DEFORMATION OF K-THEORETIC CYCLES

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ABSTRACT. By using previous results in [13], we answer the following two questions posed by Mark Green and Phillip Griffiths in chapter 10 of [10] (page 186-190):

• (1). Can one define $T Z^p(X)$ (tangent space to cycle class group) in general?
• (2). Obstruction issues.
The highlight is the appearance of negative K-groups which detects the obstructions to deforming cycles.

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1. Introduction

For $X$ a smooth projective variety over a field $k$ of characteristic 0, for each integer $p$ satisfying $1 \leq p \leq \dim(X)$, let $Z^p(X)$ denote the cycle class group,

$$Z^p(X) = \bigoplus_{y \in X^{(p)}} \mathbb{Z} \cdot \{y\}.$$

The following question is posed by Green-Griffiths:

**Question 1.1** (page 186 in [10]). Can one define $T Z^p(X)$ in general?
Here, \( TZ^p(X) \) is the tangent space to the cycle class group \( Z^p(X) \). Since the abelian group \( Z^p(X) \) is not a complex manifold or a scheme, the known deformation theory, such as Kodaira-Spencer theory or the theory of Hilbert schemes, can’t apply to this question directly. We consider \( Z^p(-) \) as a functor and attempt to define the tangent space to this functor as usual

\[
TZ^p(X) := \ker \{ Z^p(X \times \text{Spec}(k[\varepsilon]/(\varepsilon^2))) \xrightarrow{\varepsilon=0} Z^p(X) \},
\]

where \( k[\varepsilon]/(\varepsilon^2) \) is the ring of dual numbers. Unfortunately, the classical definition of algebraic cycles can’t distinguish nilpotent, \( Z^p(X \times \text{Spec}(k[\varepsilon]/(\varepsilon^2))) = Z^p(X) \), so this definition is clearly not the desirable one.

Green-Griffiths has answered this question for \( p = 1(\text{divisors}) \) and \( p = \dim(X)(0\text{-cycles}) \) in \([10]\). To give an example of what tangent spaces to cycle class groups are, we recall:

**Definition 1.2** (page 84-85 and page 141 in \([10]\)). For \( X \) a smooth projective surface over a field \( k \) of characteristic 0, the tangent space \( TZ^2(X) \) to the 0-cycles on \( X \) and the tangent subspace \( TZ^2_{\text{rat}}(X) \) to the rational equivalence class are defined to be :

\[
TZ^2(X) := \bigoplus_{x \in X^{(2)}} H^2_x(\Omega^1_{X/Q}) ,\quad TZ^2_{\text{rat}}(X) := \text{Im}(\partial_1^{1,-2}),
\]

where \( \partial_1^{1,-2} \) is the differential of the Cousin complex of \( \Omega^1_{X/Q} \):

\[
0 \to \Omega^1_{k(X)/Q} \to \bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/Q}) \xrightarrow{\partial_1^{1,-2}} \bigoplus_{x \in X^{(2)}} H^2_x(\Omega^1_{X/Q}) \to 0.
\]

It is worth noting that absolute differentials and local cohomology appear in this definition.

Moreover, Green-Griffiths points out that(page 186 in \([10]\)):

*The technical issue that arises in trying straightforwardly extend the definitions given in the text for \( p = n, 1 \) concerns cycles that are linear combinations of irreducible subvarieties

\[
Z = \sum_i n_i Z_i,
\]

where some \( Z_i \) may not be the support of a locally Cohen-Macaulay scheme.*

To handle this technical issue, we look at generic points of \( Z_i \)'s and need to use higher algebraic K-theory. In Section 2, we propose a definition of \( TZ^p(X) \) in Definition 2.6 for general \( p \), generalizing Green-Griffiths’ Definition 1.2 above.
Considering an element \( \tau \in TZ^p(X) \) as a first order deformation, Green-Griffiths asks whether we can successively deform \( \tau \) to infinite order. It is well-known that the deformation of a subvariety \( Y \), considered as an element of the Hilbert scheme \( \text{Hilb}(X) \), may be obstructed. However, Green-Griffiths predicts that we can eliminate obstructions, by considering \( Y \) as an element of \( Z^p(X) \):

**Conjecture 1.3** (page 187-190 in [10]). \( TZ^p(X) \) is formally unobstructed, see Conjecture 3.8 in Section 3.2.

We answer this conjecture in Theorem 3.11. The main idea for answering Question 1.1 and Conjecture 1.3 is to use Milnor K-theoretic cycles to replace the classical algebraic cycles. In [3], Balmer defines K-theoretic Chow groups in terms of the derived category \( D^{perf}(X) \) obtained from the exact category of perfect complexes of \( O_X \)-modules. His idea is followed by Klein [13] and the author [18]. By modifying Balmer’s K-theoretic Chow groups [3], in [18], we extend Soulé’s variant of Bloch-Quillen identification from \( X \) to its infinitesimally trivial deformations. In this note, we continue using the techniques developed in [18] and focus on the geometry behind the formal definitions of K-theoretic cycles.

This note is organized as follows. We recall Milnor K-theoretic cycles and answer Green-Griffiths’ Question 1.1 in Section 2.1, concrete examples of Milnor K-theoretic cycles from geometry (locally complete intersections) are also discussed. In Section 2.2 and Section 2.3, we explain two new aspects of Milnor K-theoretic cycles, which are different from Balmer’s [3], featuring negative K-groups and Milnor K-theory.

The relation between obstructions and negative K-groups is discussed in Section 3.1. We discuss obstruction issues and answer Green-Griffiths’ Conjecture 1.3 in Section 3.2.

**Notations and conventions.**

1. K-theory used in this note will be Thomason-Trobaugh nonconnective K-theory, if not stated otherwise.
2. For any abelian group \( M \), \( M_\mathbb{Q} \) denotes the image of \( M \) in \( M \otimes \mathbb{Q} \).
3. \( k[\varepsilon]/(\varepsilon^2) \) is the ring of dual numbers.

2. First order deformation-tangent spaces

In this section, \( X \) is a \( d \)-dimensional smooth projective variety over a field \( k \) of characteristic 0. For each positive integer \( j \), \( X_j := X \times_k \text{Spec}(k[t]/t^{j+1}) \) is the \( j \)-th order infinitesimally trivial deformation of \( X \). In particular, we use \( X[\varepsilon] \) to stand for \( X_1 \), i.e., \( X[\varepsilon] = (X, O_X[t]/(t^2)) \).

Recall that Milnor K-groups with support are rationally defined in terms of eigenspaces of Adams operations in [18].
Definition 2.1 (Definition 3.2 in [18]). Let \( x_j \in X_j^{(i)} \), for any integer \( m \), Milnor K-group with support \( K^M_m(O_{X_j,x_j} \text{ on } x_j) \) is rationally defined to be
\[
K^M_m(O_{X_j,x_j} \text{ on } x_j) := K^{(m+i)}_m(O_{X_j,x_j} \text{ on } x_j)_\mathbb{Q},
\]
where \( K^{(m+i)}_m \) is the eigenspace for \( \psi^k = k^{m+i} \) and \( \psi^k \) is the Adams operations.

For each positive integer \( p \), there exists the following variant of Gersten complex, see Theorem 3.14 in [18],
\[
0 \to \bigoplus_{x_j \in X_j^{(0)}} K^M_p(O_{X_j,x_j} \text{ on } x_j) \to \cdots \to \bigoplus_{x_j \in X_j^{(p-1)}} K^M_1(O_{X_j,x_j} \text{ on } x_j) \to \bigoplus_{x_j \in X_j^{(p)}} K^M_0(O_{X_j,x_j} \text{ on } x_j) \to \cdots
\]
\[
\xrightarrow{d^p_{1,X_j}} \bigoplus_{x_j \in X_j^{(p)}} K^M_p(O_{X_j,x_j} \text{ on } x_j) \xrightarrow{d^p_{1,X_j}} \bigoplus_{x_j \in X_j^{(p+1)}} K^M_0(O_{X_j,x_j} \text{ on } x_j) \to 0.
\]

Definition 2.2 (Definition 3.4 and Definition 3.15 in [18]). For each positive integer \( p \), the \( p \)-th Milnor K-theoretic cycles and Milnor K-theoretic rational equivalence of \( X_j \), denoted \( Z^M_p(D_{\text{Perf}}(X_j)) \) and \( Z^M_{p,\text{rat}}(D_{\text{Perf}}(X_j)) \), are defined as
\[
Z^M_p(D_{\text{Perf}}(X_j)) := \text{Ker}(d^p_{1,X_j}),
\]
\[
Z^M_{p,\text{rat}}(D_{\text{Perf}}(X_j)) := \text{Im}(d^p_{1,X_j}).
\]

The \( p \)-th Milnor K-theoretic Chow group of \( X_j \) is defined to be:
\[
CH^M_p(D_{\text{Perf}}(X_j)) := \frac{\text{Ker}(d^p_{1,X_j})}{\text{Im}(d^p_{1,X_j})}.
\]

The reasons why we take the kernel of \( d^p_{1,X_j} \) to define \( Z^M_p(D_{\text{Perf}}(X_j)) \) and why we use Milnor K-groups with support, i.e., certain eigenspaces of Thomason-Trobaugh K-groups, not the entire Thomason-Trobaugh K-groups, are explained in Section 2.2 and Section 2.3 respectively.

2.1. Definition of tangent spaces. For \( Y \subset X \) a subvariety of codimension \( p \), let \( i : Y \to X \) be the inclusion, then \( i_*O_Y \) is a coherent \( O_X \)-module and can be resolved by a bounded complex of vector bundles on \( X \). Let \( Y' \) be a first order deformation of \( Y \), that is, \( Y' \subset X[\varepsilon] \) such that \( Y' \) is flat over Spec\((k[\varepsilon])\) and \( Y' \otimes_{k[\varepsilon]} k \cong Y \). Then \( i_*O_{Y'} \) can be resolved by a bounded complex of vector bundles on \( X[\varepsilon] \), where \( i : Y' \to X[\varepsilon] \).
Let $D_{\text{perf}}(X[\varepsilon])$ denote the derived category of perfect complexes of $O_X[\varepsilon]$-modules, and let $\mathcal{L}_{(i)}(X[\varepsilon]) \subset D_{\text{perf}}(X[\varepsilon])$ be defined as
\[
\mathcal{L}_{(i)}(X[\varepsilon]) := \{ E \in D_{\text{perf}}(X[\varepsilon]) \mid \text{codim}_{\text{Krull}}(\text{supph}(E)) \geq -i \},
\]
where the closed subset $\text{supph}(E) \subset X$ is the support of the total homology of the perfect complex $E$. The resolution of $i_*O_Y^\vee$, which is a perfect complex of $O_X[\varepsilon]$-module supported on $Y$, defines an element of the Verdier quotient $\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon])$, denoted $[i_*O_Y^\vee]$.

If $Y \subset X$ is a locally complete intersection of codimension $p$, there exists an open affine $U(\subset X)$ such that $U \cap Y$ is defined by a regular sequence $(f_1, \cdots, f_p)$, where $f_i \in O_X(U)$. Locally on $U$, $Y'$ is given by lifting $f_1, \cdots, f_p$ to $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$, where $g_i \in O_X(U)$.

