A GRAY-CATEGORICAL PASTING THEOREM

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ABSTRACT. We prove that every 2-dimensional pasting diagram in a Gray-category has a unique composite up to a contractible groupoid of choices.

1. INTRODUCTION

Pasting diagrams are graphical tool to express compositions in higher dimensional categories. They can be interpreted as vertical compositions of whiskerings, e.g. the triangle identity for an adjunction \( f \dashv g \) in a 2-category can be visualized as follows:

\[
\begin{array}{ccc}
A & \xrightarrow{g} & A \\
\downarrow^g & & \downarrow^g \\
B & \xrightarrow{f} & B
\end{array}
\]

While in the situation above the composite is always uniquely determined, there are cases when it is not clear how to interpret a pasting diagram. For instance:

\[
\begin{array}{ccc}
A & \xrightarrow{g} & A \\
\downarrow^g & & \downarrow^g \\
B & \xrightarrow{f} & B
\end{array}
\]

\[
\begin{array}{ccc}
A & \xleftarrow{\tilde{\eta}} & A \\
\downarrow^{\tilde{\eta}} & & \downarrow^{\tilde{\eta}} \\
B & \xleftarrow{\tilde{\epsilon}} & B
\end{array}
\]

can be written as vertical composition in two different ways, namely

\[
m\delta \cdot \varphi d \cdot \gamma d \cdot cf\beta \cdot caa
\]

and

\[
m\delta \cdot \varphi d \cdot g\beta \cdot \gamma e a \cdot cca,
\]

which coincide for strict higher categories such as 2-categories by the middle four interchange law. In fact, in these cases the interchange law holds strictly so that the square

\[
\begin{array}{ccc}
cf e a & \xrightarrow{cf\beta} & cf d \\
\gamma e a & \Rightarrow & \gamma d \\
g e a & \xrightarrow{g\beta} & g d
\end{array}
\]

commutes. A number of results has been proven in this setting, such as in [7] and [10]. The latter has been extended to bicategories in [11]. A didactic account of these two results can be found in the book [8]. In short, [10] provides a basic algorithm to get a composite for a pasting diagram as a vertical composition of whiskered 2-cells. In each step of the algorithm we remove a 2-cell and
add it to the whiskered composite, going from top to bottom. So, for instance, in our example we start by taking off $\alpha$ but then we can either remove $\beta$ or $\gamma$. If we choose $\beta$ we will remove $\gamma$ in the next step and vice versa. We just keep removing 2-cells until none is left. This will theoretically produce different composites, but each time there is a choice between two or more 2-cells we can use the middle four interchange law so that in the end all the composites will be equal.

\[ \text{Diagram showing whiskered composites and removal of 2-cells.} \]

However for more general weak higher categories the interchange law might hold just up to coherent isomorphism so this algorithm needs to be modified. Therefore the uniqueness of the pasting composite must be interpreted in a suitable way, namely as a contractibility condition on the space of composites of the pasting diagram (see for instance [5]). This is indeed the case for the current work, where we will be dealing with the semi-strict case of a pasting diagram in a Gray-category, a particular notion of 3-dimensional category where the middle four interchange law is not strict but it is instead part of coherence data. In particular, we will prove the following theorem.

**Theorem 4.17.** Every 2-dimensional pasting diagram in a Gray-category has a unique composition up to a contractible groupoid of choices.

It has to be noticed that we are only considering pasting composites of 2-cells. The proof of this theorem uses techniques from rewriting theory, but no previous knowledge of it is required. We will recall the basics of this theory along the way.

This result makes precise a remark that can be found in the seminal work [4]. It also provides an affirmative answer to a conjecture stated in [1].
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2. Preliminaries on relations

At first let us recall some basic facts about relations, since we will use them later for rewriting. In accordance with the literature, whenever \( R \) a relation on a set \( X \) we will write \( xRy \) to denote that \((x, y) \notin R\).

**Definition 2.1.** A relation \( R \) on a set \( X \) is said to be **irreflexive** if \( \forall x \in X \ xRx \).

**Definition 2.2.** A relation \( R \) on a set \( X \) is said to be **asymmetric** if for all \( x, y \in X \) we have that \( xRy \implies yRx \). An irreflexive, asymmetric and transitive relation is called a **strict partial order**.

**Remark 2.3.** An irreflexive and transitive relation is also asymmetric, hence a strict partial order.

A set equipped with a strict partial order will be called **strict poset**. A **strict linear order** (also called **strict total order**) is a strict partial order for which any two elements are comparable.

**Definition 2.4.** We say that \((A, <^*)\) is a **strict linear extension** of a strict poset \((A, <)\) if

1. \(<^*\) is a strict linear order;
2. for every \(a, b \in A\), we have that \(a < b \implies a <^* b\).

In other words, a strict linear extension is a strict linear order that contains the given partial order.

