THE DUALIZING SHEAF ON FIRST-ORDER DEFORMATIONS OF TORIC SURFACE SINGULARITIES

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Abstract. We explicitly describe infinitesimal deformations of cyclic quotient singularities that satisfy one of the deformation conditions introduced by Wahl, Kollár–Shepherd-Barron and Viehweg. The conclusion is that in many cases these three notions are different from each other. In particular, we see that while the KSB and the Viehweg versions of the moduli space of surfaces of general type have the same underlying reduced subscheme, their infinitesimal structures are different.

1. Introduction

In order to compactify the moduli space of surfaces of general type, one has to consider singular surfaces but for a long time it was not clear which class of singularities should be allowed. Building on Mori’s program, [KS88] described such a class, named semi-log-canonical singularities. These include quotient singularities, cusps and a few others; see [Kol13b, Sec.2.2] for a complete list. A new feature of the theory is that not every flat deformation of a surface with such singularities should be allowed in moduli theory. In essence this observation can be traced back to Bertini who observed that the cone over the degree 4 rational normal curve admits two distinct smoothings. One is the Veronese surface the other is a ruled surface; see [Pin74]. For the Veronese the self-intersection of the canonical class is 9 for the ruled surface it is 8. Since we would like the basic numerical invariants to be locally constant in families, one of these deformations should not be allowed.

It is not obvious how to obtain the right class of deformations. Three variants have been investigated in the past. Their common feature is that they all study the compatibility of deformations with powers of the dualizing sheaf \( \omega \). In order to define these 3 versions, we need some definitions.

1.1. General setup. We are ultimately interested in schemes with semi-log-canonical singularities \( S \), but for the basic definitions we need to assume only that \( S \) is a pure dimensional \( S_2 \) scheme over a field \( k \) such that

(i) there is a closed subset \( Z \subset S \) of codimension \( \geq 2 \) such that \( \omega_{S\setminus Z} \) is locally free and

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(ii) there is an $m > 0$ such that $\omega_S^m$ is locally free, where $\omega_S^m$ denotes the reflexive hull of $\omega_S^\otimes m$. The smallest such $m > 0$ is called the index of $\omega_S$. (Both of these conditions are satisfied by schemes with semi-log-canonical singularities.)

Let $(0, T)$ be a local scheme such that $k(0) \cong k$ and $p : X_T \to T$ a flat deformation of $S \cong X_0$. For every $g \in \mathbb{Z}$ we have natural restriction maps

$$\mathcal{R}^g : \omega_{X_T/T}^g|_{X_0} \to \omega_{X_0}^g.$$

These maps are isomorphisms over $S \setminus Z$ and we are interested in understanding those cases when they are isomorphisms over $S$. The local criterion of flatness shows (see [Kol13a] for details) that if $T$ is Artinian then

$$\mathcal{R}^g \text{ is an isomorphism } \iff \mathcal{R}^g \text{ is surjective } \iff \omega_{X_T/T}^g \text{ is flat over } T.$$

We will denote this condition by $(*)_g$ (with $g \in \mathbb{Z}$).

1.2. Definitions of qG- and V- and VW-deformations. Let $p : X_T \to T$ be a flat deformation as in (1.1).

1.2.1. qG-deformations. We call $p : X_T \to T$ a qG-deformation if the conditions $(*)_g$ defined in (1.1) hold for every $g \in \mathbb{Z}$. It is enough to check these for $g = 1, \ldots, \text{index}(\omega_S)$. (qG is short for “Quotient of Gorenstein,” but this is misleading if $\dim S \geq 3$.)

These deformations were introduced and studied by Kollár and Shepherd-Barron [KS88] as the class most suitable for compactifying the moduli of varieties of general type. A list of log canonical surface singularities with qG-smoothings is given in [KS88]. In the key case of cyclic quotient singularities the list (of the so-called T-singularities) was earlier established by Wahl [Wah80, 2.7], though he viewed them as examples of W-deformations (see below).

1.2.2. V-deformations. We call $p : X_T \to T$ a Viehweg-type deformation (or V-deformation) if the conditions $(*)_g$ from (1.1) hold for every $g$ divisible by $\text{index}(\omega_S)$. It is enough to check this for $g = \text{index}(\omega_S)$.

These deformations form the natural class suggested by the geometric invariant theory methods used in the monograph [Vie95]. Actually, [Vie95] considers the—a priori weaker—condition: $\mathcal{R}^g$ is an isomorphism for some $g > 0$ divisible by $\text{index}(\omega_S)$. One can see that in this case $(*)_g$ holds for every $g$ divisible by $\text{index}(\omega_S)$, at least in characteristic 0; see [Kol13a]. V-deformations are problematic in positive characteristic, see [HK10, 14.7].

1.2.3. W-deformations. We call $p : X_T \to T$ a Wahl-type deformation (or W-deformation) if the condition $(*)_g$ holds for $g = -1$. These deformations were considered in [Wah80] [Wah81] and called $\omega^*$-constant deformations there.
1.2.4. VW-deformations. We call \( p : X_T \to T \) a VW-deformation if it is both a V-deformation and a W-deformation.

1.3. Relations between qG, V and VW. It is clear that every qG-deformation is also a VW-deformation. Understanding the precise relationship between the four classes (1.2.1) – (1.2.4) has been a long standing open problem. For reduced base spaces we have the following very strong result.

**Theorem 1.** A flat deformation of a log canonical scheme over a reduced, local scheme of characteristic 0 is a V-deformation if and only if it is a qG-deformation.

When \( T \) is the spectrum of a DVR, \( \dim S = 2 \) and \( S \) has quotient singularities, this was proved in [Mum78] and [Kol95, 14.2]. If \( \dim S > 2 \) and \( S \) has log terminal singularities, this is a special case of inversion of adjunction as proved in [Kol92, Sec.17] and the log canonical case similarly follows from [Kaw97] and the normality of log canonical centers [Amb03, Fuj09]; see also [Kol13b, Sec.4.3]. These imply the claim for arbitrary reduced base schemes using [Kol08]; see [Kol13a] for more details.

This raised the possibility that every V-deformation of a log-canonical singularity is also a qG-deformation over arbitrary base schemes. It would be enough to check this for Artinian bases. In this note we focus on first order deformations and prove that these two classes are quite different from each other.

**Definition 2.** Let \( S \) be a scheme satisfying the conditions (1.1)(i)-(ii). Let \( T^1(S) \) be the (possibly infinite dimensional) \( k \)-vector space of deformations of \( S \) over Spec \( k[\varepsilon] \).

We denote by \( T^1_{qG}(S) \subset T^1(S) \) the space of first order qG-deformations, \( T^1_V(S) \) the space of first order V-deformations, \( T^1_W(S) \) the space of first order W-deformations, and \( T^1_{VW}(S) \) the space of first order VW-deformations.

We have obvious inclusions

\[
T^1_{qG}(S) \subset T^1_{VW}(S) \subset T^1_V(S), T^1_W(S) \subset T^1(S),
\]

but the relationship between \( T^1_V(S) \) and \( T^1_W(S) \) is not clear.

1.4. The case of cyclic quotient singularities. We completely describe first order V-, VW- and qG-deformations of two-dimensional cyclic quotient singularities. The precise answers are stated in Sections 4 and 5. The main conclusion is that V-deformations and VW-deformations, and even more V-deformations and qG-deformations are quite different over Artinian bases.

**Theorem 3.** Let \( S_{n,q} := \mathbb{A}^2/\mathbb{Z}/n\mathbb{Z}(1,q) \) denote the quotient of \( \mathbb{A}^2 \) by the cyclic group action generated by \((x,y) \mapsto (\eta x, \eta^q y)\), where \( \eta \) is a primitive \( n \)th root of unity. Then, if \( q \neq -1 \) in \((\mathbb{Z}/n\mathbb{Z})^*\), i.e. if \( \text{embdim}(S_{n,q}) \geq 4 \),

\[
\dim T^1_V(S_{n,q}) - \dim T^1_{VW}(S_{n,q}) = \text{embdim}(S_{n,q}) - 4 \quad \text{or} \quad \text{embdim}(S_{n,q}) - 5.
\]

In particular, if \( \text{embdim}(S_{n,q}) \geq 6 \) then \( S_{n,q} \) has V-deformations that are not VW-deformations, hence also not qG-deformations.
This is a direct consequence of the more detailed Theorem 5. By contrast, qG-deformations and VW-deformations are quite close to each other, as shown by the next result. This will be proved in (5.6).

**Theorem 4.** Let \( S_{n,q} \) be as in the previous theorem. Then

1. If \( \gcd(n, q+1) = 1 \) then \( T^1_{qG}(S_{n,q}) = T^1_{VW}(S_{n,q}) = \{0\} \).
2. If \( S_{n,q} \) admits a qG-smoothing then \( T^1_{qG}(S_{n,q}) = T^1_{VW}(S_{n,q}) \).
3. In general \( \dim T^1_{qG}(S_{n,q}) \leq \dim T^1_{VW}(S_{n,q}) \leq \dim T^1_{qG}(S_{n,q}) + 1 \).

1.5. **Using the interval language.** Besides the description of cyclic quotient singularities in terms of the invariants \( n \) and \( q \), there is an alternative possibility by using rational intervals \( I = [-A, B] \subseteq \mathbb{Q} \) with uniform denominators at the end points, i.e. \( A \) and \( B \) have the same denominator in reduced form. We call \( I \) or the resulting singularity \( S_I \) grounded if \( I \) contains an integer in its interior. Since integral shifts of \( I \) will be neglected, this leads to \( A, B > 0 \). See (2.5) and (2.6) for details. This language allows a much more detailed description of the situation:

**Theorem 5.** Assume that \( \text{embdim}(S_I) \geq 4 \). Then

1. If the interval \( I \) is not grounded, then the associated surface singularity \( S_I \) has neither qG- nor VW-deformations. The dimension of \( T^1_V(S_I) \) is \( \text{embdim}(S_I) - 4 \).
2. If \( A, B > 0 \), then \( \dim T^1_V(S_I) = \text{embdim}(S_I) - 4 + [A] + [B] \).
3. If \( A, B > 0 \) with fractional parts \( \{A\} = \frac{1}{m} \) or \( \{B\} = \frac{1}{m} \), then
   \[
   \dim T^1_W(S_I) = \dim T^1_{qG}(S_I) = [A + B].
   \]
4. If \( A, B > 0 \) with both fractional parts \( \{A\} \) and \( \{B\} \) different from \( \frac{1}{m} \), then
   \[
   \dim T^1_W(S_I) = [A] + [B] + 1 \quad \text{and} \quad \dim T^1_{qG}(S_I) = [A + B].
   \]

**Proof.** The first two parts, i.e. the description of the V-deformations follows from 4.3. The remaining two parts are just another formulation of Theorem 26. □

1.6. **Implications for moduli spaces.** One can construct compactified moduli spaces for surfaces of general type using either KSB-deformations or V-deformations. Let us denote these by \( \mathbf{M}(\text{KSB}) \) and \( \mathbf{M}(\text{V}) \). By Theorem 11 the underlying reduced structures of these moduli spaces are isomorphic. As a consequence of our computations we can say that the scheme structures are not isomorphic.

