Defining and generating current in open quantum systems

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Defining current in open quantum systems can be problematic: No general description exists for the current of operators not conserved by the system-environment interaction. We fill this gap by deriving a general formula for probability current on an arbitrary graph, universally applicable to any system-environment interaction. We furthermore provide a representation of the average current, whereby the operator is first measured weakly, then strongly. When the dynamics is of Lindblad form, we derive an explicit formula for the current. We exemplify our theory by analysing a simple Smoluchowski-Feynman-type ratchet, operating deep in the quantum regime. Consisting of only two interacting particles, each moving on a three-site ring, the ratchet displays several novel quantum effects, such as tunnelling-induced current inversion, which we relate to the onset of quantum contextuality, and steady-state entanglement generation in the presence of arbitrarily hot environment. The role of spatial symmetry in current generation is also studied.

I. INTRODUCTION

Current is a central notion in transport theory and non-equilibrium thermodynamics. Defining quantum current in closed systems is straightforward \cite{1}. Yet, despite the considerable attention quantum transport and quantum walks in dissipative systems have received \cite{2–11}, there exists no general definition of quantum current in open quantum systems, which would take into account both tunnelling and environment-induced hopping. The issue has been addressed only in the specific case of the position operator current in continuous-variable systems with local, only position-dependent system-environment interaction, in the Markovian approximation \cite{3, 5, 10}. Moreover, unlike heat to work conversion, which is fairly well understood both classically \cite{12, 13} and quantumly \cite{14, 15}, motion generation solely utilising thermal disequilibrium has mainly remained in the focus of classical thermodynamics and transport \cite{12, 13, 16–24}. Only a handful of models have been studied in the quantum regime: Except for several works \cite{25–29} studying the (angular) momentum in continuous-variable thermal rotors, to our knowledge, only Ref. \cite{30} studies persistent current generation in a fully quantum model, albeit in the limit of the moving part being isolated from the environment.

In any realistic model, however, all parts of the device will, in one way or another, be coupled to the environment (think of all models inherited from the classical domain, e.g., the Smoluchowski-Feynman ratchet \cite{12, 16}). Therefore, in order to study current generation in quantum devices, one first has to deal with the fundamental problem of defining the current in open quantum systems. We solve this problem in the most general form, by deriving a surprisingly simple formula for probability current, universally applicable to systems undergoing arbitrary dynamics (be it Markovian or non-Markovian). We furthermore establish a link between the current and quantum weak measurements, which allows us to gain valuable intuition about the theory.

Turning to the problem of current generation, we illu-
is the most general representation of any part of any classical thermal machine. The probability current—which defines the mass/charge current and information transport—between vertices \( j \) and \( j' \), at any moment of time, is quite standardly defined as [13]

\[
J_{j \to j'} = p_j W_{jj'} - p_{j'} W_{j'j},
\]

(1)

where \( p_j \) is the probability of finding the particle at vertex \( j \) and \( W_{jj'} \) is the transition rate from vertex \( j \) to \( j' \).

Surprisingly, no general quantum analogue of Eq. (1) exists, even when the particle’s open-system dynamics can be described by a standard Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation (ME) [33] (e.g., the Born-Markov and secular approximations are applicable [33]):

\[
\frac{d\rho}{dt} = -i[H, \rho] + \sum_{\lambda} \gamma_{\lambda} \mathcal{S}[\Lambda_{\lambda}]|\rho|.
\]

(2)

Here, \( \rho \) is the state of the particle, \( H \) is its Hamiltonian, and \( \mathcal{S}[\Lambda]|\rho| = \Lambda_{\rho} \Lambda_1 - \frac{1}{2}\{\Lambda_{\rho} \Lambda_1, \rho\} \), with \( \{\cdot, \cdot\} \) denoting the anticommutator. To gain some intuition, let us for a moment assume that being at each vertex corresponds to a pure state, and that these states constitute an orthonormal basis of the particle’s Hilbert space (i.e., that the particle has no internal structure). Then, the classical picture, including the current being given by Eq. (1), is recovered in the trivial case when \( \rho \) and \( H \) are diagonal in the vertex basis at all times and the “jump” operators, \( \Lambda_{\lambda} \), are of the form \( |j\rangle \langle j'| \). This is due to the fact that, in such cases, the quantum evolution, determined entirely by the dynamics of \( \rho \)’s diagonal elements, exactly coincides with the classical stochastic dynamics.

However, whenever the Hamiltonian is not diagonal in the vertex basis (in other words, if tunnelling is possible between vertices), defining the current becomes problematic. Indeed, although the probabilities to find the particle at given vertices are always well defined, defining the current becomes problematic. The classical protocol—measure at \( j \), evolve, measure at \( j' \)—executed in the quantum regime [2, 7, 8, 11], excludes the coherent part of the dynamics, and hence cannot provide a complete picture.

In order to find a proper definition for the current, we will undertake a more consistent approach—standardly used in quantum transport theory—to defining current, which is based on the continuity equation. In this approach, the current of an operator \( x \), \( J \), is inferred from the fact that, in the Heisenberg picture, the continuity equation \( dx_j/dt = -\text{div}J \) must hold [3–5, 34, 35] (note that current is not to be confused with velocity [1, 36]). When the configuration space is discrete, as is the case here, the continuity equation takes the form

\[
\frac{dx_j}{dt} = \sum_{j' \neq j} J_{j' \to j},
\]

(3)

where the subscript \( j \) means that the operator \( x \) is taken at the position (vertex) \( j \). Since we are interested in the particle current, from now on, the operator \( x_j \) in the Schrödinger picture, will be the projector on the \( j \)’th vertex (for example, in the above simplified picture where “the particle is at vertex \( j \)” was equivalent to “the particle is in pure state \( |j\rangle \)”, one would have \( x_j = |j\rangle \langle j| \)). Moreover, in order to determine the current at the moment of time \( t \), the transition from the Schrödinger to Heisenberg picture has to be performed at the very same moment of time \( t \). Such a reservation has to be made in order to ensure that the current is actually being calculated at a given location (see the discussion in Appendix A on the Heisenberg picture of general non-unitary dynamics).

Now, keeping in mind that \( \sum_j x_j = I \) (where \( I \) is the identity operator on the full Hilbert space of the particle) holds at all times in the Heisenberg picture, it is straightforward to show that the quantities

\[
J_{j \to j'} = \frac{1}{2} \left\{ x_j, \frac{dx_{j'}}{dt} \right\} - \frac{1}{2} \left\{ x_{j'}, \frac{dx_j}{dt} \right\}
\]

(4)

satisfy the continuity equation (3). In view of the above, note that \( J \) can depend on time only through the time derivatives of \( x_j \) and \( x_{j'} \), and the average current is given by \( \langle J_{j \to j'}(t) \rangle = \text{Tr}[\rho(t)J_{j \to j'}(t)] \). We note that Eq. (4) applies whenever \( \{x_j\} \) is a complete set of orthogonal projectors. Eq. (4) is the main result of this subsection: it defines the probability current in arbitrary quantum systems undergoing arbitrary dynamics (as long as it is positive and trace-preserving, see also Appendix A).

When the evolution is governed by a GKSL equation, Eq. (2), the Heisenberg-picture operators evolve according to

\[
\frac{dx_{j}}{dt} = i[H, x_j] + \sum_{\lambda} \gamma_{\lambda} \left( \Lambda_{\lambda} x_j x_j \Lambda_{\lambda} - \frac{1}{2} \{ \Lambda_{\lambda} x_j, x_j \} \right).
\]

(5)

Hence, taking into account that \( x_j x_{j'} = x_j \delta_{j,j'} \), for an arbitrary GKSL ME, we obtain that \( J_{j \to j'} = J_{j \to j'}^{(\text{th})} + J_{j \to j'}^{(\text{tun})} \), where

\[
J_{j \to j'}^{(\text{tun})} = \text{Tr}(x_j H x_{j'} - x_j H x_j)
\]

(6)

is the current resulting from the vertex-to-vertex transitions caused by the internal dynamics, which is often referred to as tunnelling (hence the superscript of the operator). This operator is routinely used to describe particle current in condensed-matter physics [1, 3–6, 34]. Note that \( J_{j \to j'}^{(\text{tun})} \) is essentially the discretized version of the textbook current associated with the continuous-space Schrödinger equation \([1]: J(q) = \frac{\hbar}{2m} \{ \Psi(q) \nabla \Psi(q) - \Psi(q) \nabla \Psi^*(q) \} \), where \( \Psi(q) \) is the state in the position \( (q) \) representation, and the corresponding current operator is \( \frac{1}{2m} \{ p, (q) \} \), with \( p \) being the momentum operator (cf. Eq. (4)). The second term, \( J_{j \to j'}^{(\text{th})} \), is the current resulting from vertex-to-vertex jumps originating from the interaction with the environment, and is given by

\[
J_{j \to j'}^{(\text{th})} = \frac{1}{2} \sum_{\lambda} \gamma_{\lambda} \{ x_j, \Lambda_{\lambda}^\dagger x_j \Lambda_{\lambda} \} - \{ x_{j'}, \Lambda_{\lambda}^\dagger x_j \Lambda_{\lambda} \},
\]

(7)
where, in both $J_{j\to j'}^{(th)}$ and $J_{j\to j'}^{(tun)}$, all operators are in the Schrödinger picture (see also Appendix A). Note that Eqs. (6) and (7) apply to any GKSL ME. An analogue of Eq. (7) was derived in refs. [3, 5, 10] for the special case of continuous-variable systems which couple to their environment through local, position-dependent interaction terms (i.e., when the operator, the current of which is to be determined, commutes with the system-environment interaction Hamiltonian).

