EPR program: a local interpretation of QM

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Abstract

An alternative interpretation of QM is presented. Hidden variables prescribe the output of measurements for individual states of the ensemble, but a distribution of amplitudes of probability, instead of a distribution of probabilities, is associated to them, as in orthodox QM. Consequently, Bell’s type inequalities do not apply. In this interpretation, locality is restored in EPR experiments with entangled subsystems, as the observable relative frequencies are obtained by the usual interference for sum of amplitudes, in complete parallelism with the two slit experiment. The action reaction principle seems to be violated in some QM measurements, and the existence of hidden variables could solve this paradox too. In indirect measurements it remains obscure the way it could be restored. A version of the two slit experiment is proposed to search for the hidden reaction.

Keywords: Alternative interpretations of QM, measurement problem, non locality, path integral formalism.

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1 Introduction

Quantum Mechanics (QM), including Quantum Field Theory (QFT), is the most successful mathematical framework of physical theories, with regards to its broad scope of applications and accuracy of predictions; but it is also a
battle field of deep metaphysical debates. Besides the orthodox interpretation \cite{1, 2, 3, 4}, many different alternative schools have appeared \cite{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}; to get a wide view of the present state of the art is a though undertaking (see for example \cite{21, 22, 23, 24, 25, 26}, obviously not up to date).

Alternative theories, i.e., with experimentally distinguishable predictions from QM, either have already been ruled out or are out of scope of the present technology. Yet, the search of alternative interpretations of QM has not been left to academics in metaphysics. Why? Probably because the odd behaviours we find in QM are far away from the classical world, and our nearest understanding of nature seems to be denied by QM. We can elaborate a list of paradoxical QM behaviours, in order of relevance with regards to their challenge against classical and common sense concepts and knowledge. This list is obviously a personal matter. Mine is

1. Measurement problem and the projection rule.

Measurement and the projection of state, as a physical phenomenon different from other quantum interactions, is the source of many interpretative problems: subjectivity, non local projection of state process for entangled systems, quantum classical boundary at the measurement apparatus, unknown definition of macroscopic system, etc.

2. Non local interaction.

Special and General Relativity (SR, GR) are as firmly consolidated as QM; QFT combines QM and SR. Yet, we should agree that some natural law breaks relativistic invariance if and only if there was found unambiguous observational evidence of a physical non local interaction. Bell’s type inequalities \cite{27}, as well as inconsistencies in assignment of pre–measurement (hidden) values to some families of operators \cite{28, 29} (GHZ theorems), are considered evidence of the spooky action at distance.

3. Tunnel effect.

Again, a broken (in a limited way) fundamental law, conservation of energy. QM does not describe the process of the system from inside the potential well to outside, just probabilities. Initial and final energy match; however, the hypothetical process is not understood, energy fluctuation is perhaps the best description we can find.
4. Wave particle duality

Alternative interpretations as “wave and particle”, “nor wave neither particle”, etc., reflect our ignorance. Fact is we observe particles (spot in a screen or photographic plate, path in a cloud chamber, up to accuracy resolution) in individual measurements, and diffraction patterns, as in the two slit experiment, exclusively in statistical samples.

5. others

Since the beginning of modern QM formulation in the nineteen twenties these and other questions opened a debate about the very meaning of scientific knowledge: What a scientific theory is? What an interpretation provides? What elements of reality are represented in the mathematical formulation? What do we do of unavoidable mathematical “artifacts” without obvious physical meaning? ... The EPR program \cite{EPR} is a generic proposal about these subjects; two explicit conditions where listed, reality and completeness. Another, locality, was implicit in the arguments (“...without in any way disturbing a system”): a system can not be disturbed by interaction with another one with spatial (i.e., not causal) separation. Locality can be included in a more generic requirement of conservation laws, which could obviously be abandoned if the corresponding symmetries were violated. Reality and completeness are more conceptual, metaphysical, they represent basic requirements we would ask to a scientific theory:

1. Completeness: Every element of the physical reality must have counterpart in the physical theory. That is, some mathematical object from which we can calculate predictions about this element of reality. Mass and electromagnetic field, e.g., can be elements of reality, while their counterparts $m$ and $F_{\mu\nu}$ are a parameter and a two form representing them.

2. Reality: If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity. The physical system we are representing has some property, whose counterpart is a mathematical artifact in out theory.

The gedanken experiment presented in EPR, where position and momentum of an isolated particle could be predicted through measurements in its
entangled pair, was argued as evidence of the incompleteness of QM, where both magnitudes do not commute and can not have simultaneously precise values. Advocates of the orthodox interpretation stood behind the completeness of QM. First, the “could” was interpreted as these magnitudes not actually having values before measurement of the other particle. Second, it was increasingly apparent that QM (at least in its orthodox interpretation) is not a local theory, an implicit ingredient in the EPR argument.

Completeness, reality, and locality (possibly, other conservation laws and basic principles were also implicitly considered in the EPR spirit) are the EPR program for an interpretation of QM. However, another “implicit” condition seems to have been amply imposed by researchers on the subject, existence (or not) of a classical probability distribution associated to hidden variables. There is not any reference to this condition in EPR, the program is open to any kind of non classical theories, as far as they fulfil the three requirements.

Along the way there have appeared different evidences against hidden variables. We can group them in two main groups, the probabilistic arguments [27], and the inconsistencies when assigning hidden values to some families of “physical” magnitudes (self adjoint operators) [28, 29]. Bell’s type inequalities theorems make mathematically impossible to find classical probability distributions for hypothetical hidden variables matching the QM (and experimental [31]) results. Theories whose formulation is not probabilistic in the classical sense are not banned; QM is one of them, when using these weird objects named amplitudes of probability. GHZ and similar theorems apply to families of non local self adjoint operators. If the EPR program is to be completely fulfilled, including locality, these non local objections will be inconsistent. How can exist a pure quantum state eigenvector of non local operators if the initial conditions were reached through local interactions and the evolution is also local (a kind of superselection rule)? In an interpretation of QM fulfilling the EPR program non local self adjoint operators can not represent elementary physical magnitudes; only local magnitudes (and not necessarily all self adjoint operators, there is no explicit prescription about this point in QM) can be measured in local measurements (interactions).

