Growth of Random Trees by Leaf Attachment

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Abstract

We study the growth of a time-ordered rooted tree by probabilistic attachment of new vertices to leaves. We construct a likelihood function of the leaves based on the connectivity of the tree. We take such connectivity to be induced by the merging of directed ordered paths from leaves to the root. Combining the likelihood with an assigned prior distribution leads to a posterior leaf distribution from which we sample attachment points for new vertices. We present computational examples of such Bayesian tree growth. Although the discussion is generic, the initial motivation for the paper is the concept of a distributed ledger, which may be regarded as a time-ordered random tree that grows by probabilistic leaf attachment.

1 Introduction

In the context of this paper, a tree is an object in graph theory. In particular, we study a directed rooted tree (vertices joined by directed edges where one vertex is the root) that grows with time according to probabilistic rules. The motivation for such a study will be discussed below. In the first instance, we give a simple illustration of the growth of a directed rooted tree.

We adopt an approach to graph theory where visual representation often takes precedence over formal description. Accordingly, Figure 1 shows a sequence of ‘snapshots’ of a tree as it grows through attachment of new vertices to existing vertices (time increases to the right in each snapshot and a directed edge from one vertex to another represents attachment of the former to the latter). The tree has two vertex types:
a) deep vertex with one or more incoming attachments
b) leaf with no incoming attachments (root is also taken to be a leaf at the start).

A new vertex, created in the latest time interval, becomes part of the tree only after attaching to an existing vertex. The growth from a given root obeys the following rules:
Rule 1: Only new vertices may issue attachments, existing vertices are ‘inactive’.
Rule 2: A new vertex issues exactly one attachment (this can be generalised).
Rule 3: Leaf attachment: a new vertex may only attach to a leaf.
Rule 4: A leaf can receive multiple attachments from different new vertices created in the same time interval.
Rule 5: New vertices select attachment targets from a probability distribution over leaves.

It follows from the rules that attachments always point backward in time. The object of study is thus a time-ordered directed random tree that grows by probabilistic leaf attachment.

There are two logically distinct processes at play. The first is the creation of a number of new vertices in the current time interval. We can take this number to be drawn from a Poisson distribution, whose rate may be time-dependent. However, this is not a fundamental design feature. Alternative vertex creation mechanisms, deterministic or probabilistic, are permissible. The second process is the attachment of newly created vertices to leaves of the existing tree. It is the latter process that is of interest here. Accordingly, the problem statement is the following:

**Problem Statement**

This paper is devoted to Rule 5 – choosing a probability distribution over leaves from which new vertices select their target attachments. Attaching entirely “at random” is equivalent to choosing a uniform leaf distribution, regardless of the tree structure. The novelty of the paper is a Bayesian scheme to update any prior choice of leaf distribution, uniform or otherwise, by taking the historical connectivity of the tree into account.

We first present a motivation for the study.
1.1 Motivation

The initial inspiration for this paper is an innovation in computer science known as a distributed ledger. It is used to enable secure transactions over peer-to-peer networks (distributed networks with no central data repository or coordinating authority). There is no trusted party to keep a record of transactions under secure ‘lock and key’. Yet there remains a need for a record-keeping mechanism that can be updated but not retroactively modified. This record must necessarily also be distributed amongst network peers in the absence of central storage or coordination. A transaction originated by one peer must be broadcast to all peers in order to maintain consistency of the distributed record keeping mechanism. This is what a distributed ledger seeks to achieve.

In the language of computer science, the ledger must be an immutable “append-only” data structure. Such a data structure might logically consist of a time-ordered sequence of transactions, updated by adding the latest transaction to the end of the sequence. Immutability requires a mechanism to ensure that a negligent or hostile actor can neither insert a new transaction before the end, nor delete or modify an existing entry.

Nakamoto [23] introduced a distributed ledger to record the exchange of a digital currency called bitcoin on a peer-to-peer network – the bitcoin network. Peers gather transactions into blocks which are chained together with the aid of cryptography to attain immutability. This immutable but verifiable cryptographic sequence of blocks has come to be known as the blockchain.

A sequence is not the only imaginable append-only data structure. A tree where new entries are appended to leaves is an alternative generalisation. Indeed, blockchain is, in fact, derived from a tree. Peers can receive multiple blocks at the same (discrete) time that were broadcast from different parts of the network. A strict time-order acceptance rule no longer applies, implying simultaneous acceptance of such blocks so that the ledger branches into a tree. When one sequence from root to leaf becomes longer than the others, Nakamoto chooses it as the effective blockchain, to which new blocks must attach.

