On Chow Rings of Fine Moduli Spaces of Modules

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Abstract

Let \( \mathcal{M} \) be a complete nonsingular fine moduli space of modules over an algebra \( S \). A set of conditions is given for the Chow ring of \( \mathcal{M} \) to be generated by the Chern classes of certain universal bundles occurring in a projective resolution of the universal \( S \)-module on \( \mathcal{M} \). This result is then applied to the varieties \( G_T \) parametrizing homogeneous ideals of \( k[x, y] \) of Hilbert function \( T \), to moduli spaces of representations of quivers, and finally to moduli spaces of sheaves on \( \mathbb{P}^2 \), reinterpreting a result of Ellingsrud and Strømme.

In a recent paper [ES] Ellingsrud and Strømme identified a set of generators of the Chow ring of the moduli space of stable sheaves of given rank and Chern classes on \( \mathbb{P}^2 \) (in the case where the moduli space is smooth and projective). In this paper we formulate a part of their argument as a general theorem about fine moduli spaces of modules over an associative algebra. This provides a more widely applicable method for showing that the Chow ring of a fine moduli space is generated by the Chern classes of appropriate universal sheaves. In particular we apply the method to verify a conjecture of Iarrobino and Yaméogo concerning the Chow rings of the varieties \( G_T \) parametrizing homogeneous ideals in \( k[x, y] \) with a given Hilbert function. We also verify a conjecture of the first author concerning Chow rings of moduli spaces of representations of quivers.

Let \( S \) be an associative algebra over an algebraically closed field \( k \). By convention we will consider only left \( S \)-modules in this paper. A flat family of \( S \)-modules over a \( k \)-scheme \( X \) is a sheaf \( \mathcal{F} \) of \( S \otimes O_X \)-modules on \( X \), quasi-coherent and flat over \( O_X \). At a (closed) point \( x \in X \) the fiber \( \mathcal{F}(x) \) is an \( S \)-module. If \( \mathcal{C} \) is a class of \( S \)-modules, then a fine moduli space for \( \mathcal{C} \) is a scheme \( \mathcal{M} \) equipped with a flat family \( U \) all of whose fibers are in \( \mathcal{C} \) and with the usual universal property. Our general theorem is the following:

**Theorem 1.** Let \( \mathcal{C} \) be a class of \( S \)-modules, and \( \mathcal{M} \) a fine moduli space for \( \mathcal{C} \) which is a complete nonsingular variety. Suppose further that

(i) If \( E \in \mathcal{C} \), then \( \text{Hom}_S(E, E) \equiv k \), \( \text{Ext}^1_S(E, E) \equiv T[E]M \), and \( \text{Ext}^p_S(E, E) = 0 \) for \( p \geq 2 \);

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(ii) If \( E \not\cong F \) are in \( \mathcal{C} \), then \( \text{Hom}_S(E, F) = 0 \) and \( \text{Ext}^p_S(E, F) = 0 \) for \( p \geq 2 \).

(iii) If \( \mathcal{U} \) is the universal \( S \otimes \mathcal{O}_M \)-module on \( M \), then \( \mathcal{U} \) has a universal projective resolution of finite length:

\[
0 \to \bigoplus_j P_{rj} \otimes \mathcal{E}_{rj} \to \cdots \to \bigoplus_j P_{ij} \otimes \mathcal{E}_{ij} \to \bigoplus_j P_{0j} \otimes \mathcal{E}_{0j} \to \mathcal{U} \to 0
\]

with the \( P_{ij} \) projective \( S \)-modules such that \( \text{dim}_k \text{Hom}_S(P_{ij}, P_{ij'}) \) is always finite, and the \( \mathcal{E}_{ij} \) are all locally free \( \mathcal{O}_M \)-modules of finite rank.

Then

(a) The Chern classes of the \( \mathcal{E}_{ij} \) generate the Chow ring \( A^*(M) \) as a \( \mathbb{Z} \)-algebra.

(b) Numerical and rational equivalence coincide on \( M \). In particular, \( A^*(M) \) is a free \( \mathbb{Z} \)-module.

