THE COHOMOLOGY OF SPHERICAL VECTOR BUNDLES ON K3 SURFACES

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ABSTRACT. We find an algorithm to compute the cohomology groups of spherical vector bundles on complex projective K3 surfaces, in terms of their Mukai vectors. In many good cases, we give significant simplifications of the algorithm. As an application, when the Picard rank is one, we show a numerical condition that is equivalent to weak Brill-Noether for a spherical vector bundle.

CONTENTS

1. Introduction 1
2. Preliminaries 4
3. Global Reduction 7
4. Wall Crossing for A Single Stable Spherical Object 15
5. Local Reduction 25
6. Simplifications 39
7. Weak Brill-Noether in Picard Rank One 48
8. Asymptotic Results 50
9. Examples 56
10. References 60

1. INTRODUCTION

The cohomology groups of a vector bundle are important invariants. Classical Brill-Noether theory, which studies the cohomology of line bundles on curves, has been an important topic in algebraic geometry for a long time [ACGH85]. Much less is known for the cohomology of vector bundles on curves. On higher dimensional varieties, computing the cohomology of a vector bundle is in general a hard problem. There has been some success concerning weak Brill-Noether for some surfaces such as minimal rational surfaces and certain del Pezzo surfaces [CH18, CH20, GH98, LZ19]. On K3 surfaces, [Ley12, Ley06] studied the Brill-Noether theory, and recently there has also been progress on the cohomology for general sheaves [CNY21].

For a moduli space of sheaves, when weak Brill-Noether fails, less is known about how to compute the exact dimensions of the cohomology groups of a general vector bundle. On a K3 surface, we consider this question for spherical vector bundles. We find an algorithm to compute the cohomology of any stable spherical vector bundle in terms of its Mukai vector. In particular, the dimension of any linear system can be computed in principle. As an interesting corollary, we see that on Picard rank one K3 surfaces with a fixed degree, the
cohomology groups of stable spherical vector bundles depend only on their Mukai vectors, but not on the underlying K3 surfaces.

When the Picard rank is one, we give a numerical condition that is equivalent to weak Brill-Noether for a spherical vector bundle. This is a generalization of the results in [CNY21]. We also show a bound for the cohomology of a spherical vector bundle when the Picard rank is one. When there is no restriction on the Picard group, we prove an asymptotic estimate on the cohomology, in the sense that the Mukai vector is obtained by many spherical reflections from two other spherical Mukai vectors.

Our algorithm in fact computes the cohomology of any rigid vector bundle with a known Harder-Narasimhan filtration. It is also very likely that our algorithm can be generalized to compute the generic cohomology of moduli of sheaves on K3 surfaces for any Mukai vector.

1.1. Main Theorem. Let \((X, H)\) be any polarized K3 surface over \(\mathbb{C}\). Let \(v = (r, D, a)\) be a Mukai vector with \(v^2 = -2\), \(r > 0\) and \(D \cdot H > 0\). Let \(E \in M_H(v)\) be the unique \(H\)-Gieseker stable vector bundle. Our main theorem is an algorithm that computes the cohomology groups of \(E\).

**Theorem 1.1.** Algorithm 3.14 computes the cohomology groups of \(E\).

In particular, note that since a line bundle is stable and spherical, Theorem 1.1 can be applied.

**Corollary 1.2.** Algorithm 3.14 computes the dimension of any linear system on \(X\).

In Algorithm 3.14, every step is totally determined by the Mukai vector \(v\). Since the lattices for all Picard rank one K3 surfaces of a fixed degree are isomorphic, as a corollary we see in the Picard rank one case the cohomology groups are independent of the underlying K3 surfaces.

**Corollary 1.3.** Let \(v = (r, dH, a) \in \mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z}\) be a Mukai vector with \(v^2 = -2\), \(r > 0\) and \(d > 0\). Then for any two K3 surfaces \(X, X'\) with \(\text{Pic}(X) \cong \text{Pic}(X') \cong \mathbb{Z}H\) and \(E \in M_{X,H}(v), E' \in M_{X',H'}(v)\), we have

\[
h^i(X, E) = h^i(X', E'), \forall i \in \mathbb{Z}.
\]

In practical terms, we also make simplifications in some good cases. We will define the height (Definition 6.13) of an object that measures the complexity for carrying out this algorithm. An important simplification occurs for an object \(E\) with height \(\leq 2\). Such an object is an extension of \(S \otimes \text{Hom}(S, E)\) and \(T \otimes \text{Hom}(E, T)^*\) for some factors \(S, T\), where the connecting homomorphisms always have maximal rank.

**Theorem 1.4** (Theorem 6.14, Global simplification). Let \(E \in M_H(v)\) be a height 2 spherical vector bundle and \(S, T\) be its factors. Then in the long exact sequence induced by \(H^0(-)\) on

\[
0 \to S \otimes \text{Hom}(S, E) \to E \to T \otimes \text{Hom}(E, T)^* \to 0,
\]

the connecting homomorphism \(H^0(T) \otimes \text{Hom}(E, T)^* \to H^1(S) \otimes \text{Hom}(S, E)\) has maximal rank.

We will give an explicit formula for the cohomology of \(S\) and \(T\) in Section 4. In particular, the cohomology of \(E\) is computed by Proposition 1.4. The global simplification covers many stable spherical vector bundles \(E\) with relatively small Mukai vectors. In fact, Proposition 1.4 is a consequence of a much more general local simplification (Theorem 6.11). Among all examples that the author did, most of them can be computed by Theorem 6.11.
In the cases where we do not need the precise dimension of the cohomology, we give some straightforward results. First, we show a numerical condition that is equivalent to weak Brill-Noether for spherical vector bundles on Picard rank one K3 surfaces.

**Theorem 1.5** (Theorem 7.1, weak Brill-Noether). Let \((X, H)\) be a polarized K3 surface of Picard rank one. Let \(v = (r, dH, a)\) be a Mukai vector with \(v^2 = -2\) and \(r, d > 0\), \(E \in M_H(v)\). Let \(y\) be the largest possible value of \(\frac{\alpha_1 d - a d_1}{r_1 d - r d_1}\) where \(v_1 = (r_1, d_1 H, a_1) \neq v\) satisfies

\[
v_1^2 = -2, vv_1 < 0, \frac{\alpha_1 d - a d_1}{r_1 d - r d_1} > 0, 0 < d_1 \leq d.
\]

Then \(H^1(E) = 0\) if and only if \(y < 1\).

When the Picard rank is one and the degree is at least 4, we also give a bound on the number of global sections.

**Theorem 1.6** (Proposition 8.2, Bound of \(h^0\)). Let \(X\) be a K3 surface with \(\text{Pic}(X) = \mathbb{Z}H\) and \(H^2 \neq 2\). Then for any \(v = (r, dH, a) \in H_{\text{alg}}^0(X)\) with \(v^2 = -2\) and \(r, d > 0\), we have

\[
h^0(E) < 2(r + a)
\]

for \(E \in M_H(v)\).

When there is no restriction on the Picard group, we can give an asymptotic result, in the sense that the Mukai vector comes from many spherical transforms. A stable spherical vector bundle can be uniquely labeled as \(S_j\) for some \(j \leq 0\). It can be constructed inductively from two spherical objects \(S_0, T_1\) by

\[
0 \rightarrow S_0 \otimes \text{Ext}^1(T_1, S_0) \rightarrow S_{-1} \rightarrow T_1 \rightarrow 0,
0 \rightarrow S_{j+1} \rightarrow S_j \otimes \text{Hom}(S_{j+1}, S_j)^* \rightarrow S_{j-1} \rightarrow 0, \quad j \leq -1.
\]

We let \(\text{coev}_j : H^0(S_{j+1}) \rightarrow H^0(S_j) \otimes \text{Hom}(S_{j+1}, S_j)^*\) be the induced maps on the cohomology.

**Theorem 1.7** (Theorem 8.1, Asymptotic result). Let \(X\) be any K3 surface. Then \(\text{coev}_j\) is injective for \(j \leq -3\). If \(\text{Pic}(X) = \mathbb{Z}H\) and \(H^2 \neq 2\), then \(\text{coev}_j\) is injective for \(j \leq -1\).

When all \(\text{coev}_j\) are injective, the connecting homomorphism \(H^0(T_1) \otimes \text{Hom}(S_j, T_1)^* \rightarrow H^1(S_0) \otimes \text{Hom}(S_0, S_j)^*\) is injective. Therefore the cohomology of \(S_j\) is computed, provided the cohomology of \(S_0\) and \(T_1\) are known. The asymptotic result tells us that this could only fail for the first few values of \(j\). As \(j\) tends to \(-\infty\), this estimate has relatively small error.

### 1.2. Idea of the Algorithm.

We illustrate the idea of the algorithm. For simplicity and without affecting the gist of the central idea, let us assume the K3 surface \((X, H)\) has Picard rank one. Then as we will explain in Section 2.3, there is a complex manifold \(\text{Stab}(X)\) parametrizing stability conditions on \(D^b(X)\), and there is a slice \(\mathcal{L}(X) \subset \text{Stab}(X)\) which is an open subset of the upper half plane

\[
\mathbb{H} = \{(sH, tH) : s \in \mathbb{R}, t \in \mathbb{R}_{>0}\},
\]

and the stability conditions in \(\mathcal{L}(X)\) can be explicitly written. We consider the vertical line

\[
b_\epsilon = \{(\epsilon, t) : t > 0\} \subset \mathcal{L}(X) \quad \text{for} \quad 0 < \epsilon \ll 1.
\]

By [Bri08] (Theorem 2.8 below), when \(t \gg 0\), \(M_{\sigma_{(\epsilon, t)}}(v) = M_H(v)\). The set of such \(\sigma_{(\epsilon, t)}\) is called the Gieseker chamber of \(v\).

The point \(\sigma_0 = (0, \sqrt{2/H^2})\) is not in \(\mathcal{L}(X)\) because \(Z_{\sigma_0}(\mathcal{O}_X) = 0\), hence for any Mukai vector \(v\) which is independent from \(v(\mathcal{O}_X) = (1, 0, 1)\), the numerical wall \(W(v, \mathcal{O}_X)\) defined
by \( v \) and \( \mathcal{O}_X \) must pass through \( \sigma_0 \). This wall is called the \textit{Brill-Noether wall} for \( v \) [Fey20]. Let \( \mathcal{C} \) be the chamber below the Brill-Noether wall.

A wall for \( v \) is called \textit{totally semistable} [BM14a], if for a stability condition \( \sigma \), all members of \( M_\sigma(v) \) are destabilized when \( \sigma \) passes through the wall. If there is no totally semistable wall of \( v \in H^*_{\text{alg}}(X) \) between its Gieseker chamber and \( \mathcal{C} \), then \( v \) has weak Brill-Noether. If there are totally semistable walls between the Gieseker chamber and \( \mathcal{C} \), there is a chamber \( \mathcal{C}_0 \) such that for any \( \sigma \in \mathcal{C}_0 \), \( \sigma \) is below the Brill-Noether walls of all \( \sigma \)-Harder-Narasimhan factors of \( E \). For \( \sigma \in \mathcal{C}_0 \), the last \( \sigma \)-Harder-Narasimhan factor is \( \mathcal{O}_X[1]^{\oplus h} \) for some \( h \in \mathbb{Z}_{\geq 0} \), and \( h^1(E) = h \) (Corollary 3.8). The process to get \( \sigma \)-Harder-Narasimhan filtration for \( \sigma \in \mathcal{C}_0 \) is called the global reduction (Algorithm 3.14). Until this point we do not require that \( v \) is spherical. We deal with global reduction in Section 3.

To compute the \( \sigma \)-Harder-Narasimhan filtration for \( \sigma \in \mathcal{C}_0 \), we need to keep track of the Harder-Narasimhan filtration whenever it changes. The points on \( b_\epsilon \) such that the Harder-Narasimhan filtration changes are called the \textit{Harder-Narasimhan walls}. We want every Harder-Narasimhan wall on \( b_\epsilon \) to only involve a rank 2 sublattice of \( H^*_{\text{alg}}(X) \). This is indeed true for \( \epsilon \) sufficiently small. Now if \( v \) is spherical, then every time we cross such a wall, there are two stable spherical objects \( S, T \) that generate all semistable spherical objects in the associated rank 2 lattice. There is a local reduction process (Algorithm 5.23) that computes the Harder-Narasimhan filtration of a rigid semistable object on one side of a wall, provided that we know the Harder-Narasimhan filtration on the other side of that wall, in terms of the order of \( S \) and \( T \) appearing in the filtrations. Then we may compute the new Harder-Narasimhan filtration of \( E \) whenever we encounter a Harder-Narasimhan wall, and the \( \sigma \)-Harder-Narasimhan filtration for \( \sigma \in \mathcal{C}_0 \) can be eventually computed.

The main idea of local reduction is to find an intermediate filtration that connects the information of the Harder-Narasimhan filtrations on two sides of the wall. This notion is Definition 5.1, it is the core of the whole algorithm. We deal with local reduction in Section 4 and Section 5. In many good cases the local reduction can be significantly simplified, we discuss this in Section 6.

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2. \textbf{Preliminaries}

In this section we collect necessary preliminaries about theory of moduli spaces of sheaves and Bridgeland stability conditions on K3 surfaces. Some good references are [HL10, Bri07, Bri08, BM14a, BM14b].

2.1. \textbf{Lattices and the Mukai Pairing}. In this paper, a \textit{K3 surface} is a projective K3 surface \( X \) over \( \mathbb{C} \) together with a polarization \( H \). We let

\[
H^*_{\text{alg}}(X) = H^0(X, \mathbb{Z}) \oplus \text{Pic}(X) \oplus H^4(X, \mathbb{Z})
\]

be the algebraic part of its cohomology ring.

Let \( \mathcal{D}^b(X) \) be the bounded derived category of \( X \). For any object \( E \in \mathcal{D}^b(X) \), its Chern character is defined by

\[
\text{ch}(E) = (r(E), c_1(E), \frac{c_1(E)^2}{2} - c_2(E)) \in H^*_{\text{alg}}(X),
\]
and we define its Mukai vector
\[ v(E) = \text{ch}(E) \cdot \sqrt{\text{td}(X)} = (r(E), c_1(E), r(E) + \text{ch}_2(E)). \]

By the Riemann-Roch Theorem, the Euler characteristic of two objects can be computed in terms of their Mukai vectors. Let \( v(E) = (r, D, a) \) and \( v(F) = (r', D', a') \), then
\[ \chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \text{ext}^i(E, F) = -D \cdot D' + ra' + r'a. \]

The pairing on \( H^*_{\text{alg}}(X) \) defined by
\[ (r, D, a) \cdot (r', D', a') = D \cdot D' - ra' - r'a \]
is called the Mukai pairing. A vector \( v = (r, D, a) \in H^*_{\text{alg}}(X) \) is called positive \cite{Yos01}, if \( v^2 \geq -2 \), and
- either \( r > 0 \),
- or \( r = 0 \), \( D \) is effective, and \( a \neq 0 \),
- or \( r = 0 \), \( D = 0 \), \( a > 0 \).

A vector \( v \) is called spherical if \( v^2 = -2 \).

2.2. Stability of Sheaves. Throughout this paper, all sheaves are assumed to be coherent and pure dimensional. Two of the classical stability notions for sheaves are slope stability and Gieseker stability.

**Definition 2.1.** We call \( \mu_H(E) = \frac{c_1(E)H}{r(E)} \) the \( H \)-slope of \( E \). A sheaf \( E \) on \( X \) is \( H \)-(semi)stable if for any subsheaf \( 0 \neq F \subset E \),
\[ \mu_H(F) < (\leq) \mu_H(E). \]

In general if the underlying variety has dimension \( > 1 \), slope stability does not form good moduli spaces. A better notion is Gieseker stability. Recall that for any sheaf \( E \), the \( H \)-Hilbert polynomial of \( E \) is
\[ P_E(n) = \chi(E(nH)). \]
Let \( a_E \) be the leading coefficient of \( P_E \). The reduced Hilbert polynomial of \( E \) is \( p_E = \frac{P_E}{a_E} \).

**Definition 2.2.** A sheaf \( E \) is \( H \)-(Gieseker) (semi)stable if for any subsheaf \( 0 \neq F \subset E \),
\[ p_F(n) < (\leq) p_E(n) \text{ for } n \gg 0. \]

On a K3 surface, Gieseker stability reads explicitly in the following form

**Definition 2.3.** On a polarized K3 surface \((X, H)\), a sheaf \( E \) with Mukai vector \((r, D, a)\) is \( H \)-semistable if for any subsheaf \( 0 \neq F \subset E \) with Mukai vector \((r', D', a')\),
\[ \frac{D'H}{r'} \leq \frac{DH}{r}, \quad \text{and if } \frac{D'H}{r'} = \frac{DH}{r}, \text{ then } \frac{a'}{r'} \leq \frac{a}{r}. \]
The sheaf \( E \) is stable if the equalities above cannot all hold.

**Remark 2.4.** The different notions of stability have the following implications:

slope stable \( \implies \) Gieseker stable \( \implies \) Gieseker semistable \( \implies \) slope semistable.

For a primitive Mukai vector \( v \) (not a multiple of another integral vector), there exists a space \( M_H(v) \) parametrizing \( H \)-semistable (hence stable by primitivity) sheaves with Mukai vector \( v \) \cite{Gie77, Mar77, Mar78}. We collect some facts about these moduli spaces.
Theorem 2.5 ([BM14b, KLS06, O’G99, PR14, Yos01, Yos99]). Let \( v \in H^*_\text{alg}(X) \) be a primitive positive Mukai vector. If \( H \) is generic in the sense of [O’G97], then

1. \( M_H(v) \) is non-empty.
2. The dimension of \( M_H(v) \) is \( v^2 + 2 \).
3. \( M_H(v) \) is a smooth and irreducible, deformation equivalent to Hilbert scheme of \( \left( \frac{v^2 + 2}{2} \right) \) points on a K3 surface.

In this paper we consider the following question. Let \( v = (r, D, a) \) be a spherical Mukai vector with \( r > 0 \) with respect to a generic polarization. Since it is spherical, it is primitive. And it is positive since \( r > 0 \). By Theorem 2.5, \( M_H(v) = \{ * \} \) is a single point, let \( E \) be the unique sheaf in this moduli space. It is a vector bundle [HL10]. We would like to compute \( H^0(E) \). We will eventually answer this question by an algorithm.

2.3. Bridgeland Stability Condition on K3 Surfaces. The main tool of this paper is Bridgeland stability. We collect facts about stability conditions on K3 surfaces in this subsection, the main references are [Bri07, Bri08].

Definition 2.6. A stability condition on a triangulated category \( D \) is a pair \((Z, A)\), where \( A \) is the heart of a bounded \( t \)-structure on \( D \), and \( Z : K(A) \rightarrow \mathbb{C} \) is a group homomorphism called the central charge, such that

- For any \( 0 \neq E \in A \), \( Z(E) = r(E) \cdot e^{i \tau \phi(E)} \) for some \( r(E) > 0 \) and \( \phi \in (0, 1] \). \( \phi(E) \) is called the phase of \( E \).
- An object \( G \in A \) is called (semi)stable, if for any subobject \( 0 \neq F \subset G \), we have \( \phi(F) < (\leq) \phi(G) \). For any object \( E \in A \), there exists a filtration called the Harder-Narasimhan filtration:
  \[
  0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,
  \]
  such that \( G_i = E_i/E_{i-1} \) are semistable, and \( \phi(G_i) > \phi(G_{i+1}) \) for all \( i \).

A stability condition \( \sigma = (Z, A) \) is called numerical if the central charge \( Z : K(X) \rightarrow \mathbb{C} \) factors through the numerical group \( N(X) = K(X)/K(X)^\perp \). A stability condition \( \sigma \) is called geometric if all skyscraper sheaves \( \mathcal{O}_x \) are \( \sigma \)-stable.

There is an effective way to construct concrete stability conditions, which we recall here. For any \( \beta \in \text{Pic}(X)_{\mathbb{R}} \) and \( \omega \in \text{Amp}(X)_{\mathbb{R}} \), we construct a torsion pair \((T_{\beta, \omega}, F_{\beta, \omega})\) on the abelian category \( \text{Coh}(X) \) as follows:

- \( F_{\beta, \omega} = \{ F \in \text{Coh}(X) : \text{for all subsheaves } 0 \neq F' \subset F, \mu_\omega(F') \leq \beta \cdot \omega \} \),
- \( T_{\beta, \omega} = \{ T \in \text{Coh}(X) : T \text{ is torsion, or for all quotient } T \rightarrow T', \mu_\omega(T') > \beta \cdot \omega \} \).

Let \( A_{\beta, \omega} = \{ E \in D^b(X) : H^{-1}(E) \in F_{\beta, \omega}, H^0(E) \in T_{\beta, \omega}, H^i(E) = 0 \text{ for all } i \neq -1, 0 \} \) be the tilt of \( \text{Coh}(X) \) with respect to this torsion pair. Let \( Z_{\beta, \omega} : N(X) \rightarrow \mathbb{C} \) be

\[
Z_{\beta, \omega}(r, D, a) = -a - r \frac{\beta^2 - \omega^2}{2} + D \cdot \beta + i \omega \cdot (D - r \beta).
\]

We let \( K(X) = \{ (\beta, \omega) \in \text{Pic}(X)_{\mathbb{C}} : \omega \in \text{Amp}(X)_{\mathbb{R}} \} \), and \( L(X) \) be the locus of \((\beta, \omega) \in K(X) \) such that \( Z_{\beta, \omega}(v) \notin \mathbb{R}_{\leq 0} \) for all spherical Mukai vector \( v \) with positive rank. Then \((Z_{\beta, \omega}, A_{\beta, \omega})\) is a geometric stability condition for \((\beta, \omega) \in L(X)\).
There is a natural complex manifold structure on the set of all stability conditions Stab(X).
Let $GL_2^+(\mathbb{R})$ be the universal cover of the orientation preserving component of $GL_2(\mathbb{R})$.
Then there is a natural action of $GL_2^+(\mathbb{R})$ on Stab(X). There exists a connected component
Stab$(X) \subset$ Stab$(X)$ and an open subset $U(X) \subset$ Stab$(X)$, such that $GL_2^+(\mathbb{R})$ acts freely
on $U(X)$. A section of this action is naturally identified with $\mathcal{L}(X)$. We will carry all the
work in this paper on $\mathcal{L}(X)$.

2.4. Wall and Chamber Structure. Our main usage of $\mathcal{L}(X)$ is the wall and chamber
structure. Let $v$ be a Mukai vector. Then there is a locally finite collection of real codimension
one submanifolds $\{W_i\}_{i \in I}$ of $\mathcal{L}(X)$, such that each connected component $C$ of
$\mathcal{L}(X) \backslash \bigcup_{i \in I} W_i$ has the following property: If $E \in \mathcal{D}_c(X)$ is $\sigma$-semistable for
some $\sigma \in C$, then $E$ is $\sigma$-semistable for all $\sigma \in C$. The walls $\{W_i\}_{i \in I}$ are called the (actual) walls
for $v$, and such connected components are called chambers.

For two Mukai vectors $u, v$, the numerical wall $W(u, v)$ is
$$W(u, v) = \{ \sigma \in \mathcal{L}(X) : \phi_\sigma(u) = \phi_\sigma(v) \}.$$  
An actual wall must be numerical, but a numerical wall is not necessarily actual.

For a stability condition $\sigma = (Z, A)$ and a primitive Mukai vector $v$, the $\sigma$-semistable
objects (hence stable by primitivity) in $A$ with Mukai vector $v$ form a moduli space, which
we shall denote by $M_\sigma(v)$. The following theorem is a generalization of Theorem 2.5.

Theorem 2.7 ([BM14a, BM14b]). Let $\sigma \in \mathcal{L}(X)$ be a stability condition and $v \in H^*_a_g(X)$
be a primitive Mukai vector. Then $M_\sigma(v)$ is non-empty when $v^2 \geq -2$. If in addition $\sigma$
is generic with respect to $v$, then

1. The dimension of $M_\sigma(v)$ is $v^2 + 2$,
2. $M_\sigma(v)$ is an irreducible normal projective variety with $\mathbb{Q}$-factorial singularities.

In particular, if $v$ is spherical and $\sigma$ is generic, then $M_\sigma(v)$ consists of a single point. The
following theorem connects Gieseker stability and Bridgeland stability.

Theorem 2.8 ([Bri08], Large volume limit). Let $H$ be a polarization of $X$, $E \in \mathcal{D}_c(X)$ such
that $r(E) > 0$ and $c_1(E) \cdot H > 0$. Then $E$ is $\sigma$-semistable for $\sigma = (Z_{0,tH}, A_{0,tH}), t \gg 0$
if and only if some shift of $E$ is $H$-Gieseker semistable.

We shall refer the chamber that contains this stability condition the $H$-Gieseker chamber
of $v$.

3. Global Reduction

Let $v = (r, D, a)$ be a spherical Mukai vector with $r > 0$. Let $H$ be a generic polarization.
We first assume $D \cdot H \neq 0$, otherwise we may take some $H'$ in the ample cone that is
sufficiently close to $H$, such that $M_{H'} = M_H$. Then we may assume $D \cdot H > 0$: since
$E \in M_H(v)$ is a vector bundle [HL10], $E^* \in M_H(r, -D, a)$. If $D \cdot H < 0$, then $-D \cdot H > 0$.
By Serre duality, $h^i(E) = h^{2-i}(E^*)$, hence it suffices to compute the cohomology groups of
$E^*$. Hence from now on we assume $v$ satisfies $r > 0$ and $D \cdot H > 0$.

Since $E$ is an $H$-Gieseker stable vector bundle, $E^*$ is also stable. We have $H^2(E) = H^0(E^*)^* = 0$
since
$$\mu(E^*) = -\mu(E) = -D \cdot H < 0 = \mu(\mathcal{O}_X).$$
Hence the only possible non-zero cohomology groups of $E$ are $H^0(E)$ and $H^1(E)$. By Riemann-Roch we have
\[ h^0(E) - h^1(E) = \chi(E) = r + a. \]
Hence if we know $h^0(E)$ or $h^1(E)$, the other is also known.

3.1. **Finiteness of Harder-Narasimhan Walls.** The strategy of the algorithm is to compute the Harder-Narasimhan filtration of a spherical object at a certain stability condition. To make this precise, we have to first prove that there are only finitely many walls at which the Harder-Narasimhan filtration changes.

**Definition 3.1.** Let $E \in \mathcal{D}^b(X)$ be an object. A Harder-Narasimhan chamber $U$ for $E$ is an open subset of $\mathcal{L}(X)$ such that the $\sigma$-Harder-Narasimhan filtration of $E$ is constant for $\sigma \in U$. A Harder-Narasimhan wall is the intersection of the closures of two adjacent Harder-Narasimhan chambers.

For an ample divisor $\omega$ and any divisor $\beta$, consider the upper half plane slice $\mathbb{H} = \{(s\beta, tw)|t > 0\}$. Let $\beta_1, \ldots, \beta_\rho$ be an effective basis for $\text{Pic}(X)$, and $\omega_1, \ldots, \omega_\rho$ be an ample basis for $\text{Pic}(X)$ with $\omega_1 = H$. The $\mathbb{Q}(X)$ described in [Bri08] can be embedded into
\[ \text{Pic}(X)_C \cong \mathbb{R}^{2\rho} = \{(s_1\beta_1, s_2\beta_2, \ldots, s_\rho\beta_\rho, t_1\omega_1, t_2\omega_2, \ldots, t_\rho\omega_\rho|s_i, t_j \in \mathbb{R}\}. \]

For $\epsilon > 0$ and $0 < y_0 < y_1$ let
\[ \mathbb{B}(\epsilon; y_0, y_1) = \{(s_1, \ldots, s_\rho, t_1, \ldots, t_\rho)|0 \leq s_1, \ldots, s_\rho \leq \epsilon, y_0 \leq t_1 \leq y_1, 0 \leq t_2, \ldots, t_\rho < \epsilon\} \subset \mathbb{H}, \]
and let $\mathbb{B} = \mathbb{B}(\epsilon; y_0, y_1) = \mathbb{B}(\epsilon; y_0, y_1) \cap \text{Stab}(X)$.

Let $E$ be some object which lies in $\mathcal{A}_{0,\omega_1}$ with $c_1(E) \cdot \omega_1 > 0$, and suppose $E$ is stable with respect to $(Z_{0,t_1\omega_1}, \mathcal{A}_{0,t_1\omega_1})$ for $t_1 \gg 0$ (By [Bri08], this is clearly true for an $H$-stable vector bundle $E$). Then we may choose $\epsilon$ sufficiently small, such that for any $p \in \mathbb{B}(\epsilon; y_0, y_1)$, $E \in \mathcal{A}_p$. Let
\[ \mathcal{S}_\mathbb{B}(E) = \{v(F)|F \text{ is a } \sigma \text{ -- Jordan-H"older factor of } E \text{ for some } \sigma \in \mathbb{B}\}. \]
The following proof is essentially covered in [Bri08] Section 9.

**Proposition 3.2.** Let the Mukai vector of $E$ be $(r, D, a)$, where $D \cdot H > 0$. Then $\mathcal{S}_{\mathbb{B}(\epsilon; y_0, y_1)}(E)$ is finite for fixed $y_0, y_1$ and sufficiently small $\epsilon$.

**Proof.** Let the Mukai vector of $E$ be $(r, D, a)$. Take any $z_1 > \sqrt{2/H}$. Take sufficiently small $\epsilon'$ so that in the box
\[ B = \{(s, t)|0 \leq s_1, \ldots, s_\rho, t_2, \ldots, t_\rho \leq \epsilon', z_1 - \epsilon' \leq t_1 \leq z_1\}, \]
the Harder-Narasimhan filtration of $E$ is fixed. Let $E'$ be the first factor. Let $B'$ be a slightly smaller box inside $B$, and let $\mu' = \min\{-\frac{1}{\mu_\infty(X)}|(s, t) \in B\}$. Consider the set
\[ \mathcal{C} = \bigcup_{(s, t) \in B'} \{\text{Mukai vectors of subobjects of } E \text{ in } \mathcal{A}_{s, t}\}. \]
I claim for all $(r', D', a') \in \mathcal{C}$, $r'$ is bounded below and $a'$ is bounded above. Suppose not, let $v_i = (r_i, D_i, a_i) \in \mathcal{C}$ be a sequence such that $\lim_{i \to \infty} r_i = -\infty$. Suppose $v_i$ is the Mukai vector of a subobject $E_i \subset E$ in $\mathcal{A}_i = \mathcal{A}_{\beta(i), \omega(i)}$, let $(\beta, \omega)$ be an accumulation point of $\{((\beta(i), \omega(i))}$
and we may assume \( \lim(\beta(i), \omega(i)) = (\beta, \omega) \). Then since \( E_i \subset E \) in \( A_i \), their central charges at \( Z_{\beta(i), \omega(i)} \) have imaginary parts no larger than that of \( E \):

\[
\omega(i)(D_i - r_i \beta(i)) \leq \omega(i)(D - r \beta(i)).
\]

Note that the right hand side has a finite limit, so \( \omega(i)(D_i - r_i \beta(i)) \) is bounded. On the other hand, note that for a \( \lambda \) sufficiently close to 1, \((\beta(i), \lambda \omega(i)) \in B\)

\[
-a_i - r_i \frac{(\beta(i))^2 - (\lambda \omega(i))^2}{2} + D_i \beta(i)
\]

is universally bounded below by \( \mu' \), otherwise \( E_i \) has larger phase than \( E' \) and would have been the first Harder-Narasimhan factor. Moreover, since \( \omega(i)H(D_i - r_i \beta(i)) \) is universally bounded,

\[
-a_i - r_i \frac{(\beta(i))^2 - (\lambda \omega(i))^2}{2} + D_i \beta(i)
\]

is universally bounded below. Note that since \( \epsilon \) is small, \((\beta(i))^2 - (\lambda \omega(i))^2 \) are negative. If \( a_i \) is not bounded above or \( r_i \) not bounded below, we may perturb \( \lambda \) so that this quantity is unbounded below, a contradiction. The claim is proved.

