FUNCTIONS ON SURFACES AND
INCOMPRESSIBLE SUBSURFACES

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Abstract. Let $M$ be a smooth connected compact surface, $P$ be either a real line $\mathbb{R}$ or a circle $S^1$. Then we have a natural right action of the group $\mathcal{D}(M)$ of diffeomorphisms of $M$ on $C^\infty(M, P)$. For $f \in C^\infty(M, P)$ denote respectively by $S(f)$ and $O(f)$ its stabilizer and orbit with respect to this action. Recently, for a large class of smooth maps $f : M \to P$ the author calculated the homotopy types of the connected components of $S(f)$ and $O(f)$. It turned out that except for few cases the identity component of $S(f)$ is contractible, $\pi_i O(f) = \pi_i M$ for $i \geq 3$, and $\pi_2 O(f) = 0$, while $\pi_1 O(f)$ it only proved to be a finite extension of $\pi_1 \mathcal{D}(M) \oplus \mathbb{Z}^l$ for some $l \geq 0$. In this note it is shown that if $\chi(M) < 0$, then $\pi_1 O(f) = G_1 \times \cdots \times G_n$, where each $G_i$ is a fundamental group of the restriction of $f$ to a subsurface $B_i \subset M$ being either a 2-disk or a cylinder or a Möbius band. For the proof of main result incompressible subsurfaces and cellular automorphisms of surfaces are studied.

1. Introduction

Let $M$ be a smooth compact connected surface and $P$ be either the real line $\mathbb{R}$ or the circle $S^1$. Consider the right action of the group $\mathcal{D}(M)$ of diffeomorphisms of $M$ on $C^\infty(M, P)$ defined by

$$h \cdot f = f \circ h^{-1}$$

for $h \in \mathcal{D}(M)$ and $f \in C^\infty(M, P)$. For every $f \in C^\infty(M, P)$ let

$$\mathcal{O}(f) = \{ f \circ h \mid h \in \mathcal{D}(M) \},$$

$$S(f) = \{ h \mid f = f \circ h, \ h \in \mathcal{D}(M) \}$$

be respectively the orbit and the stabilizer of $f$ with respect to this action. We will endow $\mathcal{D}(M)$, $S(f)$, $C^\infty(M, P)$, and $\mathcal{O}(f)$ with the
corresponding topologies $C^\infty$. Denote by $S_{id}(f)$ the identity path component of $S(f)$ and by $O_f(f)$ the path component of $f$ in $O(f)$. In [10] the author calculated the homotopy types of $S_{id}(f)$ and $O_f(f)$ for all Morse maps $f : M \to P$.

Moreover, in [12] the results of [10] were extended to a large class of maps with (even degenerate) isolated critical points satisfying certain “non-degeneracy” conditions. In fact there were introduced three types of isolated critical points (called S, P, and N) and the following three axioms for $f$:

**(Bd)** $f$ takes constant value at each connected component of $\partial M$ and $\Sigma_f \subset \text{Int} M$.

**(SPN)** Every critical point of $f$ is either an S- or a P- or an N-point.

**(Fibr)** The natural map $p : D(M) \to O(f)$ defined by $p(h) = f \circ h^{-1}$ is a Serre fibration with fiber $S(f)$ in topologies $C^\infty$.

Recall that if $f : (\mathbb{C}, 0) \to (\mathbb{R}, 0)$ is a smooth germ for which $0 \in \mathbb{C}$ is an isolated critical point, then there exists a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $h(0) = 0$ and

$$f \circ h(z) = \begin{cases} \pm |z|^2, & \text{if } z \text{ is a local extremum, } \text{(3)}, \\ \text{Re}(z^n), (n \geq 1) & \text{otherwise, so } z \text{ is a saddle, } \text{(15)}. \end{cases}$$

Examples of the foliation by level sets of $f$ near 0 are presented in Figure 1.1.

![Figure 1.1. Isolated critical points](image)

From this point of view S-points are saddles, while P- and N-points a local extremes. Moreover, P-points admit non-trivial $f$-preserving circle actions (as non-degenerate local extremes do), while N-points admit only $\mathbb{Z}_n$-action preserving $f$. We will not give precise definitions but recall a large class of examples of such points.

**Example 1.1.** [10]. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial without multiple factors with $\deg f \geq 2$, so

$$f = L_1 \cdots L_a \cdot Q_1 \cdots Q_b, \quad a + 2b \geq 2,$$
where every \( L_i \) is a linear function and every \( Q_j \) is an irreducible over \( \mathbb{R} \) (i.e. definite) quadratic form such that \( L_i/L_{i'} \neq \text{const} \) for \( i \neq i' \) and \( Q_j/Q_{j'} \neq \text{const} \) for \( j \neq j' \).

If \( a \geq 1 \), so \( f \) has linear factors and thus \( 0 \) is a saddle, then the origin \( 0 \in \mathbb{R}^2 \) is an \( S \)-point for \( f \).

If \( a = 0 \) and \( b = 1 \), so \( f = Q_1 \), then the origin \( 0 \in \mathbb{R}^2 \) is a \( P \)-point for \( f \).

Otherwise, \( a = 0 \) and \( b \geq 2 \), so \( f = Q_1 \cdots Q_b \). Then the origin \( 0 \in \mathbb{R}^2 \) is an \( N \)-point for \( f \).

Lemma 1.2. [10]. Let \( f : M \to P \) be a \( C^\infty \) map satisfying \((\text{Bd})\), and such that every of its critical points belongs to the class described in Example 1.1, in particular, \( f \) also satisfies \((\text{SPN})\). Then \( f \) also satisfies \((\text{Fibr})\).

It follows from Morse lemma and Example 1.1 that non-degenerate saddles are \( S \)-points while non-degenerate local extremes are \( P \)-points.

Now the main result of [12] can be formulated as follows.

**Theorem 1.3.** [10, 12]. Suppose \( f : M \to P \) satisfies \((\text{Bd})\) and \((\text{SPN})\). If \( f \) has at least one \( S \)- or \( N \)-point, or if \( M \) is non-orientable, then \( S_{id}(f) \) is contractible.

Moreover, if in addition \( f \) satisfies \((\text{Fibr})\), then \( \pi_i O(f) = \pi_i M \) for \( i \geq 3 \), \( \pi_2 O(f) = 0 \), and for \( \pi_1 O(f) \) we have the following short exact sequence

\[
1 \to \pi_1 \mathcal{D}(M) \oplus \mathbb{Z}^l \to \pi_1 O(f) \to G \to 1,
\]

for a certain finite group \( G \) and \( l \geq 0 \) both depending on \( f \).

Thus, the information about the fundamental group \( \pi_1 O(f) \) is not complete. The aim of this note is to show that the calculation of \( \pi_1 O(f) \) can be reduced to the case when \( M \) is either a 2-disk, or a cylinder, or a Möbius band, see Theorems 1.7 and 1.8 below. The obtained results hold for a more general class of maps \( M \to P \) than the one considered in [12].

**1.4. Admissible critical points.** We will now introduce a certain type of critical points for \( f \). Let \( F \) be a vector field on \( M \), \( V \subset M \) be an open subset, and \( h : V \to M \) be an embedding. Say that \( h \) preserves orbits of \( F \) if for every orbit \( o \) of \( F \) we have that \( h(V \cap o) \subset o \).

**Definition 1.5.** Let \( f : M \to P \) be a \( C^\infty \) map and \( z \in \text{Int} M \) be an isolated critical point of \( f \) which is not a local extreme (so \( z \) is a saddle). Say that \( z \) is admissible if there exists a neighbourhood \( U \) of \( z \) containing no other critical points of \( f \) and a vector field \( F \) on \( U \) having the following properties:
(1) \(f\) is constant along orbits of \(F\) and \(z\) is a unique singular point of \(F\).

(2) Let \((F_t)\) be the local flow of \(F\) on \(U\). Then for every germ of diffeomorphisms \(h : (M, z) \to (M, z)\) preserving orbits of \(F\) there exists a \(C^\infty\) germ \(\sigma : (M, z) \to \mathbb{R}\) such that \(h(x) = F(x, \sigma(x))\) near \(z\).

This definition almost coincides with the definition of an \(S\)-point, c.f. [12]. The difference is that for \(S\)-points it is also required that the correspondence \(h \mapsto \sigma\) is continuous with respect to topologies \(C^\infty\). In particular every \(S\)-point is admissible.

