On the Gevrey regularity for Sums of Squares of vector fields, study of some models.

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Abstract. The Gevrey hypoellipticity of a class of “sums of squares” with real analytic coefficients is studied in detail. The Gevrey regularity obtained is matched in relation with the structure of the Poisson-Treves stratification of such operators. Some partial regularity result is also given.

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I. INTRODUCTION

The purpose of this paper is to discuss the Gevrey hypoellipticity properties of three model operators that are sums of squares of vector fields in four dimensions. The operators have analytic coefficients and verify the Hörmander condition: the Lie algebra generated by the vector fields as well as by their commutators has, in every point, dimension equal to the dimension of the ambient space. Hence in view of the celebrated Hörmander theorem, the operators are $C^\infty$-hypoelliptic.

In 1996, Treves, and in [13], formulated a conjecture concerning the analytic hypoellipticity for sums of squares satisfying the Hörmander condition. The Treves’ conjecture related the analytic hypoellipticity of such an operator with particular geometrical properties of its characteristic variety: for $P = X_1^2 + \cdots + X_m^2$, $X_j(x; D)$ vector fields with real analytic coefficients, to be analytic hypo-elliptic in $\Omega$, open neighborhood of the origin in $\mathbb{R}^n$, it is necessary and sufficient that every Poisson-Treves stratum of $\text{Char}(P)$ be symplectic.

We recall, without giving a definition, the main properties of the Poisson-Treves stratification for sums of squares:

**Theorem I.1** ([13], see also [5]). Let $P$ be the operator $P(x; D) = \sum_{j=1}^k X_j^2(x; D)$, $X_j(x; D)$ vector fields with real analytic coefficients on a open neighborhood of the origin in $\mathbb{R}^n$. Let $X_j(x, \xi)$ be the symbol of the vector field $X_j$. Let $\Sigma = \text{Char}(P)$ be the characteristic set of $P$ that is $\Sigma = \{(x, \xi) \in T^*\mathbb{R}^n \setminus \{0\} : X_j(x, \xi) = 0 \forall j \in \{1, \ldots, k\}\}$.

Then there is a stratification of $\Sigma$ such that

1. Each stratum is a real analytic manifold.
2. The symplectic form $\sigma$ has constant rank on each stratum.
3. There is a sequence of integers, $\nu_1 < \nu_2 < \cdots < \nu_{p-1} < \nu_p = r$ ($r$ denotes the maximum length of the Lie brackets involved in the Hörmander condition), and real analytic relatively open connected disjoint manifolds (strata) $\Sigma_{\nu_{j,l}}$, $l = 1, \ldots, l_j$, $j < p$. Here the index $l$ counts the connected components at level $\nu_j$. Moreover, all the Poisson brackets of the vector fields of length lesser than $\nu_j$ vanish on $\Sigma_{\nu_{j,l}}$, $l = 1, \ldots, l_j$, but there is at least one bracket of length $\nu_j$ which is non identically zero.

The length of a Poisson bracket of vector fields is just the number of vector fields forming the bracket; for example $X_j(x, \xi)$ is a bracket of length one while $\{X_1, \{X_1, X_2\}\}(x, \xi)$ is a bracket of length three.

In recent papers Albano, Bove and Mughetti, and Bove and Mughetti, showed that the sufficient part of the Treves’ conjecture does not hold neither locally nor microlocally. More precisely in [2] and [3] the authors produced and studied the first models which are not consistent with the Treves conjecture, suggesting moreover the necessity to define a new stratification associated to sums of squares operators. However, contrary to the cases of [2] and [3], the operators studied here have no exceptional strata because the symbols do not depend on the tangent variables of the “inner most” stratum.

As well as the generalized Oleinik-Radkevi\'c operator, the operators studied have the same characteristic variety and the bicharacteristic curves of each stratum are horizontal, that is their base projection is a “true” curve (for more details on the subject see [13]). The main difference is the shape of the deeper stratum. The operators $P_1$, and $P_2$, have the deeper stratum given by $\Sigma_{r+kp}$ and $\Sigma_{r+kq}$ respectively. In both case we have that
Theorem I.2. Our results can be stated as follows: deeper stratum, \( \Sigma^r \) involved in the Hörmander condition, that, as shown in [10], [4], [7] and [6], the regularity of the Olešnik-Radkeviš operator is given by
\[ p < q < r \]
and
\[ P \] obtained in the Olešnik-Radkeviš operator, Theorem I.2–A, and that, on the other side, this does not happen in the case of the operator \( P \), Theorem I.2–B, where the regularity found in the deeper stratum of \( P_1 \) is smaller than that in the Olešnik-Radkeviš operator. This difference would seem to suggest that the microlocal regularity in the deeper stratum would be better than in the case of the Olešnik-Radkeviš operator. However, even if at the present we are not able to prove the optimality, the regularity results obtained, Theorem [12], would show that this difference not improve always the microlocal regularity on the deeper stratum, which would not be able to prove the optimality, the regularity results obtained, Theorem I.2, would show that better than in the case of the Olešnik-Radkeviš operator. However, even if at the present we are not able to prove the optimality, the regularity results obtained, Theorem [12], would show that this difference not improve always the microlocal regularity on the deeper stratum, which would seem connected to other facts, as the precise nature of the bi characteristic curves. We have that
\[ \dim((T\Sigma_\nu + T\Sigma^\perp_\nu)/(T\Sigma_\nu \cap T\Sigma^\perp_\nu)) = 6 \]
\[ 1 \]
\[ \nu \text{ equal to } r + kp \text{ or } r + kq, \]
and the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to 4. The main difference is that in the first case, \( \Sigma_{r+kp} \), the bicharacteristic curves are parallel to the \( x_3\)-axis, in the second case, \( \Sigma_{r+kq} \), the bicharacteristic curves are parallel to the \( x_2\)-axis. In the case of the operator \( P_1 \), Theorem [12] ii, the deeper stratum is symplectic, and the operator is microlocal analytic hypoelliptic on the stratum. Instead, in the case of the Olešnik-Radkeviš operator, if we denote with \( \Sigma^r \) the deeper stratum, \( \nu \text{ equal to } r + kp \) or \( r + kq \), we always have that\[ \dim((T\Sigma_\nu + T\Sigma^\perp_\nu)/(T\Sigma_\nu \cap T\Sigma^\perp_\nu)) = 4 \]and the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to 2. Roughly speaking the non symplectic component of the deeper stratum of \( P_1 \) is smaller than that in the Olešnik-Radkeviš operator. This difference would seem to suggest that the microlocal regularity in the deeper stratum would be better than in the case of the Olešnik-Radkeviš operator. However, even if at the present we are not able to prove the optimality, the regularity results obtained, Theorem I.2, would show that this difference not improve always the microlocal regularity on the deeper stratum, which would seem connected to other facts, as the precise nature of the bicharacteristic curves. We have that the microlocal regularity of the operator \( P_1 \), Theorem I.2, in the deeper stratum, \( \Sigma_{r+kp} \), is better of that obtained in the Olešnik-Radkeviš operator, Theorem [12] A, and that, on the other side, this does not happen in the case of the operator \( P_2 \), Theorem [12] B, where the regularity found in the deeper stratum, \( \Sigma_{r+kq} \), is the same that in the case of the Olešnik-Radkeviš operator. We recall that, as shown in [10], [8] and [7], the regularity of the Olešnik-Radkeviš operator is given by the ratio between the depth of the last stratum, which is the maximum length of the Lie brackets involved in the Hörmander condition, \( \nu \), and the depth of the first stratum, the only one symplectic.

Our results can be stated as follows:

**Theorem I.2.** Let \( P_1(x, D) \) and \( P_2(x, D) \) the sums of squares given by
\[ P_1(x, D) = D_1^2 + x_1^{2(p-1)} D_2^2 + x_1^{2(q-1)} D_3^2 + x_1^{2(r-1)} D_4^2 + x_1^{2(r+\ell-1)} D_5^2 \]
and
\[ P_2(x, D) = D_1^2 + x_1^{2(p-1)} D_2^2 + x_1^{2(q-1)} D_3^2 + x_1^{2(r-1)} x_2^{2k} D_4^2 + x_1^{2(r+\ell-1)} D_5^2, \]
in \( \Omega \), open neighborhood of the origin in \( \mathbb{R}^4 \), where \( p, q, r, k \) and \( \ell \) are positive integers such that \( p < q < r \) and \( qk < \ell \). We have:

**A.** \( P_1(x, D) \) is \( G^s \)-hypoelliptic with \( s = \sup \left\{ \frac{r+kp}{q}, \frac{r+kq}{p} \right\} \). In particular if \( u \) solves the equation \( P_1 u = f \) and \( f \) is analytic then \( \rho_0 \in \Sigma_{r+kp} \) does not belong to \( WF_{\frac{r+kp}{q}}(u) \) and \( \rho_1 \in \Sigma_r \) does not belong to \( WF_{\frac{r}{p}}(u) \).

**B.** \( P_2(x, D) \) is \( G^s \)-hypoelliptic with \( s = \frac{r+kq}{p} \). In particular if \( u \) solves the equation \( P_2 u = f \) and \( f \) is analytic then \( \rho_0 \in \Sigma_{r+kq} \) does not belong to \( WF_{\frac{r+kq}{p}}(u) \) and \( \rho_1 \in \Sigma_r \) does not belong to \( WF_{\frac{r}{p}}(u) \).

Using the same strategy used to obtain the above results we can obtain the following result:

\[ T\Sigma_\nu^\perp \] denotes the annihilator of \( \Sigma_\nu \) with respect to the symplectic form \( \sigma \).
Remark I.1. If \( pk \geq \ell \) then \( P_1 \) is \( G^s \)-hypoelliptic with \( s = \sup \left\{ \frac{r + \ell}{q}, \frac{r}{p} \right\} \) and \( P_2 \) is \( G^s \)-hypoelliptic with \( s = \frac{r + \ell}{p} \). In particular if \( u_i \) solves the equation \( P_i u_i = f \), \( i = 1, 2 \), and \( f \) is analytic then \( \rho_0 \in \Sigma_{r + \ell} \) does not belong to to \( WF_{\frac{r + \ell}{q}}(u_1)/WF_{\frac{r + \ell}{p}}(u_2) \) and \( \rho_1 \) does not belong to \( WF_{\frac{r}{p}}(u_i) \), \( i = 1, 2 \).

We recall that in view of the classical result of Derridj and Zuily, \([8]\), \( P_1 \) is \( (r + kp) \)-Gevrey hypoelliptic and \( P_2 \) is \( (r + kq) \)-Gevrey hypoelliptic when \( kq < \ell \) and they are both \( (r + \ell) \)-Gevrey hypoelliptic when \( kp \geq \ell \). In particular the microlocal version of the Derridj and Zuily result, \([1]\), shows that when \( kq < \ell \) then \( P_1 \) and \( P_2 \) are both \( r \)-Gevrey microlocal hypoelliptics on \( \Sigma_r \), \( P_1 \) is \( (r + kp) \)-Gevrey microlocal hypoelliptic on \( \Sigma_{r + kp} \) and \( P_2 \) is \( (r + kq) \)-Gevrey microlocal hypoelliptic on \( \Sigma_{r + kq} \).

