We propose a construction of 1 + 1 integrable Heisenberg–Landau–Lifshitz type equations in the case. The dynamical variables are matrix elements of an $N \times N$ matrix $S$ with the property $S^2 = \text{const}$. The Lax pair with spectral parameter is constructed by means of a quantum $R$-matrix satisfying the associative Yang–Baxter equation. Equations of motion for $gl_N$ Landau–Lifshitz model are derived from the Zakharov–Shabat equations. The model is simplified when $\text{rank}(S) = 1$. In this case the Hamiltonian description is suggested. The described family of models includes the elliptic model coming from Baxter–Belavin elliptic R-matrix. In case the widely known Sklyanin’s elliptic Lax pair for XYZ Landau–Lifshitz equation is reproduced. Our construction is also valid for trigonometric and rational degenerations of the elliptic R-matrix.

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**INTRODUCTION AND NOTATION**

The Landau–Lifshitz equation [1] is 1 + 1 field theory describing behavior of the magnetization vector $S(t, x) = (S_1, S_2, S_3)$ in one-dimensional model of ferromagnetic solid ($x$ is a space variable on a unit circle and $t$ is a time variable):

$$\begin{align*}
\partial_t S &= \tilde{c}_1 S \times J(S) + \tilde{c}_2 S \times \partial_x^2 S, \\
J(S) &= (J_1 S_1, J_2 S_2, J_3 S_3),
\end{align*}$$

(1)

where $\tilde{c}_1, \tilde{c}_2$ and $J_1, J_2, J_3$ are some constants. The periodic boundary conditions $S(t, x) = S(t, x + 2\pi)$ are assumed. In addition, for simplicity we deal with the complex version of (1) so that the dynamical variables $(S_1, S_2, S_3)$ and all the constants are $\mathbb{C}$-valued (the reduction to real-valued case is available as well). The second term on the right-hand side of (1) describes the spin exchange interaction, while the first one comes from the anisotropy. In the fully anisotropic case (when $J_1, J_2, J_3$ are pairwise distinct) the model is referred to as XYZ type model, the partially anisotropic case ($J_1 = J_2$) is called XXZ, and the fully isotropic case ($J_1 = J_2 = J_3$) is known as XXX model. In the latter case $S \times J(S) = 0$ and (1) is reduced to the equation for 1 + 1 continuous classical XXX Heisenberg magnet. Equivalently, the Landau–Lifshitz equation (1) is written in the matrix form

$$\begin{align*}
\partial_t S &= c_1 [S, J(S)] + c_2 S \partial_x^2 S, \\
J(S) &= \sum_{k=1}^3 S_k \sigma_k, \\
S &= \sum_{k=1}^3 S_k J_k \sigma_k,
\end{align*}$$

(2)

where $S$ is a traceless $2 \times 2$ matrix, $S_k$ are its components in the basis of the Pauli matrices $\sigma_k$, and $c_{1,2} = 2\sqrt{-1} < 2$.

**LAX PAIR IN THE 2 × 2 CASE**

Integrability of Eqs. (1) and (2) was established in [2] (see also [3]) in the sense of the classical inverse scattering method, which is based on representing a nonlinear equation(s) in the Zakharov–Shabat (or zero-curvature) form [3–6]:

$$\begin{align*}
\partial_t U(z) - \partial_z V(z) + [U(z), V(z)] &= 0, \\
\partial_z U(z), V(z) &= \text{a pair of matrix–valued functions of the variables } t, x \text{ depending also on the spectral}
\end{align*}$$

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parameter \( z \). In \cite{2} the following Lax pair (or \( U-V \) pair) was suggested:

\[
U(z) = \sum_{k=1}^{3} S_k \phi_k(z),
\]

\[
V(z) = \sum_{k=1}^{3} \frac{S_k \phi_k(z) \phi_k(z)}{\phi_k(z)} + \sum_{k=1}^{3} W_k \sigma_k \phi_k(z),
\]

where \( \phi_k(z) \) is the following set of elliptic functions on elliptic curve with moduli \( \tau \) (with \( \text{Im}(\tau) > 0)\):

\[
\phi_1(z) = e^{\nu_1 z} \left( z + \frac{\tau}{2} \right),
\]

\[
\phi_2(z) = e^{\nu_2 z} \left( z + \frac{1 + \tau}{2} \right),
\]

\[
\phi_3(z) = \phi \left( z + \frac{1}{2} \right).
\]  

