FRAÎSSÉ LIMITS OF LIMIT GROUPS

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Abstract. We modify the notion of a Fraïssé class and show that various interesting classes of groups, notably the class of nonabelian limit groups and the class of finitely generated elementary free groups, admit Fraïssé limits.

Furthermore, we rediscover Lyndon’s $\mathbb{Z}[t]$-exponential completions of countable torsion-free CSA groups, as Fraïssé limits with respect to extensions of centralizers.

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1. Introduction

Fraïssé constructions have been introduced by Roland Fraïssé in 3, where he observed that one can see the class of finite linear orders as approximations of $(\mathbb{Q}, <)$ and extended this observation to more general structures satisfying certain properties (still in a finite relational language). He showed, in particular, how one can costruct the ordering of the rational numbers as a direct limit of finite linear orders using amalgamations. Furthermore, his construction implies the countability, the universality and the homogeneity of the limit structure, as well as its uniqueness with respect to those properties. The idea of Fraïssé has been proved extremely fruitful and it has been adapted and used to discover mathematical structures with certain universal and homogeneous properties. The applicability of Fraïssé’s ideas in many different areas of mathematics demonstrate their power and usefulness. The
random graph in graph theory and Philip Hall’s universal locally finite group in group theory are conspicuous examples amongst many.

In this paper we study generalized Fraïssé limits in the class $\mathcal{F}$ of all nonabelian limit groups. The class $\mathcal{F}$, which coincides with the class of nonabelian finitely generated residually free or universally free groups, is nowadays recognized as the main study of the algebraic geometry or model theory of free groups. One cannot apply directly Fraïssé’s methods to the class $\mathcal{F}$, put differently the class $\mathcal{F}$ is not a Fraïssé class with respect to the standard embeddings of groups. However, it is more natural and more advantageous to consider limit groups as a category with $\forall$-embeddings (after all, limit groups are groups $\forall$-equivalent to a free non-abelian group). In this case it is a ”Fraïssé class” (more precisely ”Fraïssé category”, but we stay with the old name here), so the Fraïssé limits exist, they are unique up to an isomorphism, and have nice universal properties. Furthermore, going this way, one can naturally consider a subclass $\mathcal{F}_e \subset \mathcal{F}$ of all elementary free groups (those limit groups which are elementarily equivalent to a free nonabelian group) with respect to elementary embeddings. Again, this is a Fraïssé class, so the limits exists, unique, and homogeneous. These uniquely defined groups, say $G$ and $G_e$ (the Fraïssé limits of the classes $\mathcal{F}$ and $\mathcal{F}_e$) are very interesting objects in their own rights. Their algebraic structure is a mystery, which we did not attend to in this paper.

In more detail, the first main theorem of this paper is:

**Theorem 1.** The class $\mathcal{F}$ forms a $\forall$-Fraïssé class. In particular there exists a countable group $G$ with the following properties:

- the universal age of $G$, $\forall$-age($G$), is the class of nonabelian limit groups;
- for any two finitely generated isomorphic $\forall$-subgroups of $G$, there is an automorphism of $G$ extending the isomorphism;
- the group $G$ is the union of a $\forall$-chain of limit groups.

Moreover, any other countable group with the above properties is isomorphic to $G$.

For the class of finitely generated elementarily free groups we prove the following:

**Theorem 2.** The class $\mathcal{F}_e$ of elementary free groups forms an $e$-Fraïssé class. In particular there exists a countable group $G_e$ with the following properties:

- the elementary age of $G_e$, $e$-age($G_e$), is the class of elementary free groups;
- the group $G_e$ is homogeneous, i.e. whenever two tuples from $G_e$ have the same type there is an automorphism taking one to the other;
- the group $G_e$ is the union of an elementary chain of elementary free groups.

Moreover, any other countable group with the above properties is isomorphic to $G_e$.

In addition, we prove that Lyndon’s $\mathbb{Z}[t]$-exponential completion of a countable torsion-free CSA group is homogeneous with respect to a special class of subgroups, thus can be obtained as a Fraïssé limit.

2. Preliminaries

For the benefit of the reader we collect in this section a brief overview of the classical Fraïssé theory as well as natural generalizations. The material in this section is well-known to model theorists (see [2] for example). Nevertheless, we record it.

For the rest of the section we fix a countable first-order language $\mathcal{L}$. 
2.1. **Fraïssé limits.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The age of $\mathcal{M}$, $\text{age}(\mathcal{M})$, is the class of all finitely generated substructures of $\mathcal{M}$.

**Definition 2.1.** Let $\mathcal{K}$ be a countable (with respect to isomorphism types) non-empty class of finitely generated $\mathcal{L}$-structures with the following properties:

- (IP) the class $\mathcal{K}$ is closed under isomorphisms;
- (HP) the class $\mathcal{K}$ is closed under finitely generated substructures;
- (JEP) if $A_1, A_2$ are in $\mathcal{K}$, then there is $B$ in $\mathcal{K}$ and embeddings $f_i : A_i \to B$ for $i \leq 2$;
- (AP) if $A_0, A_1, A_2$ are in $\mathcal{K}$ and $f_i : A_0 \to A_i$ for $i \leq 2$ are embeddings, then there is $B$ in $\mathcal{K}$ and embeddings $g_i : A_i \to B$ for $i \leq 2$ with $g_1 \circ f_1 = g_2 \circ f_2$.

Then $\mathcal{K}$ is a Fraïssé class.

The classical example that motivated the above definition is the class of finite linear orders in $\mathcal{L} := \{<\}$. If one is concerned with groups, in the language of groups, then it is not hard to see that finitely generated groups or even finitely presented groups do not form a Fraïssé class. On the other hand, the class of finitely generated abelian groups (or even finitely presented free abelian groups) is a Fraïssé class. It is more challenging to show that the class of finite groups is a Fraïssé class.

Fraïssé classes are important for the following reason.

**Theorem 2.2** (Fraïssé’s theorem). Let $\mathcal{K}$ be a Fraïssé class. Then there exists a countable $\mathcal{L}$-structure $\mathcal{M}$ such that:

- the age of $\mathcal{M}$ is exactly $\mathcal{K}$;
- the $\mathcal{L}$-structure $\mathcal{M}$ is ultrahomogeneous, i.e. every isomorphism between finitely generated substructures of $\mathcal{M}$ extends to an isomorphism of $\mathcal{M}$.

Moreover, any other countable $\mathcal{L}$-structure with the above properties is isomorphic to $\mathcal{M}$.

2.2. **Universal Fraïssé limits.** We generalize Fraïssé constructions by strengthening the properties embeddings preserves. We will call an embedding that preserves universal (or equivalently $\Pi_1^0$) formulas a $\forall$-embedding. We will denote $\forall$-embeddings by $\rightarrow_{\forall}$. We remark that if $A \subseteq B$, i.e. the inclusion map is a $\forall$-embedding, then $A$ is existentially closed in $B$, i.e. for any quantifier free formula $\phi(\vec{x})$ with parameters in $A$, if $B \models \exists \vec{x} \phi(\vec{x})$, then $A \models \exists \vec{x} \phi(\vec{x})$. Observe that the other direction of the previous implication holds trivially by the fact that $A$ is a substructure of $B$.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The universal age of $\mathcal{M}$, $\forall\text{-age}(\mathcal{M})$, is the class of all finitely generated $\forall$-substructures of $\mathcal{M}$ (or substructures existentially closed in $\mathcal{M}$) up to isomorphism.

