Hilbert’s Tenth problem and NP-completeness of Boolean Syllogistic with unordered cartesian product (BS⊗)

Domenico Cantone and Pietro Ursino
Dipartimento di Matematica e Informatica, Università di Catania
Viale Andrea Doria 6, I-95125 Catania, Italy.
E-mail: domenico.cantone@unict.it, pietro.ursino@unict.it
January 5, 2021

Abstract
We relate the decidability problem for BS⊗ with Hilbert’s Tenth problem and prove that BS⊗ is NP-complete.

Introduction
The well-celebrated Hilbert’s Tenth problem (HTP, for short; see [Hilbert-02]), posed by David Hilbert at the beginning of last century, asks for a uniform procedure that can determine in a finite number of steps whether any given Diophantine polynomial equation with integral coefficients is solvable in integers.

In 1970, it was shown that no algorithmic procedure exists for the solvability problem of generic polynomial Diophantine equations, as was proved by the combined efforts of M. Davis, H. Putnam, J. Robinson, and Y. Matiyasevich (DPRM theorem, see [Rob, DPR61, Mat70]).

In the early eighties, Martin Davis asked whether the decision problems for the theories MLS× and MLS⊗ can be reducible to HTP.

By considering MLS× (resp., MLS⊗) as set-theoretic counterpart of HTP, disjoint sets union and Cartesian product (unordered Cartesian product of

\footnote{MLS× is the acronym for MultiLevel Syllogistic with Cartesian product (MLS⊗ with unordered cartesian product), MLS, is the quantifier-free fragment of set theory involving the Boolean set operators and the equality and membership predicates; see [FOSS80].}
disjoint sets, in the case of $\text{MLS} \otimes$) play in some sense the roles of integer addition and multiplication, respectively, since $|s \cup t| = |s| + |t|$, for any disjoint sets $s$ and $t$, and $|s \times t| = |s| \cdot |t|$, for any sets $s$ and $t$ (whereas $|s \otimes t| = |s| \cdot |t|$, for any disjoint sets $s$ and $t$)(we will show in a future article that $\text{MLS} \otimes$ is decidable).

This connection has been fully established by Cantone, Cutello and Policriti in [CCP90].

Indeed, when $\text{MLS} \otimes$ and $\text{MLS} \times$ are extended with the two-place predicate $|\cdot| \leq |\cdot|$ for cardinality comparison, where $|s| \leq |t|$ holds if and only if the cardinality of $s$ does not exceed that of $t$, their satisfiability problems become undecidable, since Hilbert’s Tenth problem would be reducible to them.

We denote by $\text{BS} \otimes^2$ the language $\text{MLS} \otimes$ without the use of membership operator.

In the above cited reduction to HTP, membership operator plays no role [CU18], then, by extending $\text{BS} \otimes$ with the two-place predicate $|\cdot| \leq |\cdot|$ for cardinality comparison, you get again a problem reducible to HTP.

Therefore the real set-theoretic counterpart of HTP is actually $\text{BS} \otimes$.

Moreover this language can force a model to be infinite, hence there is no way to prove small model property (Definition 1).

Nevertheless, we prove in [CU14] that even theories which force a model to be infinite can be proved to be decidable by using the technique of formative processes [CU18] and the small witness-model property (Definition 2), which is a way to finitely represent the infinity.

Observe that Boolean Syllogistic ($\text{BS}$) is NP-complete.

In the present paper we prove that $\text{BS} \otimes$ is not only decidable but, rather unexpectedly, NP-complete (Theorem 5).

Actually, the real counterpart of HTP is $\text{BS} \otimes_{fin}$, the language $\text{BS} \otimes$ restricted to finite models, which is proved to enjoy small model property (Corollary 1).

The language $\text{BS}$ with cardinal inequalities is equivalent to a pure existential presburger arithmetic, which is proved to be NP-complete in [Sca] (anyway you can perform a straightforward calculation through our tools, just considering that $\text{BS}$ is NP-complete and cardinal inequalities are polynomial time verifiable).

Combining the NP-completeness of the pure existential presburger arithmetic and the main result of the present article, we can argue that undecidability of HTP arises from an interaction between unordered cartesian

\footnote{$\text{BS} \otimes$ is the acronym for Boolean Syllogistic with unordered Cartesian product.}
1 Preliminaries

1.1 The theory BS⊗

BS⊗ is the quantifier-free fragment of set theory consisting of the propositional closure of atoms of the following types:

\[ x = y \cup z, \quad x = y \cap z, \quad x = y \setminus z, \quad x = y \otimes z, \quad x \subseteq y \]

where \( x, y, z \) stand for set variables or the constant \( \emptyset \).

1.1.1 Semantics of BS⊗

The semantics of BS⊗ is defined in a very natural way. A set assignment \( M \) is any map from a collection \( V \) of set variables (called the VARIABLES DOMAIN of \( M \) and denoted \( \text{dom}(M) \)) into the von Neumann universe \( \mathcal{V} \) of all well-founded sets.

We recall that \( \mathcal{V} \) is a cumulative hierarchy constructed in stages by transfinite recursion over the class \( On \) of all ordinals. Specifically, \( \mathcal{V} := \bigcup_{\alpha \in On} \mathcal{V}_\alpha \) where, recursively, \( \mathcal{V}_\alpha := \bigcup_{\beta < \alpha} \text{pow}(\mathcal{V}_\beta) \), for every \( \alpha \in On \), where \( \text{pow}(\cdot) \) denotes the powerset operator. Based on such construction, we can readily define the RANK of any well-founded set \( s \in \mathcal{V} \), denoted \( \text{rk}(s) \), as the least ordinal \( \alpha \) such that \( s \subseteq \mathcal{V}_\alpha \). The collection of sets of finite rank, hence belonging to \( \mathcal{V}_\alpha \) for some finite ordinal \( \alpha \), forms the set \( \mathcal{H} \) of the HEREDITARILY FINITE SETS. Plainly, \( \mathcal{H} = \mathcal{V}_\omega \), where \( \omega \) is the first limit ordinal, namely the smallest non-null ordinal having no immediate predecessor.

Given a set assignment \( M \) and a collection \( W \subseteq \text{dom}(M) \), we put \( MW := \{ Mv \mid v \in W \} \). The SET DOMAIN of \( M \) is defined as the set \( \bigcup MV = \bigcup_{v \in V} Mv \). The RANK of \( M \) is the rank of its set domain, namely, \( \text{rk}(M) := \text{rk}(\bigcup MV) \) (so that, when \( V \) is finite, \( \text{rk}(M) = \max_{v \in V} \text{rk}(Mv) \)). A set assignment \( M \) is FINITE, if so is its set domain.

For \( x, y \in \text{dom}(M) \), we set

\[
M(x \cup y) := Mx \cup My, \quad M(x \cap y) := Mx \cap My, \quad M(x \setminus y) := Mx \setminus My, \\
M(x \otimes y) := Mx \otimes My = \{ \{ u, u' \} \mid u \in Mx, u' \in My \}.
\]
We also put
\[ M(x = y \star z) = \text{true} \iff Mx = M(y \star z), \]
where \( \star \in \{\cup, \cap, \setminus\} \). Finally, we put recursively
\[ M(\neg \Phi) := \neg M\Phi, \quad M(\Phi \land \Psi) := M\Phi \land M\Psi, \]
\[ M(\Phi \lor \psi) := M\Phi \lor M\Psi, \quad M(\Phi \to \psi) := M\Phi \to M\Psi, \quad \text{etc.,} \]
for all \( \text{BS} \otimes \)-formulae \( \Phi \) and \( \Psi \) such that \( \text{Vars}(\Phi), \text{Vars}(\Psi) \subseteq \text{dom}(M) \).