Now put the following two axioms for \(f\) both implied by \((\text{SPN})\):

- \((\text{Isol})\) All critical points of \(f\) are isolated.
- \((\text{SA})\) Every saddle of \(f\) is admissible.

1.6. Main result. Let \(\mathcal{D}_{\text{id}}(M)\) be the identity path component of the group \(\mathcal{D}(M)\) and

\[ S'(f) = S(f) \cap \mathcal{D}_{\text{id}}(M) \]

be the stabilizer of \(f\) with respect to the right action of \(\mathcal{D}_{\text{id}}(M)\). Thus \(S'(f)\) consists of diffeomorphisms \(h\) isotopic to \(\text{id}_M\) and preserving \(F\), i.e. \(f \circ h = f\).

For a closed subset \(X \subset M\) denote by \(S'(f, X)\) the subgroup of \(S'(f)\) consisting of diffeomorphisms fixed on some neighbourhood of \(X\).

The aim of this note is to prove the following theorem:

**Theorem 1.7.** Suppose \(\chi(M) < 0\). Let \(f : M \to P\) be a \(C^\infty\) map satisfying the axioms \((\text{Bd})\), \((\text{Isol})\), and \((\text{SA})\). Then there exists a compact subsurface \(X \subset M\) with the following properties:

1. \(f\) is locally constant on \(\partial X\) and every connected component \(B\) of \(M \setminus X\) is either a 2-disk or a 2-cylinder or a Möbius band. Moreover, \(\partial B \subset X\) and \(B\) contains critical points of \(f\).

2. Let \(h \in S'(f, X)\) and \(B\) be a connected component of \(M \setminus X\), thus \(h\) is fixed on some neighbourhood of \(\partial B\). Then the restriction \(h|_B\) is isotopic in \(B\) to \(\text{id}_B\) with respect to some neighbourhood of \(\partial B\).

3. The inclusion \(i : S'(f, X) \subset S'(f)\) induces a group isomorphism \(i_0 : \pi_0 S'(f, X) \approx \pi_0 S'(f)\).

The proof of this theorem will be given in \(\S 7\). We will now show how to simplify calculations of \(\pi_1 \mathcal{O}(f)\) using Theorem 1.7.

Let \(X\) be the surface of Theorem 1.7 and let \(B_1, \ldots, B_l\) be all the connected components of \(M \setminus X\). For every \(i = 1, \ldots, l\) denote by
\( \mathcal{D}_{\text{id}}(B_i, \partial B_i) \) the group of diffeomorphisms of \( B_i \) fixed on some neighbourhood of \( \partial B_i \) and isotopic to \( \text{id}_{B_i} \) relatively to some neighbourhood of \( B_i \). Let also \( S'(f|_{B_i}, \partial B_i) \) be the stabilizer of the restriction \( f|_{B_i} : B_i \to P \) with respect to the right action of \( \mathcal{D}_{\text{id}}(B_i, \partial B_i) \). Then we have an evident isomorphism of groups:

\[
\psi : S'(f, X) \approx \times_{i=1}^l S'(f|_{B_i}, \partial B_i), \quad \psi(h) = (h|_{B_1}, \ldots, h|_{B_l}),
\]

It is easy to show that \( \psi \) is in fact a homeomorphism with respect to the corresponding \( C^\infty \) topologies.

**Theorem 1.8.** Under assumptions of Theorem 1.7 suppose that \( f \) also satisfies \((\text{Fibr})\). Then we have an isomorphism:

\[
\pi_1 \mathcal{O}_f(f) \approx \times_{i=1}^l \pi_0 S'(f|_{B_i}, \partial B_i).
\]

*Proof.* It is easy to show that if \( f \) satisfies \((\text{Fibr})\), then \( \mathcal{O}_f(f) \) is the orbit of \( f \) with respect to the action of \( \mathcal{D}_{\text{id}}(M) \) and the projection \( p : D_{\text{id}}(M) \to \mathcal{O}_f(f) \) is a Serre fibration as well, see [11]. Hence we get the following part of exact sequence of homotopy groups

\[
\cdots \to \pi_1 \mathcal{D}_{\text{id}}(M) \to \pi_1 \mathcal{O}_f(f) \to \pi_0 S'(f) \to \pi_0 \mathcal{D}_{\text{id}}(M) \to \cdots
\]

Since \( \chi(M) < 0 \), we have \( \pi_1 \mathcal{D}_{\text{id}}(M) = 0 \), [3, 4, 7]. Moreover, \( \mathcal{D}_{\text{id}}(M) \) is path-connected, whence together with Theorem 1.7 we obtain an isomorphism:

\[
\pi_1 \mathcal{O}_f(f) \approx \pi_0 S'(f) \approx \pi_0 S'(f, X) \approx \times_{i=1}^l \pi_0 S'(f|_{B_i}, \partial B_i).
\]

Theorem is proved. \( \square \)

Thus a general problem of calculation of \( \pi_1 \mathcal{O}_f(f) \) for maps satisfying the above axioms completely reduces to the case when \( \chi(M) \geq 0 \). A presentation for \( \pi_1 \mathcal{O}_f(f) \) will be given in another paper.

1.9. **Structure of the paper.** In next four sections we study incompressible subsurfaces \( N \subset M \). §2 contains their definition and some elementary properties. In §3 we show how such subsurfaces appear in studying maps \( M \to P \) with isolated singularities. In §4 and §5 we extend results of W. Jaco and P. Shalen [8] about deformations of incompressible subsurfaces and periodic automorphisms of surfaces. §6 contains two technical statements about deformations of diffeomorphisms preserving a map \( M \to P \). Finally in §7 we prove Theorem 1.7.
2. INCOMPRESSIBLE SUBSURFACES

The following Lemma 2.1 is well-known, see e.g. [14, Pr. 2.1]. It was also implicitly formulated in [8, page 359].

**Lemma 2.1.** 1) Let $M$ be a connected surface, and $N \subset \text{Int} M$ be a proper compact (possibly not connected) subsurface neither of whose connected components is a 2-disk. Then the following conditions are equivalent:

(a) for every connected component $N_i$ of $N$ the inclusion homomorphism $\pi_1 N_i \to \pi_1 M$ is injective;

(b) none of the connected components of $M \setminus N$ is a 2-disk.

If these conditions hold, then $N$ will be called **incompressible**, see [8, Def. 3.2].

**Corollary 2.2.** If $N \subset M$ is incompressible, then $\chi(M) \leq \chi(N)$.

**Corollary 2.3.** Let $R \subset \text{Int} M$ be a proper compact connected subsurface. Then the following conditions are equivalent:

(R1) the homomorphism $\xi : \pi_1 R \to \pi_1 M$ is trivial;

(R2) $R$ is contained in some 2-disk $D \subset M$.

**Proof.** The implication (R2)$\Rightarrow$(R1) is evident.

(R1)$\Rightarrow$(R2). Suppose $R$ is not contained in any 2-disk. We will show that $\xi$ is non-trivial. Let $N$ be the union of $R$ with all of the connected components of $M \setminus N$ which are 2-disks. Then by our assumption $N$ is not a 2-disk and by Lemma 2.1 $N$ is incompressible. Notice that $\xi$ is a product of homomorphisms induced by the inclusions $R \subset N \subset M$:

$$\xi = \beta \circ \alpha : \pi_1 R \to \pi_1 N \to \pi_1 M.$$ 

Also notice that $\alpha$ is surjective and by Lemma 2.1 $\beta$ is a non-trivial monomorphism. Hence $\xi$ is also non-trivial. \hfill \Box

**Corollary 2.4.** Let $R \subset \text{Int} M$ be a proper (possibly non connected) subsurface such that neither of its connected components is contained in some 2-disk. Then every connected component $B$ of $M \setminus R$ which is not a 2-disk is incompressible.

**Proof.** Let $C$ be a connected component of $M \setminus B$. Due to Lemma 2.1 it suffices to show that $C$ is not a 2-disk. Notice that $C \cap R \neq \emptyset$, whence it contains some connected component $R_i$ of $R$. By Corollary 2.3 the product of homomorphisms $\pi_1 R_i \to \pi_1 C \to \pi_1 M$ is non-trivial, and therefore $\pi_1 C \to \pi_1 M$ is also non-trivial. This implies that $C$ is not a 2-disk. \hfill \Box
3. Incompressible subsurfaces associated to a map $M \to P$

3.1. Singular foliation $\Delta_f$ of $f$. Let $f : M \to P$ be a map satisfying axioms (Bd) and (Isol). Then $f$ induces on $M$ a one-dimensional foliation $\Delta_f$ with singularities defined as follows: a subset $\omega \subset M$ is a leaf of $\Delta_f$ if and only if $\omega$ is either a critical point of $f$ or a connected component of the set $f^{-1}(c) \setminus \Sigma_f$ for some $c \in P$. Thus the leaves of $\Delta_f$ are 1-dimensional submanifolds of $M$ and critical points of $f$. Local structure of $\Delta_f$ near critical points of $f$ is illustrated in Figure 1.1.

