Tensor products of \textit{NCDL} -- $C^*$-algebras and the $C^*$-algebra of the Heisenberg motion groups $\mathbb{T}^n \ltimes \mathbb{H}_n$.

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Abstract

We show that the tensor product $A \otimes B$ over $\mathbb{C}$ of two $C^*$-algebras satisfying the \textit{NCDL} conditions has again the same property. We use this result to describe the $C^*$-algebra of the Heisenberg motion groups $G_n = \mathbb{T}^n \ltimes \mathbb{H}_n$ as algebra of operator fields defined over the spectrum of $G_n$.

1 Introduction.

1.1

The family of $C^*$-algebras with norm controlled dual limits (\textit{NCDL}) was introduced in [11]. It is shown in section 2 that the tensor product of two \textit{NCDL}-$C^*$-algebras is again \textit{NCDL}. In section 3 we introduce the groups $G_n = \mathbb{T}^n \ltimes \mathbb{H}_n, n \in \mathbb{N}^*$, the semi-direct product of the torus $\mathbb{T}^n$ acting on the $2n + 1$ dimensional Heisenberg group $\mathbb{H}_n$. We recall the topology of the spectrum of the groups $G_n$ and the Fourier transform of their group $C^*$-algebras. In section 4 the norm control of dual limits is then computed explicitly for the group $G_1$. This is the main result of the paper. In the last section, the structure of the $C^*$-algebra of $G_n$ is obtained by combining the general results on tensor products of $C^*$-algebras and the properties of the algebra of operator fields $\mathcal{F}(C^*(G_1))$.

1.2 $C^*$-algebras with norm controlled dual limits

Definition 1.1.

- Let $S$ be a topological space. We say that $S$ is \textit{locally compact of step} $\leq d$, if there exists a finite increasing family $\emptyset \neq S_0 \subset S_1 \subset \cdots \subset S_d = S$ of closed subsets of $S$, such that the subsets $\Gamma_0 = S_0$ and $\Gamma_i := S_i \setminus S_{i-1}$, $i = 1, \ldots, d$, are locally compact and Hausdorff in their relative topologies.

- Let $S$ be locally compact of step $\leq d$, and let $\{\mathcal{H}_i\}_{i=1, \ldots, d}$ be Hilbert spaces. For a closed subset $M \subset S$, denote by $CB(M, \mathcal{H}_i)$ the unital $C^*$-algebra of all uniformly bounded operator fields $(\psi(\gamma) \in B(\mathcal{H}_i))_{\gamma \in M \cap \Gamma_i, i = 1, \ldots, d}$, which are operator norm continuous on the subsets $\Gamma_i \cap M$ for every $i \in \{0, \ldots, d\}$ with $\Gamma_i \cap M \neq \emptyset$ and such that $\gamma \mapsto \psi(\gamma)$ goes to 0 in operator norm if $\gamma$ goes to infinity on $M$. We provide the algebra $CB(M, \mathcal{H}_i)$ with the infinity-norm

$$||\varphi||_M = \sup \{||\varphi(\gamma)||_{B(\mathcal{H}_i)} \mid M \cap \Gamma_i \neq \emptyset, \gamma \in M \cap \Gamma_i\}.$$ 

- Let $S$ be a set. Choose for every $s \in S$ a Hilbert space $\mathcal{H}_s$. We define the $C^*$-algebra $l^\infty(S)$ of uniformly bounded operator fields defined over $S$ by

$$l^\infty(S) := \{((\phi(s))_{s \in S} \mid \phi(s) \in B(\mathcal{H}_s), s \in S, \sup_{s \in S} ||\phi(s)||_{op} < \infty\}.$$ 

Here $B(\mathcal{H})$ denotes the algebra of bounded linear operators on the Hilbert space $\mathcal{H}$.

Definition 1.2. Let $\mathcal{A}$ be a separable liminary $C^*$-algebra, such that the spectrum $\mathring{\mathcal{A}}$ of $\mathcal{A}$ is a locally compact space of step $\leq d$,

$$\emptyset = S_{-1} \subset S_0 \subset S_1 \subset \cdots \subset S_d = \mathring{\mathcal{A}}.$$ 

Suppose that for $0 \leq i \leq d$ there is a Hilbert space $\mathcal{H}_i$, and for every $\gamma \in \Gamma_i$ a concrete realization $(\pi_\gamma, \mathcal{H}_i)$ of $\gamma$ on the Hilbert space $\mathcal{H}_i$ and that the set $S_0$ is the collection of all characters of $\mathcal{A}$. 

1
Denote by $F : A \to l^\infty(\hat{A})$ the Fourier transform of $A$ i.e. for $a \in A$ let

$$F(a)(\gamma) = \hat{a}(\gamma) := \pi_\gamma(a) \in B(H_\gamma), \gamma \in \Gamma_i, i = 0, \cdots, d.$$ 

We say that $F(A)$ is continuous of step $\leq d$ $F(A)|_{\Gamma_i}$ is contained in $CB(\hat{A}, H_\gamma)$ for every $i$.

**Definition 1.3.** Let $A$ be a separable liminary $C^*$-algebra.

We say that the $C^*$-algebra $A$ has norm controlled dual limits (NCDL) if the spectrum $\hat{A}$ is continuous of step $\leq d$ for some $d \in \mathbb{N}$ $(0 = S_{-1} \subset S_0 \subset S_1 \subset \cdots \subset S_d = \hat{A})$ and

1. $F(A)$ is continuous of step $\leq d$.

2. For any $i = 1, \ldots, d$ and for any converging sequence contained in $\Gamma_i$ with limit set contained in $S_{i-1}$, there exists a properly converging sub-sequence $\gamma = (\gamma_k)_{k \in \mathbb{N}}$ with limit set $L(\gamma) \subset S_{i-1}$, a constant $C > 0$ and for every $k \in \mathbb{N}$ an involutive linear mapping $\sigma_{\gamma,k} : F(A)|_{L(\gamma)} \to B(H_\gamma)$, that is bounded by $C$, such that

$$\lim_{k \to \infty} \|F(a)(\gamma_k) - \sigma_{\gamma,k}(F(a))\|_{B(H_\gamma)} = 0.$$

**Remark 1.4.** It turns out that the norm control of dual limits is a consequence of the properties of liminary $C^*$-algebras with continuous Fourier transform of finite step:

**Theorem 1.5** (see [2]). Assume that for the separable liminary $C^*$-algebra $A$, its Fourier transform $F(A)$ is continuous of step $\leq d$ for some $d \in \mathbb{N}$. Then $A$ has norm controlled dual limits. Specifically let $1 \leq \ell \leq d$ be fixed, and $\gamma = (\gamma_k)_{k \in \mathbb{N}}$ be a properly convergent sequence with limit set $L(\gamma) \subset \Gamma_\ell$. Then there exists a sequence $(\sigma_{\gamma,k})$ of completely positive and completely contractive maps $\sigma_{\gamma,k} : \hat{A}|_{L(\gamma)} \to B(H_\gamma)$ such that

$$\lim_{k \to \infty} \|F(a)(\gamma_k) - \sigma_{\gamma,k}(F(a))\|_{B(H_\gamma)} = 0.$$

**Remark 1.6.** Let us mention that any NCDL $C^*$-algebra $A$ is determined up to an isomorphism by its spectrum $\hat{A}$ and the family $\sigma_\gamma (\gamma \text{ any properly converging sequence in } \hat{A})$ of norm controls (see [2] and [11]). Therefore in order to determine the structure of a given NCDL $C^*$-algebra and to understand its Fourier transform, it is essential to have a precise description of its spectrum and of its norm control of dual limits.

Let $A$ be a separable $C^*$-algebra and let $\gamma = (\gamma_k)_{k \in \mathbb{N}}$ be a properly converging sequence of irreducible unitary representations of $A$. Let $L$ be the set of limits in $\hat{A}$ of the sequence $\gamma$. By definition of the topology of $\hat{A}$, (see [3]), there exists for every $\sigma \in L$ and every element $\xi\sigma$ in the Hilbert space $H_{\sigma}$ of $\sigma$ a sequence $(\xi_k \in H_{\gamma_k})_{k \in \mathbb{N}}$, such that the sequence of coefficients $c_{\xi_k}$ converges weakly to the coefficient $c_{\xi\sigma}$. This means that

$$\langle \sigma(a)\xi\sigma, \xi\sigma \rangle_{H_{\sigma}} = \lim_{k \to \infty} \langle \gamma_k(a)\xi_k, \xi_k \rangle_{H_{\gamma_k}}, \quad a \in A.$$

By a theorem of Fell ([3]) we have that

$$\lim_{k \to \infty} \|\pi_k(a)\|_{op} = \sup_{\sigma \in L} \|\sigma(a)\|_{op}, \quad a \in A.$$

In this paper these sequences $(\xi_k)_{k}$ are explicitly determined for generic sequences $\gamma_k \subset \hat{G}_1$ and it is shown how to construct the control from these data (see the proof of theorem 1.5).

## 2 Tensor Products of $C^*$-algebras and NCDL $C^*$-algebras.

### 2.1 Tensor products

(see [1] for details) If $A$ and $B$ are $C^*$-algebras, we can form their algebraic tensor product $A \otimes B$ over $\mathbb{C}$. The vector space $A \otimes B$ has a natural structure as a $*$-algebra with multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2.$$
and involution \((a \otimes b)^* = a^* \otimes b^*\). If \(\gamma\) is a \(C^*\)-norm on \(A \otimes B\), we will write \(A \otimes \gamma B\) for the completion. If \(\pi_A\) and \(\pi_B\) are representations of \(A\) and \(B\) on Hilbert spaces \(H_1\) and \(H_2\) respectively, we can form the representation \(\pi = \pi_A \otimes \pi_B\) of \(A \otimes B\) on \(H_1 \otimes H_2\) by \(\pi(a \otimes b) = \pi_A(a) \otimes \pi_B(b)\). For any \(\pi_A\) and \(\pi_B\) we have \(||(\pi_A \otimes \pi_B)(\sum_{i=1}^n a_i \otimes b_i)|| \leq \sum_{i=1}^n ||a_i|| ||b_i||\), so the norm \(||\sum_{i=1}^n a_i \otimes b_i||_{\text{min}}\) is finite and hence a \(C^*\)-norm called the minimal \(C^*\)-norm. The completion of \(A \otimes B\) with respect to this norm is written \(A \otimes_{\text{min}} B\) and called the minimal or spatial tensor product of \(A\) and \(B\). For any representation \(\pi\) of \(A \otimes B\) we have \(||\pi(\sum_{i=1}^n a_i \otimes b_i)|| \leq \sum_{i=1}^n ||a_i|| ||b_i||\), so the norm \(||\sum_{i=1}^n a_i \otimes b_i||_{\text{max}}\) is finite and hence a \(C^*\)-norm called the maximal \(C^*\)-norm. The completion is denoted \(A \otimes_{\text{max}} B\), and called the maximal tensor product of \(A\) and \(B\).

