GROUPOID SEMIDIRECT PRODUCT FELL BUNDLES II —
PRINCIPAL ACTIONS AND STABILIZATION

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Abstract. Given a free and proper action of a groupoid on a Fell bundle (over
another groupoid), we give an equivalence between the semidirect-product and
the generalized-fixed-point Fell bundles, generalizing an earlier result where
the action was by a group. As an application, we show that the Stabilization
Theorem for Fell bundles over groupoids is essentially another form of crossed-
product duality.

1. Introduction

Let $H$ be a locally compact Hausdorff groupoid, and let $p : A \to H$ be a Fell
bundle. [KMQW13a, Corollary 3.4] shows that if a locally compact group $G$ acts
principally (i.e., freely and properly) by automorphisms on $A$, then there is a “Yam-
agami equivalence” between the semidirect-product Fell bundle $A \rtimes G \to H \rtimes G$ and
the quotient Fell bundle $G \backslash A \to G \backslash H$. The equivalence theorem of [MW08
Theorem 6.4] (based upon an unpublished preprint of Yamagami [Yam]) then gives
a Morita equivalence $C^*(H \rtimes G, A \rtimes G) \sim_M C^*(G \backslash H, G \backslash A)$ between the Fell-
bundle algebras. [KMQW13b, Theorem 3.1] shows that this should be regarded as
a version of Rieffel’s equivalence theorem [Rie90, Corollary 1.7] for a “universal”
generalized fixed-point algebra, whereas Rieffel’s original theorem is in some sense
a “reduced” version (see also [BE14, Proposition 2.2]). Similarly, $C^*(G \backslash H, G \backslash A)$
should be regarded the universal generalized fixed-point algebra for the action of
$G$ on $A$.

In this paper we generalize [KMQW13a, Corollary 3.4] to allow $G$ to be a
locally compact groupoid. This requires some preliminary work: a key tool in
[KMQW13a] is a result of Palais (see the discussion preceding [Pal61
Proposition 1.3.4]) that characterizes principal group bundles as pullbacks. This was
paralleled in [KMQW13a, Theorem A.16] to a characterization of Fell bundles carrying
a principal action of a group. Here we need to begin with a generalization (Theo-
rem 3.8) of Palais’ theorem for principal bundles associated to an action of a
groupoid, rather than just a group. We also need to generalize much of the “in-
frastructure” developed in [KMQW13a, Appendix] from groups to groupoids. This
includes, for example, quotients and semidirect products of Fell bundles by actions
of a groupoid, and a Palais-type classification (Theorem 4.15) of principal actions
of groupoids on Fell bundles.

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2. Preliminaries

We refer to [HKQW] for our conventions regarding groupoids, Fell bundles, Haar systems, actions of groupoids by isomorphisms on either groupoids or Fell bundles, and semidirect product groupoids and Fell bundles.

Throughout, $G$ and $H$ will be second countable, locally compact Hausdorff groupoids with Haar systems $\lambda_G = \{ \lambda^u_G \}_{u \in G^{(0)}}$ and $\lambda_H = \{ \lambda^v_H \}_{v \in H^{(0)}}$, respectively.

If $T$ is a left $G$-space, then we use $G\backslash T$ to denote the space of orbits with the quotient topology. Since we assume $G$ has a Haar system, the quotient map $q : T \to G\backslash T$ is open [Wil19, Proposition 2.12].

If $p : A \to G$ is a Fell bundle and $q : E \to T$ is an upper-semicontinuous Banach bundle over a $G$-space $T$, then as in [MW08, §6] we say that $A$ acts on (the left of) $E$ if there is a continuous map $(a, e) \mapsto a \cdot e$ from $A \ast E = \{(a, e) : s(a) = q(e)\}$ to $E$ that is bilinear on $A(\gamma) \times E(s(\gamma))$ and such that

- (A1) $q(a \cdot e) = p(a) \cdot q(e),$
- (A2) $a \cdot (b \cdot e) = (ab) \cdot e$ if $(b, e) \in A \ast E$ and $(a, b) \in A^{(2)},$ and
- (A3) $\|a \cdot e\| \leq \|a\|\|e\|.$

Fell Bundle Equivalence. Suppose that $T$ is a $(G, H)$-equivalence. Then there is a continuous open map $(t, t') \mapsto \tau_G(t, t')$ from $T \ast_T T \to G$ such that $\tau_G(t, t') : t' = t$ (see [Wil19 Lemma 2.42]). Similarly, there is a continuous open map $\tau_H : T \ast_T T \to H$ such that $t \cdot \tau_H(t, t') = t'.

Definition 2.1 ([MW08 Definition 6.1]). Suppose that $T$ is a $(G, H)$-equivalence and $p : A \to G$ and $\overline{p} : B \to H$ are Fell bundles. We say that an upper semicontinuous Banach bundle $q : E \to T$ is an $A - B$-equivalence if the following conditions are met.

- (E1) There is a left action of $A$ and a right action of $B$ on $E$ such that $a \cdot (e \cdot b) = (a \cdot e) \cdot b$ for all $a \in A, e \in E,$ and $b \in B.$
- (E2) There are sesquilinear maps $(e, f) \mapsto (e, f)$ from $E \ast E$ to $A$ and $(e, f) \mapsto (e, f)$ from $E \ast E$ to $B$ such that
  - (a) $p((e, f)) = \tau_G(q(e), q(f))$ and $\overline{p}(e, f) = \tau_H(q(e), q(f)),$
  - (b) $(e, f)^* = (f, e)^*$ and $(e, f)^* = (f, e)^*,$
  - (c) $(a \cdot e, f) = a(e, f)$ and $(e, f \cdot b)^* = (e, f)^* b,$ and
  - (d) $(e, f) \cdot g = e \cdot (f, g)^*.$
- (E3) With the actions coming from [E1] and the inner products from [E2] each $E(t)$ is an $A(r(t)) - B(s(t))$-imprimitivity bimodule.

The Equivalence Theorem—that is, [MW08, Theorem 6.4]—implies that if there is a $A - B$-equivalence as above, then $C^*(G, A)$ and $C^*(H, B)$ are Morita equivalent.

3. Bundles and Proper Actions

Recall that if $G$ is a groupoid and $T$ is a locally compact $G$-space, then we say that $G$ acts properly or that $T$ is a proper $G$-space if the map $\Theta : G \ast T \to T \times T$ given by $\Theta(x, t) = (x \cdot t, t)$ is proper in that the inverse image of every compact set in $T \times T$ is compact in $G \ast T.$

1Note that there is a typo in condition (c) in [MW08 §6].
Proposition 3.1. Suppose that $T$ is a locally compact $G$-space. Then the following are equivalent.

(PA1) $G$ acts properly on $T$.
(PA2) For every pair of compact sets $K$ and $L$ in $T$,
$$P(K, L) = \{ x \in G : K \cap x \cdot L \neq \emptyset \}$$

is compact in $G$.
(PA3) If $t_i \to t$ and $x_i \cdot t_i \to t'$ in $T$, then $\{ x_i \}$ has a convergent subnet in $G$.

If in addition $G$ acts freely on $T$, then the above conditions are equivalent to
(PA4) If $t_i \to t$ and $x_i \cdot t_i \to t'$ in $T$, then $\{ x_i \}$ is convergent in $G$.

Proof. This is a combination of [Will19] Proposition 2.17 and Corollary 2.26. □

Remark 3.2. It should be noted that this definition of a proper groupoid action is only appropriate for actions of locally compact groupoids on locally compact spaces. In [Pal61], for group actions, Palais writes $((K, L))$ in place of $P(K, L)$ in Proposition 3.1. If $G$ and $T$ are completely regular, then Palais calls a $G$-space $T$ proper if each point has a neighborhood $S$ such that every point in $T$ has a neighborhood $U$ such that $((S, U))$ has compact closure [Pal61, Definition 1.2.2]. Our definitions coincide for locally compact group actions on locally compact (Hausdorff) spaces by [Pal61, Theorem 1.2.9]. We will discuss proper actions on locally compact Hausdorff spaces.

Note that if $t = t'$ in (PA4), then we must have $x_i \to x$ for some $x \in G$, and $x_i \cdot t_i \to x \cdot t$. Since $T$ is Hausdorff, $x \cdot t = t$ and $x \in G(t) = \{ x \in G : x \cdot t = t \}$. Thus we have the following useful observation.

Corollary 3.3. Suppose that $T$ is a locally compact free and proper $G$-space with moment map $\rho : T \to G^{(0)}$. If $t_i \to t$ and $x_i \cdot t_i \to t$, then $x_i \to \rho(t)$.

Recall from the beginning of Section 3 of [HKQW] our conventions regarding $G$-bundles $(T, p, B)$.

Definition 3.4. We will call a $G$-space $T$ principal if the $G$-action is both free and proper. Then a $G$-bundle $(T, p, B)$ is called principal if $T$ is principal.

Lemma 3.5. If $G$ is a locally compact groupoid and $p : T \to B$ and $p' : T' \to B$ are principal $G$-bundles, then every $G$-bundle morphism $f : T \to T'$ over $B$ is an isomorphism.

Proof. To see that $f$ must be injective, suppose that $f(t) = f(t')$. Then
$$p(t) = p'(f(t)) = p'(f(t')) = p(t').$$
Thus there is a unique $x \in G$ such that $t' = x \cdot t$. But then $f(t) = f(t') = x \cdot f(t)$ and $x = \rho_T(t)$ since that action is free. Thus $t = t'$ as required.

To see that $f$ is surjective, let $t' \in T'$. Since $p$ is surjective, there is a $t \in T$ such that $p(t) = p'(t')$. Then $p'(f(t)) = p(t) = p'(t')$. Hence there is an $x \in G$ such that $x \cdot f(t) = t'$. But then $f(x \cdot t) = t'$. Thus $f$ is surjective.