Sompolinsky and Zohar [29] suggested an alternative to the longest sequence that “selects at each fork in the chain the heaviest subtree rooted at the fork”. As discussed in [29], blocks associated with greater branching enhance security, amongst other advantages.

The “longest chain” and “heaviest subtree” deterministic rules select a single leaf to which new blocks must attach, abandoning all other leaves. A less extreme approach is to define a leaf probability distribution that assigns non-zero probability to every leaf. Attachment points for new blocks can then be sampled probabilistically from this leaf distribution rather than deterministically prescribed. In this way, the whole tree continues to be the active distributed ledger rather than a selected chain leading to a single leaf.

Popov [25] adopted a variant on the method of Sompolinsky and Zohar involving weighted random walks from root to leaf, with probabilistic branching based on relative subtree weights. The terminal leaf of the walk then becomes the point of attachment for new blocks. Hence the tree grows probabilistically without abruptly abandoning all chains but one, as deterministic rules do. (To be precise, Popov immediately generalised the
tree to a graph by allowing a new vertex to make multiple attachments, in an approach
aimed at the internet-of-things where vertices represent individual transactions rather
than blocks of transactions.)

Popov’s weighted random walk aims to generate a sample from a leaf distribution without
explicitly constructing it. In this paper, we construct the explicit probability distribution
of leaves based on the connectivity of the tree and then sample attachment points for new
vertices from it. More significantly, we present a novel approach that involves exploring
suitably weighted directed paths from leaf to root rather than branching from root to leaf.

Having drawn inspiration from blockchain, we may set it aside to discuss random trees by
probabilistic leaf attachment in the abstract. The generic topic is of mathematical and
computational interest in its own right, whether or not it finds application in alternative
distributed ledger design or other context not envisaged here.

We conclude by noting that we have consciously glossed over many challenges arising
from the absence of a central authority in a peer-to-peer network. For example, there are
variable broadcast delays so that, at any instant, peers may hold different snapshots of the
ledger. This requires consensus rules on such issues as what constitutes the longest chain or
heaviest subtree sequence at any stage. Furthermore, aside from attempted cryptanalysis,
distributed ledgers have to contend with a slew of hostile attacks. For instance, Nakamoto
used the Gambler’s Ruin problem (Feller [12], p342) to model a particular attack scenario.
A comprehensive discussion of distributed ledgers would involve computer science, discrete
mathematics (graph theory, cryptography), probability theory and game theory.

1.2 Context and Scope

We have discussed the distributed ledger as an example of a random tree that grows
by probabilistic leaf attachment. The tree can be generalised to a random graph if new
vertices are allowed to make multiple attachments. We consciously limit the scope here
to a tree, from which much can be learned before tackling the more general graph.

There is another context involving random graphs that is worth commenting on to avoid
potential misinterpretation of the theme of this paper. The bitcoin network referred to
earlier is an example of a real-world network, much like biophysical networks, the internet,
social networks, scholarly citation networks etc. In a field of study known as network
science, random graphs are used as mathematical models of complex real-world networks
(Barabási [1], van der Hofstad [15], Newman [24]). Such networks are typically dynamic,
evolving through structural reconfiguration or growth. The random graph models must
correspondingly reflect such evolution in some probabilistic sense.

Pioneering work on random graphs considered a fixed number of vertices with a probability
distribution over (i) the number of edges for classes of graphs (Erdős and Rényi [10]), or
(ii) placement of edges for a given graph (Gilbert [14]). However, real networks often
grow in time through attachment of new vertices. For example, a new member of a
social network may prefer to befriend members who are already well-connected to other
members. To capture such preference in the random graph model, a new vertex may
attach to a current vertex with probability proportional to the degree of the latter (the degree of a vertex is the number of edges connecting it to other vertices: a measure of the vertex’s connectivity). Barabási and Albert [2] studied such degree-biased attachment for undirected graphs, which has come to be known as preferential attachment. The concept was first described by Yule [32] and Simon [28] and it is associated with the Yule-Simon distribution. Bollobás et al. [5] extended preferential attachment to directed graphs, where edges have a one-directional meaning, so that a vertex has two degree types: in-degree (number of incoming edge) and out-degree (number of outgoing edges).

Generalisations include attachment probability proportional to a nonlinear function of degree (Krapivsky et al. [19]) and degree weighted by, say, assigning an additional ‘fitness’ variable to vertices (Bianconi and Barabási [3]). Preferential attachment has received much interest because it leads to “scale-free” graphs, i.e. power-law degree distributions at very large scale. The prevalence of scale-free behaviour in complex real-world networks compared to their random graph counterparts is a subject of debate in network science (Broido and Clauset [6]).

