Critical role of the sign structure in the doped Mott insulator: Superconductivity vs. Fermi liquid

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Mechanism of superconductivity (SC) in a purely interacting electron system has been one of the most challenging issues in condensed matter physics. In the BCS theory, the Landau’s Fermi liquid is a normal state against which an SC instability occurs if an additional pairing force is added. We show that in the doped Mott insulator an intrinsic SC ground state (specifically a Luther-Emery state at a finite doping-two-leg t-J ladder) can be directly turned into a Fermi-gas-like state by merely switching off the hidden statistical sign structure via two schemes. It points to a new pairing paradigm, which is an “Amperian-like pairing” with a “stringlike” pairing force as shown by an adiabatic continuity to a strong anisotropic limit of the model.

Introduction.—Recently large-scale density matrix renormalization group (DMRG) calculations have shown that the t-J and Hubbard models on two- and four-leg ladders14,17 have a Luther-Emery (LE) ground state8 at finite doping, which is a superconducting (SC) state for the quasi one-dimensional (1D) system. Due to the finite spin-spin correlation at half-filling, the doped case of an even-leg spin ladder may provide a prototypical model for examining the underlying SC mechanism in a doped Mott insulator, especially that of the resonating-valence-bond (RVB) mechanism9 originally proposed for the high-Tc cuprate.

Very few exact results are known for the t-J model10,13, but it has been rigorously established that the statistical fermion sign structure in a weakly-interacting electron system will be replaced by the statistical phase-string sign structure in the bipartite t-J model at any doping, temperature, and dimensions15,19. Previously, by DMRG20,21 and variational Monte Carlo22 (VMC) studies, it has been also revealed that the phase-string sign structure plays a critical role in the pairing of two holes in the two- and four-leg t-J ladders to indicate that the pairing mechanism of RVB-type23,24 is not sufficient.

These motivate us to systematically study the ground state of the two-leg t-J ladder at finite doping by DMRG. In this paper, we shall show that it is indeed an LE liquid similar to the four-leg case at finite doping concentration.5 The single-particle Green’s function and the spin-spin correlation clearly show exponential decays indicating a gap opening in the single particle channel due to forming Cooper-pairing.

However, such an LE state can be reduced to a non-pairing Luttinger liquid (LL)25, very close to the free Fermi gas limit, as soon as the phase-string sign structure is turned off either by inserting a spin-dependent sign to the hopping term or by making the charge-string recombination in a strongly anisotropic case. By further making an adiabatic continuation of the LE state in the limit of rung hopping $t_\perp \to 0$, a hidden stringlike pairing force, originating from the phase-string sign structure, can be revealed. It resembles an “Amperian-like pairing”25,28 and is predominantly responsible for the strong pairing in the LE ground state that goes beyond the conventional RVB mechanism.

Model Hamiltonian.—The hole-doped t-J model on a square lattice is defined by $H_{c,t} = H_t + H_J$, with

\begin{align}
H_t &= -\sum_{\langle ij \rangle \sigma} t_{ij} \left( \hat{c}^\dagger_{i\sigma} \hat{c}_{j\sigma} + h.c. \right), \\
H_J &= \sum_{\langle ij \rangle} J_{ij} \left( \hat{S}_i \cdot \hat{S}_j - \frac{n_i n_j}{4} \right),
\end{align}

where $\hat{c}^\dagger_{i\sigma} \hat{c}_{i\sigma}$ is the electron creation (annihilation) operator on site $i = (x_i, y_i)$ with spin $\sigma$. $\hat{S}_i$ is the spin operator and $\hat{n}_i = \sum_{\sigma} \hat{c}^\dagger_{i\sigma} \hat{c}_{i\sigma}$ is the electron number operator. $\langle ij \rangle$ denotes nearest-neighbor (NN) sites and the Hilbert space is constrained by the no-double occupancy condition $n_i \leq 1$. $t_{ij}$ is the electron hopping integral and $J_{ij}$ is the spin exchange interaction between NN sites. Specifically, $t_{ij} = t_L$ and $J_{ij} = J_\perp$ for the intra-chain couplings, and $t_{ij} = t_L$ and $J_{ij} = J_\parallel$ for the inter-chain couplings.

For comparison, we shall also study the so-called $\sigma$-t-J model $H_{\sigma,t-J} = H_{\sigma,t} + H_J$29 with the kinetic energy term $H_t$ in Eq. (1) replaced by

\begin{align}
H_{\sigma,t} = -\sum_{\langle ij \rangle \sigma} \sigma t_{ij} \left( \hat{c}^\dagger_{i\sigma} \hat{c}_{j\sigma} + h.c. \right),
\end{align}

where an extra spin-dependent sign $\sigma = \pm 1$ is added. It can be proven that the phase-string sign structure hidden in the $t$-$J$ model is precisely removed in the $\sigma$-t-$J$ model20 (cf. Supplemental Material). Both models reduced to the same antiferromagnetic (AFM) Heisenberg model at half-filling and the difference in sign structure only shows up upon doping.

