Noncommutative Einstein equations

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Abstract
We study a noncommutative deformation of general relativity (called matrix
gravity) proposed recently by one of the authors, in which the gravitational
field is described by a matrix-valued symmetric two-tensor field instead of a
pseudo-Riemannian metric. In this theory all the constructions of Riemannian
geometry, including connection and curvature, become matrix valued, and
the action functional is constructed by using a matrix-valued generalization
of scalar curvature and a matrix-valued measure. In the present paper, we
compute the first variation of the action functional and derive the equations
of motion of matrix gravity. Interestingly, the genuine noncommutative part
of the dynamical equations is described only in terms of a particular tensor
density that vanishes identically in the commutative limit. A noncommutative
generalization of the energy–momentum tensor for the matter field is studied
as well.

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1. Introduction
The basic notions of general relativity (GR) are closely related to the geometric interpretation of
second-order elliptic partial differential operators describing the propagation of fields without
internal structure [1–3] over the spacetime. Let us consider a complete smooth n-dimensional
manifold $M$ without boundary and let $L : C^\infty(M) \rightarrow C^\infty(M)$ be a second-order partial
differential operator acting on smooth functions of the form

$$L = -a^{\mu\nu}(x)\partial_\mu \partial_\nu + b^\mu(x)\partial_\mu + c(x), \quad (1.1)$$

with the smooth real coefficients $a^{\mu\nu}$, $b^\mu$ and $c$.

In particular, $a^{\mu\nu}(x)$ is a real smooth symmetric non-degenerate matrix at each point $x$ of
the manifold $M$. Such a matrix has only real non-zero eigenvalues and the number of negative
eigenvalues is constant throughout the manifold $M$. If the number of negative eigenvalues is
equal to one, then the operator $L$ is hyperbolic and describes propagation of fields, in particular
photons, over the manifold $M$. It can be easily shown that for the operator $L$ to be covariant the matrix $a^{\mu\nu}$ should transform like a tensor of type $(2, 0)$ under diffeomorphisms of the manifold $M$. This enables us to identify the matrix $a^{\mu\nu}$ with a (pseudo)-Riemannian metric $g^{\mu\nu}$ over $M$. Once a metric $g^{\mu\nu}$ is defined one can construct all the geometrical quantities needed to develop the theory of general relativity (GR) (for more details, see [1–3]).

A natural generalization of the GR theory leads us to describe the gravitational field by a $(N \times N)$ matrix of tensor fields $(a^{\mu\nu})^{AB}$ instead of a single metric (for details, see [1–3]). Such a matrix-valued metric appears naturally if one considers the propagation of fields with an internal structure (a multiplet of fields rather than a single field). In this case the coefficients of the operator $L$, that is, the functions $a^{\mu\nu}(x)$, $b^{\mu}(x)$ and $c(x)$, become matrix valued. This follows from the idea that at short distances, or at high energies, the role played by the photon can be played by a multiplet of gauge fields like gluons [1–3].

It is important to stress here that our approach for deforming general relativity is different from those proposed by other authors in the framework of noncommutative geometry [6–8, 10], where the coordinates do not commute and the standard product between functions is replaced by the Moyal product [9].

The formalism of matrix geometry is related to the algebra-valued tangent space formulation of Mann [4] and Wald [5]. In particular, Wald introduced the notion of an algebra-valued tangent space (more generally, algebra-valued tensor fields) and generalized the whole formalism of differential geometry (that is covariant derivatives, metrics, forms and curvature) to the algebra-valued case. The main motivation of Wald was to construct a consistent theory describing the interaction of a collection of massless spin-2 fields. However, his main conclusion was that the consistency conditions can only be satisfied for associative commutative algebras. Therefore, all constructions ‘diagonalize’ (everything commutes) and the interacting theory of a collection of massless spin-2 fields is simply a sum of the usual Einstein–Hilbert actions for each field (no cross-interactions).

Thus our approach is essentially different from the schemes studied by other authors, in particular Mann [4] and Wald [5]. Our main ingredient is the matrix-valued two-tensor field, so that the components of these tensor fields do not commute with each other, in general. Our algebra is associative but noncommutative. The other difference (related to the first one) is the form of the gauge transformations. We start from the very beginning with a real manifold with real-valued coordinates and the usual diffeomorphism group, so that there is no problem of defining the finite gauge transformations. Our gauge group is simply the product of diffeomorphisms and an internal group. That is why we do not have the inconsistences studied by [5] and we can allow our algebra to be noncommutative. In our approach there is a collection of tensor spin-2 fields, but only one of them is massless, all other fields are massive. How exactly this happens depends on the details of the symmetry breaking mechanism etc, but since we have only one (the usual real-valued) diffeomorphism group, only one field is massless. The parameter of our gauge transformations is a real-valued vector field, not the algebra-valued vector fields of [5] needed to describe multiple massless spin-2 fields, which transform independently of each other.

In the previous papers [1, 2] some kind of ‘matrix geometry’ was developed. There, a matrix-valued metric, affine connection, torsion and curvature were defined and a particular form of a compatibility condition was found that enabled one to explicitly express the connection in terms of the derivatives of the metric. An action functional was constructed that: (i) contains no more than two derivatives of the fields in such a way that the equations of motion contain no more than two derivatives, (ii) is invariant under both diffeomorphisms and the gauge transformations, (iii) reduces to the sum of the Einstein–Hilbert action and the Yang–Mills action in the weak deformation limit. As it was mentioned in [3] several
choices have to be made when generalizing the real-valued geometry to the matrix-valued one. In particular, the definition of the ’measure’, raising and lowering indices, definition of higher-order covariant derivatives, torsion constraints and some others. As a result the action constructed in [1, 2] is not unique.

In the paper [3] an additional low energy limit principle was proposed and a new action functional was constructed in the spirit of the ’induced gravity’ approach. This action has all of the above properties but is also unique (or natural). It is well known that the usual Einstein–Hilbert action (with a cosmological term) is just the first two terms of the asymptotic expansion of the effective action, or, simply, the first two coefficients of the asymptotic expansion of the trace of the heat kernel for a certain invariant second-order self-adjoint elliptic Laplace-type (in Euclidean formulation) partial differential operator with a scalar leading symbol given by the Riemannian metric. That is why, in [3] it was proposed to define the action in a similar way as a linear combination of the first two coefficients of the trace of the heat kernel for a more general non-Laplace-type differential operator, with a matrix-valued leading symbol given by the matrix-valued metric.

The purpose of this paper is to derive the equations of motion for the matrix-valued metric $a^{\mu\nu}$ (noncommutative Einstein’s equations) from the action functional proposed in [1, 2]. We intend to study the action of the paper [3], which is more complicated, in a future work.

