A Probabilistic Inequality Related to Negative Definite Functions

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Abstract. We prove that for any pair of i.i.d. random vectors $X, Y$ in $\mathbb{R}^n$ and any real-valued continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ the inequality
\[ E\psi(X - Y) \leq E\psi(X + Y). \]
holds. In particular, for $\alpha \in (0, 2]$ and the Euclidean norm $\| \cdot \|_2$ one has
\[ E\|X - Y\|_2^\alpha \leq E\|X + Y\|_2^\alpha. \]
The latter inequality is due to A. Buja et al. [4] where it is used for some applications in multivariate statistics. We show a surprising connection with bifractional Brownian motion and provide some related counter-examples.

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1. Introduction

Let $X, Y$ be i.i.d. random variables with finite expectations. Then one has
\[ E|X - Y| \leq E|X + Y|. \] (1.1)
The inequality (1.1) appeared recently in an analytic context (properties of integrable functions) [8]. Since (1.1) is a nice fact in itself and since it seems not to be well known in the probabilistic community, it is desirable to search for adequate proofs and to explore possible extensions of it. For instance, for which values of $\alpha$ do we have
\[ E|X - Y|^\alpha \leq E|X + Y|^\alpha? \] (1.2)
As before, we assume that $X$ and $Y$ are i.i.d. and $E|X|^\alpha < \infty$. 
Proving (1.1) is a non-trivial exercise for a probability course. If \( X, Y \) are real-valued, one way to see this inequality is to use the identity
\[
E|X + Y| - E|X - Y| = 2\int_0^\infty [P(X > r) - P(X < -r)]^2 \, dr.
\]
For (1.2) we are, however, not aware of a similar elementary approach. On the other hand, A. Buja et al. prove in [4] even a multivariate version of (1.2): for any pair of i.i.d. random vectors \( X, Y \) in \( \mathbb{R}^n \), any \( \alpha \in (0, 2] \) and for a class of norms \( \| \cdot \| \) on \( \mathbb{R}^n \) including the Euclidean norm \( \| \cdot \|_2 \) the estimate
\[
E\|X - Y\|_\alpha \leq E\|X + Y\|_\alpha
\]
holds true. The elegance of this inequality is obvious; at the same time we stress that it arises from statistical applications. In any case it merits to be better known in the probabilistic community!

In Section 2 we give an extension of (1.3) by replacing the norm with an arbitrary negative definite function. Moreover, we show how this fact extends to an arbitrary number of i.i.d. random vectors. In Sections 3 and 4 we establish a surprising connection to some recent advances in the theory of random processes related to bifractional Brownian motion. A counterexample to (1.2) with \( \alpha \in (2, \infty) \) is given in Section 5.

2. Main result

Consider the class of continuous real-valued negative definite functions, i.e. characteristic exponents of symmetric Lévy processes. The notion of negative definite function goes back to Schoenberg; good sources are the books [3] and [10]. Recall that a continuous real-valued negative definite function is uniquely given by its Lévy-Khintchine representation
\[
\psi(\xi) = a + \frac{1}{2} \langle Q\xi, \xi \rangle + \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos \langle \xi, u \rangle) \nu(du), \quad \xi \in \mathbb{R}^n,
\]
where \( a \geq 0 \) is a constant, \( Q \in \mathbb{R}^{n \times n} \) is a symmetric positive semidefinite matrix and \( \nu \) is the Lévy measure, i.e. a measure on \( \mathbb{R}^n \setminus \{0\} \) satisfying the integrability condition
\[
\int_{\mathbb{R}^n \setminus \{0\}} \min\{\|u\|_2^2, 1\} \nu(du) < \infty.
\]
Without loss of generality, we will always assume that \( a = 0 \), i.e. \( \psi(0) = 0 \). For our discussion it is worth noticing that \( (\xi, \eta) \mapsto \sqrt{\psi(\xi - \eta)} \) is always a metric. A deep theorem of Schoenberg states that a metric space \( (\mathbb{R}^n, d) \) can be isometrically embedded into an (in general infinite-dimensional) Hilbert space \( \mathcal{H} \) if, and only if, \( d(\xi, \eta) \) is of the form \( d_\psi(\xi, \eta) = \sqrt{\psi(\xi - \eta)} \), cf. [11], [2, p. 187] as well as [7] for a discussion of metric measure spaces related to the metric \( d_\psi \).
An important subclass of continuous negative definite functions are the spherically symmetric negative definite functions. These are of the form

$$\xi \mapsto f(\|\xi\|_2^2)$$

where $f$ is a Bernstein function. \hfill (2.3)

