Celestial Current Algebra from Low’s Subleading Soft Theorem

Elizabeth Himwich and Andrew Strominger

Center for the Fundamental Laws of Nature, Harvard University, Cambridge, MA USA

Abstract

The leading soft photon theorem implies that four-dimensional scattering amplitudes are controlled by a two-dimensional (2D) $U(1)$ Kac-Moody symmetry that acts on the celestial sphere at null infinity ($\mathcal{I}$). This celestial $U(1)$ current is realized by components of the electromagnetic vector potential on the boundaries of $\mathcal{I}$. Here, we develop a parallel story for Low’s subleading soft photon theorem. It gives rise to a second celestial current, which is realized by vector potential components that are subleading in the large radius expansion about the boundaries of $\mathcal{I}$. The subleading soft photon theorem is reexpressed as a celestial Ward identity for this second current, which involves novel shifts by one unit in the conformal dimension of charged operators.
1 Introduction

In any four-dimensional (4D) theory with photons, the soft photon theorem implies the existence of a two-dimensional (2D) $U(1)$ Kac-Moody symmetry. The consequences of the symmetry become most transparent when 4D scattering amplitudes are reexpressed as correlation functions on the celestial sphere at null infinity ($I^+$), on which the 4D Lorentz group acts as the 2D Euclidean conformal group. The Kac-Moody currents act on this celestial sphere and are sourced by electromagnetic charge currents that cross it. All amplitudes are thereby highly constrained, and in particular are set to zero by infrared divergences if the associated conservation laws are violated. The celestial Kac-Moody current may be explicitly realized by a sum of the gauge potentials on the $S^2$ boundaries of $I^+$, denoted $A_z(0)$ below. This story is reviewed in [7].

In the 1950s Low and others [8, 9, 10, 11, 12] established a second, universal, relation governing the subleading term in the soft expansion of an asymptotic photon. A similar story is expected to derive from this universal relation, but so far is only partially understood [13, 14, 15].

In this paper we show that the subleading soft theorem implies a second current algebra on the celestial sphere. The currents are the constructed from the boundary values of the subleading term of the gauge potential, denoted $A_z(1)$, in the large radius expansion around $I^+$. Naively, $A_z(1)$ is determined from the leading potential $A_z(0)$ by the equations of motion and is not an independent field. However, in attempting to explicitly solve for $A_z(1)$ in terms of $A_z(0)$, one encounters an integration function on the sphere. This implies that the boundary values of $A_z(1)$ are independent fields after all, and in fact turn out to comprise an independent ‘subleading’ current algebra.

The current algebra generated on the celestial sphere by boundary values of $A_z(1)$ has interesting and unconventional features. The OPE of the subleading current with a charged operator with 2D conformal weights $(h, \bar{h})$ shifts the weights to $(h - \frac{1}{2}, \bar{h} - \frac{1}{2})$. This is possible because such operators lie in the continuous unitary principal series. Our main result is formula (25) below which describes this action. It will be interesting to eventually understand the constraints of (25) on scattering amplitudes.

In this note we make the simplifying restrictions that there are no longe range magnetic fields near spatial infinity and also that charge is carried by massless scalar fields. We expect our results
to hold in a more general context, as their form is largely dictated by symmetries.

This note is organized as follows. In Section 2, we introduce our conventions and present basic formulas. In Section 3, we rewrite the conservation law as a relation between the subleading soft currents and the hard charged currents. In Section 4, we take the quantum matrix element of this conservation law and express it as a Ward identity for a novel 2D current algebra on the celestial sphere. Appendix A gives some details of the asymptotic expansion about $I$ in Lorenz gauge.

## 2 Maxwell Equations in Lorenz Gauge

We largely employ the retarded (advanced) coordinates on flat Minkowski space

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} = -dv^2 + 2dvd\bar{r} + 2r^2\gamma_{z\bar{z}}dzd\bar{z},$$

with $u (v)$ retarded (advanced) time and $\gamma_{z\bar{z}} = 2/(1 + z\bar{z})$ the unit round metric on $S^2$. These are related to the Cartesian coordinates $(x^0, x^1, x^2, x^3)$ by

$$x^0 = u + r = v - r$$
$$x^1 + ix^2 = \frac{2rz}{1 + z\bar{z}}$$
$$x^3 = \frac{r(1 - z\bar{z})}{1 + z\bar{z}}.$$  