The outline of the paper is as follows. First we develop the necessary formalism that will be used in the derivation of the dynamical equations of the theory. More precisely we will describe a generalization of various important geometrical objects to endomorphism-valued quantities. Then we describe the construction of the action proposed in [1, 2], that is, invariant under diffeomorphisms of the manifold $M$ and gauge transformations and contains the matrix-valued scalar curvature. Finally, by varying the action with respect to the new dynamical field $a^{\mu\nu}$, we obtain the equations of motion. In conclusion we summarize the results and discuss possible physical interpretation of matrix gravity and some open problems, in particular, quantization.

2. Matrix geometry

We will describe, in this section, the formalism needed in order to write the action for matrix general relativity following [1–3] and derive the equations of motion of the theory. This formalism is a generalization of differential geometry, which is the natural language of general relativity, and it will be called matrix geometry. As already stated before, the main idea is to describe the gravitational field as a matrix-valued symmetric two-tensor field. Let $M$ be an $n$-dimensional Riemannian manifold without boundary and let $g^{\mu\nu}$ be a metric tensor defined on the tangent bundle $TM$. It is well known that general relativity is nothing but the dynamical theory of the metric two-tensor field which is, basically, an isomorphism between tangent and cotangent bundles. In our model the metric two-tensor field is replaced by an endomorphism-valued two-tensor field $a^{\mu\nu}$ which represents an isomorphism of more general bundles over the manifold $M$. The main idea here, similar to general relativity, is to develop a dynamical theory of this endomorphism-valued two-tensor field $a^{\mu\nu}$. This generality brings a much richer structure and content to the model.

Let $V$ be an $N$-dimensional Hermitian vector bundle over $M$, let $\mathcal{F} = TM \otimes V$ be the bundle constructed by taking the tensor product of the tangent bundle to the manifold $M$ with the vector bundle $V$, and let $\mathcal{F}^* = T^*M \otimes V$, where $T^*M$ is the cotangent bundle to $M$. We consider, then, the bundle $\text{Iso}(\mathcal{F}, \mathcal{F}^*)$, elements of which are isomorphisms between $\mathcal{F}$ and $\mathcal{F}^*$. In general the isomorphisms $B : \mathcal{F} \longrightarrow \mathcal{F}^*$ can be identified with the sections of the
bundle $T^* M \otimes T^* M \otimes \text{End}(V)$ and the isomorphisms $A : \mathcal{T}^* \longrightarrow \mathcal{T}$ with the sections of the bundle $TM \otimes TM \otimes \text{End}(V)$.

Let $a$ be a symmetric self-adjoint element of $TM \otimes TM \otimes \text{End}(V)$, that is,

$$a^{\mu
u} = a^{\nu\mu}, \quad (a^{\mu
u})^* = a^{\mu
u}. \quad (2.1)$$

This element is an isomorphism between the bundles $\mathcal{T}^*$ and $\mathcal{T}$ if the following equation:

$$a^{\mu\nu}\phi_{\nu} = \psi_{\mu}, \quad (2.2)$$

with arbitrary $\psi \in \mathcal{T}^*$, has a unique solution

$$\phi_{\nu} = b_{\nu\mu}\psi_{\mu}. \quad (2.3)$$

This last requirement can be cast in another form; the element $a$ is an isomorphism if the following equation:

$$a^{\mu\nu}b_{\nu\rho} = b_{\mu\nu}a_{\nu\rho} = \delta_{\rho}^{\mu} \cdot \mathbb{I}, \quad (2.4)$$

has a unique solution $b \in T^* M \otimes T^* M \otimes \text{End}(V)$.

There are some properties of the matrix $b_{\mu\nu}$ that need attention. The first property is the following: the matrix $b_{\mu\nu}$ satisfies the following equation,

$$b^{\ast}_{\mu\nu} = b_{\nu\mu}, \quad (2.5)$$

but it is not necessarily a self-adjoint matrix symmetric in its tensor indices. Moreover, one can use $a^{\mu\nu}$ and $b_{\mu\nu}$ to lower and raise indices, although particular care is required in these operations because, in general, $a^{\mu\nu}$ and $b_{\mu\nu}$ do not commute and $b_{\mu\nu}$ is not symmetric in its tensorial indices [1].

We need, now, to develop some kind of invariant calculus. For this purpose, we introduce matrix-valued Christoffel symbols $\mathcal{A}^{\mu\alpha\beta}_{\nu}$ that transform, under diffeomorphisms of $M$, as connection coefficients, namely

$$\mathcal{A}^{\mu\nu}_{\rho}(x') = \frac{\partial x^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x^{\mu}} \mathcal{A}^{\nu\gamma}_{\lambda}(x) + \frac{\partial x^{\mu}}{\partial x^{\nu}} \frac{\partial^2 x^{\rho}}{\partial x^{\mu} \partial x^{\tau}} \mathcal{B}^{\nu\beta}_{\tau}, \quad (2.6)$$

where $\mathbb{I}$ is the identity matrix. Once we have a connection on $M$ we can define a way to differentiate tensors.

Let $T^p_q(M)$ be the tensor bundle of type $(p, q)$ on the manifold $M$. We define the following linear map:

$$D : T^p_q \otimes V \longrightarrow T^{p+1}_q \otimes V \quad (2.7)$$

by

$$D_{\nu} \phi_{\mu_1...\mu_p} = \partial_{\nu} \phi_{\mu_1...\mu_p} + \sum_{j=1}^{p} \mathcal{A}^{\mu\nu}_{\lambda} \phi_{\mu_1...\lambda...\mu_j...\mu_p} - \sum_{j=1}^{q} \mathcal{B}^{\nu\rho}_{\gamma} \phi_{\gamma\mu_1...\lambda...\mu_j...\mu_p}. \quad (2.8)$$

This map is a well-defined operator between $T^p_q \otimes V$ and $T^{p+1}_q \otimes V$. It is important to stress, at this point, that the linear map defined in (2.8) is not a covariant differentiation. In fact, it is easy to show that, because of the noncommutativity, the Leibnitz rule does not hold. However, in the commutative limit, the operator (2.7) reduces to an ordinary covariant derivative on the manifold $M$.

In complete analogy with the ordinary Riemannian geometry, we shall define the matrix curvature and the matrix torsion tensors. Let $\phi \in T^* M \otimes V$, we compute

$$D_{\mu} D_{\nu}\phi_{\alpha} - D_{\nu} D_{\mu}\phi_{\alpha} = -\mathcal{R}^{\lambda}_{\mu\nu\alpha} \phi_{\lambda} + T^{\lambda}_{\mu\nu}\phi_{\alpha}, \quad (2.9)$$

where

$$\mathcal{R}^{\lambda}_{\mu\nu\alpha} = \partial_{\mu} \mathcal{A}^{\lambda\nu}_{\alpha} - \partial_{\nu} \mathcal{A}^{\lambda\mu}_{\alpha} + \mathcal{A}^{\lambda\beta}_{\mu} \mathcal{A}^{\beta\rho}_{\nu} - \mathcal{A}^{\lambda\beta}_{\nu} \mathcal{A}^{\beta\rho}_{\mu}. \quad (2.10)$$
and
\[ T_{\mu\nu}^\lambda = \xi^\rho_{\mu\nu} - \eta^\rho_{\nu\mu}. \]  

