QFT derivation of the decay law of an unstable particle with nonzero momentum

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Abstract
We present a quantum field theoretical derivation of the nondecay probability of an unstable particle with nonzero three-momentum $p$. To this end, we use the (fully resummed) propagator of the unstable particle, denoted as $S$, to obtain the energy probability distribution, called $d_pS(E)$, as the imaginary part of the propagator. The nondecay probability amplitude of the particle $S$ with momentum $p$ turns out to be, as usual, its Fourier transform: $a_pS(t) = \int_{m_{th}}^{\infty} dE d_pS(E)e^{-iEt}$ ($m_{th}$ is the lowest energy threshold in the energy frame, corresponding to the sum of masses of the decay products). Upon a variable transformation, one can rewrite it as $a_pS(t) = \int_{m_{th}}^{\infty} dm S(m)e^{-i\sqrt{m^2+p^2}t} |$ here, $d_S(m) \equiv dS(m)$ is the usual spectral function (or mass distribution) in the rest frame. Hence, the latter expression, previously obtained by different approaches, is here confirmed in an independent and, most importantly, covariant QFT-based approach. Its consequences are not yet fully explored but appear to be quite surprising (such as the fact that usual time-dilatation formula does not apply), thus its firm understanding and investigation can be a fruitful subject of future research.

1 Introduction

The study of the decay law is a fundamental part of Quantum Mechanics (QM). It is now theoretically [1, 2, 3, 4, 5, 6, 7, 8, 9] and experimentally [10, 11] established that deviations from the exponential decay exist, but they are usually small. Such deviations are also present in Quantum Field Theory (QFT) [9, 12].

An interesting question addresses the decay of an unstable particle with nonzero momentum $p$. In Refs. [13, 14, 15, 16, 17], it was shown –by using QM-based approaches enlarged to include special relativity– that the nondecay probability of an unstable particle $S$ with momentum $p$ is given by (in natural units)

$$P_p(t) = |a_pS(t)|^2 \text{ with } a_pS(t) = \int_{m_{th}}^{\infty} dm S(m)e^{-i\sqrt{m^2+p^2}t},$$

where $d_S(m)$ is the corresponding energy (or mass) distribution in its rest frame [$dm S(m)$ is the probability that the unstable state has energy (or mass) between $m$ and $m + dm$]. A review of the derivation is presented in Sec. 2. Quite remarkably, there are many interesting properties linked to this equation, which include deviations from the standard dilatation formula, see below.

The purpose of this work is straightforward: we derive Eq. (1) in a QFT framework (see Sec. 3). We thus confirm its validity and, as a consequence, its peculiar consequences. We shall start from the relativistic propagator of an unstable particle $S$. For definiteness, an underlying Lagrangian involving scalar fields shall be introduced, but our discussion is valid for any unstable field.
In this introduction, we recall some basic and striking features connected to Eq. (1). The normalization
\[
\int_{m_{th}}^{\infty} dm d_S(m) = 1
\] (2)
is a crucial feature of the spectral function, implying that \(a_S^0(0) = 1\). It must be valid both in QM and in QFT. Here, without loss of generality, we set the lower limit of the integral to \(m_{th} \geq 0\). In fact, a minimal energy \(m_{th}\) is present in each physical system; in particular, for a (relevant for us) relativistic system, it is given by the sum of the rest masses of the produced particles \(m_{th} = m_1 + m_2 + \ldots \geq 0\).

Clearly, for \(p = 0\), Eq. (1) reduces to the usual expression
\[
P_{S,rest}^P(t) = P_{S,rest}^0(t) = \left| a_S^0(t) \right|^2 = \left| \int_{m_{th}}^{\infty} dm d_S(m) e^{-i m t} \right|^2.
\] (3)

A detailed study of Eq. (1) shows peculiar features. First, the usual time dilatation does not hold:
\[
P_{S}^P(t) \neq P_{S,rest}^P(t),
\] (4)
where \(M\) is the mass of the state \(S\) (defined, for instance, as the position of the peak of the distribution \(d_S(m)\); in general, however, other definitions are possible, such as the real part of the pole of the propagator, see Sec. 3. The point is that, no matter which definition one takes, the expression (4) remains an inequality.)

