{j}$ is identical with the one induced by $\langle \cdot, \cdot \rangle$, which implies that $\mathcal{H}$, as a $\mathbb{C}_{j}$-linear vector space, endowed with the inner product $\langle \cdot, \cdot \rangle_{j}$ is a complex Hilbert space. Furthermore, the adjoint of any densely defined right quaternion-linear operator with respect to the quaternionic inner product $\langle \cdot, \cdot \rangle_{j}$ is identical with the adjoint with respect to the complex inner product $\langle \cdot, \cdot \rangle_{j}$.

From this point of view, $U(t)$ can be treated as a strongly continuous one-parameter unitary group in the complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{j})$, and $\text{Ar}_{j} (= R_{j}A)$ is exactly the classical infinitesimal generator of $U(t)$ (see, e.g., Definition 10.13 in [7]), where $R_{j}$ is the right scalar multiplication by $j$, i.e.,

\[ R_{j}(x) = xj, \quad \forall x \in \mathcal{H}. \]

We thus have $AR_{j}$ is self-adjoint in the complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{j})$ according to the original Stone’s theorem (see, e.g., Theorem 10.15 in [7]), which indicates $A$ is anti-self-adjoint in $(\mathcal{H}, \langle \cdot, \cdot \rangle_{j})$. Since the adjoint of $A$ with respect to $\langle \cdot, \cdot \rangle$ is identical with the adjoint with respect to $\langle \cdot, \cdot \rangle_{j}$, we conclude that $A$ is anti-self-adjoint in the quaternionic Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{j})$. \[ \square \]

**Proof of Theorem 3.3.** Suppose $U(t)$ is a strongly continuous one-parameter unitary group on $\mathcal{H}$, and $A$ is its infinitesimal generator. By Lemma 3.7, $A$ is anti-self-adjoint. Then by Lemma 3.6, $e^{tA}|_{\mathcal{S}}$ is a strongly continuous one-parameter unitary group with the infinitesimal generator identical with $A$.

Take $x \in \text{D}(A)$, and consider the function $g_{x}(t) = U(t)(x) - e^{tA}|_{\mathcal{S}}(x)$. From the definition of the infinitesimal generator, it follows immediately that

\[ \frac{d}{dt} U(t)(x) = AU(t)(x) = U(t)A(x), \]

and

\[ \frac{d}{dt} e^{tA}|_{\mathcal{S}}(x) = Ae^{tA}|_{\mathcal{S}}(x) = e^{tA}|_{\mathcal{S}}A(x), \]
hold in the norm topology of $\mathcal{H}$, which means $U(t)(x)$ and $e^{tA}|_S(x)$ both belong to $D(A)$, and
\[ \frac{d}{dt}g_x(t) = A(U(t)(x) - e^{tA}|_S(x)) = Ag_x(t). \]
Hence,
\[
\frac{d}{dt}(g_x(t), g_x(t)) = \langle \frac{d}{dt}g_x(t), g_x(t) \rangle + \langle g_x(t), \frac{d}{dt}g_x(t) \rangle = \langle Ag_x(t), g_x(t) \rangle + \langle g_x(t), Ag_x(t) \rangle = \langle g_x(t), -Ag_x(t) \rangle + \langle g_x(t), Ag_x(t) \rangle = 0.
\]
Since $g_x(0) = 0$, we deduce that $g_x(t) = 0$ holds for all $x \in D(A)$ and all $t \in \mathbb{R}$, or equivalently
\[ U(t)(x) = e^{tA}|_S(x), \quad \forall \, x \in D(A), \forall \, t \in \mathbb{R}. \]
In conclusion, $U(t)$ and $e^{tA}|_S$ agree on a dense subspace of $\mathcal{H}$, and thus on the whole space. \hfill $\Box$

**Proof of Theorem 3.4.** An analog of Lemma 3.6 can be obtained by replacing $e^{tA}|_S$ with $e^{tA}|_L$. Then we can adopt the same procedure as in the proof of Theorem 3.3 to carry out this one. To avoid repetition, the details of this proof are omitted. \hfill $\Box$

## 4. Bochner’s theorem for quaternionic positive definite functions

In this section we are going to show that every continuous quaternionic positive definite function is related with a spectral system on $\mathbb{R}^+$ and also related with a non-negative finite Borel measure on $\mathbb{R}^3$ in certain ways. Moreover, the spectral system and the Borel measure will induce two identical quaternion-valued measures of an unusual type that we name as slice-condensed measures. Finally, a one-to-one correspondence between continuous quaternionic positive definite functions and slice-condensed measures will be established.

**Definition 4.1.** A quaternion-valued function $\varphi$ on $\mathbb{R}$ is said to be positive definite if for any $t_1, t_2, \cdots, t_k \in \mathbb{R}$ and any $q_1, q_2, \cdots, q_k \in \mathbb{H}$, the following inequality
\[ \sum_{1 \leq i, j \leq k} \overline{p_i} \varphi(t_i - t_j) p_j \geq 0 \tag{4.1} \]
is satisfied.

Before giving a formal definition for slice-condensed measures, we introduce some notations. $\mathbb{H}_I$ denotes the set of all pure imaginary quaternions, and $\mathbb{R}^+$ the set of non-negative real number. $\mathcal{B}(X)$ stands for the Borel $\sigma$-algebra on a topological space $X$. The function $\rho : \mathbb{H}_I \rightarrow \mathbb{R}^+$ is defined as
\[ \rho(x) := |x|. \]
Evidently, $\rho$ is Borel measurable, then induces a push-forward mapping:

$$\rho_*(\Gamma)(\Omega) := \Gamma(\rho^{-1}(\Omega)), \ \forall \ \text{Borel measure } \Gamma \text{ on } \mathbb{H}_I, \ \forall \ \Omega \in \mathcal{B}(\mathbb{R}^+).$$

**Definition 4.2.** A quaternion-valued regular Borel measure $\mu$ on $\mathbb{R}^+$ is said to be slice-condensed if there exists a non-negative finite regular Borel measure $\Gamma$ on $\mathbb{H}_I$ such that the following equality holds:

$$\mu = \rho_*(\Gamma + \frac{x}{|x|} \Gamma).$$

Here we stipulate that $\frac{x}{|x|} = 0$ when $x = 0$, and consider this function as a Radon-Nikodym derivative, which means $\frac{x}{|x|} \Gamma$ is a regular Borel measure defined by

$$\frac{x}{|x|} \Gamma(\Omega) := \int_{x \in \Omega} \frac{x}{|x|} d\Gamma(x)$$

for any Borel set $\Omega \in \mathcal{B}(\mathbb{H}_I)$. This concept can also be defined equivalently as follows.