(2.11)

Once the matrix curvature (Riemann) tensor is defined we can construct the matrix Ricci tensor, namely
\[ R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}. \]  

(2.12)

In order to write the action for matrix gravity, we need to introduce the matrix scalar curvature \( R \). Since the metric \( a_{\mu\nu} \) and the Ricci tensor \( R_{\mu\nu} \) are matrices, they do not commute in general and the definition of the scalar curvature, obtained by contracting the metric tensor with the Ricci tensor on the right. In order to avoid this choice, we use a symmetrized definition of the matrix-valued scalar curvature as follows:
\[ R = \frac{1}{2} (a_{\mu\nu} R^{\mu\nu} + R^{\mu\nu} a_{\mu\nu}). \]  

(2.13)

We need, now, to relate the connection coefficients \( A_{\alpha\lambda\mu\nu} \) to the metric tensor \( a_{\mu\nu} \). We impose a compatibility condition similar to the one in Riemannian geometry as
\[ \partial_{\mu} a_{\alpha\beta} + A_{\alpha\lambda\mu} a_{\lambda\beta} + A_{\beta\lambda\mu} a_{\alpha\lambda} = 0. \]  

(2.14)

The above equation has the following solution in a closed form [1]:
\[ A_{\alpha\lambda\mu} = \frac{1}{2} b_{\lambda\sigma} (a_{\alpha\gamma} \partial_{\gamma} a_{\rho\sigma} - a_{\rho\gamma} \partial_{\gamma} a_{\alpha\sigma} - a_{\sigma\gamma} \partial_{\gamma} a_{\alpha\rho} + S_{\alpha\rho\sigma} + S_{\rho\sigma\alpha} + S_{\sigma\rho\alpha}) b_{\rho\mu}, \]  

(2.15)

where \( S \) is an arbitrary matrix-valued tensor that satisfies the symmetry property
\[ S_{\alpha\mu\nu} = -S_{\alpha\nu\mu}. \]  

(2.16)

The matrix-valued tensor \( S \) is related, in the general case, to the torsion \( T \) in (2.11). Moreover, in the commutative limit, the tensor \( S \) reduces exactly to the torsion. It is important to note that in the matrix geometry connection (2.15) is not symmetric in the two lower indices even if the tensor \( S \) vanishes. In the rest of the paper we will assume, without loss of generality, that the tensor \( S \) vanishes.

In order to write an action for the model under consideration we need a generalization of the concept of measure. As a guiding principle, any generalization of the measure \( \mu \) has to lead, in the commutative limit, to the ordinary Riemannian measure \( \sqrt{\det g_{\mu\nu}} \). Moreover the measure \( \mu \) is a density depending only on the metric \( a_{\mu\nu} \) and not on its derivatives and transforming, under diffeomorphisms of \( M \), as
\[ \mu'(x') = J(x) \mu(x), \]  

(2.17)

where
\[ J(x) = \det \left( \frac{\partial x'^\mu(x)}{\partial x^\nu} \right). \]  

(2.18)

A definition of the measure \( \mu \) which is a straightforward generalization of the Riemannian measure, is the following [1]:
\[ \mu = \frac{1}{N} \text{Tr}_V \rho, \]  

(2.19)

where \( \rho \) is a matrix-valued scalar density, which can be defined, for example, as follows:
\[ \rho = \int_{\mathbb{R}^N} \frac{d\xi}{\pi^N} \exp(-a a_{\mu\nu} \xi_{\mu} \xi_{\nu}). \]  

(2.20)

Then \( \rho \) only depends on the metric \( a \) and transforms in the correct way under diffeomorphisms of \( M \). We would like to stress, here, that the choice of the measure is not unique. However, the
definition (2.19), together with (2.20), seems to be the most natural because it represents the quantity that appears as $A_0$ coefficient in the heat kernel asymptotic of a generalized Laplace operator with matrix-valued symbol defined on the manifold $M$ under consideration [3]. Of course different choices of the measure would lead to different noncommutative limits of the theory. More precisely, the zeroth order of the expansion in the deformation parameter of the action always gives the usual GR. The second-order term, which gives the dynamical equations for the noncommutative part of the metric, instead, changes if the definition of measure is different. Further studies are required in order to fully understand how the choice of the measure affects the dynamics of the noncommutative part of the metric.

3. The action of matrix gravity and its first variation

We construct the action functional for the field $a^\mu{}^\nu$ following [1]. This functional has to be invariant under both diffeomorphisms of $M$ and gauge transformations. The infinitesimal form of these transformations is

$$\delta_\omega a^\mu{}^\nu = [\omega, a^\mu{}^\nu],$$

and

$$\delta_\xi a^\mu{}^\nu = \mathcal{L}_\xi a^\mu{}^\nu,$$

where $\omega$ is an element of the algebra of the gauge group $\delta_\omega U = \omega$, and $\xi$ is the generator of the infinitesimal coordinate transformation $\delta_\xi x^\mu = \mathcal{L}_\xi x^\mu = -\xi^\mu$.

We can construct an action functional for the field $a^\mu{}^\nu$ that satisfies the properties described above, by using the matrix-valued scalar curvature, defined in (2.13), and the measure (2.19). A good candidate for the action is the following:

$$S_{MGR}(a) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr}_V (\rho R).$$

(3.3)

It is worth noting that because of the cyclic property of the trace, the relative position of $\rho$ and the scalar curvature is irrelevant; moreover, it is easily shown that the action functional (3.3) is invariant under the diffeomorphisms (3.2) and under the gauge transformations (3.1).

Since the action has the invariant properties discussed above, the currents associated with the symmetries satisfy the identities

$$\partial_\mu \left( a^\mu{}^\nu \frac{\delta S}{\delta a^\mu{}^\nu} \right) + \frac{1}{2} (\partial_\nu a^\mu{}^\nu) \frac{\delta S}{\delta a^\mu{}^\nu} = 0,$$

(3.4)

and

$$\left[ a^\mu{}^\nu, \frac{\delta S}{\delta a^\mu{}^\nu} \right] = 0,$$

(3.5)

where (3.4) is the current generated by the invariance with respect to diffeomorphisms and (3.5) is the current generated by the internal (gauge) symmetry. The above identities represent an endomorphism-valued generalization of the Noether identities, which are related, in the usual theory, to the contracted Bianchi identities.

