An Infinite Number of Static Soliton Solutions to the 5D Einstein-Maxwell Equations with a Dilaton Field

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We study the 5D static Einstein-Maxwell equations with a dilaton field. We develop an infinite number of solutions using a soliton technique. We study the rod structure of a two-soliton solution and show that a 5D dilatonic black ring and black hole solutions are included.

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\section*{§1. Introduction}

Ever since the discovery of the black ring solution by Emparan and Reall,\textsuperscript{1)} much effort has been made to find solutions in five dimensions. 5D black holes exhibit qualitative new properties in contrast with 4D black holes. Emparan and Reall used the C-metric to obtain their particular solution. After that, many solution generating techniques have been developed.\textsuperscript{2)}–\textsuperscript{10)} One of them is the inverse scattering method (ISM),\textsuperscript{11)} and the symmetries of the theory after the dimensional reduction are also used in generating new solutions. One of the authors applied ISM to the 5D static Einstein equation.\textsuperscript{12)} We also applied the method to the 5D stationary Einstein equation.\textsuperscript{13)} The 5D rotating black ring solution is the solution in which the gravity and centrifugal forces cancel out. We expected to develop a solution to the 5D Einstein-Maxwell (EM) equations in which the gravity and electric repulsive forces cancel out in a static case, and found a charged black ring solution with a conical singularity.\textsuperscript{14)} In these series of papers, we have shown an infinite number of solutions that show that these equations are completely integrable systems, as in the 4D case.\textsuperscript{15)}–\textsuperscript{19)} Our conjecture is that black hole solutions are given by soliton solutions also in the 5D case.

The study of the 5D stationary Einstein equations with electric and dilatonic charges, which is inspired from the string theory and brane-world scenario with large extra dimensions, is interesting, but no solution has been found yet. As a first step toward it, we study a 5D static case. The purpose of the present study is to examine the 5D Einstein-Maxwell-dilaton (EMd) equations and show their complete integrability by the explicit development of an infinite number of solutions. We study

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the two-soliton solution in detail to investigate whether or not the solutions include a black ring solution with no conical singularity. We found works \(^9\), \(^20\)–\(^22\) in which the 5D EMd equations are studied and black ring solutions are obtained, but the methods used there are different from our explicit development of an infinite number of solutions.

The reason why we can apply the ISM to the 5D EMd equations is that there is a similarity between the 2D soliton equation and the higher-dimensional Einstein equation with axial symmetry. The higher-dimensional Einstein equation with axial symmetry is expressed in terms of the canonical variables \(\rho\) and \(z\). Both the soliton and the Einstein equations are written as a 2D zero-curvature equation with these variables. When we add the dilaton and electromagnetic fields to the vacuum Einstein equation, the EMd system does not seem to be cast to the 2D zero-curvature equation at first glance, but we are able to derive the 2D zero-curvature equation after introducing new functions. We already used a similar method to study the equilibrium of two dilaton black holes with electric charges in four dimensions.\(^19\)

This paper is constructed as follows. In the following section, we show an infinite number of soliton solutions to the EMd equations by introducing new functions. The new functions are fit to express them as the 2D zero-curvature equation, which enables us to yield soliton solutions. We leave the dilaton coupling strength arbitrary throughout this paper. In \(\S\)3, we discuss the two-soliton solution in detail, which generally corresponds to black ring and/or black hole solutions. We analyse the solution by studying the rod structure of the solution, and show that the two-soliton solution is a black ring solution with electric and dilatonic charges but that there is a conical singularity. By removing this conical singularity, we obtain a 5D dilatonic black hole solution. The last section is devoted to summary and discussion.

\(\S\)2. Solutions of Einstein-Maxwell-dilaton equations

The action of 5D EMd gravity is given by

\[
S = \frac{1}{16\pi} \int d^5x \sqrt{-g} \left( R - \frac{8}{3} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} e^{-\frac{4}{3} \alpha \Phi} F_{\mu\nu} F^{\mu\nu} \right),
\]

(2.1)

and the metric we consider is given by

\[
ds^2 = f(d\rho^2 + dz^2) + g_{ab} dx^a dx^b, \quad (a, b = 0, 1, 2)
\]

(2.2)

where \(f\) and \(g_{ab}\) are functions of \(\rho\) and \(z\). The 5D EMd equations are expressed as

\[
R^\mu_{\nu} = \frac{4}{3} \partial^\mu \Phi \partial_\nu \Phi + 2e^{-\frac{4}{3} \alpha \Phi} \left( F^{\mu\sigma} F_{\nu\sigma} - \frac{1}{6} \delta^\mu_{\nu} F^{\beta\sigma} F_{\beta\sigma} \right),
\]

(2.3)

\[
\nabla^2 \Phi = -\frac{\alpha}{2} e^{-\frac{4}{3} \alpha \Phi} F_{\beta\sigma} F^{\beta\sigma},
\]

(2.4)

\[
(e^{-\frac{4}{3} \alpha \Phi} F^{\mu\nu})_{;\mu} = 0,
\]

(2.5)

with

\[
F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}.
\]

(2.6)
Here, $\alpha$ is a constant. We solve these equations under the static condition with the coordinate condition $\det g = -\rho^2$. We also assume that there is only electric charge. Then, the part of the metric $g = (g_{ab})$ and the $U(1)$ gauge field are assumed to have the following forms:

$$g = \text{diag}\left(-h_1^{-1}h_2^{-1}, \left[\sqrt{\rho^2 + z^2} - z\right] h_1, \left[\sqrt{\rho^2 + z^2} + z\right] h_2\right),$$

$$A_0 = -\chi, \quad A_1 = A_2 = A_\rho = A_z = 0,$$  

where $h_1$ and $h_2$ are functions of $\rho$ and $z$, and $\chi(\rho, z)$ is the electrostatic potential. Then, the EMd equations are explicitly written as

$$[\rho(\ln h_i)_\rho],_\rho + [\rho(\ln h_i),_z] = -\frac{4\rho}{3} h_1 h_2 e^{-\frac{4}{3}\alpha\phi}(\chi_{,\rho}^2 + \chi_{,z}^2), \quad (i = 1, 2) \tag{2.9}$$

$$f(\rho, z) = \frac{\rho}{\sqrt{\rho^2 + z^2}} ([\ln h_1]_\rho - (\ln h_2),_\rho)$$

$$+ \frac{\sqrt{\rho^2 + z^2} + z}{\sqrt{\rho^2 + z^2}} (\ln h_1),_\rho \frac{\sqrt{\rho^2 + z^2} - z}{\sqrt{\rho^2 + z^2}} (\ln h_2),_\rho$$

$$+ \rho [(\ln h_1),_\rho^2 + (\ln h_2),^2 + (\ln h_1),_\rho (\ln h_2),_\rho]$$

$$- \rho [(\ln h_1),_\rho + (\ln h_2),_z + (\ln h_1),_z (\ln h_2),_\rho]$$

$$= -2\rho \left[ h_1 h_2 e^{-\frac{4}{3}\alpha\phi}(\chi_{,\rho}^2 - \chi_{,z}^2) - \frac{2}{3}(\Phi_{,\rho}^2 - \Phi_{,z}^2) \right], \tag{2.10}$$

$$f(\rho, z) + \frac{\rho}{\sqrt{\rho^2 + z^2}} ([\ln h_1],_\rho - (\ln h_2),_\rho)$$

