A note on Klein-Gordon equation in a generalized Kaluza-Klein monopole background

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Abstract

We investigate solutions of the Klein-Gordon equation in a class of five dimensional geometries presenting the same symmetries and asymptotic structure as the Gross-Perry-Sorkin monopole solution. Apart from globally regular metrics, we consider also squashed Kaluza-Klein black holes backgrounds.

1 Introduction

Recently, there has been an increasing interest in the solutions of Einstein equations involving more than four dimensions. Such configurations are important if one supposes the existence of extra dimensions in the universe, which are likely to be compact and described by a Kaluza-Klein (KK) theory.

A particularly interesting case are the \( d = 5 \) asymptotically locally flat configurations, approaching a twisted \( S^1 \) bundle over a four dimensional Minkowski spacetime. The best known example with this structure is the Gross-Perry-Sorkin (GPS) monopole solution [1, 2]. Various generalizations of this solution with similar asymptotics have been discussed in the literature, representing both globally regular and black hole configurations.

The study of classical fields in these backgrounds is a reasonable task as a first step towards their quantization. Also, the GPS background provides one of the few examples available in the literature where it is possible to solve in closed form various field equations on a curved background. The purpose of this work is to study the solutions of the scalar wave equation in a generic \( d = 5 \) asymptotically locally flat spacetime, presenting the same symmetries as the GPS monopole. We find that in the globally regular case, the properties of the solutions are rather background independent.

2 The general framework

2.1 Generalized Kaluza-Klein monopole space-times

We consider generalized \((4+1)\)-dimensional KK monopole space-times, with the metric given by line elements of the form

\[
\text{d}s^2 = g_{\mu\nu}\text{d}x^\mu\text{d}x^\nu = -h\text{d}t^2 + f (\text{d}r^2 + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2)) + g (\text{d}\chi + \cos \theta \text{d}\phi)^2,
\]

written in spherical coordinates, \( r, \theta, \phi \), commonly related with the Cartesian ones and \( \chi \) defined as usual, while \( t \) is the time coordinate. The coordinates \( \theta \) and \( \phi \) cover the sphere \( S^2 \) while \( \chi \in D_\chi = [0, 4\pi) \). We suppose that the functions \( f, g \) and \( h \) depend only on the radial coordinate \( r \). In addition we assume that on
the radial domain of the local chart, \(D_r\), these functions are positive definite. Obviously, the whole physical space domain of this chart is \(D = D_r \times S^3\).

By construction, the spaces with the metric \(\Box \) have five Killing vectors. The corresponding constants of motion consist of a conserved quantity for the cyclic variable \(\chi\), \(q = g(r)(\chi + \cos \theta \dot{\varphi})\), the energy, and the angular momentum vector \(\vec{J} = \vec{x} \times \vec{p} + g \vec{\varphi}\), with \(\vec{p} = f(r)\vec{x}\).

The line element \(\Box\) may describe two different types of configurations. The first one corresponds to (topologically nontrivial) globally regular spacetimes, with \(h > 0\) for the whole range of \(r\). The structure of these solutions is determined by the dimension of the fixed point set of the Killing vector \(\partial/\partial \chi\). In general, this \(U(1)\) isometry can have a zero-dimensional fixed point set (referred to as a "nut" solution) or a two-dimensional point set in the four dimensional Euclidean space (correspondingly referred to as "bolt" solution). The standard KK monopole solution \(\Box\) corresponds to the first case and has

\[
h = 1, \quad f = 1 + \frac{4m}{r}, \quad g = \frac{16m^2}{f}, \tag{2}
\]

with \(m\) a constant. The \(r = 0\) corresponds here to the origin of the coordinate system in \(R^4\). When taking instead the product of the \(d = 4\) Euclidean Taub-bolt solution \(\Box\) with the real line, the metric functions are

\[
h = 1, \quad f = 1 + \frac{m}{4r} + \frac{9m^2}{64r^2}, \quad g = \frac{4m^2}{f}(1 - \frac{9m^2}{64r^2})^2, \tag{3}
\]

with \(r \geq r_h = 3m/8\).