**Definition 4.3.** A quaternion-valued regular Borel measure $\mu$ on $\mathbb{R}^+$ is said to be slice-condensed if there exists a spectral system $(E : \mathcal{B}(\mathbb{R}^+) \to \mathcal{B}(\mathcal{H}), J)$ and a point $\alpha$ in a quaternionic Hilbert space $\mathcal{H}$ such that the following equality holds:

$$\mu = \langle E\alpha, \alpha \rangle + \langle J_0 E\alpha, \alpha \rangle,$$

where $J_0 = J - JE(\{0\})$ and $\langle \cdot, \cdot \rangle$ stands for the inner product on $\mathcal{H}$.

More precisely, this equality $\mu = \langle E\alpha, \alpha \rangle + \langle J_0 E\alpha, \alpha \rangle$ means

$$\mu(\omega) = \langle E(\omega)\alpha, \alpha \rangle + \langle J_0 E(\omega)\alpha, \alpha \rangle$$

is satisfied for any $\omega \in \mathcal{B}(\mathbb{R}^+)$. 

**Remark 4.4.** We would like to emphasize that Definitions 4.2 and 4.3 are equivalent, and this assertion will be illuminated in Subsection 4.3. To prevent confusion, one may ignore Definition 4.2 temporarily until reach Subsection 4.3.

Let $\mathcal{M}_S(\mathbb{R}^+)$ denote the set of all slice-condensed regular Borel measures on $\mathbb{R}^+$. The next theorem will show there exists a one to one correspondence between the continuous quaternionic positive definite functions and the slice-condensed regular Borel measures.

**Theorem 4.5** (Generalized Bochner’s theorem). If a quaternion-valued function $\varphi$ on $\mathbb{R}$ is continuous and positive definite, then there exists a unique $\mu \in \mathcal{M}_S(\mathbb{R}^+)$ such that

$$\varphi(t) = \int_{\mathbb{R}^+} \cos(tx)d\text{Re}\mu(x) + \int_{\mathbb{R}^+} \sin(tx)d(\mu - \text{Re}\mu)(x),$$

and vice versa.
4.1. The quaternionic Hilbert space associated with a positive definite function

Let $\varphi$ be a quaternionic positive definite function, and $F_0(\mathbb{R}, \mathbb{H})$ be the family of quaternion-valued functions on $\mathbb{R}$ with finite support. Evidently, $F_0(\mathbb{R}, \mathbb{H})$ has a natural right $\mathbb{H}$-linear structure that makes it a right $\mathbb{H}$-vector space. The positive definite function $\varphi$ will induce a (possibly degenerate) inner product:

$$\langle f, g \rangle := \sum_{s,t \in \mathbb{R}} g(s) \varphi(s - t) f(t),$$

for all $f, g \in F_0(\mathbb{R}, \mathbb{H})$.

Quotienting $F_0(\mathbb{R}, \mathbb{H})$ by the subspace of functions with zero norm eliminates the degeneracy. Then taking the completion gives a quaternionic Hilbert space $(\mathcal{H}_\varphi, \langle \cdot, \cdot \rangle)$.

Note that if $\varphi$ vanishes at the origin, it can be easily seen that $\mathcal{H}$ is a 0-dimensional space, and $\varphi \equiv 0$; then all the main results are trivially true. So without loss of generality, we can always assume $\varphi(0) \neq 0$.

Recall that every quaternionic Hilbert space can be transformed into a bilateral Hilbert space as follows:

1. Introduce a new inner product $(\cdot|\cdot)$, given by

$$\langle x|y \rangle = \langle x, y \rangle, \quad \forall x, y \in \mathcal{H}_\varphi.$$

2. Construct a left $\mathbb{H}$-linear structure on $\mathcal{H}_\varphi$: Let $\{x_\alpha\}$ be an orthonormal basis, the left scalar multiplication is defined as

$$qx = \sum_{\alpha} x_\alpha q\langle x, x_\alpha \rangle, \quad \forall q \in \mathbb{H}, \ x \in \mathcal{H}_\varphi.$$

Then the $\mathbb{H}$-vector space $\mathcal{H}_\varphi$ endowed with the inner product $(\cdot|\cdot)$ can be treated as a bilateral quaternionic Hilbert space. For convenience, we denote the bilateral quaternionic Hilbert space $(\mathcal{H}_\varphi, (\cdot|\cdot))$ by $\tilde{\mathcal{H}}_\varphi$.

In this very case, we choose the orthonormal basis specifically given as

$$\{x_\alpha\} := \{\delta/\|\delta\|\} \cup \{x_\beta\},$$

where $\delta$ is the finite delta function given by

$$\delta(x) = \begin{cases} 1, & x = 0; \\ 0, & x \neq 0. \end{cases}$$

and $\{x_\beta\}$ is an arbitrary orthonormal basis of $\{\delta/\|\delta\|\}^\perp$. Such choice ensures the following commutativity:

$$q\delta = \delta q, \quad \forall q \in \mathbb{H}. \quad (4.2)$$
Remark 4.6. Note that the left scaler multiplication on $\tilde{\mathcal{H}}$ is different from the conventional left scaler multiplication. In other words, for a quaternion-valued function $f$ on $\mathbb{R}$ with finite support and an arbitrary element $q \in \mathbb{H}$, the following equality
\[(qf)(x) = qf(x), \quad x \in \mathbb{R},\]
is generally non-valid in $\tilde{\mathcal{H}}$. So the commutativity in (4.2) is not trivial.

