Static properties and semileptonic decays of doubly heavy baryons in a nonrelativistic quark model.

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We evaluate static properties and semileptonic decays for the ground state of doubly heavy Ξ, Ξ’, Ξ∗ and Ω, Ω’, Ω∗ baryons. Working in the framework of a nonrelativistic quark model, we solve the three-body problem by means of a variational ansatz made possible by heavy quark spin symmetry constraints. To check the dependence of our results on the inter-quark interaction we use five different quark-quark potentials that include a confining term plus Coulomb and hyperfine terms coming from one–gluon exchange. Our results for static properties (masses, charge radii and magnetic moments) are, with a few exceptions for the magnetic moments, in good agreement with a previous Faddeev calculation. Our much simpler wave functions are used to evaluate semileptonic decays of doubly heavy Ξ, Ξ’ (J = 1/2) and Ω, Ω’ (J = 1/2) baryons. Our results for the decay widths are in good agreement with calculations done within a relativistic quark model in the quark–diquark approximation.

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I. INTRODUCTION

The subject of doubly heavy baryons has been attracting attention for a long time. Magnetic moments of doubly charmed baryons were evaluated back in the 70’s by Lichtenberg [1] within a nonrelativistic approach. The infinite heavy quark mass limit was already used in the 90’s to relate the spectrum of doubly heavy baryons to the one of mesons with a single heavy quark [2], or to analyze their semileptonic decay [3]. A factor of two error in the hyperfine

| Baryon | S | J^P | I | S^h | Quark content |
|-------|---|-----|---|-----|--------------|
| Ξ_{bc} | 0 | 1^+ | 1 | 1^+ | ccl |
| Ξ’_{bc} | 0 | 3/2^+ | 3/2 | 1^+ | ccl |
| Ω_{bc} | −1 | 1^+ | 0 | 1^+ | ccs |
| Ω’_{bc} | −1 | 3/2^+ | 3/2 | 1^+ | ccs |
| Ξ_{bb} | 0 | 1^+ | 1 | 1^+ | bbl |
| Ξ’_{bb} | 0 | 3/2^+ | 3/2 | 1^+ | bbl |
| Ω_{bb} | −1 | 1^+ | 0 | 1^+ | bbs |
| Ω’_{bb} | −1 | 3/2^+ | 3/2 | 1^+ | bbs |
| Ξ_{bc} | 0 | 1^+ | 1 | 0^+ | bcl |
| Ξ’_{bc} | 0 | 3/2^+ | 3/2 | 1^+ | bcl |
| Ω_{bc} | −1 | 1^+ | 0 | 0^+ | bcs |
| Ω’_{bc} | −1 | 3/2^+ | 3/2 | 1^+ | bcs |

TABLE I: Quantum numbers of doubly heavy baryons analyzed in this study. S, J^P are strangeness and the spin parity of the baryon, I is the isospin, and S^h is the spin parity of the heavy degrees of freedom. I denotes a light u or d quark.

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The findings of this work are summarized in Sect. III. In Sect. IV we present results for the static properties: masses (Subsect. IV A), charge densities and radii (Subsect. IV B), and magnetic moments (Subsect. IV C). Semileptonic decays are analyzed in Sect. V. After the presentation of general formulas, in Subsect. VA we relate the form factors to matrix elements and show how the latter ones are evaluated within our model. In Subsect. VB we present our results for the form factors, differential and total semileptonic decay widths, and angular asymmetry parameters. The findings of this work are summarized in Sects. III and IV.

The paper is organized as follows. In Sect. II we study the Hamiltonian of the system (Subsect. II A) and briefly introduce the different inter-quark interactions used in this work (Subsect. II B). The variational wave functions are discussed in Sect. III. In Sect. IV we present results for the static properties: masses (Subsect. IV A), charge densities and radii (Subsect. IV B), and magnetic moments (Subsect. IV C). Semileptonic decays are analyzed in Sect. V. After the presentation of general formulas, in Subsect. VA we relate the form factors to matrix elements and show how the latter ones are evaluated within our model. In Subsect. VB we present our results for the form factors, differential and total semileptonic decay widths, and angular asymmetry parameters. The findings of this work are summarized in Sects. III and IV.

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in Sect. VI. The paper also includes three appendices: in appendix A we analyze the infinite heavy quark mass limit of our radial wave functions. In appendix B we relate the form factors for semileptonic decay to the two basic integrals in terms of which all of them can be obtained. Finally, in appendix C we give explicit expressions for those basic integrals.

In Table I we summarize the quantum numbers of the doubly heavy baryons considered in this study.

II. THREE BODY PROBLEM

A. Intrinsic Hamiltonian

In the Laboratory (LAB) frame (see Fig. 1), the Hamiltonian \( H \) of the three quark \((h_1, h_2, q)\), where \( h_1, h_2 = c, b \) and \( q = l(u, d, s) \) system reads:

\[
H = \sum_{j=h_1, h_2, q} \left( m_j - \frac{\vec{\nabla}^2 \vec{x}_j}{2m_j} \right) + V_{h_1 h_2} + V_{h_1 q} + V_{h_2 q}
\]  

where \( m_{h_1}, m_{h_2}, m_q \) are the quark masses and the quark-quark interaction terms \( V_{jk} \) depend on the quark spin-flavor quantum numbers and the quark coordinates \((\vec{x}_{h_1}, \vec{x}_{h_2}, \vec{x}_q)\) for the \( h_1, h_2, q \) quarks respectively. To separate the Center of Mass (CM) free motion, we go to the light quark frame \((\vec{R}, \vec{r}_1, \vec{r}_2)\),

\[
\vec{R} = \frac{m_{h_1} \vec{x}_{h_1} + m_{h_2} \vec{x}_{h_2} + m_q \vec{x}_q}{m_{h_1} + m_{h_2} + m_q}
\]

\[
\vec{r}_1 = \vec{x}_{h_1} - \vec{x}_q \]

\[
\vec{r}_2 = \vec{x}_{h_2} - \vec{x}_q
\]

where \( \vec{R} \) and \( \vec{r}_1, \vec{r}_2 \) are the CM position in the LAB frame and the relative positions of the \( h_1, h_2 \) heavy quarks with respect to the light quark \( q \). The Hamiltonian now reads

\[
H = -\frac{\vec{\nabla}^2 \vec{R}}{2M} + H^{\text{int}}
\]  

\[1\] Note that the definitions of \( \Xi_{bc} \) and \( \Xi'_{bc} \) are interchanged in some references, with \( \Xi_{bc} \) having \( S_h = 0 \) and \( \Xi'_{bc} \) having \( S_h = 1 \). The same applies to \( \Omega_{bc} \) and \( \Omega'_{bc} \). In tables we always quote the results corresponding to the convention we use.
\[ H^{\text{int}} = M + \sum_{j=1,2} H_j^{sp} + V_{h_1h_2}(\vec{r}_1 - \vec{r}_2, \text{spin}) - \frac{\vec{\nabla}_1 \cdot \vec{\nabla}_2}{m_q} \]
\[ H_j^{sp} = -\frac{\vec{\nabla}_j^2}{2\mu_j} + V_{bq}(\vec{r}_j, \text{spin}), \quad j = 1, 2 \]

where \( M = m_{h_1} + m_{h_2} + m_q \), \( \mu_j = (1/m_{h_1} + 1/m_q)^{-1} \) and \( \vec{\nabla}_j = \partial/\partial \vec{r}_j \), \( j = 1, 2 \). The intrinsic Hamiltonian \( H^{\text{int}} \) describes the dynamics of the baryon. Apart from the sum of the quark masses \( M \), it consists of the sum of two single particle Hamiltonian \( H_j^{sp} \), each of them describing the dynamics of a heavy-light quark system, plus the heavy–heavy interaction term, including the Hughes-Eckart term \((\vec{\nabla}_1 \cdot \vec{\nabla}_2)\). We will use a variational approach to solve it.

B. Quark–Quark Interactions

We have examined five different interactions, one suggested by Bhaduri and collaborators [19] (BD) and four suggested by B. Silvestre-Brac and C. Semay [20, 21] (AL1, AL2, AP1 y AP2). All of them contain a confinement term, plus Coulomb and hyperfine terms coming from one-gluon exchange, and differ from each other in the form factors used for the hyperfine terms, the power of the confining term or the use of a form factor in the one gluon exchange Coulomb potential. All free parameters in the potentials had been adjusted to reproduce the light (\( \pi, \rho, K, K^+, \text{etc.} \)) and heavy-light (\( D, D^*, B, B^*, \text{etc.} \)) meson spectra\(^2\). All details on the above interactions can be found in Refs. [13, 20, 21].

These interactions were also used in Ref. [21] to obtain, within a Faddeev calculation, the spectrum and static properties of heavy baryons. Our simpler variational method will not only give equally good results for the observables analyzed in [21], but it will also provide us with easy to handle wave functions that can be used to evaluate other observables.

III. VARIATIONAL WAVE FUNCTIONS

For the above mentioned interactions, we have that both the total spin and the internal orbital angular momentum given as
\[ \vec{S} = (\vec{\sigma}_{h_1} + \vec{\sigma}_{h_2} + \vec{\sigma}_q)/2 \]
\[ \vec{L} = \vec{l}_1 + \vec{l}_2, \quad \text{with} \quad \vec{l}_j = -i \vec{r}_j \times \vec{\nabla}_j, \quad j = 1, 2 \]
commute with the intrinsic Hamiltonian and are thus well defined. We are interested in the ground state of baryons with total angular momentum \( J = 1/2, 3/2 \) so that we can assume the orbital angular momentum of the baryons to be \( L = 0 \). This implies that the spatial wave function can only depend on the relative distances \( r_1, r_2 \) and \( r_{12} = |\vec{r}_1 - \vec{r}_2| \). Furthermore when the heavy quark mass is infinity (\( m_h \rightarrow \infty \)), the total spin of the heavy degrees of freedom, \( \vec{S}_{\text{heavy}} = (\vec{\sigma}_{h_1} + \vec{\sigma}_{h_2})/2 \), commutes with the intrinsic Hamiltonian, since the spin–spin terms in the potentials vanish in this limit. We can then assume the spin of the heavy degrees of freedom to be well defined.

With these simplifications we have used the following intrinsic wave functions in our variational approach\(^3\)

- \( \Xi_{h_1h_2}, \Omega_{h_1h_2} \)-type baryons:
\[ |\Xi_{h_1h_2}, \Omega_{h_1h_2}; J = \frac{1}{2}; M_J \rangle = \sum_{M_{S_h}M_{S_q}} (1/2) |M_{S_h}M_{S_q}M_J \rangle |h_1h_2; 1M_{S_h} \rangle \otimes |q; 1/2M_{S_q} \rangle \]
\[ \times \Psi_{h_1h_2}(r_1, r_2, r_{12}) \]

where \( M_J \) is the third component of the baryon total angular momentum while \( |h_1h_2; S_h, M_{S_h} \rangle \) and \( |q; S_q, M_{S_q} \rangle \) represent spin states of the \( h_1h_2 \) subsystem and the light quark respectively. \((jj_1j_2|m_1m_2m)\) is a Clebsch-Gordan

\(^{2}\) To get the quark–quark interaction starting from a quark-antiquark one the usual \( V_{ij}^{qq} = V_{ij}^{qq}/2 \) prescription, coming from a \( \vec{\lambda}_i\vec{\lambda}_j \) color dependence (\( \vec{\lambda} \) are the Gell-Mann matrices) of the whole potential, has been assumed.