We will focus on the following cases in this work: (1) Isotropic case with $t_\parallel = t_\perp = t$ and $J_\parallel = J_\perp = J$; (2) Anisotropic in hopping: $t_\parallel = t$ and $t_\perp = \gamma t$, while $J_\parallel = \gamma J$; (3) Anisotropic in exchange: $J_\parallel = \gamma J$ and $J_\perp = J$; and (4) An anisotropic hopping and exchange: $t_\parallel = t$, $t_\perp = \gamma t$, and $J_\parallel = \gamma J$. For comparison, we shall also study the so-called $\sigma$-t-J model $H_{\sigma,t-J} = H_{\sigma,t} + H_J$29 with the kinetic energy term $H_t$ in Eq. (1) replaced by

\begin{align}
H_{\sigma,t} = -\sum_{\langle ij \rangle \sigma} \sigma t_{ij} \left( \hat{c}^\dagger_{i\sigma} \hat{c}_{j\sigma} + h.c. \right),
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TABLE I: Summary of the phases: “LE” (“LL”) denotes the Luther-Emery (Luttinger) liquid. Corresponding central charge $c$ and exponents ($K_G, K_{sc}$ and $K_s$) of the power-law behavior in the CDW amplitude $A_{cdw}$, single particle Green’s function $G_x$, pair-field correlation function $\Phi$ and spin-spin correlation function $F$ are shown (otherwise for an exponential decay, an length scale is marked by $\xi_G$ or $\xi_s$. Note that for $F$ of the $\sigma$-$t$-$J$ model, only the exponent for the $S_z$ component is shown, see text.)

| Parameters | Phase  | $c$ | $A_{cdw}$ | $G_x$ | $\Phi$ | $F$ |
|------------|--------|-----|----------|-------|--------|-----|
| $t$-$J$    | $t_\perp = t_{\parallel}, J_\parallel = J_\perp$ | LE | 1.27(3) | 1.06(1) | $\xi_G \sim 5$ | 0.85(2) | $\xi_s \sim 6$ |
| $t_\perp = t_{\parallel}, J_\parallel = J_\perp$ | LE | 1.27 $\sim$ 1.30 |
| $t_\parallel = 0.4t_{\perp}, J_\parallel = 0.4J_\perp$ | LL | 2.08(9) | 1.90(2) | $\sim 1$ | 2.12(1) $\sim 1.6$ |
| $\sigma$-$t$-$J$ | $t_\perp = t_{\parallel}, J_\parallel = J_\perp$ | LL | 2.04(1) | 1.99(1) | 1.10(1) | 2.82(1) | 1.93(1) |

$J_\perp = J$, where $\gamma$ is a tuning parameter; (3) Anisotropic in both hopping and superexchange terms: $t_\perp = t$ and $J_\perp = J$, while $t_{\parallel} = at$ and $J_{\parallel} = \alpha J$, where $\alpha$ is another tuning parameter. Here the system is a square lattice two-leg ladder with system size $N = L_x \times L_y$, where $L_x$ and $L_y$ ($= 2$) are the number of sites along the $\hat{x} = (1,0)$ and $\hat{y} = (0,1)$ directions, respectively, with the length up to $L_x = 192$. The doping concentration $\delta = \frac{N_h}{N}$, where $N_h$ is the number of doped holes measured from half-filling. Our calculation will mainly focus on a typical doping $\delta = 12.5\%$ without loss of generality. We set $J = 1$ as an energy unit and consider $t = 3$, and keep up to $m = 8000$ number of states in each DMRG block with truncation error $\epsilon \leq 5 \times 10^{-9}$ and perform up to 60 sweeps. This leads to excellent convergence for our results when extrapolated to $m = \infty$ limit.

Physical quantities.—The following physical quantities will be calculated by DMRG.\textsuperscript{30} The charge density $n(x) = \frac{1}{2} \sum_{\sigma=1}^{2} \langle \hat{n}(x, \sigma) \rangle$. The charge density wave (CDW) amplitude, $A_{cdw}$, inferred from $n(x)$ by

$$n(x) = A_{cdw}(L_x) \cos(Q_{cdw}x + \theta) + n_0,$$

where $Q_{cdw}$ denotes the CDW wavevector, while $\theta$ and $n_0$ are fitting parameters. Since the ends of a finite system break the translational symmetry, only the central-half region with rung indices $\frac{L_x}{4} < x \leq \frac{3L_x}{4}$ is used in the fitting to minimize the boundary effect, as shown in Fig. 1 for $L_x = 64$.

