The State of Matrix Theory

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This is a brief description of what has been accomplished and what remains to be done in the construction of a nonperturbative formulation of “The Theory Formerly Known as String”. It is culled from two short talks given by the author at SUSY97 and Strings97.

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1. Introduction

This is a brief summary of the present state of Matrix Theory given in talks at the SUSY 97 conference at the University of Pennsylvania and the Strings 97 conference in Amsterdam. The SUSY97 talk was given to an audience dominated by people with no expertise in String Theory, and technical details were kept to a minimum. In addition, since the field is in a rapid state of flux, I attempted to talk mostly about certain key features of Matrix Theory which will survive until the time when it is sufficiently well understood to receive a name which describes what it is really about. Apart from that I will primarily give only a list of key results. A fairly extensive reference list can be compiled by downloading citations of the papers [1] [2] [3] from the SPIRES database.

Matrix Theory is a nonperturbative Hamiltonian formalism for the theory formerly known as string theory/M theory (more on the ambiguous meaning of the term M theory below). Let me begin by describing its most salient defect. It is formulated in the infinite momentum frame (IMF)(sometimes called light cone gauge - though the two concepts are not precisely equivalent). Thus, it is not manifestly covariant, nor is it background independent. To understand the latter remark, remember the key feature of IMF kinematics: the dispersion relation of a particle like excitation has the form

\[ E = \sqrt{P_L^2 + P^2 + m^2} \rightarrow |P_L| + \frac{(P^2 + m^2)}{2|P_L|} \]  

(1.1)

If we define the light cone Hamiltonian by \( H = E - P_L \) then as \( P_L \rightarrow \infty \), we see that particle like excitations with positive longitudinal momenta have very small light cone energies. The key simplification of IMF dynamics is that one tries to write down a Hamiltonian which describes only these very low energies (or perhaps energies which at most remain constant as \( P_L \rightarrow \infty \)). Degrees of freedom with negative or vanishing \( P_L \), whose energy blows up in the limit, are imagined to be integrated out in the Wilsonian manner.

In conventional field theory, a background is defined by the value of a degree of freedom of the theory which is completely translation invariant, and in particular, has \( P_L = 0 \). Thus, changes in the background refer to degrees of freedom which are left out of the conventional IMF description. It is of course to be hoped that we will eventually find a covariant description of Matrix Theory. However, as I will suggest below that spacetime is only an approximate concept in Matrix Theory , it is perhaps premature to define what we mean by covariant.

The second key feature of Matrix Theory is that it is holographic[4]. Imagine that the longitudinal direction is compactified on a circle, so that longitudinal momentum is quantized in units \( \frac{1}{R} \), with the total \( P_L \) equal to \( N/R \) with \( N \rightarrow \infty \). A holographic
theory contains only degrees of freedom which carry the smallest unit of longitudinal momentum (and those connected to them by the gauge transformations to be described below). Low energy states corresponding to particles with finite fractions of the lowest $P_L$ will be composites of these fundamental objects. There should thus be at least $N$ of these fundamental degrees of freedom, which one might think of as the partons of Matrix Theory.

It is convenient to arrange these $N$ degrees of freedom as the diagonal matrix elements of an $N \times N$ matrix $Z^A$ ($A$ is a label for all of the quantum numbers that an individual parton carries. As we will see, in compact spacetimes these labels can encode all of the degrees of freedom of a field theory.). If we think of the matrix elements as the coordinates of partons, then there is a natural $S_N$ statistics symmetry, which interchanges the partons, and acts on the matrix $Z^A$ as $Z^A \rightarrow SZ^A S^\dagger$. It is tempting to view this as a residuum of a continuous $U(N)$ gauge invariance, which we can do by adding degrees of freedom to fill out the diagonal matrix to a general Hermitian matrix. Thus for example, for a supersymmetric system of $N$ particles in eleven space time dimensions, the degrees of freedom will be a nine component vector $X$ and a sixteen component spinor $\Theta$, each of whose components is an $N \times N$ Hermitian matrix. Conjugation of the matrices by $U(N)$ unitary transformations is a gauge symmetry, generalizing the $S_N$ of statistics. Remarkably, in the situations which we understand completely, this simple prescription (plus IMF Super-Galilean invariance) turns out to completely determine the interacting theory. Perhaps this should be viewed as a nonperturbative generalization of the way in which perturbative string interactions follow immediately from the laws of string propagation. It is at any rate a completely different route than that taken in quantum field theory for generalizing the Fock space description of free particle propagation to an interacting theory.

