I. INTRODUCTION

Quasinormal modes are proper oscillation frequencies of black holes, corresponding to the specific boundary conditions: purely outgoing wave at infinity and purely incoming wave at the event horizon. They do not depend on the way the perturbation was excited, but only on the black-hole parameters, which makes them a characteristic feature of the black hole geometry, a kind of "fingerprints" of black holes. Quasinormal modes play a crucial role in the current observations of gravitational waves and, being studied during the past decades in a great number of papers, become an essential characteristic of a black hole geometry [1,2]. Even though there were detected signals for which the quasinormal frequencies are known with rather a small error of about a few percents [1,2], the large uncertainty in the determination of the mass and angular momentum of the black hole allows one to ascribe the same observed frequencies to a non-Kerr solution [3] with different parameters, so that the alternative theories of gravity not only are not excluded by the current experiments, but even are not strongly constrained by observations in the gravitational [1,2] and electromagnetic [4,5] spectra.

Among alternative theories of gravity an interesting approach is connected with the Einstein-aether theory, which is a Lorentz-violating theory [6–14] endowing a spacetime with both a metric and a unit timelike vector field (aether) having a preferred time direction. It includes the Einstein relativity as a special case. Quasinormal modes of various black-hole solutions [15,16] in this theory were considered in [17–20], depending on the way the aether vector is chosen. For the first time quasinormal modes in the Einstein-aether theory were studied in [17,18], but it proved out that the black hole solution [15] considered in [17,18] did not satisfy the observed post-Newtonian behavior and, thereby, cannot describe a viable astrophysical black hole. The same is true for the so called Aether II type black hole solution considered in [19,20]. This means that those black hole models and their spectra still may be relevant for the miniature or primordial black holes, but not for large astrophysical black holes. The Aether I type considered in [19,20] is not discarded by the current experiments in the weak field regime, but, as we will show in the present paper, the data for quasinormal modes represented in [19,20] suffers from the two drawbacks:

- the lower multipoles are calculated with insufficient accuracy, so that the effect is, sometimes, smaller than the relative error and
- gravitational perturbations are reduced to the master wave-like equation in a non-self consistent way, so that it cannot describe the gravitational spectrum even approximately.

Here we will compute quasinormal modes for both types of aether with the help of two alternative methods: the higher order WKB method [23–28] with the usage of Padé approximants [24,28] and the time-domain integration [32]. Both methods are sufficiently accurate and are in good agreement with each other.

In addition, we will consider perturbations of a massive scalar field and show that, in a similar fashion with the Einstein theory, spectrum of massive fields in the Einstein-aether theory allows for arbitrarily long lived quasinormal modes, called quasi-resonances [40–52]. We will show that at asymptotically late times, the quasinormal modes are surpassed by the power-law tail, which is indistinguishable from the Schwarzschild ones.

The paper is organized as follows. In sec. II we review the essentials of the Einstein-aether theory and wave-like equations for test scalar and electromagnetic fields. Sec. III is devoted to the WKB and time-domain integration methods we used for finding quasinormal modes. In sec. IV we discuss the quasinormal modes of massless fields in the black hole background for the Einstein-aether theory, while in Sec. V the case of a massive scalar field and existence of quasi-resonances are discussed. The late time tails are presented in Sec. VI. In Sec. VII we give a brief remark on a wrong treatment of gravitational perturbations in a number of earlier publications. Finally, we summarize the obtained results and mention some open problems.
II. THE WAVE EQUATION

The Einstein aether theory under consideration is described by the action [39]

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G_{ae}} (R + \mathcal{L}_{ae}) \right], \]

where \( G_{ae} \) is the aether gravitational constant, \( \mathcal{L}_{ae} \) is the aether Lagrangian density:

\[ -\mathcal{L}_{ae} = Z^{ab}_{cd} (\nabla_a u^c) (\nabla_b u^d) - \lambda (u^2 + 1) \]

with

\[ Z^{ab}_{cd} = c_{13} g^{ab} g_{cd} + c_2 \delta^a_d \delta^b_c + c_3 \delta^a_b \delta^c_d - c_{14} u^a u^b g_{cd}, \]

where \( c_i, i = 1, 2, 3, 4, \) are coupling constants of the theory. Although there is a number of severe constraints on the coupling constants \( c_i \) (not only theoretical, but also observational), the papers [19, 20], which we consider here, deal with the following theoretical ones [16]:

\[ 0 \leq c_{13} < 1, \quad 0 \leq c_{14} < 2, \quad c_{13} \geq c_{14}/2, \]

where \( c_{13} = c_1 + c_3, c_{14} = c_1 + c_4. \)