We use $F_*(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ to denote the Koszul complex associated to the regular sequence $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$, which is a resolution of $O_X(U)[\varepsilon]/(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$:
\[
0 \longrightarrow F_0 \xrightarrow{A_0} F_1 \xrightarrow{A_1} \cdots \xrightarrow{A_2} F_{p-1} \xrightarrow{A_p} F_p,
\]
where each $F_i = \wedge^i(O_X(U)[\varepsilon])^{\oplus p}$ and $A_i : \wedge^i(O_X(U)[\varepsilon])^{\oplus p} \xrightarrow{\Delta} \wedge^{i+1}(O_X(U)[\varepsilon])^{\oplus p}$ are defined as usual. And one can define tangent to this Koszul complex, which is given by the following commutative diagram (we assume $g_2 = \cdots = g_p = 0$ for simplicity):
\[
\begin{split}
F_*(f_1, f_2, \cdots, f_p) & \quad \longrightarrow \quad O_X(U)/(f_1, f_2, \cdots, f_p) \\
F_p(\cong O_X(U)) & \quad \xrightarrow{g_1d_1f_2 \wedge \cdots \wedge d_pf_p} \quad F_0 \otimes O_{O_X(U)/Q}^{p-1}(\cong O_{O_X(U)/Q}^{p-1}),
\end{split}
\]
where $d = d_Q$.

However, in general, $Y \subset X$ may not be a locally complete intersection and the length of the perfect complex $[i_*O_Y^\vee]$, which is the resolution of $i_*O_Y^\vee$, may not equal to $p$. To modify this, instead of considering $[i_*O_Y^\vee]$ which is an element of the Verdier quotient $\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon])$, we consider its image in the idempotent completion $(\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon]))^\#$, denoted $[i_*O_Y^\vee]^\#$. And we have the following result:

**Theorem 2.3.** For each $i \in \mathbb{Z}$, localization induces an equivalence
\[
(\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon]))^\# \simeq \bigsqcup_{x[\varepsilon] \in X[\varepsilon]^{(-i)}} D_{x[\varepsilon]} \text{perf}(X[\varepsilon])
\]
between the idempotent completion of the Verdier quotient $\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon])$ and the coproduct over $x[\varepsilon] \in X[\varepsilon]^{(-i)}$ of the derived category of perfect complexes of $O_{X[\varepsilon],x[\varepsilon]}$-modules with homology supported on the closed
point \( x[\varepsilon] \in \text{Spec}(O_{X[x],x[\varepsilon]}) \). And consequently, one has

\[
K_0((\mathcal{L}(i)(X[\varepsilon])/\mathcal{L}(i-1)(X[\varepsilon]))^\#) \simeq \bigoplus_{x[\varepsilon] \in X[\varepsilon][i][-i]} K_0(D_{x[\varepsilon]}^\text{perf}(X[\varepsilon])).
\]

Let \( y \) be the generic point of \( Y \), \( Y \) is generically defined by a regular sequence of length \( p \): \( f_1, \cdots, f_p \), where \( f_1, \cdots, f_p \in O_{X,y} \). \( Y' \) is generically given by lifting \( f_1, \cdots, f_p \) to \( f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p \), where \( g_1, \cdots, g_p \in O_{X,y} \). We use \( F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p) \) to denote the Koszul complex associated to the regular sequence \( f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p \), which is a resolution of \( O_{X,y}[\varepsilon]/(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p) \).

Under the equivalence (2.2), the localization at the generic point \( y \) sends \([i_*O_{Y'}] \# \) to the Koszul complex \( F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p) \):

\[
[i_*O_{Y'}] \# \rightarrow F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p).
\]

And one can define tangent to the Koszul complex \( F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p) \) similarly as (2.1), which defines an element of \( H^p_y(\Omega^{p-1}_{X/Q}) \).

**Remark 2.4.** In general, we don’t know whether the above kind of Koszul complexes can generate the Grothendieck group \( \bigoplus_{x \in X^{(p)}} K_0^M(O_{X,x}[\varepsilon]) \) on \( x \) or not. So we can’t use only these Koszul complexes to define tangent space to cycle class groups and have to use the following formal approach.

We recall that the Milnor K-theoretic cycles and Chow groups in Definition 2.2 recover the classical ones for \( X \):

**Theorem 2.5** (Theorem 3.16 in [18]). For \( X \) a smooth projective variety over a field \( k \) of characteristic 0, for each positive integer \( p \), let \( Z^p(X), Z^p_{\text{rat}}(X) \) and \( CH^p(X) \) denote the classical \( p \)-cycles, rational equivalence and Chow groups respectively, then we have the following identifications

\[
\begin{align*}
Z^M_p(D_{\text{perf}}(X)) &= Z^p(X)_Q, \\
Z^M_{p,\text{rat}}(D_{\text{perf}}(X)) &= Z^p_{\text{rat}}(X)_Q, \\
CH^M_p(D_{\text{perf}}(X)) &= CH^p(X)_Q.
\end{align*}
\]

Recall that the tangent space to a functor \( \mathcal{F} \), denoted \( T\mathcal{F}(X) \), is defined to be

\[
T\mathcal{F}(X) := \text{Ker}\{\mathcal{F}(X[\varepsilon]) \xrightarrow{\varepsilon=0} \mathcal{F}(X)\}.
\]

Considering \( Z^M_p(D_{\text{perf}}(-)) \) as a functor, we are guided to the following definition, which answers Green-Griffiths’ Question [18].
Definition 2.6. For $X$ a smooth projective variety over a field $k$ of characteristic 0, for each positive integer $p$, the tangent space to $p$-cycles, denoted $TZ^p(X)$, is defined to be

$$TZ^p(X) := TZ^p(D_{\text{perf}}(X)) = \text{Ker}\{Z^p_{\text{M}}(D_{\text{perf}}(X)[\varepsilon])) \xrightarrow{\varepsilon=0} Z^p_{\text{M}}(D_{\text{perf}}(X))\}.$$

Similarly, the tangent space to rational equivalent classes, denoted $TZ^p_{\text{rat}}(X)$, is defined to be

$$TZ^p_{\text{rat}}(X) := TZ^p_{p,\text{rat}}(D_{\text{perf}}(X)) = \text{Ker}\{Z^p_{p,\text{rat}}(D_{\text{perf}}(X)[\varepsilon])) \xrightarrow{\varepsilon=0} Z^p_{p,\text{rat}}(D_{\text{perf}}(X))\}.$$
The following theorem has been proved in \cite{7, 18}.

**Theorem 2.7** \cite{7, Theorem 3.14 in 18}. Let $X$ be a smooth projective variety over a field $k$ of characteristic 0. For each integer $p \geq 1$, there exists the following commutative diagram in which the Zariski sheafification of each column is a flasque resolution of $\Omega_{X/Q}^{p-1}$, $K_p^M(O_X,e)$ and $K_p^M(O_X)$ respectively. The left arrows are induced by Chern character from $K$-theory to negative cyclic homology and the right ones are the natural maps sending $e$ to 0:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\bigoplus_{x \in X^{(p-1)}} H^p_{\text{rat}}(\Omega_{X/Q}^{p-1}) & \rightarrow & \bigoplus_{x \in X^{(p)}} H^p_{\text{rat}}(\Omega_{X/Q}^{p-1}) \\
\downarrow & \downarrow & \downarrow \\
\bigoplus_{x \in X^{(p+1)}} H^p_{\text{rat}}(\Omega_{X/Q}^{p+1}) & \rightarrow & \bigoplus_{x \in X^{(p+1)}} H^p_{\text{rat}}(\Omega_{X/Q}^{p+1}) \\
\downarrow & \downarrow & \downarrow \\
\vdots & \vdots & \vdots \\
\bigoplus_{x \in X^{(d)}} H^p_{\text{rat}}(\Omega_{X/Q}^{d}) & \rightarrow & \bigoplus_{x \in X^{(d)}} H^p_{\text{rat}}(\Omega_{X/Q}^{d}) \\
\downarrow & \downarrow & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

This diagram enables us to compute $TZ^p(X)$ and $TZ_{\text{rat}}^p(X)$. A quick diagram chasing shows

**Theorem 2.8.** Let $X$ be a smooth projective variety over a field $k$ of characteristic 0. For each integer $p \geq 1$, we have the following
identifications:
\[ TZ^p(X) \cong \text{Ker}(\partial_1^{p-1}), \]
\[ TZ^p_{\text{rat}}(X) \cong \text{Im}(\partial_1^{p-1}). \]

Evidently, \( TZ^p_{\text{rat}}(X) \) is a subspace of \( TZ^p(X) \). We use the quotient space to define the tangent space to Chow groups:

**Definition 2.9.** Let \( X \) be a smooth projective variety over a field \( k \) of characteristic 0. For each integer \( p \geq 1 \), the tangent space to \( \text{CH}^p(X) \), denoted \( T\text{CH}^p(X) \), is defined to be
\[ T\text{CH}^p(X) := \frac{TZ^p(X)}{TZ^p_{\text{rat}}(X)}. \]

**Theorem 2.10.** \( T\text{CH}^p(X) \) agrees with the formal tangent space \( T_f\text{CH}^p(X) \) defined by Bloch [4], where \( T_f\text{CH}^p(X) = H^p(X, \Omega_X^{p-1}/\mathbb{Q}) \).

**Proof.** It immediately follows from the fact that the Zariski sheafification of the left column in Theorem 2.7 is a flasque resolution of \( \Omega_X^{p-1}/\mathbb{Q} \).

For \( X \) a smooth projective surface over a field \( k \) of characteristic 0, by taking \( p = 2 \) in Theorem 2.8, we immediately see that

**Corollary 2.11.** For \( X \) a smooth projective surface over a field \( k \) of characteristic 0, Green and Griffiths’ definitions of \( TZ^2(X) \) and \( TZ^2_{\text{rat}}(X) \), recalled in Definition 1.2, agree with the formal Definition 2.6.

Next, we provide concrete examples of Milnor K-theoretic cycles which are from geometry. Let \( Y \subset X \) be a locally complete intersection of codimension \( p \). For a point \( x \in Y \subset X \), there exists an open affine \( U(\subset X) \) containing \( x \) such that \( U \cap Y \) is defined by a regular sequence \( f_1, \ldots, f_p \), where \( f_i \in O_{X,x} \). Let \( Y' \) be a first order deformation of \( Y \), locally on \( U \), \( Y' \) is given by lifting \( f_1, \ldots, f_p \) to \( f_1 + \varepsilon g_1, \ldots, f_p + \varepsilon g_p \), where \( g_i \in O_{X,x} \).

Let \( y \) be the generic point of \( Y \), then \( O_{X,y} = (O_{X,x})_{(f_1, \ldots, f_p)} \) and we see \( Y \) is generically defined by \( f_1, \ldots, f_p \). We use \( F_\bullet(f_1 + \varepsilon g_1, \ldots, f_p + \varepsilon g_p) \) to denote the Koszul complex associated to the regular sequence \( f_1 + \varepsilon g_1, \ldots, f_p + \varepsilon g_p \), which is a resolution of \( O_{X,y}[\varepsilon]/(f_1 + \varepsilon g_1, \ldots, f_p + \varepsilon g_p) \):
\[
0 \rightarrow F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \cdots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,
\]
where each \( F_i = \wedge^i(O_{X,y}[\varepsilon])^\oplus p \) and \( A_i : \wedge^i(O_{X,y}[\varepsilon])^\oplus p \to \wedge^{i-1}(O_{X,y}[\varepsilon])^\oplus p \) are defined as usual. Then \( F_\bullet(f_1 + \varepsilon g_1, \ldots, f_p + \varepsilon g_p) \in K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]). \)
Theorem 2.12 (Prop 4.12 of [8]). The Adams operations $\psi^k$ defined on perfect complexes, defined by Gillet-Soulé in [8], satisfy $\psi^k(F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)) = k^p F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$.

