Some remarks on the geometry of grammar

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Abstract

This paper, following [4], presents an approach to grammar description and processing based on the geometry of cancellation diagrams, a concept which plays a central role in combinatorial group theory [6]. The focus here is on the geometric intuitions and on relating group-theoretical diagrams to the traditional charts associated with context-free grammars and type-0 rewriting systems. The paper is structured as follows. We begin in Section 1 by analyzing charts in terms of constructs called cells, which are a geometrical counterpart to rules. Then we move in Section 2 to a presentation of cancellation diagrams and show how they can be used computationally. In Section 3 we give a formal algebraic presentation of the concept of group computation structure, which is based on the standard notions of free group and conjugacy. We then relate in Section 4 the geometric and the algebraic views of computation by using the fundamental theorem of combinatorial group theory [8]. In Section 5 we study in more detail the relationship between the two views on the basis of a simple grammar stated as a group computation structure. In section 6 we extend this grammar to handle non-local constructs such as relative pronouns and quantifiers. We conclude in section 7 with some brief notes on the differences between normal submonoids and normal subgroups, group computation versus rewriting systems, and the use of group morphisms to study the computational complexity of parsing and generation.

1 Introduction: grammar and geometry

In the drawing on the left of Figure 1, we give an example of a simple context-free chart, where we have assumed the edges to be conventionally oriented from left to right.

Seen in geometric terms, this chart is an oriented planar graph dividing the plane in five internal regions and one external one. Each of the internal regions
can be transformed into a disk by a topological transformation of the plane preserving orientation. The five corresponding circular graphs, or cells, are shown on the right. Relative to one of these cells, an edge can be considered oriented positively if it goes clockwise, or negatively if it goes counter-clockwise. Thus, the NP edge of the S-NP-VP cell is oriented negatively, while the NP edge of the NP-john cell is oriented positively.

It is then an obvious observation that the chart on the left can be obtained by a process of “stitching” together the five cells along identically labelled but inversely oriented edges. In the course of this operation, each cell undergoes an orientation-preserving topological deformation.

The cells of Figure 1 provide a geometric presentation of context-free productions. Each cell has the property that it has exactly one positively oriented edge, labelled with a nonterminal, and one or several negatively oriented edges, labelled with nonterminals or terminals. If, in the context-free chart, one chooses a point in each internal region and draws lines between two points belonging to adjacent regions, then one obtains a planar graph which is tree-like, and which corresponds to the usual notion of derivation tree in a context-free grammar.

If, instead of a context-free grammar, one considers a general type-0 grammar, charts can be generalized to graphs which do not have this “tree-like” property. An illustration of such a case is given in Figure 2, which shows a derivation of the French prepositional phrase “du livre”, involving the contraction “du” of the preposition “de” with the determiner “le”. The graph on the left could be called a “type-0 chart”, or a “Q-system chart”, in reference to

\[1\] This construct is related to the notion of dual graph of a planar graph.

\[2\] Q-systems were a direct predecessor of Prolog. Because of the symmetry between input and output, the formalism could be applied to both parsing and generation in machine translation and was a pioneer of grammar reversibility.
The cells associated with this chart are shown on the right. In contrast to the context-free case, one of the cells now contains two positively oriented edges.

Figure 2: A “type-0 chart”, or “Q-system chart”, and the corresponding cells.

In the computation systems we are proposing here, we will consider cells with an arbitrary number of positively or negatively oriented edges to be the building blocks from which computations are built. Our formalization will rely on the mathematical concept of a CANCELLATION DIAGRAM.

2 Cancellation Diagrams

2.1 Diagrams

Cancellation diagrams were introduced in 1933 by Van Kampen and have been playing since the seventies an increasing role in combinatorial group theory.

Definition. A CANCELLATION DIAGRAM, or simply DIAGRAM, over the vocabulary $V$ is a finite graph which is: (1) planar, that is, embedded in the plane in such a way that two edges can only intersect at a vertex; (2) connected; (3) directed, that is, the edges carry an orientation; (4) labelled, that is, each edge carries a label taken in $V$. In the limit, a graph consisting of a single vertex is also considered to be a diagram.

A diagram separates the plane in $n+1$ connected open sets: the exterior (set of points that can be connected to a point at infinity without crossing an edge), and $n$ open internal regions, called CELL REGIONS, each consisting of points
which can be connected without crossing an edge, but which are separated from the exterior. The set of points in the plane which belong to an edge or to an internal region of the diagram is called the **locus** of the diagram. The locus of a diagram is always a connected and simply connected (no holes) closed region of the plane. An edge whose points are not in the closure of a cell region is called a **thin** edge of the diagram.

