Coulomb phase in high harmonic generation

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Abstract
We derive a regularized expression for the Coulomb term in the phase of high order harmonics emitted in the interaction of intense laser pulses with atoms. The calculation is based on the formalism of quantum orbits and on the imaginary time method and allows one to match the initially divergent Coulomb integral with the phases of the atomic wave function at the instants of ionization and recombination, leading to a finite and closed analytic formula. This complex-valued Coulomb phase can considerably modify both the yield and the interference structure of high harmonic spectra. Its possible implications for numerical calculations of the high harmonic signal are discussed.

Keywords: high harmonic generation, recollision, strong field ionization, Coulomb effects

(Some figures may appear in colour only in the online journal)

1. Introduction

The generation of high order harmonics (HHG) of intense infrared or optical radiation interacting with atomic and molecular gases has remained in the scope of strong field physics for three decades. The constant interest in this phenomenon is motivated by the great potential of high harmonics (HH) as a cheap and compact source of intense coherent XUV radiation which can be used both in experimental atomic and molecular optics and for various applications (see [1–3] for review). The currently thriving area of attosecond physics is essentially based on the effect of HHG

The microscopic picture of HHG relies on the concept of recollision introduced within the simple man model (SMM) [7] of strong field laser-atom interaction (see also earlier publications [8, 9] where the significance of recollision effects has been recognized). This model predicted the celebrated $3.17U_p^2$ cut-off in HH spectra with $U_p$ being the ponderomotive electron energy in a linearly polarized laser field, as well as other features of HH which stem from classical dynamics. For a better quantitative description of HHG, the Lewenstein model [10] and other conceptually close approaches [11–13] were developed. These analytical methods may differ from each other in essential details, but all of them are based on the nonperturbative description of the highly nonlinear semiclassical dynamics of photoelectrons in a strong laser field. Similarly to the strong field approximation (SFA) [14–16] widely applied in the theory of the above threshold ionization (ATI), these nonperturbative approaches have become a standard tool in the theory of HHG. Although nowadays HH spectra emitted by a single atom or molecule in the field of infrared or optical linearly polarized lasers can be numerically calculated by solving exactly the time-dependent Schrödinger equation (TDSE) with minimal computational efforts, analytical approaches to the problem and approximate models remain in high demand for two reasons. First, they combine a simple intuitive picture of the HHG which has already gained a number of fruitful insights with reasonable quantitative accuracy usually sufficient for practical calculations. Second, the collective effect of phase matching can only be described within the macroscopic picture where individual atomic or molecular emitters play the role of elementary currents entering Maxwell’s equations. Simple and at the same time acceptably accurate analytic expressions for these currents are necessary to describe propagation and amplification of HH waves.
Semiclassical strong field models of HHG provide us with analytic instruments for the calculation of these currents. These models confirm the concept of recollision based on the idea that those electrons which can be driven back to the atom after the ionization event mostly contribute to the HHG process. Technically, semiclassical models employ the SFA formalism, so that the HHG amplitude includes the amplitude of ATI as a building block (see equations (1)–(3) in the next section). In its original formulation [14–16], the SFA discards Coulomb interaction between the valence electron and the atomic residual, and so do the SFA-based theories of HHG. Formally, this means that such theoretical approaches are quantitatively correct for the description of ionization or HHG in negative ions. Although strong field detachment of negative ions was studied experimentally in tiny detail and indeed the SFA was proven very precise in this case (see e.g. [17–19] and references therein), the macroscopic HH response of negative ions is hard to observe because of space charge effects.

