Modulated structures in a Lebwohl-Lasher model with chiral interactions

E. S. Nascimento∗1, A. Petri2, and S. R. Salinas3

1 Depto de Física, PUC-Rio, Rio de Janeiro, RJ, Brazil
2 CNR - Istituto dei Sistemi Complessi, Dipartimento di Fisica, Università Sapienza, Roma, Italy
3 Instituto de Física, USP, São Paulo, SP, Brazil

Abstract

We consider a Lebwohl-Lasher lattice model with nematic directors restricted to point along p planar directions. This XY Lebwohl-Lasher system is the nematic analogue of the standard p-state clock model. We then include chiral interactions, and introduce a chiral p-state nematic clock model. The statistical problem is formulated as a discrete non-linear map on a Cayley tree. The attractors of this map correspond to the physical solutions deep in the interior of the tree. It is possible to observe uniform and periodic structures, depending on temperature and a parameter of chirality. We find many different chiral nematic phases, and point out the effects of temperature and chirality on the modulation associated with these structures.

1 Introduction

Competing interactions are the basic mechanism to describe the onset of sequences of modulated phases in many physical systems [1–4]. In magnetism, the most investigated lattice statistical model to account for the existence of spatially modulated structures is the ANNNI model [5–7], which is an Ising system with competing ferro and antiferromagnetic interactions between first and second-neighbor spins along an axial direction. The ANNNI model exhibits one of the richest phase diagrams in the literature, with a wealth of modulated structures depending on temperature and on a parameter associated with the ratio between the strengths of the competing couplings. It is remarkable that the modulation associated with these periodic structures displays a nontrivial, staircase-like behavior, as a function of the thermodynamic field parameters [8].

There is an alternative mechanism of competition, which is suggested by an earlier proposal of Dzyaloshinskii and Moriya to explain the modulated structures of helimagnets [9][11].

∗edusantos18@esp.puc-rio.br
According to this Dzyaloshinskii-Moriya (DM) mechanism, in addition to the standard exchange interactions, first-neighbor vector spins along an axis are supposed to interact via chiral couplings. In classical statistical mechanics, this DM proposal has been mimicked by a \( p \)-state Chiral Clock (CC) model, with planar vector spin variables along \( p \) directions, and the inclusion of chiral interactions between first-neighbors along an axis. These CC models have been shown to lead to similar complex modulated structures as the ANNNI model. The phase diagrams of the CC models display sequences of many different helical ferromagnetic phases, and a complex behavior of the main wave numbers in terms of temperature \[12\] \[13\].

Spatially modulated structures with helical ordering are not exclusive of magnetic systems. In soft-matter physics, simple cholesteric phases in liquid-crystalline compounds provide perhaps the best examples of chiral nematic structures, in which the nematic director exhibits a spatial variation of helical type along a given direction \[23\]. Many physical aspects of cholesteric nematics can be studied in the framework of the phenomenological approaches, along the lines of the Landau-de Gennes theory, with the inclusion of adequate terms associated with the spatial variation of the nematic director \[16\] \[17\].

From the molecular point of view, according to an early work by Goossens \[18\], the consideration of an induced dipole-quadrupole contribution to the dispersion interactions can give rise to a cholesteric phase. This picture has been extended by some investigators \[19\] \[20\], with the formulation of a statistical model, of mean-field type, which might be able to describe the temperature effects in a cholesteric phase. In the calculations of van der Meer and collaborators \[19\], there is a modulated structure, which is not affected by the temperature. Although there is no attempt to draw a global phase diagram, the numerical mean-field calculations of Krutzen and Vertogen \[20\] point out the possibility of a temperature-dependent pitch if one includes a nematic-twist term in the original pair potential. Along Goossens’s ideas, Lin-Liu and collaborators \[21\] considered a planar model for the cholesteric phase, which does lead to the description of a temperature sensitive pitch. In these earlier statistical mechanics calculations it is difficult to point out the effects of chirality. Also, there is no attempt to draw a global phase diagram in terms of temperature and a parameter to gauge the strength of chiral interactions. In fact, by means of these earlier calculations, it is not clear to see how thermal fluctuations affect the cholesteric pitch.