In this paper we use the Lorenz gauge condition $\nabla_\mu A^\mu = 0$. The Maxwell equations $\nabla_\mu F^{\mu\nu} = e^2 j^\nu$ in this gauge in retarded coordinates are

$$-2r\partial_\nu(\partial_\mu A_\mu) + \partial_\nu(\partial_\mu A_\mu) + 2\gamma_{z\bar{z}}\partial_z\partial_{\bar{z}}A_u = e^2 r^2 j_u$$

$$-2\partial_\nu(\partial_\mu A_\mu) + 2\partial_\nu\partial_\mu(\partial_\mu A_\mu) + 2\gamma_{z\bar{z}}\partial_z\partial_{\bar{z}}A_r = e^2 r^2 j_r$$

$$-2r^2\partial_\mu\partial_\nu A_z + r^2\partial_\mu^2 A_z + 2r\partial_\nu(A_r - A_u) + 2\partial_z(\gamma_{z\bar{z}}\partial_{\bar{z}}A_z) = e^2 r^2 j_z.$$ 

See Appendix A for further details.

## 3 Conservation Law on $I^+$

Low’s subleading soft photon theorem was recently shown to be the quantum matrix element of a charge conservation law [13]. The conserved charge on $I^+$ is:

$$Q^+ = -\frac{2}{e^2} \int_{I^+} d^2zdu \partial_\mu A^{(0)}_\mu D^z Y^z - \int_{I^+} d^2zdu \left( uD_z Y_z j_u^{(2)} + Y_z j_z^{(2)} \right)$$

$$= \frac{1}{e^2} \int_{I^+} d^2zY_z \left( uD_z F_{ur}^{(2)} - 2F_{zr}^{(2)} \right),$$

where the charge is parameterized by a real vector field $Y^z$ on $I^+$, the past boundary of $I^+$. The fields in this expression are the functions of $(u, z, \bar{z})$ that appear as coefficients in the asymptotic

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1 The matter currents defined in [13] differ by those in [13] by a sign.
The order $\frac{1}{r}$ expansion about $\mathcal{I}^+$. The order $\frac{1}{r}$ at which they appear in this expansion is denoted by the superscript $(n)$. For simplicity, we restrict here to the case where there are no long range magnetic fields near spatial infinity so that $A_\pm^{(0)}$ is pure gauge and $F_{\pm 2}^{(0)} = 0$ at $\mathcal{I}^\pm$. A similar charge $Q^-$ is defined on $\mathcal{I}^-$. The conservation law is $Q^+ = Q^-.$

For the leading soft charge, the analog of the first ‘soft’ term in $Q^+$ is a total $u$-derivative and reduces to a difference between two terms on the boundaries of $\mathcal{I}^+$, signalling the central role of $\mathcal{I}$ boundary dynamics. In contrast, this total derivative structure is not manifest in the soft term in (4). However, we now show that this structure reappears when $Q^+$ is reexpressed in terms of the subleading component $A_\pm^{(1)}$ of the gauge field, which enables one to rewrite it in terms of hard currents and the $\mathcal{I}^\pm$ boundary values of $A_\pm^{(1)}$. The elimination of $A_\pm^{(0)}$ from (4) in favor of $A_\pm^{(1)}$ proceeds via the asymptotic expansion of the Maxwell equations (see Appendix A for details)

$$2\partial_u A_\pm^{(1)} + e^2\partial_u j_\mu^{(2)} + 2\partial_u D^\mu D_\mu A_\pm^{(0)} = e^2\partial_u j_\mu^{(2)}.$$ (5)

Inverting the action of $D_\pm$ on $A_\pm^{(0)}$, integrating over $u$ and assuming the hard currents vanish at the boundaries $\mathcal{I}^\pm$ gives

$$Q^+ = -\int d^2 z Y_z \left[ \int du \left( j_\mu^{(2)} + \frac{d^2 w}{2\pi (z - w)^2} \right) - 2 \int du u D_z j_\mu^{(2)} \right]$$

$$- \frac{2}{e^2} \int d^2 z Y_z D_z \int \frac{d^2 w}{2\pi} \frac{1}{z - w} \left( 1 - u \partial_u A_\mu^{(1)} \right) \bigg|_{\mathcal{I}^\pm}.$$ (6)

