Mechanism Redesign

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Abstract

This paper develops the theory of mechanism redesign by which an auctioneer can reoptimize an auction based on bid data collected from previous iterations of the auction on bidders from the same market. We give a direct method for estimation of the revenue of a counterfactual auction from the bids in the current auction. The estimator is a simple weighted order statistic of the bids and has the optimal error rate. Two applications of our estimator are A/B testing (a.k.a., randomized controlled trials) and instrumented optimization (i.e., revenue optimization subject to being able to do accurate inference of any counterfactual auction revenue).

1 Introduction

This paper develops data-driven methods that enable a principal to adjust the parameters of an auction so as to optimize its performance, i.e., for mechanism redesign. These methods are inspired by settings of online markets such as online advertising, hotel booking platforms, online auctions, etc. For a paradigmatic family of auctions, we derive estimators for the revenue and welfare of a counterfactual auction in the family from equilibrium bids in an incumbent auction. Our analysis exposes the relationship between the the error of the estimator and the tuned parameters of the counterfactual and incumbent auctions. From this analysis, we identify a family of mechanisms that are statistically efficient to redesign.

Our work is motivated by the simple observation that revenue maximization in auctions is at odds with statistical inference. Specifically, in the classic setting of Myerson (1981), an auctioneer who adopts a mechanism from the family of revenue optimal auctions, which employ reserve prices, will not generally be able to infer distribution of values of the bidders in the auction and, thus, will not generally be able to determine the revenue optimal auction from the family. Reserve pricing and ironing pool bidders with distinct values and, thereafter, no procedure for structural inference can distinguish them. Consequently, counterfactual revenue estimation from bids in revenue optimal auctions is not generally possible.

Our work focuses on a canonical family of mechanisms that generalize multi-unit auctions. Position auctions were introduced and studied by Varian (2007) and Edelman et al. (2007) as

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1We have had extensive discussions over the last decade with R&D teams at companies in this space, especially Microsoft, which brought us to the model and questions studied in this paper.
a model of auctions for selling advertisements on Internet search engines. A position auction is defined by a decreasing sequence of weights which correspond to allocation probabilities, bidders are assigned to weights assortatively by bid. The configurable parameters in this family of auctions are the weights of the positions. Position auctions generalize classical single-item and multi-unit auction models and are an important form of auction for theoretical study. We study the winner-pays-bid and all-pay variants of position auctions. The main results for the all-pay setting are stated in Section 3 and Section 4 while the winner-pays-bid variant is considered in Appendix C.

Position auctions (without reserve prices or ironing) are not revenue optimal; however, we show that they can be tuned so that their revenue is approximately optimal. For example, in a multi-unit auction for bidders with identically distributed values it is revenue optimal to impose a reserve price; however this optimal revenue is closely approximated by reducing supply instead (e.g., to sell the same number of items in expectation as under the reserve price). Moreover, in followup work to ours

Hartline and Taggart (2019) show that the linking-decisions approach of Jackson and Sonnenschein (2007) can be applied to repeated Bayes-Nash auctions with non-identically distributed bidders to give position auctions that approximate the revenue optimal auction arbitrarily closely.

Our main technical contribution is a method for counterfactual revenue estimation: given two position auctions we define an estimator for the equilibrium revenue of one (the counterfactual auction) from equilibrium bids of the other (the incumbent auction). Our estimator has three important properties that contrast with the classical approach to counterfactual revenue estimation in auctions.

1. The estimator is a linear functional of the bid distribution and can be applied to any estimator for the bid distribution.

   In contrast, the classical approach requires the estimator for the bid distribution be continuously differentiable so that the bidders’ first order condition can be inverted to infer values.

   In contrast, the classical analysis requires uniform estimation of the distribution of values to obtain error bounds on counterfactual revenue estimates derived from value estimates.

2. Applied to the empirical bid distribution, the estimator is a weighted order statistic. Specifically the order statistics of the bids in the incumbent mechanisms are mapped to a revenue estimate as a simple weighted sum with weights that are determined by the parameters of the counterfactual and incumbent auctions.

3. The estimator converges at the rate equal to the square root of the number of samples with leading coefficients that depend directly on the relationship between the tuned parameters of the counterfactual and incumbent auction; thus, we are able to identify mechanisms that are good for redesign, i.e., for which the estimator has small leading coefficients.

Our theoretical analysis suggests that the typical smoothing in the classical analysis is not necessary. We confirm suggestion with Monte Carlo simulations that show that the empirically optimal smoothing for counterfactual revenue estimation is, in fact, no smoothing. Our theoretical analysis also suggests that error can be reduced by truncating, i.e., zeroing out, the contribution

Unfortunately, our methods cannot be directly applied to position auctions with the so-called “generalized second price” payment rule of Google’s AdWords platform. Online ad auctions are, however, moving towards winner-pays-bid formats like the ones we consider (e.g., Paes Leme et al. 2020).

For numerical stability this estimator should be computed in its algebraically equivalent form as a weighted sum of the difference of consecutive bids.
to the estimator from extreme (low or high) bids. We find empirically that the level of truncation suggested by the analysis, which does not depend on distributions or mechanisms, is indeed close to the correct level.

Our detailed analysis of the error of the counterfactual revenue estimation, in terms of the parameters of the incumbent and counterfactual mechanisms allows for direct comparison to the industry standard practice of A/B testing, a.k.a., randomized controled trials (e.g., Kohavi et al. 2009). An ideal A/B test for auctions would work as follows: The auction format is randomized between formats A and B with equilibrium bids for A collected for format A and equilibrium bids for B collected for format B. Bids in A are used to estimate revenue of A and bids in B are used to estimate revenue in B. This ideal is typically not achieved in practice in online markets, often instead the bids are collected in advance of the realization of the randomized format, and thus can be assumed to be in equilibrium of format C given by the convex combination of the two formats. Our analysis allows the revenue for B (also A) to be estimated from the equilibrium bids in C and our error bounds depend on the relationship between B and C. When the A/B test mixes in B with probability $\epsilon$ (and A with probability $1-\epsilon$), our estimators dependence on epsilon is $\log 1/\epsilon$ while the ideal A/B test has dependence $\sqrt{1/\epsilon}$.

The main result of the paper is an analysis of the instrumented optimization problem where the principal aims to run a mechanism that is both good for revenue and good for subsequent inference. As a first result, we show that there is universal treatment B such that in the A/B test mechanism C, the revenue of any other position auction can be estimated with low error. A heuristic solution to the instrumented optimization problem, then, is to run the A/B test mechanism C that corresponds to the revenue optimal position auction A and this universal treatment B. Our second result incorporates a bound on the desired rate of estimation into the revenue optimization problem and derives the revenue optimal auction subject to good revenue inference. Our analysis gives a tradeoff between revenue bounds (relative to the optimal position auction) and the desired rate of convergence.

Our bounds on the error of our estimator are expressed in terms of the the number of samples (from the bid distribution, subsequently denoted by $N$), the number of bidders (in each auction, subsequently denoted by $n$, and the similarity between the incumbent and counterfactual auction by $\epsilon$). A straightforward econometric analysis might treat the number of bidders and parameters of the auctions as constants without quantifying the detailed dependence of the error on these constants. Such analysis does not preclude the possibility that there is a very large error until the number of samples is exponentially larger than this number of bidders. In contrast, our error bounds show at most polynomial dependence on the number of bidders and logarithmic dependence on the similarity between auctions $1/\epsilon$.

While much of this paper focuses on estimating and optimizing revenue, in Section 7 we extend the method to the estimation of social welfare.

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4Consider the example of online advertising market. The advertisers bid in advance; the users to whom the ads are shown arrive online; and the randomization occurs at the user level. Typically these A/B tests are to understand user behavior; however, development teams that employ A/B testing also track “key performance indicator” like revenue. Implications for revenue for A and B, of course, cannot be inferred from averages across the A and B treatments, since bids are in equilibrium for C not A nor B.

5For example, in a single-unit $n$-agent all-pay auction the bids of the median bidder are an exponential factor smaller than their value. Of course, the revenue of the $n/2$-unit all-pay auction depends very much on good estimates on the contribution to the revenue from the median bidder.
1.1 Motivating Example: Auctions for Internet Search Advertising.

Our work is motivated by the auctions that sell advertisements on Internet search engines (see historical discussion by [Fain and Pedersen, 2006]). The first-price position auction we study in this paper was introduced in 1998 by the Internet listing company GoTo. This auction was adapted by Google in 2002 for their AdWords platform, modified to have a second-price-like payment rule, and is known as the generalized second-price auction. Early theoretical studies of equilibrium in the generalized second-price auction were conducted by [Varian, 2007] and [Edelman et al., 2007]; unlike the second-price auction for which it is named, the generalized second-price auction does not admit a truth-telling equilibrium.

Internet search advertising works as follows. A user looking for web pages on a given topic specifies keywords on which to search. The search engine then returns a listing of web pages that relate to these keywords. Alongside these search results, the search engine displays sponsored results. These results are conventionally explicitly labeled as sponsored and appear in the mainline, i.e., above the search results, or in the sidebar, i.e., to the right of the search results. The mainline typically contains up to four ads and the sidebar contains up to seven ads. The order of the ads is of importance as the Internet user is more likely to read and click on ads in higher positions on the page. In the classic model of [Varian, 2007] and [Edelman et al., 2007] the user’s click behavior is exogenously given by weights associated with the positions and the weights are decreasing in position. An advertiser only pays for the ad if the user clicks on it. Thus in the classic first-price position auction, advertisers are assigned to positions in order of their bids, and the advertisers on whose ads the user clicks each pay their bids.

As described above, the ads in the mainline have higher click rates than those in the sidebar. The mainline, however, is not required to be filled to capacity (a maximum of four ads). In the first-price position auction described above, the choice of the number of ads to show in the mainline affects the revenue of the auction and, in the standard auction-theoretic model of Bayes-Nash equilibrium, this choice is ambiguous with respect to revenue ranking. For some distributions of advertiser values, showing more ads in the mainline gives more revenue, while for other distributions fewer ads gives more revenue.

The keywords of the user enable the advertisers to target users with distinct interests. For example, hotels in Hawaii may wish to show ads to a user searching for “best beaches,” while a computer hardware company would prefer users searching for “laptop reviews.” Thus, the search advertising is in fact a collection of many partially overlapping markets, with some high-volume high-demand keywords and a long tail of niche keywords. The conditions of each of these markets are distinct and thus, as per the discussion of the preceding paragraph, the number of ads to show in the mainline depends on the keywords of the search.

One empirical method for evaluating two alternatives, e.g., showing one or two mainline ads, is A/B testing. In the ideal setting of A/B testing, the auctions for a given keyword would be randomly divided into the A and B groups. In part A the advertisers would bid in Bayes-Nash equilibrium for A and in part B they would bid in equilibrium for B. Unfortunately, because we need to test both A and B in each market, ideal A/B testing would require soliciting distinct bids for each variant of the auction. This approach is impractical, both from an engineering perspective and from a public relations perspective. In practice, A/B tests are run on these ad platforms all the time and without informing the advertisers. Of course, advertisers can observe any overall change in the mechanism.

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6Endogenous click models have also been considered, e.g., [Athey and Ellison, 2011], but are less prevalent in the literature.
and adapt their bids accordingly, i.e., they can be assumed to be in equilibrium. Our approach of A/B testing, where bids are in equilibrium for auction C, the convex combination of A and B, is consistent with the industry standard practice for Internet search advertising.

Our A/B testing framework is motivated specifically by the goal of optimizing an auction to local characteristics of the market in which the auction is run. It is important to distinguish this goal from that of another framework for A/B testing which is commonly used to evaluate global properties of auctions across a collection of disjoint markets. This framework randomly partitions the individual markets into a control group (where auction A is run) and a treatment group (with auction B). From such an A/B test we can evaluate whether it is better for all markets to run A or for all to run B. It cannot be used, however, for our motivating application of determining the number of mainline ads to show, where the optimal number naturally varies across distinct markets. The work of Ostrovsky and Schwarz (2011) on reserve pricing in Yahoo!’s ad auction demonstrates how such a global A/B test can be valuable. They first used a parametric estimator for the value distribution in each market to determine a good reserve price for that market. Then they did a global A/B test to determine whether the auction with their calculated reserve prices (the B mechanism) has higher revenue on average than the auction with the original reserve prices (the A mechanism). Our methods relate to and can replace the first step of their analysis.

1.2 Related Work

Our work is motivated, in part, by field work in the past decade that considers the empirical optimization of reserve prices in auctions (e.g., Reiley 2006; Brown and Morgan 2009; and Ostrovsky and Schwarz 2011). The field study of Ostrovsky and Schwarz is most similar to our theoretical study and the motivating example of Section 1.1. They consider the generalized second-price position auction of Internet search advertising (on Yahoo!). They assume that the distribution of advertiser (bidder) values is lognormal and use structural inference to estimate the parameters of the distribution for the keywords of each search. This allows for inference of the optimal reserve price. They then suggest using a reserve price that is slightly smaller. Finally, they evaluate the method of setting reserves via a global A/B test that compares the original reserves with their reserves across all keywords. While the authors motivate the usage of reserves slightly smaller than the optimal reserves for reasons of robustness, in the context of our motivation these smaller reserves also allow future inference around the optimal reserve price where the optimal reserves do not.

The classical approach to counterfactual inference is based on recovering the values of bidders by inverting their best responses using the empirical distribution of bids. This approach was developed by Guerre et al. (2000) for single-unit first-price auctions and it has seen application broadly in auction theory (e.g. see Athey and Haile 2007; Paarsch and Hong 2006, and Marmer and Shneyerov 2012). There are several ways in which our revenue estimator improves upon this standard approach. First, Guerre et al. (2000) and subsequent works assume that values lie in a bounded range, the value density is bounded away from zero, and the density is differentiable. We also assume that values are bounded, but do not require any other assumptions on the value density. Second, this literature focuses on single item first price auctions. In contrast, our work applies to first-price and all-pay $k$-unit auctions, as well as mixtures over them. On the other hand, while Marmer and Shneyerov (2012) allow for the seller to impose a reserve price, our techniques do not extend to auctions with reserve prices. However, we show that auctions with reserve prices can be approximated in revenue by position auctions. Third, the classical approach requires selecting
an appropriate bandwidth for bid density estimation that is tuned to properties of the endogenous
bid distribution, whereas our estimator is not parameterized. Our inference method is based on
a technique similar to that in Myerson (1981) that “integrates out” best responses of agents so
that the auction revenue can be expressed directly in terms of the observable bids. Our proposed
estimator is equivalent to the plug-in estimator with no smoothing and we empirically show that
estimation error as a function of the degree of smoothing is minimized with no smoothing.

Our work focuses on position auctions with agents with identically distributed values, i.e.,
symmetric position auctions. Recently, Hartline and Taggart (2019) generalize our results to
single-dimensional mechanism design problems that are repeated with asymmetric agents within
rounds but identically distributed agents across rounds, i.e., a repeated agent-normal-form game.
Their approach employs our analysis as a black box and demonstrates that symmetric position
auctions are fundamental to the theory of mechanism design for single-dimensional agents more
generally.

Our estimator of the counterfactual auction revenue is simply a weighted order statistic of
samples from the bid distribution. Other works have proposed using similarly simple estimators to
obtain bounds on the performance a counterfactual auctions. Unlike our work, these estimators do
not use the first-order condition of Bayes-Nash equilibrium, but in exchange for a weakening of the
assumptions of the model, they obtain bounds instead of point estimates. For example, Coey et al.
(2014) consider ascending single-item auctions and use the main theorem of Bulow and Klemperer
(1996), the revenue submodularity of Dughmi et al. (2012), and the expected second- and third-
highest bids to bound the revenue of the (counterfactual) optimal auction. See their related work
section for a discussion of similar studies.

The mechanism design literature has previously considered the problem of an uninformed de-
signer who wishes optimize a mechanism under three conditions: (a) repeatedly on agents from
the same population (each agent participates only once; see Kleinberg and Leighton 2003, Blum
and Hartline 2005, and Cesa-Bianchi et al. 2015), (b) with samples from the value distribution
(see Cole and Roughgarden 2014, and Fu et al. 2014), and (c) on the fly in one mechanism (see
Goldberg et al. 2006, Segal 2003, and Baliga and Vohra 2003). These works exclusively consider
mechanisms that have truth-telling equilibria and for which, consequently, inference is trivial. The
papers listed in category (a) also consider a model where the designer only learns the revenue of
the mechanism in each round and not the individual bids. These papers adapt methods from the
multi-armed bandit literature, e.g., Auer et al. (2002), which tradeoff exploring the performance of
mechanisms that the designer is less informed about with exploiting the mechanisms which have
been learned to perform well. Our approach of instrumented optimization is similar to the ex-
ploration steps of these multi-armed bandit algorithms, except that we assume that bids are in
equilibrium for the distribution over mechanisms rather than for each individual mechanism. This
distinction is important for mechanisms that do not have truth-telling equilibria.

Finally, the theory that we develop for optimizing revenue over the class of rank-by-bid position
auctions is isomorphic to the theory of envy-free optimal pricing developed by Devanur et al. (2015).
2 Preliminaries

2.1 Auction Theory

A standard auction design problem is defined by a set \([n] = \{1, \ldots, n\}\) of \(n \geq 2\) agents, each with a private value \(v_i\) for receiving a service. The values are bounded as \(v_i \in [0, 1]\) and are independently and identically distributed according to a continuous distribution \(F\). If \(x_i\) indicates the probability of service and \(p_i\) the expected payment required, agent \(i\) has linear utility \(u_i = v_ix_i - p_i\). An auction elicits bids \(b = (b_1, \ldots, b_n)\) from the agents and maps the vector \(b\) of bids to an allocation \(\tilde{x}(b) = (\tilde{x}_1(b), \ldots, \tilde{x}_n(b))\), specifying the probability with which each agent is served, and prices \(\tilde{p}(b) = (\tilde{p}_1(b), \ldots, \tilde{p}_n(b))\), specifying the expected amount that each agent is required to pay. An auction is denoted by \((\tilde{x}, \tilde{p})\).

**Standard payment formats** In this paper we study two standard payment formats. In a first-price format, each agent pays his bid upon winning, that is, \(\tilde{p}_i(b_i) = b_i\tilde{x}_i(b)\). In an all-pay format, each agent pays his bid regardless of whether or not he wins, that is, \(\tilde{p}_i(b) = b_i\).

**Bayes-Nash equilibrium** The values are independently and identically distributed according to a continuous distribution \(F\). This distribution is common knowledge to the agents. A strategy \(s_i\) for agent \(i\) is a function that maps the value of the agent to a bid. The distribution of values \(F\) and a profile of strategies \(s = (s_1, \ldots, s_n)\) induces interim allocation and payment rules (as a function of bids) as follows for agent \(i\) with bid \(b_i\).

\[
\tilde{x}_i(b_i) = E_{v_i \sim F}[\tilde{x}_i(b_i, s_{-i}(v_{-i}))] \quad \text{and} \quad \tilde{p}_i(b_i) = E_{v_i \sim F}[\tilde{p}_i(b_i, s_{-i}(v_{-i}))].
\]

Agents have linear utility which can be expressed in the interim as:

\[
\tilde{u}_i(v_i, b_i) = v_i\tilde{x}_i(b_i) - \tilde{p}_i(b_i).
\]

The strategy profile forms a Bayes-Nash equilibrium (BNE) if for all agents \(i\), values \(v_i\), and alternative bids \(b_i\), bidding \(s_i(v_i)\) according to the strategy profile is at least as good as bidding \(b_i\). I.e.,

\[
\tilde{u}_i(v_i, s_i(v_i)) \geq \tilde{u}_i(v_i, b_i). \tag{1}
\]

A symmetric equilibrium is one where all agents bid by the same strategy, i.e., \(s\) satisfies \(s_i = s\) for all \(i\) and some \(s\). For a symmetric equilibrium of a symmetric auction, the interim allocation and payment rules are also symmetric, i.e., \(\tilde{x}_i = \tilde{x}\) and \(s_i = s\) for all \(i\). For implicit distribution \(F\) and symmetric equilibrium given by strategy \(s\), a mechanism can be described by the pair \((\tilde{x}, \tilde{p})\). Chawla and Hartline (2013) show that the equilibrium of every auction in the class we consider is unique and symmetric.

The strategy profile allows the mechanism’s outcome rules to be expressed in terms of the agents’ values instead of their bids; the distribution of values allows them to be expressed in terms of the agents’ values relative to the distribution. This latter representation exposes the geometry of the mechanism. Define the quantile \(q\) of an agent with value \(v\) to be the probability that \(v\) is larger than a random draw from the distribution \(F\), i.e., \(q = F(v)\). Denote the agent’s value as a function of quantile as \(v(q) = F^{-1}(q)\), and his bid as a function of quantile as \(b(q) = s(v(q))\). The outcome rule of the mechanism in quantile space is the pair \((x(q), p(q)) = (\tilde{x}(b(q)), \tilde{p}(b(q)))\).
Revenue curves and auction revenue. Myerson (1981) characterized Bayes-Nash equilibria and this characterization enables writing the revenue of a mechanism as a weighted sum of revenues of single-agent posted pricings. Formally, the revenue curve \( R(q) \) for a given value distribution specifies the revenue of the single-agent mechanism that serves an agent with value drawn from that distribution if and only if the agent’s quantile exceeds \( q \): \( R(q) = v(q)(1 - q) \). Myerson’s characterization of BNE then implies that the expected revenue of a mechanism at BNE from an agent facing an allocation rule \( x(q) \), notated \( P_x \), can be written as follows:

\[
P_x = R(0) x(0) + E_q [R(q) x'(q)] = R(1) x(1) - E_q [R'(q) x(q)]
\]

where \( x' \) and \( R' \) denote the derivative of \( x \) and \( R \) with respect to \( q \), respectively. For value distributions supported on \([0, 1]\), \( R(0) = R(1) = 0 \) and the constant terms in equation (2) are identically zero.

