APPROXIMATION BY MULTIVARIATE
BASKAKOV–KANTOROVICH OPERATORS IN ORLICZ SPACES

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Abstract. Utilizing some properties of multivariate Baskakov–Kantorovich
operators and using $K$-functional and a decomposition technique, the authors
find two equivalent theorems between the $K$-functional and modulus of smooth-
ness, and obtain a direct theorem in the Orlicz spaces.

1. Motivations. For proceeding smoothly, we recall from [31] some definitions and
related results.

A continuous convex function $\Phi(t)$ on $[0, \infty)$ is called a Young function if it satisfies

$$\lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty.$$ 

For a Young function $\Phi(t)$, its complementary Young function is denoted by $\Psi(t)$.

It is clear that the convexity of $\Phi(t)$ can lead to $\Phi(\alpha t) \leq \alpha \Phi(t)$ for $\alpha \in [0, 1]$. In particular, one has $\Phi(\alpha t) < \alpha \Phi(t)$ for $\alpha \in (0, 1)$.

A Young function $\Phi(t)$ is said to satisfy the $\Delta_2$-condition, denoted by $\Phi \in \Delta_2$,
if there exist $t_0 \geq 0$ and $C \geq 1$ such that $\Phi(2t) \leq C\Phi(t)$ for $t \geq t_0$. 

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Throughout the paper, we shall use the following standard notations:

\[ \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \quad \mathbb{N} = \{1, 2, 3, \ldots\}, \quad m \in \mathbb{N}, \]
\[ x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m, \quad |x| = \sum_{i=1}^m x_i, \quad k = (k_1, k_2, \ldots, k_m) \in \mathbb{N}_0^m, \]
\[ x^k = x_1^{k_1}x_2^{k_2} \cdots x_m^{k_m}, \quad k! = k_1!k_2! \cdots k_m!, \quad |k| = \sum_{i=1}^m k_i, \]
\[ \binom{n}{k} = \frac{n!}{k!(n-|k|)!}, \quad \sum_{k=0}^\infty \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty \cdots \sum_{k_m=0}^\infty, \]
\[ \mathbb{R}_0^m = \{ x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : 0 \leq x_i < \infty, 1 \leq i \leq m \}, \]
\[ D^k = D_1^{k_1}D_2^{k_2} \cdots D_m^{k_m}, \quad D_i^r = \frac{\partial^r}{\partial x_i^r} \]
for \( r \in \mathbb{N}. \)

Let \( \Phi(t) \) be a Young function. We define the Orlicz class \( L_\Phi(\mathbb{R}_0^m) \) as the collection of all Lebesgue measurable functions \( f(x) \) on \( \mathbb{R}_0^m \) such that
\[
\rho(f, \Phi) = \int_{\mathbb{R}_0^m} \Phi(|f(x)|)dx < \infty.
\]
We also define the Orlicz space \( L^*_\Phi(\mathbb{R}_0^m) \) as the collection of all Lebesgue measurable functions \( f(x) \) on \( \mathbb{R}_0^m \) such that \( \int_{\mathbb{R}_0^m} \Phi(|\alpha f(x)|)dx < \infty \) for some \( \alpha > 0 \). The Orlicz space \( L^*_\Phi(\mathbb{R}_0^m) \) is a Banach space under the Luxemburg norm
\[
\|f\|_{\Phi} = \inf_{\lambda > 0} \left\{ \lambda : \rho\left(\frac{f}{\lambda}, \Phi\right) \leq 1 \right\}.
\]
The Orlicz norm \( \|f\|_{\Phi} \) on \( L^*_\Phi(\mathbb{R}_0^m) \), which is equivalent to the Luxemburg norm on \( L^*_\Phi(\mathbb{R}_0^m) \), is given by
\[
\|f\|_{\Phi} = \sup_{\rho(g, \Phi) \leq 1} \left| \int_{\mathbb{R}_0^m} f(x)g(x)dx \right|
\]
and satisfies
\[
\|f\|_{\Phi} \leq \|f\|_{\Phi} \leq 2\|f\|_{\Phi}.
\] (1)

Throughout this paper, we use \( C \) to denote a constant, which may be not necessarily the same in different cases, independent of \( n \) and \( x \).

For \( x \in \mathbb{R}_0^m \), we introduce weight functions \( \varphi(x) = \sqrt{x(1 + x)} \) for \( m = 1 \) and \( \varphi_i(x) = \sqrt{x_i(1 + |x|)} \) for \( m > 1 \) and \( 1 \leq i \leq m \). We also define weighted Sobolev space
\[
W^r_{\varphi}(\mathbb{R}_0^m) = \left\{ f \in L^*_\Phi(\mathbb{R}_0^m) : D^k f \in A.C.loc(\mathbb{R}_0^m)^0, \varphi_i D_i^r f \in L^*_\Phi(\mathbb{R}_0^m) \right\},
\]
where \( |k| \leq r \) and \( \mathbb{R}_0^m \) is the interior of \( \mathbb{R}_0^m \).

The Peetre \( K \)-functional is defined in [6] by
\[
K_{r,\varphi}(f, t')_\Phi = \inf \left\{ \|f - g\|_{\Phi} + t' \sum_{i=1}^m \|\varphi_i D_i^r g\|_{\Phi} : g \in W^r_{\varphi}(\mathbb{R}_0^m) \right\}
\]
for $t > 0$. Now we define the modified $K$-functional as
\[
\mathcal{K}_{r, \varphi}(f, tr) = \inf \left\{ \|f - g\|_\Phi + tr \sum_{i=1}^m \|\varphi_r^i D_r^ig\|_\Phi + t^{2r} \sum_{i=1}^m \|D_r^ig\|_\Phi : g \in W_{r, \varphi}^r(\mathbb{R}_0^m) \right\}
\]
for $t > 0$.

For any vector $e \in \mathbb{R}^m$, we write
\[
\Delta^r_{he}f(x) = \begin{cases} \sum_{i=0}^r \binom{r}{i} (-1)^i f(x + ihe), & x, x + rhe \in \mathbb{R}_0^m \\ 0, & \text{otherwise} \end{cases}
\]
for the $r$th forward difference of a function $f$ in the direction of $e$. We define the modulus of smoothness of $f \in L_\Phi^r(\mathbb{R}_0^m)$ as
\[
\omega_{r, \varphi}(f, t) = \sup_{0 < h \leq t} \sum_{i=1}^m \|\Delta^r_{he}f\|_\Phi.
\]

Let
\[
p_{n,k}(x) = \binom{n + k - 1}{k} \frac{x^k}{(1 + x)^{n+k}}, \quad x \in [0, \infty), \quad n \in \mathbb{N}.
\]
The well known Baskakov operators [3] were defined by
\[
B_n(f, x) = \sum_{k=0}^\infty p_{n,k}(x) f\left(\frac{k}{n}\right)
\]
which can be used to approximate any function $f$ defined on $[0, \infty)$. In order to consider the approximation in $L_p[0, \infty)$, Ditzian and Totik [8] modified the form of the Baskakov operators as
\[
V_{n,1}(f, x) = \sum_{k=0}^\infty p_{n,k}(x) \int_{k/n}^{(k+1)/n} f(u)du
\]
which are called Baskakov–Kantorovich’s operators. There are many approximation results about one variable operator of the Baskakov type in $C[0, \infty)$ or $L_p[0, \infty)$. See [1, 2, 3, 8, 9, 10, 11, 13, 14, 15, 26, 33, 34, 35] and closely related references therein.

