Generalized Klein-Nishina formula

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The generalized Klein-Nishina formula for Compton scattering of charged particles by a finite train of pulses is derived in the framework of quantum electrodynamics. The formula also applies to classical Thomson scattering provided that frequencies of generated radiation are smaller that the cut-off frequency. The validity of the formula for incident pulses of different durations is illustrated by numerical examples. The positions of the well-resolved Compton peaks, with the clear labeling by integer orders, opens up the possibility of the precise diagnostics of properties of relativistically intense, short laser pulses. This includes their peak intensity, the carrier-envelope phase, and their polarization properties.

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I. INTRODUCTION

In order to understand the physics behind the interaction of strong laser pulses with matter, it is necessary to have well-characterized interacting pulses. With current technology, this is accomplished for laser fields in the optical regime provided that their intensity is no larger than $10^{15}$ W/cm$^2$. As pointed out by many authors (see, for instance, Refs. [1–6]), the same task becomes particularly challenging at higher intensities, where any direct measurement is prone to damaging the equipment. Therefore, different proposals were put forward to determine properties of ultra-strong and short laser pulses (e.g., the laser peak intensity [1–5] or the carrier-envelope phase [6]). As we will argue in this paper, a sensitivity to the properties of pulses comprising the train. This can be based on the generalized Klein-Nishina (GKN) formula, which we derive in this paper.

A. Klein-Nishina formula

The original Klein-Nishina formula [8] for the Compton scattering with the electron initial four-momentum $p_i [p_i \cdot p_i = (m_e c)^2]$, where ‘dot’ means the relativistic scalar product, $a \cdot b = a^i b^i - a^2 b^2 - a^3 b^3 = a^i b^i - a \cdot b$, has the form

$$\omega_K = \frac{\omega}{\frac{p_i \cdot p_K}{p_i \cdot n} + \frac{1}{\omega_{cut}}}.$$  \hspace{0.5cm} (1)

where $k = (\omega/c)n$ and $K = (\omega_K/c)n_K$ are the four-momenta of the initial and final photons, respectively, $n = (1, n)$, $n_K = (1, n_K)$, and $k \cdot k = K \cdot K = 0$. In this equation,

$$\omega_{cut} = \frac{cp_i \cdot n}{hm \cdot n_K}.$$  \hspace{0.5cm} (2)

is the maximum frequency of photons generated in the Compton process.

Consider the Compton process which takes place in an intense monochromatic plane wave of the frequency $\omega$, propagating in the direction $n$. The vector potential of, in general, the elliptically polarized plane wave equals

$$A(k \cdot x) = A_0 (\varepsilon_1 \cos(k \cdot x) \cos \delta + \varepsilon_2 \sin(k \cdot x) \sin \delta),$$  \hspace{0.5cm} (3)

where $\varepsilon_j (j = 1, 2)$ are two real polarization four-vectors normalized such that $\varepsilon_j \cdot \varepsilon_j' = -\delta_{jj'}$ and $k \cdot \varepsilon_j = 0$. Without the loss of generality, we assume that the time-components of these vectors are zero, i.e., $\varepsilon_j = (0, \varepsilon_j)$. If the electron absorbs $N = 1, 2, \ldots$ photons from the plane wave it may emit, in the direction $n_K$, the Compton photon of frequency $\omega_{K,N}$ [8],

$$\omega_{K,N} = \frac{N \omega}{\omega_{cut} + \frac{cp_i \cdot n}{m_e c} + \frac{U}{mc}}.$$  \hspace{0.5cm} (4)

where

$$U = \frac{1}{4} \mu^2 (m_e c^2)^2$$  \hspace{0.5cm} (5)

has the meaning of the ponderomotive energy of electrons in the laser field. Here, the relativistically invariant parameter,

$$\mu = \frac{|e| A_0}{m_e c},$$  \hspace{0.5cm} (6)

determines the intensity of the electromagnetic plane wave. Both $U$ and $\mu$ are the classical quantities, whereas the term proportional to $\omega/\omega_{cut}$ in (4) accounts for the quantum recoil of electrons during the Compton process. Such a recoil of electrons does not take place in the corresponding classical process, that is called the Thomson scattering. This allows to introduce into the Klein-Nishina formula the classical Thomson frequency,

$$\omega_{Th,N} = \frac{N \omega}{\omega_{cut} + U \frac{m_e c}{m_e c} + \frac{\omega_{cut}}{U}}.$$  \hspace{0.5cm} (7)

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such that

\[ \omega_{K,N} = \frac{\omega_{Th}^{K,N}}{1 + \frac{\omega_{cut}^{K,N}}{\omega_{K,N}}} \]  

(8)

Note that, for a given geometry of the process and for a
given initial electron momentum, the Thomson frequencies
are equally separated from each other. The same is
not true for the Compton frequencies. This scaling law,
which relates the quantum Compton frequency \( \omega_{K,N} \) to
its classical analog, the Thomson frequency \( \omega_{Th,K,N} \),
has been discussed recently for long laser pulses \[15–17\]. Its
extension to short laser pulses together with the investi-
gation of polarization and spin effects, and the synthesis
of ultra-short pulses have been presented in \[18\].