An example of a diagram over the vocabulary \(\{a, b, c\}\) is given in Fig. 3. This diagram is made up of three cells and contains one thin edge.

![Figure 3: A diagram. A boundary of this diagram is \(c^{-1}c^{-1}a^{-1}cacaac^{-1}\).](image)

The **boundary** of a cell is the set of edges which constitute its topological boundary. The boundary of a diagram is the set of edges which are such that all their points are connected to the exterior.

If one choses an arbitrary vertex (such as O in the figure) on the boundary of a diagram, and if one moves on the boundary in a conventional clockwise fashion, then one collects a list of edges which are either directed in the same way as the movement, or contrary to it. By producing a sequence of labels with exponent +1 in the first case, \(-1\) in the second case, one can then construct a word over the vocabulary \(V \cup V^{-1}\) (that is, a word in the **free group** over \(V\), see below); this word is said to be a **boundary word** of the diagram.

### 2.2 Reduced diagrams

We will say that a diagram is **reduced** if there does not exist a pair of edges with a common vertex \(O\), with the same label, oriented oppositely relative to \(O\) (that is, both edges point towards \(O\) or both point from \(O\)), and such that at least one of the two “angles” formed by the two edges is “free”, that is, does not “contain” another diagram edge (see Fig. 4).

### 2.3 Diagrams as computational devices

Suppose that one is given a fixed set of cells \(C\) over a vocabulary \(V\). Let’s consider the following generative process for producing diagrams (only informally described here):
Figure 4: A reduced diagram. The diagram of the previous figure was not reduced because of the two c edges outgoing from vertex O.

1. Initialize the diagram by choosing a point \( O \) in the plane;

2. Iterate an arbitrary number of times the following procedure any of the following steps:

   - Add a new oriented labelled edge to the exterior of the current diagram by connecting it to one vertex on the boundary of the diagram;
   - Add a new cell taken from \( C \) to the exterior of the current diagram, either by connecting one vertex of the cell to one vertex on the boundary of the diagram or by "pasting" the cell to the boundary of the diagram along consecutive edges having the same labels and arrow directions;
   - Reduce the current diagram by "folding together" two adjacent edges on its boundary which have the same label but opposite directions (informally, the deformation involves gradually reducing the external "angle" between the two edges until they are identified, that is, the folding is towards the interior of the diagram);

It is easy to prove that this process generates exactly the set \( D \) of diagrams \( d \) over \( V \) such that all the cells of \( d \) are elements of \( C \). The boundary words of elements of \( D \) can be seen as coding results of the computations determined by the "specification" \( C \).

For instance, going back to the example of Figure 1, the boundary word \( S \, \text{mary}^{-1} \, \text{likes}^{-1} \, \text{john}^{-1} \) is the result of a computation over the specification \( C \) consisting of the five circular cells in the figure. Any word thus obtained which is of the form \( S \, \text{word}_n^{-1} \ldots \, \text{word}_1^{-1} \) can be seen as coding the fact that \( \text{word}_1 \ldots \, \text{word}_n \) is a sentence relative to the "grammar" specified by \( C \).

This informal notion of computation with diagrams will now be made more precise by turning to its algebraic counterpart, group computation structures.
3 Computation in groups

We start by quickly reviewing some basic concepts of group theory before turning to group computation structures.

3.1 Groups, monoids, normal subsets

A **monoid** $M$ is a set $M$ together with a product $M \times M \rightarrow M$, written $(a, b) \mapsto ab$, such that:

- This product is associative;
- There is an element $1 \in M$ (the neutral element) with $1a = a1 = a$ for all $a \in M$.

A **group** is a monoid in which every element $a$ has an inverse $a^{-1}$ such that $a^{-1}a = aa^{-1} = 1$.

A **submonoid** of $G$ is a subset of $G$ containing $1$ and closed under the product of $G$. A **subgroup** of $G$ is a submonoid of $G$ which is closed under inversion.

Two elements $x, x'$ in a group $G$ are said to be **conjugate** if there exists $y \in G$ such that $x' = yxy^{-1}$.

A subset (resp. a subgroup, a submonoid) of a group $G$ is said to be a **normal subset** (resp. **normal subgroup**, **normal submonoid**) of $G$ iff when it contains $x$, it contains all the conjugates of $x$ in $G$.

If $S$ is a subset of $G$, the intersection of all normal submonoids of $G$ containing $S$ (resp. of all subgroups of $G$ containing $S$) is a normal submonoid of $G$ (resp. a normal subgroup of $G$) and is called the **normal submonoid closure** $NM(S)$ of $S$ in $G$ (resp. the **normal subgroup closure** $NG(S)$ of $S$ in $G$).

The notion of normal subgroup is central in algebra. For our purposes here, the less usual notion of normal submonoid will be the more important notion.

