On the volume of a six-dimensional polytope

A. Felikson, P. Tumarkin

Abstract. This note is a comment to the paper [7]. That paper concerns with the projective surface $S$ in $\mathbb{P}^3$ defined by the equation $x_1x_2x_3 = x_4^3$. It is shown there that the evaluation of the leading term of the asymptotic formula for the number of rational points of bounded height in $S(\mathbb{Q})$ is equivalent to the evaluation of the volume of some 6-dimensional polytope $\mathcal{P}$. The volume of $\mathcal{P}$ is known from several papers; we calculate this volume by elementary method using symmetry of the polytope $\mathcal{P}$. We also discuss a combinatorial structure of $\mathcal{P}$.

1 Introduction

Consider the projective surface $S$ in $\mathbb{P}^3$ defined by the equation

$$x_1x_2x_3 = x_4^3.$$ 

A few years ago several authors, [2], [4], [5], [7], [8], obtained an asymptotic formula for the number of rational points of bounded height in $S(\mathbb{Q})$, as a simple illustration of a rather general conjecture discussed in [1], [2], [8]. To explicitly evaluate the leading term of the asymptotic formula obtained in [2], [7], [8] one must calculate the volume $\text{Vol}(\mathcal{P})$ of the polytope $\mathcal{P} \in \mathbb{R}^6$ determined by the inequalities

$$x_{12} + x_{13} + 2(x_{21} + x_{31}) \leq 1,$$
$$x_{21} + x_{23} + 2(x_{12} + x_{32}) \leq 1,$$
$$x_{31} + x_{32} + 2(x_{13} + x_{23}) \leq 1,$$

$$x_{ij} \geq 0,$$

where

$$(x_{12}, x_{23}, x_{31}, x_{13}, x_{32}, x_{21})$$

are Cartesian coordinates in $\mathbb{R}^6$.

As it is shown in papers [2], [7] and [8],

$$\text{Vol}(\mathcal{P}) = \frac{1}{4 \cdot 6!} = \frac{1}{2880}. \quad (1)$$

The goal of this note is to prove formula (1) by a direct elementary method, using symmetry of the polytope $\mathcal{P}$. We dissect $\mathcal{P}$ into 6 congruent parts and decompose one of these parts into 6 simplices. Then we show that the volumes
of these simplices sum up to $\frac{1}{24}\text{Vol}(\Delta)$, where $\Delta$ is the standard six-dimensional simplex determined by the inequalities

$$x_{12} + x_{23} + x_{31} + x_{13} + x_{32} + x_{21} \leq 1, \quad x_{ij} \geq 0, \quad i \neq j, \quad 1 \leq i, j \leq 3.$$  

We use the notation and terminology of [6]. In particular, given a subset $X$ of $\mathbb{R}^n$, $n \in \mathbb{N}$, we write $\text{conv } X$ and $\partial X$ for the convex hull and the boundary of $X$ respectively.

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## 2 Volume of a rational polytope

Let $E^n$ be the $n$-dimensional Euclidean space with Cartesian coordinates. Let $v_0, v_1, \ldots, v_k$ be rational points in $E^n$, i.e. the coordinates $(v_{i1}, \ldots, v_{in})$ of $v_i$ are rational numbers for all $i = 0, \ldots, k$. Suppose that the points $v_0, v_1, \ldots, v_k$ are in the convex position, that is no of these points belongs to the convex hull of the others. Suppose that $k \geq n$. Then the set $\text{conv}\{v_0, v_1, \ldots, v_k\}$ is an $n$-dimensional polytope with non-zero volume $\text{Vol}(v_0, \ldots, v_k)$. In this section we describe how to find $\text{Vol}(v_0, \ldots, v_k)$.

Denote by $\text{det}(v_1, \ldots, v_n)$ the determinant

$$\text{det}(v_1, \ldots, v_n) = \begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{vmatrix}$$

If $k = n$, the polytope $\text{conv}\{v_0, v_1, \ldots, v_k\}$ is a simplex (may be degenerate). Recall that

$$\text{Vol}(v_0, v_1, \ldots, v_n) = \frac{1}{n!} \cdot \text{det}(v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0)$$

(see for example [3], Chapter 9).

