DIMENSION ESTIMATES FOR THE ATTRACTOR OF THE REGULARIZED DAMPED EULER EQUATIONS ON THE SPHERE

ALEXEI ILYIN¹, ANNA KOSTIANKO³,⁴, AND SERGEY ZELIK²,³

Abstract. We prove existence of the global attractor of the damped and driven Euler–Bardina equations on the 2D sphere and on arbitrary domains on the sphere and give explicit estimates of its fractal dimension in terms of the physical parameters.

1. Introduction and main result

The following regularized Euler system has attracted considerable attention over the last years

\[
\begin{aligned}
\partial_t u + (\bar{u}, \nabla x) \bar{u} + \gamma u + \nabla x p &= g, \\
\text{div} \bar{u} &= 0, \quad u(0) = u_0, \quad u = (1 - \alpha \Delta x) \bar{u}.
\end{aligned}
\]

Here \(g\) is the forcing term, \(\gamma u\) is the Ekman damping term that makes the system dissipative, and \(\alpha > 0\) is a small parameter of dimension \((\text{length})^2\) so that \(\bar{u}\) is a smoothed vector function with high spatial modes filtered out.

In 3D this system is studied as a subgrid scale model of turbulence and is known in literature as the inviscid Euler–Bardina model [3]. The asymptotic behavior of solutions and estimates for the number of the degrees of freedom for this model and the similar Navier–Stokes–Voight model were studied in [4, 14] (see also the references therein).

A comprehensive analysis of this system from the point of view of attractors has recently been done in [12, 13]. Explicit upper bounds for the fractal dimension of system (1.1) on the 2D torus were obtained in [12] and, furthermore, by the instability analysis on the corresponding generalized Kolmogorov flows it was also shown there that the upper estimates are optimal in the limit \(\alpha \to 0\).

2000 Mathematics Subject Classification. 35B40, 35B45, 35L70.

Key words and phrases. Regularized Euler equations, Bardina model, attractors, fractal dimension, spectral inequalities on the sphere.

The second author was partially supported by the Leverhulme grant No. RPG-2021-072 (United Kingdom).
A more difficult 3D case was studied in detail in [13] for system (1.1) on the 3D torus and in a domain $\Omega \subseteq \mathbb{R}^3$, where we again obtained explicit upper bounds for the attractor dimension and the estimate for the 3D torus is also optimal as $\alpha \to 0$.

Motivated by possible geophysical applications we study in this work the regularized Euler–Bardina system on the 2D sphere and in proper domains $\Omega$ on the sphere:

$$
\begin{cases}
\partial_t u + \nabla u \bar{u} + \gamma u + \nabla p = g, \\
\text{div } \bar{u} = 0, \quad u(0) = u_0, \quad u = (1 - \alpha \Delta)\bar{u}.
\end{cases}
$$

(1.2)

The corresponding phase space with respect to $\bar{u}$ is

$$
\bar{u} \in H^1 := \begin{cases}
H^1_0(\Omega), & \Omega \subsetneq S^2, \\
H^1(S^2), & \Omega = S^2, \\
\text{div } \bar{u} = 0,
\end{cases}
$$

(1.3)

and $\bar{u} = (1 - \alpha \Delta)^{-1}u$ in the case of $S^2$, while for $\Omega \subsetneq S^2$ we recover $\bar{u}$ by solving the Stokes problem in $\Omega$:

$$
\bar{u} - \alpha \Delta \bar{u} + \nabla q = u, \quad \text{div } \bar{u} = 0, \quad \bar{u}|_{\partial \Omega} = 0.
$$

In [12] $\nabla_u u$ is the covariant derivative of $u$ along $u$ for which we have [8]

$$
\nabla_u u = \nabla \frac{u^2}{2} - u^\perp \text{rot } u.
$$

In the vector case by the Laplace operator acting on (tangent) vector fields on $S^2$ we mean the Laplace–de Rham operator $-\delta \delta - \delta \delta$ identifying 1-forms and vectors. Then for a two-dimensional manifold we have [8]

$$
\Delta u = \nabla \text{div } u - \text{rot rot } u,
$$

(1.4)

where the operators $\nabla = \text{grad}$ and $\text{div}$ have the conventional meaning. The operator rot of a vector $u$ is a scalar and for a scalar $\psi$, rot $\psi$ is a vector: $\text{rot } u := \text{div}(u^\perp)$, rot $\psi := \nabla^\perp \psi$, where in the local frame $u^\perp = (u_2, -u_1)$, that is, $\pi/2$ clockwise rotation of $u$ in the local tangent plane. Integrating by parts we obtain

$$
(-\Delta u, u) = \| \text{rot } u \|^2_{L^2} + \| \text{div } u \|^2_{L^2}.
$$

(1.5)

We can now state the main result of this work proved in Section 3.