The functions \( \phi_k(z) \) are defined in terms of the Kronecker elliptic function \( \phi(z,u) \) and the first Jacobi theta-function \( \Theta(z) \):

\[
\Theta(z,u) = \frac{\Theta(0)(z+u)}{\Theta(z)(u)}, \quad \Theta(z) = \Theta(z|\tau)
\]

\[
= -\sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau \left( k + \frac{1}{2} \right)^2 + 2\pi i \left( z + \frac{1}{2} \right)(k + \frac{1}{2}) \right).
\]  

The theta function is odd \( \Theta(-z) = -\Theta(z) \) and has simple zero at \( z = 0 \). Therefore, all \( \phi_k(z) \) have simple pole at \( z = 0 \). Thus \( \text{Res} U(z) = S \). Plugging the Lax pair (4) into equation (3) one gets two types of terms. The first and second types contain the first and second order pole at \( z = 0 \), respectively. These two types vanish separately since (3) is assumed to be valid identically in \( z \). In this way we get two equations. From vanishing of the terms with the first order pole one gets:

\[
\partial_z S = \partial_z T + [S, J(S)]
\]

\[
= \sum_{k=1}^{3} T_k \sigma_k,
\]

where \( J_k = \phi(\omega_k) \). From vanishing of the terms with the second order pole, we obtain

\[
\partial_z S = [S, T].
\]  

The latter equation can be solved with respect to \( T \). Since \( S \) is a traceless \( 2 \times 2 \) matrix, it satisfies the characteristic equation \( S^2 = \lambda \sigma_3 \), where \( \sigma_3 \) is the identity matrix and \( \lambda \) is an eigenvalue of \( S \). Suppose \( \partial_z \lambda = 0 \). This means that the length of vector \( S \) is constant along the direction. Then, differentiating the characteristic equation with respect to \( x \) yields \( \partial_z S = [S, \partial_z S] = 0 \). One can easily verify that Eq. (8) has the solution \( T = \frac{1}{4\lambda^2} [S, \partial_z S] \). Plugging it into (7), we obtain the Landau–Lifshitz equation (2) with \( c_1 = 1, c_2 = 1/(4\lambda^2) \).



The purpose of the work is to propose higher rank generalization of the above construction. For this purpose, we represent the \( U-V \) pair in terms of \( R \)-matrix data. Let us demonstrate it for the above example. The classical \( \mathfrak{gl}_2 \) elliptic \( r \)-matrix has the form

\[
r_{ij}(z) = \sigma_i \otimes \sigma_j E_{ij}(z) + \sum_{k=1}^{3} \sigma_k \otimes \sigma_k \phi_k(z),
\]

\[
E_{ij}(z) = \frac{\partial^2(z)}{\partial(z)}.
\]  

Define

\[
U(z) = \sigma_0 S_0 E_{12}(z) + \sum_{k=1}^{3} \sigma_k S_k \phi_k(z)
\]  

and notice that it differs from the one given in (4) by only the scalar term \( i_2 S_0 E_{12}(z) \), which is not necessary since \( S_0 = \text{tr}(S)/2 \) has trivial dynamics due to (2). Expression (10) is rewritten in terms of \( r \)-matrix (9):

\[
U(z) = U(S, z) = \frac{1}{2} \text{tr}(r_{12}(z) S),
\]

\[
S = i_2 \otimes S, \quad S = \sum_{k=0}^{3} \sigma_k S_k.
\]  

Similarly, one can represent \( V(z) \) in the form:\footnote{To derive (7) one should use the identities \( \phi^2(z) = \phi(z) - \phi(\omega_0 z) \), where \( \omega_0 = \pi/2, \omega_2 = (t + 1)/2 \), \( \omega_1 = 1/2 \) and \( \phi(z) \) is the Weierstrass \( \wp \)-function.}

\[
V(z) = -\partial_z U(S, z) + U(T, z).
\]  

Therefore, the existence of the Zakharov–Shabat representation for the Landau–Lifshitz equations is based on some properties of the classical \( r \)-matrix (9) only. Below we suggest a universal construction based on certain properties of \( R \)-matrices, such as associative Yang–Baxter equation. In this way we describe not only elliptic model but also a family of trigonometric and rational models, which correspond to \( R \)-matrices satisfying the set of required relations.