For a given \( \text{BS} \otimes \)-formula \( \Phi \), a set assignment \( M \) defined over \( \text{Vars}(\Phi) \) is said to satisfy \( \Phi \) if \( M\Phi = \text{true} \) holds, in which case we also write \( M \models \Phi \) and say that \( M \) is a model for \( \Phi \). If \( \Phi \) has a model, we say that \( \Phi \) is satisfiable; otherwise, we say that \( \Phi \) is unsatisfiable. Two \( \text{BS} \otimes \)-formulae \( \Phi \) and \( \Psi \) are said to be equisatisfiable if \( \Phi \) is satisfiable if and only if so is \( \Psi \), possibly by distinct models.

The decision problem or satisfiability problem for \( \text{BS} \otimes \) is the problem of establishing algorithmically whether any given \( \text{BS} \otimes \)-formula is satisfiable or not by some set assignment.

By applying disjoint normal form and the simplification rules illustrated in [CUTS], the satisfiability problem for \( \text{BS} \otimes \) can be reduced to the satisfiability problem for normalized conjunctions of \( \text{BS} \otimes \), namely conjunctions of \( \text{BS} \otimes \)-atoms of the following restricted types:
\[ x = y \cup z, \quad x = y \setminus z, \quad x = y \otimes z, \]  \( \quad \text{(1)} \)
where \( x, y, z \) stand for set variables.

### 1.2 The “relaxed” fragment \( \text{BS} \otimes_\subseteq \)

Strictly related to the satisfiability problem for \( \text{BS} \otimes \)-normalized conjunctions is the satisfiability problem for the fragment \( \text{BS} \otimes_\subseteq \) consisting of the conjunctions of atoms of the following types:
\[ x = y \cup z, \quad x = y \setminus z, \quad x \subseteq y \otimes z. \]  \( \quad \text{(2)} \)
Plainly, \( \text{BS} \otimes_\subseteq \)-conjunctions can be expressed in the theory \( \text{BS} \otimes \), so the decidability of the satisfiability problem for \( \text{BS} \otimes_\subseteq \) will follow from that of \( \text{BS} \otimes \). However, whereas any satisfiable \( \text{BS} \otimes_\subseteq \)-conjunction always admits a finite model (in fact, a model of finite bounded rank), the same is not true
for BS⊗-conjunctions. Consider for instance the BS⊗-conjunction
\[ \Phi_\infty := x \neq \emptyset \land z = x \otimes x \land z \subseteq x. \]

Putting \( M^*x := HF \) and \( M^*z := HF \otimes HF \) (so, \( M^*z \) is the collection of all nonempty hereditarily finite sets with at most two members), it is an easy matter to check that the set assignment \( M^* \) satisfies \( \Phi_\infty \). In addition, if a set assignment \( M \) satisfies \( \Phi_\infty \), then for every \( s \in Mx \) we have \( \{s\} \in Mz \subseteq Mx \), and therefore \( \{s\} \in Mx \). For any set \( a \), define the \( n \)-iterated singleton \( \{a\}^n \) by putting recursively
\[
\begin{cases}
\{a\}^0 = 0 \\
\{a\}^{n+1} = \{\{a\}^n\}, \text{ for } n \in \mathbb{N}.
\end{cases}
\]
Thus, letting \( s \) be any member of \( Mx \) (which exists since \( Mx \neq \emptyset \)), it follows that \( Mx \) contains as a subset the infinite set \( \{\{s\}^n \mid n \in \mathbb{N}\} \), proving that \( Mx \) is infinite, and in turn showing that the conjunction \( \Phi_\infty \) is satisfied only by infinite models.

### 1.3 Small model property and small witness-model property

**Definition 1.** We say that a given quantifier-free subtheory \( T \) of set theory has the small model property if there exists a computable function \( c: \mathbb{N} \to \mathbb{N} \) such that, for any satisfiable \( T \)-formula \( \Psi \) there is a set assignment \( M \) of rank at most \( c(\vert \Psi \vert) \) that is a model for \( \Psi \).

We shall prove that the theory BS\( \otimes \subseteq \) enjoys small model property.

This definition could seem useless in case of languages have not the small model property. This is the case of the theory BS\( \otimes \). Indeed, there are satisfiable BS\( \otimes \)-conjunctions admitting only models of infinite rank, therefore the finite partition which imitates the original one cannot satisfy in any case a BS\( \otimes \)-formula. However, we have showed in other works that even if the partition does not generate any model for the given formula still the notion of imitation (see [CU18] and Section 1.5) makes sense. Indeed, it witnesses the existence of a model.

**Definition 2.** We say that a given quantifier-free subtheory \( T \) of set theory has the small witness-model property if there exists a computable function \( c: \mathbb{N} \to \mathbb{N} \) such that, for any satisfiable \( T \)-formula \( \Psi \) there is a set assignment \( M \) of rank at most \( c(\vert \Psi \vert) \) that certifies the existence of a model for \( \Psi \).
For the above reasons, the role of imitation remains important also for languages which does not enjoy small model property.

Much as before, the small witness-model property for $T$ implies the decidability of the satisfiability problem for $T$.

More specifically, for any given satisfiable $BS\otimes$-conjunction $\Phi$, any witness model $M$ for $\Phi$ will be a small model of the related relaxed conjunction $\tilde{\Phi}$ obtained from $\Phi$ by replacing each of its literals $x = y \otimes z$ by the literal $x \subseteq y \otimes z$.

A rather common strategy to solve decidability problems consists in finding for any model of the formula a finite bounded assignment which either satisfies the formula or witnesses the existence of a model for the formula. To this purpose, it becomes extremely useful finding finite bounded partitions which imitates the original model.

We shall review next the notions of *satisfiability by partitions, partition simulations, $\otimes$ graphs* (special graphs superimposed to the Venn partition of a given model), together with some of their properties (see [CU18]).

### 1.4 Partitions and their use in the satisfiability problem

A partition is a collection of pairwise disjoint non-null sets, called the blocks of the partition. The union $\bigcup \Sigma$ of a partition $\Sigma$ is the domain of $\Sigma$.

#### 1.4.1 Satisfiability by partitions

Let $V$ be a finite collection of set variables and $\Sigma$ a partition. Also, let $I : V \rightarrow \text{pow}(\Sigma)$. The map $I$ induces in a very natural way a set assignment $M_I$ over $V$ by putting

$$M_I v := \bigcup I(v), \quad \text{for } v \in V.$$  

We refer to the triple $(\Sigma, V, I)$ as a partition assignment.

**Definition 3.** Given a map $I : V \rightarrow \text{pow}(\Sigma)$ over a finite collection $V$ of set variables, with $\Sigma$ a partition, for any $BS\otimes$-formula $\Phi$ such that $\text{Vars}(\Phi) \subseteq V$, the partition $\Sigma$ satisfies $\Phi$ via the map $I$ (or, equivalently, the partition assignment $(\Sigma, V, I)$ satisfies $\Phi$), and we write $\Sigma/I \models \Phi$, if the set assignment $M_I$ induced by $I$ satisfies $\Phi$. We say that $\Sigma$ satisfies $\Phi$, and write $\Sigma \models \Phi$, if $\Sigma$ satisfies $\Phi$ via some map $I : V \rightarrow \text{pow}(\Sigma)$.

Thus, if an $BS\otimes$-formula $\Phi$ is satisfied by some partition, then it is satisfied by some set assignment. The converse holds too. Indeed, let us
assume that $M \models \Phi$, for some set assignment $M$ over a given collection $V$ of set variables such that $\text{Vars}(\Phi) \subseteq V$. Let $\Sigma_M$ be the VENN partition induced by $M$, namely

$$\Sigma_M := \left\{ \bigcap M V' \setminus \bigcup M (V \setminus V') \mid \emptyset \neq V' \subseteq V \right\} \backslash \{ \emptyset \}.$$

Thus, for any $\sigma \in \Sigma_M$ and $v \in V$, either $\sigma \cap Mv = \emptyset$ or $\sigma \subseteq Mv$. Let $\mathcal{I}_M : V \rightarrow \text{pow}(\Sigma_M)$ be the map defined by putting

$$\mathcal{I}_M(v) := \{ \sigma \in \Sigma_M \mid \sigma \subseteq Mv \}, \quad \text{for } v \in V.$$

It is an easy matter to check that the set assignment induced by $\mathcal{I}_M$ is just $M$. Thus $\Sigma_M / \mathcal{I}_M \models \Phi$, and therefore $\Sigma_M \models \Phi$, proving that $\Phi$ is satisfied by some partition, in fact by a finite partition.

