COVERINGS OVER LAX INTEGRABLE EQUATIONS AND THEIR NONLOCAL SYMMETRIES

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Abstract. Using the Lax representation with non-removable parameter, we construct two hierarchies of nonlocal conservation laws for the 3D rdDym equation \( u_{ty} = u_x u_{xy} - u_y u_{xx} \) and describe the algebras of nonlocal symmetries in the corresponding coverings.

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INTRODUCTION

The 3D rdDym equation $E, [1, 2, 3]$, is an example of nonlinear integrable equations in three independent variables. Integrability here means the existence of a Lax pair with non-removable parameter. Such equations were studied rather intensively in recent papers, [4, 5]. In particular, in our recent papers [7, 8] we gave a full description of 2D symmetry reductions for four of such equations and discussed integrability properties of the reductions.

Using the Lax integrability of the 3D rdDym equation, we construct two infinite hierarchies of two-component nonlocal conservation laws (corresponding to non-negative and non-positive powers of the spectral parameter). To these hierarchies there correspond two infinite-dimensional Abelian coverings (in the sense of [9]) which we call positive and negative and denote by $\tilde{E}^+$ and $\tilde{E}^-$, respectively, and we describe the algebras of nonlocal symmetries in these coverings.

The equation itself possesses an infinite-dimensional algebra of local symmetries parametrized by three arbitrary functions in $t$ and one in $y$ plus one standing alone scaling symmetry (which allows to assign weight to the variables under consideration), see Table 1 below. We show that all these symmetries admit lifts to both positive and negative coverings. In addition to the extensions of local symmetries, new, purely nonlocal ones arise in both cases.

For the covering $\tilde{E}^+$, the scaling symmetry becomes the terminal element of the non-positive part of the Witt algebra $W^-$, while the $y$-dependent symmetries become a part of the loop algebra $L[y]$ whose coefficients are functions in $y$; a natural action of $W^-$ on $L[y]$ is defined. No new $t$-dependent symmetry arises on $\tilde{E}^+$ and the local ones form a graded ideal in $\text{sym}(\tilde{E}^+)$. The exact formulation is given in Theorem 1.

In the case of $\tilde{E}^-$ (see Theorem 2) the scaling symmetry becomes the first element in the non-negative part of the Witt algebra $W^+$ and the $t$-dependent symmetries become a part of the loop algebra $L[t]$. The algebra $W^+$ acts on $L[t]$, while the local $y$-dependent symmetries constitute a direct summand in $\text{sym}(\tilde{E}^-)$.

Finally, we show that the mutually inverse recursion operators found by one of the authors in [6] act on $L[y]$ and $L[t]$ and accomplish ‘tunneling’ between $W^-$ and $W^+$.

The exposition is organized as follows. In Section 1 we introduce some basic preparatory definitions and facts needed below. Section 2 describes the 3D rdDym equation: local symmetries, the Lax pair, and the coverings. The main results are formulated and proved in Section 3.

1. Preliminaries

We expose here (in a simplified, local coordinate form) the basics of the geometrical approach to differential equations and differential coverings following [10] and [9].

1.1. Jets and equations. Consider $\mathbb{R}^n$ with coordinates $x^1, \ldots, x^n$ and $\mathbb{R}^m$ coordinated by $u^1, \ldots, u^m$. The space of $k$-jets $J^k(n, m)$, $k = 0, 1, \ldots, \infty$, carries the coordinates $x^1, \ldots, x^n$ and $u^j_\sigma$, where $j = 1, \ldots, m$ and $\sigma$ is a symmetrical multi-index of length $|\sigma| \leq k$, $u^j_\sigma = u^j$. If $u^j = f(x^1, \ldots, x^n)$ is a vector-function then the collection

$$u^j_\sigma = \frac{\partial^{\sigma}u^j}{\partial x^\sigma}, \quad j = 1, \ldots, m, \quad |\sigma| \leq k,$

is called its $k$-jet.

At a fixed point $\theta \in J^k(n, m)$ tangent planes to the graphs of $k$-jets passing through this point span the Cartan plane $\mathcal{C}_\theta$ and the correspondence $\mathcal{E}: \theta \mapsto \mathcal{C}_\theta$ is
called the \textit{Cartan distribution}. For \( k = \infty \), a basis of \( \mathcal{E} \) consists of the vector fields
\[
D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u_j^\sigma \frac{\partial}{\partial u_j^\sigma}, \quad i = 1, ..., n,
\]
called the \textit{total derivatives}. The total derivatives commute which amounts to the formal integrability of the Cartan distribution on \( J^\infty(n, m) \).

Consider a submanifold in \( J^k(n, m) \) given by the relations
\[
F^1(x^i, u_j^\sigma) = \cdots = F^r(x^i, u_j^\sigma) = 0. \quad (1)
\]
This is a \textit{differential equation of order} \( k \). Its \textit{infinite prolongation} \( E \subset J^\infty(n, m) \) is given by
\[
D_{\sigma}(F^j) = 0, \quad j = 1, ..., r, \quad |\sigma| \geq 0,
\]
where \( D_{\sigma} = D_{x^i_1} \circ \cdots \circ D_{x^i_k} \) for \( \sigma = i_1 \cdots i_k \). Everywhere below we deal with infinite prolongations only and identify them with differential equations.

The total derivatives are restrictable to infinite prolongations and these restrictions span the Cartan distribution on \( E \). Maximal integral manifolds of this distribution are solutions.

1.2. \textbf{Symmetries}. Consider an equation \( E \subset J^\infty(n, m) \). We shall assume below that the natural projection \( E \to J^0(n, m) = \mathbb{R}^n \times \mathbb{R}^m \) is a surjective map \textit{onto} its target\footnote{This means that the differential consequences of (1) do not contain 0-order functions.}. Consequently, the algebra \( C^\infty(J^0(n, m)) \) is embedded into the algebra \( C^\infty(E) \).

A vector field \( X : C^\infty(E) \to C^\infty(E) \) is called \textit{vertical} if \( X|_{C^\infty(J^0(n, m))} = 0 \), i.e., \( X \) does not contain components of the form \( \partial/\partial x^i \). A vertical field \( X \) is a (\textit{higher}, or \textit{generalized}) \textit{symmetry} of \( E \) if it preserves the Cartan distribution, i.e., \([X, \mathcal{E}] \subset \mathcal{E} \). Symmetries of \( E \) form a Lie algebra denoted by \( \text{sym}(\mathcal{E}) \).

A vector field is a symmetry if and only if it has the \textit{evolutionary} form
\[
E_{\varphi} = \sum D_{\sigma}(\varphi^j) \frac{\partial}{\partial u_j^\sigma}, \quad (2)
\]
where summation is taken over the \textit{internal} coordinates on \( E \). Here \( \varphi = (\varphi^1, ..., \varphi^m) \) is a vector-function on \( E \) called the \textit{generating section} (or \textit{characteristic}) of the symmetry. It must satisfy the equation
\[
\ell_E(\varphi) = 0,
\]
where \( \ell_E \) is the \textit{linearization} of \( E \) defined as the restriction of the operator
\[
\ell_F = \left\| \sum_{\sigma} \frac{\partial F^j}{\partial u_j^\sigma} D_{\sigma} \right\| \quad (3)
\]
to \( E \). Generating functions form a Lie algebra with respect to the \textit{Jacobi bracket}
\[
{\{\varphi, \psi\}}^j = \sum \left( D_{\sigma}(\varphi^j) \frac{\partial \psi^l}{\partial u_l^\sigma} - D_{\sigma}(\psi^l) \frac{\partial \varphi^j}{\partial u_l^\sigma} \right),
\]
which in the coordinate-free way can be defined as \( \{\varphi, \psi\} = E_\varphi(\psi) - E_\psi(\varphi) \).
1.3. **Differential coverings.** Consider the space $\tilde{E} = \mathbb{R}^s \times E$, $s \leq \infty$, and the natural projection $\tau: \tilde{E} \to E$. We say that $\tau$ is an $s$-dimensional (differential) covering over $E$ if $\tilde{E}$ is endowed with vector fields $\tilde{D}_{x^i}, \ldots, \tilde{D}_{x^n}$ such that

$$[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0, \quad \tau_*(\tilde{D}_{x^i}) = D_{x^i}, \quad i, j = 1, \ldots, n.$$ 

Let $\{w^\alpha\}$ be coordinates in $\mathbb{R}^s$ (they are called *nonlocal variables*). Then the covering structure is given by

$$\tilde{D}_{x^i} = D_{x^i} + X_i$$

such that

$$D_{x^i}(X_j) = D_{x^j}(X_i) + [X_i, X_j] = 0,$$

where

$$X_i = \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}$$

are $\tau$-vertical vector fields.

