A Projective Representations of the Thompson Group F and Its Lifting Problem

Jun Yang

December 7, 2018

Abstract

The Thompson group $F$ has a canonical unitary representation on $H = L^2([0, 1])$. With a special projection, we construct a projective unitary representation on a Fermionic Fock space associated with $H$. This comes from the representation of the CAR algebra of $H$. Then, by computing the 2nd cohomology group, we will be able to decide if this projective unitary representation can be lifted to an ordinary representation. We will mainly discuss the lifting problem of this projective representation.

1 Introduction

In this paper, we will mainly construct a projective unitary representation of the Thompson group $F$. It is realized through the CAR algebra of $H = L^2([0, 1])$ on a Fermionic Fock space. Then, after getting $H^2(F; S^1)$, we will discuss the lifting problem.

In chapter 2, we give a short introduction to the Thompson group $F$ including some presentations and a canonical Koopman representation. In chapter 3, there is a review of the representation theory of CAR algebra over some Fermionic Fock spaces. Then, a criterion of the implementation of the representation above is given. We define a special projection in $H$ which leads to the projective unitary representation of $F$.

In chapter 4, we will compute the cohomology groups using the classifying space given by Quillen. Then, we will go back to the lifting problem.
In chapter 5, we want to answer the question by giving a concrete computation to examine whether there is a lifting. It is given by the consideration of the vacuum vectors. (We haven’t get the final result since the computation is hard and will be run on computers.)

In chapter 6, we will generalize the special projection to the ones correspond to $\text{SO}(2)$ and $\text{U}(2)$.

## 2 The Thompson group $F$

The Thompson group $F$ is a finitely generated group presented as

$$\langle A, B | [AB^{-1}, A^{-1}BA] = [AB^{-1}, A^{-2}BA^2] = \text{id} \rangle$$

And there is also a presentation given by

$$\langle x_0, x_1, \ldots | x_i^{-1}x_nx_i = x_{n+1} \text{ for all } i < n \rangle.$$  

We review some basic facts about $F$ that we will need later. Most of these can be found in introductive materials of the Thompson group such as [2][3].

### 2.1 Dyadic automorphism presentations of $F$

**Definition 2.1** Let $I, J$ be two real intervals. A homeomorphism $f : I \to J$ is called a dyadic piecewise linear homeomorphism if it satisfies:

1. $f$ is piecewise linear with finite many singular points,
2. the coordinates of each singular points is dyadic rational, i.e. $n/2^m$ for $n, m \in \mathbb{N}$,
3. each slope of $f$ is an integral power of $2$

And we let $\text{Aut}_{\text{DPL}}(I)$ denote all the dyadic piecewise linear homeomorphisms from the interval $I$ to itself.

It is well-known that $F = \text{Aut}_{\text{DPL}}([0, 1])$. In this way, the generators $A, B$ can be presented as

$$A(x) = \begin{cases} 
  x/2, & \text{if } x \in [0, 1/2) \\
  x - 1/2, & \text{if } x \in [1/2, 3/4) \\
  2x - 1, & \text{if } x \in [3/4, 1] 
\end{cases}$$
Figure 1: The two generators $A, B$

$$B(x) = \begin{cases} 
    x, & \text{if } x \in [0, 1/2) \\
    x/2 + 1/4, & \text{if } x \in [1/2, 3/4) \\
    x - 1/2, & \text{if } x \in [3/4, 7/8) \\
    2x - 1, & \text{if } x \in [7/8, 1]
\end{cases}$$

One can check such two homeomorphisms satisfy

$$[AB^{-1}, A^{-1}BA] = [AB^{-1}, A^{-2}BA^2] = \text{id}$$

and generate the group $F$.

For each $g \in F$, it can be written as a product of powers of $A, B$ and then $g$ also gives an element in $\text{Aut}_{\text{DPL}}(I)$. This leads to the description of dyadic subdivision of $[0, 1]$ by repeating insertion of midpoint. For example, the canonical map from one dyadic subdivision to another of equal number of subintervals is an element of $\text{Aut}_{\text{DPL}}([0, 1])$.

**Definition 2.2** Given any $g \in F$, we define the minimal interval length, denoted by $\text{mil}(g)$, to be the minimal length of interval that contains no singular points.

We also define $n_g = -\log_2(\text{mil}(g))$ and call this the level of $g$.

One can easily check $\text{mil}(g)$ is always an integral power of 2 and hence $n_g \in \mathbb{N}$. For example, we have $\text{mil}(A) = 1/4$, $\text{mil}(B) = 1/8$ and $n_A = 2, n_B = 3$.

**Lemma 2.1** For any $g \in F$ with a reduced word form $g = A^{\alpha_1}B^{\beta_1} \cdots A^{\alpha_s}B^{\beta_s}$ ($\alpha_i, \beta_j \in \mathbb{Z}\setminus\{0\}$ with $\alpha_1, \beta_s = 0$ allowed), we have
1. \( n_g \leq \sum_{1 \leq k \leq s}(2|\alpha_k| + 3|\beta_k|); \)
2. \( \text{mil}(g) \geq \frac{1}{\prod_{1 \leq k \leq s} 4^{|\alpha_k|} \cdot 8^{|\beta_k|}}. \)

The proof is straightforward by induction of \( s \) and \( \sum_{1 \leq k \leq s}(|\alpha_k| + |\beta_k|) \), with the observation on the range of slopes.