Now let

\[
S'_\beta(\epsilon; y_0, y_1)(E) = \{ v(F) | F \text{ is the first Jordan-H"older factor of } E \text{ for some } \sigma \in B \}.
\]

We may choose \( \epsilon \) small enough so that for every \((\beta, \omega) \in B\), there exists a scaling \((\beta, \nu \omega) \in B \) for some \( \nu \in \mathbb{R}_+ \). Note that \( A_{\beta, \nu \omega} \) is independent of \( \nu \), we have \( S' \subset C \). For any \( v' = (r', D', a') \in S' \), since \( a' \) is bounded above and \( r' \) is bounded below, the imaginary parts of the central charges are universally bounded, and the phase is universally bounded below by the phase of \( E \), we see \( |Z_{\beta, \omega(v')}| \) is universally bounded above and below for all \((\beta, \omega) \in B\).

Hence by the geometry of the Harder-Narasimhan polygon, \( |Z_{\beta, \omega(v''')}| \) is universally bounded above and below for all \( v'' \in S(B) \). By Lemma 9.2 of [Bri08], it suffices to show that \( |Z_{\beta, \omega}(E)| \) is bounded above, but this is clear since we may extend \( Z_{s,t} \) continuously to \( \overline{B} \).

**Corollary 3.3.** The number of Harder-Narasimhan walls in \( B \) is finite.

We donote the set of Harder-Narasimhan walls by \( HN(B) \).

### 3.2. Separating Walls

The idea of the algorithm is to compute the Harder-Narasimhan filtration at a certain stability condition \( \sigma_0 \). In this subsection, we will choose a path \( b \) from a stability condition \( \sigma_1 \) in the Gieseker chamber to \( \sigma_0 \), so that the intersections of all the walls in \( HN(B) \) with \( b \) are pairwise distinct. The order on \( b \) induces an order on \( HN(B) \) that records the order of touching walls when we move from \( \sigma_1 \) to \( \sigma_0 \) along \( b \). We also show how to compute this order.

The next lemma shows that all Harder-Narasimhan walls in \( B \) are locally a function in \( s \).

**Lemma 3.4.** There is no divisor \( D \) on a K3 surface such that

\[
D^2 = -2, D \cdot H = 0.
\]

**Proof.** If \( D^2 = -2 \), by Riemann-Roch, \( \chi(O_X(D)) = 2 + \frac{1}{2}D^2 = 1 \). Either \( H^0(O_X(D)) \) or \( H^2(O_X(D)) = H^0(O_X(-D))^* \) is non-zero. Hence \( D \) or \( -D \) is effective, so \( D \cdot H \neq 0 \). □
The Harder-Narasimhan wall \( W \), on each slice \( \mathbb{H} = \mathbb{H}_{s, t, \omega_1} \), is not a vertical wall, hence a function near \( s = 0 \). Consequently, by the inverse function theorem locally near \( \{(0, t, \omega_1)|t_1 \in \mathbb{R}_+\} \) it is still a function in \((s_1, \ldots, s_{\rho}, t_2, \ldots, t_\rho)\) valued in \( t_1 \). We may write it as \( t_1 = f_w(s_1, \ldots, s_{\rho}, t_2, \ldots, t_\rho) \), where recall that we have chosen the embedding

\[
\mathcal{Q}(X) \subset \text{Pic}(X) \otimes \mathbb{C} = \{(s_1\beta_1, \ldots, s_{\rho}\beta_{\rho}, t_1\omega_1, \ldots, t_\rho\omega_\rho)|s_i, t_j \in \mathbb{R}\},
\]

\( \omega_i \) are ample, and \( \beta_i \) are chosen to be effective. Then we may write

\[
t_1 = f_0(W) + \partial s_1(W)s_1 + \cdots + \partial s_{\rho}(W)s_{\rho} + \partial t_2(W)t_2 + \cdots + \partial t_\rho(W)t_\rho + o(s, t),
\]

where \( o(s, t) \) consists of higher order terms of \( s_i, t_j \). Let

\[
\mathcal{W}(\mathbb{B}) = \{\partial W = (f_0(W), \partial s_1(W), \ldots, \partial s_{\rho}(W), \partial t_2(W), \ldots, \partial t_\rho(W))|W \in HN(\mathbb{B})\} \subset \mathbb{R}^{2\rho}.
\]

By Proposition 3.3, this is a finite set. The next lemma shows that \( \mathcal{W}(\mathbb{B}) \) separates all walls.

**Lemma 3.5.** \( HN(\mathbb{B}) \rightarrow \mathcal{W}(\mathbb{B}) \) is bijective.

**Proof.** We need to show that if \((f_0(W), \partial s_1(W), \ldots, \partial s_{\rho}(W), \partial t_2(W), \ldots, \partial t_\rho(W))\) is known, then the wall \( W \) is determined. Viewing \( W \subset \mathbb{R}^{2n} \), it has an equation

\[
F = F_0 + \sum_{i=1}^{\rho} \frac{\partial F}{\partial s_i} s_i + \sum_{i=1}^{\rho} \frac{\partial F}{\partial t_i} t_i + o(s, t).
\]

By assumption, \([F] = [F_0, \frac{\partial F}{\partial s_1}, \ldots, \frac{\partial F}{\partial s_{\rho}}, \frac{\partial F}{\partial t_2}, \ldots, \frac{\partial F}{\partial t_\rho}] \in \mathbb{P}^{2\rho}_R \) is known. Hence if we take any slice \( \mathbb{R}^2_{s, t, \omega} = \{(s, t, \omega)|s, t \in \mathbb{R}\} \) and consider \( W \cap \mathbb{R}^2_{s, t, \omega} \), it is a circle with center lying on the \( \beta \)-axis by [Mac14]. We may write its equation as \( t = g(s) \), then \( g(0), g'(0) \) are already determined by \([F]\). Since \( g \) is a circle centered on the \( \beta \)-axis, there is a unique such \( g \) with given \( g(0), g'(0) \). Now since \( \mathcal{L}(X) \subset \mathbb{R}^{2n} \) is covered by such \( \mathbb{R}^2_{s, t, \omega} \) for \( \beta \in \text{Pic}(X)_R \) and \( \omega \in \text{Amp}(X)_R \), \( W \cap \mathcal{L}(X) \) is uniquely determined. Hence in particular \( HN(\mathbb{B}) \rightarrow \mathcal{W}(\mathbb{B}) \) is bijective.

Before we proceed to separate the walls, we need a technical lemma that describes \( \mathcal{L}(X) \) near the \( \omega_1 \)-axis. Let \( \mathbb{H} := \{(s, t, \omega)|s \in \mathbb{R}, t \in \mathbb{R}_{>0}\} \) be an upper half plane. Not all points in \( \mathbb{H} \) are stability conditions, however we have the following estimate:

**Lemma 3.6.** Let \( V = \mathbb{H} \setminus \mathbb{H} \cap \mathcal{L}(X) \). Then

\[
V \cap \{s = 0\} = \{0\} \times (0, \sqrt{2/\omega^2}],
\]

\[
\lim_{t \to 0} \sup \{t : (s, t) \in V, 0 < s < \epsilon\} = 0.
\]

**Proof.** Let \( v = (r, D, a) \) be a Mukai vector such that \( v^2 = -2 \) and \( r > 0 \).

For the first claim, by definition,

\[
Z_{0,1}(v) = -a + \frac{t^2}{2} \omega^2 + it \omega \cdot D.
\]

Suppose \( Z_{0,1}(v) \in \mathbb{R}_{\leq 0} \). Then \( \omega \cdot D = 0 \). There are two cases. If \( D = 0 \), then \( v^2 = -2ra = -2 \), \( r = a = \pm 1 \). By assumption \( r > 0 \), hence \( r = a = 1 \). In this case \(-a + \frac{t^2}{2} \omega^2 \leq 0 \) is equivalent to \( t \leq \sqrt{2/\omega^2} \). If \( D \neq 0 \), then since \( D \omega = 0 \), by the Hodge Index Theorem, \( D^2 < 0 \). Since on a K3 surface the pairing is even, \( D^2 \leq -2 \). Since \(-2 = v^2 = D^2 - 2ra \leq -2 - 2ra \), \( ra \leq 0 \). Since \( r > 0 \), \( a \leq 0 \). In this case \(-a + \frac{t^2}{2} \omega^2 \) is always positive. The first claim is proved.
For the second claim, by definition,

\[ Z_{\epsilon; t}(v) = -a - \frac{\epsilon^2 \beta^2 - t^2 \omega^2}{2} + \epsilon D \cdot \beta + i t \omega(D - r \epsilon \beta). \]

Suppose \( Z_{\epsilon; t}(v) \in \mathbb{R}_{< 0} \). Write \( \beta = \omega + \gamma \), where \( \gamma \cdot \omega = 0 \). Then \( |D \gamma| \leq M \sqrt{|D^2| \sqrt{|\gamma^2|}} \) for some constant \( M > 0 \). Since \( Im(Z(v)) = 0 \), \( D \omega = r \epsilon \omega \beta \),

\[ -a - \frac{\epsilon^2 \beta^2 - t^2 \omega^2}{2} + \epsilon D \cdot \beta = -a - \frac{\epsilon^2 \beta^2 - t^2 \omega^2}{2} + r \epsilon c \omega \beta + \epsilon D \gamma \leq 0 \]

is equivalent to

\[ \frac{t^2 \omega^2}{2} \leq a + \frac{\epsilon^2 \beta^2}{2} - \epsilon c \omega \beta + \frac{D \gamma}{r}. \]

Since the left hand side is positive and

\[ \frac{D \gamma}{r} \leq M \sqrt{|D^2|} \sqrt{|\gamma^2|} = \sqrt{2/r} |a - 2/r \omega | \sqrt{|\gamma^2|} \leq \sqrt{2/r} |a - 2/r | \sqrt{|\gamma^2|} + \sqrt{2/|\gamma^2|}, \]

it suffices to prove \( a/r \to 0 \) as \( \epsilon \to 0 \). Now since \( \epsilon > 0 \), \( D \omega = r \epsilon \omega \beta \) is a positive integer, hence \( r = \frac{D \omega}{\epsilon \omega \beta} \geq \frac{1}{\epsilon \omega \beta} \). Also note that by the Hodge Index Theorem, \( \omega^2 D^2 \leq (D \omega)^2 = r^2 \epsilon^2 (\omega \beta)^2 \), namely \( D^2 \leq r^2 \epsilon^2 (\omega \beta)^2 \). Since \( \omega^2 = D^2 - 2r a = -2 \), we have

\[ \frac{a}{r} = \frac{D^2 + 2}{2r^2} \leq \frac{r^2 \epsilon^2 (\omega \beta)^2 + 2}{2r^2} \leq \frac{r^2 \epsilon^2 (\omega \beta)^2 + 2\omega^2}{2\omega^2 (1/\epsilon \omega \beta)^2} = \epsilon^2 (\omega \beta)^2 (r^2 \epsilon^2 (\omega \beta)^2 + 2\omega^2)/(2\omega^2). \]

The second claim is proved.

Now we put a lexicographic order on \( \mathcal{W}(\mathbb{B}) \). Then there exists

\[ 0 < \epsilon_1 \ll \cdots \ll \epsilon_2 \ll \epsilon \ll \cdots \ll \epsilon_1 \ll 1, \]

such that if we let \( t \) range from infinity to, say \( \sqrt{2/\omega^2_1}/2 \), the order of the stability condition

\[ \{(Z(\epsilon_1 \beta_1 + \cdots + \epsilon_1 \beta_1 \omega_1 + \epsilon_2 \omega_1 + \cdots + \epsilon_2 \omega_1 \omega_1, A(\epsilon_1 \beta_1 + \cdots + \epsilon_1 \beta_1 \omega_1 + \epsilon_2 \omega_1 + \cdots + \epsilon_2 \omega_1 \omega_1)), t > \sqrt{2/\omega^2_1}/2 \} \]

touching walls in \( HN(\mathbb{B}) \) coincide with the order of walls in \( \mathcal{W}(\mathbb{B}) \). We fix this set once and for all and call it \( \mathbb{B} \). By Lemma 3.6, \( \mathbb{B} \subset \mathcal{L}(X) \). Note that the abelian category \( \mathcal{A} = \mathcal{A}(\epsilon_1 \beta_1 + \cdots + \epsilon_1 \beta_1 \omega_1 + \epsilon_2 \omega_1 + \cdots + \epsilon_2 \omega_1 \omega_1) \) is independent of \( t \).

### 3.3. Global Reduction

In this subsection we state the global reduction, which is the algorithm to compute the cohomology of a spherical vector bundle.

We say a shape of \( E \) is a pair

\[ (E^\bullet, \partial W) \in \text{Fil}(\mathcal{A}) \times \mathcal{W}(\mathbb{B}), \]

where \( E^\bullet \) is a filtration of \( E \). Now suppose \( \sigma \in \mathbb{B} \) is a stability condition that is not on a wall in \( HN(\mathbb{B}) \). It determines a shape of \( E \)

\[ sh(\sigma) = (E^\bullet_\sigma, \partial W) \in \text{Fil}(\mathcal{A}) \times \mathcal{W}(\mathbb{B}), \]

where \( E^\bullet_\sigma \) is the \( \sigma \)-Harder-Narasimhan filtration, and \( W \in HN(\mathbb{B}) \) is the wall right below \( \sigma \). Note that by the construction of \( \mathbb{B} \), all walls are distinct points on \( \mathbb{B} \cong \mathbb{R}_+ \), this is well-defined. The next simple but important observation shows how shapes are related to the cohomology of \( E \).
Proposition 3.7. Let $σ ∈ b$ be a stability condition with $sh(σ) = (E^*, ∂W)$, and write
\[ ∂W = (f_0(W), ∂s_1(W), \cdots, ∂s_ρ(W), ∂t_2(W), \cdots, ∂t_ρ(W)). \]
If $f_0(W) < \sqrt{2/ω_1^2}$, then the last $σ$-Harder-Narasimhan factor of $E^*$ is $O_X[1]^{≥h}$ for some $h ∈ \mathbb{Z}_{≥0}$ ($h$ can be zero, in this case we still have $\text{Hom}(E, O_X[1]) = 0$ for any factor $E_i$).

Proof. Let $E_i$ be any Harder-Narasimhan factor of $E$ that is not a direct sum of $O_X[1]$. By Lemma 3.9 that we shall show below, $d_i > 0$. Then we may shrink $ε_i, ε'_j$ further such that $φ_∞(E_i) < φ_∞(O_X[1])$.

Then there are two cases. First if $φ_σ(E_i) < φ_σ(O_X[1])$ for some factor $E_i$, consider the wall
\[ W' = W(E_i, O_X[1]). \]
By assumption, $W'$ is under $σ$. By construction, $∂W'$ has $f_0(W') = \sqrt{2/ω_1^2}$, which contradicts our assumption. Hence $W' \notin HN(ℌ)$, namely when crossing $W'$ the Harder-Narasimhan filtration does not change. Hence $O_X[1]$ cannot appear in $E^*$, and $\text{Hom}(E_i, O_X[1]) = 0$. For those $E_j$ with $φ_σ(E_j) ≥ φ_σ(O_X[1])$, they are not direct sums of $O_X[1]$, hence by the assumption that $σ$ is not on a wall they have $φ_σ(E_j) > φ_σ(O_X[1])$. Hence $E_j$ also have $\text{Hom}(E_j, O_X[1]) = 0$. Hence the proposition is proved in this case.

The second case is that $φ_σ(E_i) ≥ φ_σ(O_X[1])$. By assumption $σ$ is not on a wall, hence for those $E_i \notin O_X[1]^{≥h}$, they have $φ_σ(E_i) > φ_σ(O_X[1])$. Hence if $O_X[1]$ appears in $E^*$, it must be the last Harder-Narasimhan factor. □

Corollary 3.8. Let $σ$ satisfy the condition in Proposition 3.7. Then
\[ h^1(E) = h, \]
where the last Harder-Narasimhan factor of $E^*$ is $O_X[1]^{≥h}$.

Proof. We have the exact sequence
\[ 0 → E_{m-1} → E → O_X[1]^{≥h} → 0, \]
where $O_X[1]^{≥h}$ is the last Harder-Narasimhan factor. By Proposition 3.7, $E_{m-1}$ has a filtration whose factors have no map to $O_X[1]$. Hence $\text{Hom}(E, O_X[1]) = \text{Hom}(Q_m, O_X[1])$. □

Hence to compute the cohomology groups of $E$, it suffices to compute the shape at a certain stability condition. Now suppose we know the shape $P = (E^*_ρ, ∂W_ρ)$ of $E$ at a stability condition that is not on a wall. We call the shape $P' = (E^*_ρ', ∂W_ρ')$ the next shape after $P$ if $P' = sh(σ)$ for $σ$ in the chamber right below $W_ρ$. By definition, $W_ρ' < W_ρ$. Note that $\mathcal{W}(ℌ)$ is finite, hence if we keep computing the next shapes, in finite steps it must happen that $f_0(W_ρ) < \sqrt{2/ω_1^2}$ for some shape $P$. By Corollary 3.8, the cohomology is computed and the algorithm terminates here.

We have to examine closely when shapes change. Let $P = (E^*, ∂W)$ be the shape at $σ_+$ with filtration $0 = E^+_0 \subset E^+_1 \subset \cdots \subset E^+_m = E$ and $P' = sh(σ_-)$ be the next shape. We know that by definition of the Harder-Narasimhan filtration, $G_i$ are semi-stable, and
\[ φ_+(G^+_1) > φ_+(G^+_1) > \cdots > φ_+(G^+_m). \]
It will stop being a Harder-Narasimhan filtration at $σ_-$, if either one or both of the following happen:

- Some $G_i$ become unstable;
- For some pair $j < k$, the phases become $φ_-(G_j) < φ_-(G_k)$. 
In the first case, an actual wall for some factor is in between $\sigma_+$, and in the second case, the numerical wall of a pair $G_j, G_k$ is in between $\sigma_-$. By construction of $b$, all such walls are separated, hence $W$ corresponds to a unique rank 2 lattice $\mathcal{H} \subset H^*_\text{alg}(X)$, the $G_i, G_j, G_k$ described above all have Mukai vectors lying in $\mathcal{H}$. Furthermore, let $i/j$ be the minimal/maximal index such that $G_{ij} \in \mathcal{H}$, then for any $i \leq k \leq j$ and $\sigma_0 \in W \cap b$, $\phi_0(G_i) = \phi_0(G_k) = \phi_0(G_j)$. Since all walls are separated, $G_k$ also has Mukai vector in $\mathcal{H}$. This provides us a way to determine the next wall:

Compute all numerical walls of all adjacent factors of the current shape, and all actual walls for all factors, denote them by $\{W_i\}$. Then we pick the $W_i$ with maximal $\partial W_i \in \mathcal{W}(\mathbb{B})$, that is the next wall. We may then track back the $E_j/E_i$ that produced it.

Let

$$0 = F_0^- \subset F_1^- \subset \cdots \subset F_n^- = E_j/E_{i-1}$$

be the $\sigma_-$-Harder-Narasimhan filtration of $E_j/E_i$. Then it has a lift that replaces the segment $E_j/E_{i-1}$ of the original filtration of $E$:

$$0 = E_0^+ \subset E_1^+ \subset \cdots \subset E_{i-1}^+ \subset F_1^- \subset \cdots \subset F_{n-1}^- \subset E_j^+ \subset \cdots \subset E_m^+ = E.$$ 

By continuity, it is clearly the $\sigma_-$-Harder-Narasimhan filtration for $\sigma_-$ being sufficiently close to $W$, hence the Harder-Narasimhan filtration of the next shape. Hence the rest of the algorithm is to compute the Harder-Narasimhan filtration of all possible $E_j/E_{i-1}$ that can appear above at an adjacent chamber from its known Harder-Narasimhan filtration of the given chamber. Then from a shape, we can compute its next shape, and by iterating this, eventually the algorithm terminates. Before we do the computation, we need to capture some common properties of all such possible $E_j/E_{i-1}$.

**Lemma 3.9.** Preserve the notation in Proposition 3.7. Let $G$ be a $\sigma$-Harder-Narasimhan factor. If $G$ is not a direct sum of $\mathcal{O}_X[1]$, then its Mukai vector $(r, D, a)$ satisfies $D \cdot \omega_1 > 0$.

**Proof.** We use induction on the finite ordered set $\mathcal{W}(\mathbb{B})$. Let $\partial W_0$ be the initial element in $\mathcal{W}(\mathbb{B})$. Since $E$ is stable, the corresponding shape of $E$ is $(E, \partial W_0)$, here the filtration is trivial. By assumption $E$ has $c_1(E) \cdot \omega_1 > 0$, the base case is true.

Now suppose we have proved the lemma for $sh(\sigma_+) = (E_i^+, \partial W)$ and $\mathcal{P}' = sh(\sigma_-)$ is the next shape. By the discussion above, it suffices to show $G_i^- = F_i^- / F_{i-1}^-$ has positive degree with respect to $\omega_1$. Let the Mukai vector of $E_j/E_{i-1}$ be $v = (r, D, a)$ and the Mukai vectors of any Jordan-Hölder factor of $G_i^-$ be $v_i = (r_i, D_i, a_i)$, $Z' = Z_{(0, \tilde{f}_0(W) \cdot \omega_1)}$.

If $f_0(W) > \sqrt{2/\omega_1}$, by induction $Im(Z'(v)) > 0$, and by semi-stability of $G_i^-$, $Z'(v_i) \neq 0$. Hence

$$Im(Z'(v_i)) = f_0(W) \omega_1 \cdot D_1$$

is either positive or negative, since $Z'$ is a stability condition on the wall. It cannot be negative. Otherwise let $Z = W \cap b$, then there is a continuous path $\gamma \subset W$ between $Z'$ and $Z$. Since $Im(Z(v_i)) > 0$ there must be some $Z''$ in between such that $Im(Z''(v)) = 0$. Since we have chosen $\mathbb{B}$ sufficiently small such that $Im(Z''(v)) > 0$ for any $Z'' \in \mathbb{B}$, $Z''(v) = 0$. This is impossible by the positivity of stability conditions.

If $f_0(W) = \sqrt{2/\omega_1}$, $Z'$ is no longer a stability condition, but it is still a function with $Im(Z'(v)) > 0$. The only difference is that $Im(Z'(v_i))$ could be zero. In this case, since $v$ and $v_i$ have the same phase at $Z'$, $Z'(v_i) = 0$. Explicitly,

$$-a_i + \frac{r_i \omega_1^2}{2} = r_i - a_i = 0, D_i \cdot \omega_1 = 0.$$
From the second equation, either \( D_i = 0 \) or \( D_i^2 < 0 \) by the Hodge Index Theorem. If \( D_i = 0 \), since \( v_i \) is spherical, \( -2 = D_i^2 - 2r_i a_i \), we know \( r_i = a_i = \pm 1 \). Since \( G_i^- \in \mathcal{A} \), \( r_i = a_i = -1 \) and \( G_i^- \) is a direct sum of \( \mathcal{O}_X[1] \), which is excluded. Hence \( D_i^2 < 0 \). Since the K3 lattice is even, \( D_i^2 \leq -2 \). Since \( r_i - a_i = 0 \), we have

\[-2 = D_i^2 - 2r_i a_i \leq -2 - 2r_i^2 \leq -2.\]

Hence all inequalities are equalities. In particular, \( D_i^2 = -2 \). By Lemma 3.4, this is impossible. Hence \( \text{Im}(Z'(v_i)) \neq 0 \), and all arguments for \( f_0(W) \neq \sqrt{2/\omega_1^2} \) case apply without change.

**Definition 3.10.** We say an object \( E \in \mathcal{D}^b(X) \) is rigid if \( \text{Ext}^1(E, E) = 0 \).

A very important fact for rigid objects is Mukai’s Lemma.

**Lemma 3.11** ([Bri08], Lemma 5.2). Let \( \mathcal{A} \) be the heart of a bounded \( t \)-structure of \( \mathcal{D}^b(X) \). Let

\[ 0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0 \]

be a short exact sequence in \( \mathcal{A} \) such that \( \text{Hom}(F, G) = 0 \). Then

\[ \text{ext}^1(E, E) \geq \text{ext}^1(F, F) + \text{ext}^1(G, G). \]

**Corollary 3.12.** Using the notation of Lemma 3.11, if \( E \) is rigid, then \( F \) and \( G \) are rigid.

The next lemma shows that all possible \( E_j/E_{i-1} \) that appear in the Harder-Narasimhan filtration at some stability condition are rigid.

**Lemma 3.13.** Let \( E \) be a rigid object, and

\[ 0 = E_0 \subset E_1 \subset \cdots \subset E_m = E \]

be the \( \sigma \)-Harder-Narasimhan filtration for some \( \sigma \in \text{Stab}(X) \). Then \( E_j/E_{i-1} \) is rigid for any \( i \leq j \).

**Proof.** It suffices to prove \( E_{m-1}/E_0 \) and \( E_m/E_1 \) are rigid, then we may repeat this and the lemma is proved. Since this is a Harder-Narasimhan filtration, \( \text{Hom}(E_i, G_i) = 0 \) for \( i \geq 2 \), where \( G_i = E_i/E_{i-1} \). Hence \( \text{Hom}(E_1, E_m/E_1) = 0 \). By Mukai’s Lemma (Lemma 3.11),

\[ 0 = \text{ext}^1(E, E) \geq \text{ext}^1(E_1, E_1) + \text{ext}^1(E_m/E_1, E_m/E_1), \]

\( E_m/E_1 \) is rigid. For \( E_{m-1}/E_0 \), the argument is similar. \( \square \)

Hence what we still need is a local reduction process, which we describe here. Let \( W \) be a wall and \( \mathcal{H} \) be its rank 2 lattice. Let \( \mathcal{A}_\mathcal{H} \) be the full subcategory of \( \mathcal{A} \) that consists of objects whose Mukai vectors lie in \( \mathcal{H} \). Let \( \sigma_\pm \) be two stability conditions on two sides of \( W \). Let \( E \in \mathcal{A}_\mathcal{H} \) be a rigid object with \( \omega_1 \)-degree positive. The objective of local reduction is to compute the \( \sigma_- \)-Harder-Narasimhan filtration of \( E \), provided that we know the \( \sigma_+ \)-Harder-Narasimhan filtration. We will develop the local reduction in Section 4 and Section 5 formally. Assuming local reduction, we may summarize the global reduction formally as follows.

**Algorithm 3.14** (Global reduction). Let \( E \in M_\mathcal{H}(v) \) be a spherical vector bundle with \( c_1(E) \cdot H > 0 \). By Theorem 2.8, the initial shape is \( (E, \partial W_0) \), where \( E \) is the trivial filtration, and \( W_0 \) is the largest wall of \( v \) on \( b \).

For the current shape \( (E^*, \partial W) \), let the rank 2 lattice of \( W \) be \( \mathcal{H} \).
5.23
BM14a
9.3
3.14
3.8
3
3.14
BM14a
4.1
3.3

for compute all actual walls for a spherical Mukai vector. The type is easy to compute. In this section we deal with the former type. We show how to find an actual wall for a factor, and a numerical wall defined by two adjacent factors. The latter chamber of effective, then since \( \sigma \in W \) \( \sigma \)-Harder-Narasimhan filtration of \( E \), which we denote by \( E^\sigma \).

(3) Compute all numerical walls for all adjacent factors of \( E^\bullet \), and compute all actual walls for all factors of \( E^\bullet \) by Proposition 4.1, denote the union of them by \( \{ W_i \} \).

(4) Pick the maximal \( W^- \in \{ W_i \} \) with respect to the order on \( W(\mathbb{B}) \). Let \( (E^\bullet, W^-) \) be the current shape.

(5) If \( f_0(W^-) < \sqrt{2/\omega_1^2} \), then by Corollary 3.8, the last \( \sigma \)-Harder-Narasimhan factor of \( E^\bullet \) is \( \mathcal{O}_X[1]^{\leq h} \) and \( h^1(E) = h \), the algorithm terminates. Otherwise, do (1)-(5) again.

By doing (1)-(5) each time, \( \partial W^- < \partial W \) in \( W(\mathbb{B}) \). Since \( W(\mathbb{B}) \) is finite by Corollary 3.3, the algorithm terminates in finitely many steps.

For an example of global reduction (Algorithm 3.14), see Example 9.3.

4. Wall Crossing for a Single Stable Spherical Object

In this section we compute the Harder-Narasimhan filtration of a stable spherical object in an adjacent chamber. A good reference for background is [BM14a].

4.1. Rank 2 Lattice. As mentioned in Section 3, to carry out the global reduction (Algorithm 3.14), we need to know the Harder-Narasimhan walls. There are two types of walls: an actual wall for a factor, and a numerical wall defined by two adjacent factors. The latter type is easy to compute. In this section we deal with the former type. We show how to compute all actual walls for a spherical Mukai vector.

Let \( v \in H^*_\text{alg} \) be a spherical Mukai vector. Since \( M_\sigma(v) \) is a single point, every actual wall for \( v \) is totally semistable [BM14a]. Proposition 5.7 in [BM14a] gives a numerical criterion for such walls when the associated rank 2 lattice is hyperbolic, however when \( v \) is spherical, the proposition needs the following modification.

**Proposition 4.1.** Let \( H \subset H^*_\text{alg} \) be a rank 2 lattice (not neccesarily hyperbolic). Then \( H \) defines an actual wall for \( v \) (hence totally semistable) if and only if there exists a spherical Mukai vector \( v_1 \in H \) such that \( vv_1 < 0 \) and \( v_1, v - v_1 \) are both effective.

**Proof.** If \( H \) defines an actual wall \( W \), let \( \sigma_+ \) be a generic stability condition in an adjacent chamber of \( W \). Then there is a unique object \( E \in M_+(v) \). By definition, \( E \) is destabilized at \( \sigma_0 \in W \). By Mukai’s Lemma (Lemma 3.11), at \( \sigma_0 \) all Jordan-Hölder factors are effective \( \sigma_0 \)-stable spherical objects, whose Mukai vectors are denoted by \( v_1, \cdots, v_n \). Then \( v = \sum_{i=1}^n a_i v_i \), \( a_i > 0 \). By assumption \( v \) is spherical, hence

\[-2 = v^2 = v(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i vv_i.\]

There must exist some \( i \) such that \( a_i vv_i < 0 \). Since \( a_i > 0 \), we have \( vv_i < 0 \). Furthermore, \( v - v_1 \) is clearly effective since \( E \) is not \( \sigma_0 \)-stable, and the Jordan-Hölder filtration is non-trivial.