More generally, let \( X \) be a projective variety with isolated singularities \( x_1, \ldots, x_m \). Any flat deformation of \( X \) restricts to a deformation of the singularities \( (x_i, X) \). This induces a map of the local deformation spaces

\[
\mathcal{R} : \text{Def}(X) \to \text{Def}(x_1, X) \times \cdots \times \text{Def}(x_m, X).
\]

A direct consequence of the definition of qG-deformations given in [KS88] is that \( \text{Def}_{qG}(x, X) \) is smooth for 2-dimensional quotient singularities. Our computations show that, by contrast, \( \text{Def}_V(x, X) \) is usually non-reduced but

\[
\text{red(Def}_V(x, X)) = \text{Def}_{qG}(x, X).
\]
We thus expect that if \( X \) is a surface with quotient singularities then \( \text{Def}_V(X) \) can be non-reduced but \( \text{Def}_{qG}(X) \) should be smooth. This is not true in general, but there are many examples when local-to-global obstructions vanish and the map \( \Re \) is smooth. The situation is not well understood for surfaces of general type, but [HP10, Prop 3.1] shows that local-to-global obstructions vanish for Del Pezzo surfaces. Thus we obtain that if \( S \) is a Del Pezzo surface with quotient singularities then \( \text{Def}_{qG}(S) \) is smooth but \( \text{Def}_V(S) \) is nonreduced as soon as \( S \) has at least 1 singular point of multiplicity \( \geq 5 \).

2. Five descriptions of cyclic quotient singularities

In (2.1) – (2.5) we present several ways of representing two-dimensional cyclic quotient singularities \( S = \mathbb{A}^2_k/G \), i.e. those coming from a cyclic group \( G \) acting on \( \mathbb{A}^2_k \). While most of them are quite classic, the description (2.5) seems to be not common so far. At the end, in (2.6), we introduce the notion of grounded singularities. In the language of (2.5) this becomes especially simple.

2.1. Normalizing the action. Let \( G \) denote a cyclic group of order \( n \) with \( \text{char } k \nmid n \). Then, by [Bri68, §2], every linear action of \( G \) on \( \mathbb{A}^2_k \) is isomorphic to some action \( \frac{1}{n}(1, q) \) generated by

\[
(x, y) \mapsto (\eta x, \eta^q y),
\]

where \( q \in (\mathbb{Z}/n\mathbb{Z})^* \) and \( \eta \) is a primitive \( n \)-th root of unity. The corresponding ring of invariants is \( R_{n, q} := k[x, y]^G \) and the corresponding quotient singularity is

\[
S_{n, q} := \mathbb{A}^2_k/\frac{1}{n}(1, q) = \text{Spec } R_{n, q}.
\]

While we work with this affine model, all the results apply to its localization, Henselisation or completion at the origin. We can also choose \( \eta' = \eta^a \) as our primitive \( n \)-th root of unity. This shows the isomorphism

\[
S_{n, q} \cong S_{n, q'} \text{ where } qq' = 1 \text{ in } (\mathbb{Z}/n\mathbb{Z})^*.
\]

Note that we can and will choose a representative for \( q \) such that \( 1 \leq q \leq n - 1 \). The case \( q = n - 1 \) encodes the \( A_{n-1} \)-singularities. These are exceptional for many of the subsequent formulas, so we assume from now on that \( q \neq -1 \) in \( (\mathbb{Z}/n\mathbb{Z})^* \).

2.2. The abc notation. Here we just rename the invariants \( n \) and \( q \). Denote \( b := \gcd(n, q + 1) \), \( a := n/b \), and \( c := (q + 1)/b \). Hence we know that \( \gcd(a, c) = 1 \), and \( n \) and \( q \) can be recovered as \( n = ab \) and \( q = bc - 1 \). When using these invariants, we might write \( S_{abc} = \frac{1}{ab}(1, bc - 1) \) instead of \( S_{n, q} = \frac{1}{n}(1, q) \).

Note that the case \( q = n - 1 \) which was just excluded at the end of (2.1) can be recovered in the abc language as the case \( a = 1 \).

The isomorphic singularities \( S_{n, q} \) and \( S_{n, q'} \) from (2.1) share the same \( a \) and \( b \), i.e. \( a' = a \) and \( b' = b \). This follows from the fact that \( qq' \equiv 1 \text{ mod } n \) implies \( qq' \equiv 1 \text{ mod } b \) and that \( q \equiv -1 \text{ mod } b \) becomes then equivalent to \( 1 \equiv -q' \text{ mod } b \). The
third invariants $c$ and $c'$ differ. However, it is in general not true that they are mutually inverse within \((\mathbb{Z}/a\mathbb{Z})^*\). See the discussion at the end of (2.5).

2.3. The toric nature of $S_{n,q}$. Dealing with toric varieties involves a standardized language, cf. [CLST11] for details: Assume that $N$ and $M$ are mutually dual free abelian groups of finite rank; with $M_Q$ and $M_K$ we denote the associated $\mathbb{Q}$-vector spaces; similarly we often write $N_k$ and $M_k$ for $N \otimes \mathbb{Z} k$ and $M \otimes \mathbb{Z} k$, respectively. Let $\sigma \subseteq N_Q$ be a polyhedral cone and denote by $\sigma^\vee := \{ r \in M_Q \mid \langle \sigma, r \rangle \geq 0 \}$ the dual one. Then, $\sigma^\vee \cap M$ is a finitely generated semigroup, and its $M$-graded semigroup ring (with $k$-basis $\{ x^r \mid r \in \sigma^\vee \cap M \}$) provides the affine toric variety

$$TV(\sigma, N) := \text{Spec } k[\sigma^\vee \cap M].$$

Since we are going to deal with surface singularities, $N$ and $M$ will be of rank two. Hence, the primitive generators of $\sigma$ and $\sigma^\vee$ are just pairs $\alpha, \beta \in N$ and $r^1, r^e \in M$ (with $\langle \alpha, r^1 \rangle = \langle \beta, r^e \rangle = 0$), respectively. The relation to (2.1) is the well-known

**Proposition 6.** $S_{n,q} = TV(\sigma, \mathbb{Z}^2)$ where $\sigma = \langle (1, 0), (-q, n) \rangle \subseteq \mathbb{Q}^2 = N_Q$.

Note that we use $\langle \cdot, \cdot \rangle$ to denote both the pairing $N \times M \to \mathbb{Z}$ and the generation of a polyhedral cone. Moreover, when using coordinates, we try to distinguish between $M$ and $N$ by using the different brackets $[\cdot, \cdot]$ and $(\cdot, \cdot)$, respectively. So we will write $\sigma^\vee = \langle [0, 1], [n, q] \rangle \subseteq \mathbb{Q}^2 = M_Q$. That is, $\alpha = (1, 0)$, $\beta = (-q, n)$, $r^1 = [0, 1]$, and $r^e = [n, q]$. The group order $n$ may be recovered as $\det(\alpha, \beta)$.

**Proof.** Writing $k[x, y]$ as the semigroup ring $k[\mathbb{N}^2]$, we know that $R_{n,q} = k[x, y]^G = k[\mathbb{N}^2 \cap M]$ where $M \subseteq \mathbb{Z}^2$ is the sublattice freely generated by, e.g. $[-q, 1], [n, 0] \in \mathbb{Z}^2$. Now, a linear combination $\lambda \cdot [-q, 1] + \mu \cdot [n, 0] = [-\lambda q + \mu n, \lambda]$ has non-negative entries if and only if $\langle (-q, n), [\lambda, \mu] \rangle \geq 0$ and $\langle (1, 0), [\lambda, \mu] \rangle \geq 0$. \hfill \square

2.4. Equations of $S_{n,q}$ via continued fractions. Let $S := TV(\sigma)$ for some two-dimensional cone $\sigma = \langle \alpha, \beta \rangle$ as in (2.3). Denote by $E \subset \sigma^\vee \cap M$ the set of indecomposable elements within this semigroup ("Hilbert basis"). This finite set coincides with the lattice points on the compact edges of $\text{conv}(\sigma^\vee \cap M \setminus 0)$. In particular, we can naturally list its elements as $E = \{ r^1, r^2, \ldots, r^{e-1}, r^e \}$ with $e \geq 4$ (the cases $e = 2$ and $e = 3$ refer to $S$ being smooth or an $A_{n-1}$-singularity). Any two adjacent elements of this set do always form a $\mathbb{Z}$-basis of $M \cong \mathbb{Z}^2$. Hence, for $i = 2, \ldots, e-1$, we can write

$$r^{i-1} + r^{i+1} = a_i \cdot r^i$$

with natural numbers $a_i \geq 2$. The continued fraction $[a_2, \ldots, a_{e-1}] := a_2 - \frac{1}{a_3 - \ldots}$ recovers $\frac{n}{n-q}$. Moreover, $E$ provides an embedding $S \hookrightarrow \mathbb{A}^e_k$. Among the equations one finds $x_{i-1}x_{i+1} - x_i^{e+1}$, see [Rei74] for more details.

**Remark 7.** With growing $i$, the values $\langle \alpha, r^i \rangle$ and $\langle \beta, r^i \rangle$ increase and decrease, respectively. Hence, defining $\eta_i := \min\{\frac{\langle \alpha, r^{i+1} \rangle}{\langle \alpha, r^i \rangle}, \frac{\langle \beta, r^{i+1} \rangle}{\langle \beta, r^i \rangle}\} \in \mathbb{Q}_{\geq 1}$, we obtain that $[\eta_i] = a_i - 1 \in \mathbb{Z}_{\geq 1}$. 

2.5. Replacing cones by intervals. Let \( \sigma = \langle \alpha, \beta \rangle \) and \( \sigma^\vee = \langle r^1, r^e \rangle \) be mutually dual (two-dimensional, rational) cones as before. The primitive elements \( R \in \text{int} \sigma^\vee \cap M \) (we will call them primitive degrees of \( \sigma \)) give rise to affine crosscuts \( Q(\sigma, R) := \sigma \cap [R = 1] \). Since the affine line \([R = 1]\) can be identified with the rational line \( \mathbb{Q}^1 \) (canonically, up to integral shifts), we can and will understand \( Q(\sigma, R) \) as an interval in \( \mathbb{Q}^1 \).

