A modularized algorithmic framework for interface related optimization problems using characteristic functions

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Abstract. In this paper, we consider the algorithms and convergence for a general optimization problem, which has a wide range of applications in image segmentation, topology optimization, flow network formulation, and surface reconstruction. In particular, the problem focuses on interface related optimization problems where the interface is implicitly described by characteristic functions of the corresponding domains. Under such representation and discretization, the problem is then formulated into a discretized optimization problem where the objective function is concave with respect to characteristic functions and convex with respect to state variables. We show that under such structure, the iterative scheme based on alternative minimization can converge to a local minimizer. Extensive numerical examples are performed to support the theory.

1. Introduction

Interface related optimization problem is a fundamental problem in many applications, including problems in material science [10], image processing [3], topology optimization [39], surface reconstruction [55] and so on. A lot of numerical approaches have been developed to solve such problems, including front track based methods [19, 42], phase-field based methods [10, 14, 28, 38], level set based methods [37], parametric finite element based methods [25], two-stage thresholding [6, 7], Centroidal Voronoi Tessellations (CVT) based methods [12, 13, 29], primal dual methods [1, 5, 52, 43], and many others. Usually, solving such problems includes three ingredients: 1. representation of the interface (implicit or explicit), 2. approximation of the objective functional under the representation, and 3. approaches to minimize the approximate objective functional. In particular, the representation of the interface is the most fundamental part of a model or a method for interface related optimization problems.

This paper focuses on a wide class of approximate interface related optimization problems where the interface is implicitly represented by indicator functions of corresponding domains. It is motivated by the MBO method for approximating...
mean curvature flow using indicator functions [2, 35]. Esedoglu and Otto [17] then develop a novel interpretation using minimizing movement and generalize this type of method to multiphase flow with arbitrary surface tensions. The method has subsequently been extended to deal with many other applications, including image processing [15, 33, 34, 40], problems of anisotropic interface motions [15, 16, 36], the wetting problem on solid surfaces [30, 48, 50, 51], convex object segmentation [31, 32], and so on. The method can also be considered as piecewise constant level set methods [23, 40, 41, 49, 54].

Recently, based on Esedoglu and Otto’s novel interpretation, in [46, 47], the authors develop an efficient iterative convolution thresholding method (ICTM) for image segmentation and extend into topology optimization problems [9, 22] and surface reconstructions from point clouds [44]. In general, the problem could be formulated into

\begin{equation}
\min_{\Theta_i \in S, \Omega_i} \mathcal{E} = \sum_{i=1}^{n} \int_{\Omega_i} F_i(\Theta_1, \ldots, \Theta_n) \, dx + \sum_{i=1}^{n} \lambda_i|\partial \Omega_i|
\end{equation}

Here, \( F_i \) are usually fidelity terms, \( \Theta_i = (\Theta_{i,1}, \ldots, \Theta_{i,m}) \) contains all possible parameters in fidelity terms, \( \Omega = \bigcup_{i=1}^{n} \Omega_i, S = S_1 \times S_2 \times \ldots \times S_n \cap S_1 \times \ldots \times S_n \) and \( S_i \cap S_o \) as the admissible set of \( \Theta = (\Theta_1, \Theta_2, \ldots, \Theta_n) \), \( S_i \cap S_o \) is the admissible set of \( \Theta_i \) where \( S_i \) is the admissible set for satisfying some constraints of \( \Theta_i \) which are dependent on the partition of \( \Omega = \bigcup_{i=1}^{n} \Omega_i \) and \( S_i \) is the admissible set for satisfying some constraints of \( \Theta_i \) which are independent on the partition, and \( \lambda_i \) are fixed parameters.

Denote \( u_i \) to be indicator functions of \( \Omega_i \) (\( i \in [n] \)), as pointed out in [17], when \( \tau \ll 1 \), the measure of \( \partial \Omega_i \) can be approximated by

\begin{equation}
|\partial \Omega_i| \approx \sqrt{\frac{\pi}{\tau}} \sum_{j \in [n], j \neq i} \int_{\Omega} u_i G_{\tau} * u_j \, dx,
\end{equation}

where \( * \) represents convolution and \( G_{\tau} \) is

\begin{equation}
G_{\tau}(x) = \frac{1}{4\pi \tau} \exp\left(-\frac{|x|^2}{4\tau}\right).
\end{equation}

Then, problem (1.1) can be approximately written as

\begin{equation}
\min_{\Theta_i \in S, u_i \in \mathcal{B}} \mathcal{E}^{\tau} : = \sum_{i=1}^{n} \int_{\Omega} u_i F_i(\Theta_1, \ldots, \Theta_n) \, dx + \sqrt{\frac{\pi}{\tau}} \sum_{i=1}^{n} \lambda_i \sum_{j \in [n], j \neq i} \int_{\Omega} u_i G_{\tau} * u_j \, dx
\end{equation}

where

\( \mathcal{B} : = \{ u \in BV(\Omega, \mathbb{R}) \mid u = \{0, 1\} \} \)

and \( BV(\Omega, \mathbb{R}) \) denotes the bounded variation functional space.

We note that special cases of problem (1.4) include but not limited to the approximate models for Chan-Vese model for image segmentation [3], binary local fitting model (LBF) for intensity inhomogeneous image segmentation [27], local statistical active contour model [53], topology optimization for Stokes flow [3], formulation of biological flow networks [21, 22], composite materials [11], Dirichlet partition problems [45], and surface reconstruction from point clouds [44]. In all these cases, \( F_i \) is usually taken to be convex in the admissible set \( S \) with respect to \( \Theta \).
It is not difficult to check that the second term in (1.4) is strictly concave by the following direct calculation using the fact that $G^*_{\tau}$ is a self adjoint operator:

$$
\sum_{j \in [n], j \neq i} \int_{\Omega} u_j G_{\tau}^* u_j \, dx = \int_{\Omega} u_i G_{\tau}^* (1 - u_i) \, dx = \int_{\Omega} (G_{\tau/2}^* u_i) G_{\tau/2}^* (1 - u_i) \, dx.
$$

Obeying the fact that either images are defined on discretized pixels or computational domains are discretized for fluid problems, we thus arrive at the following optimization problem in a compact form

$$
\min_{u \in C, \theta \in S} \Phi(u, \theta) := h(\theta) + \sum_i \sum_j u_{i,j} f_{i,j}(\theta) + g(u),
$$

where $h$ is a convex (can be nonsmooth) function, $f_{i,j}$ are convex functions with respect to $\theta$ for each $i$ and $j$, $h$ and $f_{i,j}$ are continuous on $S$, $S$ is the admissible set for $\theta$, $g$ is a strictly concave function and $C$ is the admissible set for $u := (u_1; u_2; \ldots; u_n) \in \mathbb{R}^{n \times p}$;

$$
C := \{ u \mid u_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,p}) \in \{0, 1\}^p, \sum_{i=1}^n u_{i,j} = 1, \forall j \in [p] \}.
$$

In particular, $n$ (usually $n \geq 2$) denotes the number of phases in different applications (for example, number segments in image segmentation or number of materials in optimal composite of materials) and $p$ usually describes the degree of freedom for problems after discretization. In addition, summation equals 1 constraint implies that at each fixed point (say point $j$) of discretization, only one index $i$ such that $u_{i,j}$ takes 1 and all others take 0.

Existing ICTM for (1.5) usually achieves stable solutions in a couple of steps, which is empirically observed in the literature; see [9, 46, 47]. However, theoretical guarantee of convergence towards solutions with “good” qualities is still lacking. In response to this limitation, in this paper, we develop a convex relaxation based alternating minimization algorithmic framework for solving (1.5). We justify the validity of our scheme by that the relaxation yields exact solutions. This new framework is flexible and modularized to handle a broad class of application-driven problems in the form of (1.5). We then ask, is it possible to achieve stationary or even local solutions? We provide a positive answer to this question by separating our discussion for different embedded modules.

In this paper, we give a theoretical and systematical study on the convergence of the ICTM. The convergence to a local minimizer is rigorously proved in accordance with three practical situations. Among existing literature of interface related optimization problems, only energy decaying property and the convergence of the value of objective function have been studied. However, there is no study on the convergence of the generated sequence itself. As for related interface motion problems where only the perimeter term is minimized, existing work mainly considers the convergence on the dynamics when the approximation parameter (i.e., $\tau$ in (1.2)) goes to 0; see [24]. Indeed, individual treatments on the three situations enable us to take advantage of their special algorithmic structures more effectively and thus to derive some new results. This is a striking feature of our study that leads to a big difference between our convergence analysis and others. To our knowledge, this is the first and systematical convergence result of the ICTM for interface related optimization problems.
The paper is organized as follows. The detail on the convergence is introduced in Section 3 and numerical experiments for different applications are performed in Section 4. We draw some conclusion and discussions in Section 5.

