Permutation interpretation of quantum mechanics

V V Kornyak
Laboratory of Information Technologies, Joint Institute for Nuclear Research
141980 Dubna, Russia
E-mail: kornyak@jinr.ru

Abstract. We analyse quantum concepts in a constructive finite background. Introduction of continuum or other actual infinities into physics leads to non-constructivity without any need for them in description of empirical observations. We argue that quantum behavior is a natural consequence of symmetries of dynamical systems. It is a result of fundamental impossibility to trace identity of indistinguishable objects in their evolution — only information about invariant combinations of such objects is available. General mathematical arguments imply that any quantum dynamics can be reduced to a sequence of permutations. Quantum phenomena, such as interferences, arise in invariant subspaces of permutation representations of the symmetry group of a system. Observable quantities can be expressed in terms of the permutation invariants. We demonstrate that for description of quantum phenomena there is no need to use such non-constructive number system as complex numbers. It is sufficient to employ the cyclotomic numbers — a minimal extension of the natural numbers which is suitable for quantum mechanics.

1. Introduction
A remarkable feature of quantum mechanics is its universality. It is applicable to systems of quite different physical nature and scales: from subatomic particles up to large molecules\(^1\). Such a universality is inherent to theories based on some \textit{a priori} mathematical principles. In the case of quantum mechanics, the leading mathematical principle is symmetry. The quantum behavior is demonstrated only by systems that contain indistinguishable particles: any deviation from the exact identity of particles destroys quantum interferences. The indistinguishability of elements of a system implies that they belong to the same orbit of the symmetry group of the system. For systems with symmetries only \textit{invariant} — i.e., independent of “relabeling” of “homogeneous” elements — relations and statements are objective. For example, no objective meaning can be attached to electric potentials \(\varphi\) and \(\psi\) or to points of space, denoted as vectors \(\mathbf{a}\) and \(\mathbf{b}\). But the combinations denoted as \(\psi - \varphi\) or \(\mathbf{b} - \mathbf{a}\) (in more general group notation \(\varphi^{-1}\psi\) and \(\mathbf{a}^{-1}\mathbf{b}\)) are meaningful.

Quantum mechanical formalism is based on unitary operators in a Hilbert space. These operators belong to the general unitary group, in more rigorous words, to the \textit{unitary representation} of the automorphism group of the Hilbert space, i.e., of the group that preserves the Hermitian inner product in the space. To make quantum concepts constructive we can replace general unitary groups by unitary representations of \textit{finite groups} without any risk to destroy the physical content of the problem. The metaphysical difference between “finite” and

\(^1\) For example, in [1] observations of the quantum interference between molecules of the fullerene \(C_{60}\) are reported.
“infinite” can not have any empirically observable consequences. Moreover, there are strong experimental evidences that finite groups of relatively small orders underlie some fundamental physical processes. The origin of these groups is unclear in the context of currently accepted theories like the Standard Model. A brief review of finite symmetries in the flavor physics phenomenology is presented in Appendix A.

Using the fact that any representation of finite group can be embedded into a permutation representation, we show that any quantum mechanical problem can be reduced to permutations, and quantum observables can be expressed in terms of permutation invariants. More detailed examination of the “permutation” approach leads to the conclusion that complex numbers in the quantum mechanical formalism should be replaced by cyclotomic numbers. If we adopt this modification of the formalism, then

- quantum amplitudes acquire a simple and natural meaning: they are projections onto invariant subspaces of vectors of “population numbers” of the elements that are subject to permutations;
- Born’s probabilities appear to be rational numbers — in accordance with the “frequency interpretation” of probability for finite sets;
- quantum phenomena arise as manifestation of fundamental impossibility to trace identity of indistinguishable elements in their evolution — only information about invariant combinations of such elements is available.

2. Classical and quantum evolution
We assume that dynamical systems evolve in the discrete time $T$: $T = \mathbb{Z}$ or $T = [0, 1, \ldots, T]$ for some $T \in \mathbb{N}$.

Classical states of dynamical system form a finite set $\Omega = \{\omega_1, \ldots, \omega_N\}$ of some entities having a symmetry group $G = \{g_1, \ldots, g_M\} \leq \text{Sym} (\Omega)$. Classical evolution or trajectory of the dynamical system is a sequence $\ldots, \omega_{t-1}, \omega_t, \omega_{t+1}, \ldots \in \Omega^T$.

For reasons that will become clear later, quantum evolution can be defined as a sequence of permutations $\ldots, p_{t-1}, p_t, p_{t+1}, \ldots$ where $p_t = \Omega a_t$ is the permutation of the set $\Omega$ by a group element $a_t \in G$. A natural condition here is that at any moment $t$ only information on invariant combinations of elements from $\Omega$ is available, whereas the permutation $p_t$ itself is unobservable.

More precisely, the information about arrangement of elements from $\Omega$ corresponding to $p_t$ does not make sense without a “reference frame” (or “observer”).

In physics the set $\Omega$ usually has a special structure of a set of functions $\Omega = \Sigma^X$ on a space $X = \{x_1, \ldots, x_X\}$ with values in a set $\Sigma = \{\sigma_1, \ldots, \sigma_{|\Sigma|}\}$ of local states.

We assume that both the space $X$ and the local states $\Sigma$ possess nontrivial groups of space $F = \{f_1, \ldots, f_F\} \leq \text{Sym} (X)$ and internal $\Gamma = \{\gamma_1, \ldots, \gamma_{|\Gamma|}\} \leq \text{Sym} (\Sigma)$ symmetries, respectively. We can combine these groups into a unified symmetry group $G$ acting on the whole set of states (1). An explicit construction of the group $G$ combining the space and internal symmetries is described in Appendix B.

An important peculiarity of dynamical systems with space is the possibility to introduce non-trivial gauge connections. The gauge structures lead to observable physical consequences: the curvatures of non-trivial connections describe forces in physical theories. Another important topic involving the space structure is the spin/statistics relation. On the other hand there are many problems, for example, quantum computing, where any underlying space is inessential.
3. Standard and finite quantum mechanics

As is well known, all approaches to quantization are equivalent to the traditional matrix formulation of quantum mechanics where the evolution of a system from an initial to a final state is described by an evolution matrix $U$: $|\psi_0\rangle \rightarrow |\psi_T\rangle = U|\psi_0\rangle$. The evolution matrix of a quantum dynamical system can be represented as the product of matrices corresponding to elementary time steps: $U = U_{T-T-1} \cdots U_{t-t-1} \cdots U_{t=0}$.

The main ingredients of the standard quantum mechanics are the following:

(i) Quantum description deals with unitary operators $U$ acting in a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathbb{C}$. The elements $|\psi\rangle \in \mathcal{H}$ of the space are called “states”, “state vectors”, “wave functions”, “amplitudes” etc. The operators $U$ belong to the general unitary group $\text{Aut}(\mathcal{H})$ acting in $\mathcal{H}$.

(ii) Quantum mechanical particles are associated with unitary representations in $\mathcal{H}$ of some symmetry groups. The representations are called “singlets”, “doublets”, “triplets” etc. in accordance with their dimensions. The multidimensional representations describe spin.

(iii) Quantum mechanical evolution is an unitary transformation of the initial state vector $|\psi_{\text{in}}\rangle$ into the final $|\psi_{\text{out}}\rangle = U|\psi_{\text{in}}\rangle$. In the continuous time, an elementary step of evolution is described by the Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle,$$

where $H$ is a Hermitian operator called energy operator or Hamiltonian.

(iv) Quantum mechanical experiment (observation, “measurement”) is a comparison of the state $|\psi\rangle$ of a system with the state $|\phi\rangle$ of an apparatus.

(v) In accordance with the Born rule, the probability $P(\phi, \psi)$ to register a particle described by $|\psi\rangle$ by apparatus tuned to $|\phi\rangle$ is equal to $\frac{|\langle \phi | \psi \rangle|^2}{\langle \phi | \phi \rangle \langle \psi | \psi \rangle}$.

(vi) Quantum observables are described by Hermitian operators acting in the Hilbert space $\mathcal{H}$.