The Klein-Nishina formula \[4\] can be also expressed
in the form

\[ N = \frac{\omega_{K,N} \omega_{cut}}{\omega(\omega_{cut} - \omega_{K,N})} \left( \frac{p_i \cdot n_K}{p_i \cdot n} + \frac{U}{c} \frac{n \cdot n_K}{p_i \cdot n} \right), \]  

(9)

which, for the given geometry of the scattering process
and for the given frequency of the emitted Compton pho-
ton, allows to determine the nonlinearity of the process,
\( N \). Note that the quantum nature of this formula is hid-
en in \( \omega_{cut} \). Therefore, we recover the classical result in
the limit when \( \omega_{K,N} \ll \omega_{cut} \). Similar formulas can be
derived for multichromatic plane-wave-fronted fields with
commensurate frequencies.

\section{B. Spectrally resolved Compton peaks induced by}
short laser pulses

If the Compton process occurs in a short and intense
laser pulse, the situation is different. It follows from the
time-frequency uncertainty principle that, if the driving
pulse lasts for time \( T_p \), the frequency scale over which the
system undergoes a significant change cannot be smaller
than roughly \( 2\pi/T_p \). For this reason, the individual
peaks in the Compton frequency spectrum are hardly visible if the process occurs in few-cycle pulses (see, e.g.,
Refs. \[15\], \[16\], \[19\], \[21\]). Now, the question arises: Is it pos-
sible to design a short laser pulse such that the individual
peaks in the laser-induced Compton spectrum are clearly
distinguished from each other, with unambiguously pre-
scribed to them integer orders \( N \)? Although the answer
to this question is in general negative, for suitably de-
signed pulses, one can achieve the limit imposed by the
above-mentioned uncertainty relation. The idea of how to
avoid the spectral broadening follows from the Fraunho-
fer diffraction theory as applied to the diffraction grating,
or from the frequency comb generation, and is based on
the application of modulated laser pulses \[22\], \[23\]. We
will demonstrate in this paper that, by using this tech-
nique, it is possible to achieve the clear spectral resolu-
tion of individual peaks in the Compton spectrum, even
if driven by few-cycle laser pulses.

The Compton scattering by short laser pulses is very
sensitive to the precise form of the driving pulse. It was
pointed out in \[9\] that this sensitivity can be used to char-
acterize the driving laser fields. In Ref. \[6\], the angular
distributions of Compton radiation were traced back to
the carrier-envelope phase of the driving pulse. As we
will point out, the spectrally resolved peaks in the Com-
pton distribution allow for determining properties of the
driving train of pulses. This is done by mapping the po-

tions of the Compton peaks to the GKN formula, that
we derive in this paper for an arbitrary, finite train of
pulses.

The paper is organized as follows. In the next sec-
tion, the theory of Compton scattering in finite plane-
wave-fronted pulses is presented, which is then followed
by the discussion of the diffraction formula (Sec. \[III\]).
In Sec. \[V\] we present the GKN formula. It describes
the major peaks in the Compton spectrum of radiation
induced by a finite train of pulses. Since the formula de-
dpends on properties of the individual pulse from the train,
one may exploit the properties of the Compton spectra
in the diagnostics of relativistically intense, short laser
pulses. This is illustrated in Sec. \[V\] for one- and
three-cycle pulses. Sec. \[VI\] contains concluding remarks.

In analytic formulas we keep \( \hbar = 1 \) and, hence, the
finite-structure constant becomes \( \alpha = e^2/(4\pi\varepsilon_0c) \). Unless
stated otherwise, in numerical analysis we use relativistic
units (rel. units) such that \( \hbar = m_e = c = 1 \), where \( m_e \) is the
electron mass.

\section{II. COMPTON SCATTERING

The probability amplitude for the Compton process,
\( e_{p_{\lambda_1}} \rightarrow e_{p_{\lambda_1}} + \gamma_K e_{\sigma} \), with the initial and final electron
momenta and spin polarizations \( p_{\lambda_1} \) and \( p_{\lambda_2} \), respec-
tively, equals \[21\]

\[ \mathcal{A}(e_{p_{\lambda_1}} \rightarrow e_{p_{\lambda_1}} + \gamma_K e_{\sigma}) = -ie \int d^4x \, j_{p_{\lambda_1}p_{\lambda_1}e_{\sigma}}^{(++)}(x) \mathcal{A}_{K\sigma}^{(-)}(x). \]  

(10)

Here, \( K\sigma \) denotes the Compton photon momentum and
polarization, and

\[ \mathcal{A}_{K\sigma}^{(-)}(x) = \sqrt{\frac{1}{2\varepsilon_0\omega_{K\sigma}}} e^{i\mathbf{K}\cdot\mathbf{x}}, \]  

(11)

where \( V \) is the quantization volume, \( \omega_{K\sigma} = cK_0 = c|K| \)
(\( K \cdot K = 0 \)), and \( \varepsilon_{K\sigma} = (0, \varepsilon_{\sigma \sigma'}) \) are the polarization
four-vectors satisfying the conditions \( K \cdot \varepsilon_{K\sigma} = 0 \)
and \( e_{K\sigma} \cdot \varepsilon_{K\sigma'} = -\delta_{\sigma\sigma'}, \) for \( \sigma, \sigma' = 1,2 \). Moreover,
\( j_{p_{\lambda_1}p_{\lambda_1}e_{\sigma}}^{(++)}(x) \) is the matrix element of the electron current
operator with its \( \nu \)-component equal to

\[ [j_{p_{\lambda_1}p_{\lambda_1}e_{\sigma}}^{(++)}(x)]^\nu = \bar{\psi}_{p_{\lambda_1}p_{\lambda_1}}^{(++)}(x) \gamma^\nu \psi_{p_{\lambda_1}p_{\lambda_1}}^{(++)}(x). \]  

