The Effect of ‘Probability Skew’ in Bell-test Experiments

Dale Hodgson

Abstract

We consider typical experiments that use Bell-inequalities to test local-realist theories of quantum mechanics and gain insight into how certain results can be obtained. We see that results against local-realism arise from some ‘quantum skew’ of the correlation between entangled qubit pairs. Furthermore, we find some conditions necessary for a conclusion against local-realism. Finally we show that the problem of ‘no-signalling’ that arises in these experiments cannot be reduced to arbitrary experimental accuracy.

Introduction

The testing of Bell-inequalities is the favoured method of investigating the relationship between local-realism and quantum mechanics. Significant results have been achieved in this area, but further work is still ongoing.

Realism is the assumption that the physical universe exists (has some ‘element of reality’) independent of any measurement or observation. Locality is the assumption that physical influences cannot travel faster than the speed of light — the cornerstone of special relativity. In their 1935 paper [5], Einstein, Podolsky and Rosen noted the disparity arising between the assumptions of local-realism and the behaviour of entangled quantum objects. John Bell’s later work on that idea [3] lead to experimentally testable results, now known as Bell-inequalities, that can test whether certain quantum mechanical systems obey the ideas of local-realism.

A popular type of Bell-test experiment involves two entangled qubits being measured under space-like separation sufficient to exclude relativistic interaction and noting the correlation of the two sets of results. The correlations become a test value which, under the assumption of local-realism, has a theoretical bound. Hence such experiments can verify or contradict any local-realist theory for the behaviour of quantum objects.

The testable theoretical bound arises from consideration of a ‘correlation coefficient’. If \( a \) is a measurement examining the state of the first qubit, and likewise \( b \) for the second, then the correlation coefficient for the joint outcome
ab is

\[ E_{ab} = \text{probability of correlation} - \text{probability of anti-correlation} \]

realised experimentally as: \( \frac{n(\text{corr})_{ab} - n(\text{anti-corr})_{ab}}{n(\text{ab trials})} \).

Bell’s work shows that if some ‘hidden variable’ that could not be directly measured exists (the assumption of reality), then the inequality:

\[ E_{ac} - E_{ba} - E_{bc} \leq 1 \] (1)

should hold (in a set-up with perfect anti-correlation).

The later work of Clauser, Horne, Shimony and Holt [4] generalised this to allow for non-perfect anti-correlation in what is now known as the CHSH inequality. With two measurements prepared for each qubit, chosen such that measurement 1 is precisely the complement of measurement 0 (e.g. measures of spin in perpendicular directions).

\[ E_{00} + E_{01} + E_{10} - E_{11} \leq 2 \] (2)

Recent work by Hensen et al. [6] [7] have shown statistically significant violation of the CHSH-inequality, suggesting to us that there is no local-realist theory that completely describes quantum mechanics.

A Bell-test experiment using CHSH

A general Bell-test experiment produces results with eight independent variables: \( N \) the total number of trials, \( a, b, c \) the number of trials with setting configurations 00, 01, 10, respectively, and the number of correlated results under each setting \( (n_{00}, n_{01}, n_{10}, n_{11}) \). (See Table 1.)

| Setting | 00 | 01 | 10 | 11 |
|---------|----|----|----|----|
| No. of Trials | \( a \) | \( b \) | \( c \) | \( d = N - a - b - c \) |
| No. of Correlated results | \( n_{00} \) | \( n_{01} \) | \( n_{10} \) | \( n_{11} \) |
| No. of Anti-correlated results | \( a - n_{00} \) | \( b - n_{01} \) | \( c - n_{10} \) | \( d - n_{11} \) |
| \( p(\text{corr}) \) | \( \frac{n_{00}}{a} \) | \( \frac{n_{01}}{b} \) | \( \frac{n_{10}}{c} \) | \( \frac{n_{11}}{d} \) |
| \( p(\text{anti-corr}) \) | \( \frac{1 - n_{00}}{a - 1} \) | \( \frac{1 - n_{01}}{b - 1} \) | \( \frac{1 - n_{10}}{c - 1} \) | \( \frac{1 - n_{11}}{d - 1} \) |

\[ E = p(\text{corr}) - p(\text{anti-corr}) \]

| 2 \( \frac{n_{00}}{a} \) - 1 | 2 \( \frac{n_{01}}{b} \) - 1 | 2 \( \frac{n_{10}}{c} \) - 1 | 2 \( \frac{n_{11}}{d} \) - 1 |

Giving test value:

\[ S = E_{00} + E_{01} + E_{10} - E_{11} = 2 \left( \frac{n_{00}}{a} + \frac{n_{01}}{b} + \frac{n_{10}}{c} - \frac{n_{11}}{d} - 1 \right) \]
Local-realism predicts $S \leq 2$, so an experiment disproving local-realist explanations of quantum entanglement will violate the CHSH-inequality — showing $S > 2$.

**The Effect of ‘Quantum probability skew’**

Suppose that the probabilities of correlation under each experimental setting are equal (to $P$ say). Then $S = 2(2P - 1)$; in which case $S > 2 \iff P > 1$. Hence uniformity of these correlations will never violate the CHSH-inequality — the nature of quantum mechanics lends some ‘unnatural skew’ to these results.