4.2. Spectral theorems for quaternionic positive definite functions

Theorem 4.7. A quaternion-valued function $\varphi$ defined on $\mathbb{R}$ is continuous and positive definite if and only if there exist a spectral system $(E : \mathcal{B}(\mathbb{R}^+) \rightarrow \mathcal{B}(\mathcal{H}), J)$ and a point $\alpha$ in a quaternionic Hilbert space $\mathcal{H}$ such that
\[\varphi(t) = \int_{\mathbb{R}^+} \cos(tx) \, d\langle E\alpha, \alpha \rangle(x) + \int_{\mathbb{R}^+} \sin(tx) \, d\langle J\alpha, \alpha \rangle(x).\]

Proof. First, we shall show the sufficiency. Assume that for a function $\varphi$, there exist a spectral system $(E : \mathcal{B}(\mathbb{R}^+) \rightarrow \mathcal{B}(\mathcal{H}), J)$ and a point $\alpha$ in a quaternionic Hilbert space $\mathcal{H}$ such that
\[\varphi(t) = \int_{\mathbb{R}^+} \cos(tx) \, d\langle E\alpha, \alpha \rangle(x) + \int_{\mathbb{R}^+} \sin(tx) \, d\langle J\alpha, \alpha \rangle(x).\]

Obviously, $\varphi$ is continuous in view of the dominated convergence theorem. We only need to check whether $\varphi$ is positive definite.

By Lemma 5.3 of [1], $\varphi(t)$ is identical with $\langle I(f_t)\alpha, \alpha \rangle$ where $f_t(x) := e^{tx}$ and $I$ is a $\ast$-homomorphism induced by the spectral system $(E, J)$. For arbitrary $t_1, t_2, \cdots, t_k \in \mathbb{R}$ and $q_1, q_2, \cdots, q_k \in \mathbb{H}$, we have
\[\sum_{1 \leq i, j \leq k} p_i \varphi(t_i - t_j)p_j = \sum_{1 \leq i, j \leq k} p_i \langle I(f_{t_i} - t_j)\alpha, \alpha \rangle p_j\]
\[= \sum_{1 \leq i, j \leq k} p_i \langle I(f_{t_i})I(f_{t_j})\alpha, \alpha \rangle p_j\]
\[= \sum_{1 \leq i, j \leq k} p_i \langle I(f_{t_i})\alpha, I(f_{t_j})\alpha \rangle p_j\]

Subsequently, since $\langle xp, yq \rangle = \overline{\langle q(x, y) \rangle}p$ for all $p, q \in \mathbb{H}$ and all $x, y \in \mathcal{H}$, the following equality holds true:
\[\sum_{1 \leq i, j \leq k} p_i \varphi(t_i - t_j)p_j = \sum_{1 \leq i, j \leq k} \langle I(f_{t_i})\alpha p_j, I(f_{t_j})\alpha p_i \rangle\]
\[= \| \sum_{1 \leq j \leq k} I(f_{-t_j})\alpha p_j \|^2\]
\[\geq 0\]

Hence $\varphi$ is positive definite.
Next, we shall verify the necessity. The quaternionic Hilbert space $\mathcal{H}_\varphi$ constructed in Subsection 4.1 will come into immediate use.

Assume that $\varphi : \mathbb{R} \mapsto \mathbb{H}$ is a continuous positive definite function. Consider a family of shift operators $U_t$ $( t \in \mathbb{R})$ on $F_0(\mathbb{R}, \mathbb{H})$ given by

$$U_t(f) = f(\cdot + t). \quad (4.3)$$

The following facts hold:

1. $U_t$ is a right $\mathbb{H}$-linear bijection preserving the (possibly degenerate) inner product $\langle \cdot, \cdot \rangle$ induced by the positive definite function $\varphi$ for all $t \in \mathbb{R}$.

2. $U_0 = I$ and $U_t U_s = U_{t+s}$ for all $t, s \in \mathbb{R}$.

3. The continuity of $\varphi$ implies that $\lim_{s \to t} \| U_s(f) - U_t(f) \| = 0$ holds for all $f \in F_0(\mathbb{R}, \mathbb{H})$ and all $t \in \mathbb{R}$.

Thus $U_t$ $(t \in \mathbb{R})$ can be uniquely extended as a strongly continuous one-parameter unitary group on $\mathcal{H}_\varphi$. It follows directly from Theorem 3.3 that the infinitesimal generator, denoted by $A$, of $U_t$ is a right linear anti-self-adjoint operator, and

$$U_t = e^{tA}|_S \text{ for all } t \in \mathbb{R}. \quad (4.4)$$

Here $e^{tA}|_S$ is defined by the functional calculus for the intrinsic slice function $e^{tx}$ on the $S$-spectrum of $A$. More precisely,

$$\langle e^{tA}|_S x, y \rangle = \int_{\sigma_S(A) \cap \mathbb{C}_j^+} \text{Re}(e^{tp}) d(E_j x, y)(p) + \int_{\sigma_S(A) \cap \mathbb{C}_j^+} \text{Im}(e^{tp}) d(J E_j x, y)(p), \quad (4.5)$$

where $(E_j : \mathcal{B}(\sigma_S(A) \cap \mathbb{C}_j^+) \mapsto \mathcal{B}(\mathcal{H}_\varphi), J)$ is the spectral system associated with $A$. Moreover,

$$\text{Re}(e^{tp}) = \cos(tp) \quad \text{and} \quad \text{Im}(e^{tp}) = \sin(tp)$$

since $\sigma_S(A)$ contains only pure imaginary quaternions.

Define a resolution of identity $E : \mathcal{B}(\mathbb{R}^+) \mapsto \mathcal{B}(\mathcal{H}_\varphi)$ as

$$E(\omega) = E_j(\sigma_S(A) \cap \mathbb{C}_j^+ \cap j \omega), \quad \omega \in \mathcal{B}(\mathbb{R}^+).$$

One may notice that $\sigma_S(A) \cap \mathbb{C}_j^+$ is in fact a subset of $j \mathbb{R}^+$. So $E$ is essentially a zero extension of $E_j$. Furthermore, (4.3) and (4.5) indicate

$$\langle U_t \delta, \delta \rangle = \int_{\mathbb{R}^+} \cos(tx) d(E\delta, \delta)(x) + \int_{\mathbb{R}^+} \sin(tx) d(J E\delta, \delta)(x), \quad (4.6)$$

where $\delta$ is the finite delta function given by

$$\delta(x) = \begin{cases} 1, & x = 0; \\ 0, & x \neq 0. \end{cases}$$

A direct calculation yields the left side of (4.6) is also equal to $\varphi(t)$. This completes the proof. \frak
Corollary 4.8. A quaternion-valued function \( \varphi \) defined on \( \mathbb{R} \) is continuous and positive definite if and only if there exist a spectral system \((E : \mathcal{B}(\mathbb{R}^+) \mapsto \mathcal{B}(\mathcal{H}), J)\) and a point \( \alpha \) in a quaternionic Hilbert space \( \mathcal{H} \) such that
\[
\varphi(t) = \int_{\mathbb{R}^+} \cos(tx) \, d\langle E\alpha, \alpha \rangle(x) + \int_{\mathbb{R}^+} \sin(tx) \, d\langle J_0 E\alpha, \alpha \rangle(x),
\]
where \( J_0 = J - JE(\{0\}) \).