Apart from these configurations, there are also squashed KK black holes, with an event horizon located at \(r = r_h\) with \(h(r_h) = 0\) and \(f, g\) nonvanishing, while \(f, g, h\) stay positive for any \(r \geq r_h\) (see e.g. \(\Box\)). For example, the line element of the Einstein-Maxwell squashed black hole found by Ishihara and Matsumo, expressed in the coordinate system used for the metric ansatz \(\Box\), reads

\[
f(r) = \frac{1}{16r^4}(r^2 + \frac{1}{2r}(2r_0 + r_m + r_p) + \frac{1}{16}(r_p - r_m)^2)((4r + r_p)^2 - 2r_m(r_p - 4r) + r_m^2),
\]

\[
g(r) = \frac{(r_0 + r_m)(r_0 + r_p)((4r + r_p)^2 - 2r_m(r_p - 4r) + r_m^2)}{16r^2 + 8r(2r_0 + r_m + r_p) + (r_m - r_p)^2}
\]

\[
h(r) = \frac{(r_m - r_p)^2 - 16r^2)^2}{(4r + r_p)^2 - 2r_m(r_p - 4r) + r_m^2},
\]

where \(r_0, r_m\) and \(r_p\) are real parameters related to the mass, electric charge and the size of the extra dimension, and \(r_h = (r_p - r_m)/4\). For completeness, we present also the \(U(1)\) potential expression (the Lagrangian in this case is \(L = R/(16\pi G) - F^2/4\), with \(F = dA\))

\[
A = 4\sqrt{\frac{3}{\pi G}} \frac{r\sqrt{m\gamma_p}}{(4r + r_m)^2 + 8rr_p - 2r_m r_p + r_p^2} dt,
\]

(the choice \(r_m = 0\) gives the vacuum black version of the GPS monopole presented in \(\Box\)). No similar closed form expression can be written for a more complicated matter content (e.g. a nonabelian field).

In the general case, the expression of the metric function \(f, g\) and \(h\) is usually fixed by the matter content of the theory and the boundary conditions we choose. We shall suppose, however, that, similar to the vacuum cases \(\Box, \Box\), the spacelike infinity is always a squashed sphere or \(S^1\) bundle over \(S^2\). Thus, for large values of \(r\), one finds

\[
f(r) = 1 + \frac{\tilde{f}_1}{r} + O\left(\frac{1}{r^2}\right), \quad h(r) = 1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right), \quad g(r) = 16m^2\left(1 + \frac{\tilde{g}_1}{r}\right) + O\left(\frac{1}{r^2}\right), \tag{5}
\]

where \(\tilde{f}_1, M, \tilde{g}_1\) and \(m\) are real constant (with \(m\) and \(M\) fixing the size of the extra dimension and total mass-energy \(\Box\)).
2.2 The Klein-Gordon equation

The equation governing the behaviour of a massless, minimally coupled scalar field is (in natural units with $\hbar = c = 1$),

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu \nu} \frac{\partial \psi}{\partial x^\nu} \right) = 0,$$

and has particular solutions

$$\psi(x) = U_E(x, \chi) e^{-iEt},$$

of given frequency $E$. The separation of variables in Eq.(6) can be done if one takes

$$U_E(x, \chi) = \mathcal{P}(r) Y_{l,m}^q(\theta, \phi, \chi),$$

where $Y_{l,m}^q(\theta, \phi, \chi)$ are the $SO(3) \otimes U(1)$ harmonics [11]. These harmonics are eigenfunctions of the operators

$${\cal L}^2 = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) - \frac{1}{\sin^2 \theta} (\partial_\phi^2 + \partial_\chi^2 - 2 \cos \theta \partial_\theta \partial_\chi), \quad L_3 = \partial_\theta, \quad Q = \partial_\chi,$$

with

$${\cal L}^2 Y_{l,m}^q = l(l+1) Y_{l,m}^q, \quad L_3 Y_{l,m}^q = m Y_{l,m}^q, \quad Q Y_{l,m}^q = q Y_{l,m}^q,$$

and satisfy the orthonormalization condition

$$\langle Y_{l,m}^q, Y_{l',m'}^q \rangle = \int_{S^2} d(\cos \theta) d\phi \int_0^{2\pi} d\chi Y_{l,m}^q(\theta, \phi, \chi) Y_{l',m'}^q(\theta, \phi, \chi) = \delta_{l,l'} \delta_{m,m'} \delta_{q,q'},$$

Notice that the boundary conditions on $S^2 \times D_\chi$ require $l$ and $m$ to be integer numbers while $q = 0, \pm 1/2, \pm 1, \ldots$ [12]. The form of these harmonics and a discussion of their properties is given in [11].

