The Real Solution to Scalar Field Equation in 5D Black String Space

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After the nontrivial quantum parameters $\Omega_n$ and quantum potentials $V_n$ obtained in our previous research, the circumstance of a real scalar wave in the bulk is studied with the similar method of Brevik (2001). The equation of a massless scalar field is solved numerically under the boundary conditions near the inner horizon $r_e$ and the outer horizon $r_c$. Unlike the usual wave function $\Psi_{\omega l}$ in 4D, quantum number $n$ introduces a new functions $\Psi_{\omega ln}$, whose potentials are higher and wider with bigger $n$. Using the tangent approximation, a full boundary value problem about the Schrödinger-like equation is solved. With a convenient replacement of the 5D continuous potential by square barrier, the reflection and transmission coefficients are obtained. If extra dimension does exist and is visible at the neighborhood of black holes, the unique wave function $\Psi_{\omega ln}$ may say something to it.

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I. INTRODUCTION

In 1974 [1] and 1975 [2] Stephen Hawking published his analysis of the effects of gravitational collapse on quantum fields, and predicted that black holes are not perfect black, but radiate thermally and eventually explode, named Hawking radiation. Since then, many people have used various methods and techniques to research black hole through the particles radiating from it, such as the simple Klein-Gordon particles and Dirac particles (for some early works, see Damour and Ruffini [3] and Chandrasekhar [4] respectively). Recently, as for the former scalar field, Higuchi et al. [5] and Grispino et al. [6] gave the scalar field solution outside a Schwarzschild black hole, Brady et al. [7], Brevik et al. [8] and Tian et al. [9] studied the Schwarzschild-de Sitter case and Guo et al. [10] made further studies in the Reissner-Nordström-de Sitter case and so on.

Regarding higher dimensional background, in order to avoid the interactions beyond any acceptable phenomenological limits, people assume that standard model fields (such as fermions, gauge bosons, Higgs fields) are confined on a $(3 + 1)$ dimensional hypersurface (3-brane) without accessing along the transverse dimensions. The branes are embedded in the higher dimensional space (bulk), in which only gravitons and scalar particles without charges could propagate under standard model gauge group. In this paper, the massless scalar field is chosen. Meanwhile, there are also many works (for a review with large extra dimensions see Ref. [11]) focusing on Hawking radiation such as Kanti, Frolov and Harris et al. [12] [13] [14] [15] [16], in which they employed both analytical and numerical techniques to calculate various particles’ greybody factors and different energy emission rates on the brane and in the bulk outside a $(4 + n)$-dimensional Schwarzschild black holes and $(4 + n)$-dimensional Schwarzschild-de Sitter black holes. These are two kinds of small higher dimensional black holes with horizon $r_H \ll L$, where $L$ is the size of the extra dimensions. Such small black holes can be treated safely as classical objects.

That the world may have more than four dimensions is due to Kaluza (1921) [17] and Klein (1926) [18], who realized that a 5D manifold could be used to unify general relativity with Maxwell’s theory of electromagnetism. After that, many people are more interested in the higher dimensional gravitational theory. Here, we consider the Space-Time-Matter (STM) theory presented by Wesson and co-workers [19] [20]. This theory is distinguished from the classical Kaluza-Klein theory for a non-compact fifth dimension, the 4D source is reduced from an empty 5D manifold. Because of this, the STM theory is also called induced matter theory and the effective 4D matter is called induced matter. That is, in STM theory, the 5D manifold is Ricci-flat while the 4D hypersurface is curved by the 4D matter. Mathematically, this approach is supported by Campbell’s theorem which states that any analytical solution of N-dimensional Einstein equations with a source can be locally embedded in an $(N + 1)$-dimensional Ricci-flat manifold [21]. In the framework of STM, people studied many astrophysical implications such as Perihelion Problem [22], Kaluza-Klein Solitons [23] [24], Black hole [25], Solar System Tests [26] and so on. So far as we know, except for
our previous work [27], no one has researched the radiation of 5D black holes in STM theory before.

This paper is organized as follows: In Section II, the 5D black string metric and the time-dependent radial equation about $R_e(r, t)$ are given. In section III, by a tortoise coordinate transformation, the radial equation becomes a Schrödinger-like equation. According to the boundary condition and the tangent approximation, a full numerical solution is presented. In section VI, with replacing the real potential barriers around black hole by square barriers, the reflection and transmission coefficients are naturally obtained. Section V is a conclusion.

