Scaling properties of physical observables have been for a long time one of the most instructive ways to get information about the underlying physics. In some cases this information could be fully quantified; here the most prominent example is Bjorken scaling of Deep Inelastic Scattering (DIS) structure functions [1]. In some other cases, like the Koba-Nielsen-Olesen scaling [2] for example, the underlying physics is still poorly understood. Some theoretically conjectured scaling laws, like Feynman scaling in high energy hadronic collisions [3], have not been experimentally confirmed. In any case both the emergence of a scaling law, and – sometimes more importantly – its violation, test our understanding of microscopic phenomena that are of importance in given experimental conditions. Again Bjorken scaling is here a benchmark, since both its emergence and violation are fully understood in terms of perturbative quantum chromodynamics (QCD) [4].

In this paper we will focus on another scaling law, called geometrical scaling (GS), which has been introduced in the context of DIS [5]. It has been shown that GS scaling is present also in high energy pp collisions [6, 7]. An onset of GS in heavy ion collisions at RHIC energies has been reported in Refs. [8].

The proton seen at low Bjorken $x < x_{\text{max}}$ is dominated by a gluon cloud characterized by a typical scale $Q_s(x)$ which is called saturation scale [9-11]. The phenomenon of saturation (for a review see [12, 13]) appears due to the nonlinearities of parton evolution at small $x$ given by so-called JIMWLK hierarchy equations [14] which in the large $N_c$ limit reduce to the Balitsky-Kovchegov equation [15]. These equations admit traveling wave solutions which explicitly exhibit GS [16]. The appearance of GS in the prototype of the BK equation obtained in double logarithmic approximation by Gribov, Levin and Ryskin [9] has been studied in Ref. [17]. An effective theory describing the small $x$ regime is Color Glass Condensate [9, 10, 18]. For the present study the details of saturation are not of primary interest, it is the very existence of the saturation scale which plays the crucial role.

Recently it has been shown [19] that GS in DIS works very well up to relatively large $x_{\text{max}} \sim 0.1$ (see also [20]). This should not come as a total surprise since it is known that GS scaling extends well above the saturation scale both in the DGLAP [21] and BFKL [22] evolution schemes not only for the boundary conditions that exhibit GS [23] to start with. For $x < x_{\text{max}}$ in the first approximation the virtual photon cross-section $\sigma_{\gamma^*g}$ in DIS is a function of a scaling variable $\tau = -q^2/Q_s^2(x)$ only, where $q^2$ is photon virtuality [5, 11]. Similarly in hadronic collisions charged particle multiplicity in central rapidity exhibits GS [9]

$$\left. \frac{dN}{dyd^2p_T} \right|_{y=0} = \frac{1}{Q_0^2} F(\tau)$$

where $F$ is a universal dimensionless function of the scaling variable

$$\tau = p_T^2/Q_s^2(x).$$

In this paper we shall address a question of applicability of Eq. (1) beyond the central rapidity region. This is an important and interesting question for the following reasons. First, Bjorken $x$’s of scattering gluons that produce a particle of a given transverse momentum $p_T$ and rapidity $y$ read

$$x_{1,2} = e^{\pm y} p_T/W$$

and can be quite different for large rapidities (we shall assume positive $y$, hence $x_2 < x_1$). Here $W = \sqrt{s}$ denotes pp scattering energy. If $x_2$ and $x_1$ are different then two different saturation scales emerge: $Q_{s1}$ and $Q_{s2}$ [24]. Whether this leads to violation of GS depends on the form of the saturation scale (for discussion of different forms of scaling variable see e.g. [25] and references therein). Second, and – as we shall argue below – more importantly, when the larger Bjorken $x = x_1$ for some $y$ leaves the domain of GS, i.e. $x_1 > x_{\text{max}}$, violation of GS should appear due to the fact that one of the scattering gluonic clouds becomes dilute and quark degrees of freedom become important. We are going to show in a model-independent way that this is what indeed does happen.