Although the theory is formally restricted to systems bound by short-range forces, it is being routinely used for description of the ATI and HHG in neutral atoms and molecules. In the case of ATI, this leads to considerable quantitative and sometimes even qualitative discrepancies between predictions made within the SFA from one side and results of exact TDSE solutions and experimental data from the other. This problem has received a detailed consideration in the literature, particularly during the latest decade when several significant effects of the Coulomb interaction in ATI spectra were identified and described quantitatively. This includes the low and very low-energy structures [20–22] and other peculiarities in the low-energy part of photoelectron spectra [23], side lobes [24] and other Coulomb-induced interference structures [25, 26] in momentum distributions, Coulomb-induced asymmetries in elliptically polarized fields [27], etc. Theoretical approaches to the description of Coulomb effects in ATI spectra are based on the idea that in a strong laser field, Coulomb interaction is relatively weak (although significant) and can therefore be treated within a specific perturbation theory for the photoelectron action [28]. In the last few years, this approach and its extensions [25, 26, 29–32] were successfully applied for the quantitative description of several significant Coulomb effects in the ATI (see, e.g. [19] for review). In terms of the SMM, the Coulomb field deforms the photoelectron trajectory so that the same initial conditions, say, the tunnel exit position and the initial electron velocity, lead to essentially different values of the detected momentum with and without the Coulomb field accounted for. On a more rigorous level of description which uses the SFA language, the Coulomb field not only disturbs photoelectron trajectories, but also generates a contribution into the phase of the photoelectron wave function which determines the quantum-mechanical ionization amplitude. This contribution, which is complex-valued and numerically large, can strongly affect photoelectron momentum distributions. A significant role of Coulomb effects in the ATI is now well documented both in experiments and in theory.

As the effect of HHG results from a very similar physics, one could expect the Coulomb interaction to be comparably significant there too. However, until now, much less attention has been paid to this problem, compared to that for the ATI. There are two reasons for this. First, classical photoelectron trajectories contributing to the HHG process return to the parent atom, i.e. so-called closed trajectories are primarily important for the HHG. Along such trajectories, the electron is first decelerated by the Coulomb field and then accelerated on its way back to the atom. These two contributions are known to cancel each other rather accurate, so that the Coulomb distortion of relevant trajectories appears to be of lower order compared to the case of the ATI where such compensation can happen only occasionally for certain values of the final momentum. Second, in contrast to photoelectron spectra where the interference pattern is determined by the local single-atom dynamic, the phase of HH consists of two parts, one resulting from the single-atom response, and the second accumulated during the propagation in the medium. In experimentally observed HH spectra, this macroscopic contribution is difficult to separate from the single-atomic phase. As a result, HH spectra obtained by numerically solving the TDSE demonstrate significantly different interference structure from that observed in a collective signal emitted by a gaseous target.

However, although screened by macroscopic effects, the Coulomb contribution to the phase of HH can be significant. Even being small compared to the phase induced by the laser field, this contribution may appear numerically large, leading to considerable corrections to the total phase of the emitting atomic dipole. In this paper, we calculate this Coulomb phase using the formalism of complex-valued quantum orbits. The main element of this calculation is in the derivation of a regularized expression for the Coulomb action which is originally divergent both in the initial and the final part of the photoelectron trajectory corresponding to the ionization and recombination events. Here we derive such a regular analytic expression and examine it in view of its possible contribution to the HH phase and magnitude. Our analysis is very close to the one presented in the recent paper by Torlina and Smirnova [33], although the final result is different as well as the aim of the calculation: in [33], the corrections to the time of ionization and recombination were derived from the Coulomb phase while we derive this phase itself assuming the aforementioned time instants fixed.