Proof. For any \( \omega \in \mathcal{B}(\mathbb{R}^+) \) with \( 0 \not\in \omega \), we have
\[
E(\{0\})E(\omega) = E(\{0\} \cap \omega) = E(\emptyset) = 0.
\]
Hence \( JE(\omega) = J_0 E(\omega) \), which means \( \langle JE(\omega)\alpha, \alpha \rangle = \langle J_0 E(\omega)\alpha, \alpha \rangle \). Then we obtain
\[
\int_{\mathbb{R}^+} \sin(tx) \, d\langle JE\alpha, \alpha \rangle(x) = \int_{\mathbb{R}^+} \sin(tx) \, d\langle J_0 E\alpha, \alpha \rangle(x),
\]
since the integrand \( \sin(tx) \) vanishes at \( x = 0 \) and these two measures \( \langle JE\alpha, \alpha \rangle \) and \( \langle J_0 E\alpha, \alpha \rangle \) are identical on \((0, +\infty)\). Therefore, this corollary follows directly from Theorem 4.7.

Proof of Theorem 4.6. We only verify the necessity. The sufficiency can be proved reversely.

Assume \( \varphi : \mathbb{R} \mapsto \mathbb{H} \) is a continuous positive definite function. By Corollary 4.8, there exist a spectral system \((E, J)\) and a point \( \alpha \) in a quaternionic Hilbert space \( \mathcal{H} \) such that
\[
\varphi(t) = \int_{\mathbb{R}^+} \cos(tx) \, d\langle E\alpha, \alpha \rangle(x) + \int_{\mathbb{R}^+} \sin(tx) \, d\langle J_0 E\alpha, \alpha \rangle(x),
\]
(4.7)
Consider a regular Borel measure given by
\[
\mu = \langle E\alpha, \alpha \rangle + \langle J_0 E\alpha, \alpha \rangle.
\]
By Definition 4.3, i.e., the second definition of a slice-condensed measure, we know \( \mu \) is slice-condensed.

Notice two facts:
1. \( E(\omega) \) is self-adjoint for all \( \omega \in \mathcal{B}(\mathbb{R}^+) \).
2. \( J \) is anti-self-adjoint, and commutes with \( E \).

Thus, \( \langle E\alpha, \alpha \rangle \) is pure real valued and \( \langle J_0 E\alpha, \alpha \rangle \) is pure imaginary valued. This implies
\[
\text{Re} \mu = \langle E\alpha, \alpha \rangle.
\]
Hence, (4.7) indicates
\[
\varphi(t) = \int_{\mathbb{R}^+} \cos(tx) d\text{Re} \mu(x) + \int_{\mathbb{R}^+} \sin(tx) d(\mu - \text{Re} \mu)(x).
\]
In addition, the uniqueness of the slice-condensed measure \( \mu \) is a direct result of the Stone-Weierstrass theorem. \( \square \)
4.3. Equivalence between the two definitions of slice-condensed measures

We claimed Definitions 4.2 and 4.3 are equivalent. Now this assertion will be verified.

First we recall some notations. \( \mathbb{H}_I \) denotes the set of pure imaginary quaternions, and \( \mathbb{R}^+ \) the set of non-negative real numbers. \( \mathcal{B}(X) \) stands for the Borel \( \sigma \)-algebra on a topological space \( X \). The function \( \rho : \mathbb{H}_I \to \mathbb{R}^+ \) is defined as

\[
\rho(x) := |x|.
\]

It induces a push-forward mapping \( \rho_* \) given by

\[
\rho_*(\Gamma)(\Omega) := \Gamma(\rho^{-1}(\Omega)),
\]

for any Borel measure \( \Gamma \) on \( \mathbb{H}_I \), and any \( \Omega \in \mathcal{B}(\mathbb{R}^+) \).

Recall Definition 4.2:

a quaternion-valued regular Borel measure \( \mu \) on \( \mathbb{R}^+ \) is said to be slice-condensed if there exists a non-negative finite regular Borel measure \( \Gamma \) on \( \mathbb{H}_I \) such that the following equality holds:

\[
\mu = \rho_*(\Gamma + \frac{x}{|x|} \Gamma).
\]

Here we stipulate that \( \frac{x}{|x|} = 0 \) when \( x = 0 \), and consider this function as a Radon-Nikodym derivative, which means \( \frac{x}{|x|} \Gamma \) is a regular Borel measure defined by

\[
\frac{x}{|x|} \Gamma(\Omega) := \int_{x \in \Omega} \frac{x}{|x|} \, d\Gamma(x)
\]

for any Borel set \( \Omega \in \mathcal{B}(\mathbb{H}_I) \).

Recall Definition 4.3:

a quaternion-valued regular Borel measure \( \mu \) on \( \mathbb{R}^+ \) is said to be slice-condensed if there exists a spectral system \( (E : \mathcal{B}(\mathbb{R}^+) \to \mathcal{B}(\mathcal{H}), J) \) and a point \( \alpha \) in a quaternionic Hilbert space \( \mathcal{H} \) such that the following equality holds:

\[
\mu = \langle E\alpha, \alpha \rangle + \langle J_0 E\alpha, \alpha \rangle,
\]

where \( J_0 \) is given by \( J_0 = J - JE(\{0\}) \) and \( \langle \cdot, \cdot \rangle \) stands for the inner product on \( \mathcal{H} \).

For ease of explanation, we call any measure that satisfies Definition 4.2 is a slice-condensed measure of type I, any measure that satisfies Definition 4.3 is a slice-condensed measure of type II.