The single-particle Green’s function is defined as

$$G_x(r) = \frac{1}{2} \sum_{y=1}^{2} \langle c_{x_0,y,\sigma}^{\dagger} c_{x_0+r,y,\sigma} \rangle,$$

where $(x_0,y)$ is the reference site and $r$ is the displacement along the $\hat{x}=(1,0)$ direction. If $G_x(r)$ is short-ranged, it is characterized by a length scale $\xi_G$: $G_x(r) \sim e^{-r/\xi_G}$. Otherwise, it is described by the Luttinger exponent $K_G$ in a power-law behavior: $G_x(r) \sim r^{-K_G}$. A diagnostic of the SC order is by the pair-field correlation function

$$\Phi_{\alpha,\beta}(r) = \frac{1}{2} \sum_{y=1}^{2} \langle \Delta_{\alpha}^{\dagger}(x_0,y) \Delta_{\beta}(x_0+r,y) \rangle.$$

Here the spin-singlet pair-field creation operator $\Delta_{\alpha}^{\dagger}(x,y) = \frac{1}{\sqrt{2}}[c_{(x,y),\uparrow}^{\dagger} c_{(x,y),\downarrow}^{\dagger} - c_{(x,y),\downarrow}^{\dagger} c_{(x,y),\uparrow}^{\dagger}]$, where bond orientations are designated $\alpha = \hat{x}, \hat{y}$, and $(x_0,y)$ is the reference bond and $r$ the displacement along the $\hat{x} = (1,0)$ direction. Similarly, the spin-spin correlation function is given by

$$F(r) = \frac{1}{2} \sum_{y=1}^{2} \langle \langle S_{x_0,y} \cdot S_{x_0+r,y} \rangle \rangle,$$

where $S_{x_0,y}$ denotes the spin operator at site $i = (x,y)$. A spin gapped or gapless state is characterized by a short-ranged, $F(r) \sim e^{-r/\xi_s}$, or quasi-long-ranged, $F(r) \sim r^{-K_s}$, respectively.

Finally, in the DMRG simulation, the central charge can be obtained by calculating the von Neumann entropy $S = -\text{Tr} \rho \ln \rho$, where $\rho$ is the reduced density matrix of a subsystem with length $l$. For a critical system, it has been established\textsuperscript{31} that $S(l) = \frac{c}{4} \ln(l) + \hat{c}$ for open systems, where $c = \frac{1}{2}$. The central charge of the conformal field theory (CFT) and $\hat{c}$ denotes a model dependent constant. For finite cylinders with length $L_x$, we may fix $l = \frac{L_x}{2}$ and use the formula $S(\frac{L_x}{2}) = \frac{c}{4} \ln(\frac{L_x}{2}) + \hat{c}$ to extract the central charge $c$, i.e., the number of gapless modes.

Table I summarizes the main results obtained by DMRG and the details will be discussed in the following.
Superconducting/Luther-Emery state.—The ground state of the two-leg $t$-$J$ ladder shows a typical LE liquid behavior characterized by the following correlators (cf. Table 1):

$$
\Phi_{\alpha\beta}(r) \propto r^{-K_{sc}},
$$

(7)

$$
A_{cdw}(L_x) \propto L_x^{-K_{cdw}/2},
$$

(8)

$$
K_c K_{sc} \sim 1.
$$

(9)

In the isotropic case, one obtains $K_{sc} = 0.85(2)$ and $K_c = 1.06(1)$, respectively, for the SC pairing and CDW amplitude, with $K_c K_{sc} \sim 1$ within the numerical error and finite size effect (cf. Fig. 2). The density oscillates with a well-defined wavevector $Q_{cdw}$ with a wavelength $\lambda = \frac{\pi}{\xi}$ as shown in Fig. 1. The single-particle correlator has a length scale $\xi_G \sim 6$, while the spin-spin correlation length $\xi_s \sim 6$ (Fig. 2):

$$
G_\sigma(r) \sim e^{-r/\xi_G},
$$

(10)

$$
F(r) \sim e^{-r/\xi_s}.
$$

(11)

Moreover, the central charge extracted from the scaling of the entanglement entropy, is $c = 1.27(3)$, which is qualitatively consistent with one gapless charge mode with $c = 1$ in the LE liquid. The above LE liquid is similar to that of the four-leg $t$-$J$ ladder and persists over a wide range of finite doping.