The metric of the spherically symmetric static Einstein aether black hole spacetime is given by:

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (\sin^2 \theta d\phi^2 + d\theta^2). \]

The metric function has the following form:

- for the first kind aether

\[ f(r) = 1 - \frac{2M}{r} - I \left( \frac{2M}{r} \right)^4, \quad I = \frac{27c_{13}}{256(1 - c_{13})}, \]

- and for the second kind aether

\[ f(r) = 1 - \frac{2M}{r} - J \left( \frac{M}{r} \right)^2, \quad J = \frac{c_{13} - c_{14}/2}{1 - c_{13}}. \]

Note that for the values \( c_{13} = 0 \) (for the first kind aether) and \( c_{13} = c_{14}/2 \) (for the second kind aether) the metric [11] reduces to the Schwarzschild black hole case.

The general covariant equations for the test scalar \( \Phi \) and electromagnetic \( A_{\mu} \) fields have the form

\[ \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \Phi \right) = 0, \]

\[ \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} F^{\mu\nu} \right) = 0, \]

where \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \). After separation of the variables Eqs. (7) and (8) take the following Schrödinger-like form (see, for instance, [21, 22])

\[ \frac{d^2 \Psi_s}{dr^2} + \left( \omega^2 - V(r) \right) \Psi_s = 0, \]

where \( s = 0 \) corresponds to scalar field and \( s = 1 \) to electromagnetic field and the "tortoise coordinate" \( r_* \) is defined by the relation

\[ dr_* = f(r)dr. \]

The effective potential is

\[ V(r) = f(r) \left( \frac{\ell (\ell + 1)}{r^2} + \frac{1 - s}{r} \cdot \frac{df(r)}{dr} \right), \]

and has the form of a potential barrier (see Fig. 1).

III. THE METHODS

A. The WKB method

The WKB method for finding quasinormal frequencies, which was first used by Schutz and Will [22] (reproducing at the first order the earlier result of Mashhoon [24]), grew very popular because of its effectiveness and was treated in numerous papers.

For finding quasinormal modes we use higher-order WKB formula [23, 28]:

\[ \omega^2 = V_0 + A_2(K^2) + A_4(K^2) + A_6(K^2) + \ldots \]

\[ - iK \sqrt{-2V_2 (1 + A_2(K^2) + A_4(K^2) + A_6(K^2) + \ldots)}, \]

where \( K = \text{sign} \text{Re}(\omega) \left( n + \frac{1}{2} \right), n = 0, 1, 2, 3, \ldots \) The corrections \( A_k(K^2) \) of order \( k \) to the first-order formula are polynomials of \( K^2 \) with rational coefficients, which depend on the values \( V_2, V_3, \ldots \) of higher derivatives of the potential \( V(r) \) in its maximum (but not on the maximum \( V_0 \) itself), whence it follows that the right-hand-side of (12) does not depend on \( \omega \).
For the polynomial Padé approximants, with $\tilde{n} \approx \hat{n}$ usually provide the best approximation. In [27] $P_{6/6}(1)$ and $P_{6/7}(1)$ were compared to the 6th-order WKB formula $P_{6/0}(1)$. In [28] it has been observed that as a rule even $P_{5/3}(1)$ gives a more accurate value for the squared frequency than $P_{6/0}(1)$. In our case we use 6th-order WKB expansion with appropriate Padé partition. The corresponding automatic code in Mathematica® is in open access [31].

