Localized states of Dirac equation

S. B. Faruque, S.D. Shuvo and P.K. Das

February 19, 2019

Department of Physics, Shahjalal University of Science and Technology, Sylhet 3114, Bangladesh.

†Corresponding author. E-mail: bzaman@sust.edu

Abstract

In this paper, we introduce an extension of the Dirac equation, very similar to Dirac oscillator, that gives stationary localized wave packets as eigenstates of the equation. The extension to the Dirac equation is achieved through the replacement of the momentum operator by a PT-symmetric generalized momentum operator. In the 1D case, the solutions represent bound particles carrying spin having continuous energy spectrum, where the envelope parameter defines the width of the packet without affecting the dispersion relation of the original Dirac equation. In the 2D case, the solutions are localized wave packets and are eigenstates of the third component of total angular momentum and involve Bessel functions of integral order. In the 3D case, the solutions are localized spherical wave packets with definite total angular momentum.

PACS number(s): 03.65.Pm

1 Introduction

Ordinarily, in quantum mechanics, the free particle solutions of the Dirac equation are plane waves with infinite uncertainty in position. But, infinite wave trains are not suitable for application. One, therefore, creates wave packets by superposing many quantum eigenstates. Wave packets are packets of wave function having finite width in position and in momentum, and as such, suitable for application. The aim of this paper is to present a particular coupling of the momentum of a Dirac particle with a position dependent dynamical operator which create eigenstates of the equation that are localized stationary wave packets. This presents within the premise of Dirac theory a way alternative to the conventional creation of wave packets by superposition of many eigenstates- the generalized momentum operator does the job of wave packing. This process is
similar to the process of nonlinear coupling between Coulomb motion of Rydberg electron and linearly polarized microwave field that generate electronic wave packets as stationary eigenstates of Schrödinger like equation [1-3].

Study of wave packets in the context of Dirac theory is itself an important task because of their use in nanophysics [4-7]. Moreover, relativistic wave packet pose a challenge to theory and as such, many authors addressed this problem [8-11]. Localized stationary wave packets in orbit of atoms is an interesting topic being studied for long (see, for example, [12] and the references therein). Such stationary wave packets have numerous potential applications [12], such as in information processing, in cavity quantum electrodynamics, and in precision spectroscopy. The present study elevates the issue to the relativistic regime where stationary localized wave packets are automatic and stable products of relativistic Dirac equation. And as such, our study opens up a new door to applications of relativistic quantum states. Experimental realization of such stationary wave packets can be anticipated as its predecessor Dirac oscillator has already been realized in experiments [13]. The present work is connected with the Dirac oscillator in the way that the Hamiltonian we employ here is derived from Dirac oscillator [14-16]. The solutions to Dirac oscillator are harmonic oscillator states, whereas, here we get qualitatively very similar states, namely, wave packet states. We present the equation and its properties in Section 2. In Section 3, we present the solutions in (1+1) freedom and discuss some of their properties. In Section 4, we present the solution in (2+1) freedom assuming the mass to be zero. In Section 5, we present the solution in (3+1) freedom. Finally, in section 6, we summarize our work.

\section{2 Dirac equation with PT-symmetric generalized momentum}

The equation that gives wave packets as eigenstates is derived from Dirac oscillator [14] suppressing the Dirac matrix \( \beta \) in the coupling operator, i.e., in the free particle Dirac equation we replace \( \vec{p} \) by \( (\vec{p} - i q \vec{r}) \) to obtain the following equation:

\[ \left( c \vec{\alpha} \cdot (\vec{p} - i q \vec{r}) + \beta mc^2 \right) \Psi = i \hbar \frac{\partial \Psi}{\partial t}, \]

