Partial self-testing and randomness certification in the triangle network

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Quantum nonlocality can be demonstrated without inputs (i.e. each party using a fixed measurement setting) in a network with independent sources. Here we consider this effect on ring networks, and show that the underlying quantum strategy can be partially characterized, or self-tested, from observed correlations. Applying these results to the triangle network allows us to show that the nonlocal distribution of Renou et al. [Phys. Rev. Lett. 123, 140401 (2019)] requires that (i) all sources produce a minimal amount of entanglement, (ii) all local measurements are entangled, and (iii) each local outcome features a minimal entropy. Hence we show that the triangle network allows for genuine network quantum nonlocality and certifiable randomness.

Introduction – Discovered by Bell in the 1960s [1], the phenomenon of quantum nonlocality has been traditionally investigated in a setting where two (or more) separated observers perform local measurements on a shared entangled state [2]. One can then prove, e.g. via Bell inequality violation, that the observed correlations are Bell nonlocal, in the sense that they are incompatible with any physical theory satisfying a natural notion of locality, such as in classical physics. Beyond fundamental aspects, quantum nonlocality is also a strong resource for black-box quantum information processing.

Networks offer an intriguing new platform for exploring quantum nonlocality; see [3] for a review. The key novelty is that the network structure features several sources, each distributing entanglement to various subsets of the parties. At each party, quantum joint measurements can be performed, which enable the distribution of strong correlations across the entire network. The main idea behind network nonlocality is to investigate the resulting correlations under the assumption that all sources in the network are independent [4, 5]. This assumption leads to a formal definition of classical (or network-local) correlations, which can be viewed as a natural generalization of the notion of Bell locality. Characterizing classical and quantum correlations in such networks is a highly challenging task, see e.g. [6–11].

A central question in this research area is to uncover novel forms of quantum nonlocal correlations inherent to the network structure. In turn, one would like to characterize such new forms of nonlocality and explore their potential for applications in quantum information processing. Our work brings progress towards this second direction.

In 2012, Fritz [12] and Branciard et al. [5] discovered that quantum nonlocality can be demonstrated in networks without the need for measurement inputs, i.e. each party performing a single fixed measurement. The example of Fritz considers a simple triangle network, where each pair of parties is connected via a bipartite source, see Fig. 1. While the construction of Fritz can be viewed as a clever embedding of a standard Bell test in the triangle network (see also [13]), Renou et al. [14] presented a strikingly different instance of quantum nonlocality (referred to as RGB4), which they argued is genuine to the triangle network; see also [15–19]. In order to formalize this intuition, the concepts of genuine network nonlocality [20] (GNNL) and full network nonlocality [21] (FNNL) where proposed. The first demonstrates the presence of non-classical joint measurements, while the second witnesses the distribution of entanglement by all sources. However, the initial question of whether the RGB4 distribution (or any other quantum nonlocal distribution without inputs) has GNNL features remained open so far.

In this work, we precisely address these questions. We develop methods for the characterization of quantum distributions in networks without inputs. This allows us to partially characterize the RGB4 distribution, and prove the following properties: (i) GNNL, all parties must perform a non-classical measurement, (ii) each source should distribute entanglement, and we obtain a lower bound on the entanglement of formation \( E_F > 2.5\% \), and (iii) certifed randomness, via a lower bound on the min-entropy \( H_{\text{min}} > 3.8\% \). Our main technical results are self-testing (or quantum rigidity) proofs that apply to quantum (Parity) Token Counting strategies on ring networks. The exposition in the main text will be focused on the triangle, the generalizations are presented at the end.
Triangle network – The triangle network depicted in Fig. 1, involves three parties A, B and C. Each pair of parties is connected by a bipartite source, labeled with \(\alpha, \beta, \gamma\). Each party receives two systems (from the neighboring sources) and produces an output \(a, b, c\). There are in total six involved systems labeled \(X_i\), with \(X \in \{A, B, C\}\) referring to the party receiving the system and \(\xi \in \{\alpha, \beta, \gamma\}\) to the source preparing it.

The set of output probability distributions \(P(a, b, c)\) possible on the triangle depends on the physical theory used to model the experiment. Classically, a source distributes (correlated) random variables, from which the measurements produce an output. In quantum theory, a Hilbert space is associated to each system, which for simplicity we assume to have an arbitrarily large but finite dimension. Without loss of generality one can assume that the sources distribute pure states \(|\psi_\alpha\rangle_{B_aC_a}, |\psi_\beta\rangle_{C_\beta A_\beta}, |\psi_\gamma\rangle_{A_\gamma B_\gamma}, \) with the global state denoted

\[
|\Psi\rangle = |\psi_\alpha\rangle_{B_aC_a} |\psi_\beta\rangle_{C_\beta A_\beta} |\psi_\gamma\rangle_{A_\gamma B_\gamma}.
\]

The Hilbert space associated to each system is taken as the support of the state, e.g. \(H_{B_a} = \text{supp}(tr_{C_a}|\psi_\alpha\rangle\langle\psi_\alpha|)\). Thus, when discussing a system in the following, we refer to the Hilbert space where \(|\Psi\rangle\langle\Psi|\) is supported, which is natural in the device-independent framework. The measurements performed by the parties are modeled as positive operator valued measurements (POVMs) \(\{E_{a}\}_a, \{E_{b}\}_b, \{E_{c}\}_c\), with the support of the state, e.g. \(\sum_a |\psi_a\rangle\langle\psi_a| = 1\). The condition implies that the dilation of measurements is trivial as summarized by the following observation.

Result 0. For unitaries \(U_X\) defined in Eq. (5), the identity (6) implies that the original measurements are projective

\[
E_{X}^\pi = \langle 0| M_X \overline{\Pi}_{X} | 0\rangle_{M_X} = \Pi_{X}^\pi.
\]

Proof sketch – The detailed proof is given in Appendix A. The condition (6) ensures that the unitaries do not change the state of the auxiliary systems and imply that the operators \(U_X = \langle 0|_{M_X} \overline{U}_{X} | 0\rangle_{M_X}\) are also unitary. But \(U_X = \sum_x e^{i\xi_x} E_x^\pi\) can only be unitary if \(E_x^\pi = \Pi_x^\pi\) is a projector valued measure.

Hence, we can rewrite Eq. (6) in a similar form \(U_{A} U_{B} U_{C} |\Psi\rangle = |\Psi\rangle\), where \(U_X = \sum_x e^{i\xi_x} \overline{\Pi}_{X}\). This condition implies that the unitaries are product.

Result 1. Consider a quantum state \(|\Psi\rangle = |\psi_\alpha\rangle_{B_aC_a} |\psi_\beta\rangle_{C_\beta A_\beta} |\psi_\gamma\rangle_{A_\gamma B_\gamma}\) on the triangle network, and local unitaries \(U_A, U_B, U_C\) acting on the systems \(A_\alpha A_\gamma, B_\beta B_\gamma\) and \(C_\beta C_\gamma\). The condition \(U_{A} U_{B} U_{C} |\Psi\rangle = |\Psi\rangle\) implies that all the unitaries are product.

\[
U_A = V_{A_\beta} \otimes W_{A_\gamma},
U_B = V_{B_\gamma} \otimes W_{B_\alpha},
U_C = V_{C_\beta} \otimes W_{A_\beta},
\]

with unitary \(V_{X_\xi}\) and \(W_{X_\xi}\) acting on the respective systems.

Proof sketch – The proof can be found in Appendix B for any ring network. We rely on the Schmidt decomposition of the states \(|\psi_\xi\rangle\) for the possibility to "move" an operator to act on the other half of an entangled state (upon transposition and rescaling). Together with the
systems quantum system accordingly to probabilities sety with the systems. Here, the unknown states of the junk may in particular decomposes the system $A$ fulfilling Eq. (4). Hence, the Hilbert space associated to the system $A_{\beta}$ (or $A_{\gamma}$) can be split as a direct sum $\mathcal{H}_{A_{\beta}} = \bigoplus_{j=0}^{N} \mathcal{H}_{A_{\beta}}^{(j)}$ of subspaces $\mathcal{H}_{A_{\beta}}^{(j)}$, on which the different $\Pi_{A_{\beta}}^{j}$ project. As it is common in self-testing, one can add enough virtual levels to rewrite the direct sum as a tensor product $\mathcal{H}_{A_{\beta}} = C_{\beta}^{N+1} \otimes \mathcal{H}_{J_{A_{\beta}}}$, with $\Pi_{A_{\beta}}^{j} = |j\rangle\langle j|_{A_{\beta}} \otimes \mathbb{1}_{J_{A_{\beta}}}$. This decomposes the system $A_{\beta}$ into a qudit $A_{\beta}$ and a "junk system" $J_{A_{\beta}}$ on which the measurements act trivially. The same decomposition can be derived for each system and imposes the following form on any quantum model fulfilling Eq. (4)

$$E_{A}^{(\alpha)} = \left( \sum_{(i,\ell) \in S(\alpha)} |j\rangle\langle j|_{A_{\beta}} \otimes |\ell\rangle\langle \ell|_{A_{\gamma}} \right) \otimes \mathbb{1}_{J_{A_{\beta}}J_{A_{\gamma}}}, \quad (8)$$

$$|\psi_{\alpha}\rangle = \sum_{i,j=0}^{N} \Psi_{ij}^{(\alpha)} |i\rangle_{B_{\alpha}}c_{\alpha} |j\rangle_{A_{\beta}}^{(ij)} \otimes \mathbb{1}_{J_{A_{\beta}}J_{C_{\alpha}}}, \quad (9)$$

Here, the unknown states of the junk may in particular contain a copy of the qudit states $|j\rangle = |j\rangle$ and remain inside the source. In this case, the quantum model becomes classical, once the junk states are traced out. Finally, with the help of the rigidity result [17] for classical TC strategies, we arrive at the following result.

**Result 2.** Consider a quantum strategy on the triangle with the global state $|\Psi\rangle = |\psi_{\alpha}\rangle_{A_{\beta}B_{\alpha}C_{\alpha}} |\psi_{\beta}\rangle_{A_{\beta}B_{\beta}} |\psi_{\gamma}\rangle_{A_{\gamma}B_{\gamma}}$ and the measurements $\{E_{A}^{(\alpha)}\}_{\alpha}, \{E_{B}^{(\beta)}\}_{\beta}, \{E_{C}^{(\gamma)}\}_{\gamma}$ acting on systems $A_{\beta}A_{\alpha}, B_{\beta}B_{\alpha} \beta$, and $C_{\beta}C_{\alpha}$. If the strategy leads to a TC distribution $P(a,b,c)$, arising from a TC strategy with the $N_{\alpha}, N_{\beta}, N_{\gamma}$ tokens distributed by the sources according to the probabilities $p_{\alpha}(i), p_{\beta}(j), p_{\gamma}(k)$, then each quantum system $X_{\xi} = X_{\xi} \otimes J_{X_{\xi}}$ can be decomposed into sub-systems $X_{\xi}$ and $J_{X_{\xi}}$ such that the quantum strategy becomes classical, once the junk states are traced out. Furthermore, for the classical TC strategies, we have that

$$E_{X_{\xi}} = \left( \sum_{i,j=x}^{N_{\xi}} |j\rangle\langle j|_{X_{\xi}} \otimes |\ell\rangle\langle \ell|_{X_{\xi}} \right) \otimes \mathbb{1}_{J_{X_{\xi}}J_{Y_{\xi}}} \quad (10)$$

$$|\psi_{\xi}\rangle_{X_{\xi}Y_{\xi}} = \sum_{i=0}^{N_{\xi}} \sqrt{p_{\xi}(i)} |i\rangle_{X_{\xi}} \otimes |j_{\xi}^{(i)}\rangle_{J_{X_{\xi}}J_{Y_{\xi}}}, \quad (10)$$

where $\xi$ and $\xi'$ denote the sources connected to the party $X$, and $X$ and $Y$ denote the parties connected to the source $\xi$.

**Proof sketch** – The full proof can be found in Appendix C for any ring network. The idea is to observe that any quantum strategy given by Eqs. (8,9) defines a unique classical strategy, where each source $\xi$ samples integer local variables $(i,j)$ accordingly to the probability distribution $|\Psi_{ij}\rangle^2$, and sends them to the neighbouring parties $X_{\xi}$ and $Y_{\xi}$. Upon receiving two such variables from the neighbouring sources, each party outputs $x(j,\ell)$ for which $(j,\ell) \in S(x)$ in Eq. (8). Classical rigidity of TC distributions implies a unique possible $|\Psi_{ij}\rangle^2$ and enforces Eq. (10).

To illustrate the power of the self-testing (or quantum rigidity) provided by Result 2, we now consider a concrete example of a nonlocal quantum distribution on the triangle.

**RGB4 distribution** – In [14] a family of tripartite distribution $P_{Q}(a,b,c)$ with four-value outcomes $a,b,c \in \{0,2,10,11\}$ has been proposed. It results from quantum models on the triangle network, where each source distributes the same maximally entangled two qubit state $|\psi^{+}\rangle = \frac{1}{\sqrt{2}} |01\rangle + |10\rangle$ and each party performs the same two-qubit projective measurement $\{\Pi_{0} = |00\rangle\langle 00|, \Pi_{2} = |11\rangle\langle 11|, \Pi_{b} = |i_{0}\rangle\langle i_{0}|, \Pi_{b} = |i_{1}\rangle\langle i_{1}|\}$, with $|i_{0}| = u_{i}|01| + v_{i}|10|$. The values $u_{i}, v_{i}$ are given by $u_{i} = -v_{i} = \cos(\theta)$ and $v_{i} = u_{i} = \sin(\theta)$ for a parameter $\theta \in [0,\pi/4]$. The resulting distributions, which we call RGB4, are given by

$$P_{Q}(1_{i},1_{j},1_{k}) = \frac{1}{8} (u_{i}u_{j}u_{k} + v_{i}v_{j}v_{k})^{2} \quad (11)$$

where $\circ$ signifies that the equation is valid up to cyclic permutations of the parties. All the other probabilities $P_{Q}(a,b,c)$ are strictly zero.