**Definition 2.3** (Universal Fraïssé class). Let $\mathcal{K}$ be a countable (with respect to isomorphism types) non-empty class of finitely generated $\mathcal{L}$-structures with the following properties:

- (IP) the class $\mathcal{K}$ is closed under isomorphisms;
- (∀-HP) the class $\mathcal{K}$ is closed under finitely generated $\forall$-substructures;
- (∀-JEP) if $A_1, A_2$ are in $\mathcal{K}$, then there is $B$ in $\mathcal{K}$ and $\forall$-embeddings $f_i : A_i \rightarrow_{\forall} B$ for $i \leq 2$;
- (∀-AP) if $A_0, A_1, A_2$ are in $\mathcal{K}$ and $f_i : A_0 \rightarrow_{\forall} A_i$ for $i \leq 2$ are $\forall$-embeddings, then there is $B$ in $\mathcal{K}$ and $\forall$-embeddings $g_i : A_i \rightarrow_{\forall} B$ for $i \leq 2$ with $g_1 \circ f_1 = g_2 \circ f_2$.

Then $\mathcal{K}$ is a universal Fraïssé class or for short a $\forall$-Fraïssé class.
In complete analogy to classical Fraïssé limits one can prove:

**Theorem 2.4.** Let $K$ be a $\forall$-Fraïssé class. Then there exists a countable $L$-structure $M$ such that:

- the $\forall$-age of $M$ is exactly $K$;
- the $L$-structure $M$ is weakly $\forall$-homogeneous, i.e. every isomorphism between finitely generated $\forall$-substructures of $M$ extends to an isomorphism of $M$;
- the $L$-structure $M$ is the union of a $\forall$-chain of $L$-structures in $K$.

Moreover, any other countable $L$-structure with the above properties is isomorphic to $M$.

Observe that in this case in order to obtain uniqueness of the limit we need to additionally assume that $M$ is a union of a $\forall$-chain. The main reason for that is that a substructure generated by a finite tuple is not necessarily a $\forall$-substructure, thus we cannot assume that a countable $L$-structure is exhausted by its finitely generated $\forall$-substructures.

As yet another difference from the classical Fraïssé theorem is that the $\forall$-Fraïssé limit is not $\forall$-homogeneous, but only weakly $\forall$-homogeneous. Recall, that an $L$-structure $M$ is $\forall$-homogeneous if whenever two finite tuples from $M$ satisfy the same universal formulas, there is an automorphism of $M$ taking one to the other. This can be “corrected” if one considers partial $\forall$-embeddings. If $A_0$ is a subset of the domain of the $L$-structure $A$, then the map $f : A_0 \to B$ is a partial $\forall$-embedding if for any quantifier-free formula $\phi(x, a)$ over $A_0$, if $A \models \forall x \phi(x, a)$ then $B \models \forall \bar{x} \phi(x, f(a))$.

On the light of the above we introduce the notion of a **strong $\forall$-Fraïssé class** by strengthening the $\forall$-AP property.

**Definition 2.5** (Strong Universal Fraïssé class). Let $K$ be a countable (with respect to isomorphism types) non-empty class of finitely generated $L$-structures with the following properties:

- (IP) the class $K$ is closed under isomorphisms;
- (\forall-HP) the class $K$ is closed under finitely generated $\forall$-substructures;
- (\forall-JEP) if $A_1$, $A_2$ are in $K$, then there is $B$ in $K$ and $\forall$-embeddings $f_i : A_i \to \forall B$ for $i \leq 2$;
- (strong $\forall$-AP) if $A_0$, $A_1$, $A_2$ are in $K$ and $f_i : \bar{a} \to \forall A_i$ for $i \leq 2$ are partial $\forall$-maps of some tuple $\bar{a} \in A_0$, then there is $B$ in $K$ and $\forall$-embeddings $g_i : A_i \to \forall B$ for $i \leq 2$ with $g_1 \circ f_1(\bar{a}) = g_2 \circ f_2(\bar{a})$.

Then $K$ is a strong universal Fraïssé class or for short a strong $\forall$-Fraïssé class.

We can prove that strong universal Fraïssé limits are $\forall$-homogeneous.

**Theorem 2.6.** Let $K$ be a strong $\forall$-Fraïssé class. Then there exists a countable $L$-structure $M$ such that:

- the $\forall$-age of $M$ is exactly $K$;
- the $L$-structure $M$ is $\forall$-homogeneous;
- the $L$-structure $M$ is the union of a $\Pi^0_1$-chain of $L$-structures in $K$.

Moreover, any other countable $L$-structure with the above properties is isomorphic to $M$.

With respect to these notions we will prove that the class of nonabelian limit groups forms a $\forall$-Fraïssé class (see Theorem 3.11) and the class of abelian limit groups, i.e. finitely generated free abelian groups, forms a strong $\forall$-Fraïssé class (see Theorem 3.3). It is an open question whether nonabelian limit groups form a strong $\forall$-Fraïssé class.
We will skip the proofs of the above results and only prove the theorems of the next subsection since the arguments are identical.

2.3. Elementary Fraïssé limits. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The elementary age of $\mathcal{M}$, $e$-age$(\mathcal{M})$, is the class of all finitely generated elementary substructures of $\mathcal{M}$ up to isomorphism.

When $\mathcal{A}$ is an elementary substructure of $\mathcal{M}$ we denote it by $\mathcal{A} \prec_e \mathcal{M}$.

**Definition 2.7.** Let $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures and $\bar{a}$ be a tuple from $\mathcal{A}$. Then a map $f : \bar{a} \to \mathcal{B}$ is called partial elementary if for any $\mathcal{L}$-formula $\phi(x)$, $\mathcal{A} \models \phi(\bar{a})$ if and only if $\mathcal{B} \models \phi(f(\bar{a}))$.

**Definition 2.8.** Let $\mathcal{K}$ be a countable (with respect to isomorphism types) non-empty class of finitely generated $\mathcal{L}$-structures with the following properties:

- (IP) the class $\mathcal{K}$ is closed under isomorphisms;
- (e-HP) the class $\mathcal{K}$ is closed under finitely generated elementary substructures, i.e. if $\mathcal{A} \in \mathcal{K}$ and $\mathcal{B}$ is a finitely generated elementary substructure of $\mathcal{A}$, then $\mathcal{B} \in \mathcal{K}$;
- (e-JEP) if $\mathcal{A}_1, \mathcal{A}_2$ are in $\mathcal{K}$, then there is $\mathcal{B}$ in $\mathcal{K}$ and elementary embeddings $f_i : \mathcal{A}_i \to \mathcal{B}$ for $i \leq 2$;
- (strong e-AP) if $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ are in $\mathcal{K}$ and $f_i : \bar{a} \to \mathcal{A}_i$, for $i \leq 2$, are partial elementary maps of some tuple $\bar{a} \in \mathcal{A}_0$, then there is $\mathcal{B}$ in $\mathcal{K}$ and elementary embeddings $g_i : \mathcal{A}_i \to \mathcal{B}$ for $i \leq 2$ with $g_1 \circ f_1(\bar{a}) = g_2 \circ f_2(\bar{a})$.

Then $\mathcal{K}$ is an elementary Fraïssé class or an $e$-Fraïssé class for short.

**Definition 2.9.** Let $\mathcal{M}$ be a countable $\mathcal{L}$-structure. Then $\mathcal{M}$ is homogeneous if there exists an automorphism taking the tuple $\bar{a}$ to the tuple $\bar{b}$ whenever $tp^\mathcal{M}(\bar{a}) = tp^\mathcal{M}(\bar{b})$.

**Definition 2.10** (strong $e$-Extension Property). An $\mathcal{L}$-structure has the strong $e$-Extension Property if for any $\mathcal{A}, \mathcal{B}$ finitely generated elementary substructures of $\mathcal{M}$ with $\mathcal{A} \prec_e \mathcal{B}$ and any partial elementary embedding $f : \bar{a} \to \mathcal{M}$, for $\bar{a} \in \mathcal{A}$, there exists an elementary embedding $g : \mathcal{B} \to \mathcal{M}$ that extends $f$.

**Lemma 2.11.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure which is the union of an elementary chain $\mathcal{M}_1 \prec_e \mathcal{M}_2 \prec_e \ldots \prec_e \mathcal{M}_n \prec_e \ldots$ of finitely generated $\mathcal{L}$-structures. Then $\mathcal{M}$ is homogeneous if and only if it has the strong $e$-Extension Property.