where \( q \) is the envelope parameter that determines the width of the resulting wave packet, \( \vec{\alpha} \), \( \beta \) are Dirac matrices, \( \vec{p} \) and \( \vec{r} \) are respectively momentum and coordinate of the fermion and \( c \) is the speed of light in free space. This equation differs from Dirac oscillator by only the matrix \( \beta \) which is present in the second factor within parentheses of the first term on the left side of Eq. (1) in case of Dirac oscillator. The operator \((\vec{p} - i q \vec{r})\) is PT-symmetric as can be easily checked. Under P (parity) transformation: \( \vec{p} \rightarrow -\vec{p}, \vec{r} \rightarrow -\vec{r} \) and under \( T \) (time) reflection: \( \vec{p} \rightarrow -\vec{p}, \vec{r} \rightarrow \vec{r}, i \rightarrow -i \). Moreover, Eq. (1) can be generated from the free Dirac Hamiltonian \( H_0 \) by the similarity transformation \( SH_0 S^{-1} \) with \( S = \exp \left( -\frac{aq^2}{2m} \right) \). Hence, Eq. (1) changes the description of free
Dirac particles from unlocalized states to localized states. Now, we proceed
from the next section to study its solutions. We reduce the equation in a way
to be studied in (1+1) freedom in the next section and find the solutions.

3 Localized states in one dimension

We assume stationary solutions of Eq.(1) in the form $\Psi = \psi(\mathbf{r})exp(-iEt/\hbar)$. Then, the equation becomes

$$[c\mathbf{\alpha}.(\mathbf{p} - iq\mathbf{r}) + \beta mc^2]\psi(\mathbf{r}) = E\psi(\mathbf{r}).$$

(2)

Solutions to this equation in (1+1) freedom will be worked out with the
assumption that the motion of the particle is along the z-direction with momen-
tum $p$. To realize this, we use $\alpha_z$ in place of $\mathbf{\alpha}$ and replace
$\mathbf{r}$ by $z$ to obtain

the governing equation as,

$$(c\alpha_z p - iq\alpha_z z + \beta mc^2)\psi(z) = E\psi(z).$$

(3)

To construct the solution of this equation, first we note that the operator
on the left of this equation, the Hamiltonian $H$, commutes with the operator
of z-component of spin, $\Sigma_z$, i.e., $[\Sigma_z, H] = 0$. Hence, our solution should be
simultaneous eigenstate of energy and spin. So, we write the solution in the
form

$$\psi(z) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$  

(4)

Inserting this into Eq.(3), we find the following coupled equations:

$$(cp - iqcz)u_3 = (E - mc^2)u_1,$$

(5)

$$(cp - iqcz)u_1 = (E + mc^2)u_3,$$

(6)

$$- (cp - iqcz)u_4 = (E - mc^2)u_2,$$

(7)

$$- (cp - iqcz)u_2 = (E + mc^2)u_4.$$  

(8)

Following traditional methods, we first assume $u_2 = u_4 = 0$ and find from
Eqs.(5) and (6),

$$(cp - iqcz)(cp - iqcz)u_1 = (E^2 - m^2c^4)u_1,$$

(9)

and

$$(cp - iqcz)(cp - iqcz)u_3 = (E^2 - m^2c^4)u_3.$$  

(10)
Therefore, we see that $u_1$ and $u_3$ satisfy the same equation and thus, will have the same structure. We can now take $u_1 = u_3 = 0$ and find for $u_2$ and $u_4$ the same governing equations as Eq.(9) or (10). Thus, we need to solve only one equation, say, Eq.(9). We obtain from Eq.(9),

$$ (p^2 - iqpz - iqzp - q^2z^2)u_1 = \left(\frac{E^2}{c^2} - m^2c^2\right)u_1. \quad (11) $$