The expected revenue of an auction is the sum over the agents of its per-agent expected revenue; for auctions with symmetric equilibrium allocation rule \( x \) this revenue is \( n P_x \).

Position environments and rank-based auctions. A position environment expresses the feasibility constraint of the auction designer in terms of position weights \( w \) satisfying \( 1 \geq w_1 \geq w_2 \geq \cdots \geq w_n \geq 0 \). A position auction assigns agents (potentially randomly) to positions 1 through \( n \), and an agent assigned to position \( i \) gets allocated with probability \( w_i \). The rank-by-bid position auction orders the agents by their bids, with ties broken randomly, and assigns agent \( i \), with the \( i \)th largest bid, to position \( i \), with allocation probability \( w_i \). Multi-unit environments are a special case and are defined for \( k \) units as \( w_j = 1 \) for \( j \in \{1, \ldots, k\} \) and \( w_j = 0 \) for \( j \in \{k+1, \ldots, n\} \). The highest-\( k \)-bids-win multi-unit auction is the special case of the rank-by-bid position auction for the \( k \)-unit environment.

In our model with agent values drawn i.i.d. from a continuous distribution, rank-by-bid position auctions with either all-pay or first-price payment semantics have a unique Bayes-Nash equilibrium and this equilibrium is symmetric and efficient, i.e., in equilibrium, the agents’ bids and values are in the same order (Chawla and Hartline, 2013).

Rank-by-bid position auctions can be viewed as convex combinations of highest-bids-win multi-unit auctions. The marginal weights of a position environment are \( w' = (w'_{1}, \ldots, w'_{n}) \) with \( w'_{k} = w_{k} - w_{k+1} \). Define \( w'_0 = 1 - w_1 \) and note that the marginal weights \( w' \) can be interpreted as a probability distribution over \( \{0, \ldots, n\} \). As rank-by-bid position auctions are efficient, the rank-by-bid position auction with weights \( w \) has the exact same allocation rule as the mechanism that draws a number of units \( k \) from the distribution given by \( w' \) and runs the highest-\( k \)-bids-win auction.

Denote the highest-\( k \)-bids-win allocation rule as \( x_{k:n} \) and its per-agent revenue as \( P_k = P_{x_{k:n}} = E_q [-R'(q) x_{k:n}(q)] \). This allocation rule is precisely the probability an agent with quantile \( q \) has one of the highest \( k \) quantiles of \( n \) agents, or at most \( k - 1 \) of the \( n - 1 \) remaining agents have quantiles greater than \( q \). Formulaically,

\[
x_{k:n}(q) = \sum_{i=0}^{k-1} \binom{n-1}{i} q^{n-1-i}(1 - q)^i.
\]
Importantly, the allocation rule (in quantile space) of a rank-by-bid position auction does not depend on the distribution at all. The allocation rule $x$ of the rank-by-bid position auction with weights $w$ is:

$$x(q) = \sum_k w'_k x_{k:n}(q).$$

By revenue equivalence (Myerson, 1981), the per-agent revenue of the rank-by-bid position auction with weights $w$ is:

$$P_x = \sum_k w'_k P_k.$$ 

Of course, $P_0 = P_n = 0$ as always serving or never serving the agents gives zero revenue.

A rank-based auction is one where the probability that an agent is served is a function only of the rank of the agent’s bid among the other bids and not the magnitudes of the bids. Any rank-based auction induces a position environment where $\bar{w}_k$ denotes the probability that the agent with the $k$th ranked bid is served. This auction is equivalent to the rank-by-bid position auction with these weights $\bar{w}$. In a position environment with weights $w$, the following lemma characterizes the weights $\bar{w}$ that are induced by rank-based auctions.

**Lemma 2.1** (e.g., Devanur et al., 2013). There is a rank-based auction with induced position weights $\bar{w}$ for a position environment with weights $w$ if and only if their cumulative weights satisfy

$$\sum_{j=1}^k \bar{w}_j \leq \sum_{j=1}^k w_j$$

for all $k$.  

### 2.2 Inference

As we discussed in the introduction, the traditional structural inference in the auction settings is based on inferring distribution of values, which is unobserved but can be inferred from the distribution of bids, which is observed. Once the value distribution is inferred, other properties of the value distribution such as its corresponding revenue curve, which is fundamental for optimizing revenue, can be obtained. In this section we briefly overview this approach.

The key idea behind the inference of the value distribution from the bid distribution is that the strategy which maps values to bids is a best response, by equation (1), to the distribution of bids. As the distribution of bids is observed, and given suitable continuity assumptions, this best response function can be inverted.

We assume that the value distribution function $F(\cdot)$, the allocation rule $x(\cdot)$, and consequently also the quantile function of bid distribution $b(\cdot)$, are monotone, continuously differentiable, and invertible.

**Inference for first-price auctions** Consider a first-price rank-based auction with a symmetric bid function $b(q)$ and allocation rule $x(q)$ in BNE. To invert the bid function we solve for the bid that the agent with any value would make. Continuity of this bid function implies that its inverse is well defined. Applying this inverse to the bid distribution gives the value distribution.

The utility of an agent with quantile $q$ as a function of his bid $z$ is

$$u(q, z) = (v(q) - z) x(b^{-1}(z)).$$

(3)
Differentiating with respect to $z$ we get:

$$\frac{d}{dz} u(q, z) = -x(b^{-1}(z)) + (v(q) - z) x'(b^{-1}(z)) \frac{d}{dz} b^{-1}(z).$$

Here $x'$ is the derivative of $x$ with respect to the quantile $q$. Because $b(\cdot)$ is in BNE, the derivative $\frac{d}{dz} u(q, z)$ is 0 at $z = b(q)$. Rarranging, we obtain:

$$v(q) = b(q) + \frac{x(q)b'(q)}{x'(q)}$$

(4)

**Inference for all-pay auctions** We repeat the calculation above for rank-based all-pay auctions; the starting equation (3) is replaced with the analogous equation for all-pay auctions:

$$u(q, z) = v(q) x(b^{-1}(z)) - z.$$  

(5)

Differentiating with respect to $z$ we obtain:

$$\frac{d}{dz} u(q, z) = v(q) x'(b^{-1}(z)) \frac{d}{dz} b^{-1}(z) - 1,$$

Again the first-order condition of BNE implies that this expression is zero at $z = b(q)$; therefore,

$$v(q) = \frac{b'(q)}{x'(q)}.$$  

(6)

**Known and observed quantities** Recall that the functions $x(q)$ and $x'(q)$ are known precisely; these are determined by the rank-based auction definition. The functions $b(q)$ and $b'(q)$ are observed. The calculations above hold in the limit as the number of samples from the bid distribution goes to infinity, at which point these observations are precise.

Equations (4) and (6) enable the value function, or equivalently, the value distribution, to be estimated from the estimated bid function and an estimator for the derivative of the bid function, or equivalently, the density of the bid distribution. Estimation of densities is standard; however, it requires assumptions on the distribution, e.g., continuity, and the convergence rates in most cases will be slower. Our main results do not take this standard approach. Below we discuss errors in estimation of the bid function.

**3 Revenue estimator and error bounds for all-pay auctions**

We will now describe our estimator for the revenue of one rank-based auction using bids from another rank-based auction. There are two advantages of the restriction of our analysis to rank-based auctions. First, the allocation rule (in quantile space) of a rank-based auction is independent of the bid and value distribution; therefore, it is known and does not need to be estimated. Second, the allocation rules that result from rank-based auctions are well behaved, in particular their slopes are bounded, and our error analysis makes use of this property.

Recall from Section 2.1 that the revenue of any rank-based auction can be expressed as a linear combination of the multi-unit revenues $P_1, \ldots, P_n$ with $P_k$ equal to the per-agent revenue of the highest-$k$-bids-win auction. Therefore, in order to estimate the revenue of a rank-based auction, it suffices to estimate each $P_k$ accurately.
In Section 3.1 we derive the counterfactual revenue estimator. We state and discuss the error bounds of this estimator in Section 3.2. Two bounds are given; the first bound holds in worst case over counterfactual and incumbent mechanisms and the second bound depends on the closeness of the allocation rules of the counterfactual and incumbent mechanisms. The main ideas of the derivation of the error bounds are given in Section 6.

3.1 The revenue estimator

Consider estimating the revenue of an auction with allocation rule \( y \) from the bids of an all-pay rank-based auction. In terms of the revenue curve \( R(\cdot) \) or inverse demand function \( v(\cdot) \), the per-agent revenue of the allocation rule \( y \) is given by:

\[
P_y = E_q[y'(q) R(q)] = E_q[y'(q) v(q) (1 - q)].
\]

Let \( x \) denote the allocation rule of the auction that we run, and \( b \) denote the bid distribution in BNE of this auction. Recall that for an all-pay auction format, we can convert the bid distribution into the value distribution as follows: \( v(q) = b'(q)/x'(q) \). Substituting this equation into the expression for \( P_y \) above we get

\[
P_y = E_q[y'(q) (1 - q) \frac{b'(q)}{x'(q)}] = E_q[Z_y(q) b'(q)] \tag{7}
\]

where \( Z_y(q) = (1 - q) y'(q) x'(q) \).

To estimate \( P_y \) the analyst obtains \( N \) samples from the bid distribution. Each sample is the corresponding agent’s best response to the true bid distribution. We can estimate the quantile function of the equilibrium bid distribution \( b(q) \) as follows. Let \( \hat{b}_1, \ldots, \hat{b}_N \) denote the \( N \) samples drawn from the bid distribution. Sort the bids so that \( \hat{b}_1 \leq \hat{b}_2 \leq \cdots \leq \hat{b}_N \) and define the quantile function of the empirical bid distribution \( \hat{b}(\cdot) \) as

\[
\hat{b}(q) = \hat{b}_i \quad \forall i \in N, q \in [i - 1, i)/N \tag{8}
\]

We further observe that truncating the bid distribution at its extremes results in a tradeoff of the variance of the resulting estimator (which can diverge at the extreme quantiles of the bid distribution) with a bias (which is bounded). Accordingly, we obtain the following estimator.

**Definition 1.** The estimator \( \hat{P}_y \) (with truncation parameter \( \delta_N \)) for the revenue of an auction with allocation rule \( y \) from \( N \) samples \( \hat{b}_1 \leq \cdots \leq \hat{b}_N \) from the equilibrium bid distribution of an all-pay auction with allocation rule \( x \) is:

\[
\hat{P}_y = \sum_{i=\delta_N N}^{N-\delta_N N} \left( 1 - \frac{i}{N} \right) \frac{y'(\frac{i}{N})}{x'(\frac{i}{N})} \left( \hat{b}_{i+1} - \hat{b}_i \right).
\]

This estimator is obtained from (7) by integration by parts over range of the quantile function \( \hat{b}(\cdot) \) of the empirical bid distribution with its support truncated to \([\delta_N, 1 - \delta_N]\) and re-grouping of the terms in the resulting sum.

Our main theorems set the truncation parameter to a specific value \( \delta_N = \max(25 \log \log N, n)/N \) and show that, with no assumptions on the distribution of values or bids, the truncated estimator’s mean absolute error is bounded. Importantly, this truncated estimator does not have any parameters that need to be tuned to the distribution of bids or values.
We refer to the estimator with truncation parameter set to zero as the untruncated estimator. We study the untruncated estimator in simulations in Section 5 and demonstrate that, when \( Z_y(0) \) and \( Z_y(1) \) are small, it can be quite accurate with very few bid samples.

### 3.2 Error bounds

Our first main result of this section is the following error bound for the estimator of Definition 1.

**Theorem 3.1.** The mean absolute error in estimating the revenue of a rank-based auction with allocation rule \( y \) using \( N \) samples from the bid distribution for an all-pay rank-based auction with allocation rule \( x \) is bounded as below. Here \( n \) is the number of positions in the two auctions, and \( \hat{P}_y \) is the estimator in Definition 1 with \( \delta_N \) set to \( \max(25 \log \log N, n)/N \).

\[
E_b \left[ |\hat{P}_y - P_y| \right] \leq \frac{16n^2 \log N}{\sqrt{N}}.
\]

Observe that the above error bound is independent of the allocation rules \( x \) and \( y \). When \( x \) and \( y \) are similar to each other, our estimator should in fact achieve a much better error rate than the one above. For example, when \( x \) and \( y \) are identical, the error in estimation should have the same dependence on the number of samples as the statistical error in bids, namely \( 1/\sqrt{N} \). Our next theorem quantifies this relationship.

In order to capture the dependence of our error bounds in estimating \( P_y \) on the relationship between the incumbent allocation rule \( x \) and the counterfactual allocation rule \( y \), we define a new quantity, \( \Phi_{x,y} \), as follows:

\[
\Phi_{x,y} := \sup_q \{y'(q)\} \max \left\{ 1, \log \sup_{q:y'(q) \geq 1} \frac{x'(q)}{y'(q)}, \log \sup_q \frac{y'(q)}{x'(q)} \right\}.
\]

We then obtain the following theorem for the special case of estimating the multi-unit revenues.

**Theorem 3.2.** Let \( x \) and \( x_k \) denote the allocation rules for any all-pay rank-based auction and the \( k \)-highest-bids-win auction over \( n \) positions, respectively. Let \( \hat{P}_k \) denote the estimator from Definition 1 for estimating the revenue \( P_k \) of the latter auction from \( N \) samples of the bid distribution of the former, with \( \delta_N \) set to \( \max(25 \log \log N, n)/N \). If \( \delta_N \leq 1/n \), the mean absolute error of the estimator \( \hat{P}_k \) is bounded as follows.

\[
E_b \left[ |\hat{P}_k - P_k| \right] \leq \frac{80}{\sqrt{N}} \Phi_{x,x_k}.
\]

We obtain a slightly worse error bound when \( y \) is a general rank-based auction:

**Corollary 3.3.** Let \( x \) and \( y \) denote the allocation rules for any two all-pay rank-based auctions over \( n \) positions. Let \( \hat{P}_y \) denote the estimator from Definition 1 for estimating the revenue of the latter from \( N \) samples of the bid distribution of the former, with \( \delta_N \) set to \( \max(25 \log \log N, n)/N \). If \( \delta_N \leq 1/n \), the mean absolute error of the estimator \( \hat{P}_y \) is bounded as follows.

\[
E_b \left[ |\hat{P}_y - P_y| \right] \leq \frac{80}{\sqrt{N}} n \log \sup_q n \frac{y'(q)}{x'(q)}.
\]
We sketch the main ideas of Theorem 3.1 in Section 6. The full proof and the refinement necessary to obtain Theorem 3.2 and Corollary 3.3 are given in Appendix A.

From Theorem 3.2, the error in the estimator depends on the slopes of the allocation rules $x$ and $y$. The maximum slope of the multi-unit allocation rules, and therefore also that of any rank-based auction, is always bounded by $n$, the number of agents in the auction (summarized as Fact 3.4, below).

**Fact 3.4.** The maximum slope of the allocation rule $x$ of any $n$-agent rank-based auction is bounded by $n$: $\sup_{q} x'(q) \leq n$. More specifically, the maximum slope of the allocation rule $x_k$ for the $n$-agent highest-$k$-bids-win auction is bounded by

$$\sup_{q} x'_k(q) \in \left[ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \right] \frac{n - 1}{\sqrt{\min\{k - 1, n - k\}}} = \Theta \left( \frac{n}{\sqrt{\min\{k, n - k\}}} \right).$$

We evaluate the error bound given by Theorem 3.2 for a few special cases of $x$ and $x_k$. For simplicity in applying Fact 3.4, we assume $k < n/2$.

- When $x = x_k$, $\Phi_{x,x_k} \leq n/\sqrt{k}$ and we get an error bound of $80 \frac{n}{\sqrt{Nk}}$, which is the same (within a constant factor) as the statistical error in bids.

- The bound in the previous case degrades smoothly when $x$ and $x_k$ are close but not identical, as in $\epsilon x'_k \leq x' \leq x'_k/\epsilon$ for $\epsilon > 0$. We have $\Phi_{x,x_k} \leq \log(1/\epsilon) n/\sqrt{k}$ and the error bound is: $80 \log(1/\epsilon) \frac{n}{\sqrt{Nk}}$.

- Finally, as long as $x' \geq \epsilon x'_k$, that is, the highest-$k$-bids-win auction is mixed in with $\epsilon$ probability into $x$, we observe via Fact 3.4 that $\sup_{q} x'_k(q) \leq 1/\epsilon$ for all quantiles $q$, we obtain an error bound of $80 \log(n/\epsilon) \frac{n}{\sqrt{Nk}}$.

### 3.3 Applications to A/B testing

Consider the setup described in the introduction where an auction house running auction $A$ would like to determine the revenue of a novel mechanism $B$. The typical approach for doing so is to run the auction $B$ with some probability $\epsilon > 0$ and $A$ with the remaining probability. Ideally, if in doing so, the auction house obtains $\epsilon N$ bids in response to the auction $B$ out of a total of $N$ bids, the revenue of $B$ can be estimated within an error bound of

$$\Theta \left( \frac{1}{\sqrt{\epsilon}} \right) \frac{\sup_{q} x'_B(q)}{\sqrt{N}}$$

where $x_B$ denotes the allocation rule corresponding to $B$. We refer to this approach as ideal A/B testing.

In practice, however, instead of obtaining bids in equilibrium for auction $B$, the analyst obtains bids in equilibrium for the aggregate mechanism $C = (1 - \epsilon)A + \epsilon B$. We can then use Definition 1 to estimate the revenue of $B$. As a consequence of Corollary 3.3, and noting that $x'_B(q)/x'_C(q) \leq 1/\epsilon$ for all quantiles $q$, we obtain the following error bound.

**Corollary 3.5.** The revenue of a rank based mechanism $B$ can be estimated from $N$ bids of a rank-based mechanism $C = (1 - \epsilon)A + \epsilon B$ with absolute error bounded by

$$\frac{80 n \log(n/\epsilon)}{\sqrt{N}}.$$
Relative to the ideal situation described above, our error bound has a better dependence on $\epsilon$ and a worse dependence on $n$. Note that when $\epsilon$ is very small, our error bound of equation (11) may be smaller than the ideal bound of equation (10). In fact, we obtain a non-trivial bound on the error even when $\epsilon = 0$, per Theorem 3.1.

**Corollary 3.6.** The revenue of a rank based mechanism $B$ can be estimated from $N$ bids of any rank-based mechanism $C$ with absolute error bounded by

$$
\frac{16n^2 \log N}{\sqrt{N}}.
$$

(12)

This is not surprising: the ideal bound ignores information that we can learn about the revenue of $B$ from the $(1 - \epsilon)N$ bids obtained when $B$ is not run.

When $B$ is a multi-unit auction, we obtain a slightly better error bound using Theorem 3.2 which is closer to the ideal bound of equation (10).

**Corollary 3.7.** The revenue of the highest-$k$-bids-win mechanism $B$ can be estimated from $N$ bids of a rank-based mechanism $C = (1 - \epsilon)A + \epsilon B$ with absolute error bounded by

$$
\frac{80 \sup_q \{x_B'(q)\} \log(n/\epsilon)}{\sqrt{N}}.
$$

(13)

## 4 Applications to instrumented optimization

In this section we consider the problem of the principal who would like a mechanism that optimizes revenue for the current distribution of agent values while simultaneously enabling the inference necessary to reoptimize the mechanism in the future, should the distribution of values change. Recall, the optimal auctions of the classical theory pool agents with distinct values and are thus not well suited to counterfactual inference. In Section 4.1 we develop a theory for optimizing revenue over the class of all rank-based auctions that resembles Myerson’s theory for optimal auction design. Importantly, the optimal rank-based auction does not require knowledge of the full distribution, instead the multi-unit revenues $(P_1, \ldots, P_n)$ are sufficient; moreover, the previous developments of this paper enable the estimation of these multi-unit revenues. Where Myerson’s theory employs ironing by value and value reserves, our approach analogously employs ironing by rank and rank reserves. In Section 4.2 we extend the A/B-testing approach of Section 3 to develop a “universal B test mechanism” that can be used to estimate all of the multi-unit revenues $(P_1, \ldots, P_n)$ simultaneously. Combined with the optimal rank-based mechanism $A$, the A/B-test mechanism with this universal B test is simultaneously good for revenue and counterfactual inference. In Section 4.3 we take a more principled approach to optimization subject to inference, and solve for the revenue-optimal mechanism subject to the constraint that all multi-unit revenues can be estimated.