The statistical mechanics calculations for the magnetic phase diagrams of the ANNNI and CC models provided the inspiration to carry out the present work. Similar calculations were still missing for the Dzyaloshinskii-Moriya picture of model systems with nematic interactions. We then formulated a \( p \)-state Chiral Clock model with head-tail symmetry, and checked the existence of long-period structures in a nematic setting. The main goal of this work is the establishment of qualitative contacts with the description of cholesteric structures in liquid crystals. In this preliminary analysis, calculations have been restricted to the simplest versions of these nematic clock models, which are already sufficient to point out the statistical origin of spatially modulated phases, to draw representative global phase diagrams, and give an indication of the change of the pitch with temperature and model parameters. We remark that it is easy to see that there is a correspondence between the essential terms in the Hamiltonians associated with the \( p \)-state Nematic Chiral Clock model and of a planar version of Goossen’s proposal, as it has been used by Krutzen and Vertogen \[20\] and by Lin-Liu and collaborators \[21\].

This paper is organized as follows. In Section 2 we consider the planar version of the
Lebwohl-Lasher model, which is obtained from the standard form of the Maier-Saupe interactions \[22–24\]. Some explicit calculations are performed in Section 3 in order to analyze the restricted 6-state Chiral Lebwohl-Lasher model on the Bethe lattice. Section 4 discusses the mean-field treatment of the problem, which is obtained by taking the limit of infinite coordination of the Bethe lattice. Finally, the conclusions as well an outlook of future work are presented in the Section 5.

2 The \(XY\) Lebwohl-Lasher model

In order to account for the head-tail symmetry of nematic liquid crystals, we define a $2 \times 2$ traceless tensor $Q$, given by

\[
Q_{\mu\nu} = \frac{1}{2} (2n_\mu n_\nu - \delta_{\mu\nu}),
\]

(1)

where $\mu, \nu = x, y$ are Cartesian coordinates, $\delta_{\mu\nu}$ is the usual Kronecker symbol, and $n_\mu$ is the $\mu$ component of a two-dimensional microscopic director,

\[
\vec{n} = \begin{pmatrix}
\cos \theta \\
\sin \theta
\end{pmatrix}.
\]

(2)

We then write the traceless tensor

\[
Q = \frac{1}{2} \begin{pmatrix}
2 \cos^2 \theta - 1 & 2 \cos \theta \sin \theta \\
2 \cos \theta \sin \theta & 2 \sin^2 \theta - 1
\end{pmatrix},
\]

(3)

where $\theta$ is the angle of the (planar) director with the $x$ axis.

According to the Maier-Saupe approach \[22–24\] to the nematic transition, the energy of interaction between a pair of nematic elements is written as

\[
\mathcal{H} = -J \sum_{(i,j)} \sum_{\mu,\nu} Q_{\mu i}^i Q_{\nu j}^j = -J \sum_{(i,j)} \mathbf{Q}_i \cdot \mathbf{Q}_j,
\]

(4)

where $J > 0$ is a positive energy parameter, and $(i, j)$ is a pair of lattice sites. Thus, given the local director elements,

\[
\vec{n}_i = \begin{pmatrix}
\cos \theta_i \\
\sin \theta_i
\end{pmatrix}, \quad \vec{n}_j = \begin{pmatrix}
\cos \theta_j \\
\sin \theta_j
\end{pmatrix},
\]

(5)

and discarding a harmless constant term, it is straightforward to write the energy of the LLL model,

\[
\mathcal{H}_{LLL} = -J \sum_{(i,j)} \cos^2 (\theta_i - \theta_j).
\]

(6)

This expression is just a (nematic) generalization of the Hamiltonian associated with the standard clock model. If we consider continuous angle variables, this is also the Hamiltonian of a nematic version of the classical \(XY\) spin model.