To show that $f$ is open, we employ Fell’s Criterion. Suppose that $t'_i \to f(t)$ in $T'$. Then $p'(t'_i) \to p'(f(t)) = p(t)$ in $B$. Since $p$ is open, we can lift $\{ p'(t'_i) \}$ using Fell’s Criterion (see [HKQW, Lemma 2.1]). Thus, after passing to a subnet and

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2It should be emphasized that we do not require our principal $G$-spaces to be locally trivial.
relabeling, we can find \( t_i \to t \) in \( T \) such that \( p(t_i) = p'(t'_i) \). Then \( p'(f(t_i)) = p'(t'_i) \) and there are \( x_i \in G \) such that \( x_i \cdot f(t_i) = t'_i \). But then \( f(t_i) \to f(t) \) while \( x_i \cdot f(t_i) \to f(t) \). By Corollary 3.3 this forces \( x_i \to \rho_T(f(t)) \). But since \( f \) is \( G \)-equivariant, \( \rho_T(f(t)) = \rho_T(t) \). Therefore \( x_i \cdot t_i \to t \) and \( f(x_i \cdot t_i) = t'_i \).

**Lemma 3.6 (Diagonal Actions).** If \( T \) and \( U \) are \( G \)-spaces, then the diagonal action of \( G \) on \( T \ast U \) is principal if either \( T \) or \( U \) is principal.

**Proof.** This follows easily from item [PA4] of Proposition 3.1. Suppose \( T \) is proper and that both \((t_1, u_1) \to (t, u)\) and \( x_i \cdot (t_i, u_i) \to (t', u') \) in \( T \ast U \). Then \( t_i \to t \) and \( x_i \cdot t_i \to t' \). Then \( \{x_i\} \) is convergent by [PA4].

**Lemma 3.7.** Let \((T, p, B)\) be a bundle and \( f : C \to B \) a continuous map. Then the pullback bundle \( f^*T \) is principal whenever \( T \) is.

**Proof.** If \( T \) is principal, then we can use item [PA4] of Proposition 3.1 to see that \( f^*T \) is principal just as in Lemma 3.6.

We have the following universal characterization for principal bundle pullbacks, which is modeled on that for groups due to Palais mentioned in the introduction. Our conventions for pull-backs are recorded immediately preceding Lemma 3.2 in [HKQW].

**Theorem 3.8 (Palais).** Suppose that \((f, g) : (T, p, B) \to (U, q, C)\) is a morphism of principal \( G \)-bundles. Then \( T \) is isomorphic to the pullback \( g^*U \). In particular, the canonical map \( h \) making the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{f} & U \\
\downarrow{h} & & \downarrow{\pi_2} \\
B & = & B \ast U \\
\downarrow{\pi_1} & & \downarrow{q} \\
C & \rightarrow & C
\end{array}
\]

commute is a \( G \)-bundle isomorphism over \( B \).

**Proof.** The pullback \( g^*U \) is principal by Lemma 3.7. Moreover, \( h \) is \( G \)-equivariant:

\[
h(x \cdot t) = (p(x \cdot t), f(x \cdot t)) = (p(t), x \cdot f(t)) = x \cdot (p(t), f(t)) = x \cdot h(t).
\]

Thus \( h \) is a \( G \)-bundle isomorphism by Lemma 3.3.

**Definition 3.9.** We say that an action of \( G \) on a bundle \( p : T \to B \) over a locally compact space \( B \) is principal if the action on the base space \( B \) is principal.

**Remark 3.10.** Suppose that \( G \) acts principally on a bundle \( p : T \to B \) as above. Then the \( G \)-action on \( T \) must be free since \( x \cdot t = t \) implies that \( x \cdot p(t) = p(t) \). Furthermore, the action on \( T \) must also satisfy item [PA4] of Proposition 3.1 if \( t_i \to t \) and \( x_i \cdot t_i \to t' \), then \( p(t_i) \to p(t) \) and \( x_i \cdot p(t_i) \to p(t') \). Hence \( \{x_i\} \) converges since the \( G \)-action on \( B \) is proper. Hence if \( T \) is locally compact, it is a principal \( G \)-space. However, we want to allow for the case that \( T \) might be neither locally compact nor Hausdorff—specifically, \( T \) may be a Fell bundle. But we still will have \( B \) locally compact and Hausdorff. Hence if \( G \)-acts principally on the bundle and
Let $x_i \rightarrow t$ while $x_i \cdot t_i \rightarrow t$, then we have $x_i \rightarrow x$ and then $x_i \cdot t_i \rightarrow x \cdot t$. But then $p(t) = x \cdot p(t)$, and since $B$ is Hausdorff, it follows that $x = \rho_B(p(t)) = \rho_T(t)$, and we recover the conclusion of Corollary 3.3 in this case.

4. Actions by isomorphisms

**Definition 4.1.** Let $H$ be a locally compact groupoid, and let $G$ act on the space $H$. We say the $G$-action on $H$ is principal and by isomorphisms if it is principal as an action on the space $H$ and is also an action by isomorphisms.

**Remark 4.2.** Actions by isomorphisms are also discussed in [BM16].

We want to extend [KMQW13a, Proposition A.10] to principal groupoid actions. Recall that if $A$ and $B$ are subsets of a groupoid $H$, then $AB = \{ab : (a,b) \in A \times B \cap H^{(2)}\}$.

**Lemma 4.3.** Suppose that $G$ acts freely on $H$ by isomorphisms. If $G \cdot s_H(h) = G \cdot \tau_H(k)$, then
\[(G \cdot h)(G \cdot k) = G \cdot (h'k')\]
for any $h' \in G \cdot s_H(h)$ and $k' \in G \cdot \tau_H(k)$ such that $s_H(h') = \tau_H(k')$.

**Proof.** Since we can replace $h$ by $h'$ and $k$ by $k'$ without changing the left-hand side of (4.1), we may as well assume $s(h) = \tau(k)$ from the start. We can assume (see [HKQW, Remark 5.2]) that $h, k, hk \in H_u$ where $u = \rho(s(h)) = \rho(\tau(k))$, and where in turn $\tau : H \rightarrow G^{(0)}$ is the moment map. Thus $G \cdot h = G_u \cdot h, G \cdot k = G_u \cdot k$ and $G \cdot (hk) = G_u \cdot (hk)$. Since $x \cdot (hk) = (x \cdot h)(x \cdot k)$, we have $G \cdot (hk) \subset (G \cdot h)(G \cdot k)$.

On the other hand, if $s(x \cdot h) = r(y \cdot k)$, then
\[x \cdot s(h) = y \cdot r(k)\]
Since $s(h) = \tau(k)$ and $G$ acts freely, we must have $x = y$, therefore $(x \cdot h)(x \cdot k) = x \cdot (hk) \in G \cdot (hk)$. Thus $(G \cdot h)(G \cdot k) \subset G \cdot (hk)$ and (4.1) holds. \qed

Let
\[(G \cdot H)^{(2)} = \{(G \cdot h, G \cdot k) \in G \cdot H \times G \cdot H : G \cdot s(h) = G \cdot \tau(k)\}\]
Using Lemma 4.3 we get a well-defined map from $(G \cdot H)^{(2)}$ to $G \cdot H$ sending $(G \cdot h, G \cdot k) \rightarrow (G \cdot h)(G \cdot k)$. We can also define $(G \cdot h)^{-1} = G \cdot h^{-1}$.

**Proposition 4.4.** Suppose that $G$ acts principally on $H$ by isomorphisms. Let $(G \cdot H)^{(2)}$ be as in (4.2). Then with multiplication and inversion defined as above, and assuming $G$ has an open range map, $G \cdot H$ is a locally compact groupoid with unit $H = G \cdot H^{(0)}$. Then range and source maps on $G \cdot H$ are given by $r(G \cdot h) = G \cdot \tau_H(h)$ and $s(G \cdot h) = G \cdot s_H(h)$.

If $H$ has an open range map, then so does $G \cdot H$.

**Proof.** It is routine to verify that $G \cdot H$ satisfies axioms (a), (b), and (c) of [Wil19, Definition 1.2]. Hence $G \cdot H$ is a groupoid as claimed. Furthermore $r_{G \cdot H}(G \cdot h) = (G \cdot h)(G \cdot h)^{-1} = G \cdot hh^{-1} = G \cdot r(h)$. Similarly $s_{G \cdot H}(G \cdot h) = G \cdot s(h)$, and we have $(G \cdot H)^{(0)} = G \cdot H^{(0)}$. 

\[\]
Since the action of $G$ on $H$ is proper, $G \backslash H$ is locally compact Hausdorff by \cite[Proposition 2.18]{Wil19}.
To see that $G \backslash H$ is a topological groupoid, we need to verify that the operations are continuous. To that end, suppose that $(G \cdot h_i, G \cdot k_i) \to (G \cdot h, G \cdot k)$ in $(G \backslash H)^{(2)}$. There is no harm in assuming $s(h_i) = r(k_i)$ and that $s(h) = r(k)$. Then we want to see that $G \cdot (h_i k_i) \to G \cdot (hk)$ in $G \backslash H$. Since it will suffice to show that every subnet has a subnet converging to $G \cdot (hk)$, we can replace $\{G \cdot (h_i k_i)\}$ with a subnet, relabel, and show that $\{G \cdot (h_i k_i)\}$ has a convergent subnet. Since $G$ has open range and source maps, the orbit map from $G$ to $G \backslash H$ is open. Hence we can pass to a subnet, relabel, and assume there are $x_i, y_i \in G$ such that $x_i \cdot h_i \to h$ and $y_i \cdot k_i \to k$. Letting $s_H(h_i) = v_i = r_H(k_i)$, we must have $x_i, y_i \in G_u$, where $u_i = \rho(v_i)$ as in the proof of Lemma 4.3.

Then we have $x_i \cdot v_i \to v$ while $y_i \cdot v_i \to v$ where $s_H(h) = v = r_H(k)$. But then $y_i \cdot v_i = (y_i x_i^{-1}) x_i \cdot v_i \to v$. It follows from Corollary 3.3 that $y_i x_i^{-1} \to \rho(v)$. Thus $y_i \cdot h_i = (y_i x_i^{-1}) x_i \cdot h_i \to h$. But then $y_i \cdot (h_i k_i) = (y_i \cdot h_i)(y_i \cdot k_i) \to hk$. Thus $G \cdot (h_i k_i) \to G \cdot (hk)$ as required. The continuity of inversion is similar, but more straightforward.