Our aim is to reproduce all this in the constructive finite background. Our strategy — in accordance with Occam’s principle — will be not to introduce entities unless we really need them. Keeping these lines we come to the following:

i’. The Hilbert space $\mathcal{H}$ over the field $\mathbb{C}$ should be replaced by a $K$-dimensional Hilbert space $\mathcal{H}_K$ over an abelian number field $\mathcal{F}$ — an extension of the rationals $\mathbb{Q}$ with an abelian Galois group $[2]$. The unitary operators $U$ belong now to an unitary representation $U$ of a finite group $G = \{g_1, \ldots, g_M\}$ in the space $\mathcal{H}_K$. The field $\mathcal{F}$ is determined by the structure of the group $G$ and its representation $U$.

ii’. The notion of quantum particle remains the same as in the standard quantum mechanics.

iii’. It is clear that now we have only finite number of possible evolutions:

$$U_j \in \{U(g_1), \ldots, U(g_j), \ldots, U(g_M)\}.$$

Obviously we do not need any analog of the Schrödinger equation at all. Though formally one can always introduce Hamiltonians by the formula $H_j = i \ln U_j \equiv \sum_{k=0}^{p-1} \lambda_k U_j^k$, where $p$ is period of $U_j$ (i.e., minimal $p > 0$ such that $U_j^p = 1$), $\lambda_k$’s are some coefficients.$^2$

---

$^2$ These coefficients contain the non-algebraic element $\pi$ which is an infinite sum of elements from $\mathcal{F}$. In other words, the $\lambda_k$’s are elements of a transcendental extension of $\mathcal{F}$ — the logarithmic function is essentially a construction from the continuous mathematics dealing with actual infinities.
iv'. The notion of observation remains without changes.

v'. The formula for Born’s probability remains the same

\[ P(\phi, \psi) = \frac{|\langle \phi | \psi \rangle|^2}{\langle \phi | \phi \rangle \langle \psi | \psi \rangle}. \]  \hspace{1cm} (2)

But some conceptual refinement is needed. In the finite background the only reasonable interpretation of probability is the frequency interpretation: the probability is the ratio of the number of singled out combinations to the total number of combinations under consideration. Dealing only with combinations of elements from finite sets, we expect that if all things are arranged correctly, then formula (2) must give rational numbers. This will be one of our guiding principles.

vi'. Hermitian operators describing observables in quantum formalism can be expressed in terms of the group algebra representation:

\[ A = \sum_{k=1}^{M} \alpha_k U(g_k). \]

To provide hermiticity appropriate conditions should be imposed on the coefficients \( \alpha_k \).

Note that other elements of the quantum theory are obtained in the finite background by a straightforward rewriting. For example, as is well known, the Heisenberg uncertainty principle follows from the Cauchy(-Bunyakovsky-Schwarz) inequality

\[ \langle A\psi | A\psi \rangle \langle B\psi | B\psi \rangle \geq |\langle A\psi | B\psi \rangle|^2. \]  \hspace{1cm} (3)

This is equivalent to the standard property of any probability \( P(A\psi, B\psi) \leq 1. \)

4. Permutations, representations and numbers

All transitive actions of a finite group \( G = \{g_1, \ldots, g_M\} \) on finite sets can easily be described \[3\]. Any such set \( \Omega = \{\omega_1, \ldots, \omega_N\} \) is in one-to-one correspondence with a set of right \( H \setminus G \) cosets of some subgroup \( H \leq G \). The set \( \Omega \) is called a homogeneous space of the group \( G \) (\( G \)-space for short). Action of \( G \) on \( \Omega \) is faithful, if the subgroup \( H \) does not contain normal subgroups of \( G \). We can write the action in the form of permutations

\[ \pi(g) = \begin{pmatrix} \omega_i \\ \omega_ig \end{pmatrix} \sim \begin{pmatrix} Ha \\ Hag \end{pmatrix}, \quad g, a \in G, \quad i = 1, \ldots, N, \]  \hspace{1cm} (4)

or, equivalently, in the form of matrix with entries 0 and 1

\[ \pi(g) \rightarrow P(g) = (P(g)_{ij}), \quad \text{where} \quad P(g)_{ij} = \delta_{\omega_i \omega_j}; \quad i, j = 1, \ldots, N. \]  \hspace{1cm} (5)

Here \( \delta_{\alpha, \beta} \) is the Kronecker delta on \( \Omega \). Mapping (5) is called the permutation representation.

Maximal transitive set \( \Omega \) is the set of all elements of the group \( G \) itself, i.e., the set of cosets of the trivial subgroup \( H = \{1\} \). The corresponding action and matrix representation are called regular. One of the central theorems in the representation theory states that any irreducible representation of a finite group is contained in the regular representation (see Appendix C).

Representation (5) makes sense over any number system with 0 and 1. The most natural number system is the semi-ring of natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots\} \). With this semi-ring we can attach counters to elements of the set \( \Omega \). These counters (natural numbers) can be interpreted
as “multiplicities of occurrences” or “population numbers” of elements \( \omega_i \) in the state of a system involving elements from \( \Omega \). Such state can be represented by the vector with natural components

\[
|n\rangle = \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix}.
\]

Thus, we come to the representation of the group \( G \) in an \( N \)-dimensional module \( H_N \) over the semi-ring \( \mathbb{N} \). Representation (5) when applied to vector (6) simply permutes its components. For further development we can turn the module \( H_N \) into an \( N \)-dimensional Hilbert space \( H_N \) by extending \( \mathbb{N} \) to some field.

The main field in the theory of representations (and hence in the quantum mechanics) is the field of complex numbers \( \mathbb{C} \). The reason for this choice is simple: the field \( \mathbb{C} \) is algebraically closed, so no complications can be expected in solving characteristic equations and, hence, in the whole linear algebra. However, the field \( \mathbb{C} \) is excessively large — most of its elements are non-constructive and, hence, useless for application to empirical reality. So let us consider the problem more carefully.

First of all, we do not need to solve arbitrary characteristic equations: any representation is subrepresentation of some permutation representation, and eigenvalues of any permutation representation are roots of unity. This is clear from the easily calculated characteristic polynomial of permutation matrix (5)

\[
\chi_{\text{P}(g)}(\lambda) = \det (\text{P}(g) - \lambda \text{I}) = (\lambda - 1)^{k_1} (\lambda^2 - 1)^{k_2} \cdots (\lambda^n - 1)^{k_n},
\]

where \( k_i \) is the number of cycles of the length \( i \) in permutation (4). To provide unitarity of representations we use square roots of their dimensions as normalizing coefficients. In fact, the irrationalities of both types (square roots of integers and roots of unity) are elements of the same nature — they are cyclotomic integers, i.e., combinations of roots of unity with integer coefficients that can be made natural by using appropriate (given below) identity for roots of unity. Thus, the basic elements of the number system, we are going to construct, are natural numbers and linear combinations (with natural coefficients) of roots of unity of some degree \( r \) depending on the structure of the group \( G \). This degree is called a conductor. Generally, the conductor is a multiple of the exponent of the group \( G \) — the least common multiple of the orders of its elements.

Any set of combinations of roots of unity and quadratic irrationalities generates an abelian number field \( F \). In particular, the minimal abelian number field containing a given set of irrationalities can be computed with the help of the computer algebra system GAP [4]. The command \text{Field\{gens\}} of this system returns the smallest field that contains all elements from the list of irrationalities \text{gens}. The Kronecker-Weber theorem states that any abelian number field is a subfield of some cyclotomic field.

Let us dwell on the construction of cyclotomic fields. A \( P \)th root of unity is a solution of the cyclotomic equation \( r^P = 1 \). Any root of unity is a power of a primitive root of unity. The primitive \( P \)th root of unity \( r_P \) is a root of unity with period that is equal exactly to \( P \). All primitive \( P \)th roots of unity (and only they) are described by the \( P \)th cyclotomic polynomial \( \Phi_P(r) \) that is an irreducible over \( \mathbb{Q} \) divisor of \( r^P - 1 \). Linear combinations of the roots of unity with integer coefficients form the ring \( \mathbb{N}_P \) of cyclotomic integers. In fact, we can always assume that coefficients at roots of unity in cyclotomic integers are natural, since negative integers can be introduced via the identity \((-1) = \sum_{k=1}^{p-1} r_P^{-k} \), where \( p \) is any divisor of \( P \). The \( P \)th cyclotomic field \( \mathbb{Q}_P \) is the field of fractions of \( \mathbb{N}_P \). The field \( \mathbb{Q}_P \) can be represented as \( \mathbb{Q}_P = \mathbb{Q} [r] / (\Phi_P(r)) \).
Figure 1. Embedding $\mathbb{N}_P$ into $\mathbb{C}$. Arrows $\nearrow$ and $\searrow$ depict primitive and nonprimitive roots, respectively.

