Strang splitting method for semilinear parabolic problems with inhomogeneous boundary conditions: a correction based on the flow of the nonlinearity

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Abstract

The Strang splitting method, formally of order two, can suffer from order reduction when applied to semilinear parabolic problems with inhomogeneous boundary conditions. The recent work [L. Einkemmer and A. Ostermann. Overcoming order reduction in diffusion-reaction splitting. Part 1. Dirichlet boundary conditions. SIAM J. Sci. Comput., 37, 2015. Part 2: Oblique boundary conditions, SIAM J. Sci. Comput., 38, 2016] introduces a modification of the method to avoid the reduction of order based on the nonlinearity. In this paper we introduce a new correction constructed directly from the flow of the nonlinearity and which requires no evaluation of the source term or its derivatives. The goal is twofold. One, it reduces the computational effort to construct the correction, especially if the nonlinearity is numerically heavy to compute. Second, numerical experiments suggest it is well suited in the case where the nonlinearity is stiff. We provide a convergence analysis of the method for a smooth nonlinearity and perform numerical experiments to illustrate the performances of the new approach.

Key words. Strang splitting, diffusion-reaction equation, non-homogeneous boundary conditions, order reduction, stiff nonlinearity.

AMS subject classifications. 65M12, 65L04

1 Introduction

In this paper, we consider a parabolic differential equation of the form

\[
\partial_t u = D u + f(u) \quad \text{in } \Omega, \quad B u = b \quad \text{on } \partial \Omega, \quad u(0) = u_0,
\]

(1.1)

where \( D \) is a linear diffusion operator and \( f \) is a nonlinearity. A natural method for approximating (1.1) are splitting methods. The idea is to divide the main equation (1.1) into two auxiliary subproblems (1.4) and (1.5) so one can use specific numerical methods to both subproblems to enhance the global efficiency of the computation of (1.1). Let \( N \in \mathbb{N} \) and let \( \tau = \frac{T}{N} \) be the time step. Then, one step of the classical Strang splitting is either

\[
u_{n+1} = \phi_\tau^D \circ \phi_\tau^f \circ \phi_\tau^D (u_n)
\]

(1.2)

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or alternatively
\[ u_{n+1} = \phi^{f}_{\frac{\tau}{2}} \circ \phi^{D}_{\tau} \circ \phi^{f}_{\frac{\tau}{2}}(u_n), \] (1.3)
where \( \phi^{f}_{\cdot}(u_0) \) is the flow after time \( t \) of
\[ \partial_t u = f(u), \quad u(0) = u_0, \] (1.4)
and \( \phi^{D}_{\cdot}(u_0) \) is the flow after time \( t \) of
\[ \partial_t u = Du \text{ in } \Omega, \quad Bu = b \text{ on } \partial\Omega, \quad u(0) = u_0, \] (1.5)
The Strang splitting, when applied to ODE with a sufficiently smooth solution, is a method of order two. However, when the Strang splitting is applied to solve the problem (1.1), a reduction of order can be observe in the case of non homogeneous boundary conditions as shown in [2, 3]. The reason is that \( Bu \) is not left invariant through the flow \( \phi^{D}_{\cdot} \) and therefore leaves the domain of \( D \) which creates a discontinuity at \( t = 0 \) in the flow \( \phi^{D}_{\cdot} \).
In this case the Strang splitting has in general a fractional order of convergence between one and two [3, Section 4.3]. In [2, 3], a modification of the Strang splitting is given to recover the order two. The main idea in [3] is to find a function \( q_n \) such that \( Bu \) is now left invariant by \( \phi^{f}_{\frac{\tau}{2}} - q_n \), the exact flow of
\[ \partial_t u = f(u) - q_n. \] (1.6)
One step of the modified splitting in [3] is then
\[ u_{n+1} = \phi^{D+q_n}_{\frac{\tau}{2}} \circ \phi^{-q_n}_{\frac{\tau}{2}} \circ \phi^{D+q_n}_{\frac{\tau}{2}}(u_n), \] (1.7)
where \( \phi^{D+q_n}_{\cdot} \) is the exact flow of
\[ \partial_t u = Du + q_n \text{ in } \Omega, \quad Bu = b \text{ on } \partial\Omega. \] (1.8)
Numerically, one can choose any smooth function \( q_n \) such that
\[ Bq_n = Bf(u_n) + O(\tau) \text{ on } \partial\Omega. \] (1.9)
Several options to construct \( q_n \) are presented in [1]. One challenge is then to find a correction \( q_n \) that is both cheap to compute and reduces at most the constant of error.
In this paper, we give a new modification that removes the order reduction and allows a cheaper construction of \( q_n \) in presence of a costly nonlinearity. As illustrated in the experiments, this new construction performs better for the case of a stiff reaction. The idea is to leave \( Bu \) unpreserved at the boundary through the flow \( \phi^{f}_{\frac{\tau}{2}} \) an then apply a correction \( q_n \) afterward that brings back the solution on the domain of \( D \). This new splitting is then
\[ S_{\tau}(u_n) = (\phi^{f}_{\frac{\tau}{2}} \circ \phi^{-q_n}_{\frac{\tau}{2}} \circ \phi^{D+q_n}_{\frac{\tau}{2}} \circ \phi^{-q_n}_{\frac{\tau}{2}} \circ \phi^{f}_{\frac{\tau}{2}})(u_n) \]
and the correction \( q_n \) is constructed such that
\[ Bq_n = \frac{2}{\tau}(B\phi^{f}_{\tau/2}(u_n) - Bu_n) \text{ on } \partial\Omega. \]
The correction \( q_n \) is now constructed from the output of the flow \( \phi^{f}_{\tau/2}(u_n) \) and not directly from the nonlinearity \( f \). Note that the computation of \( \phi^{-q_n}_{\frac{\tau}{2}} \) and \( q_n \) requires no evaluation of \( f \) which is particularly useful if \( f \) is computationally heavy to compute like for example when
f is costly. More importantly, in many situations, the flow $\phi^{\tau/2}_{\tau/2}(u_n)$ is smoother than the nonlinearity $f$ itself which can avoid the possible instability due to the eventual stiffness of the reaction.

In Section 2, we give the appropriate framework for the convergence analysis of this modified splitting. In Section 3, we describe the new modification we consider in this paper. In Section 4, we prove that the method is of global order two under the hypotheses made in Section 2. In Section 5, we present some numerical experiments to illustrate the performance of the new approach.

2 Analytical framework

In this section, we describe the appropriate analytical framework that we consider in this paper. We choose the framework described in [7, chapter 3]. The notations are similar to the one used in [7]. Let $\Omega \subset \mathbb{R}^d$ be a bounded connected open set with a $C^2$ boundary $\partial \Omega$. We consider the following semilinear parabolic problem on $\Omega \times [0, T]$, $T \geq 0$.

$$\partial_t u = Du + f(u) \quad \text{in } \Omega, \quad Bu = b \quad \text{on } \partial \Omega, \quad u(0) = u_0.$$ The differential operator $D$ is define by

$$D = \sum_{i,j=1}^{n} a_{ij}(x) \partial_{ij} + \sum_{i=1}^{n} a_i(x) \partial_i + a(x) I,$$

where the matrix $(a_{ij}(x)) \in \mathbb{R}^{d \times d}$ is assumed symmetric and there exists $\lambda > 0$ such that

$$\forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^n, \quad \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2,$$

and $a_{ij}, a_i, a$ are assumed continuous, $a_{i,j}, a_i, a \in C(\Omega, \mathbb{R})$. Let $B$ be the linear operator

$$B = \sum_{i=1}^{n} \beta_i(x) \partial_i + \alpha(x) I,$$

where we assume the uniform non tangentiality condition

$$\inf_{x \in \partial \Omega} \left| \sum_{i=1}^{n} \beta_i(x) \nu_i(x) \right| > 0,$$

where $\nu(x)$ is the exterior normal unit vector at $x \in \partial \Omega$. We assume that the functions $\beta_i$ and $\alpha$ are continuously differentiable, $\beta_i, \alpha \in C^1(\partial \Omega, \mathbb{R})$ and $b$ is continuously differentiable, $b \in C^1([0, T], W^{2,p}(\partial \Omega))$. We follow next the construction made in [3] to take benefit of homogeneous boundary conditions. Let $z \in C^1([0, T], W^{2,p}(\Omega))$ satisfying $Bz = b$ on $\partial \Omega$. We define $\tilde{u} = u - z$ and (1.1) becomes

$$\partial_t \tilde{u} = D\tilde{u} + f(\tilde{u} + z) + Dz - \partial_t z \quad \text{in } \Omega, \quad B\tilde{u} = 0 \quad \text{on } \partial \Omega, \quad \tilde{u}(0) = u_0 - z(0). \tag{2.1}$$

We define the linear operator $A$ as

$$Av = Dv \quad \forall v \in D(A) = \{ u \in W^{2,p}(\Omega) : Bu = 0 \text{ in } \partial \Omega \}. $$
Under those conditions, \(-A\) is a sectorial operator and therefore \(A\) is the generator of an analytic semigroup \(e^{tA}\) (see [1], chapter 3). In particular, the operator \(A\) satisfies the parabolic smoothing property that we use intensively throughout this paper,

\[
\|(−A)^αe^{tA}\| ≤ \frac{C}{t^α}\quad α ≥ 0\quad t > 0. \tag{2.2}
\]

We denote

\[
φ_1(τA) = \int_0^τ e^{(τ−s)A} \frac{1}{τ} ds.
\]

We observe that \(τAφ_1(τA) = O(1)\) is a bounded operator. We recall the following theorem, a direct consequence of [4, Theorem 8.1'], which states that there exists \(α > 0\) such that \(D((−A)^α)\) becomes free of the boundary conditions.