There is another presentation of \( F \) by binary trees \([2]\). Given a dyadic subdivision of the interval \([0, 1]\), there is a obvious binary tree corresponding to it. And one can shown such a correspondence is one-to-one. In this way, the group \( F \) will acts on the set of binary trees. Such notations will only be used in chapter 4.1.

### 2.2 Koopman representation on \( L^2([0, 1]) \)

There is a canonical Koopman representation \( u \) of \( F \) on the Hilbert space \( H = L^2([0, 1]) \) which is defined by

\[
(u(g)f)(x) = f(g^{-1}x)\sqrt{\frac{dg_*\mu}{d\mu}}(x), \quad f \in H \text{ and } x \in [0, 1]
\]

where \( \mu \) is the usual measure of \([0, 1]\), \( g_*\mu \) is defined by \( g_*\mu(A) = \mu(gA) \) and the quotient is the Radon-Nikodym Derivative.

Artem Dudko \([5]\) proves this representation is irreducible by introducing the measure contracting action

**Definition 2.3** A group \( G \) acts on a probability space \((X, \mu)\) that is measure class preserving. It is called measure contracting if for any measurable subset \( A \subset X \), any \( M, \varepsilon \in \mathbb{R} \), there is \( g \in G \) such that

1. \( \mu(\text{supp}(g) \setminus A) \leq \varepsilon; \)
2. $\mu(\{x \in A | \sqrt{\frac{d\mu(a(x))}{d\mu(x)}} < M^{-1}\}) > \mu(A) - \varepsilon$.

The main theorem connecting this definition with irreducibility is

**Theorem 2.2 ([5])** For any ergodic measure contraction action of a group $G$ on a probability space $(X, \mu)$, the associated Koopman representation of $G$ is irreducible.

Then, by checking the Koopman representation $\nu : F \to U(H)$ is ergodic and measure contracting, $\nu$ is irreducible as a corollary.

# 3 A Projective Representation of $F$

## 3.1 CAR algebra and Fermionic Fock space

Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. There is a Fermionic Fock space $\wedge H$ which is also a Hilbert space spanned by $f_1 \wedge \cdots \wedge f_k, k \in \mathbb{N}$ and $f_i \in H$ for $1 \leq i \leq k$ with inner product $\langle f_1 \wedge \cdots \wedge f_k, g_1 \wedge \cdots \wedge g_l \rangle = \delta_{g,l} \det(\langle f_i, g_j \rangle)$.

There is a CAR (canonical anticommutation relation) algebra $\text{CAR}(H)$ which is a complex algebra generated by the $C^{-1}$ linear symbols $\{a(f) | f \in H\}$ with the anticommutation relations

$$\{a(f), a(g)\} = 0 \text{ and } \{a(f), a(g)^*\} = \langle f, g \rangle$$

where $\{X, Y\} = XY + YX$.

The algebra $\text{CAR}(H)$ has a representation $\pi$ on the Fermionic Fock space $\wedge H$ [12] given by

1. $\pi(a(f))(g_1 \wedge \cdots \wedge g_m) = f \wedge g_1 \wedge \cdots \wedge g_m$,
2. $\pi(a(f)^*)(g_1 \wedge \cdots \wedge g_m) = \sum_{i=1}^{m} (-1)^{i-1} \langle g_i, f \rangle g_1 \wedge \cdots \wedge \hat{g_i} \wedge \cdots \wedge g_m$

**Lemma 3.1 ([1])** The representation $\pi$ of $\text{CAR}(H)$ on $\wedge H$ is irreducible.

## 3.2 Segal’s criterion

Let $P \in \mathcal{B}(H)$ be a projection. There is a corresponding Fermionic Fock space defined by

$$F_P = \wedge(PH) \otimes (P^\perp H)^*.$$
We have $\mathcal{F}_P$ is also a Hilbert space with the inner product on the tensor space of $\wedge(PH)$ and $\wedge(P^\perp H)^*$. If $\{p_i\}_{i\in\mathbb{N}}$ is a orthonormal basis for $PH$ and $\{q_i\}_{i\in\mathbb{N}}$ is a orthonormal basis for $P^\perp H$, then we have a canonical orthonormal basis of $\mathcal{F}_P$.

**Lemma 3.2** Given $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_i\}_{i\in\mathbb{N}}$ as orthonormal basis of $PH$ and $P^\perp H$ respectively, then

$$\{p_{i_1} \wedge \cdots \wedge p_{i_k} \otimes q_{j_1}^* \wedge \cdots \wedge q_{j_l}^* | l, k \in \mathbb{N} \cup \{0\}, 1 \leq i_1 < \cdots < i_k, 1 \leq j_1 < \cdots < j_l\}$$

is an orthonormal basis of $\mathcal{F}_P$, where $k, l = 0$ stands for the vacuum vector $\Omega_1, \Omega_2$ in $\wedge(PH), \wedge(P^\perp H)^*$ respectively.

**Proof:** Note that $\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_k} \otimes \eta_{j_1}^* \wedge \cdots \wedge \eta_{j_l}^*$ spans a dense subspace of $\mathcal{F}_P$ if $\epsilon_{i_k} \in PH$ and $\eta_{j_k} \in P^\perp H$.

Such a vector can be approximated by finite linear combinations of the vectors defined above. One can easily check these vectors are orthonormal, which completes the proof. 

Moreover, we have a representation $\pi_P$ of $\text{CAR}(H)$ on $\mathcal{F}_P$ defined by

$$\pi_P(a(f)) = a(P f) \otimes 1 + 1 \otimes a((P^\perp f)^*)^*.$$ 

where $a(g)$ stands for the action $\pi$ (defined above) on the corresponding exterior space $\wedge(PH)$ or $\wedge((P^\perp H)^*)$.