Hence, $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ is of eigenweight $p$ and can be considered as an element of $K_0^{(p)}(O_{X,y}[\varepsilon])$ on $y[\varepsilon]_\mathbb{Q}$:

$F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p) \in K_0^{(p)}(O_{X,y}[\varepsilon])$ on $y[\varepsilon]_\mathbb{Q} = K_0^M(O_{X,y}[\varepsilon])$ on $y[\varepsilon]$.

Moreover, we shall show $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ lies in the kernel of

$$d_{1,X[\varepsilon]}^{p,-p} : \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(p)}} K_0^{M}(O_{X,x}[\varepsilon]) \rightarrow \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(p+1)}} K_1^{M}(O_{X,x}[\varepsilon])$$

so that it is a Milnor K-theoretic $p$-cycle:

Theorem 2.13. For $X$ a smooth projective variety over a field $k$ of characteristic 0, let $Y \subset X$ be a locally complete intersection of codimension $p$. Suppose $Y$ is locally defined by a regular sequence $f_1, \cdots, f_p$, where $x$ is a point on $Y$ and $f_i \in O_{X,x}$. A first order deformation $Y'$ is locally given by lifting $f_1, \cdots, f_p$ to $f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p$, where $g_i \in O_{X,x}$. Then the Koszul complex $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p) \in \text{Ker}(d_{1,X[\varepsilon]}^{p,-p})$, i.e., $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p) \in Z_p^M(D\text{perf}(X[\varepsilon]))$.

The strategy for proving this theorem is to use the map induced by Chern character from K-theory to negative cyclic homology

$$\text{Ch} : K_0^M(O_{X,y}[\varepsilon]) \rightarrow H_y^p(\Omega_{X/\mathbb{Q}}^{p-1}),$$

mapping $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ to an element of $H_y^p(\Omega_{X/\mathbb{Q}}^{p-1})$, and then show its image under the differential

$$\partial_1^{p,-p} : \bigoplus_{x \in X^{(p)}} H_x^p(\Omega_{X/\mathbb{Q}}^{p-1}) \rightarrow \bigoplus_{x \in X^{(p+1)}} H_x^{p+1}(\Omega_{X/\mathbb{Q}}^{p-1})$$

is zero.

Proof. The map(left arrows) induced by Chern character from K-theory to negative cyclic homology in the commutative diagram of Theorem 2.7

$$\text{Ch} : K_0^M(O_{X,y}[\varepsilon]) \rightarrow H_y^p(\Omega_{X/\mathbb{Q}}^{p-1}),$$

can be described by a beautiful construction of Angénoil and Lejeune-Jalabert, see Lemme 3.1.1 on page 24 and Definition 3.4 on page 29 in [1] for details or Section 3 of [19] for a brief summary.
For our purpose, the Ch map on the Koszul complex $F_\bullet(f_1 + \varepsilon g_1, \ldots, f_p + \varepsilon g_p)$ can be described easily. For simplicity, we assume $g_2 = \cdots = g_p = 0$ in the following. To the Koszul complex,

$$0 \longrightarrow F_0 \xrightarrow{A_1} F_1 \xrightarrow{A_2} \cdots \xrightarrow{A_{p-1}} F_{p-1} \xrightarrow{A_p} F_p,$$

one defines the following class

$$\frac{1}{p!} dA_1 \circ dA_2 \circ \cdots \circ dA_p,$$

where $d = d_Q$ and each $dA_i$ is the matrix of absolute differentials. In other words,

$$dA_i \in \text{Hom}(F_i, F_{i-1} \otimes \Omega^1_{X,Y[e]/Q}).$$

The truncation map $\frac{\partial}{\partial \varepsilon} |_{\varepsilon=0}$ sends $\frac{1}{p!} dA_1 \circ dA_2 \circ \cdots \circ dA_p$ to $g_1 df_2 \wedge \cdots \wedge df_p$. So the image of $F_\bullet(f_1 + \varepsilon g_1, \ldots, f_p)$, under the Ch map, in $H^p_y(\Omega^{p-1}_{X,Y})$ is represented by the following diagram (an element of $\text{Ext}^p(\Omega_{X,Y}/(f_1, f_2, \ldots, f_p), \Omega^{p-1}_{X,Y/Q})$);

$$F_\bullet(f_1, f_2, \ldots, f_p) \xrightarrow{g_1 df_2 \wedge \cdots \wedge df_p} F_0 \otimes \Omega^{p-1}_{X,Y/Q} \cong \Omega^{p-1}_{X,Y/Q}.$$

The regular sequence $f_1, \ldots, f_p$, where $f_i \in O_{X,x}$, can be extended to be a system of parameter $f_1, \ldots, f_p, f_{p+1}, \ldots, f_d$ in $O_{X,x}$. The prime ideals $Q_i := (f_1, \ldots, f_p, f_i)$, where $i = p+1, \ldots, d$, define generic points $z_i \in X^{(p+1)}$. In the following, we consider the prime $Q_{p+1}$ and the generic point $z_{p+1}$, other cases work similarly.

Let $P = (f_1, \ldots, f_p)$ be the prime ideal defining the generic point $(of \ Y)y \in X^{(p)}$, $O_{X,Y} = (O_{X,z_{p+1}})_P$. The above diagram (2.3) can be rewritten as, denoted $[\alpha]$,

$$F_\bullet(f_1, f_2, \ldots, f_p) \xrightarrow{g_1 f_{p+1} df_2 \wedge \cdots \wedge df_p} F_0 \otimes \Omega^{p-1}_{(O_{X,z_{p+1}})_P/Q} \cong \Omega^{p-1}_{(O_{X,z_{p+1}})_P/Q}.$$

Here, $F_\bullet(f_1, f_2, \ldots, f_p)$ is of the form

$$0 \longrightarrow F_p \xrightarrow{A_1} F_{p-1} \xrightarrow{A_2} \cdots \xrightarrow{A_{p-1}} F_1 \xrightarrow{A_p} F_0,$$

where each $F_i = \bigwedge^i((O_{X,z_{p+1}})_P)^{\otimes p}$. Since $f_{p+1} \notin (f_1, \ldots, f_p)$, $f_{p+1}$ exists in $(O_{X,z_{p+1}})_P$, we can write $g_1 df_2 \wedge \cdots \wedge df_p = \frac{g_1 f_{p+1}}{f_{p+1}} df_2 \wedge \cdots \wedge df_p$. 

The image of the diagram (2.4) under the differential
\[ \partial_1^{p,-p} : \bigoplus_{x \in X^{(p)}} H^p_x(\Omega^{p-1}_{X/Q}) \rightarrow \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/Q}) \]
is represented by the following diagram
\[
\begin{array}{c}
F_\bullet(f_1, f_2, \ldots, f_p, f_{p+1}) \\
F_{p+1}(\cong O_{X,z_{p+1}}) \xrightarrow{g_1 f_{p+1} d_{2} \wedge \cdots \wedge d_{p}} F_0 \otimes \Omega^{p-1}_{O_{X,z_{p+1}}} = \Omega^{p-1}_{O_{X,z_{p+1}}}.
\end{array}
\]
The complex \( F_\bullet(f_1, f_2, \ldots, f_p, f_{p+1}) \) is of the form
\[
0 \rightarrow \wedge^{p+1}(O_{X,z_{p+1}})^{\otimes p+1} \xrightarrow{A_{p+1}} \wedge^{p}(O_{X,z_{p+1}})^{\otimes p+1} \rightarrow \cdots.
\]
Let \( \{e_1, \ldots, e_{p+1}\} \) be a basis of \((O_{X,z_{p+1}})^{\otimes p+1}\), the map \( A_{p+1} \) is
\[
e_1 \wedge \cdots \wedge e_{p+1} \rightarrow \sum_{j=1}^{p+1} (-1)^j f_j e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots e_{p+1},
\]
where \( \hat{e}_j \) means to omit the \( j \)-th term.

Noting \( f_{p+1} \) appears in \( A_{p+1} \),
\[
g_1 f_{p+1} d_{2} \wedge \cdots \wedge d_{p} \equiv 0 \in \text{Ext}_{O_{X,z_{p+1}}}^{p+1}(O_{X,z_{p+1}}/(f_1, f_2, \cdots, f_p, f_{p+1}), \Omega^{p-1}_{O_{X,z_{p+1}}} / Q),
\]
so \( \partial_1^{p,-p}(\alpha) = 0 \). There exists the following commutative diagram,
which is part of the commutative diagram in Theorem 2.7
\[
\begin{array}{c}
\bigoplus_{x \in X^{(p)}} H^p_x(\Omega^{p-1}_{X/Q}) \xleftarrow{\partial_1^{p,-p}} \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(p)}} K^M_{0}(O_{X, x[\varepsilon]} \text{ on } x[\varepsilon]) \\
\downarrow_{\partial_1^{p,-p}} \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/Q}) \xrightarrow{d_{1,X[\varepsilon]}} \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(p+1)}} K^M_{-1}(O_{X, x[\varepsilon]} \text{ on } x[\varepsilon]).
\end{array}
\]
This gives the following commutative diagram
\[
\begin{array}{c}
[\alpha] \xleftarrow{\text{Ch}} F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p) \\
\downarrow_{\partial_1^{p,-p}} \xleftarrow{d_{1,X[\varepsilon]}} d_{1,X[\varepsilon]}^{p,-p}(F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p)),
\end{array}
\]
which shows \( d_{1,X[\varepsilon]}^{p,-p}(F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p)) = 0 \).

In general, \( Y \subset X \) may not be a locally complete intersection, and
the associated Koszul complex \( F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p) \) may not be a Milnor K-theoretic \( p \)-cycle. However, we can find another subscheme \( Z \subset X \) of codimension \( p \) and \( Z' \in T_{Z}\text{Hilb}^p(X) \) such that the two
Koszul complexes associated $Y'$ and $Z'$ defines a Milnor K-theoretic $p$-cycle

To fix notations, let $W \subset Y$ be a subvariety of codimension 1 in $Y$, with generic point $w$. One assumes $W$ is generically defined by $f_1, f_2, \cdots, f_p, f_{p+1}$ and $Y$ is generically defined by $f_1, f_2, \cdots, f_p$. Let $y$ be the generic point of $Y$, one has $O_{X,y} = (O_{X,w})_P$, where $P$ is the idea $(f_1, f_2, \cdots, f_p) \subset O_{X,w}$.

$Y'$ is generically given by $(f_1 + \varepsilon g_1, f_2 + \varepsilon g_2, \cdots, f_p + \varepsilon g_p)$, where $g_i \in O_{X,y} = (O_{X,w})_P$. For simplicity, we assume $g_2 = \cdots = g_p = 0$. We can write $g_1 = \frac{a}{b}$, where $a, b \in O_{X,w}$ and $b \notin P$. $b$ is either in or not in the maximal idea $(f_1, f_2, \cdots, f_p, f_{p+1}) \subset O_{X,w}$.

**Theorem 2.14** (Theorem 4.7 in [19]). For $Y' \in T_Y \text{Hilb}^p(X)$ which is generically defined by $(f_1 + \varepsilon g_1, f_2, \cdots, f_p)$, where $g_1 = \frac{a}{b} \in O_{X,y} = (O_{X,w})_P$, we use $F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p)$ to denote the Koszul complex associated to $f_1 + \varepsilon g_1, f_2, \cdots, f_p$,

- Case 1: If $b \notin (f_1, f_2, \cdots, f_p, f_{p+1})$, then $F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p) \in Z^p_\bullet(D^\text{perf}(X[\varepsilon]))$.