**Theorem 2.1.** Let \(A\) be defined as in Section 2. Then, there exist \(α > 0\) such that

\[
W^{1,p}(Ω) \subset D((−A)^α).
\]

We ask \(f\) to satisfy the following. Let \(U \subset W^{2,p}(Ω)\) be a neighborhood of the exact solution \(u\). Then we require the nonlinearity \(f\) to be twice continuously differentiable in \(U\) with values in \(W^{2,p}(Ω)\), \(f \in C^2(U,W^{2,p}(Ω))\). We refer to the discussion in [3, Section 4] for possibly relaxing the hypotheses made on \(f\). We assume that the solution \(u\) of (1.8) is twice continuously differentiable,

\[
u \in C^2([0,T],W^{2,p}(Ω)).
\]

The exact solution of (1.1) can be expressed using the variation of constant formula,

\[
\begin{align*}
u(t_{n+1}) &= z_n(τ) + e^{τA}(u(t_n) − z_n(0)) \\
&\quad + \int_0^τ e^{(τ−s)A}(f(u(t_n + s)) + Dz_n(s) − ∂t z_n(s))ds.
\end{align*}
\tag{2.3}
\]

\[
\begin{align*}
v(t_{n+1}) &= S_n(u_n) = \phi^f_τ \circ \phi^−q_n \circ \phi^{D+q_n} \circ \phi^−q_n \circ \phi^f_\frac{τ}{2}(u_n),
\end{align*}
\tag{3.1}
\]

where \(q_n\) is independent of time, in the spirit of projection methods used in the context of geometric numerical integration, see [5, Chapter IV.4]. The splitting algorithm that we propose and analyze in this paper is given by

\[
Bq_n = \frac{2}{τ}(Bφ^{f}_{τ/2}(u_n) − B(u_n)), \tag{3.2}
\]

or alternatively

\[
Bq_n = \frac{2}{τ}(Bφ^{f}_{τ/2}(u_n) − b_n), \tag{3.3}
\]
see Remark 3.2 below. In the interior of Ω, we require \( q_n \) to be in \( W^{2,p}(\Omega) \). A possibility, to construct \( q_n \) in \( \Omega \), is to choose \( q_n \) to be harmonic if this is possible or to use a smoothing iterative algorithm. For more details on how to construct the correction \( q_n \) on the interior of the domain, see \([7]\). We also assume \( (q_n)_{n\in\{0,...,N\}} \) uniformly bounded, that is there exists a constant \( C \) independent of \( n, \tau \) and \( N \), such that \( \|q_n\|_{L^p(\Omega)} \leq C \). We observe that \( \frac{2}{\tau}(B\phi^{f}_{\tau/2}(u_n) - Bu_n) \) is a finite difference approximation of \( \partial_t \phi^{f}_{\tau/2}(u_n) = f(u_n) \), hence this new condition is close to \([1.9]\).

Remark 3.1. In contrast to the correction of \([3]\), the correction \( q_n \) for the five parts modified Strang splitting \([3.1]\) is constructed directly from the flow of the nonlinearity \( \phi^{f}_{\tau} \) and not from the nonlinearity \( f \) itself. The modified splitting of \([3]\) has a good behavior when the nonlinearity \( f \) is not stiff and cheap to compute as analyzed and illustrated numerically in \([3]\). However, in the case of a stiff nonlinearity \( f \), the modification for the splitting in \([3]\) can lead to instability in contrast to \([3.1]\) as shown in the experiments (see Section 5). Furthermore, if the nonlinearity \( f \) is very costly to compute, the correction in \([3]\) requires an additional cost that can be substantial. In comparison, the construction of the correction \( q_n \) for \([3.1]\) requires no evaluation of \( f \) or its derivatives. We also observe that, in the extreme case where the diffusion \( D \) is zero, the five parts modified Strang splitting \([3.1]\) becomes exact analogously to the classical Strang splitting methods \([1.2]\) and \([1.3]\). This later property does not hold for the modified splitting in \([3]\) (note that the flows \( \phi^{f}_{\tau} \) and \( \phi^{f/q}_{\tau} \) do not commute in general).

Remark 3.2. When implementing the classical Strang splitting, it is often computationally advantageous to compose the flows \( \phi^{f}_{\tau} \) that appear in the splitting, that is \( \phi^{f}_{\tau} \circ \phi^{D}_{\tau} = \phi^{f}_{\tau} \). The numerical approximation \( u_n \) of \( u(t_n) \) is then

\[
    u_n = \phi^{f}_{\tau} \circ \phi^{D}_{\tau} \circ (\phi^{f}_{\tau} \circ \phi^{D}_{\tau})^{n-1} \circ \phi^{f}_{\tau}(u_0),
\]

which makes the classical Strang splitting having the same cost as the Lie Trotter Splitting with only one evaluation of \( \phi^{f}_{\tau} \) per time step. If we use the correction \([3.2]\), we need then to compute \( Bu_k \) and this idea does not apply since the algorithm requires \( Bu_k \). However, if we use the correction \([3.3]\) instead, we can implement the five parts modified Strang splitting \([3.1]\) as explained above for the classical Strang splitting. Note that this is an advantageous implementation that can not be used with the method presented in \([3]\).

4 Convergence analysis

We prove in this section that, using the framework and assumptions described in Section 2, the five parts modified Strang splitting method \([3.1]\) is of global order of convergence two and thus avoid order reduction phenomena.

Theorem 4.1. Under the assumptions of Section 2 the five parts modified Strang splitting \([3.1]\) satisfies the bound

\[
    \|u_n - u(t_n)\| \leq C\tau^2 \quad 0 \leq n\tau \leq T,
\]

for all \( \tau \) small enough, and where the constant \( C \) depends on \( T \) but is independent on \( \tau \) and \( n \).
We start to show the following proposition which states that the five parts splitting is at least first order convergent.

**Proposition 4.2.** Under the assumptions of Section 2 the five parts modified Strang splitting (3.1) satisfies
\[ \|u_n - u(t_n)\| \leq C\tau \quad 0 \leq n\tau \leq T, \]
for all \( \tau \) small enough, and where the constant \( C \) depends on \( T \) but is independent on \( \tau \) and \( n \).

We need this result to justify the condition (3.2) and (3.3) for the construction of \( q_n \), that is we need to show that
\[ f(u(t_n + s)) - q_n = \phi_0 + O(\tau), \]
with \( \phi_0 \in D(A) \) and \( \phi_0 = O(1) \).

The proof of Theorem 4.1 relies on Theorem 2.1 (from [4]). The proof of Theorem 2.1 uses sophisticated tools from interpolation theory. Since all the arguments in our proofs do not require any knowledges of interpolation theory, we decide to present first the proof without using Theorem 2.1 and obtain Proposition 4.3 below. We then explain how we use Theorem 2.1.

**Proposition 4.3.** Under the assumptions of Section 2 the five parts modified Strang splitting (3.1) satisfies
\[ \|u_n - u(t_n)\| \leq C\tau^2(1 + | \log \tau |) \quad 0 \leq n\tau \leq T, \]
for all \( \tau \) small enough, and where the constant \( C \) depends on \( T \) but is independent on \( \tau \) and \( n \).

