Empirical performance of GARCH, GARCH-M, GJR-GARCH and log-GARCH models for returns volatility

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Abstract. Volatility plays an important role in the field of financial econometrics as one of the risk indicators. Many various models address the problem of modeling the volatilities of financial asset returns. This study provides a new empirical performance comparison of the four different GARCH-type models, namely GARCH, GARCH-M, GJR-GARCH, and log-GARCH models based on simulated data and real data such as the DIA, S&P 500, and S&P CNX Nifty indices on a daily period from January 2000 to December 2017. We also investigate the estimation results obtained using Solver’Excel and verify those results against the results obtained using a Markov chain Monte Carlo method. The simulation study showed that the GARCH model is outperformed by other models. Meanwhile, the empirical study provides evidence that the GJR-GARCH model provides the best fitting, followed by the GARCH-M, GARCH, and log-GARCH models. Furthermore, this study recommends the use of Excel’s Solver in practice when the parameter estimates for GARCH-type model do not close to zero.

1. Introduction

The theory and practice of economic development led to the empirical findings that financial return time series exhibit heteroscedasticity, means the volatility (standard deviation) of returns changes over time. Since Engle [1] introduced Autoregressive Conditional Heteroscedasticity (ARCH) model for modeling the current variance (squared-volatility) as a linear function of past squared-returns and generalized by Bollerslev [2] to the GARCH (Generalized ARCH) model by adding the past conditional variance, study in the financial econometrics area is dominated by the modification of ARCH and GARCH-type models and get serious attention from researchers, practitioners, and policymakers. GARCH-type models contribute to the ease of predicting the future volatility of financial time series and hence the result in financial applications can help investors to make investment decisions.

A number of modifications of the standard GARCH models have been conducted for many years such as the GARCH-in-Mean (GARCH-M), GJR-GARCH, and log-GARCH models. The GARCH-M model was proposed by Engle et al. [3] which introduces an effect of conditional volatility into the returns process. Glosten et al. [4] extended the GARCH-M and suggested a model, popularly known as the GJR-GARCH, allowing the current variance has a different response to the past return. Another class of the GARCH-type model that appears to have the same characteristics but less attention is the so-called log-GARCH models introduced by Geweke [5] and Pantula [6] independently. Asai [7] derived
this model as a representation of the simple Stochastic Volatility (SV) process (see [8] for a discussion on SV models). Recently, Sucarrat et al. [9] proposed a generalization of the log-GARCH model by adding other conditioning variables to the conditional volatility equation. Unfortunately, Asai and Sucarrat et al. do not provide an empirical study whether the log-GARCH model improves the GARCH model. Motivated by their studies, it is important to check the performance of the log-GARCH model for simulated and real data. Meanwhile, some empirical studies showed that the GARCH-M and GJR-GARCH models work well, e.g., in [10–14]. To the best of authors’ knowledge, there is no literature that provides an empirical comparison of the above models. Therefore, the main contribution of this study is to investigate the empirical performance and some characteristics of the GARCH-M, GJR-GARCH, log-GARCH models compared to the standard GARCH model. Another important contribution is the use of Excel’s Solver to estimate the model parameters. So, the main difference between this paper and others is the use of Excel’s Solver and both simulated and real data to check the modeling performance of GARCH models.

The remainder of this paper is organized as follows. Section 2 discusses the considered models and the empirical methods employed in estimation. Section 3 conducts the empirical study and its results on the basis of simulation and real data. Section 4, finally, concludes and offers some possible modifications to improve the models.

2. Four types of GARCH model and their estimation

2.1. GARCH models

Conditional variance determined through the GARCH model allows the current conditional variance depends not only on past return but also on the past conditional variance. Let \( R_t \) denote the asset returns at time \( t \) and computed by \( R_t = 100 \times (\log S_t - \log S_{t-1}) \), where \( S_t \) denote the price of an asset at time \( t \). A simple specification of GARCH-type models, the so-called GARCH(1,1), is expressed by

\[
R_t = \sigma_t \varepsilon_t, \varepsilon_t \sim N(0,1),
\]

\[
\sigma^2_t = \omega + \alpha R^2_{t-1} + \beta \sigma^2_{t-1},
\]

where \( \omega > 0, \alpha \geq 0 \) and \( \beta \geq 0 \) for the positivity of variances and \( 0 \leq \alpha + \beta < 1 \) for the stationarity of variances. On the basis of empirical study, Hansen and Lunde [15] found no evidence that the GARCH(1,1) model is inferior to other ARCH-type models.