The theoretical formula for the multiplicity density in pp collisions is known since 1981 [20]:

$$\frac{dN}{dyd^2p_T} = \frac{C}{p_T^2} \int d^2k_T \alpha_s \varphi_1(x_1, k_T^2) \varphi_2(x_2, (\vec{k} - \vec{p})^2).$$

$\varphi_1$ and $\varphi_2$ are parton distribution functions.
Here $1/p_T^2$ corresponds to the $gg \to g$ cross section, $\varphi_{1,2}$ are unintegrated gluon densities and $\alpha_s$ stands for strong coupling constant. Although in principle [4] describes gluon production, one uses parton-hadron duality [27] to argue that it can be also applied to particle multiplicities where constant $C$ takes care of the hadronization effects.

At high energies and small $x$'s unintegrated gluon densities depend on $k_T$ through the ratio $k_T^2/Q_0^2(x)$ [11] – which is an underlying mechanism behind geometrical scaling. Throughout this paper we shall use the following form of the saturation scale [11]:

$$Q_0^2(x_{1,2}) = Q_0^2(e^{\pm y p_T}/(x_0 \sqrt{s}))^{-\lambda}$$

although other possibilities have been also discussed in the literature [24]. Parameters $Q_0 \sim 1$ GeV/c and $x_0 \sim 10^{-3}$ depend on a particular model of gluon densities, whereas exponent $\lambda$ can be extracted from the data in a model-independent way. Indeed, by analyzing multiplicity spectra at CMS we have found that for pp collisions $\lambda = 0.27$ [15]. Note that for $\lambda = 0$ saturation scale is equal to $Q_0$ and scaling variable $\tau = p_T^2/Q_0^2$ is essentially transverse momentum squared in units of $Q_0^2$.

There are many sources of possible violation of GS in multiplicity spectra given by Eq. (1). First – as already discussed – one of the Bjorken $x$'s [3] can be outside the GS domain. Second, gluons of very low $k_T$ will feel nonperturbative effects of the order of $Q_0^{CD}$, Next, running of the strong coupling $\alpha_s$ will evidently produce logarithmic corrections to GS. Finally, gluon fragmentation into physical particles will introduce dependence on their masses, which should induce GS violation in spectra of identified particles.

In this paper we shall concentrate on the first two effects which can be studied by looking at rapidity dependence of the particle multiplicities in pp collisions. For particles produced with $y \neq 0$ Bjorken $x$’s [3] are different and we have in fact two different saturation scales [24]. Hence instead of $F(\tau)$ in Eq. (1) we will have some other function $\bar{G}(\tau_1, \tau_2)$. This by itself does not mean that GS is necessarily violated since we can always rewrite $\bar{G}(\tau_1, \tau_2) = G(\tau_1, \rho)$ with $\rho = Q_{s1}/Q_{s2}$. For fixed rapidity $y$ ratio $\rho$ is constant for $Q_0$ of the form given by Eq. (5) and universal dependence of multiplicity distribution upon scaling variable $\tau_1$ is retained. Note that $F(\tau) = G(\tau, 1)$.

On the other hand by looking at the spectra with increasing $y$ one can eventually reach $x_1 > x_{max}$ and GS violation should be seen. Luckily we have at our disposal very recent pp data from the NA61/SHINE experiment at CERN [28] where particle spectra at different rapidities $y = 0.1 \div 3.5$ and five different energies have been measured for $p_T = 0.025 \div 1.375$ GeV/c. For the present analysis we shall use negative pion spectra, which has an advantage that one avoids possible GS violation due to particle masses. The only problem with this piece of data is that the scattering energies are rather low: $W_1,...,5 = 17.28, 12.36, 8.77, 7.75$, and 6.28 GeV. Nevertheless, as we will show below, Bjorken $x$’s involved are small enough for GS to be present. This is mainly due to the before-mentioned fact that in DIS $x_{max} \sim 0.1$ [19, 20]; for the purpose of the present analysis we shall take $x_{max} = 0.09.$ Then GS should be seen if $x_1 < x_{max}$, i.e. for

