A Non-linear $\sigma$-model for Partially Polarized QH-states

T.H. Hansson$^{1,3}$, A. Karlhede$^{1,3}$ and J.M. Leinaas$^{2,3}$

$^1$Department of Physics, Stockholm University, Box 6730, S-11385 Stockholm, Sweden
$^2$Institute of Physics, University of Oslo, P.O. Box 1048 Blindern, N-0316 Oslo, Norway
$^3$Centre for Advanced Study, P.O. Box 7606 Skillebekk, 0205 Oslo, Norway

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Abstract

We consider a two-component quantum Hall system within a Landau-Ginzburg theory with two Chern-Simons gauge fields. From this theory we derive a sigma model covariantly coupled to one Chern-Simons field and find mean field solutions that could describe partially polarized quantum Hall states. The quasiparticles in the original model, which have quantized charge and spin, are described in the covariant sigma model by topological excitations, with the correct quantum numbers. They have finite energy due to the presence of the Chern-Simons field, and closely resemble the skyrmions in the usual non-linear sigma model. For the fully polarized states the spin is no longer quantized, but determined by Coulomb and Zeeman interactions.
There is a continuing strong interest in the Quantum Hall (QH) effect in multi component systems [1]. The components can be electrons with different spin, or electrons in different valleys (in Si systems), or in different layers in multilayer systems. We refer to all such components as spin. A new feature of these systems is that there are quasiparticle excitations with a topologically nontrivial texturing of the components. The first example of this was the “skyrmions” involving the spin [2,3]; there is now strong experimental evidence that they are the lowest energy quasiparticles in the fully polarized $\nu = 1$ QH effect [4–6]. Subsequently, a similar topological excitation, the meron, was used to explain a novel phase transition observed in double layer systems [7]. The textures have been studied within a two dimensional non-linear sigma model, describing the long distance physics, where the quasi particles are the (baby) skyrmions, as well as within a Hartree-Fock scheme [2,7,8]. For slowly varying textures the two approaches agree.

In this letter we consider textures in a more general setting, which allows for partially polarized states, and which we hope will be useful in the study of textured edges and of the response to slowly varying external fields.

Our starting point is a Chern-Simons-Landau-Ginzburg (CSLG) lagrangian for a two component system. This theory has two CS fields, which are needed to describe partially polarized states, and an integer valued coupling matrix that determines filling fractions as well as charges and statistics of the quasiparticles. The resulting effective long-distance spin theory turns out to be a nonlinear sigma model covariantly coupled to a CS field, which survives from the original CSLG theory.

The presence of a gauge field in the sigma model allows for non-singular finite energy solutions of skyrmion type. The (topological) charge of the skyrmions is quantized, just as in the usual sigma model, but the coupling to the CS field along with the requirement of finite energy also quantizes the $z$-component of the spin.\(^1\) This is important, since the

\(^1\) Technically this is possible since the coupling of the CS field breaks scale-invariance even in the
vortex solutions in the original CSLG model have quantized charge and spin, and we should emphasize that in the ordinary sigma model the spin is not quantized. We thus believe that the covariant sigma model provides the correct spin theory for the partially polarized states. It is another question whether or not the skyrmions in the long distance theory give a good quantitative descriptions of the original vortices. Only numerical calculations can answer this.

It turns out that the fully polarized case is special; the spin is not quantized, and the size of the skyrmions is determined by the Coulomb and Zeeman terms, just as in the usual sigma model aproach. We note that CSLG theories with two CS fields have been discussed before [9,10,7], and those of our results that are purely kinematical, such as filling fractions and quantum numbers, are certainly not new.

We consider a two dimensional electron gas subject to a magnetic field perpendicular to the plane of motion. Following the general strategy for mean field calculations, we start from the following lagrangian describing the electrons in terms of a two-component \textit{bosonic} field $\phi$,

$$
\mathcal{L} = \phi^\dagger (i\partial_0 - a_0^1 - a_0^2\sigma_z - A_0)\phi - \kappa |(i\vec{V} + \vec{a}_1 + \vec{a}_2\sigma_z + \vec{A})\phi|^2 \\
- \frac{1}{\pi} l_{\alpha\beta} a_0^\alpha b_\beta - \frac{1}{2} \mu B \phi^\dagger \sigma_z \phi - V(\phi^\dagger \phi).
$$

(1)

The two (Coulomb gauge) CS gauge potentials $a_1^\mu$ and $a_2^\mu$ couple to the charge and the $z$-component of the spin densities respectively. This Chern-Simons lagrangian\textsuperscript{2} is such that flux quanta of the magnetic fields $b_\alpha = \epsilon_{ij} \partial^i a_\alpha^j$ are attached to the bosons described by $\phi$ so absence of Coloumb and Zeeman terms, so the skyrmions can never become large. If one gives up the requirement that the energy of the excitations is finite then they can be large, with a large (unquantized) spin.

