Axial Anomaly and Ginsparg-Wilson fermions in the Lattice Dirac Sea picture

Srinath Cheluvaraja

Dept. of Physics and Astronomy, Louisiana State University, Baton Rouge, LA, 70808

N.D. Hari Dass

Institute of Mathematical Sciences, Chennai, 600113

ABSTRACT

The axial anomaly equation in 1+1 dimensional QED is obtained on the lattice for fermions obeying the Ginsparg-Wilson relation. We make use of the properties of the Lattice Dirac sea to investigate the connection between the anomaly and the Ginsparg-Wilson operator in the Hamiltonian picture. The correct anomaly is reproduced for gauge fields whose characteristic time is much larger than the lattice spacing, which is the regime where the adiabatic approximation applies. A non-zero Wilson $r$ parameter is necessary to get the correct anomaly. The anomaly is shown to be independent of $r$ for $r > 0.5$. The generalization to 3+1 dimensions is also discussed.

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The lattice regularization is one of the few non-perturbative methods available for defining quantum field theories. Lattice gauge theories have revealed many interesting features of gauge theories that are not easily visible in the usual perturbative approach. Nevertheless, the lattice regulator has proved problematic if fermions have to be incorporated into the theory. A naive discretization of the fermionic theory suffers from the replication of fermion modes due to the "doublers". The doublers are degenerate in energy with the originally introduced fermions and though they have lattice momenta of the order of the cut-off (1/a in lattice theories, a is the lattice spacing), they mimic ordinary low energy fermions. These double modes cannot be ignored as they participate in physical processes, for instance they can be pair created, and can affect the value of physical quantities—such as the free energy. The first method to handle these doublers was given in [1] and it uses an additional term in the action—the Wilson term—to lift the degeneracy of the fermions, thereby decoupling the doublers in the continuum limit. However, this method has the disadvantage of explicitly breaking chiral symmetry, and hinders the study of dynamical questions related to chiral symmetry breaking. A cure for the doubling problem that explicitly breaks chiral symmetry also makes the lattice regularization of chiral gauge theories, such as the standard model, much more difficult.

The Nielsen-Ninomiya no-go theorem [2] decrees that any chirally symmetric lattice Hamiltonian satisfying general properties like locality and hermiticity must result in a replication of fermion species. This theorem seems to suggest the impossibility of defining undoubled fermions on the lattice without breaking chiral symmetry. Recently, however, alternative methods for tackling these problems have emerged. One of them [3] uses the so-called Ginsparg-Wilson relation [4] for Dirac fermions. The Ginsparg-Wilson (G-W) operator is obtained by the application of block-spin transformations to a chirally invariant Dirac operator, and which therefore suffers from fermion doubling, using a chirally non-invariant blocking kernel. Although the G-W operator is not chirally invariant, it contains the information of chiral symmetry because it has been obtained after blocking a chirally invariant lattice action. Its construction, by a renormalization group transformation of a chirally invariant action, is bound to leave the low energy properties related to the chiral symmetry unchanged. This approach of formulating lattice fermions has led to many recent developments [5], such as, lattice formulations of chiral symmetry, the search for chiral gauge theories on the lattice, methods of defining a lattice topological charge, and formulation of lattice index theorems etc.
The G-W operator has to satisfy the following relation

\[ D\gamma_5 + \gamma_5 D = aD\gamma_5 D. \]  

(There are different versions of the G-W relation depending on the precise form of the blocking kernel used. The above form is one of the simpler ones and is sufficient for the ensuing discussion. Here \( a \) is the lattice spacing.) The G-W operator clearly does not satisfy chiral symmetry (because \( \{D, \gamma_5\} \neq 0 \)). Even though the G-W operator seems to share the properties of a chirally noninvariant mass term, it is a milder way to break the chiral symmetry on the lattice. This is because it is obtained by blocking a chirally symmetric action, although using a chirally non-invariant kernel. The low energy properties of the G-W operator on the lattice are the same as those of the chirally symmetric action.

Another approach to problems of chirality on the lattice is the overlap approach introduced in [15]. It captures many essential elements of domain wall fermions [17] as well as the one in [18] which requires an infinite number of auxiliary fields. The chiral determinant is expressed as an overlap of the ground states of two many body Hamiltonians and a construction of chiral gauge theories involves regularizing the overlap [16]. Though the original Ginsparg-Wilson approach and the overlap approach appear to have nothing in common, the overlap operator (which appears in the Hamiltonian) has been shown to satisfy the Ginsparg-Wilson relation [19]. The overlap is also a real time approach since it involves the quantum mechanical scalar product of the ground states of two Hamiltonians. This approach has led to many further studies of chirality on the lattice [20].