Let \(A\) a \(C^*\)-algebra, \(A\) is called nuclear if for every \(C^*\)-algebra \(B\), there is a unique \(C^*\)-norm on \(A \otimes B\), i.e \(A \otimes_{\text{min}} B = A \otimes_{\text{max}} B\). We write for this nuclear \(C^*\)-algebra \(A\) and any other \(C^*\)-algebra \(B\)

\[C = A \otimes B\]

for the \(C^*\)-algebra \(A \otimes_{\text{max}} B = A \otimes_{\text{min}} B\).

Every type I \(C^*\)-algebra is nuclear.

**Theorem 2.1.** ([1] IV.3.4.21) If \(A\) and \(B\) are arbitrary \(C^*\)-algebras, there is an injective map

\[\Pi : \widehat{A} \times \widehat{B} \to \widehat{A \otimes B}\]

given by \(\Pi(\rho, \sigma) = \rho \otimes \sigma\). This is a continuous map relative to the natural topologies and drops to a well-defined map, also denoted \(\Pi\), from \(\text{Prim}(A) \times \text{Prim}(B)\) to \(\text{Prim}(A \otimes B)\); this \(\Pi\) is injective, continuous (it preserves containment in the appropriate sense), and its range is dense in \(\text{Prim}(A \otimes B)\) since the intersection of the kernels of the representations \(\{\rho \otimes \sigma : \rho \in \widehat{A}, \sigma \in \widehat{B}\}\) is 0.

**Theorem 2.2.** ([1] IV.3.4.27) If \(\pi \in \widehat{A \otimes B}\) and \(\pi_A\) or \(\pi_B\) is of type I, the other is also type I and \(\pi \cong \rho \otimes \sigma\) for \(\rho, \sigma\) irreducible representations quasi-equivalent to \(\pi_A\) and \(\pi_B\) respectively. Thus, if \(A\) or \(B\) is type I (no separability necessary), the map \(\Pi : \widehat{A} \times \widehat{B} \to \widehat{A \otimes B}\) is surjective and it is easily verified to be a homeomorphism.

**Theorem 2.3.** Let \(A\) and \(B\) two NCDL-\(C^*\)-algebras. Then the tensor product \(A \otimes B\) over \(\mathbb{C}\) is also a NCDL-\(C^*\)-algebra.

**Proof.** Let \(A, B\) be two NCDL-\(C^*\)-algebras. Since both algebras are by definition liminary, they are both of type I. Furthermore \(C\) is separable, since so are \(A\) and \(B\). By theorem 2.2 we have that \(C \simeq A \times B\) and so \(C\) is liminary too. We can write the spectrum of \(C\) in the following way. There are increasing finite families \(S^A_0 \subset S^A_1 \subset \cdots \subset S^A_n = \widehat{A}\), \(n \in \mathbb{N}\), resp. \(S^B_0 \subset S^B_1 \subset \cdots \subset S^B_m = \widehat{B}\), \(m \in \mathbb{N}\), of closed subsets of the spectrum \(\widehat{A}\) of \(A\), resp. of the spectrum \(\widehat{B}\) of \(B\), such that the subsets \(\Gamma^A_j := S^A_j \setminus S^A_{j-1}\), \(j = 1, \cdots, m\), resp. \(\Gamma^B_j := S^B_j \setminus S^B_{j-1}\), \(j = 1, \cdots, m\) have a separated relative topology. Then the spectrum of \(C\) is the disjoint union of the subsets

\[\Gamma^C_{i,j} = \Gamma^A_i \times \Gamma^B_j, 0 \leq i \leq n, 0 \leq j \leq m,\]

and each subset \(\Gamma^C_{i,j}\) is locally compact with a Hausdorff topology. Let for \(k \in \{0, \cdots, n + m\}\)

\[T^C_k := \bigcup_{i+j \leq k} \Gamma^C_{i,j},\]

Then for every \(0 \leq i \leq n, 0 \leq j \leq m\), the subset

\[R^C_{i,j} := \Gamma^C_{i,j} \cup T^C_{i+j-1}\]

is closed in \(\widehat{C}\). For every \(0 \leq k \leq n + m\) we choose a total order on the family

\[R^C_k := \{\Gamma^C_{i,j} | i + j = k\}\]

and we say that \(\Gamma^C_{i,j} \subset \Gamma^C_{i',j'}\) whenever \(i + j < i' + j'\). This gives us a total order on the family of sets \(\{\Gamma^C_{i,j} | 0 \leq i \leq n, 0 \leq j \leq m\}\). Furthermore

\[\widehat{A} \times \widehat{B} = \bigcup_{i,j} \Gamma^C_{i,j}\]
and the subsets
\[ S_{i,j}^C := \bigcup_{(i',j') \leq (i,j)} \Gamma_{i',j'}^C \]
are closed in \( \hat{C} \) for any pair \((i,j)\).

It is easy to see that \( C \) has all the required properties to be NCDL. Indeed for \( c \in C \) it is immediately seen that the operator fields \( \hat{c} \) defined on \( \hat{C} \) by \( \hat{c}(\rho \otimes \sigma) = \rho \otimes \sigma(c) \) for \( \rho \in A \), \( \sigma \in B \) operate on the Hilbert-spaces \( \mathcal{H}_i \otimes \mathcal{H}_j \) and it is continuous on the different subsets \( \Gamma_{i,j}^C \) and tends to 0 at infinity, since this is true for elementary tensors \( a \otimes b \).

Hence the Fourier transform of \( A \otimes \hat{B} \) is continuous of some finite step and so Theorem 1.5 tells us that \( C \) has the NCDL property.

Let us see how to build the norm control of dual limits for \( C = \Lambda \otimes B \).

1. If \( (\gamma^C)_k \) is a sequence in \( \Gamma_{i,j}^C \) which admits its limits in \( T^C_{i+j-2} \), then for a properly convergent subsequence we have
\[ \gamma^C_k = \gamma^A_k \otimes \gamma^B_k \in \hat{C}, \]
where \( (\gamma^A_k, \gamma^B_k) \) is a sequence of \( \Gamma_i^A \times \Gamma_j^B \) for some \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \), which converges to its limit set \( L \left( (\gamma^A)_k \right) \times L \left( (\gamma^B)_k \right) \) in \( S_{i-1}^A \times S_j^B \subset \hat{A} \times \hat{B} \). Since \( A \) and \( B \) are nuclear, we have that
\[ \mathcal{F}(C)_{|L(\hat{C})} = \mathcal{F}(A)_{|L(\hat{A})} \boxtimes \mathcal{F}(B)_{|L(\hat{B})}. \]

Let \( (\sigma_{\gamma^C_k})_k : \mathcal{F}(A)_{|L(\hat{A})} \to B(\mathcal{H}_i)_k \), (resp. \( (\sigma_{\gamma^C_k})_k : \mathcal{F}(B)_{|L(\hat{B})} \to B(\mathcal{H}_j)_k \)) be the sequence of uniformly bounded linear mappings that comes from the NCDL property for \( A \) (resp. for \( B \)). Define then for all \( k \in \mathbb{N} \):
\[ \sigma_{\gamma^C_k}(\phi^A \otimes \phi^B) = \sigma_{\gamma^C_k}(\phi^A) \otimes \sigma_{\gamma^C_k}(\phi^B) \]
on elementary tensors. This definition can be extended in a unique way to a bounded selfadjoint linear mapping
\[ \sigma_{\gamma^C_k} : \mathcal{F}(C)_{|L(\hat{C})} \to B(\mathcal{H}_i \otimes \mathcal{H}_j), k \in \mathbb{N}. \]

Then we see that for all finite sums \( c = \sum_l a_l \otimes b_l \in C \):
\[ \lim_{k \to \infty} \| \hat{c}(\gamma^C)_k - \sigma_{\gamma^C_k}(\hat{c})_{|L(\hat{C})} \|_{op} = \lim_{k \to \infty} \| \sum_l \hat{a}_l(\gamma^A)_k \otimes \hat{b}_l(\gamma^B)_k - \sigma_{\gamma^C_k}(\sum_l \hat{a}_l \otimes \sum_l \hat{b}_l) \|_{op}, \]

Furthermore
\[ \| \sigma_{\gamma^C_k}(c) \|_{op} = \sum_l \sigma_{\gamma^C_k}(\hat{a}_l \otimes \hat{b}_l) \]
\[ \leq \sum_l \| \sigma_{\gamma^C_k}(\hat{a}_l) \|_{op} \| \sigma_{\gamma^C_k}(\hat{b}_l) \|_{op}, \]
\[ \leq \beta_{\gamma^A} \beta_{\gamma^B} \sum_l \| \hat{a}_l \|_{\gamma^A} \| \hat{b}_l \|_{\gamma^B}, \]
for some constants \( \beta_{\gamma^A} > 0 \) and \( \beta_{\gamma^B} > 0 \).

Hence
\[ \| \sigma_{\gamma^C_k}(c) \|_{op} \leq \beta_{\gamma^A} \beta_{\gamma^B} \| \mathcal{F}(c)_{|L(\hat{C})} \|, \]

2. Similarly, if \( (\gamma^C)_k \) is a sequence in \( \Gamma_{i,j}^C \) which admits its limits in \( S_{i-1}^A \otimes \Gamma_j^B \) then for a properly convergent subsequence we have
\[ \gamma^C_k = \gamma^A_k \otimes \gamma^B_k \in \hat{C}, \]
where \( (\gamma^A_k \otimes \gamma^B_k)_k \) is a sequence of \( \Gamma_i^A \times \Gamma_j^B \) for some \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \) which converges to its limit set \( L \left( (\gamma^A)_k \right) \times \{ \gamma^B \} \) in \( S_{i-1}^A \times \Gamma_j^B \subset \hat{A} \times \hat{B} \) for some \( \gamma^B \in \Gamma_j^B \).
Let \((\sigma_{\gamma_j}^j) : CB(S_{i-1}^A) \to B(H_i)_k\), be the sequence of uniformly bounded linear mappings that comes from the NCDL property for \(A\). Define then for all \(k \in \mathbb{N}\):
\begin{align*}
\gamma_k^C : CB(T_{i+j-1}^C) &\to B(H_i \times H_j) : \gamma_k^C(\phi^A \otimes \phi^B) = \gamma_k^A(\phi^A) \otimes \phi^B(\gamma^B).
\end{align*}
Then we see again for all finite sums \(c = \sum_i a_i \otimes b_i \in C\) that:
\begin{align*}
\lim_{k \to \infty} \|\bar{\sigma}_{\gamma_j}^j(c_{[r-1]}^C)\|_{op} &= \lim_{k \to \infty} \|\sum_j \bar{a}_j(\gamma_k^A) \otimes \bar{b}_j(\gamma_k^B) - \bar{\sigma}_{\gamma_j}^j(\gamma_k^A) \otimes \bar{b}_j(\gamma_k^B)\|_{op} \\
&= 0.
\end{align*}

\[\square\]

**Example 2.4.** [The \(C^*\)-algebra of a direct product of a locally compact group second countable and a second countable abelian group]

Let \(G\) a second countable locally compact and \(A\) a locally compact second countable abelian group.

