Angular dependence of the upper critical induction of clean $s$- and $d_{x^2−y^2}$-wave superconductors with self-consistent ellipsoidal effective mass and Zeeman anisotropies

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Abstract
We employ the Schrödinger-Dirac method generalized to an ellipsoidal effective mass anisotropy in order to treat the spin and orbital effective mass anisotropies self consistently, which is important when Pauli-limiting effects on the upper critical field characteristic of singlet superconductivity are present. By employing the Klemm–Clem transformations to map the equations of motion into isotropic form, we then calculate the upper critical magnetic induction $B_{c2}(\theta, \phi, T)$ at arbitrary directions and temperatures $T$ for isotropic $s$-wave and for anisotropic $d_{x^2−y^2}$-wave superconducting order parameters. As for anisotropic $s$-wave superconductors, $B_{c2}$ is largest in the direction of the lowest effective mass, and is proportional to the universal orientation factor $\alpha(\theta, \phi)$. However, for $d_{x^2−y^2}$-wave pairing and vanishing planar effective mass anisotropy, $B_{c2}(\pi/2, \phi, T)$ exhibits a four-fold azimuthal pattern with $C_4$ symmetry the maxima of which are along the crystal axes just below the transition temperature $T_c$, but these maxima are rotated by $\pi/4$ about the $z$ axis as $T$ is lowered to 0. However for $d_{x^2−y^2}$-wave pairing with weak planar effective mass anisotropy, $B_{c2}(\pi/2, \phi, T)$ exhibits a two-fold pattern with $C_2$ symmetry for all $T \leq T_c$, which also rotates by $\pi/4$ about the $z$ axis as $T$ is lowered to 0. These low planar effective mass anisotropy cases provide a new method to distinguish $s$-wave and $d_{x^2−y^2}$-wave pairing symmetries in clean unconventional superconductors.

Keywords: anisotropic Fermi Surface, Zeeman energy, the upper critical field, angle dependence

(Some figures may appear in colour only in the online journal)
1. Introduction

Many unconventional superconductors such as the heavy-fermion superconductors, ferromagnetic superconductors, high $T_c$ copper oxide superconductors, the iron-based superconductors, and magic-angle graphene superlattices have been found [1–3], and enormous efforts also have been triggered by the motivations to find higher transition temperature $T_c$ and higher upper critical field $H_{c2}$ superconductors, and to understand the nature and physical origin of the superconductivity in these materials. Recently, the then-record high $T_c$ of 203 K was obtained in nominal H$_2$S under high pressure [4], exhibiting a conventional isotope effect. Since then, a variety of hydrides have been found under high pressure to be superconducting at least up to 243 K [5, 6]. Since critical field measurements on those rather conventional superconductors have not yet been made at low temperature $T_c$, most of the presently more interesting and/or accessible materials have $T_c$ values below 100 K. These unconventional superconductors usually have strongly anisotropic structures and upper critical fields that exhibit obvious and unique orientational dependencies. Measurement of the upper critical induction $B_{c2} = \mu_0 H_{c2}$ in non-magnetic systems, where $\mu_0$ is the magnetic permeability of vacuum, is an important tool to understand various superconductors, since it is a thermodynamic measurement from which coherence lengths, anisotropy parameters, and pair-breaking mechanisms can be extracted.

In unconventional superconductors, $B_{c2}$ usually exhibits obvious and unique orientational dependencies. One of the reasons is that the Fermi surface (FS) is not always a single closed surface, and sometimes more than one electronic orbital can contribute to the overall FS, so that multi-band effects should be taken into account [7–9]. Even FS reconstruction can occur in hole-doped cuprate superconductors [10], which could be due to the onset of a charge- or spin-density wave, as are known to occur respectively in the transition metal dichalcogenides and certain organic layered superconductors [11–13]. But it is the simplest and most convenient approach at present to theoretically assume that the FS is a sphere for isotropic superconductors, ellipsoidal for anisotropic superconductors or rippled cylindrical for layered superconductors [13–15]. Here we approximate the FS structure as an ellipsoid with three different effective masses and do not consider either multi-band effects or FS reconstruction in this work [16].

In addition to FS anisotropy, the Pauli paramagnetic (or Zeeman) energy can also affect $B_{c2}$. Although for triplet pair-spin superconductors containing all three spin states, there is no paramagnetic suppression of the superconductivity if the external field is orthogonal to the direction along which the Cooper pairs have zero total spin, a magnetic field parallel to the direction of zero total spin will be pair breaking [17–23]. But this is not the case for a singlet pair state. It always maintains the singlet state, no matter which direction the state is observed, and $B_{c2}$ can be bounded by the paramagnetic effect for all field directions, provided that the magnitude of $\frac{\partial B_{c2}}{\partial T}$ at $T_c$ is sufficiently large and that counteracting effects such as spin fluctuations and spin-orbit scattering are sufficiently weak. Therefore, it is necessary to add the paramagnetic term to study singlet superconductors. However, many workers that developed theories to fit the experimental data have neglected this term or have added it by hand, without making it consistent with the intrinsic effective mass anisotropy appropriate for the material under study. In this work, the most important point is that we study the effects of the self-consistently anisotropic Zeeman energy and the orbital effective mass anisotropy from the anisotropic non-relativistic limit, that we derived previously [16].

Generally, there exist two distinct ways to induce pair-breaking of singlet-pair superconductors by an applied magnetic field, i.e. orbital [24] and spin-paramagnetic effects [25, 26]. The actual $H_{c2}$ of real materials depends upon both of these effects, both of which can be greatly modified by the same effective mass anisotropy parameters. For an isotropic superconductor, the relative importance of the orbital and spin-paramagnetic effects is described by the Maki parameter [27], $\alpha_M = \sqrt{2H_{c2}^0(0)/H_{c2}^P(0)}$. Since $\alpha_M$ is known to be on the order of $\Delta(0)/E_F$, where $\Delta(0)$ is one-half of the isotropic superconducting gap in its energy spectrum at $T = 0$, and $E_F$ is the Fermi energy, so that $\alpha_M \ll 1$, indicating that the paramagnetic effect is negligibly small in most superconductors. However, in layered superconductors, or in heavy fermion superconductors that have quite small $E_F$ values, $\alpha_M$ can be larger than unity. Or in ‘Ising’ layered superconductors, for which $B_{c2}$ parallel to the layers can well exceed the Pauli paramagnetic limit for an isotropic superconductor, the paired spins are less susceptible to Pauli pair-breaking due to the effective internal Zeeman-like field [28]. But in nearly all of the calculations to date on superconductors with anisotropic FSs, the Zeeman energy was assumed to be completely independent of the FS anisotropy parameters. As shown in the following, such assumptions violate the laws of special relativity, and must be completely revised for anisotropic materials.

We note that beyond those two primary pair-breaking mechanisms, $B_{c2}$ is affected by many additional factors: magnetic structure that may coexist or interfere with superconductivity [18], spin-orbit scattering [14, 29], flux pinning, the possibility of a Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) phase [30–34], structural asymmetry [35–37], and the spin and orbital symmetries of the superconducting order parameter. For example, in $p$-wave superconductors, the triplet pair-spin configuration can allow for strong violations of the Pauli limit [17–23]. In addition, in the case of URhGe, there are profound differences in the temperature dependencies of $B_{c2}$ for the applied field along the three crystallographic directions, and there is a reentrant phase for a yet larger applied field in one direction [18–21]. Much more subtle order parameter anisotropy effects can still arise for singlet spin superconductors, especially if strong spin-orbit scattering [14, 29] or Ising single-spin direction locking [28] are present.