Another useful way of looking at the fermion doubling problem on the lattice is to look at the chiral anomaly structure of the lattice theory and to see what it yields in the continuum limit. A symmetry is said to be anomalous if it is no longer present in the quantum theory although it is present in the classical theory. The anomaly manifests itself by a non-conservation of a classically conserved charge. Anomalies are an inescapable part of some quantum field theories and have many important physical consequences. Their origin is related to the problem of regularizing amplitudes in quantum field theories while maintaining their invariances. In the path integral formulation of quantum field theory they arise because of the non-invariance of the measure of the path integral [9]. As is well known 1+1 dimensional QED has a chiral anomaly when massless fermions are present. The anomaly arises because it is not possible to find a
regularization of the gauge theory which maintains both the gauge invariance and chiral symmetry. A simple way of demonstrating the anomaly in 1+1 dimensions is by using a gauge invariant point-split definition of the axial vector current \[ \mathbf{10} \] which can be seen not to be conserved. There are no anomalies in the naive latticisation because it is a gauge invariant regulator which also maintains chiral symmetry, but it is impossible to put only fields of one chirality on the lattice. This is consistent with the fact that no regularisation exists which simultaneously preserves gauge invariance and chiral symmetry for arbitrary matter content. The continuum limit of the lattice theory, on the other hand, must be able to reproduce the correct anomaly structure of a given theory. The naive fermionic lattice action coupled to gauge fields gives an anomaly free theory in the continuum limit because the anomalies are cancelled between the naive and doubled modes \[ \mathbf{11} \]. It was shown in \[ \mathbf{11} \] that the Wilson term reproduces the correct anomaly on the lattice provided the symmetry breaking parameter \( r \neq 0 \), the anomaly in the continuum limit being given by the co-efficient of the Wilson term. This is quite a surprising result because the Wilson term explicitly breaks the chiral symmetry but yet reproduces the anomaly which is essentially a quantum mechanical breakdown of the classical chiral symmetry.

As stressed by Nielsen and Ninomiya, and Peskin, the Hamiltonian formulation provides a much clearer physical picture of the anomaly in terms of the energy level shifting of the filled Dirac sea \[ \mathbf{12} \] (for a very clear exposition see also \[ \mathbf{13} \]). In this picture the anomaly arises because pairs of net chirality are pumped out of the infinitely filled Dirac sea. If one tries to transcribe this picture on the lattice, as was done by Ambjorn et al \[ \mathbf{14} \], one finds that the lattice Dirac sea is always finite and the anomaly always gets cancelled by the doubler modes in the absence of the Wilson term \[ \mathbf{11,14} \]. The physical picture of the anomaly presented in \[ \mathbf{12} \] can be applied on the lattice with a Wilson mass term. The role of the Wilson mass term is to suppress the contributions to the chiral charge coming from the doubler modes resulting in a non-zero anomaly on the lattice \[ \mathbf{14} \].

Our aim is to carry out a similar analysis for fermions satisfying the G-W relation and to see how a non-zero anomaly comes about on the lattice. This should complement the derivation of the anomaly from the Ginsparg-Wilson action in the Euclideanised formalism where the anomaly is showed to arise out of the measure \[ \mathbf{6} \]. We mention here that the axial anomaly in 1+1 dimensional QED is also reproduced in
the overlap formulation [13]. Our derivation, apart from being quite different from the methods employed in [6,13], also highlights the role played by the Wilson term in giving the correct anomaly. The discussion will be in the Hamiltonian framework and we will derive the anomaly equation for the abelian theory in the 1+1 dimensions. We will then comment on the extension of this picture to 3+1 dimensions.

The Lagrangian density for a Dirac fermion in Minkowski space is given by

$$L(\psi, \bar{\psi}) = \bar{\psi}(i\gamma_\mu \partial_\mu - m)\psi;$$

the gamma matrices satisfy $\gamma^\dagger_0 = \gamma_0$, $\gamma^\dagger_i = -\gamma_i$, and obey the relation $\{\gamma_\mu, \gamma_\nu\} = -2g_{\mu\nu}$. The metric $g_{\mu\nu}$ is $\text{diag}(1, -1, -1, -1)$. $\bar{\psi}(x)$ denotes the relativistic adjoint $\psi^\dagger(x)\gamma_0$. We shall use the Weyl representation for the gamma matrices. The Hamiltonian density is given by

$$H = \bar{\psi}(x)(-i\gamma_i \partial_i + m)\psi(x)$$

Before we discuss the GW fermions in the Hamiltonian picture, it is instructive to briefly review how the unwanted doublers are handled in the Euclidean formalism with the help of the Wilson mass term. In Euclidean space the Lagrangian density becomes

$$L(\psi, \bar{\psi}) = \bar{\psi}(-i\gamma_\mu \partial_\mu - m)\psi;$$

the Euclidean gamma matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$, and $\gamma^\dagger_\mu = -\gamma_\mu$. The naive lattice discretization of the Dirac action for a massless fermion in Euclidean space is given by

$$S = \sum_{x} \frac{-i}{2a} \bar{\psi}(x)\gamma_i(\psi(x + i) - \psi(x - i)).$$

In momentum space (in $d$ dimensions) this becomes

$$S = \int_{BZ} d^d k \bar{\psi}(k)(\sum_{i=1}^{d} \frac{\sin(k_i a)}{a} \gamma_i)\psi(k).$$

