Quantization of Chern-Simons topological invariants for H-type and L-type quantum systems

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In 2+1-dimensions (2+1D), a gapped quantum phase with no symmetry (i.e. a topological order) can have a thermal Hall conductance \( \kappa_{xy} = c \frac{e^2}{h} T \), where the dimensionless \( c \) is called chiral central charge. If there is a \( U_1 \) symmetry, a gapped quantum phase can also have a Hall conductance \( \sigma_{xy} = \nu \frac{e^2}{h} \), where the dimensionless \( \nu \) is called filling fraction. In this paper, we derive some quantization conditions of \( c \) and \( \nu \), via a cobordism approach to define Chern–Simons topological invariants which are associated with \( c \) and \( \nu \). In particular, we obtain quantization conditions that depend on the ground state degeneracies on Riemannian surfaces, and quantization conditions that depend on the type of spacetime manifolds where the topological partition function is non-zero.

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I. INTRODUCTION

Different phases of matter are characterized by different orders in them.[1, 2] Topological order[3–5] is a new kind of order beyond Landau symmetry breaking order.[1, 2] It cannot be characterized by the local order parameters associated with the symmetry breaking, but can be characterized by topological quantum field theories.[6] Physically, we need to use new topological quantum numbers to characterize topological orders. In 2+1D, the chiral central charge \( c \)[7, 8] (i.e. the thermal Hall conductance \( \kappa_{xy} = c \frac{e^2}{h} T \)[9]) of the edge excitations is one such topological quantum number. When there is a \( U_1 \) symmetry, the Hall conductance \( \sigma_{xy} = \nu \frac{e^2}{h} \) is another such topological quantum number. In other words, the chiral central charge \( c \) (partially) characterize 2+1D topological orders, and the dimensionless Hall conductance \( \nu \) (together with \( c \)) (partially) characterize 2+1D topological orders with \( U_1 \)-symmetry.[10]
In this paper, we like to address the issue of the quantization of those topological quantum numbers. At first sight, it seems that $c$ and $\nu$ are not quantized since they do not have to be integers. On the other hand, $c$ and $\nu$ must be rational numbers, suggesting that they do satisfy certain quantization conditions which can be complicated. A quantization condition on $\nu$ for interacting systems was first obtained by Niu–Thouless–Wu in Ref. 11. In this paper, we are going to generalize their result and discuss those complicated quantization conditions that apply to all topological orders with $U_1$-symmetry.

Before calculating those quantization conditions, we need to distinguish two types of quantum systems: H-type and L-type.12 The two types of quantum systems have different quantization conditions.

The quantum systems in condensed matter physics are all H-type, i.e. are all described by local Hamiltonians defined on smooth spatial manifolds. The quantum systems in high energy theory and quantum field theory, such as the topological quantum field theories, are L-type, i.e. are described by local Lagrangian path integrals on smooth spacetime manifolds. Mathematically, the L-type topological orders may correspond to unitary fully extended topological quantum field theories. The H-type topological orders may correspond to topless fully extended topological quantum field theories (or $d + c$ dimensional fully extended topological quantum field theories, see [13]).

The gapped liquid states8,9 with $U_1$ symmetry include $U_1$-symmetry enriched topological (SET) orders10, 16–18 and $U_1$-symmetry protected trivial (SPT) orders.19–25 For 2+1D H-type gapped states with $U_1$ symmetry, the quantization of $c$ and $\nu$ is determined from the ground state degeneracy on closed genus $g$ surfaces, and the main results are given by eqn. (20) and eqn. (63).

In particular, for H-type invertible gapped states1 with $U_1$ symmetry, $c$ and $\nu$ satisfy the following quantization conditions:

- **bosonic systems**: $c, \nu = 0 \mod 2$.
- **fermionic systems**: $c, \nu, \frac{\nu - c}{2} = 0 \mod 1$. (1)

The even Hall conductance $\nu = 0 \mod 2$ for bosonic SPT phases was pointed out in Ref. 22–25. We remark that the known H-type invertible topological orders with $U_1$ symmetry do not saturate the above quantization conditions. For example, we do not know any H-type bosonic invertible topological orders with $c = 2$. Also, we do not know any H-type fermionic $U_1^f$ symmetric invertible topological orders with $\nu - c = 2$.

For 2+1D L-type gapped states with $U_1$ symmetry, the quantization of $c$ and $\nu$ is determined by the non-vanishing topological partition functions on certain types of spacetime manifolds, such as orientable or spin$^c$. We find that, for L-type invertible gapped phases with $U_1$ symmetry, $c$ and $\nu$ satisfy the following quantization conditions:

- **bosonic systems**: $c = 0 \mod 8$, $\nu = 0 \mod 2$.
- **fermionic systems**: $c, \nu, \frac{\nu - c}{8} = 0 \mod 1$. (2)

The above L-type gapped phases with $U_1$ symmetry have a framing anomaly, i.e. the partition functions of the states dependent on the choices of the framing of the spacetime manifold.6

If the microscopic Lagrangian path integral is manifestly independent of choices of the framing (for example, only dependent on the diffeomorphism equivalent classes of spacetime metrics), then the resulting partition must also be independent of choices of the framing. The resulting phase is said to be free of framing anomaly. In this case, $c$ must be multiple of 24, and the above quantization conditions reduce to

- **bosonic systems**: $c = 0 \mod 24$, $\nu = 0 \mod 2$.
- **fermionic systems**: $c = 0 \mod 24$, $\nu = 0 \mod 8$. (3)

It is interesting to note that the bosonic $E_8$ quantum Hall state with $c = 8$ (see eqn. (21) with $K$ given by the $E_8$ matrix (A5)) is an H-type topological order. But it cannot be realized by a topological quantum field theory with no framing anomaly. In other words, it cannot be realized by a Lagrangian path integral that manifestly only depends on the diffeomorphism equivalent classes of spacetime metrics.

However, the $E_8$ state can be realized by a Lagrangian path integral with framing anomaly (i.e. depends on the additional framing structure of spacetime manifold).6 One such realization is given by the dynamical Chern–Simons theory eqn. (22) with $K = E_8$. Here we like to remark that it is highly nontrivial to define the dynamical Chern–Simons theory eqn. (22) on spacetime so that we can compute the partition function via a finite calculation. For bosonic Chern–Simons theory eqn. (22) (where $K_{II}$ is even), a non-perturbative definition was recently given in Ref. 26, by triangulating the spacetime and giving the triangulation a branching structure. It appears that the branching structure plays a role of framing structure.

On the other hand, the bosonic $E_8^3$ quantum Hall state (the stacking of three bosonic $E_8$ quantum Hall state) can be described by a topological quantum field theory with no framing anomaly, i.e. it can be realized by a Lagrangian path integral that only depends on the diffeomorphism equivalent classes of spacetime metrics (i.e. does not depend on the framing structure of spacetime). An explicit construction of the Lagrangian path integral in terms of $SO_{\infty}$ non-linear $\sigma$-model was given in Ref. 27.

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1 By definition, invertible gapped states have no fractionalized excitations. As a result, they have non-degenerate ground state on closed smooth spatial manifolds.
A. Notations and conventions

We will abbreviate the cup product $a \smile b$ and the wedge product $a \wedge b$ as $ab$. We will use $\equiv$ to mean equal up to a multiple of $n$, and use $\bar{\equiv}$ to mean equal up to $df$ (i.e. up to a coboundary). We will use $\lfloor x \rfloor$ to denote the greatest integer less than or equal to $x$, and $(l, m)$ for the greatest common divisor of $l$ and $m$ (with $(0, m) \equiv m$).

We introduce a symbol $\lambda$ to construct fiber bundle $E = F \times B$ from the fiber $F$ and the base space $B$:

$$pt \rightarrow F \rightarrow E \rightarrow B \rightarrow pt.$$  

We will also use $\lambda$ to construct group extension of $G$ by $N$ [28]:

$$1 \rightarrow N \rightarrow N \times_{e_2, \alpha} G \rightarrow G \rightarrow 1.$$  

Here $e_2 \in H^2[G; Z(N)]$ and $Z(N)$ is the center of $N$. Also $G$ may have a non-trivial action on $Z(N)$ via $\alpha : G \rightarrow Aut(N)$. $e_2$ and $\alpha$ characterize different group extensions.

Also, we will use $Z_n = \{1, e^{2\pi i n}, e^{4\pi i n}, \ldots, e^{((n-1)2\pi i)}\}$ to denote an Abelian group, where the group multiplication is “$+$”. We use $Z_n = \{0, 1, \ldots, n-1\}$ to denote an integer lifting of $Z_n$, where “$+$” is done without mod-$n$.

In this sense, $Z_n$ is not a group under “$+$”. But under a modified equality $\bar{\equiv}$, $Z_n$ is the $Z_n$ group under “$+$”.

Similarly, we will use $R/Z = [0, 1)$ to denote an $R$-lifting of $U_1$ group. Under a modified equality $\bar{\equiv}$, $R/Z$ is the $U_1$ group under “$+$”. In this paper, we have expressions containing the addition “$+$” of $Z_n$-valued or $R/Z$-valued, such as $a_1^{Z_n} + a_2^{Z_n}$ where $a_1^{Z_n}$ and $a_2^{Z_n}$ are $Z_n$-valued. Those additions “$+$” are done without mod $n$ or mod 1. In this paper, we also have expressions like $\frac{1}{n}a_1^{Z_n}$. Such an expression converts a $Z_n$-valued $a_1^{Z_n}$ to a $R/Z$-valued $\frac{1}{n}a_1^{Z_n}$, by viewing the $Z_n$-value as a $Z$-value. (In fact, $Z_n$ is a $Z$-lifting of $Z_n$.)

II. THE VECTOR BUNDLE ON THE MODULI SPACE FOR H-TYPE BOSONIC SYSTEMS WITH GAPPED LIQUID GROUND STATES

In this paper, we study topological invariants for gapped liquid [14, 15] ground states without excitations. In order to characterize different classes of gapped liquid ground states, we consider a moduli space – the space of Hamiltonians with on-site symmetry $G$ that have gapped liquid state as the ground state. Here we assume that the Hamiltonians in the moduli space depend on the metrics $g_{ij}$ of the space as well as the $G$-symmetry twist (i.e. the background $G$-gauge field $A_i$ from gauging the on-site $G$-symmetry[29]). So for a closed $n$-dimensional space $M^n$, with a certain topology as well as a $G$-principle bundle for the $G$-symmetry twist on $M^n$, the moduli space is the space of spatial metrics and $G$-connections, which is denoted as $M_{G \times M^n}$. Here $G \times M^n$ describes the $G$-principle bundle:

$$pt \rightarrow G \rightarrow G \times M^n \rightarrow M^n \rightarrow pt.$$  

Also we assume that the pairs $(g_{ij}, A_i)$ differ by gauge transformations and diffeomorphisms are equivalent and represent the same point of the moduli space $M_{G \times M^n}$.

Since on every point of the moduli space, the corresponding Hamiltonian is in a gapped liquid phase, the subspace formed by the degenerate ground states can be viewed as the fiber at the point. This way, we obtain a vector bundle over the moduli space $M_{G \times M^n}$. It was conjectured in Ref. 5 that

the topology of this vector bundle completely characterize the H-type topological orders, i.e. gapped liquid phases with symmetry.