Theorem I.3. Let the operator \( P_3(x; D) \) be given by

\[
P_3(x; D) = D_1^2 + x_1^{2(p-1)} D_2^2 + x_1^{2(q-1)} D_3^2 + x_1^{2(r-1)} x_2^k D_4^2 + x_1^{2(f-1)} x_3^q D_5^2 + x_1^{2(f+e-1)} D_6^2
\]

in \( \Omega \), open neighborhood of the origin in \( \mathbb{R}^6 \), where \( p, q, r, k, f, \ell \) and \( e \) are positive integers such that \( p < q < r < f \) and \( e > \sup\{pk, q\ell\} \). We have:

\[\begin{align*}
\text{i) } & P_3 \text{ is } G^s \text{-hypoelliptic, with } s = \sup \left\{ \frac{r + kp}{q}, \frac{r}{p} \right\} \text{ if } f > r + kp. \quad \text{In particular if } u \text{ solves the equation } P_3 u = g \text{ and } g \text{ is analytic then } \rho_3 \in \Sigma_{r + kp} \text{ does not belong to } \WF_{\frac{r + kp}{q}}(u) \text{ and } \rho_3 \in \Sigma_r \text{ does not belong to } \WF_{\frac{r}{p}}(u). \\
\text{ii) } & P_3 \text{ is } G^s \text{-hypoelliptic, with } s = \sup \left\{ \frac{f}{q}, \frac{r}{p} \right\} \text{ if } f \leq r + kp. \quad \text{In particular if } u \text{ solves the equation } P_3 u = g \text{ and } g \text{ is analytic then } \rho_3 \in \Sigma_{r + kp} \text{ does not belong to } \WF_{\frac{f}{q}}(u) \text{, the analytic wave front set, } \rho_3 \in \Sigma_f \text{ does not belong to } \WF_{\frac{r}{p}}(u) \text{, and } \rho_3 \in \Sigma_r \text{ does not belong to } \WF_{\frac{r}{p}}(u).
\end{align*}\]

We recall that in view of the classical result of Derridj and Zuily, \([8]\), \( P_3 \) is \( r + kp \)-Gevrey hypoelliptic if \( f > r + kp \) and \( \inf\{r + kp, f + q\ell\} \)-Gevrey hypoelliptic if \( f < r + kp \). In particular the microlocal version of the Derridj and Zuily result, \([1]\), shows that if \( f > r + kp \), \( \rho_3 \notin \WF_{\frac{f}{q}}(u) \) and \( \rho_2 \notin \WF_{\frac{r}{p}}(u) \) and if \( f > r + kp \), \( \rho_6 \notin \WF_{\frac{r}{p}}(u) \), \( \rho_5 \notin \WF_{\frac{f}{q}}(u) \) and \( \rho_4 \notin \WF_{\frac{f}{q}}(u) \). In particular if \( e > \sup\{pk, q\ell\} \) shows, without particular technical trouble, that:

Remark I.2. If \( e < \sup\{pk, q\ell\} \) we can distinguish two cases: \( f + e < r + kp \), the operator \( P_3 \) is a generalization of the Oleinik-Radkevich operator and it is \( G_{\frac{f+e}{r}} \)-hypoelliptic and \( kp < e < lq, \) \( i.e. \) \( r + kp < f + e, \) as in the case i) of the Theorem \([4]\), \( P_3 \) is a generalization of the operator \( P_2 \), it is \( G^s \)-hypoelliptic with \( s = \sup\{\frac{r + kp}{q}, \frac{r}{p}\} \).

Remark I.3. If \( p < q < f < r \) and \( e > \sup\{r + pk, f + q\ell\} \) we can distinguish two case: \( f + q\ell > r \), \( P_3 \) is \( G_{\frac{f + q\ell}{r}} \)-hypoelliptic, and \( f + q\ell < r \), \( P_3 \) is \( G_{\frac{f + q\ell}{r}} \)-hypoelliptic.

Even if, at the present, the proof of the optimality of the above operators is an open problem, we think that the Gevrey regularities obtained are optimal.

This is the plan of the paper: in the first section we present a detailed study of the Poisson-Treves stratification associated to the studied operators; in the second and third section we give the proof of the theorems stated above; in the fourth section following the ideas in \([3]\) we study the non-isotropic Gevrey regularity of the operators \( P_1 \) and \( P_2 \); in the last section we present, without proof, a couple of results which generalize at the n-dimensional case the results obtained in the Theorems \([2] \) and \([3]\).
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II. THE POISSON-TREVES STRATIFICATION FOR $P_1$, $P_2$ AND $P_3$

In this section we compute the stratification for the operators studied.

The operators $P_1$, $P_2$ and $P_3$ have the same characteristic variety:

$$\text{Char}(P_i) = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0\}, \quad i = 1, 2, 3.$$  

The operators $P_i$, $i = 1, 2$, in the case $q k < \ell$ have the related stratification:

$$\Sigma_p = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0, \xi_2 \neq 0\},$$

this is a symplectic stratum and the restriction of the symplectic form to $\Sigma_p$ has rank 6;

$$\Sigma_q = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0, \xi_2 = 0, \xi_3 \neq 0\},$$

this is a non symplectic stratum and the restriction of the symplectic form to $\Sigma_q$ has rank 4, dim $(T\Sigma_q \cap T\Sigma_q^\perp) = 1$, dim $((T\Sigma_q + T\Sigma_q^\perp)/(T\Sigma_q \cap T\Sigma_q^\perp)) = 6$, the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to 2 and the bicharacteristics curves are the “horizontal” lines $\mathbb{R} \ni t \mapsto (0, t, x^0_3, x^0_4, 0, 0, \xi^0_3, \xi^0_4)$, with $(x^0_3, x^0_4, \xi^0_3, \xi^0_4) \in \mathbb{R}^4$ fixed, $\xi^0_3 \neq 0$, that is are parallel to the $x_2$-axis;

$$\Sigma_r = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0, \xi_2 = 0, \xi_3 = 0, \xi_4 x_{i+1} \neq 0\} \quad i = 1, 2$$

it is a non symplectic stratum and the restriction of the symplectic form to $\Sigma_r$ has rank 2, dim $(T\Sigma_r \cap T\Sigma_r^\perp) = 2$, dim $((T\Sigma_r + T\Sigma_r^\perp)/(T\Sigma_r \cap T\Sigma_r^\perp)) = 4$, the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to 2 and the bicharacteristics curves are the “horizontal” lines parallel to the $x_2$ and $x_3$-axis.

In the case of the operator $P_1$ the last strata is given by

$$\Sigma_{r+kp} = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0, \xi_2 = 0, \xi_3 = 0, x_2 = 0, \xi_4 \neq 0\},$$

this is a non symplectic stratum and the restriction of the symplectic form to $\Sigma_{r+kp}$ has rank 2, dim $(T\Sigma_{r+kp} \cap T\Sigma_{r+kp}^\perp) = 1$, dim $((T\Sigma_{r+kp} + T\Sigma_{r+kp}^\perp)/(T\Sigma_{r+kp} \cap T\Sigma_{r+kp}^\perp)) = 6$, the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to 4 and the bicharacteristics curves are the “horizontal” lines parallel to the $x_2$-axis.

In the case of the operator $P_2$ the last strata is given by

$$\Sigma_{r+qk} = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0, \xi_2 = 0, \xi_3 = 0, x_3 = 0, \xi_4 \neq 0\},$$

this is a non symplectic stratum and the restriction of the symplectic form to $\Sigma_{r+qk}$ has rank 2, dim $(T\Sigma_{r+qk} \cap T\Sigma_{r+qk}^\perp) = 1$, dim $((T\Sigma_{r+qk} + T\Sigma_{r+qk}^\perp)/(T\Sigma_{r+qk} \cap T\Sigma_{r+qk}^\perp)) = 6$, the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to 4 and the bicharacteristics curves are the “horizontal” lines parallel to the $x_3$-axis.

We remark that in the case $pk > \ell$ the operators $P_1$ and $P_2$ have the same stratification with the
only difference that the last strata have a depth \( r + \ell \) i.e we have to replace \( \Sigma_{r+pk} \) and \( \Sigma_{r+qk} \) with \( \Sigma_{r+\ell} \).

The operator \( P_3 \) with the assumption in the Theorem \( \text{[3]} \) has the related stratification

\[
\Sigma_p = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0, \xi_2 \neq 0\},
\]

this is a symplectic stratum and the restriction of the symplectic form to \( \Sigma \) has rank 6;

\[
\Sigma_q = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0, \xi_2 = 0, \xi_3 \neq 0\},
\]

it is a non symplectic stratum and the restriction of the symplectic form to \( \Sigma_q \) has rank 4, \( \dim(T\Sigma_q \cap T\Sigma_q^{\perp}) = 1 \), \( \dim((T\Sigma_q + T\Sigma_q^{\perp})/(T\Sigma_q \cap T\Sigma_q^{\perp})) = 6 \), the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to 2 and the bicharacteristics curves are the “horizontal” lines parallel to the \( x_2 \)-axis;

\[
\Sigma_r = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0, \xi_2 = 0, \xi_3 = 0, \xi_4 \neq 0\},
\]

this is a non symplectic stratum and the restriction of the symplectic form to \( \Sigma_r \) has rank 2, \( \dim(T\Sigma_r \cap T\Sigma_r^{\perp}) = 2 \), \( \dim((T\Sigma_r + T\Sigma_r^{\perp})/(T\Sigma_r \cap T\Sigma_r^{\perp})) = 4 \), the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to 2 and the bicharacteristics curves are the “horizontal” lines parallel to the \( x_2 \) and \( x_3 \)-axis.

Now if \( f > pk + r \) i.e. case \( i) \) of the Theorem \( \text{[3]} \) there is only one more strata of depth \( r + kp \):

\[
\Sigma_{r+pk} = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0, \xi_2 = 0, x_2 = 0, \xi_3 = 0, \xi_4 \neq 0\};
\]

it is a non symplectic stratum and the restriction of the symplectic form to \( \Sigma_{r+pk} \) has rank 2 \( \dim(T\Sigma_{r+pk} \cap T\Sigma_{r+pk}^{\perp}) = 1 \), \( \dim((T\Sigma_{r+pk} + T\Sigma_{r+pk}^{\perp})/(T\Sigma_{r+pk} \cap T\Sigma_{r+pk}^{\perp})) = 6 \), the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to 4 and the bicharacteristics curves are the “horizontal” lines parallel to the \( x_2 \)-axis.

Otherwise if \( f < pk + r \), i.e. case \( ii) \) of the Theorem \( \text{[3]} \) there are other two strata of depth \( f \) and of depth \( r + kp \) if \( r + kp < f + q\ell \) or of depth \( f + q\ell \) if \( r + kp > f + q\ell \):

\[
\Sigma_f = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0, \xi_2 = 0, x_2 = 0, \xi_3 = 0, x_3 \xi_4 \neq 0\},
\]

this is a non symplectic stratum and the restriction of the symplectic form to \( \Sigma_f \) has rank 2 \( \dim(T\Sigma_f \cap T\Sigma_f^{\perp}) = 1 \), \( \dim((T\Sigma_f + T\Sigma_f^{\perp})/(T\Sigma_f \cap T\Sigma_f^{\perp})) = 6 \), the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to 4 and the bicharacteristics curves are the “horizontal” lines parallel to the \( x_3 \)-axis;

\[
\Sigma_{r+kp}(\Sigma_{f+q\ell}) = \{(x; \xi) \in T^*\mathbb{R}^4 \setminus \{0\} : x_1 = 0, \xi_1 = 0, \xi_2 = 0, x_2 = 0, \xi_3 = 0, x_3 = 0, \xi_4 \neq 0\},
\]

it is a symplectic stratum and the restriction of the symplectic form to \( \Sigma_{r+kp}(\Sigma_{f+q\ell}) \) has rank 2.