3.2 The free group over $V$.

Let’s consider an arbitrary set $V$, called the **vocabulary**, and let’s form the so-called **set of atoms on $V$**, which is notated $V \cup V^{-1}$ and is obtained by taking elements $v$ in $V$ as well as the formal inverses $v^{-1}$ of these elements.

We now consider the set $F(V)$ consisting of the empty string, notated $1$, and of strings of the form $x_1x_2...x_n$, where $x_i$ is an atom on $V$. It is assumed that such a string is **reduced**, that is, never contains two consecutive atoms which are inverse of each other: no substring $vv^{-1}$ or $v^{-1}v$ is allowed to appear in a reduced string.

When $\alpha$ and $\beta$ are two reduced strings, their concatenation $\alpha\beta$ can be reduced by eliminating all substrings of the form $vv^{-1}$ or $v^{-1}v$. It can be proven that the reduced string $\gamma$ obtained in this way is independent of the order of such eliminations. In this way, a product on $F(V)$ is defined, and it is easily shown that $F(V)$ becomes a (non-commutative) group, called the **free group** over $V$. 

6
3.3 Group computation

We will say that an ordered pair $GCS = (V, R)$ is a group computation structure if:

1. $V$ is a set, called the vocabulary, or the set of generators
2. $R$ is a subset of $F(V)$, called the lexicon, or the set of relators.

The submonoid closure $NM(R)$ of $R$ in $F(V)$ is called the result monoid of the group computation structure $GCS$. The elements of $NM(R)$ will be called computation results, or simply results.

If $r$ is a relator, and if $\alpha$ is an arbitrary element of $F(V)$, then $\alpha r \alpha^{-1}$ will be called a quasi-relator of the group computation structure. It is easily seen that the set $R_N$ of quasi-relators is equal to the normal subset closure of $R$ in $F(V)$, and that $NM(R_N)$ is equal to $NM(R)$.

A computation relative to $GCS$ is a finite sequence $c = (r_1, \ldots, r_n)$ of quasi-relators. The product $r_1 \cdots r_n$ in $F(V)$ is evidently a result, and is called the result of the computation $c$. It can be shown that the result monoid is entirely covered in this way: each result is the result of some computation. A computation can thus be seen as a “witness”, or as a “proof”, of the fact that a given element of $F(V)$ is a result of the computation structure.

For specific computation tasks, one focusses on results of a certain sort, for instance results which express a relationship of input-output, where input and output are assumed to belong to certain object types. For example, in computational linguistics, one is often interested in results which express a relationship between a fixed semantic input and a possible textual output (generation mode), or conversely in results which express a relationship between a fixed textual input and a possible semantic output (parsing mode).

If $GCS = (V, R)$ is a group computation structure, and if $A$ is a given subset of $F(V)$, then we will call the pair $GCS_A = (GCS, A)$ a group computation structure with acceptors. We will say that $A$ is the set of acceptors, or the public interface, of $GCS_A$. A result of $GCS$ which belongs to the public interface will be called a public result of $GCS_A$.

3 For readers familiar with group theory, this terminology will evoke the classical notion of group presentation through generators and relators. The main difference with our definition is that, in the classical case, the set of relators is taken to be symmetrical, that is, to contain $r^{-1}$ if it contains $r$. When this additional assumption is made, $NM(R)$ becomes equal to $NG(R)$.

4 The analogy with the view in constructive logics is clear. There what we call a result is called a formula or a type, and what we call a computation is called a proof.
4  Relating the geometric and the algebraic views: the fundamental theorem of combinatorial group theory

The two views of computation provided on the one hand by diagrams and on the other hand by group computation structures are in fact equivalent to each other. This equivalence is the consequence of the “the fundamental theorem of combinatorial group theory” [8].

4.1 Cyclically reduced words

Definition. A word $w$ on $V \cup V^{-1}$ is said to be **cyclically reduced** iff every cyclic permutation of it is reduced.

It is easy to see that:

- A reduced word is cyclically reduced iff it is not of the form $aw'a^{-1}$ with $a$ an atom (positive or negative);
- If a word is cyclically reduced, then all its cyclic permutations are cyclically reduced;
- For any word $w$, there is a conjugate of $w$ which is cyclically reduced;
- Two conjugates of a word $w$ which are cyclically reduced are cyclic permutations of each other.

It is often convenient to picture the set of all cyclic permutations of a cyclically reduced word as a circular diagram of labelled oriented edges such that no adjacent edges cancel each other. For any word $w$, such a diagram provides a canonical representation of the cyclically reduced conjugates of $w$.

