Ergodic Spectral Efficiency in MIMO Cellular Networks

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Abstract

This paper shows how the application of stochastic geometry to the analysis of wireless networks is greatly facilitated by (i) a clear separation of time scales, (ii) abstraction of small-scale effects via ergodicity, and (iii) an interference model that reflects the receiver’s lack of knowledge of how each individual interference term is faded. These procedures render the analysis both simpler and more precise and more amenable to the incorporation of subsequent features. In particular, the paper presents analytical characterizations of the ergodic spectral efficiency of cellular networks with single-user MIMO and sectorization. These characterizations, in the form of easy-to-evaluate expressions, encompass both the distribution of spectral efficiency over the network locations as well as the average thereof.

Index Terms

Stochastic geometry, cellular networks, ergodic spectral efficiency, MIMO, sectorization, Poisson point process, shadowing, interference, SINR

I. INTRODUCTION

Stochastic geometry is quickly becoming an indispensable instrument in wireless network analysis. By mapping the empirical distribution of transmitter and receiver locations to appropriate point processes, it becomes possible to apply a powerful and expanding toolkit of mathematical

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results. This offers a complement, and increasingly even an outright alternative, to the Monte-Carlo simulations that have long been the workhorse of wireless network design.

Although a stochastic modelling of transmitter and receiver locations may seem mostly amenable to ad hoc networks, which are devoid of fixed infrastructure, a seminal paper by Andrews et al. [1] proved the truly remarkable effectiveness of stochastic modelling also in cellular networks—even with simple Poisson point processes (PPPs). Indeed, while it may appear that more sophisticated spatial distributions could better capture the relative regularity of actual base station (BS) placements, because of shadowing it is the case that PPPs lead to remarkably precise characterizations of signal strengths and interference, and thus of all ensuing performance measures. In fact, as shown in [2], [3] and expounded later in this paper, PPP-based characterizations represent the limit to which actual behaviors converge as the shadowing strengthens.

Altogether, the irruption of stochastic geometry is a transcendent development in wireless research, and it is reasonable to expect its importance to grow even further as networks become denser and more heterogeneous [4], [5]. Important contributions to the advancement of the discipline in the context of wireless networks include [6]–[22].

The present paper deals with the ergodic spectral efficiency of PPP cellular networks, a quantity already tackled in works dating back to [1]. Our analysis, however, relies on a different modeling approach for the interference. This takes us on a different route, one that proves greatly advantageous because it yields solutions that are both simpler and more precise, and, most importantly, because it opens the door to accommodating key ingredients—such as MIMO and cell sectorization—that seemed previously elusive. The modeling approach that unlocks these new analytical possibilities is not arbitrary, but rather based on sound arguments:

1) A clean separation between small- and large-scale channel features, in recognition that the phenomena that give rise to these features are distinct.

2) An unwavering embrace of ergodic performance metrics with respect to the small-scale features, in recognition that such small-scale ergodicity reflects well the operating conditions of modern wireless systems [23], [24].

3) The admission that each receiver can track the fading of its intended signal, but not the fading of each individual interference term.

With small- and large-scale features decoupled, ergodicity enables abstracting out the former so as to focus the stochastic geometry analysis where it makes a difference (on the large-scale
aspects), reaping the most out of its potent machinery. As mentioned, this allows advancing the analysis on all fronts: simplicity, accuracy, and generality. In particular, and to the extent of our knowledge, the spectral efficiency expressions obtained in this paper are the first such characterizations that incorporate MIMO spatial multiplexing. Likewise, sectorization is also readily included.

Besides providing new expressions for quantities of interest, the present paper seeks to promote the importance of keeping the two foregoing arguments present when applying stochastic geometry to wireless networks. For conceptual clarity, these arguments are herein elaborated on the basis of a cellular network where each user is served by a single BS, and where only the downlink is considered. However, the arguments apply equally to networks featuring BS cooperation, and to the uplink, only with certain expressions suitably replaced by generalizations or counterparts.

II. NETWORK MODELLING

A. Separation of Scales

Rooted in extensive propagation measurements, the separation between large-scale propagation phenomena (distance-dependent path loss and shadowing) and small-scale multipath fading has been instrumental in the study of wireless networks since their onset, greatly facilitating characterizations that would otherwise be unwieldy [25]. The premise of this separation is that, over suitably small distances (tens to hundreds of wavelengths), the large-scale phenomena remain essentially unchanged and only small-scale variations transpire. This allows delineating local neighborhoods around transmitters and receivers wherein the small-scale channel behavior conforms to a stationary random process whose distribution is dictated by the large-scale features. Then, through the user velocity, this space-domain random process maps to a time-domain process. Moreover, under the very mild condition that the Doppler spectrum be free of delta functions, this time-domain process is ergodic.

For frequencies and velocities in the widest possible range of interest, the dwell time in a local neighborhood is far longer than the extension of signal codewords. Thus, large-scale features can be regarded as constant over an individual codeword. Alternatively, the small-scale fading may or may not remain constant over a codeword, depending on the coherence of such fading and on how codewords are arranged in time and frequency, and this dichotomy gives rise to two classic information-theoretic idealizations of the channel over the horizon of a codeword:
• **Nonergodic.** Fading random, but fixed over the codeword.
• **Ergodic.** Fading random and exhibiting sufficiently many values to essentially reveal its distribution over the codeword.

These two idealizations, in turn, map respectively to outage and ergodic definitions for the spectral efficiency. Although both are useful, the ergodic definition is the most representative in modern systems where codewords can be interspersed over very wide bandwidths, across hybrid-ARQ repetitions, and possibly over multiple antennas, and they can be subject to scheduling and link adaptation. As argued in [23], [24], the balance of these mechanisms is indeed best abstracted by ergodic spectral efficiencies involving expectations over the small-scale fading, with the large-scale features held fixed. It is at this point that stochastic geometry should enter the analysis, when the small-scale effects have been abstracted out and we can zoom out to cleanly examine the large-scale ones.

**B. Large-scale Modeling**

We consider the downlink of a cellular network, initially with omnidirectional antennas (to be generalized to sectorized antennas in Section VII), where the signals are subject to path loss with exponent \( \eta > 2 \) and shadowing.

Suppose that the BS positions are agnostic to the radio propagation. It has been recently shown [2], [3] that, regardless of what those BS positions are (under only a very mild homogeneity condition), an increasing shadowing standard deviation \( \sigma_{dB} \) renders the network progressively PPP-like from the vantage of any given user, i.e., it makes the powers that a user receives from any population of BSs look as if they originated from PPP-distributed BSs. This important observation strongly justifies the modelling assumption of PPP-located BSs. Ironically then, shadowing, a nuisance in the study of regular geometries, simplifies the stochastic modelling of networks by making them all look alike propagation-wise regardless of their underlying geometry. And, although the convergence to a PPP behavior is asymptotic in \( \sigma_{dB} \), values of interest suffice for networks to look essentially Poissonian. In particular, it is shown in [2] that, for a regular lattice of BSs spawning hexagonal cells and \( \eta = 4 \), \( \sigma_{dB} = 12 \) dB suffices to render the received powers indistinguishable (with 99% confidence in a Kolmogorov-Smirnov test) from those in a PPP network, 90% of the time. In Section VIII we provide further evidence supporting the suitability of a PPP model for the analysis of lattice networks with relevant values of \( \sigma_{dB} \).
The foregoing convergence is a powerful argument in favor of a PPP model for the spatial distribution of BSs, say a process $\Phi_b \subset \mathbb{R}^2$, without the need for explicit modeling of the shadowing as it is already implicitly captured by the Poisson nature of the network. The density of $\Phi_b$, say $\lambda_b$, depends on the type and strength of the shadowing in addition to the actual positions of the BSs [26]–[29].