The bitcoin network is an instance of a real-world network that can be modelled as a random graph. Kondor et al. [18] explored the ‘rich-get-richer’ consequence of preferential attachment for the bitcoin network. Javarone and Wright [16] suggested the Bianconi-Barabási random graph representation. The bitcoin network model is logically distinct from the distributed ledger that keeps a record of transactions issued by the network.

A network science model that resembles the problem of this paper is the directed time-ordered graphical model of citations of scholarly publications. It was initially studied by Price [26], who referred to preferential attachment as cumulative advantage. A new vertex represents a new publication and an attachment to an existing vertex represents citation of an older publication. Conceptually, this might be regarded as a more general variant of our problem, where a new vertex may make multiple attachments to existing vertices at any depth of the graph. However, beside restriction to leaf attachment, our context differs fundamentally from that of citation modelling or any other network science study.

Given a random graph model of a real-world network, the natural question of network science is: “Do real-world complex networks actually behave like that?” The question of interest here is: “Does the random tree/graph behave according to design specifications?” This is more of an engineering problem (designing and testing a product with prescribed characteristics) than a scientific problem (modelling an observable phenomenon). It is akin to designing an aircraft and testing a prototype’s flight capabilities against expectation. This may well be inspired by observing birds in flight but it need not try to mimic them through a mathematical model of their prowess, including properties like flock intelligence that might emerge at ‘large scale’. Quite separately, a complex network of flight paths and destinations woven by commercial use of aircraft might be modelled by random graphs, along with questions about the large scale behaviour of model versus reality.

The upshot of all this is that we shall not feel obliged to explore the asymptotic behaviour of our leaf attachment scheme.
1.3 Trees Everywhere

For completeness, we note that a directed rooted tree is a common representation of a hierarchy of concepts and actions in practically any field. The tree typically grows by branching, sprouting new vertices like a natural tree, rather than by attachment of separately created vertices to the existing tree. The branching process can be probabilistic or deterministic.

An early example of probabilistic branching is the Galton-Watson tree in genealogy. First studied in the 1800s (Bienaymé [4], Galton and Watson [13]), it is well-documented in subsequent literature on probability and trees (e.g. Drmota [9], Lyons and Peres [20]).

A simple example of deterministic branching is the binary search tree of computer science. It is, in turn, an instance of a decision tree, which finds much use in machine learning (e.g. Quinlan [27] and a wealth of more recent literature).

Unlike natural trees, abstract trees can also grow backward through merging of paths from leaf to root. An example is the Merkle tree (Merkle [22]), which starts with leaves, each being a cryptographic hash function of specified data. Proceeding from leaf to root, each inner vertex is, in turn, a hash of two preceding vertices (for a binary implementation), terminating in a hash of the whole tree at the root. Since cryptographic hash functions are one-way (non-invertible), the Merkle tree can only grow from leaf to root. It plays a fundamental role in blockchain design for cryptographic compression of transactions in a block (not to be confused with the tree structure that arises as the blockchain grows).

While trees can be studied in the abstract in graph theory, they also find application as a representational tool in mathematics. For example, the Collatz tree is a representation of the Collatz conjecture of number theory. A Quanta article [21] on a recent advance toward a proof of the conjecture by Tao [30] opened with an animation of a growing Collatz tree. The animation, due to Davies [8], displays the tree in outward-growing concentric layers. The Collatz conjecture is normally formulated so that the Collatz tree (like the Merkle tree) grows inward, governed by a simple integer function, from any starting vertex (integer \( n > 1 \)) to the root (\( n = 1 \)). But (unlike the Merkle tree), this function is invertible so that the Collatz tree can also grow by branching from the root.

1.4 Novelty

There are two steps to the growth of our tree:

1) Constructing the leaf distribution given the current snapshot of the tree
2) Growing the tree by attachment of new vertices to leaves based on the leaf distribution.

It may seem natural to expect the leaf distribution to be constructed by branching from the root. On the other hand, the Merkle and Collatz trees – deterministic though they may be – inspire the thought of inward exploration of the snapshot, gathering knowledge about the tree from leaf to root instead.

The key insight here is that we can indeed traverse the tree in the inward direction along
directed paths from leaf to root. In so doing, we learn about the history of the tree in an appropriate probabilistic sense. Then we can use Bayes’ rule to infer the leaf distribution, without further need to branch outward from root to leaf.