Interestingly, if the outputs are coarse-grained by merging $1_{a}$ and $1_{i}$ into a single outcome 1, the resulting distribution $P_{Q}(a,b,c)$ with $a,b,c = \{0,1,2\}$ becomes TC (with a single token sent left or right at random). Thus by Result 2 we know that the states and the measurement are of the form

$$|\psi_{\xi}\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle_{X_{\xi}Y_{\xi}} |\xi_{\xi}'\rangle_{X_{\xi}Y_{\xi}} + |10\rangle_{X_{\xi}Y_{\xi}} |\xi_{\xi}'\rangle_{X_{\xi}Y_{\xi}} \right) \quad (12)$$

$$\Pi_{X}^{0} = |00\rangle\langle 00|_{X_{\xi}X_{\xi'}} \otimes \mathbb{1}_{J_{X}} \quad (13)$$

$$\Pi_{X}^{1} = |11\rangle\langle 11|_{X_{\xi}X_{\xi'}} \otimes \mathbb{1}_{J_{X}}$$

Here, a dilation step is in general required to write the projectors $\Pi_{X}^{0}$ and $\Pi_{X}^{1}$ before coarse-graining, the auxiliary system is the absorbed into one of the incoming
junk systems, see appendix for details. With the help of Eqs. (12,13) we express the output probabilities as
\[
P_Q(1,1,1,1) = \frac{1}{S} \langle \Pi_A^1 \Pi_B^1 \Pi_C^1 | \psi^c \rangle^2 \\
P_Q(1,0,0,0) = \frac{1}{S} \langle \Pi_A^1 | \psi^a \rangle^2 \\
P_Q(1,1,1,0) = \frac{1}{S} \langle \Pi_C^1 | \psi^c \rangle^2
\]
where we introduced the global states
\[
|\psi^c\rangle \equiv |01,01,01\rangle_{B_aC_aC_jA_iA_jB_i} |\xi^c\rangle_{J_aJ_bJ_c} \\
|\psi^a\rangle \equiv |10,10,10\rangle_{B_aC_aC_jA_iA_jB_i} |\xi^a\rangle_{J_aJ_bJ_c}
\]
and the bound is the most stringent at \(\theta = 0\).

\[
P_Q(a,b,c) = \frac{1}{S} \langle \Pi_A^1 \Pi_B^1 \Pi_C^1 | \psi^c \rangle^2 + \langle \Pi_A^1 \Pi_B^1 \Pi_C^1 | \psi^a \rangle^2 + 2 \text{Re} \langle \psi^c | \Pi_A^1 \Pi_B^1 \Pi_C^1 | \psi^a \rangle^2
\]
are particularly interesting because they involve a coherence term between the global states \(|\psi^a\rangle\) and \(|\psi^c\rangle\), which only has a quantum interpretation. As \(\Pi_A^1 |\psi^c\rangle = (1 - \Pi_A^0 - \Pi_A^2) |\psi^c\rangle\) and \(|\psi^a\rangle\) are locally orthogonal on each party, \(\langle \psi^c | \Pi_A^1 |\psi^a\rangle = 0\), the coherence terms in Eq. (16) are equal up to a sign for all possible values \(i,j,k\). This allows us to quantify the coherence with a single value
\[
r \equiv (-1)^{i+j+k} 2 \text{Re} \langle \psi^c | \Pi_A^1 \Pi_B^1 \Pi_C^1 | \psi^a \rangle.
\]

Remarkably, by adopting the nonlocality proof of [14] we derive a lower bound on this coherence
\[
r \geq \frac{1}{2} \sin^3(\theta) \left(3 \cos(\theta) + \cos(3\theta) - 6 \sin(\theta)\right),
\]
as a function of the parameter \(\theta\), see Appendix D for full detail. The bound is the most stringent at \(\theta \approx 0.36\), where \(r \geq r_* \approx 0.025\). The idea behind the derivation is to show that if \(r\) is below the bound and Eqs. (14) hold, then \(q\langle i,j,k \rangle = \langle \psi^c | \Pi_A^1 \Pi_B^1 \Pi_C^1 | \psi^a \rangle\) and \(q\langle i,j,k \rangle = \langle \psi^c | \Pi_A^1 \Pi_B^1 \Pi_C^1 | \psi^c \rangle\) can not be valid probability distributions (not just for a triangle quantum model but in general). Since this last step of the argument ignores the network structure, it is not surprising that the bound (18) we obtain is only nontrivial for the subset of distributions with \(\theta \in (0, \theta_{\text{max}} \approx 0.48)\) – the same subset where the nonlocality of the distribution has been proven in [14]. The crucial difference is that it now applies to quantum models. Furthermore, by bounding the coherence \(r\) we obtain a partial characterization of any quantum model underlying the RGB4 distribution. Quite an insightful one, as we will now see.

Genuine network nonlocality – Let us first show that the RGB4 distribution is GNLL, i.e. cannot be simulated by wiring of bipartite quantum boxes [20]. In fact, any such wiring results in measurements \(\Pi_X^i\) that are separable for each party, e.g. \(\Pi_A^i = \sum_k p_k |\psi_{k_A}^i \rangle \langle \psi_{k_B}^i|_{A_i} \otimes |\psi_{k_C}^i |_{A_j}\) for Alice. Since these measurements also satisfy the TC conditions (13), it follows that \(\langle 00 | \Phi^+_{BC} \rangle = \langle 11 | \Phi^+_{BC} \rangle = 0\). Hence, these states are either of the form \(|\psi_{k}^i \rangle = |01\rangle_{A_iA_j} |\zeta_{k} \rangle_{J_a} \) or \(|\psi_{k}^i \rangle = |10\rangle_{A_iA_j} |\zeta_{k} \rangle_{J_a}\) for each \(k\). But such measurements do not erase the information on the direction of each token, and give no coherence \(\langle \psi^c | \Pi_A^1 \Pi_B^1 \Pi_C^1 | \psi^a \rangle = 0\) (even if only one of the measurements \(\Pi_X^i\) is separable). Hence, the distribution \(P_Q(a,b,c)\) is genuinely network nonlocal if \(r \neq 0\).

Quantifying source entanglement – Next we show that all the states distributed by the sources are entangled and quantify the amount of entanglement. To analyze the entanglement distribution by the sources we need a more precise description of the states. Let us decompose the junk system \(J_p\) into some unknown auxiliary degrees of freedom \(X_1Y_1\) that are indeed received and measured by the parties \(X\) and \(Y\), and a system \(E_1\) which can be controlled by an eavesdropper (Eve). Starting with
\[
|\psi_\xi \rangle = \frac{1}{\sqrt{2}} \left( |01\rangle_{X_1Y_1} |\xi^c\rangle_{X_1Y_1E_1} + |10\rangle_{X_1Y_1} |\xi^a\rangle_{X_1Y_1E_1} \right)
\]
and tracing out Eve’s systems we define the states
\[
\rho_{X_1Y_1E_1}^{(\xi)} = \text{tr}_{E_1} |\psi_\xi \rangle \langle \psi_\xi|
\]
received by the parties. Knowing that the measurements act trivially on the system \(E_1\) kept by the eavesdropper, we want to show that all these states are entangled.

This can be shown by noting that if one state was separable the rigidity constraints would imply \(r = 0\). Instead, we will directly proceed to bound the entanglement of the state \(\rho^{(\xi)}\) (or any of the other two) as quantified by its entanglement of formation \(\mathcal{E}_F\) [22]. \(\mathcal{E}_F\) is an entanglement measure that for a mixed bipartite state \(\rho_{BC}\) equals to the minimal average entropy of entanglement among all partitions of \(\rho_{BC}\) in pure states, that is
\[
\mathcal{E}_F(\rho_{BC}) = \min \sum_k p_k S(\text{tr}_{B} |\psi_k \rangle \langle \psi_k|) \\
\text{such that } \rho_{BC} = \sum_k p_k |\psi_k \rangle \langle \psi_k|,
\]
where \(S\) is the von Neumann entropy. The rigidity constraint (12) guarantees that each state in the partition of \(\rho^{(\xi)}\) is of the form \(|\psi_k \rangle = \sqrt{q_k} |01\rangle_{B_aC_a} |\phi_k \rangle_{B_1C_1} + \sqrt{1- q_k} |10\rangle_{B_aC_a} |\zeta_k \rangle_{B_1C_1}\) for some unknown states \(|\phi_k \rangle\) and \(|\zeta_k \rangle\) of the auxiliary systems. Furthermore, the entropy of entanglement of this state is trivially bounded.
\[ S(\text{tr}_{B_n} |\psi_k\rangle\langle\psi_k|) \geq h_{\text{bin}}(q_k) \] by the entropy \( h_{\text{bin}}(q_k) \) of the binary probability distribution \((q_k, 1 - q_k)\). Hence the entanglement of formation satisfies \( E_F(\rho^{(a)}) \geq \min_k p_k h_{\text{bin}}(q_k) \). On top of that it is not difficult to see that the inequality (17), implies \( \sum_k p_k \sqrt{q_k(1 - q_k)} \geq 2r \) for any partition of \( \rho^{(a)} \). It remains to minimize \( \sum_k p_k h_{\text{bin}}(q_k) \) under the constraint \( \sum_k p_k \sqrt{q_k(1 - q_k)} \geq 2r \) to show that the entanglement of formation is lower bounded by

\[ E_F(\rho^{(a)}) \geq h_{\text{bin}}\left( \frac{1}{2} (1 - \sqrt{1 - 16r^2}) \right). \] (21)

Hence, all sources must produce entanglement when \( r \neq 0 \). All the details of the derivation can be found in appendix E. For the maximal value \( r_s \) certified by Eq. (18), we find that \( E_F(\rho^{(a)}) > 2.5\% \).

Quantifying output randomness – Finally, let us bound the amount of randomness that is produced by the measurements. We focus on the entropy of a single output, say \( a \). It is simpler to further coarse-grain the values of \( a \) to define a bit \( \bar{a} = 0 \) (for \( a = 0, 2 \)) and \( \bar{a} = 1 \) (if \( a = 1, 1 \)) encoded in the register \( A \), since Eq. (13) guarantees the junk degrees of freedom have no influence on \( \bar{a} \). When tracing out all the systems but \( AE \) one finds a simple classical-quantum state

\[ \rho_{\bar{A}E} = \frac{1}{2} |0\rangle\langle0|_\bar{A} \otimes \rho_{E|\bar{a}=0} + \frac{1}{2} |1\rangle\langle1|_\bar{A} \otimes \rho_{E|\bar{a}=1} \] (22)

\[ \rho_{E|\bar{a}=0} = \frac{1}{2} (\rho_{E|\bar{a}=0}^{\bar{a}0} + \rho_{E|\bar{a}=1}^{\bar{a}1}, \rho_{E|\bar{a}=1} = \frac{1}{2} (\rho_{E|\bar{a}=1}^{\bar{a}1} + \rho_{E|\bar{a}=1}^{\bar{a}1}) \]

where \( \rho_{E|\bar{a}=0}^{\bar{a}0} \) and \( \rho_{E|\bar{a}=1}^{\bar{a}1} \) with \( \rho_{E|\bar{a}=0}^{\bar{a}0} = \rho_{E\gamma|\bar{a}=0}^{\bar{a}0} \otimes \rho_{E\gamma|\bar{a}=0}^{\bar{a}1} \)

\[ \text{tr}_{X_\xi Y_\xi} \langle \rho_{E\gamma|\bar{a}=0}^{\bar{a}0} \rangle \]

Eve’s conditional min-entropy [23] is related by \( H_{\text{min}}(A|E) = -\log_2 \left( \frac{1}{2} (1 + D(\rho_{E|\bar{a}=0}, \rho_{E|\bar{a}=1})) \right) \) to the trace distance \( D \) between her marginal states. Clearly, the entropy is not zero, as Eve’s perfect knowledge of the direction of tokens (\( D = 1 \)) would imply no coherence \((r = 0)\). Nevertheless, we found that the technical challenge of deriving a decent upper bound on \( D \) from a lower bound on \( r \) is not straightforward. In appendix E we show that \( D(\rho_{E|\bar{a}=0}, \rho_{E|\bar{a}=1}) \geq \sqrt{1 - 4r} \), leading to

\[ H_{\text{min}}(A|E) \geq -\log_2 \left( \frac{1}{2} (1 + \sqrt{1 - 4r}) \right). \] (23)

For the maximal value \( r_s \), we find \( H_{\text{min}}(A|E) \geq 3.8\% \).

Generalizations – The above partial self-testing results can be generalized to any ring network. The proofs of Results 1 and 2 are given in Appendix B and C, respectively, with notations given in Appendix A.

Another generalization concerns Parity Token Counting (PTC) distributions on the triangle [24], for which the equivalent of Result 2 also holds and is particularly simple to prove; see Appendix F. In a PTC strategy each source has a single token, and the parties only output the parity of the total number of received tokens.

We expect these results to be helpful to characterize various quantum distributions that become (P)TC upon coarse-graining, similarly to our analysis of RGB4.

Conclusion and Outlook – We showed that quantum nonlocal distributions on ring networks without inputs can be partially self-tested. Applying these methods to the triangle network, we prove that the nonlocal distribution of RGB4 (from Ref. [14]) has interesting properties. First, all measurements must be entangled, hence demonstrating GNNL. Also, all states must be entangled, with a lower bound on their entanglement. Finally, we obtain a lower bound on the min-entropy for a local outcome, hence quantifying the amount of randomness.

All the above results can in principle be strengthened quantitatively by obtaining tighter bounds on the parameter \( r \). This could be done by better exploiting the triangle structure, or by considering other nonlocal variants of the distribution [19].

Another interesting question is whether the RGB4 can be proven to be FNNL. Here we show a first step in this direction, namely that if the experiment abides by quantum physics then all sources must produce entanglement. But can one prove that all sources must be nonlocal, even if stronger-than-quantum non-signaling resources are accessible? A related question is to show that the RGB4 distribution is genuine network nonlocal when considering sources that produce non-signaling correlations and local wirings [20].

Finally, it would be desirable to make our results robust to noise. A first step could be to obtain approximate rigidity results for (P)TC.

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APPENDIX A. RING NETWORKS

A ring network depicted in Fig. 2 is a straightforward generalization of the triangle. For a n-partite ring, each party $k \in \{1, \ldots, n\}$ receives the systems $L_k$ and $R_k$ from the neighbouring sources and outputs $a_k$. Each source $S_k$ prepares the state of the systems $R_k$ and $L_{k+1}$. For the labels of the systems, the addition is meant modulo $n$, e.g. $L_{n+1}$ and $L_1$ are the same systems.

FIG. 2. A ring network. Each source $S_k$ prepares a state of the systems $R_k$ and $L_{k+1}$; each party measures the systems $L_k R_k$ to produce the output $a_k$. In the case of quantum models, the bipartite state produced by each source $S_k$ is denoted $|\psi_k\rangle$, and the global state of all the systems before the measurements is called $|\Psi\rangle$. 

Token Counting strategies

The classical TC strategies we mentioned for the triangle can be straightforwardly generalized to the ring. Each source $S_k$ has $N_k$ tokens. With probability $p_k(t)$ it sends $i$ tokens to $R_k$ and $(N_k - t)$ tokens to $L_{k+1}$. Each party outputs the total number of received tokens. By construction, one has

$$a_1 + \cdots + a_n = N \equiv N_1 + \cdots + N_n. \quad (24)$$

A priori quantum description of the setup

By assumption in the scenario we consider each source $S_k$ controls two finite dimensional quantum systems $R_k$ and $L_{k+1}$. It prepares them in a state which a priori can not be assumed pure. Denote it $\varrho_k$ – a density operator on the tensor product of the finite dimensional Hilbert spaces associated to the systems. By introducing an auxiliary finite-dimensional system $A_k$ the state $\varrho_k$ can be purified to $|\psi_k\rangle$, i.e. $\varrho_k = \text{tr}_{A_k} |\psi_k\rangle\langle\psi_k|_{R_kL_{k+1}A_k}$. Furthermore, without loss of generality the auxiliary system $A_k$ can be absorbed in either $R_k$ or $L_{k+1}$. The dimension of the Hilbert space that describes it is them augmented, but remains finite. In other words, in the device-independent setting we consider, the state

$$|\psi_k\rangle_{R_kL_{k+1}} \in \mathcal{H}_{R_k} \otimes \mathcal{H}_{L_{k+1}} \quad (25)$$

prepared by the source $S_k$ can be assumed pure from the beginning. Here we also introduced the Hilbert spaces $\mathcal{H}_{R_k}$ and $\mathcal{H}_{L_{k+1}}$ associated to each system.