The paper is organized as follows. In the next section, the Coulomb-free expression for the HH complex-valued phase is introduced along the Lewenstein model [10]. Then the regularized Coulomb correction to this phase is derived. Section 4 contains conclusions and outlook, including a discussion of possible ways of experimental or numerical verification of the Coulomb effect in the HHG. Atomic units $\hbar = m_e = |e| = 1$ are used throughout the text.
2. Coulomb-free HH phase

Within the Lewenstein model, the laser-driven atomic dipole is given by [10]
\[ \mathbf{d}(t) = -\langle \Psi(r) | \Psi \rangle \]
with the electron wave function
\[ \Psi(r, t) = \psi_0(r) e^{i\mathbf{p} \cdot \mathbf{r}} + \int d^3p \psi_p(r, t) \mathbf{M}(\mathbf{p}, t). \]

Here the time-dependent ionization amplitude \( \mathbf{M}(\mathbf{p}, t) \) is calculated within the SFA:
\[ \mathbf{M}(\mathbf{p}) = i \int_{-\infty}^{t} \int d^3p' \psi_p^{*}(r', t') \mathbf{E}(t') \cdot \mathbf{r}' \psi_0(r') e^{i\mathbf{p} \cdot \mathbf{r}'}, \]

where \( \psi_0(r, t) = \psi_0(r) \exp(i\mathbf{p} \cdot \mathbf{r}) \) is the initial atomic bound state wave function with the ionization potential \( I_p \), and \( \mathbf{E}(t) = -\mathbf{A}(t) \) is the electric field of the laser wave with the vector potential \( \mathbf{A}(t) \). In the length gauge and within the dipole approximation, the Volkov wave function \( \psi_p \) used for expansion of the continuum part of the electron wave packet is given by
\[ \psi_p(r, t) = (2\pi)^{-3/2} \exp \left\{ i(\mathbf{p} + \mathbf{A}(t)) \cdot \mathbf{r} - \frac{1}{2} \int_{0}^{t} \frac{1}{2} (\mathbf{p} + \mathbf{A}(t'))^2 dt' \right\}. \]

An alternative approach [11] treats the same process as spontaneous emission of HH photons with the quantum-mechanical amplitude
\[ \mathcal{A}(\Omega, \mathbf{e}) = i \int_{-\infty}^{+\infty} dt \int d^3re^{i\mathbf{e} \cdot \mathbf{r}} \psi^*(\mathbf{r}, t)(\mathbf{r} \cdot \mathbf{e}) \psi(\mathbf{r}, t). \]

Here \( \Omega \) and \( \mathbf{e} \) are the harmonic frequency and polarization vector correspondingly, \( V \) is the normalization volume and the wave function \( \psi(\mathbf{r}, t) \) is assumed to be the exact field-dressed state of the valence electron. This wave function can be explicitly calculated for a model atom approximated by the zero-range well [11, 12].

Here we focus on the HH phase which is the most important ingredient of the emission amplitude, particularly when the calculation of the collective response is considered. The two aforementioned approaches lead to identical expressions for this phase which stem from the form of the Volkov wave function (4):
\[ S(\Omega; \mathbf{p}, t, t_{1}) = \mathcal{N} - I_p(t_1 - t) - \frac{1}{2} \int_{t}^{t_{1}} (\mathbf{p} + \mathbf{A}(\tau))^2 d\tau. \]

This phase has to be taken at stationary points \( \{\mathbf{p}_\alpha, t_\alpha, t_{1}\} \) determined by the saddle-point equations (see e.g. [33] for a detailed description and references)
\[ \frac{\partial S}{\partial \mathbf{p}} = -\mathbf{p}(t_1 - t) - \int_{t}^{t_{1}} \mathbf{A}(\tau) d\tau = 0, \]
\[ \frac{\partial S}{\partial t} = I_p + \frac{1}{2} (\mathbf{p} + \mathbf{A}(t))^2 = 0, \]
\[ \frac{\partial S}{\partial t_{1}} = \Omega - I_p - \frac{1}{2} (\mathbf{p} + \mathbf{A}(t_{1}))^2 = 0. \]

These three equations introduced in [10, 11] are pivotal for the theory of HHG. In a deep tunneling regime, \( \gamma \ll 1 \), the ionization potential \( I_p \) can be omitted in (8) reducing the set of equations to those of the SMM [7] with real solutions. Here
\[ \gamma = \frac{\sqrt{2I_p \omega}}{E_0} \]

is the Keldysh parameter in a laser field of amplitude \( E_0 \) and frequency \( \omega \). Note, however, that reducing the system (7)–(9) to that with real solutions can lead to considerable errors in the determination of ionization and recombination times from HH spectra [34] even at small values of \( \gamma \).

