Slow crack propagation through a disordered medium: Critical transition and dissipation

G. Pontuale1, F. Colaiori2,3 and A. Petri1,3

1 CNR - Istituto dei Sistemi Complessi - via del Fosso del Cavaliere 100, 00133 Roma, Italy, EU
2 CNR - Istituto dei Sistemi Complessi, Dipartimento di Fisica, Sapienza Università, P.le A. Moro 5, 00189 Roma, Italy, EU
3 Dipartimento di Fisica, Sapienza Università - P.le A. Moro 5, 00189 Roma, Italy, EU

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Abstract – We show that the intermittent and self-similar fluctuations displayed by a slow crack during the propagation in a heterogeneous medium can be quantitatively described by an extension of a classical statistical model for fracture. The model yields the correct dynamical and morphological scaling, and allows to demonstrate that the scale invariance originates from the presence of a non-equilibrium, reversible, critical transition which, in the presence of dissipation, gives rise to self-organized critical behaviour.

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Introduction. – The failure of materials is a complex and complicated process exhibiting broad phenomenology. The fracture of heterogeneous media under slow external loading displays intermittent dynamics and scale invariance, features observed in different phenomena involving a huge range of length and time scales, from nano-plasticity [1] and micro-fractures [2–5] to earthquakes [6–8]. In the propagation of a slow crack, the front advances through a movement that is statistically stationary but characterized by sudden and intermittent self-similar bursts, or avalanches. Bursts area and duration are power law distributed, and the crack profile is self-affine (see, e.g., [9,10] and references therein). Besides having practical relevance, this phenomenology, known as “crackling noise” [11], characterizes also the dynamics of other interfacial phenomena in disordered media, like imbibition, wetting, friction, and hysteresis in ferromagnets.

In analogy with equilibrium phenomena, the absence of typical scales is considered a mark of some underlying critical transition [12]. This conjecture has led in the last decades to the formulation of simplified statistical models for fracture [2,13–15] with the aim of understanding the origin of scale invariance, and identifying possible universalities in different systems. However, it is diffuse opinion [16,17] that these models have until now failed in catching what are thought to be the right power laws characterizing both the failure of various materials and the seismic behaviour accompanying earthquakes. Extant models yield exponents significantly higher than the experimental observations [16], and make the right values “a target for theoretical models” [14]. Moreover, they fail in reproducing the microscopic statistics in quasi-stationary situations [14], making the link with a reversible transition problematic.

On general grounds, the observation of scale invariance in many natural phenomena has led to the concept of Self-Organized Criticality (SOC) [18,19], in which a system spontaneously sets close to a critical point, in contrast with ordinary critical phenomena where some parameter needs to be finely adjusted. Attempts have been made to relate the self-similar fluctuations observed in fracture phenomena with SOC dynamics [3,20,21], and cellular automata have been devised with the aim of reproducing the power laws observed in fractures and earthquakes [8,19,22]. SOC models have been very helpful and intuitive in illustrating the general properties of the critical behaviour; however their matching with real systems is still difficult in relation to several points, in particular the correspondence between model parameters and real ones [23]. So the critical transition beneath the fracture phenomenology has remained elusive, even if more evidences have been recently emphasized [16].

In this paper we show for the first time that the critical behaviour characterizing the slow fracture and crack propagation in a heterogeneous medium originates indeed
from the existence of a non-equilibrium critical phase transition, separating two distinct phases: an active phase, in which the fracture propagates indefinitely, and a dormant one, in which the system is quiescent. We shall introduce an extension of a widely employed statistical model, described in the next section, and use it to reproduce the propagation of a planar crack. It will be seen in the following sections that this model yields intermittent and self-similar dynamics in quantitative agreement with the dynamical and morphological scaling measured in recent experiments [16,24] and in simulations [9,25]. Moreover, a same universal scaling is found to characterize the avalanche area, stress, and energy. The nature of the critical transition will be discussed in a separate section. Finally, we will show that dissipation is crucial in generating the observed intermittency by setting the systems at the edge of criticality, and connecting the process to the SOC phenomenology.