$$+ \frac{\sqrt{\rho^2 + z^2} + z}{\sqrt{\rho^2 + z^2}} (\ln h_1),_z \frac{\sqrt{\rho^2 + z^2} - z}{\sqrt{\rho^2 + z^2}} (\ln h_2),_z$$

$$+ 2\rho [(\ln h_1),_\rho (\ln h_1),_z + (\ln h_2),_\rho (\ln h_2),_z]$$

$$+ \rho [(\ln h_1),_\rho (\ln h_2),_z + (\ln h_1),_z (\ln h_2),_\rho]$$

$$= -4\rho \left( h_1 h_2 e^{-\frac{4}{3}\alpha\phi}(\chi_{,\rho} \chi_{,z} - \frac{2}{3}\Phi,_{\rho} \Phi,_{z}) \right). \tag{2.11}$$

$$\rho(\ln h_1),_\rho + (\rho h_1 h_2 \chi_{,\rho}),_z = \frac{4\rho}{3} \alpha h_1 h_2 (\Phi,_{\rho} \chi_{,\rho} + \Phi,_{z} \chi_{,z}), \tag{2.12}$$

$$\nabla^2 \Phi = \alpha h_1 h_2 e^{-\frac{4}{3} \alpha \phi}(\chi_{,\rho}^2 + \chi_{,z}^2), \tag{2.13}$$

where $\nabla^2 = \partial_{\rho}^2 + \partial_{\rho}/\rho + \partial_{z}^2$. In order to solve these equations, we first solve Eqs. (2.9), (2.12) and (2.13) to obtain $h_1$, $\Phi$ and $\chi$, and then by substituting the solutions into Eqs. (2.10) and (2.11), we integrate them to obtain $f$. The technique entails the introduction of new functions by which the equations become simpler.

Note that Eq. (2.9) is written as

$$\nabla^2 (\ln h_i) = -\frac{4}{3} h_1 h_2 e^{-\frac{4}{3} \alpha \phi}(\chi_{,\rho}^2 + \chi_{,z}^2). \quad (i = 1, 2) \tag{2.14}$$
Then, combining these equations with Eq. (2.13), we obtain

\[
\nabla^2 \left( \ln h_i + \frac{2\alpha}{3} \phi \right) = \frac{2}{3} (\alpha^2 - 2) h_1 h_2 e^{-\frac{4}{3} \alpha \phi (\chi_{\rho}^2 + \chi_{\bar{z}}^2)}, \quad (i = 1, 2)
\]

(2.15)

\[
\nabla^2 \left( \ln h_i - \frac{2\alpha}{3} \phi \right) = -\frac{2}{3} (\alpha^2 + 2) h_1 h_2 e^{-\frac{4}{3} \alpha \phi (\chi_{\rho}^2 + \chi_{\bar{z}}^2)}. \quad (i = 1, 2)
\]

(2.16)

By defining \( H_i \) and \( \bar{H}_i \) by

\[
H_i = h_i e^{\frac{2\alpha}{3} \phi}, \quad \bar{H}_i = h_i e^{-\frac{2\alpha}{3} \phi}, \quad (i = 1, 2)
\]

(2.17)

these equations are rewritten as

\[
\nabla^2 \ln H_i = \frac{2}{3} (\alpha^2 - 2) \bar{H}_1 \bar{H}_2 (\chi_{\rho}^2 + \chi_{\bar{z}}^2), \quad (i = 1, 2)
\]

(2.18)

\[
\nabla^2 \ln \bar{H}_i = -\frac{2}{3} (\alpha^2 + 2) \bar{H}_1 \bar{H}_2 (\chi_{\rho}^2 + \chi_{\bar{z}}^2), \quad (i = 1, 2)
\]

(2.19)

and Eq. (2.12) is expressed as

\[
(\rho \bar{H}_1 \bar{H}_2 \chi_{\rho} z, \rho) + (\rho \bar{H}_1 \bar{H}_2 \chi_{\bar{z}} z, z) = 0.
\]

(2.20)

From Eq. (2.17), we have the relation

\[
\frac{H_1}{H_2} = \frac{\bar{H}_1}{\bar{H}_2} = \frac{h_1}{h_2},
\]

(2.21)

and the equations in Eq. (2.17) are inversely solved to yield the relations

\[
\phi = \frac{3}{4\alpha} (\ln H_i - \ln \bar{H}_i), \quad (i = 1, 2)
\]

(2.22)

\[
\ln h_i = \frac{1}{2} (\ln H_i + \ln \bar{H}_i). \quad (i = 1, 2)
\]

(2.23)

By using these expressions, Eqs. (2.10) and (2.11) are rewritten as

\[
(\ln f)_{\rho} + \frac{\rho}{\rho^2 + z^2} - \frac{1}{4} \left( \frac{\rho}{\sqrt{\rho^2 + z^2}} [(\ln H_1)_{z} - (\ln H_2)_{z}] - \frac{\sqrt{\rho^2 + z^2} + z}{\sqrt{\rho^2 + z^2}} (\ln H_1)_{\rho} + \frac{\sqrt{\rho^2 + z^2} - z}{\sqrt{\rho^2 + z^2}} (\ln H_2)_{\rho} + \frac{\alpha^2 + 2}{2\alpha^2} \rho [(\ln H_1)_{\rho}^2 + (\ln H_2)_{\rho}^2 + (\ln H_1)_{\rho} (\ln H_2)_{\rho} - (\ln H_1)_{z}^2 - (\ln H_2)_{z}^2 - (\ln H_1)_{z} (\ln H_2)_{z}] \right)
\]

\[
- \frac{1}{4} \left( \frac{\rho}{\sqrt{\rho^2 + z^2}} [(\ln H_1)_{z} - (\ln H_2)_{z}] - \frac{\sqrt{\rho^2 + z^2} + z}{\sqrt{\rho^2 + z^2}} (\ln H_1)_{\rho} + \frac{\sqrt{\rho^2 + z^2} - z}{\sqrt{\rho^2 + z^2}} (\ln H_2)_{\rho} + \frac{\alpha^2 + 2}{2\alpha^2} \rho [(\ln H_1)_{\rho}^2 + (\ln H_2)_{\rho}^2 + (\ln H_1)_{\rho} (\ln H_2)_{\rho} - (\ln H_1)_{z}^2 - (\ln H_2)_{z}^2 - (\ln H_1)_{z} (\ln H_2)_{z}] \right)
\]

\[
+ \frac{\sqrt{\rho^2 + z^2} + z}{\sqrt{\rho^2 + z^2}} (\ln H_1)_{\rho} + \frac{\sqrt{\rho^2 + z^2} - z}{\sqrt{\rho^2 + z^2}} (\ln H_2)_{\rho}
\]
with the equation

\[ \nabla^2 \ln N_i = 0. \quad (i = 1, 2) \] 

(2.27)