From this representation and properties of cyclotomic polynomials it is clear that the field $\mathbb{Q}_P$ is a vector space (in fact, an algebra) of dimension $\varphi(P)$ over the rationals $\mathbb{Q}$ where $\varphi$ denotes the Euler totient function — the number of positive integers $\leq P$ which are coprime to $P$ (1 is assumed to be coprime to all integers). An abelian number field is a subfield $\mathcal{F} \leq \mathbb{Q}_P$ fixed by additional symmetries called Galois automorphisms. A Galois automorphism is a linear map $^*k$ on $\mathbb{Q}_P$ that is defined by the following transformation of the root of unity $r_P \rightarrow r_P^k$, where $1 \leq k < P$ and $k$ is coprime to $P$. The cyclotomics can be embedded into the field $\mathbb{C}$ (Fig. 1 illustrates embedding of cyclotomic integers into the complex plane), but we do not need this possibility. Purely algebraic properties of cyclotomics are sufficient for all manipulations in the Hilbert space $\mathcal{H}_N$ and its subspaces. In particular, the complex conjugate of a cyclotomic number is defined by the following transformation of the roots of unity $r_P^{-m} = r_P^{P-m}$.

All irrationalities are intermediate elements of quantum description that disappear in the final expressions for quantum observables. This is a constructive refinement of the usual relationship between the complex and real numbers in the standard quantum mechanics: the intermediate values may be complex whereas the final observables are to be real.

5. Embedding quantum system into permutations

It follows from the above that any $K$-dimensional representation $U$ can be extended to an $N$-dimensional representation $\tilde{U}$ in a Hilbert space $\mathcal{H}_N$, in such a way that the representation $\tilde{U}$ corresponds to the permutation action of the group $G$ on some $N$-element set of entities $\Omega = \{\omega_1, \ldots, \omega_N\}$. This means that $T^{-1}PT = \tilde{U}$, where $P$ is permutation representation (5) and $T$ is a transformation matrix. It is clear that $N \geq K$. The case when $N$ is strictly greater than $K$ is most interesting. In this case the representation has the following structure

$$T^{-1}PT = \begin{pmatrix} 1 & U \\ V & \end{pmatrix} \equiv 1 \oplus U \oplus V.$$
It is clear that for non-vanishing natural vectors $|n\rangle$ apparatur tuned to quantum interference in the system as a whole. However, the destructive interference respectively. In accordance with the Born rule the probability to fix the system state strictly greater than zero. This means, in particular, that it is impossible to observe destructive interference here $1$ is the trivial one-dimensional representation — an obligatory component of any permutation representation. The component $V$ may be empty.

Clearly, the additional “hidden parameters” — appearing due to increase of the number of dimension of space in the case $N > K$ — in no way can effect on the data relating to the space $\mathcal{H}_K$ since both $\mathcal{H}_K$ and its complement in $\mathcal{H}_N$ are invariant subspaces of the extended space $\mathcal{H}_N$. Thus, any quantum problem in $K$-dimensional Hilbert space can be reformulated in terms of permutations of $N$ things.

Under the trivial assumption that the components of state vectors are arbitrary elements of the underlying field $\mathcal{F}$, we can set arbitrary (e.g., zero) data in the subspace $\mathcal{H}_{N-K}$ complementary to $\mathcal{H}_K$. In this case we come — up to the physically inessential difference between “finite” and “infinite” — to the standard quantum mechanics reformulated in terms of permutations.

Dropping this assumption we can attach more natural meaning to quantum amplitudes. Let us represent the (quantum) states of the system and apparatus in the permutation representation by the natural vectors

$$|n\rangle = \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix} \text{ and } |m\rangle = \begin{pmatrix} m_1 \\ \vdots \\ m_N \end{pmatrix},$$

respectively. In accordance with the Born rule the probability to fix the system state $|n\rangle$ by apparatus tuned to $|m\rangle$ is

$$P(m,n) = \frac{(\sum_i m_i n_i)^2}{\sum_i m_i^2 \sum_i n_i^2}. \quad (7)$$

It is clear that for non-vanishing natural vectors $|n\rangle$ and $|m\rangle$ expression (7) is a rational number strictly greater than zero. This means, in particular, that it is impossible to observe destructive quantum interference in the system as a whole. However, the destructive interference of the vectors with natural components can be observed in the proper invariant subspaces of the permutation representation. Let us demonstrate this by a simple example.

6. Detailed example: smallest non-commutative group $S_3$

$S_3$ is the smallest non-commutative group. Nevertheless, $S_3$ has important applications in physics. In particular, it describes the so-called tribimaximal mixing in the neutrino oscillations [5, 6]. The group consists of six elements having the following representation by permutations of three things

$$g_1 = (), \quad g_2 = (2,3), \quad g_3 = (1,3), \quad g_4 = (1,2), \quad g_5 = (1,2,3), \quad g_6 = (1,3,2).$$

The exponent of $S_3$ is equal to 6 since the group elements have degrees 2 and 3. The group can be generated by many pairs of its elements. Let us choose, for instance, $g_2$ and $g_6$ as generators. $S_3$ decomposes into the three conjugacy classes

$$K_1 = \{g_1\}, \quad K_2 = \{g_2, g_3, g_4\}, \quad K_3 = \{g_5, g_6\}.$$

3 From the algorithmic point of view, manipulations with permutations are much more efficient than the linear algebra operations with matrices. But degree $N$ of permutations might be much larger than dimension $K$ of matrices. For example, the Fischer-Griess Monster group $M$ — the largest sporadic simple group of the order $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 10^{54}$ — has minimal faithful irreducible representation of dimension $47 \cdot 59 \cdot 71 \approx 2 \cdot 10^5$ and minimal permutation representation of degree $2^4 \cdot 3^7 \cdot 5^3 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 29 \cdot 41 \cdot 59 \cdot 71 \approx 10^{20}$. However, the very possibility to reduce quantum dynamics to permutations is much more important conceptually than the algorithmic issues.
The group \( S_3 \) has the following character table (see Appendix C)

| \( \chi \) | \( K_1 \) | \( K_2 \) | \( K_3 \) |
|-----------|--------|--------|--------|
| \( \chi_1 \) | 1      | 1      | 1      |
| \( \chi_2 \) | 1      | -1     | 1      |
| \( \chi_3 \) | 2      | 0      | -1     |

In accordance with the physical tradition, we will denote irreducible representations by their dimensions in bold. Thus, we have here three irreducible representations \( 1, 1', 2 \) (the last is the only faithful). To denote permutation representations playing an important role in this paper we will use underlined dimensions in bold.

Matrices of permutation representation of generators are

\[
P_2 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \quad \text{and} \quad P_6 = \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}.
\]

The eigenvalues of \( P_2 \) and \( P_6 \) are \((1, 1, -1)\) and \((1, r_3, r_3^2)\), respectively, where \( r_3 \) is a primitive 3rd root of unity satisfying to the cyclotomic polynomial \( \Phi_3(r) = 1 + r + r^2 \).

As mentioned above, any permutation representation contains one-dimensional invariant subspace with the basis vector \((1, \ldots, 1)^T\). Therefore the only possible structure of decomposition of the permutation representation into irreducible parts is \( 3 \cong 1 \oplus 2 \) or in the explicit matrix form

\[
\tilde{U}_j = \begin{pmatrix} 1 & 0 \\ 0 & U_j \end{pmatrix}, \quad j = 1, \ldots, 6,
\]

where the matrices \( U_j \) are elements of the faithful representation \( 2 \).

