Instantons on Cylindrical Manifolds

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Abstract

We consider an instanton, $A$, with $L^2$ curvature $F_A$ on the cylindrical manifold $Z = \mathbb{R} \times M^n$, where $M$ is a compact Riemannian n-manifold, $n \geq 4$. We assume $M$ admits a 3-form $P$ and a 4-form $Q$ satisfy $dP = 4Q$ and $d\ast Q = (n - 3) \ast P$. Manifolds with these forms include nearly Kähler 6-manifolds and nearly parallel $G_2$-manifolds in dimension 7. Then we can prove the instanton must be a flat connection.

1 Introduction

Let $X$ be an $(n+1)$-dimensional Riemannian manifold, $G$ be a compact Lie group and $E$ be a principal $G$-bundle on $X$. Let $A$ denote a connection on $E$ with the curvature $F_A$. The instanton equation on $X$ can be introduced as follows. Assuming there is a 4-form $Q$ on $X$. Then an $(n-3)$-form $\ast Q$ exists, where $\ast$ is the Hodge operator on $X$. A connection, $A$, is called anti-self-dual instanton, when it satisfies the instanton equation

$$\ast F_A + \ast Q \wedge F_A = 0 \quad (1.1)$$

When $n + 1 > 4$, these equations can be defined on the manifold $X$ with a special holonomy group, i.e.the holonomy group $G$ about the Levi-Civita connection on the tangent bundle $TX$ is a subgroup of the group $SO(n + 1)$, where each solution of equation (1.1) satisfies the Yang-Mills equation. The instanton equation (1.1) is also well-defined on a manifold $X$ with non-integrable $G$-structures, but equation (1.1) implies the Yang-Mills equation will have torsion.

Instantons on the higher dimension, proposed in [3] and studied in [2, 5, 6, 12, 13], are important both in mathematics [6, 12] and string theory [8]. In this paper, we consider the cylinder manifold $Z = \mathbb{R} \times M$ with metric

$$g_Z = dt^2 + g_M$$
where $M$ is a compact Riemannian manifold. We assume $M$ admits a 3-form $P$ and a 4-form $Q$ satisfy

\[ dP = 4Q \quad (1.2) \]
\[ d \ast Q = (n - 3) \ast P. \quad (1.3) \]

On $Z$, the 4-form [9, 10] can be defined as

\[ Q_Z = dt \wedge P + Q. \]

Then the instanton equation on the cylinder is

\[ \ast F_A + \ast Q_Z \wedge F_A = 0 \quad (1.4) \]

**Remark 1.1.** Manifolds with $P$ and $Q$ satisfying equations (1.2), (1.3) include nearly Kähler 6-manifolds and nearly parallel $G_2$-manifolds.

(1) $M$ is a nearly Kähler 6-manifold. It is defined as a manifold with a 2-form $\omega$ and a 3-form $P$ such that

\[ d\omega = 3 \ast_M P \text{ and } dP = 2\omega \wedge \omega = 4Q \]

For a local orthonormal co-frame $\{e^a\}$ on $M$ one can choose

\[ \omega = e^{12} + e^{34} + e^{56} \text{ and } P = e^{135} + e^{164} - e^{236} - e^{245}, \]

where $a = 1, \ldots, 6$, $e^{a_1 \ldots a_l} = e^1 \wedge \ldots e^l$, and

\[ \ast_M P = e^{145} + e^{235} + e^{136} - e^{246}, \quad Q = e^{1234} + e^{1256} + e^{3456}. \]

Here $\ast_M$ denotes the $\ast$-operator on $M$.