Denote by $\Delta_f^{\text{reg}}$ the union of all leaves of $\Delta_f$ homeomorphic to the circle and by $\Delta_f^{\text{cr}}$ the union of all other leaves. The leaves in $\Delta_f^{\text{reg}}$ (resp. $\Delta_f^{\text{cr}}$) will be called regular (resp. critical). Similarly, connected components of $\Delta_f^{\text{reg}}$ (resp. $\Delta_f^{\text{cr}}$) will be called regular (resp. critical) components of $\Delta_f$. It follows from (Bd) that $\partial M \subset \Delta_f^{\text{reg}}$. It is also evident, that every critical leaf of $\Delta_f^{\text{cr}}$ either is homeomorphic to an open interval or is a critical point of $f$.

3.2. Atoms and canonical neighbourhoods of critical components of $\Delta_f$. For every critical component $K$ of $\Delta_f$ define its regular neighbourhood $R_K$ as follows. Let $c_1, \ldots, c_l$ be all the critical values of $f$ and the values of $f$ on $\partial M$. Since $M$ is compact, it follows from axioms (Bd) and (Isol) that $l$ is finite. For each $i = 1, \ldots, l$ let $W_i \subset P$ be a closed connected neighbourhood (i.e. just an arc) of $c_i$ containing no other $c_j$. We will assume that $W_i \cap W_j = \emptyset$ for $i \neq j$.

Now let $K$ be a critical component of $\Delta_f$. Then $f(K) = c_i$ for some $i$. Let $R_K$ be the connected component of $f^{-1}(W_i)$ containing $K$. Evidently, $R_K$ is a union of leaves of $\Delta_f$. Following [2] we will call $R_K$ an atom of $K$, see Figure 3.1.

\[ \begin{array}{ccc}
& & W_i \\
\downarrow & f & \downarrow \\
K & \rightarrow & R_K
\end{array} \]

\textbf{Figure 3.1.}

Evidently, $R_K$ is a regular neighbourhood of $K$ with respect to some triangulation of $M$. Similarly to [8] define the canonical neighbourhood $N_K$ of $K$ to be the union of $R_K$ with all the connected components of $M \setminus R_K$ being 2-disks. If $N_K$ is not a 2-disk, then by Lemma 2.1 $N_K$ is incompressible in $M$.

Notice that

\[(3.1) \quad \partial R_K = f^{-1}(\partial W_i) \cap R_K.\]
Let $K'$ be another critical component of $\Delta_f$ such that $f(K') = f(K)$. Since $R_{K'}$ is also constructed via $W'_i$, we obtain from (3.1) that $f$ takes on $\partial R_{K'}$ the same values as on $\partial R_K$. This technical assumption is not essential, however it will be useful for the proof of Theorem 1.7.

**Lemma 3.3.** Let $K$ and $K'$ be two distinct critical components of $\Delta_f$.

(i) Then $R_K \cap R_{K'} = \emptyset$, while $N_K$ and $N_{K'}$ are either disjoint or one of them, say $N_K$, is contained in $N_{K'}$. In the last case $N_K$ is a 2-disk.

(ii) Suppose $f(K) = f(K')$ and there exists $h \in S(f)$ such that $h(K) = K'$. Then $h(R_K) = R_{K'}$ and $h(N_K) = N_{K'}$.

**Proof.** (i) follows from the assumption that $W_i \cap W_j = \emptyset$ for $i \neq j$, and (ii) follows from (3.1). We leave the details for the reader. □

**Lemma 3.4.** Let $K$ be a critical component of $\Delta_f$ such that $N_K$ is a 2-disk. Then either

(i) $M$ is a 2-disk itself, or

(ii) $N_K$ is contained in a unique canonical neighbourhood $N_{K'}$ of another critical component $K'$ of $\Delta_f$ such that $N_{K'}$ is not a 2-disk.

**Proof.** Let $R$ be the union of atoms of all critical components of $\Delta_f$. Then every connected component $B$ of $M \setminus R$ is diffeomorphic to the cylinder $S^1 \times [0,1]$ and the restriction $f|_B$ has no critical points.

Notice that $M \setminus N_K$ is connected since $N_K$ is a 2-disk. Also, there exists a unique connected component $B$ (being a cylinder $S^1 \times [0,1]$) of $M \setminus R$ such that $\partial N_K \subset B$. Then $N_K \cup B$ is also a 2-disk.

Let $n$ be the total number of critical components of $\Delta_f$ in $M \setminus N_K$. If $n = 0$, then $N_K \cup B = M$. Whence $M$ is a 2-disk.

Suppose that $n \geq 1$. Let $\gamma$ be another connected component of $\partial B$ distinct from $\partial N_K$. Then there exists an atom $R_{K'}$ of some critical component $K'$ of $\Delta_f$ such that $\gamma \subset \partial R_{K'}$. Since $N_K \cup B$ is a 2-disk, we see that it is contained in $N_{K'}$. If $N_{K'}$ is not a 2-disk, then the lemma is proved. Otherwise, the number of critical components in $M \setminus N_{K'}$ is less than in $M \setminus N_K$ and the lemma holds by the induction on $n$. □

**Example 3.5.** Let $T^2$ be a 2-torus embedded in $\mathbb{R}^3$ as shown in Figure 3.2 and $f : T^2 \to \mathbb{R}$ be the projection onto the vertical line. Figure 3.2a shows the critical components of level-sets of $f$, and Figure 3.2b presents blackened canonical neighbourhoods of three critical components of $\Delta_f$ containing canonical neighbourhoods of all other critical components of $\Delta_f$.
3.6. Canonical neighbourhoods of negative Euler characteristic. Suppose $M$ is not a 2-disk. Let $K_1, \ldots, K_r$ be all the critical components of $\Delta f$ whose canonical neighbourhoods are not 2-disks. By Lemma 3.4 this collection is non-empty and by Lemma 3.3 $N_{K_i} \cap N_{K_j} = \emptyset$ for $i \neq j$. Moreover, again by Lemma 3.4 any other critical component of $\Delta f$ is contained in some $N_{K_i}$. It follows that $M \setminus \bigcup_{i=1}^r N_{K_i}$ contains no critical points of $f$, whence it is a disjoint union of cylinders $S^1 \times I$. Therefore

\begin{equation}
\chi(M) = \sum_{i=1}^r \chi(N_{K_i}).
\end{equation}

The following two statements will be used for the construction of a surface $X$ of Theorem 1.7, see §7.

**Lemma 3.7.** The following conditions are equivalent:

1. $\chi(M) < 0$;
2. $\chi(N_{K_i}) < 0$ for some $i = 1, \ldots, r$.

**Proof.** (1)$\Rightarrow$(2). As $\chi(M) < 0$, we get from (3.2) that $\chi(N_{K_i}) < 0$ for some $i$.

The implication (2)$\Rightarrow$(1) follows from Corollary 2.2. □

**Corollary 3.8.** Let $K_1, \ldots, K_k$ be all the critical components of $\Delta f$ whose canonical neighbourhoods have negative Euler characteristic and $R_{K_1}, \ldots, R_{K_k}$ be their atoms. Put $R_{<0} := \bigcup_{i=1}^k R_{K_i}$. If $R_{<0} \neq \emptyset$, then every connected component $B$ of $M \setminus R_{<0}$ is either a 2-disk, or a cylinder, or a Möbius band.

**Proof.** Since the homomorphism $\pi_1 R_{K_i} \to \pi_1 M$ is non-trivial for each $i$, it follows from Corollary 2.4 that $B$ is incompressible. Suppose $\chi(B) < 0$. Notice that $f$ takes constant values of $\partial B$. Then by Lemma 3.4 there exists a critical component $K \subset B$ of $\Delta f$ such that the canonical neighbourhood $N$ of $K$ with respect to $f|_B$ has negative Euler characteristic.
characteristic. It follows that the homomorphisms \( \pi_1 N \to \pi_1 B \to \pi_1 M \) induced by the inclusions \( N \subset B \subset M \) are monomorphisms, so \( N \) is incompressible in \( M \). This implies that \( N \) is a canonical neighbourhood of \( K \) with respect to \( f \). But since \( \chi(N) < 0 \), we should have that \( N \subset R_{<0} \), which contradicts to the assumption. \( \square \)

4. DEFORMATIONS OF INCOMPRESSIBLE SUBSURFACES

The aim of this section is to extend some results of [8] concerning incompressible subsurfaces, see Proposition 4.5.