$BZ$ denotes the range of integration to be the $d$ dimensional Brillouin zone of the lattice. As is well known, the above discretization suffers from the presence of additional fermions at the corners of the Brillouin zone ($2^d$ in $d$ Euclidean dimensions) leading to $2^d$ fermions in the $a \to 0$ limit. A method for eliminating the unwanted fermions is to give them very high masses in the continuum limit. The oldest way of achieving this is by adding a Wilson term to the massless action
\[ S_w = -\frac{r}{2a} \sum_{x,i} \bar{\psi}(x)(\psi(x+i) + \psi(x-i) - 2\psi(x)) \quad . \] (7)

(The Wilson term mimicks a mass term although in a more subtle way; the mass terms are ”momentum dependent”) This leads to the modified propagator (in momentum space)

\[ D_w(k) = \sum_{i=1}^{d} \gamma_i \left( \frac{\sin k_i a}{a} \right) + m + \frac{r}{a} \sum_{i=1}^{d} \left( 1 - \cos(k_i a) \right) \quad . \] (8)

In the above expression we have also introduced a bare mass term \( m \). The momentum dependent mass terms ensure that the modes at the corners of the Brillouin zone have masses of the order of \( 1/a \) and decouple from the low energy effects (in the limit \( a \to 0 \)). The price paid for eliminating these doublers is the lack of chiral symmetry in the fermion action (the Wilson term explicitly breaks chiral symmetry ).

In order to define the real time evolution of Ginsparg-Wilson fermions in the Hamiltonian formulation, we have to first construct a Hamiltonian operator starting from the Euclidean functional integral. In the transfer matrix formalism this is done by choosing a particular axis as the time direction (with a lattice spacing \( \tau \)) and then taking the so called \( \tau \) continuum limit. The \( \tau \) continuum limit (\( \tau \to 0 \)) is taken on an anisotropic lattice with different spacings in the space and the time directions. In the Euclidean formulation, the Ginsparg-Wilson operator can be formally understood to have arisen out of a chirally non-invariant blocking transformation. One could have used a \( \tau \)-continuum procedure to obtain the Hamiltonian version of the GW prescription. However, since we are eventually interested only in the Hamiltonian, it is sufficient to do the block spinning only in the spatial directions which already yields an (anisotropic) lattice with a blocked action. This blocked action has the same partition function as the original action, though spatial correlation lengths are halved on this lattice. We can then proceed to construct a Hamiltonian operator via the transfer matrix in the usual way for fermions \[8\]. This way we are able to study the real time evolution of Ginsparg-Wilson fermions. As is well known, the passage from the transfer-matrix formalism to the Hamiltonian formalism can be carried out in more than one way, depending on the simplicity or complexity one desires. Our Hamiltonian is invariant under the symmetry discussed by Luscher in \[6\] and is a valid starting point for carrying out calculations of wave functions, ground states, excited states, etc. The chirally invariant action on the lattice is

\[ S = \sum_{m,n} \bar{\phi}_m h_{mn} \phi_n \quad , \] (9)
where $h$ satisfies $\{ h, \gamma_5 \} = 0$. We can block only the spatial degrees of freedom by using the kernel (defined in [4])

$$K(\psi, \phi) = \exp -\xi \sum_{mn} (\bar{\psi}(m) - \bar{\chi}(m))(\psi(m) - \chi(m))$$  \hspace{1cm} (10)

The only difference is that the blocked field $\chi(m)$ is defined as

$$\chi(m, t) = \eta \sum_n \phi(n, t)$$  \hspace{1cm} (11)

where the above summation is only over fields defined over a hypercube at the same instant. This is the condition which ensures that only the spatial degrees of freedom are blocked. This leads to the following effective action for the fermions on the blocked lattice

$$\exp(-\tilde{A}[\bar{\psi}\psi]) = \int d\bar{\phi}d\phi \exp[-A(\bar{\phi}, \phi)]K(\psi, \phi)$$  \hspace{1cm} (12)

The action on the blocked lattice is given by

$$\tilde{A}[\bar{\psi}, \psi] = \sum_{m,n} \bar{\psi}(m)\tilde{h}_{mn}\psi(n)$$  \hspace{1cm} (13)

where the fields $\psi$ are defined on the blocked lattice which has twice the spacing (in the spatial direction) of the original lattice. The propagator of the blocked lattice satisfies the following relations

$$\{ \tilde{h}_t, \gamma_5 \} = 0$$

$$\{ \tilde{h}_s, \gamma_5 \} = a \tilde{h}_s \gamma_5 \tilde{h}_s$$

After the usual passage to the Hamiltonian via a transfer matrix formalism the Hamiltonian one obtains is simply

$$H = \sum_{x,y} \bar{\psi}(x)\gamma_0 \tilde{h}_s(x, y)\psi(y)$$  \hspace{1cm} (14)