Turning now to the spatial distribution of users, a good starting point is to model it as an independent PPP $\Phi_u$ with density $\lambda_u$. (This model could be refined to incorporate user clustering tendencies as well as dependences between the positions of users and BSs [30]–[33].)

Without loss of generality, the analysis can be conducted from the perspective of a user at the origin, which becomes the typical user under expectation over $\Phi_b$. Denoting by $r_k$ the distance between such user and the $k$th BS, whose location—recall—is distributed according to $\Phi_b$, we index the BSs in increasing order of $r_k$, i.e., such that $r_k < r_{k+1}$ for $k \in \mathbb{N}_0$. Since, in terms of $\Phi_b$, the only large-scale propagation mechanism at play is path loss, the typical user receives the strongest power from the BS at $r_0$, which we deem the serving BS.

C. Small-scale Modeling

Let the communication be SISO for now, i.e., with BSs and users having a single antenna, and further let each receiver be privy to the fading of only its intended signal. Denoting by $P$ the power measured at 1 m from a BS transmitter, at symbol $n$ the typical user observes

$$y[n] = \sqrt{P r_0^{-\eta}} H_0[n] s_0[n] + z[n],$$

where the leading term is the intended signal from the serving BS while

$$z[n] = \sum_{k=1}^{\infty} \sqrt{P r_k^{-\eta}} H_k[n] s_k[n] + v[n]$$

is the aggregate interference from all other BSs, plus thermal noise $v$. In turn, $s_k$ is the signal transmitted by the $k$th BS and $H_k$ is the associated fading coefficient.

The fading coefficients $\{H_k\}_{k=0}^{\infty}$ are independent and of unit power, but otherwise arbitrarily distributed, while $v \sim \mathcal{N}_\mathbb{C}(0, N_0)$. The signal is distributed as $s_k \sim \mathcal{N}_\mathbb{C}(0, 1)$, a choice that is justified later.
Conditioned on \( \{r_k\}_{k=0}^{\infty} \), which are fixed as far as the small-scale modeling is concerned, the instantaneous SINR enjoyed by the typical user at symbol \( n \) is

\[
\text{SINR}[n] = \frac{P r_0^{-\eta} |H_0[n]|^2}{P \sum_{k=1}^{\infty} r_k^{-\eta} |H_k[n]|^2 + N_0}.
\]  

(3)

III. INTERFERENCE MODELING

With \( H_0[1], \ldots, H_0[N] \) known at the typical user, the mutual information (in bits/symbol) over codewords spanning \( N \) symbols is

\[
\frac{1}{N} I(s_0[1], \ldots, s_0[N]; y[1], \ldots, y[N] \mid H_0[1], \ldots, H_0[N], \{r_k\}_{k=0}^{\infty}),
\]  

(4)

which, with IID codeword symbols, becomes

\[
\frac{1}{N} \sum_{n=1}^{N} I(s_0[n]; y[n] \mid H_0[n], \{r_k\}_{k=0}^{\infty}).
\]  

(5)

As argued earlier, codewords are nowadays long enough—thousands of symbols—and arranged in such a way—interspersed in time, frequency, and increasingly across antennas—so as to experience sufficiently many fading swings for an effective averaging of the mutual information to take place over the small-scale fading. From the stationarity and ergodicity of the small-scale fading over the codeword, the averaging in (5) becomes an expectation and confers the significance of the ergodic spectral efficiency (in bits/s/Hz)

\[
C_{\text{exact}} = \mathbb{E}_{H_0}[I(s_0; y \mid H_0, \{r_k\}_{k=0}^{\infty})] \]  

(6)

\[
= \mathbb{E}_{H_0}[I(s_0; \sqrt{P r_0^{-\eta} H_0 s_0 + z} \mid H_0, \{r_k\}_{k=0}^{\infty})].
\]  

(7)

This quantity, a baseline in the sequel, does not admit explicit expressions. Rather, the evaluation of \( C_{\text{exact}} \) requires computationally very intensive Monte-Carlo simulation [34, App. A] and a 64-core high-performance computing cluster is employed to generate the corresponding results throughout the paper; for all such results, 99% confidence intervals are given.

Let us examine the local distribution, for some given \( \{r_k\}_{k=1}^{\infty} \), of the interference-plus-noise \( z \) as defined in (2). The first thing to note is that, without further conditioning on \( \{H_k\}_{k=1}^{\infty} \), i.e., without the receiver knowing the fading coefficients from all interfering BSs, the distribution of \( z \) over the local neighborhood is generally not Gaussian. Conditioned only on \( \{r_k\}_{k=1}^{\infty} \), the distribution of \( z \) is actually highly involved; in Rayleigh fading, for instance, it involves products of Gaussians. While the non-Gaussianity of \( z \) is irrelevant to the local-average SINR, since only
the variance of \( z \) matters in that respect, it is relevant to information-theoretic derivations and chiefly that of the spectral efficiency, which does depend on the distribution of \( z \).

It is nevertheless customary to analyze \( C_{\text{exact}} \) in the form it would have if \( \{H_k\}_{k=1}^{\infty} \) were actually known by the typical user and \( z \) were consequently Gaussian, namely the form

\[
C_{\text{ub}} = \mathbb{E}_{\{H_k\}_{k=1}^{\infty}} \left[ \log_2 \left( 1 + \frac{P r_0^{-\eta}|H_0|^2}{P \sum_{k=1}^{\infty} r_k^{-\eta}|H_k|^2 + N_0} \right) \right],
\]

where the tacit—and seldom made explicit—redefinition of \( z \) as Gaussian is unmistakable from \( I(s_0; \sqrt{\gamma}s_0 + z) = \log_2(1 + \gamma) \), which holds only when \( s_0 \) and \( z \) are Gaussian. As it turns out, a Gaussian modeling of \( z \) is not unreasonable because, if a decoder is designed for Gaussian interference-plus-noise (either by design or because the distribution thereof is unknown), then the spectral efficiency is precisely as if the interference-plus-noise were indeed Gaussian, regardless of its actual distribution [35]. And, once \( z \) is taken to be Gaussian, the capacity-achieving signal distribution is also Gaussian, validating our choice for \( s_0 \). At the same time, the granting of \( \{H_k\}_{k=1}^{\infty} \) as additional side information to the receiver renders (8) an upper bound to \( C_{\text{exact}} \), hence the denomination \( C_{\text{ub}} \).

