Spacelike Curves of Constant-Ratio in Pseudo-Galilean Space

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Abstract. In the theory of differential geometry curves, a curve is said to be of constant-ratio if the ratio of the length of the tangential and normal components of its position vector function is constant. In this paper, we study and characterize a spacelike admissible curve of constant-ratio in terms of its curvature functions in pseudo-Galilean space \( G^3_1 \). Some special curves of constant-ratio such as \( T \) and \( N \) constant types are investigated. As an application of our main results, some examples are given.

Keywords: Curves of constant-ratio; Position vector; Pseudo-Galilean space; Frenet frame.

Mathematics Subject Classification: 53A04, 53A17, 53B30.

1 Introduction

From the differential geometry of the curve theory it is well known that a curve \( \alpha(s) \) in \( E^3 \) lies on a sphere if its position vector lies on its normal plane at each point. If the position vector \( \alpha \) lies on its rectifying plane then \( \alpha(s) \) is called rectifying curve [1]. Rectifying curves characterized by the simple equation

\[
\alpha(s) = \lambda(s)T(s) + \mu(s)B(s),
\]

where \( \lambda(s) \) and \( \mu(s) \) are smooth functions and \( T(s) \) and \( B(s) \) are tangent and binormal vector fields of \( \alpha \), respectively. In [2] the author provide that a twisted curve is congruent to a non constant linear function of \( s \). On the hand, in the Minkowski 3-space \( E^3_1 \), the rectifying curves are investigated in [3,4]. Besides, in [4] a characterization of the spacelike, the timelike and the null rectifying curves in the Minkowski 3-space in terms of centrodes is given. The
characterization of rectifying curves in three dimensional compact Lee groups as well as in dual spaces is given in [5], [6], respectively. For the study of constant-ratio curves, the authors give the necessary and sufficient conditions for curves in Euclidean and Minkowski spaces to become $T$ constant or $N$ constant [7–10]. In analogy with the Euclidean 3-dimensional case, our main goal in this work is to define the spacelike admissible curves of constant-ratio in the pseudo Galilean 3-space as a curve whose position vector always lies in the orthogonal complement $N^\perp$ of its principal normal vector field $N$. Consequently, $N^\perp$ is given by

$$N^\perp = \{V \in G^1_3 : <V, N >= 0\}$$

where $<·,·>$ denotes the scalar product in $G^1_3$. Hence $N^\perp$ is a 2-dimensional plane of $G^1_3$, spanned by the tangent and binormal vector fields $T$ and $B$ respectively. Therefore, the position vector with respect to some chosen origin, of a considered curve $\alpha$ in $G^1_3$, satisfies the parametric equation

$$\alpha(s) = m_0(s)T(s) + m_1(s)N(s) + m_2(s)B(s),$$

for some differential functions $m_i(s)$, $0 \leq i \leq 2$ in arc-length function $s$, and we give the necessary and sufficient conditions for the curve $\alpha$ in $G^1_3$ to be a constant-ratio.

2 Pseudo-Galilean geometry

In this section, we introduce the basic concepts, familiar definitions and notations on pseudo-Galilean space which are needed throughout this study. The pseudo-Galilean geometry is one of the real Cayley-Klein geometries of projective signature (0,0,+,-). The absolute of the pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$ where $w$ is the ideal (absolute) plane, $f$ is a line in $w$ and $I$ is the fixed hyperbolic involution of points of $f$ [12]. The geometry of the pseudo-Galilean space is similar (but not the same) to the Galilean space which is presented in [11]. The scalar product and cross product of two vectors $x = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$ in $G^1_3$ are respectively defined by:

$$g(x, y) = \begin{cases} 
x_1x_2, & \text{if } x_1 \neq 0 \lor x_2 \neq 0 \\
y_1y_2 - z_1z_2, & \text{if } x_1 = 0 \land x_2 = 0,
\end{cases}$$

$$x \times y = \begin{vmatrix} 0 & -e_2 & e_3 \\
e_1 & z_1 \\
e_2 & z_2 
\end{vmatrix}.$$
\[\|\mathbf{x}\| = \begin{cases} \frac{x}{\sqrt{|y^2 - z^2|}}, & \text{if } x = 0 \\ x, & \text{if } x \neq 0 \end{cases} \tag{2.1}\]