**Proof.** First assume that $\mathcal{M}$ is homogeneous. Let $\mathcal{A} \prec_e \mathcal{B}$ be finitely generated elementary substructures of $\mathcal{M}$ and $f : \bar{a} \to \mathcal{M}$ be a partial elementary map, where $\bar{a}$ is a tuple from $\mathcal{A}$. Then $tp^\mathcal{M}(\bar{a}) = tp^\mathcal{A}(\bar{a}) = tp^\mathcal{M}(f(\bar{a}))$. By the homogeneity of $\mathcal{M}$ there exists an automorphism $g$ taking $\bar{a}$ to $f(\bar{a})$. The restriction of $g$ on $\mathcal{B}$ is an elementary map extending $f$. Note that this direction does not use that $\mathcal{M}$ is the union of an elementary chain.

For the other direction, assume that $\mathcal{M}$ has the strong $e$-Extension Property and let $tp^\mathcal{M}(\bar{a}) = tp^\mathcal{M}(\bar{b})$. Let $\mathcal{M}_i$ be an $\mathcal{L}$-structure in the elementary chain that contains $\bar{a}$. Let $c$ be an element of $\mathcal{M}$ and $\mathcal{M}_j$ be an $\mathcal{L}$-structure in the elementary chain that contains both $\bar{a}$ and $c$. We obviously have $\mathcal{M}_i \prec_e \mathcal{M}_j$ and a partial elementary map $f : \bar{a} \to \mathcal{M}$ with $f(\bar{a}) = \bar{b}$. By the $e$-Extension Property we get an elementary map $g : \mathcal{M}_j \to \mathcal{M}$ that extends $f$. Thus, we get $tp^\mathcal{M}(\bar{a}, c) = tp^\mathcal{M}(\bar{b}, g(c))$ and we conclude the proof by a back-and-forth argument.

**Theorem 2.12.** Let $\mathcal{K}$ be a strong $e$-Fraïssé class. Then there exists a countable $\mathcal{L}$-structure $\mathcal{M}$ such that:
the e-age of \( M \) is exactly \( K \);
- the \( L \)-structure \( M \) is homogeneous;
- the \( L \)-structure \( M \) is the union of an elementary chain of \( L \)-structures in \( K \).

Moreover, any other countable \( L \)-structure with the above properties is isomorphic to \( M \).

Proof. We will construct an elementary chain \( M_1 \prec_e M_2 \prec_e \ldots \prec_e M_n \prec_e \ldots \) of \( L \)-structures in \( K \) and prove that its union \( M := \bigcup_{i<\omega} M_i \) has all the desired properties.

Let \( \{ A_i \}_{i<\omega} \) be a list of all \( L \)-structures in \( K \) (up to isomorphism). Suppose we have constructed \( M_i \). We construct \( M_{i+1} \) taking cases depending on whether \( i \) is even or odd.

- If \( i = 2n \) is even, then we apply e-JEP to \( M_i \) and \( A_n \). We set \( M_{i+1} \) to be the resulting \( L \)-structure.
- If \( i = 2n+1 \) is odd, then we apply e-AP to \( A \prec_e M_i \), \( A \prec_e B \) and partial elementary maps \( f_1 : \bar{a} \to M_i \), \( I d : \bar{a} \to B \), for some \( \bar{a} \in A \). We set \( M_{i+1} \) to be the resulting \( L \)-structure.

The even cases take care of the equality between the elementary age of \( M \) and \( K \). In order to prove the strong e-Extension Property and consequently the homogeneity of \( M \) we need to ensure that for any couple \( A \prec_e B \) of elementary substructures of \( M \) and any partial elementary map \( f : \bar{a} \to M \) there exists an elementary embedding \( g : B \to M \) that extends \( f \). We may assume that \( f(\bar{a}) \) and \( A \) are in \( M_j \) for some \( j < \omega \). Thus, we need to ensure that for some odd \( i \geq j \), we have chosen the partial map \( f \) and the inclusion map \( A \prec_e B \) to apply e-AP. This is clearly possible since there are only countably many choices of partial elementary maps and couples of elementary substructures.

The uniqueness result follows from an easy back-and-forth argument.

\[ \square \]

2.4. Finitary extensions. There is a straightforward generalization of the classical Fraïssé result when we can only define “special” extensions of structures in a finitary way. A particular case is when special extensions can only be defined for say finite \( L \)-structures. In this case it is not clear a priori how to give meaning to the notion of age of a countably infinite \( L \)-structure with respect to special embeddings. In this subsection we record well known results that deal with this case.

Let \( K \) be a class of finitely generated \( L \)-structures. Suppose \( A \subseteq B \) are in \( K \), we define a notion of special extension and denote it by \( A \preceq B \). The notion of a special embedding between structures in \( K \) is defined in the natural way. If the \( L \)-structure \( M \) is the union of a \( \preceq \)-chain \( M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n \subseteq \ldots \) we will write \( A \subseteq M \) if \( A \subseteq M_i \) for some \( i < \omega \). We will assume that \( \subseteq \) satisfies the following properties:

For any \( A, B, C \) in \( K \):

- (N1) the \( L \)-structure \( A \) is a special extension of itself, i.e. \( A \preceq A \);
- (N2) if \( A \preceq B \subseteq C \), then \( A \preceq C \);
- (N3) if \( A \subseteq C \) and \( A \preceq B \subseteq C \), then \( A \subseteq B \).

The last property, (N3), guarantees that \( A \preceq M \), as defined above, does not depend on the choice of the \( \preceq \)-chain \( M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n \subseteq \ldots \).

Definition 2.13. Let \( K \) be a countable (with respect to isomorphism types) non-empty class of finitely generated \( L \)-structures that has countably many \( \preceq \)-embeddings between any pair of elements, with the following properties:

- (IP) the class \( K \) is closed under isomorphisms;
Theorem 2.14. Let $K$ be a $\sqsubseteq$-Fraïssé class. Then there exists a countable $L$-structure $M$ such that:

- the $L$-structure $M$ is the union of a $\sqsubseteq$-chain of $L$-structures in $K$;
- the $\sqsubseteq$-age of $M$ is exactly $K$;
- the $L$-structure $M$ is $\sqsubseteq$-homogeneous, i.e. whenever $A$, $B$ are isomorphic finitely generated substructures then there is an automorphism of $M$ extending this isomorphism;

Moreover, any other countable $L$-structure with the above properties is isomorphic to $M$.

All definitions and results in this subsection go through in the category of relatively finitely generated $L$-structures, i.e. when all $L$-structures considered are extensions of a fixed $L$-structure $N$ and finitely generated with respect to it, moreover all special embeddings between them fix $N$ pointwise.

3. The class of limit groups

Limit groups have been introduced by Sela in [12]. It turned out that the class of limit groups coincides with the long studied class of finitely generated fully residually free groups introduced by Baumslag in [1]. A group $G$ is fully residually free if for any finite subset $X$ of $G$ there exists a homomorphism from $G$ to a free group that is injective on $X$. In [4] (see also [12]) it was proved that limit groups are finitely presented, thus the class of limit groups is countable. One can characterize limit groups in terms of first-order logic: they are the finitely generated models of the universal theories of free groups (abelian and nonabelian). The class of abelian limit groups is, in particular, the class of finitely generated free abelian groups.