Now that we have the action functional we can derive the equations of motion for the field $a^\mu{}^\nu$, which is the main goal and the main result of the present paper. These equations will be matrix valued and they will constitute a generalization of the ordinary Einstein equations that
we will call noncommutative Einstein equations. In order to find the dynamics of the model we vary the action (3.3) with respect to the field $a^{\mu \nu}$ considered as independent variable, namely

$$a^{\mu \nu} \rightarrow a^{\mu \nu} + \delta a^{\mu \nu}. $$

By doing so we obtain, for the variation of the action, the following:

$$\delta S = S(a^{\mu \nu} + \delta a^{\mu \nu}) - S(a^{\mu \nu}) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr} V (G_{\mu \nu} \delta a^{\mu \nu}), $$

(3.6)

where $G_{\mu \nu}$ is some matrix-valued symmetric tensor density. Then, of course, the desired equations of motion are

$$G_{\mu \nu} = 0. $$

(3.7)

It is important to note that the matrix-valued tensor density (3.7) has to coincide with the Einstein tensor in the commutative limit, more precisely we need that, in the commutative limit, the following relation holds:

$$\sqrt{g} \left( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \right) = \frac{1}{N} \text{Tr} G_{\mu \nu}. $$

(3.8)

Our main task, then, is to find the explicit form of the equations of motion that result from the variation of the action (3.3). In all the calculations that will follow the order of the terms is important, unless explicitly stated, due to their matrix nature.

First, we rewrite the action in a more explicit form which is more suitable for the subsequent variation, namely

$$S_{\text{MGR}}(a) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr} \left[ \rho \frac{1}{2} \{ a^{\mu \nu} R_{\mu \nu} + R_{\mu \nu} a^{\mu \nu} \} \right]. $$

(3.9)

By varying the terms in (3.9) with respect to the independent field $a^{\mu \nu}$, and by using the cyclic property of the trace we get

$$\delta S_{\text{MGR}}(a) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr} \left[ \delta \rho R + \frac{1}{2} \{ R_{\mu \nu}, \rho \} \delta a^{\mu \nu} + \frac{1}{2} \{ \rho, a^{\mu \nu} \} \delta R_{\mu \nu} \right], $$

(3.10)

where the curly brackets $\{, \}$ denote anti-commutation, namely $[A, B] = AB - BA$.

From expressions (2.10) and (2.12), we can evaluate the variation of the matrix-valued Ricci tensor; more precisely, we have

$$\delta R_{\mu \nu} = \partial_{(\mu} (\delta a^{\alpha \nu}) - \partial_{(\mu} (\delta a^{\nu \alpha}) + \delta a^{\alpha \mu} \partial_{(\nu} \delta a_{\alpha \mu}) + \delta a^{\mu \nu} \partial_{(\alpha} \delta a^{\lambda) \nu} - \delta a^{\alpha \mu} \delta a^{\lambda} (\mu) - \delta a^{\mu \nu} \delta a^{\lambda} (\alpha). $$

(3.11)

From now on, for simplicity of notation, we set

$$B^{\mu \nu} \equiv \{ \rho, a^{\mu \nu} \}. $$

(12)

By substituting (3.11) into (3.10), and by using the cyclic property of the trace we obtain

$$\delta S_{\text{MGR}}(a) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr} \left[ \delta \rho R + \frac{1}{2} \{ R_{\mu \nu}, \rho \} \delta a^{\mu \nu} + \frac{1}{2} B^{\mu \nu} \partial_{(\mu} (\delta a^{\alpha \nu}) - \frac{1}{2} B^{\mu \nu} \partial_{(\mu} \delta a^{\alpha \nu} + \frac{1}{2} B^{\mu \nu} \delta a^{\alpha \mu} \partial_{(\nu} \delta a^{\lambda) \mu} - \frac{1}{2} \delta a^{\mu \nu} \delta a^{\alpha \mu} \partial_{(\alpha} \delta a^{\lambda) \nu} - \frac{1}{2} B^{\mu \nu} \delta a^{\alpha \mu} \delta a^{\lambda} (\nu) - \frac{1}{2} B^{\mu \nu} \delta a^{\alpha \nu} \delta a^{\lambda} (\mu) \right]. $$

(3.13)

By integrating by parts and by collecting similar terms we get

$$\delta S_{\text{MGR}}(a) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr} \left[ \delta \rho R + \frac{1}{2} \{ R_{\mu \nu}, \rho \} \delta a^{\mu \nu} - \frac{1}{2} \left( B^{\mu \nu} + B^{\nu \mu} \delta a^{\lambda} (\mu) \right) \delta a^{\alpha \mu} \partial_{(\lambda} \delta a^{\lambda) \mu} + \frac{1}{2} \left( B^{\mu \nu} + B^{\nu \mu} \delta a^{\lambda} (\nu) \right) \partial_{(\lambda} \delta a^{\lambda) \mu} \right]. $$

(3.14)
We can rewrite the last expression in a more compact form, namely
\[
\delta S_{\text{MGR}}(a) = \frac{1}{16\pi G} \int_M \frac{1}{N} \text{Tr}_V \left[ \delta \rho R + \frac{1}{2} \{ R_{\mu \nu}, \rho \} \delta a^\mu_{\nu} - \frac{1}{2} C^{\mu \nu}_{\alpha \beta} \delta \omega^\alpha_{\mu \nu} + \frac{1}{2} D^\mu \delta \omega^\alpha_{\mu \nu} \right],
\]
(3.15)
where the matrix-valued tensor densities \( C^{\mu \nu}_{\alpha \beta} \) and \( D^\mu \) have the explicit expression
\[
C^{\mu \nu}_{\alpha \beta} = \{ a^{\mu \nu}, \rho, \omega^\lambda_{\alpha \lambda} \} - \rho \{ a^{\mu \nu}, \omega^\lambda_{\alpha \lambda} \} - \rho \{ \omega^\mu_{\nu \alpha}, a^{\mu \nu} \} - \frac{1}{2} \rho \{ a^{\mu \nu}, \omega^\nu_{\alpha \lambda} \},
\]
(3.16)
and
\[
D^\mu = \{ a^{\mu \nu}, \rho, \omega^\rho_{\nu \rho} \} - \rho \{ \omega^\mu_{\nu \rho}, a^{\mu \nu} \} - \rho \{ \omega^\mu_{\nu \rho}, a^{\mu \nu} \}.
\]
(3.17)
It is worth noting that in the commutative limit, or, in other words, when all the matrices commute, the tensor densities \( C^{\mu \nu}_{\alpha \beta} \) and \( D^\mu \) are identically zero, and the variation of the action \( \delta S_{\text{MGR}} \) simply reduces to the standard result of the general theory of relativity.

We can write, now, the variation of the connection coefficients. By using expression (2.15), and by noting that
\[
\delta b_{\mu \nu} = -b_{\mu \rho} (\delta a^{\rho \sigma}) b_{\sigma \nu},
\]
we obtain the following:
\[
\delta \omega^\alpha_{\lambda \mu} = -b_{\lambda \sigma} (\delta a^{\rho \sigma}) b_{\mu \rho} + \frac{1}{2} b_{\lambda \sigma} (\delta a^{\rho \sigma}) b_{\rho \mu}
\]
(3.18)
where in this last expression we used the cyclic property of the trace.