(12)

Here, \( \psi_{p_{\lambda}}^{(++)}(x) \) is the Volkov solution of the Dirac equation
coupled to the electromagnetic field \[24\].
The Volkov solution of the Dirac equation for electrons of the four-momentum $p = (\rho^0, p)$, $\rho \cdot p = m_e c^2$, is of the form

$$
\psi_{p\lambda}^{(+)}(x) = \sqrt{\frac{m_e c^2}{V E_p}} \left[ 1 - \frac{e}{2k \cdot p} A(k \cdot x) \right] u_{p\lambda}^{(+)} e^{-iS_p^{(+)}(x)},
$$

where

$$
S_p^{(+)}(x) = p \cdot x + \int_0^{k \cdot x} \left[ e A(\phi) \cdot p - \frac{e^2 A^2(\phi)}{2k \cdot p} \right] d\phi,
$$

and $E_p = cp^0$ and $u_{p\lambda}^{(+)}$ is the free-electron bispinor normalized such that $\bar{u}_{p\lambda}^{(+)} u_{p\lambda}^{(+)} = \delta_{\lambda\lambda}$, with $\lambda = \pm$ labeling the spin degrees of freedom. The electromagnetic vector potential $A(k \cdot x)$ is assumed to be an arbitrary function of $k \cdot x$, with $k \cdot k = 0$ and $\omega = c k^0$. Let the laser pulse lasts for time $T_p$. Therefore, by choosing $\omega = 2\pi / T_p$ we can assume that $A(k \cdot x)$ vanishes for $k \cdot x < 0$ and $k \cdot x > 2\pi$. This allows to interpret the label $p$ of the Volkov wave as the electron momentum in the remote past/future.

In analogy to the Bloch theorem in solid state physics, we can introduce the electron quasi-momentum $\bar{p}$. It describes the electron dressing by the electromagnetic field,

$$
S_p(x) = \bar{p} \cdot x + G_p(k \cdot x),
$$

where, for the most general, elliptically polarized plane-wave-fronted pulse,

$$
A(k \cdot x) = A_0 [\varepsilon_1 f_1(k \cdot x) + \varepsilon_2 f_2(k \cdot x)],
$$

the laser-dressed momentum has the form

$$
\bar{p} = p + \mu m_e c \left( \frac{p \cdot \varepsilon_1}{p \cdot n} f_1 + \frac{p \cdot \varepsilon_2}{p \cdot n} f_2 \right) n + \frac{1}{2} (\mu m_e c)^2 \left( \frac{f_1^2}{p \cdot n} + \frac{f_2^2}{p \cdot n} \right) n.
$$

Here, the parameter $\mu$ is defined according to Eq. (10). The so-called shape functions, $f_j(k \cdot x)$, $j = 1, 2$, are arbitrary functions with continuous second derivatives that vanish outside the interval $(0, 2\pi)$. For any of such functions $F(\phi)$, we define

$$
\langle F \rangle = \frac{1}{2\pi} \int_0^{2\pi} F(\phi) d\phi.
$$

Note that the pulses with plane wavefronts, which are considered in this paper, very well describe the interaction of laser fields with energetic electrons. This is provided that the kinetic energy of electrons is much larger than their ponderomotive energy in the laser field (see, e.g., Ref. 22).

Our definition of the laser-dressed momentum follows directly from the Volkov solution. For the plane wave, the polarization-dependent terms in Eq. (17) vanish, since $(f_j) = 0$. Note that the laser-dressed momentum, as the quantity defined in the laser field, cannot be a physical observable. It follows, however, from Eqs. (13) and (15) that the difference $(\bar{p} - \bar{p}_0)$ (up to a four-vector proportional to $k$) can be directly measured in an experiment, as it uniquely determines the Compton frequency $\omega_{\text{KC}}$ [see, Eq. (21) below]. This means that, in principle, we can redefine the dressed momentum by adding an arbitrary four-vector, that is independent of $p$ and vanishes outside the laser pulse. By further assuming that this four-vector is not space- and time-dependent, and that it should be determined only by the four-vectors present in the definition of the laser pulse, one can consider the following modification

$$
\bar{p} \rightarrow \bar{p} + g_1 \varepsilon_1 + g_2 \varepsilon_2 + g_0 k,
$$

with $g_j = 0$ in the absence of the laser field. It appears that this dressed momentum is on the mass-shell (i.e., $\bar{p} \cdot \bar{p}$ is independent of $p$) for a particular choice of $g_j$,

$$
g_1 = \mu m_e c (f_1), \quad g_2 = \mu m_e c (f_2), \quad g_0 = 0,
$$

for which

$$
\bar{p} \cdot \bar{p} = (\bar{m}_e c)^2 = (m_e c)^2 \left[ 1 + \frac{2U}{m_e c^2} \right],
$$

where $\bar{m}_e$ is called the electron dressed mass, and

$$
U = \frac{1}{2} \left( f_1^2 - \langle f_1 \rangle^2 + f_2^2 - \langle f_2 \rangle^2 \right). \tag{22}
$$

This result has led the authors of Ref. 27 to the conclusion that the electron mass shift in a laser field could be measured by comparing the spectrum of Compton radiation induced by two different pulses, but of the same energy. However, it follows from our analysis that the electron mass shift can be well-defined only for the particular choice of the momentum dressing [Eqs. (19) and (20)]. This can create doubts about the physical nature of this quantity.