To our awareness, such a conceptual framework is novel. Bayesian analysis has been applied to decision trees for classification in machine learning (dating back to Buntine [7]), but this is a different application from that considered here. Decision trees grow by branching rather than attachment, hence the question of constructing a leaf distribution to guide attachment does not arise.

1.5 Structure of Paper

The detail of the proposed path-based Bayesian method will be discussed in section 2. For completeness, we will also explore a degree-based branching approach in section 3. We present computational examples in section 4.

2 Probabilistic Tree Growth

Let \( \mathcal{L} \) be the set of leaves and let \( \mathcal{H} \) denote the history of the tree, i.e. the observable data about the tree. We proceed as follows:

**Prior:** Assign an initial leaf distribution \( \Pr(\ell) \) for the discrete variable \( \ell \in \mathcal{L} \) without prior knowledge of the history of the snapshot of the tree. In the absence of any other information, the natural choice is a uniform prior. If we choose to ignore the tree’s history, attachment points for new vertices may, without further ado, be sampled from this prior. Else proceed to the next step.

**Likelihood:** Learn about the history of the tree by exploring it backward from leaf to root, i.e. construct the likelihood \( \Pr(\mathcal{H}|\ell) \).

**Posterior:** Construct the posterior leaf distribution \( \Pr(\ell|\mathcal{H}) \) by making use of the product and sum rules of probability theory

\[
\Pr(\ell, \mathcal{H}) = \Pr(\mathcal{H}|\ell) \Pr(\ell) = \Pr(\ell|\mathcal{H}) \Pr(\mathcal{H}) \quad (1)
\]

\[
\Rightarrow \quad \Pr(\ell|\mathcal{H}) = \frac{\Pr(\mathcal{H}|\ell) \Pr(\ell)}{\Pr(\mathcal{H})} \quad \text{Bayes’ rule} \quad (2)
\]

where

\[
\Pr(\mathcal{H}) = \sum_{\ell \in \mathcal{L}} \Pr(\ell, \mathcal{H}) = \sum_{\ell \in \mathcal{L}} \Pr(\mathcal{H}|\ell) \Pr(\ell) \quad (3)
\]

**Growth:** Given \( n \) new vertices where \( n \) is generated from a Poisson distribution (say), sample the \( n \) attachment points for these new vertices from \( \Pr(\ell|\mathcal{H}) \).

**Remark 1.** Despite the presentation order, constructing the likelihood is independent of the choice of prior. Both are needed to construct the posterior.

**Remark 2.** In general, the use of the posterior depends on the application context. The sole aim here is to sample attachment targets from the posterior in order to grow the tree.
This contrasts with the common objective of Bayesian analysis where the posterior is used for parameter estimation or classification.

**Remark 3.** \( \mathcal{L} \) is a set of any distinct leaf labels, such as simple leaf number. We strike no distinction here between events of a sample space and corresponding random variables.

**Remark 4.** There is a single history of the tree. Hence \( \Pr(H|\ell) \) must be understood not as a probability distribution over multiple instances of history, but as a function of the leaf variable \( \ell \), known as the likelihood function. Similarly, \( \Pr(H) \) is a constant rather than a distribution over different instances of \( H \). The only probability distributions involved are the prior \( \Pr(\ell) \) and the posterior \( \Pr(\ell|H) \).

Construction of the likelihood depends on design objectives. Our preference in this paper is for new vertices to attach to leaves that are directly or indirectly connected to multiple branches of the current tree. Equivalently, we aim to discourage the growth of isolated chains of vertices. This, in turn, guides our construction of \( \Pr(H|\ell) \). It is thus appropriate at this point to be explicit about the meaning of \( H \). We shall make repeated use of the following concept:

**Definition 1.** There is unique sequence of attachments from any leaf \( \ell \in \mathcal{L} \) to the root \( r \) that we refer to as a directed ordered path, denoted by \( \ell \rightarrow r \). The sequence of connected vertices along the path is strictly ordered from later to earlier in time.

We use \( y \rightarrow x \) to denote a direct attachment of vertex \( y \) to vertex \( x \). Any direct attachment necessarily lies on at least one path and possibly multiple merged paths. The more merging there is the better connected the tree is. How we use this to construct a likelihood depends on how we interpret the paths as quantitative data. To that end, we endow an attachment and a path with a weight as follows:

**Definition 2.** The weight (or multiplicity) of an attachment \( y \rightarrow x \), denoted by \(|y \rightarrow x|\), is the number of paths from the set \( \mathcal{L} \) of leaves to the root \( r \) that pass through \( y \rightarrow x \).

