Transmission Amplitude through a Coulomb blockaded Majorana Wire

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We study coherent electronic transport through a Coulomb blockaded superconducting Rashba wire in the co-tunneling regime between conductance resonances. By varying an external Zeeman field the wire can be tuned into a topological regime, where non-local transport through Majorana zero modes is the dominant mechanism. We model coherent transport in the co-tunneling regime by using a scattering matrix formalism, and find that the transmission amplitude has a maximum as a function of Zeeman field, whose height is proportional to the wire length. We relate the transmission amplitude to the Majorana correlation length, and argue that the Zeeman field and length dependence of the transmission amplitude are unique signatures for the presence of Majorana zero modes.

I. INTRODUCTION

In recent years, Majorana zero modes (MZMs) have attracted much attention as possible candidates for the realization of topologically protected quantum bits [1–3]. MZMs can arise as localized zero energy excitations in topological superconductors under suitable conditions [4–7], and many of their predicted experimental signatures have been observed, for instance a zero-bias conductance peak [8–12] and the suppression of the even-odd splitting of Coulomb blockade resonances in the topological phase [13].

The topological nature of MZMs manifests itself in their non-local character [14, 15]. We study how the non-locality of the electronic state encoded by MZMs can be probed by phase coherent transport through a Coulomb blockaded wire with MZMs at its ends [16–22]. By embedding the Majorana wire into the arm of an electron interferometer, the amplitude of coherent transmission through the MZMs can be studied [23]. In a conductance valley in between Coulomb blockade peaks, the amplitude of the transmission through a Majorana wire is determined by the magnitude of the wave functions at the ends of the wire. For MZMs, the wave function has a large magnitude $\propto 1/\sqrt{\xi}$ near the wire end, where $\xi$ denotes the Majorana correlation length. In contrast, if the transmission is dominated by transport extended states, the magnitude of wave functions $\propto 1/\sqrt{L}$ depends on the wire length. Taking into account the decrease of the charging energy with wire length, in a conductance valley one expects an increase of coherent transmission $\propto L$ when entering the topological regime, a robust signature of MZMs which also allows to distinguish them from pseudo-MZMs [21, 24–30], which can arise in the presence of a soft confinement at the wire ends. Details of the magnetic field dependence follow from the relation $1/\xi = \Delta_{p,\text{ind}}/(\hbar v_F)$, where $\Delta_{p,\text{ind}}$ is the topological p-wave superconducting gap, in agreement with results of a recent experiment [23].
B. Transmission amplitude

In lowest order interference, the current through the interferometer is given by

\[ I(\Phi) \propto 2\text{Re}[T_{\text{ref}}]^2 + \sum_{\sigma} |T_{\sigma}|^2 + 2 \text{Re}[e^{i\Phi} T_{\sigma}] \]  

(1)

where \( T_{\text{ref}} \) is the transmission amplitude through the reference arm (assumed to be diagonal in spin). Here, \( T_{\sigma} \) is the transmission amplitude of coherently tunneling electrons with spin quantum number \( \sigma \), which is an entry of the scattering matrix determined by the overlap integral between a decaying state and the initial state by adding or removing an intermediate state

\[ (W)_{\alpha \sigma} = \sqrt{\frac{\tau_i}{2}} (\lambda_{\alpha j\sigma}(N_0, \{n_i\}), \ldots, \lambda_{\alpha j_{\text{max}} \sigma}(N_0, \{n_i\}), \lambda_{\alpha j_{\text{max}} \sigma}(N_0, \{n_i\}), \ldots, \lambda_{\alpha j_{\text{max}} \sigma}(N_0, \{n_i\})) \]  

(4)

The energies for electron(hole)-like tunneling processes \( \varepsilon_j^{(h)}(N_0, \{n_i\}) \) contain both charging energy and single particle energy levels of the wire Hamiltonian. To describe co-tunneling processes, we consider the dot in an initial state \( |N_0, \{n_i\} \rangle \). The transmission then occurs via an intermediate state \( |N_0 \pm 1, \{n_i'\} \rangle \), where the allowed occupation numbers \( \{n_i'\} \) of the intermediate state deviate from those of the initial state by adding or removing a single Bogolubon, and by adding (removing) one electron charge to the dot. The electron(hole)-like couplings \( \lambda_{\alpha j\sigma}(N_0, \{n_i\}) \) of lead \( \alpha \) to level \( j \) in the dot are obtained from the overlap \( \langle N_0, \{n_i\}; \{\alpha, \sigma\}|H_{\text{tun}}, |N_0 \pm 1, \{n_i'\}\rangle \), with \( H_{\text{tun}} \) defined in Eq. (7).

In the topological regime, exponentially localized MZMs occur, for instance at the left end of the wire with the wave function \( \chi_{\sigma, L}(y) \) with envelop \( \varepsilon^{-1/2}e^{-y/\xi} \). In the presence of a small overlap between the left and right MZM, the BdG eigenfunctions are given by \( (\chi_{\sigma, L} \pm \chi_{\sigma, R})/\sqrt{2} \). Evaluating Eq. (2) to leading order in the dot-lead couplings, one finds that the transmission amplitude through the MZMs is \( T_{\sigma} \sim \chi_{\sigma, L}(y_L)\chi_{\sigma, R}(y_R)/(E_c/2) \). Thus, the transmission amplitude provides direct information about the Majorana localization length \( \xi \).

C. Hamiltonian

We describe the proximitized semiconductor wire by the Hamiltonian

\[ H_{\text{wire}} = \tau_z \otimes \left[ -\frac{\hbar^2 \partial_y^2}{2m^*} \sigma_0 - \mu \sigma_0 - i\hbar \alpha \sigma \partial_y \right] - E_z \tau_0 \otimes \sigma_x + \Delta \tau_x \otimes \sigma_0 . \]  

(5)

Mahaux-Weidenmüller formula [37]

\[ S = 1 - 2\pi i \left( \frac{1}{\varepsilon - H_{\text{eff}} + i\pi W} \right) . \]  

(2)

The brackets denote the thermal average over occupations \( \{n_i\} \) of BdG eigenstates in the dot, defined as \( \langle O \rangle = 1/Z \sum_{\{n_i\}} e^{-\beta E_{\{n_i\}}} O(\{n_i\}) \). The average is performed for fixed total particle number \( N_0 \), which determines the number parity of occupied BdG levels \( \{n_i\} \).