Let \(\hat{G} := G \times A\) be the direct product of \(G\) and \(A\). Then
\begin{align*}
\hat{G} = \hat{G} \times \hat{A}.
\end{align*}

For \(F \in C_c(\hat{G})\) we define the application
\begin{align*}
\hat{F}^A : \hat{A} &\to L^1(G); \hat{F}^A(\chi)(g) := \int_A F(g,a)\chi(a)da, \ g \in G, \chi \in \hat{A}.
\end{align*}

Then \(\hat{F}^A\) is a continuous mapping which extends for all \(b \in C^*(\hat{G})\) into a continuous mapping
\begin{align*}
\hat{b}^F : \hat{A} &\to C^*(G).
\end{align*}

If now \(C^*(G)\) is NCDL, we can write the spectrum of \(C^*(\hat{G})\) the following way:
There is an increasing finite family \(S_0 \subset S_1 \subset \cdots \subset S_d = \hat{G}\) of closed subsets of the spectrum \(\hat{G}\) of \(G\) such that for all \(i = 1, \cdots, d\), the subset \(\Gamma_i = S_i\) and \(\Gamma_i := S_i\backslash S_{i-1}\), where \(\Gamma_i\) are separated for their relative topology. Then the spectrum of \(\hat{G}\) is the disjoint union of the subsets \(S_j := S_j \times \hat{A}\), \(j = 0, \cdots, d\) and for all \(j = 1, \cdots, d\), the subset \(\Gamma_j := S_j\backslash S_{j-1}\) has Hausdorff relative topology.

It is easy to see that \(C^*(G)\) has all the required properties to be NCDL. Indeed for \(F \in C_c(\hat{G})\) it is immediate to see that the operator fields \(\hat{F}\) defined on \(\hat{G}\) by \(\hat{F}(\pi \times \chi) = (\pi \times \chi)(F) = \pi(\hat{F}^A(\chi)), \ \pi \in \hat{G}, \ \chi \in \hat{A}\), are continuous on the different subsets \(\Gamma_j\) and they go to 0 at infinity.

### 3 The \(C^*\)-algebra of the Heisenberg motion groups \(G_n\).

The structure of the group \(C^*(G_n)\) realized as algebra of operator fields defined over the spectrum \(\hat{G}\) of \(G\) is already known for certain classes of Lie groups, such as the Heidelberg and the thread-like Lie groups (see [13]) and the \(ax + b\)-like groups (see [9]). Furthermore, the \(C^*\)-algebras of the 5-dimensional nilpotent Lie groups have been determined in [11], while those of all 6-dimensional nilpotent Lie groups have been characterized in [15] and the \(C^*\)-algebras of the two-step nilpotent Lie groups have been determined in [14]. Furthermore it follows from general principles that the \(C^*\)-algebra of any connected nilpotent Lie group has the NCDL-property (see [2]). The description of the \(C^*\)-algebra of the motion group \(SO(n) \times \mathbb{R}^n\) has been done in [10].

#### 3.1 The Heisenberg motion groups \(G_n\).

We denote by \(\text{diag}(\gamma_1, \cdots, \gamma_n)\) a diagonal matrix in \(\text{Mat}(n, \mathbb{C})\) with numbers \(\gamma_1, \cdots, \gamma_n\). Let \(\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}\) denote the \((2n+1)\)-dimensional Heisenberg group, with group law
\begin{align*}
(\mathbb{z},t)(\mathbb{z}',t') = \left(z+z', t+t' - \frac{1}{2}\text{Im}(z \cdot \bar{z}')\right), \ z, z' \in \mathbb{C}^n, \ t, t' \in \mathbb{R},
\end{align*}
where $\text{Im}(z)$ is the imaginary part of $z$ in $\mathbb{C}^n$ and $z \cdot z' := \sum_{j=1}^n z_j w_j$.

The group $\mathbb{T}^n$ acts naturally on $\mathbb{H}_n$ by automorphisms as follows

$$e^{i\theta}(z, t) := (e^{i\theta} z, t),$$

where $e^{i\theta} = (e^{i\theta_1}, \cdots, e^{i\theta_n}) \in \mathbb{T}^n$.

Let $G_n$ be the semi-direct product $\mathbb{T}^n \ltimes \mathbb{H}_n$, equipped with the following group law:

$$(e^{i\theta}, z, t) \cdot (e^{i\theta'}, z', t') := \left( e^{i(\theta + \theta')}, z + e^{i\theta} z', t + t' - \frac{1}{2} \text{Im}(z \cdot e^{i\theta} z') \right), \forall e^{i\theta}, e^{i\theta'} \in \mathbb{T}^n \text{ and } (z, t), (z', t') \in \mathbb{H}_n.$$

For $z \in \mathbb{C}^n$, we introduce the $\mathbb{R}$-linear form $z^*$ on $\mathbb{C}^n$ defined by

$$z^*(w) := \text{Im}(z \cdot \overline{w})$$

and we identify the algebraic dual of the Lie algebra $\mathfrak{t}_n$ of $\mathbb{T}^n \subset U(n)$ with $i\mathbb{R}^n$ via the scalar product

$$iA \cdot iB = -\sum A_j B_j, \quad iA, iB \in \mathfrak{t}_n.$$

We have a map

$$\times : \quad \mathbb{C}^n \times \mathbb{C}^n \quad \longrightarrow \quad \mathbb{R}^n$$

given by

$$z \times w(A) := w^*(Az) = \text{Im}(w \cdot \overline{Az}) = \sum_{j=1}^n \text{Re}(w_j \overline{A_j}) A_j, \quad A = (iA_1, \cdots, iA_n) \in \mathfrak{t}_n.$$

It follows easily from the group law in $G_n$ that the coadjoint representation $\text{Ad}^*$ of $G_n$ is given by

$$\text{Ad}^*(e^{i\theta}, z, t)(U, u, x) = \left( U + z \times (e^{i\theta} u) + \frac{x}{2} \times z, e^{i\theta} u + xz, x \right),$$

for all $(U, u, x) \in \mathfrak{g}_n$. Therefore the coadjoint orbit of $G_n$ through $(U, u, x)$ is given by

$$O_{(U, u, x)} = \text{Ad}^*(G_n)(U, u, x) = \left\{ (U + z \times (e^{i\theta} u) + \frac{x}{2} \times z, e^{i\theta} u + xz, x) ; \ e^{i\theta} \in \mathbb{T}^n, z \in \mathbb{C}^n \right\}.$$

**Definition 3.1.** Let for $0 \neq r = (r_1, \cdots, r_n) \in \mathbb{R}^n_+$

$$I_r := \{ j \in \{1, \cdots, n\} \mid r_j \neq 0 \}$$

and

$$Z_r := \{ (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_j = 0, \forall j \in I_r \}.$$

Let also

$$T_r := \{ e^{i\theta} \in \mathbb{T}^n \mid \theta = (\theta_1, \cdots, \theta_n), \theta_j \in \mathbb{Z}, \forall j \in I_r \}.$$

Then $T_r$ is a closed connected subgroup $\mathbb{T}^n$ isomorphic to $\mathbb{T}^{|I_r|}$.

Furthermore, the subgroup $T_r$ is the stabilizer group of the linear functional $\ell_r = (ir_1, \cdots, ir_n) \in \mathfrak{t}_n^*$ and $Z_r$ describes the spectrum of the group $T_r^\perp$.

It follows that the space $\mathfrak{g}_n^\perp / G_n$ of admissible coadjoint orbits of $G_n$ is the union of the set $\Gamma^2$ of all orbits

$$O_{(\lambda, \alpha)} = \left\{ (i(\lambda_1 - \frac{\alpha}{2}(x_1)^2), \cdots, i(\lambda_n - \frac{\alpha}{2}(x_n)^2), \alpha e^{i\theta} x, \alpha) ; \ e^{i\theta} \in \mathbb{T}^n, \ x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+ \right\}.$$
Theorem 3.2.

We can thus write the space \( O_{\lambda, r} = \{ i(\lambda + z \times (e^{i\theta} r)), e^{i\theta} r, 0); e^{i\theta} \in \mathbb{T}^n, z \in \mathbb{R}^n, r \in \mathbb{R}^n_+, r \neq 0, \lambda \in \mathbb{Z}^n_r \}. \) (this corrects a mistake in [8],) and of the set \( \Gamma^0 \) of all the one point orbits

\[ O_{\lambda} = \{ (i\lambda, 0, 0) \}, \lambda \in \mathbb{Z}^n. \]

In the case of \( \Gamma^1 \), we parametrize its orbits by

\[ \Gamma^1 = \{ (i\lambda, r, 0); r \in \mathbb{R}^n_+, r \neq 0, \lambda \in \mathbb{Z}^n_r \}. \]

We can thus write the space \( \mathfrak{g}^+_n/G_n \) of admissible coadjoint orbits of \( G_n \) as the disjoint union

\[ \mathfrak{g}^+_n/G_n = \Gamma^2 \cup \Gamma^1 \cup \Gamma^0 = (i\mathbb{Z}^n \times \mathbb{R}^*) \bigcup_{r \in \mathbb{R}^n_+, r \neq 0, \lambda \in \mathbb{Z}^n_r} (i\lambda, r) \bigcup_{\lambda \in \mathbb{R}} \ell_\lambda. \]

The topology of the space \( \mathcal{G}_1 \) has been determined in [5] and of the space \( G_n \) in [8] (at least partially). We have the following description of the topology of \( \mathfrak{g}^+_n/G_n \).

Theorem 3.2.

1. Let \((\mathcal{O}(\lambda^k, \alpha_k))_k \) be a sequence of admissible coadjoint orbits of \( G_n \). Then

   (a) \((\mathcal{O}(\lambda^k, \alpha_k))_k \) converges to \( \mathcal{O}_{\lambda, \alpha} \) in \( \mathfrak{g}^+_n/G_n \) if and only if the sequence \((\alpha_k)_k \) converges to \( \alpha \) and \( \lambda^k = \lambda \) for \( k \) large enough.

   (b) \((\mathcal{O}(\lambda^k, \alpha_k))_k \) converges to \( \mathcal{O}_{\lambda, r} \) in \( \mathfrak{g}^+_n/G_n \) if and only if the sequence \((\alpha_k)_k \) converge to zero, for all \( j \in I_r \), \( \alpha_k \lambda^k_j \) tends to \( \frac{r_j^2}{2} \) and for all \( j \notin I_r \), \( \alpha_k \lambda^k_j \) tends to \( 0 \) and \( \alpha_k (\lambda^k_j - \lambda_j) \geq 0 \) as \( k \to +\infty \).