The answer to the question, “Why don’t we see all of these extra, off diagonal, degrees of freedom in the real world?”, is remarkable, and comes in several parts. First we must realize that the general setup of holographic theories requires the existence of bound states of $N$ partons for any value of $N$. This is simply the requirement that physical particles come with all possible values of the longitudinal momentum. In the examples studied so far, these are always threshold bound states, so it is not easy to establish their existence.\footnote{I am trying to be very general here. Lack of time and space precluded the discussion of detailed examples. The reader should be assured that they exist and should consult the references mentioned in the next section. In eleven dimensions the threshold bound state problem has been solved only for $N = 2$, while for weakly coupled string limits, it has been solved in general.} One way of trying to ensure this is to study sequences of matrix models which differ only
in the dimensionality of the matrices. For large $N$, this should guarantee some uniformity of behavior.

Let us imagine that we have succeeded in establishing the existence of some number of such stable bound states in the $N \times N$ matrix model for every $N$. Another characteristic feature of the models under discussion is that nonderivative interactions between the matrices are always functions of their commutators $[Z^A, Z^B]$. As a consequence, the classical potential of the model has a large number of flat, zero energy directions. Any collection of block diagonal matrices, with different coefficients of the unit matrix in each block, is a minimum of the classical potential. It is extremely important that, as a consequence of SUSY, these are exact flat directions of the quantum problem - no effective potential is generated by quantum corrections\footnote{Actually, all that is necessary is that SUSY is restored at asymptotically large values of the differences between coordinates in each block. These differences play the role of relative separations between particles and asymptotic SUSY is enough to guarantee that the potential falls off at large distance.}. Now let us consider configurations of the system in which all of the coordinates within each block are put into the wave function of one of the block bound states. The coefficient of the unit matrix within this block acts as the transverse center of mass position of the corresponding particle. When the transverse distances between particles are large, the off (block) diagonal degrees of freedom get large frequencies (essentially via the Higgs mechanism) and are unobservable at low energies (thus answering the rhetorical question above). SUSY guarantees that their virtual effects do not generate a potential along the flat directions (asymptotic SUSY is enough to guarantee that any potential falls off with distance).

Thus, we establish the existence of scattering states of arbitrary numbers of particles. Put more dramatically, what we have accomplished here is the systematic derivation of an approximate spacetime picture from the dynamics of our quantum system. In the examples which have been studied, the resulting scattering states are particles and strings with relativistic dispersion relations - although the underlying Hamiltonian has only the manifest symmetries of the IMF. Space is derived as the moduli space of a supersymmetric quantum system. Approximate locality (the fact that forces due to virtual effects fall off with distance) is a consequence of SUSY (in bosonic matrix models quantum corrections generate linear confining potentials along the classical flat directions). The apparent singularity of the force law at short distances is a consequence of improperly integrating out the off diagonal matrices in a regime where their frequencies are low.

The resulting theory is \textit{au fond} nonlocal. Locality appears as an approximate property of long distance, low energy interactions. The original matrix model of \cite{1} describes flat
eleven dimensional spacetime. In a local theory this would immediately have led to a
description of compact spacetimes. In field theory, degrees of freedom are associated with
individual points. The difference between a flat infinite space and a compact one is merely
a question of boundary conditions. This led to a widespread expectation that the discovery
of eleven dimensional M theory would lead to a complete nonperturbative formulation of
“the theory formerly known as string”. Thus there was a conflation of the idea of M
theory as just another asymptotic limit of string moduli space and that of M theory as the
nonperturbative formulation of everything.