While much more tractable than \( C_{\text{exact}} \), the form of \( C_{\text{ub}} \) has the issue of depending not only on \( H_0 \), but further on \( \{H_k\}_{k=1}^{\infty} \). This still clutters its analysis considerably, as seen later.

Alternatively, what we propound in this paper is to model \( z \) as Gaussian, but forgoing the small-scale variations in its power, i.e., to use

\[
z \sim \mathcal{N}_C \left( 0, P \sum_{k=1}^{\infty} r_k^{-\eta} + N_0 \right).
\]

This model has the virtue of rendering \( z \) Gaussian without the strain of gifting the receiver with \( \{H_k\}_{k=1}^{\infty} \), and the result of this restrain is gratifyingly good. Indeed, the closeness between the distribution of \( z \) as originally defined in (2) and the distribution in (9) has been tightly bounded by means of the Kolmogorov-Smirnov distance [36]. With \( z \) as in (9), the instantaneous SINR then becomes

\[
\text{SINR} = \frac{P r_0^{-\eta}|H_0|^2}{P \sum_{k=1}^{\infty} r_k^{-\eta} + N_0},
\]

and the corresponding ergodic spectral efficiency is

\[
C = \mathbb{E}_{H_0} \left[ \log_2 \left( 1 + \frac{P r_0^{-\eta}|H_0|^2}{P \sum_{k=1}^{\infty} r_k^{-\eta} + N_0} \right) \right],
\]

which is the quantity we shall work with.
Since, with Gaussian codewords and a given $E[|z|^2]$, the mutual information is minimized when $z$ is Gaussian [37], we have that $C \leq C_{\text{exact}}$. The similarity between $C$ and $C_{\text{exact}}$, with the former characterized analytically and the latter obtained through Monte-Carlo simulation, is illustrated throughout the paper.

Contrasting the instantaneous SINR expressions in (3) and (10), the analytical virtues of our model for $z$ become evident once we rewrite the latter as $\text{SINR} = \rho |H_0|^2$, where

$$\rho = \frac{P r_0^{-\eta}}{P \sum_{k=1}^{\infty} r_k^{-\eta} + N_0}$$

is the local-average SINR at the typical user, fixed over any entire codeword and cleanly separated from the fluctuant term $|H_0|^2$; this reflects the decoupling between the large- and small-scale dependences. In interference-limited conditions ($P/N_0 \to \infty$), the local-average SINR specializes to $\rho = r_0^{-\eta}/\sum_{k=1}^{\infty} r_k^{-\eta}$. For given BS and user locations, $\rho$ becomes determined and the conditional distribution of the instantaneous SINR is then given directly by that of $|H_0|^2$, i.e., by the CDF $F_{|H_0|^2}(\cdot)$, while the ergodic spectral efficiency of a user with local-average SINR $\rho$ is

$$C(\rho) = E_{H_0}\left[ \log_2 \left( 1 + \rho |H_0|^2 \right) \right]$$

$$= \int_0^{\infty} \log_2(1 + \rho \xi) \, dF_{|H_0|^2}(\xi),$$

which, through $\rho$, sets the stage for further computations involving the geometry of the network. As anticipated, it is here where stochastic geometry can be applied with all its potency, undistracted by lingering small-scale terms.

**Example 1.** In Rayleigh fading, $F_{|H_0|^2}(\xi) = 1 - e^{-\xi}$, from which

$$C(\rho) = e^{1/\rho} E_1 \left( \frac{1}{\rho} \right) \log_2 e,$$

where $E_n(x) = \int_1^{\infty} t^{-n} e^{-xt} \, dt$ is an exponential integral.

For fading distributions other than Rayleigh, or with MIMO or other features, corresponding forms can be obtained for $C(\rho)$, always with the key property of these being a function of $\rho$ and not of the instantaneous fading coefficients.

**Example 2.** Let us examine the applicability of (15) for an interference-limited network with 100 interfering BSs. To typify the network, we set $r_k$ to the expected value of the distance to the
Fig. 1: Spectral efficiency vs. $r_0$ for $\lambda_b = 2$ BSs/km$^2$, $\eta = 3.8$ and $r_k = \frac{\Gamma(k+1.5)}{\sqrt{\pi \lambda_b} \Gamma(k+1)}$ for $k = 1, \ldots, 100$. The 99% confidence intervals around $C_{\text{exact}}$ range from $\pm 0.029$ at $r_0 = 150$ m to $\pm 0.009$ at $r_0 = 450$ m, for both SISO and MIMO.

$k$th nearest point in a PPP: $r_k = \frac{\Gamma(k+1.5)}{\sqrt{\pi \lambda_b} \Gamma(k+1)}$ for $k = 1, \ldots, 100$ [38]. We further set $\eta = 3.8$ and $\lambda_b = 2$ BSs/km$^2$ and neglect the noise. Shown in Fig. 1 is $C(\rho)$ in (15) compared against $C_{\text{exact}}$. The same comparison is provided for MIMO with two transmit and two receive antennas, whose ergodic spectral efficiency $C_{\text{MIMO}}^{2\times 2}(\cdot)$ is derived later in the paper. In both cases the differences are minute, supporting the interference modeling approach propounded in this paper.

IV. DISTRIBUTION OF THE LOCAL-AVERAGE SINR

In subsequent sections, an ingredient in our analysis of the ergodic spectral efficiency is to be the CDF $F_\rho(\cdot)$. This is an important function, extensively utilized by system designers and traditionally obtained by means of simulation over lattice networks [39]. For hexagonal cells in particular, and without shadow fading, an infinite series solution is also available [40].

Here we set out to characterize $F_\rho(\cdot)$ for PPP networks, so as to implicitly incorporate shadowing, and specifically in interference-limited conditions. Interestingly, with PPP-populated
BSs, the powers received by any given user are statistically invariant—save for a scaling of the BS density if noise were not negligible—to the distribution of the channel gains [26]–[28] and thus $F_{\rho}(\cdot)$ is equivalent to what $F_{\text{SINR}}(\cdot)$ would look like if users connected to the BS with the strongest instantaneous link. This distribution was established in [12], [41]–[43], very compactly for arguments above 1 and in a still manageable form between 1/2 and 1, but in an accelerating cumbersome fashion (involving progressively higher-dimensional integrations) as the argument dips below 1/2 [42, Sec. V.A], [43, Cor. 19]. As alternatives, [19], [41], [44] derived $F_{1/\rho}(\cdot)$ in the Laplace domain, which would then require numerical inversion, while [45] showed that the lower tail ($\rho \to 0$) of $F_{\rho}(\cdot)$ behaves as

$$F_{\rho}(\theta) \sim e^{s^*/\theta}$$

with $s^* < 0$ being the solution to

$$1F_1(-\delta, 1-\delta, -s^*) = 0,$$

where $1F_1(\cdot)$ is the confluent hypergeometric or Kummer function and, for compactness, we have introduced the shorthand notation $\delta = 2/\eta$. Since it only depends—through $\delta$—on the path loss exponent $\eta$, the parameter $s^*$ can be precomputed for all relevant values thereof. Values of $s^*$ for some typical $\eta$ are listed in Table I.

