An Extension of the Character Ring of sl(3) and Its Quantisation

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Abstract

We construct a commutative ring with identity which extends the ring of characters of finite dimensional representations of sl(3). It is generated by characters with values in the group ring $\mathbb{Z}[\hat{W}]$ of the extended affine Weyl group of $\hat{sl}(3)_k$ at $k \notin \mathbb{Q}$. The ‘quantised’ version at rational level $k + 3 = 3/p$ realises the fusion rules of a WZW conformal field theory based on admissible representations of $\hat{sl}(3)_k$.

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1. Introduction

The aim of this work is to describe the characters (one-dimensional representations) of the fusion algebra of \( \mathfrak{g} = \hat{\mathfrak{sl}}(n) \) WZW conformal models at rational (shifted) level \( \kappa := k + n = p'/p \), for \( n = 3 = p' \), \( (p, 3) = 1 \) and their ‘classical’ counterparts.

In [9] we have realised this fusion algebra as a matrix algebra \( \mathcal{F} \subset \text{Mat}_{p^2}(C) \) with integer nonnegative structure constants \( (p) N_{y,x}^z \) and a basis \( \{N_y, (N_y)_x = (p) N_{y,x}^z, N_1 = 1_{p^2} \} \). The labels of the basis run over the highest weights of the admissible representations at \( \kappa = 3/p \) [12], which conveniently are parametrised by a subset \( \mathcal{W}_p^{(+)} \) of the affine Weyl group \( W \), see the text for precise definitions. \( \mathcal{F} \) is a matrix realisation of a commutative, associative algebra with identity, a distinguished basis, and an involution \( * \). The definition of the fusion algebra – an example of C-algebra (Character - algebra) in the terminology of [2], implies that \( N_y \) are normal, hence simultaneously diagonalisable by a unitary matrix \( \psi(\mu) \label{1.1} \) labelled by some set \( E_p = \{\mu\} \) of \( p^2 \) indices,

\[ (p) N_{x,y}^z = \sum_{\mu \in E_p} \frac{\psi_x^{(\mu)}(\mu)}{\psi_1^{(\mu)}} \psi_y^{(\mu)}(\mu)^* = \sum_{\mu \in E_p} \chi_x^{(p)}(\mu) \chi_y^{(p)}(\mu) \chi_z^{(p)}(\mu) \sum_{u \in \mathcal{W}_p^{(+)}(\mu)} |\chi_u^{(p)}(\mu)|^2. \]

The eigenvalues \( \chi_x^{(p)}(\mu) = \psi_x^{(\mu)}(\mu)/\psi_1^{(\mu)}(\mu) \) of \( N_x \) provide \( p^2 \) linear representations \( N_y \rightarrow \chi_y^{(p)}(\mu) \), i.e., characters of \( \mathcal{F} \), labelled by the set \( E_p \),

\[ \chi_x^{(p)}(\mu) \chi_y^{(p)}(\mu) = \sum_{z \in \mathcal{W}_p^{(+)}(\mu)} (p) N_{x,y}^z \chi_z^{(p)}(\mu). \]

With an appropriate reinterpretation of the labels this is the general setting for any RCFT. In particular for WZW models based on the \( \hat{\mathfrak{sl}}(n)_k \) integrable representations (a subclass of the admissible representations at integer level \( k = p' - n \)), both indices of the unitary matrix \( \psi_\lambda^{(\mu)} \) belong to the alcove \( P^k_+ \), \( \psi_\lambda^{(\mu)} \) is a symmetric matrix which coincides with the modular matrix, \( \psi_\lambda^{(\mu)} = S^{(p')}_{\lambda\mu} \), while (1.1) reduces to the Verlinde formula [15] for the fusion rule (FR) multiplicities, i.e., the dimensions of the spaces of chiral vertex operators.

In [9] we have specified the above general setting to the case of the generic subseries \( \kappa = 3/p \) of the admissible representations, describing explicitly one of the fusion matrices, a ‘fundamental’ fusion matrix \( N_f \). Here \( f \) is some analog of the \( \mathfrak{g} = sl(3) \) fundamental weight \( \bar{\Lambda}_1 = (1, 0) \), but the product of \( N_f \) with any \( N_x \) produces generically seven terms, each appearing with multiplicity one. The numbers \( (p) N_{f,y}^z = 0, 1 \) were found by solving
a set of algebraic equations coming from the decoupling of singular vectors in $\mathfrak{g} = \hat{sl}(3)_k$ Verma modules. More precisely the decoupling of the ‘horizontal’ singular vectors (which exist also for generic values of the level, $\kappa \not\in \mathbb{Q}$) determines the generic seven points, while the additional truncation conditions, including the ones at rational level, were obtained making also some assumptions suggested from explicit computations at small values of $p$. Although $N_f$ together with its conjugate $N_{f^*}$ are not sufficient to build up the full polynomial fusion ring, diagonalising this matrix in an orthonormal basis, i.e., finding the eigenvector matrix $\psi_y^{(\mu)}$ (common to all fusion matrices) recovers through (1.1) all FR multiplicities. We called (1.1) Pasquier–Verlinde type formula since similar formulæ – though with different interpretation of the structure constants (no more required to be nonnegative integers), were first discussed in \cite{13} in the context of lattice ADE models. Such formulæ, together with their dual counterparts, describing the structure constants of a ‘dual’ algebra, have been furthermore exploited in the study of nondiagonal modular invariants of the integrable WZW conformal models \cite{5}, \cite{14}.

Diagonalising a $p^2 \times p^2$ matrix becomes a tedious task for big $p$ (we have done this exercise for $p = 4, 5$) so it is preferable to have an explicit analytic formula for $\psi_y^{(\mu)}$, or equivalently, for the characters $\chi_y^{(p)}(\mu)$ of the fusion algebra. The path we follow in this paper to find the characters $\chi_y^{(p)}(\mu)$ was suggested by the second formula for the admissible FR multiplicities conjectured (with slightly changed notation) in \cite{9},

$$ (p) N_{x,y}^z := \sum_{w' \in W[z]} \det(w') m_{w' z y}^{x} . $$

This formula is analogous to the formula for the integrable FR multiplicities derived in \cite{11},\cite{16}, \cite{8}, which is a truncated version of the classical Weyl–Steinberg formula for the tensor product multiplicities of finite dimensional representations of the horizontal subalgebra $\mathfrak{g}$ (with the role of the horizontal Weyl group $W$ taken by the affine Weyl group $W[z]$). In (1.3) the summation runs over the Kac–Wakimoto (KW) affine group $W[z]$, generated by Kac–Kazhdan reflections corresponding to singular vectors of the $\mathfrak{g}$ Verma module of highest weight parametrised by the element $z \in W$. In the integrable case the counterpart of $m_{w' z}^x$ is the multiplicity $m_{\mu}^\lambda$ of the weight $\mu$ of the $\mathfrak{g}$ finite dimensional module of highest weight $\lambda$. Here the integers $m_{w' z}^x, x, v \in W$ describe a generalised (finite) weight diagram. The realisation that one has to generalise the classical notion of weight diagram in order to describe the rational level FR was the main lesson of the, otherwise still incomplete, analysis of the null-decoupling equations of \cite{3}. While the $\mathfrak{g}$ weight diagrams are subsets
of the root lattice $Q$ of $\hat{\mathfrak{g}}$, attached to the highest weight $\lambda$, in our case the role of a ‘root lattice’ is taken by the affine Weyl group $W = W \ltimes t_Q$ at a generic ($\kappa \not\in \mathbb{Q}$) level. Now the question is, can one find formal characters encoding this information about the generalised weight diagrams, which furthermore are closed under multiplication and recover a ‘classical’ analogue of (1.3) at generic levels, with the affine KW groups replaced by their horizontal counterparts $\tilde{W}^{(2)}$. Then by analogy with the integrable case one can ‘quantise’ these ‘classical’ characters imposing the periodicity conditions, accounting for the rational level, and thus recover the linear representations of $F$.

Though it is not necessary, to understand our way of reasoning and the motivating idea it is helpful to imagine that there is a finite dimensional algebraic object playing the role of $\hat{\mathfrak{g}}$. The irreps of the assumed hidden algebra are labelled in general by $x \in \tilde{W}^{(+)}$, a fundamental subset of the extended affine Weyl group $\tilde{W} = \tilde{W} \ltimes t_P$ with respect to the right action of $\tilde{W}$ (equivalent to the action of the horizontal KW Weyl group, see [12] and the text below). The same set also parametrises maximally reducible Verma modules of $\mathfrak{g}$ at generic level and the corresponding irreducible quotients. In the simplest $\hat{sl}(2)_k$ case solved in [1, 3] this intrinsic algebra is the $\mathbb{Z}_2$ graded algebra $osp(1|2)$ [7]; the Weyl group $\hat{W} \simeq \mathbb{Z}_2$ of $sl(2)$ distinguishes between even and odd vacuum state finite dimensional irreps of $osp(1|2)$. Alternatively mapping $\iota : \hat{W} \rightarrow Q$ one can use the horizontal subalgebra $\mathfrak{g} = sl(2)$ itself, but restricting to its representations with highest weights on the root lattice $Q$ (integer isospins). In general this map $\iota$ (see section 2) allows to describe the generalised modules through subsets of the supports of standard modules of $\hat{\mathfrak{g}}$ (with highest weights of $n$-ality zero); the case $sl(2)$ is trivial since then $\text{Im}(\iota) \equiv Q$.

The construction of the ‘classical characters’ at generic level, $k \not\in \mathbb{Q}$, takes a considerable part of this work and is its main novel result. It is done in the first part of the paper (sections 2,3,4) by full analogy with the standard case starting with some analogs of the Verma module characters, used as ingredients in a generalised Weyl character formula. While the input about the relevant generalised supports of the finite or infinite ‘modules’ (the set of weights and their multiplicities) is essentially taken over from [7], slightly rephrased and generalised, the main effort here is to find the proper multiplicative structure of the formal characters. They are elements of the group ring $\mathbb{Z}[\hat{W}]$ of the extended affine Weyl group of $\hat{sl}(3)_k$, closed under multiplication. The structure constants of the resulting commutative subring $\mathfrak{W}$ of $\mathbb{Z}[\hat{W}]$ satisfy a generalised Weyl-Steinberg formula. As a side result one obtains an explicit formula for the cardinality of the generalised weight
diagrams, i.e., for the dimensions of the ‘finite dimensional modules’ of the unknown algebra possibly generalising $osp(1|2)$. Some steps of the construction in this first part hold, or are straightforwardly generalisable, for $\mathfrak{g} = \widehat{sl}(n)_k$, arbitrary $n$. So we keep the exposition general although we present the details fully for the case $\mathfrak{g} = \widehat{sl}(3)_k$; the general case will be elaborated elsewhere. In the second part (sections 5,6,7) we consider rational values $\kappa = 3/p$, thus a generic subseries of the admissible representations of $\mathfrak{g} = \widehat{sl}(3)_k$. The formal characters are ‘quantised’ imposing periodicity constraints and realised as $\mathbb{C}$-valued functions. They satisfy the orthogonality and completeness relations equivalent to the unitarity of $\psi_y(\mu)$, so that one recovers the Pasquier–Verlinde type formula (1.1).

Thus given the characters both formulæ for the FR multiplicities of the admissible representations (1.1) and (1.3) derive from the classical analog of the Weyl–Steinberg formula established in the first part of the paper. Furthermore one proves a third formula for the FR multiplicities, conjectured in [9], which also has a ‘classical’ counterpart,

\[ (p)N^z_{x,y} = (3p)\bar{N}_{\iota(x)\iota(y)}^{\iota(z)}, \]  

where $(3p)\bar{N}_{\iota(x)\iota(y)}^{\iota(z)}$ are structure constants of the integrable fusion algebra at $\kappa = 3p$, while $\iota$ is the map sending $\bar{W}$ to a subset of the triality zero weights. Presumably the fusion rules of $\widehat{sl}(n)$ WZW at $\kappa = n/p$ are given by the above formula with $n$ substituting 3.

2. Preliminaries.

We start with fixing some notation. Let $\Delta, \Delta_+, \Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ be, respectively, the sets of roots, positive roots, and the simple roots of $\mathfrak{g} = \mathfrak{sl}(n)$, and $\Delta^\text{re}, \Delta_+^\text{re} = \Delta_+ \cup (\Delta + \mathbb{Z}_{>0} \delta)$, $\Pi = \{\alpha_0 = \delta - \sum_i \alpha_i, \Pi\}$, – the set of real roots, real positive roots and the simple roots of the affine algebra $\mathfrak{g} = \widehat{sl}(n)_k$. Let $\Xi_i$ be the fundamental weights of $\mathfrak{g}$, i.e., $\langle \Xi_i, \alpha_j \rangle = \delta_{ij}$, with respect to the Killing-Cartan bilinear form $\langle \cdot, \cdot \rangle$ on the dual $\mathfrak{h}^*$ of the Cartan algebra. With $Q = \bigoplus_i \mathbb{Z} \alpha_i$ and $P = \bigoplus_i \mathbb{Z} \Xi_i$ we denote the root and weight lattices of $\mathfrak{g}$. Their positive cones are $Q^+ = \bigoplus_i \mathbb{Z}_{\geq 0} \alpha_i$ and $P_+ = \bigoplus_i \mathbb{Z}_{\geq 0} \Xi_i$. The negative cone $-Q^+$ is the support of the $\mathfrak{g}$ Verma module of 0 highest weight, while $P_+$ is the chamber of integral dominant weights – the highest weights of the finite dimensional representations of $\mathfrak{g}$. The form $\langle \cdot, \cdot \rangle$ extends to $\mathfrak{h}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ with $\langle \mathfrak{h}^*, \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta \rangle = 0$, $\langle \Lambda_0, \Lambda_0 \rangle = 0 = \langle \delta, \delta \rangle$, $\langle \delta, \Lambda_0 \rangle = 1$. The fundamental weights of $\mathfrak{g}$ are $\{\Lambda_i = \Lambda_0 + \Xi_i\}$, $\Xi_0 = 0$. The ‘horizontal’ projection $\langle \cdot \rangle : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ is defined as having the kernel $\mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$. The Weyl vector is $\rho = \sum_j \Lambda_j$. The Weyl group $\mathcal{W}$ of $\mathfrak{g}$ is a finite Coxeter group generated
by the simple reflections \( w_i = w_{\alpha_i} \) with relations \((w_i)^2 = 1 = (w_i w_{i+1})^3, w_i w_j = w_j w_i, j \neq i \pm 1, i, j = 1, 2, \ldots, n - 1. \) The affine Weyl group \( W \) is generated by the simple reflections \( w_j, j = 0, 1, \ldots, n - 1 \) with the same type of relations (for \( n > 2 \)), identifying \( w_n = w_0 \). These groups can be depicted by their Cayley graphs, in which the vertices correspond to elements of the group, edges to the generators, and the elementary polygons to the relations. E.g., fig. 1 depicts (a finite part of) the Cayley graph of the affine Weyl group \( \hat{W} \) of \( g = \hat{sl}(3) \), presented as \( \{w_i : (w_i)^2 = 1 = (w_i w_j)^3, \{i, j\} \subset \{0, 1, 2\}\} \), while any of the ‘12’ elementary hexagons on fig. 1 is the Cayley graph of the corresponding horizontal Weyl group \( \hat{W} \). The labels \( i \) on the edges correspond to the generators \( w_i, i = 0, 1, 2 \). The ‘origin’ of the graph, the vertex corresponding to the group unit is denoted by \( \overset{\text{o}}{\Box} \). The three types of hexagons (‘12’, ‘01’, and ‘20’) correspond to the three Artin type relations among the generators. It is convenient to introduce the following shorthand notation: \( w_{ijk\ldots} = w_i w_j w_k \ldots \).