The multivariate Baskakov–Kantorovich’s operators [5] were defined by
\[
V_{n,m}(f, x) = \sum_{k=0}^\infty p_{n,k}(x) Q_{n,k}(f),
\]
where
\[
p_{n,k}(x) = \binom{n + |k| - 1}{k} \frac{x^k}{(1 + |x|)^{n+|k|}}
\]
and
\[
Q_{n,k}(f) = \int_{k/n}^{(k+1)/n} \int_{k_2/n}^{(k_2+1)/n} \cdots \int_{k_m/n}^{(k_m+1)/n} f(u_1, u_2, \ldots, u_m)du_1du_2\cdots du_m
\]
\[= \int_{k/n}^{(k+1)/n} f(u)du.
\]
There are few results about multivariate Baskakov type operators. Cao and An introduced in [4] multivariate Baskakov–Durrmeyer operators and obtained a direct inequality in $L_p[0,\infty)$. Cao and Ding [5] established a direct theorem of multivariate Baskakov–Kantorovich operators in $L_p[0,\infty)$ as

$$
\|V_{n,m}(f) - f\|_p \leq C \left[ \omega_{2,p} \left( f, \frac{1}{n^{1/2}} \right) + \frac{\|f\|_p}{n} \right].
$$

For more information on approximation properties for operators in the Orlicz, Morrey, Baskakov–Durrmeyer–Stancu, or other type spaces, we recommend three groups of references, [21, 22, 23, 24, 25], [16, 17, 18, 19, 20], [12, 27, 28, 29, 30], to interested readers.

In this paper, basing on the above conclusions, utilizing $K$-functional and a decomposition technique, considering properties of multivariate Baskakov–Kantorovich operators in the form of Lemmas 2.1 to 2.4 in Section 2, we establish two equivalent theorems, Theorems 3.1 and 3.2 in Section 3, between the operators in the form of Lemmas 2.1 to 2.4 in Section 2, we establish two equivalent theorems, Theorems 3.1 and 3.2 in Section 3, between the

2. **Lemmas.** In order to prove the direct theorem, we need several lemmas below.

**Lemma 2.1.** Let $f \in L^*_\Phi(\mathbb{R}_n^m)$, $n > m$. Then

$$
\|V_{n,m}(f)\|_\Phi \leq C\|f\|_\Phi.
$$

**Proof.** By the decomposition formula

$$
V_{n,m}(f, \mathbf{x}) = \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)n \int_{k_1/n}^{(k_1+1)/n} du_1 \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}(\frac{x_2}{1+x_1}) \times n \int_{k_2/n}^{(k_2+1)/n} du_2 \cdots \sum_{k_m=0}^{\infty} p_{n+k_1+\cdots+k_{m-1},k_m}(\frac{x_m}{1+x_1+\cdots+x_m-1}) \times n \int_{k_m/n}^{(k_m+1)/n} du_m,
$$

Jensen’s inequality, and the double inequality (1), we obtain

$$
\|V_{n,m}(f)\|_\Phi \leq 2\|V_{n,m}(f)\|_\Phi
$$

$$
= 2 \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}_n^m} \Phi \left( \frac{1}{n} \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)n \int_{k_1/n}^{(k_1+1)/n} du_1 \right. \times \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}(\frac{x_2}{1+x_1})n \int_{k_2/n}^{(k_2+1)/n} du_2 \cdots \times \sum_{k_m=0}^{\infty} p_{n+k_1+\cdots+k_{m-1},k_m}(\frac{x_m}{1+x_1+\cdots+x_m-1}) \times n \int_{k_m/n}^{(k_m+1)/n} f(u_1, u_2, \ldots, u_m)du_m \bigg) \right\} \leq 1
$$

$$
\leq 2 \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}_n^m} \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)n \int_{k_1/n}^{(k_1+1)/n} du_1 \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}(\frac{x_2}{1+x_1}) \times \cdots \times \sum_{k_m=0}^{\infty} p_{n+k_1+\cdots+k_{m-1},k_m}(\frac{x_m}{1+x_1+\cdots+x_m-1}) \times n \int_{k_m/n}^{(k_m+1)/n} f(u_1, u_2, \ldots, u_m)du_m \right\} \leq 1
$$

$$
\leq 2 \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}_n^m} \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)n \int_{k_1/n}^{(k_1+1)/n} du_1 \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}(\frac{x_2}{1+x_1}) \times \cdots \times \sum_{k_m=0}^{\infty} p_{n+k_1+\cdots+k_{m-1},k_m}(\frac{x_m}{1+x_1+\cdots+x_m-1}) \times n \int_{k_m/n}^{(k_m+1)/n} f(u_1, u_2, \ldots, u_m)du_m \right\} \leq 1
$$
Proof. By Taylor's formula Lemma 2.2 is the Hardy-Littlewood function of 

\[ f_n(x) = \frac{1}{\lambda} |f(u_1, u_2, \ldots, u_m)| \] 

\[ \int_{k_1/n}^{(k_1+1)/n} p_{n+1, k_1} n \int_{k_1/n}^{(k_1+1)/n} \Phi \left( \frac{1}{\lambda} |f(u_1, u_2, \ldots, u_m)| \right) du_m dx \leq 1 \}

= 2 \inf_{\lambda > 0} \left\{ \lambda : \sum_{k_1=0}^{\infty} p_{n+1, k_1} \int_{k_1/n}^{(k_1+1)/n} \Phi \left( \frac{1}{\lambda} |f(u_1, u_2, \ldots, u_m)| \right) du_m \leq 1 \right\}

= 2 \inf_{\lambda > 0} \left\{ \lambda : \sum_{k_1=0}^{\infty} p_{n+1, k_1} \int_{k_1/n}^{(k_1+1)/n} \Phi \left( \frac{1}{\lambda} |f(u_1, u_2, \ldots, u_m)| \right) du_m \leq 1 \right\}

\leq 2 \inf_{\lambda > 0} \left\{ \lambda : \sum_{k_1=0}^{\infty} p_{n+1, k_1} \int_{k_1/n}^{(k_1+1)/n} \Phi \left( \frac{1}{\lambda} |f(u_1, u_2, \ldots, u_m)| \right) du_m \leq 1 \right\}

= C \|f\|_\phi \leq C \|f\|_\phi.