Indeed, the quantity defined in Eq. (22) is the direct generalization of the ponderomotive energy in the electron reference frame for the finite laser pulses; in the arbitrary reference frame the ponderomotive energy can be defined as the time-component of the ponderomotive four-momentum $A$ multiplied by the speed of light $c$, i.e., as $U = cu$, where (cf., Ref. 25 for the case when $\langle A \rangle = 0$),

$$
u^c = - \frac{c^2}{2} \frac{\langle A \cdot A \rangle - \langle A \rangle \cdot \langle A \rangle}{p \cdot k} k^\nu. \tag{23}
$$

This allows to define the dressed mass in the relativistically invariant form

$$
\bar{m}_e^2 = \frac{1}{c^2} (p + u)^2, \tag{24}
$$

which is independent of both the electron momentum $p$ and the fundamental frequency $\omega$, as well as of the laser pulse polarization vectors $\varepsilon_j$. Eq. (24), among others,
has lead Reiss \cite{29} to the critique of the concept of the electron mass shift in a laser field. Note that such a dressing does not follow from the prescription \cite{13} if the parameters $g_j$ are $p$-independent and, in general, cannot be related to the ‘quasi-momentum’ for the Volkov solution.

It is not the purpose of this paper to take part in the discussion concerning the mass shift. However, independently of the physical interpretation, both the mass shift and the ponderomotive momentum are uniquely defined by particular laser pulse characteristics; namely, by $\mu^2((f_1^2 + f_2^2))$ and $\mu(f_j)$ for $j = 1, 2$. It appears that also the Compton photon frequency depends on them (see, e.g., Eqs. (29) and (30) in Ref. \cite{18}). This indicates that there should exist an experimental method which, for example, by applying the Boca-Florescu transformation \cite{19}, the frequency-angular distribution shown in Ref. \cite{21} that, by applying the Boca-Florescu transformation, is uniquely defined by the difference of dressed momenta, $(\vec{p} - \vec{p}_i)$. For this reason one can select any form of the electron momentum dressing. Below, we adopt our definition, Eq. (17), as for such a choice the following equations hold: $\vec{p} \cdot n = p \cdot n, \vec{p} \cdot \varepsilon_j = p \cdot \varepsilon_j$ and $\vec{p}^i = \vec{p}^\perp$. That significantly simplifies analytical calculations.

In the following, we shall consider the linearly polarized laser pulse such that, for $0 \leq \phi = k \cdot x \ll 2\pi$, the vector potential has a general form $A(\phi) = A_0 \varepsilon f(\phi)$ [i.e., we put $\varepsilon = \varepsilon_1$, $f(\phi) = f_1(\phi)$ and $f_2(\phi) = 0$] and the electric field vector equals $\mathbf{E}(\phi) = -\omega A_0 \varepsilon f'(\phi)$. The shape function $f(\phi)$ is defined via its derivative,

\[
f'(\phi) = \begin{cases} 0, & \phi < 0, \\
N_f \sin^2(N_{\text{rep}} f) \sin(N_{\text{rep}} N_{\text{osc}} \phi), & 0 \leq \phi \leq 2\pi, \\
0, & \phi > 2\pi,
\end{cases}
\]

where we assume that $f(0) = 0$. Above, the integers $N_{\text{rep}}$ and $N_{\text{osc}}$ determine the number of identical pulses in a train and the number of cycles in each pulse, respectively, whereas $N_f$ is a suitably chosen normalization constant. Since the duration of the laser pulse is $T_p$, we can also define the fundamental frequency, $\omega = 2\pi/T_p$, and the central one, $\omega_L = N_{\text{rep}} N_{\text{osc}} \omega$ of the laser field, which is supposed to be fixed and equal to $\omega_L = 1.55eV \approx 3 \times 10^{-6}mc^2$ in all calculations presented here. Moreover, in the following we assume that $N_f = N_{\text{rep}} N_{\text{osc}}$, which guarantees that the time-averaged intensity of the laser pulse is independent of $N_{\text{rep}}$ and $N_{\text{osc}}$, as for this particular selection the amplitude of the electric field scales as $\omega_L \mu$. Note that, for the rectangular pulse, the sin$^2$ envelope is not present in (32), meaning that the laser pulse depends only on the product $N_{\text{rep}} N_{\text{osc}}$.