As any other quantity inferred from its divergence, the current defined by Eq. (4) can be supplemented by any zero-divergence term, and it will still satisfy the continuity equation (3). Among this infinite family of possible currents, we choose the one given by Eq. (4) for two fundamental reasons. First, for the minimal non-trivial situation—a particle on a three-vertex graph, evolving according to a GKSL ME—we show by an explicit calculation of the derivative of $x_j$ that there arise no terms other than those in Eqs. (6) and (7); see Appendix B for the derivation. Second, as we show below, its average can be rewritten in the form of Eq. (1), and, in the classical limit discussed below Eq. (2), it converges to the classical current.

The remarkably simple Eq. (4) for the current of a particle undergoing an arbitrarily general quantum evolution on an arbitrary graph, to the best of our knowledge, has not been reported in the literature before. In order to better understand its meaning, let us rewrite the derivative of $x_j$ according to a GKSL ME— we show by an explicit calculation of the derivative of $x_j$ that there arise no terms other than those in Eqs. (6) and (7); see Appendix B for the derivation. Second, as we show below, its average can be rewritten in the form of Eq. (1), and, in the classical limit discussed below Eq. (2), it converges to the classical current.

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the onset of contextuality can be detected by observing the inversion of the thermal component of the average current (see Fig. 2).

III. MODEL RATCHET

The graph has to have at least three vertices in order for the particle current to be meaningful. Therefore, the minimal system that can harbour current is a three-dimensional Hilbert space, and the vertices of the graph correspond to three mutually orthogonal pure state which constitute a basis in that Hilbert space. In other words, the minimal system is a qutrit, which can be realized by physical objects as diverse as a spin-1 particle, viewed from the perspective of the eigenstates of the angular momentum in some direction, or an atom on an optical lattice [32], confined to hop and tunnel (the confining potential cannot be infinite) between three neighbouring potential wells.

Our model rotor consists of two such minimal systems, particle $a$ and particle $b$, each interacting with their own thermal baths, respectively, at temperatures $T_a$ and $T_b$ (see Fig. 1 for an illustration). The self-Hamiltonians of these particles are taken to be translation (rotation) invariant and periodic, so that the “ratchetness”, i.e., the spatial asymmetry indispensable [12, 19, 20, 22, 24] for triggering the motion, is delegated to the interaction between the particles. Specifically,

$$H_{ab} = \tau X_a \otimes \mathbb{I}_b + \tau \mathbb{I}_a \otimes X_b + H_{\text{int}},$$

where $\mathbb{I}$ is the identity operator in the corresponding Hilbert space and, in the basis of vertex states,

$$X = \sum_{k=1}^{3} (|k+1\rangle \langle k| + |k\rangle \langle k+1|).$$

$\tau X$ determines the local rotation-symmetric self-Hamiltonians of the particles, with $\tau$ being the tunnelling rate between vertices. Here, the index is cyclic in the sense that $|k+3\rangle \equiv |k\rangle$.

We choose the interaction term to be classical, $\langle j_a, j_b | H_{\text{int}} | j_a, k_b \rangle = \delta_{j_a, k_a} \delta_{j_b, k_b} U_{\text{int}}(j_a, j_b)$, with the potential $U$ being of Potts form [31]:

$$U_{j_a, j_b} = \frac{K}{2} \cos \left[ \frac{2\pi}{3} (j_a - j_b) + \phi \right].$$

Such a potential mimics a dipole-dipole interaction between two spins, each pointing to one of the three equispaced directions specified by the angles $2\pi j/3$. It can be immediately seen that the potential is periodic in each coordinate, $U(j_a + 3, j_b) = U(j_a, j_b + 3) = U(j_a, j_b)$, however, unless $\phi = 0$, it breaks the exchange symmetry in that swapping the particles bears an energetic cost: $U_{j_a, j_b} \neq U_{j_b, j_a}$. In what follows, we will see that it is this exchange symmetry breaking that is responsible for the occurrence of thermal current in our model.

Reading from Eq. (6), we obtain that the tunnelling current of, say, the particle $a$ is given by

$$J_{j_a \rightarrow j_a+1}^{\text{tun}} = i \tau \langle j_a + 1 | - j_a + 1 \rangle \otimes \mathbb{I}_b \otimes \mathbb{I}_a.$$  (13)

Let us now couple the particles to the environment. We will consider only the limit of weak coupling to a Markovian environment, so that the evolution of the rotor is governed by a standard GKSL ME. One can use two approaches towards prescribing a GKSL equation to our model: phenomenological and that based on microscopic derivation. Phenomenologically, one wishes a GKSL ME which, in the classical limit (expressed by $\tau \to 0$), would coincide with the classical ME [24]:

$$\frac{d p_{j_a, j_b}}{dt} = \sum_{j'_a, j'_b} \gamma_{j_a, j'_a} |p_{j_a, j'_a} W_{j_a, j'_a} - p_{j_a, j_b} W_{j_a, j_b} |^2 \gamma_{j'_a, j_b}.$$  (14)

There, the sum is primed to indicate that it does not include simultaneous jumps, and the transition rates satisfy local detailed balance (e.g., for the jumps of particle $a$, $W_{j_a, j'_a} W_{j'_a, j_a} = \exp \left[ (U_{j_a, j'_a} - U_{j'_a, j_a}) / T_a \right]$). Now, it is straightforward to see that the GKSL equation (see Eq. (2)) with the jump operators of the form $|j_a \rangle \langle j'_a| \otimes |j_b \rangle \langle j'_b|$ and the dissipation rates $\gamma_{j, j'}$ coinciding with the transition rates $W_{j_a, j'_a} |j', j_a\rangle$, is equivalent to the classical ME. Thus, using these jump operators also when $\tau \neq 0$, one obtains a quantum GKSL master equation with a well-defined and correct classical limit. This is the standard quantum ME used, e.g., in the theory of quantum random walks [8, 9]. This ME falls under the category of “local” GKSL MEs since its dissipators are of product form, and hence it cannot generally grasp the global dynamics of the system. This is typically expressed by the thermal state not being a steady-state solution of the local ME when the temperatures of the baths are equal. This brings about thermodynamically inconsistent behaviour such as non-zero particle or heat flow between two thermal baths at equal temperatures [49–51], or, when the temperatures are different, spontaneous heat flow against the temperature gradient [52].

This is contrasted by the microscopic approach, when one derives the equation for the state of the system from
the exact dynamics of the system-plus-environment composite, under the Born-Markov and secular approximations [33]. The GKSL equation thus obtained takes full account of the global structure of the Hamiltonian even if the environments act locally on the subsystems, and it does not suffer from thermodynamic inconsistencies. In order to derive such a ME, one has to specify the full, system-plus-environment Hamiltonian and use the Born-Markov, rotating-wave, and secular approximations [33] in order to be able to derive a closed-form equation for the density matrix of the system. This is a standard procedure [33], and, for a general total Hamiltonian of the form

\[ H_{\text{tot}} = H_{ab} + \sum_{\alpha=a,b} A_{\alpha} \otimes B_{\alpha} + \sum_{\alpha=a,b} h_{\alpha}, \quad (14) \]

where \( h_{a} \) and \( h_{b} \) are the Hamiltonians of the baths and \( A_{\alpha} \) and \( B_{\alpha} \) are operators belonging, respectively, to the particles and the baths, yields the following ME:

\[ \frac{d\rho}{dt} = -i[H, \rho] + \sum_{\alpha=a,b} \gamma_{\alpha}(\omega) S[A_{\alpha}(\omega)][\rho] \equiv \mathcal{L}_{\text{gl}}[\rho], \quad (15) \]

with the jump operators given by

\[ A_{\alpha}(\omega) = \sum_{E_{k} - E_{m} = \omega} \Pi_{E_{m}} A_{\alpha} \Pi_{E_{k}}. \quad (16) \]

Here, \( E_{m} \) is an eigenvalue of \( H_{ab} \) and \( \Pi_{E_{m}} \) is the eigen-projector corresponding to it. Note that the \( \omega \)'s are all the gaps of the system’s bare Hamiltonian \( H_{ab} \). Furthermore, since we are working in the weak coupling regime, we will henceforth omit the Lamb shift [33] correction to the Hamiltonian, in view of its being a second-order effect [33].

The bath dissipation rates are given by bath correlation functions: \( \gamma_{\alpha}(\omega) = \int_{-\infty}^{\infty} ds e^{i\omega s} \langle B_{\alpha}^{\dagger}(s)B_{\alpha}(0) \rangle \). For a generic bosonic bath, living in a single spatial dimension, we find [33]

\[ \gamma_{\alpha}(\omega) = \frac{g|\omega|}{1 - e^{-\beta_{\alpha}|\omega|}} \times \begin{cases} e^{\beta_{\alpha}|\omega|} & \text{when } \omega \leq 0 \vspace{2mm} \text{1} & \text{when } \omega > 0 \end{cases}. \quad (17) \]

These functions satisfy the detailed balance condition: \( \gamma_{\alpha}(\omega) = e^{-\beta_{\alpha}|\omega|}, \) which, combined with Eq. (16), guarantees that, when \( \beta_{a} = \beta_{b} \), the thermal state is a steady solution for the global ME given by Eq. (15) [33]. We will use \( \gamma_{\alpha}(\omega) \) also when dealing with the classical and local GKSL MEs.