**Theorem 1.1.** Let $\Omega \subseteq S^2$. The regularized Euler system (1.2) has a global attractor $\mathcal{A}$ in $H^1$ with finite fractal dimension satisfying the following upper
bound

\[ \dim_{F, \mathcal{A}} \leq \frac{1}{8\pi} \cdot \begin{cases} \\
\frac{1}{\alpha \gamma^4} \min \left( \| \text{rot} \ g \|_{L^2}^2, \ |\| g \|_{L^2}^2 \| / 2\alpha \right), \quad \Omega = \mathbb{S}^2 \\
\| g \|_{L^2}^2 / 2\alpha^2 \gamma^4, \quad \Omega \subseteq \mathbb{S}^2.
\end{cases} \quad (1.6) \]

In Section 2 we prove dissipative estimates, write the system as an ODE in $H^1$ with bounded nonlinearity and construct the global attractor. In the Appendix (which is of an independent interest) we prove in the spirit of [17] collective Sobolev inequalities on the sphere for families of functions with orthonormal derivatives.

We point out in conclusion that estimates (1.6) are exactly the same as those in the case of the 2D torus $\mathbb{T}^2$ (and $\mathbb{R}^2$), and $\Omega \subseteq \mathbb{R}^2$, see, respectively [12, 13]. Furthermore, since as shown in [12] the estimate on $\mathbb{T}^2$ is optimal, we have a strong evidence that it is also true for $\mathbb{S}^2$.

2. A priori estimates and the global attractor

Before we prove two types of energy estimates we recall the following two orthogonality relations [8].

**Lemma 2.1.** Let $\Omega \subseteq \mathbb{S}^2$ and let $u, v \in T\mathbb{S}^2$, $\text{div} \ u = 0$ be smooth vector functions in $\Omega$. In case when $\Omega \subseteq \mathbb{S}^2$ we further suppose that $u|_{\partial \Omega} = 0$. Then

\[ \int_\Omega (\nabla_u v \cdot v) dS = 0, \quad \int_{\mathbb{S}^2} (\nabla_u u \cdot \Delta u) dS = 0. \quad (2.1) \]

**Proof.** To prove the first identity we use the following result in differential geometry, see, for instance, [7]. Namely, suppose that $M$ is a surface in $\mathbb{R}^d$ and let $u, v$ be tangent vector functions on $M$. Let $u$ and $v$ be somehow prolonged in a neighborhood of $M$ in $\mathbb{R}^d$ with fixed Cartesian system. We denote them as $\tilde{u}, \tilde{v}$. Then at a point $x \in M$

\[ \nabla_u v(x) = \pi \left( \sum_{i=1}^d \tilde{u}_i \partial_i \tilde{v}_j \right), \]

where $\pi$ is the projection on the tangent plane to $M$ at $x$. 
Using this we obtain taking into account that $u$ is tangent
\[
\int_{\Omega} (\nabla u \cdot v) dS = \int_{\Omega} \sum_{i,j=1}^{3} \tilde{u}_i \partial_i \tilde{v}_j dS = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{3} \tilde{u}_i \partial_i (\tilde{v}_j)^2 dS =
\]
\[
= \frac{1}{2} \int_{\Omega} \sum_{j=1}^{3} u \cdot \nabla (v_j)^2 dS = -\frac{1}{2} \int_{\Omega} \sum_{j=1}^{3} (v_j)^2 \text{div } u dS = 0.
\]

We point out that $\nabla$ and div here are the surface gradient and divergence.

The second identity follows since setting $\omega = \text{rot } u$ we have
\[
\int_{S^2} \nabla u \cdot \Delta u dS = \int_{S^2} \left( \frac{1}{2} \nabla u^2 - \omega u^\perp \right) \cdot \nabla^\perp \omega dS = -\int_{S^2} \frac{1}{2} u \cdot \nabla (\omega)^2 dS = 0.
\]

\[\square\]

**Proposition 2.1.** Let $u$ be a smooth solution of equation (1.2). Then for any $\Omega \subseteq S^2$ the following dissipative energy estimate holds:
\[
\| \bar{u}(t) \|^2_\alpha \leq \| \bar{u}(0) \|^2_\alpha e^{-\gamma t} + \frac{1}{\gamma^2} \| g \|^2_{L^2},
\]  
(2.2)

where
\[
\| \bar{u} \|^2_\alpha := \| \bar{u} \|^2_{L^2} + \alpha \| \text{rot } \bar{u} \|^2_{L^2}.
\]

For $\Omega = S^2$ estimate (2.2) still holds and, in addition,
\[
\| \tilde{\omega}(t) \|^2_\alpha \leq \| \tilde{\omega}(0) \|^2_\alpha e^{-\gamma t} + \frac{1}{\gamma^2} \| \text{rot } \bar{u} \|^2_{L^2},
\]  
(2.3)

where $\omega = \text{rot } u$, $\tilde{\omega} = \text{rot } \bar{u}$, $\omega = (1 - \alpha \Delta) \tilde{\omega}$ and
\[
\| \tilde{\omega} \|^2_\alpha := \| \tilde{\omega} \|^2_{L^2} + \alpha \| \nabla \tilde{\omega} \|^2_{L^2}.
\]