### B. The time domain integration

If we keep in Eq. (9) the second derivative in time instead of $\omega^2$-term, then the perturbation equations can be integrated at a fixed $r$ in the time domain. We use the technique of integration in the time domain developed by Gundlach, Price and Pullin in [32]. We shall integrate the wave-like equation rewritten in terms of the lightcone variables $u = t - r_s$ and $v = t + r_s$. The appropriate discretization scheme is:

$$
\Psi (N) = \Psi (W) + \Psi (E) - \Psi (S) - \\
- \Delta^2 \frac{V (W) \Psi (W) + V (E) \Psi (E)}{8} + \mathcal{O} (\Delta^4),
$$

### Table 1. Fundamental quasinormal modes for the first kind aether black hole spacetime

| Parameter | QNM | Effect % | Error % | QNM | Effect % | Error % |
|-----------|-----|----------|---------|-----|----------|---------|
| $c_{13}$  | $\omega$ | $\delta \Re \epsilon_1$ | $\delta \Im \epsilon_1$ | $\omega$ | $\delta \Re \epsilon_1$ | $\delta \Im \epsilon_1$ |
| 0         | 0.104647 - 0.115197i | 0 | 0 | 5.4 | 10 | 0.245870 - 0.093106i | 0 | 0 | 0.96 | 0.68 |
| 0.15      | 0.110876 - 0.104424i | 0.64 | 2.0 | 5.2 | 11 | 0.243928 - 0.094312i | 0.79 | 1.3 | 0.88 | 0.96 |
| 0.3       | 0.109637 - 0.105690i | 0.75 | 6.9 | 7.4 | 14 | 0.241266 - 0.095728i | 1.9 | 2.8 | 0.8 | 1.4 |
| 0.45      | 0.104550 - 0.107923i | 0.237420 - 0.097401i | 3.4 | 4.6 | 0.75 | 1.8 |
| 0.6       | 0.103976 - 0.117446i | 0.64 | 2.0 | 5.2 | 11 | 0.243928 - 0.094312i | 0.79 | 1.3 | 0.88 | 0.96 |
| 0.75      | 0.101739 - 0.120032i | 2.8 | 4.2 | 5.5 | 14 | 0.241266 - 0.095728i | 1.9 | 2.8 | 0.8 | 1.4 |
| 0.9       | 0.108494 - 0.109823i | 0.239207 - 0.095662i | 3.4 | 4.6 | 0.75 | 1.8 |

As the WKB method converges only asymptotically, simple increasing of the WKB formula order does not necessarily imply improving of the results (see more about the asymptotic WKB regime in [24]). So as to increase the accuracy of the higher-order WKB formula [12], we use Padé approximants [31], following Matyjasek and Opala [28]. For the order $k$ of the WKB formula [12] we define a polynomial $P_k(\epsilon)$ as

$$
P_k(\epsilon) = V_0 + A_2(\kappa^2)\epsilon^2 + A_4(\kappa^2)\epsilon^4 + A_6(\kappa^2)\epsilon^6 + \ldots \\
- iK\sqrt{-2V_2} (\epsilon + A_3(\kappa^2)\epsilon^3 + A_5(\kappa^2)\epsilon^5 + \ldots), \quad (13)
$$

whence we can obtain the squared frequency taking $\epsilon = 1$:

$$
\omega^2 = P_k(1).
$$

For the polynomial $P_k(\epsilon)$ we consider a family of the rational functions

$$
P_{\hat{n} / \tilde{n}}(\epsilon) = \frac{Q_0 + Q_1\epsilon + \ldots + Q_\hat{n}\epsilon^{\hat{n}}}{R_0 + R_1\epsilon + \ldots + R_{\tilde{n}}\epsilon^{\tilde{n}}},
$$

called Padé approximants, with $\hat{n} + \tilde{n} = k$, such that near $\epsilon = 0$

$$
P_{\hat{n} / \tilde{n}}(\epsilon) - P_k(\epsilon) = \mathcal{O}(\epsilon^{k+1}).
$$

It turns out that for finding fundamental mode ($n = 0$) Padé approximants with $\tilde{n} \approx \hat{n}$ usually provide the best approximation. In [27] $P_{6/6}(1)$ and $P_{6/7}(1)$ were compared to the 6th-order WKB formula $P_{6/0}(1)$. In [28] it has been observed that as a rule even $P_{5/3}(1)$ gives a more accurate value for the squared frequency than $P_{6/0}(1)$. In our case we use 6th-order WKB expansion with appropriate Padé partition. The corresponding automatic code in Mathematica® is in open access [31].
Table 2. Fundamental quasinormal modes for the second kind aether black hole spacetime with fixed $c_{14} = 0.2$ (presented in [19] (1st line), obtained here by WKB (2nd line) and time-domain (3rd line) methods).