Lastly, we emphasize that Eq. (15) is qualitatively different from the local ME in that the jump operators cannot be represented by a product of local terms. Moreover, since the spectrum of \( H_{ab} \) contains degeneracies and some of the gaps may be repeated, the jump operators (Eq. (16)) will generally not be of rank-1. Therefore, the global and local ME do not coincide even when the inter-particle interaction, controlled by \( K \), goes to zero. Neither do they in the “classical” limit of \( \tau \to 0 \). The discrepancy is in fact exacerbated when \( K \) or \( \tau \) go to zero, which is due to the increase in the degeneracy of \( H_{ab} \). Given that there is no interaction between the particles when \( K \ll 1 \), it is obvious that it is the description provided by the global ME that fails. Indeed, as \( K \) decreases, so do also some of the gaps in the spectrum of \( H_{ab} \), leading to the breakdown of the secular approximation [33, 53]. In fact, in the weak coupling regime of some models [53, 54], the local ME leads to a steady state closer to that obtained via an exact solution of the system as a whole, including the baths.

### A. Steady states and symmetries

Symmetries of a system strongly affect its transport properties [4, 19, 23, 55–57]. In this section, we will describe the symmetries of our machine’s Hamiltonian and show how they relate to the transport properties of the machine.

In order to make a proper bookkeeping of the role symmetry breaking plays in current generation, we exclude any source of asymmetry in the system-environment interaction, by not only choosing the total Hamiltonian to be symmetric under rigid rotations of the rotor as a whole, given by the operator \( R = \sum_{j_{a},j_{b}} |j_{a} + 1, j_{b} + 1\rangle \langle j_{a}, j_{b}| \) and its integer powers, but also by additionally enforcing the particle-bath interactions to be symmetric under local rotations, given by \( R_{a} = \sum_{j_{a} = 1}^{3} |j_{a} + 1\rangle \langle j_{a}| \) and its integer powers. It is easy to see that, by choosing

\[ A_{a} = \frac{1}{3} X \otimes I_{b} \quad \text{and} \quad A_{b} = \frac{1}{3} I_{a} \otimes X, \quad (18) \]

where \( X \) is given by Eq. (11), we satisfy both of these symmetry properties. Being the arithmetic mean of the Gell-Mann matrices \( \lambda_1, \lambda_4, \) and \( \lambda_6 \) (see, e.g., [58]), \( X \) is in fact the most natural choice for a particle-bath coupling, in that it generalizes the x-component of spin in \( SU(2) \) (the standard 2-dimensional Pauli X operator) to \( SU(3) \).

As is shown in Ref. [55], the existence of any “strong” symmetry of the total Hamiltonian, e.g., \( [H_{\text{tot}}, R] = 0 \), necessarily implies that the microscopically derived GKSL ME (see Eqs. (15) and (16)) has multiple steady states. More specifically, there are at least as many linearly independent, trace-1 steady solutions of the ME as there are distinct eigenvalues of \( R \). Importantly, this ambiguity means that the steady state of the evolution bears some memory on the initial state [55, 56]. Now, for \( \phi \neq k\pi/3 \), our numerical analysis reveals that the steady state depends on two free parameters. Given that the strong symmetry with respect to \( R \) guarantees [55] 3 trace-1 solutions to \( \mathcal{L}_{\text{gl}}[\rho] = 0 \), we thus conclude that each eigensubspace of \( R \) contains a unique steady state and that there are no non-zero traceless solutions to \( \mathcal{L}_{\text{gl}}[\rho] = 0 \). Let us now write the eigenresolution of \( R \) as \( R = \sum_{k=1}^{3} e^{\pm \pi k} R_{k} \), where the cubic roots of 1 are the eigenvalues and \( R_{k} \) are the eineprojectors \( (R_{2} = R_{1}^{*}) \). By numerically finding an arbitrary solution for the degenerate linear system \( \mathcal{L}_{\text{gl}}[\rho] = 0 \)—call it
where we determine the three mutually orthogonal steady states as \( \rho_{st}^{k} = \frac{R_{k} \rho R_{k}}{\text{Tr}(R_{k} \rho R_{k})} \), which of course do not depend on \( \hat{\rho} \). An important property of the steady states is that, although they are not invariant under particle exchange, their marginals coincide for all choices of the parameters:

\[
\text{Tr}_a \rho_{st}^k = \text{Tr}_b \rho_{st}^k, \tag{19}
\]

for all values of \( k \).

If the rotor’s initial state is \( \rho_0 \), then the steady state it eventually evolves into, will be

\[
\rho_{st}[\rho_0] = \sum_{k=1}^{3} \text{Tr}(R_k \rho_0) \rho_{st}^k. \tag{20}
\]

Indeed, keeping in mind that \( \sum_k R_k = 1 \), we can write \( \rho_0 = \sum_k \text{Tr}(R_k \rho_0) \frac{R_k \rho R_k}{\text{Tr}(R_k \rho R_k)} + \sum_{k \neq k'} R_k \rho_0 R_{k'} \). Since there are no non-zero traceless solutions to \( L_k[\rho] = 0 \) and \( R_k \rho_0 R_{k'} \) (\( k \neq k' \)) are traceless (and remain so because the evolution is trace preserving), the second sum in the decomposition will vanish as the system converges to its steady state. Whereas each of the states \( \frac{R_k \rho_0 R_k}{\text{Tr}(R_k \rho R_k)} \) will evolve into its corresponding \( \rho_{st}^k \), leaving us with Eq. (20).

We can now use Eqs. (6) and (7) to determine the currents associated to all three \( \rho_{st}^k \)'s. First of all, for all the values of the parameters, all the average currents are independent of the site (e.g., \( J_{a \rightarrow j_a + 1} \rho_{st}^k \) is the same for all \( j_a \)'s). Furthermore, we find (numerically) several other general properties that also hold for all values of the parameters. First, the average current in \( \rho_{st}^3 \) is zero for both particles \( a \) and \( b \): \( J_a[\rho_{st}^3] = 0 \). Next, the tunneling currents of the first and second steady states are always opposite, \( J_a^{(\text{tun})}[\rho_{st}^1] = -J_a^{(\text{tun})}[\rho_{st}^2] \), while their thermal currents always coincide: \( J_a^{(\text{th})}[\rho_{st}^1] = J_a^{(\text{th})}[\rho_{st}^2] \). Lastly, due to Eq. (19), we also have \( J_{a \rightarrow j_a}^{(\text{tun})} J_a[\rho_{st}^k] = J_{b \rightarrow j_b}^{(\text{tun})} J_b[\rho_{st}^k] \), for any \( k \). In view of all this, and using the numerical values of \( R_1 \) and \( R_2 \), for an arbitrary initial state \( \rho_0 \), we find

\[
J[\rho_{st}[\rho_0]] = \frac{2 \text{Im} \theta_0}{\sqrt{3}} J^{(\text{tun})}[\rho_{st}^1] + \frac{2 - 2 \text{Re} \theta_0}{3} J^{(\text{th})}[\rho_{st}^1], \tag{21}
\]

where \( \theta_0 = \sum_{j_a,j_b=1}^{3} \langle j_a,j_b | \rho_0 | j_a + 1,j_b + 1 \rangle \) is the sum of some of the non-diagonal elements of the initial density matrix.

Importantly, the average steady-state current is a \( 2\pi/3 \)-periodic function of \( \phi \). This fact can be appreciated already in Fig. 2a, and is rigorously proven in Appendix E.

Another important feature of the global GKSL ME is that, whenever \( \phi = k\pi/3 \) (\( k \) is an arbitrary integer), \( J^{(\text{th})} = 0 \) for any initial state. This nullification of the thermal current also takes place for both the classical and phenomenological quantum MEs (see Appendix G), and is related to the presence of an exchange symmetry, which is broken whenever \( \phi \neq k\pi/3 \). More specifically, if \( \phi = k\pi/3 \), then \( U_{j,i} = U_{j+3+k,i} \); i.e., the potential is symmetric under the combination of particle exchange and unilateral rotation. The same symmetry applies also to the quantum Hamiltonian of the rotor, \( H_{ab} \). Indeed, the unitary operator corresponding to the generalized exchange is given by \( \Xi_k = \text{SWAP} : (1 \otimes \mathcal{R}_k^a) \), where \( \text{SWAP} \) is the swap operator (for \( \forall |\psi \rangle \) and \( |\xi \rangle \), \( \text{SWAP} |\psi \rangle |\xi \rangle = |\xi \rangle |\psi \rangle \)), and it is easy to see that \( [H_{ab}, \Xi_k] = 0 \). In other words, in order to generate thermal current, one needs to break the generalized exchange symmetry of the rotor Hamiltonian. It is this asymmetry that provides the ratchet effect propelling the rotor both in the classical and quantum regimes. In Fig. 2a, we plot the total (thermal plus tunneling) current of particle \( a \) against the phase \( \phi \), for different values of \( \tau \). We see that, in contrast to the thermal current, the tunneling current is not zero when \( \phi = k\pi/3 \), which is the case only for the global ME.