**Definition 3.** The weighted length (or just weight) of a path \( \ell \rightarrow r \) is the sum of weights of all intermediate attachments between \( \ell \) and \( r \). We denote this weight by \(|\ell \rightarrow r|\).

Figure 2 shows root \( r \), deep vertices \( \{d_1, \ldots, d_6\} \) and leaves \( \mathcal{L} = \{\ell_1, \ldots, \ell_5\} \). Each leaf initiates a path to \( r \), e.g. path \( \ell_2 \rightarrow r \equiv \ell_2 \rightarrow d_4 \rightarrow d_2 \rightarrow r \). The label on each attachment is its weight as defined above.

### 2.1 Global Interpretation

We identify \( \Pr(H|\ell) \) with a global attribute of the path \( \ell \rightarrow r \), viz. the path length:

**Conditional 1.** The conditional probability of the path \( \ell \rightarrow r \) given \( \ell \) is taken to be the weighted path length

\[
\Pr(H|\ell) = |\ell \rightarrow r| \tag{4}
\]
Figure 2: Tree with root r, deep vertices \{d_i\} and leaves \{\ell_i\}. The attachment label is the number of paths from leaf to root passing through that attachment.

Figure 3: As in Figure 2, with unnormalised prior\(\text{posterior}\) values on leaves for two likelihood choices: (1) Global – based on weighted path length, (2) Local – based on individual attachment weights.

Hence the likelihood for the example of Figure 2 is

\[
\begin{align*}
\Pr(\mathcal{H}|\ell_1) &= 1 + 1 = 2 \\
\Pr(\mathcal{H}|\ell_2) &= 3 + 2 + 1 = 6 \\
\Pr(\mathcal{H}|\ell_3) &= 3 + 2 + 1 = 6 \\
\Pr(\mathcal{H}|\ell_4) &= 3 + 1 + 1 = 5 \\
\Pr(\mathcal{H}|\ell_5) &= 1 + 1 + 1 = 3
\end{align*}
\]

(5)

We have made no assumptions about the prior \(\Pr(\ell)\) and we can accommodate any choice. For a uniform prior, the posterior is \(\Pr(\ell|\mathcal{H}) \propto \Pr(\mathcal{H}|\ell)\). The leaves in frame 1 of Figure 3 show the prior on the left and the posterior on the right, omitting normalisation in each case. The shorter isolated chain (terminating on leaf \(\ell_1\)) has lowest probability, followed by the longer one (\(\ell_5\)). Highest probability goes to \(\ell_2\) and \(\ell_3\), which exhibit the most merging of paths between leaf and root. This is in keeping with the objective expressed earlier that the posterior must favour attachment to well-connected leaves and discourage isolated chains.
2.2 Local Interpretation

We take an individual attachment as the basic data object, so that we describe a path \( \ell \to r \) explicitly in terms of its attachments. In Figure 2, \( \ell_2 \to r \equiv \ell_2 \to d_4 \to d_2 \to r \). In this instance, we write \( \Pr(H|\ell_2) \) as \( \Pr(r, d_2, d_4|\ell_2) \) which, by the product rule, may be written as

\[
\Pr(H|\ell_2) \equiv \Pr(r, d_2, d_4|\ell_2) = \Pr(r|d_2) \Pr(d_2|d_4) \Pr(d_4|\ell_2)
\]

with analogous expressions for the other 4 paths from \( L \) to \( r \). The decomposition (6) follows from the Markov property that a vertex on a specified path is conditionally independent of vertices that indirectly attach to it, given the vertex that directly attaches to it on that path. Succinctly, \( \Pr(r|d_2, d_4, \ell_2) = \Pr(r|d_2) \), for example.

Conditional 2. The conditional probability of \( x \) given \( y \), where \( y \) directly attaches to \( x \) is taken to be the attachment weight

\[
\Pr(x|y) = |y \to x|
\]

Hence the likelihood for the example of Figure 2 is

\[
\begin{align*}
\Pr(H|\ell_1) &\equiv \Pr(r, d_1|\ell_1) = \Pr(r|d_1) \Pr(d_1|\ell_1) = 1 \times 1 = 1 \\
\Pr(H|\ell_2) &\equiv \Pr(r, d_2, d_4|\ell_2) = \Pr(r|d_2) \Pr(d_2|d_4) \Pr(d_4|\ell_2) = 3 \times 2 \times 1 = 6 \\
\Pr(H|\ell_3) &\equiv \Pr(r, d_2, d_4|\ell_3) = \Pr(r|d_2) \Pr(d_2|d_4) \Pr(d_4|\ell_3) = 3 \times 2 \times 1 = 6 \\
\Pr(H|\ell_4) &\equiv \Pr(r, d_2, d_5|\ell_4) = \Pr(r|d_2) \Pr(d_2|d_5) \Pr(d_5|\ell_4) = 3 \times 1 \times 1 = 3 \\
\Pr(H|\ell_5) &\equiv \Pr(r, d_3, d_6|\ell_5) = \Pr(r|d_3) \Pr(d_3|d_6) \Pr(d_6|\ell_5) = 1 \times 1 \times 1 = 1
\end{align*}
\]

