UNIQUE MIXING OF THE SHIFT ON THE C*-ALGEBRAS GENERATED BY THE q-CANONICAL COMMUTATION RELATIONS

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Abstract. The shift on the C*-algebras generated by the Fock representation of the q-commutation relations has the strong ergodic property of unique mixing, when |q| < 1.

1. INTRODUCTION

The q-commutation relations have been studied in the physics literature, see e.g. [8]. These are the relations

\[ a_i a_j^+ - qa_j^+ a_i = \delta_{ij} 1, \quad i, j \in \mathbb{Z} \]

where \(-1 \leq q \leq 1\). This gives an interpolation between the canonical commutation relations (Bosons) when \(q = 1\) and the canonical anticommutation relations (Fermions) when \(q = -1\), while when \(q = 0\) we have freeness (cf. [16]). In [3], (see also [9] and [7]) a Fock representation of these relations was found, giving annihilators \(a_i\) and their adjoints, the creators \(a_i^+\), acting on a Hilbert space with a vacuum vector \(\Omega\). The C*-algebras and von Neumann algebras generated by sets of these operators or by their real parts \(a_i + a_i^+\) have been much studied. The reader is referred to [11, 5, 12, 14, 15, 13] for results, applications and further details.

In the present note, we show that the shift \(\alpha_q\) on the C*-algebras generated by these \(a_i\), or by their self-adjoint parts, has the strong ergodic property of unique mixing, which was introduced in the companion paper [6], whenever \(|q| < 1\). The case of free shifts was treated in [6], but when \(q \neq 0\), this shift \(\alpha_q\) cannot be a free shift, so these results provide new noncommutative examples of unique mixing. We also observe that the examples provided by the \(a_i\) and by the \(a_i + a_i^+\) are nonconjugate. It was shown in [6] that there is no classical counterpart to this situation.

2. TERMINOLOGY AND BASIC NOTATIONS

For a (discrete) C*-dynamical system we mean a triplet \((\mathfrak{A}, \alpha, \omega)\) consisting of a unital C*-algebra \(\mathfrak{A}\), an automorphism \(\alpha\) of \(\mathfrak{A}\), and a state \(\omega \in \mathcal{S}(\mathfrak{A})\) invariant under the action of \(\alpha\). The pair \((\mathfrak{A}, \alpha)\) consisting of C*-algebra

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and an automorphism as before, is called a $C^*$–dynamical system as well. A classical $C^*$–dynamical system is simply a dynamical system such that $\mathfrak{A} \sim C(X)$, $C(X)$ being the Abelian $C^*$–algebra of all the continuous functions on the compact Hausdorff space $X$. In this situation, $\alpha(f) = f \circ T$ for some homeomorphism $T : X \to X$.

Consider for $j = 1, 2$, the $C^*$–dynamical systems $(\mathfrak{A}_j, \alpha_j, \omega_j)$ together with the canonically associated $W^*$–dynamical systems $(M_j, \hat{\alpha}_j, \hat{\omega}_j)$. Here, $M_j := \pi_{\omega_j}(\mathfrak{A}_j)^{\pi}$ and $\hat{\alpha}_j$, $\hat{\omega}_j$ are the canonical extensions of $\alpha_j$ and $\omega_j$ to $M_j$, respectively. The $C^*$–dynamical system $(\mathfrak{A}_j, \alpha_j, \omega_j)$ is said to be conjugate if there exists an automorphism $\beta : M_1 \to M_2$ intertwining the dynamics ($\beta \hat{\alpha}_1 = \hat{\alpha}_2 \beta$), and the states ($\hat{\omega}_2 \circ \beta = \hat{\omega}_1$). Suppose that $\mathfrak{A}_j \sim C(X_j)$, $X_j$ being compact spaces. Then there exist probability measures $\mu_j$ on $X_j$, and measure–preserving homeomorphisms $T_j$ of the compact spaces $X_j$ such that

$$\alpha_j(f) = f \circ T_j, \quad \omega_j(f) = \int_{X_j} f \, d\mu_j.$$ 

Thanks to a result by J. von Neumann (cf. [1], p. 69), our definition is equivalent to the following one, provided that the $X_j$ are compact metric spaces. There exist $\mu_j$–measurable sets $A_j \subseteq X_j$ of full measure such that $T_j(A_j) = A_j$, and a one–to–one measure–preserving map $S : A_1 \to A_2$ such that $T_2 = S \circ T_1 \circ S^{-1}$. The reader is referred to [10] for further details relative to the classical case.