Therefore the notions of satisfiability by set assignments and that of satisfiability by partitions coincide.

1.5 Imitating partitions

We extensively treated the above argument in [CU18]. Here we provide a short and, possibly, exhaustive resume of this.

We start by observing that satisfiability of Boolean literals of type $x = y \cup z$ and $x = y \setminus z$ by the set assignment $M_3$ depends solely on $\mathcal{I}$, as shown in the following lemma.

**Lemma 1.** Let $\Sigma$ be a partition and let $\mathcal{I} : V \rightarrow \text{pow}(\Sigma)$ be a map over a (finite) set of variables $V$. Also, let $M_3$ be the set assignment induced by $\mathcal{I}$ over $V$. Then

$$M_3 \models x = y \cup z \iff \mathcal{I}(x) = \mathcal{I}(y) \cup \mathcal{I}(z)$$

and

$$M_3 \models x = y \setminus z \iff \mathcal{I}(x) = \mathcal{I}(y) \setminus \mathcal{I}(z),$$

for any $x, y, z \in V$.

**Proof.** Let $x, y, z \in V$, and let $\ast \in \{ \cup, \setminus \}$. Since $\Sigma$ is a partition (and therefore its blocks are nonempty and mutually disjoint), we have:

$$M_3 \models x = y \ast z \iff \bigcup \mathcal{I}(x) = \bigcup \mathcal{I}(y) \ast \bigcup \mathcal{I}(z)$$

$$\iff \bigcup \mathcal{I}(x) = \bigcup (\mathcal{I}(y) \ast \mathcal{I}(z))$$

$$\iff \mathcal{I}(x) = \mathcal{I}(y) \ast \mathcal{I}(z). \quad \square$$
Remark 1. By exploiting the fact that $s \cap t = s \setminus (s \setminus t)$, for any sets $s$ and $t$, under the assumptions of Lemma 1 we have

$$M_s \models x = y \cup z \iff M_y \models x = y \setminus (y \setminus z) \iff M_{y'} \models x = y \setminus x' \land x' = y \setminus z,$$

where $x' \notin V$ and $\mathcal{I}'$ extends $\mathcal{I}$ over $V \cup \{x'\}$ by letting $\mathcal{I}'(x') := \mathcal{I}(y) \setminus \mathcal{I}(z)$.

Hence,

$$M_s \models x = y \cap z \iff \mathcal{I}'(x) = \mathcal{I}'(y) \setminus \mathcal{I}'(x') \land \mathcal{I}'(x') = \mathcal{I}'(y) \setminus \mathcal{I}'(z)$$

$$\iff \mathcal{I}(x) = \mathcal{I}(y) \setminus (\mathcal{I}(y) \setminus \mathcal{I}(z)) \iff \mathcal{I}(x) = \mathcal{I}(y) \cap \mathcal{I}(z).$$

Let $(\Sigma, V, \mathcal{I})$ and $(\widehat{\Sigma}, V, \widehat{\mathcal{I}})$ be partition assignments, with $\Sigma$ and $\widehat{\Sigma}$ partitions of the same size and $V$ a finite set of variables. As noted above, the pairs $\Sigma, \mathcal{I}$ and $\widehat{\Sigma}, \widehat{\mathcal{I}}$ induce respectively the set assignments $M_s$ and $\widehat{M}_s$ over $V$. Towards establishing the small model property for $\text{BS} \otimes \subseteq$ (and then the small witness-model property for $\text{BS} \otimes$), we prove next some results that cumulatively will provide sufficient conditions in order that

$$(*) \text{ any } \text{BS} \otimes \subseteq\text{-conjunction (resp., BS} \otimes\text{-conjunction) } \Phi \text{ such that } \text{Vars}(\Phi) \subseteq V \text{ is satisfiable by } \widehat{M}_s \text{ whenever it is satisfied by } M_s, \text{ i.e.,}$$

$$M_s \models \Phi \iff \widehat{M}_s \models \Phi.$$

Lemma 2. Let $\Sigma$ and $\widehat{\Sigma}$ be partitions and $\beta: \Sigma \rightarrow \widehat{\Sigma}$ a bijection. Let $\mathcal{I}: V \rightarrow \text{pow}(\Sigma)$ be a map over a (finite) set of variables $V$, and let $\widehat{\mathcal{I}}: V \rightarrow \text{pow}(\widehat{\Sigma})$ be the map induced by $\beta$ and $\mathcal{I}$ by letting

$$\widehat{\mathcal{I}}(x) = \beta[\mathcal{I}(x)], \quad \text{for any } x \in V.$$

Then

$$\mathcal{I}(x) = \mathcal{I}(y) \cup \mathcal{I}(z) \iff \widehat{\mathcal{I}}(x) = \widehat{\mathcal{I}}(y) \cup \widehat{\mathcal{I}}(z)$$

$$\mathcal{I}(x) = \mathcal{I}(y) \cap \mathcal{I}(z) \iff \widehat{\mathcal{I}}(x) = \widehat{\mathcal{I}}(y) \cap \widehat{\mathcal{I}}(z)$$

$$\mathcal{I}(x) = \mathcal{I}(y) \setminus \mathcal{I}(z) \iff \widehat{\mathcal{I}}(x) = \widehat{\mathcal{I}}(y) \setminus \widehat{\mathcal{I}}(z),$$

for any $x, y, z \in V.$
Proof. Since \( \beta \) is a bijection, we have
\[
\mathcal{I}(x) = \mathcal{I}(y) \star \mathcal{I}(z) \iff \mathcal{I}(x) = \beta[\mathcal{I}(x)]
\]
\[
= \beta[\mathcal{I}(y) \star \mathcal{I}(z)]
\]
\[
= \beta[\mathcal{I}(y)] \star \beta[\mathcal{I}(z)]
\]
\[
= \mathcal{I}(y) \star \mathcal{I}(z),
\]
for \( \star \in \{\cup, \cap, \setminus\} \).

From Lemmas 1, 2 and Remark 1, we have at once property \( [\star] \) but limited to Boolean set literals of types \( x = y \star z \), with \( \star \in \{\cup, \cap, \setminus\} \) over \( V \), for any two partition assignments \( (\Sigma, V, \mathcal{I}) \) and \( (\hat{\Sigma}, V, \hat{\mathcal{I}}) \) related by a bijection \( \beta: \Sigma \to \hat{\Sigma} \).

We shall express the conditions that take also care of literals in \( BS \otimes_c \) of the form \( x \subseteq y \otimes z \) by means of some useful variants of the power set operator. They are variations of the \textit{intersecting power set operator} \( \text{pow}^* \), introduced in [Can91] in connection with the solution of the satisfiability problem for a fragment of set theory involving the power set and the singleton operators. Specifically, for any set \( S \), we put
\[
\text{pow}^*(S) := \{ t \subseteq \bigcup S \mid t \cap s \neq \emptyset, \text{ for every } s \in S \},
\]
\[
\text{pow}^*_{1, 2}(S) := \{ t \in \text{pow}^*(S) \mid |t| \leq 2 \},
\]
\[
\text{pow}^*_{> 2}(S) := \{ t \in \text{pow}^*(S) \mid |t| > 2 \}.
\]
Thus,
- \( \text{pow}^*(S) \) is the collection of all subsets of \( \bigcup S \) that have nonempty intersection with all the members of \( S \);
- \( \text{pow}^*_{1, 2}(S) \) is the set of all members of \( \text{pow}^*(S) \) of cardinality 1 or 2;
- \( \text{pow}^*_{> 2}(S) \) is the set of all members of \( \text{pow}^*(S) \) of cardinality strictly greater than 2.