### III. PROOF OF THEOREM \( \text{[2]} \)

#### A. Gevrey Regularity for \( P_I(x; D) \)

The classical results of Derridj and Zuily, \( \text{[8]} \), and Rothschild and Stein, \( \text{[11]} \), prove that for the operators \( P_I \) has the following sub-elliptic estimate with loss of \( 2(1 - 1/(r + kp)) \) derivatives; we state it in the quadratic form version:

\[
\|u\|^2_{r+kp} + \sum_{j=1}^5 \|X_j u\|^2 \leq C(|\{P_I u, u\}| + \|u\|^2).
\]

(III.4)
Here $X_1 = D_1$, $X_2 = x_1^{p-1}D_2$, $X_3 = x_1^{q-1}D_3$, $X_4 = x_1^{r-1}x_2^kD_4$, $X_5 = x_1^{s-1}D_4$. $\| \cdot \|_s$ denotes the $H^s$ Sobolev norm and $\| \cdot \|_0$ denotes the $L^2$ norm on the fixed open set $\Omega$. To study the regularity of the solutions we estimate the high order derivatives of the solutions in $L^2$ norm. As a matter of fact we estimate a suitable localization of a high derivative using (III.5). Then, we can conclude that if $\varphi_N$ is independent of the $x_1$-variable since every $x_1$-derivative landing on $\varphi_N$ would leave a cut off function supported where $x_1$ is bounded away from zero, where the operator is elliptic. Moreover we may assume that $\varphi_N$ is independent of the $x_2$-variable since every $x_2$-derivative landing on $\varphi_N$ would leave a cut off function supported where $x_2$ is bounded away from zero, in this region the operator satisfies the Hörmander-Lie algebra condition at the step $r$. The operator $P_1$, in this region, has the following sub-elliptic a priori estimate with loss of $2(1 - 1/r)$ derivatives:

$$
\|u\|_2^2 + \sum_{j=1}^{5} \|X_j u\|^2 \leq C \left( \|\langle Pu, u \rangle\| + \|u\|^2 \right),
$$

where $u \in C_0^\infty(K)$ with $K \cap \{x_2 = 0\} = \emptyset$. In this region the operator is a generalization of the Oleinik-Radkevič operator then $P_1$ is $G''/p$-hypoelliptic and not better, for more details see [4] and [4]. Then, we can conclude that if $v$ solves the equation $P_1 v = f$ and $f$ is analytic then the points $\rho_1 \in \Sigma_r$ does not belong to $WF_{\pm}(v)$. Now, we are interested to the microlocal regularity in $\rho_0 \in \Sigma_{r+kp}$. To obtain this it is sufficient to study the microlocal regularity of $P_1$ in $(0, 0, 0, 0; 0, 0, 0, 1)$. Indeed the microlocal regularity in a generic point $\rho_0$ can be obtained following the same strategy below with the only difference that the cut-off function $\varphi_N(x)$ must be identically equal to 1 in $\Omega_0$ neighborhood of $\pi_x(\rho_0) = (0, 0, x_3, x_4)$, where $\pi_x$ is the projection in the space variables. Thus since we are interested to the microlocal regularity of $P_1$ in $(0, 0, 0, 0; 0, 0, 0, 1)$ we take $\varphi_N(x) = \varphi_N(x_3, x_4)$.

We replay $u$ by $\varphi_N(x)D_4^N u$ in (III.4). We have

$$
(III.5) \quad \|\varphi_N D_4^N u\|_{L^1_{r+kp}}^2 + \sum_{j=1}^{5} \|X_j \varphi_N D_4^N u\|^2 \leq C \left( \|\langle P_1 \varphi_N D_4^N u, \varphi_N D_4^N u \rangle\| + \|\varphi_N D_4^N u\|^2 \right).
$$

The scalar product in the right hand side leads to

$$
(III.6) \quad \langle \varphi_N D_4^N P_1 u, \varphi_N D_4^N u \rangle + \sum_{j=1}^{5} \langle [X_j, \varphi_N D_4^N]u, \varphi_N D_4^N u \rangle
$$

$$
= 2 \sum_{j=1}^{5} \langle [X_j, \varphi_N D_4^N]u, X_j \varphi_N D_4^N u \rangle + \sum_{j=1}^{5} \langle [X_j, \varphi_N D_4^N], X_j u, \varphi_N D_4^N u \rangle
$$

$$
+ \langle \varphi_N D_4^N P_1 u, \varphi_N D_4^N u \rangle.
$$

The last term is trivial to estimate since $P_1 u$ is analytic; we may assume without loss of generality, that is zero. Since $\varphi_N$ depends only by $x_3$ and $x_4$ we must analyze the commutators with $X_3$, $X_4$ and $X_5$. Before to give the general form of the terms which appear inside of the iterating process we begin to analyze some particular situations.
Case $X_4$. We have

\[(III.7)\quad |\langle [X_4, \varphi_N D_4^N]u, X_4 \varphi_N D_4^N u \rangle| + |\langle [X_4, \varphi_N D_4^N], X_4]u, \varphi_N D_4^N u \rangle| = 2|\langle x_1^{(r-1)} x_2^k \varphi_N(1) D_4^N u, X_4 \varphi_N D_4^N u \rangle| + |\langle x_2^{2(r-1)} x_2^k \varphi_N(2) D_4^N u, \varphi_N D_4^N u \rangle|.
\]

The first term, we have

\[(III.8)\quad |\langle x_1^{(r-1)} x_2^k \varphi_N(1) D_4^N u, X_4 \varphi_N D_4^N u \rangle| \leq \sum_{j=1}^{N} C_j \|X_4 \varphi_N^{(j)} D_4^{N-j} u\|^2
+ \sum_{j=1}^{N+1} \frac{1}{C_j} \|X_4 \varphi_N D_4^N u\|^2 + C_{N+1} \|\varphi_N^{(N+1)} u\|^2,
\]

the constants $C_j$ are arbitrary, we make the choice $C_j = \varepsilon^{-1}2^j$, $\varepsilon$ suitable small positive constant. The terms of the form $C_j^{-1} \|X_4 \varphi_N D_4^N u\|^2$ can be absorbed on the right hand side of (III.5). We have $\|\varphi_N^{(N+1)} u\| \leq C^{N+1} \alpha!$, the analytic growth. Finally we observe that the terms in the first sum have the same form as $\|X_4 \varphi_N D_4^N u\|^2$ where one or more $x_4$-derivatives have been shifted from $u$ to $\varphi_N$; on these terms we can take maximal advantage from the sub-elliptic estimate restarting the process.

With regard to the second term on the right hand side of (III.7) we have

\[
|\langle x_1^{2(r-1)} x_2^k \varphi_N(2) D_4^N u, \varphi_N D_4^N u \rangle| \leq \frac{1}{2N^2} \|X_4 \varphi_N(2) D_4^{N-1} u\|^2 + \frac{N^2}{2} \|X_4 \varphi_N D_4^{N-1} u\|^2
+ |\langle x_1^{(r-1)} x_2^k \varphi_N(2) D_4^{N-1} u, X_4 \varphi_N(1) D_4^{N-1} u \rangle|
+ |\langle x_1^{(r-1)} x_2^k \varphi_N(3) D_4^{N-1} u, N X_4 \varphi_N D_4^{N-1} u \rangle|
+ |\langle x_1^{2(r-1)} x_2^k \varphi_N(3) D_4^{N-1} u, \varphi_N(1) D_4^{N-1} u \rangle|.
\]

The last term is the same of the left hand side in which one $x_4$-derivative has been shifted from $u$ to $\varphi_N$ on both side, we can restart the above process. On the first two terms we can take maximal advantage from the sub-elliptic estimate restarting the process. We point out that the “weight” $N$ introduced above helps to balance the number of $x_4$-derivatives on $u$ with the number of derivatives on $\varphi_N$, we take the factor $N$ as a derivative on $\varphi_N$ and $N^{-1} \varphi_N(2)$ as $\varphi_N(1)$. The other two terms have the same form of the term on the left hand side of (III.8), the second one with the help of the weight $N$, we can handled both in the same way.

The same strategy can be used to handle the case involving the field $X_5$.

The case $X_3$. We have

\[
|\langle [X_3, \varphi_N D_4^N]u, X_3 \varphi_N D_4^N u \rangle| \leq C \|x_1^{q-1} \varphi_N^{(1)} D_4^N u\| + \frac{1}{C} \|X_3 \varphi_N D_4^N u\|.
\]

The second term can be absorbed on the left hand side of (III.5), if $C^{-1}$ is chosen small enough. Since the first term does not have sufficient power of $x_1$ to take maximal advantage from the sub-elliptic estimate, we will use the sub-ellipticity. To do this we will pull back $D_4^1/(r+4\varepsilon)$.

Let $\chi_N(\xi_4)$ be an Ehrenpreis-Hörmander cutoff function such that $\chi_N$ is $C^\infty(\mathbb{R})$ non negative function such that $\chi_N = 0$ for $\xi_4 < 3$ and $\chi_N = 1$ for $\xi_4 > 4$. We have

\[
\|x_1^{q-1} \varphi_N^{(1)} D_4^N u\| \leq \|x_1^{q-1} \varphi_N^{(1)} (1 - \chi_N(N^{-1} D_4)) D_4^N u\| + \|x_1^{q-1} \varphi_N^{(1)} \chi_N(N^{-1} D_4) D_4^N u\|.
\]
Since $1 - \chi_N(N^{-1}D_4)$ has support for $\xi_4 < 4N$ we can estimate the first term of the above inequality with
\[
\|x_1^{q-1}\varphi_N^{(1)}(1 - \chi_N(N^{-1}D_4))D_4^Nu\| \leq C^{N+1}N^N,
\]
where $C$ is a positive constant independent by $N$, but depending on $u$. As already mentioned, to handle the second term of the above inequality we pull back $D_4^{1/(r+kp)}$. This is well defined since $\xi_4 > 1$, but is a pseudodifferential operator, and its commutator with $\varphi_N$ needs to some care. We use Lemma B.1 and Corollary B.1 in [2]. For completeness we recall them. Let $\omega_N \in C^\infty(\mathbb{R})$ be an Ehrenpreis type cutoff such that $\omega_N = 1$ for $x > 2$ and $\omega_N = 0$ for $x < 1$, $\omega_N$ non negative and such that $\omega_N\chi_N = \chi_N$. Then we have

**Lemma III.1 (2).** Let $0 < \theta < 1$. Them

\[
(III.9) \quad \left[\omega_N \left(N^{-1}D\right)^\theta, \varphi_N(x)\right] \chi_N \left(N^{-1}D\right)D^{N-\theta} = \sum_{j=1}^{N} a_{N,j}(x,D)\chi_N \left(N^{-1}D\right)D^j,
\]

where $a_{N,j}$ is a pseudodifferential operator of order $-k$ such that

\[
(III.10) \quad |\partial_x^\alpha a_{N,k}(x,\xi)| \leq C_a^{j+1}N^{j+\alpha}\xi^{-k-\alpha}, \quad 1 \leq j \leq N, \quad \alpha \leq N.
\]

**Corollary III.1 (2).** For $1 \leq j \leq N-1$ in (III.9) we have that

\[
(III.11) \quad a_{N,k}(x,D)\chi_N \left(N^{-1}D\right)D^N = \frac{\theta(\theta-1)\cdots(\theta-j+1)}{j!}D_4^j\varphi_N(x)\chi_N \left(N^{-1}D\right)D^{N-j}.
\]

Applying these results we find that

\[
\|x_1^{q-1}\varphi_N^{(1)}\chi_N(N^{-1}D_4)D_4^Nu\| \leq \|x_1^{q-1}\varphi_N^{(1)}\chi_N(N^{-1}D_4)D_4^{N-1/kp}u\| + \sum_{j=1}^{N-1} c_j\|x_1^{q-1}\varphi_N^{(j+1)}\chi_N(N^{-1}D_4)D_4^{N-j}u\| + \|x_1^{q-1}a_{N,k}(x,D)\chi_N(N^{-1}D_4)D_4^Nu\|.
\]

The last term has analytic growth. To handle the first term on the right hand side we will apply the subelliptic estimate. Concerning the the terms in the summation, we need, as done previously, to pull back $D_4^{1/(r+kp)}$ once more in order to use the sub-elliptic estimate, this will produce either terms with analytic growth or terms of the form $c_j\|x_1^{q-1}a_{N,K}(x,D)\chi_N(N^{-1}D_4)D_4^Nu\|_1/r+kp$ which can be handled as the first term.