\(^{3}\) We omit the antisymmetric color wave function and the plane wave for the center of mass motion which are common to all cases.
The rest of the variational parameters are determined by the variational condition $\mu$ with $\delta F_{\mu}$ where $N$ is a constant, which is determined from normalization

\[ 1 = \int d^3 r_1 \int d^3 r_2 \left| \Psi_{h_1 h_2}^{\Xi}(r_1, r_2, r_{12}) \right|^2 = 8\pi^2 \int_0^{+\infty} dr_1 r_1^2 \int_0^{+\infty} dr_2 r_2^2 \int_{-1}^{+1} d\mu \left| \Psi_{h_1 h_2}^{\Xi}(r_1, r_2, r_{12}) \right|^2 \]

with $\mu$ being the cosine of the angle between the vectors $\vec{r}_1$ and $\vec{r}_2$. For simplicity, we shall assume a Jastrow–type functional form

\[ \Psi_{h_1 h_2}^{\Xi}(r_1, r_2, r_{12}) = N \left( \frac{r_{12}}{r_{12}} \right)^{\frac{1}{2}} \varphi_1(r_1) \varphi_2(r_2) \]

where $N$ is a constant.

Similarly, the functions $\varphi_1$ and $\varphi_2$ will be taken as the $S$-wave ground states $\varphi(r_j)$ of the single particle Hamiltonians $H_{12}^{\text{pp}}$ of Eq. (1) modified at large distances.

\[ \varphi_1(r_j) = (1 + \alpha_j r_j) \varphi(r_j), \quad j = 1, 2 \]

The heavy–heavy correlation function $F^B$ will be given by a linear combination of gaussians

\[ F^B(r_{12}) = \sum_{j=1}^{4} a_j e^{-b_j^2(r_{12} + \delta_j)^2}, \quad a_1 = 1 \]

The value of one of the $\alpha_j$ parameters can be absorbed into the normalization constant $N$, so that we fix $a_1 = 1$. The rest of the variational parameters are determined by the variational condition $\delta\langle B | H^{\text{int}} | B \rangle = 0$.

Although it is not self-evident from the functional form assumed, our variational wave functions are consistent with the infinite heavy quark mass limit, reducing in that limit to the product of the internal wave function for the heavy diquark times the wave function for the relative motion of the light quark with respect to a pointlike heavy diquark. All this is discussed in appendix A.

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4 A similar form lead to very good results in the case of baryons with a single heavy quark.

5 Note that $F^B$ should vanish at large distances because of the confinement potential. The confinement potential is also responsible for the non–vanishing values of the parameters $\alpha_j$, $j = 1, 2$ in Eq. (11).
calculations, ours and the relativistic calculations in Refs. [22, 25, 29] have the best overall agreement with lattice framework of a nonrelativistic quark model but without the use of an explicit Hamiltonian. Within full dynamical the calculation of Ref. [30] where they use the Feynman-Hellmann theorem and semiempirical mass formulas in the lattice results for doubly of \( M \) a few per cent. From the experimental point of view the SELEX Collaboration [5] has recently measured the value of \( M \) for states including the contribution of the three-body force. All calculations give similar results that vary within different potentials. The same presentation is used for the results obtained in Ref. [21] for which we now show their to the results obtained with the AL1 potential, while the errors quoted take into account the variation when using results agree within errors with the lattice data for baryons with two is not given. There are also different lattice determinations for baryons with two equal heavy quarks [33, 34, 35]. Our introduction, one should take this experimental value with due caution. Note also that in Ref. [5] the systematic error of \( M \) stands for the results obtained in Ref. [21] using the same interquark interactions but within a Faddeev approach. TABLE II: Doubly heavy \( \Xi \) and \( \Omega \) baryons masses in MeV. VAR stands for the results of our variational calculation. FAD stands for the results obtained in Ref. [21] using the same interquark interactions but within a Faddeev approach. IV. STATIC PROPERTIES A. Masses The mass of the baryon is simply given by the expectation value of the intrinsic Hamiltonian. Our results (VAR) appear in Table I where we compare them with the ones obtained in Ref. [21] with the use of the same inter-quark interactions but within a Faddeev approach (FAD). For that purpose we have eliminated from the latter a small three-body force contribution of the type \( V_{123} = \text{constant}/m_{b_1}m_{b_2}m_{q} \) that was also included in the evaluation of Ref. [21]. We will show their full results in the following tables. Whenever comparison is possible we find an excellent agreement between the two calculations. In some cases the variational masses are even lower than the Faddeev ones. Besides we give predictions for states that were not considered in the study of Ref. [21].

Note in Ref. [30] the \( \Xi_{bc} \) and \( \Omega_{bc} ; 1 \rangle \) are well defined, being 0 for \( \Xi_{bc} \), \( \Omega_{bc} \) and 1 for \( \Xi_{bc}^* ; 0 \rangle \), \( \Omega_{bc}^* ; 0 \rangle \). They are thus linear combinations of our states. The different spin functions are related by

\[
\begin{align*}
\left( |qc; 0 \rangle \otimes |b; \frac{1}{2} \rangle \right)_{J=1/2} &= \frac{1}{2} \left( |bc; 0 \rangle \otimes |q; \frac{1}{2} \rangle \right)_{J=1/2} - \frac{\sqrt{3}}{2} \left( |bc; 1 \rangle \otimes |q; \frac{1}{2} \rangle \right)_{J=1/2} \\
\left( |qc; 1 \rangle \otimes |b; \frac{1}{2} \rangle \right)_{J=1/2} &= -\frac{\sqrt{3}}{2} \left( |bc; 0 \rangle \otimes |q; \frac{1}{2} \rangle \right)_{J=1/2} - \frac{1}{2} \left( |bc; 1 \rangle \otimes |q; \frac{1}{2} \rangle \right)_{J=1/2}
\end{align*}
\]

In order to extract their predictions for the \( \Xi_{bc} \), \( \Xi_{bc}^* \) and \( \Omega_{bc} ; 1 \rangle \), \( \Omega_{bc}^* ; 0 \rangle \) baryons with total spin of the two heavy quarks well defined, we have assumed that the above relations, but with coefficients square, are also valid for the masses. Note this may be incorrect as we are neglecting a possible non negligible interference contribution.

| \( \Xi_{cc} \) | AL1 | AL2 | AP1 | AP2 | BD | \( \Omega_{cc} \) | AL1 | AL2 | AP1 | AP2 | BD |
|---|---|---|---|---|---|---|---|---|---|---|---|
| VAR | 3612 | 3619 | 3629 | 3630 | 3639 | \( \Omega_{cc} \) | VAR | 3702 | 3718 | 3711 | 3710 | 3743 |
| FAD [21] | 3609 | 3616 | 3625 | 3628 | 3633 | | FAD [21] | 3711 | 3718 | 3710 | 3709 | 3741 |
| \( \Xi_{bc}^* \) | VAR | 3706 | 3715 | 3722 | 3729 | 3722 | \( \Omega_{bc}^* \) | VAR | 3783 | 3802 | 3800 | 3802 | 3805 |
| \( \Xi_{bb} \) | VAR | 10197 | 10180 | 10207 | 10179 | 10202 | \( \Omega_{bb} \) | VAR | 10260 | 10249 | 10259 | 10226 | 10274 |
| FAD [21] | 10194 | 10175 | 10204 | 10176 | 10197 | | FAD [21] | 10267 | 10246 | 10258 | 10224 | 10271 |
| \( \Xi_{bb}^* \) | VAR | 10236 | 10219 | 10245 | 10219 | 10235 | \( \Omega_{bb}^* \) | VAR | 10297 | 10287 | 10301 | 10269 | 10302 |
| \( \Xi_{bc} \) | VAR | 6919 | 6912 | 6933 | 6917 | 6936 | \( \Omega_{bc} \) | VAR | 6986 | 6986 | 6990 | 6969 | 7013 |
| FAD [21] | 6916 | 6913 | 6928 | 6907 | 6934 | | FAD [21] | 7003 | 6996 | 6996 | 6971 | 7023 |
| \( \Xi_{bc}^* \) | VAR | 6948 | 6942 | 6957 | 6944 | 6965 | \( \Omega_{bc}^* \) | VAR | 7009 | 7010 | 7011 | 6994 | 7033 |
| \( \Xi_{bc}^* \) | VAR | 6986 | 6981 | 7000 | 6987 | 6993 | \( \Omega_{bc}^* \) | VAR | 7046 | 7047 | 7055 | 7037 | 7057 |

Note in Ref. [30] the \( \Xi_{bc} \), \( \Xi_{bc}^* \) and \( \Omega_{bc} ; 1 \rangle \), \( \Omega_{bc}^* ; 0 \rangle \) baryons are defined such that the total spin of the light q quark and the heavy c quark are well defined, being 0 for \( \Xi_{bc} \), \( \Omega_{bc} \) and 1 for \( \Xi_{bc}^* ; 0 \rangle \), \( \Omega_{bc}^* ; 0 \rangle \). They are thus linear combinations of our states. The different spin functions are related by

\[
\begin{align*}
\left( |qc; 0 \rangle \otimes |b; \frac{1}{2} \rangle \right)_{J=1/2} &= \frac{1}{2} \left( |bc; 0 \rangle \otimes |q; \frac{1}{2} \rangle \right)_{J=1/2} - \frac{\sqrt{3}}{2} \left( |bc; 1 \rangle \otimes |q; \frac{1}{2} \rangle \right)_{J=1/2} \\
\left( |qc; 1 \rangle \otimes |b; \frac{1}{2} \rangle \right)_{J=1/2} &= -\frac{\sqrt{3}}{2} \left( |bc; 0 \rangle \otimes |q; \frac{1}{2} \rangle \right)_{J=1/2} - \frac{1}{2} \left( |bc; 1 \rangle \otimes |q; \frac{1}{2} \rangle \right)_{J=1/2}
\end{align*}
\]
data.

| This work | Exp. [5] | Latt. [33] | Latt. [34] | Latt. [35] |
|-----------|----------|------------|------------|------------|
| Ξ_{cc}    | 3612^{+11}_{-11} | 3519 ± 1  | 3505 ± 23  | 3549 ± 9  |
| Ξ_{cc}    | 3706^{+21}_{-21}  | 3860 ± 13 | 3760 ± 7  |
| Ξ_{bb}    | 10197^{+10}_{-17} | 10207 ± 9 | 10160 ± 100 |
| Ξ_{bb}    | 10333 ± 7         |           |            |
| Ξ_{cc}    | 3706^{+23}_{-23}  | 3749 ± 10 | 3760 ± 7  |
| Ξ_{cc}    | 3650 ± 7          |           |            |

TABLE III: First panel: doubly heavy Ξ masses in MeV as obtained in different models. Our central results, and the ones of Ref. [21], have been evaluated with the AL1 potential. Second panel: we compare our results with the experimental value for $M_{\Xi_{cc}}$ measured by the SELEX Collaboration [3]. (Note the cautions that appear on this experimental mass in the introduction), and lattice results from Refs. [33, 34, 35]. Entries with † should be taken with due caution (see footnote 6).