Thus, one finds that the function $P(r)$ in (8) is a solution of the equation

$$-\frac{1}{r^2 \sqrt{fgh}} \frac{d}{dr} \left( r^2 \sqrt{fgh} \frac{dP}{dr} \right) - \frac{1}{r^2} (l(l+1) - q^2) P - \frac{q^2}{g} P + \frac{E^2}{h} P = 0.$$  

(10)

This radial equation can be transformed to a standard Schrödinger form by defining a "tortoise" radial variable $dr_* = (r \sqrt{fgh}) \frac{df}{dr}$ and taking $\mathcal{P}(r) = F(r)/F_1(r)$, with $F_1(r) = (rf^{1/2}g^{1/4})$. One finds

$$-\frac{d^2F}{dr_*^2} + V(r)F = E^2F,$$

with a potential

$$V(r) = \frac{h}{r^2 f} (l(l+1) - q^2) + \frac{q^2 h}{g} - \frac{1}{F_1^3} \sqrt{\frac{h}{f}} (r^2 \sqrt{fgh} F_1')',$$

(where a prime denotes the derivative w.r.t. $r$).

However, in practice we found more convenient to take

$$U_E(x, \chi) = \frac{1}{\rho(r)} R_{E,l}^q(r) Y_{l,m}^q(\theta, \phi, \chi),$$

where a suitable choice for the function $\rho$ is

$$\rho = r |fgh|^{1/4}.$$  

(12)
Then, after a few manipulation, we find the radial equation

\[ -\frac{d^2 R}{dr^2} + \frac{1}{r^2} [(l + 1) - q^2] + \frac{f}{g} q^2 + \frac{1}{\rho} \frac{d^2 \rho}{dr^2} \right] R_{E,l}^q(r) = E^2 \frac{f}{h} R_{E,l}^q(r) . \]  

The radial function \( R_{E,l}^q \) is normalized according to the radial scalar product

\[ \left< R_{E,l}^q, R_{E',l'}^q \right> = \int_{D_r} dr \frac{f}{h} (R_{E,l}^q)^* R_{E',l'}^q , \]

resulted from the fact that the \( SO(3) \otimes U(1) \) harmonics, \( Y_{l,m}^q \), are normalized to unity with respect to their own scalar product. The radial equation (13) is similar to those of the non-relativistic quantum mechanics apart the term \( \rho''/\rho \).

Unfortunately, there is only one solution of the equation (13) known in closed form, corresponding to a GPS monopole background [11]. However, one can analyze the properties of the general solutions by using a combination of analytical and numerical methods, which is enough for most purposes. For example, the leading order asymptotic expansion of the radial function \( R(r) \) is shared by all solutions. Taking into account the expressions (5) of the metric functions, one finds that the equation (13) presents an effective mass term in its asymptotic expansion, \( R'' + (E^2 - \frac{q^2}{4m^2})R = 0 \). Thus, for \( E > |q|/2m \), the radial function has an oscillatory behaviour at infinity

\[ R_{E,l}^q(r) \sim e^{-i\sqrt{E^2-q^2/(4m^2)r}} + s(E)e^{i\sqrt{E^2-q^2/(4m^2)r}} . \]

For \( E < |q|/2m \), the radial function decays asymptotically according to

\[ R_{E,l}^q(r) \sim e^{-\sqrt{q^2/(4m^2)} - E^2 r} . \]

3 Solutions in a globally regular background

3.1 The Iwai-Katayama background

An interesting case of globally regular backgrounds are the metrics proposed by Iwai and Katayama (IK) [13, 14, 15], with

\[ h = 1 , \quad f(r) = \frac{a}{r} + b , \quad g(r) = \frac{ar + b r^2}{1 + cr + d r^2} , \]

where \( a, b, c, d \) are constants. If one takes these constants \( c = 2b/a, \ d = b^2/a^2 \) the four dimensional generalized Taub-NUT part of the metric [11] becomes the original Euclidean Taub-NUT metric up to a constant factor.