We begin by reviewing position environments and rank-based auctions. In a rank-based auction the allocation to an agent depends solely on the ordinal rank of his bid among other agents’ bids, and not on the cardinal value of the bid. For a position environment, a rank-based auction assigns agents (potentially randomly) to positions based on their ranks. Consider a position environment given by non-increasing weights $w = (w_1, \ldots, w_n)$. For notational convenience, define $w_{n+1} = 0$. Define the cumulative position weights $W = (W_1, \ldots, W_n)$ as $W_k = \sum_{j=1}^k w_j$, and $W_0 = 0$. We
can view the cumulative weights as defining a piece-wise linear, monotone, concave function given by connecting the point set \((0, W_0), \ldots, (n, W_n)\).

Multi-unit highest-bids-win auctions form a basis for position auctions. Consider the marginal position weights \(w' = (w'_1, \ldots, w'_n)\) defined by \(w'_k = w_k - w_{k+1}\). The allocation rule induced by the position auction with weights \(w\) is identical to the allocation rule induced by the convex combination of multi-unit auctions where the \(k\)-unit auction is run with probability \(w'_k\).

A randomized assignment of agents to positions based on their ranks induces an expected weight to which agents of each rank are assigned, e.g., \(\bar{w}_k\) for the \(k\)th ranked agent. These expected weights can be interpreted as a position auction environment themselves with weights \(\bar{w}\). As for the original weights, we can define the cumulative position weights \(\bar{W}\) as \(\bar{W}_k = \sum_{j=1}^{k} \bar{w}_j\). Lemma 2.1 characterizes the position weights \(\bar{w}\) that can be induced by any rank-based auction in a position environment \(w\) as those with cumulative weights upper bounded by thoses of the position environment, i.e., \(\bar{w}\) is feasible for \(w\) if and only if \(\bar{W}_k \leq W_k\) for all \(k\).

Any feasible weights \(\bar{w}\) can be constructed from \(w\) by a (random) sequence of the following two operations (cf. [Hardy et al. (1929)], and proof in Appendix E).

**rank reserve** For a given rank \(k\), all agents with ranks between \(k + 1\) and \(n\) are rejected. The resulting weights \(\bar{w}\) are equal to \(w\) except \(\bar{w}_k' = 0\) for \(k' > k\).

**iron by rank** Given ranks \(k' < k''\), the ironing-by-rank operation corresponds to, when agents are ranked, assigning the agents ranked in an interval \(\{k', \ldots, k''\}\) uniformly at random to these same positions. The ironed position weights \(\bar{w}\) are equal to \(w\) except the weights on the ironed interval of positions are averaged. The cumulative ironed position weights \(\bar{W}\) are equal to \(W\) (viewed as a concave function) except that a straight line connects \((k'-1, \bar{W}_{k'-1})\) to \((k'', \bar{W}_{k''})\). Notice that concavity of \(W\) (as a function) and this perspective of the ironing procedure as replacing an interval with a line segment connecting the endpoints of the interval implies that \(\bar{W} \leq W\) coordinate-wise, i.e., \(\bar{W}_k \leq W_k\) for all \(k\).

### 4.1 Optimal rank-based auctions

In this section we describe how to optimize for expected revenue over the class of rank-based auctions. Recall that rank-based auctions are linear combinations over \(k\)-unit auctions. The characterization of Bayes-Nash equilibrium, cf. equation (2), shows that revenue is a linear function of the allocation rule. Therefore, the revenue of a position auction can be calculated as the convex combination of the revenue \(P_k\) from the \(k\)-highest-bids-win auction for \(k \in \{0, \ldots, n\}\). Note that \(P_0 = P_n = 0\).

Given these multi-unit revenues, \(P = (P_0, \ldots, P_n)\), the problem of designing the optimal rank-based auction is well defined: given a position environment with weights \(w\), find the weights \(\bar{w}\) for a rank-based auction with cumulative weights \(\bar{W} \leq W\) maximizing the sum \(\sum_k (\bar{w}_k - \bar{w}_{k+1}) P_k\). This optimization problem is isomorphic to the theory of envy-free optimal pricing developed by [Devanur et al. (2015)]. We summarize this theory below; a complete derivation can be found in Appendix D.

Define the *multi-unit revenue curve* as the piece-wise linear function connecting the points \((0, P_0), \ldots, (n, P_n)\). This function may or may not be concave. Define the *ironed multi-unit revenues* as \(P = (P_0', \ldots, P_n')\) according to the smallest concave function that upper bounds the multi-unit revenue curve. Define the multi-unit marginal revenues, \(P' = (P'_1, \ldots, P'_n)\) and \(P' = (P'_0, \ldots, P'_n)\),
as the left slope of the multi-unit and ironed multi-unit revenue curves, respectively. I.e., \( P_k' = P_k - P_{k-1} \) and \( \bar{P}_k' = \bar{P}_k - \bar{P}_{k-1} \). The proof of the following theorem is given in the appendix.

**Theorem 4.1.** Given a position environment with weights \( w \), the revenue-optimal rank-based auction is defined by position weights \( \bar{w} \) that are equal to \( w \), except ironed on the same intervals as \( P \) is ironed to obtain \( \bar{P} \), and set to 0 at positions \( k \) for which \( \bar{P}_k' \) is negative.

As is evident from this description of the optimal rank-based auction, the only quantities that need to be ascertained to run this auction is the multi-unit revenue curve defined by \( P \). Therefore, an econometric analysis for optimizing rank-based auctions need not estimate the entire value distribution; estimation of the multi-unit revenues is sufficient.

**4.2 Universal B test**

In Section 3.3 we discussed how to estimate the revenue of a single auction \( B \) from the bids of the A/B test mechanism \( C \). Corollary 3.6 shows that an A/B test is not necessary as long as we have enough samples from the bid distribution: the revenue of \( B \) can be estimated from any incumbent mechanism \( A \) directly. In fact, we can estimate the revenue of all rank-based mechanisms simultaneously from the bids of a single mechanism \( A \). However, the error in estimation depends suboptimally on the number of samples, as \( \log(N)/\sqrt{N} \) rather than \( 1/\sqrt{N} \). A natural question is whether it is possible to estimate all rank-based revenues simultaneously at an optimal error rate from bids of a single incumbent auction. Precisely, we now consider the problem identifying a B test mechanism for which the revenue of any position auction \( D \) can be estimated from the equilibrium bids in the A/B test mechanism \( C \). Since the revenue of \( D \) is given by the convex combination of the multi-unit revenues \( P_k \), it suffices to estimate all of these multi-unit revenues. What properties should the auction \( B \) have in order to enable this estimation? (Equivalently, what properties should \( C \) have?)

**Definition 2.** A universal B test mechanism satisfies; for any rank-based auctions \( A \) and \( D \), any \( \epsilon > 0 \), and auction \( C \) defined by \( x_C = (1 - \epsilon)x_A + \epsilon x_B \); the revenue \( P_D \) can be estimated from \( N \) equilibrium bids of \( C \) with the dependence of the mean absolute error on \( N \) and \( \epsilon \) bounded by \( O(\log(1/\epsilon)/\sqrt{N}) \).

Since the revenue of \( D \) can be estimated from the revenue of all multi-unit auctions, Corollary 3.7 implies that it suffices to mix every multi-unit auction into \( C \) with some small probability. The uniform-stair mechanism (Definition 3 in Section 5), with position weights \( w_k = \frac{n-k}{n-1} \) for each \( k \), gives a mechanism \( B \) with such a mixture.

**Corollary 4.2.** The uniform-stair position auction is a universal B test mechanism with mean absolute error bounded by \( 80n \log(n/\epsilon)/\sqrt{N} \).

Next we observe that in fact we can get similar results by mixing in just a few of the multi-unit auctions. In particular, in order to estimate \( P_k \) accurately, it suffices to mix in a multi-unit auction with no more than \( k \) units, and another one with no less than \( k \) units. This gives us a more efficient universal B test for simultaneously inferring all of the multi-unit revenues (see Corollary 4.4).

**Lemma 4.3.** The revenue of the highest-\( k \)-bids-win mechanism \( B \) can be estimated from \( N \) bids of a rank-based all-pay auction \( C = (1 - 2\epsilon)A + \epsilon B_1 + \epsilon B_2 \) where \( A \) is an arbitrary rank-based
auction, and $B_1$ and $B_2$ are the highest-$k_1$-bids-win and highest-$k_2$-bids-win auctions respectively, with $k_1 \leq k \leq k_2$. The absolute error of the estimator is bounded by

$$\frac{80}{\sqrt{N}}(n + \log(1/\epsilon)) \sup_q \{x'_k(q)\}$$

Proof. We begin by noting that for any $j$ and $k$ with $k \leq j$,

$$\frac{x'_k(q)}{x'_j(q)} = \left(\frac{n-2}{n-1}\right) \left(\frac{q}{1-q}\right)^{j-k}.$$

When $k \leq j$ and $q \leq 1/2$, this ratio is less than $2^n$. Likewise, when $k \geq j$ and $q \geq 1/2$, the ratio is less than $2^n$. Therefore, for any $q$, and $C = (1 - 2\epsilon)A + \epsilon B_1 + \epsilon B_2$ where $B_1$ and $B_2$ are the highest-$k_1$-bids-win and highest-$k_2$-bids-win auctions respectively, with $k_1 \leq k \leq k_2$, we have

$$\sup_q \frac{x'_k(q)}{x'_C(q)} \leq \frac{2^n}{\epsilon}.$$

Next we note that $\sup_q x'_C(q) \leq n$ and, therefore, $\sup_q x'_k(q) \geq 1$ if $x'_C(q) \leq n$. Putting these quantities together with Theorem 3.2 we get that the absolute error in estimating $P_k$ from bids drawn from $C$ is at most

$$\mathbb{E}_b[|\hat{P}_k - P_k|] \leq \frac{80}{\sqrt{N}} \sup_q \{x'_k(q)\} (n + \log 1/\epsilon).$$

Corollary 4.4. The rank-by-bid position auction with weights $w_1 = 1$, $w_k = 1/2$ for $1 < k < n - 1$, and $w_n = 0$ is a universal B test mechanism with mean absolute error bounded by $O(n(n + \log(1/\epsilon))/\sqrt{N})$.

4.3 Optimal rank-based auctions with strict monotonicity

Position auctions, by definition, have non-increasing position weights $w$. The ironing in the iron-by-rank optimization of Section 4.1 converted the problem of optimizing multi-unit marginal revenue subject to non-increasing position weight, to a simpler problem of optimizing multi-unit marginal revenue without any constraints. In this section, we describe the optimization of rank-based auctions (i.e., ones for which position weights can be shifted only downwards or discarded) subject to strictly decreasing position weights. In particular, we can reinterpret the decreasing position weights of the universal B test mechanism from subsection 4.2 as such a strictness requirement. The optimal mechanism with this strictness requirement will satisfy the same inference guarantee proven for the A/B test while improving its revenue.

As described by Lemma 2.1 position weights $\bar{w}$ are feasible as a rank-based auction in the position environment $w$ if the cumulative position weights satisfy $W_k \geq \bar{W}_k$ for all $k$. Suppose we would like to optimize $\bar{w}$ for position weight $w$ subject to the monotonicity constraint that the difference in successive position weights is at least that of some other position weights $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$. Formally, $\bar{w}_k = \bar{w}_k - \bar{w}_{k+1} \geq \epsilon_k - \epsilon_{k+1} = \epsilon'_k$ for all $k$. For example, $\epsilon$ could be $\epsilon$ times the position weights of the universal B test mechanism of the preceding section. We call an
allocation rule satisfying these monotonicity constraints an $\epsilon$-strictly-monotone allocation rule. As non-trivial ironing by rank always results in consecutive positions with the same weight, i.e., $\bar{w}_k' = 0$ for some $k$, the optimal rank-based mechanism with strict monotonicity will require overlapping ironed intervals.

To our knowledge, performance optimization subject to a strict monotonicity constraint has not previously been considered in the literature. At a high level our approach is the following. We start with $w$ which induces the cumulative position weights $W$ which constrain the resulting position weights $\bar{w}$ of any feasible rank-based auction via its cumulative $\bar{W}$. We view $\bar{w}$ as the combination of two position auctions. The first has weakly monotone weights $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)$; the second has strictly monotone weights $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$; and the combination has weights $\bar{w}_k = \bar{y}_k + \epsilon_k$ for all $k$. The revenue of the combined position auction is the sum of the revenues of the two component position auctions. Since the second auction has fixed position weights, its revenue is fixed. Since the first position auction is weakly monotone and the second is strictly, the combined position auction is strictly monotone and satisfies the constraint that $\bar{w}_k' \geq \epsilon_k'$ for all $k$.

This construction focuses attention on optimization of $\bar{y}$ subject to the induced constraint imposed by $w$ and after the removal of the $\epsilon$-strictly-monotone allocation rule. I.e., $w$ must be feasible for $\bar{w}$. The suggested feasibility constraint for optimization of $\bar{y}$ is given by position weights $y$ defined as $y_k = w_k - \epsilon_k$. Notice that, in this definition of $y$, a lesser amount is subtracted from successive positions. Consequently, monotonicity of $w$ does not imply monotonicity of $y$.

To obtain $\bar{y}$ from $y$ we may need to iron for two reasons, (a) to make $y$ monotone and (b) to make the multi-unit revenue curve monotone. In fact, both of these ironings are good for revenue. The ironing construction for monotonizing $y$ constructs the concave hull of the cumulative position weights $Y$. This concave hull is strictly higher than the curve given by $Y$ (i.e., connecting $(0, Y_0), \ldots, (n, Y_n)$). Similarly the ironed multi-unit revenue curve given by $P$ is the concave hull of the multi-unit revenue curve given by $P$. The correct order in which to apply these ironing procedures is to first (a) iron the position weights $y$ to make it monotone, and second (b) iron the multi-unit revenue curve $P$ to make it concave. This order is important as the revenue of the position auction with weights $\bar{y}$ is only given by the ironed revenue curve $\bar{P}$ when the $\bar{y}' = 0$ on the ironed intervals of $\bar{P}$.

**Theorem 4.5.** The optimal $\epsilon$-strictly-monotone rank-based auction for position weights $w$ has position weights $\bar{w}$ constructed by

1. defining $y$ by $y_k = w_k - \epsilon_k$ for all $k$,
2. averaging position weights of $y$ on intervals where $y$ should be ironed to be monotone,
3. averaging the resulting position weights on intervals where $P$ should be ironed to be concave to get $\bar{y}$, and
4. setting $\bar{w}$ as $\bar{w}_k = \bar{y}_k + \epsilon_k$;

and is feasible for $w$ if $\epsilon$ is feasible for $w$.

**Proof.** The proof of this theorem follows directly by the construction and its correctness.

As described previously, the rank-based auction given by $\bar{w}$ in position environment given by $w$ can be implemented by a sequence of iron-by-rank and rank-reserve operations. Such a sequence of operations can be found, e.g., via an approach of Alaei et al. (2012) or Hardy et al. (1929).
The following proposition shows that this optimal $\epsilon$-strictly-monotone mechanism inherits the inference properties of the mechanism with position weights $\epsilon$, in particular, the A/B testing results of Corollary 3.5, Corollary 3.7, Corollary 4.2, and Corollary 4.4.

**Proposition 4.6.** For position weights $\epsilon$ defined as $\epsilon$ times the position weights of a B test mechanism, if position weights $\epsilon$ are feasible for $w$ then the optimal $\epsilon$-strictly-monotone rank-based auction for position weights $w$ has the same inference guarantee as the A/B test with $\epsilon$ probability of $B$.

5 Simulation evidence

In this section we present evidence from simulations that corroborate our theoretical analyses and provide further understanding of various methods for controlling estimation error. In the first subsection, we explore the dependence of the error of the estimator on the main parameters of our theoretical analyses: the number $n$ of agents in each auction and the number $N$ of samples the analyst obtains from the bid distribution. The second subsection considers ex ante methods for controlling estimation error, a.k.a., instrumentation. The two methods compared are A/B testing, where the counterfactual mechanism B is mixed in with the incumbent mechanisms, and the universal B-test where the universal B-test mechanism is mixed in with the incumbent mechanism. For both methods, the error is considered as a function of the amount $\epsilon$ by which the B-test auction is mixed with the incumbent auction. The third subsection contrasts ex post methods of controlling estimation error. We compare a standard approach of smoothing to estimate the empirical bid distribution that is plugged into the bidder’s first order condition versus our approach of truncating the contribution to the estimator from extreme quantiles. The above analyses are conducted under the assumption that agent values are drawn from a beta distribution; the final subsection demonstrates that the same qualitative results hold for a wide range of value distributions.

Our empirical analysis focuses on the following five $n$-agent position auctions (Figure 1).
- the low supply ([n/8]-unit) auction.
- the high supply ([7n/8]-unit) auction.
- the uniform-stair auction with allocation rule \( x(q) = q \) (see Definition 3 below).
- the 1-unit auction with allocation rule \( x(q) = q^{n-1} \).
- the \((n-1)\)-unit auction with allocation rule \( x(q) = 1 - (1-q)^{n-1} \).

We will be interested in estimating the revenue of one auction from the sample of \( N \) bids in another auction. The 1-unit and \([n/8]\)-unit auctions are extremal low-unit auctions. The \([7n/8]\)-unit and \((n-1)\)-unit auctions are extremal high-unit auctions. The uniform stair auction is a position auction with uniformly decreasing weights, equivalently, constant marginal weights.

**Definition 3.** The uniform-stair allocation rule is \( x(q) = q \); it is induced by the uniform-stair auction, an \( n \)-agent position auction defined by weights \( w = (w_1, \ldots, w_n) = (1, \frac{n-2}{n-1}, \ldots, \frac{1}{n-1}, 0) \).

Our empirical study allows benchmarking the error of our estimator against the counterfactual error, i.e., the estimation error had the incumbent mechanism been the counterfactual. With this benchmark, we see the loss (or gain) in accuracy of our approach relative to the straightforward statistical task of estimating the revenue of an auction from samples from the auction’s bid distribution. Over all of the studies we ran, we did not observe mean absolute error of our estimator to exceed the counterfactual error by more than a factor of 10.

**Methodology**

We perform simulations to calculate the mean absolute deviation of our estimator \( \hat{P}_B \) for the revenue of auction \( B \) with the auction’s expected revenue \( P_B \). The allocation rules \( x_B \) and \( x_C \), their derivatives \( x'_B \) and \( x'_C \), and the revenue curve \( R \) are calculated analytically. The expected revenue \( P_B \) is calculated from the revenue curve \( R \) and \( x'_B \) by equation (2) via numerical integration (i.e., by averaging the values of \( R(q) x'_B(q) \) on a grid). The equilibrium bid distribution in auction \( C \) for values on a uniform grid are calculated from equation (6) via numerical integration on a grid. Each simulation draws \( N \) bids from this bid distribution, the estimated revenue \( \hat{P}_B \) is calculated from Definition 1, and the mean absolute deviation is calculated by averaging \(|P_B - \hat{P}_B|\) over 8000 Monte Carlo simulations.

**5.1 Empirical Evidence versus Theoretical Bound**

In this section we compare and contrast empirical evidence with the theoretical bound by exploring the dependence of the empirical error on the \( N \), the number of samples of the analyst, and \( n \), the number of agents in each auction. Recall that theoretical dependence on \( N \) is \( \Theta(\sqrt{1/N}) \) and on \( n \) is \( O(n \log n) \) in the worst-case bound of Corollary 3.5.

We consider three cases for the counterfactual auction \( B \) as the uniform-stair, the low-supply auction, and the high-supply auction. We allow auction \( A \) to be any of these auctions and also the 1-unit and \((n-1)\)-unit auctions. We fix the mixing probability of \( \epsilon = 0.001 \) and the incumbent mechanism \( C \) is \((1-\epsilon)A + \epsilon B \). (A subsequent study will explore the role of truncation.) The results of these studies are depicted the figures below. Not shown here, when mixing in the counterfactual auction with a large probability \( \epsilon \), there is little benefit from truncation.
In Figure 2, we observe that as a function of the number of samples \( N \) the error is indeed the optimal \( \Theta(\sqrt{1/N}) \) rate. Specifically, we observe that the error in the estimation of the revenue of a counterfactual auction indeed depends on the incumbent mechanism as a constant times \( \sqrt{1/N} \) and this constant is different for different incumbent mechanisms. Moreover, unlike the result obtained by our theoretical bound, this constant is always much less than 1. Moreover, this limit behavior is already achieved with \( N \approx 1000 \) bids in the sample.

In Figure 3, we observe that, within the range where our bound holds (which requires \( N > n^2 \)), the dependence on the number of agents \( n \) is at most slowly increasing and far from the \( O(n \log n) \) worst-case bound. Considering the whole range, we see that the dependence varies, and thus precise theoretical analysis may be difficult. An important consideration in our choice of counterfactual auctions is that the per-agent revenue is roughly constant in the number of agents \( n \); thus relative changes in revenue are not confounding our empirical analysis of the error.

We observe in these empirical results that the error is not increasing at the same rate as our bounds suggest. The theoretical bounds are symmetric with respect to swapping the high supply and low supply auction. The empirical results are all better than these bounds:

- \textbf{uniform-stair incumbent; low-supply counterfactual}: \( O(\sqrt{n} \log(1/\epsilon)) \)
- \textbf{high-supply incumbent; low-supply counterfactual}: \( O(\sqrt{n} \log(n/\epsilon)) \)
- \textbf{low-supply incumbent; uniform-stair counterfactual}: \( O(\log n) \).