2.2. GARCH-M models

The GARCH-M model was proposed by Engle et al. [3] by establishing a relationship between return and conditional variance directly where the current return is expressed as a linear function of the current variance. In particular, the GARCH(1,1)-M model is defined by

\[
R_t = k \sigma^2_t + \sigma_t \varepsilon_t, \varepsilon_t \sim N(0,1),
\]

\[
\sigma^2_t = \omega + \alpha R^2_{t-1} + \beta \sigma^2_{t-1},
\]

where the constraints for parameters are the same as those in the GARCH(1,1) model.

2.3. GJR-GARCH models

The GJR-GARCH model of Glosten et al. [4], also known as the threshold GARCH (T-GARCH) model, is proposed to capture an asymmetric behavior by allowing the current conditional variance has a different response to the past positive and negative returns. In particular, the model GJR-GARCH(1,1) is expressed as follows:

\[
R_t = \sigma_t \varepsilon_t, \varepsilon_t \sim N(0,1)
\]

\[
\sigma^2_t = \omega + (\alpha + \gamma I_{t-1}) R^2_{t-1} + \beta \sigma^2_{t-1},
\]

where the positivity of conditional variances is assured by \( \omega > 0, \alpha \geq 0, \beta \geq 0, \) and \( \alpha + \gamma \geq 0 \), the variances stationary is assured by \( \alpha + \beta + 0.5\gamma < 1 \), and \( I \) is an indicator function that expressed by

\[ I_t = \begin{cases} 0 & \text{if } R_{t-1} \geq 0, \quad \text{(good news)} \\ 1 & \text{if } R_{t-1} < 0. \quad \text{(bad news)} \end{cases} \]
The following interpretation is given in [16]. When \( \gamma = 0 \), the model reduces to the standard GARCH model which treats bad news \((R_{t-1} < 0)\) and good news \((R_{t-1} > 0)\) symmetrically: that is, bad news and good news have the same impact \((\alpha R^2_{t-1})\) on the conditional variance \(\sigma_t^2\). When \( \gamma \neq 0 \), the news impact is asymmetric: that is, bad news and good news have different impacts on the conditional variance. Bad news has an impact of \( \alpha + \gamma \) on conditional variance, while good news has an impact of \( \alpha \) on conditional variance. Hence, if \( \gamma > 0 \), bad news has a larger impact on conditional variance than good news.

### 2.4. Log-GARCH models

Motivated by thinking about the positivity of conditional variances, Geweke [5] and Pantula [6] gave a logarithmic version of the GARCH model by taking the logarithm of the current conditional variance as a linear function of the logarithm of past squared-return and the logarithm of past conditional variance. This study focuses on the log-GARCH(1,1) model expressed by

\[
R_t = \sigma_t \varepsilon_t, \text{ where } \varepsilon_t \sim N(0,1),
\]

\[
\log \sigma_t^2 = \omega + \alpha \log R^2_{t-1} + \beta \log \sigma_{t-1}^2,
\]

where \(|\alpha + \beta| < 1\) for the stationarity of log-variances and \((\alpha + \beta)\beta > 0\) for the positivity of variances, see [7] for the explanation.

For all the above models, the objective is to estimate parameters that maximize the conditional log-likelihood function. In general, the conditional log-likelihood function for the above models is as follows:

\[
\log L (R_t | \sigma_t^2) = -\frac{1}{2} \left[ \log(2\pi \sigma_t^2) + \frac{(R_t - k\sigma_t^2)^2}{\sigma_t^2} \right],
\]

where the above function is for the GARCH-M(1,1) model when \( k \neq 0 \) or for the others when \( k = 0 \). The models are adjusted to the simulated and real data by the Excel’s Solver tool, which is easy to use and has been widely used by academics and financial practitioners. Implementation of the Excel’s Solver for the GARCH(1,1) model was demonstrated by some researchers, e.g., Alexander [17], Christoffersen [18], and Nugroho et al [19]. Furthermore, the Adaptive Random Walk Metropolis (ARWM) method as in [20] is implemented in Matlab to verify the Excel’s Solver estimates. The method has been showed more efficient and much faster than the Hamiltonian Monte Carlo method that was used in [21–23].