Reciprocally, every closed interval \( I \subseteq \mathbb{Q}^1 \) provides a cone via \( C(I) := \mathbb{Q}_{\geq 0} \cdot (I, 1) \subseteq \mathbb{Q}^2 \) and a primitive degree \( R := [0, 1] \). These two constructions provide a natural one-one correspondence

\[
\{ \text{pairs } (\sigma, R) \} \backslash \text{SL}(2, \mathbb{Z}) \longleftrightarrow \{ \text{bounded intervals } I \subseteq \mathbb{Q}^1 \} \backslash \{ \mathbb{Z} \text{-shifts} \}.
\]

On the other hand, every cone \( \sigma \) provides a canonical primitive degree \( \overline{R} \), called the central degree. It is defined as the primitive generator of the ray \( \mathbb{Q}_{\geq 0} \cdot (r^1 + r^e) \). It is the only primitive degree such that \( \langle \sigma, \overline{R} \rangle = \langle \beta, \overline{R} \rangle \). Using coordinates via the \((n, q)/(a, b, c)\) language discussed in (2.2), one obtains that \( r^1 + r^e = [n, q + 1] = b \cdot [a, c] \), hence \( \overline{R} = [a, c] = \frac{r^1 + r^e}{b} \).

**Remark 8.** Actually, \( S = \text{TV}(\sigma) \) is \( \mathbb{Q}\)-Gorenstein with index \( a \) (and we suppose that \( a > 1 \)). The corresponding power \( \omega_S^{[a]} \) equals the ideal \((x^{\overline{R}}) \subseteq \mathcal{O}_S \) represented by the shifted semigroup \( \overline{R} + (\text{int } \sigma^\vee \cap M) \). Thus, properly speaking, not \( \overline{R} \) but the non-integral \( \frac{a}{n} \overline{R} \) is the truly canonical degree.

Using this special central degree \( \overline{R} \), the previous correspondence yields

**Proposition 9.** There is a one-one correspondence

\[
\{ \text{cones } \sigma \} \backslash \text{SL}(2, \mathbb{Z}) \longleftrightarrow \{ \text{intervals } I \subseteq \mathbb{Q} \text{ with uniform denominators} \} \backslash \{ \mathbb{Z} \text{-shifts} \}.
\]

We call \( I \) to have “uniform denominators” (at the end points) if both become equal in the reduced forms, i.e. if \( I = \left[ \frac{g}{m}, \frac{h}{m} \right] \) with \( g, h, m \in \mathbb{Z} \) and \( \gcd(g, m) = \gcd(h, m) = 1 \).

**Proof.** \((\Rightarrow)\) After a possible coordinate change, we may assume that \( \overline{R} = [0, 1] \).

Setting \( m := \langle \alpha, [0, 1] \rangle = \langle \beta, [0, 1] \rangle \) we obtain that \( \alpha = (g, m) \) and \( \beta = (h, m) \), hence \( Q(\sigma, [0, 1]) = \left[ \frac{g}{m}, \frac{h}{m} \right] \) for some \( g, h \) as asked for in the claim.

\((\Leftarrow)\) If \( I = \left[ \frac{g}{m}, \frac{h}{m} \right] \), then \( C(I) = \langle \left( \frac{g}{m}, 1 \right), (\frac{h}{m}, 1) \rangle = \langle (g, m), (h, m) \rangle \), i.e. its primitive generators are \( \alpha = (g, m) \) and \( \beta = (h, m) \). Thus, \( R = [0, 1] \) coincides with \( \overline{R} \). \( \square \)

In (2.3) we had considered cones \( \sigma = \langle \alpha, \beta \rangle = \langle (1, 0), (-q, n) \rangle \), i.e. \( n = | \det(\alpha, \beta) | \), and \( q \) was characterized by \( n|q(\alpha + \beta) \).

Alternatively we had used \( b := \gcd(n, q + 1) \) to write \( n = ab \) and \( q + 1 = bc \) in (2.2). Now, given an interval \( I = \left[ \frac{g}{m}, \frac{h}{m} \right] \) as in Proposition 9 it has length \(|I| = \frac{h - g}{m} \), and we may obtain the invariants \((a, b, c)\) for \( \sigma := C(I) \) via

**Proposition 10.** For \( I = \left[ \frac{g}{m}, \frac{h}{m} \right] \) one has \( a = m, b = h - g, \) and \( c = -1/g \in (\mathbb{Z}/m\mathbb{Z})^* \). In particular, \( b/a = n/m^2 = |I| / \text{index}(\omega_S) = m \) with \( S = \text{TV}(C(I)) \).
Proof. Let \( a, b, c \) be as in the claim. By definition, we have \( \gcd(a, c) = 1 \). We have to show that the generators \( \alpha = (g, m) \) and \( \beta = (h, m) \) of \( C(I) \) and the invariants \( n := ab, q := bc - 1 \) yield isomorphic cones: First, we clearly obtain that \(| \det(\alpha, \beta) | = (h - g)m = n \). It remains to check the characterizing relation \( n|(q\alpha + \beta) \). But this follows from

\[
q\alpha + \beta = ((h - g)c - 1) \cdot (g, m) + (h, m) = ((h - g)(cg + 1), (h - g)cm)
\]

which is indeed divisible by \( n = (h - g)m \). Finally, \( \text{index}(\omega_s) = a \) by Remark 8. \( \square \)

In (2.2) we mentioned the invariant \( c' \) associated to \( (n, q') \) as it was \( c \) to \((n, q)\). In the “interval language”, to switch \( q \) and \( q' \) means to replace \( I \) by \(-I\), i.e. to keep \( m \) and to replace \( g \) and \( h \) by \(-h \) and \(-g \), respectively. In particular, this implies that \( c' = 1/h \in (\mathbb{Z}/m\mathbb{Z})^* \).

Moreover, there is a way to visualize both \( c \) and \( c' \): The points \([c, -\frac{g+c+1}{m}]\) and \([-c', \frac{b'c-1}{m}]\) appear as the first lattice points on the two rays of the shifted cone \( C(I)' = [0, \frac{1}{m}], \) cf. the proof of Proposition 25.

2.6. Grounded cones and intervals. To represent two-dimensional cones \( \sigma \) by intervals \( I \) via Proposition 9, the central degree \( \overline{R} \) played an important role. This leads to the following notion:

Definition 11. A two-dimensional, polyhedral cone \( \sigma \) (or the associated interval \( I \), or the associated singularity \( S_{n,q} = S_{abc} = \mathbb{T} V(\sigma) \)) is called grounded :\(\Leftrightarrow\) the central degree \( \overline{R} \) belongs to the Hilbert basis \( E = \{r^1, r^2, \ldots, r^s\} \) of \( \sigma' \cap M \), i.e. \( \overline{R} \) is irreducible within this semigroup. If \( \overline{R} = r^\nu \), then \( \nu \) is called the central index.

Proposition 12. An interval \( I \subseteq \mathbb{Q} \) with uniform denominators is grounded if and only if it contains an interior integer.

Proof. (\(\Leftarrow\)) We may assume that \( 0 \in \text{int} I \), i.e. \( I = [\frac{a}{m}, \frac{b}{m}] \) with \( m > 0, g < 0 \), and \( h > 0 \). Then, the dual cone of \( \sigma = C(I) \) equals \( \sigma' = \langle [-m, h], [m, -g] \rangle \subseteq (\mathbb{Q} \times \mathbb{Q}_{>0}) \cup \{[0, 0]\} \). On the other hand, the central degree \( \overline{R} \) coincides with \([0, 1]\), and it is obvious that this is irreducible even within the semigroup \((\mathbb{Z} \times \mathbb{Z}_{>0}) \cup \{[0, 0]\}\).

(\(\Rightarrow\)) Let \( I = [\frac{a}{m}, \frac{b}{m}] \) with \( m > 0 \) and \( g < h \), i.e. \( \sigma' = \langle [-m, h], [m, -g] \rangle \). We are going to show that we can obtain \( g < 0 \) and \( h > 0 \) by an integral shift of \( I \). Obviously, we can assume that \( 0 < h < m \) implying that \([-1, 1] \in \text{int} \langle [-m, h], [0, 1] \rangle \subseteq \text{int} \sigma' \).

On the other hand, if we had \( g > 0 \), then this would similarly imply that \( [1, 0] \in \text{int} \langle [0, 1], [m, -g] \rangle \subseteq \text{int} \sigma' \).

Then \( \overline{R} = [0, 1] = [-1, 1] + [1, 0] \) would be a decomposition within \( \text{int} \sigma' \). \( \square \)

Grounded intervals can always be shifted by integers to look like \( I = [-A, B] \) with \( A, B \in \mathbb{Q}_{>0} \) (sharing the same denominator). Then, if \( \nu \) denotes the central index, we can directly express the invariants \( \eta_\nu \) and \( a_\nu \) from (2.4) in terms of \( I \):
Proposition 13. Let $I = [-A, B]$ with $A, B \in \mathbb{Q}_{>0}$ be a (grounded) interval with uniform denominators. Then

$$\eta_\nu = 1 + \min \{|A| + B, A + |B|\}, \quad \alpha_\nu = 2 + |A| + |B|, \quad \text{and} \quad |I| = A + B.$$ 

Proof. With $A = \frac{a}{m}$ and $B = \frac{h}{m}$, we have $\sigma^\nu = \langle [-m, h], [m, -g] \rangle$ as usual. Now, since $r^\nu = R = [0, 1]$ and $\{r^{\nu - 1}, r^\nu\}$ forms a basis of $\mathbb{Z}^2$, we know that $r^{\nu - 1} = [1, \bullet]$, and it has to be the lowest lattice point above the ray $\mathbb{Q}_{\geq 0} \cdot [m, -g] = \mathbb{Q}_{\geq 0} \cdot [1, 1]$. Thus, $r^{\nu - 1} = [1, [A] + 1]$ and, similarly, $r^{\nu + 1} = [-1, [B] + 1]$. Now, the claim for $\eta_\nu = \min \{\frac{\alpha \cdot r^{\nu + 1}}{(\alpha, r^\nu)}, \frac{(\beta, r^{\nu - 1})}{(\beta, r^\nu)}\}$ follows from $\alpha = m \cdot (-A, 1)$ and $\beta = m \cdot (B, 1)$. \hfill $\square$

3. The dualizing sheaf on infinitesimal deformations of $S$

The (isomorphism classes of) infinitesimal $k[\varepsilon]$-deformations (with $\varepsilon^2 = 0$) of a $k$-algebra are gathered in a vector space called $T^1$, see [Ste03] for a detailed introduction to deformation theory. In case of toric varieties such as $S = \mathbb{T}(\sigma)$ from \eqref{eq:torus}, the torus $\mathbb{T} := \text{Spec} k[M]$ acts on the variety, on the functions, and on all naturally defined modules. In particular, the vector space $T^1$ becomes $M$-graded. This can be made explicit by comparing the $M$-degrees of the defining equations $f$ with those of the perturbation $g$ arising in $f + \varepsilon g$, cf. \eqref{eq:deformations}. Thus, the distribution along the degrees of $M$ becomes the essential information. We will study the dualizing sheaf $\omega_X$ on the total spaces $X = X_\xi$ for homogeneous elements $\xi \in T^1(S)$.