To solve Eq.(11), we use $p = -i\hbar\frac{\partial}{\partial z}$ and at the same time make the coordinate $z$ dimensionless by defining a new coordinate $z' = \sqrt{\frac{\hbar}{q}}z$ and transform Eq.(11) accordingly. In what follows only $z'$ occurs and for brevity, we drop the prime and continue to write $z$. Then, we find

$$ \frac{d^2u_1}{dz'^2} + 2z\frac{du_1}{dz} + z^2u_1 + K_1u_1 = 0, \quad (12) $$

where

$$ K_1 = \frac{E^2}{\hbar c^2} - \frac{m^2c^2}{\hbar} + 1. \quad (13) $$

We now consider a solution of the form

$$ u_1(z) = \phi(z)exp\left(-\frac{1}{2}z'^2\right). \quad (14) $$

Substituting this in Eq.(12), we obtain for $\phi(z)$, the governing equation,

$$ \frac{d^2\phi}{dz'^2} + \alpha^2\phi = 0, \quad (15) $$

which immediately gives,

$$ \phi(z) = exp(i\alpha z), \quad (16) $$

where

$$ \alpha^2 = \frac{E^2}{\hbar c^2} - \frac{m^2c^2}{\hbar}. \quad (17) $$

Hence, the full solution for $u_1$ is,

$$ u_1(z) = exp(i\sqrt{\frac{q}{\hbar}}\alpha z)exp\left(-\frac{1}{2}\frac{q}{\hbar}z^2\right), \quad (18) $$

which is a localized wave packet with the first factor giving the oscillation in space and the second factor giving the envelope of the packet with $q$ governing the width of the packet. The energy associated with the wave packet is found as,

$$ E = \pm \sqrt{\alpha^2c^2hq + m^2c^4} = \pm \sqrt{\hbar^2k^2c^2 + m^2c^4} = \pm E_k, \quad (19) $$

where $k = \sqrt{\frac{E}{\hbar}}\alpha$ is the wave number. Surprisingly, $k$ is independent of $q$ as can be seen from Eq.(17). Hence, the solution can be written more lucidly as,

$$ u_1(z) = exp(ikz)exp\left(-\frac{1}{2}\frac{q}{\hbar}z^2\right). \quad (20) $$
We now turn our attention to the spinor (4). It has the two independent forms for spin up and spin down as follows:

\[
\Psi_{\text{up}}(z,t) = N \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \exp(ikz)\exp\left(-\frac{1}{2}qz^2\right),
\]

\[
\Psi_{\text{down}}(z,t) = N \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(ikz)\exp\left(-\frac{1}{2}qz^2\right),
\]

The normalization factor can be calculated by demanding \(\int \Psi^\dagger \Psi \, dz = 1\), which gives \(N = \frac{1}{\sqrt{\frac{1}{2}\sqrt{\frac{q}{4\pi\hbar}}}^2}\). The states (21) - (22) are each eigenstates of spin and energy, and represent Gaussian wave packets with minimum uncertainty product of position and momentum. The width of the packets are governed by the envelope parameter \(q\). These states have continuous energy spectrum but they are not representing freely moving particles, rather the particles are bound. As such, it is better to say that the particles are quasiparticles.

4 Solution in two dimension for massless states

To solve Eq.(2) in 2D, we use \(m = 0\), for that gives the theory an opportunity to be applied to systems like graphene. And for that matter, we use in Eq.(2) \(\vec{\sigma}\) matrices in place of \(\vec{\alpha}\) matrices and assume \(\vec{\sigma} = (\sigma_x, \sigma_y)\), \(\vec{p} = (p_x, p_y)\) and \(\vec{r} = (x, y) = (r \cos \varphi, r \sin \varphi)\). The master equation can then be written as

\[
\left( \begin{array}{cc} 0 & cP_- \\
cP_+ & 0 \end{array} \right) \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = E \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right),
\]

where

\[
P_+ = (p_x - iqy) + i(p_y - iqy),
\]

\[
P_- = (p_x - iqy) - i(p_y - iqy).
\]

We transform Eq.(23) using polar coordinates as defined above and use the ansatz that the solutions are eigenstates of \(J_z = L_z + \frac{\hbar}{2}\sigma_z\) with the eigenvalues of \(L_z\) being \(m\hbar\). Hence, we write the solution as

\[
\Psi(r, \varphi) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = e^{im\varphi} \begin{pmatrix} f(r) \\ e^{i\varphi}g(r) \end{pmatrix}.
\]