**FIG. 2. Current.** (a) The current of particle \( a \) versus the phase \( \phi \) in the first steady state \( \rho_{st}^1 \), for different values of the tunneling coefficient \( \tau \). The blue curve corresponds to near-zero tunneling (\( \tau = 0.001 \)), and the orange and green curves correspond to, respectively, \( \tau = 0.05 \) and \( \tau = 0.1 \). (b) The total average current versus the tunneling rate \( \tau \), with the phase fixed to \( \phi = \pi/6 \). The total current starts negative, and, depending on the initial state, can turn positive as \( \tau \) increases. The (inset) shows the average thermal current of particle \( a \) in the first steady state, \( J_a^{(\text{th})}[\rho_{st}^1] \) (\( \times 10^5 \)), and the time derivative of the Margenau-Hill distribution, \( \mu_a \) (see the text), as a function of \( \tau \). It is clearly seen that the weak value, which has the same sign as \( \mu_a \), turns negative before the current changes its sign. For both plots, the rest of the parameters are: \( T_a = 0.2 \), \( T_b = 1 \), \( \gamma_a = \gamma_b = 0.2 \), \( K = 2 \).
Notice also that the average current for nearly zero tunnelling rate is noticeably smaller than the average currents for larger tunnelling rates.

When \( \phi = k\pi/3 \), the Hamiltonian of the rotor is symmetric under \( \Xi_k \), whereas the coupling operators, \( A_a \) and \( A_b \) (and therefore also \( H_{\text{tot}} \)), are not. Nevertheless, \( \phi = k\pi/3 \) is marked by an increased degeneracy in the spectrum of \( H_{\text{tot}} \), and, as a consequence, of \( \mathcal{L}_{\text{gl}} \). This is expressed in the fact that, although not invariant under \( \Xi_k \), the steady state now depends on 5 free parameters, meaning that the steady-state space of \( \mathcal{L}_{\text{gl}} \) is 6-dimensional. This extra symmetry is responsible for the nullification of the thermal current. In Appendix G, we provide an extended discussion of these issues for the phenomenological quantum and classical MEs.

An interesting purely quantum effect can be read off from Fig. 2b (in which we plot the total average current for all three basis steady states, as a function of \( \tau \)): the current changes its direction as the tunnelling rate increases. Keeping Eqs. (20) and (21) in mind, we see that, unless the initial state is completely in the subspace of \( R_3 \), \( \langle J \rangle < 0 \) for \( \tau \ll 1 \), and if \( \text{Im} \theta_0 < 0 \), \( \langle J \rangle \) will become positive for large enough \( \tau \). Such a reversal of the current is somewhat counterintuitive in that the tunnelling is symmetric with respect to both global and local rotation, and it therefore does not favour a particular direction for particle flow. This is a purely quantum phenomenon, occurring only when \( T_a \) is sufficiently low. In fact, it turns out that the inversion of the thermal current is always preceded, implying (yet by no means proving) a causal relation, by the the onset of measurement-contextuality in the system, marked by the weak value in Eq (9) turning negative (see the related discussion in Sec. II and Appendix D). We illustrate this in the inset of Fig. 2b, where we show both \( \langle J^a_\phi | \rho_{1t}^a | \rangle \) and \( \mu_{j_a,j_a+1} = \lim_{\epsilon \to 0^+} \text{Re} \text{ Tr} [x_{j_a'} x_{j_a+1}(\epsilon) \rho_{1t}^a] \) against \( \tau \). We chose to plot \( \mu_{j_a,j_a'} \)—the time-derivative of the Margenau-Hill distribution (see subsection ‘Current’)—instead of the weak value since not only \( \mu_{j_a,j_a'} \) has the same sign as the weak value (cf. Eq. (9)), but it also directly appears in the expression for the average current: \( \langle J_{j_a \to j_a'} \rangle = \mu_{j_a,j_a'} - \mu_{j_a',j_a} \) (see Eq. (8)). Let us note that current inversion, although the fundamentally different mechanism of external periodic driving, was also reported in Brownian ratchet systems [20, 59].

Lastly, we note that, whenever \( T_a = T_b, J_{ab}^{(th)} = 0 \) for any initial state. The tunnelling currents, on the other hand, do not necessarily turn to zero when \( T_a = T_b \). This however does not contradict the second law of thermodynamics, and is simply due to the fact that, although the thermal state—for which the tunnelling currents are indeed zero—is a steady-state solution for the global ME, the basis steady states, \( \rho_{k}^{th} \), are not thermal.

**FIG. 3. Heat.** The steady-state heat flux from hot to cold reservoir, versus \( \phi \), for different values of the tunnelling coefficient \( \tau \). The blue curve corresponds to near-zero tunnelling (\( \tau = 0.001 \)), and the orange and green curves correspond to, respectively, \( \tau = 0.2 \) and \( \tau = 0.4 \). The rest of the parameters are the same as those for Fig. 2.

### B. Heat

In the absence of external driving, heat always flows from hot to cold. Due to clear separation between the two baths in the total Hamiltonian (see Eq. (14)), the contributions of different baths in the resulting GKSL ME are simply summed (this is true also for the phenomenological ME). Therefore, casting the generator of the evolution in the form \( \mathcal{L}_{\text{gl}}[\rho] = -i[H_{ab},\rho] + D_0[\rho] + D_b[\rho] \), where \( D_0[\rho] = \sum \omega \gamma_\alpha(\omega)S[A_\alpha(\omega)][\rho], \) the standard definition of the heat flow yields

\[
\dot{Q}_a = \text{Tr}(H_{ab}D_\alpha[\rho]).
\]  

Since the system does not exchange energy with external media, in the steady state, the energy conservation requires \( \dot{Q}_a + \dot{Q}_b = 0 \). Moreover, at thermal equilibrium, i.e., when \( T_a = T_b \), the heat flow must be zero. The first condition is trivially satisfied by any GKSL equation, whereas, in general, the second condition is guaranteed only when one uses the global ME (although, in our model, this is also respected by the phenomenological ME).

We have numerically established the following properties of the heat flow: \( \dot{Q}[\rho_{1t}^a] = \dot{Q}[\rho_{2t}^a] > \dot{Q}[\rho_{3t}^a] > 0 \) and \( \dot{Q}[\rho_{1t}^a] = \dot{Q}[\rho_{2t}^a] \propto \tau \) (for \( \tau \ll 1 \)). This means that, when the tunnelling is not zero, the heat flow can be controlled by changing the weights of the eigensubspaces of \( R \) in the initial state of the rotor; such symmetry-controlled manipulation is well-known in the literature [57]. In Fig. 3, we plot \( \dot{Q}[\rho_{1t}^a] \) (\( \dot{Q}[\rho_{2t}^a] \) has the same behaviour) against \( \phi \), for various values of \( \tau \). First, we notice that the heat flow decreases with increasing the tunnelling rate, which is related to the fact that the higher the \( \tau \), the higher the relative weight of the non-interacting component of \( H_{\text{ab}} \). We further notice that the zeroes of \( J^{(th)} \), i.e., \( \phi = k\pi/3 \), correspond to global extrema of \( \dot{Q} \). Lastly, analogously
to the current, the heat flow is also a $2\pi/3$-periodic function of $\phi$ (cf. Fig. 3 and see Appendix E for a proof).

We conclude this section by emphasizing that caution must be maintained when using the global ME for computing the heat flow for small $K$. As discussed in the subsection ‘Model ratchet’, the secular approximation, indispensable for the applicability of the global ME, is compromised when $K$ approaches $0$. Given the results in refs. [53, 54] for the Caldeira-Leggett model, it is reasonable to expect that the local ME would provide a more reliable description in that regime. This issue can be decisively settled only upon solving the global, system-plus-baths dynamics in the limit of infinitely large baths, doing which, however, appears to be infeasible [33].

**C. Entanglement**

Entanglement is a key resource in basically all aspects of quantum technology [47, 60]. Although thermal noise can sometimes be beneficial for creating and maintaining entanglement [61–63], at high temperatures, the effect of environment will typically be detrimental. Indeed, the maximally mixed (or, equivalently, infinite-temperature) state in any finite-dimensional Hilbert space is not entangled, and there exists a finite-volume convex set around it, in which all states are not entangled [60]. In particular, this means that, for any Hamiltonian, $H$, there exists a finite temperature $T[H]$, above which all thermal states are non-entangled.

On the other hand, thermal disequilibrium is known to be beneficial for entanglement generation [62–68]. Therefore, a key question in this regard is whether a thermal disequilibrium created by temperatures higher than $T[H]$ can generate entangled non-equilibrium states. Of especial interest would be the steady states, since these can be prepared and maintained reliably and robustly. Here, we will show that, in our model, a thermal initial state can be transformed into an entangled steady state only if it is already entangled. Thus, for thermal initial states, the above question has a negative answer. However, we will show that, for initially uncorrelated states, local coherence can be traded for entanglement in the steady state, for any temperatures of the baths. In other words, powered by global rotation symmetry, our machine can convert a local quantum resource (coherence) into a global quantum resource (entanglement), even when the surroundings have such high temperatures that neither resource would survive thermalization.