**TABLE IV:** Same as Table III for doubly heavy Ω baryons.

There are also independent determinations of mass splittings in lattice QCD [33, 34, 35], nonrelativistic lattice QCD [32] and pNRQCD [1]. In Table V we compare those results to the ones obtained in the present calculation and in other models. Our central results evaluated with the AL1 potential are larger than the ones obtained in lattice QCD [33, 34, 35] and lattice nonrelativistic QCD [32]. The agreement is better when we use the BD potential of Ref. [19] for which we always get the lowest results. Similar results are obtained by the relativistic calculation of Ref. [22], whereas for the relativistic calculations in Refs. [23, 24, 25] and the nonrelativistic one in Ref. [28] the agreement with lattice QCD and nonrelativistic lattice QCD data worsens. As for the calculation in Ref. [30] we do not quote their results due to the large theoretical errors involved.

**B. Charge densities and radii**

The baryon charge density at the point $\bar{r}$ (coordinate vector $\bar{r}$ in the CM frame, see Fig. 1) is given by:

$$
\rho^B_e(\bar{r}) = \int d^3r_1 d^3r_2 \left| \Psi_{h_1 h_2}(r_1, r_2, r_{12}) \right|^2 \left\{ e_{h_1} \delta^3(r - \bar{y}_{h_1}) + e_{h_2} \delta^3(r - \bar{y}_{h_2}) + e_q \delta^3(r - \bar{y}_q) \right\}
$$

$$
\rho^B_e(\bar{r}) |_{h_1} + \rho^B_e(\bar{r}) |_{h_2} + \rho^B_e(\bar{r}) |_{q}
$$

(14)
TABLE V: Mass splittings in MeV for doubly heavy Ξ and Ω baryons. Our central values have been obtained with the AL1 potential. Entries with a † should be taken with due caution (see footnote 6).

|                | This work | [22] | [23] | [25] | [28] | [29] | [4] | [32] | Latt. [33] | Latt. [34] | Latt. [35] |
|----------------|-----------|------|------|------|------|------|-----|------|------------|------------|------------|
| $M_{\Xi^*_{cc}} - M_{\Xi_{cc}}$ | 94^{+11}_{-11} | 107 | 130 | 120 | 132 | 150 | 120 | 40 | 70 ± 13 | 80 ± 11 | 87 ± 19 |
| $M_{\Xi^*_{cb}} - M_{\Xi_{cb}}$ | 39^{+2}_{-6} | 35 | 40 | 40 | 40 | 50 | 34 | 4 | 20 ± 7 | 20 ± 6 |
| $M_{\Omega^*_{cc}} - M_{\Omega_{cc}}$ | 67^{+10}_{-10} | 47 | 80 | 90 | 80 | 70 | | | 43 ± 11 |
| $M_{\Omega^*_{bc}} - M_{\Omega_{bc}}$ | 29^{+1}_{-3} | 30 | 30 | 60 | 30 | 50 | | | 9 ± 7 |
| $M_{\Omega^*_{cc}} - M_{\Omega_{cc}}$ | 81^{+11}_{-16} | 94 | 100 | 140 | 100 | 130 | | | 63 ± 9 | 68 ± 7 | 67 ± 16 |
| $M_{\Omega^*_{bb}} - M_{\Omega_{bb}}$ | 37^{+6}_{-9} | 30 | 20 | 40 | 20 | 40 | | | 19 ± 5 | 20 ± 5 |
| $M_{\Omega^*_{cc}} - M_{\Omega_{cc}}$ | 60^{+8}_{-16} | 42 | 80 | 80 | 80 | 60 | | | 39 ± 8 |
| $M_{\Omega^*_{bc}} - M_{\Omega_{bc}}$ | 23^{+2}_{-3} | 28 | 20 | 60 | 20 | 40 | | | 9 ± 6 |

FIG. 2: Charge form factor of the Ξ_{cc} and Ξ_{bb} baryons evaluated with the AL1 [21] (solid line) and BD [19] (dashed line) potentials. We show the two possible charge states. Similar results are obtained for Ξ_{bc} and Ξ_{bb}.

where $e_{h_1, h_2, q}$ are the quark charges in proton charge units $e$, and from Fig. 1 we have $\vec{y}_{h_1} = \vec{y}_q + \vec{r}_1$, $\vec{y}_{h_2} = \vec{y}_q + \vec{r}_2$ and $\vec{y}_q = -(m_{h_1} \vec{r}_1 + m_{h_2} \vec{r}_2) / M$. Since our $L = 0$ wave functions only depend on scalars ($r_1, r_2$ and $r_{12}$) the charge density is spherically symmetric ($\rho_e^B(\vec{r}) = \rho_e^B(|\vec{r}|)$).

The charge form factor is defined as usual

$$\mathcal{F}_e^B(\vec{q}) = \int d^3 r e^{i \vec{q} \cdot \vec{r}} \rho_e^B(r)$$

and it only depends on $|\vec{q}|$. Its value at $\vec{q} = 0$ gives the baryon charge in units of the proton charge.

The charge mean square radii are defined

$$\langle r^2 \rangle_e^B = \int d^3 r r^2 \rho_e^B(r) = 4\pi \int_0^{+\infty} dr r^4 \rho_e^B(r)$$

In Figs. 2, 3 and 4 we show the charge form factors for Ξ_{cc}, Ξ_{bb}, Ω_{cc}, Ω_{cb} and Ξ_{bc}, Ω_{bc} baryons. In each case similar results are obtained for the star and prime excitations. We show the calculations with both the AL1 potential of Refs. [21] [21] and the BD potential of Ref. [19]. The differences between the two calculations are minor in most cases.

In Table VI we show the charge mean square radii. With the exceptions of the $\Xi_{bb}^*$ and $\Omega_{bc}^*$, we find good agreement with the results obtained in Ref. [21] within a Faddeev calculation. The possible presence of a $S_h = 0$ contribution in the wave functions of Ref. [21] could be the possible explanation for this discrepancy. We also compare with the results obtained, for a few states, in Ref. [37] with the use a relativistic quark model in the instant form. The agreement is bad in this case.

7) There exists the obvious restriction $m_{h_1} \vec{y}_{h_1} + m_{h_2} \vec{y}_{h_2} + m_q \vec{y}_q = 0$. 

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There exists the obvious restriction $m_{h_1} \vec{y}_{h_1} + m_{h_2} \vec{y}_{h_2} + m_q \vec{y}_q = 0$. 

C. Magnetic moments

The orbital part of the magnetic moment is defined in terms of the velocities $\vec{v}$ of the quarks, with respect to the position of the CM, and it reads

$$\mu^B = \int d^3r_1 d^3r_2 (\Psi^B_{h_1 h_2}(r_1, r_2, r_{12}))^* \left\{ \frac{e_{h_1}}{2m_{h_1}} (\vec{y}_{h_1} \times m_{h_1} \vec{v}_{h_1})_z \right. \right.$$  

$$+ \left. \frac{e_{h_2}}{2m_{h_2}} (\vec{y}_{h_2} \times m_{h_2} \vec{v}_{h_2})_z \right. \left. + \frac{e_q}{2m_q} (\vec{y}_{q} \times m_q \vec{v}_{q})_z \right. \right\} \Psi^B_{h_1 h_2}(r_1, r_2, r_{12})$$  \hspace{1cm} (17)

with $m_{h_1} \vec{v}_{h_1} = -i \vec{\nabla}_1$, $m_{h_2} \vec{v}_{h_2} = -i \vec{\nabla}_2$ and $m_q \vec{v}_q = i \left( \vec{\nabla}_1 + \vec{\nabla}_2 \right)$. Since our orbital wave function has $L = 0$, the orbital magnetic moment vanishes. The magnetic moment of the baryon is then entirely given by the spin contribution.

$$\langle B; J, M_J = J | \frac{e_{h_1}}{2m_{h_1}} (\vec{\sigma}_{h_1})_z + \frac{e_{h_2}}{2m_{h_2}} (\vec{\sigma}_{h_2})_z + \frac{e_q}{2m_q} (\vec{\sigma}_q)_z | B; J, M_J = J \rangle$$  \hspace{1cm} (18)

Those matrix elements are trivially evaluated with the results

$$\Xi_{h_1 h_2}, \Omega_{h_1 h_2} \rightarrow \frac{2}{3} \left( \frac{e_{h_1}}{2m_{h_1}} + \frac{e_{h_2}}{2m_{h_2}} - \frac{1}{2} \frac{e_q}{2m_q} \right)$$

---

Note that the classical kinetic energy has a term on $\vec{v}_{h_1} \cdot \vec{v}_{h_2}$ and then the operator $m_{h_1} \vec{v}_{h_1}$ is not proportional to $-i \vec{\nabla}_{y_{h_1}}$, but it is rather given by $m_{h_1} \vec{v}_{h_1} = (\vec{M} - m_{h_1})/\vec{M} \cdot (-i \vec{\nabla}_{y_{h_1}}) - m_{h_1}/\vec{M} \cdot (-i \vec{\nabla}_{y_{h_2}}) = (-i \vec{\nabla}_{y_1})$. Similarly $m_{h_2} \vec{v}_{h_2} = (-i \vec{\nabla}_{y_2})$. 
TABLE VI: Charge mean square radii in fm\(^2\) for doubly heavy \(\Xi\) and \(\Omega\) baryons. Our central values, and the ones of Ref. [21], have been evaluated with the AL1 potential.

| \(\Xi_{1h2}\), \(\Omega_{1h2}\) | This work | [21] | [37] |
|--------------------------|-------------|--------|--------|
| \(\Xi_{cc}^+\)           | -0.030\(\pm\)0.016 | -0.038\(\pm\)0.016 | 0.1 |
| \(\Xi_{cc}^+\)           | 0.295\(\pm\)0.028 | 0.315\(\pm\)0.030 | 1.3 |
| \(\Xi_{bc}^+\)           | -0.042\(\pm\)0.007 | -0.041\(\pm\)0.001 | |
| \(\Xi_{bc}^+\)           | -0.013\(\pm\)0.016 | -0.143\(\pm\)0.018 | |
| \(\Xi_{bb}^+\)           | -0.142\(\pm\)0.018 | -0.018\(\pm\)0.018 | |
| \(\Xi_{bc}^0\)           | 0.238\(\pm\)0.035 | -0.031\(\pm\)0.031 | |
| \(\Xi_{bc}^0\)           | -0.057\(\pm\)0.006 | -0.072\(\pm\)0.008 | |
| \(\Xi_{bc}^+\)           | 0.279\(\pm\)0.026 | 0.306\(\pm\)0.011 | |
| \(\Xi_{bb}^0\)           | -0.065\(\pm\)0.015 | -0.015\(\pm\)0.015 | |
| \(\Xi_{bc}^+\)           | 0.283\(\pm\)0.025 | 0.109\(\pm\)0.029 | |
| \(\Xi_{bb}^0\)           | -0.065\(\pm\)0.010 | -0.018\(\pm\)0.018 | |
| \(\Xi_{bc}^+\)           | 0.305\(\pm\)0.031 | 0.115\(\pm\)0.031 | |

In Table VI we give our numerical results. Our central values, as the ones obtained in Ref. [21] within a Faddeev approach, have been evaluated with the use of the AL1 potential. When compared to the values obtained in Ref. [21], we find very good agreement with just a few exceptions (\(\Xi_{bc}^0\), \(\Xi_{bc}^+\), \(\Omega_{bc}^0\)). The discrepancy for the latter baryons may come from a possible non negligible \(S_b = 0\) contribution to their wave functions in the calculation of Ref. [21]. In our case we have fixed \(S_b = 1\) which we think is a very good approximation since in the limit of infinite heavy quark masses the spin of the heavy quark degrees of freedom is well defined.