**Lemma 3.3 ([1])** The representation $\pi_P$ of $\text{CAR}(H)$ on $\mathcal{F}_P$ is irreducible.

Given to projections $P, Q \in \mathcal{B}(H)$, we have the following result on the equivalence of representations $\pi_P, \pi_Q$.

**Theorem 3.4 (Segal’s equivalence criterion[1][13])** For two projections $P, Q \in \mathcal{B}(H)$, $\pi_P, \pi_Q$ are unitarily equivalent if and only if $P - Q$ is a Hilbert-Schmidt operator.

Given a unitary $u \in \mathcal{B}(H)$, the map $a(f) \mapsto a(uf)$ gives an automorphism of $\text{CAR}(H)$. An interesting question is whether this $u$ can be realized by unitary elements in $\mathcal{B}(\mathcal{F}_P)$.

**Definition 3.1** $u$ is implemented in $\mathcal{F}_P$ if there is a unitary $U \in \mathcal{B}(\mathcal{F}_P)$ such that $\pi_P(a(uf)) = U \pi_P(a(f)) U^*$ for all $f \in H$.  

6
Then we have a criterion for the implementation of any given \( u \in U(H) \).

**Corollary 3.5**  \( u \in U(H) \) is implemented in \( \mathcal{F}_P \) if \([u, P]\) is a Hilbert-Schmidt operator.

**Proof:** Let \( Q = u^* Pu \). Then we have \( P - Q = P - u^* Pu = u^*(uP - Pu) = u^*[u, P] \) is Hilbert-Schmidt. Then, by the theorem above, there is a unitary \( U \) such that \( \pi_P(a(uf)) = U \pi_Q(a(f)) U^* = U \pi_{u^* Pu}(a(f)) U^* \) for all \( f \in H \).

On the other hand, there is a unitary \( V \in B(\mathcal{F}_Q, \mathcal{F}_P) \) defined by

\[
V(Qg_1 \wedge \cdots \wedge Qg_m \otimes (Q^\perp h_1)^* \wedge \cdots \wedge (Q^\perp h_n)) = u Qg_1 \wedge \cdots \wedge ug_m \otimes (u Q^\perp h_1)^* \wedge \cdots \wedge (u Q^\perp h_n)
\]

where \( g_i, h_j \in H \) for \( 1 \leq i \leq m, 1 \leq j \leq n \).

Similarly, we have \( V^* \in B(\mathcal{F}_P, \mathcal{F}_Q) \) acts on \( \mathcal{F}_P \) by

\[
V^*(Pg_1 \wedge \cdots \wedge Pg_m \otimes (P^\perp h_1)^* \wedge \cdots \wedge (P^\perp h_n)) = u^* Pg_1 \wedge \cdots \wedge u^* Pg_m \otimes (u^* P^\perp h_1)^* \wedge \cdots \wedge (u^* P^\perp h_n) \in \mathcal{F}_Q.
\]

It implies that \( V^* \pi_P(a(uf)) V = \pi_Q(a(f)) \) for all \( f \in H \). Then, by the unitary equivalence of \( \pi_P, \pi_Q \), \( u \) is implemented as

\[
\pi_P(a(uf)) = V \pi_Q(a(f)) V^* = (V U^*) \pi_P(a(f)) (V U^*)^* \quad \text{for all } f \in H
\]

with \( V U^* \in B(\mathcal{F}_P) \) unitary. \( \square \)

Such kind of invertible (unitary) elements is mainly studied for loop groups in \([6]\) as a special subgroup of the general linear group of some complex vector space. Here we will focus on \( F \) as discrete group below.

### 3.3 A projection in \( \mathcal{B}(H) \) and the projective representation

Consider a basis of \( H = L^2([0, 1]) \) with dyadic support defined by

\[
\begin{align*}
f_{0,0} &= 1_{[0,1]} \\
f_{1,0} &= 1_{[0,1/2]} - 1_{[1/2,1]} \\
f_{2,0} &= \sqrt{2} \cdot 1_{[0,1/4]} - \sqrt{2} \cdot 1_{[1/4,1/2]}, \quad f_{2,1} = \sqrt{2} \cdot 1_{[1/2,3/4]} - \sqrt{2} \cdot 1_{[3/4,1]}
\end{align*}
\]
and \( f_{n,k} = \sqrt{2^{n-1}} \cdot 1_{[2^k/2^n,2^k+1/2^n)} - \sqrt{2^{n-1}} \cdot 1_{[2^{k+1}/2^n,2^{k+2}/2^n)} \) is defined for all \( n \in \mathbb{N}, 0 \leq k \leq 2^{n-1} - 1 \).

One can check this forms an orthonormal basis of \( H \). It can also be renumbered lexicographically as \( \{ f_i \}_{i \in \mathbb{N}} \). We will construct another orthonormal basis \( \{ p_{n,t}, q_{n,t} \}_{n \in \mathbb{N}, 0 \leq t < 2^{n-2} - 1} \) from \( \{ f_{n,k} \} \).

\[
\begin{align*}
p_{1,0} &= \frac{\sqrt{2}}{2} (f_{0,0} + f_{1,0}), & q_{1,0} &= \frac{\sqrt{2}}{2} (f_{0,0} - f_{1,0}) \\
p_{2,0} &= \frac{\sqrt{2}}{2} (f_{2,0} + f_{2,1}), & q_{2,0} &= \frac{\sqrt{2}}{2} (f_{2,0} - f_{2,1}) \\
p_{n,t} &= \frac{\sqrt{2}}{2} (f_{n,2t} + f_{n,2t+1}), & q_{n,t} &= \frac{\sqrt{2}}{2} (f_{n,2t} - f_{n,2t+1})
\end{align*}
\]

and we also renumber it lexicographically as \( \{ p_i, q_i \}_{i \in \mathbb{N}} \).