In a nonlocal theory, one may imagine that there are fundamental degrees of freedom
(and not just composite soliton states) associated with nontrivial topological submanifolds
of a compact space. If, as one might expect, the energy of such degrees of freedom goes
to infinity with the volume of the submanifold, then the decompactification limit might
not capture all of the degrees of freedom of the theory. Experience appears to show that
this is the case for the Matrix model of M theory. This is closely connected with the old
observation that the duality groups of compactified string theory grow with the number
of compactified dimensions. Thus, the Matrix model has the property that the density of
low energy states in finite volume, increases as the volume is decreased. I will discuss this
briefly, along with the rest of the state of the art, in the next section.

2. Progress and Problems

Before beginning the litany of successes of the Matrix model, I want to make one
point. Many of the successes consist of the reproduction of facts already known from string
duality. I believe that the correct way to think about this is by analogy with symmetry
arguments in QCD. String duality should be thought of as the analog of chiral symmetry
while the Matrix Theory (in its final, as yet unrealized form, which applies to arbitrary
compactifications) is the analog of the QCD Lagrangian. It embodies the symmetries, and
fills in dynamical details which cannot be understood on the basis of symmetry arguments

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3 The cautious phrasing here has to do with the large $N$ limit. For finite $N$ one certainly needs
to introduce extra degrees of freedom to describe compactification. I have speculated that these
might appear automatically in a proper organization of the large $N$ limit. With the advent of clear
evidence that supersymmetric gauge field theories are not the whole story of compactification, I
am preparing to abandon this point of view. But the evidence is not all in yet.

4 L.Susskind has suggested to me that one might have anticipated the existence of degrees of
freedom associated with compact dimensions from the fact that the entropy of a black hole with
a given large mass, increases as the number of noncompact dimensions is decreased.
alone. Time limitations preclude the detailed discussion of even one example. I will content myself with a list of results and references.

1. The matrix model describing uncompactified eleven dimensional spacetime is given by the following supercharges and Hamiltonian

\[
Q_\alpha = \sqrt{R} \text{Tr} \left[ P^i (\gamma^i)_{\alpha\beta} + i [X^i, X^j] (\gamma^{ij})_{\alpha\beta} \right] \Theta_\beta \tag{2.1}
\]

\[
\tilde{Q}_\alpha = \frac{1}{\sqrt{R}} \text{Tr} \Theta_\alpha \tag{2.2}
\]

\[
H = R \text{Tr} \left[ (P^i)^2 - [X^i, X^j]^2 + \Theta [\gamma X, \Theta] \right] \tag{2.3}
\]

The vectors here are nine dimensional transverse coordinates and their canonical conjugates, and the spinors are their sixteen component superpartners. \(R\) is the radius of the longitudinal direction of the IMF and \(N/R\) is the total longitudinal momentum. The matrices are \(N \times N\). \(l_{11}\) is the eleven dimensional Planck length, and has been set to 1. The \(N \to \infty\) limit of this model was shown in [1] to contain the Fock space of eleven dimensional SUGRA, as well as, (following the seminal work of [6]) large metastable semiclassical membranes. A low energy, zero longitudinal momentum graviton scattering amplitude was calculated and shown to agree with the prediction of SUGRA. Various membrane scattering amplitudes [7] were calculated including one with nonzero longitudinal momentum transfer and shown to agree with SUGRA. A number of infinite energy BPS brane configurations were found [8]. These are best viewed as limits of objects wrapped around compact directions. Significantly, although one is able to exhibit five branes, one only sees those wrapped around the longitudinal direction [5]. This is related to the failure of the Super Yang Mills prescription for compactification.

2. Compactification down to ten dimensions was studied, and in the limit of small compactification radius, was shown to contain the Fock space of Type IIA string field theory [4]. In particular it is proven that the strings become free in the small radius limit. This was also done in [10] and there it was shown that the correct leading order string interactions (including the correct [11] scaling of the coupling with the radius) were reproduced by the model. The procedure for compactification is to replace the matrices by infinite dimensional operators. In particular, the compactified coordinates are replaced by \(X^a \to \frac{\partial}{\partial \sigma_a} I_{N \times N} - A_a (\sigma)\), where \(A\) is a \(U(N)\) gauge potential [1] [12]. Other variables simply become matrix valued functions of \(\sigma\). When plugged into

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5 See however the very interesting suggestion for describing the transverse five brane wrapped on a three torus in Ganor et. al. [8].
the Hamiltonian (2.3), this ansatz produces the Hamiltonian of maximally supersymmetric $1+1$ dimensional super Yang Mills theory ($SYM_{1+1}$). The zero radius limit forces one onto the moduli space of this model, which was shown at large $N$ to be the same as the Type IIA Fock space.