We will also need the extended affine group \( \hat{W} \) defined as the semi-direct product \( \hat{W} = W \ltimes t_P \) (while \( W = W \ltimes t_Q \)), \( t_P \) being the subgroup of translations in the weight lattice \( P \). The elements \( t_\beta, \beta \in P \), act on \( \hat{h}^* \) as

\[
t_\beta(\Lambda) = \Lambda + \langle \Lambda, \delta \rangle \beta - \left( \langle \Lambda, \beta \rangle + \frac{1}{2} \langle \beta, \beta \rangle \langle \Lambda, \delta \rangle \right) \delta, \tag{2.1}
\]

and for \( \alpha \in \Delta, l\delta - \alpha \in \Delta^e, y \in \hat{W}, \beta \in P \), one has the properties

\[
t_{l\alpha} = w_{l\delta - \alpha} w_\alpha, \quad y t_\beta y^{-1} = t_{\tilde{y}(\beta)}, \quad y w_\alpha y^{-1} = w_{y(\alpha)}. \tag{2.2}
\]

We have denoted by overbar the projection of \( \hat{W} \) onto the horizontal subgroup \( \overline{W} \) sending the affine translations to the unit element.

The group \( \hat{W} \) can be also written as \( \hat{W} = W \ltimes A \), where \( A \) is the subgroup of \( \hat{W} \) which keeps invariant the set of simple roots \( \Pi \) of \( g \). It is a cyclic group generated by \( \gamma = t_\overline{\Lambda}, \overline{\gamma} \), where \( \overline{\gamma} = \overline{w_1} \ldots \overline{w_{n-1}} \) is a Coxeter element in \( \overline{W} \) generating the cyclic subgroup \( \overline{A} \) of \( \overline{W} \). One has \( \gamma(\alpha_j) = \alpha_j + 1 = \gamma^{j+1}(\alpha_0) \) for \( j = 0, 1, 2, \ldots, n - 1 \) identifying \( \alpha_n \equiv \alpha_0 \). In the case of \( \overline{g} = sl(3) \) we will think of the Cayley graph of \( \hat{W} \) as a 3-sheeted covering of the graph of \( W \) with, for example, the “fiber” over the edge ‘0’ connecting the vertices \( 1 \) and \( w_0 \) being the set \( \mathcal{U} = \{A, A w_0\} \) and this part of the graph of \( \hat{W} \) is depicted on fig. 2. The oriented edges correspond to \( \gamma \) and the squares – to the implementation of the automorphism of \( W, w_\alpha \to \gamma w_\alpha \gamma^{-1} = w_{\gamma(\alpha)} = w_{\gamma(\alpha)} \), \( \alpha \in \Pi \).
Introduce the set $\mathcal{P} = \{ \Lambda = y \cdot k\Lambda_0 \mid y \in \tilde{W} \}$ where the shifted action of $w \in \tilde{W}$ on $\tilde{h}^*$ is given by $w \cdot \Lambda = w(\Lambda + \rho) - \rho$. In this and the following two sections we shall assume that the level $k$ is generic, $k \not\in \mathbb{Q}$, which in particular ensures that if $y \cdot k\Lambda_0 = k\Lambda_0$ then $y \equiv 1$.

According to the general criterion of Kac-Kazhdan if $\langle \Lambda + \rho, \beta \rangle$ is a positive integer for some $\beta \in \Delta^+_x$ the Verma module $M_{\Lambda'}$, $\Lambda' \in \mathcal{P}$, of $g$ is reducible, containing a Verma submodule $M_{w,\beta} \cdot \Lambda$. In particular if for some $y = \overline{y} t_{-\lambda} \in \tilde{W}$ and $\alpha \in \tilde{\Pi}_+$ we have $y(\alpha) = \langle \lambda, \alpha \rangle \delta + \overline{y}(\alpha) \in \Delta^+_{\tilde{x}}$, then the KK condition is fulfilled for $\Lambda = y \cdot k\Lambda_0$ and the root $\beta = y(\alpha)$, and using (2.3)

$$
w_{y(\alpha)} \cdot \Lambda = y w_\alpha y^{-1} \cdot \Lambda = y w_\alpha \cdot k\Lambda_0 = \overline{y} w_\alpha t_{-\alpha^{-1}(\lambda)} \cdot k\Lambda_0 . \tag{2.3}
$$

If $y(\alpha_i) \in \Delta^+_{\tilde{y}} \forall \alpha_i \in \tilde{\Pi}$, the reflections $w_{y(\alpha_i)}$ generate an isomorphic to $\tilde{W}$ group $\tilde{W}[\Lambda]$ (to be denoted also $\tilde{W}_{[y]}$), introduced by Kac-Wakimoto [12]. We shall refer to these groups as (horizontal) KW (Weyl) groups. According to (2.3) we can identify the action of a KW group with the right action of $\tilde{W}$ on $\tilde{W}$.

Now we introduce the subset $\mathcal{P}_+ \subset \mathcal{P}$ of weights $\Lambda$ such that the corresponding Verma modules $M_{\Lambda'}$ are maximally reducible. From the above discussion it is clear that $\mathcal{P}_+ = \tilde{W}^{(+)} \cdot k\Lambda_0$, where

$$
\tilde{W}^{(+)} = \{ y \in \tilde{W} \mid y(\alpha_i) \in \Delta^+_{\tilde{y}} \text{ for } \forall \alpha_i \in \tilde{\Pi} \} . \tag{2.4}
$$

The Bruhat ordering on the orbit $\tilde{W}[\Lambda] \cdot k\Lambda_0$ describes the embedding pattern among the Verma modules $\{ M_{\Lambda'}, \Lambda' \in \tilde{W}[\Lambda] \cdot k\Lambda_0 \}$.

Denote $\mathcal{W}^{(+)} = \tilde{W}^{(+)} \cap W$. One has

**Proposition 2.1** $\mathcal{P}_+$ is a fundamental domain in $\mathcal{P}$ with respect to the action of the KW Weyl groups, or, equivalently, $\tilde{W}^{(+)} (\mathcal{W}^{(+)} )$ is a fundamental domain in $\tilde{W}$ (W) with respect to the right action of $\tilde{W}$.

**Proof:** Let us introduce some notation first. Let $H_\alpha = \{ \lambda \in \tilde{h}^* \mid \langle \lambda, \alpha \rangle = 0 \}$ be the hyperplane orthogonal to the root $\alpha$. For $w \in \tilde{W}$ let $I(w) = \{ i \mid w(\alpha_i) < 0, \alpha_i \in \tilde{\Pi} \}$ and

$$
P^{(w)}_+ := \{ \lambda \in P_+ \mid \langle \lambda, \alpha_i \rangle > 0, i \in I(w) \} = P_+ \setminus \left( \bigcup_{i \in I(w)} H_\alpha_i \right) = P_+ \setminus \sum_{i \in I(w)} \tilde{\Lambda}_i .
$$

The definition (2.4) can be obviously rewritten as

$$
\tilde{W}^{(+)} = \{ y \in \tilde{W} \mid y = wt_{-\lambda}, w \in \tilde{W}, \lambda \in P^{(w)}_+ \} . \tag{2.5}
$$
Hence the statement of the Proposition follows from \( \bigcup_{w \in \tilde{\mathcal{W}}} \tilde{\mathcal{W}}^+(w) = \bigcup_{w \in \tilde{\mathcal{W}}} \bigcup_{w' \in \tilde{\mathcal{W}}} t_{-w'}(P_+^{(w)}) \) \( w' = \bigcup_{w \in \tilde{\mathcal{W}}} t_{-w} w = \tilde{\mathcal{W}} \) once the following lemma is established:

**Lemma 2.2** \( P = \bigcup_{w \in \tilde{\mathcal{W}}} w(P_+^{(w)}) \) is a partition, i.e., a disjoint union.

To prove the lemma one has to exploit several standard properties of the Weyl group \( \tilde{\mathcal{W}} \) which can be found e.g., in [10], chapter I.

**Proof:** Any \( \lambda' \in P \) is represented as \( \lambda' = w(\lambda) \) for some \( \lambda \in P_+^\circ \), \( w \in \tilde{\mathcal{W}} \). Denote \( X = X_\lambda = \{ i | \langle \lambda, \alpha_i \rangle = 0 \} \). According to Proposition 1.10c of [10] the element \( w \) splits uniquely into a product of two elements of \( \tilde{\mathcal{W}} \), \( w = uv \) s.t. \( v \in W_X \) (the group generated by the simple reflections labelled by the subset \( X \)), and \( u(\alpha_i) > 0 \), for any \( i \in X \). We have \( v(\lambda) = \lambda \), \( I(u) \cap X = \emptyset \), hence \( \lambda \in P_+^{(u)} \) and \( \lambda' = u(\lambda) \in u(P_+^{(u)}) \). This proves that \( P \) is covered by the union of subsets \( w(P_+^{(w)}) \). The uniqueness of the above splitting proves also the disjointness. \( \square \)

We shall refer to \( \tilde{\mathcal{W}}^+(+) \) (or, equivalently, \( P_+^\circ \)) as a ‘dominant chamber’. The left action of the group \( A \) (the shifted action of \( A \)) keeps \( \tilde{\mathcal{W}}^+(+) \) invariant and \( \bigcup_{a \in A} a \tilde{\mathcal{W}}^+(+) = \tilde{\mathcal{W}}^+(+) \). Similarly \( P_+^\circ \) is \( \mathbb{Z}/n \mathbb{Z} \) graded by the \( n \)-ality \( \tau(\Lambda = \bar{\gamma}t_{-\lambda} \cdot k\Lambda_0) := \tau(\lambda) \), where \( \tau(\lambda) = \sum_i i\langle \lambda, \alpha_i \rangle = n \langle \lambda, \bar{\Lambda}_{n-1} \rangle \) mod \( n \) is the standard grading in \( P \).

In the case \( sl(3) \) the chamber \( \tilde{\mathcal{W}}^+(+) \) can be also expressed as \( \tilde{\mathcal{W}}^+(+) = U t_{-P_+} \) in terms of the subset \( U = \{ A, A w_0 \} \subset \tilde{\mathcal{W}} \), depicted on fig. 2.

Next we introduce a map \( \iota \) of \( \tilde{\mathcal{W}} \) (or \( P \)) into \( Q \)

\[
\iota: \tilde{\mathcal{W}} \ni y = \bar{\gamma}t_{-\lambda} \mapsto n\lambda + \bar{\gamma}^{-1} \cdot 0 = n\lambda - \sum_{\beta > 0, \bar{\gamma}(\beta) < 0} \beta \in Q. \tag{2.6}
\]

See [11] (exercise 3.12 of ch. 3) for the last equality. \( A \) is mapped by \( \iota \) to zero.

The map \( \iota \) has the “twisted log” property

\[
\iota(xy) = \bar{\gamma}^{-1}(\iota(x)) + \iota(y). \tag{2.7}
\]

Compare with the horizontal projection map

\[
h: \tilde{\mathcal{W}} \ni y = \bar{\gamma}t_{-\lambda} \mapsto h(y) = \bar{\gamma} \cdot k\Lambda_0 = \bar{\gamma} \cdot (-\kappa\lambda) \in \kappa P + \tilde{\mathcal{W}} \cdot 0, \quad h(xy) = h(x) + \bar{x}(h(y)). \tag{2.8}
\]
The map (2.6) provides another equivalent definition of the chamber $\tilde{W}^{(+)}$. Indeed comparing with (2.5) and using that $1 \leq \langle \rho, \alpha \rangle \leq n-1$ for any $\alpha \in \Delta_+$ one easily checks

$$\tilde{W}^{(+)} = \{ y \in \tilde{W} \mid \iota(y) \in P_+ \}.$$  \hfill (2.9)$$

The relation (2.7) implies that $\iota$ intertwines between the KW action (equivalent according to (2.3) to the right action of $\bar{W}$ on $\tilde{W}$) and the ordinary shifted action of $\bar{W}$, i.e.,

$$\iota(yw) = \overline{w}^{-1} \cdot \iota(y), \quad \overline{w} \in \overline{W}. \hfill (2.10)$$

Both subsets of $Q$, the image $\text{Im}(\iota)$ and its complement are invariant under the shifted action of the Weyl group $\bar{W}$. The $\mathfrak{g}$ Verma modules of highest weight $\iota(y)$ are reducible iff the corresponding $\mathfrak{g}$ Verma modules of highest weight $\Lambda = y \cdot k\Lambda_0$ are reducible and the pattern of embeddings of submodules in both cases is identical.

On fig. 3 we have illustrated the map $\frac{1}{3}\iota$ for the case $\mathfrak{g} = \mathfrak{sl}(3)$. It maps the even (under the gradation $\det(w) = \pm 1$) elements of $W$ to $P$ and the odd elements to $P + \theta/3$ ($\theta = \alpha_1 + \alpha_2$) in such a way that the vertices of the Cayley graph (if we make it into a ‘rigid’ geometrical graph by fixing the length of each edge to be of length $\sqrt{2}/3$ assuming as usual that the roots of $\mathfrak{sl}(3)$ are of length $\sqrt{2}$) geometrically ‘sit’ at the same places as the vertices of the two lattices $P$ and $P + \theta/3$. (See the figure.) In other words refining by 3 the weight lattice, $\overline{\Lambda}_i \rightarrow \overline{\Lambda}_i/3$, (or equivalently, rescaling $\kappa \rightarrow \kappa/3$) the points of the Cayley graph can be identified with a subset of the triality zero weights

$$\{ \lambda = \sum_i 3n_i \overline{\Lambda}_i/3 \} \cup \{ \lambda = \sum_i (3n_i + 1) \overline{\Lambda}_i/3 \}$$

in the refined lattice, while the “excluded” triality zero points $\lambda = \sum_i (3n_i - 1) \overline{\Lambda}_i/3 \in P - \theta/3$, correspond to the centers of the elementary hexagons.

Remark. The above analysis properly extends to the larger than $\mathcal{P}$ region

$$\{ \Lambda = y \cdot (\lambda' + k\Lambda_0), \quad y \in \tilde{W}, \quad \lambda' \in P; \quad k \notin \mathbb{Q} \}.$$  

It contains a subset with $y \in \tilde{W}, \lambda' \in P_+$, providing highest weights of ‘maximally reducible’ $\mathfrak{g}$ Verma modules. For our purposes it is sufficient to choose $\lambda' = 0$ thus restricting to weights $\Lambda$ parametrised by $y \in \tilde{W}$.
3. Characters.