Lorenz gauge $\nabla^\mu A_\mu = 0$ leaves unfixed residual gauge transformations of the form $A_\mu \to A_\mu + \partial_\mu \epsilon$ with $\Box \epsilon = 0$. The solution to this equation in retarded coordinates requires two pieces of free data, at different orders in the asymptotic expansion: the free function $\epsilon^{(0)}(z, \bar{z})$, which is related to the leading soft theorem, and the free function $\epsilon^{(1)}(u, z, \bar{z})$, which is independent free data. This latter residual freedom enables us to fix the subsidiary gauge condition

$$A_\mu^{(1)} = 0,$$ (7)

which implies that $\partial_u \epsilon^{(1)} = 0$. We are left with a free function $\epsilon^{(1)}(z, \bar{z})$. The gauge transformations are parametrized as

$$\epsilon = \epsilon^{(0)}(z, \bar{z}) + \frac{u}{2} D^2 \epsilon^{(0)}(z, \bar{z}) \frac{\log r}{r} + \frac{\epsilon^{(1)}(z, \bar{z})}{r} + ...$$ (8)

At early and late times along future null infinity, where the matter current is zero, the field configurations return to pure gauge. Hence the asymptotic behavior near $\mathcal{I}^\pm$ is

$$A_\pm^{(0)} = D_z \varepsilon_\pm^{(0)}(z, \bar{z}), \quad A_\pm^{(1)} = \frac{u}{2} D_z D^2 \varepsilon_\pm^{(0)}(z, \bar{z}), \quad A_\pm^{(1)} = D_z \varphi_\pm^{(1)}(z, \bar{z}),$$ (9)

where the tilde denotes a log $r$ dependence (see Appendix A for details) and where the boundary fields $\varphi$ shift under gauge transformations as $\varphi_\pm^{(0)} \to \varphi_\pm^{(0)} + \epsilon^{(0)}$ and $\varphi_\pm^{(1)} \to \varphi_\pm^{(1)} + \epsilon^{(1)}$. The difference
in their values at \( I^+_z \) and \( I^-_z \) is determined by the hard charges and cannot be gauge-fixed to zero. To underscore this, we rewrite (6) as

\[
Q^+ = -\int d^2 z D_z Y \left[ \int du \int \frac{d^2 w}{2\pi} \left( \frac{j_w^{(2)}}{z - w} - \frac{j_{\bar{w}}^{(2)}}{\bar{z} - w} \right) + 2 \int dv \ u j_u^{(2)} - \frac{2}{e^2} \varphi^{(1)} \right] \bigg|_{I^+_z}.
\]

(10)

Similarly, on \( I^-_z \),

\[
Q^- = -\int d^2 z D_z Y \left[ -\int dv \int \frac{d^2 w}{2\pi} \left( \frac{j_w^{(2)}}{z - w} - \frac{j_{\bar{w}}^{(2)}}{\bar{z} - w} \right) - 2 \int dv \ v j_v^{(2)} - \frac{2}{e^2} \varphi^{(1)} \right] \bigg|_{I^-_z}.
\]

(11)

To write the conservation law as a shift, we need to look at both \( I^+_z \) and \( I^-_z \). Charge conservation is the identity \( Q^+ = Q^- \). Setting \( Y = \frac{1}{2\pi(z - \bar{z})} \) so that \( D_z Y \) is a delta function, and using the antipodal matching

\[
\varphi^{(1)} \bigg|_{I^+_z} = \varphi^{(1)} \bigg|_{I^-_z},
\]

the conservation law is

\[
\varphi^{(1)} \bigg|_{I^+_z} - 2\varphi^{(1)} \bigg|_{I^-_z} + \varphi^{(1)} \bigg|_{I^-_z} = e^2 \int du \int \frac{d^2 w}{4\pi} \left( \frac{j_w^{(2)}}{z - w} - \frac{j_{\bar{w}}^{(2)}}{\bar{z} - \bar{w}} \right) + e^2 \int dv \int \frac{d^2 w}{4\pi} \left( \frac{j_w^{(2)}}{z - w} - \frac{j_{\bar{w}}^{(2)}}{\bar{z} - \bar{w}} \right) + 2e^2 \left( \int du \ u j_u^{(2)} + \int dv \ v j_v^{(2)} \right).
\]

(13)