Limit groups do not form a Fraïssé class because, although they satisfy the $HP$ and $JEP$ conditions, they do not satisfy the $AP$ condition. To see this consider the groups $\langle z \rangle$, $\mathbb{F}_2 := \langle a, b \rangle$, $\mathbb{F}_2' := \langle c, d \rangle$ and the embeddings $f_1 : \mathbb{Z} \to \mathbb{F}_2$, given by $z \mapsto a^2b^2$, and $f_2 : \mathbb{Z} \to \mathbb{F}_2'$, given by $z \mapsto c^2$. The amalgamated free product $\mathbb{F}_2 *_{\mathbb{Z}} \mathbb{F}_2'$ using the maps $f_1, f_2$ is the free product $\langle a, b, c, d | a^2b^2 = c^2 \rangle$. This free product is not a limit group since $\langle a, b, c | a^2b^2 = c^2 \rangle$ is not a limit group. The latter is true because in limit groups, by a result of Lyndon for free groups, whenever the product of three squares is trivial, then the elements of the product commute. Now we can see that the class of limit groups does not have $AP$, since for any group $G$ and any $g_1 : \mathbb{F}_2 \to G$, $g_2 : \mathbb{F}_2' \to G$ such that $g_1 \circ f_1 = g_2 \circ f_2$, we get that $g_1 \circ f_1(z) = g_1(a)^2g_1(b)^2 = g_2(c)^2 = g_2 \circ f_2(z)$. But the latter implies that $g_1(a)$, $g_1(b)$, $g_2(c)$ commute in $G$, hence $a, b$, commute in $\mathbb{F}_2$, a contradiction.
3.1. Free Abelian groups. In this subsection we show that the class of finitely generated free abelian groups forms a strong \( \forall \)-Fraïssé class. We remark that in this class the notions of pure subgroup, existentially closed subgroup and direct factor coincide. All properties follow rather easily, therefore we only give the details for proving the strong \( \forall \)-AP.

**Lemma 3.1.** Let \( \bar{a} \in \mathbb{Z}^n \) and \( \bar{b} \in \mathbb{Z}^m \) such that \( tp_{\mathbb{Z}^n}^{\mathbb{Z}^m}(\bar{b}) = tp_{\mathbb{Z}^m}^{\mathbb{Z}^n}(\bar{c}) \). Then, the isomorphism between \( \langle \bar{b} \rangle \) and \( \langle \bar{c} \rangle \) extends to the smallest pure subgroups that contain \( \bar{b} \) and \( \bar{c} \) respectively.

**Proposition 3.2.** The class of finitely generated free abelian groups has the strong \( \forall \)-AP condition.

**Proof.** Let \( \bar{a} \) be a tuple from \( \mathbb{Z}^m \) and \( f_i : \bar{a} \rightarrow_{\forall} \mathbb{Z}^{m_i} \) be partial \( \forall \)-maps for \( i = 1, 2 \). Then \( tp_{\mathbb{Z}^{m_i}}^{\mathbb{Z}^{m_2}}(\bar{b}) = tp_{\mathbb{Z}^{m_2}}^{\mathbb{Z}^{m_1}}(\bar{c}) \) where \( \bar{b} = f_1(\bar{a}) \) and \( \bar{c} = f_2(\bar{a}) \). By the previous lemma there is a direct factorization of \( \mathbb{Z}^{m_i} \) as \( B \times \mathbb{Z}_1 \) and one of \( \mathbb{Z}^{m_2} \) as \( C \times \mathbb{Z}_2 \) such that \( f : C \cong B \) with \( f(\bar{c}) = \bar{b} \). Consider the group \( D = \mathbb{Z}_1 \times B \times \mathbb{Z}_2 \) and the embeddings \( g_1 = Id \) and \( g_2 = (f, Id) \), where \( f \) is the isomorphism from \( B \) to \( A \). The embeddings are in particular \( \forall \)-embeddings and moreover \( g_1 \circ f_1(\bar{a}) = g_2 \circ f_2(\bar{a}) \).

Thus, we have the following theorem.

**Theorem 3.3.** There exists a countable group \( G \) with the following properties:

- the \( \forall \)-age of \( G \) is the class of finitely generated free abelian groups;
- the group \( G \) is \( \forall \)-homogeneous;
- the group \( G \) is the union of a \( \forall \)-chain of finitely generated free abelian groups.

Moreover, any countable group with the above properties is isomorphic to \( G \).

It is not hard to see in this case that the strong \( \forall \)-Fraïssé limit is \( \mathbb{Z}^{(\omega)} \), the direct sum of \( \omega \) copies of \( \mathbb{Z} \).

As a matter of fact the class of finitely generated free abelian groups forms a Fraïssé class. In this case the Fraïssé limit is \( \mathbb{Q}^{(\omega)} \), the direct sum of \( \omega \) copies of \( \mathbb{Q} \).

3.2. Nonabelian limit groups. In this subsection we will make use of some basic constructions in combinatorial group theory, in particular amalgamated free products and HNN-extensions. We refer the reader to the classical books [7], [6].

Nonabelian limit groups in contrast to the abelian limit groups do not form a Fraïssé class. In particular the \( HP \) condition fails.

We now prove that the class \( \mathcal{F} \) of nonabelian limit groups forms a \( \forall \)-Fraïssé class. The \( \forall \)-HP condition is rather obvious, since a finitely generated subgroup of a limit group is itself a limit group, in addition an existentially closed subgroup of a nonabelian group cannot be abelian.

It will be useful to remark that in the class of limit groups one can characterize \( \forall \)-embeddings in the following way:

**Lemma 3.4.** Let \( L, M \) be limit groups. Then \( L \preceq_{\forall} M \) if and only if for every finite subset \( X \subset M \), there exists a retraction \( r_X : M \rightarrow L \) that is injective on \( X \).

We also recall that abelian subgroups of limit groups are finitely generated and thus in particular they are free abelian.

For proving the \( \forall \)-JEP and \( \forall \)-AP conditions we will use the following theorems. We recall that an extension of a centralizer of a group \( G \) is the group \( \langle G, t \mid [c, t] = 1, c \in C_G(u) \rangle \). A finite iterated extension of centralizers of a group is obtained by finitely many applications of the previous construction.
**Theorem 3.5** ([1]). Let \( L \) be a limit group (respectively nonabelian limit group). Then \( L \) embedds into some finite iterated extension of centralizers of a free group (respectively nonabelian free group).

For notation purposes when \( K \) is a subgroup of both \( L \) and \( M \) we call a map from \( L \) to \( M \) a \( K \)-map if it is the identity on \( K \).

**Theorem 3.6** ([5]). Let \( L, M \) be limit groups and \( L \leq \forall M \). Then \( M \) embedds into some finite iterated extension of centralizers of \( L \).

**Lemma 3.7.** Let \( L \) be a limit group and \( M \) be a finite iterated extension of centralizers of \( L \). Then \( L \leq \forall M \).

We first recall the following well known result:

**Lemma 3.8.** Let \( g \) be a nontrivial element of a nonabelian free group \( F \). Suppose \( g \) has the following form:
\[
a_0 b_1^\epsilon_1 a_1 b_2^\epsilon_2 a_2 \ldots a_{n-1} b_n^\epsilon_n a_n
\]
where \( n \geq 1 \) and \([a_i,b] \neq 1\) for all \( 0 < i < n \). Then \([g,b] \neq 1\) whenever \( \min k | i_k | \) is sufficiently large.

The above property easily passes to nonabelian limit groups using the fact that they are fully residually free.

**Proof of Lemma 3.7.** It is enough to show that if \( L \) is a limit group and \( M \) is an extension of centralizers, then \( L \leq \forall M \). If \( L \) is (free) abelian, then \( M \) is free abelian as well and has \( L \) as a direct factor. We may therefore assume that \( L \) is nonabelian. Using Lemma 3.4 we need to find for each finite set \( X \subseteq M \) a retraction \( r_X : M \to L \) such that \( r_X \) is injective on \( X \). Since \( M \) is an extension of centralizers of \( L \), it is in particular an HNN-extension \( L*C \) where \( C \) is a maximal abelian subgroup of \( L \).

It follows from the normal form theorem for HNN extensions that any element of \( M \) is either trivial or of the form \( a_0 t^\epsilon_1 a_1 \ldots a_{n-1} t^\epsilon_n a_n \) where \( a_i \) is in \( L \setminus C \) for all \( 0 < i < n \) and \( \epsilon_i \in \{0,1\} \) for all \( i \leq n \). Now consider the retractions \( r : M \to L \), that fix \( L \) and send the stable letter \( t \) to high powers of some generator of the free abelian group \( C \). If \( X \) is a singleton the result follows easily from Lemma 3.8. In the case \( |X| > 1 \), one has to be more careful but still the result is an easy exercise on applying Lemma 3.8.