4.2 Relator cells

Consider a group computation structure $GCS = (V, R)$. Without loss of generality, it can be assumed that the relators in $R$ are cyclically reduced, because the result monoid is invariant when one considers a new set of relators consisting of conjugates of the original ones. From now on, unless stated otherwise, this assumption will be made for all relators considered.

Take any such cyclically reduced relator $r = x_1^{e_1} \ldots x_n^{e_n}$, where $x_i \in V$ and $e_i = \pm 1$, and construct a labelled cell in the following way: take a circle and divide it in $n$ arcs; label the clockwise-$i$th arc $x_i$ and orient it clockwise if $e_i = 1$, anti-clockwise otherwise. The labelled cell thus obtained is called the **relator cell** associated with $r$.

Rather than presenting the GCS through a set of relator words as we have done before, it is now possible to present it through a set of relator cells; if one gives such a set, a standard presentation of the GCS can be derived by taking an arbitrary origin on each cell and “reading” the relator word clockwise from this.
origin; the origin chosen does not matter: any other origin leads to a conjugate relator, and this does not affect the notion of result.

4.3 Fundamental theorem of combinatorial group theory

We are now able to state what J. Rotman calls “the fundamental theorem of combinatorial group theory” \[8\]. We give the theorem in a slightly extended form, adapted to the case of a GCS, that is, using normal sub-monoid closure rather than normal subgroup closure; the subgroup case follows immediately by taking a set or relators containing $r^{-1}$ along with $r$.

**Theorem 1** Let $GCS = (V, R)$ be a group computation structure such that all relators $r \in R$ are cyclically reduced. If $w$ is a cyclically reduced word in $F(V)$, then $w \in NM(R)$ if and only if there exists a reduced diagram having boundary word $w$ and whose regions are relator cells associated with the elements of $R$.

The proof is not provided; it can easily be recovered from the property demonstrated in \[6\] (chapter 5, Section 1). The proof involves the following remark. If one considers a product

$$u_1 r_1 u_1^{-1} \ldots u_n r_n u_n^{-1}$$

with $r^{-1} \in R$ and $u_i$ arbitrary elements of $F(V)$, this product can be read as the boundary word of the “star” diagram represented in Fig. \[5\] starting at $O$ and progressing clockwise.

This star diagram is in general not in reduced form, but it can be reduced by a stepwise process of “stitching together” edges which do not respect the definition of a reduced diagram.

**Example.** Let’s consider a GCS with vocabulary $V = \{a, b, c\}$ and set of (cyclically reduced) relators

$$R = \{c^{-1}c^{-1}a^{-1}c^{-1}, acb^{-1}, baa\}$$

The cyclically reduced word $c^{-1}aac^{-1}$ is an element of $NM(R)$, for it can be obtained by forming the product

$$c^{-1}c^{-1}a^{-1}c^{-1} \cdot cacb^{-1}c^{-1} \cdot cbaac^{-1}.$$
Figure 6: Transformation of a diagram into reduced form (adapted from [7]).
If we form the star diagram for this product, we obtain the first diagram shown in Fig. 6.

This diagram is not reduced, for instance the two straight edges labelled $c$ are offending the reduction condition. If one “stitches” these two edges together, one obtains the second diagram in the figure. This stitching corresponds to a one-step reduction of the boundary word of the first diagram, $c^{-1}c^{-1}a^{-1}c^{-1} cacb^{-1}c^{-1} cbaac^{-1}$ into the boundary word of the second $c^{-1}c^{-1}a^{-1}c^{-1} cacb^{-1}baac^{-1}$. By continuing in this way, one obtains the fifth diagram of the figure, which is reduced, and whose boundary is the desired result $c^{-1}aac^{-1}$.

5 Applications to grammar

5.1 A simple grammar

We will now show how the formal concepts introduced above can be applied to the problems of grammatical description and computation. We start by introducing a simple grammar $G$-Grammar for a fragment of English. In Figure 7, this grammar is presented algebraically in terms of relators, and in Figure 8 the same grammar is presented geometrically in terms of relator cells.

Formally, $G$-Grammar is a group computation structure with acceptors over a vocabulary $V = V_{\text{log}} \cup V_{\text{phon}}$ consisting of a set of logical forms $V_{\text{log}}$ and a disjoint set of phonological elements (in the example, words) $V_{\text{phon}}$. Thus $john$, $saw$ are phonological elements, $j$, $s(j, l)$ are logical forms.

The grammar lexicon, or set of relators, $R$ is given as a list of “lexical schemes”. Thus, in Figure 6, each line is a lexical scheme and represents a set of relators in $F(V)$. The first line is a ground scheme, which corresponds to the single relator $j$ $john^{-1}$, and so are the next four lines. The sixth line is a non-ground scheme, which corresponds to an infinite set of relators, obtained by instanciating the term meta-variable $A$ (notated in uppercase) to a logical form. So are the remaining lines.