Let us start by recalling the supports and characters of ordinary Verma and finite dimensional modules of $\mathfrak{g}$. The latter characters are elements of the group ring $\mathbb{Z}[t_P]$ of the group of translations by the weight lattice. Keeping with tradition, the translations $t_{-\lambda}$ from the previous section will be written as formal exponentials $e^{-\kappa \lambda}$; $\kappa = -1$ recovers the standard notation, see, e.g., [3] for standard definitions. The character of the Verma module $V_\lambda$ of highest weight $\lambda$ is given by

$$
\text{ch}(V_\lambda) = \sum_{\mu \in \lambda - Q^+} K_{\mu}^\lambda e^{-\kappa \mu} = e^{-\kappa \lambda} \sum_{\beta \in Q^+} K_{\beta} e^{\kappa \beta} = e^{-\kappa \lambda} \prod_{\alpha \in \Delta} (1 - e^{\kappa \alpha})^{-1} = e^{-\kappa(\lambda + \rho)} d_{\kappa},
$$

(3.1)

where the multiplicity $K^\lambda_{\mu}$ of an weight $\mu$ is expressed via the Kostant partition function $K^\lambda_{\mu} := K_{\lambda - \mu} \in \mathbb{Z}_{\geq 0}$, while $d_{\kappa}$ is a $W$ invariant (up to a sign) quantity $w(d_{\kappa}) = \det(w) d_{\kappa}$ for $w(e^{\kappa \beta}) := e^{\kappa w(\beta)}$. The support of a module is the set of weights $\mu$ of nonzero multiplicity $K_{\mu}^\lambda$, thus supp$V_\lambda = \lambda - Q^+$.

The irreducible finite dimensional modules can be resolved in terms of Verma modules, i.e., each Verma module $V_{w \cdot \lambda}$, with $w \in W$ and $\lambda \in P_+$, contains submodules of weight $V_{w' \cdot \lambda}$ for all $w' > w$ in the Bruhat ordering of $W$ (in a convention in which $1$ is the smallest element) and grading $\mathbb{Z}$ by the reduced length of words the Verma module inclusions organize in a BGG (Bernstein–Gelfand–Gelfand) resolution. For the characters $\chi_\lambda$, $\lambda \in P_+$ of irreducible finite dimensional modules the BGG resolution gives immediately the Weyl character formula

$$
\chi(\lambda) = \sum_{w \in W} \det(w) \text{ch}(V_{w \cdot \lambda}) = d_{\kappa} \sum_{w \in W} \det(w) e^{-\kappa w(\lambda + \rho)} = \sum_{\mu \in P} m_{\mu}^\lambda e^{-\kappa \mu},
$$

(3.2)

$$
\overline{m}_\mu^\lambda = \sum_{w \in W} \det(w) K_{w \cdot \lambda - \mu},
$$

and from $\chi_1 = 1$ the factor $d_{\kappa}$ is expressed as $1/d_{\kappa} = \sum_{w \in W} \det(w) e^{-\kappa w(\rho)}$. The support (weight diagram) is $\Gamma_\lambda = \{\mu \in P | m_{\mu}^\lambda \neq 0\}$. From (3.2) it follows that

$$
w(\chi_\lambda) = \chi_\lambda, \quad \overline{m}_{w(\mu)}^\lambda = \overline{m}_\mu^\lambda,
$$

(3.3)

and, extending the first line of (3.2) to $\lambda \in P$,

$$
\chi_{w \cdot \lambda} = \det(w) \chi_\lambda.
$$

(3.4)
The map \( \iota \) introduced in (2.6) establishes a correspondence between (highest weights of) Verma modules of the affine algebra \( \mathfrak{g} \) and of the horizontal subalgebra \( \mathfrak{g}_0 \), with identical reducibility structure. Now we shall introduce another class of ‘Verma modules’ \( V_y \) (of yet unknown finite dimensional algebra) described through its supports, i.e., weights and their multiplicities. The supports are parametrised by elements of \( aW \subset \tilde{W} \) (for a fixed \( a \in A \)) and mapped by \( \iota \) into subsets of supports of \( \mathfrak{g} \) Verma modules. Motivated by the intertwining property (2.10) we shall call such a module reducible if its \( \iota \) image is a reducible Verma module. In particular the maximally reducible ‘Verma modules’ have highest weights \( \Lambda = y \cdot k\Lambda_0, y \in \tilde{W}^{(+)} \), whence the name ‘dominant chamber’ for the latter subset of \( \tilde{W} \). Furthermore in full analogy with the representation theory of \( \mathfrak{g} \) we shall introduce ‘finite dimensional modules’ obtained factorising maximal ‘Verma submodules’ of reducible ‘Verma modules’ by an analog of the BGG resolution in which the role of \( \overline{W} \) is replaced by the action of the KW group, equivalent according to (2.3) to the right action of \( \overline{W} \) on \( \tilde{W} \). We have already explained on the example of \( \mathfrak{g} = sl(3) \) this construction in [9] on the level of multiplicities of weights, here we add a realisation of the characters of these ‘Verma- and finite dimensional modules’. It recovers the prescribed multiplicities but furthermore allows to consider a tensor product of ‘finite dimensional modules’ realised by a multiplication of their characters. The result is a new commutative ring which extends the ring of \( \mathfrak{g} \) characters. It will be used in sections 6,7 as a basis for the construction of quantised ‘\( q \)-characters which realise the fusion rules of the admissible representations at rational level.

We can describe the analogs of the multiplicities in (3.1) via the map \( \iota \), so what remains to be done is to generalise the formal exponentials entering (3.1). The naive extension of the map \( \iota \) to the characters, thus leading to ordinary formal exponentials with arguments involving the weights \( \iota(y) \) (or the horizontal projections \( \Lambda = h(y) \in \mathfrak{h}^* \)) is possible but is not consistent (except in the case \( \mathfrak{g} = sl(2) \)) since in general a sum of such weights goes beyond the image of \( \iota \). (This is in agreement with the fact that there is no nontrivial subset of the \( n \)-ality zero representations of \( \mathfrak{g} \) closed under tensor products). So our main idea is, instead of exponentials (elements of the group algebra of the group of translations \( t_P \)) assigned to weights, to consider the elements \( \overline{w} e^{-\kappa \lambda} = y \in \tilde{W}, i.e., the generating elements of the group algebra of the group \( \tilde{W} \) (the extension of \( t_P \) over \( \overline{W} \)). (We shall use the same notation for both interpretations denoting as before sometimes the affine
translations by formal exponentials. Unlike the formal exponentials these generating elements do not commute any more (rather satisfy the multiplication rules of $\tilde{W}$ in (2.2), but nevertheless the ‘finite dimensional module’ characters we obtain, do commute and multiply according to the rules conjectured in our previous paper.

Now we introduce the supports, $V_u$ and $G_u$, and characters, $ch(V_u)$ and $\chi_u$, of, respectively, ‘Verma’ and ‘finite dimensional modules’. The supports are certain infinite or finite subsets of the extended affine Weyl group $\tilde{W}$ while the characters are certain (formal) series or finite sums with integer coefficients of elements of $\tilde{W}$.

Let $\Lambda = y \cdot k\Lambda_0$, $y \in aW$, $a \in A$. We define

$$V_y := \{z \in aW| \iota(z) \in \iota(y) - Q^+\} = \{xy| x \in W, \iota(x) \in -\bar{y}(Q^+)\}.$$  (3.5)

Alternatively, using that any $\nu \in Q^+$ can be represented uniquely as $\nu = n\beta + \lambda$, with some $\beta \in Q^+$ and $\lambda \in Q^+/nQ^+ = \{\sum_{i=1}^{n-1} k_i\alpha_i| 0 \leq k_i \leq n - 1\}$, we can bring $V_y$ into the form

$$V_y = T^{\bar{y}} y t_{Q^+} = \{t_{\bar{y}^{-1}\beta} uy| u \in T^{\bar{y}}, \beta \in Q^+\}.$$  (3.6)

Here the finite subsets $T^{\bar{y}} \subset W$, which project horizontally to $\bar{W}$, are subject to the condition

$$T^{\bar{y}} = \{u \in W| -\bar{y}^{-1}(\iota(u)) \in Q^+/nQ^+\}.$$  (3.7)

E.g., if an element $u \in T^{\bar{y}}$ projects to a reflection $\bar{w}_\alpha \in \bar{W}$, $\alpha \in \underline{\Delta}_+$ then it equals $u = \bar{w}_\alpha$, or $u = \bar{w}_\alpha t_{-\alpha} = w_{\delta-\alpha}$ if $\bar{y}^{-1}(\alpha) \in \underline{\Delta}_+$, or $\bar{y}^{-1}(-\alpha) \in \underline{\Delta}_+$, respectively.

Thus the support $V_y$ in (3.5) naturally generalises the support of a $\bar{g}$ Verma module, being defined as a collection of $|\bar{W}|$ “positive” (for a choice of a set of simple roots) root lattice cones $t_{\bar{y}^{-1}\beta} t_{\lambda}$ applied to $uy \in T^{\bar{y}} y$. The two descriptions of $V_y$ reflected in (3.6) and in (3.5) – as a collection of supports of ordinary $\bar{g}$ modules of highest weights $h(u) = uy \cdot k\Lambda_0$, or as the support of $\bar{g}$ module of highest weight $\iota(y)$ with some ‘excluded’ points, will be both useful in what follows. Next we define the multiplicity of an weight $z$ through its $\iota$ image,

$$K^y_z := K_{\iota(y) - \iota(z)} \text{ or } K^y_{xy} = K_{\iota^{-1}(\iota(x))}.$$  (3.8)

\[1\] From now on $w e^{\kappa\lambda}$ will stand for the multiplication of two elements in the group ring $\mathbb{Z}[\tilde{W}]$; accordingly the standard notation for the action of the (horizontal) Weyl group on the formal exponentials, $w(e^{-\kappa\lambda}) = e^{-\kappa w(\lambda)}$, will be replaced by $e^{-\kappa w(\lambda)} = w e^{-\kappa\lambda} w^{-1}$, cf. (2.2).
where $K_\beta$ is the Kostant partition function in (3.1). Then the characters generalising (3.1) are defined according to

$$
\text{ch}(\mathcal{V}_y) := \sum_{z \in \tilde{W}, zy^{-1} \in W} \sum_{\iota(z) \in \iota(y) - Q^+} \sum_{x \in W, \iota(x) \in -\overline{\mathcal{Y}}(Q^+)} \iota(z) \in \iota(y) - Q^+ \sum_{\iota(x) \in \iota(y) - Q^+} \iota(x) \in \iota(y) - Q^+ xy K_{-\overline{\mathcal{Y}}^{-1}(\iota(x))}.
$$

(3.9)

Equivalently, using that $-\bar{y}^{-1}(\iota(x)) = -\bar{y}^{-1}(\iota(u)) + n\beta$ for $xy = uyt_\beta$, and denoting

$$
P_r e^{-\kappa \overline{\mathcal{Y}}} d_\kappa := \sum_{\beta \in Q^+} K_{n\beta + r} e^{\kappa \beta},
$$

(3.10)

we rewrite (3.9) as

$$
\text{ch}(\mathcal{V}_y) = \sum_{u \in \mathcal{T}_\overline{\mathcal{Y}}} uy P_{-\overline{y}^{-1}(\iota(u))} e^{-\kappa \overline{\mathcal{Y}}} d_\kappa.
$$

(3.11)

Multiplying both sides of (3.10) by $e^{\kappa n \tau}$, summing over $r \in Q^+/nQ^+$ and making a change of variables $n\beta + r = \beta'$, we recover in the r.h.s. the factor $e^{-\kappa \overline{\mathcal{Y}}} d_\kappa/n$, cf. (3.1). Thus $P_r$ are polynomials of translations in $Q^+$ determined through the quotient of the standard denominators, $d_{\kappa/n}$ and $d_\kappa$.

$$
\sum_{r \in Q^+/nQ^+} P_r e^{\kappa \overline{\mathcal{Y}}} = \prod_{\alpha > 0} \sum_{k_\alpha = 0}^{n-1} e^{\kappa k_\alpha} \alpha = \sum_{\mu \in Q^+} (n) K_{\mu} e^{\kappa \mu},
$$

(3.12)

$$
P_r = \sum_{\nu \in Q^+} (n) K_{r + n\nu} e^{\kappa \nu}.
$$

(3.13)

The partition function $(n) K_{\mu}$ defined through the last equality in (3.12) is apparently nonzero for a finite subset of $Q^+$, i.e., the summation in (3.13) is finite. The relations (3.10), (3.13) imply

$$
K_{n\beta + r} = \sum_{\nu \in Q^+} (n) K_{n\nu + r} K_{\beta - \nu}.
$$

(3.14)

We have the symmetry properties

$$
\text{ch}(\mathcal{V}_{ay}) = a \text{ch}(\mathcal{V}_y), \quad a^{-1} \mathcal{T}^{-1} \mathcal{Y} a = \mathcal{T} \mathcal{Y}, \quad P_{-(\overline{\mathcal{Y})}^{-1}(\iota(aua^{-1}))} = P_{-\overline{y}^{-1}(\iota(u))},
$$

(3.15)

using that $\iota(a) = 0$, $\iota(a^{-1}xa) = a^{-1}(\iota(x))$ for $a \in A$.

Now consider $\Lambda = y \cdot k\Lambda_0 \in \mathcal{P}_+$, $y \in \tilde{W}(\overline{\mathcal{Y}})$, hence $\iota(y) \in \mathcal{P}_+$. According to our definition $\mathcal{V}_y$ is reducible with submodules $\mathcal{V}_{yw}$, $w \in \mathcal{W}$, since $V_{\iota(y)}$ is reducible with
submodules \( V_{w^{-1} \cdot \iota(y)} \). In parallel with (3.2) we define the characters of ‘finite dimensional modules’ by a ‘resolution’ formula with respect to the KW Weyl group

\[
\chi_y := \sum_{w' \in W^{[A]}} \det(\bar{w}') \text{ch}(V_{w' y}) = \sum_{\bar{w} \in \bar{W}} \det(\bar{w}) \text{ch}(V_{\bar{w} y}) = \sum_{z \in \bar{W}, zy^{-1} \in W} m_z^y \cdot \iota(y). \tag{3.16}
\]

We extend this definition to the whole \( \bar{W} \), i.e.,

\[
\chi_{y \bar{w}} = \det(\bar{w}) \chi_y, \quad y \in \bar{W}^{(+)}, \quad \bar{w} \in \bar{W}. \tag{3.17}
\]

Using (3.8) and the intertwining property (2.10) of the map \( \iota \) (3.16) gives for the multiplicities \( m_z^y \) (cf. (3.2))

\[
m_z^y = \sum_{\bar{w} \in \bar{W}} \det(\bar{w}) K_z^y \bar{w} = \sum_{\bar{w} \in \bar{W}} \det(\bar{w}) K_{\iota(\bar{w} y) - \iota(z)} = \sum_{\bar{w} \in \bar{W}} \det(\bar{w}) K_{\bar{w} \cdot \iota(y) - \iota(z)} = \tilde{m}_{\iota(z)}^y. \tag{3.18}
\]

Having an explicit description for the multiplicities we can introduce the supports \( G_y \) of ‘finite dimensional modules’ as \( G_y = \{ z \in \bar{W} | m_z^y \neq 0 \} \). From (3.18) and from the definition of the map \( \iota \) it follows that these ‘generalised weight diagrams’ have the structure of n-ality zero \( sl(n) \) weight diagrams \( \Gamma_{\iota(y)} \) with the points \( \mu \notin \text{Im}(\iota) \) excluded. An \( sl(3) \) example is illustrated on fig. 4, see [9] for more examples. Let us point out some symmetry properties of the generalised weight diagrams. We have from (3.15)

\[
\chi_{a y} = a \chi_y, \quad a \in A, \tag{3.19}
\]

which implies that \( m^y_{a z} = m^y_z \). The invariance of the \( \bar{w} \) multiplicities \( \tilde{m}_{a(z)}^{(y)} = \tilde{m}_{\iota(z)}^{(y)} = \tilde{m}_{\iota(a z a^{-1})}^{(y)} \) implies

\[
m_{a^{-1} z a} = m^y_z, \quad a \in A \tag{3.20}
\]

and hence \( a \chi_y a^{-1} = \chi_y \). For \( z = a \cdot t - \mu \), \( a \in A \), the symmetry property (3.20) extends to

\[
m^y_{a \cdot t - \bar{w}(\mu)} = m^y_{a \cdot t - \mu}, \quad \bar{w} \in \bar{W}, \tag{3.21}
\]

using once again (2.11) to obtain \( \iota(aw^{-1} z \bar{w}^{-1}) = \bar{w}(\iota(z)) \) as well as the invariance of the \( \bar{w} \) multiplicities \( \tilde{m}_{\bar{w}(\iota(z))}^{(y)} = \tilde{m}_{\iota(z)}^{(y)} \) for any \( \bar{w} \in \bar{W} \).