In the above bounds, we could swap the low-supply auction for the 1-unit auction and the high-supply auction for the 1-unit auction and the bounds by replacing the \( \sqrt{n} \) term with an \( n \) term.

One perhaps unexpected outcome that is present in these empirical results is the large error for small \( n \) and similar incumbent and counterfactual mechanisms, specifically, the low-supply counterfactual with the 1-unit incumbent or the high-supply counterfactual with the \((n - 1)\)-unit incumbent. These auctions have allocation rules that are near zero for low values, near 1 for high values, and at some point in between transitioning from 0 to 1 (see Figure 1). For small \( n \) these
transitions overlap and this results in higher error. For large \( n \) these transitions do not overlap and the error is small.

As a final note, in Figure 3c the error is trending upwards with large \( n \) even when the counterfactual and incumbent are identical. This trend is from truncation which is increasing with \( n \) (relative to the fixed sample size \( N \)) at these parameter settings.

## 5.2 Ex Ante Error Control (A.k.a., Instrumentation)

A main focus of this paper is identifying properties of auctions that make them good for inference. Specifically, our A/B-testing method suggests that better estimates of the revenue of mechanism B are possible by running mechanism C that mixes in B with mechanism A. See Figure 4. These empirical result should be compared with Theorem 3.2 which gives the dependence on \( \epsilon \) as \( O(\log 1/\epsilon) \) when \( N > 1/\epsilon \), i.e., to the left of the vertical dashed line. (When \( N > 1/\epsilon \) the theoretical bound of Theorem 3.1 with term \( \log N/\sqrt{N} \) becomes the better bound.) In the relevant region we observe that the dependence on \( 1/\epsilon \) is sub-logarithmic except when the counterfactual mechanism is the uniform-stair auction where the dependence is \( \Theta(\log 1/\epsilon) \).

As described in Section 4.2, there is a universal B-test mechanism which is a mixture of the 1-unit auction and the \((n-1)\)-unit auction. Mixing this auction with any other position auction makes it possible to infer the revenue of that position auction. See Figure 4. Comparing the A/B-test with a universal B-test empirically we see that there is not much improvement from the A/B-test. As described in preceding sections, the benefit of the universal B-test is that instrumentation with it makes it possible to infer the performance of any other position auction.

## 5.3 Ex Post Error Control

The classical econometric approach to estimation in auctions is to use a consistent estimator for the distribution of values of bidders and then to estimate revenue in a counterfactual auction from this distribution. To obtain a consistent estimator for the distribution of values, the derivative of the
Figure 4: This figure depicts the error as a function of the A/B-test mixing probability $\epsilon$ (fixed parameters $N = 10000$, $n = 16$, $F = \text{Beta}(2, 2)$). The counterfactual auction $B$ is fixed, and the error is depicted as a function of $\epsilon$. The left column is the A/B test where the counterfactual is mixed in with probability $\epsilon$. The right column is the universal B-test where a mix of the 1-unit and $(n - 1)$-unit auction is mixed in with probability $\epsilon$. The vertical line corresponds to $\epsilon = 1/N$. 

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Figure 5: This figure depicts on a log-log scale the error as a function of the number of samples $N$ for several choices of the number of agents $n$ (fixed parameters: $F = \text{Beta}(2, 2)$). The counterfactual auction B is the high supply auction, the incumbent auction C is the low supply auction (the incumbent does not mix in B). The trivial error bound of 1 is depicted with a solid line. The thick line in each figure shows the error when the counterfactual and the incumbent are the same, a.k.a., the counterfactual error.

bid function needs to be estimated and error of this estimator is typically controlled by smoothing, i.e., averaging bids with adjacent bids in the sorted order. In contrast, the estimators of this paper do not employ smoothing of the bid distribution, and instead errors are controlled by truncation, i.e., zeroing out the contribution to the estimated revenue from potentially-ill-behaved extremal quantiles of the bid distribution.

This section compares these approaches to controlling error and makes several empirical findings. First, we observe that ex post methods for error control are necessary in some scenarios. This observation comes from comparing the truncated estimator described above with the same estimator with no truncation. We see in Figure 5 that when the mechanisms are extreme and opposite that our estimator with truncation has low error while without truncation the error is generally worse than the trivial bound.

As we have discussed previously, when the B-test probability $\epsilon$ in an A/B-test is large, truncation has limited benefit. Indeed, Figure 4 shows error as a function of $\epsilon$ for the truncated estimator. In fact, the same plots result from the untruncated estimator. Nonetheless, when we consider very small $\epsilon$, the truncated estimator gives a non-trivial error, while the untruncated estimator does not. This comparison is depicted in Figure 5. The constant-in-$\epsilon$ bound on the truncated estimator is guaranteed by Theorem 3.1.

The truncation we use zeros out the contribution to the estimator from extreme quantiles. The truncation parameter does not depend on fundamentals of the environment and instead was selected to integrate with theoretical guarantees from statistics. To show that this statistically-motivated choice is good, we empirically evaluate the extent to which other truncation parameters give better

---

7We have assumed values to be bounded on $[0, 1]$; thus, the per-agent revenue is at most 1 and error bounds that exceed 1 are trivial.
Figure 6: This figure depicts the error as a function of the A/B-test mixing probability $\epsilon$ (fixed parameters $N = 10000$, $n = 16$, $F = \text{Beta}(2, 2)$). The counterfactual auction B is fixed, and the error is depicted as a function of $\epsilon$ when the data is drawn from the bid distribution of the A/B-test mechanism C that corresponds to several incumbent mechanisms A. The thick line in each figure shows the error when the counterfactual and the incumbent are the same, a.k.a., the counterfactual error. Note that the $y$-axes in the two figures are different.

We consider a counterfactual of the $(n-1)$-unit auction, an incumbent of the 1-unit auction, and various small numbers of agents $n$ (Figure 7a). We see that for three selected auction sizes our truncation has at most four times the error of the optimal truncation; moreover, for a broad range of sample size $N$ our truncation is at most 50% worse than the optimal truncation.

We compare truncation to the classical approach for controlling error in auctions which is smoothing the bid distribution. We consider a natural approach smoothing, namely averaging for each bid the $k$ adjacent bids in the sorted order. The classical approach, which asks for a uniform bound on the error in estimates of values to plug into the revenue estimator, would tune $k$ depending on properties of the bid distribution (which is endogenous to the environment). Here we show that with the plug-in estimator the optimal smoothing is no smoothing. Thus, estimation of the bid distribution via smoothing used by the classical approach of controlling error is unhelpful for estimating revenue. See Figure 7b. Meanwhile, as we have seen, truncation both controls error and does not require tuning to endogenous properties of the bid distribution.

### 5.4 Distribution Robustness

We have experimented with a number of distributions over values and the qualitative results observed above continue to hold. Here we repeat the study of truncation with value distributions intended to stress the estimation procedure. We observe that there are no significant changes (Figure 8). The distributions considered are the equal revenue distribution on interval $[0.1,1]$, the uniform distribution on interval $[0.3,1]$ and a bimodal distribution. For example our interest in the bimodal distribution is that the low-supply auction and high-supply auctions have revenue driven from different modes. We were unable to identify any distribution that resulted in significantly different outcomes from what we observed for the beta distribution.
Figure 7: On the left, the ratio of our selected truncation parameter is compared to the optimal truncation parameter and the ratio of the errors is reported. Ratios closer to 1 show that our truncation parameter is nearly optimal. On the right, the error as a function of the number of adjacent bids that are smoothed in the classical approach to controlling value estimation error in auctions. It is optimal not to smooth when using value estimates with the plug-in estimator for revenue. On the right, fixed parameters are $N = 1000$, $n = 5$, $F = \text{Beta}(2, 2)$. The counterfactual auction $B$ is the $(n - 1)$-unit auction, the incumbent auction $A$ is the 1-unit auction (the incumbent does not mix in $B$).

Figure 8: This figure depicts on a log-log scale the error as a function of the number of samples $N$ (fixed parameters $n = 16$, $F = \text{Beta}(2, 2)$). The counterfactual auction $B$ is the high supply auction, the incumbent auction $A$ is the low supply auction (the incumbent does not mix in $B$). The trivial error bound of 1 is depicted with a solid line. The thick line in each figure shows the error when the counterfactual and the incumbent are the same, a.k.a., the counterfactual error.
6 Derivation of error bounds (Proof of Theorems 3.1 and 3.2)

We will now derive the error bounds stated in Theorems 3.1 and 3.2 for the revenue estimator \( \hat{P} \).

Since the rate of estimation of bid density is generally worse than that of the bid distribution, we will first express the revenue estimator directly in terms of the empirical bid distribution. Written in this manner, the estimator will turn out to be a weighted order statistic of the empirical bids. We will then use standard bounds on the error in estimating bids to bound the error in estimating revenue. We begin by describing the standard statistical error bounds we use in our analysis.

6.1 Statistical Model and Methods

Our framework for counterfactual auction revenue analysis is based on directly using the distribution of bids for inference. The main error in estimation of the bid distribution is the sampling error due to drawing only a finite number of samples from the bid distribution. Evaluation of the auction revenue requires the knowledge of the quantile function of bid distribution. While estimation of empirical distributions is standard, quantile functions can be significantly more difficult to estimate especially if the distribution density can approach zero on its support since the distribution function is non-invertible at those points. As we show further, estimation of the counterfactual auction revenues requires the knowledge of the density-weighted quantile function which can be robustly estimated despite the potential non-invertibility of the distribution function. In this subsection, we overview the uniform absolute error bound of the density-weighted quantile function of the bid distribution of a multi-unit auction based on the results in Cs"org"o (1983).

Definition 4. For function \( b(\cdot) \), estimator \( \hat{b}(\cdot) \) and the weighting function \( \omega(\cdot) \geq 0 \), the weighted uniform mean absolute error is defined as

\[
E_{\hat{b}} \left[ \sup_q \omega(q) \left| b(q) - \hat{b}(q) \right| \right].
\]

The main object that will arise in our subsequent analyses will be the weighted quantile function of the bid distribution where the weights are determined by the allocation rules of the auctions under consideration, e.g., \( E_q[\omega(q) b(q)] \) for some quantile weighting function given by \( \omega(\cdot)^8 \). The important insight is that while the estimation of the quantile function of the bid distribution \( b(\cdot) \) maybe problematic around the points where the density of the bid distribution is close to zero, the estimation of the density-weighted quantile function is a lot more robust. As we will show further, estimation of auction revenues involves such a density-weighted form of the quantile function. Our error bounds are based on the uniform convergence of quantile processes and weighted quantile processes in Csorgo and Revesz (1978), Cs"org"o (1983), and Cheng and Parzen (1997). For the quantile weighting function \( \omega(q) = 1/b'(q) \), i.e., the inverse derivative of the bid function, the \( \sqrt{N} \)-normalized mean absolute error is bounded by a universal constant.

Lemma 6.1. Suppose that \( b \) and \( b' \) exist on \((0, 1)\) and \( \sup_{q \in (0, 1)} q(1 - q)b'(q) < \infty \). Then the density-weighted uniform mean absolute error of the empirical quantile function \( \hat{b}(\cdot) \) on \( q \in [\delta_N, 1 - \delta_N] \) with \( \delta_N = \frac{25 \log \log N}{N} \) is bounded almost surely as

\[
E_{\hat{b}} \left[ \sup_{q \in [\delta_N, 1 - \delta_N]} \left| \sqrt{N} b'(q)^{-1} (b(q) - \hat{b}(q)) \right| \right] < 1 + 16 \frac{\log \log N}{\sqrt{N}} \sup_q q(1 - q) b'(q).
\]

8Estimators of these functions, i.e., replacing the bid distribution with the empirical bid distribution, are called L-statistics in the statistics literature.
This result is a consequence of statement (3.2.3) in Theorem 3.2.1 in Csörgő (1983). For all-pay auctions by equation (6), the term $\sup_q \{q(1-q)b'(q)\}$ is bounded by $\frac{1}{4} \sup_q \{x'(q)\}$. For first-price auctions by equation (4), it is bounded by $\sup_q \{(1-q)x'(q)/x(q)\}$.

6.2 Alternate formulation of the revenue estimator

This section expresses the revenue estimator $\hat{P}$ directly as a function of the empirical bids. This alternate formulation is based on the same integration by parts technique that was used to obtain the estimator for the revenue, but without the subsequent grouping of the terms that resulted in the weighted sum of differences $\hat{b}_{i+1} - \hat{b}_i$ in the form of the estimator.

Recall from Section 3.1 that we have

$$P_y = \mathbb{E}_q \left[ y'(q)(1-q)\frac{b'(q)}{x'(q)} \right] = \mathbb{E}_q \left[ Z_y(q) b'(q) \right]$$

(14)

where $Z_y(q) = (1-q)\frac{y'(q)}{x'(q)}$. Treating this expectation as an integral and integrating it by parts, when the constant terms are zero, gives:

$$P_y = \mathbb{E}_q \left[ -Z_y'(q) b(q) \right].$$

(15)

The subsequent analysis will include consideration of the constant terms when they are not zero.

This analysis gives two ways to write counterfactual revenue. Equation (7) writes the revenue as linear in the derivative of the bid function while equation (15) writes it as linear in the bid function. We will define our estimator in terms of former for extreme quantiles and in terms of the latter for moderate quantiles. The reason for this definition is that the latter gives a simple and well behaved estimator in terms of the bid function, but might diverge at the extremes while the former at the extremes introduces only modest bias when approximated by zero.

**Lemma 6.2.** The per-agent counterfactual revenue of a rank-based auction with allocation rule $y$ can be expressed in terms of the bid function $b$ of an all-pay mechanism $x$ as:

$$P_y = \frac{\mathbb{E}_{q \not\in \Lambda} \left[ -Z_y'(q) b(q) \right]}{\mathbb{E}_{q \in \Lambda} \left[ Z_y(q) b'(q) \right]} + Z_y(1 - \delta_N) b(1 - \delta_N) - Z_y(\delta_N) b(\delta_N) + Z_y(1 - \delta_N) b(1).$$

(16)

where $Z_y(q) = (1-q)\frac{y'(q)}{x'(q)}$, extreme quantiles are $\Lambda = [0, \delta_N] \cup [1 - \delta_N, 1]$, and the truncation parameter is $\delta_N \in [0, 1/2]$. For bid functions that are constant on the extreme quantiles, the counterfactual revenue can be written as

$$P_y = \mathbb{E}_{q \not\in \Lambda} \left[ -Z_y'(q) b(q) \right] + Z_y(1 - \delta_N) b(1).$$

(17)

In this latter case or when $\delta_N = 0$, the expressed revenue is linear in the bid function.

---

Both $Z_y(q) = (1-q)\frac{y'(q)}{x'(q)}$ and $Z_y'(q)$ can be infinite at the boundary $q \in \{0, 1\}$ when $x$ and $y$ are polynomials of different degrees.
Proof. The first part of the lemma follows from plugging the all-pay inference equation (6) into the revenue equation (2) and integrating by parts on moderate quantiles $[\delta_N, 1 - \delta_N]$. The second part of the lemma simplifies the first part using $b'(q) = 0$ for extremal $q \in \Lambda$, $b(0) = b(\delta_N) = 0$, and $b(1 - \delta_N) = b(1)$.

This formulation allows the estimation of $P_y$ directly as a weighted order statistic of the observed bids, with $b(\cdot)$ replaced by the estimated bid distribution $\hat{b}(\cdot)$. Lemma 6.1 tells us that, except at the extreme quantiles, the estimated bid distribution $\hat{b}(\cdot)$ closely approximates $b(\cdot)$. At the extreme quantiles there is a bias-variance tradeoff. The variance from including the contribution to the revenue from these quantiles in the estimator can greatly exceed the bias from excluding them entirely. Thus, to prevent the larger error at the extreme quantiles from degrading the accuracy of the estimator, these estimated bids are rounded down to zero and up to the maximum observed bid at the low and high extremes, respectively. Recall that the estimated bid distribution $\hat{b}(\cdot)$ is defined, in equation (8), as a piecewise constant function with $N$ pieces. Thus, the estimator $\hat{P}_y = E_q[-Z'_y(q) \hat{b}(q)]$ can be simplified as expressed in the following definition.

Definition 5. The estimator $\hat{P}_y$ (with truncation parameter $\delta_N$) for the revenue of an auction with allocation rule $y$ from $N$ samples $\hat{b}_1 \leq \cdots \leq \hat{b}_N$ from the equilibrium bid distribution of an all-pay auction with allocation rule $x$ can be rewritten as:

$$
\hat{P}_y = \sum_{i=\delta_N N}^{N-\delta_N N} \left[ \left(1 - \frac{i-1}{N}\right) \frac{y'(i/N)}{x'(i/N)} - \left(1 - \frac{i}{N}\right) \frac{y'(i/N)}{x'(i/N)} \right] \hat{b}_i + \delta_N \frac{y'(1-\delta_N)}{x'(1-\delta_N)} \hat{b}_N.
$$

This alternate formulation is in fact numerically identical to the original definition of the revenue estimator, Definition 1, but uses a different grouping of the terms in the sum. From a computational perspective, Definition 1 turns out to be better behaved as the differences $Z'_y((i-1)/N) - Z'_y(i/N)$ can diverge for large $N$. But the alternate Definition 5 is better suited to analyzing the statistical error, as the error in bids is better behaved than the error in bid derivatives.

6.3 Derivation of error bounds

We will now give the main ideas behind the proof Theorem 3.1. This analysis is non-trivial because the estimator, which is a weighted order statistic, is based on weights with magnitudes that can be exponentially large. Importantly $Z_k(q) = (1 - q)y'(q)/x'(q)$ and, while the numerator $y'(q)$ is bounded by Fact 3.4 by $n$ (the number of agents), the denominator can be exponentially small. Thus, both $Z_k(q)$ and its derivative $Z'_k(q)$ can be exponentially big, specifically $N^c n$ for absolute constant $c \in (0, 1)$ (see Example 1 below). The revenue estimator is a weighted order statistic with weights proportional to $Z'_k$ and thus straightforward analyses will not give good bounds on the error. We begin with one such analysis that gives an error bound that is linear in the maximum of $Z_k(q)$ and modify it to reduce the dependence on this term to be logarithmic. For non-extremal quantiles $q$, $Z_k(q)$ is bounded by $N^c n$ and thus $\log Z_k(q)$ is at most the $n \log N$ term that appears in the error bound of Theorem 3.1.

Example 1. The allocation rules and derivatives for the $k = 1$ unit auction and the $k = n - 1$ unit auction are:

$$
x_{n-1}(q) = 1 - (1 - q)^{n-1}; \quad x'_n(q) = (n - 1)(1 - q)^{n-2}.
x_1(q) = q^{n-1}; \quad x'_1(q) = (n - 1)q^{n-2}.
$$
Consider the estimator for the revenue of the $(n-1)$-unit auction from bids in the one-unit auction, i.e., $y = x_{n-1}$ and $x = x_1$, at the lower extreme quantile $q = \log \log N/N$ and with number of samples $N \gg n$. We get $Z_k(q) = (1-q)y'(q)/x'(q) = (1-q)^{n-1} q^{2-n} \approx q^{2-n}$ and $Z'_k(q) \approx (2-n)q^{1-n}$; thus the magnitudes of both $Z_k(q)$ and $Z'_k(q)$ at quantile $q = \log \log N/N$ are on the order of $[N(\log \log N)]^n$ which is upper and lower bounded by $N^c n$ for appropriate absolute constants $c$. Recall that terms from extreme quantiles below $\delta_N = O(\log \log N/N)$ are rounded down to zero in the estimator; thus, this upper bound is tight for the subsequent analysis.

The remainder of this section sketches the main ideas in deriving the above logarithmic bound on the error. The full proof of Theorem 3.1 as well as the more detailed analysis that gives Theorem 3.2 is given in Appendix A. Assume that the counterfactual auction $y$ is the highest-$k$-bids-win auction for some $k$; denote the allocation rule of this auction by $x_k$, and let $Z_k = Z_{x_k}$. We will prove Theorem 3.1 for this special case. Then, by virtue of the fact that $P_y$ is a weighted average of the constituent $P_k$’s, the theorem trivially extends to all rank-based auctions $y$.

As in the statement of the theorem, let $\delta_N = \max(25 \log \log N, n)/N$, and let $\Lambda = [0, \delta_N] \cup [1 - \delta_N, 1]$ denote the set of extreme quantiles. Apply equation (16) and equation (17) from Theorem 6.2 to the true bid function and truncated empirical bid function to write the counterfactual revenue and the estimated revenue, respectively, as:

$$P_k = \mathbb{E}_{q \notin \Lambda}[-Z'_k(q) b(q)] + \mathbb{E}_{q \in \Lambda}[Z_k(q) b'(q)] + Z_k(1 - \delta_N) b(1 - \delta_N) - Z_k(\delta_N) b(\delta_N).$$

$$\hat{P}_k = \mathbb{E}_{q \notin \Lambda}[-Z'_k(q) \hat{b}(q)] + Z_k(1 - \delta_N) \hat{b}_N.$$

The mean absolute error is bounded by the expected value of the absolute value of the difference in these two quantities:

$$|\hat{P}_k - P_k| \leq |\mathbb{E}_{q \notin \Lambda}[-Z'_k(q)(\hat{b}(q) - b(q))]| + |\mathbb{E}_{q \in \Lambda}[Z_k(q) b'(q)]|$$

$$+ |Z_k(1 - \delta_N)(b(1 - \delta_N) - \hat{b}_N)| + |Z_k(\delta_N) b(\delta_N)|$$

(18)

There are now two steps to the analysis. The first step is an analysis of the contribution to the error from moderate quantiles, i.e., the first term in equation (18). We will sketch this step below. The second step is analysis of the contribution to the error from extreme quantiles, i.e., the remaining terms in equation (18). In Appendix A we show that the error from these terms is dominated by the error in the first term. As a summary of this deferred analysis, e.g., for the final term $Z_k(\delta_N) b(\delta_N)$, the denominator of $Z_k(q)$ can be very small, but it is approximately proportional to $b(q)$ in the numerator and can be canceled. The other terms are similarly bounded.