We now assume that chiral effects are mimicked by a rotation of the nematic director. As a result, we have

\[
\vec{n}_{rot} = \begin{pmatrix}
\cos \Delta & \sin \Delta \\
-\sin \Delta & \cos \Delta
\end{pmatrix} \begin{pmatrix}
\cos \theta \\
\sin \theta
\end{pmatrix} = \begin{pmatrix}
\cos (\theta - \Delta) \\
\sin (\theta - \Delta)
\end{pmatrix},
\]

(7)
where the angle $\Delta$ is the parameter that can be used to gauge chirality. The associated rotated tensor is given by

$$Q_{\text{rot}} = \frac{1}{2} \begin{pmatrix} 2 \cos^2 (\theta - \Delta) - 1 & 2 \cos (\theta - \Delta) \sin (\theta - \Delta) \\ 2 \cos (\theta - \Delta) \sin (\theta - \Delta) & 2 \sin^2 (\theta - \Delta) - 1 \end{pmatrix}.$$ (8)

Consequently, the pair interaction between nematogenic elements at sites $i$ and $j$ is given by

$$Q_i \cdot Q_{\text{rot},j} = \cos^2 (\theta_i - \theta_j + \Delta) - \frac{1}{2}.$$ (9)

which leads to the Hamiltonian of the CLLL model,

$$H_{\text{CLLL}} = -J \sum_{(i,j)} \left[ \cos^2 (\theta_i - \theta_j + \Delta) - \frac{1}{2} \right].$$ (10)

This last expression is a natural nematic extension of the well-known CC model [12, 13], which has been suitably generalized to describe chiral nematic-like phases.

### 3 The 6-state CLLL model

It is important to take into account that the description of macroscopic nematic structures requires the assumption of a head-tail symmetry of the microscopic states. This assumption suggests that we should restrict the considerations to $p$-state CC models with even values of the integer $p$. However, the simplest case, with $p = 4$, does not present complex modulated structures, as we explicitly show in Appendix A.

We then consider a 6-state model, with a discrete choice of angular variables,

$$\theta (k) = \frac{\pi}{3} (k - 1), \quad k = 1, ..., 6.$$ (11)

Due to the head-tail symmetry of the nematic systems, the pair of angular variables $\theta (1)$ and $\theta (4)$ leads to a single tensor variable, which we call $Q (1)$. Also, $\theta (2)$ and $\theta (5)$ lead to $Q (2)$, and the angular variables $\theta (3)$ and $\theta (6)$ lead to $Q (3)$. We then use equation (3) to write the microscopic tensor variables,

$$Q (1) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Q (2) = \frac{1}{4} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

$$Q (3) = \frac{1}{4} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$ (12)

In this nematic context, the problem is reduced to the consideration of just these three local states, as schematically represented in Figure 1.

We use the tensor variables (12) to analyze the statistical problem on a Cayley tree of ramification $r \geq 1$. It is quite well known that problems on a Cayley tree can be formulated
in terms of discrete non-linear dissipative mappings, whose attractors lead to solutions of physical interest. Similar calculations on a Cayley tree have been carried out for a $p$-state chiral clock model \cite{25} and for some analogs of the ANNNI model \cite{26}. Along the lines of these earlier calculations, we write the partial partition function of a tree of $n+1$ generations, $Z_{n+1}(i)$, with state $Q(i)$ for $i = 1, 2, 3$, at the root site, in terms of the three partial partition functions, $Z_n(1)$, $Z_n(2)$, and $Z_n(2)$, of the connected trees of the previous $n$ generation.

