An analytic cylindrically symmetric solution for collapsing dust

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Abstract

Dust configurations are the simplest models for astrophysical objects. Here we examine the gravitational collapse of an infinite cylinder of dust and give an analytic interior solution. Surprisingly, starting with a cylindrically symmetric ansatz one arrives at a 3-space with constant curvature, i.e. the resulting metric describes a piece of the Friedman interior of the Oppenheimer-Snyder collapse. Indeed, by introducing double polar coordinates, a 3-space of constant curvature can be interpreted as a cylindrically symmetric space as well. This result shows afresh that topology is not fixed by the Einstein equations.
I. INTRODUCTION

Astrophysical collapse processes are one of the main sources of gravitational radiation. However, the complexity of the Einstein equations makes it very difficult to model such processes mathematically. This is why we turn here to a dust model. Dust is the simplest material for astrophysical and cosmological studies and may be interpreted as a many-particle system with only gravitational interaction between its particles.

Unfortunately, one cannot study gravitational radiation in spherically symmetric space-times (e.g. the Oppenheimer-Snyder collapse) since the Birkhoff theorem shows radiation to be absent. For this reason, much work has been devoted to studying cylindrical symmetry, e.g.\(^2,3,4,5,6\).

There is an interesting suggestion for the description of cylindrically symmetric dust models by Singh and Vaz. In\(^7\) they examine the collapse of an infinite cylinder of dust making the assumption that the axial and azimuthal metric functions are connected by a simple relation. The authors discuss the resulting field equations qualitatively and note a striking analog to the Friedman equations that describe the interior of the Oppenheimer-Snyder collapse.

This paper is meant to show that this statement is not accidental. By solving the differential equations we obtain a geometry that is a piece of the Friedman Universe (with a 3-space of constant curvature). Nevertheless, by introducing double polar coordinates the geometry takes the form of a cylindrically symmetric space-time as well.

II. CYLINDRICALLY SYMMETRIC LINE ELEMENT

We want to find solutions of the Einstein equations which describe an infinite cylinder of dust (no pressure: \(p = 0\)). In this case, the stress-energy tensor of an ideal fluid simplifies to

\[
T_{ij} = \mu(\rho, t)u_iu_j,
\]

where \(\rho\) and \(t\) are the radial and the time component of the co-moving coordinates \((\rho, \varphi, z, t)\) and \(u_i\) is the four-velocity with the special structure

\[
(u^i) = (0, 0, 0, 1)
\]
(we choose units in which $G = c = 1$). We discuss the general cylindrically symmetric line element
\[ ds^2 = L^2(\rho, t) \, d\rho^2 + B^2(\rho, t) \, d\varphi^2 + M^2(\rho, t) \, dz^2 - dt^2 , \] (3)
but with the additional assumption that for the functions $B$ and $M$ of the line element the condition
\[ B = \rho M \] (4)
holds. This assumption generalizes a property of the surface geometry of cylinders in flat 3-spaces: Surfaces $\rho = \text{constant}, \ t = \text{constant}$ have the line element $ds^2 = M^2(\rho^2 d\varphi^2 + dz^2)$ which differs only by a conformal factor (a constant for every slice) from the usual expression $ds^2 = \rho^2 d\varphi^2 + dz^2$.

III. SOLUTION OF THE FIELD EQUATIONS

From $T^{ij}_{\ j} = 0$ and by using eq. (1) one finds
\[ (\mu u^i)_{,i} = \frac{1}{\sqrt{-g}} (\mu u^i \sqrt{-g})_{,i} = \frac{1}{\sqrt{-g}} (\rho LM^2 \mu)_{,t} = 0 \] (5)
with the solution
\[ \mu(\rho, t) = \frac{2\psi(\rho)}{\kappa \rho LM^2} , \] (6)
where $\kappa$ is Einstein’s gravitational constant, $\kappa = 8\pi$. Using this result, the Einstein equations are (see (7))
\[ \frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} = - \frac{1}{L^2} \left( \frac{B'^2}{B^2} - \frac{B'}{\rho B} \right) , \] (7)
\[ \frac{2\ddot{B}}{B} + \frac{\ddot{L}}{L} = - \frac{\psi(\rho)}{\rho LM^2} , \] (8)
\[ \frac{2\ddot{M}'}{M} + \frac{\dot{M}}{\rho M} = \frac{\dot{L}}{L} \left( \frac{2M'}{M} + \frac{1}{\rho} \right) , \] (9)
\[ \frac{2M'}{M} = L' , \] (10)
where a dot and prime denote derivatives with respect to $t$ and $\rho$ respectively. Eq. (10) has the solution
\[ L(\rho, t) = h(t)M^2(\rho, t) \] (11)
and eq. (9) has the first integral

\((\sqrt{\rho}M)' = g(\rho)L(\rho, t)\).  