We first prove the \( \forall \)-JEP condition.

**Proposition 3.9.** Let \( L, M \) be limit groups and \( L \) be nonabelian. Then \( L \leq \forall L * M \).

**Proof.** It is not hard to see, using the same arguments as in the proof of Lemma 3.7, that \( L \leq \forall L * F \), where \( F \) is a free group. Now by Theorem 3.3 \( M \) embedds in a finite iterated extension of centralizers of some free group \( F \), say \( \Gamma \). In addition, \( L * F \leq \forall L * \Gamma \) as \( L * \Gamma \) can be obtained as a finite iterated extension of centralizers of \( L * F \). Therefore, \( L \leq \forall L * \Gamma \). Now observe that since \( L * M \) is a subgroup of \( L * \Gamma \) that contains \( L \) we have that \( L \leq \forall L * M \), proving the result.

It is easy to see that the free product of two limit groups is a limit group. Lemma 3.9 in addition proves that a nonabelian limit group is existentially closed in the free product.
of itself with another limit group, thus making the $\forall$-JEP condition true for the class of nonabelian limit groups.

We now prove the $\forall$-AP condition.

**Proposition 3.10.** Let $K, L, M$ be nonabelian limit groups such that $K \leq \forall L$ and $K \leq \forall M$. Then $L \leq \forall L \ast_K M$.

**Proof.** It is enough to prove the proposition for the special case when $M$ is a finite iterated extension of centralizers of $K$. To see this, one uses that, by Theorem 3.6, $M$-embeds in a finite iterated extension of centralizers of $K$, say $\Gamma$. But then $L \ast_K M$ is a subgroup of $L \ast_K \Gamma$ that contains $L$. Therefore, $L \leq \forall L \ast_K M$.

We now prove by induction on the length of the finite iterated extension of centralizers $K = \Gamma_0 < \Gamma_1 < \ldots < \Gamma_n = M$ that the proposition holds.

For the base case, suppose $M = K \ast_C$ where $C$ is the centralizer of a nontrivial element $c$. Since $K$ is existentially closed in $L$, we have that the centralizer of $c$ in $L$ is actually the centralizer of $c$ in $C$. Therefore, the amalgamated free product $L \leq \forall L \ast_K M$ is actually the following extension of centralizers $(L \ast_K K) \ast_C$. Hence, it is an extension of centralizers of $L$ and consequently, by Lemma 3.7, $L \leq \forall L \ast_K M$.

Assume it holds for all finite iterated extension of centralizers of length $n$. We show that it holds for all finite iterated extension of centralizers of length $n + 1$. By the induction hypothesis we have that $L \leq \forall L \ast_K \Gamma_n$, so it is enough to show that $L \ast_K \Gamma_n \leq \forall L \ast_K \Gamma_{n+1}$. The latter is true considering the following two cases. First, if the element, say $c \in \Gamma_n$, whose centralizer $C$ we extended is in $\Gamma_n \setminus K$, then $C_{\Gamma_n}(c) = C_{L \ast_K \Gamma_n}(c)$ and therefore $L \ast_K \Gamma_{n+1}$ is $(L \ast_K \Gamma_n) \ast_C$, an extension of centralizers of $L \ast_K \Gamma_n$. Hence, by Lemma 3.7, we get $L \ast_K \Gamma_n \leq \forall L \ast_K \Gamma_{n+1}$. On the other hand, if $c \in K$, since $K$ is existentially closed in both $L$ and $\Gamma_n$ we have that $C_K(c) = C_L(c) = C_{\Gamma_n}(c) = C_{L \ast_K \Gamma_n}(c)$, therefore $L \ast_K \Gamma_{n+1}$ is $(L \ast_K \Gamma_n) \ast_C$ once more an extension of centralizers of $L \ast_K \Gamma_n$. Hence, once again by Lemma 3.7, we get $L \ast_K \Gamma_n \leq \forall L \ast_K \Gamma_{n+1}$ and this completes the inductive argument.  

Proposition 3.10 proves that when we amalgamate two nonabelian limit groups over an existentially closed subgroup, then the result is a limit group and in addition the natural embeddings of nonabelian limit groups into the amalgamated free product are $\forall$-embeddings.

Finally, combining the results above we get:

**Theorem 3.11.** Class $\mathcal{F}$ is a $\forall$-Fraïssé class.

In particular there exists a countable group $G$ with the following properties:

- the $\forall$-age of $G$ is the class $\mathcal{F}$;
- the group $G$ is weakly $\forall$-homogeneous;
- the group $G$ is a union of a $\forall$-chain of nonabelian limit groups.

Moreover, any countable group with the above properties is isomorphic to $G$.

We next show that the $\forall$-Fraïssé limit of the class of nonabelian limit groups is a fully residually free group.

**Proposition 3.12.** Let $G$ be a union of a $\forall$-chain of limit groups. Then $G$ is fully residually free.

**Proof.** Let $G$ be the union of the following $\forall$-chain $G_1 \leq \forall G_2 \leq \forall \ldots \leq \forall G_n \leq \ldots$. Let $X$ be a finite subset of $G$. Then $X$ is a subset of $G_n$ for some $n$. It is not hard to see that for each $n$ and each finite set $X$ there is a retract $r_X : G_{n+1} \to G_n$ which is injective on $X$. Using this we obtain a chain of morphisms $f_m : G_m \to F$ for all $m \geq n$, where $F$ is a nonabelian
free group, such that each $f_m$ is injective on $X$. The union $f := \bigcup f_m$ is a morphism from $G$ to $F$ that is injective on $X$. \qed

4. Lyndon’s completions

4.1. Hierarchy of extensions of centralizers. In this section we consider $G$-groups, i.e., extensions of $G$, where $G$ is a countable nonabelian torsion-free CSA-group. We relate the free Lyndon $\mathbb{Z}[t]$-group $G^{\mathbb{Z}[t]}$ with Fraïssé limits with respect to extensions of centralizers.

We denote by $ICE(G)$ the class of $(G)$-groups obtained from the given group $G$ as finite iterated extensions of centralizers of $G$. We allow the empty sequence of centralizer extensions, so that $G \in ICE(G)$. We say that an embedding $\phi : A \to B$ is an $ICE$-embedding if $B$ can be obtained from $\phi(A)$ as a finite iterated extension of centralizers. In this event we write $\rightarrow_{ICE}$. We will apply the results of Section 2.4 with $K$ to be $ICE(G)$ and $\subseteq$ to be $ICE$-extensions.

Lemma 4.1. Let $G$ be a countable non-abelian torsion-free CSA group (not necessarily finitely generated). Then the class $ICE(G)$ satisfies JEP and AP relative to $ICE$-embeddings.

Proof. Since $G$ is $ICE$-embedded in every group from $ICE(G)$ it suffices to prove AP.

Suppose that $A, B, C \in ICE(G)$ and $\phi : C \to A$ and $\psi : C \to B$ are $ICE$-embeddings.

Case 1. Suppose $A$ and $B$ are obtained from $C$ by a single centralizer extension, i.e., $A = C(u_1, t_1), B = C(u_2, t_2)$. If the centralizers $C(u_1)$ and $C(u_2)$ are conjugate in $C$, say $d^{-1}C(u_1)d = C(u_2)$, then the groups $A$ and $B$ are isomorphic with the isomorphism $\alpha : A \to B$ such that $\alpha_C = id_C$ and $t_1 \to dt_2$. If $C(u_1)$ and $C(u_2)$ are not conjugate in $C$ then the centralizer extensions $C(u_1, t_1)$ and $C(u_2, t_2)$ “do not overlap”, so

$$A(u_2, t_2) \simeq C(u_1, t_1)(u_2, t_2) \simeq C(u_2, t_2)(u_1, t_1) \simeq B(u_1, t_1),$$

hence the group $A(u_2, t_2)$ (as well as $B(u_1, t_1)$) gives the required amalgamation.