There exists a distinguished class of coverings that are associated with two-component conservation laws of $E$. Fix two integers $i$ and $j$, $1 \leq i < j \leq n$, and consider a differential form

$$\omega = X_i dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n + X_j dx^1 \wedge \cdots \wedge \hat{dx^j} \wedge \cdots \wedge dx^n,$$

where ‘hat’ means that the corresponding term is omitted, closed with respect to the horizontal de Rham differential, i.e., such that

$$D_{x^i}(X_j) = (-1)^{i+j-1} D_{x^j}(X_i).$$

Consider the Euclidean space $V$ with the coordinates $w^\sigma$, where $\sigma$ is symmetric multi-index whose entries are any integers $1, \ldots, n$ except for $i$ and $j$. Thus, $\dim V = 1$ if $n = 2$ and $\dim V = \infty$ otherwise. Then the system of vector fields

$$\tilde{D}_{x^k} = D_{x^k} + \sum_\sigma w^{\sigma k} \frac{\partial}{\partial w^\sigma}, \quad k \neq i, j,$$

$$\tilde{D}_{x^i} = D_{x^i} + \sum_\sigma \tilde{D}_\sigma(X_j) \frac{\partial}{\partial w^\sigma},$$

$$\tilde{D}_{x^j} = D_{x^j} + (-1)^{i+j-1} \sum_\sigma \tilde{D}_\sigma(X_i) \frac{\partial}{\partial w^\sigma}$$

define a covering structure on $\tilde{E}_\omega = V \times E$. Coverings of this type are called Abelian.

1.4. **Nonlocal symmetries.** Denote by $\mathcal{C}$ the distribution on $\tilde{E}$ spanned by the fields $\tilde{D}_{x^1}, \ldots, \tilde{D}_{x^n}$ and let $X$ be a field vertical with respect to the composition $\tilde{E} \to E \to \mathbb{R}^n$. Such a field is called a *nonlocal symmetry* if it preserves $\mathcal{C}$. These symmetries form a Lie algebra denoted by $\text{sym}_\tau(E)$. The restriction $X|_{C^\infty(E)}: C^\infty(E) \to C^\infty(\tilde{E})$ is called a nonlocal $\tau$-shadow. A nonlocal symmetry is said to be *invisible* if its shadow vanishes.

In local coordinates, any $X \in \text{sym}_\tau(E)$ is of the form

$$X = \tilde{E}_\varphi + \sum_\alpha \psi^\alpha \frac{\partial}{\partial w^\alpha},$$

where $\varphi = (\varphi^1, \ldots, \varphi^n)$, $\psi^\alpha$ are functions on $\tilde{E}$ satisfying the equations

$$\tilde{E}_{\tilde{E}}(\varphi) = 0,$$

$$\tilde{D}_{x^i}(\psi^\alpha) = \sum_{j, \sigma} \frac{\partial X_i^\alpha}{\partial w^\sigma} \tilde{D}_\sigma(\varphi^j) + \sum_\beta \frac{\partial X_i^\alpha}{\partial w^\beta} \psi^\beta,$$
where $\tilde{E}_\varphi$ and $\tilde{E}_\psi$ are obtained from the expressions (2) and (3), respectively, by changing $D_{x_\varphi}$ to $D_{x_\psi}$. Nonlocal shadows are the operators $\tilde{E}_\varphi$ while invisible symmetries are obtained from general ones by setting $\varphi = 0$. 

In particular, for coverings of the form $\tilde{E}_\omega$, where $\omega$ is a 2-component conservation law, the symmetries acquire the form

$$X = \tilde{E}_\varphi + \sum_{\sigma} D_\sigma(\psi) \frac{\partial}{\partial \psi^\sigma},$$

where $\varphi$ and $\psi$ satisfy

$$\tilde{\ell}_E(\varphi) = 0,$$

$$\tilde{D}_{x_\varphi}(\psi) = \sum_{\sigma,k} \frac{\partial X_j}{\partial u^k_\sigma} \tilde{D}_\sigma(\varphi^k) + \sum_{\sigma} \frac{\partial X_j}{\partial \psi^\sigma} \tilde{D}_\sigma(\psi),$$

$$\tilde{D}_{x_i}(\psi) = (-1)^{i+j-1} \left( \sum_{\sigma,k} \frac{\partial X_j}{\partial u^k_\sigma} \tilde{D}_\sigma(\varphi^k) + \sum_{\sigma} \frac{\partial X_j}{\partial \psi^\sigma} \tilde{D}_\sigma(\psi) \right).$$

1.5. Bäcklund transformations and recursion operators. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be equations. A Bäcklund transformation between $\mathcal{E}_1$ and $\mathcal{E}_2$ is the diagram

$$\begin{array}{ccc}
\mathcal{E}_1 & \xrightarrow{\tau_1} & \mathcal{E}_2 \\
\downarrow & & \downarrow \\
\mathcal{E} & & \mathcal{E}
\end{array}$$

where $\tau_1$, $\tau_2$ are coverings. When $\mathcal{E}_1 = \mathcal{E}_2$, it is called a Bäcklund auto-transformation. If $\tau_1$ is finite-dimensional and $\gamma \subset \mathcal{E}_1$ is a graph of solution then, generically, $\tau_2(\tau_1^{-1}(\gamma))$ is a finite-dimensional manifold endowed with an integrable $n$-dimensional distribution whose integral manifolds are solutions of $\mathcal{E}_2$.

Consider now an equation $\mathcal{E}$ given by (1) and the system

$$F(x^i, u^k) = 0, \quad \ell_F(x^i, u^\alpha, q^\beta) = 0,$$

where $F = (F^1, \ldots, F^r)$. This system is called the tangent equation to $\mathcal{E}$ and denoted by $\mathcal{T}\mathcal{E}$, while the projection $t: \mathcal{T}\mathcal{E} \to \mathcal{E}$ is called the tangent covering. Sections of this covering that preserve the Cartan distribution are identified with generating functions of symmetries.

Let $\mathcal{R}$ be a Bäcklund transformation between $\mathcal{T}\mathcal{E}_1$ and $\mathcal{T}\mathcal{E}_2$. Then it follows from the above said that it accomplishes a correspondence between symmetries of the two equations. If $\mathcal{E}_1 = \mathcal{E}_2$ then such a correspondence is called a recursion operator; $\mathcal{R}$.

2. The equation

The 3D rdDym equation $\mathcal{E}$ is of the form

$$u_{ty} = u_x u_{xy} - u_y u_{xx}. \quad (4)$$

For internal coordinates in $\mathcal{E}$ one can choose the functions

$$u_k = u_x \ldots x, \quad u_{k,l} = u_x \ldots x f \ldots f, \quad u_{k,l}^y = u_x \ldots x y \ldots y, \quad k \geq 0, \quad l \geq 0.$$ 

Thus, $u_0 = u$, $u_1 = u_x$, $u_{0,1} = u_y$, $u_{0,1} = u_{tt}$, etc. The total derivatives acquire the form

$$D_x = \frac{\partial}{\partial x} + \sum_k u_{k+1} \frac{\partial}{\partial u_k} + \sum_{k,l} \left( u_{k+1,l}^y \frac{\partial}{\partial u_k} + u_{k+1,l} \frac{\partial}{\partial u_{k,l}} \right),$$
there corresponds the evolutionary vector field $\theta$ of

\[ \frac{\partial}{\partial x} + \sum_{k,l} D_k D_l y = -\theta_1(A), \quad \theta_2(A), \]

in these coordinates.

2.1. Local symmetries. Local symmetries of Equation (1) are solutions to the linearized equation

\[ \ell_{\varphi}(\varphi) \equiv D_t D_y(\varphi) - u_x D_x D_y(\varphi) + u_y D_y^2(\varphi) - u_{xy} D_x(\varphi) + u_{xx} D_y(\varphi) = 0. \]  (5)

The space of solutions is spanned by the functions

$\psi_0 = xu_x - 2u,$  
$v_0(B) = Bu_y,$  
$\theta_0(A) = Au_t + A'(xu_x - u) + \frac{1}{2} A'' x^2,$  
$\theta_1(A) = Au_x + A'x,$  
$\theta_2(A) = A,$

where $A = A(t), B = B(y)$ and ‘prime’ denotes the $t$-derivative. To any solution $\varphi$ there corresponds the evolutionary vector field

\[ E_{\varphi} = \sum_{k} D_k^k(\varphi) \frac{\partial}{\partial u_k} + \sum_{k,l} \left( (D_k D_l y) \frac{\partial}{\partial u_{k,l}} + D_k D_l^l y(\varphi) \frac{\partial}{\partial u_{k,l}} \right). \]  (6)

on $\tilde{\mathcal{E}}$.