In the exponential limit the spectral function of the state \(S\) reads [18, 19]
\[
d_{BW}^S(m) = \frac{\Gamma}{2\pi} \left[ (m - M)^2 + \frac{\Gamma^2}{4} \right]^{-1},
\] (5)
where \(M\) is the ‘mass of the unstable state’ corresponding to the peak. Even if the spectral function \(d_{BW}^S(m)\) is clearly unphysical because there is no minimal energy \((m_{th} \to -\infty)\), in many physical cases it is a good approximation for a quite broad energy range. Here, the decay amplitude and the decay law in the rest frame of the decaying particle notoriously read
\[
a_{BW,0}^S(t) = e^{-i M t - \Gamma t/2} \to P_{S,rest}^{BW,rest}(t) = e^{-\Gamma t}.
\] (6)

When a nonzero momentum is considered, the nondecay probability is still an exponential given by
\[
P_{S,BW,p}^P(t) = e^{-\Gamma_M t}
\] (7)
where the width is [17]:
\[
\Gamma_p = \sqrt{2} \sqrt{\left[ \left( M^2 - \frac{\Gamma^2}{4} + p^2 \right)^2 + M^2 \Gamma^2 \right]^{1/2} - \left( M^2 - \frac{\Gamma^2}{4} + p^2 \right)}.
\] (8)

Clearly: \(\Gamma_{p=0} = \Gamma\). One realizes that \(\Gamma_p\) differs from the naively expected standard time-dilatation formula, according to which the decay width of an unstable state with momentum \(p\) should simply be
\[
\frac{\Gamma M}{\sqrt{p^2 + M^2}} = \gamma \Gamma.
\] (9)

Namely, the quantity \(\gamma = \sqrt{p^2 + M^2} / M = 1 / \sqrt{1 - v^2}\) is the usual dilatation factor for a state with (definite) energy \(M\). Deviations between Eq. (8) and (9) are very small, see the numerical discussion in Ref. [17], and surely not measurable by current experiments [20]. Yet, the very fact that deviations exist is very interesting and deserves further study.
It should be stressed that in this work we consider unstable states with a definite momentum \( p \). This is a subtle point: while for a state with definite energy, a boost and a momentum translation are equivalent, this is not so for an unstable state, since it is not an energy eigenstate. Even more surprisingly, a boost of an unstable state is a quantum state whose nondecay probability is actually zero: it is already decayed (on the contrary, its survival probability present a peculiar time contraction \([21]\)). In other words, a boosted muon consists of an electron and two neutrinos \([15, 17]\). In this sense, the boost mixes the Hilbert subspace of the undecayed states with the subspace of the decay products, see Ref. \([17]\) for details. There, it is also discussed why the basis of unstable states is contains states with definite three-momentum. Indeed, the investigation of this paper also confirms this aspect: unstable states with definite momentum naturally follow from the study of its propagator in QFT.

The paper is organized as follows: Sec. 2 we recall the QM derivation of Eq. (1), while in Sec. 3 – the key part of this paper– we present this derivation in a QFT context. In the end, in Sec. 4 we describe our conclusions.

# Recall of the QM-based derivation of Eq. (1)

For completeness, we report here the “standard” derivation of Eq. (1). To this end, we use the arguments presented in Ref. \([17]\), but similar ones can be found in Refs. \([13, 14, 15, 16]\).

We consider a system described by the Hamiltonian \( H \), whose eigenstates are denoted as \( |m, p\rangle = U_p |m, 0\rangle \), where \( U_p \) is the unitary operator associated to the translation in momentum space. Standard normalization expressions are assumed:

\[
\langle m_1, p_1 | m_2, p_2 \rangle = \delta(m_1 - m_2) \delta(p_1 - p_2) .
\]

The state \( |m, p\rangle \) has definite energy,

\[
H |m, p\rangle = \sqrt{p^2 + m^2} |m, p\rangle ,
\]

definite momentum,

\[
P |m, p\rangle = p |m, p\rangle ,
\]

as well as definite velocity \( p/\sqrt{p^2 + m^2} \). Note, assuming that the energy of \( |m, p\rangle \) is \( \sqrt{p^2 + m^2} \), we have a relativistic spectrum.