Henceforth, the operator $\tilde{h}_s$ will be called $D$ and it satisfies the G-W relation. Any $\tilde{h}_s$ satisfying the Ginsparg-Wilson relation can be used for defining our Hamiltonian. An explicit choice for $D$ satisfying the Ginsparg-Wilson relation is the overlap operator given by Neuberger's construction [15]

$$D = \frac{1}{a} (1 - \frac{A}{\sqrt{A^\dagger A}})$$  \hspace{1cm} (15)
A is defined in terms of the Wilson operator $D_w$ as

$$A = 1 - aD_w.$$  \hspace{1cm} (16)

It can be easily checked that the operator $D$ satisfies the G-W relation. It is worth adding here that any $A$ satisfying the following properties: $A^\dagger = \gamma_5 A \gamma_5$ and $A^\dagger A$ commutes with $A$ will give a $D$ satisfying the GW relation. This may be useful in a more general context.

Though any $D$ satisfying the Ginsparg-Wilson relation can be used to construct our Hamiltonian, we have used the above explicit form proposed by Neuberger et al for our calculations. It should be stressed that apart from this choice, the considerations of this paper are independent of the overlap formalism. The above operator relations can be translated into momentum space and it is in momentum space that we will make most of our manipulations. It should be emphasised that in general, with external fields, momentum space description is not very economical. But in the $1+1$ dimensional abelian case with only an electric field, and in the $3+1$ dimensional case with uniform electric and magnetic fields, momentum space description is still useful.

The Hamiltonian of the lattice field theory is

$$H = \sum_{x,y} \bar{\psi}(x) D(x,y) \psi(y).$$  \hspace{1cm} (17)

$\bar{\psi}$ and $\psi$ can be interpreted as field operators in the usual sense. The wave equation for the fermion fields is

$$i \frac{\partial \psi(x,t)}{\partial t} = \sum_y \gamma_0 D(x,y) \psi(y).$$  \hspace{1cm} (18)

Using the properties of $D$ and $\gamma_0$ it is easy to show that the Hamiltonian is hermitian, and therefore the evolution is unitary. We are basically interested in how the anomaly arises in this model. To get the anomaly we must of course couple the fermions to an external gauge field and then look for non-conservation of the chiral charge. It is well known that the anomaly can be extracted by treating the gauge fields as a classical variable and quantizing only the fermions. In order to be able to study the problem of fermions in an external field we will have to make some approximations which will be described shortly. The analysis will be presented in $1+1$ dimensions.
In 1+1 dimensions the only effect of the external gauge field is to shift the momentum variable \( k \) to \( k - ga(t) \) where \( a(t) \) is the time dependent component of the vector potential \( A_1(x, t) = a(t) \) and \( A_0 = 0 \). This is true only in the Hamiltonian picture, even in the 1+1 case, because in the Euclidean case, \( D^\dagger D = (k - ea(t))^2 + \gamma_5 \mathcal{E} \), where \( \mathcal{E} \) is the electric field. We define the chiral charge operator on the lattice in the usual manner as

\[
Q_n = a \sum_x \psi(x, t)^\dagger \gamma_5 \psi(x, t) \ . 
\]  

All the operators are defined in the Heisenberg representation. The time dependence of the chiral charge is given by

\[
\dot{Q}_n = \frac{\partial Q_n}{\partial t} + i[H, Q_n] \ . 
\]  

Since only the second term on the righthand side contributes to \( \dot{Q}_n \) we have

\[
\dot{Q}_n = i[H, Q_n] \ . 
\]  

Evaluating the commutator this becomes

\[
\dot{Q}_n = \sum_{x,y} \bar{\psi}(x, t) \{ D, \gamma_5 \} \psi(y, t) \ . 
\]  

Using the Ginsparg-Wilson relation this becomes

\[
\dot{Q}_n = ia \sum_{x,y} \bar{\psi}(x, t) D \gamma_5 D \psi(y, t) \ . 
\]  