(2) $M$ is a nearly parallel $G_2$ manifold. It is defined as a manifold with a 3-form $P$ (a $G_2$ structure [11]) preserved by the $G_2 \subset SO(7)$ such that

\[ dP = \gamma \ast_M P \]

for some constant $\gamma \in \mathbb{R}$. For a local orthonormal co-frame $e^a$, $a = 1, \ldots, 7$, on $M$ one can choose

\[ P = e^{123} + e^{145} - e^{367} + e^{246} + e^{257} + e^{347} - e^{356} \]

and therefore

\[ \ast_M P =: Q = e^{4567} + e^{2367} - e^{2345} + e^{1357} + e^{1346} + e^{1256} - e^{1247}. \]

It is easy to check $dP = 4Q$. 

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Constructions of solutions to the instanton equations on cylinders over nearly Kähler 6-manifolds and nearly parallel $G_2$ manifold were considered in [10]. In this paper, we assume the instanton $\mathbf{A}$ with $L^2$ curvature $F_{\mathbf{A}}$, then we have the following theorem.

**Main Theorem.** Let $Z = \mathbb{R} \times M$, where $M$ is a compact Riemannian $n$-manifold, $n \geq 4$, which admits a 3-form $P$ and a 4-form $Q$ those satisfy equations (1.2) and (1.3). Let $\mathbf{A}$ be a instanton over $Z$. Assume the curvature $F_{\mathbf{A}} \in L^2(Z)$ i.e.
\[
\int_Z F_{\mathbf{A}} \wedge \ast F_{\mathbf{A}} < +\infty
\]
then the instanton is a flat connection.

## 2 Preliminaries

The main aim of this section is to get the relationship about gauge theory between an $n$-dimensional manifold $M$ and the $n+1$-dimensional manifold $Z = \mathbb{R} \times M$. The main idea is that a connection on $\mathbb{R} \times M$ can form one-parameter families of connections on $M$ by local trivialisation and local coordinates. Let $t$ be the standard parameter on the factor $\mathbb{R}$ in the $\mathbb{R} \times M$ and let $\{x^j\}_{j=1}^n$ be local coordinates in $M$. A connection $\mathbf{A}$ over the cylinder $Z$ is given by a local connection matrix
\[
\mathbf{A} = A_0 \, dt + \sum_{i=1}^n A_i \, dx^i.
\]
where $A_0$ and $A_i$ dependence on all $n + 1$ variable $t, x^1, \ldots, x^n$. We take $A_0 = 0$ (sometimes called a temporal gauge). In this situation, the curvature in a mixed $x_i$-plane is given by the simple formula
\[
F_{0i} = \frac{\partial A_i}{\partial t}.
\]
We denote $A = \sum_{i=1}^n A_i \, dx^i$ and $\dot{A} = \frac{\partial A}{\partial t}$, then the curvature is given by
\[
F_{\mathbf{A}} = F_A + dt \wedge \dot{A}.
\]
$M$ has a Riemannian metric and $\ast$-operator $*_M$. If $\phi$ is a 1-form on $M$ then, for the $n + 1$-dimensional $\ast$-operator defined respect to the product metric on $\mathbb{R} \times M$,
\[
*(dt \wedge \phi) = *_M \phi.
\]
Then the instanton equation is equivalent to
\[ *_M \dot{A} = - *_M P \wedge F_A, \quad (2.1) \]
\[ *_M F_A = - \dot{A} \wedge *_M P - *_M Q \wedge F_A. \quad (2.2) \]

We define the Chern-Simons function,
\[ CS(A) := \int_M (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \wedge *_M P \]
\[ = \frac{1}{n-3} \int_M Tr(F_A \wedge F_A) \wedge *_M Q, \quad (2.3) \]
the second formula holds by equation (1.3). This function is gauge-invariant when \( n \geq 4 \). The space of all connections \( A \) on \( M \) can be given an \( L^2 \) metric, then the gradient flow equation for \( CS \) is the first equation (2.1).