Let \( K_n \) be the subspace of \( H \) spanned by \( \{ p_{n,t} \}_{0 \leq t < 2^{n-2}-1} \). Let \( K \) be the subspace of \( H \) spanned by \( \{ p_i \}_{i \in \mathbb{N}} \) and \( P \in \mathcal{B}(H) \) is the projection onto \( K \). We have that \( \ker P = \text{span}\{ q_i \}_{i \in \mathbb{N}} \). The question is whether the Koopman representation of \( u : F \to \mathcal{U}(H) \) (via \( g \mapsto u_g \)) can be implemented in \( \mathcal{F}_P \).

**Lemma 3.6** For any \( g \in F \), there is a \( n_g \in \mathbb{N} \) such that \( u_g(p_{n,t}) \in K \) and \( u_g(q_{n,t}) \in K^\perp \) for all \( n \geq n_g \).

**Proof:** Given \( g \in F \), suppose \( g \) can be presented by a reduced form of \( A^{\alpha_1} B^{\beta_1} \cdots A^{\alpha_s} B^{\beta_s} \). By lemma 2.1, We have that the minimal interval length \( \text{mil}(g) \geq \frac{1}{\prod_{1 \leq i \leq s} |4^{\alpha_k} |8^{|\beta_k}|} \). Take \( N_g \) to be the \( 2 + \min\{ n \in \mathbb{N} | \frac{1}{n} < \text{mil}(g) \} \).

Then, for \( p_{n,t} \in K_n \) with \( n \geq N_g \), we have all \( \text{supp} p_{n,t} \) are the dyadic intervals of \( g \) without singularities. This implies \( u_g(p_{n,t}) \in \bigoplus_{i=-n_g}^{n_g} K_{n+i} \subset K \).

and the proof is similar for \( q_{n,t} \). \( \square \)

**Lemma 3.7** For any \( g \in F \), \([u_g, P]\) is a Hilbert-Schmidt operator.

**Proof:** By the lemma above, there is \( n_g \in \mathbb{N} \) such that \( u_g(p_{n,t}) \in K \) and \( u_g(q_{n,t}) \in K^\perp \) for all \( n \geq n_g \). After renumbering, there is \( N_g \in \mathbb{N} \) such that \( u_g(p_k) \in K \) and \( u_g(q_k) \in K^\perp \) for all \( n \geq N_g \). Then we can get the Hilbert-Schmidt norm of \([u_g, P]\) is bounded by the following inequality.
\[ \| [u_g, P]\|_2^2 = \sum_{k=1}^{\infty} \| [u_g, P]p_k \| + \sum_{k=1}^{\infty} \| [u_g, P]q_k \| \]
\[ = \sum_{k=1}^{\infty} \| (u_g - Pu_g)p_k \| + \sum_{k=1}^{\infty} \| Pu_gq_k \| \]
\[ = \sum_{k=1}^{N_g} (\| (u_g - Pu_g)p_k \| + \| Pu_gq_k \|) + \sum_{k=N_g+1}^{\infty} (\| (u_g - Pu_g)p_k \| + \| Pu_gq_k \|) \]
\[ = \sum_{k=1}^{N_g} (\| (u_g - Pu_g)p_k \| + \| Pu_gq_k \|) < \infty. \]

So \([u_g, P]\) is a Hilbert-Schmidt operator. \(\Box\)

Then, by the corollary 3.5, one can directly get:

**Corollary 3.8** The Thompson group \(F\) is implemented in \(\mathcal{F}_P\) through its Koopman representation \(\{u_g| g \in F\} \in \mathcal{U}(H)\).

Up to now, for any \(g \in F\), there is a unitary \(U_g\) such that

\[ \pi_P(a(u_gf)) = U_g \pi_P(a(f)) U_g^* \text{ for all } f \in H. \]

As \(\pi_P\) is irreducible, such a \(U_g\) is unique up to a phase when it exists. So, by passing to the projective unitary group \(PU(\mathcal{F}_P) = U(\mathcal{F}_P)/(S^1: I)\), we have a projective unitary representation

\[ \rho : F \to PU(\mathcal{F}_P) \]

as the composition of

\[ \rho : F \xrightarrow{\text{u}} \mathcal{U}(H) \xrightarrow{U} U(\mathcal{F}_P) \xrightarrow{/(S^1: I)} PU(\mathcal{F}_P) \]

by \(g \mapsto u_g \mapsto U_g \mapsto \rho_g \in PU(\mathcal{F}_P)\).

Remark: The map \(U\) is just one between sets, not a group homomorphism.