The following straightforward analysis gives a bound on the error in the estimator from moderate quantiles that is linear in $\sup_{q \notin \Lambda} Z_k(q)$. Specifically, the expected error is bounded as:

$$|\mathbb{E}_{q \notin \Lambda}[Z'_k(q)(\hat{b}(q) - b(q))]| \leq \mathbb{E}_{q \notin \Lambda}[|Z'_k(q)|] \cdot \sup_{q \notin \Lambda} |\hat{b}(q) - b(q)|.$$ 

For the second term in this expression, Lemma 6.3 provides a uniform bound on the absolute error in bids $|\hat{b}(q) - b(q)|$. For the first term, the following lemma shows that $Z_k$ is single-peaked and, thus, $\mathbb{E}_{q \notin \Lambda}[|Z'_k(q)|] \leq 2 \sup_{q \notin \Lambda} Z_k(q)$. The proof of this lemma, which is formally given in Appendix A, follows from the fact that $x$ is a convex combination of multi-unit auctions and that the ratio of the derivatives of the allocation rules of two multi-unit auctions is single-peaked.
Lemma 6.3. For any rank-based auction and \( k \)-highest-bids-win auction with allocation rules \( x \) and \( x_k \), respectively, the function \( Z_k(q) = \left(1 - q\right) \frac{x_k(q)}{x(q)} \) achieves a single local maximum for \( q \in [0, 1] \).

As described in Example 1, \( \sup_{q \not\in \Lambda} Z_k(q) \) can be very large. In order to obtain a better bound, we observe that the error in bids is large precisely at quantiles where \( Z_k \) is small and vice versa: \( Z_k \) depends inversely on the slope of the allocation rule of the incumbent auction, \( x' \), whereas, the error in bids is directly proportional to the bid density \( b' \), which in turn is proportional to \( x' \). We utilize this observation as follows:

\[
\left| E_{q \not\in \Lambda} \left[ Z_k'(q) \right] \right| \leq E_{q \not\in \Lambda} \left[ \left| \frac{Z_k'(q)}{Z_k(q)} \right| Z_k(q) \left( \hat{b}(q) - b(q) \right) \right] \leq E_{q \not\in \Lambda} \left[ \sup_{q} \left| \frac{Z_k'(q)}{Z_k(q)} \right| \right] \sup_{q} \left| Z_k(q) \left( \hat{b}(q) - b(q) \right) \right| .
\]

(19)

As the integral of \( Z_k'(q)/Z_k(q) \) is \( \log Z_k(q) \), this analysis and the single-peaked-ness of \( Z_k(q) \) gives an error bound that is logarithmic instead of linear in \( \sup_{q \not\in \Lambda} Z_k(q) \). The following lemma, formally proved in Appendix A, summarizes the bound on the error from moderate quantiles.

Lemma 6.4. For \( Z_k \) and \( \Lambda \) defined as above, the first error term in equation (18) of the estimator \( \hat{P}_k \) is bounded by:

\[
E_b \left[ E_{q \not\in \Lambda} \left[ Z_k'(q) \left( \hat{b}(q) - b(q) \right) \right] \right] \leq \frac{8n \log N}{\sqrt{N}} \sup_{q \not\in \Lambda} \{ x_k'(q) \}.
\]

This lemma combines with analyses of the contribution to the error of extremal quantiles to give Theorem 3.1. The refined bound of Theorem 3.1 comes from improved factoring of the error term over that of equation 19 in Lemma 6.4.

7 Inference for social welfare

We now consider the problem of estimating the social welfare of a rank-based auction using bids from another rank-based all-pay auction. Consider a rank-based auction with induced position weights \( w \). By definition, the expected per-agent social welfare obtained by this auction is as below, where \( V_k \) is the expected value of the \( k \)th highest value agent, or the \( k \)th order statistic of the value distribution.

\[
S_w = \frac{1}{n} \sum_{k=1}^{n} w_k V_k.
\]

We note that the value order statistics, \( V_k \), are closely related to the expected revenues of the multi-unit auctions. The \( k \)-unit second-price auction serves the top \( k \) agents with probability 1, and charges each agent the \( k + 1 \)th highest value. Its expected revenue is therefore \( n P_k = k V_{k+1} \). We therefore obtain:

\[
S_w = w_1 V_1 + \sum_{k=1}^{n-1} w_{k+1} \frac{P_k}{k}.
\]
The methodology developed in the previous sections can be used to estimate the $P_k$’s in the above expression. The first order statistic of the values, $V_1$, cannot be directly estimated in this manner. Notate the expected value of an agent as

$$V = \mathbb{E}_q[v(q)] = \frac{1}{n} \sum_{k=1}^{n} V_k.$$ 

Therefore, we can calculate the social welfare of the position auction with weights $w$ as

$$S_w = w_1 V - \sum_{k=2}^{n} \left( w_1 - w_k \right) \frac{V_k}{n} = w_1 V - \sum_{k=1}^{n-1} \left( w_1 - w_{k+1} \right) \frac{P_k}{k}.$$ (20)

We now argue that $V$ can be estimated at a good rate from the bids of another rank-based all-pay auction. Let $x$ denote the allocation rule of the auction that we run, and $b$ denote the bid distribution in BNE of this auction. Then we note that

$$V = \mathbb{E}_q[v(q)] = \mathbb{E}_q \left[ \frac{b'(q)}{x'(q)} \right] = \mathbb{E}_q \left[ \bar{Z}(q)b'(q) \right]$$

where $\bar{Z}(q) = \frac{1}{x'(q)}$. We might now try to directly apply Theorems 3.1 or 3.2 to bound the error in our estimate of $V$. This does not immediately work, as Lemma 6.3 fails to hold for $\bar{Z}$. Instead, we observe that since $x'(q)$ is a degree $n - 1$ polynomial and has fewer than $n$ local minima, therefore $\bar{Z}$ has fewer than $n$ local maxima. We can therefore adapt the arguments for the aforementioned theorems to obtain the following lemma:

**Lemma 7.1.** The mean absolute error in estimating the expected value $V$ using $N$ samples from the bid distribution for an all-pay rank-based auction with allocation rule $x$ is bounded as given by the two expressions below. Here $n$ is the number of positions in the position auction.

$$\mathbb{E}_b \left[ |\hat{V} - V| \right] \leq \frac{8n^2 \log N}{\sqrt{N}}$$

$$\mathbb{E}_b \left[ |\hat{V} - V| \right] \leq \frac{40n}{\sqrt{N}} \max \left\{ 1, \log \sup_{q \notin \Lambda} x'(q), \log \sup_{q \notin \Lambda} \frac{1}{x'(q)} \right\}$$

As an example application of Lemma 7.1, we adapt Corollary 4.2 to bound the error from estimating the social welfare of any position auction using bids from another position auction that is mixed with the uniform-stair auction. Recall that the uniform-stair auction is a universal B test. Using the universal B test of Corollary 4.4 instead of the uniform-stair auction gives a slightly worse error bound, because the slope of the allocation rule for that auction can be as small as $N^{-O(n)}$.

Other revenue estimation results can be similarly adapted to estimate social welfare.

**Theorem 7.2.** For any rank-based auction $A$; uniform-stair auction $B$ with position weights $w_k = \frac{n-k}{n}$ for each $k \in [1, n]$; and all-pay rank-based auction $C$ with $x_C = (1 - \epsilon)x_A + \epsilon x_B$; the mean absolute error for estimating the social welfare of any rank-based auction $D$ from $N$ samples from the bid distribution of $C$ is bounded by:

$$O \left( \frac{n \log n \log(n/\epsilon)}{\sqrt{N}} \right) = O \left( \frac{n \log n \log(n/\epsilon)}{\sqrt{N}} \right).$$
The theorem follows by combining Lemma 7.1 with equation (20) and Corollary 4.2. The first term follows from Lemma 7.1 by noting that the uniform-stair auction satisfies $x'(q) = 1$ for all $q$. The second term follows from the error bounds on $P_k$ given by Corollary 4.2; The extra factor of $\log n$ (relative to the statement of the corollary) arises from the fact that the total weight of the multipliers for the terms in equation (20) can be as large as $\sum_{k=1}^{n} 1/k \approx \log n$.

References

Alaei, S., Fu, H., Haghpanah, N., Hartline, J., and Malekian, A. (2012). Bayesian optimal auctions via multi- to single-agent reduction. In ACM Conference on Electronic Commerce.

Athey, S. and Ellison, G. (2011). Position auctions with consumer search. The Quarterly Journal of Economics, 126:1213–1270.

Athey, S. and Haile, P. A. (2007). Nonparametric approaches to auctions. Handbook of econometrics, 6:3847–3965.

Auer, P., Cesa-Bianchi, N., Freund, Y., and Schapire, R. E. (2002). The nonstochastic multiarmed bandit problem. SIAM Journal on Computing, 32(1):48–77.

Baliga, S. and Vohra, R. (2003). Market research and market design. Advances in Theoretical Economics, 3(1).

Blum, A. and Hartline, J. D. (2005). Near-optimal online auctions. In Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms, pages 1156–1163. Society for Industrial and Applied Mathematics.

Brown, J. and Morgan, J. (2009). How much is a dollar worth? tipping versus equilibrium coexistence on competing online auction sites. Journal of Political Economy, 117(4):668–700.

Bulow, J. and Klemperer, P. (1996). Auctions vs negotiations. American Economic Review, 86(1):180–194.

Cesa-Bianchi, N., Gentile, C., and Mansour, Y. (2015). Regret minimization for reserve prices in second-price auctions. IEEE Transactions on Information Theory, 61(1):549.

Chawla, S., Hartline, J., and Nekipelov, D. (2014). Mechanism design for data science. In Proceedings of the fifteenth ACM conference on Economics and computation, pages 711–712. ACM.

Chawla, S., Hartline, J., and Nekipelov, D. (2016). A/b testing of auctions. In Proceedings of the 2016 ACM Conference on Economics and Computation, pages 19–20. ACM.

Chawla, S. and Hartline, J. D. (2013). Auctions with unique equilibria. In Proceedings of the Fourteenth ACM Conference on Electronic Commerce, EC ’13, pages 181–196, New York, NY, USA. ACM.

Cheng, C. and Parzen, E. (1997). Unified estimators of smooth quantile and quantile density functions. Journal of statistical planning and inference, 59(2):291–307.
Coey, D., Larsen, B., and Sweeney, K. (2014). The bidder exclusion effect. Technical report, National Bureau of Economic Research.

Cole, R. and Roughgarden, T. (2014). The sample complexity of revenue maximization. In Proceedings of the 46th Annual ACM Symposium on Theory of Computing, pages 243–252. ACM.

Csörgő, M. (1983). Quantile processes with statistical applications. SIAM.

Csorgo, M. and Revesz, P. (1978). Strong approximations of the quantile process. The Annals of Statistics, pages 882–894.

Devanur, N., Hartline, J., and Yan, Q. (2015). Envy freedom and prior-free mechanism design. Journal of Economic Theory, 156:103–143.

Devanur, N. R., Ha, B. Q., and Hartline, J. D. (2013). Prior-free auctions for budgeted agents. In ACM Conference on Electronic Commerce, pages 287–304.

Dughmi, S., Roughgarden, T., and Sundararajan, M. (2012). Revenue submodularity. Theory of Computing, 8:95–119.

Edelman, B., Ostrovsky, M., and Schwarz, M. (2007). Internet advertising and the generalized second-price auction: Selling billions of dollars worth of keywords. The American Economic Review, 97(1):242–259.

Fain, D. and Pedersen, J. (2006). Sponsored search: A brief history. Bulletin of the American Society for Information Science and Technology, 32(2):12–13.

Fu, H., Haghpanah, N., Hartline, J., and Kleinberg, R. (2014). Optimal auctions for correlated buyers with sampling. In Proceedings of the fifteenth ACM conference on Economics and computation, pages 23–36. ACM.

Goldberg, A. V., Hartline, J. D., Karlin, A. R., Saks, M., and Wright, A. (2006). Competitive auctions. Games and Economic Behavior, 55(2):242–269.

Guerre, E., Perrigne, I., and Vuong, Q. (2000). Optimal nonparametric estimation of first-price auctions. Econometrica, 68(3):525–574.

Hardy, G., Littlewood, J., and Pólya, G. (1929). Some simple inequalities satisfied by convex functions. Messenger of Math, 58:145–152.

Hartline, J. and Taggart, S. (2019). Sample complexity for non-truthful mechanisms. In Proceedings of the 2019 ACM Conference on Economics and Computation, pages 399–416.

Jackson, M. O. and Sonnenschein, H. F. (2007). Overcoming incentive constraints by linking decisions 1. Econometrica, 75(1):241–257.

Kleinberg, R. and Leighton, T. (2003). The value of knowing a demand curve: Bounds on regret for online posted-price auctions. In Foundations of Computer Science, 2003. Proceedings. 44th Annual IEEE Symposium on, pages 594–605. IEEE.

Kohavi, R., Longbotham, R., Sommerfield, D., and Henne, R. M. (2009). Controlled experiments on the web: survey and practical guide. Data mining and knowledge discovery, 18(1):140–181.
A  Proofs for Section 3

In this section we prove the results from Section 3 which analyze the error of the counterfactual revenue estimator for both multi-unit and (more generally) rank-based auctions with all-pay payment semantics.

Recall that for all-pay auctions with allocation rule \( x(q) \), the equilibrium bid function \( b(q) \) satisfies \( b'(q) = v(q) x'(q) \). From \( N \) bids in a mechanism with allocation rule \( x \) we are estimating the counterfactual revenue of a mechanism with allocation rule \( y \). Recall that for an implicit allocation rule \( x \) and another allocation rule \( y \), we define the function \( Z_y(q) = (1 - q)^{y'(q)} x'(q) \). When \( y \) is the allocation rule corresponding to a \( k \)-unit auction, we let \( Z_k(q) \) denote \( Z_{x_k}(q) \). Our analysis treats the contribution to the error from extreme quantiles \( q \in \Lambda = [0, \delta_N] \cup [1 - \delta_N, 1] \) for \( \delta_N = \max(25 \log \log N, n)/N \) and moderate quantiles \( q \not\in \Lambda \) separately. In equation (18), restated below, the first term is the error from moderate quantiles and the latter three terms is the error from extremal quantiles.

\[
|\hat{P}_k - P_k| \leq |E_{q \not\in \Lambda}[-Z'_k(q)(\hat{b}(q) - b(q))]| + |E_{q \in \Lambda}[Z_k(q) b'(q)]| + |Z_k(1 - \delta_N)(b(1 - \delta_N) - \hat{b}_N)| + |Z_k(\delta_N) b(\delta_N)|
\]  

The proofs in this appendix are organized as follows. The error in our estimator for the revenue \( P_k \) of a \( k \)-unit auction from moderate quantiles is analyzed in Section A.1. Section A.2 proves some basic properties of allocation rules and bid functions for rank-based auctions that will be employed in Section A.3 where the error from extremal quantiles, specifically the three latter terms of equation (18), are analyzed. The main results from Section 3.2, namely Theorems 3.1 and 3.2 and Corollary 3.3 are proven in Section A.4.
A.1 Bounding the error from moderate quantiles

We will now restate and prove Lemmas 6.3 and 6.4 bounding the contribution to the error of the estimator from moderate quantiles, $E_{q \notin \Lambda} \left[ |Z_k(q) (\hat{b}(q) - b(q))| \right]$. The first lemma proves that $Z_k$ has a single local maximum.

**Lemma 6.3** For any rank-based auction and $k$-highest-bids-win auction with allocation rules $x$ and $x_k$, respectively, the function $Z_k(q) = (1 - q) x_k'(q) / x_k(q)$ achieves a single local maximum for $q \in [0, 1]$. 

**Proof.** Consider the function $A(q) = 1/Z_k(q) = x'(q) / (1 - q) x'_k(q)$. Recall that $x'(q)$ is a weighted sum over $x'_j(q)$ for $j \in \{1, \cdots, n - 1\}$. Thus, $A(q)$ is a weighted sum over terms $x'_j(q)/(1 - q)x'_k(q)$. Let us look at these terms closely.

$$
\frac{x'_j(q)}{(1 - q)x'_k(q)} = \alpha_{k,j} q^{j-k} (1 - q)^{j-k-1}
$$

where coefficient $\alpha_{k,j}$ is a constant. The functions $q^{j-k} (1 - q)^{j-k-1}$ are convex. This implies that $A(q)$ which is a weighted sum of convex functions is also convex. Consequently, it has a unique minimum. Therefore, $Z_k(q) = 1/A(q)$ has a unique maximum. \hfill \square

The following lemma gives the basic analysis of the error from moderate quantiles. A key aspect of this proof is that its dependence on $\sup_{q \notin \Lambda} Z_k(q)$ is logarithmic. Immediately following this proof we give a more refined analysis that enables better bounds when estimating the revenue of counterfactual mechanism $y$ from bids in $x$ when the allocation rules of $x$ and $y$ are related.

**Lemma 6.4** For $Z_k$ and $\Lambda$ defined as above, the first error term in equation (18) of the estimator $\hat{P}_k$ is bounded by:

$$
E_k \left[ E_{q \notin \Lambda} \left[ Z'_k(q) (\hat{b}(q) - b(q)) \right] \right] \leq \frac{8n \log N}{\sqrt{N}} \sup_q \{ x'_k(q) \}
$$

**Proof.** Recall from Section 6 that we can write the error on the moderate quantiles as:

$$
E_{q \notin \Lambda} \left[ Z'_k(q) (\hat{b}(q) - b(q)) \right] \leq E_{q \notin \Lambda} \left[ Z'_k(q) / Z_k(q) \right] \sup_q Z_k(q) (\hat{b}(q) - b(q)). \quad (19)
$$

Using Lemma 6.3, the first term on the right in equation (19), $E_{q \notin \Lambda} \left[ Z'_k(q) / Z_k(q) \right]$, is bounded by $2(\sup_{q \notin \Lambda} \log Z_k(q) - \inf_{q \notin \Lambda} \log Z_k(q))$. 

We note that for $q \notin \Lambda$, and any rank-based allocation rule $y$, $y'(q) \in (\delta_N^n, n]$. Therefore, $Z_k(q) \in [\delta_N^n / n, n\delta_N^n] \in (N^{-n}, N^n)$. Therefore, we have:

$$
E_{q \notin \Lambda} \left[ Z'_k(q) / Z_k(q) \right] < 4 \log N^n = 4n \log N.
$$

To bound the second term on the right in equation (19), we write:

$$
\sup_q Z_k(q) (\hat{b}(q) - b(q)) \leq \sup_q x'_k(q) \sup_q \frac{1}{x'(q)} (\hat{b}(q) - b(q)) 
\leq \sup_q x'_k(q) \sup_q \frac{1}{y'(q)} (\hat{b}(q) - b(q)).
$$

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Invoking Lemma 6.1, the expected value of this term for random samples from the bid distribution is bounded as:

\[ E_b \left[ \sup_q \left| Z_k(q) \left( \hat{b}(q) - b(q) \right) \right| \right] \leq \sup_q x'_k(q) \frac{1}{\sqrt{N}} \left( 1 + \frac{4n \log \log N}{\sqrt{N}} \right). \]

Putting the two bounds together, we get,

\[ E_b \left[ E_q \xi \left[ Z'_k(q) \left( \hat{b}(q) - b(q) \right) \right] \right] \leq \frac{4n \log N}{\sqrt{N}} \sup_q \{x'_k(q)\} \left( 1 + \frac{4n \log \log N}{\sqrt{N}} \right). \]

We may assume without loss of generality that \( 4n \log N < \sqrt{N} \), otherwise the first term, and therefore the entire error bound, exceeds 1 and is trivially true. Under this assumption, the term in brackets is no more than 2, and the lemma follows.

The following lemma gives a refinement of Lemma 6.4 that enables better bounds when estimating the revenue of counterfactual mechanism \( y \) from bids in \( x \) when the allocation rules of \( x \) and \( y \) are related.