In addition, as is well known in the high-$T_c$ community, there has been an ongoing battle regarding the orbital symmetry of the superconducting order parameter in the cuprates.
The difficulties in such determinations are complicated by the ubiquitous presence of a ‘pseudogap’, which is most likely either a charge-density wave (CDW), as occurs, even with a node, in the hole bands of 2H-TaS2, one of the transition metal dichalcogenides [12, 13, 49], or a spin-density wave (SDW), as can occur in the organic layered superconductors and the pnictides, or possibly a combination of both [11]. In the case of 2H-TaS2, the nodal CDW disappears upon intercalation, so it requires more than one layer to exist [50]. Recently, a single crystal of Bi2Sr3CaCu2O8+δ (Bi2212) was cleaved in vacuum and covered with a monolayer of CuO2 to protect the surface against reacting with the remaining atoms, etc. in the vacuum, before scanning tunnelling microscopy (STM) studies were performed [51]. Those authors found that the Bi2212 pseudogap and the superconducting order parameter separate into rather distinct spatial nanodomains. The pseudogap was found by STM studies to have a node that persists up to its onset well above the superconducting Tc, but the STM-observed superconducting gap was nodeless and essentially isotropic, with a flat STM bottom and large coherence peaks, that disappears at or near to Tc [51]. We note that the Postech and Harvard groups were unaware of the nodal ‘pseudogap’ (most likely a CDW) in underdoped and optimally doped Bi2212 [52], but which disappears in the overdoped region of the Bi2212 phase diagram, as was studied in the original c-axis twist experiment on that compound [41]. This nodal CDW has confused all previous angle resolved photoemission experiments that average over an area of about 1 mm2, which covers thousands of such nanodomains [38]. Since either CDWs or SDWs can open up gaps in the FS, changing its topology from an anisotropic spheroid to that containing nodal points, lines, or areas, we therefore only consider Bc2 calculations for anisotropic systems in regions of their compositional phase diagrams in which CDWs and SDWs are absent.

In addition, the narrow linewidth of the emission averaged over all directions at 2.4 THz from a disk-shaped device of Bi2212 is difficult to explain unless the quasiparticle energy has a minimum gap of at least 9.8 meV [53, 54]. Hence, these new experiments bring the published interpretations of many previous experiments into serious question, and new and repeated tests of order parameter symmetry on better quality samples are strongly warranted. Here we show that for clean anisotropic type-II superconductors, the angular and temperature dependence of the upper critical field Hc2 can provide additional evidence for the orbital symmetry of the superconducting order parameter.

Recently, the number of clean layered and two-dimensional (2D) superconductors has increased substantially [2, 3]. In addition, a model based upon isotropic in-plane conduction and coherent Josephson c-axis tunnelling was studied for the d2−x−y wave order parameter neglecting non-local effects [15], as in previous models of layered superconductors [13, 14]. Those authors found a very interesting azimuthal φ angular dependence of Bc2(π/2, φ, T), where Bc2(θ, φ, T) is the upper critical induction for the applied magnetic field \( \mathbf{H} = H(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi) \). Just below \( T_c \), \( B_{c2}(\pi/2, \phi, T) \) has a four-fold symmetry with maxima along the crystal axes and minima half-way between the maxima, but for \( 0 < T < T' < T_c \), the maxima and minima are rotated about the c-axis by \( \pi/4 \) from their former positions. As shown in the following, our calculations based upon an ellipsoidal FS, confirm their results. But that model was only marginally applicable to the cuprate YBa2Cu3O7−δ, for which \( B_{c2}(\theta, \phi, T) \) studies have been made or attempted [55–58], as detwinned samples have an in-plane effective mass anisotropy of about a factor of 2, depending upon the precise oxygen stoichiometry, which in our model could be large enough to smear out that four-fold anisotropy.

More recent interesting \( B_{c2}(\theta, \phi, T) \) measurements on CeCu2Si2 were also made [59]. Since those pioneering measurements of \( B_{c2}(\theta, \phi, T) \), some of the world’s high magnetic field laboratories have improved their capabilities, and can now operate at sufficiently low T ranges with dc applied fields as large as 45 T, so that accurate experimental measurements of \( B_{c2}(\theta, \phi, T) \) can now be obtained well below \( T_c \), in materials of interest. Since a complete consensus on c-axis Josephson tunnelling experiments on hole-doped cuprates has not yet been attained, and the in-plane Josephson tunnelling experiments [39] have not yet been successfully explained in a manner consistent with the majority of the c-axis Josephson experiments, other experiments to test the orbital symmetry of the superconducting order parameter in unconventional superconductors are warranted. But since it was recently shown that a proper relativistically consistent treatment of the orbital and spin effective mass anisotropies can be made [16], such experiments should also be relativistically consistent with those properties.

This paper is organized as follows. In section 2, the non-relativistic limit of the Hamiltonian in an orthorhombically anisotropic medium is presented. The superconducting pairing interactions are also presented, and the general gap functions for all reduced temperatures \( t = T/T_c \) for s-wave and \( d_{x^2−y^2} \)-wave are derived. The numerical results based on the equations presented in section 2 are presented and discussed in section 3. Details of the effects of the self-consistent orbital effective mass and Zeeman energy anisotropies combined with the two order parameter anisotropies are presented and discussed. Finally, the main results are presented in section 4.

### 2. The model

#### 2.1. The transformation

We begin with the non-relativistic limit of the kinetic energy of an electron in an orthorhombically:

\[
H_0 = \sum_{\mu=1}^{3} \frac{1}{\sqrt{2m_{\mu}}} \sigma_{\mu} (-i \partial_{\mu} + eA_{\mu})^2,
\]

where \( \mathbf{B}(r) = \nabla \times \mathbf{A}(r) \) is the magnetic induction, \( \mathbf{A} \) is the vector potential, \( e \) is the absolute value of the electron charge, the \( \sigma_{\mu} \) are the Pauli matrices, \( \partial_{\mu} \equiv \partial/\partial x_{\mu} \), \( x_{\mu} \) is \( x, y, z \) for
\[ B = B(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \]
in spherical coordinates. Expanding the squared quantities in \( H_0 \), we obtain [16]:

\[
H_0 = \sum_{\mu} \left[ \frac{(-i\partial_\mu + eA_\mu)^2}{2m_\mu} + \frac{e}{2m_\mu} \sigma_\mu \sqrt{\mu B_\mu} \right],
\]
where \( m_\mu = (m_1 m_2 m_3)^{1/3} \) is the geometric mean effective mass, \( \mu = m_\mu/m_\sigma \), and \( \sigma_\mu \) is the matrix representing the \( \mu \)th component of the electron spin. We emphasize that the effective masses \( m_\mu \) appear in the orbital part of the electron motion, and are not necessarily related to the rest mass \( m \) of an electron in vacuum.

We then make the Klemm–Clem transformations [13, 60], and figure 1 provides a schematic diagram of the transformations. The first transformation is an anisotropic scale transformation,

\[ x_\mu = (\bar{m}_\mu)^{-1/2} x'_\mu, \]
for \( \mu = 1, 2, 3 \) that also preserves the Maxwell equation of vanishing magnetic monopoles, and the second transformation is a rotation that takes \( B' \) to \( B'' = \hat{e}' B'', \) and changes the coordinate variables \( x'_\mu \) to the rotated \( x''_\mu \) variables, all followed by an isotropic scale transformation [13, 60]. After these transformations, \( H_0 \) reduces to:

\[
\tilde{H}_0 = \frac{1}{2m_\mu} \left( -i\nabla'' + eA'' \right)^2 + \frac{e}{2m_\mu} \sigma_\mu \tilde{B}_\mu,
\]
where \( \tilde{B}_\mu = B''/\alpha(\theta, \phi) \). We note that the transformed Zeeman energy is proportional to \( \alpha(\theta, \phi) \), the direction of the magnetic induction \( B(\theta, \phi) \) has changed to \( (\theta', \phi') \), the Landau-level orbits scale equivalently with the transformed FS.