Formally, the Hamiltonian can be written as

\[
H = \int d^3 p \int_{m_{th}}^{\infty} dm \sqrt{p^2 + m^2} \langle m, p | H_p |m, p\rangle,
\]

where

\[
H_p = \int_{m_{th}}^{\infty} dm \sqrt{p^2 + m^2} \langle m, p | H_p |m, p\rangle
\]

is the effective Hamiltonian in the subspace of states with definite momentum \( p \).

Let us now consider an unstable state \( S \) in its rest frame. The corresponding quantum state at rest is assumed to be

\[
|S, 0\rangle = \int_{m_{th}}^{\infty} d\alpha_S(m) |m, 0\rangle ,
\]

where \( \alpha_S(m) \) is the probability amplitude that the state \( S \) has energy \( m \). Hence, it is natural that the quantity \( d_S(m) = |\alpha_S(m)|^2 \) is the mass distribution: \( d_S(m) dm \) is the probability that the unstable
particle \( S \) has a mass between \( m \) and \( m + \text{dm} \). As a consequence, \( \int_{m}^{\infty} \text{dmd}\rangle_{\text{S}}(m) = 1 \), as already discussed in the introduction.

For the states of zero momentum, the Hamiltonian \( H_{n=0} \) can be expressed in terms of the undecayed state \( |S,0\rangle \) and its decay products in the form of a Lee Hamiltonian \( [22] \) (similar effective Hamiltonians are used also in quantum mechanics \([6, 19]\) and quantum field theory \([9, 23]\)):

\[
H_{\text{p}=0} = \int_{m_{\text{th}}}^{\infty} \text{dmm} \ |m,0\rangle \langle m,0|
= M_{0} |S,0\rangle \langle S,0| + \int \text{d}^{3}\omega(k) \ |k,0\rangle \langle k,0| + \int \text{d}^{3}k \frac{\omega(k)}{2\pi^3/2} \sigma \cdot f(k) \ [|S,0\rangle \langle k,0| + |k,0\rangle \langle S,0|], \tag{16}
\]

where \( |k,0\rangle \) represents a decay product with vanishing total momentum: in the two-body decay case, \( |k,0\rangle \) describes two particles, the first with momentum \( k \) and the second with momentum \(-k\), hence

\[
\omega(k) = \sqrt{k^2 + m^2} + \sqrt{k^2 + m^2}.
\tag{17}
\]

The last term in Eq. (16) represents the “mixing” between \( |S,0\rangle \) and \( |k,0\rangle \), which cause the decay of the former into the latter. Moreover, \( \sigma \) is a coupling constant and \( f(k) \) encodes the dependence of the mixing on the momentum of the produced particles. The explicit expressions connecting the states \( |k,0\rangle \) to \( |m,0\rangle \) formally reads

\[
|k,0\rangle = \int_{m_{\text{th}}}^{\infty} \text{dmm} \beta_{k}(m) \ |m,0\rangle \tag{18}
\]

where \( \beta_{k}(m) \) can be found by diagonalization of the Hamiltonian \( (16) \).

Let us now consider an unstable state with definite momentum \( p \), which is denoted as \( |S,p\rangle = U_{p} |S,0\rangle \):

\[
|S,p\rangle = \int_{m_{\text{th}}}^{\infty} \text{dmm} \alpha_{S}(m) \ |m,p\rangle.
\tag{19}
\]

The normalization

\[
\langle S,p_{1}|S,p_{2}\rangle = \delta(p_{1} - p_{2})
\tag{20}
\]

follows. Note, Eq. (19) is not a state with definite velocity. This is due to the fact that each state \( |m,p\rangle \) in the superposition has a different velocity \( p/\sqrt{p^2 + m^2} \). The subset of Hilbert space given by \( \{|S,p\rangle |p \subset \mathbb{R}^{3}\} \) represents the set of all undecayed quantum states of the system under study.