Further properties of \( \text{pow}^* \) are listed in [CU18, pp. 16–20].

Some useful properties of the operators \( \text{pow}^*_{1, 2}, \text{pow}^*_{> 2}, \) and \( \otimes \) are contained in the following lemmas.

The \( \text{pow}^*_{1, 2} \) operator is strictly connected with the unordered Cartesian operator \( \otimes \), as shown next.
Lemma 3. For all sets $s$ and $t$ (not necessarily distinct), we have
\[ \text{pow}_{1,2}^*(\{s,t\}) = s \otimes t. \]

Proof. Plainly, $s \otimes t \subseteq \text{pow}_{1,2}^*(\{s,t\})$. Indeed, if $u \in s \otimes t$, then
\[ 1 \leq |u| \leq 2, \quad u \subseteq s \cup t, \quad \text{and} \quad u \cap s \neq \emptyset \neq u \cap t, \]
so that $u \in \text{pow}_{1,2}^*(\{s,t\})$.

Conversely, let $\{u,v\} \in \text{pow}_{1,2}^*(\{s,t\})$. Then
\[ \{u,v\} \subseteq s \cup t \quad \text{and} \quad \{u,v\} \cap s \neq \emptyset \neq \{u,v\} \cap t. \]

Without loss of generality, let us assume that $u \in s$. If $v \in t$, we are done. Otherwise, if $v \notin t$, then $v \in s$ (since $\{u,v\} \subseteq s \cup t$) and $u \in t$ (since $\{u,v\} \cap t \neq \emptyset$). Hence, $\{u,v\} \in s \otimes t$, proving that also the inverse inclusion $\text{pow}_{1,2}^*(\{s,t\}) \subseteq s \otimes t$ holds. \hfill \Box

The following is a simple yet useful property of the unordered Cartesian operator $\otimes$.

Lemma 4. For any sets $s_1, s_2, t_1, t_2$,
\[ (s_1 \otimes s_2) \cap (t_1 \otimes t_2) \neq \emptyset \rightarrow (t_1 \cap s_i \neq \emptyset \land t_2 \cap s_{3-i} \neq \emptyset), \]
for some $i \in \{1, 2\}$.

Proof. Preliminarily, we observe that $(s_1 \otimes s_2) \cap (t_1 \otimes t_2) \neq \emptyset$ plainly implies the following inequalities:
\[ s_1 \cap (t_1 \cup t_2) \neq \emptyset, \quad s_2 \cap (t_1 \cup t_2) \neq \emptyset, \]
\[ t_1 \cap (s_1 \cup s_2) \neq \emptyset, \quad t_2 \cap (s_1 \cup s_2) \neq \emptyset. \]

Thus, if $s_i \cap t_j = \emptyset$ for some $i, j \in \{1, 2\}$, then $s_i \cap t_{3-j} \neq \emptyset$ and $s_{3-i} \cap t_j \neq \emptyset$, and we are done. On the other hand, if $s_i \cap t_j \neq \emptyset$ for all $i, j \in \{1, 2\}$, we are immediately done. \hfill \Box

Then, in order to get property [\ref{item:*}] also for literals of type $x \subseteq y \otimes z$, it is enough to require that the bijection $\beta: \Sigma \rightarrow \hat{\Sigma}$ relating two given partition assignments $(\Sigma, V, \mathcal{J})$ and $(\hat{\Sigma}, \hat{V}, \hat{\mathcal{J}})$ satisfies the following conditions, for every $X \subseteq \Sigma$ and $\sigma \in \Sigma$:
\[ (C_2) \quad \text{pow}_{1,2}^*(X) \cap \sigma = \emptyset \rightarrow \text{pow}_{1,2}^*(\beta[X]) \cap \beta(\sigma) = \emptyset, \]
(C₃) $\text{pow}^s_2(X) \cap \sigma = \emptyset \rightarrow \text{pow}^s_2(\beta[X]) \cap \beta(\sigma) = \emptyset$,

and that the partition $\hat{\Sigma}$ is weakly $\otimes$-transitive, i.e., for all $\otimes$ place $p$, $\bigcup \hat{p} \subseteq \hat{\Sigma}$. A place $p$ such that $p \subseteq \bigcup_{A \subseteq \Sigma} \text{pow}^s_1(A)$ is an $\otimes$ place.

For a given partition $\Sigma$ an $\otimes$-place $q$ is such that $q \subseteq \bigcup A \subseteq \Sigma \text{pow}^s_1$, $\bigcup A_2(B)$.

This defines a labelling $\otimes$ on the set of $\mathcal{P}$. The collection of all $\otimes$ places are denoted by $\otimes \mathcal{P}$.

**Lemma 5.** Let $\Sigma$ and $\hat{\Sigma}$ be partitions related by a bijection $\beta: \Sigma \rightarrow \hat{\Sigma}$ such that $\hat{\Sigma}$ is weakly $\otimes$-transitive and $V$ a set of variables. In addition, let us assume that, for all $\sigma \in \Sigma$ and $X \subseteq \Sigma$, conditions (C₂) and (C₃) hold.

Then, for all $X, Y, Z \subseteq \Sigma$ and every $\sigma \in \Sigma$,

(a) $\sigma \subseteq \bigcup_{A \subseteq \Sigma} \text{pow}^s_1(A) \rightarrow \beta(\sigma) \subseteq \bigcup_{A \subseteq \Sigma} \text{pow}^s_1(\beta[A])$,

(b) $\bigcup X \subseteq \bigcup Y \otimes \bigcup Z \rightarrow \bigcup \beta[X] \subseteq \bigcup \beta[Y] \otimes \bigcup \beta[Z]$,

(c) Let $\mathcal{I}: V \rightarrow \text{pow}(\Sigma)$ and let $\hat{\mathcal{I}}: V \rightarrow \text{pow}(\hat{\Sigma})$ be the map induced by $\mathcal{I}$ and $\beta$. Also, let $M_\mathcal{I}$ and $\hat{M}_\hat{\mathcal{I}}$ be the set assignments over $V$ induced by $(\Sigma, V, \mathcal{I})$ and $(\hat{\Sigma}, V, \hat{\mathcal{I}})$ respectively, then for all $x, y, z \in V$ we have:

$$M_\mathcal{I} | x \subseteq y \otimes z \Rightarrow \hat{M}_\hat{\mathcal{I}} | x \subseteq y \otimes z.$$  

**Proof.** Concerning (a), let $\sigma \subseteq \bigcup_{A \subseteq \Sigma} \text{pow}^s_1(A)$, so that

$$\sigma \cap \bigcup_{A \subseteq \Sigma} \text{pow}^s_2(A) = \emptyset. \quad (3)$$

Being $\sigma$ an $\otimes$ place, from weak $\otimes$-transitive of $\hat{\Sigma}$ and the bijectivity of $\beta$, it follows that

$$\beta(\sigma) \subseteq \hat{\Sigma} \subseteq \text{pow}(\hat{\Sigma}) = \bigcup_{A \subseteq \hat{\Sigma}} \text{pow}^s(A) = \bigcup_{A \subseteq \Sigma} \text{pow}^s(\beta[A]). \quad (4)$$

