Homogeneous linear ion crystal in a hybrid potential

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Received: 9 June 2021 / Accepted: 30 December 2021 / Published online: 22 January 2022
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Abstract
We investigate the properties of a linear ion crystal in a combination of quadratic and quartic potentials. Both the discrete and the continuous models are employed to explore the homogeneity of a linear ion crystal by tuning the proportional parameter between the quadratic and quartic components. It is found that the size of the uniform region is significantly larger than that in either a purely quadratic or a quartic potential. The zigzag transition is also investigated in the hybrid potential, and its critical condition and phase diagram are determined numerically, which agrees well with previous theoretical and experimental results. Our results pave the way for experimental investigation of superlarge linear ion crystals in the combination of quadratic and quartic potentials.

Keywords Homogeneous ion crystal · Hybrid potential

Ming-shen Li, Xin-Xin Rao and Zhao Wang contribute equally to this work.

This work is supported by the Key-Area Research and Development Program of Guangdong Province under Grant No. 2019B030330001, the National Natural Science Foundation of China under Grant No. 11774436, No. 11974434, No. 11904423, No. 12074439. Le Luo received supports from Guangdong Province Youth Talent Program under Grant No. 2017GC010656, Sun Yat-sen University Core Technology Development Fund. Yang Liu acknowledges the support from the Natural Science Foundation of Guangdong Province under Grant 2020A1515011159.

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1 Introduction

Quantum computers have the potentials to tackle many problems hard to be solved or even unsolvable using classical computers [1,2]. Owing to the long-lived internal qubit coherence, the ion trap becomes one of the most attractive candidates for the realization of a quantum computer. Today’s prototype quantum computers containing tens of qubits have been demonstrated in several systems including ion traps. To realize the promise of quantum computing, intensive efforts had been paid to construct quantum computer with a large scale of ion qubits [3–5]. Different types of ion traps had been investigated, such as the linear ion trap designed to trap a long ion chain with even space [6,7], the ion trap on chips [8] to facilitate ion transportation in different areas, and the collection of small-scale ion traps integrated by optical methods [9].

A linear chain of trapped ions in a harmonic potential has been demonstrated as a quantum computer of tens of qubits [6,10–12]. However, in the harmonic potential it is difficult to address single ion due to the unequal spacing. Moreover, the structure of the trapped ions is determined by the details of the confining potential and the number of ions [13]. The transition from a linear chain to a zigzag structure will occur if the radial potential reaches a critical value, dependent on the number of ions [14]. This will make the operation on quantum gate much more difficult. Therefore, it is requisite to realize the uniform distribution of a long ion chain for a large-scale quantum information system. The uniform ion chain is critical for the single- and multiple-qubit processing and gate operation. It would also provide an ideal test bed for the study of many-body effects such as topological phase transition [15]. In addition, under the same radial confinement, the evenly spaced linear ion chain can accommodate more ions than in the harmonic potential before the zigzag transition occurs [16].

To achieve a uniform distribution of a long chain, ion distributions in anharmonic potentials have been studied [11]. The results showed that the distribution of the ions in an anharmonic potential is more uniform at the center than that in a harmonic potential, but it changes quickly at both edges. Additionally, the combination of the harmonic and anharmonic potentials was proposed as the trapping potential [16], in order to scale up the qubits and to operate the entangling gates more efficiently. Recently, using anharmonic axial potentials, 100 ions in a linear configuration have been realized and up to 44 ions have nearly equidistant spacings [17]. There is an exciting prospect that system with even larger uniform distribution appears in the near future.

In this paper, we present a homogeneous crystal with a large number of ions trapped in a hybrid potential, consisting of harmonic and anharmonic potentials with a dimensionless ratio. First, we use the discrete model to calculate the requisites for making a homogeneous ion chain by using the hybrid potential. Then, we test our calculations using a universal variance criterion. To compare with the discrete model, we further apply variational method to the hybrid potential by using the continuous model. Almost identical conditions are obtained, which confirms the accuracy of our calculations and the demanding of a well-designed hybrid potential in order for a large homogeneous ion chain. At the end, we discuss the zigzag transition in the hybrid potential.
2 Discrete model

The discrete model has been applied to the linear ion chain in a purely quadratic (i.e., harmonic) or quartic (i.e., anharmonic) potential [2,10,11]. The potential energy of the system in a quadratic axial potential can be described as [2,10]

\[ e = \sum_{i=1}^{N} \frac{1}{2} m \omega^2 x_i^2 + \sum_{i,j=1, i<j}^{N} \frac{q^2}{|x_i - x_j|} \]  

(1)

where \( m \) and \( q \) are the mass and charge of an ion, \( \omega \) is the axial trap frequency, \( x_i \) is the axial position of each ion, and \( N \) is the total number of ions. When replacing \( m \omega^2 x_i^2 / 2 \) with \( ax_i^4 / 4 \), the confinement becomes a purely quartic potential, in which \( a \) is the coefficient.