(12)

The combination of these two relations gives

\((\sqrt{\rho}M)' = h(t)\frac{g(\rho)}{\rho}(\sqrt{\rho}M)^2\)

(13)

This equation can be integrated. One obtains

\[
\frac{1}{\sqrt{\rho}M} = h(t)\left[G(\rho) + d(t)\right],
\]

(14)

where

\[
G(\rho) := -\int \frac{g(\rho)}{\rho} d\rho.
\]

(15)

Using these results, the metrical coefficients take the form

\[
M = \frac{1}{h(t)\sqrt{\rho}[G(\rho) + d(t)]}, \quad B = \frac{\sqrt{\rho}}{h(t)[G(\rho) + d(t)]}, \quad L = \frac{1}{h(t)\rho[G(\rho) + d(t)]^2}.
\]

(16)

We have not yet ensured that equations (7) and (8) are fulfilled. If one substitutes \(B\) and \(L\) from (16) into eq. (8) then one finds

\[
-\rho\psi(\rho) = \left[\frac{3\dot{H}^2 - 3\ddot{H}}{(G(\rho) + d(t))^4} + \frac{8\dot{H}\dot{d} - 4\ddot{d}}{(G(\rho) + d(t))^5} + \frac{10\dot{d}^2}{(G(\rho) + d(t))^6}\right] e^{-3H}
\]

(17)

with the definition \(H(t) := \ln h(t)\). The derivative with respect to time \(t\) of this equation leads to \(\dot{d} = 0\). Hence one can set, without loss of generality,

\[d \equiv 0,\]

(18)

because \(d\) only appears in the form \(G(\rho) + d\) and would thus merely lead to a redefinition of the metric function \(G\). Then (17) simplifies to

\[
-\rho\psi(\rho)G^4(\rho) = (3\dot{H}^2 - 3\ddot{H})e^{-3H} = -K_0 = \text{constant}.
\]

(19)

That this expression is constant, follows from the fact that the left-most expression is a function of \(\rho\) alone, whereas the expression to its right is a function \(t\) alone. The mass density then is given by

\[
\mu = \frac{2K_0}{\kappa}e^{3H(t)}.
\]

(20)
Thus $\mu$ is a spatial constant for every time $t$. $H$ has to satisfy the equation

$$\ddot{H} - \dot{H}^2 = \frac{K_0}{3} e^{3H}. \tag{21}$$

We have yet to satisfy eq. (7). By using (16) with $d = 0$ one finds

$$-(2\ddot{H} - 3\dot{H}^2)e^{-2H} = \rho^2 G^2 G'' - \frac{1}{4} G^4 = K_1 = \text{constant}. \tag{22}$$

Again the left hand side is a function of another variable than the right hand side and so both sides have to be constant. This leads to another differential equation for $H$,

$$\ddot{H} - \frac{3}{2} \dot{H}^2 = -\frac{K_1}{2} e^{2H}. \tag{23}$$

Instead of the system of equations (21), (23) one could use the two combinations

$$\ddot{H} = K_0 e^{3H} + K_1 e^{2H}, \quad \dot{H}^2 = \frac{2}{3} K_0 e^{3H} + K_1 e^{2H}. \tag{24}$$

The first equation in (24) follows from the second one by differentiating with respect to time $t$ (for the case $\dot{H} \neq 0$, but the case $H = \text{constant}$ is not possible, because then one would find $K_0 = 0$ from (21) and from this $\mu = 0$ from (20) — i.e. a vacuum).

Due to the freedom of coordinate transformations, we need only study the cases $K_1 = -\varepsilon$ with $\varepsilon = 0, \pm 1$. With the definitions $a(t) := e^{-H(t)} = 1/h(t)$ and $a_m = \frac{2}{3} K_0$, the second equation in (24) takes the form

$$\dot{a}^2 - \frac{a_m}{a} + \varepsilon = 0, \quad \varepsilon = 0, \pm 1. \tag{25}$$