From (3) and (C₃) we have

$$\beta(\sigma) \cap \bigcup_{A \subseteq \Sigma} \text{pow}^s_2(\beta[A]) = \emptyset,$$
which, in view of (4), yields

$$\beta(\sigma) \subseteq \bigcup_{A \subseteq \Sigma} \text{pow}^{*}_{1,2}(\beta[A]),$$

and therefore

$$\sigma \subseteq \bigcup_{A \subseteq \Sigma} \text{pow}^{*}_{1,2}(A) \rightarrow \beta(\sigma) \subseteq \bigcup_{A \subseteq \Sigma} \text{pow}^{*}_{1,2}(\beta[A]).$$

Concerning (b), let

$$\bigcup X \subseteq \bigcup Y \otimes \bigcup Z,$$

for some $X, Y, Z \subseteq \Sigma$, and let $t \in \bigcup \beta[X]$. Hence, $t \in \beta(\sigma)$, for some $\sigma \in X$. By (5) and Lemma 3, we have $\sigma \subseteq \bigcup_{A \subseteq \Sigma} \text{pow}^{*}_{1,2}(A)$, so that, by (a) $\beta(\sigma) \subseteq \bigcup_{A \subseteq \Sigma} \text{pow}^{*}_{1,2}(\beta[A])$. Hence, $t = \{a, b\}$, for some sets $a$ and $b$ not necessarily distinct. Since $\hat{\Sigma}$ is weakly $\otimes$ transitive, $t \subseteq \Sigma$. Thus, $t \subseteq \beta(\sigma_1) \cup \beta(\sigma_2)$, for some $\sigma_1, \sigma_2 \in \Sigma$ not necessarily distinct such that $t \cap \beta(\sigma_1) \neq \emptyset$ and $t \cap \beta(\sigma_2) \neq \emptyset$. Plainly, $t \in \text{pow}^{*}_{1,2}(\{\beta(\sigma_1), \beta(\sigma_2)\})$, and so $\text{pow}^{*}_{1,2}(\{\beta(\sigma_1), \beta(\sigma_2)\}) \cap \beta(\sigma) \neq \emptyset$. Therefore, by (C2), $\text{pow}^{*}_{1,2}(\{\sigma_1, \sigma_2\}) \cap \sigma \neq \emptyset$ and, by Lemma 3, $(\sigma_1 \otimes \sigma_2) \cap \sigma \neq \emptyset$, so that a fortiori $(\sigma_1 \otimes \sigma_2) \cap (\bigcup Y \otimes \bigcup Z) \neq \emptyset$. Thus, by Lemma 3, $\bigcup Y \cap \sigma_i \neq \emptyset$ and $\bigcup Z \cap \sigma_{3-i} \neq \emptyset$, for some $i \in \{1, 2\}$. But then, $\sigma_1 \in Y$ and $\sigma_{3-i} \in Z$, so that $\beta(\sigma_1) \subseteq \bigcup \beta[Y]$ and $\beta(\sigma_{3-i}) \subseteq \bigcup \beta[Z]$. Hence, $\beta(\sigma_1) \otimes \beta(\sigma_2) \subseteq \bigcup \beta[Y] \otimes \bigcup \beta[Z]$ and so $t \in \bigcup \beta[Y] \otimes \bigcup \beta[Z]$. By the arbitrariness of $t \in \bigcup \beta[X]$, it follows that

$$\bigcup \beta[X] \subseteq \bigcup \beta[Y] \otimes \bigcup \beta[Z],$$

and therefore

$$\bigcup X \subseteq \bigcup Y \otimes \bigcup Z \rightarrow \bigcup \beta[X] \subseteq \bigcup \beta[Y] \otimes \bigcup \beta[Z]$$

holds.
Finally, concerning \( [c] \) we have

\[
M_2 \models x \subseteq y \otimes z \implies M_2x \subseteq M_2y \otimes M_2z
\]

\[
\implies \bigcup \mathcal{I}(x) \subseteq \bigcup \mathcal{I}(y) \otimes \bigcup \mathcal{I}(y)
\]

\[
\implies \bigcup \beta[\mathcal{I}(x)] \subseteq \bigcup \beta[\mathcal{I}(y)] \otimes \bigcup \beta[\mathcal{I}(y)] \quad \text{(by \([b]\))}
\]

\[
\implies \bigcup \tilde{\mathcal{I}}(x) \subseteq \bigcup \tilde{\mathcal{I}}(y) \otimes \bigcup \tilde{\mathcal{I}}(y)
\]

\[
\implies \hat{M}_2x \subseteq \hat{M}_2y \otimes \hat{M}_2z
\]

\[
\implies \hat{M}_2 \models x \subseteq y \otimes z.
\]

The following definition and theorem summarize the above considerations.

**Definition 4** (BS\( \otimes \)-imitation). A weakly \( \otimes \)-transitive partition \( \hat{\Sigma} \) is said to weakly BS\( \otimes \)-imitates another partition \( \Sigma \), when there exists a bijection \( \beta: \Sigma \to \hat{\Sigma} \) such that, for all \( X \subseteq \Sigma \) and \( \sigma \in \Sigma \),

\[
(C_1) \quad \text{pow}^*_1(X) \cap \sigma = \emptyset \implies \text{pow}^*_1(\beta[X]) \cap \beta(\sigma) = \emptyset,
\]

\[
(C_2) \quad \text{pow}^*_2(X) \cap \sigma = \emptyset \implies \text{pow}^*_2(\beta[X]) \cap \beta(\sigma) = \emptyset.
\]

Theorem 1 contains sufficient conditions to achieve property \((\star)\) above (just before Lemma 2) for BS\( \otimes \_\_\_\_\-\)conjunctions.

**Theorem 1.** Let \( \Sigma \) and \( \hat{\Sigma} \) be partitions such that \( \hat{\Sigma} \) is weakly \( \otimes \)-transitive and weakly BS\( \otimes \)-imitates \( \Sigma \), when there exists a bijection \( \beta: \Sigma \to \hat{\Sigma} \) such that, for every \( X \subseteq \Sigma \),

\[
M_2 \models \Phi \implies \hat{M}_2 \models \Phi,
\]

where \( M_2 \) and \( \hat{M}_2 \) are the set assignments over \( V \) induced by the partition assignments \((\Sigma, V, \mathcal{I})\) and \((\hat{\Sigma}, V, \hat{\mathcal{I}})\), respectively.

1.5.1 (Strong) BS\( \otimes \)-imitation of a partition

**Definition 5** (Strong BS\( \otimes \)-imitation). A weakly \( \otimes \)-transitive \( \hat{\Sigma} \) is said to (strong) BS\( \otimes \)-imitates another partition \( \Sigma \), when it weakly BS\( \otimes \)-imitates \( \Sigma \) via a bijection \( \beta \) and the following additional \( \otimes \)-saturatedness condition holds, for every \( X \subseteq \Sigma \):

\[
M_3 \models x \subseteq y \otimes z \implies M_3x \subseteq M_3y \otimes M_3z
\]

\[
\implies \bigcup \mathcal{I}(x) \subseteq \bigcup \mathcal{I}(y) \otimes \bigcup \mathcal{I}(y)
\]

\[
\implies \bigcup \beta[\mathcal{I}(x)] \subseteq \bigcup \beta[\mathcal{I}(y)] \otimes \bigcup \beta[\mathcal{I}(y)] \quad \text{(by \([b]\))}
\]

\[
\implies \bigcup \tilde{\mathcal{I}}(x) \subseteq \bigcup \tilde{\mathcal{I}}(y) \otimes \bigcup \tilde{\mathcal{I}}(y)
\]

\[
\implies \hat{M}_3x \subseteq \hat{M}_3y \otimes \hat{M}_3z
\]

\[
\implies \hat{M}_3 \models x \subseteq y \otimes z.
\]
In general, a node $A$ is said to be saturated if $\text{pow}_{1,2}(A) \subseteq \bigcup \Sigma$ holds.