Conversely if there is some effective spherical \( v_1 \) with \( vv_1 < 0 \) and \( v_1, v - v_1 \) both being effective, then since \( \sigma_+ \) is generic, there is a unique \( F \in M_+(v_1) \). Since

\[0 < -vv_1 = \chi(v, v_1) = \hom(E, F) - \ext^1(E, F) + \ext^2(E, F),\]
either \( \text{Hom}(E, F) \) or \( \text{Hom}(E, F) = \text{Ext}^2(E, F)^* \) is non-zero, say \( \text{Hom}(F, E) \). Consider any non-zero map \( F \to E \). First note that it cannot be surjective, otherwise \( v - v_1 \) is not effective, since \( E, F \in \mathcal{A} \). Now let \( F_1, \ldots, F_n \) be Jordan-Hölder factors of \( F \) at \( \sigma_0 \), whose Mukai vectors are denoted by \( v_1, \ldots, v_n \). By Mukai’s Lemma (Lemma 3.11), they are all spherical. Then since \( \text{Hom}(F, E) \neq 0 \), there exists some \( i \) such that \( \text{Hom}(F_i, E) \neq 0 \). Since any map \( F \to E \) is not surjective, the map \( F_i \to E \) cannot be surjective. On the other hand, since \( F_i \) is \( \sigma_0 \)-stable and \( \phi_0(E) = \phi_0(F_i) \), the map \( F_i \to E \) must be injective. Hence \( E \) is destabilized by \( F_i \), and the wall is actual. 

The next goal is to classify all such possible rank 2 lattices \( \mathcal{H} \). Now if \( \mathcal{H} \) is hyperbolic, then everything in [BM14] applies. Let \( v_1 \) be any Mukai vector as in Proposition 4.1, and let \( -\chi = vv_1 \). If \( \chi \geq 3 \), then \( \mathcal{H} \) is hyperbolic. By Proposition 4.1, \( \chi > 0 \). Hence there are two cases left: \( \chi = 1 \) or \( \chi = 2 \).

We first look at the case \( \chi = 2 \). In this case \( \eta' = v - v_1 \) is an isotropic vector in \( \mathcal{H} \). Take any spherical \( w \in \mathcal{H} \), then \( w = av + bv_1 \), \( a, b \in \mathbb{Q} \). We have

\[
-2 = (av + bv_1)^2 = -2a^2 - 2b^2 + 2ab(-2) = -2(a + b)^2,
\]

hence \( a + b = \pm 1 \), \( w = \pm v + tr \eta' \) for some \( t \in \mathbb{Q} \). Now let \( F_1, F_2, F_3 \) be three \( \sigma_+ \)-stable spherical objects with Mukai vectors \( w_1, w_2, w_3 \). Then there exist two of them whose difference is isotropic, say \( w_1, w_2 \). Then

\[
0 = (w_1 - w_2)^2 = -2 - 2w_1w_2 - 2,
\]

namely \( w_1w_2 = -2 \). Hence \( F_1, i = 1, 2, 3 \), cannot all be \( \sigma_0 \)-stable, there are at most two \( \sigma_0 \)-stable spherical objects whose Mukai vectors lie in \( \mathcal{H} \). On the other hand, there are at least two such, since \( \mathcal{H} \) defines an actual wall. Hence there are exactly two, denoted by \( s_0, t_1 \), and let \( \eta = s_0 + t_1 \). We have \( s_0t_1 = 2 \) and \( \eta^2 = 0 \).

In this case, there are infinitely many spherical classes in \( \mathcal{H} \), and all of them still come from spherical reflections, as in [BM14]. For any \( w = at_1 + b\eta \in \mathcal{H} \), \( a, b \in \mathbb{Q} \), since the Mukai pairing is even, we have

\[
w^2 = a^2t_1^2 = -2a^2 \in 2\mathbb{Z},
\]

hence \( a \in \mathbb{Z} \), \( b\eta \in \mathcal{H} \). If \( b \notin \mathbb{Z} \), since \( \eta \in \mathcal{H} \), there exists \( 0 < b' < 1 \) such that \( b'\eta \in \mathcal{H} \). Then \( t'_1 = t_1 - b'\eta \) is an effective class. Let \( T_1' \in M_+(t'_1) \) and \( T_1 \in M_0(t_1) \), then \( \chi(T_1, T_1') = -t_1t'_1 = 2 \), either \( \text{Hom}(T_1, T_1') \) or \( \text{Hom}(T_1', T_1) \) is non-zero, say \( \text{Hom}(T_1', T_1) \). Since \( t_1 - t'_1 = b'\eta \) is effective, the map \( T_1' \to T_1 \) cannot be surjective. On the other hand since \( T_1 \) is \( \sigma_0 \)-stable and \( \phi_0(T_1') = \phi_0(T_1) \), the image of this non-zero map must be \( T_1 \) itself, a contradiction. Hence \( b \in \mathbb{Z} \), we see \( s_0, t_1 \) is an integral basis of \( \mathcal{H} \). Then all effective spherical classes are given recursively: \( t_2 = s_0 + (s_0t_1)t_1 \), \( t_i = -(t_{i-1}t_{i-2})t_{i-1} - t_{i-2} \) for \( i \geq 3 \). Similarly \( s_{-1} = t_1 + (t_1s_0)s_0 \), \( s_j = -s_{j+2} - (s_{j+2}s_{j+1})s_{j+1} \) for \( j \leq -2 \).

We are left with the case where \( \chi = 1 \). In this case \( \mathcal{H} \) is negative definite. We claim there are exactly 3 effective spherical objects whose Mukai vectors lie in \( \mathcal{H} \), among which exactly two are \( \sigma_0 \)-stable. Note that the discriminant

\[
\text{disc}(v, v_1) = \begin{pmatrix} v^2 & v_1v \\ vv_1 & v_1^2 \end{pmatrix} = 4 - 1 = 3
\]

is square free, hence \( v, v_1 \) is a basis for \( \mathcal{H} \).

Now suppose \( w = xv + yv_1 \), \( x, y \in \mathbb{Z} \) is spherical, then

\[
-2 = w^2 = (xv + yv_1)^2 = -2x^2 - 2y^2 - 2xy = -x^2 - y^2 - (x + y)^2.
\]
We have either \( x = 0, y = 0 \), or \( x + y = 0 \). Solving this we get
\[
\begin{align*}
x &= 0, y = \pm 1 \\
x &= \pm 1, y = 0 \\
x &= 1, y = -1 \\
x &= -1, y = 1
\end{align*}
\]

Hence there are exactly 6 spherical Mukai vectors, with 3 of them being effective. By assumption \( \mathcal{H} \) defines a wall for \( v \), hence \( E \) is not \( \sigma_0 \)-stable. However there are at least two effective \( \sigma_0 \)-stable objects, hence there are exactly two. Denote them by \( S_0, T_1 \) whose Mukai vectors are \( s_0, t_1 \). In this case we see \( v = s_0 + t_1 \). Explicitly, \( E \in M_+(v) \) is a general extension
\[
0 \longrightarrow S_0 \longrightarrow E \longrightarrow T_1 \longrightarrow 0.
\]

We summarize the section by separate the rank 2 lattices into two cases in the following proposition.

**Proposition 4.2.** Let \( \mathcal{H} \) be the rank 2 lattice associated to a wall of \( v \), whose Mukai vectors of the two stable spherical objects are \( s_0, t_1 \). Let \( v_1 \) be any Mukai vector as in Proposition 4.1, and let \( \chi = -vv_1 \). Then
\[
(1) \text{ If } \chi \geq 2, \text{ then there are infinitely many effective spherical Mukai vectors in } \mathcal{H}, \text{ constructed recursively by }
\]
\[
t_2 = s_0 + (s_0 t_1) t_1, \quad t_i = -(t_{i-1} t_{i-2}) t_{i-1} - t_{i-2} \text{ for } i \geq 3,
\]
\[
s_{-1} = t_1 + (t_1 s_0) s_0, \quad s_j = -s_{j+2} - (s_{j+2} s_{j+1}) s_{j+1} \text{ for } j \leq -2.
\]
\[
(2) \text{ If } \chi = 1, \text{ then there are exactly 6 spherical Mukai vectors in } \mathcal{H}, \text{ with 3 of them being effective, which are } s_0, t_1, v = s_0 + t_1. \text{ Let } E \in M_{\sigma_+}(v), \text{ then } E \text{ fits into the unique non-trivial extension }
\]
\[
0 \longrightarrow S_0 \longrightarrow E \longrightarrow T_1 \longrightarrow 0.
\]

Note that item (2) in Proposition 4.2 can happen. See Example 9.8.

**4.2. Construction of Spherical Objects.** Let \( v \) be a spherical Mukai vector. Let \( W \) be a wall for \( v \), and \( \mathcal{H} \) its rank 2 lattice. Let \( \sigma_0 \in W \), and \( \sigma_+ \), \( \sigma_- \) be two sufficiently close stability conditions near \( \sigma_0 \) on two sides of \( W \). Then there are two stable spherical objects \( S_0, T_1 \) on \( W \). We assume that \( \phi_+(T_1) > \phi_+(S_0) \).

In this section we explicitly construct all \( \sigma_- \)-stable spherical objects whose Mukai vectors are in \( \mathcal{H} \) from \( S_0, T_1 \). From this construction, we get the \( \sigma_- \)-(resp. \( \sigma_+ \))-Harder-Narasimhan filtration of a \( \sigma_-(\text{resp. } \sigma_+) \)-stable spherical object (Corollary 4.13).

If \( \mathcal{H} \) is negative definite, then the description is already given in Proposition 4.2. Hence in the following we assume \( \mathcal{H} \) is degenerate or hyperbolic. By Proposition 4.2 there are infinitely many \( \sigma_+ \)-stable objects whose Mukai vectors are in \( \mathcal{H} \). Let \( T_i \in M_{\sigma_+}(t_i), i \geq 1 \) and \( S_j \in M_{\sigma_+}(s_j), j \leq 0 \).

**Proposition 4.3.** The \( T_i \) are constructed as follows:
\[
0 \longrightarrow S_0 \longrightarrow T_2 \longrightarrow T_1 \otimes \text{Ext}^1(T_1, S_0) \longrightarrow 0,
\]
\[
0 \longrightarrow T_{m+1} \longrightarrow T_m \otimes \text{Hom}(T_m, T_{m-1}) \longrightarrow T_{m-1} \longrightarrow 0, m \geq 2.
\]
where $T_2$ is a general extension, and $T_i \otimes \text{Hom}(T_i, T_{i-1}) \to T_{i-1}$ are evaluation maps. Similarly, $S_j$ are constructed as follows:

$$0 \to S_0 \otimes \text{Ext}^1(T_1, S_0) \to S_{-1} \to T_1 \to 0, $$

$$0 \to S_{m+1} \to S_m \otimes \text{Hom}(S_{m+1}, S_m)^* \to S_{m-1} \to 0, m \leq -1. $$

where $S_{-1}$ is a general extension, and $S_{j+1} \to S_j \otimes \text{Hom}(S_{j+1}, S_j)^*$ are coevaluation maps.

**Proof.** We only prove the statement for $T_i, S_j$ are similar. First note that the Mukai vectors of $T_i$’s are spherical, hence it suffices to prove $T_i$ are $\sigma_+$-stable. We start with $T_2$. It is in $\mathcal{A}$, since it is an extension of two objects in $\mathcal{A}$. Applying $\text{Hom}(T_2, -)$ to the sequence

$$0 \to S_0 \to T_2 \to T_1 \otimes \text{Ext}^1(T_1, S_0) \to 0$$

we get

$$\text{Ext}^1(T_2, S_0) \to \text{Ext}^1(T_2, T_2) \to \text{Ext}^1(T_2, T_1) \otimes \text{Ext}^1(T_1, S_0).$$

To show $\text{Ext}^1(T_2, T_2) = 0$, it suffices to show $\text{Ext}^1(T_2, T_1) = \text{Ext}^1(T_2, S_0) = 0$. Applying $\text{Hom}(T_1, -)$ to the same sequence, we have

$$\text{Hom}(T_1, T_1) \otimes \text{Ext}^1(T_1, S_0) \to \text{Ext}^1(T_1, T_2) \to \text{Ext}^1(T_1, T_1) \otimes \text{Ext}^1(T_1, S_0) = 0.$$ 

Note that $\delta$ is an isomorphism, $\text{Ext}^1(T_1, T_2) = 0$. By a similar argument we also have $\text{Ext}^1(S_0, T_2) = 0$. Hence $\text{Ext}^1(T_1, T_2) = 0$.

Let $T$ be the first Jordan-Hölder factor of the first Harder-Narasimhan factor, by Mukai’s Lemma (Lemma 3.11), $T$ is $\sigma_+$-stable spherical object. Hence $\phi_+(T) \geq \phi_+(T_2)$. The only possibilities are $v(T) = v(T_1)$ or $v(T) = v(T_2)$. By stability of $T$, any non-zero map $T \to T_2$ is an injection. If $v(T) = v(T_2)$, then it is an isomorphism. If $v(T) = v(T_1)$, then since $T_1$ is $\sigma_+$-stable, $T = T_1$. Now apply $\text{Hom}(T_1, -)$ to

$$0 \to S_0 \to T_2 \to T_1 \otimes \text{Ext}^1(T_1, S_0) \to 0,$$

we get

$$0 = \text{Hom}(T_1, S_0) \to \text{Hom}(T_1, T_2) \to \text{Hom}(T_1, T_1) \otimes \text{Ext}^1(T_1, S_0) \to \text{Hom}(T_1, S_0[1]),$$

where $\delta'$ is an isomorphism. Hence $\text{Hom}(T_1, T_2) = 0, T = T_2$, and therefore $T_2$ is $\sigma_+$-stable.

Suppose we have proven that for $n \leq i$, the $T_n$ that are constructed as in the claim are $\sigma_+$-stable. Now consider the evaluation map

$$T_i \otimes \text{Hom}(T_i, T_{i-1}) \to T_{i-1}$$

I claim that this map is surjective if $\mathcal{H}$ is hyperbolic or degenerate. Denote the image by $I$ and assume for contradiction that $I \neq T_{i-1}$. Let $I_1$ be the first Jordan-Hölder factor at $\sigma_+$. By Mukai’s Lemma (Lemma 3.11), $I_1$ is $\sigma_+$-stable spherical object. If $I_1 \neq T_{i-1}$, by semistability of $T_i \otimes \text{Hom}(T_i, T_{i-1})$ and stability of $T_{i-1}$, we have

$$\phi_+(T_{i-1}) > \phi_+(I_1) \geq \phi_+(I) \geq \phi_+(T_i).$$

Since there is no spherical Mukai vectors in $\mathcal{H}$ whose $\sigma_+$-phase is strictly in between $\phi_+(T_i)$ and $\phi_+(T_{i-1})$, the equalities have to hold, namely $I$ is semistable whose Jordan-Hölder factors all have Mukai vectors equal to $v(T_i)$. Since $M_+(v(T_i)) = \{T_i\}$, all Jordan-Hölder factors of $I$ are $T_i$. Since $\text{Ext}^1(T_i, T_i) = 0$, $I = T^{\oplus a}_i$. Since $\mathcal{H}$ is hyperbolic or degenerate, $v(T_i) \text{hom}(T_i, T_{i-1}) - v(T_{i-1}) = v(T_{i+1})$ is effective, $a < \text{hom}(T_i, T_{i-1})$. This is impossible, since $T_i \otimes \text{Hom}(T_i, T_{i-1}) \to T_{i-1}$ is the evaluation map. If $a < \text{hom}(T_i, T_{i-1})$ then there is a non-zero $f \in \text{Hom}(T_i, T_{i-1})$ such that $T_i \otimes f$ is mapped to zero in $T^{\oplus a}_i$, a contradiction.
Hence $I = I_1 = T_{i-1}$, the evaluation map $T_i \otimes \text{Hom}(T_i, T_{i-1}) \to T_{i-1}$ is surjective. Hence $T_{i+1}$ is the kernel of this map, which is in $\mathcal{A}$, since $\mathcal{A}$ is abelian.

Now we need to prove $T_{i+1}$ is $\sigma_+$-stable. First I claim $\text{Ext}^1(T_{i+1}, T_{i+1}) = 0$. Applying $\text{Hom}(T_i, -)$ to the exact sequence

$$0 \longrightarrow T_{i+1} \longrightarrow T_i \otimes \text{Hom}(T_i, T_{i-1}) \longrightarrow T_{i-1} \longrightarrow 0$$

we get

$$\text{Hom}(T_i, T_i) \otimes \text{Hom}(T_i, T_{i-1}) \to \text{Hom}(T_i, T_{i-1}) \to \text{Ext}^1(T_i, T_{i+1}) \to \text{Ext}^1(T_i, T_i) \otimes \text{Hom}(T_i, T_{i-1}).$$

the first map is an isomorphism, and $\text{Ext}^1(T_i, T_i) = 0$ by the induction hypothesis, hence $\text{Ext}^1(T_{i+1}, T_i) = \text{Ext}^1(T_i, T_{i+1})^* = 0$.

Applying $\text{Hom}(-, T_i)$ we get

$$0 = \text{Hom}(T_{i-1}, T_i) \to \text{Hom}(T_i, T_{i-1})^* \to \text{Hom}(T_{i+1}, T_i) \to \text{Ext}^1(T_i, T_{i-1}) = 0.$$

Hence there is a natural identification

$$\text{Hom}(T_i, T_{i-1})^* = \text{Hom}(T_{i+1}, T_i).$$

The map of functors $\text{Hom}(-, T_i) \otimes \text{Hom}(T_i, T_{i-1})) \to \text{Hom}(-, T_{i-1})$ applied to the evaluation map $T_{i+1} \longrightarrow T_i \otimes \text{Hom}(T_i, T_{i-1})^*$ gives rise to a commutative diagram

$$\begin{array}{ccc}
\text{Hom}(T_i, T_i) \otimes \text{Hom}(T_i, T_{i-1}) & \to & \text{Hom}(T_{i+1}, T_i) \otimes \text{Hom}(T_i, T_{i-1}) \\
\downarrow & & \downarrow \\
\text{Hom}(T_i, T_{i-1}) \otimes \text{Hom}(T_i, T_{i-1})^* & \to & \text{Hom}(T_{i+1}, T_{i-1}) \\
\downarrow & & \downarrow \\
0 & & \text{Ext}^1(T_{i+1}, T_{i+1}) \\
\downarrow & & \downarrow \\
& & \text{Ext}^1(T_{i+1}, T_i) \otimes \text{Hom}(T_i, T_{i-1}) = 0
\end{array}$$

Hence $\text{Ext}^1(T_{i+1}, T_{i+1}) = 0$. Let $T$ be the first Jordan-Hölder factor of the first Harder-Narasimhan factor of $T_{i+1}$, by Mukai’s Lemma (Lemma 3.11), $T$ is $\sigma_+$-stable spherical object. If $v(T) = v(T_{i+1})$, then by stability of $T$ we have $T = T_{i+1}$. Otherwise, $T = T_j$ for some $j \leq i - 1$, by stability $\text{Hom}(T_j, T_i) = 0$, and

$$0 = \text{Ext}^{-1}(T_j, T_{i-1}) \to \text{Hom}(T_j, T_{i+1}) \to \text{Hom}(T_j, T_i) \otimes \text{Hom}(T_i, T_{i-1}) = 0.$$

Hence $\text{Hom}(T_j, T_{i+1}) = 0$, a contradiction. Hence $T_{i+1}$ is $\sigma_+$-stable. By induction, the $T_n$ that are constructed in the claim are $\sigma_+$-stable for all $n \geq 1$.

\[\square\]

**Remark 4.4.** The map $T_{i+1} \to T_i \otimes \text{Hom}(T_i, T_{i-1})$ is the coevaluation map.

**Proof.** Let $V$ be a vector space. In general, if we have a map $A \to B \otimes V$, then to check it is the coevaluation map, we need to check the map

$$\text{id} \otimes V^* \subset \text{Hom}(B, B) \otimes V^* = \text{Hom}(B \otimes V, B) \to \text{Hom}(A, B)$$

is an isomorphism. We apply $\text{Hom}(-, T_i)$ to the sequence in Proposition 4.3 and get

$$0 = \text{Hom}(T_{i-1}, T_i) \to \text{Hom}(T_i, T_i) \otimes \text{Hom}(T_i, T_{i-1})^* \to \text{Hom}(T_{i+1}, T_i) \to \text{Ext}^1(T_{i-1}, T_i)$$

By the proof of Proposition 4.3, $\text{Ext}^1(T_{i-1}, T_i) = 0$. Hence the remark is proved. \[\square\]
Then there is a natural identification \( \epsilon_i : \text{Hom}(T_{i+j}, T_i)^* \xrightarrow{\cong} \text{Hom}(T_i, T_{i-1}) \). We shall denote

\[
m_i : \text{Hom}(T_i, T_{i-1}) \otimes \text{Hom}(T_{i+j}, T_i) \overset{id \otimes (\epsilon_i^*)^{-1}}{\longrightarrow} \text{Hom}(T_i, T_{i-1}) \otimes \text{Hom}(T_i, T_{i-1})^* \rightarrow \mathbb{C}
\]

where the second map is the natural pairing. There is also a comultiplication induced by the sequence itself. Since \( T_{i+j} \rightarrow T_i \otimes \text{Hom}(T_i, T_{i-1}) \) is the coevaluation map, applying \( \text{Hom}(T_{i+j}, -) \) to this sequence gives us a map

\[
\delta_i : \mathbb{C} = \text{Hom}(T_{i+j}, T_i) \rightarrow \text{Hom}(T_i, T_{i-1}) \otimes \text{Hom}(T_i, T_{i-1}).
\]

The identifications here satisfy the following commutativity:

**Lemma 4.5.** Let \( \delta : \mathbb{C} \rightarrow \text{Hom}(T_i, T_{i-1})^* \otimes \text{Hom}(T_i, T_{i-1}) \) be the diagonal map. Then \( (\epsilon_i \otimes \text{id}) \circ \delta = \delta_i \), namely there is a commutative diagram

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\delta} & \text{Hom}(T_i, T_{i-1})^* \otimes \text{Hom}(T_i, T_{i-1}) \\
\downarrow{\delta_i} & & \downarrow{\epsilon_i \otimes \text{id}} \\
\text{Hom}(T_{i+j}, T_i) \otimes \text{Hom}(T_i, T_{i-1}) & & \\
\end{array}
\]

Dually, let \( m : \text{Hom}(T_i, T_{i-1})^* \otimes \text{Hom}(T_i, T_{i-1}) \rightarrow \mathbb{C} \) be the natural pairing. Then \( m \circ (\text{id} \otimes \epsilon_i) = \delta_i^* \), namely there is a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(T_i, T_{i-1})^* \otimes \text{Hom}(T_{i+j}, T_i) & \xrightarrow{id \otimes \epsilon_i} & \text{Hom}(T_i, T_{i-1})^* \otimes \text{Hom}(T_i, T_{i-1}) \\
& \downarrow{\delta_i^*} & \downarrow{m} \\
\mathbb{C} & & \\
\end{array}
\]

**Proof.** We prove the first claim, the second follows by dualizing. Perhaps the most direct way to prove this is to calculate with an explicit basis. Let \( e_1, \cdots, e_n \) be a basis for \( \text{Hom}(T_i, T_{i-1}) \). Applying \( \text{Hom}(-, T_i \otimes \text{Hom}(T_i, T_{i-1})) \) to \( \Phi : T_{i+1} \rightarrow T_i \otimes \text{Hom}(T_i, T_{i-1}) \) gives the isomorphism

\[
\phi : \text{Hom}(\text{Hom}(T_i, T_{i-1}), \text{Hom}(T_i, T_{i-1})) \rightarrow \text{Hom}(T_{i+1}, T_i) \otimes \text{Hom}(T_i, T_{i-1}).
\]

Since \( \Phi : T_{i+1} \rightarrow T_i \otimes \text{Hom}(T_i, T_{i-1}) \) is the coevaluation map, we may write \( \Phi \in \text{Hom}(T_{i+1}, T_i) \otimes \text{Hom}(T_i, T_{i-1}) \) as \( \sum f_i \otimes e_k \), where \( f_1, \cdots, f_n \) is a basis for \( \text{Hom}(T_{i+1}, T_i) \). Suppose \( A = (a_{kl})_{1 \leq k, l \leq n} \in \text{Hom}(\text{Hom}(T_i, T_{i-1}), \text{Hom}(T_i, T_{i-1})) \) is a matrix, then it is naturally identified with \( \sum a_{kl} \epsilon_i^* \otimes e_k \in \text{Hom}(T_i, T_{i-1})^* \otimes \text{Hom}(T_i, T_{i-1})^* \otimes \text{Hom}(T_i, T_{i-1})^* \). By definition, \( \phi(A) = A \circ \Phi \in \text{Hom}(T_{i+1}, T_i) \otimes \text{Hom}(T_i, T_{i-1}) \), explicitly

\[
A \circ \Phi = \sum f_i \otimes Ae_i = \sum f_i \otimes a_{kl} e_k = \sum a_{kl} f_i \otimes e_k,
\]

hence the identification \( \epsilon_i^* : \text{Hom}(T_i, T_{i-1})^* \rightarrow \text{Hom}(T_{i+1}, T_i) \) sends \( \epsilon_i^* \) to \( f_i \) for all \( 1 \leq l \leq n \).

On the other hand, by definition applying \( \text{Hom}(T_{i+1}, -) \) to \( \Phi : T_{i+1} \rightarrow T_i \otimes \text{Hom}(T_i, T_{i-1}) \) sends \( \text{id} \in \text{Hom}(T_{i+1}, T_{i+1}) \) to \( \Phi \in \text{Hom}(T_{i+1}, T_i) \otimes \text{Hom}(T_i, T_{i-1}) \), namely \( (\epsilon_i^* \otimes \text{id}) \). We have

\[
(\epsilon_i^* \otimes \text{id}) \circ \delta(1) = (\epsilon_i^* \otimes \text{id})(\sum e_k^* \otimes e_k) = \sum f_k \otimes e_k = \Phi = \delta_i(\text{id}),
\]

the lemma is proved. \( \square \)
The next goal is to compute homomorphisms between $T_i$ and $T_1$ or $S_0$.

**Lemma 4.6.** Consider the maps

$$coev_{i+1} : T_{i+1} \rightarrow T_i \otimes \text{Hom}(T_{i+1}, T_i)^*,$$

$$ev_{i+1} : T_{i+1} \otimes \text{Hom}(T_{i+1}, T_i) \rightarrow T_i.$$

Denote $m : \text{Hom}(T_{i+1}, T_i)^* \otimes \text{Hom}(T_{i+1}, T_i) \rightarrow \mathbb{C}$ the natural pairing. Then we have

$$(id \otimes m) \circ (coev_{i+1} \otimes id) = ev_{i+1}.$$

In other words, we have the following commutative diagram

$$
\begin{array}{ccc}
T_{i+1} \otimes \text{Hom}(T_{i+1}, T_i) & \xrightarrow{coev_{i+1} \otimes id} & T_i \otimes \text{Hom}(T_{i+1}, T_i)^* \otimes \text{Hom}(T_{i+1}, T_i) \\
& \downarrow{ev_{i+1}} & \downarrow{id \circ m} \\
& T_i & \\
\end{array}
$$

**Proof.** Let $f_1, \ldots, f_n$ be a basis of $\text{Hom}(T_{i+1}, T_i)$, and let $f_1^*, \ldots, f_n^*$ be the dual basis. Take any non-zero $f \in \text{Hom}(T_{i+1}, T_i)$. Restricting to $T_{i+1} \otimes f$, we have

$$coev_{i+1} \otimes id = f_1(-) \otimes f_1^* \otimes f + \cdots + f_n(-) \otimes f_n^* \otimes f,$$

$$(id \otimes m)(f_1(-) \otimes f_1^* \otimes f + \cdots + f_n(-) \otimes f_n^* \otimes f) = f(-).$$

On the other hand, the restriction of $ev_{i+1}$ on $T_{i+1} \otimes f$ is equal to $f$ by definition. \qed 

Note that $\text{Hom}(T_1, T_i) = 0$ for $i \geq 2$, $\text{Hom}(T_j, S_j) = 0$ for all $j$. Hence the only relevant functors are $\text{Ext}^1(T_1, -)$ and $\text{Hom}(-, T_1) = \text{Ext}^2(T_1, -)^*$.

We compute $\text{Ext}^1(T_1, T_i)$ for $1 \leq i \leq 4$ by directly chasing the long exact sequences. For $i = 1$, $\text{Ext}^1(T_1, T_1) = 0$ by definition. For $i = 2$, we also have $\text{Ext}^1(T_1, T_2) = 0$.

For $i = 3$, we have $\text{Ext}^1(T_1, T_3) \cong \text{Hom}(T_1, T_1) = \mathbb{C}$. For $i = 4$, we have $\text{Ext}^1(T_1, T_4) = \text{Hom}(T_3, T_2)$. For larger $i$, we have the following

**Proposition 4.7.** For $i \geq 3$,

$$\text{Ext}^1(T_1, T_{i+1}) = \bigcap_{j=3}^{i-1} \ker(m_j) \subset \bigotimes_{j=3}^{i} \text{Hom}(T_j, T_{j-1}),$$

where by abuse of notation $m_j$ denotes $id \otimes \cdots \otimes id \otimes m_j \otimes id \otimes \cdots \otimes id$.

**Proof.** We use induction on $i$. The proposition is true for $i = 3$. Suppose it is proven for $3 \leq n \leq i$, and denote the embedding $\text{Ext}^1(T_1, T_{i+1}) \subset \bigotimes_{j=3}^{i} \text{Hom}(T_j, T_{j-1})$ by $\iota$. We have an exact sequence

$$0 \rightarrow \text{Ext}^1(T_1, T_{i+1}) \rightarrow \text{Ext}^1(T_1, T_i) \otimes \text{Hom}(T_i, T_{i-1}) \xrightarrow{ev_i} \text{Ext}^1(T_1, T_{i-1}).$$

Hence $\text{Ext}^1(T_1, T_{i+1})$ is identified with

$$\ker(ev_i) \subset \text{Ext}^1(T_1, T_i) \otimes \text{Hom}(T_i, T_{i-1}) \subset \bigotimes_{j=3}^{i} \text{Hom}(T_j, T_{j-1}).$$

By induction, an element in $\bigotimes_{j=3}^{i} \text{Hom}(T_j, T_{j-1})$ is inside $\text{Ext}^1(T_1, T_i) \otimes \text{Hom}(T_i, T_{i-1})$ precisely if it is inside $\ker(m_j)$ for $3 \leq j \leq i - 2$. On the other hand, we have a commutative
diagram

\[
\begin{array}{c}
\Ext^1(T_1, T_i) \otimes \Hom(T_i, T_{i-1}) \xrightarrow{ev_i} \Ext^1(T_1, T_{i-1}) \\
\downarrow \langle \id \otimes \text{coev}_{i-1} \rangle \otimes \id \\
\Ext^1(T_1, T_{i-1}) \otimes \Hom(T_{i-1}, T_{i-2}) \otimes \Hom(T_i, T_{i-1}) \xrightarrow{\id \otimes m_{i-1}} \Ext^1(T_1, T_{i-1}) \\
\downarrow \otimes \id \otimes \id \\
\bigotimes_{j=3}^{i} \Hom(T_j, T_{j-1}) \xrightarrow{m_{i-1}} (\bigotimes_{j=3}^{i-2} \Hom(T_j, T_{j-1}))
\end{array}
\]

where commutativity of the top square is by Lemma 4.6 and the definition of \(m_{i-1}\), and commutativity of the bottom square is clear. Since all vertical maps are inclusions, being mapped to zero by \(ev_i\) is equivalent to being inside \(\ker(m_{i-1})\). The theorem is proved.