In analogy to the original Klein-Nishina formula [Eqs. (4) and (5)], we present the frequency spectrum as a function of $N$,

\[
N = \frac{\omega K \omega_{\text{cut}}}{\omega L (\omega_{\text{cut}} - \omega K)} \left( \frac{\vec{p}_i \cdot n K}{\vec{p}_i \cdot n} + \frac{U n \cdot n K}{c \vec{p}_i \cdot n} \right).
\]

In this case, we expect that in the limit of a very long pulse the peaks in the spectrum will appear for $N$’s very close to integers, as it indeed takes place for a monochromatic plane wave.
In Fig. 1 we present the angular-resolved distributions of Compton radiation [Eq. (27)] generated by a single pulse, with $N_{osc} = 30$ field oscillations. This distribution is presented as the function of $\mathcal{N}$. One could expect that, for such a long driving pulse, the Klein-Nishina formula could be successfully used and that the dominant peaks would correspond to integer values of $\mathcal{N}$. Clearly, we do not observe such a behavior as the peaks are red-shifted, independently of the Compton photon polarization. As we will show in the next section, the situation is changed when a train of incident pulses is considered.

III. DIFFRACTION FORMULA

It was shown in [23] that, for a finite train of pulses, the Compton probability amplitude has the diffraction-type form,

$$\mathcal{A}_{C,\sigma}(\omega_{\mathbf{K}}, \lambda_i, \lambda_f) = \exp \left[ \Phi_{C,\sigma}(\omega_{\mathbf{K}}, \lambda_i, \lambda_f) \right] \times \frac{\sin(\pi \bar{Q}^+/k^0)}{\sin(\pi \bar{Q}^+/k^0 N_{rep})} \left| \mathcal{A}_{C,\sigma}^{(1)}(\omega_{\mathbf{K}}, \lambda_i, \lambda_f) \right|,$$

where $\mathcal{A}_{C,\sigma}^{(1)}$ is the Compton amplitude for a single pulse and $\Phi_{C,\sigma}(\omega_{\mathbf{K}}, \lambda_i, \lambda_f)$ is the Compton global phase. In the above equation $\bar{Q}^+ = \bar{p}^+_1 - \bar{p}^+_2 - K^+$, where, for an arbitrary four-vector $a$, we define $a^+ = a^0 - (a \cdot n)/2 = (a^0 + a \cdot n)/2$. For particular frequencies of emitted photons ($\omega_{\mathbf{K},N}$ with integer $N$), that satisfy the condition

$$\pi \bar{Q}^+ = -\pi NN_{rep}k^0,$$

we observe the coherent enhancement of the Compton amplitude. This, in turn, leads to the quadratic, $N_{rep}$, enhancement of the respective probability distribution. In contrast to the classical Thomson process, these frequencies are not exactly equally spaced in the allowed frequency region, $0 < \omega_{\mathbf{K}} < \omega_{cut}$. When $\omega_{\mathbf{K}}$ approaches the cut-off value $\omega_{cut}$ [Eq. (2)], i.e., when the quantum recoil of the scattered electron cannot be neglected, the spectrum of $\omega_{\mathbf{K},N}$ becomes increasingly denser. This means that one can generate the Compton-based frequency combs with equidistant peak frequencies only within limited frequency intervals.

The Compton global phase equals

$$\Phi_{C,\sigma}(\omega_{\mathbf{K}}, \lambda_i, \lambda_f) = -\frac{\pi \bar{Q}^+}{k^0} + \Phi_{C,\sigma}^{dyn}(\omega_{\mathbf{K}}, \lambda_i, \lambda_f),$$

where $\Phi_{C,\sigma}^{dyn}$ is the so-called dynamic phase [23]. For arbitrary laser pulses and polarizations of emitted photons, the dynamic phase can only be calculated numerically. It happens that, for pulses considered in this paper, the dynamic phase is independent of $\omega_{\mathbf{K}}$. This means that, for frequencies $\omega_{\mathbf{K},N}$ satisfying the condition (33), the global phase is equal to

$$\Phi_{C,\sigma}(\omega_{\mathbf{K},N}, \lambda_i, \lambda_f) = \pi N N_{rep} + \Phi_{C,\sigma}^{dyn}(\omega_{\mathbf{K},N}, \lambda_i, \lambda_f),$$

and, hence, it takes on the same values modulo $\pi$. The selection of these particular phases for the peak frequencies leads not only to the enhancement of the frequency spectrum, but also to the synthesis of ultra-short pulses of radiation generated during the Compton scattering [30].

As an illustration, we consider the same laser pulse as in Fig. 1 but repeated $N_{rep}$ times. In Fig. 2 we show the Compton spectra for $N_{rep} = 1, 2$, and $4$, when divided by $N_{rep}$. The results are presented as the functions of $N_{osc}/\mathcal{N}$. The clearly visible peaks occur for $N_{rep} > 1$. Their positions are almost independent of $N_{rep}$ and they correspond to the integer values of $N_{osc}/\mathcal{N}$. Note that $\omega = \omega_N/N_{osc}$ is the fundamental frequency of the individual laser pulse from the train. Such a pulse can be approximately interpreted as a coherent superposition of at most $N_{osc}$ photon states of frequencies $K\omega$. 