The group of motions of the pseudo-Galilean \(G^1_3\) is a six-parameter group given (in affine coordinates) by

\[
\bar{x} = a + x, \quad \bar{y} = b + cx + y \cosh \varphi + z \sinh \varphi, \quad \bar{z} = d + ex + y \sinh \varphi + z \cosh \varphi.
\]

It leaves invariant the pseudo-Galilean length of the vector \([12]\). According to the motion group in pseudo-Galilean space, a vector \(x(x, y, z)\) is said to be non isotropic if \(x \neq 0\). All unit non-isotropic vectors are of the form \((1, y, z)\). For isotropic vectors \(x = 0\) holds. There are four types of isotropic vectors: spacelike \((y^2 - z^2) > 0\), timelike \((y^2 - z^2) < 0\) and two types of lightlike \((y = \pm z)\) vectors. A non-lightlike isotropic vector is a unit vector if \(y^2 - z^2 = \pm 1\).

A trihedron \((T; e_1, e_2, e_3)\) with a proper origin \(T_o(x_o, y_o, z_o) = (T; x_o : y_o : z_o)\) is orthonormal in pseudo-Galilean sense if the vectors \(e_1, e_2, e_3\) are of the following form: \(e_1 = (1, y_1, z_1)\), \(e_2 = (0, y_2, z_2)\) and \(e_3 = (0, \varepsilon z_2, \varepsilon y_2)\) with \(y^2 - z^2 = \delta\), where \(\varepsilon, \delta\) is +1 or -1. Such trihedron \((T; e_1, e_2, e_3)\) is called positively oriented if for its vectors \(\det(e_1, e_2, e_3)\) holds; that is if \(y^2 - z^2 = \varepsilon\).

Let \(\alpha(t) : I \subset R \to G^1_3\) be a curve given first by \(\alpha(t) = (x(t), y(t), z(t))\), where \(x(t), y(t), z(t) \in C^3\) (the set of three-times continuously differentiable functions) and \(t\) run through a real interval \([12]\).

**Definition 2.1** A curve \(\alpha\) given by \(\alpha(t) = (x(t), y(t), z(t))\) is admissible if \(\dot{x}(t) \neq 0\).

If \(\alpha\) is taken as

\[\alpha(x) = (x, y(x), z(x)), \tag{2.2}\]

with the condition that

\[y''^2(x) - z''^2(x) \neq 0, \tag{2.3}\]

then the parameter of arc-length \(s\) is defined by

\[ds = |\dot{x}(t)| \, dt = dx. \tag{2.4}\]

For simplicity, we assume \(ds = dx\) and \(s = x\) as the arc-length of the curve \(\alpha\) \([12]\). The vector

\[T(s) = \alpha'(s),\]
is called the tangential unit vector of the curve $\alpha$ in a point $P(s)$. Also the unit vector

$$N(s) = \frac{\alpha''(s)}{\sqrt{|y''^2(s) - z''^2(s)|}}, \quad (2.5)$$

is called the principal normal vector of the curve $\alpha$ in the point $P$. The binormal vector of the given curve in the point $P$ is given by

$$B(s) = \frac{(0, \varepsilon z''(s), \varepsilon y''(s))}{\sqrt{|y''^2(s) - z''^2(s)|}}. \quad (2.6)$$

It is orthogonal in pseudo-Galilean sense to the osculating plane of $\alpha$ spanned by the vectors $\alpha'(s)$ and $\alpha''(s)$ in the same point. The curve $\alpha$ given by (2.2) is spacelike (resp. timelike) if $N(s)$ is a timelike (resp. spacelike) vector. The principal normal vector or simply normal is spacelike if $\varepsilon = +1$ and timelike if $\varepsilon = -1$. Here $\varepsilon = +1$ or $-1$ is chosen by the criterion $\det(T, N, B) = 1$. That means

$$|y''^2(s) - z''^2(s)| = \varepsilon(y''^2(s) - z''^2(s)). \quad (2.7)$$

By the above construction, the following can be summarized [12].

**Definition 2.2** In each point of an admissible curve in $G^1_3$, the associated orthonormal (in pseudo-Galilean sense) trihedron $\{T(s), N(s), B(s)\}$ can be defined. This trihedron is called pseudo-Galilean Frenet trihedron.