In particular, let us consider a closed subspace $B$ in $M_{G \times M^n}$. The vector bundle on $M_{G \times M^n}$ reduces to a vector bundle on $B$. In the next section, we concentrate on this vector bundle on $B$, and try to relate the vector bundle to topological term in the partition function of the quantum system under consideration. We will follow closely the approach proposed in Ref. 12 and 30.

III. CHERN–SIMONS INVARIANTS IN 2+1D H-TYPE BOSONIC U1-SET AND U1-SPT ORDERS

A. Bosonic gapped liquids with $U_1$ symmetry in 2-dimensional space

In this section, we consider gapped liquids with $U_1$ symmetry in 2-dimensional space for bosonic systems. Let $\Sigma_g$ be a 2-dimensional closed spatial manifold with a $U_1$ connection, that describes the twisted global $U_1$ symmetry. The corresponding $U_1$-bundle is given by $U_1 \times \Sigma_g$. In this case, the moduli space $M_{U_1 \times \Sigma_g}$ of the system is the space of metrics on $\Sigma_g$ and the $U_1$-connections. The degenerate ground states on $\Sigma_g$ give rise to a vector bundle over the moduli space $M_{U_1 \times \Sigma_g}$, where the dimension of the vector is given by the ground state degeneracy $D_{U_1 \times \Sigma_g}$.

Let us consider a loop $S^1$ in the moduli space $M_{U_1 \times \Sigma_g}$. The holonomy of the vector bundle around the loop is given by a unitary matrix of $D_{U_1 \times \Sigma_g}$ dimension: $W_{S^1}$. We note that the unitary matrix $W_{S^1}$ is the non-Abelian geometric phase[31] of the degenerate ground states under the adiabatic deformation around the loop $S^1$.

From the vector bundle, we can obtain a determinant bundle, whose holonomy around the loop is given by the determinant: $Det(W_{S^1})$. This phase factor is directly related to the topological term in the effective action.
\[ S_{\text{eff}} = \int_{\Sigma_g \times S^1} d^3x \mathcal{L}_{\text{eff}}: \]

\[
\text{Det}(W_{S^1}) = \left( e^{i \int_{S^1 \times S^1} d^2x \mathcal{L}_{\text{eff}}^0} \right)^{D_{U_1, \Sigma_g}} (7)
\]

The effective action may contain a gravitational Chern–Simons term and a \( U_1 \) Chern–Simons term:[32–34]

\[
\mathcal{L}_{\text{eff}} d^3x = -2\pi c \frac{e^2}{24} \omega_3 + \frac{\nu}{4\pi} (A + sA^{SO_2}) d(A + sA^{SO_2})
\]

\[= -2\pi c \frac{e^2}{24} \omega_3 + \frac{\nu}{4\pi} A_{\text{eff}} dA_{\text{eff}} \] (8)

where \( A \) is the \( U_1 \) connection 1-form describing the total electromagnetic field, \( s \) is the orbital spin carried by the charged bosons,[32, 33] and \( A^{SO_2} \) is the time-dependent connection 1-form describing the \( SO_2 \) tangent bundle of the curved space. \( A_{\text{eff}} \equiv A + sA^{SO_2} \) is the effective \( U_1 \) connection 1-form. Also \( \omega_3 \) is the gravitational Chern–Simons term that satisfy \( d\omega_3 = p_1 \), where \( p_1 \) is the first Pontryagin class of the tangent bundle. In the above, \( e \) is the chiral central charge of the edge theory, and \( \nu \) is proportional to the Hall conductance:

\[
\sigma_{xy} = \frac{\nu}{2\pi} = \frac{e^2}{h}
\] (9)

(in \( e = \hbar = 1 \) unit).

We note that \( \frac{1}{\pi} dA_{\text{eff}} = j \) is the 2-cocycle describing the conserved density and current of charged bosons in the ground state. Thus \( A_{\text{eff}} \) satisfy the quantization condition

\[
\int_{\Sigma_g} \frac{\nu}{2\pi} dA_{\text{eff}} \in \mathbb{Z}
\] (10)

for the closed 2-dimensional space \( \Sigma_g \). In fact, \( \int_{\Sigma_g} \frac{\nu}{2\pi} dA_{\text{eff}} \) is the number of bosons in the ground state.

Now assume the \( S^1 \) to be the boundary of a 2-dimensional subspace \( B^2 \) in \( \mathcal{M}_{U_1, \Sigma_g} \). Eqn. (7) can be rewritten as[35]

\[
\text{Det}(W_{S^1}) = e^{D_{U_1, \Sigma_g} \frac{12\pi}{2\pi} \int_{\Sigma_g \times B^2} \frac{1}{2} j, p_1 + \frac{\nu}{2} c_1^2},
\] (11)

where \( a = \frac{A_{\text{eff}}}{2\pi} \) and \( c_1 = da \) is the first Chern class. If we shrink \( S^1 \) to a point, \( B^2 \) becomes a closed 2-dimensional subspace. In this case, \( \text{Det}(W_{S^1}) = 1 \) and \( \nu \). The combination of all those quantization conditions gives us the strictest constraint on the possible values of \( c \) and \( \nu \). In the following, we will choose some special combination of surface bundles \( \Sigma_g \times B^2 \) and \( U_1 \) bundles to obtain concrete quantization conditions of \( c \) and \( \nu \).

We note that \( \int_{\Sigma_g \times B^2} p_1 = 0 \mod 12 \) for any orientable surface bundles (also called \( \Sigma_g \)-bundles), by (B3). If the genus \( g \) of the fiber \( \Sigma_g \) is equal or less than 2, then \( \int_{\Sigma_g \times B^2} p_1 = 0 \).[36] If \( g \geq 3 \), then we can always find a base manifold \( B^2 \) with a genus equal or less than 111, such that there is a surface bundle \( \Sigma_g \times B^2 \) with \( \int_{\Sigma_g \times B^2} p_1 = \pm 12 \).[37] If we choose the \( U_1 \) bundle to be trivial \( c_1 = 0 \), eqn. (12) gives us the quantization condition on \( c \) first proposed in Ref. 12 and 30:

\[
\frac{c}{2} D_{U_1, \Sigma_g} \in \mathbb{Z}, \quad \text{for } g \geq 3.
\] (13)

Next, we consider a trivial surface bundle with \( S^2 = \Sigma_0 \) as the fiber: \( \Sigma_g \times B^2 = \Sigma_0 \times B^2 \). On such a surface bundle, \( c_1 \) has a form \( c_1 = c_1^B + c_1^\Sigma \) where \( c_1^\Sigma \) lives on \( \Sigma_0 \). We have

\[
\frac{1}{2} \nu \int_{\Sigma_0 \times B^2} c_1^2 = \frac{1}{2} \nu \int_{\Sigma_0 \times B^2} (c_1^B + c_1^\Sigma)^2
\]

\[= \nu \int_{\Sigma_0} c_1^\Sigma \times \int_{B^2} c_1^B
\] (14)

Therefore, for \( U_1 \) bundles whose Chern class \( c_1 \) satisfying \( \nu c_1^\Sigma \in \mathbb{Z}, \frac{1}{2} \nu c_1^\Sigma \times c_1^B \) is always an integer. No constraint on \( \nu \) is obtained. Also, \( \int_{\Sigma_0 \times B^2} p_1 = 0 \). Therefore, the trivial surface bundle \( \Sigma_g \times B^2 = \Sigma_0 \times B^2 \) does not give us any non-trivial quantization conditions for \( c \) and \( \nu \). So in the following, we will assume \( g > 0 \) for our space \( \Sigma_g \).

Next, we consider the surface bundle \( \Sigma_g \times B^2 = S^1_a \times S^1_b \times S^1 \), where \( S^1_a \times S^1_b \times S^1 = \Sigma_0 \) is the space and \( S^1_a \times S^1_b = B^2 \). We consider the \( U_1 \) bundle whose first Chern class has the following form \( c_1 = c_1^{\Sigma a} + c_1^{\Sigma b} \), where \( c_1^{\Sigma a} \) lives on \( S^1_a \times S^1_a \) and \( c_1^{\Sigma b} \) lives on \( S^1_b \times S^1_b \). Since \( \int_{\Sigma_g \times B^2} p_1 = 0 \) and \( \int_{\Sigma_0} c_1^{\Sigma a} + c_1^{\Sigma b} = \int_{S^1_a \times S^1_a} c_1^{\Sigma a} + c_1^{\Sigma b} = 0 \), the quantization condition eqn. (12) becomes

\[
\nu \frac{1}{2} D_{U_1, \Sigma_g} \int_{S^1_a \times S^1_a \times S^1} (c_1^{\Sigma a} + c_1^{\Sigma b})^2
\]

\[= \nu D_{U_1, \Sigma_g} \int_{S^1_a \times S^1_a} c_1^{\Sigma a} \int_{S^1_b \times S^1_b} c_1^{\Sigma b}.
\] (15)

We can choose the \( U_1 \) bundles such that \( \int_{S^1_a \times S^1_a} c_1^{\Sigma a} = 0, \pm 1 \) and \( \int_{S^1_b \times S^1_b} c_1^{\Sigma b} = 0, \pm 1 \). We obtain the following quantization condition

\[
\nu D_{U_1, \Sigma_g} \in \mathbb{Z}.
\] (16)

The above result was first obtained by Ref. 11, using a similar consideration. Because \( \Sigma_g \) is the connected sum
of $g$ $\Sigma_1$’s, the above reasoning can be generalized to the case $\Sigma_g \times B^2 = \Sigma_g \times S^1_a \times S^1_b$, which allows us to obtain
\[
\nu D_{U_1, \Sigma_g} \in \mathbb{Z}, \quad g \geq 1.
\] (17)

This result generalizes that of Ref. 11.

To obtain an even stronger result, we consider a surface bundle over surface $\Sigma_g \times \Sigma_h$. In Appendix B, we show that, for each $g \geq 5$, there are many surface bundles $\Sigma_g \times \Sigma_h$ such that each of them has a 2-cocycle $c_1 \in H^2(\Sigma_g \times \Sigma_h; \mathbb{Z})$ satisfying $\int_{\Sigma_g} c_1 = 2 \nu \sigma$ and $\int_{\Sigma_h} p_1 = 3 \eta \nu$, where $\nu$ and $\eta$ are integers. We find that the set of allowed $(\nu, \eta)$ is given by (see Appendix B)
\[
\{(\nu, \eta) \mid \nu \in 4\mathbb{Z}, \eta \in \mathbb{Z}\}.
\] (18)

We obtain a stronger quantization condition on $\nu$ from eqn. (12):
\[
\frac{\nu}{2} D_{U_1, \Sigma_g} \in \mathbb{Z}, \quad g \geq 5.
\] (19)

To summarize

| for a 2+1D bosonic gapped phase with $U_1$ symmetry, its chiral central charge $c$ and filling fraction $\nu$ satisfy the following quantization conditions |
|---|
| $\frac{c}{2} D_{U_1, \Sigma_g} = 0 \text{ mod } 1$, \quad $g \geq 3$, |
| $\nu D_{U_1, \Sigma_g} = 0 \text{ mod } 1$, \quad $g \geq 1$, |
| $\frac{\nu}{2} D_{U_1, \Sigma_g} = 0 \text{ mod } 1$, \quad $g \geq 5$. |

where $D_{U_1, \Sigma_g}$ is the ground state degeneracy on closed space $\Sigma_g$.