3. This compactification procedure can be generalized to two and three dimensional tori, producing $SYM_{d+1}$ for $d = 2, 3$. The theory has the full duality group of compactified M theory, which arises as a combination of obvious geometric transformations and electric-magnetic duality transformations\[13\]. In particular, one can study in some detail the Aspinwall-Schwarz\[14\] limit of compactification on a zero area two torus, in which a new dimension grows and gives the ten dimensional Type IIB string theory. Using the matrix model one can explicitly derive the rotation symmetry between this new dimension and the existing noncompact dimensions - something which is completely mysterious from all other points of view\[2\],\[15\]. One can also establish that zero longitudinal momentum ten dimensional graviton scattering emerges in the correct manner\[16\]. Finally, one obtains a prediction for wrapped membrane-antimembrane production in graviton scattering\[17\].

3. In four compact dimensions, the proposed SYM theory is nonrenormalizable, and also fails to reproduce the correct $SL(5)$ duality group. The authors of\[18\] argued that a definition of the $SYM_{4+1}$ theory as a compactification of the $5+1$ dimensional $(2,0)$ supersymmetric fixed point theory with a spectrum of tensor charges in the $U(N)$ weight lattice\[19\] repaired all of these difficulties. A pattern begins to emerge in which the spectrum of “momenta” in the auxiliary quantum theory which defines the compactified matrix model, is the spectrum of finite energy BPS charges coming from branes wrapped around the longitudinal and transverse directions\[20\]. The new dimension of the $(2,0)$ theory is associated with the longitudinally wrapped five brane.

4. Above 4 compact dimensions, arguments have been advanced that the auxiliary quantum theory can no longer be a local quantum field theory\[21\]. It has been identified with the theory of $N$ Type II Neveu-Schwarz fivebranes wrapped around a five dimensional transverse torus, in the limit of string theory in which the coupling goes to zero. Various low energy limits of this as yet rather mysterious theory, as well as its spectrum of BPS charges, seem to fit the known data. In particular, the purely transverse wrapped five brane can be exhibited. This gives an existence proof, though not yet a construction, of the matrix model compactified on $T^5$ and perhaps $T^6$. There is a strong suspicion that the logarithmic infrared singularities of a holographic theory with only two noncompact transverse dimensions will drastically change the story of compactification on spaces of dimension 7 and higher. Thus, compactifications with four or fewer large spacetime dimensions are beyond the limits of current knowledge.
5. All of the above refers to compactifications with maximal SUSY. Compactification of M theory with half maximal SUSY down to ten dimensions has been successfully carried out [22]. Heterotic string theory has been derived in the ten dimensional limit, and the Horava-Witten [23] picture of its strong coupling limit has been verified by explicit matrix model calculations. Compactification on further circles in this formalism leads to unresolved problems [24] with anomalies. There are strong indications that these problems can be resolved, and a full picture of the moduli space obtained, only by using the $(2, 0)$ fixed point theory compactified on $K3 \times S^1$ and taking various limits [25]. There exists a proposal for compactifications on five dimensional manifolds with half maximal SUSY via the weak coupling limit of heterotic Neveu-Schwarz fivebranes [26] (see also [27]).

6. Finally, I would like to note that the work of [28] shows that when there are no strong SUSY nonrenormalization theorems, correct answers in Matrix Theory will only be obtained after the large $N$ limit is taken. This points up the need for some sort of large $N$ renormalization group to isolate the dynamics of those states with energies of order $1/N$. This is very different from standard large $N$ physics (essentially because of the SUSY flat directions). At the moment, we only have a general treatment of this problem in the weakly coupled string theory limit, where it is solved by the renormalization group of two dimensional field theory.