Substituting Eq. (2.26) into Eqs. (2.24) and (2.25), we have

\[
(\ln f)_{\rho} + \frac{\rho}{\rho^2 + z^2} - \frac{1}{\alpha^2 + 2} \left( \frac{\rho}{\sqrt{\rho^2 + z^2}} [(\ln \tilde{H}_1)_{\rho} - (\ln \tilde{H}_2)_{\rho}] \right)
\]
In order to solve these equations, we introduce \( f_N \) and \( f_{em} \) by

\[
f = \sqrt{f_N \cdot f_{em}},
\]

and require that \( f_N \) should satisfy

\[
\begin{align*}
\langle \ln f_N \rangle_{\rho} + \frac{\rho}{\rho^2 + z^2} & - \frac{1}{2} \left( \frac{\rho}{\sqrt{\rho^2 + z^2}} \langle (\ln N_1)_{\rho} \rangle_{z} - \langle (\ln N_2)_{\rho} \rangle_{z} \right) \\
& + \frac{\sqrt{\rho^2 + z^2 + z}}{\sqrt{\rho^2 + z^2}} \langle \ln H_1 \rangle_{z} + \frac{\sqrt{\rho^2 + z^2 - z}}{\sqrt{\rho^2 + z^2}} \langle \ln H_2 \rangle_{z} \\
& + \rho \left[ 2 \langle (\ln H_1)_{\rho} \rangle_{z} + 2 \langle (\ln H_2)_{\rho} \rangle_{z} \right]
\end{align*}
\]

\[
= -4 \rho \bar{H}_1 \bar{H}_2 \chi_{\rho} \chi_{z}.
\]
\[ + \frac{\alpha^2 + 2}{2\alpha^2} \rho \left[ (\ln N_1)_\rho \rho + (\ln N_2)_\rho \rho + (\ln N_1)_\rho (\ln N_2)_\rho \right] - (\ln N_1)_z^2 - (\ln N_2)_z^2 - (\ln N_1)_z (\ln N_2)_z \right] = 0, \quad (2.31) \]

\[ (\ln f_N)_z + \frac{z}{\rho^2 + z^2} - \frac{1}{2} \left( - \frac{\rho}{\sqrt{\rho^2 + z^2}} [(\ln N_1)_\rho - (\ln N_2)_\rho] \right. \]
\[ \left. + \frac{\sqrt{\rho^2 + z^2} + z}{\sqrt{\rho^2 + z^2}} (\ln H_1)_z + \frac{\sqrt{\rho^2 + z^2} - z}{\sqrt{\rho^2 + z^2}} (\ln H_2)_z \right. \]
\[ \left. + \rho \left[ (\ln H_1)_\rho (\ln H_2)_\rho + (\ln H_1)_\rho (\ln H_2)_\rho \right] - (\ln H_1)_z^2 - (\ln H_2)_z^2 - (\ln H_1)_z (\ln H_2)_z \right] \]
\[ = -4\rho H_1 H_2 (\chi^2 - \chi z), \quad (2.33) \]

\[ (\ln f_{em})_z + \frac{z}{\rho^2 + z^2} - \frac{2}{\alpha^2 + 2} \left( - \frac{\rho}{\sqrt{\rho^2 + z^2}} [(\ln H_1)_\rho - (\ln H_2)_\rho] \right. \]
\[ \left. + \frac{\sqrt{\rho^2 + z^2} + z}{\sqrt{\rho^2 + z^2}} (\ln H_1)_z + \frac{\sqrt{\rho^2 + z^2} - z}{\sqrt{\rho^2 + z^2}} (\ln H_2)_z \right. \]
\[ \left. + \rho \left[ 2(\ln H_1)_\rho (\ln H_1)_z + 2(\ln H_2)_\rho (\ln H_2)_z \right] + (\ln H_1)_\rho (\ln H_2)_z + (\ln H_1)_z (\ln H_2)_\rho \right] = -8\rho H_1 H_2 \chi \rho \chi z, \quad (2.34) \]

which have similar forms to those of the corresponding equations in the 5D static vacuum case. Then, \( f_{em} \) should satisfy

In order to solve Eqs. (2.19) and (2.20), we assume that the functions \( \tilde{H}_1 \) and \( \tilde{H}_2 \) have the forms

\[ \tilde{H}_1 = \left[ 1 - \frac{4}{3} (\alpha^2 + 2) c\chi + \frac{2}{3} (\alpha^2 + 2) \chi^2 \right]^{-1/2} N^{1/2}, \quad (2.35) \]
\[ \tilde{H}_2 = \left[ 1 - \frac{4}{3} (\alpha^2 + 2) c\chi + \frac{2}{3} (\alpha^2 + 2) \chi^2 \right]^{-1/2} N^{-1/2}, \quad (2.36) \]
where \( N \) is a function of \( \rho \) and \( z \), and \( c \) is a constant. Then, Eqs. (2.19) and (2.20) are both put into the form

\[
\nabla^2 \chi = \frac{4(\alpha^2 + 2)(\chi - c)}{3 - 4(\alpha^2 + 2)c\chi + 2(\alpha^2 + 2)\chi^2(\chi_{,\rho}^2 + \chi_{,z}^2)},
\]

(2.37)

with the Laplace equation for \( \ln N \) expressed as

\[
\nabla^2 \ln N = 0.
\]

(2.38)

Next, we introduce a new function \( R(\rho, z) \) through the relation

\[
\chi = \frac{e}{R + m},
\]

(2.39)

where \( e \) and \( m \) are constants satisfying

\[
m = \frac{2(\alpha^2 + 2)}{3}ce.
\]

(2.40)

Then, Eqs. (2.35) and (2.36) lead to

\[
\bar{H}_1 = \frac{m + R}{\sqrt{R^2 - d^2}}N^{1/2},
\]

(2.41)

\[
\bar{H}_2 = \frac{m + R}{\sqrt{R^2 - d^2}}N^{-1/2},
\]

(2.42)

where

\[
d = \sqrt{\frac{3m^2 - 2(\alpha^2 + 2)e^2}{3}}.
\]

(2.43)

Equation (2.37) can also be expressed in terms of \( R \) as

\[
\nabla^2 R = 2R(R^2 - d^2)^{-1}(R_{,\rho}^2 + R_{,z}^2).
\]

(2.44)

Then, introducing the function \( h(\rho, z) \) through the relation

\[
R = d \frac{1 + h}{1 - h},
\]

(2.45)

we obtain the Laplace equation for \( \ln h \) as

\[
\nabla^2 \ln h = 0.
\]

(2.46)

Note that we can construct the solution to Eq. (2.37) for the electric potential \( \chi \) via \( R \) given in Eq. (2.45). We rewrite Eqs. (2.33) and (2.34) in terms of \( \chi \) and \( N \). Equation (2.33) is written as

\[
(\ln f_{em})_{,\rho} = -\frac{\rho}{\rho^2 + z^2} - \frac{8(\chi - c)\chi_{,\rho}}{3 - 4(\alpha^2 + 2)c\chi + 2(\alpha^2 + 2)\chi^2}.
\]
Eqs. (2.47) and (2.48) are respectively expressed as

\[ \frac{12\rho [2(\alpha^2 + 2)c^2 - 3](\chi_{,\rho}^2 - \chi_{,z}^2)}{[3 - 4(\alpha^2 + 2)c\chi + 2(\alpha^2 + 2)\chi^2]^2} \]

\[ + \frac{2}{\alpha^2 + 2} \left( \frac{z(\ln N)_{,\rho} + \rho(\ln N)_{,z}}{\sqrt{\rho^2 + z^2}} + \frac{\rho}{4} [(\ln N)_{,\rho}^2 - (\ln N)_{,z}^2] \right), \tag{2.47} \]

and Eq. (2.34) as

\[ (\ln f_{em})_{,z} = -\frac{z}{\rho^2 + z^2} - \frac{8(\chi - c)\chi_{,z}}{3 - 4(\alpha^2 + 2)c\chi + 2(\alpha^2 + 2)\chi^2} \]

\[ + \frac{24\rho [2(\alpha^2 + 2)c^2 - 3](\chi_{,\rho}^2 - \chi_{,z}^2)}{[3 - 4(\alpha^2 + 2)c\chi + 2(\alpha^2 + 2)\chi^2]^2} \]

\[ + \frac{2}{\alpha^2 + 2} \left( -\frac{\rho(\ln N)_{,\rho} - z(\ln N)_{,z}}{\sqrt{\rho^2 + z^2}} + \frac{\rho}{2} (\ln N)_{,\rho} (\ln N)_{,z} \right). \tag{2.48} \]