**Remark 2.5.** In general there is more than one linear extension of a given poset, for instance \((\{x, y\}, =)\) with \(x \neq y\) can be extended to a linear order either by choosing \(x < y\) or \(y < x\).

Given an irreflexive relation \( R \) we can ask if its transitive closure is still irreflexive, so that by Remark 2.3 it is a strict partial order. A sufficient condition for this to happen is that \( R \) is **acyclic**, i.e. there are no \(x_1, x_2, \ldots, x_n \in X\) s.t. \(x_1Rx_2\) and \(x_2Rx_3\) and \(\ldots\) and \(x_nRx_1\) \((x_1Rx_2Rx_3\ldots x_nRx_1\) for short). Whenever this condition holds, we define a strict linear extension of \( R \) to be a strict linear extension of its transitive closure. The following proposition guarantees that in such a case a linear extension always exists.

**Proposition 2.6.** Let \((X, R)\) be a finite set endowed with an irreflexive relation. The following are equivalent:

(a) \( R \) is acyclic;

(b) \( R \) is well founded, i.e. \( \forall S \subseteq X, \ S \neq \emptyset, \ \exists m \in S \ \forall s \in S \ sRm \) (called a minimal element);

(c) \( R \) admits a strict linear extension \(<\).

**Proof.** \((a) \implies (b)\) We prove the contrapositive. Since \( S \neq \emptyset \) there exists \( m_1 \in S \). If \( \forall s \in S \ sRm_1 \), we have \( m = m_1 \), otherwise there exists \( m_2 \in S \) s.t. \( m_2Rm_1 \). For the same reason, either \( m = m_2 \) or there exists \( m_3 \in S \) with \( m_3Rm_2 \). In the latter case, iterating this argument eventually gives an element \( m_i \) we already visited (since \( S \) is finite) and therefore a cycle \( m_iRm_{i+1}\ldots m_1Rm_i \).
(b) \( \Rightarrow \) (c) The whole set \( X \) is a subset of itself, so it has a minimal element \( x_1 \). The set \( X \setminus \{ x_1 \} \) is contained in \( X \), therefore has a minimal element \( x_2 \). We put \( x_1 < x_2 \) in the linear extension. This choice is allowed since \( x_2 \mathcal{R} x_1 \) for the minimality of \( x_1 \) in \( X \). We can go on with this procedure and build a descending chain

\[
X_1 = X \triangleright X_2 = X \setminus \{ x_1 \} \triangleright X_3 = X \setminus \{ x_1, x_2 \} \triangleright \cdots \triangleright X_n = \emptyset
\]

of finite length since \( X \) is finite, corresponding to the linear order on \( X = \{ x_1, \ldots, x_n \} \) given by

\[
x_1 < x_2 < \cdots < x_n
\]

which is compatible with \( \mathcal{R} \) by construction.

(c) \( \Rightarrow \) (a). We prove the contrapositive. A cycle \( y_1 \mathcal{R} y_2 \mathcal{R} \cdots y_l \mathcal{R} y_1 \) cannot be ordered: a linear extension of \( \mathcal{R} \) would have to satisfy \( y_1 < y_2 < \cdots < y_n < y_1 \) and then, by transitivity of \( < \), we have \( y_1 < y_1 \) contradicting the irreflexivity of \( < \). Therefore if \( \mathcal{R} \) has a cycle, it cannot be extended to a strict linear order. \( \square \)

3. Rewriting Systems

Rewriting theory is the main tool we will be using to prove the pasting theorem for Gray-categories. It is indeed useful to think about the groupoid appearing in the claim of the theorem in terms of generators and relations, so that it can be studied using rewriting. For this reason, here we will briefly introduce the fundamental notion of rewriting system and the most important results related to it.

**Definition 3.1.** A rewriting system is a set \( A \) equipped with a binary relation \( \rightarrow \), called reduction.

The idea is that if \( a \rightarrow b \), we can substitute any occurrence of \( a \) with \( b \). For instance, in the theory of groups we have a reduction \( gg^{-1} \rightarrow e \) so we can replace every consecutive product of an element by its inverse with the identity element.

We will denote by \( \overset{*}{\rightarrow} \) the reflexive transitive closure of \( \rightarrow \), namely the smallest preorder containing \( \rightarrow \).

**Definition 3.2.** An element \( a \in A \) is said to be confluent if for all \( b, c \in A \) s.t. \( a \overset{*}{\rightarrow} b \) and \( a \overset{*}{\rightarrow} c \) there exists \( d \in A \) s.t. \( b \overset{*}{\rightarrow} d \) and \( c \overset{*}{\rightarrow} d \). A rewriting system is called confluent if all its elements are confluent.