We adopt the signature $(+,-,-,-,-)$ and put $h, c, G$ equal to unity. Lowercase Greek indices $\mu, \nu \ldots$ will be taken to run over $0, 1, 2, 3$ as usual, while capital indices $A, B, C, \ldots$ run over all five coordinates $(0, 1, 2, 3, 4)$.

II. THE KLEIN-GORDON EQUATION IN THE 5D SCHWARZSCHILD-DE SITTER SOLUTION

Within the framework of STM theory, a class of exact 5D solutions, presented by Mashhoon, Wesson and Liu [28] [29], describes a 5D black hole. The line element takes the form

$$dS^2 = \frac{\Lambda \xi^2}{3} \left[ f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] - d\xi^2. \quad (1)$$

In this metric

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2, \quad (2)$$

where $\xi$ is the open non-compact extra dimension coordinate, $\Lambda$ is the induced cosmological constant and $M$ is the central mass. The part of this metric inside the square bracket is exactly the same line-element as the 4D Schwarzschild-de Sitter solution, which is bounded by two horizons — an inner horizon (black hole horizon) and an outer horizon (one may call this cosmological horizon). This metric (1) satisfies the 5D vacuum equation $R_{AB} = 0$ and, therefore, there is no cosmological constant when viewed from 5D. However, when viewed from 4D, there is an effective cosmological constant $\Lambda$. So one can actually treat this $\Lambda$ as a parameter which comes from the fifth dimension. This solution has been studied in many works [30] [31] [32] [33] focusing mainly on the induced constant $\Lambda$, the extra force and so on.

We redefine the fifth dimension in this model,

$$\xi = \sqrt{\frac{3}{\Lambda} \sqrt{\pi} y}, \quad (3)$$

Then we use (1) ~ (3) to build up a RS type brane model in which one brane is at $y = 0$, and the other brane is at $y = y_1$. Hence the fifth dimension becomes finite. It could be very small as RS I brane model [35] or very large as RS II model [34]. It has been noted that the relevancy between STM and brane-worlds theories and some embedding of 5D solutions to brane models are studied in [36] [37] [38] [39]. For the present brane model, when viewed from a ($\xi$ or $y = constant$) hypersurface, the 4D line-element represents exactly the Schwarzschild-de Sitter black hole. However, when viewed from 5D, the horizon does not form a 4D sphere — it looks like a black string lying along the fifth dimension. Usually, people call the solution to the 5D equation $5(G_{AB} = \Lambda_5 (5) g_{AB})$ ($\Lambda_5$ is the 5D cosmological constant) as the 5D Schwarzschild-de Sitter solution. Therefore, to distinguish it, we call the solution (1) a black string, or more precisely, a 5D Ricci-flat Schwarzschild-de Sitter solution.

In our previous paper [27], we have introduced a massless scalar field to stabilizing this black string brane model. Considering a single mode of the scalar field, the wave function for this mode may reach the maximum value but keep smooth and finite at the brane. Hence, a steady standing wave is constructed. A suitable superposition of some of the quantized and continuous components of $L(y)$, which is one component of scalar field, may provide a wave function which is very large at $y = 0$ and drop rapidly for $y \neq 0$. Naturally, a practical 3-brane is formed at the $y = 0$ hypersurface.

With the redefinition (3), the metric (1) can be rewritten as

$$dS^2 = e^{\sqrt{\pi} y} \left[ f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - dy^2 \right], \quad (4)$$

where $y$ is the new fifth dimension. Expression (2) can be recomposed as follows

$$f(r) = \frac{\Lambda}{3r} (r - r_e)(r_e - r)(r - r_o). \quad (5)$$
The singularity of the metric (4) is determined by \( f(r) = 0 \). Here we only consider the real solutions. The solutions to this equation are inner horizon \( r_c \), outer horizon \( r_e \) and a negative solution \( r_o = -(r_c + r_e) \). The last one has no physical significance, and \( r_c \) and \( r_e \) are given as

\[
\begin{align*}
  r_c &= \frac{2}{\sqrt{\Lambda}} \cos \eta, \\
  r_e &= \frac{2}{\sqrt{\Lambda}} \cos(120^\circ - \eta),
\end{align*}
\]

where \( \eta = \frac{1}{3} \arccos(-3M\sqrt{\Lambda}) \) with \( 30^\circ \leq \eta \leq 60^\circ \). The real physical solutions are accepted only if \( \Lambda \) satisfy \( \Lambda M^2 \leq \frac{1}{9} \).