We also compare our results to the ones obtained in Refs. [1, 36, 37, 38, 39, 40] using different approaches. The differences between different calculations are in some cases large. Being \(L = 0\) a good approximation the magnetic moments are essentially determined by the spin contribution of the quarks. With \(m_s \gg m_u, m_d, m_s\), the contribution from the \(b\) quarks is negligible compared to the one of the light quark. This is also true to a lesser extent for the \(c\) quark.

V. SEMILEPTONIC DECAY

In this section we shall use the wave functions obtained with the variational method to study different doubly \(B(1/2^+) \rightarrow B'(1/2^+)\) baryon semileptonic decays involving a \(b \rightarrow c\) transition at the quark level.

The differential decay width reads

\[
d\Gamma = 8|V_{cb}|^2 m_B G_F^2 \frac{d^3 p'}{(2\pi)^3 E_{B'}} \frac{d^3 k}{(2\pi)^3 2E_{B}} \frac{d^3 k'}{(2\pi)^3 2E_{l}} (2\pi)^4 \delta^4(p - p' - k - k') \mathcal{L}^{\alpha\beta}(k, k') \mathcal{H}_{\alpha\beta}(p, p')
\]

where \(|V_{cb}|\) is the modulus of the corresponding Cabibbo–Kobayashi–Maskawa matrix element, \(m_B\) is the mass of the final baryon, \(G_F = 1.16637(1) \times 10^{-11} \text{MeV}^{-2}\) is the Fermi decay constant, \(p, p', k, k'\) are the four-momenta of the initial baryon, final baryon, final anti-neutrino and final lepton respectively, and \(\mathcal{L}\) and \(\mathcal{H}\) are the lepton and hadron tensors.

The lepton tensor is given as

\[
\mathcal{L}^{\mu\nu}(k, k') = k'^\mu k'^\nu + k'^\sigma k'^\mu - g^{\mu\nu} k' \cdot k' + i\epsilon^{\mu\nu\alpha\beta} k'^\alpha k'^\beta
\]

(21)
The four momentum transfer ($B$ of the initial with $|u$ where
where we use the convention $\epsilon^{0123} = -1$, $g^{\mu\nu} = (+, -, -, -)$.

The hadron tensor is given as

$$H_{\mu\nu}(p, p') = \frac{1}{2} \sum_{r, r'} \left< B, r' | \bar{p}' \gamma_{\mu} (I - \gamma_5) \Psi^b(0) \right| B, r \bar{p} \right> \left< B', r' | \bar{p}' \gamma_{\nu} (I - \gamma_5) \Psi^b(0) \right| B, r \bar{p} \right>$$

where $u^{B', B}$ are dimensionless Dirac spinors, normalized as $\bar{u} u = 1$, and $v_\mu (p) = p_\mu / m_B (v'_\mu = p'_\mu / m_{B'})$ is the four momentum of the initial $B$ (final $B'$) baryon. The form factors are functions of the velocity transfer $w = v \cdot v'$ or equivalently of the four momentum transfer ($q = p - p'$) square $q^2 = m_B^2 + m_{B'}^2 - 2m_B m_{B'} w$. In the decay $w$ ranges from $w = 1$, corresponding to zero recoil of the final baryon, to a maximum value given by $w = w_{\text{max}} = (m_B^2 + m_{B'}^2)/(2m_B m_{B'})$ which depends on the transition.

|      | This work | [21] | [1] | [36] | [37] | [38] | [39] | [40] |
|------|-----------|------|-----|------|------|------|------|------|
| $\Xi^+_c$ | $0.785^{+0.009}_{-0.010}$ | $0.784^{+0.009}_{-0.029}$ | 0.806 | 0.72 | 0.72 | 0.89 | 0.98 | 0.778 $\sim$ 0.790 | 0.86 |
| $\Xi^{+0}_{cc}$ | $-0.208^{+0.035}_{-0.086}$ | $-0.206^{+0.034}_{-0.086}$ | -0.124 | 0.13 | -0.10 | -0.47 | $-0.172 \sim -0.154$ | 0.17 |
| $\Xi^{++}_{cc}$ | $-0.311^{+0.052}_{-0.130}$ | $-0.186$ | -1.17 $\sim$ -0.98 | 0.20 |
| $\Xi^{++}_{cc}$ | $2.67^{+0.27}_{-0.15}$ | 2.60 | 3.16 $\sim$ 3.18 | 2.54 |
| $\Xi^0_{bb}$ | $-0.742^{+0.044}_{-0.091}$ | $-0.742^{+0.044}_{-0.091}$ | -0.53 | $-0.726 \sim -0.705$ | 0.61 |
| $\Xi^0_{bb}$ | $2.501^{+0.045}_{-0.021}$ | $0.251^{+0.046}_{-0.021}$ | 0.18 | $0.226 \sim 0.236$ | 0.14 |
| $\Xi^*_0_{bb}$ | $1.8^{+0.27}_{-0.13}$ | $-1.1^{+0.27}_{-0.14}$ | 1.37 |
| $\Xi^{*-0}_{bb}$ | $-1.1^{+0.27}_{-0.14}$ | -0.95 |
| $\Xi^0_{bc}$ | $0.518^{+0.048}_{-0.020}$ | $0.058^{+0.059}_{-0.056}$ | 0.42 |
| $\Xi^+_{bc}$ | $-0.475^{+0.045}_{-0.088}$ | $-0.198^{+0.057}_{-0.066}$ | -0.12 |
| $\Xi^{+0}_{bc}$ | $-0.993^{+0.065}_{-0.137}$ | $-0.76$ | -0.385 $\sim$ -0.366 |
| $\Xi^{+-}_{bc}$ | $1.99^{+0.13}_{-0.13}$ | 1.52 | $1.50 \sim 1.54$ |
| $\Xi^{*+}_{bc}$ | $-0.712^{+0.059}_{-0.133}$ | -0.39 |
| $\Xi^{*-+}_{bc}$ | $2.27^{+0.14}_{-0.14}$ | 2.04 |

TABLE VII: Magnetic moments, in nuclear magnetons ($|e|/2m_p$, with $m_p$ the proton mass), of doubly heavy $\Xi$ and $\Omega$ baryons. Our central values, and the ones of Ref. [21], have been evaluated with the AL1 potential.
Neglecting lepton masses, we have for the differential decay rates from transversely \((\Gamma_T)\) and longitudinally \((\Gamma_L)\) polarized \(W\)’s (the total width is \(\Gamma = \Gamma_L + \Gamma_T\)) the form factors are given by

\[
\frac{d\Gamma_T}{dw} = \frac{G_F^2 |V_{cb}|^2}{12\pi^3} \frac{m_B^3}{\sqrt{w^2 - 1}} q^2 \left\{ (w - 1)|F_1(w)|^2 + (w + 1)|G_1(w)|^2 \right\}
\]

\[
\frac{d\Gamma_L}{dw} = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \frac{m_B^3}{\sqrt{w^2 - 1}} \left\{ (w - 1)|F^V(w)|^2 + (w + 1)|F^A(w)|^2 \right\}
\]

\[
F^{V,A}(w) = \left[ (m_B \pm m_B')F^{V,A}(w) + (1 \pm w) \left( m_B'F_2^{V,A}(w) + m_BF_3^{V,A}(w) \right) \right],
\]

\[
F^V_j = F_j(w), \ F^A_j = G_j(w), \ j = 1, 2, 3
\]

(24)

One can also evaluate the polar angle distribution [43]:

\[
\frac{d^2\Gamma}{dw \, d\cos \theta} = \frac{3}{8} \left( \frac{d\Gamma_T}{dw} + 2 \frac{d\Gamma_L}{dw} \right) \left\{ 1 + 2\alpha' \cos \theta + \alpha'' \cos^2 \theta \right\}
\]

(25)

where \(\theta\) is the angle between \(\vec{k}'\) and \(\vec{p}'\) measured in the off–shell \(W\) rest frame, and \(\alpha'\) and \(\alpha''\) are asymmetry parameters given by

\[
\alpha' = - \frac{G_F^2 |V_{cb}|^2}{6\pi^3} \frac{m_B^3}{\sqrt{w^2 - 1}} q^2 \left( w^2 - 1 \right) F_1(w) G_1(w)
\]

\[
\frac{d\Gamma_T/dw}{d\Gamma_T/dw + 2 d\Gamma_L/dw}
\]

\[
\alpha'' = \frac{d\Gamma_T/dw - 2 d\Gamma_L/dw}{d\Gamma_T/dw + 2 d\Gamma_L/dw}
\]

(26)

These asymmetry parameters are functions of the velocity transfer \(w\) and on averaging over \(w\) the numerators and denominators are integrated separately and thus we have

\[
\langle \alpha' \rangle = - \frac{G_F^2 |V_{cb}|^2}{6\pi^3} \frac{m_B^3}{\sqrt{w^2 - 1}} q^2 \left( w^2 - 1 \right) F_1(w) G_1(w) dw
\]

\[
\frac{w_{\text{max}}}{\Gamma_T} \frac{1}{1 + 2 R_{L/T}}, \quad \langle \alpha'' \rangle = \frac{1 - 2 R_{L/T}}{1 + 2 R_{L/T}}, \quad R_{L/T} = \frac{\Gamma_L}{\Gamma_T}
\]

(27)