The Lie algebra structure in the space $\text{sym}(\tilde{\mathcal{E}})$ is presented in Table 1.

| $\psi_0$ | $v_0(B)$ | $\theta_0(A)$ | $\theta_1(A)$ | $\theta_2(A)$ |
|----------|-----------|---------------|---------------|---------------|
| $\psi_0$ | 0         | 0             | $-\theta_1(A)$ | $-2\theta_2(A)$ |
| $v_0(B)$ | $v_0(B)(B' - B')$ | 0             | 0             | 0             |
| $\theta_0(A)$ | $\theta_0(\bar{A}A' - AA')$ | $\theta_1(\bar{A}A' - AA')$ | $\theta_2(\bar{A}A' - AA')$ | 0 |
| $\theta_1(A)$ | $\theta_1(\bar{A}A' - AA')$ | $\theta_1(\bar{A}A' - AA')$ | $\theta_2(\bar{A}A' - AA')$ | 0 |
| $\theta_2(A)$ | $\theta_2(\bar{A}A' - AA')$ | $\theta_2(\bar{A}A' - AA')$ | $\theta_2(\bar{A}A' - AA')$ | 0 |

Table 1. The Lie algebra structure of $\text{sym}(\tilde{\mathcal{E}})$

2.2. Coverings. The 3D rdDym equation (1) possesses the linear Lax representation

\[ u_t = (u_x - \lambda)w_x, \quad w_y = \lambda^{-1}u_y w_x, \]  (7)

where $\lambda \neq 0$ is a non-removable parameter. Expanding $w$ in formal series in $\lambda$

\[ w = \sum_{i=-\infty}^{+\infty} w_i \lambda^i, \]

yields, cf. [2],

\[ w_{i,t} = u_x w_{i,x} - w_{i-1,x}, \quad w_{i,y} = u_y w_{i+1,x}. \]  (8)

This system is infinite in both directions and thus the nonlocal quantities $w_i$ are not defined in a proper way. To improve the setting, consider two reductions of (8):
(a) \( w_i = 0 \) for \( i < 0 \) and (b) \( w_i = 0 \) for \( i > 0 \). Two hierarchies of nonlocal two-component conservation laws arise in such a way. They will be called the positive and the negative ones, respectively. Our aim is to describe nonlocal symmetries of the corresponding Abelian coverings.

Note that the positive hierarchy corresponds to the Taylor expansion of \( w \), while the negative one is related to the Laurent expansion.

2.2.1. The positive hierarchy. Assume \( w_i = 0 \) for \( i < 0 \) and rewrite (8) in the form

\[
\begin{align*}
  w_{i,t} &= \frac{u_x}{u_y} w_{i-1,y} - w_{i-1,x}, \\
  w_{i,x} &= \frac{w_{i-1,y}}{u_y}.
\end{align*}
\]

Then, due to the assumption, \( w_{0,t} = w_{0,x} = 0 \), or \( w_0 = G(y) \) and the defining equations of the covering are

\[
\begin{align*}
  w_{1,t} &= \frac{u_x}{u_y} G', \\
  w_{1,x} &= \frac{G'}{u_y}; \quad w_{i,t} = \frac{u_x}{u_y} w_{i-1,y} - w_{i-1,x}, \\
  w_{i,x} &= \frac{w_{i-1,y}}{u_y},
\end{align*}
\]

where \( i > 0 \) and ‘prime’ denotes the \( y \)-derivative.

Without loss of generality we may assume \( G' \neq 0 \) and make the change of variables \( y \mapsto G(y) \). This transformation preserves our equation (due to the symmetry \( \upsilon_0(B) \)). Denoting the resulting nonlocal variables by \( q_i, i > 0 \), we arrive to the covering defined by

\[
\begin{align*}
  q_{1,t} &= \frac{u_x}{u_y}, \\
  q_{1,x} &= \frac{1}{u_y} \tag{9}
\end{align*}
\]

and

\[
\begin{align*}
  q_{i,t} &= \frac{u_x}{u_y} q_{i-1,y} - q_{i-1,x}, \\
  q_{i,x} &= \frac{q_{i-1,y}}{u_y} \tag{10}
\end{align*}
\]

Note that the quantities \( q_i \) do not form a complete set of nonlocal variables in the covering under consideration. To have a complete collection, let us introduce the functions \( q_i^{(j)} \) such that

\[
q_i^{(0)} = q_i, \quad q_i^{(j+1)} = \left( q_i^{(j)} \right)_y.
\]

Then the total derivatives on the space \( \tilde{E^+} \) of the covering are given by

\[
\begin{align*}
  \tilde{D}_x &= D_x + \sum_{j=0}^{\infty} \tilde{D}_y \left( \frac{1}{u_y} \right) \frac{\partial}{\partial q_i^{(j)}} + \sum_{i=2}^{\infty} \sum_{j=0}^{\infty} \tilde{D}_y \left( \frac{q_i^{(j-1)}}{u_y} \right) \frac{\partial}{\partial q_i^{(j)}}, \\
  \tilde{D}_y &= D_y + \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} q_i^{(j+1)} \frac{\partial}{\partial q_i^{(j)}}, \\
  \tilde{D}_t &= D_t + \sum_{j=0}^{\infty} \tilde{D}_y \left( \frac{u_x}{u_y} \right) \frac{\partial}{\partial q_i^{(j)}} + \sum_{i=2}^{\infty} \sum_{j=0}^{\infty} \tilde{D}_y \left( \frac{u_x}{u_y} q_i^{(j-1)} - \tilde{D}_x \left( q_i^{(0)} \right) \right) \frac{\partial}{\partial q_i^{(j)}},
\end{align*}
\]

where \( D_x, D_y, D_t \) are the total derivatives on \( E \) given above.
2.2.2. The negative hierarchy. We have \( w_i = 0 \) for \( i > 0 \) now. Then from (5) it follows that
\[
\begin{align*}
\forall w_{0,x} &= 0; \quad \forall w_{-1,x} = u_x w_{0,x} - u_{0,t} = w_{-1,x} = u_x w_{-1,x} - w_{-1,1} = 0; \\
\forall w_{0,y} &= 0; \quad \forall w_{-1,y} = u_y w_{0,y}; \quad \forall w_{-2,y} = u_y w_{-1,x}.
\end{align*}
\]
Consequently,
\[
\begin{align*}
\forall w_0 &= \tilde{F}(t), \quad \forall w_{-1} = -xF'(t) + G(t), \quad \forall w_{-2} = -\tilde{F}'u + \frac{1}{2}x^2\tilde{F}'' - G'x + H(t).
\end{align*}
\]
Without loss of generality we can assume \( G = H = 0 \). Then, after relabeling \( r_i = w_{-i-2}, \ i = 1, 2, \ldots \), we get the following defining equations for the negative hierarchy:
\[
\begin{align*}
\forall r_{1,x} &= F(u_t - u_x^2) + F'(u + xu_x) - \frac{1}{2}x^2 F''; \quad \forall r_{1,y} = u_y(xF' - Fu_x); \\
\forall r_{1,y} &= u_y r_{1,y} = u_y r_{1,x}.
\end{align*}
\]
for \( i > 1 \), where \( F = \tilde{F}' \). The defining equations can be simplified:

**Proposition 1.** There exists a gauge transformation of the space \( \tilde{E}^- \) that ‘kills’ the function \( F \), i.e., transforms (11) to
\[
\forall r_{1,x} = u_x^2 - u_t; \quad \forall r_{1,y} = u_x u_y; \\
\forall r_{1,y} = u_x r_{1,y} = u_x r_{1,x}.
\]

**Proof.** Define the new nonlocal variable \( \tilde{r}_1 \) by
\[
\forall \tilde{r}_1 = -F\tilde{r}_1 - F'u + \frac{1}{6}F''x^3. \tag{13}
\]
Substituting (13) to the left equations in (11), we immediately see that
\[
\forall \tilde{r}_{1,x} = u_x^2 - u_t; \quad \forall \tilde{r}_{1,y} = u_x u_y.
\]
Let us now introduce the operator
\[
\forall y_- = -x \frac{\partial}{\partial t} + 2u \frac{\partial}{\partial x} - 3\tilde{r}_1 \frac{\partial}{\partial u} + \sum_{i \geq 1} (i + 3)\tilde{r}_{i+1} \frac{\partial}{\partial \tilde{r}_i}
\]
and set by induction
\[
\forall r_k = \frac{1}{k+2}y_-(r_{k-1}) \tag{14}
\]
for \( k \geq 2 \). Obviously,
\[
\forall r_k = F\tilde{r}_k + o(k - 1),
\]
where \( o(k - 1) \) denotes the terms that depend on \( \tilde{r}_1, \ldots, \tilde{r}_{k-1} \) only.

Assume now that \( k > 1 \) and the statement is valid for the defining equations on \( \tilde{r}_1, \ldots, \tilde{r}_{k-1} \). Then, substituting (14) to the equations on \( r_k \), we see that it transforms to
\[
\forall F\tilde{r}_{k,x} = F(u_x\tilde{r}_{k-1,x} - \tilde{r}_{k-1,t}), \quad \forall F\tilde{r}_{k,y} = Fu_y\tilde{r}_{k-1,x}
\]
by the induction assumption.

We forget about the ‘old’ \( r_i \)’s and change the notation from \( \tilde{r}_k \) to \( r_k \).
A complete set of nonlocal variables consists of the quantities \( r_i^{(j)} \) defined by
\[
\forall r_i^{(0)} = r_i, \quad r_i^{(j+1)} = \sum_{i=2}^{j} \frac{\partial}{\partial \tilde{r}_i}.
\]
The total derivatives on the covering space \( \tilde{E}^- \) are of the form
\[
\forall \tilde{D}_x = D_x + \sum_{j=0}^{\infty} \tilde{D}_j^l (u_x^2 - u_t) \frac{\partial}{\partial r_i^{(j)}} + \sum_{i=2}^{\infty} \sum_{j=0}^{\infty} \tilde{D}_j^l (u_x r_{i-1,x} - r_{i-1,t}) \frac{\partial}{\partial r_i^{(j)}},
\]
Weight: \[ \begin{array}{ccc} -2 & -1 & 0 \\ \theta^{-2}(A) & \theta^{-1}(A) & \theta_0(A) \\ \psi_0(B) \end{array} \]

Table 2. Distribution of local symmetries along weights

\[
\tilde{D}_y = D_y + \sum_{j=0}^{\infty} \tilde{D}^j_y (u_x u_y) \frac{\partial}{\partial q_{j+1}} + \sum_{i=2}^{\infty} \sum_{j=0}^{\infty} \tilde{D}^j_i (u_y r_{i-1,x}) \frac{\partial}{\partial r^{(j)}}, \\
\tilde{D}_t = D_t + \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} r^{(j+1)} \frac{\partial}{\partial r^{(j)}}
\]

in these coordinates.

