Exciton edge states and the microwave edge exciton absorption of a 2D topological insulator subject to the in-plane magnetic field are studied. The magnetic field forms a narrow gap in electron edge states that allows the existence of edge exciton. The exciton binding energy is found to be much smaller than the energy of a 1D Coulomb state. Phototransitions exist on the exciton states with even numbers, while odd exciton states are dark.

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We start from the Hamiltonian of a single electron in the edge state in the presence of external homogeneous magnetic field \( \mathbf{B} = (B_x, B_y, B_z) \)
\[ H_e = s\sigma \cdot \mathbf{p}_x + g|B_\perp|\mathbf{e} \cdot \mathbf{B}, \] (1)
where \( \mathbf{e} = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli matrices, \( \mathbf{p}_x \) is the one-dimensional momentum, \( \mu_B \) is the Bohr magneton, \( g \) and \( g_\perp \) are in- and out-of-plane components of the \( g \)-factor. The second term in Eq. (1) represents the Zeeman interaction. Hamiltonian (1) can be deduced from the 2D Hamiltonian for a CdTe/HgTe/CdTe quantum well (see Appendix).

In the absence of magnetic field at \( p_x = 0 \) Eq. (1) gives the double-degenerate state that linearly splits around the edge. The eigenvectors and eigenenergies of the Hamiltonian (1) are
\[ \epsilon_\pm = \pm \sqrt{s^2 |p_x|^2 + \Delta^2}, \]
\[ \Psi_{p_x}^\pm = \frac{1}{\mathcal{L}} e^{ip_x x} \left( s p_x \pm \sqrt{\Delta^2 + s^2 |p_x|^2} \right) \frac{1}{N_\pm}, \]
with \( \Delta = \sqrt{|g_\parallel B_\parallel|^2 + |g_\perp B_\perp|^2} \), \( N_\pm = \sqrt{\Delta^2 + s^2 |p_x|^2 \pm 2s \sqrt{\Delta^2 + s^2 |p_x|^2} \pm s^2 |p_x|^2} \), \( \mathcal{L} \) is the edge length; we set \( h = 1 \). This spectrum has a “relativistic” shape with a magneto-induced gap \( \Delta \) which can be made arbitrarily narrow.

Using Eq. (1), one can write the effective Hamiltonian for an electron–hole pair taking into account the Coulomb interaction \( V(x) \):
\[ H_{eh} = s(\sigma_e p_x - \sigma_h p_x) + g|B_\perp|\mathbf{e} \cdot \mathbf{B} + V(x_e - x_h). \]
(4)
Here, \( \mathbf{e} = \sigma \otimes \mathbf{I}, \sigma_\mathbf{I} = \mathbf{I} \otimes \sigma, \mathbf{I} \) is the 2 \( \times \) 2 identity matrix, \( x_e(h) \) is the electron (hole) coordinate, \( p_{eh}(x) = -i\hbar \partial_{eh}/\partial x \) is the electron (hole) momentum operator. The Coulomb interaction \( V(x_e - x_h) \) here depends on the coordinates along the edge.

Introducing the variables of the center of mass and relative motion \( X = (x_e + x_h)/2, x = x_e - x_h \), we find from Eq. (4)
\[ H_{eh} = -i\frac{s}{2}(\sigma_e x - \sigma_h x)\partial_X - is(\sigma_e + \sigma_h)\partial_X + g|B_\perp|\mathbf{e} \cdot \mathbf{B} + V(x). \]
(5)
It is evident enough that the solution of the Schrödinger equation will not depend on the magnetic field direction relative to the edge. To be convinced, one should perform the rotation clockwise through an angle \( \phi \) (\( \phi \) is the angle between \( \mathbf{B} \) and the \( x \) axis) around the \( z \) axis in spin spaces. Making the corresponding unitary transformation of the Hamiltonian given by Eq. (4), we arrive at
\[ H_{eh} = -i\frac{s}{2}(\sigma_e x - \sigma_h x)\partial_X - is(\sigma_e + \sigma_h)\partial_X + g|B_\perp|\mathbf{e} \cdot \mathbf{B} + V(x). \]
(6)
The solution of the Schrödinger equation is searched in the form of \( \Phi(x) e^{iPx/\sqrt{2E}} \), where \( \Phi(x) \) is the four-component spinor \( \Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4) \) and \( P \) is the total momentum. The equation for \( \Phi(x) \) reads
\[ 2s(\gamma\partial_x)\Phi_{1,4} = (E - V)\Phi_{1,4} - g|B_\perp|B(\Phi_2 + \Phi_3); \]
\[ \pm sP\Phi_{2,3} = (E - V)\Phi_{2,3} - g|B_\perp|B(\Phi_1 + \Phi_4), \]
(7)
where \( E \) is the energy.