We want to compute the dimension of \(\Ext^1(T_1, T_{i+1})\). By the long exact sequence

\[
\Ext^1(T_1, T_i \otimes \Hom(T_i, T_{i-1})) \to \Ext^1(T_1, T_{i-1}) \to \Ext^2(T_1, T_{i+1}) \to \Ext^2(T_1, T_i \otimes \Hom(T_i, T_{i-1})),
\]

\(\Ext^1(T_1, T_{i+1})\) can be computed by an inductive formula if we know the surjectivity of \(\Ext^1(T_1, T_i \otimes \Hom(T_i, T_{i-1})) \to \Ext^1(T_1, T_{i-1})\), which is equivalent to the surjectivity of

\[
\Hom(T_i \otimes \Hom(T_{i+1}, T_i)^*, T_1) = \Hom(T_{i+1}, T_i) \otimes \Hom(T_i, T_1) \to \Hom(T_{i+1}, T_1).
\]

By Remark 4.4, this is the composition map, since it is the dual map of the coevaluation map. By induction, we have a surjection

\[
\Hom(T_{i+1}, T_i) \otimes \Hom(T_i, T_{i-1}) \otimes \cdots \otimes \Hom(T_2, T_1) \to \Hom(T_{i+1}, T_1),
\]

By an argument similar to the one in Theorem 4.7, the kernel of this map is generated by all diagonals \(\delta_j, 2 \leq j \leq i\).

**Proposition 4.8.** For \(i \geq 2\),

\[
\bigotimes_{j=i}^{1} \Hom(T_{j+1}, T_j) \to \Hom(T_{i+1}, T_1) = \text{coker}(\sum_{j=i}^{2} \delta_j),
\]

where by abuse of notation \(\delta_j\) denotes \(\id \otimes \cdots \otimes \id \otimes \delta_j \otimes \id \otimes \cdots \otimes \id\).

Hence now we know the dimension of \(\Ext^1(T_1, T_{i+1})\) and \(\Hom(T_{i+1}, T_1)\) by an inductive formula. We make the following definition.

**Definition 4.9.** Let \(W\) be a wall whose corresponding rank 2 lattice is \(\mathcal{H}\). Let \(S_0, T_1\) be the two \(\sigma_0\)-stable objects in \(\mathcal{H}\) for general \(\sigma_0 \in \mathcal{H}\). Write \(g(\mathcal{H}) = \ext^1(T_1, S_0)\). The following sequence \(\{a_n(\mathcal{H})\}_{n \in \Z \geq 0}\) completely determined by \(\mathcal{H}\) is called the fundamental sequence of \(\mathcal{H}\):

\[
a_0(\mathcal{H}) = 1, \ a_1(\mathcal{H}) = g(\mathcal{H}), \ a_n(\mathcal{H}) = g(\mathcal{H}) \cdot a_{n-1}(\mathcal{H}) - a_{n-2}(\mathcal{H}), n \geq 2.
\]

We will write \(g\) for \(g(\mathcal{H})\) and \(a_n\) for \(a_n(\mathcal{H})\) if the context is clear.

Then the argument above and a symmetric argument yield the following theorem.

**Theorem 4.10.** For all \(i \geq 1\), we have

\[
\text{hom}(T_i, T_1) = a_{i-1}, \ \text{hom}(S_0, T_i) = a_{i-2}, \ \text{ext}^1(T_i, T_1) = a_{i-3}, \ \text{ext}^1(T_i, S_0) = a_i.
\]
Similarly, for all $j \geq 0$, we have
\[ \text{hom}(S_0, S_{-j}) = a_j, \quad \text{hom}(S_{-j}, T_1) = a_{j-1}, \quad \text{ext}^l(S_0, S_{-j}) = a_{j-2}, \quad \text{ext}^l(T_1, S_{-j}) = a_{j+1}. \]

So the dimensions of these groups are all known in terms of $\text{ext}^l(T_1, S_0)$.

4.3. The Harder-Narasimhan Filtration. In this subsection we compute the $\sigma_-$-Harder-Narasimhan filtration of a $\sigma_+$-stable spherical object.

We first prove a commutativity lemma:

**Lemma 4.11.** For any $i \geq j$, the following diagram commutes:

\[
\begin{array}{ccc}
T_j \otimes \text{Hom}(T_i, T_j)^* & \xrightarrow{\text{coev}_{j,i} \otimes \text{id}} & T_j \otimes \text{Hom}(T_j, T_{j-1})^* \otimes \text{Hom}(T_i, T_j)^* \\
\downarrow \text{id} \otimes \text{ev}_{j+1,i} & & \downarrow \text{id} \otimes \text{coev}_{j+1,i} \\
T_j \otimes \text{Hom}(T_{j+1}, T_j)^* \otimes \text{Hom}(T_i, T_{j+1})^* & \xrightarrow{(\text{ev}_{j,j-1} \circ \text{id}) \otimes \text{id}} & T_j \otimes \text{Hom}(T_i, T_{j+1})^* \\
\end{array}
\]

Similarly, for any $i \geq j$, the following diagram commutes:

\[
\begin{array}{ccc}
S_{i+1} \otimes \text{Hom}(S_{i-1}, S_j)^* & \xrightarrow{\text{ev}_{i+1,i} \otimes \text{id}} & \text{Hom}(S_{i+1}, S_i) \otimes \text{Hom}(S_{i-1}, S_j)^* \\
\downarrow \text{id} \otimes \text{coev}_{i+1,i} & & \downarrow \text{id} \otimes \text{coev}_{i+1,i} \\
S_{i+1} \otimes \text{Hom}(S_{i+1}, S_i) \otimes \text{Hom}(S_{i-1}, S_j)^* & \xrightarrow{\text{ev}_{i+1,i} \otimes \text{id}} & \text{Hom}(S_{i+1}, S_i) \otimes \text{Hom}(S_{i-1}, S_j)^* \\
\end{array}
\]

**Proof.** We prove the commutativity of the first diagram; the second is similar. By Theorem 4.8, for any $k \geq l$,

\[
\text{Hom}(T_k, T_l) \subset \bigotimes_{j=l}^{k-1} \text{Hom}(T_{j+1}, T_j)^*,
\]

where the embedding is given by the composition of the duals of the evaluation maps. Hence it suffices to check the commutativity of the following diagram

\[
\begin{array}{ccc}
T_j \otimes \bigotimes_{k=j}^{i-1} \text{Hom}(T_{k+1}, T_k)^* & \xrightarrow{\text{coev}_{j,i} \otimes \text{id}} & T_{j-1} \otimes \bigotimes_{k=j}^{i-1} \text{Hom}(T_{k+1}, T_k)^* \\
\downarrow \text{id} & & \downarrow \text{id} \otimes \delta_i^* \\
T_j \otimes \bigotimes_{k=j}^{i-1} \text{Hom}(T_{k+1}, T_k)^* & \xrightarrow{\text{ev}_{j,j-1} \circ \text{id} \otimes \text{ev}_{i,j}} & T_{j-1} \otimes \bigotimes_{k=j}^{i-1} \text{Hom}(T_{k+1}, T_k)^* \\
\end{array}
\]

By Lemma 4.5 and Lemma 4.6,

\[
(\text{id} \otimes \delta_i^*) \circ (\text{coev}_{j,j-1} \otimes \text{id}) = (\text{id} \otimes m) \circ (\text{id} \otimes \epsilon_j) \circ (\text{coev}_{j,j-1} \otimes \text{id})
\]

\[
= (\text{id} \otimes m) \circ (\text{coev}_{j,j-1} \otimes \text{id}) \circ (\text{id} \otimes \epsilon_j) = \text{ev}_{j,j-1} \circ (\text{id} \otimes \epsilon_j),
\]

namely a commutative diagram

\[
\begin{array}{ccc}
T_j \otimes \text{Hom}(T_{j+1}, T_j)^* & \xrightarrow{\text{coev}_{j,i} \otimes \text{id}} & T_{j-1} \otimes \text{Hom}(T_{j-1}, T_{j-1})^* \otimes \text{Hom}(T_{j+1}, T_j)^* \\
\downarrow \text{id} \otimes \text{ev}_{j} & & \downarrow \text{id} \otimes \text{ev}_{j} \\
T_{j-1} \otimes \text{Hom}(T_{j-1}, T_{j-1})^* \otimes \text{Hom}(T_{j+1}, T_j)^* & \xrightarrow{\text{id} \otimes \text{ev}_{j}} & T_{j-1} \\
\end{array}
\]

commutativity of the left and upper triangles are clear, commutativity of the right triangle is by Lemma 4.6, and commutativity of the right triangle is by Lemma 4.5.
We let

\[ \text{conn}_{(i,j),(i,j-1)} : T_j \otimes \text{Hom}(T_i,T_j)^* \to T_{j-1} \otimes \text{Hom}(T_i,T_{j+1})^* \]

\[ \text{conn}_{(i+1,j),(i,j)} : S_{i+1} \otimes \text{Hom}(S_{i-1},S_j) \to S_i \otimes \text{Hom}(S_i,S_j) \]

be the composition maps described in Lemma 4.11. We sometimes simply write \( \text{conn} \) if the context is clear.

**Theorem 4.12.** There are short exact sequences

\[ 0 \to T_i \to T_j \otimes \text{Hom}(T_i,T_j)^* \to T_{j-1} \otimes \text{Hom}(T_i,T_{j+1})^* \to 0, \]

\[ 0 \to S_{i+1} \otimes \text{Hom}(S_{i-1},S_j) \to S_i \otimes \text{Hom}(S_i,S_j) \to S_j \to 0, \]

for all \( i \geq j \), where \( T_0 := S_0[1] \), \( S_1 := T_1[-1] \).

**Proof.** We prove the first sequence, the second is similar. Use induction on \( i - j \). When \( i - j = 0 \), \( \text{Hom}(T_i,T_j)^* = \text{Hom}(T_i,T_1)^* = \mathbb{C} \), and \( \text{Hom}(T_i,T_{j+1})^* = \text{Hom}(T_i,T_{i+1})^* = 0 \), hence the claim reads

\[ 0 \to T_i \to T_i \to 0 \]

which is clearly true. When \( i - j = 1 \), \( \text{Hom}(T_i,T_j)^* = \text{Hom}(T_i,T_{i-1})^* = \text{Hom}(T_{i-1},T_{i-2}) \), and \( \text{Hom}(T_i,T_{j+1})^* = \text{Hom}(T_i,T_i)^* = \mathbb{C} \), hence the sequence reads

\[ 0 \to T_i \to T_{i-1} \otimes \text{Hom}(T_{i-1},T_{i-2}) \to T_{i-2} \to 0 \]

which is true by construction.

Now suppose the claim is true for up to \( i - j \). We have the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & T_{i+1} & \to & T_j \otimes \text{Hom}(T_{i+1},T_j)^* & \to & \text{coker} \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_{j+1} \otimes \text{Hom}(T_{i+1},T_{j+1})^* & \to & T_j \otimes \text{Hom}(T_{i+1},T_{j+1})^* & \to & T_{j-1} \otimes \text{Hom}(T_{i+1},T_{j+1})^* \\
\downarrow \text{conn} & & \downarrow \text{id} & & \downarrow \text{id} & & \\
0 & \to & T_j \otimes \text{Hom}(T_{i+1},T_{j+2})^* & \to & T_j \otimes \text{Hom}(T_{i+1},T_{j+2})^* & \to & 0 \\
\end{array}
\]

The commutativity of the top square is clear, and the commutativity of the bottom square is by Lemma 4.11. The first vertical sequences is exact by induction hypothesis. The second vertical sequence is exact by an argument similar to the one in Proposition 4.7. Hence by the Snake Lemma, \( \text{ker} = 0 \) and \( \text{coker} = T_{j-1} \otimes \text{Hom}(T_{i+1},T_{j+1})^* \). By definition, the composition of \( T_j \otimes \text{Hom}(T_{i+1},T_j)^* \xrightarrow{\text{id} \otimes \text{ev}_{j+1,i,j}} T_j \otimes \text{Hom}(T_{i+1},T_j)^* \otimes \text{Hom}(T_{i+1},T_{j+1})^* \)

\( \text{and} \) \( T_j \otimes \text{Hom}(T_{i+1},T_j)^* \otimes \text{Hom}(T_{i+1},T_{j+1})^* \xrightarrow{\text{ev}_{j+1,i,j} \otimes \text{id} \otimes \text{ev}_{i+1,j}} T_{j-1} \otimes \text{Hom}(T_{i+1},T_{j+1})^* \) is exactly \( \text{conn}_{(i+1,j),(i+1,j-1)} \). By induction, the theorem is proved.

\[ \square \]

**Corollary 4.13.** For all \( i \geq 1 \), the Jordan-Hölder filtration of \( T_i \) is given by

\[ 0 \to S_0 \otimes \text{Hom}(T_i,T_2)^* \to T_i \to T_1 \otimes \text{Hom}(T_i,T_1)^* \to 0, \]
similarly, for all $j \leq 0$, the Jordan-Hölder filtration of $S_j$ is given by
\[ 0 \to S_0 \otimes \text{Hom}(S_0, S_j) \to S_j \to T_1 \otimes \text{Hom}(S_{-1}, S_j) \to 0. \]

**Proof.** For the first claim, let $j = 1$ in Theorem 4.12, for the second claim, let $i = 0$ in Theorem 4.12. \qed

**Remark 4.14.** Since $\text{Hom}(S_0, T_1) = \text{Hom}(T_1, S_0) = 0$, we see $S_0 \otimes \text{Hom}(T_i, T_2) \to T_i$ is the evaluation map, and $S_j \to T_1 \otimes \text{Hom}(S_{-1}, S_j)$ is the coevaluation map.

## 5. Local Reduction

In this section, we develop the local reduction algorithm that appeared in the global reduction (Algorithm 3.14). The objective is to find the $\sigma_-$ (resp. $\sigma_+$)-Harder-Narasimhan filtration of a rigid object (not necessarily stable), from its $\sigma_+$ (resp. $\sigma_-$)-Harder-Narasimhan filtration. This can be viewed as a generalization of Corollary 4.13. For a rough idea of local reduction, see Section 3.

### 5.1. Exhaustive Filtration

In this subsection we introduce the notion of an exhaustive filtration. This notion is the central idea of the whole algorithm. It connects the information of the Harder-Narasimhan filtrations at $\sigma_+$ and $\sigma_-$. Let $W$ be a wall and $\mathcal{H}$ be the rank 2 lattice of $W$ with the two $\sigma_+$-stable spherical objects $S_0, T_1$. Assume $\phi_+(T_1) > \phi_+(S_0)$. Let $\mathcal{A}_W$ be the full subcategory of $\mathcal{A}$ consisting of objects whose Mukai vectors are in $\mathcal{H}$.

**Definition 5.1 (Exhaustive filtration).** For any rigid object $E \in \mathcal{A}_W$, a finite filtration
\[ 0 \subset E_1 \subset F_1 \subset \cdots \subset E_n \subset F_n \subset \cdots \subset E \]
of $E$ is called $S$-exhaustive, if $\text{Hom}(S_0, Q_i) = 0$ for all $Q_i = E/E_i, i \geq 1$, and $\text{Hom}(T_1, R_j) = 0$ for all $R_j = E/F_j, j \geq 1$. Similarly, it is called $T$-exhaustive if $\text{Hom}(T_1, Q_i) = 0$ for all $i \geq 1$ and $\text{Hom}(S_0, R_j) = 0$ for all $j \geq 1$.

**Lemma 5.2.** Exhaustive filtrations exist.

**Proof.** We construct an $S$-exhaustive filtration, the other case is similar. Let $E_1 = S_0 \otimes \text{Hom}(S_0, E)$. This is the first $\sigma_+$-Harder-Narasimhan factor of $E$, and let $Q_1 = E/E_1$. Then clearly $\text{Hom}(S_0, Q_1) = 0$. Suppose we have constructed $Q_i$, to construct $R_i$, just note that $T_1 \otimes \text{Hom}(T_1, Q_i)$ is the first $\sigma_-$-Harder-Narasimhan factor of $Q_i$ and we let $F_i/E_i = T_1 \otimes \text{Hom}(T_1, Q_i)$. Similarly to construct $R_{i+1}$, we let $E_{i+1}/F_i = S_0 \otimes \text{Hom}(S_0, R_i)$. Since the length of $E$ is finite, this process terminates and we get an $S$-exhaustive filtration. \qed

The next lemma shows that the exhaustive filtration of a rigid object is determined by its Jordan-Hölder factors.

**Lemma 5.3.** Let $E \in \mathcal{A}_W$ be rigid and
\[ 0 \subset E_1 \subset F_1 \subset \cdots \subset E_n \subset F_n \subset \cdots \subset E \]
a rigid filtration of $E$. Then $Q_i, R_i$ are all rigid, and
\[ 0 \to F_i/E_i \to Q_i \to R_i \to 0, \]
\[ 0 \to E_{i+1}/F_i \to R_i \to Q_{i+1} \to 0 \]
are general extensions.
Proof. It suffices to prove the statement for \( i = 1 \). Note that \( E_1 \) is the first \( \sigma_+ \) or \( \sigma_- \) Harder-Narasimhan factor of \( E \), hence by Lemma 3.13, \( E_1 \) and \( Q_1 \) are rigid. Since \( E \) is rigid, the extension

\[
0 \rightarrow E_1 \rightarrow E \rightarrow Q_1 \rightarrow 0
\]

is general. A similar argument proves the exactness of

\[
0 \rightarrow F_1/E_1 \rightarrow E/E_1 \rightarrow R_1 \rightarrow 0.
\]

\[\square\]

5.2. Admissible Extensions.

Notation 5.4. From now on we write

\[
0 \subset \cdots \subset E_2 \subset E_1 \subset F_1 \subset E_0 = E
\]

for an exhaustive filtration, \( Q_i = E/E_i, R_i = E/F_{i+1}, G_i = F_i/E_i, H_i = E_i/F_{i+1} \).

Then \( H_0 = T \otimes B_0 \) or \( H_0 = S \otimes A_0 \), where \( B_0, A_0 \) are some finite dimensional vector spaces.

For simplicity, we treat the case \( H_0 = T \otimes B_0 \). The other case is similar. Then \( G_i = S \otimes A_i \) and \( H_i = T \otimes B_i \). Let \( V = \text{Ext}^1(T_1, S_0) \).

Notation 5.5. For the rest of the paper, for two linear spaces \( V, W \), we sometimes write \( VW \) for \( V \otimes W \).

Consider \( E_i/E_{i+1} \in B_i^* V A_{i+1} \) or \( F_i/F_{i+1} \in A_i^* V^* B_i \), they are \( V \) or \( V^* \)-Kronecker modules of certain fixed dimension vector. In this subsection we want to describe all possible Kronecker modules that arise from a fixed exhaustive filtration. Since we do not address stability of these Kronecker modules nor the geometry of these loci in the moduli spaces, we will work in the representation spaces \( B_i^* V A_{i+1} \) and \( A_i^* V^* B_i \). For more details about Kronecker modules, see [Kin94].

Consider \( \text{Ext}^1(R_i, G_{i+1}) \). We have the following commutative diagram.

\[
\begin{array}{ccc}
\text{Ext}^2(R_{i-1}, G_{i+1}) & \rightarrow & 0 \\
\downarrow & & \\
\text{Ext}^1(R_i, G_{i+1}) & \rightarrow & \text{Ext}^1(H_i, G_{i+1}) \\
& \phi & \downarrow \\
& & \text{Ext}^2(G_i, G_{i+1}) \\
& & \downarrow \\
& & 0
\end{array}
\]

where the map \( \phi \) is induced by the previous extension class \( \delta_{i,i} \in \text{Ext}^1(G_i, H_i) = A_i^* V^* B_i \).

Since \( G_{i+1} = S \otimes A_{i+1} \), \( \phi \) is the adjoint map

\[
\delta_{i,i} \otimes \text{id} : B_i^* V A_{i+1} \rightarrow A_i^* A_{i+1}.
\]

Definition 5.6. We say an extension \( \tilde{\delta}_{i,i+1} \in \text{Ext}^1(H_i, G_{i+1}) = B_i^* V A_{i+1} \) is admissible, if it is the image of some \( \delta_{i,i+1} \in \text{Ext}^1(R_i, G_{i+1}) \) under the map above, where the middle term of \( \tilde{\delta}_{i,i+1} \) is isomorphic to \( Q_{i+1} \).
Similarly, we say $\delta_{i,i} \in \text{Ext}^1(G_i, H_i) = A^1V^*B_i$ is admissible, if it is the image of some $\tilde{\delta}_{i,i} \in \text{Ext}^1(Q_i, H_i)$, where the middle term of $\tilde{\delta}_{i,i}$ is isomorphic to $R_i$. We denote the set of admissible extensions by $\text{Adm}_{i,i}$.

There are quasi-projective varieties in $B_i^*VA_{i+1}$ or $A_i^*V^*B_i$ that bound the set of admissible extensions.

**Lemma 5.7.** There exists a closed subvariety $\text{ADM}_{i,i+1} \subset B_i^*VA_{i+1}$ and an open subvariety $\text{adm}_{i,i+1} \subset \text{ADM}_{i,i+1}$, such that

$$\text{adm}_{i,i+1} \subset \text{Adm}_{i,i+1} \subset \text{ADM}_{i,i+1}.$$  

A similar conclusion holds for $\text{Adm}_{i,i}$.

**Proof.** We do induction, clearly this is true for $\text{Adm}_{0,1}$. Consider $\text{adm}_{i,j} \times X$ with projections $p, q$, respectively. Then we have the relative version of the long exact sequence

$$R^1 p_*\text{Hom}(q^*R_i, q^*G_{i+1}) \to R^1 p_*\text{Hom}(q^*H_i, q^*G_{i+1}) \to R^2 \text{Hom}(q^*G_i, q^*G_{i+1})$$

of sheaves on $\text{adm}_{i,j}$. Now $R^1 p_*\text{Hom}(q^*R_i, q^*G_{i+1})$ is a trivial vector bundle whose fibers are $\text{Ext}^1(R_i, G_{i+1})$. By the rigidity of $Q_{i+1}$, there is an open subvariety $U_{i,i+1} \subset \text{Ext}^1(R_i, G_{i+1})$, whose elements as extensions have middle term isomorphic to $Q_{i+1}$. Then $f(U_{i,i+1} \times \text{adm}_{i,i}) \subset R^1 p_*\text{Hom}(q^*H_i, q^*G_{i+1})$ contains an open subvariety $U'_{i,i+1}$ of the image of $f$, denote by $\mathcal{F}$. Here, $f$ is viewed as a morphism on the total spaces of vector bundles. Since $R^1 p_*\text{Hom}(q^*H_i, q^*G_{i+1}) \cong \text{adm}_{i,j} \times \text{Ext}^1(H_i, G_{i+1})$ is a trivial vector bundle, denote the projection to $\text{Ext}^1(H_i, G_{i+1})$ by $\pi$. Then we may take

$$\text{ADM}_{i,i+1} = \pi(\mathcal{F}) \subset \text{Ext}^1(H_i, G_{i+1}).$$

This subvariety is equal to the closure of

$$\bigcup_{\delta_{i,i} \in \text{Adm}_{i,i}} \ker(\delta_{i,i} \otimes \text{id}),$$

which contains $\text{Adm}_{i,i+1}$. Now by construction, $\text{Adm}_{i,i+1}$ contains $\pi(U'_{i,i+1})$, which contains an open subvariety of $\text{ADM}_{i,i+1}$.

□

We will show how to compute the ideals of $\text{ADM}_{j,j}$ and $\text{ADM}_{i,i+1}$ inductively. For this, we need the following lemma.

**Lemma 5.8.** Let $f : X \to Y$ be a surjective morphism between varieties, and $Y_1 \subset Y$ an irreducible component. Suppose a general fiber over $Y_1$ is a linear space of fixed dimension. Then there is a unique irreducible component $X_1 \subset X$ that dominates $Y_1$.

**Proof.** By assumption, over an open subset $U \subset Y_1$, $f$ is a topological fibration with isomorphic irreducible fibers. Let $X_1$ be the closure of $f^{-1}(U)$, clearly it is irreducible. Let $x \in X_1$ be a general point, $y = f(x)$ be its image. Since $x$ is general, $y \in Y_1$ is also general. Take an open neighborhood $V$ of $x$, by shrinking if necessary we have $f(V) \subset U$. Then $V \subset f^{-1}(U) \subset X_1$, we see that any specialization to $x$ must come from $X_1$. Hence $X_1$ is an irreducible component. □
We describe how to compute $\text{ADM}_{i,i+1}$ assuming that the ideal of $\text{ADM}_{i,i}$ is known. The method for $\text{ADM}_{j,j}$ is similar. Consider the incidence correspondence

$$\Gamma = \{ (\delta_{i,i}, \delta_{i,i+1}) : \delta_{i,i} \in \text{ADM}_{i,i}, m(\delta_{i,i}, \delta_{i,i+1}) = 0 \} \subset A_i^* V^* B_i \times B_i^* V A_{i+1},$$

where $m : A_i^* V^* B_i \otimes B_i^* V A_{i+1} \rightarrow A_i^* \otimes A_{i+1}$ is the natural pairing. Suppose that we know the ideal of $\text{ADM}_{i,i}$. The ideal of $\Gamma$ can be computed, since it is the intersection of the pre-image of $\text{ADM}_{i,i}$ with the vanishing locus of the paring $m$, which can be written down explicitly in coordinates. By Lemma 5.8, $\Gamma$ has a distinguished component $\Gamma_1$ that dominates $\text{ADM}_{i,i}$. By doing primary decomposition, we may find the ideal of this component $\Gamma_1$. By the proof of Lemma 5.7, the image of $\Gamma_1$ under the second projection is $\text{ADM}_{i,i+1}$ set-theoretically. Since we only care about the admissible locus set-theoretically, we may take the scheme-theoretic image of $\Gamma_1$ under the second projection. If the ideal of the image is not prime, then we take its radical. The resulting ideal is the ideal of $\text{ADM}_{i,i+1}$. Hence in principle, the ideal of $\text{ADM}_{i,i+1}$ can be computed, given the ideal of $\text{ADM}_{i,i}$. We summarize this procedure in the following figure:

$$\text{ADM}_{i,i} \xrightarrow{\text{incidence correspondence}} \Gamma \xrightarrow{\text{primary decomposition}} \Gamma_1 \xrightarrow{\text{projection + radical}} \text{ADM}_{i,i+1}.$$ 

Now $\text{ADM}_{0,1} = B_i^* V A_1$ is known. Hence to compute $\text{ADM}_{i,i+1}$, the only needed input is the dimensions of $A_k, B_l$ for $k \leq i + 1, l \leq i$. We summarize this in the following proposition.

**Proposition 5.9.** Using Notation 5.4, the ideal of $\text{ADM}_{i,i+1} \subset B_i V A_{i+1}$ can be computed in terms of the dimensions of $A_k, B_l$ for $k \leq i + 1, l \leq i$. A similar conclusion holds for $\text{ADM}_{j,j}$.

By Lemma 5.3, an exhaustive filtration is determined by the dimensions of all $A_k, B_l$ under Notation 5.4. Hence we introduce the following definition.

**Definition 5.10.** Using Notation 5.4, the shape of an exhaustive filtration is the sequence of numbers

$$(\cdots, \dim A_{i+1}, \dim B_i, \dim A_i, \cdots, \dim A_1, \dim B_0).$$

Hence from now on we assume all admissible loci are computable in terms of the shape of the exhaustive filtration.

The next observation describes the symmetry of $\text{ADM}$.

**Lemma 5.11.** The variety $\text{ADM}_{i,i+1}$ is stable under $GL_{B_i} \times GL_{A_{i+1}}$. Similarly, $\text{ADM}_{i,i}$ is stable under $GL_{A_i} \times GL_{B_i}$.

**Proof.** We only prove the lemma for $\text{ADM}_{i,i+1}$, the other case is similar.

First note that $\text{Adm}_{i,i+1}$ is clearly stable under $GL_{A_{i+1}}$, because a change of basis on $A_{i+1}$ does not change the middle term of the extension. Since $\text{ADM}_{i,i+1}$ is irreducible and $\text{Adm}_{i,i+1}$ contains an open subset of $\text{ADM}_{i,i+1}$, $\text{ADM}_{i,i+1}$ is stable under $GL_{A_{i+1}}$.

For $GL_{B_i}$ part, take any element $\beta \in GL_{B_i}$. Under the notations above, consider

$$\pi(\mathcal{F}) = \bigcup_{\delta_{i,i} \in \text{adm}_{i,i}} \ker(\delta_{i,i} \otimes \text{id}).$$

By the induction hypothesis on $\text{ADM}_{i,i}$, we may shrink $\text{adm}_{i,i}$ if necessary so that it is preserved under $GL_{B_i}$. Now then clearly

$$\beta(\pi(\mathcal{F})) = \pi(\beta(\mathcal{F})) = \pi(\mathcal{F}).$$
since \( \beta(\delta_{i,i} \otimes \text{id}) = \beta(\delta_{i,i}) \otimes \text{id} \) and \( \beta(\delta_{i,i}) \in \text{adm}_{i,i} \) by construction. \( \square \)

**Remark 5.12.** One may be tempted to expect the varieties \( \text{ADM}_{i,i} \) or \( \text{ADM}_{i,i+1} \) are determinantal varieties, but in fact they can be more complicated.