Before to analyze the first term on the right hand of the above inequality we remark that

\[
\|(X_3,\varphi_ND_4^N)[X_3]u,\varphi_ND_4^Nu]\| = \|x_1^{2q-1}\varphi_N^{(2)}D_4^Nu,\varphi_ND_4^Nu\| \leq \frac{1}{2N^2}\|x_1^{q-1}\varphi_N^{(1)}D_4^Nu\|^2 + \frac{N^2}{2}\|x_1^{q-1}\varphi_ND_4^Nu\|^2.
\]

As above we use the “weight” $N$ to balance the number of $x_4$-derivatives on $u$ with the number of derivatives on $\varphi_N$. The two terms on the right hand side have the same form as $\|x_1^{q-1}\varphi_N^{(1)}D_4^Nu\|$, we can use the same strategy to analyze these two terms.

Then the only term that we have to handle is the term $\|x_1^{q-1}\varphi_N^{(1)}\chi_N(N^{-1}\xi_4)D_4^{N-1/kp}u\|_1/r+kp$.

\[9\]
To estimate this term we use the sub-elliptic estimate (III.4) replacing $u$ with $x_1^{q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u$. We have

(III.12) 
\[
\|x_1^{q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 + \sum_{j=1}^{5} \|X_j x_1^{q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 
\leq \|x_1^{q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} P u\|_2^2 + \|x_1^{q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 
+ 2 \sum_{j=1}^{5} \|\langle X_j, x_1^{q-1} \varphi_N^1 \rangle \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u, x_1^{q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 
+ \sum_{j=1}^{5} \|\langle X_j, [X_j, x_1^{q-1} \varphi_N^1] \rangle \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u, x_1^{q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 
\leq C(q - 1)^2 \|x_1^{q-2} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 + 2C \|x_1^{2q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 
+ \frac{1}{N^2} \|x_1^{2q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 + N^2 \|x_1^{2q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 
+ 2C \|x_1^{r+\ell + q-1} a_{N,x}^2 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 + 2C \|x_1^{r+\ell - q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 
+ \|x_1^{2q-2} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u, x_1^{q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 
+ \|x_1^{2q-2} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u, x_1^{q-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 
\]
modulo terms which can be absorbed on the left hand side or which give analytic growth. We remark that on the last four terms we can take maximal advantage from the sub-elliptic estimate restarting the processes; moreover in view of the role of the weight $N$ the third and the fourth term have the same form of the second one. Before to give the general form of the terms which appear inside of the iterating process we analyze the particular situations. To handle the first term on the right hand side of (III.12) we must use the sub-ellipticity, i.e. we pull back $D_4^{1/(r + kp)}$. Using the Lemma (III.1) and the Corollary (III.3) we have

\[
\|x_1^{q-2} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 \leq \|x_1^{q-2} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{2}{1/(r + kp)}} u\|_2^2 
+ \sum_{j=1}^{N-1} c_j \|x_1^{q-2} \varphi_N^{j+1} \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 
+ \|x_1^{q-2} a_{N,x}^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{1}{r + kp}} u\|_2^2 
\]

The last term has analytic growth. To handle the first term on the right hand side we will apply the subelliptic estimate. Concerning the the terms in the summation, we need, as done previously, to pull back $D_4^{1/(r + kp)}$ once more in order to use the sub-elliptic estimate, this will produce either terms with analytic growth or terms of the form $c_j \|x_1^{q-2} \varphi_N^{j+1} \chi_N(N^{-1} \xi_4) D_4^{N - \frac{2}{1/(r + kp)}} u\|_2^2$, which can be handled as the first term.

Iterating the above strategy at the $j$-th step we obtain a term of the form

\[
\|x_1^{q-j-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{j+1}{r + kp}} u\|_2^2 \leq \|x_1^{q-j-1} \varphi_N^1 \chi_N(N^{-1} \xi_4) D_4^{N - \frac{j+1}{r + kp}} u\|_2^2 
\]
When $j = q - 1$ we have $\|\varphi_N^{(1)} \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h}{r + kp} - \beta} u\|^2$. Iterating this cycle $s$-times we obtain a term of the form
\[
C^s \|\varphi_N^{(s)} \chi_N(N^{-1} \xi_4) D_4^{N - s \frac{h}{r + kp} - \beta} u\|^2.
\]
Using up all $x_4$-derivatives we estimate this term, hence the right hand side of (III.5), with $C^2(N+1)N^{2N(r+kp)/q}$. We have a growth corresponding to $G^{(r+kp)/q}$.

We focus on the second term on the right hand side of (III.12), $\|x_1^{2(q-1)} \varphi_N^{(2)} \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h}{r + kp} - \beta} u\|^2$. To handle this term we must use the sub-ellipticity, that is using the Lemma III.1 and the Corollary III.1 we pull back $D_4^{1/(r+kp)}$ restarting the process, i.e replacing $u$ with $x_1^{2(q-1)} \varphi_N \chi_N(N^{-1} \xi_4) D_4^{N - 2/(r+kp)} u$ in (III.4). Iterating this strategy at the $h$-th step we obtain a term of the form
\[
\|x_1^{h(q-1)} \varphi_N^{(h)} \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h-1}{r+kp} - \beta} u\|^2.
\]
Let $0 < \beta < 1$ a parameter that will be chosen later. Using the Lemma III.1 and the Corollary III.1 we pull back $D_4^j$; we can estimate the above quantity with
\[
\|x_1^{h(q-1)} D_4 \varphi_N \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h-1}{r+kp} - \beta} u\|^2
\]
modulo terms of the form $C_j \|x_1^{h(q-1)} D_4 \varphi_N^{(j)} \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h-1}{r+kp} - j} u\|^2$, $j = 1, \ldots, N - 1$, which can be handled restarting the process i.e. pulling back $D_4^j$ and using the same procedure to estimate (III.13) that we will show below, and $\|x_1^{h(q-1)} a_{N,N}(x,D) \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h-1}{r+kp} - \beta} u\|^2$ which gives analytic growth. The term (III.13) can be estimate by
\[
\|x_1^{h(q-1) - (p-1)} x_2 D_4 \varphi_N \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h-1}{r+kp} - \beta} u\|^2 + \|x_2 \varphi_N \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h-1}{r+kp} - \beta} u\|^2
\]
\[
\leq C_1 \|x_1^{(m+1)(q-1) - m(p-1)} x_2 D_4^{(m+1)\beta} \varphi_N \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h-1}{r+kp} - \beta} u\|^2
\]
\[
+ \|x_2 \varphi_N \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h-1}{r+kp} - \beta} u\|^2 + C_2 \|x_1^{h(q-1)} D_4^{\beta} \varphi_N \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h-1}{r+kp} - \beta} u\|^2
\]
where $C_2$ is a small suitable constant. The last term can be absorbed on the left hand side. Choosing $m = k$, $\beta = (k+1)^{-1}$ and $h = (r - 1 + k(p - 1))/((k+1)(q-1))$ we obtain
\[
\|X_1 \varphi_N \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h-1}{r+kp} - \beta} u\|^2 + \|X_2 \varphi_N \chi_N(N^{-1} \xi_4) D_4^{N - \frac{h-1}{r+kp} - \beta} u\|^2.
\]
Restarting the process, taking maximum advantage from the sub-elliptic estimate we obtain after $s$ step
\[
\|X_1 \varphi_N^{(s)} \chi_N(N^{-1} \xi_4) D_4^{N - s \left( \frac{h-1}{r+kp} + \beta \right)} u\|^2 + \|X_2 \varphi_N^{(s)} \chi_N(N^{-1} \xi_4) D_4^{N - s \left( \frac{h-1}{r+kp} + \beta \right)} u\|^2.
\]
Iterating until all the $x_4$-derivatives are used up, that is until $N - s((h-1)(r+kp)^{-1} + \beta) \sim 0$, we have the growth corresponding to $G^{(r+kp)/q}$.

Combining and iterating the above processes more time, removing powers of $x_1$ and $x_2$ with $D_4$ and taking profit from the sub-ellipticity we may estimate the left hand side of (III.5) with terms
of the form
\[ N^{-2m_0} \left( \| X_1 \varphi_N^{m_1 + hm_2 + m_3 + m_6} \right) \chi_N(N^{-1} \xi_4) D_4^{N-(m_1 + m_2 h) \frac{r}{r+k} - m_3} u^2 + \| X_2 \varphi_N^{m_1 + hm_2 + m_3 + m_6} \right) \chi_N(N^{-1} \xi_4) D_4^{N-(m_1 + m_2 h) \frac{r}{r+k} - m_3} u^2 + \| x_1^{m(q-1) - m_5} \varphi_N^{m_0 + m_1 + m_2 h + m_3 + m_4} \right) \chi_N(N^{-1} \xi_4) D_4^{N-m_1 + \frac{(m_3 + m_5 h) q}{r+k} - m_6 + \frac{m_6}{r+k} u^2} \]

where \( h \) is as above, \((h-1)(r+k)^{-1} - \beta = q h(r+k)^{-1} \) and \( m_4(q-1) - m_5 \leq (q-1) \). Iterating until all \( x_4 \)-derivatives are used up, that is \( N - (m_1 + m_2 h) q (r+p)^{-1} - m_3 \sim 0 \) and \( N - m_1 + (m_3 + m_2 h) q (r+k)^{-1} - (m_4 + m_3) (r+k)^{-1} - m_6 + m_6 (r+k)^{-1} \sim 0 \) we have that \( m_1 + hm_2 + m_3 \) and \( m_1 + hm_2 + m_3 + m_4 \), since \( m_3 \geq 1 \) and \( m_6 \geq 1 \), are small or equal to \((r+k) N q^{-1}\). We can conclude
\[ \| \varphi_N D_4^N u^2 \|_{L^2} + \sum_{j=1}^5 \| X_j \varphi_N D_4^N u^2 \| \leq C^{2(N+1)}(N)^{2N \frac{r+k}{q}} \]

where \( C \) is independent by \( N \) but depends on \( u \). This conclude the proof.

**B. Gevrey Regularity for \( P_2(x;D) \)**

The classical results of Derridj and Zuily, [8], and Rothschild and Stein, [11], prove that for the operators \( P_2 \) has the following sub-elliptic estimate with loss of \( 2(1 - 1/(r+k)) \) derivatives; we state it in the quadratic form version:

\[ (\text{III}14) \]
\[ \| u \|_{L^2}^2 + \sum_{j=1}^5 \| X_j u \|^2 \leq C \left( \| P_2 u \| + \| u \| \right). \]

Here \( X_1 = D_1, X_2 = x_1^{-1} D_2, X_3 = x_1^{-1} D_3, X_4 = x_1^{-1} x_3^k D_4, X_5 = x_1^{r+l-1} D_4. \| \cdot \|_s \) denotes the \( H^s \) Sobolev norm and \( \| \cdot \| = \| \cdot \|_0 \) denotes the \( L^2 \) norm on the fixed open set \( \Omega \).

To study the regularity of the solutions we estimate the high order derivatives of the solutions in \( L^2 \) norm. As a matter of fact we estimate a suitable localization of a high derivative using \((\text{III}14)\).

For \( x_1 \neq 0 \) the operator \( P_2 \) is elliptic and we shall not examine this region, elliptic operators are Gevrey hypoelliptic in any class \( G^s \) for \( s \geq 1 \).

Let \( \varphi_N(x) \) be a cutoff function of Ehrenpreis-Hörmander type with the same properties described in the beginning of the previous paragraph.

We assume that \( \varphi_N \) is independent of the \( x_1 \)-variable for the same reason described in the proof of the regularity of \( P_1 \). Moreover we may assume that \( \varphi_N \) is independent of the \( x_3 \)-variable since every \( x_3 \)-derivative landing on \( \varphi_N \) would leave a cut off function supported where \( x_3 \) is bounded away from zero, in this region the operator satisfies the Hörmander-Lie algebra condition at the step \( r \). The operator \( P_2 \) is sub-elliptic with loss of \( 2(1-1/r) \) derivatives. In this region the operator is a generalization of the Oleinik-Radkevič operator then \( P_2 \) is \( G^{r/p} \)-hypoelliptic and not better, for more details see [8] and [11]. Thus we can conclude that if \( v \) solves the equation \( P_1 v = f \) and \( f \) is analytic then the points \( \rho \in \Sigma_r \) does not belong to \( WF(v) \).