To show the robustness of the LE phase, we further examine the anisotropy in the hopping: $t_{\parallel} = t$ and $t_{\perp} = \gamma t$, while the superexchange coupling remains isotropic. From $\gamma = 1 \rightarrow 0$, we find that the ground state always remains an LE liquid with the same modulation wavelengths of the charge density and spin density, and the product $K_c K_{sc} \sim 1$ in the whole regime of $0 \leq \gamma \leq 1$ (cf. Fig. 3). The correlation lengths, $\xi_G$ and $\xi_s$, slightly decrease with reducing $\gamma$ as shown in Fig. 3(a). In the inset of Fig. 3(a), the central charge $c = 1.27 \sim 1.30$ is qualitatively consistent with one gapless charge mode with $c = 1$.

Fermi liquid/Luttinger liquid state: Disappearance of the phase string sign structure.—In sharp contrast, in the two-leg $\sigma$-$t$-$J$ model at the same doping, the ground state is qualitatively changed to an LL state characterized by (cf. Fig. 2):

$$
G_\sigma(r) \sim r^{-K_G},
$$

(12)

$$
F(r) \sim r^{-K_s},
$$

(13)

with the dominant Luttinger exponent $K_G = 1.10(1)$, very close to the Fermi gas limit, while both the density-density and SC correlators become sub-leading, with $K_c = 1.99(1)$ and $K_{sc} = 2.82(1)$, respectively. The spin-spin correlation also changes to a power-law behavior with $K^{zz}_s = 1.93(1)$ and $K^{zz,yy}_s = 0.43(1)$, respectively (note the absence of spin rotation symmetry in the hopping term of the $\sigma$-$t$-$J$ model, see below). Correspondingly the charge modulation wavelength becomes $\lambda = \frac{1}{2\delta}$ and spin modulation wavelength $\frac{1}{2\delta}$.

![FIG. 2: (Color online) Finite-size scalings of (a) CDW amplitude and (b) superconducting pair-field correlator for the isotropic $t$-$J$ and $\sigma$-$t$-$J$ models, respectively. Insets: Finite-size scaling of the spin-spin and single-particle correlators for (a) $t$-$J$ model in semi-logarithmic scales and (b) $\sigma$-$t$-$J$ model in double-logarithmic scales, respectively.](image)

Furthermore, similar Fermi gas/LL ground state is also identified even in the $t$-$J$ ladder in a strong rung coupling case. With $t_{\perp} = t$ and $J_{\parallel} = J$, while $t_{\parallel} = \alpha t$ and $J_{\parallel} = \alpha J$, a transition to an LL state is found at $\alpha < \alpha_c \sim 0.68$ (for $t/J = 3$) in the $t$-$J$ ladder. For example, as shown in Fig. 4 at $\alpha = 0.4$, the Luttinger exponent $K_G \sim 1$, while the decay of the CDW amplitude $A_{cdw}(L_x)$ with length $L_x$ and the pair-field correlator $\Phi(r)$ at large distance also become sub-leading: $K_c = 1.90(2)$ and $K_{sc} = 2.12(1)$, respectively, with the central charge $c = 2.08(9)$, which are all similar to the $\sigma$-$t$-$J$ case. However, it is noted that the spin SU(2) rotation symmetry is still maintained here and the spin-spin correlator is specified by a single exponent $K_s \sim 1.6$.

As a matter of fact, a transition to a conventional quasiparticle state has been previously identified in the same anisotropic $t$-$J$ ladder in the strong rung coupling at $\alpha_c \sim 0.68$ for the single-hole-doped case. There it has been shown that due to the strong recombination of the hole and its spin partner, the phase string sign structure is indeed effectively removed in the strong rung limit $\alpha < \alpha_c$. In the present finite doping at $\delta = 0.125$, the ground state at $\alpha < \alpha_c$ still remains in a Fermi gas.
state, with \( \alpha_c \sim 0.68 \) essentially independent of doping. Here \( \alpha_c \) is determined by the second-order derivative of the ground state energy density as shown in the inset (a2) of Fig. 3(a) where one more hole is added on top of the all paired \( \delta = 12.5\% \) holes. For comparison, a much smoother peak at \( \alpha_c \sim 0.64 \) is also shown in the second-order derivative of the energy for the paired ground state at \( \delta = 12.5\% \) holes in the insets of Fig. 3(a). Such an “SC” transition point coincides with the critical point for binding between two doped holes \( \delta = 0.125 \) (and with one more hole which determines \( \alpha_c \sim 0.68 \) at \( t/J = 3 \)) (see text).