**Definition.** A triangulation of a polytope is a decomposition of this polytope into finite number of simplices $T_1, \ldots, T_r$, such that if $T_i \cap T_j \neq \emptyset$, then $T_i \cap T_j$ is a face of both $T_i$ and $T_j$.

The formula above reduces the problem to the question of triangulation of the polytope $\text{conv}\{v_0, v_1, \ldots, v_k\}$.

Lemmas 1 through 3 are either well-known or elementary.

**Lemma 1.** Let $v_i$, $1 \leq i \leq n$ be $n$ points in general position in $E^n$ (i.e. no $k$ of these points belong to $(k - 2)$-plane). Let $H$ be the hyperplane spanned by
Consider any \( n \) points \( v_1, \ldots, v_n \). Let \( a \) and \( b \) be two points in \( \mathbb{E}^n \). The points \( a \) and \( b \) belong to different open half-spaces with respect to \( H \) if and only if 
\[
\det(v_1 - a, v_2 - a, \ldots, v_n - a) \cdot \det(v_1 - b, v_2 - b, \ldots, v_n - b) < 0.
\]

**Lemma 2.** Let 
\[
S := \{Q, Q_i \mid 1 \leq i \leq m\}
\]
be a set of \( m + 1 \) polytopes satisfying the following conditions:

a) \( Q_i \subseteq Q \) for \( 1 \leq i \leq m \);

b) each facet of \( Q_i \) is a facet of one the polytopes belonging to the set 
\( Q \setminus \{Q_i\} \) for \( 1 \leq i \leq m \).

Then 
\[
Q = \bigcup_{i=1}^{m} Q_i
\]

**Definition.** Let \( P \) be a polytope in \( \mathbb{E}^n \). A point \( y \in P \) is visual from the point \( x \notin P \), if \( [x, y] \cap P = y \) (where \( [x, y] \) is the segment \( \{\lambda a + (1 - \lambda) b \mid 0 \leq \lambda \leq 1\} \). A facet \( f \) of \( P \) is visual from the point \( x \notin P \), if each point of \( f \) is visual from \( x \).

If \( P \) is convex any facet containing at least one inner visual point is visual.

**Lemma 3.** Let \( P \) be a convex polytope, and \( x \notin P \) be a point. Any point \( y \in P \) visual from \( x \) is contained in at least one visual facet of \( P \).  

**Theorem 1.** Let \( T = \bigcup_{i=1}^{m} T_i \) be a triangulation of \( \text{conv}\{v_1, \ldots, v_k\} \). Let \( v_0 \notin \text{conv}\{v_1, \ldots, v_k\} \). Let \( f_1, \ldots, f_r \) be all facets of \( T_1, \ldots, T_m \) visual from \( v_0 \). Denote \( T_{m+j} := \text{conv} v_0 \cup f_j \), \( 1 \leq j \leq r \). Then 
\[
T' = \bigcup_{i=1}^{m+r} T_i
\]
is a triangulation of \( \text{conv}(v_0, v_1, \ldots, v_k) \).  

The proof is obvious.

Theorem \( \text{I} \) suggests an algorithm of triangulation of \( \text{conv}\{v_0, v_1, \ldots, v_k\} \): Consider any \( n + 1 \) non-coplanar points (without loss of generality we may assume that these points are \( v_0, v_1, \ldots, v_n \)). To check that these points are non-coplanar make sure that \( \det(v_1 - v_0, \ldots, v_n - v_0) \neq 0 \). Then \( T_1 = \text{conv}\{v_0, v_1, \ldots, v_n\} \) is a simplex and \( T_1 \) is triangulated by itself. Use Theorem \( \text{I} \) to successively triangulate \( \text{conv}\{v_0, v_1, \ldots, v_i\} \) for \( i = n + 1, n + 2, \ldots, k \).