For all the convergence analysis, we highlight that the asymptotic notation \( O(\tau^k) \), \( k = 0, 1, \ldots \) means that \( \|O(\tau^k)\|_{L^p(\Omega)} \leq C\tau^k \) for \( \tau \to 0 \) with \( C \) independent of \( \tau \). We always assume that \( \tau \) is small enough.

### 4.1 Quadrature error analysis

The main idea of the convergence analysis is to approximate the integrals of the form
\[ \int_0^\tau e^{(\tau-s)A}\hat{\psi}(s)ds \]
with quadrature formulas. This idea is not new in the literature and is used, for example, in [6, 3, 2]. One can not use quadrature formulas naively to such integrals since \( e^{(\tau-s)A}\hat{\psi}(s) \) is not necessarily continuous. Therefore such quadrature formulas can be less accurate than in the classical case. We show however in Lemma 4.4 below, with the help of the parabolic smoothing property, that if \( \hat{\psi} \) is close to the domain of \( A \), that is if
\[ \hat{\psi}(s) = \phi_0 + O(\tau), \quad \phi_0 \in D(A), \quad \phi_0 = O(1), \]
is satisfied, then first and second order quadrature formulas regain partially their accuracy. In [3], the authors prove this statement is true for the left rectangle quadrature formula and the midpoint rule. Since we need such results for various quadrature formulas, we prove instead it is true for a general quadrature formula since it adds no difficulties to the proof. The first of the two lemmas that follow deals with quadrature formulas of order one. The second lemma deals with quadrature formulas of order two.
Lemma 4.4. Let $\hat{\psi} : \mathbb{R} \to L^p(\Omega), 1 < p < \infty,$ be a continuously differentiable function and let $\psi(s) = e^{(\tau-s)A}\hat{\psi}(s)$. Let $Q(\psi) = \tau \sum_{k=1}^{m} b_k \psi(\tau c_k)$ be a quadrature formula that approaches the integral $\int_0^\tau \psi(s)ds$ such that $\sum_{k=1}^{m} b_k = 1$.

Then the quadrature error $E$ satisfies

$$E := \int_0^\tau \psi(s)ds - \tau \sum_{k=1}^{m} b_k \psi(\tau c_k) = O(\tau).$$

If additionally $\hat{\psi}(s) = \phi_0 + O(\tau)$ with $\phi_0 \in D(A)$ and $\phi_0 = O(1)$, then

$$E = O(\tau^2).$$

Proof. Since $\psi$ is uniformly bounded on $[0, \tau]$, the first result follows. Let us assume that the condition (4.1) is satisfied.

Let us compute the first derivative of $\psi$,

$$\psi'(s) = -A e^{(\tau-s)A}\hat{\psi}(s) + e^{(\tau-s)A}\hat{\psi}'(s).$$

Let $Q_l(\psi) = \tau \psi(0)$, be the left rectangle quadrature formula. We prove that every first order quadrature formula $Q$ satisfies

$$Q_l(\psi) - Q(\psi) = O(\tau^2). \quad (4.2)$$

First, we observe that $\|\tau A \varphi_1(\tau A)\hat{\psi}(0)\| = O(\tau)$. Indeed

$$\|\tau A \varphi_1(\tau A)\hat{\psi}(0)\| \leq \tau \|\varphi_1(\tau A)\| \|A\phi_0\| + \|\tau A \varphi_1(\tau A)\| \|O(\tau)\| \leq C\tau.$$ 

We extend $\psi(c\tau)$ around 0,

$$\psi(0) = e^{\tau A}\hat{\psi}(0) = \hat{\psi}(0) + \tau A \varphi_1(\tau A)\hat{\psi}(0) = \hat{\psi}(c\tau) + O(\tau) = \psi(c\tau) + O(\tau),$$

which proves (4.2) since

$$\tau \psi(0) - \tau \sum_{k=0}^{m} b_k \psi(\tau c_k) = \tau \psi(0) - \tau \psi(0) \sum_{k=1}^{m} b_k + O(\tau^2) = O(\tau^2).$$

Therefore, we only need to show that

$$\int_0^\tau \psi(s)ds - Q_l(\psi) = O(\tau^2). \quad (4.3)$$

We write the quadrature error as follows using the Peano kernel representation of the error for a first order quadrature formula,

$$\int_0^\tau \psi(s)ds - Q(\psi) = \tau^2 \int_0^{1} (1-s)\psi'(\tau s)ds - \tau^2 \sum_{i=1}^{m} b_i \int_0^{c_i} \psi'(\tau s)ds,$$

which gives for $Q_l$,

$$\int_0^\tau \psi(s)ds - Q_l(\psi) = \tau^2 \int_0^{1} (1-s)\psi'(\tau s)ds.$$
We need to bound the integral $\int_0^1 (1 - s) \psi'(\tau s)ds$. We first bound $\| - A e^{(1-s)A} \hat{\psi}(\tau s) \|$. We get
\[
\| - A e^{(1-s)A} \hat{\psi}(\tau s) \| \leq \| e^{(1-s)A} \| - A \phi_0 \| + \| - A e^{(1-s)A} \| \| O(\tau) \| \leq C + \frac{\tau C}{\tau(1-s)}.
\]
Therefore
\[
\| \psi'(\tau s) \| = \| - A e^{(1-s)A} \hat{\psi}(\tau s) + e^{(1-s)A} \hat{\psi}(\tau s) \| \leq C(1 + \frac{1}{1-s}).
\]
We can now compute the error of the quadrature formula $Q_\tau$ and show (4.3). This follows from the inequality
\[
\left\| \tau^2 \int_0^1 (1 - s) \psi'(\tau s)ds \right\| \leq \tau^2 \int_0^1 (1 - s)C(1 + \frac{1}{1-s})ds = C\tau^2.
\]
Which gives us, with (4.2), the desired result for any first order quadrature formula,
\[
\int_0^\tau \psi(s)ds - Q(\psi) = O(\tau^2),
\]
which concludes the proof of Lemma 4.4.

**Lemma 4.5.** Let $Q$ and $\psi$ be as in Lemma 4.4. We assume that $\hat{\psi}$ is twice continuously differentiable and that $Q$ is a second order quadrature formula ($\sum_{k=1}^m c_kb_k = \frac{1}{2}$). Then
\[
E = AO(\tau^2) + O(\tau^2). \tag{4.4}
\]
If, in addition, $\hat{\psi}(s) = \phi_0 + O(\tau)$ with $\phi_0 \in D(A)$ and $\phi_0 = O(1)$, then
\[
E = AO(\tau^3) + O(\tau^3). \tag{4.5}
\]

**Proof.** The second derivative of $\psi$ is
\[
\psi''(s) = A^2 e^{(\tau-s)A} \hat{\psi}(s) - 2A e^{(\tau-s)A} \hat{\psi}'(s) + e^{(\tau-s)A} \hat{\psi}''(s).
\]
If $Q$ is a second order quadrature formula we write the quadrature error as follows using the Peano kernel representation of the error for a second order quadrature formula.
\[
\int_0^\tau \psi(s)ds - Q(\psi) = \tau^3 \int_0^1 (1 - s)^2(1 - 2\psi''(\tau s)ds - \tau^3 \sum_{i=1}^m b_i \int_0^{c_i} (c_i - s)\psi''(\tau s)ds.
\]
It remains to estimate
\[
P_1(\tau s) = A e^{(1-s)A} \hat{\psi}(\tau s) - 2e^{(1-s)A} \hat{\psi}'(\tau s)
\]
and
\[
P_2(\tau s) = e^{(1-s)A} \hat{\psi}''(\tau s).
\]
We first bound $\|A e^{(1-s)A} \hat{\psi}(\tau s)\|$. We get
\[
\left\| A e^{(1-s)A} \hat{\psi}(\tau s) \right\| \leq \left\| A e^{(1-s)A} \right\| \left\| \hat{\psi}(\tau s) \right\| \leq C \frac{1}{\tau(1-s)}.
\]
Then
\[ \| P_1(\tau s) \| = \left\| A e^{\tau(1-s)A} \hat{\psi}(\tau s) - 2e^{\tau(1-s)A} \hat{\psi}'(\tau s) \right\| \leq C(1 + \frac{1}{\tau(1-s)}) \] (4.6)
and
\[ \| P_2(\tau s) \| = \left\| e^{\tau(1-s)A} \hat{\psi}''(s) \right\| \leq \left\| e^{\tau(1-s)A} \right\| \left\| \hat{\psi}''(s) \right\| \leq C. \]