2. Let \((\mathcal{O}(\lambda^k, r_k))_k \) be a sequence of admissible coadjoint orbits of \( G_n \) such that \( I_{r_k} = I \) is constant. Then \((\mathcal{O}(\lambda^k, r_k))_k \) converges to \( \mathcal{O}_{\lambda, r} \in \Gamma^1 \) if and only if \( \lim_{k \to \infty} r_k = r \) and \( \lambda^k_j = \lambda_j \) for all \( j \notin I_r \),

3. Let \((\mathcal{O}(\lambda^k, r_k))_k \) be a sequence of admissible coadjoint orbits of \( G_n \) such that \( I_{r_k} = I \) is constant. Then \((\mathcal{O}(\lambda^k, r_k))_k \) converges to \( \mathcal{O}_{\lambda, r} \in \Gamma^0 \) if and only if \( \lim_{k \to \infty} r_k = 0 \) and \( \lambda^k_j = \lambda_j \) for all \( j \notin I_r \).

Proof.

1. (a) See Theorem 5 in [8].

   (b) i. If \( I_r = \{1, \cdots, n\} \), see Theorem 4 in [8].

      ii. If \( I_r = \emptyset \), see Theorem 3 in [8].

      iii. If \( I_r \subseteq \{1, \cdots, n\} \), assume that \((\mathcal{O}(\lambda^k, \alpha_k))_k \) converges to \( \mathcal{O}_{\lambda, r} \). Then there exist two sequences \((e^{i\theta})_k \subset \mathbb{T}^n \) and \((x^k)_k \subset \mathbb{R}^n_+ \) such that

\[
\begin{align*}
\alpha_k & \to 0, \\
\alpha_k e^{i\theta^k} x^k_j & \to r_j, \forall j \in I_r \\
\alpha_k e^{i\theta^k} x^k_j & \to 0, \forall j \notin I_r \\
\lambda^k_j - \frac{\alpha_k}{2} (x^k_j)^2 & \to 0, \forall j \in I_r \\
\lambda^k_j - \frac{\alpha_k}{2} (x^k_j)^2 & \to \lambda_j, \forall j \notin I_r
\end{align*}
\]

(3.1)

We have \( |\alpha_k| x^k_j \to r_j \) for all \( j \in I_r \) and \( |\alpha_k| x^k_j \to 0 \) for all \( j \notin I_r \). Since \( \alpha_k (\lambda^k_j - \frac{\alpha_k}{2} (x^k_j)^2) \to 0 \) for all \( j \in \{1, \cdots, n\} \), we immediately see that the sequence \((\alpha_k \lambda^k_j)_k \) tends to \( \frac{r^2_j}{2} \) for all \( j \in I_r \) and \((\alpha_k \lambda^k_j)_k \) tends to zero for all \( j \notin I_r \). We also have \( \alpha_k (\lambda^k_j - \lambda_j) \geq 0 \) for large \( k \) for all \( j \notin I_r \).

Conversely, let us assume that \((\alpha_k)_k \) converges to zero, for all \( j \in I_r \), \((\alpha_k \lambda^k_j)_k \) converges to \( \frac{r^2_j}{2} \) and for all \( j \notin I_r \), \((\alpha_k \lambda^k_j)_k \) converges to zero and \( \alpha_k (\lambda^k_j - \lambda_j) \geq 0 \). Then for \( k \) large enough we can define for all \( j \in I_r \) the sequence \( x^k_j = \sqrt{\frac{2\lambda_j}{\alpha_k}} \). We see that for all \( j \in I_r \) \( \alpha_k e^{i\theta^k} x^k_j \to r_j \). For all \( j \notin I_r \) we have assumed that the sequence \((\alpha_k \lambda^k_j)_j \) converges to zero and \( \alpha_k (\lambda^k_j - \lambda_j) \geq 0 \) for large \( k \). Take for all \( j \notin I_r \) \( x^k_j = \sqrt{\frac{2\lambda_j}{\alpha_k}} (\lambda^k_j - \lambda_j) \).

This the sequence \((\mathcal{O}(\lambda^k, \alpha_k))_k \) converges to \( \mathcal{O}_{\lambda, r} \) in \( \mathfrak{g}^+_n/G_n \).
2. The orbits $O_{\lambda, r}$ are direct products of orbits of $T \times H_1$. It suffices to apply Lemma 6 in [5].

As a consequence we obtain (see [4], [10] and [8])

\textbf{Theorem 3.3.} The unitary dual $\hat{G}_n$ is homeomorphic to the space of admissible coadjoint orbits $\mathfrak{g}_n^*/G_n$.

\section{The Fourier Transform.}

According to Theorem 3.3 the spectrum $\hat{G}_n$ is determined by the space of admissible coadjoint orbits. Hence we have irreducible representations of the form $\pi_{\lambda, \alpha}$, $(\alpha \in \mathbb{R}^+, \lambda \in \mathbb{Z}^n)$, $\pi_{\lambda, r}$, $(r \in \mathbb{R}_{>0}, \lambda \in \mathbb{Z}^n)$ and characters $\chi_{\lambda, \lambda} \in \mathbb{Z}^n$.

\subsection{The generic representations $\pi_{\lambda, \alpha}$}

They are extensions of the infinite dimensional irreducible representations $\pi_\alpha, \alpha \in \mathbb{R}^+$, to $G_n$.

The irreducible representation $\pi_\alpha$ of $\mathbb{H}_n$ acting on the space $F_\alpha(n)$ for $\alpha > 0$ is given by Folland in [7]. Let for $u, v \in \mathbb{C}^n$

$$uv = u \cdot v := \left( \sum_{j=1}^{n} u_j v_j \right), \quad u = (u_1, \cdots, u_n), \quad v = (v_1, \cdots, v_n).$$

For $\alpha > 0$, the Hilbert space of $\pi_\alpha$ is the space

$$F_\alpha(n) := \left\{ f : \mathbb{C}^n \rightarrow \mathbb{C}; \; f \text{ holomorphic }, \; \|f\|^2_\alpha = \alpha \int_{\mathbb{C}^n} |f(w)|^2 e^{-\frac{\alpha}{2}|w|^2} dw < \infty \right\}.$$

by, taking for the character $\chi_\alpha$ the expression

$$\chi_\alpha(t) = e^{i\alpha t}, \; t \in \mathbb{R}.$$

We have

$$\pi_\alpha(z, t) \xi(w) = e^{i\alpha t} e^{-\frac{\alpha}{2}|z|^2} e^{-\frac{\alpha}{2}w^2} \xi(z + w).$$

On the other hand, if $\alpha < 0$, the Fock space $F_\alpha(n)$ consists of antiholomorphic functions $f : \mathbb{C}^n \rightarrow \mathbb{C}$ such that

$$\|f\|^2_\alpha := |\alpha| \int_{\mathbb{C}^n} |f(w)|^2 e^{-\frac{|w|^2}{2}} dw < \infty.$$

The representation $\pi_\alpha$ takes the form

$$\pi_\alpha(z, t) f(w) = e^{i\alpha t} e^{-\frac{\alpha}{2}|z|^2} e^{-\frac{\alpha}{2}w^2} \xi(w + \overline{z}).$$

Therefore the representation $\pi_{\lambda, \alpha}$ acts for $\alpha > 0$ on $F_\alpha(n)$ by

$$\pi_{\lambda, \alpha}(e^{i\theta}, z, t) f(w) = e^{i\lambda \theta} e^{i\alpha t} e^{-\frac{\alpha}{2}|z|^2} e^{-\frac{\alpha}{2}w^2} \xi(e^{-i\theta} w + e^{-i\theta} z),$$

and for $\alpha < 0$:

$$\pi_{\lambda, \alpha}(e^{i\theta}, z, t) f(w) = e^{i\lambda \theta} e^{i\alpha t} e^{-\frac{\alpha}{2}|z|^2} e^{-\frac{\alpha}{2}w^2} \xi(e^{i\theta} w + e^{i\theta} \overline{z}).$$

For $F \in L^1(G_n), f \in F_\alpha(n), \alpha > 0$ we then have that

$$\pi_{\lambda, \alpha}(F) f(w) = \int_{\mathbb{T}^n} \int_{\mathfrak{g}_n^*} e^{i\lambda \theta} e^{i\alpha t} e^{-\frac{\alpha}{2}|w|^2} e^{-\alpha/2w^2 \pi} F((e^{i\theta}, z, t)) f(e^{-i\theta} w + e^{-i\theta} z) dz d\theta dt \quad (3.2)$$

$$= \int_{\mathbb{T}^n} \int_{\mathbb{C}^n} e^{i\lambda \theta} e^{-\frac{\alpha}{2}|z-w|^2} e^{-\alpha/2w^2(z-w)} F^3(e^{i\theta}, z-w, \alpha) f((e^{-i\theta} z) dz d\theta$$

$$= \int_{\mathbb{T}^n} \int_{\mathbb{C}^n} e^{i\lambda \theta} e^{-\frac{\alpha}{2}|z|^2 + \frac{\alpha}{2}|w|^2 - \frac{\alpha}{2} |w-\overline{z}|^2} \text{Im}(w \overline{z}) \overline{F^3(e^{i\theta}, z-w, \alpha) f((e^{-i\theta} z) dz d\theta.$$
3.2.2 The infinite dimensional representations $\pi_{\lambda,r}$ (which are trivial on the centre of $G_n$).

Let $\chi_{\lambda,r}(0 \neq r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n, \lambda \in \mathbb{Z}_n^\ast)$ be the unitary character

$$\chi_{\lambda,r}(e^{i\theta} z, t) := e^{i\theta \lambda_R e^{iR_r/2}} e^{i\theta} \in \mathbb{T}_r, z \in \mathbb{C}^n, t \in \mathbb{R}.$$  

The representation $\pi_{\lambda,r} = \text{ind}_{\mathbb{T}_r^n \times \mathbb{H}_n} \chi_{\lambda,r}$ acts then on the space

$$\mathcal{H}_{\lambda,r} := L^2(\mathbb{T}_r^n / \mathbb{T}_r^n, \chi_{\lambda,r})$$

by

$$\pi_{\lambda,r}(e^{i\theta}, z, t) \xi(e^{i\mu}) = e^{i\theta \lambda_R e^{-iR_r/2}} \xi(e^{i\mu}).$$

Hence, for $F \in L^1(G_n), \xi \in L^2(\mathbb{T}_r^n / \mathbb{T}_r^n, \lambda), e^{i\mu} \in \mathbb{T}_r^n / \mathbb{T}_r^n$,

$$\pi_{\lambda,r}(F) \xi(e^{i\mu}) = \int_{\mathbb{T}_r^n} \int_{\mathbb{C}^n} e^{i\theta \lambda_R e^{-iR_r/2}} \xi((e^{i\mu} - \xi) d\theta dz$$

$$= \int_{\mathbb{T}_r^n / \mathbb{T}_r^n} \hat{F}^{1,2,3}(e^{i\mu - \theta/2}, \lambda, e^{i\mu} \cdot e^{i\theta}, 0) \xi(e^{i\theta}) d\theta, \xi$$

where

$$\hat{F}^{1,2,3}(e^{i\theta}, \lambda, r, 0) := \int_{\mathbb{T}_r^n} e^{i\lambda R} e^{-iR_r/2} F(e^{i\theta + \varphi}, z, t) d\varphi dz dt.$$

3.2.3 The characters.

Let for $\lambda \in \mathbb{Z}_n$

$$\chi_{\lambda}(e^{i\theta}, z, t) := e^{i\lambda_R e^{iR_r/2}} e^{i\theta} \in \mathbb{C}^n, t \in \mathbb{R}.$$  

Then the set $\{\chi_{\lambda}, \lambda \in \mathbb{Z}\}$ is the collection of all unitary characters of the group $G_n$.