Now we examine the case \( \mathfrak{g} = sl(3) \) in details. The 6 element sets \( T_{\bar{w}} \) are \( T_{\bar{a}} = a \bar{W} a^{-1} \), while \( T_{\bar{a} \bar{w} \bar{a}} = a T^{w} a^{-1} \), and \( T^{w} = \{ 1, w_{010}, w_{020}, w_{110}, w_{201}, w_{001} \} \), see figs. 5,6, where the two basic supports are ‘visualized’ as certain subgraphs of the Cayley graph of
The set of polynomials $P_{-\vec{y}^{-1}(\mu(u))}$ in (3.11) associated with $\mathcal{T}_{\vec{y}}$ and the corresponding characters for $\vec{y} = 1, w_\theta$ read

$$
\text{ch}(\mathcal{V}_{t_{-\lambda}}) = \sum_{w \in \mathcal{W}} w P_{-t_{w}} e^{-\kappa(\lambda + \vec{y})} d_\kappa
$$

$$
= [(1 + 2e^{\kappa \theta}) + w_{12}(2 + e^{\kappa \alpha_1}) + w_{21}(2 + e^{\kappa \alpha_2})
+ w_1(1 + e^{\kappa \theta} + e^{\kappa \alpha_2}) + w_2(1 + e^{\kappa \theta} + e^{\kappa \alpha_1}) + 3w_\theta] e^{-\kappa(\lambda + \vec{y})} d_\kappa,
$$

(3.22)

$$
\text{ch}(\mathcal{V}_{w_\theta t_{-\lambda}}) = \sum_{u \in T_{w_\theta}} u w_\theta P_{-\theta^{-1}(\mu(u))} e^{-\kappa(\lambda + \vec{y})} d_\kappa
$$

$$
= [w_0 w_\theta (2 + e^{\kappa \theta}) + w_{010} w_\theta (1 + 2e^{\kappa \alpha_2}) + w_{020} w_\theta (1 + 2e^{\kappa \alpha_1})
+ w_{20} w_\theta (2 + e^{\kappa \alpha_1}) + w_{10} w_\theta (2 + e^{\kappa \alpha_2}) + w_\theta (1 + 2e^{\kappa \theta})] e^{-\kappa(\lambda + \vec{y})} d_\kappa.
$$

(3.23)

Now we rewrite the ‘finite dimensional module’ characters (3.16) expressing them in terms of the ordinary $sl(3)$ characters, the elements of $A$ and the ‘class’ element

$$
F \equiv \sum_{a \in A} a w_0 a^{-1} = w_0 + w_1 + w_2.
$$

(3.24)

**Proposition 3.1** For any $x = \vec{x} t_{-\lambda} = t_{-\nu} \vec{x} \in \mathcal{W}^{(+)}$, $\nu = \vec{x}(\lambda)$ we have

$$
\chi_x = \det(\vec{x}) \left( \chi_\nu + \gamma \chi_{\nu-2\vec{1}_1} + \gamma^{-1} \chi_{\nu-2\vec{1}_2} + (F + 2) (\chi_{\nu-\vec{1}} + \gamma \chi_{\nu-\vec{1}_2} + \gamma^{-1} \chi_{\nu-\vec{1}_1}) \right)
$$

$$
= \chi_{\lambda + \vec{1}_0} + \gamma \chi_{\lambda + \vec{1}_1} + \gamma^{-1} \chi_{\lambda + \vec{1}_2} + (F + 2) \left( \chi_{\lambda + \vec{1}_1} + \gamma \chi_{\lambda + \vec{1}_2} + \gamma^{-1} \chi_{\lambda + \vec{1}_1} \right).
$$

(3.25)

**Proof:** First the ‘Verma module’ characters (3.22), (3.23) can be rewritten for any $x = \vec{x} t_{-\mu} \in \mathcal{W}$ as (recall that $\gamma = w_{12} e^{-\kappa \vec{1}_2}$)

$$
\text{ch}(\mathcal{V}_x) = [(1 + \gamma e^{\kappa \vec{1}_1}) + \gamma^{-1} e^{\kappa \vec{1}_1} (2\vec{1}_2)]
+ (F + 2) (\gamma e^{\kappa \vec{1}_1} (\vec{1}_2) + \gamma^{-1} e^{\kappa \vec{1}_1} (\vec{1}_1)) d_\kappa e^{-\kappa(\mu + \vec{x}(\lambda))}.
$$

(3.26)

Next apply the ‘resolution’ formula (3.16) in the form

$$
\chi_x = \sum_{w' \in \mathcal{W}} \det(\vec{w}') \text{ch}(\mathcal{V}_{w' x}) = \sum_{w \in \mathcal{W}} \det(\vec{w}) \text{ch}(\mathcal{V}_{\vec{w} t_{-\vec{1}^{-1}(\lambda)}})
$$
To obtain the second equality in (3.25), i.e., to express the characters in terms of \( sl(3) \) characters \( \chi_\lambda \) with \( \lambda \in P_+ \), we have used (3.4), so that each term \( \det(\varphi) \chi_{\nu, -\nu} \) turns into \( \chi_{\lambda + \varphi^{-1}, -(-\nu)} \).

Taking into account the same property (3.4) (which in particular implies that \( \chi_\lambda = 0 \) for weights on the shifted reflecting hyperplanes, i.e., \( \bar{w}_\alpha \cdot \lambda = \lambda \) for some \( \alpha \in \Delta_+ \)) some terms in the general expression (3.25) may vanish or compensate each other for some \( \text{‘boundary’ weights.} \)

The character formula (3.25) can be also rewritten as a decomposition over \( \bar{W} \), or \( \mathcal{U} \)

\[
\chi_y = \sum_{\mu \in P} P^\mu \chi_\mu e^{-\kappa_\mu} := \sum_{a \in A, \mu \in P} \left( (m^\mu_\mu + \varphi^{-1}, -\nu_a) + 2m^{\lambda + \varphi^{-1}, -\nu_a'} a + m^{\lambda + \varphi^{-1}, -\nu_a} F a \right) e^{-\kappa_\mu}
\]

\[
= \sum_{a \in A, \mu \in P} \left( (m^\mu_\mu + \varphi^{-1}, -\nu_a) + 2m^{\lambda + \varphi^{-1}, -\nu_a'} a + (\sum_{b \in A} m^{\lambda + \varphi^{-1}, -\nu_a} b w_b) \right) e^{-\kappa_\mu}
\]

\[
= \sum_{z = a t_{-\mu}} m^x_z a e^{-\kappa_\mu} + \sum_{a \in A, \mu \in P} m^x_z a w_0 e^{-\kappa_\mu}
\]

(3.27)

where the weights \( \nu_a, \nu_a' \in P \) can be read from (3.25), and the polynomials \( P^\mu_\mu = \frac{\chi_\mu}{\varphi(\mu)} \) are invariant under the nonshifted action of \( \bar{W} \). The weights \( \mu_{b,a} \in P_+ \) result from the resummation of the ‘odd’ term in (3.25), setting \( w_j a = b w_0 t_{-\mu_{a,b}}, b \in A \) (since \( \bar{W} = \mathcal{U} t_\mathcal{P} \)), which allows comparing the second and the third lines in (3.27) to get an expression for the multiplicities \( m^x_z \) in terms of the \( sl(3) \) ones.

Let us give some examples

\[
\chi_a = a, \quad a \in A,
\]

\[
\chi_{w_0} = 2 + w_0 + w_1 + w_2 = 2 + F,
\]

\[
\chi_f \equiv \chi_{t_{-\lambda_1}} = \chi_{\lambda_1} + (1 + F) \gamma^{-1} = \chi_{\gamma^{-1} \chi_{w_2}} = \gamma^{-1} (w_2 + w_1 + w_0 + w_0 + w_1 + w_2 + 1),
\]

(3.28)

\[
\chi_{f^*} \equiv \chi_{t_{-\lambda_2}} = \chi_{\lambda_2} + (1 + F) \gamma = \chi_\gamma \chi_{w_{10}} = \gamma (w_{10} + w_{21} + w_{02} + w_0 + w_1 + w_2 + 1).
\]
Finally we define dimensions of the ‘finite dimensional modules’ by setting in (3.16) every (generating) element \( \varpi e^{-\kappa_\mu} \) in \( \mathbb{Z}[\tilde{W}] \) to 1, i.e., for \( y = \tilde{y} t_{-\lambda} \in \tilde{W}^\vee \)

\[
D_y = \sum_{z \in \tilde{W}} m^y_z.
\]

(3.29)

Using the decomposition formula (3.25) the dimension can be expressed as a sum of dimensions \( D_{\lambda} \) of \( \mathfrak{sl}(3) \) finite dimensional representations – the final expression can be cast into the form

\[
D_y = 9 \prod_{\alpha > 0} \left( \langle \tilde{y}(\lambda) + \frac{\varpi}{3}, \tilde{y}(\alpha) \rangle + \frac{\det(\tilde{y})}{3} \right) \nonumber
\]

\[
= \frac{1}{3} \prod_{\alpha > 0} \left( \langle \nu(y) + \varpi, \alpha \rangle + \frac{\det(\tilde{y})}{3} \right) = \frac{2}{3} D_{\nu(y)} + \frac{\det(\tilde{y})}{3}.
\]

(3.30)

Since \( \nu(a) = 0, \quad a \in A \) we have \( D_y = D_{a y} \). Here are the first few numbers produced from (3.30): 1, 5, 7, 19, 23, 43, 83, 103, ....

Remark. In the case \( \mathfrak{g} = \mathfrak{sl}(2) \) we have \( \nu(W) \equiv Q = 2P \) and \( \tilde{W}^\vee \equiv A t_{-P_+} \). The supports of the ‘Verma modules’ and the weight diagrams \( G_y \) are isomorphic to the supports of \( \mathfrak{g} = \mathfrak{sl}(2) \) Verma modules \( V_{\nu(y)} \) and weight diagrams \( \Gamma_{\nu(y)} \), respectively. Alternatively the map \( \frac{1}{2} \nu \) recovers the supports of modules of highest weight \( \frac{1}{2} \nu(a t_{-\lambda}) = \lambda \in P_+ \) of the superalgebra \( \mathfrak{osp}(1|2) \), with \( a = 1, \gamma \), labelling the two types of modules for a given \( \lambda \). One has \( \chi_{a t_{-\lambda}} = a(\chi_\lambda + \gamma \chi_{\lambda - \alpha/2}) \) and we can replace \( \gamma \) simply by a sign \( \epsilon = -1 \), thus recovering the supercharacters of the finite dimensional representations of \( \mathfrak{osp}(1|2) \). The formula (3.30) is replaced by \( D_y = \langle \nu(y) + \varpi, \alpha \rangle = 2 \langle \lambda + \frac{\varpi}{2}, \alpha \rangle, \quad y = a t_{-\lambda} \).

4. Character ring and its structure constants

Denote by \( \tilde{\mathfrak{m}} \), ring of ‘characters of finite dimensional modules’, the subring of \( \mathbb{Z}[\tilde{W}] \) generated by the set of characters \( \{ \chi_y | y \in \tilde{W}^\vee \} \) with \( \chi_y \) defined in (3.16). Thus the multiplication in \( \tilde{\mathfrak{m}} \) is inherited from the multiplication in the group ring \( \mathbb{Z}[\tilde{W}] \). The main aim of this section is to obtain a formula for the structure constants of the ring.

Proposition 4.1. The ring \( \tilde{\mathfrak{m}} \) is a commutative ring.

Proof: The statement follows from the fact that \( A \) is commutative and commutes with \( F \), and (3.3) is equivalent to \( w \tilde{\chi}_\lambda w^{-1} = \tilde{\chi}_\lambda \), i.e., the ordinary characters commute with elements of \( \tilde{W} \).
Next we have

**Lemma 4.2**

\[
\begin{align*}
\overline{X}_1 x_y &= \sum_{i=1,2,3} \chi_{t-e_i} y, \\
\overline{X}_2 x_y &= \sum_{i=1,2,3} \chi_{t+e_i} y, \\
F x_y &= \sum_{j=0,1,2} \chi_{w_j} y.
\end{align*}
\] (4.1)

**Proof:** The first two of these equalities (in which \(\sum_i e_i = 0, e_1 = \Lambda_1, e_3 = -\Lambda_2\)) follow from the decomposition (3.25) of \(\chi_y\) in \(sl(3)\) characters and the analogous multiplication rules of the latter with the characters of the fundamental representations \(\Lambda_i\); recall that \(\chi_{\Lambda_1} = \sum_i e^{-\kappa e_i}\). The derivation of the third is based on the straightforward relation

\[
F^2 = 3 + \gamma \overline{X}_{\Lambda_1} + \gamma^{-1} \overline{X}_{\Lambda_2},
\] (4.2)

and the use of (3.19) and the first two of the equalities in (4.1), taking into account the splitting of (3.25) into ‘even’ and ‘odd’ part, \(\chi_y = \chi_y^{(+)} + F \chi_y^{(-)}, \chi_y^{(\pm)}\) being linear combinations of \(A\) with \(sl(3)\) characters as coefficients (or, elements in the group ring \(W[A]\) of \(A\), over the ring \(W\) of \(sl(3)\) characters). □

The quantities in (4.1) are the basic ingredients of the examples (3.28) in the previous section and thus from the above Lemma and from (3.19) we have

**Corollary 4.3**

\[
\begin{align*}
\chi_\gamma x_y &= \chi_\gamma y, \\
\chi_{w_0} x_y &= 2 \chi_y + \chi_{w_0} y + \chi_{w_1} y + \chi_{w_2} y = \sum_{x \in G_{w_0}} m_x^{w_0} \chi_x y = \sum_{z \in G_{w_0} y} m_x^{w_0} y, \chi_z, \\
\chi_f x_y &= \chi_f y + \chi_{w_1} f y + \chi_{w_2} f y + \chi_{w_{12}} f y + \chi_{w_{012}} f y + \chi_{w_{021}} f y + \chi_{w_{201}} f y, \\
&= \sum_{z \in G_f} \chi_{z y} = \sum_{z \in G_{f y}} \chi_z, \quad f = t_{-\Lambda_1},
\end{align*}
\] (4.3)

\[
\begin{align*}
\chi_{f^*} x_y &= \chi_{f^*} y + \chi_{w_2} f^* y + \chi_{w_{12}} f^* y + \chi_{w_{021}} f^* y + \chi_{w_{012}} f^* y + \chi_{w_{012}} f^* y, \\
&= \sum_{z \in G_{f^*}} \chi_{z y} = \sum_{z \in G_{f^* y}} \chi_z, \quad f^* = t_{-\Lambda_2}.
\end{align*}
\]

These Pieri type formulæ hold for generic \(y\), in general there could be cancellations on \(KW\) orbits due to (3.17).