### B. Examples

1. **Bosonic K-matrix Abelian topological orders**

   We check $\frac{c}{2} D_{U_1, \Sigma_g} \in \mathbb{Z}$ for 2+1D bosonic Abelian topological orders described by the following multi-layer wave function characterized by symmetric integral matrix $K$ with even diagonal:

   \[
   \Psi(z_i) = \prod_{i,j} (z_i - z_j)^{K_{ij}} \prod_{i,j} (z_i - z_j)^{K_{ij}} e^{-\Sigma_{i,j} |z_i - z_j|^2},
   \] (21)

   where $z_i$ the complex coordinates of the $i$th boson in $I$th-layer. The effective field theory of the above state is described by $K$-matrix Abelian Chern–Simons theory:

   \[
   S_K = \int K_{ij} \frac{q_I}{4\pi} da_i da_j + q_I \frac{q_I}{2\pi} A da_j.
   \] (22)

   where the charge-vector $q_I$ are integers, describing the $U_1$ charge of the bosons in $I$th-layer. It is believed that the $K$-matrix Abelian Chern–Simons theory can realize all the 2+1D Abelian topological orders.

   After integrating out the matter field $a_I$, we obtain the effective theory eqn. (8) with

   \[
   c = \text{sgn}(K) = n_+ - n_-,
   \nu = q \top K^{-1} q.
   \]

   where $n_+$ are the number of eigenvalues of $K$ with sign $\pm$. The ground state degeneracy is given by

   \[
   D_{U_1, \Sigma_g} = |\text{Det}K|^g.
   \]

   So eqn. (12) states that the product $\text{sgn}(K)|\text{Det}K|^g$ is an even integer for $g > 2$. To check this, note that $\text{sgn}(K) = n \text{ mod } 2$, where $n$ is the dimension of $K$. So it suffices to check that

   \[
   \text{Det}K \in 2\mathbb{Z} \quad \text{for odd } n.
   \] (23)

   To see this, write

   \[
   \text{Det}K = \sum_{\sigma} (-1)^\sigma \prod_{I=1}^{n} K_{I\sigma(I)}
   \] (24)

   which is summed over the permutations $\sigma$ of $\{1, \ldots, n\}$ and $(-1)^\sigma$ is its sign. Note that any $\sigma$ which involves a diagonal element of $K$ contributes an even term to the sum, so we only need to consider the permutations where $\sigma(I) \neq I$ for all $I$. Every permutation $\sigma$ can be expressed as a number of disjoint cycles. The sum of all the lengths of the cycles is $n$, which is odd by assumption. So there exists a cycle whose length is odd. Such cycle cannot have length 1, since we assumed $\sigma(I) \neq I$. So it has length $\geq 3$. The existence of such a cycle implies $\sigma \neq \sigma^{-1}$, since reversing the cycle gives a different permutation. Moreover $\sigma$ and $\sigma^{-1}$ contributes equally to the sum (24), since

   \[
   \prod_{I=1}^{n} K_{I\sigma^{-1}(I)} = \prod_{I=1}^{n} K_{I\sigma^{-1}(I)} = \prod_{I=1}^{n} K_{I\sigma(I)}.
   \]

   So the two terms together is even. So (24) is even.

   The conditions on $\nu$, eqn. (17) and eqn. (19), now become

   \[
   \prod_{I=1}^{n} K_{I\sigma^{-1}(I)} = \prod_{I=1}^{n} K_{I\sigma^{-1}(I)} = \prod_{I=1}^{n} K_{I\sigma(I)}.
   \]

   for all integral vectors $q$. The first expression is satisfied because $\text{Det}(K)K^{-1}$ is an integer matrix. For the second expression, we only need to consider the case when $\text{Det}(K)$ is odd. By (23) this implies the $n$, the dimension of $K$, is even. So the submatrix of $K$ by deleting the $i$th row and $i$th column has odd dimension for all $i$. Applying (23) to this submatrix, we see that it has even determinant. Hence the diagonals of the cofactor matrix of $K$ are even, which means $\text{Det}(K)K^{-1}$ has even diagonals.
From this, the second expression reads
\[
|\text{Det}(K)|^5 \frac{q_I K^{-1}_{IJ} q_J}{2} = \frac{1}{2} |\text{Det}(K)|^5 \left( \sum_{I} q_I^2 K^{-1}_{II} + \sum_{I \neq J} q_I K^{-1}_{IJ} q_J \right) = |\text{Det}(K)|^5 \left( \sum_{I < J} q_I^2 K^{-1}_{IJ} + \sum_{I < J} q_I K^{-1}_{IJ} q_J \right) \in \mathbb{Z}.
\]

From the above result, we find that, for Abelian bosonic topological orders with \( U_1 \) symmetry, the filling fraction is quantized as
\[
\nu = \frac{2 \times \text{integer}}{\gcd(2D_1, D_1^5)} = \begin{cases} \frac{D_1}{2}, & \text{if } D_1 = \text{even} \\ \frac{D_1}{6}, & \text{if } D_1 = \text{odd} \end{cases} (26)
\]
where \( D_1 = |\text{Det}(K)| \) is the ground state degeneracy on torus. In particular, for bosonic \( U_1 \)-SPT order, \( D_1 = |\text{Det}(K)| = 1 \) and \( \nu = 0 \mod 2 \). We note that the bosonic \( \nu = 1/m \) Laughlin state has \( D_1 = m = \text{even} \), and saturates the above quantization condition.

2. Bosonic non-Abelian topological orders described by \( SU(2)_k \) Chern–Simons Theory

The following wave function
\[
\Psi(z_i) = (\chi_k(z_i))^2 (27)
\]
describes a bosonic non-Abelian topological orders of charge-1 bosons, where \( \chi_k(z_i) \) is the fermion wave function of \( k \)-filled Landau levels,\(^{[39]}\) The low energy effective theory is the \( SU(2)_k \) Chern–Simons theory. The edge is described by a \( U^{2k}/SU(2)_k \) WZW theory, whose chiral central charge and filling fraction are\(^{[39]}\)
\[
c = 2k - \frac{3k}{k+2}, \quad \nu = \frac{k}{2}. (28)
\]

Using the \( SU(2)_k \) Chern–Simons theory, we find that there are \( k+1 \) anyons in the bulk labeled by \( s = m/2 \) where \( m = 0, 1, \ldots, k \). The ground state degeneracy is given by
\[
D_{U_1, \Sigma_g} = \sum_m S_{0m}^{-2(g-1)} \in \mathbb{Z}
\]

where
\[
S_{mn} = \sqrt{\frac{2}{k+2}} \sin \left( \frac{(m+1)(n+1)\pi}{k+2} \right)
\]

So eqn. (20) reads
\[
\nu = -\frac{3k}{2(k+2)} \left( \frac{k+2}{2} \right)^{g-1} \sum_{m=1}^{k+1} \csc^2(g-1) \frac{m\pi}{k+2} \in \mathbb{Z} (29)
\]
which is verified in Appendix C for all integers \( k \geq 0 \) and \( g \geq 3 \). We also have
\[
\nu D_{U_1, \Sigma_g} = \frac{k}{2} \left( \frac{k+2}{2} \right)^{g-1} \sum_{m=1}^{k+1} \csc^2(g-1) \frac{m\pi}{k+2} \in \mathbb{Z} (30)
\]
for \( g \geq 1 \), and
\[
\nu D_{U_1, \Sigma_g} = \frac{k}{4} \left( \frac{k+2}{2} \right)^{g-1} \sum_{m=1}^{k+1} \csc^2(g-1) \frac{m\pi}{k+2} \in \mathbb{Z} (31)
\]
for \( g \geq 5 \). We also verify the above two relations in Appendix C.

C. A general point of view

In general, we may consider a boson system in \( d \)-dimensional space with an internal symmetry described by group \( G_b \). The bosons may carry non-zero integer spin and may transform non-trivially under the spatial rotation \( SO_d \). We embed \( SO_d \) into \( SO_\infty \equiv SO \). Thus, bosons transform under the full symmetry group \( G_{SO} = G_b \times SO \). On curved spacetime with \( G_b \)-symmetry twist, we have a \( G_{SO} \) connection \( a^{G_{SO}} \) on the spacetime. The connection \( a^{G_{SO}} \) is a special one. When we project \( G_{SO} \) to \( SO \): \( G_{bSO} \overset{\pi}{\rightarrow} SO \), the \( G_{bSO} \) connection is projected into an \( SO \) connection: \( a^{G_{SO}} \overset{\pi}{\rightarrow} a^{SO} \), and such an \( SO \) connection \( a^{SO} \) must be the connection for the tangent bundle of the spacetime. The low energy effective Lagrangian \( L_{eff} (8) \) is a Chern–Simons term for group \( G_{bSO} \). This point of view will help us to understand the effective Lagrangian for fermion system. Since \( G_{bSO} = G_b \times SO \), the low energy effective Lagrangian \( L_{eff} (8) \) is a sum of the gravitational Chern–Simons term and a Chern–Simons term for group \( G_b \).

IV. TOPOLOGICAL INVARIANTS IN 2+1D H-TYPE FERMIONIC ENRICHED TOPOLOGICAL ORDERS WITH \( U_1 \) SYMMETRY

A. Symmetry twist of fermion system

We have seen that to probe the topological properties of a bosonic gapped liquid phase with symmetry \( G_b \), we can use the symmetry twist described by a \( G_b \)-connection \( a^{SO} \), plus the curved spacetime described by the \( SO \)-connection \( a^{SO} \) of the tangent bundle. In other words, we can use the \( G_{bSO} \)-connection \( a^{G_{SO}} \) and effective action \( S_{eff}(a^{G_{SO}}) \) to probe the topological properties. Here, we will discuss the symmetry twist for fermion systems, which turns out to be mixed with spacetime curvature.

For a fermion system in \( d \)-dimensional space with an internal symmetry, its symmetry group \( G_f \) has a central \( Z_2 \) subgroup:
\[
id \rightarrow Z_2 \rightarrow G_f \rightarrow G_b \rightarrow id. (32)
\]
We also write $G_f = Z^I_f \times_{e_2} G_b$, where $e_2$ is a group 2-cocycle
\[ e_2 \in H^2(G_b; \mathbb{Z}_2^f) \]  
(33)
describing the group extension (32) (see Appendix D). The fermion carry half-integer spins and transform non-trivially under the spatial rotation $SO_2$, which is enlarged to $SO$. Thus, fermions transform under the full symmetry group $G_{fSO} = G_f \times SO$. On curved spacetime with $G_f$-symmetry twist, we have a $G_{fSO}$ connection $A_{fSO}$ on the spacetime. The connection $a_{fSO}$ is a special one. When we project $G_{fSO}$ to $SO$: $G_{fSO} \xrightarrow{\pi} SO$, the $G_{fSO}$ connection is projected into an $SO$ connection: $a_{fSO} \xrightarrow{\pi} a_{SO}$, and such an $SO$ connection $a_{SO}$ must be the connection for the tangent bundle of the spacetime. The low energy effective Lagrangian describing the group extension $(\pi)$ to label the group elements of $G_f$.