$$p_T < p_{T_{max}}(W, y) = x_{max} W e^{-y}.$$  

On the other hand transverse momenta of produced gluons should be larger than some nonperturbative scale $\Lambda$:

$$p_T > p_{T_{min}} = \Lambda$$

for which we take $\Lambda = 250$ MeV [29]. Having fixed the kinematical range where GS should be present, we can have a look at kinematical range of NA61/SHINE spectra shown in Fig. [4] The shaded area below $p_{T_{min}} = \Lambda$ is excluded for all rapidities. The shaded region in an upper part of the plot is excluded for $y = 0.1$, and dashed lines represent $p_{T_{max}}(W, y)$ as functions of $W$ for different rapidities $y$. Therefore for given rapidity only points below these lines should exhibit GS. Experimental points for different energies are represented by stars (17.28 GeV), squares (12.36 GeV), circles (8.77 GeV), and triangles (7.75 and 6.28 GeV). One can see from Fig. [4] that for $y = 0.1$ GS region extends towards the smallest NA61/SHINE energy. This is due to the fact that $x_{max}$ is as large as 0.09. By increasing $y$ we see that some points fall outside the GS window and finally for $y \gtrsim 1.7$ no GS should be present in NA61/SHINE data.

One may note that both $x_{max}$ and $\Lambda$ together with exponent $\lambda$ could be determined in a self-consistent way from

FIG. 1. Kinematic range of NA61/SHINE experiment in the $(W, p_T)$ plane. Data points are shown as stars, squares, circles and triangles. Shaded areas denote regions where GS is expected to be violated for $y = 0.1$. For other rapidities (marked at the r.h.s. of the plot) GS violation takes place above the respective dashed line.
the NA61/SHINE data. This is beyond the scope of the present note and will be presented elsewhere. Therefore our analysis has to be considered mostly as a qualitative one.

Limits (6) and (7) define GS window \( \tau_{\text{min}} < \tau < \tau_{\text{max}} \) in scaling variable \( \tau \).

\[
\tau_{\text{max}} = e^{-2y_{\text{max}}^2 W^2/Q_0^2 (x_{\text{max}}/x_0)^\lambda}, \\
\tau_{\text{min}} = e^{\lambda y \Lambda^2/Q_0^2 (\Lambda/(x_0 W))^\lambda}. \\
\tag{8}
\]

We see from (8) that for fixed \( W \), \( \tau_{\text{max}} \) falls exponentially with rapidity \( y \) much faster than \( \tau_{\text{min}} \) is rising with \( y \). When \( \tau_{\text{min}} \approx \tau_{\text{max}} \) window for GS closes, and we get from (8) that the larger \( W \) the larger \( y \) when this happens. For fixed \( y \) the GS window is growing with \( W \).

Now we come to the formulation of a model-independent criterion that allows one to assess whether GS is present or not. To this end we follow Refs. (9) and (19) and define ratios of multiplicities

\[
R_{ik}(\tau; y) = \frac{\frac{dN}{dyd^2p_T}(W_i, \tau; y)}{\frac{dN}{dyd^2p_T}(W_k, \tau; y)} \\
\tag{9}
\]

which, according to (1), should be equal to unity if GS is present. Since GS window is the largest for \( W_1 = 17.28 \) GeV, we form four ratios \( R_{1k} \) with \( k = 2, \ldots, 5 \) where index \( k \) refers to NA61/SHINE energies mentioned above.

In Fig. 2 we plot ratios \( R_{1k} \) as functions of \( \sqrt{\tau} \) for the lowest rapidity \( y = 0.1 \) first for \( \lambda = 0 \), i.e. essentially as functions of \( p_T \), and second for \( \lambda = 0.27 \). We see that for \( \lambda = 0 \) all ratios \( R_{1k} \) are larger than 1 and rise with \( \tau \) up to \( R_{1k} \sim 1.5 \div 6 \). In contrast, for \( \lambda = 0.27 \) for all ratios \( R_{1k} \) there exists a domain in \( \sqrt{\tau} \) where, within the errors, they are equal to 1. We see that \( \tau_{\text{min}} \) for \( W_2 = 12.36 \) GeV is approximately 0.6 while for \( W_5 = 6.28 \) GeV it is 0.8 or so. On the other hand for \( W_5 \) the GS windows closes at \( \sqrt{\tau} \approx 1.5 \), whereas for \( W_2 \) it extends further than 2.1. This is in qualitative agreement with limits (8), and as we will see in the following, this trend continues once we increase \( y \).