**Definition 3.3.** An element \( a \in A \) is said to be locally confluent if for all \( b, c \in A \) s.t. \( a \rightarrow b \) and \( a \rightarrow c \) there exists \( d \in A \) s.t. \( b \overset{*}{\rightarrow} d \) and \( c \overset{*}{\rightarrow} d \). A rewriting system is called locally confluent if all its elements are locally confluent.

**Remark 3.4.** The difference between local confluence and confluence is that in the former we have a one-step reduction from \( a \) to \( b \) and \( c \), while in the latter we can reach \( b \) and \( c \) in more than one step.

**Definition 3.5.** A rewriting system is called terminating if there is no infinite chain of the form

\[
a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow \cdots
\]
A trick that is often useful in showing that a rewriting system is terminating is to define a measure \( \rho: X \to \mathbb{N} \) that is reduced by any application of the rewrite \( \to \), namely \( x \to y \Rightarrow \rho(x) > \rho(y) \).

**Lemma 3.6** (Newman’s Lemma). *A terminating rewriting system is confluent if and only if it is locally confluent.*

A short proof of this lemma, using induction, can be found in [6]. For a given \( a \in A \) and rewrites \( a \to b \) and \( a \to c \), a key point in proving that a diamond

\[
\begin{array}{c}
  a \\
  \downarrow \\
  b \\
  \downarrow \\
  d \\
  \downarrow \\
  c
\end{array}
\]

does indeed exist is to use induction on the derivation length and tessellate it as follows

\[
\begin{array}{c}
  a \\
  \downarrow \\
  b' \\
  \downarrow \\
  b \\
  \downarrow \\
  d' \\
  \downarrow \\
  d \\
  \downarrow \\
  c
\end{array}
\]

obtaining (I) by local confluence, while (II) and (III) follow by inductive hypothesis. This idea will also be important in the proof of the pasting theorem, where the smaller diagrams are actually commutative.

An important consequence of Newman’s lemma is the existence and unicity of a minimal element in every connected component of a relation \( p \xrightarrow{\mathcal{R}} q \).

### 4. The pasting theorem

In this section we will provide the proof of the main result, namely Theorem 4.17, using rewriting techniques. Before that, we review the notion of Gray-category and the graph-theoretical concepts needed to formalize the intuition behind pasting diagrams.

A Gray-category is a special example of enriched category. In particular, we can define it in a very concise way as follows.

**Definition 4.1.** A Gray-category is a category enriched over the monoidal category \((\mathbf{2-Cat}, \otimes, 1)\) of 2-categories and strict 2-functors equipped with the Gray tensor product.

\(^1\)We define a connected component of a relation \((X, \mathcal{R})\) as a connected component of the corresponding directed graph having \( V = X \) and a directed edge \( x \to y \) whenever \( x \mathcal{R} y \).
Unpacking this definition, a Gray-category $\mathcal{K}$ consists of the following data:

i) a class of objects $\mathcal{K}_0$,

ii) for each couple of objects $a$ and $b$ in $\mathcal{K}$, a 2-category $\mathcal{K}(a, b)$,

iii) for every $a \in \mathcal{K}$ an identity 1-cell $\text{id}_a : a \to a$,

iv) a composition 2-functor $\mathcal{K}(b, c) \otimes \mathcal{K}(a, b) \to \mathcal{K}(a, c)$ from the Gray tensor product between the 2-categories $\mathcal{K}(b, c)$ and $\mathcal{K}(a, b)$ satisfying associativity and unitality rules.

Explicitly, the 2-category $\mathcal{K}(b, c) \otimes \mathcal{K}(a, b)$ is defined as follows:

- $\text{Ob}(\mathcal{K}(b, c) \otimes \mathcal{K}(a, b)) = \text{Ob}(\mathcal{K}(b, c)) \times \text{Ob}(\mathcal{K}(a, b))$,
- 1-cells generated by $(\alpha, g) : (f, g) \to (f', g)$ and $(f, \beta) : (f, g) \to (f, g')$ with $\alpha : f \to f'$ in $\mathcal{K}(b, c)$ and $\beta : g \to g'$ in $\mathcal{K}(a, b)$ subject to the relations $(\alpha'\alpha, g) = (\alpha', g)\alpha, (f, \beta'\beta) = (f, \beta')(f, \beta)$ whenever these pairs are composable pairs and $\text{id}_{(f, g)} = (\text{id}_f, g) = (f, \text{id}_g)$.
- 2-cells generated by

\[
\begin{array}{c}
\xymatrix{(f, g) \ar[rr]^{(\alpha', g)} & & (f', g) \ar[dd]_{(f, \beta')} \ar[ll]_{(f, \beta)} \\
(f, g) \ar[rr]^{(\alpha, g)} & & (f', g') \ar[ll]_{(f, \beta)}
\end{array}
\]

for any $\Phi : \alpha \Rightarrow \alpha'$ in $\mathcal{K}(b, c)$ and $\Psi : \beta \Rightarrow \beta'$ in $\mathcal{K}(a, b)$ satisfying relations for vertical and horizontal composition similar to the ones we have for 1-cells. In addition, we have generating 2-cell (sometimes called Gray cells):