In (4.3) the shifted weight diagram \(G_f y\), consisting generically of 7 points, appears, thus we recover the multiplication rule of the ‘fundamental’ representation \(f\) obtained in [9]. We recall that it was found solving the null-decoupling equations resulting from a pair
of singular vectors of weight \( w_{f(\alpha_i)} \cdot \Lambda, \ i = 1, 2 \) (i.e., \( w_{\delta+\alpha_i} \cdot \Lambda, \ w_{\alpha_2} \cdot \Lambda \)) in the \( \mathfrak{g} \) Verma module of highest weight \( \Lambda = t_{-k_1} \cdot k \Lambda_0 \). Similarly the general property (3.19) (first line in (4.3)) was confirmed in [9] in the case \( sl(3) \) analysing the decoupling conditions corresponding to singular vectors of weight \( w_{-\gamma(\alpha_i)} \cdot \Lambda \), i.e., \( w_0 \cdot \Lambda, \ w_2 \cdot \Lambda \), in the \( \mathfrak{g} \) Verma module of h.w. \( \Lambda = \gamma \cdot k \Lambda_0 \).

The second of the product rules in (4.3) appeared in [9] as a consequence of a conjectured general Weyl-Steinberg type formula for the structure constants. We shall now prove that this formula indeed holds for the multiplication of the characters constructed in the previous section. We proceed in full analogy with the proof of its standard \( sl(3) \) analog. First we establish

**Lemma 4.4** For any \( x, y \in \mathcal{W}^{(+)} \) we have

\[
\chi_x \chi_y = \sum_{z \in W} m_x^z \chi_{zy} = \sum_{z \in W \cap G_x y} m_{zy}^x \chi_z. \tag{4.4}
\]

The formula (4.4) is the analog of the \( sl(3) \) geometric algorithm for the decomposition of the product, namely one takes the generalized ‘weight diagram’ \( G_x \) and ‘translates’ it by \( y \). The proof of Lemma 4.4 relies on the decomposition (3.23) of the characters \( \chi_y \) in terms of ordinary characters \( \chi \). (Below we will give an alternative proof.) Recall that the proof of the standard analogs of (4.4) is based on the invariance of the multiplicities \( m^{\mu}_{\mu}(\mu) = m_{\mu}^{\mu} \) (3.3) under the ordinary (unshifted) action of \( \hat{W} \), so that

\[
\overline{\chi}_\lambda = \sum_{\mu \in P_+} \overline{m}_{\mu}^{\mu} \sum_{\pi \in \hat{W}} e^{-\kappa_{\mu}(\mu)} \chi_{\pi(\mu) + \lambda}, \tag{4.5}
\]

Let us now prove Lemma 4.4.

**Proof:** Using that \( \sum_{\pi \in \hat{W}} e^{-\kappa_{\mu}(\mu)} \chi_{\pi(\mu)} \) commutes with any \( w \in \hat{W} \) (4.3) implies

\[
\sum_{\pi \in \hat{W}} e^{-\kappa_{\mu}(\mu)} \chi_y = \sum_{\pi \in \hat{W}} \chi_{\pi^{-1}(\mu)} y. \tag{4.6}
\]

Now we start from the decomposition (3.27) for \( \chi_x \) and we insert (4.6) in \( \chi_x \chi_y \) using furthermore (4.3),

\[
\chi_x \chi_y = \sum_{\mu \in P} P_{\mu}^{\lambda} e^{-\kappa_{\mu}} \chi_y = \sum_{\mu \in P} \sum_{\pi \in \hat{W}} P_{\mu}^{\lambda} e^{-\kappa_{\mu}(\mu)} \chi_y = \sum_{\mu \in P} \sum_{\pi \in \hat{W}} \chi_{\pi^{-1}(\mu)} y = \sum_{\mu \in P} P_{\mu}^{\lambda} \chi_{t^{-\mu} y} \tag{4.7}
\]

\[
= \sum_{\alpha \in A, \mu \in P} \left( (m_{\mu}^{\lambda} + \overline{W}^{-1}(-\nu) + 2m_{\mu}^{\lambda} + \overline{W}^{-1}(-\nu') \sum_{j=0,1,2} \chi_{w_{\alpha,at^{-\mu} y}} \right)
\]

\[
= \sum_{\alpha \in A, \mu \in P} \left( (m_{\mu}^{\lambda} + \overline{W}^{-1}(-\nu) + 2m_{\mu}^{\lambda} + \overline{W}^{-1}(-\nu') \chi_{at^{-\mu} y} + \sum_{b \in A} m_{\mu+b \cdot \alpha}^{\lambda} + \overline{W}^{-1}(-\nu') \chi_{awq t^{-\mu} y} \right),
\]

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repeating the resummation in the second line of (3.27); it remains to use the last line in (3.27) to recover (4.4). □

The lemma extends to \( x, y \in \mathcal{W}^{(+)}. \) using (3.19), (3.20). Next we have

**Proposition 4.5** Let \( x \in a W^{(+)}, y \in a' W^{(+)}, a, a' \in A. \) Then

\[
\chi_x \chi_y = \sum_{z \in aa' W^{(+)}} N_{x,y}^z \chi_z, \quad (4.8)
\]

\[
N_{x,y}^z = \sum_{w' \in \overline{W}[z]} \det(w') m_{w', z y^{-1}}^x = \sum_{w \in \overline{W}} \det(w) m_{z^{-1} w y^{-1}}^x = \sum_{w \in \overline{W}} \det(w) m^{(x)}_{\iota(z w y^{-1})}. \quad (4.9)
\]

Note that the summation in (1.8) runs effectively over the shifted weight diagram \( \mathcal{W}^{(+)} \cap G_{x,y} \) (of 'shifted highest weight' \( xy \)) since from the expression of the structure constants \( N_{x,y}^z \) it follows that \( z w \in G_{x,y} \) for any \( w \in \overline{W}. \) To make contact with the notation in [9], where we have used the horizontal projections of the weights \( \Lambda_y = y \cdot k \Lambda_0, \) note that the 'shifted highest weight' was denoted in [9] by \( \Lambda_x \circ \Lambda_y = h(x) \circ h(y) := h(x) + \bar{x}(h(y)) \) which coincides, according to (2.8), with \( h(xy). \)

**Proof:** Using Lemma 4.4 it remains to account for cancellations on KW orbits due to (3.17), using that \( \tilde{W}^{(+)}. \) is a fundamental domain in \( \tilde{W} \) with respect to the KW Weyl group. □

Finally we prove another property announced in [1]. The structure constants \( N_{x,y}^z \) of the character ring \( \tilde{W} \) can be expressed by the structure constants of the \( sl(3) \) character ring \( W \) through the map \( \iota: \)

**Proposition 4.6**

\[
N_{x,y}^z = \tilde{N}_{\iota(x) \iota(y), \iota(z)}. \quad (4.10)
\]

**Proof:** Using (2.3) we have \( \iota(z w y^{-1}) = \overline{y}(\iota(z w)) + \iota(y^{-1}) = \overline{y}(\iota(z w) - \iota(y)). \) Hence \( m_{z^{-1} w y^{-1}}^x = m^{(x)}_{\iota(z w y^{-1})} = m^{(x)}_{\overline{y}(\iota(z w) - \iota(y))} = m^{(x)}_{\overline{y}(\iota(z) - \iota(y))}. \) (using (2.10) in the last step), which inserted in (1.9) converts it into the classical Weyl–Steinberg formula for the \( sl(3) \) tensor product multiplicities \( \tilde{N}_{\iota(x) \iota(y)}. \) □

**Corollary 4.7** The structure constants \( N_{x,y}^z \) are nonnegative integers.

---

2 Strictly speaking the summation over \( \overline{W} \) in the first line of (4.7), which is rather a summation over orbits, should contain an additional factor for weights \( \mu \) with nontrivial stationary subgroup of \( \overline{W}; \) the same remark applies to the summation in the last line before (4.3).
We introduce an involution in $\hat{W}$ induced by the $\mathbb{Z}_2$ automorphism of the $\mathfrak{g}$ Dynkin diagram $\alpha_i \to \alpha_i^* := \alpha_{n-i}$, $i = 1, 2, \ldots, n-1$, according to $w_{\alpha_i}^* = w_{\alpha_i^*}$, $t_\lambda^* = t_{\lambda^*}$, $\langle \lambda^*, \alpha_i \rangle = \langle \lambda, \alpha_i^* \rangle$, and then $\iota(x^*) = (\iota(x))^*$. The involution extends to an automorphism of the ring $\hat{W}$ with $\chi_y^* = \chi_{y^*}$. Proposition 4.6 implies the standard properties of the structure constants

$$N_{x,y} = \delta_{x,y^*}, \quad N_{x,y}^z = N_{x,z^*} = N_{z^*,y}^z,$$  \hfill (4.11)

along with

$$N_{x,y}^z = N_{ax,y}^z, \quad a \in A. \tag{4.12}$$

Remark. The ‘classical’ dimensions $D_x = D_{x^*}$ (3.29) provide a numerical realisation of the product rule (4.8)

$$D_x D_y = \sum_z N_{x,y}^z \ D_z = \sum_z \bar{N}_{\iota(x) \iota(y)}^z \ D_z.$$ The ring $\hat{W}$ has a set of ‘fundamental’ characters that generate it as a polynomial ring. One possibility for this fundamental set is $\{\chi_{w_0}, \chi_f, \chi_{f^*}, \chi_\gamma = \gamma\}$. The group $A$ is like a set of simple currents or in the language of rings it is a group of units of the ring $\hat{W}$. Thus $\hat{W} = W[A]$ where $W$ is the triality zero subring $W$ of $\hat{W}$ having as a fundamental set $\{\chi_{w_0}, \chi_{w_20}, \chi_{w_10}\} = \{\chi_{w_0}, \gamma \chi_f, \gamma^{-1} \chi_{f^*}\}$. It is convenient to introduce the notation $f_0 = \chi_{w_0}$, $f_1 = \chi_{w_20}$, $f_2 = \chi_{w_10}$. The weight diagrams $G_{f_j}$, $j = 0, 1, 2$ are depicted on fig. 7. Since $\bar{\chi}_{A_1} = \gamma^{-1} \sum_{a \in A} a \gamma a^{-1}$ and $\bar{\chi}_{A_2} = \gamma \sum_{a \in A} a \gamma^{-1} a^{-1}$ the relations (4.1) read equivalently

$$\sum_{a \in A} a x a^{-1} \chi_y = \sum_{a \in A} \chi_{axa^{-1}y},$$

for any $y \in \hat{W}$ and $x = w_0, w_{10}, w_{20}$. Similarly the Pieri type formulæ (4.3) can be rewritten for the triality zero counterparts $\{f_j, j = 0, 1, 2\}$. With the shorthand notation $M_0^u = m_{w_0}^u$, $M_1^u = m_{w_20}^u$, $M_2^u = m_{w_{10}}^u$, $u \in W$, we have

$$f_j \chi_y = \sum_{u \in G_{f_j}} M_j^u \chi_{uy} = \sum_{z \in G_{f_j}} M_j^{z_{y^{-1}}} \chi_z.$$  \hfill (4.13)

Before we proceed with the next proposition we introduce on the chambers $W^{(+)}$ and $\hat{W}^{(+)}$ a filtration and related gradation using the reduced length of words $W_{\leq k}^{(+)} = \{x \in W^{(+)} : \ell(x) \leq k\}$ and $W_{= k}^{(+)} = \{x \in W^{(+)} : \ell(x) = k\}$.  

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Lemma 4.8 Let \( x \in \mathcal{W}_{=k}^{(+)} \) then the sets \( \mathcal{G}_f, x \), for \( i = 1, 2 \), have a single element in \( \mathcal{W}_{=k+2}^{(+)} \) while the rest are in \( \mathcal{W}_{\leq k+1}^{(+)} \).

Proof: The statement is proved by a direct check, cf. figs.1,7, using (4.13). \( \square \)

Proposition 4.9 The ring \( \mathfrak{W} \) is generated as a polynomial ring by \( \chi_{w_0}, \chi_{w_{10}}, \chi_{w_{20}} \), subject to one algebraic relation

\[
\chi_{w_0} \chi_{w_0} = 2 \chi_{w_0} + 1 + \chi_{w_{10}} + \chi_{w_{20}}.
\]

Proof: We want to show that any \( \chi_y, y \in \mathcal{W}^{(+)} \), can be represented as

\[
\chi_y = \sum_{\varepsilon=0,1} \sum_{n_1,n_2 \in \mathbb{Z}_{\geq 0}} c_{\varepsilon,n_1,n_2}^y f_0 \prod_{i=1}^{n_1} f_{1,i} \prod_{j=1}^{n_2} f_{2,j}, \quad c_{\varepsilon,n_1,n_2}^y \in \mathbb{Z}. \tag{4.15}
\]

First note that \( \mathcal{W}_{\leq 1}^{(+)} = \{1, w_0\} \), and \( \mathcal{W}_{=2}^{(+)} = \{w_{10}, w_{20}\} \). Using Lemma 4.8 and (4.13), (4.14) we see that these polynomials are determined inductively going up the gradation. \( \square \)

Due to the relation (4.14) the ring can be generated also by only two fundamental characters, i.e., either by \( f_0, f_1 \) or by \( f_0, f_2 \).