To see that the range map on $G \backslash H$ is open, we use Fell’s Criterion. Suppose that $G \cdot v_i \to G \cdot r_H(h) = r(G \cdot h)$. Since the range map on $G$ is open, we can pass to a subnet, relabel, and assume that there are $x_i \in G$ such that $x_i \cdot v_i \to r_H(h)$. Since $r_H$ is open, we can pass to another subnet and find $h_i \to h$ such that $r_H(h_i) = x_i \cdot v_i$. Then $G \cdot h_i \to G \cdot h$ and $r(G \cdot h_i) = G \cdot v_i$. \hfill $\Box$

Remark 4.5. Observe that when $G$ acts principally and by isomorphisms on $H$, the $G$-bundle map $H \to G \backslash H$ is a surjective groupoid homomorphism.

Example 4.6. Let $H$ be the action groupoid for the right action of $G$ on itself as in \cite{HKQW} Example 5.6. Then $(x, y, xy) \mapsto y$ induces an isomorphism of $G \backslash H$ onto $G$.

Proposition 4.7. Suppose that $H$ has a Haar system $\lambda_H$, that $G$ acts principally on $H$ by isomorphisms, and that the action is invariant as in \cite{HKQW} Definition 5.7. Then $G \backslash H$ has a Haar system $\lambda_{G \backslash H}$ given by

$$
\int_{G \backslash H} g(G \cdot h) \ d\lambda_{G \backslash H}^G(h) = \int_H g(G \cdot h) \ d\lambda_H(h)
$$

for $g \in C_c(G \backslash H)$ and $v \in H^{(0)}$.

Proof. Let $r_H : H \to H^{(0)}$. Then $\lambda_H = \{ \lambda_H^v \}_{v \in H^{(0)}}$ is an equivariant $r_H$-system. Thus $\lambda_{G \backslash H} = \{ \lambda_{G \backslash H}^v \}_{G \cdot v \in G \backslash H}$ is $r_{G \backslash H}$-system by \cite[Proposition 3.14]{Wil19}. The rest is routine. \hfill $\Box$

Definition 4.8. If an action of $G$ on a Fell bundle $p : A \to H$ is principal as a bundle action and is also an action by isomorphisms, we say that $G$ acts principally and by isomorphisms.

Assume that $p : A \to H$ is a Fell bundle and that $G$ acts on $A$ principally and by isomorphisms as above. Then $G \backslash H$ is a locally compact Hausdorff groupoid by Proposition 4.4. Since $p$ is equivariant, we have a continuous map

$$
\overline{p} : G \backslash A \to G \backslash H
$$
given by $G \cdot a \mapsto G \cdot p(a)$. Here $G \backslash A$ is the orbit space with the quotient topology. Using Fell’s Criterion, we can see that $\overline{p}$ is also open: suppose $G \cdot h_i \to G \cdot p(a) = \overline{p}(a)$ in $G \backslash H$. Then we may as well assume that $h_i \to p(a)$. Since $p$ is open, we can pass to subnet, relabel, and assume that there are $a_i \to a$ in $A$ such that $p(a_i) = h_i$. Then $G \cdot a_i \to G \cdot a$ and $\overline{p}(G \cdot a_i) = G \cdot h_i$.

To see that $\overline{p} : G \backslash A \to G \backslash H$ is an upper semicontinuous Banach bundle, we first have to equip each fibre $\overline{p}^{-1}(G \cdot h)$ with a Banach space structure. But $a \mapsto G \cdot a$ is a bijection of $p^{-1}(h') = A(h')$ onto $\overline{p}^{-1}(G \cdot h)$ for any $h' \in G \cdot h$. Since $G$ acts by isomorphisms as in [HKQW] Definition 5.9(b)], we can use this map to impose a well-defined Banach space structure on $\overline{p}^{-1}(G \cdot h)$: define $\|G \cdot a\| = \|a\|$, $G \cdot a + G \cdot b = G \cdot (a + b)$, and $\alpha G \cdot a = G \cdot (\alpha a)$. Since $G$ acts by isomorphisms, these operations are independent of our choice of $h' \in G \cdot h$.

**Proposition 4.9.** Suppose that $p : A \to H$ is a Fell bundle on which $G$ acts principally and by isomorphisms. Let $\overline{p} : G \backslash A \to G \backslash H$ be as above. Then the map $\overline{p} : G \backslash A \to G \backslash H$ introduced above is a Fell bundle with operations $(G \cdot a)(G \cdot b) = G \cdot (ab)$ provided $s(a) = r(b)$, and $(G \cdot a)^* = G \cdot a^*$.

**Proof.** First, to see that $\overline{p} : G \backslash A \to G \backslash H$ is an upper-semicontinuous Banach bundle, we need to verify axioms (B1)–(B4) of [HKQW] Definition 2.4. This can be done almost exactly as in [KMRW98] Proposition 2.15 with minor alterations.

To verify (B1), suppose that $G \cdot a_i \to G \cdot a$ with $\|a_i\| \geq \varepsilon > 0$ for all $i$. Then, after passing to a subnet and relabeling, we can assume that there are $x_i \in G$ such that $x_i \cdot a_i \to a$. Since the action of $G$ must be isometric, we have $\|a\| \geq \varepsilon$. This verifies (B1).

(B2): Suppose that $(G \cdot a_i, G \cdot b_i) \to (G \cdot a, G \cdot b)$ in $G \backslash A \times G \backslash A$. Then $\overline{p}(G \cdot a_i) = G \cdot p(a_i) = G \cdot p(b_i) = \overline{p}(G \cdot b_i)$ for all $i$. By adjusting $a_i$ and $b_i$ within their orbits, we can assume that $p(a_i) = p(b_i)$ and $p(a) = p(b)$. Thus we need to see that $G \cdot (a_i + b_i) \to G \cdot (a + b)$ in $G \backslash A$. If this is not the case, then we can pass to a subnet, relabel, and assume that there is a neighborhood $U$ of $G \cdot (a + b)$ that contains no $G \cdot (a_i + b_i)$. Since $G$ has open range and source maps, the orbit map from $A$ to $G \backslash A$ is open [W110] Proposition 2.12. Hence we can pass to another subnet, relabel, and assume that there are $x_i \in G$ such that $x_i \cdot (a_i + b_i) \to (a + b)$. But then $G \cdot (a_i + b_i) \to G \cdot (a + b)$ which leads to a contradiction. This proves (B2).

The validity of (B3) and (B4) follow by similar, albeit easier, arguments.

To see that the indicated operations give a Fell bundle, we first need to check that the multiplication is well-defined: Suppose that $(G \cdot a, G \cdot b) \in G \backslash A^{(2)}$: that is, $(G \cdot p(a), G \cdot p(b)) \in G \backslash H^{(2)}$. As in the proof of Lemma 4.3 we can adjust $p(a)$ and $p(b)$ in their orbits, and hence $a$ and $b$ in theirs, so that $(p(a), p(b)) \in H^{(2)}$. Then the “product” $(G \cdot a)(G \cdot b)$ is the orbit $G \cdot (ab)$. Hence, just as in Proposition 4.4 we get a well-defined map from $G \backslash A^{(2)} \to G \backslash A$ sending $(G \cdot a, G \cdot b)$ to $G \cdot (ab)$. Properties (FB1)–(FB3) of [HKQW] Definition 2.5 are routine. Since $a \mapsto G \cdot a$ is an isomorphism of $A(h)$ onto $(pb)^{-1} \cdot (G \cdot h)$, (FB4)–(FB5) follow since $A(h)$ is given to be an $A(r(h)) - A(s(h))$-imprimitivity bimodule. □

**Lemma 4.10.** Let a locally compact Hausdorff space $T$ have commuting actions by locally compact groupoids $G$ and $K$ such that the moment map $p_K : T \to K^{(0)}$

---

3Note that $A$ is almost never locally compact—the relative topology on the fibres is the Banach space topology—and may not even be Hausdorff (see [W107] Example C.27]).
is a principal $G$-bundle. Then we can view $K \times T = K \ast T$ as the pullback by $s_K: K \to K^{(0)}$.

(a) The pullback action $G \curvearrowright K \times T$ given by $x \cdot (k, t) = (k, x \cdot t)$ is principal and by isomorphisms.

(b) The map $\theta: G\backslash (K \times T) \to K$ given by $\theta([k, t]) = k$ is a groupoid isomorphism.

(c) The right action of $K$ on $K \times T$ given by $(k, t) \cdot \ell = (k\ell, \ell^{-1} \cdot t)$, with moment map $\sigma(k, t) = s(k)$, is principal.

Proof. For part (a), it follows from Lemma 3.7 that the pullback $G$-action is principal, and it is easy to see that the action is by isomorphisms:

$$x \cdot (k\ell, t) = (k\ell, x \cdot t) = (k, \ell \cdot (x \cdot t))(\ell, x \cdot t) = (k, \ell \cdot t)(\ell, x \cdot t) = (x \cdot (k, \ell \cdot t))(x \cdot (\ell, t)).$$

For part (b), note that the coordinate projection $\pi_1: (k, t) \mapsto k$ factors through $G\backslash (K \times T)$ and we get a continuous bijection $\theta$ making the diagram

\[
\begin{array}{ccc}
K \times T & \xrightarrow{q} & G\backslash (K \times T) \\
\downarrow{\pi_1} & & \\
K & \xrightarrow{\theta} & K \backslash (K \times T)
\end{array}
\]

commute. Then $\theta$ is a homeomorphism because $\pi_1$ and $q$ are open, and is a groupoid isomorphism because $\pi_1$ and $q$ are groupoid homomorphisms.

Part (c) is an immediate consequence of Lemma 3.6 (adapted to right actions). □

Our next result shows that the situation in Lemma 4.10 is generic for principal groupoid actions. This result parallels the corresponding result in the group case proved in [KMQW13a, Theorem A.15]. Note that the unit space of the action groupoid $H \times T$ can be identified with $T$.