The proof of Lemma 2.1 is complete. \hfill \Box

Lemma 2.2 ([24]). For \( f \in L^*_\Phi[0, \infty) \) and \( \Psi \in \Delta_2 \), we have

\[ \|\theta(f)\|_\phi \leq C \|f\|_\phi, \]

where

\[ \theta(f, x) = \sup_{0 \leq t < \infty} \left[ \frac{1}{t-x} \int_x^t f(u) du \right] \]

is the Hardy-Littlewood function of \( f(x) \).

Lemma 2.3. Let \( f \in L^*_\Phi[0, \infty) \). Then

\[ |B_n(f, x) - f(x)| \leq \frac{C}{n} \theta(f^n, x). \]

Proof. By Taylor's formula

\[ f(t) = f(x) + f'(x)(t-x) \]
in \[\mathbb{R}^2\] and the inequality

\[
\left| \frac{t - x - \tau}{(x + \tau)(1 + x + \tau)} \right| \leq \begin{cases} \frac{4|t - x|}{x(1 + x)}, & t \geq \frac{x}{2}, x \leq 1 \\ \frac{2|t - x|}{x(1 + t)}, & 0 \leq t < \frac{x}{2}, x > 1 \end{cases}
\]

in [32, Eq. (6.1)], one acquires

\[
|B_n(f,x) - f(x)| \leq CB_n\left(\frac{(t - x)^2}{\varphi^2(x)} \max\left\{1, \frac{1 + x}{1 + t}\right\}, x\right) \theta(\varphi^2|f''|, x) \\
\leq \frac{C}{n} \theta(\varphi^2|f''|, x),
\]

where we used

\[
B_n(1,x) = 1, \quad B_n(t - x, x) = 0, \quad B_n((t - x)^2, x) = \frac{\varphi^2(x)}{n}.
\]

and

\[
B_n\left(\frac{(t - x)^2}{\varphi^2(x)} \max\left\{1, \frac{1 + x}{1 + t}\right\}, x\right) \leq \frac{C}{n}.
\]

The proof of Lemma 2.3 is complete. \(\square\)

**Lemma 2.4.** Let \(f \in L^2_\Phi(\mathbb{R}^2)\) and \(\Psi \in \Delta_2\). Then

\[
\|V_{n,2}(f) - f\|_\Phi \leq \frac{C}{n} \left(\|f\|_\Phi + \sum_{i=1}^2 \|\varphi_i^2D_2^2f\|_\Phi\right).
\]

**Proof.** Let

\[
z = \frac{x_2}{1 + x_1} \quad \text{and} \quad g_{u_1}(t) = f(u_1, (1 + u_1)t)
\]

for \(0 \leq t < \infty\). Utilizing the decomposition formula

\[
V_{n,2}(f, x) = \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)n \int_{k_1/n}^{(k_1+1)/n} \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}\left(\frac{x_2}{1 + x_1}\right) \\
\times n \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} f(u_1, (1 + u_1)u_2)(1 + u_1)du_2du_1
\]

in [5] concludes

\[
V_{n,2}(f, x) - f(x) = \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)n \int_{k_1/n}^{(k_1+1)/n} \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}(z)n \\
\times \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} [g_{u_1}(t) - g_{u_1}(z)](1 + u_1)dtdu_1 \\
+ V_{n,1} \left( f\left(u_1, (1 + u_1)\frac{x_2}{1 + x_1}\right) \right) - f(x, x_2) \\
\triangleq J_1 + J_2, \quad 0 \leq u_1 < \infty.
\]
Now we start out to estimate $J_1$. Using Jensen’s inequality and the convexity of $\Phi(t)$, it follows

$$
\int_0^\infty \int_0^\infty \Phi \left( \frac{1}{\lambda} J_1 \right) \, dx_1 \, dx_2 = \int_0^\infty \int_0^\infty \Phi \left( \frac{1}{\lambda} \sum_{k_1=0}^\infty p_{n,k_1} (x_1) n \int_{k_1/n}^{(k_1+1)/n} \right.

\sum_{k_2=0}^\infty p_{n+k_1,k_2} \left( \frac{x_2}{1+x_1} \right) n(1+u_1) \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} [g_{u_1}(t) - g_{u_1}(z)] \, dt \, du_1 \, dx_1 \, dx_2

\leq \int_0^\infty \int_0^\infty \sum_{k_1=0}^\infty p_{n,k_1} (x_1) n \int_{k_1/n}^{(k_1+1)/n} \Phi \left( \frac{1}{\lambda} \sum_{k_2=0}^\infty p_{n+k_1,k_2} \left( \frac{x_2}{1+x_1} \right) n(1+u_1)

\times \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} [g_{u_1}(t) - g_{u_1}(z)] \, dt \right) \, du_1 \, dx_1 \, dx_2

= \sum_{k_1=0}^\infty \frac{n(n+k_1-1)}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \Phi \left( \frac{1}{\lambda} \sum_{k_2=0}^\infty p_{n+k_1,k_2} (z) n(1+u_1)

\times \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} [g_{u_1}(t) - g_{u_1}(z)] \, dt \right) \, du_1 \, dx_1 \, dx_2

= \sum_{k_1=0}^\infty \frac{n(n+k_1-1)}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \Phi \left( \frac{1}{\lambda} \sum_{k_2=0}^\infty p_{n+k_1,k_2} (z) 2n(1+u_1)

\times \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} [g_{u_1}(t) - g_{u_1}(z)] \, dt \right) \, du_1 \, dx_1 \, dx_2

= \frac{1}{2} \sum_{k_1=0}^\infty \frac{n(n+k_1-1)}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \left\{ \Phi \left( \frac{1}{\lambda} \sum_{k_2=0}^\infty p_{n+k_1,k_2} (z) 2n(1+u_1)

\times \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} g_{u_1}'(s) \, ds \, dt \right) \right\} \, du_1 \, dx_1 \, dx_2

= \frac{1}{2} \sum_{k_1=0}^\infty \frac{n(n+k_1-1)}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \left\{ \Phi \left( \frac{1}{\lambda} \sum_{k_2=0}^\infty p_{n+k_1,k_2} (z) 2n(1+u_1)

\times \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} g_{u_1}'(s) \, ds \, dt \right) \right\} \, du_1 \, dx_1 \, dx_2