The vocabulary and the set of relators that we have just specified define a group computation structure $GCS = (V, R)$. Let’s now describe a set of acceptors $A$ for this computation structure. We take $A$ to be the set of elements of $F(V)$ which are products of the following form:

$$SW_n^{-1}W_{n-1}^{-1}\ldots W_1^{-1}$$

where $S$ is a logical form (S stands for “semantics”), and where each $W_i$ is a phonological element (W stands for “word”). The expression above is a way of encoding the ordered pair consisting of the logical form $S$ and the phonological string $W_1W_2\ldots W_n$ (that is, the inverse of the product $W_n^{-1}W_{n-1}^{-1}\ldots W_1^{-1}$).
Figure 7: G-Grammar given in algebraic terms: relator schemes

Figure 8: G-Grammar given in geometric terms: cells schemes.
5.2 Computation

We now show how to compute a proof of the fact that $i(s(j,l),p) \overset{\text{paris}^{-1}}{\overset{-1}{\overset{\text{in}^{-1}}{\overset{\text{louise}^{-1}}{\overset{\text{saw}^{-1}}{\overset{\text{john}^{-1}}{\text{is}}}}}}}$ is a public result for G-Grammar, or, in other words, that the logical form $i(s(j,l),p)$ and the phonological form $john \ saw \ louise \ in \ paris$ are in correspondance relative to G-Grammar.

We start by an informal geometric computation of this result and follow by an algebraic computation of it. The geometric computation if the more intuitive of the two, the algebraic one the one for which we have the more precise formal definition and which more directly displays group-theoretical characteristics.

5.2.1 Geometric computation

Consider the diagram of Figure 9. The cells of this diagram are instances of the cell schemes of Figure 8. From theorem 1, this means that $i(s(j,l),p) \overset{\text{paris}^{-1}}{\overset{-1}{\overset{\text{in}^{-1}}{\overset{\text{louise}^{-1}}{\overset{\text{saw}^{-1}}{\overset{\text{john}^{-1}}{\text{is}}}}}}}$ is a public result of G-Grammar.

The diagram in Figure 9 can be obtained by several computations, in the sense of Section 2.3. One such computation, corresponding to a generation mode, starts from the top edge ($i(s(j,l),p)$) and progressively adds cells in a top-down way. A second computation, corresponding to a parsing mode, starts by building a diagram consisting of the sequence of the five phonological edges at the bottom and progressively adds cells in a bottom-up way. Still another geometric computation, establishing the connection with the algebraic computation, will be shown below.

5.2.2 Algebraic computation

Consider the following relators, instanciations of relator schemes of Figure 6:

$r_1 = i(s(j,l),p) \overset{\text{p}^{-1}}{\overset{\text{in}^{-1}}{\overset{s(j,l)^{-1}}{}}}$
$r_2 = s(j,l) \overset{\text{l}^{-1}}{\overset{saw^{-1}}{\overset{j^{-1}}{}}}$
$r_3 = j \overset{\text{john}^{-1}}{\overset{}}$
$r_4 = l \overset{\text{louise}^{-1}}{\overset{}}$
\[ r_5 = p \text{ par}is^{-1} \]

and the quasi-relators:

\[
\begin{align*}
  r_1' &= r_1 \\
  r_2' &= r_2 \\
  r_3' &= r_3 \\
  r_4' &= (john \ saw) \ r_4 \ (john \ saw)^{-1} \\
  r_5' &= (john \ saw \ louise \ in) \ r_5 \ (john \ saw \ louise \ in)^{-1}
\end{align*}
\]

Then we have:

\[
\begin{align*}
  r_1' \ r_2' \ r_3' \ r_4' \ r_5' &= \\
  i(s(j,l),p) \text{ par}is^{-1} \text{ in}^{-1} \text{ louise}^{-1} \text{ saw}^{-1} \text{ john}^{-1}
\end{align*}
\]

and therefore \( i(s(j,l),p) \text{ par}is^{-1} \text{ in}^{-1} \text{ louise}^{-1} \text{ saw}^{-1} \text{ john}^{-1} \) is the result of a computation \((r_1', r_2', r_3', r_4', r_5')\), as announced.

What is the relationship of this computation with the geometric view of Figure 9? The answer is given by the geometric computation indicated by the diagram in Figure 10. If we read the boundary of this diagram clockwise starting from O, we obtain the unreduced expression \( r_1' \ r_2' \ r_3' \ r_4' \ r_5' \). But if we perform further geometric reduction steps on the diagram of Figure 10, we obtain the diagram of Figure 9.