For the pseudo-Galilean Frenet trihedron of an admissible curve $\alpha$, the following derivative Frenet formulas are true [12].

$$T' = \kappa N,$$

$$N' = \tau B,$$

$$B' = \tau N. \quad (2.8)$$

The spacelike $T$, the timelike $N$ and the spacelike $B$ are called the vectors of the tangent, principal normal and the binormal line, respectively. The function $\kappa$ is the pseudo-Galilean curvature given by

$$\kappa(s) = \sqrt{|y''^2(s) - z''^2(s)|}. \quad (2.9)$$

And $\tau$ is the pseudo-Galilean torsion of $\alpha$ defined by

$$\tau(s) = \frac{y''(s)z''(s) - y''(s)z''(s)}{\kappa^2(s)}. \quad (2.10)$$

It can be written as

$$\tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2(s)}. \quad (2.11)$$
In a matrix form, the Serret-Frenet equations (2.8) are

\[
\frac{d}{ds} \begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.
\]

The Pseudo-Galilean sphere is defined by

\[ S^2_\pm = \{ u \in G^1_3 : g(u, u) = \pm r^2 \} , \]

where \( r \) is the radius of the sphere.

3 Spacelike curves of constant-ratio in \( G^1_3 \)

Let \( \alpha : I \subset R \to G^1_3 \) be an arbitrary spacelike admissible curve. In the light of which introduced in [13–17], we consider the following:

**Theorem 3.1** The position vector of \( \alpha \) with curvatures \( \kappa(s) \) and \( \tau(s) \neq 0 \) with respect to the Frenet frame in the pseudo-Galilean space \( G^1_3 \) can be written as:

\[
\alpha(s) = (s + c_o)T + e^{-\int \tau(s)ds} \left( c_1 e^{2 \int \tau(s)ds} + \frac{\kappa(s)(s + c_o)}{2} e^{-\int \tau(s)ds} \right) - \frac{\kappa(s)(s + c_o)}{2} e^{\int \tau(s)ds} + c_2 N + e^{-\int \tau(s)ds} \left( c_1 e^{2 \int \tau(s)ds} + c_2 \right) B
\]

where \( c_o, c_1 \) and \( c_2 \) are arbitrary constants.

**Proof.** Let \( \alpha \) be an arbitrary spacelike curve in the pseudo-Galilean space \( G^1_3 \), then we may express its position vector as

\[
\alpha(s) = m_o(s)T(s) + m_1(s)N(s) + m_2(s)B(s).
\]

Differentiating the last equation with respect to the arc-length parameter \( s \) and using the Serret-Frenet equations (2.8), we obtain

\[
\alpha'(s) = m'_o(s)T(s) + (m'_1(s) + \kappa(s)m_o(s) + \tau(s)m_2(s))N(s)
\]

\[
+ (m'_2(s) + \tau(s)m_1(s))B(s).
\]

It follows that

\[
m'_o(s) = 1,
\]

\[
m'_1(s) + \kappa(s)m_o(s) + \tau(s)m_2(s) = 0,
\]

\[
m'_2(s) + \tau(s)m_1(s) = 0.
\]
From the first equation of (3.2), we have

\[ m_o(s) = s + c_o. \] (3.3)

It is useful to change the variable \( s \) by the variable \( t = \int \tau(s)ds \). So that all functions of \( s \) will transform to functions of \( t \). Here, we will use dot to denote derivation with respect to \( t \) (prime denotes derivative with respect to \( s \)). The third equation of (3.2) can be written as

\[ m_1(t) = -\dot{m}_2(t), \text{ where } \dot{m}_2 = \frac{dm_2}{dt}. \] (3.4)

Substituting the above equation in the second equation of (3.2), we have the following equation for \( m_2(t) \)

\[ \ddot{m}_2(t) - m_2(t) = \frac{y(t)\kappa(t)}{\tau(t)}, \quad y(t) = m_o(s) = s + c_o. \] (3.5)

The general solution of this equation is

\[ m_2(t) = e^{-t} \left[ c_1 e^{2t} + e^{2t} \int \frac{\kappa(t)y(t)}{2\tau(t)} e^{-t}dt + \int \frac{\kappa(t)y(t)}{2\tau(t)} e^{t}dt - c_2 \right], \] (3.6)

where \( c_1 \) and \( c_2 \) are arbitrary constants. From (3.4) and (3.6), the function \( m_1(t) \) is obtained

\[ m_1(t) = e^{-t} \left[ c_1 e^{2t} + e^{2t} \int \frac{\kappa(t)y(t)}{2\tau(t)} e^{-t}dt - \int \frac{\kappa(t)y(t)}{2\tau(t)} e^{t}dt + c_2 \right]. \] (3.7)