Let us now consider the description of a measurements performed by the $k$-th party on the system $X_k = L_kR_k$ to produce the output $a_k$. A priori they are given by a positive operator valued measure (POVM), that is a set of non-negative hermitian operators $E^a_{X_k} : \mathcal{H}_{L_k} \otimes \mathcal{H}_{R_k} \rightarrow \mathcal{H}_{L_k} \otimes \mathcal{H}_{R_k}$

$$\{E^a_{X_k}\}_a, \text{ such that } E^a_{X_k} = E^a_{X_k} \geq 0 \quad \sum_a E^a_{X_k} = 1_{X_k} \quad (26)$$

By Steinspring dilation theorem this POVM can be dilated to a projective measurement. This requires the introduction of an auxiliary system $M_k$ prepared in the state $|0\rangle$, with associated finite dimensional Hilbert space $\mathcal{H}_{M_k}$. For any state $\rho_{L_kR_k}$ the probability of an outcome $a_k$ is then given by

$$E^a_{X_k} = \text{tr}_{M_k} \bar{\Pi}^a_{X_k} |0\rangle\langle 0|_{M_k}. \quad (27)$$

Here $\{\bar{\Pi}_{a_k}\}_{a_k}$ is a projector valued measure (PVM), i.e. in addition to being a POVM it fulfills $(\bar{\Pi}^a_{X_k})^2 = \bar{\Pi}^a_{X_k}$, with the projector acting on the systems $L_kR_kM_k$. Below we will also deal with unitaries $\bar{U}_k$ acting on the same systems

$$\bar{U}_k = \sum_x e^{i\varphi_x} \bar{\Pi}^x_{X_k}, \quad (28)$$

here it is only important that the phases $e^{i\varphi_x}$ are different for different $x$.

Proof of RESULT 0

**Result 0** For a network state $|\Psi\rangle = |\psi_1\rangle \cdots |\psi_n\rangle$ and unitaries $\bar{U}_k = \sum_x e^{i\varphi_x} \bar{\Pi}^x_{X_k}$ with $e^{i\varphi_x} \neq e^{i\varphi_{x'}}$ for $x \neq x'$ the identity

$$\left( \bigotimes_{k=1}^n \bar{U}_k \right) |\Psi\rangle \bigotimes_k |0\rangle_{M_k} = |\Psi\rangle \bigotimes_k |0\rangle_{M_k} \quad (29)$$

implies that the original measurements are projective

$$E^x_X = |0\rangle_{M_k} \bar{\Pi}^x_X |0\rangle_{M_k} = \bar{\Pi}^x_X. \quad (30)$$
Consider the action of a single unitary $\bar{U}_k$ on the states $|\psi_{k-1}\rangle_{R_{k-1}L_{k}} |\psi_k\rangle_{R_kL_{k+1}} |0\rangle_{M_k}$. It is the only unitary acting on the system $M_k$ and by virtue of Eq. (29) we know that it leaves it in the state $|0\rangle_{M_k}$. Formally, this can be expressed as

$$\bar{U}_k |\Psi\rangle \bigotimes_i |0\rangle_{M_i} = \left( \bigotimes_j \bar{U}_j \right) |\Psi\rangle \bigotimes_i |0\rangle_{M_i} = |0\rangle_{M_k} \left( \bigotimes_j \bar{U}_j \right) |\Psi\rangle \bigotimes_i |0\rangle_{M_i}$$  \hspace{1cm} (31)

and therefore

$$\left( 1_{R_{k-1}L_{k+1}} \otimes \bar{U}_k \right) |\psi_{k-1}\rangle_{R_{k-1}L_{k}} |\psi_k\rangle_{R_kL_{k+1}} |0\rangle_{M_k} = |\xi_k\rangle_{R_{k-1}L_{k}} R_k L_{k+1} |0\rangle_{M_k}.$$  \hspace{1cm} (32)

for some state $|\xi_k\rangle_{R_{k-1}L_{k}} R_k L_{k+1}$. Since the density operators $\text{tr}_{R_{k-1}} |\psi_{k-1}\rangle_{R_{k-1}L_{k}} |\psi_k\rangle_{R_kL_{k+1}}$ and $\text{tr}_{L_{k+1}} |\psi_k\rangle_{R_kL_{k+1}}$ have full support on the Hilbert spaces $H_{L_k}$ and $H_{R_k}$ (by definition of these Hilbert spaces), Eq. (32) implies that for any state $|\ell\rangle_{L_k} \in H_{L_k}$ and $|r\rangle_{R_k} \in H_{R_k}$

$$\bar{U}_k |\ell\rangle_{L_k} |r\rangle_{R_k} |0\rangle_{M_k} = |\xi_{\ell,r}\rangle_{L_k} R_k R_{k+1} |0\rangle_{M_k},$$  \hspace{1cm} (33)

with some state $|\xi_{\ell,r}\rangle_{L_k} R_k R_{k+1} \in H_{L_k} \otimes H_{R_k}$. It is the easiest to see with state $|\ell\rangle_{L_k}$ and $|r\rangle_{R_k}$ that appear in the Schmidt decomposition of $|\psi_{k-1}\rangle_{R_{k-1}L_{k}}$ and $|\psi_k\rangle_{R_kL_{k+1}}$ respectively.

Starting with two orthonormal bases $\{|\ell\rangle_{L_k}\}_\ell$ of $H_{L_k}$ and $\{|r\rangle_{R_k}\}_r$ of $H_{R_k}$ we see that the states $\{|\xi_{\ell,r}\rangle\}_r,\ell$ form an orthonormal basis of $H_{L_k} \otimes H_{R_k}$

$$\delta_{\ell',\ell} \delta_{r',r} = \langle \ell, r, 0 | \ell', r', 0 \rangle_{L_k R_k R_{k+1}} = \langle \ell, r, 0 | \bar{U}_k^\dagger \bar{U}_k | \ell', r', 0 \rangle = \langle \xi_{\ell,r} | \xi_{\ell',r'} \rangle$$  \hspace{1cm} (34)

This guarantees that the operator $U_k$ defined as the restriction

$$U_k = \langle 0 |_{M_k} \bar{U}_k | 0 \rangle_{M_k} = \sum_{\ell,r} |\xi_{\ell,r}\rangle_{L_k} \bar{U}_k |r\rangle_{R_k} |0\rangle_{M_k} \sum_{\ell,r} |\xi_{\ell,r}\rangle_{L_k} \bar{U}_k |r\rangle_{R_k} |0\rangle_{M_k}$$  \hspace{1cm} (35)

is also unitary (a basis change). By construction this operator is also equal to

$$U_X = \langle 0 |_{M_k} \bar{U}_k | 0 \rangle_{M_k} = \sum_x e^{i\varphi_x} \langle 0 |_{M_k} \bar{\Pi}_x | 0 \rangle_{M_k} = \sum_x e^{i\varphi_x} \bar{E}_x |_{M_k}.$$  \hspace{1cm} (36)

The following lemma shows that an operator $U_X = \sum_x e^{i\varphi_x} \bar{E}_x |_{M_k}$ is unitary if and only if $\{\bar{E}_x |_{M_k}\}_x$ is a PVM. Which proves the result $\Box$.

**Lemma 0** Consider a POVM $\{E_x\}_x$ and the operator $U = \sum_x e^{i\varphi_x} E_x$ with all $e^{i\varphi_x}$ different for different $x$. $U$ is unitary if and only if $\{E_x |_{M_k}\}_x$ is a PVM.

Since $U$ is unitary it can be diagonalized. Hence, there is a basis $\{|k\rangle\}_k$ of the underlying Hilbert space such that

$$U |k\rangle = e^{i\lambda_k} |k\rangle$$  \hspace{1cm} (37)

for some real $\lambda_k \in [0,2\pi)$. For all $k$ we thus have

$$e^{i\lambda_k} = \langle k | U |k\rangle = \langle k | \sum_x e^{i\varphi_x} E_x |k\rangle = \sum_x e^{i\varphi_x} p_k(x),$$  \hspace{1cm} (38)

where $p_k(x) = \langle k | E_x |k\rangle$ is a probability distribution. In particular, it implies

$$1 = |e^{i\lambda_k}| = \sum_x |e^{i\varphi_x} p_k(x)|,$$  \hspace{1cm} (39)

since all $e^{i\varphi_x}$ are different and $\sum_x p_k(x) = 1$, this equality is only possible if $p_k(x) = \delta_{x,y(k)}$ i.e. nonzero for only a single $x = y(k)$. Therefore, $\|\sqrt{E_x} |k\rangle\| = \delta_{x,y(k)}$ and

$$E_x |k\rangle = \begin{cases} |k\rangle & x = y(k) \\ 0 & x \neq y(k). \end{cases}$$  \hspace{1cm} (40)

Hence the operators $E_x$ orthogonal projector diagonal in the basis $\{|k\rangle\}_k$. This shows the "only if" direction. The converse is trivial as any $\sum_x e^{i\varphi_x} \bar{E}_x$ is unitary. $\Box$
APPENDIX B. PROOF OF RESULT 1.

Result 1. Consider a quantum state \( |\Psi\rangle = |\psi_1\rangle \ldots |\psi_n\rangle \) of a \( n \)-partite ring network, where each party receives two systems \( L_k \) and \( R_k \) from two different sources, and a collection of unitary operators \( U_1, \ldots, U_n \) applied by each party. The condition

\[
\bigotimes_{k=1}^{n} U_k |\Psi\rangle = |\Psi\rangle
\]

implies that all the unitaries are products

\[
U_k = V_{L_k}^{(k)} \otimes W_{R_k}^{(k)}.
\]

We prove this statement in this section. To start we will introduce some notation and useful operators. We put each state \( |\psi_k\rangle \) in the Schmidt diagonal form

\[
|\psi_k\rangle = \sum_{i=1}^{d_k} \lambda_i^{(k)} |i, i\rangle_{R_k L_{k+1}}
\]

for some finite \( d_k \) and \( \lambda_i^{(k)} > 0 \), it defines a basis for each of the systems. It also allows us to rewrite the states as

\[
|\psi_k\rangle = \Lambda^{(k)} \otimes I |\Omega\rangle = I \otimes \Lambda^{(k)} |\omega_k\rangle,
\]

where \( |\omega_k\rangle = \frac{1}{\sqrt{d_k}} \sum_{i=1}^{d_k} |i, i\rangle_{R_k L_{k+1}} \) and \( \Lambda^{(k)} \) is diagonal in the Schmidt basis \( \Lambda^{(k)} |i\rangle_{R_k} = \sqrt{d_k} \lambda_i^{(k)} |i\rangle_{R_k} \) is diagonal in the Schmidt basis and invertible. With the help of the identity \( I \otimes M |\omega\rangle = M^T \otimes I |\omega\rangle \) for a maximally entangled state \( |\omega\rangle \) and the transpose with respect to the Schmidt basis, we can now move the unitaries to act on different systems. Concretely, we can write

\[
(U_k)_{L_k R_k} |\psi_{k-1}\rangle_{R_{k-1} L_k} |\psi_k\rangle_{R_k L_{k+1}} = \Lambda_{R_{k-1}}^{(k-1)} \otimes (U_k)_{L_k R_k} \otimes \Lambda_{L_{k+1}}^{(k)} |\omega_{k-1}\rangle |\omega_k\rangle
= (\Lambda_{R_{k-1}}^{(k-1)} \otimes \Lambda_{L_{k+1}}^{(k)}) (U_k^T)_{R_{k-1} L_{k+1}} |\omega_{k-1}\rangle |\omega_k\rangle
= (G_k)_{R_{k-1} L_{k+1}} |\omega_{k-1}\rangle |\omega_k\rangle
\]

where in the last line we defined the invertible operator

\[
G_k \equiv (\Lambda^{(k-1)} \otimes \Lambda^{(k)}) U_k^T
\]

acting on the systems \( R_{k-1} L_{k+1} \). Similarly, we rewrite

\[
(U_k)_{L_k R_k}^\dagger |\psi_{k-1}\rangle_{R_{k-1} L_k} |\psi_k\rangle_{R_k L_{k+1}} = (U_k)_{L_k R_k}^\dagger (\Lambda_{L_k}^{(k-1)} \otimes \Lambda_{R_k}^{(k)}) |\omega_{k-1}\rangle |\omega_k\rangle
= (F_k)_{L_k R_k} |\omega_{k-1}\rangle |\omega_k\rangle
\]

where we introduced the invertible operator

\[
F_k \equiv U_k^\dagger (\Lambda^{(k-1)} \otimes \Lambda^{(k)})
\]

acting on the systems \( L_k R_k \). For the following, it is convenient to introduce the notation

\[
|\Omega\rangle \equiv |\omega_1\rangle \ldots |\omega_n\rangle.
\]

Next, we will consider even and odd rings separately in the next two sections. The reasoning leading to the proof is graphically summarized in Fig. 3, which can be a useful guide to the reader. In both cases (even and odd) to make the final step, we invoke one of the two technical lemmas on the rigidity of a chain of operators acting on several systems. The lemmas will be stated at the moment where we use them, while their proofs are given at the very end of this appendix.
FIG. 3. (Square) Even rings – minimal example of the square network: A graphical summary of transformations leading from the equation $U_1 U_3 |\Psi\rangle = U_2^\dagger U_4^\dagger |\Psi\rangle$ (50) – the left column – to the equation $(F_1)_{R_1, L_3} (F_3)_{R_3, L_1} |\Omega\rangle = (G_2)_{L_2, R_4} (G_4)_{L_4, R_2} |\Omega\rangle$ (51,52) – the right column. (Triangle) Odd rings – minimal example of the triangle network: A graphical summary of transformations leading from the equation $U_1^\dagger U_3^\dagger |\Psi\rangle = U_2 |\Psi\rangle$ (57) – the left column – to the equation $(F_1)_{L_1, R_1} Q_{L_3, L_1} |\Omega\rangle = (G_2)_{R_1, L_3} |\Omega\rangle$ (67) – the right column. (Both) $|\omega_k\rangle$ are maximally entangled states and $\Lambda^{(k)}$ are invertible local operators such that $\mathbb{1} \otimes \Lambda^{(k)} |\omega_k\rangle = \Lambda^{(k)} \otimes \mathbb{1} |\psi_k\rangle$. We use the property $\mathbb{1} \otimes M |\omega\rangle = M^T \otimes \mathbb{1} |\omega\rangle$ to change the systems on which the unitaries act.

Even rings

For even rings ($n = 2m, m \geq 2$), by multiplying both side of the equation (41) by $U_2^T U_4^T \ldots U_{2m-1}^T$ we rewrite it as

$$U_1 U_3 \ldots U_{2m-1} |\Psi\rangle = U_2^T U_4^T \ldots U_{2m-1}^T |\Psi\rangle. \quad (50)$$

Here, the unitaries only act on odd parties (lhs) or even parties (rhs). Now, using the definitions (45,47) we rewrite
both sides of the equation as
\[
U_1 U_3 \cdots U_{2m-1} |\Psi\rangle = (G_1)_{R_{2m}} L_2 (G_3)_{R_2 L_4} \cdots (G_{2m-1})_{R_{2m-2} L_{2m}} |\Omega\rangle ,
\]
\[
U_2^\dagger U_4^\dagger \cdots U_{2m}^\dagger |\Psi\rangle = (F_2)_{L_2 R_2} (F_4)_{L_4 R_4} \cdots (F_{2m})_{L_{2m} R_{2m}} |\Omega\rangle .
\]
Note that on the right hand side of these equations the operators only act on the even parties 2k. Using Choi–Jamiołkowski duality, \( 1 \otimes A |\omega\rangle = 1 \otimes B |\omega\rangle \implies A = B \), we combine Eqs. (50,51,52) to obtain the following equality between the chains of operators
\[
(F_2)_{L_2 R_2} \otimes (F_4)_{L_4 R_4} \cdots \otimes (F_{2m})_{L_{2m} R_{2m}} = (G_1)_{R_{2m}} L_2 \otimes (G_3)_{R_2 L_4} \cdots \otimes (G_{2m-1})_{R_{2m-2} L_{2m}} .
\]
In order to continue, we need the following Lemma.