\textsuperscript{2} We shall consistently use Coulomb gauge, but it is straightforward to introduce (redundant) longitudinal parts of the CS fields to obtain the usual form $\sim l_{\alpha\beta} \epsilon_{\mu\nu\rho} a_\alpha^\mu \partial^\nu a_\beta^\rho$ for the CS lagrangian\textsuperscript{[11]}. 

3
that they effectively become fermions. One can show that the most general way to achieve this is to take \( l_{\alpha \beta} = K_{\alpha \beta}^{-1} \) where \( K_{\alpha \beta} \) is a symmetric \( 2 \times 2 \) matrix with integer entries whose diagonal elements are both either even or odd. \( B \hat{e}_z = \nabla \times \vec{A} \) is the external magnetic field, \( A_0 \) is the external scalar potential and the “effective mass”, \( 1/2\kappa \), and the magnetic moment, \( \mu \), are phenomenological parameters.

To disentangle the charge and spin degrees of freedom we decompose \( \phi \) as,

\[
\phi = \varphi \chi \quad ,
\]

where \( \chi^\dagger \chi = 1 \) and the real field \( \varphi \) is related to the density, \( \rho \), by \( \varphi = \sqrt{\varphi^\dagger \varphi} = \sqrt{\rho} \). The CP(1) field \( \chi \) is related to the spin (unit)vector \( \hat{n} \) by \( \hat{n} = \chi^\dagger \vec{\sigma} \chi \). We also introduce the gauge potential \( \tilde{a}_\mu = i\chi^\dagger \partial_\mu \chi \), \( \mu = 0,1,2 \). Note that \( \tilde{a} \) is not a separate dynamical field, but determined by \( \chi \). (The degrees of freedom in \( \chi \) can conveniently be thought of as the two angles describing the direction of \( \hat{n} \) plus an additional overall phase.) transformations \( \chi \rightarrow e^{i\alpha(x,t)} \chi \) corresponding to \( a_1^\mu \rightarrow a_1^\mu + \partial^\mu \alpha \), but under the transformations \( \chi \rightarrow e^{i\beta(x,t)} \chi \), corresponding to \( a_2^\mu \rightarrow a_2^\mu + \partial^\mu \beta \), it rotates around the \( z \)-axis as \( \hat{n} \rightarrow e^{2i\beta(x)L_z} \hat{n} \), where the \( 3 \times 3 \) matrix \( L_z \) is the \( z \)-component of the angular momentum. Also note that \( \tilde{a}^\mu \) transforms as \( \tilde{a}^\mu \rightarrow \tilde{a}^\mu - \partial^\mu \alpha - n_z \partial^\mu \beta \), so the combination \( a_1^\mu - n_z a_2^\mu - \tilde{a}^\mu \) is gauge invariant. It is now merely a matter of algebraic manipulations to rewrite (1) as,

\[
\mathcal{L} = \varphi (i\partial_0 + \kappa \nabla^2)\varphi - V(\rho) - \rho (A_0 + a_1^0 + a_2^0 n_z + \tilde{a}_0) - \kappa \rho (\vec{\nabla} + \vec{\tilde{a}}_1 + \vec{\tilde{a}}_2 n_z + \vec{\tilde{a}})^2 \\
+ \frac{\kappa}{4} \rho (\vec{\tilde{D}}^{ab} \vec{\nabla}_b)^2 - \frac{1}{\pi} l_{\alpha \beta} a_\alpha^0 b_\beta - \frac{1}{2} \mu B \rho n_z 
\]

where the covariant derivative is defined by \( \vec{\tilde{D}}^{ab} = \delta^{ab} \vec{\nabla} + 2i\vec{\tilde{a}}_2 L_z^{ab} \). From the transformation properties of \( \tilde{a}^\mu \), it is easily established that, except for the gauge fixed CS-lagrangian, all terms in (3) are invariant under each of the two gauge transformations.