Our rotor is a $3 \times 3$-dimensional system, and in 9 (or higher) dimensions it is generally very hard to tell whether a state is entangled or not [60]. In this paper, we will use a very strong necessary condition, called positive partial transpose (PPT) criterion [60]. The partial transpose of $\rho$ with respect to partition $b$, $\rho^{T_b}$, is defined as the matrix with entries $\langle j_a, j_b | \rho^{T_b} | j'_a, j'_b \rangle = \langle j_a, j_b | \rho | j'_a, j'_b \rangle$. Now, if $\rho$ is not entangled, then $\rho^{T_b}$ will be a non-negative matrix. Hence, $\rho$ will necessarily be entangled if $\rho^{T_b}$ has a negative eigenvalue. This condition is also sufficient when the joint Hilbert space dimension is $\leq 6$. However, in higher dimensions, e.g., for our rotor, this condition is not sufficient: $\rho$ can be entangled even when $\rho^{T_b} \geq 0$ (it is then said to be “bound entangled” [60]).

The entanglement detected by the PPT criterion can be measured by the negativity, $N(\rho)$, defined as the sum of the absolute values of all negative eigenvalues of $\rho^{T_b}$ [69]:

$$N(\rho) = \frac{1}{2} \left( ||\rho^{T_b}||_1 - 1 \right), \text{ where } ||O|| = \text{Tr} \sqrt{O^\dagger O} \text{ is the trace norm} \text{ [47].}$$

Similarly to the heat flux, we find numerically that (i) $N(\rho_0^t) = N(\rho_0^t) > N(\rho_0^t) > 0$ and (ii) the global extrema of the negativity correspond to the zeroes of the thermal current ($\phi = k\pi/3$). In Fig. 4, we plot $N(\rho_0^t)$ against $\phi$, for three different values of tunnelling: $\tau = 10^{-3}$, $\tau = 0.2$, and $\tau = 0.4$. It shows, in particular, that the negativity is a monotonically decreasing function of $\tau$. This seemingly counterintuitive phenomenon can be easily understood by first observing that the presence of entanglement even when $\tau \rightarrow 0$ is caused by the fact that, in order to obtain the basis steady state, we project on the highly entangled eigensubspaces of the global rotation matrix $R$; and the decrease of entanglement with the increase of $\tau$ is caused by the increased weight of the non-interacting part of the Hamiltonian, $H_{ab} - H_{int}$ (see Eq. (10)). The standard intuition is recovered when one considers realistic initial states. We show this in the inset of Fig. 4, where we plot the steady-state negativity versus $\tau$. There, we fix $K = 2$ and $\phi = \pi/3$, and, for any $\tau$, choose the initial state to be the thermal state at the temperature of the hot bath: $\rho_0 \propto e^{-\beta_0 H_{ab}}$. As expected, we see that there is no entanglement for weak tunnelling. Then, having reached a maximum at an intermediate value of $\tau$, the entanglement decays as $H_{ab}$ be-
comes more and more dominated by the non-interacting tunnelling term.

Note also that, as is visible in Fig. (4), similarly to the current and heat flux, the negativity is a $2\pi/3$-periodic function of $\phi$. See Appendix E for the proof.

The dependence of the steady-state entanglement of global GKSL MEs on the temperature difference of the baths has been extensively studied in qubit systems [62–66]. In highly asymmetric problems, situations when entanglement can be generated only when there is temperature difference have been reported [63]. In our model, we find that, for any values of all the other parameters fixed, if we set the cold temperature $T_a$ constant, then the negativity of all three basis steady states is a monotonically decreasing function of $T_b$. Furthermore, the negativity at $T_b = T_a$ is a monotonically decreasing function of $T_a$. Interestingly, we find that, when both keeping $T_a$ constant and varying $T_b$ or fixing $T_b = T_a$ and varying $T_a$, the entanglement does not experience sudden death: the negativity decays gradually, reaching zero only asymptotically. Therefore, by choosing the initial state appropriately, we can guarantee the presence of entanglement in the steady state for arbitrarily large values $T_a$ and $T_b$. More importantly, we can achieve this with uncorrelated initial states, albeit paying for that with local coherence. Indeed, as we prove in Appendix F, for any $T_a$ and $T_b$, there exist two single-particle states, $\sigma_a$ and $\sigma_b$, such that the machine evolves $\sigma_a \otimes \sigma_b$ into an entangled steady state. As $\min\{T_a, T_b\}$ increases, $\sigma_a$ and $\sigma_b$ become monotonically less mixed and more coherent. In the limit of infinite temperatures, $\sigma_a$ and $\sigma_b$ necessarily become pure and maximally coherent (with respect to the measure of coherence defined as $C[\sigma] = \sum_{j \neq j'} |\langle j | \sigma | j'\rangle|^2$ [70]). The family $\sigma_a(\delta) = (1 - \delta)\sigma_a(\delta) + \frac{\delta}{3}$, with $|\sigma_a\rangle = \frac{1}{\sqrt{3}} [(1 + e^{-2\pi i/3}|2 + e^{2\pi i/3}|3)]$ and $|\sigma_b\rangle = \frac{1}{\sqrt{3}} [(1 + |2 + |3)]$, represents an example of such states: for any $T_a$ and $T_b$, there exists $\delta > 0$ ($\delta \to 0$ as $\min\{T_a, T_b\} \to \infty$) such that $\rho^{st}[\sigma_a(\delta) \otimes \sigma_b(\delta)]$ is entangled.

D. Work content

The steady states of our machine can be useful not only in quantum information, but also in thermodynamics. In particular, the non-equilibrium steady states of the rotor store work that can be extracted from it in a cyclic Hamiltonian process. In other words, the average energy of the rotor, with respect to its Hamiltonian $H_{ab}$, can be decreased by unitarily evolving its state $\rho^{st}$. The unitary operation that extracts maximal work is the one that diagonalizes $\rho^{st}$ in the basis of $H_{ab}$ in such a way that, if $H_{ab}$’s eigenbasis is chosen so that $E_1 \leq E_2 \leq \cdots$, then the eigenvalues of $\rho^{st}$, $\{r_k\}$, have the reverse ordering: $r_1 \geq r_2 \geq \cdots$ [71]. The average work thus extracted, $E$, is called ergotropy [71] and is equal to

$$E[\rho^{st}] = \text{Tr}(H_{ab}\rho^{st}) - \sum_m E_m r_m. \quad (23)$$

The states with zero ergotropy are called passive [71, 72], with the most prominent example being provided by the thermal states [72].

When the rotor’s initial state is thermal at temperature $T_{in}$, the ergotropy of the steady state will be a decreasing function of $T_{in}$, turning to zero somewhere in between $T_a$ and $T_b$. Now, given the infinite size of the baths, thermal states at the temperatures of the baths can be considered to be free—one just puts the system in contact with the bath and waits for it to thermalize. Therefore, if $T_{in} = T_a$, the machine essentially takes in a free (and useless from the perspective of work extraction) state and evolves (and maintains for an arbitrarily long time) it into a state “charged” by work. In Fig. 5, we plot $E[\rho^{st} [e^{-H_{ab}/T_a}/Z]]$, where $Z = \text{Tr}e^{-H_{ab}/T_a}$ is the partition function, as a function of $\tau$. We see that, in addition to a temperature gradient, a sufficiently high tunnelling rate is necessary for the machine to work as a charger. The plot also shows the total steady-state current, in order to indicate the surprising fact that the work content of the steady state is not related to (let alone being caused by) the current it harbours.

IV. CONCLUSIONS

In this work, we have derived a general formula (Eq. (4)) for the current of a quantum particle on an arbitrary graph, undergoing an arbitrary open-system dy-
namics. Our formula recovers all known definitions of particle current [1, 3–6, 10], in the corresponding regimes of their validity. In the classical limit, it reduces to the standard Eq. (1). Usable in all situations, our formula describes the current even when the position operator does not commute with the system-environment interaction, which, until now, was an uncharted territory. We further deepen the formalism by providing an interpretation for our formula in terms of quantum weak measurements. We show that the average current is a result of two position measurements—one weak and one strong. This sheds new light also on well-known and widely-used formulas of particle current, such as, e.g., Eq. (6).

As a next step, we used the developed formalism to study current generation on a minimal, two-particle toy model, autonomously operating between two thermal baths. There, by carefully accounting for all relevant spatial symmetries of the full system-environment Hamiltonian, we established that the breaking of particle exchange symmetry governs current generation both classically and deep in the quantum regime. Counter-intuitively, the model exhibits reversal of the direction of the steady-state current as the rotationally invariant tunnelling rate is varied. We argued that this is a purely quantum effect by showing that it appears only when the system exhibits quantum contextuality.

We also showed that our device can be useful in such practical tasks as converting uncorrelated states into entangled states in the presence of arbitrarily hot dissipative environment and evolving initially thermal states, which are useless for work extraction, into non-passive steady states, from which non-zero work can be extracted. Both tasks are enabled by purely quantum resources. The entanglement generation is made possible by feeding the machine with coherent local states. In other words, the evolution to the steady state is a realization of a completely positive trace preserving map, converting local coherence into entanglement which depends on the amount of coherence at the input, akin to the analysis in Ref. [73]. In the second case, “charging” of the initially thermal rotor is possible only if the tunnelling rate is sufficiently high.

