CONFIGURATION SPACES OF $\mathbb{C} \setminus K$

CHRISTOPH SCHIESSL

Abstract. In this note, we collect mostly known formulas and methods to compute the standard and virtual Poincaré polynomials of the configuration spaces of the plane $\mathbb{C} \setminus k$ with $k$ deleted points and compare the answers.

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References

For any complex quasi-projective algebraic variety $X$, the virtual Poincaré polynomial $S(X) \in \mathbb{Z}[x]$ is defined [DK87], [Tot02] by the properties

- $S(X) = \sum \text{rk} H^i(X) x^i$ for smooth, projective $X$,
- $S(X) = S(X \setminus C) + S(C)$ for a closed subvariety $C \subset X$,
- $S(X \times Y) = S(X) S(Y)$.

In contrast, we write

$$P(X) = \sum \text{rk} H^i(X) x^i$$

for the standard Poincaré polynomial.

For any space $X$, the ordered configuration space

$$F_n(X) = \{x_1, \ldots, x_n \in X^n | x_i \neq x_j\}$$

is the space of $n$ distinct points in $X$. The symmetric group $S_n$ acts on $F_n(X)$ by permuting the points and the quotient

$$C_n(X) = F_n(X)/S_n$$

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is the unordered configuration space. Computing their cohomology is a classical, hard problem. As the configuration space is the complement of diagonals, determining the virtual Poincaré polynomials is simpler and was done for example by Getzler \cite{Get95, Get99}.

We look at the cohomology of ordered and unordered configuration spaces of $\mathbb{C} \setminus k$. We compute their normal and virtual Poincaré Polynomials by existing methods and see that Stirling and pyramidal numbers show up. The calculation for $C_n(\mathbb{C} \setminus k)$ seems not to be in the literature in this form.

1. Pyramidal Numbers

The $k$-dimensional pyramidal numbers are integers $P_{k,i}$ for $i \geq -1$, $k \geq -1$. They satisfy the recursions

$$P_{-1,i} = \begin{cases} 1 & i = 0 \\ 0 & \text{otherwise} \end{cases} \quad P_{k+1,i} = \sum_{j=0}^{i} P_{k,j}.$$  

An equivalent recursion would be

$$P_{k,0} = 1 \quad P_{k+1,i+1} = P_{k,i+1} + P_{k+1,i}.$$  

Some examples are

$$P_{0,i} = 1 \quad P_{1,i} = i + 1 \quad P_{2,i} = \frac{(i+1)(i+2)}{2}.$$  

Some pyramidal numbers $P_{k,i}$:

\begin{center}
\begin{tabular}{c|cccc}
\hline
$k$ & $i$ & $0$ & $1$ & $2$ & $3$ & $4$ \\
\hline
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 3 & 4 & 5 \\
2 & 1 & 3 & 6 & 10 & 15 \\
3 & 1 & 4 & 10 & 20 & 35 \\
\hline
\end{tabular}
\end{center}

The recursion allows us to compute the generating function

$$\sum P_{k,i} x^i = (1 + x + x^2 + x^3 + x^4 + \ldots)^{k+1} = \frac{1}{(1-x)^{k+1}}.$$  

By standard manipulation of generating series for $k \geq 0$:

$$\frac{1}{(1-x)^{k+1}} = \frac{1}{k!} \frac{d^k}{dx^k} \frac{1}{1-x} = \frac{1}{k!} \sum_{i \geq 0} (i+k) \ldots (i+2)(i+1)x^i = \sum_{i \geq 0} \binom{i+k}{i} x^i.$$  

The result

$$P_{k,i} = \binom{i+k}{i}.$$  

also holds for \( k = 0 \) and can be proved directly using the recursion
\[
P_{k+1,i+1} = \binom{i + k + 2}{i + 1} = \binom{i + k + 1}{i} + \binom{i + k + 1}{i} = P_{k,i+1} + P_{k+1,i}.
\]

The definition could be extended by setting
\[P_{k,i} = 0 \text{ for } i < 0.\]

In this way, all recursions stay valid for \( i < 0\).