The remarkable result of Iwai and Katayama is that the generalized Taub-NUT space (17) admits a hidden symmetry represented by a conserved vector, quadratic in \( 4 \)-velocities, analogous to the Runge-Lenz vector of the following form

\[ \vec{K} = \vec{p} \times \vec{J} + \kappa \vec{F} \]

The constant \( \kappa \) involved in the Runge-Lenz vector (18) is \( \kappa = -a E + \frac{1}{2} c q^2 \) where the conserved energy \( E \) is \( \vec{E} = \vec{p}^2/2f(r) + q^2/2g(r) \). The components \( K_i = \kappa \mu_{i
\nu} p_\mu p_\nu \) of the vector \( \vec{K} \) (18) involve three Stäckel-Killing tensors \( k^{\mu
\nu}_i, \ i = 1, 2, 3 \).

1 As usual in metric backgrounds with \( g_{00} = -1 \), the expression of the scalar field \( \Psi \) can be read from the solutions of the Schrödinger equation in [11] by replacing \( E \rightarrow E^2/2 \) in the relations there.

2 These configurations are of interest mainly because they admit a Kepler-type symmetry. Other generalized Taub-NUT metrics could have a self-dual Weyl curvature tensor, be conformally flat [13], or produce complete Einstein self-dual metrics on 4-balls [16], etc.
By rescaling the radial coordinate \(r\), one can always set \(b = 1\) in the general metric functions \([17]\). It is also convenient to define \(c = Ca\), \(d = Da\), the line element \([1]\) becoming

\[
ds^2 = -dt^2 + \left(1 + \frac{a}{r}\right)(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)) + \frac{a^2(1 + \frac{a}{r})}{D + \frac{2Ca}{r} + \frac{a^2}{r^2}}(d\chi + \cos \theta d\phi)^2 ,
\]

(18)

We suppose that the coefficients \(C\), \(D\) are restricted such that the function \(D + \frac{2Ca}{r} + \frac{a^2}{r^2}\) takes only positive values on \(D_r\). Also, to simplify the general picture we shall suppose \(a > 0\).

### 3.1.1 A perturbative solution

The radial equation \([13]\) can be written as \(\hat{O} R(r) = 0\), with the operator

\[
\hat{O} = -\frac{d^2}{dr^2} + \frac{1}{r^2}(l(l + 1) - q^2) + \frac{f}{g} q^2 + \frac{1}{\rho} \frac{d^2 \rho}{dr^2} - E^2 \frac{f}{h} \rho.
\]

(20)

The choice \(C = D = 1\) in the generic line element \([13]\) corresponds to a standard GPS monopole background. This suggest a perturbative approach for the equation \([13]\), by taking the expansion

\[
C = \sum_{k=0}^{\infty} c_k e^k, \quad D = \sum_{k=0}^{\infty} d_k e^k, \quad R(r) = \sum_{k=0}^{\infty} R_k(r) e^k,
\]

(19)

with \(c_0 = d_0 = 1\) and \(\varepsilon\) a small parameter. This implies the following decomposition of the operator \(\hat{O}\)

\[
\hat{O} = \hat{O}_0 + \sum_{k>0} \hat{O}_k e^k
\]

(21)

with

\[
\hat{O}_0 = -\frac{d^2}{dr^2} + \frac{1}{r^2}(l(l + 1) - q^2) + \frac{(1+a/r)^2}{a^2} q^2 - E^2 (1 + \frac{a}{r}),
\]

\[
\hat{O}_k = \frac{1}{a^2} (d_k + \frac{2Ca}{r}) + \left(\frac{a'}{\rho}\right)_k , \quad \text{with } k > 1,
\]

where the first terms in the general expression of \((\rho''/\rho)_k\) are

\[
\left(\frac{a'}{\rho}\right)_{(1)} = -\frac{a^2}{2r(a+r)^2} (2ac_1 + (3d_1 - 4c_1)r), \quad \left(\frac{a'}{\rho}\right)_{(2)} = \frac{a^2}{4r(a+r)^2} ( -4a^3 c_2
\]

\[
+a^2(13c_1^2 - 6d_2)r - 2a(6d_2 - 6c_2 + 13c_1(c_1 - d_1))r^2 + (5c_1^2 + 8c_2 - 18c_1d_1 + 11d_1^2 - 6d_2)r^3).
\]