4.1. ±-twist. Let \( \gamma \subset \text{Int} M \) be a two-sided simple closed curve, \( U \) be its regular neighbourhood diffeomorphic to \( S^1 \times [-1,1] \) so that \( \gamma \) correspond to \( S^1 \times 0 \). Take a function \( \mu : [-1,1] \to [0,1] \) such that \( \mu = 0 \) near \( \{ \pm 1 \} \) and \( \mu = 1 \) on some neighbourhood of 0. Define the following homeomorphism \( g_\gamma : M \to M \) by

\[
g_\gamma(x) = \begin{cases} (ze^{2\pi i \mu(t)}, t), & x = (z,t) \in S^1 \times [-1,1] \cong U \\ x, & x \in M \setminus U, \end{cases}
\]

see Figure 4.1. Then \( g_\gamma \) is fixed on some neighbourhood of \( M \setminus U \) and isotopic to \( \text{id}_M \) via an isotopy supported in \( \text{Int} U \). Evidently, \( g_\gamma \) is a product of Dehn twists in opposite directions along the curves parallel to \( \gamma \). Therefore we will call \( g_\gamma \) a ±-twist near \( \gamma \).

**Figure 4.1. ±-twist**

The following lemma is a particular case of [6, Lm. 6.1].

**Lemma 4.2.** [6, Lm. 6.1]. Suppose \( \chi(M) < 0 \). Let \( \gamma \subset \text{Int} M \) be a simple closed curve which does not bound a 2-disk nor a M"obius band, \( h : M \to M \) be a homeomorphism homotopic to \( \text{id}_M \) and such that \( h(\gamma) = \gamma \). Let also \( H : M \times I \to M \) be any homotopy of \( \text{id}_M \) to \( h \). Then there exists another homotopy \( G_t : M \times I \to M \) of \( \text{id}_M \) to \( h \) such that \( G_t(\gamma) = \gamma \) and \( G_t = H_t \) on \( M \setminus U \) for all \( t \in I \).

Moreover, there exists \( m \in \mathbb{Z} \) and a homotopy \( G'_t : M \times I \to M \) of \( \text{id}_M \) to \( g_\gamma^m \circ h \) such that \( G'_t = G \) outside \( U \) and \( G'_t \) is fixed on \( \gamma \) for all \( t \in I \).

The following statement is also well-known.
Lemma 4.3. Let $M$ be a surface with $\chi(M) < 0$. Suppose $\partial M \neq \emptyset$ and let $\gamma_1, \ldots, \gamma_l$ be all the connected components of $\partial M$. For each $i = 1, \ldots, l$ let $\tau_i$ be a Dehn twist along the curve parallel to $\gamma_i$ and fixed on $\partial M$. Let $m_1, \ldots, m_l \in \mathbb{Z}$ be integer numbers not all of which are equal to zero. Then the homeomorphism $\tau_1^{m_1} \circ \cdots \circ \tau_l^{m_l}$ is not homotopic to $\text{id}_M$ via a homotopy fixed on $\partial M$.

4.4. Deformations of incompressible subsurfaces. Let $M$ be a surface distinct from the 2-sphere $S^2$ and the projective plane $\mathbb{R}P^2$, $N \subset M$ be an incompressible subsurface, and $N_1, \ldots, N_k$ be all of its connected components. Let also $h : M \to M$ be a homeomorphism homotopic to $\text{id}_M$ and $H : M \times I \to M$ be any homotopy of $\text{id}_M$ to $h$.

The following Proposition 4.5 follows the line of [8, Lm. 4.2]. In fact the first part of statement (B) is a particular case of that lemma.

Proposition 4.5. c.f. [8, Lm. 4.2] (A) If $N_j$ is not a cylinder for some $j$, then $h(N_j) \cap N_j \neq \emptyset$.
(B) Suppose $\chi(N_j) < 0$ and $h(N_j) \subset N_j$ for some $j$. Then there exists a homotopy $G : N_j \times I \to N_j$ of the identity map $\text{id}_{N_j}$ to the restriction $h|_{N_j}$ such that $G_t(x) = H_t(x)$ whenever $H(x \times I) \subset N_j$.
Moreover, suppose $H(\gamma \times I) \subset \gamma$ for each connected component $\gamma$ of $\partial N_j$. Extend $G$ to a map $G : M \times I \to M$ by $G_t = H_t$ on $M \setminus N_j$. Then $G$ is a homotopy of $\text{id}_M$ to $h$.
(C) Suppose $\chi(N_j) < 0$ and $h(N_j) = N_j$ for all $j = 1, \ldots, k$. Then there exists a homotopy $G : M \times I \to M$ of $\text{id}_M$ to $h$ such that $G(N_j \times I) \subset N_j$ for all $j = 1, \ldots, k$ and $G(B \times I) \subset B$ for every connected component $B$ of $M \setminus N$.
(D) Suppose $\chi(N_j) < 0$ and $h$ is fixed on $N$ for all $j = 1, \ldots, k$. Then there exists a homotopy of $\text{id}_M$ to $h$ fixed on $N$.

Proof. First we make the following remark which repeats the key arguments of [8, Lm. 4.2]. For $j = 1, \ldots, k$ let $p_j : \tilde{M}_j \to M$ be the covering map corresponding to the subgroup $\pi_1 N_j$ of $\pi_1 M$. Then the embedding $i : N_j \subset M$ lifts to the embedding $i^* : N_j \to \tilde{M}_j$ which induces an isomorphism between $\pi_1 N_j$ and $\pi_1 \tilde{M}_j$. Denote $\tilde{N}_j = i^*(N_j)$.

Then we have the following commutative diagram:

\[ \begin{array}{ccc}
\tilde{N}_j & \approx & \tilde{M}_j \\
\downarrow & & \downarrow \pi_j \\
N_j & \to & M
\end{array} \]
Since $\tilde{M}_j$ and $N_j$ are aspherical, it follows from Whitehead’s theorem that $\tilde{N}_j$ is a strong deformation retract of $\tilde{M}_j$. Then every connected component of $\text{Int}(\tilde{M}_j \setminus \tilde{N}_j)$ is an open cylinder. Let $H : N_j \times I \to M$ be any homotopy between the identity embedding $H_0 = i : N_j \subset M$ and $H_1 = h|_{N_j}$. Then there exists a lifting $\tilde{H} : N_j \times I \to \tilde{M}_j$ such that $\tilde{H}_0 = i^*$ and $p_j \circ \tilde{H} = H$. Denote $\tilde{N}'_j = \tilde{H}_1(N_j)$. Since both $\tilde{N}_j$ and $\tilde{N}'_j$ are deformational retracts of $\tilde{M}_j$, they are incompressible in $\tilde{M}_j$.

(A) Suppose $h(N_j) \cap N_j = \emptyset$. Then $\text{Int}(\tilde{N}'_j)$ is included into some connected component $C$ of $\text{Int}(\tilde{M}_j \setminus \tilde{N}_j)$ being a cylinder. Since $\tilde{N}'_j$ is incompressible in $M$, it is also incompressible in $C$, whence $\tilde{N}'_j$ and therefore $N_j$ are cylinders. Thus if $N_j$ is not a cylinder, then we obtain that $h(N_j) \cap N_j \neq \emptyset$.

(B) Let $r_j : \tilde{M}_j \to \tilde{N}_j$ be any retraction. Then the map

$$G = p_j \circ r_j \circ \tilde{H} : N_j \times I \to N_j$$

is a homotopy of $\text{id}_{N_j}$ to $h|_{N_j}$ in $N_j$. It is easy to see that $G_t(x) = H_t(x)$ whenever $H(x \times I) \subset N_j$.

Suppose that $H(\gamma \times I) \subset \gamma \subset N_j$ for each connected component $\gamma$ of $\partial N_j$. Then by the construction $G_t = H_t$ on $\partial N_j$. Notice that $\partial N_j$ separates $M$. Extend $G$ to all of $M \times I$ by $G = \tilde{H}$ of $(M \setminus N_j) \times I$. Then $G$ is continuous, $G_0 = \text{id}_M$ and $G_1 = h$.