### 5.3. Relation to the Harder-Narasimhan Filtration.

In this subsection we relate exhaustive filtration to the \( \sigma_- \)-Harder-Narasimhan filtration. Suppose

\[
0 = E_n \subset F_n \subset E_{n-1} \subset F_{n-1} \subset \cdots \subset E_1 \subset F_1 \subset E_0 = E
\]

is an \( S \)-exhaustive filtration of a rigid object \( E \in \mathcal{A}_W \). We want to compute the \( \sigma_- \)-Harder-Narasimhan filtration of \( E \). The first thing to note is that

**Lemma 5.13.** The first \( \sigma_- \)-Harder-Narasimhan factor is

\[
gr_1^-(E) = F_n/E_n = G_n = S \otimes A_n.
\]

**Proof.** Apply \( \text{Hom}(S, -) \) to the short exact sequence

\[
0 \rightarrow G_n \rightarrow E \rightarrow R_{n-1} \rightarrow 0,
\]

we have

\[
0 \rightarrow \text{Hom}(S, G_n) \rightarrow \text{Hom}(S, E) \rightarrow \text{Hom}(S, R_{n-1}).
\]

By the definition of \( S \)-exhaustive filtration, \( \text{Hom}(S, R_{n-1}) = 0 \). Hence \( \text{Hom}(S, E) \cong \text{Hom}(S, G_n) = A_n \). Then \( G_n \subset E \) is the evaluation map

\[
S \otimes \text{Hom}(S, E) \rightarrow E.
\]

Note that \( S = S_0 \) has maximal \( \sigma_- \)-phase among all \( \sigma_- \)-stable spherical objects in \( \mathcal{H} \), hence \( S \otimes \text{Hom}(S, E) \) is the first Harder-Narasimhan factor. \( \square \)

Hence our task is to compute the first \( \sigma_- \)-Harder-Narasimhan factor of \( R_{n-1} = E/\text{gr}_1^-(E) \). For simplicity we write \( O_i = S_i^- \) for \( i \leq 0 \) and \( O_i = T_i^- \) for \( i \geq 1 \). Our next observation makes crucial use of the notion of exhaustive filtration.

**Proposition 5.14.** Under Notation 5.4, let \( R_{n-1} \) be an object equipped with a \( T \)-exhaustive filtration. Then for any \( i \in \mathbb{Z} \),

\[
\text{Hom}(O_i, R_{n-1}) = \text{Hom}(O_i, F_{n-1}/F_n).
\]

**Proof.** Apply \( \text{Hom}(O_i, -) \) to the short exact sequence

\[
0 \rightarrow F_n/F_{n-1} \rightarrow R_{n-1} \rightarrow R_{n-2} \rightarrow 0,
\]

we get

\[
0 \rightarrow \text{Hom}(O_i, F_n/F_{n-1}) \rightarrow \text{Hom}(O_i, R_{n-1}) \xrightarrow{f} \text{Hom}(O_i, R_{n-2}).
\]

It suffices to show the map \( f \) is zero.

We know \( O_i \) fits into

\[
0 \rightarrow T \otimes \text{Hom}(T, O_i) \rightarrow O_i \rightarrow S \otimes \text{Hom}(O_i, S)^* \rightarrow 0.
\]

Let

\[
N_i = \text{Hom}(T, O_i), \quad M_i = \text{Hom}(O_i, S)^*.
\]

(1)
We have the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(S \otimes M_i, R_{n-1}) & \longrightarrow & \text{Hom}(O_i, R_{n-1}) \\
\downarrow & & \downarrow f \\
\text{Hom}(S \otimes M_i, R_{n-2}) & \longrightarrow & \text{Hom}(O_i, R_{n-2}) \\
\end{array}
\]

Since \( R_{n-2} \) is a \( T \)-exhaustive filtration, we have \( \text{Hom}(S, R_{n-2}) = 0 \). Also note that the factorization of the map \( R_{n-1} \to R_{n-2} \) into the composition of \( R_{n-1} \to Q_{n-1} \) and \( Q_{n-1} \to R_{n-2} \) induces the factorization of the map \( g : \text{Hom}(T \otimes N_i, R_{n-1}) \to \text{Hom}(T \otimes N_i, R_{n-2}) \) into \( \text{Hom}(T \otimes N_i, R_{n-1}) \to \text{Hom}(T \otimes N_i, Q_{i-1}) \) and \( \text{Hom}(T \otimes N_i, Q_{i-1}) \to \text{Hom}(T \otimes N_i, R_{n-2}) \).

By the definition of a \( T \)-exhaustive filtration, \( \text{Hom}(T, Q_{i-1}) = 0 \). Hence \( g = 0 \), and the commutative diagram now reads

\[
\begin{array}{ccc}
\text{Hom}(O_i, R_{n-1}) & \longrightarrow & \text{Hom}(T \otimes N_i, R_{n-1}) \\
\downarrow f & & \downarrow 0 \\
0 & \longrightarrow & \text{Hom}(O_i, R_{n-2}) \\
\end{array}
\]

Hence the map \( f \) must be zero.

Hence to get the first Harder-Narasimhan factor for \( R_{n-1} \), it remains to find the index \( i \) for \( O_i \) such that \( \text{Hom}(O_i, R_{n-1}) \neq 0 \) and \( \phi_-(O_i) \) is maximal, and to compute the dimension of \( \text{Hom}(O_i, R_{n-1}) \). For simplicity, we will sometimes drop the indices of \( O_i \) and \( R_{n-1} \) when there is no ambiguity.

Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(O, T \otimes B) = 0 & \longrightarrow & \text{Hom}(T \otimes N, T \otimes B) \\
\downarrow & & \downarrow \delta \otimes \text{id}_B \\
0 = \text{Hom}(S \otimes M, \Gamma) & \longrightarrow & \text{Hom}(S \otimes M, S \otimes A) \\
\downarrow & & \downarrow \text{id}_M \otimes \delta \\
0 = \text{Hom}(O, T \otimes B) & \longrightarrow & \text{Hom}(O, \Gamma) \\
\downarrow & & \downarrow \text{id}_A \\
0 = \text{Hom}(T \otimes N, S \otimes A) & \longrightarrow & \text{Ext}^1(O, T \otimes B) \\
\end{array}
\]

where \( \delta \in \text{ADM} \subseteq A^*V^*B \) is a general admissible extension, and

\[
[0 \longrightarrow T \otimes N \longrightarrow O \longrightarrow S \otimes M \longrightarrow 0] = \delta_0 \in M^*V^*N
\]

is general, hence in particular the image of \( \text{Hom}(T \otimes N, T \otimes B) \) under \( \delta_0 \otimes \text{id}_B \) is of the form \( W \otimes B \), where \( W \subseteq M^*V^* \) is general.

\( J = \text{Hom}(O, \Gamma) \) can be identified with

\[
\text{Hom}(O, \Gamma) = \text{Hom}(S \otimes M, S \otimes A) \cap \text{Hom}(T \otimes N, T \otimes B) = M^*\delta(A) \cap \delta_0(N^*)B \subset M^*V^*B.
\]

The dimension of \( \delta_0(N^*)B \) is known in terms of the dimensions of \( M, N \), since \( \delta_0 \in M^*V^*N \) is general. The dimension of \( M^*\delta(A) \) is also known in principle, since we know the ideal
of ADM by Proposition 5.9. We would like to compute the dimension of their intersection. Unfortunately, in general they do not intersect transversely.

Hence we need to know the locus of \( \delta(A) \subset V^*B \) for a general \( \delta \in ADM \). Let \( a = \dim \delta(A) \) for a general \( \delta \in ADM \) and \( A' \subset A \) be any subspace of dimension \( a \), and \( i : A' \longrightarrow A \) be the inclusion. Let \( ADM' \subset A'^*V^*B \) be the image of ADM under the projection \( i^* : A'^*V^*B \longrightarrow A'^*V^*B \). Since ADM is stable under \( GL_A \times GL_B \), \( ADM' \) is independent of the choice of \( A' \). Then for a general \( \delta' \in ADM' \), the map \( \delta' : A' \longrightarrow V^*B \) is injective.

Let \( x_1, \ldots, x_a \) be a basis of \( A' \). Then there is a rational map \( l : ADM' \longrightarrow G(a, V^*B) \), defined on the locus \( \{ \delta' \in ADM : \delta' : A' \longrightarrow V^*B \) is injective \}, by

\[
(2) \quad l(\delta') = \delta'(x_1) \wedge \delta'(x_2) \wedge \cdots \wedge \delta'(x_a)
\]

under the Plücker embedding. Denote the image by \( l(ADM) \). Let

\[
\sigma(N, k) = \{ \Lambda \in G(ma, M^*V^*B) : \dim(\Lambda \cap \delta_0(N) \otimes B) \geq k \}.
\]

The \( \sigma(N, k) \) are given by Proposition 4.7 and Proposition 4.8, hence in principle we may compute the maximal integer \( k \) such that \( M^* \otimes l(ADM') \subset \sigma(N, k) \). This number is determined by the index \( i \) of \( O_i \), which we shall denote by \( k(i) \).

In order to find the first Harder-Narasimhan factor, we need to compare the \( \phi_- \)-phases of \( O_i \). Recall that we defined \( a_i \) to be the fundamental sequence of \( H \) in Definition 4.9. Now we define a function \( \phi_- \) on \( \mathbb{Z} \) by

\[
\phi_-(i) = \begin{cases} 
\frac{a_i - 2 + a_{i-1}}{a_{i+1}} , & i \geq 1 , \\
\frac{a_i - a_{i-1}}{a_{i+1} + a_{i-1}} , & i \leq 0 .
\end{cases}
\]

Then \( \phi_-(O_i) > \phi_-(O_j) \) if and only if \( \phi_-(i) > \phi_-(j) \). Hence we have the following proposition.

**Proposition 5.15.** Under the assumptions in Proposition 5.14, the first Harder-Narasimhan factor of \( R_{n-1} \) is \( O_i \otimes J \) such that \( \phi_-(i) \) is maximal and \( k(i) > 0 \). The vector space \( J_i = \text{Hom}(O_i, R_{n-1}) \) has dimension \( k(i) \).

**Remark 5.16.** At first glance there are infinitely many \( i \) such that \( k(i) > 0 \), and it is not clear that if we can find the \( i \) such that \( \phi_-(i) \) is maximal.

However, in practice there are only finitely many \( i \) to check. Using the notation in Proposition 5.15, if \( O_i \) is the first \( \sigma_-\)-Harder-Narasimhan factor, then \( O_i \otimes J_i \longrightarrow R_{n-1} \) is injective. Hence we must have

\[
\dim(M_i) \leq \dim(M_i) \cdot k(i) \leq \dim(A) , \quad \dim(N_i) \leq \dim(N_i) \cdot k(i) \leq \dim(B) .
\]

Since the dimensions of \( A \) and \( B \) are data in the exhaustive filtration of \( E \), the possible range of \( i \) is computed by Theorem 4.10. There are only finitely many such \( i \), since the fundamental sequence (Definition 4.9) \( a_i \) tends to infinity.

**Remark 5.17.** The first \( \sigma_-\)-Harder-Narasimhan factor is of the form \( O_i^\oplus n \). Hence the information consists of two numbers \( i \) and \( n \). Proposition 5.15 is where we find out the two numbers. Since the dimensions of \( M_k, N_i \) are given by Theorem 4.10, the \( i \) and \( n \) are determined by the admissible locus \( ADM \subset A^*V^*B \), which is eventually determined by the shape of the exhaustive filtration by Proposition 5.9.
5.4. **Receptive Extensions.** Recall that our objective of the local reduction is to compute the \( \sigma_-\)-Harder-Narasimhan filtration of \( E \). In the last subsection, we showed how to compute the first \( \sigma_-\)-Harder-Narasimhan factor \( E_1 \) of \( E \) from the shape of an exhaustive filtration of \( E \). In order to find the \( \sigma_-\)-Harder-Narasimhan filtration of \( E \), we would like to know the shape of an exhaustive filtration of \( E/E_1 \), so that we may use Proposition 5.15 again. In this subsection, we introduce a necessary notion called *receptive extension*.

The original \( S \)-exhaustive filtration on \( E \) is indeed a \( T \)-exhaustive filtration on \( R_{n-1} \), hence we only need to consider \( \text{Hom}(O_i, F_{n-1}/F_n) \). Let \( \Gamma = F_{n-1}/F_n \). Then \( \Gamma \) fits into an extension
\[
[0 \rightarrow T \otimes B_{n-1} \rightarrow \Gamma \rightarrow S \otimes A_{n-1} \rightarrow 0] = \delta_{n-1,n-1} \in A^*_{n-1}V^*B_{n-1},
\]
where \( \delta_{n-1,n-1} \in \text{Adm}_{n-1,n-1} \). If \( i \) is the integer so that \( O_i \otimes \text{Hom}(O_i, R_{n-1}) \) is the first \( B_-\)Harder-Narasimhan factor of \( R_{n-1} \), then \( O_i \otimes \text{Hom}(O_i, R_{n-1}) \rightarrow R_{n-1} \) is injective. Let \( \delta = \text{Hom}(O_i, R_{n-1}) \). We have the following commutative diagram
\[
\begin{array}{ccc}
T \otimes K_1 & \longrightarrow & S \otimes K_2 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
T \otimes N_i J & \longrightarrow & O_i J \\
\downarrow & & \downarrow \\
0 & \longrightarrow & S \otimes M_i J \\
\downarrow & & \downarrow \\
T \otimes B' & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
0 & \longrightarrow & S \otimes A' \\
\downarrow & & \downarrow \\
T \otimes B'' & \longrightarrow & \Gamma'' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & S \otimes A''
\end{array}
\]
where \( B'', A'' \) are some fixed finite dimensional vector spaces. By the Snake Lemma, \( K_1 = 0 \). Note that \( \text{Hom}(S, T) = 0 \), hence \( K_2 = 0 \). Hence we get an extension
\[
[0 \rightarrow T \otimes B'' \rightarrow \Gamma'' \rightarrow S \otimes A'' \rightarrow 0] = \delta'' \in A'^*V^*B''.
\]
Hence there is a rational map
\[
\pi_{n-1,n-1} : \text{ADM}_{n-1,n-1} \rightarrow A'^*V^*B''.
\]
In this subsection we study the image of \( \pi = \pi_{n-1,n-1} \). We call the closure of the image the receptive locus \( \text{REC}_{n-1,n-1} \) and a general element in it receptive. Note that \( \text{REC} \) is stable under \( \text{GL}_{A''} \times \text{GL}_{B''} \) action.

We describe the rational map \( \pi = \pi_{n-1,n-1} \) now. Recall that
\[ J = \text{Hom}(O, \Gamma) = \text{Hom}(S \otimes M, S \otimes A) \cap \text{Hom}(T \otimes N, T \otimes B) = M^*\delta(A) \cap \delta_0(N^*)B \subset M^*V B. \]
The difficulty here is that, with fixed vector spaces \( A, B \), the embeddings \( N J \hookrightarrow B \) and \( M J \hookrightarrow A \) depends on \( \delta \). However, this problem can be solved if we restrict to an open subset of \( \text{ADM} = \text{ADM}_{n-1,n-1} \), because it does not affect the image of \( \pi \).

The map factors as \( \pi_B : \text{ADM} \rightarrow \text{ADM}' \subseteq A^*V B'' \) and \( \pi_A : \text{ADM}' \rightarrow A'^*V^* B'' \). We first compute the image of \( \text{ADM} \) under \( \pi_B \). For a general \( \delta \in \text{ADM} \), let \( q_\delta : B \rightarrow B'' \) be the family of projections so that \( B = NJ \oplus \delta B'' \). Then the map \( \pi_B|_U \) is defined as
\[
\pi_B(\delta) = (\text{id} \otimes q_\delta)(\delta) \in A'^*V^*B''.
\]
Fix a projection \( q : B \rightarrow B'' \), then since \( \text{ADM}' \) is stable under the \( \text{GL}_{B''} \) action, for a general \( \delta \in \text{ADM} \), \( (\text{id} \otimes q)(\delta) \in \text{ADM}' \). Conversely, for a general \( \delta \), there is some \( g \in \text{GL}_B \)
such that \((\text{id} \otimes q_\delta)(\delta) = (\text{id} \otimes q)(g(\delta))\), and clearly \(g(\delta) \in \text{ADM}\) since it is stable under \(\text{GL}_B\). Hence \(\text{ADM}' = q(\text{ADM})\).

Similarly for every \(\delta' \in \text{ADM}' \subset A^*V^*B''\), there exists a subspace \((A'')^* \subset A''^*\) such that \(\delta \in (A'')^*V^*B''\). Fix a subspace \(A''^* \subset A^*\) and a projection \(p : A^* \rightarrow A''^*\). Let \(U \subset \text{ADM}' \subset A^*V^*B''\) be an open subset such that for \(\delta \in U\), \(\delta\) has maximal \((A,VB)\)-rank and \((AV,B)\)-rank, and \(p|_{(A'')^*}\) is an isomorphism. Then by a similar trick as above, we see the image of \(\pi_A\) is \(p(\text{ADM}')\). Hence in summary we have showed:

**Proposition 5.18.** Let \(\pi : \text{ADM} \rightarrow A^*V^*B''\) be the rational map defined in (3). Choose any projections \(p : A^* \rightarrow A''^*\) and \(q : B \rightarrow B''\). Then the image of \(\pi\) is

\[
\text{REC} = (p \times \text{id} \times q)(\text{ADM}).
\]

Since \(\text{ADM}\) is stable under \(\text{GL}_A \times \text{GL}_B\), \(\text{REC}\) is independent of the choice of \(p, q\).

By Proposition 5.18, the recepitional locus \(\text{REC}\) can be computed, assuming that we know the admissible locus \(\text{ADM}\). By Proposition 5.9, \(\text{ADM}\) can be computed from the shape of the exhaustive filtration. Hence at this stage we may assume the recepitional loci \(\text{REC}\), produced by taking quotient from \(\Gamma\) by the first Harder-Narasimhan factor, are all known in terms of the shape of the exhaustive filtration.

### 5.5. Concatenation

In this subsection we determine the shape of an exhaustive filtration of an object, that is, a general extension of a known exhaustive filtration by a receptive extension.

We know \(O \otimes J \subset R_{n-1}\) is the first Harder-Narasimhan factor of \(R_{n-1}\), by Mukai’s Lemma (Lemma 3.11), \(R_{n-1}/(O \otimes J)\) is rigid. On the other hand, \(O \otimes J \rightarrow R_{n-1}\) factors through \(\Gamma\), we have the following commutative diagram

\[
\begin{array}{ccc}
O \otimes J & \rightarrow & O \otimes J \\
\downarrow & & \downarrow \\
0 & \rightarrow & \Gamma \\
\downarrow & & \downarrow \\
0 & \rightarrow & \Lambda \\
\end{array}
\]

\[
\begin{array}{ccc}
& & R_{n-1} \rightarrow R_{n-2} \rightarrow 0 \\
\downarrow & & \downarrow \\
& & R_{n-1}/(O \otimes J) \rightarrow R_{n-2} \rightarrow 0,
\end{array}
\]

where \(\Lambda = \Gamma/(O \otimes J)\) can fit into a general receptive extension

\[
[0 \rightarrow T \otimes B'' \rightarrow \Lambda \rightarrow S \otimes A'' \rightarrow 0] = \delta'' \in \text{REC}_{n-1,n-1} \subset A''^*V^*B''.
\]

By rigidity of \(Q_1^- = R_{n-1}/(O \otimes J)\), \(Q_1^-\) fits into a general extension in \(\text{Ext}^1(R_{n-2},\Lambda)\).

Our next task is to compute the \(T\)-exhaustive filtration of \(Q_1^-\). We slightly generalize the situation to the following questions:

**Question T:**

Let \(P\) be a rigid object that fits into an exact sequence

\[
0 \rightarrow \Lambda \rightarrow P \rightarrow R_m \rightarrow 0,
\]

where \(R_m\) is a rigid object equipped with a known \(T\)-exhaustive filtration labeled as in Definition 5.1, and \(\Lambda\) fits into a general receptive extension

\[
[0 \rightarrow T \otimes B \rightarrow \Lambda \rightarrow S \otimes A \rightarrow 0] = \delta \in \text{REC} \subset \text{Ext}^1(S \otimes A, T \otimes B) = A^*V^*B.
\]

Compute the \(T\)-exhaustive filtration of \(P\).
Similarly, we can formulate
Question S:
Let $P$ be a rigid object that fits into an exact sequence

$$0 \to \Lambda \to P \to Q_m \to 0,$$

where $Q_m$ is a rigid object equipped with a known $S$-exhaustive filtration labeled as in Definition 5.1, and $\Lambda$ fits into a general receptive extension

$$[0 \to S \otimes A \to \Lambda \to T \otimes B \to 0] = \delta \in \text{REC} \subset \text{Ext}^1(T \otimes B, S \otimes A) = B^*VA.$$

Compute the $S$-exhaustive filtration of $P$.
We now solve Question T. First we consider the following exact sequence

$$0 \to \text{Hom}(T, \Lambda) \to \text{Hom}(T, P) \to \text{Hom}(T, R_m) \xrightarrow{f} \text{Ext}^1(R_m, \Lambda),$$

let $B' = \ker(f)$ and $B'' = \text{Hom}(T, R_m)/B'$. Let $R'_m = R_m/(T \otimes B')$. Then we have the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & T \otimes B & \to & T \otimes \text{Hom}(T, P) & \to & T \otimes B' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda & \to & P & \to & R_m & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
S \otimes A & \to & P' & \to & R'_m & \to & 0.
\end{array}
\]

Now $\text{Hom}(T, P') = 0$, hence if we can find the $S$-exhaustive filtration for $P'$, then the $T$-exhaustive filtration for $P$ is also known. Since $P$ is rigid and $\text{Hom}(T \otimes \text{Hom}(T, P), P') = 0$, by Mukai’s Lemma (Lemma 3.11), $P'$ is also rigid. Now consider the following pullback diagram

\[
\begin{array}{cccccc}
0 & \to & S \otimes A & \to & \Lambda' & \to & T \otimes B'' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S \otimes A & \to & P' & \to & R'_m & \to & 0.
\end{array}
\]

$P'$ fits into an exact sequence

$$0 \to \Lambda' \to P' \to Q_m \to 0,$$
where \( Q_m \) is a rigid object equipped with a known \( S \)-exhaustive filtration. We want to formulate Question S for \( P' \), now the only missing input is the receptional locus that \( \Lambda' \) belongs to. We formalize this as follows.

There is a vector bundle \( \mathcal{E} \) defined over an open subset of \( \text{REC} \subset \text{Ext}^1(S \otimes A, T \otimes B) \), whose fiber over an extension 

\[
[0 \rightarrow T \otimes B \rightarrow \Lambda \rightarrow S \otimes A \rightarrow 0] = \delta \in \text{Ext}^1(S \otimes A, T \otimes B)
\]

is naturally identified with \( \text{Ext}^1(R_m, \Lambda) \). On that fiber, a general extension 

\[
[0 \rightarrow \Lambda \rightarrow P \rightarrow R_m \rightarrow 0] = (\Delta, \delta) \in \text{Ext}^1(R_m, \Lambda)
\]
determines an extension class

\[
[0 \rightarrow S \otimes A \rightarrow \Lambda' \rightarrow T \otimes B'' \rightarrow 0] = \delta' \in \text{Ext}^1(T \otimes B'', S \otimes A).
\]

Hence there is a rational map \( \pi : \mathcal{E} \rightarrow \text{Ext}^1(T \otimes B'', S \otimes A) \), where \( \mathcal{E} \) is understood as the total space of the vector bundle. We define the receptive locus \( \text{REC}' \subset \text{Ext}^1(T \otimes B'', S \otimes A) \) for \( \Lambda \) to be the closure of \( \pi(\mathcal{E}) \).

Let \( N = \text{Hom}(T, R_m) \) and \( M = \text{Hom}(S, Q_m) \). The rational map \( \pi \) factors through \( \text{Ext}^1(T \otimes N, S \otimes A) \) by cutting down \( T \otimes B \) and \( Q_m \) from \( P \), denote the image by \( \text{REC} \subset N^*V\Lambda \). Note that the map \( \text{Ext}^1(R_m, \Lambda) \rightarrow \text{Ext}^1(T \otimes N, S \otimes A) \) is actually linear and independent of the extension class \( \Delta \in \text{Ext}^1(R_m, \Lambda) \). We can describe its image by the following commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^1(Q_m, S \otimes A) & \xrightarrow{\cdot} & \text{Ext}^2(Q_m, T \otimes B) \\
\downarrow & & \downarrow \\
\text{Ext}^1(R_m, \Lambda) & \rightarrow & \text{Ext}^1(R_m, S \otimes A) & \rightarrow & \text{Ext}^2(R_m, T \otimes B) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Ext}^1(T \otimes N, S \otimes A) & \rightarrow & \text{Ext}^2(T \otimes N, T \otimes B) \\
\downarrow & & \downarrow \\
\text{Ext}^2(Q_m, S \otimes A) & & 0
\end{array}
\]

The image of \( \text{Ext}^1(R_m, \Lambda) \) in \( N^*V\Lambda \) is the intersection

\[
\ker(\text{id} \otimes \delta_1 : N^*VA \rightarrow N^*B) \cap \ker(\delta_2 \otimes \text{id} : N^*VA \rightarrow M^*A),
\]

where \( \delta_2 \in \text{ADM}_{m,m} \subset M^*V^*N \) and \( \delta_1 \in \text{REC} \subset A^*V^*B \) are general. Consider the incidence correspondence in \( M^*V^*N \times N^*V\Lambda \times A^*V^*B \):

\[
\Gamma = \{ (\delta_1, \delta, \delta_2) \in \text{REC} \times N^*V\Lambda \times \text{ADM} : m_1(\delta_1, \delta) = 0, m_2(\delta, \delta_2) = 0 \},
\]

where \( m_1 : M^*V^*N \times N^*V\Lambda \rightarrow M^*A \) and \( m_2 : N^*V\Lambda \times A^*V^*B \rightarrow N^*B \) are natural pairings. Let \( p_1, p, p_2 \) be projections to \( M^*V^*N, N^*V\Lambda, A^*V^*B \) respectively. Since \( \{ (\delta_1, 0, \delta_2) : \delta_1 \in \text{REC}, \delta_2 \in \text{ADM} \} \) is contained in \( \Gamma \), the projection \( p_1 \times \text{id} \times p_2 \) is dominant. By Lemma 5.8, there is a distinguished component of \( \Gamma \), this component is clearly \( \text{REC} \). We have shown

**Proposition 5.19.** Under the notations as above, \( \text{REC} \) is the distinguished component of \( \Gamma \) over \( \text{REC} \times \text{ADM} \) under the projection \( p_1 \times p_2 \).
Finally, we use the same trick as before: shrink an open set in \( \text{REC} \) if necessary, we may choose any decomposition \( N = B' \oplus B'' \) and let \( q : N^* \to B''^* \) be the projection. Then \( \text{REC}' \) is the closure of the image of \( \text{REC} \) under the projection \( q \otimes \text{id} \). We have

**Proposition 5.20.** Let \( \pi : E \to \text{Ext}^1(T \otimes B'', S \otimes A = B''^*VA) \) as above. Choose any projection \( q : N^* \to B''^* \). Then the image of \( \pi \) is

\[
\text{REC}' = (q \times \text{id})(\text{REC}) \subset B''^*VA,
\]

where \( \text{REC} \) is computed in Proposition 5.19.

Hence we reduced solving Question T for \( P \) to solving Question S for \( P' \), which is defined in (4). For a \( \sigma_0 \)-semistable object \( E \), let \( l(E) \) be the number of \( \sigma_0 \)-Jordan-Hölder factors of \( E \). Note that \( l(P') < l(P) \), hence in finite steps we have \( l(P) = 0 \). Question S and Question T are solved. We summarize this as follows:

**Algorithm 5.21** (Concatenation). Let \( R_m \in A_H \) be a rigid object with a known \( T \)-exhaustive filtration labeled as in Definition 5.1, let \( N = \text{Hom}(T, R_m) \) and \( M = \text{Hom}(S, Q_m) \). Let \( \Lambda \in \text{REC} \subset \text{Ext}^1(S \otimes A, T \otimes B) \) be a general receptive extension in the known receptive locus \( \text{REC} \). Let \( P \in \text{Ext}^1(R_m, \Lambda) \) be a general extension. The objective is to compute the \( T \)-exhaustive filtration of \( P \).

1. Compute \( \text{REC} \subset N^*VA \) by Proposition 5.19.
2. Compute \( \text{REC}' \subset B''^*VA \) by Proposition 5.20.
3. Let \( \Lambda' \in \text{REC}' \subset B''^*VA \) be a general receptive extension. Under Notation 5.4, \( Q_m \) is a rigid object with a known \( S \)-exhaustive filtration. Then \( Q = P/(T \otimes \text{Hom}(T, P)) \) is a general extension in \( \text{Ext}^1(Q_m, \Lambda') \). An \( S \)-exhaustive filtration of \( Q \) lifts to a \( T \)-exhaustive filtration of \( P \).
4. If \( Q = 0 \), the algorithm terminates. Otherwise, do (1)-(4) for \( Q \).

Since by doing (1)-(4) each time, \( l(Q) < l(P) \), the algorithm terminates in finitely many steps.

5.6. Local Reduction. In this subsection we state the local reduction that is used in the global reduction (Algorithm 3.14).

First we need to know the \( S \)-exhaustive filtration of \( E \), provided its \( \sigma_+ \)-Harder-Narasimhan filtration.

**Algorithm 5.22** (Initial filtration). Let

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_n = E
\]

be the \( \sigma_+ \)-Harder-Narasimhan filtration of \( E \). Let \( Q_i = E/E_i, G_i = E_i/E_{i-1} \). The objective is to find the \( S \)-exhaustive filtration of \( E = Q_0 \).

Clearly \( Q_n = 0 \) is an \( S \)-exhaustive filtration.

1. Suppose we know the \( S \)-exhaustive filtration of \( Q_i \). Let \( \Lambda_i = G_i, A_i = \text{Hom}(S, \Lambda_i), B_i = \text{Hom}(\Lambda_i, T)^* \).
2. Then \( \Lambda_i \in \text{REC} = \text{Ext}^1(T \otimes B_i, S \otimes A_i) \) is a general extension, \( Q_{i-1} \in \text{Ext}^1(Q_i, G_i) \) is a general extension, we do the concatenation (Algorithm 5.21) on \( Q_{i-1} \) to get the \( S \)-exhaustive filtration of \( Q_{i-1} \).
3. If \( i - 1 = 0 \), the algorithm terminates. Otherwise, do (1)-(3) again.

Now we can state local reduction formally.
**Algorithm 5.23** (Local reduction). Let $E \in \mathcal{A}_H$ be a rigid object whose $\sigma_+$-Harder-Narasimhan filtration is known. The objective is to find its $\sigma_-$-Harder-Narasimhan filtration.

1. Initially we find the $S$-exhaustive filtration of $E$ by using Algorithm 5.22.
2. The first $\sigma_-$-Harder-Narasimhan factor $E_1^-$ is the first factor of the $S$-exhaustive filtration. Let $Q = Q_1^- = E/E_1^-$. It has a known $T$-exhaustive filtration
   
   $$0 = F_n \subset E_{n-1} \subset F_{n-1} \subset \cdots \subset E_1 \subset F_1 \subset E_0 = Q.$$  

3. We find the first $\sigma_-$-Harder-Narasimhan factor $O$ of $Q$ by Proposition 5.15, and note that $O \otimes \text{Hom}(O, Q) \subset \Gamma = F_{n-1}/F_n$. Let $\Lambda = \Gamma/O \otimes \text{Hom}(O, Q)$, then
   
   $$\Lambda \in \text{REC} \subset \text{Ext}^1(S \otimes A'', T \otimes B'')$$

   is a general receptive extension, where $A'', B''$ are vector spaces with known dimension by Proposition 5.15. The receptive locus REC is given by Proposition 5.18.

