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connective K-theory of the classifying
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by

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THE ETA INVARIANT AND THE REAL CONNECTIVE
K-THEORY OF THE CLASSIFYING SPACE FOR QUATERNION GROUPS

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Abstract. We express the real connective $K$ theory groups $\tilde{ko}_{4k-1}(BQ_\ell)$ of the quaternion group $Q_\ell$ of order $\ell = 2^j \geq 8$ in terms of the representation theory of $Q_\ell$ by showing $\tilde{ko}_{4k-1}(BQ_\ell) = \tilde{K}Sp(S^{4k+3}/\tau Q_\ell)$ where $\tau$ is any fixed point free representation of $Q_\ell$ in $U(2k + 2)$.

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1. Introduction

A compact Riemannian manifold $(M, g)$ is said to be a spherical space form if $(M, g)$ has constant sectional curvature $+1$. A finite group $G$ is said to be a spherical space form group if there exists a representation $\tau : G \to U(k)$ for $k \geq 2$ which is fixed point free - i.e. $\det(I - \tau(\xi)) \neq 0 \forall \xi \in G - \{1\}$. Let

$$M^{2k-1}(G, \tau) := S^{2k-1}/\tau(G)$$

be the associated spherical space form; $G$ is then the fundamental group of the manifold $M^{2k-1}(G, \tau)$. Every odd dimensional spherical space form arises in this manner; the only even dimensional spherical space forms are the sphere $S^{2k}$ and real projective space $\mathbb{R}P^{2k}$. The spherical space form groups all have periodic cohomology; conversely, any group with periodic cohomology acts without fixed points on some sphere, although not necessarily orthogonally. We refer to [18] for further details concerning spherical space form groups.

Any cyclic group is a spherical space form group since the group of $\ell^{th}$ roots of unity acts without fixed points by complex multiplication on the unit sphere $S^{2k-1}$ in $\mathbb{C}^k$. Let $H = \text{span}_\mathbb{R}\{1, I, J, K\}$ be the quaternions, let $\ell = 2^j \geq 8$, and let $\xi := e^{4\pi I / \ell} \in H$ be a primitive $(\frac{\ell}{2})^{th}$ root of unity. The quaternion group $Q_\ell$ is the subgroup of $H$ of order $\ell$ generated by $\xi$ and $J$:

$$Q_\ell := \{1, \xi, ..., \xi^{\ell/2-1}, J, \xi J, ..., \xi^{\ell/2-1}J\}.$$

Let $BG$ be the classifying space of a finite group and let $ko_w(BG)$ be the associated real connective $K$ theory groups; we refer to [2, 3, 7, 9, 14] for a further discussion of connective $K$ theory and related matters.

The $p$ Sylow subgroup of a spherical space form group $G$ is cyclic if $p$ is odd and either cyclic or a quaternion group $Q_\ell$ for $\ell = 2^j \geq 8$ if $p = 2$. This focuses attention on these two groups. We showed previously in [4] that:

**Theorem 1.1.** Let $\mathbb{Z}_\ell$ be the cyclic group of order $\ell = 2^j > 1$. Let $k \geq 1$. Let $\tau : \mathbb{Z}_\ell \to U(2k + 2)$ be a fixed point free representation. Then

$$\tilde{ko}_{4k-1}(B\mathbb{Z}_\ell) = \tilde{K}Sp(M^{4k+3}(\mathbb{Z}_\ell, \tau)).$$

In this paper, we generalize Theorem 1.1 to the quaternion group:

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Theorem 1.2. Let $Q_\ell$ be the quaternion group of order $\ell = 2^j \geq 3$. Let $k \geq 1$. Let $\tau : Q_\ell \to U(2k+2)$ be a fixed point free representation. Then

$$\tilde{ko}_{4k-1}(BQ_\ell) = \tilde{K}Sp(M^{4k+3}(Q_\ell, \tau)).$$

The quaternion (symplectic) $K$ theory groups $\tilde{K}Sp(M^{4k+3}(Q_\ell, \tau))$ are expressible in terms of the representation theory - see Theorem 4.1. Thus Theorem 1.2 expresses $ko_{4k-1}(BQ_\ell)$ in terms of representation theory. If $\ell = 8$, then these groups were determined previously [3, 5].

Here is a brief outline to this paper. In Section 2, we review some facts concerning the representation theory of $Q_\ell$ which we shall need. In Section 3, we review some results concerning the eta invariant. In Section 4, we use the eta invariant to study $\tilde{K}Sp(M^{4k+3}(Q_\ell, \tau))$. In Section 5, we use the eta invariant to study $\tilde{ko}(BQ_\ell)$ and complete the proof of Theorem 1.2.

The proof of Theorem 1.2 is quite a bit different from the proof of Theorem 1.1 given previously; the extension is not straightforward. This arises from the fact that unlike the classifying space $BZ_\ell$, the 2 localization of $BQ_\ell$ is not irreducible.

Let $SL_2(F_q)$ be the group of $2 \times 2$ matrices of determinant 1 over the field $F_q$ with $q$ elements where $q$ is odd. Then the 2-Sylow subgroup of $SL_2(F_q)$ is $Q_\ell$ for $\ell = 2^j$ where $j$ is the power of 2 dividing $q^2 - 1$. There is a stable 2-local splitting of the classifying space $BQ_\ell$ in the form

$$BQ_\ell = BSL_2(F_q) \vee \Sigma^{-1}BS^3/BN \vee \Sigma^{-1}BS^3/BN$$

where $N$ is the normalizer of a maximal torus in $S^3$ [16, 15]. It is necessary to find a corresponding splitting of $\tilde{K}Sp(M^{4k+3}(Q_\ell, \tau))$ that mirrors this decomposition; see Remark 5.2.