- Case 2: If $b \in (f_1, f_2, \cdots, f_p, f_{p+1})$, we reduce to considering $b = f_{p+1}$. Then there exists $Z \subset X$ which is generically defined by $(f_{p+1}, f_2, \cdots, f_p)$ and exists $Z' \in T_Z \text{Hilb}^p(X)$ which is generically defined by $(f_{p+1} + \varepsilon \frac{a}{f_1}, f_2, \cdots, f_p)$ such that $F_\bullet(f_1 + \varepsilon \frac{a}{f_1}, f_2, \cdots, f_p) + F_\bullet(f_{p+1} + \varepsilon \frac{a}{f_1}, f_2, \cdots, f_p) \in Z^p_\bullet(D^\text{perf}(X[\varepsilon]))$.

### 2.2. Why take kernel

In this subsection, we explain the reasons why we use the kernel of $d^p_{1,X_j}$ to define $Z^M_p(D^\text{perf}(X_j))$ in Definition 2.2 instead of taking $\bigoplus_{x_j \in X_j^{(p)}} K_0^M(O_{X_j,x_j}$ on $x_j$).

1. As explained in the beginning of Section 2.1, in general, the length of the perfect complex $i_*O_{Y'}$, which is the resolution of $i_*O_{Y'}$, may not equal to $p$. To modify this, we need to look at its image $[i_*O_{Y'}]^\#$ in the idempotent completion $(\mathcal{L}_{(-p)}(X[\varepsilon]))/\mathcal{L}_{(-p-1)}(X[\varepsilon])^\#$. From the K-theoretic viewpoint, taking idempotent completion can result in the appearance of negative K-groups. We should include negative K-groups into the study of deformation of cycles, so we use the kernel of $d^p_{1,X_j}$ to define $Z^M_p(D^\text{perf}(X_j))$. In Section 3.2, negative K-groups will be used for obstruction issues.
2). From the geometric viewpoint, taking the kernel of $$d_{1,X^{[\varepsilon]}}^p: \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{[p]}} K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \to \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{[p+1]}} K_1^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])$$ to define $$Z^M_p(D_{\text{perf}}(X[\varepsilon]))$$ can produce the desirable tangent space. This can be explained by the following example.

Let $$X$$ be a smooth projective surface over a field $$k$$ of characteristic 0, we consider the 1-cycles $$Z^1(X)$$ on $$X$$ and study its tangent space $$TZ^1(X)$$. For simplicity, we look at the sheaf level, that is, we look at the tangent sheaf $$TZ^1(X)$$ to the 1-cycles $$Z^1(X)$$.

Let $$Z^1(X)$$ be the Zariski sheaf of 1-cycles on $$X$$, we have the following short exact sequence of sheaves:

$$0 \to O_X^* \to K(X)^* \to \bigoplus_{y \in X^{(1)}} i_{y,*}H^1_y(O_X) \xrightarrow{\partial_{1,-2}} \bigoplus_{x \in X^{(2)}} i_{x,*}H^2_x(O_X) \to 0,$$

where $$K(X)$$ is the function field of $$X$$. It is known that the tangent sheaves to $$O_X^*$$ and $$K(X)^*$$ are $$O_X$$ and $$K(X)$$ respectively, and there exists the following short exact sequence of sheaves:

$$0 \to O_X \to K(X) \to PP_X \to 0,$$

where $$PP_X$$ is the sheaf of principal parts. This suggests that

**Definition 2.15** (page 100 [10]). The tangent sheaf $$TZ^1(X)$$ to the 1-cycles $$Z^1(X)$$ is defined to be

$$TZ^1(X) := PP_X.$$

To related this definition with the formal Definition 2.6, we note that the Cousin resolution of $$O_X$$ is

$$0 \to O_X \to K(X) \to \bigoplus_{y \in X^{(1)}} i_{y,*}H^1_y(O_X) \xrightarrow{\partial_{1,-2}} \bigoplus_{x \in X^{(2)}} i_{x,*}H^2_x(O_X) \to 0.$$

For $$X$$ a smooth projective surface over a field $$k$$ of characteristic 0, taking $$p = 1$$ in Theorem 2.8, we see the tangent sheaf is $$\ker(\partial_{1,-2})$$. The two exact sequences (2.5)(2.6) show that

$$PP_X \cong K(X)/O_X \cong \ker(\partial_{1,-2}).$$

This proves:

**Corollary 2.16.** For 1-cycles $$Z^1(X)$$ on $$X$$, the formal Definition 2.6 (at the sheaf level) agrees with the Definition 2.15 by Green-Griffiths.
If we don’t use the kernel of $d_{1,X}^{-1}$, but use $\bigoplus_{y \in X^{(1)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$, to define Milnor K-theoretic 1-cycles, then the tangent sheaf becomes $\bigoplus_{y \in X^{(1)}} H^1_y(O_X)$, which is obviously not the desirable one.

In the next, combining with Green-Griffiths’ results in [10], we construct a concrete element of the kernel of $d_{1,X}^{-1}$.

**Theorem 2.17.** Let $X$ be a smooth projective surface over a field $k$ of characteristic 0, for $p = 1$ in Theorem 2.7, we have the following commutative diagram. The left arrows are induced by Chern character from $K$-theory to negative cyclic homology and the right ones are the natural maps sending $\varepsilon$ to 0:

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
k(X) & K_1(k(X)[\varepsilon])_\mathbb{Q} & K_1(k(X))_\mathbb{Q} \\
\downarrow & \downarrow & \downarrow \\
\bigoplus_{y \in X^{(1)}} H^1_y(O_X) & \bigoplus_{y \in X^{(1)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_\mathbb{Q} & \bigoplus_{y \in X^{(1)}} K_0(O_{X,y} \text{ on } y)_\mathbb{Q} \\
\downarrow & \downarrow & \downarrow \\
\bigoplus_{x \in X^{(2)}} H^2_x(O_X) & \bigoplus_{x \in X^{(2)}} K_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon])_\mathbb{Q} & \bigoplus_{x \in X^{(2)}} K_{-1}(O_{X,x} \text{ on } x)_\mathbb{Q} = 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
\]

Let’s explain why one can use $K_0(O_{X,y} \text{ on } y)_\mathbb{Q}$ to replace $K_0^M(O_{X,y} \text{ on } y)$ (defined in Definition 2.1) in the above diagram. One notes that $K_0^{(j)}(O_{X,y} \text{ on } y)_\mathbb{Q} \cong K_0^{(j-1)}(k(y)) = 0$, except for $j = 1$. That is,

$$K_0^{(1)}(O_{X,y} \text{ on } y)_\mathbb{Q} = K_0(O_{X,y} \text{ on } y)_\mathbb{Q}.$$ 

This says $K_0^M(O_{X,y} \text{ on } y) = K_0(O_{X,y} \text{ on } y)_\mathbb{Q}$. Similar arguments for other $K$-groups in the middle and right columns in the above diagram.

Let $Y_1$ and $Y_2$ be two curves on $X$ with generic point $y_1$ and $y_2$ respectively. For simplicity, we work locally in Zariski topology and assume $Y_1$ and $Y_2$ intersect transversely at a point $x$. Around the point $x$, we can write

$$Y_1 = \text{div}(f_1); \ Y_2 = \text{div}(f_2).$$
Take \( g \in O_{X,x} \) such that \( g(x) \neq 0 \), we consider \( O_{X,x}[\varepsilon]/(f_1 f_2 + \varepsilon g) \). The Koszul resolution of \( O_{X,x}[\varepsilon]/(f_1 f_2 + \varepsilon g) \),

\[
L^\bullet: 0 \to O_{X,x}[\varepsilon] \xrightarrow{f_1 f_2 + \varepsilon g} O_{X,x}[\varepsilon],
\]
defines an element of \( K_0(\mathcal{L}_-^{-1}(X[\varepsilon])/\mathcal{L}_-^{-2}(X[\varepsilon]))^\#) \).

**Theorem 2.18.** \( L^\bullet \in \text{Ker}(d_{1,X[\varepsilon]}^{-1}) \), i.e., \( L^\bullet \in Z_1^M(D_{\text{perf}}(X[\varepsilon])) \).

**Proof.** Under the isomorphism in Theorem 2.3

\[
K_0((\mathcal{L}^{-1}_-)(X[\varepsilon])/\mathcal{L}^{-2}_-(X[\varepsilon]))^\#) \simeq \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K_0(D_{\text{perf}}(X[\varepsilon])),
\]

\( L^\bullet \) decomposes into the direct sum of

\[
L^1_\bullet: 0 \to (O_{X,x})(f_1)[\varepsilon] \xrightarrow{f_1 + \varepsilon g/ f_2} (O_{X,x})(f_1)[\varepsilon]
\]

and

\[
L^2_\bullet: 0 \to (O_{X,x})(f_2)[\varepsilon] \xrightarrow{f_2 + \varepsilon g/ f_1} (O_{X,x})(f_2)[\varepsilon].
\]

Noting \( O_{X,y_1} = (O_{X,x})(f_1) \), we have \( L^1_\bullet \in K_0(O_{X,y_1}[\varepsilon] \text{ on } y_1[\varepsilon]) \). Similarly, \( L^2_\bullet \in K_0(O_{X,y_2}[\varepsilon] \text{ on } y_2[\varepsilon]) \).

The following diagram, associated to \( L^1_\bullet \),

\[
\begin{align*}
(O_{X,x})(f_1) & \xrightarrow{f_1} (O_{X,x})(f_1) \xrightarrow{g} (O_{X,x})(f_1)/(f_1) \xrightarrow{0} \\
(O_{X,x})(f_1) & \xrightarrow{f_2} (O_{X,x})(f_1)/(f_1)
\end{align*}
\]
gives an element \( \alpha \) in \( \text{Ext}^1_{O_{X,y_1}}(O_{X,y_1}/(f_1), O_{X,y_1}) \), which further defines an element in \( H^1_{y_1}(O_X) \) and it is the image of \( L^1_\bullet \) under the map in Theorem 2.17.

(2.7) \( \text{Ch} : \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_Q \to \bigoplus_{y \in X^{(1)}} H^1_y(O_X) \).

Similarly, the following diagram, associated to \( L^2_\bullet \),

\[
\begin{align*}
(O_{X,x})(f_2) & \xrightarrow{f_2} (O_{X,x})(f_2) \xrightarrow{g} (O_{X,x})(f_2)/(f_2) \xrightarrow{0} \\
(O_{X,x})(f_2) & \xrightarrow{f_1} (O_{X,x})(f_2)/(f_2)
\end{align*}
\]
gives an element \( \beta \) in \( \text{Ext}^1_{O_{X,y_2}}(O_{X,y_2}/(f_2), O_{X,y_2}) \), which further defines an element in \( H^1_{y_2}(O_X) \) and it is the image of \( L^2_\bullet \) under the Ch map.
where Green-Griffiths observes that $H$ in diagram:

\[ \Ext_{X,x}^{p+2} \longrightarrow \Ext_{X,x}^{p+1} \longrightarrow \Ext_{X,x}^{p} \longrightarrow \Ext_{X,x}^{p-1} \longrightarrow 0 \]

Similarly, $\partial^{1,-1}$ maps $\beta$ in $H^2_x(O_X)$ to:

\[ \begin{array}{rcl}
O_{X,x} & \xrightarrow{(f_1,-f_2)^T} & O_{X,x}^{\oplus 2} \\
O_{X,x} & \xrightarrow{g} & O_{X,x}.
\end{array} \]

Noting the commutative diagram below

\[ \begin{array}{ccc}
O_{X,x} & \xrightarrow{(f_1,-f_2)^T} & O_{X,x}^{\oplus 2} \\
\downarrow & & \downarrow \\
O_{X,x} & \xrightarrow{(f_2,-f_1)^T} & O_{X,x}^{\oplus 2} \\
\downarrow & & \downarrow \\
O_{X,x} & \xrightarrow{g} & O_{X,x}
\end{array} \]

where $M$ stands for the matrix

\[ \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}. \]

Green-Griffiths observes that $\partial^{1,-1}(\alpha)$ and $\partial^{1,-1}(\beta)$ are negative of each other in $\Ext^2_{O_{X,x}}(O_{X,x}/(f_1, f_2), O_{X,x})$. Hence, $\partial^{1,-2}(\alpha + \beta)$ is 0 in $H^2_x(O_X)$. Therefore, $d^{1,-1}_{1,X[\varepsilon]}(L^*) = 0$ because of the commutative diagram:

\[ \begin{array}{ccc}
\bigoplus_{y \in X^{(1)}} H^1_y(O_X) & \xleftarrow{\partial^{1,-1}} & \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K_0(O_{X,y[\varepsilon]} \text{ on } y[\varepsilon]) \\
\bigoplus_{x \in X^{(2)}} H^2_x(O_X) & \xleftarrow{\approx} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(2)}} K_{-1}(O_{X,x[\varepsilon]} \text{ on } x[\varepsilon]).
\end{array} \]

The above argument seems formal, so it's convenient to intuitively explain the meaning of taking the kernel of $d^{1,-1}_{1,X[\varepsilon]}$. This has been done by using residue by Green-Griffiths [10].