**Lemma 4.9.** Any slice-condensed measure of type I is a slice-condensed measure of type II.

**Proof.** Assume \( \mu \) is a slice-condensed measure of type I, then there exists a non-negative finite regular Borel measure \( \Gamma \) on \( \mathbb{H}_I \) such that the following equality holds:

\[
\mu = \rho_*(\Gamma + \frac{x}{|x|} \Gamma)
\]
Let $L^2(\mu, \mathbb{H})$ denote the quaternionic Hilbert space consisting of Borel measurable quaternion-valued functions on $\mathbb{H}$ which are square-integrable with respect to the measure $\Gamma$. The inner product on $L^2(\mu, \mathbb{H})$ is naturally given by

$$\langle f, g \rangle := \int_{\mathbb{H}} \overline{g(x)} f(x) d\Gamma(x).$$

Consider a resolution of identity $E'$ on $\mathbb{H}$ and an anti-self-adjoint unitary operator $J$ defined as follows:

$$E'(\omega)f(x) := \chi_{\omega}(x)f(x),$$

for all $\omega \in \mathcal{B}(\mathbb{H})$, and all $f \in L^2(\mu, \mathbb{H})$. Here $\chi_{\omega}$ is the characteristic function of the set $\omega$.

$$Jf(x) := \begin{cases} \frac{x}{|x|} f(x), & x \neq 0; \\ jf(0), & x = 0; \end{cases}$$

where $j$ is an arbitrary imaginary unit. It can be easily seen that $E'$ commutes with $J$.

Applying the push-forward mapping $\rho_*$ given by (4.8) to $E'$ produces a resolution of identity $E$ on $\mathbb{R}^+$ as

$$\langle Ef, g \rangle := \rho_* \langle E' f, g \rangle, \quad \forall f, g \in L^2(\mu, \mathbb{H}).$$

Equivalently, the resolution of identity $E$ is defined by

$$E(\omega) := E'(\rho^{-1}_*(\omega)), \quad \forall \omega \in \mathcal{B}(\mathbb{R}^+),$$

where $\rho(x) = |x|, x \in \mathbb{H}$. We notice that $E$ also commutes with $J$. Thus $(E, J)$ is a spectral system according to Definition 2.3. Furthermore, direct calculations yield

$$\langle E(\omega) \alpha, \alpha \rangle = \rho_* \Gamma(\omega),$$

and

$$\langle J_0 E(\omega) \alpha, \alpha \rangle = \rho_* \left(\frac{x}{|x|}\right) \Gamma(\omega),$$

for all $\omega \in \mathcal{B}(\mathbb{R}^+)$, where $\alpha$ is given as the characteristic function of $\mathbb{H}$, and $J_0 = J - JE(\{0\})$. Substituting the two equalities above into $\mu = \langle E(\omega) \alpha, \alpha \rangle + \langle J_0 E(\omega) \alpha, \alpha \rangle$, we thus obtain

$$\mu = \langle E \alpha, \alpha \rangle + (J_0 E \alpha, \alpha).$$

Therefore, $\mu$ is a slice-condensed measure of type II.

**Lemma 4.10.** Any slice-condensed measure of type II is a slice-condensed measure of type I.
Proof. This proof is lengthy. We would like to outline it, then give the details.

Let $\mu$ be a slice-condensed measure of type II. Corollary 4.8 indicates that the following function

$$
\varphi(t) := \int_{\mathbb{R}^+} \cos(tx)d\text{Re} \mu(x) + \int_{\mathbb{R}^+} \sin(tx)d(\mu - \text{Re} \mu)(x),
$$

(4.10)
is continuous and positive definite. As shown in Subsection 4.1, there is a quaternionic Hilbert space $H_\varphi$ associated with $\varphi$. Applying Theorem 3.4 to the unitary group $U_t$ on $H_\varphi$ given by (4.3), we obtain

$$
U_t = e^{tA}|_L
$$

with $A$ being the infinitesimal generator of $U(t)$. Definition 2.9, along with the commutativity shown in (4.2), yields two facts:

1. 

$$
(U_t \delta \mid \delta) = \sum_{l=0}^{3} \int_{\sigma_L(A)} \cos t|x|i_l d\mu_{i_0,i_l}[\delta,\delta](x) - \sum_{k=1}^{3} \sum_{l=0}^{3} \int_{\sigma_L(A)} \frac{x_k}{|x|} \sin t|x|i_l d\mu_{i_0,i_l}[\delta,\delta](x),
$$

where $\delta$ is the finite delta function given by

$$
\delta(x) = \begin{cases} 
1, & x = 0; \\
0, & x \neq 0;
\end{cases}
$$

and $\mu_{i_0,i_l}[\delta,\delta]$ ($l = 0, 1, 2, 3$) are regular Borel measures on $\sigma_L(A)$ determined by $A$ and $\delta$.

2. 

$$
\mu_{i_0,i_l}[\delta,\delta] = \begin{cases} 
is finite and non-negative, & l = 0, \\
0 & l = 1, 2, 3.
\end{cases}
$$

We thus have

$$
(U_t \delta \mid \delta) = \int_{\mathbb{H}_I} \cos t|x|d\Gamma(x) + \int_{\mathbb{H}_I} \frac{x}{|x|} \sin t|x|d\Gamma(x),
$$

(4.11)

where $\Gamma$ is a non-negative finite regular Borel measure on $\mathbb{H}_I$, the set of pure imaginary quaternions, defined by

$$
\Gamma(\omega) := \mu_{i_0,i_0}[\delta,\delta](\sigma_L(A) \cap (-\omega)), \quad \forall \omega \in \mathcal{B}(\mathbb{H}_I).
$$

By the definitions of $U_t$ and $\delta$, it is easy to see that the left side of (4.11) equals $\varphi(t)$. On the other hand, a direct calculation yields the right side of (4.11) equals

$$
\int_{\mathbb{R}^+} \cos tx \, d\text{Re} \mu'(x) + \int_{\mathbb{R}^+} \sin tx \, d(\mu' - \text{Re} \mu')(x)
$$
where \( \mu' \) is a slice-condensed measure of type I given by
\[
\mu' := \rho_\ast (\Gamma + \frac{x}{|x|} \Gamma).
\]

Hence,
\[
\varphi(t) = \int_{\mathbb{R}^+} \cos tx \, d\text{Re}\mu'(x) + \int_{\mathbb{R}^+} \sin tx \, d(\mu' - \text{Re}\mu')(x)
\]

Comparing this equality with (4.10) we conclude that the slice-condensed measure \( \mu \) of type II is identical with the slice-condensed measure \( \mu' \) of type I. It completes the proof.