### 3 Combinatorial structure of the polytope \( \mathcal{P} \)

Denote by \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) the hyperplanes
\[
\begin{align*}
x_{12} + x_{13} + 2(x_{21} + x_{31}) &= 1, \\
x_{21} + x_{23} + 2(x_{12} + x_{32}) &= 1, \\
x_{31} + x_{32} + 2(x_{13} + x_{23}) &= 1
\end{align*}
\]
respectively. Denote by \( \pi_{ij} \) the hyperplane \( x_{ij} = 0 \) \( (i, j = 1, 2, 3, \ i \neq j) \). Denote by \( \sigma \) the hyperplane \( x_{12} + x_{23} + x_{31} + x_{13} + x_{32} + x_{21} = 1 \).
It is easy to see, that if \( X \in \sigma \) and \( X \in P \) then \( X \in \alpha_i \) for \( i = 1, 2, 3 \). Therefore, \( \sigma \) is not a facet of \( P \). Thus, \( P \) is bounded by \( \alpha_i \), \( i = 1, 2, 3 \) and six facets \( \pi_{ij} \). Any vertex of \( P \) is an intersection of at least 6 facets. Thus, any vertex of \( P \) is a subject to at least 6 equations under consideration. A straightforward calculation shows that \( P \) is a convex hull of 21 points. We list these points in Table 3. The right column of the table shows, to which of the hyperplanes \( \alpha_j \), \( 1 \leq j \leq 3 \), if any, the corresponding point belongs. Note, that the indices indicate non-zero coordinates.

We present below the Gale diagram of \( P \), describing the combinatorial type of the polytope (see Figure 1).

\( P \) is a 6-dimensional polytope with 9 facets. The combinatorics of a convex \( n \)-polytope with \( n + 3 \) facets can be described by 2-dimensional Gale diagram (see [6]). This consists of \( n + 3 \) points \( a_1, \ldots, a_{n+3} \) of unit circle in \( \mathbb{R}^2 \) centered at the origin. The combinatorial type of a convex polytope can be read off from the Gale diagram in the following way. Each point \( a_i \) corresponds to the facet \( f_i \) of \( P \). For any subset \( J \) of the set of facets of \( P \) the intersection of facets \( \{ f_j | j \in J \} \) is a face of \( P \) if and only if the origin is contained in the set \( \text{conv}\{a_j | j \notin J \} \).

![Gale diagram of \( P \).](image)

### 4 Triangulation of the polytope \( P \)

In this section we use the group of symmetries of \( P \) to find a nice triangulation of \( P \) containing relatively small number of simplices.

Let \( P_j := \text{conv}\{O \cup \alpha_j\}, \ 1 \leq j \leq 3 \).
### Table 1: Vertices of \( P \).

| Notation | Coordinates \( (x_{12}, x_{23}, x_{31}, x_{13}, x_{32}, x_{21}) \) | \( \alpha_1 \) | \( \alpha_2 \) | \( \alpha_3 \) |
|----------|-------------------------------------------------|----------------|-------------|-------------|
| \( O \)  | \((0, 0, 0, 0, 0, 0)\)                          | \(-\)          | \(-\)       | \(-\)       |
| \( Q_{23}^{1} \) | \((0, \frac{1}{2}, 0, 0, 0, \frac{1}{2})\)       | \(+\)          | \(+\)       | \(+\)       |
| \( Q_{32}^{1} \) | \((0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0)\)       | \(+\)          | \(+\)       | \(+\)       |
| \( Q_{13}^{12} \) | \((\frac{1}{2}, 0, 0, \frac{1}{2}, 0, \frac{1}{2})\) | \(+\)          | \(+\)       | \(+\)       |
| \( P_{12} \) | \((\frac{1}{2}, 0, 0, 0, 0, 0)\)                | \(-\)          | \(+\)       | \(-\)       |
| \( P_{31} \) | \((0, 0, \frac{1}{2}, 0, 0, 0)\)                | \(+\)          | \(-\)       | \(-\)       |
| \( P_{13} \) | \((0, 0, 0, \frac{1}{2}, 0, 0)\)                | \(-\)          | \(-\)       | \(+\)       |
| \( P_{32} \) | \((0, 0, 0, 0, \frac{1}{2}, 0)\)                | \(-\)          | \(-\)       | \(+\)       |
| \( P_{21} \) | \((0, 0, 0, 0, 0, \frac{1}{2})\)                | \(+\)          | \(-\)       | \(-\)       |
| \( R_{1} \) | \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)\) | \(+\)          | \(+\)       | \(+\)       |
| \( R_{2} \) | \((0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})\) | \(+\)          | \(+\)       | \(+\)       |
| \( T_{23}^{23} \) | \((\frac{1}{3}, 0, 0, 0, 0, \frac{1}{3})\)       | \(+\)          | \(+\)       | \(+\)       |
| \( T_{21}^{32} \) | \((0, \frac{1}{3}, 0, 0, \frac{1}{3}, 0)\)       | \(-\)          | \(+\)       | \(+\)       |
| \( T_{31}^{13} \) | \((0, 0, \frac{1}{3}, \frac{1}{3}, 0, 0)\)       | \(+\)          | \(+\)       | \(+\)       |
| \( V_{21}^{32} \) | \((0, 0, 0, 0, \frac{1}{2}, \frac{1}{2})\)       | \(+\)          | \(+\)       | \(+\)       |
| \( V_{12}^{31} \) | \((\frac{1}{2}, 0, \frac{1}{3}, 0, 0, 0)\)       | \(+\)          | \(+\)       | \(+\)       |
| \( V_{13}^{23} \) | \((0, 0, 0, \frac{1}{2}, 0, \frac{1}{2})\)       | \(+\)          | \(+\)       | \(+\)       |
| \( V_{13}^{32} \) | \((0, 0, 0, \frac{1}{2}, \frac{1}{2}, 0)\)       | \(+\)          | \(+\)       | \(+\)       |
| \( V_{23}^{12} \) | \((\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0)\)       | \(+\)          | \(+\)       | \(+\)       |