(c) If \( k = \mathbb{C} \), then the cycle map \( A^*(M) \to H^*(M, \mathbb{Z}) \) is an isomorphism. In particular, there is no odd-dimensional cohomology.

If \( S \) is a graded algebra, then one may formulate a graded version of the theorem by replacing “module” by “graded module” throughout and using the degree-zero parts of Hom and Ext. More generally, one should be able to formulate a version of Theorem 1 for moduli spaces of objects in an abelian category of \( k \)-vector spaces in any situation where one has a suitable notion of a family of objects of the category.

Our first application of Theorem 1 is to the Iarrobino varieties \( G_T \) which parametrize homogeneous ideals \( I \subset k[x, y] \) of Hilbert function \( T \). Here \( T = (t_0, t_1, \ldots) \) is a sequence of nonnegative integers such that \( t_n = 0 \) for \( n \gg 0 \), and the points of \( G_T \) correspond to those \( I \) such that \( \text{dim}_k(k[x, y]/I)_n = t_n \) for all \( n \). These smooth projective varieties were originally constructed in [I] in order to study the Hilbert-Samuel function stratification of the punctual Hilbert scheme of a surface. They have since been studied in several papers including [G], [IY], and [Y]. The fact that these \( G_T \) are fine moduli spaces was addressed formally in [Go] Kap. 2, Lemma 4.

Having fixed the Hilbert function \( T \), the degree-\( n \) graded pieces of the quotient rings form a family of quotients of dimension \( t_n \) of the space of binary forms of degree \( n \). This induces a natural morphism from \( G_T \) to the Grassmannian of quotient spaces \( \text{Gr}(t_n, n+1) \). Let \( \mathcal{A}_n \) denote the pullback to \( G_T \) of the universal quotient bundle on \( \text{Gr}(t_n, n+1) \). Our result is

**Theorem 2.** Let \( G_T \) be the Iarrobino variety parametrizing homogeneous ideals \( I \subset k[x, y] \) of Hilbert function \( T \), and let the \( \mathcal{A}_n \) be the universal bundles defined above. Then

(a) The Chern classes of the \( \mathcal{A}_n \) generate the Chow ring \( A^*(G_T) \) as a \( \mathbb{Z} \)-algebra.

(b) Numerical and rational equivalence coincide on \( G_T \). In particular, \( A^*(G_T) \) is a free \( \mathbb{Z} \)-module.

(c) If \( k = \mathbb{C} \), then the cycle map \( A^*(G_T) \to H^*(G_T, \mathbb{Z}) \) is an isomorphism. In particular, there is no odd-dimensional cohomology.

Parts (b) and (c) were already known because \( G_T \) has a cell decomposition corresponding to the initial ideals with Hilbert function \( T \) (cf. [G] or [IY]). Nevertheless, our methods give a new proof.

Our second application is to fine moduli spaces of representations of a quiver without oriented cycles. These moduli spaces were constructed in [K].
To fix notation, we recall that a quiver $Q$ is a directed graph, specified by a finite set of vertices $Q_0$ and a finite set of arrows $Q_1$ between the vertices together with two maps $h, t: Q_1 \rightarrow Q_0$ specifying the head and tail of each arrow. A representation of $Q$ consists of vector spaces $W_i$ for each $i \in Q_0$ and $k$-linear maps $\phi_a : W_{ta} \rightarrow W_{ha}$ for each $a \in Q_1$. A subrepresentation is a collection of subspaces $W'_i \subset W_i$ such that $\phi_a(W'_{ta}) \subset W'_{ha}$ for all $a$. The dimension vector $\alpha \in \mathbb{N}^{Q_0}$ of a representation $(W_i, \phi_a)$ is given by $\alpha_i = \dim_k(W_i)$.