Physical solutions, in the deep interior of a large tree, correspond to the attractors of the non-linear map. This deep interior of a very large tree is known as a Bethe lattice, since the physical solutions are expected to agree with a simple pair or Bethe-Peierls approximation. In the limit of infinite coordination (and vanishing interactions) we are supposed to recover standard mean-field results \cite{25,26}. It should be clear that we take advantage of the structure of the Cayley tree, as represented in Figure 2, to investigate the occurrence of modulated structures.

We now adopt a more compact notation, and write a set of three recurrence relations between the partial partition functions of a tree of $n+1$ generations, $Z'_1 = Z_{n+1}(1)$, $Z'_2 = Z_{n+1}(2)$, and $Z'_3 = Z_{n+1}(3)$, and the partition functions of a tree of $n$ generations. Although it demands some algebraic effort, it is not difficult to obtain

$$
Z'_1 = [AZ_1 + BZ_2 + CZ_3]^r,
Z'_2 = [CZ_1 + AZ_2 + BZ_3]^r,
Z'_3 = [BZ_1 + CZ_2 + AZ_3]^r,
$$

(13)

where

$$
A = \exp \left\{ \beta J \left( \cos^2 \Delta - \frac{1}{2} \right) \right\},
B = \exp \left\{ \beta J \left[ \cos^2 \left( \frac{\pi}{3} + \Delta \right) - \frac{1}{2} \right] \right\},
C = \exp \left\{ \beta J \left[ \cos^2 \left( \frac{\pi}{3} - \Delta \right) - \frac{1}{2} \right] \right\},
$$

(14)

and $\beta$ is the inverse of temperature.

It is interesting to rewrite this map in terms of density variables,

$$
\rho_i = \frac{Z_i}{Z_1 + Z_2 + Z_3},
$$

(15)
for $i = 1, 2, 3$. Taking into account that

$$\rho_1 + \rho_2 + \rho_3 = 1,$$

we further reduce the problem to a two-dimensional map, in terms of two densities only,

$$\rho_1' = \frac{1}{D} \left[ C + (A - C) \rho_1 + (B - C) \rho_2 \right]^r,$$
$$\rho_2' = \frac{1}{D} \left[ B + (C - B) \rho_1 + (A - B) \rho_2 \right]^r,$$

with

$$D = \left[ C + (A - C) \rho_1 + (B - C) \rho_2 \right]^r +$$
$$+ \left[ B + (C - B) \rho_1 + (A - B) \rho_2 \right]^r +$$
$$+ \left[ A + (B - A) \rho_1 + (C - A) \rho_2 \right]^r,$$

where $A$, $B$, and $C$, are given by \cite{14}. The analysis of this two-dimensional system of equations leads to the stability borders of the phase diagrams in terms of temperature and the chiral parameter $\Delta$. It is easy to see that there is a trivial (disordered) fixed point, $\rho_1^* = \rho_2^* = 1/3$, which is linearly stable at sufficiently high temperatures.

Instead of working with the densities, we can change to more convenient variables from the physical point of view. We then consider the average value of the elements of the symmetric
and traceless tensor $Q$, given by equation (3), and define two new variables,

$$q_1 = \langle Q_{xx} \rangle = \langle \cos^2 \theta \rangle - \frac{1}{2},$$

$$q_2 = \langle Q_{xy} \rangle = \frac{1}{2} \langle \sin 2\theta \rangle. \tag{19}$$

Using the densities $\rho_1$, $\rho_2$, and $\rho_3$, we write

$$q_1 = \frac{3}{4} \rho_1 - \frac{1}{4},$$

$$q_2 = \frac{\sqrt{3}}{4} (-1 + \rho_1 + 2 \rho_2), \tag{20}$$

from which we have the densities in terms of new variables, $q_1$ and $q_2$,

$$\rho_1 = \frac{4}{3} q_1 + \frac{1}{3},$$

$$\rho_2 = \frac{1}{3} - \frac{2}{3} q_1 + \frac{2}{\sqrt{3}} q_2. \tag{21}$$

Therefore, if we insert these expressions in (17), we can as well work with a two-dimensional map in terms of the alternative variables $q_1$ and $q_2$, which are directly related to the tensor order parameter. It is immediate to see that $q_1^* = q_2^* = 0$ corresponds to the disordered fixed point ($\rho_1^* = \rho_2^* = 1/3$) of this model system.