| Parameter | Scalar field ($l = 0$) | Electromagnetic field ($l = 1$) |
|-----------|------------------------|-------------------------------|
| $c_{13}$  | $\omega$ $\delta_{Re}$ $\delta_{Im}$ $\varepsilon_{Re}$ $\varepsilon_{Im}$ | $\omega$ $\delta_{Re}$ $\delta_{Im}$ $\varepsilon_{Re}$ $\varepsilon_{Im}$ |
| 0.10      | 0.104647 $-$ 0.1115197i 0 0 5.4 10 0.245870 $-$ 0.093106i 0 0 0.96 0.68 |
| 0.25      | 0.100755 $-$ 0.114893i 3.7 0.26 5.9 11 0.236985 $-$ 0.091929i 3.6 1.3 1.0 0.78 |
| 0.40      | 0.102441 $-$ 0.101957i 8.4 0.98 6.5 12 0.225711 $-$ 0.090122i 8.2 3.2 1.2 0.92 |
| 0.55      | 0.102778 $-$ 0.101465i | 0.228342 $-$ 0.089303i |
| 0.70      | 0.080354 $-$ 0.108123i 15 2.6 7.1 13 0.210705 $-$ 0.087219i 14 6.3 1.3 1.1 |
| 0.85      | 0.085688 $-$ 0.097327i 37 16 8.5 17 0.152828 $-$ 0.071437i 38 23 2.1 1.8 |

The following designations were used: $N = (u + \Delta, v + \Delta)$, $W = (a + \Delta, v)$, $E = (u, v + \Delta)$ and $S = (u, v)$. The initial data are given on the null surfaces $u = u_0$ and $v = v_0$. The values of the quasinormal modes were obtained by Prony’s method (see, e.g., [13]) and fitting the signal by a sum of damped exponents.

IV. QUASINORMAL MODES

We considered a fundamental ($n = 0$) quasinormal mode for the scalar and electromagnetic perturbations of the Einstein-aether black hole spacetime. We were interested in the lower multipole numbers ($l = 0$ for the scalar and $l = 1$ for the electromagnetic field) because of their dominating role in the signal.

First of all, we looked at the quasinormal frequencies from [19] which correspond to the Szwarzchild limit (c_{13} = 0 for the first kind aether in Tables I, II and c_{13} = c_{14}/2 = 0.1 for the second kind aether in Tables III, IV). As these frequencies differed from the accurate values in the second digit after the point already (for the scalar field case), we recalculated them. For this we used two methods: the 6th order WKB formula with Padé approximants $P_{5/1}(1)$ and the time domain integration. The results obtained by the both methods turned out to be in a good agreement with the accurate values for the Szwarzchild case. Therefore we went on with our calculations, keeping the methods’ parameters (such as the order of WKB series and the orders of Padé approximants) unchanged, for the rest of the values of the parameter $c_{13}$, considered in Tables I, II and III, IV.

At each step we also found a relative effect and a relative error of the results presented in [19]. A relative effect is defined as

$$\delta_{Re} = \frac{|Re\omega_l - Re\omega_f|}{Re\omega_l} \cdot 100\%$$  \hspace{1cm} (16)

$$\delta_{Im} = \frac{|Im\omega_l - Im\omega_f|}{Im\omega_l} \cdot 100\%$$  \hspace{1cm} (17)

where $\omega_l$ is the current value of the quasinormal mode and $\omega_f$ is the value of the quasinormal mode, which corresponds to the Szwarzchild limit. A relative error is defined by

$$\varepsilon_{Re} = \frac{|Re\omega_l - Re\omega_0|}{Re\omega_0} \cdot 100\%$$  \hspace{1cm} (18)

$$\varepsilon_{Im} = \frac{|Im\omega_l - Im\omega_0|}{Im\omega_0} \cdot 100\%$$  \hspace{1cm} (19)

where $\omega_l$ denotes the result from [19] and $\omega_0$ denotes our new result at each step.
All the obtained results are presented in Tables 1 and 2. The values of the fundamental quasinormal mode are placed one under the other: the result from \[19\] (1st line) and the results obtained here by WKB (2nd line) and time domain (3rd line) methods. The additional 4th line (for \(c_{13} = 0\) in Table 1 and for \(c_{13} = 0.1\) in Table 2) contains accurate values of the fundamental quasinormal mode for the Szchwarzchild case. The effect and the error are calculated for the real and imaginary parts of the quasinormal frequencies obtained in \[19\].