2. The Representation Theory of $Q_\ell$

We say that $f : Q_\ell \to \mathbb{C}$ is a class function if $f(\ell g x^{-1}) = f(g)$ for all $x, g \in Q_\ell$; let Class$(Q_\ell)$ be the Hilbert space of all class functions with the $L^2$ inner product

$$\langle f_1, f_2 \rangle = \ell^{-1} \sum_{g \in Q_\ell} \bar{f}_1(g)f_2(g).$$

Let Irr$(Q_\ell)$ be a set of representatives for the equivalence classes of irreducible unitary representations of $Q_\ell$. The orthogonality relations show that $\{\text{Tr} (\sigma)\}_{\sigma \in \text{Irr}(Q_\ell)}$ is an orthonormal basis for Class$(Q_\ell)$, i.e. we may expand any class function:

$$f = \sum_{\sigma \in \text{Irr}(Q_\ell)} \langle f, \text{Tr} (\sigma) \rangle \text{Tr}(\sigma).$$

The unitary group representation ring $RU(Q_\ell)$ and the augmentation ideal $RU_0(Q_\ell)$ are defined by:

$$RU(Q_\ell) = \text{Span}_\mathbb{Z}\{\sigma\}_{\sigma \in \text{Irr}(Q_\ell)},$$

$$RU_0(Q_\ell) = \{\sigma \in RU(Q_\ell) : \dim \sigma = 0\}.$$
The $\ell + 3$ conjugacy classes of $Q_\ell$ have representatives:

$$\{1, \xi, ..., \xi^{\ell/4} = -1, \mathcal{J}, \xi\mathcal{J}\}.$$ 

There are $\ell + 3$ irreducible inequivalent complex representations of $Q_\ell$. Four of these representations are the 1 dimensional representations defined by:

$$\rho_0(\xi) = 1, \quad \kappa_1(\xi) = -1, \quad \kappa_2(\xi) = 1, \quad \kappa_3(\xi) = -1,$$

$$\rho_0(\mathcal{J}) = 1, \quad \kappa_1(\mathcal{J}) = 1, \quad \kappa_2(\mathcal{J}) = -1, \quad \kappa_3(\mathcal{J}) = -1.$$ 

We define representations $\gamma_u : Q_\ell \to U(2)$ by setting:

$$\gamma_u(\xi) = \begin{pmatrix} \xi^u & 0 \\ 0 & \xi^{-u} \end{pmatrix}, \quad \gamma_u(\mathcal{J}) = \begin{pmatrix} 0 & (-1)^u \\ 1 & 0 \end{pmatrix}.$$ 

The representations $\gamma_u$, $\gamma_{-u}$, and $\gamma_{\frac{\ell}{4} + \frac{s}{2}}$ are all equivalent. The representations $\gamma_u$ are irreducible and inequivalent for $1 \leq u \leq \frac{\ell}{4} - 1$; $\gamma_0$ is equivalent to $\rho_0 + \kappa_2$ and $\gamma_{\frac{\ell}{4}}$ is equivalent to $\kappa_1 + \kappa_3$. We have:

$$\text{Irr} (Q_\ell) = \{\rho_0, \kappa_1, \kappa_2, \kappa_3, \gamma_1, ..., \gamma_{\frac{\ell}{4} - 1}\}.$$ 

If $\bar{s} = (s_1, ..., s_k)$ is a $k$ tuple of odd integers, then

$$\gamma_{\bar{s}} := \gamma_{s_1} \oplus ... \oplus \gamma_{s_k}$$

is a fixed point free representation from $Q_\ell$ to $U(2k)$; conversely, every fixed point free representation of $Q_\ell$ is conjugate to such a representation. The associated spherical space forms are the quaternion spherical space forms.

The representations $\{\rho_0, \kappa_1, \kappa_2, \kappa_3\}$ are real, the representations $\gamma_{2i}$ are real, and the representations $\gamma_{2i+1}$ are quaternion. We have:

$$RO(Q_\ell) = \text{span}_\mathbb{C} \{\rho_0, \kappa_1, \kappa_2, \kappa_3, 2\gamma_1, \gamma_2, ..., 2\gamma_{\ell/4 - 1}\},$$

$$RSp(Q_\ell) = \text{span}_\mathbb{C} \{2\rho_0, 2\kappa_1, 2\kappa_2, 2\kappa_3, \gamma_1, 2\gamma_2, ..., \gamma_{\ell/4 - 1}\}.$$ 

We define:

$$\Theta_1 (g) := \begin{cases} \frac{\ell}{4} & \text{if } g = \pm \mathcal{I}, \\ -2 & \text{if } g = \xi^{\ell/4} \mathcal{J}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\Theta_2 (g) := \begin{cases} \frac{\ell}{4} & \text{if } g = \pm \mathcal{I}, \\ -2 & \text{if } g = \xi^{2i+1} \mathcal{J}, \\ 0 & \text{otherwise}. \end{cases}$$

(2.2)

The two class functions $\Theta_i$ will be used to mirror in $RU(Q_\ell)$ the splitting of $BQ_\ell$ given in equation (1.2).

We identify virtual representations with the class functions they define henceforth. Let

$$\Delta := 2\rho_0 - \gamma_1; \quad \text{Tr} (\Delta) = \det (I - \gamma_1).$$

Lemma 2.1.

1. We have $\Theta_1 \in RO_0(Q_\ell)$ and $\Theta_2 \in RO_0(Q_\ell)$.
2. Let $c_{i} := \ell^{-1} \sum_{g \in Q_\ell - \{1\}} \Delta (g)^i$. We have $c_{0} = \frac{\ell-1}{2}$. If $i > 0$, then $c_{2i} \in \mathbb{Z}$ and $c_{2i-1} \in 2\mathbb{Z}$.