3.1. Degrees carrying $T^1$. Let $\sigma$ be a two-dimensional cone – we will adopt the notation of \eqref{eq:torus} and \eqref{eq:cone}. The dimensions of the homogeneous components $T^1(S, -R)$ ($R \in M$) of the finite-dimensional vector space $T^1(S)$ (abbreviated as $T^1(-R) \subseteq T^1$) are, \eqref{eq:degrees}:

\begin{itemize}
  \item[(i)] $R = r^2$ or $R = r^{e - 1}$: $\dim_k T^1(-R) = 1$,
  \item[(ii)] $R = r^i$ for $i = 3, \ldots, e - 2$: $\dim_k T^1(-R) = 2$, and
  \item[(iii)] $R = k \cdot r^i$ for $i = 2, \ldots, e - 1$ with $2 \leq k \leq a_i - 1$: $\dim_k T^1(-R) = 1$.
\end{itemize}

We would like to recall Pinkham’s method to obtain this – this approach will also provide the major tool for our own calculations of $\omega_X$. However, unlike the original reference, we will consequently use the toric language. It leads to a slightly more structured description than just naming the dimensions.

3.1.1. Puncturing. The main point is to consider deformations of the smooth, but non-affine $S \setminus 0$ first. They are always locally trivial, and some of them lift to deformations of $S$. The exact statement for the $k[\varepsilon]$-level is encoded in the exact sequence

$$0 \rightarrow T^1 \rightarrow H^1(S \setminus 0, \mathcal{O}_S) \rightarrow H^1(S \setminus 0, \mathcal{O}_S^*)$$

where the latter map is given by $\sum_{i=1}^e dx^{r^i}$. For the upcoming calculations it is helpful to use this sequence for redoing the calculation of $T^1(-R)$ for $R \in M$. Moreover, since we have a very nice open affine covering $S \setminus 0 = \mathbb{T}(\alpha) \cup \mathbb{T}(\beta)$
where we identify $\alpha$ and $\beta$ with the rays they are generating, hence $TV(\alpha)$ and $TV(\beta)$ are defined similarly to $TV(\sigma)$ in \([2.3]\). Since

$$TV(\alpha) \cap TV(\beta) = TV(\alpha \cap \beta) = TV(0) = \text{Spec } k[M] = T,$$

this is easily done by using Čech cohomology:

### 3.1.2. \(H \) coboundaries are generated by the monomials $x^r$ (iii) where $(\beta, R)$ are defined similarly to $(\alpha, R)$ in \([2.3]\).

Note that these cases include $(\alpha, R) = (0, 1)$ or $(\beta, R) = (0, 1)$.

### 3.1.3. \(H \) coboundaries are generated by the monomials $x^r$ (ii) where this does not happen at all yielding $H^1(\sigma^\vee)$ (and $= 0$ otherwise).

### 3.1.4. The kernel. The $i$-th summand $dx^i$ maps $x^r \partial_a$ to $(a, r^i) \cdot x^{-r^i+1}$. In particular, whenever $R - r^i \in \text{int } \sigma^\vee$ enforces $a = 0$. Using the numbering of the beginning of this section (iii) where this does not happen at all yielding $T^1(-R) = N_k$, and

(iii) where $(k \cdot r^i) - r^i = (k - 1) \cdot r^i \in \text{int } \sigma^\vee$ leads to the single condition $\langle a, r^i \rangle = 0$.

There are left with a one-dimensional $T^1(-R) = (r^i)^\perp N_k$.

### Case 2. Assume that $\langle \alpha, R \rangle = 1$ and $\langle \beta, R \rangle \geq 2$. Then, either $\langle \beta, R \rangle \geq \langle \beta, r^1 \rangle = n$, i.e. $R - r^i \in \text{int } \sigma^\vee$ implying the condition $\langle a, r^1 \rangle = 0$ forcing $a \in N/\alpha \mathbb{Z}$ to become $0$, or, using the numbering of (3.1) again,

(i) $R = r^2$ with $T^1(-R) = N_k/k \cdot \alpha$.

The case $\langle \alpha, R \rangle \geq 2$, $\langle \beta, R \rangle = 1$ (yielding $R = r^e-1$) works similar.

### 3.2. The construction of $X_k \setminus 0$. Let $\xi \in H^1(S \setminus 0, \theta_S)$ be given by the 1-Čech cocycle $\xi = x^r \partial_a \in \Gamma(T, \theta_S|_T) = k[M] \otimes N$, cf. \([3.1.3]\). The associated infinitesimal
deformation \( X_\xi \setminus 0 \) of \( S \setminus 0 \) arises from glueing the trivial pieces \( TV(\alpha) \otimes k[\varepsilon] \) and \( TV(\beta) \otimes k[\varepsilon] \) along the \( k[\varepsilon] \)-algebra map

\[
\begin{array}{ccc}
k[\alpha^\vee \cap M, \varepsilon] & \xrightarrow{\varphi_\xi} & k[\beta^\vee \cap M, \varepsilon] \\
\downarrow & & \downarrow \\
k[M, \varepsilon] & \xrightarrow{\varphi_\xi} & k[M, \varepsilon]
\end{array}
\]

with \( \varphi_\xi(x^\tau) := x^\tau + \varepsilon \cdot \xi(x^\tau) = x^\tau + \varepsilon \cdot \langle a, r \rangle \cdot x^{R-\tau} \). Note that we have decided to use the notation \( X_\xi \setminus 0 \) even in the case when there is no extension of this to some deformation \( X_\xi \) of the non-punctured \( S \).

3.3. The dualizing sheaf on \( X_\xi \setminus 0 \). Let \( \xi = x^{-R} \partial_a \) as before. Since \( X_\xi \setminus 0 \) is smooth over \( \text{Spec} \, k[\varepsilon] \) and since \( \omega_{k[\varepsilon]} = k[\varepsilon] \), it follows from \([\text{Har66}, \text{p.140}]\) that

\[
\omega_{X_\xi,0} = \omega_{(X(0)|k[\varepsilon]} = \Lambda^2 \Omega^1_{(X(0)|k[\varepsilon]}.
\]

Choosing a \( \mathbb{Z} \)-basis \( \{ A, B \} \) of \( M \), the local pieces of the latter equal

\[
\omega_\alpha = \oplus_{\langle a, r \rangle \geq 1} k[\varepsilon] \cdot x^\tau \cdot \frac{dx^A}{x^A} \wedge \frac{dx^B}{x^B} \cong \oplus_{\langle a, r \rangle \geq 1} k[\varepsilon] \cdot x^\tau \subseteq k[M, \varepsilon]
\]

and similarly for \( \omega_\beta \), cf. \([\text{CLS11}, \text{Prop. 8.2.9}]\). Note that the isomorphism does, up to sign, not depend on the choice of \( \{ A, B \} \). Now, we determine the impact of the \( k[\varepsilon] \)-algebra isomorphism \( \varphi_\xi^{[0]} := \varphi_\xi \) on the glueing \( \varphi_\xi^{[1]} \) of the modules \( \omega_\alpha |_T \) and \( \omega_\beta |_T \).

Since

\[
\frac{dx^A}{x^A} \mapsto \frac{dx^A + \varepsilon \langle a, A \rangle x^{A-R}}{x^A + \varepsilon \langle a, A \rangle x^{A-R}} = \frac{dx^A}{x^A} + \varepsilon \langle a, A \rangle \, dx^{R}.
\]

we obtain that \( x^\tau \cdot \frac{dx^A}{x^A} \wedge \frac{dx^B}{x^B} \) maps to

\[
\left( x^\tau \cdot \frac{dx^A}{x^A} \wedge \frac{dx^B}{x^B} \right) + \varepsilon x^{R} \left( \langle a, r \rangle \frac{dx^A}{x^A} \wedge \frac{dx^B}{x^B} + \langle a, A \rangle \frac{dx^{R}}{x^A} \wedge \frac{dx^B}{x^B} + \langle a, B \rangle \frac{dx^A}{x^A} \wedge \frac{dx^{R}}{x^B} \right).
\]

Expressing \( R \) within the basis \( \{ A, B \} \) (and suppressing \( \frac{dx^A}{x^A} \wedge \frac{dx^B}{x^B} \)) finally yields

\[
\varphi_\xi^{[1]} : \omega_\alpha |_T \ni x^\tau \longmapsto x^\tau + \varepsilon \cdot \langle a, r - R \rangle \cdot x^{R} \in \omega_\beta |_T.
\]

This description enables us to determine the class \( [\omega_{X_\xi,0}] \in H^1(S \setminus 0, \mathcal{O}_X^\ast) \). If \( x^s \) and \( x^t \) are generators of \( \omega_\alpha \) and \( \omega_\beta \), respectively, i.e. if \( \langle \alpha, s \rangle = \langle \beta, t \rangle = 1 \), then \( [\omega_{X_\xi,0}] \) is represented by the 1-Cech cocycle \( \varphi(x^s)/x^t \in \Gamma(T, \mathcal{O}_T^s \otimes k[\varepsilon]) \). It is equal to

\[
\psi_1 := x^{s-t} + \varepsilon \cdot \langle a, s - R \rangle \cdot x^{s-t-R}.
\]

Similarly, we might consider the glueing map \( \varphi_\xi^{[g]} \) for a reflexive power \( \omega_{X_\xi,0}^{[g]} \) instead of just for \( \omega_{X_\xi,0} \). Then, the previous calculations yield

**Lemma 14.** \( \varphi_\xi^{[g]} : x^\tau \mapsto x^\tau + \varepsilon \cdot \langle a, r - gR \rangle \cdot x^{R-\tau} \), and the 1-Cech cocycle becomes

\[
\psi_g := x^{s(g)-t(g)} + \varepsilon \cdot \langle a, s(g) - gR \rangle \cdot x^{s(g)-t(g)-R} \in \Gamma(T, \mathcal{O}_T^s \otimes k[\varepsilon])
\]

with \( s(g), t(g) \in M \) satisfying \( \langle \alpha, s(g) \rangle = \langle \beta, t(g) \rangle = g \).
Note that one might take, if some \( s = s(1) \) and \( t = t(1) \) are available, the multiples \( s(g) = g \cdot s \) and \( t(g) = g \cdot t \) for a general \( g \in \mathbb{Z} \). However, in (4.3) we will prefer a different choice for \( g = m \).

### 3.4. Extending functions along codimension two

Let \( S = TV(\sigma) \) be as before. Since it is normal, it carries the Hartogs property \( S_2 \) as it was asked for in (1.1). Now, if \( A \) is an Artinian \( k \)-algebra and \( X \) is a deformation of \( S \) over \( A \) (we just need the case \( A = k[\varepsilon] \) here), we would like to keep this property.

**Lemma 15.** If \( F \) is a reflexive \( O_X \)-module, i.e. if \( F = \text{Hom}_{O_X}(G, O_X) \) for some \( O_X \)-module \( G \), then \( \Gamma(S, F) \to \Gamma(S \setminus 0, F) \) is an isomorphism.