Recall that a **Bernstein function** is a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ which admits the following Lévy-Khintchine representation

$$f(\lambda) = a + b\lambda + \int_0^\infty \left(1 - e^{-t\lambda}\right) \mu(dt);$$

here $a, b \geq 0$ are constants and $\mu$ is a measure on $(0, \infty)$ satisfying the integrability condition $\int_0^\infty \min\{t, 1\} \mu(dt) < \infty$. In probability theory Bernstein functions arise as the characteristic exponents of the Laplace transform of subordinators, i.e. increasing one-dimensional Lévy processes. Bernstein functions, many examples and their connections to various fields of mathematics are discussed in the monograph [10]. It is easy to see that Bernstein functions are infinitely many times differentiable, increasing, concave; moreover, they grow at most linearly. Typical examples are $\lambda \mapsto \log(1 + \lambda)$ and $\lambda \mapsto f_\beta(\lambda) := \lambda^\beta$ for $0 < \beta \leq 1$. Note that the composition $f \circ \psi$ of a Bernstein function $f$ with a continuous real-valued negative definite function $\psi$ is again a continuous real-valued negative definite function. At the level of stochastic processes this corresponds to **Bochner’s subordination** of the Lévy process with characteristic exponent $\psi$ by the subordinator with the Laplace exponent $f$.

Using the Bernstein functions $f_\beta$ with $\beta = \alpha/2$ and $0 < \alpha \leq 2$ we obtain

$$\xi \mapsto \|\xi\|_2^\alpha = f_{\alpha/2}(\|\xi\|^2), \quad 0 < \alpha \leq 2,$$

$$\xi \mapsto d_\psi(\xi, 0)^\alpha = \sqrt{\psi(\xi)}^\alpha = f_{\alpha/2}(\psi(\xi)), \quad 0 < \alpha \leq 2,$$

as examples for real-valued continuous negative definite functions. Note that the functions defined by (2.3) are characteristic exponents of subordinate Brownian motions.

We prove the following result extending (1.3).

**Theorem 2.1.** Let $\psi$ be a real-valued continuous negative definite function on $\mathbb{R}^n$. For any pair of i.i.d. random vectors $X, Y$ in $\mathbb{R}^n$ it is true that

$$\mathbb{E} \psi(X - Y) \leq \mathbb{E} \psi(X + Y).$$

(2.4)

**Proof.** Without loss of generality we may assume that $a = 0$ and $Q = 0$ – in both cases the inequality (2.4) is elementary.
Using the Lévy-Khintchine representation of $\psi$ we get
\[
\mathbb{E} \psi(X + Y) = \mathbb{E} \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - \cos \langle X + Y, u \rangle \right) \nu(du)
\]
\[
= \mathbb{E} \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - \operatorname{Re} \exp(i \langle X + Y, u \rangle) \right) \nu(du)
\]
\[
= \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - \operatorname{Re} \mathbb{E} \exp(i \langle X + Y, u \rangle) \right) \nu(du)
\]
\[
= \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - \operatorname{Re} \left[ \mathbb{E} \exp(i \langle X, u \rangle) \right]^2 \right) \nu(du).
\]

A similar calculation yields
\[
\mathbb{E} \psi(X - Y) = \mathbb{E} \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - \cos \langle X - Y, u \rangle \right) \nu(du)
\]
\[
= \mathbb{E} \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - \operatorname{Re} \exp(i \langle X - Y, u \rangle) \right) \nu(du)
\]
\[
= \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - \operatorname{Re} \mathbb{E} \exp(i \langle X - Y, u \rangle) \right) \nu(du)
\]
\[
= \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - \left| \mathbb{E} \exp(i \langle X, u \rangle) \right|^2 \right) \nu(du).
\]

Using the elementary estimate $\operatorname{Re}(z^2) \leq |z|^2 = |z|^2$ we obtain (2.4). □

**Remark 2.2.** Let $X_1, \ldots, X_{2m}$ be i.i.d. random variables in $\mathbb{R}^n$ and $\varepsilon_j = \pm 1$ (non-random, or even random but independent of the $X_1, \ldots, X_{2m}$) constants satisfying $\sum_{j=1}^{2m} \varepsilon_j = 0$. Then
\[
\mathbb{E} \psi \left( \sum_{j=1}^{2m} \varepsilon_j X_j \right) \leq \mathbb{E} \psi \left( \sum_{j=1}^{2m} X_j \right) .
\]

(2.5)

This follows if we use Theorem 2.1 for $X = \sum_{j=1}^{2m} \varepsilon_j^+ X_j$ and $Y = \sum_{j=1}^{2m} \varepsilon_j^- X_j$.