To recall the definition from [6], a $C^*$–dynamical system $(\mathfrak{A}, \alpha)$ is said to be uniquely mixing if

$$\lim_{n \to +\infty} \varphi(\alpha^n(x)) = \varphi(1)p(x), \quad x \in \mathfrak{A}, \varphi \in \mathfrak{A}^*$$

for some $\omega \in \mathcal{S}(\mathfrak{A})$.

It can readily seen that $\omega$ is invariant under $\alpha$. In addition, it is unique among the invariant states for $\alpha$.

Let $\mathcal{H} := \ell^2(\mathbb{Z})$, with $e_i \in \mathcal{H}$ the function taking value 1 at $i$ and zero elsewhere. The $q$–deformed Fock space $\mathcal{F}_q$ is the completion of the algebraic linear span of the vacuum vector $\Omega$, together with vectors

$$f_1 \otimes \cdots \otimes f_n, \quad f_j \in \mathcal{H}, j = 1, \ldots, n, n = 1, 2, \ldots$$

with respect to the inner product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_q := \delta_{n,m} \sum_{\pi \in \mathbb{P}_n} q^{i(\pi)} \langle f_1, g_{\pi(1)} \rangle \cdots \langle f_n, g_{\pi(n)} \rangle,$$

$\mathbb{P}_n$ being the symmetric group of $n$ elements, and $i(\pi)$ the number of inversions of $\pi \in \mathbb{P}_n$. We have $\langle f, g \rangle_q = \langle P_q f, g \rangle_0$, where $P_q$ is determined by

$$P_q \Omega = \Omega, \quad P_q f_1 \otimes \cdots \otimes f_n = q^{i(\pi)} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}.$$  \hspace{1cm} (1)

The creator $a_i^+$ acts on $\mathcal{F}_q$ by

$$a_i^+ \Omega = e_i, \quad a_i^+ (f_1 \otimes \cdots \otimes f_n) = e_i \otimes f_1 \otimes \cdots \otimes f_n,$$
and its adjoint is the annihilator $a_i$ given by
\[ a_i \Omega = 0, \]
\[ a_i (f_1 \otimes \cdots \otimes f_n) = \sum_{k=1}^{n} q^{k-1} \langle f_k, e_i \rangle f_1 \otimes \cdots \otimes f_{k-1} \otimes f_{k+1} \otimes \cdots \otimes f_n. \]

Denote by $R_q$ the $C^*$–algebra generated by $\{a_i \mid i \in \mathbb{Z}\}$, and by $G_q$ the $C^*$–algebra generated by $\{a_i + a_i^+ \mid i \in \mathbb{Z}\}$. The right shift $\alpha = \alpha_q$ acting on $R_q$ is uniquely determined by
\[ \alpha(a_i) := a_{i+1}, \quad i \in \mathbb{Z} \]
on the generators. The Fock vacuum expectation $\omega := \langle \cdot \Omega, \Omega \rangle$ is invariant for the shift $\alpha$. The restriction of the vacuum expectation to $G_q$ is a faithful trace. For further details, we refer to [2, 4] and the literature cited therein.

### 3. Unique Mixing of the $q$–Shifts

In the present section we prove the announced result on the unique mixing of the shift on the $q$–canonical commutation relations. We start with the following

**Lemma 1.** Let $\{\xi_j\}_{j=1}^{n} \subset \mathcal{H}^{\otimes n}$, and $\{f_j\}_{j=1}^{n} \subset \mathcal{H}$ be an orthonormal set. Then
\[ \left\| \sum_{j=1}^{n} a^+(f_j) \xi_j \right\| \leq \sqrt{\frac{n}{1-|q|}} \max_{1 \leq j \leq n} \| \xi_j \|. \]