\[
\begin{array}{c}
\xymatrix{(f, g) \ar[rr]^{(\alpha, g)} & & (f', g) \\
(f, g) \ar[d]_{(f, \beta)} \ar[r]_{(\alpha', g)} & (f', g') \ar[d]_{(f, \beta')} & (f, g') \ar[l]_{(\alpha, g')} \ar[r]^{(f, \beta')} & (f', g') \\
(f, g) \ar[r]_{(\alpha, g')} & (f', g') & (f, g) \ar[r]_{(\alpha', g')} & (f', g')
\end{array}
\]

which are invertible for the pseudo version and oriented in either way for the lax/colax version. These 2-cells are subject to the relations

\[
\begin{array}{c}
\xymatrix{(f, g) & (f', g) \\
(f, g) \ar[d]_{(f, \beta)} \ar[r]_{(\alpha', g)} & (f', g') \ar[d]_{(f, \beta')} \ar[r]_{(\alpha, g')} & (f', g') \ar[d]_{(f, \beta')} \ar[r]_{(\alpha', g')} & (f', g') \ar[d]_{(f, \beta')} \ar[r]_{(\alpha, g')} & (f', g')
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{(f, g) \ar[r]^{(\text{id}_f, g)} & (f, g) \\
(f, g) \ar[d]_{(f, \beta)} \ar[r]_{(\text{id}, f, g)} & (f, g') \ar[d]_{(f, \beta')} \ar[r]_{(\text{id}_f, g')} & (f, g') \ar[d]_{(f, \beta')} \ar[r]_{(\text{id}, f, g')} & (f, g') \ar[d]_{(f, \beta')} \ar[r]_{(\text{id}_f, g')} & (f, g')
\end{array}
\]
We turn now to make precise the notion of pasting diagram inside a Gray-category. The key idea is to provide the structure of our pasting diagrams using a graph which will be labelled in components of a Gray-category.

**Definition 4.2.** A graph is *finite* if both the vertex and edge sets are finite.

**Definition 4.3.** A directed graph is *connected* if its underlying undirected graph is connected.

In the following, we will assume that every graph is directed, connected and finite.

**Definition 4.4.** A plane graph $G$ with source $s$ and sink $t$ is a plane graph with distinct vertices $s$ and $t$ such that for every vertex $v$ there exist a directed path from $s$ to $v$ and from $v$ to $t$.

With these prerequisites in mind, we can now introduce one of fundamental graph-theoretical tools used to formalize pasting diagrams. This specific model has been defined in [10] and we will employ it in the rest of the paper.

**Definition 4.5.** A pasting scheme $G$ is an acyclic plane graph with source $s$ and sink $t$.

It is important to notice that the acyclicity condition for pasting schemes refers to the lack of directed cycles. The underlying undirected graph of a pasting scheme has generally many cycles, namely the boundaries of the internal faces of the pasting scheme. In [10] the author also proves the following characterization of pasting schemes.

**Proposition 4.6.** A plane graph $G$ with source $s$ and sink $t$ is a pasting scheme if and only if for every interior face $F$ there exist distinct vertices $s_F$ and $t_F$ and directed paths $\sigma_F$ and $\tau_F$ from $s_F$ to $t_F$ such that the boundary of $F$ is $\sigma_F \tau_F^*$, where by $\tau_F^*$ we mean that $\tau_F$ is traversed in the opposite way.

A pasting scheme can be seen as a “free-living pasting diagram”, which can be then interpreted inside some higher category through a labelling.

**Definition 4.7.** A labelling of a pasting scheme $G$ in a Gray-category $\mathcal{K}$ (or pasting diagram for short) is an assignment of a 0-cell to each vertex, a 1-cell to each edge and a 2-cell to each face of $G$ in a way that preserves domains and codomains.

Namely,

1. we label each vertex $u$ of $G$ with an object $l(u)$ of $\mathcal{K}$,
2. we label each edge $e$ with a 1-cell $l(e)$ of $\mathcal{K}$ so that $\text{dom}_0(l(e)) = l(\text{source}(e))$ and $\text{cod}_0(l(e)) = l(\text{target}(e)),$
(3) given a directed path \( p = e_1 e_2 \ldots e_n \) we define \( l(p) := l(e_n) \circ l(e_{n-1}) \circ \ldots \circ l(e_2) \circ l(e_1) \) which is uniquely defined because horizontal composition of 1-cells is associative,

(4) we label a face \( F \) with a 2-cell \( l(F) \) such that \( \text{dom}_1(l(F)) = l(\sigma_F) \) and \( \text{cod}_1(l(F)) = l(\tau_F) \).