The spatial indices of \( D \) and all Dirac indices have been suppressed for ease in reading. Since \( D \gamma_5 D \) is non-zero we see that the chiral charge is not in general conserved, as expected. It remains to be seen if this non-conservation of the chiral charge reproduces the correct anomaly. For this the Dirac equation in Eq. 18 has to be solved in the presence of an external potential and the solutions have to be examined. We consider a time dependent potential which rises from zero to a constant value \( A_\tau \) in a time \( \tau \). \( \tau \) is a time scale in the problem and two cases can be easily analyzed, the sudden limit \( \tau \rightarrow 0 \) and the adiabatic limit \( \tau \rightarrow \infty \), as was also done in [14]. First one writes the free field \( \psi(x, t) \) as a superposition of positive and negative energy spinors

\[
\psi(x, t) = \int_{BZ} \frac{1}{2\pi} \left[ b(k) u(k) \exp(-ikx) + d^\dagger(k) v(k) \exp(ikx) \right] \ . 
\]  

9
is short for \( k_0 x_0 - k_1 x_1 \). \( E = k_0 \). As usual \( u(k) \) and \( v(k) \) represent positive and negative energy spinors. The operator \( b(k) \) destroys an electron of momentum \( k_1 \) and the operator \( d^\dagger(k) \) creates a positron of momentum \( -k_1 \). Putting these spinors in the equation for the axial charge we have

\[
\langle |\dot{Q}_n| \rangle = \int_{k_z} \frac{dk}{2\pi} \bar{v}(k,t)D(k,t)\gamma_5 D(k,t)v(k,t)\langle d(k)d^\dagger(k) \rangle .
\] (25)

The angular brackets denote expectation values in the vacuum which is the state with zero electrons and positrons. Using the definition of \( D \) and the properties of the gamma matrices it is easy to show that

\[
D(k)\gamma_5 D(k) = \gamma_5 D^\dagger(k)D(k) .
\] (26)

Now, \( D^\dagger(k)D(k) \) is a c-number and acts trivially on the Dirac spinors. \( D^\dagger D \) is not a c-number in general. For the specific class where \( D \) is of the form \( A\gamma_i + B \) with \( A_i, B \) commuting, this is true. Already in the 3 + 1 dimensional case this is no longer true even for a uniform magnetic field.) In order to evaluate \( \bar{v}(k,t)\gamma_5 v(k,t) \) we have to determine the evolution of the negative energy spinor in the external field \( a(t) \).

Before we calculate this quantity it is instructive to calculate the same without any field. In the absence of an external field the positive and negative energy states evolve as

\[
u(k,t) = \exp(-iE(k)t)\bar{u}(k)
\]
\[
v(k,t) = \exp(iE(k)t)\bar{v}(k) .
\]

The spinors \( \bar{u}(k) \) and \( \bar{v}(k) \) satisfy the time independent Schrödinger equations with positive and negative energies.

\[
\gamma_0 D(k)\bar{u}(k) = E(k)\bar{u}(k)
\]
\[
\gamma_0 D(-k)\bar{v}(k) = -E(k)\bar{v}(k) .
\]

The Weyl representation for the 2 dimensional \( \gamma \) matrices is

\[
\gamma_1 = i\gamma_5 = i\sigma_3 \quad \gamma_0 = \sigma_1 \quad \gamma_5 = \sigma_2 .
\] (27)

\( \sigma_1 \) and \( \sigma_3 \) are the Pauli matrices. The eigen values of the spinors are given by \( E(k)^2 = D^\dagger(k)D(k) \). Using the definition of \( D(k) \) we get \( D^\dagger(k)D(k) \) to be a c-number. \( D(k) \) can be written as
\[
D(k) = \frac{1}{a} (1 - \frac{1 - \text{am}(k)}{\sqrt{A^2(k)}}) + \frac{1}{a} \left( \frac{\sin(k_1 a)}{\sqrt{A^2(k)}} \right) \gamma_1 .
\]  

(28)

If we now write \( D(k) \) as

\[
D(k) = g(k, t) + \gamma_1 f(k, t) ,
\]

then \( E_2^2(k) = D^\dagger(k)D(k) = f^2(k, t) + g^2(k, t) \). \( f \) and \( g \) are functions given by

\[
g(k) = \frac{1}{a} (1 - \frac{1 - \text{am}(k)}{\sqrt{A^2(k)}})
\]

\[
f(k) = \frac{1}{a} \left( \frac{\sin(k_1 a)}{\sqrt{A^2(k)}} \right) .
\]

The function \( f \) is an odd function of \( k \) whereas the function \( g \) is an even function of \( k \). The time independent spinors can be normalized to satisfy

\[
\tilde{u}(k) \gamma_5 \tilde{u}(k) = 0
\]

\[
\tilde{v}(k) \gamma_5 \tilde{v}(k) = 0
\]

\[
\tilde{u}(k) \gamma_5 \tilde{v}(k) = 1 .
\]

This means that \( \langle \dot{Q}_n \rangle = 0 \) in the absence of an external field. This is as expected, there is no anomaly in zero external field even though the Hamiltonian is not chirally invariant.