Proof: We use equation (2.2) to compute:

- For any $\ell$, $$(\Theta_1, \rho_0) = 0, \quad (\Theta_1, \gamma_1) = 0, \quad (\Theta_1, \gamma_{2i+1}) = 0,$$

$$ (\Theta_2, \rho_0) = 0, \quad (\Theta_2, \gamma_1) = 0, \quad (\Theta_2, \gamma_{2i+1}) = 0.$$ 

- For $\ell = 8$, $$(\Theta_1, \kappa_1) = -1, \quad (\Theta_1, \kappa_2) = 1, \quad (\Theta_1, \kappa_3) = 0,$$

$$ (\Theta_2, \kappa_1) = 0, \quad (\Theta_2, \kappa_2) = 1, \quad (\Theta_2, \kappa_3) = -1.$$ 

- For $\ell > 8$, $$(\Theta_1, \kappa_1) = 0, \quad (\Theta_1, \kappa_2) = 1, \quad (\Theta_1, \kappa_3) = 1,$$

$$ (\Theta_2, \kappa_1) = 1, \quad (\Theta_2, \kappa_2) = 1, \quad (\Theta_2, \kappa_3) = 0.$$
We use equation (2.1) to complete the proof of assertion (1):

$$\Theta_1 = \begin{cases} 
\mathsf{Tr}\{\kappa_2 - \kappa_1\} & \text{if } \ell = 8, \\
\mathsf{Tr}\{\kappa_2 + \kappa_3 + \sum_{1 \leq i < \ell/8} (-1)^i \gamma_{2i}\} & \text{if } \ell \geq 16,
\end{cases}$$

$$\Theta_2 = \begin{cases} 
\mathsf{Tr}\{\kappa_2 - \kappa_3\} & \text{if } \ell = 8, \\
\mathsf{Tr}\{\kappa_2 + \kappa_1 + \sum_{1 \leq i < \ell/8} (-1)^i \gamma_{2i}\} & \text{if } \ell \geq 16.
\end{cases}$$

The first identity of assertion (2) is immediate. Let $r > 0$. As $\mathsf{Tr}(\Delta^r)(1) = 0$,

$$c_r = t^{-1} \sum_{g \in Q_\ell-\{1\}} \mathsf{Tr}(\Delta^r)(g) = (\Delta^r, \rho_0) \in \mathbb{Z}.$$ 

If $r$ is odd, then $\gamma_r^*$ is quaternion so $\langle \gamma_1^r, \rho_0 \rangle \in 2\mathbb{Z}$. Since $\Delta^r \equiv \gamma_1^r \mod 2RU(Q_\ell)$, $\langle \Delta^r, \rho_0 \rangle \in 2\mathbb{Z}$ if $r$ is odd. \boxed{} 

3. The eta invariant, K theory, and bordism

Let $V$ be a smooth complex vector bundle over a compact Riemannian manifold $M$. Let $V$ be equipped with a unitary (Hermitian) inner product. Let

$$P : C^\infty(V) \to C^\infty(V)$$

be a self-adjoint elliptic first order partial differential operator. Let $\{\lambda_i\}$ denote the eigenvalues of $P$ repeated according to multiplicity. Let

$$\eta(s, P) := \sum_i \text{sign}(\lambda_i)|\lambda_i|^{-s}.$$ 

The series defining $\eta$ converges absolutely for $\Re(s) > 0$ to define a holomorphic function of $s$. This function has a meromorphic extension to the entire complex plane with isolated simple poles. The value $s = 0$ is regular and one defines

$$\eta(P) := \frac{1}{2i} \langle \eta(s, P) + \dim(\ker P) \rangle|_{s=0}$$

as a measure of the spectral asymmetry of $P$; we refer to [11] for further details concerning this invariant which was first introduced by [1] and which plays an important role in the index theorem for manifolds with boundary.

We say that $P$ is quaternion if $V$ has a quaternion structure and if the action of $P$ commutes with this structure. We say that $P$ is real if $V$ is the complexification of an underlying real vector bundle and if $P$ is the complexification of an underlying real operator.

**Lemma 3.1.** Let $M$ be a spin manifold of dimension $m$.

1. If $m \equiv 3, 4 \mod 8$, then the Dirac operator is quaternion.
2. If $m \equiv 7, 8 \mod 8$, then the Dirac operator is real.

**Proof:** Let $\text{Clif}(m)$ be the real Clifford algebra on $\mathbb{R}^m$. We have:

- $\text{Clif}(3) = \mathbb{H} \oplus \mathbb{H}$,
- $\text{Clif}(4) = M_2(\mathbb{H})$,
- $\text{Clif}(7) = M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$,
- $\text{Clif}(8) = M_{16}(\mathbb{R})$, and
- $\text{Clif}(m + 8) = \text{Clif}(m) \otimes_{\mathbb{R}} M_{16}(\mathbb{R})$.

Therefore, the fundamental spinor representation of $\text{Clif}(m)$ is quaternion if we have $m \equiv 3, 4 \mod 8$ and real if we have $m \equiv 7, 8 \mod 8$. The Lemma now follows. \boxed{} 

The following deformation result will be crucial to our investigations:

**Lemma 3.2.** Let $P_u$ be a smooth 1 parameter family of self-adjoint first order elliptic partial differential operators on a compact manifold $M$.

1. The reduction mod $\mathbb{Z}$ of $\eta(P_u)$ is a smooth $\mathbb{R}/\mathbb{Z}$ valued function.
2. The variation $\partial_u \eta(P_u)$ is locally computable.
3. If the operators $P_u$ are quaternion, then the reduction mod $2\mathbb{Z}$ of $\eta(P_u)$ is a smooth $\mathbb{R}/2\mathbb{Z}$ valued function.
Proof: We sketch the proof briefly and refer to [1] Theorem 1.13.2 for further details. Since \( \text{sign}(u) \) has an integer jump when \( u = 0 \), \( \eta(P_u) \) can have integer valued jumps at values of \( u \) where \( \dim(\ker(P_u)) > 0 \). However, in \( \mathbb{R}/\mathbb{Z} \), the jump disappears so the mod \( \mathbb{Z} \) reduction of \( \eta(P_u) \) is a smooth \( \mathbb{R}/\mathbb{Z} \) valued function of \( u \); one uses the pseudo-differential calculus to construct an approximate resolvant and to show that the variation \( \partial_u \eta(P_u) \) is locally computable. Assertions (1) and (2) then follow. If \( P_u \) is quaternion, then the eigenspaces of \( P_u \) inherit quaternion structures. Thus \( \dim(\ker(P_u)) \) is even so \( \eta(P_u) \) has twice integer jumps as eigenvalues cross the origin. Consequently the reduction mod \( 2\mathbb{Z} \) of \( \eta(P_u) \) is smooth and assertion (3) follows. \( \square \)