2. Algorithmic framework

The nonconvex optimization problem (1.5) is in general computationally challenging. The difficulty is mainly due to the binary variables. To develop efficient solution scheme for solving problem (1.5), we first include a convex relaxation of the binary constraint:

\[
\min_{u \in C, \theta \in S} \Phi(u, \theta) := h(\theta) + \sum_{i} \sum_{j} u_{i,j} f_{i,j}(\theta) + g(u),
\]

where \( S \) is a closed convex set and \( \hat{C} \) is the convex hull of the binary constraint \( C \),

\[
\hat{C} := \{ u \mid u_i = (u_{i,1}, u_{i,2}, \cdots, u_{i,n}) \in [0,1]^p, \sum_{i=1}^{n} u_{i,j} = 1, \forall j \in [p] \}.
\]

Before we consider continuous optimization algorithms for solving problem (2.1), it is hoped that the convex relaxation yields exact solution. Thanks to the strict concavity of \( g \), the following results exhibit the relaxation exactness by showing that any relaxed local solution maps to a true solution.

**Lemma 1.** For any \( \bar{\theta} \in S \), let \( \bar{u} \) be a local minimum of problem

\[
\arg\min_{u \in \hat{C}} \Phi(u, \bar{\theta}),
\]

that is, there exists \( \epsilon > 0 \) such that

\[
\Phi(\hat{u}, \bar{\theta}) \leq \Phi(u, \bar{\theta}), \quad \forall u \in \hat{C} \cap \mathbb{B}_\epsilon(\bar{u}),
\]

where \( \mathbb{B}_\epsilon(\bar{u}) := \{ u \mid \| u - \bar{u} \| \leq \epsilon \} \). Then \( \bar{u} \in C \).

**Proof.** We prove the result for by contradiction. Assume that there exist \( i \) and \( j \) such that \( \bar{u}_{i,j} \notin \{0,1\} \), then the set

\[
A := \{ j \mid \bar{u}_{i,j} \notin \{0,1\} \text{ for some } i \}
\]

is nonempty. Since \( \sum_{i} \bar{u}_{i,j} = 1 \), there exist \( \bar{j} \in A \) and \( i_1, i_2 \in \{1, \ldots, n\} \) such that \( \bar{u}_{i_1, \bar{j}}, \bar{u}_{i_2, \bar{j}} \in (0,1) \). And we can find \( c_0 \in (0, \frac{1}{2}) \) such that

\[
c_0 \leq \bar{u}_{i_1, \bar{j}}, \quad \bar{u}_{i_2, \bar{j}} \leq 1 - c_0.
\]

Define \( u(t) = (u_1(t); u_2(t); \ldots; u_n(t)) \), where

\[
u_{i_1, \bar{j}}(t) = \bar{u}_{i_1, \bar{j}} + t, \quad u_{i_2, \bar{j}}(t) = \bar{u}_{i_2, \bar{j}} - t, \quad u_{i,j}(t) = \bar{u}_{i,j}, \quad \text{if } i \notin \{i_1, i_2\}, j \neq \bar{j}.
\]

When \( -c_0 \leq t \leq c_0 \), we have that \( 0 \leq u_{i,j}(t) \leq 1 \) for all \( i \) and \( j \), and \( \sum_{i} u_{i,j}(t) = 1 \) for all \( j \), which yields that \( u(t) \in \hat{C} \).

Define further \( \varphi(t) := \Phi(u(t), \bar{\theta}) \), then \( \varphi(t) \) is strictly concave since \( \Phi \) is strictly concave with respect to \( u \). Since \( \bar{u} \) is a local minimum to \( \min_{u \in \hat{C}} \Phi(u, \bar{\theta}) \), immediately \( 0 \) is a local minimum to problem

\[
\min_{t \in [-c_0, c_0]} \varphi(t),
\]

which contradicts to the strict concavity of \( \varphi \). \( \square \)

The relaxation exactness regarding local minimums follows immediately.
Proposition 1. Let $(\bar{u}, \bar{\theta})$ be a local minimum of problem (2.1), then $\bar{u}_{i,j} \in \{0, 1\}$ for any $i$ and $j$.

We are now in the position to solve the relaxed problem (2.1). We present an alternating minimization algorithmic framework. That is, the algorithms are based on generating the following updating sequence starting with an initial guess $u^0$:

$$\quad \theta^0, u^1, \theta^1, u^2, \theta^2, \ldots, u^n, \theta^n, \ldots$$

where

$$\quad \theta^k = \arg \min_{\theta \in S} \Phi(u^k, \theta) \quad \text{(2.2)}$$

and

$$\quad u^{k+1} = \arg \min_{u \in \hat{C}} \Phi(u, \theta^k) \quad \text{(2.3)}$$

3. Convergence results

The algorithmic framework stated in the preceding section consists of sequentially solving subproblems (2.2) and (2.3) with respect to one variable when the other variable is fixed in an alternative manner. Naturally, for different structures of $h, f_{i,j},$ and $g$, we may utilize efficiently their structural information and hence call suitable solution schemes to tackle subproblems (2.2) and (2.3). Thereby, our convergence analysis in this section mainly relates to three cases where we employ appropriate method for minimizations which aims at finding for each $k$,

1. a global minimizer of convex subproblem (2.2) and a global minimizer of subproblem (2.3),
2. a global minimizer of convex subproblem (2.2) and a local minimizer of subproblem (2.3),
3. an approximate stationary solution of convex subproblem (2.2) and a thresholding approach for subproblem (2.3).

The main convergence results for the three situations, namely **Global $\theta$ & Global $u$ case, Global $\theta$ & Local $u$ case and inexact Stationary $\theta$ & Thresholding $u$ case** are presented in Theorems 3, 5, 7 and 9.

3.1. Convergence analysis for Global $\theta$ & Global $u$ case. We first analyze the alternating minimization method on problem (1.5) in the situation where both global minimizers of subproblems (2.2) and (2.3) are accessible. The algorithm update is summarized in Algorithm 1.

Note that, as shown in Lemma 1, that argmin$_{u \in \hat{C}} \Phi(u, \theta^k) \subset C$. Hence, condition (5.1) is equivalent to that $u^k \in \arg\min_{u \in \hat{C}} \Phi(u, \theta^k)$. Furthermore, we can derive the following result immediately.

Lemma 2. Let \{$(u^k, \theta^k)$\} be the sequence generated by (3.2) and (3.3), then $u^k_{i,j} \in \{0, 1\}$ for any $i, j$ and $k$.

The subsequent result establishes the stability of the solution set of $u$-minimization problem with respect to variable $\theta$, denoted as $C^*(\theta) := \arg\min_{u \in \hat{C}} \Phi(u, \theta)$.

Lemma 3. For any $\bar{\theta} \in S$ and $\bar{u} \in \arg\min_{u \in \hat{C}} \Phi(u, \bar{\theta})$, then there exists $\epsilon > 0$ such that for any $\theta_{\epsilon} \in B_{\epsilon}(\bar{\theta}) \cap S$, where $B_{\epsilon}(\bar{\theta}) := \{\theta \mid \|\theta - \bar{\theta}\| \leq \epsilon\}$, argmin$_{u \in \hat{C}} \Phi(u, \theta_{\epsilon}) \subseteq C^*(\bar{\theta}) := \arg\min_{u \in \hat{C}} \Phi(u, \bar{\theta})$. 

Algorithm 1 An iterative scheme with Global $\theta$ & Global $u$ for (1.5)

**Input:** $(u^0, \theta^0)$

**Output:** $(\bar{u}, \bar{\theta})$ that approximately minimizes (1.5).

Set $k = 0$

**While** the following condition fail to hold:

(3.1) $u^k \in \text{argmin}_{u \in \mathbb{C}} \Phi(u, \theta^k)$,

1. Find $u^{k+1}$ such that

(3.2) $u^{k+1} \in \text{argmin}_{u \in \mathbb{C}} \Phi(u, \theta^k) = \sum_i \sum_j u_{i,j} f_{i,j}(\theta^k) + g(u)$. 

2. Find $\theta^{k+1}$ such that

(3.3) $\theta^{k+1} = \text{argmin}_{\theta \in \mathbb{S}} \Phi(u^{k+1}, \theta) = h(\theta) + \sum_i \sum_j u_{i,j}^{k+1} f_{i,j}(\theta)$. 

