Research Article

Analysis of the Chaotic Dynamics Duopoly Game of ISPs Bounded Rational

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We propose in this paper an investigation of the dynamical behaviors of a duopoly model with two Internet services providers (ISPs). The competition between them is assumed to take place in terms of their pricing decisions and their number of cached items. According to the ISPs’ rationality level, we consider a scenario where both ISPs are bounded rational. Each ISP in any period uses the marginal profit observed from the previous period to choose its strategies. We compute explicitly the steady states of the dynamic system induced by bounded rationality and establish a necessary and sufficient condition for the stability of its Nash equilibrium (NE). Numerical simulations show that if some parameters of the model are varied, the stability of the NE point is lost and the complex (periodic or chaotic) behavior occurs. The chaotic behavior of the system is stabilized on the NE point by applying control.

1. Introduction

The coexistence of interests leads to conflict situations; the latter can be studied by game theory; interacting players choose their actions based on the actions of other players. Moreover, each player makes its decisions in order to maximize its utility function, which generally depends on the actions of all the players.

In last years, game theory has been used to solve many problems in communication networks [1–7]. It has been used to propose new pricing strategies for Internet services [8, 9]. Many other issues relating to wireless networks have been modeled and analyzed using game theory, such as resource allocation [10, 11], power control [12], network routing [13], network caching [14], security [15], etc.

Many researches have paid great attention to the dynamics of games (see [16–20]). Researchers tried to make the theory more realistic, by exhibiting various bounded rationality behaviors, by combining game theory with dynamic systems, and by introducing a delay in terms of bounded rationality. All these aspects gave birth to the bounded rational dynamic game. Expectations are important factors when formulating such games. There are several expectation techniques that have been studied in the literature for a player to update and adjust his strategies, namely: adaptive expectations, naive expectations, bounded rationality expectations, and best response expectations [21–23]. In [21], the authors investigated the dynamics of the Cournot duopoly game with heterogeneous players, in which case, one player adopts bounded rational expectation, and the other takes naive expectation. In [24], the authors studied the dynamics of the duopoly game model. The authors in [25] studied duopoly game with homogenous players in the electric power industry, where both players adjust their strategies according to bounded rational expectation. The authors in [16] assumed that one player adopts adaptive expectation and the other uses bounded rational expectation. In [21], the authors investigated the dynamics of a Cournot duopoly game in which bounded rational players adopt a gradient adjustment mechanism to update their outputs in each period. The authors in [26] considered bounded rationality in the telecommunication...
networks. The authors in [8, 27, 28] studied competition between service providers where end-users are bounded rational. In [29], the authors investigated a batch matching system with boundedly rational end-users. The authors in [30] developed a duopoly game with heterogeneous players participating in carbon emission trading and investigated the asymptotic stability of the equilibrium points of the game. In [31, 32], the authors investigated the long-term price competition in the multichannel supply chain and showed complex phenomena such as bifurcation and chaos. The authors in [33] developed the chaotic phenomenon in the closed-loop supply chain and performed effective chaotic control for the system. In [34] the authors studied a dynamic Stackelberg game model of the supply chain and analyzed the influences of market parameters on the stability of a dual-channel supply chain. The authors in [35] considered a dynamic epiphytic supply chain game model with two players and showed that the system had two routes to chaos: Neimark-Sacker bifurcation and flip bifurcation. In [36], the authors studied the local stability of the Nash equilibrium under quantity or price competition among firms with differentiated expectations. The dynamics of the game can lead to complex behaviors such as chaos. The authors in [37] showed complex dynamics such as bifurcation and chaos, in a duopoly remanufacturing game with bounded rationality. In [38], the authors considered a duopoly model of technological innovation with bounded rationality and studied the stability of the equilibrium points. They showed, through numerical simulation, a series of chaotic phenomena cause period-doubling bifurcation and strange attractors. The main idea in [39] is synchronization of fractional-order uncertain chaotic systems in the finite time. In [40], the authors design and implement a fast reaching finite-time synchronization technique for chaotic systems along with its application to medical image encryption. For the robust synchronization of uncertain delayed chaotic systems, in [41], the authors propose a disturbance observer-based Sliding Mode Control approach. In [42], the authors investigate a chaotic secure communication system between mobile equipment and a base transmitter station based on hybrid chaotic encryption method and finite-time chaos synchronization.

Social media proliferation rapidly increases Internet traffic. This high increase in demand for content on the Internet and the need for new approaches that control the massive volume of information have motivated the development of a new approach called Information-Centric Network (ICN). This last is a new communication paradigm which aims to increase the efficiency of content delivery and content availability. In this new concept, the network infrastructure actively contributes to content caching and distribution and every ICN node can cache and serve the requested content. The authors in [43] studied a new business model where an ISP charges CP for caching service because CP benefits from more efficient content distribution. A pricing model is proposed in [44] to study the economic incentive for caching and sharing content in ICN, where ICN consists of access ICN, transit ICN, and CP. In [45] the authors considered a noncooperative game between CP and ISP in ICN, where CP and ISP are playing with their pricing strategies and fixing the caching policy. The authors in [46] analyzed the impact of caching cost in joint caching and pricing strategies in ICN with one CP, two access ICNs, and one transit ICN. In [47], the authors studied a noncooperative game between one CP and one ISP in ICN, where ISP cached content. It shows that caching investment is beneficial for ISP and CP. The authors in [48] modeled the caching game between CP and ISP as a cooperative game, where ISP cached a fraction of content.

In the above literature on oligopoly games, most papers focused on games with a bounded rational player. Namely, players adopted different expectations (Naive expectations, adaptive expectations, and bounded rationality) to update their output; another branch of the literature studied competition among ISPs in ICN market where ISPs are rational and adopt best response expectation. However, we can hardly find a few papers that considered bounded rational ISP in an ICN market. On the other hand we followed the methods of [19]. The present paper moves toward this less explored direction.