**Proof.** Indeed, taking the scalar product of (1.2) and $\bar{u}$, using (1.5), that is,
\[
(u, \bar{u}) = \| \bar{u} \|^2_{L^2} + \| \text{rot } \bar{u} \|^2_{L^2}
\]
and the first identity in (2.1), we obtain
\[
\frac{d}{dt} (\| \bar{u} \|^2_{L^2} + \alpha \| \text{rot } \bar{u} \|^2_{L^2}) + 2\gamma (\| \bar{u} \|^2_{L^2} + \alpha \| \text{rot } \bar{u} \|^2_{L^2}) = 2(g, \bar{u}) \leq
\]
\[
\leq 2\| g \|_{L^2} \| \bar{u} \|_{L^2} \leq \gamma \| \bar{u} \|^2_{L^2} + \frac{1}{\gamma} \| g \|^2_{L^2}. \tag{2.4}
\]

Applying the Gronwall inequality, we obtain estimate (2.2).
The proof of (2.3) is similar, and we take the scalar product of (1.2) and $-\Delta \bar{u}$ instead. Using the second orthogonality relation we obtain
\[
\frac{d}{dt} \left( \|\bar{\omega}\|_{L^2}^2 + \alpha \|\nabla \bar{\omega}\|_{L^2}^2 \right) + 2\gamma \left( \|\bar{\omega}\|_{L^2}^2 + \alpha \|\nabla \bar{\omega}\|_{L^2}^2 \right) = 2(g, \text{rot rot} \bar{u}) = 2(\text{rot} g, \bar{\omega}) \leq \gamma \|\bar{\omega}\|_{L^2}^2 + \frac{1}{\gamma} \|\text{rot} g\|_{L^2}^2
\]
and complete the proof as before. \(\square\)

The following time averaged estimates are essential in Section 3.

**Corollary 2.1.** For any $\Omega \subsetneq S^2$
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \| \text{rot} \bar{u}(s) \|_{L^2}^2 \, ds \leq \frac{1}{\gamma \sqrt{2\alpha}} \|g\|_{L^2}. \tag{2.5}
\]
For $\Omega = S^2$ estimate (2.5) still holds and, in addition,
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \| \text{rot} \bar{u}(s) \|_{L^2}^2 \, ds \leq \frac{1}{\gamma} \| \text{rot} g \|_{L^2}. \tag{2.6}
\]

**Proof.** Estimate (2.6) immediately follows from (2.3). To see that (2.5) holds we integrate (2.4) from 0 to $t$, divide by $t$ and let $t \to \infty$. Since in view of (2.2), $\|\bar{u}(t)\|_\alpha$ is bounded, we obtain that
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \| \text{rot} \bar{u}(s) \|_{L^2}^2 \, ds \leq \frac{1}{2\alpha \gamma^2} \|g\|_{L^2}^2.
\]
Using Hölder inequality
\[
\frac{1}{t} \int_0^t \| \text{rot} \bar{u}(s) \|_{L^2} \, ds \leq \left( \frac{1}{t} \int_0^t \| \text{rot} \bar{u}(s) \|_{L^2}^2 \, ds \right)^{1/2},
\]
we obtain (2.5). \(\square\)

We now write equation (1.2) as an ODE in a Hilbert space with bounded nonlinearity. Applying to (1.2) the operator
\[
A_\alpha := (1 - \alpha A)^{-1} \Pi
\]
where $A = \Pi \Delta$ is the Stokes operator in $\Omega$ and $\Pi$ is the Helmholtz–Leray projection, we obtain
\[
\partial_t \bar{u} + \gamma \bar{u} + B(\bar{u}, \bar{u}) = \bar{g}, \quad \bar{u}|_{t=0} = \bar{u}_0, \tag{2.7}
\]
where $B(\bar{u}, \bar{v}) := A_\alpha \Pi (\nabla \bar{u} \bar{v}), \bar{g} = (1 - \alpha A)^{-1} \Pi g$.

Arguing as in [12] and also using the elliptic regularity for the Stokes operator we see that $B$ is bounded from $H^1$ to $H^{2-\varepsilon}$, $\varepsilon > 0$. 

As a result, we have written (1.2) as an ODE in $H^1$ with bounded nonlinearity. Therefore the local existence and uniqueness of a solution as well as (an infinite) differentiability of the corresponding local solution semigroup are straightforward corollaries of the Banach contraction principle. The global existence follows from a priori estimates obtained above, so that we have proved the following theorem.

**Theorem 2.1.** Let $\bar{u}_0 \in H^1(\Omega)$. Then there exists a unique global solution $\bar{u} \in C([0, \infty), H^1)$ of problem (2.7) (which is simultaneously the unique solution of (1.2)). In other words, a dissipative solution semigroup

$$S(t) : H^1 \rightarrow H^1, \quad S(t)\bar{u}_0 := \bar{u}(t), \quad t \geq 0$$

is well defined. Moreover, $S(t)$ is $C^\infty$-differentiable for every fixed $t$.

Concluding this section we construct the main object of our interest, namely, the global attractor of the solution semigroup $S(t)$.