4. Let $R_{n-1} = Q/F_{n-1}$ and $Q' = Q/(O \otimes \text{Hom}(O, Q))$. Then $Q' \in \text{Ext}^1(R_{n-1}, \Lambda)$ is a general extension, where $\Lambda \in \text{REC} \subset A'' B''$ is a general receptive extension whose receptive locus is computed in step (3), and $R_{n-1}$ has the induced $T$-exhaustive filtration that is computed in step (2). We do the concatenation (Algorithm 5.21) to get the $T$-exhaustive filtration of $Q'$.

5. If $Q' = 0$, the algorithm terminates. Otherwise, let $Q = Q'$ and do (3)-(5) again.

We briefly summarize local reduction (Algorithm 5.23) in words. We start with a rigid object $E = Q_0^-$ with a known $\sigma_+$-Harder-Narasimhan filtration. First we solve Question $S$ repeatedly to get an $S$-exhaustive filtration of $E$. Quotient out the first factor, we get $Q_1^-$, which has a known $T$-exhaustive filtration, which is the original $S$-exhaustive filtration with the first factor deleted. For any $i \geq 1$, once we computed the first $\sigma_-$-Harder-Narasimhan factor of $Q_i^-$ and take quotient $Q_{i+1}^-$, we fall into Question $T$. This can be solved, and we get a $T$-exhaustive filtration for $Q_{i+1}^-$, and the process is repeated. Since the length of the rigid object reduces at every step, finally this terminates and we get the $\sigma_-$-Harder-Narasimhan filtration of $E$.

Next we show an example of carrying out local reduction.

**Example 5.24.** Let $(X, H)$ be a K3 surface with $\text{Pic}(X) = \mathbb{Z}H$ and $H^2 = 2$. Let $S \in \mathcal{M}_H((5, 2H, 1))$, $T = \mathcal{O}_X[1]$ and $W = W(T, S)$ be the numerical wall defined by $T$ and $S$, and $\sigma_+$ (resp. $\sigma_-$) a stability condition on $b$ (see Section 3) that is right above (resp. below) $W$. Let $V = \text{Ext}^1(T, S) \cong \mathbb{C}^6$, $B_0 = \mathbb{C}^{35}$, $A_1 = \mathbb{C}^6$, $B_1 = \mathbb{C}^{155}$. Let $Q_1 \in \mathcal{M}_{\sigma_+}((35, -12H, -29))$ and $E_1 = T \otimes B_1$.  

Let $E$ be a general extension of $Q_1$ by $E_1$. One may check that $E$ is rigid, by noticing that $E$ is the quotient of $E' \in \mathcal{M}_H((305, 477H, 746))$ by its first $\sigma_+$-Harder-Narasimhan factor. The shape (Definition 5.10) of $\sigma_+$-Harder-Narasimhan filtration of $E$ is

$$(155T, Q_1).$$

We run local reduction (Algorithm 5.23) to compute the $\sigma_-$-Harder-Narasimhan filtration of $E$. The $\sigma_-$-Harder-Narasimhan filtration of $Q_1$ has shape $(6S, 35T)$, hence there is a filtration of $E$ (not necessarily exhaustive at this moment) whose shape is

$$(155T, 6S, 35T).$$

We first show that $E^*$ is exhaustive. By the construction of exhaustive filtration (Lemma 5.2), it suffices to show $\text{Hom}(T, E) = 155$. Then the claim follows from Corollary 4.13 that
the $\sigma_-$-Harder-Narasimhan filtration of $Q_1$ is exhaustive. We have

$$0 \to \text{Hom}(T, T \otimes B_1) \to \text{Hom}(T, E) \to \text{Hom}(T, Q_1) = 0.$$  

Hence $E^\bullet$ is exhaustive. This finishes step (1) in the local reduction (Algorithm 5.23): the initial filtration (Algorithm 5.22) is $E^\bullet$.

The $Q'_1$ in step (2) of local reduction (Algorithm 5.23) is $T \otimes B_0$. We now run step (3). Let $\delta_{0,1} \in \text{ADM}_{0,1} \subset B_0 V A_1$ be a general element. Then the ker$(\delta_{0,1})$ is a general subspace of $A_1^* V^*$ of dimension 1, where $\delta_{0,1}$ is viewed as the adjoint map $\delta_{0,1} : A_1^* V^* \to B_0^*$. By the definition of admissible extensions, ADM$_{1,1} \subset A_1^* V^* B_1$ is the closure of the union of ker$(\delta_{0,1}) \otimes B_1$ for general $\delta_{0,1}$. Since ker$(\delta_{0,1})$ is a general 1 dimensional subspace, we have

$$\text{ADM}_{1,1} = \{ \delta_{1,1} \in A_1^* V^* B_1 : \text{rk}_{(A_1 V, B_1)}(\delta_{1,1}) \leq 1 \},$$

where $\text{rk}_{(A_1 V, B_1)}(\delta_{1,1})$ is the rank of $\delta_{1,1}$ viewed as a 2-tensor in $(A_1^* V^*) \otimes B_1$.

Next we compute the $k(i)$ in Proposition 5.15. Note that $\text{rk}_{(A_1 V, B_1)}(\delta_{1,1}) = 1$ for a general $\delta_{1,1} \in \text{ADM}_{1,1}$, the map $\delta_{1,1} : A_1 \to V^* B_1$ is the composition of a general map $\delta_{1,1}' : A_1 \to V^* B_1'$ and the inclusion $i \otimes \text{id} : V^* B_1' \to V^* B_1$, where $i : B_1' \to B_1$ is a one dimensional subspace. Using the notation of Proposition 5.15, we have

$$J_i = (\delta(N_i) \otimes B_1) \cap (M_i V^* B_1') \cap (M_i \otimes \delta_{1,1}'(A_1')) = (\delta(N_i) \otimes B_1') \cap (M_i \otimes \delta_{1,1}'(A_1')).$$

Since dim$B_1' = 1$, $\delta_{1,1}'(A_1')$ can be viewed as a general subspace of $V$. Since dim$A_1 = \text{dim} V = 6$, $\delta_{1,1}'(A_1) = V \otimes B_1'$. We omit writing $B_1'$ since its dimension is 1. Then $J_i = \delta(N_i) \cap M_i V = \delta(N_i)$. Since $E^\bullet$ is exhaustive, Hom$(O_0, E) = 0$. Hence the $i$ with maximal $\phi_-(i)$ is $i = -1$, and $O_{-1} \in M_{\sigma_-}((29, 12H, 5))$. By Theorem 4.10, we have

$$J_{-1} = \delta(N_{-1}) = C,$$

hence $k(-1) = 1$. Using the notation in step (3), $\Lambda = T^{\oplus 154}$ and there is no receptive locus. We completed step (3) in local reduction (Algorithm 5.23).

Then in step (4), $Q'$ is a general extension of $R = T \otimes B_0$ by $\Lambda = T^{\oplus 154}$. Hence $Q' = T^{\oplus 189}$. We run the local reduction again for $Q'$, but there is nothing to do. Hence the algorithm terminates. The $\sigma_-$-Harder-Narasimhan filtration of $E$ has shape

$$(S_{-1}^-, 189T),$$

where $S_{-1}^- \in M_{\sigma_-}((29, 12H, 5))$.

**Remark 5.25.** The local reduction (Algorithm 5.23) gives a theoretical way to compute the $\sigma_-$-Harder-Narasimhan filtration of a rigid object from its $\sigma_+$-Harder-Narasimhan filtration in principle. However as one may have noticed in Example 5.24, in practice it could be hard to carry out by hand. The main computational difficulties are writing down the ideal of the admissible locus (Proposition 5.9) and computing the functions $k(i)$ (Proposition 5.15).

In many cases, we may overcome these computational difficulties by using a simplified version (Theorem 6.11) of local reduction, which is discussed in the next section. We will revisit Example 5.24 in Example 6.12 by using the simplified local reduction.
6. Simplifications

In many good cases, the algorithm can be significantly simplified. This includes two aspects. Locally, if at each wall the \( \sigma_+ \)-Harder-Narasimhan filtration has only two factors, then the local reduction (Algorithm 5.23) is significantly simplified (Theorem 6.11) and can be computed by hand. The local complexity of a spherical object is dominated by the global complexity, which is measured by a notion called height (Definition 6.13). For a spherical object of height \( \leq 2 \), we show a straightforward inductive formula (Theorem 6.14).

In practical terms, among all examples that the author did, Theorem 6.11 can be applied to most of them. However, there exist Mukai vectors that need to fully invoke the global reduction (Algorithm 3.14), see Example 9.4.

6.1. Local Simplification. The goal of this subsection is to find a simple way to compute the \( \sigma_- \)-Harder-Narasimhan filtration of those rigid objects \( E \) that fits into the following exact sequence

\[
0 \to F^q \to E \to G^p \to 0,
\]

where \( F, G \in \mathcal{H} \) are \( \sigma_+ \)-stable spherical objects with \( \phi_+(F) > \phi_+(G) \). This is Theorem 6.11.

Let \( W \) be a wall whose corresponding rank 2 lattice is \( \mathcal{H} \). Let \( S_0, T_0 \) be the two \( \sigma_0 \)-stable objects in \( \mathcal{H} \) where \( \sigma_0 \in \mathcal{H} \). Let \( \sigma_+, \sigma_- \) be two stability conditions on two sides of \( W \) and near \( W \), with \( \phi_+(T_1) > \phi_+(S_0) \). Recall that the fundamental sequence of \( \mathcal{H} \) is defined in Definition 4.9 as

\[
a_0 = 1, \ a_1 = g = \text{ext}^1(T_0, S_0), \ a_n = g \cdot a_{n-1} - a_{n-2} \text{ for } n \geq 2.
\]

For the rest of the paper, we sometimes write \( S = S_0 \) and \( T = T_0 \).

We first consider rigid objects \( E \) that can fit into an exact sequence

\[
0 \to S^p \to E \to T^q \to 0.
\]

Since \( E \) is rigid, we may assume

\[
[0 \to S^p \to E \to T^q \to 0] = \delta \in \text{Ext}^1(T^q, S^p)
\]

is a general extension. We call such rigid objects of type I.

**Lemma 6.1.** Let \( E \) be a rigid object that fits into an exact sequence

\[
0 \to S^p \to E \to T^q \to 0.
\]

Then

\[
pq \cdot g < q^2 + p^2.
\]

**Proof.** Let

\[
\eta_i = [0 \to S^p \to E_i \to T^q \to 0] \in \text{Ext}^1(T^q, S^p), \ i = 1, 2
\]

be two general extensions. Since \( E \) is rigid and \( \delta \) is general, \( E_1 \cong E_2 \). Consider the following diagram

\[
\begin{array}{ccc}
0 & \to & S^p \overset{f_1}{\to} E_1 \overset{g_1}{\to} T^q \overset{\phi}{\to} 0 \\
\downarrow \phi & & \downarrow \psi \\
0 & \to & S^p \overset{f_2}{\to} E_2 \overset{g_2}{\to} T^q \overset{\psi}{\to} 0.
\end{array}
\]

Since \( \text{Hom}(S, T) = 0 \), the composition map

\[
S^p \overset{f_1}{\to} E_1 \overset{g_1}{\to} T^q
\]
is zero. Since $S^p \xrightarrow{f_2} E_2$ is the kernel of $E_2 \xrightarrow{g_2} T^q$, there exists a unique map $\phi : S^p \rightarrow S^p$ such that the diagram commutes. Swap the two rows in the diagram, we see that $\phi$ is an isomorphism. Since $S$ is simple, $\phi$ is an element in $GL_p$. Similarly, there exists a unique $\psi$ fitting into the diagram above, which is an element of $GL_q$.

Modulo scalars, we see that an orbit of $\mathbb{P}GL_p \times \mathbb{P}GL_q$ action contains an open subset of $\mathbb{P}\text{Ext}^1(T^q, S^p)$. By comparing dimensions, we have

\[
\text{ext}^1(T^q, S^p) - 1 = \dim \mathbb{P}\text{Ext}^1(T^q, S^p) \leq \dim \mathbb{P}GL_p \times \mathbb{P}GL_q = p^2 - 1 + q^2 - 1,
\]

namely

\[
\text{ext}^1(T^q, S^p) < q^2 + p^2.
\]

\[\square\]

We note a simple property of the fundamental sequence in the following lemma.

**Lemma 6.2.** Assume $g(\mathcal{H}) \geq 2$. Then

\[
0 < \frac{a_{n+1}}{a_n} < \frac{a_n}{a_{n-1}}.
\]

**Proof.** For $n \in \mathbb{Z}_{\geq 1}$, let $b_n = \frac{a_n}{a_{n-1}}$. Clearly $b_n > 0$. From the definition

\[
a_{n+1} = g \cdot a_n - a_{n-1},
\]

we have

\[
b_{n+1} = g - \frac{1}{b_n}.
\]

Consider

\[
b_{n+1} - b_n = -\frac{1}{b_n} + \frac{1}{b_{n-1}},
\]

if $b_n < b_{n-1}$, then $b_{n+1} < b_n$. Now $b_2 = g - \frac{1}{g} < g = b_1$, the lemma is proved by induction. \[\square\]

**Remark 6.3.** An immediate consequence of the lemma is that \(\{\frac{a_{n+1}}{a_n}\}\) monotonically approaches its limit from above. The limit is the smaller root of the equation $x = g - \frac{1}{x}$, which we denote by $\rho(\mathcal{H}) = \rho$.

**Proposition 6.4.** Assume $g(\mathcal{H}) \geq 2$. Let $E$ be a rigid object that fits into an exact sequence

\[
0 \rightarrow T_i^q \rightarrow E \rightarrow S_0^p \rightarrow 0.
\]

If $q \leq p$, then the $\sigma_-$-Harder-Narasimhan filtration of $E$ is given by

\[
E = (S_{-i})^m \oplus (S_{-i-1})^n,
\]

where $i$ is the unique integer such that $\frac{a_{i+1}}{a_i} \leq \frac{p}{q} < \frac{a_i}{a_{i-1}}$, and $m,n$ are the unique integers such that

\[
m \cdot v(S_{-i}) + n \cdot v(S_{-i-1}) = v(E).
\]

Symmetrically if $q \geq p$, then the $\sigma_-$-Harder-Narasimhan filtration of $E$ is given by

\[
E = (T_{-i+1})^m \oplus (T_{-i+2})^n,
\]

where $i$ is the unique integer such that $\frac{a_{i+1}}{a_i} \leq \frac{q}{p} < \frac{a_i}{a_{i-1}}$, and $m,n$ are the unique integers such that

\[
m \cdot v(T_{-i+1}) + n \cdot v(T_{-i+2}) = v(E).
\]
Hence we have the following exact sequence
\[ p, q \]
Since \( \frac{a_{i+1}}{a_i} \) approaches \( p \) monotonically from above, we may divide the interval \((\rho, +\infty)\) into disjoint subintervals
\[ (\rho, +\infty) = \prod_{i=0}^{\infty} \left[ \frac{a_{i+1}}{a_i}, \frac{a_i}{a_{i-1}} \right], \]
here we set \( \frac{a_0}{a_{-1}} = +\infty \). Since \( \frac{a}{p} > \rho \), there exists a unique \( i \) such that \( \frac{a_{i+1}}{a_i} \leq \frac{a}{p} < \frac{a_i}{a_{i-1}} \). In other words, on \( \mathbb{Z}^2 \) the point \((p, q)\) is in between the positive rays spanned by \((a_{i-1}, a_i)\) and \((a_i, a_{i+1})\).

Next we observe that \((a_{i-1}, a_i), (a_i, a_{i+1})\) form a basis of \( \mathbb{Z}^2 \). To see this, we compute
\[
\begin{align*}
\det \begin{pmatrix} a_{i-1} & a_i \\ a_i & a_{i+1} \end{pmatrix} &= a_{i-1}a_{i+1} - a_i^2 \\
&= (ga_i - a_{i-1})a_{i+1} - (ga_{i-1} - a_{i-2})a_i \\
&= a_{i-2}a_i - a_{i-1}^2 \\
&= \det \begin{pmatrix} a_{i-2} & a_{i-1} \\ a_{i-1} & a_i \end{pmatrix} \\
&\vdots \\
&= \det \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} \\
&= 1 \cdot (g^2 - 1) - g^2 = -1.
\end{align*}
\]
Hence there exists a unique pair of non-negative integers \( m, n \) such that \((p, q) = m(a_{i-1}, a_i) + n(a_i, a_{i+1})\).

Next we consider the object \((T_{i+1})^m \oplus (T_{i+2})^n\). By Corollary 4.13, for any \( j \),  \( T_j^- \) fits into a general extension
\[ 0 \rightarrow T_1^{aj-1} \rightarrow T_j^- \rightarrow S_0^{a_j-2} \rightarrow 0. \]
Hence we have the following exact sequence
\[ 0 \rightarrow T_1^{ma_i+na_{i+1}} \rightarrow (T_{i+1}^-)^m \oplus (T_{i+2}^-)^n \rightarrow S_0^{ma_i+na_i} \rightarrow 0. \]
Since \((p, q) = m(a_{i-1}, a_i) + n(a_i, a_{i+1})\), the sequence above also reads
\[ 0 \rightarrow T_1^q \rightarrow (T_{i+1}^-)^m \oplus (T_{i+2}^-)^n \rightarrow S_0^p \rightarrow 0. \]
Note that \( \Ext^1(T_{i+1}^-, T_{i+2}^-) = 0 \), hence
\[ \Ext^1((T_{i+1}^-)^m \oplus (T_{i+2}^-)^n, (T_{i+1}^-)^m \oplus (T_{i+2}^-)^n) = 0, \]

namely \((T_{i+1}^-)^n \oplus (T_{i+2}^-)^n\) is rigid, hence it fits into a general extension
\[ [0 \rightarrow T_1^q \rightarrow (T_{i+1}^-)^m \oplus (T_{i+2}^-)^n \rightarrow S_0^p \rightarrow 0] \in \Ext^1(S_0^p, T_1^q). \]
On the other hand, since $E$ is rigid, it also fits into a general extension

$$[0 \to T_i^q \to E \to S_0^p \to 0] \in \text{Ext}^1(S_0^p, T_i^q).$$

Hence $E$ and $(T_{i+1}^-)^m \oplus (T_{i+2}^-)^n$ are isomorphic.

When one Harder-Narasimhan factor of $E$ is not $S$ or $T$, we have the following proposition.

**Proposition 6.5.** Let $E$ be a rigid object.
If $E$ fits into the following exact sequence

$$0 \to T_i^q \to E \to G^p \to 0,$$

where $G$ is some $\sigma_+$-stable spherical object in $\mathcal{H}$. Then the $\sigma_-$-Harder-Narasimhan filtration of $E$ is of the form

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E,$$

where $E_3/E_2 = \mathcal{T}_i^{\text{hom}(G^p, T_i)+\varepsilon}$ and $\varepsilon = \max\{q - \text{ext}^1(G^p, T_i), 0\}$, and $E_2$ is a rigid object of type I: $E_2$ fits into

$$0 \to T_i^{q-\varepsilon} \to E_2 \to S_0 \otimes \text{Hom}(S_0, G^p) \to 0.$$  

Similarly, if $E$ fits into the following exact sequence

$$0 \to F^q \to E \to S_0^p \to 0,$$

where $F$ is some $\sigma_+$-stable spherical object in $\mathcal{H}$. Then the $\sigma_-$-Harder-Narasimhan filtration of $E$ is of the form

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E,$$

where $E_1 = S_0^{\text{hom}(S_0, F^q)+\varepsilon}$ and $\varepsilon = \max\{p - \text{ext}^1(S_0, F^q), 0\}$, and $E_3/E_1$ is a rigid object of type I: $E_3/E_1$ fits into

$$0 \to T_1 \otimes \text{Hom}(F^q, T_1)^* \to E_3/E_1 \to S_0^{p-\varepsilon} \to 0.$$  

We shall refer to the rigid objects $E$ in Proposition 6.5 as type II.

**Proof.** We only prove the first case, the other case is similar. We may assume $G \neq S_0$, otherwise $E$ is of type I and the Harder-Narasimhan filtration is already known. In particular, $\text{Hom}(E, T_i) \neq 0$.

By Mukai’s Lemma (Lemma 3.11), $E_3/E_2$ is a direct sum of a $\sigma_-$-stable spherical object in $\mathcal{H}$, hence $\phi_-(E_3/E_2) \geq \phi_-(T_1)$. On the other hand, $\phi_-(E_i/E_{i-1}) > \phi_-(E_3/E_2) \geq \phi_-(T_1)$ for $i < 3$, hence $\text{Hom}(E_2, T_1) = 0$. Since $\text{Hom}(E, T_1) \neq 0$, we know $\text{Hom}(E_3/E_2, T_1) \neq 0$, $\phi_-(E_3/E_2) \leq \phi_-(T_1)$, hence $E_3/E_2$ is naturally isomorphic to $T_1 \otimes \text{Hom}(E, T_1)^*$. Applying $\text{Hom}(\cdot, T_1)$ to

$$0 \to T_i^q \to E \to G^p \to 0$$

and note that we may assume this extension $\delta \in \text{Ext}^1(G^p, T_i^q)$ is general since $E$ is rigid, we get

$$0 \to \text{Hom}(G^p, T_1) \to \text{Hom}(E, T_1) \to \text{Hom}(T_i^q, T_1) \xrightarrow{\delta} \text{Ext}^1(G^p, T_1).$$

Note that we have the natural identification

$$\text{Ext}^1(G^p, T_i^q) \cong \text{Hom}(\text{Hom}(T_i^q, T_1), \text{Ext}^1(G^p, T_1))$$
sending $\delta$ to $\delta^*$. Since $\delta^*$ is general, the kernel has dimension $\varepsilon = \max\{q - \text{ext}^1(G^p, T_1), 0\}$, and $\text{Hom}(E, T_1) \cong \mathbb{C}^{p \cdot \text{hom}(G, T_1) + \varepsilon}$. Now we have the following commutative diagram

\[
\begin{array}{ccccccccc}
T_1^{q-\varepsilon} & \rightarrow & E_2 & \rightarrow & S_0 \otimes \text{Hom}(S_0, G^p) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & T_1^q & \rightarrow & E & \rightarrow & G^p & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & T_1^\varepsilon & \rightarrow & T_1 \otimes \text{Hom}(E, T_1)^* & \rightarrow & T_1 \otimes \text{Hom}(G^p, T_1)^* & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & ,
\end{array}
\]

where the rows and columns are all exact. Hence $E_2$ fits into

\[
0 \rightarrow T_1^{q-\varepsilon} \rightarrow E_2 \rightarrow S_0 \otimes \text{Hom}(S_0, G^p) \rightarrow 0.
\]

By Mukai’s Lemma (Lemma 3.11), $E_2$ is rigid, hence a rigid object of type I. □

When neither Harder-Narasimhan factors are $S$ or $T$, the key technical proposition is the following

**Proposition 6.6.** Let $E$ be a rigid object that fits into the following exact sequence

\[
0 \rightarrow F^q \rightarrow E \rightarrow G^p \rightarrow 0,
\]

where $F, G \in \mathcal{H}$ are $\sigma_+$-stable spherical objects with $\phi_+(F) > \phi_+(G)$ and $\text{Ext}^1(G, F) \neq 0$. Then either

\[
\text{Hom}(S_0, G^p) \rightarrow \text{Ext}^1(S_0, F^q)
\]

is injective, or

\[
\text{Hom}(F^q, T_1) \rightarrow \text{Ext}^1(G^p, T_1)
\]

is injective, where the two maps are the connecting homomorphisms in the long exact sequences.

We refer to the rigid objects in Proposition 6.6 as two step rigid objects. Before proving Proposition 6.6, we first state a lemma confirming that the dimensions of the spaces allow the map to be injective. It is also needed for the proof of Proposition 6.6.

**Lemma 6.7.** If $\text{Ext}^1(G, F) \neq 0$, then either

\[
\text{hom}(S_0, G^p) \leq \text{ext}^1(S_0, F^q),
\]

or

\[
\text{hom}(F^q, T_1) \leq \text{ext}^1(G^p, T_1).
\]

**Proof.** Based on the condition that $\phi_+(F) > \phi_+(G)$, we separate into three cases. Case TS: $F = T_i$ and $G = S_j$, where $i \geq 1, j \leq 0$. Case SS: $F = S_i$ and $G = S_j$, where $i + 2 \leq j \leq 0$. Case TT: $F = T_i$ and $G = T_j$, where $1 \leq i \leq j - 2$. (In case SS and case TT, $j \neq i + 1$ because by assumption $\text{Ext}^1(G, F) \neq 0$.)

We first analyze the the case TS. Recall that we have

\[
\text{hom}(S_0, S_j) = a_{-j}, \ \text{ext}^1(S_0, T_i) = a_i, \ \text{hom}(T_i, T_1) = a_{i-1}, \ \text{ext}^1(S_j, T_1) = a_{-j+1},
\]
where \( \{a_n\} \) is the fundamental sequence of \( \mathcal{H} \). Now suppose the lemma is not true, then we must have
\[
p \cdot a_j > q \cdot a_i,
q \cdot a_{i-1} > p \cdot a_{j+1}.
\]
We may rewrite this as
\[
\frac{a_i}{a_{j+1}} < \frac{p}{q} < \frac{a_{i-1}}{a_{j}}.
\]
This is clearly impossible, since \( \frac{a_i}{a_{j+1}} > 1 > \frac{a_{i-1}}{a_{j}} \). The lemma is proven for the case TS.

The case SS and the case TT are similar, we only prove for the case TT. We have
\[
\text{hom}(S_0, T_j) = a_{j-2}, \quad \text{ext}^1(S_0, T_i) = a_i, \quad \text{hom}(T_i, T_1) = a_{i-1}, \quad \text{ext}^1(T_j, T_1) = a_{j-3}.
\]
If the lemma is not true, then we must have
\[
p \cdot a_{j-2} > q \cdot a_i
q \cdot a_{i-1} > p \cdot a_{j-3},
\]
which can be rewritten as
\[
\frac{a_i}{a_{j-2}} < \frac{p}{q} < \frac{a_{i-1}}{a_{j-3}}.
\]
Hence \( \frac{a_i}{a_{j-1}} < \frac{a_{i-2}}{a_{j-3}} \). By Lemma 6.2, this is equivalent to \( j - 2 < i \), which contradicts the assumption that \( i \leq j - 2 \).

In the following we will always assume \( \text{hom}(S_0, G^p) \leq \text{ext}^1(S_0, F^q) \), the other case can be treated in the same manner.

For any finite dimensional linear spaces \( K \subset A \otimes B \), we let
\[
\overline{K} = \bigcap_{K \subset A' \otimes B' \subset A \otimes B} A' \otimes B'.
\]
Then \( \overline{K} \) is also of the form \( A_0 \otimes B_0 \) for some subspaces \( A_0, B_0 \).

In general, if \( V, W \) are two linear spaces with \( \dim V \leq \dim W \), and \( f \in W \otimes V^* = \text{Hom}(V, W) \) is general, then \( f : V \to W \) is injective. The following lemma shows that we may slightly weaken the condition.

**Lemma 6.8.** Let \( V, W \) be two linear spaces with \( \dim V \leq \dim W \). Let \( K \subset W \otimes V^* \) be a subspace such that \( \overline{K} = W \otimes V^* \). Then for a general \( f \in K \), \( f : V \to W \) is injective.

**Proof.** For any \( f \in K \), we may choose a base for \( V \) and \( W \) such that \( f \) is a diagonal matrix. If \( f \) is not full rank, then below the, say \( n \)-th row, all entries of the matrix are zero. Since \( \overline{K} = W \otimes V^* \), there exists \( g \in K \), such that in matrix form some entries below the \( n \)-th row is not zero. Then for sufficiently small \( \varepsilon > 0 \), \( f + \varepsilon \cdot g \in K \) has rank strictly bigger than \( f \). We may then iterate this process to get an element in \( K \) of full rank. Since \( \dim V \leq \dim W \), a full rank map is injective. \( \square \)

Applying \( \text{Hom}(G, -) \) to
\[
0 \longrightarrow S_0 \otimes \text{Hom}(S_0, F) \longrightarrow F \longrightarrow T_1 \otimes \text{Hom}(F, T_1)^* \longrightarrow 0,
\]
we get long exact sequence
\[
\text{Ext}^1(G, F) \to \text{Ext}^1(G, T_1) \otimes \text{Hom}(F, T_1)^* \to \text{Hom}(S_0, G)^* \otimes \text{Hom}(S_0, F),
\]
let \( K \) be the image of \( \text{Ext}^1(G, F) \to \text{Hom}(G, T_1) \otimes \text{Hom}(F, T_1)^* \).
Let $V = \text{Ext}^1(T_1, S_0)$. We write $V^{(n)}$ to be $V \otimes V^* \otimes \cdots \otimes V$ or $V \otimes V^* \otimes \cdots \otimes V^*$, where the total number of $V$ and $V^*$ is $n$. Let $K_n$ be the intersection of all kernels of pairings on consecutive factors.

**Lemma 6.9.** Under the notations above, we have

$$K = \text{Ext}^1(S_0, F) \otimes \text{Hom}(S_0, G)^*.$$

**Proof.** We separate into three cases. Case TS: $F = T_i$ and $G = S_j$, where $i \geq 1, j \leq 0$. Case SS: $F = S_i$ and $G = S_j$, where $i + 2 \leq j \leq 0$. Case TT: $F = T_i$ and $G = T_j$, where $1 \leq i \leq j - 2$.

In the case TS, $K$ is the kernel of the map

$$\text{Ext}^1(S_j, T_i) \otimes \text{Hom}(T_i, T_i)^* \longrightarrow \text{Hom}(S_0, S_j)^* \otimes \text{Hom}(S_0, T_i).$$

The map is the composition of

$$\text{ev}^* \otimes \text{id} : \text{Ext}^1(S_j, T_i) \otimes \text{Hom}(T_i, T_i) \longrightarrow \text{Hom}(S_0, S_j)^* \otimes \text{Ext}^1(S_0, T_i) \otimes \text{Hom}(T_i, T_i)^*$$

and

$$\text{id} \otimes \delta : \text{Hom}(S_0, S_j)^* \otimes \text{Ext}^1(S_0, T_i) \otimes \text{Hom}(T_i, T_i)^* \longrightarrow \text{Hom}(S_0, S_j)^* \text{Hom}(S_0, T_i).$$

By similar arguments as in Proposition 4.7 and 4.8, $\text{ev}^* : \text{Ext}^1(S_j, T_i) \longrightarrow \text{Hom}(S_0, S_j)^* \otimes \text{Ext}^1(S_0, T_i)$ is injective and $\text{Ext}^1(S_j, T_i) = K_{-j+1}$ under the inclusion into $V^{(-j+1)}$.