By further introducing \( Q_{em}(\rho, z) \) through

\[ f_{em} = \frac{1}{2\sqrt{\rho^2 + z^2}} \left[ 1 - \frac{4}{3}(\alpha^2 + 2)c\chi + \frac{2}{3}(\alpha^2 + 2)\chi^2 \right]^{-2/(\alpha^2+2)} Q_{em}, \tag{2.49} \]

Eqs. (2.47) and (2.48) are respectively expressed as

\[ (\ln Q_{em})_{,\rho} = \frac{2}{\alpha^2 + 2} \left( \frac{3\rho}{4} [(\ln h)_{,\rho}^2 - (\ln h)_{,z}^2] + \frac{\rho}{4} [(\ln N)_{,\rho}^2 - (\ln N)_{,z}^2] \right. \]

\[ + \left. \frac{z(\ln N)_{,\rho} + \rho(\ln N)_{,z}}{\sqrt{\rho^2 + z^2}} \right) \tag{2.50} \]

and

\[ (\ln Q_{em})_{,z} = \frac{2}{\alpha^2 + 2} \left( \frac{3\rho}{2} (\ln h)_{,\rho} (\ln h)_{,z} + \frac{\rho}{2} (\ln N)_{,\rho} (\ln N)_{,z} \right. \]

\[ - \left. \frac{\rho(\ln N)_{,\rho} - z(\ln N)_{,z}}{\sqrt{\rho^2 + z^2}} \right). \tag{2.51} \]

We next rewrite Eqs. (2.31) and (2.32) in terms of \( N \) and \( N_2 \). We rewrite these equations by introducing \( Q_N(\rho, z) \) through

\[ f_N = \frac{1}{2\sqrt{\rho^2 + z^2}} Q_N, \tag{2.52} \]

and by taking account of the relation

\[ \frac{N_1}{N_2} = N^{2\alpha^2/(\alpha^2+2)}, \tag{2.53} \]

Eqs. (2.31) and (2.32) are reduced to

\[ (\ln Q_N)_{,\rho} = (\ln N_2)_{,\rho} + \frac{3(\alpha^2 + 2)\rho}{4\alpha^2} [(\ln N_2)_{,\rho}^2 - (\ln N_2)_{,z}^2] \]
\[
\begin{align*}
&\frac{3\rho}{2} [(\ln N_2),_\rho (\ln N),_\rho - (\ln N_2),_z (\ln N),_z ] \\
&+ \frac{\alpha^2}{\alpha^2 + 2} \left( (\ln N),_\rho + \frac{z(\ln N),_\rho + \rho(\ln N),_z}{\sqrt{\rho^2 + z^2}} + \rho \left[ (\ln N),^2_\rho - (\ln N),^2_z \right] \right) ,
\end{align*}
\]
(2.54)

\[
(\ln Q_N),_z = (\ln N_2),_z + \frac{3(\alpha^2 + 2)\rho}{2\alpha^2} (\ln N_2),_\rho (\ln N_2),_z \\
+ \frac{3\rho}{2} [(\ln N_2),_\rho (\ln N),_z + (\ln N_2),_z (\ln N),_z ] \\
+ \frac{\alpha^2}{\alpha^2 + 2} \left( (\ln N),_z - \frac{\rho(\ln N),_\rho - z(\ln N),_z}{\sqrt{\rho^2 + z^2}} + 2\rho(\ln N),_\rho (\ln N),_z \right) .
\]
(2.55)

As is seen from Eqs. (2.27), (2.38) and (2.46), \(N_2\), \(N\) and \(h\) all satisfy the Laplace equation. When we restrict ourselves only to the soliton solutions, they can have solutions with different soliton numbers in general. However, they have too many parameters to analyse their structure. In order to clarify the structure of the solutions, we simplify the solution by assuming the ansatz that \(\ln N_2\) and \(\ln N\) can be expressed in terms of \(\ln h\):

\[
\ln N_2 = \frac{2\alpha^2 k_1}{\alpha^2 + 2} \ln h,
\]
(2.56)

\[
\ln N = k_2 \ln h,
\]
(2.57)

where \(k_1\) and \(k_2\) are constants. Then, the functions \(h_1\) and \(h_2\) in Eq. (2.23) are written in terms of \(h\) through Eqs. (2.26), (2.41), (2.42) and (2.53) as

\[
h_1 = \left[ \frac{d(1 + h) + m(1 - h)}{2d} \right]^{2/\alpha^2 + 2} h^{\alpha^2 k_1 + (\alpha^2 + 1)k_2 - 1}/(\alpha^2 + 2),
\]
(2.58)

\[
h_2 = \left[ \frac{d(1 + h) + m(1 - h)}{2d} \right]^{2/\alpha^2 + 2} h^{\alpha^2 k_1 - k_2 - 1}/(\alpha^2 + 2).
\]
(2.59)

From Eqs. (2.30), (2.49) and (2.52), we find that \(f\) is now expressed as

\[
f = \frac{1}{2\sqrt{\rho^2 + z^2}} \left[ 1 - \frac{4}{3} (\alpha^2 + 2) c\chi + \frac{2}{3} (\alpha^2 + 2) \chi^2 \right]^{-1/(\alpha^2 + 2)} \sqrt{Q_N Q_{em}}.
\]
(2.60)

The quantity \(Q\) defined by \(Q = \sqrt{Q_N Q_{em}}\) is simply expressed using the above ansatz as

\[
(\ln Q),_\rho = \frac{\alpha^2 (2k_1 + k_2)}{2(\alpha^2 + 2)} (\ln h),_\rho + \frac{\rho^2}{2} [\ln h],^2_\rho - (\ln h),^2_z \\
+ \frac{k_2}{2\sqrt{\rho^2 + z^2}} [z(\ln h),_\rho + \rho(\ln h),_z ],
\]
(2.61)

\[
(\ln Q),_z = \frac{\alpha^2 (2k_1 + k_2)}{2(\alpha^2 + 2)} (\ln h),_z + \rho \nu (\ln h),_\rho (\ln h),_z \\
- \frac{k_2}{2\sqrt{\rho^2 + z^2}} [\rho(\ln h),_\rho - z(\ln h),_z ],
\]
(2.62)
where

$$\nu = \frac{1}{\alpha^2 + 2} \left( \frac{3}{2} + 3\alpha^2 k_1^2 + 3\alpha^2 k_1 k_2 + \frac{2\alpha^2 + 1}{2} k_2^2 \right).$$

(2.63)

The $n$-soliton solution for $h$ is given by

$$h = \frac{\prod^n_k(i\mu_k)}{\rho^n},$$

(2.64)

with the pole trajectories

$$\mu_k = w_k - z + (-1)^{k-1} \sqrt{(w_k - z)^2 + \rho^2}, \quad (k = 1, 2, \cdots, n)$$