One can see that \(R_0(r)\) solves the equation \([13]\) for a standard GPS monopole background, \(\hat{O}_0 R_0(r) = 0\). The solutions of this equation have been discussed in \([11]\). The \(k\)–th order equation reads

\[
\sum_{i=0}^{k} \hat{O}_{(i)} R_{k-i} = 0, \quad \text{or} \quad \hat{O}_0 R_k = S_k(r) = -(\hat{O}_{(1)} R_{k-1}(r) + \cdots + \hat{O}_{(k)} R_0(r)) .
\]

(22)

The general solution of the above equation compatible with the required boundary conditions reads \([17]\)

\[
R_k(r) = R_0(r) + \int_{D_r} G(r,r') S_k(r') dr',
\]

(23)

where \(G(r,r')\) is the Green function associated with radial equation \([13]\) for a GPS monopole background, \(\hat{O}_0 G(r,r') = -\delta(r,r')\), i.e.

\[
G(r,r') = \frac{f_>(r_>) f_<(r_<)}{W(f_>, f_<)} ,
\]

(24)
where

\[ f_1(r) = e^{-\sqrt{q^2 - a^2 E^2}r/a} r^{1+l} L(-1-l + \frac{-q^2 + a^2 E^2 / 2}{\sqrt{q^2 - a^2 E^2}}, 1 + 2l, \frac{2\sqrt{q^2 - a^2 E^2 / 2}}{a}, r), \]

\[ f_2(r) = e^{-\sqrt{q^2 - a^2 E^2}r/a} r^{1+l} U(1+l + \frac{q^2 - a^2 E^2 / 2}{\sqrt{q^2 - a^2 E^2}}, 2(1+l), \frac{2\sqrt{q^2 - a^2 E^2 / 2}}{a}, r), \]

are the independent solutions to the equation \( \dot{\cal O}(0) f(r) = 0 \), with \( U \) and \( L \) the confluent hypergeometric functions and the generalized Laguerre polynomial, respectively \[17\]. \( f_\rightarrow(r_\rightarrow) = f_\rightarrow(r_\rightarrow) \) satisfies the boundary condition of finiteness at infinity and \( f_\leftarrow(r_\leftarrow) = f_\leftarrow(r_\leftarrow) \) is similar finite as \( r \) goes to zero; \( W \) is the wronskian of \( f_\rightarrow \) and \( f_\leftarrow \) (with \( r_\leftarrow = \min(r, r') \), \( r_\rightarrow = \max(r, r') \)).

### 3.1.2 Numerical solutions

One can also look for nonperturbative solutions of the equation \[13\]. Here we need the asymptotic expansion of the solution near the origin and at infinity. These asymptotics are very similar to the case of solutions in the background \[2\]. A systematic analysis gives

\[ R(r) \sim r^s, \quad \text{as} \quad r \to 0, \quad \text{(24)} \]

where the parameter \( s \) is a solution of the equation \( s(s - 1) = l(l + 1) \). The modes with \( s = l + 1 \), called in \[11\] regular modes are similar to those of the usual nonrelativistic case. Apart from these, there are also irregular modes with \( s = -l \), which are relevant in the \( a < 0 \) case (not considered here).