We introduce the following definition, which will be useful in order to simplify the notation,
\[
F^{\beta \alpha \rho} = b_{\beta \nu} C^{\mu \nu}_{\alpha} b_{\mu \rho}.
\]
(3.20)
By using the above definition, the expression in (3.19) can be rewritten as follows:
\[
-\frac{1}{2} \int_M \text{d}x \text{Tr}_V \left( C^{\mu \nu}_{\alpha \beta} \delta \omega^\alpha_{\mu \nu} \right)
\]
\[
= \frac{1}{2} \int_M \text{d}x \text{Tr}_V \left[ \delta \omega^\alpha_{\beta \nu} C^{\mu \nu}_{\alpha \beta} b_{\mu \rho} \delta a^{\rho \sigma} b_{\sigma \nu} + b_{\beta \nu} C^{\mu \nu}_{\alpha \beta} b_{\nu \rho} \delta \omega^\rho_{\mu \rho} \right]
\]
(3.19)
where in this last expression we used the cyclic property of the trace.

We introduce the following definition, which will be useful in order to simplify the notation,
\[
F^{\beta \rho \alpha} = b_{\beta \nu} C^{\mu \nu}_{\alpha \beta} b_{\mu \rho}.
\]
(3.20)
By using the above definition, the expression in (3.19) can be rewritten as follows:
\[
-\frac{1}{2} \int_M \text{d}x \text{Tr}_V \left( C^{\mu \nu}_{\alpha \beta} \delta \omega^\alpha_{\mu \nu} \right)
\]
\[
= \frac{1}{2} \int_M \text{d}x \text{Tr}_V \left[ \delta \omega^\alpha_{\beta \nu} C^{\mu \nu}_{\alpha \beta} b_{\mu \rho} \delta a^{\rho \sigma} b_{\sigma \nu} + F^{\beta \rho \alpha} \delta a^{\rho \sigma} \right]
\]
(3.19)
\[-\frac{1}{2} (\partial_\tau a^\mu) F_{\mu\rho\sigma} \delta a^{\rho\sigma} + \frac{1}{2} (\partial_\rho a^\mu) F_{\mu\sigma\tau} \delta a^{\sigma\tau} + \frac{1}{2} (\partial_\sigma a^\mu) F_{\mu\tau\rho} \delta a^{\tau\rho}\]
\[-\frac{1}{2} F_{\mu\rho\sigma} a^{\rho\sigma} (\partial_\tau \delta a^{\mu\sigma}) + \frac{1}{2} F_{\mu\sigma\tau} a^{\sigma\tau} (\partial_\rho \delta a^{\mu\sigma}) + \frac{1}{2} F_{\mu\tau\rho} a^{\tau\rho} (\partial_\sigma \delta a^{\mu\tau})\]. (3.21)

where the first two terms in the last expression has been derived by using the relation
\[\delta a^{\mu\tau} C_{\mu\sigma} a_{\tau\nu} b_{\rho\lambda} \delta a^{\lambda\beta} = \delta a^{\mu\tau} a^{\rho\nu} b_{\rho\lambda} C_{\mu\sigma} a_{\lambda\beta} \delta a^{\lambda\beta}. (3.22)\]

By integrating by parts and by relabeling dummy indices we find the final expression for (3.21), namely
\[-\frac{1}{2} \int_M \delta x Tr\left( C_{\mu\nu} a \delta a^{\mu\nu} \right) = \frac{1}{2} \int_M \delta x Tr\left[ \delta a^{\mu\tau} \rho a^{\tau\nu} F_{\mu\nu} A_{\rho\alpha} + \delta a^{\mu\tau} a^{\nu\rho} G_{\rho\alpha} \delta a^{\lambda\beta} \right] \delta a^{\lambda\beta}
+ \frac{1}{4} \left[ \partial_\gamma \left( (F_{\rho\sigma} (G_{\lambda\tau} - G_{\lambda\rho} - G_{\tau\rho}) a^{\lambda\tau} \right) \right] \delta a^{\sigma\tau}. (3.23)\]

For the last term in the variation of the action (3.15), we use similar arguments which lead us to the expression (3.23). In this case we introduce the following definition:
\[G_{\rho\alpha} = b_{\beta\alpha} D^\mu b_{\mu\rho}. (3.24)\]

By using the definition above and the cyclic property of the trace we obtain
\[\frac{1}{2} \int_M \delta x Tr\left( D^\mu \delta a^{\mu\nu} \right) = \frac{1}{2} \int_M \delta x Tr\left[ \delta a^{\mu\nu} \rho a^{\nu\mu} G_{\gamma\nu} \delta a^{\gamma\beta} + G_{\beta\mu} a^{\gamma\mu} \delta a^{\lambda\beta} \right] \delta a^{\lambda\beta}
- \frac{1}{2} (\partial_\tau a^{\mu}) G_{\rho\sigma\tau} \delta a^{\rho\sigma} + \frac{1}{2} (\partial_\rho a^{\mu}) G_{\rho\sigma\tau} \delta a^{\rho\sigma} + \frac{1}{2} (\partial_\sigma a^{\mu}) G_{\rho\sigma\tau} \delta a^{\rho\sigma}
- \frac{1}{2} G_{\rho\sigma\tau} a^{\tau\rho} (\partial_\sigma \delta a^{\mu\rho}) + \frac{1}{2} G_{\rho\sigma\tau} a^{\tau\rho} (\partial_\rho \delta a^{\mu\sigma}) + \frac{1}{2} G_{\rho\sigma\tau} a^{\tau\rho} (\partial_\tau \delta a^{\mu\sigma})\]. (3.25)

By integrating by parts and relabeling dummy indices we get
\[\frac{1}{2} \int_M \delta x Tr\left( D^\mu \delta a^{\mu\nu} \right) = -\frac{1}{2} \int_M \delta x Tr\left[ \delta a^{\mu\nu} \rho a^{\nu\mu} G_{\gamma\nu} \delta a^{\gamma\beta} + G_{\beta\mu} a^{\gamma\mu} \delta a^{\lambda\beta} \right] \delta a^{\lambda\beta}
- \frac{1}{2} (\partial_\tau a^{\mu}) (G_{\rho\sigma\tau} - G_{\lambda\sigma\rho} - G_{\tau\rho\lambda}) \delta a^{\rho\sigma}
- \frac{1}{4} \left[ \partial_\gamma \left( (G_{\rho\sigma} (G_{\lambda\tau} - G_{\lambda\rho} - G_{\tau\rho}) a^{\lambda\tau} \right) \right] \delta a^{\sigma\tau}. (3.26)\]