Let \( \tilde{M} \) be the universal cover of a connected manifold \( M \) and let \( \sigma \) be a representation of \( \pi_1(M) \) in \( U(k) \). The associated vector bundle is defined by:

\[
V^\sigma := \tilde{M} \times \mathbb{C}^k / \sim \quad \text{where we identify}
\]

\[
(\tilde{x}, z) \sim (g \cdot \tilde{x}, \sigma(g) \cdot z) \quad \text{for} \ g \in \pi_1(M), \ \tilde{x} \in \tilde{M}, \ \text{and} \ z \in \mathbb{C}^k.
\]

The trivial connection on \( \tilde{M} \times \mathbb{C}^k \) descends to define a flat connection on \( V^\sigma \). The transition functions of \( V^\sigma \) are locally constant; they are given by the representation \( \sigma \). Thus the bundle \( V^\sigma \) is said to be locally flat. Let \( P : C^\infty(V) \to C^\infty(V) \) be a self-adjoint elliptic first order operator on \( M \);

\[
P^\sigma : C^\infty(V \otimes V^\sigma) \to C^\infty(V \otimes V^\sigma)
\]

is a well defined operator which is locally isomorphic to \( k \) copies of \( P \). Define \( \eta^\sigma(P) := \eta(P^\sigma) \); we extend by linearity to \( \sigma \in \mathcal{RU}(\pi_1(M)) \).

This invariant is a homotopy invariant.

Lemma 3.3. Let \( P_u \) be a smooth \( 1 \) parameter family of elliptic first order self-adjoint partial differential operators over \( M \).

1. If \( \sigma \in \mathcal{RU}_0(\pi_1(M)) \), then the mod \( \mathbb{Z} \) reduction of \( \eta^\sigma(P_u) \) is independent of the parameter \( u \).

2. If all the operators \( P_u \) are quaternion and \( \sigma \in \mathcal{RO}_0(\pi_1(M)) \) or if all the operators \( P_u \) are real and \( \sigma \in \mathcal{RSP}_0(\pi_1(M)) \), then the mod \( 2\mathbb{Z} \) reduction of \( \eta(P_u, \sigma) \) is independent of the parameter \( u \).

Proof: If \( \sigma \) is a representation of \( \pi_1(M) \), then the mod \( \mathbb{Z} \) reduction of \( \eta^\sigma(P_u) \) is smooth a smooth function of \( u \) by Lemma 3.2. Since \( P^\sigma_u \) is locally isomorphic to \( \dim \sigma \) copies of \( P_u \) and since the variation is locally computable,

\[
\partial_u \eta^\sigma(P_u) = \dim \sigma \cdot \partial_u \eta(P_u).
\]

This formula continues to hold for virtual representations. In particular, if we have that \( \sigma \in \mathcal{RU}_0(\pi_1(M)) \), then \( \dim \sigma = 0 \) so \( \partial_u \eta^\sigma(P_u) = 0 \); (1) follows.

If \( P_u \) is quaternion and \( \sigma \) is real or if \( P_u \) is real and if \( \sigma \) is quaternion, then \( P^\sigma_u \) is quaternion and \( \eta^\sigma(P_u) \) is a smooth \( \mathbb{R}/2\mathbb{Z} \) valued function of \( u \). The same argument shows that \( \partial_u \eta^\sigma(P_u) = 0 \). \( \square \)

We can use the eta invariant to construct invariants of \( K \) theory. Let \( P : C^\infty(V) \to C^\infty(V) \) be a first order self-adjoint elliptic partial differential operator with leading symbol \( p \). Let \( W \) be a unitary vector bundle over \( M \). We use a partition of unity to construct a self-adjoint elliptic first order operator \( P^W \) on \( C^\infty(V \otimes W) \) with leading symbol \( p \otimes \text{id} \); this operator is not, of course, canonically defined.

We can extend the invariant \( \eta^\sigma \) to the the reduced unitary unitary and quaternion (symplectic) \( K \) theory groups \( \tilde{K}U \) and \( \tilde{K}Sp \).

Theorem 3.4. Let \( P \) be an elliptic self-adjoint first order partial differential operator. Let \( \sigma \in \mathcal{RU}_0(\pi_1(M)) \).
The map $W \to \eta^o(P^W)$ extends to a map $\eta^o_p : KU(M) \to \mathbb{R}/\mathbb{Z}$.

(2) Suppose that $P$ and $\sigma$ are both real or that $P$ and $\sigma$ are both quaternion. The map $W \to \eta^o(P^W)$ extends to a map $\eta^o_p : KSp(M) \to \mathbb{R}/\mathbb{Z}$.

Proof: Let $P^W$ and $\tilde{P}^W$ be two first order self-adjoint partial differential operators on $C^\infty(V \otimes W)$ with leading symbol $p \otimes \text{id}$. Set:

$$P_u := uP^W + (1 - u)\tilde{P}^W.$$  

This is a smooth 1 parameter family of first order self-adjoint partial differential operators. As the leading symbol of $P_u$ is $p \otimes \text{id}$, the operators $P_u$ are elliptic. By Lemma 3.3, $\eta^o_p(P_u) \in \mathbb{R}/\mathbb{Z}$ is independent of $u$. Consequently $\eta^o_p(W) := \eta^o(P^W) \in \mathbb{R}/\mathbb{Z}$ only depends on the isomorphism class of the bundle $W$. As the eta invariant is additive with respect to direct sums, we may extend $\eta^o_p$ to $KU(M)$ as an $\mathbb{R}/\mathbb{Z}$ valued invariant. Let $W$ be quaternion. By Lemma 3.3, $\eta^o_p(P_u) \in \mathbb{R}/\mathbb{Z}$ is independent of $u$ if both $P$ and $\sigma$ are real or if both $P$ and $\sigma$ are quaternion and thus $\eta^o_p$ extends to $KSp$ as an $\mathbb{R}/\mathbb{Z}$ valued invariant in this instance. $\square$

We can use the Atiyah-Patodi-Singer index theorem [1] to see that the eta invariant also defines bordism invariants. Let $G$ be a finite group. A $G$ structure $f$ on a connected manifold $M$ is a representation $f$ from $\pi_1(M)$ to $G$. Equivalently, $f$ can also be regarded as a map from $M$ to the classifying space $BG$. We consider tuples $(M, g, s, f)$ where $(M, g)$ is a compact Riemannian manifold with a spin structure $s$ and a $G$ structure $f$. We introduce the bordism relation $[[M, g, s, f]] = 0$ if there exists a compact manifold $N$ with boundary $M$ so that the structures $(g, s, f)$ extend over $N$; this induces an equivalence relation and the equivariant bordism groups $\text{MSpin}_*(BG)$ consists of bordism classes of these triples. Disjoint union defines the group structure.