### 3. Simulation and empirical application

This section investigates the fitting performance of the competing models on the basis of simulated and real data. In this application, we follow the experimental procedures and algorithms demonstrated by Nugroho et al [19] and Nugroho [20].

#### 3.1. Simulation study

The simulations are carried out by generating 1,000 returns data from each model, excluding the GARCH(1,1) model, using the true parameter values presented in tables 1–3. These values are set based on the empirical studies in the literature. In this case, the Excel’s Solver is initialized by setting the initial values \( \omega = 0.03, \alpha = 0.04, \beta = 0.89, k = 0.5 \) for the GARCH-M(1,1) model, \( \omega = 0.06, \alpha = 0.38, \gamma = -0.31, \beta = 0.4 \) for the GJR-GARCH(1.1) model, and \( \omega = 0.04, \alpha = 0.04, \beta = 0.89 \) for the log-GARCH(1,1) model. The estimation results by the Excel’s Solver are presented in tables 1–3 for the GARCH(1,1)-M, GJR-GARCH(1,1), and log-GARCH(1,1) models, respectively. The log\((L)\) and LRT Stat represent log-likelihood and Log-likelihood Ratio Test statistic, respectively. The LRT statistic is computed by twice the difference in log-likelihoods.

In terms of relative errors, the authors found that Excel’s Solver is reliable to estimate all considered models. However, the authors noted that Excel’s Solver is very sensitive to the initial guess value of a parameter. If the parameter estimate cannot be found, the author uses an initial value that is close to the true value. Furthermore, all cases provide evidence supporting the extended models indicated by the values of LRT statistic which are larger than the chi-square critical values at any conventional level with
degrees of freedom equal to the difference in the number of parameters. It means that all extended models fit significantly better than the standard model. This result shows that all extended GARCH models have the potential to outperform the standard GARCH model.

### Table 1. Simulation results from the GARCH(1,1)-M models.

| Parameter | True value | GARCH(1,1)-M | GARCH(1,1) |
|-----------|------------|--------------|------------|
|           |            | Estimate     | Relative error | Estimate | Relative error |
| $\omega$  | 0.04       | 0.035        | 13.6%       | 0.026    | 35.4%         |
| $\alpha$  | 0.05       | 0.044        | 11.4%       | 0.069    | 38.0%         |
| $\beta$   | 0.90       | 0.914        | 1.6%        | 0.914    | 1.6%          |
| $k$       | 0.50       | 0.437        | 12.7%       |          |               |
| Total log(L) |          | −1473.15    |             | −1577.38 |
| LRT Stat  |           | 208.46       |             |          |

Source: Authors’ calculations.

### Table 2. Simulation results from the GJR-GARCH(1,1) and GARCH(1,1) models.

| Parameter | True value | GJR-GARCH(1,1) | GARCH (1,1) |
|-----------|------------|----------------|-------------|
|           |            | Estimate       | Relative error | Estimate | Relative error |
| $\omega$  | 0.07       | 0.071          | 1.17%       | 0.082    | 16.97%         |
| $\alpha$  | 0.40       | 0.399          | 0.29%       | 0.280    | 30.13%         |
| $\gamma$  | −0.30      | −0.303         | 0.94%       |          |               |
| $\beta$   | 0.50       | 0.502          | 0.40%       | 0.428    | 14.35%         |
| Total log(L) |          | −717.13       |             | −726.19  |
| LRT Stat  |           | 18.12         |             |          |

Source: Authors’ calculations.

### Table 3. Simulation results from the log-GARCH(1,1) and GARCH(1,1) models.

| Parameter | True value | log-GARCH(1,1) | GARCH(1,1) |
|-----------|------------|----------------|-------------|
|           |            | Estimate       | Relative error | Estimate | Relative error |
| $\omega$  | 0.05       | 0.061          | 22.22%      | 0.070    | 39.35%         |
| $\alpha$  | 0.05       | 0.066          | 32.05%      | 0.110    | 120.82%        |
| $\beta$   | 0.90       | 0.857          | 4.74%       | 0.800    | 11.11%         |
| Total log(L) |          | −1254.30      |             | −1264.55 |
| LRT Stat  |           | 20.50         |             |          |

Source: Authors’ calculations.