**Theorem 2.2.** The semigroup $S(t)$ has a global attractor in $\mathcal{A} \subset H^1$ which, by definition, is a set that is

1) compact in $H^1$, $\mathcal{A} \Subset H^1$;
2) strictly invariant $S(t)\mathcal{A} = \mathcal{A}$;
3) attracts bounded sets in $H^1$: for every bounded set $B \subset H^1$ and every neighborhood $O(\mathcal{A})$

$$S(t)B \subset O(\mathcal{A}) \text{ for } t \geq T(B; O(\mathcal{A})).$$

**Proof.** The semigroup $S(t)$ is continuous and dissipative in view of (2.2). To apply a general and by now standard result on the existence of the attractor, see, for instance, [2, 15] we only need to establish the asymptotic compactness of the semigroup $S(t)$. This is also achieved by the standard splitting of $S(t)$ into a exponentially decaying part and a uniformly compact part:

$$S(t) = \Sigma(t) + S_2(t), \quad S_2(t) = S(t) - \Sigma(t),$$

where $v(t) = \Sigma(t)\bar{u}_0$ is the decaying solution of the linear equation

$$\partial_t v + \gamma v = 0, \quad v(0) = \bar{u}_0,$$

and $w(t) = S_2(t)\bar{u}_0$ is the solution of the equation

$$\partial_t w + \gamma w = G(t) := -B(\bar{u}, \bar{u}) + \bar{g}, \quad w(0) = 0$$

with zero initial condition and right-hand side uniformly bounded in $H^{2-\varepsilon}$. Therefore $w$ is uniformly bounded in $H^{2-\varepsilon}$ and since $\bar{u} = v + w$ the asymptotic compactness of the semigroup $S(t)$ is established. □
3. **Upper Bound for the Dimension of the Attractor**

**Proof of Theorem 3.1.** The solution semigroup $S(t) : H^1 \to H^1$ is differentiable with respect to the initial data so we only need to estimate the global Lyapunov exponents for the linearization of equation (2.7) on the trajectories lying on the attractor. The linearized system is:

\[
\begin{aligned}
\partial_t \bar{\theta} &= -\gamma \bar{\theta} - B(\bar{u}(t), \bar{\theta}) - B(\bar{\theta}, \bar{u}(t)) =: L_{u(t)} \bar{\theta}, \\
\text{div} \bar{\theta} = 0, \quad \bar{\theta}|_{t=0} = \bar{\theta}_0 \in H^1(\Omega),
\end{aligned}
\]

where $B(\bar{u}, \bar{v}) := (1 - \alpha A)^{-1} \Pi (\nabla \bar{u} \bar{v})$. It is convenient to define the scalar product in $H^1$ induced by the operator $1 - \alpha A$, namely,

\[
(\bar{\theta}, \bar{\xi})_\alpha = (\bar{\theta}, \bar{\xi}) + \alpha (\text{rot} \bar{\theta}, \text{rot} \bar{\xi}) = ((1 - \alpha A) \bar{\theta}, \bar{\xi}) \quad (3.1)
\]

Then, using that $\Pi A_\alpha = A_\alpha$ and $\Pi \bar{\theta} = \bar{\theta}$, we obtain

\[
(B(\bar{u}, \bar{\theta}), \bar{\theta})_\alpha = ((1 - \alpha A)^{-1} \Pi \nabla \bar{u} \bar{\theta}, (1 - \alpha \Delta_x) \bar{\theta}) =
\]

\[
= ((1 - \alpha A)^{-1} \Pi \nabla \bar{u} \bar{\theta}, (1 - \alpha \Pi \Delta_x \bar{\theta}) = (\Pi \nabla \bar{u} \bar{\theta}, \bar{\theta}) = (\nabla \bar{u} \bar{\theta}, \bar{\theta}) \equiv 0. \quad (3.2)
\]

Following the general strategy, see e.g. [15], the sums of the first $n$ global Lyapunov exponents, which control the dimension, can be estimated from above by the following numbers:

\[
q(n) := \limsup_{t \to \infty} \sup_{u(t) \in A} \sup_{(\bar{\theta}_j)_{j=1}^n} \frac{1}{t} \int_0^t \sum_{j=1}^n (L_{u(\tau)} \bar{\theta}_j, \bar{\theta}_j)_\alpha d\tau,
\]

where the first (inner) supremum is taken over all orthonormal families $(\bar{\theta}_j)_{j=1}^n$ with respect to the scalar product (3.1) in $H^1$ and the second (middle) supremum is taken over all trajectories $u(t)$ on the attractor $A$. Then, using (3.2) we obtain

\[
\sum_{j=1}^n (L_{u(t)} \bar{\theta}_j, \bar{\theta}_j)_\alpha = - \sum_{j=1}^n \gamma \|\bar{\theta}_j\|^2_\alpha - \sum_{j=1}^n \int_\Omega (\nabla_{\bar{\theta}_j} \bar{u} \cdot \bar{\theta}_j) dS \leq
\]

\[
\leq -\gamma n + \frac{1}{\sqrt{2}} \|\text{rot} \bar{u}(t)\|_{L^2} \|\rho\|_{L^2},
\]

where

\[
\rho(s) = \sum_{j=1}^n |\bar{\theta}_j(s)|^2,
\]
and where we used the following inequality special for the spherical geometry (and similar to that in the 2D flat case), see [10, Lemma 3.2]:

\[ \sum_{j=1}^{n} \int_{\Omega} (\nabla v_j u(s) \cdot v_j(s))dS \leq 2^{-1/2}\|\rho\|_{L^2}\|\text{rot } u\|_{L^2}, \quad \rho(s) = \sum_{j=1}^{n} |v_j(s)|^2, \quad \text{div } u = 0. \]