As \( r \to \infty \) one finds the following asymptotic form

\[ R(r) \sim e^{-\lambda r}, \quad \text{with} \quad \lambda = \sqrt{\frac{Dq^2}{a^2} - E^2}, \quad \text{(25)} \]

the general solution being on the form

\[ R(r) = r^s e^{-\lambda r} P(r). \quad \text{(26)} \]

Taking \( s = l + 1 \), one finds that satisfies \( P(r) \) the equation

\[ -P'' + \frac{2(l + 1)}{r} P' + 2 \sqrt{\frac{Dq^2}{a^2} - E^2} P(r) + K(r) P(r) = 0 \quad \text{(27)} \]

where \( K(r) \) has a complicated expression. However, as \( r \to 0 \) one can write

\[ K(r) = \frac{1}{ar} (1 - a^2 E^2 + C(2q^2 - 1) + 2a \sqrt{Dq^2 - a^2 E^2 (l + 1)}) \]

\[ - \frac{5 + (2 - 13C)C + 6D}{4a^2} + \frac{r}{2a^2} (3 - D + C(1 + 2(1 - 9C)C + 13D)) + O(r^2), \quad \text{(28)} \]

its asymptotic expansion as \( r \to \infty \) being

\[ K(r) = \frac{2}{ar} (-a^2 E^2 / 2 + Cq^2 + \sqrt{Dq^2 - a^2 E^2 (l + 1)}) + O(1/r^3). \quad \text{(29)} \]

One can see from \[25\] that, in agreement with the relation \[13\], for \( E^2 > Dq^2 / a^2 \) there is only a continuous energy spectrum, the levels of which are infinite degenerate.

The case \( E^2 < Dq^2 / a^2 \) is more involved, since one finds also a discrete sector of the spectrum. The arguments here are similar to the case \( C = D = 1 \) \[11\]. Following the standard approach \[17\], we suppose that \( P(r) \) admits a power series expansion

\[ P(r) = \sum_k p_k r^k, \quad \text{(30)} \]
with $p_k$ real coefficients. Supposing that this series has $n$ terms and replacing in (27), one finds the quantization of the energy

$$E^2 = \frac{2}{a^2}(Cq^2 - (1 + n + l)^2 + (1 + n + l)\sqrt{(1 + n + l)^2 - 2Cq^2 + Dq^2}),$$

with $\lim_{n \to \infty} E = 2Cq^2/a^2$. This result is found by taking the expression (28) for $K(r)$, or the large $r$ form (29).

Although the existence of the global solution (26), (30) (with $E$ fixed by (31)) still requires an existence proof, this agrees with the closed form solution known for $C = D = 1$ [11]. The expansion (30) has only one free coefficient, which is fixed by the normalization condition (14). However, the coefficients $p_k$ satisfy a complicated recurrence relation, which we could not establish in the general case.

In practice, the solutions of the equation (13) interpolating between (24) and (25) can be constructed numerically by using a standard ordinary differential equation solver. A plot of two typical radial functions with quantum numbers $l = 1, n = 2, q = 1$ is given in Figure 1 for $C = D = 1$ (the case of a standard KK monopole background) and a IK geometry with $C = 1.1, D = 2.7$. These solutions decay exponentially and have a frequency fixed by (31). The solution with an asymptotic oscillatory behaviour looks very similar to that presented in Figure 2 for a metric background given by (3).

One should also remark that, although we have restricted to a IK background because of their geometrical relevance, a quantization relation similar to (31) is found for any regular geometry with a "nut" in the $t = \text{const.}$ section, provided that $E < |q|/(4n)$.

### 3.2 A globally regular "bolt" background

Although no closed form solution is available in this case, our results indicate that the properties of the radial function $R(r)$ are rather similar to the case discussed above. Here we shall restrict to the metric form (3), which is the only case considered so far in the literature. The radial function vanishes near $r = r_b$, with

$$R(r) = (r - r_b)^{(1 + 4|q|)/2}c_0\left(1 - \frac{2}{3m}(1 - 4|q|)(r - r_b) + \ldots\right),$$

while the asymptotic form is still given by (15), (16).

When taking a general solution on the form $R(r) = (r - r_b)^{(1 + 4|q|)/2}e^{-\sqrt{q^2/(4m^2) - E^2}}P(r)$, and suppose a power series expansion for $P(r)$, one finds that, for $E < q/2m$, the spectrum is quantized again according
Figure 2: The radial function $R(r)$ is plotted for a $d = 5$ metric which is the product of the $d = 4$ Euclidean Taub-bolt solution with the real line. The parameters here are $m = 1$, $l = 1$, $q = 0$ and $E = 3$.

to

\[ E^2 = \frac{q^2}{4m^2} - \left( \frac{2n - 1 + 4|q|}{5m} - \sqrt{\frac{(2n - 1 + 4|q|)^2}{25m^2} - \frac{q^2}{4m^2}} \right)^2. \]  

(33)

For $E > q/2m$, the spectrum is continuous, with an oscillatory behaviour of the radial wave function given by (16).