FIG. 1. (Color online) Shows the spectra of Compton radiation [Eq. (27)] resulting from the head-on collision of the linearly polarized laser pulse and an electron of momentum $p_i = -10^3 m_e c \epsilon_z$. The laser pulse ($\mu = 1$ and $\omega_i = 1.55$ eV) propagates along the $z$ direction and is polarized along the $x$ direction. These two axes determine the scattering plane. The pulse has a $\sin^2$ envelope [Eq. (2)] with $N_{osc} = 30$ and $N_{rep} = 1$. The Compton photon is emitted in the direction of $\theta_{\mathbf{K}} = 0.9999\pi$ and $\varphi_{\mathbf{K}} = \pi$, with the polarization vector either parallel (upper panel) or perpendicular to the scattering plane (lower panel). The energy spectra are presented as functions of $\mathcal{N}$, where the vertical lines mark the integer values of this argument.
K = 1, . . . , Nosc (of course, photons with other frequencies are also present, but with smaller amplitudes). Therefore, in the course of the Compton scattering, the electron can absorb these photons with the total energy Nω and emit a single photon of frequency ω(K,N). However, due to the time-frequency uncertainty relation and an incoherent interference of probability amplitudes, the spectrum is smeared out such that it is not possible to clearly prescribe peaks to orders N. This is clearly seen in Figs. 1 and 2. The situation changes if we consider the sequence of at least two such pulses. Now, due to the constructive interference, the processes with integer NoscN are coherently enhanced. As a result, we observe in the spectrum the clearly resolved peaks already for Nrep = 2.

The coherent enhancement of the Compton frequency spectra does not take place for a single pulse with the time-varying envelope. It appears, however, that important features of a single pulse can be precisely determined from positions of the main diffraction peaks in the Compton spectrum, when generated by a train of such pulses.

IV. GENERALIZED KLEIN-NISHINA FORMULA

As we have demonstrated above, the application of the pulse train with two subpulses already allows to increase the resolution of the frequency spectrum of Compton radiation such that one can unambiguously prescribe an integer number to the individual peaks. We have shown this for a long pulse (Nosc = 30), for which (f) is negligibly small, so that the original Klein-Nishina formula may be applied. For shorter pulses, (f) starts to be significantly different than zero and the generalization of the Klein-Nishina formula, that accounts for this fact, is necessary. We apply the diffraction formula and determine the Compton photon frequency by solving the system of equations (39) and (40). After some algebra, we arrive at the following GKN formula valid for a pulse train of an arbitrary polarization,

\[
\omega_{K,N} = \frac{\langle N/Nosc \rangle \omega_L}{(p_i \cdot n_K)} + \frac{\nu n \cdot \epsilon + g_1 p_{i,1} + g_2 p_{i,2}}{(p_i \cdot n)^2} + \frac{|N/Nosc| \omega_L}{\omega_{cut}}.
\]  

(38)

Here, g1 and g2 are defined in Eq. (20),

\[
\nu = \frac{1}{2} (\mu m_e c)^2 (\langle f_1^2 \rangle + \langle f_2^2 \rangle),
\]  

(39)

and (for j = 1, 2)

\[
p_{i,j} = (p_i \cdot n)(n_K \cdot \epsilon_j) - (p_i \cdot \epsilon_j)(n \cdot n_K).
\]  

(40)

As for the original Klein-Nishina formula, the quantum signature is hidden in the definition of ωcut [Eq. (2)]. The frequencies determined by Eq. (38) mark the positions of main peaks in the Compton spectrum. Similarly, one can find frequencies of the secondary peaks (if \(N_{\text{rep}} > 2\)) and zeros (if \(N_{\text{rep}} > 1\)) in the angular-resolved frequency distributions. It is worth noting that now the polarization-dependent terms appear not in the unphysical dressing of the electron initial and final momenta, but in the definition of the directly measurable quantity; in other words, they affect the peak frequencies of the Compton spectrum. Similarly to the original Klein-Nishina formula we define the quantity

\[
N_{\text{GKN}} = \frac{N_{\text{osc}} \omega_{K,N} \omega_{cut}}{\omega_L(\omega_{cut} - \omega_K)} \frac{(p_i \cdot n_K)}{(p_i \cdot n)} + \frac{\nu n \cdot n_K + g_1 p_{i,1} + g_2 p_{i,2}}{(p_i \cdot n)^2},
\]  

(41)

which acquires integer values for peak frequencies ω_{K,N}. For \(f_{j,1} = 0\) and \(N_{\text{osc}} = 1\), this formula reduces to the original one, given by Eq. (33).

The derivation of the formula (41) shows that, in order to observe a coherent enhancement of the Compton spectra, the modulations of the driving pulse cannot be
arbitrary\[23\]. Both the integral of the electric field over the time duration of a single modulation and the vector potential in the beginning and at the end of it have to vanish. In other words, we have to deal with a train of pulses. This precludes the application of the diffraction formula not only to single pulses with varying in time envelopes, but also to the rectangular pulses of an arbitrary polarization. The exception is the linearly polarized rectangular pulse, but even in this case one has to redefine the vector potential such that \(f \neq 0\). This means that, irrespectively of the duration of such a rectangular pulse, the original Klein-Nishina formula \(\text{Eq. (4)}\) is not applicable, unless the particular geometry is selected such that \(p_i \neq 0\) for \(g_j \neq 0\). For instance, in the electron reference frame this happens if \(n \cdot \varepsilon = 0\), as \(p_i \varepsilon = 0\). This corresponds to the case when the Compton photon is ejected perpendicular to the laser field polarization vector.