Subjective interpretations, even if they are not popular, make very relevant the task of finding an interpretation of measurement as a regular interaction, not different from others, and to restore objectivity in Physics, at least as a plausible hypothesis of work. Measurement presents difficulties not only when the projection rule must be applied, where the output state is different from the input; in the simplest measurement of a physical magni-
tude in an eigenstate of the corresponding operator we find a paradox, there is a change of state in the measurement system (the pointer) while, according to QM, input and output states of the measured system coincide. The action reaction principle seems to be violated. Also in indirect measurements, when some “virtual” path of the system is obstructed; even when, in an individual event, there is not interaction with the additional ingredient (obstacle or measurement apparatus), the system under study changes of state, for example with a behaviour that was forbidden in absence of the obstruction. Action reaction should be added to locality causality as possibly broken laws in the orthodox interpretation of QM.

Let us review some evidences. One, electrons are point particles. This is scientific, inductive evidence. Of course it must be understood as approximate, with finite accuracy, etc. But, individual electrons (and photons) imprint a point like spot on a screen or photographic plaque, and a unidimensional path on a cloud chamber; we sometimes find wavelike patterns for an ensemble, not individual particles. QM is so puzzling that we are uncertain of applying the scientific method of induction here. Each time the mass of an electron is measured, with different techniques, similar values are found. Of course direct evidence is just about these individual electrons actually measured, but each new measurement increases (by induction) our confidence in the existence of a universal constant $m_e$ (apart from many indirect evidences, obviously). It would be enough one different measured value (and I am sure it would be analysed and repeated with the most extreme care) to abandon this hypothesis. Scientific rules of the game are this severe, and that is why scientific knowledge is so robust. A wavelike (or any other not particle like) imprint of an individual electron will erase the particle like hypothesis. Meanwhile, electrons are point particles.

Of course, electrons are point particles when we observe them. This is almost a tautology, but again QM paradoxes have made us insanely careful. When we use detectors in the two slit experiment to observe the electron in the slits, it always goes through one or the other slit, never through both or any other way. This is consistent with the previous evidence of electrons as particles. Yet, we avoid to assert that this is also the case in absence of detectors. In QM there is not detailed description of the process.

The two slit experiment is also scientific evidence of a wave like behaviour of electrons (individual electrons, not an ensemble of them in interaction as fluid waves). These scientific facts, which seem contradictory, are a challenge to our ability to built models, but to deny raw facts, or to state that things
are different when we close our eyes (a strongly subjective statement) does not seem a satisfactory solution. The EPR program has not expired. The point particle property is an element of reality, and to incorporate it into QM, even if all its predictions are preserved (that is, just as a metaphysical addendum) has scientific interest and could be source of inspiration for new theories.

Even if scientifically unsuccessful, the debate is sometimes valuable because it fosters a deep analysis of many interesting points that usually do not deserve our attention, but become instructive after a more observant study. As an example, consider the concept of (or alternative approaches to) probability [32]: either as relative frequencies in a finite universe of elements with non identical properties, or as “propensities”, probabilities that identical elements of a considered universe entail to show different properties when observed. In QM, a pure quantum state is a complete description of an individual physical system with propensities, not an ensemble of different physical systems with relative frequencies.

Imagine a large black box with $2^N$ balls; you are informed that either (1) $N$ are red and $N$ green, or (2) each one is transparent but has equal probability of becoming red or green as soon as it is illuminated. If after a number of extractions you get $N + 1$ red or green balls, the first case is discarded. You can end up with $N$ red and $N$ green balls in both cases, but this is very improbable in the second case for large $N$, so you could guess (1). There is a practical impossibility to perform the former method for very large $N$, say of the order of $10^{23}$; with a limited sample, just a small fraction of the total, there is no way to distinguish (1) from (2). OK, open the box and have a look. You will see, at least in the top where light illuminates, some red and green balls; that is the problem with properties hidden by law.

In EPR experiments (entangled particles) it seems the propensity concept wins, because there are no local hidden variables with a classical distribution of relative frequencies able to match the QM and experimental correlations. After Alice measurement there is some non local transmission of propensities to the second particle through the projection rule, a non local process. However, it could also be transmitted, again in a non local way, the output of Bob’s measurements in arbitrary directions, following a table of relative frequencies in a draw. EPR experiment is about local versus non local behaviour (and completeness of QM, and elements of reality), not about propensities (represented in the quantum state) versus relative frequencies (represented through, e.g., non local hidden variables).
In order to review the main facts that challenge our interpretation of QM it seems advisable to focus on some paramount experiments. After an analysis of step interactions (e.g. classical measurement) in section 2, particles with entangled spin experiment and the two slit experiment will be the subject of the following two sections. The action reaction principle is a central issue in the development of the proposal presented in this paper; in section 5 we consider an hypothetical (artifact) accompanying system in interaction with the point particle, of stochastic character, Brownian like, instead of the deterministic quantum potential in the de Broglie Bohm theory (dBB). Last section presents an interpretation of QM with “minimal” ingredients, grounded in the path integral formalism, with a hidden label attached to a particular path in each individual quantum system. Because of the accompanying system, the labelled path of the particle is not a complete description of the individual system. The pure quantum state is understood as a statistical (ensemble) representation of the individual particle plus accompanying system compounds. We will finally summarise how this interpretation reads some of the paradoxical phenomena of QM previously presented.

2 Step interaction, and measurement

We want to interpret measurement as any other interaction in QM, with the obvious property that the measurement apparatus modifies its pre–measurement state into a macroscopically different post–measurement state (the pointer), independently of the presence of an observer. If the projection rule should be applied to all interactions of small quantum systems with macroscopic systems (precisely defined), with explicit and objective quantum rules, the problem of measurement would not exist. But this is not the orthodox interpretation of QM. The projection rule represents either a subjective process associated to the observer, or, in some research lines, a physical phenomenon external to QM, where decoherence, entropy or some unknown boundary between CM and QM, explains the change of quantum state as a short walk outside QM, the system returning to the quantum world in a point different from its departure, giving way to the projection of state, non unitary jump.