Set $k = k + 1$

**Proof.** From the definition of $\mathbb{C}$ we have $|\mathbb{C}| < \infty$. As $\bar{u} \in C^*(\bar{\theta})$, 

$\Phi(\bar{u}, \bar{\theta}) < \Phi(u, \bar{\theta}), \forall u \in C \setminus C^*(\bar{\theta})$. 

Furthermore, because $|\mathbb{C}| < \infty$, there exists $\delta > 0$ such that $\Phi(\bar{u}, \bar{\theta}) < \Phi(u, \bar{\theta}) - \delta$ for any $u \in C \setminus C^*(\bar{\theta})$.

Since $h$ and $f_{i,j}$ are continuous on $\mathbb{S}$, for each $u \in C \setminus C^*(\bar{\theta})$, there exists $\epsilon_u > 0$ such that $\Phi(\bar{u}, \theta_e) < \Phi(u, \theta_e) - \delta/2$ for any $\theta_e \in \mathbb{B}_{\epsilon_u}(\bar{\theta}) \cap \mathbb{S}$. In addition, because $|\mathbb{C}| < \infty$, there exists $\epsilon > 0$ such that $\min_{u \in C \setminus C^*(\bar{\theta})} \epsilon_u \geq \epsilon$ and thus

$\Phi(\bar{u}, \theta_e) < \Phi(u, \theta_e) - \delta/2, \forall u \in C \setminus C^*(\bar{\theta})$

and $\theta_e \in \mathbb{B}_{\epsilon_u}(\bar{\theta}) \cap \mathbb{S}$.

Combining above with Lemma 1 yields that 

$\text{argmin}_{u \in \mathbb{C}} \Phi(u, \theta_e) \subseteq C^*(\bar{\theta}), \forall \theta_e \in \mathbb{S} \cap \mathbb{B}_{\epsilon}(\bar{\theta})$. 

To analyze the convergence property of the sequence generated by alternating minimization scheme (3.2) and (3.3), we introduce the following assumption on the problem (1.5). This assumption is considered mild, particularly for the problem (1.5) derived from the problem (1.1), where the functions $f_{i,j}(\theta)$ typically vary distinctively for each $i, j$.

**Assumption 1.** For any $u_1, u_2 \in C$ and $\theta \in \mathbb{S}$, if $u_1 \neq u_2$, then $\Phi(u_1, \theta) \neq \Phi(u_2, \theta)$.

It can be established that, under Assumption 1, $C^*(\theta)$ is a singleton for any $\theta \in \mathbb{S}$.

**Proposition 2.** Suppose Assumption 1 is satisfied. Let $\{(u^k, \theta^k)\}$ be the sequence generated by alternating minimization scheme (3.2) and (3.3), suppose that condition (3.1) is satisfied for some $k$, then $(u^k, \theta^k)$ is a local minimizer of problem (1.5) in the sense that there exists $\epsilon > 0$ such that

$\Phi(u^k, \theta^k) \leq \Phi(u, \theta), \forall (u, \theta) \in \hat{C} \times (\mathbb{S} \cap \mathbb{B}_{\epsilon}(\theta^k))$. 

PROOF. Let \((\bar{u}, \bar{\theta})\) denote the iterate \((u^k, \theta^k)\) satisfying conditions \((3.1)\). Given the fulfillment of Assumption \(1\), \(C^*(\bar{\theta})\) is a singleton, yielding \(C^*(\bar{\theta}) = \{\bar{u}\}\). Lemma \(3\) shows that there exists \(\epsilon > 0\) such that
\[
\text{argmin}_{u \in \mathcal{C}} \Phi(u, \theta) \subseteq C^*(\bar{\theta}) = \{\bar{u}\}, \quad \forall \theta \in \mathcal{S} \cap B_\epsilon(\bar{\theta}).
\]
If there exists \((u_0, \theta_0) \in \hat{\mathcal{C}} \times (\mathcal{S} \cap B_\epsilon(\bar{\theta}))\) such that
\[
\Phi(u_0, \theta_0) < \Phi(\bar{u}, \bar{\theta}),
\]
then \(\Phi(\bar{u}, \theta_0) \leq \Phi(u_0, \theta_0)\) because of the fact that
\[
\text{argmin}_{u \in \mathcal{C}} \Phi(u, \theta_0) \subseteq C^*(\bar{\theta}) = \{\bar{u}\}.
\]
In addition, because \(\bar{\theta} = \text{argmin}_{\eta \in \mathcal{S}} \Phi(\bar{u}, \eta)\), we have
\[
\Phi(\bar{u}, \bar{\theta}) \leq \Phi(\bar{u}, \theta_0) \leq \Phi(u_0, \theta_0) < \Phi(\bar{u}, \bar{\theta}),
\]
which leads to a contradiction and we get the conclusion. \(\square\)

Next, we have following result for the fixed point of Algorithm \(1\).

THEOREM 3. Suppose Assumption \(1\) is satisfied. Let \(\{(u^k, \theta^k)\}\) be the sequence generated by alternating minimization scheme \((3.2)\) and \((3.3)\), then \(u^k_j \in \{0, 1\}\) for any \(i, j\) and \(k\), and
\[
\Phi(u^{k+1}, \theta^{k+1}) < \Phi(u^k, \theta^k),
\]
provided condition \((3.1)\) is not satisfied at iteration \(k\). Furthermore, if \(\Phi\) is bounded below on \(\mathcal{C} \times \mathcal{S}\), there exists \(K \geq 0\) such that \((u^K, \theta^K)\) satisfies condition \((3.1)\), and is a local minimizer of problem \((1.5)\) in the sense that there exists \(\epsilon > 0\) such that
\[
\Phi(u^K, \theta^K) \leq \Phi(u, \theta), \quad \forall (u, \theta) \in \mathcal{C} \times \mathcal{S} \cap B_\epsilon(\theta^K).
\]

PROOF. Given the fulfillment of Assumption \(1\), \(C^*(\theta)\) is a singleton for any \(\theta \in \mathcal{S}\). Because \(u^{k+1} \in \text{argmin}_{u \in \mathcal{C}} \Phi(u, \theta^k)\), we have that \(C^*(\theta^k) = \{u^{k+1}\}\). When condition \((3.1)\) is not satisfied, i.e., \(u^{k+1} \neq u^k\), it follows that
\[
\min_{\theta \in \mathcal{S}} \Phi(u^{k+1}, \theta) = \Phi(u^{k+1}, \theta^k) \leq \Phi(u^{k+1}, \theta^k) < \Phi(u^k, \theta^k) = \min_{\theta \in \mathcal{S}} \Phi(u^k, \theta).
\]

The fact that \(u^k \in \{0, 1\}\) for any \(i, j\) and \(k\) follows from Lemma \(1\). Thus \(u^k \in \mathcal{C}\) for any \(k\). Finally, recall the fact that \(|\mathcal{C}|\) is finite, combining with Proposition \(2\) the conclusion easily follows. \(\square\)

3.2. Convergence analysis for Global \(\theta\) & Local \(u\) case. A weakness of Proposition \(2\) is that it relates exclusively to global (as opposed to local) minima of subproblem \((2.3)\). This part remedies the situation somewhat. We replace the \(u\)-update \((3.2)\) by only reaching a local minimum on \(\mathcal{C}\) in sense of
\[
\text{(3.4) } u^{k+1} \in C^*_r(\theta^k),
\]
where
\[
C^*_r(\theta) := \{\bar{u} \in \mathcal{C} \mid \Phi(\bar{u}, \theta) \leq \Phi(u, \theta), \forall u \in \mathcal{C} \cap B_r(\bar{u})\},
\]
with given \(r > 0\). We then accordingly replace condition \((3.1)\) by
\[
\text{(3.5) } u^k \in C^*_r(\theta^k).
\]
The algorithm is thus illustrated in Algorithm \(2\).