YANG–BAXTER EQUATIONS. QUANTUM AND CLASSICAL \( R \)-MATRICES

The Landau–Lifshitz equation is the classical continuous version of a certain quantum spin chain, which is described by the underlying quantum \( R \)-matrix. Quantum \( \mathfrak{gl}_N \) \( R \)-matrix in the fundamental representation is a matrix valued function\footnote{For this purpose, one should use the identities \( \partial_x \psi(z) = -\phi(z) \phi_k(z) \psi_k(z) \) valid for any set of distinct \( i, j, k \in \{1, 2, 3\} \).}

\[
R_{\alpha,\beta}(z) = \sum_{i<j}^{N} R_{\alpha,\beta}(h(z) E_{ij} \otimes E_{ji}) \in \text{Mat}(N, \mathbb{C})^{\otimes 2}
\]  

of the reduced Planck constant \( \hbar \) and the spectral
parameter \( z \). By definition, \( R \)-matrix satisfies the quantum Yang–Baxter equation [7–10]:

\[
R^{h}_{12}(z_1 - z_2)R^{h}_{13}(z_1 - z_3)R^{h}_{23}(z_2 - z_3) = R^{h}_{23}(z_2 - z_3)R^{h}_{13}(z_1 - z_3)R^{h}_{12}(z_1 - z_2).
\]

(12)

Here, \( R^{h}_{ab}(z_a - z_b) \) is considered as an element of \( \text{Mat}(N, C) \). It is nontrivial in two tensor components (the \( a \)th and \( b \)th), and the third one is filled by identity matrix. For example, \( R^{h}_{12}(z) = \sum_{ijkl=1}^{N} R_{ijkl}(h, z) E_{ij} \otimes 1_{N} \otimes E_{kl} \). We consider a class of \( R \)-matrices, which also satisfy the unitarity property

\[
R^{h}_{12}(z)R^{h}_{21}(-z) = N^2 \phi(N h, z) \phi(N h, -z) 1_{N} \otimes 1_{N},
\]

(13)

and the skew-symmetry\(^4\)

\[
R^{h}_{12}(z) = -R^{h}_{21}(-z).
\]

(14)

A solution of (12) is defined up to multiplication by a function, but this freedom is fixed on the right-hand side of (13), where the function \( \phi \) is from (6). In the trigonometric or rational case this function turns into \( \phi^{\text{trig}}(h, z) = \pi \cot(pz) + \pi \cot(p h) \) and \( \phi^{\text{rat}}(h, z) = h^{-1} + z^{-1} \). With the normalization (13) we also have the property that \( R^{h}_{12}(z) \) has single simple pole in \( z = 0 \) and the residue is the permutation operator:

\[
\text{Res} R^{h}_{12}(z) = NP_{12}.
\]

In the classical limit \( h \to 0 \) we have the expansion

\[
R^{h}_{12}(z) = \frac{1}{h} 1_{N} \otimes 1_{N} + r_{12}(z) + hm_{12}(z) + O(h^2),
\]

(15)

where \( r_{12}(z) \) is the classical \( r \)-matrix. Plugging (15) into (12) provides the classical Yang–Baxter equation

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,
\]

(16)

The classical \( r \)-matrix has the following expansion near \( z = 0 \):

\[
r_{12}(z) = \frac{1}{z} NP_{12} + r^{(0)}_{12} + O(z).
\]

(17)