Here, \( \varepsilon \) is the energy of incoming electrons, and the effective dot Hamiltonian \( H_{\text{eff}} \) and the matrix of dot-lead couplings \( W \) with lead index \( \alpha \) and spin \( \sigma \) are given by

\[ H_{\text{eff}} = \begin{pmatrix}
\text{diag} \left[ \varepsilon_j^0(N_0, \{n_i\}) \right]_{j=1..j_{\text{max}}} & 0 \\
0 & \text{diag} \left[ \varepsilon_j^0(N_0, \{n_i\}) \right]_{j=1..j_{\text{max}}}
\end{pmatrix} , \]  

(3)

\[ (W)_{\alpha \sigma} = \sqrt{\frac{\tau_i}{2}} (\lambda_{\alpha j\sigma}(N_0, \{n_i\}), \ldots, \lambda_{\alpha j_{\text{max}} \sigma}(N_0, \{n_i\}), \lambda_{\alpha j_{\text{max}} \sigma}(N_0, \{n_i\}), \ldots, \lambda_{\alpha j_{\text{max}} \sigma}(N_0, \{n_i\})) . \]  

(4)

Here, \( \tau_k \) and \( \sigma_k \) are Pauli matrices in particle-hole and spin space, respectively, and the Nambu basis spinor is given by \( (d_{\uparrow}^k(y), d_{\downarrow}^k(y), d_{\uparrow}^k(y), -d_{\downarrow}^k(y)) \). The parameter \( m^* \) is the effective mass of the electrons in the wire, \( \alpha_R \) is the Rashba spin-orbit coupling strength, \( E_z \) the Zeeman energy due to the perpendicular magnetic field \( B_z \), and \( \Delta \) the proximity induced s-wave superconducting gap, which we choose to be real. The operator \( d_{\uparrow}^k \) creates an electron in the \( j \)-th eigenstate of \( H_{\text{wire}} \) in the absence of superconductivity. We treat the charging term

\[ H_{\text{ch}} = \sum_j \left[ -eV_g + \frac{E_c}{2} \sum_{i \neq j} d_{i \uparrow}^j d_{i \downarrow}^j \right] \]  

(6)

in the Hartree approximation, which yields \( E_c(N_0 - 1) - eV_g \) for the expectation value of the expression in brackets in case of a hole-like co-tunneling process, and \( E_c N_0 - eV_g \) for an electron-like process. Here, \( E_c \) is the charging energy needed to add an electron to the dot, which is proportional to the inverse of the wire length, and \( V_g \) is the gate voltage. Coupling between dot and leads is described by the tunneling Hamiltonian

\[ H_{\text{tun}} = \sum_{j_{\alpha \sigma}} t_{\alpha j\sigma} \chi_{\sigma\alpha}(y_L) d_j + \text{h.c.} , \]  

(7)

where the couplings \( t_{\alpha j\sigma} = \tau_0 \int dy \Psi_{\alpha \sigma}^\dagger(y) \varphi_j(y) \) are approximated as the overlap integral between a decaying wave \( \Psi_{\alpha \sigma} \) from lead and the eigenfunction \( \varphi_j \) of the Hamiltonian \( H_{\text{wire}} \) for \( \Delta = 0 \) (see appendix). This approximation is relaxed later where we use a microscopic model to compute the couplings. In the topological regime, the weight of wave function \( \Psi_{\alpha \uparrow}^\dagger \) dominates over \( \Psi_{\alpha \downarrow} \), such that the tunneling barrier effectively filters one spin direction [30]. In the first part, we therefore focus
of the number of particles in the wire is given by consistently determined such that the expectation value \(\epsilon\) energies. From the Hamiltonians Eq. (5), (6), (7) we determine the \(V\) field. Here \(V_{g,mid}\) is the center between amplitude resonances corresponding to a particle number \(N_w\) = 35. The dashed gray line shows the decay of the amplitude according to Eq. (10), where the constant factor is obtained from a fit. We use \(\Delta = 2E_{so}\) and \(L = 32.5l_{so}\).

on the transmission amplitude for spin-up electrons \(|T_{↑↑}|\). From the Hamiltonians Eq. (5), (6), (7) we determine the energies \(\epsilon_j^{↑}(N_0, \{n_i\})\) and couplings \(\lambda_j^{↑}(N_0, \{n_i\})\) of lead \(\alpha\) by numerically solving the BdG equation, combined with analytical arguments for the spatial parity of wave functions (see appendix). We take into account the particle-hole redundancy in the solutions of the BdG equation by including only the eigenstates with positive energy [38]. The quantum dot contains an integer number \(N_0\) of electrons. On the other hand, the particle number in the wire \(N_w\) is fractional in general. We therefore describe the proximity effect using a mean-field superconductivity term in the wire Hamiltonian, but distinguish \(N_0\) from \(N_w\) [22]. The chemical potential \(\mu\) is self-consistently determined such that the expectation value of the number of particles in the wire is given by \(N_w\), but we take into account the total number of particles in the dot \(N_0\) when we determine the charging contribution to the effective single particle energies. When varying the gate voltage one observes a conductance resonance whenever a level crosses the Fermi level. We increase \(N_0\) by one after each such resonance. However, it is assumed that only a charge \(\Delta N_w \ll 1\) is added to the wire.

D. Parameters

For the numerical calculations, we use a spin orbit coupling strength of \(\hbar a_R = 0.2\) eV Å and an effective mass \(m^* = 0.02m_e\), which are typical for semiconductor structures such as InAs [8, 39]. From these parameters we obtain the characteristic energy and length scales \(E_{so} = a_R^2m^*/2 = 0.05\) meV and \(l_{so} = \hbar/(a_Rm^*) = 0.19\) μm, respectively. We discretize the wire Hamiltonian Eq. (5) to \(N\) lattice sites with lattice constant \(a = L/N = 0.026l_{so}\), where \(L\) denotes the wire length. We assume that each electron that is added to the dot contributes a charge \(\Delta N_w = 1/20\) to the wire. We use a charging energy \(E_c = 8E_{so}(32.5l_{so})/L\). For computing the average of the scattering matrix Eq. (2) we use finite temperature \(T = 34\) mK [23] corresponding to \(\beta = 18E_{so}\).