3. Conclusions

It is fairly clear that Matrix theory is a nonperturbative formulation of a theory which underlies string theory. It is also fairly clear that this theory is completely nonlocal in nature, and incompletely understood. Spacetime is only an approximate concept in the theory and it is clear that much of our difficulty in formulating it is related to our reliance on spacetime intuitions. More fundamental to the theory is the supertranslation algebra, including charges associated with wrapped two branes and fivebranes. Some of the approximate space-time limits of the theory utilize these brane charges as momenta, so it is clear that we should treat them on an equal footing with ordinary momenta. Perhaps the ultimate shape of the theory will be related to allowed representations of this algebra. The charge spectrum is generally constrained to lie in a lattice, via arguments related to the Dirac quantization condition. We should find a derivation of these constraints which does not rely on spacetime notions. I would suggest that ultimately, compactifications will
be described simply in terms of the allowed spectra of BPS charges. P. Aspinwall and D. Morrison have assured me that at least for complex dimensions two and three, the energies of BPS states associated with supersymmetric cycles determine the geometry of a Calabi Yau manifold up to a few finite choices. I am suggesting that in the final formulation of the theory, there will be no other measure of geometry which is unconnected with some sort of low energy approximation.

Matrix theory is undergoing very rapid development. I believe strongly that our view of it a year from now will be radically different from anything we are now saying. I believe that the supertranslation algebra, the general construction of multibody states from block diagonal matrices, and the concept of space as the moduli space of a SUSY quantum system will remain, but that much of the rest of what has been done up till now will be seen as a series of special approximations. I am virtually certain that we will have to understand the large $N$ limit much more throughly than we do today. I suspect that the field theoretically unheard of way in which the density of low energy states changes with the volume will lie at the heart of the black hole information paradox and the problem of the cosmological constant. I am curious to see what the covariant formulation of the Matrix Theory will look like, and how cosmology gets into the picture (more than ever this seems to be a question of the dynamics of moduli). I wonder whether the apparent existence of a global definition of time is an artifact of the light cone gauge or a sign that our notions of general covariance are wrong (almost inevitable if spacetime does not really exist). And does the fact that we live in four large dimensions have to do with the fact that in a holographic theory, two noncompact space dimensions is the first place where transverse potentials grow at infinity?

My purpose in this talk has been to show you that something exciting is going on. As string theorists we are used to this happening every couple of years. At the risk of seeming immodest, I would like to propose that the present situation has within it the seeds of something more striking than anything we’ve seen before. In our study of various limits of moduli space we’ve succeeded in coding much of what we learn into low energy effective field theories. This beautiful and efficient tool has also been the means which allowed us to squeeze string theory into our low energy prejudices about geometry and

\[6\] The first hint that other BPS charges should be considered on an equal footing with momentum comes from perturbative T duality. The work of Aspinwall and Schwarz cited above gave another indication of this, as did various other results in string duality. P.K. Townsend [29] was the first to advocate the importance of the full M theory SUSY algebra, and I. Bars [30] was the first to emphasize that all charges should be put on the same footing as momenta. Hints of this approach to geometry can be found in the last section of [1]
locality. Matrix theory, as ugly and unwieldy as it is, has the virtue of refusing to be forced into this mold. Instead it forces us to give up our blindsers and face the bizarre nature of the beast we have snared straight on. To the present author at least it seems quite clear that the fundamental rules of the new theory will seem outlandish to anyone with a background in quantum field theory or general relativity. The challenge will be to find the correspondence principle by which the old rules of geometry and locality emerge from these new axioms. At the moment it appears that the only things which may remain unscathed are the fundamental principles of quantum mechanics. I invite you to join in the search for these rules. It’s lots of fun, and it leaves your mind panting like a long distance runner after a marathon. Maybe in a few years we’ll even find a decent name for the damned thing.