For the following, we introduce notations for the stationary points \( t_1 \equiv t_1 \) and \( t_{1} \equiv t_{1} \), meaning their common interpretation as the times of ionization and recombination correspondingly. The system (7)–(9) has been extensively analyzed in the literature and needs no detailed description here. It is sufficient to note that for HH within the plateau, \( I_p < \mathcal{O} \leq I_p + 3.17I_p \) with \( U_p = E_0^2/4a^2 \) being the ponderomotive energy in a linearly polarized field, the ionization time is essentially complex with \( \text{Im}(\omega t_1) \approx \text{Im}(\omega t) \approx \text{Im}(\omega t) \approx 1. \), \( \gamma \), in agreement with the physical interpretation of the equations: the ionization process is quantum and therefore can proceed, in terms of classical equations of motion, only in complex time, so that the respective analytical technique is frequently referred to as the imaginary time method [35]. In contrast, the laser-driven return of the electron to its parent ion and the recombination are classically allowed, and solutions of (7) and (9) are almost real with small complex parts generated by the complex ionization time. Near and beyond the classical cut-off of the HH plateau where the SMM does not apply, imaginary parts of \( t_1 \) and \( I_p \) start growing in absolute values.

Below, we consider the case of a linearly polarized monochromatic field, \( \mathbf{E}(t) = (E_0 \cos(\omega t), 0, 0) \) where for each HH frequency \( \mathcal{O} \) there is a set of pairs of solutions corresponding to the ionization time located within one laser half-cycle. These solutions can be classified according to the value of the propagation time \( R(t_{1} - t) \) and to the number of returns \( k \) to the parent atom the electron experienced before the recombination event. For a fixed \( k = 0, 1, 2, \ldots \), two trajectories solving the system (7)–(9) exist, which are usually referred to as the short and the long trajectory, according to the relation between their travel times. The shortest trajectory (i.e., the short trajectory with no returns before the recombination, \( k = 0 \)) is favored by the phase matching conditions making the most significant contribution to the macroscopic signal. When the solutions to the system (7)–(9) are found, the emitting dipole takes the form of a sum over the saddle-point contributions
\[ d_\alpha(\Omega) \sim \sum \mathcal{P}_\alpha \exp \{iS(\Omega; \mathbf{p}_\alpha, t_{1\alpha}, t_{1\alpha})\}, \]
polarization vector $\mathbf{e}$ and is determined by the choice of gauge and of the model. Saddle-point solutions depend on two dimensionless parameters which can be chosen as the Keldysh parameter $\gamma$ and the harmonic number $l = 2j + 1$, $\Omega = \omega_c$. A complex-valued trajectory associated with the saddle-point solutions has the form

$$r_0(t) = \mathbf{p}(t - t_0) + \int_{t_0}^{t} \mathbf{A}(\tau) d\tau, \quad r_0(t_0) = r_0(t) = 0. \quad (12)$$

Concluding this section, note the scaling of the Coulomb-free phase (6). Its real part is proportional to the strong field parameter $z_F$ [16]:

$$\text{Re} S(\Omega) = z_F s(\gamma, l), \quad z_F = \frac{E_0^2}{\omega_c^3} \gg 1. \quad (13)$$

The function $s$ is about unity and grows with increasing of the travel time, so that the real part of the phase appears smaller for shorter trajectories. In the tunneling regime, $\gamma < 1$, the imaginary part of the phase (6) is smaller in absolute value compared to (13), and mostly determined by the ionization time [19, 35]:

$$\text{Im} S \simeq \frac{2(E_0^2)^{2/3}}{3|E[\text{Re}(t_0)]|} \gg 1. \quad (14)$$