There is another way to describe full fermionic symmetry group $G_{fSO} = G_f \times SO$:
\[ G_{fSO} = (G_f \times Spin) / \mathbb{Z}_2^f \]  
(34)
We note that the spin group $Spin \equiv Spin_{\infty} = Z^I_f \times_{w_2} SO$ contains a central subgroup $Z_2^f$ such that $Spin / Z_2^f = SO$, where $w_2 \in H^2(SO; \mathbb{Z}_2^f)$ is the group 2-cocycle describing the group extension
\[ id \to Z_2^f \to Spin \to SO \to id. \]  
(35)
$G_f = Z^I_f \times_{e_2} G_b$ also contains a central subgroup $Z_2^f$ such that $G_f / Z_2^f = G_b$. $G_{fSO}$ is obtained by identifying these two $Z_2^f$ subgroups in $G_f \times Spin$. Let us described the $G_{fSO}$-connection using this point of view.

We first triangulate the spacetime, where the vertices are labeled by $i, j, k, \ldots$, the links labelled by $ij, jk, \ldots$, etc. A $G_f$-connection on the triangulation is described by the group elements of $G_f$ on the links: $a_{ij}^{G_f} \in G_f$. We can view $G_f = Z^I_f \times_{e_2} G_b$ and use the pair $(a_{ij}^{G_f}, a_{ij}^Z)$ to label the group elements of $G_f$, where $a_{ij}^{G_f} \in G_b$ and $a_{ij}^Z \in Z^I_f$. Similarly, a $Spin$-connection on the triangulation is described by the group elements of $Spin$ on the links: $a_{ij}^{Spin} \in Spin$. We can view $Spin = Z^I_f \times_{w_2} SO$ and use the pair $(a_{ij}^{SO}, a_{ij}^Z)$ to label the group elements of $Spin$, where $a_{ij}^{SO} \in SO$ and $a_{ij}^Z \in Z^I_f$. Now, the $G_{fSO}$-connection is obtained by requiring $a_{ij}^{Z_f} = a_{ij}^Z$. Also the $SO$-connection $a_{ij}^{SO}$ describes the tangent bundle of the spacetime.

Now, let us describe a nearly flat $G_{fSO}$-connection that describe the curved spacetime and twisted symme-try. The nearly flat condition is given by
\[ a_{ij}^{fSO} A_{fSO} a_{ij}^{fSO} \approx a_{ik}^{fSO} \]  
(36)
for all the triangles $ijk$. It implies that the $G_f$-connection is nearly flat
\[ (a_{ij}^{G_f}, a_{ij}^Z) \circ (a_{jk}^{G_f}, a_{jk}^Z) \approx (a_{ik}^{G_f}, a_{ik}^Z), \]  
(37)
and the $Spin$-connection is nearly flat
\[ (a_{ij}^{SO}, a_{ij}^Z) \circ (a_{jk}^{SO}, a_{jk}^Z) \approx (a_{ik}^{SO}, a_{ik}^Z). \]  
(38)
In other words (see Appendix D)
\[ a_{ij}^{SO} a_{ij}^{G_f} \approx a_{ij}^{G_f}, \]  
\[ a_{ij}^{SO} a_{jk}^{SO} \approx a_{ik}^{SO}, \]  
(39)
and
\[ a_{ij}^{Z_f} + a_{jk}^{Z_f} - a_{ik}^{Z_f} \approx e_2(a_{ij}^{G_f}, a_{jk}^{G_f}) = (e_2)_{ijk}, \]  
\[ a_{ij}^{Z_f} + a_{jk}^{Z_f} - a_{ik}^{Z_f} \approx w_2(a_{ij}^{SO}, a_{jk}^{SO}) = (w_2)_{ijk}. \]  
(40)
The above can be rewritten as
\[ da_{ij}^{Z_f} \approx e_2(a_{ij}^{G_f}), \]  
\[ da_{ij}^{Z_f} \approx w_2(a_{ij}^{SO}). \]  
(41)
Since $a_{SO}$ is the connection of the tangent bundle of the spacetime, $w_2(a_{SO}) = w_2$ is the second Stiefel–Whitney class of the tangent bundle of the spacetime. We see that the $G_b$-connection and the $SO$-connection on spacetime are not arbitrary. They must satisfy the constraint
\[ e_2(a_{ij}^{G_b}, a_{jk}^{G_b}) \approx w_2(a_{ij}^{SO}, a_{jk}^{SO}) = w_2. \]  
(42)
In other words, the nearly flat $G_b$ bundle on the spacetime (describing the symmetry twist) is not arbitrary. Its nearly flat connection $a_{SO}$ must satisfy
\[ e_2(a_{ij}^{G_b}) \approx w_2. \]  
(43)

B. $U_f^1$ symmetry and spin$^C$ structure

Let us consider an example of the above result. A fermion system with fermion number conservation has a $U_f^1$ symmetry. Assume all the odd charges of $U_f^1$ are fermionic, then $U_f^1$ has a $Z_2^f$ subgroup generated by the $\pi$-rotation, which corresponds to the fermion-number-parity. It is more convenient to view $U_f^1$ as a group extension of $U_1$ by $Z_2^f$, i.e. $U_f^1 = Z_2^f \times_{e_2} U_1$ where $e_2$ is the $Z_2$ valued group 2-cocycle that generates $H^2(U_1, Z_2)$ (see Appendix D).

Using $\mathbb{R}/\mathbb{Z}$-valued $a^{(\mathbb{R}/\mathbb{Z})}$ to label group elements in $U_f^1$, $\mathbb{R}/\mathbb{Z}$-valued $a^{\mathbb{R}/\mathbb{Z}}$ to label group elements in $U_1$, and $\mathbb{Z}_2$-valued $a^{Z_2^f}$ to label group elements in $Z_2^f$, we find that
\[ a^{(\mathbb{R}/\mathbb{Z})} = \frac{1}{2} a^{\mathbb{R}/\mathbb{Z}} + \frac{1}{2} a^{Z_2^f}, \]  
(44)
Here \( \mathbb{R}/\mathbb{Z} = [0, 1) \) and \( \mathbb{Z}_n = \{0, 1, \ldots, n - 1\} \). Now the group multiplication in \( U^f_1 \) can be rewritten as

\[
\begin{align*}
  a_1^{(R/Z)} + a_2^{(R/Z)} &= \frac{1}{2} a_1^{R/Z} + \frac{1}{2} a_2^{R/Z} + \frac{1}{2} z_1 + \frac{1}{2} z_2 \\
  &= \frac{1}{2} (a_1^{R/Z} + a_2^{R/Z} - [a_1^{R/Z} + a_2^{R/Z}]) \\
  + (\frac{1}{2} a_1^{R/Z} + \frac{1}{2} a_2^{R/Z} + [a_1^{R/Z} + a_2^{R/Z}]) \\
  &= \frac{1}{2} (a_1^{R/Z} + a_2^{R/Z} - [a_1^{R/Z} + a_2^{R/Z}]) \\
  + (\frac{1}{2} a_1^{R/Z} + \frac{1}{2} a_2^{R/Z} + c_2(a_1^{R/Z}, a_2^{R/Z})),
\end{align*}
\]

where \( [x] \) denotes the largest integer smaller than or equal to \( x \), and

\[
e_2(a_1^{R/Z}, a_2^{R/Z}) = [a_1^{R/Z} + a_2^{R/Z}]
\]

which is a \( \mathbb{Z}_2 \)-valued group 2-cocycle \( e_2 \in H^2(U_1; \mathbb{Z}_2) \) that characterizes the group extension of \( U_1 \) by \( \mathbb{Z}_2^2 \):

\[
U^f_1 = \mathbb{Z}_2^2 \rtimes_{c_2} U_1.
\]

Note that \( e_2(a_1^{R/Z}, a_2^{R/Z}) \) is a smooth function of \( a_1^{R/Z}, a_2^{R/Z} \) when \( 0 < a_1^{R/Z} < \frac{1}{2} \) and \( 0 < a_2^{R/Z} < \frac{1}{2} \). But it has discontinuities in other places. It turns out that

\[
e_2 \equiv c_1^{U_1}
\]

where \( c_1^{U_1} \in H^2(U_1, \mathbb{Z}) \) is the first Chern class of \( U_1 \) viewed as group 2-cocycle.

The “Chern Class” for \( U^f_1 \) and the Chern Class for \( U_1 \) are related (see eqn. (44))

\[
c_1^{U^f_1} = \frac{1}{2} c_1^{U_1},
\]

where \( c_1^{U_1} = da^{R/Z} \) and \( c_1^{U_1} = da^{(R/Z)/f} \). We like to point out that the \( U^f_1 \)-connection \( a^{(R/Z)/f} \) corresponds to the electromagnetic gauge potential plus the connection of the curved space:

\[
A_{eff} = 2\pi a^{(R/Z)/f},
\]

and the Chern class \( c_1^{U^f_1} \) corresponds to the field strength of the effective \( U^f_1 \)-connection

\[
F_{eff} = dA_{eff} = 2\pi c_1^{U^f_1}.
\]

We find that for fermion system with \( U^f_1 \) symmetry, the Chern number on a closed 2-dimensional space \( C^2 \) satisfies

\[
\int_{C^2} c_1^{U^f_1} = \frac{1}{2\pi} \int_{C^2} F_{eff} = \frac{1}{2} \int_{C^2} w_2.
\]

In other words, on a spin manifold \( w_2 = 0 \), the Chern number \( \frac{1}{2\pi} \int_{C^2} F_{eff} \) is quantized as integers. On non-spin manifold, the fermion system with \( U^f_1 \) symmetry can still be defined, provided that the Chern number \( \frac{1}{2\pi} \int_{C^2} F_{eff} \) is quantized as half-integers when \( \int_{C^2} w_2 = \frac{1}{2} \).

We see that for fermion systems, there is a constraint on the \( U^f_1 \) connection \( a^{(R/Z)/f} = \frac{1}{2\pi} A_{eff} \) and the curved spacetime. When \( A_{eff} = 0 \), this constraint requires that \( w_2 \leq 0 \), i.e. the spacetime to be spin. This leads to the general impression that the appearance of fermions requires the spacetime manifold to be spin. Here we see that the appearance of \( U^f_1 \) fermions requires the spacetime manifold to be spin only when there is no background effective \( U^f_1 \) gauge field. In the presence of effective background \( U^f_1 \) gauge field, the spacetime may not be spin. This result can be summarized more precisely by the following statement: the appearance of \( U^f_1 \) fermions requires the spacetime manifold to be spin\(^C \), and the \( U^f_1 \) connection \( a^{(R/Z)/f} \) is a spin\(^C \) structure.