Now, we are interested to the microlocal regularity in \( \rho \in \Sigma_{r+kp} \), to obtain this it is sufficient to study the microlocal regularity of \( P_2 \) in \((0,0,0,0;0,0,0,1)\). The microlocal regularity in a generic point \( \rho \) can be obtained following the same strategy below with the only difference that the cutoff function \( \varphi_N(x) \) will be identically equal to 1 in \( \Omega_0 \) neighborhood of \( \pi_x(\rho_0) = (0, x_2^0, 0, x_3^0) \).
Thus since we are interesting to the microlocal regularity of $P_2$ in $(0,0,0;0,0,1)$ we take
\[ \varphi_N(x) = \varphi_N(x_2, x_4). \]
We replay $u$ by $\varphi_N(x)D^N_4u$ in (III.14). We have
\[
(\text{III.15}) \quad \|\varphi_N D^N_4u\|_2^{r+kq} + \sum_{j=1}^5 \|X_j \varphi_N D^N_4u\|^2 \leq C \left( \|P_2\varphi_N D^N_4u, \varphi_N D^N_4u\| + \|\varphi_N D^N_4u\|^2 \right).
\]
As in the case of the operator $P_1$ we want to estimate terms of the form:
\[
(\text{III.16}) \quad \langle [X_j, \varphi_N D^N_4u], X_j \varphi_N D^N_4u \rangle \quad \text{and} \quad \langle [X_j, \varphi_N D^N_4u], [X_j]u, \varphi_N D^N_4u \rangle, \quad j = 1, 2, 3, 4, 5.
\]
Since $\varphi_N$ depends only by $x_2$ and $x_4$ we will analyze the commutators with $X_2, X_4$ and $X_5$. The cases $X_4$ and $X_5$ give analytic growth, they can handled in same way as done in the study of $P_1$; in these cases we can take maximal advantage from the sub-elliptic estimate. The case $X_2$. In this case we have to estimate the term
\[
\|x_2^{p-1} \varphi_N^{(1)} D^N_4u\|.
\]
Since it does not have sufficient power of $x_2$ to take maximal advantage from the sub-elliptic estimate, we will use the sub-ellipticity. To do this we will pull back $D^N_4(x/r+kq)$. Using the same strategy employed to study the case of the vector field $X_3$ in the study of the regularity of $P_1$, here we have $x_2^{p-1}$ instead of $x_2^{p-1}$. Following the same strategy used to deduce the regularity of $P_1$, we conclude that
\[
\|\varphi_N D^N_4u\|_2^{r+kq} + \sum_{j=1}^5 \|X_j \varphi_N D^N_4u\|^2 \leq C^{2(N+1)}(N)^{2N^{r+kq/p}},
\]
where $C$ is independent by $N$ but depends on $u$. We have that the point $(0; e_4)$ and more in general that the points $\rho_0 \in \Sigma_{r+kq}$ do not belong to \(WF_{r+kq}(u)\). This conclude the proof of the theorem.

IV. PROOF OF THEOREM [1.3]

Part i) Theorem [1.3] case $f > r + kp$: In this case the Hörmander condition is satisfied at the step $r + kp$. The classical results of Derridj and Zuily, [8], and Rothschild and Stein, [11], prove that for the operator $P_1$ we have the following sub-elliptic estimate with loss of $2(1 - 1/(r + kp))$ derivatives; we state it in the quadratic form version:
\[
(\text{IV.17}) \quad \|u\|^2 + \sum_{j=1}^6 \|X_j u\|^2 \leq C\left(\|P_3u\| + \|u\|^2\right).
\]
Here $X_1 = D_4$, $X_2 = x_2^{p-1} D_2$, $X_3 = x_2^{q-1} D_3$, $X_4 = x_2^{p-1} x_2^4 D_4$, $X_5 = x_2^{p-1} x_2^4 D_4$ and $X_6 = x_2^{p-1} x_2^4 D_4$. The result can be archived following the same strategy used to characterize the regularity of the operator $P_1(x,D)$, Theorem [1.2] A. In fact the presence of the additional vector field $X_5 = x_2^{p-1} x_2^4 D_4$ gives, in the algorithm developed to handle the operator $P_1$, only a negligible contribution, i.e. analytic growth: to estimate the terms $\langle x_2^{p-1} x_2^4 \varphi_N^{(1)} D^N_4u, X_5 \varphi_N D^N_4u \rangle \text{ and } \langle x_2^{p-1} x_2^4 \varphi_N^{(2)} D^N_4u, \varphi_N D^N_4u \rangle$ can take maximal advantage from the sub-elliptic estimate. This conclude the proof of the part i.
Part ii) Theorem 4.3, case $f < r + kp$: In this case we distinguish two different situations: $r + kp < f + \ell q$ and $f + \ell q < r + kp$. Since the only difference between the two cases is the subelliptic index that is in the first case the Hörmander condition is satisfied at the step $r + kp$ in the other at the step $f + \ell q$ we will analyze only the first one.

**Case $r + kp < f + \ell q$:** The operator $P_3$ is sub-elliptic with loss of $2(1 - 1/(r + kp))$ derivatives, as above the sub-elliptic a priori estimate \((\text{IV.17})\) holds.

Let $\varphi_N(x)$ be a localizing cutoff function of Ehrenpreis-Hörmander type. We may assume that $\varphi_N$ is independent of the $x_1$-variable since every $x_1$-derivative landing on $\varphi_N$ would leave a cutoff function supported where $x_1$ is bounded away from zero, where the operator is elliptic. We can also assume that $\varphi_N$ is independent of the $x_2$-variable. If $x_3 \neq 0$ the operator $P_3$ is an operator of Oleinik-Radkević type, \([11]\), in view of the result obtained in \([6]\) the operator is $G^{r/p}$-hypoelliptic we can conclude that if $u$ solves the equation $P_3u = g$ and $g$ is analytic then the points $\rho_6 \in \Sigma_r$ does not belong to $WF_{r/p}(u)$.

Moreover we may assume that $\varphi_N$ is independent of the $x_3$-variable. Every $x_3$-derivative landing on $\varphi_N$ would leave a cut off function supported where $x_3$ is bounded away from zero, in this region the Hörmander condition is satisfied at the step $f$. The operator $P_3$ has the same form of the operator $P_1$, \((\text{I.1})\), in the Theorem \((\text{II.2})\) with $pk > \ell$. We can conclude that if $u$ solves the equation $P_3u = g$ and $g$ is analytic then the points $(0, 0, 0, x_3^0, x_4^0, 0, 0, 0, 0, \xi_4^0) \in \Sigma_j$, $x_3^0 \neq 0$, do not belong to $WF_{f/p}(u)$.

We assume that $\varphi_N(x) = \varphi_N(x_4)$. We replay $u$ by $\varphi_N(x_4)D_q^N u$ in \((\text{IV.17})\). We have

\[(\text{IV.18}) \quad \|\varphi_N D_q^N u\|_{r+kp}^2 + \sum_{j=1}^6 \|X_j\varphi_N D_q^N u\|^2 \leq C \left( \|P_3\varphi_N D_q^N u, \varphi_N D_q^N u\| + \|\varphi_N D_q^N u\|^2 \right). \]

We have to estimate terms of the form:

\[(\text{IV.19}) \quad \langle [X_j, \varphi_N D_q^N u], X_j\varphi_N D_q^N u \rangle \quad \text{and} \quad \langle [[X_j, \varphi_N D_q^N], X_j]u, \varphi_N D_q^N u \rangle, \quad j = 1, 2, 3, 4, 5, 6. \]

Since $\varphi_N$ depends only by $x_4$, $X_4$, $X_1$, $X_2$ and $X_3$ commute with $\varphi_N$. We must only analyze the commutators with $X_4$, $X_5$ and $X_6$. These cases give analytic growth, in these cases we can take maximal advantage from the sub-elliptic estimate. They can be handled as the field $X_4$, $X_5$, $X_6$, in the proof of the Theorem \((\text{II.2})\) we conclude that the point $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \xi_4^0) \in \Sigma_{r+kp}$, $\xi_4^0 \neq 0$, do not belong to $WF_{f}(u)$.

**V. A PARTIALLY REGULARITY RESULT FOR P₁ AND P₂**

In this section, following the ideas in \([4]\), we analyze the partial regularity in a neighborhood of the origin of the operators $P_1$, \((\text{I.1})\), and $P_2$, \((\text{II.2})\). We recall the definition of the non-isotropic Gevrey classes:

**Definition V.1.** A smooth function $f(x_0, x_1, \ldots, x_n)$ belongs to the Gevrey space $G^{(\alpha_0, \alpha_1, \ldots, \alpha_n)}$ at the point $x_0$ provided that there exists a neighborhood, $U$, of $x_0$ and a constant $C_f$ such that for all multi-indices $\beta$

\[|D^\beta f| \leq C_f^{|\beta|+1} \beta!^\alpha \quad \text{in} \quad U, \]

where $\beta!^\alpha = \beta_0!^{\alpha_0} \beta_1!^{\alpha_1} \ldots \beta_n!^{\alpha_n}$.

Our result can be stated as follows:
Proposition V.1. Let $P_1$ be as in the Theorem 1.2, where $pk < \ell$. If $u$ solves the problem $P_1 u = f$ and $f$ is analytic then $u \in G^{(s_1, s_2, s_3, s_4)}$ where $s_4 \geq \sup \left\{ \frac{r+q}{q}, s_2 \geq \frac{k}{k+1} + \frac{1}{r} \frac{r+q}{q} \right\}$, $s_3 \geq \frac{\text{r}(q-1)}{r(p-1)+q-p}$ and $s_4 \geq 1 + \sup \left\{ \frac{1}{\text{p}(k+1)} \left( \frac{r+q}{q} - 1 \right), \frac{1}{r} \left( \frac{r+q}{q} - 1 \right), \frac{1}{r} \left( \frac{r}{p} - 1 \right), \frac{(r-1)(q-p)}{q(r(p-1)+q-p)} \right\}$.

The same strategy used in the proof of the above Proposition shows that:

Remark V.4. If $pk > \ell$ then $u \in G^{(s_1, s_2, s_3, s_4)}$ where $s_4 \geq \sup \left\{ \frac{r+q}{q}, s_2 \geq \frac{k}{k+1} + \frac{1}{r} \frac{r+q}{q} \right\}$, $s_3 \geq \frac{\text{r}(q-1)}{r(p-1)+q-p}$ and $s_4 \geq 1 + \sup \left\{ \frac{1}{\text{p}(k+1)} \left( \frac{r+q}{q} - 1 \right), \frac{1}{r} \left( \frac{r+q}{q} - 1 \right), \frac{1}{r} \left( \frac{r}{p} - 1 \right), \frac{(r-1)(q-p)}{q(r(p-1)+q-p)} \right\}$.

Remark V.5. Let $P_2(x; D)$ as in the Theorem 1.2. If $qk < \ell$ and $u$ solves the problem $P_2 u = f$, $f$ analytic, then $u \in G^{(s_1, s_2, s_3, s_4)}$ where $s_4 \geq \sup \left\{ \frac{r+q}{q}, s_2 \geq 1, s_3 \geq \frac{k}{k+1} + \frac{1}{r} \frac{r+q}{q} \right\}$ and $s_4 \geq 1 + \sup \left\{ \frac{1}{\text{p}(k+1)} \left( \frac{r+q}{q} - 1 \right), \frac{1}{r} \left( \frac{r+q}{q} - 1 \right), \frac{1}{r} \left( \frac{r}{p} - 1 \right), \frac{(r-1)(q-p)}{q(r(p-1)+q-p)} \right\}$.