We now look for solutions to the equations of motion following from \( \mathcal{L} \). Following the usual line of arguments we impose the condition \( \vec{\tilde{A}} + \vec{\tilde{a}}_1 + \vec{\tilde{a}}_2 n_z + \vec{\tilde{a}} = 0 \), which implies that the external magnetic field is cancelled by the internal fields, \( i.e. \ B + b_1 + \hat{b}_2 + \hat{b} = 0 \), where \( \hat{b}_2 = \epsilon_{ij} \partial^j (a_2^i n_z) \). Varying \( \mathcal{L} \) with respect to \( a_\alpha^0 \) gives the constraints \( \pi \rho = -l_{11} b_1 - l_{12} b_2 \) and
\[ \pi \rho n_z = -l_{12}b_1 - l_{22}b_2. \]

For vanishing Zeeman energy (\( \mu = 0 \)) one easily verifies that a solution to the equations of motion is:

\[ \rho = l_{11}B/\pi \equiv \bar{\rho} \] (where \( V'(\bar{\rho}) = 0 \)),

\[ a^0_1 = a^0_2 = \bar{a}_2^\mu = \bar{a}^\mu = 0, \]

and the spin vector is an arbitrary constant unit vector with fixed z-component \( n_z = \frac{\bar{n}_z}{l_{11}} \).

This is a quantum Hall state with filling fraction, \( \nu = 2\pi \bar{\rho}/B \), and polarization, \( \bar{n}_z \),

\[ \nu = 2l_{11} = \frac{2K_{22}}{K_{11}K_{22} - K_{12}^2} \]

\[ \bar{n}_z \equiv \frac{l_{12}}{l_{11}} = -\frac{K_{12}}{K_{22}}. \]

(4)

The corresponding filling fractions for the spin up and spin down states are \( \nu^\uparrow = l_{11} + l_{12} \) and \( \nu^\downarrow = l_{11} - l_{12} \), as discussed in e.g. [10].

When the Zeeman energy is included, the solution above is modified. Instead of pointing in a fixed direction, the (still \( \vec{x} \)-independent) spin vector \( \hat{n} \) precesses around the magnetic field; the filling fraction and polarization still being given by (4). Note that these states are partially polarized.

We now derive our final form of the low energy lagrangian. Using \( b_1 = -(B + \hat{\mathbf{b}} + \tilde{\mathbf{b}}) \), and making the same type of approximation as in [3,2,7], namely neglecting terms \( \sim \partial_0 \varphi \) and \( \sim \vec{\nabla} \varphi \), we are left with the following lagrangian (still in Coulomb gauge) for the fields \( \rho, \hat{n} \) and \( a_2^\mu \):

\[ L = \rho \left[ \bar{a}^0 + \frac{\kappa}{4} (\vec{D} a^n_b \hat{n}_b)^2 - \frac{1}{2} \mu B n_z - A^0 \right] - V(\rho) \]

\[ - 2a^0_2 \delta \rho - \frac{1}{\pi} a^0_2 [l_{22}b_2 - l_{12}(\hat{b}_2 + \tilde{b})] \].

(5)

Here,

\[ \rho = \bar{\rho} + \delta \rho = \bar{\rho} + \frac{1}{\pi} [l_{11}(\hat{b}_2 + \tilde{b}) - l_{12}b_2] \]

\[ \rho n_z = \bar{\rho} \bar{n}_z + 2 \delta \rho \]

(6)

is the charge and spin density respectively, with the ground state values, \( \bar{\rho} \) and \( \bar{n}_z \) given above. In deriving (5) from (3) we integrated out \( a^0_1 \), but kept \( a^0_2 \). Note that although \( \hat{b}_2 \) and \( \tilde{b} \) are not separately invariant under the remaining gauge transformation, the combination \( \hat{b}_2 + \tilde{b} \) is. The mean field solution is regained from (5) by first minimizing the spin-stiffness

\[ ^3 \text{Also note that this combination is the curl of the covariant version of } \hat{a}^\mu, \text{ given by } \bar{a}^\mu = i \chi^\dagger D^\mu \chi = \]
term by taking \( \hat{n} \) constant and \( \vec{a}_2 = 0 \), this implies \( \hat{b}_2 = 0 \) and minimizing \( V(\rho) \) then also gives \( \vec{a} = 0 \), and, solving the \( a_0^2 \) constraint, \( \delta \rho_z = 0 \).

The lagrangian (3) contains terms \( \sim \delta \rho (\vec{D}^{ab} \hat{n}_b)^2 \) and \( \sim a_0^2 \rho (n_z - \hat{n}_z) \) that break scale invariance. The first is usually neglected since it is higher in derivatives of \( \hat{n} \) but in our case this might not be allowed since the skyrmions will be rather small. The second term depends explicitly on the CS field and is important for determining the size of the quasiparticles. Note that since the spin is quantized, the Zeeman term will be a constant independent of the profile of the skyrmion.