We know that an orientable manifold is spin if and only if its \( w_2 = 0 \). Similarly, an orientable manifold is spin\(^C \) if and only if \( \beta_2 w_2 = 0 \). Here \( \beta_2 = \frac{1}{2} d \) is the the Bockstein homomorphism. In word orders, an orientable manifold \( M \) is spin\(^C \) if and only if its \( w_2 \) is the mod 2 reduction of a \( Z \)-valued 2-cocycle in \( H^2(M; \mathbb{Z}) \). Also a complex vector bundle of rank \( n \) can be viewed as a real vector bundle of rank \( 2n \). The \( n^{th} \) Chern class \( c_{2n}^{U_1} \) of the complex vector bundle and the \( 2n^{th} \) Stiefel–Whitney class are related

\[
w_{2n} \equiv c_{2n}^{U_1}.
\]

This implies that a complex manifold is always spin\(^C \).

### C. Fermionic gapped liquids with \( U^f_1 \) symmetry in 2-dimensional space

Now, we consider a gapped liquids with \( U^f_1 \) symmetry in 2-dimensional space for fermionic systems. The effective action may contain a gravitational Chern–Simons term and a \( U^f_1 \) Chern–Simons term:

\[
\mathcal{L}_{eff} d^3x = -2\pi c^{U_1} \omega_3 + \frac{\nu}{4\pi} A_{eff} dA_{eff}
\]

where \( A_{eff} \) is the \( U^f_1 \) connection 1-form describing the total electromagnetic field, plus the connection from the curved space due to the orbital spin of the fermions. Follow the same discussion for the bosonic case, we find that the quantization conditions of \( c \) and \( \nu \) for fermionic case can be obtained from (see eqn. (12))

\[
D_{U_1 \times \Sigma_g} \int_{\Sigma_g \times B^2} \frac{c^{U_1}}{24} \pi_1 + \frac{\nu}{8} (c_1^{U_1})^2 \in \mathbb{Z},
\]

for any orientable surface bundle \( \Sigma_g \times B^2 \) and for any \( U_1 \) bundle whose first Chern class \( c_1^{U_1} \) satisfies \( \frac{1}{2\nu} \int_{\Sigma_g} c_{1}^{U_1} \in \mathbb{Z} \) and \( c_{1}^{U_1} \equiv \frac{1}{2} \).
Here \( c_{U_1}^{\nu_1} = da^{R/Z}, \ a^{R/Z} = \frac{2A_{\mathcal{U}}}{g} \) is the \( U_1 \)-connection for charge-2 bosons (fermion pairs), and \( w_2 \) is the second Stiefel–Whitney class of the tangent bundle of \( \Sigma_g \times B^2 \).

Next, we choose some special surface bundles \( \Sigma_g \times B^2 \) and the allowed \( U_1 \) bundles to obtain quantization conditions of \( \nu \) and \( c \). Let us choose the surface bundle to be \( \Sigma_g \times B^2 = S^1 \times S^1 \times S^1 \times S^1 \), where \( S^1 \times S^1 \) is the space and \( S^1 \times S^1 = B^2 \). The second Stiefel–Whitney class of such a surface bundle is trivial \( w_2 = 0 \). Thus the allowed \( U_1 \) bundles satisfy \( \int c_{U_1}^{\nu_1} = 0 \).

We consider the following form of \( c_{U_1}^{\nu_1} = c_1^a + c_1^b \), where \( c_1^a \) lives on \( S^1 \times S^1 \) and \( c_1^b \) lives on \( S^1 \times S^1 \). Since \( \int_{\Sigma_g \times B^2} p_1 = 0 \) and \( \int_{S_g^1 \times S^1} c_1^b = 0 \), the quantization condition eqn. (55) becomes

\[
\nu D_{U_1,\Sigma_g} = \int_{S_g^1 \times S^1} \left( c_1^a + c_1^b \right)^2
\]

We can choose the \( U_1 \) bundle such that \( \int_{S_g^1 \times S^1} c_1^a = 0, \pm 2 \) and \( \int_{S_g^1 \times S^1} c_1^b = 0, \pm 2 \). We obtain the following quantization condition

\[
\nu D_{U_1,\Sigma_g} \in \mathbb{Z}, \quad g \geq 1.
\]

If \( \Sigma_g \times B^2 \) is spin, then \( \int_{\Sigma_g \times B^2} p_1 = 0 \mod 48 \) by (B4). We explain in Appendix B that as long as \( g \geq 9 \) there is such a spin surface bundle with \( \int_{\Sigma_g \times B^2} p_1 = \pm 48 \), in which case we may choose the \( U_1 \)-bundle to be trivial. This gives us a quantization of \( c \) (see Ref. 30):

\[
2c D_{U_1,\Sigma_g} \in \mathbb{Z}, \quad g \geq 9.
\]

To obtain additional quantization condition, we consider a more general surface bundle over surface \( \Sigma_g \times B^2 \). In Appendix B, we show that, for each \( g \geq 5 \), there are many surface bundles \( \Sigma_g \times B^2 \) equipped with a 2-coycle \( c_{U_1}^{\nu_1} \in H^2(\Sigma_g \times B^2; \mathbb{Z}) \) satisfying (1) \( \int c_{U_1}^{\nu_1} = 0 \) and (2) \( c_{U_1}^{\nu_1} \simeq w_2 \). Let \( \int_{\Sigma_g \times B^2} (c_{U_1}^{\nu_1})^2 = \eta_c \) and \( \int_{\Sigma_g \times B^2} p_1 = 3\eta_p \) where \( \eta_c \) and \( \eta_p \) are integers. Eqn. (55) becomes

\[
D_{U_1,\Sigma_g} \left( -\eta_p \frac{c}{8} + \eta_c \frac{\nu}{8} \right) \in \mathbb{Z},
\]

In Appendix B, we find that the set of allowed \((\eta_p, \eta_c)\) is exactly given by the integer pairs that satisfy

\[
\eta_c = \eta_p = 0.
\]

Now eqn. (60) implies that \( \nu D_{U_1,\Sigma_g} \in \mathbb{Z} \) and two new quantization conditions

\[
c D_{U_1,\Sigma_g} \in \mathbb{Z}, \quad g \geq 5,
\]

\[
\left( -\frac{c}{2} + \frac{\nu}{2} \right) D_{U_1,\Sigma_g} \in \mathbb{Z}, \quad g \geq 5.
\]

To summarize

for a 2+1D fermionic gapped phase with \( U_1 \)-symmetry, its chiral central charge \( c \) and filling fraction \( \nu \) satisfy the following quantization conditions

\[
c D_{U_1,\Sigma_g} = 0 \mod 1, \quad g \geq 5,
\]

\[
\nu D_{U_1,\Sigma_g} = 0 \mod 1, \quad g \geq 1,
\]

\[
(-c + \nu) D_{U_1,\Sigma_g} = 0 \mod 2, \quad g \geq 5
\]

where \( D_{U_1,\Sigma_g} \) is the ground state degeneracy on closed space \( \Sigma_g \).

D. H-type invertible fermionic \( U_1 \)-enriched topological orders

Let us discuss a simple example. The invertible fermionic \( U_1 \)-enriched topological order is classified by \( K \)-matrix (see eqn. (22))

\[
K = (1)^{\oplus m} \oplus (-1)^{\oplus n}
\]

and charge vector \( q \) with odd-integer components. Its \( c \) and \( \nu \) are given by

\[
c = \text{Tr}(K), \quad \nu = \sum_{i=1}^{m} q_i^2 - \sum_{i=m+1}^{m+n} q_i^2.
\]

The ground state degeneracy is always \( D_{U_1,\Sigma_g} = |\text{Det}(K)| = 1 \). We see that \(-c + \nu = 0 \mod 8 \), which satisfies (but does not saturate) eqn. (63).

V. THE TOPOLOGICAL INVARIANTS FOR L-TYPE TOPOLOGICAL ORDERS

A. Topological partition function for \( L \)-type topological order

Consider a bosonic system in gapped liquid phase described by a path integral on \( n \)D spacetime \( M^n \). After integrating out all the dynamical degrees of freedom, we will obtain a partition function that has the following form

\[
Z(M^n) = e^{-\int_{M^n} d^n x \epsilon(x) Z^{\text{top}}(M^n)}
\]

where \( \epsilon(x) \) is the energy density of the ground state. The term \( Z^{\text{top}}(M^n) \) is called the topological partition function and is a topological invariant of the 2+1D topological order. To be more precise, \( Z^{\text{top}}(M^n) \) is a complex

\[

\text{Z}(M^n) = e^{-\int_{M^n} d^n x \epsilon(x) Z^{\text{top}}(M^n)}
\]
function on $\mathcal{M}_{M^n}$, which is the moduli space of the spacetime $M^n$, i.e. $\mathcal{M}_{M^n}$ is a space of metrics $g_{\mu\nu}$ of a closed manifold. The $g_{\mu\nu}$’s differ by diffeomorphisms are equivalent and represent the same point of the moduli space $\mathcal{M}_{M^n}$.

We like to point out that for certain spacetime topologies, the spacetime $M^n$ must contain world-line of point-like excitations, and/or world-sheet of string-like excitations, etc. For those certain spacetime topologies $Z^{\text{top}}(M^n) = 0$ on the corresponding moduli space $\mathcal{M}_{M^n}$. For other spacetime topologies, the spacetime $M^n$ does not contain any excitations. In this case $Z^{\text{top}}(M^n)$ is always non-zero on the corresponding moduli space $\mathcal{M}_{M^n}$. If $Z^{\text{top}}(M^n)$ was zero at an isolated point (or isolated lower dimensional subspace of $\mathcal{M}_{M^n}$), then a small perturbation will make $Z^{\text{top}}(M^n)$ non-zero. This causes the diverging change in the effective action $S_{\text{eff}} = \ln(Z^{\text{top}}(M^n))$, which indicates that the system to be gapless. Thus[12]

| on the moduli space of gapped systems, $Z^{\text{top}}(M^n)$ is either always zero (when the spacetime must contain excitations), or always non-zero (when the spacetime is filled by gapped ground state). |

We also like to conjecture that

| on the moduli space of gapped systems, $|Z^{\text{top}}(M^n)|$ is a constant. |

Physically, since the system is gapped, the $Z^{\text{top}}(M^n)$ should not depend on the size (i.e. the metrics) of the spacetime, we naively expect $Z^{\text{top}}(M^n)$ to be constant. However, the such a naive expectation is not totally correct. We will see that the phase of $Z^{\text{top}}(M^n)$ can depend on the metrics. So we conjecture that $|Z^{\text{top}}(M^n)|$ is a constant.

The metrics dependence of the phase of $Z^{\text{top}}(M^n)$ has a topological reason: they come from the gravitational Chern–Simons terms $\omega_n(a^{SO})$ that depend on the $SO$ connection $a^{SO}$ of the tangent bundle

$$Z^{\text{top}}(M^n) = |Z^{\text{top}}(M^n)| e^{2\pi i \int_{M^n} \kappa \omega_n(a^{SO})}.$$  \hspace{1cm} (67)

The gravitational Chern–Simons terms exist only in spacetime dimensions $n \leq 3$.