**Definition 4.8.** For a couple of 2-cells \( \alpha \) and \( \beta \) in a labelling of a pasting scheme \( G \), we define a relation \( \alpha \prec \beta \) if and only if the paths underlying \( \text{cod}_1 \alpha \) and \( \text{dom}_1 \beta \) share at least one edge.

**Remark 4.9.** The relation \( \prec \) has the following properties:

- it is irreflexive, by Corollary 2.7 (2) of [10]
- it can be extended to a strict linear order, by Theorem 3.3 of [10].

Hence, by Proposition 2.6, it is also acyclic and well founded.

**Definition 4.10.** Given two 2-cells \( \alpha \) and \( \beta \), we define a relation \( \alpha \underline{\prec} \beta \) if and only if there exists a (possibly empty) path \( t_{\alpha} \leadsto s_{\beta} \) from the 0-dimensional codomain of \( \alpha \) to the 0-dimensional domain of \( \beta \).

**Proposition 4.11.** The relation \( \alpha \underline{\prec} \beta \) is a strict partial order.

**Proof.** We write \( t_{\alpha} \alpha \underline{\prec} \beta \underline{\prec} s_{\beta} \) to represent a (possibly empty) path from \( t_{\alpha} \) to \( s_{\beta} \) that witnesses their relationship. Let us prove that the relation \( \underline{\prec} \) satisfies irreflexivity, transitivity and asymmetry.

**Irreflexivity:** if not, we would have the directed cycle
\[
t_{\alpha} \alpha \underline{\prec} \beta \underline{\prec} \alpha \quad \text{dom}_1 \alpha \underline{\prec} \text{dom}_1 \beta \underline{\prec} t_{\alpha}
\]

**Transitivity:** suppose \( \alpha \prec \beta \) and \( \beta \prec \gamma \) therefore there exist \( t_{\alpha} \leadsto s_{\beta} \) and \( t_{\beta} \leadsto s_{\gamma} \), giving a directed path
\[
t_{\alpha} \alpha \underline{\prec} \beta \underline{\prec} \gamma \quad \text{dom}_1 \beta \underline{\prec} \text{dom}_1 \gamma \underline{\prec} t_{\beta}
\]
and so \( \alpha \prec \gamma \).

**Asymmetry:** we have to show that \( \alpha \prec \beta \Rightarrow \beta \not\prec \alpha \). Suppose by contradiction that \( \beta \prec \alpha \), then there exists a directed path \( t_{\beta} \leadsto s_{\alpha} \) which on the other hand would give a directed cycle
\[
s_{\alpha} \alpha \underline{\prec} \beta \underline{\prec} \alpha \quad \text{dom}_1 \beta \underline{\prec} \text{dom}_1 \alpha \underline{\prec} t_{\beta}
\]
that contradicts the acyclicity of the pasting scheme. \( \square \)

**Remark 4.12.** If \( \alpha \prec \beta \) then they must both lie on some path from source to sink of the pasting scheme. That is, their 1-dimensional domains must both be sub-paths of some path from the source to the sink.

**Example 4.13.** The relations \( \prec \) and \( \prec \) are unrelated, in the sense that neither one is contained in the other. For instance in the pasting diagram
\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]
we have \( \alpha_1 \not\prec \alpha_3 \), \( \alpha_2 \not\prec \alpha_3 \) and \( \alpha_1 \prec \alpha_2 \).
These relations are indeed a way to capture two different features of the process that returns a composite of a given pasting diagram, which is a vertical composition of whiskered 2-cells. The relation $<$ specifies the order in which the 2-cells appear in this vertical composition: if $\alpha < \beta$ in the pasting diagram, then any composite will have a term of the form $\beta \cdot \alpha$ (or a suitable whiskering of it) in this precise order. The relation $<$ has instead to do with the application of the middle four interchange law to a pair of 2-cells that are $\prec$-compatible. Nevertheless, there exists a connection between the two relations. In fact, the following holds.

**Proposition 4.14.** Let $\prec^t$ denote the transitive closure of $\prec$. We have that $\alpha \prec \beta$ and $\beta \prec \alpha$ if and only if $\alpha < \beta$ or $\beta < \alpha$.

**Proof.** $\Rightarrow$) Define $G_\alpha = \{ \gamma \mid \gamma \prec^t \alpha \}$, $G_\beta = \{ \gamma \mid \gamma \prec^t \beta \}$ and $G_{\alpha, \beta} = G_\alpha \cup G_\beta$. By Remark 4.9 we know that $\prec$ is both irreflexive and acyclic, hence $\prec^t$ is still irreflexive. In addition, $\beta \not\in G_\alpha$ and $\alpha \not\in G_\beta$ by assumption. Therefore, $\alpha$ and $\beta$ are not in $G_{\alpha, \beta}$. Since $\prec$ is well founded and $G_{\alpha, \beta}$ is a subset of the sets of 2-cells of the pasting diagram, there exists a minimal element $\delta_1$ in $G_{\alpha, \beta}$. For the same reason, there exists a minimal element $\delta_2 \in G_{\alpha, \beta} \setminus \{ \delta_1 \}$. If we keep removing the $\delta_i$ for $1 \leq i \leq |G_{\alpha, \beta}|$ we get a subpasting scheme of $G$ that has no 2-cells related to either $\alpha$ or $\beta$ in $\prec^t$, meaning that the domains of $\alpha$ and $\beta$ are now contained in the top boundary of this subpasting scheme. Therefore $\alpha < \beta$ or $\beta < \alpha$.