**Alternative explanation by using residue, due to Green-Griffiths [10]** (page 103-104 and the summary on page 119) To fix notations, let $Y_1$ and $Y_2$ be two curves on $X$. It is well-known that tangent vectors to the curves $Y_1$ and $Y_2$ are given by normal vector fields,

\[ v_1 \in H^0(N_{Y_1/X}), v_2 \in H^0(N_{Y_2/X}). \]
For simplicity, we work locally in Zariski topology and assume $Y_1$ and $Y_2$ intersect transversely at a point $x$. Around the point $x$, we can write
\[ Y_1 = \text{div}(f_1); \quad Y_2 = \text{div}(f_2). \]
Then $v_1$ and $v_2$ can be expressed as
\[ v_1 = w_1 \frac{\partial}{\partial f_1}, \quad v_2 = w_2 \frac{\partial}{\partial f_2}, \]
for some functions $w_1$ and $w_2$. For our purpose, we take $w_1 = \frac{g}{f_2}$ and $w_2 = \frac{h}{f_1}$, then
\[ v_1 = \frac{g}{f_2} \frac{\partial}{\partial f_1}, \quad v_2 = \frac{h}{f_1} \frac{\partial}{\partial f_2}. \]

For $\omega = df_1 \wedge df_2$, we consider the Poincaré residue:
\[
\left\{ \begin{array}{l}
v_1 \omega = \text{Res}_{Y_1} \left( \frac{g df_1 \wedge df_2}{f_1 f_2} \right) = \frac{g df_2}{f_2} \in \Omega^1_{K(Y_1)/\mathbb{C}}; \\
v_2 \omega = \text{Res}_{Y_2} \left( \frac{h df_1 \wedge df_2}{f_1 f_2} \right) = -\frac{h df_1}{f_1} \in \Omega^1_{K(Y_2)/\mathbb{C}}.
\end{array} \right.
\]
We further take the residue at $x$:
\[ \text{Res}_x \left( \frac{g df_2}{f_2} \right) = g, \quad \text{Res}_x \left( -\frac{h df_1}{f_1} \right) = -h. \]
The sum of the residues is
\[ \text{Res}_x \left( \frac{g df_2}{f_2} \right) + \text{Res}_x \left( -\frac{h df_1}{f_1} \right) = g - h. \]
When $g = h$, the sum of the residues is 0.

\textbf{Conclusion:} for $v_1 = \frac{g}{f_2} \frac{\partial}{\partial f_1}$ and $v_2 = \frac{h}{f_1} \frac{\partial}{\partial f_2}$,
\[ \text{Res}_x(v_1 \omega) + \text{Res}_x(v_1 \omega) = 0. \]

How does this connect to K-groups?
For normal vectors
\[ v_1 = \frac{g}{f_2} \frac{\partial}{\partial f_1}, \quad v_2 = \frac{h}{f_1} \frac{\partial}{\partial f_2}, \]
$v_1$ corresponds to $f_1 + \varepsilon \frac{g}{f_2}$ and $v_2$ corresponds to $f_2 + \varepsilon \frac{g}{f_1}$. In other words, $v_1$ corresponds to the complex
\[ L^*_1 : 0 \to (O_{X,x})(f_1)[\varepsilon] \xrightarrow{f_1 + \varepsilon \frac{g}{f_2}} (O_{X,x})(f_1)[\varepsilon]. \]
and \( v_2 \) corresponds to the complex

\[
L^\bullet_2: 0 \to (O_{X,x})_{(\varepsilon)} \xrightarrow{f_2 + \varepsilon g} (O_{X,x})_{(\varepsilon)}.
\]

**Conclusion:** \( \text{Res}_x(v_1|\omega) + \text{Res}_x(v_2|\omega) = 0 \) corresponds to \((L^\bullet_1 + L^\bullet_2) \in \text{Ker}(d_1^{p-1})\) in Theorem 2.18

**Remark 2.19.** One may ask why there is no necessary to take kernel in Quillen’s or Soulé’s proofs of Bloch’s formula in [16, 17]. That’s because negative K-groups are zero in this case, \( K_{-1}(k(x)) = 0 \). If we take kernel, the cycles class group \( Z^p(X) \) is still identified with \( \bigoplus_{x \in X^{(p)}} K_0(k(x)) \).

**2.3. Why use Milnor K-theory.** In the following, we explain why we use Milnor K-groups with support, i.e., certain eigenspaces of Thomason-Trobaugh K-groups, not the entire Thomason-Trobaugh K-groups, to define cycles and Chow groups in Definition 2.2.

In 2012 Fall, the author met a question on describing certain eigenspaces of K-groups and he E-mailed this question to Christophe Soulé for help. Christophe Soulé suggested that if the author’s question is true, then it should be only true for Milnor K-theory and guided the author to read Theorem 5 in [17]:

In our setting, \( X \) is smooth projective over \( k \), so the Gersten complex has the form of

\[
0 \to K_p(k(X)) \to \cdots \to \bigoplus_{x \in X^{(p-1)}} K_1(O_{X,x} \text{ on } x) \to \bigoplus_{x \in X^{(p)}} K_0(O_{X,x} \text{ on } x) \to 0,
\]

which agrees with the Gersten complex by Quillen [16] because of Dévissage:

\[
0 \to K_p(k(X)) \to \cdots \to \bigoplus_{x \in X^{(p-1)}} K_1(k(x)) \to \bigoplus_{x \in X^{(p)}} K_0(k(x)) \to 0.
\]

For \( x \in X^{(p)} \), Adams operations can decompose \( K_0(O_{X,x} \text{ on } x) \) and \( K_0(k(x)) \) into direct sums of eigenspaces respectively. Moreover, Riemann-Roch without denominator, due to Soulé [17], says

\[
K_0^{(j)}(O_{X,x} \text{ on } x)_Q = K_0^{(j-p)}(k(x))_Q.
\]

For \( j = p \),

\[
K_0^{(p)}(O_{X,x} \text{ on } x)_Q = K_0^{(0)}(k(x))_Q = K_0(k(x))_Q.
\]

This forces to

\[
K_0^{(j)}(O_{X,x} \text{ on } x)_Q = 0, \text{ for } j \neq p.
\]
So only $K^p_0(O_{X,x}$ on $x)_Q$ is needed to study $Z^p(X)_Q$.

To give an example, for $X$ a smooth projective three-fold over a field $k$ of characteristic 0, a point $x \in X^{(3)}$ is defined by $(f, g, h)$ and a first order deformation of $x$ is given by $(f + \varepsilon f_1, g + \varepsilon g_1, h + \varepsilon h_1)$. According to Theorem 2.12 the Koszul complex $F_\bullet(f + \varepsilon f_1, g + \varepsilon g_1, h + \varepsilon h_1)$ associated to $(f + \varepsilon f_1, g + \varepsilon g_1, h + \varepsilon h_1)$ is of weight 3:

$$F_\bullet(f + \varepsilon f_1, g + \varepsilon g_1, h + \varepsilon h_1) \in K_0^{(3)}(O_{X,x}[\varepsilon] \text{ on } x)_Q,$$

and $F_\bullet(f + \varepsilon f_1, g + \varepsilon g_1, h + \varepsilon h_1) \notin K_0^{(2)}(O_{X,x}[\varepsilon] \text{ on } x)_Q$.

So we ignore $K_0^{(2)}(O_{X,x}[\varepsilon] \text{ on } x)_Q(\cong H^3_x(O_X) \neq 0)$, and use only $K_0^{(3)}(O_{X,x}[\varepsilon] \text{ on } x)_Q$ to define Milnor K-theoretic 3-cycles $Z_3^M(D^{Perf}(X[\varepsilon]))$, which is the first order deformation of $Z^3(X)_Q$.

### 3. Higher order deformation-obstructions

Let $X$ be a smooth projective variety over a field $k$ of characteristic 0. For each positive integer $j$, $X_j = X \times_k \text{Spec}(k[t]/t^{j+1})$ is the $j$-th order infinitesimally trivial deformation of $X$. For any integer $m$, let $K_m^M(O_{X_j,x_j}$ on $x_j, t)$ denote the relative K-group, that is, the kernel of the natural projection

$$K_m^M(O_{X_j,x_j}$ on $x_j) \xrightarrow{t=0} K_m^M(O_{X,x}$ on $x)$.

Recall that we have proved the following isomorphisms in [18]:

**Theorem 3.1 (Corollary 3.11 in [18]).** Let $X$ be a smooth projective variety over a field $k$ of characteristic 0 and let $x \in X^{(i)}$. Chern character induces the following isomorphisms between relative K-groups and local cohomology groups:

$$K_m^M(O_{X_j,x_j}$ on $x_j, t) \cong H^i_x((\Omega^{m+i-1}_X)_Q)^{\oplus j})$.