Before we study quasiparticle solutions, we comment on the special case of a fully polarized state. One can then use the lagrangian of Lee and Kane with only one CS field. This is obtained from (1) by letting \( a_{2\mu} = 0 \). All manipulations leading to (3) go through as before and the final result is obtained by letting \( a_0^0 = 0 \) in (3). This gives the lagrangian used by Sondhi et. al., if we use that \( \tilde{\rho} \) is the Pontryagin density, \( 8\pi \tilde{\rho} = e^{ij} \hat{n} \cdot (\partial_i \hat{n} \times \partial_j \hat{n}) \). Note that (3) contains the (Coulomb gauge) Hopf term,

\[
\delta \rho = \frac{1}{\pi} l_{11} \tilde{b} = 2l_{11} \tilde{\rho}
\]

which is needed to give the correct statistics to the quasiparticles. That such a Hopf term should be present in the effective lagrangian was proposed earlier, but to our knowledge

\[
i \chi^\dagger (\partial^\mu - i\sigma^z a_2^\mu) \chi = \tilde{a}^\mu + n_z a_2^\mu.
\]

To get the covariant form of the Hopf term, one must first introduce the longitudinal parts of the CS field to get the term \( \frac{1}{2\pi} \epsilon_{ij} a_1^i \partial_0 a_1^j \), and then note that (for \( a_2^0 = 0 \), the mean field condition implies \( \partial_0 a_1^i = -\partial_0 \tilde{a}^i \)).
it has not previously been derived. (The Hopf term is usually ignored since it is high in derivatives.)

When discussing the skyrmion type quasiparticles, we must treat the fully polarized case separately. We start with the partially polarized case where $|n_z| < 1$, and consider a general, static, rotationally symmetric, vortex solution:

$$\chi(\vec{r}) = \begin{pmatrix} \cos \frac{\alpha(r)}{2} e^{i n \phi} \\
\sin \frac{\alpha(r)}{2} e^{i n \phi} \end{pmatrix}, \quad \vec{a}_2(\vec{r}) = a(r) \hat{e}_\phi, \quad a_0^0(\vec{r}) = a_0^0(r), \quad (9)$$

where $(r, \phi)$ are polar coordinates. This ansatz for $\chi$ implies

$$\hat{n} = (\sin \alpha(r) \cos[(m - n)\phi], \sin \alpha(r) \sin[(m - n)\phi], \cos \alpha(r)) \quad , (10)$$

and $\vec{a} = -(n \cos^2 \frac{\alpha(r)}{2} + m \sin^2 \frac{\alpha(r)}{2}) \hat{e}_r \rho(r)$. It is straightforward to substitute (8) in the e.o.m. derived from the lagrangian (3) to get three differential equations for the functions $\alpha(r)$, $a(r)$ and $a_0(r)$. The detailed expressions will be presented elsewhere, and here we shall only discuss some general properties of the solutions. First we notice that only those with either $n$ or $m$ equal to zero are consistent with a constant $\phi = \sqrt{\rho}$. If there is vorticity in both the upper and lower spinor component, the density must go to zero at the center of the vortex to avoid singularities. However, if e.g. $n = 0$, we can avoid the singularity by taking $\alpha(0) = 0$, while still keeping $\rho(0)$ finite. These are the smooth “skyrmion” solutions referred to earlier. To see that the configuration given by (8) can have finite energy, it is sufficient to note that

$$\vec{D}^{a \bar{b}} \hat{n}^b = \hat{e}_r \partial_r \hat{n}^a + \hat{e}_\phi \left[ \frac{\partial_\phi}{r} \delta^{a \bar{b}} + 2ia(r) L^{ab}_z \right] \hat{n}^b, \quad (11)$$

so if

$$a(r) \to \frac{n - m}{2r} \quad \text{and} \quad \cos \alpha(r) \to n_z \quad , (12)$$

for $r \to \infty$, the covariant derivative in (11) vanishes, and $n_z$ takes its asymptotic value, which are the conditions for having finite energy. Note that the gauge potential is crucial in order to have finite energy “skyrmions”. It is at this point the logic will differ for a fully
polarized state, since in this case the field $\vec{a}_2$ decouple from $\hat{n}$ in the ground state, and there will be no extra condition of the type (12). In the usual sigma model, configurations of the type (4) are logarithmically divergent except for $\alpha(\infty)$ equal to 0 or $\pi$ corresponding to $n_z = \pm 1$, i.e. a fully polarized state having the usual skyrmions. This logarithmic divergence was discussed in [7].