Using the distance function $d_{\psi}(\xi, \eta) := \sqrt{\psi(\xi - \eta)}$ related to a real-valued continuous negative definite function $\psi$ we get the following counterpart of (1.3).

**Corollary 2.3.** Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a real-valued continuous negative definite function, $d_{\psi}(\xi, \eta) = \sqrt{\psi(\xi - \eta)}$ the associated metric and $0 < \alpha \leq 2$. For any pair of i.i.d. random vectors $X, Y$ in $\mathbb{R}^n$ it is true that
\[
\mathbb{E} d_{\psi}^\alpha(X - Y) \leq \mathbb{E} d_{\psi}^\alpha(X + Y).
\]

(2.6)

**Remark 2.4.** Assume that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that $\psi(0) = 0$ and $\psi(\xi) = \psi(-\xi)$. If (2.4) holds for this $\psi$ and any random variable $X$ (and an independent copy $Y$ of $X$), then one can show that the kernel $K_{\psi}(\xi, \eta) := \psi(\xi + \eta) - \psi(\xi - \eta)$ is positive definite. We wonder whether this already entails that $\psi$ is a continuous negative definite function.
3. A relation to random processes

We will show now that the inequality (2.4) has an interesting relation to
Gaussian processes. Let \( \psi : \mathbb{R}^n \to \mathbb{R} \) be a real-valued continuous negative
definite function defined on \( \mathbb{R}^n \).

Lemma 3.1. The kernel \( K^\psi(\xi, \eta) = \psi(\xi + \eta) - \psi(\xi - \eta) \) is positive definite.

Proof. By the Lévy-Khintchine formula (2.1) we get
\[
K^\psi(\xi, \eta) = 2\langle Q\xi, \eta \rangle + \int_{\mathbb{R}^n \setminus \{0\}} \left( \cos(\langle \xi - \eta, u \rangle) - \cos(\langle \xi + \eta, u \rangle) \right) \nu(du).
\]

Using the elementary trigonometric identity
\[
\cos(\langle \xi - \eta, u \rangle) - \cos(\langle \xi + \eta, u \rangle) = 2 \sin(\langle \xi, u \rangle) \sin(\langle \eta, u \rangle),
\]
we see that
\[
K^\psi(\xi, \eta) = 2\langle Q\xi, \eta \rangle + 2 \int_{\mathbb{R}^n \setminus \{0\}} \sin(\langle \xi, u \rangle) \sin(\langle \eta, u \rangle) \nu(du).
\]

Now let \( S \) be a finite set and \( (\lambda_\xi, \xi \in S) \) be complex numbers. Then
\[
\sum_{\xi, \eta \in S} K^\psi(\xi, \eta) \lambda_\xi \lambda_\eta
= 2 \sum_{\xi, \eta \in S} \lambda_\xi \lambda_\eta \langle Q\xi, \eta \rangle + 2 \int_{\mathbb{R}^n \setminus \{0\}} \left( \sum_{\xi, \eta \in S} \lambda_\xi \sin(\langle \xi, u \rangle) \lambda_\eta \sin(\langle \eta, u \rangle) \right) \nu(du)
\]
\[
= 2 \left( Q \sum_{\xi \in S} \lambda_\xi \xi, \sum_{\xi \in S} \lambda_\xi \xi \right) + 2 \int_{\mathbb{R}^n \setminus \{0\}} \left( \sum_{\xi \in S} \lambda_\xi \sin(\langle \xi, u \rangle) \right)^2 \nu(du)
\]
\[
\geq 0,
\]
which means that \( K^\psi(\cdot, \cdot) \) is positive definite. \( \square \)

Remark 3.2. A special case of Lemma 3.1 for powers of \( \ell_p \)-norms is proved in [4].

Probabilistic proof of Theorem 2.1. Since \( K^\psi(\xi, \eta) \) is positive definite, there
is a centred Gaussian process \( (G_\xi^\psi, \xi \in \mathbb{R}^n) \) whose covariance function is
\( K^\psi(\xi, \eta) \).