Unfortunately, \( E_q \xi \left[ Z'_k(q) \left( \hat{b}(q) - b(q) \right) \right] \) can be quite large, as \( Z_k(q) \) can take on exponentially large values at extreme quantiles (see Example 6 in Section 6). The main idea in the refined analysis is a better factoring in the error from moderate quantiles in equation (19). We instead factor this error term as follows, for an appropriate function \( h(Z_k) \) which is just slightly sublinear in \( Z_k \).

\[ \left| E_q \xi \left[ Z'_k(q) \left( \hat{b}(q) - b(q) \right) \right] \right| \leq E_q \xi \left[ Z'_k(q) \left( \hat{b}(q) - b(q) \right) \right] \sup_q h(Z_k) \left( \hat{b}(q) - b(q) \right). \]

This factoring gives greater control in balancing the error generated from the two terms. For an appropriate choice of the function \( h(\cdot) \), we obtain the following lemma.

**Lemma A.1.** For \( Z_k \) and \( \Lambda \) defined as above, the first error term in equation (18) of the estimator \( \hat{P}_k \) is bounded by:

\[ E_b \left[ E_q \xi \left[ Z'_k(q) \left( \hat{b}(q) - b(q) \right) \right] \right] \leq \frac{40}{\sqrt{N}} \left( 1 + \frac{4n \log \log N}{\sqrt{N}} \right) \sup_q \{x'_k(q)\} \max \left\{ 1, \log \sup_q x'_k(q), \log \sup_q x'_k(q) \right\}. \]

**Proof.** For any \( \alpha > 0 \) we can write

\[ |\hat{P}_k - P_k| \leq E \left[ \frac{(\log(1 + Z_k(q)))^\alpha}{Z_k(q)} |Z'_k(q)| \right] \sup_q \left| \frac{Z_k(q)}{(\log(1 + Z_k(q)))^\alpha} (\hat{b}(q) - b(q)) \right|. \]

We start by considering the first term. Lemma 6.3 shows that \( Z'_k(\cdot) \) changes sign only once. Consider the region where the sign of \( Z'_k(\cdot) \) is constant and make the change of variable \( t = Z_k(q) \). Denote \( Z_k^* = \sup_q Z_k(q) \), and note that \( \inf_q Z_k(q) \geq 0 \). The first term evaluates as

\[ E \left[ \frac{(\log(1 + Z_k(q)))^\alpha}{Z_k(q)} |Z'_k(q)| \right] \leq 2 \int_0^{Z_k^*} \frac{(\log(1 + t))^\alpha}{t} dt. \]
Note that for any $t > 0$, $\log(1 + t) \leq t$. Thus,

$$\int_0^\delta \frac{(\log(1 + t))^\alpha}{t} dt < \frac{\delta^\alpha}{\alpha}.$$ 

Now split the integral into two pieces as

$$\int_0^{Z_k^*} \frac{(\log(1 + t))^\alpha}{t} dt = \int_0^1 \frac{(\log(1 + t))^\alpha}{t} dt + \int_1^{Z_k^*} \frac{(\log(1 + t))^\alpha}{t} dt.$$ 

We just proved that the first piece is at most $1/\alpha$. Now we upper bound the second piece and consider the integrand at $t \geq 1$. First, note that

$$(\log(1 + t))^\alpha = \left(\log t + \log(1 + \frac{1}{t})\right)^\alpha \leq \left(\log t + \frac{1}{t}\right)^\alpha \leq (\log t + 1)^\alpha.$$ 

Thus, the integral behaves as

$$\int_1^{Z_k^*} \frac{(\log(1 + t))^\alpha}{t} dt \leq \int_1^{Z_k^*} \frac{(\log t + 1)^\alpha}{t} dt = \frac{1}{1 + \alpha} (\log Z_k^* + 1)^{1 + \alpha}.$$ 

Thus, we just showed that

$$\mathbb{E} \left[ \frac{(\log(1 + Z_k(q)))^\alpha}{Z_k(q)} | Z'_k(q) \right] \leq \frac{2}{\alpha} + \frac{2}{1 + \alpha} (\log Z_k^* + 1)^{1 + \alpha},$$

which is at most $2(1 + e)/\alpha$ for $\alpha < 1/\log Z_k^*$. 

Now consider the term

$$\sup_q \left| \frac{Z_k(q)}{(\log(1 + Z_k(q)))^\alpha} (\hat{b}(q) - b(q)) \right|.$$ 

Note that $\log(1 + t) \geq \min\{1, t\}/2$. So the first term can be bounded from above as

$$\frac{Z_k(q)}{(\log(1 + Z_k(q)))^\alpha} \leq 2^\alpha \max \left\{ Z_k(q), (Z_k(q))^{1 - \alpha} \right\}.$$ 

Thus using Lemma 6.1

$$\mathbb{E} \left[ \sup_q \left| \frac{Z_k(q)}{(\log(1 + Z_k(q)))^\alpha} (\hat{b}(q) - b(q)) \right| \right] \leq \sup_q \left| \frac{Z_k(q)}{(\log(1 + Z_k(q)))^\alpha} b'(q) \right| \mathbb{E} \left[ \sup_q \left| \frac{\hat{b}(q) - b(q)}{b'(q)} \right| \right]$$ 

$$\leq 2^\alpha \sup_q \left( \max \left\{ x'_k(q), (x'_k(q))^{1 - \alpha} \right\} \right) \left( 1 + \frac{16 \log \log N}{\sqrt{N}} \sup_q (1 - q) b'(q) \right)$$ 

$$\leq 2^\alpha \sup_q \left( x'_k(q) \right) \left( \max \left( 1, \sup_{q:x'_k(q) \geq 1} \frac{x'(q)}{x_k(q)} \right) \right) \frac{1}{\sqrt{N}} \left( 1 + \frac{4n \log \log N}{\sqrt{N}} \right).$$
where the last inequality follows by noting that $b'(q) \leq x'(q) \leq n$, and $q(1 - q) \leq 1/4$.

Now we combine the two evaluations together and pick $\alpha = \min\{1, 1/\log A, 1/\log Z^*\}$, with $A$ defined as above, to obtain

$$
\mathbb{E}[|\hat{P}_k - P_k|] \leq \frac{2(1 + e)}{\alpha} 2^\alpha A \alpha \sqrt{\frac{1}{N}} \sup_q (x'_k(q)) \leq \frac{40}{\sqrt{N}} \sup_q (x'_k(q)) \max \left\{ 1, \log A, \log \sup_q \left\{ \frac{x'_k(q)}{x'(q)} \right\} \right\} \left( 1 + \frac{4n \log \log N}{\sqrt{N}} \right).
$$

A.2 Bounds for the allocation rules and bid distributions of rank-based auctions

In this section we prove some basic properties of allocation rules for rank-based auctions. These properties will be useful, in Section A.3, for analyzing the error of the estimator at extreme quantiles. As described in Section 2, the allocation rule and its derivative for the $n$-agent $k$-unit auction are

$$
x_k(q) = \sum_{i=0}^{k-1} \binom{n-1}{i} q^{n-i-1}(1 - q)^i,
$$

$$
x'_k(q) = (n-1) \binom{n-2}{k-1} q^{n-k-1}(1 - q)^{k-1}.
$$

We will be interested in the behavior of allocation rule $x_k$ and its derivative $x'_k$ at the extremes, specifically for $q \in [0, 1/n]$ and $q \in [1 - 1/n, 1]$. The allocation rule is steepest at $q = (k-1)/(n-2)$ and is convex before this point and concave after it. Specifically, $x_1$ is steepest at $q = 1$ and is convex and $x_{n-1}$ is steepest at $q = 0$ and is concave. For all other $k \in \{2, \ldots, n-2\}$, the allocation rule derivative $x'_k$ is maximized between $1/(n-2) > 1/n$ and $(n-3)/(n-2) < 1 - 1/n$.

The following two lemmas bound the derivative of the allocation of multi-unit auctions at extreme quantiles. Combining them we obtain the subsequent theorem.

Lemma A.2. For $k \in \{2, n-2\}$ units and $\delta < 1/n$, the allocation rule derivative $x'_k$ satisfies:

1. $\sup_q (x'_k(q)) = x'_k(\delta)$ and
2. $\sup_q (x'_k(q)) = x'_k(1 - \delta)$.

Proof. This lemma follows from convexity of the allocation rule $x_k$ on $[0, 1/n]$ and concavity on $[1 - 1/n, 1]$. \qed

Lemma A.3. For $k \in \{1, n-1\}$ units and $\delta < 1/n$, the allocation rule derivative $x'_k$ satisfies:

1. $\sup_q (x'_{n-1}(q)) \leq e \ x'_{n-1}(\delta)$ and
2. $\sup_q (x'_{1}(q)) \leq e \ x'_{1}(1 - \delta)$.
Lemma A.7. For any $b$ and $x_{n-1}(q) = (n-1)(1-q)^{n-2}$. Thus, $x'_1(1) = x'_{n-1}(0) = n-1$ and

$$x'_1(1-\delta) = x'_{n-1}(\delta) = (n-1) (1-\delta)^{n-2} \geq \frac{1}{e} (n-1) \quad \Rightarrow \quad \frac{1}{e} x'_1(1) = \frac{1}{e} x'_{n-1}(0).$$

Concavity of $x_{n-1}$ and convexity of $x_1$, then, imply the result. 

**Theorem A.4.** For any $n$-agent rank-based mechanism with allocation rule $x$ and $\delta < 1/n$, the allocation rule derivative $x'$ satisfies:

1. $\sup_{q<\delta} x'(q) \leq e x'(\delta)$ and
2. $\sup_{q>1-\delta} x'(q) \leq e x'(1-\delta)$.

The bid function $b(\cdot)$ can be bounded by the allocation rule $x(\cdot)$ and its derivative $x'(\cdot)$ via the following lemma. The subsequent theorem follows from the lemma via Theorem A.4.

**Lemma A.5.** For any all-pay mechanism with allocation rule $x$ and $\delta \in [0,1]$, the equilibrium bid function $b$ satisfies

1. $b'(\delta) \leq x'(\delta)$,
2. $b(\delta) \leq \delta \sup_{q<\delta} x'(\delta)$, and
3. $b(1) - b(1-\delta) \leq \delta \sup_{q>1-\delta} x'(\delta)$.

**Proof.** The equilibrium bid function is defined by $b'(q) = v(q)x'(q)$ and $b(0) = 0$ (where $v(q) \in [0,1]$ is the value function). Part (1) follows from the upper bound $v(q) \leq 1$. Parts (2) and (3) follow by upper bounding $x'(q)$ by its supremum on the interval of the integral and integrating the bound of part (1). For example for part (2), $b(\delta) = \int_0^\delta v(r) x'(r) dr \leq \int_0^\delta \sup_{q<\delta} x'(q) dr = \delta \sup_{q<\delta} x'(q)$. □

**Theorem A.6.** For any $n$-agent all-pay rank-based mechanism with allocation rule $x$ and $\delta < 1/n$, the equilibrium bid function $b$ satisfies

1. $b(\delta) \leq \delta e x'(\delta)$, and
2. $b(1) - b(1-\delta) \leq \delta e x'(1-\delta)$.

**A.3 Bounding the error at extreme quantiles**

We now bound the remaining terms in equation 18. Once again these bounds rely on the observation that for any quantile $q$, $Z_k(q)b(q)$ is bounded, because $Z_k$ depends inversely on $x'(q)$, whereas $b(q)$ is roughly proportional to it.

**Lemma A.7.** For $Z_y$ and $\Lambda$ as defined above, if $\delta_N \leq 1/n$, the second error term of the estimator $\hat{P}_y$ is bounded as follows.

$$E_{q \in \Lambda} [Z_y(q) b'(q)] \leq e \delta_N y'(\delta_N) + e \delta_N^2 y'(1-\delta_N).$$
Proof. Apply part (1) of Lemma A.5 and the definition of \( Z_y(q) = (1 - q) y'(q)/x'(q) \) to obtain the following upper bound:

\[
E_{q \in \Lambda}[Z_y(q) b'(q)] \leq E_{q \in \Lambda}[(1 - q) y'(q)].
\]

For \( q < \delta_N \), bound this expectation by \( e \delta_N y'(\delta_N) \) from Theorem A.4. For \( q > 1 - \delta_N \), bound this expectation by \( e \delta_N y'(1 - \delta_N) \).

Note, we could alternatively obtain the bound \( \delta_N n \) by using the fact that \( \sup_q y'(q) \leq n \) (Fact 3.4).

**Lemma A.8.** For \( Z_y \) and \( \Lambda \) as defined above, if \( \delta_N \leq 1/n \), the third error term of the estimator \( \hat{P}_y \) is bounded as follows.

\[
E_b[|Z_y(1 - \delta_N)(b(1 - \delta_N) - \hat{b}_N)|] \leq \delta_N y'(1 - \delta_N)(e \delta_N + \frac{\delta}{N}).
\]

**Proof.** Let \( \hat{q} \) be the quantile of the highest of the \( N \) observed bids, i.e., \( b(\hat{q}) = \hat{b}_N \).

Conditioned on \( \hat{q} > 1 - \delta_N \), bid \( \hat{b}_N = b(\hat{q}) \) is upper bounded by \( b(1) \). Applying Theorem A.6 to bound \( b(1) - b(1 - \delta_N) \) gives conditional error bound of

\[
Z_y(1 - \delta_N)(b(1) - b(1 - \delta_N)) \leq e \delta_N y'(1 - \delta_N).
\]

Now condition on \( \hat{q} < 1 - \delta_N \). For this conditioning, Lemma A.5 shows that \( b(1 - \delta_N) - b(\hat{q}) \leq x(1 - \delta_N) - x(\hat{q}) \). We will now bound \( E_q[x(1 - \delta_N) - x(\hat{q}) | \hat{q} < 1 - \delta_N] \Pr[\hat{q} \leq 1 - \delta_N] \) which is at most \( E_q[1 - x(\hat{q})] \).

We first analyze \( E_q[1 - x(\hat{q})] \) in the case that \( x = x_k \) is the allocation rule for the \( k \)-unit auction. We have,

\[
E_q[1 - x(\hat{q})] = \int_0^1 (1 - x_k(q))N q^{N-1} dq
\]

\[
= N \int_0^1 q^{N-1} \left( \sum_{i=k}^{n-1} \binom{n-1}{i} q^{n-1-i}(1-q)^i \right) dq
\]

\[
= N \sum_{i=k}^{n-1} \binom{n-1}{i} \int_0^1 q^{N+n-2-i}(1-q)^i dq
\]

\[
= N \sum_{i=k}^{n-1} \binom{n-1}{i} \frac{(N+n-2-i)!i!}{(N+n-1)!}
\]

\[
= \frac{N}{N+n-1} \sum_{i=k}^{n-1} \binom{n-1}{i} \frac{N}{N+n-1}^i
\]

\[
\leq \frac{N}{N+n-1} \sum_{i=k}^{n-1} \left( \frac{n}{N} \right)^i
\]

\[
\leq \frac{N}{N+n-1} \left( \frac{n}{N} \right)^k \frac{1}{1-n/N}
\]

\[
\leq 2 \left( \frac{n}{N} \right)^k,
\]

where the last inequality uses \( N > 1.5n \).
Substituting this back, we get for \( x = x_k \):

\[
\mathbf{E}_b[|Z_y(1 - \delta_N)(b(1 - \delta_N) - \hat{b}_N)|] \\
\leq \delta_N \frac{y'(1 - \delta_N)}{x'_k(1 - \delta_N)} \left\{ e\delta_N x'_k(1 - \delta_N) + 2 \left( \frac{n}{N} \right)^k \right\} \\
= e\delta_N^2 y'(1 - \delta_N) + 2\delta_N y'(1 - \delta_N) \left\{ \frac{1}{(n-1)(n-2)(1-\delta_N)^{n-1-k}} \left( \frac{n}{N} \right)^{k-1} \right\} \\
\leq e\delta_N^2 y'(1 - \delta_N) + \frac{2}{N} \left( \frac{n}{n-1} \right) \delta_N y'(1 - \delta_N) \left( \frac{n}{N\delta_N} \right)^{k-1} \frac{1}{(n-2)(1-\delta_N)^{n-1-k}} \\
\leq e\delta_N^2 y'(1 - \delta_N) + \frac{8}{N} \delta_N y'(1 - \delta_N).
\]

Here the last inequality follows by noting that \( \left( \frac{n}{n-1} \right) \geq 1 \), \( (1 - \delta_N)^n > 1/4 \), and using \( \delta_N \geq n/N \), \( \frac{n}{n\delta_N} \leq 1 \).

Finally, since \( x \) is a linear combination of the \( x_k \)'s, we have,

\[
\mathbf{E}_b[|Z_y(1 - \delta_N)(b(1 - \delta_N) - \hat{b}_N)|] \\
\leq \delta_N \frac{y'(1 - \delta_N)}{x'_k(1 - \delta_N)} \left\{ e\delta_N x'_k(1 - \delta_N) + \mathbf{E}_b[|1 - x(q)|]\right\} \\
\leq \max_k \delta_N \frac{y'(1 - \delta_N)}{x'_k(1 - \delta_N)} \left\{ e\delta_N x'_k(1 - \delta_N) + \mathbf{E}_b[|1 - x_k(q)|]\right\} \\
\leq e\delta_N^2 y'(1 - \delta_N) + \frac{8}{N} \delta_N y'(1 - \delta_N).
\]

**Lemma A.9.** For \( Z_y \) and \( \Lambda \) as defined above, if \( \delta_N \leq 1/n \), the fourth error term of the estimator \( \hat{P}_y \) is bounded as follows.

\[
Z_y(\delta_N)b(\delta_N) \leq e\delta_N y'(\delta_N).
\]

**Proof.** The lemma follows directly from the definition of \( Z_y \) with the upper-bound on \( b(\delta_N) \) of Theorem A.6. \( \square \)

**A.4 Proofs of main theorems**

This section gives the complete proofs for the main theorems of Section 3.2. These theorems follow fairly directly from the previous lemmas.

**Theorem 3.1.** The mean absolute error in estimating the revenue of a rank-based auction with allocation rule \( y \) using \( N \) samples from the bid distribution for an all-pay rank-based auction with allocation rule \( x \) is bounded as below. Here \( n \) is the number of positions in the two auctions, and \( \hat{P}_y \) is the estimator in Definition 7 with \( \delta_N \) set to \( \max(25 \log \log N, n)/N \).

\[
\mathbf{E}_b[|\hat{P}_y - P_y|] \leq \frac{16n^2 \log N}{\sqrt{N}}.
\]
Proof. As in the proof of Lemma 6.4, we may assume without loss of generality that \(4n \log N < \sqrt{N}\), and indeed, \(16n^2 \log N < \sqrt{N}\). This implies \(\delta_N < 1/n\), and then Lemmas 6.4, A.7, A.8, and A.9 together imply that the error in \(P_k\) is bounded by:

\[
E_b \left[ |\hat{P}_k - P_k| \right] \leq \frac{8n \log N}{\sqrt{N}} \sup_{q \notin \Lambda} \{ x'_k(q) \} + 2e\delta_N x'_k(\delta_N) + (2e + 8)\delta_N^2 x'_k(1 - \delta_N)
\]

Further, \(16n^2 \log N < \sqrt{N}\) also implies that the second and third terms together are no larger than the first. The theorem then follows by recalling that \(\sup_q x'_k(q) \leq n\).

We will now prove the improved error bounds of Theorem 3.2 and Corollary 3.3. Recall the definition of \(\Phi_{x,y}\) from equation 9 in Section 3.2.

\[
\Phi_{x,y} := \sup_q \{ y'(q) \} \max \left\{ 1, \log \sup_{q:y'(q) \geq 1} \frac{x'(q)}{y'(q)}, \log \sup_q \frac{y'(q)}{x'(q)} \right\}. \tag{9}
\]

Theorem 3.2 follows from Lemma A.1 in much the same way as Theorem 3.1 does from Lemma 6.4. We may assume, without loss of generality, that \(\sqrt{N} < 80\), in which case the errors from the extreme quantiles get absorbed into the error from the moderate quantiles.

**Theorem 3.2.** Let \(x\) and \(x_k\) denote the allocation rules for any all-pay rank-based auction and the \(k\)-highest-bids-win auction over \(n\) positions, respectively. Let \(\hat{P}_k\) denote the estimator from Definition 7 for estimating the revenue \(P_k\) of the latter auction from \(N\) samples of the bid distribution of the former, with \(\delta_N\) set to \(\max(25 \log \log N, n)/N\). If \(\delta_N \leq 1/n\), the mean absolute error of the estimator \(\hat{P}_k\) is bounded as follows.

\[
E_b \left[ |\hat{P}_k - P_k| \right] \leq \frac{80}{\sqrt{N}} \Phi_{x,x_k}.
\]

We now generalize error bound to estimate the revenue \(P_y\) of an arbitrary rank-based auction with allocation rule \(y\) from the bids of another rank-based auction with allocation rule \(x\).