When the tangent bundle of the spacetime is non-trivial, the $SO$ connection $a^{SO}$ of the tangent bundle is not globally defined, and as the result the gravitational Chern–Simons action $\int_{M^n} \kappa \omega_n(a^{SO})$ is not well defined. In this paper, we use a cobordism approach trying to define the difference of two gravitational Chern–Simons actions as

$$\int_{M^n_1} \kappa \omega_n(a^{SO}) - \int_{M^n_2} \kappa \omega_n(a^{SO}) \equiv \int_{N^{n+1}} \kappa p(a^{SO}) = \int_{N^{n+1}} \kappa p(a^{SO})$$  \hspace{1cm} (68)

where $M^n_1$ and $M^n_2$ are the boundary of $N^{n+1}$: $\partial N^{n+1} = M^n_1 \sqcup - M^n_2$, and $p(a^{SO}) = \omega_n(a^{SO})$ is a product of Pontryagin classes. Note that $a^{SO}$ in $p(a^{SO})$ is the $SO$-connection of the tangent bundle for $N^{n+1}$. Since $N^{n+1}$ satisfying $\partial N^{n+1} = M^n_1 \sqcup - M^n_2$ is not unique, the different choices of $N^{n+1}$ may give different $\int_{M^n} \kappa \omega_n(a^{SO}) - \int_{M^n_2} \kappa \omega_n(a^{SO})$. However, if the ambiguity is an integer, then $\frac{\int_{M^n} \kappa \omega_n(a^{SO})}{\int_{M^n_2} \kappa \omega_n(a^{SO})}$ is well defined. This leads to a quantization of $\kappa$:

$$\int_{N^{n+1}} \kappa p(a^{SO}) \in \mathbb{Z}, \quad \partial N^{n+1} = \emptyset. \hspace{1cm} (69)$$

So according to the cobordism approach, the phase of the partition function $e^{2\pi i \int_{M^n} \kappa \omega_n(a^{SO})}$ can be well defined only when $\kappa$ satisfy a quantization condition eqn. (69).

B. The quantization of $c$ for 2+1D bosonic topological orders with non-zero partition function

Let us apply the above result for 2+1D L-type bosonic topological orders. Here we make a crucial assumption that the partition function is non-zero for all closed orientable spacetime manifolds

$$Z^{\text{top}}(M^3) = |Z^{\text{top}}(M^3)| e^{i \frac{2\pi c}{24} \int_{M^3} \omega_3} \neq 0. \hspace{1cm} (70)$$

Not all 2+1D L-type bosonic topological orders satisfy this condition. But at least, the 2+1D L-type bosonic invertible topological orders[12, 40, 41] are believed to satisfy this condition.

For 2+1D L-type bosonic topological orders satisfying this condition, their chiral central charges satisfy

$$\frac{c}{24} \int_{N^4} p_1 \in \mathbb{Z}, \quad \forall \partial N^4 = \emptyset. \hspace{1cm} (71)$$

Since the partition function is non-zero on all closed orientable 3-manifold, here we choose $N^4$ to an arbitrary closed orientable 4-manifold. This is a key assumption in our cobordism approach.

We know that for all different closed orientable 4-manifolds $N^4$, the set $\{f_{N^4} p_1 \}$ is given by 3Z. We find that for those L-type bosonic topological orders, their chiral central charges are quantized as

$$\frac{c}{8} \in \mathbb{Z}. \hspace{1cm} (72)$$

We conclude that, at least for the 2+1D L-type bosonic invertible topological orders, their chiral central charges satisfy $c = 0 \mod 8$.

C. The quantization of $c$ for 2+1D bosonic topological orders with vanishing partition functions only on non-spin manifolds and for 2+1D fermionic invertible topological orders

It was pointed out that a bosonic topological order with emergent fermions must have vanishing topological
partition functions on non-spin manifolds. Here we consider a subclass of 2+1D bosonic topological order with emergent fermions, such that topological partition functions vanish only on non-spin manifolds. The topological partition functions are non-zero on all closed orientable spin manifolds. We believe that the partition functions for fermionic invertible topological orders satisfy this condition. Also the bosonic topological order obtained by gauging the $\mathbb{Z}_2^f$ symmetry in fermionic invertible topological order satisfy this condition. In this case, the quantization condition becomes

$$\frac{c}{24} \int_{N^4} p_1 \in \mathbb{Z}, \quad \forall \partial N^4 = \emptyset, \quad w_2 = 0. \quad (73)$$

i.e. we also require $N^4$ to be orientable and spin. As explained in Appendix A, the set of integers which may be realized as $\int_N p_1$ for closed spin 4-manifolds $N^4$ is $48 \mathbb{Z}$. We find that for those simple L-type bosonic topological orders with emergent fermions, their chiral central charges are quantized as

$$2c \in \mathbb{Z}. \quad (74)$$

We conclude that, at least for the 2+1D L-type fermionic invertible topological orders, their chiral central charges satisfy $c = 0 \mod \frac{1}{2}$.

**D. Framing anomaly**

We have mentioned that the partition function is a function on the moduli space $M_{M^3}$ of different closed spacetime manifolds $M^3$. In fact, in the presence of gravitational Chern–Simons term, such a statement is incorrect. The partition function is actually a function on the moduli space $M_{M^3}$ of different closed spacetime manifolds $M^3$ with framing structure. In other words, the partition functions for the same manifold but with different framing structures may have different values.

So, what is a framing structure? Give a $n$-manifold $M^n$ and its tangent bundle $TM$, its stabilized tangent bundle is given by $TM \oplus \mathbb{R}^\infty$. The manifold $M^n$ is framable if the stabilized tangent bundle is trivial. A trivialization of the stabilized tangent bundle (i.e. a choice of a global basis of the stabilized tangent bundle) is a framing structure.

It turns out that all orientable 3-manifold $M^3$ is framable. This gives us a way to define the gravitational Chern–Simons term. If the tangent bundle $TM$ of $M^3$ is non-trivial, the $SO_3$ connection $a^{SO_3}$ for the tangent bundle $TM$ cannot be globally defined on $M^3$, which makes the gravitational Chern–Simons term $\int_{M^3} \omega_3 (a^{SO_3})$ not well defined. But if we embed $SO_3$ into $SO_\infty \equiv SO$, then the corresponding $a^{SO}$ connection can be globally defined on $M^3$ (since the stabilized tangent bundle is trivial), which makes the gravitational Chern–Simons term $\int_{M^3} \omega_3 (a^{SO})$ well defined. But there are different ways to turn $a^{SO}$ connection into a globally defined $SO$-connection $a^{SO}$, which corresponds to different trivializations (i.e. different choices of framing structures). Such different choices of framing structures can change the gravitational Chern–Simons term $\int_{M^3} \omega_3 (a^{SO})$ by an arbitrary integer. We see that if $c \neq 0 \mod 24$, then the phase of partition function $e^{i \frac{A_{F^3}}{24}} \int_{M^3} \omega_3 (a^{SO})$ will depend on the framing structures and the partition function is not a function on the moduli space $M_{M^3}$ of 3-manifolds. This phenomenon is called framing anomaly. In this case, the partition function is a function on the moduli space $M_{M^3}$ of 3-manifolds with framing structures.

In general, H-type topological orders have $c \neq 0 \mod 24$, and thus correspond to L-type topological orders with framing anomaly. The L-type topological orders without framing anomaly must have $c = 24$. In this paper, we will mainly discuss L-type topological orders with framing anomaly. Their partition function is a function on the moduli space $M_{M^3}$ of 3-manifolds with framing structures.

**VI. THE TOPOLOGICAL INVARIANTS FOR L-TYPE TOPOLOGICAL ORDERS WITH $U_1$ SYMMETRY**

**A. Quantization of $c$ and $\nu$ for bosonic $U_1$-SET orders with non-zero partition functions**

A L-type bosonic $U_1$-enriched topological orders may contain two Chern–Simons terms given by eqn. (8). Here we assume that the partition function is non-zero for all closed orientable spacetime with any $U_1$-bundle on it. Repeating the arguments in the last section, but for spacetime with a $U_1$ connection $a^{U_1/2} = \frac{A_{U_1}}{2\pi}$, we obtain the following quantization for $c$ and $\nu$, for bosonic $U_1$-enriched topological orders whose topological partition functions are non-zero on any closed orientable spacetime:

$$\int_{N^4} -\frac{c}{24} p_1 + \frac{\nu}{2} c_2^2 \in \mathbb{Z}, \quad \forall \partial N^4 = \emptyset \text{ and } \forall U_1 \text{ bundles.} \quad (75)$$

Here $N^4$ is an arbitrary orientable 4-manifold with an arbitrary $U_1$-bundle on it. Let $\eta_p = \int_{N^4} p_1$ and $\eta_c = \int_{N^4} c_2^2$, where $\eta_p$ and $\eta_c$ are integers. In Appendix A we show that the pairs $(\eta_p, \eta_c)$ which may be realized is given by $(Z, Z)$. This leads to the quantization of $c$ and $\nu$:

$$c = 0 \mod 8, \quad \nu = 0 \mod 2. \quad (76)$$

The above result, at least, applies to L-type invertible bosonic $U_1$-enriched topological orders and L-type bosonic $U_1$-SPT orders.
B. Quantization of \( c \) and \( \nu \) for fermionic \( U_1^c \)-enriched topological orders with non-zero partition functions

A L-type fermionic \( U_1^c \)-enriched topological orders also contain two Chern–Simons terms given by eqn. (8). For fermionic \( U_1 \)-SET orders whose topological partition functions are non-zero on any closed smooth \( \text{orientable spin}^C \) spacetime manifolds, the quantization for \( c \) and \( \nu \) is given by

\[
\int_{N^4} -\frac{c}{24} p_1 + \frac{\nu}{8} (\epsilon U_1)^2 \in \mathbb{Z}, \quad \forall \partial N^4 = \emptyset \text{ and } \forall \epsilon U_1 \equiv \epsilon \mod w_2.
\]  

(77)

Here \( N^4 \) is an arbitrary closed smooth \( \text{orientable spin}^C \) 4-manifold, as implied by the condition \( \epsilon U_1 \equiv \epsilon \mod w_2 \). Let \( 3\eta_p = \int_{N^4} p_1 \) and \( \eta_c = \int_{N^4} (\epsilon U_1)^2 \), where \( \eta_p \) and \( \eta_c \) are integers. As explained in Appendix A, we may find such \( \text{spin}^C \) 4-manifolds realizing any integers \( \eta_p \) and \( \eta_c \) as long as they satisfy

\[
\eta_p \equiv \eta_c. \quad \text{(78)}
\]

This leads to the quantization of \( c \) and \( \nu \):

\[
c = 0 \mod 1, \quad \nu = 0 \mod 1, \quad -c + \nu = 0 \mod 8.
\]  

(79)

The above result, at least, applies to L-type invertible fermionic \( U_1^c \)-enriched topological orders and L-type fermionic \( U_1^c \)-SPT orders.

For L-type fermionic \( U_1^c \)-SPT orders, the central charge vanishes \( c = 0 \) and \( \nu = 0 \mod 8 \). This agrees with the result in Ref. 42. We also note that the fermionic invertible \( U_1^c \)-SET states discussed in Sec. IV D satisfy the above quantization condition, and actually saturate the quantization condition.

We remark that the combination of invertible fermionic \( U_1^c \)-enriched topological orders and fermionic \( U_1^c \)-SPT orders (i.e. the invertible fermionic gapped liquid phases with \( U_1^c \) symmetry) is classified via \( \text{spin}^C \) cobordism in Ref. 43, which is given by \( \mathbb{Z}^2 \) in 2+1D. This agrees with our result that those state are labeled by two integers \( \{(\eta_p, \eta_c) \mid \eta_p \equiv \eta_c\} \). However, the values of \( c \) and \( \nu \) are not discussed in Ref. 43.