**Lemma 1e** Consider \( n = 2m \) quantum systems and two sets \( m \) nonzero bipartite operators \( A_{1,2}, A_{3,4}, \ldots, A_{2m-1,2m} \) and \( B_{2,3}, B_{4,5}, \ldots, B_{2m,1} \) where the labels indicate on which pair of systems the operators act. If the following identity is satisfied
\[
\bigotimes_{k=1}^m A_{2k-1,2k} = \bigotimes_{k=1}^m B_{2k,2k+1} ,
\]
then all the operators are product
\[
A_{2k-1,2k} = A_{2k-1} \otimes A'_{2k},
\]
\[
B_{2k,2k+1} = B_{2k} \otimes B'_{2k+1}.
\]
A graphical illustration of the Lemma is given in Fig. 4(a), while its proof can be found in Sec. .

To apply Lemma 1e, note that in Eq. (53) we are precisely dealing with \( n = 2m \) quantum systems, that can be arranged as \( L_2, R_2, L_4, R_4, \ldots, L_{2m}, R_{2m} \) and two collections of operators \( F_2, \ldots, F_{2m} \) (playing the roles of \( A_{2k-1,2k} \)) and \( G_3, \ldots, G_{2m-1}, G_1 \) (playing the roles of \( B_{2k,2k+1} \)) acting on neighbouring pairs of systems. Hence, we conclude that \( F_k \) and \( G_k \) are products, i.e.
\[
(G_{2k+1})_{R_{2k} L_{2k+2}} = (G_{2k+1})_{L_{2k}} \otimes (G'_{2k+1})_{R_{2k+2}}
\]
\[
(F_{2k})_{L_{2k} R_{2k}} = (F_{2k})_{L_{2k}} \otimes (G'_{2k})_{R_{2k}}.
\]
From the definition of \( G_k \) and \( F_k \) it is straightforward to see that all the \( U_k \) are then also products (recall that all the operators are invertible). This proves result 1 for even rings.

**Odd rings**

Let us now consider odd rings \( (n = 2m+1, n \geq 3) \), which includes the triangle network. We rewrite the equation (41) as
\[
U_2 U_4 \cdots U_{2m} |\Psi\rangle = U_1^\dagger U_3^\dagger \cdots U_{2m+1}^\dagger |\Psi\rangle .
\]
Similarly, to the even ring case, the left hand side can be rewritten as
\[
U_2 U_4 \cdots U_{2m} |\Psi\rangle = (G_2)_{R_1 L_3} (G_4)_{R_3 L_5} \cdots (G_{2m})_{R_{2m-1} L_{2m+1}} A^{(2m+1)}_{R_{2m+1}} |\Omega\rangle ,
\]
using Eq. (45).

A notable difference with the case of even rings, is that the right hand side of Eq. (57) involves the operators \( U_1^\dagger \) and \( U_{2m+1}^\dagger \) that act of the same state \( |\psi_{2m+1}\rangle \). In order to bring it to the desired form we proceed in steps. First rewrite
\[
U_3^\dagger \cdots U_{2m-1}^\dagger |\Psi\rangle = (F_3)_{L_3 R_3} \cdots (F_{2m-1})_{R_{2m-1} L_{2m-1}} |\Omega\rangle ,
\]
where the operators \( (F_3)_{L_3 R_3} \cdots (F_{2m-1})_{R_{2m-1} L_{2m-1}} \) act trivially on the systems \( R_2 L_2, R_{2m-1} L_{2m-1} \) and \( R_1 L_2 \) prepared by the sources \( S_{2m}, S_{2m+1} \) and \( S_1 \).
For the unitaries $U^\dagger_1$ and $U^\dagger_{2m+1}$ acting on these systems we have
\[
U_1^\dagger U_{2m+1}^\dagger \left| \psi_{2m} \right\rangle_{R_{2m+1} L_{2m+1}} \left| \psi_{2m+1} \right\rangle_{R_{2m+1} L_1} = U_1^\dagger U_{2m+1}^\dagger (A_{L_{2m+1}}^{(2m)} \otimes A_{L_1}^{(2m+1)} \otimes A_{R_1}^{(1)}) \left| \omega_{2m} \right\rangle \left| \omega_{2m+1} \right\rangle \left| \omega_1 \right\rangle = (U_1^\dagger A_{L_1}^{(2m)} \otimes A_{R_1}^{(1)})(U_{2m+1}^\dagger A_{L_{2m+1}}^{(2m)}) \left| \omega_{2m} \right\rangle \left| \omega_{2m+1} \right\rangle \left| \omega_1 \right\rangle = (F_1)_L R_1 (U_{2m+1}^\dagger A_{L_{2m+1}}^{(2m)}) \left| \omega_{2m} \right\rangle \left| \omega_{2m+1} \right\rangle \left| \omega_1 \right\rangle
\]

We want to rewrite this expression in such a way that nothing acts on the system $R_{2m+1}$. This can be done by defining
\[
(U_{2m+1}^\dagger)_{L_{2m+1} R_{2m+1}} A_{L_{2m+1}}^{(2m)} \left| \omega_{2m} \right\rangle_{R_{2m+1} L_{2m+1}} \left| \omega_{2m+1} \right\rangle_{R_{2m+1} L_1} = Q_{L_{2m+1} L_1} \left| \omega_{2m+1} \right\rangle_{R_{2m+1} L_1} \left| \omega_1 \right\rangle_{R_1 L_2}
\]

with $Q_{AB} = ((U_{2m+1}^\dagger)_{AB} A^{(2m)})^T_B$ is the partially transposed operator (with respect to the system $B$, in our case $R_{2m+1}$). $Q_{L_{2m+1} L_1}$ is a product of invertible operators, and is thus invertible. With the help of this operator we write
\[
U_1^\dagger U_{2m+1}^\dagger \left| \psi_{2m} \right\rangle_{R_{2m+1} L_{2m+1}} \left| \psi_{2m+1} \right\rangle_{R_{2m+1} L_1} \left| \psi_1 \right\rangle_{R_1 L_2} = (F_1)_L R_1 Q_{L_{2m+1} L_1} \left| \omega_{2m} \right\rangle \left| \omega_{2m+1} \right\rangle \left| \omega_1 \right\rangle.
\]

Finally, combing this realtion with Eq. (59) in order to obtain
\[
U_1^\dagger U_3^\dagger \cdots U_{2m+1}^\dagger \left| \Psi \right\rangle = \left( \bigotimes_{k=1}^m (F_{2k-1})_{L_{2k-1} R_{2k-1}} \right) Q_{L_{2m+1} L_1} \left| \Omega \right\rangle.
\]

Coming back to the condition (57) we get the equality,
\[
(G_2)_{R_1 L_3} (G_4)_{R_3 L_5} \cdots (G_{2m})_{R_{2m-1} L_{2m+1}} A_{R_{2m+1}}^{(2m+1)} \left| \Omega \right\rangle = \left( \bigotimes_{k=1}^m (F_{2k-1})_{L_{2k-1} R_{2k-1}} \right) Q_{R_{2m+1} R_1} \left| \Omega \right\rangle
\]

on both side of the equation all the operators act trivially on the systems $L_{2k}$, $R_{2k}$, and $L_1$, prepared by the sources $S_{2k-1}, S_{2k}$ (for $k = 1, \ldots, m$) and $S_{2m+1}$ respectively. In other words, for all the involved bipartite maximally entangled states $|\omega_k\rangle$ the operators only act on one of the two systems. By Choi–Jamiołkowski duality we thus arrive at the operator identity
\[
(G_2)_{R_1 L_3} (G_4)_{R_3 L_5} \cdots (G_{2m})_{R_{2m-1} L_{2m+1}} A_{R_{2m+1}}^{(2m+1)} = \left( \bigotimes_{k=1}^m (F_{2k-1})_{L_{2k-1} R_{2k-1}} \right) Q_{R_{2m+1} R_1}
\]

Next, we multiply both sides by $(A_{R_{2m+1}}^{(2m+1)})^{-1}$, and define
\[
\tilde{Q}_{R_{2m+1} R_1} \equiv (A_{R_{2m+1}}^{(2m+1)})^{-1} Q_{R_{2m+1} R_1}
\]

to write
\[
\left( \bigotimes_{k=1}^m (F_{2k-1})_{L_{2k-1} R_{2k-1}} \right) \tilde{Q}_{R_{2m+1} R_1} = (G_2)_{R_1 L_3} (G_4)_{R_3 L_5} \cdots (G_{2m})_{R_{2m-1} L_{2m+1}}.
\]

To continue the reasoning we use the following lemma.

**Lemma 10** Consider $n = 2m + 1$ quantum systems, a set of $m + 1$ nonzero bipartite operators $A_{1,2}, A_{3,4}, \ldots, A_{2m-1,2m}, A_{2m+1,1}$ and a set of $m$ nonzero operators $B_{2,3}, B_{4,5}, \ldots, B_{2m,2m+1}$ where the labels indicate on which pair of systems the operators act. If the following identity is satisfied
\[
\left( \bigotimes_{k=1}^m A_{2k-1,2k} \right) A_{2m+1,1} = \left( \bigotimes_{k=1}^m B_{2k} \right)_{2k,2k+1}
\]

then all the operators are product
\[
A_{2k-1,2k} = A_{2k-1} \otimes A'_{2k}
B_{2k,2k+1} = B_{2k} \otimes B'_{2k+1}
\]
The proof of the lemma is postponed to the end of the current appendix, while an illustration is given in Fig. 4(b).

To apply Lemma 1o to our situation, given by Eq. (67), one identifies the $n = 2m + 1$ quantum system of the Lemma with $R_1, L_3, R_3, \ldots, L_{2m-1}, R_{2m-1}$. Then the bipartite operators $F_1, \ldots, F_{2m-1}, Q$ play the role of $A_{1,2}, \ldots, A_{2m-1,2m}, A_{2m+1,1},$ and $G_{2,3}, \ldots, G_{2m,2m+1}$ play the role of $B_{2,3}, \ldots, B_{2m,2m+1}$. Lemma 1o guarantees that all of these operators are products. Finally, it is straightforward to conclude that all of the unitaries $U_1, \ldots, U_{2m+1}$ are also products for odd rings. This concludes the proof of Result 1 $\Box$.

**Proof of Lemma 1e**

Consider $n = 2m$ quantum systems and two sets $m$ of nonzero bipartite operators $A_{1,2}, A_{3,4}, \ldots, A_{2m-1,2m}$ and $B_{2,3}, B_{4,5}, \ldots, B_{2m,1}$ which satisfy

$$\bigotimes_{k=1}^{m} A_{2k-1,2k} = \bigotimes_{k=1}^{m} B_{2k,2k+1}, \tag{70}$$

where the labels indicate on which pair of systems the operators act.

To show that the operators are product introduce an orthonormal operator basis $\{X^{(i)}_{2k}\}$ (such that $\text{tr} X^{(i)}_{2k} X^{(j)\dagger}_{2k} = \delta_{ij}$) for each even system. It allows us to decompose each operator as

$$A_{2k-1,2k} = \sum_{i} Y^{(i)}_{2k-1} \otimes X^{(i)}_{2k}, \tag{71}$$

$$B_{2k,2k+1} = \sum_{i} X^{(i)}_{2k} \otimes Z^{(i)}_{2k+1},$$

where the operators $Y^{(i)}_{2k-1}$ and $Z^{(i)}_{2k+1}$ are arbitrary. The condition (54) implies

$$\sum_{i=1}^{m} \bigotimes_{k=1}^{m} X^{(i_k)}_{2k} \otimes Y^{(i_k)}_{2k-1} = \sum_{i=1}^{m} \bigotimes_{k=1}^{m} X^{(i_k)}_{2k} \otimes Z^{(i_k)}_{2k+1}. \tag{72}$$

Because the operators $X^{(i)}_{2k}$ are orthonormal it follows that

$$\bigotimes_{k=1}^{m} Y^{(i_k)}_{2k-1} = \bigotimes_{k=1}^{m} Z^{(i_k)}_{2k+1} \tag{73}$$

for all values of the indices $i_1, \ldots, i_m$. This implies that for fixed indices values the operators acting on the same system, e.g. $Y^{(i_k+1)}_{2k+1}$ and $Z^{(i_k)}_{2k+1}$, are equal up to a constant the operators are equal up to a multiplicative factor. Now, chose a value $i_k = j$ such that $Z^{(j)}_{2k+1} \neq 0$, such a value exists because $B_{2k,2k+1} \neq 0$ by assumption. For any value $i_{k+1}$ we thus have $Y^{(i_{k+1})}_{2k+1} = c^{(i_{k+1})}_{2k+1} Z^{(j)}_{2k+1}$ for some scalar $c^{(i_{k+1})}_{2k+1}$. In turn, since at least one $Y^{(i_{k+1})}_{2k+1}$ is nonzero, we obtain
that \( Z_{2k+1}^{(i_k)} = z_{ik}^{(i_k)} Z_{2k+1}^{(j)} \). We conclude that for each system there is a nonzero operator \( C_{2k+1} \) and scalar values \( y_{ik} \) and \( z_{ik} \) such that

\[
Y_{2k-1}^{(i)} = y_{ik} C_{2k-1} 
\]
\[
Z_{2k+1}^{(i)} = z_{ik} C_{2k+1} \quad (74)
\]

For the original operators in Eq. (71) this gives

\[
A_{2k-1,2k} = \sum_i y_{ik} C_{2k-1} \otimes X_{2k}^{(i)} = C_{2k-1} \otimes \left( \sum_i y_{ik} X_{2k}^{(i)} \right) 
\]
\[
B_{2k,2k+1} = \sum_i X_{2k}^{(i)} \otimes z_{ik} C_{2k+1} = \left( \sum_i z_{ik} X_{2k}^{(i)} \right) \otimes C_{2k+1}. \quad (75)
\]

We have thus proven that all \( A_{2k-1,2k} \) and \( B_{2k,2k+1} \) are product. □

**Proof of Lemma 1o**

Consider \( n = 2m+1 \) quantum systems, a set of \( m+1 \) bipartite nonzero operators \( A_{1,2}, A_{3,4}, \ldots, A_{2m-1,2m}, A_{2m+1,1} \) and a set of \( m \) operators \( B_{2,3}, B_{4,5}, \ldots, B_{2m,2m+1} \) which satisfy

\[
\left( \bigotimes_{k=1}^m A_{2k-1,2k} \right) A_{2m+1,1} = \bigotimes_{k=1}^m B_{2k,2k+1}, \quad (76)
\]

with the labels indicating on which pair of systems the operators act.