Let $\text{MSpin}_* := \text{MSpin}_*(B\{1\})$ be defined by the trivial group. Cartesian product makes $\text{MSpin}_*(BG)$ into an $\text{MSpin}_*$ module. Let $\mathcal{F}$ be the forgetful homomorphism which forgets the $G$ structure $f$. The reduced bordism groups are then defined by:

$$\tilde{M}\text{Spin}_*(BG) := \ker(\mathcal{F}) : \text{MSpin}_*(BG) \to \text{MSpin}_*.$$  

Since the eta invariant vanishes on $\text{MSpin}_*$, we restrict henceforth to the reduced groups.

If $s$ is a spin structure on $(M, g)$, let $P_{(M,g,s)}$ be the associated Dirac operator. If $\sigma \in RU_0(G)$, then $f^*\sigma \in RU_0(\pi_1(M))$ and we may define:

$$\eta^o(M, g, s, f) := \eta^o(P_{(M,g,s)}).$$  

Theorem 3.5. Let $G$ be a finite group. Assume either that $m \equiv 3 \mod 8$ and that $\sigma \in RO_0(G)$ or that $m \equiv 7 \mod 8$ and that $\sigma \in RSp_0(G)$. Then the map $(M, g, s, f) \to \eta^o(M, g, s, f)$ extends to a map $\eta^o : \tilde{M}\text{Spin}_m(BG) \to \mathbb{R}/\mathbb{Z}$.

Proof: We sketch the proof and refer to [6] for further details. Suppose that $m \equiv 3 \mod 4$ and that $[[M, g, s, f]] = 0$ in $\text{MSpin}_m(BG)$. Then $M = dN$ where the spin and $G$ structures on $M$ extend over $N$. We may also extend the given Riemannian metric on $M$ to a Riemannian metric on $N$ which is product near the boundary.

Let $\sigma \in RU_0(G)$. The Dirac operator $P_{(M,g,s)}$ on $M$ is the tangential operator of the spin complex $\overline{Q}_{(N,g,s)}$ on $N$. We twist these operators by taking coefficients in the locally flat virtual bundle $Vf^*\sigma$.

Let $\tilde{A}(N, g, s)$ be the $A$-roof genus and let $ch(Vf^*\sigma)$ be the Chern character. By the Atiyah-Patodi-Singer index theorem [1]:

$$\text{index}(Q_{(N,g,s)}^f(\sigma)) = \int_N \tilde{A}(N, g, s) \wedge ch(Vf^*\sigma) + \eta(P_{(M,g,s)}).$$
Since $V^f\sigma$ is a virtual bundle of virtual dimension $0$ which admits a flat connection, the Chern character of $V^\sigma$ vanishes. Consequently:

$$\eta^\sigma(M, g, s, f) = \eta(P_{(M, g, s)}) = \text{index}(Q_{k, s, f}).$$

The dimension of $N$ is $m + 1$. We apply Lemma 3.1 to see that if $m \equiv 3$ mod $8$ and if $\sigma$ is real or if $m \equiv 7$ mod $8$ and if $\sigma$ is quaternion, then $Q_{k, s, f}$ is quaternion. Thus $\text{index}(Q_{k, s, f}) \in \mathbb{Z}$ so $\eta^\sigma(M, g, s)$ vanishes as an $\mathbb{R}/\mathbb{Z}$ valued invariant if $[(M, g, s, f)] = 0$ in $\text{MSpin}_m(BG)$. □

There is a geometric description of the real connective $K$ theory groups $\tilde{k}\text{ono}(BG)$ in terms of the spin bordism groups. Let $\mathbb{HP}^2$ be the quaternionic projective plane. Let $\tilde{T}_m(BG)$ be the subgroup of $\text{MSpin}_m(BG)$ consisting of bordism classes $[(E, g, s, f)]$ where $E$ is the total space of a geometrical $\mathbb{HP}^2$ spin fibration and where the $G$ structure on $E$ is induced from a corresponding $G$ structure on the base. The following theorem is a special case of a more general result [17]:

**Theorem 3.6.** Let $G$ be a finite group. There is a 2 local isomorphism between $k\text{ono}(BG)$ and $\text{MSpin}_m(BG)/\tilde{T}_m(BG)$.

We use Theorem 3.6 to draw the following consequence:

**Corollary 3.7.** Assume either that $m \equiv 3$ mod $8$ and $\sigma \in \text{RO}_0(Q_1)$ or that $m \equiv 7$ mod $8$ and $\sigma \in \text{RS}_0(Q_1)$. Then $\eta^\sigma$ extends to a map from $k\text{ono}(BQ_1)$ to $Q/\mathbb{Z}$.

**Proof:** If $[(E, s, f)] \in T_m(BQ_1)$, then $\eta^\sigma(P_{(E, g, s)}) = 0$; see [6] Lemma 4.3 or [13] Lemma 2.7.10 for details. Thus by Theorems 3.5 and Theorem 3.6, the eta invariant extends to $k\text{ono}(BQ_4)$. By [6] Theorem 2.4, $k\text{ono}_{4k-1}(BQ_1)$ is a finite group. Thus it is not necessary to localize at the prime 2 and the eta invariant takes values in $Q/\mathbb{Z}$. □

The eta invariant is combinatorially computable for spherical space forms. The following theorem follows from [8].