Following the work [22], the massless scalar field \( \Phi \) in the 5D black string space, satisfies the Klein-Gordon equation

\[
\Box \Phi = 0,
\]

where \( \Box = \frac{1}{\sqrt{g}} \partial_x \left( \sqrt{g} g^{AB} \partial_x g_{AB} \right) \) is the 5D d'Alembertian operator. We suppose that the separable solutions to Eq. (7) are of the form

\[
\Phi = \frac{1}{\sqrt{4\pi\omega r}} R_\omega(r,t) L(y) Y_{lm}(\theta, \phi),
\]

where \( R_\omega(r,t) \) is the radial time-dependent function, \( Y_{lm}(\theta, \phi) \) is the usual spherical harmonic function, and \( L(y) \) is the function about the fifth dimension. The differential equation of \( R_\omega(t,r) \) is

\[
- \frac{1}{f(r)^2} \frac{\partial^2}{\partial t^2} \left( \frac{R_\omega(r)}{r} \right) + \frac{\partial}{\partial r} \left( r^2 f(r)^2 \frac{\partial}{\partial r} \left( \frac{R_\omega(r)}{r} \right) \right) - \left[ \Omega r^2 + l(l+1) \right] \frac{R_\omega(r)}{r} = 0,
\]

where \( \Omega \) is a constant which is adopted to separate variables \( (t, r, \theta, \phi) \). Meanwhile, in our previous study [27], we have solved successfully the evolution equation \( L(y) \) in branes model. Its norm \( |L(y)|^2 \), which represents the probability of finding the massless scalar particle, gets an extremum on the branes. According to the standing wave condition in the bulk, the spectrum of \( \Omega \) is broken into two parts: one is the continuous spectra below \( \frac{3}{4} \Lambda \) and the other is the discrete spectra above \( \frac{1}{4} \Lambda \). The quantum parameter \( \Omega_n \) is

\[
\Omega_n = \frac{n^2 \pi^2}{y_1^2} + \frac{3}{4} \Lambda,
\]

where \( n = 1, 2, 3 \cdots \) and \( y_1 \) is the thickness of the bulk.

### III. THE BOUNDARY VALUE PROBLEM IN THE BULK

#### A. The Schrödinger-like Equation

A more important fact is the aspect of radial direction. In Eq. (9) time variable can be eliminated by the Fourier component \( e^{-i\omega t} \) via

\[
R_\omega(r,t) \rightarrow \Psi_{\omega ln}(r)e^{-i\omega t},
\]

where the subscript \( n \) presents a new wave function unlike the usual 4D case \( \Psi_{\omega l} \). Eq. (9) can be rewritten as

\[
\left[ -f(r) \frac{d}{dr} \left( f(r) \frac{d}{dr} \right) + V(r) \right] \Psi_{\omega ln}(r) = \omega^2 \Psi_{\omega ln}(r),
\]

whose potential function is given by

\[
V(r) = f(r) \left[ \frac{1}{r} \frac{df(r)}{dr} + \frac{l(l+1)}{r^2} + \Omega \right].
\]

Now we introduce the tortoise coordinate

\[
x = \frac{1}{2M} \int \frac{dr}{f(r)}.
\]
The tortoise coordinate can be expressed with the surface gravity as follows

\[ x = \frac{1}{2M} \left[ \frac{1}{2K_e} \ln \left( \frac{r}{r_e} - 1 \right) - \frac{1}{2K_c} \ln \left( \frac{1 - r}{r_c} \right) + \frac{1}{2K_o} \ln \left( 1 - \frac{r}{r_o} \right) \right], \]  

(15)

where

\[ K_i = \frac{1}{2} \left| \frac{df}{dr} \right|_{r=r_i}. \]  

(16)

Explicitly, we have

\[ K_e = \frac{(r_e - r_c)(r_e - r_o)}{6r_e} \Lambda, \]  

(17)

\[ K_c = \frac{(r_c - r_e)(r_c - r_o)}{6r_c} \Lambda, \]  