**Proof.** It suffices to check this for \( F = O_X \). We proceed by induction. Choosing a non-trivial element \( \varepsilon \in A \) with \( \varepsilon \cdot m_A = 0 \), we obtain an exact sequence of \( A \)-modules

\[
0 \to k \xrightarrow{\varepsilon} A \to \overline{A} \to 0.
\]

Denoting \( \overline{X} := X \otimes_A \overline{A} \), flatness, restriction to \( S \setminus 0 \), and taking global sections provides the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Gamma(S, O_S) & \xrightarrow{\varepsilon} & \Gamma(S, O_X) & \longrightarrow & \Gamma(S, O_{\overline{X}}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(S \setminus 0, O_S) & \xrightarrow{\varepsilon} & \Gamma(S \setminus 0, O_X) & \longrightarrow & \Gamma(S \setminus 0, O_{\overline{X}}) & \longrightarrow & \Omega^1(S \setminus 0, O_S).
\end{array}
\]

Now, the claim follows from the 5-lemma. \( \square \)

### 3.5. The dualizing sheaf on \( X_\xi \)

In contrast to (3.2) we now start with a

\[
\xi = x^{-R} \partial_a \in T^1(-R) \subseteq \Omega^1(S \setminus 0, \theta_S)(-R) \leftrightarrow k[M] \otimes_\mathbb{Z} N
\]

yielding a true \( X = X_\xi \) and not just a punctured \( X_\xi \setminus 0 \). The extension theorem along two-codimensional subsets gives us the right tool to understand \( \omega_X^{[g]} \) out of \( \omega_{X \setminus 0}^{[g]} \), we have obtained in (3.3), namely

\[
\omega_X^{[g]} = \Gamma(S \setminus 0, \omega_X^{[g]}) = \{ f = \sum_{a \geq g} (c_r + \varepsilon d_r) x^r | \varphi^{[g]}_\xi(f) \in \oplus_{a \geq g} k[\varepsilon] \cdot x^r \}.
\]

From Lemma 14 we know that

\[
\varphi^{[g]}_\xi\left( \sum_r (c_r + \varepsilon d_r) x^r \right) = \sum_r c_r x^r + \varepsilon \cdot \left( \sum_r d_r x^r + c_r(a, r - gR) \cdot x^{r-R} \right),
\]

and the combination of these two statements yields

**Lemma 16.** The sum \( f = \sum_{r \in M} (c_r + \varepsilon d_r) x^r \) belongs to \( \omega_X^{[g]} \) if and only if the following two assertions hold

(i) \( \langle a, r \rangle < g \) implies \( c_r = d_r = 0 \)

(ii) \( \langle \beta, r \rangle < g \) implies \( c_r = 0 \) and \( d_r = c_{r+R} \cdot \langle a, (g-1)R - r \rangle \).
Lemma 17. $\xi = x^{-R} \partial_a$ satisfies $(*)_g \iff$ each $r \in M$ with both 

$$g \leq \langle \alpha, r \rangle < g + \langle \alpha, R \rangle \quad \text{and} \quad g \leq \langle \beta, r \rangle < g + \langle \beta, R \rangle$$

leads to $\langle a, gR - r \rangle = 0$.

Proof. The second part of Condition (ii) for $f \in \omega^g_X$ in Lemma 16 can be read as that 

$$\langle \beta, r - R \rangle < g$$

implies $d_{r-R} = c_r \cdot \langle a, gR - r \rangle$. Hence, together with $\langle \alpha, r - R \rangle < g$, this would enforce that $c_r \cdot \langle a, gR - r \rangle = 0$. On the other hand, if $\langle \alpha, r \rangle, \langle \beta, r \rangle \geq g$, then $c_r \neq 0$ is allowed in $\omega^g_S$.

For given $R \in M$ and $g \in \mathbb{Z}$ we define the following zones within $M_Q$:

$$Z_{R,g} := \{ r \in M_Q \mid g \leq \langle \alpha, r \rangle < g + \langle \alpha, R \rangle \quad \text{and} \quad g \leq \langle \beta, r \rangle < g + \langle \beta, R \rangle \}. $$

Then, for $\xi = x^{-R} \partial_a$, the previous lemma says that

$$(*)_g \iff \mathcal{R}^g_\xi \text{ is surjective} \iff Z_{R,g} \cap M \subseteq a^\perp + gR.$$

Actually, up to the point that $X_\xi$ does not make sense otherwise, we did not use $\xi \in T^1$ so far. This property is equivalent to $(*)_0$, and it will be discussed in (4.2).

4. V-DEFORMATIONS

Let $\sigma = \langle \alpha, \beta \rangle$ be as before, e.g. it can be obtained as the cone $C(I)$ over an interval with uniform denominators $I = \left[ \frac{a}{m}, \frac{b}{m} \right]$ as in (2.5).

In the present section, we will approach the V-deformations of $S = TV(\sigma)$ defined in (1.2.2). Since $\text{index}(\omega_S) = m$, we will mostly study the property $(*)_m$ for a given infinitesimal deformation $\xi = x^{-R} \partial_a$.

4.1. Shifting the zones. Recall from Remark 8 that $\frac{r^\perp + r}{n} = \frac{1}{m} \cdot \overline{R} \in M_Q$ is the truly canonical (but rational) degree. Moreover, depending on $R \in M$ we denote

$$Z_R := Z_{R,0} = \sigma^\perp \cap (R - \text{int } \sigma^\perp) \subseteq M_Q.$$ 

The degrees $R$ we are interested in are always elements of $\text{int } \sigma^\perp$. In particular, $Z_R$ is then a bounded region – it is a half-open parallelogram having 0 and $R$ as opposite vertices. While these two vertices belong to the lattice $M$, the remaining ones usually do not. The relation to the zones $Z_{R,g}$ from (3.6) is

$$Z_{R,g} = \frac{g}{m} \overline{R} + Z_R \quad \text{for every } g \in \mathbb{Z}.$$
In particular, $Z_{R,g+m} = \overline{R} + Z_{R,g}$, i.e. the zones $Z_{R,g+m}$ just differ by integral translation. This gives rise to define the “stable” condition

$$(*)_g := \bigcap_{\ell \in \mathbb{Z}} (*)_g + \ell m$$

(still being a condition for $\xi = x^{-R} \partial_a$).

That is, the condition that $(*)_g$ is true for all $g \in \mathbb{Z}$ can be replaced by the finite one asking for $(*)_g$ for all $g \in \mathbb{Z}/m\mathbb{Z}$. Moreover, to be a V-deformation in the sense of (1.2.2) means to fulfill the condition $(*)_0$.

**Proposition 18.** 1) If $Z_{R,g} \cap M = \emptyset$, then $(*)_g$ is fulfilled for each $a \in \mathbb{N}$.

2) Assume that $Z_{R,g} \cap M \neq \emptyset$. Then the following conditions are equivalent:

$$(*)_g \iff (*)_g + \ell m \text{ for two different } \ell \in \mathbb{Z} \iff (*)_g \text{ and } a \in (\overline{R} - mR)^\perp.$$ 

**Proof.** Let $\ell \in \mathbb{Z}$. If $r \in Z_{R,g} \cap M$, then $r + \ell \overline{R} \in Z_{R,g+\ell m}$. Hence, the conditions $(*)_g$ and $(*)_g + \ell m$ mean that

$$\langle a, gR - r \rangle = 0 \quad \text{and} \quad \langle a, (g+\ell m)R - (r + \ell \overline{R}) \rangle = 0,$$

respectively. However, the difference of the two left hand sides equals $\langle a, -\ell mR + \ell \overline{R} \rangle = \ell \cdot \langle a, \overline{R} - mR \rangle$. \hfill $\square$

**Corollary 19.** 1) If $\xi = x^{-R} \partial_a$ is a V-deformation, then $a \in (\overline{R} - mR)^\perp$. (See the upcoming Corollary 21 for a stronger statement.)

2) Assume that $a \in (\overline{R} - mR)^\perp \setminus \{0\}$. Then, for any $g \in \mathbb{Z}$, $(*)_g$ (or even $(*)_g$) is equivalent to $(Z_{R,g} \cap M) - gR \subseteq \mathbb{Q} \cdot (\overline{R} - mR)$. Likewise, this condition is equivalent to $(Z_{R,g} \cap M) - \frac{a}{m} \overline{R} \subseteq \mathbb{Q} \cdot (\overline{R} - mR)$.

**Proof.** (1) follows from the fact that there are non-empty $(Z_{R,\ell m} \cap M)$ whenever $R \in \text{int} \sigma^\vee$ (and only those $R$ matter for $T^1(-R) \neq 0$): Just take $\ell = 0$.

(2) $(*)_g$ means that for each $r \in Z_{R,g} \cap M$ we have $a \in (r - gR)^\perp$. Together with $a \in (\overline{R} - mR)^\perp$ this means that $a$ can be non-trivial if and only if both $r - gR$ and $\overline{R} - mR$ are collinear. Moreover, $\overline{R} - mR$ does never vanish (since $T\overline{V}(\sigma) \neq A_k$). \hfill $\square$

### 4.2. Focusing on $T^1$-degrees.

For investigating the $(*)_g$ property we did not use yet that the set of degrees $R \in M$ with $T^1(-R) \neq 0$ is very restricted. Taking this into account implies

**Lemma 20.** Every deformation $x^{-R} \partial_a \in T^1(-R)$ satisfies $(*)_0$.

**Proof.** Actually, this statement is trivial – the condition $(*)_0$ means that $\omega_X^{[0]} = O_X$ is flat over $k[\varepsilon]$, i.e. it even characterizes the elements of $T^1(-R)$. Nevertheless, e.g. to practice our new language involving the zones $Z_R$, we would like to present a direct argument, too:

Condition $(*)_0$ means $Z_R \cap M \subseteq a^\perp$ with $Z_R = \sigma^\vee \cap (R - \sigma^\vee)$. According to (3.1), we distinguish between two cases:

(i)+(ii) $R = r^i$ with $i = 2, \ldots, e - 1$: Since these elements are irreducible in the
semigroup $\sigma^\vee \cap M$, we obtain $Z_R \cap M = \{0\}$, and this belongs to every $a\perp$.

(iii) $R = k \cdot r^i$ for $i = 2, \ldots, e - 1$ with $2 \leq k \leq a_i - 1$: Here we have

$$Z_R \cap M = \{0, r^i, \ldots, (k - 1)r^i\},$$

i.e. Condition $(*)_0$ means $\langle a, r^i \rangle = 0$. However, by (3.1.4), Case 1, exactly this is ensured to hold true within $T^1(-k \cdot r^i)$.