Defining the subleading soft photon current

\[
J_z^{(1)} = \frac{2}{e^2} \left( D_z \varphi^{(1)} \bigg|_{I^+_z} - 2D_z \varphi^{(1)} \bigg|_{I^-_z} + D_z \varphi^{(1)} \bigg|_{I^-_z} \right),
\]

(14)

\( J_z^{(1)} \) can finally be written

\[
J_z^{(1)} = -\left( \int du + \int dv \right) \left[ \int \frac{d^2 w}{2\pi} \frac{j_w^{(2)}}{(z - w)^2} + j_z^{(2)} \right] + 2 \left( \int du \ u D_z j_u^{(2)} + \int dv \ v D_z j_v^{(2)} \right).
\]

(15)

We see that, if there is any charge flux \( j_z \) and \( j_u \), it is impossible to set \( A_z^{(1)} = D_z \varphi^{(1)} \) to zero on all boundaries of \( I \), just as the leading soft theorem makes it impossible to set \( A_z^{(0)} = D_z \varphi^{(0)} \) to zero on all boundaries. Hence \( \varepsilon^{(1)} \), as well as \( \varepsilon^{(0)} \), is a large gauge transformation, and maps one vacuum to a physically inequivalent one. The Goldstone boson (in the chosen gauge) is simply \( A_z^{(1)} \), which transforms inhomogeneously under the large gauge transformations generated by \( \varepsilon^{(1)} \), just as \( A_z^{(0)} \) is the Goldstone boson for the leading large gauge transformations generated by \( \varepsilon^{(0)} \).

### 4 Celestial Current Ward Identity

In a quantum setting, the conservation law (15) becomes operator identities whose S-matrix elements are equivalent to the subleading soft photon theorem. We denote this

\[
\langle z_n, \ldots | Q^+ S - SQ^- | z_1, \ldots \rangle = 0,
\]

(16)

\( ^2 \text{This current can be shown from the Lie action to have 2D conformal weight } (h, \bar{h}) = (\frac{1}{4}, -\frac{1}{4}). \)
where we consider a state \(|z_1, \ldots\rangle\), with \(n\) massless hard scalar particles of energies \(\omega_k\), charges \(\epsilon q_k\) and momenta \(p_k^\mu\). The total charge current for massless scalars \(\Phi_k\) with charge \(Q_k\) is

\[
 j_\mu = i \sum_k Q_k(\bar{\Phi}_k \partial_\mu \Phi_k - \Phi_k \partial_\mu \bar{\Phi}_k). \tag{17}
\]

The canonical commutator of the current component \(j_w^{(2)}\) with the leading term in \(\Phi_k\) at \(I^+\) is

\[
\left[ j_w^{(2)}(u, w, \bar{w}), \Phi_k^{(1)}(u', z_k) \right] = \frac{Q_k}{2} \Theta(u - u') \delta^2(w - z_k) Dw^{(1)}(u, w, \bar{w}). \tag{18}
\]

where \(Dw \equiv \gamma^{w\bar{w}} \partial_{w\bar{w}}\). In terms of the Fourier transform

\[
\Phi_{k\omega}^{(1)} = \int du e^{i\omega u} \Phi_{k\omega}^{(1)}(u), \tag{19}
\]

the action of the hard currents in the charge \(I^0\) becomes

\[
\left[ Q_H^+, \Phi_{k\omega}^{(1)}(z_k, \bar{z}_k) \right] = i Q_k e^2 \frac{D\omega^2 \Phi_{k\omega}^{(1)}(z_k, \bar{z}_k)}{2\pi \omega_k (z - z_k)^2}. \tag{20}
\]

It is illuminating to rewrite the scattering amplitudes as correlation functions on the celestial sphere, adopting the compact notation \(\mathcal{I}\) and expanding

\[
\langle z_{n+1}, \ldots |S| z_1, \ldots \rangle \rightarrow \langle \mathcal{O}_{\omega_1}^{(1)}(z_1, \bar{z}_1) \cdots \mathcal{O}_{\omega_n}^{(1)}(z_n, \bar{z}_n) \rangle. \tag{21}
\]

The subleading soft theorem then becomes

\[
\langle J_\omega^{(1)} \mathcal{O}_{\omega_1}^{(1)}(z_1, \bar{z}_1) \cdots \mathcal{O}_{\omega_n}^{(1)}(z_n, \bar{z}_n) \rangle = \sum_{k=1}^n \frac{i Q_k}{2\pi \omega_k (z - z_k)^2} D\omega^2 \langle \mathcal{O}_{\omega_1}^{(1)}(z_1, \bar{z}_1) \cdots \mathcal{O}_{\omega_n}^{(1)}(z_n, \bar{z}_n) \rangle. \tag{22}
\]