Due to (14) one can easily get the (skew) symmetry properties for the coefficients of expansions (15) and (17):

\[
r_{12}(z) = -r_{21}(-z),
\]

\[
m_{12}(z) = m_{21}(-z),
\]

\[
r^{(0)}_{12} = -r^{(0)}_{21},
\]

\[
m^{(0)}_{12} = m^{(0)}_{21}.
\]

ASSOCIATIVE YANG–BAXTER EQUATION

Besides the quantum Yang–Baxter equation (12) (which is valid for any \( R \)-matrix) the \( R \)-matrices under consideration satisfy also the quadratic relation

\[
R^{h}_{12}R^{h}_{23} = R^{h}_{13}R^{h}_{23} + R^{h}_{12}R^{h}_{32},
\]

(19)

known as the associative Yang–Baxter equation [11, 12]. Together with Eqs. (13) and (14), Eq. (19) provides the quantum Yang–Baxter equation (12), that is \( R \)-matrices under consideration belong to a special subset of quantum \( R \)-matrices, see details in [13–17]. As shown in [18], Eq. (19) degenerates in a certain limiting case into \([m_{13}(z) + m_{23}(z), r_{12}(z - z_2)] = [m_{12}(z - z_2) + m_{32}(z_1), r_{32}(z_1)]\) and in the limit \( z_2 \to z_1 \) one gets

\[
[m_{13}(z), r_{12}(z)] = [r_{12}(z), m_{32}(0)]
\]

\[
- \partial_z m_{12}(z), NP_{23} + [m_{12}(z), r^{(0)}_{23}] + [m_{13}(z), r^{(0)}_{23}].
\]

Another important relation coming from (19) is as follows:

\[
r_{12}(z)r_{13}(z + w) - r_{23}(w)r_{12}(z) + r_{13}(z + w)r_{23}(w)
\]

\[
= m_{12}(z) + m_{23}(w) + m_{32}(z) + m_{13}(z) + m_{23}(w).
\]

In particular, one can derive the classical Yang–Baxter equation (16) from (21) using the symmetry properties (18). We are going to use degeneration of (21) in the limit \( w \to 0 \):

\[
r_{12}(z) r_{13}(z) = r^{(0)}_{12}(z) r_{13}(z) - r_{13}(z) r^{(0)}_{12}(z)
\]

\[
- N \partial_z r_{13}(z) P_{23} + m_{12}(z) + m_{23}(0) + m_{32}(z).
\]

LAX PAIRS AND ZAKHAROV–SHABAT EQUATION THROUGH R-MATRICES

Classical Mechanics of Integrable Top

We begin with the 0 + 1 example considered in [18]. Introduce

\[
L(S, z) = \frac{1}{N} \text{tr}_2(r_{12}(z) S),
\]

(23)

\[
M(S, z) = \frac{1}{N} \text{tr}_2(m_{12}(z) S), \quad S = 1_N \otimes S,
\]

where \( S \) is an arbitrary \( N \times N \) matrix, which matrix elements are dynamical variables in the model of inte-
Equations of motion have the form $S = [S, J(S)]$, where

$$J(S) = \frac{1}{N} \text{tr}_2(m_2(0)S) = M(S, 0). \quad (24)$$

In order to show that the equations of motion are represented in the Lax for $L(z) = [L(z), M(z)]$ one needs to compute $[L(z), M(z)]$. For this purpose, we multiply both parts of (20) by $S$ and take trace (of both parts) $\text{tr}_{2,3}$ over the second and the third tensor components. Then the last three commutators on the right-hand side of (20) are cancelled out (see details in [18]) and one gets

$$[L(S, z), M(S, z)] = L([S, J(S)], z). \quad (25)$$

In this way we conclude that the Lax equations hold true on the equations of motion. To prove converse statement, we mention that for any matrix $A$ the map $A \rightarrow L(A, z)$ is linear and $L(A, z) = 0$ if $A = 0$.

**ANSATZ FOR THE $U$–$V$ PAIR**

In contrast to the above example in mechanics, in $1 + 1$ case the set of matrices of dynamical variables $S$ is restricted by the condition

$$S^2 = cS, \quad (26)$$

where $c \in \mathbb{C}$ is some constant. The condition (26) means that the eigenvalues of the matrix $S$ are equal to either 0 or $c$. In particular case, when $N - 1$ eigenvalues coincide, we come to the matrix $S$ of rank 1. This case will be considered below separately.