This gives the following estimation for the integrals \( \int_0^1 \frac{(1-s)^2}{2} P_1(\tau s) ds, \)
\[ \left\| \int_0^1 \frac{(1-s)^2}{2} P_1(\tau s) ds \right\| \leq C \int_0^1 \frac{(1-s)^2}{2} (1 + \frac{1}{\tau(1-s)}) ds \leq \frac{C}{\tau}, \]
\[ \left\| \int_0^1 \frac{(1-s)^2}{2} P_2(\tau s) ds \right\| \leq \int_0^1 \frac{(1-s)^2}{2} C ds \leq C. \]

We show that \( \| \sum_{i=1}^m b_i \int_0^{c_i} (c_i - s) P_1(\tau s) ds \| \leq \frac{C}{\tau}. \)
\[ \left\| \sum_{i=1}^m b_i \int_0^{c_i} (c_i - s) P_1(\tau s) ds \right\| \leq \sum_{i=1}^m b_i \int_0^{c_i} (c_i - s) \| P_1(\tau s) \| ds \]
\[ \leq C \sum_{i=1}^m b_i c_i + C \sum_{i=1}^m b_i \int_0^{c_i} (c_i - s) \frac{1}{\tau(1-s)} ds. \]

If \( c_i = 1, \) we have
\[ \int_0^1 \frac{1}{\tau(1-s)} ds = \int_0^1 \frac{1}{\tau} ds = \frac{C}{\tau}. \]
If \( c_i \neq 1, \) then
\[ \int_0^{c_i} (c_i - s) \frac{1}{\tau(1-s)} ds = \frac{C}{\tau}. \]
Therefore
\[ \left\| \sum_{i=1}^m b_i \int_0^{c_i} (c_i - s) P_1(\tau s) ds \right\| \leq \frac{C}{\tau}. \]

For the integral of \( P_2, \) we obtain
\[ \left\| \sum_{i=1}^m b_i \int_0^{c_i} (c_i - s) P_2(\tau s) ds \right\| \leq \sum_{i=1}^m b_i \int_0^{c_i} (c_i - s) \| P_2(\tau s) \| ds \leq C, \]
which gives the desired bound for the error,
\[ \int_0^\tau \psi(s) ds - Q(\psi) = A\mathcal{O}(\tau^2) + \mathcal{O}(\tau^2). \]

If condition (4.1) is satisfied we can obtain a better bound for \( \| A e^{\tau(1-s)A} \hat{\psi}(\tau s) \| \) and thus also for \( \| P_1(\tau s) \|, \)
\[ \| A e^{\tau(1-s)A} \hat{\psi}(\tau s) \| \leq \| e^{\tau(1-s)A} \| \| A \phi_0 \| + \| A e^{(1-s)A} \| \| \mathcal{O}(\tau) \| \leq C + C\tau \frac{1}{\tau(1-s)}. \]
We obtain the following estimation for $P_1(\tau s)$,

$$
\|P_1(\tau s)\| = \|Ae^{\tau (1-s)A}\hat{\psi}(\tau s) - 2e^{\tau (1-s)A}\hat{\psi}'(\tau s)\| \leq C(1 + \frac{1}{1 - s}), \quad (4.7)
$$

which gives us the estimation $\|\sum_{i=1}^{m} b_i \int_0^{c_i} (c_i - s) P_1(\tau s) ds\| \leq C$. Finally, we have the desired error bound

$$
\int_0^{\tau} \psi(s) ds - \tau Q_1(\psi(\tau s)) = A\mathcal{O}(\tau^3) + \mathcal{O}(\tau^3),
$$

which concludes the proof of Lemma 4.5.

Using Theorem 2.1, we can improve Lemma 4.5 as follows.

**Lemma 4.6.** Under the hypotheses of Lemma 4.5, there exists $\alpha > 0$ such that

$$
\int_0^{\tau} \psi(s) ds - \tau Q(\psi(\tau s)) = (-A)^{1-\alpha}\mathcal{O}(\tau^2) + \mathcal{O}(\tau^2).
$$

If additionally condition (4.1) is satisfied, then

$$
\int_0^{\tau} \psi(s) ds - \tau Q(\psi(\tau s)) = (-A)^{1-\alpha}\mathcal{O}(\tau^3) + \mathcal{O}(\tau^3).
$$

**Proof.** We use Theorem 2.1 which states that for sufficiently small $\alpha > 0$, $W^{1,p}(\Omega)$ is included in the domain of $(-A)^\alpha$ which does not involve any condition on the boundary. One then obtains

$$
\|(A)^{1+\alpha}e^{\tau (1-s)A}\hat{\psi}(\tau s)\| \leq \|Ae^{\tau (1-s)A}\|\|(-A)^\alpha\hat{\psi}(\tau s)\| \leq C \frac{1}{\tau(1 - s)}
$$

and

$$
\|2(-A)^\alpha e^{\tau (1-s)A}\hat{\psi}'(\tau s)\| \leq \|2e^{\tau (1-s)A}\|\|(-A)^\alpha\hat{\psi}'(\tau s)\| \leq C
$$

which gives

$$
\|P_1(\tau s)\| = \|(A)^{1+\alpha}e^{\tau (1-s)A}\hat{\psi}(\tau s) - 2(-A)^\alpha e^{\tau (1-s)A}\hat{\psi}'(\tau s)\| \leq C(1 + \frac{1}{\tau(1 - s)})
$$

instead of (4.6). Similarly, if condition (4.1) is satisfied, one obtains

$$
\|P_1(\tau s)\| = \|(A)^{1+\alpha}e^{\tau (1-s)A}\hat{\psi}(\tau s) - 2(-A)^\alpha e^{\tau (1-s)A}\hat{\psi}'(\tau s)\| \leq C(1 + \frac{1}{1 - s})
$$

instead of (4.7), and this concludes the proof. □
4.2 Order one error estimate for the five parts Strang splitting

In this section, we prove that the five parts modified splitting (3.1) is of global order one because this is needed in the proof of the global order of the method. We start to give two estimations for the local error. To perform our convergence analysis, we need an exact formula for (3.1). We expand each flow that appears in the Strang splitting,

\[ w_n = \phi_{\frac{\tau}{2}}(u(t_n)) = u(t_n) + \int_{0}^{\tau/2} f(\phi_{s}^{\frac{\tau}{2}}(u(t_n))) ds, \]

\[ \tilde{w}_n = \phi_{\frac{\tau}{2}}^{-q_n}(w_n) = u(t_n) + \int_{0}^{\tau/2} f(\phi_{s}^{\frac{\tau}{2}}(u(t_n))) - q_n ds, \]

\[ v_n = \phi_{\frac{\tau}{2}}^{D+q_n}(\tilde{w}_n) = z_n(\tau) + e^{\tau A}(\tilde{w}_n - z_n(0)) + \int_{0}^{\tau} e^{(\tau-s)A}(q_n + Dz_n(s) - \partial_t z_n(s)) ds, \]

\[ \tilde{v}_n = \phi_{\frac{\tau}{2}}^{-q_n}(v_n) = v_n - \int_{0}^{\tau/2} q_n ds, \]

\[ u_{n+1} = \phi_{\frac{\tau}{2}}^{f}(\tilde{v}_n) = v_n + \int_{0}^{\tau/2} f(\phi_{s}^{\frac{\tau}{2}}(\tilde{v}_n)) - q_n ds. \]

We obtain the following exact formula for the numerical flow,

\[ S_{\tau}(u(t_n)) = z_n(\tau) + e^{\tau A} \left( u(t_n) + \int_{0}^{\tau/2} \left( f(\phi_{s}^{\frac{\tau}{2}}(u(t_n))) - q_n \right) ds - z_n(0) \right) \]

\[ + \int_{0}^{\tau} e^{(\tau-s)A}(q_n + Dz_n(s) - \partial_t z_n(s)) ds + \int_{0}^{\tau/2} \left( f(\phi_{s}^{f}(\tilde{v}_n)) - q_n \right) ds. \quad (4.8) \]

We define the local error at time \( t_{n+1}, \delta_{n+1}, \) as follows,

\[ \delta_{n+1} := S_{\tau}u(t_n) - u(t_{n+1}). \]

Using the formula (2.3) of the exact solution and formula (4.8) of the numerical solution we obtain

\[ \delta_{n+1} = e^{\tau A} \frac{1}{2} \int_{0}^{\tau} f(\phi_{s/2}^{f}(u(t_n))) - q_n ds \]

\[ + \int_{0}^{\tau} e^{(\tau-s)A}(q_n - f(u(t_n + s))) ds + \frac{1}{2} \int_{0}^{\tau} f(\phi_{s/2}^{f}(\tilde{v}_n)) - q_n ds. \quad (4.9) \]

Since all the integrands are uniformly bounded on \([0, \tau]\), we obtain the following result, which states that the five parts modified Strang splitting (3.1) is locally of first order.