**Theorem 3.8.** Let $\tau : G \to SU(2k)$ be fixed point free, let $P$ be the Dirac operator on $M^{4k-1}(G, \tau)$, and let $\sigma \in \text{RU}_0(G)$. Then

$$\eta^\sigma(P) = \ell^{-1} \sum_{g \in G-\{1\}} \text{Tr}(\sigma(g)) \det(I - \tau(g))^{-1}.$$

4. THE GROUPS $\tilde{K}\text{Sp}(M^{4\nu-1}(Q_1, \nu \cdot \gamma_1))$

Let $\Delta = \det(I - \gamma_1) \in \text{RS}_0(Q_1)$. By equation (2.1):

$$\Delta^\nu \text{RS}_0(Q_1) \subset \text{RS}_0(Q_1) \quad \text{if } \nu \text{ is even},$$

$$\Delta^\nu \text{RO}(Q_1) \subset \text{RO}(Q_1) \quad \text{if } \nu \text{ is odd}.$$  

The following Theorem is well known - see, for example [10, 12]:

**Theorem 4.1.** Let $\tau : Q_1 \to U(2\nu)$ be fixed point free. Then

$$\tilde{K}\text{Sp}(M^{4\nu-1}(Q_1, \tau)) = \begin{cases} \text{RS}_0(Q_1)/\Delta^\nu \text{RS}_0(Q_1) & \text{if } \nu \text{ is even}, \\ \text{RS}_0(Q_1)/\Delta^\nu \text{RO}(Q_1) & \text{if } \nu \text{ is odd}. \end{cases}$$

By Theorem 4.1, the particular representation $\tau$ plays no role and we therefore set $\tau = \nu \cdot \gamma_1$. We use the eta invariant to study these groups. Let $\eta^\nu(W)$ be the invariant described in Theorem 3.4 for the Dirac operator $P$ on $M^{4\nu-1}(Q_1, \nu \cdot \gamma_1)$. We define:

$$\eta^\nu(W) := \begin{cases} (\eta^\nu_0, \eta^\nu_1, \eta^\nu_2, \ldots, \eta^\nu_{\Delta^2}, \eta^\nu_{2\Delta^2}, \ldots, \eta^\nu_{\Delta^\nu-2}, \eta^\nu_{2\Delta^\nu-2}, \eta^\nu_{\Delta^\nu-1})(W) & \text{if } \nu \text{ is even}, \\ (\eta^\nu_0, \eta^\nu_1, \eta^\nu_2, \ldots, \eta^\nu_{\Delta^2}, \eta^\nu_{2\Delta^2}, \ldots, \eta^\nu_{\Delta^\nu-2}, \eta^\nu_{2\Delta^\nu-2}, \eta^\nu_{\Delta^\nu-1})(W) & \text{if } \nu \text{ is odd}. \end{cases}$$

**Lemma 4.2.** Let $M := M^{4\nu-1}(Q_1, \nu \cdot \gamma_1)$. Then

$$\eta^\nu : \tilde{K}\text{Sp}(M) \to (Q/\mathbb{Z})^{\nu+1}.$$
Proof: We apply Lemma 3.1 and Theorem 3.4. We distinguish two cases:

(1) If \( \nu \) is even, then \( P \) is real. Thus \( \eta_\nu^r : \tilde{K}Sp(M) \to Q/2\mathbb{Z} \) for real \( \sigma \) and the Lemma follows as we have used the real representations \( \{ \Theta_1, \Theta_2, 2\Delta, 2\Delta^2, \ldots, 2\Delta^{\nu-2}, 2\Delta^{\nu-1} \} \) to define \( \eta_\nu^r \).

(2) If \( \nu \) is odd, then \( P \) is quaternion. Thus \( \eta_\nu^r : \tilde{K}Sp(M) \to Q/2\mathbb{Z} \) if \( \sigma \) is quaternion and the Lemma follows as we have used the quaternion representations \( \{ 2\Theta_1, 2\Theta_2, \Delta, 2\Delta^2, \ldots, 2\Delta^{\nu-2}, 2\Delta^{\nu-1} \} \) to define \( \eta_\nu^r \). \( \square \)

Let \( \varepsilon_{2i} = 2 \) and \( \varepsilon_{2i-1} = 1 \): \( \{ 2\Theta_1, 2\Theta_2, \Delta, 2\Delta^2, \ldots, \varepsilon_{\nu-1}\Delta^{\nu-1} \} \) are quaternion. In Lemma 2.1, we defined constants

\[
e_i := \ell^{-1} \sum_{g \in Q_{\ell-1}} \det(I - \gamma_1(g))^i.
\]

Since \( \Delta(g) = \det(I - \gamma_1(g)) \), we use Theorem 3.8 to compute:

\[
(4.1) \quad \eta_\nu^{\Delta^s}(\Theta_1) = \ell^{-1} \sum_{g \in Q_{\ell-1}} \Delta(g)^r \Delta(g)^{r+s} \Delta^{\nu-r} = e_{r+s-\nu}.
\]

Since \( \Theta_1 \) and \( \Theta_2 \) are supported on the elements of order 4 in \( Q_\ell \) and since \( \Delta(g) = 2 \) for such an element, we may use Theorem 3.8 and equation (2.2) to see:

\[(4.2) \quad \eta_\nu^{\Theta_1}(\Theta_1) = \ell^{-1} \sum g \in Q_{\ell-1} \frac{2}{\ell^2} \text{Tr}(\Theta_1(g)) = 0,
\]

\[
\eta_\nu^{\Theta_1}(\Theta_2) = \ell^{-1} \sum_{g \in Q_{\ell-1}} \frac{2}{\ell^2} \text{Tr}(\Theta_1(g)) \text{Tr}(\Theta_2(g)) = \ell^{-1} \frac{2}{\ell^2} \cdot \frac{2}{\ell^2}.
\]