The Chern-Simons function is obtained by integrating the Chern-Simons 1-form
\[ \Gamma(\beta)_A = \Gamma_A(\beta_A) = 2 \int_M Tr(F_A \wedge \beta_A) \wedge *_M P \]
\( CS \) actually is integration \( \Gamma \) over paths \( A(t) = tA \):
\[ CS(A) = 2 \int_0^1 \Gamma_{A(t)}(\dot{A}(t)) \, dt \]
\[ = 2 \int_0^1 \left( \int_M Tr((t \, dA + t^2 A \wedge A) \wedge A) \wedge *_M P \right) \, dt \]
\[ = \int_M Tr(\, dA \wedge A + \frac{2}{3} A \wedge A \wedge A) \wedge *_M P. \]

The co-closed condition \( d * P = 0 \) implies that the Chern-Simons 1-form is closed, so it does not depend on the path \( A(t) \) [4, 7].

3 Asymptotic Behavior And Conformal Transformation

3.1 Asymptotic Behavior

Let \( Z = \mathbb{R} \times M \) be an \((n+1)\)-manifold and \( M \) be an \( n \)-manifold. Let \( A \) be an instanton on \( Z \) with finite energy, i.e. \( \int_Z |F_A|^2 < \infty \). We use the Chern-Simons function to study the decay of instantons over the cylinder manifold. We will see that, an instanton with \( L^2 \) curvature can be represented by a connection form which decays exponentially on the tube.
Proposition 3.1. Under these conditions, at the end of $Z$ there is a flat connection $\Gamma$ over $M$ such that $A$ converges to $\Gamma$, i.e. the restriction $A_{M \times \{ T \}}$ converge (modulo gauge equivalence) in $C^\infty$ over $M$ as $T \to \infty$.

We consider a family of bands $B_T = (T - 1, T) \times M$ which we identify with the model $B = (0,1) \times M$ by translation. Let $A_T$ be the connection over $B$ obtaining from the restriction of $A$ on $B_T$, so the integrability of $\| F_A \|^2$ over the end implies that

$$\| F_{A_T} \|_{L^2(B)} \to 0 \quad \text{as} \quad T \to \infty.$$ 

Uhlenbeck’s weak compactness \cite{14} implies that for any sequence $T_i \to \infty$, there are a sequence $T'_i$ and a flat connection $\Gamma$ over $B$ such that, after suitable gauge transformations,

$$A_{T'_i} \to \Gamma$$

in $C^\infty$ over compact subsets of $B$. In particular the restriction of $[A_{T'_i}]$ to the cross-section $M \times \{ \frac{1}{2} \}$ converges in $C^\infty$ to $\Gamma$.

Lemma 3.2. Let $A$ be an instanton with temporal gauge, then

$$CS(T) - CS(T') = - \int_{[T', T] \times M} F_A \wedge *F_A - (n - 3) \int_{T'}^T CS(A(t)) \quad (3.1)$$

Proof. Using the method above section, we have

$$\frac{dCS(A(t))}{dt} = \Gamma_{A(t)}(A'(t))$$

Then

$$CS(A_T) - CS(A_{T'}) = \int_{T'}^T dCS(A(t)) = \int_{T'}^T \Gamma_{A(t)}(A'(t)) \, dt$$

$$= 2 \int_{[T', T] \times M} Tr(F_{A(t)} \wedge dt \wedge A'(t)) \wedge *P$$

$$= \int_{[T', T] \times M} Tr(F_A \wedge F_A) \wedge *Q_Z$$

$$- (n - 3) \int_{T'}^T \left( \int_M Tr(F_A \wedge F_A) \wedge *P \right) \, dt$$

$$= - \int_{[T', T] \times M} F_A \wedge *F_A - (n - 3) \int_{T'}^T CS(A(t))$$

\hfill \qed
We set
\[ J(T) = \int_T^\infty \| F_A \|_{L^2}^2. \]

On the one hand, we can express \( J(T) \) as the integration of \( -Tr(F_A^2) \wedge *Q_Z \) since \( A \) is an instanton.
\[ J(T) = \int_{[T,\infty) \times M} -Tr(F_A^2) \wedge *Q_Z \]