**4 The Lifting Problem**

In this section, we will discuss whether the projective unitary representation \(\rho : F \to PU(\mathcal{F}_P)\) can be lifted up to a unitary one into \(U(\mathcal{F}_P)\). This is equivalent to the problem of central extensions of \(F\).
Let \( \tilde{F} = \{(A,g) \in U(F_P) \times F | \tilde{A} = \rho(g) \} \) where \( \tilde{A} \) is the image of \( A \) under the quotient map \( U(F_P) \to PU(F_P) \). We have a unitary representation of \( \tilde{F} \) by the projection on the first factor: \( \tilde{\rho}(A,g) = A \), which can be denoted by the following diagram.

\[
\begin{array}{c}
1 \rightarrow S^1 \rightarrow \tilde{F} \rightarrow F \rightarrow 1 \\
\downarrow \text{id} \downarrow \tilde{\rho} \downarrow \rho \\
1 \rightarrow S^1 \rightarrow U(F_P) \rightarrow PU(F_P) \rightarrow 1
\end{array}
\]

More generally, the central extension of \( F \) by \( S^1 \) is a group \( E \) with a short exact sequence:

\[ 1 \rightarrow S^1 \rightarrow E \rightarrow F \rightarrow 1. \]

And we say two central extensions \( E_1, E_2 \) are equivalent if there is an isomorphism \( \phi : E_1 \rightarrow E_2 \). There is a result from the well-known classification theorem [14] of central extensions of any arbitrary group (up to isomorphism).

**Theorem 4.1** The equivalence classes of central extensions are in 1-1 correspondence with the second cohomology group \( H^2(F; S^1) \).

In particular, if \( H^2(F; S^1) \) is trivial, every projective unitary representation of \( F \) can be lifted to a unitary one.

### 4.1 Group cohomology of \( F \)

We will use classifying space to get the homology and cohomology groups of \( F \). Given a group \( G \), there is a classifying space \( BG \) whose homology and cohomology groups are the same as \( G \) [8]. But we will mainly discuss the classifying space of a given small category defined by Quillen (see [7] or [10]).

Let \( C \) be a small category. The nerve of \( C \), denoted \( NC \), is defined to be the following simplicial set:

\[
A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_1} \cdots A_{n-1} \xrightarrow{f_{n-1}} A_n,
\]

where \( A_i \)'s are objects in \( C \) and each \( f_i : A_i \rightarrow A_{i+1} \) is a morphism. Then the geometric realization of \( NC \) is defined to be \( BC \).

The most basic example is the category \( C_0 \) of a partially ordering set of two element \( \{0,1\} \) with \( 0 \leq 1 \). Then there is only one simplex \( 0 \rightarrow 1 \) and hence we have \( BC_0 = [0,1] \).

Kenneth Brown [3] gave the structure of cohomology ring \( H^*(F; \mathbb{Z}) \) by the homology of classifying spaces of several categories. Here, as a review, we will give a quick and direct outline of the proof with [7].
Firstly, there are three categories related to $F$

1. The category $\mathcal{F}$

Let $\mathcal{F}$ be a (small) category whose objects are intervals $[0,n]$ ($n \geq 1$) and morphisms are the dyadic piecewise linear homeomorphisms. Let $|\mathcal{F}|$ be its geometric realization for small categories defined by Quillen [7]. It is obvious the automorphism group of one object is exactly $F$. Then, by [7] again, $|\mathcal{F}|$ is an Eilenberg-MacLane complex of type $K(F,1)$.

2. The category $\mathcal{S}$

Let $\mathcal{S}$ be a (small) category whose objects are the same as $\mathcal{F}$, but its morphisms are restricted to be subdivision maps: For any two intervals $[0, n+k], [0, n]$ ($k \leq 1 - n$), take dyadic subdivision (section 2.2) of $[0, n+k]$ into $[0, 1], \ldots, [n+k-1, n+k]$ and $[0, n]$ into any dyadic subdivision with $n+k$ parts, then the dyadic piecewise linear homeomorphisms is defined by correspondence of two pairs of $n+k+1$ points in $[0, n+k], [0, n]$ respectively. $|\mathcal{S}|$ is an Eilenberg-MacLane complex of type $K(F,1)$.

3. The category $\mathcal{B}$

The category $\mathcal{B}$ is a POSET whose objects are any binary trees. And for two objects $B, C$, $B \leq C$ if $C$ can be extended from $B$ by the usual binary tree expansion. There is an obvious action of $F$ on $\mathcal{B}$. And the geometric realization $|\mathcal{B}|$ is contractible with a free $F$ action and the quotient is just $|\mathcal{S}|$.

With these three categories and geometric realizations above, we will introduce a new complex $X$, from which we can compute the homology and cohomology groups of $F$.

Let $L$ be a binary forest. An elementary expansion of $L$ is expansion of $L$ of some different nodes belong to $L$, but not any expansion at a new node. This implies the height will plus at most one after this expansion. We will write $L \preceq M$ if $M$ is an elementary expansion of $L$. Note that $L = L_0 \preceq L_1 \preceq \cdots \preceq L_p = M$ will not imply $L \preceq M$. But $L = L_0 \preceq L_1 \preceq \cdots \preceq L_p = M$ with $L \preceq M$ will imply $L = L_0 \preceq L_1 \preceq \cdots \preceq L_p = M$. By [7] again, the elementary expansions form the elementary simplices $\tilde{X}$ which is an $F$-invariant subcomplex of $|\mathcal{B}|$.

Then, by passing to the quotient of action by $F$, we get a subcomplex $X \subset |\mathcal{S}|$. It has one cell for each chain $L = L_0 \preceq L_1 \preceq \cdots \preceq L_p = M$ with $L \preceq M$. And by [9], we can decompose $X$ into cubes.
Recall that the geometric realization of the POSET \{0, 1\} is \([0, 1]\).
Given any \(L \leq M\) where \(M\) is a \(k\)-fold elementary expansion of \(L\), let 
\([L, M]\) denotes all chains of elementary expansions from \(L\) to \(M\). We have the following interesting result:

the geometric realization \(|[L, M]| = [0, 1]^k\) as \([L, M] \simeq \{0, 1\}^k\).