The form of the Hamiltonian \( H_{p} \) in term of the states \( |S,p\rangle \) and \( U_{p} |k,0\rangle = |k,p\rangle \) can be in principle derived by using the expressions above. Together with Eq. (19), one shall also take Eq. (18) and apply \( U_{p} \) in order to get:

\[
U_{p} |k,0\rangle = |k,p\rangle = \int_{m_{\text{th}}}^{\infty} \text{dmm} \beta_{k}(m) \ |m,p\rangle.
\tag{21}
\]

Then, once should invert Eqs. (19) and (21) and insert it into \( H_{p} \) of Eq. (14). However, its explicit expression is definitely not trivial but, fortunately, also not needed in the present work. Hence, we do not attempt to write it down here.

We now turn to the nondecay amplitudes. By starting from a properly normalized state with zero momentum, \( |S,0\rangle / \sqrt{\delta(0)} \), one obtains the usual expression

\[
a_{S}^{0}(t) = \frac{1}{\delta(0)} \langle S,0|e^{-iHt}|S,0\rangle = \frac{1}{\delta(0)} \int_{m_{\text{th}}}^{\infty} \text{dmm}_{1} \text{dmm}_{2} \langle m_{1},0|e^{-iHt}|m_{2},0\rangle
= \int_{m_{\text{th}}}^{\infty} \text{dmd}\rangle_{\text{S}}(m)e^{-imt},
\tag{22}
\]
in agreement with Eq. (3). The theory of decays is discussed in great detail for the case \( p = 0 \) in Refs. [4, 6, 7, 9] and refs. therein. Note, here the nondecay probability coincides with the survival probability (that is, the probability that the state did not change), but in general this is not the case [17].

Next, we consider a normalized unstable state \( S \) with nonzero momentum: \( |S, p\rangle / \sqrt{\delta(0)} \). The resulting non-decay probability amplitude

\[
a_S^p(t) = \frac{1}{\delta(0)} \langle S, p | e^{-iHt} | S, p \rangle = \frac{1}{\delta(0)} \int_{m_{th}}^{\infty} dm_1 dm_2 \langle m_1, p | e^{-iHt} | m_2, p \rangle
\]

\[
= \int_{m_{th}}^{\infty} dm_S(m)e^{-i\sqrt{m^2 + p^2}t}
\]

(23)

coincides with Eq. (1), hence concluding our derivation.

In principle, one could also start from the Hamiltonian \( H_p \) and obtain the energy distribution associated to this state, denoted as \( d_S^p(E) \). Then, \( a_S^p(t) \) should also emerge as the Fourier transform of the latter. This is hard to do here, since the explicit expression of \( H_p \) in terms of \( |S, p\rangle \) and \( |k, p\rangle \) was not written down (this is not an easy task). Quite interestingly, in the framework of QFT the function \( d_S^p(E) \) can be easily determined, see Sec. 3.

As a last comment of this section, we recall that the general nondecay probability of an arbitrary state \( |\Psi\rangle \) reads:

\[
P_{|\Psi\rangle}(t) = \int d^3p \left| \langle S, p | e^{-iHt} | \Psi \rangle \right|^2,
\]

(24)

whose interpretation is straightforward: we project \( |\Psi\rangle \) onto the basis of undecayed states. In general, \( P_{|\Psi\rangle}(0) \) is not unity. Notice also that \( P_{|\Psi\rangle}(t) \) is not the survival probability of the state \( |\Psi\rangle \) (a state can change with time, but still be undecayed if it is a different superposition of \( |S, p\rangle \)).