In particular for the group $G_1$ we obtain the partition of $\hat{G}_1$ into three Hausdorff subsets

$$\hat{G}_1 = \Gamma_2 := \{\pi_{\lambda,0} | \lambda \in \mathbb{Z}, \alpha \in \mathbb{R}_+^* \} \cup \Gamma_1 := \{\pi_{\lambda,0} | r > 0, \lambda \in \mathbb{Z}\} \cup \Gamma_0 := \{\chi_{\lambda} | \lambda \in \mathbb{Z}\}.$$  

4 The NCDL property for $C^*(G_1)$.

4.1 Some definitions

1. Let

$$b_{N,0}(z) := \sqrt{\frac{\alpha^N}{2^N N!^2}} z^N$$

be the $N$'th orthonormal vector of the canonical Hilbert basis of $F_\alpha(1)$.

2. We define for $N \in \mathbb{N}, M \in \mathbb{R}_+, N \geq M$ the orthogonal projection $P_N$ of $L^2(\mathbb{T})$ by

$$P_N \left( \sum_{j \in \mathbb{Z}} c_j \chi_j \right) := \sum_{j \geq -N} c_j \chi_j.$$  

and

$$P_{N,M} \left( \sum_{j \in \mathbb{Z}} c_j \chi_j \right) := \sum_{\substack{j \geq -N \\epsilon M}} c_j \chi_j.$$
Let also

\[ L^2(T)_N := P_N(L^2(T)) = \left\{ \sum_{j \geq -N} c_j \chi_j; \sum_{j \in \mathbb{Z}} |c_j|^2 < \infty \right\}, \]

\[ L^2(T)_{N,M} := P_{N,M}(L^2(T)) = \left\{ \sum_{|j| \leq M} c_j \chi_j; \sum_{j \in \mathbb{Z}} |c_j|^2 < \infty \right\}. \]

Define for \( \eta = \sum_{j \in \mathbb{Z}} c_j \chi_j \in L^2(T) \) the element

\[ V_k(\eta) := \sum_{j=-\lambda_k}^{\infty} i^j c_j b_j + \lambda_k, \alpha_k \]

of the Fock space \( \mathcal{F}_\alpha(1) \). We see that the mapping \( \eta \to V_k(\eta) \) from the space

\[ L^2(T)_{\lambda_k} := \left\{ \sum_{j=-\lambda_k}^{\infty} c_j \chi_j; \sum_{j} |c_j|^2 < \infty \right\} \]

onto the Fock space \( \mathcal{F}_{\alpha_k}(1) \) is linear, isometric and surjective. We have that

\[ V_k^*(f) = \sum_{j=0}^{\infty} (-i)^j \gamma_j \chi_{j-\lambda_k}; \quad \hat{f}_k = \sum_{j=0}^{\infty} \gamma_j b_j, \alpha_k. \]

Then

\[ V_k \circ V_k^* = \mathbb{I}_{\mathcal{F}_{\alpha_k}(1)}, \quad k \in \mathbb{N}. \] (4.1)

3. We shall make an essential use of Bessel functions in the proof Theorem 4.5 (see [16] for the definition and properties of Bessel functions).

**Definition 4.1.** Let for \( n \in \mathbb{Z} \) and \( z \in \mathbb{C} \)

\[ J_n(z) := \left( \frac{z}{2} \right)^n \sum_{k+n \geq 0, k \geq 0} \frac{(-1)^k \left( \frac{1}{2} z^2 \right)^k}{k! (k+n)!}. \]

Then,

\[ J_n(z) = \frac{i^{-n}}{\pi} \int_{0}^{\pi} e^{iz \cos(\theta)} \cos(n \theta) d\theta, \quad z \in \mathbb{C}, \quad n \in \mathbb{Z}. \]

Write for \( z \in \mathbb{C} \),

\[ z = e^{i\nu |z|}. \]

Then for \( v \in \mathbb{C} \),

\[ J_n(-|v|) = \frac{i^{-n}}{2\pi} \int_{-\pi}^{\pi} e^{-i|v| \cos(\theta)} e^{in\theta} e^{-in\theta} d\theta = \frac{i^{-n}}{2\pi} \int_{-\pi}^{\pi} e^{-i \Re(|v| e^{i\theta})} e^{-in\theta} d\theta. \]
4.2 Convergence to \( \pi_r \)

**Lemma 4.2.** For \( F \in L^1(G_1) \) and \( j, l, k \in \mathbb{Z} \) we have that

\[
\langle \pi_{\lambda_k, \alpha_k}(F) V_k(\chi_j), V_k(\chi_l) \rangle = \sum_{l+j-q \geq 0, 0 \leq q \leq j + \lambda_k} (-1)^{q+l-j-l} \frac{(\lambda_k \alpha_k/2)^{q+(l-j)/2}}{l^q j^l ((l + \lambda_k)! (j + \lambda_k)!)} \int_{|z|^2 \leq \lambda_k} \frac{1}{\lambda_k} e^{-\frac{\alpha_k}{2} |z|^2} \tilde{F}^{1,3}((-j, z, \alpha_k))dz
\]

and for \( j, l, k \in \mathbb{Z} \)

\[
\langle \pi_r(F)(\chi_j), \chi_l \rangle = \sum_{l+j-q \geq 0, 0 \leq q \leq j + \lambda_k} \int_{|z|^2 \leq \lambda_k} (-1)^{q+l-j-l} \frac{1}{\lambda_k} e^{-\frac{\alpha_k}{2} |z|^2} \tilde{F}^{1,3}((-j, z, 0))dz.
\]

**Proof.** For \( F \in L^1(G_1) \), such that \( \tilde{F}^3 \in C_c(T \times \mathbb{C} \times \mathbb{R}) \) and \( l, j \in \mathbb{Z}, l + \lambda_k \geq 0, j + \lambda_k \geq 0 \), we have that

\[
\langle \pi_{\lambda_k, \alpha_k}(F) V_k(\chi_j), V_k(\chi_l) \rangle = \alpha_k \int_T \int_T \int_{\mathbb{R}} e^{i\lambda_k \theta} e^{i\alpha_k t} e^{-\frac{\alpha_k}{2} |z|^2} e^{-\frac{\alpha_k}{2} w^2} F((e^{i\theta}, z, t)) V_k(\chi_j)(w) e^{-\pi \alpha_k |w|^2} dw d\theta dt
\]

\[
= \alpha_k l^j \int_{T \times \mathbb{R}} e^{i\lambda_k \theta} e^{i\alpha_k t} e^{-\frac{\alpha_k}{2} |z|^2} e^{-\frac{\alpha_k}{2} w^2} F((e^{i\theta}, z, t))(w) \int_{\mathbb{R}} e^{-\pi \alpha_k |w|^2} dw d\theta dt
\]

\[
= \alpha_k l^j \int_{T \times \mathbb{R}} e^{i\lambda_k \theta} e^{i\alpha_k t} e^{-\frac{\alpha_k}{2} |z|^2} e^{-\frac{\alpha_k}{2} w^2} F((e^{i\theta}, z, \lambda_k)(w + z)) \int_{\mathbb{R}} e^{-\pi \alpha_k |w|^2} dw d\theta dz
\]

\[
= \alpha_k l^j \int_{T \times \mathbb{R}} e^{i\lambda_k \theta} e^{i\alpha_k t} e^{-\frac{\alpha_k}{2} |z|^2} e^{-\frac{\alpha_k}{2} w^2} F((e^{i\theta}, z, \alpha_k)) \int_{\mathbb{R}} e^{-\pi \alpha_k |w|^2} dw d\theta dz
\]

(orthogonality relations \( \Rightarrow m = l + \lambda_k - q \))

\[
= l^j \sum_{l+j-q \geq 0, 0 \leq q \leq j + \lambda_k} \frac{(\alpha_k/2)^{\lambda_k+j}}{(j + \lambda_k)!} \frac{(\alpha_k/2)^{\lambda_k+l}}{(l + \lambda_k)!} \int_T e^{-ij\theta} e^{-\frac{\alpha_k}{2} |z|^2} e^{-\frac{\alpha_k}{2} w^2} \tilde{F}^{3}((e^{i\theta}, z, \alpha_k)) dz d\theta
\]
\[= \sum_{l+\ell_k+q \geq 0 \atop 0 \leq q \leq j + \ell_k} (-1)^{\ell_k+l-q} \frac{1}{\lambda^{q+l}} \frac{\sqrt{(l+\lambda_k)(j+\lambda_k)}}{q!(j+\lambda_k-q)!} \frac{\alpha_k^q/2}{(j+\lambda_k-q)!} \frac{(\lambda_k-\alpha_k)^{q+l+1/2}}{(l+\lambda_k-q)!} \]

\[
\int_T \int_{C} e^{-i\theta} \int_{\mathbb{C}} e^{-\frac{1}{2}\lambda} (e^{i\theta}, \alpha_k) d\zeta d\theta
\]

\[
(q \rightarrow \lambda_k + j - q)
\]

\[= \sum_{l+\ell_k+q \geq 0 \atop 0 \leq q \leq j + \ell_k} (-1)^{\ell_k+l-q} \frac{1}{\lambda^{q+l}} \frac{\sqrt{(l+\lambda_k)(j+\lambda_k)}}{q!(j+\lambda_k-q)!} \frac{\alpha_k^q/2}{(j+\lambda_k-q)!} \frac{(\lambda_k-\alpha_k)^{q+l+1/2}}{(l+\lambda_k-q)!} \]

\[
\int_{C} e^{-\frac{1}{2}\lambda} (e^{i\theta}, \alpha_k) d\zeta
\]

Furthermore, we have that:

\[
\int_{C} (l-j) \frac{1}{\sqrt{\lambda}} \frac{1}{(q!)^2} \frac{1}{(q+1) \cdots (q+l-j)} \frac{1}{(q+1) \cdots (q+l-j)} e^{-i\theta} \frac{1}{\lambda} \frac{1}{(q!)^2} \frac{1}{(q+1) \cdots (q+l-j)} d\zeta d\theta
\]

\[
\sum_{l+\ell_k+q \geq 0 \atop 0 \leq q \leq \infty} \frac{(l-j) \lambda^q}{(q!)^2} \frac{1}{(q+1) \cdots (q+l-j)} e^{-i\theta} \frac{1}{\lambda} \frac{1}{(q!)^2} \frac{1}{(q+1) \cdots (q+l-j)} d\zeta d\theta
\]

\[
= \frac{1}{(q!)^2} \frac{1}{(q+1) \cdots (q+l-j)} e^{-i\theta} \frac{1}{\lambda} \frac{1}{(q!)^2} \frac{1}{(q+1) \cdots (q+l-j)} d\zeta d\theta
\]

\[
= \frac{1}{(q!)^2} \frac{1}{(q+1) \cdots (q+l-j)} e^{-i\theta} \frac{1}{\lambda} \frac{1}{(q!)^2} \frac{1}{(q+1) \cdots (q+l-j)} d\zeta d\theta
\]

\[
(\pi_{\ell_k}, (\chi_j, \chi_l)).
\]

**Remark 4.3.** Let \((\pi_{\ell_k}, \alpha_k)\) be a properly converging sequence in \(\hat{G}_1\). By Theorem 3.2, there exists a convergent subsequence (for simplicity of notations we denote it also by \(\ell_k\) and \(\alpha_k\)) such that \((\alpha_k)_{k \in \mathbb{N}}\) tends to 0 and that \(\lim_{k \to \infty} \lambda_k \alpha_k = \omega\), where \(\omega = c^2, r > 0\), or \(\lim_{k \to \infty} \lambda_k \alpha_k = 0\). If \(\omega > 0\), then, if \(\alpha_k\) is positive (respectively negative) for every \(k\), we have that \(\lambda_k > 0\) (resp. \(\lambda_k < 0\)) for \(k\) large enough.