Similar to Proposition \(2\) and Theorem \(3\) we may derive convergence results as follows. The proofs are purely technical and thus omitted.
Set $k$ minimizer of problem (1.5) minimizer of problem (3.8)

2. Find $u$ such that $u^{k+1} = \arg\min_{\theta \in S} \Phi(u^{k+1}, \theta) = h(\theta) + \sum_{i} \sum_{j} u_{i,j}^{k+1} f_{i,j}(\theta)$. Set $k = k + 1$

Proposition 4. Suppose Assumption [4] is satisfied. Let $\{(u^k, \theta^k)\}$ be the sequence generated by alternating minimization scheme (3.7) with given $r > 0$ and (3.8), suppose that condition (3.6) is satisfied for some $k$, then $(u^k, \theta^k)$ is a local minimizer of problem (1.5) in the sense that there exists $\epsilon > 0$ such that

$$\Phi(u^k, \theta^k) \leq \Phi(u, \theta), \quad \forall (u, \theta) \in (C \cap B_r(u^k)) \times (S \cap B_r(\theta^k)).$$

Theorem 5. Suppose Assumption [4] is satisfied. Let $\{(u^k, \theta^k)\}$ be the sequence generated by alternating minimization scheme (3.7) with given $r > 0$ and (3.8), then

$$\Phi(u^{k+1}, \theta^{k+1}) < \Phi(u^k, \theta^k),$$

provided conditions (3.6) is not satisfied at iteration $k$. Furthermore, if $\Phi$ is bounded below, there exists $K \geq 0$ such that $(u^K, \theta^K)$ satisfies condition (3.6) and is a local minimizer of problem (1.5) in the sense that there exists $\epsilon > 0$ such that

$$\Phi(u^K, \theta^K) \leq \Phi(u, \theta), \quad \forall (u, \theta) \in (C \cap B_r(u^K)) \times (S \cap B_r(\theta^K)).$$

3.3. Convergence analysis for inexact Stationary $\theta$ & Thresholding $u$ case. Both Algorithms [1] and [2] assume that a global or local minimum $u^*$ of subproblem (3.2) or (3.7), and a global minimum $\theta^*$ of subproblem (2.2) can be found exactly. In this part, assuming that $\Phi$ is continuously differentiable with respect to $u$, and denoting $\nabla_{u,i,j} \Phi$ as the gradient of $\Phi$ with respect to $u_{i,j}$, we update the variable $u$ by applying the very simple thresholding approach to solve the $u$-subproblem

$$\min_{u \in C} \Phi(u, \theta^k).$$

To elaborate, given $u^{k,t} \in C$, for any $j \in [p]$, we choose $I(j) \in \arg\min_{i \in [n]} \nabla_{u,i,j} \Phi(u^{k,t}, \theta^k)$ and set

$$u_{i,j}^{k,t+1} = \begin{cases} 1 & \text{for } i = I(j), \\ 0 & \text{otherwise}. \end{cases}$$

This $u^{k,t+1}$ is also a solution to the following approximation problem of $\min_{u \in C} \Phi(u, \theta^k)$, where the objective is the linearization of $\Phi(u, \theta^k)$ at $u^{k,t}$,

$$u^{k,t+1} \in \arg\min_{u \in C} \Phi(u^{k,t}, \theta^k) + \langle \nabla_{u} \Phi(u^{k,t}, \theta^k), u - u^{k,t} \rangle.$$
On the other hand, unconstrained convex minimization method is usually solved iteratively, and only approximate first-order stationary solutions can be pursued by internal iterations. As a consequence, it is mathematically important to involve a suitable inexactness criterion for solving convex subproblem (2.2) and study the convergence with \( \theta \) being accessed inexactly. To be specific, in \( \theta \)-update, we compute an approximate stationary point to subproblem (2.2)

\[
\min_{\theta \in \mathcal{S}} \Phi(u^{k+1}, \theta) = h(\theta) + \sum_{i} \sum_{j} u^{k+1}_{i,j} f_{i,j}(\theta)
\]

satisfying

\[
(3.10) \quad \Phi(u^{k+1}, \theta^{k+1}) \leq \Phi(u^{k}, \theta^{k}) + \text{dist}_{\theta}(0, \partial \theta \Phi(u^{k+1}, \theta^{k+1})) + N_{\mathcal{S}}(\theta^{k+1}) \leq \varepsilon_{k+1},
\]

where in the sense of convex analysis, \( \partial \theta \Phi \) denotes the subgradient of \( \Phi \) with respect to \( \theta \), \( N_{\mathcal{S}}(\theta) \) denotes the normal cone of \( \mathcal{S} \) at \( \theta \), i.e., \( N_{\mathcal{S}}(\theta) := \{ \xi \mid \langle \xi, s - \theta \rangle \leq 0, \forall s \in \mathcal{S} \} \), and \( \{ \varepsilon_{k} \} \) is a positive vanishing sequence which controls the accuracy. The algorithm is illustrated in Algorithm 3.

**Algorithm 3** An iterative scheme with inexact Stationay \( \theta \) & Local \( u \) for (1.5)

**Input:** \((u^{0}, \theta^{0}) \in \mathcal{C} \times \mathcal{S}, \) and sequence \( \{ \varepsilon_{k} \} \) that \( \varepsilon_{k} \to 0 \)

**Output:** \((\bar{u}, \bar{\theta}) \) that approximately minimizes (1.5) .

Set \( k = 0 \)

**While** \( \| u^{k+1} - u^{k} \| + \| \theta^{k+1} - \theta^{k} \| > \text{tol} \)

1. Find \( u^{k+1} \in \mathcal{C} \) by using (3.9) with \( u^{k,0} = u^{k} \) until \( u^{k,T+1} = u^{k,T} \). Set \( u^{k+1} = u^{k,T} \).

2. Find \( \theta^{k+1} \) being an approximate stationary point to problem

\[
\min_{\theta \in \mathcal{S}} \Phi(u^{k+1}, \theta)
\]

satisfying (3.10).

Set \( k = k + 1 \)

3.3.1. Analysis on the thresholding on \( u \). The following lemma presents the basic property of the thresholding on \( u \) given in (3.9).

**Lemma 4.** Given \( u \in \mathcal{C} \) and \( \theta \in \mathcal{S} \), let

\[ u^{+} \in \text{argmin}_{v \in \mathcal{C}} \Phi(u, \theta) + \langle \nabla_{u} \Phi(u, \theta), v - u \rangle, \]

which, for any \( j \in [p] \), can be expressed as

\[
u_{i,j}^{+} = \begin{cases} 1 & \text{for } i = I(j), \\ 0 & \text{otherwise}, \end{cases}
\]

where \( I(j) \in \text{argmin}_{i \in [n]} \nabla_{u_{i,j}} \Phi(u, \theta) \). Then

\[ \Phi(u^{+}, \theta) < \Phi(u, \theta), \]

if \( u^{+} \neq u \). Moreover, \( u \) is a strict local minimum of \( \min_{u \in \mathcal{C}} \Phi(u, \theta) \) if \( u^{+} = u \) and \( \text{argmin}_{i \in [n]} \nabla_{u_{i,j}} \Phi(u, \theta) \) is a singleton for any \( j \in [p] \).

**Proof.** Given \( u \in \mathcal{C} \), if \( u^{+} \neq u \), since \( \Phi(u, \theta) \) is strictly concave with respect to the variable \( u \), it follows from [4] Exercise 16(a), Section 3.1 that

\[ \Phi(u^{+}, \theta) < \Phi(u, \theta) + \langle \nabla_{u} \Phi(u, \theta), u^{+} - u \rangle \leq \Phi(u, \theta), \]
where the last inequality follows from \( u^+ \in \arg\min_{v \in C} \Phi(u, \theta) + \langle \nabla_u \Phi(u, \theta), v - u \rangle \).

Next, for any \( u \in C \), if \( u^+ = u \), we will have

\[
I_u(u, j) := \{ i \in [n] \mid u_{i,j} = 1 \} \subseteq \arg\min_{i \in [n]} \nabla_{u_{i,j}} \Phi(u, \theta).
\]

According to the assumption that \( \arg\min_{i \in [n]} \nabla_{u_{i,j}} \Phi(u, \theta) \) is a singleton for any \( j \in [p] \), we have

\[
\nabla_{u_{I_u(u, j)}} \Phi(u, \theta) < \nabla_{u_{i,j}} \Phi(u, \theta)
\]

for any \( j \in [p] \) and \( i \neq I_u(u, j) \). Then for any \( d \in C - u := \{ v - u \mid v \in C \} \)
satisfying \( d \neq 0 \), it holds that

\[
\langle \nabla_u \Phi(u, \theta), d \rangle = \sum_{j=1}^{p} (\nabla_{u_{I_u(v, j)}} \Phi(u, \theta) - \nabla_{u_{I_u(u, j)}} \Phi(u, \theta)) > 0,
\]

where \( I_u(v, j) := \{ i \in [n] \mid v_{i,j} = 1 \} \). Therefore, there exists \( \epsilon_d > 0 \) such that

\[
\Phi(u + s d, \theta) > \Phi(u, \theta),
\]

for any \( s \in (0, \epsilon_d) \).