**Non-BCS nature of pairing in the t-J model.**—We have found that the LE liquid as a prototypical SC phase in the quasi-1D system can make a transition to a non-pairing LL phase by switching off the phase-string sign structure either in the \( \sigma \)-t-J model or at \( \alpha < \alpha_c \) in an anisotropic t-J ladder. In the following, let us explicitly show how such a novel sign structure plays a critical role in the pairing of the LE state.

First, let us recall that the LE state remains smooth as a function of \( \gamma \) and persists in the limit of \( \gamma \to 0 \) as the hopping \( t_\perp \) diminishes while the \( J\)-term remains isotropic in the t-J model (cf. Fig. 3). In this limit, a duality transformation \( e^{i \Theta} \) can be explicitly performed to turn the t-J model into the phase-string-free \( \sigma \cdot t-J \) model plus an additional “stringlike” pairing term as previously shown for the two-hole case, which still holds true in an arbitrary many-hole case

\[
H_{t-J} \to \tilde{H}_{t-J} = H_{\sigma \cdot t-J} + H_{t-J}^{\text{string}}
\]

where

\[
H_{t-J}^{\text{string}} = \frac{1}{2} J \sum_{x_i} (\tilde{\phi}_{(x_i,1)}^+ \tilde{\phi}_{(x_i,2)}^- + \tilde{\phi}_{(x_i,1)}^- \tilde{\phi}_{(x_i,2)}^+) (1 - \Delta \Lambda_i^h)
\]

in which the summation over \( i \) is along the chain direction, and \( \Delta \Lambda_i^h = \exp \left[ -i \pi \sum_{x_i < x_j} (\hat{n}_{(x_i,1)}^h - \hat{n}_{(x_j,2)}^h) \right] \) describes the nonlocal phase shift effect created by the doped holes at both chains (legs). Since the transverse spin at each rung: \( \langle \tilde{\phi}_{(x_i,1)}^+ \tilde{\phi}_{(x_i,2)}^- + \tilde{\phi}_{(x_i,1)}^- \tilde{\phi}_{(x_i,2)}^+ \rangle < 0 \) as ensured by \( J \) at half-filling and finite doping, one finds that doped holes will generally acquire a string-like strong pairing potential in Eq. (14), in addition to the usual
$J$-term in $H_{t-J}$. It will then result in an strong pairing ground state $|\Psi_{\text{BCS}}\rangle$ for $H_{l-J}$.

The true LE ground state of the original $H_{l-J}$ (at $t_{\parallel} = 0$) is then written by

$$|\Psi_G\rangle = e^{i\Theta}|\Psi_{\text{BCS}}\rangle,$$

(16)

where the duality transformation $\Theta \equiv -\sum n_{(x,y)}^b \hat{\Theta}_i$

and $\hat{\Theta}_i = \pm \pi \sum_{x>y} n_{(x,y)}^i$, where $n_{(x,y)}^i$ is the number operator of down spin at site $(x,y)$. Therefore, the phase string sign structure as represented by $e^{i\Theta}$ is topological and non-perturbative, which gives rise to a non-BCS form [Eq. (S13)] of the ground state with an Amperean-like novel pairing force shown in Eq. (S15). Alternatively in the supplemental material, a bosonization method has been applied to treat the LE ground state in the large $\beta \equiv J_{\perp}/J_{\parallel}$ limit.

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1. SUPPLEMENTARY MATERIAL

This supplementary material contains two parts. In the first part, we outline the rigorous phase string sign structures in a bipartite \( t\)-\( J \) model and the absence of this novel statistical sign structure in the \( \sigma\)-\( t\)-\( J \) model, and discuss the implication for the comparative study in the main text. In the second part, we provide an analytic study of the Luther-Emery liquid ground state for the \( t\)-\( J \) two-leg ladder in the limit of \( t_\perp = 0 \) and \( J_\perp \gg J_\parallel \).

2. SIGN STRUCTURE

The \( t\)-\( J \) and \( \sigma\)-\( t\)-\( J \) models with \( H_{t,J} = H_t + H_J \) and \( H_{\sigma\cdot t,J} = H_{\sigma\cdot t} + H_J \)

\[
H_t = -\sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + \text{h.c.}, \quad (S1)
\]

\[
H_{\sigma\cdot t} = -\sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + \text{h.c.}, \quad (S2)
\]

\[
H_J = \sum_{\langle ij \rangle} J_{ij} \left( \hat{S}_i \cdot \hat{S}_j - \frac{1}{4} \hat{n}_i \hat{n}_j \right) \quad (S3)
\]

The many-body Hilbert space is subject to a no-double occupancy constraint

\[
\sum_i \hat{n}_i \leq 1. \quad (S4)
\]

Here \( \hat{c}_{i\sigma} \) annihilates an electron with spin \( \sigma \) at site \( i \). And \( \hat{S}_i \) and \( \hat{n}_i \) are spin and electron number operators respectively at site \( i \). Specifically, \( t_{ij} = t_\parallel \) and \( J_{ij} = J_\parallel \) for the intra-chain couplings and \( t_{ij} = t_\perp \) and \( J_{ij} = J_\perp \) for the inter-chain couplings.