From \ref{3.1}, taking the limit over finite tubes \((T', T) \times M\) with \( T' \to +\infty \) we see that
\[ J(T) = CS(A_T) - CS(A_\infty) - (n - 3) \int_T^\infty CS(A(t)) \, dt \] (3.2)

where \( A_T \) is the connection over \( M \) obtain by restriction to \( M \times \{ T \} \). In fact, from the Proposition \ref{3.1} and \ref{2.3}, \( CS(A_\infty) \) is vanishing. Form \ref{3.2} we can obtain the \( T \) derivative of \( J \) is
\[ \frac{dJ(T)}{dT} = \frac{dCS(A_T)}{dT} + (n - 3)CS(A_T) \] (3.3)

On the other hand, the \( T \) derivative of \( J(T) \) can be expressed as minus the integral over \( M \times \{ T \} \) of the curvature density \( |F_A|^2 \), and this is exactly the n-dimensional curvature density \( |F_A|^2 \) plusing the density \( |A|^2 \). By the relation \ref{1.2} and \ref{1.3} between the two components of the curvature for an instanton, we have
\[ \| F_{A_T} \|_{L^2(M)}^2 = \| A \|_{L^2(M)}^2 - (n - 3)CS(A_T) \]

Thus
\[ \frac{dJ(T)}{dT} = -2\| F_{A_T} \|_{L^2(M)}^2 - (n - 3)CS(A_T) \] (3.4)
\[ = -2\| A \|_{L^2(M)}^2 + (n - 3)CS(A_T) \] (3.5)

From \ref{3.3} and \ref{3.4}, we have
\[ \frac{dCS(A_T)}{dT} + 2(n - 3)CS(A_T) \leq 0 \]

From \ref{3.3} and \ref{3.5}, we have
\[ \frac{dCS(A_T)}{dT} \leq 0 \]
It is easy to see these imply that \( CS(T) \) is non-negative and decays exponentially,

\[
0 \leq CS(T) \leq Ce^{-(2n-6)T}
\]

(3.6)

We introduce a parameter \( \delta \) and set

\[
L_\delta(T) := \int_T^\infty e^{\delta t} \| F_A \|_{L^2(M)}^2 \, dt
\]

**Theorem 3.3.** Let \( A \) be an instanton with \( L^2 \) curvature on \( Z \), then there is a constant \( C \), such that

\[
L_\delta(t) \leq Ce^{(\delta-2n+6)t}
\]

where \( \delta < 2n - 6 \).

**Proof.** From (3.3)

\[
\| F_A \|_{L^2(M)}^2 = \frac{dCS(A_T)}{dT} + (n-3)CS(A_T)
\]

Then

\[
L_\delta(T) = \int_T^\infty e^{\delta t} \frac{dCS(A(t))}{dt} + (n-3) \int_T^\infty e^{\delta t} CS(A(t))
\]

\[
= e^{\delta T} CS(A_T) + (n-3) \int_T^\infty e^{\delta t} CS(A(t))
\]

\[
\leq Ce^{(\delta-2n+6)T}
\]

\[\square\]

### 3.2 Conformal Transformation

We consider \( \tilde{Z} = C(M) \), where \( C(M) \) is a cone over \( M \) with metric

\[
g_{\tilde{Z}} = dr^2 + r^2 g_M = e^{2t} (dt^2 + g_M),
\]

where \( r := e^t \).

It means that the cone \( C(M) \) is conformally equivalent to the cylinder

\[
Z = \mathbb{R} \times M
\]

with the metric

\[
g_Z = dt^2 + g_M.
\]
Furthermore, we can show that the instanton equation on the cone $\tilde{Z} = C(M)$ is related with the instanton equation on the cylinder $Z = \mathbb{R} \times M$,

$$\bar{*}F_A + \bar{*}Q_Z \wedge F_A = e^{(n-3)t}(*F_A + *Q_Z \wedge F_A) = 0,$$

where $\dim C(M) = \dim Z = n + 1$, $\bar{*}$ is the $*$-operator in $C(M)$. And

$$Q_Z = e^{4t}(dt \wedge P + Q).$$

(3.7)

In the other word, equation on $C(M)$ is equivalent to the equation on $\mathbb{R} \times M$ after rescaling of the metric. So we can only consider the instanton equation

$$\bar{*}F_A + \bar{*}Q_Z \wedge F_A = 0$$

on the cone $C(M)$ over $M$.