The presented theory unlocks the possibility of describing particle current in most general physical scenarios, and, in particular, to devise practical quantum thermal motors without the limitation of having the rotor isolated from the environment. It may also open a new perspective on the study of coherent energy [74] and information transport [75] in open quantum systems undergoing general non-Markovian dynamics.

The facts that particle current can witness the onset of quantum contextuality and noise-resistant entanglement can be produced in our simple model, add constructively to the ongoing discussion on the genuinely quantum effects in the thermodynamic context [14, 43] and the practical usefulness of nanoscale thermal machines [14, 63, 67, 68].

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Appendix A: Open-system Heisenberg picture

When the dynamics is unitary, the Heisenberg picture is used to shift the time dependence from the state to the operators (see any textbook [1] on quantum mechanics): the pair \((\rho(t), O)\), where \(O\) is an observable and \(\rho(t) = U(t)\rho U(t)\dagger\) is the time-dependent state, is taken to \((\rho, O(t))\), where \(O(t) = U(t)\dagger O U(t)\). The two pictures are equivalent in that \(\text{Tr}[\rho(t)O(t)] = \text{Tr}[\rho O(t)]\) for any \(t\) and any \(O\), and the ultimate convenience of the Heisenberg picture stems from the simple fact that it preserves the algebra of observables: \((O_1 O_2)(t) = O_1(t) O_2(t)\).

The situation is more complicated for non-unitary dynamics. For simplicity, let us focus on the case when the evolution, \(\rho(t) = \mathcal{E}_t[\rho]\), is given by a completely positive, trace-preserving (CPTP) map [47]:
\[
\mathcal{E}_t[\rho] = \sum_{\mu} K_{\mu}(t)\rho K_{\mu}^\dagger(t),
\]
where \(K_{\mu}\)—the so-called Kraus operators—satisfy \(\sum_{\mu} K_{\mu}^\dagger(t)K_{\mu}(t) = I\). Here, as above, the Heisenberg picture is defined through the requirement of \(\text{Tr}[\rho(t)O] = \text{Tr}[\rho O(t)]\), for any \(\rho\), which yields \(O(t) = \sum_{\mu} K_{\mu}^\dagger(t)O K_{\mu}(t)\).

Unlike the unitary case (i.e., when there is more than one non-zero Kraus operator), the algebra of observables is not preserved: \((O_1 O_2)(t) \neq O_1(t) O_2(t)\). When the evolution is not given by a CPTP map, the transition to the Heisenberg picture, although still determined through requiring the averages to be the same, will have to be done on an ad hoc basis, and one will generally be even further from preserving the observable algebra as one was in the case of CPTP evolution. Nevertheless, one general statement can be made: the identity operator remains unchanged upon transitioning to the Heisenberg picture. Indeed, any quantum dynamics, completely positive or not, preserves Hermiticity, positivity, and trace. The latter is thus equivalent to the preservation of the identity operator.

Coincidentally, the requirement that the transition to the Heisenberg picture in Eq. (4) has to be done at the same moment of time as the current is calculated, exempts the operator products in Eq. (4) from the problem of the Heisenberg picture not preserving the observable algebra. Indeed, all the operators in Eq. (4) are calculated at the zeroth second of the Heisenberg picture, meaning that \(x_j\) and \(x_j'\), coincide with their Shrödinger-picture values. When it comes to the derivatives of \(x_j\) and \(x_j'\), the situation is tractable when the dynamics is given by a CPTP map which is divisible [76] at the moment of time \(t\), which is when we are calculating the current. Indeed, the divisibility means that, for any \(\epsilon > 0\), the decomposition \(\mathcal{E}_{t+\epsilon} = \mathcal{E}_t \circ \mathcal{C}_\epsilon\), where \(\mathcal{C}_\epsilon\) is a CPTP map, holds. Let \(\kappa_{\mu}(\epsilon)\) be the Kraus operators of \(\mathcal{C}_\epsilon\). Then,
\[
x_j(\epsilon) = \sum \kappa_{\mu}(\epsilon) x_j \kappa_{\mu},
\]
whence we can calculate \(\frac{dx_j(\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}\), which is the derivative required in Eq. (4). When the dynamical map is not divisible, e.g., when the evolution is non-Markovian, the time-derivative of \(x_j\) may depend on the past of the dynamics, which can make the calculation of the current intractable.

Quantum master equations of the GKSL form (see Eq. (2)) constitute an important class of divisible maps [33, 76]. When the dynamics is described by a GKSL master equation, one can readily bypass the Kraus representation and work directly with the Heisenberg equation, Eq. (5) [33], irrespective of the moment of time at which one transitions from the Shrödinger picture to that of Heisenberg. Therefore, the particle current operator, as defined by Eq. (4), is time independent, and is given by Eqs. (6) and (7). Note that the GKSL equation holds only on the coarse-grained time scale where the fast environmental degrees of freedom are averaged out [33]. For example, this means that the limit \(\epsilon \to 0\) in Eq. (8) is to be understood as \(\epsilon\) being much smaller than
the time-scale of the system evolution, but much larger
than the relaxation time-scale of the environment.

Appendix B: Current of a Markovian particle on a
triangle

The graph is a triangle, therefore the continuity equa-
tion reduces to
\[ \frac{dx_j}{dt} = J_{j-1\rightarrow j} - J_{j\rightarrow j+1}. \]  
(B1)

Moreover, we can write \( x_{j+3} = x_j \), and, therefore, for
any \( j \), we have that
\[ x_{j-1} + x_j + x_{j+1} = I. \]  
(B2)

Now, the time derivative of \( x_j \) is given by Eq. (5), and,
using the above identity resolution in \( [H, x_j] = \mathbb{H} x_j - x_j \mathbb{H} \),
we immediately obtain that
\[ \dot{i}[H, x_j] = J_{j-1\rightarrow j}^{(\text{tun})} - J_{j\rightarrow j+1}^{(\text{tun})}, \]  
where \( J_{j-1\rightarrow j}^{(\text{tun})} \) is given by Eq. (6).

Turning to the dissipative part of Eq. (5), \( \mathcal{D}^*[x_j] = \sum_{\lambda} \gamma_{\lambda} \begin{bmatrix} \Lambda_{\lambda}^1 x_j x_{\lambda} - \frac{1}{2} \left\{ \Lambda_{\lambda}^1 \Lambda_{\lambda}, x_j \right\} \end{bmatrix} \), we substitute the
identity resolution (B2) in the trivial identities \( \Lambda_{\lambda} = \Lambda_{\lambda}^1 \) and
\( \Lambda_{\lambda}^1 = \mathbb{I} \), in order to represent \( \mathcal{D}^*[x_j] \) as a sum
of terms of the form \( x_j \Lambda_{\lambda}^1 x_{\lambda}, \Lambda_{\lambda} x_{\lambda} \). As can be checked
by direct inspection, all these terms can be rearranged in
such a way that
\[ \mathcal{D}^*[x_j] = J_{j-1\rightarrow j}^{(\text{th})} - J_{j\rightarrow j+1}^{(\text{th})}, \]  
(B4)

with
\[ J_{j-1\rightarrow j}^{(\text{th})} = \sum_{\lambda} \gamma_{\lambda} \left[ (j, j+1, j) - (j+1, j, j+1) \right] \]  
(B5)
\[ + \frac{1}{2} (j, j+1, j) - \frac{1}{2} (j, j, j+1) \]  
(B6)
\[ + \frac{1}{2} (j, j+1, j+1) - \frac{1}{2} (j+1, j, j) \]  
(B7)
\[ + \frac{1}{2} (j, j+1, j+2) - \frac{1}{2} (j+1, j, j+1) \]  
(B8)
\[ + \frac{1}{2} (j+2, j+1, j) - \frac{1}{2} (j+2, j, j+1) \].  
(B9)

where the symbol \((j', j'', j''')\) is introduced for ease of nota-
tion and stands for \( x_j \Lambda_{\lambda}^1 x_{j'}, \Lambda_{\lambda} x_{j''}, \Lambda_{\lambda} x_{j'''} \). Using the identity
resolution (B2) again, we can further massage the above
expression into
\[ J_{j-1\rightarrow j+1}^{(\text{th})} = \frac{1}{2} \sum_{\lambda} \gamma_{\lambda} \left[ \{ x_j, \Lambda_{\lambda}^1 x_{j+1} \Lambda_{\lambda} \} - \{ x_{j+1}, \Lambda_{\lambda}^1 x_j \Lambda_{\lambda} \} \right], \]  
(B10)

which exactly coincides with Eq. (7).

This calculation thus demonstrates that no terms other
than those in Eq. (4) arise when one derives the current
of a particle on a triangle, directly stemming from the
equations of motion.

