First-order perturbative approach to Q-balls with massive gauge field

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1. Introduction

Nontopological solitons [1] or Q-balls [2] represent stationary localized solutions in flat space possessing a finite mass. In contrast to topological solitons, nontopological solitons are stable not because of topological charge, but due to a globally conserved Noether charge that comes from a continuous symmetry existing in the system. In the simplest case of nontopological solitons, Q-balls consist of complex scalar field that has suitable self-interaction. The global phase invariance of scalar field theory is connected to Noether charge, symbolized by Q, corresponding to particle number of the Q-balls. Generally when the gravity is invoked, the Q-balls solutions will be boson stars (BSs) solutions [3–5]. The scalar field on Q-balls actually can be a model to describe dark matter [7] or on BSs to describe dark energy (DE) [6, 14].

These Q-balls solutions appear with possessing a global continuous $U(1)$ symmetry. When the $U(1)$ symmetry is local or gauged, the product between Noether charge and the constant defining the coupling between scalar field and the gauge one can be interpreted as the physical charge of the solitons [4, 8] but for convenience we refer to [1, 5, 9, 10] that there is no product with coupling constant to obtain the charge. For the non-gauged Q-balls, the solutions exist only in certain frequency range, i.e., $\omega_{\text{min}} < \omega < \omega_{\text{max}}$ determined by the properties of the potential [1]. At the highest critical value of the frequency, both mass and charge of the Q-balls are assumed in their maximum values. If we go across the limit of $\omega$, indeed, we will see that the solutions have some nodes.

It is clear that, for the small gauge field on the scalar field, the properties of these Q-balls solutions will not differ considerably from the non-gauged Q-balls. So, the interesting case happens if the gauge field considerably affects the properties of the Q-balls. In general case, the gauge field can possess mass hence the gauge invariance is not satisfied [11]. In this paper, we...
examine gauged Q-balls solutions with no self-interacting term in potential because we assume
the interaction that holds the Q-balls to be stable comes from interaction between scalar field
and gauge field. In addition, we add the term of mass of the gauge field. We find the solutions by
using perturbative approach where the coupling constant \( q \) is used as a perturbative parameter,
and we just examine to the first-order solutions. Next, we find the mass-charge relation for this
solution and compare it with the massless case.

The paper is organized as follows. In Section 2, we present the general setup and introduce
the notations that will be used throughout the paper. In Section 3, we present analytically
perturbative approach results of the field solutions related to (massive) gauged Q-balls. In
Section 4, we examine the solutions, pressures, mass and charge of the Q-balls. The results are
discussed in the last section.

2. General setup and Lagrangian
We describe Q-balls that consist of scalar field \( \Phi \) and gauge field \( A_{\alpha} \) with the Lagrangian
density, in natural units with \( \hbar = c = 1 \), as
\[
\mathcal{L} = g^{\alpha\beta}(D_{\alpha}\Phi)^{*}(D_{\beta}\Phi) - m_{b}^{2}\Phi^{*}\Phi - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}\mu^{2}A_{\alpha}A^{\alpha}.
\]  
(1)
Here \( m_{b} \) is the scalar field mass, \( \mu \) is the gauge field mass, \( D_{\alpha} = \partial_{\alpha} + iqA_{\alpha} \), where \( q \) is coupling
constant of the interaction strength and \( F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha} \). Then we set stationary scalar
field ansatz [2]
\[
\Phi(r, t) = \phi(r)e^{i\omega t}.
\]  
(2)
Equation (2) describes a spherically symmetric bound state of scalar fields with eigenfrequency
\( \omega \). Accordingly, we set the gauge field to having only time-component, so we have
\[
A_{\alpha}dx^{\alpha} = A(r)dt.
\]  
(3)
In contrast to the electromagnetic field with no coupling, which is gauge invariant, our massive
gauge field will not satisfy any gauge conditions [11]. We do the analytic calculations to obtain
some differential equations of motion related to Langrangian (1), as done in [3,12]. The conserved
current can be defined in the following
\[
J_{\alpha} = \left[i(\Phi^{*}\partial_{\alpha}\Phi - \Phi\partial_{\alpha}\Phi^{*}) - 2qA_{\alpha}\Phi^{*}\Phi\right],
\]  
(4)
so energy-momentum tensor \( T_{\beta}^{\alpha} \) becomes
\[
T_{\beta}^{\alpha} = g^{\alpha\alpha}\left[(D_{\sigma}\Phi)^{*}(D_{\beta}\Phi) + (D_{\beta}\Phi)^{*}(D_{\sigma}\Phi) - g^{\alpha\beta}F_{\alpha\sigma}F_{\beta\rho} + \mu^{2}A_{\sigma}A_{\rho}\right] - \delta_{\beta}^{\alpha}\mathcal{L}.
\]  
(5)
From (4), we obtain the number density defined by
\[
J_{0} = 2(\omega + qA)\phi^{2}.
\]  
(6)
By using Euler-Lagrange equation, we obtain the system of equations for scalar and gauge fields,
i.e.,
\[
\phi'' + \frac{2}{r}\phi' + (\omega + qA)^{2}\phi - \frac{1}{2}\frac{dV(\phi)}{d\phi} = 0,
\]  
(7)
\[
A'' + \frac{2}{r}A' - 2q(\omega + qA)\phi^{2} - \mu^{2}A = 0,
\]  
(8)
where we choose $V(\phi) = m_b^2 \phi^2$. Next, the energy density $\rho$ is defined by