III. MAGNETIC FIELD DEPENDENCE OF THE TRANSMISSION AMPLITUDE

A. Magnetic field independent induced gap

We consider the transmission amplitude \(|T_{↑↑}(V_{g,mid})|\) as a function of Zeeman energy \(E_z\), computed at a gate voltage \(V_{g,mid}\) in the middle between the two conductance resonances for a fixed particle number \(N_w = (L/l_{so})(14/13)\), such that the particle density is the same for all wire lengths.

We first consider a magnetic field independent proximity gap \(\Delta = 2E_{so}\). By increasing \(E_z\), the transition to the topological phase takes place, in which an eigenstate close to zero energy is formed, separated from the second level by the topological gap (see Fig. 2). The transmission amplitude strongly increases when entering the topological phase at \(E_{z,top}\), reaches a peak value, and then decreases. In the topological regime the tunneling matrix element for a Majorana wave function is \(\propto 1/\sqrt{\xi}\).
where the correlation length

$$\xi = \frac{\hbar v_F}{\Delta_{p,\text{ind}}}$$  \hspace{1cm} (8)$$

with $v_F = \hbar k_F(1/m^* - \alpha_R^2(E_z^2 + \hbar^2 k_F^2)^{-1/2})$ is determined by the induced effective p-wave gap at the Fermi points in the hybrid wire [35, 40, 41]

$$\Delta_{p,\text{ind}} = \frac{\hbar k_F \alpha_R \Delta}{\sqrt{E_z^2 + \alpha_R^2 \hbar^2 k_F^2}}.$$  \hspace{1cm} (9)$$

With this, we obtain the Zeeman field dependence of the transmission amplitude as

$$|T_{\uparrow\uparrow}(V_{g,\text{mid}})| \sim \frac{m^* \alpha_R \Delta}{\hbar} \frac{1}{\sqrt{E_z^2 + \alpha_R^2 \hbar^2 k_F^2 - \alpha_R^2 m^*}}.$$  \hspace{1cm} (10)$$

proportional to the inverse field strength for large $E_z$ (dashed gray line in Fig. 2, in very good agreement with the numerical result taking a single level into account). When comparing the result for transmission through $j_{\text{max}} = 200$ levels (solid blue line) with that for a single level (Fig. 2, dotted red line) it becomes apparent that the amplitude at the beginning of the topological range is mostly determined by the lowest level, i.e. the MZMs. For very large Zeeman energy, the Majorana modes are split more strongly, and there is a small correction due to taking into account many higher levels. In the trivial regime for $E_z < E_{z,\text{top}}$, however, where the spacing between the lowest energy Bogoliubons is small, many levels contribute to the transmission amplitude, and interfere destructively.

**B. Magnetic field dependent induced gap**

For a thin superconductor subject to a parallel field, we describe the suppression of the induced s-wave superconducting gap by the magnetic field via [42]

$$\Delta(E_z) = \Delta(0) \left(1 - \left(\frac{E_z}{E_{z,c}}\right)\right)^{1/2},$$  \hspace{1cm} (11)$$

where $E_{z,c}$ is the critical Zeeman energy at which superconductivity is destroyed. Entering the topological region at $E_{z,\text{top}}$ is again accompanied by an increase in transmission amplitude (see Fig. 3). Further within the topological regime, the proximity gap $\Delta$ is reduced, and the correlation length $\xi \propto 1/|\Delta_{p,\text{ind}}|$ increases, i.e. the Majorana wave function delocalizes. Therefore, the amplitude drops to the normal-conducting value over a relatively narrow range of magnetic field values. Using Eq. (11) in Eq. (9), we find an amplitude dependence

$$|T_{\uparrow\uparrow}| \sim \frac{m^* \alpha_R \Delta(0)}{\hbar} \frac{\sqrt{1 - (E_z/E_{z,c})^2}}{\sqrt{E_z^2 + \alpha_R^2 \hbar^2 k_F^2 - \alpha_R^2 m^*}}.$$  \hspace{1cm} (12)$$

This dependence is depicted by the dashed gray line and fits well in the region where the amplitude decays to the normal-conducting value (see Fig. 3). For $E_z > E_{z,c}$, the wire is normal-conducting, and the amplitude is approximately constant. These results for the amplitude are in agreement with the recent experiment [23].
Figure 6. Dot-lead couplings for both spin directions using a wire of length $L = 45.5 \ell_{so}$. We use the Zeeman field dependent induced gap $\Delta(E_z)$ Eq. (11) with a critical field $E_{z,c} = 10E_{so}$. (a) Couplings to the lowest dot level using the steep potential and particle number $N_w = 47L/(39\ell_{so})$ in the wire. (b) Couplings to the lowest dot level using the smooth potential for $N_w = 53L/(39\ell_{so})$.

C. Wire length dependence

The non-locality of MZMs is expected to have a profound consequence when considering wires of varying lengths. In the inset of Fig. 4, the value of the amplitudes at the maximum and in the normal-conducting region are depicted as a function of the wire length $L$. From our scattering matrix analysis using a charging energy that is proportional to the inverse of the wire length, we find that the transmission amplitude is indeed proportional to the wire length in the topological region, while it is independent of the wire length in the normal-conducting range (see Fig. 4).

IV. DISORDER IN THE WIRE

The proposed experiment for establishing the wire length dependence of the transmission amplitude in the presence or absence of MZMs requires the comparison of different wires. Since these wires may differ from each other in terms of their detailed composition, we study how robust our results for the transmission amplitude are in the presence of on-site disorder. We use a Gaussian disorder distribution with zero mean and standard deviation $W$. Disorder is strong when the elastic scattering rate $\hbar/\tau$ from the impurities is on the order of the induced effective gap $\Delta_{p,\text{ind}}$ in the wire [43–50]. We define $W_m$ such that $\hbar/\tau = \Delta_{p,\text{ind}}$ for $W = W_m$, i.e.

$$W_m = \frac{\sqrt{2\ell_{so}E_{so}}}{\Delta_{p,\text{ind}}}.$$  \hspace{1cm} (15)

Numerical results of the amplitude for various disorder strengths are depicted in Fig. 5. When the disorder strength is smaller but of the order of $W_m$, the transmission amplitude is reduced at its maximum. This reduction is however much smaller that the peak height such that the proposed experiment is robust against disorder $W < W_m$. When using a disorder strength close to or larger than $W_m$, the amplitude is significantly reduced.