**Theorem 4.11.** Let $G$ and $H$ be locally compact groupoids, let $G$ act principally and by isomorphisms on $H$, and let $K = G\backslash H$ be the quotient groupoid. Then there is an action of $K$ on $H^{(0)}$ such that

(a) $H$ is $G$-equivariantly isomorphic to the action groupoid $K \times H^{(0)}$, and

(b) the restricted action $G \curvearrowleft H^{(0)}$ commutes with the action of $K$.

Proof. Let $q: H \to K$ be the quotient map, and $q|: H^{(0)} \to K^{(0)}$ the restriction to unit spaces. We have a commuting diagram

\[
\begin{array}{ccc}
H^{(0)} & \xleftarrow{r_H} & H & \xrightarrow{s_H} & H^{(0)} \\
\downarrow{q|} & & \downarrow{q} & & \downarrow{q|} \\
K^{(0)} & \xleftarrow{r_K} & K & \xrightarrow{s_K} & K^{(0)}
\end{array}
\]

where $(s_H, s_K)$ and $(r_H, r_K)$ are principal $G$-bundle morphisms. So by Theorem 3.8 we can form the pullbacks $r_K^* K^{(0)} = K \ast H^{(0)} = \{(k, u) : r_K(u) = q|((u))\}$ and
\[ s^*_K K = K * s H^{(0)} = \{ (k, u : s_K(k) = q(u) \} \], and the canonical \( G \)-bundle isomorphisms \( \theta_r \) and \( \theta_s \) such that the diagram

\[
\begin{array}{ccc}
H^{(0)} & \xleftarrow{\pi_2} & K * r H^{(0)} \\
\downarrow q & & \downarrow \pi_1 \\
K^{(0)} & \xleftarrow{s_K} & K
\end{array}
\begin{array}{ccc}
& \xleftarrow{\theta_r} & \\
& & H \\
& \downarrow \theta_s & \\
& \downarrow \pi_2 & \\
& K * s H^{(0)} & \\
& r_K & \\
& K & \xleftarrow{r_K} K^{(0)}
\end{array}
\]

commutes. At this point, the only structure on the pullbacks \( K * s H^{(0)} \) and \( K * r H^{(0)} \) is the \( G \)-bundle structure coming from Theorem 3.8. We use the homeomorphisms \( \theta_r \) and \( \theta_s \) to impose groupoid structures on these pullbacks. Then, by definition,

\[
\theta := \theta_r \circ \theta_s^{-1} : K * s H^{(0)} \to K * r H^{(0)}
\]

is an isomorphism. If \( \theta(k, u) = (k', u') \), then \( k' = \pi_1 \circ \theta_r \circ \theta_s^{-1} = q(\theta^{-1}_s(k, u)) = k \).

It follows that \( \theta \) has the form

\[
\theta(k, u) = (k, k \cdot u)
\]

for some continuous map

\[
(k, u) \mapsto k \cdot u : K * s H^{(0)} \to H^{(0)}.
\]

We claim that this gives an action of the groupoid \( K \) on the space \( H^{(0)} \) with respect to the moment map \( \rho = q| : H^{(0)} \to K^{(0)} \). Thus we need to show that \( q|(u) \cdot u = u \), and that \( k \cdot (\ell \cdot u) = (k \ell) \cdot u \) for composable \( k, \ell, \) and \( u \).

We claim that in the imposed groupoid structure on \( K * s H^{(0)} \),

\[
s(k, u) = (s_K(k), u) \quad \text{and} \quad r(k, x) = (r_K(k), k \cdot x).
\]

To verify the formula for the range map, we use the commutativity of (4.3); let \( h = \theta_s^{-1}(k, u) \). Then

\[
\pi_2(r(k, u)) = \pi_2(r(\theta_s(h)))
\]

\[
= \pi_2(\theta_s(r_H(h)))
\]

\[
= s_H(r_H(h))
\]

\[
= r_H(h)
\]

\[
= \pi_2(\theta_r(h))
\]

\[
= \pi_2(\theta_r \circ \theta_s^{-1}(k, u))
\]

\[
= k \cdot u.
\]

Similarly,

\[
\pi_1(r(k, u)) = \pi_1(r(\theta_s(h)))
\]

\[
= \pi_1(\theta_s(r_H(h)))
\]

\[
= q(r_H(h)) = q(r_H(h))
\]

\[
= r_K(q(h)) = r_K(k).
\]

The formula for the source map is proved similarly.

\footnote{Note that \( q| : H^{(0)} \to K^{(0)} = G \backslash H^{(0)} \) is open since \( G \) has an open range map.}
Then \( (k, u, (\ell, v)) \in (K * H^{(0)})^{(2)} \) if and only if \( (k, \ell) \in K^{(2)} \) and \( u = \ell \cdot v \).

We claim
\[
(k, u)(\ell, v) = (k, \ell \cdot v)(\ell, v) = (k\ell, v).
\]
The verify the claim, let \( (k, u) = \theta_\ell(h) \) and \( (\ell, v) = \theta_s(k) \). We have
\[
\pi_1((k, u)(\ell, v)) = \pi_1 \circ \theta_\ell(hk) = q(hk) = q(h)q(k) = k\ell,
\]
and similarly
\[
\pi_2((k, u)(\ell, v)) = \pi_2 \circ \theta_s(hk) = s_H(hk) = s_H(k) = v.
\]
This establishes (4.4).

We now see that
\[
r(k, \ell \cdot u) = r((k, \ell \cdot u)(\ell, u)) = r(k\ell, u),
\]
consequently
\[
k \cdot (\ell \cdot u) = (k\ell) \cdot u.
\]
Since \( s(q|(u), u) = (q|(u), u) \), we must also have \( (q|(u), u) = r(q|(u), u) = (q|(u), q|(u) \cdot u) \), so \( q|(u) \cdot u = u \) and we have an action of \( K \) on \( H^{(0)} \). Moreover,
\[
\theta_s : H \to K * H^{(0)}
\]
is an isomorphism of locally compact groupoids.

We verify that this isomorphism is \( G \)-equivariant:
\[
x \cdot \theta_s(h) = x \cdot (q(h), s_H(h)) = (q(h), x \cdot s_H(h)) = (q(h), s_H(x \cdot h))
\]
\[
= (q(x \cdot h), s_H(x \cdot h)) = \theta_s(x \cdot h).
\]
This establishes part (a), and then (b) follows from the computation
\[
(r_H(h), h \cdot (x \cdot t)) = r(h, x \cdot t)
\]
\[
= r((x \cdot (h, t))
\]
\[
= x \cdot r(h, t)
\]
\[
= x \cdot (r_H(h), h \cdot t)
\]
\[
= (r_H(h), x \cdot (h \cdot t)).
\]

When \( G \) acts principally and by isomorphisms on a locally compact groupoid \( H \), we will find it useful to have formulas describing various constructions when \( H \) is replaced by an action groupoid \( K \rtimes T \) as described in Theorem 4.11. Moreover, it is convenient to apply Lemma 4.10 along with Theorem 4.11. For example, we already mentioned in Lemma 4.10 that
\[
G \setminus (K \rtimes T) \simeq K
\]
via the map \( G \cdot (k, t) \mapsto k \).

**Remark 4.12.** If \( h \in H \) and \( q : H \to K \) is the quotient map, then \( \theta_s(h) = \theta_q(h) = (q(h), s_H(h)) \). Thus \( \theta(q(h), s_H(h)) = \theta_r(h) = (q(h), r_H(h)) \). Hence the action of the quotient groupoid \( K = G \setminus H \) on \( H^{(0)} \) in the previous result satisfies
\[
r_H(h) = q(h) \cdot s_H(h)
\]
for \( h \in H \).

We want to promote Theorem 4.11 to a classification of principal actions by isomorphisms on Fell bundles, thus providing a generalization to groupoid actions of [KMQW13a, Theorem A.16]. For the proof, we will need to introduce the concept of an action Fell bundle. Let \( \mathcal{B} \to K \) be a Fell bundle over a groupoid \( K \) and assume that \( K \) acts on a space \( T \). As usual, we let \( K \rtimes T = K \rtimes T \) be the action
groupoid with unit space identified with $T$ so that $r(k, t) = k \cdot t$, $s(y, t) = t$, and $(k, t)(f, t) = (k f, t)$. Then $(k, t) \mapsto k$ is a groupoid homomorphism and we can form the pullback Fell bundle

$$\pi^* B = \{ (b, k, t) : \varphi(b) = k \}$$

over $K \times T$. Clearly, we can condense the notation and consider elements of $\pi^* B$ to be pairs $B \times T = \{ (b, t) : s(\varphi(b)) = \rho(t) \}$. Then $s(b, t) = t$ and $r(b, t) = \varphi(b) \cdot t$. Moreover,

$$(b, \varphi(b') \cdot t)(b', t) = (bb', t) \quad \text{and} \quad (b, t)^* = (b^*, \varphi(b) \cdot t).$$

Then in an analogy with Lemma 4.10 we have the following.

**Lemma 4.13.** Let $G$ be a groupoid, $p : B \to K$ a Fell bundle, and suppose that we have a locally compact Hausdorff space $T$ which admits commuting actions by $G$ and $K$ such that the moment map $\rho_K : T \to K^{(0)}$ is a principal $G$-bundle (as in Lemma 4.10).

(a) The pullback action $G \curvearrowright B \times T$ given by $x \cdot (b, t) = (b, x \cdot t)$ is principal and by isomorphisms.

(b) The map $\Theta : G\backslash(B \times T) \to B$ given by $\Theta([b, t]) = b$ is a Fell-bundle isomorphism.

(c) $B$ acts on the right of $B \times T$ by $(b, t) \cdot c = (bc, p(c)^{-1} \cdot t)$.

**Remark 4.14.** Note that the statement in item (c) above includes the assertion that the action $(B \times T) \curvearrowright B$ covers the principal groupoid action $(K \times T) \curvearrowright K$ of Lemma 4.10.