Using (4.13) and (4.14) one can give an alternative proof of Lemma 4.4:

Proof: Let us do it for \( \mathfrak{W} \). By Proposition 4.9 we have

\[
\chi_x = \sum_{\varepsilon=0,1} \sum_{n_1,n_2 \in \mathbb{Z}_{\geq 0}} c_{\varepsilon,n_1,n_2}^x f_0 \prod_{i=1}^{n_1} f_{1,i} \prod_{j=1}^{n_2} f_{2,j}
\]

\[
= \sum_{\varepsilon=0,1} \sum_{n_1,n_2 \in \mathbb{Z}_{\geq 0}} c_{\varepsilon,n_1,n_2}^x \sum_{\{u,\{u_{i,j}\}\}} (M_u^0)^{\varepsilon} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} M_{u_{i,j}}^i u_{1,1} \ldots u_{1,n_1} u_{2,1} \ldots u_{2,n_2},
\]

using (3.16) for each \( f_j \) with the notation for the multiplicities as in (4.13). Note that all of the above sums are actually finite sums. Writing \( \chi_x = \sum_z m_z^x z \) we get

\[
m_z^x = \sum_{n_1,n_2 \in \mathbb{Z}_{\geq 0}} (c_{0,n_1,n_2}^x \sum_{\{u_{i,j}\}} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} M_{u_{i,j}}^i + c_{1,n_1,n_2}^x \sum_{\{u,\{u_{i,j}\}\}} M_u^0 \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} M_{u_{i,j}}^i),
\]

where the two internal sums are over subsets \( \{u_{i,j}\} \) and \( \{u,\{u_{i,j}\}\} \) of \( W \) such that \( u_{1,1} \ldots u_{1,n_1} u_{2,1} \ldots u_{2,n_2} = z \) and \( uu_{1,1} \ldots u_{1,n_1} u_{2,1} \ldots u_{2,n_2} = z \) respectively. Using repeatedly associativity together with the Pieri type formulæ (4.13), e.g.,

\[
(\sum_{u_{i,j}} M_{u_{i,j}}^i u_{i,j}) \chi_{u_{i,j}u_{i,j+1} \ldots y} = \sum_{u_{i,j}} M_{u_{i,j}}^i \chi_{u_{i,j}u_{i,j+1} \ldots y},
\]

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we are done. □

Remark. The ring $\hat{\mathfrak{W}}$ can be looked as a ‘quadratic’ extension of the ring of $sl(3)$ characters $\hat{\mathfrak{M}}$ – a subring of $\hat{\mathfrak{W}}$ according to (3.28). I.e., $\hat{\mathfrak{W}}$ is a polynomial ring with coefficients in $\hat{\mathfrak{M}} := \hat{\mathfrak{M}}[A]$ modulo a quadratic relation, $\hat{\mathfrak{W}} = \hat{\mathfrak{M}}[F]/\{F^2 - C = 0\}; \ C \in \mathfrak{W}$ is given by the r.h.s. of (1.2). Here $\mathfrak{W} \subset \hat{\mathfrak{M}}[A]$ is a triality zero subring of $\hat{\mathfrak{W}}$. Similarly $\mathfrak{W} = \hat{\mathfrak{W}}[F]/\{F^2 - C = 0\}.$

5. Alcove of admissible weights at level $k + 3 = 3/p$.

Beginning with this section we will analyse the case of $g = \hat{\mathfrak{sl}}(3)_k$ at rational levels, namely $\kappa = k + 3 = 3/p$ with $p \in \mathbb{Z}_{\geq 2}\backslash 3\mathbb{Z}$. This selects a subseries of the general set of admissible weights of $[12]$ which we will describe in more detail. \footnote{This subset is generic since it is expected that as in the $\hat{\mathfrak{sl}}(2)$ case there is an effective factorisation of the FR multiplicities for the general admissible representations at $\kappa = p'/p$ into the multiplicities for the two subseries – at $\kappa = n/p$ and the integrable one at $\kappa = p'$, the former represented by the r.h.s. of (1.4), which extends to arbitrary $p \in \mathbb{Z}_{\geq 1}$.}

The rationality of $\kappa$ has two consequences – for $y \in \hat{W}$ the map $y \mapsto y \cdot k\Lambda_0$ for $y \in \hat{W}$ is not injective and the KW groups are isomorphic to the affine Weyl group $W$. Let $\Pi^{[p]} = \{\alpha^{[p]} = p\delta - \theta\} \cup \Pi_1$, and $\Delta_{++}^{re} = \Delta_+ \cup \{mp\delta + \alpha, \alpha \in \Delta, m \in \mathbb{Z}_{>0}\}$. Denote by $W^{[p]}$ the isomorphic to $W$ subgroup of $W$, generated by the reflections $\{w_\alpha, \alpha \in \Pi^{[p]}\}$. We have $\hat{W}^{[p]} = t_p^\Lambda \times \mathfrak{W}$. The subgroup $A^{[p]}$ of $\hat{W}$ generated by $\gamma^{[p]} := t_p \lambda_1 \gamma = t_{(p-1)\lambda_1} \gamma$ keeps invariant the set $\Pi^{[p]}$ and hence $a W^{[p]} a^{-1} = W^{[p]}$ for $a \in A^{[p]}$. We have $\hat{W} = W \rtimes A^{[p]}$ and for $\kappa = 3/p$

$$A^{[p]} \cdot k\Lambda_0 = k\Lambda_0. \quad (5.1)$$

Let $y \in \hat{W}$ and $\mathcal{P} = \{\Lambda = y \cdot k\Lambda_0 | y \in \hat{W}\}$. From the Kac-Kazhdan condition and from the analog of (2.3) with $\alpha \in \Pi^{[p]}$ it is clear that if $y(\alpha) \in \Delta_{++}^{re}, \forall \alpha \in \Pi^{[p]}$ the reflections $\{w_{y(\alpha)}, \alpha \in \Pi^{[p]}\}$ generate a KW group $W^{[\Lambda]}$ (to be denoted also $W^{[y]}$) such that its shifted action on $\mathcal{P}$ gives the weights of the Verma submodules of $M_\Lambda$. As in (2.3) the shifted action of $W^{[y]}$ on the weights in $\mathcal{P}$ is intertwined with the right action of $W^{[p]}$ on $\hat{W}$. Moreover $M_\Lambda$ is a maximally reducible Verma module with infinitely many singular vectors. Hence we are led to the definition of the alcove of admissible weights as $\mathcal{P}_{+,p} = \hat{W}_p^{(+)} \cdot k\Lambda_0 = W_p^{(+)} \cdot k\Lambda_0$ where

$$\hat{W}_p^{(+)} = \{y \in \hat{W} | y(\Pi^{[p]}) \subset \Delta_{++}^{re}\} \quad \text{and} \quad W_p^{(+)} = \hat{W}_p^{(+)} \cap W. \quad (5.2)$$
Denote $P_{+,p} = \{ \lambda \in P_{+,p} \mid \langle \lambda, \theta \rangle < p \text{ if } w(-\theta) < 0 \text{ or } \langle \lambda, \theta \rangle \leq p \text{ if } w(-\theta) > 0, w \in \widehat{W} \}$. In particular $P_{+,p}^{(1)}$ coincides with the integrable alcove $P_{+,k}$ at level $k = p - 1$. It is easy to see that the definition of $\tilde{\mathcal{W}}_{p}^{(+)}$ is equivalent to

$$\tilde{\mathcal{W}}_{p}^{(+)} = \{ y = 7 y t_{-\lambda} \in \widehat{W} \mid \lambda \in P_{+,p}^{(7)} \}$$

$$= A t_{-p_{+}^{p-1}} \cup A w_0 t_{-p_{+}^{p-2}} = t_{-p_{+}^{p-1}} A[p] \cup w_0 t_{-p_{+}^{p-2}} A[p]$$,

$$\mathcal{W}_{p}^{(+)} = \{ y = 7 y t_{-\lambda} \in W \mid \lambda \in P_{+,p}^{(7) \cap Q} \}, \quad \tilde{\mathcal{W}}_{p}^{(+)} = \cup_{a \in A[p]} \mathcal{W}_{p}^{(+) a}$$.

The second equality in (5.3), representing the alcove as a disjoint union of two leaves, parametrised by the two alcoves $P_{+,p}^{p-1}$ and $P_{+,p}^{p-2}$, takes into account the equivalence of elements in $\tilde{\mathcal{W}}_{p}^{(+)}$ implemented by the right action of the group $A[p]$, or, more explicitly,

$$t_{-\lambda} \gamma_{[p]} = \gamma t_{-\sigma_{[p]}^{-1}(\lambda)}, \quad \lambda \in P_{+,p}^{p-1},$$

$$w_0 t_{-\lambda} \gamma_{[p]} = \gamma^{-1} w_0 t_{-\sigma_{[p]}^{-1}(\lambda)}, \quad \lambda \in P_{+,p}^{p-2}.$$ (5.5)

Here $\sigma_{[k]}(\lambda) := \gamma(\lambda + k\Lambda_0) = w_{12}(\lambda) + k\Lambda_1$ denotes the automorphism of the alcove $P_{+,k}$ at integer level $k$ induced by the action of $A$. Alternatively, due to (5.1), the admissible alcove is parametrised by the elements of the fundamental domain $\mathcal{W}_{p}^{(+)}$ of $W$ (i.e., triality zero points on any orbit of $A[p]$ in $\mathcal{W}_{p}^{(+)}$), as indicated in (5.4).

In analogy with Lemma 2.2 one can show that $P = \left( \cup_{w \in \widehat{W}} w(P_{+,p}^{(w)}) \right) + pQ$ is a partition and hence one has that $\tilde{\mathcal{W}}_{p}^{(+)}$, respectively $\mathcal{W}_{p}^{(+)}$, is a fundamental domain in $\widehat{W}$, respectively $W$, for the right action of $W[p]$. Again the map $\iota$ intertwines the right action of $W[p]$ on $W$ with the action of the affine Weyl group at level $3p - 3$; it is sufficient to check, taking into account (2.10), that

$$\iota(y w_{p^3 - 3}) = w_0 \cdot (\iota(y) + (3p - 3)\Lambda_0).$$ (5.6)

Accordingly $\tilde{\mathcal{W}}_{p}^{(+)}$ is represented by a formula analogous to (2.9), with $P_{\pm}$ replaced by $P_{+,p}^{3p-3}$.

As an example we depict the alcove of admissible weights $P_{+,p}$, for $p = 5$, on fig. 8. It is parametrised by $\left\{ w t_{-\lambda} \in \mathcal{W}_{p}^{(+)} \mid w \in \{ e, w_{\theta} \} \right\}$ with a circle or box in case of $w = e$ or $w = w_{\theta} = w_0 t_{\theta}$ respectively, the numbers inside being the labels of $\lambda$. Equivalently, keeping only the triality zero labels $\lambda$, the same figure depicts the alternative representation of the admissible alcove through the elements of the fundamental domain $\mathcal{W}_{p}^{(+)}$. Unlike [9] the latter choice will be mostly used in what follows. Sometimes it will be also useful to
work with the full domain $\tilde{W}^{(+)}_p$ imposing the constraints implemented by the right action of $A^{[p]}$.

Define a triality preserving order 3 automorphism of $\tilde{W}$ (and hence of $W$)

$$\sigma_p(x) := \gamma x \gamma_{[p]} \gamma_p, \quad x \in \tilde{W},$$

$$\iota(\sigma_p(x)) = \sigma^{[3p-3]}_{[3p-3]}(\iota(x)). \quad (5.7)$$

Geometrically $\sigma_p$ fixes the ‘middle’ point of the alcove $W^{(+)}_p$, or $w_0 t_{-\frac{3p}{2}} = \omega \omega \omega$, cf. (5.3), and “rotates” it sending the “corners” into one another, i.e., it behaves like the usual ‘simple current’ automorphism of an integrable alcove.

6. Quantised ‘q’-characters

Recall first the integrable case where the ‘classical’ $g$ characters $\chi_\lambda$, $\lambda \in P_+$ are converted into $C$-valued ‘q’-characters, labelled by the set $\{\lambda \in P^k_+\}$ of integrable highest weights at (positive) integer level $k$. Essentially one turns the formal exponentials $e^\lambda$, $\lambda \in P$ into ‘true’ exponentials,

$$e^\lambda \rightarrow e^\lambda(\mu) := e^{-2\pi i \frac{\langle \lambda,\mu+p \rangle}{k+n}}, \quad \mu \in P. \quad (6.1)$$

This ‘quantises’ the ‘classical’ $g$ characters into ‘periodic’ characters, $\tilde{\chi}_\lambda(\mu) = \chi_{\lambda+kQ}(\mu)$, $h = k+n$, i.e., (skew-)invariant under the full affine Weyl group at level $k$,

$$\det(\Xi) \tilde{\chi}_\lambda(\mu) = \chi_{\lambda+kQ}(\mu) = \chi_{\lambda}(w \cdot (\mu + k\Lambda_0)), \quad w \in W, \quad (6.2)$$

so that we can restrict the ‘dual’ set (the set of $\mu$’s) to the integrable alcove $P^k_+$ itself.

The ‘q’-characters are given explicitly by a ratio $\tilde{\chi}_\lambda(\mu) = S_{\lambda\mu}/S_0\mu$ of matrix elements of the integrable modular matrix $S_{\lambda\mu}$, a unitary, symmetric matrix. It is recovered up to an overall constant by the second equality in (3.2), with $\kappa = -1$ and exponentials transformed as in (6.1), i.e., (3.2) turns into the Kac–Peterson formula [11]. Thus the complex numbers $\{\tilde{\chi}_\lambda(\mu), \mu \in P^{h-n}_+\}$ can be interpreted as eigenvalues of the matrix $N_{\lambda \mu}$ of fusion rule coefficients $N_{\lambda \mu}^{\beta \alpha} = N_{\lambda \mu}^{\beta \alpha}$ of the integrable WZW conformal models. This relates the Verlinde formula for $N_{\lambda \mu}^{\beta \alpha}$ to the classical Weyl-Steinberg formula [11],

\[\sum_{i=1}^n e_i = 0, e_i = \bar{\Lambda}_i - \bar{\Lambda}_{i-1}, \bar{\Lambda}_0 = 0 = \bar{\Lambda}_n.\]

---

4 Alternatively the ‘q’-characters are obtained restricting the standard group characters to the discrete subset of elements $\{\text{diag}(e^{2\pi i (e_i, \mu + p)}), i = 1, 2, \ldots, n), \mu \in P^{h-n}_+\}$ in the Cartan subgroup of $SU(n)$. Here $\sum_{i=1}^n e_i = 0, e_i = \bar{\Lambda}_i - \bar{\Lambda}_{i-1}, \bar{\Lambda}_0 = 0 = \bar{\Lambda}_n.
In what follows we shall also need $\chi^{(h)}_\lambda(\mu)$ for $\mu$ belonging to some of the shifted hyperplanes $H^{(h)}_\alpha := \{ \mu \in \bar{h}^* | \langle \mu + \rho, \alpha \rangle = l h \}$, $\alpha \in \bar{\Sigma}^+$, $l \in \mathbb{Z}$. While the Kac-Peterson formula has no sense on these hyperplanes, since both the numerator and the denominator vanish, the characters $\chi^{(h)}_\lambda(\mu)$ are well defined through the analog of the last equality in (3.2), or any of the standard determinant formulae for the classical $sl(3)$ characters.

Following the analogy with the integrable case the idea is to replace the affine Weyl group with the affine KW group at level $\kappa - 3 = 3/p - 3$, i.e., to extend the invariance (3.17) of the ‘classical’ characters with respect to the right action of the horizontal Weyl group $W$ to invariance with respect to the right action of the affine group $W^{[p]}$. This will lead to (1.2) with the structure constants given by the conjectured in [9] formula (1.3), which now derives from the ‘classical’ formula (4.9). Finally inverting (1.2) we will recover in section 7 the Pasquier–Verlinde type formula (1.1).

