Witness to detect quantum correlation of bipartite states in arbitrary dimension

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In this work we introduce a nonlinear witness that is a sufficient condition for detecting the vanishiment of quantum correlation of bipartite states. Our result directly generalizes the result of [J. Maziero, R. M. Serra, arXiv:1012.3075] to arbitrary dimension based on the Bloch representation of density matrices.

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INTRODUCTION

Entanglement is an essential resource in almost all quantum computing and informational processing tasks [1, 2]. However, there are quantum correlations beyond entanglement, i.e., entanglement is not necessarily needed to illustrate the non-localities in a quantum system. It has been shown that the quantum correlation, even without entanglement, can lead to the speedup of quantum computing[3]. Furthermore, it is more robust than entanglement in resisting environment-induced decoherence, which makes the quantum computation based on quantum correlation more robust than those based on the entanglement [9–11].

If a bipartite quantum state is in a product state, $\rho = \rho_A \otimes \rho_B$, with $\rho_A$ ($\rho_B$) being the reduced density matrix of subsystem A (B), the state has no quantum correlation. However, a state with zero quantum correlation is not always a product state.

Detecting whether the quantum correlation of a bipartite state, quantified by Ollivier and Zurek’s quantum discord [12], is zero or not is fundamentally important, e.g., it has been proven that zero quantum discord between a quantum system and its environment is necessary and sufficient for describing the evolution of the system through a completely positive map [13, 14]. In addition, a quantum state can be locally broadcasted if and only if it has zero quantum discord [15, 16].

In fact, a system is classically correlated only if its state can be written as

$$\sum_{ij} p_{ij} |a_i \rangle \langle a_i| \otimes |b_j \rangle \langle b_j|,$$

with $\{|a_i \rangle\}$ and $\{|b_j \rangle\}$ forming orthonormal basis for the two subsystems and $\{p_{ij}\}$ being a probability distribution.

In this work, we will try to solve the problem of detect whether a state is classical or not.

MAIN RESULT

It is well known that every $N \times N$ density matrix can be represented by the $(N^2 - 1)$-dimensional Bloch vector as: $\rho(u) = \frac{1}{N^2} (I + \sqrt{N(N-1)-\vec{u} \cdot \vec{\lambda}}), \vec{u}$, but the converse is not true, i.e., not all operator of the form $\frac{1}{N}(I + \sqrt{N(N-1)} \vec{u} \cdot \vec{\lambda})$ is a density matrix, where $\vec{u}$ is an arbitrary $(N^2 - 1)$-dimensional Bloch vector. Note that a density matrix must satisfy three conditions: (a). Trace unity, $\text{Tr}(\rho(u)) = 1$. (b). Hermitian, $\rho(u)^+ = \rho(u)$; and (c). positivity, i.e., all eigenvalues of $\rho(u)$ are non-negative.

Indeed, the operator $\frac{1}{N}(I + \sqrt{N(N-1)} \vec{u} \cdot \vec{\lambda})$ automatically satisfies the conditions (a) and (b). However, not every vector $\vec{u}$, $|\vec{u}| \leq 1$, satisfies the positive condition (c), for example, see [22-23].

In the case of bipartite quantum systems $(H = \mathbb{C}^n \otimes \mathbb{C}^n)$ composed of subsystems A and B, we can analogously represent the density operators as

$$\rho = \frac{1}{n^2} (I_n \otimes I_n + \sum_{i=1}^{n^2-1} r_i \lambda_i \otimes I_n + \sum_{j=1}^{n^2-1} s_j I_n \otimes \tilde{\lambda}_j + \sum_{i,j=1}^{n^2-1} t_{ij} \lambda_i \otimes \tilde{\lambda}_j),$$

(2)

where $\lambda_i$ are the generators of $SU(n)$. Notice that $\vec{r} \in \mathbb{R}^{n^2-1}$ and $\vec{s} \in \mathbb{R}^{n^2-1}$ are the coherence vectors of the subsystems, so that they can be determined locally,

$$\rho_A = \text{Tr}_B \rho = \frac{1}{n} (I_n + r_i \lambda_i),$$

$$\rho_B = \text{Tr}_A \rho = \frac{1}{n} (I_n + s_i \tilde{\lambda}_i).$$

(3)

The coefficients $t_{ij}$, responsible for the possible correlations, form the real matrix $T \in \mathbb{R}^{(n^2-1) \times (n^2-1)}$, and, as before, they can be easily obtained by

$$t_{ij} = n^2 \text{Tr}(\rho \lambda_i \otimes \tilde{\lambda}_j) = n^2 \langle \lambda_i \otimes \tilde{\lambda}_j \rangle.$$

(4)

Now consider observables represented by the following set of hermitian operators:

$$\hat{O}_k = \lambda^a_k \otimes \lambda^b_k,$$

$$\hat{O}_{(n^2-1)^2+1} = \hat{X}^a \otimes \hat{I}^b + \hat{I}^a \otimes \hat{X}^b,$$

(5)
where $i, j = 1, 2, 3...n^2 - 1$, for $i = j = 1, k = 1$, for $i = 1, j = 2, k = 2$, and so on, so $k = 1, 2...(n^2 - 1)^2$. And $\vec{z}, \vec{w} \in \mathbb{R}^{n^2-1}$ with $||\vec{z}|| = ||\vec{w}|| = 1$. We observe that the directions $\vec{z}$ and $\vec{w}$ can be picked out randomly. Now we consider a relation among these observables as follows

$$W_\rho = \sum_{i<j} (n^2 - 1)^2 + 1 ||\langle \hat{O}_i \rangle_\rho \hat{O}_j \rangle_\rho,$$

where $\langle \hat{O}_i \rangle_\rho = \text{Tr}(\hat{O}_i \rho)$ and $|x|$ is the absolute value of $x$. We see that $W_\rho = 0$ if and only if the average value of at least $(n^2 - 1)^2$ of the $(n^2 - 1)^2 + 1$ observables defined above is zero.

So, if $W_\rho = 0$, then $\rho$ must be of the following form:

$$X_{ij} = \rho = \frac{1}{n^2}(I_n \otimes I_n + t_{ij} \lambda_i \otimes \lambda_j),$$

$$X_{(n^2-1)^2+1} = \frac{1}{n^2}(I_n \otimes I_n + \sum_{i=1}^{n^2-1} r_i \lambda_i \otimes I_n + \sum_{j=1}^{n^2-1} s_j I_n \otimes \lambda_j),$$

where $i, j = 1, 2...n^2 - 1$.

The above $X_i$ are all classical states, i.e., they are the form of Eq. (1).

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