To obtain a moduli space of representations of $Q$ of dimension vector $\alpha$ one needs to introduce a notion of stability. Having chosen $\theta = (\theta_i) \in \mathbb{R}^{Q_0}$ such that $\sum_i \theta_i \alpha_i = 0$, we say that a representation $(W_i, \phi_a)$ is $\theta$-stable if all (proper) subrepresentations $(W'_i)$ satisfy $\sum_i \theta_i \dim(W'_i) > 0$. When $\alpha$ is an indivisible dimension vector and $\theta$ is generic, there is a smooth fine moduli space $M_Q(\alpha, \theta)$ of $\theta$-stable representations of $Q$ of dimension vector $\alpha$ ([K] Proposition 5.3). If the quiver $Q$ has no oriented cycles, then this fine moduli space is projective ([K] Proposition 4.3). Note that this moduli space may actually be empty. The conditions on $\alpha$ and $\theta$ which make it non-empty are more subtle (cf. [K] Remark 4.5).

The universal representation over $M_Q(\alpha, \theta)$ consists of vector bundles $U_i$ of rank $\alpha_i$ together with the universal morphisms. We use Theorem 1 to prove the following, confirming the conjecture made in Remark 5.4 of [K].

**Theorem 3.** Let $Q$ be a quiver without oriented cycles, and $M = M_Q(\alpha, \theta)$ be a smooth projective fine moduli space of $\theta$-stable representations of $Q$ of dimension vector $\alpha$. Let $U_i$ be the universal bundles on $M$ described above. Then

(a) The Chern classes of the $U_i$ generate the Chow ring $A^*(M)$ as a $\mathbb{Z}$-algebra.

(b) Numerical and rational equivalence coincide on $M$. In particular, $A^*(M)$ is a free $\mathbb{Z}$-module.

(c) If $k = \mathbb{C}$, then the cycle map $A^*(M) \rightarrow H^*(M, \mathbb{Z})$ is an isomorphism. In particular, there is no odd-dimensional cohomology.

The outline of the paper is as follows. In the first section we prove Theorem 1 by adapting the method of Ellingsrud and Strømme. In the second and third sections we apply Theorem 1 to prove Theorems 2 and 3. In the fourth section we explain how Theorem 1 may be used to prove Ellingsrud and Strømme’s original result for sheaves on $\mathbb{P}^2$.

## 1 Proof of the Main Theorem

In this section we prove Theorem 1 by adapting a method of Ellingsrud and Strømme.

Let $\delta$ be the class of the diagonal in $A^*(M \times M)$, and let $p_1$ and $p_2$ denote the projections from $M \times M$ onto its two factors. We will adapt the methods of [ES] §2 to show that $\delta$ can be written as a polynomial in the Chern classes of the $p_1^*(E_{ij})$ and the $p_2^*(E_{ij})$. The theorem will then follow from the following result, which Ellingsrud and Strømme describe as “a well-known observation on varieties with decomposable diagonal class”:

**Theorem 4.** ([ES] Theorem 2.1) Let $X$ be a nonsingular complete variety. Assume that the rational equivalence class $\delta$ of the diagonal $\Delta \subset X \times X$ decomposes in the form

$$\delta = \sum_{i \in J} p_1^* \alpha_i \, p_2^* \beta_i$$

(1)
where \( p_1 \) and \( p_2 \) are the projection of \( X \times X \) onto its factors, and \( \alpha_i, \beta_i \in A^*(X) \). Then

(a) The \( \alpha_i \) generate \( A^*(X) \) as a \( \mathbb{Z} \)-module.

(b) Numerical and rational equivalence coincide on \( X \). In particular, \( A^*(X) \) is a free \( \mathbb{Z} \)-module.

(c) If \( k = \mathbb{C} \), then the cycle map \( A^*(X) \to H^*(X,\mathbb{Z}) \) is an isomorphism. In particular, there is no odd-dimensional cohomology.

(d) Suppose the set \( \{\alpha_i\} \) in (\( \mathbb{I} \)) is minimal. Then \( \{\alpha_i\} \) and \( \{\beta_i\} \) are dual bases with respect to the intersection form on \( A^*(X) \).