**Lemma 4.7.** Under the assumption of Section 2, the five parts modified Strang splitting (3.1) satisfies the following local error estimate.

\[ \delta_{n+1} = O(\tau). \]

We prove the next local error estimate we use in the theorem for the global error.

**Lemma 4.8.** Under the assumption of Section 2, the five parts modified Strang splitting (3.1) satisfies the following local error estimate.

\[ \delta_{n+1} = A O(\tau^2) + O(\tau^2). \]
Proof. In formula (4.9) of the local error, we use the trapezoidal quadrature formula to approximate the integrals. By Lemma 4.5, the quadrature error made to approximate \( \int_0^t f(\phi^f_{\tau/2}(u(t_n))) - q_n \, ds \) is equal to \( A\mathcal{O}(\tau^2) + \mathcal{O}(\tau^2) \). We get

\[
\delta_{n+1} = e^{rA} \frac{T}{4} \left( f(u(t_n)) - q_n + f(\phi^f_{\tau/2}(u(t_n))) - q_n \right)
+ \frac{T}{2} \left( e^{rA}(q_n - f(u(t_n))) + q_n - f(u(t_n + \tau)) \right)
+ \frac{T}{4} \left( f(\tilde{v}_n) - q_n + f(\phi^f_{\tau/2}(\tilde{v}_n)) - q_n \right) + A\mathcal{O}(\tau^2) + \mathcal{O}(\tau^2).
\]

Since \( \phi^f_{\tau/2}(u(t_n)) = u(t_n) + \mathcal{O}(\tau), f(u(t_n + \tau)) = f(u(t_n)) + \mathcal{O}(\tau) \) and \( \tilde{v}_n = u(t_n) + \mathcal{O}(\tau) \), and expending \( e^{rA} = Id + \tau A \varphi_1(\tau A) \), we obtain

\[
\delta_{n+1} = \frac{T}{2} ( f(u(t_n)) - q_n ) + (q_n - f(u(t_n))) + \frac{T}{2} ( f(u(t_n)) - q_n )
+ \frac{T^2}{2} A \varphi_1(\tau A) ( f(u(t_n)) - q_n ) + \frac{T^2}{2} A \varphi_1(\tau A) ( q_n - f(u(t_n)) ) + A\mathcal{O}(\tau^2) + \mathcal{O}(\tau^2)
= A\mathcal{O}(\tau^2) + \mathcal{O}(\tau^2),
\]
which concludes the proof. \( \square \)

Using Theorem 2.1 and Lemma 4.6 we can improve Lemma 4.8 as follows.

Lemma 4.9. Under the assumption of Section 2, the five parts modified Strang splitting (3.1) satisfies the following. There exist \( \alpha > 0 \) such that

\[
\delta_{n+1} = (-A)^{1-\alpha} \mathcal{O}(\tau^2) + \mathcal{O}(\tau^2).
\]

Proof. Using Lemma 4.6, we can obtain that the quadrature error made to approximate \( \int_0^t f(\phi^f_{\tau/2}(u(t_n))) - q_n \, ds \) is equal to \( (-A)^{1-\alpha} \mathcal{O}(\tau^2) + \mathcal{O}(\tau^2) \). We then use Theorem 2.1 to bound \( \frac{T^2}{2} A \varphi_1(\tau A) ( f(u(t_n)) - q_n ) + \frac{T^2}{2} A \varphi_1(\tau A) ( q_n - f(u(t_n)) ) \). We obtain

\[
\| A \varphi_1(\tau A) ( f(u(t_n)) - q_n ) \| \leq \| (-A)^{1-\alpha} \varphi_1(\tau A) \| \| (-A)^{\alpha} ( f(u(t_n)) - q_n ) \|.
\]
This gives us the desired result. \( \square \)

Using the previous results for the local error, we can now prove the following order estimation for the global error.

Proposition 4.10. Under the assumption of Section 2, the five parts modified Strang splitting (3.1) satisfies

\[
\| u_n - u(t_n) \| \leq C \tau (1 + | \log \tau |) \quad 0 \leq n \tau \leq T.
\]

The constant \( C \) depends on \( T \) but is independent on \( \tau \) and \( n \).
Proof. The global error is defined as \( e_n = u_n - u(t_n) \).

\[
e_{n+1} = S_r u_n - u(t_{n+1}) = S_r u_n - S_r u(t_n) + S_r u(t_n) - u(t_{n+1}) = S_r u_n - S_r u(t_n) + \delta_{n+1}.
\]

Using the exact formula \( (4.8) \) for \( S_r u_n \) and \( S_r u(t_n) \), we obtain for \( e_{n+1} \),

\[
e_{n+1} = e^{\tau A} e_n + E(u(t_n), u_n) + \delta_{n+1},
\]

with

\[
E(u(t_n), u_n) = e^{\tau A} \int_0^\tau (f(\phi^\tau_s (u(t_n))) - f(\phi^\tau_s (u_n))) ds \\
+ \int_0^\tau f(\phi^\tau_s \circ \phi^{-q_n} \circ \phi^{D+q_n}_r \circ \phi^{-q_n}_r \circ \phi^\tau_s (u(t_n))) ds \\
- \int_0^\tau f(\phi^\tau_s \circ \phi^{-q_n} \circ \phi^{D+q_n}_r \circ \phi^{-q_n}_r \circ \phi^\tau_s (u_n)) ds.
\] (4.10)

Let us bound \( E(u(t_n), u_n) \). We use the Lipschitz continuity of \( f \) and \( \phi^\tau_s \). For the first integral in (4.10), we have

\[
\| \int_0^\tau (f(\phi^\tau_s (u(t_n))) - f(\phi^\tau_s (u_n))) ds \| \leq C \tau \| e_n \|.
\]

For the second integral that appears in (4.10), we observe that

\[
\phi^{q_n} s (u) - \phi^{q_n} s (v) = u - sq_n - v + sq_n = u - v.
\]

We obtain

\[
\| \int_0^\tau f(\phi^\tau_s \circ \phi^{D+q_n}_r \circ \phi^{-q_n}_r \circ \phi^\tau_s (u(t_n))) - f(\phi^\tau_s \circ \phi^{D+q_n}_r \circ \phi^{-q_n}_r \circ \phi^\tau_s (u_n)) ds \| \leq \\
\leq C \tau \| \phi^{D+q_n}_r \circ \phi^{-q_n}_r \circ \phi^\tau_s (u(t_n)) - \phi^{D+q_n}_r \circ \phi^{-q_n}_r \circ \phi^\tau_s (u_n) \|.
\]

Writing the exact formula for \( \phi^{D+q_n}_r \circ \phi^{-q_n}_r \circ \phi^\tau_s \), we have

\[
C \tau \| \phi^{D+q_n}_r \circ \phi^{-q_n}_r \circ \phi^\tau_s (u(t_n)) - \phi^{D+q_n}_r \circ \phi^{-q_n}_r \circ \phi^\tau_s (u_n) \| \\
= C \tau \| e^{\tau A} (u(t_n) - u_n) + e^{\tau A} \int_0^{\tau/2} f(\phi^\tau_s (u(t_n))) - f(\phi^\tau_s (u_n)) ds \| \\
\leq C \tau \| e_n \|.
\]

Therefore

\[
\| E(u(t_n), u_n) \| \leq C \tau \| e_n \|.
\]

The global error \( e_n \) satisfies the following recursive formula,

\[
e_n = e^{n \tau A} e_0 + \sum_{k=0}^{n-1} e^{(n-k-1) \tau A} \delta_{k+1} + \sum_{k=0}^{n-1} e^{(n-k-1) \tau A} E(u_k, \tilde{u}_k).
\]
This gives us, thanks to previous estimation for \( \|E(u_k, \tilde{u}_k)\| \) and \( \|\delta_k\| \) and since \( \|e_0\| = 0 \),

\[
\|e_n\| \leq \|e^{n\tau A}\||e_0\| + \sum_{k=0}^{n-1} \|e^{(n-k-1)\tau A}\|\|\delta_{k+1}\| + \sum_{k=0}^{n-1} \|e^{(n-k-1)\tau A}\||E(u_k, \tilde{u}_k)\|
\]

\[
\leq \sum_{k=0}^{n-2} \|e^{(n-k-1)\tau A}(A\mathcal{O}(\tau^2) + \mathcal{O}(\tau^2))\| + \|\delta_n\| + C\tau \sum_{k=0}^{n-1} \|e_k\|.
\]