Proof Proposition V.1. Since the regularity in the direction $D_4$ has been obtained in the Theorem 1.2 we have only to analyze the direction $D_1$, $D_2$ and $D_3$. The primary tool will be once again the subelliptic estimate 3.1. Roughly speaking the strategy will be to transform the derivatives in the directions $D_2$ and $D_1$ in powers of the derivative in the direction $D_4$, this will allow us to use the result in the Theorem 1.2. Concerning the direction $D_3$ we will obtain the result directly.

Direction $D_3$: Let $\varphi_N(x_3, x_4)$ be a cut off function of Ehrenpreis-Hörmander type described in the proof of the Theorem 1.2 A to analyze the direction $x_4$. We replace $u$ by $\varphi_N D_3^N u$ in (3.1). We have

$$\quad \| \varphi_N D_3^N u \|_0^2 + \sum_{j=1}^5 \| X_j \varphi_N D_3^N u \|_0^2 \leq C \left( \| P_1 \varphi_N D_3^N u, \varphi_N D_3^N u \| + \| \varphi_N D_3^N u \|_0^2 \right).$$

The scalar product in the right hand side leads to

$$\quad 2 \sum_{j=1}^5 \langle X_j [X_j, \varphi_N] D_3^N u, \varphi_N D_3^N u \rangle + \sum_{j=1}^5 \langle [X_j, [X_j, \varphi_N]] D_3^N u, \varphi_N D_3^N u \rangle + \langle \varphi_N D_3^N P_1 u, \varphi_N D_3^N u \rangle.$$

The last term has a trivial estimate since $P_N u$ is analytic. Without loss of generality we can assume that it is zero. We focus our attention only on the vector field $X_3$, the case $X_4$ and $X_5$ can be handled in the same way, these vector fields have coefficients with power of $x_1$ greater than $q - 1$. We have

$$\quad \langle [X_3, \varphi_N] D_3^N u, X_3 \varphi_N D_3^N u \rangle = \langle x_1^{q-1} \varphi_N^{(1)} D_3^N u, X_3 \varphi_N D_3^N u \rangle \leq \langle x_1^{q-1} \varphi_N^{(1)} D_3^{N-1} u, X_3 \varphi_N D_3^N u \rangle + \| x_1^{q-1} \varphi_N^{(1)} D_3^{N-1} u, X_3 \varphi_N D_3^N u \| \leq C_1 \| x_3 \varphi_N D_3^{N-1} u \| + \frac{1}{C_1} \| x_3 \varphi_N D_3^N u \|^2 + \| x_1^{q-1} \varphi_N^{(2)} D_3^{N-1} u, X_3 \varphi_N D_3^N u \| \leq \cdots \leq \sum_{j=1}^N C_j \| x_3 \varphi_N^{(j)} D_3^{N-j} u \|^2 + \frac{1}{C_j} \| x_3 \varphi_N D_3^N u \|^2 + C_{N+1} \| \varphi_N^{(N+1)} u \|^2,$$
The constant $C_j$ are arbitrary, we make the choice $C_j = \delta^{-1} 2^j$, for a suitable fixed small $\delta$. We can absorb each term of the form $C_j^{-1} \|X_3 \varphi_N D_3^j u\|^2$ on the left hand side of (V.22). The term $C_{N+1} \|\varphi_N^{N+1} u\|^2$ is smaller than $C2(N+1)!N^2$, that is it gives analytic growth. To estimate the terms $C_j \|X_3 \varphi_N D_3^{N-j} u\|^2$, we observe that for each of them there has been a shift of one or more $x_3$-derivatives from $u$ to $\varphi_N$, but they have the same form as $\|X_3 \varphi_N D_3^j u\|^2$. We have to estimate the sum

$$
\sum_{j=1}^{N} \frac{2^j}{\delta} \|X_3 \varphi_N^{(j)} D_3^j u\|^2 = \frac{2^j}{\delta} \|X_3 \varphi_N^{(1)} D_3^{N-1} u\|^2 + \sum_{j=2}^{N} \frac{2^j}{\delta} \|X_3 \varphi_N^{(j)} D_3^{N-j} u\|^2.
$$

We start from the first term in the sum. We use the Rothschild-Stein sub-elliptic estimate replacing $u$ with $D_3^{N-1} u$, repeating the above procedure we have

$$
\|X_3 \varphi_N^{(1)} D_3^{N-1} u\|^2 \leq \sum_{j=1}^{N-1} \left( \frac{2^j}{\delta} \|X_3 \varphi_N^{(j+1)} D_3^{N-j-1} u\|^2 + \delta \|X_3 \varphi_N^{(1)} D_3^{N-1} u\|^2 \right)
$$

modulo terms which give analytic growth or which have the following form $\|([X_3, \varphi_N^{(1)}] D_3^{N-1} u, \varphi_N^{(1)} D_3^{N-1} u)\|$; we remark that for each of them there has been a shift of $x_3$-derivatives from $u$ to $\varphi_N$, but essentially they have the same form as $\|([X_3, \varphi_N] D_3^{N-1} u, \varphi_N^{(1)} D_3^{N-1} u)\|$ in (V.21), for the discussion of these terms see in the continuations of the proof. As before we may absorb the second term in the left hand side of the estimate. Repeating the above process $s$ times we have

$$
\sum_{j=1}^{N} \frac{1}{\delta} 2^j \|X_3 \varphi_N^{(j)} D_3^{N-j} u\|^2 \leq \sum_{j=s}^{N} \frac{1}{\delta} \left( 1 + \frac{1}{\delta} \right)^{s-1} 2^j \|X_3 \varphi_N^{(j)} D_3^{N-j} u\|^2
$$

modulo terms which can be absorbed on the left hand side or which give analytic growth or which have the form $\|([X_3, \varphi_N] D_3^{N-1} u, \varphi_N^{(1)} D_3^{N-1} u)\|$, $1 \leq j \leq s - 1$. With the same procedure, after $N - 1$ iterates, we obtain a term of the form

$$
\frac{1}{\delta} \left( 1 + \frac{1}{\delta} \right)^{N-1} 2^N \|X_3 \varphi_N^{(N)} u\|^2.
$$

This term can be estimate by $C2(N+1)!(N!)^2$, we have analytic growth.

On the other hand we have

$$
\|([X_3, \varphi_N] D_3^{N} u, \varphi_N^{(1)} D_3^{N-1} u)\| = \|x_3^{(q-1)} \varphi_N^{(2)} D_3^q u, \varphi_N D_3^q u\| \\
\leq \|x_3^{q-1} D_3 \varphi_N^{(2)} D_3^{N-1} u, x_3^{q-1} D_3 \varphi_N D_3^{N-1} u\| \\
+ \|x_3^{q-1} D_3 \varphi_N^{(2)} D_3^{N-1} u, x_3^{q-1} \varphi_N^{(1)} D_3^{N-1} u\| \\
+ \|x_3^{q-1} \varphi_N^{(3)} D_3^{N-1} u, x_3^{q-1} D_3 \varphi_N D_3^{N-1} u\| \\
+ \|x_3^{q-1} \varphi_N^{(3)} D_3^{N-1} u, x_3^{q-1} \varphi_N^{(1)} D_3^{N-1} u\| \\
= H_0 + H_1 + H_2 + H_3.
$$

We study any single term. Term $H_0$:

$$
H_0 \leq \frac{2}{N^2} \|X_3 \varphi_N^{(2)} D_3^{N-1} u\|^2 + 2N^2 \|X_3 \varphi_N D_3^{N-1} u\|^2.
$$

\[16\]
As done previously the weight $N$ is introduced to balance the number of $x_3$-derivatives on $u$ with the number of derivatives on $\phi_N$. The terms on the right hand side have the same form as $\|X_3\phi_N D_3^N u\|^2$. We can restart the process.

The term $H_1$:

$$H_1 \leq C_1 \|X_3\phi_N^{(2)} D_3^{N-2} u\|^2 + \frac{1}{C_1} \|X_3\phi_N^{(1)} D_3^{N-1} u\|^2 + \frac{1}{C_j} \|X_3\phi_N^{(1)} D_3^{N-1} u\|^2.$$

The above sum can be handled with the same strategy used to estimate the sum \(^{(1)}\). The last term gives analytic growth. To estimate the terms in the sums, we observe that with the help of the weight $N$ we have essentially, on each of them, shifted one or more $x_3$-derivatives from $u$ to $\phi_N$; they have the same form as $\|X_3\phi_N D_3^N u\|^2$.

The term $H_2$:

$$H_2 \leq \frac{C_1}{N^4} \|X_3\phi_N^{(3)} D_3^{N-1} u\|^2 + \frac{N^4}{C_1} \|X_3\phi_N D_3^{N-2} u\|^2 + \frac{C_2}{N^2} \|X_3\phi_N^{(3)} D_3^{N-2} u\|^2.$$

The last term gives analytic growth. To estimate the terms in the sums, we observe that with the help of the weight $N$ we have essentially, on each of them, shifted one or more $x_3$-derivatives from $u$ to $\phi_N$; they have the same form as $\|X_3\phi_N D_3^N u\|^2$.

The term $H_3$:

$$H_3 \leq \langle x_1^{q-1} \phi_N^{(3)} D_3^{N-2} u, x_1^{q-1} \phi_N^{(1)} D_3^{N-2} u \rangle + \langle x_1^{q-1} \phi_N^{(3)} D_3^{N-2} u, x_1^{q-1} \phi_N^{(1)} D_3^{N-2} u \rangle + \langle x_1^{q-1} \phi_N^{(4)} D_3^{N-2} u, x_1^{q-1} \phi_N^{(1)} D_3^{N-2} u \rangle.$$

Iterating we obtain

$$H_3 \leq \sum_{q=1}^{N} \langle x_1^{q-1} \phi_N^{(j+2)} D_3^{N-(j+1)} u, x_1^{q-1} \phi_N^{(j)} D_3^{N-(j+1)} u \rangle + \sum_{j=1}^{N} \langle x_1^{q-1} \phi_N^{(j+3)} D_3^{N-(j+1)} u, x_1^{q-1} \phi_N^{(j)} D_3^{N-(j+1)} u \rangle + \sum_{j=1}^{N} \langle x_1^{q-1} \phi_N^{(j+3)} D_3^{N-(j+1)} u, x_1^{q-1} \phi_N^{(j)} D_3^{N-(j+1)} u \rangle.$$
We observe that the terms in the first sum have the same form as \( H_0 \), the terms in the second sum have the same form as \( H_1 \) and those in the third sum have the same form as \( H_2 \), we can handle each of them as above. Finally, the last term gives analytic growth. Using the estimate (III.3) with \( u \) replaced by \( N^{i} \varphi_N^{(j)} D_{N}^{-(j+i)} u \) or \( N^{i} \varphi_N^{(j)} D_{N}^{N-j} u \) and applying recursively the same strategy followed above we are able to shift all free derivatives on \( \varphi_N \).

As previously observed, to analyze the case \( X_4 \) and \( X_5 \) we can use the same strategy used to study the case \( X_3 \). Indeed since the commutators \([X_4, \varphi_N], [X_5, \varphi_N], [X_4, [X_4, \varphi_N]]\) and \([X_5, [X_5, \varphi_N]]\) give terms with powers of \( x_1 \) greater than \( q-1 \), we can take again maximum advantage from the sub-elliptic estimate. Also in these cases we have analytic growth.