As the values of the modes are placed one under the other, it is easy to compare them and see that the discrepancy of our results and the accurate values starts at the 4th (scalar field) or even the 5th (electromagnetic field) digit after the point, while for the results from \[19\] these digits are respectively the 2nd and the 3rd. For the rest of the considered values of the parameter \(c_{13}\) this tendency is kept: the deviation of the results of \[19\] from both of our results is considerably larger than the difference between our results as such.

The error of the quasinormal frequencies obtained in \[19\] is rather large even for the values of the parameter \(c_{13} = 0\), which correspond to the Szchwarzchild limit (\(\varepsilon_{Rs} = 5.4\%\) and \(\varepsilon_{Im} = 10\%\) for the scalar field). It can be seen that in the case of the scalar field for the values of the parameter \(c_{13}\) near the Szchwarzchild limit the error is greater than the effect, for the imaginary part even by an order. For the larger values of \(c_{13}\), which cannot promise too much accuracy, even if the error becomes less than the effect, it still remains comparable to it. Although in the case of the electromagnetic field the situation is not so extreme, the error as yet can come to 50 or even 110\%.

![Image](image-url)

**FIG. 2.** An example of the time domain profile: scalar perturbations of the second kind aether black hole (\(\ell = 0, s = 0, c_{13} = 0.45, c_{14} = 0.2\)).

The eikonal formulas (\(\ell \to \infty\)) for the quasinormal modes in the Einstein-aether theory were obtained in \[34\] for both types of aether.

## V. QUASIRESONANCE

For a massive scalar field \(\Phi\) of the mass \(\mu\), general covariant equation having the form

\[
\frac{1}{\sqrt{-g}}\partial_{\mu} \left( \sqrt{-g}g^{\mu\nu} \partial_{\nu} \Phi \right) - \mu^2 \Phi = 0, \tag{20}
\]

there exists a phenomenon of so-called quasiresonance \[41\]: increasing of the field mass \(\mu\) causes decreasing of the lower overtones damping rate, which means that infinitely long lived modes appear in the spectrum.

Figs. 3, 4 show dependance of the real and imaginary parts of the fundamental quasinormal mode on the mass \(\mu\) of the scalar test field for the first and the second kind aether black hole spacetime. As WKB method works accurately when \(\ell\) is much larger than \(\mu M\) \[51\] (although it cannot be applied in the regime of quasi-resonances), the extrapolation of the WKB data can indicate the existence of quasi-resonances. The red part of the lines marks the values of the quasinormal modes obtained by the 6th order WKB method with Padé approximation and checked by the time domain integration (they turned out to coincide at least up to the second digit after the point). Therefore Figs. 3, 4 indicate that for the considered case of the massive scalar field in the Einstein-aether black hole spacetime the phenomenon of quasiresonance exists.

## VI. LATE TIME TAILS

The incompleteness of the quasinormal modes set implies that at sufficiently late times the quasinormal modes are suppressed by exponential or power-law tails. Fig. 2 demonstrates an example of the time domain profile for the scalar perturbations (\(s = 0, \ell = 0\)) of the second kind Einstein aether black hole spacetime, where it can be seen that the late-times tails for some fixed values of the black-hole parameters and \(\ell = 0\) \(|\Psi| \sim t^{-3}\) are the same that those for the Szchwarzchild black hole case. Indeed, for a scalar field in the Schwarzschild background we have the following general law:

\[
|\Psi| \sim t^{-(2\ell+3)}. \tag{21}
\]

## VII. REMARK ON GRAVITATIONAL PERTURBATIONS

In a few previously published works not only on the Einstein-aether gravitational perturbations \[18, 20\], but also for the \[53\] on the Einstein-Maxwell theory the Einstein equations

\[
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \tag{22}
\]

were perturbed in such a way that perturbations of the right hand side of the Einstein equations, containing the
energy momentum tensor of the matter fields, were neglected. Thus, instead of the full perturbation equations
\[ \delta (R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu}) = \kappa \delta T_{\mu \nu}, \] (23)
the reduced set of equations was considered
\[ \delta (R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu}) = 0. \] (24)