Using standard procedure, we obtain the second order differential equation satisfied by \(f(r)\) given by,

\[
\frac{d^2 f}{dr^2} + \left(\frac{1}{r} + \frac{2q}{\hbar r}\right)\frac{df}{dr} + \frac{q^2}{\hbar^2}r^2 f - \frac{m^2}{\hbar^2} f + K_2 f = 0,
\]
where \( K_2 = \frac{E^2}{mc^2} + \frac{2}{m} \). Now, an exactly similar equation is satisfied by \( g(r) \) with only \( m \rightarrow m + 1 \). So, solving Eq.(27) only suffices for both the functions. Next, using an alternative coordinate as defined by \( r' = \sqrt{\frac{\rho}{q}} r \) and continuing with the use of \( r \) for \( r' \), we get from Eq.(27),

\[
d\frac{2}{dr^2} + \left( \frac{1}{r} + 2r \right) \frac{df}{dr} + \left( r^2 - \frac{m^2}{r^2} + K_3 \right) f = 0,
\]

where \( K_3 = \frac{E^2}{mc^2} + 2 \). Now, we write \( f(r) = v(r)e^{-\frac{1}{2}r^2} \) and obtain for \( v(r) \), the following equation:

\[
d\frac{2}{dr^2} v + \frac{1}{r} \frac{dv}{dr} + \left( \rho^2 - \frac{m^2}{r^2} \right) v = 0,
\]

where \( \rho^2 = \frac{E^2}{mc^2} \). Solution of this equation are Bessel functions of integral order and one may choose any one from three types of Bessel functions of integral order. Here we choose the first kind of Bessel functions \( J_m \) and find the solution of Eq. (29) as

\[
\Psi(r, \varphi) = Ne^{im\varphi} e^{-\frac{1}{2}r^2} \left( J_m(\rho r) e^{i\rho} J_{m+1}(\rho r) \right),
\]

where \( k = \rho \sqrt{\frac{\rho}{q}} = \frac{E}{mc} \) are the wave numbers. The dispersion relation is thus \( E = \pm \hbar kc \). Normalizatin constant \( N \) in Eq.(30) can be evaluated using the results of Ref.[17]. We obtain

\[
N^2 \pi \hbar q \exp \left( -\frac{\hbar k^2}{2q} \right) \left[ I_m \left( \frac{\hbar k^2}{2q} \right) + I_{m+1} \left( \frac{\hbar k^2}{2q} \right) \right] = 1,
\]

where \( I_m(z) \) and \( I_{m+1}(z) \) are modified Bessel functions. The functions \( \Psi(r, \varphi) \) of Eq.(30) are eigenstates of \( J_z \), here the total angular momentum, and of energy. The solutions in the present case are wave packets as is evident from the structure of Eq.(30). Moreover, the solutions (30) are similar in form as those found for graphene quantum dots in [18] except the appearance of the Gaussian factor in our case. Hence, we can assume that our extension of the Dirac equation affects only the extent of the wave function in space without affecting the energy level spectrum.

### 5 Solution in three dimension

We now solve Eq.(2) in full. To do so, we decompose \( \Psi(\mathbf{r}) \) as

\[
\Psi(\mathbf{r}) = \begin{pmatrix} \psi_1(\mathbf{r}) \\ \psi_2(\mathbf{r}) \end{pmatrix}.
\]

Using the standard representation of \( \sigma \) through the Pauli matrices \( \sigma \) and using Eq.(32) in Eq.(2), we obtain the two coupled equations given by

\[
c \mathbf{\sigma} \cdot \mathbf{p} \psi_2 - iqc \mathbf{\sigma} \cdot \mathbf{\tau} \psi_2 = (E - mc^2) \psi_1,
\]

where \( \psi_1 \) and \( \psi_2 \) are the wave functions of the electron and the hole, respectively.
\[ c \vec{\sigma} \cdot \vec{p} \psi_1 - iq \vec{c} \cdot \vec{r} \psi_1 = (E - mc^2) \psi_2. \]  

(34)