We have quantized the rotated spins along the \( z'' \) axis, the direction of the transformed \( \tilde{B} \). Thus, in second quantization, the full Hamiltonian \( H' = H - \mu N \) may now be written in transformed real space as:

\[
H' = \sum_{\sigma = \pm} \int d^3 r'' \left[ \psi^\dagger \sigma (r'') \left[ \frac{1}{2m_\mu} \left( -i\nabla'' + eA'' \right)^2 - \mu + \frac{e}{2m_\mu} \sigma_\mu \tilde{B}_\mu \right] \psi \sigma (r'') \right. \\
+ \frac{1}{2} \int d^3 r'' \left[ \psi^\dagger \sigma (r'') \psi_\sigma (r'') \right] \times V(r''_x - r'_x) \psi^\dagger \sigma (r''_x),
\]
where \( \mu \) is the chemical potential, \( \tilde{r} = \alpha r'' \), and the untransformed pairing interactions in reciprocal space may be written as [19]:

\[
V(\tilde{k}, \tilde{k}') = \begin{cases} 
V_0, & \text{s} \\
V_0 \left( \tilde{k}'_x - \tilde{k}_x \right) \left( \tilde{k}'_y - \tilde{k}_y \right), & \text{d}, \text{e} - \text{e}'.
\end{cases}
\]

We note that the anisotropic scale transformation also causes the magnitude and direction of \( k(\theta_k, \phi_k) \) to be changed into \( k'(\theta'_k, \phi'_k) \), since \( k' = \sqrt{\mu} k \), and the overall magnitude of the wave vector changes according to \( k' = \beta(\theta_k, \phi_k) k \), where \( \beta(\theta_k, \phi_k) \) is given by:

\[
\beta = \left[ \frac{\sin^2 \theta_k \cos^2 \phi_k}{m_1} + \frac{\sin^2 \theta_k \cos^2 \phi_k}{m_2} + \frac{\cos^2 \theta_k}{m_3} \right]^{1/2},
\]
and the angular relationships are:

\[
\sin \theta_k' = \frac{\beta(\theta_k, \phi_k)}{\beta(\tilde{\theta}_k, \tilde{\phi}_k)} \sin \theta_k; \cos \theta_k' = \frac{1}{\beta(\tilde{\theta}_k, \tilde{\phi}_k)} \cos \theta_k;
\]
\[
\sin \phi_k' = \frac{1}{\beta(\tilde{\theta}_k, \tilde{\phi}_k)} \sin \phi_k; \cos \phi_k' = \frac{1}{\beta(\tilde{\theta}_k, \tilde{\phi}_k)} \cos \phi_k.
\]

However, at the chemical potential (or \( \cos E_F \)), we have \( k' = 2m_\mu \), so in integrations about the Fermi energy, we may safely approximate:

\[
\beta^2 k^2 \approx 2m_\mu.
\]
2.2. Gap function

All of the representations of physical quantities are shown in the new coordinate system with double primes, so it is convenient to omit the double primes in the following. We define the superconducting order parameter to be:

\[
\Delta_{\sigma,\sigma'}(r, r') = V(r, r')F_{\sigma,\sigma'}(r, r', 0^+),
\]

in which \(V(r, r')\) is the pairing interaction in weak-coupling theory. It may be written as \(V(r, r') = \lambda_0(r - r')\), and then \(\Delta_{\sigma,\sigma'}(r, r') = \Delta(0)\delta(r - r')\delta_{\sigma,-\sigma'}\), for traditional s-wave pairing. For non-s-wave superconductors, it is easier to define the gap function in wave vector space [19]. Using Gor’kov’s description of weakly coupled superconductors, we can derive the fully transformed equations of motion for the normal and anomalous temperature Green functions of anisotropic superconductors [61],

\[
\begin{align*}
\left[ \omega_n - \frac{(\nabla / i - eA)^2}{2m_g} + E_F - \frac{e}{2m_g}\sigma_z B_0 \right] & \times \sum_p \int d^3 \xi \Delta_{\sigma,\sigma'}(\xi, r) F^\dagger_{\rho,\sigma'}(\xi, r', \omega_n) = \delta_{\sigma,\sigma'}\delta(r - r'), \\
\left[ -i\omega_n - \frac{(i\nabla - eA)^2}{2m_g} + E_F - \frac{e}{2m_g}\sigma_z B_0 \right] & \times \sum_p \int d^3 \xi \Delta_{\sigma,\sigma'}(\xi, r) G_{\rho,\sigma'}(\xi, r', \omega_n) = 0,
\end{align*}
\]

where \(\omega_n = (2n + 1)\pi T\) are the Matsubara frequencies. For a charged anisotropic superfluid in a magnetic field, we use the method introduced by Scharnberg and Klemm in their study of p-wave superconductors to obtain the gap function before the transformations [19]:

\[
\Delta_{\sigma,\sigma'}(R, k_F) = 2\pi TN(0) \int \frac{d\Omega_{k_F}}{4\pi} V(k_F - k_F)
\]

\[
\times \int_0^\infty d\xi e^{-2|\omega_\xi|}\cos \left[ \frac{e}{2m_g}\sigma_z (\sigma_z') B_0 \xi \right] \times e^{-i\text{sgn}(\omega_\xi)\epsilon_{k_F} \Pi(R) \Delta_{\sigma,\sigma'}(R, k_F)}.
\]

where \(\Pi(R) = \nabla_R / i + 2eA(R)\), \(R\) denotes the center of mass position of the paired electrons, and \(N(0)\) is the electronic density of states at \(E_F\).

2.3. Particular models

The most notable quality of an anisotropic superconductor, especially for a heavy fermion superconductor (HFS), is its anisotropic gap. For anisotropic p-wave superconductors, one may have a nodeless gap, or a gap with either planar nodes (such as a polar state), or point nodes (such as an axial state), where the gap vanishes on the FS. We consider s-wave and \(d_{x^2-y^2}\)-wave superconductors with ellipsoidally anisotropic masses and Zeeman energies in this section.

For a conventional s-wave superconductor, the gap function \(\Delta_s(R, \hat{k}_F) = \Delta_s(R)\) and the interaction \(V(k, \hat{k}) = V_0\) is a constant. But for a \(d_{x^2-y^2}\)-wave superconductor, the interaction in equation (6) leads to:

\[
\Delta_d(R, \hat{k}_F) = \Delta_d(R) \left( \hat{k}_z^2 - k_y^2 \right),
\]

in which \(\hat{k}_F\) denotes the unit vector of the wave vector \(k_F\) on the FS, and the \(k_i\) are its components.

We then transform the unit wave vector components, \(\hat{k}_i(\theta_k, \phi_k)\) into the \(\hat{k}'_i(\theta_k', \phi_k')\), obtaining:

\[
\hat{k}_x = \sqrt{\bar{m}_1}\alpha_\beta \left( \cos \theta' \cos \phi' \hat{k}_x - \sin \phi' \hat{k}_y + \sin \theta' \cos \phi' \hat{k}_z \right),
\]

\[
\hat{k}_y = \sqrt{\bar{m}_2}\alpha_\beta \left( \cos \theta' \sin \phi' \hat{k}_x + \cos \phi' \hat{k}_y + \sin \theta' \sin \phi' \hat{k}_z \right),
\]

\[
\hat{k}_z = \sqrt{\bar{m}_3}\alpha_\beta \left( - \sin \theta' \hat{k}_x + \cos \theta' \hat{k}_y \right),
\]

and the \(d_{x^2-y^2}\) pairing interaction transforms according to:

\[
V(\hat{k}, \hat{k}') = V_0 f(k)/k',
\]

\[
f(k) = \left( \hat{k}_x^2 - \hat{k}_y^2 \right) \rightarrow \alpha_2^2\beta_2^2 f(\hat{k}),
\]

where

\[
f(\hat{k}) = m_1 \left( \cos \theta' \cos \phi' \hat{k}_x - \sin \phi' \hat{k}_y + \sin \theta' \cos \phi' \hat{k}_z \right)^2 - m_2 \left( \cos \theta' \sin \phi' \hat{k}_x + \cos \phi' \hat{k}_y + \sin \theta' \sin \phi' \hat{k}_z \right)^2.
\]

It is then elementary to transform the gap function:

\[
\Delta_d(R, \hat{k}_F) = \Delta_d(R) f(\hat{k}) \rightarrow \Delta_d(R) \alpha_2^2\beta_2^2 f(\hat{k}),
\]

where \(f(\hat{k})\) is given by equation (18).