(18)

\[ K_o = \frac{(r_o - r_c)(r_o - r_e)}{6r_o} \Lambda. \]  

(19)

So under the tortoise coordinate transformation (14), the radial equation (12) can be written as

\[ \left[ - \frac{d^2}{dx^2} + 4M^2V(r) \right] \Psi_{\omega \ln}(x) = 4M^2\omega^2 \Psi_{\omega \ln}(x), \]  

(20)

which is of the form of Schrödinger equation in quantum mechanics. Because there are two different coordinates — radial coordinate \( r \) and tortoise coordinate \( x \) in it, people usually call it Schrödinger-like equation. The incoming or outgoing particle flow between inner horizon \( r_e \) and outer horizon \( r_c \) is reflected and transmitted by the potential \( V(r) \). Substituting the quantum parameters \( \Omega_n \) (10) into Eq. (13), the quantum potentials are obtained as follows

\[ V_n(r) = f(r) \left[ \frac{1}{r} \frac{df}{dr} + \frac{l(l+1)}{r^2} + \frac{n^2\pi^2}{y^2} + \frac{3}{4} \Lambda \right]. \]  

(21)

This means that the potential \( V(r) \) (21) determines the evolution of the field \( \Psi_{\omega \ln} \).

### B. The numerical solution

Near the horizons \( r_e \) and \( r_c \), the tortoise coordinate \( x \rightarrow \pm \infty \). According to Eq. (5) and Eq. (21), the boundary conditions are as follows

\[ V(r_e) = V(r_c) = 0. \]  

(22)

So near the horizons, Eq. (20) reduces to

\[ \left[ - \frac{d^2}{dx^2} + 4M^2\omega^2 \right] \Psi_{\omega \ln}(x) = 0, \]  

(23)

where the potential vanishes. Obviously, its solutions are \( e^{\pm i2M\omega x} \). In this paper, taking into account only real field, we choose the solution \( \Psi_{\omega \ln} = \cos(2M\omega x) \)

(24)

as a boundary condition near the two horizons (19).

Because there are two coordinates — radial coordinate \( r \) and tortoise coordinate \( x \) coupled in Eq. (20), the source transformation expression (15) is too complex to be solved. In order to solve Eq. (20) conveniently, it is necessary to use an approximate method for transition

\[ \tilde{r}(x) = f(x). \]  

(25)

As far as we know, there are two methods used frequently — one is tangent approximation [8], and the other is polynomial approximation [9]. The former method is convenient for theoretical analysis and is adopted here. The...
FIG. 1: The first three potential barriers $V_n(x)$ of 5D black string with $n = 1$ (solid), $n = 2$ (dashed) and $n = 3$ (dash-dot). We use $l = 1$, $M = 1$, $\Lambda = 10^{-3}$ and $y_1 = 10^{3/2}$ (a very large 5th dimension). Meanwhile, the potential barrier of the corresponding 4D solution ($\Omega = 0$) is plotted for comparison.

latter one involves too many polynomials approximation and we do not consider it in this paper. To the parameter $\Lambda$, we can always find an appropriate approximate method from the above. For a widely separated horizons model, we adopt the same value $\Lambda = 10^{-3}$ as appeared in [8] [9]. Then putting $\Lambda$ into the Eq. (6), we find the inner horizon $r_e = 2.00268M$ and the outer horizon $r_c = 53.7435M$. We employ the useful tangent approximation

$$\tilde{r}(x) = \frac{1}{b} \arctan \left[ \frac{1}{10}(x - 18) \right] + d,$$

in which $b = 2.6/(r_c - r_e)$ and $d = (r_c + r_e)/2 - 3$ are two parameters to linearly fit $r$ and $x$. Thus, the potentials $V_n(r)$ (21) can be converted into $V_n(x)$, which are plotted in Fig. 1. Clearly those potentials are highly localized near $x = 0$ and fall off exponentially at both inner horizon $r_e(x)$ and outer horizon $r_c(x)$. It is characteristic in our higher dimensional scenario that potentials are higher with bigger quantum number $n$.