(2.65)

where $w_k (k = 1, 2, \cdots, n)$ are constants. Using the formulas

$$\frac{\rho^2((\ln h)_\rho)^2 - (\ln h)_z^2}{2\sqrt{\rho^2 + z^2}} = \frac{\partial}{\partial \rho} \ln \left[ \frac{\rho^{n/2} \prod (\mu_k - \mu)_{\nu}^2 \prod_{k > l} (\mu_k - \mu_l)^2 \sqrt{\rho^2 + z^2 + z + \mu}}{\prod_k (\sqrt{\rho^2 + z^2 + z + \mu_k})^k \prod_k (\mu_k^2 + \rho^2)^{\nu}} \right],$$

(2.66)

$$\frac{(z(\ln h)_\rho + \rho(\ln h)_z)}{2\sqrt{\rho^2 + z^2}} = \frac{\partial}{\partial \rho} \ln \left[ \frac{(\sqrt{\rho^2 + z^2 + z})^n} {\prod_k (\sqrt{\rho^2 + z^2 + z + \mu_k})^k} \right],$$

(2.67)

we obtain

$$Q = C^{(n)} \frac{\rho^n \prod_k^n(i\mu_k)^{\nu^2} \prod_{k > l}^n(\mu_k - \mu_l)^{2\nu} \sqrt{\rho^2 + z^2 + z} n_{k_2/2}}{\prod_k^k \prod_{k > l}^k (\mu_k^2 + \rho^2)^{\nu}},$$

(2.68)

where $C^{(n)}$ is a constant to be determined by the asymptotic flatness condition and

$$\nu_1 = \frac{n^2}{2\nu} - \frac{n}{\alpha^2 + 2}[2\alpha^2 k_1 + (\alpha^2 + 1)k_2],$$

(2.69)

$$\nu_2 = -(n - 2)\nu + \frac{1}{\alpha^2 + 2}[2\alpha^2 k_1 + (\alpha^2 + 1)k_2].$$

(2.70)

Here, we summarize our result. The metric components are given by

$$g_{00} = -K^{-1}h - (2\alpha^2 k_1 + \alpha^2 k_2 - 2)/(\alpha^2 + 2),$$

(2.71)

$$g_{11} = (\sqrt{\rho^2 + z^2 - z})\sqrt{K} h^{[\alpha^2 k_1 + (\alpha^2 + 1)k_2 - 1]/(\alpha^2 + 2)},$$

(2.72)

$$g_{22} = -(\sqrt{\rho^2 + z^2 + z})\sqrt{K} h^{(\alpha^2 k_1 - k_2 - 1)/(\alpha^2 + 2)},$$

(2.73)

$$g_{pp} = g_{zz} = \frac{C^{(n)}}{2\sqrt{\rho^2 + z^2}} \frac{\sqrt{K}}{\prod_k^n(i\mu_k)^{\nu^2} \prod_{k > l}^n (\mu_k - \mu_l)^{2\nu} \sqrt{\rho^2 + z^2 + z} n_{k_2/2}} \prod_k^k \prod_{k > l}^k (\mu_k^2 + \rho^2)^{\nu},$$

(2.74)

and the dilaton field and electric potential are respectively given by

$$\Phi = -\frac{3\alpha}{8} \ln K + \frac{3\alpha(1 + 2k_1 + k_2)}{4(\alpha^2 + 2)} \ln h,$$

(2.75)

$$\chi = \frac{e(1 - h)}{2d} K^{-(\alpha^2 + 2)/4},$$

(2.76)
where $K$ is given by

$$K = \left[ \frac{d(1 + h) + m(1 - h)}{2d} \right]^{4/(\alpha^2 + 2)}, \quad (2.77)$$

and $h$ is given by Eq. (2.64).

We illustrate the solution by taking the $n = 2$ or two-soliton solution case. In this case, $h$ and $Q$ are given by

$$h = -\frac{\mu_1\mu_2}{\rho^2}, \quad (2.78)$$

$$Q = \left[ \frac{2(z_0 + d)(\sqrt{\rho^2 + z^2 + z})}{(\sqrt{\rho^2 + z^2 + z + \mu_1})(\sqrt{\rho^2 + z^2 + z + \mu_2})} \right]^{k_2} \times \left( \frac{\mu_1\mu_2}{\rho^2} \right)^{(\alpha^2 k_1 + (\alpha^2 + 1)k_2)/\alpha^2 + 2} \left[ \frac{\rho^2(\mu_2 - \mu_1)^2}{(\mu_1^2 + \rho^2)(\mu_2^2 + \rho^2)} \right]^\nu, \quad (2.79)$$

where we have put $w_1 = z_0 - d$ and $w_2 = z_0 + d$, and so $\mu_1$ and $\mu_2$ are given by

$$\begin{cases} 
\mu_1 = z_0 - z - d + \sqrt{(z_0 - z - d)^2 + \rho^2}, \\
\mu_2 = z_0 - z + d - \sqrt{(z_0 - z + d)^2 + \rho^2}. 
\end{cases} \quad (2.80)$$

By transforming the variable $z$ to $z + z_0$, we rewrite these poles as

$$\begin{cases} 
\mu_1 = -(z + d) + \sqrt{(z + d)^2 + \rho^2}, \\
\mu_2 = -(z - d) - \sqrt{(z - d)^2 + \rho^2}, 
\end{cases} \quad (2.81)$$

and define new poles by

$$\begin{cases} 
\mu = -(z + z_0) - \sqrt{(z + z_0)^2 + \rho^2}, \\
\mu^* = -(z + z_0) + \sqrt{(z + z_0)^2 + \rho^2}. 
\end{cases} \quad (2.82)$$

With these expressions, the two-soliton solution is given by

$$g_{00} = -K^{-1}h^{-2(\alpha^2 k_1 + \alpha^2 k_2 - 2)/\alpha^2 + 2}, \quad (2.83)$$

$$g_{11} = \mu^*\sqrt{K}h[(\alpha^2 k_1 + (\alpha^2 + 1)k_2 - 1)/\alpha^2 + 2], \quad (2.84)$$

$$g_{22} = -\mu\sqrt{K}h[(\alpha^2 k_1 - k_2 - 1)/\alpha^2 + 2], \quad (2.85)$$

$$f = \frac{1}{2\sqrt{(z + z_0)^2 + \rho^2}} \sqrt{K} \times \left[ \frac{\rho^2(\mu_2 - \mu_1)^2}{(\mu_1 - \mu)(\mu_2 - \mu)} \right]^{k_2} \left[ \frac{\rho^2(\mu_2 - \mu_1)^2}{(\mu_1^2 + \rho^2)(\mu_2^2 + \rho^2)} \right]^\nu. \quad (2.86)$$

We note that $\mu$ and $\mu^*$ have the same forms as the poles $\mu_1$ and $\mu_2$. Therefore, by regarding them as poles at different positions, we note that the metric components are completely expressed in terms of these poles. In the following section, we analyse the rod structure of this two-soliton solution. By requiring the horizon to exist, we determine the parameters $k_1$ and $k_2$. 