(C) Suppose $\chi(N_j) < 0$ and $h(N_j) = N_j$ for all $j = 1, \ldots, k$. Let $\gamma_1, \ldots, \gamma_l$ be all the connected components of $\partial N$. Since $N$ is incompressible, we have by Corollary 2.2 that $\chi(M) \leq \chi(N_j) < 0$ as well. Moreover, by (B) for each $j$ the restriction $h|_{N_j}$ is a homeomorphism of $N_j$ homotopic in $N_j$ to $\text{id}_{N_j}$. This, in particular, implies that $h(\gamma_i) = \gamma_i$ for $i = 1, \ldots, l$.

Then by Lemma 4.2 we can suppose that $H(\gamma_i \times I) \subset \gamma_i$ for all $i = 1, \ldots, l$ as well. Moreover, due to (B) it can be additionally assumed that $H(N_j \times I) \subset N_j$.

Let $B$ be a connected component of $M \setminus N$. Since $N$ is incompressible, $B$ is not a 2-disk. Then by Corollary 2.4 $B$ is incompressible. Therefore we can apply statement (B) to $B$ and change the homotopy $G$ on $B \times I$ so that $G(B \times I) \subset B$.

(D) Suppose $h$ is fixed on $N$. For each $i$ let $U_i$ be a regular neighbourhood of $\gamma_i$, and $g_i$ be a ±-twist near $\gamma_i$ supported in $U_i$. We can assume that $U_i \cap U_j = \emptyset$ for $i \neq j$. Then by Lemma 4.2 there exist integer numbers $m_1, \ldots, m_l \in \mathbb{Z}$ and a homotopy $G : M \times I \to M$ of
id_M to a homeomorphism $h' := g_1^{m_1} \circ \cdots \circ g_l^{m_l} \circ h$ such that $G_t$ is fixed on $L$ for each $t \in I$. By (C) we can also assume that $G(N_j \times I) \subset N_j$ and $G(B \times I) \subset B$ for every connected component of $M \setminus N$ and each $j = 1, \ldots, k$.

In particular, we see that the restriction $h'|_N$ is homotopic to id$_N$ relatively $\partial N$. But this restriction is evidently a product of Dehn twists along boundary components of $N$. Since $\chi(N_j) < 0$ for all $j$, we get from Lemma 4.3 that $m_i = 0$ for all $i = 1, \ldots, l$. Hence $h' = h$. Thus $G$ is in fact a homotopy between id$_M$ and $h$ relatively $\partial N$. Since $\partial N$ separates $M$, and id$_M$ and $h$ are fixed on $N$, we can change $G$ on $N \times I$ by $G_t(x) = x$. This gives a homotopy between id$_M$ and $h$ relatively to $N$. $\square$

5. AUTOMORPHISMS OF CELLULAR SUBDIVISIONS

Let $N$ be a compact surface and $\Xi = \{e_\lambda\}_{\lambda \in \Lambda}$ be some partition of $N$ into a disjoint family of connected orientable submanifolds. Say that a homeomorphism $h : N \to N$ is a $\Xi$-homeomorphism provided it yields a permutation of elements of $\Xi$, that is for each $e \in \Xi$ its image $h(e)$ also belongs to $\Xi$. An element $e \in \Xi$ will be called $h$-invariant if $h(e) = e$. Say that $e$ is $h^+$-invariant ($h^-$-invariant) if the restriction $h|_e : e \to e$ is a preserving (reversing) orientation map. We will also say that $h$ is $\Xi$-trivial if each $e \in \Xi$ is $h^+$-invariant.

Remark 5.1. Notice that we can say that a map $h : e \to e$ preserves or reverses orientation only if $\dim e \geq 1$. To each 0-dimensional element $e \in \Xi$ (being of course a point) we formally assign a “positive orientation” and assume that by definition every cellular homeomorphism preserves orientation of each invariant 0-element of $\Xi$.

Example 5.2. Let $M$ be a connected surface and $K \subset \Int M$ be an embedded finite connected graph. Assume that $K$ is a subcomplex of $M$ with respect to some triangulation of $M$. By $R_K$ we will denote a regular neighbourhood of $K$. Following define a canonical neighbourhood $N_K$ of $K$ to be the union of a regular neighbourhood $R_K$ of $K$ with those connected components of $M \setminus R_K$ which are 2-disks. Notice that $N_K \setminus K$ is a disjoint union of open 2-disks and half-open cylinders $S^1 \times (0,1]$ with $S^1 \times \{1\}$ corresponding boundary components of $\partial N_K$. Thus we obtain a natural partition of $N_K$ by vertexes and edges of $K$ and connected components of $N_K \setminus K$. We shall denote this partition by $\Xi_K$.

Now let $\Xi$ be a cellular subdivision of $N$. Denote by $N_i$ ($i = 0, 1, 2$) the $i$-th skeleton of $N$. In particular, $N_1$ is a finite connected subgraph
in \(N\) such that \(N \setminus N_1\) is a disjoint union of 2-disks. Let \(c_i\) \((i = 0, 1, 2)\) be the total number of \(i\)-cells of \(\Delta\). Then of course \(\chi(N) = c_0 - c_1 + c_2\).

Let \(C = \{C_i, \partial_i\}\) be the \(\mathbb{R}\)-chain complex of \(N\) corresponding to a given cellular subdivision. Thus \(C_i\) is a real vector space of dimension \(c_i\) with the oriented \(i\)-cells of \(\Xi\) as a basis. Then every \(\Xi\)-homeomorphism \(h\) induces a chain automorphism \(\{h_i : C_i \to C_i, \ i = 0, 1, 2\}\) of \(C\).

Recall that for each continuous mapping \(h : N \to N\) we can define its Lefschetz number \(L(h)\) by the formula:

\[
L(h) = \text{tr}(\bar{h}_0) - \text{tr}(\bar{h}_1) + \text{tr}(\bar{h}_2),
\]

where \(\bar{h}_i : H_i(N, \mathbb{R}) \to H_i(N, \mathbb{R})\) is the induced homomorphism of the corresponding homology groups and \(\text{tr}\) is the trace of this homomorphism. If \(h\) is cellular, then \(L(h)\) can also be calculated via the chain homomorphisms \(h_i\) by:

\[
L(h) = \text{tr}(h_0) - \text{tr}(h_1) + \text{tr}(h_2).
\]

The following theorem is relevant to [8, Lm. 4.4] being a statement about periodic homeomorphisms.

**Theorem 5.3.** c.f. [8, Lm. 4.4]. Let \(M\) be a compact surface, \(K \subset M\) a connected subgraph, \(N_K\) be a canonical neighbourhood of \(K\). Let also \(h : M \to M\) a homeomorphism such that \(h\) is homotopic to \(\text{id}_M\), \(h(K) = K\), and \(h\) preserves the set of vertexes of \(K\) of degree 2, and \(h(N_K) = N_K\). In particular, \(h|_{N_K}\) is a \(\Xi_K\)-homeomorphism.

1. If \(\chi(N_K) < 0\), then \(h\) is \(\Xi_K\)-trivial.
2. Suppose that \(N_K = M\), \(M\) is orientable, and \(\chi(M) \geq 0\). Then every annulus \(a \in \Xi_K\) is \(h^+\)-invariant, and the total number of \(h\)-invariant cells of \(\Xi_K\) is equal to \(\chi(M)\).

The proof of Theorem 5.3 will be given in §5.7. It is based on Proposition 4.5 and on the following statement.

**Proposition 5.4.** Let \(N\) be a closed, orientable surface endowed with some cellular subdivision \(\Xi\) and \(h : N \to N\) be a \(\Xi\)-homeomorphism preserving orientation of \(N\) and being not \(\Xi\)-trivial, i.e. \(h(e) \neq e\) for some cell \(e \in \Xi\). Then the number of \(h\)-invariant cells of \(\Xi\) is precisely equal to \(L(h)\). In particular, \(L(h) \geq 0\).

**Proof.** Let \(k_i\) \((i = 0, 1, 2)\) be the number of \(h\)-invariant \(i\)-cells of \(\Xi\) and \(k := k_0 + k_1 + k_2\). We will show that

\[
k_i = (-1)^i \text{tr}(h_i),
\]
which will imply

\[ k = \sum_{i=0}^{2} k_i = \sum_{i=0}^{2} (-1)^i \text{tr}(h_i) = L(h). \]

To prove (5.1) we have to show that there are no \( h^- \)-invariant 0- and 2-cells and no \( h^+ \)-invariant 1-cells. For 0-cells this holds by Remark 5.1 and for 2-cells from the assumptions that \( N \) is orientable and \( h \) preserves orientation.