And the relative interior of this \(k\)-cube is the union of the open simplices corresponding to the chain \(L = L_0 \leq L_1 \leq \cdots \leq L_p = M\) with \(L \preceq M\).

Moreover, there is a natural product \(\mathcal{F} \times \mathcal{F} \to \mathcal{F}\) by just gluing the objects and connect the two morphisms. This makes \(\mathcal{F}\) a semigroup with \(|S|, X\) as subsemigroups. And as (3), one can check \(X\) is finite generated as semigroups by two elements \(v, e\) (Figure 3):

Let \(C = C(X)\) be the cellular chain complex of \(X\). Then \(C\) is a differential graded ring without identity. One can check \(\deg(v) = 0, \deg(e) = 1\) and the differential rule:

\[
\partial(e) = v^2 - v \quad \text{and} \quad \partial(xy) = \partial(x) \cdot y + (-1)^{\deg(x)} x \cdot \partial(y).
\]

Then, follows (3), we have:

**Proposition 4.2** There is are ring isomorphisms

1. \(H_*(F; \mathbb{Z}) \cong \mathbb{Z}[\varepsilon, \zeta]\) with relations \(\varepsilon^2 = \varepsilon, \varepsilon \zeta = \zeta - \varepsilon \zeta\).
2. \(H^*(F; \mathbb{Z}) \cong \Gamma(u) \otimes (a, b)\) with \(\deg a = \deg b = 1\) and \(\deg u = 2\), where \(\Gamma(u)\) is the ring generated by \(u^{(i)} = u^i/i!\) over \(\mathbb{Z}\).

So there is \(H^2(F; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}\) from the degree= 2 terms. Then, by the universal coefficient theorem (14)

\[
0 \to \Ext^1_R(H_{n-1}(P), M) \to H^n(\Hom_R(P, M)) \to \Hom_R(H_n(P), M) \to 0
\]
with $R = Z, M = S^1, n = 2$ and $\text{Ext}_Z^1(H_1(F), S^1) = 0, \text{Hom}_Z(H_2(F), S^1) = S^1 \times S^1$ from the proposition above. Then we have

**Corollary 4.3** $H^2(F; S^1) = S^1 \times S^1$.

That is to say not every projective unitary representation of $F$ can be lifted to a unitary one. We will check in details whether the representation $\rho$ can be lifted.

### 4.2 Criterion for lifting

Now, we suppose there is a lifting $U : F \to U(F_P)$ of $\rho : F \to PU(F_P)$. That is to say $\pi(U(g)) = \rho(g)$ for all $g \in F$, where $\pi : U(F_P) \to PU(F_P)$ is the quotient map.

Since $F$ is finitely generated, it is enough to consider the lifting of $\rho_A, \rho_B$. We can choose two arbitrary $R, S \in U(F_P)$ such that

\[ \pi(R) = \rho_A, \pi(S) = \rho_B. \]

Then there must be two complex numbers $c_A, c_B \in S^1$ such that

\[ U_A = c_A \cdot R, \quad U_B = c_A \cdot S. \]

And we will also have the lifting of $\rho_A^*, \rho_B^*$ are $U_A^{-1} = c_A^{-1} \cdot R^{-1}, U_B^{-1} = c_B^{-1} \cdot S^{-1}$ respectively. Then we have:

**Proposition 4.4** With the lifting of $\rho_A, \rho_B$ (hence also $\rho_A^{-1}, \rho_B^{-1}$) fixed as $U_A, U_B$, there is a lifting $U : F \to U(F_P)$ of $\rho : F \to PU(F_P)$ if and only if the following conditions are satisfied:

1. $[U_A U_B^{-1}, U_A^{-1} U_B U_A] = 1$;
2. $[U_A U_B^{-1}, U_A^{-1} U_B U_A^2] = 1$.

Moreover, whether it is satisfied is independent on the choice of $c_A, c_B$.

**Proof:** Given $U_A, U_B, \rho$ can be lifted to a unitary $U$ is equivalent whether $U : F \to U(F_P)$ is a homomorphism. Obviously, this is equivalent to the two conditions above.

Moreover, as the two relations are both commutators. The commutators are always $[R S^{-1}, R^{-1} S R], [R S^{-1}, R^{-2} S R^2]$, which are independent with the choice of $c_A, c_B$. \[\square\]
5 The Vacuum Vectors and Explicit Action Formulas

5.1 The vacuum vector

Consider the vacuum vector $\Omega = \Omega_1 \otimes \Omega_2 \in \mathcal{F}_P$, by lemma 3.2 and action of $\pi_P(a(f))$, we have

$$\{\pi_P(a(p_{i_1})) \cdots \pi_P(a(p_{i_k}))\pi_P(a(q_{j_1})^*) \cdots \pi_P(a(q_{j_l})^*)\Omega | l, k \in \mathbb{N} \cup \{0\}, 1 \leq i_1 < \cdots < i_k, 1 \leq j_1 < \cdots < j_l\}$$

forms the orthonormal basis of $\mathcal{F}_P$. That is to say $\pi_P(CAR(H))\Omega \| \cdot \| = \mathcal{F}_P$.

Lemma 5.1 The vacuum vector $\Omega = \Omega_1 \otimes \Omega_2 \in \mathcal{F}_P$ is the unique vector (up to a scalar) that annihilated by $\pi_P(a(p_i)^*),\pi_P(a(q_j))$ for all $i, j \in \mathbb{N}$.

Proof: It is easy to check $\Omega$ is annihilated by these operators.