Hence the above equations take the following forms

\[ m_1 = e^{-\int \tau(s)ds} \left[ c_1 e^{2\int \tau(s)ds} + e^{2\int \tau(s)ds} \int \frac{(s + c_o)\kappa}{2} e^{-\int \tau(s)ds}ds - \int \frac{(s + c_o)\kappa}{2} e^{\int \tau(s)ds}ds + c_2 \right], \] (3.8)

\[ m_2 = e^{-\int \tau(s)ds} \left[ c_1 e^{2\int \tau(s)ds} + e^{2\int \tau(s)ds} \int \frac{(s + c_o)\kappa}{2} e^{-\int \tau(s)ds}ds + \int \frac{(s + c_o)\kappa}{2} e^{\int \tau(s)ds}ds - c_2 \right]. \] (3.9)

Substituting from (3.3), (3.8) and (3.9) in (1.2), we obtain Eq.(3.1), so the proof of the theorem is completed.

**Theorem 3.2** Let \( \alpha : I \subset R \rightarrow G_3^1 \) be a spacelike curve with \( \kappa \neq 0 \) and \( \tau \neq 0 \) in \( G_3^1 \). Then the position vector and curvatures of \( \alpha \) satisfy a vector differential equation of third order.

**Proof.** Let \( \alpha : I \subset R \rightarrow G_3^1 \) be a spacelike curve with curvatures \( \kappa \neq 0 \) and \( \tau \neq 0 \) in \( G_3^1 \). From Frenet equations (2.8), one can write

\[ N = \frac{T'}{\kappa}, \] (3.10)

\[ B = \frac{N'}{\tau}. \] (3.11)
Substituting (3.10) in the third equation of (2.8), we get

$$B' = \frac{\tau}{\kappa} T'. \quad (3.12)$$

After differentiating (3.10) with respect to $s$ and substituting in (3.11), we find

$$B = \frac{1}{\tau} \left[ \left( \frac{1}{\kappa} \right)' T' + \left( \frac{1}{\kappa} \right)' T'' \right]. \quad (3.13)$$

Similarly, taking the differentiation of (3.13) and equalize with the third equation of (2.8), we obtain

$$\frac{1}{\tau \kappa} T''' + \left[ \frac{2}{\tau} \left( \frac{1}{\kappa} \right)' - \left( \frac{1}{\tau} \right)' \frac{1}{\kappa} \right] T'' + \left[ \frac{1}{\tau} \left( \left( \frac{1}{\kappa} \right)'' - \frac{\tau^2}{\kappa} \right) - \left( \frac{1}{\tau} \right)' \left( \frac{1}{\kappa} \right)'' \right] T' = 0. \quad (3.14)$$

It proves the theorem. ■

**Theorem 3.3** The position vector $\alpha(s)$ of a spacelike admissible curve with curvature $\kappa(s)$ and torsion $\tau(s)$ in the Pseudo-Galilean space $G^1_3$ is computed from the intrinsic representation form

$$\alpha(s) = \left( s, -\int \left[ \int \kappa(s) \sinh \left[ \int \tau(s) ds \right] ds \right] ds, \int \left[ \int \kappa(s) \cosh \left[ \int \tau(s) ds \right] ds \right] ds \right),$$

with tangent, principal normal and binormal vectors respectively, given by

$$T(s) = \left( 1, -\int \kappa(s) \sinh \left[ \int \tau(s) ds \right] ds, \int \kappa(s) \cosh \left[ \int \tau(s) ds \right] ds \right),$$

$$N(s) = \left( 0, -\sinh \left[ \int \tau(s) ds \right], \cosh \left[ \int \tau(s) ds \right] \right),$$

$$B(s) = \left( 0, -\cosh \left[ \int \tau(s) ds \right], \sinh \left[ \int \tau(s) ds \right] \right).$$

Now, similar to the Euclidean case [18], we consider the following definition:

For each given $\alpha : I \subset R \rightarrow G^1_3$, there is a natural orthogonal decomposition of the position vector $\alpha$ at each point on $\alpha$; namely,

$$\alpha = \alpha^T + \alpha^N, \quad (3.15)$$

where $\alpha^T$ and $\alpha^N$ denote the tangential and normal components of $\alpha$ at the point, respectively. Let $\|\alpha^T\|$ and $\|\alpha^N\|$ denote the length of $\alpha^T$ and $\alpha^N$, respectively. In what follows we introduce the notion of constant-ratio curves

**Definition 3.1** A curve $\alpha$ of a pseudo-Galilean space $G^1_3$ is said to be of constant-ratio curve if the ratio $\|\alpha^T\| : \|\alpha^N\|$ is constant on $\alpha(I)$. 