We now use estimate (4.9) from the Appendix and obtain

\[ \sum_{j=1}^{n} (L_{u(s)} \bar{\theta}_j, \bar{\theta}_j)_{\alpha} \leq -\gamma n + \frac{1}{2\sqrt{2\pi} \alpha^{1/2}} \|\text{rot } \bar{u}(t)\|_{L^2}. \]

Finally, using (2.5) and (2.6) we arrive at

\[ q(n) \leq -\gamma n + \frac{1}{2\sqrt{2\pi} \alpha^{1/2}} \cdot \left\{ \frac{1}{\gamma} \min \left( \|\text{rot } g\|_{L^2}, \frac{\|g\|_{L^2}}{\sqrt{2\alpha}} \right), \quad \Omega = \mathbb{S}^2, \right. \]

\[ \left. \frac{\|g\|_{L^2}}{\gamma^{1/2}}, \quad \Omega \subsetneq \mathbb{S}^2. \right\]  

It only remains to recall that, according to the general theory, any number \( n^* \) for which \( q(n^*) < 0 \) is an upper bound both for the Hausdorff [2, 15] and the fractal [5, 6] dimension of the global attractor \( \mathcal{A} \). This gives estimate (1.6). □

4. Appendix

In this section we prove the following result that makes it possible to write the estimates for the dimension of the attractor in the explicit form.

**Theorem 4.1.** Let \( \Omega \subseteq \mathbb{S}^2 \) be a (curved) domain on \( \mathbb{S}^2 \). Let a family \( \{v_j\}_{j=1}^{n} \in H^1_0(\Omega) \), \( \text{div } v_j = 0 \), be orthonormal in \( H^1_0(\Omega) \) with respect to the scalar product

\[ m^2(v_i, v_j)_{L^2} + (\text{rot } v_i, \text{rot } v_j)_{L^2} = \delta_{ij}. \]  

Then the function \( \rho(s) := \sum_{j=1}^{n} |v_j(s)|^2 \) satisfies the inequality

\[ \|\rho\|_{L^2} \leq \frac{1}{2\sqrt{\pi}} m^{-1} n^{1/2}. \]  

**Proof.** We first recall the basic facts concerning the spectrum of the scalar Laplace operator \( \Delta = \text{div } \nabla \) on the sphere \( \mathbb{S}^2 \):

\[ -\Delta Y^k_n = n(n+1)Y^k_n, \quad k = 1, \ldots, 2n + 1, \quad n = 0, 1, 2, \ldots \]  

Here the \( Y^k_n \) are the orthonormal real-valued spherical harmonics and each eigenvalue \( \Lambda_n := n(n+1) \) has multiplicity \( 2n + 1 \).
The following identity is essential in what follows: for any \( s \in S^2 \)
\[
\sum_{k=1}^{2n+1} Y_n^k(s)^2 = \frac{2n + 1}{4\pi}.
\] (4.4)

In the vector case identity (4.4) is replaced by its vector analogue:
\[
\sum_{k=1}^{2n+1} |\nabla Y_n^k(s)|^2 = n(n+1) \frac{2n + 1}{4\pi}.
\] (4.5)

Turning to the proof we first consider the whole sphere \( \Omega = S^2 \). Corresponding to the eigenvalue \( \Lambda_n = n(n+1) \), where \( n = 1, 2, \ldots \), there is a family of \( 2n+1 \) orthonormal vector-valued eigenfunctions \( w_n^k(s) \) of the vector Laplacian on the invariant space of divergence free vector-functions, that is, the Stokes operator on \( S^2 \)
\[
w_n^k(s) = (n(n+1))^{-1/2} \nabla Y_n^k(s), \quad -\Delta w_n^k = n(n+1)w_n^k, \quad \text{div} w_n^k = 0;
\] (4.6)
where \( k = 1, \ldots, 2n+1 \), and (4.5) implies the following identity:
\[
\sum_{k=1}^{2n+1} |w_n^k(s)|^2 = \frac{2n + 1}{4\pi}.
\] (4.7)

Let us define two operators
\[
\mathbb{H} = V^{1/2}(m^2 - \Delta)^{-1/2} \Pi, \quad \mathbb{H}^* = \Pi(m^2 - \Delta)^{-1/2} V^{1/2}
\]
acting in \( TS^2 \), where \( V \in L^1 \cap L^\infty \) is a non-negative scalar function and \( \Pi \) is the Helmholz–Leray projection. Then \( \mathbb{K} = \mathbb{H}^* \mathbb{H} \) is a compact self-adjoint operator acting from \( L^2(S^2) \) to \( L^2(S^2) \) and
\[
\text{Tr} \mathbb{K}^2 = \text{Tr} \left( \Pi(m^2 - \Delta)^{-1/2} V(m^2 - \Delta)^{-1/2} \Pi \right)^2 \leq \text{Tr} \left( \Pi(m^2 - \Delta)^{-1} V^2(m^2 - \Delta)^{-1} \Pi \right) = \text{Tr} \left( V^2(m^2 - \Delta)^{-2} \Pi \right),
\]
where we used the Araki–Lieb–Thirring inequality for traces \([1, 18, 19]\):
\[
\text{Tr}(BA^2B)^p \leq \text{Tr}(B^p A^{2p} B^p), \quad p \geq 1,
\]
and the cyclicity property of the trace together with the facts that \( \Pi \) commutes with the Laplacian and that \( \Pi \) is a projection: \( \Pi^2 = \Pi \). Using the basis of orthonormal eigenfunctions of the Laplacian (4.6) along with (4.7)
in view of the key estimate (4.10) proved below we find that

$$\text{Tr } K^2 \leq \text{Tr } \left( V^2 (m^2 - \Delta)^{-2} \right) =$$

$$= \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{2n + 1}{(m^2 + n(n + 1))^2} \int_{S^2} V^2(s) dS \leq \frac{1}{4\pi} \frac{1}{m^2} \| V \|_{L^2}^2.$$