For the uniqueness, let us consider the space $\Omega^\perp$ which is spanned (densely) by

$$\{v = \pi_P(a(p_{i_1})) \cdots \pi_P(a(p_{i_k}))\pi_P(a(q_{j_1})^*) \cdots \pi_P(a(q_{j_l})^*)\Omega | l, k \in \mathbb{N} \cup \{0\}, l \neq 0, 1 \leq i_1 < \cdots < i_k, 1 \leq j_1 < \cdots < j_l\}$$

that at least one of $a(p_i)$ or $a(q_j)^*$ appears. If there is such a vector $\Omega'$, there must be $\langle \Omega', v \rangle = 0$ for any $v$ described above. So $\Omega' \in \mathbb{C}\Omega$.

Now, let us go back to the implementation of $\{u_g | g \in F\}$. Suppose each $u_g$ is implemented by $U_g \in U(\mathcal{F}_P)$ such that

$$\pi_P(a(u_gf))U_g = U_g\pi_P(a(f))$$

for all $f \in H$.

We have its action on $\Omega$ as

$$\pi_P(a(u_gf))U_g\Omega = U_g\pi_P(a(f))\Omega$$

for all $f \in H$.

Now, replace $a(f)$ by $a(p_i)^*$ and $a(q_j)$ with $i, j \in \mathbb{N}$ and define

$$\Omega_g \overset{\text{def}}{=} U_g\Omega$$

As $U_g$ is invertible, we have

Corollary 5.2 $\Omega_g$ is the unique vector (up to a scalar) that is annihilated by

$$a(u_g p_i)^* \text{ and } a(u_g q_j)$$

This give us a first description of the action $U_g$. 

14
5.2 The explicit formula of the lifting representation

Once we get the action of a $U_g$ on $\Omega$, i.e. $\Omega_g = U_g\Omega$, one may wonder the action of $U_g$ on the whole space $\mathcal{F}_P$.

Now, let us go back to the equation

$$\pi_P(a(u_g f))U_g\Omega = U_g\pi_P(a(f))\Omega$$

for all $f \in H$.

Take an arbitrary vector from the basis mentioned above, i.e.

$$v = \pi_P(a(p_{i_1})) \cdots \pi_P(a(p_{i_k})) \pi_P(a(q_{j_1})^*) \cdots \pi_P(a(q_{j_l})^*)\Omega$$

with $l, k \in \mathbb{N} \cup \{0\}$, $1 \leq i_1 < \cdots < i_k$, $1 \leq j_1 < \cdots < j_l$. Let $U_g$ act on this vector, we have

$$U_g v = U_g\pi_P(a(p_{i_1})) \cdots \pi_P(a(p_{i_k})) \pi_P(a(q_{j_1})^*) \cdots \pi_P(a(q_{j_l})^*)\Omega$$

$$= \pi_P(a(u_g p_{i_1})) U_g \pi_P(a(p_{i_2})) \cdots \pi_P(a(p_{i_k})) \pi_P(a(q_{j_1})^*) \cdots \pi_P(a(q_{j_l})^*)\Omega$$

$$= \cdots = \pi_P(a(u_g p_{i_1})) \cdots \pi_P(a(u_g p_{i_k})) \pi_P(a(u_g q_{j_1})^*) \cdots \pi_P(a(u_g q_{j_l})^*) U_g\Omega$$

$$= \pi_P(a(u_g p_{i_1})) \cdots \pi_P(a(u_g p_{i_k})) \pi_P(a(u_g q_{j_1})^*) \cdots \pi_P(a(u_g q_{j_l})^*)\Omega_g$$

which is to say

**Corollary 5.3** Once $\Omega_g$ is known, the action of $U_g$ on $\mathcal{F}_P$ is explicit by its action of the orthonormal basis given above.

5.3 An elementary example

Before going to $\Omega_g$ for any $g \neq 1 \in \mathcal{F}$, let us look at a simple case. Let $u_D \in U(H)$ is given by the matrix

$$u_D = \begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \ddots \\
\cdots & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & \cdots \\
\cdots & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & \cdots \\
\ddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Remark: this $u_D$ is not in the $u(F) \subset U(H)$.

One can check $[u_D, P]$ is also a Hilbert Schmidt operator. By corollary 3.5, it can be implemented in $PU(\mathcal{F}_P)$ as $\rho_D$. Let $U_D$ be the lifting of $\rho_D$. 

By corollary 5.2, $\Omega_D = U_D \cdot \Omega$ is the unique vector (up to a scalar) that annihilated by

$$\pi_P(a(u_D p_i)^*) \text{ and } \pi_P(a(u_D q_j))$$

with $i, j \in \mathbb{N}$,

which is $\pi_P(a(q_i)^*)$, $\pi_P(a(p_1))$ and $\{\pi_P(a(p_i)^*), \pi_P(a(q_i))\}_{i \geq 2}$. Then one can get $\Omega_D$ is given by

$$c_D \cdot p_1 \otimes q_1^*$$

with $c_D \in S^1$.

**5.4 $\Omega_A$**

We know that $\Omega_A$ is the unique vector (up to a scalar) annihilated by $a(u_A p_i)^*$ and $a(u_A q_j)$ with $i, j \in \mathbb{N}$. The $\{u_A p_i\}_{i \in \mathbb{N}}$ and $\{u_A q_j\}_{j \in \mathbb{N}}$ can be denoted clearly as below.