Apparently there are two problems to be solved. We have to find an analog of the discrete set $\{ \mu \in P^k_+ \}$ and furthermore the elements of the group algebra of $\tilde{W}$ have to be converted into some $C$-valued functions on this set.

Denote by $E_p$ the ‘double alcove’ region

$$E_p = \{ \mu \in P_+ | 0 \leq \langle \mu, \alpha_i \rangle \leq p - 1, i = 1, 2 \} = P^{p-3}_+ \cup (w_\theta(P^{p-3}_+) + (p - 2)\theta) \cup_{\alpha \in \bar{\Sigma}_+} (H^{(p)}_\alpha \cap P^{2p-2}_+) \subset P^{3p-3}_+ .$$

This set, which can be also looked as $P^{p-1}_+ \cup \{ w_\theta(P^{p+1}_+) + p\theta \}$, contains $p^2$ weights, $|E_p| \equiv |P_{+,p}|$, and we shall argue below that it is the analog for $k = 3/p - 3$ of the integrable ‘dual’ set $\{ \mu \in P^{k-3}_+ \}$, see figs. 9a, 9b, where $E_p$ is depicted for $p = 5$ and $p = 4$ (the dotted lines indicate the hyperplanes $H^{(p)}_\alpha$). For $p = 2$ $E_p$ consists of the alcove $P^1_+$ and the weight $(p - 1, p - 1) = (1, 1)$ and thus represents the $\mathbb{Z}_3$ factorisation of the integrable alcove at level $3p - 3$, $P^{3p-3}_+$, obtained after identifying the points $\sigma_{[3p-3]}^i(\lambda)$ along an orbit of the $\sigma$ automorphism of $P^{3p-3}_+$, including the $\sigma$ stable point $(p - 1, p - 1)$. For $p > 2$ this factorisation leads to a subset of the alcove $P^{3p-3}_+$ which is of cardinality $|E_p| + |P^{p-3}_+| > |E_p|$.  

We look for a solution of the invariance condition

$$\chi_{y w}(\cdot) = \det(w) \chi_y(\cdot), \quad w \in W^{[p]} ,$$

(6.4)

together with

$$\chi_{ya}(\cdot) = \chi_y(\cdot), \quad a \in A^{[p]}, \quad y \in \tilde{W}_p^{(+)} .$$

(6.5)
Accounting for the invariance of the characters with respect to the horizontal Weyl group \( W \) (3.17) the requirement (3.4) reduces to the periodicity condition

\[
\chi_y t_p \nu(\cdot) = \chi_y(\cdot), \quad \nu \in Q. \tag{6.6}
\]

The formula (3.25) for the characters \( \chi_y(\cdot) \) involves the three basic ingredients – the elements of the group \( A \), the \( sl(3) \) characters \( \chi_\lambda \), and the combination \( F \) in (3.24), so we have to give meaning to some \( \mathbb{C} \)-valued counterparts \( \gamma(\cdot) \), \( \chi_\lambda(\cdot) \), \( F(\cdot) \). The natural realisation for the generator of the group \( A \) – isomorphic to the cyclic group \( \mathbb{Z}_3 \), reads

\[
\gamma \rightarrow \gamma(\mu) := e^{2\pi i m \tau(\mu)} , \quad m = 1, 2, \text{ mod } 3 \tag{6.7}
\]

The periodicity requirement (6.6) suggests to look for a realisation of the \( sl(3) \) characters in (3.25) in terms of the integrable characters \( \chi_{\lambda}(\cdot) \), \( \lambda \in P \), determined for \( \gamma(\cdot) \), \( \chi_{\lambda}(\cdot) \), \( F(\cdot) \). We can choose

\[
\varepsilon_{\lambda, \mu} = e^{-\frac{2\pi i l \tau(\mu)}{3}}, \quad l = 1, 2, \text{ mod } 3 \tag{6.8}
\]

In (6.8) we have allowed for an arbitrary overall phase constant \( \varepsilon_{\lambda, \mu} \), invariant with respect to both indices under the shifted action of the affine Weyl group. We can choose

\[
\varepsilon_{\lambda, \mu} = e^{-\frac{2\pi i l \tau(\mu)}{3} \tau(\lambda)}, \quad l = 1, 2, \text{ mod } 3 \tag{6.9}
\]

which effectively leads to the realisation of the formal exponentials as

\[
e^{-\kappa \lambda} \rightarrow e^{-\kappa \lambda}(\mu) := e^{-\frac{2\pi i l \tau(\mu)}{3} \tau(\lambda)} e^{-\frac{2\pi i}{3} (\lambda, \mu + \rho)} \tag{6.10}
\]

The need for this phase is dictated by the requirement (5.5), which combined with (3.19), (5.5) reads for each of the parts \( \chi_y(\cdot) \) in \( \chi_y = \chi_y(+) + F \chi_y(-) \) (treating for the time being \( F(\mu) \) as a formal variable)

\[
\gamma(\mu) \chi_y(\cdot)(\mu) = \chi_{t-\sigma[p-1](\lambda)}^{(\pm)}(\mu) \quad \chi_{t-\sigma[p-3](\lambda-2\lambda_2)}^{(\pm)}(\mu) \quad \text{for } y = t-\lambda,
\]

\[
\gamma(\mu) \chi_y(\cdot)(\mu) = \chi_{w_0t-\sigma^2[p-2](\lambda)}^{(\pm)}(\mu) \quad \chi_{w_0t-\sigma^2[p-3](\lambda-\lambda_1)}^{(\pm)}(\mu) \quad \text{for } y = w_0 t-\lambda . \tag{6.11}
\]

The above conditions and the corresponding standard property of the integrable ‘\( q \)’-characters

\[
\chi_{\sigma[p-3]}(\lambda)(\mu) = e^{2\pi i \tau(\mu)} \chi_{\lambda}^{(p)}(\mu), \tag{6.12}
\]
fix the integer \( l \) to \( l = p \mod 3 \) (using that \( p^2 - 1 = 0 \mod 3 \)), and keeps arbitrary the power \( m \) in the phase in (6.7). Without lack of generality we can choose \( m = l = p \) since otherwise the remaining phases can be absorbed using the analogous to (6.12) symmetry with respect to the index \( \mu \),

\[
\chi^{(p)}_{\lambda}(\sigma_{[p-3]}(\mu)) = e^{2\pi i \tau(\lambda)} \overline{\chi^{(p)}_{\lambda}(\mu)},
\]

thus changing the value of \( \mu \) to \( \mu' = \sigma_{[p-3]}(\mu) \in P^p_+ \); we can do this since the three terms in each of \( \chi^y(\pm) \) are described by \( sl(3) \) characters of weights of different triality \( \tau = 0,1,2 \).

Now we turn to the operator \( F = w_0 + w_1 + w_2 \). We recall that it commutes with the elements of \( A \) as well as with the \( sl(3) \) characters. Preserving the relation (4.2) – which is the basic relation used to derive the character ring structure constants, we see that the square of \( F(\cdot) \) can be determined by the (fundamental) integrable characters, i.e.,

\[
F^2 \to F^2(\mu) := 3 + \chi^{(p)}_{\lambda_1}(\mu) + \chi^{(p)}_{\lambda_2}(\mu)
\]

for any \( \mu \in P \). This determines \( F(\mu) \) up to a sign, \( F(\mu) = \varepsilon(\mu) \sqrt{F^2(\mu)} \), \( \varepsilon(\mu) = \pm 1 \).

The r.h.s of (6.14) is equivalently reproduced by

\[
F^2(\mu) = |R(\mu)|^2,
\]

\[
R(\mu) = \sum_{\tilde{a} \in \tilde{A}} e^{-2\pi i \langle \tilde{a}(\theta), \mu + \overline{\rho} \rangle} = e^{-2\pi i \langle \theta, \mu + \overline{\rho} \rangle} + e^{2\pi i \langle \alpha_1, \mu + \overline{\rho} \rangle} + e^{2\pi i \langle \alpha_2, \mu + \overline{\rho} \rangle}.
\]

One has the relations

\[
i 3p\sqrt{3} S_0^{(3p)}(\mu) = 1/d_{\kappa/3}(\mu) = R(\mu) - \overline{R}(\mu),
\]

\[
i p\sqrt{3} S_0^{(p)}(\mu) = 1/d_{\kappa}(\mu) = \sum_{\bar{a} \in \bar{A}} e^{-\bar{a}(\theta)\kappa}(\mu) - \sum_{\bar{a} \in \bar{A}} e^{-\bar{a}(\theta)\kappa}(\mu) = (R(\mu))^3 - (\overline{R}(\mu))^3
\]

\[
= i 3p\sqrt{3} S_0^{(3p)} \left( R(\mu) + \overline{R}(\mu) - |R(\mu)| \right) \left( R(\mu) + \overline{R}(\mu) + |R(\mu)| \right).
\]

(Here \( \overline{R}(\mu) \) is the complex conjugation of \( R(\mu) \).)

It remains to determine the sign of \( \varepsilon(\mu) \). Since the parts \( \chi^y(\pm) \) in \( \chi_y = \chi^y(+) + F \chi^y(-) \), as well as \( F^2(\mu) \), coincide for \( \mu \) and its reflected images according to (6.12), we can assign \( \varepsilon(\mu) = 1 \) for \( \mu \in P^p_+ \) and \( \varepsilon(\mu) = -1 \) for \( \mu \) sitting on the ‘mirror’ (with respect to the hyperplane \( H^{(p)}_\theta \) alcove in \( E_p \)). On the intersection of \( E_p \) with the reflection hyperplanes \( H^{(p)}_\alpha \) we choose \( \varepsilon(\mu) = 1 \) for \( \alpha = \theta \), \( \varepsilon(\mu) = -1 \) for \( \alpha = \alpha_1, \alpha_2 \) and the justification of
this choice will become clear below. The domain $E_p$ splits into two disjoint subsets $E_p^{(\pm)}$, $E_p^{(\pm)} := P^{p-2}_\pm$, thus
\[ \varepsilon(\mu) := \pm 1 \text{ for } \mu \in E_p^{(\pm)}. \] (6.18)

Summarising we are led to the following expression for the quantised characters $\chi_y^{(p)}(\mu), y = \gamma t_{-\lambda} \in \mathcal{W}_p^{(+)}$:
\[
\chi_y^{(p)}(\mu) := e^{-\frac{2\pi i p}{3}} \tau(\mu) \tau(\lambda) \left[ \chi(p)_{\lambda+\gamma}^{-1,-1}(0) (\mu) + \chi(p)_{\lambda+\gamma}^{-1,-1,-2\Lambda_1}(\mu) + \chi(p)_{\lambda+\gamma}^{-1,-1,-2\Lambda_2}(\mu) \right] \\
+ (F(\mu) + 2) \left( \chi(p)_{\lambda+\gamma}^{-1,-1,-2\Lambda_2}(\mu) + \chi(p)_{\lambda+\gamma}^{-1,-1,-2\Lambda_2}(\mu) \right). \] (6.19)

For $y \in \mathcal{W}_p^{(+)}$ the overall phase in (6.19) disappears. Taking $\mu = 0$ we define ‘$q$’-dimensions $D_y^{(p)} := \chi_y^{(p)}(0)$ expressed by the ‘$q$’-dimensions of the integrable level $p - 3$ case.

**Proposition 6.1** Let $x, y \in \mathcal{W}_p^{(+)}$, $\mu \in E_p$. Then
\[
\chi_x^{(p)}(\mu) \chi_y^{(p)}(\mu) = \sum_{z \in \mathcal{W}_p^{(+)}} (p) N_{x,y}^{z} \chi_z^{(p)}(\mu), \] (6.20)

where
\[
(p) N_{x,y}^{z} = \sum_{w' \in W[x \cdot k \Lambda_0]} \det(w') \ m_{w',z} \cdot y^{-1} = \sum_{w \in W[p]} \det(w) \ m_{z,w} \cdot y^{-1} = \sum_{w \in W[p]} \det(w) \ N_{z,w}^{x,y}. \] (6.21)

Furthermore the equality (1.4) holds true.

**Proof:** Since the basic relations (4.1), (4.2) are conserved the map $\chi_y \rightarrow \chi_y^{(p)}(\mu)$ is a ring homomorphism, so (1.4) holds and it remains to use (6.4) to recover (6.20), (6.21). Finally the derivation of (1.4) parallels that of (1.11) using (1.6). \(\square\)

The statement extends to $\chi_y^{(p)}(\mu), y \in \mathcal{W}_p^{(+)}$. Given $y \in \mathcal{W}_p^{(+)}$ take $\gamma^m y \in \mathcal{W}_p^{(+)}$ with the appropriate $m$. Then $\chi_y^{(p)}(\mu) = e^{-\frac{2\pi i p}{3} m \tau(\mu)} \chi_{\gamma^m y}^{(p)}(\mu)$ and the product of characters $\chi_y^{(p)}(\mu), y \in \mathcal{W}_p^{(+)}$ reduces to (6.20), (6.21) due to the symmetry $(3p) \bar{N}_{a(x)}^{(a z)} = (3p) \bar{N}_{x}^{(a z)}, a \in A$, i.e., the symmetries (1.11), (1.12) extend to $(p) N_{x,y}^{z}$,
\[
(p) N_{x,y}^{z} = (p) N_{x,ay}^{az} = (p) N_{x,z}^{y^*} = (p) N_{x,z}^{y^*}, \quad (p) N_{x,y}^{1} = \delta_{x,y^*}. \] (6.22)

The action of the involution $*$ on the characters coincides with the complex conjugation
\[
\chi_y^{(p)}(\mu) = \chi_y^{(p)}(\mu^*) = \chi_y^{(p)}(\bar{\mu}) (= \chi_y^{(p)}(\mu)). \] (6.23)
The second equality follows from $\varepsilon(\mu) = \varepsilon(\mu^*)$ and the analogous equality for the integrable characters.

Using (3.7) the first relation in (6.22) can be also rephrased in terms of elements of $\mathcal{W}^{(+)}_p$ only, since $\chi^{(p)}_{\gamma y} = \chi^{(p)}_{\sigma_p(y)}(\sigma_p(y), y \in \mathcal{W}^{(+)}_p$, being the triality zero representative of $\gamma y \in \mathcal{W}^{(+)}_p$ on its $A_p$ orbit),

$$\chi^{(p)}_{\gamma y} = \chi^{(p)}_{\sigma_p(y)}.$$ (6.24)

The analogs of the basic examples in (3.28) read

$$\chi^{(p)}_{\gamma}(\mu) = e^{2\pi i p \tau(\mu)} = \chi^{(p)}_{t_{-\sigma_p(\mu)}}(\mu) = \chi^{(p)}_{\gamma} t_{-\sigma_p(\mu)} = \chi^{(p)}_{\gamma} + 2 + F(\mu) = \chi^{(p)}_{\gamma} + 1 + F(\mu) = \chi^{(p)}_{\gamma} + 1 + F(\mu).$$ (6.25)

In (6.25) we have expressed the characters in terms of the integrable characters $\chi^{(p)}_{\gamma(y)}(\mu)$ at (shifted) level 3. Since $E_p \subset P^{3p-3}$, taking $\mu \in E_p$ gives well defined expressions. On the hyperplanes $H^{(p)}_\alpha \cap E_p$ these characters reduce (up to a sign) to the corresponding integrable characters $\chi^{(p)}_{\gamma(y)}(\mu)$ at (shifted) level 3. Indeed one proves

**Lemma 6.2** Let $\mu \in E_p \cap \left( \cup_{\alpha \in \pi} H^{(p)}_\alpha \right)$. Then

$$r_{\varepsilon}(\mu) := R(\mu) + \bar{R}(\mu) - \varepsilon(\mu) |R(\mu)| = 0$$ (6.26)

for $\varepsilon(\mu)$ as in (6.18).