When a boost \( U_{\gamma} \) on the state with zero momentum (and hence with zero velocity) \( |S, 0\rangle \) is considered, the resulting state reads [17]:

\[
|\varphi_{\gamma}\rangle = U_{\gamma} |S, 0\rangle = \int_{m_{th}}^{\infty} dm_S(m)\sqrt{m\gamma^3/2} |m, m\gamma v\rangle,
\]

(25)

where \( \gamma = (1 - v^2)^{-1/2} \). In fact, each element of the superposition, \( |m, m\gamma v\rangle \), has velocity \( v \). Of course, \( |\varphi_{\gamma}\rangle \) is not an eigenstate of momentum, since each element in Eq. (25) has a different momentum \( p = m\gamma v \). In this respect the state \( |S, 0\rangle \) is special: it is the only state which has at the same time definite momentum and definite velocity (both of them vanishing). As mentioned in the Introduction, the nondecay probability associated to \( |\varphi_{\gamma}\rangle \) vanishes:

\[
P_{|\varphi_{\gamma}\rangle}(t) = 0 \ \forall v \neq 0.
\]

As soon as a nonzero velocity is considered, the state has decayed. This result is quite surprising but also rather ‘delicate’: when a wave packet is considered, \( P_{|\varphi_{\gamma}\rangle}(t) \) is nonzero (even if it is not 1 for \( t = 0 \) ) [17].

3 Covariant QFT derivation of Eq. (1)

Let us consider an unstable particle described by the field \( S(x) = S(t, x) \). For simplicity one can take a scalar field \( S \) with bare mass \( M_0 \) coupled to two scalar fields \( \varphi_1 \) (with mass \( m_1 \)) and \( \varphi_2 \) (with mass \( m_2 \)) via the interaction term \( g S \varphi_1 \varphi_2 \), leading to the QFT Lagrangian

\[
\mathcal{L} = \frac{1}{2} \left[ (\partial_{\mu} \varphi_1)^2 - m_1^2 \varphi_1^2 \right] + \frac{1}{2} \left[ (\partial_{\mu} \varphi_2)^2 - m_2^2 \varphi_2^2 \right] + \frac{1}{2} \left[ (\partial_{\mu} S)^2 - M_0^2 S^2 \right] + g S \varphi_1 \varphi_2.
\]

(27)

This is the QFT counterpart of the previous section. However, our discussion is in no way limited to this scalar theory.
The (full) propagator of the state $S$ (details in Ref. [24]) reads:

$$\Delta_S(p^2) = \frac{1}{p^2 - M_0^2 + i\varepsilon} + \Pi(p^2) + i\varepsilon \quad \text{with} \quad p^2 = E^2 - \mathbf{p}^2,$$

(28)

where $E = p^0$ is the energy and $\mathbf{p}$ the three-momentum. Because of covariance, $\Delta_S(p^2)$ depends only on $p^2$. The quantity $\Pi(p^2)$ is the one-particle irreducible diagram. Its calculation is of course non-trivial (it requires a proper regularization), but it is not needed for our purposes. The imaginary part entering in Eq. (28) are possible: $\text{Im} \Pi(p^2)$ is the one-particle irreducible diagram. Its calculation is of course non-trivial (it requires a proper regularization), but it is not needed for our purposes. The imaginary part

$$\text{Im} \Pi(p^2) = \sqrt{p^2} \Gamma(\sqrt{p^2}) = \frac{|k|}{8\pi \sqrt{-\mathbf{p}^2}} g^2 f_A^2(|k|) + ..., \quad (29)$$

where dots refer to higher orders, which are however typically very small [24]. Once $\text{Im} \Pi(p^2)$ is fixed, $\text{Re} \Pi(p^2)$ can be determined by dispersion relations (for an example of this technique, see e.g. Ref. [26]). The quantity $\Gamma^\text{tl} = \Gamma(\sqrt{p^2} = M)$, is the usual tree-level decay width, hence in the exponential limit the decay law $P_S(t) = e^{-\Gamma^\text{tl} t}$ must be reobtained. As mentioned in the Introduction, an unstable state has not a definite mass: this is why different definitions for $M$ (which is not the bare mass $M_0$ entering in Eq. (27)) are possible: $\text{Re} \Delta_S^{-1}(p^2 = M^2) = 0$ (zero of the real part of the denominator), or $\text{Re} \sqrt{s_\text{pole}}$, with $\Delta_S^{-1}(s_\text{pole}) = 0$ (real part of the pole), or the maximum of the spectral function defined below.