So we have the following split exact sequence

$$0 \rightarrow H^i_x((\Omega^{m+i-1}_X)_Q)^{\oplus j}) \rightarrow K_m^M(O_{X_j,x_j}$ on $x_j) \xrightarrow{t=0} K_m^M(O_{X,x}$ on $x) \rightarrow 0.$$

Moreover, it is known that from the computation of Hochschild(cyclic) homology of truncated polynomials, $H^i_x((\Omega^{m+i-1}_X)_Q)^{\oplus j})$ carries additional structure:

$$H^i_x((\Omega^{m+i-1}_X)_Q)^{\oplus j}) \cong tH^i_x(\Omega^{m+i-1}_X)_Q) \oplus \cdots \oplus t^iH^i_x(\Omega^{m+i-1}_X)_Q).$$

To simplify the notations, we use $A$ to denote $K_m^M(O_{X,x}$ on $x)$ and $B$ to denote $H^i_x(\Omega^{m+i-1}_X)_Q$, then we have

$$K_m^M(O_{X_j,x_j}$ on $x_j) \cong A \oplus tB \oplus \cdots t^iB.$$
The natural map
\[ f_j : X_j \to X_{j+1}, \]
induces \( f_j^* : K^M_m(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \to K^M_m(O_{X_j,x_j} \text{ on } x_j) \). Moreover, there exists the following commutative diagram
\[
\begin{array}{c}
K^M_m(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \\
\downarrow \\
A \oplus tB \oplus \cdots \oplus t^j B \oplus t^{j+1} B \\
\end{array}
\begin{array}{c}
f_j^* \\
\approx \\
f_j^* \\
\approx \\
\end{array}
\begin{array}{c}
K^M_m(O_{X_j,x_j} \text{ on } x_j) \\
A \oplus tB \oplus \cdots \oplus t^j B \\
\end{array}
\]
and exists the following short exact sequence of abelian groups:
\[
0 \to B \to A \oplus tB \oplus \cdots \oplus t^j B \oplus t^{j+1} B \to 0.
\]
This shows that

**Lemma 3.2.** For \( X \) a smooth projective variety over a field \( k \) of characteristic 0, for each positive integer \( j \) and \( x \in X^{(j)} \), there exists the following short exact sequence of abelian groups, where \( m \) is any integer,
\[
0 \to H^j_x(\mathcal{O}^m_{X/Q}) \to K^M_m(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \xrightarrow{f_j^*} K^M_m(O_{X_j,x_j} \text{ on } x_j) \to 0.
\]

**3.1. Obstructions and negative K-groups.** The natural map \( f_j : X_j \to X_{j+1} \) induces the following commutative diagram:
\[
\begin{array}{c}
\bigoplus_{x_{j+1} \in X_{j+1}^{(p+1)}} K^M_0(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \\
\downarrow \quad d^p_{1,X_{j+1}} \\
\bigoplus_{x_j \in X_j^{(p)}} K^M_0(O_{X_j,x_j} \text{ on } x_j) \\
\end{array}
\begin{array}{c}
f_j^* \\
d^p_{1,X_j} \\
\end{array}
\begin{array}{c}
\bigoplus_{x_{j+1} \in X_{j+1}^{(p+1)}} K^M_{-1}(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \\
\downarrow \quad d^p_{1,X_{j+1}} \\
\bigoplus_{x_j \in X_j^{(p+1)}} K^M_{-1}(O_{X_j,x_j} \text{ on } x_j) \\
\end{array}
\]
so it further induces \( f_j^* : Z^M_p(D^{\text{perf}}(X_{j+1})) \to Z^M_p(D^{\text{perf}}(X_j)) \), recall that \( Z^M_p(D^{\text{perf}}(X_j)) \) is defined as the kernel of \( \text{Ker}(d^p_{1,X_j}) \) in Definition 2.2.

**Definition 3.3.** Given \( \xi_j \in Z^M_p(D^{\text{perf}}(X_j)) \), an element \( \xi_{j+1} \in Z^M_p(D^{\text{perf}}(X_{j+1})) \) is called a deformation of \( \xi_j \), if \( f_j^*(\xi_{j+1}) = \xi_j \).

\( \xi_j \) and \( \xi_{j+1} \) can be formally written as finite sums
\[
\sum_{x_j} \lambda_j \cdot \{x_j\}_{\text{red}} \quad \text{and} \quad \sum_{x_{j+1}} \lambda_{j+1} \cdot \{x_{j+1}\}_{\text{red}},
\]
where \( \sum_{x_j} \lambda_j \in \ker(d_{1,X_j}^{\mathbb{P}^N}) \subset \bigoplus_{x_j \in X_j^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j) \) and \( \overline{x_j} \) is the closed reduced scheme associated to \( \{x_j\} \).

Since \( \overline{x_j} = \overline{x_{j+1}} \), when we deform from \( \xi_j \) to \( \xi_{j+1} \), we deform the coefficients, i.e., we deform from \( \sum_{x_j} \lambda_j \) to \( \sum_{x_{j+1}} \lambda_{j+1} \).

Since \( f_j^* : \bigoplus_{x_{j+1} \in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \rightarrow \bigoplus_{x_{j} \in X_{j}^{(p)}} K_0^M(O_{X,x_j} \text{ on } x_j) \) is surjective, see lemma 3.2 given any \( \xi_j \in Z^M_p(D_{\text{perf}}(X_j)) \), there exists \( \xi_{j+1} \in \bigoplus_{x_{j+1} \in X_{j+1}^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \) such that \( f_j^*(\xi_{j+1}) = \xi_j \). We would like to know whether \( \xi_{j+1} \in Z^M_p(D_{\text{perf}}(X_{j+1})) \).

An easy diagram chasing shows \( f_j^* d_{1,X_{j+1}}^{\mathbb{P}^N}(\xi_{j+1}) = 0 \), so \( d_{1,X_{j+1}}^{\mathbb{P}^N}(\xi_{j+1}) \in \ker(f_j^*) = \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(O_{X/Q}^{p-1}) \), see lemma 3.2 take \( m = -1 \) and \( i = p+1 \). If \( d_{1,X_{j+1}}^{\mathbb{P}^N}(\xi_{j+1}) = 0 \), then we can lift \( \xi_j \) to \( \xi_{j+1} \).

**Definition 3.4.** The obstruction space for lifting elements in \( Z^M_p(D_{\text{perf}}(X_j)) \) to \( Z^M_p(D_{\text{perf}}(X_{j+1})) \) is defined to be \( \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(O_{X/Q}^{p-1}) \).

### 3.2. Obstruction issues-versus Hilbert scheme.

For each positive integer \( j \), let \( X_j \) denote the \( j \)-th trivial deformation of \( X \). Let \( Y \subset X \) be a subvariety of codimension \( p \). Obstruction issues asks whether it is possible to lift \( Y \) to \( Y_j \) successively, where \( Y_j \subset X_j \) with suitable assumptions.

It is a common phenomenon that obstructions can occur in deformation, though the deformation of \( X \) is trivial. It is well known that, considering \( Y \) as an element of \( \text{Hilb}(X) \), the tangent space \( T_Y \text{Hilb}(X) \) may be obstructed.

However, Green-Griffiths predicts that, considering \( Y \) as an element of \( Z^p(X) \), we can eliminate obstructions in their program [10]:

**Obstruction issues** (page 187-190 in [10]): There are essentially four (not mutually exclusive) possibilities:

- (i) \( TZ^p(X) \) may be obstructed. That is, there exists some \( \tau \in TZ^p(X) \) such that, thinking of \( \tau \) as a map
  \[ \text{Spec}(k[\varepsilon]/(\varepsilon^2)) \rightarrow Z^p(X) \],
this map cannot be lifted to a map
\[ \text{Spec}(k[[\varepsilon]]/(\varepsilon^{k+1})) \to Z_p(X) \]
for some \( k \geq 2 \).

- (ii) \( TZ^p(X) \) is formally unobstructed. That is, for any \( \tau \in TZ^p(X) \), \( \tau \) may be lifted to a map
\[ \text{lim}(\text{Spec}(k[[\varepsilon]]/(\varepsilon^{k+1}))) \to Z_p(X). \]

- (iii) \( TZ^p(X) \) is formally unobstructed, but there exists \( \tau \in TZ^p(X) \) which is not the tangent to a geometric arc in \( Z_p(X) \).

- (iv) Every \( \tau \in TZ^p(X) \) is the tangent to a geometric arc in \( Z_p(X) \).

For \( p = 1 \), this question was solved by TingFai Ng in his Ph.D thesis,

\textbf{Theorem 3.5} (Theorem 1.3.3 in [15]). \textit{Every }\( \tau \in TZ^1(X) \text{ is the tangent to a geometric arc in } Z^1(X). \)

For \( p \geq 2 \), Green-Griffiths observes that

\textbf{Proposition 3.6} ((10.11) on page 189 in [10]). \textit{For }\( p \geq 2 \text{, there exists } X \text{ and } \tau \in TZ^p(X) \text{ which is not the tangent to a geometric arc in } Z_p(X). \)

This means only possibilities (i)-(iii) can occur for \( p \geq 2 \). Green-Griffiths conjectures that

\textbf{Conjecture 3.7} (page 190 in [10]). (ii) and (iii) above are the only possibilities that actually occur for \( p \geq 2 \).

Because of the Proposition 3.6 above, all we need to show is \( TZ^p(X) \) is formally unobstructed. The above question(ii) is expressed in a way, as if \( Z^p(X) \) were a scheme. In fact, we know \( Z^p(X) \) can’t be treated as a scheme. So we restate this conjecture as follows:

\textbf{Conjecture 3.8}. [10] \textit{Let }\( X \text{ be a smooth projective variety over a field } k \text{ of characteristic } 0 \text{. For each positive integer } p \text{, } TZ^p(X) \text{ is formally unobstructed. That is, for any } \tau \in TZ^p(X) \text{, } \tau \text{ can be lifted to } \tau_j \in Z^p(D^\text{perf}(X_j)) \text{ successively, where } j = 1, 2, \ldots. \)

To get a feeling of how to eliminate obstructions to deforming cycles, we firstly look at locally complete intersections.

For \( X \) a smooth projective variety over a field \( k \) of characteristic 0 and \( Y \subset X \) a subvariety, which is a locally complete intersection of
codimension \( p \). We assume that, on an open affine \( U_i \subset X \), \( Y \cap U_i \) is defined by a regular sequence \((f^i_1, \cdots, f^i_p)\), where \( f^i_j \in O_X(U_i) \). On another open affine \( U_j \subset X \), \( Y \cap U_j \) is defined by a regular sequence \((f^j_1, \cdots, f^j_p)\), where \( f^j_i \in O_X(U_j) \).

Let \( Y' \) be a first order deformation of \( Y \) in \( X[t] \), then \( Y' \cap U_i \) is given by lifting \((f^i_1, \cdots, f^i_p)\) to \((f^i_1 + \varepsilon g^i_1, \cdots, f^i_p + \varepsilon g^i_p)\), where \( g^i_j \in O_X(U_i) \). And \( Y' \cap U_j \) is given by lifting \((f^j_1, \cdots, f^j_p)\) to \((f^j_1 + \varepsilon g^j_1, \cdots, f^j_p + \varepsilon g^j_p)\), where \( g^j_i \in O_X(U_j) \).

On the intersection \( U_{ij} = U_i \cap U_j \), there exists two liftings which defines an element of \( \alpha_{ij} \in \Gamma(U_{ij}, \mathcal{N}_{Y/X}) \), where \( \mathcal{N}_{Y/X} \) is the normal sheaf. On the intersection \( U_{ijk} = U_i \cap U_j \cap U_k \) of three open affine subschemes, there are three liftings which defines \( \alpha_{ij}, \alpha_{jk}, \alpha_{ik} \). One checks \((\alpha_{ij})\) is a Čech 1-cocycle, which is the obstruction of finding a global lifting \( Y' \), see Theorem 6.2(page 47) of [12] for details.

Let \( y \in Y \) be the generic point, then \( y \in U_i \). One has \( O_{X,Y} = O_{U_i,y} = O_X(U_i)(f^i_1, \cdots, f^i_p) \), with maximal ideal \((f^i_1, \cdots, f^i_p)\). So \( Y \) is generically generated by \((f^i_1, \cdots, f^i_p)\) and the Koszul complex \( F_\bullet(f^i_1, \cdots, f^i_p) \in K^{(b)}_0(O_{X,y} on y) \subset Z^M_p(D_{\text{perf}}(X)) \).