To construct decomposition (8) we determine matrices \( U_j \) and \( T \) such that \( \tilde{U}_j = T^{-1}P_jT \). Additionally we impose unitarity on all the matrices. Clearly, it suffices to perform the procedure only for matrices of generators. There are different ways to construct decomposition (8). If we start with the diagonalization of \( P_6 \), we come to the following

\[
U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & r_3^2 \\ r_3 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & r_3 \\ r_3 & 0 \end{pmatrix},
\]

\[
U_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_5 = \begin{pmatrix} r_3 & 0 \\ 0 & r_3 \end{pmatrix}, \quad U_6 = \begin{pmatrix} r_3 & 0 \\ 0 & r_3 \end{pmatrix}.
\]

The transformation matrix (up to inessential degrees of freedom for its entries) takes the following form

\[
T = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & r_3^2 \\ r_3 & 1 & r_3 \\ r_3^2 & r_3 & 1 \end{pmatrix}, \quad T^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & r_3 & r_3^2 \\ r_3 & r_3^2 & 1 \end{pmatrix}.
\]

The minimal abelian number field containing all entries of matrices (9) and (10) is the cyclotomic field \( \mathbb{Q}_{12} \). Thus we can rewrite, say, matrix (10) in terms of elements from \( \mathbb{Q}_{12} \)

\[
T = \frac{1}{3} \begin{pmatrix} 2r_{12} + r_{12}^9 & 2r_{12} + r_{12}^9 & r_{12}^7 + r_{12}^9 \\ 2r_{12} + r_{12}^9 & r_{12}^7 + r_{12}^9 & 2r_{12} + r_{12}^9 \\ 2r_{12} + r_{12}^9 & 2r_{12}^3 + r_{12}^7 & 2r_{12}^3 + r_{12}^7 \end{pmatrix}.
\]
where \( r_{12} \) is a primitive 12th root of unity, i.e., an arbitrary solution of the equation \( \Phi_{12}(r) = 1 - r^2 + r^4 = 0 \). We use in (11) the “natural” representation of cyclotomic integers. Reduction \textit{modulo} the cyclotomic polynomial \( \Phi_{12}(r) \) leads to the “integer” representation with negative coefficients but with minimal degrees of \( r_{12} \):

\[
T = \frac{1}{3} \begin{pmatrix}
2r_{12} - r_{12}^3 & 2r_{12} - r_{12}^3 & -r_{12} - r_{12}^3 \\
2r_{12} - r_{12}^3 & -r_{12} - r_{12}^3 & 2r_{12} - r_{12}^3 \\
2r_{12} - r_{12}^3 & -r_{12} + 2r_{12}^3 & -r_{12} - r_{12}^3
\end{pmatrix}.
\]

Otherwise, the diagonalization of \( P_2 \) leads to another second component of decomposition (8). For the generator matrices we have

\[
U'_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U'_6 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}.
\]

The transformation matrix in this case takes the form

\[
T' = \begin{pmatrix} 1 & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \quad T'^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}.
\]

The matrix \( T' \) is known in particle physics (in description of neutrino oscillation data; see Appendix A) under the names \textit{Harrison-Perkins-Scott} or \textit{tribimaximal} mixing matrix.

The minimal abelian number field \( \mathcal{F} \) containing all entries of matrices (12) and (13) is a subfield of the cyclotomic field \( \mathbb{Q}_{24} \) fixed by the Galois automorphism \( r_{24} \rightarrow r_{24}^2 \). Here \( r_{24} \) is the primitive 24th root of unity. The 24th cyclotomic polynomial is \( \Phi_{24}(r) = 1 - r^4 + r^8 \). Thus, in terms of cyclotomics we have

\[
U'_6 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -2r_{24} + r_{24}^6 & 2r_{24}^2 - r_{24}^6 \end{pmatrix}
\]

and

\[
T' = \frac{1}{6} \begin{pmatrix} 4r_{24}^2 - 2r_{24}^6 & 2r_{24} + 2r_{24}^3 + 2r_{24}^5 - 4r_{24}^7 & 0 \\ 4r_{24}^2 - 2r_{24}^6 & -r_{24} - r_{24}^5 - r_{24}^7 + 2r_{24}^3 & -6r_{24} + 6r_{24}^5 \end{pmatrix}.
\]

The information about “quantum behavior” is encoded, in fact, in transformation matrices like (10) or (13).

Let \( |n\rangle = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \) and \( |m\rangle = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \) be system and apparatus state vectors in the “permutation” basis. Transformation of these vectors from the permutation to “quantum” basis with the help of (10) leads to

\[
\tilde{|\psi\rangle} = T^{-1}|n\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} n_1 + n_2 + n_3 \\ n_1 + n_2r_3 + n_3r_3^2 \\ n_1r_3 + n_2 + n_3r_3^2 \end{pmatrix},
\]

\[
\tilde{|\phi\rangle} = T^{-1}|m\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} m_1 + m_2 + m_3 \\ m_1 + m_2r_3 + m_3r_3^2 \\ m_1r_3 + m_2 + m_3r_3^2 \end{pmatrix}.
\]
Projections of the vectors onto two-dimensional invariant subspace are:

\[ |\psi\rangle = \frac{1}{\sqrt{3}} \left( n_1 + n_2 r_3 + n_3 r_2^2 \right), \quad |\phi\rangle = \frac{1}{\sqrt{3}} \left( m_1 + m_2 r_3 + m_3 r_2^2 \right). \]

Constituents of Born’s probability (2) for the two-dimensional subsystem are

\[ \langle \psi | \psi \rangle = Q_3 (n, n) - \frac{1}{3} L_3 (n)^2, \quad (14) \]
\[ \langle \phi | \phi \rangle = Q_3 (m, m) - \frac{1}{3} L_3 (m)^2, \quad (15) \]
\[ |\langle \phi | \psi \rangle|^2 = \left( Q_3 (m, n) - \frac{1}{3} L_3 (m) L_3 (n) \right)^2, \quad (16) \]

where \( L_N (n) = \sum_{i=1}^{N} n_i \) and \( Q_N (m, n) = \sum_{i=1}^{N} m_i n_i \) are linear and quadratic permutation invariants, respectively. These invariants are common to all permutation groups.

Note that:

(i) Expressions (14)–(16) are combinations of the invariants of permutation representation.

(ii) Conditions for destructive quantum interference — vanishing Born’s probability — are determined by the equation

\[ 3 (m_1 n_1 + m_2 n_2 + m_3 n_3) - (m_1 + m_2 + m_3) (n_1 + n_2 + n_3) = 0. \]

This equation has infinitely many “natural” solutions, e.g., \(|n\rangle = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad |m\rangle = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}. \]

Thus, we have obtained essential features of quantum behavior from “permutation dynamics” and “natural” interpretation (6) of quantum amplitude by a simple transition to invariant subspaces.

We can slightly generalize this example. Any permutation representation contains \((N - 1)\)-dimensional invariant subspace. The inner product in this subspace can be expressed in terms of the permutation invariants by the formula

\[ \langle \phi | \psi \rangle = Q_N (m, n) - \frac{1}{N} L_N (m) L_N (n). \]

The identity \( Q_N (n, n) - \frac{1}{N} L_N (n)^2 \equiv \frac{1}{N^2} \sum_{i=1}^{N} (L_N (n) - n_i)^2 \) shows explicitly that \( \langle \psi | \psi \rangle > 0 \) for \(|n\rangle \) with different components \(n_i\). This inner product does not contain irrationalities for natural \(|n\rangle \) and \(|m\rangle \).

7. The icosahedral group \( A_5 \)

The icosahedral group \( A_5 \) is the smallest simple non-commutative group. It consists of 60 elements and its exponent is equal to 30. The group plays so important role in mathematics and applications that F. Klein devoted a whole book to it [7]. In the physical literature the group is often denoted as \( \Sigma (60) \). The group has a “physical incarnation”: the carbon molecule fullerene \( C_{60} \) (“buckyball”) has the structure of the Cayley graph of \( A_5 \) (see Fig. 2). This is clear from the
Figure 2. The Cayley graph of $A_5$. The pentagons, hexagons and the links connecting adjacent pentagons correspond to the relators $a^5$, $(ab)^3$ and $b^2$ in presentation (17), respectively.