Using the coefficient $r_2^{(0)}$ in the expansion (17) introduce the following linear map:

$$A \rightarrow E(A) = \frac{1}{N} \text{tr}_2(r_2^{(0)} A)^2, \quad (27)$$

$$A = 1_N \otimes A, \quad A \in \text{Mat}(N, \mathbb{C}).$$

It is important to mention that in the $N = 2$ case explicit evaluation provides $r_2^{(0)} = 0$, that is $E(A) \neq 0$ for the higher rank cases $(N \geq 3)$ only. Define

$$U(z) = L(S, z) = \frac{1}{N} \text{tr}_2(r_2(z) S), \quad V(z) = V_1(z) + V_2(z), \quad (28)$$

$$V_1(z) = -c \partial_z L(S, z) + L(SE(S), z) + L(E(S)S, z), \quad V_2(z) = -c L(T, z). \quad (29)$$

**DERIVATION OF EQUATIONS**

Multiplying both sides of the identity (22) by $AB$ and taking trace $\text{tr}_{2,3}$ we get

$$L(A, z)L(B, z) = L(AE(B), z) + L(E(AB), z) \quad - \frac{\partial_z L(AB, z) + \text{tr}(B)}{N} M(A, z)$$

$$+ \frac{\text{tr}(A)}{N} M(B, z) + \frac{1}{N} \text{tr}_2(m_{23}(0) A B), \quad (30)$$

where $A, B \in \text{Mat}(N, \mathbb{C})$ are arbitrary. Plugging $A = B = S$ into (30) yields

$$L^2(S, z) = L(SE(S), z) + L(E(S)S, z) - \partial_z L(S^2, z) + 2 \frac{\text{tr}(S)}{N} M(S, z) + \frac{1}{N} \text{tr}_2(m_{23}(0) S S).$$

Taking into account (26), this relation provides an alternative representation for $V_i(z)$ defined in (29):

$$V_1(z) = L^2(S, z) - 2s_v M(S, z)$$

$$- \frac{1}{N} \text{tr}_2(m_{23}(0) S S) - s_0 = \frac{\text{tr}(S)}{N}. \quad (31)$$

In addition, it is easy to see from (30) that

$$[L(S, z), L(T, z)] = -\partial_z L([S, T], z)$$

$$+ L([S, E(T)], z) + L([E(S), T], z). \quad (32)$$

Plugging the ansatz for $U$–$V$ pair (28)–(29) into the Zakharov–Shabat equation (3) we get

$$\partial_z S + c \partial_z T = -\partial_z (SE(S) + E(S)S)$$

$$= 2s_v [S, J(S)] + c[S, E(T)] + c[E(S), T] \quad (33)$$

and

$$-\partial_z S = [S, T]. \quad (34)$$

The last equation comes as vanishing of $\partial_z L(*, z)$ terms with the second order pole at $z = 0$, while (33) appears from vanishing of $L(*, z)$ terms with the first order pole at $z = 0$. Notice that for $\partial_z V(z)$ term in the Zakharov–Shabat equation we used $V_i(z)$ as given in (29), and in the term $[U(z), V(z)]$ we used $V_i(z)$ in the form (31). It is useful because from representation (31) we have $[U(z), V_i(z)] = -2s_v L(S, z), M(S, z)] = -2s_v L([S, J(S)], z)$. In order to solve (34) with respect to $T$ we mention that $(S - (c/2)1_N)^2 = (c/2)^2 1_N$, that is similarly to $N = 2$ case we obtain

$$T = -c^2 [S, \partial_z S]. \quad (35)$$

Finally, we substitute Eq. (35) into Eq. (33):

$$\partial_z S - \frac{1}{c} [S, \partial_z S] - \partial_z (SE(S) + E(S)S)$$

$$= 2s_v [S, J(S)] - \frac{1}{c} [S, E([S, \partial_z S])] - \frac{1}{c} [E(S), [S, \partial_z S]], \quad (36)$$

where $s_v = \text{tr}(S)/N$. This is the higher rank Landau–Lifshitz equation. It is formulated in terms of the linear maps $J(S)$ (24), $E(A)$ (27), which arise in
the $R$-matrix expansions (15) and (17). The dynamical matrix $S$ is assumed to satisfy condition (26) with some constant $c$.