We have shown that, in Theorem 2.13 the Koszul complex \( F_\bullet(f^i_1 + \varepsilon g^i_1, \cdots, f^i_p + \varepsilon g^i_p) \in Z^M_p(D_{\text{perf}}(X_1)) \), which lifts \( F_\bullet(f^i_1, \cdots, f^i_p) \). So the obstructions of gluing (as a subscheme) \( Y' \cap U_i \) and \( Y' \cap U_j \) along the intersection \( U_{ij} = U_i \cap U_j \) obvious vanishes (To lift K-theoretic cycles, we don’t need to glue). By mimicking the proof of Theorem 2.13 we can show that the Koszul complex \( F_\bullet(f^i_1 + \varepsilon g^i_1 + \varepsilon^2 h^i_1, \cdots, f^i_p + \varepsilon g^i_p + \varepsilon^2 h^i_p) \), where \( \varepsilon^2 \neq 0, \varepsilon^3 = 0 \), and \( h^i_p \in O_X(U_i) \), is a Milnor K-theoretic \( p \)-cycle and lifts \( F_\bullet(f^i_1 + \varepsilon g^i_1, \cdots, f^i_p + \varepsilon g^i_p) \). Furthermore, we can lift \( F_\bullet(f^i_1 + \varepsilon g^i_1 + \varepsilon^2 h^i_1, \cdots, f^i_p + \varepsilon g^i_p + \varepsilon^2 h^i_p) \) to higher order successively. In summary, we have shown that

**Lemma 3.9.** For \( X \) a smooth projective variety over a field \( k \) of characteristic 0 and \( Y \subset X \) a subvariety, which is locally defined by a regular sequence \((f_1, \cdots, f_p)\), let \( F_\bullet(f_1, \cdots, f_p) \) denote the associated Koszul complex, which defines a K-theoretic cycle in \( Z^M_p(D_{\text{perf}}(X)) \), we can lift this K-theoretic cycle to higher order successively.

In general, \( Y \subset X \) may not be a locally complete intersection. To eliminate the obstructions to lifting \( Y \) to higher order, we need to use the following strategy which has been known to Green-Griffiths [10] (page 187-189) and Ng [13] for the divisor case. We should introduce another cycle \( Z \) to help \( Y \) to eliminate obstructions. As a cycle,

\[ Y = (Y + Z) - Z, \]
and the cycle $Z$ should satisfy that

- (1) One can lift $(Y + Z)$ to higher order successively, i.e., $Z$ helps $Y$ to eliminate obstructions.
- (2) $Z$ doesn’t introduce new obstructions.

To illustrate the idea, we sketch an example of curves on a three-fold and refer the readers to [20] for details. For $X$ a nonsingular projective 3-fold over a field $k$ of characteristic 0, let $Y \subset X$ be a curve with generic point $y$. For a point $x \in Y \subset X$ which is defined by $(f, g, h)$, we assume $Y$ is generically defined by $(f, g)$. The Koszul complex $F_\bullet(f, g)$ is a K-theoretic 2-cycle:

$$F_\bullet(f, g) \in \mathbb{K}_0(O_{X,y} \text{ on } y) \subset Z^M_2(D_{\text{perf}}(X)).$$

For a first order deformation $Y'$ which is generically given by $(f + \frac{\varepsilon}{h} g)$, the Koszul complex $F_\bullet(f + \frac{\varepsilon}{h} g)$ associated to $(f + \frac{\varepsilon}{h} g)$ is in $K_0(O_{X,y} [\varepsilon] \text{ on } y)$, but we can show it is not in $Z^M_2(D_{\text{perf}}(X[\varepsilon]))$, see Example 4.4 in [19]. So $F_\bullet(f + \frac{\varepsilon}{h} g)$ is not a first order deformation of $F_\bullet(f, g)$. To modify this, we consider the curve $Z$ on $X$ which is generically defined by $(h, g)$. As a cycle,

$$Y = (Y + Z) - Z.$$

As a K-theoretic cycle,

$$F_\bullet(f, g) = (F_\bullet(f, g) + F_\bullet(h, g)) - F_\bullet(h, g).$$

To lift $F_\bullet(f, g)$ is equivalent to lifting $(F_\bullet(f, g) + F_\bullet(h, g))$ and $F_\bullet(h, g)$ respectively. We can show that $(F_\bullet(f + \frac{\varepsilon}{h} g) + F_\bullet(h + \frac{\varepsilon}{f} g))$ is in $Z^M_2(D_{\text{perf}}(X[\varepsilon]))$. And it is a first order deformation of $(F_\bullet(f, g) + F_\bullet(h, g))$, and can be lifted to higher order successively. On the other hand, $F_\bullet(h, g)$ is always a first order deformation of itself, which means we fix $F_\bullet(h, g)$ so it doesn’t introduce new obstructions. Consequently,

$$(F_\bullet(f + \frac{\varepsilon}{h} g) + F_\bullet(h + \frac{\varepsilon}{f} g)) - F_\bullet(h, g)$$

is a first order deformation of $F_\bullet(f, g)$, and can be lifted to higher order successively.

However, as pointed out in Remark 2.4, in general, we don’t know whether the Milnor K-theoretic cycles $Z^M_p(D_{\text{perf}}(X_j))$ are generated by these Koszul complexes or not. So, to answer Green-Griffiths’ Conjecture 3.8, we have to give a formal argument which relies on the following theorem:
Theorem 3.10 ([7], Theorem 3.14 in [18]). For $X$ a smooth projective variety over a field $k$ of characteristic 0, $X_j = X \times_k \text{Spec}(k[t]/(t^{j+1}))$, where $j$ is any positive integer. For each integer $p \geq 1$, there exists the following commutative diagram in which the Zariski sheafification of each column is a flasque resolution of $(\Omega^{p-1}_{X/k})^j$, $K^M_p(O_{X_j})$ and $K^M_p(O_X)$ respectively. The left arrows are induced by Chern characters from $K$-theory to negative cyclic homology and the right ones are the natural maps sending $\varepsilon$ to 0:

\[
\begin{array}{cccc}
\ldots & \longleftarrow & \ldots & \longleftarrow \\
\oplus H^p_x((\Omega^{p-1}_{X/k})^j) & \longleftarrow & \oplus K^{M,(p)}_{j-1}(O_{X_j}, x_j) & \longleftarrow \\
\oplus H^p_x((\Omega^{p-1}_{X/k})^j) & \longleftarrow & \oplus K^{M,(p)}_{j-1}(O_{X_j}, x_j) & \longleftarrow \\
\oplus H^p_x((\Omega^{p-1}_{X/k})^j) & \longleftarrow & \oplus K^{M,(p)}_{j-1}(O_{X_j}, x_j) & \longleftarrow \\
\oplus H^p_x((\Omega^{p-1}_{X/k})^j) & \longleftarrow & \oplus K^{M,(p)}_{j-1}(O_{X_j}, x_j) & \longleftarrow \\
\oplus H^p_x((\Omega^{p-1}_{X/k})^j) & \longleftarrow & \oplus K^{M,(p)}_{j-1}(O_{X_j}, x_j) & \longleftarrow \\
\end{array}
\]

Using this theorem, we answer Green-Griffiths’ Conjecture 3.8 affirmatively:

Theorem 3.11. The Conjecture 3.8 is true, that is, $T^pZ(X)$ is formally unobstructed.
**Proof.** For any positive integer \( j \) and given any \( \xi_j \in Z_p^M(D_{\text{perf}}(X_j)) \), we need to show \( \xi_j \) can be lifted to an element of \( Z_p^M(D_{\text{perf}}(X_{j+1})) \). There exists the commutative diagram (part of the diagram in Theorem 3.10),

\[
\begin{align*}
&\bigoplus_{x \in X^{(p)}} H^p_x((\Omega_{X/Q}^{-1})^\otimes_j) & \xleftarrow{\text{Ch}} & \bigoplus_{x \in X^{(p)}} K^M_0(O_{X_j,x_j} \text{ on } x_j) \\
&\downarrow{\partial^{p,-p}_{1,j}} & & \downarrow{\partial^{p,-p}_{1,j}} \\
&\bigoplus_{x \in X^{(p+1)}} H^{p+1}_x((\Omega_{X/Q}^{-1})^\otimes_j) & \xleftarrow{\text{Ch}} & \bigoplus_{x \in X^{(p+1)}} K^{M}_{-1}(O_{X_{j+1},x_j} \text{ on } x_j),
\end{align*}
\]

where the maps \( \text{Ch} \) are induced by Chern characters from K-theory to negative cyclic homology. It is obvious that \( \text{Ch}(\xi_j) \in \text{Ker}(\partial^{p,-p}_{1,j}) \).

There exists a similar commutative diagram for \( j+1 \):

\[
\begin{align*}
&\bigoplus_{x \in X^{(p)}} H^p_x((\Omega_{X/Q}^{-1})^\otimes_{j+1}) & \xleftarrow{\text{Ch}} & \bigoplus_{x \in X^{(p)}} K^M_0(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \\
&\downarrow{\partial^{p,-p}_{1,j+1}} & & \downarrow{\partial^{p,-p}_{1,j+1}} \\
&\bigoplus_{x \in X^{(p+1)}} H^{p+1}_x((\Omega_{X/Q}^{-1})^\otimes_{j+1}) & \xleftarrow{\text{Ch}} & \bigoplus_{x \in X^{(p+1)}} K^{M}_{-1}(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}).
\end{align*}
\]

As explained on page 20 (isomorphism (3.2)), \( \bigoplus_{x \in X^{(p)}} H^p_x((\Omega_{X/Q}^{-1})^\otimes_j) \) carries additional structure:

\[
\bigoplus_{x \in X^{(p)}} H^p_x((\Omega_{X/Q}^{-1})^\otimes_j) \cong t \bigoplus_{x \in X^{(p)}} H^p_x((\Omega_{X/Q}^{-1})) \oplus \cdots \oplus t^j \bigoplus_{x \in X^{(p)}} H^p_x((\Omega_{X/Q}^{-1})).
\]

And the differential

\[
\partial^{p,-p}_{1,j} : \bigoplus_{x \in X^{(p)}} H^p_x((\Omega_{X/Q}^{-1})^\otimes_j) \rightarrow \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x((\Omega_{X/Q}^{-1})^\otimes_j)
\]

is \( t\partial^{p,-p}_{1,0} \oplus \cdots \oplus t^j \partial^{p,-p}_{1,j} \):

\[
\begin{align*}
&\bigoplus_{x \in X^{(p)}} H^p_x((\Omega_{X/Q}^{-1})^\otimes_j) \xrightarrow{\cong} t \bigoplus_{x \in X^{(p)}} H^p_x((\Omega_{X/Q}^{-1})) \oplus \cdots \oplus t^j \bigoplus_{x \in X^{(p)}} H^p_x((\Omega_{X/Q}^{-1}) \\
&\downarrow{\partial^{p,-p}_{1,j}} \quad \quad \quad \quad \quad \downarrow{t\partial^{p,-p}_{1,0} \oplus \cdots \oplus t^j \partial^{p,-p}_{1,j}} \\
&\bigoplus_{x \in X^{(p+1)}} H^{p+1}_x((\Omega_{X/Q}^{-1})^\otimes_j) \xrightarrow{\cong} t \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x((\Omega_{X/Q}^{-1})) \oplus \cdots \oplus t^j \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x((\Omega_{X/Q}^{-1})),
\end{align*}
\]

where \( \partial^{p,-p}_{1,p} : \bigoplus_{x \in X^{(p)}} H^p_x((\Omega_{X/Q}^{-1})) \rightarrow \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x((\Omega_{X/Q}^{-1})) \).
Under these isomorphisms, $\text{Ch}(\xi_j)$ can be written as $ta_1 + \cdots + t^ja_j$, where each $a_i \in \bigoplus_{x \in X^{(p)}} H^p_x((\Omega^{p-1}_{X/Q}))$ and $\partial^{p,-p}_t(a_i) = 0$. There exists a similar isomorphism for $j + 1$:

$$\bigoplus_{x \in X^{(p)}} H^p_x((\Omega^{p-1}_{X/Q})^\oplus j+1) \cong \bigoplus_{x \in X^{(p)}} H^p_x((\Omega^{p-1}_{X/Q})) \oplus \cdots \oplus t^{j+1} \bigoplus_{x \in X^{(p)}} H^p_x((\Omega^{p-1}_{X/Q})).$$

And the differential

$$\partial^{p,-p}_{i,j+1} : \bigoplus_{x \in X^{(p)}} H^p_x((\Omega^{p-1}_{X/Q})^\oplus j+1) \to \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x((\Omega^{p-1}_{X/Q})^\oplus j+1)$$

is $t\partial^{p,-p}_i \oplus \cdots \oplus t^{j+1}\partial^{p,-p}_i$.