Frame 2 of Figure 3 shows the prior and posterior. The shape of the posterior is indeed as intended, with isolated chains having the same lowest probability regardless of length.

2.3 Discussion

In summary, the history \( H \) of the tree is the set of paths \( \{ \ell \to r : \ell \in L \} \) from leaves to root, where a path consists of connected attachments from \( \ell \) to \( r \). An attachment is weighted by the number of coincident paths sharing that attachment. The likelihood \( \Pr(H|\ell) \) depends only on the path \( \ell \to r \) or, more precisely, on some quantitative attribute thereof. In the global interpretation, we take this attribute to be the sum of weights of direct attachments along the path (i.e. the weight of the path), which we directly equate to \( \Pr(H|\ell) \). In the local interpretation, we take the weight of an attachment to be the conditional probability between the two vertices associated with the attachment. Then \( \Pr(H|\ell) \) is the product of such pairwise conditionals along the path.

The path-based characterisation of a tree generalises to a time-ordered directed graph, where a vertex may make multiple attachments. Hence, there may be multiple directed paths from any leaf to the root, where the paths may merge with other paths as well as split into several subsidiary paths. In our view, characterising a directed ordered graph in
terms of directed ordered paths is more natural and useful than describing it as a directed acyclic graph (DAG), as per standard practice. The acyclic property is a consequence of being directed and ordered rather than a fundamental attribute. It is like describing time evolution by saying that history does not repeat itself. True though this may be (except in a purely figurative sense), it is not self-evidently useful.

Karrer and Newman [17] similarly commented that the term DAG is “perhaps slightly misleading, focusing our attention, as it does, on the acyclic property rather than the more fundamental ordering”. While conceding that “directed ordered graph” would be unlikely to find wide adoption, they explicitly “incorporate an underlying ordering of the vertices that then drives the acyclic structure”. Thus they described their model as a “random ordered graph”.

In their discussion of the citation graph, Evans et al. [11] refer to a fundamental “arrow of time” because a document can only cite documents older than itself. This necessarily means that there can be no directed cycles (closed loops) in the graph as this can only arise from an earlier document paradoxically citing a later one. Hence the graphical model of a citation network is an instance of a DAG. Although the concept of ordering is more generally applicable than time-ordering, we may implicitly take DAG – whenever we use the term – to be a shorthand for “directed arrow-of-time graph”.

Our contention is that, at least for our leaf attachment problem, directed ordered paths offer the natural way to characterise directed ordered graphs, or to make explicit use of the arrow-of-time. The paths may then be endowed with probabilities, where the basic probabilistic construct is the conditional probability between two vertices rather than the probability of a vertex. In fact, the leaves are the only vertices that make up the ‘parameter space’ and are thus endowed with prior and posterior probabilities. All other vertices are part of the likelihood.

That said, as noted in section 1.2, graphs are often endowed with probabilities based on relative vertex degree. A distinction is made between in-degree or out-degree for directed graphs, but this remains silent on ordering. Nonetheless, a question that arises is whether we can construct a leaf distribution with desired attributes based on placing probabilities on vertices rather than attachments or paths. We turn to this question next.

3 Degree-based Branching

First consider assigning some weights \( \{w\} \) to vertices. Then, starting at the root, traverse the tree by branching from any vertex to directly reachable vertices using the relative weights of the latter as branching probabilities. The products of such successive branching probabilities induce a leaf distribution (equivalently, the destination proportions of a large number of probabilistically weighted random walks from the root to the leaves).

In the simplest case, every vertex is taken to have the same unit weight. Frame 1 of Figure 4 shows the example considered earlier, with unit weights on deep vertices and the left label on the leaves. The right label on leaves is the unnormalised leaf probability
induced by branching from the root as described, i.e. taking the product of successive branching probabilities. The leaves at the ends of isolated chains are allocated the highest probability, contrary to intended behaviour.