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Clearly, for a constant-ratio curve in pseudo-Galilean space $G^1_3$, we have

$$\frac{m_2^2}{m_2^2 - m_1^2} = c_3,$$  \hspace{1cm} (3.16)

for some constant $c_3$.

**Definition 3.2** Let $\alpha : I \subset R \to G^1_3$ be an admissible curve in $G^1_3$. If $\|\alpha^T\|$ is constant, then $\alpha$ is called a $T$–constant curve. Further, a $T$–constant curve $\alpha$ is called of first kind if $\|\alpha^T\| = 0$, otherwise second kind [19].

**Definition 3.3** Let $\alpha : I \subset R \to G^1_3$ be an admissible curve in $G^1_3$. If $\|\alpha^N\|$ is constant, then $\alpha$ is called a $N$–constant curve. For a $N$–constant curve $\alpha$, either $\|\alpha^N\| = 0$ or $\|\alpha^N\| = \mu$ for some non-zero smooth function $\mu$. Further, a $N$–constant curve $\alpha$ is called of first kind if $\|\alpha^N\| = 0$, otherwise second kind [19].

For a $N$–constant curve $\alpha$ in $G^1_3$, we can write

$$\|\alpha^N(s)\|^2 = m_2^2(s) - m_1^2(s) = c_4,$$ \hspace{1cm} (3.17)

where $c_4$ is a real constant.

In what follows, we characterize the admissible curves in terms of their curvature functions $m_i(s)$ and give the necessary and sufficient conditions for these curves to be $T$–constant or $N$–constant curves.

**Theorem 3.4** Let $\alpha : I \subset R \to G^1_3$ be a spacelike curve in $G^1_3$. Then $\alpha$ is of constant-ratio if and only if

$$\left(\frac{\kappa' - \kappa^3c_3(s + c_o)}{c_3\kappa^2\tau}\right)' = \frac{-\tau}{c_3\kappa}.$$

**Proof.** Let $\alpha : I \subset R \to G^1_3$ be a spacelike curve given with the invariant parameter $s$. As we know from the above

$$m_o(s) = s + c_o,$$

where $c_o$ is an arbitrary constant. Also, from (3.16), the curvature functions $m_i(s), 0 \leq i \leq 2$ satisfy

$$m_2(s)m_2'(s) - m_1(s)m_1'(s) = \frac{s + c_o}{c_3}.$$  \hspace{1cm} (3.18)

By using Eqs.(3.2) with (3.18), we obtain

$$m_1 = \frac{1}{c_3\kappa}.$$  

It follows that

$$m_2 = \frac{\kappa' - \kappa^3c_3(s + c_o)}{c_3\kappa^2\tau}.$$  

So, we get the result. ■
3.1 T-constant spacelike curves in $G^1_3$

**Proposition 3.1** There are no $T - constant$ spacelike curves in pseudo-Galilean space $G^1_3$.

**Proof.** Let $\alpha : I \subset R \to G^1_3$ be a spacelike curve in $G^1_3$. Then $\|\alpha^T\| = m_o$, where $m_o$ is equal to zero or a nonzero constant. Since $m_o = x + c_o$, this contradicts the fact of value of $m_o$. ■

3.2 N-constant spacelike curves in $G^1_3$

**Lemma 3.1** Let $\alpha : I \subset R \to G^1_3$ be a spacelike curve in $G^1_3$. Then $\alpha$ is a $N - constant$ curve if and only if the following condition

$$m_2(s)m_2'(s) - m_1(s)m_1'(s) = 0,$$

is hold together Eqs.(3.2), where $m_i(s)$, $0 \leq i \leq 2$ are differentiable functions.

**Proposition 3.2** Let $\alpha : I \subset R \to G^1_3$ be a spacelike curve in $G^1_3$. Then $\alpha$ is a $N - constant$ curve of first kind if $\alpha$ is a straight line in $G^1_3$.