2. Poincaré Polynomials of \( C_n(\mathbb{C} \setminus k) \)

Let \( M \) be a connected manifold. Napolitano [Nap03, Theorem 2] proved the following relation between the cohomology of unordered configuration spaces of \( M \setminus 1 \) and \( M \setminus 2 \):
\[
H^j(C_n(M \setminus 2), \mathbb{Z}) = \bigoplus_{t=0}^n H^{j-t}(C_{n-t}(M \setminus 1, \mathbb{Z})�)
\]

We use the conventions
\[H^0(C_0(M \setminus 1), \mathbb{Z}) = \mathbb{Z} \quad H^j(C_0(M \setminus 1), \mathbb{Z}) = 0 \text{ if } j > 0.\]

In general, this relation does not hold between the cohomology of the configuration spaces of \( M \setminus 1 \) and \( M \) as the proof works by pushing in points from the missing point.

**Theorem 2.1.** We have
\[
\text{rk } H^i(C_n(\mathbb{C} \setminus k), \mathbb{Z}) = \begin{cases} 
P_{k-1,i} & i = n \\
P_{k-1,i} + P_{k-1,i-1} & 0 \leq i < n \\
0 & \text{otherwise} \end{cases}
\]

or
\[
\sum_{n \geq 0} P(C_n(\mathbb{C} \setminus k)) y^n = \frac{1 + xy^2}{(1 - y)(1 - xy)^k}.
\]

**Proof.** Write
\[
Q_k(x, y) = \sum_{n,i \geq 0} \text{rk } H^i(C_n(\mathbb{C} - k), \mathbb{Z}) x^i y^n.
\]

Then applying Napolitano’s recursion to \( M = S^2 \setminus k + 1 \) we get
\[
Q_{k+1}(x, y) = Q_k(x, y) \frac{1}{1 - xy}.
\]

Arnold’s computation
\[
H^0(C_n(\mathbb{C}), \mathbb{Q}) = \mathbb{Q} \quad H^1(C_n(\mathbb{C}), \mathbb{Q}) = \mathbb{Q} \text{ if } n \geq 2 \quad H^i(C_n(\mathbb{C}), \mathbb{Q}) = 0 \text{ if } i \geq 2
\]
in [Arn70] provides initial values for \( k = 0 \):
\[
Q_0(x, y) = 1 + y + (1 + x)y^2 + (1 + x)y^3 + \cdots = \frac{1 + xy^2}{1 - y}
\]

Hence we have shown
\[
Q_k(x, y) = \frac{1 + xy^2}{(1 - y)(1 - xy)^k}.
\]
Expansion now proves the theorem. □

This theorem can also be deduced from [DK16, Prop. 3.5]. As $C_1(\mathbb{C} \setminus k) = \mathbb{C} \setminus k$, the reality check for $n = 1$ works:

$$\text{rk} H^j(C_1(\mathbb{C} \setminus k), \mathbb{Z}) = \begin{cases} 1 & \text{for } j = 0 \\ k & \text{for } j = 1 \\ 0 & \text{otherwise} \end{cases}.$$ 

We can conclude that $\text{rk} H^j(C_n(\mathbb{C} \setminus k), \mathbb{Z})$ stabilizes (seen as a function of $n$) for $n > j$.

**Corollary 2.2.** In the limit we get

$$\text{rk} H^j(C_\infty(\mathbb{C} \setminus k), \mathbb{Z}) = P_{k-1,j} + P_{k-1,j-1}$$

or as a generating series

$$P(C_\infty(\mathbb{C} \setminus k)) = \frac{1 + x}{(1 - x)^k}.$$ 

Taking stability for granted, this can be deduced by the stable version of Napolitano’s recursion:

$$H^j(C_\infty(\mathbb{C} \setminus k + 1), \mathbb{Z}) = \bigoplus_{t=0}^j \text{rk} H^t(C_\infty(\mathbb{C} \setminus k), \mathbb{Z}).$$

Vershinin [Ver99, Cor. 11.1] showed that

$$H^*(C_\infty(\mathbb{C} \setminus k) \simeq H^*(\Omega^2 S^3) \otimes \left(H^*(\Omega S^2)\right)^k$$

extending the May-Segal formula [Seg73, Ver99, Th. 8.11]

$$H^*(C_\infty(\mathbb{C}) \simeq H^*(\Omega^2 S^3).$$

Combining the results of Arnold and the cohomology of the loop spaces of a sphere

$$H^i(\Omega S^2) = \mathbb{Z}$$

for $i \geq 0$ [Hat04, Example 1.5]), this gives back corollary (2.2).