Since, by Lemma 4.7, \( \|\delta_n\| = \mathcal{O}(\tau) \) and using the parabolic smoothing property, we get

\[
\|e_n\| \leq C\tau^2 \sum_{k=0}^{n-2} \frac{1}{(n-k-1)\tau} + nC\tau^2 + C\tau + C\tau \sum_{k=0}^{n-1} \|e_k\|.
\]

We rearrange the second sum and observe that \( nC\tau^2 = C\tau \), which gives

\[
\|e_n\| \leq C\tau \sum_{k=0}^{n-1} \|e_k\| + C\tau^2 \sum_{k=1}^{n-1} \frac{1}{k\tau} + C\tau.
\]

The second sum can be bounded as

\[
C\tau^2 \sum_{k=1}^{n-1} \frac{1}{k\tau} \leq C\tau \int_{\tau}^{n\tau} \frac{1}{x} \,dx \leq C\tau \int_{\tau}^{T} \frac{1}{x} \,dx \leq C\tau(1 + |\log(\tau)|).
\]

Using the discrete Gronwall’s lemma and we obtain the desired result.

\[
\|e_n\| \leq C\tau(1 + |\log(\tau)|) e^{(n-1)C\tau} \leq C\tau(1 + |\log(\tau)|) e^{CT} = C\tau(1 + |\log(\tau)|),
\]

which concludes the proof. □

Using Theorem 2.1 and Proposition 4.10 we are now in position to prove Proposition 4.2, which provides a first order estimation for the global error.

Proof of Proposition 4.2 We use Lemma 4.6 to remove the term \( \log(\tau) \) in the global error estimate. Indeed, we obtain,

\[
\|e_n\| \leq C\tau \sum_{k=0}^{n-1} \|e_k\| + C\tau^2 \sum_{k=1}^{n-1} \frac{1}{k\tau^{1-\alpha}} + C\tau.
\]

We then estimate

\[
C\tau^2 \sum_{k=1}^{n-1} \frac{1}{k\tau^{1-\alpha}} \leq C\tau \int_{\tau}^{n\tau} \frac{1}{x^{1-\alpha}} \,dx \leq C\tau \int_{\tau}^{T} \frac{1}{x^{1-\alpha}} \,dx = C\tau(T^{\alpha} - \tau^{\alpha}) \leq C\tau,
\]

which concludes the proof. □
Remark 4.11. In Lemma 4.12, to show that a function $q_n$ satisfying the boundary condition (3.2) or (3.3) satisfies the condition (4.1), we need to use Proposition 4.2 which we prove using Theorem 2.1. To prove $\|u_n - u(t_n)\| \leq C\tau^2(1 + |\log \tau|)$ without using Theorem 2.1, we need a weaker condition that $q_n$ must satisfy, for example $f(u(t_n + s)) - q_n = \phi_0 + O(\tau(1 + |\log(\tau)|))$ with $\phi_0 \in D(A)$ and $\phi_0 = O(1)$, instead of (4.1). We can then prove, with the help of the bound, $\|u_n - u(t_n)\| \leq C\tau(1 + |\log \tau|)$, that a function satisfying (3.2) satisfies this new condition. We decide not to follow this approach as we think it simplifies our arguments to only have condition (4.1) throughout the paper.

4.3 Analysis of the corrector function

We show that the conditions (3.2) and (3.3) for $q_n$ are properly chosen, that is $\hat{\psi}(s) = q_n - f(u(t_n + s))$ satisfies the hypothesis (4.1) when one of those conditions is satisfied. For that purpose, we use Proposition 4.2 which states that $u(t_n) - u_n = O(\tau)$. We stress out that no condition on $q_n$ is required in the proof of Proposition 4.2. We first consider the boundary condition (3.2) for $q_n$.

Lemma 4.12. Let $q_n$ be chosen such that (3.2) is satisfied,

$$Bq_n = \frac{2}{\tau}(B\phi_{\tau/2}(u_n) - B(u_n)).$$

Let $\hat{\psi}(s) = q_n - f(u(t_n + s))$. Then $\hat{\psi}(s) = \phi_0 + O(\tau)$ with $\phi_0 \in D(A)$ and $\phi_0 = O(1)$.

Proof. We observe that

$$f(u_n) = \frac{2}{\tau}\phi_{\tau/2}(u_n) - \frac{2}{\tau}u_n + O(\tau).$$

We obtain, with Proposition 4.2 that

$$f(u(t_n + s)) = f(u_n) + O(\tau) = \frac{2}{\tau}\phi_{\tau/2}(u_n) - \frac{2}{\tau}u_n + O(\tau).$$

We define $\phi_0$ as follows,

$$\phi_0 = q_n - \frac{2}{\tau}\phi_{\tau/2}(u_n) - \frac{2}{\tau}u_n.$$

Since $B(q_n - (\frac{2}{\tau}\phi_{\tau/2}(u_n) - \frac{2}{\tau}u_n)) = 0$, $\phi_0$ is in $D(A)$. Furthermore

$$\|\phi_0\| = \|q_n - f(u(t_n + s)) + O(\tau)\| \leq C + C\tau.$$

Therefore $\phi_0 = O(1)$.

We now consider the boundary condition (3.3) for the corrector functions.

Lemma 4.13. Let $q_n$ be chosen such that (3.3) is satisfied,

$$Bq_n = \frac{2}{\tau}(B\phi_{\tau/2}(u_n) - b_n).$$

Let $\hat{\psi}(s) = q_n - f(u(t_n + s))$. Then $\hat{\psi}(s) = \phi_n + O(\tau)$, with $\phi_n \in D(A)$ and $\phi_n = O(1)$.
Proof. The proof is conducted by induction. Since $Bu_0 = b_0$, we know by Lemma 4.12 that the result is true for $n = 0$.

We assume that the statement is true for $n = 0, \ldots, k - 1$. Let us show that it is true for $n = k$. We write the exact formula for $\phi_{\frac{\tau}{2}}^k(u_k)$ in function of $v_{k-1}$ and $q_k$.

$$\phi_{\frac{\tau}{2}}^k(u_k) = v_{k-1} - \frac{\tau}{2}q_{k-1} + \int_0^\tau f(\phi_{\frac{\tau}{2}}^k(v_{k-1} - q_{k-1}))ds.$$ 

We then apply the midpoint quadrature formula to the integral and obtain an error of size $O(\tau^2)$ since $f$ is twice continuously differentiable.

$$\phi_{\frac{\tau}{2}}^k(u_k) = v_{k-1} + \tau f(\phi_{\frac{\tau}{2}}^k(v_{k-1} - q_{k-1})) - \frac{\tau}{2}q_{k-1} + O(\tau^2).$$

We observe that $\phi_{\frac{\tau}{2}}^k(v_{k-1} - q_{k-1}) = u_k$. Since by Proposition 4.2, $u_k = f(u(t_k)) + O(\tau)$ and since $f(u(t_k)) = f(u(t_{k-1})) + O(\tau)$, we obtain

$$\phi_{\frac{\tau}{2}}^k(u_k) = v_{k-1} + \frac{\tau}{2}f(u(t_k)) + \frac{\tau}{2}f(u(t_{k-1})) - \frac{\tau}{2}q_{k-1} + O(\tau).$$

Since $Bv_{k-1} = b_k$, and since by hypothesis $f(u(t_{k-1})) - q_{k-1} = \phi_{k-1} + O(\tau)$ with $\phi_{k-1} \in D(A)$ and $\phi_{k-1} = O(1)$, we obtain

$$Bq_k = 2\tau(Bv_{k-1} - b_k) + Bf(u(t_k)) + Bf(u(t_{k-1})) - Bq_{k-1} + BO(\tau) = Bf(u(t_k)) + BO(\tau).$$

We can therefore decompose $q_k$ as $q_k = \tilde{q}_k + r_k$, where $B\tilde{q}_k = \frac{2\tau}{\tau}Bf(u(t_k))$ and $r_k = O(\tau)$ and choose $\phi_k = \tilde{q}_k - f(u(t_k))$. Then $B\phi_k = 0$ and $\|\phi_k\| \leq \|f(u(t_k + s))\| + \|q_k\| + \|r_k\| \leq C$. This concludes the proof. \hfill $\square$

### 4.4 Order two error estimate for the five steps modified Strang Splitting

The following lemma is an estimation of the local error for the modified Strang splitting which states that the five parts modified Strang splitting (3.1) is locally a method of second order.