It is paradoxical that CM can be both an approximation of QM and an independent physical theory (at least when applied to quantum measurement with macroscopic systems). It is paradoxical that Statistical Mathematics can give way to some fundamental physical law, outside QM. Can an en-
semble violate the fundamental interaction and evolution rules (as unitary evolution) of its simple constituents?

2.1 Classical Hamiltonian step interaction

Let us consider a step interaction in Classical Mechanics, along a short time interval $\tau$. In Hamiltonian formulation (the Lagrangian approach is equivalent; in the context of Newtonian Mechanics we should take into account the action reaction principle) and for small $\tau$, a simple model is the interaction Hamiltonian $H_{\text{int}} = \lambda qy$, where $q$ and $y$ are variables of both systems in interaction respectively, and $\lambda$ represents an intensity of interaction; equivalently, $\lambda = 0$ for $|t| > \tau/2$. Hamiltonians $H_1$ and $H_2$ determine the dynamics of both isolated systems, but we will not consider the additional “free” evolution along $\tau$ associated to them, neither do we consider it to be negligible, although in a measurement it usually is. Our interest is focused on the additional evolution associated to the interaction, more precisely on its first order approximation.

There is a step evolution for all variables $p_q$ and $\pi_y$ of each system which do not commute with $q$ and $y$, $\Delta p_q = \{p_q, H_{\text{int}}\} \tau = \{p_q, q\}(\lambda \tau)y$, and $\Delta \pi_y = \{\pi_y, H_{\text{int}}\} \tau = \{\pi_y, y\}(\lambda \tau)q$. As we see, this step is proportional to the interacting variable of the other system, so it can be used to distinguish among states of each system with different $q$ and $y$ values. $q$ and $y$ are constant (up to their isolated systems dynamics) along $\tau$, but for $H_1(p_q)$ and $H_2(\pi_y)$ there will be an evolution of $q$ and $y$ after the interaction different from the isolated system one, because of the increments $\Delta p_q$ and $\Delta \pi_y$. If we consider the second system as a measurement apparatus, it is designed to transform $y$ or $\pi_y$ into a new macroscopic state of the pointer. It is unavoidable that the output state of the measured system will be different from the input. It is a technological challenge to minimise this perturbation, $\lambda \tau \to 0$, and yet get a macroscopic response. There is, however, no fundamental limit of accuracy in CM.

Of course, $H_{\text{int}}$ is an idealisation, and real interactions are more complex $H_{\text{int}} + H_c$. We can apply a step interaction on an ensemble of systems 1, and classify them into sub–ensembles according to the output. It is clear that the specific protocol we choose to do it will in general give way to different sub–ensembles, even though the “relevant” term is an ideal $H_{\text{int}}$. The probably unavoidable $H_c$ introduces different outputs for different techniques, no matter if the measured magnitude is $q$ in all of them.
2.2 Quantum step interaction

In QM we face a quite different phenomenon. First, there is a minimal (inter)action, \( H \tau \geq h \), Planck’s constant; this property makes explicit the discrete and discontinuous character of QM interactions. Second, perhaps related to the former, ideal measurements can be performed; when a physical magnitude is measured, the final quantum state is common to different measurement techniques, at least for discrete magnitudes. So, there is something fundamental in these interactions, independent of particular details. If there is a discrete, discontinuous step, it seems reasonable that small differences between measurement processes for the same magnitude do not (can not) give way to forbidden small differences in the output. Quantum interaction (and therefore measurement) becomes quite different from its classical counterpart at this small, quantum, scale; at larger scales we can recover the classical approximation when the discrete, discontinuous, phenomenon can be approached by the continuum. Magnitudes as spin have, on the other hand, no classical counterpart.

The problem with the projection rule under measurement is not about perturbation of the measured system, this perturbation is also a classical phenomenon; neither it is about discontinuities without classical counterpart, some quantum discontinuities could be represented by unitary step maps. The problem is that measurement is not like other interactions, it is not a unitary map. The best prescription at hand for the projection rule is that everybody knows when a measurement happens.

Hidden variables theories state that measurement is like any other interaction, that a hidden difference between individual systems in the same pure quantum state determines the difference in the outputs. No role of observer, no unknown boundary between QM and other worlds. Although there are well known difficulties with hidden variables theories and their interpretation, it should be acknowledged its merit in this particular point. And measurement interpretation is a very relevant issue.

Some people reject hidden variables because they can not give a local description of EPR, according to Bell’s inequalities. But the measurement problem has enough entity by itself to ponder alternative interpretations, even if they do not solve all other paradoxes in QM, as non locality. We can reject non local hidden variables theories because of Ockham’s razor or whatever other reason, not because their reach is limited to some, and not all, paradoxes.
There is an argument in favour of hidden variables I have not found in the (too large to study) literature. It does not refer to the projection rule when the input system is not an eigenstate of the measured magnitude (the difficult challenge of violated unitary evolution under measurement), but to the simples measurement of a pure quantum eigenstate. The output quantum system equals the input, according to QM rules. It is something impossible at the classical level, there is necessarily some change of state. Hidden variables could change at each individual measurement, in such a way that the overall description of the ensemble, the quantum state, was maintained. Hypothetical perturbed variables, not commuting with the measured magnitude, do not have (in QM) definite values before and after the interaction; if they had a hidden value and it was modified by the measurement, it would remain hidden. Does the action reaction principle not apply to this process in QM? The pointer has moved after measurement because of the interaction, not as an isolated evolution of the apparatus. Where is the additional step, unavoidably associated to an interaction, in the measured system?

3 Entangled spins, and non local interaction

Let us analyse now the experiment (we will denote it EPR) of generation, at event O, of two particles \( a \) and \( b \) in a null total spin state, followed by measurement of spins at spatially separated events A and B. I propose three different interpretations, not to conclude that one of them is preferred but because there are alternatives and perhaps they tells us something about which to ponder. A preliminary version of the content of this section was presented in [33].