\triangleq J_{11} + J_{12}. \quad (4)
Employing Lemmas 2.2 and 2.3 yields

\[
J_{11} = \frac{1}{2} \sum_{k_1=0}^{\infty} \frac{n(n + k_1 - 1)}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \int_{0}^{\infty} \Phi \left( \frac{2}{\lambda} |B_{n+k_1}(g_{u_1}, z) - g_{u_1}(z)| \right) dzdu_1
\]

\[
\leq \frac{1}{2} \sum_{k_1=0}^{\infty} \frac{n(n + k_1 - 1)}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \int_{0}^{\infty} \Phi \left( \frac{1}{\lambda(n+k_1)} \theta(\varphi^2 g''_{u_1}, z) \right) dzdu_1
\]

\[
\leq \frac{1}{2} \sum_{k_1=0}^{\infty} \frac{n(n + k_1 - 1)}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \int_{0}^{\infty} \Phi \left( \frac{C}{\lambda(n+k_1)} \varphi^2(z)|g''_{u_1}(z)| \right) dzdu_1.
\]

On the other hand, by definition, we can deduce

\[\varphi^2(t)g''_{u_1}(t) = t(1+t)(1+u_1)^2D_2^2f(u_1, (1+u_1)t) = (\varphi^2 D_2^2f)(u_1, (1+u_1)t)\]

and

\[
J_{11} \leq \frac{1}{2} \sum_{k_1=0}^{\infty} \frac{n(n + k_1 - 1)}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \int_{0}^{\infty} \Phi \left( \frac{C}{\lambda(n+k_1)} \left( (\varphi^2 D_2^2f)(u_1, (1+u_1)z) \right) \right) dzdu_1
\]

\[
\leq \sum_{k_1=0}^{\infty} \int_{k_1/n}^{(k_1+1)/n} \int_{0}^{\infty} \frac{1}{1+u_1} \Phi \left( \frac{C}{\lambda n} \left( (\varphi^2 D_2^2f)(u_1, (1+u_1)z) \right) \right) d((1+u_1)z)du_1
\]

\[
\leq \int_{0}^{\infty} \int_{0}^{\infty} \Phi \left( \frac{C}{\lambda n} \left( (\varphi^2 D_2^2f)(u_1, u_2) \right) \right) du_1du_2. \quad (5)
\]

Using Jensen’s inequality, we derive

\[
J_{12} = \frac{1}{2} \sum_{k_1=0}^{\infty} \frac{n(n + k_1 - 1)}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \int_{0}^{\infty} \Phi \left( \frac{1}{\lambda} \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}(z)n(1+u_1) \right)
\]

\[
\times \int_{k_2/[n(1+u_1)]}^{t} \left( 2g''_{u_1}(s) ds \right) dzdu_1
\]

\[
\leq \sum_{k_1=0}^{\infty} \frac{n(n + k_1 - 1)}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \int_{0}^{\infty} \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}(z)n(1+u_1)
\]

\[
\times \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} \Phi \left( \frac{1}{\lambda} \int_{k_2/[n(1+u_1)]}^{t} 2g''_{u_1}(s) ds \right) dzdu_1
\]

\[
= \sum_{k_1=0}^{\infty} \frac{n}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \sum_{k_2=0}^{\infty} n(1+u_1)
\]

\[
\times \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} \Phi \left( \frac{1}{\lambda} \int_{k_2/[n(1+u_1)]}^{t} 2g''_{u_1}(s) ds \right) dzdu_1
\]

\[
\leq \sum_{k_1=0}^{\infty} \frac{n^2}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \sum_{k_2=0}^{\infty} (1+u_1)
\]

\[
\times \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} \Phi \left( \frac{1}{\lambda} \int_{k_2/[n(1+u_1)]}^{(k_2+1)/[n(1+u_1)]} 2g''_{u_1}(s) ds \right) dzdu_1
\]
By the above inequalities and

\begin{equation}
\frac{n^2}{(n-1)(n-2)} \sum_{k_1=0}^{k_1/n} \frac{1}{n} \Phi \left( \frac{1}{\lambda} \int_{k_2/(n+k_1)}^{(k_2+1)/[n(1+u_1)]]} 2g_{u_1}'(s) ds \right) du_1
\end{equation}

From Lemma 2.3 in [25], we obtain

\begin{equation}
\leq \sum_{k_1=0}^{\infty} \frac{n^2}{(n-1)(n-2)} \int_{k_1/n}^{(k_1+1)/n} \sum_{k_2=0}^{k_2/n} \frac{n+k_1}{n} \ \Phi \left( \frac{2n}{\lambda(n+1)(n-2)} g_{u_1}'(s) \right) ds du_1
\end{equation}

\begin{equation}
\leq \sum_{k_1=0}^{\infty} \int_{k_1/n}^{(k_1+1)/n} \sum_{k_2=0}^{(k_1+1)/n+1} \Phi \left( \frac{C}{\lambda n} g_{u_1}'(s) \right) ds du_1
\end{equation}

\begin{equation}
\leq \int_0^\infty \int_0^\infty \Phi \left( \frac{C}{\lambda n} |g_{u_1}'(u_2)| \right) du_2 du_1.
\end{equation}

From Lemma 2.3 in [25], we obtain

\begin{equation}
\int_0^\infty \Phi \left( \frac{1}{\lambda} |g_{u_1}'(u_2)| \right) du_2 \leq \int_0^\infty \Phi \left( \frac{1}{\lambda} |g_{u_1}'(u_2)| \right) du_2
\end{equation}

By the above inequalities and

\begin{equation}
\varphi^2(s)g_{u_1}''(s) = s(1+s)(1+u_1)^2D_2^2 f(u_1, (1+u_1)s) = \varphi^2 D_2^2 f(u_1, (1+u_1)s),
\end{equation}

we obtain

\begin{equation}
J_{12} \leq \int_0^\infty \int_0^\infty \left[ \Phi \left( \frac{C}{\lambda n} |g_{u_1}'(u_2)| \right) + \Phi \left( \frac{C}{\lambda n} \varphi^2(u_2) |g_{u_1}''(u_2)| \right) \right] du_1 du_2
\end{equation}

\begin{equation}
\leq \int_0^\infty \int_0^\infty \Phi \left( \frac{C}{\lambda n} |f(u_1, u_2)| \right) du_1 du_2
\end{equation}

\begin{equation}
+ \int_0^\infty \int_0^\infty \Phi \left( \frac{C}{\lambda n} |\varphi^2(u_1, u_2) D_2^2 f(u_1, u_2)| \right) du_1 du_2.
\end{equation}