Since set \( C \) is with finite elements, we can find a constant \( \epsilon > 0 \) such that \( \Phi(u + sd, \theta) > \Phi(u, \theta) \) for any \( d \in C - u \) and \( s \in (0, \epsilon) \). As \( \hat{C} = \text{conv}(C) \), thus \( \hat{C} - u = \text{conv}(C - u) \) and hence there exists \( \epsilon_0 > 0 \) such that

\[
\hat{C} \cap \mathbb{B}_{\epsilon_0}(u) - u = (\hat{C} - u) \cap \mathbb{B}_{\epsilon_0}(0) \subseteq \text{conv}(C - u) \cap \mathbb{B}_s(0).
\]

Then because \( \Phi(u + s d, \theta) > \Phi(u, \theta) \) for any \( d \in C - u \) and \( s \in (0, \epsilon) \) and \( \Phi \) is strictly concave with respect to \( u \), we get that \( \Phi(v, \theta) > \Phi(u, \theta) \) for any \( v \in \hat{C} \cap \mathbb{B}_{\epsilon_0}(u) \setminus \{ u \} \).

\[\square\]

**Proposition 6.** Let \( \{ u^{k,t} \} \) be the sequence generated by (3.9) with \( u^{k,0} \in C \). Then, there exists a finite \( T > 0 \) such that \( u^{k,T+1} = u^{k,T} \) and \( \Phi(u^{k,T}, \theta^k) \leq \Phi(u^{k,0}, \theta^k) \). Additionally, if \( \arg\min_{i \in [n]} \nabla_{u_{i,j}} \Phi(u^{k,T}, \theta^k) \) is a singleton for any \( j \in [p] \), then \( u^{k,T} \) is a local minimum of \( \min_{u \in C} \Phi(u, \theta^k) \).

**Proof.** As shown in Lemma 4, if \( u^{k,t+1} \neq u^{k,t} \), we will have

\[
\Phi(u^{k,t+1}, \theta^k) < \Phi(u^{k,t}, \theta^k).
\]

Then, since \( u^{k,t} \in C \) and \( |C| \) is finite, there must exist \( T > 0 \) such that \( u^{k,T+1} = u^{k,T} \), and the conclusion follows from Lemma 4. \[\square\]

**3.3.2. Convergence analysis.** In this part, we present the basic convergence result for Algorithm 3. Firstly, as established in Proposition 6, \( \Phi(u^{k+1}, \theta^k) \leq \Phi(u^k, \theta^k) \). This, combined with the fact that \( \theta^{k+1} \) satisfies (3.10), ensures that \( \Phi(u^{k+1}, \theta^{k+1}) \leq \Phi(u^k, \theta^k) \), indicating a non-increasing energy trend throughout the iteration. Considering that \( u^k \) is within the bounded set \( C \), once \( \Phi(u, \theta) \) is coercive with respect to variable \( \theta \) for any \( u \in C \), that is, for any \( u \in C \), \( \Phi(u, \theta) \to \infty \) as \( \| \theta \| \to \infty \), the boundedness of the generated sequence \( \{ (u^k, \theta^k) \} \) is ensured, thereby guaranteeing the existence of the limit point of the sequence \( \{ (u^k, \theta^k) \} \) generated by Algorithm 3.

**Theorem 7.** Let \( \{ (u^k, \theta^k) \} \) be the sequence generated by alternating minimization Algorithm 3 that satisfies condition (3.10) with \( \varepsilon_k \to 0 \). Consider any limit point \( (\bar{u}, \bar{\theta}) \) of sequence \( \{ (u^k, \theta^k) \} \). Then \( \bar{u} \in C \). Suppose that \( \| \theta^{k+1} - \theta^k \| \to 0 \),
and that \( \arg\min_{i \in [n]} \nabla_{u_i} \Phi(u, \theta) \) is a singleton for any \( j \in [p] \), then \((\bar{u}, \bar{\theta})\) is a local minimizer of problem \((1.5)\) in the sense that there exists \( \epsilon > 0 \) such that

\[
\Phi(\bar{u}, \bar{\theta}) \leq \Phi(u, \theta), \quad \forall (u, \theta) \in (\bar{C} \cap \bar{B}_e(\bar{u})) \times (S \cap \bar{B}_e(\bar{\theta})).
\]

**Proof.** Let \((\bar{u}, \bar{\theta})\) be any limit point of sequence \(\{(u^k, \theta^k)\}\) and \(\{(u^l, \theta^l)\}\) be the subsequence of \(\{(u^k, \theta^k)\}\) such that \((u^l, \theta^l) \to (\bar{u}, \bar{\theta})\). Since for each \(k\), \(\theta^k\) satisfies \((3.10)\), there exists \(\xi^k \in \partial_\theta \Phi(u^k, \theta^k) + N_S(\theta^k)\) such that \(\|\xi^k\| \leq \epsilon_k\). And by the convexity of \(\Phi\) with respect to \(\theta\), for any \(\theta \in S\),

\[
\Phi(u^l, \theta) + \langle \xi^l, \theta - \theta^l \rangle \leq \Phi(u^l, \theta).
\]

Since \(\Phi\) is assumed to be continuous on \(S\), \((u^l, \theta^l) \to (\bar{u}, \bar{\theta})\) and \(\xi^l \to 0\), by taking \(l \to \infty\), we can obtain from the above inequality that

\[
\Phi(\bar{u}, \bar{\theta}) \leq \Phi(u, \theta), \quad \forall \theta \in S.
\]

Next, since \(u^l\) satisfies

\[
u^l \in \arg\min_{u \in C} \Phi(u^l, \theta^l) + \langle \nabla_u \Phi(u^l, \theta^l), u - u^l \rangle,
\]

it follows that, for any \(j \in [p]\) and \(I_u(u^l, j) := \{i \in [n] \mid u^l_{i,j} = 1\}\),

\[
\nabla_{u_{I_u(u^l, j), j}} \Phi(u^l, \theta^l) \leq \nabla_{u_{I_u(\bar{u}, j), j}} \Phi(\bar{u}, \bar{\theta}), \quad \forall i \in [n].
\]

Because \(u^l \in C\), and thus there exists \(\bar{u} \in C\) such that \(u^l = \bar{u}\) for all sufficiently large \(l\). By taking \(l \to \infty\) in \((3.11)\), it follows from the continuity of \(\Phi\), \(\|\theta^{k+1} - \theta^k\| \to 0\) and \(\theta^l \to \bar{\theta}\), that for any \(j \in [p]\) and \(I_u(\bar{u}, j) := \{i \in [n] \mid \bar{u}_{i,j} = 1\}\),

\[
\nabla_{u_{I_u(\bar{u}, j), j}} \Phi(\bar{u}, \bar{\theta}) \leq \nabla_{u_{I_u(\bar{u}, j), j}} \Phi(\bar{u}, \bar{\theta}), \quad \forall i \in [n].
\]

This implies that

\[
\bar{u} \in \arg\min_{u \in C} \Phi(\bar{u}, \bar{\theta}) + \langle \nabla_u \Phi(\bar{u}, \bar{\theta}), u - \bar{u} \rangle.
\]

Then it follows from the assumption that \(\arg\min_{i \in [n]} \nabla_{u_{i,j}} \Phi(\bar{u}, \bar{\theta})\) is a singleton for any \(j \in [p]\) and Lemma \ref{lem:singleton} that there exists \(\epsilon_u > 0\) such that

\[
\Phi(\bar{u}, \bar{\theta}) < \Phi(v, \bar{\theta}), \quad \forall v \in \bar{C} \cap \bar{B}_e(\bar{u}) \setminus \{\bar{u}\}.
\]

We now show that there exists \(\epsilon > 0\) such that for any \(\theta \in S \cap \bar{B}_e(\bar{\theta})\),

\[
\bar{u} \in \arg\min_{u \in \bar{C} \cap \bar{B}_e(\bar{u})} \Phi(u, \theta).
\]

If this is not the case, according to Lemma \ref{lem:continuity}, there must be sequence \(\{(u_\ell, \theta_\ell)\}\) such that \(u_\ell \in (\bar{C} \cap \text{bdy} \, \bar{B}_e(\bar{u})) \cup (\bar{C} \setminus \{\bar{u}\} \cap \bar{B}_e(\bar{u}))\), \(\theta_\ell \in S \cap \theta_\ell \to \bar{\theta}\) and

\[
\Phi(u_\ell, \theta_\ell) < \Phi(\bar{u}, \theta_\ell).
\]

Since \(\{u_\ell\}\) is bounded, let \(\bar{u}\) be a limit point of \(\{u_\ell\}\). We have \(\bar{u} \in (\bar{C} \cap \text{bdy} \, \bar{B}_e(\bar{u})) \cup (\bar{C} \setminus \{\bar{u}\} \cap \bar{B}_e(\bar{u}))\). Taking \(l \to \infty\) in above inequality, the continuity of \(\Phi\) on its domain and \(\theta_\ell \to \bar{\theta}\) yields that

\[
\Phi(\bar{u}, \bar{\theta}) \leq \Phi(\bar{u}, \bar{\theta}),
\]

which contradicts \((3.13)\).