The following presentation of $A_5$ by generators and relators (products of generators that are equal to the group identity)

$$A_5 \cong \langle a, b \mid a^5, b^2, (ab)^3 \rangle.$$  \hspace{1cm} (17)

The group $A_5$ decomposes into the five conjugacy classes $K_1, K_{15}, K_{20}, K_{12}, K_{12'}$. To mark the classes we use here their sizes as subscripts. The character table of the group is

|    | $K_1$ | $K_{15}$ | $K_{20}$ | $K_{12}$ | $K_{12'}$ |
|----|-------|----------|----------|----------|-----------|
| $\chi_1$ | 1  | 1  | 1  | 1  | 1  |
| $\chi_3$ | 3  | -1 | 0  | $\phi$ | $1 - \phi$ |
| $\chi_3'$ | 3  | -1 | 0  | $1 - \phi$ | $\phi$ |
| $\chi_4$ | 4  | 0  | 1  | -1 | -1 |
| $\chi_5$ | 5  | 1  | -1 | 0  | 0  |

where $\phi = \frac{1 + \sqrt{5}}{2}$ is the “golden ratio”. Note that $\phi$ and $1 - \phi$ are cyclotomic integer (or “cyclotomic naturals”): $\phi = -r_5^2 - r_5^3 \equiv 1 + r_5 + r_5^2$ and $1 - \phi = -r_5 - r_5^2 \equiv 1 + r_5^2 + r_5^3$, where $r_5$ is a primitive 5th root of unity. We see that the group has five irreducible representations: the trivial representation $1$ and four faithful representations $3, 3', 4, 5$.

As to the permutations, the group has three primitive actions on sets with 5, 6 and 10 elements. The corresponding permutation representations have the following decompositions into the irreducible components

$$5 \cong 1 \oplus 4, \quad 6 \cong 1 \oplus 5, \quad 10 \cong 1 \oplus 4 \oplus 5.$$
Recall that transitive action of a group $G$ on a set $\Omega$ is called \textit{imprimitive} \cite{8} if the group leaves invariant a non-trivial partition of the set $\Omega$. The \textit{trivial partitions} are: partition into one-element blocks and partition into empty set and the whole set $\Omega$. An action is called \textit{primitive} if only trivial partitions are invariant. The primitive actions are considered as most fundamental among all permutation actions.

Let us consider the action of $A_5$ on the set $\Omega_{12}$ of icosahedron vertices. This action is transitive but \textit{imprimitive}. The non-trivial invariant partition — called \textit{system of imprimitivity} (or \textit{block system}) — of $\Omega_{12}$ is the following

$$\{|B_1| \cdots |B_i| \cdots |B_6|\} \equiv \{|1,7| \cdots |i,i+6| \cdots |6,12|\}$$

where, the vertex numbering of Fig. 3 is assumed. Each block $B_i$ is a pair of two opposite vertices of the icosahedron and $A_5$ permutes the blocks amongst themselves as whole entities. We shall denote the correspondence ("complementarity") between opposite vertices by the symbol $c$, i.e., if $B_i = \{p,q\}$ then $q = p^c$ and $p = q^c$. For the vertex numbering depicted in Fig. 3 the complementarity can be expressed by the formula $p^c = 1 + (p + 5 \mod 12)$.

The permutation representation of the action of $A_5$ on the vertices of icosahedron has the following decomposition into irreducible components

$$\mathbf{12} \cong \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{5} \quad \text{or} \quad T^{-1} \left( \mathbf{12} \right) T = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{5}. \quad (18)$$

With the notations

$$\alpha = \frac{\phi}{4} \sqrt{10 - 2\sqrt{5}}, \quad \beta = \frac{\sqrt{5}\sqrt{10 - 2\sqrt{5}}}{20}, \quad \gamma = \frac{\sqrt{3}}{8} \left(1 - \frac{\sqrt{5}}{3}\right), \quad \delta = -\frac{\sqrt{3}}{8} \left(1 + \frac{\sqrt{5}}{3}\right)$$
a particular form of unitary transformation matrix \( T \) from (18) can be written as

\[
T = \frac{\sqrt{3}}{6} \begin{pmatrix}
\frac{\sqrt{3}}{6} & \alpha & \beta & 0 & \alpha & \beta & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{2} & 0 & 0 & \frac{\sqrt{15}}{12} \\
0 & \alpha & \beta & 0 & \alpha & \beta & 0 & 0 & 0 & \frac{1}{4} & 0 & -\frac{1}{2} & 0 & \gamma \\
\frac{\sqrt{3}}{6} & \beta & 0 & 0 & -\alpha & -\beta & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \delta \\
\frac{\sqrt{3}}{6} & 0 & \alpha & -\beta & 0 & -\alpha & -\beta & 0 & 0 & \frac{1}{4} & 0 & -\frac{1}{2} & 0 & \gamma \\
\frac{\sqrt{3}}{6} & -\beta & 0 & \alpha & 0 & \alpha & -\beta & 0 & 0 & 0 & \frac{1}{2} & 0 & \delta \\
\frac{\sqrt{3}}{6} & 0 & -\alpha & -\beta & 0 & 0 & -\alpha & -\beta & 0 & 0 & \frac{1}{2} & 0 & \gamma \\
\frac{\sqrt{3}}{6} & \beta & 0 & 0 & -\alpha & 0 & -\alpha & 0 & 0 & \frac{1}{2} & 0 & \delta \\
\frac{\sqrt{3}}{6} & 0 & -\alpha & 0 & \alpha & 0 & \alpha & 0 & 0 & \frac{1}{2} & 0 & \delta \\
\frac{\sqrt{3}}{6} & 0 & -\alpha & 0 & \alpha & \beta & 0 & 0 & 0 & \frac{1}{2} & 0 & \gamma \\
\frac{\sqrt{3}}{6} & -\beta & 0 & -\alpha & 0 & \alpha & \beta & 0 & 0 & 0 & \frac{1}{2} & 0 & \gamma \\
\frac{\sqrt{3}}{6} & 0 & -\alpha & 0 & \alpha & \beta & 0 & 0 & 0 & \frac{1}{2} & 0 & \gamma \\
\frac{\sqrt{3}}{6} & -\beta & 0 & -\alpha & 0 & \alpha & \beta & 0 & 0 & 0 & \frac{1}{2} & 0 & \gamma \\
\frac{\sqrt{3}}{6} & 0 & -\alpha & 0 & \alpha & \beta & 0 & 0 & 0 & \frac{1}{2} & 0 & \gamma \\
\end{pmatrix}.
\]

Note that the standard computer algebra systems like Maple or Mathematica can not handle such matrices since they can not simplify complicated expressions with irrationalities (especially with nested roots) properly.

But rewriting the matrix entries via elements of suitable abelian number field reduces the problem of simplification to a simple one-variable polynomial algebra modulo corresponding cyclotomic polynomial. In our case the minimal abelian number field \( \mathcal{F} \) is a subfield of the cyclotomic field \( \mathbb{Q}_{60} \). \( \mathcal{F} \) is fixed in \( \mathbb{Q}_{60} \) by the Galois automorphism \( r_{60} \rightarrow r_{60}^{59} \), where \( r_{60} \) denotes the primitive 60th root of unity. The 60th cyclotomic polynomial is \( \Phi_{60}(r) = 1 + r^2 + r^6 + r^8 + r^{10} + r^{14} + r^{16} \). The entries of transformation matrix (19) can be rewritten in terms of cyclotomics:

\[
\frac{\sqrt{3}}{6} = \frac{1}{6} \left( 2r_{60}^5 - r_{60}^{15} \right), \\
\alpha = \frac{1}{2} \left( r_{60} + r_{60}^3 - r_{60}^9 - r_{60}^{11} + r_{60}^{15} \right), \\
\beta = \frac{1}{2} \left( r_{60} + 4r_{60}^3 - 3r_{60}^9 - r_{60}^{11} + 2r_{60}^{15} \right), \\
-\frac{\phi}{4} = \frac{1}{4} \left( -r_{60}^4 + r_{60}^{14} \right), \\
\phi - \frac{1}{4} = \frac{1}{4} \left( -1 + r_{60}^4 + r_{60}^8 - r_{60}^{14} \right), \\
\frac{\sqrt{15}}{12} = \frac{1}{12} \left( -2r_{60} + 2r_{60}^5 + 4r_{60}^7 + 2r_{60}^9 + r_{60}^{11} + 2r_{60}^{13} + 3r_{60}^{15} \right), \\
\gamma = \frac{1}{12} \left( r_{60} + 2r_{60}^5 - 2r_{60}^7 - r_{60}^9 + r_{60}^{11} + 2r_{60}^{13} \right), \\
\delta = \frac{1}{12} \left( r_{60} - 4r_{60}^5 - 2r_{60}^7 - r_{60}^9 - r_{60}^{11} + 2r_{60}^{13} + 3r_{60}^{15} \right).
\]