The general solution is again constructed numerically and looks rather similar to the case of a IK background. A typical oscillatory solution for a "bolt" background with $r_b = 8/3$ is plotted in Figure 2.

4 Solutions in a squashed Kaluza-Klein black hole background

Apart from the geometries with a regular origin considered in the previous section, the metric ansatz (1) presents another type of configurations. These are the squashed KK black holes, which enjoyed recently some interest, following the discovery by Ishihara and Matsuno (IM) (4) of a charged solution in the five dimensional Einstein-Maxwell theory with some special properties. The horizon of the IM black hole has $S^3$ topology, and its spacelike infinity is a squashed sphere or $S^1$ bundle over $S^2$. The mass and thermodynamics of this solution have been discussed in (18). A vacuum black hole solution with similar properties has been presented in (8). The IM black hole has been generalized in several directions, including KK rotating solutions with squashed horizon (6), an Einstein-Maxwell-dilaton version (7) and solutions with nonabelian matter fields (9).

As mentioned above, the expression of the metric functions are fixed by the matter content of the theory, the Einstein-Maxwell solution (4), (5) being the most studied case in the literature. However, in all cases, the squashed KK black holes approach asymptotically a twisted $S^1$ bundle over a four dimensional Minkowski spacetime, with the approximate form of the metric functions given by (18). For such solutions, the spacetime behaves as a five-dimensional black hole near the horizon, while the dimensional reduction to four is realized in the far region.

The regular horizon is located at $r = r_h$, where the following expansion holds (here we consider non-extremal solutions only)

\[ f(r) = f_h + \sum_{k \geq 1} f_k (r - r_h)^k, \quad g(r) = g_h + \sum_{k \geq 1} g_k (r - r_h)^k, \quad h(r) = h_2 (r - r_h)^2 + \sum_{k \geq 3} h_k (r - r_h)^k. \]  

(34)
where \( f_h, f_k, g_h, g_k, h_2, h_k \) are a set of constants which are fixed by the Einstein-matter field equations (with \( f_h, g_h, h_2 > 0 \)). The Hawking temperature of the black holes, as computed from the surface gravity \( T_H = \kappa/(2\pi) \) (with \( \kappa^2 = -\frac{1}{2}g^{tt}g^{rr}(\partial_t g_{tt})^2 \)), is \( T_H = \sqrt{h_2/f_h/(2\pi)} \).

It is interesting to study the KG equation in this case and to see the differences with respect to the globally regular case. The equation (6) is solved here with a suitable set of boundary conditions at the inner \( r = r_h \) boundary and at infinity. Some properties of the KG equation in a IM background have been discussed recently in Ref. [19]. Here we consider instead a generic line element, supposing the metric functions satisfy the asymptotic expansions (34), (5). As expected, no closed form solution of the KG equation is available in the black hole case. Also, these solutions cannot be considered as a perturbation around a GPS background.

As usual with black holes [20], the general solution of the Schrödinger-type eq. (11) has the following form near the horizon of the black hole

\[
F = \lambda_1 e^{-iEr_h} + \lambda_2 e^{+iEr_h}.
\]

However, only those solutions are retained which represent no particles coming out of the horizon of the black hole; this is satisfied if we set \( \lambda_2 = 0 \). Expressing this result in terms of the radial function \( R(r) \), one finds the approximate form of the solutions as \( r \to r_h \)

\[
R(r) = \lambda_1 \sqrt{r - r_h} \exp \left( -i \frac{E}{2\pi T_H} \log(r - r_h) \right) (1 + c_1(r - r_h) + c_2(r - r_h)^2 + \ldots),
\]

where the coefficients \( c_k \) are fixed by the geometry and the parameters \( E, l \) and \( q \), e.g. one finds for the lowest order term

\[
c_1 = \frac{h_2 (f_1 g_h h_2 r_h + f_k (4 g_h h_2 + g_1 h_3 r_h + g_2 h_3 r_h) + 4 f_h g_h (f_k (h_3 - f_1 h_2) r_h E^2))}{4 f_h g_h h_2 r_h (h_2 + 2i \sqrt{f_h h_2 E})}.
\]