Case 2. Suppose now that $A$ and $B$ are obtained from $C$ by some finite sequences of CE, so

$$C = A_0 \leq_{CE} A_1 \leq_{CE} \cdots \leq_{CE} A_k = A,$$

$$C = B_0 \leq_{CE} B_1 \leq_{CE} \cdots \leq_{CE} B_m = B$$

for some $k, m$. By Case 1 we can amalgam $A_1, B_1$ into some $D_1$, and then amalgam $A_2$ and $D_1$ into some $D_2$, after $k$ steps we will get the following diagram:

Now the subdiagram formed by the chains $B_1 \to D_1 \to \cdots \to D_k$ and $B_1 \to B_2 \to \cdots \to B_m$ is smaller, so by induction on $k + m$ one gets the required amalgamation. \qed

Lemma 4.2. Let $G$ be a countable non-abelian torsion-free CSA group.

1) (ICE-HP) If $G \rightarrow_{ICE} A, G \leq B, B \rightarrow_{ICE} A$, then $G \rightarrow_{ICE} B$.

2) (ICE-N3) If $A \leq B \leq C \in ICE(G)$ and $A \rightarrow_{ICE} C$, then $A \rightarrow_{ICE} B$.

Proof. 1) We can assume that both $A$ and $B$ are freely indecomposable relative to $G$.

Suppose

$$G = A_0 \leq_{CE} A_1 \leq_{CE} \cdots \leq_{CE} A_k = A$$

and on each step the corresponding centralizer is extended by one letter. We will use induction on $k$. 

□
If \( k = 1 \), \( A = A_1 = \langle G, t_1 | [u_1, t_1] = 1 \rangle \). Either \( B = G \) or \( G \) is a proper subgroup of \( B \). In the latter case if \( B \) does not contain the stable letter \( t_1 \), then \( A \) must be obtained from \( B \) by freely extending the centralizer of \( u_1 \) by \( t_1 \). But in this case \( B \) cannot have elements containing \( t_1 \) in the normal form in \( A \). Therefore this case is impossible. Hence if \( G \) is a proper subgroup of \( B \), then \( B = A \).

Consider now the general case. Let \( A_{j+1} = \langle A_j, t_j | [C(u_{j+1}), t_{j+1}] = 1 \rangle \) for \( j \leq k - 1 \). Suppose that it is possible to change the order of centralizer extensions such a way that \( B \) is conjugate into \( A_{k-1} \). Since \( G \leq B \), \( B \leq A_{k-1} \). Then by mapping \( t_k \) into \( u_k \) we see that \( A_{k-1} \) can be obtained by a sequence of centralizer extensions from \( B \). By induction, \( B \) is obtained from \( G \) by a sequence of extensions of centralizers.

If it is not possible to change the order of centralizer extensions such a way that \( B \) is conjugate into \( A_{k-1} \), then every conjugate \( B^9 \) contains elements that have \( t_k \) in the normal form, therefore it must contain the conjugate \( t_k^9 \) (because to obtain \( A_k \) from \( B \) we can only use centralizer extensions). If \( t_k \) belongs to the centralizer of some \( u_k \in G \) we can consider \( \langle G, t_k | [u_k, t_k] = 1 \rangle \) instead of \( G \) and apply induction. Otherwise, we map \( t_k \) to \( u_k^9 \). Then the image of \( A_k \) is \( A_{k-1} \) and the image of \( B \) is some subgroup \( B \). And \( A_{k-1} \) is obtained from \( B \) by a series of centralizer extensions. Then by induction, \( B \) is obtained from \( G \) by a series of centralizer extensions, and \( B \) is obtained from \( B \) by extending some centralizer by \( t_k \).

2) Let

\[
A = A_0 \leq_{CE} A_1 \leq_{CE} \ldots \leq_{CE} A_k = C.
\]

Then \( B \in A_i \) for some \( i \leq k \). Taking \( A_i \) instead of \( C \) we can assume that \( i = k \). We use induction on \( k \). If \( k = 1 \), then either \( B = A \) or \( B \) contains elements with \( t_1 \) in the normal form. But since \( B \in CE(G) \) it must be \( B = C \) in the latter case. For the general case we can suppose that \( B \nleq A_{k-1} \). If there are elements in \( B \) that contain \( t_k \) in the normal form, then \( B \) contains \( t_k \) because \( B \) is obtained from \( G \) by centralizer extensions. Therefore \( B = \langle B_1, t_k | [C(u_k) \cap B, t_k] = 1 \rangle \), where \( B_1 \in A_{k-1} \). By induction, \( A \to_{ICE} B_1 \), and, therefore, \( A \to_{ICE} B \).

If \( H \) is the direct limit of a chain of ICE-embeddings of groups from \( ICE(G) \), namely

\[
G = G_0 \leq_{CE} G_1 \leq_{CE} \ldots \leq_{CE} G_i \leq_{CE} \ldots ,
\]

then for \( A \in ICE(G) \) we can define \( A \to_{ICE} H \) if \( A \to_{ICE} G_i \) for some \( i \). It follows from Lemmas 1.1 [1.2] and the results of Section 2.4 that this definition does not depend on the chain. It also follows that \( A \to_{ICE} H \) if \( H \) can be obtained as the direct limit of a chain of ICE-embeddings of groups from \( ICE(A) \). It also follows that \( ICE(G) \) is a Fraïssé class with respect to ICE-embeddings. Theorem 2.14 implies the following

**Theorem 4.3.** Let \( G \) be a countable non-abelian torsion-free CSA group. Then there exists a countable group \( H \) with the following properties:

- the group \( H \) is the union of an ICE-chain of groups in \( ICE(G) \);
- the ICE-age of \( H \) is exactly \( ICE(G) \);
- Any \( G \)-isomorphism between two groups \( A, B \leq_{ICE} H \) can be extended to an automorphism of \( H \).

Moreover, any other countable group with the above properties is \( G \)-isomorphic to \( G \).
**Theorem 4.4.** The group $H$ in Theorem 4.3 is isomorphic to the Lyndon’s completion $G^Z[t]$ of the group $G$. In particular when $G$ is a nonabelian free group $F$, then $H$ is isomorphic to $F^Z[t]$.

Proof. In [8] it was proved that the Lyndon’s completion $G^Z[t]$ of the group $G$ can be constructed as follows. We begin with $G$ and extend all non conjugate centralizers obtaining the group $G_1$. Then we extend all non-conjugate centralizers in $G_1$ and so on. To obtain $G_i$ we extend all non-conjugate centralizers of $G_i$. It is clear that $G^Z[t]$ can be obtained as the direct limit of a chain of extensions of centralizers:

$$G = \tilde{G}_0 \leq_{CE} \tilde{G}_1 \leq_{CE} \ldots \leq_{CE} \tilde{G}_i \leq_{CE} \ldots$$

At the same time $H$ is the direct limit of a chain of ICE-embeddings of groups from $ICE(G)$, namely

$$G = G_0 \leq_{CE} G_1 \leq_{CE} \ldots \leq_{CE} G_i \leq_{CE} \ldots$$

Since $ICE-age(G^Z[t]) = ICE(G)$, for each $i$ there are $j, k, m$ such that

$$\tilde{G}_i \leq G_j \leq \tilde{G}_k \leq G_m.$$ 

Property ICE-N3 implies that

$$\tilde{G}_i \rightarrow_{ICE} G_j \rightarrow_{ICE} \tilde{G}_k \rightarrow_{ICE} G_m$$

and $H$ is isomorphic to $G^Z[t]$. This proves the first statement.

We remark that if $G$ is a fully residually free group then the class of finitely generated subgroups of $G^Z[t]$ is exactly the class of limit groups. Indeed, since $F \leq G$ we get, by [4], that $G^Z[t]$ contains all limit groups. On the other hand, since the class of limit groups is closed under extensions of centralizers and finitely generated subgroups, all finitely generated subgroups of $G^Z[t]$ are limit groups.