2.3. **Weights.** Let us assign the following weights

\[ |x| = 1, \quad |u| = 2, \quad |y| = |t| = 0 \]

to the dependent and independent variables. Then

\[ |u_k| = |u_{k,l}| = |u_{k,l}| = 2 - k \]

and to any monomial in jet variables we assign the summarized weight of its factors.

We say that a vector field \( X \) is **homogeneous** if

\[ |X(f)| = |X| + |f| \]

for any homogeneous function \( f \), where the integer \( |X| \) depends on \( X \) only and is the weight of \( X \). All local symmetries are homogeneous in this sense and their weight are presented in Table 2. Obviously,

\[ ||X,Y|| = |X| + |Y| \]

for any homogeneous \( X \) and \( Y \).

From Equations (9), (10) and (11) we immediately deduce the weights

\[ |q_i| = -i, \quad |r_i| = i + 2, \quad i = 1, 2, \ldots, \]

of the nonlocal variables of nonlocal variables in \( \tilde{E}^+ \) and \( \tilde{E}^- \).

3. **Symmetries**

We describe here the Lie algebras \( \text{sym } \tilde{E}^+ \) and \( \text{sym } \tilde{E}^- \).

3.1. **Symmetries in the positive hierarchy.** Any symmetry of \( \tilde{E}^+ \) is a vector field

\[
X_\Phi = \tilde{E}_\varphi + \sum_{i=1}^{\infty} \left( \varphi_i \frac{\partial}{\partial q_i} + \sum_{j=1}^{\infty} \tilde{D}^j_y (\varphi_i) \frac{\partial}{\partial q_{j+1}} \right),
\]

where \( \tilde{E}_\varphi \) is given by (12) with the total derivatives \( \tilde{D}_* \) instead of \( D_* \) and the collection of functions

\( \Phi = (\varphi_0 = \varphi, \varphi_1, \ldots, \varphi_i, \ldots), \quad \varphi_0, \varphi_i \in C^\infty(\tilde{E}^+) \),

satisfies the equations

\[
\tilde{D}_y (\varphi) \equiv \tilde{D}_t \tilde{D}_y (\varphi) - u_x \tilde{D}_y (\varphi) + u_y \tilde{D}_x (\varphi) = 0, \\
\tilde{D}_y (\varphi) q_{1,t} + u_y \tilde{D}_t (\varphi_1) = \tilde{D}_x (\varphi), \\
\tilde{D}_y (\varphi) q_{1,x} + u_y \tilde{D}_x (\varphi_1) = 0,
\]

\[
\tilde{D}_y (\varphi) q_{1,t} + u_y \tilde{D}_t (\varphi_1) = \tilde{D}_x (\varphi), \\
\tilde{D}_y (\varphi) q_{1,x} + u_y \tilde{D}_x (\varphi_1) = 0,
\]

\[
\tilde{D}_y (\varphi) q_{1,t} + u_y \tilde{D}_t (\varphi_1) = \tilde{D}_x (\varphi), \\
\tilde{D}_y (\varphi) q_{1,x} + u_y \tilde{D}_x (\varphi_1) = 0,
\]
\[ \tilde{D}_y(\varphi) = D_y(\varphi) + \varphi \tilde{D}_y, \]
\[ D_y(\varphi)(q_{i,1} + q_{i-1,x}) + u_y D_x(\varphi) = \tilde{D}_x(\varphi)q_{i-1,y} + u_x \tilde{D}_x(\varphi), \quad (19) \]
\[ \tilde{D}_y(\varphi)q_{i,x} + u_y D_x(\varphi) = \tilde{D}_y(\varphi). \quad (20) \]

For any two symmetries \( \Phi \) and \( \Psi \) their Jacobi bracket \( \{ \Phi, \Psi \} \) is defined by
\[ X_{\{ \Phi, \Psi \}} = \{ X_{\Phi}, X_{\Psi} \}. \]

### 3.1.1. Lifts of local symmetries and hierarchies of nonlocal ones.

We begin with the following statement:

**Proposition 2.** The local symmetries \( \psi_0, \theta_{-2}(A), \theta_{-1}(A), \) and \( \theta_0(A) \) can be lifted to \( \tilde{\mathcal{E}}^+ \).

**Proof.** Let us denote the desired lifts by
\[ \Psi_0 = \langle \psi_0, \psi_1, \ldots, \psi_i, \ldots \rangle, \]
\[ \Theta_{-2}(A) = \langle \theta_{-2}(A), \theta_{-2}^2(A), \ldots, \theta_{-2}^{i-2}(A), \ldots \rangle, \]
\[ \Theta_{-1}(A) = \langle \theta_{-1}(A), \theta_{-1}^2(A), \ldots, \theta_{-1}^{i-1}(A), \ldots \rangle, \]
\[ \Theta_0(A) = \langle \theta_0(A), \theta_0^2(A), \ldots, \theta_0^i(A), \ldots \rangle \]

and set
\[ \psi_0^i = \varphi_{i} + xq_{i,x}, \quad i \geq 1, \]
\[ \theta_{-2}^i(A) = 0, \quad i \geq 1, \]
\[ \theta_{-1}^i(A) = \lambda q_{i,x}, \quad i \geq 1, \]
\[ \theta_0^i(A) = \theta_{-1}(A)q_{i,x} - \lambda q_{i-1,x}, \quad i > 1. \]

To establish that the above introduced functions are symmetries, we straightforwardly check that they satisfy Equations \( 17 - 20 \). For example, let us prove that \( \Psi_0 \) is a symmetry.

For Equation \( 17 \) one has
\[ D_y(xu_x - 2u)q_{1,x} + u_y D_x(q_1 + xq_{1,x}) = (xu_{xy} + 2u_y)q_{1,x} + u_y(2q_{1,x} + xq_{1,xx}) \]
\[ = xu_{xy}q_{1,x} + u_y x \left( \frac{1}{u_y} \right) = xu_{xy} - xu_{xy} u_y u_y = 0. \]

Now, Equation \( 18 \) reads
\[ D_y(xu_x - 2u) + u_y D_x(q_1 + xq_{1,x}) - D_x(xu_x - 2u) = \]
\[ (xu_{xy} + 2u_y)q_{1,t} + u_y q_{1,t} + xq_{1,tx} + u_x - xu_{xx} = (xu_{xy} - u_y) \frac{u_x}{u_y} + xu_{y} \left( \frac{u_x}{u_y} \right) x \]
\[ + u_y - xu_{xx} = (xu_{xy} - u_y) \frac{u_x}{u_y} + xu_{y} \frac{u_x u_y - u_x u_x u_x}{u_y} + u_x = 0. \]

Equation \( 19 \) acquires the form
\[ D_y(xu_x - 2u)q_{i,1} + u_y D_x(q_i + xq_{i,x}) - D_x(xu_x - 2u) = \]
\[ (xu_{xy} + 2u_y)q_{i,x} + u_y q_{i,x} + xq_{i,xx} - (i - 1)q_{i-1} + xq_{i-1,x} = \]
\[ (xu_{xy} + (i - 1)u_y) \frac{q_{i-1,x} - q_{i-1}}{u_y} + u_y x \left( \frac{q_{i-1,y}}{u_y} \right) - (i - 1)q_{i-1,xy} = \]
\[ xu_{xy} \frac{q_{i-1,y}}{u_y} + u_y \frac{q_{i-1,x} - q_{i-1}}{u_y} - xu_{xy} u_y - xu_{xy} u_y = -xq_{i-1,xy} = \]
\[ xu_{xy} \frac{q_{i-1,y}}{u_y} + u_y \frac{q_{i-1,x} - q_{i-1}}{u_y} - xu_{xy} u_y - xu_{xy} u_y = -xq_{i-1,xy} = 0. \]

\(^2\)Here and below, the boxed terms cancel each other.
Finally, for Equation (20) one has
\[
D_y(x_{ux} - 2u_x \frac{ux}{uy} q_{i-1,y} + u_y D_t (iq_i + xq_{i,xx}) + D_x ((i-1)q_{i-1} + xq_{i-1,xx})) - \\
D_x(x_{ux} - 2u_y \frac{uy}{ux} q_{i-1,y} + (iq_i + xq_{i,tx} + iq_{i,1,x} + xq_{i,1xx}) + \\
(u_x - xu_{xx})q_{i-1,y} - u_x((i-1)q_{i-1,y} + xq_{i-1,xy}) = \\
(x_{uxy} - 2u_y \frac{uy}{ux} q_{i-1,y} + u_y \frac{ux}{uy} q_{i-1,y} + xq_{i,tx} + xq_{i,1xx}) + \\
(u_x - xu_{xx})q_{i-1,y} - u_x((i-1)q_{i-1,y} + xq_{i-1,xy}) = \\
x_{uxy} \frac{ux}{uy} q_{i-1,y} + xu_y \frac{ux}{uy} q_{i-1,y} \frac{ux}{uy} q_{i-1,xy} - x_{uxy} q_{i-1,y} - u_x xq_{i-1,xy} = \\
x_{uxy} \frac{ux}{uy} q_{i-1,y} + xu_y \frac{ux}{uy} q_{i-1,y} - u_x xq_{i-1,xy} = \\
x_{uxy} \frac{ux}{uy} q_{i-1,y} + xu_y \frac{ux}{uy} q_{i-1,y} - u_x xq_{i-1,xy},
\]
and this finishes the proof.