3.2. **Empirical results for exchange rates**

This section provides the empirical analysis of four considered models using daily returns for the Dow Jones Industrial Average (DJIA), Standard and Poors 500 (S&P 500), and S&P CNX Nifty stock indices.
The sample period is from January 2000 to December 2017, excluding the zero returns. These data have been downloaded from the Oxford-Man Institute of Quantitative Finance (https://realized.oxfordman.ox.ac.uk).

Table 4. Empirical results.

| Stock      | Model                | Tool      | $\omega$ | $\alpha$ | $\beta$ | $\gamma$ | $k$         | $\log(L)$       |
|------------|----------------------|-----------|----------|----------|---------|----------|-------------|-----------------|
|            | GARCH(1,1)           | Solver    | 0.0107   | 0.1038   | 0.8890  |          |            | −5732.74       |
|            |                      | Matlab    | 0.0111   | 0.1053   | 0.8873  |          |            | −5734.12       |
| DJIA       | GARCH(1,1)-M         | Solver    | 0.0107   | 0.1037   | 0.8891  | 0.0178   | 0.0589     | −5731.84       |
|            |                      | Matlab    | 0.0116   | 0.1072   | 0.8848  |          |            | −5723.68       |
|            | GJR-GARCH(1,1)       | Solver    | 0.0143   | 0.0000   | 0.8977  | 0.1825   | 0.0589     | −5640.75       |
|            |                      | Matlab    | 0.0149   | 0.0053   | 0.8919  | 0.1838   |            | −5642.85       |
|            | log-GARCH(1,1)       | Solver    | 0.0900   | 0.0635   | 0.9209  |          |            | −5806.16       |
|            |                      | Matlab    | 0.0934   | 0.0659   | 0.9175  |          |            | −5807.93       |
|            | GARCH(1,1)           | Solver    | 0.0103   | 0.0947   | 0.8979  |          |            | −5868.08       |
|            |                      | Matlab    | 0.0109   | 0.0964   | 0.8956  |          |            | −5869.50       |
| S&P 500    | GARCH(1,1)-M         | Solver    | 0.0104   | 0.0949   | 0.8975  | 0.0075   | 0.0470     | −5862.74       |
|            |                      | Matlab    | 0.0115   | 0.0989   | 0.8925  |          |            | −5862.89       |
|            | GJR-GARCH(1,1)       | Solver    | 0.0154   | 0.0000   | 0.8974  | 0.1754   | 0.0790     | −5772.04       |
|            |                      | Matlab    | 0.0161   | 0.0033   | 0.8923  | 0.1790   |            | −5779.07       |
|            | log-GARCH(1,1)       | Solver    | 0.0782   | 0.0544   | 0.9331  |          |            | −5948.55       |
|            |                      | Matlab    | 0.0806   | 0.0560   | 0.9308  |          |            | −5950.27       |
|            | GARCH(1,1)           | Solver    | 0.0193   | 0.1039   | 0.8850  |          |            | −5422.50       |
|            |                      | Matlab    | 0.0225   | 0.1142   | 0.8734  |          |            | −5424.29       |
| S&P CNX Nifty | GARCH(1,1)-M         | Solver    | 0.0193   | 0.1038   | 0.8851  | 0.0157   | 0.0367     | −5421.80       |
|            |                      | Matlab    | 0.0228   | 0.1130   | 0.8739  |          |            | −5419.90       |
|            | GJR-GARCH(1,1)       | Solver    | 0.0229   | 0.0558   | 0.8788  | 0.1040   | 0.1071     | −5395.81       |
|            |                      | Matlab    | 0.0244   | 0.0582   | 0.8744  | 0.1071   |            | −5396.74       |
|            | log-GARCH(1,1)       | Solver    | 0.0367   | 0.0245   | 0.9730  |          |            | −5533.61       |
|            |                      | Matlab    | 0.0372   | 0.0248   | 0.9725  |          |            | −5535.13       |

Source: Authors’ calculations.