For respective s-wave and \(d_{x^2-y^2}\)-wave order parameters, we have:

\[
\Delta_s(R) = 2\pi TN(0) \int \frac{d\Omega_{k_F}}{4\pi} V_0 \int_0^\infty d\xi e^{-2|\omega_\xi|\xi} \times \cos \left( \frac{2e}{2m_g}\alpha B_0 \xi \right) e^{-\text{sgn}(\omega_\xi)\epsilon_{k_F} \Pi(R) \Delta_s(R)},
\]

\[
\Delta_d(R) = 2\pi TN(0) \int \frac{d\Omega_{k_F}}{4\pi} V_0 \alpha_2^2\beta_2^2 \int_0^\infty d\xi e^{-2|\omega_\xi|\xi} \times \epsilon_{k_F} \Pi(R) \Delta_d(R),
\]

where

\[
\Pi(R) = -i\alpha \nabla_R + 2eA(R).
\]
As shown previously [19, 21], the gap functions $\Delta_i(R)$ may be expanded:

$$\Delta_i(R) = \sum_n a_{n,i} |n(R)|,$$  \hspace{1cm} (23)

for $i = s, d$ in terms of the complete orthonormal set of one-dimensional harmonic oscillator eigenfunctions $|n\rangle$, where $\langle n'|n\rangle = \delta_{n,n'}$, we respectively obtain a closed expression for the $a_{ns}$ and a recursion relation for the $a_{nd}$,

$$a_{n,s} = a_{n,s} e^{2\pi TN(0)} \sum_{\omega_n} \int_{V_0} \frac{d\kappa'}{4\pi} \int_0^\infty d\xi e^{-2|\omega_n|\xi} \times \cos \left( 2 \frac{2}{2m_e} \alpha B^2 \xi \right) e^{-\frac{3}{2} |\eta|^2} \int_0^\infty d\xi e^{-2|\omega_n|\xi} \cos \left( 2 \frac{2}{2m_e} \alpha B^2 \xi \right)$$

$$a_{n,d} = a_{n,d} e^{2\pi TN(0)} \sum_{\omega_n} \int_{V_0} \frac{d\kappa'}{4\pi} \int_0^\infty d\xi e^{-2|\omega_n|\xi} \times \cos \left( 2 \frac{2}{2m_e} \alpha B^2 \xi \right) e^{-\frac{3}{2} |\eta|^2} \int_0^\infty d\xi e^{-2|\omega_n|\xi} \cos \left( 2 \frac{2}{2m_e} \alpha B^2 \xi \right)$$

where

$$\eta = -\text{isgn}(\omega_m) \sin \theta \alpha \beta \sqrt{e//e} B,$$  \hspace{1cm} (26)

the $L_0^x(x)$ are the Laguerre polynomials, and the matrix elements in equation (25) are evaluated exactly as by Lörscher et al [21], although the integrations over $f^2(\kappa')$ are more complicated than in that $p$-wave case with broken symmetry.

From equation (25), we derived a recursion relation for the $a_{n,d}$ corresponding to the d$_{s-z}$-wave order parameter,

$$0 = C_{n,n,-4a_{n-4,d}} + C_{n,n,-24a_{n-2,d}} + C_{n,n,a_{n,d}}$$

$$+ C_{n,n,24a_{n+2,d}} + C_{n,n,a_{n+4,d}},$$  \hspace{1cm} (27)

where the coefficients $C_{n,n'}$ are given in the appendix.

3. Numerical results

Here we present our numerical results for the reduced upper critical induction $b_{2c}$ and reduced temperature $t$ defined by:

$$b_{2c}(\theta, \phi, t) = 2eB_{2c}(\theta, \phi, t) r_T^2 / (2\pi T_c)^2,$$  \hspace{1cm} (28)

for s-wave and d$_{s-z}$-wave superconductors, and discuss the influences of the anisotropic effective mass, Zeeman energy, and the order parameter (OP) symmetry.

First, we present our results for $B_{2c}$ of an s-wave superconductor with an anisotropic effective mass and Zeeman tensor, but which has an isotropic superconducting gap on the basis of section 2. In figure 2(a), $b_{2c}$ for an s-wave superconductor is plotted versus $t$ for a spherical FS and for a FS with ellipsoidal anisotropy. Clearly, $b_{2c}$ is independent of the direction of the s-wave OP on an isotropic FS, as shown by the solid black line in figure 2(a). However, that is not the case for an s-wave OP on an anisotropic FS, and one example is presented in figure 2(a) for $\theta_1 = 0.5, \theta_2 = 2.0, \theta_3 = 1.0$. The reduced upper critical field $b_{2c}$ depends upon the field direction, and $b_{2c}$ is largest in the direction corresponding to the smallest effective mass. Actually, the difference arises from the first step of the Klemm–Clem transformations, in which $B'(\theta, \phi)$ is $\alpha(\theta, \phi)$ times the magnitude of $B(\theta, \phi)$, but the gap function is a constant in wave vector space. So it is clear that when the FS is isotropic, $\alpha = 1$, and $b_{2c}$ for an isotropic s-wave superconductor is independent of the field direction.

We note that the $b_{2c}$ as shown in figure 2(a) is larger in the direction with the smaller effective mass. We therefore investigated the effects of the much stronger effective mass anisotropy but also with an isotropic s-wave pairing interaction. In figures 2(b) and (c), the $b_{2c}$ for the fields along the x, y, and z directions are depicted respectively in solid red, dashed blue, and dotted black curves. In figure 2(b), the reduced effective mass values are $m_1 = 10^{-4}$, $m_2 = m_3 = 1.0 \times 10^2$, which is for a very flat pancake-shaped FS, and in figure 2(c), $m_1 = m_2 = 1.10^5$, $m_3 = 4.0 \times 10^4$, which is for a very narrow cigar-shaped FS. For the field along the largest effective mass direction, $b_{2c}(t)$ is so small that we displayed the results in the figure insets.

Helfand and Werthamer [24] investigated the upper critical field for an s-wave superconductor on an isotropic FS in the clean limit, and found that its slope at $T_c$ was given by $H_{c2} = 0.73 |dH_{c2}/dT|_{T_c}$. We also obtained the same value, for which the slope for the reduced upper critical field $b_{2c}$ at $t = 1$ is $-1.426$ for an isotropic FS. We also found that the slope of $B_{2c}(t)$ at $T_c$ for an anisotropic FS is $B_{2c}(t) = H_{c2}(t) / \cos(\alpha) \pi T_c^2 / |dB_{c2}/dT|_{T_c}$. There is a chance that if this slope were to persist to low temperatures, $B_{2c}(0)$ could exceed the conventional Pauli limiting $B_{2c}(T) = 1.76 T_c$. However, without any effective mass anisotropy, provided that $B_{2c}(0)$ is greater than 1.76, where $\alpha$ is given by equation (4).