3. Coulomb phase

Equations of the previous section discard the Coulomb interaction between the photoelectron and the parent ion. We will now introduce the Coulomb term in the phase, employing the technique earlier developed for the analysis of Coulomb effects in the ATI [19, 29, 30, 32]. This method roots back to the pivotal paper by Perelomov and Popov [28] where the Coulomb field was accounted for within the semiclassical perturbation theory for the action (see [19, 35] for details). Taking the phase (6) and the corresponding trajectory (12) as a zero approximation, one may calculate a first-order correction generated by the Coulomb potential energy $U_C(t)$ as

$$S_C(\Omega) = -\int_{t_0}^{t} U_C[r_0(t)] dt = Z \int_{t_0}^{t} \frac{dt}{\sqrt{r_0^2(t)}}, \quad (15)$$

where $Z$ is the residual atomic charge ($Z = 1$ for neutral atoms). The trajectory $r_0(t)$ is a complex-valued function, and the square root in (15) is determined such that its real part is positive, so that the Coulomb force always remains attractive. Mathematically this corresponds to a certain choice of a leaf on the Riemann surface of the multivalued square root function. Compared to the case of the ATI where the Coulomb correction is determined by a similar integral, for the HHG its calculation differs due to the closed character of the trajectory: so that the integral (15) diverges both at the lower and upper limits. Similarly to the ATI, the Coulomb integral diverges logarithmically at $t \to t_0$, as the trajectory starts from the origin and behaves in its initial part in a universal way [19]:

$$r_0(t) \approx v_0(t - t_0)^2, \quad v_0 = \sqrt{-2I_0}. \quad (16)$$

This ionization divergency can be removed by matching the phase (15) at the lower integration limit with the phase of the bound state wave function in the Coulomb potential [28]. This procedure is described in detail in [19, 36] and leads to the renormalized expression

$$S_C'(\Omega) = -i\nu \ln [2iI_0(t_0 - t)] + \int_{t_0}^{t} \left\{ \frac{Z}{\sqrt{r_0^2(t)}} + \frac{i\nu}{t - t_0} \right\} dt. \quad (17)$$

where the effective principal quantum number of the bound state $\nu = Z/\sqrt{2I_0}$ is introduced. The recombination divergency at $t \to t_0$ can be eliminated by a similar method.

To this end, we note that close to the recombination event ($t \to t_0$) the electron trajectory can be approximated by

$$r_0^2(t) \approx v_0^2(t - t_0)^2, \quad v_0^2 = 2(\Omega - I_p) > 0. \quad (18)$$

The electron velocity $v_0$ upon the return is always real, in contrast to the initial velocity $v_0 = \pm i\sqrt{2I_0}$. The sign of $v_0$ is not important for the following calculations: it is sufficient to take into account that the return condition implies

$$\text{Re}(v_0 \cdot r_0) < 0. \quad (19)$$

According to (18), the Coulomb integral (15) diverges at the upper integration limit $t \to t_0$ as

$$S_C(t) \approx -\nu I \ln \frac{t - t_0}{t_0 - t}, \quad \nu I = \frac{Z}{|v_0|}, \quad (20)$$

where the Sommerfeld parameter $\nu I$ [37] is introduced. This divergency can be eliminated by matching with the asymptotic of the Coulomb scattering wave function [37]

$$\Psi^{(+)}(z) = \exp \left\{ \frac{ikr}{k} \ln(kr - k\mathbf{r}) \right\}
= \exp \left\{ i|v_0|kx_0 - iv_0 \ln 2|v_0|kx_0 \right\}, \quad (21)$$

where the second equality assumes (19) and $k = |v_0|$, and the trajectory is taken 1D, $r_0 \to x_0$, because of $\mathbf{p}_x = 0$ in a linearly polarized field as follows from equation (7). After a simple algebra similar to the one used for the matching at the ionization time, $t \to t_0$, we arrive at the doubly regularized expression for the Coulomb phase:

$$S_C^R = -i\nu \ln [2iI_0(t_0 - t)] + \nu I \left( \frac{2v_0^2(t_0 - t)}{t_0 - t} \right) dt, \quad (22)$$

which is the main result of this work. It allows for the calculation of the contribution of the Coulomb interaction into the complex phase of the laser-driven atomic dipole responsible for emission of HH. This expression is similar to the one introduced by Torlina and Smirnova (see equation (27) in [33]). The main difference is in the method of regularization at the upper integration limit, $t \to t_0$. In contrast to the procedure applied here, in the paper integration limit $t_{\text{cut}}$ in [33]...
was chosen to satisfy the condition $x(t_{\text{end}}) = r_0$ with $r_0$ determined by the electron velocity $v_e$ upon recombination (see equations (22)–(24) in [33]). As a result, the logarithmic part of the Coulomb phase expressed by the second term in equation (25) below differs from that of [33].

The Coulomb correction (22) is finite and can be numerically calculated for arbitrary atom and field parameters. In the limit of low frequencies, $\gamma \rightarrow 0$, corresponding to the HHG in mid-infrared laser fields, this calculation can be simplified by omitting the upper integration limit and expanding the trajectory in series with respect to $\gamma$. This procedure was realized in [38] for a bircircular laser field. However, such tunneling asymptotic is only capable of correctly reproducing the imaginary part of the Coulomb term (22) which is formed at the initial part of the electron trajectory. In contrast, the real part of (22) forms during the whole photoelectron excursion, so that no expansion can be applied for its calculation even in the tunneling regime, and no approximate analytic expression for the time integral can be provided.

For a practical calculation, one has to make sure that the integration path in complex time is properly chosen. The difficulty here originates from the fact that the function $r_0^2(t)$ may have simple or second-order zeros in the complex time plane generating branch points and branch cuts of the integrand $1/\sqrt{r_0^2(t)}$ [39–41]. Physically, solutions to the equation

$$r_0^2(t_n) = 0$$

(23)

correspond to photoelectron returns to the parent ion happened prior the recombination event. If the trajectory was real as it is in the SMM [7], such returns could be directly interpreted as recollisions. For complex-valued trajectories, these recollisions occur in (generally) complex time, allowing one to interpret them as an under-barrier phenomenon [41]. For nonvanishing lateral momenta $p_\perp$ the complexness of the integration paths allows one to navigate them through the gaps between the branch points keeping the contour of integration on the physical leaf determined by the condition

$$\text{Re} \{ \sqrt{r_0^2(t)} \} > 0.$$  

(24)

For the ATI, this procedure was developed in [39–41]. In the case of the HHG in a linearly polarized field the photoelectron trajectory becomes 1D, and the pairs of branch points merge into first-order poles, because the condition $x_0^2(t_n) = 0$ implies also $x_0(t_n) = 0$. The calculation of the Coulomb integral (22) is simplified for the shortest trajectories which experience no returns before the recombination event and therefore do not cross branch cuts. For such trajectories corresponding to $k = 0$, the integration contour can be chosen in a ‘standard’ way, as is shown in figure 1(a), and only its last segment 3 coincides with the cut side. Figure 1(b) shows the case of a longer trajectory which returns to the parent ion once before the recombination. For this configuration, a pole with an infinite cut separates the initial and the final integration points, so a proper calculation must also involve integration along the cut (segments 3–8 on figure 1(b)). This more complicated case mostly important for the ATI will be considered elsewhere [42].