Here, we first use the discrete model to calculate the ground-state distribution of the ion chain in the hybrid potential. The energy of the system can be written in the following form: [16]:

\[ e = \sum_{i=1}^{N} \frac{\alpha_4}{4} x_i^4 - \sum_{i=1}^{N} \frac{\alpha_2}{2} x_i^2 + \sum_{i,j=1, i<j}^{N} \frac{q^2}{|x_i - x_j|} \]  

(2)

This equation can be rescaled by \( z_i = x_i / l \) with \( x_i \) with a length scale \( l^5 = q^2 / \alpha_4 \). Through a dimensionless processing, Eq. (2) can be rewritten as

\[ E = \sum_{i=1}^{N} \frac{1}{4} z_i^4 - \sum_{i=1}^{N} \frac{D}{2} z_i^2 + \sum_{i,j=1, i<j}^{N} \frac{1}{|z_i - z_j|} \]  

(3)

using a dimensionless ratio \( D = |\alpha_2/q^2|^{2/5} (\alpha_2/\alpha_4)^{3/5} \), which is introduced to characterize the strength of the quadratic potential relative to the quartic potential.

The ground-state positions of the ions can be determined by the following conditions:

\[ \frac{\partial E}{\partial z_i} = 0 \]  

(4)

By solving totally \( N \) nonlinear differential equations, we obtain the equilibrium position of each ion. The positions of 20 ions in different potentials are compared in Fig. 1. The ions in the hybrid potential are much more evenly distributed across the chain compared to the quadratic or quartic potential.

By defining the ion density as follows:

\[ n(\tilde{z}_i) = \frac{1}{|\tilde{z}_{i+1} - \tilde{z}_i|} \]  

(5)
Fig. 1 Positions of 20 ions in (a) quadratic, (b) quartic and (c) the hybrid potentials. Notice that the ions in the hybrid potential are more uniformly distributed. In these plots, the position $z$ has been rescaled to be dimensionless.

Fig. 2 The ground-state ion density $n$ in hybrid potentials for different $D$ in discrete model, when the total number of ions is $N = 1000$. In the middle part of the ion chain (i.e., around $z = 0$), the density decreases with the increase in $D$. a represents the midpoint of the ion chain, b represents the point where the density reaches the maximum, and c represents the point where the density is the same as point A. Both $n$ and $z$ are rescaled to be dimensionless.

where $z_i$ is the coordinate of the $i$th ion, $\tilde{z}_i$ is the midpoint between $z_i$ and $z_{i+1}$, and the resultant ion density distribution is shown in Fig. 2 for different $D$ parameters when $N = 1000$. It can be seen from Fig. 2 that the density in the middle part of the ion chain decreased with the increase in the $D$ parameters. This is because when the parameter $D$ increases, the extreme point of the hybrid potential deviates farther from the center, resulting in an increase in the ion spacing at the center.

In order to find an appropriate combination of the quadratic and quartic potentials to maximize the uniform distribution of the ions, the variance $s(z) = \frac{1}{N} \sum_{i=1}^{N} (\Delta z_i - \bar{\Delta z})^2$ of $\Delta z$ in $[0, z_0]$ is introduced. Figure 3a shows the variance as a function of the position $z_0$. The optimal distribution is set by the criterion of $s(z) \leq \eta$. The optimization of $D$ parameter is shown for $\eta = 6 \times 10^{-7}$ and $N = 1000$. We define the optimal number of ions that satisfy the optimization criterion to be $N'$. As shown in Fig. 3b, the optimal number of ions fluctuates little for $10 < D < 18$ since these curves intersect with the threshold in a small range. The optimal ion numbers of $N'$ have a maximum of 888 when $D \in [11.5, 15.5]$. Under the same conditions, $N'$ is 396 or 844 in the quadratic or the quartic potential, respectively, from our calculation, which is significantly lower than the optimal ion number in the hybrid potential.
3 Continuous model