The model. – The model we adopt is an extension of the classical Fiber Bundle Model (FBM), which in its original formulation [26] consists of a set of N parallel fibers, each one having a random, quenched, breaking strength drawn from some identical probability distribution. Applying a slowly increasing load parallel to the fibers axis, more and more fibers break and redistribute their stress among those still intact, which in turn can break, generating avalanches in a domino effect. The probability distribution for the number of fibers broken in an avalanche is a power law, with a cut-off at large values. The dynamics is non-stationary, as the number of intact fibers decreases with increasing load, and at a critical stress the system undergoes global and irreversible failure. While in its original formulation FBM is a mean-field model [27,28], several variants have been also devised to take into account dimensionality and different stress redistribution rules among fibers [14,15,29–33].

In order to describe the propagation of a crack front during its quasi-stationary regime, we have complemented the classical FBM with two rules: 1) Fiber regeneration: each time a fiber breaks, it is replaced by another one, with new random breaking strength and zero initial stress. 2) Energy dissipation: when a fiber breaks, part of its elastic energy is dissipated. Thus only a fraction of its stress is transferred to other fibers, while a complementary fraction is lost.

These rules characterize the Dissipative Regenerating Fiber Bundle Model (DRFBM) investigated in the present paper, and are motivated by the following considerations: 1) Regeneration. The replacement of broken with intact fibers mimics the crack propagation, in which regions previously hedged from the stress get involved as the overloaded zones yield. The propagation of a planar crack, orthogonal to the fibers, can thus be described by a 1d version of the model. During each avalanche broken fibers are replaced by new ones, each failure event representing an onwards local motion of the crack, as will be explained in detail in the section dedicated to the roughness.

2) Dissipation. The dissipation of elastic energy in real fractures may happen by several mechanisms: in Griffith’s description of a perfectly brittle medium it is due to the cost of opening the crack. In the general case other processes can contribute, such as the emission of elastic waves and the plastic deformation. Whereas the introduction of dissipation in the standard FBM would just cause a slow-down of the dynamics, and a delay of the final breakdown, we will show that in the DRFBM it is crucial for the onset of intermittence in the motion of the crack front.

We consider here a bundle of harmonic fibers with identical unitary elastic modulus, each one having a random, quenched, breaking threshold t extracted from an identical probability density p(t). The dynamics of the DRFBM can be summarized as follows: starting from zero initial stress, all the fibers are subjected to a same slow increase of strain u, until the weakest fiber breaks and it is replaced by an intact fiber having zero stress and a new random threshold. Soon after, a fraction (1 − δ) of the released stress is redistributed within the bundle according to some rule, while the remaining fraction δ is lost. The redistributed stress may cause the breaking of one or more other fibers, which in turn will redistribute part of their stress, and so on. Different fibers broken from a same redistribution process are all regenerated at the same time. In the model we assume a separation between the time scales of the internal avalanches dynamics and the driving field rate, keeping the external strain fixed during avalanches. An avalanche stops when the stress transferred from the broken fibers causes no further failures. At this point the bundle is subjected to a new strain increase until a fiber breaks and so on, and the dynamics evolves under such an adiabatic driving. It is worth noting that each fiber bears a history-dependent stress, resulting from summing, from the epoch of its generation, the internally redistributed stress and the externally applied strain.

The propagation of a planar crack. – Here we consider the DRFBM model in d = 1, with the fibers placed on a line and subjected to periodic boundary conditions. Each site is identified by its discrete coordinate x_i, to which it is associated a set of successive fibers i_α. Each time a fiber i_α breaks it is replaced by a new one, i_{α+1}, and the crack front moves onwards. According to the rules that will be specified in the section devoted to the roughness, this gives rise to an irregular crack front profile, like the one sketched in the inset of fig. 1. Since only one fiber at time is present at a given site, we shall from now on identify the current intact fiber with only the site index i and the related stress with s_i, unless otherwise specified.