In the special one-dimensional case, with ramification $r = 1$, we have

$$\begin{pmatrix} Z_1' \\ Z_2' \\ Z_3' \end{pmatrix} = M \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}, \tag{22}$$

where $M$ is a cyclic matrix,

$$M = \begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix}. \tag{23}$$

The eigenvalues of this cyclic matrix are given by

$$\Lambda_0 = A + B + C,$$

$$\Lambda_\pm = A - \frac{1}{2} (B + C) \pm \frac{\sqrt{3}}{2} (B - C). \tag{24}$$

Note the presence of the complex conjugate eigenvalues $\Lambda_+$ and $\Lambda_-$, which lead to an oscillating decay of the pair correlation function for most values of $\Delta$. In fact, this is already an indication of the existence of spatially modulated structures in larger dimensional systems. The fixed point, $\rho_1^* = \rho_2^* = 1/3$, is linearly stable, except at zero temperature. It is easy to analyze a number of statistical properties of the one-dimensional model. In particular, the ground state is highly degenerate at $\Delta = \pi/6$. 
4 Mean-field limit of the $p$-state CLLL model

We now resort to a well known technique to obtain mean-field results from the recursion relations on a Cayley tree. We take an “infinite coordination limit”, $r \to \infty$ and $J \to 0$, while keeping the product $rJ$ fixed. In this limit, the nonlinear map becomes somewhat more feasible to deal with, which allows us to grasp the main qualitative features of this problem.

In the infinite coordination limit, equations (17) can be rewritten as

$$\rho'_1 = \frac{\exp(\beta J r M_1)}{\sum_{i=1}^3 \exp(\beta J r M_i)}, \quad \rho'_2 = \frac{\exp(\beta J r M_2)}{\sum_{i=1}^3 \exp(\beta J r M_i)},$$

(25)

with

$$M_1 = -\frac{1}{4} \left( \cos 2\Delta + \sqrt{3} \sin 2\Delta \right) + \frac{1}{4} \left( 3 \cos 2\Delta + \sqrt{3} \sin 2\Delta \right) \rho_1 +$$

$$+ \frac{\sqrt{3}}{2} (\sin 2\Delta) \rho_2,$$

$$M_2 = -\frac{1}{4} \left( \cos 2\Delta - \sqrt{3} \sin 2\Delta \right) - \frac{\sqrt{3}}{2} (\sin 2\Delta) \rho_1 +$$

$$+ \frac{1}{4} \left( 3 \cos 2\Delta - \sqrt{3} \sin 2\Delta \right) \rho_2,$$

$$M_3 = \frac{1}{2} \cos (2\Delta) - \frac{1}{4} \left( 3 \cos 2\Delta - \sqrt{3} \sin 2\Delta \right) \rho_1 -$$

$$- \frac{1}{4} \left( 3 \cos 2\Delta + \sqrt{3} \sin 2\Delta \right) \rho_2.$$ (26)

The system of nonlinear equations (25) can be solved iteratively, and the solutions allow us to draw the global phase diagram exhibited in Figure 3.