**Theorem 2.** Let $\Sigma$ and $\hat{\Sigma}$ be partitions such that $\hat{\Sigma}$ is weakly $\otimes$-transitive and $\text{BS} \otimes$-imitates $\Sigma$ via a bijection $\beta : \Sigma \rightarrow \hat{\Sigma}$. Also, let $\mathcal{J} : V \rightarrow \text{pow}(\Sigma)$ be any map over a given finite collection $V$ of variables, and let $\hat{\mathcal{J}}$ be the map over $V$ induced by $\mathcal{J}$ and $\beta$. Then, for every $\text{BS} \otimes$-conjunction $\Phi$ such that $\text{Vars}(\Phi) \subseteq V$, we have

$$M_3 \models \Phi \implies \hat{M}_3 \models \Phi,$$

where $M_3$ and $\hat{M}_3$ are the set assignments over $V$ induced by the partition assignments $(\Sigma, V, \mathcal{J})$ and $(\hat{\Sigma}, V, \hat{\mathcal{J}})$, respectively.

**Proof.** In view of Theorem [1] it is enough to prove that for every literal of the form $y \otimes z \subseteq x$, with $x, y, z \in V$, we have

$$M_3 \models y \otimes z \subseteq x \implies \hat{M}_3 \models y \otimes z \subseteq x.$$  

(6)

Thus, let us assume that $M_3 \models y \otimes z \subseteq x$, so that

$$\bigcup \mathcal{J}(y) \otimes \bigcup \mathcal{J}(z) \subseteq \bigcup \mathcal{J}(x),$$

(7)

and let $t \in \bigcup \hat{\mathcal{J}}(y) \otimes \bigcup \hat{\mathcal{J}}(z) = \bigcup \beta[\mathcal{J}(y)] \otimes \bigcup \beta[\mathcal{J}(z)]$. Hence, $t \in \beta(\sigma_1) \otimes \beta(\sigma_2)$, for some $\sigma_1 \in \mathcal{J}(y)$ and $\sigma_2 \in \mathcal{J}(z)$. By (7) and Lemma [3] we have

$$\text{pow}_{1,2}((\sigma_1, \sigma_2)) = \sigma_1 \otimes \sigma_2 \subseteq \bigcup \mathcal{J}(y) \otimes \bigcup \mathcal{J}(z) \subseteq \bigcup \mathcal{J}(x) \subseteq \bigcup \Sigma.$$  

(8)

Hence, by condition (7) and Lemma [3] again, we have

$$\beta(\sigma_1) \otimes \beta(\sigma_2) = \text{pow}_{1,2}((\beta(\sigma_1), \beta(\sigma_2))) \subseteq \bigcup \hat{\Sigma},$$

so that $t \in \bigcup \hat{\Sigma}$.

Let $\gamma \in \Sigma$ be such that $t \in \beta(\gamma)$. Since $\text{pow}_{1,2}((\beta(\sigma_1), \beta(\sigma_2))) \cap \beta(\gamma) \neq \emptyset$, from condition $[C_1]$ of Definition [4] it follows that $\text{pow}_{1,2}((\sigma_1, \sigma_2)) \cap \gamma \neq \emptyset$. Hence, by (8), we have $\gamma \cap \bigcup \mathcal{J}(x) \neq \emptyset$, and therefore $\gamma \in \mathcal{J}(x)$, so that $t \in \beta(\gamma) \subseteq \bigcup \beta[\mathcal{J}(x)]$, which in turn implies $t \in \bigcup \beta[\mathcal{J}(x)] = \hat{\mathcal{J}}(x)$.

By the arbitrariness of $t \in \bigcup \mathcal{J}(y) \otimes \bigcup \mathcal{J}(z)$, it follows that $\bigcup \hat{\mathcal{J}}(y) \otimes \bigcup \hat{\mathcal{J}}(z) \subseteq \bigcup \hat{\mathcal{J}}(z)$, namely $\hat{M}_3 \models y \otimes z \subseteq x$, and therefore $\hat{M}_3 \models y \otimes z \subseteq x$. This completes the proof of (6), and in turn of the theorem. \qed
1.6 $\otimes$-graphs

By relying on the results in Theorem [1] our next task will be to address the problem of how to generate a transitive partition $\hat{\Sigma}$ of bounded rank that weakly $\text{BS}\otimes$-imitates a given finite partition $\Sigma$.

The basic idea consists in a progressive copying of the structure of the graph linked to the partition. The basic idea behind the generation of a suitable imitating partition $\hat{\Sigma}$ of a given partition $\Sigma$ is to single out a representation of the structure of a partition through a graph structure. Since the only operator we take in account is the unordered cartesian product our graph structure will do the same.

The reason to move from partitions towards graphs lies in a greater flexibility of this last representation in building a new model.

The path to reach a decidability test for languages which involve unordered cartesian product operator, requires a construction procedure of $\Sigma$ conveniently modified so as to obtain another transitive $\hat{\Sigma}$ that imitates $\Sigma$.

We describe in which way to create a graph in order to take into account unordered cartesian operator. We call such a graph related to a partition as $\otimes$-graph.

The idea is to associate with every transitive partition $\Sigma$ a bipartite graph with two types of vertices, places and nodes.

We consider a non-empty finite set $\mathcal{P}$, whose elements are called places (or syntactical Venn regions) and whose subsets are called nodes. We denote by $\mathcal{N}$ the collection of nodes and assume that $\mathcal{P} \cap \mathcal{N} = \emptyset$, so that no node is a place, and vice versa. We shall use these places and nodes as the vertices of a directed bipartite graph $\mathcal{G}$ of a special kind, called $\otimes$-graph.

The edges issuing from each place $q$ are exactly all pairs $\langle q, B \rangle$ such that $q \in B \subseteq \mathcal{P}$: these are called membership edges. The remaining edges of $\mathcal{G}$, called distribution edges, go from nodes to places; hence, $\mathcal{G}$ is fully characterized by the function

$$ T \in \text{pow}(\mathcal{P})^{\text{pow}(\mathcal{P})} $$

associating with each node $B$ the set of all places $t$ such that $\langle B, t \rangle$ is an edge of $\mathcal{G}$. The elements of $T(B)$ are the targets of $B$, and $T$ is the $\otimes$ target function of $\mathcal{G}$. Thus, we usually represent $\mathcal{G}$ by $T$.

Edges $B \rightarrow_{\otimes} q$ of a $\mathcal{P}$-graph, where $q$ is a $\otimes$-place, will be referred to as $\otimes$-edges.

When $B$ is a subset of $\mathcal{P}$ we denote by $\mathcal{G} \mid_B$ the subgraph restricted to
vertices $B$ (and obviously the corresponding nodes).\footnote{Intuitively speaking, only elements in $\text{pow}_{1,2}(B^{(*)})$ can flow from node $B$ to a place $q$ along any $\otimes$-edge $B \rightarrow q$ (see Definition 6).}

**Definition 6** (Compliance with a $\otimes$-graph). Given a $\otimes$-graph $\mathcal{G}$, a transitive partition $\Sigma$ and a , $\Sigma$ is said to COMPLY WITH $\mathcal{G}$ (and, symmetrically, $\mathcal{G}$ is said to be INDUCED BY $\Sigma$) via the map $q \mapsto q^{(*)}$, where $|\Sigma| = |\mathcal{P}|$ and $q \mapsto q^{(*)}$ belongs to $\Sigma^P$, if

(a) the map $q \mapsto q^{(*)}$ is bijective,

(b) the target function $\mathcal{T}$ of $\mathcal{G}$ satisfies

$$\mathcal{T}(B) = \{q \in \mathcal{P} \mid q^{(*)} \cap \text{pow}_{1,2}(B^{(*)}) \neq \emptyset\}$$

for every $B \subseteq \mathcal{P}$, and

(c) for every $\otimes$-place $q \in \mathcal{P}$, the set $q^{(*)}$ may contain only singletons and doubletons, hence:

$$q^{(*)} \subseteq \bigcup \{\text{pow}_{1,2}(B) \mid q \in \mathcal{T}(B)\}.$$

A $\otimes$-graph is realizable if it is induced by some partition.