The \( t\)-\( J \) model is considered to be one of the simplest model to describe the spin full doped Mott insulator. The strong correlation nature of the Mott physics may be well represented by the novel statistical sign structure hidden in the \( t\)-\( J \) model, which has been demonstrated in Ref. \[14\]-[16]. For a bipartite lattice of any dimensions, doping concentration, and temperature, the partition function of the \( t\)-\( J \) model can be generally expressed as \[16\]

\[
Z_{t,J} = \sum_c \tau_c Z[c] \quad (S5)
\]

with each path \( c \) composed of a set of closed loops of the spatial trajectories of all holes and \( Z[c] \geq 0 \). The general sign factor \( \tau_c \) in Eq. \[S5\] is given by

\[
\tau_c \equiv (-1)^{N_h^\uparrow[c] + N_h^\downarrow[c]}. \quad (S6)
\]

Here the Berry-phase-like sign factor \((-1)^{N_h^\uparrow[c]}\) is associated with the hopping processes of the exchanging between holes and spin-\( \downarrow \), which is known as the phase string enforced on each hole closed loop in Eq. \[S5\]. Such a novel sign structure replaces the conventional fermion signs for the electrons and implies an intrinsic mutual statistics between holes and spins, which further suggests a new type of fractionalization\[17\]. Furthermore, there is another sign factor \((-1)^{N_h^\downarrow[c]}\) in Eq. \[S6\], which resembles a conventional Fermi-Dirac statistical signs associated with hole-hole exchange process as intrinsic particles.

The novel phase-string sign in Eq. \[S6\] can be switched off by inserting a sign \( \sigma \) in each hopping term of \( H_t \) to result in \( H_{\sigma\cdot t} \) as given in Eq. \[S2\]. Then one obtains a phase-string-free \( \sigma\)-\( t\)-\( J \) model whose partition function reduces to

\[
Z_{\sigma\cdot t,J} = \sum_c (-1)^{N_h^\downarrow[c]} Z[c]. \quad (S7)
\]

where the fermion signs between holes are unchanged and \( Z[c] \geq 0 \) remains the same as in Eq. \[S5\]. But the phase string sign factor is precisely removed.

To understand the two-dimensional (2D) \( \sigma\)-\( t\)-\( J \) model further, let us introduce a transformation

\[
c_{(x,\nu)\uparrow} \rightarrow c_{(x,\nu)\uparrow}^\dagger, \quad (S8)
\]

\[
c_{(x,\nu)\downarrow} \rightarrow (-1)^x (\pm 1)^x c_{(x,\nu)\downarrow}, \quad (S9)
\]

where each site \( i = (x, \nu) \) is specified by two coordinates in 2D. Then the \( \sigma\)-\( t\)-\( J \) model is transformed into a 2D doping XXZ model:

\[
H_{\sigma\cdot t,J} \rightarrow H_{\text{dXXZ}} \quad (S10)
\]

where \( H_{\text{dXXZ}} = H_t + H_{\text{XXZ}} \), with

\[
H_{\text{XXZ}} = \sum_{\langle ij \rangle} J_{ij} \left( \hat{S}_i^z \hat{S}_j^z - \frac{1}{4} \hat{n}_i \hat{n}_j \right) - J_{ij} \left( \hat{S}_i^x \hat{S}_j^x + \hat{S}_i^y \hat{S}_j^y \right). \quad (S11)
\]

Compared with \( H_{t,J}, \) \( H_{\text{dXXZ}} \) now has an antiferromagnetic spin background with a ferromagnetic interaction in the XY-plane. In other words, the doped holes now feel like as if they are moving in a much less frustrated quantum spin background where the spins in the easy (XY)-plane are in ferromagnetic array. Of course, in the original \( \sigma\)-\( t\)-\( J \) model, the \( J\)-term still remains the same as in the \( t\)-\( J \) model case. It is the hopping term that is changed to remove the phase string frustration.