It is immediate to show that the mean-field equations (25) lead to a disordered fixed point,

$$\rho^*_1 = \rho^*_2 = \frac{1}{3},$$

(27)

for all values of temperature and chiral parameter. A linear analysis of stability indicates that this disordered solution is unstable below a certain limiting reduced temperature,

$$T \equiv \frac{1}{\beta rJ} = \frac{1}{4},$$

(28)

for all values of $\Delta$. The eigenvalues of the recursion relations (25) about this trivial fixed point are a pair of complex conjugates,

$$\lambda = \frac{\beta J r}{4} (\cos 2\Delta \pm i \sin 2\Delta),$$

(29)

which is an indication of the transition to a modulated structure.
Figure 3: Phase diagram of the $p = 6$ nematic version of the Chiral Clock model; $T$ is the reduced temperature and $\Delta$ is the chiral field. We indicate an isotropic phase (ISO), a nematic phase ($N$), and a modulated nematic phase with wave number $q = 2\pi/3$. Region $N_c$ is occupied by a multitude of chiral nematic phases.

In order to check the properties of the isotropic phase, we have performed some numerical calculations to explore the solutions of the mean-field equations (25). It is easy to confirm the limit of stability of the trivial, disordered, fixed point. We do expect the onset of modulation below $T = 1/4$. However, as we can study in detail for $\Delta = 0$, there should be a small overlap between regions of stability of disordered and ordered solutions, which is an indication of a (weak) first-order transition, as in the three-state Potts model.

In the absence of chirality ($\Delta = 0$), recursion relations (25) are reduced to the simple expressions

$$
\rho_1' = \frac{\exp (c\rho_1)}{\exp (c\rho_1) + \exp (c\rho_2) + \exp [c (1 - \rho_1 - \rho_2)]},
$$

$$
\rho_2' = \frac{\exp (c\rho_2)}{\exp (c\rho_1) + \exp (c\rho_2) + \exp [c (1 - \rho_1 - \rho_2)]},
$$

(30)

where $c = 3\beta Jr/4$. We have already shown that the disordered fixed point, $\rho_1^* = \rho_2^* = 1/3$ is stable at high temperatures, $\beta Jr < 4$ (see Figure 3). There is also an ordered fixed point, $\rho_1^* = \rho_2^* = \rho^* \neq 1/3$, which comes from the equation

$$
\frac{9}{4} \beta Jrw = \ln \frac{1 + 6w}{1 - 3w},
$$

(31)

where

$$
\rho^* = \frac{1}{3} - w,
$$

(32)

which is the standard mean-field equation for the three-state Potts model.
It is immediate to see that there is always a disordered solution, $w = 0$. At low temperatures, for $\beta J r < 4$, it is also simple to see that there is another (ordered) solution, $w \neq 0$. However, a careful numerical analysis of equation (31) shows that the ordered solution is already present in a small range of temperatures above $k_B T / r J = 1/4$, which is a typical feature of first-order transitions.

![Figure 4: Periodic orbits for chiral field $\Delta = 0.4$ and different reduced temperatures. Thermal fluctuations affect the cholesteric pitch in a nontrivial fashion.](image)

It is remarkable that this simple model already displays a quite complex modulated behavior as a function of the “chiral field” $\Delta$ and the temperature $T$. For given values of $\Delta$ and $T$, the mean-field equations (25) can be solved iteratively. Depending on the values of the model parameters, it is possible to notice the presence of modulated structures, which are represented by periodic orbits in the $\rho_1 \times \rho_2$ plane, as shown in Figure 4. These orbits indicate the periodic behavior of the nematic order parameter along the generations of the tree.

It is possible to calculate the main harmonic component of the modulated structures as a function of the temperature, for all values of $\Delta$. In fact, our preliminary study clearly indicates that the temperature plays an important role on the modulation or cholesteric pitch. It is straightforward to perform numerical calculations to show that this nematic model system exhibits a phase lock-in behavior, which is also present in the ANNNI and CC models. In other words, the modulation is constant for some intervals of temperature or chiral parameter. This suggests a staircase-like variation of the pitch as a function of temperature in these nematic CC models.