VII. SUMMARY

For 2+1D gapped liquid phases with \( U_1 \) symmetry, the central charge \( c \) and the dimensionless Hall conductance \( \nu \) are described by the Chern–Simons terms in the partition function. However, for arbitrary smooth spacetime 3-manifolds and for arbitrary \( U_1 \)-bundle over spacetime 3-manifolds, it is highly non-trivial to define the Chern–Simons terms. In this paper, we use a cobordism approach to define the Chern–Simons terms. We find that for H-type quantum systems, the 4-manifolds used in the cobordism approach must be a surface bundle. For L-type quantum systems, the 4-manifolds must have the same type as that of the spacetime 3-manifolds, such that the partition functions are non-zero. This leads to different quantization conditions for \( c \) and \( \nu \). In particular, for the H-type quantum systems, the quantization of \( c \) and \( \nu \) depends on the ground state degeneracies on Riemannian surfaces. While for the L-type quantum systems, the quantization of \( c \) and \( \nu \) depends on the type of spacetime manifolds where the topological partition function is non-zero.

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Appendix A: Characteristic numbers of 4-manifolds

The most fundamental invariant of a closed oriented 4-manifold \( M \) is its intersection form

\[
Q_M : H^2(M^4;\mathbb{Z}) \times H^2(M^4;\mathbb{Z}) \to \mathbb{Z}
\]

\[
(a, b) \mapsto \langle a \sim b, [M^4] \rangle = \int_M ab.
\]

Under connected sum the intersection form satisfies

\[
Q_M \# N = Q_M \oplus Q_N.
\]

By Poincaré duality it is unimodular, and by commutativity of the cup-product it is symmetric and therefore has a signature, denoted \( \sigma(M) \). By a theorem of Thom, \( M \) is the boundary of a 5-manifold if and only if its signature is 0.

Hirzebruch’s signature theorem relates the signature to Pontrjagin classes, and for a 4-manifold gives

\[
\int_M p_1(TM) = 3 \cdot \sigma(M). \quad \text{(A1)}
\]

By definition of the Wu class \( u_2 \) for any \( \mathbb{Z}_2 \)-cohomology class \( x \) we have \( \int_M x^2 = \int_M x \cdot u_2 \). As \( M \) is oriented, so its first Stiefel–Whitney class vanishes, and \( u_2 = w_2 + w_1^2 = w_2 \), so

\[
\int_M x^2 = \int_M x \cdot w_2. \quad \text{(A2)}
\]

Thus if \( M \) is spin, i.e. \( w_2 = 0 \), then the form \( Q_M \) is even. If \( M \) is spin then by Rochlin’s theorem its signature is
divisible by 16, and so

$$\int_M p_1(TM) \cong 0. \quad (A3)$$

Finally consider spin\(^C\) 4-manifolds \(M\), with \(c_1 \in H^2(M; \mathbb{Z})\) the associated Chern class. Then \(c_1\) reduces modulo 2 to \(w_2\), so is a characteristic element of \(Q_M\). In this case we may also form the characteristic number \(\int_M c_1^2\). By an elementary property of symmetric forms over \(\mathbb{Z}\) (see II.5.2 of [44]) we have

$$\int_M c_1^2 \cong \sigma(M). \quad (A4)$$

We now describe to what extent these characteristic numbers may be realized.

**Oriented 4-manifolds.** The manifold \(\mathbb{C}P^2\) has \(Q_{\mathbb{C}P^2}\) given by the 1-by-1 matrix (1), so has signature 1; similarly \(\overline{\mathbb{C}P^2}\) has signature \(-1\). By taking connected sums we may therefore realize every element of \(\mathbb{Z}\) as the signature of an oriented 4-manifold, so may realize every element of \(3\mathbb{Z}\) as \(\int_M p_1(TM)\).

**Oriented 4-manifolds with \(U_1\)-bundle.** The manifold \(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}\) has signature 0, and cohomology ring

$$H^*(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2 + y^2, x^3, y^3).$$

Taking the \(U_1\)-bundle with \(c_1 = x\) it has \(\int_{\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}} c_1^2 = 1\), and taking instead the \(U_1\)-bundle with \(c_1 = y\) it has \(\int_{\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}} c_1^2 = -1\). By forming connected sums of \(\mathbb{C}P^2\) and \(\overline{\mathbb{C}P^2}\) with trivial \(U_1\)-bundle, and \(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}\) with the \(U_1\)-bundles just described, we can realize any element of \(\mathbb{Z}\) as \((\sigma(N), \int_N c_1^2)\) for an oriented 4-manifold \(N\) with a \(U_1\)-bundle over it.

**Spin 4-manifolds** The 4-manifold \(K3\) is spin, and has intersection form

$$Q_{K3} = -E_8^{\oplus 2} \oplus H^{\oplus 3},$$

where

$$H = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

and

$$E_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

so has signature \(-16\). By taking connected sums of \(K3\) and its orientation-reversed analogue \(\overline{K3}\) we may realize every element of \(16\mathbb{Z}\) as the signature of a spin 4-manifold, so may realize every element of \(48\mathbb{Z}\) as \(\int_M p_1(TM)\).

**Spin\(^C\) 4-manifolds.** The manifold \(\mathbb{C}P^2\) has cohomology ring \(H^*(\mathbb{C}P^2; \mathbb{Z}) = \mathbb{Z}[x]/(x^3)\). Its usual complex structure induces a spin\(^C\)-structure having \(c_1 = 3x\), and so having

$$\sigma(\mathbb{C}P^2) = 1$$

and

$$\frac{1}{8} \left( \sigma(\mathbb{C}P^2) - \int_{\mathbb{C}P^2} c_1^2 \right) = -1.$$

The manifold \(\mathbb{C}P^1 \times \mathbb{C}P^1\) has cohomology ring \(H^*(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2, y^2)\). Its usual complex structure induces a spin\(^C\)-structure having \(c_1 = 2x + 2y\), and so having

$$\sigma(\mathbb{C}P^1 \times \mathbb{C}P^1) = 0$$

and

$$\frac{1}{8} \left( \sigma(\mathbb{C}P^1 \times \mathbb{C}P^1) - \int_{\mathbb{C}P^1 \times \mathbb{C}P^1} c_1^2 \right) = -1.$$
Using these classes we may form \( \int_E p_1 \), which by (B1) agrees with \( \int_E e^2 \), known as the first Miller–Morita–Mumford class. By (A1) we have \( \int_E p_1 = 3 \cdot \sigma(E) \) so \( \int_E p_1 \) is always divisible by 3, but in fact more is true: as a consequence of the Atiyah–Singer index theorem applied to the fiberwise signature operator we have

\[
\int_E p_1 \equiv 0. \tag{B3}
\]

This may be obtained from p. 555 of Ref. 45, using that \( B_2 = \frac{1}{6} \). If in addition \( w_2 = 0 \), then (A3) gives

\[
\int_E p_1 \equiv 0. \tag{B4}
\]

Now let us suppose that we are further given a \( c_1 \in H^2(E; \mathbb{Z}) \). Using this we may form \( \int_E c_1^2 \) and \( \int_E e \cdot c_1 \). As \( e \) reduces modulo 2 to \( w_2 \) by (B2), (A2) gives

\[
\int_E c_1^2 = \int_E e \cdot c_1. \tag{B5}
\]

Let us now suppose in addition that \( r_2(c_1) = w_2 \), i.e. that \( c_1 \) provides a fiberwise spin^\text{c}\-structure, and investigate its consequences. As the Euler class \( e \) also reduces to \( w_2 \) modulo 2, it follows from the Bockstein sequence that \( c_1 = e + 2\bar{c}_1 \) for some \( \bar{c}_1 \in H^2(E; \mathbb{Z}) \). Then we have \( \int_E c_1^2 = \int_E e^2 + 4\int_E (e + \bar{c}_1)^2 \) which by the divisibility results above we can write in terms of integers as \( \int_E c_1^2 = 12 \cdot \left( \frac{\int_E p_1}{4} \right) + 8 \cdot \left( \frac{\int_E (e + \bar{c}_1)^2}{2} \right) \). This implies

\[
\int_E c_1^2 = \frac{\int_E p_1}{3}, \tag{B6}
\]

recovering (A4). Similarly, \( \int_E e \cdot c_1 = 12 \cdot \left( \frac{\int_E p_1}{12} \right) + 2 \cdot \int_E e \cdot \bar{c}_1 \) so we have

\[
\int_E e \cdot c_1 \equiv 0. \tag{B7}
\]

2. **Realising characteristic numbers**

The Madsen–Weiss theorem [46] and its variants [47, 48] can be used to establish the existence of surface bundles with given geometric structure and characteristic numbers in a highly indirect way, assuming that the genus \( g \) is sufficiently large in comparison with the dimension of the base. For the structure considered here, of a surface bundle equipped with a \( U_1 \)-bundle (equivalently a second integral cohomology class) on the total space and 2-dimensional base, this has been carried out in [49]. The conclusion is as follows.

Firstly, for any \( g \geq 5 \) there are oriented \( \Sigma_g \)-bundles \( \pi : E^4 \to B^2 \) and classes \( c_1 \in H^2(E; \mathbb{Z}) \) which realize any values of

\[
\int_{\Sigma_g} c_1, \int_E p_1, \int_E c_1^2, \int_E e \cdot c_1
\]

as long as they satisfy (B3) and (B5). This is obtained by combining Theorem A (iv) and Theorem C of Ref. 49 and using that every second homology class may be represented by a map from a surface \( B^2 \).

Secondly, for any \( g \geq 5 \) there are oriented \( \Sigma_g \)-bundles \( \pi : E^4 \to B^2 \) and classes \( c_1 \in H^2(E; \mathbb{Z}) \) satisfying \( r_2(c_1) = w_2 \) which realize any values of

\[
\int_{\Sigma_g} c_1, \int_E p_1, \int_E c_1^2, \int_E e \cdot c_1
\]

as long as they satisfy (B3), (B6), and (B7). This is obtained by taking \( c_1 = e + 2\bar{c}_1 \) and translating the conditions on \( \int_{\Sigma_g} \bar{c}_1, \int_E p_1, \int_E c_1^2, \int_E e \cdot \bar{c}_1 \) from the previous paragraph to conditions on \( \int_{\Sigma_g} c_1, \int_E p_1, \int_E c_1^2, \int_E e \cdot c_1 \).

Thirdly, the analogous analysis for spin surface bundles though not equipped with an additional line bundle, has been carried out in [50]. It follows from Example 1.11 of loc. cit. that for any \( g \geq 9 \) there are Spin \( \Sigma_g \)-bundles \( \pi : E^4 \to B^2 \) realising any value of \( \int_E p_1 \) satisfying (B4).