The potential remedy is to allocate higher weight to vertices exhibiting more branching. As in the previous section, we need first to traverse the tree backward from the leaves to construct weights. The difference is that we now assign branch weights to vertices rather than multiplicity weights to attachments. This is not a likelihood construction exercise since we no longer retain conditioning on leaves as we traverse the tree backward, assigning vertex weights. Hence, once we reach the root, we need to traverse the tree again from root to leaf to construct the leaf distribution.

Frame 2 of Figure 4 shows each vertex labelled by weighted in-degree, i.e. the in-degree sum of all vertices that directly attach to it (having assigned starting values of 1 to all leaves). In effect, this takes the weights we associated with attachments earlier and places them on the source vertex. This time the branching distribution at the leaves is also uniform, we have an exact inversion process.

At this point we can try to enforce the qualitative differentiation we seek through further sharpening of relative branch weights. For example, we might sharpen by raising the weights to some power \( \{w^\alpha\}, \alpha > 1 \). Frame 3 of Figure 4 shows the case \( \alpha = 2 \), i.e. squaring every weight in Frame 2. We finally attain a leaf distribution that is keeping with design objectives. Notably, isolated chains are assigned the same probability regardless of length. However, this has come at the expense of imposing non-linear structure on linear weighted in-degree (like Krapivsky et al. [19]).

We consider next using the weight of the subtree rooted at each vertex, as referred to earlier when we discussed the blockchain method due to Sompolinsky and Zohar [29]. We define it as the total number of vertices that the branch carries, i.e. the cumulative in-degree of the vertex as a root of the subtree (we could choose to include the latter vertex so that weight = cumulative in-degree+1).

Frame 2 of Figure 5 shows vertices weighted by cumulative in-degree. The leaf terminating
the longer isolated vertex still receives highest probability. Despite having lower branch weight, it does not have to allocate that weight to subsidiary branches, resulting in the single leaf inheriting all of it. Once again, further sharpening of relative branch weights is indicated. Frame 3 of Figure 5 shows the case $\alpha = 2$, i.e. squaring every subtree weight in Frame 2. Only then do we finally attain a leaf distribution that is keeping with design objectives, although in this case longer isolated chains are preferred over shorter ones. Such ‘nonlinear sharpening’ of cumulative in-degree weights is the approach described by Popov [25], where the subtree weights are exponentiated, $\{\exp(\alpha w)\}$.

In both examples, we may think of the 3 successive frames as corresponding to exponents $\{w^\alpha\}$, $\alpha = 0, 1, 2$ respectively for the particular choice of vertex weights. In our view, introducing nonlinear functions of weighted degree or cumulative degree in order to meet design objectives is rather ad hoc. We have included this section for conceptual completeness but we shall not pursue degree-based branching further here.

## 4 Computational Examples

The growth of random trees by leaf attachment is best illustrated by visual animation. Here, we can only display still frames from animations available on the site Sibisi Movies. We wrote the tree generation code in C++ and created the animations using gnuplot [31]. Although the code is quite general, the examples here are of modest scale to aid visual interpretation. The runtime parameters are:

- $N$ The number of time intervals $\{t = 0, 1 \ldots N - 1\}$
- $\mu$ The mean of the Poisson distribution used to generate the number of new vertices in each time interval $t > 0$. In the examples here, $\mu$ is held constant for all $t > 0$.
- The initial state at $t = 0$ is the root as the sole new vertex.
- Bayes The choice of leaf distribution used to sample attachment points to grow the tree:
  - Case 0: The prior $\Pr(\ell)$, taken to be uniform
Case 1: The posterior $\Pr(\ell|\mathcal{H})$, using the ‘global’ likelihood of subsection 2.1
Case 2: The posterior $\Pr(\ell|\mathcal{H})$, using the ‘local’ likelihood of subsection 2.2

4.1 First Example

Choose $\mu = 2$, $N = 10$ and use a fixed seed to generate the same Poisson sequence of new vertices for each of the 3 Bayes cases.

Figure 6 shows consecutive movie frames 1 to 10 as the tree grows from the root by uniform leaf attachment (Bayes=0). Frame 11 shows the complete tree after the last set of new vertices have been converted to leaves. Frame 12 shows the longest path between leaf and root (where this is not unique, we simply choose one). The intention is not to abandon the rest of the tree but to find a proxy for the connectivity structure of the tree that is easy to visualise. This is a particularly helpful attribute for trees much denser than the sparse, low scale (small $\mu$, $N$) example chosen here for illustrative purposes.