The approach we take is to apply the exact form given in [42, Sec. V.A], [43, Cor. 19] down to $\theta = 1/2$ and then (16) for $\theta < 1/2$. This combination gives

$$F_{\rho}(\theta) \approx \begin{cases} e^{s^*/\theta} & 0 \leq \theta < \frac{s^*}{\log A_\delta} \\
A_\delta & \frac{s^*}{\log A_\delta} \leq \theta < 1/2 \\
1 - \theta^{-\delta}\text{sinc} \delta + B_\delta\left(\frac{\theta}{1-\theta}\right) & 1/2 \leq \theta < 1 \\
1 - \theta^{-\delta}\text{sinc} \delta & \theta \geq 1,\end{cases}$$

### Table I: Parameter $s^*$ for typical values of the path loss exponent $\eta$.  

| $\eta$ | $s^*$  | $\eta$ | $s^*$  |
|--------|-------|--------|-------|
| 3.5    | -0.672| 3.9    | -0.819|
| 3.6    | -0.71 | 4.0    | -0.854|
| 3.7    | -0.747| 4.1    | -0.888|
| 3.8    | -0.783| 4.2    | -0.922|
where

\[ A_\delta = 1 - 2^\delta \text{sinc} \delta + B_\delta(1) \tag{19} \]

and

\[ B_\delta(x) = \frac{\delta \text{sinc}^2(\delta) \Gamma^2(\delta + 1) \, _2F_1(1, \delta + 1; 2\delta + 2; -1/x)}{x^{1+2\delta} \Gamma(2\delta + 2)} \tag{20} \]

with \(_2F_1(\cdot)\) the Gauss hypergeometric function. When the path loss exponent is \(\eta = 4\), we have that \(\delta = 1/2\) and the above specialize to

\[ F_\rho(\theta) \approx \begin{cases} 
  e^{-0.854/\theta} & 0 \leq \theta < 0.457 \\
  0.154 & 0.457 \leq \theta < 1/2 \\
  1 - \frac{4\sqrt{\theta - \theta - 1}}{\pi \theta} & 1/2 \leq \theta < 1 \\
  1 - \frac{2}{\pi \sqrt{\theta}} & \theta \geq 1.
\end{cases} \tag{21} \]

An even simpler, slightly less accurate expression for \(F_\rho(\cdot)\) can be obtained using the exact form only down to \(\theta = 1\) while stretching the lower tail expansion in (16) up to \(\theta = 1\). This gives

\[ F_\rho(\theta) \approx \begin{cases} 
  e^{s^*/\theta} & 0 \leq \theta < \frac{s^*}{\log(1 - \text{sinc} \delta)} \\
  1 - \text{sinc} \delta & \frac{s^*}{\log(1 - \text{sinc} \delta)} \leq \theta < 1 \\
  1 - \theta^{-\delta \text{sinc} \delta} & \theta \geq 1.
\end{cases} \tag{22} \]

**V. DISTRIBUTION OF THE SPECTRAL EFFICIENCY**

The CDF of \(C\) provides a complete description of the ergodic spectral efficiency offered by the network over all locations. By mapping the applicable function \(C(\rho)\) onto the expressions for \(F_\rho(\cdot)\) put forth in the previous section, \(F_C(\cdot)\) is readily characterized. In interference-limited conditions, such \(F_C(\cdot)\) is bound to depend only on the path loss exponent, \(\eta\).

**A. SISO**

In Rayleigh-faded SISO channels, \(C(\rho)\) is given by (15). By resorting to the invertible approximation

\[ e^{\nu} E_1(\nu) \log_2 e \approx 1.4 \log \left( 1 + \frac{0.82}{\nu} \right), \tag{23} \]
it becomes possible to write $\rho \approx \frac{c^{1.4-1}}{0.82}$ and subsequently, by means of (18),

$$F_C(\gamma) \approx F_\rho\left(\frac{e^{\gamma/1.4} - 1}{0.82}\right)$$

$$= \begin{cases} 
0 & 0 \leq \gamma < 1.4 \log\left(1 + \frac{0.82 s^*}{\log A_3}\right) \\
\frac{e^{0.82 s^*}}{0.82} & 1.4 \log\left(1 + \frac{0.82 s^*}{\log A_3}\right) \leq \gamma < 0.48 \\
A_\delta & 0.48 \leq \gamma < 0.84 \\
B_\delta & \gamma \geq 0.84,
\end{cases}$$

whose goodness is validated in the following example.

**Example 3.** Consider an interference-limited network with $\eta = 3.8$. Shown in Fig. 2 are (25), as well as the numerical mapping of $C(\rho)$—without the bypass of its invertible approximation—onto (18), both solutions contrasted against $C_{\text{exact}}$. 
For $\eta = 4$, (25) specializes to

$$F_C(\gamma) \approx \begin{cases} e^{-\exp(0.7(\gamma/1.4)^{-1})} & 0 \leq \gamma < 0.44 \\ 0.154 & 0.44 \leq \gamma < 0.48 \\ 1 - 4 \pi \sqrt{0.82/(\gamma/1.4-1)} + \frac{\exp(\gamma/1.4-0.18)}{\pi(\gamma/1.4-1)} & 0.48 \leq \gamma < 0.84 \\ 1 - 2 \pi \sqrt{0.82/(\gamma/1.4-1)} & \gamma \geq 0.84. \end{cases}$$

(26)

In contrast to these pleasing results, without the model for $z$ propounded in this paper the distribution of the spectral efficiency over the network locations is far more inaccessible. Indeed, the corresponding $C_{ub}$ can be rewritten (cf. Appendix A) as

$$C_{ub} = \log_2 e \int_0^\infty e^{-x r_0^\eta N_0/P} \prod_{k=1}^\infty \frac{1}{1 + x (r_0/r_k)^\eta} \, dx,$$

(27)

which is no longer a functional of singly $\rho$, whose distribution was established in the previous section; rather, (27) is a more involved function of $\{r_k\}_{k=0}^\infty$ and offers no obvious way of disentangling these dependences. Faced with this obstacle, some authors choose to instead characterize the distribution of $\log_2 \left( 1 + \frac{P r_0^\eta |H_0|^2}{P \sum_{k=1}^\infty r_k^\eta |H_k|^2 + N_0} \right)$ over $\{H_k\}_{k=0}^\infty$ as well as $\{r_k\}_{k=0}^\infty$ [47, Sec. VII.A]. However, the mixing of small- and large-scale variations within this quantity clutters potential observations. Moreover, the generalization to more involved settings, say MIMO, appears arduous or outright hopeless. (Prior stochastic geometry analyses of spectral efficiency featuring MIMO were restricted to beamforming or SDMA, rather than spatial multiplexing.)