**Proof.** Suppose that $\alpha$ is a $N - constant$ curve of first kind in $G^1_3$, then

$$m_2^2(s) - m_1^2(s) = 0.$$

So, we have two cases to be discussed:

**Case1.**

$$m_2(s) = m_1(s).$$

Using Eqs.(3.2), we get

$$\kappa = 0.$$

**Case2.**

$$m_2(s) = -m_1(s).$$

Also, from Eqs.(3.2), we obtain

$$\kappa = 0.$$

It means that the curve $\alpha$ is a straight line in $G^1_3$. ■

**Theorem 3.5** Let $\alpha : I \subset R \to G^1_3$ be a spacelike curve in $G^1_3$. If $\alpha$ is a $N - constant$ curve of second kind, then the position vector $\alpha$ has the parametrization of the form

$$\alpha(s) = (s + c_o)T(s) + \left[\frac{1}{4}e^{-u(s)}(-4c_4 + e^{2u(s)}) - \frac{1}{2}e^{u(s)}\right]N(s)$$

$$+ \left[\frac{1}{4}e^{-u(s)}(-4c_4 + e^{2u(s)})\right]B(s),$$

(3.19)

where $u(s) = \int \tau(s)ds + c_5$, $c_5$ is integral constant.
Proof. From (3.3), we have
\[ m_\alpha(s) = (s + c_\alpha). \]

Besides, the third equation of (3.2) together (3.17), yield
\[ m'^2_2(s) - \tau^2_2m^2_2(s) - c_4\tau^2_2 = 0, \]
where \( c_4 \neq 0 \) is a real constant. The solution of this equation is
\[ m_2(s) = \frac{1}{4} e^{-u(s)} \left( -4c_4 + e^{2u(s)} \right). \quad (3.20) \]

If we substitute Eq.(3.20) in the third equation of (3.2), it can be obtained
\[ m_1(s) = \frac{1}{4} e^{-u(s)} \left( -4c_4 + e^{2u(s)} \right) - \frac{1}{2} e^{u(s)}, \quad (3.21) \]

Eqs.(3.3), (3.20) and (3.21) give the required result of the theorem. \[ \blacksquare \]

Theorem 3.6 Let \( \alpha \) be a spacelike curve in the Pseudo-Galilean space \( G^1_3 \) with its pseudo-Galilean trihedron \( \{T(s), N(s), B(s)\} \). If the curve \( \alpha \) lies on a pseudo-Galilean sphere \( S^2_\pm \), then it is a \( N - \) constant curve of second kind and the center of a pseudo-Galilean sphere of \( \alpha \) at the point \( c(s) \) is given by
\[ c(s) = \alpha(s) + m_1(s)N(s) + m_2(s)B(s). \]

Proof. Let \( S^2_\pm \) be a sphere in \( G^1_3 \), then \( S^2_\pm \) is given by
\[ S^2_\pm = \{u \in G^1_3 : g(u, u) = \pm r^2\}, \]
where \( r \) is the radius of the pseudo-Galilean sphere and it is constant.

Let \( c \) be the center of the pseudo-Galilean sphere, then we have
\[ g(c(s) - \alpha(s), c(s) - \alpha(s)) = \pm r^2. \]

Differentiating this equation with respect to \( s \), we get
\[ g(-T(s), c(s) - \alpha(s)) = 0. \quad (3.22) \]

If we repeat the derivation, we have
\[ g(-T'(s), c(s) - \alpha(s)) + g(-T(s), -T(s)) = 0. \]

From (2.8), we get
\[- \kappa(s)g(N(s), c(s) - \alpha(s)) + 1 = 0, \quad (3.23)\]
and since we know that $c(s) - \alpha(s) \in Sp\{T(s), N(s), B(s)\}$, so we can write
\[
c(s) - \alpha(s) = m_o(s)T(s) + m_1(s)N(s) + m_2(s)B(s).
\] (3.24)

Now, from (3.23) and (3.24), we have
\[
\kappa(s)m_1(s) + 1 = 0.
\]

From which
\[
m_1(s) = -\frac{1}{\kappa(s)}.
\]

Also, from (3.22) and (3.24), one can write
\[
g(T(s), c(s) - \alpha(s)) = m_o(s),
\]
which gives
\[
m_o(s) = 0,
\]
and then Eq.(3.24) becomes
\[
c(s) - \alpha(s) = m_1(s)N(s) + m_2(s)B(s).
\]

Besides, the derivation of Eq.(3.23) leads to
\[
m_2(s) = \frac{-m'_1(s)}{\tau(s)}.
\]

From the calculations, we can obtain
\[
m_2^2(s) - m_1^2(s) = \pm r^2 = const.
\]

which completes the proof. ■

**Theorem 3.7** Let $\alpha$ be a $N$ – constant curve of second kind which lies on a pseudo-Galilean sphere $S^2_{\pm}$ with constant radius $r$ in $G^1_3$. Then we have the following equation
\[
m'_2(s) - \tau(s)m_1(s) = 0,
\]
where $m_2(s) \neq 0, \tau(s) \neq 0$.