In a crack front each point generates stress on other points proportionally to the inverse square of the relative distance [34,35]. Therefore we adopt a distance-dependent stress redistribution rule [32] such that if the fiber at x_i breaks under the stress s_i, the stress of the fiber at x_j,
increases by an amount
\[ \Delta s_j = \kappa \frac{\tilde{s}_i}{(x_i - x_j)^2} \text{ for } j \neq i \text{ and } \Delta s_i = 0, \]
where \( x_i - x_j \) is taken modulo \( N \), and \( \kappa \) is a \( \delta \)-dependent normalization factor assuring that a fraction \( \delta \) of \( \tilde{s}_i \) is dissipated: \( \sum_j \Delta s_j = (1 - \delta)\tilde{s}_i \).

Figure 1 shows the behaviour of the 1d DRFBM at increasing applied strain, obtained from the simulation of a system with \( N = 10^4 \), \( \delta = 10^{-3} \), and random fiber thresholds \( t \) extracted uniformly in \([0, 1]\). The lower curve represents the energy of avalanches \( E = \sum_j \tilde{s}_j^2 / 2 \), where the sum is over all the failures in the avalanche. We observe that at low strain there are only a few rare and small failure events, but eventually a large avalanche starts, leading the systems to a state where avalanches of any size occur. At that stage each increase in stress due to the externally applied strain balances in average the dissipated stress, and the total bundle stress \( \sigma = \sum_i \tilde{s}_i / N \) reaches a statistically stationary value \( \langle \sigma \rangle \), as shown by the upper curve in the figure.

We have extensively investigated the 1d DRFBM with different values of dissipation \( \delta \) and size \( N \). For \( \delta \neq 0 \), under increasing strain the system reaches a state in which the bundle stress \( \sigma \) is statistically stationary. The case \( \delta = 0 \) is special and will be discussed in a devoted section. Besides the energy \( E \), we have computed several quantities for each avalanche, and their probability distributions: the area \( A \), defined as the total number of failed fibers; the duration \( T \), as the number of regeneration processes occurring within the avalanche, counting as single process the simultaneous regeneration of more fibers; the stress \( S \), as the sum of the stress born by all the fibers broken in the avalanche. In addition, since isolated avalanche clusters can take place at the same time in different regions of the crack front [16,24,25], we have also considered the statistics of the area \( C \) of isolated clusters, i.e. of locally connected but mutually disconnected failures in the same avalanche.

It is seen that the bursts display self-similar features in all the quantities considered. For large enough avalanches the distribution for each quantity, \( y \), is well described by
\[ p(y) \simeq y^{-\tau_y} f_y(y/y_o). \]
The exponents \( \tau_y \) do not depend on the dissipation rate \( \delta \), which instead determines the cut-off values \( y_o \). By assuming a power law dependence, \( y_o \simeq \delta^{-\psi_y} \), the distributions of a same quantity for different \( \delta \) can be collapsed onto a unique curve. Figure 2 shows, on double logarithmic scale, the collapsed probability distributions of a) the cluster avalanche area \( C \), b) the total avalanche area \( A \), c) the avalanche duration \( T \), as obtained from numerical simulations of the 1d DRFBM with \( N = 10^4 \) and for different values of \( \delta \).

By means of the data collapse we have got as best estimates for the exponents: \( \tau_A = 1.2 \) and \( \tau_C = 1.5 \) for the total and cluster area, respectively, and \( \tau_T = 1.5 \) for the duration. These values, resumed in the table 1, strictly agree with those found in the experiments [9,24], in linear fracture models [16] with following rediscussions [9,10,25], and in depinning models (see, e.g., [16] and references therein). The rescaling cut-off exponents \( \psi_y \) are also reported in the table 1. Since it is expected that the

![Fig. 1: (Colour on-line) Avalanche energy (lower curve) and instantaneous bundle stress (upper curve) in the DRFBM as functions of the applied external strain, from numerical simulations with \( N = 10^4 \) and \( \delta = 10^{-3} \). Inset: a snapshot of the crack front obtained by the 1d DRFBM, as described in the section discussing the roughness.](image)

![Fig. 2: (Colour on-line) Collapsed probability distributions for the avalanches in the planar crack front dynamics as resulting from the DRFBM, relative to: a) cluster avalanche area \( C \); b) total area \( A \); c) duration \( T \). Relevant exponents are reported in table 1.](image)
average avalanche obeys the scaling $(y) \approx \delta^{-\psi_y}$, the exponents must be consistent with the relation $\phi_y = \psi_y (2 - \tau_y)$, that is indeed the case.