B. MIMO

Our approach, in contrast, only requires mapping the appropriate $C(\rho)$ onto $F_\rho(\cdot)$. Whenever $C(\rho)$ does not lend itself to inversion, even approximately, it is straightforward to perform this mapping numerically.

Example 4. Reconsider Example [3] but now with two transmit and two receive antennas. If $H_0$ is replaced by a $2 \times 2$ channel matrix $\mathbf{H}_0$ having IID Rayleigh-faded entries, then [49]

$$C_{2x2}^{\text{MIMO}}(\rho) = 2 e^{2/\rho} \left[ E_1 \left( \frac{2}{\rho} \right) + E_3 \left( \frac{2}{\rho} \right) \right] \log_2 e$$

(28)

$$= \left[ 2 e^{2/\rho} E_1 \left( \frac{2}{\rho} \right) \left( 1 + \frac{2}{\rho^2} \right) + \left( 1 - \frac{2}{\rho} \right) \right] \log_2 e.$$  

(29)

Fig. 2 depicts the numerical mapping of $C_{2x2}^{\text{MIMO}}(\rho)$ onto (18), as well as the corresponding $C_{\text{exact}}$. 

Fig. 3: Empirical PDF of the logarithm of the 2 × 2 MIMO ergodic spectral efficiency for η = 4 and its normal fit.

C. Lognormal Fit

Inspecting the distribution of C(ρ) we observe that, interestingly, it closely resembles a lognormal function, i.e., that log C(ρ) admits a rather precise Gaussian fit. This opens the door to an alternative to $F_C(\cdot)$ as obtained by mapping $C(\rho)$ onto $F_\rho(\cdot)$, namely the alternative $log C(\rho) \sim N(\mu, \sigma^2)$ with

$$
\mu = \int_0^\infty \log C(\theta) \, dF_\rho(\theta) \tag{30}
$$

$$
\sigma^2 = \int_0^\infty \left[ \log C(\theta) \right]^2 \, dF_\rho(\theta) - \mu^2. \tag{31}
$$

Example 5. Let η = 4. For $C_{2\times2}^{\text{MIMO}}(\rho)$ as given in (29) and $F_\rho(\cdot)$ as given in (21), the numerical integrations in (30)–(31) yield $\mu = 0.92$ and $\sigma^2 = 0.8$. Fig. 3 presents the empirical PDF of $log C_{2\times2}^{\text{MIMO}}(\rho)$, generated via Monte-Carlo, and a Gaussian PDF with $\mu = 0.92$ and $\sigma^2 = 0.8$.

Thanks to this lognormal behavior, which holds also for SISO, a Gaussian distribution with mean $\mu$ and variance $\sigma^2$ can provide a quick idea of the disparity of the user experiences throughout the network. While its tails differ from those of $dF_{\log C(\rho)(\gamma)}/d\gamma$ via (24), this
Gaussian distribution may be used to determine the fraction of users whose spectral efficiency lies within a certain interval of the average, or it may simplify further calculations that require averaging with respect to the distribution of $C(\rho)$.

VI. SPATIAL AVERAGE OF THE SPECTRAL EFFICIENCY

Sometimes, it is of interest to condense $F_C(\cdot)$ down to a single quantity, and in that case the average is the logical choice. Under spatial ergodicity, which holds for the PPP and many other point processes [7], this quantity equals the average of all per-user spectral efficiencies in any realization of the network.

Here again, the approach propounded in this paper proves advantageous. For any setting for which $C(\rho)$ is available, our expressions for $F_\rho(\cdot)$ enable computing

$$
\bar{C} = \int_0^\infty C(\theta) \, dF_\rho(\theta),
$$

which, in interference-limited conditions, again depends only on the path loss exponent.

A. SISO

In Rayleigh-faded SISO channels, with (15) and (22) plugged into (32), the integration yields (cf. Appendix B)

$$
\bar{C} \approx -s^* \log_2 e \left[ E_1(-s*/D_\delta) - e^{(1+s^*)/D_\delta} E_1(1/D_\delta) \right] + \frac{\sin(\pi \delta) \log_2 e}{\pi} G_{2,3}^{2,2} \left( \begin{array}{c} 0, 1 - \delta \\ 0, 0, -\delta \end{array} \right)
$$

(33)

where

$$
G_{p,q}^{m,n} \left( z \begin{array}{c} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{array} \right)
$$

(34)

is the Meijer-G function while $D_\delta = s^*/\log(1 - \text{sinc} \delta)$. An even more precise, albeit also more involved expression for $\bar{C}$ can be obtained using (18) in lieu of (22).

For extreme values of $\delta$ ($\delta \to 0$ or $\delta \to 1$),

$$
\frac{\sin(\pi \delta)}{\pi} G_{2,3}^{2,2} \left( \begin{array}{c} 0, 1 - \delta \\ 0, 0, -\delta \end{array} \right) \sim \frac{1}{\delta} - 1
$$

(35)

but, for typical path loss exponents, this behavior is not sufficiently precise and the Meijer-G function needs to be evaluated.
Example 6. Fig. 4 compares \( \bar{C}_{\text{exact}} \) against \( \bar{C} \) as in (33), the Monte-Carlo average of \( C_{\text{exact}} \), for \( 0.48 \leq \delta \leq 0.57 \) corresponding to \( 3.5 \leq \eta \leq 4.2 \). For \( \eta = 4 \) in particular,

\[
\bar{C} \approx 0.187 + \frac{\log_2 e \, \delta}{\pi} G_{2,2}^{2,2} \left( 1 \left| \begin{array}{c} 0, 1/2 \\ 0, 0, -1/2 \end{array} \right. \right) = 1.99,
\]

while \( \bar{C}_{\text{exact}} \) is 2.01.

It is worthwhile to contrast (33) with its counterpart obtained without the model for \( z \) propounded in this paper, namely the average of \( C_{\text{ub}} \) given by [1]

\[
\bar{C}_{\text{ub}} = \int_0^\infty \frac{\log_2 e}{1 + (e^t - 1)^{2/\eta}} \frac{1}{1 + \eta/2} dt,
\]

or, by means of Pfaff’s transformation [50], equivalently by

\[
\bar{C}_{\text{ub}} = \int_0^\infty \frac{\log_2 e}{2 F_1(1, 1; 1 - \delta; \frac{x}{1 + \gamma})} d\gamma.
\]
Example 7. Included in Fig. 4 alongside its SISO counterparts $\bar{C}$ and $\bar{C}_{\text{exact}}$, is also $\bar{C}_{\text{ub}}$.

Besides being further from $C_{\text{exact}}$ than our solution $\bar{C}$, and requiring either a double integration or a single integral over a hypergeometric function, neither of the expressions for $\bar{C}_{\text{ub}}$ offers a viable path to MIMO generalization. With our approach, in contrast, the analysis of the spatial average becomes feasible also with MIMO.