The main contributions of this work can, therefore, be summarized as follows:

(i) We study the competition between two ISPs' bounded rational in the ICN market.
(ii) We present new features in the mathematical modeling that include the price to access the content in the cache, the number of items in the cache, and the ISPs revenues.
(iii) We model the interactions in prices, quality of service, and caching among ISPs in ICN, where each ISP is bounded rational. Each ISP acts under bounded rationality and adjusts its strategies according to its expected marginal profit.
(iv) Users' behavior is modeled as a function of ISP strategies (i.e., network access price, the price for access to the content in the cache, quality of service (QoS), and the number of items in the cache).
(v) The existence and stability of the equilibrium solution of the game and the bifurcation of the system have been further discussed by the nonlinear dynamics theory.
(vi) We complement our analysis with numerical results that demonstrate that the variation of the system parameters causes a period-doubling bifurcation and chaotic behavior. Then an appropriate controlling method can be used to force the system back to the stability state.

The rest of this paper is as follows: in Section 2, we develop our model, and we provide a detailed description of the duopoly game with bounded rationality model. In Sections 3–5, we have presented two prices and the number of cached items games with bounded rationality and existence and stability of Nash equilibrium are studied. Section 6 presents a numerical study to validate our claims. In Section 7, we apply the control scheme on the proposed system to suppress the chaotic behavior that appeared. Finally, we conclude the paper in Section 8.
2. Model

We consider a hierarchical ICN market model with two ISPs and an arbitrary number of end-users who can switch from one ISP to another. Each ISP decides its network access price \( p_{s_j} \), price to access the content in cache \( p_{c_j} \), quality of service \( q_{s_j} \), and number of cached items \( k_j \) in order to maximize their profits.

2.1. Demand Model. The demand of ISP \( j \) is a linear function with respect to the network access price \( p_{s_j} \), price to access the content in cache \( p_{c_j} \), quality of service \( q_{s_j} \), and number of cached items \( k_j \).

The demand of ISP \( j \) is given by

\[
D_j(p_{s_j}, p_{c_j}, q_{s_j}, k_j) = d_j - \alpha_j p_{s_j} + \beta_j q_{s_j} + \gamma_j p_{c_j} + \sum_{m=1, m \neq j}^M (\alpha_j^m p_{s_m} - \beta_j^m q_{s_m} - \gamma_j^m p_{c_m} + \sigma_j^m k_m + \rho_j^m p_{s_m}).
\]

(1)

The parameter \( d_j \) expresses the total potential demand of end-users. \( \alpha_j^m, \beta_j^m, \gamma_j^m, \) and \( \sigma_j^m \) are positive constants representing, respectively, the responsiveness of the ISP to price \( p_{s_j} \), QoS \( q_{s_j} \), price \( p_{c_j} \), and number of cached items \( k_m \).

2.2. Utility. In the system model, the revenue of an ISP gained by providing services to end-users can be represented by a utility function. Each ISP prefersentially uses the content cached to satisfy end-user requests; when the requested content does not exist in the cache, end-user requests will be forwarded to the content provider. According to those mentioned above, the utility function of an ISP is given as

\[
U_j = \frac{(p_{s_j} - p_t)(N - k_j)}{B_j} + \left(p_{s_j} + p_{c_j} - c_j\right)k_jD_j - v_j(N - k_j)B_j,
\]

(2)

where \( c_j \) is the caching cost. \( p_t \) is the transmission price. The first term \( (p_{s_j} - p_t)(N - k_j) \) is the revenue of network access. The second term \( (p_{s_j} + p_{c_j} - c_j)k_jD_j \) is the revenue of caching. \( v_j \) is the cost of a unit of backhaul bandwidth. \( B_j \) is the backhaul bandwidth needed to serve the demand \( D_j \). The third term \( v_j (N - k_j)B_j \) is the bandwidth cost. The quality of service \( q_{s_j} \) is defined as the expected delay calculated according to the Kleinrock function (see [50, 51]):

\[
q_{s_j} = \frac{1}{\sqrt{\text{Delay}}} = \sqrt{B_j - D_j},
\]

(3)

and it means that

\[
B_j = q_{s_j}^2 + D_j.
\]

(4)

Then, the utility of the ISP \( j \) is given by

\[
U_j = \frac{(p_{s_j} - p_t)(N - k_j)}{B_j} + \left(p_{s_j} + p_{c_j} - c_j\right)k_jD_j - v_j(N - k_j)(q_{s_j}^2 + D_j).
\]

(5)

3. Price \( P \) Game

We suppose both ISPs have bounded rationality and use the marginal profit method to update their strategy in the next period, as assumed in the existing work on the classical Cournot games for output competition [52–54]. It means that each ISP will increase its strategies in a period \( t + 1 \) if the marginal profit in the current period \( t \) is positive; otherwise, the firm will decrease its investment. Then the investment adjustment mechanism of a player can be modeled as

\[
\begin{align*}
\Delta p_{s_j}(t + 1) &= p_{s_j}(t) + \theta_1 p_{s_j}(t) \frac{\partial U_1(p_{s_j}(t), p_{s_j}(t + 1))}{\partial p_{s_j}}, \\
\Delta p_{c_j}(t + 1) &= p_{c_j}(t) + \theta_2 p_{c_j}(t) \frac{\partial U_2(p_{c_j}(t + 1), p_{s_j}(t))}{\partial p_{c_j}}.
\end{align*}
\]

(6)

As it was considered by Gibbons in [55], it makes sense to assume that, in the bounded rationality term, the expected price \( p_s^e(t + 1) \) decided by ISP \( j \) is equal to its previous value \( p_s(t) \). However, to anticipate \( p_s^e(t + 1) \), it may make more sense to assume that each ISP has a memory for storing prices decided in previous slot time before time \( t \). To anticipate the expected price of his competitor at \( (t + 1) \), ISP \( j \) uses previous prices of ISP \( m \), i.e., \( p_{s_m}(t - 1), p_{s_m}(t - 2), \ldots, p_{s_m}(t - T), \) \( j \neq m \) with different weights; this point of view has been studied [56, 57] in different contexts. Generally, the expected price of ISP \( m \) becomes

\[
p_s^e_m(t + 1) = \sum_{l=0}^T \omega_l p_{s_m}(T - l), \quad \omega_l \geq 0 \quad \text{and} \quad \sum_{l=0}^T \omega_l = 1,
\]