Therefore, the essential Mott physics, which is normally associated with the no double occupancy constraint in Eq. \[S4\], is really given by the phase string sign structure of Eq. \[S6\]. What we have shown above is that the novel phase string effect can be precisely distinguished by the difference between the \( t\)-\( J \) and \( \sigma\)-\( t\)-\( J \) models, which is shown in the DMRG study as given in the main text for the two-leg ladder case at finite doping.

Finally, we point out that in the one-dimensional (1D) case, the \( t\)-\( J \) and \( \sigma\)-\( t\)-\( J \) models can be further connected by a unitary transformation. Namely, under the open boundary condition, one finds \( H_{\sigma\cdot t,J} = U_{1D} H_{t,J} U_{1D}^\dagger \) by

\[
U_{1D} = \prod_l \exp \left( -i \pi \sum_{j \succ l} n_j^\dagger n_j \right) \quad (S12)
\]
where \( n^h_i = 1 - n_i \) and \( n^↓_i \) are number operators of holes and down spin at site \( l \). Similar transformation can be then constructed in a two-leg ladder system with \( t_\perp = 0 \) such that the hopping term reduces to the 1D like two-decoupled chains but the superexchanger \( J_\perp \) remains finite in the rung of the ladder. It is similar to the 1D version of Eq. \((S12)\) as given by

\[
|\Psi_{t=J}\rangle = e^{i\hat{\Theta}}|\Psi_{t=J}\rangle, \tag{S13}
\]

where

\[
\hat{\Theta} = -\sum_i \hat{n}^h_{(x_i,y_i)} \hat{\Omega}_i \tag{S14}
\]

where \( \hat{\Omega}_i = \pm \pi \sum_{y=1}^{L_y} \sum_{x_i>x} n^↓_{(x_i,y)} \) where \( n^↓_{(x_i,y)} \) is the number operator of down spin at site \( (x_i,y) \). Then it is straightforward to obtain

\[
H_{t,J} \rightarrow H_{\sigma,J} + H_{t,J}^{\text{string}} \tag{S15}
\]

where

\[
H_{t,J}^{\text{string}} = -\frac{1}{2} J \sum_{x_i} (\hat{S}^+_{(x_i,1)} \hat{S}^-_{(x_i,2)} + \hat{S}^-_{(x_i,1)} \hat{S}^+_{(x_i,2)}) (1-\Delta \Lambda^h) \tag{S16}
\]

in which the summation over \( i \) is along the chain direction, and \( \Delta \Lambda^h \) describes the nonlocal phase shift effect created by the doped holes at both legs.

It is easy to see that \( H_{t,J}^{\text{string}} \) vanishes if \( J_\perp = 0 \), but with \( J_\perp \neq 0 \) in a two-leg ladder coupling, a string-like or “Amperean-like” pairing force emerges in additional to the usual J term in \( H_{\sigma,J} \). Since the transverse spin at each rung: \( \hat{S}^+_{(x_i,1)} \hat{S}^-_{(x_i,2)} + \hat{S}^-_{(x_i,1)} \hat{S}^+_{(x_i,2)} < 0 \) as ensured by \( J_\perp \neq 0 \), one finds that doped holes will generally acquire a string-like strong pairing potential in Eq. \((S16)\), which will then result in an strong pairing ground state \( |\Psi_{t,J}\rangle \) in the transformed \( t-J \) ladder.

Alternatively, in the following we shall treat the strong string-like pairing potential in a straightforward way in the large \( J_\perp/J_\parallel \gg 1 \) limit, where a bosonization method can be applied to show that the ground state is indeed an LE liquid, consistent with the DMRG in the main text.

### 3. The Ground State for \( H_{t,J} \) at \( J_\perp/J_\parallel \gg 1 \)

The ground state of the two-leg \( t-J \) ladder stays stable from large \( \gamma \) to \( \gamma \rightarrow 0 \) limit (cf. Fig. 3 in the main text) as an LE liquid. It stays smooth also at \( \gamma = 0 \) over a large range of \( \beta \equiv J_\perp/J_\parallel \). For the following analytic analysis, we shall consider the limit \( \beta \gg 1 \) at finite doping case.