The Mellin transform to a conformal basis for particles with helicity \(s\) with conformal weights

\[(h, \bar{h}) = \frac{1}{2}(\Delta + s, \Delta - s) = \frac{1}{2}(-E \partial_E + s, -E \partial_E - s)\]

is simply

\[
\mathcal{O}_{(h, \bar{h})}(z, \bar{z}) = \int d\omega \omega^{\Delta - 1} e^{\pm ip\cdot x - \epsilon \omega} \mathcal{O}_\omega^{(1)}(z, \bar{z}). \tag{24}
\]

In this conformal basis, \(22\) becomes the current algebra relation

\[
\langle J_\omega^{(1)} \mathcal{O}_{(h_1, \bar{h}_1)} \cdots \mathcal{O}_{(h_n, \bar{h}_n)} \rangle = i \sum_k Q_k \frac{D\omega^2 \langle \mathcal{O}_{(h_1, \bar{h}_1)} \cdots \mathcal{O}_{(h_k - \frac{1}{2}, \bar{h}_k - \frac{1}{2})} \cdots \mathcal{O}_{(h_n, \bar{h}_n)} \rangle}{(z - z_k)^2}. \tag{25}
\]

This is the celestial representation of the subleading soft theorem.

\[\text{4Up to irrelevant contact terms which will vanish after contour integration.}\]

\[\text{4Using } -i\omega \Phi_{k\omega}^{(2)} = \gamma^{w\bar{w}} D_w \Phi_{k\omega}^{(1)}, \text{ the right hand side can be rewritten as } \frac{Q_k}{2\pi \omega_k} \mathcal{O}_{\omega_1}^{(2)}(z_k, \bar{z}_k). \text{ This suggests a connection with the identification in } 14 \text{ of subleading soft symmetries with gauge transformations that diverge linearly with } r. \text{ It would be interesting to understand this better.}\]
The operators $O$ which create spacetime particles in a conformal basis appearing in celestial amplitudes are in different types of representations - typically the continuous unitary principal series - than those we are accustomed to in standard 2D CFT. The corresponding amplitudes take a rather different form often involving delta functions on the sphere \[16, 17, 18, 19\], which makes possible relations between amplitudes with shifted conformal weights. Relations of this general type were noted in the gravitational context in \[20\] and verified by Stieberger and Taylor \[21\] in some special cases. It would be of interest to examine (25) in explicit examples.

Finally, we note that integrating around a contour $C$ weighted by a holomorphic function $\varepsilon(z)$, the subleading soft theorem takes the alternate form

$$\left\langle \oint_C \frac{dz}{2\pi i} \varepsilon(z) J_z^{(1)} O_{(h_1, \bar{h}_1)} \cdots O_{(h_n, \bar{h}_n)} \right\rangle = i \sum_{k \in C} Q_k \langle O_{(h_1, \bar{h}_1)} \cdots \partial_z \varepsilon(z_k) D^{z_k} O_{(h_k - \frac{1}{2}, \bar{h}_k - \frac{1}{2})} \cdots O_{(h_n, \bar{h}_n)} \rangle,$$

where the sum is restricted to operators inside the contour.

**Acknowledgements**

This work was funded partially by DOE grant [de-sc0007870]. E.H. is funded by NSF grant 1745303. We are grateful to Laura Donnay, Slava Lysov, and Dan Kapec for useful correspondence, and to Monica Pate, Ana Raclariu, and Sabrina Pasterski for discussion and advice throughout this work.