However, the situation is considerably more complex for a d$_{s-z}$-wave OP than for an s-wave OP, especially for an anisotropic FS. Firstly, after the transformations, the interaction and the d$_{s-z}$ OP both depend upon the directions in the pairing plane, as can be seen in equations (17) and (20). In figure 3, we exhibit $b_{2c}(t)$ for a d$_{s-z}$-wave superconductor with the applied field along the x, y, and z directions for three different cases, each with $m_1 = m_2 = m_3 = 1$. In figure 3(a), the FS is isotropic, $b_{2c}(t)$ is identical for the field along the x- and y-axes, which are larger than that for the field along the z-axis. The strong anisotropy in slope just below $t = 1$ is solely due to the OP anisotropy, as the FS is isotopic. In fact, $b_{2c}$ for a d$_{s-z}$-wave superconductor with an anisotropic effective mass becomes more complicated and difficult to describe, as shown in figures 3(b) and (c), where $b_{2c}(\theta, \phi)(t)$ crossing points exist. In figure 3(b), $b_{2c}(t)$ along the x and y directions are identical, due to the equal effective masses in the xy plane, but cross the $b_{2c}(t)$ curve for the field along the
Figure 2. Plots of the reduced upper critical induction $b_{c2}(t)$ defined in equation (28) for an $s$-wave superconductor with an anisotropic FS characterized by different effective masses and a correspondingly anisotropic Zeeman energy. The isotropic ($m_1 = m_2 = m_3 = 1$) case is identical to the solid black curve. For an $s$-wave OP with an anisotropic effective mass characterized by $m_1 = 0.5$, $m_2 = 2.0$, and $m_3 = 1.0$, $b_{c2}(t)$ curves for the field along the $x$ (dashed red), $y$ (dotted blue), and $z$ (dash dot magenta) directions, respectively, are also shown in (a). Plots of the reduced $b_{c2}$ for a highly anisotropic effective mass with strong spin-orbit interactions for an $s$-wave OP are shown in (b) and (c). In both figures, $b_{c2}$ along the $x$ direction is in solid red, along the $y$ direction is in dashed blue (just above the horizontal axis), and along the $z$ direction is in dotted magenta shown in the insets. In (b), $m_1 = 1.0 \times 10^{-4}$, $m_2 = m_3 = 1.0 \times 10^{2}$. In (c), $m_1 = m_2 = 0.0050$, and $m_3 = 4.0 \times 10^{4}$.

Figure 3. The reduced upper critical induction $b_{c2}(t)$ for a $d_{x^2-y^2}$-wave superconductor with the applied field along the $x$ (solid red), $y$ (dashed blue), and $z$ (dotted magenta) directions, all for which $m_1 m_2 m_3 = 1$. (a) $m_1 = m_2 = m_3 = 1$. (b) $m_1 = m_2 = 1.26$, $m_3 = 0.63$. (c) $m_1 = 0.5$, $m_2 = 2.0$, $m_3 = 1.0$.

Figure 4. The reduced $b_{c2}(t)$ for a $d_{x^2-y^2}$-wave superconductor without the Zeeman energy. Figures (a)–(c) correspond to the same sets of $m_i$ values as in figure 3, and the applied field directions are also indicated exactly as in figure 3.
argument requires further study. We only show a few examples in figures 4(b) and (c), for which the crossing points vanish. The crossing point exists for a pancake FS \((\tilde{m}_1 = \tilde{m}_2 > \tilde{m}_3)\) with the ratio \(m_1/m_3\) between 1.6 and 3.2. Therefore, the Zeeman energy, which is intrinsically anisotropic, plays a crucial role in determining \(b_{\tilde{2}}(\pi/2, \phi, t)\) for a \(d\tilde{z}-\tilde{y}\)-wave superconductor. At least, it suppresses the \(B_{\tilde{2}}(t)\) for a \(d\tilde{z}-\tilde{y}\)-wave superconductor more strongly than for an \(s\)-wave superconductor. But there remains the possibility that the conventional Pauli limit could be exceeded in some field directions for a \(d\tilde{z}-\tilde{y}\)-wave superconductor with a highly anisotropic FS.

It is interesting to investigate \(b_{\tilde{2}}(\pi/2, \phi, t)\) in the \(xy\) plane. Not only have many researchers recently paid more attention to measuring \(b_{\tilde{2}}\) in that plane \([62, 63]\), many superconductors, such as the cuprates, have been thought by many authors to exhibit \(d\tilde{z}-\tilde{y}\) OP symmetry \([38, 39]\). Here we present results for \(b_{\tilde{2}}(\pi/2, \phi, t)\) in the \(xy\) plane, with and without the Zeeman interaction. It is intriguing to find that \(b_{\tilde{2}}(\pi/2, \phi, t)\) including the Zeeman energy at different \(t\) values does not always follow the commonly held belief that the maxima of \(b_{\tilde{2}}(\pi/2, \phi, t)\) always occur along the antinodal directions of the OP. Instead, there is a \(\pi/4\) azimuthal shift in these maxima if this slope were to persist to low \(t\). That is, the maxima in \(b_{\tilde{2}}(\pi/2, \phi, t)\) for a \(d\tilde{z}-\tilde{y}\)-wave superconductor just below \(t = 1\) is indeed along the antinodal directions, but at low \(t\), the maxima are along the nodal directions.

This change in symmetry is shown in figures 5(a) and (b), which are the results for an isotropic FS at \(t = 0\) and \(t = 0.9\), respectively. Figure 5(c) presents the results for \(b_{\tilde{2}}(\pi/2, \phi, t)\) without the Zeeman energy in the \(xy\) plane at low \(t\). The shape of \(b_{\tilde{2}}(\pi/2, \phi, t)\) in figure 5(c) without the Zeeman energy actually is similar to that of figure 5(b), but the four-fold anisotropy is greatly reduced in magnitude by the Zeeman energy in figure 5(b). This is easy to understand from equation (21). The Zeeman energy suppresses the upper critical field for a \(d\tilde{z}-\tilde{y}\)-wave superconductor and enters into the gap equation in the \(\cos[(\sigma - \sigma') \varphi/2 \tilde{B} \xi]\) term, which can be expanded in powers of \(\tilde{B}\), yielding the factor \(1 - [(\sigma - \sigma') \varphi/2 \tilde{B} \xi]^2/2\), which is nearly unity near to \(t = 1\). However, at large \(b_{\tilde{2}}\) at low \(t\), the \(\pi/4\) phase shift demonstrates that the Zeeman energy is very important at low \(t\).

Because \(b_{\tilde{2}}(\pi/2, \phi, t)\) for a \(d\tilde{z}-\tilde{y}\)-wave superconductor with an isotropic FS with or without the Zeeman interaction is greatly different at \(t = 0\) and \(t = 0.9\), we explored the process of the \(\pi/4\) phase change in its \(\phi\)-dependence, and found that it changes rapidly in the mid-\(t\) range, as shown in figure 6 with the Zeeman interaction to emphasize the changes in the azimuthal dependence of \(b_{\tilde{2}}(\pi/2, \phi, t)\) with decreasing \(t\). As \(t\) is lowered from 0.9, \(b_{\tilde{2}}(\pi/2, \phi, t)\) initially increases very slightly along the antinodal directions down to 0.421, and then the intensity along the nodal directions starts to grow quickly, finally taking over around \(t = 0.415\).