**Sketch of the Proof.** The proof of (a)–(b) follows the same lines as Lemma 4.10. For part (c), we check the conditions in the discussion preceding [MW08, Definition 6.1]: first of all, the pairing

$$(B \times T) \ast B \to B \times T$$

is continuous since the left coordinate is the multiplication in $B$ and the right coordinate is the action of $K$ on $T$. Routine computations show that the pairing covers the action of $K$ on $K \times T$, $(s(t) \cdot c) \cdot d = (b, t) \cdot (cd)$, and $\||b, t) \cdot c|| \leq ||(b, t)|| ||c||$. \qed

Even though we did not think it necessary to give any details in the above proof, it will be convenient to have the lemma for reference. For example, the isomorphism $G\backslash(B \times T) \simeq B$ is useful in our proof of the following classification theorem.

**Theorem 4.15.** Let a locally compact groupoid $G$ act principally and by isomorphisms on a Fell bundle $p : A \to H$, and let $\varphi : B \to K$ be the quotient Fell bundle $G\backslash A \to G\backslash H$. Then there is an action of $K$ on $H^{(0)}$ such that $A$ is $G$-equivariantly isomorphic to the action Fell bundle

$$B \times H^{(0)} \to K \times H^{(0)}.$$

**Proof.** Let $\varphi : A \to B$ and $q : H \to K$ be the quotient maps. As above we can form the action Fell bundle $B \times H^{(0)}$ for the $K$-action on $H^{(0)}$ from Theorem 3.8. Then

$$B \times H^{(0)} = \{ (b, v) \in B \times H^{(0)} : s_K(\varphi(b)) = q(v) \}.$$
Thus, topologically at least, \( \mathcal{B} \rtimes H^{(0)} \) is the pullback \((s_K \circ p) \ast K^{(0)}\):

\[
\begin{array}{ccc}
\mathcal{B} \rtimes H^{(0)} & \longrightarrow & H^{(0)} \\
\pi_2 & \simeq & q| \\
\pi_1 & \simeq & s_K \circ p \\
\mathcal{B} & \longrightarrow & K^{(0)}.
\end{array}
\]

Since \((H^{(0)}, q|, K^{(0)})\) is a principal \(G\)-bundle, it follows exactly as in the proof of Lemma 3.7 that \((\mathcal{B} \rtimes H^{(0)}, \pi_1, \mathcal{B})\) is a \(G\)-bundle with respect to the action of \(G\) on its second factor. Then we get a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{s_H \circ p} & H^{(0)} \\
\downarrow & & \downarrow \\
\mathcal{B} \times H^{(0)} & \xrightarrow{\pi_2} & q| \\
\downarrow & & \downarrow \\
\mathcal{B} & \xrightarrow{s_K \circ p} & K^{(0)}.
\end{array}
\]

where \(\Theta\) is the uniquely determined map given by \(\Theta(a) = (q(a), s_H(p(a)))\). We claim that \(\Theta\) is a \(G\)-equivariant homeomorphism. It is clearly \(G\)-equivariant: since \(G\) acts by automorphisms of \(H\), we have \(\Theta(x \cdot a) = (q(x \cdot a), s_H(p(x \cdot a))) = (q(a), x \cdot s_H(p(a))) = x \cdot \Theta(a)\).

The proof that \(\Theta\) is a bijection follows exactly as in the proof of Lemma 3.5. Since \(\Theta\) is clearly continuous, we just need to verify it is open. For this, since our bundles are not locally compact or necessarily Hausdorff, we have to modify the proof of Lemma 3.5 slightly. We still use Fell’s Criterion. Suppose that \((b_i, v_i) \to \Theta(a) = (\overline{q}(a), s_H(p(a)))\) in \(\mathcal{A} \times H^{(0)}\). Since \(\overline{q}\) is open, we can pass to a subnet, relabel, and assume there are \(a_i \to a\) in \(A\) such that \(\overline{q}(a_i) = b_i\). Since \(\pi_1(\Theta(a_i)) = b_i\), there are \(x_i \in G\) such that \((b_i, v_i) = x_i \cdot \Theta(a_i)\). Then \(x_i \cdot s_H(p(a_i)) = s_H(p(a_i))\) and \(s_H(p(a_i)) \to s_H(p(a))\). Since the action of \(G\) on \(H^{(0)}\) is principal, we have \(x_i \to \rho_H(s_H(p(a_i))) = \rho_H(p(a)) = \rho_A(a)\). Hence \(x_i \cdot a_i \to a\) and \(\Theta(x_i \cdot a_i) = (b_i, v_i)\). Therefore \(\Theta\) is a homeomorphism.

Then we get a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\Theta} & \mathcal{B} \times H^{(0)} \\
\downarrow & & \downarrow \overline{q} \times \text{id} \\
H & \xrightarrow{\theta_s} & K \ast_s H^{(0)}
\end{array}
\]

where \(\overline{q} \times \text{id}(b, u) = (\overline{q}(b), u)\) and \(\theta_s(h) = (\overline{q}(h), s_H(h))\) is the groupoid isomorphism from the proof of Theorem 4.11. Since \(\Theta\) is a \(G\)-equivariant homeomorphism and \(\theta_s\) is a groupoid isomorphism, \(\Theta\) will be a Fell bundle isomorphism provided it preserves multiplication and the involution.
Since \( \theta_s(p(b)) = (\overline{\theta}(p(b)), s_H(p(b))) = (\overline{\theta}(q(b)), s_h(p(b))) \), it follows that \( r_H(p(b)) = \overline{\theta}(q(b)) \cdot s_H(p(b)) \). Hence, if \( a, b \in A \) with \( s \circ p(a) = r \circ p(b) \), then
\[
\Theta(a)\Theta(b) = (q(a), s \circ p(a)) (q(b), s \circ p(b)) = (q(a)q(b), s \circ p(b)) = (q(ab), s \circ p(ab)) = \Theta(ab),
\]
where the multiplication on the right-hand side of the top line is defined since
\[
s_H(p(a)) = r_H(p(b)) = \overline{\theta}(q(b)) \cdot s_H(p(b)).
\]
For the involution, we have
\[
\Theta(a)^* = (q(a), s_H(p(a)))^* = (q(a)^*, \overline{\theta}(q(a)) \cdot s_H(p(a))) = (q(a)^*, r_H(p(a))) = (q(a)^*, s_H(p(a)^{-1})) = (q(a)^*, s_H(p(a^*)) = \Theta(a^*). \]

5. Semidirect-product actions

Give an action of \( G \) on \( H \) by isomorphisms, we want to consider how to build actions of their semidirect product \( S(H, G) \) on a space \( T \) from actions of \( H \) and \( G \) on \( T \). For motivation, consider the case where \( H \) and \( G \) are groups acting on a space \( T \) and \( S(H, G) \) is the semidirect product of groups. The following notation may be helpful. Let \( \alpha : G \to \text{Aut} H \) be a homomorphism and let \( L(x) \) and \( \pi(h) \) be the homeomorphisms of \( T \) induced by \( x \in G \) and \( h \in H \) respectively. It is basic algebra to verify that we get an action of \( S(H, G) \) restricting to the given actions on \( H \) and \( G \) exactly when the pair \((\pi, L)\) is covariant in that the diagram
\[
\begin{array}{ccc}
T & \xrightarrow{L(x)} & T \\
\downarrow{\pi(h)} & & \downarrow{\pi(\alpha_x(h))} \\
T & \xrightarrow{L(x)} & T
\end{array}
\]
commutes for all \( x \in G \) and \( h \in H \). Alternatively,
\[
(5.1) \quad L(x) \circ \pi(h) = \pi(\alpha_x(h)) \circ L(x),
\]
which is a formula that is familiar to that for covariant representations of dynamical systems. (This is the motivation for our terminology below.)

Now we return to the case where \( G \) and \( H \) are groupoids. We let \( \rho_G : T \to G^{(0)} \) and \( \rho_H : T \to H^{(0)} \) be the moment maps for the actions on \( T \), and \( \rho_H^G : H \to G^{(0)} \) the moment map for the action of \( G \) on \( H \) by isomorphisms. As above, we use notations \( L(x) \) and \( \pi(h) \) for the partially defined maps on \( T \) induced by \( x \in G \) and \( h \in H \), and \( \alpha_x \) for the isomorphism of \( T_{\pi(x)} \) onto \( T_{\pi(x)} \). Then we want these actions to satisfy a similar covariance condition \([5.1]\) even though both sides are
not everywhere defined. Thus, written in the usual notation for groupoid actions, we want
\[(5.2) \quad x \cdot (h \cdot t) = (x \cdot h) \cdot (x \cdot t)\]
to hold when both sides are defined. More precisely, let
\[
L = \{ (x, h, t) : s_H(h) = \rho_H(t) \text{ and } s_G(x) = \rho_G(h \cdot t) \}
\]
be the subset of \(G \times H \times T\) where the left-hand side of \(5.2\) is defined, and let
\[
R = \{ (x, h, t) : s_G(x) = \rho_H^H(h), s_G(x) = \rho_G(t), \text{ and } s_H(x \cdot h) = \rho_H(x \cdot t) \}
\]
be the subset where the right-hand side is defined. Then we require that \(L = R\), and that \(5.2\) holds on this common set.

While this might seem overly difficult to establish, the following observation greatly simplifies checking the validity of these requirements.

**Lemma 5.1.** Suppose that the G- and H-actions on T are compatible in that the diagram
\[
\begin{array}{ccc}
G^{(0)} & \xrightarrow{\rho_G} & T \\
\rho_H \downarrow & & \downarrow \rho_H \\
H^{(0)} & \xleftarrow{\rho_G} & 
\end{array}
\]
commutes and \(\rho_H\) is G-equivariant in that
\[
(5.4) \quad \rho_H(x \cdot t) = x \cdot \rho_H(t) \quad \text{if } s_G(x) = \rho_G(t).
\]
Then the sets \(L\) and \(R\) defined above coincide and the left-hand side of \(5.2\) is defined precisely when the right-hand side is. Moreover, if \(x \cdot h\) and \(h \cdot t\) are both defined, then \((x, h, t) \in L = R\). Conversely, if \(R\) and \(L\) coincide, then \(\rho_G(t) = \rho_G^H(\rho_H(t))\) and \(\rho_H(x \cdot t) = x \cdot \rho_H(t)\) for all \((x, h, t) \in L = R\).