Firstly, we observe that when the action of $u_A$ on $p_{n,t}$ can be described for $n \geq 4$ as:

1. $u_A(p_{n,t}) = p_{n+1,t}$ if $\text{supp} p_{n,t} \subset [0, 1/2]$;
2. $u_A(p_{n,t}) = p_{n,t-2^{n-4}}$ if $\text{supp} p_{n,t} \subset [1/2, 3/4]$;
3. $u_A(p_{n,t}) = p_{n-1,t-2^{n-4}}$ if $\text{supp} p_{n,t} \subset [0, 1/4]$.

After renumbering, we have

$$\{u_A p_i\}_{i \in \mathbb{N}} = \{u_A p_i\}_{i \geq 5} \cup \{u_A p_1, u_A p_2, u_A p_3, u_A p_4\}$$

and

$$\{u_A p_i\}_{i \geq 4} \cup \{u_A p_1, u_A p_2, u_A p_4\}$$

**6 Generalization to $SO(2)$ and $U(2)$**

In chapter 3.3, we define a new orthonormal basis $\{p_{n,t}, q_{n,t}\}_{n \in \mathbb{N}, 0 \leq t \leq 2^{n-2}-1}$ given by

$$\begin{pmatrix} p_{n,t} \\ q_{n,t} \end{pmatrix} = M \cdot \begin{pmatrix} f_{n,2t} \\ f_{n,2t+1} \end{pmatrix}$$

where

$$M = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$$
We may let $M$ to be any matrix in $\text{SO}(2)$ or $\text{U}(2)$. Now, assume $M \in \text{U}(2)$ and there is another orthonormal basis $B_M = \{p_1^M, q_1^M\} \cup \{p_{n,t}^M, q_{n,t}^M\}_{n \geq 2, 0 \leq t \leq 2^{n-2} - 1}$ given by

\[
\begin{pmatrix}
    p_{n,t} \\
    q_{n,t}
\end{pmatrix} = M \cdot \begin{pmatrix}
    f_{n,2t} \\
    f_{n,2t+1}
\end{pmatrix}
\]

with the $p_1^M = p_1, q_1^M = q_1$ defined in section 3.3.

Now, let $P_M \in \mathcal{B}(H)$ to be the projection onto the subspace $K_M = \text{span}\{p_1^M, \{p_{n,t}^M\}_{n \geq 2, 0 \leq t \leq 2^{n-2} - 1}\}$. And there is also a Fermionic Fock space

\[
\mathcal{F}_M = \mathcal{F}_{P_M} = \wedge (P_M H) \hat{\otimes} \wedge (P_M^\perp H)^*.
\]

Every result in section 3.3 follows similarly. So that we get a projective unitary representation

\[
\rho_M : F \to PU(\mathcal{F}_M).
\]

According to proposition 4.4, once the lifting of $\rho_M(A), \rho_M(B)$ are fixed as $U_M(A), U_M(B) \in U(\mathcal{F}_M)$, it gives a value $\Psi_M = (\alpha_M, \beta_M) \in S^1 \times S^1$ by

1. $\alpha_M = [U_M(A)U_M(B)^{-1}, U_M(A)^{-1}U_M(B)U_M(A)];$
2. $\beta_M = [U_M(A)U_M(B)^{-1}, U_M(A)^{-2}U_M(B)U_M(A)^2].$

Let $M \in \text{U}(2)$ be arbitrary, we get a map

\[
\Psi : \text{U}(2) \to S^1 \times S^1 \text{ by } \Psi(M) = (\alpha_M, \beta_M)
\]

defined above. Moreover, when $M \subset \text{SO}(2)$, one can always obtain

\[
\Psi : \text{SO}(2) \to \mathbb{R}^2 \cap (S^1)^2 = \{-1, 1\} \times \{-1, 1\}.
\]

We will discuss the map $\Psi$ (in the future?).

References

[1] Baez J C, Segal I E, Zhou Z. Introduction to algebraic and constructive quantum field theory[M]. Princeton University Press, 2014.

[2] J. M. Belk, Thompsons group F, Ph.D. Thesis, Cornell University, 2004.
[3] Brown K S. The homology of Richard Thompson’s group F[J]. Contemporary Mathematics, 2006, 394: 47.

[4] Cannon J W, Floyd W J, Parry W R. Introductory notes on Richard Thompson’s groups[J]. Enseignement Mathmatique, 1996, 42: 215-256.

[5] Dudko A. On irreducibility of Koopman representations of Higman-Thompson groups[J]. arXiv preprint arXiv:1512.02687, 2015.

[6] Pressley A, Segal G B. Loop groups[M]. Clarendon Press, 1986.

[7] Quillen D. Higher algebraic K-theory: I[M]//Higher K-theories. Springer, Berlin, Heidelberg, 1973: 85-147.

[8] Rosenberg J. Algebraic K-theory and its applications[M]. Springer Science and Business Media, 1995.

[9] Stein M. Groups of piecewise linear homeomorphisms[J]. Transactions of the American Mathematical Society, 1992, 332(2): 477-514.

[10] Srinivas V. Algebraic K-theory[M]. Springer Science and Business Media, 2007.

[11] Swierczkowski S. On isomorphic free algebras[J]. Fundamenta Mathematicae, 1961, 1(50): 35-44.

[12] Toledano-Laredo V. Fusion of positive energy representations of lspin (2n)[J]. arXiv preprint math/0409044, 2004.

[13] Wassermann A. Operator algebras and conformal field theory III. Fusion of positive energy representations of LSU (N) using bounded operators[J]. Inventiones mathematicae, 1998, 133(3): 467-538.

[14] Weibel C A. An introduction to homological algebra[M]. Cambridge university press, 1995.