**Lemma 4.** \( \text{Vol}(P_{1}) = \text{Vol}(P_{2}) = \text{Vol}(P_{3}) = \frac{1}{3} \text{Vol}(P) \).  

*Proof.* A permutation \( \varphi \) of coordinates

\[ \varphi = (x_{12}x_{23}x_{31})(x_{13}x_{21}x_{32}) \]
induces an order 3 orthogonal transformation $I_\varphi : \mathbb{R}^6 \to \mathbb{R}^6$, such that $I_\varphi(P_1) = P_2$, $I_\varphi(P_2) = P_3$ and $I_\varphi(P_3) = P_1$. Therefore,

$$Vol(P_1) = Vol(P_2) = Vol(P_3).$$

By construction, the set $\{P_i | 1 \leq i \leq 3\}$ satisfies the conditions of Lemma 2, hence,

$$P_1 \cup P_2 \cup P_3 = P.$$  

Moreover, if $1 \leq i < j \leq 3$, then $P_i \cap P_j$ is a common facet of $P_i$ and $P_j$; consequently,

$$Vol(P_i \cap P_j) = 0, \ 1 \leq i < j \leq 3.$$  

The lemma follows from (2)–(4).

In view of Lemma 4, it suffices to calculate the volume of $P_1$. By construction, $P_1$ is a convex hull of the points

$$Q_{23}^{21}, T_{12}^{21}, P_{21}, R_1, V_{21}^{32}, V_{13}^{21},$$

$$Q_{32}^{31}, T_{13}^{31}, P_{31}, R_2, V_{31}^{23}, V_{12}^{31},$$

$$Q_{13}^{12}, O.$$  

Note that all vertices of $P_1$ except for $Q_{13}^{12}$ and $O$ split into pairs of similar vertices denoted by one letter with different indices. This will be used in the proof of Lemma 5.

Denote by $\theta_1$ and $\theta_2$ respectively the hyperplanes

$$x_{12} - 2x_{23} + x_{31} = x_{13} + x_{32} - 2x_{21}$$

and

$$x_{12} + x_{23} - 2x_{31} = x_{13} - 2x_{32} + x_{21}.$$  

It is easy to check that $P_1$ has the following facets: $\pi_{ij}, \alpha_1, \theta_1$ and $\theta_2$.

Denote by $\delta$ the hyperplane

$$x_{12} + x_{23} + x_{31} = x_{13} + x_{32} + x_{21}.$$  

**Lemma 5.** $\delta$ divides $P_1$ into two congruent parts.

**Proof.** It is easy to check that $P_{31}P_{21}$ is the only edge of $\alpha_1$ which is intersected by $\delta$ transversally.