1. First, the orthodox interpretation. A statistical ensemble of measurement events A and their outputs determine projection of state of the compound system and, as a consequence, a modified table of probabilities for an ensemble of measurements in B, or vice versa. If the projection rule is a real physical process the phenomenon is non local. It is difficult to state an \textbf{individual} change of state in particle \( b \) when this state is characterised by a table of probabilities, we necessarily need a statistical sample to detect the change of state. Does change each individual state of particle \( b \) after each individual measurement of particle \( a \), or are we selecting a conditional ensemble in the statistical
sample? The heart of EPR is about this dilemma. If we could state unambiguously an individual change of state in particle $b$ there would be no doubt about non locality. But we can not. Even when a physical state is described by a table of probabilities, it is possible to detect a physical change of state of an individual system in two ways:

(a) Some output that was forbidden in the initial state becomes possible (not necessarily certain) later on. An individual process in which this previously forbidden output occurs shows unambiguously a change of state. We do not need statistics here.

(b) Some output that was certain becomes uncertain after the process. Each individual process in which the previously certain output does not occur also shows unambiguously a change of state.

None of these ways can be applied in EPR experiment. There are no, previous to the measurement event $A$, forbidden or certain outputs for spin measurement at $B$. A change in the table of probabilities can not be, up to the previous cases, stated unambiguously in a single measurement event. Notice that certainty, probability null or unit, for some output of measurement in $B$ after (better once known) measurement output in $A$, is not conclusive about a change of state, as far as this output was possible beforehand.

2. Local hidden variables are ruled out because of Bell’s type inequalities. There are not local probability distributions able to match the QM and experimental correlations. This is mathematical fact, you can not beat a theorem. However, it is quite easy to solve the linear equations for a table of hypothetical quasi probabilities reproducing QM correlations. In the simplest non trivial case of three directions of spin, $\theta_j, j = 1, 2, 3$, the solution is

$$W(s_1, s_2, s_3) = \frac{1}{8}[1 + s_1 s_2 \cos(\theta_2 - \theta_1) +$$

$$+ s_1 s_3 \cos(\theta_3 - \theta_1) + s_2 s_3 \cos(\theta_3 - \theta_2)]$$

where $s_j = \pm$ represent the spin state in direction $\theta_j$. We understand that the state of particle $b$, $(-s_1, -s_2, -s_3)$, is opposite to that of $a$ in
a total null spin state. $W$, and not $P$, is used to represent weights (or Wigner), because they can be negative. $W$ is a mathematical artifact that can not be measured. Observable relative frequencies are correctly obtained

$$P(s_1, s_2) = W(s_1, s_2, +1) + W(s_1, s_2, -1) = \frac{1}{4}(1 + s_1 s_2 \cos(\theta_2 - \theta_1))$$

so, $\frac{1}{2}\cos^2((\theta_2 - \theta_1)/2)$ for $s_1 s_2 = +1$, or $\frac{1}{2}\sin^2((\theta_2 - \theta_1)/2)$ for $s_1 s_2 = -1$.

It could be argued (but I do not stand behind it) that it is better to loose positivity of $W$ in order to preserve locality. $W$ represents the spin equivalent of Wigner’s quasi probability distribution in phase space [34], reproducing orthodox QM at the expense of a $W(q, p)$ distribution function with both positive and negative values.

3. We can maintain the hidden variables description $(s_1, s_2, s_3)_a$, $(−s_1, −s_2, −s_3)_b$, and instead of a table of probabilities, assign them a table of amplitudes of probability. After all, this is the orthodox QM way, but now with additional variables. The solution is even simpler,

$$\Psi(s_1, s_2, s_3) = \sum_{j=1}^{3} s_j e^{i\theta_j}$$

in a not normalised representation. As usual with amplitudes,

$$P(s_1, s_2) = \frac{|\Psi(s_1, s_2, +) + \Psi(s_1, s_2, −)|^2}{\sum_{s'_1, s'_2 = ±} |\Psi(s'_1, s'_2, +) + \Psi(s'_1, s'_2, −)|^2}$$

is the probability, relative frequency of the output. If we could measure three spin directions, the corresponding hypothetical probabilities would be $|\Psi(s_1, s_2, s_3)|^2$, adequately normalised. It is interesting to notice that the negative weight $W(+, −, +)$ for angles $\{0, \pi/3, 2\pi/3\}$ becomes $\Psi(+, −, +) = 0$. We can not measure simultaneously two directions, but one of them is indirectly measured in the companion particle, “without in any way disturbing the system”.

What we have found is a description of EPR with hidden variables and amplitudes of probability as machinery to calculate relative frequencies of outputs. If we understand $(s_1, s_2, s_3)$ as fixed (but hidden) in
an individual experimental event, the output of measurement is fixed from the generation event $O$ of the entangled pair; so, no non local interaction between A and B measurement events. However, as we are calculating relative frequencies through the amplitudes of probability, and it is the “interference” term in $\Psi(s_1, s_2, +) + \Psi(s_1, s_2, -)$ that allows to reproduce the QM correlations, we should perhaps conclude that there is an interference interaction between different generation events, i.e., $O_1$ with $O_2$, etc.

I am sure this interpretation will not be popular, but it is interesting that it exists. After all, while A and B are spatially separated if we want to check non locality (they can be at both far ends of the observable universe), $O_k$ are usually in a temporal, causal space time relation. But this is not the point, we could obtain the same result if, instead of using a unique experimental set up and repeated measurements with it, we prepare a large number of identical copies of the apparatus and perform just one experiment with each one; now these $O_k$ can also be spatially separated.