Similarly, we may examine Bell’s original inequality (\(\Pi\)). Let $n_{ac}$ denote the number of correlated results under setting $ac$, $N_{ac}$ the total number of trials under setting $ac$, $n_{ba}$ the number of correlated results under measurement setting $ba$ etc. Then correlation values take the form

$$E_{ac} = \frac{n_{ac} - (N_{ac} - n_{ac})}{N_{ac}} = 2n_{ac}/N_{ac} - 1$$

and Bell’s inequality asserts that a local-realist theory implies the following inequality:

$$\frac{n_{ac}}{N_{ac}} - \frac{n_{ba}}{N_{ba}} - \frac{n_{bc}}{N_{bc}} \leq 0 .$$

Again, if these three fractions (each equal to the probability of correlation under a given setting) are all equal, then the inequality will always be satisfied.

**Approximating this Experiment for Large Number of Trials**

Since the experimental settings are determined randomly, we would expect that the number of trials under each setting would tend to a uniform distribution for suitable large $N$. Supposing this is the case, let $a = b = c = \frac{N}{4}$, then we get

$$S = \frac{8}{N} \left( n_a + n_b + n_c - n_d - \frac{N}{4} \right) .$$

Let us denote the range of correlation counts $\sigma$.

$$\sigma := \text{range} \{ n_a, n_b, n_c, n_d \} = n_{\text{max}} - n_{\text{min}}$$

Where $n_{\text{max}} = \max \{ n_a, n_b, n_c, n_d \}$ and $n_{\text{min}} = \min \{ n_a, n_b, n_c, n_d \}$. Let us only consider $\sigma \geq 1$ (for if $n_{\text{max}} - n_{\text{min}} = 0$, then we will always get $S < 2$). Furthermore, we must have that $n_a + n_b + n_c + n_d \leq N$; consequently, it is clear that we must have $n_{\text{min}} \leq \frac{N}{4}$.

Let us denote $(n_a + n_b + n_c - n_d)$ by $S'$. Then we will observe violation of the CHSH inequality $(S > 2)$ if and only if $S' > \frac{N}{2}$. Note also that $S'$ is bounded above and below by $S'_{\text{max}} = 3n_{\text{max}} - n_{\text{min}} = 2n_{\text{min}} + 3\sigma$ and $S'_{\text{min}} = 3n_{\text{min}} - n_{\text{max}} = 2n_{\text{min}} - \sigma$. 

3
Violation of the CHSH inequality is only possible if $S'_\text{max} > \frac{N}{2}$, that is to say: $2n_{\text{min}} + 3\sigma > \frac{N}{2}$ is necessary (but not sufficient) for violation. If we desire violation of a certain magnitude $\delta$, then we may say that $S'_\text{max} > \frac{N}{2} + \delta \Leftrightarrow 2n_{\text{min}} + 3\sigma > \frac{N}{2} + \delta$ is necessary. From this we may see that for a violation of magnitude $\delta$, the ‘quantum skew’ (range of correlation counts) $\sigma$ must be strictly greater than $\frac{\delta}{3}$:

$$\sigma > \frac{N}{6} + \frac{\delta}{3} - \frac{2}{3}n_{\text{min}} \quad \Rightarrow \quad \sigma > \frac{\delta}{3}.$$ 

If we desire $S > 2 + \Delta$, then we are in fact asking that $S' > \frac{N}{2} + \frac{N\Delta}{8}$, so $\delta = \frac{N\Delta}{8}$; hence $\sigma > \frac{N\Delta}{24}$ is necessary.

The No-signalling Problem

A crucial factor in these experiments is that the two measurements must be made independent of one another. Khrennikov’s analysis of the first Hensen et al. experiments [1] noted that the independence of the measurements could be verified by analysing marginal probabilities available from the data. Bednorz [2] also reviewed recent experiments in a similar way.

That is to say, if the measurements are truly independent, then we should observe:

$$p(\text{corr}|s_1 = 0) = p(\text{corr}|s_1 = 0 \land s_2 = 0) = p(\text{corr}|s_1 = 0 \land s_2 = 1),$$

and similarly for other settings.

In the experimental set-up we are considering, all such probability differences look like

$$p(\text{corr}|0s_b) - p(\text{corr}|00) = \frac{n_a + n_b}{a+b} - \frac{n_a}{a} = \frac{an_b - bn_a}{a(a+b)}.$$

The family of all these equations can be described by $\frac{an_\beta - bn_\alpha}{\alpha(\alpha + \beta)}$ and $\frac{\beta n_\alpha - \alpha n_\beta}{\alpha(\alpha + \beta)}$, where $\alpha, \beta \in \{a, b, c, d\}$ and $\alpha \neq \beta$.

We can say that we have achieved experimental results with a good no-signalling accuracy if all these probability differences are small. All such probabilities will be ‘small’ (to some $\varepsilon$), if

$$\left| \frac{\alpha n_\beta - \beta n_\alpha}{\alpha + \beta} \right| < \varepsilon \cdot \min \{\alpha, \beta\}.$$ 

In the case of uniform experimental test settings considered above $(a = b = c = \frac{N}{4})$, this condition becomes: $|n_\alpha - n_\beta| < \frac{\varepsilon N}{2} \forall \alpha, \beta$. This is satisfied if

$$n_{\text{max}} - n_{\text{min}} < \frac{\varepsilon N}{2}.$$
So $\sigma < \frac{\varepsilon N}{2}$ is necessary to achieve results with a good no-signalling accuracy. Since $\sigma \geq 1$ is necessary for a CHSH-inequality violation, $N > \frac{\varepsilon}{\sigma}$ trials must be performed if an experiment violating the CHSH-inequality is to achieve no-signalling tolerance of accuracy $\varepsilon$.

Combining this with our knowledge that $\sigma > \frac{N\Delta}{24}$ is necessary for violation of the CHSH inequality to magnitude $\Delta$, we conclude that

$$\varepsilon > \frac{\Delta}{12}$$

is a limit on our no-signalling tolerance — independent of the number of trials we perform.

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