To show that the operators are products we again introduce an orthonormal operator basis \( \{ X_{2k}^{(i)} \} \) for each even system. This allows us to express

\[
A_{2k-1,2k} = \sum_i Y_{2k-1}^{(i)} \otimes X_{2k}^{(i)} \quad k \leq m 
\]
\[
B_{2k,2k+1} = \sum_i X_{2k}^{(i)} \otimes Z_{2k+1}^{(i)} \quad (77)
\]

We then rewrite the condition of the lemma as

\[
\left( \sum_{i_1, \ldots, i_m} \bigotimes_{k=1}^m X_{2k}^{(i_k)} \bigotimes_{k=1}^m Y_{2k-1}^{(i_k)} \right) A_{2m+1,1} = \sum_{i_1, \ldots, i_m} \bigotimes_{k=1}^m X_{2k}^{(i_k)} \bigotimes_{k=1}^m Z_{2k+1}^{(i_k)}. \quad (78)
\]

By linear independence of all \( X_{2k}^{(i)} \) this implies

\[
\left( \bigotimes_{k=1}^m Y_{2k-1}^{(i_k)} \right) A_{2m+1,1} = \mathbb{1}_1 \otimes \bigotimes_{k=1}^m Z_{2k+1}^{(i_k)}. \quad (79)
\]

Next, let us chose the values of the coefficients \( i_1, \ldots, i_m \) such that the \( Z_{2k+1}^{(i_k)} \) are all nonzero (we known that \( B_{2k,2k+1} \) are nonzero hence such values exist), and relabel these values to \( i_1 = \ldots = i_m = 1 \). We find

\[
(Y_{1}^{(1)} \otimes Y_{3}^{(1)} \otimes \ldots Y_{2m-1}^{(1)} \otimes \mathbb{1}_{2m+1}) A_{2m+1,1} = \mathbb{1}_1 \otimes Z_{3}^{(1)} \otimes \ldots Z_{2m+1}^{(1)}. \quad (80)
\]

Now multiply both sides of the equation by \( Z_{3}^{(1)} \otimes \ldots Z_{2m+1}^{(1)} \) from te right and trace out all the systems but the first one. We find

\[
Y_{1}^{(1)} \left( \text{tr} A_{2m+1,1} Z_{2m+1}^{(1)} \right) \prod_{k=1}^{m-1} \text{tr} Y_{2k+1}^{(1)} Z_{2k+1}^{(1)} = \mathbb{1}_1 \prod_{k=1}^{m} \text{tr} Z_{2k+1}^{(1)} z_{2k+1}^{(1)} \quad (81)
\]
where the right hand side is nonzero since we have chosen the operators $Z_{2k+1}^{(1)}$ that are all nonzero. We can thus conclude that
\[ Y_1^{(1)}(\text{tr} A_{2m+1,1} Z_{2m+1}^{(1)}) = \xi \mathbb{I}_1 \] (82)

with a nonzero constant $\xi = \prod_{k=1}^{m} \text{tr} Z_{2k+1}^{(1)} Z_{2k+1}^{(1)\dagger}$. The last equation guarantees that the operator $Y_1^{(1)}$ is inevitable. Multiplying the Equation (80) by $(Y_1^{(1)})^{-1}$ we find
\[ Y_3^{(1)} \otimes \ldots Y_{2m-1}^{(1)} \otimes A_{2m+1,1} = Z_3^{(1)} \otimes \ldots Z_{2m-1}^{(1)} \otimes Z_{2m+1}^{(1)} \otimes (Y_1^{(1)})^{-1}, \] (83)
and in particular
\[ A_{2m+1,1} = a Z_{2m+1}^{(1)} \otimes (Y_1^{(1)})^{-1}, \] (84)
where we defined a nonzero constant $a$. This shows that $A_{2m+1,1}$ is product.

Finally, the last identity allows us to rewrite the Eq. (79) as
\[ a Y_1^{(i)} (Y_1^{(1)})^{-1} \bigotimes_{k=2}^{m} Y_{2k-1}^{(i)} Z_{2m+1}^{(1)} = \mathbb{I}_1 \bigotimes_{k=1}^{m} Z_{2k+1}^{(i)}, \] (85)
again this guarantees that all the operators acting on the same system are equal up to a multiplicative factor. With identical arguments to those in the even lemma we arrive to
\[ Y_{2k+1}^{(i)} = y_{ik} C_{2k+1} \quad Z_{2k-1}^{(i)} = z_{ik} C_{2k-1}. \] (86)
Plugging these relations in the Eq. eq app: lemma odd A B we conclude that
\[ A_{2k-1,2k} = \sum_i y_{ik} C_{2k-1} \otimes X_{2k}^{(i)} = C_{2k-1} \otimes \left( \sum_i y_{ik} X_{2k}^{(i)} \right) \quad k < m \] (87)
\[ B_{2k,2k+1} = \sum_i X_{2k}^{(i)} \otimes Z_{ik} C_{2k+1} = \left( \sum_i z_{ik} X_{2k}^{(i)} \right) \otimes C_{2k+1}. \]

Since we already know that $A_{2m+1,1}$ is product this concludes the proof. □

APPENDIX C. PROOF OF RESULT 2

**Result 2** Consider a quantum strategy on the n-partite ring network with the global state $|\Psi\rangle = |\psi_1\rangle_{R_1 L_2} \cdots |\psi_n\rangle_{R_n L_1}$ and the measurements given by POVMs $\{E_{X_k}^{a_k}\}_{a_k}$ acting on systems $L_k R_k$. If the strategy leads to a TC distribution $P(a_1, \ldots, a_n)$, arising steaming from a TC strategy with the $N_1, \ldots, N_n$ tokens distributed by each source accordingly to the probability distributions $p_1(t_1), \ldots, p_n(t_n)$, then each quantum system $R_k = R_k J_k^R$ and $L_k = L_k J_k^L$ can be decomposed in subsystems such that the quantum strategy takes the form
\[ |\psi_k\rangle_{R_k L_{k+1}} = \sum_{t=0}^{N_k} \sqrt{p_k(t)} |t, N_k - t\rangle_{R_k L_{k+1}} |j_k^{(t)}\rangle_{J_k^R J_k^L}, \] (88)
\[ E_{X_k}^{a_k} = \Pi_{X_k}^{a_k} = \left( \sum_{t, t' = a_k} |t\rangle_{L_k} \otimes |t'\rangle_{L_k}^\dagger \right) \otimes \mathbb{I}_{J_k^R J_k^L}. \]

To each system $R_k (L_k)$ let us associate an Hilbert space $\mathcal{H}_{R_k (L_k)}$ on which the state $|\psi_k\rangle (|\psi_k-1\rangle)$ is supported, it is assumed to have an arbitrary but finite dimension. Next, by introduction an auxiliary system $M_k$ in the state $|0\rangle \in \mathcal{H}_{M_k}$ we dilate each POVMs $\{E_{X_k}^{a_k}\}$ to a PVM $\{\Pi_{X_k}^{a_k}\}$, with the projectors $\Pi_{X_k}^{a_k}$ acting on the systems $R_k L_k M_k$ and
\[ E_{X_k}^{a_k} = \text{tr}_{M_k} \Pi_{X_k}^{a_k} (1_{M_k} R_k \otimes |0\rangle\langle 0|_{M_k}) = |0\rangle_{M_k} \bar{\Pi}_{X_k}^{a_k} |0\rangle_{M_k}. \] (89)
For short we collect all the states of the auxiliary systems into \( |0\rangle = |0\rangle_M \cdots |0\rangle_M \).

Following the main text, this allows us to define a unitary operator for each party \( k \)

\[
\tilde{U}_k = \sum_{x=0}^{N} e^{ix_a} \tilde{\Pi}_{X_k}^x \quad \text{with} \quad e^{ix} = \exp \left( i(x + \frac{1}{n}) \frac{2\pi}{N+1} \right)
\]  

(90)

Note that in general not all outcomes \( x = a_k \in [0, \ldots, N] \) are possible, hence some of the projectors \( \{\tilde{\Pi}_{X_k}^x\}_x \) can be zero. The TC correlations satisfy

\[
a_1 + \cdots + a_n = N.
\]  

(91)

For our quantum model it implies

\[
\tilde{\Pi}_{X_1}^{a_1} \cdots \tilde{\Pi}_{X_n}^{a_n} |\Psi\rangle |0\rangle = 0 \quad \text{if} \quad a_1 + \cdots + a_n \neq N
\]  

(92)

and guarantees that

\[
\bigotimes_{k=1}^{n} \tilde{U}_k |\Psi\rangle |0\rangle = |\Psi\rangle |0\rangle.
\]  

(93)

To see this expand the unitaries as \( \bigotimes_{k=1}^{n} \tilde{U}_k = \sum_{a_1, \ldots, a_n} \exp \left( \frac{1+\sum_{i=1}^{n} a_i}{N+1} 2\pi \right) \tilde{\Pi}_{X_1}^{a_1} \cdots \tilde{\Pi}_{X_n}^{a_n} \) and note that

\[
\exp \left( \frac{1+\sum_{i=1}^{n} a_i}{N+1} 2\pi \right) \tilde{\Pi}_{X_1}^{a_1} \cdots \tilde{\Pi}_{X_n}^{a_n} = 0
\]  

(94)

is zero unless \( a_1 + \cdots + a_n = N \), or \( \exp \left( \frac{1+\sum_{i=1}^{n} a_i}{N+1} 2\pi \right) = 1 \). Therefore \( |\Psi\rangle |0\rangle \) is an eigenstate of \( \bigotimes_{k=1}^{n} \tilde{U}_k \) with eigenvalue 1.

Applying our result 0 to Eq. (93) guarantees that the original measurements \( \{E_{X_k}^{a_k} = \Pi_{X_k}^{a_k}\}_{a_k} \) are projective. We can thus rewrite the constraint (93) in a simpler form

\[
\bigotimes_{k=1}^{n} U_k |\Psi\rangle = |\Psi\rangle \quad \text{with}
\]  

(95)

\[
\bigotimes_{k=1}^{n} U_k |\Psi\rangle = |\Psi\rangle \quad \text{with}
\]  

(96)

Result 1 then implies that all the unitaries \( U_X \) are product

\[
U_X = \sum_{x=0}^{N} \exp \left( i(x + \frac{1}{n}) \frac{2\pi}{N+1} \right) \Pi_{X_k}^x = V_{L_k}^{(k)} \otimes W_{R_k}^{(k)}.
\]  

(97)

Let us now focus on the eigenvalues of the unitaries \( V_{L_k}^{(k)} \) and \( W_{R_k}^{(k)} \), which we know have to fulfill the identity

\[
\exp \left( i(v_j + w_x) \right) = \exp \left( i(a_k + \frac{1}{n}) \frac{2\pi}{N+1} \right)
\]  

(98)

for some \( a_k \). That is, for each pair of eigenvalues with \( v_j \) and \( w_x \) there is an \( a_k \) fulfilling Eq. (98), and conversely for each possible output value \( a_k \) there is a pair \( v_i \) and \( w_j \) satisfying Eq. (98). By adjusting the relative phase of the unitaries in the decomposition (98), \( \left(V_{L_k}^{(k)}, W_{R_k}^{(k)}\right) \rightarrow (e^{iw} V_{L_k}^{(k)}, e^{-iw} W_{R_k}^{(k)}) \) we are free to shift the eigenvalues \( v_j \rightarrow v_j + \omega \) and \( w_j \rightarrow -\omega \). Let us pick a decomposition such that \( V_{L_k}^{(k)} \) admits the eigenvalue \( e^{iv_0} = 1 \), and ask what are the possible values of \( e^{iw_j} \). Form \( e^{i(v_0+w_j)} = e^{i(a_k+\frac{1}{n}) \frac{2\pi}{N+1}} \) we conclude that the possible values are

\[
e^{i(w_j)} = e^{i(a_k+\frac{1}{n}) \frac{2\pi}{N+1}}
\]  

for some possible value of \( a_k \). In turn, the same argument with any of the \( w_x \) guarantees the
possible values of $e^{i\alpha}$ are of the form $e^{i\alpha j \frac{2\pi}{N+1}}$. Both can take at most $N+1$ values. We thus obtain a decomposition

$$ U_k = V_L^{(k)} \otimes W_R^{(k)} $$

with

$$ V_L^{(k)} = \sum_{j=0}^{N} e^{ij \frac{2\pi}{N+1}} \Pi_L^j $$

$$ W_R^{(k)} = e^{i \frac{2\pi}{N+1} \sum_{\ell=0}^{N}} e^{i\ell \frac{2\pi}{N+1}} \Pi_R^\ell. $$

(99)

Here again, some of the projectors can be zero because not all of the values $j$ and $\ell$ are generally possible.

In any case this decomposition allows us to split the Hilbert space associated with each system as e.g.

$$ \mathcal{H}_{L_k} = \bigoplus_{j=0}^{N} \mathcal{H}_{L_k}^{(j)}, $$

(100)

where each $\mathcal{H}_{L_k}^{(j)}$ is the subspace on which $\Pi_L^j$ projects. It is more insightful to write this decomposition as a tensor product instead of a direct sum. While the subspaces $\mathcal{H}_{L_k}^{(j)}$ might have different dimensions, it is always possible to complete them with virtual levels (that do not support $|\Psi\rangle$) to make their dimensions match. Then we can write

$$ \mathcal{H}_{L_k} = \mathcal{G}_{L_k}^{N+1} \otimes \mathcal{H}_{J_k}, $$

(101)

where $\mathcal{G}_{L_k}^{N+1}$ describes the quit ($d = N + 1$) that carries the value $j$, and $\mathcal{H}_{J_k}$ collects all the other degrees of freedom necessary to describe $L_k$ but that do not influence the measurement outcome. The same logic can be applied to the system $R_k$. By construction we obtain

$$ V_L^{(k)} = \left( \sum_{j=0}^{N} e^{ij \frac{2\pi}{N+1}} |j\rangle \langle j|_{L_k} \right) \otimes 1_{J_k} $$

$$ W_R^{(k)} = e^{i \frac{2\pi}{N+1} \sum_{\ell=0}^{N}} e^{i\ell \frac{2\pi}{N+1}} |\ell\rangle \langle \ell|_{R_k} \right) \otimes 1_{J_k}. $$

(102)

Combining them, we obtain

$$ U_k = \left( \sum_{j,\ell=0}^{N} e^{i(j+\ell+\frac{1}{2}) \frac{2\pi}{N+1}} |j,\ell\rangle \langle j,\ell|_{L_k R_k} \right) \otimes 1_{J_k}. $$

(103)

Finally, comparing with the definition (90) we obtain a decomposition of the measurement operators

$$ \Pi_{X_k}^{(k)} = \left( \sum_{j,\ell=0}^{N} e^{i(j+\ell+\frac{1}{2}) \frac{2\pi}{N+1}} |j,\ell\rangle \langle j,\ell|_{L_k R_k} \right) \otimes 1_{J_k}. $$

(104)

The decomposition (101) can also be used to express the states prepared by the source as

$$ |\psi_k\rangle = \sum_{i,j=0}^{N} \Psi_{ij}^{(k)} |i\rangle_{R_k} |L_k\rangle \otimes |j\rangle_{L_k} |r_{k+1} \rangle_{J_k} |l_{k+1} \rangle_{J_k} \otimes |j_{k+1}\rangle_{J_k} |l_{k+1}\rangle_{J_k} \rangle $$

(105)

Here, the states of the junk systems $|j_{k+1}\rangle_{J_k} |l_{k+1}\rangle_{J_k}$ have no influence on the measurement outcomes and are completely arbitrary. Without loss of generality the amplitudes $\Psi_{ij}^{(k)}$ can be taken real positive, by absorbing any complex phase inside $|j_{k+1}\rangle_{J_k} |l_{k+1}\rangle_{J_k}$.
So far we have only used the equality $\sum_i a_k = N$ to derive rather strict restrictions (104,105) on the form of any the quantum model leading to a TC distribution. Let us now use the classical rigidity property TC distributions $P(a_1, \ldots, a_n)$ to show that an equal probability of tokens $N_1, \ldots, N_n$ and the probability distribution $p_k(i)$ that the source $S_k$ send $i$ tokens to $R_k$ and $N_k - i$ tokens to $L_{k+1}$ for $i \in \{0, \ldots, N_k\}$.