Although the discrete model is effective in obtaining requisites for a homogeneous distribution of an ion chain, it is very time-consuming, especially when the number of ions is hundreds or thousands. Thus, in order to solve the properties of the ion chain more efficiently and also to confirm our calculations with discrete model, we implement continuous model based on Dubins’ variational method [10], which revolves around the assumption of the local density approximation [18], where the distribution of each ion in an ion chain is approximated by a continuous density function \( n(z) \). Time consumption using this method does not relate to the scale of the total ion number \( N \), thus making the calculation much faster and more flexible. The variational method was extended to an quartic potential recently [11], and the potential energy equation of the system under a hybrid potential can be expressed as
Fig. 4 The ground-state density functions calculated by the variational method when \( N = 1000 \). The shapes of the curves are identical to the corresponding ones calculated by the discrete model in Fig. 2. Both \( n \) and \( z \) are rescaled to be dimensionless

\[
E[n] = \int_{-\infty}^{\infty} dz \left\{ \frac{1}{4} z^4 n(z) - \frac{D}{2} z^2 n(z) + \gamma n(z)^2 
- \frac{1}{2} n(z) \int_{0}^{\infty} dy \ln[yn(z)] \frac{d}{dy} [n(z - y) + n(z + y)] \right\}
\]

The first term in the integral is the energy due to the quartic trapped potential, while the second term comes from the quadratic potential. The other term is the Coulomb energy of the ion chain and \( \gamma \approx 0.57721 \) is Euler’s constant.

The principle of minimum energy is used to minimize Eq. (6), and the density function \( n(z) \) can be obtained. According to the local density approximation, the density function of the ions in a hybrid potential can be expressed as \( n(z) = Az^4 + Bz^2 + C \) \cite{10,11}. By substituting the boundary condition and the normalization condition, it becomes

\[
n(z) = -\frac{5(-2C + 6CL - 3N)}{2L^4(-5 + 3L)} z^4 + \frac{3(8CL - 5N)}{2L^2(-5 + 3L)} z^2 + C,
\]

which is shown in Fig. 4. It is worth to notice that both \( L \) and \( C \) are functions of \( N \) and \( D \), since the distribution of the ions is determined by \( N \) and \( D \) according to the principle of the minimum energy. It is found that these parameters have extremely complicated representations with \( N \) and \( D \) as variables. However, we find simple expressions for these two parameters, by using numerical methods, which is, gradient search algorithm, to find best \( C \) and \( L \) such that the potential energy of the system \cite{11} reaches its minimum. When \( N \) is in the order of hundreds or thousands, the \( C \) and \( L \) can be formulated as

\[
C = 0.26N^{0.80} - 0.36D^{1.23} + 3.80
\]

\[
L = 0.70N^{0.30} + 0.04D^{1.13} + 2.16
\]

Consequently, the density function is determined, as shown in Fig. 4.
Using global density function (7), it is convenient to analyze the whole distribution uniformity to find the number of the evenly distributed ions. In order to demonstrate the characteristic of the general distribution, we define the variance criterion as

\[ s(z_0) = \frac{1}{z_0} \int_0^{z_0} [\Delta(z) - \frac{1}{z_0} \int_0^{z_0} \Delta(z) dz]^2 dz \]  

(10)

where \( \Delta(z) = \frac{1}{n(z)} \) represents the distance between the adjacent ions. The variance criterion is a function of position \( z_0 \), as shown in Fig. 5a.

With the continuous model, we have \( N(z_0) = \int_0^{z_0} n(z) dz \), where \( N(z_0) \) represents the particle number in the range of \([0, z_0]\). Once we determine the appropriate \( z_0 \), we get the number of particles. The variance criterion of the density function can have different requirements restricted by the design of the experiment. Figure 5b shows the number of ions with homogeneous distribution that depends heavily on \( D \) parameter, the same as the trend shown in Fig. 3b. If the variance criterion \( s \leq 6 \times 10^{-7} \), we have \( N' = 872 \) when \( D \in [10, 12] \), while \( N' = 796 \) for quartic potential and 380 for quadratic potential from our calculation. We also notice that there are small differences between critical \( D \) value between continuous model and discrete model,
which is also shown in [11] and is due to the differences introduced by the integration by the variational method.