V. MICROSCOPIC MODEL FOR COUPLINGS

In the first part, we assumed that the couplings between lead and dot are determined by the dot wave functions at the ends of the wire. To validate this assumption, we consider a tight-binding model of leads and wire which are separated by tunnel barriers of shape $V_{\sigma,i}(y) = V_0 \exp(-y^2/(2\sigma_i^2))$. The potential at the left
lead are normal-conducting and without spin-orbit coupling. At position $y_1$ and energy $E_s$ the narrow Gaussian peak transitions continuously into the wide peak in the case of the smooth potential. The height of the peak is given by $V_0 = 65 E_{so}$. (b) Numerical results for the amplitude $|T_{↑↑}(V_{µ,mid}) + T_{↑↓}(V_{µ,mid})|$ with microscopic couplings as a function of the Zeeman field using a steep confinement potential and ground state particle number $N_w = 47L/(39 l_{so})$. We consider wires of length $L = 32.5 l_{so}, 39 l_{so}, 45.5 l_{so}, 52 l_{so}$, and $58.5 l_{so}$. The results are in good agreement with Fig. 4 where we used the more paradigmatic model for the couplings.

We define microscopic couplings as matrix elements $\lambda^u_{ασ} = \langle Ψ_{ασ}^u | H | Ψ_{ασ}^v \rangle$ and $\lambda^v_{ασ} = \langle Ψ_{ασ}^v | H | Ψ_{ασ}^u \rangle$ of the combined Hamiltonian of wire and leads. Here, $Ψ_i$ is the $i$-th BdG level in the wire, and due to the particle-hole symmetry and the absence of superconductivity in the leads we can write $\Phi_{ασ}^u = (\varphi^{(e,p,σ)}_α, \varphi^{(e,p,σ)}_α, 0, 0)$ and $\Phi_{ασ}^v = (0, 0, \varphi^{(e,p,σ)}_α, -\varphi^{(e,p,σ)}_α)$ where $(\varphi^{(e,p,σ)}_α, \varphi^{(e,p,σ)}_α)$ is the wave function localized in lead $α$ with spin $σ$ that is closest to the Fermi level. To numerically obtain the wave function localized in one region, we fix the potential at height $V_0$ in all the other regions. Fig. 6 depicts the couplings between lead and first dot level for spin-$↑$ and spin-$↓$ electrons. For both types of confinement, the couplings of spin-$↑$ electrons are dominant. It is therefore justified to only consider $|T_{↑↑}|$ in the more paradigmatic model in the first part. Due to the effective time-reversal symmetry $T = σ K$ with $K$ denoting complex conjugation, $T^2 = +1$, and $|H, T| = 0$, the transmission amplitudes $T_{↑↑}$ and $T_{↑↓}$ have both the same phase (modulo $π$), such that the magnitude of the interference term in Eq. (1) is given by $|T_{↑↑} + T_{↑↓}|$. We compute this magnitude for various wire lengths.

We distinguish two types of barrier potentials (i) only a narrow Gaussian peak and (ii) a narrow Gaussian peak together with a potential decaying smoothly into the wire (see Fig. 7(a)). Parameters for (i) the steep potential are given by $σ_1 = σ_2 = 0.1 l_{so}$, $E_s = V_0$ and $V_0 = 65 E_{so}$, and for (ii) the smooth confinement $σ_1 = 0.1 l_{so}$, $σ_2 = 6 l_{so}$, $E_s = 10 E_{so}$ and $V_0 = 65 E_{so}$. In case (i) there are zero-energy states only in the topological region, which are the MZMs (see inset of Fig. 7(b)). In case (ii), the Fourier decomposition of the smooth potential does not contain large momenta, so that in the trivial region each of the two bands contributes a pair of MZMs, which however are not coupled among each other by the potential [24, 25, 30]. Therefore, in addition to the MZMs in the topological region, two quasi-degenerate, quasi-zero-energy Andreev bound states (also called pseudo-MZMs) occur in the trivial region for $5 E_{so} < E_z < 7.6 E_{so}$ (see inset of Fig. 7(b)). Since they are nearly degenerate, there are two ground states with equal Boltzmann weight in the thermal average. For even $N_0$ the degenerate ground states for $E_z = E_z^0$ are states where either all $N_0$ electrons are in the condensate or $N_0 - 2$ electrons form the condensate and both pseudo-MZMs are occupied. In the case of odd $N_0$ there are $N_0 - 1$ electrons in the condensate and either the first or the second pseudo-Majorana level is occupied. In both cases the thermally averaged amplitude is proportional to $\sum_{j=1}^2 (\lambda_{νj,↓}^α \lambda_{νj,↑}^u + \lambda_{νj,↑}^v \lambda_{νj,↓}^u) = 0$. The anti-unitary reflection symmetry $\Pi \varphi_j(y) = K \varphi_j(L - y)$ (where $\varphi_j$ are eigenfunctions of $H_{wire}$) ensures that both terms
Figure 8. (a) Numerical results for the transmission amplitude $|T_{||}(V_{g,\text{mid}}) + T_{\perp}(V_{g,\text{mid}})|$ using tunnel couplings obtained for the smooth barrier potential as a function of the Zeeman energy. (b) Conductance through the wire without an interferometer in the case of a smooth potential for $N_w = 53L/(39 l_0)$. A comparison with (a) shows that the interferometer is crucial to distinguish MZMs from pseudo-MZMs. We consider wires of length $L = 32.5 l_0, 39 l_0, 45.5 l_0, 52 l_0$, and $58.5 l_0$. We use a Zeeman field dependent induced gap Eq. (11) with a critical field $E_{z,c} = 10 E_{so}$ and a particle number $N_w = 53L/(39 l_0)$. The insets depict the lowest two energy eigenvalues of the wire Hamiltonian for $L = 45.5 l_0$.

VI. COMPARISON BETWEEN INTERFEROMETER SETUP AND DIRECT CONDUCTANCE MEASUREMENT

In this section, we compare signatures from the interferometer setup (Fig. 1), with an easier to implement direct conductance measurement through the dot, without interferometer. In the calculation of the transmission amplitude through the dot (interferometer case) or the transmission probability (direct conductance), the main difference is how the thermal average is performed. For the calculation of the amplitude of conductance oscillations through the interferometer, the thermal average is performed over the complex transmission amplitude (see Eq. (2)), so that the transmission phase contributes to such an average. In the case of a direct conductance measurement, the squared absolute value of the transmission amplitude is averaged, and the phase information does not contribute. Fig. 8(b) depicts the direct conductance through the dot for a smooth confinement potential, analogous to Fig. 8(a). For MZMs we also find a maximum at the beginning of the topological region, whose height scales with the wire length. The crucial difference is that the conductance is not suppressed for pseudo-MZMs and thus this maximum is not a unique signature for the presence of MZMs.