We have \( \ell = 2^j \). We use equation (4.1), equation (4.2), and Lemma 2.1 to see:

\[
\eta_\nu = \begin{pmatrix}
2\Theta_1 \\
2\Theta_2 \\
\Delta \\
2\Delta^2 \\
\cdots \\
\varepsilon_{\nu-1}\Delta^{\nu-1}
\end{pmatrix} = \begin{pmatrix}
A_{\nu} \\
0
\end{pmatrix} \in M_{\nu+1}(Q/2\mathbb{Z})
\]

where \( A \) is the \( 2 \times 2 \) matrix given by

\[
A_{\nu} = \begin{pmatrix}
2^{1-\nu} & \frac{2^{j-3} + 1}{2^{j-3}} & \frac{2^{j-3} + 1}{2^{j-3}} \\
0 & 1 & 1
\end{pmatrix}
\]

if \( \nu \) is even

\[
A_{\nu} = \begin{pmatrix}
2^{2-\nu} & \frac{2^{j-3} + 1}{2^{j-3}} & \frac{2^{j-3} + 1}{2^{j-3}} \\
0 & 0 & 0
\end{pmatrix}
\]

if \( \nu \) is odd

and where \( B \) is the \( \nu - 1 \times \nu - 1 \) matrix given by:

\[
B_{\nu} = \begin{pmatrix}
2c_{2-\nu} & c_{3-\nu} & 2c_{4-\nu} & \cdots & 2c_{-2} & c_{-1} & 2c_0 \\
4c_{3-\nu} & 2c_{4-\nu} & 4c_{5-\nu} & \cdots & 4c_{-1} & 2c_0 & 0 \\
2c_{4-\nu} & c_{5-\nu} & 2c_{6-\nu} & \cdots & 2c_0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2c_{-2} & c_{-1} & 2c_0 & \cdots & 0 & 0 & 0 \\
4c_{-1} & 2c_0 & \cdots & 0 & 0 & 0 & 0 \\
2c_0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}
\]

if \( \nu \) is even.
Theorem 4.3. Let $B_\nu$ be the subgroup of $(\mathbb{Q}/2\mathbb{Z})^{\nu-1}$ spanned by the rows of the matrix $B_\nu$ defined above. Let $M = M^{4\nu-1}(Q_\ell, \nu \cdot \gamma_1)$. Then

$$\tilde{K}Sp(M) = \begin{cases} \mathbb{Z}_{2\nu} \oplus \mathbb{Z}_{2\nu} \oplus B_\nu & \text{if } \nu \text{ is even}, \\ \mathbb{Z}_{2\nu-1} \oplus \mathbb{Z}_{2\nu-1} \oplus B_\nu & \text{if } \nu \text{ is odd}. \end{cases}$$

Proof: Let $\mathcal{K}_\nu$ be the subspace of $\tilde{K}Sp(M)$ spanned by the virtual vector bundles defined by $\{2\Theta_1, 2\Theta_2, \Delta, 2\Delta^2, \ldots, \nu-1 \Delta^{\nu-1}\}$. It is then immediate from the definition and from the form of the matrix $A_\nu$ that

$$\tilde{\eta}_\nu(\mathcal{K}_\nu) = \begin{cases} \mathbb{Z}_{2\nu} \oplus \mathbb{Z}_{2\nu} \oplus B_\nu & \text{if } \nu \text{ is even}, \\ \mathbb{Z}_{2\nu-1} \oplus \mathbb{Z}_{2\nu-1} \oplus B_\nu & \text{if } \nu \text{ is odd}. \end{cases}$$

We use Lemma 2.1 to see $c_0 = \frac{\ell-1}{2}$. Thus $2c_0$ is an element of order $\ell$ in $\mathbb{Q}/2\mathbb{Z}$. We use the diagonal nature of matrix $B_\nu$ to see that:

$$|\tilde{\eta}_\nu(\mathcal{K}_\nu)| \geq \begin{cases} 4^\nu \ell^{\nu-1} & \text{if } \nu \text{ is even}, \\ 4^{\nu-1} \ell^{\nu-1} & \text{if } \nu \text{ is odd}. \end{cases}$$

The $E_2$ term in the Atiyah-Hirzebruch spectral sequence for the $K$ theory groups $\tilde{K}Sp^v(M)$ is

$$\oplus_{v+v=0} \tilde{H}^u(M; \text{KSp}^v(pt)).$$

We take $w = 0$ and study the reduced groups to obtain the estimate:

$$|\tilde{K}Sp(M)| \leq |\oplus_{u+v=0} \tilde{H}^u(M; \text{KSp}^v(pt))|.$$ 

We have that:

$$\begin{align*}
\text{KSp}^v(pt) &= \mathbb{Z} & \text{if } v \equiv 0, 4 \mod 8, \\
\text{KSp}^v(pt) &= \mathbb{Z}_2 & \text{if } v \equiv -5, -6 \mod 8, \\
\text{KSp}^v(pt) &= 0 & \text{otherwise,} \\
\tilde{H}^u(M; \mathbb{Z}) &= \mathbb{Z}_{\ell} & \text{if } u \equiv 0, 4 \mod 8, \ u < 4\nu - 1, \\
\tilde{H}^u(M; \mathbb{Z}_2) &= \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } u \equiv 1, 2, 5, 6 \mod 8, \ u \leq 4\nu - 1. 
\end{align*}$$

Equations (4.5) and (4.6) then imply:

$$|\tilde{K}Sp(M)| \leq \begin{cases} 4^\nu \ell^{\nu-1} & \text{if } \nu \text{ is even}, \\ 4^{\nu-1} \ell^{\nu-1} & \text{if } \nu \text{ is odd}. \end{cases}$$