**Proof.** Suppose that \(L = R\) and \((x, h, t) \in L = R\). Since \(G\) acts by automorphisms, \(\rho_G^H(s_H(h)) = \rho_G^H(h), \) and hence
\[
\rho_G^H(\rho_H(t)) = \phi_G^H(s_H(h)) = \phi_G^H(h) = s_G(x) = \rho_G(t).
\]
On the other hand,
\[
\rho_H(x \cdot t) = s_H(x \cdot h) = x \cdot s_H(h) = x \cdot \rho_H(t).
\]
For the other direction, now assume \(\rho_G = \rho_G^H \circ \rho_H\) and that \(\rho_H\) is G-equivariant. Assume \((x, h, t) \in L\). Then, since \(G\) acts by isomorphisms,
\[
s_G(x) = \rho_G(h \cdot t) = \rho_G^H(\rho_H(h \cdot t)) = \rho_G^H(r_H(h)) = \rho_G^H(h).
\]
Then
\[
s_G(x) = \rho_G^H(h) = \rho_G^H(s_H(h)) = \rho_G^H(\rho_H(t)) = \rho_G(t).
\]
Lastly,
\[
s_H(x \cdot h) = x \cdot s_H(h) = x \cdot \rho_H(t) = \rho_H(x \cdot t).
\]
Therefore \((x, h, t) \in R\).
Now suppose that \((x, h, t) \in R\). Then
\[
x \cdot s_H(h) = s_H(x \cdot h) = \rho_H(x \cdot t) = x \cdot \rho_H(t).
\]
But then \(s_H(h) = \rho_H(t)\). Then we also have
\[
s_G(x) = \rho_G(t) = \rho_G^{H}(\rho_H(t)) = \rho_G^{H}(s_H(h))
= \rho_G^{H}(r_H(h)) = \rho_G^{H}(\rho_H(h \cdot t))
= \rho_{G}(h \cdot t).
\]
Thus \((x, h, t) \in L\). Thus \(L = R\) as claimed.

Now suppose that \(x \cdot h\) and \(h \cdot t\) are defined. That means \(s_G(x) = \rho_G^{H}(h)\) and \(s_H(h) = \rho_H(t)\). Then
\[
s_H(x \cdot h) = x \cdot s_H(h) = x \cdot \rho_H(t) = \rho_H(x \cdot t).
\]
Therefore \((x, h, t) \in R\) and we already know \(L = R\).

Lemma 5.1 and the preceding discussion motivate the following definition.

**Definition 5.2.** Let \(G\) act on \(H\) by isomorphisms. We define actions of \(G\) and \(H\) on \(T\) to be **covariant** if the moment maps commute as in (5.3), if \(\rho_H\) is \(G\)-equivariant as in (5.4), and if the covariance condition
\[
x \cdot (h \cdot t) = (x \cdot h) \cdot (x \cdot t)
\]
holds whenever \(x \cdot h\) and \(h \cdot t\) are defined.

**Example 5.3.** Suppose that \(G\) acts on \(H\) by isomorphisms. We can also let \(H\) act on itself by left translation—so that \(H\) also plays the role of \(T\) in Definition 5.2. Then \(\rho_H = r_H\) and \(\rho_G^{H} = \rho_G\). The actions are covariant because \(G\) acts by isomorphisms.

Covariant actions allow us to build actions of semidirect products.

**Lemma 5.4.** Let \(G\) and \(H\) be locally compact groupoids, let \(T\) be a locally compact Hausdorff space, let \(G\) act on \(H\) by isomorphisms, and let \(G\) and \(H\) act covariantly on \(T\). Then the semidirect product groupoid \(S(H, G)\) acts on \(T\) by
\[
(h, x) \cdot t = h \cdot (x \cdot t) \quad \text{if } s(x) = \rho_G(t) \text{ and } s(h) = x \cdot \rho_H(t),
\]
where \(\rho_G\) and \(\rho_H\) are the moment maps for the actions of \(G\) and \(H\) on \(T\), respectively. The moment map for the action \((x, h)\) is \(\rho(t) = (\rho_H(t), \rho_G(t))\). If \(\rho_H\) is open, then so is \(\rho\).

**Proof.** Recall that
\[
S(H, G)^{(0)} = \{(v, u) \in H^{(0)} \times G^{(0)} : \rho_G^H(v) = u\},
\]
where \(\rho_G^H : H \to G^{(0)}\) is the moment map for the action of \(G\) on \(H\). The map \(\rho : T \to S(H, G)^{(0)}\) defined by
\[
\rho(t) = (\rho_H(t), \rho_G(t))
\]
is clearly continuous. If \(\rho_H\) is open, we show that \(\rho\) is as well using Fell’s Criterion. Suppose that \((v_n, u_n) \to (v, u) = \rho(t)\). Then \(v_n \to v = \rho_H(t)\), and we can pass to subnet, relabel, and assume there are \(t_n \in T\) such that \(t_n \to t\) and \(\rho_H(t_n) = v_n\). Since \((v_n, u_n) \in S(H, G)^{(0)}\), we have \(\rho_G^H(v_n) = u_n\). Then since the actions are
compatible, \( \rho_G(t_n) = \rho^H_G(\rho_H(t_n)) = \rho^H_G(\rho_H(t_n)) = u_n \). It now follows that \( \rho \) is open by Fell’s Criterion. Since \( \rho_G(t_n) = \rho^H_G(\rho_H(t_n)) \), we have \( \rho(t) \) if and only if \( \rho_G(s(x)) = \rho_G(t) \) and \( \rho_H(h) = x \cdot \rho_H(t) \).

Since we are assuming \( x \cdot \rho_H(t) = \rho_H(x \cdot t) \), the operation in \( \mathcal{S} \) is well-defined.

Moreover, we can easily check that \( \rho(t) \cdot t = t \). To check the action condition, we first need to see that \( \rho \) is equivariant. But
\[
\rho((h, x) \cdot t) = \rho(h \cdot (x \cdot t)) = (\rho_H(h \cdot (x \cdot t)), \rho_G(h \cdot (x \cdot t))) = (\rho_H(h), \rho_G(h \cdot (x \cdot t))) = (\rho_H(h), \rho^H_G(\rho_H(h \cdot (x \cdot t)))) = (\rho_H(h), \rho^H_G(\rho_H(h))).
\]

Since \( G \) acts by isomorphisms on \( H \), this is
\[
= (\rho_H(h), \rho^H_G(h))
\]
so that, since \( (h, x) \in S(H, G) \), this is
\[
= (\rho_H(h), \rho^H_G(h)) = r(h, x)
\]
as required. Then, on the one hand, if \( ((h, x), (h', y)) \in S(H, G)^{(2)} \), then
\[
(h, x) \cdot ((h', y) \cdot t) = (h, x) \cdot (h' \cdot (y \cdot t)) = h \cdot ((x \cdot (h' \cdot (y \cdot t))).
\]
On the other hand,
\[
((h, x)(h', y)) \cdot t = (h(x \cdot h'), xy) \cdot t = (h(x \cdot h')) \cdot ((xy) \cdot t) = h \cdot [(x \cdot h') \cdot (x \cdot t)]
\]
which, since the actions are covariant, is
\[
= h \cdot [x \cdot (h' \cdot (y \cdot t))] = h \cdot [x \cdot ((y \cdot t))]
\]
as required.

The continuity is routine: if \( (h_i, x_i) \rightarrow (h, x) \) in \( S(H, G) \) and \( t_i \rightarrow t \) in \( T \), then
\[
(h_i, x_i) \cdot t_i = h_i \cdot (x_i \cdot t_i) \rightarrow h \cdot (x \cdot t) = (h, x) \cdot t,
\]
because \( x_i \cdot t_i \rightarrow x \cdot t \).

\[\square\]

**Corollary 5.5.** Let \( G \) and \( H \) be locally compact groupoids, let \( G \) act on \( H \) by isomorphisms, and let \( H \) act on itself as a space by left translation. Then the semidirect product groupoid \( S(H, G) \) acts on \( H \) as a space by
\[
(h, x) \cdot k = h(x \cdot k)
\]
whenever \( k \in H_{s(x)} \) and \( s(h) = x \cdot r(k) \). Moreover, if \( G \) acts on \( H \) principally, then so does \( S(H, G) \). Furthermore, the moment map \( \rho'(k) = (r(k), \rho(k)) \) for \( S(H, G) \sim H \) is open provided the range map on \( H \) is open.
Moreover, the coordinate projection $x$ and the moment map for the action $\rho_S$ of the semidirect product actions on Fell bundles.

Proposition 3.11: If $S$ is a semidirect product action of $H \rtimes K$ on $\mathcal{E}$, then both sides of (3.10) are defined.

Note that in Definition 5.6, when we replace $H$ by $\mathcal{E}$, we assume that $\mathcal{E}$ acts on $\mathcal{E}$ by isomorphisms. Thus $S(H, G)$ acts on $H$ by Lemma 5.4.

Suppose that $G$ acts principally on $H$. Then by Theorem 4.11 we may replace $H$ by an action groupoid $K \rtimes T$, where $K$ is a locally compact groupoid acting on a locally compact Hausdorff space $T$, the moment map $\rho_K : T \to K^{(0)}$ is a principal $G$-bundle, the actions of $G$ and $K$ on $T$ commute, and (by Lemma 4.10) the action $G \rtimes K \rtimes T$ is given by $x \cdot (k, t) = (k, x \cdot t)$. Note that $S(K \times T, G)$ can be described, with a mild abuse of notation, as

$$\{ (k, t, x) \in K \times T \times G : \rho_K(t) = s(k) \text{ and } \rho_G(t) = r(x) \}.$$  

Moreover, $(k, t, x)$ is a unit if and only if $k$ and $x$ are. In particular, $(k, t, x)$ acts on $(l, u)$ if $\rho_G(u) = s_G(x)$ and $t = l \cdot (x \cdot u)$, in which case $(k, t, x) \cdot (l, u) = (kl, x \cdot u)$.