We also recall that

$$|k| = \sqrt{p^4 + (m_1^2 - m_2^2)^2 - 2p^2(m_1^2 + m_2^2)}$$

(30)

coincides, for the on-shell decay, with the three-momentum of one of the outgoing particles. The vertex function $f_A(|k|)$ is a proper regularization which fulfills the condition $f_A(|k| \to 0) = 1$ and describes the high-energy behavior of the theory (its UV completion), hence the parameter $\Lambda$ is some (very) high energy scale; $f_A(|k|)$ is formally not present in Eq. (27) since it appears in the regularization procedure, but it can be included directly in the Lagrangian by rendering it nonlocal [27] in a way that fulfills covariance [28]. In a renormalizable theory (such as the one of Eq. (27)), the dependence on $\Lambda$ disappears in the low-energy limit.

The properties outlined above, although in general very important in specific calculations, turn out to be actually secondary to the proof that we present below, where only the formal expression of the propagator of Eq. (28) is relevant. Moreover, even when the unstable particle is not a scalar, one can always define a scalar part of the propagator which looks just as in Eq. (27), then the outlined properties apply, mutatis mutandis, to each QFT Lagrangian.

As a next step, upon introducing the Mandelstam variable $s = p^2$, the function $F(s)$ defined as

$$F(s) = \frac{1}{\pi} \text{Im} \left[ \Delta_S(p^2 = s) \right]$$

(31)

fulfills the normalization condition:

$$\int_{s_{th}}^{\infty} ds F(s) = 1,$$

(32)

where $s_{th} = m_{th}^2$ is the minimal squared energy. For the case of Eq. (27), one has obviously $s_{th} = (m_1 + m_2)^2$. The normalization is a consequence of the Källén–Lehmann representation [29]

$$\Delta_S(p^2) = \int_{s_{th}}^{\infty} ds \frac{F(s)}{p^2 - s + i\varepsilon}, \quad (33)$$

in which the propagator $\Delta_S(p^2)$ has been rewritten as the ‘sum’ of free propagators $[p^2 - s + i\varepsilon]^{-1}$, each one of them weighted by $F(s)$: $\text{ds}F(s)$ is the probability that the squared mass lies between $s$
and $s + ds$. Of course, the normalization (32) is a central feature. For the detailed proof of it, we refer to Ref. [30]. Here we recall a simple version of it, which is obtained by assuming the rather strong requirement $\Pi(p^2) = 0$ for $p^2 > \Lambda^2$, where $\Lambda$ is a high-energy scale (no matter how large). Under this assumption

$$
\Delta_S(p^2) = \frac{1}{p^2 - M_0^2 + \Pi(p^2) + i\varepsilon} = \int_{s_{th}}^{\Lambda^2} ds \frac{F(s)}{p^2 - s + i\varepsilon}.
$$

(34)

Then, upon taking a certain value $p^2 \gg \Lambda^2$, the previous equation reduces to

$$
\frac{1}{p^2} = \int_{s_{th}}^{\Lambda^2} \frac{ds}{p^2} F(s) \int_{s_{th}}^{\Lambda^2} F(s) = 1.
$$

(35)

The general case in which $\Pi(p^2 \to \infty) = 0$ requires more steps, but the final result of Eq. (32) still holds [30].

Let us now consider the rest frame of the decaying particle: $p = 0$, $s = E^2 = m^2$. Here, upon a simple variable change ($m = \sqrt{s}$), we obtain the mass distribution (or spectral function) $d_S^{p=0}(m)$ through the equation

$$
dm d_S^{p=0}(m) = ds F(s),
$$

(36)

out of which:

$$
ds(m) = d_S^{p=0}(m) = 2mF(s = m^2).
$$

(37)