Let \( e \) be an \( h \)-invariant 1-cell and \( f_0 \) and \( f_1 \) be two 2-cells that are incident to \( e \). It is possible of course that \( f_0 = f_1 \). Since \( h \) preserves orientation, it follows that

(a) either \( h_2(f_j) = +f_j \) for \( j = 0, 1 \), and \( h_1(e) = +e \),
(b) or \( h_2(f_j) = +f_{1-j} \) for \( j = 0, 1 \), and \( h_1(e) = -e \).

The following Claim 5.5 implies that in the case (a) \( h \) is \( \Xi \)-trivial. Since \( h \) is not \( \Xi \)-trivial, we will get from (b) that all \( h \)-invariant 1-cells are \( h^- \)-invariant.

Claim 5.5. Suppose that there exists a 1-cell \( e \in \Xi \) such that

(i) \( h_1(e) = +e \in C_1 \) and
(ii) \( h \) preserves each 2-cell which is adjacent to \( e \).

Then \( h \) is \( \Xi \)-trivial.

Proof. Notice that for each vertex \( v \in N_0 \) the inclusion \( N_1 \subseteq N \) induces a cyclic ordering of edges that are incident to \( v \).

Let \( v \) be a vertex of \( e \). Then it follows from (i) and (ii) that all of the 1- and 2-cells incident to \( v \) are \( h^+ \)-invariant. Moreover, for each 1-cell that is incident to \( v \) the conditions (i) and (ii) also hold true. Since \( N \) is connected, it follows that \( h \) is \( \Xi \)-trivial. \( \square \)

Proposition 5.4 is completed. \( \square \)

Corollary 5.6. Let \( N \) be a closed surface, \( \Xi \) be a cellular subdivision of \( M \), and \( h : N \to N \) be a \( \Xi \)-homeomorphism. If \( h \) is isotopic to \( \text{id}_N \), then each of the following conditions implies that \( h \) is \( \Xi \)-trivial:

(1) \( \chi(N) < 0 \);
(2) \( \chi(N) \geq 0 \) and the total number of \( h^+ \)-invariant 2-cells is greater than \( \chi(N) \).

Proof. Since \( h \) is isotopic to \( \text{id}_N \), we have that \( L(h) = L(\text{id}_N) = \chi(N) \).

If \( N \) is orientable, then \( h \) preserves orientation and by Proposition 5.4 \( h \) is either \( \Xi \)-trivial or has exactly \( \chi(N) \geq 0 \) invariant cells. Each of the conditions (1) and (2) implies that the number of \( h \)-invariant cells is not equal to \( \chi(N) \). Hence \( h \) is \( \Xi \)-trivial.
Suppose that $N$ is non-orientable and let $p : \tilde{N} \to N$ be its oriented double covering. Then $\Xi$ lifts to some cellular subdivision $\tilde{\Xi}$ of $\tilde{N}$ and $h$ lifts to a unique $\tilde{\Xi}$-cellular homeomorphism $\tilde{h}$ of $\tilde{N}$ which is isotopic to $\text{id}_{\tilde{N}}$. Therefore $L(\tilde{h}) = L(\text{id}_{\tilde{N}}) = \chi(\tilde{N}) = 2\chi(N)$.

We claim that every of the conditions (1) and (2) implies that $\tilde{h}$ is $\tilde{\Xi}$-trivial, whence $h$ will be $\Xi$-trivial.

(1) If $\chi(N) < 0$, then $\chi(\tilde{N}) < 0$, whence $\tilde{h}$ is $\tilde{\Xi}$-trivial.

(2) Suppose that $\chi(N) \geq 0$ and the total number $b$ of $h^+$-invariant 2-cells is greater than $\chi(N)$. Let $e$ be an $h^+$-invariant 2-cell of $\Xi$ and $\tilde{e}_1$ and $\tilde{e}_2$ be its liftings in $\tilde{\Xi}$. Then they are $\tilde{h}^+$-invariant. Hence $\tilde{h}$ has at least $2b > 2\chi(N) = \chi(\tilde{N})$ invariant cells. Then by Proposition 5.4 $\tilde{h}$ is $\tilde{\Xi}$-trivial. 

\begin{proof}
5.7. Proof of Theorem 5.3

Let $h : M \to M$ be a homeomorphism homotopic to the identity and such that $h|_{N_K}$ is a $\Xi_K$-homeomorphism. Let $\gamma_1, \ldots, \gamma_b$ be all the connected components of $\partial N_K$, and $a_1, \ldots, a_b$ be the annuli of $\Xi_K$ corresponding to them, so that $\gamma_i \subset a_i$. Shrink every $\gamma_i$ to a point $x_i$ and denote the obtained surface by $\hat{N}_K$. Then $\hat{N}_K$ is a closed orientable surface and $\Xi_K$ yields an evident cellular partition $\Xi$ of $\hat{N}_K$ such that each annulus $a_i$ corresponds to a certain 2-cell $\hat{a}_i \in \Xi$.

Also notice that $\chi(\hat{N}_K) = \chi(N_K) + b$.

Claim 5.8. Suppose that either $\chi(N_K) < 0$ or $N_K = M$. Then

(a) $h|_{N_K}$ is homotopic to $\text{id}_{N_K}$ in $N_K$.
(b) $h(\gamma_i) = \gamma_i$ for $i = 1, \ldots, b$ and $h$ preserves orientation of $\gamma_i$;
(c) $h$ induces some $\Xi$-homeomorphism $\hat{h} : \hat{N}_K \to \hat{N}_K$ homotopic to $\text{id}_{\hat{N}_K}$ with respect to $\{x_1, \ldots, x_b\}$, in particular, every 2-cell $a_i \in \Xi$ is $\hat{h}^+$-invariant;
(d) $L(\hat{h}) = L(\text{id}_{\hat{N}_K}) = \chi(\hat{N}_K) = \chi(N_K) + b$.

Proof. (a) For $N_K = M$ this statement is trivial. If $\chi(N_K) < 0$, then by (B) of Proposition 4.5 (or directly by \cite{S} Lm. 4.1) $h|_{N_K}$ is homotopic to $\text{id}_{N_K}$ in $N_K$. All other statements (b)-(d) follow from (a).

Now we can complete Theorem 5.3

(1) Suppose that $\chi(N_K) < 0$. If also $\chi(\hat{N}_K) < 0$, then by (1) of Corollary 5.6 $\hat{h}$ is $\tilde{\Xi}$-trivial, whence $h$ is $\Xi_K$-trivial as well.

Let $\chi(\hat{N}_K) \geq 0$. By Claim 5.8 $\hat{h}$ has at least $b$ $\hat{h}^+$-invariant 2-cells $a_1, \ldots, a_b$. Moreover, since $\chi(\hat{N}_K) - b = \chi(N_K) < 0$, we obtain that...
b > \chi(\hat{N}_K)$, whence by (2) of Corollary 5.6 $\hat{h}$ is \( \tilde{\Xi} \)-trivial. Therefore $h$ is \( \Xi_K \)-trivial.

(2) Suppose that $N_K = M$ and $M$ is orientable. It follows from (c) of Claim 5.8 and Proposition 5.4 that $\hat{h}$ is either \( \tilde{\Xi} \)-trivial or has exactly $\chi(\hat{N}_K)$ invariant cells. Therefore, $h$ is either \( \Xi_K \)-trivial or has exactly $\chi(N_K) - b = \chi(M)$ invariant cells.

6. DEFORMATIONS OF DIFFEOMORPHISM NEAR CRITICAL COMPONENTS OF $\Delta_f$

The following two propositions will be crucial for the proof of Theorem 1.7. Suppose $f : M \to P$ satisfies (Bd), (isol), and (SA).

**Proposition 6.1.** Let $K$ be a critical component of $\Delta_f$ such that every $z \in K \cap \Sigma_f$ is admissible, $R$ be its atom, and $U$ be any neighbourhood of $R$. Let also $h \in S(f)$. Suppose that $h(\omega) = \omega$ for each leaf $\omega$ of $\Delta_f$ contained in $K$ and that $h$ preserves orientation of $\omega$ whenever $\dim \omega = 1$. Then $h$ is isotopic in $S(f)$ to a diffeomorphism $h' \in S(f)$ such that $h' = h$ on $M \setminus U$, and $h'$ is the identity on some neighbourhood of $R$ in $U$.