B. MIMO

For $2 \times 2$ MIMO, the integration of $C_{2 \times 2}^{\text{MIMO}}(\rho)$ as given in (29) over $F_\rho(\cdot)$ as given in (22) returns (cf. Appendix B)

\[
\bar{C}_{2 \times 2}^{\text{MIMO}} \approx 2 s^* e^{\frac{2 + s^*}{D_\delta}} \left[ 2(2 + s^*)^2 - 4(2 + s^*)D_\delta + (8 + s^*(4 + s^*))D_\delta^2 \right] E_1(2/D_\delta) \frac{(2 + s^*)^3D_\delta^2 \log 2}{(2 + s^*)^3D_\delta^2 \log 2} + s^*D_\delta \left[ 2(8 + s^*(4 + s^*))D_\delta E_1(-s^*/D_\delta) - e^{s^*/D_\delta}(2 + s^*)((6 + s^*)D_\delta - 2(2 + s^*)) \right] \frac{(2 + s^*)^3D_\delta^2 \log 2}{(2 + s^*)^3D_\delta^2 \log 2} + \frac{\sin(\pi \delta) \log_2 e}{\pi} \left[ 2 C_{2,3}^{2,2}(1, 0, 1 - \delta) + 4 C_{2,3}^{2,2}(1, 0, 1 - \delta) + \frac{1 - \delta}{(1 + \delta) \delta} \right].
\]

An alternative expression not involving the Meijer-G function can be obtained using $E_3(x) \approx \frac{e^{-3x/2}}{2}$ and $2 e^{\frac{3}{2} E_1\left(\frac{2}{\rho}\right)} \log_2 e \approx 2.8 \log(1 + 0.41 \rho)$ to simplify $C_{2 \times 2}^{\text{MIMO}}(\rho)$ in (28) into

\[
C_{2 \times 2}^{\text{MIMO}}(\rho) \approx 2.8 \log(1 + 0.41 \rho) + e^{-1/\rho} \log_2 e.
\]

Using this form in place of (29) in the integration (cf. Appendix B),

\[
\bar{C}_{2 \times 2}^{\text{MIMO}} \approx 2 \log_2 e \frac{s^*}{(s^* - 1)e^{(s^* - 1)/D_\delta}} + _1F_1(\delta, 1 + \delta, -1) \sin(\delta) \log_2 e + 2.8 \left[ e^{-0.41 s^*} E_1(-s^*(0.41 + 1/D_\delta)) - E_1(-s^*/D_\delta) + e^{s^*/D_\delta} \log(1 + 0.41 D_\delta) \right] + 2.8 \left[ _2F_1(1, \delta, 1 + \delta, -2.44) + \delta \log(1.41) \right] \frac{\sin(\delta)}{\delta}.
\]

Example 8. Fig. 4 compares (39) and (41) against $\bar{C}_{\text{exact}}$. For $\eta = 4$ in particular, (41) returns

\[
\bar{C}_{2 \times 2}^{\text{MIMO}} \approx 0.26 + \log_2 e \frac{\sqrt{\pi}}{\pi} \text{erf}(1) + \frac{5.6}{\pi} \left[ \sqrt{0.41} \left( \pi - 2 \arctan(\sqrt{0.41}) \right) + \log 1.41 \right] = 3.84
\]

while the Monte-Carlo average of its $C_{\text{exact}}$ counterpart is 3.87.

Combining Examples 6 and 8, two-antenna single-user MIMO is seen from our analysis to provide a 93% increase in the spectral efficiency of an entire interference-limited network, a determination that would classically have entailed very extensive simulations.
VII. SECTORIZATION

Let us now incorporate cell sectorization to the model. Each BS, now allowed to comprise $S$ sector antennas uniformly staggered in azimuth, communicates with one user per channel and per sector (cf. Fig. 5). From the vantage of the typical user, the downlink signals from the sectors of any given BS undergo the same path loss and shadowing but different antenna gains.

Given an arbitrary azimuth pattern for the sector antennas, [19] characterized the local-average SINR distribution in the Laplace domain. Differently, we provide direct expressions that rely only on the antenna front-to-back ratio $Q \geq 1$ and on the number of sectors, $S$. Given an in-sector gain $G = \frac{Q S}{Q + S - 1}$ and an out-of-sector gain $g = \frac{S}{Q + S - 1}$, our model for the antenna pattern as a function of the azimuth $\phi$ is (cf. Fig. 6)

$$g_s(\phi) = \begin{cases} G & \phi_0 - \pi/S \leq \phi < \phi_0 + \pi/S \\ g & \text{elsewhere,} \end{cases}$$

where $\phi_0$ indicates the orientation of the antenna. This way, it is ensured that the total radiated power is preserved, i.e., that

$$\int_{0}^{2\pi} \frac{g_s(\phi)}{2\pi} \, d\phi = 1. \quad (44)$$
Fig. 6: Antenna pattern for $S = 3$ with in-sector gain $G$ and out-of-sector gain $g$.

Setting $S = 1$ we recover an unsectorized network where $g_1(\phi) = 1$. In turn, for $Q \to \infty$, the $S$ sectors become ideal as $G \to S$ and $g \to 0$.

Under the foregoing model, the intended signal from the serving sector has gain $G$ while the $(S - 1)$ interfering transmissions from other sectors of the same BS have gain $g$. The $S$ transmissions from every other BS add to the interference.

The small-scale fading in the link from each sector is independent and of unit-power, with the receiver knowing only the fading experienced by the intended signal.

A. Local-Average SINR

With $P$ now denoting the per-sector transmit power, the total power that each interfering BS launches towards the typical user is $P \left( G + (S - 1) g \right) = PS$. Since the useful signal launched by the intended BS is $PG$, the local-average SINR at the typical user is

$$\rho_s = \frac{PG r_0^{-\eta}}{P (S - 1) g r_0^{-\eta} + P S \sum_{k=1}^{\infty} r_k^{-\eta} + N_0},$$

irrespective of how the sectors are oriented at each BS. (This orientation-invariance is not an artifact of our model; rather, it has been shown that the distribution of $\rho_s$ is insensitive to the sector orientations regardless of the antenna patterns [19].)