It is worth noting that in the above expressions, (3.23) and (3.26), the tensor densities \(F\) and \(G\) always appear in the same combination. This observation justifies the following definitions:
\[X_{\rho\lambda\sigma} = F_{\rho\lambda\sigma} - F_{\lambda\sigma\rho} - F_{\sigma\rho\lambda}, \quad (3.27)\]

and
\[Y_{\rho\lambda\sigma} = G_{\rho\lambda\sigma} - G_{\lambda\sigma\rho} - G_{\sigma\rho\lambda}. \quad (3.28)\]
By using the two definitions above we can rewrite the arguments of the traces in (3.23) and in (3.26), respectively, as

$$-\frac{1}{2} C^{\mu\nu}_{\phi\lambda} \delta_{\alpha\beta} a^\alpha A^\beta_{\mu\nu} = \frac{1}{2} \left[ \delta^\alpha_{\beta\gamma\delta} a^\beta F_{\phi\alpha\lambda} + F_{\phi\lambda\sigma} a^\sigma_{\beta\gamma} \right] X_{\phi\lambda\sigma} - \frac{1}{2} (\partial_\beta a^\sigma_{\gamma\delta}) X_{\phi\lambda\sigma} + \frac{1}{2} \delta_\gamma (X_{\phi\lambda\sigma} a^\sigma_{\gamma\delta}),$$

(3.29)

and

$$\frac{1}{2} D^{\mu}_{\phi\lambda} \delta_{\alpha\beta} a^\alpha_{\mu\nu} = -\frac{1}{2} \left[ \delta^\alpha_{\beta\gamma\delta} a^\beta G_{\phi\alpha\lambda} + G_{\phi\lambda\sigma} a^\sigma_{\beta\gamma} \right] Y_{\phi\lambda\sigma} - \frac{1}{2} (\partial_\beta a^\sigma_{\gamma\delta}) Y_{\phi\lambda\sigma} + \frac{1}{2} \partial_\gamma (Y_{\phi\lambda\sigma} a^\sigma_{\gamma\delta}).$$

(3.30)

By combining the results (3.29) and (3.30) we obtain the expression for the last two terms in the variation of the action, namely

$$-\frac{1}{2} C^{\mu\nu}_{\phi\lambda} \delta_{\alpha\beta} a^\alpha A^\beta_{\mu\nu} + \frac{1}{2} D^{\mu}_{\phi\lambda} \delta_{\alpha\beta} a^\alpha_{\mu\nu} = \frac{1}{2} \left\{ A^\alpha_{\beta\rho\gamma} (Y_{\alpha\lambda\gamma} - X_{\alpha\lambda\gamma}) + A^\alpha_{\beta\rho\gamma} (Y_{\lambda\gamma\alpha} - X_{\lambda\gamma\alpha}) + H_{\alpha\gamma\beta} a_{\rho\gamma} A^\rho_{\alpha_{\mu\nu}} + H_{\gamma\beta\alpha} a_{\rho\gamma} A^\rho_{\alpha_{\mu\nu}} + (\partial_\beta a^\sigma_{\gamma\delta}) H_{\rho\lambda\sigma} - \partial_\gamma (H_{\rho\lambda\sigma} a^\sigma_{\gamma\delta}) \right\} [\delta a^\beta_{\rho\sigma}].$$

(3.31)

4. Noncommutative Einstein equations

With expression (3.31) for the last two terms in (3.15), the variation of the action has the form (3.6) which is suitable for the derivation of the dynamical equations of the model. Before writing the complete dynamical equations, we will simplify further expression (3.31).

Definition (3.27) gives a linear relation between the matrix-valued tensor density $X$ and a particular combination of matrix-valued tensor density $F$; a similar linear relation between $Y$ and $G$ is given in (3.28). By using simple tensor algebra, it can be easily shown that those relations can be inverted, namely we can write

$$F_{\rho\lambda\sigma} = -\frac{1}{2} (Y_{\rho\lambda\sigma} + X_{\rho\lambda\sigma}),$$

(4.1)

and

$$G_{\rho\lambda\sigma} = -\frac{1}{2} (Y_{\rho\lambda\sigma} - X_{\rho\lambda\sigma}).$$

(4.2)

By substituting equations (4.1) and (4.2) into expression (3.31) we obtain the following:

$$-\frac{1}{2} C^{\mu\nu}_{\phi\lambda} \delta_{\alpha\beta} a^\alpha_{\mu\nu} + \frac{1}{2} D^{\mu}_{\phi\lambda} \delta_{\alpha\beta} a^\alpha_{\mu\nu} = \frac{1}{2} \left\{ \delta^\alpha_{\beta\gamma\delta} a^\beta F_{\phi\alpha\lambda} + \delta^\alpha_{\beta\gamma\delta} a^\beta G_{\phi\alpha\lambda} + (\partial_\beta a^\sigma_{\gamma\delta}) (Y_{\rho\lambda\sigma} - X_{\rho\lambda\sigma}) \right\} [\delta a^\beta_{\rho\sigma}].$$

(4.3)

We can see, in the last formula, that the tensor densities $X$ and $Y$ enter always in the same combination. It is useful, therefore, to define the following tensor density:

$$H_{\mu\nu\rho} = Y_{\mu\nu\rho} - X_{\mu\nu\rho}.$$
Since $H_{\mu\nu}$ is a tensor density, we can write

$$D_{\lambda} H_{\rho\beta} = \delta_{\lambda} H_{\rho\beta} - \delta^\alpha_{\rho} \lambda_{\gamma} H_{a\alpha\beta} - \delta^\alpha_{\rho} \beta_{\gamma} H_{a\alpha\beta} - \delta^a_{\rho} \gamma_{\alpha} H_{a\alpha\beta} - \delta^a_{\rho} \alpha_{\lambda} H_{a\alpha\beta}. \tag{4.8}$$

By using the results obtained in (4.6), (4.7) and (4.8) we can express (4.5) as follows:

$$-\frac{1}{3} C_{\mu\nu}^\rho \delta \delta^\rho_{\mu\nu} + \frac{1}{4} D_{\lambda} \delta \delta^\rho_{\mu\nu} = \frac{1}{4} \left\{ 2 \delta_{\rho} (\rho_{\beta}) a_{\alpha} \beta_{\gamma} H_{\lambda} + 2 H_{\alpha\beta} a_{\gamma} (\gamma_{\rho}) - (D_{\gamma} H_{\rho\beta}) a_{\rho\gamma} \right\}
- \left[ \delta_{\rho} a_{\gamma}, H_{\lambda\rho} \right] a_{\rho\gamma} - \delta_{\rho} a_{\gamma} [H_{a\alpha\rho}, a_{\rho\gamma}] - \delta_{\rho} a_{\gamma} [H_{a\alpha\rho}, a_{\rho\gamma}] \right\} \delta a^\lambda_{\beta}. \tag{4.9}$$

At this point we introduce the operator $P$ defined as

$$P_{\gamma} H_{\lambda\rho\beta} = D H_{\rho\beta} + \left[ \delta_{\gamma} a_{\lambda\rho}, H_{\rho\beta} \right] + \left[ \delta_{\gamma} a_{\rho\beta}, H_{\lambda\rho} \right] + \left[ \delta_{\gamma} a_{\rho\beta}, H_{\lambda\rho} \right] + \left[ \delta_{\gamma} a_{\rho\beta}, H_{\rho\beta} \right]. \tag{4.10}$$