The map $\delta : \text{Ext}^1(S_0, T_i) \otimes \text{Hom}(T_i, T_i)^* \longrightarrow \text{Hom}(S_0, T_i)$ is represented by the class

$$[0 \longrightarrow S_0 \otimes \text{Hom}(S_0, T_i) \longrightarrow T_i \longrightarrow T_1 \otimes \text{Hom}(T_i, T_i)^* \longrightarrow 0] = \delta$$

inside $\text{Hom}(T_i, T_i) \otimes \text{Ext}^1(T_1, S_0) \otimes \text{Hom}(S_0, T_i)$. Moreover, we may choose a perfect pairing on $\text{Ext}^1(S_0, T_i) \otimes \text{Hom}(T_2, T_i)^*$, so that $\ker(\delta) = K_i \subset V^{(i)}$. Hence inside $\text{Hom}(S_0, S_j)^* \otimes \text{Ext}^1(S_0, T_i) \otimes \text{Hom}(T_i, T_i)^*$, $K$ is the intersection

$$K = (\text{Ext}^1(S_j, T_i) \otimes \text{Hom}(T_i, T_i)^*) \cap (\text{Hom}(S_0, S_j)^* \otimes \ker(\delta)).$$

Under the inclusion

$$\text{Hom}(S_0, S_j)^* \otimes \text{Ext}^1(S_0, T_i) \otimes \text{Hom}(T_i, T_i)^* \subset V^{(-j+i)},$$

we see $K = K_{-j+i}$, $\text{Ext}^1(S_j, T_i) \otimes \text{Hom}(T_i, T_i)^* = K_{-j+1} \otimes K_{i-1}$. By linear algebra, one may check explicitly that $K_{-j+i} \subset K_{-j+1} \otimes K_{i-1}$ is full on both factors.

The case TT and the case SS can be treated in similar manners. \qed

**Remark 6.10.** An immediate consequence of Lemma 6.9 is that, for any positive integers $p, q$, we have

$$\overline{K^{\otimes pq}} = \text{Ext}^1(S_0, F^q) \otimes \text{Hom}(S_0, G)^*.$$

Now we are ready to prove Proposition 6.6.

**Proof.** By Lemma 6.7, we may assume $\text{hom}(S_0, G^p) \leq \text{ext}^1(S_0, F^q)$, the other case is similar. Since $E$ is rigid, we may assume the extension

$$[0 \longrightarrow F^q \longrightarrow E \longrightarrow G^p \longrightarrow 0] = \delta \in \text{Ext}^1(F^q, G^p)$$

is general. Under the map

$$\text{Ext}^1(G^p, F^q) \longrightarrow \text{Ext}^1(S_0, F^q) \otimes \text{Hom}(S_0, G)^* \cong \text{Hom}(\text{Hom}(S_0, G^p), \text{Ext}^1(S_0, F^q)),$$
\( \delta \in \text{Ext}^1(G^p, F^q) \) is sent to the connecting homomorphism \( \delta_* : \text{Hom}(S_0, G^p) \to \text{Ext}^1(S_0, F^q) \). Since \( \delta_* \) is general and by Remark 6.10

\[
K^{\text{trip}} = \text{Ext}^1(S_0, F^q) \otimes \text{Hom}(S_0, G^p),
\]

we may apply Lemma 6.8, the proposition is proved.

Now we are ready to state and prove the main result of this subsection.

**Theorem 6.11** (Local simplification). Let \( E \) be a rigid object that fits into the following exact sequence

\[
0 \to F^q \to E \to G^p \to 0,
\]

where \( F, G \in \mathcal{H} \) are \( \sigma^+ \)-stable spherical objects with \( \phi_+(F) > \phi_+(G) \) and \( \text{Ext}^1(G, F) \neq 0 \). Then the \( \sigma^- \)-Harder-Narasimhan filtration of \( E \) is of the form

\[
0 = E_0 \subset E_1 \subset E_2 \subset E_3 \subset E_4 = E,
\]

where either: (a) \( E_1 = S_0 \otimes \text{Hom}(S_0, F^q) \) and \( E_4/E_1 \) is rigid of type II, or (b) \( E_4/E_3 = T_1 \otimes \text{Hom}(G^p, T_1)^* \) and \( E_3 \) is rigid of type II.

**Proof.** By Proposition 6.6, we assume \( \text{Hom}(S_0, G^p) \to \text{Ext}^1(S_0, F^q) \) is injective, the other case is similar. Hence we have an isomorphism

\[
\text{Hom}(S_0, F^q) \cong \text{Hom}(S_0, E).
\]

At \( \sigma^- \), the first Harder-Narasimhan factor is \( E_1 = S_0 \otimes \text{Hom}(S_0, F^q) \). By Corollary 4.13, \( F^q \) has \( \sigma^- \)-Harder-Narasimhan filtration

\[
0 \to S_0 \otimes \text{Hom}(S_0, F^q) \to F^q \to T_1 \otimes \text{Hom}(F^q, T_1)^* \to 0.
\]

Hence the quotient \( E/E_1 \), which is rigid, fits into

\[
0 \to T_1 \otimes \text{Hom}(F^q, T_1)^* \to Q_1 \to G^p \to 0.
\]

By Proposition 6.5, \( E/E_1 \) is of type II, the proof is completed.

**Example 6.12** (Example 5.24, revisit). We revisit Example 5.24 by using the simplified local reduction (Theorem 6.11). Keeping the notations in Example 5.24, the rigid object \( E \) is two step, in fact of type II. Then by Proposition 6.5, the connecting homomorphism

\[
\text{Hom}(T \otimes B_1, T) \to \text{Ext}^1(Q_1, T)
\]

has maximal rank. By Theorem 4.10, \( \text{Ext}^1(Q_1, T) = \mathbb{C} \), and

\[
\text{Hom}(T \otimes B, T) = \mathbb{C}^{155}.
\]

Hence \( \text{hom}(E, T) = 35 + 154 = 189 \), and the \( \sigma^- \)-Harder-Narasimhan filtration has shape

\[
(E_2, 189T),
\]

where \( E_2 \) is a rigid object of type I that fits into the exact sequence

\[
0 \to T \to E_2 \to S^{\oplus 6} \to 0.
\]

By Proposition 6.4, the \( \sigma^- \)-Harder-Narasimhan filtration of \( E_2 \) consists of a single factor

\[
S^{-1}_1 \in M_{\sigma^-}((29, 12H, 5)).
\]

Hence the \( \sigma^- \)-Harder-Narasimhan filtration of \( E \) has shape

\[
(S^{-1}_1, 189T).
\]
6.2. Global Simplification. We introduce a notion called the height (Definition 6.13) to measure the complexity of a stable spherical object with respect to a generic stability condition. In this subsection we give a simple method to compute the cohomology of a height 2 object.

Recall that a stable spherical object is represented by a pair \((v, \sigma)\), where \(v \in H^*(X)\) is spherical and \(\sigma \in \mathfrak{b}\), defined in Section 3. We define the height as follows.

**Definition 6.13 (Height).** Let \((v, \sigma)\) be a pair such that \(\sigma\) is generic, and \(v = (r, D, a)\) satisfies \(r > 0, D \cdot H > 0\). Let \(W\) be the wall of \(v\) that is right below \(\sigma\). Let \(s, t\) be Mukai vectors of the two stable spherical objects on \(W\) and let \(\sigma_-\) be a stability condition right below \(W\). Then we say \((v, \sigma)\) splits into \((s, \sigma_-)\) and \((t, \sigma_-)\) and \((s, \sigma_-), (t, \sigma_-)\) are factors of \((v, \sigma)\).

We say \((v, \sigma)\) is of height zero, if \(\sigma\) is right below the numerical wall \(W(v, \mathcal{O}_X[1])\) or \(v = v(O_X[1]) = (-1, 0, -1)\). We define the height of \((v, \sigma)\) to be the length of the longest chain of splitting so that every factor has height zero.

By definition, a \(\sigma_{++}\)-stable spherical object \(E \in M_{\sigma_{++}}(v)\) has height 1, if the next wall below \(\sigma_{++}\) is the Brill-Noether wall. In this case, the whole algorithm has only one step, the result is given by Corollary 4.13.

Let \(E \in M_{\sigma}(v)\) be a height 2 object. Let \(W'\) be the wall below \(\sigma\) and \(\mathcal{H}'\) be the rank 2 lattice. Then the two factors \(S', T'\) have height 1 or 0. By Corollary 4.13, we have

\[
0 \rightarrow S' \otimes \text{Hom}(S', E) \rightarrow E \rightarrow T' \otimes \text{Hom}(E, T')^* \rightarrow 0.
\]

Let \(\sigma_+\) be a stability condition below but near \(W'\). First note that if \((S', \sigma_+)\) has height 0, \(S'\) cannot be \(O_X[1]\), because \(v(E) = (r, D, a)\) satisfies \(DH > 0\), hence \(\text{Hom}(O_X[1], E) = 0\).

Hence \(S'\) has \(H^1(S') = 0\), we have

\[
0 \rightarrow H^0(S') \otimes \text{Hom}(S', E) \rightarrow H^0(E) \rightarrow H^0(T') \otimes \text{Hom}(E, T')^* \rightarrow 0.
\]

Now \((T', \sigma_+)\) has height at most 1, hence its \(h^0\) is computed by Corollary 4.13.

Hence we may assume \((S', \sigma_+)\) has height 1. The next claim is that the connecting homomorphism \(\delta: H^0(T') \otimes \text{Hom}(E, T')^* \rightarrow H^1(S') \otimes \text{Hom}(S', E)\) always has maximal possible rank. From this we may compute \(H^0(E)\) inductively from \(H^0(S')\) and \(H^0(T')\).

Let \(W_{S'}, W_{T'}\) be the Brill-Noether walls for \(S', T'\), respectively. First we note that \(W_{S'} > W_{T'}\) under the order on \(W(\mathbb{B})\) defined in Section 3. Suppose not, then \(W_{T'} > W_{S'}\). Let \(\sigma\) be a stability condition that is below \(W_{T'}\), but above \(W_{S'}\). Then we have

\[
\phi_\sigma(O_X[1]) < \phi_\sigma(T') < \phi_\sigma(S') < \phi_\sigma(O_X[1]),
\]

a contradiction. Hence \(W_{S'} > W_{T'}\). Let \(\sigma_-\) be a stability condition below \(W_{S'}\) but above \(W_{T'}\). The \(\sigma_-\)-Harder-Narasimhan filtration of \(S'\) is

\[
0 \rightarrow S_- \otimes \text{Hom}(S_-, S') \rightarrow S' \rightarrow O_X[1] \otimes H^1(S') \rightarrow 0.
\]

At \(\sigma_-\), we have

\[
\phi_-(S_-) > \phi_-(O_X[1]) > \phi_-(T'),
\]

hence the first \(\sigma_-\)-Harder-Narasimhan factor of \(E\) is \(S_- \otimes \text{Hom}(S_-, S')\), and the quotient \(Q_1 = E/(S_- \otimes \text{Hom}(S_-, S'))\) is a rigid object of type II (Proposition 6.5), it fits into an exact sequence

\[
0 \rightarrow O_X[1] \otimes H^1(S') \otimes \text{Hom}(S', E) \rightarrow Q_1 \rightarrow T' \otimes \text{Hom}(E, T')^* \rightarrow 0,
\]
where $\mathcal{O}_X[1], T' \in \mathcal{H}_T$, the rank 2 lattice of $W_T$. Since $H^1(-) = \text{Hom}(-, \mathcal{O}_X[1])^*$, by Proposition 6.5 the connecting homomorphism
\[
H^1(T') \otimes \text{Hom}(E, T')^* \longrightarrow H^1(\mathcal{O}_X[1]) \otimes H^1(S') \otimes \text{Hom}(S', E)
\]
is a general linear map, hence has maximal possible rank. Note that $H^1(\mathcal{O}_X[1]) \otimes H^1(S')$ is naturally identified with $H^1(S')$. We have proved:

**Theorem 6.14** (Global simplification). Let $E \in M_{\rho}(v)$ be a height 2 spherical object and $S, T$ be its factors. Then in the long exact sequence induced by $H^0(-)$ on
\[
0 \longrightarrow S \otimes \text{Hom}(S, E) \longrightarrow E \longrightarrow T \otimes \text{Hom}(E, T)^* \longrightarrow 0,
\]
the connecting homomorphism $H^0(T) \otimes \text{Hom}(E, T)^* \longrightarrow H^1(S) \otimes \text{Hom}(S, E)$ has maximal rank.

We show the use of Theorem 6.14 explicitly in Example 9.2.

**Remark 6.15.** The height zero and height one cases are clear. If $E$ is a height zero stable spherical vector bundle with $c_1(E) \cdot H > 0$, we have $h^1(E) = 0$ by Corollary 3.8. If $E$ has height one, the cohomology of $E$ is computed by Theorem 4.10 (see Example 9.1).

### 7. Weak Brill-Noether in Picard Rank One

In this section we show a numerical condition that is equivalent to weak Brill-Noether for a spherical vector bundle on a K3 surface with Picard rank one.

For the rest of this section, we assume the polarized K3 surface $(X, H)$ has $\text{Pic}(X) = \mathbb{Z}H$ and $H^2 = 2n$. Let $\mathbb{H} = \{(sH, tH) : s \in \mathbb{R}, t \in \mathbb{R}_{>0}\}$, and let $\mathcal{A}_0 = \mathcal{A}_{0, tH}, \forall t > 0$. Recall that we defined the line $b \subset \mathbb{H}$ in Section 3. When the Picard rank is one, $b = \{(\epsilon H, tH)\}$ for a sufficiently small $\epsilon > 0$. Let $\mathcal{A}_s$ be the heart of any stability condition in $b$. The main theorem of this section is the following.

**Theorem 7.1.** (Weak Brill-Noether) Let $(X, H)$ be a polarized K3 surface of Picard rank one. Let $v = (r, dH, a)$ be a Mukai vector with $v^2 = -2$ and $r, d > 0, E \in M_{H}(v)$. Let $y$ be the largest possible value of $\frac{a_1d - ad_1}{r_1d - rd_1}$ where $v_1 = (r_1, d_1H, a_1) \neq v$ satisfies
\[
v_1^2 = -2, vv_1 < 0, \frac{a_1d - ad_1}{r_1d - rd_1} > 0, 0 < d_1 \leq d.
\]
Then $H^1(E) = 0$ if and only if $y < 1$.

We show the use of Theorem 7.1 in Example 9.5 and Example 9.6.

The crucial observation is the following lemma.

**Lemma 7.2.** Let $\mathcal{H}$ be the rank 2 lattice of a wall, whose two stable spherical objects have Mukai vectors $v_1, v_2$. Assume $v_0 = (-1, 0, -1) \notin \mathcal{H}$. Then $r(v_1) \cdot r(v_2) < 0$.

**Proof.** Write $v_i = (r_i, d_iH, a_i), i = 1, 2$. First note that $r_i \neq 0$. If $r_i = 0$, then
\[
-2 = v_i^2 = d_i^2H^2 - 2r_ia_i = 2na_i^2 \geq 0,
\]
a contradiction. Hence it suffices to show $r_1, r_2$ cannot both be positive or negative. We prove by contradiction that they cannot both be positive, the other case is similar. Let $\chi_i = v_0v_i = r_i + a_i$. Note that $r_i > 0$ is equivalent to $a_i > 0$, which is also equivalent to $\chi_i > 0$, because $2r_ia_i = 2nd_i^2 + 2 > 0$. 
Let $\chi = v_1 v_2$. Since $v_1, v_2$ are the Mukai vectors of the two stable spherical objects in $\mathcal{H}$, we have $\chi > 0$. Note that $\mathcal{H}$ cannot be negative definite, since the lattice $H^*_{alg}(X) \cong U \oplus \mathbb{Z}H$ where $U$ is a hyperbolic plane. The signature of $U \oplus \mathbb{Z}H$ is $(2, 1)$, it cannot contain a negative definite plane. Hence by Proposition 4.2, $\chi \geq 2$.

Next we consider the sublattice $N = \langle v_0, v_1, v_2 \rangle \subset H^*_{alg}(X) \cong U \oplus \mathbb{Z}H$. By assumption, $v_0 \notin \mathcal{H}$. And since $\mathcal{H}$ is the rank 2 lattice of a wall, it is primitive, hence $v_0 \notin \mathcal{H}_Q$, $N$ is a sublattice of full rank. In particular, its discriminant has the same sign as $\text{disc}(U \oplus \mathbb{Z}H) = -2n$, which is negative. Now we compute the discriminant of $N$:

$$\text{disc}(N) = \det \begin{pmatrix} v_0^2 & v_0 v_1 & v_0 v_2 \\ v_1 v_0 & v_1^2 & v_1 v_2 \\ v_2 v_0 & v_2 v_1 & v_2^2 \end{pmatrix}$$

$$= \det \begin{pmatrix} -2 & \chi_1 & \chi_2 \\ \chi_1 & -2 & \chi \\ \chi_2 & \chi & -2 \end{pmatrix}$$

$$= 2(\chi^2 - 4) + \chi_1(2 \chi_1 + \chi \chi_2) + \chi_2(\chi_1 \chi + 2 \chi_2) .$$

Since $\chi \geq 2$, $\chi^2 - 4 \geq 0$. If $\chi_1, \chi_2 > 0$, then $\text{disc}(N) > 0$, a contradiction. \hfill \square

**Proof of Theorem 7.1.** Let $\sigma = \sigma_{(H,T,W)} \in \mathfrak{b}$ with $t \gg 0$. By Theorem 2.8, $E \in M_\sigma(v)$. Using the notation in Proposition 3.7, let $sh(\sigma) = (E^*, \partial W)$. Since $E \in M_\sigma(v)$, the filtration $E^* = E$ is trivial. By Proposition 4.1, $y = \frac{\text{f}_{0}(W)}{\sqrt{2/H^2}}$. We separate into three cases.

**Case I:** $y < 1$. Then $\sigma$ satisfies the condition in Proposition 3.7. Since $E^* = E$ is trivial, under the notations in Proposition 3.7, $h = 0$. Hence by Corollary 3.8, $h^1(E) = 0$.

**Case II:** $y = 1$. Let $W'$ be the numerical wall defined by $v$ and $v(\mathcal{O}_X[1]) = (-1, 0, -1)$. Then $\frac{\text{f}_{0}(W')}{\sqrt{2/H^2}} = 1$. By [Mac14], numerical walls of $v$ are nested, hence $W = W'$. Let $S, T$ be the two stable spherical objects in $\mathcal{H}$, the rank 2 lattice of $W$. Since $\mathcal{O}_X[1]$ is $\sigma$-stable for all $\sigma \in \mathfrak{b}$, we have $T = \mathcal{O}_X[1]$. Let $\sigma_- \in \mathfrak{b}$ be a stability condition right below $W$. Since $\text{Hom}(\mathcal{O}_X[1], E) = 0$, by Corollary 4.13, the $\sigma_-$-Harder-Narasimhan filtration of $E$ is

$$0 \longrightarrow S^p \longrightarrow E \longrightarrow \mathcal{O}_X[1]^q \longrightarrow 0$$

for some $p, q > 0$. By Corollary 3.8, $h^1(E) = q > 0$.

**Case III:** $y > 1$. Same notations as in Case II. By Corollary 4.13, the $\sigma_-$-Harder-Narasimhan filtration of $E$ is

$$0 \longrightarrow S^p \longrightarrow E \longrightarrow T^q \longrightarrow 0$$

for some $p, q > 0$. Note that $S \in \mathcal{A}_r$ is a sheaf (complex concentrated in degree 0): by taking the long exact sequence of cohomology associated to (5), we have

$$0 \longrightarrow \mathcal{H}^{-1}(S^p) \longrightarrow \mathcal{H}^{-1}(E) \longrightarrow \cdots.$$

Since $E$ is a sheaf, $\mathcal{H}^{-1}(E) = 0$, hence $\mathcal{H}^{-1}(S) = 0$. Therefore $r(S) > 0$. Since $y \neq 1$, $v(\mathcal{O}_X[1]) = (-1, 0, -1) \notin \mathcal{H}$. By Lemma 7.2, $r(T) < 0$, equivalently $\chi(T) < 0$. Hence $H^1(T) \neq 0$. Consider the long exact sequence of cohomology on (5):

$$\cdots \longrightarrow H^1(E) \longrightarrow H^1(T^q) \longrightarrow H^2(S^p) \longrightarrow \cdots.$$

Since $y > 1$, $\sigma_-$ is above the Brill-Noether wall of $S$, hence $H^2(S) = 0$. Since $H^1(T) \neq 0$, we have $H^1(E) \neq 0$. \hfill \square
8. ASYMPTOTIC RESULTS

In this section we prove an asymptotic estimation (Theorem 8.1) of the cohomology of a spherical vector bundle, in the sense that the error become relatively small when the Mukai vector is obtained by many spherical reflections from the two stable spherical Mukai vectors.

Let $\mathcal{H}$ be the rank 2 lattice of a wall. Recall from Proposition 4.2 that every effective spherical Mukai vector in $\mathcal{H}$ can be written as $t_i$ or $s_j$. In this section we give an asymptotic result for the cohomology of $E \in M_H(t_i)$ or $M_H(s_j)$ for $i \gg 0$ or $j \ll 0$.

Using the notation in Proposition 4.3, let
ev_{i-1} : H^0(T_{i-1}) \otimes \text{Hom}(T_{i-1}, T_{i-2}) \rightarrow H^0(T_{i-2}),
coev_j : H^0(S_{j+1}) \rightarrow H^0(S_j) \otimes \text{Hom}(S_{j+1}, S_j)^*\

be the induced maps on the evaluation map or coevaluation map. If we know the ranks of $ev_i, coev_j$ for all $i \geq 4$ and $coev_j$ is injective for $j \leq -3$. If Pic$(X) = \mathbb{Z}H$ and $H^2 \neq 2$, then $ev_i$ is surjective for $i \geq 2$ and $coev_j$ is injective for $j \leq -1$.

8.1. A Bound for $h^0$. In this subsection we find a bound for $h^0$ of a spherical vector bundle on a K3 surface with Picard rank one and degree at least 4. The bound itself is interesting, it is also needed for the proof of Theorem 8.1.

Proposition 8.2 (Bound of $h^0$). Let $X$ be a K3 surface with Pic$(X) = \mathbb{Z}H$ and $H^2 \neq 2$. Then for any $v = (r, dH, a) \in H_{alg}^0(X)$ with $v^2 = -2$ and $\sigma \in \text{Stab}(X)$ above the Brill-Noether wall, if $r > 0$ we have

$h^0(S) < 2\chi(S)$

for $S \in M_\sigma(v)$. Symmetrically, if $r < 0$, then

$h^1(T) < -2\chi(T)$

for $T \in M_\sigma(v)$.

Given the proposition, we have an immediate corollary

Corollary 8.3. Let $S, T$ be two spherical objects as in the proposition. Then

$-\chi(S, T) > h^1(S) + 2,$

$-\chi(S, T) > h^0(T) + 2.$

Proof. Write $v(S) = (r_0, d_0H, a_0)$ and $v(T) = (r_1, d_1H, a_1)$. By assumption $d_0, d_1$ are positive integers. By Proposition 8.2, $h^1(S) = h^0(S) - \chi(S) < 2\chi(S) - \chi(S) = \chi(S)$. On the other hand

$-\chi(S, T) = d_0d_1H^2 - r_0a_1 - r_1a_0 \geq H^2 + r_0 + a_0 \geq 2 + r_0 + a_0 = 2 + \chi(S) > 2 + h^1(S).$

The proof for $T$ is similar.

We only prove the proposition for $r > 0$, the other case is similar.
Lemma 8.4. Let $X$ be a K3 surface with $\text{Pic}(X) = \mathbb{Z}H$. Let $v = (r, dH, a) \in H^0_{\text{alg}}(X)$ with $v^2 = -2$, $r > 0$, $W$ an actual wall for $v$ that is above the Brill-Noether wall, and $S_0, T_1$ the two stable spherical objects at the wall. Let $\sigma_+, \sigma_- \in \text{Stab}(X)$ be above/below but near $W$ respectively. Then
\[
\frac{h^0(S_{-m})}{h^1(S_{-m}) + 2} \geq \frac{h^0(S_0)}{h^1(S_0) + 2},
\]
where $S_{-m} \in M_+(v)$ is constructed as in Proposition 4.3.

For any positive integer $n$, denote by $P_n$ Proposition 8.2 for $d \leq n$, denote by $L_n$ Lemma 8.4 for $d \leq n$. We will prove Proposition 8.2 and Lemma 8.4 simultaneously by showing that $L_d$ implies $P_d$ and $P_{d-1}$ implies $L_d$.

Proof of $L_d$ implies $P_d$. Let $S$ be as in Proposition 8.2. Since we know $h^0(S) > h^1(S)$, $P_d$ is equivalent to $\frac{h^0(S)}{h^1(S)} > 2$.Assuming $L_d$ is true, we prove a stronger claim that $\frac{h^0(S)}{h^1(S) + 2} \geq 2$ by induction on the height (Definition 6.13) of $S$.

If $\text{height}(S) = 0$, then $S$ satisfies weak Brill-Noether. Write $v(S) = (r, dH, a)$, we have
\[
\frac{h^0(S)}{h^1(S) + 2} = \frac{\chi(S)}{2} = \frac{r + a}{2}.
\]
It suffices to prove $r + a \geq 4$. Since $H^2 \geq 4$, we have
\[
r + a \geq 2\sqrt{ra} = 2\sqrt{d^2H^2/2 + 1} \geq 2\sqrt{3} > 3.
\]
hence $r + a \geq 4$.

If $\text{height}(S) \geq 1$, let $W$ be the first wall of $S$ that is below $\sigma$. Let $S_0, T_1$ be the two stable spherical objects at $W$, and label $S = S_{-m}$ as in Proposition 4.3. By definition, $\text{height}(S_0) < \text{height}(S)$. Since $\text{height}(S) \geq 1$, we have $c_1(S_0) < c_1(S_{-m})$. Then by $L_d$, we have
\[
\frac{h^0(S)}{h^1(S) + 2} = \frac{h^0(S_0)}{h^1(S_0) + 2} + 2 \geq 2.
\]

Proof of $P_{d-1}$ implies $L_d$. First note that
\[
h^0(S_{-m}) - h^1(S_{-m}) = \chi(S_{-m}) = r + a \geq 2\sqrt{ra} = 2\sqrt{d^2H^2/2 + 1} \geq 2,
\]
by the same argument in the proof of $P_d$ from $L_d$, equality cannot hold, hence $h^0(S_{-m}) \geq h^1(S_{-m}) + 2$. The function
\[
\frac{h^0(S_{-m}) + x}{h^1(S_{-m}) + 2 + x}
\]
is non-increasing for positive $x$. Now $S_{-m}$ fits into
\[
0 \rightarrow S_0 \otimes \text{Hom}(S_0, S_{-m}) \rightarrow S_{-m} \rightarrow T_1 \otimes \text{Hom}(S_{-m}, T_1)^* \rightarrow 0.
\]
By Theorem 4.10, we know $\text{hom}(S_0, S_{-m}) = a_m$, $\text{hom}(S_{-m}, T_1) = a_{m-2}$, where $a_j$ is the fundamental sequence of $W$ (Definition 4.9) given by
\[
a_{-1} = 0, a_0 = 1, a_1 = -\chi(S_0, T_1) =: g, a_j = ga_{j-1} - a_{j-2}.
\]
Taking the long exact sequence, let $t$ be the rank of the connecting homomorphism
\[
H^0(T_1) \otimes \text{Hom}(S_{-m}, T_1)^* \rightarrow H^1(S_0) \otimes \text{Hom}(S_0, S_{-m}).
\]
Then we have
\[
\frac{h^0(S_m)}{h^1(S_m) + 2} = \frac{a_nh^0(S_0) + a_m-2h^0(T_1) - t}{a_nh^1(S_0) + a_m-2h^1(T_1) + 2 - t} > \frac{a_nh^0(S_0) + a_m-2h^0(T_1)}{a_nh^1(S_0) + a_m-2h^1(T_1) + 2}.
\]
Since \(T_1\) has smaller degree than \(S_m\), \(P_{d-1}\) is true for \(T_1\). Write \(v(S_0) = (r_0, d_0H, a_0)\) and \(v(T_1) = (r_1, d_1H, a_1)\), we have \(h^1(T_1) < -2(r_1 + a_1)\) by \(P_{d-1}\). On the other hand
\[
g = -\chi(S_0, T_1) = d_0d_1H^2 - r_0a_1 - r_1a_0 > -(r_1 + a_1) > \frac{h^1(T_1)}{2}
\]
hence
\[
\frac{h^0(S_m)}{h^1(S_m) + 2} > \frac{a_nh^0(S_0) + a_m-2h^0(T_1)}{a_nh^1(S_0) + a_m-2h^1(T_1) + 2} > \frac{a_nh^0(S_0)}{a_nh^1(S_0) + 2ga_{m-2} + 2},
\]
it suffices to show \(2ga_{m-2} + 2 \leq 2a_m\). Since \(a_i\) are integers, it suffices to show
\[
am_m - ga_{m-2} = ga_{m-1} - a_{m-2} - ga_{m-2} = ga_{m-1} - (g + 1)a_{m-2} > 0,
\]
namely \(\frac{a_{m-1}}{a_{m-2}} > \frac{g+1}{g}\). If \(m = 1\), \(2ga_1 + 2 = 2 < 2g = 2a_1\). If \(m \geq 2\), note that \(g \geq 3\), by Lemma 6.2, we have
\[
\frac{a_{m-1}}{a_{m-2}} > \rho(W) > 2 > \frac{g+1}{g},
\]
where \(\rho(W)\) is the bigger solution of the equation \(x + \frac{1}{x} = g\) (Remark 6.3). \(\square\)

8.2. Asymptotic Result. In this subsection we prove Theorem 8.1. First, we need the following simple observation.

**Lemma 8.5.** Let \(X = \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^N\) be the Segre embedding, \(H \subset \mathbb{P}^N\) a hyperplane. Then \(H \cap X\) is non-degenerate in \(H\) set theoretically.

**Proof.** We first prove that for any non-degenerate subvariety \(X \subset \mathbb{P}^N\), \(\tilde{X} = H \cap X \subset H\) is non-degenerate in \(H\) scheme theoretically. The non-degeneracy of \(X\) in \(\mathbb{P}^N\) is equivalent to injectivity of the map \(H^0(O_{\mathbb{P}^N}(H)) \rightarrow H^0(O_X(H))\). Denote \(\tilde{X} = H \cap X\). Then we have
\[
0 \rightarrow H^0(O_X) \rightarrow H^0(O_X(H)) \rightarrow H^0(O_{\tilde{X}}(H))
\]
Since \(X\) is closed projective, \(H^0(O_X) = \mathbb{C}\). Together with the injectivity of \(H^0(O_{\mathbb{P}^N}(H)) \rightarrow H^0(O_X(H))\) shows that the composition map
\[
H^0(O_{\mathbb{P}^N}(H)) \rightarrow H^0(O_{\tilde{X}}(H))
\]
has one dimensional kernel, namely \(H\). Hence \(\tilde{X}\) is non-degenerate in \(H\).