A. Form factors

To obtain the form factors we have to evaluate the matrix elements

\[
\langle B', r' \vec{p}' \mid \bar{\Psi}(0) \gamma_\mu (I - \gamma_5) \Psi^b(0) \mid B, r \vec{p} \rangle
\]

(28)

which in our model are given by

\[
\sqrt{\frac{E_B(-\vec{q})}{m_B'}} \sqrt{\frac{E_{B'}(-\vec{q}'')}{m_{B'}}} \langle B', r' \vec{p}' \mid \bar{\Psi}(0) \gamma_\mu (I - \gamma_5) \Psi^b(0) \mid B, r \vec{p} \rangle_{NR}
\]

(29)

where the suffix “NR” denotes our nonrelativistic states and the factors \(\sqrt{E/m}\) take into account the different normalization. We shall work in the initial baryon rest frame so that \(\vec{p} = 0, \vec{p}' = -\vec{q}, \) and take \(\vec{q}\) in the positive \(z\) direction. Furthermore we shall use the spectator approximation. Having all this in mind we have in momentum space

\[
\sqrt{\frac{E_{B'}(-\vec{q}'')}{m_{B'}}} \langle B', r' \vec{p}' - \vec{q} \mid \bar{\Psi}(0) \gamma_\mu (I - \gamma_5) \Psi^b(0) \mid B, r \vec{p} \rangle_{NR}
\]

\[
= \sqrt{2} \sqrt{\frac{E_{B'}(-\vec{q}'')}{m_{B'}}} \sum_{s_1} \sum_{s_2} \left( \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} s_h \begin{pmatrix} s_1, s_2 - s_1, s_2 \end{pmatrix} \begin{pmatrix} s_h \frac{1}{2} \frac{1}{2} \\ s_2, r - s_2, r \end{pmatrix} \right)
\]

\[
\times \left( \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} s_h \begin{pmatrix} r' - r + s_1, s_2 - s_1, r' - r + s_2 \end{pmatrix} \begin{pmatrix} s_h \frac{1}{2} \frac{1}{2} \\ r' - r + s_2, r - s_2, r' \end{pmatrix} \right)
\]
\[
\times \int \frac{d^3 q_1}{m_b} \frac{d^3 q_2}{m_c} \left( \Phi_{\mathbf{h}_2}^B(\vec{q}_1 - \frac{m_{h_2} + m}{\sqrt{2} E_c(\vec{q}_1 - \vec{q})}, \vec{q}_2 + \frac{m_{h_2}}{\sqrt{2} E_c(\vec{q}_1 - \vec{q})}) \right)^* \Phi_{\mathbf{h}_2}^B(\vec{q}_1, \vec{q}_2)
\]

where \( \Phi_{\mathbf{h}_2}^B(\vec{q}_1, \vec{q}_2) \) is the Fourier transform of the coordinate space wave function \( \Psi_{\mathbf{h}_2}^B(r_1, r_2, r_{12}) \) with \( \vec{q}_1, \vec{q}_2 \) being the conjugate momenta to the space variables \( \vec{r}_1, \vec{r}_2 \). The factor of two comes from the fact that: i) for \( bc \)-baryon decays, the charm quark resulting from the \( b \to c \) transition could be either the particle 1 or the particle 2 in the final \( cc \) baryon, while ii) for \( bb \)-baryon decays, there exist two equal contributions resulting for the decay of each of the two bottom quarks of the initial baryon.

The actual calculation is done in coordinate space where we have

\[
\sqrt{\frac{E_{B'}(-\vec{q})}{m_{B'}}} N_R \left\langle B', r' - \vec{q} \left| \bar{\Psi}^0(0) \gamma_\mu (I - \gamma_5) \Psi^B \right| B, r \right\rangle_{NR}
\]

\[
= \sqrt{2} \left( \frac{d^3}, m_{B'} \right) \sum_{s_1} \sum_{s_2} \frac{1}{2} \left( \Xi_{h_2} \right| s_{1,2} - s_{1,2} \right) \left( S_{h_2} \frac{1}{2} \right| s_{2,2} - s_{2,2} \right) \left( \Xi_{h_2} \frac{1}{2} \right| r_{1,2} - r_{1,2} \right) \left( S_{h_2} \frac{1}{2} \right| r_{1,2} - r_{1,2} \right)
\]

\[
\times \int d^3 r_1 d^3 r_2 \Psi_{\mathbf{c}_2}^B(r_1, r_2, r_{12}) e^{i \frac{m_b}{m_c} \vec{q} \cdot \vec{r}_2} e^{-i \frac{m_b}{m_c} \vec{q} \cdot \vec{r}_1}
\]

\[
\times \sqrt{\frac{m_b}{E_c(\vec{r})}} \frac{m_c}{E_c(\vec{q} - \vec{q})} \bar{u}_{c', -r_{s_1}} (\vec{l} - \vec{q}) \gamma_\mu (I - \gamma_5) u_{s_1}(\vec{l}) \Psi_{\mathbf{h}_2}^B(r_1, r_2, r_{12})
\]

where \( \vec{l} = -i \vec{\nabla} \) represents an internal momentum which is much smaller than the heavy quark masses \( m_b, m_c \). On the other hand \( |q| \) can be large\(^9\). Thus, to evaluate the above expression we have made use of an expansion in \( \vec{l} \), introduced in Ref. [6], where second order terms in \( \vec{l} \) are neglected, while all orders in \( |q| \) are kept. For instance \( E_c(\vec{l} - \vec{q}) \) is approximated by \( E_c(q) \times (1 - \vec{l} \cdot \vec{q} / E_c(\vec{q})) \) with \( E_c(\vec{q}) = \sqrt{m_c^2 + \vec{q}^2} \).

The three vector and three axial form factors can be extracted from the set of equations\(^10\)

\[
\left\langle B', 1/2 - \vec{q} \left| \bar{\Psi}^0(0) \gamma_1 \Psi^B(0) \right| B, 1/2 \vec{0} \right\rangle = \sqrt{\frac{E_{B'}(-\vec{q}) + m_{B'}}{2m_{B'}}} \frac{|\vec{q}|}{m_{B'}(-\vec{q}) + m_{B'}} F_1(|\vec{q}|)
\]

\[
\left\langle B', 1/2 - \vec{q} \left| \bar{\Psi}^0(0) \gamma_3 \Psi^B(0) \right| B, 1/2 \vec{0} \right\rangle = \sqrt{\frac{E_{B'}(-\vec{q}) + m_{B'}}{2m_{B'}}} \frac{|\vec{q}|}{m_{B'}(-\vec{q}) + m_{B'}} \left( \frac{F_1(|\vec{q}|) + F_3(|\vec{q}|)}{m_{B'}} \right)
\]

\[
\left\langle B', 1/2 - \vec{q} \left| \bar{\Psi}^0(0) \gamma_0 \Psi^B(0) \right| B, 1/2 \vec{0} \right\rangle = \sqrt{\frac{E_{B'}(-\vec{q}) + m_{B'}}{2m_{B'}}} \left( F_1(|\vec{q}|) + F_2(|\vec{q}|) + \frac{E_{B'}(-\vec{q}) + m_{B'}}{m_{B'}} F_3(|\vec{q}|) \right)
\]

for the vector form factors and

\[
\left\langle B', 1/2 - \vec{q} \left| \bar{\Psi}^0(0) \gamma_1 \gamma_5 \Psi^B(0) \right| B, 1/2 \vec{0} \right\rangle = \sqrt{\frac{E_{B'}(-\vec{q}) + m_{B'}}{2m_{B'}}} (-G_1(|\vec{q}|))$

\[
\left\langle B', 1/2 - \vec{q} \left| \bar{\Psi}^0(0) \gamma_3 \gamma_5 \Psi^B(0) \right| B, 1/2 \vec{0} \right\rangle = \sqrt{\frac{E_{B'}(-\vec{q}) + m_{B'}}{2m_{B'}}} \left( -G_1(|\vec{q}|) + \frac{|\vec{q}|^2 G_3(|\vec{q}|)}{m_{B'}E_{B'}(-\vec{q}) + m_{B'}} \right)
\]

\(^9\) At \( q^2 = 0 \) one has \( |\vec{q}| = (m_3^2 - m_2^2) / 2m_B \) which is \( \approx m_B / 3 \) for the transitions under study.

\(^{10}\) Remember \( \vec{q} \) is in the \( z \) direction. Notice also that, for \( \vec{p} = \vec{0} \), \( w \) is just a function of \( |\vec{q}| \).
\[
\left\langle B', 1/2 - q \right| \overline{\Psi}^{(0)}(0) \gamma_0 \gamma_5 \Psi^{b(0)} | B, 1/2 \right\rangle = \sqrt{\frac{E_{B'}(-q) + m_{B'}}{2m_{B'}}} \frac{|q|}{E_{B'}(-q) + m_{B'}} \left( -G_1(|q|) + G_2(|q|) \right) + \frac{E_{B'}(-q)}{m_{B'}} G_3(|q|)
\]

(33)

for the axial ones. All the left hand side terms can be evaluated using Eq. (31) with the approximation mentioned above.

For each transition there are only two different coordinate space integrals from which all different matrix elements can be evaluated. Those integrals are

\[
\mathcal{I}^{B|B'}(|q'|) = \int d^3r_1 d^3r_2 e^{i \frac{m_{h_2}}{M} \cdot \vec{r}_1} e^{-i \frac{m_{h_2}}{M} \cdot \vec{r}_2} \left[ \Psi_{Bh_2}^{B'}(r_1, r_2, r_{12}) \right]^* \Psi_{Bh_2}^B(r_1, r_2, r_{12})
\]

(34)

In appendix B we relate the form factors to the integrals \( \mathcal{I}^{B|B'}(|q'|) \) and \( \mathcal{K}^{B|B'}(|q'|) \) for the different \( S_h, S'_h \) combinations, while in appendix C we give the actual expressions we use to evaluate those integrals.

1. Current conservation

In the limit \( m_h = m_c \) and for \( B' = B \) (and thus \( S_h = S'_h \)) vector current conservation provides a relation among the vector \( F_2 \) and \( F_3 \) form factors, namely

\[
F_2(w) = F_3(w)
\]

(35)

On the other hand, if the two quarks in the current had the same flavour, vector current conservation would fix the normalization of the zeroth component of the vector current matrix element at \( w = 1 \), since this just counts the number of heavy quarks, two, in that case. For quarks with distinct flavours, but still in the limit \( m_h = m_c \), the forward vector matrix element has an extra Clebsch-Gordan factor of \( 1/\sqrt{2} \), and thus we have

\[
F_1(1) + F_2(1) + F_3(1) = \sqrt{2}
\]

(36)

In this limiting situation the integrals \( \mathcal{I}^{BB}(|q'|) \) and \( \mathcal{K}^{BB}(|q'|) \) are related by\(^\text{11}\)

\[
\mathcal{K}^{BB}(|q'|) = \frac{m_{h_2} + m_q}{2M} \mathcal{I}^{BB}(|q'|)
\]

(37)

Besides one has that \( \mathcal{I}^{BB}(0) = 1 \).