If $G$ is a toral relatively hyperbolic group, then, it follows from [5], that the class of finitely generated subgroups of $G^Z[t]$ is exactly the class of fully residually $G$ groups.

Theorem 4.4 explains the model-theoretic nature of Lyndon’s completions $G^Z[t]$ for a countable non-abelian torsion free CSA-group $G$.

**4.2. Hierarchy of free products and extensions of centralizers.** Denote by $H$ the smallest set of groups obtained from the trivial group by finitely many operations of free products with $\mathbb{Z}$ and extensions of centralizers. We refer to $H$ as the hierarchy of centralizer extensions. We say that an embedding $\phi : A \rightarrow B$ is an $FPCE$-embedding if $B$ can be obtained from $\phi(A)$ by a finite sequence of free products with $\mathbb{Z}$ and extensions of centralizers.

**Lemma 4.5.** The class $H$ satisfies JEP, AP, HP and (N3) conditions with respect to $FPCE$-embeddings.

Proof. Observe first that for any groups $A$ and $B$ one has $A(u, t) \ast B \simeq (A \ast (B))(u, t)$. Indeed, the centralizer of any non-trivial $u$ in $A$ is equal to the centralizer of $u$ in $A \ast B$. Now the claim follows from observing the defining relations of the groups. This shows that the result of applying a finite sequence of operations FP (free product) and CE (centralizer extension) to a group $C$ is the same as applying some FP and after that some EC.

Now let $A$ and $B$ are $FPCE$-extensions of some $C$ in $H$. Then by the observation above $A$ is $ICE$-extension of $C \ast F(X)$ for some free group $F(X)$ with finite $X$, and $B$ is an $ICE$-extension of $C \ast F(Y)$ for some finite $Y$. Consider the $FPCE$-extensions $A_1 = A \ast F(Y)$
and $B_1 = B \ast F(X)$ and put $C_1 = C \ast F(X) \ast F(Y)$. Then again by the observation above $A_1, B_1 \in ICE(C_1)$. Applying AP in the class $ICE(C_1)$ one gets the required amalgamation of $A_1$ and $B_1$ with respect to $FPCE$-embeddings $C_1 \to A_1$ and $C_1 \to B_1$.

The operation $FP$ gives JEP in $\mathcal{H}$.

Conditions HP and (N3) are proved as in Lemma 4.2 □

Lemma 4.5 shows that $\mathcal{H}$ forms an $FPCE$-Fraïssé class. Hence we have the following result.

**Theorem 4.6.** There exists a unique up to isomorphism group $H$ such that

1) $H$ is countable,

2) Finitely generated subgroups of $H$ are exactly limit groups.

3) $H$ is the union of an $FPCE$-chain of groups from $\mathcal{H}$,

4) $age_{FPCE}(H) = \mathcal{H}$,

5) $H$ is $FPCE$-homogeneous.

Furthermore, the group $H$ is isomorphic to the Lyndon’s completion $\mathbb{F}[\omega]$ of the free group $\mathbb{F}_\omega$ of countable rank.

5. The free group

The basis of this section is the following celebrated result.

**Theorem 5.1 (Kharlampovich-Myasnikov, Sela).** The chain $\mathbb{F}_2 \subset \mathbb{F}_3 \subset \ldots \subset \mathbb{F}_n \subset \ldots$, under the natural embeddings, is an elementary chain. In particular, all nonabelian free groups are elementarily equivalent.

The **theory of the free group** has attracted much attention from both communities, model theorists and group theorists. Apart from its natural models, nonabelian free groups, it is satisfied by groups which are not free. As a matter of fact, since the theory of the free group admits uncountably many countable models, the majority of the countable models are not free. Despite the complexity of the theory of the free group Kharlampovich-Myasnikov and Sela characterized, group theoretically, its finitely generated models.

In this section we first give the above mentioned description and then prove that the class of finitely generated elementary free groups is an elementary Fraïssé class.

The following subsections make heavy use of Bass-Serre theory. We refer the unfamiliar reader to the classical book of Serre [13], or the quick introduction in [14].

5.1. Hyperbolic towers.

**Definition 5.2 (Surface type vertices).** Let $G$ be a group acting on a tree $T$ without inversions and $(T_1, T_0, \{\gamma_e\})$ be a Bass-Serre presentation for this action. Then a vertex $v \in T^0$ is called a surface type vertex if the following conditions hold:

- $\text{Stab}_G(v) = \pi_1(\Sigma)$ for a connected compact surface $\Sigma$ with non-empty boundary;
- For every edge $e \in T_1$ adjacent to $v$, $\text{Stab}_G(e)$ embeds onto a maximal boundary subgroup of $\pi_1(\Sigma)$, and this induces a one-to-one correspondence between the set of edges (in $T^1$) adjacent to $v$ and the set of boundary components of $\Sigma$.

**Definition 5.3 (Hyperbolic Floor).** Let $G$ be a group and $H$ be a subgroup of $G$. Then $G$ is a **hyperbolic floor over $H$**, if $G$ acts minimally on a tree $T$ and the action admits a
Bass-Serre presentation \((T^1, T^0, \{\gamma_e\})\), where the set of vertices of \(T^1\) is partitioned in two subsets, \(V_S\) and \(V_R\), such that:

- each vertex in \(V_S\) is a surface type vertex;
- the tree \(T^1\) is bipartite between \(V_S\) and \(V_R\);
- the subgroup \(H\) of \(G\) is the free product of the stabilizers of vertices in \(V_R\);
- either there exists a retraction \(r : G \to H\) that, for each \(v \in V_S\), sends \(\text{Stab}_G(v)\) to a non abelian image or \(H\) is cyclic and there exists a retraction \(r' : G * \mathbb{Z} \to H * \mathbb{Z}\) that, for each \(v \in V_S\), sends \(\text{Stab}_G(v)\) to a non abelian image.

A hyperbolic tower is a sequence of hyperbolic floors and free products with finite rank free groups.

**Definition 5.4 (Hyperbolic Tower).** A group \(G\) is a hyperbolic tower over a subgroup \(H\) if there exists a sequence \(G = G^m > G^{m-1} > \ldots > G^0 = H\) such that for each \(i, 0 \leq i < m\), one of the following holds:

- (i) the group \(G^{i+1}\) has the structure of a hyperbolic floor over \(G^i\), in which \(H\) is contained in one of the vertex groups that generate \(G_i\) in the floor decomposition of \(G^{i+1}\) over \(G^i\);
- (ii) the group \(G^{i+1}\) is a free product of \(G^i\) with a finite rank free group.

Hyperbolic towers over a given subgroup amalgamate well in the following sense.

**Lemma 5.5.** Suppose \(H_1, H_2\) are hyperbolic towers over \(A\). Then the amalgamated free product \(H_1 *_A H_2\) is a hyperbolic tower over \(H_i\) for each \(i \leq 2\).

Hyperbolic towers describe finitely generated elementary free groups (or even more generally finitely generated elementary torsion free hyperbolic groups) as well as elementary embeddings amongst torsion free hyperbolic groups.

**Theorem 5.6.** A finitely generated group \(G\) is elementarily equivalent to a nonabelian free group if and only if \(G\) is a nonabelian hyperbolic tower over \(\{1\}\).

**Theorem 5.7** (Kharlampovich-Myasnikov, Sela). Let \(G\) be torsion-free hyperbolic. Let \(G\) be a hyperbolic tower over a nonabelian subgroup \(H\). Then \(H\) is an elementary subgroup of \(G\).

**Theorem 5.8** (Perin [10]). Let \(G\) be a torsion-free hyperbolic group. Let \(H\) be an elementary subgroup of \(G\). Then \(H\) is a hyperbolic tower over \(G\).
Convention: From now on we restrict our attention to finitely generated elementary free groups. A hyperbolic tower, unless otherwise specified, will always mean a hyperbolic tower over $\{1\}$.