**Corollary 3.3.** Let \(x\) and \(y\) denote the allocation rules for any two all-pay rank-based auctions over \(n\) positions. Let \(\hat{P}_y\) denote the estimator from Definition 7 for estimating the revenue of the latter from \(N\) samples of the bid distribution of the former, with \(\delta_N\) set to \(\max(25 \log \log N, n)/N\). If \(\delta_N \leq 1/n\), the mean absolute error of the estimator \(\hat{P}_y\) is bounded as follows.

\[
E_b \left[ |\hat{P}_y - P_y| \right] \leq \frac{80}{\sqrt{N}} n \log \sup_q n \frac{y'(q)}{x'(q)}.
\]

**Proof.** Write \(y\) as a rank-based auction with weights \(w\):

\[
y = \sum_k w'_k x_k, \quad \text{and,} \quad P_y = \sum_k w'_k P_k.
\]
Accordingly, the error in $P_y$ is bounded by a weighted sum of the error in $P_k$ which are bounded by Theorem 3.2. The weighted sum of these errors is simplified by observing that $x'_k(q) \leq y'(q)/w'_k$ for all $k$ and $q$:

$$E_{\hat{b}}[|\hat{P}_y - P_y|] \leq \sum_k w'_k E_{\hat{b}}[|\hat{P}_k - P_k|] \leq \frac{80}{\sqrt{N}} \sum_k w'_k \Phi_{x,x_k} \leq \frac{80}{\sqrt{N}} \sum_k w'_k \sup_q \{x'_k(q)\} \max \left\{ \log n, \frac{1}{w'_k} + \log \sup_q \frac{y'(q)}{x'(q)} \right\}.$$  

We now simplify the terms one at a time. Recall that $\sup_q \{x'_k(q)\} \leq n$ for all $k$. The first and third terms can therefore be simplified using $\sum_k w'_k \leq 1$. For the second term, we observe $\sum_k w'_k \log \frac{1}{w'_k} \leq \log n$. We therefore have:

$$E_{\hat{b}}[|\hat{P}_y - P_y|] = \frac{80}{\sqrt{N}} n \log \sup_q \frac{y'(q)}{x'(q)}. \quad (21)$$  

**B Proofs for Section 3.3**

We will now prove Theorem B.4, restated here for convenience.

**Theorem B.4.** For arbitrary $n$-agent rank-based auctions $A$, $B_1$, and $B_2$ and $N$ bids from the equilibrium bid distribution of mechanism $C = \epsilon B_1 + \epsilon B_2 + (1 - 2\epsilon)A$, the estimator for the binary classifier $\gamma = 1\{P_{B_1} - \alpha P_{B_2} > 0\}$, that establishes whether the revenue of mechanism $B_1$ exceeds $\alpha$ times the revenue of mechanism $B_2$, has error rate bounded by

$$\exp \left( -O \left( \frac{N a^2}{\alpha^2 n^3 \log(n/\epsilon)} \right) \right),$$

where $a = |P_{B_1} - \alpha P_{B_2}|$, as long as $N \gg n/\epsilon a$.

Whereas we bound the expected absolute error of our revenue estimator in Section 3 in this section we will require a concentration result for the error. We state this concentration result below and prove it in Section B.1. We focus on the main term in our error bound, $E_{q \notin \Lambda} \left[ -Z'_y(q)(\hat{b}(q) - b(q)) \right]$. We can split this error into two components, one corresponding to the bias in the estimated bid function and the other corresponding to the deviation of the estimated bids from their mean:

$$\left| E_{q \notin \Lambda} \left[ -Z'_y(q)(\hat{b}(q) - b(q)) \right] \right| \leq \left| E_{q \notin \Lambda} \left[ Z'_y(q)(\hat{b}(q) - \bar{b}(q)) \right] \right| + \left| E_{q \notin \Lambda} \left[ Z'_y(q)(\bar{b}(q) - b(q)) \right] \right|. \quad (22)$$

Here, $\bar{b}$ is a step function that equals the expectation of the empirical bid function $\hat{b}$: $\bar{b}(q) = E \left[ \hat{b}(q) \right]$.

The bias of the estimator, i.e., the second term above, is small:
Lemma B.1. With \( \tilde{b} \) defined as above,
\[
\left| \mathbb{E}_{q \in \Lambda} \left[ Z_y(q)(\tilde{b}(q) - b(q)) \right] \right| = \frac{O(1)}{N} \sup_q \left\{ x' \right\} \sup_q \left\{ y'(q) \right\}.
\]

The deviation from the mean, i.e., the first term in equation (22), is concentrated.

Lemma B.2. Let \( \Delta = \sup_{q \in \Lambda} \left| (b'(q))^{-1}(\tilde{b}(q) - b(q)) \right| \). Then for any \( a > 0 \),
\[
\Pr \left[ \left| \mathbb{E}_{q \in \Lambda} \left[ Z'_y(q)(\tilde{b}(q) - b(q)) \right] \right| \geq a \right| \Delta \right] \leq \exp \left( -\frac{a^2}{n(80\Delta \Phi_{x,y})^2} \right).
\]

The proofs of Lemmas B.1 and B.2 are deferred to the next subsection. We are now ready to prove Theorem B.4.

Proof of Theorem B.4. We need to bound the probability that the error in estimating \( \hat{P}_{B_1} - \alpha \hat{P}_{B_2} \) is greater than \( |P_{B_1} - \alpha P_{B_2}| \). This error can in turn be decomposed into the error in estimating \( P_{B_1} \) and that in estimating \( P_{B_2} \). Denote \( a = |P_{B_1} - \alpha P_{B_2}| > 0 \). Then,
\[
\Pr \left[ (\hat{P}_{B_1} - \alpha \hat{P}_{B_2}) - (P_{B_1} - \alpha P_{B_2}) \right] \geq a
\]
\[
\leq \Pr \left[ \hat{P}_{B_1} - P_{B_1} \right] + \Pr \left[ \hat{P}_{B_2} - P_{B_2} \right] > a/2 \alpha.
\]

Let \( x \) denote the allocation rule of the mechanism \( C \) that we are running, and let \( b \) be the corresponding bid function. Now, recall that for
\[
\Delta = \sup_q \left| (b'(q))^{-1}(\tilde{b}(q) - b(q)) \right|
\]
and
\[
\Phi_{x,x_B} = \sup_q \left\{ x'_{B_1}(q) \right\} \max \left\{ 1, \log \sup_q \left( \frac{x'(q)}{x'_{B_1}(q)} \right), \log \sup_q \left( \frac{x'_{B_1}(q)}{x'(q)} \right) \right\}
\]
equations (18) and (22) bound the error in estimation as a sum of five terms. Of these, all but the first term in equation (22) can be bounded by \( O(n/\epsilon N) \) using Lemmas A.7, A.8, A.9, and B.1. Then, Lemma B.2 implies that, conditioned on \( \Delta \),
\[
\Pr \left[ \hat{P}_{B_1} - P_{B_1} \right] > a/2 \leq 2 \exp \left( -\frac{1}{n(80\Delta \Phi_{x,x_B})^2} \left( \frac{a}{2} - O \left( \frac{n}{\epsilon N} \right) \right)^2 \right).
\]
Finally, \( \Phi_{x,y} < n \log(n/\epsilon) \), and with high probability \( \Delta \) is at most a constant times \( 1/\sqrt{N} \) (Lemma 6.1). Consequently, for \( N \gg n/\epsilon a \),
\[
\Pr \left[ |\hat{P}_{B_1} - P_{B_1}| > a/2 \right] \leq \exp \left( -O \left( \frac{N a^2}{n^3 \log(n/\epsilon)} \right) \right).
\]
Likewise,
\[
\Pr \left[ |\hat{P}_{B_2} - P_{B_2}| > a/2 \alpha \right] \leq \exp \left( -O \left( \frac{N a^2}{\alpha^2 n^3 \log(n/\epsilon)} \right) \right).
\]
\( \square \)

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B.1 Concentration bound for the revenue estimator

Lemma B.1. With $\tilde{b}$ defined as above,

$$\left| E_{q \in \Lambda} \left[ Z_y(q)(\tilde{b}(q) - b(q)) \right] \right| = \frac{O(1)}{N} \sup_q \{x'(q)\} \sup_q \left\{ y'(q) \right\}.$$

Proof. We can write the function $\tilde{b}(i/N)$ as

$$\tilde{b}(i/N) = \frac{N!}{(i-1)!(N-i)!} \int G(t)^{i-1}(1 - G(t))^{N-i} g(t) dt$$

$$= \frac{N!}{(i-1)!(N-i)!} \int t^{i-1}(1 - t)^{N-i} b(t) dt.$$

Note that

$$\frac{N!}{(i-1)!(N-i)!} t^{i-1}(1 - t)^{N-i}$$

is the density of the beta distribution with parameters $\alpha = i$ and $\beta = N - i + 1$. Denote this density $f(t; \alpha, \beta)$. Then we can write

$$\tilde{b}(i/N) = \int_0^1 b(t) f(t; \alpha, \beta) dt.$$

Now let $q \in [i/N, (i+1)/N]$, and consider an expansion of $b(t)$ at $q$ such that

$$b(t) = b(q) + b'(q)(t - q) + O((t - q)^2).$$

Now we substitute this expansion into the formula for $\tilde{b}(\cdot)$ above to get

$$\tilde{b}(i/N) = b(q) + b'(q) \int_0^1 (t - q) f(t; \alpha, \beta) dt + O(\int_0^1 (t - q)^2 f(t; \alpha, \beta) dt).$$

The mean of the beta distribution is $\alpha/\alpha + \beta$ and the variance is $\alpha\beta/((\alpha + \beta)^2(\alpha + \beta + 1))$. This means that

$$\tilde{b} \left( \frac{i}{N} \right) - b(q) = b'(q) \left( \frac{i}{N+1} - q \right) + O \left( \frac{1}{N^2} \right).$$

Thus

$$\sup_{q \in [i/N, (i+1)/N]} \left| \tilde{b}(i/N) - b(q) \right| \leq \sup_q b'(q) \frac{2}{N} + O \left( \frac{1}{N^2} \right).$$

Therefore, the expectation $|\tilde{P}_y - E[\tilde{P}_y]|$ is at most $O(1)/N \sup_q \{x'(q)\} \sup_q Z_y(q)$. \qed

We now focus on the deviation of our estimator from its mean. In order to obtain a concentration bound, we express the estimator as a sum over many independent terms.

To this end, we first identify the set of quantiles at which the function $\tilde{b}$ “crosses” the function $\tilde{b}$ from below. This set is defined inductively. Define $i_0 = \delta_N N$. Then, inductively, let $i_\ell$ be the smallest integer strictly greater than $i_{\ell-1}$ such that

$$\tilde{b} \left( \frac{i_{\ell-1}}{N} \right) \leq \tilde{b} \left( \frac{i_{\ell}}{N} \right) \quad \text{and} \quad \tilde{b} \left( \frac{i_{\ell}}{N} \right) > \tilde{b} \left( \frac{i_{\ell-1}}{N} \right).$$
Let \( i_{m-1} \) be the last integer so defined, and let \( i_m = (1 - \delta_N)N \). Let \( I \) denote the set of indices \( \{i_0, \ldots, i_m\} \). Let \( T_{i,j} \) denote the following integral:

\[
T_{i,j} = \int_{q=i/N}^{q=j/N} Z_y'(q) (\hat{b}(q) - \overline{b}(q)) \, dq
\]

Then, our goal is to bound the quantity \( E_b[|T_{0,N}|] \) where \( T_{0,N} \) can be written as the sum:

\[
T_{0,N} = \sum_{\ell=0}^{m-1} T_{i_{\ell}, i_{\ell}+1}.
\]

We now claim that conditioned on \( I \) and the maximum weighted bid error, this is a sum over independent random variables.

**Lemma B.3.** Conditioned on the set of indices \( I \) and \( \Delta = \sup_{q \notin \Lambda} |(b'(q))^{-1}(\hat{b}(q) - \overline{b}(q))| \), over the randomness in the bid sample, the random variables \( T_{i_{\ell}, i_{\ell+1}} \) are mutually independent.

**Proof.** Fix \( I \) and \( \ell \), and note that the function \( \hat{b} \) is fixed (that is, it does not depend on the empirical bid sample). Then, the sum \( T_{i_{\ell}, i_{\ell+1}} \) depends only on the empirical bid values \( \hat{b}(q) \) for quantiles in the interval \([i_{\ell}/N, i_{\ell+1}/N)\). By the definition of \( I \), we know that the smallest \( i_{\ell} \) bids in the sample are all smaller than \( \hat{b}((i_{\ell} - 1)/N) \leq \hat{b}(i_{\ell}/N) \), and the largest \( N - i_{\ell+1} \) bids in the sample are all larger than \( \hat{b}(i_{\ell+1}/N) \geq \hat{b}((i_{\ell+1} - 1)/N) \). On the other hand, the empirical bids \( \tilde{b}(q) \) for \( q \in [i_{\ell}/N, i_{\ell+1}/N) \) lie within \([\tilde{b}(i_{\ell}/N), \tilde{b}((i_{\ell+1} - 1)/N)]\). Therefore, conditioned on \( i_{\ell} \) and \( i_{\ell+1} \), the latter set of empirical bids is independent of the former set of empirical bids.

Since within each interval \((i_{\ell}, i_{\ell+1})\) the multiplier \( \hat{b}(q) - \tilde{b}(q) \) changes sign only once, we can apply the approach of Section 6 to bound each individual \( T_{i_{\ell}, i_{\ell+1}} \) by \( 40\Delta \Phi_{x,y} \). We then apply Chernoff-Hoeffding bounds to obtain a bound on the probability that \( E_b[|T_{0,N}|] | I, \Delta \) exceeds some value \( a > 0 \).

**Lemma B.2.** Let \( \Delta = \sup_{q \notin \Lambda} |(b'(q))^{-1}(\hat{b}(q) - \overline{b}(q))| \). Then for any \( a > 0 \),

\[
\Pr\left[ |E_b \left[ Z_y'(q) (\hat{b}(q) - \overline{b}(q)) \right] | \geq a \right] \leq \exp\left( -\frac{a^2}{n(80\Delta \Phi_{x,y})^2} \right).
\]

**Proof.** We will use Chernoff-Hoeffding bounds to bound the expectation of \( T_{0,N} \) over the bid sample, conditioned on \( I \) and \( \Delta \). We first note that \( T_{0,N} \) has mean zero because for any integer \( i \in [0, N] \),

\[
E_{\text{samples}}[\hat{b}(i/N)] = \tilde{b}(i/N).
\]

Next we note that the \( T_{i,j} \)’s are bounded random variables. Specifically, let \( Q \) be an interval of quantiles over which the difference \( \hat{b}(q) - \overline{b}(q) \) does not change sign. Then, following the proof of Lemma A.1, we can bound

\[
|T_Q| = \left| \int_Q Z_y'(q) (\hat{b}(q) - \overline{b}(q)) \, dq \right| 
\leq 40\Delta \sup_{y'(q)} \left\{ 1, \log \sup_{y'(q) \geq 1} \frac{x'(q)}{y'(q)}, \log \sup_{x'(q)} \frac{y'(q)}{x'(q)} \right\}.
\]

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Theorem B.4. For arbitrary probability of incorrect output decreases exponentially with $N$.

We now apply Chernoff-Hoeffding bounds to bound the probability that the sum $\sum_{\ell=0}^{\ell=m-1} T_{i\ell,i\ell+1}$ exceeds some constant $a$. With $\tau_\ell$ denoting the upper bound on $|T_{i\ell,i\ell+1}|$, this probability is at most

$$
\exp\left(-\frac{a^2}{\sum_{\ell} \tau^2_\ell}\right).
$$

By our observations above, for all $\ell$, $\tau_\ell \leq 80\Delta \Phi_{x,y}$, and $\sum_{\ell} \tau_\ell \leq \int_0^1 |Z'_y(q)(\hat{b}(q) - \hat{b}(q))| \, dq \leq 80n\Delta \Phi_{x,y}$. Therefore, $\sum_{\ell} \tau^2_\ell \leq n(80\Delta \Phi_{x,y})^2$. Since the bound does not depend on $I$, we can remove the conditioning on $I$.

B.2 Comparing revenues

We have considered the case where the empirical task was to recover the revenues for one mechanism $(y)$ using the sample of bids responding to another mechanism $(x)$. In many practical situations the empirical task is simply the verification of whether the revenue from a given mechanism is higher than the revenue from another mechanism. Or, equivalently, the task could be to verify whether one mechanism provides revenue which is a certain percentage above that of another mechanism. We now demonstrate that this is a much easier empirical task in terms of accuracy than the task of inferring the revenue.

Suppose that we want to compare the revenues of mechanisms $B_1$ and $B_2$ by mixing them in to an incumbent mechanism $A$, and running the composite mechanism $C = \epsilon B_1 + \epsilon B_2 + (1 - 2\epsilon)A$. Specifically, we would like to determine whether $P_{B_1} > \alpha P_{B_2}$ for some $\alpha > 0$. Consider a binary classifier $\hat{\gamma}$ which is equal to 1 when $P_{B_1} > \alpha P_{B_2}$ and 0 otherwise. Let $\gamma = 1\{P_{B_1} - \alpha P_{B_2} > 0\}$ be the corresponding “ideal” classifier for the case where the distribution of bids from mechanism $C$ is known precisely. To evaluate the accuracy of the classifier, we need to evaluate the probability $\Pr[\hat{\gamma} = 1|\gamma = 0]$, and likewise, $\Pr[\hat{\gamma} = 0|\gamma = 1]$. The classifier will give the wrong output if the sampling noise in estimating $P_{B_1} - \alpha P_{B_2}$ is greater than $|P_{B_1} - \alpha P_{B_2}|$.

Our main result of this section says that fixing the number of positions $n$, $\alpha$, and the difference $|P_{B_1} - \alpha P_{B_2}|$, with the number of samples from the bid distribution, $N$, being large enough, the probability of incorrect output decreases exponentially with $N$.

Theorem B.4. For arbitrary $n$-agent rank-based auctions $A$, $B_1$, and $B_2$ and $N$ bids from the equilibrium bid distribution of mechanism $C = \epsilon B_1 + \epsilon B_2 + (1 - 2\epsilon)A$, the estimator for the binary classifier $\gamma = 1\{P_{B_1} - \alpha P_{B_2} > 0\}$, that establishes whether the revenue of mechanism $B_1$ exceeds $\alpha$ times the revenue of mechanism $B_2$, has error rate bounded by

$$
\exp\left(-O\left(\frac{N \alpha^2}{\alpha^2 n^3 \log(n/\epsilon)}\right)\right),
$$

where $a = |P_{B_1} - \alpha P_{B_2}|$, as long as $N \gg n/\epsilon a$. 

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We obtain a similar error bound when our goal is to estimate which of \( r \) different novel mechanisms obtains the most revenue, for any \( r > 1 \):

**Corollary B.5.** Suppose that our goal is to determine which of \( r \) rank-based auctions, \( B_1, B_2, \ldots, B_r \), obtains the most revenue while running incumbent mechanism \( A \), by running each of the novel mechanisms with probability \( \epsilon/r \). Then the error probability of the corresponding classifier constructed using \( N \) bids from composite mechanism \( C = \sum_{i=1}^{r} \epsilon/r B_i + (1-\epsilon)A \) is bounded from above by

\[
 r \exp \left( -O \left( \frac{Na^2}{n^3 \log (rn/\epsilon)} \right) \right),
\]

where \( a \) is the absolute difference between the revenue obtained by the best two of the \( r \) mechanisms.

## C  Inference methodology and error bounds for first-price auctions

In this section we define and analyze an estimator for counterfactual revenue the bids in first-price auctions. Our approach will be to reduce this estimation problem to the all-pay estimation problem that we solved previously. Recall that the all-pay estimator is a weighted order statistic of the empirical all-pay bid function. Our first-price estimator will map the empirical first-price bid function to an empirical all-pay bid function and then apply to it the all-pay estimator.

Recall that the Bayes-Nash equilibrium bid function of first-price auction and all-pay auction are related by the payment identity. Specifically an all-pay bid is deterministically equal to the expected payment of the payment identity, while in a first-price auction an agent only pays upon winning. To facilitate comparison to previous results we notate the equilibrium bid function of the all-pay auction as \( b \) and the equilibrium bid function of the first-price auction as \( c \). Given the allocation rule \( x \), the payment identity requires \( b(q) = x(q) c(q) \). Consequently, an empirical all-pay bid function can be defined from the empirical first-price bid function as \( \hat{b}(q) = x(q) \hat{c}(q) \). Note that while in previous sections the empirical all-pay bid function is piece-wise constant (similarly the empirical first-price bid function is piece-wise constant), this empirical all-pay bid function is not piece-wise constant.

Partition the quantile range into extreme quantiles \( \Lambda = [0, \delta_N] \cup [1 - \delta_N, 1] \) and the moderate quantiles \( [\delta_N, 1 - \delta_N] \). Recall that truncation trades off a (potentially diverging) variance of the estimator suggested by [Theorem 6.2](#) at the extreme quantiles with a bias that can be bounded. Specifically, truncation replaces bids at low quantiles with zero and bids at high quantiles with the upper bound \( \hat{b}(1) \) (which, in terms of the first-price bids, is \( x(1) \hat{c}(1) \)).

As in [section 3](#) the estimator for counterfactual revenue plugs the truncated empirical bid function into the counterfactual revenue equation 17 of Lemma 6.2. We obtain the following estimator in terms of the empirical first-price bids:

\[
\hat{P}_y = \mathbf{E}_{q \notin \Lambda} \left[ -Z'_y(q) x(q) \hat{c}(q) \right] + Z_y(1 - \delta_N) x(1) \hat{c}(1).
\]

This estimator is a weighted order statistic as formalized in the following definition.