**Lemma 4.14.** Under the assumption of Section 2, the five parts modified Strang splitting (3.1) satisfies

$$\delta_{n+1} = O(\tau^2).$$

**Proof.** In the exact formula of $\delta_{n+1}$ (4.9), we use the left rectangle quadrature formula for the first and third integral and a right rectangle formula for the second integral. By Lemma 4.3, the quadrature error is $O(\tau^2)$. We get

$$\delta_{n+1} = e^{\tau A} \frac{\tau}{2}(f(u(t_n)) - q_n) + \tau (q_n - f(u(t_n + \tau))) + \frac{\tau}{2}(f(\bar{u}_n) - q_n) + O(\tau^2).$$

Expanding $e^{\tau A}$ and $u(t_n)$ around $\tau$, that is $e^{\tau A} \frac{\tau}{2}(f(u(t_n)) - q_n) = \frac{\tau}{2}(f(u(t_{n+1})) - q_n) + O(\tau^2)$, we obtain the following result,

$$\delta_{n+1} = \frac{\tau}{2}(f(\bar{u}_n) - f(u(t_{n+1})) + O(\tau^2).$$
Under the assumption of Section 2, the five parts modified Strang split-
for the global convergence of the method.

error made to approximate
approximate the integrals in formula (4.9) of the local error. By Lemma 4.5, the quadrature
Proof. We start as in the proof of Lemma 4.8, by using trapezoidal quadrature formulas to

The following lemma gives the second local error estimates that we need in the proof
for the global convergence of the method.

Lemma 4.15. Under the assumption of Section 2, the five parts modified Strang splitting (3.1) satisfies
\[ \delta_{n+1} = A\mathcal{O}(\tau^3) + \mathcal{O}(\tau^3). \]

Proof. We start as in the proof of Lemma 4.3 by using trapezoidal quadrature formulas to
approximate the integrals in formula (4.9) of the local error. By Lemma 4.5, the quadrature
error made to approximate \( \int_0^\tau f(\phi^{f}_{s/2}(u(t))) - q_n ds \) is equal to \( A\mathcal{O}(\tau^3) + \mathcal{O}(\tau^3) \). We get

\[
\delta_{n+1} = e^{\mathcal{A}\tau} \left( f(u(t_n)) - q_n + f(\phi^{f}_{\tau/2}(u(t))) - q_n \right) + \frac{\tau}{2} e^{\mathcal{A}\tau} \left( q_n - f(u(t_n)) + q_n - f(u(t_n + \tau)) \right) + \frac{\tau}{4} \left( f(\bar{v}_n) - q_n + f(\phi^{f}_{\tau/2}(\bar{v}_n)) - q_n \right) + A\mathcal{O}(\tau^3) + \mathcal{O}(\tau^3)
\]

By Lemma 4.14 we have the following equality.

\[
f(\phi^{f}_{\tau/2}(\bar{v}_n)) = f(S_{\tau}(u(t_n))) = f(u(t_n + \tau)) + \mathcal{O}(\tau^2).
\]

We obtain

\[
\delta_{n+1} = e^{\mathcal{A}\tau} \left( f(\phi^{f}_{\tau/2}(u(t))) - q_n \right) - \frac{\tau}{4} e^{\mathcal{A}\tau} \left( f(u(t_n)) - q_n \right) + \frac{\tau}{4} \left( f(\bar{v}_n) - q_n \right) - \frac{\tau}{4} \left( f(u(t_n + \tau)) - q_n \right) + A\mathcal{O}(\tau^3) + \mathcal{O}(\tau^3).
\]

We observe that \( \frac{\tau}{2} e^{\mathcal{A}\tau} \left( f(u(t_n)) - q_n \right) + \frac{\tau}{2} \left( f(u(t_n + \tau)) - q_n \right) \) is the trapezoidal quadrature formula for \( \int_0^\tau e^{(\tau-s)A}(f(u(t_n+s)) - q_n) ds \) and that \( e^{\mathcal{A}\tau} \left( f(\phi^{f}_{\tau/2}(u(t))) - q_n \right) + \frac{\tau}{2} \left( f(\bar{v}_n) - q_n \right) \) is the trapezoidal quadrature formula for \( \int_0^\tau e^{(\tau-s)A}(f(\phi^{f}_{\tau/2}(u(t))) - q_n) + \frac{\tau}{2} \left( f(\bar{v}_n) - q_n \right) \) is the trapezoidal quadrature formula for \( \int_0^\tau e^{(\tau-s)A}(f(\phi^{f}_{\tau/2}(u(t))) - q_n) + \frac{\tau}{2} \left( f(\bar{v}_n) - q_n \right) \) is the trapezoidal quadrature formula for \( \int_0^\tau e^{(\tau-s)A}(f(\phi^{f}_{\tau/2}(u(t))) - q_n) + \frac{\tau}{2} \left( f(\bar{v}_n) - q_n \right) \) is the trapezoidal quadrature formula for \( \int_0^\tau e^{(\tau-s)A}(f(\phi^{f}_{\tau/2}(u(t))) - q_n) + \frac{\tau}{2} \left( f(\bar{v}_n) - q_n \right) \) is the trapezoidal quadrature formula for \( \int_0^\tau e^{(\tau-s)A}(f(\phi^{f}_{\tau/2}(u(t))) - q_n) + \frac{\tau}{2} \left( f(\bar{v}_n) - q_n \right) \) is the trapezoidal quadrature formula for the second integrand satisfies the hypotheses of Lemma 4.3 by Lemma 4.12 and Lemma 4.13. Applying Lemma 4.5, we therefore have a quadrature error of the form \( A\mathcal{O}(\tau^3) + \mathcal{O}(\tau^3) \).

\[
\begin{align*}
\delta_{n+1} &= \frac{1}{2} \int_0^\tau e^{(\tau-s)A}(f(\phi^{f}_{\tau/2} \circ \phi^{q}_{s/2} \circ \phi^{D+q}_{s} \circ \phi^{q}_{s/2} \circ \phi^{f}_{1/2}(u(t))) - q_n) ds \\
&\quad - \frac{1}{2} \int_0^\tau e^{(\tau-s)A}(f(u(t_n + s))) - q_n) ds + A\mathcal{O}(\tau^3) + \mathcal{O}(\tau^3).
\end{align*}
\]
Applying the midpoint quadrature method to both integrals, we obtain

\[ \delta_{n+1} = \tau \frac{1}{4} e^{\tau A} (f(S_{\tau}^n(u(t_n))) - q_n) - \tau \frac{1}{4} e^{\tau A} (f(u(t_n + \tau/2)) - q_n) + A\mathcal{O}(\tau^3) + \mathcal{O}(\tau^3). \]

Using Lemma 4.14, the Lipschitz continuity of \( f \) and the boundedness of \( e^{\tau A} \) we have the desired result,

\[ \delta_{n+1} = A\mathcal{O}(\tau^3) + \mathcal{O}(\tau^3), \]

which concludes the proof.

We now improve Lemma 4.15 as follows using Lemma 4.6 instead of Lemma 4.5 for the quadrature error.

**Lemma 4.16.** Under the assumption of Section 2, the five parts modified Strang splitting (3.1) satisfies the following. There exists \( \alpha > 0 \) such that

\[ \delta_{n+1} = (-A)^{1-\alpha} \mathcal{O}(\tau^3) + \mathcal{O}(\tau^3). \]

Using the previous results for the local error, we can prove Proposition 4.3. It is the main global error estimate that we obtain without using Theorem 2.1.

**Proof of Proposition 4.3.** The proof is similar to the proof of the Proposition 4.10 except for the bounds of the local errors \( \delta_n \), since if (3.2) is satisfied, \( \delta_k = A\mathcal{O}(\tau^3) + \mathcal{O}(\tau^3) \) and \( \delta_n = \mathcal{O}(\tau^2) \). Hence

\[ \|e_n\| \leq C\tau^2(1 + |\log(\tau)|)e^{C\tau(1 + |\log(\tau)|)} = C\tau^2(1 + |\log(\tau)|), \]

which concludes the proof.

**Proof of Theorem 4.7.** We follow the proof of Proposition 4.2 using Lemma 4.16 to remove the \( \log(\tau) \) term in the global error estimate of Proposition 4.3.