In the tunneling limit, $\gamma \ll 1$, and for the case shown in figure 1(a), the integral in (22) can be essentially simplified owing to the smallness of the imaginary parts of the saddle points, $|\text{Im}(\omega t_{\text{Reg}})| \ll 1$. Then the integrals over the vertical segments 1 and 3 can be analytically calculated leading to the expression

$$S_\gamma^\nu = -i\nu \ln(4L^\nu) + \nu_i \ln(2i\nu^2 \tau_i) + \int_{t_{\text{Reg}}}^{t_{\text{Ret}}(i)} Zdt \sqrt{x_0^2(t)}$$

(25)

where the integration is taken along the real part of the contour represented on figure 1(a) by segment 2. Here we denote the imaginary parts of the stationary points as $\tau_i$ and $\tau_r$ correspondingly.

4. Conclusions and outlook

Phase (22) is in general numerically large (although small compared to (6)) and can therefore considerably modify HH spectra. Its value depends on the laser and atom parameters as well as on the type of trajectory. Figure 2 shows this phase calculated numerically for the HHG in lithium and xenon at different laser wavelengths and intensities for the two shortest trajectories corresponding to the case $k = 0$ (no returns before the recombination). The imaginary part of the phase which influences the probability of harmonic emission remains almost constant both with the variation of the harmonic order and of the laser wavelength. This is not surprising because it is mainly determined by the contribution from the ‘ionization’ part of the trajectory (segment 1 in figure 1(a)). In the tunneling limit, this contribution reduces to the well-known static Coulomb correction to the rate of ionization [19, 35] given by the first term in (25). Deviations from a constant value are only visible at shorter wavelengths on figure 2(b) where the Keldysh parameter approaches unity.

The real part of the phase is mostly given by the integral term in (22) or (25) and appears numerically large and sensitive to the wavelength and, to less extent, to the harmonic order. The dependence on the atomic specie and on the wavelength clearly visible on figure 2(b) can be qualitatively understood by noticing that the integral contribution in (25) is proportional to the parameter

$$\frac{Z\omega}{E_0} = \nu\gamma$$

(26)

which, at the intensities chosen for the calculation, appears to be about three times larger for lithium than for xenon. This observation allows one to conclude that the Coulomb effect on the HH phase will be more significant in the case of the HHG in atoms with relatively low ionization potentials.

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1 For the HHG in hydrogen subject to a bircircular laser field with the fundamental frequency $\omega = 0.775$ eV, the frequency ratio 1: 3 and intensity $10^{13}$ W/cm$^2$ equal for the two components the factor varies between 1 and 0.44 along the HH spectrum, see figure 5 in [38].
As was shown in [40, 41] for the ATI, for trajectories of the type presented on figure 1(b), the contribution of branch points can lead to an order of magnitude enhancement in the ionization probability for laser and atom parameters making the product (26) numerically large (due to the exponential structure of the Coulomb factor it is sufficient to have this product above unity, to make the effect of enhancement significant). The same effect should be expected in HHG spectra generated in vapors of alkali metals [42, 43] with relatively small ionization potentials and correspondingly large values of the effective principle quantum number $\nu$. When excited states with even higher $\nu$ are involved into the HHG process [43], one could expect a more pronounced effect of enhancement in the probability of emission.
In contrast to the ATI, where photoelectron spectra calculated within approximate methods or by numerically solving the TDSE can be directly compared to experimental data, the single-atom HHG response does not immediately relate to the results of measurements, as the latter also include a significant effect of phase matching. This makes the verification of the obtained result nontrivial. The most realistic way for the application of the Coulomb phase (22) in the theory of HHG consists of its inclusion into macroscopic numerical codes as a correction of the phase of single-atom emitters. Although this procedure is straightforward, it can lead to a considerable deceleration of a numerical code because the integral term in (22) requires a numerical calculation of the complex-time integral along the path which depends on the type of trajectory and on the local field parameters changing through the laser focus. As a first step to considerably simplify numerical efforts, a calculation of the HH signal for the case of an ultrashort laser pulse supporting only the shortest trajectories with no returns before recombination can be suggested.

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