We also considered the effects of different effective masses on \(b_{\tilde{2}}(\pi/2, \phi, t)\) for a \(d\tilde{z}-\tilde{y}\)-wave superconductor at low \(t\). When the effective mass anisotropy within the pairing \(xy\) plane becomes strong, as in detwinned orthorhombic \(\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}\), the in-plane azimuthal angular dependence of \(b_{\tilde{2}}\) not only deviates strongly from the \(C_4\) symmetry that the nominal \(d\tilde{z}-\tilde{y}\) Cooper pair obeys if the crystal were to be tetragonal, its shape changes from a four-leaf clover with \(C_4\) symmetry to that of a dumbbell with \(C_2\) symmetry. The precise form of the in-plane azimuthal anisotropy of \(b_{\tilde{2}}(\pi/2, \phi, 0)\) for an orthorhombic \(d\tilde{z}-\tilde{y}\)-wave superconductor changing with the \(m_1/m_2\) effective mass anisotropy is pictured in figure 7. In this figure, we show the results of our calculations for a fixed geometric mean \(m_1m_2m_3 = 1\) and \(m_2 = 1\), with \(m_1 = 0.9, 0.8,\) and 0.7 in figures 7(a)–(c), respectively. For the least in-plane anisotropy pictured in figure 7(a), the predicted \(\pi/4\) rotation of the maxima in the four-fold azimuthal anisotropy shown in figures 5(a) and 6(a) is partly present for two of the maxima, but the figure shown in figure 7(a) has overall \(C_2\) symmetry. At \(m_1/m_2 = 0.7\), the azimuthal anisotropy of \(b_{\tilde{2}}(\pi/2, \phi, 0)\) has rotated to the \(x\) axis, and has the simpler dumbbell shape.
Here we present some details of the region of rapid change in the symmetry of $b_{c2}(\pi/2, \phi, t)$ for a $d_{x^2-y^2}$-wave order parameter with an isotropic FS and with the Zeeman interaction. From left to right, $t = 0.415, 0.419, 0.421$ and 0.423.

Azimuthal dependence at $t = 0$ of $b_{c2}(\pi/2, \phi, 0)$ for a $d_{x^2-y^2}$-wave superconductor with decreasing $m_1/m_2$ at fixed $m_1/m_2, m_3 = 1$ and $m_2 = 1$. From left to right: $m_1 = 0.9, 0.8,$ and 0.7. In all three of these figures, $b_{c2}(\pi/2, \phi, 0)$ has a dumbbell shape exhibiting $C_2$ symmetry, but the directions of the maxima rotate from the $x$ axis for $m_1 = 0.7$ to increasingly off that axis as $m_1$ increases.

Plots of the polar ($\theta$) angular dependence of $b_{c2}(\theta, \phi, t)/b_{c2}(0, 0, 0)$ normalized to its $\theta = 0^\circ$ value for a $d_{x^2-y^2}$-wave superconductor as $t \to 0$, for $m_1 = m_2$, and $m_1/m_3 = 0.6, 0.8, 1.0, 1.6$ and 1.8 respectively. For each tetragonal effective mass anisotropy pictured, $b_{c2}(\theta, \phi, t)$ is a monotonically increasing or decreasing function of $\theta$ at fixed $t$ and $\phi$.

Since a peak in the polar dependence of $b_{c2}$ was predicted for $p$-wave superconductors with completely broken symmetry for anisotropies corresponding to $m_3/\alpha^2(\pi/2, \phi) > 3$ [21], we tried to study whether a peak in $b_{c2}(\theta, \phi, t)$ at some fixed $\phi$ and $t$ might occur for some effective mass anisotropy values. As shown in figures 8(a) and (c), for $m_1/m_3 = 1.6$ and 1.8, $b_{c2}(\theta, \phi, t)/b_{c2}(0, 0, 0)$ decreases monotonically with increasing $\theta$, but increases in figure 8(b), and for $m_1/m_3 = 0.6, 0.8$, and 1.0, the isotropic FS case, $b_{c2}(\theta, \phi, t)/b_{c2}(0, 0, 0)$ is a monotonically increasing function of $\theta$. For all of the cases plotted in figure 8, there is no peak in $b_{c2}(\theta, \phi, t)$ for $0 < \theta < \pi/2$, but it does not imply that $b_{c2}$ is a monotonic function of $\theta$ strictly. Recalling that there exits a crossing point in figure 3(b), we note that it does not stay monotonic near to the crossing point, but the change is slow and a little flat with increasing $\theta$.

4. Conclusions

In this paper we studied the temperature and angular dependence of the upper critical induction $B_{c2}(\theta, \phi, T)$ of $s$-wave and $d_{x^2-y^2}$-wave superconductors based on the Gorkov equation.
which includes the anisotropic Zeeman energy arising from the ellipsoidal anisotropy of the FS (with effective masses \(m_i\) along the three Cartesian coordinates for an orthorhombic crystal structure) that is treated self-consistently using the anisotropic Schrödinger–Dirac single particle Hamiltonian. While we have derived the upper critical field for \(s\)-wave and \(d\)-wave superconductors with anisotropic effective masses, this work could easily be applied to any anisotropic pairing function. It is analogous to the treatment of \(p\)-wave superconductors with completely broken symmetry and ellipsoidal effective mass symmetry, except for the addition of the Pauli pair-breaking effects in singlet-spin superconductors.

For an \(s\)-wave superconductor, we find the reduced upper critical induction \(b_{d2}(\theta, \phi, t)\) is angularly modulated by the universal orientation factor \(\alpha(\theta, \phi)\) [60], the Pauli limiting can be exceeded by adjusting the effective mass, and the Zeeman energy almost can be neglected when comparing it with that for a \(d_{xy}\) OP. But it is not the case for \(d_{xy}\)-wave superconductors, whose interaction and gap function depend upon the unit wave vector components in the \(xy\) plane, which become more complicated after the transformation of the single particle Hamiltonian to isotropic form. \(b_{d2}(\theta, \phi, t)\) for a \(d_{xy}\)-wave superconductor not only changes along the axial \(\phi\) direction with reduced temperature \(t = T/T_c\), it shows an interesting \(\phi\) dependence in the \(xy\) plane. There is a \(\pi/4\) shift in the \(C_4\)-symmetric calculated \(b_{d2}(\pi/2, \phi, t)\) patterns between low and high \(t\) values when the Zeeman interaction is included. The effective mass anisotropy in the plane of the pairing interaction is significant, as for untwinned orthorhombic YBa\(_2\)Cu\(_3\)O\(_{7−\delta}\), the \(\phi\) dependence of \(b_{d2}(\pi/2, \phi, t)\) in \(xy\) plane at low \(t\) changes from exhibiting \(C_4\) symmetry to \(C_2\) symmetry. In addition, the variation of \(b_{d2}(\theta, 0, t)\) with \(\theta\) at low \(t\) is a monotonic function of \(\theta\).

Although we are not aware of any azimuthal (\(\phi\)) measurements of \(b_{d2}(\pi/2, \phi, t)\) in high-transition temperature layered superconductors at low \(t\), since nearly all angular measurements have been made of \(b_{d2}(\theta, 0, t)\) for fixed \(\phi_0\), and comparisons were made to the Tinkham thin film formula for Bi2212 and to the anisotropic Ginsburg–Landau model for YBa\(_2\)Cu\(_3\)O\(_{7−\delta}\) [55–57]. Pulsed-field measurements of \(b_{d2}(\pi/2, \phi_0, t)\) at a fixed but unspecified \(\phi_0\) were made for \(t\) from 0.05 to 1 in that material [58]. In order to probe \(b_{d2}(\pi/2, \phi, t)\) for \(t < 0.5\), one needs to have a rather low value of \(T_c\). In addition, detwinned samples of YBa\(_2\)Cu\(_3\)O\(_{7−\delta}\) are too anisotropic in the \(ab\) plane, so that if it were a clean \(d_{xy}\)-wave superconductor, the only planar that could be measured would have \(C_2\) symmetry fixed to the \(a\) axis, which would be difficult to distinguish from effective mass anisotropy [13]. Bi2212 is too dirty to see such effects [13]. The only such experiment of which we are aware is that of the low-\(T_c\) layered superconductor CeCu\(_2\)Si\(_2\) [59]. In that material, \(T_c = 0.67\) K, so it was possible to measure \(b_{d2}(\pi/2, \phi, t)\) at low \(t\) values. Those authors provided evidence that their measured sample was in the clean limit. They measured \(B_{c2}\) at 40 mK as a function of \(\phi\) (defined as the planar angle relative to the crystalline \(a\) axis), and noted that their experimental results for \(b_{d2}(\pi/2, \phi, 0.06)\) had maxima at \(\phi = 0°, 90°,\) and \(180°\), which they identified as being due to

the superconducting order parameter having \(d_{xy}\)-wave symmetry, not \(d_{x^2−y^2}\)-wave symmetry [59]. We would like to see other \(b_{d2}(\pi/2, \phi, t)\) experiments made on other compounds. Recently, scanning tunneling microscopy measurements on the electron-doped cuprate Sr\(_{1−\delta}\)Nd\(_\delta\)Cu\(_2\)O\(_y\) with a \(T_c\) value of about 40 K were made, and the superconductivity was claimed to be nodeless, exhibiting three phonon modes [64]. Since it has neither a CDW nor an SDW, our model should apply, and it would be interesting to see the results of \(b_{d2}(\pi/2, \phi, t)\) measurements in that material.