**Definition 7.** Let $\Sigma$, $\hat{\Sigma}$ two partitions such that $\hat{\Sigma}$ is weakly $\otimes$-transitive and $\mathcal{G}$, $\hat{\mathcal{G}}$ the induced $\otimes$-graphs. An bijective map $q \mapsto \hat{q}$ naturally extends to the nodes $B \mapsto \hat{B} = \{\hat{q} \mid q \in B\}$ and obviously to the $\otimes$-graphs. We define a map $\beta : \mathcal{G} \rightarrow \hat{\mathcal{G}}$ a weak isomorphism between $\otimes$-graphs when

$$(C_1) \ v \rightarrow \otimes w \leftrightarrow \beta(v) \rightarrow \otimes \beta(w),$$

and we denote this relation in the following way $\mathcal{G} \simeq \hat{\mathcal{G}}$ and we say that the $\otimes$-graphs are weakly isomorphic.

Moreover, if the following statement

$$(I_4) \ \text{pow}_{1,2}(X) \subseteq \bigcup \Sigma \leftrightarrow \text{pow}_{1,2}(\beta[X]) \subseteq \bigcup \hat{\Sigma},$$

is fulfilled then we say that the map $\beta : \mathcal{G} \rightarrow \hat{\mathcal{G}}$ is an isomorphism between $\otimes$-graphs, and we write $\mathcal{G} \simeq \hat{\mathcal{G}}$.

Obviously the following holds

**Theorem 3.** Let $\Sigma$ and $\hat{\Sigma}$ be partitions such that $\mathcal{G}$ and $\hat{\mathcal{G}}$ are (weak) isomorphic then $\hat{\Sigma}$ and $\Sigma$ (weakly) BS$\otimes$-imitates each other.
2 The decidability of BS⊗

Consider a BS⊗⊆-conjunction Φ satisfied by a partition Σ with ⊗-graph G. Assume that the longest path without repetitions through ⊗-arrows is k and |P| = P.

Theorem 4. BS⊗⊆ is decidable.

Proof. Using the procedure BuilderPartitionMLImitate, we create a rank bounded partition that (weakly) BS⊗⊆-imitates Σ.

Theorem 1 and Assert A4 imply the small model property for BS⊗⊆ and, by-product, our result.

In the prosecution we denote by p a non ⊗-place, by q an ⊗-place. If the procedure has passed through a path of length h starting from a place p until a node A we write p⇝h A.

We define minrank(q) = min{α | t ∈ q, rank(t) = α}.

Looking the status of Assert A5 at the end of execution of procedure BuilderPartitionMLImitate, it results ˆG ≃ G. Then, by Theorem 3 ˆΣ (weakly)-BS⊗⊆-imitates Σ.

We are left to prove inductively the asserts A1 – A5.
procedure BuilderPartitionMLImitate (Σ, ordered sequence of minimal ranks \(i_1, \ldots, i_\ell\));

1. - We denote by \(p\) places not in \(\otimes P\), for each \(p\) charge \(\hat{p}\) with \(P^k\) elements of rank \(H\),
2. - Label signed node \(A\) such that \(p \in A\)
3. - Pick \(i_j\) and all signed nodes \(A\) not distributed such that for each \(q \in A\) \(\text{minrank}(q) \leq i_j\)
4. - let \(q \mapsto \nabla(\hat{q})\) be a set-valued map over \(\otimes P\) such that
   - (a) \(\{\nabla(\hat{q}^{[i]}) | q \in \otimes P\} \setminus \{\emptyset\}\) is a partition of a non-null subset of \(\text{pow}_{1,2}((\hat{A}) \setminus \hat{P})\),
   - (b) \(|\nabla(\hat{q})| \geq P^{k-1-j}\) (using Assert \(A_2\))
   - (c) \(\hat{q} = \hat{q} \cup \nabla(\hat{q})\)
5. - Label as distributed node \(A\).
6. - Label as signed all nodes \(B\) such that \(B \cap T(A) \neq \emptyset\)
   - Assert \(A_1\): for each \(q \in \otimes P\) and \(\text{minrank}(q) \leq i_{j+1}\), \(\hat{q} \neq \emptyset\)
   - Assert \(A_2\): if \(p \sim A A\) signed then \(\text{pow}_{1,2}(\hat{A}) \setminus \hat{P} \geq P^{k-j}\)
   - Assert \(A_3\): \(\hat{\Sigma}\) is a partition,
   - Assert \(A_4\): The rank of \(\hat{\Sigma}\) does not exceed \(H + j\)
   - Assert \(A_5\): \(\hat{G} |_{p,q,\text{minrank}(q) \leq i_j} \simeq - G |_{p,q,\text{minrank}(q) \leq i_j}\),

end procedure;

Table 1: A procedure to create a partition of bounded rank which weakly \(BS \otimes\)-imitates a given partition \(\Sigma\).

[Base Step 0]

[Proof-Assert \(A_3\)] At the beginning of procedure \(\{\hat{p}\}_{p \in \otimes P}\) is a partition, by construction \(\{\hat{p}, \hat{q}\}_{p \in \otimes P, \text{minrank}(q) = i_1}\) is a partition.

[Proof-Assert \(A_5\)] At the beginning of procedure both \(\hat{\Sigma}\) and \(G |_{p_1, \ldots, p_r}\) has no arrows. After the first execution of the procedures all outgoing arrows starting from nodes composed only by \(p\) type of places have been activated. Since for any place \(q\) with \(\text{minrank}(q) = i_1\) there must be an incoming arrow from nodes composed only by \(p\) type of places, \(A_5\) holds.

[Proof-Assert \(A_1\)] A straightforward consequence of \(A_5\).

[Proof-Assert \(A_2\)] The only case is \(p \sim A\). By construction of \(p\) they have all \(P^k\) elements of rank \(H\). A node composed only by \(p\) type of places have at least \(P^k\) pairs, that necessarily have rank \(H + 1\), therefore they have intersection null with all \(\hat{p}\), hence \(\text{pow}_{1,2}(\hat{A}) \setminus \hat{P} \geq P^k\) and \(A_2\) holds.

[Proof-Assert \(A_4\)] As observed above, the rank of places of type \(p\) is \(H\).

[Inductive Step \(j + 1\)]
THE DECIDABILITY OF $\text{BS}^\otimes$

[Proof-Assert $A_3$] It is true by inductive hypothesis and the construction of $\nabla$.

[Proof-Assert $A_5$] We have to prove that for all $q$ such that $\text{minrank}(q) = i_{j+1}$ if $A = \{q_1, q_2\}$, $\text{minrank}(q_1), \text{minrank}(q_1) < i_j$ and $A \rightarrow^\otimes q$ then $\widehat{A} \rightarrow^\otimes \widehat{q}$. By construction all these nodes are distributed then they have activated all their outgoing $\otimes$-arrows.

[Proof-Assert $A_1$] If $\text{minrank}(q) = i_{j+1}$ there exists a node $\{q_1, q_2\} \rightarrow^\otimes q$ such that $\text{minrank}(q_1), \text{minrank}(q_2) < i_{j+1}$. By Assert $A_5$ this arrow exists in $\widehat{G}$, hence $\widehat{q} \neq \emptyset$.