In the presence of an external field the only change in the operator \( D \) is a replacement of \( k \) by \( k - g a(t) \). Since the structure of \( D \) is not affected by an external field the Ginsparg-Wilson relation is still satisfied. Although the evolution of Dirac spinors in an arbitrary external field can only be analyzed numerically, two limiting cases admit a simpler analysis. These are the adiabatic limit and the sudden limit, and we shall examine these two cases separately. When an external field is turned on slowly (compared to the time scales in the system) we can use the adiabatic approximation. The sudden approximation is useful when the field is turned on faster than the fastest time scale in the system. The rate at which the field is turned on can be controlled by introducing a parameter \( \tau \) defined as follows

\[
A(t) = 0 \quad t \leq 0
\]

\[
A(t) = A \quad t > \tau .
\]

(30)

(31)
The precise form of $A(t)$ for $0 < t < \tau$ is not very important. The sudden limit corresponds to $\tau \to 0$ and the adiabatic limit corresponds to $\tau \to \infty$. In the adiabatic approximation the form of the positive and negative energy spinors for times $0 < t < \tau$ is given by

$$u(k, t) = a(k, t)\xi^*(t)u^0(k, t) + b(k, t)\xi(t)v^0(k, t) \quad (32)$$

$$v(k, t) = c(k, t)\xi^*(t)u^0(k, t) + d(k, t)\xi(t)v^0(k, t) \quad . \quad (33)$$

The above equation is written for the individual fourier components of the positive and negative energy spinors in an external field, $u^0(k, t)$ and $v^0(k, t)$ are the positive and negative energy spinors satisfied by the instantaneous Schrodinger equation at the instant $t$. We closely follow the notation of [14] and we have also corrected some of the misprints which occur therein. $\xi(t)^*$ and $\xi(t)$ are the phase factors for the positive and negative energy spinors. The phase factor $\xi(t)$ is given by

$$\xi(t) = \exp(i \int_0^t E(k, t')dt') \quad . \quad (34)$$

$a, b, c,$ and $d$ are called Bogoulobov coefficients. The Bogoulobov co-efficients are chosen to satisfy the following boundary conditions

$$a(0) = d(0) = 1 \quad \quad b(0) = c(0) = 0 \quad . \quad (35)$$

These boundary conditions ensure that we are looking at the evolution of the positive and the negative energy states before the external field is switched on. As mentioned before, $u^0(k, t)$ and $v^0(k, t)$ are positive and negative energy spinors having momentum $k$ and $-k$ respectively, and they satisfy the instantaneous Schrodinger equations given by

$$\gamma_0 D(k)u^0(k, t) = E(k, t)u^0(k, t)$$

$$\gamma_0 D(-k)v^0(k, t) = -E(k, t)v^0(k, t) \quad .$$

Substituting the expressions in Eq. 32 and Eq. 33 in the wave equation for the fermions and using the previously mentioned boundary conditions we get

$$c(k, t) = \int_0^t \alpha(k, t')d(k, t')\xi^2(t')dt' \quad (36)$$
\begin{equation}
d(k, t) = 1 - \int_0^t \alpha(k, t')c(k, t')\xi^*(t')dt'
\end{equation}

\begin{equation}
b(k, t) = -c^*(k, t)
\end{equation}

\begin{equation}
a(k, t) = d^*(k, t)
\end{equation}

The quantity $\alpha(k, t)$ is defined by

\begin{equation}
\dot{u}^0(k, t) = \alpha(k, t)v^0(k, t)
\end{equation}

and is

\begin{equation}
\alpha(k, t) = \frac{(g\dot{f} - f\dot{g})}{2E^2(k, t)}
\end{equation}

After using the stated normalizations of the spinors, $\bar{v}(k, t)\gamma_5 v(k, t)$ is given

\begin{equation}
\bar{v}(k, t)\gamma_5 v(k, t) = c^*d\xi^2 - c.c.
\end{equation}

The co-efficients $c(k, t)$ and $d(k, t)$ can be approximated by (for small values of $\alpha(k, t)$)

\begin{equation}
c(k, t) = -i\frac{\alpha(k, t)}{2E(k, t)}\xi^2(t) \quad d(k, t) = 1 + O(\alpha^2)
\end{equation}

In the adiabatic approximation the quantities inside the integrand on the right hand side of Eq. 36 and Eq. 37 are evaluated at $t = 0$. Substituting for the values of $f(k, t)$ and $g(k, t)$ and using the relation

\begin{equation}
D(k)\gamma_5 D(k) = \gamma_5 D\gamma_5 D(k) = \gamma_5 E^2(k)
\end{equation}

the r.h.s of Eq. 25 becomes

\begin{equation}
\int_{BZ} \frac{1}{2\pi} dk \ C(k) \ (a a \dot{g}(t))
\end{equation}

where $C(k)$, a complicated expression, is given in the appendix along with the expressions for $f, g, \dot{f}, \dot{g}$. The function $C(k)$ can be plotted and the integral of $C(k)$ over the Brillouin zone can be estimated numerically.