(7)

where \( \omega_l \) is the weight given to previous prices, and \( T \) represents the size of memory, so (6) becomes

\[
\Delta p_{s_j}(t + 1) = p_{s_j}(t) + \theta_1 p_{s_j}(t) \frac{\partial U_1(p_{s_j}(t), p_{s_j}(t + 1))}{\partial p_{s_j}},
\]

\[
\Delta p_{c_j}(t + 1) = p_{c_j}(t) + \theta_2 p_{c_j}(t) \frac{\partial U_2(p_{c_j}(t + 1), p_{s_j}(t))}{\partial p_{c_j}}.
\]
as a fourth-dimensional system in the form

\begin{align*}
 p_{s_1} (t + 1) &= p_{s_1} (t) + \theta_1 p_{s_1} (t) \left( l_1 + p_{s_1} (t) \alpha_1^1 (-2N + k_1 - 1) + \alpha_1^2 (N - k_1 + 1) \sum_{f=0}^{T} \omega_2 p_{s_2} (t - f) \right), \\
 p_{s_2} (t + 1) &= p_{s_2} (t) + \theta_2 p_{s_2} (t) \left( l_2 + p_{s_2} (t) \alpha_2^1 (-2N + k_2 - 1) + \alpha_2^2 (N - k_2 + 1) \sum_{f=0}^{T} \omega_1 p_{s_1} (t - f) \right),
\end{align*}

(7)

where \( l_1 = N d_1 + \beta_1^1 N q_{s_1} + \sigma_1^1 N k_1 - \gamma_1^1 N p_{c_1} - \beta_1^2 N q_{s_2} - \sigma_2^1 N k_2 - N p_{c_1} + \alpha_1^1 + N v_1 \alpha_1^1 + k_1 \alpha_1^1 (p_{s_1} - v_1 - p_{c_1} + c_1) \), \( l_2 = N d_2 + \beta_2^1 N q_{s_2} + \sigma_2^1 N k_2 - \gamma_2^1 N p_{c_2} - \beta_2^2 N q_{s_1} - \sigma_2^1 N k_1 - N p_{c_2} + \alpha_2^1 + N v_2 \alpha_2^1 + k_2 \alpha_2^1 (p_{s_2} - v_2 - p_{c_2} + c_2) \).

For simplicity, we set \((T = 1); in this case, the previous dynamic model, with one step \((T = 1)\), is given by

\begin{align*}
 p_{s_1} (t + 1) &= p_{s_1} (t) + \theta_1 p_{s_1} (t) \left( l_1 + p_{s_1} (t) \alpha_1^1 (-2N + k_1 - 1) + \alpha_1^2 (N - k_1 + 1) (\omega_2 p_{s_2} (t) + (1 - \omega_2) p_{s_2} (t - 1)) \right), \\
 p_{s_2} (t + 1) &= p_{s_2} (t) + \theta_2 p_{s_2} (t) \left( l_2 + p_{s_2} (t) \alpha_2^1 (-2N + k_2 - 1) + \alpha_2^2 (N - k_2 + 1) (\omega_1 p_{s_1} (t) + (1 - \omega_1) p_{s_1} (t - 1)) \right).
\end{align*}

(8)

To study the stability of dynamic system (8), we rewrite it as a fourth-dimensional system in the form

\begin{align*}
 x_1 (t + 1) &= p_{s_1} (t), \\
 x_2 (t + 1) &= p_{s_2} (t), \\
 p_{s_1} (t + 1) &= p_{s_1} (t) + \theta_1 p_{s_1} (t) \left( l_1 + p_{s_1} (t) \alpha_1^1 (-2N + k_1 - 1) + \alpha_1^2 (N - k_1 + 1) (\omega_2 p_{s_2} (t) + (1 - \omega_2) x_2 (t)) \right), \\
 p_{s_2} (t + 1) &= p_{s_2} (t) + \theta_2 p_{s_2} (t) \left( l_2 + p_{s_2} (t) \alpha_2^1 (-2N + k_2 - 1) + \alpha_2^2 (N - k_2 + 1) (\omega_1 p_{s_1} (t) + (1 - \omega_1) x_1 (t)) \right).
\end{align*}

(9)

Equilibrium points of (9) can be calculated by setting \( p_{s_i} (t + 1) = p_{s_i} (t), \) \( i = 1, 2, \) in (9). This gives the following algebraic two equations:

\begin{align*}
 \theta_1 p_{s_1} (t) \left( l_1 + p_{s_1} (t) \alpha_1^1 (-2N + k_1 - 1) + \alpha_1^2 (N - k_1 + 1) p_{s_2} (t) \right) &= 0, \\
 \theta_2 p_{s_2} (t) \left( l_2 + p_{s_2} (t) \alpha_2^1 (-2N + k_2 - 1) + \alpha_2^2 (N - k_2 + 1) p_{s_1} (t) \right) &= 0.
\end{align*}

(10)

Solving equations in (10), we obtain four equilibrium states of dynamics (9), which are listed as follows:
Theorem 1. Steady states $E_1$, $E_2$, and $E_3$ are unstable equilibrium points of system (9).

To analyze the local stability properties of the equilibrium points of system (9), the eigenvalues of the Jacobian matrix need to be calculated. The equilibrium point of a nonlinear system is stable if and only if the modules of all the eigenvalues of the Jacobian matrix evaluated at the equilibrium points are less than one.

The Jacobian matrix of system (9) corresponding to the state variables $(r_1, r_2, p_s, p_a)$ is calculated as follows:

$$
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
p_1(1 - \omega_a) \theta_4 p_n & \theta_1 l_4 + 2 \theta_4 p_n p_4 + \theta_2 \omega_a \theta_4 p_4 p_4 + \theta_4 (1 - \omega_a) x_2 p_4 + 1 & \omega_2 \omega_4 \theta_4 p_n & \theta_4 l_4 + 2 \theta_2 \omega_a \theta_4 p_4 + \theta_2 \omega_a \theta_4 p_4 + \theta_4 (1 - \omega_a) x_2 p_4 + 1
\end{pmatrix}
$$

where $p_1 = a_1^2(N - k_1 + 1)$, $p_2 = a_1^2(-2N + k_1 - 1)$, $p_3 = a_2^2(N - k_2 + 1)$, and $p_4 = a_2^2(-2N + k_2 - 1)$.

The proof of the above theorem can be found in Appendix A.