**Definition 6.** The estimator \( \hat{P}_y \) (with truncation parameter \( \delta_N \)) for the revenue of an auction with allocation rule \( y \) from \( N \) samples \( \hat{c}_1 \leq \cdots \leq \hat{c}_N \) from the equilibrium bid distribution of a first price auction with allocation rule \( x \) is:

\[
\hat{P}_y = \sum_{i=\delta_N N}^{N-\delta_N N} \mathbf{E}_{q \in [i,i+1]/N} \left[ -Z'_y(q) x(q) \right] \hat{c}_i + Z_y(q) x(1) \hat{c}_N.
\]
To obtain a bound on the mean absolute error of the estimator judiciously plug the identity relating first-price and all-pay equilibrium bids into the error bound of equation (13) to get:

\[
|\hat{P}_k - P_k| \leq |E_{q \in A}[\bar{Z}_k(q)x(q)(\hat{c}(q) - c_q)]| + |E_{q \in A}[Z_k(q)b'(q)]| \\
+ |Z_k(1 - \delta_N)(b(1 - \delta_N) - x(1)\hat{c}_N)| + |Z_k(\delta_N)b(\delta_N)|
\]

(23)

It is clear that terms that depend only on the equilibrium bid functions and not the empirical bid functions need no further analysis. Specifically Theorem A.7 and Theorem A.9 bound the contribution from the first and third terms. These bounds come from relatively minor adjustments to the analogous bounds for all-pay auctions.

For the second part, write

\[
\hat{q} \rightarrow q
\]

for \(\hat{q}\), i.e., \(\hat{c}_N = c(\hat{q})\). There are two parts of the analysis, the first part is for the case \(\hat{q} \geq 1 - \delta_N\) and the second part is for the case \(\hat{q} < 1 - \delta_N\).

For the first part, the proof of Theorem A.8 upper bounds \(\hat{b}_N\) by \(b(1)\). We can do the same for \(x(1)\)\(\hat{c}_N\): \(x(1)c(\hat{q}) \leq x(1)c(1) = b(1)\). The first inequality follows from the monotonicity of the equilibrium bid function, i.e., \(c(\hat{q}) \leq c(1)\) for \(\hat{q} \leq 1\). Thus, we can upper bound the error in the case that \(\hat{q} \geq 1 - \delta_N\) by \(Z_k(1 - \delta_N)(b(1) - b(1 - \delta_N))\) which was bounded already in the proof of Theorem A.8.

For the second part, write

\[
x(1)c(\hat{q}) = x(1)b(\hat{q})/x(\hat{q}) \\
\geq x(\hat{q})b(\hat{q})/x(\hat{q}) \\
= b(\hat{q}),
\]

where inequality follows from monotonicity of \(x\). Thus, we can upper bound the error in the case that \(\hat{q} < 1 - \delta_N\) by \(Z_k(1 - \delta_N)(b(1 - \delta_N) - b(\hat{q}))\) which was bounded already by Theorem A.8.

To analyze the first term in the error bound of equation (23), we begin with the following upper bound:

\[
|E_{q \in A}[\bar{Z}_k(q)x(q)(\hat{c}(q) - c(q))]| \leq E_{q \in A}\left[\frac{Z_k(q)}{h(Z_k(q))}\right]|x(q)h(Z_k(q))(\hat{c}(q) - c(q))| \\
\leq \sup_q E_{q \in A}\left[\frac{Z_k(q)}{h(Z_k(q))}\right]|h(Z_k(q))(\hat{c}(q) - c(q))|.
\]

We can then carry out an analysis identical to the proof of Theorem A.1 with an appropriate choice of \(h(\cdot)\). The only difference is in the application of Theorem 6.1. Whereas for all-pay auctions the lemma bounds the weighted error in bids in terms of \(\sup_q\{q(1 - q)x'(q)\} \leq n/4\), in the case of first-price auctions, this term is replaced by \(\sup_q\{q(1 - q)x'(q)/x(q)\}\), which is no more than \(n\) for rank-based allocation rules. We obtain the following theorem.

**Theorem C.1.** The expected absolute error in estimating the revenue of a position auction with allocation rule \(y\) using \(N\) samples from the bid distribution for a first-pay position auction with allocation rule \(x\) is bounded by both of the expressions below; Here \(n\) is the number of positions in
the two position auctions.

\[ E_b[|\hat{P}_y - P_y|] \leq \frac{28n^2 \log N}{\sqrt{N}}, \]

\[ E_b[|\hat{P}_y - P_y|] \leq \frac{80}{\sqrt{N}} n \log \sup q \, y'(q) x'(q). \]

When \( y \) is the highest-\( k \)-bids-win allocation rule, the latter bound improves to:

\[ E_b[|\hat{P}_k - P_k|] \leq \frac{80}{\sqrt{N}} \Phi_{x,x_k} \]

with \( \Phi_{x,x_k} \) as defined in equation (9).

Because the error bounds in Theorem C.1 are identical up to constant factors to those in Theorems 3.1, 3.2 and Corollary 3.3, other results in Lemma 4.3, Corollaries 3.5, 3.7, 4.2, 4.4, and Theorems B.4 and 7.2 continue to hold when bids are drawn from a first-price auction.

D Finding the optimal iron by rank auction

Recall that iron by rank auctions are weighted sums of multi-unit auctions. Therefore, their revenue can be expressed as a weighted sum over the revenues \( P_k \) of \( k \)-unit auctions. We consider a position environment given by non-increasing weights \( w = (w_1, \ldots, w_n) \), with \( w_0 = 0 \), \( w_1 = 1 \), and \( w_{n+1} = 0 \). Define the cumulative position weights \( W = (W_1, \ldots, W_n) \) as \( W_k = \sum_{j \leq i} w_j \).

Define the multi-unit revenue curve as the piece-wise constant function connecting the points \( (0, P_0, \ldots, (n, P_n)) \). This function may or may not be concave. Define the ironed multi-unit revenue curve as \( \bar{P} = (\bar{P}_1, \ldots, \bar{P}_n) \) the smallest concave function that upper bounds the multi-unit revenue curve. Define the multi-unit marginal revenues as \( P' = P'_1, \ldots, P'_n \) and \( \bar{P}' = \bar{P}'_1, \ldots, \bar{P}'_n \) as the left slope of the multi-unit and ironed multi-unit revenue curves, respectively. I.e., \( P'_k = P_k - P_{k-1} \) and \( \bar{P}'_k = \bar{P}_k - \bar{P}_{k-1} \).

We now see how the revenue of any position auction can be expressed in terms of the multi-unit revenue curves and marginal revenues.

\[ E[\text{revenue}] = \sum_{k=0}^{n} P_k w'_k = \sum_{k=0}^{n} P'_k w_k \]

\[ \leq \sum_{k=0}^{n} \bar{P}_k w'_k = \sum_{k=0}^{n} \bar{P}'_k w_k. \]

The first equality follows from viewing the position auction with weights \( w \) as a convex combination of multi-unit auctions (where its revenue is the convex combination of the multi-unit auction revenues). The second and final inequality follow from rearranging the sum (an equivalent manipulation to integration by parts). The inequality follows from the fact that \( \bar{P} \) is defined as the smallest concave function that upper bounds \( P \) and, therefore, satisfies \( \bar{P}_k \geq P_k \) for all \( k \). Of course the inequality is an equality if and only if \( w'_k = 0 \) for every \( k \) such that \( \bar{P}'_k > P'_k \).

We now characterize the optimal ironing-by-rank position auction. Given a position auction weights \( w \) we would like the ironing-by-rank which produces \( \bar{w} \) (with cumulative weights satisfying
\(W \geq \tilde{W}\) with optimal revenue. By the above discussion, revenue is accounted for by marginal revenues, and upper bounded by ironed marginal revenues. If we optimize for ironed marginal revenues and the condition for equality holds then this is the optimal revenue. Notice that ironed revenues are concave in \(k\), so ironed marginal revenues are monotone (weakly) decreasing in \(k\). The position weights are also monotone (weakly) decreasing. The assignment between ranks and positions that optimizes ironed marginal revenue is greedy with positions corresponding to ranks with negative ironed marginal revenue discarded. Tentatively assign the \(k\)th rank agent to slot \(k\) (discarding agents that correspond to discarded positions). This assignment indeed maximizes ironed marginal revenue for the given position weights but may not satisfy the condition for equality of revenue with ironed marginal revenue. To meet this condition with equality we can randomly permute (a.k.a., iron by rank) the positions that corresponds to intervals where the revenue curve of revenue with ironed marginal revenue. The position weights are also monotone (weakly) decreasing. The assignment between ranks and positions that optimizes ironed marginal revenue is greedy with positions corresponding to ranks \((\bar{r}_i - Y_i)/i\) for all \(i \in [n]\). Let \(\bar{r}_i = Y_i\) for all \(k \leq i \leq k\). We will now present a (randomized) transformation that will increase the value of \(i\). Specifically, we will reassign some agents to positions in a manner such that the resulting position weights \(\tilde{y}\) satisfy: \(\tilde{W}_k = \tilde{Y}_k\) for \(k \leq i + 1\) and \(Y_k \geq \tilde{Y}_k \geq W_k\) for \(k > i + 1\).

Consider the operation of ironing by rank over the interval \([i, \ldots, i']\) for some index \(i' > i\). Recall that this operation averages out the position weights over this interval, setting each such weight equal to \((\bar{r}_i - Y_i)/(i' - i)\), while leaving all other position weights intact. It also preserves cumulative weights at positions \(k \leq i\) and positions \(k \geq i'\). Note also that the larger that \(i'\) is, the smaller is the average weight \((\bar{r}_i - Y_i)/(i' - i)\). In particular, for any \(i' > i + 1\), this operation strictly decreases the \(i + 1\)th position weight.

Suppose that there exists an index \(i'\), with \(i + 1 < i' < n\), such that \((\bar{r}_i - Y_i)/(i' - i) \geq \tilde{w}_{i+1}\) and \((\bar{r}_{i' + 1} - Y_i)/(i' + 1 - i) < \tilde{w}_{i+1}\). Let \(A := (\bar{r}_i - Y_i)/(i' - i)\) and \(B := (\bar{r}_{i' + 1} - Y_i)/(i' + 1 - i)\). Let

**E Constructing a position auction with a target vector of position weights**

In this section we show that in any position auction environment given by position weights \(w\), we can construct a rank based auction with induced position weights \(\tilde{w}\) satisfying \(\tilde{W} \leq W\). The allocation rule of the auction is constructed as a (random) sequence of iron by rank and rank reserve operations.

**Lemma E.1.** In any position auction environment with position weights \(w\) and target weights \(\tilde{w}\) satisfying \(\tilde{W} \leq W\), there exists a (random) sequence of iron by rank and rank reserve operations following which the induced position weights are exactly \(\tilde{w}\).

**Proof.** Suppose that we have \(\tilde{W} \leq W\). We will describe how to assign agents to slots so as to obtain position weights \(\tilde{w}\). Let \(y\) denote the position weights corresponding to an intermediate assignment. We begin by assigning the agent with the \(i\)th largest bid to the \(i\)th slot for all \(i \in [n]\). The position weights for this assignment are given by \(y = w\). We will then construct a series of transformations or reassignments of agents to positions, each time making a small change to the weights \(y\), so as to bring them closer to the target \(\tilde{w}\). Each transformation is either a rank-based ironing operation or a rank reserve.

Let \(i\) denote the largest index such that \(\tilde{W}_k = Y_k\) for all \(k \leq i\). We will now present a (randomized) transformation that will increase the value of \(i\). Specifically, we will reassign some agents to positions in a manner such that the resulting position weights \(\tilde{y}\) satisfy: \(\tilde{W}_k = \tilde{Y}_k\) for \(k \leq i + 1\) and \(Y_k \geq \tilde{Y}_k \geq W_k\) for \(k > i + 1\).
α ∈ (0, 1] be defined such that αA + (1 − α)B = ̃w_{i+1}. Now consider the following transformation. With probability α, we iron over the rank interval \{i, \ldots, i′\} and with probability 1 − α, we iron over the rank interval \{i, \ldots, i′ + 1\}. Let \( \tilde{y} \) and \( \tilde{Y} \) denote the new positions weights and cumulative position weights at the end of the (randomized) ironing operation. Note that both of these ironing operations preserve the position weights over positions \( k \leq i \) and \( k > i′ + 1 \). Over positions \( k \in \{i, \ldots, i′\} \), the new position weight \( \tilde{y}_k \) is exactly \( \alpha A + (1 − \alpha)B = ̃w_{i+1} \). Finally, both ironing operations maintain the same cumulative weight at position \( i′ + 1 \). Since \( \tilde{Y}_{i′} = Y_{i′} = \tilde{W}_{i′} \) and \( \tilde{Y}_{i′+1} = Y_{i′+1} > \tilde{W}_{i′+1} \), we get that the new position weight at \( i′ + 1 \) is at least \( ̃w_{i′+1} \). This completes one step of our transformation.

Alternately, suppose that for \( i′ = n \), we have \( (Y_{i′} − Y_i)/(i′ − i) \geq ̃w_{i+1} \). Let \( A := (Y_{i′} − Y_i)/(i′ − i) \) and let \( \alpha \in [0, 1] \) be defined such that \( \alpha A = ̃w_{i+1} \). Now consider the following transformation. With probability α, we iron over the rank interval \{i, \ldots, n\} and with probability 1 − α, we set a rank reserve of \( i \), that is, we reject every agent with rank \( > i \). Note that both of these operations preserve the position weights over positions \( k \leq i \). For \( k > i \), the new position weights are exactly \( \alpha A = ̃w_{i+1} \). Therefore, once again, we obtain \( \tilde{W}_k = \tilde{Y}_k \) for \( k \leq i + 1 \) and \( Y_k \geq \tilde{Y}_k \geq \tilde{W}_k \) for \( k > i + 1 \).

To summarize, we described a sequence of randomized operations. Each step of the sequence increases the number of positions over which the position weights corresponding to our current assignment, \( y \), match the target position weights, \( \tilde{w} \). After at most \( n \) such operations we obtain a randomized assignment of agents to positions achieving the target position weights.

\[ \square \]

### F Approximation via rank-based auctions

In this section we show that the revenue of optimal rank-based auction approximates the optimal revenue (over all auctions) for position environments. Instead of making this comparison directly we will instead identify a simple non-optimal rank-based auction that approximates the optimal auction. Of course the optimal rank-based auction of Theorem 4.1 has revenue at least \( \frac{1}{k} \) of that of this simple rank-based auction, thus its revenue also satisfies the same approximation bound.

Our approach is as follows. Just as arbitrary rank-based mechanisms can be written as convex combinations over \( k \)-highest-bids-win auctions, the optimal auction can be written as a convex combination over optimal \( k \)-unit auctions. We begin by showing that the revenue of optimal \( k \)-unit auctions can be approximated by multi-unit highest-bids-win auctions when the agents’ values are distributed according to a regular distribution (Lemma F.1 below). In the irregular case, on the other hand, rank-based auctions cannot compete against arbitrary optimal auctions. For example, if the agents’ value distribution contains a very high value with probability \( o(1/n) \), then an optimal auction may exploit that high value by setting a reserve price equal to that value; on the other hand, a rank-based mechanism cannot distinguish very well between values correspond to quantiles above \( 1 − 1/n \). We show that rank-based mechanisms can approximate the revenue of any mechanism that does not iron or reserve price within the quantile interval \([1 − 1/n, 1]\) (but may arbitrarily optimize over the remaining quantiles). Theorem F.3 presents the precise statement.

**Lemma F.1.** For regular \( k \)-unit \( n \)-agent environments, there exists a \( k′ ≤ k \) such that the highest-bid-wins auction that restricts supply to \( k′ \) units (i.e., a rank reserve) obtains at least half the revenue of the optimal auction.

**Proof.** This lemma follows easily from a result of Bulow and Klemperer (1996) that states that
for agents with values drawn i.i.d. from a regular distribution the revenue of the \(k'\)-unit \(n\)-agent highest-bid-wins auction is at least the revenue of the \(k'\)-unit \((n - k')\)-agent optimal auction. To apply this theorem to our setting, let us use \(\text{OPT}(k, n)\) to denote the revenue of an optimal \(k\)-unit \(n\)-agent auction, and recall that \(nP_k\) is the revenue of a \(k\)-unit \(n\)-agent highest-bids-win auction.

When \(k \leq n/2\), we pick \(k' = k\). Then,

\[
P_k \geq \text{OPT}(k, n - k) \geq \frac{(n - k)}{n} \text{OPT}(k, n) \geq \frac{1}{2} \text{OPT}(k, n),
\]

and we obtain the lemma. Here the first inequality follows from Bulow and Klemperer’s theorem and the third from the assumption that \(k \leq n/2\). The second inequality follows via by lower bounding \(\text{OPT}(k, n - k)\) by the following auction which has revenue exactly \(\frac{(n - k)}{n} \text{OPT}(k, n)\):

simulate the optimal \(k\)-unit \(n\)-agent on the \(n - k\) real agents and \(k\) fake agents with values drawn independently from the distribution. Winners of the simulation that are real agents contribute to revenue and the probability that an agent is real is \(\frac{n - k}{n}\).

When \(k > n/2\), we pick \(k' = n/2\). As before we have:

\[
P_{n/2} \geq \text{OPT}(n/2, n/2) = \frac{1}{2} \text{OPT}(n, n) \geq \frac{1}{2} \text{OPT}(k, n).
\]

**Lemma F.2.** For (possibly irregular) \(n\)-agent environments with revenue curve \(R(\cdot)\) and quantile \(q \leq 1 - 1/n\), there exists an integer \(k \leq (1 - q)n\) such that the revenue of the \(k\)-highest-bids-win auction is at least a quarter of \(nR(q)\), the revenue from posting a price of \(v(q)\).

**Proof.** First we get a lower bound on \(P_k\) for any \(k\). For any value \(z\), the total expected revenue of the \(k\)-highest-bids-win auction is at least \(zk\) times the probability that at least \(k + 1\) agents have value at least \(z\). The median of a binomial random variable corresponding to \(n\) Bernoulli trials with success probability \((k + 1)/n\) is \(k + 1\). Thus, the probability that this binomial is at least \(k + 1\) is at least 1/2. Combining these observations by choosing \(z = v(1 - (k + 1)/n)\) we have,

\[
P_k \geq v(1 - (k + 1)/n) k/2.
\]

Choosing \(k = \lfloor(1 - q)n\rfloor - 1\), for which \(v(1 - (k + 1)/n) \geq v(q)\), the bound simplifies to,

\[
P_k \geq v(q) k/2.
\]

The ratio of \(P_k\) and \(R(q) = (1 - q) v(q)\) is therefore at least

\[
\frac{k}{2(1 - q)n} \geq \frac{k}{2(k + 2)}.
\]

For \(q \leq 1 - 3/n\) (or, \(k \geq 2\)) this ratio is at least 1/4.

For \(q \in (1 - 3/n, 1 - 1/n]\), we pick \(k = 1\). Then, \(P_1\) is at least 1/n times \(v(q)\) times the probability that at least two agents have a value greater than or equal to \(v(q)\). We can verify for \(n \geq 2\) that

\[
P_1 \geq \frac{v(q)}{n} (1 - q^n - n(1 - q)q^{n-1}) \geq \frac{1}{4} (1 - q) v(q).
\]
Theorem F.3. For regular value distributions and position environments, the optimal rank-based auction obtains at least half the revenue of the optimal auction. For any value distribution (possibly irregular) and position environments, the optimal rank-based auction obtains at least a quarter of the revenue of the optimal auction that does not iron or set a reserve price for the highest $1/n$ measure of values i.e., $q \in [1 - 1/n, 1]$.

Proof. In the regular setting, the theorem follows from Lemma F.1 by noting that the optimal auction (that irons by value and uses a value reserve) in a position environment is a convex combination of optimal $k$-unit auctions: since the revenue of each of the latter can be approximated by that of a $k'$-unit highest-bids-win auction with $k' \leq k$, the revenue of the convex combination can be approximated by that of the same convex combination over $k'$-unit highest-bids-win auctions; the resulting convex combination over $k'$-unit auctions satisfies the same position constraint as the optimal auction.

In the irregular setting, once again, any auction in a position environment is a convex combination of optimal $k$-unit auctions. The expected revenue of any $k$-unit auction is bounded from above by the expected revenue of the optimal auction that sells at most $k$ items in expectation. The per-agent revenue of such an auction is bounded by $\bar{R}(1 - k/n)$, the revenue of the optimal allocation rule with ex ante probability of sale $k/n$. Here $\bar{R}(\cdot)$ is the ironed revenue curve (that does not iron on quantiles in $[1 - 1/n, 1]$). $\bar{R}(1 - k/n)$ is the convex combination of at most two points on the revenue curve $R(a)$ and $R(b)$, $a \leq 1 - k/n \leq b < 1 - 1/n$. Now, we can use Lemma F.2 to obtain an integer $k_a < n(1 - a)$ such that $P_{k_a}$ is at least a quarter of $R(a)$, likewise $k_b$ for $b$. Taking the appropriate convex combination of these multi-unit auctions gives us a 4-approximation to the optimal auction $k$-unit auction (that does not iron over the quantile interval $[1 - 1/n, 1]$). Finally, the convex combination of the multi-unit auctions with $k_a$ and $k_b$ corresponds to a position auction with that is feasible for a $k$ unit auction (with respect to serving the top $k$ positions with probability one; service probability is only shifted to lower positions). \qed