**ELLiptic case**

The most general is the elliptic case. It comes from the elliptic Baxter–Belavin $R$-matrix [7] in the fundamental representation of $GL_N$. Introduce the special matrix basis in Mat($N, \mathbb{C}$): $T_a = T_{apq} = \exp(\frac{\pi i}{N} a_p a_q) Q_1^{a_p} Q_2^{a_q}$, where $a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$, and $(Q_1)_{kl} = \delta_{kl} \exp(2\pi i k/N)$, $(Q_2)_{kl} = \delta_{k-l+n \mod N}$. In particular, $T_{(0,0)} = I_N$. The basis has the property $\text{tr}(T_a T_b) = N\delta_{a+b,(0,0)}$. See details in [7] (also Appendix in [19]). The quantum $R$-matrix is as follows:

$$R_{12}^N(z) = \sum_{a \in \mathbb{Z}_N} T_a \otimes T_{-a} \exp\left(\frac{2\pi i a \tilde{z}}{N}\right) \phi\left(z, \frac{a_1 + a_2 \tau}{N}\right) + \frac{\ell}{N} \otimes 1_N \tag{37}$$

The classical limit (15) provides

$$r_{12}(z) = E_1(z) 1_N \otimes 1_N + \sum_{a \neq (0,0)} T_a \otimes T_{-a} \exp\left(\frac{2\pi i a \tilde{z}}{N}\right) \phi\left(z, \frac{a_1 + a_2 \tau}{N}\right) \tag{38}$$

and

$$m_{12}(z) = \rho(z) 1_N \otimes 1_N + \sum_{a \neq (0,0)} T_a \otimes T_{-a} \exp\left(\frac{2\pi i a \tilde{z}}{N}\right) f\left(z, \frac{a_1 + a_2 \tau}{N}\right) \tag{39}$$

where $f(z, u) = \partial_u \phi(z, u)|_{u=-u}$ and $\rho(z) = (E_1^2(z) - \phi(z))/2$. Then one finds

$$r_{12}^{(0)} = \sum_{a \neq (0,0)} T_a \otimes T_{-a} \exp\left(\frac{2\pi i a \tilde{z}}{N}\right) E_1\left(\frac{a_1 + a_2 \tau}{N}\right), \tag{40}$$

$$E(A) = \sum_{a \neq (0,0)} T_a A_a \exp\left(\frac{2\pi i a \tilde{z}}{N}\right) \tag{41}$$

$$m_{12}(0) = \frac{\partial^{(0)}}{\partial \theta^{(0)}} \frac{1}{2} \delta_{1_N \otimes 1_N}$$

$$E_2(x) = -E_1'(z) = -\partial_z^2 \log \partial(z), \tag{42}$$

In the last line we also used relation $E_2(z) = \phi(z) - \partial^{(0)}/(\partial^0(0))$. Let us mention that in $N = 2$ case the classical $r$-matrix (38) turns into (9). Another important remark is that in $N = 2$ case $E(A) = 0$ for any $A \in \text{Mat}_2$, since $E_1(1/2) = 0$ and $E_1(\tau/2) = E_1((\tau + 1)/2) = -\pi i$.

**Case rank($S$) = 1.** If rank($S$) = 1, the matrix $S$ is represented as $S = \xi \otimes \psi$, where $\xi$ and $\psi$ are $N$-dimensional vector column and row, respectively. Then $S^2 = \text{tr}(S) S$, so that $c = \text{tr}(S) = N s_0$ in (26) and (36).