To study the local stability of the Nash equilibrium $E_4$ of (9), we estimate the Jacobian matrix $J$ at $E_4$, which is
The characteristic polynomial of $J(E_j)$ can be described as

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + b_1\lambda + c_1\lambda + d_1,$$  

(14)

where

\(a_1 = 2 + \theta_1l_1 + \theta_2l_2 + (2\theta_1p_2 + \theta_2p_2)\theta_3s_1 + (p_1\theta_1 + 2\theta_2p_3)\)

\(b_1 = 1 + \theta_1l_1 + \theta_1l_1l_2 + (\theta_1\theta_1l_2 + \theta_2p_1p_2 + 2\theta_1p_2 + \theta_1\theta_2p_2p_3)\theta_3s_1 + (\theta_1\theta_1l_2 + \theta_2p_1p_2 + 2\theta_1p_2 + \theta_1\theta_2p_2p_3 + \theta_1\theta_2)\theta_3s_1 + \omega_1p_3\theta_2s_3 + \omega_2p_1\theta_2s_3 + \omega_3p_3\theta_2s_3 + \omega_4p_3\theta_3s_3 + \omega_5p_3\theta_3s_3.

\(c_1 = \theta_1\theta_2p_1p_2(\omega_1 + \omega_2 - \omega_3),\)

\(d_1 = -p_1p_2p_3\theta_1(\omega_1 - \omega_2).\)

Following a standard stability analysis, a sufficient and necessary condition for the local stability of Nash equilibrium $E_j$ is that the eigenvalues of the Jacobian matrix $J(E_j)$ are inside the unit circle of the complex plane; this is true if and only if the following Jury’s conditions [58] hold.

1. \(1 + a_1 + b_1 + c_1 + d_1 > 0\)
2. \(1 - a_1 + b_1 - c_1 + d_1 > 0\)
3. \((1 - d_1)(1 - (d_1)^2) - b_1(1 - d_1)^2 + (a_1 - c_1)(c_1 - a_1d_1) > 0\)
4. \(3 + 3d_1 > b_1\)
5. \(|d_1| < 1\)

### 4. Price $P_c$ Game

According to the above section, a bounded rational player modifies his price according to his marginal profit: \(\partial U_i(p_c)/\partial p_c, \ i \in \{1, 2\}\). So, the dynamical system for decision of ISPs has this form:

\[
\begin{align*}
    p_{ci}(t + 1) &= p_{ci}(t) + \theta_1 p_{ci}(t) \left( i_2 - 2y^1 p_{ci}(t) + y^2 k_2 \omega_{ci} + (1 - \omega_{ci}) p_{ci}(t - 1) \right), \\
    p_{cj}(t + 1) &= p_{cj}(t) + \theta_2 p_{cj}(t) \left( i_4 - 2y^2 p_{cj}(t) + y^1 k_2 \omega_{cj} + (1 - \omega_{cj}) p_{cj}(t - 1) \right),
\end{align*}
\]

(15)

where $\theta_1$ represents the speed of adjustment of price of ISP $i$, and $p_{ci}^e(t + 1)$ is the expected price of the opponent in the next time step $(t + 1)$.

As explained above, and if we consider that $T = 1$, the expected price $p_{ci}^e(t + 1)$ is given by

$$p_{ci}^e(t + 1) = \omega_i p_{ci}(t) + (1 - \omega_i) p_{ci}(t - 1),$$

(16)

where the positive constants $\omega_i$ are the weights given to the previous prices.

Thus, the dynamical model (13) becomes

\[
\begin{align*}
    p_{ci}(t + 1) &= p_{ci}(t) + \theta_1 p_{ci}(t) \left( i_2 - 2y^1 p_{ci}(t) + y^2 k_2 \omega_{ci} + (1 - \omega_{ci}) p_{ci}(t - 1) \right), \\
    p_{cj}(t + 1) &= p_{cj}(t) + \theta_2 p_{cj}(t) \left( i_4 - 2y^2 p_{cj}(t) + y^1 k_2 \omega_{cj} + (1 - \omega_{cj}) p_{cj}(t - 1) \right),
\end{align*}
\]

(17)

where $i_4 = -\left( (p_{ci} - p_{cj}) y^1(N - k_1) - k_1 y^1 p_{ci} - c_1 \right) + v_1 y^1(N - k_2) + v_2 y^2(N - k_2) + \beta_1 y^1 p_{ci} + \alpha_2 p_{ci} + \beta_2 y^2 p_{ci} + \omega_{ci} p_{cj} + \omega_{ci} y^2 p_{cj} + \beta_1 y^1 p_{cj} + \alpha_2 p_{cj} + \beta_2 y^2 p_{cj}.$

To study the dynamic system (17), we write it as a fourth-dimensional system in the form:

\[
\begin{align*}
    y_1(t + 1) &= p_{ci}(t), \\
    y_2(t + 1) &= p_{cj}(t), \\
    p_{ci}(t + 1) &= p_{ci}(t) + \theta_1 p_{ci}(t) \left( i_2 - 2y^1 p_{ci}(t) + y^2 k_2 \omega_{ci} + (1 - \omega_{ci}) y^1(t) \right), \\
    p_{cj}(t + 1) &= p_{cj}(t) + \theta_2 p_{cj}(t) \left( i_4 - 2y^2 p_{cj}(t) + y^1 k_2 \omega_{cj} + (1 - \omega_{cj}) y^2(t) \right).
\end{align*}
\]

(18)

The steady states of the dynamic system (18) are
\[ E_5 = (0, 0, 0, 0), \]
\[ E_6 = \left( \frac{l_1}{2y_1^2 k_1}, 0, \frac{l_3}{2y_1^2 k_1}, 0 \right), \]
\[ E_7 = \left( 0, \frac{l_4}{2y_2^2 k_2}, 0, \frac{l_5}{2y_2^2 k_2} \right). \]
\[ E_8 = \left( \begin{array}{cccc}
    k_1 l_1 y_1^2 + 2k_1 l_2 y_2^2 & k_1 l_1 y_1^2 + 2k_1 l_2 y_2^2 \\
    k_2 k_2 (4y_1^2 y_2^2 - y_1^2 y_1^2) & k_2 k_2 (4y_1^2 y_2^2 - y_1^2 y_1^2) \\
    k_1 l_1 y_1^2 + 2k_1 l_2 y_2^2 & k_1 l_1 y_1^2 + 2k_1 l_2 y_2^2 \\
    k_2 k_2 (4y_1^2 y_2^2 - y_1^2 y_1^2) & k_2 k_2 (4y_1^2 y_2^2 - y_1^2 y_1^2)
\end{array} \right). \]