We can now argue as in [17]. We observe that

$$\int_{S^2} \rho(s) V(s) dS = \sum_{i=1}^{n} \| H\psi_i \|_{L^2}^2,$$

where

$$\psi_j = (m^2 - \Delta)^{1/2} v_j, \quad j = 1, \ldots, n.$$

Next, in view of (4.1) the $\psi_j$'s are orthonormal in $L^2$ and in view of the variational principle

$$\sum_{i=1}^{n} \| H\psi_i \|_{L^2}^2 = \sum_{i=1}^{n} (K\psi_i, \psi_i) \leq \sum_{i=1}^{n} \lambda_i,$$

where $\lambda_i$ are the eigenvalues of the operator $K$. Therefore

$$\int_{S^2} \rho(s) V(s) dS \leq \sum_{i=1}^{n} \lambda_i \leq n^{1/2} \left( \sum_{i=1}^{n} \lambda_i^2 \right)^{1/2} \leq$$

$$\leq n^{1/2} \left( \text{Tr } K^2 \right)^{1/2} \leq \frac{n^{1/2} m^{-1}}{2 \sqrt{\pi}} \| V \|_{L^2}.$$

Setting $V(x) := \rho(x)$ we complete the proof of (4.2) for $\Omega = S^2$.

Finally, if $\Omega \subsetneq S^2$ is a proper domain on $S^2$, we extend by zero the vector functions $v_j$ outside $\Omega$ and denote the results by $\tilde{v}_j$, so that $\tilde{v}_j \in H^1(S^2)$ and $\text{div } \tilde{v}_j = 0$. We further set $\tilde{\rho}(x) := \sum_{j=1}^{n} |\tilde{v}_j(x)|^2$. Then setting $\tilde{\psi}_i := (m^2 - \Delta)^{1/2} \tilde{v}_i$, we see that the system $\{ \tilde{\psi}_j \}_{j=1}^{n}$ is orthonormal in $L^2(S^2)$ and $\text{div } \tilde{\psi}_j = 0$. Since clearly $\| \tilde{\rho} \|_{L^2(S^2)} = \| \rho \|_{L^2(\Omega)}$, the proof of the estimate (4.2) reduces to the case of the whole sphere and therefore is complete. $\Box$

A word for word translation of the above proof gives a similar inequality in the scalar case, the only difference being that we have to consider the case of the whole sphere and impose the zero mean condition.

**Theorem 4.2.** Let a family $\{ \varphi_j \}_{j=1}^{n} \in H^1(S^2)$, $\int_{S^2} \varphi_j(s) dS = 0$ be orthonormal with respect to the scalar product

$$m^2 (\varphi_i, \varphi_j)_{L^2} + (\nabla \varphi_i, \nabla \varphi_j)_{L^2} = \delta_{ij}.$$
Then the function \( \rho(s) := \sum_{j=1}^{n} |\varphi_j(s)|^2 \) satisfies the inequality
\[
\|\rho\|_{L^2} \leq \frac{1}{2\sqrt{\pi}} m^{-1/2} n^{1/2}.
\]

**Remark 4.1.** The difference in the formulations of Theorems 4.1 and 4.2 is due to the fact that the vector Laplacian is positive \( -\Delta \geq 2I \), while the scalar Laplacian \( -\Delta \geq 0 \) is just non-negative.

In terms of the scalar product (4.8) estimate (4.2) is as follows.

**Corollary 4.1.** Under the assumptions of Theorem 4.1 let the family \( \{v_j\}_{j=1}^{n} \), \( \text{div} v_j = 0 \), be orthonormal with respect to
\[
(v_i, v_j)_{L^2} + \alpha (\text{rot} v_i, \text{rot} v_j)_{L^2} = \delta_{ij}.
\]
Then the function \( \rho(s) = \sum_{j=1}^{n} |v_j(s)|^2 \) satisfies
\[
\|\rho\|_{L^2} \leq \frac{1}{2\sqrt{\pi}} \alpha^{1/2} n^{1/2}.
\]

**Proposition 4.1.** The following inequality holds for \( m > 0 \)
\[
F(m) := m^2 \sum_{n=1}^{\infty} \frac{2n+1}{(m^2 + n(n+1))^2} < 1.
\]

**Proof.** We write \( F(m) \) as follows
\[
F(m) = \frac{1}{m^2} \sum_{n=1}^{\infty} (2n+1) f\left( \frac{n(n+1)}{m^2} \right), \quad f(t) = \frac{1}{(1+t)^2}.
\]