**A Asymptotic Expansion**

This appendix gives a few details of the large $r$ expansion about $\mathcal{I}^+$. A massless scalar field has $\frac{1}{r}$ expansion near $\mathcal{I}^+$ as

$$\Phi(u, r, z) = \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(u, z, \bar{z})}{r^n}. \quad (27)$$

The matter currents

$$j_\mu = iQ (\bar{\Phi} \partial_\mu \Phi - \Phi \partial_\mu \bar{\Phi}) \quad (28)$$

fall off as

$$j_u \sim \mathcal{O}\left(\frac{1}{r^2}\right), \quad j_z, j_{\bar{z}} \sim \mathcal{O}\left(\frac{1}{r^2}\right), \quad j_r \sim \mathcal{O}\left(\frac{1}{r^3}\right). \quad (29)$$

Finite energy flux and charge suggest the falloffs

$$A_u \sim \mathcal{O}\left(\frac{1}{r}\right), \quad A_z, A_{\bar{z}} \sim \mathcal{O}(1), \quad A_r \sim \mathcal{O}\left(\frac{1}{r^2}\right). \quad (30)$$
In order to consistently solve the Maxwell equations in $\nabla^\mu A_\mu = 0$ gauge we must allow logarithmic falloffs in the gauge fields. This gives the expansion

$$A_u = \sum_{n=2}^\infty \frac{A_u^{(n)}}{r^n} + \sum_{m=1}^\infty \frac{\tilde{A}_u^{(m)}}{r^m} \log r$$

$$A_r = \sum_{n=2}^\infty \frac{A_r^{(n)}}{r^n} + \sum_{m=2}^\infty \frac{\tilde{A}_r^{(m)}}{r^m} \log r$$

$$A_z = \sum_{n=0}^\infty \frac{A_z^{(n)}}{r^n} + \sum_{m=1}^\infty \frac{\tilde{A}_z^{(m)}}{r^m} \log r$$

$$A_{\bar{z}} = \sum_{n=0}^\infty \frac{A_{\bar{z}}^{(n)}}{r^n} + \sum_{m=1}^\infty \frac{\tilde{A}_{\bar{z}}^{(m)}}{r^m} \log r.$$  \hspace{1cm} (31)

Our gauge condition leaves unfixed gauge transformations of the form $\Box \varepsilon = 0$, among which are residual gauge transformations with falloff $O(r^{-1})$ which, like a radiative massless scalar field, have an arbitrary boundary dependence. We have used this freedom above to set $A_u^{(1)} = 0$.

The Maxwell equations $\nabla_\mu F^{\mu\nu} = e^2 j^\nu$ in retarded coordinates, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, are

$$(\partial_u - \partial_r)(r^2 F_{ru}) + \gamma^{zz}(\partial_z F_{zu} + \partial_z F_{u\bar{z}}) = e^2 r^2 j_u$$

$$-\partial_r(r^2 F_{ru}) + \gamma^{zz}(\partial_z F_{rz} + \partial_z F_{z\bar{z}}) = e^2 r^2 j_r$$

$$r^2(\partial_u - \partial_r)F_{rz} - r^2 \partial_r F_{u\bar{z}} - \partial_z(\gamma^{zz} F_{z\bar{z}}) = e^2 r^2 j_z$$

$$r^2(\partial_r - \partial_u)F_{r\bar{z}} - r^2 \partial_r F_{u\bar{z}} - \partial_{\bar{z}}(\gamma^{zz} F_{z\bar{z}}) = e^2 r^2 j_{\bar{z}},$$  \hspace{1cm} (32)

while the Lorenz gauge condition reads

$$-\partial_u(r^2 A_r) - \partial_r(r^2 A_u - r^2 A_r) + \gamma^{zz}(\partial_z A_{\bar{z}} + \partial_{\bar{z}} A_z) = 0 .$$  \hspace{1cm} (33)

Together these imply

$${\mathcal O}(\log r) :$$

$$2\partial_u \tilde{A}_{\bar{z}}^{(1)} - 2\partial_{\bar{z}} \tilde{A}_u^{(1)} = 0$$  \hspace{1cm} (34)

$${\mathcal O}(1) :$$

$$-2 \partial_u \tilde{A}_u^{(1)} = e^2 j_u^{(2)}$$  \hspace{1cm} (35)

$$2\partial_u A_\bar{z}^{(1)} - 2\partial_{\bar{z}} A_u^{(1)} + 2D_{z\bar{z}} A_{\bar{z}}^{(0)} = e^2 j_{\bar{z}}^{(2)} .$$  \hspace{1cm} (36)

where we have used that $\mathcal{O}(\log r)$ expression for $\tilde{j}_{\bar{z}}^{(2)}$ is set to zero because the currents should not have logarithmic falloff. Note that $j_u^{(2)}$ would be incorrectly set to zero if log terms were not included in the expansion. We use these equations to arrive at (5).

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