Obviously, \( E_5, E_6, \) and \( E_7 \) are boundary equilibrium points and \( E_8 \) is the unique Nash equilibrium point. The study of local stability of a fixed point in the four-dimensional system (18) depends on the eigenvalues of its Jacobian matrix. The Jacobian matrix \( J(y_1, y_2, p_c, p_c) \) at any point \( (y_1, y_2, p_c, p_c) \) in the dynamic system (18) takes the form

\[
J(y_1, y_2, p_c, p_c) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\theta_1 p_c y_1^2 k_1 (1 - \omega_2) 1 + \theta_1 l_3 - 4y_1^2 \theta_1 k_1 p_c + \theta_1 y_1^2 k_1 \omega_2 p_c + (1 - \omega_2) y_2 & \theta_1 y_1^2 k_1 \omega_2 p_c \\
0 & \omega_1 y_2 k_2 (1 - \omega_1) \theta_2 p_c \\
\end{pmatrix}.
\]

**Theorem 2.** The trivial equilibrium points \( E_5, E_6, \) and \( E_7 \) of system (18) are unstable.

And by direct calculation, we obtain the characteristic polynomial \( P(\lambda) \) for \( J(E_8) \) as follows:

\[ P(\lambda) = \lambda^4 + a_2 \lambda^3 + b_2 \lambda^2 + c_2 \lambda + d_2, \]

where \( (1) \quad a_2 = 2v_1 l_1 + v_2 l_4 - (4y_1^2 v_1 k_1 + v_2 k_2 y_2^2) p_c + (-4v_2 y_2^2 k_2 + v_1 y_1^2 k_1) p_c \)
\[ (2) \quad b_2 = 1 + v_2 l_4 + v_1 l_3 + v_1 v_2 l_3 l_4 + (v_2 k_2 y_2^2 + v_1 v_2 l_3 k_2 y_2^2 - 4v_1 v_2 y_2^2 l_4 + v_1 v_2 l_3 k_3 y_3^2 - 4v_1 v_2 l_4 k_3 y_3^2 + v_1 v_2 l_5 k_5 y_5^2) p_c + (-4y_2^2 v_2 k_2 - 4v_2 l_4 k_2 y_2^2 + v_1 v_2 y_2^2 k_1 + v_1 v_2 l_4 k_1 y_1^2) p_c \)
\[ (3) \quad c_2 = v_1 v_2 y_2^2 k_2 k_1 (\omega_1 + \omega_2 - 2(\omega_1 \omega_2) p_c, p_c \)
\[ (4) \quad d_2 = y_1^2 y_2^2 k_2 v_1 v_2 (1 - \omega_1 - \omega_2 + \omega_1 \omega_2) \]

The system (18) will be stable only if the regions match Jury conditions [58], that is,

\[ (1) \quad 1 + a_2 + b_2 + c_2 + d_2 > 0 \]
\[ (2) \quad 1 - a_2 + b_2 - c_2 + d_2 > 0 \]
\[ (3) \quad (1 - d_2)(1 - (d_2)^4) - b_2 (1 - d_2)^2 + (a_2 - c_2)(c_2 - a_2 d_2) > 0 \]
\[ (4) \quad 3 + 3d_2 > b_2 \]
\[ (5) \quad |d_2| < 1 \]

### 5. Cache Game

According to the above section, the adjustment mechanism of number of cached items over time of the \( j \)th ISPs is described by
where \( \theta_j, j = 1, 2, \) are positive constants which stand for the speed of adjusting to market variations.

As explained above, and if we consider that \( T = 1 \), the expected price \( k_j(t+1) \) is given by

\[
k_j(t+1) = \omega_j k_j(t) + (1 - \omega_j) k_j(t-1).
\] (24)

\[
k_j(t+1) = \frac{\partial U_j(k_j(t), k_m(t+1))}{\partial k_j}, \quad j \neq m \in \{1, 2\},
\] (23)

The positive constants \( \omega_j \) are the weights given to the previous number of cached items.

Thus, the dynamical model with bounded rationality ISPs is defined as

\[
\begin{align*}
k_1(t+1) &= k_1(t) + \eta_1 k_1(t) (l_5 + 2e_1 \sigma_1^2 k_1(t) - \epsilon_1 \sigma_1^2 \sum_{j=0}^{T} w_2 k_2(t-f), \\
k_2(t+1) &= k_2(t) + \eta_2 k_2(t) (l_6 + 2e_2 \sigma_2^2 k_2(t) - \epsilon_2 \sigma_2^2 \sum_{j=0}^{T} w_1 k_1(t-f),
\end{align*}
\] (25)

where \( e_1 = p_t + p_q - c_t + v_1 \), \( e_2 = p_t + p_c - c_t + v_2 \), \( l_5 = (p_t + p_q - c_t + v_1)(d_1 - \alpha_1 p_t + \beta_1 q_t - \gamma_1 p_t + \alpha_2 p_c - \beta_2 q_c + \gamma_2 p_c) + (p_t - p_c) N \sigma_1^2 + (p_t - c_t) \sigma_1^2 + v_1 q_c - v_1 N \sigma_1^2 \) and \( l_6 = (p_t + p_c - c_t + v_2)(d_2 - \alpha_2 p_c + \beta_2 q_c - \gamma_2 p_c) + v_2 q_c - v_2 N \sigma_2^2 \).