Combining the above inequality with (4) and (5) acquires

\begin{equation}
\int_0^\infty \int_0^\infty \Phi \left( \frac{1}{\lambda} |J_1| \right) dx_1 dx_2 \leq \int_0^\infty \int_0^\infty \Phi \left( \frac{C}{\lambda n} |f(u_1, u_2)| \right) du_1 du_2
\end{equation}

\begin{equation}
+ 2 \int_0^\infty \int_0^\infty \Phi \left( \frac{C}{\lambda n} |\varphi^2(u_1, u_2) D_2^2 f(u_1, u_2)| \right) du_1 du_2.
\end{equation}

Now we can start off to estimate \(J_2\). Let

\begin{equation}
h(u_1) = h(u_1, x) = f \left( u_1, (1+u_1) \frac{x_2}{1+x_1} \right), \quad 0 \leq u_1 < \infty.
\end{equation}

From [5, 8], we obtain

\begin{equation}
|V_{n, 1}(f, x) - f(x)| \leq \frac{C}{n} (|f'(x)| + \varphi^2(x) |f''(x)|).
\end{equation}
Using inequalities (3) and (7), Lemma 2.2, and the convexity of \( \Phi(t) \) arrives at
\[
\int_{\mathbb{R}^2} \Phi\left( \frac{1}{\lambda} |J_2| \right) dx = \int_0^\infty \int_0^\infty \Phi\left( \frac{1}{\lambda} |V_{n,1}(h(\cdot), x_1) - h(x_1)| \right) dx_1 dx_2
\]
\[
\leq \int_{\mathbb{R}^2} \Phi\left( \frac{C}{\lambda n} \left( |h'(x_1)| + \varphi^2(x_1) |h''(x_1)| \right) \right) dx
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^2} \Phi\left( \frac{C}{\lambda n} |h'(x_1)| \right) dx + \frac{1}{2} \int_{\mathbb{R}^2} \Phi\left( \frac{C}{\lambda n} \varphi^2(x_1) |h''(x_1)| \right) dx
\]
\[
\leq \int_{\mathbb{R}^2} \Phi\left( \frac{C}{\lambda n} |h'(x_1)| \right) dx + \int_{\mathbb{R}^2} \Phi\left( \frac{C}{\lambda n} \varphi^2(x_1) |h''(x_1)| \right) dx.
\]
When denoting \( \varphi_{12}(x) = \varphi_{21}(x) = \sqrt{x_1 x_2} \), \( D^2_{12} = \frac{\partial^2}{\partial x_1 \partial x_2} \), and \( D^2_{21} = \frac{\partial^2}{\partial x_2 \partial x_1} \), we can write
\[
|\varphi^2(u)h''(u)| = \left| u(1 + u) \left[ D^2_{12} f + \frac{x_2^2}{1 + x_1} D^2_{12} f + \frac{x_2^2}{1 + x_1} D^2_{21} f \right.ight.
\]
\[
+ \left. \frac{x_2^2}{(1 + x_1)^2} D^2_{21} f \right] \left( - \frac{1 + x_2}{1 + x_1 + x_2} \varphi_1^2 D^2_{12} f + \varphi_1^2 D^2_{12} f + \varphi_2^2 D^2_{21} f \right.
\]
\[
+ \left. \frac{x_2}{1 + u} \frac{x_2}{1 + x_1 + x_2} \varphi_2^2 D^2_{21} f \right) \left( - \frac{1 + x_2}{1 + x_1 + x_2} \right) \right|.
\]
By virtue of the facts that \( |\varphi_{12}(x)| \) is not bigger than \( \varphi_1(x) \) or \( \varphi_2(x) \) and that
\[
|D^2_{12} f(x)| \leq \sup \{ |D^2_{12} f(x)|, |D^2_{21} f(x)| \}
\]
in [7, Lemma 2.1], we obtain
\[
\int_{\mathbb{R}^2} \Phi\left( \frac{C}{\lambda n} (\varphi^2(x_1) |h''(x_1)|) \right) dx \leq \int_{\mathbb{R}^2} \Phi\left( \frac{C}{\lambda n} \sum_{i=1}^2 |\varphi_i^2 D^2_{i} f| \right) dx
\]
and
\[
\int_{\mathbb{R}^2} \Phi\left( \frac{1}{\lambda} |J_2| \right) dx \leq \frac{1}{2} \int_{\mathbb{R}^2} \Phi\left( \frac{C}{\lambda n} (|f(x_1, x_2)|) \right) dx + \int_{\mathbb{R}^2} \Phi\left( \frac{C}{\lambda n} \sum_{i=1}^2 |\varphi_i^2 D^2_{i} f| \right) dx,
\]
where we used the convexity of \( \Phi(t) \). Combining these two inequalities with (3) and (6) and paying attention to computation of norm and the inequality (1) yield
\[
\|V_{n,2}(f) - f\|_{\Phi} \leq \|J_1\|_{\Phi} + \|J_2\|_{\Phi} \leq \frac{C}{n} \left( \|f\|_{\Phi} + \sum_{i=1}^2 \|\varphi_i^2 D^2_{i} f\|_{\Phi} \right).
\]
The proof of Lemma 2.4 is complete. \( \square \)

3. Equivalent theorems. There exist the following equivalent theorems between the modulus of smoothness and the K-functional.

**Theorem 3.1.** Let \( f \in L^p_\Phi(\mathbb{R}^m) \) and \( r \in \mathbb{N} \). Then there exist some constants \( C \) and \( t_0 \) such that
\[
\frac{\omega_{r,\Phi}(f, t)_{\Phi}}{C} \leq K_{r,\Phi}(f, t')_{\Phi} \leq C \omega_{r,\Phi}(f, t)_{\Phi}, \quad 0 \leq t \leq t_0.
\]
Proof. We shall reduce the proof to the one dimension. Some ideas come from [6]. For \( x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}_0^m \), we write \( x^* = (x_2, \ldots, x_m) \) and \( \mathbb{R}_0^{m*} = \{ x^* : x = (x_1, x^*) \in \mathbb{R}_0^m \} \). Let \( x_1 = (1 + |x^*|)z \) for \( 0 \leq z < \infty \) and \( F(z) = F(z, x^*) = f((1 + |x^*|)z, x^*) \). Then

\[
\varphi_1(x) = (1 + |x^*|)\varphi(z), \quad D^r_f(x) = F^{(r)}(z) (1 + |x^*|)^r, \quad \Delta^r_{\varphi_1}(x) f(x) = \Delta^r_{\varphi(z)} F(z).
\]