**Lemma 4.4.** Let \((\pi_{\ell_k}, \alpha_k)\) be a properly converging sequence in \(\hat{G}_1\). Suppose that \(\lim_{k \to \infty} \lambda_k \alpha_k = \frac{c^2}{2} > 0\). Let \(F \in L^1(G_1)\), such that \(\hat{F}^{1,3} \in C_0^\infty(T \times \mathbb{C}^2)\). Then we have that

\[
\lim_{k \to \infty} \|\pi_{\ell_k, \alpha_k}(F) \circ V_k \circ (I - P_{\lambda_k \sqrt{\lambda_k}})\|_{op} = 0.
\]
Furthermore, we have that
\[ \lim_{k \to \infty} \| \pi_r(F) \circ (I - P_{\lambda_k, \sqrt{|\lambda_k|}}) \|_{\text{op}} = 0. \]

Proof. Suppose that \( \alpha_k > 0 \) for \( k \in \mathbb{N} \). By (2.22) we have for \( j \in \mathbb{Z}, \ |j| \geq \sqrt{\lambda_k}, \ j \geq -\lambda_k \) that
\[
\pi_{\lambda_k, \alpha_k}(F)b_{\lambda_k+j}(w) = \int_T \int_C \int C \int C \int e^{-\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{2} \Im(w \pi) \hat{F}^3 (e^{i\theta}, z - w, \alpha) b_{\lambda_k+j}(z) dz d\theta = \pi_{\alpha_k}(\hat{F}^{1,3}(-j)) b_{\lambda_k+j}(w).
\]

Now, since \( \| \hat{F}^{1,3}(j) \|_1 \leq C_F (1 + |j|)^3, j \in \mathbb{Z} \), for some \( C_F > 0 \) it follows that for \( k \) large enough, for any for
\[
\eta = \sum_{|j| > \sqrt{\lambda_k}, j \geq -\lambda_k} c_j \chi_{\lambda_k+j}
\]
we have that
\[
\| \pi_{\alpha_k}(F)\eta \|_2 = \| \sum_j c_j \pi_{\alpha_k}(\hat{F}^{1,3}(-j)) b_{\lambda_k+j} \|_2 \leq \left( \sum_j |c_j|^2 \right)^{1/2} \left( \sum_j \frac{C_F}{(\sqrt{\lambda_k})^2 (1 + |j|)^2} \right) \leq \frac{1}{\sqrt{\lambda_k}} \| \eta \|_2.
\]

Similarly for \( \pi_r(F) \):
\[
\pi_r(F)(\chi_j)(e^{i\theta}) = \int_T \hat{F}^{2,3}(e^{i(\mu - \theta)}, e^{i\mu r}, 0) e^{i\theta} d\mu.
\]

Then
\[
|\pi_r(F)(\chi_j)(e^{i\theta})| \leq \frac{C_F}{(1 + |j|)^3}, \ j \in \mathbb{Z}, \ \theta \in [0, 2\pi],
\]
and so for \( k \) large enough and any \( \eta = \sum_{|j| > \sqrt{\lambda_k}, j \geq -\lambda_k} c_j \chi_{\lambda_k+j} \) we have that
\[
\| \pi_r(F)\eta \|_2 \leq \frac{1}{\sqrt{\lambda_k}} \| \eta \|_2.
\]

\[ \square \]

**Theorem 4.5.** Let \( (\pi_{\lambda_k, \alpha_k})_k \) be a properly converging sequence in \( \widehat{G}_1 \). Suppose that \( \lim_{k \to \infty} \lambda_k \alpha_k = \frac{\pi^2}{2} > 0 \). Define:
\[
\sigma_{r,k}(a) := V_k \circ \pi_r(a) \circ V_k^*, a \in C^*(G_1), k \in \mathbb{N}.
\]

Then we have that
\[
\lim_{k \to \infty} \| \pi_{\lambda_k, \alpha_k}(a) - \sigma_{r,k}(a) \|_{\text{op}} = 0, a \in C^*(G_1).
\]

Proof. Suppose that \( \alpha_k > 0 \) for all \( k \in \mathbb{N} \) (the case \( \alpha_k < 0 \) is similar). For \( k \) large enough it follows for any \( l \geq j \in \mathbb{Z} \) and \( q \geq 0, q + l - j \geq 0 \) that:
\[
\sqrt{(\lambda_k + j + 1) \cdots (\lambda_k + l)(j + \lambda_k - q + 1) \cdots (j + \lambda_k)} \left( \frac{\lambda_k \alpha_k}{2} \right)^{q(l-j)/2} \left( \frac{\lambda_k \alpha_k}{2} \right)^{(q+1) \cdots (q+l-j)} |z|^{2q} \leq \left( \frac{1}{\sqrt{\pi}} \right) (\sqrt{\pi} + 1)^{q+1} |z|^{2q} \leq \left( \frac{1}{\sqrt{\pi}} \right) (\sqrt{\pi} + 2)^{q+1} |z|^{2q} \leq \frac{q^2 (q+1)^l}{q^2 (q+1)^l} |z|^{2q}.
\]
Let \( l, j \in \mathbb{Z} \), \( l \geq -\lambda_k \), \( |l| \leq \sqrt{\lambda_k} \), \( j \geq -\lambda_k \), \( |j| \leq \sqrt{\lambda_k} \). Then by (4.2) and (4.3):

\[
|\langle \pi_{\lambda_k, \alpha_k}(F)V_k(\chi_j), V_k(\chi_l) \rangle - \langle \pi_r(F)(\chi_j), \chi_l \rangle| = \\
\left| \sum_{l-j+q \geq 0} (-1)^{q+l-j} \frac{1}{(q!)^2} \frac{(r^2|z|^2/4)^q}{(q+1) \cdots (q+l-j)} \hat{F}^{1,3}(j, z, 0)dz \right|
\]
Then,

$$\left|\langle \pi_{\lambda_k, \alpha_k} (F) V_k (\chi_j), V_k (\chi_i) \rangle - \langle \pi_r (F) (\chi_j), \chi_i \rangle \right|$$

$$= \left| \sum_{l-j+q \geq 0 \atop 0 \leq q \leq \lambda_k} (-1)^q \sqrt{\frac{(1 + \frac{j+1}{l}) \cdots (1 + \frac{j+q+1}{l}) \cdots (1 + \frac{j}{l})}{(q!)^2}} \frac{(\lambda_k \alpha_k / 2)^{q+(l-j)/2}}{(q+1) \cdots (q+l-j)} \right|$$

$$+ \left| \sum_{l-j+q \geq 0 \atop 0 \leq q \leq \lambda_k} \int_{\mathbb{C}} (\bar{z})^{l-j} |z|^{2q} e^{-\frac{\alpha_k}{2} |z|^2} \tilde{F}^{1,3}((j,z,\alpha_k)) dz \right|$$

$$- \left| \sum_{l-j+q \geq 0 \atop 0 \leq q \leq \lambda_k} \int_{\mathbb{C}} (\bar{z})^{l-j} |z|^{2q} \tilde{F}^{1,3}((j,z,\alpha_k)) dz \right|$$

$$\leq \left| \sum_{l-j+q \geq 0 \atop 0 \leq q \leq \lambda_k + j} \int_{\mathbb{C}} (\bar{z})^{l-j} |z|^{2q} \frac{\sqrt{(1 + \frac{j+1}{l}) \cdots (1 + \frac{j+q+1}{l}) \cdots (1 + \frac{j}{l})}}{(q!)^2} \frac{(\lambda_k \alpha_k / 2)^{q+(l-j)/2}}{(q+1) \cdots (q+l-j)} \right|$$

$$\left| \sum_{l-j+q \geq 0 \atop q=\lambda_k + j+1} (r^2/4)^{q+(l-j)/2} \int_{\mathbb{C}} (\bar{z})^{l-j} |z|^{2q} (e^{-\frac{\alpha_k}{2} |z|^2} - 1) \tilde{F}^{1,3}((j,z,\alpha_k)) dz \right|$$

$$+ \sum_{l-j+q \geq 0 \atop 0 \leq q \leq \lambda_k + j} \int_{\mathbb{C}} (\bar{z})^{l-j} |z|^{2q} \frac{\sqrt{(1 + \frac{j+1}{l}) \cdots (1 + \frac{j+q+1}{l}) \cdots (1 + \frac{j}{l})}}{(q!)^2 (q+1) \cdots (q+l-j)} (r^2/4 + 1)^{-1+(l-j)/2} \right|$$

$$\left| \lambda_k \alpha_k (r^2/4) \right| \left| \sum_{l-j+q \geq 0 \atop 0 \leq q \leq \lambda_k + j} (q + (l-j) - 1/2) \sqrt{(1 + \frac{j+1}{l}) \cdots (1 + \frac{j+q+1}{l}) \cdots (1 + \frac{j}{l})} (r^2/4 + 1)^{-1+(l-j)/2} \right|$$

$$+ \sum_{l-j+q \geq 0 \atop q=\lambda_k + j+1} \int_{\mathbb{C}} (\bar{z})^{l-j} |z|^{2q} \left( e^{-\frac{\lambda_k}{2} |z|^2} - 1 \right) \tilde{F}^{1,3}((j,z,\alpha_k)) dz \right|$$

$$+ \int_{\mathbb{C}} (\bar{z})^{l-j} |z|^{2q} \tilde{F}^{1,3}((j,z,\alpha_k)) dz \right|$$

$$+ \left| \sum_{l-j+q \geq 0 \atop 0 \leq q \leq \lambda_k} \sqrt{(\lambda_k + j + 1) \cdots (\lambda_k + l)(j + \lambda_k - q + 1) \cdots (j + \lambda_k)} \frac{(\lambda_k \alpha_k / 2)^{q+(l-j)/2}}{(q+1) \cdots (q+l-j)} \right|$$