We have also computed the distribution of the energy $E$ and stress $S$, for both total and cluster avalanches, and we have found that they are described by the same exponents of the respective avalanche areas $\tau_C$ and $\tau_s$, indicating that the scaling of the size fluctuations is a universal feature, irrespective of the measured quantity.

Finally, we have checked that all results do not depend on the probability distribution $p(t)$ from which the fiber threshold strengths are extracted, provided that this vanishes rapidly enough as $t \to \infty$. Finite-size effects are not generally displayed as they appear only for very small values of $N \cdot \delta$.

**The roughness of the front.** – The self-similar features of crack profiles have been subject of study since when they were first observed in the surface of fractured metals [36]. A wide amount of work has then been devoted to characterizing different morphological scalings, clarifying their origin and looking for classes of universality [10,37]. It also required time to explain some apparent contradictions between different experiments, and discrepancies between some experimental results and models for elastic depinning (see, e.g., [10,38]), but it seems now rather well established that different scaling regimes can be attributed to the interplay between elasticity, disorder and possible spatial correlations in the disorder strength [39,40].

In order to define the position of the crack front in the 1d DRFBM, we assume that each time a fiber yields, the crack advances locally—perpendicularly to the fibers and to the $x$-axis—a distance proportional to the stress $\delta$ stored in the broken fiber. Since fibers are harmonic, the total perpendicular displacement at $x_i$ is given by $z_i = \sum_\alpha \delta \alpha_i$, where $\alpha_i$ counts all the consecutive failures occurred at site $i$ up to the current epoch.

The scaling properties of the roughness can be evaluated through different quantities. Here we compute

$$R_\ell^2 = \langle (z_{i+\ell} - z_i)^2 \rangle$$

that is expected to scale as $\ell^{2\zeta}$. For the 1d DRFBM we find a non-trivial power law behaviour of $R_\ell$. Figure 3 shows the values of $R_\ell$ as function of $\ell$ for uniform initial distribution of the fiber strength, $N = 2^{10}$ and different dissipations $\delta$. The crack profile is self-similar in a wide range and it can be noticed that even in this case the dissipation plays the fundamental role of setting the upper cut-off $\ell_o$ of the scaling range. Different curves corresponding to different values of dissipation can in fact be collapsed assuming the scaling $(2)$ with a dependence of the form $\ell_o \approx \delta^{-\psi_y}$. A best fit yields $\zeta = 0.35$ in agreement with those found in experiments and simulations for uncorrelated and not too strong disorder [39,40]. We also find $\psi_y = 1.00$.

![Fig. 3: (Colour on-line) Scaling of the crack front roughness $R$ in the 1d DRFBM. Curves from systems with different dissipation have been collapsed by assuming the same dependence of the upper cut-off on $\delta$. The dashed line has slope $\zeta = 0.35$.](Image)

Very recently, the transition between two different scaling regimes has been also observed in a different FBM based model [41] in which the fibers interact through a 2d kernel. The model shows that, as the Young modulus increases, the front becomes rougher and overhangs and damage islands appear, the exponent in the soft elasticity regime being in agreement with that observed in the DRFBM.

**The critical transition and the effect of the dissipation.** – As discussed in the introduction, a major point is to establish whether a critical transition is at the origin of the scale-free fluctuations observed in the propagation of the crack front. In fact, linking irreversible intermittent phenomena to a reversible transition may be of help in the identification of different universality classes and of key working mechanisms.

In a phase transition, a system must exist in at least two different states separated by a critical point. In order to address this issue we have investigated the system behaviour with null dissipation ($\delta = 0$). Note that in this case all the energy fed remains in the system, and the bundle stress $\sigma = \sum_i s_i$ equals the totally applied strain.