In interference-limited conditions, the local-average SINR becomes

$$\rho_s = \frac{Q r_0^{-\eta}}{(S - 1) r_0^{-\eta} + (Q + S - 1) \sum_{k=1}^{\infty} r_k^{-\eta}},$$

which, with ideal sectorization, i.e., for $Q \to \infty$, converges to its unsectorized self, $\rho = r_0^{-\eta} / \sum_{k=1}^{\infty} r_k^{-\eta}$. It follows that, under ideal sectorization, all the results derived for unsectorized
networks continue to apply, only on a per-sector rather than a per-BS basis. Conversely, under nonideal sectors, the CDF of $\rho_S$ can be obtained as

$$F_{\rho_S}(\theta) = \mathbb{P}\left[\frac{Q r_0^{-\eta}}{(S - 1) r_0^{-\eta} + (Q + S - 1) \sum_{k=1}^{\infty} r_k^{-\eta}} < \theta\right]$$  \hfill (47)

$$= \mathbb{P}\left[\frac{r_0^{-\eta}}{\sum_{k=1}^{\infty} r_k^{-\eta}} < \frac{Q + S - 1}{Q/\theta - S + 1}\right]$$  \hfill (48)

$$= \left\{\begin{array}{ll}
F_{\rho}\left[\frac{Q+S-1}{Q/\theta - S + 1}\right] & 0 \leq \theta < \frac{Q}{S-1} \\
1 & \frac{Q}{S-1} \leq \theta \leq \frac{Q}{S-1}
\end{array}\right.$$  \hfill (49)

which is capped at $\frac{Q}{S-1}$ because of interference among same-BS sectors. Plugging (18) into (49),

$$F_{\rho_S}(\theta) \approx \begin{cases}
e^{-0.854 \frac{Q/\theta - S + 1}{Q+S-1}} A_0 \\
1 - \frac{\delta}{\frac{Q}{\theta} - S + 1} + B_0 \left[\frac{Q+S-1}{Q(1/\theta - 1/2 (S-1))}\right] & 0 \leq \theta < \frac{0.457 Q}{Q+1.457 (S-1)} \\
1 - \frac{\delta}{\frac{Q}{\theta} - S + 1} \delta & \frac{0.457 Q}{Q+1.457 (S-1)} \leq \theta < \frac{Q}{2Q+3 (S-1)} \\
1 & \frac{Q}{2Q+3 (S-1)} \leq \theta < \frac{Q}{Q+2 (S-1)} \\
& \frac{Q}{Q+2 (S-1)} \leq \theta < \frac{Q}{S-1} \\
& \frac{Q}{S-1} \leq \theta \leq \frac{Q}{S-1}
\end{cases}$$  \hfill (50)

which, for $\eta = 4$, specializes to

$$F_{\rho_S}(\theta) \approx \begin{cases}
e^{-0.854 \frac{Q/\theta - S + 1}{Q+S-1}} & 0 \leq \theta < \frac{0.457 Q}{Q+1.457 (S-1)} \\
0.154 & \frac{0.457 Q}{Q+1.457 (S-1)} \leq \theta < \frac{Q}{2Q+3 (S-1)} \\
\frac{0.457 Q}{Q+1.457 (S-1)} & \frac{Q}{2Q+3 (S-1)} \leq \theta < \frac{Q}{Q+2 (S-1)} \\
\frac{Q}{Q+2 (S-1)} & \frac{Q}{Q+2 (S-1)} \leq \theta < \frac{Q}{S-1} \\
& \frac{Q}{S-1} \leq \theta \leq \frac{Q}{S-1}
\end{cases}$$  \hfill (51)

Alternatively, plugging (22) into (49), a simpler and slightly less accurate expression is obtained, precisely

$$F_{\rho_S}(\theta) \approx \begin{cases}
e^{-0.854 \frac{Q/\theta - S + 1}{Q+S-1}} A_0 & 0 \leq \theta < \frac{s^* Q/\log(1-\delta)}{Q + [1 + s^*] / \log(1-\delta)} \\
1 - \delta & \frac{s^* Q/\log(1-\delta)}{Q + [1 + s^*] / \log(1-\delta)} \leq \theta < \frac{Q}{Q+1.457 (S-1)} \\
\frac{s^* Q/\log(1-\delta)}{Q + [1 + s^*] / \log(1-\delta)} & \frac{Q}{Q+1.457 (S-1)} \leq \theta < \frac{Q}{Q+2 (S-1)} \\
\frac{Q}{Q+2 (S-1)} & \frac{Q}{Q+2 (S-1)} \leq \theta < \frac{Q}{Q+1.457 (S-1)} \\
& \frac{Q}{Q+1.457 (S-1)} \leq \theta < \frac{Q}{S-1} \\
& \frac{Q}{S-1} \leq \theta \leq \frac{Q}{S-1}
\end{cases}$$  \hfill (52)

**Example 9.** Let $\eta = 4$. Shown in Fig. 7 are $F_{\rho}(\cdot)$ without sectorization and $F_{\rho_S}(\cdot)$ with $Q = 20$ dB, which is a rather typical front-to-back ratio.
Fig. 7: $F_{\rho}(\cdot)$ without sectorization and $F_{\rho_S}(\cdot)$ with $Q = 20$ dB, both for $\eta = 4$.

Although, as Fig. 7 visualizes, nonideal sectorization puts a hard ceiling on the SIR, this only affects the distribution modestly. In exchange, the available bandwidth gets to be reused $S$ times per cell and thus sectorization is decidedly advantageous.

B. Ergodic Spectral Efficiency

As in Section V, the CDF of the ergodic spectral efficiency $F_C(\cdot)$ is characterized by mapping the applicable function $C(\rho_S)$ onto $F_{\rho_S}(\cdot)$. For Rayleigh-faded SISO channels in particular, we can invoke (24) to explicitly express this mapping as

$$F_C(\gamma) \approx F_{\rho_S} \left( \frac{e^{\gamma+1}}{0.82} \right).$$

(53)

Example 10. Reconsider Examples 3 and 4, but now with $S = 3$ sectors having front-to-back ratio $Q = 20$ dB. Shown in Fig. 8 are (53), the numerical mapping of $C(\rho)$ onto (50) for SISO and MIMO, and the simulated $C_{\text{exact}}$.

For any setting for which $C(\rho_S)$ is available, we can also compute the spatially averaged
ergodic spectral efficiency as
\[
\bar{C} = \int_{0}^{\frac{Q}{S-1}} C(\theta) \, dF_{\rho_{S}}(\theta).
\] (54)

**Example 11.** Let $\eta = 3.8$, $S = 3$ and $Q = 20$ dB. For SISO, $\bar{C} = 1.53$ b/s/Hz per sector, which adds up to 4.59 b/s/Hz per BS. For $S = 1$, in contrast, a read-out of Fig. 4 gives $\bar{C} = 1.84$ b/s/Hz per BS. With the sector nonideality accounted for, the overall spectral efficiency of a SISO network gets multiplied by 2.5 when cells are split in three sectors.

For $2 \times 2$ MIMO, in turn, $\bar{C} = 2.95$ b/s/Hz per sector adding up to 8.85 b/s/Hz per BS. The sectorization multiplier is 2.49, almost unchanged from its SISO value.

**VIII. Application to Lattice Networks**

Before wrapping up the paper, we can close the loop and verify the initial premise whereby a PPP model was invoked for the BS locations, confirming that such a model is always representative because of shadowing. To that end, we compare our PPP-based analytical results with
Monte-Carlo simulations for settings at the extreme end of the point process spectrum, namely regular lattices.

**Example 12.** Shown in Fig. 9 is $F_C(\cdot)$ as given in (26) for SISO, in comparison with the results for a lattice of 977 hexagonal cells with $\eta = 4$ and various values of $\sigma_{\text{dB}}$. The convergence is conspicuous and, most importantly, the agreement is excellent for typical outdoor values of $\sigma_{\text{dB}}$, in the range of 10–12 dB.