Now we check that the action is proper. For this, we use (PA4) of Proposition 3.1.

First we verify that this action is free: suppose $(k, t, x) \cdot (l, u) = (l, u)$. Then

$$k(1, u) = (l, u)$$

by definition of the action. Thus $k \in K^{(0)}$, and $x \in G^{(0)}$ since $G$ acts freely on $T$. Thus $(k, t, x)$ is a unit.

Now we check that the action is proper. For this, we use (PA4) of Proposition 3.1.

Suppose that $(l_i, u_i) \to (l, u)$ and $(k, l_i \cdot (x_i \cdot u_i), x_i) \cdot (l_i, u_i) \to (l', u')$. Then $(k, l_i, x_i \cdot u_i) \to (l', u')$. Since the $G$-action on $T$ is principal, we can assume that $x_i \to x$ in $G$. Since $l_i \to l$, and $k_i l_i \to l'$, we must have $k_i \to k = l' l^{-1}$. Thus

$$\{ (k_i, l_i \cdot (x_i \cdot u_i), x_i) \}$$

converges and the action is proper. \qed

Note that in Corollary 5.5 we replace $H$ by $K \rtimes T$, the range and source maps are

$$r(k, t, x) = (r(k), k \cdot t, r(x))$$

$$s(k, t, x) = (s(k), x^{-1} \cdot t, s(x)),$$

and the moment map for the action $S(K \times T, G) \rtimes K \times T$ is given by

$$\rho(k, t) = (r(k), k \cdot t, \rho_G(t)).$$

Moreover, the coordinate projection

$$\pi_2 : S(K \times T, G)^{(0)} \to T$$

is a homeomorphism.

Semidirect product actions on Fell bundles. Let $p : A \to H$ be a Fell bundle and assume that $G$ acts on $A$ by isomorphisms. Let $q : \mathcal{E} \to T$ be an upper semicontinuous Banach bundle on which $G$ acts. We assume that the actions $e \mapsto x \cdot e$ on the fibres are Banach space isomorphisms. We also want $A$ to act on $\mathcal{E}$ as in Definition 2.1.

Definition 5.6: We say that the actions of $G$ and $A$ on $\mathcal{E}$ are covariant if the underlying actions of $G$ and $H$ on $T$ are covariant and

$$x \cdot (a \cdot e) = (x \cdot a) \cdot (x \cdot e) \quad \text{whenever } (x, a) \in G \ast A \text{ and } (a, e) \in A \ast \mathcal{E}.$$

Note that it follows as in Lemma 5.1 that if $(x, a) \in G \ast A$ and $(a, e) \in A \ast \mathcal{E}$, then both sides of (5.6) are defined.
Corollary 5.7. With the above notation, if the actions of $G$ and $A$ on $E$ are covariant, then the semidirect-product Fell bundle $S(A,G)$ acts on $E$ by

$$(a,x) \cdot e = a \cdot (x \cdot e)$$

whenever the right-hand side makes sense.

Proof: First of all, to see that the pairing is continuous, let $((a_i, x_i), e_i) \rightarrow ((a, x), e)$ in $S(A,G) \ast E$. Then $a_i \rightarrow a$, $x_i \rightarrow x$, and $e_i \rightarrow e$, so $x_i \cdot e_i \rightarrow x \cdot e$, and then $a_i \cdot (x_i \cdot e_i) \rightarrow a \cdot (x \cdot e)$, so $(a_i, x_i) \cdot e_i \rightarrow (a, x) \cdot e$. Routine exercises in the definitions show that the pairing covers the action $H \rtimes G \supset T$, $(a, x) (b, y)) \cdot e = (a, x) \cdot ((b, y) \cdot e)$, and $\|(a, x) \cdot e\| \leq \|(a, x)\| \|e\|$. □

6. Quotient Action

Proposition 6.1. Let $G$ and $H$ be locally compact Hausdorff groupoids, and let $G$ act on $H$ principally and by isomorphisms. Then there is a principal right action of the quotient groupoid $G \backslash H$ on $H$, given by

$$(6.1) \quad h \cdot (G \cdot k) = h k'$$

whenever $s(h) \in G \cdot r(k)$, where $k'$ is the unique representative of $G \cdot k$ with $r(k') = s(h)$. Moreover, the associated moment map $H \rightarrow (G \backslash H)^{(0)}$ is open.

Proof. Note first of all that (6.1) makes sense because $G$ acts freely by isomorphisms. By Theorem 4.11 we can replace $H$ by $K \rtimes T$, where $K = G \backslash H$, $T = H^{(0)}$, $G$ acts on $K \rtimes T$ by $x \cdot (k, t) = (k, x \cdot t)$, and the actions of $G$ and $K$ on $T$ commute. Then by Lemma 4.10, $K$ acts principally on the right of $K \rtimes T$ by

$$(k, t) \cdot \ell = (k \ell, \ell^{-1} \cdot t).$$

Moreover, the isomorphism $G \backslash (K \times T) \simeq K$ of Lemma 4.10 transforms the action $(K \times T) \acts K$ into the action of $G \backslash (K \times T)$ described in the statement of the proposition, and hence this action is principal.

Furthermore, under the isomorphism the moment map $\sigma : K \times T \rightarrow K^{(0)}$ is

$$(k, t) \mapsto s(k),$$

which is open because it is a composition of two open maps. □

Corollary 6.2. Let $G$ and $H$ be locally compact groupoids, and let $p : A \rightarrow H$ be a Fell bundle. Suppose that $G$ acts principally and by isomorphisms on $A$. Then the orbit Fell bundle $G \backslash A$ acts on the right of the Banach bundle $A$ by

$$(6.2) \quad a \cdot (G \cdot b) = ab'$$

whenever $s(a) \in G \cdot r(b)$, and where $b'$ is the unique representative of $G \cdot b$ with $s(a) = r(b)$.

Proof. The proof follows the lines of Proposition 6.1. Note first that (6.2) makes sense because the range map $r : A \rightarrow H^{(0)}$ is equivariant for free $G$-actions. By Theorem 4.11 we can replace $A$ by $B \times T$, and by Lemma 4.10, $B$ acts on the right of $B \times T$ by

$$(b, t) \cdot c = (bc, p(c) \cdot t).$$

Moreover, the isomorphism $G \backslash (B \times T) \simeq B$ of Lemma 4.13 transforms the action $(B \times T) \acts B$ into the action of $G \backslash (B \times T)$ described in the statement of the proposition. □
Proposition 6.3. Let $G$ and $H$ be locally compact Hausdorff groupoids, and let $G$ act on $H$ principally and by isomorphisms. Then $H$ is a $S(H,G) - G\backslash H$-equivalence.

Remark 6.4. In view of [HKQW] Example 6.2, if $H$ is a principal $G$-space, then we recover the well-known result that in this case the action groupoid $G \times H$ is equivalent to the orbit space $G\backslash H$ (cf., [Wil19 Ex. 2.35]).

Proof. Note that by [HKQW] Corollary 8.5 and Proposition 6.1, $H$ has principal left and right actions by $S(H,G)$ and $G\backslash H$, respectively. As in the proof of Proposition 6.1, we use Theorem 4.11 to replace $H$ by $K \rtimes T$, where $K = G\backslash H$, $T = H^{(0)}$, $G$ acts on $K \rtimes T$ by $x \cdot (k,t) = (k,x \cdot t)$, and the actions of $G$ and $K$ on $T$ commute, and moreover $K$ acts on the right of $K \rtimes T$ by $(k,t) \cdot \ell = (k\ell, \ell^{-1} \cdot t)$. Since $S(K \rtimes T, G)$ acts on $K \rtimes T$ by

$$(k,t,x) \cdot (\ell,s) = (k,t)(x \cdot (\ell,s)) = (k,t)(\ell,x \cdot s) = (k\ell, x \cdot s)$$

if $t = \ell \cdot x \cdot s$, and since the actions of $G$ and $K$ on $T$ commute, it is clear from the formulas that the actions of $S(K \rtimes T, G)$ and $K$ on $K \rtimes T$ commute.

If $\rho'$ and $\sigma$ are the moment maps for the $S(K \rtimes T, G)$- and $K$-actions, respectively, then they are open by [HKQW] Corollary 8.5 and Proposition 6.3. Since $K$ and $K \rtimes T$ have open range and source maps by Proposition 4.1, the quotient maps for the left and right actions are also open. Hence it suffices to show that the induced maps $\overrightarrow{\sigma}$ and $\overrightarrow{\tau}$ making the diagram

\[
\begin{array}{ccc}
S(K \rtimes T, G)^{(0)} & \xleftarrow{\rho'} & K \rtimes T \\
\uparrow \sigma & & \uparrow \sigma \\
(K \rtimes T)/K & \xrightarrow{Q} & K^{(0)} \setminus (K \rtimes T)
\end{array}
\]

commute are bijections, where the $Q$'s are the appropriate quotient maps. (See [Wil19 Remark 2.31].) Thus it suffices to show that for all $(k,t), (\ell,s) \in K \rtimes T$, if $\rho'(k,t) = \rho'(\ell,s)$ then $(\ell,s) \in (k,t) \cdot K$, and if $\sigma(k,t) = \sigma(\ell,s)$ then $(\ell,s) \in S(K \rtimes T, G) \cdot (k,t)$.

For the first, we have $r(k) = r(\ell)$, so $\ell = kk^{-1}\ell$, and $k \cdot t = \ell \cdot s$, so $s = \ell^{-1} k \cdot t$, and hence $(\ell,s) = (k,t) \cdot k^{-1} \ell$.

For the second, we have $s(k) = s(\ell)$, so $\ell = \ell k^{-1} k$. We also have $\rho_K(k) = \rho_K(s)$. Since $\rho_K$ is the restriction of the quotient map to $T = H^{(0)}$ for the principal $G$ action on $H$, we have $s = x \cdot t$ for a unique $x \in G$. Hence $(\ell,s) = (lk^{-1}, r,x) \cdot \sigma(k,t)$ with $r = k \cdot x \cdot t$. □

7. Principal Action Groupoids

In this section we prove our main result, the equivalence theorem, in two forms: Theorems 7.1 and 7.2, since each is appropriate in its own setting.