Using now the relations in Eq. (34) in appendix B we see that our model satisfies the constraint in Eq. (36) exactly. On the other hand we have a small violation of vector current conservation due to binding effects. For instance, and again using the relations in Eq. (34), we obtain for \( w = 1 \)

\[
F_2(1) = F_3(1) + \sqrt{2}(1 - \frac{m_B}{M})
\]

(38)

which shows that current conservation is violated by a term proportional to the binding energy of the baryon divided by the sum of the masses of its constituents. This violation disappears in the infinite heavy quark mass limit. This deficiency is avoided in most calculations by the neglect of binding effects between the heavy diquark and the light quark\(^\text{12}\). Improvements on vector current conservation would require at minimum the introduction of two–body currents \(^\text{17}\), going thus beyond the spectator approximation, that we have not considered in this analysis.

\(^{11}\) One just has to integrate by parts in the \( \mathcal{K}^{BB}(|q'|) \) expression

\(^{12}\) In Refs. \(^\text{18, 19}\) the currents are constructed at the diquark level. At the baryon level, vector current is conserved because light quark-heavy diquark binding effects are neglected. The calculation in Ref. \(^\text{16}\) uses the infinite heavy quark mass limit thus canceling binding effects.
Note also that, for transitions that do not conserve the spin of the heavy quark subsystem $S_h$ (i.e. $\Xi^{bc}_b \to \Xi^{cc}_c \bar{l} \nu_l$) we have in the $m_b = m_c$ limit and at zero recoil that

$$F_1(1) + F_2(1) + F_3(1) = 0$$

due to the orthogonality of the initial and final baryon wave–functions.

B. Results

In Fig. 5 we show the form factors for $\Xi$ decays evaluated with the AL1 potential. Variations when using a different potential are at the level a few per cent at most. The results for doubly heavy $\Omega$ decays are almost identical to the corresponding ones for doubly heavy $\Xi$ decays. The fact that we have two heavy quarks and that the light one acts as a spectator makes the results almost independent of the light quark mass.

In Fig. 6 we show now our results for the differential $d\Gamma_T/dw$, $d\Gamma_L/dw$ and $d\Gamma/dw$ decay widths evaluated with the AL1 and BD potentials. The differences between the results obtained with the two inter-quark interactions could reach 30% for some transitions and for some regions of $w$. As a consequence of the apparent SU(3) symmetry in the form factors we also find that the results for doubly heavy $\Xi$ and $\Omega$ decays (not shown) are very close to each other. This apparent SU(3) symmetry goes over to the integrated decay widths and asymmetry parameters.

In Table VIII we give our results for the semileptonic decay width (transverse $\Gamma_T$, longitudinal $\Gamma_L$ and total $\Gamma$) for the different processes under study. Our central values have been evaluated using the AL1 potential while the errors show the variations when changing the interaction. The biggest variations appear for the BD potential for which one obtains results which are larger by $7 \sim 12\%$. In Table IX we compare our results to the ones calculated in different models. For that purpose we need a value for $|V_{cb}|$ for which we take $|V_{cb}| = 0.0413$. Our results are in
reasonable agreement with the ones in Ref. [41] where they use a relativistic quark model evaluated in the quark-diquark approximation\(^\text{13}\). The result for \(\Gamma(\Xi_{bc} \rightarrow \Xi_{cc})\) obtained in Ref. [44] using HQET is a factor of 4 larger than ours\(^\text{14}\). We can not directly compare our result for the latter transition with the one published in Ref. [42], the reason being that in this reference the \(\Xi_{bc}\) state is defined differently, with the \(c\) and the light quark coupled to total spin 1. Had they defined the state with the \(bc\) pair coupled to spin 1 they would had obtained roughly a factor of 4 larger result [50], in reasonable agreement with our calculation or the one by Ebert et al.\(^\text{15}\). Finally in Ref. [43], where they use the Bethe–Salpeter equation applied to a quark-diquark system, they obtain much larger results for all transitions.

In Table X we compile our results for the average angular asymmetries \(\alpha'\) and \(\alpha''\), as well as the \(R_{L/T}\) ratio, introduced in Eq.(27). The central values have been obtained with the AL1 potential. Being all quantities ratios the variation when changing the inter-quark interaction are in most cases small.

\(^{13}\) Note we have divided by a factor 2 the results originally published in Ref. [41] as there, and as we initially did, the authors forgot a symmetry factor \(1/\sqrt{2}\) for the case of diquarks with two equal quarks [48].

\(^{14}\) Note we have multiplied by a factor 2 the result originally published in Ref. [44] as the author overlooked a normalization factor \(\sqrt{2}\) [49].

\(^{15}\) Note the actual physical states would be an admixture of what we call \(\Xi_{bc}\) and \(\Xi'_{bc}\), and similarly in the \(\Omega\) case. A measurement of the decay width could give information on the admixtures.
The central values have been obtained with the AL1 potential.

Table VIII: Semileptonic decay widths in units of $|V_{cb}|^2 \cdot 10^{-11}$ GeV. $\Gamma_T$ and $\Gamma_L$ stand for the transverse and longitudinal contributions to the width $\Gamma$. The central values have been obtained with the AL1 potential. $l$ stands for a light charged lepton, $l = e, \mu$.

| Decay               | $\Gamma_T$        | $\Gamma_L$        | $\Gamma$        |
|---------------------|--------------------|--------------------|-----------------|
| $\Xi_{bb} \to \Xi_{bc} \bar{l} \nu_l$ | $0.49^{+0.5}_{-0.01}$ | $0.64^{+0.10}_{-0.02}$ | $1.13^{+0.15}_{-0.03}$ |
| $\Xi_{bc} \to \Xi_{cc} \bar{l} \nu_l$ | $0.58^{+0.04}_{-0.03}$ | $0.93^{+0.11}_{-0.01}$ | $1.51^{+0.15}_{-0.02}$ |
| $\Xi_{bb} \to \Xi_{bc} \bar{l} \nu_l$ | $0.37^{+0.04}_{-0.03}$ | $0.26^{+0.04}_{-0.01}$ | $0.62^{+0.08}_{-0.02}$ |
| $\Xi_{bc} \to \Xi_{cc} \bar{l} \nu_l$ | $0.45^{+0.03}_{-0.01}$ | $0.35^{+0.03}_{-0.01}$ | $0.80^{+0.06}_{-0.02}$ |

Table IX: Semileptonic decay widths in units of $10^{-14}$ GeV. We have used a value $|V_{cb}| = 0.0413$. $l$ stands for a light charged lepton, $l = e, \mu$. For results with †, ‡ and § see text for details.

| Decay               | This work [41]† | [42]‡ | [43] | [44]§ |
|---------------------|-----------------|------|------|------|
| $\Gamma(\Xi_{bb} \to \Xi_{bc} \bar{l} \nu_l)$ | $1.92^{+0.25}_{-0.05}$ | 1.63 | 28.5 |
| $\Gamma(\Xi_{bc} \to \Xi_{cc} \bar{l} \nu_l)$ | $2.57^{+0.26}_{-0.04}$ | 2.30 | 89.3 | 8.0 |
| $\Gamma(\Xi_{bb} \to \Xi_{bc} \bar{l} \nu_l)$ | $1.06^{+0.14}_{-0.03}$ | 0.82 | 4.28 |
| $\Gamma(\Xi_{bc} \to \Xi_{cc} \bar{l} \nu_l)$ | $1.36^{+0.10}_{-0.01}$ | 0.88 | 7.76 |

Table X: Averaged values of the asymmetry parameters $\alpha'$ and $\alpha''$ evaluated as indicated in Eq. [29]. We also show the ratio $R_{L/T} = \Gamma_L/\Gamma_T$. The central values have been obtained with the AL1 potential. $l$ stands for a light charged lepton, $l = e, \mu$.

| Decay               | $\langle \alpha' \rangle$ | $\langle \alpha'' \rangle$ | $R_{L/T}$ |
|---------------------|---------------------------|---------------------------|-----------|
| $\Xi_{bb} \to \Xi_{bc} \bar{l} \nu_l$ | $-0.13^{+0.01}_{-0.05}$ | $-0.45^{+0.10}_{-0.02}$ | $1.33^{+0.06}_{-0.01}$ |
| $\Xi_{bc} \to \Xi_{cc} \bar{l} \nu_l$ | $-0.12^{+0.01}_{-0.05}$ | $-0.53^{+0.10}_{-0.01}$ | $1.62^{+0.07}_{-0.02}$ |
| $\Xi_{bb} \to \Xi_{bc} \bar{l} \nu_l$ | $-0.19$ | $-0.17^{+0.01}_{-0.01}$ | $0.71^{+0.01}_{-0.01}$ |
| $\Xi_{bc} \to \Xi_{cc} \bar{l} \nu_l$ | $-0.19$ | $-0.23^{+0.01}_{-0.01}$ | $0.79^{+0.02}_{-0.01}$ |

| Decay               | $\langle \alpha' \rangle$ | $\langle \alpha'' \rangle$ | $R_{L/T}$ |
|---------------------|---------------------------|---------------------------|-----------|
| $\Omega_{bb} \to \Omega_{bc} \bar{l} \nu_l$ | $-0.13^{+0.01}_{-0.05}$ | $-0.47^{+0.10}_{-0.02}$ | $1.37^{+0.06}_{-0.01}$ |
| $\Omega_{bc} \to \Omega_{cc} \bar{l} \nu_l$ | $-0.12^{+0.01}_{-0.05}$ | $-0.53^{+0.10}_{-0.01}$ | $1.63^{+0.07}_{-0.02}$ |
| $\Omega_{bb} \to \Omega_{bc} \bar{l} \nu_l$ | $-0.19^{+0.01}_{-0.05}$ | $-0.18^{+0.01}_{-0.01}$ | $0.72^{+0.02}_{-0.01}$ |
| $\Omega_{bc} \to \Omega_{cc} \bar{l} \nu_l$ | $-0.19^{+0.01}_{-0.05}$ | $-0.23^{+0.01}_{-0.01}$ | $0.79^{+0.02}_{-0.01}$ |

VI. SUMMARY

We have evaluated static properties and semileptonic decays for the ground state of doubly heavy $\Xi$ and $\Omega$ baryons. The calculations have been done in the framework of a nonrelativistic quark model with the use of five different inter-quark interactions. The use of different quark-quark potentials allows us to obtain an estimation of the theoretical uncertainties related to the quark-quark interaction. In order to build our wave functions we have made use of the constraints imposed by the infinite heavy quark mass limit. In this limit the spin–spin interactions vanish and the total spin of the two heavy quarks is well defined. With this approximation we have used a simple variational approach, with Jastrow type orbital wave functions, to solve the involved three-body problem.