The following asymptotic expansion holds for \( F(m) \) (see [16, Lemma 3.5])
\[
F(m) = \int_{0}^{\infty} f(t) dt - \frac{2}{m^2 3} f(0) + O(m^{-4}), \quad \text{as } m \to \infty.
\]

which in view of \( f(0) = 1 \), and \( \int_{0}^{\infty} f(x) dx = 1 \) gives that
\[
F(m) = 1 - \frac{1}{m^2 3} + O(m^{-4}).
\]

This shows that (4.10) holds for all \( m \in [m_0, \infty) \), where \( m_0 \) is sufficiently large. A general method for proving this type of inequalities is to somehow specify the value of \( m_0 \) and show that (4.10) also holds on \( m \in [0, m_0] \) by a reliable computer calculation. However, in this specific case we can prove inequality (4.10) completely rigorously, and for this purpose we use a refinement of the method proposed in [9, 11].
Let
\[ a_1 = 0, \quad a_n = a_n(m) := \frac{(n - 1)n}{m^2}, \quad n = 2, \ldots. \]

Then
\[ \sum_{n=1}^{\infty} n f(n(n + 1)/m^2) = \frac{1}{2} \sum_{n=1}^{\infty} f(a_{n+1})(a_{n+1} - a_n), \]
\[ m^{-2} \sum_{n=1}^{\infty} (n + 1) f(n(n + 1)/m^2) = \frac{1}{2} \sum_{n=1}^{\infty} f(a_{n+1})(a_{n+2} - a_{n+1}). \]

Therefore
\[ F(m) = \frac{1}{2} f(a_2)(a_2 - a_1) + \sum_{n=2}^{\infty} \frac{f(a_n) + f(a_{n+1})}{2} (a_{n+1} - a_n), \]
and inequality (4.10) is equivalent to
\[ \frac{1}{2} f(a_2)(a_2 - a_1) + \sum_{n=2}^{\infty} \frac{f(a_n) + f(a_{n+1})}{2} (a_{n+1} - a_n) < \]
\[ \sum_{n=1}^{\infty} \int_{a_n}^{a_{n+1}} f(x)dx = \int_{a_1}^{a_2} f(x)dx + \sum_{n=2}^{\infty} \int_{a_n}^{a_{n+1}} f(x)dx = 1, \]
or, equivalently, to
\[ \int_{a_1}^{a_2} f(x)dx - \frac{1}{2} f(a_2)(a_2 - a_1) > \]
\[ \sum_{n=2}^{\infty} \left( \frac{f(a_n) + f(a_{n+1})}{2} (a_{n+1} - a_n) - \int_{a_n}^{a_{n+1}} f(x)dx \right) =: \sum_{n=2}^{\infty} R_n(m). \]

Next, for \( f(x) = 1/(1 + x)^2 \) we have
\[ \int_{a_1}^{a_2} f(x)dx - \frac{1}{2} f(a_2)(a_2 - a_1) = \frac{m^2 + 4}{(m^2 + 2)^2}, \]
and
\[ \frac{f(a) + f(b)}{2} (b - a) - \int_{a}^{b} f(x)dx = \frac{1}{2} \frac{(b - a)^3}{(1 + a)^2(1 + b)^2}. \]

Therefore
\[ R_n(m) = \frac{4}{m^2} \frac{1}{m} \frac{(n/m)^3}{((1 + n(n - 1)/m^2) (1 + n(n + 1)/m^2))^2}. \]
and we see that inequality (4.10) holds if and only if
\[
m^2 \left[ \frac{m^2 + 4}{(m^2 + 2)^2} + R_1(m) \right] = m^2 \left[ \frac{m^2 + 4}{(m^2 + 2)^2} + \frac{4}{m^2(m^2 + 2)} \right] = 1 > \frac{1}{4} \sum_{n=1}^{\infty} \frac{(n/m)^3}{((1 + n(n-1)/m^2)^2(1 + n(n+1)/m^2)^2)}.
\]
that is,
\[
R(m) := \frac{1}{m} \sum_{n=1}^{\infty} \frac{(n/m)^3}{((1 + n(n-1)/m^2)^2(1 + n(n+1)/m^2)^2)} < \frac{1}{4}.
\]
Next, we find a lower bound for the denominator in terms of a function depending on \(n/m\). Taking into account that
\[
F'(m) = 2m \sum_{n=1}^{\infty} \frac{(2n + 1)(n(n+1) - m^2)}{(m^2 + n(n+1))^2} > 0
\]
for \(m \in [0, \sqrt{2}]\), we see that it suffices to prove inequality (4.10) for \(m \in [\sqrt{2}, \infty)\).