The inner products in the invariant subspaces can be expressed in terms of permutation
invariants as follows:

\[
\langle \Phi_1 | \Psi_1 \rangle = \frac{1}{12} L_{12}(m) L_{12}(n),
\]

(20)

\[
\langle \Phi_3 | \Psi_3 \rangle = \frac{1}{20} \left( 5Q_{12}(m,n) - 5A(m,n) + \sqrt{5} (B(m,n) - C(m,n)) \right) ,
\]

(21)

\[
\langle \Phi_3' | \Psi_3' \rangle = \frac{1}{20} \left( 5Q_{12}(m,n) - 5A(m,n) - \sqrt{5} (B(m,n) - C(m,n)) \right) ,
\]

(22)

\[
\langle \Phi_5 | \Psi_5 \rangle = \frac{1}{12} (5Q_{12}(m,n) + 5A(m,n) - B(m,n) - C(m,n)),
\]

(23)

where

\[
A(m,n) = A(n,m) = \sum_{k=1}^{12} m_k n_{k'},
\]

(24)

\[
B(m,n) = B(n,m) = \sum_{k=1}^{12} m_k \sum_{i \in N(k)} n_i,
\]

(25)

\[
C(m,n) = C(n,m) = \sum_{k=1}^{12} m_k \sum_{i \in N(k')} n_i.
\]

(26)

In formulas (25) and (26) \( N(k) \) denotes the “neighborhood” of the icosahedron vertex \( k \), i.e., the set of vertices adjacent to \( k \). For example, \( N(1) = \{2, 3, 4, 5, 6\} \) in Fig. 3. Thus, the inner sum in (25) is the sum of data on the neighborhood of the vertex \( k \), whereas analogous sum in (26) is the sum of data on the neighborhood of the vertex opposite to \( k \).

Quadratic invariants (24)–(26) are not independent. There is an identity among them:

\[
A(m,n) + B(m,n) + C(m,n) + Q_{12}(m,n) = L_{12}(m) L_{12}(n).
\]

Inner products (21) and (22) when considered separately lead to the mentioned above conceptual problem with probability. Born’s probabilities computed separately for the representations 3 and 3’ contain irrationals. This contradicts the frequency interpretation of probability for finite sets. Of course, this is a consequence of the imprimitivity: one cannot move a vertex of the icosahedron without simultaneous moving of its complement in the block.

To resolve the contradiction we should consider the complementary representations 3 and 3’ together. The inner product for the representation \( 3 \oplus 3' \)

\[
\langle \Phi_{3\oplus3'} | \Psi_{3\oplus3'} \rangle = \frac{1}{2} (Q_{12}(m,n) - A(m,n))
\]

always gives rational Born’s probabilities for vectors with natural “population numbers”.

8. Conclusions

Let us summarize the main ideas of the paper

(i) Quantum mechanics is, in fact, an a priori mathematical scheme based on the fundamental impossibility to trace identity of indistinguishable objects in their evolution — some kind of “calculus of indistinguishables”, a subject of combinatorics.

(ii) Any quantum mechanical problem can be reduced to permutations.

(iii) Quantum interferences are appearances observable in the invariant subspaces of permutation representations. They can be expressed in terms of the permutation invariants.
Natural interpretation of quantum amplitudes ("waves") as vectors of "population numbers" of underlying entities ("particles") leads to the rational quantum probabilities — in accordance with the frequency interpretation of probability for finite sets.

The idea of natural quantum amplitudes is very promising. In particular, it allows to interpret the elements of the underlying set as "particles" and vectors of multiplicities of these elements as "waves". This idea leads to a simple and self-consistent picture of quantum behavior. However, it requires verification. If it is valid, then quantum phenomena in different invariant subspaces reveal different manifestations — visible in different "observational settings" — of a single process of permutations of underlying things. One has to interpret the data corresponding to different invariant subspaces of the same permutation representation. For example, the trivial one-dimensional subrepresentation contained in any permutation representation can be interpreted as the "counter of particles": the permutation invariant \( L_N(n) \) corresponding to this subrepresentation is the total number of particles. Interpretation of data in other invariant subspaces requires further careful deliberation.

**Acknowledgments**

The work was partially supported by the grants 01-01-00200 from the Russian Foundation for Basic Research and 3810.2010.2 from the Ministry of Education and Science of the Russian Federation.

**Appendix A. Finite symmetry groups in particle physics**

At present, all observations concerning fundamental particles [9] are compatible with the Standard Model (SM). The SM is a gauge theory with the group of internal (gauge) symmetries \( \Gamma = SU(3) \times SU(2) \times U(1) \). In the context of Grand Unified Theory (GUT) \( \Gamma \) is assumed to be a subgroup of some larger (hypothetically simple) group. With respect to space-time symmetries, the elementary particles are divided into two classes: bosons, responsible for physical forces (roughly speaking, they are elements of the gauge group) and fermions, usually treated as particles of matter. The fermions of the SM are divided into three generations of quarks and leptons as follows (antiparticles are omitted for brevity):

| Generations     | 1     | 2     | 3     |
|-----------------|-------|-------|-------|
| Up-quarks       | Up    | c     | Top   |
| Down-quarks     | Down  | d     | Bottom|
| Charged leptons | Electron, e^- | Muon, \( \mu^- \) | Tau, \( \tau \) |
| Neutrinos       | Electron neutrino, \( \nu_e \) | Muon neutrino, \( \nu_\mu \) | Tau neutrino, \( \nu_\tau \) |

Between generations particles differ only by their mass and quantum property called flavor. The flavor changing transitions — taking place in such phenomena as weak decays of quarks and neutrino oscillations — are described by \( 3 \times 3 \) unitary mixing matrices. The outputs of experiments allow to calculate magnitudes of elements of these matrices.

In the case of quarks ("in the quark sector"), the mixing matrix describing transitions between up- and down-type quarks is the Cabibbo–Kobayashi–Maskawa (CKM) matrix

\[
V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix},
\]
where $|V_{\alpha\beta}|^2$ represents the probability that the quark (of flavor) $\beta$ decays into a quark $\alpha$. The current experimental data rounded to three significant digits are:

$$
\begin{pmatrix}
|V_{ud}| & |V_{us}| & |V_{ub}| \\
|V_{cd}| & |V_{cs}| & |V_{cb}|
\end{pmatrix} =
\begin{pmatrix}
0.974 & 0.225 & 0.004 \\
0.225 & 0.974 & 0.041 \\
0.009 & 0.040 & 0.999
\end{pmatrix}.
$$

More precise values can be found in [9].