### 4 Zigzag phase transition

During the discussion above, we focus on the axial potential and its effect on the axial distribution of the trapped ions, while the effect of radial potential is ignored since the radial motion is frozen out by a tight radial potential which is much stronger than the axial potential. However, when the radial confinement is not that strong, it is energetically favorable for the ions to be displaced in the radial direction, and the zigzag transition occurs [15,19,20], which is proven to be a second-order phase transition. With the results of the continuous model, the potential energy of the system is described by

\[
e = \sum_{i=1}^{N} \left( \frac{\beta^2}{2} x_i^2 + \frac{1}{4} z_i^4 - \frac{D}{2} z_i^2 \right) + \sum_{i,j=1; i<j}^{N} \frac{1}{| (x_i - x_j)^2 + (z_i - z_j)^2 |^{\frac{3}{2}}}
\]

(11)

where we ignore the radial y-potential by assuming that the zigzag transition occurs in the xz plane and the y coordinates of the ions are the same. \( \beta \) represents the strength of the radial x-potential relative to axial z-potential. The anisotropy of the trap can be approximated by the \( \alpha(z \to 0) = \frac{\alpha_x^2(z \to 0)}{\alpha_z^2} \approx \frac{\beta^2}{D} \), because it is always the case the phase transition first occurs at the center of the ion chain (\( z = 0 \)).

In order to better analyze the zigzag transition, we use the Lagrangian [2,21,22], i.e., the full Hamiltonian to describe the motion of the ions:

\[
L = \frac{k}{2} \sum_{m=1}^{N} (\dot{q}_m)^2 - \frac{1}{2} \sum_{m,n=1}^{N} q_m q_n \left[ \frac{\partial^2 E}{\partial x_m \partial x_n} \right]_0
\]

(12)

where \( A_{mn} = \frac{\partial^2 E}{\partial x_m \partial x_n} \bigg|_{groundstate} \) is the matrix element of the Hessian matrix [11]. By using Eq. (11), \( A_{mn} \) is further expressed by

\[
\]
Fig. 6 The impact of different $D$ parameters on $k$ and $\lambda$. When $D = 0$, the potential corresponds to a quartic potential with $k = 0.47$ and $\lambda = 1.15$, which is in agreement with Ref. [11]

$$A_{mn} = \frac{\partial}{\partial x_m} \left( \sum_{n=1}^{N} \beta^2 x_n + \sum_{p=1; p \neq n}^{N} \frac{-(x_n-x_p)}{|(x_n-x_p)^2+(z_n-z_p)^2|} \right)_{\text{groundstate}}$$  \hspace{1cm} (13)

$$= \begin{cases} 
(\beta^2 - \sum_{p \neq n} \frac{1}{|z_p - z_n|^3}), & m = n, \\
\frac{1}{|z_m - z_n|^3}, & m \neq n
\end{cases} \hspace{1cm} (14)$$

which defines the normal mode of oscillation of the ions along axial $z$ direction. The eigenvectors $b_{m}^{(p)}$ are defined by $\sum_{m=1}^{N} A_{mn} b_{m}^{(p)} = \mu_{p} b_{n}^{(p)}$, where eigenvalue $\mu_{p} \geq 0$ and $p$ is the mode index, which can take the value of 1, \cdots, $N$. The eigenvectors are normalized and enumerated in order of increasing eigenvalue.

In order to find the critical value $\beta_c$ at which the zigzag transition occurs, we search when the minimum eigenvalue $A = 0$ as $\beta$ decreases. We obtain the critical point $\beta_c$ for the phase transition as a function of given $N$. Then, we use a well-known simple power form relationship [14,16,19,21,23,24]:

$$\beta_c = k N^{\lambda} \hspace{1cm} (15)$$

for fitting, since this relationship exists in both purely quadratic and purely quartic potential. During the calculation, the location of each particle $z_i$ is determined from the continuous model by using the reciprocal function of the integration of the density function. In Fig. 6, we demonstrate the impact of $D$ parameters on $k$ and $\lambda$.

Since a huge system with hundreds of ions would be impossible to fully simulate, approximations are made that the ions are at zero temperature and their thermal fluctuations as well as quantum fluctuations which can induce tunneling between the double-well minima are ignored. Then, we can map out the phase diagram for zigzag phase transition with relations in Eq. (15), which is shown in Fig. 7.