**Proof.** This proposition follows the line of [10, Th. 6.2]. For the convenience of the reader we will recall the key arguments for the case when $M$ is orientable. A non-orientable case can be deduced from the orientable one similarly to [10, Th. 6.2].

As $M$ is orientable, it has a symplectic structure. Let $H$ be the Hamiltonian vector field of $f$. Then $f$ is constant along orbits of $H$, the set of singular points of $H$ coincides with the set of critical points of $f$, and the foliation by orbits of $H$ coincides with $\Delta_f$. In particular, $H$ is tangent to $\partial M$ and therefore generates a flow $H : M \times \mathbb{R} \to M$.

We will now change $H$ on neighbourhoods of admissible critical points of $f$ similarly to [10, Lm. 5.1]. Let $z \in \Sigma_f$ be such a point and $F_z$ be a vector field on some neighbourhood $U_z$ of $z$ satisfying assumptions of Definition 1.5. Then it follows from (i) of Definition 1.5 that for every $x \in U_z$ the vectors $H(x)$ and $F_z$ are parallel each other. Therefore, using partition unity technique and changing (if necessary) the signs of $F_z$, we can change $H$ near each $z \in R \cap \Sigma_f$ and assume that $H = F_z$ on $U_z$.

**Claim 6.2.** There exists a neighbourhood $U$ of $R$ and a unique $C^\infty$ function $\sigma : U \to \mathbb{R}$ such that $h(x) = H(x, \sigma(x))$ for all $x \in U$.

**Proof.** Let $z \in K \cap \Sigma_f$. By assumption $h$ preserves leaves of $\Delta_f$ (i.e. orbits of $H$) in $K$ with their orientations. Since $F_z = H$ near $z$, it follows from (ii) of Definition 1.5 that there exists a neighbourhood $V_z$ of
z and a unique $C^\infty$ function $\sigma_z : V_z \to \mathbb{R}$ such that $h(x) = H(x, \sigma_z(x))$. Then the functions $\{\sigma_z\}_{z \in K \cap \Sigma_f}$ yield a unique $C^\infty$ function $\sigma$ on the union $\bigcup_{z \in K \cap \Sigma_f} V_z$. It remains to note that $K \setminus \Sigma_f$ is a disjoint union of open intervals, whence $\sigma$ uniquely extends to a $C^\infty$ function on $R$ such that $h(x) = H(x, \sigma(x))$, see [10, Lm. 6.4] for details.

Then the desired isotopy of $h$ to $h'$ in $S(f)$ can be constructed similarly to [10, Lm. 4.14]. Take any $C^\infty$ function $\mu : M \to [0, 1]$ such that $\mu = 0$ on some neighbourhood of $M \setminus U$, $\mu = 1$ on $R$, and $\mu$ is constant along orbits of $F$. Then the function $\nu = \mu \sigma$ is $C^\infty$ and well-defined on all of $M$. Consider the following homotopy

$$g : M \times I \to M, \quad g_t(x) = F(x, t\nu(x)).$$

Then $g_0 = \text{id}_M$, $g_t$ is fixed on $M \setminus U$, and $g_1 = h$ on $R$. Since $\mu$ is constant along orbits of $F$ and $h$ is a diffeomorphism, it follows from [10, Lm. 4.14] that $g$ is an isotopy. Hence $g_t^{-1} \circ h : M \to M$, $(t \in I)$, is an isotopy in $S(f)$ supported in $U$ and deforming $h$ to a desired diffeomorphism $h' = g_1^{-1} \circ h$. \hfill \Box

**Proposition 6.3.** Let $X \subset M$ be a compact subsurface such that $\partial X$ consists of (regular) leaves of $\Delta_f$. Suppose $h \in S_{\text{id}}(f)$ is fixed on some neighbourhood $U$ of $X$. Then there exists an isotopy of $h$ to $\text{id}_M$ in $S(f)$ fixed on some neighbourhood of $X$.

**Proof.** Again we will consider only the case when $M$ is orientable. Let $H : M \times \mathbb{R} \to M$ be the flow constructed in the proof of Proposition 6.1. Since $h \in S_{\text{id}}(f)$, there exists an isotopy $G : M \times I \to M$ of $\text{id}_M$ to $h$ in $S(f)$. Then it is easy to show that each $G_t$ preserves orbits of $H$ on some neighbourhood of $X$, see [10, Lm. 3.4]. Now it follows from [9, Th. 25], see also [13], that there exists a continuous function $\Lambda : (M \setminus \Sigma_f) \times I \to \mathbb{R}$ such that $\Lambda_t$ is $C^\infty$ for each $t \in I$, $\Lambda_0 = 0$, and $G_t(x) = H(x, \Lambda_t(x))$ for all $x \in M \setminus \Sigma_f$. Let $\mu : M \to [0, 1]$ be a $C^\infty$ function constant along orbits of $H$, $\mu = 0$ on $X$, and $\mu = 1$ on some neighbourhood of $M \setminus U$. Define the following map $a : M \times I \to M$ by

$$a(x, t) = \begin{cases} H(x, \mu(x)\Lambda(x, t)), & x \in U \\ G_t(x), & x \in M \setminus U. \end{cases}$$

We claim that $a$ is an isotopy between $\text{id}_M$ and $h$ in $S(f)$ fixed on some neighbourhood of $X$. 

Since $\mu = 1$ on some neighbourhood of $M \setminus U$, we see that $a$ is continuous and $a_t$ is $C^\infty$ for each $t$. Moreover,

$$a(x, 0) = \begin{cases} H(x, 0) = x, & x \in U \\ G_0(x) = x, & x \in M \setminus U. \end{cases}$$

Since $h$ is fixed on $U$, it follows that $\Lambda(x, 1) = 0$ on $U$. Therefore $\mu \Lambda_1 = \Lambda_1$ and $a_1 = h$. As $\mu = 0$ on $X$, we obtain that $a_t$, $(t \in I)$, is fixed on $X$. \hfill $\Box$

7. Proof of Theorem 1.7

Suppose $\chi(M) < 0$ and that $f : M \to P$ satisfies (Bd), (Isol), and (SA). We have to find a compact subsurface $X \subset M$ satisfying conditions (1)-(3) of Theorem 1.7.

Construction of $X$. Let $K_1, \ldots, K_k$ be all the critical components of level-sets of $f$ whose canonical neighbourhoods $N_{K_i}$ have negative Euler characteristic: $\chi(N_{K_i}) < 0$. Since $\chi(M) < 0$, we have by Lemma 3.4 that this collection is non-empty. Denote

$$K = \bigcup_{i=1}^k K_i.$$ 

For each $i = 1, \ldots, k$ choose an atom $R_i$ for $K_i$ in a way described in §3.2 and let $N_i$ be the corresponding canonical neighbourhood of $K_i$. Then we can assume that conditions (i) and (ii) of Lemma 3.3 hold. In particular, $R_i \cap R_j = N_i \cap N_j = \emptyset$ for $i \neq j$.

Denote $R_{<0} := \bigcup_{i=1}^k R_i$. Let also $B_1, \ldots, B_q$ be all the connected components of $M \setminus R_{<0}$ such that every $B_i$ is a cylinder and $f$ has no critical points in $B_i$. Put

$$X = R_{<0} \cup B_1 \cup \cdots \cup B_q.$$ 

We will show that $X$ satisfies the statement of Theorem 1.7.

Example 7.1. Let $M$ be an orientable surface of genus 2 embedded in $\mathbb{R}^3$ in a way shown in Figure 7.1a) and $f : M \to \mathbb{R}$ be the projection to the vertical line. Critical components of level-sets of $f$ whose canonical neighbourhoods have negative Euler characteristic are denoted by $K_1$ and $K_2$. The corresponding surface $X$ is shown in Figure 7.1b).

Before proving Theorem 1.7 we establish the following statement.

Claim 7.2. (i) Let $h \in S'(f)$. Then $h$ preserves every leaf $\omega \subset R_{<0}$ of $\Delta_f$ and its orientation.
Suppose $h$ is fixed on a neighbourhood of $R_{<0}$. Then for every connected component $B$ of $M \setminus R_{<0}$ the restriction $h|_B$ is isotopic to $\text{id}_B$ with respect to a neighbourhood of $\partial B \cap R_{<0}$.