Surprisingly, the scale factor $a$ satisfies the dynamical equation of the Friedman cosmology. Hence, its solutions are

$$\varepsilon = +1 : \quad a(\eta) = \frac{a_m}{2} (1 - \cos \eta), \quad t(\eta) = \frac{a_m}{2} (\eta - \sin \eta)$$

$$\varepsilon = 0 : \quad a(t) = \left(\frac{3}{2} \sqrt{a_m} \right)^{2/3}$$

$$\varepsilon = -1 : \quad a(\eta) = \frac{a_m}{2} (\cosh \eta - 1), \quad t(\eta) = \frac{a_m}{2} (\sinh \eta - \eta). \tag{26}$$

Solving eq. (22) one obtains

$$\varepsilon = +1 : \quad G^2(\rho) = \frac{\rho}{\rho_0} + \frac{\rho_0}{\rho}$$

$$\varepsilon = 0 : \quad G^2(\rho) = \frac{\rho_0}{\rho}$$

$$\varepsilon = -1 : \quad G^2(\rho) = \left| \frac{\rho}{\rho_0} - \frac{\rho_0}{\rho} \right|. \tag{27}$$
with an integration constant $\rho_0 > 0$. Finally, with the substitution $\rho = \rho_0 \tan \chi$ (in the case $\varepsilon = +1$) or $\rho = \rho_0 \tanh \chi$ (in the case $\varepsilon = -1$) one arrives (after stretching the coordinates to eliminate the constant $\rho_0$) at the line elements

$$
\varepsilon = +1 : \quad ds^2 = a^2(t) \left( d\chi^2 + \sin^2 \chi \, d\varphi^2 + \cos^2 \chi \, dz^2 \right) - dt^2
$$

$$
\varepsilon = 0 : \quad ds^2 = a^2(t) \left( d\rho^2 + \rho^2 \, d\varphi^2 + dz^2 \right) - dt^2
$$

$$
\varepsilon = -1 : \quad ds^2 = a^2(t) \left( d\chi^2 + \sinh^2 \chi \, d\varphi^2 + \cosh^2 \chi \, dz^2 \right) - dt^2
$$

and the mass density is (see (20))

$$
\mu = \frac{3a_m}{\kappa a^3(t)}.
$$

IV. INTERPRETATION OF THE SOLUTIONS

With the line elements (28) and the dynamical equation (25) we finally arrived at the geometry of the Friedman models, here described in double polar coordinates $\chi, \varphi, z$, cf. [8]. This is quite amazing since we started with cylindrical symmetry and tried to obtain collapse models different from the spherically symmetric Oppenheimer-Snyder model.

In double polar coordinates the $z$-coordinate is limited by $z \in [0, 2\pi]$. However, for a cylinder, $z$ should take on all real values. Here, we provide a suitable topological interpretation to circumvent this contradiction: For $\varphi = \text{constant}, t = \text{constant}$, the first line element in (28), for example, reduces to

$$
ds^2 = a^2(d\chi^2 + \cos^2 \chi \, dz^2) \, .
$$

Of course, this is the line element for a sphere of radius $a$ with the angular variables $\chi$ and $z$. Now, instead of only one sphere, we consider an infinite number of connected shells (as sketched in fig. [1]). If $z$ increases by $2\pi$ one moves from one skin onto another.

This topology leads to a cylindrical interpretation of the solution.

V. DISCUSSION

The intention of this paper was to find a cylindrically symmetric solution for collapsing dust. It turned out that the assumption that the axial and azimuthal metric functions
fulfill eq. (4) leads to only one class of solutions: collapsing Friedman Universes. The apparent contradiction between the spherical symmetry of the Friedman models and the required cylindrical symmetry was eliminated using a new topological interpretation. Thus we have found an analytic solution describing a collapsing infinite cylinder of dust. To have a complete solution of the Einstein equations, one has to compute the exterior vacuum space-time that matches to the given interior solution (if such a solution exists, which is not clear a priori). For the vacuum region one should use the line element

\[ ds^2 = e^{-2U} \left[ e^{2k}\left(\rho^2 - \rho^2 \phi^2\right) + \rho^2 \rho^2 d\phi^2 \right] + e^{2U} dz^2. \] (31)

Then one has to solve the field equations

\[ U'' + \frac{1}{\rho} U' - \ddot{U} = 0, \quad k' = \rho(U'^2 + \dot{U}^2), \quad \dot{k} = 2\rho U' \dot{U}. \] (32)

For matching a vacuum region the picture in FIG. 1 changes: Only in the interior region, \( \chi \leq \chi_0 \), is the metric the Friedman solution in double polar coordinates. So one has to cut off a segment around the south pole of the hemisphere of FIG. 1 and connect the inner solution from there to the vacuum region.

Unfortunately, first numerical attempts seem to run into singularities.
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