Because such approximation (26) does not allow very large $|x|$, we shorten the distance of $x$ to $[-10,180]$, and hence boundary condition (24) can be rewritten as

$$\Psi_{\omega ln}(-10) = \cos(20M\omega), \quad \Psi_{\omega ln}(180) = \cos(360M\omega).$$

By using Mathematica software, combining the Schrödinger-like equation (20) and boundary conditions (27), we can solve Eq. (20) numerically as a boundary value problem. The amplitude versus the tortoise coordinate is illustrated in Fig. 2. It can be seen that the solution $\Psi_{\omega ln}(x)$ is similar to a harmonic wave without considering the decay factor $1/r$ in expression [8]. Taking into account the real surrounding, we use the tortoise transformation (15) and plot the amplitude versus $r$ in Fig. 3. Since the penetrating power becomes weaker, the wave is sparser than classical case [8] near the black horizon. From the view of penetration of a square barrier, the difference can be demonstrated more clearly in the next section.

**IV. THE REFLECTION AND TRANSMISSION**

We assume that the particle flux with energy $E$ bursts towards a square well along the positive direction of $x$ axis, where the potential is

$$\hat{V}(x) = \begin{cases} 
V_0, & x_1 < x < x_2, \\
0, & x < x_1 \text{ or } x > x_2.
\end{cases}$$
FIG. 2: Variation of the field amplitude versus tortoise coordinate $x$ with $M = 1$, $n=1$, $\omega = 1$, $l = 1$, $\Lambda = 10^{-3}$ and $y_1 = 10^{3/2}$ (a very large 5th dimension).

FIG. 3: Variation of the field amplitude versus $r$ with $M = 1$, $n=1$, $\omega = 1$, $l = 1$, $\Lambda = 10^{-3}$ and $y_1 = 10^{3/2}$ (a very large 5th dimension). The waves pile up near the outer horizon.

From the view of quantum mechanics, considering the wave behavior of the particles, this process is similar to scattering on the surface of propagation medium with thickness of $|x_2 - x_1|$. Parts of them are transmitted and parts of them are reflected back. According to statistical interpretation of wave function, whether the energy $E > V_0$ or not, there is definite probabilities to transmit and reflect by the potential. The reflection and transmission coefficients denote the magnitude of those probabilities.

As mentioned above, it is necessary to replace the continuously varying potential barrier with a discontinuous barrier of constant height in analytical work. Therefore, the usual reflection and transmission coefficients can be obtained. With the method of [40] and [8], We suppose a scalar wave propagates from $-\infty$ to $+\infty$, which is illustrated in Fig 6. The same denotation is cited here, namely associating "1" with the incoming wave in the region $-\infty < x < x_1$, "2" with the potential plateau $x_1 < x < x_2$, and "3" with the outgoing wave in the region $x_2 < x < +\infty$. Hence, potential $V(x)$ in Eq.(20) reduces to

$$V(x) = \begin{cases} \hat{V}_1, & -\infty < x < x_1, \\ \hat{V}_2, & x_1 < x < x_2, \\ \hat{V}_3, & x_2 < x < +\infty. \end{cases}$$

(29)

Following those square barriers, the solutions to the Eq.(20) are

$$\Psi_{\omega ln} = \begin{cases} a_1 e^{ik_1 x} + b_1 e^{-ik_1 x}, & -\infty < x < x_1, \\ a_2 e^{ik_2 x} + b_2 e^{-ik_2 x}, & x_1 < x < x_2, \\ a_3 e^{ik_3 x}, & x_2 < x < +\infty, \end{cases}$$

(30)

where $k_i = \sqrt{4M^2(\omega^2 - \hat{V}_i)}$ ($i = 1, 2, 3$) are the wave numbers, $a_i$ and $b_i$ are the undetermined coefficients to the
FIG. 4: The reflected log $R$ versus the height $H$ (or $V_2$). The same width $d = 14$ with $n = 2$ is adopted. Here, $M = 1$, $l = 1$, $\Lambda = 10^{-3}$, and $y_1 = 10^{1/2}$. The resonant points are marked by the star symbol ($\ast$). The n=2 case is represented by a black point.