![Diagram](image-url)

Figure 10: An unreduced version of the diagram of Figure 9, showing explicitly the role of conjugates.
6 Multi-relators and non-local dependencies

6.1 Multi-relators

The linguistic examples that we have presented up to now have been rather simple. We will now introduce an extension of group computation structure which gives rise to forms of non-local grammar dependencies.

Let \( mr = \langle w_1; \ldots; w_n \rangle \) be a finite multiset (unordered list) of words in \((V \cup V^{-1})^*\). We will call such an expression a multi-relator over \( V \). We will say that a word \( w \) is a multi-conjugate of \( mr \) iff \( w \) can be expressed as a product \( \alpha_1 w_1 \alpha_1^{-1} \ldots \alpha_n w_n \alpha_n^{-1} \), where \( \alpha_1, \ldots, \alpha_n \) are elements of \( F(V) \). Let \( MC(mr) \) be the set of multi-conjugates of \( mr \).

\[ \text{Remark: This notion is well-defined, for any ordering of the multiset } \text{mr} \text{ leads to the same set of multi-conjugates (by a simple property of conjugacy). It is also easy to check that replacing any } w_i \text{ by a cyclic permutation of } w_i \text{ does not change } MC(mr), \text{ and furthermore, that } MC(mr) \text{ is a normal subset of } F(V). \]

If we are given a (finite or infinite) collection \( MR \) of multi-relators, we can consider the group computation structure obtained by taking as set \( R \) of relators the set:

\[ R = \bigcup_{mr \in MR} MC(mr); \]

We will call the GCS thus obtained the group computation structure with multi-relators \( GCS-MR = (V, MR) \).

When presenting a group computation structure with multi-relators, we will sometimes use the notation:

\[ w_1; w_2; \ldots; w_n \]

for presenting a multi-relator. A multi-relator consisting of only one relator \( w_1 \) will be written simply \( w_1 \); specifying such a multi-relator has exactly the same effect on the computation structure as specifying the simple relator \( w_i \), because \( MC(\langle w_i \rangle) \) is normal in \( F(V) \).

6.2 Multi-relators, multi-cells, and diagrams

We will now state a theorem which is an extension of the fundamental theorem of combinatorial group theory for the case of GCS’s with multi-relators. We first need the notion of multi-cell associated with a multi-relator.

We first remark that we can always assume that a multi-relator \( \langle w_1; w_2; \ldots; w_n \rangle \) is such that each \( w_i \) is cyclically reduced, because taking the cyclically reduced conjugate of \( w_i \) does not change the notion of multi-conjugate. We will assume this is always the case in the sequel.

If \( \langle w_1; w_2; \ldots; w_n \rangle \) is a multi-relator, and if \( \Gamma_i \) is the cell associated with \( w_i \) in the manner of \( \Gamma_i \), then we will call the multiset of cells \( \Theta = \{ \Gamma_1, \ldots, \Gamma_n \} \) the multi-cell associated with this multi-relator.
Figure 11: G-Grammar given in algebraic terms.

Consider a finite multiset of multi-relators and take the multi-set obtained by forming the multiset union $\Omega = \biguplus_k \Theta_k$ of the multi-cells associated with these multi-relators. A diagram whose cells are exactly (that is, taking account of the cell counts) those of the multiset $\Omega$ will be said to be a DIAGRAM RELATIVE TO the GCS with multi-relators under consideration.

One can easily prove the following extension of theorem 1.

**Theorem 2.** Let $GCS-MR = (V, MR)$ be a group computation structure with multi-relators. If $w$ is a cyclically reduced word in $F(V)$, then $w$ is in the result monoid of $GCS-MR$ iff there exists a reduced diagram relative to $GCS-MR$ having boundary $w$.

### 6.3 Linguistic examples

Let’s consider the extension $G-Grammar'$ of $G-Grammar$ presented in Figure 11. The first nine entries are the same as before, but three new multi-relator schemes have been added to the end. They correspond to definitions of the quantifiers “every” and “some” and to the definition of the relative pronoun “that”.

The notation $P[x]$ is employed to express the fact that a logical form containing an argument identifier $x$ is equal to the application of the abstraction $P$ to $x$. The identifier meta-variable $X$ in $P[X]$ ranges over such identifiers ($x, y, z, ...$), which are notated in lower-case italics (and are always ground).

The meta-variable $P$ ranges over logical form abstractions missing one argument (for instance $\lambda z. s(j, z)$). When matching meta-variables in logical forms, use of higher-order unification will be allowed. For instance, one can match $P[X]$ to $s(j, x)$ by taking $P = \lambda z. s(j, z)$ and $X = x$.