Another interesting point of this interpretation is that non locality can be generalised, democratically extended to other QM interactions and measurements, e.g., the two slit experiment. We usually interpret it as an interference phenomenon between both components (left and right slit) of the amplitude function, without any relationship with a non local phenomenon. We can split both components into different individual events (an individual electron arriving to the screen); after all we know that, when we look, the electron always goes through one or the other slit, never through both. The two slit experiment can also be performed with $N$ copies of the experimental set up, spatially separated. We could say there is just one two slit experiment (after rescaling of relevant distances and momentum of the electrons) that is successively performed every year at different universities and research centres, for example for pedagogical reasons. If we collect all data of two slit experiments to the present time (an rescale appropriately) there would certainly be a much better sample size than usual.

This line of reasoning can be put upside down. If other quantum interference phenomena can be read as non local like EPR, we could also read EPR as a local interference phenomenon. Each particle could have spin wavelike behaviour, as it has spatial wavelike behaviour. We do
not propose waves of particle \( a \) interfering with waves of particle \( b \), a non local phenomenon; waves of particle \( a \) among themselves, and \textbf{independently} waves of particle \( b \) among themselves, each \( a \) and \( b \) having a copy of the family of amplitudes (waves) \( \{ \Psi(s_1, s_2, s_3) \} \) together with the particle spin state \( (s_1, s_2, s_3)^a, (−s_1, −s_2, −s_3)^b \).

Notice the equivalence with the two slit experiment. The electron going through one slit, and wave interference of both amplitudes. Left and right slits are in correspondence with the hidden \( s_3 = \pm \) in a third not measured direction. The probability distribution is \( |\Psi_L + \Psi_R|^2 \), and interference gives way to some regions of the screen being avoided by the electrons, although \( |\Psi_L|^2 \) and \( |\Psi_R|^2 \) do not vanish there. We do not need Bell’s type inequalities to see that no classical probability distribution exists with two \textbf{independent} strictly positive probabilities that add to zero.

4 Two slit experiment, and action–reaction principle

In the path integral formalism \cite{4} the sum of virtual paths from right slit into a particular region of the final screen determines \( \Psi_R \) and the sum of paths from the left one to the same region determines \( \Psi_L \). When there is not measurement in the slits determining which slit goes the electron through, the relative frequency of spots in the region of the screen is \( |\Psi_R + \Psi_L|^2 \). On the other hand, if we have detected the electron in the slits the distribution is \( |\Psi_R|^2 + |\Psi_L|^2 \), or we can split it into both components by sending the electrons in large enough time intervals to be able to assign each impact to R or L slit.

We can perform a measurement restricted to R slit, e.g., by using a light beam along it in such a way that all electrons going through R are detected (in an ideal approach) while no electron through L is detected. In case an electron arrives to the screen without having being detected we conclude it has gone through L, and apply the projection rule even though there has been no direct detection; this is an indirect measurement, and in orthodox QM the projection rule applies to both direct and indirect measurements.

Let us consider a small region of the screen where \( |\Psi_R + \Psi_L|^2 \simeq 0 \), and in a particular event in which, with the light beam switched on, a non detected electron arrives to this region. This is one of the cases in which we have unambiguously detected a change of state. But there has been no reaction
on the additional system, the light beam. Is the action reaction principle violated in this process?

Similar experiments have been performed with spin. Four Stern–Gerlach systems are prepared, numbers 1 and 4 in $X$ direction and numbers 2 and 3 in $Y$ and $-Y$ directions respectively. The electron beam out of 1 corresponding to + spin in $X$ direction (the $-X$ beam is discarded) goes then through 2 and 3, an finally through 4. If we do not block one of the beams (with + and $-\text{spin}$ in $Y$ direction) between 2 and 3, and being both processes opposite, the initial $+X$ state is reconstructed (no projection rule) and final measurement after 4 gives always $+X$ output.

If we block one of the beams, say $-Y$, electrons arriving to 3 are in $+Y$ state, and measurement in 4 gives $+X$ and $-X$ with even probabilities. Let us concentrate in a single event. An electron does not hit the initial obstacle after 1, neither the intermediate obstacle (both obstacles able to detect the particle) between 2 and 3, and after 4 it hits the $-X$ detector. An event that was forbidden in absence of the intermediate obstacle (that allows the indirect measurement) happens in its presence, so we can unambiguously state a change of state associated to the presence of the obstacle. Again, no reaction anywhere associated to the action on the electron, its change of state.

There seems to be something odd with the action reaction principle in QM, at least for indirect measurements. In the two slit experiment the theories of hidden variables will point to L or R as the hidden variable in $\Psi_R + \Psi_L$ for an individual event, with 1/2 relative frequency for each; but we can not understand $\Psi_R$ or $\Psi_L$ as a complete description of the individual electron state. Even with L as hidden variable we need $\Psi_R$ as a statistical representation of something real, an accompanying wavelike system. The reaction on the light beam for the indirect measurement in the former experimental set up could be exerted by this hidden companion. Taking into account that the diffraction pattern disappears, we could make the hypothesis that it is this wavelike companion which suffers a phase shift (action), so that the hypothetical spatial hidden diffraction pattern also shifts by an unknown distance. We only see one spot of each individual diffraction pattern. The shift is stochastic, different at each individual event, and the overall diffraction pattern, built spot by spot, disappears in the statistical battery of interactions.

If the action were a phase shift on the wavelike hidden companion, the reaction could be a phase shift on the photon. A laser light beam of coherent
photons could then be used to detect a phase shift (decoherence) whenever an undetected electron arrives to the screen. In QM the component of the quantum state discarded after projection is simply forgotten, although we could formally follow its evolution (wave function without associated particle) as far as it has no physical interaction with the surviving component; there is no statistical difference between discarding this component or maintaining it with a stochastic phase shift, that destroys coherence and the diffraction pattern in a statistical sample.

The question of what is the reaction on the light beam associated to the unambiguous action over the electron (and experimental set up around it, and hypothetical companion) remains obscure. But facts are conclusive; without light beam electrons do not hit on some forbidden areas of the screen; with the light beam switched on some undetected electrons do. Also detected ones, but we are focused on the “hidden” action reaction. This is not just a modified table of relative frequencies, it is an individual effect on an individual electron, whose actual end point on the screen was ruled out in absence of the additional system. There is (inter)action. It is the light beam that acts on the previous system and modifies its behaviour. Yet, no observable reaction appears.