It was shown in [25] that

\[
\int_0^\infty \Phi \left( \frac{|\Delta^r_{\varphi(z)} F(z)|}{\lambda} \right) dz \leq \begin{cases} 
\int_0^\infty \Phi \left( \frac{|F(z)|}{\lambda} \right) dz, & F \in L^*_\varphi [0, \infty); \\
\int_0^\infty \Phi \left( \frac{|h^r \varphi^{(r)}(z) F^{(r)}(z)|}{\lambda} \right) dz, & F \in W^{r, \varphi} [0, \infty).
\end{cases}
\]

Consequently, it follows that, for \( f \in L^*_\varphi (\mathbb{R}^m_0) \),

\[
\| \Delta^r_{\varphi_1} f \|_{(\Phi)} = \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}^m_0} dx^* \int_0^\infty \Phi \left( \frac{1}{\lambda} |\Delta^r_{\varphi_1} f(x)| \right) dx_1 \leq 1 \right\} 
\]

\[
= \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}^m_0} (1 + |x^*|) \int_0^\infty \Phi \left( \frac{1}{\lambda} |\Delta^r_{\varphi(z)} F(z)| \right) dz dx^* \leq 1 \right\}
\]

\[
\leq \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}^m_0} (1 + |x^*|) \int_0^\infty \Phi \left( \frac{C}{\lambda} |F(z)| \right) dz dx^* \leq 1 \right\}
\]

\[
= \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}^m_0} \int_0^\infty \Phi \left( \frac{C}{\lambda} |F(z)| \right) dx_1 dx^* \leq 1 \right\}
\]

\[
= C \| f \|_{(\Phi)}.
\]

For \( f \in W^{r, \varphi} (\mathbb{R}^m_0) \), we arrive at

\[
\| \Delta^r_{\varphi_1} f \|_{(\Phi)} = \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}^m_0} dx^* \int_0^\infty \Phi \left( \frac{1}{\lambda} |\Delta^r_{\varphi_1} f(x)| \right) dx_1 \leq 1 \right\}
\]

\[
\leq \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}^m_0} (1 + |x^*|) \int_0^\infty \Phi \left( \frac{C}{\lambda} h^r \varphi^{(r)}(z) F^{(r)}(z) \right) dz dx^* \leq 1 \right\}
\]

\[
= \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}^m_0} \int_0^\infty \Phi \left( \frac{C}{\lambda} h^r \varphi^{(r)}(\varphi_1 D^r_1 f(x_1, x^*)) \right) dx_1 dx^* \leq 1 \right\}
\]

\[
= C h^r \| \varphi_1 D^r_1 f \|_{(\Phi)}.
\]

Similarly, for \( i = 2, 3, \ldots, m \), we have

\[
\| \Delta^r_{\varphi_i} f \|_{(\Phi)} \leq C \left\{ \begin{array}{ll}
\| f \|_{(\Phi)}, & f \in L^*_\varphi (\mathbb{R}^m_0); \\
h^r \| \varphi_i D^r_i f \|_{(\Phi)}, & f \in W^{r, \varphi} (\mathbb{R}^m_0).
\end{array} \right.
\]

Adding these inequalities and applying (1) yields the lower bound in (8).

To estimate the upper bound in (8), we shall again reduce it to the one-dimensional case. First we note that, for fixed \( x^* \) and \( t > 0 \), there exists a function \( G_t \in W^{r, \varphi} (\mathbb{R}_0^m) \) such that

\[
\int_0^\infty \Phi \left( \frac{1}{\lambda} |F(z) - G_t(z)| \right) dz \leq \frac{C}{t} \int_0^t \int_0^\infty \Phi \left( \frac{C}{\lambda} |\Delta^r_{\varphi(z)} F(z)| \right) dz dz.
\]
Then \( g_t \in W^r\varphi^2(\mathbb{R}_0^m) \) and
\[
\|f - g_t\|_{\Phi} = \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}_0^m} (1 + |x^*|) \int_0^\infty \Phi \left( \frac{1}{\lambda} |F(z) - G_t(z)| \right) dz dx^* \leq 1 \right\}
\]
\[
\leq \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}_0^m} (1 + |x^*|) \frac{C}{t} \int_0^t \int_0^\infty \Phi \left( \frac{C}{\lambda} |\Delta_{r,\varphi^1} F(z)| \right) dz d\tau dx^* \leq 1 \right\}
\]
\[
= \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}_0^m} (1 + |x^*|) \int_0^\infty \frac{C}{t} \int_0^t \Phi \left( \frac{C}{\lambda} |\Delta_{r,\varphi^1} F(z)| \right) d\tau dx^* \leq 1 \right\}
\]
\[
\leq \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}_0^m} \Phi \left( \frac{C}{\lambda} |\Delta_{r,\varphi^1} f(x^*)| \right) dx \leq 1 \right\}
\]
\[
= C \|\Delta_{r,\varphi^1} f\|_{\Phi}, \quad 0 \leq \tau_1 \leq t.
\]

and
\[
t^r \|\varphi^2_t D^r_t g_t\|_{\Phi} = \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}_0^m} (1 + |x^*|) \int_0^\infty \Phi \left( \frac{t^r}{\lambda} |\varphi^r (z) G_t^{(r)} (z)| \right) dz dx^* \leq 1 \right\}
\]
\[
\leq \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}_0^m} (1 + |x^*|) \frac{C}{t} \int_0^t \int_0^\infty \Phi \left( \frac{C}{\lambda} |\Delta_{r,\varphi^1} F(z)| \right) dz d\tau dx^* \leq 1 \right\}
\]
\[
= \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}_0^m} (1 + |x^*|) \int_0^\infty \frac{C}{t} \int_0^t \Phi \left( \frac{C}{\lambda} |\Delta_{r,\varphi^1} F(z)| \right) d\tau dx^* \leq 1 \right\}
\]
\[
\leq \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}_0^m} \Phi \left( \frac{C}{\lambda} |\Delta_{r,\varphi^1} f(x^*)| \right) dx \leq 1 \right\}
\]
\[
= C \|\Delta_{r,\varphi^1} f\|_{\Phi}, \quad 0 \leq \tau_1 \leq t.
\]

Similarly, we can prove that, for each \( i \) and \( t > 0 \), there are functions \( g_t \in W^r\varphi^i(\mathbb{R}_0^m) \) and \( \tau_i \in [0, t] \) such that
\[
\|f - g_t\|_{\Phi} \leq C \|\Delta_{r,\varphi^i} f\|_{\Phi} \quad \text{and} \quad t^r \|\varphi^i_t D^r_t g_t\|_{\Phi} \leq C \|\Delta_{r,\varphi^i} f\|_{\Phi}.
\]

By the double inequality (1), we obtain
\[
\|f - g_t\|_{\Phi} \leq C \|\Delta_{r,\varphi^i} f\|_{\Phi} \quad \text{and} \quad t^r \|\varphi^i_t D^r_t g_t\|_{\Phi} \leq C \|\Delta_{r,\varphi^i} f\|_{\Phi}. \quad (9)
\]