In Figs. 3 and 4 we consider a generic case when \(p_i \neq 0\) for the linearly polarized laser field. In these figures we compare the same spectrum of emitted radiation, but we present it as the function of \(N\) or \(N_{\text{GKN}}\). We see that the main peaks of this spectrum correspond exactly to the integer values of \(N_{\text{GKN}}\). We also observe a dramatic difference in numerical values for \(N\) and \(N_{\text{GKN}}\). For instance, the first peak corresponds to \(N = 39\) or \(N_{\text{GKN}} = 14\). If the original Klein-Nishina formula had been used for interpreting this peak one would ascribe it to the process with absorption of 39 laser photons (not accounting for the fact that the majority of the peaks could not be interpreted this way, since they do not match the integer values of \(N\)). In contrast, for the GKN formula, all the main peaks can be interpreted as the result of absorption of an integer number of laser photons. Note that the positions of these peaks are the same with the increasing \(N_{\text{rep}}\), whereas their widths become increasingly narrower.

V. FEW-CYCLE LASER PULSES

A closer look at Fig. 2 shows that for \(N_{\text{rep}} > 1\) some peaks in the spectrum do not scale as \(N_{\text{rep}}^2\). This concerns peaks located close to the frequencies for which the distribution for the single pulse vanishes. As one can see, after dividing these distributions by \(N_{\text{rep}}^2\), the spectrum for \(N_{\text{rep}} = 1\) represents the envelope for the main diffraction peaks (i.e., observed for \(N_{\text{rep}} > 1\)). This means that the spectra are tangent to each other for frequencies close to the main peaks. If the Compton spectrum for a single pulse shows rapid modulations and the diffraction peak is located at the edge of a particular modulation, then the peak frequency does not correspond to the one for which the spectra are tangent. In these cases, it may happen that the application of a pulse train does not enhance, but rather suppresses, the generated radiation. This is the reason why in Fig. 3 some of the diffraction peaks are hardly visible. To avoid the suppression of emitted radiation, it is advisable to use such laser pulses, or such scattering kinematics, that the spectrum originating from a single pulse exhibits a broad structure; the so-called supercontinuum. Note that the formation of supercontinua was discussed recently in the context of Thomson and Compton scattering, and the synthesis of zepto- and yoctosecond pulses of radiation \[30\]. It appears that the best choice for their generation is to use few-cycle laser pulses. We shall illustrate this below for \(N_{\text{osc}} = 1\) and 3.

In Fig. 5, we present the Compton spectrum induced by a single-cycle pulse. The spectrum consists of the
of the GKN formula. For larger values of $N_{\text{rep}}$, the positions of the main peaks stay the same but their widths become more narrow. The energy separation between the adjacent peaks is nearly the same ($\sim 1.79 m_e c^2$). Note that the pulse train under consideration is the superposition of two plane waves of frequencies $\omega_L$ and $2\omega_L$. Also, the peak intensity of the laser field is not very large, as it does not exceed $10^{19} \text{W/cm}^2$. This suggests that our theoretical predictions could be verified experimentally, for instance at the ELI facility [31].

For pulses with more oscillations, the situation is similar. In Fig. 6 we show this for $N_{\text{osc}} = 3$. The only difference is that now the distribution of the diffraction peaks is denser, with the energy separation of roughly $0.33 m_e c^2$. As above, the main peaks (for $N_{\text{rep}} = 2$) are only the main diffraction peaks and the weaker secondary ones show up for $N_{\text{rep}} > 2$, as presented in Fig. 2 correspond to the clearly prescribed integer values of $N_{\text{GKN}}$. This, again, proves the validity of the GKN formula derived in this paper.

It follows from Eq. (11) that, knowing the geometry of the Compton scattering and the electron initial energy, measuring frequencies of only three consecutive peaks in the spectrum (for $N_{\text{rep}} > 1$) leads to the determination of the two important parameters of linearly polarized pulses which comprise the train: $\mu^2 \langle f^2 \rangle$ and $\mu \langle f \rangle$. If the form of the envelope is known such measurements allow to determine the peak intensity of incident pulses, which is characterized by the parameter $\mu$. Another possibility is to map the positions of the Compton peaks to the carrier envelope phase, assuming that the envelope type and the peak intensity is known. Similar measurements for two different geometries can extract the values of $\mu^2 (\langle f_j^2 \rangle + \langle f_j \rangle^2)$ and $\mu \langle f_j \rangle$ ($j = 1, 2$) for elliptically polarized driving pulses and, hence, also their polarization properties.

VI. CONCLUSIONS

We have demonstrated that, by using a train consisting of a finite number of identical pulses, one can generate the Compton radiation with well-resolved peaks. In other words, we propose the mechanism to reduce the spectral broadening of the emitted radiation which typically occurs if a few-cycle pulse interacts with the electron (see, for instance, Refs. [15, 16, 18–21]). This is a complementary proposal to the one presented in Ref. [22, 23], where a single but chirped initial pulse was used in order to compensate for the spectral broadening. Based on this result, we have derived the generalized Klein-Nishina formula.