Let $K_{21}^{31} := \delta \cap P_{31}P_{21} = (0, 0, \frac{1}{4}, 0, 0, \frac{1}{4})$. Then $\delta$ divides $P_1$ into two polytopes $P_1^1$ and $P_1^2$, where $P_1^1$ is a convex hull of

$$P_{31}, R_1, V_{31}^{23}, V_{12}^{31} \text{ and } O, Q_{23}^{21}, Q_{32}^{31}, Q_{13}^{12}, T_{12}^{21}, T_{13}^{31}, K_{21}^{31}$$

and $P_1^2$ is a convex hull of

$$P_{21}, R_2, V_{21}^{32}, V_{13}^{21} \text{ and } O, Q_{23}^{21}, Q_{32}^{31}, Q_{13}^{12}, T_{12}^{21}, T_{13}^{31}, K_{21}^{31}.$$  

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Table 2: Triangulation of $\mathcal{P}_1^1$.

|        | Vertices                                                                 | Neighbors                                                                 | Volume             |
|--------|--------------------------------------------------------------------------|---------------------------------------------------------------------------|--------------------|
| $\Delta_1$ | $O, P_31, Q^{31}_{12}, Q^{31}_{23}, T^{13}_{31}, T^{12}_{21}, K^{31}_{21}$ | $\alpha_1, \delta, \pi_{32}, \pi_{23}, \pi_{13}, \pi_{12}, \Delta_2$ | $\frac{1}{3} \cdot \frac{2}{3^7} \frac{1}{6!}$ |
| $\Delta_2$ | $O, P_31, Q^{31}_{12}, Q^{31}_{23}, T^{13}_{31}, T^{12}_{21}, Q^{13}_{12}$ | $\alpha_1, \delta, \pi_{32}, \pi_{23}, \Delta_3, \Delta_4, \Delta_1$ | $\frac{1}{3} \cdot \frac{1}{2^5} \frac{1}{6!}$ |
| $\Delta_3$ | $O, P_31, Q^{31}_{12}, Q^{21}_{23}, T^{13}_{31}, T^{12}_{21}, Q^{13}_{23}$ | $\alpha_1, \theta_2, \pi_{32}, \Delta_5, \Delta_2, \Delta_4, \pi_{13}$ | $\frac{1}{3} \cdot \frac{2}{3^7} \frac{1}{6!}$ |
| $\Delta_4$ | $O, P_31, Q^{31}_{12}, Q^{21}_{23}, T^{13}_{31}, T^{12}_{21}, Q^{13}_{23}$ | $\alpha_1, \theta_1, \pi_{32}, \pi_{21}, \Delta_3, \Delta_2, \Delta_6$ | $\frac{1}{3} \cdot \frac{1}{2^5} \frac{1}{6!}$ |
| $\Delta_5$ | $O, P_31, Q^{31}_{12}, V^{31}_{12}, T^{13}_{31}, T^{12}_{21}, Q^{13}_{23}$ | $\alpha_1, \theta_2, \pi_{32}, \Delta_3, \pi_{23}, \pi_{21}, \pi_{13}$ | $\frac{1}{3} \cdot \frac{1}{2^5} \frac{1}{6!}$ |
| $\Delta_6$ | $O, P_31, Q^{31}_{12}, V^{31}_{12}, T^{13}_{31}, R_1, V^{23}_{31}$       | $\alpha_1, \theta_1, \pi_{32}, \pi_{21}, \pi_{13}, \pi_{12}, \Delta_4$ | $\frac{1}{3} \cdot \frac{2}{3^7} \frac{1}{6!}$ |

Note that $\mathcal{P}_1^1$ and $\mathcal{P}_2^1$ differ by 4 vertices only.

Consider an involution $I_\psi$ induced by the following permutation of coordinates: $\psi = (x_{12} x_{13})(x_{21} x_{23})(x_{32} x_{23})$. Clearly, $I_\psi$ preserves $\alpha_1$ and $\delta$. More precisely, $I_\psi$ fixes the points $Q^{13}_{12}$ and $O$ and interchanges six pairs of other points spanning $\mathcal{P}_1$. Thus, $I_\psi$ interchanges $\mathcal{P}_1^1$ and $\mathcal{P}_2^1$. Since $I_\psi$ is an isometry, $\mathcal{P}_1^1$ is congruent to $\mathcal{P}_2^1$.