In Fig. 7a, the dependences of critical anisotropy $\alpha_{cr}$ on the $D$ and $N$ parameter are shown. Clearly, $\alpha_{cr}$ is much larger at large $N$ and small $D$. While it is much smaller at small $N$ and large $D$. From this phase diagram, the phase transition from linear chain to zigzag configuration can be precisely controlled by parameter $D$ for a fixed number of ions $N$. Figure 7b shows $\alpha_{cr}$ as a function of $N$ when $D = 10$ for the hybrid...
Fig. 7  a The phase diagram showing the dependence of critical anisotropy $\alpha_{cr}$ on both $D$ and $N$ parameter. This critical anisotropy $\alpha_{cr}$ determines when the phase transition from linear chain to zigzag phase occurs. 

b When $D = 10$, the critical anisotropy $\alpha_{cr}$ as a function of $N$ for the hybrid potential (solid black line), pure quadratic potential (solid red line), [14,19,21,23] and pure quartic potential (solid blue line), [11] as well as the function (solid green line) obtained from experimental data. [21] c When $N = 800$, the critical anisotropy $\alpha_{cr}$ as a function of $D$ for the hybrid potential potential (solid black line), pure quadratic potential (solid red line) [14,19,21,23], and pure quartic potential (solid blue line) [11]. The function obtained from experimental data [21] as well as its extension for larger $N$ with the same $k$ and $\lambda$ is also shown in solid green line. This comparison shows that the calculation with quartic potential deviate more and more from experimental results for increasing $N$, and our simulation results get very close to the extension of the experimental results for large number of
ions $N > 200$ although it has some deviation from experimental results for small $N$. Figure 7c shows $\alpha_{cr}$ as a function of $D$ when $N = 800$ for the hybrid potential. As it is shown, $\alpha_{cr}$ increases with $N$ in a power form, more importantly, $\alpha_{cr}$ in our cases lies between that of pure quadratic and quartic potential, and reach closely to the value inferred from experimental data [21] for large $N$. Such a good agreements verifies the accuracy and the universality of our proposal.

5 Conclusion

We have studied the properties of the one-dimensional ion crystal in a hybrid potential. Both the density distribution and the optimal parameter $D$ are attained by both the discrete model and the continuous model. Large homogeneous ion chain with ion number more than 800 has been achieved, which has never been realized before. In principle, the number of ions can be increased to tens of thousands, even millions in such hybrid potential, if the trapping space allowed. Based on the result from continuous model, we calculate the critical condition for the zigzag transition. When the zigzag transition occurs, the $\beta_c$ can be controlled by adjusting the parameter $D$, as Fig. 6 illustrates.

Such potential can accommodate more ions uniformly distributed than both quadratic potential and quartic potential, avoiding zigzag transitions at the center of the chain. Additionally, it can prevent the ions in the center coming too close, thus reducing cross talk in ion state detection and enabling single ion manipulation with focused laser beams. Particularly, it offers unique advantages in performing coherent operations on large ion chain [16,25]. It can also be used to investigate the effects of the quartic term in the hybrid potential on the normal mode structure [6,26]. As the spin–spin interaction approximately scales to the second power of the distance, a homogeneous ion chain can be used for quantum simulation of Ising models [27].