The geometric presentation of $G-Grammar'$ is given in Figure 12, for the three new multi-relators (we have omitted reproducing the cells of Figure 8, which also belong to the specification). The dotted lines indicate which cells belong to
a given multi-cell. The cells belonging to a multi-cell always work in solidarity: they have to appear together in a diagram or not at all.

Figure 12: \textit{G-Grammar}' given in geometric terms (omitting the cells already described in Figure 8).

6.4 Linguistic computation with multi-relators

The figures 13, 14, and 15 illustrate some computations with \textit{G-Grammar}'.

The first diagram consists of seven cells: three “mono”-cells, and two “bi”-cells (a) and (b), corresponding respectively to the entries for “every” and for “some” in the grammar. The boundary of the diagram can be read $\text{ev(m,x,sm(w,y,s(x,y))) woman}^{-1} \text{some}^{-1} \text{saw}^{-1} \text{man}^{-1} \text{every}^{-1}$. Because of theorem 2, this proves the correspondence relative to \textit{G-Grammar}' of the logical form $\text{ev(m,x,sm(w,y,s(x,y)))}$ with the sentence \textit{every man saw some woman}.

The second diagram is similar to the first, but with a different layout between the multi-cells for “every” and “some”. It establishes a different scoping for the quantifiers of the same sentence.

The third diagram establishes the correspondence between the sentence \textit{the man that louise saw ran} and the logical form $r(t(t(t(t(m,x,s(1,x))))))$.

\footnote{For a more detailed discussion, and for the algebraic counterparts of these computations, see 1.}
Figure 13: A diagram using multi-cells establishing a correspondence between the sentence *every man saw some woman* and the logical form \( \text{ev}(m, x, \text{sm}(w, y, s(x, y))) \).

Figure 14: A diagram establishing the correspondence between the same sentence and the differently scoped logical form \( \text{sm}(w, y, \text{ev}(m, x, s(x, y))) \).
Figure 15: A diagram using multi-cells establishing a correspondence between the sentence \textit{the man that louise saw ran} and the logical form \( r(t(t(t(t(m,x,s(l,x)))))) \).

7 Final remarks

7.1 Normal submonoids versus subgroups

In Section 3.1, we stated without justification that, for the purpose of presenting group computation structures, we would use normal submonoids of a free group rather than normal subgroups, which are much more central in algebra. We would like now to give some support to this claim.

Suppose that we were to define computation results as being elements of the normal subgroup closure \( \text{NG}(R) \) of a set of relators, rather than of the normal submonoid closure \( \text{NM}(R) \). Then \( \text{NG}(R) \) would partition the elements of \( F(V) \) into equivalence classes: two elements \( x, y \in F(V) \) being equivalent iff \( xy^{-1} \in \text{NG}(R) \). For all purposes two such elements would become undistinguishable. For linguistic formalization, where we need to model the relationship between a logical form and a phonological form, this approach would neutralize important differences: if we had, say, the results \( \text{sem}_1\text{string}_1^{-1} \), \( \text{sem}_2\text{string}_2^{-1} \), and \( \text{sem}_3\text{string}_3^{-1} \) — meaning that \( \text{sem}_1 \) is associated with \( \text{string}_1 \), and \( \text{sem}_2 \) is associated with both \( \text{string}_1 \) and \( \text{string}_2 \), then \( \text{sem}_1, \text{sem}_2, \text{string}_1, \text{string}_2 \) would all be in the same equivalence class, meaning in particular that \( \text{sem}_1 \) would be associated with \( \text{string}_2 \), counter-intuitively. This problem does not

\footnote{Of course, if it is deemed useful to identify two expressions completely, such as in case of synonymy between two logical forms \( \text{sem}_a \) and \( \text{sem}_b \), it would be possible to specify both relators \( \text{sem}_a\text{sem}_b^{-1} \) and \( \text{sem}_b\text{sem}_a^{-1} \) in the group computation structure. One might then perhaps even think of \textit{quotienting} the group computation structure by the equivalence relation thus created.}
occur with the choice of $\text{NM}(R)$: in this case the relation between $x, y$ defined by $xy^{-1} \in \text{NM}(R)$ is only a preorder, not an equivalence relation (see [4]).

7.2 Group computation versus rewriting

We started this paper in Section 1 by taking a geometric viewpoint on conventional rewriting systems such as context-free grammars or type-0 grammars. Before ending, we would like to consider the following question: what, if any, is the difference between group computation and these rewriting systems? A detailed discussion of this question is outside the scope of this paper (see [4]), and only some brief indications will be given here.