Lemmas 4 and 5 show that the permutation group $S_3$ acts on $\mathcal{P}$. Moreover, $S_3$ is a group of symmetries of $\mathcal{P}$ and $\mathcal{P}_1^1 = \mathcal{P}/S_3$.

Lemma 3 implies that $Vol(\mathcal{P}_1^1) = Vol(\mathcal{P}_2^1)$. By Lemmas 4 and 5 it is sufficient to triangulate $\mathcal{P}_1^1$. This polytope is bounded by $\pi_{ij}$, $\alpha_1$, $\theta_1$, $\theta_2$ and $\delta$.

We decompose $\mathcal{P}_1^1$ into 6 simplices $\Delta_1, \ldots, \Delta_6$ (see Table 2). We represent these simplices by their vertices. Each facet of $\Delta_1$ belongs either to some $\Delta_j$ ($i \neq j$) or to one of the hyperplanes bounding $\mathcal{P}_1^1$. Note, that a facet $f$ of a simplex corresponds to the vertex $v(f)$ opposite to this facet and vice versa. For each vertex $v(f)$ of $\Delta_i$ we indicate the simplex $\Delta_j$ or the facet of $\mathcal{P}_1^1$ containing the facet $f$ (see the third column).

**Theorem 2.** 1) $\mathcal{P}_1^1$ is triangulated by $\Delta_1, \ldots, \Delta_6$.
2) $Vol(\mathcal{P}) = \frac{1}{3^7} = \frac{1}{3} Vol(\Delta)$. 

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Proof. The set vertices of $\Delta_1, \ldots, \Delta_6$ coincides with the set of vertices of $\mathcal{P}_1$. Hence, the union of $\Delta_i$ ($i = 1, \ldots, 6$) lies inside $\mathcal{P}_1$. The intersection $\Delta_i \cap \Delta_j$ is a face of $\Delta_i$ and $\Delta_j$. It is left to show that any point of $\mathcal{P}_1$ belongs to some of $\Delta_i$. The third column of Table 4 shows that each facet of $\Delta_i$, $i = 1, \ldots, 6$ either belongs to $\partial \mathcal{P}_1$ or to some $\partial \Delta_j$, $j \neq i$. By Lemma 2, $\mathcal{P}_1 = \bigcup_{i=1}^{6} \Delta_i$ and the first statement is proved.

By lemmas 4 and 5

$$Vol(\mathcal{P}) = 6 \cdot Vol(\mathcal{P}_1) = 6 \sum_{i=1}^{6} Vol(\Delta_i).$$

In view of triangulation shown in Table 4 this equals to $\frac{1}{4} \cdot \frac{1}{6}$. This proves the first equality. The second equality is trivial. 

References

[1] V. V. Batyrev and Yu. I. Manin, Sur le nombre des points rationnelles de hauteur borné des variétés algébriques. Math. Ann., 286 (1990), 27–43.

[2] V. V. Batyrev and Yu. Tschinkel, Tamagawa numbers of polarized algebraic varieties. Astérisque, 251 (1998), 299–340.

[3] M. Berger, Geometry I, Springer-Verlag Berlin Heidelberg, 1987.

[4] R. de la Bretèche, Sur le nombre des points de hauteur borné d’une certaine surface cubique singulière. Astérisque, 251 (1998), 51–77.

[5] E. Fouvry, Sur la hauteur des points d’une certaine surface cubique singulière. Astérisque, 251 (1998), 31–49.

[6] B. Grünbaum, Convex Polytopes. Graduate Texts in Mathematics, Springer-Verlag, 2003.

[7] D. R. Heath-Brown and B. Z. Moroz, The density of rational points on the cubic surface $X^3_0 = X_1X_2X_3$. Math. Proc. Camb. Phil. Soc. 125 (1999), 385–395.

[8] P. Salberger, Tamagawa measures on universal torsors and points of bounded height on Fano varieties. Astérisque, 251 (1998), 91–258.

Independent University of Moscow, Russia
Max-Planck Institut für Mathematik Bonn, Germany
e-mail: felikson@mccme.ru pasha@mccme.ru