For given i.i.d. random vectors \( X, Y \in \mathbb{R}^n \) set
\[
Z^\psi := \int_{\mathbb{R}^n} G_\xi^\psi P(d\xi),
\]
where \( P \) stands for the common distribution of \( X \) and \( Y \). Then
\[
0 \leq \text{Var}(Z^\psi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K^\psi(\xi, \eta) P(d\xi) P(d\eta)
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \psi(\xi + \eta) - \psi(\xi - \eta) \right) P(d\xi) P(d\eta)
\]
\[
= \mathbb{E}(\psi(X + Y) - \psi(X - Y)),
\]
and we obtain again \( E \psi(X - Y) \leq E \psi(X + Y) \). \( \square \)

4. Relation to bifractional Brownian motion

In some most important cases it is possible to identify the Gaussian process \((G^\psi_\xi, \xi \in \mathbb{R}^n)\) of Section 3 with bifractional Brownian motion \((bBm)\). The latter process was introduced by Houdré and Villa in \([6]\) as a centred Gaussian process \(B^{H,K} = (B^{H,K}_t, t \in \mathbb{R}^n)\) with covariance function

\[
R^{H,K}(t, s) := \mathbb{E} \left( B^{H,K}_t B^{H,K}_s \right) = 2^{-K} \left( (||t||_2^{2H} + ||s||_2^{2H}) K - ||t - s||_2^{2HK} \right),
\]

where \( s, t \in \mathbb{R}^n \). For \( n = 1, K = 1 \) we get the usual fractional Brownian motion \(B^H\) with Hurst index \(H\). Originally, the process was defined for the parameters \( H \in (0, 1] \) and \( K \in (0, 1] \). Bardina and Es-Sebaiy \([1]\) recently proved that \(B^{H,K}\) exists for all \((H,K) \in D\), where

\[
D := \{H, K : 0 < H \leq 1, 0 < K \leq 2, H \cdot K \leq 1\}.
\]

(The possibility of such an extension was already indicated in the earlier work by Lei and Nualart \([9]\) who established an integral representation relating \(B^{H,K}\) with fractional Brownian motion \(B^{HK}\)).

For \( \psi(\xi) := ||\xi||^\alpha, 0 < \alpha \leq 2, \) and

\[
G^\psi_\xi := 2^{\alpha/2} \text{sgn}(\xi) B^{1,\alpha}_{||\xi||}, \quad \xi \in \mathbb{R},
\]

it is trivial to see that

\[
\mathbb{E} \left( G^\psi_\xi G^\psi_\eta \right) = \text{sgn}(\xi) \text{sgn}(\eta) 2^\alpha \mathbb{E} \left( B^{1,\alpha}_{||\xi||}, B^{1,\alpha}_{||\eta||} \right) = ||\xi + \eta||^\alpha - ||\xi - \eta||^\alpha = K^\psi(\xi, \eta).
\]

Therefore, we are led to a probabilistic interpretation of the inequality \((1.2)\) through \(B^{1,\alpha}\).

Remark 4.1. In higher dimensions bi-fractional Brownian motion does not show up in the context of our inequalities (nor do we rely on bBm with \(H \neq \frac{1}{2}\)); therefore it becomes natural to search for the extensions of bBm based upon general negative definite functions. This will be done elsewhere.

5. A counterexample

The inequality \((1.2)\) trivially extends to the case \(\alpha = \infty\) in the following sense. Let

\[
M = \sup \{r : \mathbb{P}(X < r) < 1\} = \text{ess sup} X;
\]

\[
m = \sup \{r : \mathbb{P}(X < r) = 0\} = \text{ess inf} X.
\]

Then

\[
||X - Y||_\infty = M - m \leq 2 \max\{|M|, |m|\} = ||X + Y||_\infty.
\]

Without further assumptions the inequality \((1.2)\) will, in general, not hold, for \(2 < \alpha < \infty\). To see this, fix \(\alpha \in (2, \infty)\) and \(c > 0\). For any \(M \geq c\)
set \( q := c/M \) and \( p := 1 - q \). Let \( X_M, Y_M \) be i.i.d. random variables such that

\[
\begin{align*}
\mathbb{P}(X_M = 1) &= \mathbb{P}(Y_M = 1) = p; \\
\mathbb{P}(X_M = -M) &= \mathbb{P}(Y_M = -M) = q.
\end{align*}
\]

If \( M \geq 1 \), then

\[
\mathbb{E}|X_M - Y_M|^\alpha - \mathbb{E}|X_M + Y_M|^\alpha
= 2pq [(M + 1)^\alpha - (M - 1)^\alpha] - 2^\alpha M^\alpha q^2 - 2^\alpha p^2
\geq 4pqM^{-1} - 2^\alpha M^\alpha q^2 - 2^\alpha p^2
= M^{-2}(4pq - 2^\alpha c^2) - 2^\alpha p^2.
\]

Hence, whenever \( c < 2^{-\alpha} \) and \( M \) is large enough,

\[
\mathbb{E}|X_M - Y_M|^\alpha - \mathbb{E}|X_M + Y_M|^\alpha > 0,
\]

and (1.2) fails.

**Remark 5.1.** Further counterexamples are presented in [4].

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