The spin and charge (electric charge = $-eQ$), of a quasiparticle are given by

$$Q = \int d^2x \delta \rho \quad \text{and} \quad S = \int d^2x \delta \rho_s \quad ,$$

(13)

Note that although the total charge is given by the topological charge, just as in the case of the usual skyrmions considered in [2,7], the charge density $\delta \rho$ in (8) is no longer given simply by the Pontryagin density $\tilde{b}/2\pi$.

Combining (8) and (13) with the asymptotic values in (12), it is easy to derive,

$$Q = \frac{l_{11}}{\pi} \int d^2x \tilde{b} = -\frac{\nu}{2}(1 + \bar{n}_z)n - \frac{\nu}{2}(1 - \bar{n}_z)m \quad ,$$

(14)

so the quasiparticle charge is determined by the filling fraction and the polarization, or equivalently, by the filling fractions of the two spin levels. Similarly, by combining the $a_2^0$ constraint equation from (8) with the asymptotic condition (12), we get the spin of the quasiparticle as

$$S = -\frac{1}{2\pi} \int d^2x [l_{22}b_2 - l_{12}(\hat{b}_2 + \tilde{b})] = -\frac{1}{4}(\nu n_z + 2l_{22})n - \frac{1}{4}(\nu \bar{n}_z - 2l_{22})m \quad ,$$

(15)

which depends on the extra parameter $l_{22}$. Note that both charge and spin of the quasi particles are quantized, a result that was obtained earlier [10]. This is a consequence of having two U(1) CS fields and requiring the quasiparticle energy to be finite.

The fully polarized case, $n_z = \pm 1$, needs special attention since, as we have already mentioned, the finite energy condition then does not determine the asymptotic form of $\vec{a}_2$, and thus leaves the $b_2$ flux undetermined. If we assume the condition $n_z = \frac{l_{12}}{l_{11}}$ to be satisfied as in the partially polarized case, there is now a restriction on the parameters, $l_{12} = \pm l_{11}$. The filling fraction is one over an odd integer, just as in the mean field description with a
single (odd integer) statistics parameter. Furthermore, the quasiparticle charge $Q$ in this case is quantized whereas the spin $S$ is instead determined dynamically by the Zeeman term. It is pleasing that our generalized sigma model description of the fully polarized case does not qualitatively differ from the one given by the standard sigma model, when it comes to the quasi particles. It is however not excluded that other properties, like correlation functions, and edge excitations, will be qualitatively different.

However, there are also fully polarized ground states where the condition $l_{12} = \pm l_{11}$ is not satisfied. These are somewhat different. In these cases neither the charge nor the spin of the quasiparticles are quantized. Instead a linear combination of these, with coefficients determined by the matrices $K_{\alpha\beta}$, will be identical to the (integer) topological charge of the $\vec{n}$-field.

We end with a few comments:
1. A separate analysis is needed to check the stability of the mean field ground states. For sufficiently large Zeeman coupling one would expect the states which are not fully polarized in the direction of the magnetic field to become unstable. It is of interest to examine the conditions for stability relative to small oscillations as well as relative to quasiparticle creation.
2. One of the motivations for this work was to find a mean field theory for the edge excitations. It is believed that the number and properties of the (gapless) edge excitations can be inferred from the properties of the bulk state [13]. In particular, in a (abelian) CS description (related to the one employed here via a duality transformation [14,15]) a simple gauge argument due to Wen shows that there are equally many edge modes as CS fields [13]. It has recently been shown that the ground state at $\nu = \vec{n}_z = 1$ has a spin texture of the skyrmion type along the edge for suitable strength of the confining potential, and it is proposed that there is a related gapless excitation, in addition to the usual gapless density wave [16]. It is thus rather natural that the effective CS theory should contain two gauge fields, and in this context it would be interesting to study the dual CS theory corresponding to [13].
3. An important question which we have not addressed in this paper concerns the collective modes. For the polarized state with \( n_z = \frac{I_n}{I_1} \), we expect a spin wave with a gap given by the Zeeman energy, just as in the usual sigma model description. For the partially polarized states we expect a gap for the spin wave proportional to the cyclotron energy. As discussed in [7], this can not be correct since there are empty states in the lowest Landau level so the gap should be determined by the Zeeman and Coulomb energies. We have no resolution to this puzzle.

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