§3. Analysis of solutions

We now study the $\rho \sim 0$ behaviors of the metric components, or the rod structure of the two-soliton solution obtained in the previous section. We divide the $z$-axis into four intervals separated by $z = -z_0$ and $z = \pm d$ and assume that these parameters are ordered as follows, without loss of generality:

$$-\infty < -d < d < -z_0 < \infty. \quad (3.1)$$

If $g_{ii} \propto \rho^2 \ (i = 1, 2)$ in some intervals, the rod is spacelike and in the $\partial / \partial x^i$ direction, and the $\rho$-$i$ plane is perpendicular to the $z$-axis. Going around the $z$-axis in the $i$-th direction, we study the period of the coordinate $x^i$. The behavior of the metric components for $\rho \sim 0$ in each region is as follows:

(i) $\infty > z > -z_0$

$$g_{00} \sim -\left(\frac{z + m}{z + d}\right)^{-4/(\alpha^2 + 2)} \left(\frac{z - d}{z + d}\right)^{-2(\alpha^2 k_1 + \alpha^2 k_2 - 2)/(\alpha^2 + 2)}, \quad (3.2)$$

$$g_{11} \sim \frac{\rho^2}{2(z + z_0)} \left(\frac{z + m}{z + d}\right)^{2/(\alpha^2 + 2)} \left(\frac{z - d}{z + d}\right)^{[\alpha^2 k_1 + (\alpha^2 + 1) k_2 - 1]/(\alpha^2 + 2)}, \quad (3.3)$$

$$g_{22} \sim 2(z + z_0) \left(\frac{z + m}{z + d}\right)^{2/(\alpha^2 + 2)} \left(\frac{z - d}{z + d}\right)^{[\alpha^2 k_1 - k_2 - 1]/(\alpha^2 + 2)}, \quad (3.4)$$

$$f \sim \frac{1}{2(z + z_0)} \left(\frac{z + m}{z + d}\right)^{2/(\alpha^2 + 2)} \left(\frac{z - d}{z + d}\right)^{[\alpha^2 k_1 + (\alpha^2 + 1) k_2 - 1]/(\alpha^2 + 2)}. \quad (3.5)$$

In this interval, we note that $g_{11} \sim \mathcal{O}(\rho^2)$. Since the rod is in the $\partial / \partial x^1$ direction, we study the period $\Delta x^1$ to find that

$$\Delta x^1 = 2\pi \lim_{\rho \to 0} \sqrt{\frac{\rho^2 f}{g_{11}}} = 2\pi, \quad (3.6)$$

which shows no conical singularity in this interval.

(ii) $-z_0 > z > d$

$$g_{00} \sim -\left(\frac{z + m}{z + d}\right)^{-4/(\alpha^2 + 2)} \left(\frac{z - d}{z + d}\right)^{-2(\alpha^2 k_1 + \alpha^2 k_2 - 2)/(\alpha^2 + 2)}, \quad (3.7)$$

$$g_{11} \sim -2(z + z_0) \left(\frac{z + m}{z + d}\right)^{2/(\alpha^2 + 2)} \left(\frac{z - d}{z + d}\right)^{[\alpha^2 k_1 + (\alpha^2 + 1) k_2 - 1]/(\alpha^2 + 2)}, \quad (3.8)$$

$$g_{22} \sim -\frac{\rho^2}{2(z + z_0)} \left(\frac{z + m}{z + d}\right)^{2/(\alpha^2 + 2)} \left(\frac{z - d}{z + d}\right)^{[\alpha^2 k_1 - k_2 - 1]/(\alpha^2 + 2)}, \quad (3.9)$$

$$f \sim -\frac{1}{2(z + z_0)} \left(\frac{z + m}{z + d}\right)^{2/(\alpha^2 + 2)} \left(\frac{z - d}{z + d}\right)^{[\alpha^2 k_1 + (\alpha^2 + 1) k_2 - 1]/(\alpha^2 + 2)} \times \left[\frac{(z + d)(z_0 + d)}{(z - d)(z_0 - d)}\right]^{k_2}. \quad (3.10)$$
In this interval, we note that \( g_{22} \sim \mathcal{O}(\rho^2) \), and the rod is in the \( \partial/\partial x^2 \) direction. Therefore, we study the period \( \Delta x^2 \) to find that

\[
\Delta x^2 = 2\pi \lim_{\rho \to 0} \sqrt{\frac{\rho^2 f}{g_{22}}} = 2\pi \left( \frac{z_0 + d}{z_0 - d} \right)^{k_2/2},
\]

which shows that the parameters should be adjusted to avoid the conical singularity.

(iii) \( d > z > -d \)

\[
g_{00} \sim -\left( \frac{d + m}{2d} \right)^{-4/(\alpha^2 + 2)} \left[ 4(d^2 - z^2) \right]^{(2\alpha^2 k_1 + \alpha^2 k_2 - 2)/(\alpha^2 + 2)} \rho^{-2(2\alpha^2 k_1 + \alpha^2 k_2 - 2)/(\alpha^2 + 2)},
\]

\[
g_{11} \sim -2(z + z_0) \left( \frac{d + m}{2d} \right)^{2/(\alpha^2 + 2)} \left[ 4(d^2 - z^2) \right]^{-[\alpha^2 k_1 + (\alpha^2 + 1) k_2 - 1]/(\alpha^2 + 2)} \rho^{2(\alpha^2 k_1 + (\alpha^2 + 1) k_2 - 1)/(\alpha^2 + 2)},
\]

\[
g_{22} \sim -\frac{1}{2(z + z_0)} \left( \frac{d + m}{2d} \right)^{2/(\alpha^2 + 2)} \left[ 4(d^2 - z^2) \right]^{-\alpha^2 k_1 - k_2 - 1}/(\alpha^2 + 2) \rho^{2(\alpha^2 k_1 - k_2 - 1)/(\alpha^2 + 2) + 2\nu},
\]

\[
f \sim -\frac{1}{2(z + z_0)} \left( \frac{d + m}{2d} \right)^{2/(\alpha^2 + 2)} \left[ 4(d^2 - z^2) \right]^{-\alpha^2 k_1 + (\alpha^2 + 1) k_2 - 1}/(\alpha^2 + 2) \rho^{2(\alpha^2 k_1 - k_2 - 1)/(\alpha^2 + 2) + 2\nu}.
\]

The behavior in this interval depends on \( \alpha, k_1 \) and \( k_2 \).

(iv) \(-d > z > -\infty\)

\[
g_{00} \sim -\left( \frac{z - m}{z - d} \right)^{-4/(\alpha^2 + 2)} \left( \frac{z + d}{z - d} \right)^{-2\alpha^2 k_1 + \alpha^2 k_2 - 2)/(\alpha^2 + 2)},
\]

\[
g_{11} \sim -2(z + z_0) \left( \frac{z - m}{z - d} \right)^{2/(\alpha^2 + 2)} \left( \frac{z + d}{z - d} \right)^{[\alpha^2 k_1 + (\alpha^2 + 1) k_2 - 1]/(\alpha^2 + 2)}
\]

\[
g_{22} \sim -\rho^2 \left( \frac{z - m}{z - d} \right)^{2/(\alpha^2 + 2)} \left( \frac{z + d}{z - d} \right)^{(\alpha^2 k_1 - k_2 - 1)/(\alpha^2 + 2)}
\]

\[
f \sim -\frac{1}{2(z + z_0)} \left( \frac{z - m}{z - d} \right)^{2/(\alpha^2 + 2)} \left( \frac{z + d}{z - d} \right)^{(\alpha^2 k_1 - k_2 - 1)/(\alpha^2 + 2)}
\]