First, remark that the equations (104,105) also define a unique classical strategy. Each source $S_k$ samples a pair of integers $(i,j) \in \{0, \ldots, N\}^2$ from the probability distribution

$$P^{(k)}(i,j) = |\Psi_{ij}^{(k)}|^2,$$

the value $i$ and $j$ define the states of the classical systems $R_k$ and $L_{k+1}$ sent to the neighbouring parties. The party $k$ reads the values $j$ from $L_k$ and $\ell$ for $R_k$, and outputs

$$a_k(j, \ell) = (j + \ell) \mod (N + 1).$$

The rigidity of classical TC strategies [17] guarantees that for all systems $R_k$ and $L_k$ there exist "token functions"

$$T_k^R : \{0, \ldots, N\} \to \{0, \ldots, N_k\}$$

$$T_{k+1}^L : \{0, \ldots, N\} \to \{0, \ldots, N_k\}$$

such that

(i) $T_k^R(i) + T_{k+1}^L(j) = N_k$ if $P^{(k)}(i,j) \neq 0$.

(ii) $a_k(j, \ell) = T_{k+1}^L(j) + T_k^R(\ell)$ for all possible values $j$ and $\ell$ (nonzero probability).

(iii) The tokens are distributed in the same way that in the TC strategy. That is

$$\sum_{i,j} P^{(k)}(i,j) \delta_{T_k^R(i),i'} \delta_{T_{k+1}^L(j),N_k-i'} = p_k(i').$$

For each system the token function define disjoint subsets $T_d^{R,L}(L_k) \subset \{0, \ldots, N\}$, such that $T_k^R(i) = t$ if $i \in T_d^{R}(L_k)$. For a quantum model this defines a block diagonal structure for our qudits $C_{R_k}^d$ and $C_{L_{k+1}}^d$ in $N_k + 1$ blocks. As before we can embed the qudits $C_{R_k}^d, C_{L_{k+1}}^d$ into tensor product spaces $C_{T_k^R}^{(N_k+1)} \otimes M_{R_k}$ and $C_{T_{k+1}^L}^{(N_{k+1})} \otimes M_{L_{k+1}}$ in order to write

$$|i\rangle_{R_k} = |t\rangle \langle T_k^R(i) |\alpha_i \rangle_{M_k^R}$$

$$|j\rangle_{L_{k+1}} = |T_{k+1}^L(j) \rangle \langle t_{J_{k+1}} |\alpha_j \rangle_{M_{k+1}^L}$$

By the property (ii) we know that the response functions $a_k(j, \ell) = t_j + t_\ell$ only depend on the the values $t_j$ and $t_\ell$ but not on the multiplicities. Hence, we can rewrite Eq. (104) as

$$\Pi_{N_k}^{a_k} = \sum_{t+t'=a_k} |t,t'\rangle\langle t,t'|_{T_k^R T_{k+1}^L} \otimes 1_{M_k^R M_{k+1}^L} \otimes 1_{J_{k}^R J_{k+1}^L}.$$  

In turn the state of Eq. (105) can be rewritten as

$$\langle \psi_k \rangle = \sum_{i,j=0}^{N} \Psi_{ij}^{(k)} |i\rangle_{R_k} \otimes |j\rangle_{L_{k+1}} \otimes |\alpha_i \rangle_{M_k^R} \otimes |\alpha_j \rangle_{M_{k+1}^L}$$

$$= \sum_{i,j=0}^{N} \Psi_{ij}^{(k)} |t_{i,j} \rangle \otimes |\alpha_i \rangle_{M_k^R} \otimes |\alpha_j \rangle_{M_{k+1}^L} \otimes |j_{i,j} \rangle_{J_{k}^R J_{k+1}^L}$$

$$= \sum_{t,t'=0}^{N_k} |t,t'\rangle_{T_k^R T_{k+1}^L} \sum_{i \in T_{R_k}} \sum_{j \in T_{L_{k+1}}} \Psi_{ij}^{(k)} |\alpha_i \rangle_{M_k^R} \otimes |\alpha_j \rangle_{M_{k+1}^L} \otimes |j_{i,j} \rangle_{J_{k}^R J_{k+1}^L} \equiv \Gamma_{t,t'} |j_{t,t'} \rangle_{J_{k}^R J_{k+1}^L}.$$  

(112)
Compute the normalization of the state defined in the last line
\[
|\Gamma_{t,t'}|^2 = \sum_{i \in T_{h_k}} \sum_{j \in T_{l_{k+1}}} \Psi_{ij}^{(k)} |\alpha_i,\alpha_j\rangle_{M^{k}_{X} M^{L}_{X}} |j_{k}^{(ij)}\rangle_{J^{R}_{k} J^{L}_{k}}^j \tag{113}
\]
by (iii). Furthermore, by (i) we know that in Eq. (112) \(t_i + t_j = N_k\) whenever \(\Psi_{ij}^{(k)}\) is nonzero. Therefore,
\[
|\psi_k\rangle = \sum_{t=0}^{N_k} \sqrt{p_k(t)} |t, N_k - t\rangle_{T^R_k T^L_{k+1}} |j_{k}^{(t,t')}\rangle_{M^{R}_{k} M^{L}_{k+1} J^{R}_{k} J^{L}_{k+1}}. \tag{114}
\]
It remains to absorb the systems \(M^{R}_{k}\) and \(M^{L}_{k+1}\) inside the junk to obtain the desired decomposition
\[
|\psi_k\rangle = \sum_{t=0}^{N_k} \sqrt{p_k(t)} |t, N_k - t\rangle_{T^R_k T^L_{k+1}} |j_{k}^{(t)}\rangle_{J^{R}_{k} J^{L}_{k+1}} \tag{115}
\]
\[
\Pi_{X_k}^{1} = \left( \sum_{t+t' = a_k} |t,t'\rangle \langle t,t'|_{T^R_k T^L_{k}} \right) \otimes \mathbb{1}_{J^{R}_{k} J^{L}_{k}}. \tag{116}
\]

**APPENDIX D. DERIVATION OF THE BOUND ON THE COHERENCE \(r\) OF QUANTUM MODELS SIMULATING THE RGB4 DISTRIBUTION.**

**Dilation of the measurements \(\{\Pi_{X}^{0}, E_{X}^{0}, E_{X}^{1}, \Pi_{X}^{1}\}\)**

We start by commenting on the dilation needed to write \(\Pi_{X}^{1}\) as projectors. Note that result 2 only guarantees that the coarse-grained measurements are projective
\[
\Pi_{X}^{0} = |00\rangle_{x,x'} \otimes \mathbb{1}_{J_X}, \quad \Pi_{X}^{1} = |11\rangle_{x,x'} \otimes \mathbb{1}_{J_X}, \quad \Pi_{X}^{1} = (|01\rangle_{01} + |10\rangle_{10})_{x,x'} \otimes \mathbb{1}_{J_X}. \tag{117}
\]
This is not necessarily the case of the operators \(E_{X}^{0}\) and \(E_{X}^{1}\), that only need to satisfy
\[
E_{X}^{0} + E_{X}^{1} = \Pi_{X}^{1}. \tag{118}
\]
Nevertheless, these measurements can be dilated to projectors For the sake of clarity we present it explicitly here. First, diagonalize the operators
\[
E_{X}^{0} = \sum_{k} p_k |\phi_k\rangle \langle \phi_k| \quad E_{X}^{1} = \sum_{k} (1 - p_k) |\phi_k\rangle \langle \phi_k|
\]
where the states \(|\phi_k\rangle\) define a basis of the Hilbert space \(\mathcal{H}_{X} = \text{span} \left( |01\rangle_{x,x'}, |10\rangle_{x,x'} \right) \otimes \mathcal{H}_{J_X} \otimes \mathcal{H}_{J_R}. \) Next, introduce an auxiliary qubit \(M_X\) initially in the state \(|0\rangle\), and define a unitary
\[
V = \sum_{k} \left( \sqrt{p_k} |\phi_k,0\rangle + \sqrt{1 - p_k} |\phi_k,1\rangle \right) \langle \phi_k,0| + \left( \sqrt{1 - p_k} |\phi_k,0\rangle - \sqrt{p_k} |\phi_k,1\rangle \right) \langle \phi_k,1| \tag{119}
\]
acting on \(\mathcal{H}_{X} \otimes \mathbb{C}_{M_X}^{2}\), and the two projectors
\[
\bar{\Pi}_{X}^{0} = |0\rangle \langle 0|_{M_X} V |0\rangle \langle 0|_X \quad \bar{\Pi}_{X}^{1} = |0\rangle \langle 0|_{M_X} V |1\rangle \langle 1|_X \langle 0|_X . \tag{120}
\]
One verifies that this is indeed a dilation of the original operators $E^1_X = \langle 0 | \tilde{\Pi}^1_X | 0 \rangle$, as

\[
(P^1_X \otimes |0\rangle\langle 0|_M_X)V | 0 \rangle = \sum_k \sqrt{p_k} |\phi_k, 0 \rangle |\phi_k\rangle \tag{121}
\]

\[
(P^1_X \otimes |1\rangle\langle 1|_M_X)V | 0 \rangle = \sum_k \sqrt{p_k} |\phi_k, 1 \rangle |\phi_k\rangle .
\]

Furthermore,

\[
\tilde{\Pi}^1_X = \tilde{\Pi}^0_X + \tilde{\Pi}^1_X = V^\dagger (P^1_X \otimes I_{M_X})V = P^1_X \otimes I_{M_X} \tag{122}
\]

Finally, we get the complete PVM \{\tilde{\Pi}^0_X, \tilde{\Pi}^0_X, \tilde{\Pi}^1_X, \tilde{\Pi}^2_X\} by defining

\[
\tilde{\Pi}^0_X = P^0_X \otimes I_{M_X} \quad \tilde{\Pi}^1_X = P^1_X \otimes I_{M_X} \tag{123}
\]

So that

\[
\tilde{\Pi}^0_X = |00\rangle\langle 00|_{X, X'} \otimes I_{J_X} \otimes I_{M_X}, \quad \tilde{\Pi}^1_X = |11\rangle\langle 11|_{X, X'} \otimes I_{J_X} \otimes I_{M_X} \tag{124}
\]

\[
\tilde{\Pi}^1_X = \tilde{\Pi}^0_X + \tilde{\Pi}^1_X = (|01\rangle\langle 01| + |10\rangle\langle 10|)_{X, X'} \otimes I_{J_X} \otimes I_{M_X}.
\]

Let us now simply absorb each qubit $M_X$ into one of the junk system, say the one received by the party from the left $J^L_k = J^L_k M_X$, and $\langle j^{c(a)}_\xi / j^{c(a)}_\xi \rangle = \langle j^{c(a)}_\xi / j^{c(a)}_\xi \rangle |0\rangle_{M_X}$. We obtain the states

\[
|\psi_\xi\rangle = \frac{1}{\sqrt{2}} (|01\rangle_{X, Y, \xi} |j^{c(a)}_\xi \rangle_j + |10\rangle_{X, Y, \xi} |j^{c(a)}_\xi \rangle_j ) \tag{125}
\]

and $|\Psi\rangle = |\psi_\alpha\rangle |\psi_\beta\rangle |\psi_\gamma\rangle$ such that $P_Q(a, b, c) = \|\Pi^a_A \Pi^b_B \Pi^c_C |\Psi\rangle\|^2$ with projective measurements. At this point we forget the bars over the projectors and the junk systems, and simply write

\[
P_Q(a, b, c) = \|\Pi^a_A \Pi^b_B \Pi^c_C |\Psi\rangle\|^2. \tag{126}
\]

**Derivation of the bound for $r$**

Let us now compute the probabilities of the equation (14) in the main text, starting with $P_Q(1, 1, 1, 1)$. Since $\Pi^1_A \Pi^1_B \Pi^1_C = (P^1_A \Pi^1_B \Pi^1_C)^2$ are projectors we express

\[
P_Q(1, 1, 1, 1) = \|\Pi^1_A \Pi^1_B \Pi^1_C |\Psi\rangle\|^2
\]

\[
= \frac{1}{8} \|\Pi^1_A \Pi^1_B \Pi^1_C (|01\rangle_{B_a, C_a} |j^{c(a)}_\alpha \rangle_j + |10\rangle_{B_a, C_a} |j^{c(a)}_\alpha \rangle_j ) \otimes (|01\rangle_{A, B, C_a} |j^{c(a)}_\alpha \rangle_j + |10\rangle_{A, B, C_a} |j^{c(a)}_\alpha \rangle_j ) \|^2. \tag{127}
\]

Each $\Pi^1_X$ is only supported on the subspace where the the party receives a single token span\{|00\rangle_{X, X'}, |10\rangle_{X, X'}\}, therefore in the above expression only two terms are nonzero.

\[
P_Q(1, 1, 1, 1) = \frac{1}{8} \|\Pi^1_A \Pi^1_B \Pi^1_C (|01\rangle_{B_a, C_a} |j^{c(a)}_\alpha \rangle_j |01\rangle_{A, B, C_a} |j^{c(a)}_\alpha \rangle_j |01\rangle_{A, B, C_a} |j^{c(a)}_\alpha \rangle_j |10\rangle_{A, B, C_a} |j^{c(a)}_\alpha \rangle_j ) \|^2
\]

\[
= \frac{1}{8} \|\Pi^1_A \Pi^1_B \Pi^1_C (|\Psi^c\rangle + |\Psi^a\rangle) \|^2 \tag{128}
\]

where we defined the two global states $|\Psi^c\rangle \equiv |01, 01, 01\rangle_{B_a, C_a, B_a B_a} |j^{c(a)}_\alpha \rangle_j |j^{c(a)}_\alpha \rangle_j |j^{c(a)}_\alpha \rangle_j j_{\alpha, \beta, \gamma}$ and $|\Psi^a\rangle \equiv |10, 10, 10\rangle_{B_a, C_a, B_a B_a} |j^{a(a)}_\alpha \rangle_j |j^{a(a)}_\alpha \rangle_j |j^{a(a)}_\alpha \rangle_j j_{\alpha, \beta, \gamma}$. A priori, these are arbitrary quantum states

\[
|\Psi^c\rangle = \langle j^{c(a)}_\alpha / j^{c(a)}_\alpha \rangle |01\rangle_{B_a, C_a} \langle j^{c(a)}_\beta / j^{c(a)}_\beta \rangle |01\rangle_{C_a A_a} \langle j^{c(a)}_\gamma / j^{c(a)}_\gamma \rangle |01\rangle_{A_a B_a}.
\]

\[
|\Psi^a\rangle = \langle j^{a(a)}_\alpha / j^{a(a)}_\alpha \rangle |10\rangle_{B_a, C_a} \langle j^{a(a)}_\beta / j^{a(a)}_\beta \rangle |10\rangle_{C_a A_a} \langle j^{a(a)}_\gamma / j^{a(a)}_\gamma \rangle |10\rangle_{A_a B_a}.
\]
on a triangle network, with the important property that for each system $X_\xi$ the state $|\Psi^o\rangle$ and $|\Psi^c\rangle$ live on orthogonal subspaces (labeled by the state of the qubit $X_\xi$). With the help of the coherence $r = (-1)^{i+j+k} 2 \Re \langle \Psi^c | \Pi_A^1 \Pi_B^1 \Pi_C^1 | \Psi^o \rangle$ introduced in the main text, we obtain

$$P_Q(1, 1, 1) = \frac{1}{8} \left( \| \Pi_A^1 \Pi_B^1 \Pi_C^1 | \Psi^o \rangle \|^2 + \| \Pi_A^1 \Pi_B^1 \Pi_C^1 | \Psi^c \rangle \|^2 + (-1)^{i+j+k} r \right).$$

(130)

Next, let us compute $P_Q(1, 0, 2)$. We obtain have

$$P_Q(1, 0, 2) = \frac{1}{8} \| \Pi_A^1 | \Psi^c \rangle \|^2,$$

and any cyclic permutation of these two equations. We obtain

$$\| \Pi_A^1 \Pi_B^1 \Pi_C^1 | \Psi^c \rangle \|^2 = \| \Pi_A^1 | \Psi^o \rangle \|^2 + (-1)^{i+j+k} r = (u_i u_j u_k + v_i v_j v_k)^2$$

$$\| \Pi_A^1 | \Psi^o \rangle \|^2 = u_i^2$$

$$\| \Pi_A^1 | \Psi^c \rangle \|^2 = v_i^2.$$  

(132)

Remark that the state $|\Psi^o\rangle$ and $|\Psi^c\rangle$ belong to the subspace where each party receive a single token, i.e. $(\Pi_A^1 + \Pi_B^1) |\Psi^{(a)}\rangle = |\Psi^{(a)}\rangle$. In this subspace, each pair of projectors $(\Pi_A^0, \Pi_B^1)$ defines a PVM for the corresponding party. Accordingly, let us define the probability distributions

$$q_c(i, j, k) = \| \Pi_A^1 \Pi_B^1 \Pi_C^1 | \Psi^c \rangle \|^2 = \| \Pi_A^1 \Pi_B^1 \Pi_C^1 | \Psi^o \rangle \|^2,$$

(133)

that both describe some quantum correlations on the triangle with binary outputs. These distributions have to satisfy $q_c(i, j, k) + q_o(i, j, k) + (-1)^{i+j+k} = (u_i u_j u_k + v_i v_j v_k)^2$ and $q_c(i) = \sum_{jk} q_c(i, j, k) = u_i^2$ and $q_o(i) = \sum_{jk} q_o(i, j, k) = u_i^2$ by virtue of Eqs. (132).