$\Leftarrow$ On the other hand, if $\alpha$ and $\beta$ are $\prec$-comparable, their domains lie in the same path from source to sink by definition. This path is the top boundary of a subpasting scheme in which $\alpha$ and $\beta$ are minimal elements with respect to the relation $\prec^t$ restricted to the 2-cells of the subpasting scheme. It follows that $\alpha \prec \beta$ and $\beta \prec \alpha$. \hfill \Box

**Definition 4.15.** Given a pasting diagram $G$ in a Gray-category $\mathcal{K}$, let $\mathcal{C}_G$ be the groupoid with

1. **objects**: strings of 2-cells $\alpha_1 \alpha_2 \cdots \alpha_n$ of the pasting diagram corresponding to strict linear extensions of the relation $\prec$;

2. **generating morphisms**:

$$\alpha_i \alpha_{i+1} : \alpha_1 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_n \to \alpha_1 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_n$$

and

$$\alpha_i \alpha_{i+1} : \alpha_1 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_n \to \alpha_1 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_n$$

with $\alpha_i$ and $\alpha_{i+1}$ not comparable in $\prec^t$, subject to the relations

- (a) $\alpha_i \alpha_{i+1} \alpha_{i+1} \alpha_i = \text{id}$ and $\alpha_{i+1} \alpha_i \alpha_i \alpha_{i+1} = \text{id}$;
- (b) $\alpha_j \alpha_{j+1} \alpha_i \alpha_{i+1} = \alpha_i \alpha_{i+1} \alpha_j \alpha_{j+1}$ for $j < i - 1$;
- (c) $\alpha_{i+1} \alpha_i \alpha_i + 2 \alpha_i \alpha_{i+1} \alpha_{i+1} \alpha_{i+2} = \alpha_i \alpha_{i+1} \alpha_{i+1} \alpha_i \alpha_i + 2 \alpha_i \alpha_{i+1} \alpha_{i+1}$.

**Remark 4.16.** Every extension of $\prec$ to a strict linear order can be interpreted as uniquely determining a composite of the pasting diagram. The idea is that each vertical composite of whiskered 2-cells gives rise to a strict linear order extending $\prec$, that is the order in which they appear in the vertical composition (which has to respect the constraints imposed by $\prec$) and viceversa. For
instance, considering the example in the Introduction

we have that \( \alpha \triangleleft \beta, \alpha \triangleleft \gamma, \beta \triangleleft \delta \) and \( \gamma \triangleleft \varphi \). Therefore in this situation the relation \( \triangleleft \) can be extended to the strict linear orders \( \alpha \beta \gamma \varphi \delta \) corresponding to the composite \( m\delta \cdot \varphi d \cdot \gamma d \cdot cf \beta \cdot coa \) and \( \alpha \gamma \beta \varphi \delta \) corresponding to \( m\delta \cdot \varphi d \cdot g \beta \cdot \gamma ea \cdot coa \) (notice that compositions are read from right to left).

**Theorem 4.17.** The groupoid \( \mathcal{C}_G \) is contractible, i.e. it is equivalent to the terminal category \( 1 \).

We will show this result using rewriting (see also [2] and [3] for a detailed account of rewriting theory in Gray-categories). In order to be able to apply rewriting techniques we need to choose an orientation for the arrows of \( \mathcal{C}_G \). We will do it by extracting a category \( \mathcal{C}_G' \) which is not a groupoid such that \( \mathcal{C}_G \) is the groupoid reflection of \( \mathcal{C}_G' \), that is the image of \( \mathcal{C}_G' \) under the left adjoint to the inclusion \( \text{Gpd} \hookrightarrow \text{Cat} \). The problem then reduces to proving that \( \mathcal{C}_G' \) has a terminal object. In fact, \( \mathcal{C}_G' \) has a terminal object if and only if the unique functor \( \mathcal{C}_G' \to 1 \) has a right adjoint \( 1 \to \mathcal{C}_G' \) picking out the terminal object. The groupoid reflection is a 2-functor \( \text{Cat} \to \text{Gpd} \) because \( \text{Gpd} \) is closed under cotensors and hence the adjunction between the inclusion and the groupoid reflection can be lifted to a 2-adjunction by Theorem 4.85 of [4]. It therefore sends adjunctions in \( \text{Cat} \) to adjunctions in \( \text{Gpd} \), but every adjunction in \( \text{Gpd} \) is an equivalence because its unit and counit are natural isomorphisms (their components are morphisms inside groupoids, thus are invertible).

