Some Exact Results of Hopfield Neural Networks and Applications

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Abstract

A set of fixed points of the Hopfield type neural network was under investigation. Its connection matrix is constructed with regard to the Hebb rule from a highly symmetric set of the memorized patterns. Depending on the external parameter the analytic description of the fixed points set had been obtained. And as a conclusion, some exact results of Hopfield neural networks were gained.

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Over the past decade, there has been an explosion of interest in so-called artificial neural network(ANN) technology. ANNs are a model of computing inspired by the brain[1]-[5], consisting of a collection of model "neurons" connected by model "synapses." Computation in the network is performed in a distributed fashion, by propagating excitatory and inhibitory activations across the synapses, and computing the neuronal outputs as a nonlinear (typically sigmoidal) function of total synaptic input. These networks also have a capacity to "learn" to perform a given computation by adjusting real-valued synaptic connection strengths (weight values) between units. ANNs are of considerable interest both for biological modeling of information processing in the nervous system, and for solving many classes of complex real-world applications.

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J.J. Hopfield boosted neural network research at the beginning of the 1980s with the publication of a famous paper on artificial neural networks, which he used for pattern completion and to solve optimization problems[4]. These networks consist of one layer of neurons that are completely connected with each other. Hopfield analysed the behavior of networks belonging to that type, and could prove mathematically that stable behavior may be achieved under certain conditions.

It can be shown that the dynamic behavior of Hopfield type neural networks is described by an energy surface. Each network state corresponds to a certain position on that surface. Through external clamping, neurons may be forced to certain states of activity, and thus the whole network may be forced to move to a well defined point on the energy surface. If the network is released, i.e. external clamping is removed, it will change its state in such a way that it moves on the energy surface towards new states of lower energy. Finally, neuron states will stop changing if a local minimum in the energy surface is reached. Through careful selection of weights, oscillations will be avoided. A set of fixed points of the Hopfield type neural network[4][6] is under investigation. Its connection matrix is constructed with regard to the Hebb rule from a \((p \times n)\)-matrix \(S\) of memorized patterns:

\[
S = \begin{pmatrix}
1 - x & 1 & \ldots & 1 & 1 & \ldots & 1 \\
1 & 1 - x & \ldots & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1 - x & 1 & \ldots & 1
\end{pmatrix}.
\]

Here \(n\) is the number of neurons, \(p\) is the number of memorized patterns \(\vec{s}^{(l)}\), which are the rows of the matrix \(S\), and \(x\) is an arbitrary real number.

Depending on \(x\) the memorized patterns \(\vec{s}^{(l)}\) are interpreted as \(p\) distorted vectors of the standard

\[
\vec{\varepsilon}(n) = \underbrace{(1, 1, \ldots, 1)}_{n}.
\]

We denote by \(\vec{\varepsilon}(k)\) the configuration vector which is collinear to the bisectrix of the principle orthant \textit{standard-vector}. Next, \(n\) is the number of the spin variables, \(p\) is the number of the memorized patterns and \(q = n - p\) is the number of the nondistorted coordinates of the standard-vector. Configuration vectors are denoted by small Greek letters. We use small Latin letters to denote vectors whose coordinates are real.
The problem is as follows: the network has to be learned by p-times showing of the standard (1), but a distortion has slipped in the learning process. How does the fixed points set depends on the value of this distortion $x$?

Depending on the distortion parameter $x$ the analytic description of the fixed points set has been obtained. It turns out to be very important that the memorized patterns $\vec{s}^{(l)}$ form a highly symmetric group of vectors: all of them correlate one with another in the same way:

$$(\vec{s}^{(l)}, \vec{s}^{(l')}) = r(x),$$

where $r(x)$ is independent of $l, l' = 1, 2, \ldots, p$. Namely this was the reason to use the words "highly symmetric" in the title.

It is known [7], that the fixed points of a network of our kind have to be of the form:

$$\vec{\sigma}^* = (\sigma_1, \sigma_2, \ldots, \sigma_p, 1, \ldots, 1), \quad \sigma_i = \{\pm 1\}, \quad i = 1, 2, \ldots, p.$$  (3)

Let’s join into one class $\Sigma^{(k)}$ all the configuration vectors $\vec{\sigma}^*$ given by Eq.(3), which have $k$ coordinates equal to "–1" among the first $p$ coordinates. The class $\Sigma^{(k)}$ consists of $C_p^k$ configuration vectors of the form (3), and there are $p + 1$ different classes ($k = 0, 1, \ldots, p$).

Our main result can be formulated as a Theorem.

**Theorem.** As $x$ varies from $-\infty$ to $\infty$ the fixed points set is exhausted in consecutive order by the classes of the vectors[8]

$$\Sigma^{(0)}, \Sigma^{(1)}, \ldots, \Sigma^{(K)},$$

and the transformation of the fixed points set from the class $\Sigma^{(k-1)}$ into the class $\Sigma^{(k)}$ occurs when $x = x_k$:

$$x_k = p \frac{n - (2k - 1)}{n + p - 2(2k - 1)}, \quad k = 1, 2, \ldots, K.$$

If $\frac{p - 1}{n - 1} < \frac{1}{3}$, according this scheme all the $p$ transformations of the fixed points set are realized one after another and $K = p$. If $\frac{p - 1}{n - 1} > \frac{1}{3}$, the transformation related to

$$K = \left\lfloor \frac{n + p + 2}{4} \right\rfloor$$

is the last. The network has no other fixed points.
The Theorem makes it possible to solve a number of practical problems. We would like to add that the Theorem can be generalized onto the case of arbitrary vector

\[ \vec{u} = (u_1, u_2, \ldots, u_p, 1, \ldots, 1), \quad \sum_{i=1}^{p} u_i^2 = p \]

being a standard instead the standard (1). Here memorized patterns \( \vec{s}^{(l)} \) are obtained by the distortion of the first \( p \) coordinates of the vector \( \vec{u} \) with regard to the fulfillment of Eqs.(2).

The obtained results can be interpreted in terms of neural networks, Ising model and factor analysis.

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