After rescaling of the metric,

$$F_A \wedge \bar{*}F_A = e^{(n-3)t}F_A \wedge *F_A.$$ 

The curvature $F_A$ is belong to $L^2$ in $Z$, is it also belong to $L^2$ in $C(M)$? Next, we will see the question is right.

**Lemma 3.4.** Let $A$ be a instanton on $Z$ with $L^2$ curvature $F_A$, i.e. \( \int_{\mathbb{R} \times M} F_A \wedge *F_A < +\infty \), then

$$\int_{\mathbb{R} \times M} F_A \wedge \bar{*}F_A < +\infty$$

It means $F_A$ also belong to $L^2$ in $C(M)$.

**Proof.** From theorem 3.3, we have

$$L_{n-3}(T) = \int_T^\infty \|F_A\|_{L^2(M)}^2 \leq Ce^{-(n-3)T}$$

Then

$$\int_{\mathbb{R} \times M} e^{(n-3)t}F_A \wedge *F_A = \int_{(-\infty,T) \times M} + \int_{(T,\infty) \times M} e^{(n-3)t}F_A \wedge *F_A$$

$$\leq e^{(n-3)T} \int_{\mathbb{R} \times M} F_A \wedge *F_A + L_{n-3}(T) < +\infty$$

Next, we only consider instantons with $L^2$ curvature on the cone of $M$. And $*\bar{Q}_Z$ is closed, this implies instantons also satisfy Yang-Mills equation. In the next section, we will give an vanishing theorem for Yang-Mills connection with finite energy on cone of $M$. 

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4 Vanishing Theorem for Yang-Mills

In this section, notations may be different from the above sections. We use the conformal technique to give the vanishing theorem for Yang-Mills connection on the cone of $M$.

Let $M$ be a Riemannian $n + 1$-manifold. Suppose $X \in \Gamma(TM)$ is a conformal vector field on $(M, g)$, namely,

$$\mathcal{L}_X g = 2fg$$

where $f \in C^\infty(M)$. Here $\mathcal{L}_X$ denotes the Lie derivative with respect to $X$.

The vector field $X$ generates a family of conformal diffeomorphism.

$$F_t = \exp(tX) : M \to M$$

This conformal diffeomorphism can induce a bundle automorphism, $\tilde{F}_t$, of the principal bundle $P$. One can see the detailed process in [11].

We will consider the variation of the Yang-Mills functional under the family of diffeomorphism.

$$Y_M(A, g) = \int_M |F_A|^2 \, dg$$

where $dg = \sqrt{\det g} \, dx$ is the volume form of $M$.

$$\lambda := |F_A|^2 \, dg$$

is an $n$-form on $M$. For any $\eta \in C^\infty_0(M)$

$$0 = \int_M d[(i_X \lambda)\eta] = \int_M \eta d(i_X \lambda) + \int_M d\eta \wedge i_X \lambda$$

that is

$$\int_M \eta \mathcal{L}_X \lambda = - \int_M d\eta \wedge i_X \lambda \quad (4.1)$$

where $i_X$ stands for the inner product with the vector $X$. Now, let us compute $\mathcal{L}_X \lambda$.