The hypothesis of a hidden companion of the particle in QM is not new. The quantum potential of the de Broglie Bohm theory represents an interaction, and therefore there must be another system present. We briefly consider this possibility in next section.

5 “Free” particle, and accompanying system

The starting point of the de Broglie Bohm (dBB) theory [6] is well known. If we write the Schroedinger equation of the free electron for the modulus and phase variables we find a continuity equation for the probability density function $\rho = |\Psi|^2$, and a Hamilton–Jacoby (HJ) like equation for the phase, $\Psi = |\Psi|e^{(iS/\hbar)}$, identified with the action $S$ of usual HJ. This last HJ is that of a free classical particle plus a quantum potential $V_Q$, an odd function of $\rho$ and derivatives. Instead of understanding $V_Q$ as a deterministic interaction, it seems more reasonable (we will consider the hypothesis that the equation for the phase is a true, but stochastic, HJ) to interpret it as a stochastic term, after all it depends on a probability density. Stochastic interpretations of QM are developed in this or similar ways, e.g. [20].
We can describe qualitatively some consequences of this quantum interaction. First is the existence of an accompanying system, in interaction with the isolated electron. We can denote it æther or vacuum. Of course, not the classical æther associated to an absolute rest frame. It should be a Lorentz invariant vacuum. There is a Lorentz invariant vacuum in QFT. It is a physical system (although there is also debate around this statement), playing a fundamental role in the formulation, and it has observable effects, as the Casimir energy. On the other side of length scale there is overwhelming agreement that dark energy, responsible of a large fraction of the cosmic energetic content, is vacuum (zero point) energy. It sounds reasonable, vacuum is dark, it stores energy and there is a lot of vacuum in the universe \[33\]. Perhaps also dark matter could be associated to the vacuum, although it is not the favourite candidate, at least among particle physicist.

We can not understand in classical terms a Lorentz invariant fluid of mass \(m\) particles with infinite spatial density of particles and energy, like

\[
\Omega = \frac{mc}{p_0} d^3p \quad x^0 = ct
\]

which is obtained by pulling back, in the momentum space, Minkowsky’s metric into the mass shell \((mc)^2 = p_0^2 - \mathbf{p}^2\), and then calculating the associated 3–volume, an explicitly Lorentz invariant definition. Its limit \(m \to 0\) obviously vanishes (null vectors in the light cone), but we could, for example, substitute \(mc\) by a “zero point” momentum \(\pi_0\), associated to non vanishing statistical dispersion of vacuum energy. It could be a classical relativistic approximation of vacuum. We have vacuum playing a physical role at cosmological (dark energy) and perhaps galactic (dark matter) scale, and also at high energy and short length scale of QFT. Vacuum could be the accompanying system of the electron, in the intermediate scale of non relativistic QM; some early proposals about the role of vacuum zero point energy are \[35, 36, 37, 38\].

There are some analogies, just qualitative ones. Vacuum, as a fluid of “virtual” particles, interacting with the isolated electron, can be compared to Brownian motion. In Brownian motion there is a kind of uncertainty principle, dispersion on the displacement variable is \(\sigma_X = D\sqrt{t}\), so that for the average momentum \(<P(t)> = m(X(t) - 0)/t\) (null initial condition) we find \(\sigma_{<P>} = mD/\sqrt{t}\), and \(\sigma_X\sigma_{<P>} = mD^2\) constant.

In Brownian motion we are usually interested in the dynamics of the pollen grain, but there is obviously a reaction on the fluid at each collision,
giving way to a wavelike motion. A pollen grain in fluid water could be an attractive image of the wave particle duality, with interacting systems having properties one of wave and the other of particle. Also the tunnel effect finds here a representation; even if at large time intervals each subsystem preserves energy (statistically), there are fluctuations allowing the pollen grain to overcome a potential barrier. Statistical conservation of energy was proposed very early, in 1924 [39], although quickly rejected. In the isolated compound particle plus wave system total energy is conserved; yet, fluctuations of energy can occur between both subsystems. As far as they interact as a whole, these hypothetical internal fluctuations are hidden. Hypothetical existence of hidden variables, including a hidden companion, are the ingredients of the following alternative interpretation of QM.

6 A path integral formalism interpretation, with additional label

In the phase space of a classical point particle, there are individual and real classical paths corresponding to a classical deterministic evolution. In the path integral formalism of QM we consider all virtual paths corresponding to a physical state, and associate an elementary amplitude $\exp(iS_{\text{path}}/\hbar)$ to each of them. From it, we calculate the wave function by addition of these amplitudes, not probabilities. When we can distinguish (through measurement) between macroscopically different processes we apply classical probabilities to independent alternatives. There is no understanding, at the classical level, of this mysterious mechanism of the sum of amplitudes. A pure quantum state is described by a family of virtual paths

$$\mathcal{S} = \{\text{path}_l, l = 1, \ldots\}$$

The wave function, in the spatial representation, takes the value

$$\Psi(q_k) = \sum_l e^{iS_l(q_k)}/\hbar$$

at point $q_k$, where the sum is over all paths with endpoint $q_k$. The quantum state becomes in this representation

$$|\mathcal{S} > = \sum_k \Psi(q_k)|q_k >$$
6.1 Alternative interpretation

The goal of the following interpretation (and of most of them) is to describe measurement as an objective process, indistinguishable from other interactions. The first step is to select one of the paths as the actual one in an individual system, so that outputs of any measurement are determined by values of the physical magnitudes in the selected path. The pure quantum state becomes an ensemble of possible paths, similar to Brownian motion. The second ingredient of the interpretation is the accompanying system; the actual path of the point particle is not a complete description of the individual system. There is no much point in trying to develop a specific quantum stochastic dynamics of particle and vacuum wave. We have experimental evidence exclusively of the ensemble properties, and the vacuum is definitely different from classical fluids. So, no classical “explanation” of the sum of amplitudes rule, here linked to the hidden properties of a Lorentz invariant æther.