As time evolves, \( k_j(t+1) \) are approximately equal to \( k_j(t) \) when the market structure is stable enough at time \( t \). Set \( k_1(t+1) = k_1(t) \) and \( k_2(t+1) = k_2(t) \); the dynamic system (26) turns to be

\[
\begin{align*}
z_1(t+1) &= k_1(t), \\
z_2(t+1) &= k_2(t), \\
k_1(t+1) &= k_1(t) + \eta_1 k_1(t) (l_5 + 2e_1 \sigma_1^2 k_1(t) - \epsilon_1 \sigma_1^2 (\omega_j k_j(t) + (1 - \omega_j) z_2(t))), \\
k_2(t+1) &= k_2(t) + \eta_2 k_2(t) (l_6 + 2e_2 \sigma_2^2 k_2(t) - \epsilon_2 \sigma_2^2 (\omega_j k_j(t) + (1 - \omega_j) z_1(t)));
\end{align*}
\] (26)

\[
\begin{align*}
\eta_1 k_1(t) (l_5 + 2e_1 \sigma_1^2 k_1(t) - \epsilon_1 \sigma_1^2 k_2(t)) &= 0, \\
\eta_2 k_2(t) (l_6 + 2e_2 \sigma_2^2 k_2(t) - \epsilon_2 \sigma_2^2 k_1(t)) &= 0.
\end{align*}
\] (27)

Solving equations in (27), we obtain four equilibrium states of dynamic system (26), which are listed as follows:

\[
\begin{align*}
E_9 &= (0, 0, 0, 0), \\
E_{10} &= \left( \frac{l_5}{2 \sigma_1^2 e_1}, 0, \frac{l_3}{2 \sigma_1^2 e_1}, 0 \right), \\
E_{11} &= \left( 0, \frac{l_6}{2 \sigma_2^2 e_2}, 0, \frac{l_4}{2 \sigma_2^2 e_2} \right), \\
E_{12} &= \left( \frac{-e_1 l_6 \sigma_1^2 + 2e_2 l_5 \sigma_2^2}{e_1 e_3 (4 \sigma_1^2 \sigma_2^2 - \sigma_1^4 \sigma_1^2)} \right), \\
E_{13} &= \left( \frac{e_1 l_6 \sigma_1^2 + 2e_2 l_5 \sigma_2^2}{e_1 e_3 (4 \sigma_1^2 \sigma_2^2 - \sigma_1^4 \sigma_1^2)} \right).
\end{align*}
\] (28)

The research of local stability at the points concluded above replies on the Jacobian matrix of the dynamic system (26), which takes the unified form.
Theorem 3. The trivial equilibrium points $E_0$, $E_{10}$, and $E_{11}$ of system (26) are unstable.

The characteristic polynomial of $J(E_{12})$ is

$$P(\lambda) = \lambda^3 + a_3 \lambda^2 + b_2 \lambda + c_1 \lambda + d_3,$$

where

(i) $a_3 = 2 + \eta_1 l_5 + \eta_2 l_6 + (4\eta_1^2 e_1 - \eta_2 e_3^2) k_1 + (\eta_1^2 e_1^2 + 4\eta_2 e_3^2) k_2$

(ii) $b_2 = 1 + \eta_1 l_5 + \eta_1 l_6 + \eta_1 l_5 + (-\eta_2 e_3^2 - \eta_1 l_6) e_2 \sigma_2^1 + 4\eta_1^2 e_1 e_3 e_4 + 4\eta_1 e_1 e_1 k_1 + (4\eta_2^2 e_2 + 4\eta_2 e_1 k_2 + \eta_2 l_5 \sigma_2^2 e_2 - \eta_1 l_6 \sigma_2^2 e_1 - \eta_2 \sigma_2^1 e_1 k_2 - 4\eta_1^2 \eta_1 l_2 e_3 e_2 k_1^2 + (\eta_2 l_5 e_2 + 16\eta_1^2 \sigma_2^1 e_1 e_1 e_1 + 16\eta_1^2 \sigma_2^1 e_1 e_1 e_1 + 16\eta_2^2 l_2 e_2 + 4\eta_2 \omega_1 \omega_2 e_1 e_1 e_1)$

(iii) $c_1 = \eta_1 l_2 e_3^2 \sigma_1^1 e_2 e_2 (\omega_1 + \omega_2 - 2\omega_1 \omega_2)$

(iv) $d_3 = -\eta_1 l_2 e_1 e_3^2 \sigma_1^2 e_1 e_2 (1 - \omega_1 - \omega_2 + \omega_1 \omega_2)$

The local stability of Nash equilibrium is given by using Jury’s conditions [58] which are

1. $1 + a_3 + b_2 + c_1 + d_3 > 0$
2. $1 - a_3 + b_2 - c_1 + d_3 > 0$
3. $(d_3 - (d_3 - a_3 - c_1)) (d_3 - (d_3 - a_3 - c_1)) > 0$
4. $|d_3|^2 > 1$
5. $|d_3|^3 < 1$

6. Numerical Simulation

In this section, we will provide the numerical evidence for the complex dynamical behaviors of systems (9), (18), and (26) when losing stability and exhibit how the system evolves when the model parameters take different levels of values.

In Figure 1(a), the bifurcation scenario occurs; if $\theta_1$ is small, then there exists a stable equilibrium point. As one can see, the Nash equilibrium point $E_4$ is locally stable for small values of $\theta_1$. As $\theta_1$ increases, the Nash equilibrium becomes unstable, the system appears for a two-period bifurcation, and then the system goes into two cycles, after four times for each cycle and eight period-doubling bifurcations; then the final system enters into a state of chaos.

Figure 2(a) presents the bifurcation diagrams of price $p_c$ with respect to the adjustment speed $\theta_1$. From Figure 2(a) we see that a low adjustment speed $(\theta_1 < 0.04)$ can make the system stable. With $\theta_1$ increasing, a flip bifurcation for system (18) takes place at $\theta_1 = 0.04$ and 2-period bifurcate as $\theta_1 > 0.04$. As long as the parameter $\theta_1$ increases, the Nash equilibrium point $E_4$ becomes unstable and the bifurcation scenario occurs and ultimately leads to unpredictable (chaotic) motions that are observed.

Figure 3(a) presents the bifurcation diagram of the number of cached items $k_1$ with change of $\eta_1$. It is clear that the Nash equilibrium point $E_{12}$ of system (26) is stable when $\eta_1 < 0.025$. As $\eta_1$ increases the Nash equilibrium of point $E_{12}$ becomes unstable and complex dynamic behavior occurs, including two-period bifurcation and chaos. From the perspective of economics, the ISPs adjustment speed should be in a certain range; otherwise, the system moves to the period-doubling bifurcation point and then four times bifurcation point until chaos while growing, which means irregularity, unpredictability, sensitivity to initial values, and harm to the economy.