5 Conclusions

We have shown that the Lebwohl-Lasher model used to describe the nematic-isotropic transition in liquid crystals, with the restriction of the microscopic nematic directors to point along $p$ planar directions, gives rise to a nematic counterpart of a $p$-state Clock model. We then
introduced chiral interactions between first-neighbor sites along a lattice direction, according
to a mechanism of chirality that has been used to explain different forms of helimagnetism.
The resulting XY chiral Lebwohl-Lasher statistical model, which we call CLLL model, is the
nematic counterpart of the well-investigated (magnetic) Chiral Clock models.

On the basis of previous calculations for the analogous magnetic models, we formulated
the statistical problem as a non linear mapping along the branches of a Cayley tree. It is
known that attractors of this map correspond to solutions of physical interest. Also, in a
suitable limit of infinite coordination of the tree, equations are somewhat simplified, and we
are supposed to recover standard mean-field results. We take full advantage of the Cayley
tree to investigate the presence of stable modulated structures.

In this preliminary work, we performed some explicit calculations for a 6-state Chiral
Clock model with head-tail symmetry. Besides the isotropic and nematic phases, we have
shown the existence of sequences of modulated structures, with a pitch that depends on
temperature and a parameter of chirality. A global phase diagram is presented, where it is
possible to identify uniform ordered states as well spatially modulated phases. These sketchy
calculations, however, are sufficient to illustrate the origins and the main qualitative features
of modulation (cholesteric behavior) in liquid-crystalline systems. We hope this investiga-
tion will be a guide for future (and more evolved) work, including connections with earlier
calculations for statistical molecular models [19–21] and predictions of the phenomenological
elastic theories [16].

Acknowledgement

This work is part of the research program of the INCT-FCx, which is funded by the Brazilian
organizations CNPq and Fapesp. E. S. Nascimento thanks CAPES for the financial support.
A. Petri is grateful for the kind hospitality of IFUSP.

A 4-state Nematic Chiral Clock model

We now use a similar approach to analyze a four-state XY chiral Lebwohl-Lasher model. We
consider just two traceless tensor microscopic states,

\[
Q(1) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q(2) = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]

from which we write the recursion relations

\[
Z_1' = \left[ Z_1 \exp \left( \frac{1}{2} \beta J \cos 2\Delta \right) + Z_2 \exp \left( -\frac{1}{2} \beta J \cos 2\Delta \right) \right]^r
\]
\[
Z_2' = \left[ Z_1 \exp \left( -\frac{1}{2} \beta J \cos 2\Delta \right) + Z_2 \exp \left( \frac{1}{2} \beta J \cos 2\Delta \right) \right]^r.
\]

We then introduce the density variables

\[
\rho_i = \frac{Z_i}{Z_1 + Z_2},
\]
for $i = 1, 2$, so that the problem is reduced to a one-dimensional map,

$$
\rho'_i = \frac{1}{D_4} \left[ \exp \left( -\frac{1}{2} \beta J \cos 2\Delta \right) + 2\rho_1 \sinh \left( \frac{1}{2} \beta J \cos 2\Delta \right) \right]^r,
$$

(36)

with

$$
D_4 = \left[ \exp \left( -\frac{1}{2} \beta J \cos 2\Delta \right) + 2\rho_1 \sinh \left( \frac{1}{2} \beta J \cos 2\Delta \right) \right]^r \right. +
$$

$$
\left. + \left[ \exp \left( \frac{1}{2} \beta J \cos 2\Delta \right) - 2\rho_1 \sinh \left( \frac{1}{2} \beta J \cos 2\Delta \right) \right]^r. \right.
$$

(37)

Taking into account that

$$
Q = \langle Q_{11} \rangle = \frac{1}{2} \rho_1 - \frac{1}{2} \rho_2 = \rho_1 - \frac{1}{2},
$$

(38)

this map can be written as $Q' = f(Q)$.