**Proof:** One easily checks that for $\mu \in E_p \cap \left( \cup_{\alpha \in \pi} H^{(p)}_\alpha \right)$ and $\varepsilon(\mu)$ chosen as in (6.18) $R(\mu)$ can be cast into the form $R(\mu) = -\varepsilon(\mu) e^{2\pi i p \tau(\mu)} |R(\mu)|$ which implies the lemma. □

The alternative expressions in (6.25) representing the characters $\chi^{(p)}_{\gamma(y)}(\mu)$ in terms of the integrable ‘$q$’-characters at level $3p - 3$ generalise to arbitrary $y \in \mathcal{W}^{(+)}_p$, $\mu \in E_p$. To simplify notation we shall omit the explicit dependence on $\mu$ denoting the overall phase in (6.19) by $\varepsilon_y$. Thus for any $y = \tilde{y} t_\lambda \in \mathcal{W}^{(+)}_p$ we obtain by straightforward computation using (6.13), (6.16), (6.17), (6.25),

$$\chi^{(p)}_y = \varepsilon_y \left( \chi^{(p)}_{\gamma} \chi^{(p)}_{\gamma + \pi} \right).$$ (6.27)
The second term in (6.27) admits also a representation entirely in terms of integrable ‘q’-characters $\chi^{(3p)}_{\nu}(\mu)$ at level $3p-3$, with weights $\nu \not\in \text{Im}(\iota)$, using that $\chi^{(3p)}_{\lambda+2\iota} = r_{\iota}^{-} r_{\iota}^{+} \chi^{(p)}_{\lambda}$.

From Lemma 6.2 and the relations (6.16), (6.17) it follows that $r_{\iota}^{-}(\mu) \neq 0$ for any $\mu \in E_p$.

Finally we can also cast (6.27) into the form

$$
\chi_y^{(p)}(\mu) = \varepsilon_y(\mu) d_{\kappa}(\mu) \left( (R^2 + FR)(\mu) \sum_{w \in \tilde{A}} e^{\frac{2\pi i}{3p} (w(y)+\overline{\tau}, \mu+\overline{\tau})} 
- (\overline{R}^2 + FR)(\mu) \sum_{w \in \tilde{A}} e^{\frac{2\pi i}{3p} (w(y)^*, \mu+\overline{\tau})} \right). 
$$

(6.28)

Lemma 6.2 and (6.27) imply

**Corollary 6.3** For any $y \in W^{(+)}_p$ and $\mu \in E_p \cap \left( \bigcup_{\alpha \in \Delta^+} H^{(p)}_{\alpha} \right)$,

$$
\chi_y^{(p)}(\mu) = \overline{\chi^{(3p)}_{\iota(y)}(\mu)}.
$$

(6.29)

Despite of the relation (1.4) between the structure constants the product of characters $\chi_y^{(p)}(\mu)$ differs in general from that of the integrable characters $\chi^{(3p)}_{\iota(y)}(\mu)$ at level $3p-3$ since the decomposition of the latter contains also terms $\chi^{(3p)}_{\lambda}(\mu)$ with $\lambda \not\in \text{Im}(\iota)$. On the other hand the equality (1.4) together with (6.29) – the latter property being enforced by the choice (6.18) of the sign of $F(\mu)$, require that on the intersection of the hyperplanes $H^{(p)}_{\alpha}$ with $E_p$, the product of the triality zero integrable characters at shifted level $3p$ has to reduce to that of the characters (6.19). Otherwise we run into contradiction, i.e., the choice of sign (6.18) will appear to be inconsistent. However it is easy to prove the above property of the standard integrable characters at level $3p-3$, thus justifying a posteriori the choice (6.18). Namely we have

**Lemma 6.4** For $\mu \in P_{+}^{3p-3} \cap \left( \bigcup_{\alpha \in \Delta^+, l \in \mathbb{Z}} H^{(1p)}_{\alpha} \right)$

$$
\chi^{(3p)}_{\iota(x)}(\mu) \overline{\chi^{(3p)}_{\iota(y)}(\mu)} = \sum_{\lambda \in \text{Im}(\iota)}^{(3p)} N_{\iota(x)\iota(y)}^{\lambda} \chi^{(3p)}_{\lambda}(\mu), \ x, y \in \tilde{W}_p^{(+)}.
$$

(6.30)

**Proof:** The proof of the Lemma reduces to the proof of the following property of the integrable characters at level $3p-3$, $p > 2$:

For $\mu \in P_{+}^{3p-3} \cap H^{(1p)}_{\alpha}$, $\alpha \in \Delta^+, l \in \mathbb{Z}$, and $\lambda \in P_{+}^{3p-3}$, $\tau(\lambda) = 0$, $\lambda \not\in \text{Im}(\iota)$,

$$
\overline{\chi^{(3p)}_{\lambda}(\mu)} = 0.
$$

(6.31)
If \( \tau(\lambda) = 0 \), and \( \lambda \notin \text{Im}(\iota) \) then \( \lambda + \overline{\rho} = 3\lambda' \), for some \( \lambda' \in P^{p-3}_+ + \overline{\rho}. \) Hence \( \overline{\chi}_\lambda^{(3p)}(\mu) = \frac{S_\lambda^{(p)}(\mu)}{3S_{\mu,\overline{\mu}}^{(3p)}} \) and (6.31) follows from the vanishing of \( S_\lambda^{(p)}(\mu) \) for \( \mu \in H_\alpha^{(lp)} \cap P^{3p-3}_+. \) □

Remark. The case \( p = 2 \) is degenerate (trivial) since the solutions of (6.20) coincide with the whole \( E_p \) and accordingly the triality zero points in \( P^{3p-3}_+ = P^3_+ \) are all in \( \text{Im}(\iota) \). Hence the characters (6.19) with \( y = 1, w_{20}, w_{10}, w_0 \) coincide with the corresponding integrable characters at level \( 3p - 3 = 3 \) – they realise the triality zero fusion subalgebra at this level labelled by \( \{ \lambda = (0, 0), (3, 0), (0, 3), (1, 1) \} \). Thus the \( \widehat{sl}(3)_k \) case \( \kappa = k + 3 = 3/2 \) is analogous to the \( \widehat{sl}(2)_k \) case at \( \kappa = k + 2 = 2/p \), \( p \) – odd, where the admissible ‘\( q \)' - characters \( \chi_y^{(p)}(\mu) = \overline{\chi}_{\iota(y)}^{(2p)}(\mu) \), \( y \in W^{(+)}_p \), close the integer isospin \( (\tau(\lambda) = 0) \) fusion subalgebra of the \( \widehat{sl}(2) \) integrable representations at shifted level \( 2p \); the representative \( W^{(+)}_p \) of the admissible alcove is defined as in (5.4), the latter formula being universal for any \( \widehat{sl}(n)_k \) and \( k + n = n/p \).

7. Pasquier–Verlinde type formula

We have found \( p^2 \) vectors \( \chi(\mu) = \{ \chi_y^{(p)}(\mu) ; y \in W^{(+)}_p \} \) with \( \mu \in E_p \), which according to (5.20) provide eigenvectors common to all fusion matrices \( N_y \), \( y \in W^{(+)}_p \), \( (N_y)_x^z = (p)N^{z}_{y,x} \), and for any \( y \) the numbers \( \chi_y^{(p)}(\mu) \) are eigenvalues of \( N_y \) labelled by the set \( E_p \).

Lemma 7.1 Let \( \mu, \mu' \in E_p \). If \( \chi(\mu) = \chi(\mu') \) then \( \mu = \mu' \).

Proof: Recall that the domain \( E_p \) splits into two disjoint subsets \( E^{(\pm)}_p \) each being a subset of a fundamental domain in \( P \) with respect to the shifted action of \( W \) at level \( p - 3 \).

From \( \chi_{f_j}^{(p)}(\mu) = \chi_{f_j}^{(p)}(\mu') \), \( j = 0, 1, 2 \) it follows that:

i) \( \varepsilon(\mu) = \varepsilon(\mu') \),

which implies that both \( \mu, \mu' \in E^{(+)}_p \), or \( \mu, \mu' \in E^{(-)}_p \),

ii) \( \overline{\chi}_\Lambda^{(p)}(\mu) = \overline{\chi}_\Lambda^{(p)}(\mu') \), \( i = 1, 2 \),

which implies that \( \mu' = w \cdot (\mu + (p - 3)\Lambda_0) \), \( w \in W \). Hence \( \mu = \mu' \). □

Following standard arguments and taking into account the properties (5.22) of the structure constants \( (p)N^z_{x,y} \) the Lemma immediately leads to:

Corollary 7.2

\[
\sum_{y \in W^{(+)}_p} \chi_y^{(p)}(\mu) \chi_y^{(p)*}(\mu') = 0, \quad \forall \mu, \mu' \in E_p \quad \mu \neq \mu',
\]  

(7.1)
and hence \( \{ \chi(\mu), \mu \in E_p \} \) is a linearly independent set of (common) eigenvectors.

Normalising the eigenvectors \( \chi(\mu) \) (recall that \( \chi^{(p)}_1(\mu) = 1 \))

\[
\psi^{(\mu)}_y = \chi^{(p)}_y(\mu) \psi^{(\mu)}_1, \quad \frac{1}{|\psi^{(\mu)}_1|^2} = \sum_{y \in W^{(p)}_p} |\chi^{(p)}_y(\mu)|^2, \quad (7.2)
\]

we can choose \( \psi^{(\mu)}_1 \) real positive, so that \( \psi^{(\mu)^*}_y = \psi^{(\mu^*)}_y = \psi^{(\mu)}_y \), (see also (3.23)). Due to (7.4) the square matrix \( \psi^{(\mu)}_y \) is nonsingular and hence both its column and row vectors are linearly independent. Thus we obtain a unitary matrix \( \psi^{(\mu)}_y \),

\[
\sum_{y \in W^{(p)}_p} \psi^{(\mu)}_y \psi^{(\mu')}_{y*} = \delta_{\mu,\mu'}, \quad \sum_{\mu \in E_p} \psi^{(\mu)}_y \psi^{(\mu')}_{x*} = \delta_{yx}, \quad (7.3)
\]

which diagonalises all \( N_y \). Indeed using the second (completeness) relation in (7.3) the formula (6.20) converts into (7.1), providing equivalent expression for the ‘q’ analog of the Weyl-Steinberg type formula (6.21). Hence we recover the Pasquier–Verlinde type formula for the fusion rule multiplicities of the admissible representations at level \( k + 3 = 3/p \) proposed in [4] with now explicitly determined eigenvector matrix \( \psi^{(\mu)}_y \).

A remaining technical problem is to perform explicitly the summation in (7.2). At least for \( \mu \in E_p \cap \left( \cup_{\alpha \in \Delta_+} H^{(p)}_\alpha \right) \) this can be easily done, getting an explicit expression for the constant \( \psi^{(\mu)}_1 \), and hence for the corresponding matrix elements of \( \psi^{(\mu)}_y \), for this particular subset of weights in \( E_p \). Indeed we have

**Lemma 7.3** Let \( y = \bar{\eta} t_{-\lambda} \in W^{(p)}_p \) and \( \mu \in E_p \cap \left( \cup_{\alpha \in \Delta_+} H^{(p)}_\alpha \right) \). Then

\[
\psi^{(\mu)}_y = \sqrt{3} S^{(3p)}_{\lambda(y) \mu}, \quad \mu + \bar{p} \neq \bar{p}; \\
\psi^{(\mu)}_y = S^{(3p)}_{\tau(\lambda) \mu}, \quad \mu + \bar{p} = \bar{p}.
\]

**Proof:** According to (7.2) and (6.29) it is sufficient to prove the statement for \( y = 1 \). From (6.23), (6.31), it follows that for \( \mu \in E_p \cap \left( \cup_{\alpha \in \Delta_+} H^{(p)}_\alpha \right) \) one has

\[
\sum_{y \in W^{(p)}_p} |\chi^{(p)}_y(\mu)|^2 = \sum_{\lambda \in P^{3p-3}_+} |\chi^{(3p)}_\lambda(\mu)|^2 = \frac{1}{(\sqrt{3} S^{(3p)}_{0 \mu})^2} \sum_{l=0}^2 \delta_{\mu,\sigma_{3p-3}(\mu)}.
\]

The last equality holds for any \( \mu \in P^{3p-3}_+ \) exploiting standard properties of the modular matrices \( S^{(3p)}_{\lambda \mu} \); see, e.g., [14]. Since the point \( \mu + \bar{p} = \bar{p} \) is a fixed point for the \( \sigma_{3p-3} \) automorphism, the factor \( \sqrt{3} \) does not appear in the second equality of (7.4). \[\square\]
According to the last Remark in the previous section in the case $p = 2$ the formulæ (7.4) describe all matrix elements of the eigenvector matrix $\psi^{(\mu)}_y$ and analogous formulæ (with the factor 3 substituted by 2) hold for the whole $sl(2)$ subseries at level $k + 2 = 2/p$.

We conclude with the remark that the character ring constructed here is an extension of the ring of integrable ‘$q$’-characters at shifted level $p$, with the two roots of the quadratic polynomial (6.14) of $F$. The latter characters are elements of the subring $\mathbb{Z}[\omega]$ of the cyclotomic extension $\mathbb{Q}[\omega]$ of the rational numbers for $\omega^{3p} = 1$, see [4].

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**Note added:**

The multiplication rule encoded in the product of the character $\chi_{w_0}$ with any $\chi_y$, see (4.3), has been reproduced – including the multiplicity two contribution, by an explicit solution of the singular vectors decoupling equations at generic level [17]. Thus filling a gap in the computations in [9] all Pieri type formulae (4.3) are now confirmed.
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fig. 1. Parts of the Cayley graph of $W$, the chamber $\mathcal{W}^{(+)}$ and the KW action on it.

fig. 2. The set $\mathcal{U}$.

fig. 3. The map $\iota$. 
fig. 4 The weight diagram $G_{\nu_{1210}}$ with weight multiplicities indicated in the circles.

fig. 5 The beginning of the 'Verma' support $\mathcal{V}_{t-\lambda}$ with multiplicities indicated in circles.

fig. 6 The beginning of the 'Verma' support $\mathcal{V}_{w_0 \cdot t-\lambda}$ with multiplicities in circles.
fig. 7. The supports of the three ‘fundamentals’.

fig. 8. The $p = 5$ alcove.
fig. 9a. The dual set $E_p$ for $p = 5$.

fig. 9b. The dual set $E_p \subset P_{+}^{3p-3}$ for the case $p = 4$. 