Hence we have

\[
\|\varphi D_3^{N} u\|^2_{L^p_{\varphi}} + \sum_{j=1}^{5} \|X_j \varphi_N D_3^{N} u\|^2 \leq C^{2(N+1)} N^{2N}.
\]

To obtain the result we need to consider when \( x_2 \neq 0 \). To do it since when \( x_2 \neq 0 \) the operator \( P_1 \) is an operator of OleÁšnik-Radkeviš type, (10), we use the following result in (6):

**Theorem V.4** (6). Let \( P \) be the operator given by

\[
(V.23) \quad P(x, D_x) = D^2_{x_1} + \sum_{j=2}^{n} x_1^{2(r_j-1)} D^2_{x_j}.
\]

We have that \( P \) is \( G^{n/r_1} \) hypoelliptic and not better. More precisely we have that if \( u \) solves the equation \( Pu = f \) and \( f \) is analytic then if \( \rho_j \in \Sigma_{r_j-1} \) then \( \rho_j \notin WF_{r_j/r_1}(u) \) and moreover \( u \in G^{(s_0, s_1, \ldots, s_n)} \) where

\[
s_1 \geq r^*, \quad s_j = \beta_j \geq \frac{r(n)(r_j-1)}{r(n)(r_j-1) + r_j - r_1} \quad \text{with} \quad j = 2, \ldots, n;
\]

where \( r^* = \sup_j \left\{ 1 - \frac{1}{r_j} + \frac{\beta_j}{r_j} \right\} \), in particular \( s_2 \geq 1 \) and \( s_n \geq r_n/r_1 \).

We can conclude that we have in the direction \( x_3 \) a growth corresponding to \( G^{\frac{r(q-1)}{r(q-1) + q-1 - p}} \).

**Direction \( D_2 \).** Once again our primary tool will be the sub-elliptic estimate (III.3). As in the study of the direction \( x_3 \), we replace \( u \) by \( \varphi_N D_2^{N} u \) in (III.4). We recall that \( \varphi_N \) does not depend on \( x_1 \) and \( x_2 \). We have

\[
(V.24) \quad \|\varphi_N D_2^{N} u\|^2_{L^p_{\varphi}} + \sum_{j=0}^{5} \|X_j \varphi_N D_2^{N} u\|^2 \leq C \left( \|P_1 \varphi_N D_2^{N} u, \varphi_N D_2^{N} u\| + \|\varphi_N D_2^{N} u\|^2 \right).
\]

We consider the scalar product in the right hand side of the above inequality. We have to study terms of the type

\[
\|\varphi_N D_2^{N} u\|_{L^p_{\varphi}} \quad j = 3, 4, 5.
\]

Since \( X_3 = x_1^{-1} D_3, X_5 = x_1^{r-\ell-1} D_4 \) and \( q \) and \( r \) are strictly greater than \( p \), as seen in the study of the direction \( x_3 \), we can take maximum advantage from the sub-elliptic estimate shifting one derivative from \( u \) to \( \varphi_N \). If we focus our attention only on these terms and we iterate the process
we will obtain analytic growth.

The case $X_4 = x_1^{r-1}x_2^kD_4$. We have

$$[X_4, \varphi_N D_2^N]u = [x_1^{r-1}x_2^kD_4, \varphi_N D_2^N]u = x_1^{r-1}x_2^k \varphi_N^{(1)} D_2^N u + x_1^{r-1}x_2^k \varphi_N[x_2^k, D_2^N]D_4 u$$

$$= x_1^{r-1}x_2^k x_1^{-\varphi_N^{(1)}} D_2^N u - x_1^{r-1} \varphi_N \sum_{j=1}^{N/k!} (i)^j j!(N-j)! (k-j)! x_2^{k-j} D_2^{N-j} D_4 u.$$

Without loss of generality we analyze one of the terms; a similar method can be used to handle the other terms. We consider the first one: $Nkix_1^{r-1}x_2^k D_2^{N-2} D_4 u$. We have to estimate $Nk \|X_2^2 x_2^k D_2^{N-2} D_4 u\|$. We apply the sub-elliptic estimate with $u$ replaced by $Nk \varphi_N x_2^k D_2^{N-2} D_4 u$, arguing as above, we study the first term coming from the commutator with $X_4$. We obtain the term $k^2 N(N-2)x_1^{r-1}x_2^{2(k-1)} \varphi_N D_2^{N-3} D_4^2 u$. We have to estimate $k^2 N(N-2) \|X_4 x_2^{k-2} \varphi_N D_2^{N-3} D_4 u\|$. Hence after two steps we have

$$\|X_4 \varphi_N D_2^N u\| \rightarrow k^2 N(N-2) \|X_4 x_2^{k-2} \varphi_N D_2^{N-3} D_4 u\|.$$

Repeating the process $j$-times, we have

$$\|X_4 \varphi_N D_2^N u\| \rightarrow \cdots \rightarrow C \frac{N!}{(N-1)!(N-(j+1))!} \|X_4 x_2^{k-j} \varphi_N D_2^{N-(j+1)} D_4 u\|.$$

Here the constant $C$ depend by $k$. We stress that $N![(N-1)!(N-(j+1))!]^{-1} \sim N^j$. In this way after $k$ iterates we have to analyze a term of the form $C R(kN)![(N-1)!(N-(k+1))!]^{-1} \varphi_N D_2^{N-(k+1)} D_4 u$. Arguing in the same way after $m$ steps we have

$$\|X_4 \varphi_N D_2^N u\| \rightarrow \cdots \rightarrow C R(kN)^m \|X_4 \varphi_N D_2^{N-m(k+1)} D_4^m u\|.$$

Iterating the cycle $N/(k+1)$-times we use up all free derivatives in $x_2$-direction and we are left with

$$C R(kN)^N \|X_4 \varphi_N D_4^N u\|.$$

As well as it was done in the proof of the Theorem[2] we introduce $\chi_N(\xi_4)$ an Ehrenpreis-Hörmander cutoff function such that $\chi_N$ is $C^\infty(\mathbb{R})$ non negative function such that $\chi_N = 0$ for $\xi_4 < 3$ and $\chi_N = 1$ for $\xi_4 > 4$. We have

$$\|X_4 \varphi_N \chi_N(N^{-1} \xi_4) D_4^N u\| \leq \|X_4 \varphi_N (1 - \chi_N(N^{-1}D_4)) D_4^N u\| + \|X_4 \varphi_N \chi_N(N^{-1} D_4) D_4^N u\|.$$

Since $1 - \chi_N(N^{-1} D_4)$ has support for $\xi_4 < 4N$ we have

$$C R(kN)^N \|X_4 \varphi_N (1 - \chi_N(N^{-1} D_4)) D_4^N u\| \leq C R(kN)^N \|X_4 \varphi_N D_4^N u\| \leq C R(kN)^N \|X_4 \varphi_N D_4^N u\|,$$

where $C$ is a positive constant independent by $N$, but depending on $u$ and $k$. To estimate $\|X_4 \varphi_N \chi_N(N^{-1}D_4) D_4^N u\|$ we use the same strategy used in the proof of the Theorem[2]. Therefore since in the direction $x_4$ we have a growth corresponding to $G^{r+k}$ we can estimate this term with $C R(kN)^N \|X_4 \varphi_N D_4^N u\|$. We can estimate the left hand side of $\|X_4 \varphi_N D_4^N u\|$ with this quantity, we have the growth corresponding to $G^{r(k+1)+\varepsilon}$, where $\varepsilon$ depends on $N$. More in general applying the sub-elliptic estimate and iterating the above processes more time, we may estimate the left hand side of $\|X_4 \varphi_N D_4^N u\|$ with terms of the form

$$N^{N-jm\varepsilon} \|X_4 \varphi_N D_4^{N-j} D_4^N u\|.$$
Iterating the procedure until all the $x_2$-derivatives are used up we have to apply the sub-elliptic estimate to terms of the form

$$\langle N \rangle^{(N-j)} \frac{k}{r+1} \phi(j) D_4^{N+\frac{r}{k}} u.$$ 

To handle these terms we argue as before that is we introduce the cut-off $\chi_N$ and we apply the strategy used in the proof of the Theorem [10] to obtain the Gevrey regularity in the direction $x_4$. Since $(r + kp)/q > 1$ we can conclude

$$\|\phi_N D_2^N u\|_r^2 + \sum_{j=0}^{5} \|X_j \phi_N D_2^N u\|_0^2 \leq C^{N+1} \langle N \rangle^{\frac{1}{k+1} \left( \frac{r+kp}{q} + k \right)}.$$ 

To gain the result we need to consider when $x_2 \neq 0$. To do it since when $x_2 \neq 0$ the operator $P_1$ is an operator of Oleinik-Radkevič type, [10], we use Theorem [13]. We have that when $x_2 \neq 0$ in the direction $D_2$ we have analytic growth. We conclude that in this direction the growth corresponding to $G^{(r + k(p + q))/q(k+1)}$.

**Direction $D_1$:** As in the study of the other directions, we replace $u$ by $\phi_N(x) D_1^N u$ in (III.4). We have

$$\|\phi_N D_1^N u\|_r^2 + \sum_{j=0}^{5} \|X_j \phi_N D_1^N u\|_0^2 \leq C \langle \{P_1 \phi_N D_1^N u, \phi_N D_1^N u\} \rangle + \|\phi_N D_1^N u\|_0^2.$$ 

We consider the scalar product in the right hand side of the above inequality. We have to study terms of the type

$$[\phi_N D_1^N u, X_j \phi_N D_1^N u], \quad j = 2, 3, 4, 5.$$ 

We describe the case $X_2$, the other cases can be handled using the same strategy. We have

$$[X_2, \phi_N D_1^N u] = \phi_N \sum_{j=1}^{p-1} \frac{N! (p-j)!}{(p-j)!(N-j)!(p-j)!} x_1^{p-j} D_1^{N-j} D_2^j u.$$ 

Without loss of generality we analyze one of the terms. A similar method can be used to handle the other terms. Consider $N(p-1) D_1 x_1^{p-2} \phi_N D_1^{N-2} D_2^1 u$ that is we have to estimate a term of the form $N(p-1)\|X_1 x_1^{p-2} \phi_N D_1^{N-2} D_2^1 u\|$. Applying the sub-elliptic estimate with $u$ replaced by $x_1^{p-2} \phi_N D_1^{N-2} D_2^1 u$ and arguing as above, we study the first term coming from the commutator with $X_2$. We obtain the term $N(N-2)(p-1)^2 x_1^{2(p-2)} D_1^{N-3} D_2^1 u$. We have to estimate $N(N-2)(p-1)^2\|X_2 x_1^{p-3} D_1^{N-3} D_2^1 u\|$. Hence after two step we have

$$\|X_2 \phi_N D_1^N u\| \to (p-1)^2 \frac{N!}{(N-1)(N-3)!} \|X_2 x_1^{p-3} \phi_N D_1^{N-3} D_2^1 u\|.$$ 

Repeating the process $s$-times, we have

$$\|X_2 \phi_N D_2^N u\| \to \cdots \to C_p \left( \frac{N!}{(N-1)(N-(s+1))!} \right) \|X_2 x_1^{p-(s+1)} \phi_N D_1^{N-(s+1)} D_2^1 u\|.$$ 

We stress that $N!(N-1)(N-(j+1))!^{-1} \sim N^j$. In this way after $s = p - 1$ iterates we have to analyze a term of the form $C_p N^{p-1} \|X_2 \phi_N D_1^{N-p} D_2^1 u\|$. Arguing in the same way after $m$ steps we have

$$\|X_2 \phi_N D_1^N u\| \to \cdots \to C_p^m N^{m(p-1)} \|X_2 \phi_N D_1^{N-mp} D_2^1 u\|.$$
Iterating the cycle $N/p$-times we use up all free derivatives in $x_1$-direction and we are left with

$$C_p^N N^N (1-\frac{1}{p}) \| X_2 \varphi_N D_2^N u \|.$$ 

Since in the direction $x_2$ we have a growth as $G^r/k(k+l)$ we can estimate the above term with

$$C^{N+1} (N!)^{1+\frac{1}{p}} \left( \frac{r+k(p+q)}{n(k+l)} - \frac{1}{k+l} \right).$$

We have the growth $G^{1+\frac{r+k(p+q)}{n(k+l)}}$.