**Proof.** Let $\alpha$ be a $N$ – constant curve in $G^1_3$, so we have
\[
m_2^2(s) - m_1^2(s) = \pm r^2.
\]

Since $r$ is const., then the differentiation of the last equation gives
\[
m_2(s)m'_2(s) - m_1(s)m'_1(s) = 0.
\]

Substituting by $m_2(s) = \frac{m'_1(s)}{\tau(s)}$ in this equation, we get
\[
m'_2(s) - \tau(s)m_1(s) = 0.
\]

So the proof is completed. ■
Theorem 3.8 Let $\alpha(s)$ be a spacelike curve in the Pseudo-Galilean space $G^1_3$ with $\kappa(s) \neq 0,$ $\tau(s) \neq 0$. The image of the $N$–constant curve $\alpha$ lies on a pseudo-Galilean sphere $S^2_\pm$ if and only if for each $s \in I \subset R$, its curvatures satisfy the following equalities:

$$
\frac{1}{4} e^{-u(s)} (-4c_4 + e^{2u(s)}) = \frac{1}{2} e^{u(s)} = \frac{1}{\kappa(s)}, \\
\frac{1}{4} e^{-u(s)} (-4c_4 + e^{2u(s)}) = \frac{\kappa'(s)}{\kappa^2(s) \tau(s)}.
$$

(3.25)

where $u(s) = \int \tau(s) ds + c_5$ and $c_o, c_4$ and $c_5 \in R$.

Proof. By assumption, we have

$$
g(\alpha(s), \alpha(s)) = r^2,
$$

for every $s \in I \subset R$ and $r$ is radius of the pseudo-Galilean sphere. Differentiation this equation with respect to $s$ gives

$$
g(T(s), \alpha(s)) = 0. \quad (3.26)
$$

By a new differentiation, we find that

$$
g(N(s), \alpha(s)) = -\frac{1}{\kappa(s)}.
$$

(3.27)

One more differentiation in $s$ gives

$$
g(B(s), \alpha(s)) = \frac{\kappa'(s)}{\kappa^2(s) \tau(s)}. \quad (3.28)
$$

Using Eqs.(3.26)-(3.28) in (3.19), we obtain (3.25).

Conversely, we assume that Eq.(3.25) holds for each $s \in I \subset R$, then from (3.19) the position vector of $\alpha$ can be expressed as

$$
\alpha(s) = -\frac{1}{\kappa(s)} N(s) + \frac{\kappa'(s)}{\kappa^2(s) \tau(s)} B(s),
$$

which satisfies the equation $g(\alpha(s), \alpha(s)) = r^2$. It means that the curve $\alpha$ lies on a pseudo-Galilean sphere $S^2_\pm$. □

Theorem 3.9 Let $\alpha$ be a spacelike curve in pseudo-Galilean space $G^1_3$. If $\alpha$ is a circle then $\alpha$ is $N$–constant curve of second kind.

Proof. If $\alpha$ is a circle, then we have

$$
\kappa(s) = \text{const} \quad \text{and} \quad \tau(s) = 0.
$$
Also, from theorem 3.4, one can write

\[ m_1 = \frac{1}{c_3 \kappa} = \text{const.}, \]

\[ m_2 = \int \left( \frac{-\tau}{c_3 \kappa} \right) ds = \text{const.}, \]

which leads to

\[ m_2^2(s) - m_1^2(s) = \text{const.} \]

So the proof is completed.

4 Examples

In this section, we give some examples to illustrate our main results.