$$\left| \sum_{l-j+q \geq 0 \atop q=\lambda_k + j+1} (r^2/4)^{q+(l-j)/2} \int_{\mathbb{C}} (\bar{z})^{l-j} |z|^{2q} \left( e^{-\frac{\lambda_k}{2} |z|^2} - 1 \right) \tilde{F}^{1,3}((j,z,\alpha_k)) dz \right|$$

$$+ \sum_{l-j+q \geq 0 \atop 0 \leq q \leq \lambda_k} \int_{\mathbb{C}} (\bar{z})^{l-j} |z|^{2q} \tilde{F}^{1,3}((j,z,\alpha_k)) \tilde{F}^{1,3}((j,z,\alpha_k)) dz \right|$$
\[ \leq \left| (\lambda_k \alpha_k/2 - r^2/4) \right| \int_{C} \sum_{0 < q < \infty} \frac{2^q(r^2/4 + 1)^q|z|^{2q}}{q!} \hat{F}^{1,3}((j, z, \alpha_k))dz \]

\[ + \frac{1}{j + \lambda_k + 1} \int_{C} \sum_{q=0}^{\infty} \frac{2^q(r^2/4 + 1)|z|^{2q}}{q!} |\hat{F}^{1,3}((j, z, \alpha_k))|dz \]

\[ + 2\alpha_k \epsilon \int_{C} \sum_{q=0}^{\infty} \frac{2^q(r^2/4 + 1)|z|^{2q}}{q!} |\hat{F}^{1,3}((j, z, \alpha_k))|dz + \int_{C} \sum_{q=0}^{\infty} \frac{2^q(r^2/4 + 1)|z|^{2q}}{q!} |\hat{F}^{1,3}((j, z, \alpha_k)) - \hat{F}^{1,3}((j, z, 0))|dz. \]

Hence

\[ |(\pi_{\lambda_k, \alpha_k}(F)V_k(\chi_j), V_k(\chi_i)) - \langle \pi_r(F)(\chi_j), \chi_i \rangle| \leq \frac{\delta_k}{(1 + |j|)^{4}}, k \in \mathbb{N}. \]

Finally

\[ |\langle \pi_{\lambda_k, \alpha_k}(F)V_k(\chi_j), V_k(\chi_i) \rangle - \langle \pi_r(F)(\chi_j), \chi_i \rangle| \leq \frac{\delta_k}{(1 + |j|)^{2}(1 + |j|)^{2}}, k \in \mathbb{N}, |\ell|, |j| \leq \sqrt{\lambda_k}. \]

Hence for any \( \eta_k = \sum_{j=0}^{\sqrt{\lambda_k}} \frac{1}{\sqrt{\lambda_k}} a^k_j \chi_j, \psi_k = \sum_{j=-\sqrt{\lambda_k}}^{\sqrt{\lambda_k}} a^k_j \chi_j \in L^2(T) \) we have that

\[ |\langle \pi_{\lambda_k, \alpha_k}(F)V_k(\eta_k), V_k(\psi_k) \rangle - \langle \pi_r(F)\eta_k, \psi_k \rangle| \leq C\|\eta_k\|_2\|\psi_k\|_2\delta_k \]

for \( C := \sqrt{\sum_{j=0}^{\infty} \frac{1}{(1 + |j|)^{4}}} \). This shows together with Lemma 4.3 that

\[ \lim_{k \to \infty} \|\pi_{\lambda_k, \alpha_k}(F) \circ V_k - V_k \circ \sigma_{r,k}(F)\|_{op} = 0. \]

Hence

\[ \lim_{k \to \infty} \|\pi_{\lambda_k, \alpha_k}(F) - \sigma_{r,k}(F)\|_{op} = 0. \]

**4.3 Convergence to characters.**

If the sequence \( (\lambda_k \alpha_k) \) tends to 0, \( \alpha_k > 0 \) (resp. \( \alpha_k < 0 \)) for \( k \in \mathbb{N} \) and the sequence \( (|\lambda_k|) \) has a bounded subsequence, then we find a subsequence such that \( \lambda_k = \lambda_\infty \in \mathbb{Z} \) for all \( k \). The limit set \( L \) of this subsequence is according to Theorem 3.2 given by

\[ L = \{ \chi_j; j \in \mathbb{Z}, j \leq \lambda_\infty \} \text{ (resp. } L = \{ j \in \mathbb{Z}, j \geq \lambda_\infty \}). \]

If the sequence \( (|\lambda_k|) \) is unbounded, the fact that the limit set of the sequence \( (\pi_{\lambda_k, \lambda_k}) \) is not empty forces \( \lim_{k \to \infty} \lambda_k = +\infty =: \lambda_\infty \) if \( \alpha_k > 0 \) (resp. \( \lim_{k \to \infty} \lambda_k = -\infty =: \lambda_\infty \) if \( \alpha_k < 0, k \in \mathbb{N} \)).

Let for \( \lambda_\infty \in \mathbb{Z} \cup \{ +\infty, -\infty \} \)

\[ \pi_{\lambda_\infty, 0} := \oplus_{\lambda \leq \lambda_\infty} \chi_\lambda \text{ (resp. } \pi_{\lambda_\infty, 0} := \oplus_{\lambda \geq \lambda_\infty} \chi_\lambda) \]

be the direct sum of the characters \( \chi_\lambda, \lambda \leq \lambda_\infty \) (resp. of the characters \( \lambda \geq \lambda_\infty \)).
Theorem 4.6. Let $(\pi_{\alpha_k, \sigma_k})_k$ be a properly converging sequence in $\hat{G}_1$. Suppose that $\lim_{k \to \infty} \alpha_k = 0$ and that $\lim_{k \to \infty} \alpha_k \lambda_k = 0$. Define:

$$\sigma_{\lambda_\infty, k}(a) := V_k \circ \pi_{\lambda_\infty, 0}(a) \circ V_k^*, a \in C^*(G_1), k \in \mathbb{N}. $$

Then for every $a \in C^*(G_1)$ we have that

$$\lim_{k \to \infty} \|\pi_{\lambda_k, \sigma_k}(a) - \sigma_{\lambda_\infty, k}(a)\|_{op} = 0. $$

Proof. We consider only the case $\alpha_k > 0, k \in \mathbb{N}$. Suppose first that $\hat{F}^{1,3}$ (and so also $(\hat{F}^*)^{1,3}$) has finite support in the first variable $j$ and compact support in the variable $z$. Then we have for some $m \in \mathbb{N}$ and some compact ball $K = B_R \subset \mathbb{C}$ that

$$|\hat{F}^{1,3}(j, z, \lambda_k)| \leq C_F 1_{[-m, m]}(j)1_K(z), j \in \mathbb{Z}, z \in \mathbb{C}, \alpha \in \mathbb{R}. $$

Let $l, j \in \mathbb{Z}, l > j, l \geq -\lambda_\infty, j \geq -\lambda_\infty$. Lemma 4.2 implies that

$$\langle \pi_{\lambda_k, \alpha_k}(F)V_k(\chi_j), V_k(\chi_l) \rangle = \sum_{l-j+q \geq 0 \atop 0 \leq q \leq j + \lambda_k} (-1)^{j+q-l-j} \frac{\lambda_k^{q+lj}j^{(l-j)/2}}{\lambda_k^{q+lj}(j+1) \cdots (j+q)(j+q+1) \cdots (j+q+1)} \cdot \int_{|z|^2} \frac{\chi_k(z)}{z^q e^{-\frac{z^2}{\lambda_k}}} \hat{F}^{1,3}((-j, z, \alpha_k))dz. $$

In particular we see that

$$\langle \pi_{\lambda_k, \alpha_k}(F)V_k(\chi_j), V_k(\chi_l) \rangle = 0, $$

if $|j| > m$ or $|j| > m$. It follows then that

$$|\langle \pi_{\lambda_k, \alpha_k}(F)V_k(\chi_j), V_k(\chi_l) \rangle| \leq \sum_{l-j+q \geq 0 \atop 0 \leq q \leq j + \lambda_k} |\lambda_k^{q+lj}j^{(l-j)/2}(1 + \frac{m}{|\lambda_k|})^{q+lj}/2 \cdot \frac{1}{(q!)^2}| \leq C_F R^{2q+lj} 1_{[-m, m]}(j)1_{[-m, m]}(l) \int_{\mathbb{C}} e^{-\frac{|z|^2}{\lambda_k}} 1_K(z)dz \leq |\lambda_k|^{q+lj} R^{2q+lj} C_F \int_{K} dz 1_{[-m, m]}(j)1_{[-m, m]}(l) \sum_{q=0}^\infty (\lambda_k^{q+lj}j^{(l-j)/2}/q!)^{1/(q!)^2} $$

(similarly if $j > l$). For $l = j \in \mathbb{Z}, l \geq -\lambda_\infty$ we see that

$$\langle \pi_{\lambda_k, \alpha_k}(F)V_k(\chi_j), V_k(\chi_l) \rangle = \int_{\mathbb{C}} e^{-\frac{|z|^2}{\lambda_k}} \hat{F}^{1,3}((-j, z, \alpha_k))dz + \sum_{q \geq j + \lambda_k} (-1)^q (j+\lambda_k)! (\lambda_k^{q+lj}j^{(l-j)/2}/q!)^{1/(q!)^2} \int_{\mathbb{C}} |z|^{2q} e^{-\frac{|z|^2}{\lambda_k}} \hat{F}^{1,3}((-j, z, \alpha_k))dz $$

and that

$$\langle \pi_{\lambda_\infty, 0}(F)(\chi_j), \chi_l \rangle = \int_T \int_T \hat{F}^{2,3}((\theta, 0, 0)) e^{i(j-q)\theta} e^{-i\mu} d\theta d\mu = \int_T \hat{F}^{1,2,3}((-j, 0, 0)) e^{i(j-q)\mu} d\mu = \delta_{j,l} \hat{F}^{1,2,3}((-j, 0, 0)) $$