**Proof.** (i). It follows from the definition of $K$ that $h(K) = K$. We claim that in fact $h(K_i) = K_i$ for all $i = 1, \ldots, k$.

Indeed, suppose that $h(K_i) = K_j$ for some $j$. Then by Lemma 3.3 $h(R_i) = R_j$ and $h(N_i) = N_j$. On the other hand, since $N_i$ is incompressible, $\chi(N_i) < 0$, and $h$ is isotopic to $\text{id}_M$, it follows from (1) of Proposition 4.5 that $h(N_i) \cap N_i \neq \emptyset$. But $N_i \cap N_j = \emptyset$ for $i \neq j$. Hence $h(N_i) = N_i$ for each $i = 1, \ldots, k$.

Denote by $\Xi_i$ the corresponding partition of $N_i$, see §5. Since $h$ preserves the set of critical points of $f$, it follows that $h$ preserves the set of vertexes of degree 2 of $K_i$. This implies that the restriction of $h$ to $N_i$ yields a certain automorphism $h^*$ of the partition $\Xi_i$. As $\chi(N_i) < 0$ and $h$ is isotopic to $\text{id}_M$, we get from Theorem 5.3 that $h$ yields a trivial automorphism of $\Xi_i$. In particular, each (critical) leaf $\omega$ of $\Delta_f$ in $K_i$ is $h^+$-invariant.

Let $\omega \subset R_i$ be a regular leaf of $\Delta_f$ and $e \subset N_i$ be the corresponding element of $\Xi_i$ containing $\omega$, so $e$ is either an open 2-disk or a half-open cylinder $S^1 \times (0, 1]$. Then

$$\omega = e \cap f^{-1} \circ f(\omega).$$

Notice that $h(e) = e$, since $h$ is $\Xi$-trivial. Moreover, $f \circ h = f$ implies that $h \circ f^{-1} \circ f(\omega) = f^{-1} \circ f(\omega)$, whence $h(\omega) = \omega$. It remains to note that $h$ preserves orientation of $\omega$ since it preserves orientation of leaves in $K_i$.

(ii) Let $B$ be a connected component of $M \setminus R_{<0}$. Then it follows from Corollary 3.8 that $B$ is either

![Figure 7.1.](image)
(a) a 2-disk, or
(b) a Möbius band, or
(c) a cylinder such that one of its boundary components belongs to $\mathcal{R}_{<0}$ and another one to $\partial M$, or
(d) a cylinder with $\partial B \subset \mathbb{R}_{<0}$.

If $B$ is of type (a)-(c), then it is well-known that $h$ is isotopic to $\text{id}_B$ with respect to a neighbourhood of $\partial B \cap \mathcal{R}_{<0}$. See [1, 16] for the 2-disk, and [6] for the cases (b) and (c).

Let $Q$ be the union of $\mathcal{R}_{<0}$ with all the components of types (a)-(c). Then we can assume that $h$ is fixed on $Q$.

It also follows that $Q$ is incompressible and every connected component $Q'$ of $Q$ contains some $N_j$. This implies that $\chi(Q') \leq \chi(N_j) < 0$. Then by (D) of Proposition [4.5] $h$ is homotopic to $\text{id}_M$ via a homotopy fixed on $Q$. In particular, the restriction of $h$ to every connected component $B$ of type (d) is homotopic in $B$ to $\text{id}_B$ relatively $\partial B$. □

Now we can complete Theorem [1.7].

(1) It follows from the definition of $\mathcal{R}_{<0}$ that $\partial X$ consists of some regular leaves of $\Delta_f$, whence $f$ is locally constant of $\partial X$. Moreover by Corollary [3.8] every connected component $B$ of $M \setminus \mathcal{R}_{<0}$ and therefore of $M \setminus X$ is either a 2-disk, or a cylinder, or a Möbius band.

It is also easy to see that $B$ contains critical points of $f$. Indeed, suppose $B$ is either a 2-disk or a Möbius band. Since $f$ is constant on $\partial B$, it follows that $f|_{B}$ is null-homotopic. Hence $f$ must have local extremes in Int$B$.

On the other hand, if $B$ is a cylinder containing no critical points of $f$, then by the construction of $X$ we should have that $B \subset X$ which is impossible.

Statement (2) is a particular case of (ii) of Claim [7.2]

(3) We have to show that the inclusion $i : \mathcal{S}'(f, X) \subset \mathcal{S}'(f)$ yields a bijection $i_0 : \pi_0\mathcal{S}'(f, X) \approx \pi_0\mathcal{S}'(f)$.

Claim 7.3. The map $i_0 : \pi_0\mathcal{S}'(f, X) \to \pi_0\mathcal{S}'(f)$ is an epimorphism.

Proof. Let $h \in \mathcal{S}'(f)$. We have to show that $h$ is isotopic in $\mathcal{S}'(f)$ to a diffeomorphism fixed on $X$.

By (i) of Claim [7.2] $h$ preserves the foliation of $\Delta_f$ on $\mathcal{R}_{<0}$. Hence by Proposition [6.1] applied to each critical component $K_i$, $(i = 1, \ldots, k)$, $h$ is isotopic in $\mathcal{S}'(f)$ to a diffeomorphism fixed on some neighbourhood of $\mathcal{R}_{<0}$, so we can assume that $h$ itself is fixed near $\mathcal{R}_{<0}$.

Let $B_i$, $(i = 1, \ldots, q)$, be a connected component of $X \setminus \mathcal{R}_{<0}$. By the construction $B_i$ is a cylinder being a union of regular leaves of $\Delta_f$
and containing no critical points of \( f \). Choose an orientation for \( B_i \). Then we can define a Hamiltonian flow \( H : B_i \times \mathbb{R} \to B_i \) of \( f \) on \( B_i \) whose orbits are leaves \( \Delta_f \) belonging to \( B_i \). Notice that \( h \) is fixed on some neighbourhood of \( \partial B_i \cap \mathcal{R}_{<0} \) and by (ii) the restriction of \( h \) to \( B_i \) is homotopic to \( \text{id}_{B_i} \) relatively \( \partial B_i \). Then by [10, Lm. 4.12] there exists a \( C^\infty \) function \( \alpha : B_i \to \mathbb{R} \) such that \( \alpha = 0 \) on some neighbourhood of \( \partial B_i \cap \mathcal{R}_{<0} \) and \( h(x) = H(x, \alpha(x)) \) for all \( x \in B_i \).

Notice that \( \partial B_i \cap \mathcal{R}_{<0} \) separates \( M \). Then the map

\[
(7.1) \quad a : M \times I \to M, \quad a(x, t) = \begin{cases} 
H(x, t\alpha(x)), & x \in B_i, \\
h(x), & x \in M \setminus B_i 
\end{cases}
\]

is an isotopy of \( h \) in \( \mathcal{S}(f) \) to a diffeomorphism fixed on \( B_i \). Applying this to each \( B_i \) we will made \( h \) fixed on all of \( X \). \( \square \)

**Claim 7.4.** \( i_0 : \pi_0 \mathcal{S}'(f, X) \to \pi_0 \mathcal{S}'(f) \) is a monomorphism.

**Proof.** Let \( \mathcal{S}'_{\text{id}}(f) \) and \( \mathcal{S}'_{\text{id}}(f, X) \) be the identity path components of \( \mathcal{S}'(f) \) and \( \mathcal{S}'(f, X) \) respectively. It is clear that \( \mathcal{S}'_{\text{id}}(f) = \mathcal{S}_{\text{id}}(f) \). Hence an injectivity of \( i_0 \) means that

\[
\mathcal{S}'_{\text{id}}(f, X) = \mathcal{S}'(f, X) \cap \mathcal{S}'_{\text{id}}(f) = \mathcal{S}'(f, X) \cap \mathcal{S}_{\text{id}}(f).
\]

Evidently, \( \mathcal{S}'_{\text{id}}(f, X) \subset \mathcal{S}'(f, X) \cap \mathcal{S}_{\text{id}}(f) \).

Conversely, let \( h \in \mathcal{S}'(f, X) \cap \mathcal{S}_{\text{id}}(f) \), so \( h \) is fixed on some neighbourhood of \( X \) and there exists an isotopy \( g_t : M \to M \) in \( \mathcal{S}(f) \) between \( h_0 = \text{id}_M \) and \( h_1 = h \). Then by Proposition 6.3 this isotopy can be made fixed on some neighbourhood of \( X \). Hence \( h \in \mathcal{S}'_{\text{id}}(f, X) \). \( \square \)

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