The function $C(k)$ depends on $r, k, a, m$. We plot $C(k)$ as a function of $k$ in Fig. to Fig. We first plot it for zero mass and then for a non-zero value ($m = 5$). The eigenvalues of the spinors (in a zero external field) are also plotted in Fig. to Fig. The first thing we observe is that when $r = 0$ the eigenvalue
spectrum does not distinguish between the modes at \( k = 0 \) and \( k = \pi/a \), and the integral of \( C(k) \) over the Brillouin zone is zero. For \( r \neq 0 \) the modes at \( k = 0 \) and \( k = \pi/a \) have different energies and an asymmetry develops in the function \( C(k) \). For \( r = 1 \), the integral of \( C(k) \) over the Brillouin Zone has the value \(-2\).

Substituting this in Eq. 25 we get the anomaly equation

\[
\langle \dot{Q}_n \rangle = -\frac{g}{2\pi} \int d^2 x \epsilon_{\mu\nu} F_{\mu\nu}
\]

in the continuum limit.

We have studied the function \( C(k) \) for different values of \( r \) and we find that the anomaly is independent of \( r \) for large \( r \) but vanishes for a smaller and, in particular, a zero value of \( r \). The value \( r = 0.5 \) seems to separate the region with and without the anomaly. A non-zero Wilson \( r \) parameter is necessary to get the anomaly in the continuum limit. This means that a Neuberger like operator for \( D \) where the naive Dirac operator is used in place of \( D_w \) in Eq. 26 will not reproduce the correct anomaly in the continuum limit inspite of satisfying the Ginsparg-Wilson relation (the operator \( D \) with \( D_w \) replaced by \( D_{\text{naive}} \) also satisfies the Ginsparg-Wilson relation). The case \( m \neq 0 \) can also be analyzed along the same lines and it turns out that the integral of \( C(k) \) is very small, consistent with zero. It appears that in this case we have an exact cancellation of the bare mass term with the anomaly term to give a zero rate of change of chiral charge. Nevertheless, the anomaly term is still present and so is the mass term, but the two appear with opposite signs. The adiabatic approximation is justified when the switching time \( \tau \) is much greater than the characteristic time periods of the system. In our example \( 2\pi/E(k) \) is the characteristic time period of the system and the adiabatic approximation is justified when \( \tau \gg a \). To summarize, a zero bare mass term with a non-zero \( r \) parameter gives an anomaly independent of \( r \) for \( r > 0.5 \). When a bare mass term is included we have a cancellation of the anomaly term with the mass term though both terms are still present.

The rate of change of the chiral charge can also be calculated in the sudden approximation in which an external field is turned on infinitely fast. This approximation corresponds to the limit \( \tau \to 0 \). In this limit the spinors \( u_0(k,t), v_0(k,t) \) are unchanged immediately after the field is turned on and only their evolution is governed by the new Hamiltonian (with a constant field). The normalization of the spinors in a constant external field (with value \( A_\tau \)) can be made just as in the zero field case and there is no anomaly in this
limit. This limit corresponds to the case \( \tau \ll a \).

The time \( \tau \) is a characteristic time associated with the gauge fields on the lattice and our calculation clearly shows that in order to get the correct anomaly we have to ensure that we are not in the regime of the sudden approximation. If we are in the intermediate region we will see a crossover from one limit to another limit. So far we have only studied fermions and (abelian) gauge fields in 1+1 dimensions. It was pointed out in [12] that the anomaly in 3+1 dimensions factorizes into a 1+1 dimensional part and an extra factor coming from the additional dimensions. We briefly review the argument in [12]. To get the anomaly in 3+1 dimensions we first turn on a magnetic field in the, say z, direction. This leads to the usual Landau levels for the fermions which are labelled by integers. We then turn on an electric field parallel to the magnetic field. The important point is that the fermions in the lowest Landau level in the presence of this electric field behave like fermions in the 1+1 dimensional case that we have just analyzed. Hence the same 1+1 dimensional anomaly is present but with an additional degeneracy factor coming from the Landau levels. As shown in [14] the argument goes through for the lattice Dirac sea case for the case of an uniform magnetic field. The degeneracy factor is a geometrical quantity that is in general dependent on the details of the lattice Hamiltonian which is more complicated for Ginsparg-Wilson fermions. However, when the Ginsparg-Wilson operator is constructed as \( aD_{GW} = (1 - \frac{1}{\sqrt{A}}) \) with \( A = 1 - aD_W \), the degeneracies of \( D_{GW} \) and \( D_W \) are the same, and in the limit of zero lattice spacing the degeneracy is just

\[
L_1 L_2 gH/(2\pi)
\]

the number of states in the square \( L_1 L_2 \) perpendicular to the magnetic field \( (H) \). The above factor simply multiplies the 1+1 dimensional anomaly and gives the correct anomaly in 3+1 dimensions.