Now let \(X = \mathbb{P}^n \times \mathbb{P}^n\) be the Segre embedding. First observe that \(\tilde{X}\) is generically reduced: suppose not, note that \(\tilde{X} \subset \mathbb{P}^n \times \mathbb{P}^n\) has class \((1, 1)\) in \(\text{Pic}(\mathbb{P}^n \times \mathbb{P}^n)\), if \(\tilde{X}\) is not generically reduced, then \(\tilde{X}_{\text{red}}\) has class \((0, 1)\) or \((1, 0)\), say \((0, 1)\). Then \(\tilde{X}\) has class at least \((0, 2)\), which is impossible. Now if \(H\) is not tangent to \(X\), then \(\tilde{X}\) is reduced. If \(H\) is tangent to \(X\), by generic reducedness of \(\tilde{X}_{\text{red}}\), \(\tilde{X}\) has to be \(\mathbb{P}^n \times \mathbb{P}^{n-1} \cup \mathbb{P}^{n-1} \times \mathbb{P}^n\), which is also reduced. Hence \(\tilde{X}_{\text{red}} = \tilde{X}\), the lemma is proved. \(\square\)

As corollaries, we get the following linear algebra lemmas

**Lemma 8.6.** Let \(V_1, V_2\) be linear spaces, and \(m : V_1 \otimes V_2 \rightarrow \mathbb{C}\) be a bilinear form. If \(\dim V_1 + \dim V_2 \geq 3\), then every element in \(\ker(m)\) can be written as a sum of pure tensors in \(\ker(m)\).
Proof. The $H = \mathbb{P} \ker(m) \subset \mathbb{P}(V_1 \otimes V_2)$ is a hyperplane in $\mathbb{P}(V_1 \otimes V_2)$. The space of pure tensors is naturally identified with the Segre embedding $X = \mathbb{P}V_1 \times \mathbb{P}V_2 \subset \mathbb{P}(V_1 \otimes V_2)$, whose dimension is $\dim V_1 + \dim V_2 - 2$. If $\dim V_1 + \dim V_2 - 2 \geq 1$, then $H \cap X$ is non-empty. By Lemma 8.5, $H \cap X$ is non-degenerate in $H$ set theoretically, namely every element in $\ker(m)$ is a sum of pure tensors in $\ker(m)$. \hfill \square

Lemma 8.7. Let $V_1, V_2, \ldots, V_n, \ldots$ be a sequence of linear spaces over a field of characteristic 0, such that for $n \leq m$ there are arbitrary bilinear forms $m_{n-1,n} : V_{n-1} \otimes V_n \rightarrow W_{n-1,n}$ for some non-zero linear spaces $W_{n-1,n}$, and for $n \geq m + 1$ there are perfect pairings $m_{n-1,n} : V_{n-1} \otimes V_n \rightarrow \mathbb{C}$. Let

$$K_n := \bigcap_{j=2}^n \ker(m_{j-1,j}) \subset \bigotimes_{j=1}^n V_j$$

where by abuse of notation $m_{j-1,j}$ means $\text{id} \otimes \cdots \otimes \text{id} \otimes m_{j-1,j} \otimes \text{id} \otimes \cdots \otimes \text{id}$. If $\dim V_n \geq 2$ for $n \geq m$, then for $n \geq m + 1$ there is a commutative diagram

$$
\begin{array}{ccc}
K_{n+1} \otimes V_{n+2} & \rightarrow & \bigotimes_{j=1}^{n+2} V_j \\
\downarrow m_{n+1,n+2} & & \downarrow m_{n+1,n+2} \\
K_n & \leftarrow & (\bigotimes_{j=1}^n V_j)
\end{array}
$$

where the vertical maps are surjective.

Proof. Commutativity is clear. For $n \geq m$, the image of the composition map

$$K_{n+1} \otimes V_{n+2} \rightarrow \bigotimes_{j=1}^{n+2} V_j \rightarrow \bigotimes_{j=1}^n V_j$$

is contained in $K_n$. Hence it suffices to check that for $n \geq m$, the left vertical map is surjective. For any $1 \leq j \leq n$, let $\{v_{ij}^j\}_{1 \leq i_j \leq \dim v_j}$ be a basis of $V_j$. Let

$$\eta = \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} v_{i_1}^1 \otimes \cdots \otimes v_{i_n}^n \in K_n \subset \bigotimes_{j=1}^n V_j$$

be any element in $K_n$. We need to find an element in $\tilde{\eta} \in K_{n+1} \otimes V_{n+2}$ such that $m_{n+1,n+2}(\tilde{\eta}) = \eta$.

The condition that $\eta \in K_n$ spells out

$$m_{j,j+1}(\eta) = \sum_{i_k} \sum_{k \neq j, j+1} a_{i_1, \ldots, i_n} m_{j,j+1}(v_{i_j}^j \otimes v_{i_{j+1}}^{j+1}) v_{i_1}^1 \otimes \cdots \otimes v_{i_{j-1}}^{j-1} \otimes v_{i_{j+2}}^{j+2} \otimes \cdots v_{i_n}^n = 0.$$ 

Since $v_{i_1}^1 \otimes \cdots v_{i_{j-1}}^{j-1} \otimes v_{i_{j+2}}^{j+2} \otimes \cdots v_{i_n}^n$ form a basis for $V_1 \otimes \cdots \otimes V_{j-1} \otimes V_{j+2} \otimes \cdots \otimes V_n$, we know that

$$\sum_{i_j, i_{j+1}} a_{i_1, \ldots, i_n} m_{j,j+1}(v_{i_j}^j \otimes v_{i_{j+1}}^{j+1}) = 0$$

for any $i_1, i_2, \ldots, i_{j-1}, i_{j+2}, \ldots, i_n$. We take $j = n - 1$. Since $n \geq m + 1$, by assumption $m_{n-1,n} : V_{n-1} \otimes V_n \rightarrow \mathbb{C}$ is a perfect pairing. By Lemma 8.6, there is a basis of $\ker(m_{n-1,n})$.
that consists of pure tensors, say \( \{u_{i_{n-1}}^{n-1} \otimes u_{i_n}^n\} \). Since
\[
\sum_{i, j, i_j+1} a_{i_1, \cdots, i_n} v_{i_j}^j \otimes v_{i_{j+1}}^{j+1} \in \ker(m_{n-1, n}),
\]
it can be written as a linear combination of this new basis. By extending \( \{u_{i_{n-1}}^{n-1} \otimes u_{i_n}^n\} \) to a basis \( \{u_{i_{n-1}}^{n-1} \otimes u_{i_n}^n\}_{1 \leq i \leq \dim V} \), we may rewrite
\[
\eta = \sum_{i_1, \cdots, i_n} b_{i_1, \cdots, i_n} v_{i_1}^1 \otimes \cdots \otimes v_{i_{n-2}}^{n-2} \otimes u_{i_{n-1}}^{n-1} \otimes u_{i_n}^n,
\]
where \( u_{i_{n-1}}^{n-1} \otimes u_{i_n}^n \in \ker(m_{n-1, n}) \). By the same argument as above, we have
\[
(6) \sum_{i, j, i_j+1} b_{i_1, \cdots, i_n} m_{j, j+1}(v_{i_j}^j \otimes v_{i_{j+1}}^{j+1}) = 0
\]
for all \( 1 \leq j \leq n - 2 \).

Since \( \dim V \geq 2 \), let \( \tau \) be any permutation on \( \{1, 2, \cdots, \dim V\} \) without fixed point. Let
\[
\tilde{\eta} = \frac{1}{\dim V} \sum_{i_1, \cdots, i_n} b_{i_1, \cdots, i_n} v_{i_1}^1 \otimes \cdots \otimes v_{i_{n-2}}^{n-2} \otimes u_{i_{n-1}}^{n-1} \otimes u_{i_n}^n (u_{\tau(i_n)})^*,
\]
where \( (u_{\tau(i_n)})^* \in V_{n+1} \) are the dual elements of \( u_{\tau(i_n)}^n \) the the identification \( V_{n+1} = V_n^* \) via the perfect pairing \( m_{n, n+1} \). By \( (6) \), \( m_{j, j+1}(\tilde{\eta}) = 0 \) for \( 1 \leq j \leq n - 2 \). Since \( u_{i_{n-1}}^{n-1} \otimes u_{i_n}^n \in \ker(m_{n-1, n}) \), we also have \( m_{n-1, n}(\tilde{\eta}) = 0 \). Since \( \tau \) has no fixed point, we have
\[
m_{n, n+1}(u_{i_n}^n \otimes (u_{\tau(i_n)})^*) = 0
\]
for any \( i_n \), we have \( \tilde{\eta} \in \ker(m_{n, n+1}) \). Hence we showed \( \tilde{\eta} \in K_{n+1} \). Finally, we have
\[
m_{n+1, n+2}(\tilde{\eta} \otimes (\sum_{i_n} (u_{i_n}^n)^*))) = \eta,
\]
where \( (u_{i_n}^n)^* \in V_{n+2} \) are dual elements of \( u_{i_n}^n \) under the identification \( V_{n+2} \cong V_{n+1}^* \) via the perfect pairing \( m_{n+1, n+2} \).

\( \square \)

**Proof of Theorem 8.1.** We prove the claims for \( ev_i, coev_j \) are similar.

Let \( V_1 = H^0(T_1) \), \( V_n = \Hom(T_n, T_{n-1})^* \) for \( n \geq 2 \). We define \( m_{1, 2} \) as the connecting homomorphism
\[
m_{1, 2} : H^0(T_1) \otimes \Hom(T_2, T_1)^* \cong H^0(T_1) \otimes \Ext^1(T_1, S_0) \rightarrow H^1(S_0).
\]

For \( n \geq 3 \), let
\[
m_{n-1, n} : V_{n-1} \otimes V_n \rightarrow \mathbb{C}
\]
be the perfect pairing induced by the dual of \( \delta_{n-1, n} : \mathbb{C} \rightarrow \Hom(T_{n-1}, T_{n-1}) \otimes \Hom(T_{n-1}, T_{n-2}) \) which was introduced in Lemma 4.5. Let \( W_{1, 2} = H^1(S_0) \). By Proposition 4.8, we have the natural identification
\[
\Hom(T_n, T_1)^* = \bigcap_{j=3}^n \ker(m_{n-1, n}).
\]

Since we have the exact sequence
\[
H^0(T_n) \rightarrow H^0(T_1) \otimes \Hom(T_n, T_1)^* \rightarrow H^1(S_0) \otimes \Hom(S_0, T_n),
\]
the image of $H^0(T_n) \rightarrow H^0(T_1) \otimes \text{Hom}(T_n, T_1)^*$ is hence identified with $K_n$ using the notation in Lemma 8.7. By setting $m = 2$ in Lemma 8.7, we have
\[ K_{n+1} \otimes V_{n+2} \rightarrow K_n \]
is surjective for $n \geq 3$. Hence in the following commutative diagram
\[
\begin{array}{c c c c c c}
0 & \rightarrow & H^0(S_0) & \otimes \text{Hom}(S_0, T_{n+1}) & \otimes \text{Hom}(T_{n+1}, T_n) & \rightarrow & H^0(T_{n+1}) & \otimes \text{Hom}(T_{n+1}, T_n) & \rightarrow & K_{n+1} & \otimes & V_{n+2} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & H^0(S_0) & \otimes \text{Hom}(S_0, T_n) & \rightarrow & H^0(T_n) & \rightarrow & K_n & \rightarrow & 0
\end{array}
\]
the third vertical map is surjective for $n \geq 3$. The first vertical map is also surjective, hence by the Five Lemma the middle vertical map is also surjective. We proved that
\[ ev_i : H^0(T_1) \otimes \text{Hom}(T_i, T_{i-1}) \rightarrow H^0(T_{i-1}) \]
is surjective for $i \geq 4$, which is the first part of Theorem 8.1.

Now we prove the second part. Assume Pic($X$) = $\mathbb{Z}H$. First we consider $ev_2$. Consider the exact sequence
\[ H^0(T_2) \rightarrow H^0(T_1) \otimes \text{Ext}^1(T_1, S_0) \rightarrow H^1(S_0). \]
For any $f \in H^0(T_1)$, consider the map
\[ ev_1(f \otimes -) : \text{Ext}^1(T_1, S_0) \rightarrow H^1(S_0). \]
By Corollary 8.3, $\text{ext}^1(T_1, S_0) = q > h^1(S_0) + 2 > h^1(S_0)$, hence the map has a non-trivial kernel that contains some non-zero $g \in \text{Ext}^1(T_1, S_0)$. Hence there exists an $h \in H^0(T_2)$ that maps to $f \otimes g$ under the coevaluation map. Take $h \otimes g^* \in H^0(T_2) \otimes \text{Hom}(T_2, T_1)$, we have
\[ ev_2(h \otimes g^*) = f \cdot \langle g, g^* \rangle = f. \]
We proved the surjectivity of $ev_2$.

Finally we consider $ev_3$. It suffices to show that
\[ K_3 \otimes V_4 \xrightarrow{m_{3,4}} K_2 \]
is surjective. Take any element
\[ \eta = \sum_{i_1, i_2} a_{i_1, i_2} v_{i_1}^{1} \otimes v_{i_2}^{2} \in K_2, \]
where $\{v_{i_j}^{j}\}_{1 \leq i_j \leq \dim V_j}$ is a basis of $V_j$ for $j = 1, 2$. Consider
\[ \tilde{\eta} = \eta \otimes \gamma \in K_2 \otimes V_3 \]
for some non-zero $\gamma \in \text{Hom}(T_3, T_2)$. Then $\tilde{\eta} \in K_3$ if the following equations are satisfied for all $1 \leq i_1 \leq h^1(T_1)$:
\[ \sum_{i_2} a_{i_1, i_2} m_{2,3}(v_{i_2}^{2} \otimes \gamma) = 0. \]
The number of equations is $h^0(T_1)$, but $\text{hom}(T_3, T_2) = q > h^0(T_1)$ by Corollary 8.3, hence there exists a non-zero solution $\gamma \in \text{Hom}(T_3, T_2)$. Consider
\[ \tilde{\eta} \otimes \gamma^* \in K_3 \otimes V_4, \]
where $\gamma^* \in \text{Hom}(T_4, T_3)$ is the dual element of $\gamma$ under the identification $\text{Hom}(T_4, T_3) = \text{Hom}(T_3, T_2)^*$ via the perfect pairing $m_{3,4}$. Then we have $m_{3,4}(\tilde{\eta} \otimes \gamma^*) = \eta$. The theorem is proved.
Remark 8.8. In Theorem 8.1, we said nothing about $ev_1$ (resp. $cov_0$). They are in general not surjective (resp. injective). We will exhibit one such in Example 9.7.

9. Examples

In this section we show many explicit examples. For simplicity and without affecting the gist of the idea, in the following we let $(X, H)$ be a polarized K3 surface with $\text{Pic}(X) = \mathbb{Z}H$ and $H^2 = 2$, unless otherwise stated.

First, we consider the following example where there are no walls between the Gieseker chamber and the Brill-Noether wall.

Example 9.1 (Fibonacci bundles, [CNY21] Example 6.3). Let $s = (1, H, 2)$, $t = (-1, 0, -1)$. Let $S_0 \in M_H(s)$, $T_1 = \mathcal{O}_X[1]$. Let $T_i, S_j$ be constructed as in Proposition 4.3. By Proposition 4.1, $S_j$ are $H$-Gieseker stable vector bundles. By Theorem 4.10, we have

\[ h^0(S_j) = f_{-2j+3}, \quad h^1(S_j) = f_{-2j-1}. \]

Here $f_n$ are the Fibonacci numbers:

\[ f_0 = f_1 = 1, \quad f_{n+1} = f_n + f_{n-1} \text{ for } n \geq 1. \]

Explicitly, $v(S_j) = (f_{-2j}, f_{-2j+1}H, f_{-2j+2})$, $j \leq 0$. The vector bundles $S_j$ are called the Fibonacci bundles.

Next, we consider a slightly complicated example: a height 2 (Definition 6.13) spherical vector bundle.

Example 9.2 (A height 2 bundle). Let $v = (305, 477H, 746)$ and $E \in M_H(v)$. Then the next wall (see Section 3) $W$ for $E$ has stable spherical objects

\[ S_0 \in M_H((2, 3H, 5)), \quad T_1 \in M_H((-5, 12H, -29)). \]

Using the notation of Proposition 4.3, $E = S_{-1}$. By Definition 6.13, $S_0$ and $T_1$ are of height one, hence $E$ has height two.

We have $g(W) = -\chi(S_0, T_1) = 155$ (Definition 4.9). By Corollary 4.13, the Harder-Narasimhan filtration of $E$ under $W$ is

\[ 0 \rightarrow S_0^{155} \rightarrow E \rightarrow T_1 \rightarrow 0. \]

By Theorem 6.14, the connecting homomorphism $H^0(T_1) \rightarrow H^1(S_0^{155})$ has maximal possible rank. By Example 9.1, $h^0(S_0) = 8, h^1(S_0) = 1$. By a similar argument as Example 9.1, $h^0(T_1) = 1$. Hence the connecting homomorphism is injective, we have

\[ h^0(E) = 155 \cdot h^0(S_0) = 1240, \quad h^1(E) = h^0(E) - \chi(E) = 189. \]

The spherical vector bundle in the next example has height greater than two, so we cannot use Theorem 6.14. We recall the notion of shape (Definition 5.10): If

\[ 0 = E_0 \subset E_1 \subset \cdots \subset E_m = E \]

is some filtration of $E$ such that $G_i = E_i/E_{i-1} \cong F_i^{\oplus n_i}$, then the shape of the filtration is

\[ (n_1 F_1, n_2 F_2, \ldots, n_m F_m). \]
Example 9.3 (A height 3 bundle). Let \( v = (1340641, 1733695 H, 2241986) \) and \( E \in M_H(v) \). One may check that \( E \) has height 3, hence in the following we invoke the global reduction (Algorithm 3.14).

The next wall \( W \) for \( E \) has stable spherical objects
\[
E_0 \in M_H((58, 75 H, 97)), \quad T_1 \in M_H((-29, 70 H, -19)),
\]
and \( g(W) = -\chi(S_0, T_1) = 23115 \). Using the notation of Proposition 4.3, \( E = S^-_v \). The Harder-Narasimhan filtration of \( E \) under \( W \) is
\[
E^*_W = (g(W) \cdot S_0, T_1).
\]

The next wall for \( E^*_W \) is an actual wall of \( S_0 \), we denote it by \( W^S \). It has two stable spherical objects
\[
S_0^S \in M_H((1, H, 2)), \quad T_1^S \in M_H((-5, 12 H, -29)),
\]
and \( g(W^S) = -\chi(S_0^S, T_1^S) = 63 \). Using the notation of Proposition 4.3, \( S_0 = S^{-}_S \). The Harder-Narasimhan filtration of \( E \) under \( W^S \) has shape
\[
E^*_W = (g(W)g(W^S) \cdot S_0^S, g(W) \cdot T_1^S, T_1).
\]
The question is to find the \( \sigma_- \)-Harder-Narasimhan filtration of \( F \). By Theorem 6.11, we need to know which inequality in Lemma 6.7 holds. By Theorem 4.10, we have
\[
\text{hom}(S_0^T, T_3^T) = a_2(W^T) = 35, \quad \text{ext}^1(S_0^T, (T_3^T)^{\oplus g(W)}) = (g(W) \cdot a_3(W^T) = 4715460 > 35.
\]
Hence by Proposition 6.6, \( \text{Hom}(S_0^T, T_3^T) \longrightarrow \text{Ext}^1(S_0^T, (T_3^T)^{\oplus g(W)}) \) is injective, the first \( \sigma_- \)-Harder-Narasimhan factor for \( F \) has shape
\[
F_1 = g(W) \cdot \text{hom}(S_0^T, T_3^T) \cdot S_0^T = g(W) a_1(W^T) \cdot S_0^T .
\]
By Theorem 6.11, \( Q_1 = F/F_1 \) is of type II (Proposition 6.5). The \( \sigma_+ \)-Harder-Narasimhan filtration of \( Q_1 \) is
\[
(Q_1)^* \big|_+ = (g(W) \text{hom}(T_3^T, T_1^T) \cdot T_1^T, T_4^T) = (g(W) a_2(W^T) \cdot T_1^T, T_4^T).
\]
Applying \( \text{Hom}(-, T_1^T) \), we have
\[
0 \longrightarrow \text{Hom}(T_4^T, T_1^T) \longrightarrow \text{Hom}(Q_1, T_1^T) \longrightarrow \text{Hom}((T_1^T)^{\oplus g(W) a_2(W^T)}, T_1^T) \xrightarrow{\delta} \text{Ext}^1(T_4^T, T_1^T).
\]
By Proposition 6.5, the connecting homomorphism \( \delta \) has maximal possible rank. By Theorem 4.10, we have
\[
\text{hom}(T_4^T, T_1^T) = a_3(W^T) = 204, \quad \text{ext}^1(T_4^T, T_1^T) = a_1(W^T) = 6,
\]
\[
\text{Hom}((T_1^T)^{\oplus g(W) a_2(W^T)}, T_1^T) = g(W) a_2(W^T) = 809025.
\]
Hence the last \( \sigma_- \)-Harder-Narasimhan factor of \( Q_1 \) is
\[
F_3/F_2 = (a_3(W^T) + g(W) a_2(W^T) - a_1(W^T)) \cdot T_1^T = 809223 \cdot T_1^T.
\]
Therefore the $\sigma_+$-Harder-Narasimhan filtration of $F_3/F_1$ has shape
\[(F_3/F_1)^*_+ = (6T_1^T, \text{hom}(S_0^T, T_4^T) \cdot S_0^T) = (6T_1^T, 35S_0^T).\]

By Proposition 6.4, the $\sigma_-$-Harder-Narasimhan filtration of $F_3/F_1$ consists of only one factor $(S_{T_2}^T)_- \in M_{\sigma_-(s_{T_2}^T)}$, where $s_{T_2}^T = (169, 70H, 29)$. Hence the $\sigma_-$-Harder-Narasimhan filtration of $F$ has shape
\[F^-_\sigma = (g(W)a_1(W^T) \cdot S_0, (S_{T_2}^T)_-, 809223 \cdot T_1^T).\]

The $\sigma_-$-Harder-Narasimhan filtration of $E$ has shape
\[E^-_\sigma = (g(W)g(W^S) \cdot S_0^S, g(W)a_1(W^T) \cdot S_0^T, (S_{T_2}^T)_-, 809223 \cdot O_X[1]).\]

The next wall for $E^+_\sigma$ is below the Brill-Noether wall of any factor, hence the global reduction (Algorithm 3.14) terminates. By Corollary 3.8, we have
\[h^1(E) = 809223, h^0(E) = \chi(E) + h^1(E) = 4391850.\]

As mentioned in Section 6, there are indeed spherical vector bundles whose cohomology cannot be computed by Theorem 6.11. In the next example we exhibit one such.

**Example 9.4** (Theorem 6.11 fails). Let $v = (42687466, 66760513H, 104409245)$ and $E \in M_{H}(v)$. We run the global reduction (Algorithm 3.14).

The next wall $W$ for $E$ has stable spherical objects
\[S_0 \in M_H((305, 477H, 746)), T_1 \in M_H((-29, 70H, -169)),\]
and $g(W) = -\chi(S_0, T_1) = 139959$. Using the notation of Proposition 4.3, $E = S_{-1}$. The Harder-Narasimhan filtration of $E$ under $W$ has shape
\[E^+_{W} = (g(W) \cdot S_0, T_1).\]

The next wall for $E^+_{W}$ is an actual wall of $S_0$, we denote it by $W^S$. It has two stable spherical objects
\[S_0^S \in M_H((2, 3H, 5)), T_1^S \in M_H((-5, 12H, -29)),\]
and $g(W^S) = -\chi(S_0^S, T_1^S) = 155$. Using the notation of Proposition 4.3, $S_0 = S_{-1}^S$. The Harder-Narasimhan filtration of $E$ under $W^S$ has shape
\[E^+_{W^S} = (g(W)g(W^S) \cdot S_0^S, g(W) \cdot T_1^S, T_1).\]

The next wall for $E^+_{W^S}$ is the Brill-Noether wall of $S_0^S$, we denote it by $W^{SS}$. It has two stable spherical objects
\[S_0^{SS} \in M_H((1, H, 2)), T_1^{SS} = O_X[1] \in M_H((-1, 0, -1)),\]
and $g(W^{SS}) = -\chi(S_0^{SS}, T_1^{SS}) = 3$. Using the notation of Proposition 4.3, $S_0 = S_{-1}^{SS}$. By Example 9.1, the Harder-Narasimhan filtration of $E$ under $W^{SS}$ has shape
\[E^+_{W^{SS}} = (g(W)g(W^{SS})g(W^{SS}) \cdot S_0^{SS}, g(W)g(W^S) \cdot T_1^{SS}, g(W) \cdot T_1^S, T_1).\]

The next wall for $E^+_{W^{SS}}$ is an actual wall for $T_1$, but it is also the numerical wall defined by $T_1$ and $T_1^S$. Furthermore, $T_1^{SS} = O_X[1]$ is also on the wall. We denote this wall by $W^T$. Then the two stable spherical objects of $W^T$ are
\[S_0^T \in M_H((5, 2H, 1)), T_1^T = O_X[1] \in M_H((-1, 0, -1)),\]
and \( g(W^T) = -\chi(S^T, T^T_1) = 6 \). Using the notation of Proposition 4.3, \( T^{SS}_1 = T^T_1, T^S_1 = T^T_3, \) and \( T_1 = T^T_4 \). The filtration

\[
Q^*_1 = (g(W) g(W^S) \cdot T^T_1, g(W) \cdot T^T_3, T^T_4)
\]

is not two-step (Proposition 6.6), hence Theorem 6.11 cannot be applied.

The next two examples use Theorem 7.1 to test weak Brill-Noether.

**Example 9.5** (weak Brill-Noether fails). Let \( v = (10, 13H, 17) \). Under the notations in Theorem 7.1, one checks that \( v_1 = (1, H, 2) \) satisfies the conditions. In this case, \( \frac{a_{13} - a_{17}}{a_{17} - a_{13}} = 3 \), hence \( y \geq 3 > 1 \). By Theorem 7.1, weak Brill-Noether fails for \( E \in M_H(v) \). In fact, we have \( h^0(E) = 33 \) and \( h^1(E) = 6 \).

**Example 9.6** (weak Brill-Noether holds). Let \( v = (195562, 59615H, 18173) \). Under the notations in Theorem 7.1, one may check that those \( v_1 \) satisfying the conditions are \((10, 3H, 1), (269, 82H, 1), \) and \((7253, 2211H, 674) \). The corresponding \( y \) values are all \( \frac{7}{13} \). Since \( \frac{7}{13} < 1 \), by Theorem 7.1, weak Brill-Noether holds for \( E \in M_H(v) \): we have \( h^0(E) = \chi(E) = 213735 \) and \( h^1(E) = 0 \).

Note that the relation of weak Brill-Noether and the complexity of the Mukai vector is delicate. Weak Brill-Noether can fail for a small Mukai vector such as Example 9.5, or even some smaller ones such as \((2, 3H, 5) \). Weak Brill-Noether can also hold for a large Mukai vector, such as Example 9.6.

As mentioned in Remark 8.8, next we give an example where \( ev_1 \) fails to be surjective.

**Example 9.7** (Non-surjective \( ev_1 \)). Let \( v = (-37666, 22095H, -12961) \) and \( E \in M_H(v) \). Then the next wall \( W \) for \( E \) has stable spherical objects

\[
S_0 \in M_H((12, 5H, 29)), T_1 \in M_H((-29, 17H, -10)),
\]

and \( g(W) = -\chi(S_0, T_1) = 1299 \). Under the notations in Proposition 4.3, \( E = T_2 \). The Harder-Narasimhan filtration of \( E \) under \( W \) is

\[
0 \longrightarrow S_0 \longrightarrow E \longrightarrow T_1 \otimes \text{Ext}^1(T_1, S_0) \longrightarrow 0.
\]

Under the notations in Theorem 8.1, \( ev_1 \) is the connecting homomorphism

\[
ev_1 : H^0(T_1) \otimes \text{Ext}^1(T_1, S_0) \longrightarrow H^1(S_0).
\]

The height of \( S_0 \) and \( T_1 \) are 1 and 2, respectively. By similar arguments as Example 9.1 and Example 9.2, we have

\[
h^0(T_1) = 6, \ h^1(S_0) = 1.
\]

Hence the maximal possible rank of \( ev_1 \) is 1. However, we will show that \( ev_1 \) is in fact zero, which is equivalent to the adjoint map \( f : \text{Ext}^1(T_1, S_0) \longrightarrow H^0(T_1)^* \otimes H^1(S_0) \) being zero.

The next wall \( W^T \) for \( T_1 \) has stable spherical objects

\[
S^T_0 \in M_H((1, 2H, 5)), T^T_1 \in M_H((-2, H, -1)),
\]

and the next wall for \( S_0 \) has stable spherical objects

\[
S^S_0 \in M_H((1, 2H, 5)), T^S_1 = \mathcal{O}_X[1].
\]

Hence \( S^T_0 = S^S_0 \). Under the notations in Proposition 4.3, \( S_0 = S^S_0 \). Hence by Theorem 4.10, we have

\[
\text{Ext}^1(S^T_0, S_0) = \text{Ext}^1(S^S_0, S^S_0) = 0.
\]
Also note that \( H^0(T_1^T) = 0 \) by a direct computation.

By Corollary 4.13, the map \( f \) fits into a commutative diagram

\[
\begin{array}{c}
\text{Hom}(T_1, T_1^T) \otimes \text{Ext}^1(T_1^T, S_0) \\
\downarrow \\
\text{Hom}(S_0^T, T_1)^* \otimes \text{Ext}^1(S_0^T, S_0) = 0
\end{array}
\]

\[
0 = \text{Hom}(T_1, T_1^T) \otimes H^0(T_1^T)^* \otimes H^1(S_0) \\
\downarrow \quad f \\
H^0(T_1)^* \otimes H^1(S_0) \quad \text{Hom}(S_0^T, T_1)^* \otimes H^0(S_0^T) \otimes H^1(S_0).
\]

Hence \( f = 0 \), \( ev_1 \) is not surjective.

The following example shows that in Proposition 4.2, item (2) can happen.

**Example 9.8** (Negative definite rank 2 lattice). Let \((X, H)\) be a K3 surface that contains a line \( L \). Then \( L \) has no deformation, \( \dim |L| = 0 \). Consider the following short exact sequence in the heart \( \mathcal{A}_b \) (Section 3):

\[
0 \to \mathcal{O}_X(L) \to \mathcal{O}_L(-2) \to \mathcal{O}_X[1] \to 0.
\]

By Proposition 4.1, the sequence defines an actual wall for \( \mathcal{O}_L(-2) \), whose stable spherical objects are \( \mathcal{O}_X(L) \) and \( \mathcal{O}_X[1] \). We have \( \chi(\mathcal{O}_X(L), \mathcal{O}_X[1]) = -1 \), hence the associated lattice is negative definite.

10. References

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