□

Remark. Actually, the condition

$$(*)_0 \iff Z_R \cap M \subseteq a\perp \iff a \in (Z_R \cap M)\perp$$

should be understood as an alternative description of $T^1(-R)$. However, this is not new – it coincides with the description in [Alh00] (2.2). There, one has defined the finite subsets

$$E^R_\alpha := \{ r \in E \mid \langle \alpha, r \rangle < \langle \alpha, R \rangle \} \quad \text{and} \quad E^R_\beta := \{ r \in E \mid \langle \beta, r \rangle < \langle \beta, R \rangle \}$$

of $M$, and this lead to an exact sequence

$$0 \to T^1(-R) \to (\text{span}_k E^R_\alpha \cap \text{span}_k E^R_\beta)^* \to \text{span}_k(E^R_\alpha \cap E^R_\beta)^* \to 0.$$

In particular, $T^1(-R)$ is a subquotient of $N_k$, and $a \in T^1(-R)$ if and only if $a \in (E^R_\alpha \cap E^R_\beta)^\perp$. Now, the relation to our condition $(*)_0$ is that $E^R_\alpha \cap E^R_\beta = Z_R \cap E$.

Corollary 21. $\xi = x^{-R}\partial_\alpha \in T^1(-R)$ is a $V$-deformation, i.e. it fulfills the stable condition $(*)_0$, if and only if $a \in (R - mR)^\perp$.

Proof. This follows from Proposition 13(2). The implication $(\Rightarrow)$ was already stated in Corollary 19. The reversed implication $(\Leftarrow)$ makes use of Lemma 20 □

4.3. Counting $V$-deformations. We run through the list (i)-(iii) of (3.1) and especially (3.1.4) to determine $(R - mR)^\perp = (r^1 + r^e - nR)^\perp$, i.e. the $V$-deformations within each homomorphic summand $T^1(-R)$.

(i) $R = r^2$ (and similarly $R = r^{e-1}$): $T^1(-r^2) = N_k/k \cdot \alpha$. The element $r^1 + r^e - nr^2$ is contained in $\alpha \perp \subseteq M$, hence it provides a linear map $N/\mathbb{Z} \cdot \alpha \to \mathbb{Z}$ where $T^1_k(-r^2)$ is generated by the kernel. Since we had excluded the $A_{n-1}$-singularity, this linear map is also non-trivial, i.e. there is no $V$-deformations in degree $-r^2$ and $-r^{e-1}$.

(ii) $R = r^i$ for $i = 3, \ldots, e - 2$: $T^1(-r^i) = N_k$. We know that $r^1 + r^e - nr^i$ is again non-trivial, hence it provides a one-dimensional kernel within the two-dimensional $T^1(-r^i)$. Altogether, this yields an $(e - 4)$-dimensional space of $V$-deformations.

(iii) $R = k \cdot r^i$ for $i = 2, \ldots, e - 1$ with $2 \leq k \leq a_i - 1$: $T^1_k(-kr^i) = (r^1 + r^e, r^i)^\perp$, i.e. this is non-trivial if and only if $r^1 + r^e \in \mathbb{N} \cdot r^i$, i.e. if $\sigma$ is grounded (see Definition 11) and $r^i = \overline{R}$ is the central degree (i.e. $i = \nu$ is the central index). If this is the case, and if $\sigma$ stems from an interval $I = [-A, B]$ as in Proposition 13 then we gather another $a_\nu - 2 = |A| + |B|$ $V$-deformations.
4.4. Representing $T^1_V$ as a kernel. An alternative approach to visualize the $V$-deformations of $S = TV(\sigma)$ is to consider the following map $\Phi : T^1 \to H^2_0(S, \mathcal{O}_S)$: If $\xi \in T^1$ is represented by an infinitesimal deformation $X = X_\xi \to \text{Spec} \ k[\varepsilon]$, then there is an exact sequence
\[
0 \to \mathcal{O}_S \to \mathcal{O}_X^* \to \mathcal{O}_S^* \to 1
\]
and
\[
f \mapsto 1 + \varepsilon f.
\]
Since the map $H^0(S \setminus 0, \mathcal{O}_X^*) \to H^0(S \setminus 0, \mathcal{O}_S^*)$ equals $\mathcal{O}_X^* \to \mathcal{O}_S^*$ on the affine $S$, i.e. it is notably surjective, this implies the exactness of
\[
0 \to H^1(S \setminus 0, \mathcal{O}_S) \to H^1(S \setminus 0, \mathcal{O}_X^*) \to H^1(S \setminus 0, \mathcal{O}_S^*)
\]
Thus, using that $\omega_{[m]}^S$ is trivial, we may define $\Phi(\xi)$ as the class of $\omega_{[m]}^X$ in
\[
H^2_0(S, \mathcal{O}_S) = H^1(S \setminus 0, \mathcal{O}_S) = \ker \left( H^1(S \setminus 0, \mathcal{O}_X^*) \to H^1(S \setminus 0, \mathcal{O}_S^*) \right)
\]
Comparing with the flatness part of the definition at the end of (1.2), it follows that the kernel $\ker \Phi \subseteq T^1$ consists exactly of the deformations satisfying $(*)_m$, i.e., of the $V$-deformations, cf. (1.2.2).

It turns out that $\Phi$ can be extended to $H^1(S \setminus 0, \theta_S)$, i.e. we consider (locally trivial) deformations $X_\xi \setminus 0$ of (the smooth) $S \setminus 0$ again. Using the descriptions of $H^1(S \setminus 0, \theta_S)$ and $H^1(S \setminus 0, \mathcal{O}_S)$ given in (3.1.2) and (3.1.3) respectively, the final result fits perfectly with Corollary 21.

**Proposition 22.** Let $R \in \text{int} \, \sigma' \cap M$. Then, the degree $-R$ part of $\Phi$
\[
\Phi(-R) : H^1(S \setminus 0, \theta_S)(-R) \to H^1(S \setminus 0, \mathcal{O}_S)(-R)
\]
is given by $x^{-R} \partial_s \mapsto \langle a, \overline{R} - mR \rangle \cdot x^{-R}$. In other words, using the natural maps $N_k \to H^1(S \setminus 0, \theta_S)(-R)$ and $H^1(S \setminus 0, \mathcal{O}_S)(-R) = k$, the map $\Phi(-R) : N_k \to k$ equals $\overline{R} - mR \in M$.

**Proof.** In (3.3) we have dealt with 1-cocycles of $\mathcal{O}_X^*$, and in Lemma 14 we have obtained an element $\psi_m$ describing the class of $\omega_{[m]}^X$ after using the surjection $\Gamma(T, \mathcal{O}_{T \otimes k[\varepsilon]}) \to H^1(S \setminus 0, \mathcal{O}_X^*)$. Restricting $\psi_m$ via $\varepsilon \mapsto 0$ to $\Gamma(T, \mathcal{O}_T^*) \to H^1(S \setminus 0, \mathcal{O}_S^*)$ yields $x^{s(m) - t(m)}$.

Since $\omega_{[m]}^S = \mathcal{O}_S$, this is a 1-Cech coboundary. One can see this directly by the possibility of choosing $s(m) = t(m) = \overline{R}$ then $x^{s(m) - t(m)}$ becomes 1 right away. Applying this recipe to the original cocycle $\psi_m$ of Lemma 14 as well, we obtain that
\[
\psi_m = 1 + \varepsilon \cdot \langle a, \overline{R} - mR \rangle \cdot x^{-R}.
\]
Recall that the second map within the exact sequence
\[
0 \to \mathcal{O}_{S \setminus 0} \to \mathcal{O}_{X \setminus 0}^* \to \mathcal{O}_{S \setminus 0}^* \to 1
\]
sends $f \mapsto 1 + \varepsilon \cdot f$. Thus, $\Phi(\xi) = [\omega^{[m]}_{X\setminus 0}] \in H^1(S \setminus 0, \mathcal{O}_S)(-R)$ is given by the 1-Čech cocycle $\langle a, \overline{R} - mR \rangle \cdot x^{-R} \in k[M] = \Gamma(T, \mathcal{O}_S)$. □

5. QG- and VW-deformations

V-deformations are understood, by Corollary 21 and (4.3). Next we will turn to the stronger qG- and VW-deformations, cf. (1.2.1) and (1.2.4) for the definition of these notions.

5.1. Extending the lattice $M$. We adopt the notation from (2.5). For $I = [g_m, h_m]$ we know that $1 + \varepsilon \cdot R \in M$ is the truly canonical, but rational degree. This gives rise to an enlargement of our lattice $M$, namely

$$\tilde{M} := M + \mathbb{Z} \cdot \frac{1}{m} \overline{R}.$$ 

Let us assume that $\xi = x^{-R}(\partial_a) \in T^1(-R)$ is a V-deformation, i.e. $\langle a, \overline{R} - mR \rangle = 0$. Using the zone $Z_R = \sigma^\vee \cap (R - \text{int } \sigma^\vee)$ defined in (4.1), we obtain

**Proposition 23.** 1) $\xi$ is a qG-deformation $\iff \tilde{M} \cap Z_R \subseteq \mathbb{Q} \cdot (\overline{R} - mR)$, and 2) $\xi$ is a VW-deformation $\iff (M + \frac{1}{m} \overline{R}) \cap Z_R \subseteq \mathbb{Q} \cdot (\overline{R} - mR)$.

Note that the difference between both cases just arises from the tiny difference between $\tilde{M} = M + \mathbb{Z} \cdot \frac{1}{m} \overline{R}$ and $M + \frac{1}{m} \overline{R}$.

**Proof.** Recall from (4.1) that $Z_{R,g} = \frac{g}{m} \overline{R} + Z_R$ for $g \in \mathbb{Z}$. Thus, Corollary 19(2) says that the conditions $(*)_g$ and $(*)_{g'}$ are equivalent to

$$Z_R \cap (M - \frac{g}{m} \overline{R}) = ((Z_R + \frac{g}{m} \overline{R}) \cap M) - \frac{g}{m} \overline{R} \subseteq \mathbb{Q} \cdot (\overline{R} - mR).$$

While $g = -1$ directly leads to (2), one uses $\bigcup_{g \in \mathbb{Z}} (M - \frac{g}{m} \overline{R}) = \tilde{M}$ for (1). □

Now, we are going to scan the degrees of $T^1_V$ listed in (4.3)(ii) and (iii) for qG- and VW-deformations. (Note that the deformations in (4.3)(i) are not even V-deformations.)

Actually, it is convenient to proceed with a minor change to the division into the two cases: We will shift (and this applies only to the grounded case) the central degree $\overline{R} = r^\nu$ from Class (ii) to (iii). Thus, in (ii) we now collect exactly the non-central $R = r^i$ ($i = 3, \ldots, e - 2$), and Class (iii) will gather all $R = k \cdot r^i$ with $1 \leq k \leq a_\nu - 1$. Note that this set is empty unless $\sigma$ is grounded, i.e. $r^\nu = \overline{R}$.