Appendix C: Two-strong-measurement current

Let us see what the current of a particle described by
a GKSL master equation will looks like, if we substitute
the first weak measurement (see Eq. (9) and the dis-
sussion around it) by a standard (i.e., non-weak) quantum
von Neumann measurement [33, 47]. To find the current,
we first measure the particle at vertex \( j \), at the moment
of time \( t \). The probability of finding the particle at the
measured vertex will be given by \( p_j(t) = \text{Tr}(x_j \rho(t)) \), and,
immediately after the measurement, the state of the system
will collapse into \( \rho_j = \frac{\rho(t) \mathbb{I}}{p_j} \). Within the period
of time \( \epsilon \), the state will evolve into \( \rho_j + \epsilon \mathcal{L}[\rho_j] + O(\epsilon^2) \), where
\( \mathcal{L} \) is the GKSL generator. Therefore, the probability
of finding the particle at site \( x_{j'} \), at the moment of time
\( t + \epsilon \), will be \( \text{Tr}[x_{j'} (\rho_j + \epsilon \mathcal{L}[\rho_j] + O(\epsilon^2))] \). Taking the reverse
flow into account, we thus find the two-strong-
measurement current to be given by
\[ \langle J_{j\rightarrow j'}^{(\text{TSM})} \rangle = \text{Tr}[\mathcal{L}[x_j \rho x_j x_{j'}] - (j \leftrightarrow j') \rangle \]  
(C1)

For the particle on a triangle, we immediately see that
\( \langle J_{j\rightarrow j'}^{(\text{TSM})} \rangle \) coincides with the RHS of Eq. (B5). In
other words, making the first measurement strong elimi-
nates all coherent contributions to the current, given by
Eqs. (6), (B6)-(B9). Note that the latter include also the
tunnelling current.

Appendix D: Contextuality

Here, we will briefly outline the main concepts behind
the notion of contextuality. See refs. [43–46] for an
extensive discussion. An operational theory (e.g., quantum
mechanics) consists of two processes: preparations \( \Pi \)
and measurements \( M \). Given a preparation (which, in
quantum mechanics, is fully characterized by a density
matrix), the theory predicts the probability distribution of
the outcomes \( o \) of measurements: \( \mathcal{P}(o|M, \Pi) \). A clas-

cical stochastic system is said to be a hidden-variable
(aka ontological) model for an operational theory, if, to
any system, it associates a space of real (or ontic) states,
\( \{ \zeta \} \), such that each state in that space fully character-
izes the system. To any preparation \( \Pi \) of the system, it
associates a probability distribution over the state space,
\( \nu_{\Pi}(\zeta) \) (called epistemic state). And since the ontic states
fully characterize the system, for each ontic state, there
exists a probability distribution of outcomes of a measure-
ment: \( \pi_{\Pi|M}(\zeta) \) (called response function). Finally,
the hidden variable model should reproduce the statistics
of measurements of the system:
\[ \mathcal{P}(o|M, \Pi) = \int \zeta d\pi_{\Pi|M}(\zeta) \nu_{\Pi}(\zeta). \]  
(D1)

Furthermore, a hidden-variable model is measurement
noncontextual, if to operationally equivalent measure-
ments, $M' \sim M$, it prescribes identical response functions: $\pi_{\delta M'}(\zeta) = \pi_{\delta M}(\zeta)$. Analogously, a hiddenvariable model is preparation noncontextual, if to operationally equivalent preparations, $\Pi' \sim \Pi$, it prescribes identical epistemic states: $\nu_{\Pi'}(\zeta) = \nu_{\Pi}(\zeta)$.

Now, if the operational theory is capable of producing a probability distribution $\mathcal{P}$ such that no measurement (preparation) noncontextual hidden variable model can reproduce it, then that situation is said to be measurement (preparation) contextual.

Taking a slightly less formal stance, in the effort to separate quantum from classical, let us assume that the ontic model is represented by classical mechanics. Since in classical mechanics full characterization excludes probabilistic description, then response functions should be either 0 or 1. Hidden-variable models with such response functions are called outcome deterministic. In short, we expect classical hidden-variable models to be measure- and preparation noncontextual and outcome deterministic.

The following theorem (first proven for pure states in Ref. [44], then generalized to mixed state in Ref. [43]) holds: If $\rho$ is a quantum state and $x$ and $x'$ are projectors, such that $\text{Tr}(\rho x x') < 0$, then there exists a POVM, $\{V_{\nu}\}$, such that no measurement-noncontextual outcome-deterministic hidden-variable model can simulate the preparation $\rho$ and postselection $x_j$. See refs. [43, 44] for the proof.

Appendix E: $2\pi/3$-periodicity of current, heat, and entanglement

Let us prove that the steady-state current, heat flux, and negativity are all $2\pi/3$-periodic function of $\phi$. All these stem from the fact that, since the potential satisfies $U_{j,a,j}(\phi+\pi/3) = U_{j,a,j+1}(\phi)$ and the tunnelling parts of the rotor Hamiltonian are invariant under local rotation,

$$H_{ab}(\phi + 2\pi/3) = I \otimes R_b^1 H_{ab} I \otimes R_b. \quad \text{(E1)}$$

Furthermore, Eq. (E1) and $[X, R_b] = 0$ mean that $\phi \rightarrow \phi + 2\pi/3$ leads to $\Lambda_{\phi}^a(\omega) \rightarrow \Lambda_{\phi}^a(\omega) I \otimes R_b$ (see Eqs. (16) and (18)). Hence, it also holds that

$$\rho^{st}(\phi + 2\pi/3) = \Lambda_{\phi}^a(\omega) \rho^{st} I \otimes R_b. \quad \text{(E2)}$$

In view of the fact that the average local current in the steady state is invariant under local rotation, this proves the $2\pi/3$-periodicity of the average current.

For the heat flux, let us notice that Eq. (E2) implies that $\phi \rightarrow \phi + 2\pi/3$ leads to $D_{\sigma}(\rho^{st}) \rightarrow \Lambda_{\phi}^a(\omega) D_{\sigma}(\rho^{st}) I \otimes R_b$, which, through Eq. (22), proves that $\phi \rightarrow \phi + 2\pi/3$ indeed leaves $Q$ unchanged.

Lastly, in view of the fact that the negativity is invariant under local unitary transformations [60], Eq. (E2) immediately proves that the negativity is indeed $2\pi/3$-periodic.

FIG. 6. Entanglement and initial purity. The minimal purity (as quantified by $\delta_{\text{max}}$) and the negativity $\mathcal{N}$ of the steady state for $\delta = \delta_{\text{max}}/2$, plotted against $T_a$ ($T_b = 2T_a$). The tunnelling rate $\tau$ is chosen to be 0.1. For this value of $\tau$, $T[H] \approx 0.3948$, which is much smaller than the bath temperature in the plot. The phase $\phi$ is set to the value $\pi/3$ and all the other parameters are the same as those in Fig. 2.

Appendix F: Trading local coherence for entanglement

As is discussed around Eq. (21), for arbitrary initial state $\rho_0$, $\rho^{st} = \sum_{k=1}^{3} \lambda_k \rho_k^{st}$, with $\lambda_{1,2} = \frac{1}{3} - \frac{2}{3} \text{Re}(e^{\pm i\pi/3} \theta_0)$, $\lambda_3 = 1 - \lambda_1 - \lambda_2$, where $\theta_0 = \sum_{j,k} \langle j_a,j_b | \rho_0 | j_a + 1,j_b + 1 \rangle$. For uncorrelated initial states, i.e., when $\rho_0 = \sigma_a \otimes \sigma_b$, we have that $\theta_0 = \theta_a \theta_b$, where $\theta_a = \sigma_{1a} + \sigma_{2a} + \sigma_{3a}$.

When mixing the basis steady states, the entanglement vanishes, especially when the states are only weakly entangled. Therefore, in the limit of high $T_a$ and $T_b$, in order for $\rho^{st}$ to be entangled, one of the $\lambda_k$’s has to be close to 1. For definiteness, let us pick $\lambda_1$; the analysis for $\lambda_2$ and $\lambda_3$ will be identical. Then, requiring $\lambda_1 \approx 1$ will be equivalent to

$$|\theta_a| |\theta_b| \cos(\theta_a + \theta_b + 4\pi/3) \approx 1. \quad \text{(F1)}$$

On the other hand, for any density matrix $\sigma$ in a 3-dimensional Hilbert space, we have that

$$|\theta_a| \leq \sum_{j \neq j'} |\sigma_{jj'}| \leq \sqrt{3} \sqrt{\sum_{j \neq j'} |\sigma_{jj'}|^2} = \sqrt{\frac{3}{2} \text{Tr}(\sigma^2) - \frac{3}{2}} \leq 1.$$
\(|\sigma_0\rangle = \frac{1}{\sqrt{3}} (|1\rangle + |2\rangle + |3\rangle)\) provide an example saturating all the above upper bounds. Let us now add noise to the states, e.g., via \(\sigma(\delta) = (1-\delta)|\sigma\rangle\langle\sigma| + \frac{\delta}{3}I\) (the purity of this state is equal to \(1 - (4\delta - \delta^2)/3\)), so that they become slightly mixed. As we have just proved, there always exists an \(\delta > 0\) \((\delta \to 0\) as \(T_a \to \infty\)) such that \(\sigma_\delta(\delta) \otimes \sigma_\delta(\delta)\) evolves into an entangled steady state. For a given pair of bath temperatures, \(T_a\) and \(T_b\), let \(\delta_{\text{max}}(T_a, T_b)\) be the supremum of all \(\delta\)'s for which the steady state is entangled. In Fig. 6, we plot \(\delta_{\text{max}}(T_a, 2T_a)\) and the negativity of the steady state for \(\delta = \delta_{\text{max}}(T_a, 2T_a)/2\) against \(T_a\). The tunnelling rate for Fig. 6 is chosen to be \(\tau = 0.1\), and note that, for this value of \(\tau\), \(T[H_{ab}] \approx 0.3948\). Lastly, let us note that \(\theta_\sigma\) is related to the measure of coherence in the state \(\sigma\), defined as \(C[\sigma] = \sum_{j \neq j'} |\sigma_{jj'}| [70]\), via \(\theta_\sigma \equiv \frac{1}{2}C[\sigma]\), and, when \(\theta_\sigma\) tends to its maximum (= 1), \(C[\sigma]\) is also maximized.