Among the static properties, our results for the masses are in very good agreement with previous results obtained with the same inter-quark interactions but within a more complicated Faddeev approach [21]. In some cases we even get lower, and thus better, masses\(^\dagger\). We have calculated mass densities and charge densities (charge form factors) finding that the corresponding mean square radii are again in good agreement with the Faddeev calculation of Ref. [21]. We have also evaluated magnetic moments. Being the total orbital angular momentum of the baryon

\(^\dagger\) For a given Hamiltonian a variational mass is an upper bound of the true ground state mass.
$L = 0$, the magnetic moments come from the spin contributions alone. With the exception of \( \Xi_{bc}^0, \Xi_{bc}^+ \) and \( \Omega_{bc}^0 \) we agree perfectly with the Faddeev calculation in Ref. [21]. For the magnetic moments of \( \Xi_{bc}^0, \Xi_{bc}^+ \) and \( \Omega_{bc}^0 \), the discrepancies between the two calculation are very large. The origin might be attributed to the presence of a non-negligible \( S_h = 0 \) component in the wave functions of Ref. [21]. In our case we have \( S_h = 1 \) which we think is a good approximation based on the infinite heavy quark mass limit. This assertion seems to be corroborated by the results obtained in the relativistic calculation of Ref. [32], at least for the \( \Xi_{bc}^0 \) and \( \Omega_{bc}^0 \) cases.

We have used our wave functions to study the semileptonic decay of doubly \( \Xi(J = 1/2) \) and \( \Omega(J = 1/2) \) baryons. We have worked in the spectator approximation with one-body currents alone. In the \( m_h = m_c \) case and for \( B = B' \) baryons we have checked that our model satisfies baryon number conservation. On the other hand we have a small vector current violation by an amount given by the binding energy over the mass of the baryon, violation that disappears in the infinite heavy quark mass limit. Small vector current violations due to binding effects seems to be present in most calculations to date. With this model we have evaluated form factors, asymmetry parameters, differential decay widths and total decay widths. Our results for the latter are in reasonable agreement with the ones obtained in Ref. [41] using a relativistic quark model in the quark–diquark approximation, while they are much smaller than the ones obtained in Ref. [43] by means of the Bethe–Salpeter equation applied to a quark-diquark system.

For the weak form factors the results exhibit an apparent SU(3) symmetry when going from \( \Xi \) to \( \Omega \) baryons. This is due to the fact that we have two heavy quarks and the light one acts as a spectator in the weak transition. This apparent symmetry appears also in the decay widths and asymmetry parameters. On the other hand SU(3) violating effects are clearly visible in some static quantities like the charge form factors and radii, and the magnetic moments, that depend strongly on the light quark charge and/or mass.

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APPENDIX A: INFINITE HEAVY QUARK MASS LIMIT OF OUR VARIATIONAL WAVE FUNCTIONS

In the infinite heavy quark mass limit the wave function of the system should look like the one for a “meson” composed of a light quark and a heavy diquark. The two heavy quarks bind into a 3 color source diquark that to the light degrees of freedom appears to be pointlike [3]. In our model the pointlike nature of the heavy diquark comes about through the one-gluon exchange Coulomb potential which binds the two heavy quarks into a distance\(^\text{17}\)

\[
r_{h_1h_2} \propto \frac{1}{\mu_{h_1h_2}}; \quad \mu_{h_1h_2} = \frac{m_{h_1}m_{h_2}}{m_{h_1} + m_{h_2}}
\]

that tends to zero if both quark masses go to infinity.

For our wave functions we can define the probability distribution \( P_{h_1h_2} \) for the two heavy quarks to be found at a distance \( r_{h_1h_2} \)

\[
P_{h_1h_2}(r_{h_1h_2}) = \int d^3r_1 \int d^3r_2 \; \delta(r_{12} - r_{h_1h_2}) |\Psi_{h_1h_2}^B(r_2, r_2, r_{12})|^2
\]

In Fig. 7 we show the \( P_{h_1h_2} \) probability distributions for the \( \Xi_{cc}, \Xi_{bc}, \Xi_{bb} \) and \( \Omega_{cc}, \Omega_{bc}, \Omega_{bb} \) baryons evaluated for the AL1 potential. We see how the maximum moves, as expected, to lower \( r_{h_1h_2} \) values as the quark masses increase.

On the other hand the relative wave function for the light quark with respect to the heavy quark subsystem should tend to the one of a light quark relative to a pointlike diquark. This limit is not evident in the coordinates we work as one has first to separate the diquark internal orbital wave function from the total one. To see that this limiting

\(^{17}\) The relation can only be approximate due to confinement and the interaction with the light quark.
FIG. 7: $P_{h_1h_2}$ probabilities (see Eq. (A2)) for the $\Xi_{cc}$, $\Xi_{bc}$, $\Xi_{bb}$ and $\Omega_{cc}$, $\Omega_{bc}$, $\Omega_{bb}$ baryons evaluated with the AL1 potential.

situation is in fact reached within our variational ansatz let us do the following: introduce the new set of coordinates

$$\vec{R} = \frac{m_{h_1} \vec{x}_{h_1} + m_{h_2} \vec{x}_{h_2} + m_q \vec{x}_q}{m_{h_1} + m_{h_2} + m_q}$$

$$\vec{r}_{12} = \vec{x}_{h_1} - \vec{x}_{h_2}$$

$$\vec{r}_q = \frac{m_{h_1} \vec{x}_{h_1} + m_{h_2} \vec{x}_{h_2}}{m_{h_1} + m_{h_2}} - \vec{x} = \frac{m_{h_1} \vec{r}_1 + m_{h_2} \vec{r}_2}{m_{h_1} + m_{h_2}}$$

in terms of which the Hamiltonian now reads

$$H = -\frac{\vec{\nabla}_{R}^2}{2M} + H^{\text{int}}$$

$$H^{\text{int}} = \frac{1}{2} \left( \frac{1}{m_{h_1} + m_{h_2}} + \frac{1}{m_q} \right) \vec{\nabla}_{q}^2 + V_{h_{1h_2}}(\vec{r}_{12}, \text{spin}) + \frac{m_{h_2}}{m_{h_1} + m_{h_2}} \vec{r}_{12}, \text{spin}$$

$$+ V_{h_{2q}}(\vec{r}_q - \frac{m_{h_1}}{m_{h_1} + m_{h_2}} \vec{r}_{12}, \text{spin})$$

With $\vec{\nabla}_{12} = \partial/\partial r_{12}$ and $\vec{\nabla}_{q} = \partial/\partial r_{q}$. Defining now

$$H^{0}_{q_{h_1h_2}} = -\frac{1}{2} \left( \frac{1}{m_{h_1} + m_{h_2}} + \frac{1}{m_q} \right) \vec{\nabla}_{q}^2 + V_{h_{1q}}(\vec{r}_q, \text{spin}) + V_{h_{2q}}(\vec{r}_q, \text{spin})$$

one would have

$$H^{\text{int}} = H_{h_1h_2} + H^{0}_{q_{h_1h_2}} + (H_{q_{h_1h_2}} - H^{0}_{q_{h_1h_2}})$$

$H_{h_1h_2}$ is the Hamiltonian for the relative movement of the two heavy quarks while $H^{0}_{q_{h_1h_2}}$ is the Hamiltonian for the relative movement of the light quark with respect to a pointlike heavy diquark where the two heavy quarks are located in their center of mass. Both movements are coupled through the term $(H_{q_{h_1h_2}} - H^{0}_{q_{h_1h_2}})$. If the heavy quark masses get arbitrarily large the average $r_{12}$ value tends to zero and thus one can neglect the effect of $H_{q_{h_1h_2}} - H^{0}_{q_{h_1h_2}}$. In that limiting situation the light and heavy quark degrees of freedom decouple completely and the internal Hamiltonian...
and the values for $|\mathcal{P}|^2$ that we obtain for the $\Xi_{cc}, \Xi_{bc}, \Xi_{bb}$ and $\Omega_{cc}, \Omega_{bc}, \Omega_{bb}$ baryons using the AL1 potential are given in Table XI. We see how $|\mathcal{P}|^2$ increases with increasing heavy quark masses indicating that for very heavy quark masses the total wave function tends to the one of a light quark relative to a pointlike diquark times the diquark internal wave function. In summary, our variational wave functions respect the infinite heavy quark mass limit.

\[ |\mathcal{P}|^2 = 0.974, 0.975, 0.991 \begin{array}{ccc} \Xi_{cc} & \Xi_{bc} & \Xi_{bb} \\ \Xi_{bc} & \Xi_{cc} & \Xi_{bb} \\ \Xi_{bb} & \Xi_{bc} & \Xi_{cc} \end{array} \]

Table XI: Absolute value square of the $\mathcal{P}$ projection coefficient defined in Eq. (A8)

We note by passing that the product $\Phi_{h_{12}}(r_{12}) \cdot \Phi_{\bar{q}_{h_{12}}}(r_{q})$ corrected by a correlation function in the variable $\vec{r}_{12} \cdot \vec{r}_{q}$ would have been a good variational orbital wave function. For instance, a calculation using the AL1 potential and an orbital wave function given just by that product $\Phi_{h_{12}}(r_{12}) \cdot \Phi_{\bar{q}_{h_{12}}}(r_{q})$ gives for the expectation value of $H_{\text{int}}$ for the $\Xi_{cc}, \Xi_{bc}, \Xi_{bb}$ baryons

\[ \langle H_{\text{int}} \rangle_{\Xi_{cc}}^{AL1} = 3640 \text{ MeV}; \langle H_{\text{int}} \rangle_{\Xi_{bc}}^{AL1} = 6943 \text{ MeV}; \langle H_{\text{int}} \rangle_{\Xi_{bb}}^{AL1} = 10198 \text{ MeV} \]

which are respectively 28 MeV, 24 MeV, and 1 MeV larger, and therefore worse (see footnote 13), than our best values in Table II. The correlation function would clearly be needed, being its role more important for the “cc” and “bc” systems. The improvement in going from a “cc” to a “bc” system is not as good as one would naively expect, the reason being that, considering $\vec{r}_{12}$ to be a small quantity, $H_{\bar{q}_{h_{12}}} - H_{\bar{q}_{h_{12}}}$ is first order in $\vec{r}_{12}$ for the “bc” system while, for symmetry reasons, it is necessarily of second order for the “cc” and “bb” ones. In any case one sees how the expectation values obtained with this simple ansatz get closer to our best values in Table II as the heavy quark masses increase.