The alternative interpretation of QM is summarised in the following points

1. A pure quantum state $S$ (the previous set of virtual paths) is an ensemble of individual states, each one with a label attached to one of the paths.

2. The labelled path is not a complete description of the individual physical state. There is an accompanying system.

3. Outputs are determined, at each individual measurement, by the values of the physical magnitude in the labelled path. Relative frequencies are calculated through the usual QM machinery of amplitudes of probability.

In the previous notation,

$$S = \cup\{[\text{path}_i; \mathcal{W}_i]\}$$

with $\mathcal{W}_i$ denoting the accompanying system or wave. As in classical Brownian motion, the wave depends on the path, not just the end point state of the particle. Each collision modifies the momentum of the particle, and it is also a source of the accompanying wave. Amplitudes of probability encode an ensemble description of particle plus wave, including interference, fluctuations, etc.
Measurement is not different from any other interaction; there is no external theory or agent (conscience, thermodynamics, etc.) applied to measurement. The quantum system is observed in our macroscopic world through the label (if an interaction becomes enhanced up to macroscopic size); only a physical magnitude (or family of compatible ones) can be measured at a time because of the unavoidable interaction Hamiltonian properties. So, only partial information of the labelled path is found.

There are inconsistencies between hidden variables and non local self adjoint operators (GHZ). In a QM theory with non local interactions we can not reject the idea of non local measurements. However, if it is the case that interactions are local, then there is obviously no way to perform non local measurements.

6.2 Hidden states in spatial phase space

The following recipe could be a way to translate the previous qualitative interpretation of QM from path integral formalism into the Schroedinger representation. We consider a spinless point particle (hidden spin states are presented below), with associated Hilbert space $\mathcal{H}_{QM}$. Definite position is represented by a vector $|r>$, or better by the associated ray in the projective space. If we want to assign a momentum to the particle at $r$ (as before, the companion wave is not explicitly formulated, only its statistical effects) we need another coordinate. Let us denote $\mathcal{H}_r$ a Hilbert space corresponding to the vector $|r>$, and $\mathcal{H}_E = \bigoplus_r$ the direct sum for all points in space. Orthonormal vectors $|r,p>$ generate $\mathcal{H}_r$ (for fixed $r$) and $\mathcal{H}_E$. The correspondence

$$\mathcal{P} : \mathcal{H}_E \rightarrow \mathcal{H}_{QM} \quad \mathcal{P}(|r,p>) = \exp\left(\frac{i}{\hbar}p \cdot r\right)|r>$$

projects vectors in $\mathcal{H}_r$ onto $|r>$. A generic vector in $\mathcal{H}_E$ is

$$|S> = \int d^3r d^3p \Omega(r,p)|r,p>$$

But this would give way to an heterodox theory, out of QM. Instead, given $|S> = \int d^3r \Psi(r)|r>$ in $\mathcal{H}_{QM}$ we define

$$|S> = \int d^3r d^3p \frac{1}{\hbar^{3/2}} \Xi(p)|r,p>$$

20
such that \( \mathcal{P}(|S\rangle\rangle) = |S\rangle\rangle \), with \( \Xi \) the Fourier transform of \( \Psi \). For these particular types of vectors in \( H_E \) it is easy to check that operators \( X_{\mathbf{r}}, \mathbf{p} \) and \( P_{\mathbf{r}}, \mathbf{p} \) project at \( H_{QM} \) onto \( X_{\mathbf{r}}\Psi(\mathbf{r}) = x\Psi(\mathbf{r}) \) and \( P_{\mathbf{r}}\Psi(\mathbf{r}) = -i\hbar \partial_{\mathbf{r}}\Psi(\mathbf{r}) \).

For a definite position vector \( |\mathbf{r}_0\rangle \), \( \Psi = \delta(\mathbf{r} - \mathbf{r}_0) \), \( \Xi = \frac{1}{\hbar^{3/2}} \exp\left(-\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{r}_0\right) \), we have the hidden state

\[
|\mathbf{r}_0\rangle\rangle = \int d^3 \mathbf{p} \frac{1}{\hbar^3} \exp\left(-\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{r}_0\right)|\mathbf{r}_0\rangle, \mathbf{p} \rangle\rangle
\]

Formally, all momenta have equal probability in this state, \( \Omega^*\Omega = \frac{1}{\hbar^6} \), and \( |\mathbf{r}_0\rangle\rangle \) can not be normalised. The vacuum distribution presented beforehand, proportional to \( \int d^3 \mathbf{p} / p_0 \), has a non relativistic limit proportional to \( \int d^3 \mathbf{p} \), uniform. What we have defined is a non relativistic limit of the vacuum as the hidden distribution of momenta for a point particle with definite position. Then, \( \frac{1}{\hbar^{3/2}} \Xi(\mathbf{p}) \) could be understood as a statistical representation of the modified vacuum background distribution in a generic state.

This purely formal analogy could underpin the role of the vacuum as accompanying system.

### 6.3 Hidden spin states

Spin states in QM are described by vectors in a two dimensional Hilbert space \( H = \langle |+\rangle, |-\rangle \). In section 3 we have considered hidden spin states \( (s_1, s_2, \ldots, s_N) \), \( s_j = \pm \), determining the output of measurement in an arbitrary direction \( \theta_j \) on a plane; they can be represented by vectors in \( H_{\text{spin}} = H \otimes H \times \cdots \), \( |s_1\rangle \otimes |s_2\rangle \otimes \cdots \). A complete quantum state is then a linear combination with complex coefficients \( s_1 e^{i\theta_1} + s_2 e^{i\theta_2} + \cdots \). The generalisation to dimension three consist of defining \( H \) as a two dimensional vector space over the quaternions, and assign to the state \( |s_n\rangle \) (spin \( s \) in direction \( \mathbf{n} \)), the elementary spin “path” amplitude \( \Psi(s\mathbf{n}) = s(n_x I + n_y J + n_z K) \), \( I^2 = -1 \), \( IJ = K \), etc, with the algebraic properties of Pauli matrices. The total amplitude associated to the hidden state \( (s_1, s_2, \ldots, s_N) \) is then

\[
\Psi(s_1, s_2, \ldots, s_N) = \sum_{j=1}^N \Psi(s_j \mathbf{n}_j) = (\sum s_j n_{xj}) I + (\sum s_j n_{yj}) J + (\sum s_j n_{zj}) K
\]