Considering now the solution in the large \( r \) region, one finds that the the asymptotic forms (15), (16) hold in this case, too. The \( M/r \) coefficient which enters the expression of \( g_{tt} \) affects there only the next to leading order terms. Different e.g. from the Schwarzschild black hole case, the spectrum of a KK black hole with squashed horizon contains an exponentially decaying part for small enough frequencies \( (E < |q|/2m) \). No quantization of frequencies occurs in the black hole case, however. For any given background, the solution interpolating between (30) and (15), (16) can be constructed numerically by employing similar methods to those used in the globally regular case.

We close this Section by remarking that the Ref. [19] presented arguments that the luminosity of the Hawking radiation of the Einstein-Maxwell black hole solution (4) encodes information about the size of the extra-dimension. This result is found by computing the absorption probability by matching the asymptotic expansions of the solutions of the KG equation and is likely to hold for any KK squashed black hole.

5 Further remarks

A considerable number of attempts have been carried out in order to solve the matter field equations in a four dimensional curved background. However, the corresponding task for higher dimensional solutions is still in its early stages, especially for spacetimes with a more complicated topological structure.

In this work we have analyzed the basic properties of the scalar wave equation in a generalized KK monopole spacetime. Our results are relevant when considering the scalar field quantization in these metric backgrounds. As a characteristic feature, we noticed the existence of an exponentially damped part of the spectrum, for small enough frequencies. In the globally regular case, the frequencies are quantized, the properties of the KG equation in a GPS monopole background being generic in this case. For large enough frequencies, a continuum wave spectrum is found instead. In the well-known GPS case, these results are in agreement with the behaviour of classical test particles. It would be interesting to consider the geodesic motion in a general background [11].

\[\text{Note that, however, a Schwarzschild-like coordinate system is used in [19].}\]
Similar solutions can be constructed for higher spin fields, the case of Dirac equation in a general metric background (1) being currently under study.

It is interesting to compare the properties of the scalar field we have found for a background (1) approaching a twisted $S^1$ bundle over a four dimensional Minkowski spacetime with the case of a topologically trivial KK background. Such geometries are described by a metric ansatz:

$$ds^2 = -h(r)dt^2 + f(r)(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)) + g(r)d\chi^2,$$

(38)

and approach at infinity the four dimensional Minkowski spacetime times a circle (here $0 \leq \chi \leq L$, with $L$ arbitrary). The simplest solution of this type is found by trivially extending to five dimensions the Schwarzschild black hole solution to vacuum Einstein equations in four dimensions, and corresponds to a uniform black string with horizon topology $S^2 \times S^1$. (Note that nontrivial globally regular solutions are found within the metric ansatz (38) only in the presence of nonlinear matter fields, see e.g. the solutions in [21]). Separating the variables in the KG equation (6) by writing (with $\rho$ still given by (12))

$$\psi(x) = \frac{1}{\rho(r)} R^2_{E,l}(r) Y_{l,m}(\theta, \phi) e^{i(q\chi - Et)} ,$$

(39)

where $Y_{l,m}(\theta, \phi)$ are the spherical harmonics and $q = 2n\pi/L$ (with $n$ an integer), one obtains the radial equation for $R$ in the form

$$\left[ -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{f}{g} q^2 + \frac{1}{\rho} \frac{d^2\rho}{dr^2} \right] R^2_{E,l}(r) = E^2 \frac{f}{h} R^2_{E,l}(r) .$$

(40)

Comparing with (13) reveals that the main effect at this level of a KK monopole asymptotic structure of the metric background is to add a new term to the potential in the Schrödinger-like equation satisfied by the radial wave function. This term decreases the height of the potential and becomes relevant for high energies. Note also the existence in this case, too, of an exponentially decaying part of the spectrum for $E < |q|$.

For the case of a vacuum black string, the solutions of the equation (40) are found by taking $E \rightarrow \sqrt{E^2 - q^2}$ in the solutions corresponding to a $d = 4$ Schwarzschild black hole background.

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