Following [14], we also define the symmetrized distributions

$$\tilde{q}_c(i, j, k) = \frac{1}{6} (q_c(i, j, k) + q_o(j, k, i) + q_o(k, i, j) + q_o(k, j, i) + q_o(i, k, j) + q_o(i, j, k))$$

$$\tilde{q}_o(i, j, k) = \frac{1}{6} (q_c(i, j, k) + q_c(j, k, i) + q_c(k, i, j) + q_c(k, j, i) + q_c(i, k, j) + q_c(i, j, k))$$

(134)

The symmetrized distributions no longer describe quantum correlations on the triangle, as the implementation of the symmetrization procedure would require global shared randomness, nevertheless by convexity they are valid probability distributions and should satisfy the same constraints as the original distribution

$$\tilde{q}_c(i, j, k) + \tilde{q}_o(i, j, k) + (-1)^{i+j+k} r = (u_i u_j u_k + v_i v_j v_k)^2$$

$$\tilde{q}_c(i) = \sum_{jk} \tilde{q}_c(i, j, k) = v_i^2$$

$$\tilde{q}_o(i) = \sum_{jk} \tilde{q}_o(i, j, k) = u_i^2.$$  

(135)

Now let us also define

$$\xi_{ijk} = \frac{1}{2} (\tilde{q}_c(i, j, k) - \tilde{q}_c(i, j, k))$$

$$\tilde{q}(i, j, k) = \frac{1}{2} (\tilde{q}_o(i, j, k) + \tilde{q}_c(i, j, k))$$

(136)
with the short notation $\xi_x = \xi_{ijk}$ for $i + j + k = x$ (all of them are equal by symmetrization). Note that the average distribution $\tilde{q}(i, j, k)$ satisfies

$$\tilde{q}(i, j, k) = \frac{(u_{x}u_{y}u_{z} + v_{x}v_{y}v_{z})^2 - (-1)^{i+j+k}r}{2}.$$  \hfill (137)

or

$$q_a(i, j, k) = \tilde{q}(i, j, k) + \xi_{ijk} \quad q_c(i, j, k) = \tilde{q}(i, j, k) - \xi_{ijk}. \hfill (138)$$

Now we have the following equities

$$\tilde{q}_a(i) = \sum_{jk} \tilde{q}_a(i, j, k) = \sum_{jk}(\tilde{q}(i, j, k) + \xi_{i,j,k}) \hfill (139)$$

\begin{align*}
&= \frac{1}{2} \sum_{jk}((u_{x}u_{y}u_{z} + v_{x}v_{y}v_{z})^2 - r(-1)^{i+j+k}) + \sum_{jk} \xi_{ijk} \\
&= \sum_{jk} \frac{(u_{x}u_{y}u_{z} + v_{x}v_{y}v_{z})^2}{2} + \sum_{jk} \xi_{ijk} \\
&= \frac{u^2 + v^2}{2} + \sum_{jk} \xi_{ijk} \\
&= \frac{1}{2} + \sum_{jk} \xi_{ijk}.
\end{align*}

Similarly, we have $q_c(i) = \frac{1}{2} - \sum_{jk} \xi_{ijk}$.

Using $\sum_{jk} \xi_{0jk} = \xi_0 + 2\xi_1 + \xi_2$ and $\sum_{jk} \xi_{1jk} = \xi_1 + 2\xi_2 + \xi_3$ we rewrite the above conditions together with the probability constraints as

$$u^2 = \frac{1}{2} + \xi_0 + 2\xi_1 + \xi_2 \hfill (140)$$

$$1 - u^2 = \frac{1}{2} + \xi_1 + 2\xi_2 + \xi_3.$$

That we use to write down

$$\xi_0 = u^2 - \frac{1}{2} - 2\xi_1 - \xi_2 \hfill (141)$$

$$\xi_3 = \frac{1}{2} - u^2 - \xi_1 - 2\xi_2.$$

Furthermore, we have the following positivity conditions for the probabilities

$$0 \leq \tilde{q}_c(0, 0, 0) = \tilde{q}(0, 0, 0) - \xi_0 = \frac{(u^3 + v^3)^2 - r}{2} - \xi_0$$

$$0 \leq \tilde{q}_a(1, 1, 1) = \tilde{q}(1, 1, 1) + \xi_3 = \frac{(u^3 - v^3)^2 + r}{2} + \xi_3$$

$$0 \leq \tilde{q}_c(0, 0, 1) = \tilde{q}(0, 0, 1) - \xi_1 = \frac{(u^2v - v^2u)^2 + r}{2} - \xi_1$$

We use the first inequality of Eq. (142) to get

$$\xi_0 \leq \frac{(u^3 + v^3)^2 - r}{2}$$

$$u^2 - \frac{1}{2} - 2\xi_1 - \xi_2 \leq \frac{(u^3 + v^3)^2 - r}{2}$$

$$u^2 - \frac{1}{2} + (u^3 + v^3)^2 - r \leq 2\xi_1 \leq \xi_2.$$
The second inequality to bound

\[ \xi_3 \geq -\frac{(u^3 - v^3)^2 + r}{2} \]

\[ \frac{1}{2} - u^2 - \xi_1 - 2\xi_2 \geq -\frac{(u^3 - v^3)^2 + r}{2} \]

\[ \frac{1 + (u^3 - v^3)^2 + r}{2} - u^2 - \xi_1 \geq 2\xi_2 \]

Together the two conditions imply

\[ 2u^2 - (1 + (u^3 + v^3)^2 - r) - 4\xi_1 \leq \frac{1 + (u^3 - v^3)^2 + r}{2} - u^2 - \xi_1 \]

\[ \xi_1 \geq \frac{6u^2 - 2(u^3 + v^3)^2 - (u^3 - v^3)^2 - 3 + r}{6} \]

and the very last inequality of Eq. (142) to simply write

\[ \xi_1 \leq \frac{(u^2v - v^2u)^2 + r}{2} \]

Combining the equations (145) and (146) gives

\[ r \geq R_\theta \equiv \frac{1}{2} \sin^3(\theta)(-6\sin(\theta) + 3\cos(\theta) + \cos(3\theta)) \]

in the notation \( u = \cos(\theta) \) and \( v = \sin(\theta) \).

**APPENDIX E. SOURCE ENTANGLEMENT AND OUTPUT RANDOMNESS OF THE RGB4 DISTRIBUTION.**

For the source states

\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|01\rangle \chi_{x_i} \chi_{y_i} |j_{\gamma_i}^{(c)} \rangle_{X_i' Y_i' E_{\xi_i}} + |10\rangle \chi_{x_i} \chi_{y_i} |j_{\gamma_i}^{(a)} \rangle_{X_i' Y_i' E_{\xi_i}}) \]

introduce the Schmidt decompositions

\[ |j_{\gamma_i}^{(c)} \rangle_{X_i' Y_i' E_{\xi_i}} = \sum_k \lambda_{k} |\gamma_{(k)}\rangle_{X_i' Y_i'} |\gamma_{(k)}\rangle_{E_{\xi_i}} \]

\[ |j_{\gamma_i}^{(a)} \rangle_{X_i' Y_i' E_{\xi_i}} = \sum_k \mu_{k} |\gamma_{(k)}\rangle_{X_i' Y_i'} |\gamma_{(k)}\rangle_{E_{\xi_i}} \]

To shorten the equations we denote the overlaps between Eve’s states as \( \Gamma_{kj}^{\xi} = \frac{\langle \sigma_{(k)}^{(a)} | \kappa_{(j)}^{(c)} \rangle_{E_{\xi_i}}}{\langle \sigma_{(k)}^{(a)} \rangle_{E_{\xi_i}}} \). With the explicit parametrization of the junk states, the coherence terms take the form

\[ \langle \Psi^{a} | \Pi_{A}^{x} \Pi_{B}^{x} \Pi_{C}^{y} |\Psi^{c}\rangle = \sum_{k,k',k''} \mu_{k} \mu_{k'} \lambda_{k} \lambda_{k'} \lambda_{k''} \lambda_{j, j', j''} \Gamma_{kj}^{\xi} \Gamma_{kj'}^{\xi} \Gamma_{kj''}^{\xi} \Gamma_{k'j''}^{\xi} \Gamma_{k''j'}^{\xi} \langle \gamma^{c} \rangle_{\gamma^{c}} \langle \gamma^{a} \rangle_{\gamma^{a}} |\Psi^{a} | \Pi_{B}^{x} \Pi_{C}^{y} |\gamma^{c}\rangle_{\gamma^{c}} |\gamma^{a}\rangle_{\gamma^{a}} \].

Where the state \( |\gamma^{c}\rangle_{\gamma^{c}} = |10, 10, 10\rangle_{A_{1}A_{2}A_{3}A_{4}B_{1}B_{2}B_{3}B_{4}C_{1}C_{2}C_{3}C_{4}} \) denotes the state where all the tokens are sent clockwise, and \( |\gamma^{c}\rangle_{\gamma^{c}} \) where they are sent anti-clockwise. Note that the two projectors \( \{\Pi_{X}^{x}, \Pi_{Y}^{x}\} \) define a PVM in the subspace \( \langle \gamma^{c} \rangle_{\gamma^{c}} \otimes \mathcal{H}_{J_{X}} \), in which \( |\Psi^{a}\rangle \) and \(|\Psi^{c}\rangle \) belong.

Now consider the coherence \( r \) defined in the main text

\[ 2 \text{Re} \langle \Psi^{a} | \Pi_{A}^{x} \Pi_{B}^{x} |\Psi^{c}\rangle = (-1)^{1+j+k} r, \]

\[ \text{(151)} \]
and define the following global operator

\[
V = \sum_{i,j,k=0,1} (-1)^{i+j+k} \Pi_A^i \Pi_B^j \Pi_C^k
\]  

(152)

which is unitary on the subspace where \(|\Psi\rangle\) and \(|\Phi\rangle\) belong. We can now compute the coherence

\[
|\langle \Psi | V | \Phi \rangle | \geq \text{Re} \langle \Psi | V | \Phi \rangle = \text{Re} \langle \Psi_a | \sum_{i,j,k=0,1} (-1)^{i+j+k} \Pi_A^i \Pi_B^j \Pi_C^k | \Phi \rangle = \sum_{i,j,k} \frac{r}{2} = 4r.
\]  

(153)

On the other hand, with the above parametrization introduced in Eq. (149), can we express it as

\[
\langle \Psi | V | \Phi \rangle = \sum_{k,k',j,j'} \mu_{k,k'}^{\gamma_j} \mu_{k,k'}^{\gamma_j} \mu_{k,k'}^{\gamma_j} \Gamma_{j,j'} \langle \gamma_j | \Gamma_{j,j'}^{\gamma_j} | \gamma_j \rangle |\langle \gamma_j | \Gamma_{j,j'}^{\gamma_j} | \gamma_j \rangle | \leq 1 \text{ allow us to get}
\]

\[
4r \leq |\langle \Psi | V | \Phi \rangle| \leq \sum_{k,k',j,j'} \mu_{k,k'}^{\gamma_j} \mu_{k,k'}^{\gamma_j} \mu_{k,k'}^{\gamma_j} \Gamma_{j,j'} \langle \gamma_j | \Gamma_{j,j'}^{\gamma_j} | \gamma_j \rangle |\langle \gamma_j | \Gamma_{j,j'}^{\gamma_j} | \gamma_j \rangle | \leq \sum_{k,j} \mu_{k}^{\gamma_j} \Gamma_{j,j'}^{\gamma_j} |\langle \gamma_j | \Gamma_{j,j'}^{\gamma_j} | \gamma_j \rangle| \]

(155)

In particular, we use \(\sum_{k,j} \mu_{k}^{\gamma_j} \Gamma_{j,j'}^{\gamma_j} \leq 1\) and \(\sum_{k,j} \mu_{k}^{\gamma_j} \Gamma_{j,j'}^{\gamma_j} \leq \sum_{k,j} \mu_{k}^{\gamma_j} \Gamma_{j,j'}^{\gamma_j}\) to get a bound useful for the next section

\[
\left(\sum_{j} \mu_{k}^{\gamma_j} \Gamma_{j,j'}^{\gamma_j}\right) \left(\sum_{j} \mu_{k}^{\gamma_j} \Gamma_{j,j'}^{\gamma_j}\right) \geq 4r
\]  

(156)

**Randomness**

To quantify the randomness produced by the measurement we focus on the conditional entropy of a single output, say \(a\), with respect to an eavesdropper. For simplicity we further coarse-grain the values of \(a\) to define a bit \(\bar{a} = 0\) (for \(a = 0, 2\)) and \(\bar{a} = 1\) (if \(a = 1, 1\)) encoded in the register \(A\). We are interested in the classical-quantum state \(\rho_{\bar{a}E}\) shared between the register \(A\) and the eavesdropper. The outcomes \(\bar{a}\) are determined by the states of the systems \(A_{\bar{a}}\) and \(A_{\gamma}\), which are only correlated with \(E_{\bar{a}}\) and \(E_{\gamma}\). For the state of interest, we obtain

\[
\theta_{\bar{a}E} = \frac{1}{4} |0\rangle\langle 0|_\bar{a} \left( \rho_{E_0}^{(c)} \otimes \rho_{E_+}^{(a)} + \rho_{E_+}^{(a)} \otimes \rho_{E_0}^{(c)} \right) + \frac{1}{4} |1\rangle\langle 1|_\bar{a} \left( \rho_{E_0}^{(c)} \otimes \rho_{E_-}^{(a)} + \rho_{E_-}^{(a)} \otimes \rho_{E_0}^{(c)} \right),
\]  

(157)

where the eavesdropper’s states are

\[
\rho_{E_0}^{(c)} = \sum_k (\lambda_\xi^{(k)})^2 |\lambda_\xi^{(k)}\rangle \langle \lambda_\xi^{(k)}|,
\]

\[
\rho_{E_0}^{(a)} = \sum_k (\mu_\xi^{(k)})^2 |\mu_\xi^{(k)}\rangle \langle \mu_\xi^{(k)}|,
\]