**Lemma 4.1.** Let $\lambda = Tr(F_A \wedge *F_A)$ and $X$ be a smooth vector field on $M$ satisfying $\mathcal{L}_X g = 2fg$, then

$$\mathcal{L}_X \lambda = (n - 3)f \lambda + 2Tr(d_A(i_X F_A) \wedge *F_A)$$
Proof. In local coordinates \(\{x^i\}_{i=1}^n\), the n-form \(\lambda\)
\[
\lambda = Tr(F_A \wedge \ast F_A) = \sum g^{ij} g^{kl} tr F_{ik} F_{jl} \sqrt{\det g} \, dx^1 \wedge \ldots \wedge dx^n
\]
is conformal of weight \(n - 3\), i.e. \(\lambda(A, e^{2t} g) = e^{(n-3)t} \lambda(A, g)\) for any \(f \in C^\infty\).

The vector field \(X\) satisfies \(L_X g = 2f g\), so

\[
F_t^* g = \exp(2 \int_0^t F_s^* f) g.
\]

Since \(\lambda\) is conformal of weight \(n - 3\),

\[
(F_t^* \lambda)(A, g) = \lambda(\tilde{F}_t^* A, F_t^* g) = \exp((n - 3) \int_0^t F_s^* f) \cdot \lambda(\tilde{F}_t^* A, g) = (1 + (n - 3) tf + o(t^2))
\]
\[
\times Tr(F_A \wedge \ast F_A + 2t d_A(i_X F_A) \wedge \ast F_A + o(t^2)) = Tr(F_A \wedge \ast F_A + t(n - 3) f F_A \wedge \ast F_A)
\]
\[
+ 2t Tr(d_A(i_X F_A) \wedge \ast F_A) + o(t^2)
\]

where we used the fact \(F_t^* f = f + o(t)\) and \(\tilde{F}_t^* A = A + ti_X F_A + o(t^2)\). By the definition of Lie derivative

\[
\mathcal{L}_X \lambda = \frac{d}{dt}(F_t^* \lambda)|_{t=0} = (n - 3) f \lambda + 2 Tr(d_A(i_X F_A) \wedge \ast F_A).
\]  

\(\square\)

We consider \(M = \mathbb{R} \times N\) with metric

\[
g_M = e^{2t}(dt^2 + g_N)
\]

where \(N\) is a compact Riemannian n-manifold, \(n \geq 4\), with metric \(g_N\). Then the vector field \(X = \frac{\partial}{\partial t}\) satisfying

\[
\mathcal{L}_X g_M = X \cdot e^{2t}(dt^2 + g_N) + e^{2t}(\mathcal{L}_X dt^2) = 2g_M,
\]

and in this case, \(f = 1\).

**Theorem 4.2.** Let \((M, g_M)\) be a Riemannian manifold as above. Assume \(A\) to be a Yang-Mills connection with \(L^2\) curvature \(F_A\), i.e.

\[
\int_M |F_A|^2 < +\infty
\]

over \(M\). Then \(A\) must be flat.
Proof. From (4.1) and (4.2), we have
\[
\int_M \eta(n-3)\lambda = -\int_M d\eta \wedge i_X\lambda - 2\int_M \eta Tr\left(d_A(i_XF_A) \wedge F_A\right) \\
= -\int_M d\eta \wedge i_X\lambda - 2\int_M Tr\left(d_A(\eta i_XF_A) \wedge F_A\right) \\
+ 2\int_M Tr\left(d\eta \wedge i_XF_A \wedge F_A\right) \\
\leq 3\int_M |d\eta| \cdot |X| \cdot \lambda
\] (4.3)
The second term in the second line vanishes since \(A\) is a Yang-Mills connection.

We choose the cut-off function with \(\eta(t) = 1\) on the interval \(|t| \leq T\), \(\eta(t) = 0\) on the interval \(|t| \geq 2T\), and \(|d\eta| \leq 2T^{-1}\). Then \(d\eta\) has support in \(T \leq |t| \leq 2T\) and \(|X(t)| = 1\),
\[
\int_M \eta(n-3)\lambda \leq \frac{6}{T} \int_{\{T \leq |t| \leq 2T\} \times N} \lambda.
\]
Letting \(T \to \infty\) we get
\[
\int_M (n-3)\lambda = 0,
\]
Then \(F_A = 0\).

\[\square\]

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