So we show how to write \( \delta \) as a polynomial in the Chern classes of the \( p_1^*(\mathcal{E}_{ij}) \) and the \( p_2^*(\mathcal{E}_{ij}) \) using a method similar to [ES] \( \S 2 \). First we write \( \mathcal{P} \) for the projective resolution of the universal family of modules \( \mathcal{U} \) in unaugmented form

\[
0 \to \bigoplus_j P_{rj} \otimes \mathcal{E}_{rj} \to \cdots \to \bigoplus_j P_{0j} \otimes \mathcal{E}_{0j} \to 0.
\]

Then let \( \mathcal{L}^\bullet = \mathcal{H}om_{\mathcal{O}}(p_1^*\mathcal{P}, p_2^*\mathcal{P}) \). Since

\[
\mathcal{L}^p = \bigoplus_{i' = p} \left( \bigoplus_{j,j'} \text{Hom}_{\mathcal{O}}(P_{ij}, P_{i'j'}) \otimes p_1^*\mathcal{E}_{ij} \otimes p_2^*\mathcal{E}_{i'j'} \right),
\]

\( \mathcal{L}^\bullet \) is a finite complex of locally free modules of finite rank with the universal property that for any morphism of \( k \)-schemes of the form \( \phi: X \to M \times M \), we have \( H^p(\phi^*\mathcal{L}^\bullet) = \mathcal{E}xt^p_{\mathcal{O}}(\phi^*\mathcal{P}, \phi^*\mathcal{P}) \) for all \( p \). In particular \( \mathcal{L}^\bullet \) is exact except in degrees 0 and 1. Indeed, if \( d^0: \mathcal{L}^p \to \mathcal{L}^{p+1} \) is the differential of \( \mathcal{L}^\bullet \), then \( \mathcal{L}^\bullet \) is quasi-isomorphic to the short complex

\[
0 \to \text{cok}(d^{-1}) \xrightarrow{\phi} \text{ker}(d^1) \to 0
\]

where \( \phi \) is a map between locally free sheaves whose degeneracy locus is exactly the diagonal. Our complex \( \mathcal{L}^\bullet \) now has all the essential properties of the complex \( \mathcal{C}^\bullet \) of [ES] Lemma 2.4. Hence by the same argument we have

\[
\delta = c_{\dim M}(\text{ker}(d^1)) - \text{cok}(d^{-1})) = c_{\dim M}(\sum (-1)^{p+1}[\mathcal{L}^p]).
\]

The formula [\( \mathbb{I} \)] and standard formulas for Chern classes now permit us to write \( \delta \) as a polynomial in the Chern classes of the \( p_1^*\mathcal{E}_{ij} \) and the \( p_2^*\mathcal{E}_{i'j'} \). This completes the proof of Theorem [\( \mathbb{I} \)]. \( \square \)

### 2 Iarrobino Varieties

In this section we apply the main theorem to the study of the Iarrobino varieties \( G_T \) which parametrize homogeneous ideals in \( k[x,y] \) with Hilbert function \( T \). We prove Theorem [\( \mathbb{II} \)].

**Proof of Theorem [\( \mathbb{II} \)].** Let \( S = k[x,y] \) and \( \mathcal{S} = S \otimes_k \mathcal{O}_{G_T} \). The fact that \( G_T \) is a fine moduli space means that there is a universal sheaf of homogeneous \( \mathcal{S} \)-ideals \( \mathcal{I} \) and a universal quotient sheaf of graded \( \mathcal{S} \)-modules \( \mathcal{A} = \mathcal{S}/\mathcal{I} \). According to our construction, we have \( \mathcal{A} = \bigoplus_n \mathcal{A}_n \).