**Definition 4.18.** Define a category \( \mathcal{C}_G' \) in such a way that \( \text{Ob}(\mathcal{C}_G') = \text{Ob}(\mathcal{C}_G) \) and the morphisms are generated by \( \alpha_{i-1} \alpha_i \) with \( \alpha_{i+1} < \alpha_i \), subject to the relations (b) and (c).

**Example 4.19.** The relation (b) specifies that the diagram
commutes whenever there are four 2-cells $\alpha < \beta < \gamma < \delta$, while the relation (c) ensure the commutativity of the diagram

\[
\begin{array}{c}
\overset{\eta\zeta\epsilon}{\uparrow} & \overset{\zeta\eta\epsilon}{\rightarrow} & \overset{\epsilon\zeta\eta}{\rightarrow} \\
\overset{\zeta\epsilon\eta}{\rightarrow} & \overset{\eta\epsilon\zeta}{\downarrow} & \overset{\epsilon\zeta\eta}{\rightarrow} \\
\overset{\zeta\epsilon\eta}{\rightarrow} & \overset{\eta\epsilon\zeta}{\downarrow} & \overset{\epsilon\zeta\eta}{\rightarrow}
\end{array}
\]

for three 2-cells $\epsilon < \zeta < \eta$. Notice that the indices need to be updated, so that the string on top of the hexagon is of the form $\alpha_1\alpha_2\alpha_3 = \eta\zeta\epsilon$ (in which $\alpha_2 = \zeta$) but the string immediately under it on the left is of the form $\alpha_1\alpha_2\alpha_3 = \zeta\eta\epsilon$ (in which $\alpha_2 = \eta$).

We interpret these morphisms as rewrite rules on the set $\text{Ob}(\mathcal{C}_G)$.

**Lemma 4.20.** $\mathcal{C}_G$ is the groupoid reflection of $\mathcal{C}_G'$.

**Proof.** First of all, notice that $\mathcal{C}_G'[\text{Mor}(\mathcal{C}_G')^{-1}] = \mathcal{C}_G'[\text{GenMor}(\mathcal{C}_G')^{-1}]$, where by $\text{GenMor}(\mathcal{C}_G')$ we mean the set of generating morphisms. In fact, every morphism $f$ in $\mathcal{C}_G'$ can be written as composition of some generating morphisms and so $f^{-1}$ is just a composition of the inverses of the generators. Clearly, $\mathcal{C}_G'[\text{GenMor}(\mathcal{C}_G')^{-1}] \subseteq \mathcal{C}_G$ because $\mathcal{C}_G$ contains every inverse of the generating morphisms of $\mathcal{C}_G'$ and in fact it does not contain any additional morphism, so the two categories are actually isomorphic. \qed

**Proposition 4.21.** $\mathcal{C}_G'$ has a terminal object.

**Proof.** The proof uses Newman’s lemma.

*Termination:* Let $X_\alpha = \{(\alpha_i, \alpha_j) | i < j \text{ and } \alpha_j < \alpha_i\}$, where $\alpha = \alpha_1 \cdots \alpha_n \in \text{Ob}(\mathcal{C}_G')$. Define the measure

\[
\rho : \text{Ob}(\mathcal{C}_G') \rightarrow \mathbb{N} \\
\alpha_1 \cdots \alpha_n \mapsto |X_\alpha|
\]

which is reduced by 1 by any rewrite $\alpha_i \alpha_{i+1}$. In fact, for $\alpha' = \alpha_1 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_n$ we have $X_\alpha = X_{\alpha'} \bigsqcup \{(\alpha_i, \alpha_{i+1})\}$ since the only pair of 2-cells that have changed their positions relative to each other is $(\alpha_i, \alpha_{i+1})$. This implies that the rewriting system is terminating because reductions reduce the measure, which is bounded below by 0.

*Local confluence:* We have to show that every fork (sometimes called critical pair) can be closed. Recall that if we can apply a rewrite $\alpha_i \alpha_{i+1}$ then we know that $\alpha_j < \alpha_i$. This is done case by case. Consider the sets of 2-cells $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ s.t. the elements of each set are $\prec$-comparable. If $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$, suppose without loss of generality that $\alpha < \beta$, $\gamma < \delta$, $\alpha <^t \gamma$, $\alpha <^t \delta$, $\beta <^t \gamma$, $\beta <^t \delta$. Then \(\alpha_i \alpha_{i+1} \vdash \alpha_i \alpha_j \alpha_{i+1} \alpha_{i+2} \vdash \alpha_i \alpha_{i+1} \alpha_i \alpha_j \alpha_{i+1} \alpha_{i+2} \alpha_{i+3} \vdash \alpha_i \alpha_{i+1} \alpha_i \alpha_j \alpha_{i+1} \alpha_{i+2} \alpha_{i+3} \alpha_{i+4} \)}
\[ \gamma, \beta <^t \gamma, \text{ so that the pasting diagram looks like the following} \]