Appendix G: Symmetries and current in classical and local quantum MEs

Let us start by analysing the steady-state regime in the classical setting. There, the evolution is given by

\[
\frac{\partial p_{j_a,j_b}}{\partial t} = \sum_{j'_a=j_a \pm 1} [p_{j'_a,j_b}W_{j'_a,j_b|j_a,j_b} - p_{j_a,j_b}W_{j_a,j_b|j'_a,j_b}] + \sum_{j'_b=j_b \pm 1} [p_{j_a,j'_b}W_{j_a,j'_b|j_a,j_b} - p_{j_a,j_b}W_{j_a,j_b|j'_a,j_b}],
\]

where \(W_{j_a,j_b|j'_a,j'_b}\) are the transition rates satisfying local detailed balance conditions:

\[
W_{j_a,j_b|j'_a,j'_b} = e^{-\beta(U_{j'_a,j'_b} - U_{j_a,j_b})} W_{j_a,j_b|j'_a,j'_b},
\]

where \(U_{j_a,j_b}\) is the potential.

Upon collecting \(p_{j,a,b}\)'s and the transition rates into, respectively, the vector \(p\) and matrix \(W\), Eq. (G1) can be rewritten as \(\frac{dp}{dt} = Wp\).

The stochastic matrix \(W\) depends on the potential \(U\) through Eq. (G2) (these can be chosen to be given by, e.g., Eq. (17) in the main text).

Now, since the phase space is discrete, all transformations are given by permutation matrices. A configuration function, e.g., the steady state \(p\), is symmetric under a transformation \(P\) if \(Pp = p\). Importantly, Eqs. (G1)-(G2) ensure that \(U\) and \(p\) have the same symmetries. Note, however, that, although \(W\) and \(U\) might have common symmetry transformations, due to the nonlinear relation between \(W\) and \(U\), not all symmetries of \(U\) will in general be respected by \(W\). The global “rigid” rotation symmetry of the potential (namely, \(U_{i+k,j+k} = U_{i,j}\)) provides an example of the first kind: \(W\) is also symmetric under the global rotation, which is expressed in the fact that \(W_{j_a,j_b|j'_a,j'_b} = W_{j_a,j_b|j'_a,j'_b}\). Interestingly, \(W\) maintains a single steady state despite this symmetry. As is discussed below, this is related to the fact that, although the local GKSL equation giving rise to the classical evolution is, as a whole, symmetric under global rotations, its individual jump operators are not. Following the terminology of Ref. [55], one may say that the classical dynamics is hence only “weakly” symmetric under global rotations.

Now, it can be shown numerically [24] that there is no current in the system whenever \(\phi = k\pi/3\) \((k\) is an arbitrary integer), and the current is non-zero whenever \(\phi \neq k\pi/3\) (see Fig. 7). At the same time, we notice that, whenever \(\phi = k\pi/3\), \(U_{i,j} = U_{i+1,j+k}\). In other words, when \(\phi = k\pi/3\), the potential is symmetric under the combination of particle exchange and a unilateral rotation. The disappearance of the current in this case can be explained by a simple physical argument. Indeed, exchanging the particles is equivalent to swapping the temperatures of the baths. On the other hand, taking, for simplicity, \(\phi = 0\), we see that the system and its state remain intact (recall that the steady state inherits all \(U\)'s symmetries). Put otherwise, inverting the direction of the temperature gradient does not alter the currents. Now, drawing our intuition from phenomenological non-equilibrium thermodynamics [77], where, for small temperature differences, the current is a linear function of the temperature gradient, we thus conclude that the current must be zero.

Interestingly, the generator of the evolution, \(W\), is not symmetric under the above “generalized” exchange sym-
metry. Nevertheless, the increased degeneracy in $U$’s spectrum ($U$ has 2 distinct eigenvalues when $\phi = k\pi/3$, and 3 when $\phi \neq k\pi/3$) brings about degeneracies in $W$, leaving it with only 5 distinct eigenvalues (contrasting $W$'s having no degeneracies when $\phi \neq k\pi/3$), which in turn means that $W$ is more symmetric when $\phi = k\pi/3$.

The classical ME and phenomenological quantum ME,

$$\frac{\partial \rho}{\partial t} = L_{\text{loc}}[\rho] = -i[H, \rho] + \sum_{j_a, j_a', j_b} W_{j_a j_b} \cdot \{L_{j_a, j_a', j_b} S[L_{j_a, j_a', j_b}][\rho] + \sum_{j_a, j_b, j_b'} W_{j_a j_b} \cdot \{L_{j_a, j_b, j_b'} S[L_{j_a, j_b, j_b'}][\rho] \} \tag{G3}$$

where $L_{j_a, j_a', j_b} \equiv |j_a, j_b, j_b'| \langle j_a' | j_a | j_b' \rangle$, $L_{j_a, j_b, j_b'} \equiv |j_a, j_b, j_b'| \langle j_a | j_b' | j_b \rangle$, share many similarities. Indeed, the phenomenological ME is only weakly symmetric under global rotation $R$: It is easy to see that $R$ does not commute with the jump operators $L_{j_a, j_b}$, whereas it can be seen by direct inspection that $RL_{\text{loc}}[\rho] R^\dagger = L_{\text{loc}}[R\rho R^\dagger]$. A GKSL ME, only weakly symmetric under a non-trivial unitary transformation, does not necessarily have multiple steady states [55], and our phenomenological ME, given by Eq. (G3), indeed has a unique steady state for any value of $\tau$. This can be seen as an artifact of the global (system plus baths) Hamiltonian not being symmetric under global rotation.

As we can see in Fig. 7, where the average of $J_{j_a \rightarrow j_a + 1}$ in the steady state (the same for all $j_a$'s) is plotted against $\phi$, the current is again zero whenever $\phi = k\pi/3$. The generalized exchange symmetry of the potential, discussed in the previous section and, in the quantum case, given by $\Xi_k = \text{SWAP} \cdot (1 \otimes R_x^k)$, is also a symmetry of $H_{\text{ab}}$: $[H, \Xi_k] = 0$. Similarly to the classical case, although the generator of the evolution is not symmetric under $\Xi_k (\Xi_k L_{\text{loc}}[\rho] \Xi_k^\dagger \neq L_{\text{loc}}[\Xi_k \rho \Xi_k^\dagger])$, the steady state is. Moreover, taking into account that, in the steady state, unilateral rotation does not affect the average local current ($J_{j_b \rightarrow j_b + 1} = J_{j_b + 1 \rightarrow j_b}$), we conclude that the transformation $\Xi_k$ is equivalent to swapping the temperatures of the baths. Hence, by the same physical argument as in the classical case, we restore the intuition as to why the current is zero for $\phi = k\pi/3$. Note that the situation is slightly different for global GKSL ME (see, e.g., Fig. 2 in the main text): only the thermal current nullifies at $\phi = k\pi/3$, whereas the tunnelling current can be non-zero. This shows that, when all quantum effects are properly taken into account, the rigid link between symmetry and current is no longer present, and the link is broken by the purely quantum component – the tunnelling current.

Importantly, we note that the average steady-state current is $2\pi/3$-periodic. Indeed, the potential satisfies $U_{j_a, j_b}(\phi + 2\pi/3) = U_{j_a, j_b+1}(\phi)$, which, along with the fact that the tunnelling parts of the Hamiltonian are local-rotation-invariant, means that a $2\pi/3$ phase shift is equivalent to a unilateral rotation. On the other hand, the average local current in the steady state is invariant under local rotation, which proves the $2\pi/3$-periodicity of the average current. This periodicity is evident in Fig. 7.

Interestingly, as can be read off from Fig. 7, the current inversion phenomenon observed for the global ME in the main text, takes place also in this case. This is a purely quantum phenomenon, and it occurs only when $T_a$ is sufficiently low.

As a final remark, let us note that the thermodynamic inconsistency of the local GKSL manifests itself when $T_a = T_b$. Indeed, when $\tau \neq 0$, the steady-state solution, $\rho_{\text{st}}^{\text{loc}}$, is not a thermal state at temperature $T_a$; in fact, $\| \rho_{\text{st}}^{\text{loc}} - \frac{1}{Z} e^{-H_{\text{ab}}/T_a} \| \propto \tau$ (where $Z = \text{Tr} e^{-H/T_a}$), so the deviation cannot be considered to be small. Interestingly, however, this inconsistency affects the current only mildly: $\langle J_{\text{a}} \rangle \propto \tau^2$, which is only a second-order effect, and, moreover, $\langle J_{\text{a}} \rangle + \langle J_{\text{b}} \rangle = 0$. 