Then

\[
\Psi(s_1, s_2) = \sum_{j\neq 1, 2} \sum_{s_j = \pm} \Psi(s_1 \mathbf{n}_1, s_2 \mathbf{n}_2, s_3 \mathbf{n}_3, \ldots, s_N \mathbf{n}_N)
\]
equals $\Psi(s_1 n_1) + \Psi(s_2 n_2)$, and

$$P(s_1, s_2) = \mathcal{N}|s_1 n_1 + s_2 n_2|^2 = \mathcal{N}2(1 + s_1 s_2 n_1 \cdot n_2)$$

$\mathcal{N}$ is the adequate normalisation, and we have used the identity

$$n_1^* n_2 + n_2^* n_1 = 2 n_1 \cdot n_2$$

where the left hand side is a quaternionic expression and the right hand side is vectorial, with the usual scalar product.

A $|+_z>$ spin state in orthodox QM is represented here by the ensemble of all hidden states with $s_z = 1$, each individual system with a label on one state determining the particular output in arbitrary directions. An accompanying “spin wave” coupled to the particle is statistically represented by the quaternionic amplitudes of the ensemble, with the corresponding interference phenomenon.

The orthodox total null spin state of the entangled particles of EPR is represented here by the union of both spin states with all outputs (not a sub ensemble as in $|+_z>$), such that the labels are correlated at the generation process, $(s_1, s_2, \ldots, s_N)_a^\lambda$ with $(-s_1, -s_2, \ldots, -s_N)_b^\lambda$. Hypothetical probability distributions for two or more spin directions in an individual particle are calculated as usual through the sum of amplitudes rule. In EPR, the measurement performed on $a$ is also an indirect measurement for $b$, and $P(s_1, s_2)$ becomes observable. But each measurement process is independent, i.e., after $A$ measurement in direction $n_1$, particle $a$ changes state to $|s_1^a>$, but particle $b$ remains in the initial state; we have just partial information about the position of $b$ label.

By restricting the analysis to $A$ measurement events in $n_1$ direction, we are determining two sub–ensembles of particles $b$, those whose label is attached to states containing $-s_1^b$ when $s_1^a$, respectively $s_1^b$ when $-s_1^a$. Interference of amplitudes associated to particle $b$ individual state determines the distribution of probabilities. Quantum interference of amplitudes is “we do not know what”, but not necessarily a non local phenomenon.

### 7 Summary

Probably the most celebrated alternative interpretation of QM is the de Broglie Bohm Theory, with a quantum potential and deterministic trajectories of point particles. All non hidden variables obstruction find here an
explicit example of possible alternative. However, it is non local as is QM, and no physical description of the interacting system, responsible of the quantum potential, is proposed.

Wigner’s quasi probability distribution in phase space appeared very early but, because of non positivity of the distribution function, it was rejected. We have seen that the same idea can be applied to spin states. There is not really any scientific argument against the previous interpretations, they are equivalent to orthodox QM. In particular, quasi probability distributions are the price to pay in order to substitute the mysterious sum of amplitudes rule by a rule closer to classical probabilities, and to introduce the point particle into the framework. Typical interference phenomena are here linked to the heterodox quasi probability, where independent events can add to a null probability. We probably prefer amplitudes because all the machinery has been developed for this formalism and we are accustomed to it.

Neither orthodox QM, nor dBB (no coupled system for reaction) or quasi probabilities confront the paradox with the action reaction principle. The accompanying wave in æther to the point particle opens the possibility to look for some reaction, at least in indirect measurements.

The alternative interpretation of QM here presented preserves the quantum machinery for calculating relative frequencies, the mysterious amplitudes, and simply adds a label (in the path integral formalism). This label prescribes the output of measurements. An orthodox quantum state is interpreted as a statistical ensemble of different physical systems; an individual system is not completely determined by the labelled path, there is a hidden wavelike companion, determined by the path and not by the final particle state. A complete description of individual systems (particle plus companion) is not proposed, the individual evolution being stochastic, Brownian like.

Measurement is an interaction in which one of the systems changes of macroscopic state, which simply means the observer can detect it. The projection rule is applied according to the found “position” of the label in a measurement; information is necessarily incomplete, so the final state belongs also to an ensemble.

The use of amplitudes of probability allows to interpret EPR correlations as local phenomena of interference, in complete analogy with the two slit experiment. Wave particle duality is described as the compound system of point particle plus accompanying wave (in the æther). Although QM does not describe in detail processes as the tunnel effect, fluctuations of energetic
interchange between both subsystems could be a qualitative interpretation. Total energy is obviously conserved, and interaction with a third system happens as a whole. As in Brownian motion, energy of each subsystem is statistically preserved.

The sum of amplitudes rule, taken for granted in orthodox QM and in this alternative interpretation, remains a mystery from the classical point of view; to assign it to the vacuum is just to rephrase the unknown. Moreover, the action reaction principle remains obscure too, at least in indirect measurements. In direct measurement there is no apparent paradox if the system is not an eigenstate of the physical magnitude; for eigenstates, if we add a hidden label the paradox can be explained by a hidden reaction (jump of the label) to the action on the pointer. Indirect measurement contradicts the action reaction principle unless there is hidden reaction of some kind.

It is proposed to look for a reaction in the modified two slit experimental set up with a laser light beam. The hypothetical action (phase shift) on the accompanying system (vacuum wave) could generate a similar reaction on the photon of the beam (decoherence). Just a possibility. But nature is not necessarily the way we think it is, less it is the way we want it to be.

8 Bibliography

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