In this interval, we note that \( g_{22} \sim \mathcal{O}(\rho^2) \) as in the interval (ii), and the period \( \Delta x^2 \) is given by

\[
\Delta x^2 = 2\pi \lim_{\rho \to 0} \sqrt{\frac{\rho^2 f}{g_{22}}} = 2\pi,
\]

which shows that there is no conical singularity.
We first determine \( k_1 \) and \( k_2 \) by limiting ourselves to finding only physically interesting solutions. We investigate the black hole solutions where the metric components should behave in the interval (iii) as \( g_{00} \sim \rho^2, \ g_{11} \sim \rho^0, \ g_{22} \sim \rho^0, \ f \sim \rho^0 \). This turns out to require that

\[
\begin{align*}
\alpha^2(2k_1 + k_2 + 1) &= 0, \\
\alpha^2k_1 + (\alpha^2 + 1)k_2 - 1 &= 0, \\
\alpha^2k_1 - k_2 + \alpha^2 + 1 &= 0,
\end{align*}
\]

which determine the parameters as

\[
\{\alpha^2 = 0, \ k_2 = 1\} \text{ or } \{k_1 = -1, \ k_2 = 1\}. \tag{3.24}
\]

Note that the solution with \( \alpha^2 = 0 \) is reduced to the 5D EM solution. We fix \( k_1 \) and \( k_2 \) as \( k_1 = -1 \) and \( k_2 = 1 \). Then, the rod in \([-d, \ d]\) represents the event horizon and is sandwiched by the rods in the same \( \partial/\partial x^2 \) direction. This shows that the topology of the event horizon is \( S^2 \times S^1 \) as in the static black ring and static charged black ring cases. Therefore, the solution in Eqs. (2.83)–(2.86) with \( k_1 = -1 \) and \( k_2 = 1 \) is a black ring solution with electric and dilatonic charges. The solution can be written as

\[
\begin{align*}
g_{00} &= -K^{-1}h, \\
g_{11} &= \mu^*\sqrt{K}, \\
g_{22} &= -\mu\sqrt{Kh}^{-1}, \\
f &= -\frac{(z_0 + d)\sqrt{K}\mu\rho^2(\mu_2 - \mu_1)^2}{\sqrt{(z + z_0)^2 + \rho^2(\mu_1 - \mu)(\mu_2 - \mu)(\mu_1^2 + \rho^2)(\mu_2^2 + \rho^2)}},
\end{align*}
\]

and the dilaton fields and electric potential are given by

\[
\begin{align*}
\Phi &= -\frac{3\alpha}{8} \ln K, \\
\chi &= \frac{e}{2d}(1 - h)K^{-(\alpha^2 + 2)/4}.
\end{align*}
\]

Note that when \( m = d \) and accordingly \( e = 0 \) and \( K = 1 \), the solution is reduced to the static black ring solution.

In order to elucidate the structure of the solution given here, we introduce the prolate spheroidal coordinates \( x \) and \( y \) by

\[
\begin{align*}
\rho &= d\sqrt{(x^2 - 1)(1 - y^2)}, \\
z &= dx y.
\end{align*}
\]

In these coordinates, we have the expressions

\[
\begin{align*}
h &= \frac{x - 1}{x + 1}, \\
K &= \left[ \frac{dx + m}{d(x + 1)} \right]^{4/(\alpha^2 + 2)},
\end{align*}
\]
\[ \mu_1 = d(x - 1)(1 - y), \quad (3.35) \]
\[ \mu_2 = -d(x - 1)(1 + y), \quad (3.36) \]
\[ \mu = -(dxy + z_0) - \sqrt{(dxy + z_0)^2 + d^2(x^2 - 1)(1 - y^2)}, \quad (3.37) \]
\[ \mu^* = -(dxy + z_0) + \sqrt{(dxy + z_0)^2 + d^2(x^2 - 1)(1 - y^2)}. \quad (3.38) \]

Using these quantities, we can rewrite the metric components \( g_{00}, g_{11} \) and \( g_{22} \) in terms of \( x \) and \( y \); we obtain the components \( g_{xx} \) and \( g_{yy} \) in the metric as follows:

\[ g_{xx} = \frac{-d^2(z_0 + d)\sqrt{K\mu}}{\sqrt{(dxy + z_0)^2 + d^2(x^2 - 1)(1 - y^2)(\mu_1 - \mu)(\mu_2 - \mu)}}, \quad (3.39) \]
\[ g_{yy} = \frac{-d^2(z_0 + d)\sqrt{K\mu(x^2 - 1)}}{\sqrt{(dxy + z_0)^2 + d^2(x^2 - 1)(1 - y^2)(\mu_1 - \mu)(\mu_2 - \mu)(1 - y^2)}}. \quad (3.40) \]

The point \( z = -z_0 \) on the \( z \)-axis in the coordinates \((\rho, z)\) corresponds to \((-z_0/d, 1)\) or \((z_0/d, -1)\) in the coordinates \((x, y)\). The horizon appearing in the interval \([-d, d]\) on the \( z \)-axis exists at the region specified by \( x = 1 \) and \(-1 < y < 1\) in \((x, y)\). The region specified by \(-1 < x < 1\) and \(-1 < y < 1\), which cannot be expressed by the coordinates \((\rho, z)\), represents the inside of the horizon. We find that a curvature singularity lies at \( x = -1 \), which can be shown from the curvature invariant behaving as

\[ R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \sim \text{constant} \times \frac{z_0 - dy}{(x + 1)^6}. \quad (3.41) \]

This curvature singularity is covered by the horizon of a dilaton black ring.

We next study the asymptotic behaviors of the metric components, dilaton fields and electric potential. Using

\[ h \sim 1 - \frac{2d}{\sqrt{\rho^2 + z^2}}, \quad (3.42) \]
\[ K \sim 1 + \frac{4}{\alpha^2 + 2\sqrt{\rho^2 + z^2}} \frac{m - d}{m}, \quad (3.43) \]

Fig. 1. The horizon and curvature singularity are depicted in the \(x-y\) plane. The hatched region characterized by \(-1 < x < 1\) and \(-1 < y < 1\) is the inside of the horizon at \( x = 1 \).
as $\sqrt{\rho^2 + z^2} \to \infty$, we have

$$
\begin{align*}
g_{00} & \sim -1, \\
g_{11} & \sim -z + \sqrt{\rho^2 + z^2}, \\
g_{22} & \sim z + \sqrt{\rho^2 + z^2}, \\
f & \sim \frac{1}{2\sqrt{\rho^2 + z^2}},
\end{align*}
$$

(3.44)

which shows that the metric becomes asymptotically flat, and that the fields approach

$$
\Phi \sim -\frac{3\alpha}{2(\alpha^2 + 2)} \frac{m - \mu}{\sqrt{\rho^2 + z^2}},
$$

(3.45)

$$
\chi \sim \frac{e}{\sqrt{\rho^2 + z^2}}.
$$

(3.46)

From Eq. (3.11), we find that there is a conical singularity in $[d, -z_0]$. Two methods can be used to get rid of this conical singularity. The first is to set $d = -z_0$, which gets rid of the interval itself. The second is to make $\Delta x^2 = 2\pi$ by imposing $d = 0$. We first compute the $d = -z_0$ case. In this case, the metric components $g_{00}, g_{11}$ and $g_{22}$ are given by the same forms as in Eqs. (3.25)–(3.27) with

$$
h = -\frac{\mu_1 \mu}{\rho^2},
$$

(3.47)

and the component $f$ is given by

$$
f = \frac{\sqrt{K}\rho^2(\mu_1 - \mu)}{(\mu_1^2 + \rho^2)(\mu + \rho^2)},
$$

(3.48)

where we have used the limit

$$
\lim_{d \to -z_0} \frac{z_0 + d}{\mu_2 - \mu} = -\frac{\sqrt{(z + z_0)^2 + \rho^2}}{\mu}.
$$