The main aim of this note was to show that a Hamiltonian analysis of Ginsparg-Wilson fermions leads to a non-zero rate of chiral charge and gives the anomaly equation in the continuum limit. The doubler modes are suppressed by the Wilson parameter, in fact the Wilson parameter \( r \) plays a crucial role in yielding the correct anomaly. A quantum mechanical analysis supplemented by an adiabatic approximation was necessary to get the anomaly. It is noteworthy that if we are not in the adiabatic regime we will get other contributions (\( \dot{\gamma}(t) \) and higher time derivatives) to the chiral charge and this will not reproduce the anomaly equation. It may be useful to compare our derivation with that of the overlap method. In the
overlap method the anomaly is extracted by looking at the scalar product of the ground states of two
different many body Hamiltonians, whereas in our approach we study the dynamical picture behind the
anomaly by using the properties of the Dirac sea in an external electric field in the adiabatic limit.
Appendix

In this appendix we collect together some expressions which are necessary to get the function \(C(k)\).

\[
f(k) = \frac{1 - a \left( m + \frac{r(1 - \cos(a k))}{a} \right)}{\sqrt{1 - 2a \left( m + \frac{r(1 - \cos(a k))}{a} \right) + a^2 \left( m + \frac{r(1 - \cos(a k))}{a} \right)^2 + \sin(a k)^2}}
\]

(48)

\[
g(k) = \frac{\sin(a k)}{a \sqrt{1 - 2a \left( m + \frac{r(1 - \cos(a k))}{a} \right) + a^2 \left( m + \frac{r(1 - \cos(a k))}{a} \right)^2 + \sin(a k)^2}}
\]

(49)

\[
f(k) = \frac{\left( 1 - m - \frac{r(1 - \cos(a k))}{a} \right) \left( -2r \sin(a k) + 2ar \left( m + \frac{r(1 - \cos(a k))}{a} \right) \sin(a k) + 2 \cos(a k) \sin(a k) \right)}{2a \left( 1 - 2a \left( m + \frac{r(1 - \cos(a k))}{a} \right) + a^2 \left( m + \frac{r(1 - \cos(a k))}{a} \right)^2 + \sin(a k)^2 \right)^{\frac{3}{2}}}
\]

\[
\hat{f}(k) = \frac{r \sin(a k)}{a \left( 1 - 2a \left( m + \frac{r(1 - \cos(a k))}{a} \right) + a^2 \left( m + \frac{r(1 - \cos(a k))}{a} \right)^2 + \sin(a k)^2 \right)}
\]

\[
g(k) = \frac{-\sin(a k)}{2a \left( 1 - 2a \left( m + \frac{r(1 - \cos(a k))}{a} \right) + a^2 \left( m + \frac{r(1 - \cos(a k))}{a} \right)^2 + \sin(a k)^2 \right)^{\frac{3}{2}}}
\]

\[
\hat{g}(k) = \frac{\cos(a k)}{a \left( 1 - 2a \left( m + \frac{r(1 - \cos(a k))}{a} \right) + a^2 \left( m + \frac{r(1 - \cos(a k))}{a} \right)^2 + \sin(a k)^2 \right)}
\]

\[
E^2(k) = \frac{\sin(a k)^2}{a^2 \left( 1 - 2a \left( m + \frac{r(1 - \cos(a k))}{a} \right) + a^2 \left( m + \frac{r(1 - \cos(a k))}{a} \right)^2 + \sin(a k)^2 \right)} + \frac{\left( 1 - a \left( m + \frac{r(1 - \cos(a k))}{a} \right) \right)^2}{\sqrt{1 - 2a \left( m + \frac{r(1 - \cos(a k))}{a} \right) + a^2 \left( m + \frac{r(1 - \cos(a k))}{a} \right)^2 + \sin(a k)^2}}
\]

\[
C(k) = (1/(2E(k))) \hat{f}(k) - f(k) \hat{g}(k)
\]

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FIG. 1. $C(k)$ at $m=0$ and $r=0$.

FIG. 2. $C(k)$ at $m=0$ and $r=0.4$

FIG. 3. $C(k)$ at $m=0$ and $r=0.6$

FIG. 4. $C(k)$ at $m=0$ and $r=1.0$

FIG. 5. $C(k)$ at $m=0$ and $r=5.0$

FIG. 6. $E(k)$ at $m=0$ and $r=0.0$

FIG. 7. $E(k)$ at $m=0$ and $r=0.4$

FIG. 8. $E(k)$ at $m=0$ and $r=0.6$

FIG. 9. $E(k)$ at $m=0$ and $r=1.0$

FIG. 10. $E(k)$ at $m=0$ and $r=5.0$

FIG. 11. $E(k)$ at $m=5$ $r=0$

FIG. 12. $E(k)$ at $m=5$ $r=5$