The other cases, that is the terms involving the commutators with $X_3$, $X_4$ and $X_5$, can be handled in the same way achieving analytic growth, $1+(r+kp-q)/rq$-Gevrey growth and $1+(r+kp-q)/(r+\ell)q$-Gevrey growth respectively. We remark that in these three situations, arguing as above, we obtain terms of the form $C_3^N (N!)^{(r-1)/q} \| X_3 \varphi_N D_3^N u \|$, $C_4^N (N!)^{(r-1)/p} \| X_4 \varphi_N D_4^N u \|$ and $C_5^N (N!)^{(r-1)/(r+\ell)} \| X_5 \varphi_N D_5^N u \|$. Moreover we point out that also in the general situation we will obtain a Gevrey growth less than or equal to that obtained by analyzing the individual cases. We have obtained a growth corresponding to $G^{s_1}$ where $s_1 = \sup \{1 + \frac{r+kp-q}{rq(n+k+1)}, 1 + \frac{r+kp-q}{rq} \}$. To obtain the result we need to consider when $x_2 \neq 0$. To do it since when $x_2 \neq 0$ the operator $P_1$ is an operator of Ole\v{c}nik-Radkevi\v{c} type, [10], we use Theorem V.4. We have that when $x_2 \neq 0$ in the direction $D_2$ we have a growth corresponding to $G^{s_2}$ where $s_2 = \sup \{1 + \frac{1}{q} \left( \frac{r(q-1)}{r(p-1)+q-p} - 1 \right), 1 + \frac{1}{p} \left( \frac{r-\ell}{r} \right) \}$. We conclude that in the direction $x_2$ we have a growth corresponding to $G^s$ where $s = \sup \{s_1, s_2 \}$. We point out that the case $x_2 \neq 0$ can be directly considered taking the cutoff function $\varphi_N$ depending also on the $x_2$-variable from the beginning.

VI. ADDITIONAL MATERIAL: THE $n$–DIMENSIONAL CASE

Following the same ideas used to archive the Theorems 1.2 and 1.3 we can extend without particular difficulties such results to the following $n$-dimensional cases, $n \geq 5$. We omit the proofs.

Theorem VI.5. Let $P_{i,n}(x; D)$ be the operator given by

\begin{equation}
(P_{i,n}(x; D) = D_i^2 + \sum_{j=2}^{n-1} x_1^{2(r_j-1)} D_j^2 + \left( x_2^{2(r_n-1)} x_1^{2k} + x_1^{2(r_n+\ell-1)} \right) D_{n,j}^2, \quad 2 \leq i \leq n-1,
\end{equation}

in $\Omega$, open neighborhood of the origin in $\mathbb{R}^n$, where $r_j$, $j = 1, \ldots, n$, $k$ and $\ell$ are positive integers such that $r_1 < r_2 < \cdots < r_n$. We have:

i) if $kr_i < \ell$, $P_{i,n}(x; D)$ is $G^s$-hypoelliptic with $s = \sup \left\{ \frac{r_n + kr_i}{r_3}, \frac{r_n}{r_2} \right\}$ if $i = 2$ and $s = \frac{r_n + kr_i}{r_2}$ if $i \neq 2$. In particular if $u$ solves the equation $P_{i,n} u = f$ and $f$ is analytic then the point $(0, e_n)$ in $\text{Char}(P_{i,n})$ does not belong to $\text{WF}_{(r_n + kr_i)/r_2}(u)$ if $i = 2$ and it does not belong to $\text{WF}_{(r_n + kr_i)/r_2}(u)$ if $i \neq 2$.

ii) if $kr_i \geq \ell$, $P_{i,n}(x; D)$ is $G^s$-hypoelliptic with $s = \sup \left\{ \frac{r_n + \ell}{r_3}, \frac{r_n}{r_2} \right\}$ if $i = 2$ and $s = \frac{r_n + \ell}{r_2}$ if $i \neq 2$. In particular if $u$ solves the equation $P_{i,n} u = f$ and $f$ is analytic then the point $(0, e_n)$ in $\text{Char}(P_{i,n})$ does not belong to $\text{WF}_{(r_n + \ell)/r_2}(u)$ if $i = 2$ and it does not belong to $\text{WF}_{(r_n + \ell)/r_2}(u)$ if $i \neq 2$. 

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Theorem VI.6. \( P_{1,n} \) has characteristic variety given by
\[
\text{Char}(P_{1,n}) = \{(x; \xi) \in T^*\mathbb{R}^n \setminus \{0\} : x_1 = 0, \xi_1 = 0\}.
\]

Case \( kr_i < \ell \). The deeper stratum of its Poisson-Treves stratification is given by
\[
\Sigma_{r_n+kr_i} = \{(x; \xi) \in T^*\mathbb{R}^n \setminus \{0\} : x_1 = 0, \xi_1 = \cdots = \xi_{n-1} = 0, x_i = 0, \xi_n \neq 0\}, \quad 2 \leq i \leq n-1.
\]

It is a non symplectic stratum and the restriction of the symplectic form to \( \Sigma_{r_n+kr_i} \) has rank 2, \( \dim(\Sigma_{r_n+kr_i} \cap T\Sigma_{r_n+kr_i} \cap T\Sigma_{r_n+kr_i}) = n - 3 \), \( \dim((T\Sigma_{r_n+kr_i} + T\Sigma_{r_n+kr_i})/T\Sigma_{r_n+kr_i} \cap T\Sigma_{r_n+kr_i}) = 6 \), the matrix of the Poisson bracket of the defining function of the stratum has rank equal to 4 and the bicharacteristics curves are the “horizontal” lines parallel to the \( x_j \)-axis, where \( j = 3, \ldots, n-1 \) if \( i = 2 \) and \( j = 2, \ldots, i-1, i+1, n-1 \) if \( i \neq 2 \).

Remark VI.6. Let \( \tilde{P}_{i,n}(x; D) \) be the operator given by
\[
(\text{VI.27}) \quad \tilde{P}_{i,n}(x; D) = D_1^2 + \sum_{j=2}^{m-1} x_1^{2(r_j-1)} D_j^2 + \left( x_1^{2(r_n-1)} x_i^{2k} + x_1^{2(r_m+\ell-1)} \right) D_m^2 + \sum_{j=m+1}^{n} x_1^{2(r_j-1)} D_j^2,
\]
in \( \Omega \), open neighborhood of the origin in \( \mathbb{R}^n \), where \( m \geq 3, 2 \leq i \leq m-1, r_j, j = 1, \ldots, n, k \) and \( \ell \) positive integers such that \( r_1 < r_2 < \cdots < r_n \) and \( r_n > r_m + \sup\{r_j, \ell\} \). We have that \( \tilde{P}_{i,n}(x; D) \) is \( r_n/r_j \)-Gevrey hypoelliptic. In particular if \( u \) solves the equation \( P_{2,n}(x; D)u = f \) and \( f \) is analytic then the point \((0, e_n) \in \text{Char}(P_{2,n}) \) does not belong to \( WF_{r_n/r_j}(u) \).

Theorem VI.6. Let \( P_{m,n}(x, D) \) be the operator given by
\[
(\text{VI.28}) \quad P_{m,n}(x; D) = D_1^2 + \sum_{i=2}^{m+1} x_1^{2(r_i-1)} D_i^2 + \sum_{i=m+2}^{n} \left( x_1^{2(r_i-1)} x_i^{2k_i/m} + x_1^{2(r_\ell+\ell_m-1)} \right) D_m^2,
\]
in \( \Omega \), open neighborhood of the origin in \( \mathbb{R}^n \), where \( r_i, i = 2, \ldots, n, k_i/m \) and \( \ell_i/m \), \( i = m+2, \ldots, n \), are positive integers such that \( r_2 < \cdots < r_n, k_2 < \cdots < k_{n-m}, \ell_2 < \cdots < \ell_{n-m} \) and \( r_i k_i/m < \ell_i/m \) for every \( i, i = m+2, \ldots, n \), then \( P_{m,n}(x; D) \) is \( G^s \)-hypoelliptic with \( s = r_n + r_n/mk_{n-m} \). Moreover if \( u \) solves the equation \( P_{m,n}u = f \) and \( f \) is analytic then the point \((0, e_n) \in \text{Char}(P_{m,n}) \) does not belong to \( WF_{r_n/r_m k_{n-m}/r_{n-m+1}}(u) \).

Remark VI.7. Let \( \tilde{P}_{m,n}(x, D) \) be the operator given by
\[
\tilde{P}_{m,n}(x; D) = D_1^2 + \sum_{i=2}^{m+2} x_1^{2(r_i-1)} D_i^2 + \sum_{i=m+3}^{n} \left( x_1^{2(r_i-1)} x_i^{2k_i/m} + x_1^{2(r_\ell+\ell_m-1)} \right) D_m^2, \quad m \geq \left\lceil \frac{n}{2} \right\rceil,
\]
in \( \Omega \), open neighborhood of the origin in \( \mathbb{R}^n \), where \( r_i, i = 2, \ldots, n, k_i/m+1 \) and \( \ell_i/m \), \( i = m+3, \ldots, n \), are positive integers such that \( r_2 < \cdots < r_n, k_3 < \cdots < k_{n-m}, \ell_3 < \cdots < \ell_{n-m} \) and \( r_i k_i/m < \ell_i/m \) for every \( i, i = m+3, \ldots, n \), then the point \((0, e_n) \in \text{Char}(P_{m,n}) \) does not belong to \( WF_{r_n/r_m k_{n-m}/r_{n-m+2}}(u) \).

The operators \( P_{m,n}(x, D) \) and \( \tilde{P}_{m,n}(x, D) \) have characteristic variety given by
\[
\text{Char}(P_{m,n}(x; D)) = \text{Char}(\tilde{P}_{m,n}(x; D)) = \{(x; \xi) \in T^*\mathbb{R}^n \setminus \{0\} : x_1 = 0, \xi_1 = 0\}.
\]
Operator $P_{m,n}$, the deeper stratum, $\Sigma_{\nu}$, $\nu = r_n + r_{n-m}k_{n-m}$, of its Poisson-Treves stratification is given by

$$\{(x; \xi) \in T^*\mathbb{R}^n \setminus \{0\} : \xi_1 = \cdots = \xi_{n-1} = 0, x_1 = \cdots = x_{n-m} = 0, \xi_n \neq 0\}.$$ 

It is a non symplectic stratum, the restriction of the symplectic form to $\Sigma_{\nu}$ has rank 2, we have that $\dim(T\Sigma_{\nu} \cap T\Sigma_{\nu}^\perp) = m + 1$, the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to $2(n-m)$, $\dim((T\Sigma_{\nu} + T\Sigma_{\nu}^\perp)/(T\Sigma_{\nu} \cap T\Sigma_{\nu}^\perp)) = 2(n-m+1)$, and the bicharacteristics curves are the “horizontal” lines parallel to the $x_j$-axis, with $j = n-m+1, \ldots, n-1$.

Operator $\tilde{P}_{m,n}$, the deeper stratum, $\Sigma_{\nu}$, $\nu = r_n + r_{n-m}k_{n-m}$, of its Poisson-Treves stratification is given by

$$\{(x; \xi) \in T^*\mathbb{R}^n \setminus \{0\} : \xi_1 = \cdots = \xi_{n-1} = 0, x_1 = 0, x_3 = \cdots = x_{n-m} = 0, \xi_n \neq 0\}.$$ 

It is a non symplectic stratum, the restriction of the symplectic form to $\Sigma_{\nu}$ has rank 2, we have that $\dim(T\Sigma_{\nu} \cap T\Sigma_{\nu}^\perp) = m$, the matrix of the Poisson bracket of the defining functions of the stratum has rank equal to $2(n-m-1)$, $\dim((T\Sigma_{\nu} + T\Sigma_{\nu}^\perp)/(T\Sigma_{\nu} \cap T\Sigma_{\nu}^\perp)) = 2(n-m)$, and the bicharacteristics curves are the “horizontal” lines parallel to the $x_2$-axis and $x_1$-axis, with $j = n-m+1, \ldots, n-1$.

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