Example 4.1 Let us consider the following spacelike curve \( \alpha : I \subset R \to G^1_3 \),

\[ \alpha(s) = \left( s, \frac{s}{6} [2 \sinh(2 \ln s) - \cosh(2 \ln s)], \frac{s}{6} [2 \cosh(2 \ln s) - \sinh(2 \ln s)] \right). \quad (4.1) \]

Differentiating (4.1), we get

\[ \alpha'(s) = \left( 1, \frac{1}{2} \cosh(2 \ln s), \frac{1}{2} \sinh(2 \ln s) \right). \quad (4.2) \]

Pseudo-Galilean inner product follows that \( \langle \alpha', \alpha' \rangle = 1 \). So the curve is parameterized by the arc-length and the tangent vector is (4.2).

In order to calculate the first curvature of \( \alpha \), let us express

\[ T' = \left( 0, \frac{1}{s} \sinh(2 \ln s), \frac{1}{s} \cosh(2 \ln s) \right), \]

taking the norm of both sides, we have \( \kappa(s) = \frac{1}{s} \). Thereafter, we have

\[ N = (0, \sinh(2 \ln s), \cosh(2 \ln s)), \]

and binormal vector

\[ B = (0, -\cosh(2 \ln s), -\sinh(2 \ln s)). \]

From Serret-Frenet equations (2.8), one can obtain \( \tau(s) = \frac{-2}{s} \). Moreover, the curvature functions \( m_i(s) \) are

\[ m_o = s, \quad m_1 = \frac{s}{c_3}, \quad m_2 = -\Omega s, \quad \Omega = \left( \frac{1 + c_3}{2c_3} \right) = \text{const.} \]
So, from (3.16), we get
\[ \frac{m_0^2}{m_2^2 - m_1^2} = \Gamma, \quad \Gamma = \frac{4(c_3)^2}{(c_3 + 1)^2 - 4} = \text{const.} \]

Under the above considerations, \( \alpha \) is of constant-ratio and the ratio is equal \( \Gamma \). Also, since
\[ \|\alpha^N(s)\|^2 = m_2^2(s) - m_1^2(s) = \left( \frac{(c_3 + 1)^2 - 4}{4(c_3)^2} \right) s^2 \neq \text{const.}, \]
then the curve \( \alpha \) is not a \( N \) - constant curve.

Figure 1: Curve with Constant-Ratio and not a \( N \) - constant.

**Example 4.2** Let \( \alpha \) be a spacelike curve in \( G^1_3 \) given by
\[ \alpha(s) = \left( s, -a \int \left( \int \sinh\left( \frac{s^2}{2} \right) ds \right) ds, a \int \left( \int \cosh\left( \frac{s^2}{2} \right) ds \right) ds \right), \]
where \( a \in \mathbb{R} \). Then we have
\[ \alpha'(s) = T(s) = \left( 1, -a \int \sinh\left( \frac{s^2}{2} \right) ds, a \int \cosh\left( \frac{s^2}{2} \right) ds \right), \]
\[ T'(s) = \left( 0, -a \sinh\left( \frac{s^2}{2} \right), a \cosh\left( \frac{s^2}{2} \right) \right). \]

By a straightforward computation, we can obtain
\[ N(s) = \left( 0, -\sinh\left( \frac{s^2}{2} \right), \cosh\left( \frac{s^2}{2} \right) \right). \]
\[ B(s) = \left( 0, -\cosh\left(\frac{s^2}{2}\right), \sinh\left(\frac{s^2}{2}\right) \right), \]

where \( \kappa(s) = a = \text{const} \) and \( \tau(s) = s \).

Since the curve has a constant curvature and non-constant torsion, so it is a Salkowski curve.

Also, from theorem 3.4 we have the curvature functions
\[
m_1 = \frac{1}{c_3 \kappa} = \frac{1}{ac_3},
\]
\[
m_2 = \frac{\kappa' - \kappa^3 c_3 s}{c_3 \kappa^2 \tau} = -a, \quad a \text{ is constant},
\]

which leads to
\[
m_2^2(s) - m_1^2(s) = (-a)^2 - \left( \frac{1}{ac_3} \right)^2 = \text{const}.
\]

It follows that \( \alpha \) is a \( N \)-constant curve in \( G_3^{13} \).

![Figure 2: N−constant Salkowski curve and not a constant − ratio.](image)

## 5 Conclusion

In the three-dimensional pseudo-Galilean space, spacelike admissible curves of constant-ratio and some special curves such as \( T - \text{constant} \) and \( N - \text{constant} \) curves have been studied. Furthermore, the spherical images of these curves are given. Some interesting results of \( N - \text{constant} \) curves are obtained. Finally as an application for this work, two examples have been given and plotted to confirm our main results.
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