$$\rho = T_0^0 = \left[ m_b^2 + (\omega + qA)^2 \right] \phi^2 + \frac{A^2}{2} + \phi'^2 + \frac{1}{2} \mu^2 A^2; \tag{9}$$

and the other nonvanishing components are

$$T_1^1 = \left[ m_b^2 - (\omega + qA)^2 \right] \phi^2 + \frac{A^2}{2} - \phi'^2 - \frac{1}{2} \mu^2 A^2, \tag{10}$$

$$T_2^2 = T_3^3 = \left[ m_b^2 - (\omega + qA)^2 \right] \phi^2 - \frac{A^2}{2} + \phi'^2 - \frac{1}{2} \mu^2 A^2, \tag{11}$$

which correspond to the pressures of the Q-balls in directions of $r$ and $\theta$, respectively. The charge of the Q-balls can be defined by $[1, 5, 9, 10]$

$$Q = 8\pi \int_0^\infty (\omega + qA) \phi^2 r^2 dr, \tag{12}$$

which is also physically defined as the particle number of the Q-balls. In addition, the physical charge for gauged Q-balls is $\overline{Q} = qQ \ [4, 8]$. The mass can also be given by

$$M = 4\pi \int_0^\infty m_b^2 \phi^2 + (\omega + qA)^2 \phi^2 + \frac{A^2}{2} + \phi'^2 + \frac{1}{2} \mu^2 A^2 \right) r^2 dr. \tag{13}$$

Next, fields will be given in the following rescaling parameters

$$\phi \rightarrow \phi m_b, \ A \rightarrow Am_b, \ \omega \rightarrow \omega m_b, \ \mu \rightarrow \mu m_b, \ r \rightarrow r/m_b. \tag{14}$$

### 3. Perturbative approximation for the fields

We use perturbative approach to obtain the solutions of KGP (also for KGM). First, we will assume that $q$ is a small constant so it will be our perturbative parameter. So, the expansions of the fields become

$$\phi = \phi_0 + q\phi_1 + q^2 \phi_2 + ... \tag{15}$$

$$A = A_0 + qA_1 + q^2 A_2 + ... \tag{16}$$

Then by inserting (15) and (16) to (7) and (8), we will obtain

$$(\phi''_0 + 2q\phi''_1 + ...) + \frac{2}{r}(\phi'_0 + q\phi'_1 + ...) - m^2(\phi_0 + q\phi_1 + ...)$$

$$= -2q\omega(A_0 + qA_1 + ...) (\phi_0 + q\phi_1 + ...) - q^2(A_0 + qA_1 + ...)^2(\phi_0 + q\phi_1 + ...), \tag{17}$$

$$(A''_0 + qA''_1 + ...) + \frac{2}{r}(A'_0 + qA'_1 + ...) - \mu^2(A_0 + qA_1 + ...)$$