The details are as follows:

**Step 1:**
Assume \( \mu \) is a slice-condensed measure of type II. Then there exists a spectral system \((E : B(\mathbb{R}^+) \to B(H), J)\) and a point \( \alpha \) in a quaternionic Hilbert space \( H \) such that the following equality holds:
\[
\mu = \langle E\alpha, \alpha \rangle + \langle J_0 E\alpha, \alpha \rangle,
\]
where \( J_0 = J - JE(\{0\}) \). It can be seen easily that \( \text{Re}\mu(x) = \langle E\alpha, \alpha \rangle \), since \( E \) is self-adjoint and \( J_0 \) is anti-self-adjoint.

Corollary 4.8 yields that the function \( \varphi \) given by
\[
\varphi(t) := \int_{\mathbb{R}^+} \cos(tx) d\text{Re}\mu(x) + \int_{\mathbb{R}^+} \sin(tx) d(\mu - \text{Re}\mu)(x),
\]
(4.12)
is continuous and positive definite. As shown in Subsection 4.1, there is a quaternionic Hilbert space \( H_\varphi \) associated with \( \varphi \); moreover, \( H_\varphi \) can be transformed into a bilateral quaternionic Hilbert space \( \tilde{H}_\varphi \).

In the proof of 4.7 it has been illuminated that the family of operators \( U_t \) (\( t \in \mathbb{R} \)) defined by (4.3) is a strongly continuous one-parameter unitary group on \( H_\varphi \). It follows immediately from Theorem 3.4 that
\[
U_t = e^{tA}|_L \text{ for all } t \in \mathbb{R},
\]
Here, \( A \) is the infinitesimal generator of \( U(t) \), and \( e^{tA}|_L \) is defined by the functional calculus for the function \( e^{tx} \) on the left spectrum of \( A \) in the bilateral Hilbert space \( \tilde{H}_\varphi \).

**Step 2:**
Because \( A \) is anti-self-adjoint, the left spectrum \( \sigma_L(A) \) must be a subset of the pure imaginary space \( \mathbb{H}_I := i_1 \mathbb{R} + i_2 \mathbb{R} + i_3 \mathbb{R} \). Due to this fact, we know the function \( e^{tx} \) can be composed as
\[
e^{tx} = \cos t|x| + i_1 \frac{x_1}{|x|} \sin t|x| + i_2 \frac{x_2}{|x|} \sin t|x| + i_3 \frac{x_3}{|x|} \sin t|x|,
\]
for all \( x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 \) in the left spectrum. Definition 2.9 thus leads to the following equality:
\[
(e^{tA}|_L \delta)(\delta) = \langle \phi(\cos t|x|) \delta|\delta \rangle + \sum_{k=1}^3 (i_k \phi(\frac{x_k}{|x|} \sin t|x|) \delta|\delta),
\]
(4.13)
where $\delta$ is the finite delta function given by

$$
\delta(x) = \begin{cases} 
1, & x = 0; \\
0, & x \neq 0;
\end{cases}
$$

since $\phi$ is a $\mathbb{H}$-algebra homomorphism from $\mathcal{B}(\sigma L(A), \mathbb{H})$ to $\mathcal{B}(\tilde{\mathcal{H}}_\mathcal{E})$. Moreover, the definition of left scalar multiplication (one may refer to Subsection 4.1 for more details) indicates that $L_{i_k}$, i.e., the left scalar multiplication by $i_k$ ($k = 1, 2, 3$), is an anti-self-adjoint right $\mathbb{H}$-linear bounded operator. Hence, by shifting the position of $i_k$ in (4.13) we have

$$
(e^{tA} | L_{i_k} | \delta) = (\phi(cos t|x|)\delta | \delta) - \sum_{k=1}^{3} (\phi(\frac{x_k}{|x|} \sin t|x|)\delta | i_k \delta),
$$

which, along with (4.2), implies

$$
(e^{tA} | L_{i_k} | \delta) = (\phi(cos t|x|)\delta | \delta) - \sum_{k=1}^{3} (\phi(\frac{x_k}{|x|} \sin t|x|)\delta | i_k \delta),
$$

(4.14)

By Definition 2.9 there exist regular $\mathbb{R}$-valued Borel measure $\mu_{i_v,i_l}^{\alpha,\beta}$ ($v, l = 0, 1, 2, 3; \alpha, \beta \in \tilde{\mathcal{H}}_\phi$) such that

$$
(\phi(f) | \alpha | \beta) = \sum_{v,l=0}^{3} \int_{\sigma L(A)} f_v i_l d\mu_{i_v,i_l}^{\alpha,\beta},
$$

(4.15)

where $f = \sum_{v=0}^{3} f_v i_v \in \mathcal{B}(\sigma L(A), \mathbb{H})$ and $f_v$ is $\mathbb{R}$-valued. Applying (4.15) to (4.14) yields

$$
(e^{tA} | L_{i_k} | \delta) = \sum_{l=0}^{3} \int_{\sigma L(A)} \cos t|x|i_l d\mu_{i_l,i_l}^{\delta,\delta}(x) - \sum_{k=1}^{3} \sum_{l=0}^{3} \int_{\sigma L(A)} \frac{x_k}{|x|} \sin t|x|i_l i_k d\mu_{i_l,i_l}^{\delta,\delta}(x).
$$

(4.16)

**Step 3:**

Since $\phi$ is an involution preserving $\mathbb{H}$-algebra homomorphism, any $\mathbb{R}$-valued bounded measurable function $f$ on $\sigma L(A)$ satisfies the following equations:

$$
(\phi(f) | \delta - i_k \delta) = (i_k \phi(f) | \delta)
$$

$$
= (\phi(i_k f) | \delta)
$$

$$
(\phi(f) | \delta | i_k \delta) = (-i_k \phi(f) | \delta)
$$

$$
= (\phi(i_k f) | \delta) \quad k = 1, 2, 3.
$$

$$
\text{Re}(\phi(i_k f) | \delta) = \text{Re}(\delta | \phi(i_k f) | \delta)
$$

$$
= \text{Re}(\phi(i_k f) | \delta)
$$

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It thus follows that
\[ \text{Re}(\phi(f)\delta - i_k\delta) = \text{Re}(\phi(f)\delta|i_k\delta), \quad k = 1, 2, 3. \]
holds true. Applying (4.2) to the equality above leads to
\[ \text{Re}\left(\phi(f)\delta(-i_k)\right) = \text{Re}\left(\phi(f)\delta|i_k\delta\right), \quad k = 1, 2, 3. \quad (4.17) \]
We combine (4.15) and (4.17) to deduce that
\[ \int_{\sigma^L(A)} f d\mu_{i_0,k}[\delta, \delta] = -\int_{\sigma^L(A)} f d\mu_{i_0,k}[\delta, \delta], \quad k = 1, 2, 3. \]
is valid for any \( \mathbb{R} \)-valued bounded measurable function \( f \) on \( \sigma^L(A) \). Due to the randomness of \( f \), we come to the vital result that
\[ \mu_{i_0,k}[\delta, \delta] = 0, \quad k = 1, 2, 3. \]
Moreover, the following equality
\[ \int_{\sigma^L(A)} f^2 d\mu_{i_0,i_0}[\delta, \delta] = \text{Re}(\phi(f^2)\delta|\delta) = \text{Re}(\phi(f)\delta) \geq 0 \quad (\neq +\infty), \]
implies that \( \mu_{i_0,i_0}[\delta, \delta] \) is non-negative and finite.

Substituting the equalities above to (4.16) yields
\[ (e^{tA})|_L\delta = \int_{\sigma^L(A)} \cos t|x|d\mu_{i_0,i_0}[\delta, \delta](x) - \int_{\sigma^L(A)} \frac{x}{|x|} \sin t|x|d\mu_{i_0,i_0}[\delta, \delta](x). \quad (4.18) \]
Set \( \gamma \) to be the zero extension of \( \mu_{i_0,i_0}[\delta, \delta] \) to the pure imaginary space \( \mathbb{H}_I = \mathbb{R}_{i_1} + \mathbb{R}_{i_2} + \mathbb{R}_{i_3} \). Let \( \Gamma \) be a non-negative finite regular Borel measure on \( \mathbb{H}_I \) defined by
\[ \Gamma(\omega) := \gamma(-\omega), \quad \forall \omega \in \mathcal{B}(\mathbb{H}_I). \]
Then (4.18) can be rewritten as
\[ (U_t\delta|_L\delta) = (e^{tA}|_L\delta) = \int_{\mathbb{H}_I} \cos t|x|d\Gamma(x) + \int_{\mathbb{H}_I} \frac{x}{|x|} \sin t|x|d\Gamma(x). \]
By the definition of \( U_t \), we obtain
\[ \varphi(t) = (U_t\delta|_L\delta) = \int_{\mathbb{H}_I} \cos t|x|d\Gamma(x) + \int_{\mathbb{H}_I} \frac{x}{|x|} \sin t|x|d\Gamma(x). \quad (4.19) \]

**Step 4:**
Consider the slice-condensed measure \( \mu' \) of type I given by
\[ \mu' := \rho_\Lambda(\Gamma + \frac{x}{|x|}\Gamma). \]
A direct calculation yields the right side of (4.19) is equal to
\[
\int_{\mathbb{R}^+} \cos tx \, d\text{Re}\mu'(x) + \int_{\mathbb{R}^+} \sin tx \, d(\mu' - \text{Re}\mu')(x)
\]
Then according to the expression of \(\varphi(t)\) in (4.12), we obtain
\[
\int_{\mathbb{R}^+} \cos tx \, d\text{Re}\mu(x) = \int_{\mathbb{R}^+} \cos tx \, d\text{Re}\mu'(x),
\]
and
\[
\int_{\mathbb{R}^+} \sin tx \, d(\mu - \text{Re}\mu)(x) = \int_{\mathbb{R}^+} \sin tx \, d(\mu' - \text{Re}\mu')(x),
\]
for all \(t \in \mathbb{R}\). Hence, by the Stone-Weierstrass theorem, we have
\[
\int_{\mathbb{R}^+} f \, d\text{Re}\mu = \int_{\mathbb{R}^+} f \, d\text{Re}\mu'
\]
holds for any \(f \in C_0(\mathbb{R}^+)\); and
\[
\int_{\mathbb{R}^+} f \, d(\mu - \text{Re}\mu) = \int_{\mathbb{R}^+} f \, d(\mu' - \text{Re}\mu')
\]
holds for any \(f \in C_0(\mathbb{R}^+)\) with \(f(0) = 0\). Thus we know
\[
\text{Re}\mu = \text{Re}\mu' \quad \text{on } \mathbb{R}^+,
\]
and
\[
\mu - \text{Re}\mu = \mu' - \text{Re}\mu' \quad \text{on } \mathbb{R}^+ \setminus \{0\}.
\]
Since \(\mu\) and \(\mu'\) are slice-condensed of type II and type I respectively, by definition \(\mu - \text{Re}\mu\) and \(\mu' - \text{Re}\mu'\) both vanish at the origin, namely,
\[
\mu(\{0\}) - \text{Re}\mu(\{0\}) = \mu'(\{0\}) - \text{Re}\mu'(\{0\}) = 0.
\]
Therefore, we come to the final conclusion: \(\mu\) is identical with \(\mu'\) on \(\mathbb{R}^+\). In other words, any slice-condensed measure of type II is a slice-condensed measure of type I. \(\square\)

Then the equivalence of Definitions 4.2 and 4.3 follows immediately from Lemmas 4.9 and 4.10.

**Theorem 4.11.** Any slice-condensed measure of type I is a slice-condensed measure of type II, and vice versa.
5. An application to quaternionic random processes

In terms of applications, a growing popularity of quaternionic random processes has arisen in the field of signal processing (see, e.g. [2, 13, 14]). In this section, we shall reveal some mathematical properties of a special family of quaternionic random processes via the spectral analysis.

Let $\mathbb{E}(Y)$ denote the mean of a quaternionic random variable $Y$. The covariance $\text{cov}(Y_1, Y_2)$ of arbitrary quaternionic random variables $Y_1, Y_2$ is given as

$$\text{cov}(Y_1, Y_2) := \mathbb{E}(Y_1 Y_2);$$

and the variance of $Y$ is defined as

$$\text{var}(Y) := \text{cov}(Y, Y).$$

One may refer to [13, 14] for more basic notations.