For other symmetries the proofs are similar. □

We shall now need a description of invisible symmetries in \( \tilde{\mathcal{E}}^+ \). We say that \( \Phi \) is an invisible symmetry of depth \( k \) if its first \( k \) components vanish, i.e.,
\[
\Phi = \underbrace{(0, \ldots, 0, \varphi_{i}^{\text{inv}}, \ldots, \varphi_{i}^{\text{inv}}, \ldots)}_{\text{\( k \) times}}
\]

The defining equations for invisible symmetries are
\[
\tilde{D}_x(\varphi_{i}^{\text{inv}}) = 0, \quad \tilde{D}_t(\varphi_{i}^{\text{inv}}) = 0; \\
u_y \tilde{D}_x(\varphi_{i}^{\text{inv}}) = \tilde{D}_x(\varphi_{i-1}^{\text{inv}}), \quad u_y \tilde{D}_t(\varphi_{i}^{\text{inv}}) + \tilde{D}_x(\varphi_{i-1}^{\text{inv}})) = u_x \tilde{D}_x(\varphi_{i-1}^{\text{inv}}), \quad \text{\( i > 1 \)}.
\]

Then \( \varphi_{1}^{\text{inv}} = B(y) \) and any homogeneous symmetry of depth \( k \) is completely determined by the function \( B \). Denote such a symmetry by \( \Upsilon_k(B) \). One has
\[
|\Upsilon_k(B)| = k.
\]

**Proposition 3.** For any integer \( k \geq 1 \) and a function \( B = B(y) \), the symmetry \( \Upsilon_k(B) \) does exist.

**Proof.** Consider the operator
\[
\mathcal{X} = q_1 \frac{\partial}{\partial y} + \sum_{i=1}^{\infty} (i+1)q_{i+1} \frac{\partial}{\partial q_i}
\]
and define
\[
\varphi_{1}^{\text{inv}} = B(y), \quad \varphi_{i}^{\text{inv}} = \frac{1}{i-1} \mathcal{X}(\varphi_{i-1}^{\text{inv}}), \quad \text{\( i > 1 \)}.
\]
(21)
Note that the defining equations for invisible symmetries can be rewritten in the form
\[
\frac{\partial \varphi_{i}^{\text{inv}}}{\partial q_1} = \frac{\partial B}{\partial y},
\]
\[
\ldots
\]
Let us prove by induction the equalities

\[ \frac{\partial \phi_{\text{inv}}^i}{\partial q_j} = \frac{\partial \phi_{\text{inv}}^{i-1}}{\partial q_{j-1}} \]

(we formally set \( q_0 = y \)). The case \( i = 2 \) is checked by straightforward computations. Assume now that the statement is valid for some \( i > 2 \) and note that

\[ \left[ \frac{\partial}{\partial q_j}, X \right] = j \frac{\partial}{\partial q_j} - (j-1) \frac{\partial}{\partial q_{j-1}} \]

Then

\[ \frac{\partial \phi_{\text{inv}}^{i+1}}{\partial q_j} = \frac{1}{i} \left( j \frac{\partial \phi_{\text{inv}}^i}{\partial q_{j-1}} + X \left( \frac{\partial \phi_{\text{inv}}^i}{\partial q_j} \right) \right) = \frac{1}{i} \left( j \frac{\partial \phi_{\text{inv}}^i}{\partial q_{j-1}} + \frac{\partial X (\phi_{\text{inv}}^i)}{\partial q_j} - \frac{\partial \phi_{\text{inv}}^{i-1}}{\partial q_{j-1}} \right) \]

and this finishes the proof.

Now, direct computations show that the functions

\[ \psi_{-1} = q_1 u_y + x, \quad \psi_{-2} = (2q_2 - q_1 q_1^{(1)}) u_y \]

are shadows in the positive covering, i.e., they satisfy Equation (16).

**Proposition 4.** The shadows \( \psi_{-1}, \psi_{-2} \) can be extended to symmetries of \( \tilde{\mathcal{E}}^+ \).

**Proof.** Let us set

\[ \Psi_{-1} = \langle \psi_{-1}, \psi_{-1}', \ldots, \psi_{-1}', \ldots \rangle, \quad \Psi_{-2} = \langle \psi_{-2}, \psi_{-2}', \ldots, \psi_{-2}', \ldots \rangle, \]

where

\[ \psi_{-1}' = -(i+1) q_{i+1} + q_i^{(1)} q_1, \quad \psi_{-2}' = -(i+2) q_{i+2} + q_1 q_{i+1}^{(1)} + (2q_2 - q_1 q_1^{(1)}) q_i^{(1)}. \]

The rest of the proof is similar to that of Proposition 2. \( \Box \)

Obviously,

\[ |\Psi_{-1}| = -1, \quad |\Psi_{-2}| = -2. \]

We now define two hierarchies of nonlocal symmetries by

\[ \Psi_{-k} = \text{ad}_{-1}^{k-2} (\psi_{-2}), \quad k \geq 3, \]

\[ \Upsilon_{-k}(B) = \{ \Psi_{-k-1}, \Upsilon_1(B) \}, \quad k \geq 0, \]

where

\[ \text{ad}_{-1}(\Phi) = \{ \Phi, \Psi_{-1} \}. \]

Obviously,

\[ |\Psi_{-k}| = |\Upsilon_{-k}(B)| = -k \]

and \( \Upsilon_0(B) \) is an extension of the local symmetry \( \psi_0(B) \) to \( \tilde{\mathcal{E}}^+ \). Elements of the algebra \( \text{sym}(\tilde{\mathcal{E}}^+) \) are distributed along weights as it is indicated in Table 3.
3.1.2. The Lie algebra structure. To compute the commutators, we shall need asymptotic estimates for coefficient of symmetries that constitute a basis of sym(\(\tilde{\mathcal{E}}^+\)).

We begin with the symmetries \(\Psi_{-k}, k \geq 1\), and we are interested in the higher order terms (with respect to \(q_j\)) of the coefficients at \(\partial/\partial q_i\). Using the notation [15], we have by definition

\[
X_{\Psi_{-1}} = \cdots + \left(- (i + 1) q_{i+1} + q_1 q_1^{(1)} + o(i - 1)\right) \frac{\partial}{\partial q_i} + \ldots,
\]

\[
X_{\Psi_{-2}} = \cdots \left(- (i + 2) q_{i+2} + q_1 q_1^{(1)} + o(i)\right) \frac{\partial}{\partial q_i} + \ldots,
\]

where \(o(k)\) denotes the terms that contain \(q_j\) with \(j \leq k\). Assume now that

\[
X_{\Psi_{-k}} = \cdots + \left(a^1_k q_{i+k} + b^1_k q_1 q_1^{(1)} + o(i + k - 2)\right) \frac{\partial}{\partial q_i} + \ldots
\]

Then

\[
X_{\Psi_{-k-1}} = [X_{\Psi_{-k}}, X_{\Psi_{-1}}] = \cdots + \left((i + k + 1) a^1_k - (i + 1) a^{i+1}_k q_{i+k+1}\right) \frac{\partial}{\partial q_i} + \ldots
\]

Thus

\[
a^1_k = -(k - 2)! (k + i), \quad b^1_k = (k - 2)!
\]

and by elementary induction with the base \(a^1_2 = -(i + 2), b^1_2 = 1\) we immediately obtain

\[
a^1_k = -(k - 2)! (k + i), \quad b^1_k = (k - 2)!
\]

for all \(i \geq 1\) (we formally set \((-1)! = 1\)). To comply with this result, we change the basic element \(\Psi_0\) by \(\Psi_0 \rightarrow -\Psi_0\).