If there exists \((u_0, \theta_0) \in (\bar{C} \cap \bar{B}_e(\bar{u})) \times (S \cap \bar{B}_e(\bar{\theta}))\) such that \(\Phi(u_0, \theta_0) < \Phi(\bar{u}, \bar{\theta})\), since \(\bar{u} \in \arg\min_{u \in \bar{C} \cap \bar{B}_e(\bar{u})} \Phi(u, \theta_0)\), then

\[
\Phi(\bar{u}, \bar{\theta}) \leq \Phi(\bar{u}, \theta_0) \leq \Phi(u_0, \theta_0) < \Phi(\bar{u}, \bar{\theta}),
\]

which arrives at a contradiction and we get the conclusion. \(\Box\)
Because the functions $f_{i,j}(\theta)$ usually vary distinctly for each $i,j$ in the problem [1.5] derived from [1.1], we introduce the following mild assumption.

**Assumption 2.** For any $u \in C$, $\theta \in \mathcal{S}$, $j \in [p]$ and $i_1, i_2 \in [n]$, if $i_1 \neq i_2$, then
\[
\nabla_{u_{i_1,j}} \Phi(u, \theta) \neq \nabla_{u_{i_2,j}} \Phi(u, \theta).
\]

With Assumption 2 fulfilled, the assumption that $\text{argmin}_{i \in [n]} \nabla_{u_{i,j}} \Phi(\bar{u}, \bar{\theta})$ is a singleton for any $j \in [p]$ in Theorem 7 holds. Consequently, we can straightforwardly derive the following result from Theorem 7.

**Corollary 8.** Suppose Assumption 2 is satisfied. Let $\{(u^k, \theta^k)\}$ be the sequence generated by alternating minimization Algorithm 3 that satisfies condition (3.10) with $\varepsilon_k \to 0$. Consider any limit point $(\bar{u}, \bar{\theta})$ of sequence $\{(u^k, \theta^k)\}$. Then $\bar{u} \in C$. Suppose that $\|\theta^{k+1} - \theta^k\| \to 0$, then $(\bar{u}, \bar{\theta})$ is a local minimizer of problem (1.5) in the sense that there exists $\epsilon > 0$ such that
\[
\Phi(\bar{u}, \bar{\theta}) \leq \Phi(u, \theta), \quad \forall (u, \theta) \in \left(\hat{C} \cap \mathbb{B}_\epsilon(\bar{u})\right) \times \left(\mathcal{S} \cap \mathbb{B}_\epsilon(\bar{\theta})\right).
\]

**3.3.3. Analysis on updating $\theta$ with one-step acceleration.** Finding the approximate stationary point during $\theta$ update in Algorithm 3 can be challenging due to the vanishing accuracy $\{\varepsilon_k\}$. We therefore propose a simple acceleration scheme for the $\theta$ update given that $\Phi$ satisfies the following Assumption 3.

**Assumption 3.** $\Phi$ is a continuously differentiable and $\nabla_\theta \Phi$ is $L_\theta$-Lipschitz continuous.

The idea is, during $\theta$ update in Algorithm 3 to approximate the stationary solution to problem
\[
\text{min}_{\theta \in \mathcal{S}} \Phi(u^{k+1}, \theta)
\]
by using only a single projected gradient iteration step, without solving the above subproblem until the accuracy $\{\varepsilon_k\}$ is met as in (3.10). The iterative procedure is outlined in Algorithm 4.

**Algorithm 4** An one-step acceleration version of Algorithm 3

**Input:** $(u^0, \theta^0) \in C \times \mathcal{S}$, $\alpha_0 \in (0, 2/L_\theta)$
**Output:** $(\bar{u}, \bar{\theta})$ that approximately minimizes (1.5).

Set $k = 0$

**While** $\|u^{k+1} - u^k\| + \|\theta^{k+1} - \theta^k\| > tol$

1. Find $u^{k+1} \in C$ by using (3.9) with $u^{k,0} = u^k$ until $u^{k,T+1} = u^{k,T}$. Set $u^{k+1} = u^{k,T}$.
2. Update $\theta^{k+1}$ by a single projected gradient step as
\[
\theta^{k+1} = \text{Proj}_{\mathcal{S}} \left(\theta^k - \alpha_0 \nabla_\theta \Phi(\theta^k, u^{k+1})\right).
\]
Set $k = k + 1$

**Theorem 9.** Suppose Assumption 2 is satisfied and $\Phi$ is bounded below on $\hat{C} \times \mathcal{S}$. Let $\{(u^k, \theta^k)\}$ be the sequence generated by alternating minimization Algorithm 4. Consider any limit point $(\bar{u}, \bar{\theta})$ of sequence $\{(u^k, \theta^k)\}$. Then $\bar{u} \in C$. Suppose that $\text{argmin}_{i \in [n]} \nabla_{u_{i,j}} \Phi(\bar{u}, \bar{\theta})$ is a singleton for any $j \in [p]$, or Assumption 4 is satisfied.
Then \((\bar{u}, \bar{\theta})\) is a local minimizer of problem (1.5) in the sense that there exists \(\epsilon > 0\) such that
\[
\Phi(\bar{u}, \bar{\theta}) \leq \Phi(u, \theta), \quad \forall (u, \theta) \in \left(\hat{C} \cap \mathbb{B}_r(\bar{u})\right) \times \left(\mathcal{S} \cap \mathbb{B}_r(\bar{\theta})\right).
\]

**Proof.** Since \(\Phi\) is convex with respect to \(\theta\) and \(\nabla_\theta \Phi\) is \(L_\theta\)-Lipschitz continuous, by the celebrated sufficient decrease lemma for convex function, we have that
\[
\Phi(u_{k+1}, \theta_{k+1}) \leq \Phi(u_k, \theta_k) - \left(\frac{1}{\alpha_\theta} - \frac{L_\theta}{2}\right) \|\theta_{k+1} - \theta_k\|^2.
\]
Combining with the fact that \(\Phi(u_{k+1}, \theta_{k+1}) \leq \Phi(u_k, \theta_k)\) yields that
\[
\Phi(u_{k+1}, \theta_{k+1}) \leq \Phi(u_k, \theta_k) - \left(\frac{1}{\alpha_\theta} - \frac{L_\theta}{2}\right) \|\theta_{k+1} - \theta_k\|^2.
\]
As \(\Phi\) is assumed to be bounded below on \(\hat{C} \times \mathcal{S}\), we have
\[
\lim_{k \to \infty} \|\theta_{k+1} - \theta_k\| = 0.
\]
Next, since
\[
\nabla_\theta \Phi(\theta_{k+1}, u_{k+1}) - \nabla_\theta \Phi(\theta_k, u_{k+1}) + \frac{1}{\alpha_\theta} (\theta_{k+1} - \theta_k) \in \nabla_\theta \Phi(u_{k+1}, \theta_{k+1}) + \mathcal{N}_\mathcal{S}(\theta_{k+1}),
\]
we have
\[
\text{dist}(0, \nabla_\theta \Phi(u_{k+1}, \theta_{k+1}) + \mathcal{N}_\mathcal{S}(\theta_{k+1})) \to 0
\]
and then the conclusion follows from Theorem 7 and Corollary 8 immediately. \(\square\)

3.3.4. **Interpretation for the theoretical validity of ICTM.** While people are not currently aware of the convergence guarantees for ICTM towards solutions with good qualities, in practice it is usually able to reach a local minimizer. We shall provide an interpretation for the theoretical validity of ICTM from the perspective of \(u\) update in (3.9). Recall that in the ICTM, an important step is to adopt a simple thresholding to update \(u\). The precise scheme is based on finding an approximate solution via solving the minimization of the linearization of \(\Phi(u, \theta_k)\) at \(u_k\),
\[
\min_{u \in \hat{C}} \sum_{i=1}^n \sum_{j=1}^p (u_{i,j} - u_{k,i,j}) \nabla_{u_{i,j}} \Phi(u_k, \theta_k)
\]
which is further equivalent to
\[
\min_{u \in \hat{C}} \sum_{i=1}^n \sum_{j=1}^p u_{i,j} \nabla_{u_{i,j}} \Phi(u_k, \theta_k).
\]
This can be done for each \(j\) independently because of the constraints \(\sum_{i=1}^n u_{i,j} = 1\) and \(u_{i,j} \geq 0\) as defined in \(\hat{C}\). In this regard, the update of \(u_{i,j}\) for a fixed \(j\) in the ICTM reads as
\[
(3.15) \quad u_{i,j} = \begin{cases} 
1 & \text{if } i = \min \{ \arg \min_m \nabla_{u_{i,j}} \Phi(u_k, \theta_k) \}, \\
0 & \text{otherwise}.
\end{cases}
\]
3.4. Summary of the convergence results. To the end of this section, we summarize our convergence results for the three cases discussed above in following:

1. In global minimizer for \( \theta \)-update and global minimizer for \( u \)-update case, alternating minimization scheme generates a local minimizer \((\bar{u}, \bar{\theta})\) of problem (1.5) in the sense that there exists \( \epsilon > 0 \) such that
\[
\Phi(\bar{u}, \bar{\theta}) \leq \Phi(u, \theta), \quad \forall (u, \theta) \in \hat{C} \times (S \cap B_\epsilon(\bar{\theta})).
\]

2. In global minimizer for \( \theta \)-update and local minimizer for \( u \)-update case, alternating minimization scheme generates a local minimizer \((\bar{u}, \bar{\theta})\) of problem (1.5) in the sense that there exists \( \epsilon > 0 \) such that
\[
\Phi(\bar{u}, \bar{\theta}) \leq \Phi(u, \theta), \quad \forall (u, \theta) \in (C \cap B_r(\bar{u})) \times (S \cap B_\epsilon(\bar{\theta})).
\]

3. In approximate stationary solution for \( \theta \)-update and thresholding for \( u \)-update case, alternating minimization scheme generates a local minimizer \((\bar{u}, \bar{\theta})\) of problem (1.5) in the sense that there exists \( \epsilon > 0 \) such that
\[
\Phi(\bar{u}, \bar{\theta}) \leq \Phi(u, \theta), \quad \forall (u, \theta) \in \left(\hat{C} \cap B_r(\bar{u})\right) \times \left(S \cap B_\epsilon(\bar{\theta})\right).
\]

4. Numerical experiments

In this section, we illustrate the performance of the algorithms for different examples via several numerical examples. We implemented the algorithms in MATLAB. All reported results were obtained on a laptop with a 3.8GHz Intel Core i7 processor and 8GB of RAM. We apply our methods to different models and numerical results show a clear advantage of the method in terms of simplicity and efficiency.

4.1. Application to the implicit surface reconstruction problem. In the first example, we consider the following implicit surface reconstruction model from a point cloud,

\[
\begin{align*}
\min_{u(x) \in \{0, 1\}} E_{ISC}(u) := & \sqrt{\pi} \int_{\mathbb{R}^n} |d| \bar{d} u G_{\tau} * \left( |d| \bar{d} (1 - u) \right) \, dx. \\
\end{align*}
\]

(4.1)

In this model, \( d(x) \) is the distance function from arbitrary points in the space to the point cloud, \( u(x) \) is the decision variable which is an indicator function to implicitly determine the inside and outside of the reconstructed surface. Because the point cloud is fixed for the reconstruction, \( d(x) \) is then fixed and the only decision variable in this case is \( u \).

After discretization, the problem reads,

\[
\begin{align*}
\min_{u \in \{0, 1\}^p} \Psi(u) = & [\tilde{d} \circ (1 - u)] K_{\tau}(\tilde{d} \circ u)^T \\
\end{align*}
\]

(4.2)

where \( u = (u_1, u_2, \cdots, u_p) \) is the indicator function at \( p \) discrete points in the computational domain, \( K_{\tau} \) is a symmetric positive definite matrix comes from the discretization and rearrangement of scaled Gaussian kernel, \( \circ \) represents the Hadamard product, and \( \tilde{d} \) is the distance from discrete points in the computational domain to the point cloud (see Figure 1). We refer more details on the model and related work to [44].
Figure 1. A diagram to several level sets of the distance function $\tilde{d}$ to a discrete point cloud. See Section 4.1.

It is clear that the problem (4.2) is concave with respect to $u$, we simple apply the thresholding step as that in (3.9) to general 2- and 3-dimensional point clouds as displayed in Figure 2:

$$u_{i}^{k+1} = \begin{cases} 1 & \text{if } (K_{\tau}(\tilde{d} \circ (1 - 2u))^{T})_{i} < 0 \\ 0 & \text{otherwise} \end{cases}.$$  

(4.3)

All converge in fewer than 100 iteration steps to the minimizer, which implies the efficiency of the algorithm and is consistent with our analysis.

Figure 2. Results obtained from (4.3) for different point clouds. See Section 4.1.

Furthermore, we use several simple examples to compare the efficiency between the method and level set approaches. A recent work in [20] has carefully studied the efficiency comparison between the semi-implicit method (SIM) and the classic level set approach in [56] and showed a big improvement in the efficiency. Therefore, we simply compare the efficiency between our method and SIM in [20].
We consider different point clouds (different \( m \)) generated using \( N = 200 \) uniform points \( \theta_i \) in \([0, 2\pi]\):

\[
\begin{align*}
  x_i &= r_i \cos(\theta_i), \\
  y_i &= r_i \sin(\theta_i),
\end{align*}
\]

where \( r_i = 1 + 0.4 \sin(m\theta_i) \) with different \( m = 3, 4, 5, 6, 7, \) and 8. We use the same initial guess as shown in Figure 3 and same discretization (128 \( \times \) 128 grids) of the computational domain for two methods. The table at the bottom of Figure 3 shows the dramatical acceleration in the computational CPU time and figures indicate the improvement on the accuracy.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{SIM in [20]} & 3.2 s & 4.47 s & 5.26 s & 6.40 s & 7.18 s & 4.56 s \\
\hline
\text{Our method} & 0.029 s & 0.036 s & 0.029 s & 0.026 s & 0.031 s & 0.030 s \\
\hline
\end{array}
\]

**Figure 3.** Comparisons between our method and SIM in [20]. Blue curve: results obtained from our method. Black curve: results obtained from SIM in [20]. Green curve: initial guess. Red points: point cloud with 200 points. Table: CPU times for the computations. See Section 4.1.

### 4.2. Application to the Chan–Vese model for image segmentation

In this experiment, we consider the well-known Chan–Vese (CV) model [8] for image segmentation approximated using characteristic functions. Specifically, in the CV model, the objective functional for the \( n \)-segment case is

\[
E_{CV}(\Omega_1, \ldots, \Omega_n, \theta_1, \ldots, \theta_n) = \lambda \sum_{i=1}^{n} |\partial \Omega_i| + \sum_{i=1}^{n} \int_{\Omega_i} |\theta_i - I|^2 \, dx
\]

(4.4)

where \( \partial \Omega_i \) is the boundary of the \( i \)-th segment \( \Omega_i \), \( |\partial \Omega_i| \) denotes the perimeter of the domain \( \Omega_i \), \( I : \Omega \to [0, 1]^d \) is the image, \( d \) is the channel of the image (e.g., \( d = 1 \): gray image, \( d = 3 \): RGB image), and \( \lambda \) is a positive parameter.
Using the approximation to $|\partial \Omega_i|$ by using characteristic functions, $u_i(x)$, of each $\Omega_i$, the approximate objective functional is written into

\[
E_{CV}^\tau = \lambda \sqrt{\frac{\pi}{\tau}} \sum_{i=1}^{n} \int_{\Omega} u_i G_\tau * (1 - u_i) \, dx + \sum_{i=1}^{n} \int_{\Omega} u_i |\theta_i - I|^2 \, dx.
\]

(4.5)

After discretization using the information on each pixel, the discretized problem is then written into

\[
\min_{u \in C, \theta \in \mathbb{R}} \Psi(u, \theta) = \sum_{i=1}^{n} \sum_{j=1}^{p} u_{i,j} |\theta_i - I_j|^2 + \lambda \sum_{i=1}^{n} (1 - u_i) K_\tau u_i^T
\]

(4.6)

where $u_i = (u_{i,1}, u_{i,2}, \cdots, u_{i,p})$ here is the $i$-th row in $u$, $K_\tau$ is a symmetric positive definite matrix comes from the discretization and rearrangement of scaled Gaussian kernel, $p$ here is the number of pixels of the image, and $\theta_i$ ($i = 1, 2, \cdots, n$) is the state variable for each segment. (More details can be referred to [46].)