Figure 4(a) shows the bifurcation diagrams of price $p_c$ with respect to the number of cached items $k_1$. With the rise of the number of cached items $k_1$, the Nash equilibrium point $E_4$ increases gradually and is locally stable for small values of the parameter $k_1$. As $k_1$ increases, the Nash equilibrium point $E_4$ becomes unstable, and complex dynamic behavior occurs such as chaos.
Figure 5(a) shows the bifurcation diagram of the number of cached items $k_1$ with respect to the caching cost $c$. From Figure 5(a), we can see that the Nash equilibrium $E_{12}$ is locally stable for small values of $c$. As $c$ increases, the Nash equilibrium point $E_{12}$ becomes unstable and complex dynamic behavior occurs, including chaos. Economic meanings of different states of this system are as follows: stabilization means that the market is regular and fixed; every ISP can get fixed profit with charging fixed prices and offering fixed number of cached items in every time period in this situation. Stable state is usually good for ISPs to make long-term strategies and can help them to get away from the trouble of changing strategies frequently. 2-period is regular but fluctuant; ISPs can know the development of the market but have to change their strategies frequently in this situation. Chaos means that the market is irregular and fluctuant; it is difficult for ISPs to make long-term strategies in this situation as the system depends on initial values sensitivity.

7. Chaos Control

From the above section, we can see that the dynamical behavior of systems (9), (18), and (26) may be chaotic. The chaotic motion is always irregular and unpredictable. In practical application, the chaotic motion is always not desired, so we need to avoid the occurrence of chaos. Therefore, all ISPs in the ICN market always hope that the market can operate stably. Over the past decade, a good deal of methods for controlling chaos has been put forward. Here, we employ the widely used control method, that is, time-delayed feedback control method to control the chaotic phenomenon (e.g., [54, 59]).

Using the time-delayed feedback control method, we adjust system (9) by inserting the control action
\[ \kappa_j(p_{s_j}(t) - p_{s_j}(t + 1)) \] in the right hands of the last two equations, where \( p_{s_j}(t) \) is the time-delayed state variable and \( \kappa_j > 0 \) is the controlling coefficient which can express the control over the price adjustment speed or the learning adaptability of market competitors. Then we get the controlled system as follows:

\[
\begin{align*}
    x_1(t + 1) &= p_{s_1}(t), \\
    x_2(t + 1) &= p_{s_2}(t), \\
    p_{s_1}(t + 1) &= p_{s_1}(t) + \theta_1 p_{s_1}(t) (l_1 + p_{s_1}(t)) \alpha_1^1 (-2N + k_1 - 1) + \alpha_2^1 (N - k_1 + 1) \left[ (\alpha_2^1 p_{s_1}(t) + (1 - \omega_1) x_2(t)) + \kappa_1 (p_{s_1}(t) - p_{s_1}(t + 1)) \right], \\
    p_{s_2}(t + 1) &= p_{s_2}(t) + \theta_2 p_{s_2}(t) (l_2 + p_{s_2}(t)) \alpha_2^2 (-2N + k_2 - 1) + \alpha_2^2 (N - k_2 + 1) \left[ (\omega_2 p_{s_1}(t) + (1 - \omega_2) x_1(t)) + \kappa_2 (p_{s_2}(t) - p_{s_2}(t + 1)) \right].
\end{align*}
\]

Figure 2: (a) Bifurcation diagrams of \( p_{s_1} \) with respect to \( \theta_1 \); (b) TE corresponding to the largest Lyapunov exponent for \( p_{s_1} \); (c) convergence speed.
It can be rewritten as

\[
\begin{aligned}
    x_1(t+1) &= p_{s_1}(t), \\
    x_2(t+1) &= p_{s_2}(t), \\
    p_{s_1}(t+1) &= p_{s_1}(t) + \frac{\theta_1 p_{s_1}(t)}{\kappa_1 + 1} (l_1 + p_{s_1}(t) \alpha_1^1 (-2N + k_1 - 1) + \alpha_1^2 (N - k_1 + 1) (\omega_2 p_{s_2}(t) + (1 - \omega_2) x_2(t))), \\
    p_{s_2}(t+1) &= p_{s_2}(t) + \frac{\theta_2 p_{s_2}(t)}{\kappa_2 + 1} (l_2 + p_{s_2}(t) \alpha_2^2 (-2N + k_2 - 1) + \alpha_2^1 (N - k_2 + 1) (\omega_1 p_{s_1}(t) + (1 - \omega_1) x_1(t))).
\end{aligned}
\]  

(33)

When \( \kappa_i = 0 \), the controlled system degenerates into the initial system (9).

Figure 3: (a) Bifurcation diagrams of \( k_1 \) with respect to \( \eta_1 \); (b) Te corresponding to the largest Lyapunov exponent for \( k_1 \); (c) convergence speed.

Adding the control action \( \xi_i (p_{s_i}(t) - p_{s_i}(t+1)) \) to system (18) and simplifying them, we obtained the following form of the controlled dynamical system:
Adding the control action $\zeta_j(k_j(t) - k_j(t+1))$ to system (26) and simplifying them, we obtained the following form of the controlled dynamical system:

\[
\begin{align*}
  y_1(t+1) &= p_{c_1}(t), \\
  y_2(t+1) &= p_{c_2}(t), \\
  p_{c_1}(t+1) &= p_{c_1}(t) + \frac{\delta_1 p_{c_1}(t)}{\xi_1 + 1} \left( l_3 - 2\gamma_1^1 k_1 p_{c_1}(t) + \gamma_1^2 k_1 \left( (\omega_2 p_{c_2}(t) + (1 - \omega_2) y_2(t)) \right) \right), \\
  p_{c_2}(t+1) &= p_{c_2}(t) + \frac{\delta_2 p_{c_2}(t)}{\xi_2 + 1} \left( l_4 - 2\gamma_2^2 k_2 p_{c_2}(t) + \gamma_2^3 k_2 \left( \omega_1 p_{c_1}(t) + (1 - \omega_1) y_1(t) \right) \right).
\end{align*}
\]
\[
\begin{align*}
    z_1(t+1) &= k_1(t), \\
    z_2(t+1) &= k_2(t), \\
    k_1(t+1) &= k_1(t) + \frac{\eta_1 k_1(t)}{c_1 + 1} (l_5 + 2\epsilon_1 \sigma_1^1 k_1(t) - \epsilon_1 \sigma_1^2 (\omega_1 k_2(t) + (1 - \omega_2) z_2(t)), \\
    k_2(t+1) &= k_2(t) + \frac{\eta_2 k_2(t)}{c_2 + 1} (l_6 + 2\epsilon_2 \sigma_2^2 k_2(t) - \epsilon_2 \sigma_2^1 (\omega_1 k_1(t) + (1 - \omega_1) z_1(t)). \\
\end{align*}
\]