Now, we estimate the elements \(\Upsilon_k(B)\). For \(k > 0\) we use the Definition [21] and by simple computations obtain that

\[
\varphi_{i}^{\text{inv}} = B' q_{i-1} + B'' q_1 q_{i-2} + o(i - 3)
\]

and consequently

\[
X_{\Upsilon_k(B)} = \varphi_{i}^{\text{inv}} \frac{\partial}{\partial q_k} + \cdots + \varphi_{i-1}^{\text{inv}} \frac{\partial}{\partial q_{i-1}} + \cdots = B \frac{\partial}{\partial q_k} + \cdots + (B' q_{i-k} + B'' q_1 q_{i-k-1} + o(i - k - 2)) \frac{\partial}{\partial q_i} + \cdots
\]

Further,

\[
X_{\Upsilon_{-k}(B)} = [X_{\Psi_{-k-1}}, X_{\Upsilon_k(B)}]
\]

\[
= \cdots + \left( a^1_{i+1} q_{i+k+1} + b^1_{i+1} q_1 q_1^{(1)} + o(i + k - 1)\right) \frac{\partial}{\partial q_i} + \ldots,
\]

\[
B \frac{\partial}{\partial q_1} + \cdots + (B' q_{i-1} + B'' q_1 q_{i-2} + o(i - 3)) \frac{\partial}{\partial q_i} + \ldots
\]
\[ = \cdots + \left( (a_{k+1}^{i-1} - a_k^{i+1})B'q_{i+k} - b_k^{i+1}Bq_{i+k} + (a_{k+1}^{i-2} - a_k^{i+1} - b_k^{i+1})B''q_{i+k-1} + (b_k^{i+1} - b_k')B'q_{i+k}q_{i+k-1} + o(i + k - 2) \right) \frac{\partial}{\partial q_i} + \cdots \]

Using the obtained estimates, we are ready to compute the commutators now.\(^3\)

**Proposition 5.** One has the following commutator relations:

\[
\{ \Psi_{-k}, \Psi_{-l} \} = \frac{(k - 2)!(l - 2)!}{(k + l - 2)!} \Psi_{-k-l}, \quad k, l \geq 0,
\]

\[
\{ \Psi_{-k}, \Upsilon_l(B) \} = \frac{l(-l - 1)!(k - 2)!}{(l - k - 1)!} \Upsilon_{l-k}(B), \quad k \geq 0, \quad l \in \mathbb{Z},
\]

\[
\{ \Upsilon_k(B), \Upsilon_l(B') \} = \frac{(-k - 1)!(l - 1)!}{(-l - k - 1)!} \Upsilon_{k+l}(B'B' - B'B), \quad k, l \in \mathbb{Z}.
\]

**Proof.** A neat use of the above deduced estimates. \(\square\)

Let us change the initial basis by

\[
\Psi_{-k} \mapsto \frac{1}{(k - 2)!} \Psi_{-k}, \quad \Upsilon_l(B) \mapsto \frac{1}{(-l - 1)!} \Upsilon_l(B)
\]

and recall a standard construction. Let \(\mathfrak{g}\) be a Lie \(\mathbb{R}\)-algebra and \(\mathbb{R}_n[z] = \mathbb{R}[z]/(z^n)\) be the ring of truncated polynomials. Then the Lie algebra \(\mathfrak{g}_n = \mathbb{R}_n[z] \otimes \mathbb{R} \mathfrak{g}\) with the bracket

\[
[a \otimes g, b \otimes h] = ab \otimes [g, h], \quad g, h \in \mathfrak{g} \quad a, b \in \mathbb{R}_n[z],
\]

is a graded Lie algebra with \(\mathfrak{g}_0 = \cdots = \mathfrak{g}_{n-1} = \mathfrak{g}\) and all other components being trivial. For polynomials in \(z^{-1}\) the similar construction is denoted by \(\mathfrak{g}_{[-n]}\). Denote also by \(\mathfrak{V}[t]\) the Lie algebra of vector fields \(A(t)\partial/\partial t\) on \(\mathbb{R}\). Then the following result is valid:

**Theorem 1.** The Lie algebra \(\text{sym}(\hat{\mathcal{L}}^+)\) is isomorphic to the semi-direct product of the non-positive part

\[
\mathfrak{W}^- = \left\{ Z_k = z^{-k+1} \frac{\partial}{\partial z} \mid k \in \mathbb{N} \cup \{0\} \right\}
\]

of the Witt algebra with the direct sum \(\mathfrak{L}[y] \oplus \mathfrak{W}_{[-3]}[t]\) of

\[
\mathfrak{L}[y] = \left\{ Y_m(B) = z^m B(y) \frac{\partial}{\partial y} \mid m \in \mathbb{Z}, \quad B \in C^\infty(\mathbb{R}) \right\}
\]

and

\[
\mathfrak{W}[t]_{[-3]} = \left\{ X_s(A) = z^s A(t) \frac{\partial}{\partial t} \mid s \in \{0, 1, 2\}, \quad A \in C^\infty(\mathbb{R}) \right\}
\]

with the natural action of \(z^{-k+1}\partial/\partial z\) on \(\mathfrak{L}[y]\) and \(\mathfrak{W}[t]_{[-3]}\).

In the theorem above the isomorphism maps \(\Psi_{-k}\) to \(Z_k\), \(\Upsilon_m(B)\) to \(Y_m(B)\) and \(\Theta_{-s}(A)\) to \(X_s(A)\).

\(^3\)Everywhere below we assume \(s! = 1\) when \(s < 0\).
3.2. **Symmetries in the negative hierarchy.** Using Proposition 1, we set $F = 1$ in the defining equations of the negative hierarchy. After such a simplification, the study of the negative case becomes quite similar to that of the positive one. Any symmetry in $\tilde{E}^-$ is a vector field

$$
X_\varphi = \mathcal{E}_\varphi + \sum_{i=1}^{\infty} \left( \varphi_i \frac{\partial}{\partial r_i} + \sum_{j=1}^{\infty} \tilde{D}_t(\varphi_i) \frac{\partial}{\partial \tilde{t}_j} \right),
$$

where $\mathcal{E}_\varphi$ with the total derivatives on $\tilde{E}^-$ and

$$\Phi = \langle \varphi_0 = \varphi, \varphi_1, \ldots, \varphi_i, \ldots \rangle,$$

satisfies the equations

$$
\tilde{t}_x(\varphi) \equiv \tilde{D}_t \tilde{D}_y(\varphi) - u_x \tilde{D}_x \tilde{D}_y(\varphi) + u_y \tilde{D}_x^2(\varphi) - u_{xy} \tilde{D}_y(\varphi) + u_{xx} \tilde{D}_y(\varphi) = 0,
$$

$$
\tilde{D}_x(\varphi_1) = \tilde{D}_t(\varphi) - 2u_x \tilde{D}_x(\varphi),
$$

$$
\tilde{D}_y(\varphi_1) = -u_y \tilde{D}_x(\varphi) - u_x \tilde{D}_y(\varphi),
$$

$$
\tilde{D}_x(\varphi_i) = r_{i-1,x} \tilde{D}_x(\varphi) + u_x \tilde{D}_x(\varphi_{i-1}) - \tilde{D}_t(\varphi_{i-1}),
$$

$$
\tilde{D}_y(\varphi_i) = r_{i-1,x} \tilde{D}_y(\varphi) + u_y \tilde{D}_x(\varphi_{i-1}),
$$

$i > 1$. Like in Section 5.1 for any two symmetries $\Phi$ and $\Psi$ their Jacobi bracket $\{\Phi, \Psi\}$ is defined by

$$
X_{\{\Phi, \Psi\}} = [X_\Phi, X_\Psi].
$$

3.2.1. **Lifts of local symmetries and hierarchies of nonlocal ones.** In what follows, we shall need the operator

$$
y_+ = -x \frac{\partial}{\partial t} + 2u \frac{\partial}{\partial x} + 3r_1 \frac{\partial}{\partial u} + \sum_{i \geq 1} (i + 3)r_{i+1} \frac{\partial}{\partial r_i}
$$

**Proposition 6.** The symmetries $\psi_0$, $v(B)$, and $\theta_{-2} (A)$ can be lifted to $\tilde{E}^-$.  

**Proof.** We denote the lifts by

$$
\Psi_0 = \langle \psi_0, \psi^1_0, \ldots, \psi^i_0, \ldots \rangle,$$

$$
\Upsilon_0 (B) = \langle v_0 (B), v^1_0 (B), \ldots, v^i_0 (B), \ldots \rangle,$$

$$
\Theta_{-2} (A) = \langle \theta_{-2} (A), \theta^1_{-2} (A), \ldots, \theta^i_{-2} (A), \ldots \rangle
$$

and set

$$
\psi_0^i = -(i + 2)r_i + x r_{i,x},
$$

$$
v^i_0 (B) = B r_{i,y},
$$

$$
\theta^i_{-2} (A) = -x A', \quad \theta^i_{-2} (A) = \frac{1}{i} y_+ (\theta^{i-1}_{-2} (A))
$$

$i > 1$.  

The rest of the proof is similar to that of Proposition 2.

The next step is to describe invisible symmetries. These symmetries must satisfy

$$
\tilde{D}_x(\varphi_1) = 0,
$$

$$
\tilde{D}_y(\varphi_1) = 0,
$$

$$
\tilde{D}_x(\varphi_i) = u_x \tilde{D}_x(\varphi_{i-1}) - \tilde{D}_t(\varphi_{i-1}),
$$

$$
\tilde{D}_y(\varphi_i) = u_y \tilde{D}_x(\varphi_{i-1}),
$$

where $i > 1$.

**Proposition 7.** For every $A = A(t)$ and $k \geq 3$ there exists a unique invisible symmetry $\Theta_{-k} (A)$ of weight $|\Theta_{-k} (A)| = -k$.  

Proof. Let us use the notation (24) and set
\[ X_{\Theta_{-k}(A)} = \varphi_1^{\text{inv}} \frac{\partial}{\partial r_{k-2}} + \cdots + \varphi_i^{\text{inv}} \frac{\partial}{\partial r_{k+i-3}} + \cdots, \]
where \( \varphi_i^{\text{inv}} = A \)
\[ \varphi_i^{\text{inv}} = \frac{1}{i-1} \gamma_+ (\varphi_{i-1}^{\text{inv}}), \quad i > 1. \]

The proof is accomplished by induction on \( i \).
\[ \square \]

Consider now two functions
\[ \psi_1 = 3r_1 + xu_t - 2ux_x, \quad \psi_2 = 4r_2 + xr_1^{(1)} + 2uu_t - (xu_t + 3r_1)u_x. \]
It is straightforwardly checked that they are shadows in \( \tilde{E}^- \), i.e., satisfy Equation (25).