We now wish to apply Theorem [\( \mathbb{I} \)] using \( \mathcal{A} \) as the universal module. To do so we need to verify the various hypotheses.
First the cohomological ones. Because we are working with graded modules, we must examine the degree 0 graded pieces of the internal Hom and Ext. If $A = S/I$ and $B = S/J$ are two graded quotients of $S$ with the same Hilbert function, then there are no nonzero morphisms of degree 0 between $A$ and $B$ unless $A = B$, in which case $\text{Hom}_S(A, A)_0 \cong k$, the homotheties. Some standard exact sequences show that $\text{Ext}^1_S(A, A)_0 \cong \text{Hom}_S(I, A)_0$ which is the tangent space to $G_T$ at $[I]$ by the graded analog of Grothendieck’s formula for the tangent space of the Hilbert scheme (cf. [PS] §4). Since $S$ is of global dimension 2, the functor $\text{Ext}^2_S$ is right exact, so the surjection $S \to B$ induces a surjection $\text{Ext}^2_S(A, S)_0 \to \text{Ext}^2_S(A, B)_0$.

But by local duality ([S] n° 72, Théorème 1) we have $\text{Ext}^2_S(A, S)_0 \cong H^0_m(A)_{-2} = A_{-2}$, which vanishes. So $\text{Ext}^2_S(A, B)_0 = 0$. Finally $\text{Ext}^2_S(A, B) = 0$ for all $p \geq 3$ because $S$ has global dimension 2. Thus the cohomological hypotheses of Theorem 1 are fulfilled.

Now we exhibit a universal projective resolution of $A$. Since $A = \bigoplus A_n$ is a $k[x, y] \otimes \mathcal{O}_{G_T}$-module, multiplication by $x$ and $y$ define morphisms $\xi, \eta: A_n \to A_{n+1}$. Then the universal projective resolution is

$$0 \to \bigoplus_n S(-n - 2) \otimes_k A_n \xrightarrow{\alpha} \bigoplus_n S(-n - 1)^3 \otimes_k A_n \xrightarrow{\beta} \bigoplus_n S(-n) \otimes_k A_n \to A \to 0$$

where the morphisms are the standard ones

$$\alpha = \begin{bmatrix} -y \otimes 1 + 1 \otimes \eta \\ x \otimes 1 - 1 \otimes \xi \end{bmatrix}, \quad \beta = \begin{bmatrix} x \otimes 1 - 1 \otimes \xi \\ y \otimes 1 - 1 \otimes \eta \end{bmatrix}.$$

Note that the sums are finite because each $A_n$ is of rank $t_n$ which vanishes for $n \gg 0$. Finally the $\text{Hom}_S(S(-i), S(-j))_0$ are all finite-dimensional. So the remaining hypotheses of Theorem 1 are fulfilled.

The theorem now follows directly from Theorem 1.

\section{Representations of Quivers}

In this section we prove Theorem 3 by applying the main theorem to fine moduli spaces of representations of quivers without oriented cycles. To do this, we start with the well-established observation (cf. for example [B]) that representations of a quiver $Q$ are the same as modules for the path algebra $kQ$. This algebra is generated over $k$ by a set of orthogonal idempotents $\{e_i \mid i \in Q_0\}$ and a further set of generators $\{x_a \mid a \in Q_1\}$ such that $x_a = e_{ha} x_a e_{ta}$. A left $kQ$-module $E$ corresponds to the representation of $Q$ consisting of the vector spaces $W_i = e_i E$ for each $i \in Q_0$, and the $k$-linear maps $\phi_a : W_{ta} \to W_{ha}$ giving multiplication by $x_a$ for each $a \in Q_1$.

The algebra $kQ$ is always hereditary, i.e. of global dimension $\leq 1$, and is finite-dimensional if and only if the quiver $Q$ has no oriented cycles.

\textbf{Proof of Theorem 3.} Let $S = kQ$. Then the indecomposable projective $S$-modules are $Se_i$ for $i \in Q_0$, and $S$ has the following minimal projective resolution as an $S, S$-bimodule (or $S \otimes S^{\text{op}}$-module)

$$0 \to \bigoplus_{a \in Q_1} Se_{ha} \otimes e_{ta} S \xrightarrow{d} \bigoplus_{i \in Q_0} Se_i \otimes e_i S \xrightarrow{\mu} S \to 0$$
where \( \mu \) is multiplication and \( d(e_{ha} \otimes e_{ta}) = x_a \otimes e_{ta} - e_{ha} \otimes x_a \). We can use this resolution to calculate the derived functors of \( \text{Hom}_S \), because \( \text{Hom}_S(E, F) = \text{Hom}_S(S, \text{Hom}_k(E, F)) \) for any \( E \) and \( F \). We see immediately see that \( \text{Ext}^i_S(E, F) = 0 \), for \( i \geq 2 \).