As already mentioned, $dm d_S(m)$ is the probability that the particle $S$ has a mass between $m$ and $m + dm$ [21, 31]. In this context, the normalization

$$
\int_{m_{th}}^{\infty} dm d_S(m) = 1
$$

(38)

follows from Eq. (32). Once the function $d_S(m)$ is identified as the mass distribution of the undecayed quantum state, the non-decay probability’s amplitude $a_{S}^{0}(t)$ can be obtained by repeating the steps of Sec. 2. The result coincides, as expected, with Eq. (6). Yet, it should be stressed that the unstable quantum state $|S, 0\rangle$ characterized by the distribution $d_S(m)$ is not simply given by $a_{S}^{0} |0_{PT}\rangle$, where $|0_{PT}\rangle$ is the perturbative vacuum and $a_{S}^{0}$ the creator operator of the non-interacting field $S$. The case of neutrino oscillations shows a similar situation: the state corresponding to a certain flavour, such as the neutrino $\nu_e$, must be constructed with due care by making use of Bogolyubov transformations [32]. Along this line, the exact and formal determination of the state $|S, 0\rangle$, corresponding to the mass distribution $d_S(m)$, in the context of QFT requires a generalization of Bogolyubov transformations and is not an easy task (it is left for the future). Nevertheless, it is not needed for the purpose of this paper.

Let us now consider the particle $S$ moving with a certain momentum $p$. Upon using $s = E^2 - p^2$, the energy distribution as function of $E$ is obtained by

$$
dE d_S^{p}(E) = ds F(s),
$$

(39)

leading to

$$
d_S^{p}(E) = 2EF(s = E^2 - p^2) = \frac{E}{\sqrt{E^2 - p^2}} d_S(\sqrt{E^2 - p^2}^2).
$$

(40)

The quantity $dE d_S^{p}(E)$ is the probability that the particle $S$ with definite momentum $p$ has an energy between $E$ and $E + dE$ (clearly, $d_S^{p=0}(E) = d_S(m = E)$). Also in this case, the normalization

$$
\int_{\sqrt{m_{th}^2 + p^2}}^{\infty} dE d_S^{p}(E) = 1
$$

(41)

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is a consequence of Eq. (32). When $d_S(m)$ has a maximum at $M$, then $d_p^p(E)$ has a maximum at $\sim \sqrt{M^2 + p^2}$. Note, the very fact that the propagator depends on $p^2 = E^2 - p^2$ allows to determine the spectral function $d_p^p(E)$ for a definite momentum $p$, that corresponds to the state $|S, p\rangle$ of Sec. 2.

The nondecay probability’s amplitude for a state $S$ moving with momentum $p$ is then given by

$$a_p^p(t) = \int_{\sqrt{m_{th}^2 + p^2}}^\infty dE d_p^p(E) e^{-iEt},$$  \hspace{1cm} (42)$$

where we have taken into account that the minimal energy is given by $\sqrt{m_{th}^2 + p^2}$.

This expression can be manipulated by using Eq. (40) and via a change of variable:

$$a_p^p(t) = \int_{\sqrt{m_{th}^2 + p^2}}^\infty dE d_p^p(E) e^{-iEt} = \int_{\sqrt{m_{th}^2 + p^2}}^\infty dE \frac{E}{\sqrt{E^2 - p^2}} d_S(\sqrt{E^2 - p^2}) e^{-iEt}$$

$$= \int_{m_{th}}^\infty dm d_S(m) e^{-i\sqrt{m^2 + p^2}t},$$  \hspace{1cm} (43)$$

which coincides exactly with Eq. (1), as we wanted to demonstrate. Thus, we confirm the validity of Eq. (1) in a covariant QFT-based framework.

4 Conclusions

The decay law of moving unstable particles is an interesting subject that connects special relativity to QM and QFT. An important aspect is the validity of Eq. (1), which expresses the nondecay probability of a state with nonzero momentum and whose standard derivation is reviewed in Sec. 2.

The main contribution of this paper has been the derivation of a quantum field theoretical proof of Eq. (1). To this end, we started from the (scalar part of the) propagator of an unstable quantum field, denoted as $S$. Then, we have determined the energy distribution of the state $S$ with definite momentum $p$, out of which the survival’s probability amplitude is calculated.

As discussed in the Introduction, there are interesting and peculiar consequences of Eq. (1). Future studies are definitely needed to further understand the properties of a decay of a moving unstable particle and to look for feasible experimental tests.

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