We work towards proving.

**Theorem 5.9.** The class $\mathcal{F}_e$ of finitely generated models of the theory of the free group is an elementary Fraïssé class.

**Definition 5.10.** Let $H$ be a hyperbolic tower. A subgroup $T$ of $H$ is a hyperbolic subtower if $H$ is a hyperbolic tower over $T$.

We remark that a free factor of a hyperbolic tower is a subtower because:

**Theorem 5.11** (Kharlampovich-Myasnikov, Sela). Let $G := G_1 * G_2 * \ldots * G_n$ be a model of the theory of the free group. Then for each $i$, either $G_i$ is a model of the theory of the free group or it is infinite cyclic.

We define a partial order on the class of hyperbolic subtowers of a given tower $H$: if $T_1, T_2$ are hyperbolic subtowers of $H$ we say that $T_1$ is smaller than $T_2$ if $T_2$ is a hyperbolic tower over $T_1$. It is not hard to see, using the descending chain condition for limit groups, that this partial order admits minimal elements. We prove that minimal hyperbolic subtowers (relative to any set of parameters) are unique up to isomorphism.

**Proposition 5.12.** Let $H$ be a tower and $\bar{a}, \bar{b}$ be tuples from $H$. Suppose $H_1$ (respectively $H_2$) is a minimal hyperbolic subtower of $H$ that contains $\bar{a}$ (respectively $\bar{b}$). Let $tp^{H_1}(\bar{a}) = tp^{H_2}(\bar{b})$. Then there exists an isomorphism $h : H_1 \to H_2$ taking $\bar{a}$ to $\bar{b}$.

In order to prove Proposition 5.12 we use the following results.

**Lemma 5.13** ([11]). Let $H_1, H_2$ be torsion-free hyperbolic groups. Suppose $H_1$ is freely indecomposable with respect to a nontrivial subgroup $\langle \bar{a} \rangle$ and $tp^{H_1}(\bar{a}) = tp^{H_2}(\bar{b})$ for some $\bar{b} \in H_2$. Then either there exists an embedding $h : H_1 \to H_2$ that sends $\bar{a}$ to $\bar{b}$ or $H_1$ admits the structure of a hyperbolic floor over a subgroup that contains $\langle \bar{a} \rangle$.

**Lemma 5.14** (relative co-Hopf property). Let $H$ be a torsion-free hyperbolic group freely indecomposable with respect to a subgroup $A$. Then any injective endomorphism of $H$ that fixes $A$ is surjective.

**Proof of Proposition 5.12.** We may assume that $\bar{a}$ is non trivial, as the minimal subtower that contains the trivial element is the trivial subgroup.

Since $H_1$ is a minimal subtower that contains $\bar{a}$ it must be freely indecomposable with respect to $\langle \bar{a} \rangle$. Therefore, Lemma 5.13 applies. Since $H_1$ is minimal (relative to $\bar{a}$) it cannot have a hyperbolic floor structure over a subgroup that contains $\bar{a}$, hence there is an injective morphism $h_1 : H_1 \to H_2$ sending $\bar{a}$ to $\bar{b}$. Similarly, using the minimality of $H_2$, we can find an injective morphism $h_2 : H_2 \to H_1$ sending $\bar{b}$ to $\bar{a}$. Their composition $h_2 \circ h_1$ is an injective endomorphism of $H_1$ that fixes $\bar{a}$, thus, by Lemma 5.14 it is surjective. In particular, the injective morphism $h_1$ is also surjective.

We can now prove that the class $\mathcal{F}_e$ forms an $e$-Fraïssé class.

The $e$-HP condition is immediate: a finitely generated elementary subgroup of an elementary free group is an elementary free group itself.

The $e$-JEP condition follows rather easily from the characterization of finitely generated models of the theory of the free group, together with Theorems 5.7 and 5.11.
Lemma 5.15. Suppose $H_1, H_2$ are finitely generated models of the theory of the free group. Then $H_1 \ast H_2$ is a finitely generated model of the theory of the free group. Moreover, each $H_i$ for $i \leq 2$ is an elementary subgroup of $H_1 \ast H_2$.

We finally prove the strong $e$-AP condition for the class of finitely generated elementary free groups.

Lemma 5.16. Let $H_0, H_1, H_2 \in \mathcal{F}_e$. Let $\bar{a}$ be a tuple from $H_0$ and $f_i : \bar{a} \to H_i$ for $i \leq 2$ be partial elementary embeddings. Then there exist a group $D \in \mathcal{F}_e$ and elementary embeddings $g_i : H_i \to D$ for $i \leq 2$, such that $g_1 \circ f_1(\bar{a}) = g_2 \circ f_2(\bar{b})$.

Proof. We may assume that $\bar{a}$ is nontrivial. The hypotheses imply that there are $\bar{b} \in H_1$ and $\bar{c} \in H_2$ such that $tp^{H_1}(\bar{b}) = tp^{H_2}(\bar{c}) = tp^{H_0}(\bar{a})$. We take cases according to whether the smallest subtower $T_1$ of $H_1$ that contains $\bar{a}$ is abelian or not.

Suppose $T_1$ is abelian, hence cyclic. We prove that $T_2$ the smallest subtower that contains $\bar{c}$ is cyclic as well. Suppose not, then Theorem 5.7 implies $tp^{T_1}(\bar{b}) = tp^{T_2}(\bar{c})$ and since $T_2$ is minimal it must embedd into $H_1$, in particular it embedds into the smallest free factor that contains $\bar{b}$, but this factor is cyclic, a contradiction.

Thus, we may assume that both $T_1, T_2$ are nonabelian. Hence, Theorem 5.7 implies $tp^{T_1}(\bar{b}) = tp^{T_2}(\bar{c})$. By Proposition 5.12 applied for $H = H_1 \ast H_2$, we obtain an isomorphism $h : T_1 \to T_2$ sending $\bar{b}$ to $\bar{c}$.

Finally, by Lemma 5.5, the amalgamated free product $H_1 \ast T_1 H_2$ (using $f$ for the amalgamation) is a finitely generated elementary free group. Hence is the group $D$ we wanted. □

We thus obtain a countable elementary free group with interesting properties.

Theorem 5.17. There exists a countable group $G_e$ with the following properties:

- the $e$-age of $G_e$ is the class of finitely generated elementary free groups;
- the group $G_e$ is homogeneous;
- the group $G_e$ is the union of an elementary chain of finitely generated elementary groups.

Moreover, any countable group with the above properties is isomorphic to $G_e$.

Finally as a by-product of the already proved theorems in this section we obtain.

Theorem 5.18. The class of finite rank nonabelian free groups is a strong elementary Fraïssé class. In particular $\mathcal{F}_\omega$ is homogeneous.

This last result also follows from the facts that finite rank nonabelian free groups are homogeneous (see [11], [9]) and $\mathcal{F}_\omega$ is the union of their elementary chain.

6. Further remarks and open questions

Lemma 6.1. There are countable elementary free groups which are not limits of $\forall$-chains of finitely generated elementary free groups.

Proof. The result follows by the following observation in [13]. Let $P$ be the set of primes. For any $X \subseteq P$, the following set of formulas $\Gamma_X := \{ \exists y (x = y^p) \mid p \in X \} \cup \{ \forall y (x \neq y^q) \mid q \in P \setminus X \}$ is consistent. Moreover, for any two $X,Y \subseteq P$ with $X \neq Y$ the union $\Gamma_X \cup \Gamma_Y$ is incoherent.

□

Questions:
• Do finitely generated elementary free groups (or finite rank nonabelian free groups) form a (strong) ∀-Fraïssé class?
• Does there exist a countable homogeneous elementary free group whose elementary age is the class of all finitely generated elementary free groups and is not isomorphic to the strong e-Fraïssé limit of this class?
• Do the nonabelian limit groups form a strong ∀-Fraïssé class?
• Are all countable elementary free groups obtained as the union of a chain of finitely generated elementary free groups?

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