Moreover, the large ratio $\alpha_{cr}$ of the radial potential and the axial potential provided by the homogeneous ion chain, given the required hybrid potential, would enable significant improvement on the gate fidelity, since the thermal ion motion can be reduced by a factor ranging from $\alpha_{cr}^2$ to $\alpha_{cr}^3$ [28]. Our proposal is also a promising route for scaling up toward a large-scale quantum computer in a single harmonic trap and could potentially open up a new avenue for the study of large-scale quantum information processing.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Kaye, P., Laflamme, R., Mosca, M., et al.: An Introduction to Quantum Computing. Oxford University Press, Oxford (2007)
2. James, D.F.V.: Quantum computation and quantum information theory. Appl. Phys. B 66, 181–190 (1998)
3. Cirac, J.I., Zoller, P.: Quantum computations with cold trapped ions. Phys. Rev. Lett. 74(20), 4091 (1995)
4. Sørensen, Anders, Mølmer, Klaus: Quantum computation with ions in thermal motion. Phys. Rev. Lett. 82(9), 1971 (1999)
5. Milburn, G.J., Schneider, S., James, D.F.V.: Ion trap quantum computing with warm ions. Fortschritte der Physik: Progress of Physics 48(9–11), 801–810 (2000)
6. Johannsen, Michael: Isospaced linear ion strings. Appl. Phys. B 122(4), 71 (2016)
7. Xie, Yi., Zhang, Xinfang, Baoquan, Ou., Chen, Ting, Zhang, Jie, Chunwang, Wu., Wei, Wu., Chen, Pingxing: Creating equally spaced linear ion string in a surface-electrode trap by feedback control. Phys. Rev. A 95(3), 032341 (2017)
8. Sørensen, Anders, Mølmer, Klaus: Quantum computation with ions in thermal motion. Phys. Rev. Lett. 82(9), 1971 (1999)
9. Milburn, G.J., Schneider, S., James, D.F.V.: Ion trap quantum computing with warm ions. Fortschritte der Physik: Progress of Physics 48(9–11), 801–810 (2000)
10. Dubin, Daniel H.E.: Minimum energy state of the one-dimensional coulomb chain. Phys. Rev. E 55(4), 4017 (1997)
11. Bastin, S.R., Lee, T.E.: Ion crystals in anharmonic traps. J. Appl. Phys. 121(1), 014312 (2017)
12. Ryutaro O., Shota K., Kyoichi T., Silpa M., Hiroki T., and Kenji T. Confinement and retrieval of local phonons in a trapped-ion chain. arXiv preprint arXiv:2003.03962 (2020)
13. Dubin, Daniel H.E., Oneil, T.M.: Trapped nonneutral plasmas, liquids, and crystals (the thermal equilibrium states). Rev. Mod. Phys. 71(1), 87 (1999)
14. Schiffer, J.P.: Phase transitions in anisotropically confined ionic crystals. Phys. Rev. Lett. 70(6), 818 (1993)
15. Pedregosa-Gutierrez, J., Mukherjee, M.: Defect generation and dynamics during quenching in finite size homogeneous ion chains. New J. Phys. 22(7), 073044 (2020)
16. Lin, G.-D., Zhu, S.-L., Islam, R., Kim, K., Chang, M.-S., Korenblit, S., Monroe, C., Duan, L.-M.: Large-scale quantum computation in an anharmonic linear ion trap. Europhys. Lett. 86(6), 6004 (2009)
17. Pagano, G., Hess, P.W., Kaplan, H.B., Tan, W.L., Richerme, P., Becker, P., Kyprianidis, A., Zhang, J., Birckelbaw, E., Hernandez, M.R., et al.: Cryogenic trapped-ion system for large scale quantum simulation. Quantum Sci. Technol. 4(1), 014004 (2018)
18. Hohenberg, Pierre, Kohn, Walter: Inhomogeneous electron gas. Phys. Rev 136(3B), B864 (1964)
19. Dubin, Daniel H.E.: Theory of structural phase transitions in a trapped coulomb crystal. Phys. Rev. Lett. 71(17), 2753 (1993)
20. Nigmatullin, R., Campo, A., Chiara, G.D., Morigi, G., Plenio, M.B., Retzker, A.: Formation of helical ion chains. Phys. Rev. B 93(1), 014106 (2016)
21. Enzer, D.G., Schauer, M.M., Gomez, J.J., Gulley, M.S., Holzschmeier, M.H., Kwiat, P.G., Lamoreaux, S.K., Peterson, C.G., Sandberg, V.D., Tupa, D., et al.: Observation of power-law scaling for transitions in linear trapped ion crystals. Phys. Rev. Lett. 85(12), 2466 (2000)
22. Kielpinski, D., King, B.E., Myatt, C.J., Sackett, C.A., Turchette, Q.A., Itano, W.M., Monroe, C., Wineland, D.J., Zurek, W.H.: Sympathetic cooling of trapped ions for quantum logic. Phys. Rev. A 61(3), 032310 (2000)
23. Hughes, Richard J., James, Daniel FV., Gomez, J.J., Gulley, M.S., Holzschmeier, M.H., Kwiat, Paul G., Lamoreaux, S.K., Peterson, C.G., Sandberg, V.D., Schauer, M.M., et al.: The los alamos trapped ion quantum computer experiment. Fortschritte der Physik: Progress of Phys 46(4–5), 329–361 (1998)
24. Morigi, Giovanna, Fishman, Shmuel: Dynamics of an ion chain in a harmonic potential. Phys. Rev. E 70(6), 066141 (2004)
25. Lin, Guin-Dar., Duan, L.-M.: Sympathetic cooling in a large ion crystal. Quantum Inf. Process. 15(12), 5299–5313 (2016)
26. Home, Jonathan P., Hannke, David, Jost, John D., Leibfried, Dietrich, Wineland, David J.: Normal modes of trapped ions in the presence of anharmonic trap potentials. New J. Phys. 13(7), 073026 (2011)
27. Porras, D., Cirac, J.I.: Effective quantum spin systems with trapped ions. Phys. Rev. Lett. 92(20), 207901 (2004)
28. Zhu, S.-L., Monroe, C., Duan, L.-M.: Trapped ion quantum computation with transverse phonon modes. Phys. Rev. Lett. 97(5), 050505 (2006)
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