Let’s consider a type-0 rewriting system and a type-0 chart $CH$ relative to this system. We have seen in Section 1 that such a chart can be obtained by “pasting” together cells corresponding to the rules of the rewriting system. That is, the chart $CH$ can be seen as a diagram relative to the GCS defined by this collection of cells. But such a diagram has an interesting property among all diagrams that could be produced relative to the GCS: the cells of $CH$ are partially ordered by the relation which considers a cell $c_1$ of a diagram to immediately precede a cell $c_2$ of the same diagram iff there is a common edge between $c_1$ and $c_2$ which is negatively oriented relative to $c_1$ and positively oriented relative to $c_2$. For a standard presentation of the chart $CH$, this partial order is just the usual “top-down” order between the cells.

The fact that the cells of the diagram $CH$ are partially-ordered by the precedence relation just defined is a global property of the diagram which means that the precedence relation does not create cycles among cells. It is a property which is foreign to group computation, where cells are assembled relative to a purely local criterion. This means that, in general, the translation of a type-0 system into a GCS produces diagrams that cannot be interpreted as derivations of the original type-0 system. However, in the case where the relator cells resulting from the translation of the rewriting system can be statically partially-ordered by the precedence relation where $c < c'$ iff $c$ has a negatively oriented edge labelled $l$ and $c$ has a positively oriented edge with the same label $l$, then no cycles can ever appear in diagrams of the GCS, and then the two notions of computation become identical.

The static condition just described is very restrictive for type-0 systems stricto sensu, that is, having a finite vocabulary of non-terminals, because it precludes recursivity. It is, however, much more interesting in the case of rewriting systems defined by rule-schemes over terms, such as DCGs [7], or their type-0 counterparts. An example of such a situation is provided by the cells of Figure 8, which can be seen to correspond to an “extended” type-0 system of this kind, and where the instanciated relator cells can be partially ordered by the precedence relation just described (see [4]).

Remark: the several total orders which are compatible with the partial order of precedence in a chart are in one-to-one correspondence with the derivations associated with the chart.

One could say that group computation ignores the directionality of time, contrarily to a rewriting system.
7.3 Group morphisms and computational complexity

One important consequence of working within a group-theoretical framework is that group theory provides powerful tools for studying invariant properties of its objects.

Let us give an illustration of this fact on the basis of our example grammar $G$-Grammar. Let's first define the semantic size $ss$ of a logical form term as the number of nodes in this term which are different from argument identifiers such as $x, y, z...$ (thus, $ss(ev(m, x, sm(w, y, s(x, y)))) = 5$), and the semantic size of a phonological word as 0. Let's also define the phonological size $ps$ of a phonological word as 1, and the phonological size of a logical form as 0.

The functions $ss$ and $ps$ can be extended to morphisms from $F(V)$ to $Z$ in the standard way. Let's then consider the morphism $h$ from $F(V)$ to $(Z \times Z, +)$ defined by $h(w) = (ss(w), ps(w))$.

If one looks at the multi-relator schemes of Figure 11, it can be checked that any grounded multi-conjugate instance $w$ of each multi-relator scheme is such that $h(w) = (1, -1)$. For example, taking the multi-relator scheme $ev(N, X, P[X])^{-1}; X^{-1}$ we see that any multi-conjugate instance $w$ of this scheme is such that

$h(w) = h(ev(N, X, P[X])) + h(P[X]^{-1}) + h(X) + h(N^{-1}) + h(every^{-1})$,

because taking conjugates does not change the value of a morphism like $h$ which takes its values in a commutative group. Thus we have:

$h(w) = (1 + ss(N) + 0 + ss(P[X]) - ss(P[X]) + 0 - ss(N), -1) = (1, -1)$,

and so on for the other relator schemes.

Consider now a computation of a result $w$ involving $n$ multi-relators (in geometric terms, a diagram with boundary $w$ involving $n$ multi-cells). The value of $h(w)$ is then $(1, -1) + \ldots + (1, -1)$ taken $n$ times, that is, $h(w) = (n, -n)$. This has the following consequences (stated informally here) for the complexity of parsing and generation.

1. If a phonological string and a logical form are in correspondence relative to the grammar, then it is possible to bound the phonological size of the string as a function of the semantic size of the logical form (in fact they are equal).

2. If we have a string of phonological size $n$ to parse, then any computation will involve at most (and in fact exactly) $n$ multi-cells; this implies that parsing is decidable and of bounded complexity.

3. If we have a logical form of semantic size $n$ to generate, then any computation will involve at most (and in fact exactly) $n$ multi-cells; this implies that generation is decidable and of bounded complexity.

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In the terminology of [3], the properties 1,2 and 3 mean that the grammar is inherently reversible.
Although the grammar $G$-Grammar is rather remarkable in admitting a morphism such as $h$ which has the same value on all the multi-relators, what is really needed for the complexity properties to hold is a less demanding requirement (see [4]) of finding a morphism which realize some reasonable “exchange” of phonological material for semantic material on each multi-relator.

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