FIG. 5: The reflected log $R$ versus the width $d$. The same width $H = 0.1093$ with $n = 2$ is adopted too. Here, $M = 1$, $l = 1$, $\Lambda = 10^{-3}$, and $y_1 = 10^{1/2}$. The resonant points are marked by the star symbol ($\ast$). The n=2 case is represented by a black point.

solutions. Then we define the reflection coefficients for the plane interfaces dividing two media

$$R_{ij} = \left( \frac{1 - Z_{ij}}{1 + Z_{ij}} \right)^2,$$

where $Z_{ij} = \frac{k_j}{k_i}$ are the real impedance ratios between medium $i$ and $j$. So the width of the barrier is $d = x_2 - x_1$ and the height of the square barrier is $H = V_2$. So in this model the reflection coefficients $R$ and transmission coefficients
FIG. 6: Replacement of the real 5D quantum potential barrier $V_1$ (solid) by square barrier with $M = 1$, $l = 1$, $\Lambda = 10^{-3}$, and $y_1 = 10^{3/2}$ (a very large 5th dimension). The replacement of the corresponding 4D solution ($\Omega = 0$) is also plotted by dashed line for comparison.

$T$ are given as \(^1\)

\[
R = \left| \frac{b_1}{a_1} \right|^2 = \frac{R_{12} + R_{23} + 2\sqrt{R_{12}R_{23}}\cos(2k_2d)}{1 + R_{12}R_{23} + 2\sqrt{R_{12}R_{23}}\cos(2k_2d)}.
\]

\[
T = \left| \frac{a_3}{a_1} \right|^2 = \frac{1}{(1 + Z_{12})^2(1 + Z_{23})^2} \frac{16}{1 + R_{12}R_{23} + 2\sqrt{R_{12}R_{23}}\cos(2k_2d)}.
\]

It is well known that in quantum mechanics \(^{41}\), the resonant effect frequently occurs in the square barrier. After the particle flux enters into the square barrier, the particle flux is reflected and transmitted again both by the walls $x_2$ and $x_1$ (Fig. 6). If wavelength satisfies the resonant condition, the resonant tunnelling would be arisen here. As a matter of fact, the waves after scattering many times have the same phasic. Their coherent states are stacked together. Hence, the amplitude of transmission waves increase evidently. Naturally, the reflection is weak more. Also, one can read this feature directly from Eq. (32) and Eq. (33), which contain cosine functions. So there is no surprise that Fig. 4 and Fig. 5 which describe $\log R$ vs. $H$ and $\log R$ vs. $d$ respectively, take oscillating like forms. When $\log R$ gets the extremum points (stars), the resonant tunnelling is arisen in the square barrier. At the same time, the case of $n = 2$ (two black points) is presented in the both figures, too.

In order to get reflection and transmission coefficients in this 5D black string, we have to find the approximative square potentials to replace the continuous potentials $V_n$. So the replacements is plotted in Fig. 6. For clearness of the figure, here we only consider the 4D ($\Omega = 0$) and the first 5D quantum potential ($n = 1$). In the case of $n = 1$, we read $V_2 = 0.1000$ as a reasonable value for the potential plateau in Fig. 6 and choose the incoming wave number

| mode | $x_1$ | $x_2$ | d | $V_2$ | $R$ | $T$ |
|------|------|------|---|------|----|----|
| 4D ($\Omega = 0$) | -4  | 6 | 10 | 0.0950 | $2.4 \times 10^{-3}$ | 0.9975 |
| 5D ($n = 1$) | -4 | 6 | 10 | 0.1000 | $2.7 \times 10^{-3}$ | 0.9973 |
| 5D ($n = 2$) | -4 | 10 | 14 | 0.1093 | $2.5 \times 10^{-4}$ | 0.9997 |
| 5D ($n = 3$) | -4 | 20 | 24 | 0.1267 | $1.9 \times 10^{-3}$ | 0.9981 |

\(^1\) See Ref \(^8\) for detail.
to be $k_1 = 2 (\hat{V}_1 = 0)$ and $k_3 = 2 (\hat{V}_3 = 0)$. For the case of 4D ($\Omega = 0$), we employ the same width and read $V_2 = 0.0950$. Meanwhile, using the same method, we can get the width and altitude of square barrier ($n = 2$) and ($n = 3$), respectively. In the end, every parameter in those modes is given in Table I. Substituting those into Eqs. (32) and (33), the reflection and transmission coefficients ($R$, $T$) are obtained and listed in Table I too. Because of the resonant effect, it is difficult to judge directly whether $R$ (or $T$, notice $R+T=1$) would be smaller or larger with the change of $n$ only by considering the width $d$ or the height $H$. Careful contrast of the Fig. 4 and Fig. 5 shows up some key differences that although they all oscillate integrality, the average order of $R$ varies obviously with the height $H$ rather than the width $d$.