**SPECIAL PROPERTY AND SIMPLIFICATION**

Up till now, we have not used one more property of the elliptic $R$-matrix (37). This is the Fourier symmetry (see, e.g., [20]) $R_{12}(z) P_{12} = R_{12}^{(0)}(N/\hbar)$. Using the expansions (15) and (17) it provides the property $r_{12}^{(0)} = r_{12}^{(0)} P_{12}$. Together with the skew-symmetry (18) it simplifies the Landau–Lifshitz equation (36) for the rank 1 case. Namely, the following identity holds:

$$\frac{\partial S}{\partial \theta}(S E(S) + E(S) S) + 2\frac{\partial S}{\partial \theta}(S, J(S)) - 2[S, E(\partial_j S)]. \tag{43}$$

We will give its proof in our future papers. Using (43) the Landau–Lifshitz equation (36) takes the form:

$$\frac{\partial S}{\partial \theta} = \frac{1}{c} [S, \partial^2 \theta] + 2\frac{\partial}{\partial \theta}(S, J(S)) - 2[S, E(\partial_j S)]. \tag{44}$$

For $N = 2$ case the last term on the right-hand side vanishes and we come back to the original Eq. (2).

**HAMILTONIAN DESCRIPTION**

Equation (44) can be easily described in the Hamiltonian formalism. The Poisson structure is given by

$$\{S(x), S(y)\} = (S_{1i}(x) \delta_{ij} - S_{ij}(x) \delta_{i1}) \delta(x-y) \tag{45}$$

$$H = \int y \delta y \text{tr}(S J(S)) - \frac{1}{2c} \text{tr}(\partial_j S \partial_j S), \tag{46}$$

The equations of motion $\partial_j S(x) = \{H, S(x)\}$ provide (44) for the following Hamiltonian:

$$H = \int y \delta y \text{tr}(S J(S)) - \frac{1}{2c} \text{tr}(\partial_j S \partial_j S), \tag{47}$$

For example, in this way one can deduce the properties $SE(S) = 0, SE(\partial_j S) = 0, SE(S \partial_j S) = -c(\partial_j S) E(S)$.

---

5 On should use the expansion of $\phi(z, u)$ near $u = 0$: $\phi(z, u) = u^{-1} + E_1(z) + u \rho(z) + O(u^2)$.

6 Namely, the following identity holds:

$$\partial_j S = \frac{1}{c} [S, \partial^2 \theta] + 2\frac{\partial}{\partial \theta}(S, J(S)) - 2[S, E(\partial_j S)]. \tag{48}$$

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The equations of motion $\partial_j S(x) = \{H, S(x)\}$ provide (47) for the following Hamiltonian:

$$H = \int y \delta y \text{tr}(S J(S)) - \frac{1}{2c} \text{tr}(\partial_j S \partial_j S), \tag{49}$$

For example, in this way one can deduce the properties $SE(S) = 0, SE(\partial_j S) = 0, SE(S \partial_j S) = -c(\partial_j S) E(S)$.
CONCLUDING REMARKS

We described a special class of higher rank generalizations of the Landau–Lifshitz equation. Main idea was to use a set of $R$-matrix identities coming from the associative Yang–Baxter equation. As a result, we deduce equations of motion (36) and the Lax pairs given by Eqs. (28) and (29) with spectral parameter on elliptic curve (or its degeneration). In the particular case $\text{rank}(S) = 1$, Eq. (36) is simplified to (44). The latter admits quite simple Hamiltonian description (45), (46).

It should be mentioned that the study of general matrix valued Lax pairs for the Zakharov–Shabat equation on elliptic curves is known from [21, 22]. Notice that in the general case the Lax matrix has a set of poles (not a single one). Then the Gaudin type models [23, 24] arise. We hope our construction can be generalized to this case as well.

Explicit formulas in the elliptic case are given in (37)–(42). Details of trigonometric and rational degenerations will be given in our next paper. The underlying finite-dimensional models (of type (23)–(25)) were studied in [13, 15, 25–27]. cases were considered in our previous paper [28], where a gauge equivalence between the Landau–Lifshitz equations and the Calogero–Moser field theories was described. In view of similar relation between the XYZ chain and the Ruijsenaars–Schneider model [19] we also expect to extend our construction to (semi)discrete equations.