Hence for some constant $D_F > 0$, we have that

$$|\langle \pi_{\lambda_k, \alpha_k}(F)V_k(\chi_j), V_k(\chi_l) \rangle - \delta_{j,l} \hat{F}^{1,2,3}((-j, 0, 0))| \leq |\alpha_k\lambda_k|D_F 1_{[-m, m]}(j)1_{[-m, m]}(l).$$
This shows that for any $\eta = \sum_{j \in \mathbb{Z}} c_j^k \chi_j$ we have that
\[
\|\pi_{\lambda, \alpha_k} (F) \circ V_k (\eta_k) - V_k \circ \pi_{\lambda, \alpha} (F)(\eta_k)\|_2^2
\]
\[
= \sum_{l \geq 0} \frac{c_l \pi_l (\pi_{\lambda, \alpha_k} (F) - \pi_{\lambda, \alpha} (F) \circ V_k - \delta_{l,j} \delta_{j,l} (l, 0, 0)) (V_k (\chi_j), V_k (\chi_l))}{\| V_k \circ \pi_{\lambda, \alpha} (F)(\eta_k) \|_2^2}
\]
\[
= \sum_{l \geq 0} \frac{c_l \pi_l (\pi_{\lambda, \alpha_k} (F) - \pi_{\lambda, \alpha} (F) \circ V_k - \delta_{l,j} \delta_{j,l} (l, 0, 0)) (V_k (\chi_j), V_k (\chi_l))}{\| V_k \circ \pi_{\lambda, \alpha} (F)(\eta_k) \|_2^2}
\]
\[
- \delta_{j,l} \delta_{j,l} (l, 0, 0) (V_k (\chi_j), V_k (\chi_l)) + \delta_{j,l} (F \circ F^{1,2,3}) (l, 0, 0)
\leq E_k [\alpha_k \lambda_k \| \eta_k \|^2]
\]
for some new constant $E_k > 0$. Therefore
\[
\lim_{k \to \infty} \|\pi_{\lambda, \alpha_k} (F) - \sigma_{\lambda, \alpha_k} (F)\|_{\text{op}} = 0.
\]
Since these $F$’s are dense in $C^* (G_1)$ the theorem follows.

\[\Box\]

4.4 The $C^*$-algebra of the group $G_1$.

**Definition 4.7.** Let us recall that
\[
\hat{G}_1 = \{ \pi_{\lambda, \alpha} \mid \lambda \in \mathbb{Z}, \alpha \in \mathbb{R}^* \} \cup \{ \pi_r \mid r > 0 \} \cup \{ \pi_\lambda \mid \lambda \in \mathbb{Z} \}
\]
\[
= \Gamma_2 \cup \Gamma_1 \cup \Gamma_0.
\]

**Definition 4.8.** Let $D_1$ be the family consisting of all operator fields $A \in \ell^\infty (\hat{G}_1)$ satisfying the following conditions
1. $A(\gamma)$ is a compact operator on $\mathcal{H}_{\pi, \gamma}$ for every $\gamma \in \mathfrak{g}_1^1 / G_1$.
2. The mapping $\lambda_1 / G_1 \to \mathcal{B} (\mathcal{H}_{\pi, \gamma}) : \lambda \mapsto A(\gamma)$ is norm continuous on $\Gamma_2, \Gamma_1$ and on $\Gamma_0$.
3. $\lim_{\gamma \to \infty} \| A(\gamma) \|_{\text{op}} = 0$.
4. $\lim_{r \to 0} \| A(r) - A(0) \|_{\text{op}} = 0$ uniformly in $\lambda$, where
\[
A(0) := \oplus_{\lambda \in \mathbb{Z}} A(\lambda).
\]
5. For every properly converging sequence $(\pi_{\lambda_k, \alpha_k})_k$ such that $\lim_{k \to \infty} \lambda_k \alpha_k = \frac{r^2}{2} > 0$, we have
\[
\lim_{k \to \infty} \| A(\lambda_k, \alpha_k) - V_k \circ A(r) \circ V_k^* \|_{\text{op}} = 0.
\]
For every properly converging sequence $(\pi_{\lambda_k, \alpha_k})_k$ such that $\lim_{k \to \infty} \lambda_k = \lambda_\infty$ and $\lim_{k \to \infty} \alpha_k = 0$, $\lim_{k \to \infty} \lambda_k \alpha_k = 0$ and $\varepsilon \alpha_k > 0$, $k \in \mathbb{N}, \varepsilon = \pm 1$ for all $k$ or $\varepsilon = -1$ for all $k$, and that we have
\[
\lim_{k \to \infty} \| A(\lambda_k, \alpha_k) - V_k \circ A(\lambda_\infty, 0) \circ V_k^* \|_{\text{op}} = 0,
\]
where
\[
A(\lambda_\infty) := \oplus_{\lambda \leq \lambda_\infty} A(\lambda) \quad \text{(if $\alpha_k > 0$ for all $k$)} \quad \text{resp.} \quad A(\lambda_\infty) := \oplus_{\lambda \geq \lambda_\infty} A(\lambda) \quad \text{(if $\alpha_k < 0$ for all $k$)}.
\]
As a consequence of relation [3], Theorem [4], Theorem [5] and Theorem [6.10] in [10], we can apply Theorem 3.5 in [11] and therefore we have

**Theorem 4.9.** The $C^*$-algebra of $G_1$ is isomorphic to $D_1$ under the Fourier transform and $C^* (G_1)$ fulfills the NCDL condition.
5 The NCDL-property of $C^*(G_n)$.

For $n \geq 1$, let $G^n = G_1 \times \cdots \times G_1$, $n$ times.

**Theorem 5.1.** The $C^*$-algebra of the group $G^n$ is isomorphic to the tensor product of $C^*(G_1) \otimes \cdots \otimes C^*(G_1)$ ($n$ times) and $C^*(G^n)$ has the NCDL condition.

*Proof.* Since $C^*(G_1)$ is liminary, we have by Theorem 2.3 that $C^*(G_1) \simeq C^*(G_1) \times \cdots \times C^*(G_1)$. It suffices to apply Theorem 2.3.

For $a \in C^*(G^n)$ the Fourier transform $F$ is thus defined by

$$F(a)(\pi_1 \times \cdots \times \pi_n) = \widehat{a}(\pi_1 \times \cdots \times \pi_n) = \pi_1 \otimes \cdots \otimes \pi_n(a) =, (\pi_i)_{1 \leq i \leq n} \subset C^*(G_1).$$

In particular for elementary tensors $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ we obtain then

$$F(a)(\pi_1 \times \cdots \times \pi_n) = \pi_1(a_1) \otimes \cdots \otimes \pi_n(a_n) \in \mathcal{B}(\mathcal{H}_{\pi_1} \otimes \cdots \otimes \mathcal{H}_{\pi_n}).$$

We consider now the centre $Z^n$ of the groups $G^n$. This subgroup $Z^n$ of $G^n$ is given by

$$Z^n = \{(1, 0, t_1) \times \cdots \times (1, 0, t_n); t_1, \cdots, t_n \in \mathbb{R}\}.$$

Let $\{Z_1, \cdots, Z_n\}$ be the canonical basis of the centre $\mathfrak{z}$ of the Lie algebra $\mathfrak{h}_n$ of the group $\mathbb{H}_n$. This means that

$$(0, 1) \text{ appearing at the } j\text{th position. We denote by } \mathfrak{z}_0 \text{ the subspace}$$

$$\mathfrak{z}_0 := \left\{ \sum_{j=0}^{n} z_j Z_j; \sum_{j=0}^{n} z_j = 0 \right\}$$

of $\mathfrak{z}$. Then $\mathfrak{z}_0$ is of codimension 1 in $\mathfrak{z}$ and

$$Z^n_0 := \exp(\mathfrak{z}_0) = \left\{ (1, 0, t_1) \times \cdots \times (1, 0, t_n); \sum_{j=1}^{n} t_j = 0 \right\}$$

is a closed connected and central subgroup of $G^n$. The quotient group $G^n/\mathfrak{z}_0^n$ is then isomorphic to $G_n = T^n \ltimes \mathbb{H}_n$. To see this, it suffices to consider the canonical basis $\{T_j, X_j, Y_j, Z_j; j = 1, \cdots, n\}$ of $\mathfrak{g}^n$ and compute the non trivial brackets in this Lie algebra modulo $\mathfrak{z}_0$. Let $Z := \frac{1}{n} (\sum_{j=1}^{n} Z_j)$. We have

$$[T_j, X_j] = Y_j, \ [T_j, Y_j] = -X_j \text{ modulo } \mathfrak{z}_0,$$

$$[X_j, Y_j] = Z_j = Z + (Z_j - Z) = Z \text{ modulo } \mathfrak{z}_0$$

since $Z_j - Z \in \mathfrak{z}_0$. The spectrum of the group $G_n$ can now be identified with the spectrum of $\widehat{G^n}/\mathfrak{z}_0^n$, which is the subset of $\widehat{G^n}$ consisting of the $n$ -tuples $\pi_1 \times \cdots \times \pi_n \in \widehat{G^n}$ such that $\pi_1 \otimes \cdots \otimes \pi_n((1, 0, z_1) \times \cdots \times (1, 0, z_n)) = I$, if $\sum_{j=1}^{n} z_j = 0$. This means that

$$\widehat{G^n} \simeq \widehat{\mathfrak{z}_0^n} := \{((\lambda_1, \cdots, \lambda_n), (\alpha, \cdots, \alpha)) ; \alpha \in \mathbb{R}^n, \lambda_j \in \mathbb{Z}, j = 1, \cdots, n\} \cup \{\pi \in \widehat{G^n}, \pi = I \text{ on } Z_0^n\}.$$ 

Let

$$K = \{a \in C^*(G^n); \pi(a) = 0, \forall \pi \in \widehat{G^n} \text{ which are trivial on } Z_0^n\}.$$

**Theorem 5.2.** The $C^*$-algebra of the Heisenberg motion group $C^*(G_n)$ is isomorphic to $C^*(G^n)/K$ and $C^*(G_n)$ satisfies the NCDL condition.
Proof. The ideal $\mathcal{K} = \{a \in C^*(G^n); \pi(a) = 0, \forall \pi \in \hat{G}^n\} \text{ is the kernel in } C^*(G^n) \text{ of the canonical surjective homomorphism } \delta^n: C^*(G^n) \to C^*(G_n), \text{ which is defined on } L^1(G^n) \text{ by }
\delta^n(F)(g) := \int_{Z_n^0} F(gz)dz, g \in G^n.

Then the $C^*$-algebra of the Heisenberg motion group $C^*(G_n)$ is isomorphic to $C^*(G^n)/\mathcal{K}$.

Let $\rho^n$ be the restriction map

$$
\rho^n: \ell^\infty(\hat{G}^n) \to \ell^\infty(\hat{G}_0^n)
$$

for any uniformly bounded operator field $\phi$ defined on $\hat{G}^n$.

Then the $C^*$-algebra of the group $G_n$ can be identified with the sub-algebra $\rho^n(F(C^*(G^n)))$ of the algebra $\ell^\infty(\hat{G}_0^n)$.

In particular, if we have a properly convergent sequence $\gamma = (\gamma_k)_{k \in \mathbb{N}}$ in $\hat{G}_1$, then we can write

$$
\gamma_k = \pi_{k_1}^1 \otimes \cdots \otimes \pi_{k_n}^n, k \in \mathbb{N},
$$

where $\gamma = (\pi_{k_j}^j)_{k_j}$ is a properly convergent sequence in $\hat{G}_1$ with limit set $L_j$, $j = 1, \cdots, n$. Let $\sigma_{\gamma_j,k}, k \in \mathbb{N}$, be the corresponding norm control. Then,

$$
\sigma_{\gamma} = \sigma_{\gamma^1} \otimes \cdots \otimes \sigma_{\gamma^n}.
$$

□

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