Because $\Psi(u, \theta)$ is strictly convex with respect to $\theta_i$, subproblem (2.2) can be explicitly given by solving

\[
\frac{\partial \Psi(u^k, \theta)}{\partial \theta_i} = 0
\]

which is

\[
\theta_{i+1} = \frac{\sum_{j=1}^{p} u_{i,j}^k I_j}{\sum_{j=1}^{p} u_{i,j}^k}.
\]

We then apply Algorithm 3 to the above minimization problem and the subproblem (2.3) for $u_{i,j}$ for each $j = 1, 2, \cdots, p$ is iteratively solved by (3.15):

\[
u_{i,j}^{k+1} = \begin{cases} 
1 & \text{if } i = \min \left\{ \arg \min_m \left| \theta_m^k - I_{m,j} \right|^2 + \lambda \left( K_\tau (1 - 2u_{m,j}^k)^T \right)_j \right\}, \\
0 & \text{otherwise.}
\end{cases}
\]

(4.7)

In the practical implementation, the matrix multiplication $K_\tau (1 - 2u_{m,j}^k)^T$ is computed effectively by fast Fourier transform (FFT). In the follows, we apply the above algorithm into the segmentation on both gray images and colorful images. In Figure 4, we list the initial guesses ($u^0$) for different images with different number of segments in the left, the final converged solution in the middle, and the decaying curve of the objective function value with respect to the iteration steps. $\lambda$ are set to be 0.03 for the gray image and 0.1 for the RGB image.

The initial guesses are the characteristic functions of the squares bounded by blue lines. For example, in the gray image, the initial guess $u^0 = (u_{1}^0; u_{2}^0)$ is given as below,

\[
u_{1,j}^0 = \begin{cases} 
1 & \text{if } j\text{-th pixel is inside the region bounded by the blue curve,} \\
0 & \text{Otherwise,}
\end{cases}
\]

and $u_{2,j} = 1 - u_{1,j}$.

One can observe that the algorithm converges to an optimal solution in just 4 steps, implying the efficiency of the algorithm.
4.3. Application to the Local Intensity Fitting (LIF) model for image segmentation. In this section, we apply Algorithm 3 to the LIF model \[26\] for the two-phase case with the objective functional

\[
E_{LIF}(\Omega_1, \Omega_2, \theta_1, \theta_2) = \lambda \sum_{i=1}^2 |\partial \Omega_i| + \mu \sum_{i=1}^2 \int_{\Omega_i} \int_{\partial \Omega_i} G_\sigma(x - y) \theta_i(x) - I(y)^2 \, dy \, dx.
\]

where \(\mu_i\) are given constants.

Using the characteristic representation, approximation similar to that in (4.5), and discretization on pixels, one arrives at the following problem

\[
\min_{u \in C, \theta_i \in \mathbb{R}^p} \Psi(u, \theta) = \sum_{i=1}^2 \lambda (1 - u_i) K_\tau u_i^T + \mu \left( u_i K_\sigma (\theta_i^2 + I^2)^T - 2(u_i \circ I) K_\sigma \theta_i^2 \right)
\]

where \(\theta_i \in \mathbb{R}^p\) is a vector for \(\theta_i(x)\) on discrete pixels, \(K_\tau\) and \(K_\sigma\) are two \(p \times p\) matrices whose \((i, j)\)-th entries are \(G_\tau(x_i - x_j)\) and \(G_\sigma(x_i - x_j)\), \(\theta_i^2\) and \(I^2\) are square of values at each entry, and \(\circ\) denotes the Hadamard product.

In this case, using the fact that \(K_\sigma\) is a symmetric positive definite matrix and

\[
(u_i \circ I) K_\sigma \theta_i^T = \theta_i K_\sigma (u_i \circ I)^T \quad \text{and} \quad u_i K_\sigma (\theta_i^2)^T = \theta_i^2 K_\sigma u_i^T.
\]

subproblem (2.2) can also be explicitly solved by

\[
\theta_{i,j}^{k+1} = \frac{(K_\sigma (u_i \circ I)^T)_j}{(K_\sigma u_i^T)_j} + \varepsilon,
\]

Here, the subscript \(j\) denotes the \(j\)-th entry in matrix vector multiplication. To make the calculation of \(\theta_{i,j}\) to be stable, one may use a regularized formula

\[
\theta_{i,j}^{k+1} = \frac{(K_\sigma (u_i \circ I)^T)_j + \varepsilon}{(K_\sigma u_i^T)_j + \varepsilon}.
\]
One can apply Algorithm 3 to problem (4.9) and the subproblem (2.3) for $u_{i,j}$ for each $j = 1, 2, \ldots, p$ at each iteration can be explicitly solved by iterating the follows:

\begin{equation}
    u_{i,j}^{k+1} = \begin{cases} 
        1 & \text{if } i = \min \left\{ \arg \min_m (\psi_m^k) \right\}, \\
        0 & \text{otherwise}
    \end{cases}
\end{equation}

where

\begin{equation*}
(\psi_m^k)_j = \mu \left[ K_\sigma \left( (\theta_m^{k+1})^2 + I^2 \right)^T - 2I^T \circ K_\sigma (\theta_m^{k+1})^T \right]_j + \lambda \left( K_\tau (1 - 2u_m^k)^T \right)_j.
\end{equation*}

We apply the algorithm into several intensity inhomogeneous images as listed in Figure 5. In the first row, we use blue curve to implicitly denote the initial guesses. The segment results are shown in the second row with red curves. In both rows, we set $u_{1,j} = 1$ at the pixel $j$ if it is located inside the curves and $u_{2,j} = 1$ at the pixel $j$ if it is located outside the curves. The third row lists the corresponding objective function value decaying curves for the iterate starting from the initial guess. Again, one can observe that the algorithm can find the solution in very few steps. In five images, the parameters $(\tau, \lambda, \mu, \sigma)$ are set to be $(5, 1, 150, 3)$, $(3, 1, 245, 3)$, $(10, 1, 110, 3)$, $(2, 1, 90, 3)$, $(3, 1, 80, 3)$, respectively.

In the table of Figure 5, we compare Algorithm 3 and the level-set method used by Li et al. [26] in terms of the number of iterations for convergence and the CPU time. One can easily observe that Algorithm 3 converges in significantly fewer iterations and a shorter CPU time, demonstrating its higher efficiency. The results obtained by Li et al. [26] are similar to those in Figure 5. More details and comparison can be referred to [47].

5. Conclusion and discussions

In this paper, we gave a rigorous study on the convergence of the ICTM for solving a general problem raised from the discretization of many interface related optimization problems, including image segmentation, topology optimization, surface reconstruction, and so on. We studied three different situations where different solutions of minimization can be obtained. To our knowledge, this is the first systematical study on the convergence of the ICTM, which verifies the empirical observation from numerical experiments.

Similar analysis could be extended to the iterative method for Dirichlet partition problems [45], where a concave objective functional is minimized on a $L^2$ circle. The algorithmic framework can also be applied into data classification problems when one considers corresponding operators on point clouds. In addition, because of the small number of iterations for the minimization, it is straightforward to combine the algorithm with deep neural networks via the deep unfolding/unrolling ideas to have a mechanism and data driven methods for problems in image processing, surface reconstruction and topology optimization.

Acknowledgments

Dr. Wang acknowledges support from National Natural Science Foundation of China grant (Grant No. 12101524), Guangdong Basic and Applied Basic Research Foundation (Grant No. 2023A1515012199) and Shenzhen Science and Technology Innovation Program (Grant No. JCYJ20220530143803007, RCYX20221008092843046).
Figure 5. First row: the initial guesses for the iteration. Second row: the converged results. Third row: curves for the objective function values with respect to iteration step. In five images, the parameters \((\tau, \lambda, \mu, \sigma)\) are set to be (5, 1, 150, 3), (3, 1, 245, 3), (10, 1, 110, 3), (2, 1, 90, 3), (3, 1, 80, 3). Table: comparison of the number of iterations and CPU time between Algorithm 3 and the level-set method. See Section 4.3.

| # of iterations | Algorithm 3 | 6 | 12 | 19 | 17 | 5 |
|------------------|-------------|---|----|----|----|---|
| Method in \([26]\) | 256         | 131| 117| 209| 29 |

| CPU time (second) | Algorithm 3 | 0.08 | 0.07 | 0.15 | 0.14 | 0.06 |
|-------------------|-------------|------|------|------|------|------|
| Method in \([26]\) | 0.59        | 0.43 | 0.48 | 1.03 | 0.15 |

Dr. Zhang acknowledges support from NSFC 12222106, Shenzhen Science and Technology Program (No. RCYX20200714114700072) and Guangdong Basic and Applied Basic Research Foundation 2022B1515020082.

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