5.2. The degrees of (4.3)(ii). Let $R = r^i$ with $i = 3, \ldots, e - 2$ be a non-central degree. The latter property can be expressed by

$$\frac{1}{m} \overline{R} \notin \mathbb{Q} \cdot (\overline{R} - mr^i).$$

On the other hand, we know that

$$\langle \alpha, \frac{1}{m} \overline{R} \rangle = \langle \beta, \frac{1}{m} \overline{R} \rangle = 1 \text{ and } \langle \alpha, r^i \rangle, \langle \beta, r^i \rangle > 1.$$
which implies that \( \frac{1}{m} R \in \sigma' \cap (r^i - \text{int } \sigma') = Z_{r^i} \). Hence,

\[
\frac{1}{m} R \in (M + \frac{1}{m} R) \cap Z_{r^i}.
\]

Applying Proposition 23(2), this shows that the deformations of degree \( r^i \) cannot be VW-deformations, let alone qG-deformations. In other words, the property of being a grounded singularity is a necessary condition for the existence of VW- or qG-deformations.

5.3. The degrees of (4.3)(iii). Let \( \sigma \) be a grounded cone with central degree \( R = r^\nu \). From (2.5) and (2.6) we know that \( \sigma = \langle \alpha, \beta \rangle \) can be obtained as \( C(I) = ((g, m), (h, m)) \) from a grounded interval \( I = [\frac{g}{m}, \frac{h}{m}] = [-A, B] \) with \( m > 0, g < 0 < h \), and \( \gcd(g, m) = \gcd(h, m) = 1 \). In particular, \( A, B \in \mathbb{Q}_{>0} \), the central degree \( R \in M \) becomes \( [0, 1] \in \mathbb{Z}^2 \), and \( \sigma' = \langle [-m, h], [m, -g] \rangle \).

Let \( R = k \cdot R \) with \( k = 1, \ldots, (a_\nu - 1) = 1 + [A] + [B] \) (cf. Proposition 13). Note that we have included the case \( k = 1 \) originally belonging to (4.3)(ii). The zone \( Z_{kR} \) is the half open parallelogram in \( M_Q = \mathbb{Q}^2 \) with the vertices

\[
[0,0], \quad \frac{k}{h-g} \cdot [m, -g], \quad [0,k], \quad \frac{k}{h-g} \cdot [-m, h].
\]

Moreover, the line \( Q \cdot (R - mR) = Q \cdot R \) we are interested in by Proposition 23 is given by the diagonal \( [0,0][0,k] \).

5.3.1. qG-deformations. From (4.3)(iii) we know that each \( k = 1, \ldots, (a_\nu - 1) \) gives rise to a one-dimensional \( T^1_{qG}(-k \cdot R) = R^1 \subseteq N_k \). For each of these \( k \) we have to decide whether \( T^1_{qG}(-k \cdot R) = 0 \) or \( R^1 \).

**Proposition 24.** The qG-deformations of \( S \) consist exactly of the one-dimensional subspaces \( R^1 \subseteq T^1(-k \cdot R) \) with \( 1 \leq k \leq \min\{a_\nu - 1, |I|\} \).

**Proof.** We consider the embedding \( \iota : \tilde{M} \rightarrow \mathbb{Z}^2 \) obtained by evaluating \( \langle \alpha, \beta \rangle \). Actually, restricting to \( M = \mathbb{Z}^2 \), this reflects the original situation of \( M = (\mathbb{Z}^2)^G \), and \( \iota|_M \) is given by the matrix \( \begin{pmatrix} g & m \\ h & m \end{pmatrix} \). The rational \( \iota_Q \) is an isomorphism, we can detect \( \iota_Q(\sigma^\nu) = Q_{\geq 0} \), and the new, truly canonical degree \( \frac{1}{m} R = [0, \frac{1}{m}] \in \tilde{M} \) maps to \([1,1] \).

We are going to apply Proposition 23. The description by \( \iota \) implies that \( \tilde{M} \cap Z_{kR} \) \( \subseteq \mathbb{Q}^2 \) is non-empty if and only if \( Z_{kR} \) contains an \( \tilde{M} \)-lattice point on the boundary \( \partial \sigma^\nu \setminus \{0\} \) — just subtract \( \iota(\frac{1}{m} R) = [1,1] \) whenever the boundary is not reached yet.

While \( r^1 = [m, -g] \) used to be a primitive generator of one ray of \( \sigma^\nu \) within the lattice \( M = \mathbb{Z}^2 \), this is no longer true in \( \tilde{M} = \mathbb{Z} \times \mathbb{Z}_{m}^1 \). Here, the element \([1, \frac{-g}{m}] \) does the job instead. Thus, it remains to check whether this generator belongs to \( Z_{kR} \). Since \( \langle \alpha, [1, \frac{-g}{m}] \rangle = 0 \) and \( \langle \beta, [1, \frac{-g}{m}] \rangle = h - g \), this leads to the condition
As before, we are in the grounded case, and we consider a $k \in \{1, \ldots, a_\nu - 1\}$ with $a_\nu - 1 = 1 + [A] + [B]$.

**Proposition 25.** $T^1_{VW}(-k \cdot \overline{R}) \not\equiv 0 \iff k \leq \min\{\frac{m+1}{a}, \frac{n+1}{a}\} = \min\{c \cdot |I|, c' \cdot |I|\}$.

**Proof.** By Corollary 19(2) or Proposition 23, a degree $k\overline{R}$ fails to meet the VW-property if and only if $(M + \frac{1}{m}\overline{R}) \cap Z_{k\overline{R}}$ or, equivalently, $M \cap (Z_{k\overline{R}} - \frac{1}{m}\overline{R})$ has points outside the diagonal $Q \cdot \overline{R}$.

First, we check that $M \cap Z_{k\overline{R}} \subseteq M \cap Z_{(a_\nu - 1)\overline{R}}$ (i.e., without the translation) is always contained in the diagonal. If not, then we could find $r^i, r^j \in \mathcal{E}$ with, w.l.o.g., $i < \nu$ such that $(a_\nu - 1)r^\nu - (r^i + r^j) \in \sigma^\nu$. This implies $\nu < j$, and we choose an element $\gamma \in \text{int} \sigma$ such that $\langle \gamma, r^i \rangle = \langle \gamma, r^j \rangle (> 0)$. Since $r^1, \ldots, r^\nu$ run along the boundary of the convex polygon $\text{conv} (\sigma^\nu \cap M \setminus 0)$, it follows that $r^\nu - 1, r^\nu, r^\nu + 1 \in \text{conv}\{r^i, r^j\}$. Thus

$$\langle \gamma, \frac{r^i + r^j}{2} \rangle \leq \langle \gamma, r^i \rangle = \langle \gamma, r^j \rangle = \langle \gamma, \frac{r^i + r^j}{2} \rangle,$$

and we obtain a contradiction via

$$\langle \gamma, r^\nu - 1 + r^\nu + 1 \rangle \leq \langle \gamma, r^i + r^j \rangle \leq \langle \gamma, (a_\nu - 1)r^\nu \rangle < \langle \gamma, a_\nu r^\nu \rangle = \langle \gamma, r^\nu - 1 + r^\nu + 1 \rangle.$$

Hence, $(M \cap Z_{k\overline{R}, -1}) \setminus Q\overline{R}$ is non-empty if and only if $Z_{k\overline{R}, -1} = Z_{k\overline{R}} - \frac{1}{m}\overline{R}$ contains an $M$-lattice point on the boundary $\partial \sigma^\nu - \frac{1}{m}\overline{R}$. So we have to determine the smallest $\lambda \in \mathbb{Q}_{>0}$ such that

$$\lambda \cdot [m, -g] - [0, \frac{1}{m}] \in M \quad (and \ \text{similarly} \ \text{with} \ [-m, h]).$$

This condition is equivalent to $\lambda m \in \mathbb{Z}$ and $\lambda \cdot g + \frac{1}{m} \in \mathbb{Z}$, i.e. $m|\lambda m \cdot g + 1$. By Proposition 10, this means $\lambda m = c$. Hence, the first lattice point on the shifted ray $\mathbb{Q}_{>0} \cdot r^1 - \frac{1}{m}\overline{R}$ is $[c, -\frac{c+1}{m}]$. Its value under $\beta$ is

$$\langle \beta, c, -\frac{c+1}{m} \rangle = \langle (h, m), [c, -\frac{c+1}{m}] \rangle = c(h - g) - 1.$$

This leads to the condition $c(h - g) - 1 < km - 1$ for $Z_{k\overline{R}, -1}$-membership. Thus, the VW-condition coming from the ray $r^1$ is exactly the opposite, namely $k \cdot m \leq c \cdot (h - g)$. Similarly, the first lattice point on the shifted ray $\mathbb{Q}_{>0} \cdot r^c - \frac{1}{m}\overline{R}$ is $[-c', \frac{bc' - 1}{m}]$. It leads to the inequality $k \cdot m \leq c' \cdot (h - g)$.

□
5.4. **Comparison of qG- and VW-deformations.** In [KSSS] Definition 3.7] the so-called T-singularities are defined as those cyclic quotient singularities that admit a Q-Gorenstein one-parameter smoothing. Their toric characterization can be found in [Alt95 (7.3) and Alt98 (1.1)]: The toric variety $\mathbb{T}V(\sigma)$ is a T-singularity with Milnor number $\mu$ if and only if $\sigma$ is the cone over a rational interval of integral length $\mu + 1$ placed in height one.

Since an integral length does automatically imply the uniform denominator property of (2.5), this description of T-singularities can directly be compared to our Proposition 24. Looking at $k = 1$, it implies that $S = \mathbb{T}V(C(I))$ allows a qG-deformation at all if and only if $|I| \geq 1$. Altogether, we obtain the following chain of properties of an interval $I \not\sim [0, 1]$ with uniform denominators:

$$(|I| = 1) \Rightarrow (|I| \in \mathbb{Z}_{\geq 1}) \Rightarrow (|I| \geq 1) \Rightarrow (\text{int}(I) \cap \mathbb{Z} \neq \emptyset)$$

translating into

$$(T_0 \text{-singularity}) \Rightarrow (T \text{-singularity}) \Rightarrow (\exists \text{ qG-deformation}) \Rightarrow (\text{grounded CQS}).$$

5.5. **The last deformation.** Let $I = [\frac{g}{m}, \frac{h}{m}] = [-A, B]$ be a grounded interval as in (5.3). By Proposition 13 we know that $|I| \geq a_\nu - 2$ (with equality exactly for the T-singularities). Hence, Proposition 24 implies that all subspaces $\mathbb{R}^\perp \subseteq T^1(-k \mathbb{R})$ with $k = 1, \ldots, a_\nu - 2$ are qG-deformations (hence VW-deformations, too).

We will call the remaining deformation in degree $-(a_\nu - 1) \cdot \mathbb{R}$ the “last deformation”. This is the only degree where qG- and VW-deformations might differ at all. Note that the last deformation might also be the first one, i.e. $k = 1$. This happens if and only if $a_\nu = 2$, i.e. if and only if $0 < A, B < 1$.

For the following theorem, we will denote by $\{C\} := C - \lfloor C \rfloor$ the fractional part of a (positive, rational) number $C$. Recall that $A, B \in \mathbb{Q}_{>0}$.