Thus equations (4.4) and (4.7) show $|\tilde{K}Sp(M)| \leq |\tilde{\eta}_\nu(\mathcal{K}_\nu)|$. As the opposite inequality is immediate, we have

$$\tilde{\eta}_\nu(\mathcal{K}_\nu) = \mathcal{K}_\nu = \tilde{K}Sp(M).$$

The Theorem now follows from equation (4.3). $\Box$

5. THE GROUPS $\tilde{k}_Q(4k-1)(BQ_\ell)$

Let $x = (M, g, s, f)$ where $s$ is a spin structure and $f$ is a $G$ structure on a compact Riemannian manifold $(M, g)$ of dimension $4k - 1$. Let $\eta^v(x)$ be the eta invariant of the associated Dirac operator with coefficients in $f^* \sigma$. We reverse the parities of the invariant defined in the previous section to define:

$$\tilde{\eta}_\nu(x) := \begin{cases} (\eta^{2\Theta_1}(x), \eta^{2\Theta_2}(x), \eta^\Delta(x), \eta^{2\Delta^2}(x), \ldots, \eta^{2\Delta^k}(x)) & \text{if even} \\
(\eta^{\Theta_1}(x), \eta^{\Theta_2}(x), \eta^\Delta(x), \eta^{2\Delta^2}(x), \ldots, \eta^{2\Delta^k}(x)) & \text{if odd}. \end{cases}$$
We have used real representations if $k$ is odd and quaternion representations if $k$ is even. Therefore, by Corollary 3.7, $\tilde{\eta}_k$ extends to:

$$\tilde{\eta}_k : \tilde{k}_0^{4k-1}(BG) \to (Q/2Z)^{k+2}.$$ 

The group $Q_\ell$ has 3 non-conjugate elements of order 4: $\{\mathcal{I}, \mathcal{J}, \mathcal{J}^\prime\}$ which generate the 3 non-conjugate subgroups $\{\mathcal{I}, \mathcal{J}, \mathcal{J}^\prime\}$ of order 4. The representation $\gamma_1$ restricts to a fixed point free representation of any subgroup of $Q_\ell$. We define the following spherical space forms:

$$M^k_{\mathcal{I}^1} := M^{4k-1}_\mathcal{I}(Q_\ell, k\gamma_1), \quad M^k_{\mathcal{J}^1} := M^{4k-1}_\mathcal{J}(Q_\ell, k\gamma_1)$$

$$M^k_{\mathcal{J}^1} := M^{4k-1}_\mathcal{J}(Q_\ell, k\gamma_1).$$

Give the lens spaces $M^k_{\mathcal{I}^1}$ the $Q_\ell$ structure induced by the natural inclusion $\langle g \rangle \subset Q_\ell$. We project into the reduced group $\tilde{M}Spin^{4k-1}(Q_\ell)$; this does not affect the eta invariant as $\eta^\sigma(MSpin_\gamma(pt)) = 0$. Let $i > 0$. By Theorem 3.8:

$$(\eta^{\theta_1}, \eta^{\theta_2}, \eta^{\Delta^1})(M^k_{\mathcal{I}^1} - M^k_{\mathcal{J}^1}) = \begin{cases} 2^{-k}(2, 1, 0) & \text{if } \ell = 8, \\
2^{-k}(1, 0, 0) & \text{if } \ell > 8,
\end{cases}$$

$$(\eta^{\theta_1}, \eta^{\theta_2}, \eta^{\Delta^1})(M^k_{\mathcal{J}^1} - M^k_{\mathcal{J}^1}) = \begin{cases} 2^{-k}(1, 2, 0) & \text{if } \ell = 8, \\
2^{-k}(0, 1, 0) & \text{if } \ell > 8,
\end{cases}$$

$$(\eta^{\theta_1}, \eta^{\theta_2}, \eta^{\Delta^1})(M^k_{\mathcal{J}^1}) = (0, 0, c_{i-k}) \text{ any } \ell.$$ 

Let $K^4$ be a spin manifold with $\tilde{A}(K^4) = 2$ and let $B^8$ be a spin manifold with $\tilde{A}(B^8) = 1$. Let $Z^{4k-4} := K^4 \times B^{4k-8}$ and $Z^{4k} = (B^8)^k$. Standard product formulas [10] then show

$$\eta^\sigma(M^{4k-1} \times Z^4) = \eta^\sigma(M^{4k-1})\tilde{A}(Z^4) = \begin{cases} 2\eta^\sigma(M^{4k-1}) & \text{if } j \text{ is odd}, \\
\eta^\sigma(M^{4k-1}) & \text{if } j \text{ is even}.
\end{cases}$$

Let $B_\nu$ and $B_\nu^\prime$ be as defined in Section 4. There is a dimension shift involved as we must set $\nu = k+1$. We use the same arguments as those given previously to see

$$\tilde{\eta}_k \begin{pmatrix} M^k_{\mathcal{I}^1} - M^k_{\mathcal{J}^1} \\ M^k_{\mathcal{J}^1} - M^k_{\mathcal{J}^1} \\ M^k_{\mathcal{J}^1} - M^k_{\mathcal{J}^1} \\ M^k_{\mathcal{J}^1} - M^k_{\mathcal{J}^1} \\ M^k_{\mathcal{J}^1} - M^k_{\mathcal{J}^1} \end{pmatrix} = \begin{pmatrix} C_k & 0 \\ 0 & B_{k+1} \end{pmatrix} \in M_{k+2}(Q/2Z)$$

where $C_k$ is the $2 \times 2$ matrix given by

$$C_k = \begin{cases} 2^{-k} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } \ell = 8 \text{ and } k \text{ is even}, \\
2^{-k} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } \ell > 8 \text{ and } k \text{ is even}, \\
2^{-k} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } \ell = 8 \text{ and } k \text{ is odd}, \\
2^{-k} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } \ell > 8 \text{ and } k \text{ is odd},
\end{cases}$$

Theorem 1.2 will follow from Theorem 4.3 and from the following:

**Theorem 5.1.** We have

$$\tilde{k}_0^{4k-1}(BQ_\ell) = \begin{cases} \mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k} \oplus B_{k+1} & \text{if } k \text{ is even}, \\
\mathbb{Z}_{2^k+1} \oplus \mathbb{Z}_{2^k+1} \oplus B_{k+1} & \text{if } k \text{ is odd}.
\end{cases}$$
Remark 5.2. Let

We use the same argument used to prove Theorem 4.3. Let

Proof: We use the same argument used to prove Theorem 4.3. Let

We then have that

By Lemma 2.1 we have

We may use equation (1.2) to decompose:

We use [6] Theorem 2.4 see:

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