To check the other cohomological conditions, we first note that, by a standard Schur’s Lemma style argument, the stability condition implies that for any two \( \theta \)-stable modules \( E \) and \( F \)

\[
\text{Hom}_S(E, F) = \begin{cases} 
  k & \text{if } E \cong F, \\
  0 & \text{otherwise.}
\end{cases}
\]

Now \( M \) is constructed as a GIT quotient of the representation space

\[
\mathcal{R}(Q, \alpha) = \bigoplus_{a \in Q_1} \text{Hom}(W_{ta}, W_{ha})
\]

by the reductive group \( GL(\alpha) = \prod_{i \in Q_0} GL(W_i) \), where \( W_i \) is a fixed vector space of dimension \( \alpha_i \). If \( E = (W_i, \phi_a) \), then the tangent space \( T_E(M) \) is isomorphic to normal space at \( \phi \in \mathcal{R}(Q, \alpha) \) to the \( GL(\alpha) \)-orbit. This is the cokernel of

\[
d_\phi : \bigoplus_{i \in Q_0} \text{Hom}(W_i, W_i) \longrightarrow \bigoplus_{a \in Q_1} \text{Hom}(W_{ta}, W_{ha})
\]

where \( (d_\phi \gamma)_a = \phi_a \gamma_{ta} - \gamma_{ha} \phi_a \). But this is exactly the complex which calculates \( \text{Ext}^*_S(E, E) \), using the projective resolution of \( S \) above. Thus \( T_E(M) \cong \text{Ext}^1_S(E, E) \).

Finally, we obtain the necessary universal projective resolution by tensoring \( U \) with the projective bimodule resolution of \( S \), giving

\[
0 \rightarrow \bigoplus_{a \in Q_1} S e_{ha} \otimes U_{ta} \rightarrow \bigoplus_{i \in Q_0} S e_i \otimes U_i \rightarrow U \rightarrow 0
\]

This completes the verification of the hypotheses for Theorem \( \text{l} \) and hence the proof of Theorem \( \text{l} \).

\[
\end{proof}
\]

### 4 Sheaves on \( \mathbb{P}^2 \)

In [ES] Ellingsrud and Strømme proved that the Chow ring of a moduli space \( M \) of stable sheaves on \( \mathbb{P}^2 \) of fixed rank and Chern classes is generated by the Chern classes of three bundles on \( M \) in those cases where \( M \) is smooth and projective. We show how their result can be viewed as an application of our Theorem \( \text{l} \).

We use notation derived from a recent paper of Le Potier [L]. Let \( r, c_1, \chi, \) and \( m \) be integers such that \(-r < c_1 \leq 0, \chi \leq 0, \chi \leq r + 2c_1, \) and \( m \gg 0 \). Write \( n = -\chi + r + c_1 \). We consider representations of the quiver with triple edges labeled \( x_1, y_1, z_1 \) and \( x_2, y_2, z_2 \)

\[
\begin{align*}
\alpha : & \quad n + c_1 \\
\theta : & \quad -(r + c_1)m + n \quad (r + 2c_1)m - 2n + r \quad -c_1m + n
\end{align*}
\]

with dimension vector \( \alpha \) as marked. Those representations satisfying the symmetric relations \( x_1y_2 = y_1x_2, \ x_1z_2 = z_1x_2, \) and \( y_1z_2 = z_1x_2 \) form a closed subvariety \( \mathcal{R}^{\text{sym}}(Q, \alpha) \) of the representation space \( \mathcal{R}(Q, \alpha) \). Let \( I \) be the two-sided ideal of \( kQ \) generated by the symmetric
relations above, and let $S = kQ/I$. For any $\alpha$ (resp. $\theta$) we say that an $S$-module is of dimension vector $\alpha$ (resp. is $\theta$-stable) if it is so as a $kQ$-module. Then for any $\theta$ the image of $\mathcal{R}^{sym}(Q, \alpha)$ in the moduli space $M_Q(\alpha, \theta)$ is a fine moduli space $M_S(\alpha, \theta)$ of $\theta$-stable $S$-modules of dimension vector $\alpha$.