$$= 2q\omega(\phi_0 + q\phi_1 + ...)^2 + 2q^2(A_0 + qA_1 + ...) (\phi_0 + q\phi_1 + ...)^2, \tag{18}$$

where $\omega^2 - 1 = -m^2$. We take from (17) and (18) to the first-order. So we will have

$$\phi''_0 + \frac{2}{r}\phi'_0 - m^2 \phi_0 = 0, \tag{19}$$

$$A''_0 + \frac{2}{r}A'_0 - \mu^2 A_0 = 0, \tag{20}$$

$$\phi''_1 + \frac{2}{r}\phi'_1 - m^2 \phi_1 = 0, \tag{21}$$

$$A''_1 + \frac{2}{r}A'_1 - \mu^2 A_1 = 0. \tag{22}$$
for the zeroth-order and
\[\phi''_1 + \frac{2}{r}\phi'_1 - m^2\phi_1 = -2\omega A_0\phi_0,\]
(21)
\[A''_1 + \frac{2}{r}A'_1 - \mu^2 b = 2\omega\phi^2_0,\]
(22)
for the first-order. To obtain the arbitrary solutions of KGP and KGM equations, we need the boundary conditions that shows a localized particle distribution and remove the singularity on \(r = 0\). We impose the following regular boundary conditions:
\[\phi(r \to \infty) = 0, \quad \phi'(r = 0) = 0.\]
(23)
We also impose the gauge field to be monotonically decreasing so that
\[A(r \to \infty) = 0, \quad A'(r = 0) = 0.\]
(24)
By applying (23) and (24) to (19) - (22), we obtain the solutions for the first-order KGP, they are
\[\phi(r) = \frac{ae^{-mr}}{r} + \frac{qe^{-mr}}{r} \left[ c + \frac{ab\omega}{m} \left( E_i(-\mu r) - e^{2mr}E_i[-(2m + \mu)r] \right) \right],\]
(25)
and
\[A(r) = \frac{be^{-mr}}{r} + \frac{qe^{-mr}}{r} \left[ d - \frac{a^2\omega}{\mu} \left( E_i[-(2m + \mu)r] - e^{2mr}E_i[-(2m + \mu)r] \right) \right],\]
(26)
where \(E_i(-at) = \int_{\infty}^{\infty} e^{-at}/t \, dt\) and \(a, b, c, d\) are integration constants. For this case, there is an upper limit for the frequency \(\omega\) that leads to the following condition [8]
\[\omega^2 < \frac{1}{2} V''(0) = 1.\]
(27)
For Maxwell case, the solutions are
\[\phi(r) = \frac{ae^{-mr}}{r} + \frac{qe^{-mr}}{r} \left[ c + \frac{ab\omega}{m} \left( \ln[r] - e^{2mr}E_i[-2mr] \right) \right],\]
(28)
and
\[A(r) = \frac{b}{r} + \frac{q}{r} \left[ d + \frac{a^2\omega}{m} \left( 2mr E_i[-2mr] + e^{-2mr} \right) \right].\]
(29)
4. Q-balls solutions
We show plots of the field profiles in figures 1 and 2 with \(\omega\) in the lowest limit, near upper limit and between them. We also set constant values of \(q\) and \(\mu\). Figure 1 shows that the different values of \(\omega\) make the scalar field change, with larger \(\omega\) making larger \(\phi(r)\). However, in Proca case, the near upper limit is dominant as shown in the figure. Scalar field profile for Maxwell case is also bigger than the Proca (massive gauge field) case. The initial value of the scalar field, \(\phi(0)\), can be determined by setting the values of \(a, b, c\) and \(d\). Figure 2 shows the gauge field profiles where the configurations are same with the scalar fields. The properties of the gauge fields are increasing simultaneously with the addition of the \(\omega\). Indeed, the increasing of the gauge mass \(\mu\) makes the solutions smaller. It is because the gauge mass behaves like a damping
Figure 1. Scalar field profile for Maxwell (left) and Proca (right) with three different values of $\omega$.

factor that decreases the value of the field. Again, we can set the initial value of $A(r)$, i.e., $A(0)$ with the same constants.