**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

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**Appendix**

In the following, we present the forms of the recursion coefficients \(C_{n,n'}\) for a \(d_{xy}\)-wave superconductor. In these expressions, \(\eta\) is given by equation (26) in the text, and the \(\hat{L}_n^\alpha(x)\) are the associated Laguerre polynomials:

\[
C_{n,n+1} = \pi T \sum_{\omega_n} \int_0^\pi \sin \theta_k \, d\theta_k \int_0^\infty d\xi e^{-2|\omega_n|\xi} \xi^{-\frac{1}{2}} |\eta|^2 \\
\times \cos \left[ 2 \frac{e^\omega}{2m_e} \alpha \hat{B}_z \right] \left[ L_n^\alpha(|\eta|^2) \right] \\
\times \left[ \sin^4 \theta_k \left( \frac{3A^2 + 3B^2}{8} + \frac{D^2 + 2AB}{8} \right) + \sin^2 \theta_k \cos^2 \theta_k \right] \\
\times \left[ E^2 + 2AC + F^2 + 2BC + \cos^4 \theta_k C^2 \right] - \frac{1}{N_0 V_0}, \tag{29}
\]

\[
C_{n,n+2} = \pi T \sum_{\omega_n} \int_0^\pi \sin \theta_k \, d\theta_k \int_0^\infty d\xi e^{-2|\omega_n|\xi} \xi^{-\frac{1}{2}} |\eta|^2 \\
\times \cos \left[ 2 \frac{e^\omega}{2m_e} \alpha \hat{B}_z \right] \left[ \frac{-|\eta|^2}{\sqrt{(n + 2)(n + 1)}} \right] \left[ L_n^\alpha(|\eta|^2) \right] \\
\times \left[ \sin^4 \theta_k \left( \frac{A^2 - B^2}{4} - \frac{2AD + 2BD}{8i} \right) + \frac{1}{4} \sin^2 2\theta_k \right] \\
\times \left[ E^2 + 2AC - F^2 - 2BC - \frac{2EF + 2CD}{4i} \right], \tag{30}
\]
\begin{align}
C_{n,n-2} & = \pi T \sum_{\omega_n} \int_0^\infty d\xi e^{-2|\omega_n|\xi} e^{-\frac{1}{2}\xi^2} \\
& \times \left[ \sin^4 \theta_k \left( \frac{A^2 + B^2 - D^2 - 2AB}{4} + \frac{2AD + 2BD}{8i} \right) \\
& \times \left( \sqrt{1 + |\eta|^2} \right) \right],
\end{align}

(31)

\begin{align}
C_{n,n+4} & = \pi T \sum_{\omega_n} \int_0^\infty d\xi e^{-2|\omega_n|\xi} e^{-\frac{1}{2}\xi^2} \\
& \times \left[ \sin^4 \theta_k \left( \frac{A^2 + B^2 - D^2 - 2AB}{4} + \frac{AD - BD}{8i} \right) \\
& \times \left( \sqrt{1 + |\eta|^2} \right) \right],
\end{align}

(32)

\begin{align}
C_{n,n-4} & = \pi T \sum_{\omega_n} \int_0^\infty d\xi e^{-2|\omega_n|\xi} e^{-\frac{1}{2}\xi^2} \\
& \times \left[ \sin^4 \theta_k \left( \frac{A^2 + B^2 - D^2 - 2AB}{4} + \frac{AD - BD}{8i} \right) \\
& \times \left( \sqrt{1 + |\eta|^2} \right) \right],
\end{align}

(33)

where

\begin{align}
A^2 & = \beta^4 (\overline{m}_1 \cos^2 \phi' - \overline{m}_2 \sin^2 \phi')^2 \cos^4 \theta', \\
B^2 & = \beta^4 (\overline{m}_1 \sin^2 \phi' - \overline{m}_2 \cos^2 \phi')^2, \\
C^2 & = \beta^4 (\overline{m}_1 \cos^2 \phi' - \overline{m}_2 \sin^2 \phi')^2 \sin^4 \theta', \\
D^2 + 2AB & = \beta^4 (\overline{m}_1 + \overline{m}_2) \cos^2 \theta' \sin^2 2\phi' \\
& + 2\beta^4 \cos^2 \theta'(\overline{m}_1 \cos^2 \phi' - \overline{m}_2 \sin^2 \phi'), \\
E^2 + 2AC & = \frac{3}{2} \beta^4 \sin^2 2\theta' (\overline{m}_1 \cos^2 \phi' - \overline{m}_2 \sin^2 \phi')^2, \\
F^2 + 2BC & = \beta^4 \sin^2 \theta' \sin^2 2\phi' (\overline{m}_1 + \overline{m}_2)^2 + 2\beta^4 \sin^2 \theta' \\
& \times (\overline{m}_1 \sin^2 \phi' - \overline{m}_2 \cos^2 \phi') \\
& \times (\overline{m}_1 \cos^2 \phi' - \overline{m}_2 \sin^2 \phi'), \\
2AD & = -2\beta^4 \cos^3 \theta' \sin 2\phi' (\overline{m}_1 + \overline{m}_2) \\
& \times (\overline{m}_1 \cos^2 \phi' - \overline{m}_2 \sin^2 \phi'), \quad (40)
\end{align}

\begin{align}
2BD & = -2\beta^4 \cos \theta' \sin 2\phi' (\overline{m}_1 + \overline{m}_2) \\
& \times (\overline{m}_1 \sin^2 \phi' - \overline{m}_2 \cos^2 \phi'), \quad (41)
\end{align}