Le Potier has established a result (¶ Théorème 3.1) which may be interpreted in this language as saying that for $\alpha$ and $\theta$ as marked in the diagram above, $M_S(\alpha, \theta)$ is isomorphic to the moduli space $M_{\mathbb{P}^2}(r, c_1, \chi)$ of Gieseker-Maruyama stable sheaves on $\mathbb{P}^2$ of rank $r$, determinant $c_1$ and Euler characteristic $\chi$. The restrictions to $M_S(\alpha, \theta)$ of the universal bundles $U_i$ on $M_Q(\alpha, \theta)$ may be identified with the universal bundles $R^1\pi_*(\mathcal{E}(-i))$ on $M_{\mathbb{P}^2}(r, c_1, \chi)$. (Note that with these conventions, the vertices of the quiver are labeled $2, 1, 0$ from left to right.) Lemma 2.2 of [ES] may be interpreted as saying that the Ext groups for $\theta$-stable $S$-modules are isomorphic to the Ext groups for the corresponding stable sheaves on $\mathbb{P}^2$. Hence the cohomological hypotheses in our Theorem 1 can be verified for $\theta$-stable $S$-modules by using properties of stable sheaves on $\mathbb{P}^2$.

Let $\tau_i$ be the images in $S$ of the idempotents $e_i$ of $kQ$ corresponding to the three vertices $2, 1, 0 \in Q_0$ (cf. [ES]). Then the indecomposable projective $S$-modules are $S\tau_i$. As in [ES] the minimal projective resolution of $S$ as an $S, S$-bimodule yields a projective resolution of the universal $S$-module $U$ on $M_S(\alpha, \theta)$ which is now of the form

$$0 \to \bigoplus_{3 \text{ relations}} S\tau_0 \otimes U_2 \to \bigoplus_{a \in Q_1} S\tau_a \otimes U_a \to \bigoplus_{i \in Q_0} S\tau_i \otimes U_i \to U \to 0.$$ 

We could therefore apply Theorem 1 to retrieve [ES] Theorem 1.1.

Remark. It is also possible to prove Theorem 2 by regarding the Iarrobino varieties $G_T$ as moduli spaces for representations of a quiver “with relations” and thus as moduli spaces for modules over a finite-dimensional non-commutative algebra. If $T = (t_0, t_1, \ldots, t_q, 0, 0, \ldots)$, then the algebra will be of the form $R = kQ/I$ where $Q$ is the quiver

$$\alpha: \quad \bullet \rightarrow t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \cdots \rightarrow t_{q-1} \rightarrow t_q$$

$$\theta: \quad - \rightarrow + \rightarrow + \rightarrow +$$

with $2q$ edges $x_1, y_1, x_2, y_2, \ldots, x_q, y_q$, and $I$ is generated by the relations $x_iy_{i+1} - y_ix_{i+1}$. There is a clear correspondence between $R$-modules of dimension vector $\alpha$ (as marked) and graded $k[x, y]$-modules of Hilbert function $T$. When $t_0 = 1$ and the coefficients of $\theta$ have the signs indicated above, then the $\theta$-stable $R$-modules correspond exactly to $k[x, y]$-modules which are generated in degree 0, i.e. to modules isomorphic to $k[x, y]/J$ for some homogeneous ideal $J$ of $k[x, y]$. Hence $M_R(\alpha, \theta) \cong G_T$. The universal bundles $U_i$ on $M_R(\alpha, \theta)$ are exactly the universal bundles $A_i$ on $G_T$.

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