VII. CONNECTION TO EXPERIMENT

In a recent experiment by Whiticar et al. [23], the transmission amplitude through a Coulomb blockaded Majorana wire was measured as a function of the Zeeman field. The experimental transmission amplitude shows a rapid growth upon entering the topological regime, followed by a pronounced maximum. Here, we discuss in detail how these features are explained by the localization properties of MZMs, which determine the transmission amplitude in the topological regime.

In the case of sufficiently long wires, in which the Majorana wave functions of opposite wire ends have negligible overlap, an analytical solution for the MZM wave functions can be found (see Appendix B). Moreover, since the transmission amplitude in the topological region is deter-
The behavior of the transmission amplitude at the beginning of the topological regime can be understood by the localization length $\xi_s$ alone, i.e. $|T_{\alpha\alpha}(V_{q,\text{mid}})| \propto 1/\xi_s$ (dashed gray line in the beginning of the topological regime). On the other hand, the behavior near the transition into the normal-conducting region, is due to the p-wave localization length $\xi_t$, i.e. $|T_{\alpha\alpha}| \propto 1/\xi_t$ (dashed gray line at the end of the topological regime). The maximum occurs where the magnitude of the localization lengths is roughly comparable.

The picture described above allows to explain the magnetic field dependence of transmission amplitude found by Whiticar et al. [23]. In the experiment, the transmission amplitude depends only weakly on the magnetic field in the region of small Zeeman fields, as predicted for the trivial phase. Above a device-specific value of the magnetic field, a rapid increase of the transmission amplitude is observed, which can be explained by the magnetic field dependence of $1/\xi_s$ at the beginning of the topological phase. Due to the divergence of $\xi_t$ at the phase transition $E_z = E_{z,\text{top}}$, the transmission amplitude increases linearly $|T_{\alpha\alpha}| \propto E_z - E_{z,\text{top}}$ in the topological regime. For larger Zeeman fields, a well-defined maximum of the amplitude arises in the experiment, which can be understood in terms of the concurrence of both correlation lengths $\xi$ and $\xi_s$. When superconductivity is destroyed by the magnetic field, Whiticar et al. observe a rapid decline of the transmission amplitude. This decrease can be explained in our model by the divergence of the coherence length $\xi$ due to the vanishing of the induced p-wave gap when approaching the critical magnetic field.

Since the amplitude of coherent transmission does not exhibit a maximum in the case of pseudo-MZMs, we believe that it is very likely that genuine topological MZMs were observed in the experiment. This is further supported by the observation that together with the appearance of the maximum also the even-odd splitting of the conductance resonances is suppressed. While the behavior of the transmission amplitude in the topological regime can be understood with our one-dimensional model, it is currently not possible to explain the large ratio between the value of the transmission amplitude at the maximum and the value in the normal-conducting regime for Device 2 measured by Whiticar et al. This could be because the amplitude in the experiment is not corrected for the influence of the transmission through the reference arm. On the other hand, it might be necessary to include the influence of orbital effects and several transverse subbands in the theoretical calculations for quantitative agreement between theory and experiment.
We believe that the experimental results are a promising step towards a proof for the presence of MZMs. Further evidence that MZMs can be consistently observed in these devices would be provided by a systematic study of wires with different lengths in future experiments.

VIII. CONCLUSION

We have studied coherent transport of electrons through a system hosting MZMs. We find that the Zee-mann field and length dependence of the transmission amplitude provide unique signatures of MZMs. When considering wires of varying lengths, the non-locality of MZMs yields a stable maximum of the amplitude at the onset of the topological regime, whose height is proportional to the wire length. In contrast, the amplitude is independent of the wire length if no localized MZM is present.

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APPENDIX A: DETAILS ON SCATTERING MATRIX FORMALISM

Truncation of the Hilbert space

As described in the main text, we do not explicitly model the superconductor but account for the proximity effect by including the induced superconducting gap directly into the Hamiltonian of the wire. However, we distinguish the particle number in the wire \( N_w \) from that in the dot \( N_0 \) consisting of wire and superconductor. Due to the Coulomb repulsion, simultaneous tunneling of more than one electron or hole is suppressed. We therefore truncate the Hilbert space to states of \( N_0 \) particles and states of \( N_0 + 1 \) electrons for electron-like co-tunneling processes and \( N_0 - 1 \) for hole-like co-tunneling, respectively, but take into account many BdG eigenstates. We denote the occupation number of the \( j \)-th BdG eigenstate by \( n_j \). We introduce states \( | N_0, \{ n_i \} \rangle \) and \( | N_0 \pm 1, \{ n'_i \} \rangle \) where the former is the initial dot state with \( N_0 \) electrons and occupation of BdG quasi-particle states \( \{ n_i \} \). The latter is the intermediate, excited state with \( N_0 \pm 1 \) electrons, and occupation numbers \( \{ n'_i \} \). As a result of the mean-field treatment of the interaction in the BCS approach, the theory does not describe a definite particle number \( N_0 \) in the dot. However, fixed-\( N_0 \) superconducting systems can even in case of small \( N_0 \) be adequately described in the grand-canonical BCS theory by choosing the chemical potential \( \mu \) such that the mean particle number \( \langle \hat{N}_0 \rangle \) is given by \( N_0 \) [51]. We determine the chemical potentials self-consistently for the particle number \( N_w \) in the wire and use the dot particle number \( N_0 \) for computing the charging energy. We note that as the couplings only depend on \( N_w \) and since we evaluate the transmission amplitude for a gate voltage in between conductance resonances, which are determined by the lowest effective hole-like and electron-like level, the amplitude does not depend on \( N_0 \) but only on \( N_w \). We therefore define the self-consistently determined chemical potential \( \mu \equiv \mu(N_0, \{ n_i \}) \) such that

\[
\langle \hat{N} \rangle \mu \equiv \int dy \left[ \sum_{j=1}^{2N} |v_j(y, \mu)|^2 + \sum_{j,n_j=1} \left( -|v_j(y, \mu)|^2 + |u_j(y, \mu)|^2 \right) \right] = N_w . \tag{A1}
\]