The largest constant \(k\) in the inequality
\[
\frac{((m^2 + n(n-1))^2(m^2 + n(n+1))^2}{m^8} \geq k \left( 1 + \frac{n^2}{m^2} \right)^4
\]
holding for \(m^2 \geq 2\) and \(n \in \mathbb{N}\) satisfies
\[
\sqrt{k} = 1 - \max_{m^2 \geq 2, \ n \in \mathbb{N}} \frac{n^2}{(m^2 + n^2)^2} = 1 - \max_{n \in \mathbb{N}} \frac{n^2}{(2 + n^2)^2} = \frac{8}{9}.
\]
This gives that for \(m \geq \sqrt{2}\)
\[
R(m) < \frac{81}{64} m \sum_{n=1}^{\infty} g(n/m), \quad g(x) = \frac{x^3}{(1 + x^2)^4}.
\]
The function \(g(x)\) has a global maximum at \(x_0 = \left( \frac{3}{7} \right)^{1/2}\) and is decreasing for \(x > x_0\). Therefore it is geometrically clear that for all \(m > 0\)
\[
\frac{1}{m} \sum_{n=1}^{\infty} g(n/m) < x_0 g(x_0) + \int_{x_0}^{\infty} g(x) dx = \frac{225}{4096} + \frac{175}{3072} = \frac{1375}{12288}.
\]
The proof is now complete, since for \(m \geq \sqrt{2}\)
\[
R(m) < \frac{81}{64} m \sum_{n=1}^{\infty} g(n/m) < \frac{81}{64} \frac{1375}{12288} = \frac{37125}{262144} = 0.14162 < \frac{1}{4}.
\]


REFERENCES

[1] H. Araki, On an inequality of Lieb and Thirring. Lett. Math. Phys., 19:2 (1990), 167–170.

[2] A. Babin and M. Vishik, Attractors of Evolution Equations. Studies in Mathematics and its Applications, vol 25. North-Holland Publishing Co., Amsterdam, 1992.

[3] J. Bardina, J. Ferziger, and W. Reynolds, Improved subgrid scale models for large eddy simulation, in Proceedings of the 13th AIAA Conference on Fluid and Plasma Dynamics, (1980).

[4] Y. Cao, E. M. Lunasin, and E.S. Titi, Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models. Commun. Math. Sci. 4:4 (2006), 823–848.

[5] V. V. Chepyzhov and A. A. Ilyin, A note on the fractal dimension of attractors of dissipative dynamical systems. Nonlinear Anal. 44 (2001), 811–819.

[6] V. V. Chepyzhov and A. A. Ilyin, On the fractal dimension of invariant sets; applications to Navier–Stokes equations. Discrete Contin. Dyn. Syst. 10: 1-2 (2004), 117–135.

[7] B.A. Dubrovin, S.P. Novikov, and A.T. Fomenko, Modern Geometry. Methods and Applications: Part I. Nauka, Moscow, 1979; English translation, Graduate texts in mathematics, 93, Springer, New York, 1984.

[8] A.A. Ilyin, The Navier–Stokes and Euler equations on two dimensional closed manifolds. Mat. Sbornik 181:4 (1990), 521–539; English transl. in Mathematics of the USSR-Sbornik 69:2 (1991).

[9] A.A. Ilyin, Best constants in Sobolev inequalities on the sphere and in Euclidean space. J. London Math. Soc.(2) 59 (1999), 263-286.

[10] A. A. Ilyin, A. Miranville, and E. S. Titi, Small viscosity sharp estimates for the global attractor of the 2-D damped-driven Navier-Stokes equations. Commun. Math. Sci. 2 (2004), 403–426.

[11] A.A. Ilyin, Lieb–Thirring inequalities on some manifolds. J. Spectr. Theory 2 (2012), 57–78.

[12] A.A. Ilyin and S.V. Zelik, Sharp dimension estimates of the attractor of the damped 2D Euler–Bardina equations. In book: EMS Series of Congress Reports Vol. 18. Partial Differential Equations, Spectral Theory, and Mathematical Physics, EMS Press, Berlin, 2021, p. 209–229.

[13] A.A. Ilyin, A.G. Kostianko, and S.V. Zelik, Sharp upper and lower bounds of the attractor dimension for 3D damped Euler–Bardina equations. arXiv: 2106.09077.

[14] V. K. Kalantarov and E. S. Titi, Global attractors and determining modes for the 3D Navier–Stokes–Voight equations, Chin. Ann. Math. 30:6 (2009), 697–714.

[15] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, 2nd ed. Springer-Verlag, New York 1997.

[16] S.V. Zelik and A.A. Ilyin, Green’s function asymptotics and sharp interpolation inequalities. Uspekhi Mat. Nauk 69:2 (2014), 23–76; English transl. in Russian Math. Surveys 69:2 (2014).
[17] E. H. Lieb, An $L^p$ bound for the Riesz and Bessel potentials of orthonormal functions, *J. Func. Anal.* 51 (1983), 159–165.

[18] E. Lieb and W. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities, Studies in Mathematical Physics. Essays in honor of Valentine Bargmann, Princeton University Press, Princeton NJ, 269–303 (1976).

[19] B. Simon, *Trace Ideals and Their Applications*, 2nd ed. Amer. Math. Soc., Providence RI, 2005.

*Email address: ilyin@keldysh.ru*
*Email address: a.kostianko@imperial.ac.uk*
*Email address: s.zelik@surrey.ac.uk*

1 Keldysh Institute of Applied Mathematics, Moscow, Russia

2 University of Surrey, Department of Mathematics, Guildford, GU2 7XH, United Kingdom.

3 School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, P.R. China

4 Imperial College, London SW7 2AZ, United Kingdom.