Based on the symmetries of the Dirac equation, we use the spin-angle functions [19] defined in two-component form as

\[
y^{jm}_{j-\frac{1}{2}}(\hat{r}) = \left( \begin{array}{c} \sqrt{\frac{j+m}{2j}} Y_{j+\frac{1}{2}m-\frac{1}{2}} \\ \sqrt{\frac{j-m+1}{2j}} Y_{j+\frac{1}{2}m+\frac{1}{2}} \end{array} \right), \]

(35)

\[
y^{jm}_{j+\frac{1}{2}}(\hat{r}) = \left( \begin{array}{c} -i \sqrt{\frac{j-m+1}{2j+1}} Y_{j+\frac{1}{2}m-\frac{1}{2}} \\ \sqrt{\frac{j+m+1}{2j+1}} Y_{j+\frac{1}{2}m+\frac{1}{2}} \end{array} \right), \]

(36)

where \( Y \)'s are spherical harmonics with \( j \) the total angular momentum quantum number and \( m \) being the magnetic quantum number associated with \( j \). The functions given by Eqs. (35) and (36) are simultaneous eigenfunctions of \( L^2, S^2, J^2, J_z \). Then, using standard procedure [19], we write

\[
\psi_1(\hat{r}) = u(r)y^{jm}_{j-\frac{1}{2}},
\]

(37)

and

\[
\psi_2(\hat{r}) = -iv(r)y^{jm}_{j+\frac{1}{2}},
\]

(38)

where in Eq. (38), the factor -i is included for later convenience. Now, we can write [19]

\[
\vec{c} \cdot \vec{r} \psi_1(\hat{r}) = \left( \begin{array}{c} \vec{\sigma} \cdot \vec{r} + \frac{\lambda+1}{r} \end{array} \right) u(r),
\]

(39)

where \( \vec{L} \) is the orbital angular momentum operator. Now,

\[
(\vec{\sigma} \cdot \vec{L}) y^{jm}_{l} = \kappa y^{jm}_{l},
\]

(40)

where \( \kappa = -(\lambda + 1) \) for \( l = j + \frac{1}{2} \) and \( \kappa = (\lambda - 1) \) for \( l = j - \frac{1}{2} \), where \( \lambda = j + \frac{1}{2} \). It is to be noted that [19]

\[
\vec{c} \cdot \vec{r} y^{jm}_{l=\pm\frac{1}{2}} = -y^{jm}_{l=\mp\frac{1}{2}}.
\]

(41)

Inserting Eqs. (37)-(41) in Eq. (33) and rearranging, we obtain

\[
\left( \frac{d}{dr} + \frac{\lambda+1}{r} + \frac{q}{\hbar} \right) v(r) = \left( \frac{E - mc^2}{\hbar c} \right) u(r).
\]

(42)

Similarly, we obtain from Eq. (34),

\[
\left( \frac{d}{dr} - \frac{\lambda-1}{r} + \frac{q}{\hbar} \right) u(r) = - \left( \frac{E + mc^2}{\hbar c} \right) v(r).
\]

(43)
We can reduce Eqs.(42) and (43) into uncoupled form by using simple algebra. We do this and use the dimensionless variable \( r' = \sqrt{\frac{q}{\hbar}} r \) and find the following equations (where we keep on using \( r \) which is actually \( r' \)):

\[
\frac{d^2 u}{dr^2} + \left( \frac{2}{r} + 2r \right) \frac{du}{dr} + \left( r^2 - \frac{\lambda(\lambda - 1)}{r^2} \right) u + K_4 u = 0, \tag{44}
\]

\[
\frac{d^2 v}{dr^2} + \left( \frac{2}{r} + 2r \right) \frac{dv}{dr} + \left( r^2 - \frac{\lambda(\lambda + 1)}{r^2} \right) v + K_4 v = 0, \tag{45}
\]

where \( K_4 = \frac{E^2 - m^2 c^4}{\hbar c^2} + 3 \). Using the ansatz

\[
u(r) = \xi(r) \exp \left( -\frac{1}{2} r^2 \right), \tag{46}\]

and using this in Eq.(44), we obtain

\[
\frac{d^2 \xi}{dr^2} + \frac{2}{r} \frac{d\xi}{dr} + \left( \gamma^2 - \frac{\lambda(\lambda - 1)}{r^2} \right) \xi = 0. \tag{47}\]