Let us introduce new quantities \( F_\pm = \Phi_1 \pm \Phi_4, G_\pm = \Phi_2 \pm \Phi_3 \), satisfying the equations
\[ -4s^2 \left( \frac{F_+}{E - V} \right)' = (E - V)F_+, \]
\[ + \frac{4\Delta^2 (E - V)F_+}{s^2 p_x^2 - (E - V)^2}, \]
\[ F_- = -2is \frac{F_+'}{E - V}, \]
\[ G_+ = \frac{2\Delta(E - V)F_+}{(E - V)^2 - s^2 p_x^2}, \]
\[ G_- = \frac{2\Delta sP F_+}{(E - V)^2 - s^2 p_x^2}. \]
(8)
(9)
Note, that Eqs. (8) and (9), unlike Eq. (7), are valid if at least one of \( V, E, P \) is nonzero.

Intuitively, due to the one-dimensional character of the edge states, one can use strict 1D \( e^--h \) Coulomb interaction \( V_0(x) = -e^2/k\kappa |x| \) (\( \kappa \) is the background dielectric constant) for the subsequent consideration. However, due to divergence of the energy of the 1D Coulomb ground level, it is insufficient to use this potential. More accurately, the function \( V(x) \) is obtained by the integration of the 2D Coulomb function \(-e^2/(x|r_e - r_h|)\) with transversal wavefunctions \( g(y) \) of the edge problem:
\[ V(x) = \frac{-e^2}{k} \int \frac{dy_y dy_y g^2(y_x)g^2(y_y)}{\sqrt{x^2 + (y_x - y_y)^2}}. \]
(10)
The function \( g(y) \) is given by Eq. (20) in the Appendix. Let \( P = 0 \). Equations (8) and (9) have a “non-relativistic” limit when the characteristic Coulomb and, hence, exciton binding energies are less than \( \Delta \). In this case, we can modify Eqs. (8) and (9):
\[ \frac{1}{2m}F_+'' + (E - 2\Delta - V)F_+ = 0, \]
\[ G_+ = F_+, \quad G_- = 0, \quad F_- \ll F_+, \]
(11)
(12)
where \( m = \Delta/(2e^2) \) is the exciton mass and \( E - 2\Delta \ll 2\Delta \). Equation (11) has a simple physical meaning. Let us consider the electron and hole near the bottom (top) of the corresponding bands. Then, the case of the Hamiltonian of the pair can be written as \( p^2/2m_e + p^2/2m_h + 2\Delta + V(x_e - x_h) \), where \( m_e = m_h = 2m \). For a pair with zero total momentum, we get the Schrödinger equation \([p^2/2m + V(x) + 2\Delta - E]\psi = 0\), which coincides with Eq. (11).

Generally speaking, the divergence of the energy is limited not only by the above-mentioned finiteness of the Coulomb potential, but also by the finite width of the forbidden band \( 2\Delta \). Hence, the transition to the “non-relativistic” case is unjustified. However, it follows from the numerical calculations below that the calculated exciton energies in the entire considered region of the parameters are close to the non-relativistic result.

**MICROWAVE ABSORPTION**

The edge state absorption can be characterized by the real part of the edge 1D conductivity \( \sigma \) at light frequency \( \omega \). The absorbing power per edge unit length is \( \sigma(\omega)\langle \varepsilon^2(t) \rangle = \sigma(\omega)\varepsilon_0^2/2 \), where \( \varepsilon_0 = \varepsilon_0\cos(t) \) is the alternating electric field. If the considered system is a planar grating of HgTe strips, the light absorption through it can also be expressed in terms of the absorption coefficient is given by the interband matrix element of the velocity operator \( s_\alpha \) between single-electron wavefunctions at zero momentum \( \psi_0^\pm = (1, \pm 1)/\sqrt{2} \) and the scalar exciton wavefunction \( \psi(x) = F_e(x) \) at \( P = 0 \) and \( x = 0 \):

\[
\sigma(\omega) = \frac{2\pi e^2}{\omega} \sum_n |V_{\alpha n}|^2 |F_e(0)|^2 \delta(\omega - E_n)
\]

\[
= \sum_n \gamma_n \delta(\omega - E_n), \quad \gamma_n = \frac{2\pi e^2 s^2}{E_n} |F_e(0)|^2.
\]

The exciton absorption has delta-functional peaks at the energies of the motionless exciton \( E_n = 2\Delta - E_n \).