Here, due to the particle hole symmetry \( P = \tau_y \otimes \sigma_y K \), we only need eigenfunctions with non-negative eigenenergies \( E_j(\mu) \geq 0 \), which solve the BdG equation

\[
H_{\text{wire}}(\mu) \begin{pmatrix} u_j(\mu) \\ v_j(\mu) \end{pmatrix} = E_j(\mu) \begin{pmatrix} u_j(\mu) \\ v_j(\mu) \end{pmatrix} . \tag{A2}
\]

We rank order the energies and corresponding wave functions such that \( E_1 = \min \{ E_j \} \) and \( E_{j+1} > E_j \). An important exception to this rule occurs when the wire is in the topological regime \( | \mu | < \sqrt{E_F^2 - \Delta^2} \), where two Majorana sub-gap states are present in the full BdG spectrum. As the two Majorana wave functions overlap in a wire of finite length \( L \), they hybridize to form a finite energy sub-gap BdG state \( E_1 \). Increasing the chemical potential, one observes that this energy adiabatically evolves into a negative excitation energy \( -|E_1| \), which corresponds to a change in parity of the ground state. We then need to take the solution with \( E_{-1} = -E_1 \) and \( (u_{-1}, v_{-1}) = (v_1, u_1) \) as the lowest level, because it corresponds to the odd parity ground state in the topological regime [22]. We denote this solution again by \( E_1, (u_1, v_1) \) and use it for the corresponding chemical potentials in the computation of effective couplings and the effective energies. We next express the wire and tunneling Hamiltonian in terms of BdG operators \( \beta_j \). In order to do so, we first relate annihilation and creation operators \( d_{j, \sigma}, \dagger_{j, \sigma} \) to eigenfunctions \( \varphi_{j, \sigma} \) with spin \( \sigma \) of \( H_{\text{wire}} \) for \( \Delta = 0 \) via \( d_{j, \sigma} = \sum_\sigma \int dy \varphi_{j, \sigma}^*(y) \Psi_\sigma(y) \).
\[ \sum_{j} e^{-i\frac{2\pi}{L}j} \left[ u_{j\sigma}(y, \mu) \beta_j(\mu) + v_{\sigma}^*(y, \mu) \beta_j^*(\mu) \right] \] one finds

\[ H_{\text{wire}} = \sum_{\varepsilon_j \geq 0} \varepsilon_j(\mu) \beta_j^*(\mu) \beta_j(\mu) \]  \tag{A3}

\[ H_{\text{tun}} = \sum_{j, \alpha} c_j^{\dagger}(y_\alpha) e^{-i\frac{2\pi}{L}j} \left[ \lambda_{\alpha \sigma \sigma}(\mu) \beta_j(\mu) \\
+ \lambda_{\alpha \sigma \sigma}^*(\mu) \beta_j^*(\mu) \right] + \text{h.c.} \]  \tag{A4}

Here the effective couplings are defined by

\[ \lambda_{j, \alpha \sigma}(\mu) = \sum_{i, \sigma} \int dy \ t_{i, \sigma \sigma} \varphi_{i, \sigma}^*(y) u_{j, \sigma}(y, \mu) , \]  \tag{A5}

\[ \lambda_{j, \sigma \alpha}^*(\mu) = \sum_{i, \sigma} \int dy \ t_{i, \sigma \sigma} \varphi_{i, \sigma}^*(y) v_{j, \sigma}(y, \mu) . \]

In the first part, the couplings \( t_{i, \alpha \sigma} = t_0 \int dy \Psi_{\alpha, \sigma}(y) \varphi_i(y) \) are approximated as the overlap integral between a decaying wave \( \Psi_{\alpha, \sigma} \) from lead \( \alpha \) and the eigenfunction \( \varphi_i \) of the Hamiltonian \( H_{\text{wire}} \) for \( \Delta = 0 \). We take \( \Psi_{L, \sigma} \propto \exp(-y/\lambda) \) with \( \lambda = 0.26 l_0 \) and similar for the right end. Since the couplings \( t_{j, \alpha \sigma} \) are therefore mostly determined by the values of the wave functions \( \varphi_{j, \sigma}(y) \) at the end \( y_\alpha \) of the wire, and these wave function form an orthonormal set, the effective couplings are determined by the BdG wave functions \( u_{j, \sigma}, v_{j, \sigma} \) at the ends of the wire. This is why we can relate the localization of the Majorana wave functions to the couplings that determine the transmission amplitude.

### Coupling matrix elements and energy levels

To obtain the effective couplings \( \lambda_{j, \sigma \alpha}(\mu) (\lambda_{j, \alpha \sigma}^*(\mu)) \) for electron and hole like co-tunneling processes, we consider a tunneling event in which the dot is initially in the state \( |N_0, \{n_i\} \rangle \) and where co-tunneling takes place via an excited state \( |N_0 \pm 1, \{n_i'\} \rangle \). The couplings are given by the overlap \( \langle N_0, \{n_i\}; \{\alpha, \sigma\}|H_{\text{tun}}|N_0 \pm 1, \{n_i'\} \rangle \). In principle, we would need to consider overlap of condensate wave functions with different numbers of Cooper pairs, which however are not easily accessible in the BdG formalism. We hence neglect this contribution and use Eq. (A5) to determine the couplings, and we choose the chemical potential \( \mu \) in the computation of the wave functions such that it corresponds to intermediate BdG state with \( N_0 \pm 1 \) electrons through which the tunneling occurs.

We separately consider electron-like and hole-like processes and distinguish between even and odd particle number in the ground state. As only pairs of electrons can enter the condensate of the superconductor, the transmission depends on the number parity of \( N_0 \). For even \( N_0 \), the \( T = 0 \) ground state is given by \( |N_0, \{n_i = 0\}; \{\alpha, \sigma\}\rangle \), i.e. all electrons are in the condensate. For odd \( N_0 \), we assume that one electron resides in the first BdG eigenstate such that the ground state is given by \( |N_0, \{n_1 = 1, n_{\neq 1} = 0\}; \{\alpha, \sigma\}\rangle \). Electron-like intermediate states are \( |N_0 + 1, \{n'_i = 1, n'_{\neq i} = 0\}\rangle \) for even \( N_0 \) and \( \{\{n_0 + 1, \{n'_i = 0\}, N_0 + 1, \{n'_i = 1, n''_i = 1, n''_{\neq i} = 0\}\}\rangle \) for odd \( N_0 \). For hole-like co-tunneling, intermediate states have \( N_0 - 1 \) electrons and the occupancy of bogoliubons changes in an analogous way to the electron excited states described above. In this way, we find the following effective couplings for \( T = 0 \)