\begin{align}
2EF + 2CD & = -3\beta^4 \sin \theta' \sin 2\theta' \sin 2\phi' \times (\overline{m}_1 + \overline{m}_2) \\
& \times (\overline{m}_1 \cos^2 \phi' - \overline{m}_2 \sin^2 \phi'), \quad (42)
\end{align
[16] Zhao A, Gu Q, Haugan T J and Klemm R A 2021 The Zeeman, spin-orbit and quantum spin Hall interactions in anisotropic and low-dimensional conductors J. Phys.: Condens. Matter 33 085802
[17] Mineev V P and Samokhin K V 1999 Introduction to Unconventional Superconductivity (New York: Gordon and Breach)
[18] Aoki D and Flouquet J 2012 Ferromagnetism and superconductivity in uranium compounds J. Phys. Soc. Japan 81 011003
[19] Scharnberg K and Klemm R A 1980 p-wave superconductors in magnetic fields Phys. Rev. B 22 5233
[20] Scharnberg K and Klemm R A 1985 Upper critical field in p-wave superconductors with broken symmetry Phys. Rev. Lett. 54 2445
[21] Lörscher C, Zhang J, Gu Q and Klemm R A 2013 Anomalous angular dependence of the upper critical induction of orthorhombic ferromagnetic superconductors with completely broken p-wave symmetry Phys. Rev. B 88 024504
[22] Zhang J, Lörscher C, Gu Q and Klemm R A 2014 Is the anisotropy of the upper critical field of SrRuO3 consistent with a helical p-wave state? J. Phys.: Condens. Matter 26 252201
[23] Zhang J, Lörscher C, Gu Q and Klemm R A 2014 First-order chiral to non-chiral transition in the upper critical induction of the Scharnberg–Klemm p-wave pair state J. Phys.: Condens. Matter 26 252202
[24] Helfand E and Werthamer N R 1966 Temperature and purity dependence of the superconducting critical field, Hc2, II Phys. Rev. 147 288
[25] Clogston A M 1962 Upper limit for the critical field in hard superconductors Phys. Rev. Lett. 9 266
[26] Chandrasekhar B S 1962 A note on the maximum critical field Phys. Rev. 91 452
[27] Fulde P and Ferrell R A 1964 Superconductivity in a strong magnetic field Phys. Rev. 135 A550
[28] Agosta C C, Fortune N A, Hannahs S T, Gu S, Liang L, Park J H and Schluter J A 2017 Calorimetric measurements of magnetic-field-induced inhomogeneous superconductivity above the paramagnetic limit Phys. Rev. Lett. 118 267001
[29] Matsuda Y and Shimahara H 2007 Fulde–Ferrell–Larkin–Ovchinnikov state in heavy fermion superconductors J. Phys. Soc. Japan 76 051005
[30] Gruenberg L W and Gunther L 1966 Fulde–Ferrell effect in type-II superconductors Phys. Rev. Lett. 16 996
[31] Larkin A I and Ovchinnikov Y N 1964 Nonuniform state of superconductors Zh. Eksp. Teor. Fiz. 47 1136
[32] Larkin A I and Ovchinnikov Y N 1965 Sov. Phys. - JETP 20 762
[33] Yip S 2015 Noncentrosymmetric superconductors Annu. Rev. Condens. Matter Phys. 5 15
[34] Néel L 1951 Some nuclear superfluids C. R. Acad. Sci. 233 1291
[35] Koch S K et al 2012 Anomalous upper critical field in CeCoIn5/YbCoIn5 superlattices with a Rashba-type heavy fermion interface Phys. Rev. Lett. 109 157006
[36] Damascelli A, Hussain Z and Shen Z X 2003 Angle-resolved photoemission studies of the cuprate superconductors Rev. Mod. Phys. 75 473
[37] Tsuei C C and Kirtley J R 2000 Pairing symmetry in cuprate superconductors Rev. Mod. Phys. 72 969
[38] Hole P and Schrieffer J R 1964 Superconductivity in a strong magnetic field Phys. Rev. Lett. 59 4846
[39] Li Q, Tsay Y N, Suenaga M, Klemm R A, Gu G D and Koshizuka N 1999 Bi2Sr2CaCu2O8+δ, bvcrystal c-axis twist Josephson junctions: a new phase-sensitive test of order parameter symmetry Phys. Rev. Lett. 83 4160
[40] Takano Y, Hatano T, Fukuyou A, Ishii A, Ohmori M, Arisawa S, Togano K and Tachiki M 2002 d-like symmetry of the order parameter and intrinsic Josephson effects in Bi2Sr2CaCu2O8+δ cross-whisker junctions Phys. Rev. B 65 140513
[41] Takano Y et al 2003 Cross-whisker intrinsic Josephson junction as a probe of symmetry of the superconducting order parameter J. Low Temp. Phys. 131 533
[42] Latsyhev Y I, Orlov A P, Nikitina A M, Monceau P and Klemm R A 2004 c-axis transport in naturally grown Bi2Sr2CaCu2O8+δ cross-whisker junctions Phys. Rev. B 70 094517
[43] Klemm R A 2005 The phase-sensitive c-axis twist experiments on Bi2Sr2CaCu2O8+δ and their implications Phil. Mag. 85 801–53
[44] Zhu Y et al 2011 Presence of s-wave pairing in Josephson junctions made of twisted ultrathin Bi2Sr2CaCu2O8+δ flakes Phys. Rev. X 1 031011
[45] Lee J et al 2021 Twisted van der Waals Josephson junction based on a high Tc superconductor Nano Lett. 21 10469–77
[46] Zhao Y et al 2021 Emergent interfacial superconductivity between twisted cuprate superconductors (arXiv:2108.13455v1)
[47] Tonjes W C, Grenaya W A, Liu R, Olson C G and Moliné P 2001 Charge-density wave mechanism in the 2H – NbSe2 family: angle-resolved photoemission study Phys. Rev. B 63 235101
[48] Gamble F R, DiSalvo F J, Klemm R A and Geballe T H 1970 Superconductivity in layered structure organometallic crystals Science 168 568
[49] Zhong Y et al 2016 Nodeless pairing in superconducting copper-oxide monolayer films on Bi2Sr2CaCu2O8+δ Sci. Bull. 61 1239
[50] The second corresponding author communicated this privately and publically, respectively, to the corresponding authors of [47, 48], after those experiments were respectively published and publicized
[51] Kashiwagi T et al 2015 A high Tc, intrinsic Josephson junction emitter tunable from 0.5 to 2.4 THz Appl. Phys. Lett. 107 082601
[52] Rain J R, Cai P Y, Baekey A, Reinhard M A, Vasquez R I, Silverman A C, Cain C L and Klemm R A 2021 Wave functions for high-symmetry, thin microstrip antennas and two-dimensional quantum boxes Phys. Rev. A 104 062205
[53] Naughton M J, Yu R C, Davies P K, Fischer J E, Chamberlin R V, Wang Z Z, Jing T W, Ong N P and Chaikin P M 1998 Orientational anisotropy of the upper critical field in single-crystal YBa2Cu3O7−δ and Bi22CaSr1−xTixCu2O8+δ Phys. Rev. B 58 9280
[54] Welp U, Kwok W K, Crabtree G W, Vandervoort K G and Liu J Z 1989 Magnetic measurements of the upper critical field of YBa2Cu3O7−δ single crystals Phys. Rev. Lett. 62 1908
[55] Welp U, Grimsditch M, You H, Kwok W K, Fang M M, Crabtree G W and Liu J Z 1989 The upper critical field of untwinned crystals Physica C 161 1
[58] Miura N, Nakagawa H, Sekitani T, Naito M, Soto H and Enomoto Y 2002 High-magnetic-field study of high-$T_c$ cuprates Physica B 319 310
[59] Vierya H A, Oeschler N, Seiro S, Jeevan H S, Geibel C, Parker D and Steglich F 2011 Determination of gap symmetry from angle-dependent $H_{c2}$ measurements on CeCu$_2$Si$_2$ Phys. Rev. Lett. 106 207001
[60] Klemm R A and Clem J R 1980 Lower critical field of an anisotropic type-II superconductor Phys. Rev. B 21 1868
[61] Abrikosov A A, Gor’kov L N and Dzyaloshinskii I E 1975 Methods of Quantum Field Theory in Statistical Physics (New York: Dover)
[62] Zuo H, Bao J K, Wang J, Jin Z, Xia Z, Li L, Xu Z, Kang J, Zhu Z and Cao G H 2017 Temperature and angular dependence of the upper critical field in K$_2$Cr$_3$As$_3$ Phys. Rev. B 95 014502
[63] Vorontsov A B and Vekhter I 2010 Vortex state in $d$-wave superconductors with strong paramagnetism: transport and specific heat anisotropy Phys. Rev. B 81 094527
[64] Fan J-Q et al 2022 Direct observation of nodeless superconductivity and phonon modes in electron-doped copper oxide Sr$_{1-x}$Nd$_x$CuO$_2$ Nat. Sci. Rev. 9 nwab225