(3.49)

Introducing the hyper-spherical coordinates $r$ and $\theta$ by

$$
\rho = \frac{1}{2}\sqrt{(r^2 - 2m)^2 - 4d^2 \sin 2\theta},
$$

(3.50)

$$
z = \frac{1}{2}(r^2 - 2m) \cos 2\theta,
$$

(3.51)

we have

$$
\mu = -(r^2 - 2m - 2d) \cos^2 \theta,
$$

(3.52)

$$
\mu^* = (r^2 - 2m + 2d) \sin^2 \theta,
$$

(3.53)

$$
\mu_1 = (r^2 - 2m - 2d) \sin^2 \theta,
$$

(3.54)

$$
h = \frac{r^2 - 2m - 2d}{r^2 - 2m + 2d},
$$

(3.55)

$$
K = \left(\frac{r^2}{r^2 - 2m + 2d}\right)^{\frac{4}{(\alpha^2 + 2)}}.
$$

(3.56)
By using these expressions the metric components are reduced to

\[
g_{00} = -\frac{r^2 - 2m - 2d}{r^2 - 2m + 2d} \left( \frac{r^2 - 2m + 2d}{r^2} \right)^{4/(\alpha^2+2)}, \tag{3.57}
\]

\[
g_{11} = (r^2 - 2m + 2d) \sin^2 \theta \left( \frac{r^2}{r^2 - 2m + 2d} \right)^{2/(\alpha^2+2)}, \tag{3.58}
\]

\[
g_{22} = (r^2 - 2m + 2d) \cos^2 \theta \left( \frac{r^2}{r^2 - 2m + 2d} \right)^{2/(\alpha^2+2)}, \tag{3.59}
\]

\[
f = \frac{r^2 - 2m + 2d}{(r^2 - 2m)^2 - 4d^2 \cos^2 2\theta} \left( \frac{r^2}{r^2 - 2m + 2d} \right)^{2/(\alpha^2+2)}. \tag{3.60}
\]

Then, noting the relation

\[
dp^2 + dz^2 = \left[ (r^2 - 2m)^2 - 4d^2 \cos^2 2\theta \right] \left[ \frac{r^2 dr^2}{(r^2 - 2m)^2 - 4d^2} + d\theta^2 \right], \tag{3.61}
\]

we obtain

\[
ds^2 = -\left[ 1 - \frac{2(m + d)}{r^2} \right] Y^{-(\alpha^2-2)/(\alpha^2+2)} (dx^0)^2
+ \left[ 1 - \frac{2(m + d)}{r^2} \right]^{-1} Y^{-2/(\alpha^2+2)} dr^2
+ Y^{\alpha^2/(\alpha^2+2)} r^2 [d\theta^2 + \sin^2 \theta (dx^1)^2 + \cos^2 \theta (dx^2)^2], \tag{3.62}
\]

where

\[
Y = 1 - \frac{2(m - d)}{r^2}. \tag{3.63}
\]

This metric has an event horizon at \( r^2 = 2(m + d) \) and a curvature singularity
at \( r^2 = 2(m - d) \). Because this singularity is covered by the horizon, the metric
represents a 5D dilaton black hole. This type of solution was first given by Gibbons
and Maeda.\(^{23} \) The difference from their solution results from the imposition of the
coordinate condition \( \det g = -\rho^2 \) in our solution. We find that

\[
g_{00} \cdot g_{rr} = -Y^{-(\alpha^2+2)/2}, \tag{3.64}
\]

which is not equal to \(-1\), showing a violation of the equivalence principle. The
extremal case obtained in the limit \( d = 0 \) has no event horizon but has a curvature
singularity at \( r^2 = 2m \). This should be compared with the 5D EM case where the
metric becomes the 5D Majumdar-Papapetrou-type solution.

The second possibility of removing the conical singularity is by setting \( d = 0 \).
When \( d = 0 \) and \( z_0 \neq 0 \), we get the metric components

\[
g_{00} = -K_0^{-1}, \tag{3.65}
\]

\[
g_{11} = \mu^* \sqrt{K_0}, \tag{3.66}
\]

\[
g_{22} = -\mu \sqrt{K_0}, \tag{3.67}
\]

\[
f = \frac{\sqrt{K_0}}{2\sqrt{(z + z_0)^2 + \mu^2}} \tag{3.68}
\]
where

\[ K_0 = \left( 1 + \frac{m}{\sqrt{\rho^2 + z^2}} \right)^{4/(\alpha^2+2)}. \] (3.69)

We find that the solution has a ringlike structure at \( z = 0 \), and near the ring the metric components behave as

\[ g_{00} \sim \rho^{4/(\alpha^2+2)}, \] (3.70)
\[ g_{22} \sim \rho^{2(\alpha^2+1)/(\alpha^2+2)}. \] (3.71)

This shows that there are two eigenvectors belonging to the zero eigenvalue for \( g \), which leads to a curvature singularity at \( \rho = 0 \) and \( z = 0 \).

\section*{§4. Summary and discussion}

In this study, we obtain an infinite number of static solutions to the 5D EMd equations, which shows that 5D EMd equations are completely integrable despite their apparent complexity. The solutions are specified by their soliton numbers. We examined the two-soliton solution in detail by studying its rod structure. By requiring the existence of a horizon, we determined the parameters that come in when we assume an ansatz. The rod structure of the two-soliton solution thus obtained is the same as those of the static black ring case and the static charged black ring case, that is, there is a conical singularity.

We get rid of the conical singularity by imposing some constraints between the positions of the poles. We thus obtain the 5D dilaton black hole solution. This solution is a comprehensive solution in a sense that the EM solution is obtained by setting \( \alpha = 0 \) and the static black ring solution is obtained by setting \( m = d \). The difference between the EMd black hole solution and EM black hole solution arises in the extremal case where \( d = 0 \). In the EM case, we have a 5D Majumdar-Papapetrou-type solution. On the other hand, the EMd solution has a curvature singularity and so it is not a black hole solution. The same situation can be also found in the 4D case.

It is well known that the Majumdar-Papapetrou (MP) solution is expressed in terms of the multi-black-hole solution in four dimensions, but, in the 5D case, such a solution does not exist in the EM case, and neither in the present case with the dilaton field. In the 4D case, we can obtain the MP solution by imposing the extremal condition, which enables us to remove the conical singularities aligned along the \( z \)-axis. The corresponding condition in the present case is \( d = 0 \), by which naked singularities appear in the replacement of the vanishing of conical singularities. A possible solution of evading conical singularities might be found in the stationary case such as a black saturn\(^{10}\) and there may be a dilatonic black saturn solution in the stationary case.

Finally, we list up the subjects that should be studied in future. In addition to the stationary case mentioned above, the higher number of soliton solutions should
be investigated to clarify their physical significance. Since our development of the solutions depends on Eqs. (2.56) and (2.57), it might be expected to find a new type of physically interesting solution by removing these assumptions and allowing for an arbitrary soliton numbers for $\ln N_2$, $\ln N$ and $\ln h$. In the development of an $n$-soliton solution, we start with a vacuum solution in this study. When we adopt a more complicated solution as the seed solution, we might obtain some other interesting solutions.

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