Looking at Fig. 1 we see that there is no qualitative difference between rapidities 0.1 and 0.3, and this is what indeed can be seen in Fig. 3a. However, already for \( y = 0.7 \) the lowest energy data are no longer in the scaling region, and we see this clearly in Fig. 3a. Further on, for rapidity \( y = 0.9 \) the second lowest energy data leave the GS window, and this is again nicely substantiated by the behavior of \( R_{14} \) shown in Fig. 3c. Finally for \( y = 1.3 \) none of the tested energies \( W_2 \div W_5 \) should exhibit GS, and this is clearly confirmed by Fig. 3d. Note that in all these cases the reference points at scattering energy \( W_k \) remain in the GS window, therefore scaling violations of ratios \( R_{1k} \) should be attributed to the points at lower energies \( W_k < W_1 \).

It is somewhat surprising that relatively low energy NA61/SHINE data do show GS at all. This is due to the fact that GS scaling works up to much higher \( x \)'s than originally anticipated: \( x_{\text{max}} \sim 0.1 \). This has been confirmed by direct analysis of DIS in Ref. (19) and also indirectly by the present study.

In this paper we have explored NA61/SHINE negative pion spectra in pp collisions taken at different rapidities \( y \). In order to study GS in multiplicity distributions we have formed ratios (9) that should be equal to 1 for scaling variable \( \tau \) within the GS window (8). We have argued that behavior of ratios \( R_{1k} \) with increasing \( y \) is in qualitative agreement with the following picture of particle production in hadronic collisions. For low \( p_T \)'s GS is violated due to the existence of the nonperturbative energy scale \( \Lambda \). For high transverse momenta, the larger of two Bjorken \( x \)'s, namely \( x_1 \), crosses the maximal \( x \) for which GS is present, \( x_1 > x_{\text{max}} \), and GS is again violated. For rapidities larger than 1.3, the window of GS closes for all energies \( W_k \) but \( W_1 \), and no GS is seen in ratios \( R_{1k} \). This behavior of GS for different energies with varying \( y \) is strikingly in qualitative agreement with the kinematical constraints shown in Fig. 1 and more quantitatively with the range of the GS window given by Eq. (8).
Our aim here was to point out that model-independent analysis of the multiplicity distributions at different rapidities can provide an interesting insight into the production mechanism of low and medium \( p_T \) pions from the point of view of gluon saturation and geometrical scaling. One obvious reservation is that for higher rapidities one can force ratios \( R_{1k} \) to approach unity at the expense of adjusting \( \lambda \). However, very soon \( \lambda \) gets bigger than 1 and for rapidities of the order of 2 it has to be bigger than 3. The quality of such "geometrical scaling" is very poor, and – more importantly – there is no physical picture supporting such big values of \( \lambda \). This is one of the reasons for which one should perform a more quantitative analysis of GS in pp collisions, which will be presented elsewhere.

Let us finish by a remark that experimental data at high LHC energies would be best suited for an analysis outlined in this paper. For the LHC energies GS closes very high rapidities up to \( y \sim 8 \). Since one needs data at different rapidities, perhaps the best detector for this purpose would be LHC-b.

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[29] This scale is higher than commonly used \( \sqrt{s_N} \)c, however, it corresponds to a physical low momentum cutoff and has been chosen by a rough fit to NA61/SHINE data.
FIG. 3. Ratios $R_{1k}$ as functions of $\sqrt{\tau}$ for $\lambda = 0.27$ and for different rapidities a) $y = 0.3$, b) $y = 0.7$, c) $y = 0.9$ and d) $y = 1.3$. With increase of rapidity, gradual closure of the GS window can be seen.