In the limit of infinite coordination ($r \to \infty, J \to 0$, with fixed values of $rJ$), we have the simple form

$$
Q' = \frac{1}{2} \tanh \left[ \beta J r \cos (2\Delta) Q \right],
$$

(39)

from which we see that the disordered fixed point is linearly stable for

$$
\frac{k_B T}{r J} > \left| \frac{1}{2} \cos 2\Delta \right|.
$$

(40)

In the $T - \Delta$ phase diagram, besides a trivial ferromagnetic structures, there is only a quite trivial antiferromagnetic arrangement.

**References**

[1] M. Seul and D. Andelman, *Science*, 267, 476–483 (1995).

[2] D. Andelman and R. E. Rosensweig, *J. Phys. Chem. B* 113, 3785–3798 (2009).

[3] A. Giuliani, Joel L. Lebowitz and Elliott H. Lieb, *AIP Conference Proceedings* 1091, 44-54 (2009).

[4] W. Selke, “Spatially modulated structures in systems with competing interactions” in *Phase Transitions and Critical Phenomena*, Vol. 15, p.1-72, ed. by C. Domb and J.L. Lebowitz (Academic Press, 1992).

[5] P. Bak, *Repts. Progr. Phys.* 45, 587 (1982).

[6] W. Selke, *Phys. Rep.* 170, 213 (1988).

[7] J. Yeomans, *Sol. State Phys.* 41, 151 (1988).

[8] E. S. Nascimento, J. P. de Lima and S. R. Salinas, *Physica A* 409, 78–86 (2014).
[9] Yu. A. Izyumov, Sov. Phys. Usp. 27, 845 (1984).

[10] J. Kishine, K. Inoue, Y. Yoshida, Progr. Theor. Phys. Suppl. 159, 82 (2005).

[11] M. Shinozaki, S. Hoshino, Y. Masaki, J. Kishine, Y. Kato, J. Phys. Soc. Jpn. 85, 074710 (2016); T. Toretsume, T. Kikuchi, R. Anita, J. Phys. Soc. Jpn. 87, 041011 (2018).

[12] D. A. Huse, Phys. Rev. B 24, 5180 (1981).

[13] H. C. Öttinger, J. Phys. C 16, L257 (1983); J. Phys. C 16, L597 (1983).

[14] M. Siegert and H. U. Everts, Z. Phys. B 60, 265 (1985).

[15] M. Pleimling, B. Neubart, R. Siems, J. Phys. A 31, 4871 (1998).

[16] R. D. Kamien and J. V. Selinger, J. Phys.: Condens. Matter 13, R1 (2001).

[17] J. V. Selinger, Introduction to the Theory of Soft Matter: From Ideal Gases to Liquid Crystals (Springer, New York, 2016).

[18] W. J. A. Goossens, Mol. Cryst. Liq Cryst. 12, 237-244 (1971).

[19] B. W. van der Meer, G. Vertogen, A. J. Dekker and J. G. J. Ypma, J. Chem. Phys. 65, 3935 (1976).

[20] B. C. H. Krutzen and G. Vertogen, Liq. Cryst. 6, 211 (1989).

[21] Y. R. Lin-Liu, Yu Ming Shih, Chia-Wei Woo, and H. T. Tan, Phys. Rev. A 14, 445 (1975).

[22] P. A. Lebwohl and G. Lasher, Phys. Rev. A 6, 426 (1972).

[23] P. G. de Gennes and J. Prost, The Physics of Liquid Crystals (Clarendon Press, Oxford, 1993).

[24] S R. Salinas and E. S. Nascimento, Mol. Cryst. Liq. Cryst. 657, 27 (2017).

[25] C. S. O. Yokoi and M. J. de Oliveira, J. Phys. A 18, L153 (1985); A. T. Bernardes and M. J. de Oliveira, J. Phys. A 25, 1405 (1992).

[26] M. J. de Oliveira and S. R. Salinas, J. Phys. A 18, L1157 (1985); C. S. O. Yokoi, M. J. de Oliveira, and S. R. Salinas, Phys. Rev. Lett. 54, 163 (1985).