Adding these inequalities results in the upper bound in (8).

\[\square\]

Remark 3.1. For \( m = 1 \), Theorem 3.1 coincides with corresponding 1-dimensional one in [25].
Theorem 3.2. Let \( f \in L^*_\Phi[0, \infty) \) and \( r \in \mathbb{N} \). Then there exist some constants \( C \) and \( t_0 \) such that

\[
\frac{\omega_{r, \varphi}(f, t)^\Phi}{C} \leq K_{r, \varphi}(f, t')^\Phi \leq C \omega_{r, \varphi}(f, t)^\Phi, \quad 0 < t \leq t_0.
\]  
(10)

Proof. Since \( K_{r, \varphi}(f, t')^\Phi \leq K_{r, \varphi}(f, t)^\Phi \), we only need to prove the upper estimate. By those inequalities in (9), we only need to prove

\[
t^{2r} \| D_t^r g_t \|_\Phi \leq C \| \Delta_{r, \varphi}^r (x)e, f \|_\Phi
\]

for each \( i \).

In [23], it was obtained that

\[
\int_0^\infty \Phi \left( \frac{1}{\lambda} | t^{2r} G_t^r (z) | \right) dz \leq C \int_0^t \int_0^\infty \Phi \left( \frac{C}{\lambda} | \Delta_{r, \varphi}^r (z) | \right) dzdr.
\]

For \( F(z) = F(z, x^*) \), we have \( G_t(z) = G_t(z, x^*) \) as well. Let

\[
g_t(x) = G_t \left( \frac{x_1}{1 + |x^*|}, x^* \right), \quad x \in \mathbb{R}^m_0.
\]

Then \( g_t \in W_{\varphi, \Phi}^r(\mathbb{R}^m) \) and

\[
t^{2r} \| D_t^r g_t \|_\Phi = \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}^m_0} (1 + |x^*|) \int_0^\infty \Phi \left( \frac{1}{\lambda} | t^{2r} G_t^r (z) | \right) dzdx^* \leq 1 \right\}
\]

\[
\leq \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}^m_0} (1 + |x^*|) \int_0^t \int_0^\infty \Phi \left( \frac{C}{\lambda} | \Delta_{r, \varphi}^r (z) | \right) dzdrdx^* \leq 1 \right\}
\]

\[
\leq \inf_{\lambda > 0} \left\{ \lambda : \int_{\mathbb{R}^m_0} (1 + |x^*|) \int_0^\infty \Phi \left( \frac{C}{\lambda} | \Delta_{r, \varphi}^r (z) | \right) dzdx^* \leq 1 \right\}
\]

\[
= C \| \Delta_{r, \varphi}^r (x)e, f \|_\Phi, \quad 0 \leq \tau_1 \leq t.
\]

Similarly, we can prove that, for each \( i \) and \( t > 0 \), there are functions \( g_t \in W_{\varphi, \Phi}^r(\mathbb{R}^m) \) and \( \tau_i \in [0, t] \) such that

\[
t^{2r} \| D_t^r g_t \|_\Phi \leq C \| \Delta_{r, \varphi}^r (x)e, f \|_\Phi.
\]

Using (1), it follows that

\[
t^{2r} \| D_t^r g_t \|_\Phi \leq C \| \Delta_{r, \varphi}^r (x)e, f \|_\Phi.
\]

Combining this inequality with (9) and adding these inequalities lead to (10). The proof of Theorem 3.2 is complete. \( \square \)

Remark 3.2. For \( m = 1 \), Theorem 3.2 coincides with corresponding 1-dimensional one in [23].

4. A direct theorem. We now in a position to state and prove the direct theorem.

Theorem 4.1 (Direct theorem). Let \( f \in L^*_\Phi(\mathbb{R}^m_0) \), \( n > m \), and \( \Psi \in \Delta_2 \). Then

\[
\| V_{n,m}(f) - f \|_\Phi \leq C \left[ \omega_{n, \varphi} \left( f, \frac{1}{n^{1/2}} \right)^\Phi + \frac{\| f \|_\Phi}{n} \right].
\]  
(11)
Proof. Our proof is based on induction on the dimension \( m \) and on a decomposition for Baskakov–Kantorovich’s operator. For \( m = 1 \), the inequality (11) can be rewritten as
\[
\|V_{n,1}(f) - f\|_\Phi \leq C\omega_{2,\phi}\left(f, \frac{1}{n^{1/2}}\right)_\Phi \leq C\left[\omega_{2,\phi}\left(f, \frac{1}{n^{1/2}}\right)_\Phi + \frac{\|f\|_\Phi}{n}\right]
\]
which has been proved in [21].

For \( m \geq 2 \), the proof of Theorem 4.1 follows from combining Lemmas 2.1 and 2.4 with the estimates
\[
\|V_{n,m}(f) - f\|_\Phi \leq C \left\{ \frac{1}{n} \sum_{k=1}^{m} \|\varphi_i^2 D_i^2 f\|_\Phi + \|f\|_\Phi \right\}, \quad f \in L^\phi_0(\mathbb{R}_0^m);
\]
\[
\|V_{n,m}(f) - f\|_\Phi \leq C \left\{ \frac{1}{n} \sum_{k=1}^{m} \|\varphi_i^2 D_i^2 f\|_\Phi + \|f\|_\Phi \right\}, \quad f \in W^{2,\phi}(\mathbb{R}_0^m).
\]