Figures 7 and 8 show the analogous growth for Bayes=1 and Bayes=2 respectively. For these two cases, the final frame shows the path to the root starting at the leaf of maximum posterior probability (if this is not unique, we display one such path). In this case of a uniform prior, the maximum posterior path coincides with the maximum likelihood path, equivalently, the heaviest path. We can already visually detect the preference in Figure 8 for attachment to well-connected leaves and tending to leave isolated chains behind.
4.2 Second Example

Choose $\mu = 2, N = 25$ and use a fixed seed to generate the same Poisson sequence of new vertices for each of the three Bayes cases.

Figure 9 shows the final frame of the 25-frame growth sequence for the Bayes=0 case. Since the leaf attachment distribution is uniform and independently of the historical connectivity of the tree, one can expect attachments of any length to leaves of any connectivity or age. Hence the longest path unsurprisingly contains a mix of short and long attachments and is thus rather sparse (10 attachments) compared to the time sequence of 25 intervals.

Figure 10 shows the final frame of the sequence for the Bayes=1 case. The maximum posterior path is denser (15 attachments) than the longest path of the Bayes=0 case, in keeping with the preference for attaching to better connected leaves as the tree grows.

Figure 11 shows the final frame of the sequence for the Bayes=2 case. There are now hardly any long attachments and the maximum posterior path is even denser (17 attachments) than the Bayes=1 case.

4.3 Discussion

As the examples illustrate, sampling from a uniform prior leaf distribution $Pr(\ell)$ allows long attachments to old leaves, which enjoy the same probability of attachment as any
other leaf. By contrast, the posterior leaf distribution \( \Pr(\ell|H) \) tends to discourage long attachments to old vertices (particularly for \( \text{Bayes}=2 \)) and encourage short attachments. This leads to dense, well-connected paths at the expense of isolated paths that are left behind (but not definitively abandoned because all leaves have non-zero probability).

The caveat is that we have only presented a few animations to indicate the behaviour of the different growth schemes. Further exploration would need to be guided by specific questions. For example, we might wish to quantify the attachment length of the longest path (\( \text{Bayes}=1 \)) or highest probability path (\( \text{Bayes}=1,2 \)), as an average over an ensemble of animations at given \( \mu, N \). We might then study the evolution of such an average path length with increasing \( N \) and compare the efficiency of the different attachment schemes at creating dense paths.

A compromise between growth properties induced by \( \Pr(\ell) \) and \( \Pr(\ell|H) \), should we deem it necessary, might be to use a linear combination of the two

\[
q \Pr(\ell) + (1 - q) \Pr(\ell|H) \quad \text{for} \quad 0 \leq q \leq 1
\]  

(9)

We might even imagine a time-dependent \( q(t) \), alternately favouring \( \Pr(\ell|H) \) to encourage attachment to well-connected leaves and \( \Pr(\ell) \) to give all leaves equal chance of receiving attachments. We might think of a smoothly oscillating \( q(t) \) as a source of ‘breathing’ growth, where (9) periodically tightens to favour \( \Pr(\ell|H) \) and relaxes to favour \( \Pr(\ell) \).
5 Conclusion

We have presented a novel Bayesian method to grow a rooted time-ordered directed tree by probabilistic leaf attachment. The connectivity of the tree induces the likelihood function which, combined with an assigned prior distribution, leads to the posterior leaf distribution from which we draw attachment points.

The concept of a directed time-ordered path has played a fundamental role in defining the connectivity of the tree, which is effectively defined as the merging of such paths from leaves to the root. The directed time-ordered path will remain the fundamental construct as we generalise from a tree to a graph where a new vertex may make multiple attachments. There will then be multiple paths from leaf to root, with both merging and splitting of paths in between.

It is also worth exploring the effect of an incomplete path history from leaves, possibly terminating after a specified number of attachments before reaching the root.

A further question that arises naturally is whether we can modify the leaf attachment methodology in order to attach to vertices at any depth. Given a deep vertex $v$, we might then consider both
a) directed paths from $v$ to the root (as we did for the case where $v$ is a leaf) and
b) reverse directed paths from $v$ to the leaves of the current snapshot of the tree.
Both sets of paths would then inform construction of the likelihood.

Such a construction would equip us to model the citation network discussed in section 1.2. We intend to pursue these ideas further in a dedicated paper.
Figure 10: Growth using posterior leaf distribution with global likelihood. Poisson mean 2, 25 time intervals. (1) Last frame in sequence. (2) Frame showing path of highest posterior.

Figure 11: Growth using posterior leaf distribution with local likelihood. Poisson mean 2, 25 time intervals. Note absence of long attachments to old leaves. (1) Last frame in sequence. (2) Frame showing path of highest posterior.
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