By using the last definition in (4.9) one obtains

$$-\frac{1}{3} C_{\mu\nu}^\rho \delta \delta^\rho_{\mu\nu} + \frac{1}{4} D_{\lambda} \delta \delta^\rho_{\mu\nu} = \frac{1}{4} \left\{ 2 \delta_{\rho} (\rho_{\beta}) a_{\alpha} \beta_{\gamma} H_{\lambda} + 2 H_{\alpha\beta} a_{\gamma} (\gamma_{\rho}) - (P_{\gamma} H_{\rho\beta}) a_{\rho\gamma} \right\}
+ \left[ \delta_{\rho} a_{\gamma}, H_{\lambda\rho} \right] a_{\rho\gamma} - \delta_{\rho} a_{\gamma} [H_{a\alpha\rho}, a_{\rho\gamma}] - \delta_{\rho} a_{\gamma} [H_{a\alpha\rho}, a_{\rho\gamma}] \right\} \delta a^\lambda_{\beta}. \tag{4.11}$$

We finally have all the ingredients that we need in order to write down the equations of the theory. Now we only have to find an expression for the variation $\delta \rho$. The definition of $\rho$ is given in (2.20), and its variation can be straightforwardly evaluated as follows:

$$\delta \rho = \int_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} \int_0^1 ds e^{-(1-s)A(\xi)} \delta a^{\mu\nu} \xi_{\mu} \xi_{\nu} e^{-sA(\xi)}, \tag{4.12}$$

where

$$A(\xi) = a^{\mu\nu} \xi_{\mu} \xi_{\nu}. \tag{4.13}$$

Once we have expression (4.12) for the variation, we can use the cyclic property of the trace to write that

$$\mathrm{Tr}_V(\delta \rho \mathcal{R}) = \mathrm{Tr}_V \left[ - \int_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} \int_0^1 ds e^{-(1-s)A(\xi)} \mathcal{R} e^{-sA(\xi)} \delta a^{\mu\nu} \xi_{\mu} \xi_{\nu} \right]. \tag{4.14}$$

By combining (4.14), (4.11) and (3.15) we obtain the noncommutative Einstein equations in absence of matter, namely

$$\mathcal{G}_{\mu\nu} = 0, \tag{4.15}$$

where

$$\mathcal{G}_{\mu\nu} = \frac{1}{3} \left\{ \rho, \mathcal{R}_{\mu\nu} \right\} + \mathcal{F}_{\mu\nu} + \frac{1}{4} \delta_{\rho} a_{\alpha} (\alpha_{\rho} a_{\beta} H_{\gamma} + 2 H_{\alpha\beta} a_{\gamma} (\gamma_{\rho}) - (P_{\gamma} H_{\rho\beta}) a_{\rho\gamma} \right\}
+ \left[ \delta_{\rho} a_{\gamma}, H_{\lambda\rho} \right] a_{\rho\gamma} - \delta_{\rho} a_{\gamma} [H_{a\alpha\rho}, a_{\rho\gamma}] - \delta_{\rho} a_{\gamma} [H_{a\alpha\rho}, a_{\rho\gamma}] \right\} \delta a^\lambda_{\beta}. \tag{4.16}$$

is the noncommutative Einstein tensor, $\mathcal{F}_{\mu\nu}$ is defined by

$$\mathcal{F}_{\mu\nu} = - \int_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} \int_0^1 ds e^{-(1-s)A(\xi)} \mathcal{R} e^{-sA(\xi)} \delta a^{\mu\nu} \xi_{\mu} \xi_{\nu}, \tag{4.17}$$

and the tensor density $H$ has the explicit form

$$H_{\alpha\gamma} = b_{\alpha\gamma} (\delta^\rho_{\alpha} \rho_{\mu} - C_{\alpha\rho}^\mu) b_{\mu\gamma} - b_{\alpha\gamma} (\delta^\rho_{\alpha} \rho_{\mu} - C_{\alpha\rho}^\mu) b_{\mu\gamma} - b_{\alpha\gamma} (\delta^\rho_{\alpha} \rho_{\mu} - C_{\alpha\rho}^\mu) b_{\mu\gamma}.$$

These equations are the main result of the present paper. One can show that the first two terms in equations (4.16) represent a straightforward generalization of Einstein’s equation to endomorphism-valued objects and the rest of the terms can be considered as a genuine noncommutative part which is not present in Einstein’s equation. It is interesting to note that
the pure noncommutative part is completely described by the tensor density $H_{\mu\nu\rho}$ defined in (4.18).

Moreover, equation (4.15) satisfies requirement (3.8), which, in words, expresses the necessity that our model reduces, in the commutative limit, to the standard theory of general relativity. In fact, the trace of the pure noncommutative terms vanishes, because of the presence of the commutators, and the first two terms just give

$$
\frac{1}{N} \text{Tr}_V \left( \frac{1}{2} \{ \rho, R_{\mu\nu} \} + F_{\mu\nu} \right) = \sqrt{g} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right). \tag{4.19}
$$

For an arbitrary matrix algebra equation (4.15) becomes more complicated than the ordinary Einstein equation due to presence of the new tensor density $H_{\mu\nu\rho}$. We mention, now, a particular case in which (4.15) simplifies. The formalism used so far deals with geometric quantities which are endomorphism valued, namely they take values in $\text{End}(V)$. By choosing a basis in the vector space $V$ we can represent $\text{End}(V)$ by means of matrices. Let us suppose that the algebra under consideration is Abelian; in this case all the elements commute with each other and the tensor density $H_{\mu\nu\rho}$ vanishes identically and equation (4.15) becomes

$$
R_{\mu\nu} - \frac{1}{2} b_{\mu\nu} R = 0. \tag{4.20}
$$

Therefore, in the case of a commutative matrix algebra, the equation of motion of our model have the same form as Einstein’s equation, with the only difference that (4.20) is matrix valued.

5. Scalar fields in matrix gravity

In order to have a complete theory for the gravitational field we need to describe the dynamics of the matter field in the framework of matrix general relativity. The main idea is to extend the general results of classical field theory. We will consider, in the following, the dynamics of a multiplet of free scalar fields propagating on a manifold $M$. We can construct an invariant action by using the matrix-valued metric $a_{\mu\nu}$ and the measure $\rho$. A typical action is

$$
S_{\text{matter}}(a, \varphi) = \frac{1}{4} \int_M dx \left\{ -\langle \partial_\mu \varphi, \{ \rho, a_{\mu\nu} \} \partial_\nu \varphi \rangle - \langle \varphi, \{ \rho, Q \} \varphi \rangle \right\}, \tag{5.1}
$$

where $\langle , \rangle$ denotes the fiber inner product on the vector bundle $V$, and $Q$ is a constant mass matrix determining the masses of the scalar fields. The equations of motion of the scalar fields are then obviously

$$
[-\partial_\mu \{ \rho, a_{\mu\nu} \} \partial_\nu + \{ \rho, Q \}] \varphi = 0. \tag{5.2}
$$