The first estimate in (12) is valid for \( m = 2 \). The second estimate in (12) is valid for \( m = 2 \). If the second estimate in (12) is valid for \( m = r \geq 2 \), that is,
\[
\|V_{n,r}(f) - f\|_\Phi \leq C \left( \sum_{i=1}^{r} \|\varphi_i^2 D_i^2 f\|_\Phi + \|f\|_\Phi \right),
\]
then we have to further verify its validity for \( m = r + 1 \). We claim that the decomposition formula
\[
V_{n,r+1}(f, x) = \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1) \int_{k_1/n}^{(k_1+1)/n} \frac{k^*}{k^*+1} \frac{x_1^*}{1+x_1} \, du_1
\]
\[
\times \int_{k^*/[n(1+u_1)]}^{(k^*/[n(1+u_1)])} f(u_1, (1+u_1)x^*_{1+u_1}) \, du^*_{1+u_1}
\]
\[
= \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1) \int_{k_1/n}^{(k_1+1)/n} \frac{k^*}{k^*+1} \frac{x_1^*}{1+x_1} \, du_1
\]
\[
\times \int_{k^*/[n(1+u_1)]}^{(k^*/[n(1+u_1)])} f(u_1, (1+u_1)x^*_{1+u_1}) \, du^*_{1+u_1}
\]
is valid, where \( x^* = (x_2, x_3, \ldots, x_{r+1}) \), \( x = (x_1, x^*) \in \mathbb{R}_0^{r+1} \), \( k^* = (k_2, k_3, \ldots, k_{r+1}) \), \( k = (k_1, k^*) \in N_0^{r+1} \), and \( \sum_{k^*} = \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \cdots \sum_{k_{r+1}=0}^{\infty} \). This formula can be directly checked up and will take an important role in the following proof. Let
\[
g_{u_1}(t) \triangleq f(u_1, (1+u_1)t), \quad 0 \leq t < \infty
\]
and
\[
z = (z_2, z_3, \ldots, z_r) = \left( \frac{x_2}{1+x_1}, \frac{x_3}{1+x_1}, \ldots, \frac{x_{r+1}}{1+x_1} \right) = \frac{x^*}{1+x_1}.
\]
From the formula (14), it follows that
\[
V_{n,r+1}(f, x) - f(x) = \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1) \int_{k_1/n}^{(k_1+1)/n} \left[ V_{n,k_1,r}(g_{u_1}(\cdot), z) - g_{u_1}(z) \right] du_1
\]
\[
+ \left[ V_{n,1}(h(\cdot), x_1) - h(x_1) \right] \triangleq T_1 + T_2,
\]
where
\[
h(u_1) \triangleq h(u_1, x) \triangleq f(u_1, (1+u_1)x^*/(1+x_1)), \quad 0 \leq u_1 < \infty.
\]
By the inequality
\[ \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \leq \int_{\mathbb{R}^n} \Phi \left( \sum_{i=1}^{r} \left( \phi_i^2 D_i^2 f(x) \right) \right) dx + \int_{\mathbb{R}^n} \Phi \left( \frac{C}{\lambda n} |f(x)| \right) dx, \]
which can be obtained from (13), and Jensen’s inequality, we arrive at
\[ \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \leq \sum_{k=0}^{\infty} p_{n,k}(x_1) \frac{\lambda}{n} \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \]
\[ \leq \sum_{k=0}^{\infty} p_{n,k}(x_1) \frac{\lambda}{n} \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) \right) dx \]
\[ \leq \sum_{k=0}^{\infty} \frac{n(n + k_1 - 1)}{(n - 1)(n - 2)} \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \]
\[ \leq \sum_{k=0}^{\infty} \frac{n(n + k_1 - 1)}{(n - 1)(n - 2)} \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \]

On the other hand, by definition, we can deduce
\[ \varphi_i^2(x)D_i^2 g_{u_1}(x) = x_i (1 + |x|)(1 + u_1) D_i^2 f(u_1, (1 + u_1)x) \]
\[ = (\varphi_i^2 D_i^2 f)(u_1, (1 + u_1)x). \]

So, we obtain
\[ \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |T_1| \right) dx \leq \sum_{k=0}^{\infty} \frac{n(n + k_1 - 1)}{(n - 1)(n - 2)} \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \]
\[ \leq \sum_{k=0}^{\infty} \frac{n(n + k_1 - 1)}{(n - 1)(n - 2)} \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \]
\[ \leq \sum_{k=0}^{\infty} \frac{n(n + k_1 - 1)}{(n - 1)(n - 2)} \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \]
\[ \leq \sum_{k=0}^{\infty} \frac{n(n + k_1 - 1)}{(n - 1)(n - 2)} \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \]
\[ \leq \sum_{k=0}^{\infty} \frac{n(n + k_1 - 1)}{(n - 1)(n - 2)} \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \]
\[ \leq \sum_{k=0}^{\infty} \frac{n(n + k_1 - 1)}{(n - 1)(n - 2)} \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \]
\[ \leq \sum_{k=0}^{\infty} \frac{n(n + k_1 - 1)}{(n - 1)(n - 2)} \int_{\mathbb{R}^n} \Phi \left( \frac{1}{\lambda} |V_{r+1}(f, x) - f(x)| \right) dx \]
Recalling that \( \varphi_{ij}^{2}(x) = \varphi_{i}^{2}(x) \varphi_{j}^{2}(x) \) is not bigger than \( \varphi_{i}(x) \) or \( \varphi_{j}(x) \) and the fact
\[ |D_{ij}^{2}f(x)| \leq \sup_{1 \leq r \leq r+1} |D_{ij}^{2}f(x)| \]
in [7, Lemma 2.1], we obtain
\[ \int_{R_{0}^{r+1}} \Phi \left( \frac{1}{\lambda} |T_{2}| \right) \, dx \leq \frac{1}{2} \int_{R_{0}^{r+1}} \Phi \left( \frac{C}{n\lambda} |f(x)| \right) \, dx + \int_{R_{0}^{r+1}} \Phi \left( \frac{C}{n\lambda} \sum_{i=1}^{r+1} |\varphi_{i}^{2}D_{ij}^{2}f(x)| \right) \, dx. \] (17)

Combining (15), (16), and (17) and paying attention to computation of norm and the inequality (1), we obtain the second estimate of (12) for any \( m \geq 2 \).
For $g \in W^{2,\Phi}(\mathbb{R}^m)$, combining (12) with Lemma 2.1 and Theorem 3.1 gives
\[\|V_{n,m}(f) - f\|_{\Phi} \leq \|V_{n,m}(f) - V_{n,m}(g)\|_{\Phi} + \|V_{n,m}(g) - g\|_{\Phi} + \|f - g\|_{\Phi}\]
\[\leq C \|f - g\|_{\Phi} + C \left(\|g\|_{\Phi} + \frac{m}{n} \sum_{i=1}^{m} \|\phi_i^2 D_i^2 g\|_{\Phi}\right)\]
\[\leq C \left(\|f - g\|_{\Phi} + \frac{1}{n} \sum_{i=1}^{m} \|\phi_i^2 D_i^2 g\|_{\Phi}\right) + C \|f\|_{\Phi}\]
\[\leq C \left[\varpi_{2,\Phi}(f, \frac{1}{n^{1/2}})_{\Phi} + \frac{1}{n} \|f\|_{\Phi}\right].\]
The proof of Theorem 4.1 is complete.

5. Conclusions. In this paper, using $K$-functional and a decomposition technique and considering some properties of multivariate Baskakov–Kantorovich operators in the form of Lemmas 2.1 to 2.4, we presented two equivalent theorems, Theorems 3.1 and 3.2, between the $K$-functional and modulus of smoothness, and obtained a direct theorem, Theorem 4.1, in the Orlicz spaces $L^*_\Psi(\mathbb{R}^m)$.

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