---

**A. Effective model**

At \( \beta \gg 1 \), one has the following perturbative scheme for the \( t-J \) ladder. Define

\[
H_0 = H_{J_\perp}, \quad H_{\text{int}} = H_{t,J} + H_{J_\parallel} \tag{S17}
\]

then a large \( \beta \) makes it possible to compress two legs into a single chain (cf. Fig. S1). More explicitly, with a Nakajima transformation, we obtain the effective Hamiltonian \( H_{\text{eff}} \) as follows:

\[
H_{\text{eff}} = \sum_i \frac{8}{3J_\perp} \left( b_i^\dagger b_{i+1}^\dagger b_{i+1} b_i \right) - \frac{J_\parallel^2}{4J_\perp} n_i n_{i+1} - \frac{3}{4} J_\parallel n_i \tag{S18}
\]

where the operator \( b_i \) annihilates a pair of electrons in the original system

\[
b_i \leftrightarrow \sum_{\sigma} \sigma \hat{\epsilon}_{1\sigma} \hat{\epsilon}_{21-\sigma} \tag{S19}
\]

and \( \hat{n}^b_i \) is the number of pairs on the rung \( i \) that only can take value \( 1 \) or \( 0 \). The effective model \( H_{\text{eff}} \) in Eq. \((S18)\) describes a 1D hard-core boson system with a nearest neighbour attractive interaction. One can then solve it via the bosonization method.

We introduce the Jordan-Wagner transformation,

\[
\psi_i = b_i \exp \left( -i\pi \sum_{l<i} n_l \right) \tag{S20}
\]

We obtain a fermionic form of Eq. \((S18)\)

\[
H_{\text{eff}} = -t_{\text{eff}} \left( \psi_i^\dagger \psi_{i+1} + \psi_{i+1}^\dagger \psi_i \right) - V n_i n_{i+1} - U n_i \tag{S21}
\]
With the bosonized field,
\[
\psi_i = \frac{\eta}{\sqrt{2\pi a}} e^{i k_b x} e^{-i (\phi_b - \theta_b)} + \frac{\bar{\eta}}{\sqrt{2\pi a}} e^{-i k_b x} e^{-i (-\phi_b - \theta_b)} \tag{S22}
\]
with \(\eta\) and \(\bar{\eta}\) Klein factor obeying an anticommutation relation, the corresponding bosonic Hamiltonian
\[
H_{\text{eff}} = \frac{u_b}{2\pi} \int dx \left[ K_b (\partial_x \phi_b (x))^2 + \frac{1}{K_b} (\partial_x \phi_b (x))^2 \right] - \frac{U}{\pi} \partial_x \phi_b \tag{S23}
\]
describes low energy fluctuations of the pairing field \(\phi\) with the stiffness constant
\[
K_b = \sqrt{\frac{u_0}{u_0 - V^2 \sin^2 (k_f a)}} = 1 + V^2 \frac{\sin^2 (k_f a)}{u_0}. \tag{S24}
\]
Here \(K_b\) is the Luttinger parameter \(k_b = \pi (1 - \delta)\) is the effective fermi momentum. The constant potential term \(U/\pi \partial_x \phi\) can be ignored.

**B. Luther-Emery Liquid**

1. Density-density correlation

The operators in the \(t-J\) model should also be mapped into effective operators in Eq. (S18). For example, the hole number operator \(n^h (x) \equiv \frac{1}{2} \sum_\nu (1 - n_{(x,\nu)})\) for \(\hat{c}_{(x,\nu)}\) is mapped into the pairing number operator \(n^b_x\) for \(b_x\). From the pair density-density correlator \(\langle n^b_x n^b_y \rangle\)
\[
\langle n^h (x) n^h (y) \rangle = \langle n^b_x n^b_y \rangle \tag{S25}
\]
\[
= - \frac{1}{\pi^2} \frac{K_b}{r^2} + \frac{2}{(2\pi)^2} |r|^{-2K_b} \cos (2K_b r) \tag{S26}
\]
with the formula for the density operator
\[
n^b_x = - \frac{1}{\pi} \partial_x \phi_b + \frac{1}{2\pi} \left[ e^{2k_b x} e^{-2i \phi_b} + \text{h.c.} \right]. \tag{S27}
\]
We can deduce \(K_c = 2K_b\).

2. Superconducting pairing correlation

The singlet superconducting pairing correlation \(\langle b_y b^\dagger_x \rangle\) can be described in the fermionic representation by (suppose \(x > y\))
\[
\langle b_y b^\dagger_x \rangle = \left\langle \exp \left( - i\pi \sum_{l>y} \hat{n}_l \right) \psi_y \psi^\dagger_x \exp \left( + i\pi \sum_{l>x} \hat{n}_l \right) \right\rangle \sim e^{ik_f r |r| - \frac{1}{2} K_b^{-1}} \tag{S28}
\]
such that \(K_{sc} = \frac{1}{2} K_b^{-1}\).

Finally we obtain an important relation for \(K_c\) and \(K_{sc}\) in Sec 3B1 and Sec 3B2 of the main text:
\[
K_c K_{sc} = 1 \tag{S29}
\]
which establishes an Luther-Emery liquid.