![Diagram](image)

where \( \phi, \rho \) and \( \tau \) are the composites of the corresponding sub-pasting schemes under their obvious sub-linear orderings of \( \alpha \) and \( \beta \). These 2-cells are fixed by the rewrites. The goal is to prove that the diagram

![Diagram](image)

commutes. This diamond simply expresses the unicity of pasting for the diagram

![Diagram](image)

in the hom-2-category \( \mathcal{K}(s, t) \), in which the middle four interchange law holds strictly (see the Introduction). When \{\alpha, \beta\} \cap \{\gamma, \delta\} \neq \emptyset, the cardinality of the intersection can be either 1 or 2. If this cardinality is 2, then the two sets coincide so that there is nothing to prove because there is only one possible swap so no branchings occur. If the cardinality is 1, we have the following possibilities:

1. \( \alpha = \gamma \) (with \( \beta \neq \delta \)),
2. \( \alpha = \delta \) (with \( \beta \neq \gamma \)),
3. \( \beta = \gamma \) (with \( \alpha \neq \delta \)),
4. \( \beta = \delta \) (with \( \alpha \neq \gamma \)).
In any of these cases all of the 2-cells are \(<\)-comparable since in the two sets the domains of the 2-cells lie on the same path and the two sets have a 2-cell in common. Up to renaming the 2-cells, we can assume that our pasting diagram looks like the following

where \(f, g, h, k, l\) and \(m\) are the composites of the 1-cells labelling those paths and \(\rho\) and \(\tau\) are the composites of the sub-pasting schemes specified by the corresponding sub-sequences of \(\alpha, \beta\) and \(\gamma\). This gives rise to the span

\[
\begin{align*}
\rho\gamma\beta\alpha\tau & \xrightarrow{\beta}\rho\beta\gamma\alpha\tau \\
\rho\beta\gamma\alpha\tau & \xrightarrow{\gamma}\rho\gamma\alpha\beta\tau
\end{align*}
\]

in the category \(\mathcal{C}_G\), that can be closed in the following way

\[
\begin{align*}
\rho\gamma\beta\alpha\tau & \xrightarrow{\beta}\rho\beta\gamma\alpha\tau \\
\rho\beta\gamma\alpha\tau & \xrightarrow{\gamma}\rho\gamma\alpha\beta\tau \\
\rho\beta\alpha\gamma\tau & \xrightarrow{\beta}\rho\alpha\beta\gamma\tau
\end{align*}
\]

thanks to the relation (c) or in other words by whiskering the cube identity

of the Gray tensor product (which is in turn a consequence of the relations for the Gray cells that define the Gray tensor product) with the two 2-cells \(\rho\) and \(\tau\), where we omitted the whiskerings with \(g, h, k, l\) for the sake of clarity (e.g. \(\gamma b_0 a_0\) stands for \(l\gamma k b_0 h a_0 g\), \(\gamma b_1 \alpha\) stands for the swap \((\gamma k b_1 h)\alpha\) between the whiskered 2-cell \(\gamma k b_1 h\) and \(\alpha\) etc.). The convergence of every critical pair is witnessed by commutative diagrams, therefore for every object in \(\mathcal{C}_G\) there exists a unique
morphism to the minimal element. In other words, the minimal element for the rewriting system is a terminal object in $\mathcal{C}_G'$, which is what we wanted to show. □

Remark 4.22. It is well known that in a category the terminal element, if it exists, is unique up to a unique isomorphism. In $\mathcal{C}_G'$ this is even stronger: the terminal element is unique. Suppose there exist two different terminal objects $m$ and $m'$, then we have a span

$$
\begin{array}{c}
\bullet \\
m & \downarrow & m' \\
\end{array}
$$

that can be closed to a commutative diagram

$$
\begin{array}{c}
\bullet \\
m & \downarrow & m' \\
& \downarrow & n \\
n & \downarrow & m' \\
\end{array}
$$

because the rewriting system is confluent. But $m$ and $m'$ are minimal, therefore $m = n = m'$.

Remark 4.23. This proof can be also interpreted inside a lax Gray-category. In this case the composition of a pasting diagram is no longer unique up to a contractible groupoid of choices but there is still a “minimal choice” for it, namely the terminal object of the category $\mathcal{C}_G'$. In fact, when we defined $\mathcal{C}_G'$ we chose an orientation for the rewrites and this can be seen in turn as choosing a direction to the Gray-cell expressing the middle four interchange isomorphism for our Gray-category.

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