Figure 6 shows that, with the control coefficient $\kappa_1$ increasing, the controlled systems (33) are gradually controlled from the chaotic state, 4-period bifurcation, and 2-period bifurcation to a stable state. When $\kappa_1 > 0.024$, the controlled system (33) stabilizes at the Nash equilibrium point.

Figure 7 shows the bifurcation diagram of controlled system (34) with respect to control parameter $\xi_1$. We can see
that, with an increase in the control parameter, the system is controlled at 8-cycle, 4-cycle, 2-cycle, and the Nash equilibrium point. One can conclude that the controlled system (34) is being gradually controlled with an increase in the control parameter $\xi_1$, and the system will be led to stability when $\xi_1$ is large enough.

Figure 8 shows that the period-doubling bifurcation disappears gradually and the controlled system (35) gets rid of chaos to be stable when the controlling parameter $\varsigma_1$ is appropriately significant. So, the above control method can control chaos in the game, and the market game can switch from chaotic trajectories to regular periodic orbits or equilibrium state.

8. Conclusion

In this paper, we established the duopoly game model regarding the prices and the number of cached items competition in an open market, based on the theory of a particular economic model. The dynamic of the duopoly game model with bounded rationality has been analyzed. The equilibrium points have been obtained as functions of the system parameters. A stability analysis of those points has been investigated. Numerical simulations show that while varying the model parameters, complex dynamic behaviors would occur, such as chaos. In addition, the control scheme has been applied to force the system back to its stable state.

As part of future work, we plan to add content popularity to our analysis of duopoly game in the ICN market with ISPs and CPs.

Appendix

A. Proof of Theorem 1

Proof. At the equilibrium point $E_1$ the Jacobian matrix (12) becomes

$$
J(E_1) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & \theta_1 l_1 + 1 & 0 \\
0 & 0 & 0 & \theta_2 l_2 + 1
\end{pmatrix}.
$$

(A.1)

The eigenvalues of $J(E_1)$ are $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = \theta_1 l_1 + 1 < -1$, and $\lambda_4 = \theta_2 l_2 + 1 < -1$. Thus, the equilibrium point $E_1$ is unstable.

At the equilibrium point $E_2$, the Jacobian matrix is
with eigenvalues $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = (\theta_1 I_1 + 2\theta_1 p_1 p_2 + 1) > 1$, and $\lambda_4 = (\theta_2 I_2 + 3\theta_2 p_2 p_3 + 1) > 1$, meaning that the equilibrium point $E_2$ is unstable. Similarly, we can prove that $E_3$ is also an unstable point.

**B. Proof of Theorem 2**

**Proof.** The Jacobian matrix (20) at the equilibrium point $E_3$ takes the form

$$J(E_3) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & p_1(1 - \omega_2)\theta_1 p_1 & \theta_2 I_1 + 2\theta_1 p_1 p_2 + 1 & \omega_2 p_1 \theta_1 p_1 \\ 0 & 0 & 0 & \theta_2 I_2 + 3\theta_2 p_2 p_3 + 1 \end{pmatrix}, \quad (A.2)$$

The eigenvalues of $J(E_3)$ are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1 + \theta_1 I_1 < 1$, and $\lambda_4 = \theta_2 I_2 + 3\theta_2 p_2 p_3 + 1 < 1$. Thus, the equilibrium point $E_3$ is also an unstable point.

At the equilibrium point $E_6$, the Jacobian matrix is

$$J(E_6) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & p_1(1 - \omega_2)\theta_1 p_1 & \theta_1 I_1 - 4\theta_1 p_1 & \omega_1 \theta_1 p_1 \\ 0 & 0 & 0 & \theta_2 I_2 + 3\theta_2 p_2 p_3 + 1 \end{pmatrix}, \quad (A.3)$$

The eigenvalues of $J(E_6)$ are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1 + \theta_1 I_1 < 1$, and $\lambda_4 = \theta_2 I_2 + 3\theta_2 p_2 p_3 + 1 < 1$. Thus, the equilibrium point $E_6$ is also an unstable point.

**C. Proof of Theorem 3**

**Proof.** The Jacobian matrix (29) at the equilibrium point $E_9$ takes the form

$$J(E_9) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & p_1(1 - \omega_2)\theta_1 p_1 & \theta_1 I_1 - 4\theta_1 p_1 & \omega_1 \theta_1 p_1 \\ 0 & 0 & 0 & \theta_2 I_2 + 3\theta_2 p_2 p_3 + 1 \end{pmatrix}, \quad (A.4)$$

The eigenvalues of $J(E_9)$ are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1 + \eta_1 I_5 > 1$, and $\lambda_4 = \eta_2 I_6 + 1 > 1$. Thus, the equilibrium point $E_9$ is unstable.

At the equilibrium point $E_{10}$, the Jacobian matrix is

$$J(E_{10}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & p_1(1 - \omega_2)\theta_1 p_1 & \theta_1 I_1 - 4\theta_1 p_1 & \omega_1 \theta_1 p_1 \\ 0 & 0 & 0 & \theta_2 I_2 + 3\theta_2 p_2 p_3 + 1 \end{pmatrix}, \quad (A.5)$$

The eigenvalues of $J(E_{10})$ are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1 + \eta_1 I_5 > 1$, and $\lambda_4 = \eta_2 I_6 + 1 > 1$. Thus, the equilibrium point $E_{10}$ is also an unstable point.

**Data Availability**

Our study is based on a mathematical model that was valuated by numerical results, and it was compared to other studies in the literature (cited in bibliography).

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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