We can always lift $ta_1 + \cdots + t^ja_j$ to $\eta_{j+1} := ta_1 + \cdots + t^ja_j + t^{j+1}a_{j+1}$ (note $t^{j+1} \neq 0$ here), where $a_{j+1} \in \text{Ker}(\partial^{p,-p}_i)$. So $\eta_{j+1} \in \text{Ker}(\partial^{p,-p}_{i,j+1})$. Hence, we can always lift $\text{Ch}(\xi_j)$ to $\eta_{j+1} \in \text{Ker}(\partial^{p,-p}_{i,j+1})$.

Since the map

$$\text{Ch} : \bigoplus_{x_{j+1} \in X^{(p)}_{j+1}} K^M_0(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \to \bigoplus_{x \in X^{(p)}} H^p_x((\Omega^{p-1}_{X/Q})^\oplus j+1)$$

is surjective, there exists $\xi_{j+1} \in \bigoplus_{x_{j+1} \in X^{(p)}_{j+1}} K^M_0(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1})$ such that $\text{Ch}(\xi_{j+1}) = \eta_{j+1}$.

By the naturality of Chern character, there exists the following commutative diagram:

$$
\begin{array}{ccc}
\bigoplus_{x \in X^{(p)}} H^p_x((\Omega^{p-1}_{X/Q})^\oplus j+1) & \xleftarrow{\text{Ch}} & \bigoplus_{x_{j+1} \in X^{(p)}_{j+1}} K^M_0(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \\
\downarrow{t^{j+1}=0} & & \downarrow{t^{j+1}=0} \\
\bigoplus_{x \in X^{(p)}} H^p_x((\Omega^{p-1}_{X/Q})^\oplus j) & \xleftarrow{\text{Ch}} & \bigoplus_{x \in X^{(p)}} K^M_0(O_{X,x} \text{ on } x_{j+1})
\end{array}
$$

So we have the following commutative diagram:

$$
\begin{array}{ccc}
\eta_{j+1} = \text{Ch}(\xi_{j+1}) & \xleftarrow{\text{Ch}} & \xi_{j+1} \\
\downarrow{t^{j+1}=0} & & \downarrow{t^{j+1}=0} \\
\eta_{j+1}|_{t^{j+1}=0} & \xleftarrow{\text{Ch}} & \xi_{j+1}|_{t^{j+1}=0}.
\end{array}
$$

This says $\eta_{j+1}|_{t^{j+1}=0} = \text{Ch}(\xi_{j+1}|_{t^{j+1}=0})$. On the other hand, since $\eta_{j+1}$ lifts $\text{Ch}(\xi_j)$, $\eta_{j+1}|_{t^{j+1}=0} = \text{Ch}(\xi_j)$. Hence, $\xi_{j+1}|_{t^{j+1}=0} - \xi_j$ is in the kernel.
of the map

$$\text{Ch} : \bigoplus_{x_j \in X^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j) \to \bigoplus_{x \in X^{(p)}} H^p_x((\Omega^{p-1}_{X/Q})^\otimes j),$$

which is $\bigoplus_{x \in X^{(p)}} K_0^M(O_{X,x} \text{ on } x)$. In other words, $\xi_{j+1}|_{t^{j+1}=0} = \xi_j + W$, for some $W \in \bigoplus_{x \in X^{(p)}} K_0^M(O_{X,x} \text{ on } x)$.

As a cycle, $\xi_j$ can be written as a formal sum

$$(3.4) \quad \xi_j = (\xi_j + W) - W.$$  

Here, since $\bigoplus_{x \in X^{(p)}} K_0^M(O_{X,x} \text{ on } x)$ is a direct summand of $\bigoplus_{x_j \in X^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j)$, both $W$ and $\xi_j + W$ are in $\bigoplus_{x_j \in X^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j)$.

Similarly, since $\bigoplus_{x \in X^{(p)}} K_0^M(O_{X,x} \text{ on } x)$ is also a direct summand of $\bigoplus_{x_j \in X^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j)$, and the cycle $\xi_{j+1} - W \in \bigoplus_{x_j \in X^{(p)}} K_0^M(O_{X_j,x_j} \text{ on } x_j)$ satisfies

$$(\xi_{j+1} - W)|_{t^{j+1}=0} = \xi_{j+1}|_{t^{j+1}=0} - W = \xi_j + W - W = \xi_j.$$  

Moreover, $\text{Ch}(\xi_{j+1} - W) = \text{Ch}(\xi_{j+1}) = \eta_{j+1} \in \text{Ker}(\partial^{p-p}_{1,j+1})$, hence, $\xi_{j+1} - W \in Z^M_p(D_{\text{perf}}(X_{j+1})) := \text{Ker}(d^{p-p}_{1,X_{j+1}})$ because of the commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{x \in X^{(p)}} H^p_x((\Omega^{p-1}_{X/Q})^\otimes j) & \xrightarrow{\text{Ch}} & \bigoplus_{x_{j+1} \in X^{(p)}} K_0^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}) \\
\downarrow \partial^{p-p}_{1,j+1} & & \downarrow d^{p-p}_{1,X_{j+1}} \\
\bigoplus_{x \in X^{(p+1)}} H^{p+1}_x((\Omega^{p-1}_{X/Q})^\otimes j) & \xrightarrow{\text{Ch}} & \bigoplus_{x_{j+1} \in X^{(p+1)}} K^M(O_{X_{j+1},x_{j+1}} \text{ on } x_{j+1}).
\end{array}
$$

In conclusion, $\xi_{j+1} - W \in Z^M_p(D_{\text{perf}}(X_{j+1})) := \text{Ker}(d^{p-p}_{1,X_{j+1}})$ can lift $\xi_j$.

In Section 4 of [9], Green-Griffiths conjectures that

**Conjecture 3.12** (page 506 [9]). Let $X$ be a smooth projective variety over a field $k$ of characteristic 0, for each positive integer $p$, $TZ^p_{\text{rat}}(X)$ is formally unobstructed.
For any positive integer \( j \) and given any \( \eta_j \in \mathbb{Z}_{p,\text{rat}}^M(D_{\text{perf}}(X_j))(\equiv \text{Im}(d_{1,X_j}^{0,-p})) \), we want to know whether \( \eta_j \) can be lifted to \( \eta_{j+1} \in \mathbb{Z}_{p,\text{rat}}^M(D_{\text{perf}}(X_{j+1})) \).

By definition, \( \eta_{j} = d_{1,X_{j+1}}^{0,-p}(\xi_{j}) \), for some \( \xi_{j} \in \bigoplus_{x_j \in X_j^{(p-1)}} K_1^M(O_{X_j,x_j} \text{ on } x_j) \).

Since \( \xi_{j} \) is surjective, see lemma 3.2, we can always lift \( d \). Then \( \text{gram} : \)

\[
\begin{align*}
\text{Theorem 3.13.} & \quad \text{The conjecture 3.12 is true, i.e., } TZ^p_{\text{rat}}(X) \text{ is unobstructed. So we have} \\
& \text{Recall that in Definition 2.2, the } p \text{-th Milnor K-theoretic Chow group is defined to be:} \\
& CH_p^M(D_{\text{perf}}(X_j)) \equiv \frac{Z_p^M(D_{\text{perf}}(X_j))}{Z_{p,\text{rat}}^M(D_{\text{perf}}(X_j))}. \\
& \text{The proof of Theorem 3.11 says, for any given } [\xi_j] \in CH_p^M(D_{\text{perf}}(X_j)), \text{ we can lift it to } [\xi_{j+1}] \in CH_p^M(D_{\text{perf}}(X_{j+1})). \\
& \text{Recall that we have shown that, in } [18], CH_p^M(D_{\text{perf}}(X_j)) \text{ satisfies Soule’s variant of Bloch-Quillen identification:} \\
& CH_p^M(D_{\text{perf}}(X_j)) = H^p(X, K_p^M(O_{X_j}))_Q, \\
& \text{where } K_p^M(O_{X_j}) \text{ is the Milnor K-theory sheaf associated to the presheaf} \\
& U \rightarrow K_p^M(O_{X_j}(U)).
\end{align*}
\]
So we have proved the following fact, which is already known to Green-Griffiths and can be deduced from Proposition 2.6 of [9] (recalled below),

**Corollary 3.14.** [9] For each positive integer \( j \), \( X_j = X \times_k \text{Spec}(k[[t]]/(t^{j+1})) \), for any given \([\xi_j] \in H^p(X, K_p^M(O_{X_j}))_\mathbb{Q}\), we can lift it to \([\xi_{j+1}] \in H^p(X, K_p^M(O_{X_{j+1}}))_\mathbb{Q}\).

We briefly explain how to prove this Corollary by Green-Griffiths [9]. As a trivial version of (2.1) of [9] (page 498) or (2.8) of Proposition 2.3 of [6], there exists the following short exact sequence of sheaves

\[
0 \to \Omega_{X/Q}^{p-1} \to K^M_p(O_{X_{j+1}}) \to K^M_p(O_{X_j}) \to 0.
\]

The associated long exact sequence is of the form (3.5)

\[
\cdots \to H^p(X, K^M_p(O_{X_{j+1}}))_\mathbb{Q} \to H^p(X, K^M_p(O_{X_j}))_\mathbb{Q} \xrightarrow{\delta} H^{p+1}(X, \Omega_{X/Q}^{p-1}) \to \cdots.
\]

The arithmetic cycle mapping

\[
\eta : H^p(X, K^M_p(O_{X_j})) \to H^p(X, \Omega_{X_j/Q}^p)
\]

is induced by the dlog map

\[
K^M_p(O_{X_j}) \to \Omega_{X_j/Q}^p
\]

\[
\{r_1, \ldots, r_p\} \to \frac{dr_1}{r_1} \wedge \cdots \wedge \frac{dr_p}{r_p},
\]

where \( d = d_Q \).

Let \( \theta_j \) denote the \( j \)-th Kodaira-Spencer class, see Section 3.1 (page 492) of [9] for the definition.

**Lemma 3.15** (Proposition 2.6 of [9] (page 502)). The coboundary map \( \delta \) in the above long exact sequence (3.5) is given by

\[
\delta(\xi_j) = \theta_j | \eta(\xi_j).
\]

In other words, the obstruction to lifting \( \xi_j \in H^p(X, K^M_p(O_{X_j})) \) to \( H^p(X, K^M_p(O_{X_{j+1}})) \) is given by

\[
\delta(\xi_j) = \theta_j | \eta(\xi_j),
\]

where \( \eta(\xi_j) \) is the arithmetic cycle class of \( \xi_j \).

In our situation, \( X_j = X \times \text{Spec}(k[[t]]/(t^{j+1})) \), the Kodaira-Spencer class \( \theta_j = 0 \) (see page 492 of [9]), so the coboundary map \( \delta = 0 \):

\[
\cdots \to H^p(X, K^M_p(O_{X_{j+1}}))_\mathbb{Q} \to H^p(X, K^M_p(O_{X_j}))_\mathbb{Q} \xrightarrow{\delta=0} H^{p+1}(X, \Omega_{X/Q}^{p-1})
\]

This proves Corollary 3.14 above.
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