The generalization of Eq. (14) to the case of strong \( e-h \) interaction requires accounting for the mixing of electron and hole states with zero momenta. Within Eq. (7) one should find the transition probability between the exciton state and the “vacuum” state with zero energy and no potential. The vacuum state can be combined from the zero-energy solutions of Eq. (7) at \( V = 0, P = 0 \). Due to the degeneracy (double degeneracy of spinors and multiple degeneracy in momentum \( p \)), there are many zero-energy states. These states are

\[
F_p^{(1)} = \frac{1}{2\sqrt{\Delta^2 + s^2 p^2}} (-\Delta, sp, sp, \Delta)e^{ipx},
\]

\[
F_p^{(2)} = \frac{1}{2\Delta} (0, 1, -1, 0)e^{ipx}.
\]

However, the vacuum corresponds to only one combination of these states:

\[
F^{\text{vac}} = \sum_p F_p^{(1)},
\]

which gives the right expression for a transition amplitude to the free electron–hole state, coinciding with the result obtained in the single-electron approach. With the use of Eqs. (16) and (15) we have

\[
F^{\text{vac}} = \frac{1}{2} \left[ -D(x), -\frac{isD'(x)}{\Delta}, -\frac{isD'(x)}{\Delta}, D(x) \right],
\]

where \( D(x) = (\Delta/\pi s)K_0(\Delta|x|/s) \) (\( K_0 \) is the modified Bessel function of the second kind).

As a result, we get

\[
\sigma(\omega) = \frac{2\pi e^2 s^2}{\omega} \sum_n \left[ \int_{-\infty}^{\infty} dx D(x) F_e(x) \right]^2 \delta(\omega - E_n).
\]

Near the threshold, Eq. (18) goes to Eq. (14). However, the presence of function \( D(x) \) expands the applicability of this formula to the case of strong \( e-h \) interaction.

**NUMERICAL RESULTS**

We have done our calculations with the use of the CdTe/HgTe/CdTe system parameters from [4]. We consider the 7-nm-wide quantum well for which \( \mathcal{A} = 3.645 \text{ eV Å}, \mathcal{B} = -68.6 \text{ eV Å}^2, \mathcal{D} = -51.2 \text{ eV Å}^2, \mathcal{M} = -0.010 \text{ eV} \). The dielectric constant of CdTe, according
to [21], is $\kappa = 10.2$; the $g$-factor, according to [1], is $g_0 = -20.5$.

The dependence of the effective interaction potential (see Eq. (10)) on the $e^-h$ distance is illustrated in Fig. 1. We see that, at a large $x$ value, the potential approaches the strict Coulomb potential $V_0(x) = -\frac{e^2}{\kappa|x|}$, but, at a small distance, it is essentially weaker than $V_0(x)$. At the same time, the potential conserves the singularity (weaker than $1/|x|$) at $x \to 0$. This fact is conditioned by the edge wavefunction singularity. The behavior of the potential at $x \to 0$ is especially important for the exciton ground state which energy diverges in the strict Coulomb case.

We have found the exciton wavefunctions and energies and optical exciton excitation/recombination probabilities. The results are presented in Figs. 1–3. The energies of the ground and excited states at $P = 0$ versus magnetic field are shown in Fig. 2. The excited states for a large $n$ approach Coulomb values $me^4/\kappa^2n^2$. They become twins with the distance in pair much less than the distance between twins. This corresponds to the observation made in [17]: the wavefunction divergence in the one-dimensional Coulomb problem leads to the separation of the $x$ axis into domains $x > 0$ and $x < 0$, where the states are independent from each other. The account of the potential finiteness at $x = 0$ mixes left and right states and lifts the degeneracy converting the levels into close pairs.

The magnetic field dependence of coefficients $\gamma_n$ describing the exciton absorption is shown in Fig. 3. The odd $n$ states are dark states with zero absorption in the dipole approximation. The even states are optically active. The absorption coefficients grow with $B$ and fall with $n$.

At $B = 5$ T, the characteristic values of the ground exciton energy and coefficient $\gamma_0$ are $\varepsilon_0 = 3.528$ meV, $2\Delta - \varepsilon_0 = 45.386$ meV, and $\gamma_0 = 7.29 \times 10^{-4}$ eV cm$^2$/s.

This value of $\varepsilon_0$ is sufficient for experimental observation at temperatures less than $|\varepsilon_0|/3 = 13$ K.