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References

[1] T. Banks, W. Fischler, S. Shenker and L. Susskind, [hep-th/9610043], Phys. Rev. D55, (1997), 112.
[2] T. Banks, N. Seiberg, [hep-th/9702187].
[3] R. Dijkgraaf, E. Verlinde, H. Verlinde, [hep-th/9703030].
[4] C. B. Thorn, in Proceedings of Sakharov Conference on Physics, Moscow, (1991), 447-454, [hep-th/9405069]. G. ’t Hooft, Dimensional Reduction in Quantum Gravity, Utrecht preprint THU-93/26, gr-qc/9310026. L. Susskind, J. Math. Phys. 36, (1995), 6377, hep-th/940989.
[5] S. Sethi, M. Stern, [hep-th/9705046].
[6] B. de Wit, J. Hoppe, H. Nicolai, Nucl. Phys. B305, [FS 23], (1988), 545; P. K. Townsend, Phys. Lett. B373, (1996), 68, hep-th/9512062.
[7] J. Polchinski, P. Pouliot, [hep-th/9704023]. N. Dorey, V. Khoze, M. Mattis, [hep-th/9704197]. O. Aharony, M. Berkooz, Nucl. Phys. B491, (1997), 184, hep-th/9611215. G. Lifschytz, S. Mathur, hep-th/9612087.
[8] T. Banks, N. Seiberg, S. Shenker, Nucl. Phys. B490, (1997), 91, [hep-th/9612157]. O. J. Ganor, S. Ramgoolam, W. Taylor IV, [hep-th/9611202]. M. Berkooz, M. Douglas, Phys. Lett. B395, (1997), 196, hep-th/9610236.
[9] L. Motl, [hep-th/9701025]. T. Banks, N. Seiberg, [hep-th/9702187].
[10] R. Dijkgraaf, E. Verlinde, H. Verlinde, [hep-th/9703030].
[11] E. Witten, Nucl. Phys. B443, (1995), 85, hep-th/9503124.
[12] W. Taylor IV, Phys. Lett. B394, (1997), 283, hep-th/9611042.
[13] L. Susskind, [hep-th/9611164]. O. J. Ganor, S. Ramgoolam, W. Taylor IV, op. cit.
[14] P. Aspinwall, in Proceedings of ICTP Trieste Conf., Jun. 1995, Nucl. Phys. Proc. Suppl. 46, (1996), 30, hep-th/9508154. J. Schwarz, Phys. Lett. B367, (1996), 97, hep-th/9510086.
[15] S. Sethi, L. Susskind, [hep-th/9702101].
[16] T. Banks, W. Fischler, N. Seiberg, L. Susskind, [hep-th/9705190].
[17] T. Banks, S. Shenker, In preparation.
[18] M. Berkooz, M. Rozali, N. Seiberg, [hep-th/9704089]. M. Rozali, [hep-th/9702136].
[19] N. Seiberg, [hep-th/9705117].
[20] T. Banks, N. Seiberg, op. cit., T. Banks, N. Seiberg, S. Shenker, op. cit., N. Seiberg, [hep-th/9705221].
[21] M. Berkooz, M. Rozali, N. Seiberg, op. cit., N. Seiberg, [hep-th/9705221].
[22] U. H. Danielsson, G. Ferretti, [hep-th/9610082]. S. Kachru, E. Silverstein, Phys. Lett. B396, (1997), 70, hep-th/9612162. L. Motl, [hep-th/9612198]. S. J. Rey, T. Banks, N. Seiberg, E. Silverstein, [hep-th/9703052]. T. Banks, L. Motl, [hep-th/9703218]. D. A. Lowe, [hep-th/9702006]. hep-th/9704041. S. J. Rey, hep-th/9704158. P. Horava, hep-th/9705053.
| Reference | Authors | Journal | Volume | Year | arXiv Number |
|-----------|---------|---------|--------|------|--------------|
| [23]      | P.Horava, E.Witten | Nucl. Phys. B | 475    | 1996 | hep-th/9603142 |
| [24]      | T.Banks, L.Motl | op. cit., P.Horava |          |      |              |
| [25]      | S.Govindarajan | hep-th/9705113; M.Berkooz, M.Rozali | hep-th/9705175 |
| [26]      | N.Seiberg | hep-th/9705221 |
| [27]      | A. Fayyazudin, D.J.Smith | hep-th/9703208; N.Kim, S.J. Rey | hep-th/9705132 |
| [28]      | M.Douglas, H.Ooguri, S.Shenker | hep-th/9702203; W.Fischler, A.Rajaraman | hep-th/9704086 |
| [29]      | P.K.Townsend | hep-th/9507048 |
| [30]      | I. Bars | hep-th/9604139 |