**Proposition 8.** The shadows \( \psi_1 \) and \( \psi_2 \) are extended to nonlocal symmetries of \( \tilde{E}^- \).

**Proof.** It suffices to set
\[ \Psi_1 = (\psi_1, \psi_1^1, \ldots, \psi_1^i, \ldots) \]
with
\[ \psi_1^i = (i + 3) r_{i+1} + xr_1^{(1)} - 2ur_{i,x} \]
and
\[ \Psi_2 = (\psi_2, \psi_2^2, \ldots, \psi_2^i, \ldots) \]
with
\[ \psi_2^i = (i + 4) r_{i+2} + xr_1^{(1)} + 2ur_1^{(1)} - (xu_t + 3r_1)r_{i,x}. \]
The rest of the proof is a straightforward check of Equations (26)–(29).
\[ \square \]

Obviously,
\[ |\Psi_1| = 1, \quad |\Psi_2| = 2. \]

Similar to the positive case, we define now the first hierarchy of nonlocal symmetries by setting
\[ \Psi_k = \text{ad}^{k-2}_{i+1} (\Psi_2), \quad k \geq 3, \]
where \( \text{ad}_{i+1} (\Phi) = \{ \Psi_1, \Phi \} \).

One has,
\[ |\Psi_k| = k. \]

The second hierarchy will be defined in the next subsection.

### 3.2.2. The Lie algebra structure.

As above, we need asymptotic estimates to compute the commutators. Similar to the positive case, we establish by induction the following estimates for the symmetries \( \Psi_k \):
\[ X_{\Psi_k} = \cdots + \left( a_k^i r_{i+k} + b_k^i xr_1^{(1)}_{i+k-1} + o(i + k - 2) \right) \frac{\partial}{\partial r_i} + \cdots, \]
where
\[ a_k^i = (k - 2)! (i + k + 2), \quad b_k^i = (k - 2)! \]
To have the unified signs, we also rescale \( \Psi_0 \mapsto -\Psi_0 \). Using this estimate, we easily prove the following

**Proposition 9.** One has the following commutator relations
\[ \{ \Psi_k, \Psi_l \} = \frac{(l - 2)! (k - 2)! (l - k)}{(k + l - 2)!} \Psi_{k+l} \]
for all \( k, l \geq 0 \).
Thus, for the new $\Psi$ obtain the commutators

\[ \{\Psi_k, \psi_l\} = (l-k)\psi_{k+l}. \]

Thus, for the new $\Psi_k$ the estimate is

\[ X_{\psi_k} = \cdots + \left( (i+k+2)r_{i+k} + x^{(i)}_{i+k-1} + o(i+k-2) \right) \frac{\partial}{\partial r_i} + \cdots. \]

We now complete the sequence of symmetries \{\Theta_k(A)\}, $k \leq 0$, by setting

\[ \Theta_k(A) = -\frac{1}{3}\{\psi_{k+3}, \psi_{-3}(A)\}, \quad k \geq -2. \]  

One has $|\Theta_k(A)| = k$, and elements of $\text{sym}(\hat{E}^-)$ are distributed along the weights as indicated in Table 4.

The coefficients of invisible symmetries are

\[
\begin{align*}
\varphi_{\text{inv}}^1 &= A, \\
\varphi_{\text{inv}}^2 &= -x A', \\
\varphi_{\text{inv}}^3 &= -u A + \frac{1}{2} x^2 A'', \\
\varphi_{\text{inv}}^4 &= -r_1 A' + u x A'' - \frac{1}{6} x^3 A'''.
\end{align*}
\]

while for $i \geq 5$ we have the estimates

\[ \varphi_{\text{inv}}^i = -A' r_{i-3} + x A'' r_{i-4} + o(i-5). \]

Thus,

\begin{align*}
X_{\Theta_k(A)} &= A \frac{\partial}{\partial r_{k-2}} + \cdots + \varphi_{\text{inv}}^{i-k+3} \frac{\partial}{\partial r_i} + \cdots \\
&= A \frac{\partial}{\partial r_{k-2}} + \cdots + (-A' r_{i-k} + x A'' r_{i-k-1} + o(i - k - 2)) \frac{\partial}{\partial r_i} + \cdots
\end{align*}

for $k \geq 3$.

Using the obtained estimates for $\psi_k$ and $\Theta_{-3}(A)$, we get

\[ X_{\Theta_k(A)} = -\frac{1}{3} [X_{\psi_{k+3}}, X_{\Theta_{-3}(A)}] \]

\[
\begin{align*}
&= \left[ \cdots + \left( (i+k+2)r_{i+k} + x^{(i)}_{i+k-1} + o(i+k-2) \right) \frac{\partial}{\partial r_i} + \cdots, \\
&\quad \cdots + (-A' r_{i-3} + x A'' r_{i-4} + o(i-5)) \frac{\partial}{\partial r_i} + \cdots \right] \\
&= \cdots + (-A' r_{i+k} + x A'' r_{i+k-1} + o(i+k-2)) \frac{\partial}{\partial r_i} + \cdots
\end{align*}
\]

for all $k \geq -2$. These estimates lead directly to

**Proposition 10.** One has

\[ \{\psi_k, \Theta_l(A)\} = l \Theta_{k+l}(A) \]
for all \( k \geq 0, \ l \in \mathbb{Z} \).

Finally, we have

**Proposition 11.** One has

\[ \{ \Theta_k(A), \Theta_l(\tilde{A}) \} = \Theta_{k+l}(A\tilde{A}' - A'\tilde{A}) \]

for all \( k, \ l \in \mathbb{Z} \) and smooth functions \( A = A(t), \ \tilde{A} = \tilde{A}(t) \).

**Proof.** The result easily follows from the above estimates when \( k \) or \( l \leq -3 \), but the method does not work when both \( k \) and \( l > -3 \). Nevertheless, one has in this case

\[ \{ \Theta_k(A), \Theta_l(\tilde{A}) \} = -\frac{1}{3} \left( \{ \Psi_{k+3}, \Theta_{-3}(A) \}, \Theta_l(\tilde{A}) \right) = -\frac{1}{3} \left( \{ \Psi_{k+3}, \Theta_l(\tilde{A}) \}, \Theta_{-3}(A) \right) \]

\[ + \left( \{ \Psi_{k+3}, \Theta_{-3}(A) \}, \Theta_l(\tilde{A}) \right) \]

\[ = -\frac{1}{3} \left( -l \Theta_{k+l+3}(A\tilde{A}' - A'\tilde{A}) + (l - 3) \Theta_{k+l}(A\tilde{A}' - A'\tilde{A}) \right) = \Theta_{k+l}(A\tilde{A}' - A'\tilde{A}) \]

and this finishes the proof. \( \Box \)

Thus we have the result similar to Theorem [11]

**Theorem 2.** The Lie algebra \( \text{sym}(\mathcal{E}^-) \) is isomorphic to the direct sum \( \mathfrak{W}^+ \ltimes \mathcal{L}[t] \oplus \mathfrak{W}[y] \) of the semi-direct product of the positive part

\[ \mathfrak{W}^+ = \left\{ z^{k+1} \frac{\partial}{\partial z} \mid k \in \mathbb{N} \cup \{ 0 \} \right\} \]

of the Witt algebra with

\[ \mathcal{L}[t] = \left\{ z^{m} A(t) \frac{\partial}{\partial t} \mid m \in \mathbb{Z}, \ A \in C^\infty(\mathbb{R}) \right\}, \]

where the vector fields \( z^{k+1} \frac{\partial}{\partial z} \) act naturally on \( \mathcal{L}[t] \), and

\[ \mathfrak{W}[y] = \left\{ B(y) \frac{\partial}{\partial y} \mid B \in C^\infty(\mathbb{R}) \right\} \]

is the Lie algebra of vector fields on the line.

4. **Action of the recursion operators**

We discuss here the action of recursion operators in the hierarchies of nonlocal symmetries described above.