(158)

The randomness of \(\bar{a}\) can be quantified by the min-entropy \(H_{\min}(\bar{A}|E) = -\log_2 P_{\text{guess}}(\bar{A}|E)\), where \(P_{\text{guess}}(\bar{A}|E)\) is the probability that Eve guesses the value \(\bar{a}\) correctly. It is related by \(P_{\text{guess}}(\bar{A}|E) = \frac{1}{2}(1 + D(\rho_{E|\bar{a}=0} \otimes \rho_{E|\bar{a}=1}))\) to the trace distance \(D\) between the conditional states of Eve

\[
\rho_{E|\bar{a}=0} = \frac{1}{2} \left( \rho_{E_0}^{(c)} \otimes \rho_{E_+}^{(a)} + \rho_{E_+}^{(a)} \otimes \rho_{E_0}^{(c)} \right),
\]

\[
\rho_{E|\bar{a}=1} = \frac{1}{2} \left( \rho_{E_0}^{(c)} \otimes \rho_{E_-}^{(a)} + \rho_{E_-}^{(a)} \otimes \rho_{E_0}^{(c)} \right).
\]  

(159)
In turn, the trace distance $D \leq \sqrt{1 - F^2}$ is bounded by the fidelity $F(\rho, \sigma) = \text{tr} |\sqrt{\rho} \sqrt{\sigma}|$ between the states, which gives us

$$H_{\text{min}}(\hat{A} | E) \geq - \log_2 \left( \frac{1}{2} (1 + \sqrt{1 - F^2(\rho_E|\tilde{a}=0, \rho_E|\tilde{a}=1)}) \right)$$  \hspace{1cm} (160)

Using the strong convexity of $F$ we obtain

$$F(\rho_E|\tilde{a}=0, \rho_E|\tilde{a}=1) \geq \frac{1}{2} \left( F(\rho_E^{(c)}|\tilde{a}=0, \rho_E^{(c)}|\tilde{a}=1) + F(\rho_E^{(a)}|\tilde{a}=0, \rho_E^{(a)}|\tilde{a}=1) \right).$$  \hspace{1cm} (161)

Then we can use the strong convexity again together with the Eqs. (158) to obtain

$$F(\rho_E^{(c)}|\tilde{a}=0, \rho_E^{(a)}|\tilde{a}=1) \geq \sum_{k,j} \lambda_{k_j}^l \mu_{k_j}^l \left| \big| E_{k_j} \big| \big| \right. \left. + \sum_{k,j} \lambda_{k_j}^r \mu_{k_j}^r \left| \big| F_{k_j} \big| \right. \right).$$  \hspace{1cm} (162)

Therefore

$$F(\rho_E|\tilde{a}=0, \rho_E|\tilde{a}=1) \geq \frac{1}{2} \left( \sum_{k,j} \lambda_{k_j}^l \mu_{k_j}^l \left| \big| E_{k_j} \big| \big| \right. \left. + \sum_{k,j} \lambda_{k_j}^r \mu_{k_j}^r \left| \big| F_{k_j} \big| \right. \right).$$  \hspace{1cm} (163)

And it remains to lower bound the right hand side, given the constant $\left( \sum_{k,j} \lambda_{k_j}^l \mu_{k_j}^l \left| \big| E_{k_j} \big| \big| \right. \left. + \sum_{k,j} \lambda_{k_j}^r \mu_{k_j}^r \left| \big| F_{k_j} \big| \right. \right) \geq 4r$ derived in the previous section (Eq. 156). Denoting $X = \sum_{k,j} \lambda_{k_j}^l \mu_{k_j}^l \left| \big| E_{k_j} \big| \big| \right.$ and $Y = \sum_{k,j} \lambda_{k_j}^r \mu_{k_j}^r \left| \big| F_{k_j} \big| \right.$ the problem is simply

$$\min_{X,Y} \frac{1}{2} (X + Y)$$

such that $X \cdot Y \geq 4r$  \hspace{1cm} (164)

Without big surprise the minimum is attained at $X = Y = \sqrt{4r}$. This allows us to conclude

$$F(\rho_E|\tilde{a}=0, \rho_E|\tilde{a}=1) \geq \sqrt{4r},$$  \hspace{1cm} (165)

and finally get the desired bound on the min entropy

$$H_{\text{min}}(\hat{A} | E) \geq - \log_2 \left( \frac{1}{2} (1 - \sqrt{1 - 4r}) \right).$$  \hspace{1cm} (166)

At the optimal value $\theta$ this given about 4% of a bit.

**Entanglement**

Assume that the entanglement of formation of $\rho^{(a)}$ is at most $E_F$. Then one can decompose this state as

$$\rho^{(a)} = \sum_{k} p_{k} \left| \tilde{\psi}_{a}^{(k)} \right\rangle \left\langle \tilde{\psi}_{a}^{(k)} \right|,$$  \hspace{1cm} (167)

where each $\tilde{\psi}_{a}^{(k)}$ has entanglement entropy of entanglement $S_k$, and $\sum_{k} p_{k} S_k \leq E_F$. By Eq. (12) we know $\rho^{(a)}$ and each $\tilde{\psi}_{a}^{(k)}$ are supported in the subspace projected by $(|01\rangle|01\rangle + |10\rangle|10\rangle)_{B_a' C_a'}$. Hence, each state in the decomposition is of the form

$$\left| \tilde{\psi}_{a}^{(k)} \right\rangle = c_{k} |01\rangle_{B_a' C_a'} \left| \phi_k \right\rangle_{B_a C_a} + s_{k} e^{i \phi_{k}} |10\rangle_{B_a' C_a'} \left| \zeta_k \right\rangle_{B_a' C_a'},$$  \hspace{1cm} (168)

with some real positive $c_k$ and $s_k$ positive satisfying $c_k^2 + s_k^2 = 1$ by normalization. The entropy of entanglement of each state is lower bounded by

$$S_k = S \left( \text{tr}_{B_a B_a'} \left| \tilde{\psi}_{a}^{(k)} \right\rangle \left\langle \tilde{\psi}_{a}^{(k)} \right| \right) \geq S \left( \text{tr}_{B_a B_a'} \left| \tilde{\psi}_{a}^{(k)} \right\rangle \left\langle \tilde{\psi}_{a}^{(k)} \right| \right) = S \left( \left( \begin{array}{cc} c_k^2 \\ s_k^2 \end{array} \right) \right) \equiv h_{\text{bin}}(c_k^2)$$  \hspace{1cm} (169)
Now recall that $\rho^{(\alpha)} = \text{tr}_{E_{\alpha}} \vert \psi_{\alpha} \rangle \langle \psi_{\alpha} \vert$, hence the decomposition in Eq. (167) can be purified, implying for the original state

$$
\vert \psi_{\alpha} \rangle = \sum_k \sqrt{p_k} \vert \tilde{\psi}_{\alpha}^{(k)} \rangle_{B_{\alpha} C_{\alpha} B_{\alpha}' C_{\alpha}'} \vert k \rangle_{E_{\alpha}}
$$

with orthogonal states $\vert k \rangle_{E_{\alpha}}$.

The bound $\vert \langle \Psi^a \vert V \vert \Psi^c \rangle \vert \geq 4r$ implies

$$
4r \leq \vert \langle \Psi^a \vert V \vert \Psi^c \rangle \vert = \vert \langle 10, 10, 10 \rangle_{B_{\alpha} C_{\alpha} C_{\alpha} A_{\beta} A_{\gamma} B_{\gamma}} \langle j_{\alpha}^a, j_{\beta}^a, j_{\gamma}^a \rangle_{j_{\alpha} J_{\beta} J_{\gamma}} V \vert 01, 01, 01 \rangle_{B_{\alpha} C_{\alpha} C_{\alpha} A_{\beta} A_{\gamma} B_{\gamma}} \vert \psi^c \rangle_{B_{\alpha} C_{\alpha} B_{\alpha}' C_{\alpha}'} \langle j_{\beta}^c, j_{\gamma}^c \rangle_{j_{\beta} J_{\gamma}} \vert
$$

$$
\leq \sum_k p_k \vert c_k s_k \vert \vert \langle 10, 10, 10 \rangle \vert \langle j_{\alpha}^a, j_{\beta}^a, j_{\gamma}^a \rangle_{j_{\alpha} J_{\beta} J_{\gamma}} V \vert 01, 01, 01 \rangle \vert \psi^c \rangle_{B_{\alpha} C_{\alpha} B_{\alpha}' C_{\alpha}'} \langle j_{\beta}^c, j_{\gamma}^c \rangle_{j_{\beta} J_{\gamma}} \vert
$$

$$
\leq 2 \sum_k p_k \vert c_k s_k \vert.
$$

The coherence value thus guarantees $\sum_k p_k \vert c_k s_k \vert \geq 2r$.

Now let use denote $c_k^2 = q_k$ and minimize the average entropy of entanglement under the constraint imposed by the coherence. Formally, it amounts to solve the minimization problem

$$
\min_{q_k \in \{0, 1/2\}} \sum_k p_k h_{\text{bin}}(q_k)
$$

such that $\sum_k p_k \sqrt{q_k (1 - q_k)} \geq 2r,$

$$
(172)
$$

where each $q_k$ is between zero and one, and $p_k$ define a probability distribution. To solve it we define new variables

$$
R_k = \sqrt{q_k (1 - q_k)} \in [0, 1/2],
$$

with $q_k = \frac{1}{2} \left( 1 - \sqrt{1 - 4R_k^2} \right)$, and solve

$$
\min_{R_k \in \{0, 1/2\}} \sum_k p_k \frac{1}{2} \left( 1 - \sqrt{1 - 4R_k^2} \right)
$$

such that $\sum_k p_k R_k \geq 2r$.

$$
(173)
$$

One can verify that

$$
\frac{d^2 h_{\text{bin}} \left( \frac{1 - \sqrt{1 - 4R_k^2}}{2} \right)}{dR_k^2} \geq 0,
$$

so the goal function is convex. It is thus minimized by setting all $R_k = 2r$. Therefore for any decomposition (Eq. 167) the average entropy of entanglement satisfies

$$
\sum_k p_k h_{\text{bin}}(R_k) \geq h_{\text{bin}} \left( \frac{1 - \sqrt{1 - 4(2r)^2}}{2} \right).
$$

By definition the same bound holds for the entanglement of formation of the mixed state $\rho^{(\alpha)}$

$$
\mathcal{E}_F \geq h_{\text{bin}} \left( \frac{1 - \sqrt{1 - 16r^2}}{2} \right).
$$

For the optimal value one finds $\mathcal{E}_F \geq 2.5\%$.

**APPENDIX F. SELF-TESTING PARITY TOKEN COUNTING DISTRIBUTIONS ON THE TRIANGLE**

In the PTC strategy on the triangle network, each source $S_\alpha, S_\beta$, and $S_\gamma$ have a single token. The source $\xi$ sends the token to the left with probably $p_\xi$ and to the right with probability $1 - p_\xi$. Each party outputs the parity $a, b, c \in \{0, 1\}$ of the number of received tokens. By construction, one has

$$
a \oplus b \oplus c = 1
$$

(175)
with $\oplus$ denoting addition modulo 2. A PTC distribution can also be obtained from any TC counting strategy if each party only outputs the parity of the total number of received tokens.

Classical rigidity of PTC distributions on the triangle was shown in [24] for all distributions steaming from strategies with $p_\alpha, p_\beta, p_\gamma \neq \frac{1}{2}$. Whenever a distribution corresponds to $p_\mu = \frac{1}{2}$ for at least one of the sources, it turns out that it can be simulated with a whole family of nonequivalent PTC strategies. However, in this case, rigidity holds, but only up to this freedom to choose the nonequivalent PTC strategy; see [24] for details.

Here we extend this result to quantum models. In the PTC task, the measurements have binary outputs and correspond to the PVMs $\{\Pi_X^0, \Pi_X^1\}$. Now, similar to the TC case, let us define a unitary operator for each party $X$ as following

$$U_X = \Pi_X^1 - \Pi_X^0.$$  \tag{176}$$

Eq. (175) guarantees that

$$U_A U_B U_C |\Psi\rangle = |\Psi\rangle,$$  \tag{177}$$

from which, by result 1, we conclude that all the unitaries are product

$$U_A = V_{A_\beta} \otimes W_{A_\gamma},$$
$$U_B = V_{B_\gamma} \otimes W_{B_\alpha},$$
$$U_C = V_{C_\alpha} \otimes W_{A_\beta}.$$

Without loss of generality we can chose the eigenvalues of $V_{A_\beta}$ and $W_{A_\gamma}$ such that $V_{A_\beta} = \Pi_{A_\beta}^1 - \Pi_{A_\beta}^0$ and $W_{A_\gamma} = \Pi_{A_\gamma}^1 - \Pi_{A_\gamma}^0$, which guarantees

$$\Pi_A^1 = \Pi_{A_\alpha}^1 \otimes \Pi_{A_\beta}^1 + \Pi_{A_\alpha}^0 \otimes \Pi_{A_\beta}^0 = \sum_{a,\beta = a_\alpha} \Pi_{A_\beta}^a \otimes \Pi_{A_\beta}^a,$$
$$\Pi_A^0 = \Pi_{A_\alpha}^1 \otimes \Pi_{A_\beta}^0 + \Pi_{A_\alpha}^0 \otimes \Pi_{A_\beta}^1 = \sum_{a,\beta = a_\alpha} \Pi_{A_\beta}^a \otimes \Pi_{A_\beta}^a.$$  \tag{178}$$

This allows us to split the Hilbert decompose associated to each system as $\mathcal{H}_{A_\alpha} = \mathbb{C}_2^2 \otimes \mathcal{H}_{J_{A_\alpha}}$ in such a way that

$$\Pi_A^a = \left( \sum_{j,\ell = a} |j, \ell\rangle\langle j, \ell| \right)_{A_\alpha} \otimes 1_{J_{A_\beta} J_{A_\gamma}},$$
$$|\psi_\alpha\rangle_{B_\alpha C_\alpha} = \sum_{i, j \in \{0, 1\}} \Psi_{ij}^{(a)} |i, j\rangle_{B_\alpha C_\alpha} |j\rangle_{J_{B_\alpha} J_{C_\alpha}}.$$  \tag{179}$$

Note that this form corresponds to a classical strategy with the response function $a = j \oplus \ell$ that is explicitly PTC. The same decomposition can be guaranteed for each party and each source. It remains to show that the amplitudes $\Psi_{ij}^{(a)}$ (that can be taken real by absorbing the phase inside the junk states) are enforced to be unique. As in the TC case, this follows from the classical rigidity of PTC distributions – which guarantee that for the generic case with $p_\alpha, p_\beta, p_\gamma \neq \frac{1}{2}$, the probabilities $P^{(a)}(i, j) = |\Psi_{ij}^{(a)}|^2$ are essentially unique upon relabeling 0 and 1, see [24]. Therefore, we have

$$|\psi_\alpha\rangle_{B_\alpha C_\alpha} = \sqrt{p_\alpha} |01\rangle_{B_\alpha C_\alpha} |j^{(a)}_{J_{B_\alpha} J_{C_\alpha}} + \sqrt{1 - p_\alpha} |10\rangle_{B_\alpha C_\alpha} |j^{(a)}_{J_{B_\alpha} J_{C_\alpha}},$$  \tag{180}$$

which illustrates the self-testing (or quantum rigidity) result, similar to result 2, for PTC distributions on the triangle.