**Proof.** Denote as in [4], $P_q^{(n)} := P_q[\mathcal{H}^{\otimes n}]$, where $P_q$ is given in (1). By taking into account Section 3 of [4] (see also [2], Section 1), we get
\[ \left\langle \sum_{j=1}^{n} a^+(f_j) \xi_j, \sum_{j=1}^{n} a^+(f_j) \xi_j \right\rangle \]
\[ \leq \frac{1}{1-|q|} \left\langle 1 \otimes P_q^{(k+1)} \sum_{j=1}^{n} f_j \otimes \xi_j, \sum_{j=1}^{n} f_j \otimes \xi_j \right\rangle_0 \]
\[ \leq \frac{1}{1-|q|} \sum_{i,j=1}^{n} \left\langle f_i \otimes P_q^{(k)} \xi_i, f_j \otimes \xi_j \right\rangle_0 \]
\[ = \frac{1}{1-|q|} \sum_{i,j=1}^{n} \left\langle f_i, f_j \right\rangle \left\langle P_q^{(k)} \xi_i, \xi_j \right\rangle_0 \]
\[ = \frac{1}{1-|q|} \sum_{i,j=1}^{n} \left\langle f_i, f_j \right\rangle \left\langle \xi_i, \xi_j \right\rangle_0 \]
\[ = \frac{1}{1-|q|} \sum_{i,j=1}^{n} \left\langle \xi_i, \xi_i \right\rangle_0 \]
\[ \leq \frac{n}{1-|q|} \max_{1 \leq j \leq n} \| \xi_j \|^2. \]
Proposition 2. Let \(0 \leq k_1 < k_2 < \cdots < k_n < \cdots\) be a sequence of increasing natural numbers, and \(e_{\sigma_1}, \ldots, e_{\sigma_i}, e_{\rho_1}, \ldots, e_{\rho_j}\) elements of the canonical basis of \(\ell^2(\mathbb{Z})\). We have
\[
\left\| \sum_{l=1}^{n} \alpha^{k_l} (a^+(e_{\sigma_1}) \cdots a^+(e_{\sigma_i})a(e_{\rho_1}) \cdots a(e_{\rho_j})) \right\| \leq \frac{n}{\sqrt{(1 - |q|)^{i+j}}}
\]
if at least either \(i\) or \(j\) is nonnull.

Proof. Suppose first \(i > 0\). It is enough consider unit vectors \(\xi \in \mathcal{H}^\otimes m, m = j, j + 1, \ldots\). Put
\[
\xi_l := a^+(e_{\sigma_{2+k_l}}) \cdots a^+(e_{\sigma_{t+k_l}})a(e_{\rho_{1+k_l}}) \cdots a(e_{\rho_{j+k_l}})\xi.
\]
Notice that, by Theorem 3.1 of [2], \(\|\xi_l\| \leq 1/\sqrt{(1 - |q|)^{i+j-1}}\). In addition, \(\langle e_{\sigma_{1+k_l}}, e_{\sigma_{1+k_l}} \rangle = \delta_{l,t}\). By applying Lemma 1, we get
\[
\left\| \sum_{l=1}^{n} \alpha^{k_l} (a^+(e_{\sigma_1}) \cdots a^+(e_{\sigma_i})a(e_{\rho_1}) \cdots a(e_{\rho_j}))\xi \right\|^2
= \left\langle \sum_{l=1}^{n} a^+(e_{\sigma_{1+k_l}})\xi_l, \sum_{l=1}^{n} a^+(e_{\sigma_{1+k_l}})\xi_l \right\rangle \leq \frac{n}{(1 - |q|)^{i+j}}.
\]
If \(i = 0\) and then \(j \neq 0\) (i.e. we have only annihilators), the assertion follows by the first part as
\[
\sum_{l=1}^{n} \alpha^{k_l} (a(e_{\rho_1}) \cdots a^+(e_{\rho_j})) = \left( \sum_{l=1}^{n} a^+(e_{\rho_{j+k_l}}) \cdots a^+(e_{\rho_{j+k_l}}) \right)^*.
\]

The following theorem is the announced result on the strong ergodic property enjoined by the \(q\)-shift.

Theorem 3. The dynamical system \((\mathcal{R}_q, \alpha)\) is uniquely mixing, with the vacuum expectation \(\omega\) as the unique invariant state.