The Landau–Lifshitz equation (1) has multicomponent generalization suggested by I.Z. Golubchik and V.V. Sokolov [29], as well as by T.V. Skrypnyk [30, 31]. We hope to clarify relation of our construction to the results of [29] in our future works.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. L. D. Landau and E. M. Lifshitz, Phys. Zs. Sowjet. 8, 153 (1935).
2. E. K. Sklyanin, Preprint LOMI, E-3-79 (LOMI, Leningrad, 1979).
3. L. D. Faddeev and L. A. Takhtajan, Hamiltonian Methods in the Theory of Solitons (Springer, Berlin, 1987).
4. V. E. Zakharov and A. B. Shabat, Funct. Anal. Appl. 8, 226 (1974).
5. V. E. Zakharov and A. B. Shabat, Funct. Anal. Appl. 13, 166 (1979).
6. What is Integrability?, Ed. by V. E. Zakharov, Springer Series in Nonlinear Dynamics (Springer, Berlin, 1991).
7. R. J. Baxter, Ann. Phys. 70, 193 (1972).
8. A. A. Belavin, Nucl. Phys. B 180, 189 (1981).
9. L. Takhtajan and L. Faddeev, Russ. Math. Surv. 34 (5), 11 (1979).
10. E. K. Sklyanin, J. Sov. Math. 46, 1664 (1989).
11. S. Fomin and A. N. Kirillov, in Advances in Geometry, Prog. Math. 172, 147 (1999).
12. A. Polishchuk, Adv. Math. 168, 56 (2002).
13. A. Levin, M. Olshanetsky, and A. Zotov, J. High Energy Phys. 1407, 012 (2014); arXiv: 1405.7523 [hep-th].
14. A. Levin, M. Olshanetsky, and A. Zotov, Nucl. Phys. B 887, 400 (2014); arXiv: 1406.2995 [math-ph].
15. A. Levin, M. Olshanetsky, and A. Zotov, J. High Energy Phys. 10, 109 (2014); arXiv: 1408.6246 [hep-th].
16. A. Levin, M. Olshanetsky, and A. Zotov, Theor. Math. Phys. 184, 924 (2015); arXiv: 1501.07351.
17. A. Levin, M. Olshanetsky, and A. Zotov, J. Phys. A: Math. Theor. 49, 014003 (2016); arXiv: 1507.02617.
18. A. Levin, M. Olshanetsky, and A. Zotov, J. Phys. A: Math. Theor. 49, 395202 (2016); arXiv: 1603.06101.
19. A. Zabrodin and A. Zotov, arXiv: 2107.01697 [math-ph].
20. A. Zotov, Mod. Phys. Lett. A 32, 1750169 (2017); arXiv: 1706.05601 [math-ph].
21. I. V. Cherednik, Theor. Math. Phys. 47, 422 (1981).
22. I. V. Cherednik, J. Sov. Math. 38, 1989 (1987).
23. A. Zotov, SIGMA 7, 067 (2011); arXiv: 1012.1072 [math-ph].
24. A. Levin, M. Olshanetsky, and A. Zotov, arXiv: 2202.10106 [hep-th].
25. G. Aminov, S. Artamonov, A. Smirnov, and A. Zotov, J. Phys. A: Math. Theor. 47, 305207 (2014); arXiv: 1402.3189 [math-ph].
26. A. Grekov, I. Sechin, and A. Zotov, J. High Energy Phys. 10, 081 (2019); arXiv: 1905.07820 [math-ph].
27. T. Krasnov and A. Zotov, Ann. Henri Poincaré 20, 2671 (2019); arXiv: 1812.04209 [math-ph].
28. K. Atalikov and A. Zotov, J. Geom. Phys. 164, 104161 (2021); arXiv: 2010.14297 [math-ph].
29. I. Z. Golubchik and V. V. Sokolov, Theor. Math. Phys. 124, 909 (2000).
30. T. V. Skrypnyk, Theor. Math. Phys. 142, 275 (2005).
31. T. Skrypnyk, J. Math. Phys. 54, 103507 (2013).