Theorem 7.1. Suppose that locally compact groupoids $G$ and $K$ have commuting actions on a locally compact Hausdorff space $T$ such that the moment map $\rho_K : T \to K^{(0)}$ is a principal $G$-bundle. Suppose further that $\mathcal{B} \to K$ is a Fell bundle. Then the action Fell bundle $\mathcal{B} \rtimes T$ becomes a $S(\mathcal{B} \rtimes T, G) - \mathcal{B}$-equivalence with operations given as follows:
Proof. Since the actions of $S$ that let $K$ acts on the action groupoid $\Gamma^\times T$, the crossed product $C^*(\Gamma^\times T, B \rtimes T)$ is Morita equivalent to the Fell-bundle $C^*$-algebra $C^*(K, B)$.

Consequently, if the Haar system on the action groupoid $K \rtimes T$ is $G$-invariant, then letting $\alpha$ denote the associated action of $G$ on $C^*(K \rtimes T, B \rtimes T)$ from [HKQW, Lemma 7.2], the crossed product $C^*(K \rtimes T, B \rtimes T) \rtimes_\alpha G$ is Morita equivalent to the Fell-bundle $C^*$-algebra $C^*(K, B)$.

We showed in the proof of Proposition 6.3 that $K \rtimes T$ is a $(S(K \rtimes T, G), K)$-imprimitivity bimodule.

We write the other two bundle projections as

\[ p' : B \rtimes T \to K \rtimes T \quad \text{given by} \quad p'(b, t) = (p(b), t) \]
\[ p'' : (B \rtimes T, G) \to (K \rtimes T, G) \quad \text{given by} \quad p''(b, t, x) = (p(b), t, x). \]

Routine computations verify the conditions of [MW08, Section 6]:

(a) $p''(\langle (b, t), (c, s) \rangle) = \langle p'(b, t), p'(c, s) \rangle$ and $p\langle (b, t), (c, s) \rangle = [p'(b, t), p'(c, s)].$

(b) $\ast \langle (b, t), (c, s) \rangle' = \ast \langle (c, s), (b, t) \rangle$ and $\langle (b, t), (c, s) \rangle = \langle (c, s), (b, t) \rangle.$

(c) $\langle (b, t, x) \cdot (c, s), (d, u) \rangle = (b, t)(c, s), (d, u) \rangle$ and $\langle (b, t), (c, s) \cdot d \rangle = \langle (b, t), (c, s) \rangle \cdot d.$

(d) with the above module actions and inner products, for each $(k, t) \in K \rtimes T, (B \rtimes T)_{(k, t)}$ is a $(B \rtimes T, G)_{p'(k, t)} - B_{s(k, t)}$ imprimitivity bimodule.

Note that

\[ (B \rtimes T)_{(k, t)} = B_k \times \{ t \} \]
\[ S(B \rtimes T, G)_{(k, t, x)} = B_k \times \{ (t, x) \} \]
\[ \rho(b, t)' = (r(b), p(b) \cdot t, \rho_G(t)) \]
\[ \sigma(b, t) = s(b). \]

so (d) follows immediately from the Fell-bundle axiom that $B_k$ is a $B_{r(k)} - B_{s(k)}$ imprimitivity bimodule.

Now assume that the Haar system on the action groupoid $K \rtimes T$ is $G$-invariant. Then $S(K \rtimes T, G)$ admits a Haar system by [HKQW, Theorem 6.4]. Thus [MW08, Theorem 6.4] applies, and we have a Morita equivalence between $C^*(S(K \rtimes T, G), S(B \rtimes T, G))$ and $C^*(K, B).$ Then by [HKQW, Theorem 7.3], we have a Morita equivalence

\[ C^*(K \rtimes T, B \rtimes T) \rtimes_\alpha G \sim C^*(K, B), \]

as required. \qed
The following alternative version of Theorem 7.1 is a groupoid version of [KMQW13a, Corollary 3.9].

**Theorem 7.2.** Let a locally compact groupoid $G$ act on a Fell bundle $p : A \to H$ principally and by isomorphisms. Then $A$ becomes an $S(A, G) - G \backslash A$-equivalence with operations given as follows:

1. $S(A, G)$ acts on the left of the Banach bundle $A$ by
   $$(a, x) \cdot b = a(x \cdot b) \quad \text{if} \quad s(a) = x \cdot r(b);$$

2. the left inner product is given by
   $$L\langle a, b \rangle = (a(x \cdot b^*), x) \quad \text{if} \quad G \cdot s(a) = G \cdot s(b),$$
   where $x$ is the unique element of $G$ such that $s(a) = x \cdot s(b);$  

3. $G \backslash A$ acts on the right of $A$ by
   $$a \cdot (G \cdot b) = ab \quad \text{if} \quad s(a) = r(b);$$

4. the right inner product is given by
   $$\langle a, b \rangle_R = G \cdot a^* b \quad \text{if} \quad r(a) = r(b).$$

Consequently, if the Haar system on $H$ is $G$-invariant, then letting $\alpha$ denote the associated action of $G$ on $C^*(H, A)$ from [HKQW, Lemma 7.2], the crossed product $C^*(H, A) \rtimes_\alpha G$ is Morita equivalent to the Fell-bundle $C^*$-algebra $C^*(G \backslash H, G \backslash A)$.

**Remark 7.3.** One could prove a symmetric version of Theorem 7.2 as in [KMQW13a, Theorem 3.1], but we have no applications of such a result, so we omit it.

**Proof.** Note that by [HKQW] Corollary 8.7, and Corollary 6.2, $A$ has left and right actions by $S(A, G)$ and $G \backslash A$, respectively. Items (1) and (3) in the current theorem just reiterate the formulas for convenient reference.

We use Theorem 4.15 to replace $A$ by $B \rtimes T$, where $B = G \backslash A$, $T = H(0)$, $G$ acts on $B \rtimes T$ by $x \cdot (b, t) = (b, x \cdot t)$, and the actions of $G$ and $K$ on $T$ commute, and moreover $B$ acts on the right of $B \rtimes T$ by $(b, t) \cdot c = (bc, p(c)^{-1} \cdot t)$. Then by Theorem 7.1 $B \rtimes T$ is a $S(B \rtimes T, G) - B$-equivalence. The isomorphism of Theorem 4.15 transforms the formulas (1), (4) of Theorem 7.1 into the formulas (1), (4) of the current theorem.

Since $A$ is isomorphic to $B \rtimes K$ and $S(B \rtimes K, G)$ is isomorphic to $S(A, G)$, we conclude that $A$ is an $S(A, G) - G \backslash A$ equivalence as claimed.

Now assume that the Haar system on $H$ is $G$-invariant. Then $S(H, G)$ and $G \backslash H$ admit Haar systems by [HKQW] Lemma 6.4 and Proposition 4.7 respectively. Thus [MW08, Theorem 6.4] applies, and we have a Morita equivalence between $C^*(S(H, G), S(A, G))$ and $C^*(G \backslash H, G \backslash A)$. Then by [HKQW] Theorem 7.3, we have a Morita equivalence
\[ C^*(H, A) \rtimes_\alpha G \sim C^*(G \backslash H, G \backslash A). \]
as required. \[\square\]
8. Stabilization

The Ionescu-Kumjian-Sims-Williams stabilization theorem \[\text{IKSW18, Theorem 3.7}\] (which was based upon \[\text{Kum98, Corollary 4.5}\] and \[\text{Muh01, Theorem 15}\]) says roughly that every Fell bundle over a groupoid is equivalent to a semidirect product. In this section we will apply Theorem 7.1 to give a new approach to the stabilization theorem.

Start with a Fell bundle \(p : \mathcal{B} \to G\). Let \(G\) act on itself by left translation, and let \(\mathcal{B} \rtimes G \to G \rtimes G\) be the associated action Fell bundle. Now define another left action of \(G\) on itself by \(x \cdot y = yx^{-1}\).

Call left translation the \textit{first action}, and this new action the \textit{second action}. Then these two actions of \(G\) on \(G\) commute, and the moment map \(\rho_G = r_G\) for the first action is a principal \(G\)-bundle for the second action. Thus Theorem 7.1 applies, and Theorem 8.1 below shows how we recover the stabilization theorem \[\text{IKSW18, Theorem 3.7}\].

**Theorem 8.1** (\[\text{IKSW18}\]). With the above notation, \(\mathcal{B} \rtimes G\) is an \(\mathcal{S}(\mathcal{B} \rtimes G, G) - \mathcal{B}\) equivalence.

**Remark 8.2.** We were lead to our approach to the stabilization theorem by the following idea: the above equivalence should be regarded as a groupoid form of crossed-product duality. To see why, suppose \(G\) is a group. Then by \[\text{KMQW10, Theorems 5.1, 7.1}\] we have

\[
C^*(S(G \rtimes G, G), S(\mathcal{B} \rtimes G, G)) \cong (C^*(G, \mathcal{B}) \rtimes \delta G) \rtimes \tilde{\delta} G,
\]

where \(\delta\) is the coaction of \(G\) on \(C^*(G, \mathcal{B})\) canonically associated to the Fell bundle \(B \to G\) as in \[\text{KMQW10, Proposition 3.1}\], and \(\tilde{\delta}\) is the dual action of \(G\) on the crossed product \(C^*(G, \mathcal{B}) \rtimes \delta G\). Thus in this case the Morita equivalence

\[
C^*(G \rtimes G, \mathcal{B} \rtimes G) \rtimes \alpha G \sim C^*(G, \mathcal{B})
\]

that we get by applying the Yamagami-Muhly-Williams Equivalence Theorem \[\text{MW08, Theorem 6.4}\] to the Fell-bundle equivalence of Theorem 8.1 gives (the Morita-equivalence form of) Katayama’s duality theorem for maximal coactions and full dual crossed products \[\text{KMQW10, Theorem 8.1}\] (see \[\text{Kat84, Theorem 8}\] for the original version).

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