The pressures of the Q-balls are shown in figures 3 and 4. Figure 3 shows the radial pressure of the Q-balls obtained by inserting the solutions (25)–(29) to (10). Radial pressure approaches zero at large $r$, which shows that the Q-balls have a finite mass. From figure 3, we can see that there is no significant effect of the gauge mass $\mu$ and $\omega$ for the radial pressure in all range of radius. However, different values of $\omega$ affect the initial value of the pressure because it has significant initial value.

Figure 4 shows the pressure in tangential direction. This pressure has similar properties with the radial pressure but the initial values are different. Tangential pressure has negative values. Mathematically, this negativeness is caused by the value of the scalar field derivatives (that have negative sign) that are larger than the fields itself. Possibly, the physical meaning of this negativeness is the existence of DE in the Q-balls like in the dark-energy star [14] but just only in the tangential pressure. In the case of Q-balls, we also consider that there is no gravitational term such that the Lagrangian of the matter does not couple to the gravity. In our universe, DE makes it accelerate and develope because the pressures are negative [6, 16]. DE also can be modeled by scalar fields with arbitrary potential; hence it yields the negative pressure and equation of state $w_\phi = P_\phi/\rho_\phi < -1/3$, such in the Quintessence model [16]. In our case of

Figure 2. Gauge field profile for Maxwell (left) and Proca (right) with three different values of $\omega$. 
Q-balls, we obtain the equation of state \( w_t = P_t/\rho \simeq -0.76 \), which satisfies the DE condition. To find the value of \( w_t \), we take the sum of tangential pressure from every point given in figure 4. We then divide it by the number of points of the tangential pressure. We do the similar way for the energy density. As the result, we get the the average values of the tangential pressure and the energy density, which we can use to obtain the value of \( w_t \). Generally, our Q-balls are suitable with all energy conditions [14, 15] where the more detailed discussion are prepared for another manuscript.

We step to the other characteristics of the Q-balls, i.e., mass and charge. Mass \( M \) and charge \( Q \) can be easily integrated from (12) and (13). These are shown in figure 5. Mass profile \( M(r) \) are shown in the left panel of figure 5 with three different values of \( \omega \) for Maxwell and Proca cases. As we see from mass-profile figure, charge profile \( Q(r) \) of Q-balls has similar properties where its values are increasing with the addition of the radius of Q-balls, see figure 5 (middle). The values also increase when \( \omega \) and \( \mu \) increase. The relation between \( M \) and \( Q \) can be obtained qualitatively from the right panel of figure 5. Every reference gives their argument about the relation between mass (or energy) end charge. As discussed in [17], it must satisfy linear relation

\[
M = \omega Q, \quad (30)
\]

for gauged Q-balls that is similar with ordinary Q-balls [5]. In [10] for non-massive gauge the
Figure 5. Profiles of mass $M$ (left), charge $Q$ (middle), and mass-charge relation $M–Q$ (right) with three different values of $\omega$ for Maxwell and Proca cases.

relation becomes

$$M = \omega Q + \frac{4\pi}{3} \int_0^\infty dr r^2 \left[ \phi'(r)^2 + q^2 A'(r)^2 \right].$$

(31)

But (31) will not reduce to ordinary Q-balls when $A = 0$. The detailed description about this relation in our case will be explained in another manuscript. In our qualitative observation, for our Q-balls (Maxwell and Proca), the condition for the relation between $M$ and $Q$ is almost linear and satisfies the condition (31) where the value of $\omega$ is limited under the value of $m_b$. But we still have to calculate the second term in right-hand side of (31) that plays a role as (may be) nonlinear term.

5. Conclusions

As we have demonstrated above, we have addressed gauged Q-balls with massless gauged field (Maxwell) and massive gauged field (Proca). We find the solution by using perturbative approach to the first-order. Field profiles from those two cases have similar properties such as we can see from the variation of position $r$ and $\omega$. As the consequence of the field solutions, the pressure, mass and charge also have similar properties. In our model of Q-balls, we also find that the tangential pressure behaves like DE model of scalar fields. Qualitatively, we obtain that $M$ and $Q$ have a linear relation such in some references where the value of $\omega$ is limited under the value of $m_b$.

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