---

**Appendix C: Checking (29), (30), (31)**

1. **Spread polynomials \( S_n(x) \)**

The spread polynomials \( S_n(x) \) are defined by \( S_n(\sin^2 \theta) = \sin^2(n\theta) \). They have the explicit form [51]:

\[
S_n(x) = x \sum_{m=0}^{n-1} \frac{n}{n-m} \binom{2n-1-m}{m} (-4x)^{n-m-1}
\]

which has roots \( \{ \sin^2 \left( \frac{m\pi}{n} \right) : m = 0, \ldots, n-1 \} \). Putting \( n = k + 2 \) and replacing \( x \to 1/x \), the polynomial

\[
x^{k+2} S_{k+2} \left( \frac{1}{x} \right) = \sum_{m=0}^{k+1} \alpha_m x^m
\]

\[
\alpha_m := \frac{k + 2}{k + 2 - m} \binom{2k + 3 - m}{m} (-4)^{k-m+1}
\]

has roots \( \{ \csc^2 \frac{m\pi}{k+2} : m = 1, \ldots, k+1 \} \).

2. **Symmetric polynomials \( e_i \) and \( p_i \)**

Let \( \xi_m := \csc^2 \frac{m\pi}{k+2} \). Their elementary symmetric polynomials are given by

\[
e_i := \sum_{1 \leq m_1 < \cdots < m_i \leq k+1} \xi_{m_1} \cdots \xi_{m_i} = \frac{(-1)^i \alpha_{k-l+1}}{\alpha_{k+1}}
\]

\[
= \frac{4^l}{(k+2)(l+1)} \binom{k + 2 + l}{2l + 1}
\]
for \( l \leq k + 1 \). For \( l > k + 1 \), \( e_l := 0 \). Define
\[
E_l := \frac{(k + 2)^l}{2^l} e_l = 2^l (k + 2)^{l-2} (k + 2 + l) \binom{k + 2 + l}{2l + 1}
\]
\[
= 2^l (k + 3) \binom{k + 2 + l}{2l + 2} \binom{k + 2 + l}{2l + 2}.
\]
Note that \( E_1 = \frac{(k+3)}{3} \in \mathbb{Z} \), which is odd iff \( k = 0 \mod 4 \), so \( kE_1 \in 4 \mathbb{Z} \) for \( l \geq 2 \), \( E_1 \in 4 \mathbb{Z} \). Moreover \( \frac{3E_1}{k} \in 2 \mathbb{Z} \) and \( \frac{3E_1}{k} \in 2 \mathbb{Z} \) for \( l \geq 3 \).

Define the \( l \)-th power sums \( p_l := \sum_{m=1}^{k+1} \alpha^{l} e_m \). They are related to \( e_l \)'s via Newton–Girard Formulas \( [52] \):
\[
p_l = (-1)^{l+1} k e_l + \sum_{j=1}^{l-1} (-1)^{j+1} e_j p_{l-j}.
\]

3. Checking (29)

Define \( n_l = \frac{3(k+2)^l}{k+2} p_l \). The expression eqn. (29) to be checked is
\[
n_{g-1} \in 2 \mathbb{Z}.
\]
We will show this by induction for \( g \geq 3 \). From (C1) we have
\[
n_l = (-1)^{l+1} k \left( \frac{3E_l}{k+2} \right) + \sum_{j=1}^{l-1} (-1)^{j+1} E_j n_{l-j}.
\]
Note that \( n_1 = \frac{(k+3)(k+4)}{4} \in \mathbb{Z} \), which is odd iff \( k = 2 \mod 4 \). For \( l = 2 \), we have
\[
n_2 = -2k \left( \frac{3E_2}{k+2} \right) + E_1 n_1.
\]
where the first term is even since the factor in bracket is even as discussed before. The second term is even since \( E_1 \) and \( n_1 \) cannot be odd at the same time. For \( l \geq 3 \), it can be seen that \( n_l \in 2 \mathbb{Z} \) inductively since both terms of (C2) are even integers.

4. Checking (30)

Define \( m_l = \frac{k(k+2)^l}{2^l} p_l \). The expression eqn. (30) to be checked is
\[
m_{g-1} \in 2 \mathbb{Z}.
\]
We will show this by induction for \( g \geq 1 \). From (C1) we have
\[
m_l = (-1)^{l+1} k E_l + \sum_{j=1}^{l-1} (-1)^{j+1} E_j m_{l-j}.
\]
Note that \( m_0 = k(k+1) \in 2 \mathbb{Z} \). For \( l \geq 1 \), the first term in (C3) is even since \( kE_l \) is even. Hence \( m_l \in 2 \mathbb{Z} \) for all \( l \geq 0 \) by induction.

5. Checking (31)

The expression eqn. (31) to be checked is
\[
\frac{m_{g-1}}{2} \in 2 \mathbb{Z}.
\]
for \( g \geq 6 \). We will show a stronger result that this holds for all \( g \geq 2 \) by induction. (C3) can be rewritten as
\[
\frac{m_l}{2} = (-1)^{l+1} \left( \frac{kE_l}{2} \right) + \sum_{j=1}^{l-1} (-1)^{j+1} E_j \left( \frac{m_{l-j}}{2} \right)
\]
\[
\frac{m_l}{2} \geq \sum_{j=1}^{l-1} E_j \left( \frac{m_{l-j}}{2} \right) \geq \begin{cases} E_1 \left( \frac{m_{l-1}}{2} \right) & \text{for } l \geq 2 \\ 0 & \text{for } l = 1 \end{cases}
\]
where we used the fact that \( \frac{kE_l}{2} \in 2 \mathbb{Z} \) for \( l \geq 1 \) and \( E_j \in 4 \mathbb{Z} \) for \( j \geq 2 \). Note that \( \frac{m_{l-j}}{2} = \frac{k(k+1)}{4} \) is odd iff \( k = 1, 2 \mod 4 \). Also recall that \( E_1 \in 2 \mathbb{Z} \). It is straightforward to read off that \( \frac{m_{l-j}}{2} \in 2 \mathbb{Z} \) and hence \( \frac{m_{l-j}}{2} \in 2 \mathbb{Z} \) for all \( l \geq 1 \) which follows by induction.

Appendix D: Group extension and trivialization

Consider an extension of a group \( H \)
\[
A \rightarrow G \rightarrow H
\]
where \( A \) is an Abelian group with group multiplication given by \( x + y \in A \) for \( x, y \in A \). Such a group extension is denoted by \( G = A \times H \). It is convenient to label the elements in \( G \) as \((h, x)\), where \( h \in H \) and \( x \in A \). The group multiplication of \( G \) is given by
\[
(h_1, x_1)(h_2, x_2) = (h_1 h_2, x_1 + \alpha(h_1) \circ x_2 + e_2(h_1, h_2)).
\]
where \( e_2 \) is a function
\[
e_2 : H \times H \rightarrow A,
\]
and \( \alpha \) is a function
\[
\alpha : H \rightarrow \text{Aut}(A).
\]
We see that group extension is defined via \( e_2 \) and \( \alpha \). The associativity
\[
[(h_1, x_1)(h_2, x_2)](h_3, x_3) = (h_1, x_1)[(h_2, x_2)](h_3, x_3)
\]
requires that
\[
x_1 + \alpha(h_1) \circ x_2 + e_2(h_1, h_2) + \alpha(h_1 h_2) \circ x_3 + e_2(h_1 h_2, h_3)
\]
\[
= \alpha(h_1) \circ x_2 + \alpha(h_1) \alpha(h_2) \circ x_3 + \alpha(h_1) \circ e_2(h_2, h_3)
\]
\[
+ x_1 + e_2(h_1, h_2 h_3)
\]
or
\[
\alpha(h_1) \alpha(h_2) = \alpha(h_1 h_2)
\]
and
\[ e_2(h_1, h_2) - e_2(h_1, h_2 h_3) + e_2(h_1 h_2, h_3) - \alpha(h_1) \circ e_2(h_2, h_3) = 0. \]  
(D8)

Such a \( e_2 \) is a group 2-cocycle \( e_2 \in H^2(\mathcal{H}; A_\alpha) \), where \( H \) has a non-trivial action on the coefficient \( A \) as described by \( \alpha \). Also, \( \alpha \) is a group homomorphism \( \alpha : H \to \text{Aut}(A) \). We see that the \( A \) extension from \( H \) to \( G \) is described by a group 2-cocycle \( e_2 \) and a homomorphism \( \alpha \). Thus we can more precisely denote the group extension by \( G = A \rtimes_{\alpha} H \).

Note that the homomorphism \( \alpha : H \to \text{Aut}(A) \) is in fact the action by conjugation in \( G \),
\[ (h, 0)(1, x) = (h, \alpha(h) \circ x)(h, 0), \]
\[ \Rightarrow (h, 0)(1, x)(h, 0)^{-1} = (1, \alpha(h) \circ x). \]  
(D9)

Thus, \( \alpha \) is trivial if and only if \( A \) lies in the center of \( G \). This case is called a central extension, where the action \( \alpha \) will be omitted.

Our way to label group elements in \( G \):
\[ g = (h, x) \in G \]  
(D10)
defines two projections of \( G \):
\[ \pi : G \to H, \quad \pi(g) = h, \]
\[ \sigma : G \to A, \quad \sigma(g) = x. \]  
(D11)

\( \pi \) is a group homomorphism while \( \sigma \) is a generic function.

Using the two projections, \( g_1 g_2 = g_3 \) can be written as
\[ (\pi(g_1), \sigma(g_1)) (\pi(g_2), \sigma(g_2)) = [\pi(g_1) \pi(g_2), \sigma(g_1) + \alpha(\pi(g_1)) \circ \sigma(g_2) + e_2(\pi(g_1), \pi(g_2))] \]
\[ = (\pi(g_3), \sigma(g_3)) = (\pi(g_1 g_2), \sigma(g_1 g_2)) \]  
(D12)

We see that the group cocycle \( e_2(h_1, h_2) \) in \( H^2(\mathcal{H}; A) \) can be pullback to give a group cocycle \( e_2(\pi(g_1), \pi(g_2)) \) in \( H^2(\mathcal{B}G; A) \), and such a pullback is a coboundary
\[ e_2(\pi(g_1), \pi(g_2)) = -\sigma(g_1) + \sigma(g_1 g_2) - \alpha(\pi(g_1)) \circ \sigma(g_2), \]  
(D13)
i.e., an element in \( B^2(\mathcal{B}G; A) \), where \( G \) has a non-trivial action on the coefficient \( A \) as described by \( \alpha \).

The above result can be put in another form. Consider the homomorphism
\[ \varphi : \mathcal{B}G \to \mathcal{H} \]  
(D14)
where \( G = A \rtimes_{\alpha} H \) and \( e_2 \) is a \( A \)-valued 2-cocycle on \( \mathcal{B}H \). The homomorphism \( \varphi \) sends an link of \( \mathcal{B}G \) labeled by \( a_{ij}^G \in G \) to an link of \( \mathcal{B}H \) labeled by \( a_{ij}^H = \pi(a_{ij}^G) \in H \). The pullback of \( e_2 \) by \( \varphi \), \( \varphi^* e_2 \), is always a coboundary on \( \mathcal{B}G \).

The above discussion also works for continuous group, if we only consider a neighborhood near the group identity \( 1 \). In this case, \( e_2(h_1, h_2) \) and \( \alpha(h) \) are continuous functions on such a neighborhood. But globally, \( e_2(h_1, h_2) \) and \( \alpha(h) \) may not be continuous functions.
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