Similarly, using

\[
v(r) = \chi(r) \exp \left( -\frac{1}{2} r^2 \right), \tag{48}\]

we obtain from Eq.(45),

\[
\frac{d^2 \chi}{dr^2} + \frac{2}{r} \frac{d\chi}{dr} + \left( \gamma^2 - \frac{\lambda(\lambda + 1)}{r^2} \right) \chi = 0, \tag{49}\]

where in Eqs.(47) and (49), \( \gamma^2 = \frac{E^2 - m^2 c^4}{\hbar c^2} \). Solutions of Eq.(47) are spherical Bessel functions \( j_{\lambda'}(\gamma r) \) and \( n_{\lambda'}(\gamma r) \), where, \( \lambda' = \lambda - 1 \). Solutions of Eq.(49) are also spherical Bessel functions, namely, \( j_\lambda(\gamma r) \) and \( n_\lambda(\gamma r) \). Hence, we can write explicitly, using only regular solutions and restoring the original variable \( r \),

\[
\psi_1(r) = N j_{\lambda'}(kr) \exp \left( -\frac{1}{2} \frac{q}{\hbar} r^2 \right) \gamma^m_{\lambda'}, \tag{50}\]

and

\[
\psi_2(r) = -i N j_\lambda(kr) \exp \left( -\frac{1}{2} \frac{q}{\hbar} r^2 \right) \gamma^m_\lambda, \tag{51}\]

where

\[
E^2 = \hbar^2 k^2 c^2 + m^2 c^4, \tag{52}\]

or,

\[
k = \pm \sqrt{\frac{E^2 - m^2 c^4}{\hbar c}}. \tag{53}\]
Evidently, $k$ is independent of $q$, the envelope parameter. Next, we normalize the wavefunction (32) with $\psi_1$ and $\psi_2$ given by Eqs.(50) and (51) where $N$ is the normalization constant. Using results of Ref.[17], we obtain

$$N^2 \left( \frac{\hbar}{q} \right)^{\frac{3}{4}} \sqrt{\frac{\pi}{4}} \exp \left( -\frac{\hbar k^2}{2q} \right) \left[ f_\lambda \left( \frac{\hbar k^2}{2q} \right) + f_\lambda \left( \frac{\hbar k^2}{2q} \right) \right] = 1,$$

(54)

where $f_n(z) = \sqrt{\frac{\pi}{2z}} I_{n+\frac{1}{2}}(z)$ are the modified spherical Bessel function of the first kind. For completeness, we now write explicitly the spinor for $j = \frac{3}{2}$ and $m = \frac{3}{2}$, which is

$$\Psi_{\frac{3}{2} \frac{3}{2}} = N \exp \left( -\frac{1}{2} \frac{q}{\hbar} r^2 \right) \begin{pmatrix}
  j_1(kr)Y_{11} \\
  0 \\
  ij_2(kr)\sqrt{\frac{3}{2}}Y_{21} \\
  -ij_2(kr)\sqrt{\frac{3}{2}}Y_{22}
\end{pmatrix}.$$  

(55)

The solutions found, namely, Eq.(32) with $\psi_1$ and $\psi_2$ given by Eqs.(50) and (51) with the specific example given by Eq.(55) are wave packets in three dimensions carrying total angular momentum and its $z$-component given by the quantum numbers $j$ and $m$ as conserved quantities. Hence, we get here stationary spherical Bessel wave packets carrying angular momentum. Thus, we have found a complete picture of the solutions of Eq.(2) which is an extension of the Dirac equation very similar to Dirac oscillator.