\[ \lambda_{\alpha, \sigma, \sigma}(N_0, \{n_i\}) = \begin{cases} \lambda_{\alpha, \sigma, \sigma}^w(\mu_n^w(\{n_i\})) & \text{for } n_j = 1 \\
\lambda_{\alpha, \sigma, \sigma}^v(\mu_n^v(\{n_i\})) & \text{for } n_j = 0 \end{cases} \]  \tag{A6}

\[ \lambda_{\alpha, \sigma, \sigma}(N_0, \{n_i\}) = \begin{cases} \lambda_{\alpha, \sigma, \sigma}^v(\mu_n^v(\{n_i\})) & \text{for } n_j = 1 \\
\lambda_{\alpha, \sigma, \sigma}^w(\mu_n^w(\{n_i\})) & \text{for } n_j = 0 \end{cases} \]  \tag{A7}

where the superscript \( e \) denotes electron-like and \( h \) hole-like couplings, \( \alpha \) is a lead index, \( \sigma \) the spin of the tunneling electron in the lead, and \( n_j \) is the state through which the tunneling occurs.

In addition, we consider excited initial states \( |N_0, \{n_i\} \rangle \) with energy \( E(\{n_i\}) = \sum_{n_j, n_j} E_j \), whose statistical weight is described by the Boltzmann factor \( e^{-\beta E(\{n_i\})} \) with \( \beta = 1/k_B T \). For even \( N_0 \), an even number of BdG states has to be occupied, and an odd number of BdG states for odd \( N_0 \). Numerically computing the excitation energies reveals that excited states with occupied levels \( n_j > 10 = 1 \) have too small Boltzmann factors to significantly contribute to the transmission amplitude. We therefore restrict ourselves to excited states where only levels with energies among the ten smallest ones can be occupied. Also states with more than three occupied levels yields negligible weights and are therefore neglected. In addition, we do not recompute the chemical potential for each excited state and instead use the chemical potential of the respective \( T = 0 \) state.

In addition to the tunneling matrix elements, the effective energies of the intermediate states are needed. We find

\[ \varepsilon_{\text{eff}, j}^e = \begin{cases} +\varepsilon_j(\mu_n^w(\{n_i\})) - eV_g + E_c(N_0 - 1) & \text{for } n_j = 1 \\
-\varepsilon_j(\mu_n^w(\{n_i\})) - eV_g + E_c(N_0 - 1) & \text{for } n_j = 0 \end{cases} \]  \tag{A8}

\[ \varepsilon_{\text{eff}, j}^e = \begin{cases} +\varepsilon_j(\mu_n^v(\{n_i\})) - eV_g + E_c N_0 & \text{for } n_j = 1 \\
+\varepsilon_j(\mu_n^v(\{n_i\})) - eV_g + E_c N_0 & \text{for } n_j = 0 \end{cases} . \]  \tag{A9}

Using these, we obtain the transmission amplitude from the scattering matrix Eq. (2) via \( t_{\uparrow \uparrow} = S(1,3) \) and \( t_{\downarrow \downarrow} = S(2,4) \). In the thermal average, we define \( Z = \sum_{\{n_j\}, \{n'_j\}} e^{-\beta E(\{n_i\})} \) where number parity of the \( n_i \) is determined by the number parity of \( N_0 \). The anti-unitary reflection symmetry \( \Pi \varphi_j(y) = K \varphi_j(L - y) \) (where \( \varphi_j \) are eigenfunctions of \( H_{\text{wire}} \)) ensures that \( T_{\sigma \overline{\sigma}} \) is imaginary in the middle between resonances where the real part of the denominator \( \approx E_j/2 \) is large compared to the level broadening. We consider transmission through the first \( j_{\max} \) levels.
APPENDIX B: ANALYTIC SOLUTION FOR MAJORANA WAVE FUNCTION

The BdG equations of a semi-infinite Rashba wire can in fact be solved analytically for an exact zero energy state [52]. Therefore, assuming a sufficiently long wire such that the Majorana wave functions of both ends have negligible overlap, one can derive an analytical expression for the Majorana wave functions. In this section, we present the analytical solution and a series of approximations that help to understand the emergence of the maximum of the transmission amplitude at the onset of the topological regime.

The solution of the BdG equation $H_{\text{wire}}\psi = 0$, $\psi(0) = 0$ for a zero energy state with $H_{\text{wire}}$ defined in Eq. (5) has the form $\psi = (\chi_\uparrow, \chi_\downarrow, i\chi_\downarrow, i\chi_\uparrow)$. By making use of the real functions $\exp(i\pi/4)\chi_\uparrow = \tilde{\chi}_\uparrow \in \mathbb{R}$, $i\exp(i\pi/4)\chi_\downarrow = \tilde{\chi}_\downarrow \in \mathbb{R}$ the BdG equation reduces to the two equations

$$
\begin{align*}
-\partial_y^2 \tilde{\chi}_\uparrow - \mu \tilde{\chi}_\uparrow - \tilde{E}_z \tilde{\chi}_\uparrow + 2\partial_y \tilde{\chi}_\downarrow + \tilde{\Delta} \tilde{\chi}_\downarrow = 0 \\
-\partial_y^2 \tilde{\chi}_\downarrow - \mu \tilde{\chi}_\downarrow + \tilde{E}_z \tilde{\chi}_\downarrow + 2\partial_y \tilde{\chi}_\uparrow - \tilde{\Delta} \tilde{\chi}_\uparrow = 0 .
\end{align*}
$$

(B1)

Here, we use reduced quantities $\tilde{\mu} = \mu/E_{so}$, $\tilde{E}_z = E_z/E_{so}$, $\tilde{\Delta} = \Delta/E_{so}$, $\tilde{y} = y/l_{so}$, and $\tilde{k} = kl_{so}$. Since Majorana modes are expected to be exponentially localized at the end of the semi-infinite wire, we use an ansatz

$$
\begin{align*}
\tilde{\chi}_\uparrow = e^{-A_y\tilde{y}} \left( \begin{array}{c} \theta_1 \\ \theta_1 \end{array} \right) .
\end{align*}
$$