4.1. **Action of the recursion operator to local symmetries and shadows.**

The algebra \( \text{sym}(\mathcal{E}) \) admits a recursion operator \( \tilde{\chi} = \mathcal{R}_+(\chi) \) defined by the system

\[ D_t(\tilde{\chi}) = u_y^{-1} \left( u_y D_x(\chi) - u_x D_y(\chi) + (u_x u_{xy} - u_y u_{xx})\tilde{\chi} \right), \]

\[ D_x(\tilde{\chi}) = u_y^{-1} (u_{xy} \tilde{\chi} - D_y(\chi)) \]  \hspace{2cm} (32)

see [3]. This means that \( \tilde{\chi} \) is a solution to (31) whenever \( \chi \) is. Another recursion operator \( \chi = \mathcal{R}_-(\tilde{\chi}) \) is given by the system

\[ D_x(\chi) = D_t(\tilde{\chi}) - u_x D_x(\tilde{\chi}) + u_{xx} \tilde{\chi} \]

\[ D_y(\chi) = -u_y D_x(\tilde{\chi}) + u_{xy} \tilde{\chi} \]  \hspace{2cm} (33)

The operators \( \mathcal{R}_+ \) and \( \mathcal{R}_- \) are mutually inverse.
The actions of $\mathcal{R}_+$ and $\mathcal{R}_-$ on $\text{sym}(\mathcal{E})$ may be prolonged to the shadows of nonlocal symmetries from $\text{sym}(\mathcal{E}^+)$ and $\text{sym}(\mathcal{E}^-)$ if we replace the derivatives $D_t$, $D_x$ and $D_y$ in (32) and (33) by $\hat{D}_t$, $\hat{D}_x$ and $\hat{D}_y$ defined as

\[
\hat{D}_x = D_x + \sum_{j=0}^{\infty} \hat{D}^j_y \left( \frac{1}{u_y} \right) \frac{\partial}{\partial q^{(j)}_1} + \sum_{i=2}^{\infty} \sum_{j=0}^{\infty} \hat{D}^j_y \left( \frac{q^{(1)}_{i-1}}{u_y} \right) \frac{\partial}{\partial q^{(j)}_i},
\]

\[
+ \sum_{j=0}^{\infty} \hat{D}^j_y (u_x^2 - u_t) \frac{\partial}{\partial q^{(j)}_i} + \sum_{i=2}^{\infty} \sum_{j=0}^{\infty} \hat{D}^j_y (u_x r_{i-1,x} - r_{i-1,t}) \frac{\partial}{\partial q^{(j)}_i},
\]

\[
\hat{D}_y = D_y + \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \hat{D}^j_y \left( \frac{u_x}{u_y} \right) \frac{\partial}{\partial q^{(j)}_i} + \sum_{j=0}^{\infty} \hat{D}^j_y (u_x u_y) \frac{\partial}{\partial q^{(j)}_i} + \sum_{i=2}^{\infty} \sum_{j=0}^{\infty} \hat{D}^j_y (u_y r_{i-1,x}) \frac{\partial}{\partial q^{(j)}_i},
\]

\[
+ \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \hat{D}^j_y \left( u_x \psi^{(1)}_{i-1} - \hat{D}_x \left( \psi^{(0)}_i \right) \right) \frac{\partial}{\partial q^{(j)}_i},
\]

\[
+ \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \hat{D}^j_y \left( u_x \psi^{(1)}_{i-1} - \hat{D}_x \left( \psi^{(0)}_i \right) \right) \frac{\partial}{\partial q^{(j)}_i},
\]

that is, consider the Whitney product of the coverings $\hat{\mathcal{E}}^+$ and $\hat{\mathcal{E}}^-$. The results of the replacement will be also denoted by $\mathcal{R}_+$ and $\mathcal{R}_-$. Note that the operators act nontrivially on ‘vacuum’:

\[
\mathcal{R}_+(0) = \theta_{-2}(A), \quad \mathcal{R}_-(0) = \upsilon_0(B),
\]

which immediately follows from Equations (32) and (33), thus the actions are reasonable to consider modulo $\theta_{-2}(A)$ for $\mathcal{R}_+$ and $\upsilon_0(B)$ for $\mathcal{R}_-$. Taking into account this remark, we have the following

**Proposition 12.** Modulo images of the trivial symmetry, the action of recursion operators is of the form

\[
\mathcal{R}_+(\theta_i(A)) = \left\{ \begin{array}{ll} \alpha^+_i \theta_{i-1}(A), & i > -2, \\
0, & i = -2, \end{array} \right. \quad \mathcal{R}_-(\theta_i(A)) = \alpha^-_i \theta_{i+1}(A), \quad i \geq -2,
\]

\[
\mathcal{R}_+(\upsilon_i(B)) = \beta^+_i \upsilon_{i+1}(B), \quad i \leq 0, \quad \mathcal{R}_-(\upsilon_i(B)) = \left\{ \begin{array}{ll} \beta^-_i \upsilon_{i+1}(B), & i < 0, \\
0, & i = 0, \end{array} \right.
\]

\[
\mathcal{R}_+(\psi_i) = \gamma^+_i \psi_{i-1}, \quad \mathcal{R}_-(\psi_i) = \gamma^-_i \psi_{i+1}, \quad i \in \mathbb{Z},
\]

where $\alpha^\pm_i$, $\beta^\pm_i$, and $\gamma^\pm_i$ are nonzero constants.

**Proof.** It suffices to notice that the weights of $\mathcal{R}_+$ and $\mathcal{R}_-$ are $-1$ and $+1$, respectively, that their action (Modulo images of 0) does not change the dependence of shadows on $y$ and $t$, and that the only shadows that may be taken to 0 are $\theta_{-2}(A)$ and $\upsilon_0(B)$.

Note that the recursion operators $\mathcal{R}_+$ and $\mathcal{R}_-$ ‘glue together’ the shadows $\psi_m$ of nonlocal symmetries in coverings $\hat{\mathcal{E}}^+$ and $\hat{\mathcal{E}}^-$ and ‘tunnel’ from the series of $\theta_k(A)$ to that of $\upsilon_j(B)$, see Table 5.

**4.2. Recursion relations for symmetries of the positive covering.** In this section we find an operator that provides an alternative way to construct elements of $\text{sym}(\hat{\mathcal{E}}^+)$. To this end, we express $u_x$, $u_y$ from (34):

\[
u_x = \frac{q_{1,t}}{q_{1,x}}, \quad u_y = \frac{1}{q_{1,x}}.
\]
where systems

Corollary 1. The linearization

Thus systems (9) and (34) define a Bäcklund auto-transformation between (4) and (35), holds. This equation is known as the universal hierarchy equation, see [11, 12].

Proof. The compatibility conditions of (34), (36) are definitions for (35) and (37).

Thus systems (9) and (34) define a Bäcklund transformation between (4) and (35), see [13]. Substituting (34) to (10) yields

where \( k \geq 2 \). The compatibility conditions for this system after renaming \( k - 1 \mapsto k \) get the form

\[
q_{k,xx} = q_{1,x} q_{k,xy} - q_{1,x} q_{k,ty}, \quad k \geq 2.
\]  

(37)

**Proposition 13.** Systems (34) and (36) define a Bäcklund auto-transformation for the infinite system of PDEs (35), (37).

**Proof.** The compatibility conditions of (34), (36) are definitions for (35) and (37). From (36) we have the inverse transformation

\[
q_{k-1,y} = \frac{q_{k,x}}{q_{1,x}},
\]

whose compatibility conditions also coincide with (37).

\( \square \)

**Corollary 1.** The linearization

\[
D_t(\hat{\chi}_1) = u_y^{-2} \left( u_y D_x(\chi_0) - u_x D_y(\chi_0) \right),
\]

(38)

\[
D_x(\hat{\chi}_1) = -u_y^{-2} D_y(\chi_0),
\]

(39)

\[
D_t(\hat{\chi}_k) = q_{1,t}D_y(\chi_{k-1}) + q_{k-1,y}D_t(\chi_1) - D_x(\chi_{k-1}),
\]

(40)

\[
D_x(\hat{\chi}_k) = q_{1,x}D_y(\chi_{k-1}) + q_{k-1,y}D_x(\chi_1)
\]

(41)

of (10) and (36) defines a recursion operator

\[
\Omega((\chi_0, \chi_1, \chi_2, \ldots, \chi_k, \ldots)) = (\chi_0, \hat{\chi}_1, \hat{\chi}_2, \ldots, \hat{\chi}_k, \ldots)
\]

for sym(\( \hat{\xi}^+ \)).

Note that symmetries \( \xi \) and \( \Omega(\xi) \) have the same shadows and consequently differ by an invisible symmetry. Thus, at first glance, the recursion operator \( \Omega \) seems to be useless, but this is not the case: it provides an alternative way for lifting shadows to nonlocal symmetries in \( \hat{\xi}^+ \). Namely, take a local symmetry or a shadow \( \chi_0 \), then (34), (36) give \( \chi_1 \), applying (10), (11) with \( k = 2 \) to \( \chi_1 \) gives \( \chi_2 \), etc., applying (10), (11) with \( k = m \) to \( \chi_{m-1} \) gives \( \chi_m \), etc.
Proposition 14. The following relations are valid:

\[ Q(\psi_0) = \psi_0, \quad Q(\psi_k) = \psi_{k+1}, \]

\[ Q(\theta_{-j}(A)) = \theta_1 - j(A), \quad Q(\theta_k(A)) = \theta_{k+1}(A) \]

for all \( k \geq 1 \) and \( j = 0, 1, 2 \). One also has

\[ Q(\upsilon_0(B)) = \upsilon_0(B), \quad Q(\upsilon_k(B)) = \upsilon_{k+1}(B). \]

The proof is fulfilled along the same lines as that of Proposition 2.

Unfortunately, we could not find a similar recursion operator for nonlocal symmetries in the negative covering.

5. Conclusion

We gave a complete description of nonlocal symmetries associated to the Lax representation of the 3D rdDym equation. The revealed Lie algebra structure of these symmetries seems quite interesting and we intend to study nonlocal symmetries of other Lax integrable equations from \([5]\) in the forthcoming research.

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