**Theorem 26.** The one-dimensional subspaces $\mathbb{R}^\perp \subseteq T^1(-k \cdot \mathbb{R})$ for a grounded $S = \mathbb{T}V(C(I))$ with $k = 1, \ldots, a_\nu - 2 = \lfloor A \rfloor + \lfloor B \rfloor$ are qG- and VW-deformations. Moreover, the “last” deformation from $\mathbb{R}^\perp$ in degree $-k \cdot \mathbb{R}$ with $k = a_\nu - 1$ is a qG- or VW-deformation in the following cases:

1) The last deformation of $S = \mathbb{T}V(C(I))$ is qG if and only if $\{A\} + \{B\} \geq 1$.
2) If $\{A\}, \{B\} \neq \frac{1}{m}$, then the last deformation is VW.
3) Otherwise, i.e. if $\{A\} = \frac{1}{m}$ or $\{B\} = \frac{1}{m}$, then the last deformation is VW if and only if it is qG. Hence, every VW-deformation is qG in this case.

**Proof.** (1) By Proposition 13 and 24, both sides are equivalent to $|I| \geq a_\nu - 1$.

(3) The condition $\{A\} = \frac{1}{m}$ means $g \equiv -1 \pmod{m}$, and since $c \cdot (-g) = 1$ in $(\mathbb{Z}/m\mathbb{Z})^*$, this translates into $c = 1$. Similarly, $\{B\} = \frac{1}{m}$ is equivalent to $c' = 1$. Thus, the bounds in Proposition 24 and 25 coincide.
(2) We distinguish two cases. First, if \(a_\nu \geq 3\), then \(|I| \geq 1\). Hence
\[ a_\nu - 1 \leq |I| + 1 \leq \min\{c \cdot |I|, c' \cdot |I|\} \quad \text{since} \quad c, c' \geq 2.\]
Otherwise, if \(a_\nu = 2\), then \(c \geq 2\) together with \(c \cdot (-g) \equiv 1 (m)\) implies that
\[ c \cdot (h - g) \geq c \cdot (-g) \geq m + 1 > m,\]
hence \(c \cdot |I| > 1 = a_\nu - 1\). Similarly we use \(c' \cdot h \equiv 1 (m)\) to obtain \(c' \cdot |I| > a_\nu - 1\).

5.6. **Proof of Theorem 4.** We are going to proof Theorem 4 of the introduction.

(1) Since \(b = \gcd(n, q + 1)\), the assumption implies \(b = 1\), hence \(a = m = n\). Thus, \(|I| = \frac{1}{m}\) and this does not leave space for \(I = [\frac{a}{m}, \frac{h}{m}]\) to become grounded, i.e. to allow an integer as an interior point of \(I\).

(2) Singularities admitting a qG-smoothing are called T-singularities. In (5.4) we have seen that they correspond exactly to the intervals of integral length, i.e. \(\{A\} + \{B\} = 1\). Now, the claim follows directly from Theorem 26 (1).

(3) This follows because the qG- and VW-deformations can at most differ by the “last deformation”. This was just addressed in (5.5). Alternatively, it follows directly from Theorem 5.

5.7. **An example of a VW-deformation which is not qG.** Let \(I = [-\frac{2}{5}, \frac{2}{5}]\), i.e. \(A = B = \frac{2}{5}\). This implies that \(\{A\} = \{B\} = \frac{2}{5}\), i.e. by Theorem 26 (1), the last deformation is not qG. Another way to see this is the criterion from (5.4): Since \(|I| < 1\), there is no qG-deformation at all.

On the other hand, both \(\{A\}\) and \(\{B\}\) are different from \(\frac{1}{5}\). Thus, Theorem 26 (2) implies that the last deformation is VW. Moreover, since there is no qG-deformation at all, this has to be the “first” deformation \(R^+ \subsetneq T^1(-R)\) (i.e. with \(k = 1\)) as well.

The other invariants are \(n = 20, \quad q = 11, \quad m = a = 5, \quad b = 4, \quad \text{and} \quad c = c' = 3\).

The continued fraction \(\frac{n}{n-q} = \frac{20}{9}\) yields \([a_2, \ldots, a_6] = [3, 2, 2, 2, 3]\), i.e. \(e = 7\) and \(R = r^4\). The associated \(a_4 = 2\) was already known from our observation that the “first” equals the “last” deformation. Finally, we obtain the following dimensions:

\[ \dim T^1 = 10 \quad \text{with} \quad \dim T^1(-k \cdot r^i) = \begin{cases} 1 & \text{if} \ k = 1, 2 \text{ and } i = 2, 6 \\ 2 & \text{if} \ k = 1 \text{ and } i = 3, 4, 5, \end{cases} \]
\[ \dim T^1_V = 3 \quad \text{(degrees } -r^3, -r^4, -r^5)\text{, and } \dim T^1_{VW} = 1 \quad \text{(in degree } -r^4).\]

5.8. **Unobstructed qG-families.** While the focus of the paper is on the infinitesimal level, we would just like to add how the first order qG-deformations of \(S = TV(\sigma)\) extend to an unobstructed global family. Assume that \(I = [-A, B]\) is an interval with uniform denominators giving rise to a cyclic quotient singularity...
From Theorem 5 we know that \( d := \dim T^1_{qG}(S_I) \) equals \( \lfloor A + B \rfloor \). This number vanishes unless \( I \) is grounded. In particular, we may write
\[
I = I' + d \cdot [0, 1]
\]
for some interval \( I' \) (with uniform denominators) of length \( |I'| < 1 \). In \( [Alt00, (3.2)] \), such decompositions gave rise to so-called homogeneous toric deformations of \( S_I \) over the parameter space \( \mathbb{A}^d_k \). Its total space arises from the cone \( \tilde{\sigma} \) taken over the Cayley-construction, i.e. from
\[
\tilde{\sigma} := \mathbb{Q}_{\geq 0} \cdot (I', e^0) + \sum_{j=1}^d \mathbb{Q}_{\geq 0} \cdot ([0, 1], e^j) \subseteq \mathbb{Q} \times \mathbb{Q}^{d+1}
\]
where \( \{e^j \mid j = 0, \ldots, d\} \) denotes the canonical basis of \( \mathbb{Q}^{d+1} \). As it is \( S_I = TV(\sigma) \), also \( TV(\tilde{\sigma}) \) is \( \mathbb{Q} \)-Gorenstein. The (non-toric) flat map \( TV(\tilde{\sigma}) \to \mathbb{A}^d_k \) arises from the toric map \( TV(\tilde{\sigma}) \to \mathbb{A}^d_k \) assigned to the projection \( \mathbb{Z} \times \mathbb{Z}^{d+1} \to \mathbb{Z}^{d+1} \) composed with the linear projection \( \mathbb{A}^{d+1}_k \to \mathbb{A}^{d+1}_k/k \cdot (1, 1, \ldots, 1) \cong \mathbb{A}^d_k \).

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References

[Alt95] Klaus Altmann. Minkowski sums and homogeneous deformations of toric varieties. *Tohoku Math. J. (2),* 47(2): 151–184, 1995.
[Alt98] Klaus Altmann. P-resolutions of cyclic quotients from the toric viewpoint. In *Singularities. The Brieskorn anniversary volume. Proceedings of the conference dedicated to Egbert Brieskorn on his 60th birthday, Oberwolfach, Germany, July 1996*, pages 241–250. Basel: Birkhäuser, 1998.
[Alt00] Klaus Altmann. One parameter families containing three dimensional toric Gorenstein singularities. In *Explicit birational geometry of 3-folds*, pages 21–50. Cambridge: Cambridge University Press, 2000.
[Amb03] Florin Ambro. Quasi-log varieties. *Tr. Mat. Inst. Steklova,* 240: 220–239, 2003.
[Bri68] Egbert Brieskorn. Rationale Singularitäten komplexer Flächen. *Invent. Math.,* 4: 336–358, 1968.
[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties.* Providence, RI: American Mathematical Society (AMS), 2011.
[Fuj09] Osamu Fujino. Introduction to the log minimal model program for log canonical pairs. arXiv: 0907.1506, 2009.
[Har66] Robin Hartshorne. Residues and duality. Appendix: Cohomologie à support propre et construction du foncteur \( f^! \) par P. Deligne. Lecture Notes in Mathematics. 20. Berlin-Heidelberg-New York: Springer-Verlag, 423 p. (1966)., 1966.
[HK10] Christopher D. Hacon and Sándor Kovács. *Classification of higher dimensional algebraic varieties.* Basel: Birkhäuser, 2010.
[HP10] Paul Hacking and Yuri Prokhorov. Smoothable del Pezzo surfaces with quotient singularities. *Compos. Math.,* 146(1):169–192, 2010.
[Kaw07] Masayuki Kawakita. Inversion of adjunction on log canonicity. *Invent. Math.,* 167(1): 129–133, 2007.
THE DUALIZING SHEAF ON DEFORMATIONS OF TORIC SURFACE SINGULARITIES

[Col92] János Kollár, editor. Flips and abundance for algebraic threefolds. A summer seminar at the University of Utah, Salt Lake City, 1991. Paris: Sociétè Mathématique de France, 1992.

[Col95] János Kollár. Flatness criteria. J. Algebra, 175(2): 715–727, 1995.

[Col08] János Kollár. Hulls and husks. arXiv: 0805.0576, 2008.

[Col13a] János Kollár. Moduli of varieties of general type. In Handbook of moduli. Volume II, pages 131–157. Somerville, MA: International Press; Beijing: Higher Education Press, 2013.

[Col13b] János Kollár. Singularities of the minimal model program. With the collaboration of Sándor Kovács. Cambridge: Cambridge University Press, 2013.

[KS88] János Kollár and Nicholas Shepherd-Barron. Threefolds and deformations of surface singularities. Invent. Math., 91(2): 299–338, 1988.

[Mum78] David Mumford. Some footnotes to the work of C. P. Ramanujam. C.P. Ramanujam. - A tribute. Collect. Publ. of C.P. Ramanujam and Pap. in his Mem., Tata Inst. fundam. Res., Stud. Math. 8, 247-262 (1978)., 1978.

[Pin74] Henry C. Pinkham. Deformations of algebraic varieties with \( G_m \) action. Société Mathématique de France, Paris, 1974. Astérisque, No. 20.

[Pin77] Henry C. Pinkham. Deformations of quotient surface singularities. Several complex Variables, Proc. Symp. Pure Math. 30, Part 1, Williamstown 1975, 65-67, 1977.

[Rie74] Oswald Riemenschneider. Deformationen von Quotientensingularitäten (nach zyklischen Gruppen). Math. Ann., 209: 211–248, 1974.

[Ste03] Jan Stevens. Deformations of singularities. Berlin: Springer, 2003.

[Vie95] Eckart Viehweg. Quasi-projective moduli for polarized manifolds. Berlin: Springer-Verlag, 1995.

[Wah80] Jonathan M. Wahl. Elliptic deformations of minimally elliptic singularities. Math. Ann., 253: 241–262, 1980.

[Wah81] Jonathan M. Wahl. Smoothings of normal surface singularities. Topology, 20: 219–246, 1981.

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