As usual, an external strain was supplied in a quasi-static mode, starting with zero initial stress. The $\delta = 0$ case displayed initially a transient phase similar to the case $\delta \neq 0$. As the bundle stress was increased, larger and larger avalanches appeared, until eventually the bundle stress $\sigma$ reached a maximum (as in fig. 1 for $\delta \neq 0$) and a very large avalanche was triggered. While in the case $\delta \neq 0$ this peak is necessarily followed by a decay and $\sigma$ sets at a lower stationary value, in the $\delta = 0$ case the domino effect provokes a neverending avalanche, and the system remains indefinitely in the active state, the redistributed stress bouncing among the fibers. Hence, for null dissipation,
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The fraction of active fibers $\rho$ suddenly departs from zero above a critical value of the bundle stress $\sigma$. The variance $\sigma^2_c$ shows a peak that increases with the system size.

The system is driven to a critical point where the crack propagation sustains indefinitely, whilst in the presence of dissipation, however small, it goes back below the critical point, and only new a stress increase will trigger more avalanches.

We have simulated the 1d DRFBM with no dissipation and different sizes, and computed the fraction of active fibers $\rho$, i.e. the average number of fibers simultaneously breaking during avalanches, at varying bundle stress. The results are shown in fig. 4, where it is seen that close to a critical value $\sigma_c$, the activity suddenly departs from zero and monotonically increases. We have also computed the variance of $\rho$, that is proportional to the system susceptibility. It is seen to display a peak close to $\sigma_c$, and to increase linearly with the system size, as expected from finite-size scaling in $d = 1$.

These features are the hallmark of a critical transition and show that for $\delta \neq 0$ the system is at the lower edge of criticality. In fact, in this case, the peak stress attained before the stationary state, visible in fig. 1, corresponds to the critical value $\sigma_c$.

Finite-size scaling study and detailed analysis of this transition will be reported elsewhere. We stress here that the resulting picture is the one described by Dickman et al. [12] for the onset of Self-Organized Criticality (SOC), where the system oscillates continuously between an active phase, to which it is driven by the external field, and an absorbing phase where it falls because of the dissipation. This demonstrates in a clear and definite way that crack propagation is a realization of SOC, as also recently conjectured [16].

Dissipation, that has been treated as an irrelevant parameter in linear elastic fracture models [16,25,35] and as a useful regularizing factor in depinning models [42], turns out to be the key factor for the onset of intermittence, establishing a clear link between fracture processes and SOC. In the DRFBM it occurs in the bulk, at variance with usual Self-Organized-Critical (SOC) models (e.g., sandpiles) where it is limited to the boundaries in order to obtain the cut-off to disappear when taking the thermodynamic limit. As a matter of fact, internal dissipation is hardly unavoidable in real macroscopic systems, and there is a growing evidence that it governs real manifestations of SOC [16,43–45]. It does not affect the exponents and may well be at the origin of the large scale cut-off always present in real systems, in alternative, or in addition, to finite-size effects. Finally, it is also noticeable that, to our knowledge, a diverging susceptibility had never been observed in SOC models, whereas it characterizes natural phenomena [45].

Discussion and summary. – We have described the propagation of a slow crack front in heterogeneous materials by means of a stationary Fiber Bundle Model which includes internal dissipation and fiber regeneration (DRFBM). In the absence of dissipation the system can be in either of two distinct phases, one active and the other quiescent, separated by a critical value of the system stress at which the susceptibility diverges. The presence of dissipation, however small, drives an active system to a slightly subcritical state whereas, by supplying stress, the system can be driven from the quiescent to the active phase. The alternance of these two processes leads the system to a statistically stationary state where the dynamics is scale invariant and intermittent.

The 1d DRFBM with the stress redistribution rule eq. (1) yields the correct experimental scaling of both dynamical and morphological fluctuations of the planar crack. Universal statistics characterizes the burst area, energy and stress, whereas dissipation sets the upper cut-off to the self-similar range.

The DRFBM establishes a link between SOC models and real systems and can be easily extended to describe systems in higher dimension, like non-planar cracks and earthquakes, or can be employed for implementing pre-existing models [46]. It also presents interesting analogies with other breaking-healing contact models, recently introduced to describe tribological experiments, earthquakes [47] and biological systems [48], that would deserve to be investigated.

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