### 3. Poincaré Polynomials of $F_n(\mathbb{C} \setminus k)$

Arnold’s calculation of $H^*(F_n(\mathbb{C}), \mathbb{Z})$ can be extended to $H^*(F_n(\mathbb{C} \setminus k), \mathbb{Z})$ via the fiber bundles

$$F_n(\mathbb{C} \setminus k) \mapsto F_{n-1}(\mathbb{C} \setminus k)$$

with fiber $\mathbb{C} \setminus (k + n - 1)$.

**Theorem 3.1.** [Ver98 Thm. 7.1] We have

$$P(F(\mathbb{C} \setminus k, n)) = (1 + kx)(1 + (k + 1)x) \cdots (1 + (n + k - 1)x).$$
4. Virtual Poincaré Polynomials of $F_n(\mathbb{C} \setminus k)$

We have

$$S(\mathbb{C} \setminus k) = S(\mathbb{CP}^1 \setminus k + 1) = x^2 + 1 - (k + 1) = x^2 - k.$$  

Using the same fiber bundles or [Get95, Theorem, page 2] we get

**Theorem 4.1.** The virtual Poincaré polynomials of $F_n(\mathbb{C} \setminus l)$ is given by

$$S(F_n(\mathbb{C} \setminus k)) = (x^2 - k)(x^2 - k - 1) \cdots (x^2 - k - n + 1).$$

5. Virtual Poincaré Polynomials of $C_n(\mathbb{C} \setminus k)$

As $S(\mathbb{C} \setminus k) = (x^2 - k)$, the calculations of Getzler [Get95, Cor. 5.7] allow us to conclude

$$\sum_{n \geq 0} S(C_n(\mathbb{C} \setminus k)) y^n = \frac{(1 - y^2 x^2)(1 - y)^k}{(1 - y x^2)(1 - y^2)^k},$$

which simplifies to

**Theorem 5.1.** [Get95] The virtual Poincaré polynomials of $C_n(\mathbb{C} \setminus k)$ are given by the following generating series:

$$\sum_{n \geq 0} S(C_n(\mathbb{C} \setminus k)) y^n = \frac{(1 - y^2 x^2)}{(1 - y x^2)(1 + y)^k}.$$  

6. Comparison

We observe that under the variable transformation

$$x \to -1/x^2, y \to y x^2$$

the respective generating series

$$\sum_{n \geq 0} P(C_n(\mathbb{C} \setminus k)) y^n \quad \sum_{n \geq 0} P(F_n(\mathbb{C} \setminus k)) y^n$$

transform into

$$\sum_{n \geq 0} S(C_n(\mathbb{C} \setminus k)) y^n \quad \sum_{n \geq 0} P(F_n(\mathbb{C} \setminus k)) y^n.$$

This means, in this case the classical and virtual Poincaré polynomials are in some sense dual to each other.

**Example 6.1.** We look 3-pointed configuration spaces of $\mathbb{C} \setminus 2$:

\[
\begin{align*}
P(C_3(\mathbb{C} \setminus 2)) &= 4x^3 + 5x^2 + 3x + 1 \\
P(F_3(\mathbb{C} \setminus 2)) &= 24x^3 + 26x^2 + 9x + 1 \\
S(C_3(\mathbb{C} \setminus 2)) &= x^6 - 3x^4 + 5x^2 - 4 \\
S(F_3(\mathbb{C} \setminus 2)) &= x^4 - 9x^4 + 26x^2 - 24
\end{align*}
\]
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ETH ZÜRICH, DEPARTMENT OF MATHEMATICS
E-mail address: christoph.schiessl@math.ethz.ch