[Proof-Assert $A_2$] By inductive hypothesis and construction for all $q \in \mathcal{T}(A)$, $\text{pow}_{1,2}(\widehat{A}) \supseteq \nabla(\widehat{q})$ and $|\nabla(\widehat{q})| \geq P^{k-j-1}$. Let $B = \{q, m\}$, obviously when it is called by the procedure $p \sim^+ j+1 B$. Observe that for a given partition different nodes generates disjoint sets of pairs. Since $\{\nabla(\widehat{q}), \widehat{m}\} \neq \widehat{X}$ for any node $X$ of $\Sigma$, this implies $\text{pow}_{1,2}(\nabla(\widehat{q}), \widehat{m}) \cap \text{pow}_{1,2}(\widehat{X}) = \emptyset$ and, in particular for any place $\sigma$, $\text{pow}_{1,2}(\nabla(\widehat{q}), \widehat{m}) \cap \sigma = \emptyset$. Therefore $|\text{pow}_{1,2}(\nabla(\widehat{q}), \widehat{m})| \geq P^{k-j-1}$.

[Proof-Assert $A_4$] By inductive hypothesis the new pairs are composed of elements of rank at most $H + j$ then the rank of all elements in $\widehat{\Sigma}$ cannot exceed $H + j + 1$.

Observe that if $\text{minrank}(q) = m$ there exists a node $\{q_1, q_2\} \rightarrow^\otimes q$ such that $\text{minrank}(q_1), \text{minrank}(q_2) < \text{minrank}(q)$.

Moreover, for each $\otimes$-place $\widehat{q}$, since at the beginning of procedure these kind of places are empty, $\bigcup \widehat{q} \subseteq \widehat{\Sigma}$. This last fact, together with Theorem 3, Theorem 4 and Assert $A_4$, imply our result. 

The above construction implies the following:

**Theorem 5.** $\text{BS}^\otimes$ is NP-complete.

**Proof.** Since $\text{BS}$ is NP-complete you can verify in polynomial time if an assignment, that makes non-empty all places, is a model or not. On the other hand, by procedure $\text{BuilderPartitionMLImitate}$, in order to verify that this model satisfies $\otimes$ literals, it is sufficient to check whether each $\otimes$-place $q$ is reachable from a non $\otimes$-place, which is a polynomial time research.

In order to solve decidability problem for $\text{BS}^\otimes$ we have to fulfill property

$$\text{pow}_{1,2}(X) \subseteq \bigcup \Sigma \quad \rightarrow \quad \text{pow}_{1,2}(\beta[X]) \subseteq \bigcup \widehat{\Sigma}$$

For this purpose we introduce the procedure $\text{CartSaturatePartition}$. If this procedure does not terminate this implies that there is at least one cycle.
procedure CartSaturatePartition \((\Sigma, \text{Stack } S \text{ of not cart-saturated nodes})\);

1. \(-\) \text{Pop}(S) = A,
2. \(-\) let \(q \mapsto \nabla(\hat{q})\) be a set-valued map over \(\otimes P\) such that
   - (a) \(\{\nabla(\hat{q}^{[i]}) \mid q \in \otimes P\} \setminus \{\emptyset\}\) is a partition of a non-null subset of \(\text{pow}^*_1(\hat{A}) \setminus P\),
   - (c) \(\hat{q} = \hat{q} \cup \nabla(\hat{q})\)
3. \(-\) \text{Push}(B) all not cart-saturated \(B\) not in \(S\).
4. \(-\) If \(S\) is empty exit.

end procedure;

Table 2: A procedure to cart-saturate a partition \(\Sigma\).

in \(G\). Otherwise there is not any cycle.

In both cases the assignment \(\hat{\Sigma} = \bigcup_{i \in \alpha} \hat{q}^{[i]} \mid q \in \Sigma\) where \(\hat{q}^{[i]}\) is the place \(\hat{q}\) at the step \(i\) of the procedure \(\text{CartSaturatePartition}\) satisfies

\[
\text{pow}^*_1(X) \subseteq \bigcup \Sigma \quad \longrightarrow \quad \text{pow}^*_1(\beta[X]) \subseteq \bigcup \hat{\Sigma}
\]

Therefore, by Theorem \ref{thm:NP} and Theorem \ref{thm:BS} the partition \(\hat{\Sigma}\) \(\text{BS}^{\otimes}\)-imitates \(\Sigma\).

In case the procedure terminates, \(\alpha \in N\) therefore there are no cycles, you have a finite construction and a small model. Otherwise, \(\alpha = \omega\), the first infinite ordinal and you have a transfinite construction and the assignment built at the end of the procedure \(\text{BuilderPartitionMLImitate}\) witnesses the existence of a model. This in particular implies NP-completeness of \(\text{BS}^{\otimes}\).

Consider \(\text{BS}^{\otimes}_{\text{fin}}\), as \(\text{BS}^{\otimes}\) restricted to finite models. Since a finite model cannot have a cycle, this, in particular, implies \(\text{BS}^{\otimes}_{\text{fin}}\) has the small model property.

Resuming,

**Corollary 1.** \(\text{BS}^{\otimes}\) is NP-complete and \(\text{BS}^{\otimes}_{\text{fin}}\) has the small model property.
3 Remark on HTP

Consider a \( \otimes \) graph \( G \) with a set of cardinal constraints of the type \(|p| \leq |q|\) with \( p, q \) vertices of \( G \). The following problem

**Problem 1.** \( G \) is realizable?

is undecidable, since this problem is reducible to HTP.

On the other side, we conjecture the following result.

**Conjecture 1.** For an algorithm \( A \) and an input \( x \) there exists a \( \otimes \) graph \( G \) with cardinal inequalities such that

\( A \) terminates on input \( x \) iff \( G \) is realizable.

If the conjecture were proved it would lead to a straightforward reduction from the HALTING problem to HTP and, contextually, to a completely alternative proof to that provided by Matyasevich.

4 Acknowledgments

The authors are grateful to Martin Davis who kindly gave his permission to cite his personal communication.

References

[Can91] D. Cantone. Decision procedures for elementary sublanguages of set theory. X. Multilevel syllogistic extended by the singleton and powerset operators. *Journal of Automated Reasoning*, 7(2):193–230, 1991.

[CCP90] D. Cantone, V. Cutello, and A. Policriti. Set-theoretic reductions of Hilbert’s tenth problem. In *Proc. of 3rd Workshop “Computer Science Logic” 1989*, pages 65–75, 1990. Lecture Notes in Computer Science, 440.

[CU14] Domenico Cantone and Pietro Ursino. Formative processes with applications to the decision problem in set theory: II. Powerset and singleton operators, finiteness predicate. *Inf. Comput.*, 237:215–242, 2014.

[CU18] Domenico Cantone and Pietro Ursino. An Introduction to the Technique of Formative Processes in Set Theory. *Springer International Publishing*, 2018.
[DPR61] Davis, M., Putnam, H., Robinson, J. . The decision problem for exponential Diophantine equations. *Annals of Mathematics*, 74(2): 425–436, 1961.

[FOS80] A. Ferro, E.G. Omodeo, and J.T. Schwartz. Decision procedures for elementary sublanguages of set theory. I: Multilevel syllogistic and some extensions. *Comm. Pure Appl. Math.*, 33:599–608, 1980.

[Mat70] Matiyasevich, Yu. V. (1970). Enumerable sets are Diophantine (in Russian). *Dokl. AN SSSR*, 191(2), 278–282; Translated in: Soviet Math. Doklady, 11(2), 354–358. Correction Ibid 11 (6) (1970), vi. Reprinted on pp. 269–273 in: Mathematical logic in the 20th century, G. E. Sacks, (Ed.), (2003). Singapore University Press and World Scientific Publishing Co., Singapore and River Edge, NJ.

[Hilbert-02] Hilbert, D. . Mathematical Problems. *Bulletin of the American Mathematical Society*, 8(10), 437–479, 1902.

[Rob] Robinson, R. M. . Arithmetical representation of recursively enumerable sets. *Journal of Symbolic Logic*, 21(2), 162–186, 1956.

[Sca] B. Scarpellini. Complexity of subcases of Presburger Arithmetic. *Transactions of the American Mathematical Society*, 284(1):93–119, 1984.