With this ansatz, we obtain the system of equations

$$
\begin{align*}
-\tilde{A}^2 + \tilde{\mu} - \tilde{E}_z &= 2A + \tilde{\Delta} \\
-2A - \tilde{\Delta} &= -\tilde{A}^2 - \tilde{\mu} + \tilde{E}_z .
\end{align*}
$$

(B3)

The requirement for a non-trivial solution, i.e. a vanishing determinant of the coefficient matrix, yields the quartic equation

$$
0 = A^4 + A^2(2\tilde{\mu} + 4) + A(4\tilde{\Delta}) + \tilde{\mu}^2 - \tilde{E}_z^2 + \tilde{\Delta}^2
$$

(B4)

in $A$ which is already in the reduced form $A^4 + A^2\alpha + A\beta + \gamma = 0$ and can be solved analytically. By factorizing the polynomial $0 = (A - A_1)(A - A_2)(A - A_3)(A - A_4)$ using its four roots, and comparing to the above equation, one finds that $0 = A_1 + A_2 + A_3 + A_4$ and $A_1A_2A_3A_4 = \tilde{\mu}^2 - \tilde{E}_z^2 + \tilde{\Delta}^2$. The four solutions are given by

$$
A_i = \frac{1}{2} \left[ \pm_1 W \pm_2 \sqrt{W^2 - 4(\alpha + Y \pm_1 Z)} \right] .
$$

(B5)

with the abbreviations $P = -\alpha^2/12 - \gamma$, $Q = -\alpha^3/108 + \alpha\gamma/3 - \beta^2/8$, $U = ( -Q/2 + \sqrt{Q^2/4 + P^3/27})^{1/3}$, $Y = -5\alpha/6 + U - P/(3U)$, $W = \sqrt{\alpha + 2Y}$, and $Z = \beta/(2W)$. Here, $\pm_1$ and $\pm_2$ can individually be $+1$ or $-1$ to give rise to four solutions $A_i$. In the topological regime $\gamma = \tilde{\mu}^2 - \tilde{E}_z^2 + \tilde{\Delta}^2 < 0$, it can be shown that a solution with Re$A_1$, Re$A_2$, Re$A_3 > 0$, $A_1 = A_2^* < 0$, Im$A_3 = 0$, and Re$A_4 < 0$ exists. To be able to normalize the solution, the coefficient of the $A_4$ term needs to vanish. Then, Eq. (B3) has the solution

$$
\begin{align*}
\left( \begin{array}{c} \theta_1 \\ \theta_1 \end{array} \right) = N_i \left( \begin{array}{c} 2A_i + \tilde{\Delta} \\ A_i^2 + \tilde{\mu} + \tilde{E}_z \end{array} \right) ,
\end{align*}
$$

(B6)

where $N_i$ are normalization constants. In the topological regime, we define

$$
\begin{align*}
A_1 &= \tilde{\xi}_1^{-1} + i\tilde{k}_{\text{eff}} \\
A_2 &= \tilde{\xi}_1^{-1} - i\tilde{k}_{\text{eff}} \\
A_3 &= \tilde{\xi}_2^{-1} .
\end{align*}
$$

(B7) (B8) (B9)
Therefore, the Majorana wave function

\[ \hat{\chi}_L(y) = N \left[ e^{-y/\xi_2} \left( \frac{2\xi_2^{-1} + \Delta}{\xi_2^{-2} + \mu + E_z} \right) + e^{-y/\xi_1} \left\{ e^{ik_{\text{eff}}y} \left( \frac{2(\xi_1^{-1} + i k_{\text{eff}}) + \Delta}{(\xi_1^{-1} + ik_{\text{eff}})^2 + \mu + E_z} \right) \right\} \right]. \tag{B10} \]

In the following, we refer to this solution as "analytic solution". To compute the transmission amplitude, we use that the Majorana wave function at the right wire end is given by \( \chi_R(y) \propto \chi_A^\dagger(L-y) \) and evaluate the overlap with decaying wave functions from the leads.

We find that the analytic expression is in very good agreement with the numerical results (Fig. 10) in the topological regime, even when taking transport through many levels into account. However, without some approximations it is difficult to gain much insight into the lengthy analytical expression. In order to make progress, we first use that the Majorana wave function at the right wire end in good agreement with the numerical results for the localization lengths (dashed lines in Fig. 11) are in excellent agreement with the exact analytical expressions (solid lines in Fig. 11) in the whole topological regime. For the approximation in the main text, the Majorana wave functions are reduced to the sum of the envelops of oscillating and evanescent term, neglecting the spin dependence and the oscillations. As the couplings are determined by the wave function weights at the ends of the wire, the oscillations are less important. However, in the evanescent term, the spin-\( \downarrow \) component can be larger than the spin-\( \uparrow \) component at the beginning of the topological regime. Nevertheless, this rough approximation is still in good agreement with the numerical results for the case where \( k_F \) is approximately constant (as for a fixed particle number in the wire) and allows to understand the occurrence of the amplitude maximum.

\[ a = b^* = \frac{(i\xi_2 + \xi_1(-i + k_{\text{eff}}\xi_2))(-2 + 2(E_z + \mu)\xi_1\xi_2 - \Delta(\xi_1 + \xi_2) + ik_{\text{eff}}\xi_1(2 + \Delta\xi_2))}{4k_{\text{eff}}(1 + \xi_1(\Delta - (E_z - k_{\text{eff}}^2 + \mu)\xi_1))}\xi_2^{-1}. \tag{B11} \]

The evanescent term however, has a different correlation length whose divergence at the topological phase transition is governed by the closing of the topological gap at \( E_z = E_{z,\text{top}} \)

\[ \xi_2 = A_3^{-1} l_{\text{so}} \approx \xi_s = \left( -\xi^{-1} + \sqrt{\frac{\mu^2 + \Delta^2 - E_z^2}{\xi^{-2} + k_F^2}} \right)^{-1} \approx \frac{1}{E_z - \sqrt{\Delta^2 + \mu^2}}. \tag{B14} \]

Here, we used the relations between the \( A_i \) above Eq. (B5) to express \( A_3 \) in terms of \( A_1, A_2 \) and ultimately in terms of \( \xi \). We find that the approximations for the localization lengths (dashed lines in Fig. 11) are in excellent agreement with the exact analytical expressions (solid lines in Fig. 11) in the whole topological regime.

\[ \chi = \frac{h v_F}{\Delta_{\text{p,ind}}} \]
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