In the lepton sector the weak interaction processes are described by the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) mixing matrix

$$
U_{PMNS} =
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix}.
$$

Here indices $e, \mu, \tau$ correspond to the neutrino flavors — this means that the neutrinos $\nu_e, \nu_\mu, \nu_\tau$ are produced with $e^+, \mu^+, \tau^+$ (or produce $e^-, \mu^-, \tau^-$), respectively, in weak processes. The indices 1, 2, 3 correspond to the mass eigenstates, i.e., neutrinos $\nu_1, \nu_2, \nu_3$ with definite masses $m_1, m_2, m_3$. Numerous experiments with solar, atmospheric, reactor, and accelerator neutrinos indicate the existence of discrete symmetries that can not be deduced from the SM. The phenomenological pattern is the following [10]:

(i) $\nu_\mu$ and $\nu_\tau$ flavors are presented with equal weights in all three mass eigenstates $\nu_1, \nu_2, \nu_3$ (this is called “bi-maximal mixing”):

$$
|U_{\mu i}|^2 = |U_{\tau i}|^2, \quad i = 1, 2, 3;
$$

(ii) all three flavors are presented equally in $\nu_2$ (“trimaximal mixing”):

$$
|U_{e2}|^2 = |U_{\mu 2}|^2 = |U_{\tau 2}|^2;
$$

(iii) $\nu_e$ is absent in $\nu_3$: $|U_{\mu 3}|^2 = 0$.

These relations together with the normalization condition for probabilities allow to determine moduli-squared of all matrix elements:

$$
(|U_{li}|^2) =
\begin{pmatrix}
\frac{2}{3} & \frac{1}{3} & 0 \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{2}
\end{pmatrix}. \quad (A.1)
$$

A particular form of unitary matrix satisfying data (A.1) was suggested by Harrison, Perkins, and Scott in [5]:

$$
U_{TB} =
\begin{pmatrix}
\frac{\sqrt{2}}{3} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}. \quad (A.2)
$$

This so-called tribimaximal (TB) mixing matrix coincides — up to the trivial permutation of two columns corresponding to the renaming $\nu_1 \rightleftharpoons \nu_2$ of states — with transformation matrix (13) decomposing the natural permutation representation of the group $S_3$ into irreducible components. This means that we can identify the flavor basis with the representation basis of permutations of three things, and the mass basis is a basis of irreducible decomposition of this
representation. In [6] Harrison and Scott study in detail connections of the neutrino mass matrix with the character table and class algebra of the group $S_3$.

At present, much effort is devoted to the construction and study of models based on finite flavor symmetries (for recent reviews, see, for example, [11, 12]). The most popular groups for constructing such models are ($\rtimes$ denotes the semidirect product):

- $T = A_4$ — the tetrahedral group;
- $T' = \text{the double covering of } A_4$;
- $O = S_4$ — the octahedral group;
- $I = A_5$ — the icosahedral group;
- $D_N$ — the dihedral groups ($N$ even);
- $Q_N$ — the quaternionic groups ($4$ divides $N$);
- $\Sigma (2N^2)$ — the groups in this series have the structure $(Z_N \times Z_N) \rtimes Z_2$;
- $\Delta (3N^2)$ — the structure $(Z_N \times Z_N) \rtimes Z_3$;
- $\Delta (6N^2)$ — the structure $(Z_N \times Z_N) \rtimes S_3$.

As to the quark sector, observations do not give such sharp picture as in the lepton case. In [13] the $D_{14}$ symmetry was suggested for explanation of the value of the Cabibbo angle (one of the parameters of the CKM matrix), but without any connection with the leptonic symmetries. The natural attempts to find discrete symmetries unifying leptons and quarks still remain not very successful, though there are some encouraging observations, for example, the quark-lepton complementarity (QLC) — observation that the sum of quark and lepton mixing angles is equal approximately to $\pi/4$.

The origin of finite symmetries among fundamental particles is unclear. There are different attempts to explain — sometimes looking a bit complicated and artificial, for example, these symmetries are treated as symmetries of manifolds arising at compactification of a higher dimensional theory to four spacetime dimensions [14]. The idea that symmetries at the most fundamental level are per se finite looks more attractive in our opinion. In this approach, unitary groups used in physical theories can be treated simply as repositories of all finite groups having faithful representation of corresponding dimensions, e.g., $U(n)$ contains all finite groups with faithful $n$-dimensional representations. Of course, due to redundancy of the field $\mathbb{C}$, $U(n)$ is not a minimal group with this property.

Such small groups as $S_3$, $A_4$, etc. are most likely only remnants of large combinations of more fundamental finite symmetries that are expected to exist at the GUT scale. Unfortunately the GUT scale ($10^{16}$ GeV) being close to the Planck scale ($10^{19}$ GeV) is out of reach of experiments (the most powerful colliders to date can provide only about $10^4$ GeV). Thus, the only practical way at present is to construct models, study them by the computational group theory methods, and compare consequences of these models with available experimental data.

Appendix B. Unification of space and internal symmetries

Having the groups $F$ and $\Gamma$ acting on $X$ and $\Sigma$, respectively, we can combine them into a single group $G \leq \text{Sym}(\Sigma^X)$ which acts on the states $\Omega = \Sigma^X$ of the whole system. The group $G$ can be identified, as a set, with the Cartesian product $\Gamma^X \otimes F$, where $\Gamma^X$ is the set of $\Gamma$-valued functions on $X$. That is, every element $u \in G$ can be represented in the form $u = (\alpha(x), \ a)$, where $\alpha(x) \in \Gamma^X$ and $a \in F$. A priori there are different possible ways to combine $F$ and $\Gamma$ into a single group. So selection of possible combinations should be guided by some natural (physical) reasons. General arguments convince that the required combination $G$ should be a split extension of the group $F$ by the group $\Gamma^X$. In physics, it is usually assumed that the space
and internal symmetries are independent, i.e., $G$ is the direct product $\Gamma^X \times F$ with action on $\Sigma^X$ and multiplication rules:
\[
\sigma(x)(\alpha(x), a) = \sigma(x)\alpha(x) \quad \text{action,}
\]
\[
(\alpha(x), a)(\beta(x), b) = (\alpha(x)\beta(x), ab) \quad \text{multiplication.} \tag{B.1}
\]

Another standard construction is the wreath product $\Gamma \wr_X F$ having a structure of the semidirect product $\Gamma^X \rtimes F$ with action and multiplication:
\[
\sigma(x)(\alpha(x), a) = \sigma(xa^{-1})\alpha(xa^{-1}),
\]
\[
(\alpha(x), a)(\beta(x), b) = (\alpha(x)\beta(xa), ab). \tag{B.2}
\]

These examples are generalized by the following

**Statement.** There are equivalence classes of split group extensions

\[
1 \to \Gamma^X \to G \to F \to 1 \tag{B.3}
\]

determined by antihomomorphisms\(^4\) $\mu : F \to F$. The equivalence is described by arbitrary function $\kappa : F \to F$. The explicit formulas for main group operations — action on $\Sigma^X$, multiplication and inversion — are
\[
\sigma(x)(\alpha(x), a) = \sigma(x\mu(a))\alpha(x\kappa(a)), \tag{B.4}
\]
\[
(\alpha(x), a)(\beta(x), b) = (\alpha(x\kappa(ab)^{-1}\mu(b)\kappa(a))\beta(x\kappa(ab)^{-1}\kappa(b)), ab), \tag{B.5}
\]
\[
(\alpha(x), a)^{-1} = \left(\alpha(x\kappa(a)^{-1}\mu(a)^{-1}\kappa(a))^{-1}, a^{-1}\right). \tag{B.6}
\]

This statement follows from the general description of the structure of split extensions of a group $F$ by a group $H$: all such extensions are determined by the homomorphisms from $F$ to Aut ($H$) (see, e.g., [15]). Specializing this description to the case when $H$ is the set of $\Gamma$-valued function on $X$ and $F$ acts on arguments of these functions we obtain our statement. The equivalence of extensions with the same antihomomorphism $\mu$ but with different functions $\kappa$ is expressed by the commutative diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & \Gamma^X & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 1 \\
\downarrow & & \downarrow \kappa & & \downarrow \kappa & & \downarrow \kappa & & \downarrow \kappa \\
1 & \longrightarrow & \Gamma^X & \longrightarrow & G' & \longrightarrow & F & \longrightarrow & 1
\end{array} \tag{B.7}
\]

where the mapping $K$ takes the form $K: (\alpha(x), a) \mapsto (\alpha(x\kappa(a)), a)$.

Note that the standard direct and wreath products are obtained from this general construction by choosing antihomomorphisms $\mu(a) = 1$ and $\mu(a) = a^{-1}$, respectively. As to the arbitrary function $\kappa$, the choices $\kappa(a) = 1$ and $\kappa(a) = a^{-1}$, respectively, are generally used in the literature.