V. CONCLUSION

In this paper we have solved the real scalar field $\Psi_{\omega\ell m}$ and obtained reflection coefficients and transmission coefficients in the 5D black string space. We summarize what has been achieved, and make some further comments.

1. The 5D black string solution presented by Mashhoon, Liu and Wesson in Ref. [29] is exact in higher dimensional gravitational theory. In the work of paper [27], the Klein-Gordon equation has been successfully separated into three parts, $\Psi \sim R_{\omega}(r, t)L(y)Y_{lm}(\theta, \phi)$, corresponding to the variables $(r, t)$, the fifth coordinate $y$, and the usual spherical harmonic coordinates $(\theta, \phi)$. We notice that the extra dimension $y$ or $\xi$ affects the usual 4D by the parameter $\Omega$, which is contained in the potential (17). The wave function $L(y)$ gets its extremum on two branes. Then by using the steady standing wave condition, the quantum potential $V_n$ are obtained after quantizing the parameter $\Omega$. So the effective 5D potential (21), which is different from the usual 4D case, determines the evolution of massless particles around the black hole (note that when $\Omega = 0$, the potential (13) reduce to the usual 4D Schwarzschild-de Sitter black hole potential [8]). Naturally, we can get a new radial wave function $\Psi_{\omega\ell m}$ being homologic with $\Psi_{\omega\ell}$ in 4D.

2. One purpose of this paper is to find the difference between $\Psi_{\omega\ell m}$ and $\Psi_{\omega\ell}$. We study scalar particles scattering around our black string solution (11) in the bulk and get the scalar field solution under the tangent approximation. Eq. (20) describes the evolution of one dimensional transmission of a wave through the potential barrier. According to Eq. (20), effective potential (21) and boundary condition (24), a full boundary value problem is presented. Because of the complex potential (21) and the approximative replacement (26), we only give the numerical solution instead of an analytical one.

3. From the view of mathematics, an oscillatory cosine function $\cos(2k_2d)$ or $\cos(2\sqrt{4M^2(\omega^2 - H)d})$ is contained both in $R$ (32) and $T$ (33). It is obviously that the function $R(H, d)$ and $T(H, d)$ are not monotonic function. We can see it clearly from Fig. 4 and Fig. 5. Considering the different replacement of square barrier obtained through the different quantum potentials, we only show the case $n=2$ (black points) in the figures. It is obvious that it is not easy to distinguish which one is smaller or bigger just by $H$ or $d$. Since that our replacement is a approximate method and its width and height are adopted very roughly, it is not necessary to give this resonant condition.

4. It is known that the hierarchy problem, which is why the characteristic scale of gravity ($M_p \sim 10^9$ GeV) is about 16 orders of magnitude larger than the Electro-Weak scale ($M_{EW} \sim 1$ TeV), is solved successfully by assuming the existence of extra dimensions both in ADD model [42], [43] and R-S model [34], [35]. It is considered that the extra dimensions would be really visible at the neighborhood of black holes and in the early epoch of universe. If it does so, regarding the former, the extra dimensionality of space would emerge out. In the strong gravitational field — black hole or black string (this model), the best way to prove this is the Hawking radiation. It contains a lot of information such as the mass of black hole or black string, the topological structure of ourbrane-world, the magnitudes of effective cosmological constant $\Lambda$ and so on. The appearance of wave function $\Psi_{\omega\ell m}$ is purely a quantum effect of the fifth dimensions. In this paper we employ a approximative method to give the real scalar field solution in the bulk roughly.

5. In this paper we only consider the quantum number $n$ depended property of scalar field. Unlike the work [16], we do not take into account the dependence on the effective cosmological constant $\Lambda$. We take $\Lambda = constant$ instead. The reason is that in order to solve the Schödinger-like Eq. (20), which has two coordinates variables $r$ and $x$, we need to find a quasi function (25). Nevertheless, we have to choose the value of effective cosmological constant firstly in the key function (15). If the value of $\Lambda$ varies, the approximate function has to be changed too. Furthermore, it is difficult to find out the approximate relation between the radial coordinate $r$ and the tortoise coordinate $x$. So in this work, the previous value $\Lambda = 10^{-3}$ [8] and the tangent approximation (20) is employed. Anyway, it is interesting to research this case and the further work is needed.
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