APPENDIX B: FORM FACTORS IN TERMS OF THE $I^{B' B}(|\vec{q}'|)$ AND $K^{B' B}(|\vec{q}'|)$ INTEGRALS

In this appendix we relate the vector $F_1, F_2, F_3$ and axial $G_1, G_2, G_3$ form factors, that we evaluate in the center of mass of the decaying baryon, to the integrals $I^{B' B}(|\vec{q}'|), K^{B' B}(|\vec{q}'|)$ defined in Eq. (34). To simplify the expressions it is convenient to introduce

\[ \hat{F}_j(|\vec{q}'|) = \sqrt{\frac{E_{B'}(-\vec{q}) + m_{B'}}{2E_{B'}(-\vec{q})}} \sqrt{\frac{2E_{c}(\vec{q})}{E_c(\vec{q}) + m_c}} F_j(|\vec{q}'|) , \quad j = 1, 2, 3 \]

\[ \hat{G}_j(|\vec{q}'|) = \sqrt{\frac{E_{B'}(-\vec{q}) + m_{B'}}{2E_{B'}(-\vec{q})}} \sqrt{\frac{2E_{c}(\vec{q})}{E_c(\vec{q}) + m_c}} G_j(|\vec{q}'|) , \quad j = 1, 2, 3 \]

\[ (B1) \]

- Cases $S_h = 1, S'_h = 0$ or $S_h = 0, S'_h = 1$:

\[ \frac{\hat{F}_1(|\vec{q}'|)}{E_{B'}(-\vec{q}) + m_{B'}} = -\sqrt{\frac{2}{3}} \left( \frac{I^{B' B}(|\vec{q}'|)}{E_c(\vec{q}) + m_c} - \frac{K^{B' B}(|\vec{q}'|)}{2} \left( m_c - \frac{m_h}{m_h} \right) \right) \]

18 Note $r_q$ can be expressed in terms of $r_1, r_2, r_{12}$ and that $d^3r_1 d^3r_2 = d^3r_{12} d^3r_q$
\[
\hat{F}_1(|\vec{q}|) + \hat{F}_2(|\vec{q}|) + \frac{E_{B'}(-\vec{q})}{m_{B'}} \hat{F}_3(|\vec{q}|) = 0
\]

where \( E_c(\vec{q}) = \sqrt{m_c^2 + q^2} \). From the above expressions we have that \( F_2 = F_3 \).

\[
\begin{align*}
- \hat{G}_1(|\vec{q}|) &= \sqrt{\frac{2}{3}} \left( \frac{\mathcal{I}^{B'B}(|\vec{q}|) + |\vec{q}|^2 \mathcal{K}^{B'B}(|\vec{q}|)}{2(E_c(\vec{q}) + m_c)} \left( \frac{m_c}{E_c^2(\vec{q})} + \frac{1}{m_b} \right) \right) \\
\left( -\hat{G}_1(|\vec{q}|) + \frac{|\vec{q}|^2 \hat{G}_3(|\vec{q}|)}{m_{B'}(E_{B'}(-\vec{q}) + m_{B'})} \right) &= \sqrt{\frac{2}{3}} \left( \frac{\mathcal{I}^{B'B}(|\vec{q}|) + |\vec{q}|^2 \mathcal{K}^{B'B}(|\vec{q}|)}{2(E_c(\vec{q}) + m_c)} \left( \frac{m_c}{E_c^2(\vec{q})} - \frac{1}{m_b} \right) \right) \\
- \hat{G}_1(|\vec{q}|) + \frac{\hat{G}_2(|\vec{q}|) + E_{B'}(-\vec{q}) \hat{G}_3(|\vec{q}|)}{E_{B'}(-\vec{q}) + m_{B'}} &= \sqrt{\frac{2}{3}} \left( \frac{\mathcal{I}^{B'B}(|\vec{q}|) + |\vec{q}|^2 \mathcal{K}^{B'B}(|\vec{q}|)}{2(E_c(\vec{q}) + m_c)} \left( \frac{m_c}{E_c^2(\vec{q})} + \frac{1}{m_b} \right) \right)
\end{align*}
\]

for \( S_h = 1, S'_h = 1 \)

\[
\begin{align*}
\frac{\hat{F}_1(|\vec{q}|)}{E_{B'}(-\vec{q}) + m_{B'}} &= \frac{4}{3\sqrt{2}} \left( \frac{\mathcal{I}^{B'B}(|\vec{q}|) - \mathcal{K}^{B'B}(|\vec{q}|)}{E_c(\vec{q}) + m_c} \left( \frac{m_c}{E_c^2(\vec{q})} - \frac{1}{m_b} \right) \right) \\
\frac{\hat{F}_1(|\vec{q}|)}{E_{B'}(-\vec{q}) + m_{B'}} + \frac{\hat{F}_2(|\vec{q}|)}{m_{B'}} &= \sqrt{2} \left( \frac{\mathcal{I}^{B'B}(|\vec{q}|) - \mathcal{K}^{B'B}(|\vec{q}|)}{E_c(\vec{q}) + m_c} \left( \frac{m_c}{E_c^2(\vec{q})} + \frac{1}{m_b} \right) \right) \\
\hat{F}_1(|\vec{q}|) + \frac{E_{B'}(-\vec{q}) \hat{F}_3(|\vec{q}|)}{m_{B'}} &= \sqrt{2} \left( \frac{\mathcal{I}^{B'B}(|\vec{q}|) + |\vec{q}|^2 \mathcal{K}^{B'B}(|\vec{q}|)}{2(E_c(\vec{q}) + m_c)} \left( \frac{m_c}{E_c^2(\vec{q})} - \frac{1}{m_b} \right) \right)
\end{align*}
\]

\[
\begin{align*}
\hat{G}_1(|\vec{q}|) &= \frac{4}{3\sqrt{2}} \left( \frac{\mathcal{I}^{B'B}(|\vec{q}|) + |\vec{q}|^2 \mathcal{K}^{B'B}(|\vec{q}|)}{2(E_c(\vec{q}) + m_c)} \left( \frac{m_c}{E_c^2(\vec{q})} + \frac{1}{m_b} \right) \right) \\
\left( -\hat{G}_1(|\vec{q}|) + \frac{|\vec{q}|^2 \hat{G}_3(|\vec{q}|)}{m_{B'}(E_{B'}(-\vec{q}) + m_{B'})} \right) &= -\frac{4}{3\sqrt{2}} \left( \frac{\mathcal{I}^{B'B}(|\vec{q}|) + |\vec{q}|^2 \mathcal{K}^{B'B}(|\vec{q}|)}{2(E_c(\vec{q}) + m_c)} \left( \frac{m_c}{E_c^2(\vec{q})} - \frac{1}{m_b} \right) \right) \\
- \hat{G}_1(|\vec{q}|) + \frac{\hat{G}_2(|\vec{q}|) + E_{B'}(-\vec{q}) \hat{G}_3(|\vec{q}|)}{E_{B'}(-\vec{q}) + m_{B'}} &= -\frac{4}{3\sqrt{2}} \left( \frac{\mathcal{I}^{B'B}(|\vec{q}|) + |\vec{q}|^2 \mathcal{K}^{B'B}(|\vec{q}|)}{2(E_c(\vec{q}) + m_c)} \left( \frac{m_c}{E_c^2(\vec{q})} + \frac{1}{m_b} \right) \right)
\end{align*}
\]

**APPENDIX C: \( \mathcal{I}^{B'B}(|\vec{q}|) \) AND \( \mathcal{K}^{B'B}(|\vec{q}|) \) INTEGRALS**

To evaluate \( \mathcal{I}^{B'B}(|\vec{q}|) \) and \( \mathcal{K}^{B'B}(|\vec{q}|) \) we use a partial wave expansion of the orbital wave functions:

\[
\begin{align*}
\Psi_{h_2}^{B}(r_1, r_2, r_{12}) &= \sum_{l=0}^{\infty} f_l^B(r_1, r_2) P_l(\mu) \\
\Psi_{h_2}^{B'}(r_1, r_2, r_{12}) &= \sum_{l=0}^{\infty} f_l^{B'}(r_1, r_2) P_l(\mu)
\end{align*}
\]
where \( \mu \) is the cosine of the angle made by \( \vec{r}_1 \) and \( \vec{r}_2 \) and \( P_l(\mu) \) is a Legendre polynomial of rank \( l \). The radial functions \( f_l^B(r_1, r_2) \), \( f_{l'}^B(r_1, r_2) \), are evaluated as

\[
\begin{align*}
&f_l^B(r_1, r_2) = \frac{2l+1}{2} \int_{-1}^{+1} d\mu \ P_l(\mu) \Psi_{b_{h_2}}^B(r_1, r_2, r_{12}) \\
&f_{l'}^B(r_1, r_2) = \frac{2l'+1}{2} \int_{-1}^{+1} d\mu \ P_{l'}(\mu) \Psi_{c_{h_2}}^B(r_1, r_2, r_{12})
\end{align*}
\]

(C2)

In terms of those we have

\[
\begin{align*}
\mathcal{I}^B B(|\vec{q}|) &= 16\pi^2 \sum_l \sum_{l'} \sum_{\nu} (ll'\nu' | 000)^2 \int_0^\infty dr_1 r_1^2 j_{\nu'}(\frac{m_{h_2} + m_q}{M}|\vec{q}|r_1) \int_0^\infty dr_2 r_2^2 j_{\nu}(\frac{m_{h_2}}{M}|\vec{q}|r_2) \\
&\times f_{l'}^B(r_1, r_2) f_l^B(r_1, r_2)
\end{align*}
\]

(C3)

with \( j \)'s being spherical Bessel functions.

\[
\begin{align*}
\mathcal{K}^B B(|\vec{q}|) &= \frac{-16\pi^2}{\sqrt{3}|\vec{q}|} \sum_l \sum_{l'} \sum_{\nu} \sum_{\nu'} (ll'\nu' | 000)^2 (-1)^{(\nu'+\nu'+1)/2} \sqrt{(2L+1)(2l'+1)(2l+1)} \\
&\times (ll'\nu' | 000) (ll'\nu' | 000) (ll'\nu' | 000) W(l''l'1L; l'l'') \\
&\times \int_0^\infty dr_1 r_1^2 j_{\nu'}(\frac{m_{h_2} + m_q}{M}|\vec{q}|r_1) \int_0^\infty dr_2 r_2^2 j_{\nu}(\frac{m_{h_2}}{M}|\vec{q}|r_2) f_{l'}^B(r_1, r_2) \Omega_L f_l^B(r_1, r_2)
\end{align*}
\]

(C4)

with \( W(l''l'1L; l'l'') \) being a Racah coefficient and \( \Omega_L \) the differential operator\(^{19}\)

\[
\begin{align*}
\Omega_{L=+1} &= -\sqrt{\frac{l+1}{2l+1}} \left( \frac{\partial}{\partial r_1} - \frac{l}{r_1} \right) \\
\Omega_{L=-1} &= \sqrt{\frac{l}{2l+1}} \left( \frac{\partial}{\partial r_1} + \frac{l+1}{r_1} \right)
\end{align*}
\]

(C5)

For the actual evaluation we restrict the \( l, l' \) values to \( l, l' = 0, \cdots, 6 \)

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\(^{19}\) Note that the Racah and Clebsch-Gordan coefficients restrict \( L \) to the two possible values \( L = l \pm 1 \).
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