**Definition 5.1.** A quaternionic process $X = \{X_t : t \geq 0\}$ is said to be weakly stationary if for all $t, s \geq 0$, and $h > 0$, the following equalities holds:

$$\mathbb{E}(X_t) = \mathbb{E}(X_s),$$

and

$$\text{cov}(X_t, X_s) = \text{cov}(X_{t+h}, X_{s+h}).$$

Thus, $X$ is weakly stationary if and only if it has a constant mean and its auto-covariance $\text{cov}(X_t, X_s)$ is a function of $s - t$ only. For a weakly stationary process $X$, such function is called the auto-covariance function of $X$, and denoted by $c_X$. More precisely,

$$c_X(t) := \begin{cases} 
\text{cov}(X_0, X_t), & t \geq 0; \\
\text{cov}(X_{-t}, X_0), & t < 0.
\end{cases}$$

Auto-covariance functions have the following property.

**Theorem 5.2.** Assume that $X$ is a weakly stationary quaternionic processes, then its auto-covariance function $c_X(t)$ is positive definite.

**Proof.** For all $t_1, t_2, \cdots , t_k \in \mathbb{R}$ and all $q_1, q_2, \cdots , q_k \in \mathbb{H},$

$$\sum_{1 \leq i,j \leq k} p_i c_X(t_i - t_j) p_j = \sum_{1 \leq i,j \leq k} p_i c_X(t'_j - t'_i) p_j$$

$$= \sum_{1 \leq i,j \leq k} p_i \text{cov}(X_{t'_i}, X_{t'_j}) p_j$$

$$= \text{var}(Y) \geq 0,$$

where, $t'_i = -t_i + \max \{t_i\}$, and $Y := \sum_{i=1}^{k} p_i X_{t'_i}$. \qed
**Theorem 5.3** (Spectral theorem for auto-variance functions). Assume $X$ is a weakly stationary quaternionic process, there exists a unique slice-condensed measure $\mu$ on $\mathbb{R}^+$ such that

$$c_X(t) = \int_{\mathbb{R}^+} \cos(tx)d\operatorname{Re}\mu(x) + \int_{\mathbb{R}^+} \sin(tx)d(\mu - \operatorname{Re}\mu)(x),$$

wherever the auto-variance function $c_X$ is continuous at the origin.

**Proof.** This follows immediately from Theorems 5.2 and 4.5. We need only demonstrate that $c_X$ is continuous on $\mathbb{R}^+$. Without loss of generality, we assume $\mathbb{E}(X_t) = 0, t \geq 0$. Then a simple application of the Cauchy-Schwarz inequality yields:

$$|c_X(t + \Delta t) - c_X(t)| = |\mathbb{E}(X_0(X_t + \Delta t - X_t))|$$

$$\leq \sqrt{\mathbb{E}(|X_0|^2)\mathbb{E}(|X_t + \Delta t - X_t|^2)}$$

$$= \sqrt{c(0)[2c(0) - c(\Delta t) - c(-\Delta t)]}$$

holds for all $t, t + \Delta t \geq 0$, which indicates that $c_X$ is continuous on $\mathbb{R}^+$. What’s more, it can be easily seen that

$$c_X(-t) = \mathbb{E}(X_tX_0) = \mathbb{E}(X_0X_t) = c_X(t)$$

holds for all $t \geq 0$. Hence, $c_X$ is continuous on $\mathbb{R}$.

**Corollary 5.4.** Assume $X$ is a weakly stationary quaternionic process, there exists a (not necessarily unique) non-negative finite regular Borel measure $\Gamma$ on the imaginary plane $\mathbb{H}_I$ such that

$$c_X(t) = \int_{\mathbb{H}_I} e^{tx}d\Gamma(x),$$

whenever the auto-variance function $c_X$ is continuous at the origin.

**Proof.** It follows from Theorem 5.3 and Definition 4.2 that there exists a non-negative regular finite Borel measure $\Gamma$ on $\mathbb{H}_I$ such that

$$c_X(t) = \int_{\mathbb{R}^+} \cos(ty)d\mu_1(y) + \int_{\mathbb{R}^+} \sin(ty)d\mu_2(y),$$

(5.1)

where $\mu_1 = \rho_\star \Gamma$ and $\mu_2 = \rho_\star \left( \frac{x}{|x|}\Gamma \right)$.

Direct calculations yield:

$$\int_{\mathbb{R}^+} \cos(ty)d\mu_1(y) = \int_{\mathbb{H}_I} \cos(t|x|)d\Gamma(x),$$

$$\int_{\mathbb{R}^+} \sin(ty)d\mu_2(y) = \int_{\mathbb{H}_I} \sin(t|x|) \frac{x}{|x|}d\Gamma(x).$$
Then we substitute these two equalities into (5.1) to obtain

\[ c_X(t) = \int_{\mathbb{H}_I} \cos(t|x|) + \sin(t|x|) \frac{x}{|x|} d\Gamma(x) \]

\[ = \int_{\mathbb{H}_I} e^{tx} d\Gamma(x). \]

**Remark 5.5.** The spectral theorems for auto-variance functions (Theorem 5.3 and Corollary 5.4) are also valid for the class of wide-sense stationary quaternion random signals introduced by C. C. Took and D. P. Mandic [14], because wide-sense stationarity is stronger than weak stationarity.

6. Final remark

What we achieve in this paper:

By the generalized Stone’s theorems, we manage to reveal some vital spectral characteristics of quaternionic positive definite functions on the real line. We also find an application to weakly stationary quaternionic random processes.

What we want to achieve in the future:

Our work may shed some light on the general theory of quaternionic positive definite functions on topological groups. Based on what is already known about the simplest case, we can make some reasonable speculations about the general case. For example, one particular speculation is as follows:

Assume \( G \) is a locally compact abelian group with Pontryagin dual \( \hat{G} \). Then for any continuous quaternionic positive definite function \( \varphi \) on \( G \), there exists a (not necessarily unique) non-negative finite regular Borel measure \( \Gamma \) on \( \hat{G} \times S \) such that

\[ \varphi(x) = \int_{\hat{G} \times S} \text{Re}(\xi(x)) + s\text{Im}(\xi(x)) d\Gamma(\xi, s). \]

Here \( S \) denotes the set of quaternionic imaginary units, \( \text{Re} \) and \( \text{Im} \) mean extracting the real part and the imaginary part respectively.

This speculation, along with others, will be discussed in our future work.

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