To observe the exciton absorption, one should produce the system with multiple edges, for example, a grating with non-overlapping exciton wavefunctions. These conditions will be satisfied for a grating of strips with width 0.5 $\mu$m and period 2 $\mu$m. Assuming the resonance width of delta-function in Eq. (18) $\Gamma = 5 \times 10^{-2}|\varepsilon_0|$, we obtain the exciton absorbance of the grating $a = 0.17$. This value looks quite large to be observed.
DISCUSSION

Thus, we have calculated the exciton energy levels and absorption coefficients on exciton transitions for the edge excitons in the 2D topological insulator HgTe with the in-plane magnetic field. The $e-h$ interaction is strong enough to produce edge excitons with the binding energy of the order of $3.5 \times 10^{-3}$ eV and total energy $\sim 4.5 \times 10^{-2}$ eV. The edge exciton states are strongly affected by the smearing of their transversal wavefunctions, which essentially decreases the exciton energies. At the same time, the mixing of electron and hole states by the Coulomb interaction does not affect them essentially. The excited even and odd exciton states appear in pairs. The absorption on odd excitons is suppressed.

Note that we have dealt with a two-particle approach only. In fact, the strong enough $e-e$ interaction in the 1D system absolutely reconstructs the many-body system ground state. In this case, many-body effects should be taken into account by the bosonization procedure. The reconstruction of the ground state means accounting for virtual processes with the excitation of multiple $e-h$ pairs. These processes are weak if excitation energy $\Delta$ is large as compared to the Coulomb energy (the validity of this assumption is supposed throughout the present paper). Otherwise, the problem should be considered within the Luttinger liquid approach going beyond the scope of the paper.

Note also, that the edge exciton is the lowest-lying branch of the 2D exciton in a 2D topological insulator. The 2D exciton energy is bound to the 2D topological insulator gap being essentially larger than the edge exciton energy. One can predict that the edge excitons should play the role of collectors for electron–hole states in the 2D topological insulator. Exciton energy $\sim 4.5 \times 10^{-2}$ eV. The edge exciton states are in pairs. The absorption on odd excitons is suppressed.

In the first order of the magnetic field, the 1D Hamiltonian proposed in [8] and complete it by the vector-potential of the magnetic field $A$ by replacement of electron momentum $p \rightarrow p - eA/c$. Here, $B_z = B_z \pm iB_x$, $g_{EL}$, $g_{HL}$ are in-plane and out-of-plane $g$-factors of a 2D topological insulator.

$\psi_{s}(y) = \frac{g(y)}{\sqrt{1 + \eta^2}}, \quad \psi_{s}(y) = \frac{g(y)}{\sqrt{1 + \eta^1}}, \quad \psi_{s}(y) = \frac{g(y)}{\sqrt{1 + \eta^2}}$

where $\eta = \sqrt{(B + D)/(B - D)}$, $g(y) = (e^{-\lambda_1^g} - e^{-\lambda_2^g})/2\lambda_1^g\lambda_2^g(-\lambda_2^g + \lambda_2^g)$, $\lambda_1^g = \lambda_2^g = \sqrt{\mathcal{A}^2/(\mathcal{B}^2 - \mathcal{D}^2)} - \mathcal{M}$.

Here, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, $\mathcal{M}$ are the HgTe layer parameters (they are determined by the material parameters and the quantum well width). As a result, we have

$$H_\epsilon = \sigma_z \left( p_x - \frac{e}{c} \overline{A}_x \right) + \mu_B g_{\parallel} \sigma \cdot B_{\parallel} + g_{\parallel} \sigma \cdot B_{\parallel}.$$  

Here, $\overline{A}_x$ is the mean value of the vector-potential averaged with the transversal edge-state wavefunctions $g(y)$. The second term in Eq. (21) represents the Zeeman part of the Hamiltonian, where $g_{\parallel} = (g_{EL} + g_{HL})/2 + (g_{HL} - g_{EL})\mathcal{D}/(2\mathcal{B})$.

In the case of a constant homogeneous magnetic field $A_z$ and $B_z$ can be excluded from Hamiltonian (1) by a non-essential shift of momentum $p_x \rightarrow p_x + eA_z/c - \mu_B g_{\parallel} B_{\parallel}$.

Note that, in the chosen approximation neglecting the bulk induced anisotropy, the $B_z$-induced gap is absent; besides, the states have the isotropic in-plane $g$-factor. A more general case (see discussion in [20]) can be considered similarly.

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