5 Numerical experiments

In this section we perform several numerical experiments to illustrate the performance of the five parts modified Strang splitting (3.1) when applied to diffusion problems with various nonlinearities. The norm we use to compute the numerical error is

\[ \|u\|_{L^\infty([0,T], L^2(\Omega))} = \sup_{t \in [0,T]} \|u(t)\|_{L^2(\Omega)}. \]
Figure 1: Comparison between the Classical and modified Strang splitting with 3 and 5 steps, denoted respectively Strang, StrangM3 and StrangM5, when applied to equation $\partial_t u = \partial_{xx} u + mu^2$ for $m = 1$ and $m = 5$ on $[0,1]$ with inhomogeneous mixed boundary conditions. StrangM5a and StrangM5b denote respectively the use of correction $Bq_n = \frac{2}{\tau} (B\phi_{\tau/2}(u_n) - B(u_n))$ and $Bq_n = \frac{2}{\tau} (B\phi_{\tau/2}(u_n) - b_n)$, whose convergence curves are superpose. Reference slopes one and two are given in dotted lines.

A quadratic nonlinearity In the following experiment, we compare the five parts modified Strang splitting (3.1) with the modified splitting proposed in [2, 3]. We first consider a problem given in [3, Example 5.2]. We then change the nonlinearity to see how both methods behave. The non linearities we consider are $f(u) = u^2$ and $f(u) = 5u^2$. The case $f(u) = u^2$ is the one presented in [3, Example 5.2]. We perform the experiment with mixed boundary conditions, $u(0) = 1$, $\partial_n u(1) = 1$. We choose a smooth initial condition that satisfies the prescribed boundary conditions. We obtain the following equation with $m = 1$ and $m = 5$.

$$\partial_t u(x,t) = \partial_{xx} u(x,t) + mu(x,t)^2,$$

$$u(0, t) = 1, \quad \partial_n u(1, t) = 1,$$

$$u(x, 0) = 1 + \frac{2}{\pi} - \frac{2}{\pi} \cos \left( \frac{1}{2} \pi x \right). \quad (5.1)$$

The correction we use for the modified Strang splitting in [3] is $q_n = m + 2m xu_n(1)$. We use 500 spatial points to discretize the interior of $\Omega$. We compute the solution at final time $t = 0.1$. The chosen time steps are $\tau = 0.02 \cdot 2^{-k}$, $k = 0, \ldots, 6$. The reference solution is computed with the five parts modified Strang splitting (3.1) and a time step $\tau = 0.02 \cdot 2^{-9}$. We use the exact solutions of $\phi_{\tau/2}^f(u_n)$ and $\phi_{\tau/2}^{f-q}(u_n)$ in the splitting algorithms. We observe that the five parts modified Strang splitting (3.1) is of order two. It has a slightly worse constant of error compared to the modified Strang splitting given in [3] for $f(u) = u^2$. However, for $f(u) = 5u^2$, the five parts modified Strang splitting (3.1) becomes more accurate.

A meteorology model with an integral source term We apply the five parts modified splitting (3.1) and the modified Strang splitting in [3] to a problem presented in [5]...
Figure 2: Comparison between the classical Strang splitting, the modified Strang splitting [3] and the splitting (3.1), denoted respectively Strang, StrangM3 and StrangM5 when applied to the integro-differential equation $\partial_t u(x, t) = \partial_{xx} u(x, t) - \int_0^1 u(s, t) \frac{1}{(1 + |x-s|)^2} ds$ with time dependent boundary conditions. StrangM5a and StrangM5b denote respectively the use of correction $Bq_n = \frac{2}{\tau}(B\phi_{\tau/2}^f(u_n) - B(u_n))$ and $Bq_n = \frac{2}{\tau}(B\phi_{\tau/2}^f(u_n) - b_n)$, whose convergence curves are superpose. Reference slope ones and two are drawn in dotted line. The numerical solution for $\tau = 0.02 \cdot 2^{-8}$ is displayed on the right picture.

Equation 3.1, where we replace the left Dirichlet boundary condition $u(0, t) = 2(2 - \sqrt{t})$ by $u(0, t) = 2(2 - t)$ to have a time continuously differentiable boundary condition. The considered differential equation is the following

$$
\partial_t u(x, t) = \partial_{xx} u(x, t) - \int_0^1 u(s, t) \frac{1}{(1 + |x-s|)^2} ds,
$$

$$
u(0, t) = 2(2 - t), \quad \partial_n u(1, t) = 0,
$$

$$
u(x, 0) = 2(\cos(\pi x) + 1), \quad (5.2)
$$

for $\Omega = [0, 1]$. We choose 500 points to discretize the interior of $[0, 1]$. We then apply the splitting methods with different time steps $\tau = 2 \cdot 10^{-2} \cdot 2^{-k}$, $k = 0, \ldots, 6$. A reference solution is computed with the splitting (3.1) for $\tau = 0.02 \cdot 2^{-8}$. To solve the integral, we use the trapezoidal quadrature formula with the 502 nodes given by the space discretization. To solve $\partial_t u = f(u)$ and $\partial_t u = f(u) - q_n$, we use the classical order four explicit Runge-Kutta method with time step $\tau$. We compute the solution at final time $t = 0.1$. We observe that the modified splitting methods given in [3] has a better constant of error. Since the nonlinearity is non local and since condition (3.2) and (3.3) give the same constant of error, it is advantageous to use condition (3.3) and the construction explained in Remark 3.2.

In this case the conditions for the five parts modified Strang splitting (3.1) require less computational cost since in the modification given in [3], one has to evaluate $f$ on the boundary at each step of the algorithm.

**Case of a stiff nonlinearity** In the following experiments, we compare the five parts modified splitting (3.1) with the modified splitting given in [2] and [3] when applied to a stiff problem. We choose the nonlinearities $f(u) = (1 - \sin(\pi x)) u^2$ and $f(u) = (1 -
Nonstiff case $f(u) = (1 - \sin(\pi x))u^2$.

Stiff case $f(u) = (1 - 100\sin(\pi x))u^2$.

Figure 3: Comparison between the Classical and modified Strang splitting with 3 and 5 steps, denoted respectively Strang, StrangM3 and StrangM5, when applied to equation $\partial_t u = \partial_{xx} u + (1 - M \sin(\pi x))u^2$, $M = 1$ and $M = 100$, on $[0, 1]$ with inhomogeneous mixed boundary conditions. StrangM5a and StrangM5b denote respectively the use of correction $Bq_n = \frac{2}{\tau}(B\phi^f_{\tau/2}(u_n) - B(u_n))$ and $Bq_n = \frac{2}{\tau}(B\phi^f_{\tau/2}(u_n) - b_n)$, whose convergence curves are superpose for $M = 1$. Reference slopes one and two are given in dotted lines.

We perform the experiment with mixed boundary conditions, $u(0) = 1$, $\partial_n u(1) = 1$. We choose a smooth initial condition that satisfies the prescribed boundary conditions. We obtain the following equation

$$
\partial_t u(x, t) = \partial_{xx} u(x, t) + (1 - M \sin(\pi x))u(x, t)^2,
$$

$$
u(0, t) = 1, \quad \partial_n u(1, t) = 1,$$

$$
u(x, 0) = 1 + \frac{2}{\pi} - \frac{2}{\pi} \cos\left(\frac{1}{2} \pi x\right),$$

with $M = 1$ or $M = 100$. The correction function we use for the modified Strang splitting in $[3]$ is $q_n = u_n(0)^2 + (M \pi u_n(1)^2 + 2u_n(1))x$. We recall that the method in $[3]$ was proposed for nonstiff nonlinearities; indeed note that $q_n$ becomes large for the stiff case $M = 100$.

We use 500 points to discretize the interior of $\Omega$. We compute the solution at final time $t = 0.1$. The chosen time steps are $\tau = 0.05 \cdot 2^{-k}$, $k = 0, \ldots, 7$. The reference solution is computed with the splitting (3.1) and a time step $\tau = 0.05 \cdot 2^{-10}$. We use the exact solution for the flow of the nonlinearity $\phi^f_{\tau/2}(u_n)$ and $\phi^f_{\tau/2}(u_n)$. We observe that the modified Strang splitting in $[3]$ is slightly more accurate than the five parts modified Strang splitting (3.1) when $M = 1$ is chosen. However, when $M = 100$, the splitting in $[3]$ becomes worse than the classical splitting for the chosen time steps. In comparison the five parts modified Strang splitting remains more accurate than the classical Strang splitting for $\tau \leq 0.05 \cdot 2^{-4}$.

For large enough time steps, we observe that the classical Splitting is the most accurate of the methods and that the condition (3.2) gives a better constant of error compare to the condition (3.3).

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