The following specialization is convenient in applications: $\mu(a) = a^{-m}$ and $\kappa(a) = a^{k}$. For such a choice formulas (B.4-B.6) take the form
\[
\sigma(x)(\alpha(x), a) = \sigma(xa^{-m})\alpha(xa^{k}), \tag{B.8}
\]
\[
(\alpha(x), a)(\beta(x), b) = \left(\alpha(x(ab)^{-k-m}a^{k+m})\beta(x(ab)^{-k}a^{k}), ab\right), \tag{B.9}
\]
\[
(\alpha(x), a)^{-1} = \left(\alpha(xa^{2k+m})^{-1}, a^{-1}\right). \tag{B.10}
\]

\(^4\) The term ‘antihomomorphism’ means that $\mu(a)\mu(b) = \mu(ba)$.
Here $k$ is arbitrary integer, but $m$ is restricted only to two values: $m = 0$ and $m = 1$, i.e., such specialization does not cover other than, respectively, direct and wreath types of split extensions. On the other hand, the antihomomorphisms $\mu(a) = 1$ and $\mu(a) = a^{-1}$ exist for any group, while others depend on the particular structure of the space group $F$. Note that actions of $F$ on any function $f(x)$ are called trivial and natural for $\mu(a) = 1$ and $\mu(a) = a^{-1}$, respectively.

Appendix C. Linear representations of finite groups

Any linear representation of a finite group is equivalent to unitary, since one can always construct invariant inner product from an arbitrary one by “averaging over the group”. Starting from, e.g., the standard inner product in $K$-dimensional Hilbert space $\mathcal{H}$

$$\langle \phi \mid \psi \rangle \equiv \sum_{i=1}^{K} \overline{\phi^i} \psi^i$$  \hspace{1cm} (C.1)

we can come via the averaging to the invariant inner product:

$$\langle \phi \mid \psi \rangle \equiv \frac{1}{|G|} \sum_{g \in G} \langle U(g) \phi \mid U(g) \psi \rangle.$$  \hspace{1cm} (C.2)

Here $U$ is a representation of a group $G$ in the space $\mathcal{H}$.

An important transformation of group elements — an analog of change of coordinates in physics — is the conjugation: $a^{-1}ga \rightarrow g'$, $g, g' \in G$, $a \in \text{Aut}(G)$. Conjugation by an element of the group itself, i.e., if $a \in G$, is called an inner automorphism. The equivalence classes with respect to the inner automorphisms are called conjugacy classes. The starting point in study of representations of a group is its decomposition into conjugacy classes

$$G = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_m.$$  \hspace{1cm} (C.3)

The group multiplication induces multiplication on the classes. The product of $K_i$ and $K_j$ is the multiset of all possible products $ab$, $a \in K_i$, $b \in K_j$, decomposed into classes. This multiplication is obviously commutative, since $ab$ and $ba$ belong to the same class: $ab \sim a^{-1}(ab)a = ba$. Thus, the multiplication table for classes is given by

$$K_iK_j = K_jK_i = \sum_{k=1}^{m} c_{ijk} K_k.$$  \hspace{1cm} (C.3)

The natural integers $c_{ijk}$ — multiplicities of classes in the multisets — are called class coefficients.

This is a short list of main properties of linear representations of finite groups:

(i) Any irreducible representation is contained in the regular representation. More specifically, there exists matrix $T$ transforming simultaneously all matrices of the regular representation to the form

$$T^{-1}P(g)T = \begin{pmatrix} D_1(g) \\ \vdots \\ D_2(g) \\ \vdots \\ \vdots \\ D_m(g) \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_2 \\ \vdots \\ \vdots \\ d_m \end{pmatrix}$$  \hspace{1cm} (C.4)
and any irreducible representation is one of $D_i$'s. The numbers of non-equivalent irreducible representation and conjugacy classes coincide. The number $d_i$ is the dimension of the irreducible component $D_i$ and simultaneously the multiplicity of its occurrence in the regular representation. It is clear from (C.4) that for the dimensions of irreducible representations the following relation holds: $d_1^2 + d_2^2 + \cdots + d_m^2 = |G| = M$. The dimensions of irreducible representations divide the group order: $d_j | M$.

(ii) Any irreducible representation $D_j$ is determined uniquely by its character $\chi_j$ defined as the trace of the representation matrix: $\chi_j (g) = \text{Tr} D_j (g)$. This is a function on the conjugacy classes since $\chi_j (g) = \chi_j (a^{-1} ga)$. Obviously, $\chi_j (1) = d_j$.

(iii) A compact form of recording all irreducible representations is the character table. The columns of this table are numbered by the conjugacy classes, while its rows contain values of characters of non-equivalent representation:

| $\chi_1$ | $K_1$ | $K_2$ | $\cdots$ | $K_m$ |
|---------|-------|-------|------|-------|
| $\chi_2$ | $\chi_2 (K_1) = d_2$ | $\chi_2 (K_2)$ | $\cdots$ | $\chi_2 (K_m)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\chi_m$ | $\chi_m (K_1) = d_m$ | $\chi_m (K_2)$ | $\cdots$ | $\chi_m (K_m)$ |

By convention, the 1st column corresponds to the identity class, and the 1st row contains the trivial representation. Both columns and rows of the character table are pairwise orthogonal. Important data about a group and its representations can be read immediately from the character table. For example, an irreducible representation is faithful if and only if the value of its character at the 1st column (i.e., dimension) is unique in the whole row. Character tables determine groups almost entirely, more precisely, up to isoclinism [16]. Two groups are said to be isoclinic if their quotients by their centers are isomorphic. Isoclinic groups have identical character tables since characters do not “feel” centers. For example, the dihedral and quaternion groups of order 8 are isoclinic.

References

[1] Nairz O Arndt M and Zeilinger A 2003 Quantum Interference Experiments with Large Molecules Am. J. Phys. 71 No. 4 pp 319–325
[2] Shafarevich I R 1990 Basic Notions of Algebra (Berlin, New York: Springer)
[3] Hall M Jr 1959 The Theory of Groups (Springer-Verlag, Berlin-New York)
[4] GAP – Groups, Algorithms, Programming – a System for Computational Discrete Algebra http://www.gap-system.org/
[5] Harrison P F, Perkins D H and Scott W G 2002 Tri-bimaximal mixing and the neutrino oscillation data Phys. Lett. B 530 p 167 (Preprint arXiv: hep-ph/0202074)
[6] Harrison P F and Scott W G 2003 Permutation symmetry, tri-bimaximal neutrino mixing and the S3 group characters Phys. Lett. B 557 p 76 (Preprint arXiv: hep-ph/0302025)
[7] Klein F 1884 Vorlesungen über das Ikosaeder (Leipzig: Teubner)
[8] Wielandt H 1964 Finite Permutation Groups (Academic Press, New York and London)
[9] Nakamura K et al 2010 (Particle Data Group): The review of particle physics. J. Phys. G 37 075021 pp 1–1422
[10] Smirnov A Yu 2011 Discrete Symmetries and Models of Flavor Mixing Preprint arXiv:1103.3461
[11] Ishimori H, Kobayashi T, Ohki H, Okada H, Shimizu Y and Tanimoto M 2010 Non-abelian discrete symmetries in particle physics Prog. Theor. Phys. Suppl. 183 pp 1–173 (Preprint arXiv:1003.3552)
[12] Ludl P O 2009 Systematic Analysis of Finite Family Symmetry Groups and Their Application to the Lepton Sector Preprint arXiv:0907.5587
[13] Blum A and Hagedorn C 2009 The Cabibbo angle in a supersymmetric D14 model Nucl. Phys. B821 pp 327–353
[14]Altarelli G and Feruglio F 2010 Discrete flavor symmetries and models of neutrino mixing Rev. Mod. Phys. 82(3) pp 2701–2729
[15] Kirillov A A 1976 Elements of the Theory of Representations (Springer-Verlag, Berlin-New York)
[16] Conway J H et al 1985 Atlas of Finite Groups (Clarendon Press, Oxford)