Proof. Let \(X \in \mathcal{R}_q\) have vanishing vacuum expectation. It is norm limit of elements as those treated in Proposition 2. By Proposition 2.3 of [6], and a standard approximation argument, it is enough to prove that
\[
\frac{1}{n} \left\| \sum_{l=1}^{n} \alpha^{k_l}(X) \right\| \longrightarrow 0
\]
for each \(X\) given by
\[
X = a^+(e_{\sigma_1}) \cdots a^+(e_{\sigma_i})a(e_{\rho_1}) \cdots a(e_{\rho_j})
\]
for which either \(i\) or \(j\) is nonnull, and for each sequence \(0 \leq k_1 < k_2 < \cdots < k_n < \cdots\) of increasing natural numbers. The result directly follows by Proposition 2. \(\square\)
Corollary 4. The dynamical system \((G_q, \alpha |_{g_q})\) is uniquely mixing, with the vacuum expectation \(\omega |_{g_q}\) as the unique invariant state.

Remark 5. The dynamical systems \((R_q, \alpha^{-1}), (G_q, (\alpha |_{g_q})^{-1})\) are uniquely mixing as well, with the vacuum expectation as the unique invariant state.

This can be shown by taking into account that 
\[
\theta \alpha = \alpha^{-1} \theta, \quad \omega \circ \theta = \omega,
\]
where \(\theta(a(e_k)) := a(e_{-k}), k \in \mathbb{Z}\), is the "time reversal", \(a(e_k)\) being the \(k\)-annihilator.

It is known from [2] that \(G''_q\) is a \(\text{II}_1\)-factor. Moreover, it is well known and easily seen that \(R''_q\) is all of \(B(F_q)\). (A proof in the case of the Fock representation of finitely many \(a_i\) is found in [5]). For convenience, we provide a proof of this fact below. In any case, from this it follows that \((R_q, \alpha)\) is not conjugate to \((G_q, \alpha |_{g_q})\)

Proposition 6. For all \(-1 < q < 1\), the von Neumann algebra \(R''_q\) is all of \(B(F_q)\).

Proof. It will suffice to show that the rank-one projection \(P_\Omega\) onto the span of the vacuum vector belongs to \(R''_q\), because \(\Omega\) is cyclic for the action of \(R_q\) on \(F_q\). In the case of \(q = 0\), let us write \(v_i\) for \(a_i\) acting on \(F_0\). Thus, \(\{v_i^*\}_{i \in \mathbb{Z}}\) is a family of isometries with orthogonal ranges, and the sum \(\sum v_i^* v_i\) converges in strong operator topology to \(I - P_\Omega\). For the general case, we fix \(-1 < q < 1\) and we will refer to Propositions 3.2 and 3.4 and Remark 3.3 of [5]. These show that there is a unitary \(U : F_q \rightarrow F_0\) that sends the \(n\)-particle space to the \(n\)-particle space, and there is a positive operator \(M\) on \(F_q\) given by \(M \Omega = \Omega\) and
\[
M(f_1 \otimes \cdots \otimes f_n) = \sum_{k=1}^{n} q^{k-1} f_k \otimes f_1 \otimes \cdots \otimes f_{k-1} \otimes f_{k+1} \otimes \cdots \otimes f_n.
\]
Proposition 3.2 of [5] shows that \(M\) has range equal to all of \(F_q\). Moreover, we have \(U a_i U^* = v_i^* R\), where \(R = UM^{1/2} U^*\). Thus,
\[
U(\sum_{i \in \mathbb{Z}} a_i^+ a_i) U^* = R(\sum_{i \in \mathbb{Z}} v_i^* v_i) R = R(I - P_\Omega) R,
\]
where the sums converge in strong operator topology. From this, we see that the range projection of \(\sum_{i \in \mathbb{Z}} a_i^+ a_i\) is \(I - P_\Omega\). □

Notice that \((R_q, \alpha, \omega)\) is not conjugate to \((G_q, \alpha |_{g_q}, \omega |_{g_q})\). Indeed, \(\pi_\omega(R_q)'' \equiv R_q''\) is not isomorphic to \(\pi_\omega(G_q)'' \equiv G_q''\) as we have shown. Thus, we provide nontrivial examples of uniquely mixing \(C^*\)-dynamical systems for which the unique invariant state is faithful (case of the \(C^*\)-algebra \(G_q\) generated by the self-adjoint part for which the restriction of vacuum state is a faithful trace),
or when it is not faithful (case of the $C^*$--algebra $\mathcal{R}_q$). However, for all cases the associated GNS covariant representation is faithful. It was shown in [6] that there is no classical counterpart to this situation.

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