The estimation results are summarized in table 4. To verify the results of Excel’s Solver in the case of real data, the Matlab estimates are also provided. In all cases, the results show that Excel’s Solver and Matlab produce similar estimates. The only exception is the parameter estimate of $k$ for the GARCH(1,1)-M model and the parameter estimate of $\alpha$ for the GJR-GARCH(1,1) model in the case of DJIA and S&P 500 stocks. In particular, authors found that the Excel’s Solver produce a value of zero when the estimate tends to zero. However, the authors note that the value of zero for $\alpha$ does not greatly affect other estimates. This result indicates the Excel’s Solver is not reliable when the estimate of a
parameter is too close to zero. For this point of view, the next analysis only focuses on the results of the Matlab.

In the case of DJIA data, the values of LRT statistic for the GARCH(1,1)-M, GJR-GARCH(1,1), and log-GARCH(1,1) models against the GARCH(1,1) model are 20.88 (significant), 182.54 (significant), −147.62 (not significant), respectively. In the case of S&P 500, the values of LRT statistic for the GARCH(1,1)-M, GJR-GARCH(1,1), and log-GARCH(1,1) models against the GARCH(1,1) model are 13.22 (significant), 180.86 (significant), −161.54 (not significant), respectively. In the case of S&P 500 CNX Nifty data, the values of LRT statistic for the GARCH(1,1)-M, GJR-GARCH(1,1), and log-GARCH(1,1) models against the GARCH(1,1) model are 8.78 (significant), 55.10 (significant), −221.68 (not significant), respectively. These results indicate that the GJR-GARCH(1,1) model provide the best fit for all considered data, followed by the GARCH(1,1)-M, GARCH(1,1), and log-GARCH (1,1) models.

Considering the GJR-GARCH(1,1) model, the estimate of $\gamma$ in all data cases is positive. It indicates the presence of the asymmetric effect of the past returns on the current conditional variance. Comparing the variance persistence, which is denoted by $\phi$ and given by $\phi = \alpha + \beta$ for the GARCH(1,1) and GARCH(1,1)-M models, by $\phi = \alpha + \beta + 0.5\gamma$ for the GJR-GARCH(1,1) model, and by $\phi = [\alpha + \beta]$ for the log-GARCH(1,1) model, the persistences implied by the GARCH(1,1) and GARCH(1,1)-M models are very close to each other and greater than those implied by the GJR-GARCH(1,1) model. These results show that adding the conditional variance in the returns process does not greatly affect on the variance persistence. Meanwhile, the presence of asymmetric effect causes the conditional variance is less persistent and more volatile. Furthermore, according to the variance half-life defined by log($0.5)/\log(\phi)$ in [24], the conditional variance implied by the GJR-GARCH(1,1) model is faster to move halfway back towards its unconditional mean than those implied by the GARCH(1,1) and GARCH(1,1)-M models. For example, in the case of adopting the S&P 500 returns, the conditional variance takes about 47 days implied by the GJR-GARCH(1,1) model, which is shorter than 81 days implied by the GARCH(1,1)-M model and than 87 days implied by the GARCH(1,1) model. Finally, considering the log-GARCH(1,1) model, the estimation results show that the logarithmic transformation does not greatly affect the parameter estimates.

4. Conclusion and future works
This study compared empirical performance of the four different GARCH(1,1)-type models, including the standard GARCH(1,1), GARCH(1,1)-M, GJR-GARCH(1,1), and log-GARCH(1,1) models, on the basis of simulated data and real data, namely DJIA, S&P 500, and S&P 500 CNX Nifty stock indices. The results are as follows. First, on the basis of the simulation and empirical results, the Excel’s Solver is reliable to estimate the GARCH(1,1)-type models when the estimate of $\alpha$ does not too close to zero. It confirms the result of Nugroho et al. [25]. Second, the simulation results show that all extended GARCH(1,1) models have the potential to better fit than the standard GARCH(1,1) model. In the case of real application, the first and second best fitting specification are respectively provided by the GJR-GARCH(1,1) and GARCH(1,1)-M models, meanwhile, the log-GARCH(1,1) model is outperformed by the GARCH(1,1) model.

There are several possible modifications of the above models that deserve further study. The future study could focus on the models’ comparison along with two distributions (normal and Student-t). It would be interesting to investigate the application of Box-Cox transformations for the return series and the lagged variance as in [26,27].

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