**Example 13.** The comparison of Fig. 9 is repeated, for $S = 3$ sectors and $Q = 20$ dB, in Fig. 10, with similar observations in terms of the convergence.

**Example 14.** Shown in Fig. 11 is $\bar{C}$ as given in (33), for SISO, in comparison with the results for a lattice of 977 hexagonal cells with various values of $\sigma_{\text{dB}}$ and $\eta$. 

---

**Fig. 9:** CDF of SISO ergodic spectral efficiency with no sectorization ($S = 1$): PPP-based analytical result vs Monte-Carlo over a lattice network of hexagonal cells with $\sigma_{\text{dB}} = 0$ dB, 10 dB and 14 dB. The dashed curve corresponds to using $F_\rho(\theta/2.188)$ in lieu of $F_\rho(\theta)$ and proceeding as if the network conformed to a PPP. In all cases, $\eta = 4$. 

- $\sigma_{\text{dB}} = \{0, 10, 14\}$
- Simulation, hexagonal grid

- (26)
- (26), SIR scaled by 2.18

$\gamma$ (bits/s/Hz) vs $F_C(\gamma)$
Example 15. The comparison of Fig. [11] is repeated, for $S = 3$ and $Q = 20$ dB, in Fig. [12].

Further reinforcing the relevance of PPP-based results to lattice networks, it has been argued in [51]–[53] that the SINR distribution of a shadowless lattice network is essentially a shifted version of its PPP counterpart, i.e., a shifted version of the SINR distribution for asymptotically strong shadowing. Moreover, this shift does not depend on the path loss exponent, but only on the type of lattice.

Example 16. For a triangular lattice (hexagonal cells), the shift to be applied is $10 \log_{10} 2.188 = 3.4$ dB [51]–[53]. Shown in Fig. [9] is $F_C(\cdot)$ recomputed with $F_{\rho}(\theta/2.188)$ in lieu of $F_{\rho}(\theta)$, and the agreement with $F_C(\cdot)$ for a lattice network of hexagonal cells and no shadowing is highly satisfactory.

Altogether then, PPP results enable characterizing the distributions of SINR and spectral efficiency both with (asymptotically strong) and without shadowing, bracketing the range of
Fig. 11: Spatially averaged SISO ergodic spectral efficiency as function of $\eta$ with no sectorization ($S = 1$): PPP-based analytical result vs Monte-Carlo over a lattice network of hexagonal cells with $\sigma_{dB} = 0$ dB, 10 dB and 14 dB. The error bars around the simulation results indicate the 99% confidence interval.

values that a given network can exhibit over all possible shadowing strengths.

IX. SUMMARY

By decoupling small- and large-scale channel features and abstracting the former via local ergodicity, the stochastic geometry analysis of wireless networks can focus crisply on the large-scale properties. Jointly with a Gaussian model for the aggregate interference that recognizes that the fading of each term therein is unknown, this enables circumventing analytical roadblocks and deriving expressions that are simpler, more accurate, and more open to generalization, readily accommodating aspects such as MIMO or sectorization. The obtained spectral efficiencies lower-bound the exact values with a degree of accuracy that justifies writing $C \lesssim C_{\text{exact}}$.

Thanks to the PPP-likeness brought about by shadowing, the obtained expressions apply to any stationary and ergodic network model exhibiting reasonable shadowing strengths. Furthermore, with a proper shift, these expressions apply also to shadowless networks, which are interesting
insofar as they can model planned deployments where the BS locations are dependent on the radio propagation. Thus, PPP analysis can serve to bracket the entire performance range in a given environment.

Although we have largely concentrated on interference-limited networks, the approach propounded in this paper is extensible to settings where noise is significant, and the accuracy of the results could only improve even further given the Gaussian and unfaded nature of noise.

Besides the addition of noise, numerous other extensions are invited, for instance multiuser MIMO or BS cooperation [54]. When dealing with MIMO, care must be exercised whenever \( N_t > N_i \) and, especially, whenever \( N_t \gg N_i \); then, if the fading of dominant interferer(s) can be tracked, spatial color can be exploited [55]. This can be accounted for by circumscribing our unfaded interference model to the rest of the interference, separately incorporating the terms that correspond to interferers whose fading is known.
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APPENDIX A

DERIVATION OF (27)

With SINR given by (3),

\[
\mathbb{E} \left[ \log_2 \left( 1 + \text{SINR} \right) \right] = \int_0^\infty \mathbb{P} \left[ \log_2 \left( 1 + \text{SINR} \right) > y \right] dy \quad (55)
\]

\[
= \int_0^\infty \frac{\log_2 e}{1 + x} F_{\text{SINR}}(x) dx \quad (56)
\]

where (56) follows from the variable change \( y = \log_2 (1 + x) \) and the CCDF \( F_{\text{SINR}}(x) \) can be computed as

\[
F_{\text{SINR}}(x) = \mathbb{P} \left[ \frac{r_0^{-\eta} |H_0|^2}{P \sum_{k=1}^\infty r_k^{-\eta} |H_k|^2 + N_0} > x \right] \quad (57)
\]

\[
= \mathbb{E} \left[ e^{-x r_0^{-\eta} \left( \sum_{k=1}^\infty r_k^{-\eta} |H_k|^2 + N_0/P \right)} \right] \quad (58)
\]

\[
= e^{-x r_0^{-\eta} N_0/P} \mathbb{E} \left[ \prod_{k=1}^\infty e^{-x r_0^{-\eta} |H_k|^2} \right] \quad (59)
\]

\[
= e^{-x r_0^{-\eta} N_0/P} \prod_{k=1}^\infty \frac{1}{1 + x \left( r_0/r_k \right)^{\eta}} \quad (60)
\]

where (58) follows from the exponential distribution of \( |H_0|^2 \) and the expectation is over \( \{H_k\}_{k=0}^\infty \). In turn, (60) follows from the fact that \( \{H_k\}_{k=1}^\infty \) are IID.

APPENDIX B

DERIVATIONS OF (33), (39) AND (41)

Plugging the PDF obtained by differentiating (22) into (32),

\[
\tilde{C} \approx \int_0^{\log(1 + \sin \delta)} C(\theta) \frac{-e^{s^*/\theta} S^*}{\theta^2} d\theta + \int_1^\infty C(\theta) \frac{\delta \sin \delta}{\theta^{\delta+1}} d\theta. \quad (61)
\]

From \( C(\cdot) \) as given in (15) and (29), the above integrals yield (33) and (39), respectively. These integrations are facilitated by invoking \( E_1(x) = -E_1(-x) \), where \( E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt \), in conjunction with the identities given in [56] with appropriate variable changes.
Similarly, from $C(\cdot)$ as given in (40), integration by parts in (61) using the identities [57, 2.325.6], [57, 2.325.7], [57, 2.728.1] and [57, 3.194.2] with appropriate variable changes gives the expression claimed in (41).

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