The GKN formula (11) predicts the positions of well-resolved peaks in the Compton spectrum, when driven by a finite train of pulses. We argue that, by analyzing the positions of the peaks in the frequency domain of Compton photons, it is possible to determine laser pulse parameters, $\mu^2 (\langle f_j^2 \rangle + \langle f_j \rangle^2)$ and $\mu \langle f_j \rangle$ for both linear polarizations. This means that the proposed method can be applied, for instance, to determine polarization properties of such pulses and either their peak intensity or their carrier-envelope phase. Note that a similar analysis can be carried out for other fundamental processes of strong-field quantum electrodynamics, like the laser-induced Breit-Wheeler and Bethe-Heitler pair creation. These possibilities are under investigations now.

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[1] V. Yanovsky, V. Chvykov, G. Kalinchenko, P. Rousseau, T. Planchon, T. Matsuoka, A. Maksimchuk, J. Nees, G. Chieriaux, G. Mourou, and K. Krushelnick, Opt. Express
[16] 2109 (2008).

[2] A. Link, E. A. Chowdhury, J. T. Morrison, V. M. Ovchinnikov, D. Offermann, L. Van Woerkom, R. R. Freeman, J. Pasley, E. Shipton, F. Beg, P. Rambo, J. Schwarz, M. Geissel, A. Edens, and J. L. Porter, Rev. Sci. Instrum. 77, 10E723 (2006).

[3] S.-W. Bahk, P. Rousseau, T. A. Planchon, V. Chvykov, G. Kalintchenko, A. Maksimchuk, G. A. Mourou, and V. Yanovsky, Opt. Lett. 29, 2837 (2004).

[4] H. G. Hetzheim and C. H. Keitel, Phys. Rev. Lett. 102, 083003 (2009).

[5] J. Gao, Appl. Phys. Lett. 88, 091105 (2006).

[6] F. Mackenroth, A. Di Piazza, and C. H. Keitel, Phys. Rev. Lett. 105, 063903 (2010).

[7] O. Har-Shemesh and A. Di Piazza, Opt. Lett. 37, 1352 (2012).

[8] O. Klein and Y. Nishina, Z. Physik 52, 853 (1929).

[9] L. S. Brown and T. W. B. Kibble, Phys. Rev. 133, A705 (1964).

[10] V. I. Ritus and A. I. Nikishov, Quantum Electrodynamics Phenomena in the Intense Field, Trudy FIAN 111, 5 (1979).

[11] V. I. Ritus, J. Sov. Laser Res. 6, 497 (1985).

[12] F. Ehlotzky, K. Krajewska, and J. Z. Kamiński, Rep. Prog. Phys. 72, 046401 (2009).

[13] A. Di Piazza, C. Müller, K. Z. Hatsagortsyan, and C. H. Keitel, Rev. Mod. Phys. 84, 1177 (2012).

[14] V. N. Nedoreshta, S. P. Roshchupkin, and A. I. Voroshilo, Laser Phys. 23, 055301 (2013).

[15] T. Heinzl, D. Seipt, and B. Kämpfer, Phys. Rev. A 81, 022125 (2010).

[16] D. Seipt and B. Kämpfer, Phys. Rev. A 83, 022101 (2011).

[17] D. Seipt and B. Kämpfer, Phys. Rev. ST Accel. Beams 14, 040704 (2011).

[18] K. Krajewska and J. Z. Kamiński, Phys. Rev. A 90, 052117 (2014).

[19] M. Boca and V. Florescu, Phys. Rev. A 80, 053403 (2009).

[20] F. Mackenroth and A. Di Piazza, Phys. Rev. A 83, 032106 (2011).

[21] K. Krajewska and J. Z. Kamiński, Phys. Rev. A 85, 062102 (2012).

[22] K. Krajewska and J. Z. Kamiński, Laser Phys. Lett. 11, 035301 (2014).

[23] K. Krajewska, M. Twardy, and J. Z. Kamiński, Phys. Rev. A 89, 052123 (2014).

[24] D. M. Volkov, Z. Phys. 94, 250 (1935).

[25] K. Lee, S. Y. Chung, S. H. Park, Y. U. Jeong, and D. Kim, Europhys. Lett. 89, 64006 (2010).

[26] K. Krajewska and J. Z. Kamiński, Phys. Rev. A 86, 052104 (2012).

[27] C. Harvey, T. Heinzl, A. Ilderton, and M. Marklund, Phys. Rev. Lett. 109, 100402 (2012).

[28] K. Krajewska and J. Z. Kamiński, Phys. Rev. A 85, 043404 (2012).

[29] H. S. Reiss, Phys. Rev. A 89, 022116 (2014).

[30] K. Krajewska, M. Twardy, and J. Z. Kamiński, Phys. Rev. A 89, 032125 (2014).

[31] www.eli-laser.eu

[32] I. Ghebregziabher, B. A. Shadwick, and D. Umstadter, Phys. Rev. ST Accel. Beams 16, 030705 (2013).

[33] B. Terzić, K. Detrick, A. S. Hofler, and G. A. Krafft, Phys. Rev. Lett. 112, 074801 (2014).

[34] D. Seipt, S. G. Rykov, A. Surzhykov, and S. Fritzsche, Phys. Rev. A 91, 033402 (2015).