This reduction was usually justified by relatively small energy content of matter fields. However, the linearized values on the right and left hand sides must be of the same order and cannot be ignored. There is a simple way to check whether our supposition is correct. For this we will consider the full set of perturbation equations given by [23] for the Reissner-Nordstrom spacetime as a solution of the Einstein-Maxwell equations and the corresponding reduced set given by eq. [24]. The effective potential for axial perturbations within the reduced procedure [24] can be found, for example, in [20]:
\[ V(r) = f(r) \left( \frac{(\ell + 2)(\ell - 1) + 2f(r)}{r^2} - \frac{1}{r} \frac{df(r)}{dr} \right), \] (25)
while one of the two axial potentials for the full set perturbations of the Einstein-Maxwell field for the Reissner-Nordström black hole is:
\[ V(r) = \frac{\frac{1}{4} \left( \frac{16}{r^2} - \frac{12}{r} \right) - 5 \sqrt{\frac{1}{r^2} + 6} \left( \frac{1}{r^2} - \frac{2}{r} + 1 \right)}{r^2}. \] (26)

From Table 3 one can see that for every value of the electric charge $Q$ the effect given by the non-zero charge in comparison with the Schwarzschild limit is smaller than or of the same order as the error due to neglecting perturbations of the energy-momentum tensor. Therefore, we conclude that such neglecting cannot be used to provide any reliable results. Thus, the full set of perturbation equations is necessary to complement the quasinormal spectrum of the Einstein-aether black holes and to conclude about their stability.
VIII. CONCLUSIONS

In the present paper we have shown that purblind considerations of quasinormal spectrum of black holes in the Einstein-aether theory[8,9,10] suffer from the two main drawbacks: insufficient accuracy of reported quasinormal frequencies at lower multipoles $\ell$, such that the effect is frequently smaller than the error, and inconsistency of treatment of gravitational perturbations for which the linearization of the energy-momentum tensor cannot be neglected. Here we compute accurate quasinormal modes of massless test electromagnetic and gravitational fields and, in addition, consider a massive scalar field for which we demonstrate the existence of the arbitrarily long lived quasinormal modes called quasi-resonances. We also study asymptotic tails and time domain profiles of the Einstein-aether theory and show that at asymptotic times the tails are identical to those of the Einstein theory.

Our paper can be extended in a number of ways. First of all, we showed that consideration of the full set of perturbations equations is necessary to analyze the gravitational spectrum and, therefore, to conclude about the stability of the black hole in the Einstein-aether theory. In addition, the fermionic perturbations can be further considered in a similar way to the bosonic ones studied in this paper.

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Table 3. Fundamental quasinormal modes for the gravitational perturbations of the Reissner-Nordström black hole spacetime ($\ell = 2, M = 1$).

| Q    | Neglected                  | Accurate                     | Effect % | Error % |
|------|----------------------------|------------------------------|----------|---------|
| 0    | $0.373620 - 0.088933i$      | $0.373620 - 0.088933i$       | 0        | 0       |
| 0.1  | $0.374273 - 0.088986i$      | $0.373880 - 0.088962i$       | 0.07     | 0.11    |
| 0.2  | $0.376260 - 0.089142i$      | $0.374691 - 0.089046i$       | 0.29     | 0.42    |
| 0.3  | $0.379675 - 0.089399i$      | $0.376142 - 0.089185i$       | 0.68     | 0.94    |
| 0.4  | $0.384687 - 0.089748i$      | $0.378381 - 0.089371i$       | 1.27     | 1.67    |
| 0.5  | $0.391573 - 0.090164i$      | $0.381624 - 0.089584i$       | 2.14     | 2.61    |
| 0.6  | $0.400778 - 0.090592i$      | $0.386173 - 0.089781i$       | 3.36     | 3.78    |
| 0.7  | $0.413048 - 0.090900i$      | $0.392475 - 0.089872i$       | 5.05     | 5.24    |
| 0.8  | $0.429717 - 0.090796i$      | $0.401211 - 0.089621i$       | 7.38     | 7.70    |
| 0.9  | $0.453363 - 0.089298i$      | $0.413568 - 0.088311i$       | 10.69    | 9.62    |
| 1    | $0.490129 - 0.081661i$      | $0.431344 - 0.083440i$       | 15.45    | -6.18   |

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