6 Summary and conclusion

In this paper, we have presented an extension of the Dirac equation, very similar to the Dirac oscillator, given by Eq.(1). We have solved the (1+1) case of the equation and the solutions are given by Eqs.(21)-(22). The solutions are spinor wave packets carrying definite spin (1/2 or -1/2) and continuous energy spectrum. They represent bound quasiparticles although the spectrum is continuous. The states are of minimum position-momentum uncertainty product, the widths in position and momentum space are determined by the envelope parameter $q$. This parameter entering Eq.(1) via the operator $(-i\hbar \nabla)$ does not affect the dispersion relation, given by Eq.(19), but only packs the otherwise sinusoidal waves into a Gaussian envelope. This is why we call Eq.(1) the “wave packing Dirac equation”. The solutions can also be looked at as representing freely moving particles, but in that case they will suffer dispersion owing to the nonlinear dispersion relation (19). Extension of the system to the massless case is easy and the solutions remain same. Finally, in section 5, we have solved the equation in full, using spherical polar coordinates and spin-angle functions. The same envelope function shows up and it envelopes the free particle functions given by spherical Bessel functions. In all the three cases, namely, one to three dimensions, we obtain wave packets carrying angular momentum. In
the 1D case, it is the spin that is conserved; in the 2D case, it is the total angular momentum, which in this case is the z-component of total angular momentum, is conserved. In the 3D case, it is the total angular momentum and its z-component that are conserved quantities in the solution. And in all the cases, the solutions are stationary wave packets. In conclusion, we have found a procedure to localize the otherwise unlocalized free Dirac spinors by transforming the free Dirac Hamiltonian to a new form via a similarity transformation.

Acknowledgment

This work is supported by Grant No. PS/2018/2/22 endowed by the Research Centre of Shahjalal University of Science and Technology, Sylhet, Bangladesh.

References

[1] I. Bialynicki-Birula, M. Kalinski and J.H. Eberly, Phys. Rev. Lett. 73, 1777 (1994).
[2] A. Buchleitner and D. Delande, Phys. Rev. Lett. 75, 1487 (1995).
[3] D. Delande, J. Zakrzewski and A. Buchleitner, Europhys. Lett. 32, 17 (1995).
[4] L.Z. Szabo, M.G. Benedict, A. Czirjak and P. Foldi, Phys. Rev. B 88, 075438 (2013).
[5] V. Ya. Demikhovskii, G.M. Maksimova, A.A. Perov and E.V. Frolova, Phys. Rev. A 82, 052115 (2010).
[6] V. Ya. Demikhovskii, G.M. Maksimova and E.V. Frolova, Phys. Rev. B 78, 115401 (2008).
[7] T.M. Rusin and W. Zawadzki, Phys. Rev. B 78, 125419 (2008).
[8] S.T. Park, Phys. Rev. A 86, 062105 (2012).
[9] P. Strange, Phys. Rev. A 96, 052103 (2017).
[10] Q. Zhang and J. Gong, Phys. Rev. A 93, 032117 (2016).
[11] I. Bialynicki-Birula and Z. Bialynicka-Birula, Phys. Rev. Lett. 118, 114801 (2017).
[12] J.J. Mestayer et al., Phys. Rev. Lett. 100, 243004 (2008).
[13] J.A. Franco-Villafune, Phys. Rev. Lett. 111, 170405 (2013).
[14] M. Moshinsky and Szczepaniak, J. Phys. A 22, L817 (1989).
[15] J. Benitez, R.P.M. Romero, H.N. Nunez-Yepez and A.L. Salas-Brito, Phys. Rev. Lett. 64, 1643 (1990).

[16] O. Castanos, A. Frank, R. Lopez and L.F. Urrutia, Phys. Rev. D 43, 544 (1991).

[17] Th. A. Rijken, arXiv:1409.5593 v1[nucl-th].

[18] P. Hewageegana and V. Apalkov, Phys. Rev. B 77, 245426 (2008).

[19] J.J. Sakurai, in Modern Quantum Mechanics (Addison-Wesley, Reading, 1994).