The complete action of the gravity and matter is described then by

$$
S(a, \varphi) = S_{\text{MGR}}(a) + S_{\text{matter}}(a, \varphi). \tag{5.3}
$$

By varying the above action with respect to $a_{\mu\nu}$ one obtains the noncommutative Einstein equation in the presence of matter

$$
G_{\mu\nu} = 8\pi G N T_{\mu\nu}, \tag{5.4}
$$

where $T_{\mu\nu}$ is the matrix energy–momentum tensor defined by

$$
T_{\mu\nu} = -\frac{1}{2} \frac{\delta S_{\text{matter}}}{\delta a_{\mu\nu}}. \tag{5.5}
$$

By using the explicit Lagrangian (5.1) for the matter field, we obtain the expression for the energy–momentum tensor

$$
T_{\mu\nu} = \frac{1}{8}[\{ \rho, \partial_\mu \varphi \otimes \partial_\nu \varphi \} + M_{\mu\nu} + N_{\mu\nu}] + (\mu \leftrightarrow \nu), \tag{5.6}
$$

where the explicit form of $\mathcal{M}_{\mu\nu}$ and $\mathcal{N}_{\mu\nu}$ is obtained by using the variation of the scalar density $\rho$ in (4.12), namely

$$
\mathcal{M}_{\mu\nu} = - \int_{\mathbb{R}^3} \frac{d\xi}{\pi^2} \int_0^1 ds \, e^{-sA(\xi)} \left\{ \partial_{\alpha} \phi \otimes \partial_{\beta} \phi \right\} \left( e^{-(1-s)A(\xi)} \xi_\mu \xi_\nu - \sqrt{g} \right) \left( e^{-(1-s)A(\xi)} \xi_\mu \xi_\nu - \sqrt{g} \right),
$$

(5.7)

$$
\mathcal{N}_{\mu\nu} = - \int_{\mathbb{R}^3} \frac{d\xi}{\pi^2} \int_0^1 ds \, e^{-sA(\xi)} \left\{ Q, \phi \otimes \phi \right\} \left( e^{-(1-s)A(\xi)} \xi_\mu \xi_\nu - \sqrt{g} \right),
$$

(5.8)

It is worth remarking, here, that the above formula (5.6) for the energy–momentum tensor $T_{\mu\nu}$ reduces, in the commutative limit, to the standard result, e.g. [6].

6. Conclusions

In this paper we studied a noncommutative deformation of general relativity, called matrix gravity, proposed in [1, 2]. The main idea of matrix gravity is to describe the gravitational field by a matrix-valued two-tensor field $a_{\mu\nu}$ instead of the usual Riemannian metric $g_{\mu\nu}$. The action functional proposed in [1, 2] is a straightforward generalization of the Einstein–Hilbert action. A careful analysis shows that there is, however, some intrinsic arbitrariness in the model, related, in particular, to a choice of the matrix-valued measure $\rho$, for example, as in equation (2.20). In principle, different choices would lead to slightly different equations. Our choice is motivated by a natural measure appearing in the approach of ‘induced gravity’ advocated in [3]. It is not difficult to modify our formulae to other choices of the density $\rho$.

The main result of this paper is the derivation of the equations of motion without matter (4.15) and with scalar matter fields (5.4). It is interesting that the noncommutative part of the modified equations only depends on a specific tensor density $H_{\mu\nu\rho}$ and on a linear combination of its commutators.

With an explicit expression for the noncommutative Einstein equation, it will be possible to study some particular simple solutions of (4.15), in particular, a static and spherically symmetric solution (‘noncommutative black hole’), or a homogeneous solution (‘noncommutative cosmology’). These simple examples are very interesting and we plan to study them systematically in a future work.

We would like to make some final remarks about the possible physical applications of this mathematical model. Matrix gravity is just some field theory for the tensor $a_{\mu\nu}$. Since the kinetic part depends on the fields, in the field-theoretic language it is some sort of a generalized nonlinear sigma model. Like any other field theory, for example, general relativity or Yang–Mills theory, it can be studied both at the classical and the quantum level. The equations of motion obtained in this paper are classical field equations and should be studied as such. Of course, they also play a role in the quantization (a mass shell). In general, the matrix-valued metric $\tilde{a}^{\mu\nu} = a^{\mu\nu} + \psi^{\mu\nu}$ has a non-zero classical part $a^{\mu\nu}$ and some quantum part $\psi^{\mu\nu}$. The classical part $a^{\mu\nu}$ can be decomposed as $a^{\mu\nu} = g^{\mu\nu}I + h^{\mu\nu}$, where $g^{\mu\nu} = N^{-1} tr a^{\mu\nu}$ is the commutative part and $h^{\mu\nu}$ is a noncommutative part; of course, $tr h^{\mu\nu} = 0$. The classical matrix-valued metric should satisfy the classical equations of motion. Whether or not the noncommutative part $h^{\mu\nu}$ is zero depends on the solution of these equations. If there is a non-trivial noncommutative solution, then the noncommutative part of the matrix-valued metric can have a classical non-zero part. Then, of course, one has to come up with a physical interpretation of those classical fields (one can speculate here about the dark matter, the dark energy, Pioneer anomaly, etc).

Further, the matrix geometry described above is intimately related to Finsler geometry rather than to Riemannian geometry (for details, see [1, 3]). Since there is no preferred metric
the structure of the geodesic flow in Finsler geometry is radically different—instead of a single geodesic there is a bunch of trajectories close to each other. In a sense, a Riemannian geodesic 'splits' or becomes 'fuzzy'. One can hope that the same applies to the singularities of the Riemannian geometry; if this is correct then this would mean that: (i) classically a collapsing body forms a dense compact object but without singularity and (ii) the ultraviolet divergences at the quantum level should be weaker than in general relativity. Of course, at the present stage this is just a speculation. This is an interesting question that requires further study.

Although the main purpose of the present paper is just the derivation of the classical equations of motion, let us make some remarks about the quantization following the discussion in [3]. It seems that this model will have the standard renormalization problems of general relativity. At the present time we can only speculate on possible avenues for quantization. An interesting idea is to introduce some kind of spontaneous breakdown of symmetry, so that in the broken phase in the vacuum there is just one tensor field, which is identified with the metric of the spacetime; note that in this case $g^{\mu\nu} \neq N^{-1} \text{tr} a^{\mu\nu}$ but rather $a^{\mu\nu} = g^{\mu\nu} \Pi$ with some constant matrix $\Pi \neq \mathbb{I}$. All other tensor fields must have zero vacuum expectation values. One could expect that the behavior of our model at higher energies should be different from the Einstein gravity since there is no preferred metric in the unbroken phase. Moreover, one could expect the noncommutative degrees of freedom to be confined within some short characteristic scales (say, Planck scale), so that only the invariants are visible at large distances (like in QCD). Then the metric and the curvature would only be effective characteristics of the spacetime at large distances. Finally, an interesting question is the limit $N \to \infty$, which can serve as a toy model for quantum gravity. All these ideas are interesting and important questions that require further study.

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