Empirical Processes and Schatte Model

Marko Raseta

Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of independent, identically distributed random variables and \(S_k = X_1 + \cdots + X_k\). Put \(f_a(x) = I\{x \leq a\} - a\). Furthermore, let \(F_n(t) = F_n(t, x)\) stand for the empirical distribution function of the sample \(\{S_1x\}, \ldots, \{S_nx\}\).

Theorem. Let \((X_n)_{n \in \mathbb{N}}\) be as above and suppose \(X_1\) is bounded with bounded density. Let

\[
\Gamma(s, t) = s(1 - t) + \sum_{\varrho=1}^{\infty} \mathbb{E}f_s(U)f_t(U + S_\varrho x) + \sum_{\varrho=1}^{\infty} \mathbb{E}f_t(U)f_s(U + S_\varrho x),
\]

where \(U\) is a \(U(0, 1)\) variable independent of \((X_n)_{n \in \mathbb{N}}\). Then for all fixed \(x \neq 0\), and after suitably enlarging the probability space, there exist mean zero Gaussian processes \(K_n(s)\) with covariance function \(\Gamma_n(s, t)\) such that

\[
\sup_{s, t \in [0, 1]^2} \left| \Gamma_n(s, t) - \Gamma(s, t) \right| = O(n^{-\alpha}) \quad \text{for all } \alpha < 1/8 \tag{1}
\]

and

\[
\sup_{0 \leq s \leq 1} \left| \sqrt{n}(F_n(s) - s) - K_n(s) \right| = O(n^{-\gamma}) \quad \text{a.s. for all } \gamma < 1/16. \tag{2}
\]

Without loss of generality, it suffices to prove the theorem for \(x = 1\). We give some lemmas of which the first was proved in our previous manuscript.

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\(^1\)Department of Mathematics, University of York Email: marko.raseta@york.ac.uk
Lemma 1. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) denote a bounded measurable function satisfying

\[
f(x + 1) = f(x), \quad \int_0^1 f(x) \, dx = 0 \quad \text{and} \quad \left( \int_0^1 |f(x + h) - f(x)|^2 \, dx \right)^{1/2} \leq Ch \quad (3)
\]

for some positive constant \( C \). Fix an integer \( \ell \geq 1 \) and define a sequence of sets by:

\[
I_1 := \{1, 2, \ldots, b\}
\]

\[
I_2 := \{p_1, p_1 + 1, \ldots, p_1 + b_1\} \quad \text{where} \quad p_1 \geq b + \ell + 2
\]

\[
\vdots
\]

\[
I_n := \{p_{n-1}, p_{n-1} + 1, \ldots, p_{n-1} + b_{n-1}\} \quad \text{where} \quad p_{n-1} \geq p_{n-2} + b_{n-2} + \ell + 2.
\]

Then there exists a sequence \( \delta_1, \delta_2, \ldots \) of random variables, not depending on \( f \), satisfying the following properties:

(i) \( |\delta_n| \leq Ce^{-\lambda \ell} \) for all \( n \in \mathbb{N} \) and some positive constants \( C, \lambda \).

(ii) The random variables

\[
\sum_{i \in I_1} f(S_i), \sum_{i \in I_2} f(S_i - \delta_1), \ldots, \sum_{i \in I_n} f(S_i - \delta_{n-1})
\]

are independent.

Partition the interval \([0, n] \) into disjoint blocks,

\[
[0, n] = I_1 \cup J_1 \cup I_2 \cup J_2 \cup \cdots \cup I_\ell \cup J_\ell
\]

where \( |I_k| = \lfloor n^\alpha \rfloor, \ |J_k| = \lfloor n^\beta \rfloor, \ \ell \sim n^{1-\alpha} \) for some positive reals \( \alpha, \beta \) to be chosen later. Using Lemma 1 we can construct sequences \( (\Delta_k)_{k \in \mathbb{N}} \) and \( (\Pi_k)_{k \in \mathbb{N}} \) of random variables such that

\[
\Delta_0 = \Pi_0 = 0, \quad |\Delta_k| \leq Ce^{-\lambda \lfloor n^\beta \rfloor}, \quad |\Pi_k| \leq Ce^{-\lambda \lfloor n^\alpha \rfloor}
\]

for \( k = 1, \ldots, \ell \) and

\[
T_k^{(f_t)} := \sum_{j \in I_k} f_t(S_j x - \Delta_{k-1}) \quad \text{and}
\]
are sequences of independent random variables. Put

\[ A(f) := \|f\|^2 + 2 \sum_{\rho=1}^{\infty} \mathbb{E} f(U) f(U + S_{\rho} x) \]  \hspace{1cm} (4)

where \( U \) is a uniform \((0,1)\) random variable, independent of \((X_j)_{j \geq 1}\). Moreover, let

\[ m_k = k \left( \lfloor n^\alpha \rfloor + \lfloor n^\beta \rfloor \right) \text{ for } 1 \leq k \leq n. \]

**Lemma 2.** We have

\[ \sum_{k=1}^{\ell} \text{Var} \left( T_k^{(f_t)}(f_t) \right) \sim A(f_t)n, \quad \sum_{k=1}^{\ell} \text{Var} \left( T_k^{*}(f_t) \right) \sim A(f_t)n^{1-\beta} \]

where \( A(f_t) \) is defined in (4) above.

**Proof.** To simplify the formulas, we shall drop \( f_t \) from the upper index because it is constant throughout the argument. We will establish the result for the long blocks, the corresponding result for short blocks can be obtained in an identical fashion. We have

\[ \text{Var}(T_k) = \sum_{j \in I_k} \mathbb{E} f^2(S_j - \Delta_{k-1}) + 2 \sum_{\rho=1}^{\lfloor |I_k| \rfloor} \sum_{q=m_k-1+1}^{|I_k|} \mathbb{E} f(S_q - \Delta_{k-1})(S_{q+\rho} - \Delta_{k-1}) - L^{(k)} \]

where

\[ L^{(k)} := \left( \sum_{j=m_k-1+1}^{m_k-1+|I_k|} \mathbb{E} f(S_j - \Delta_{k-1}) \right)^2. \]

By the above assumptions and the bounds on \( \Delta_k \) we have

\[ \|f(S_j - \Delta_{k-1}) - f(S_j)\| \leq C e^{-\lambda \lfloor n^\beta \rfloor} \]

and

\[ |\mathbb{E} f(S_j)| \leq C e^{-\lambda j}. \]

Thus

\[ L^{(k)} \leq C n^{2\alpha} e^{-\lambda \lfloor n^\beta \rfloor}. \]

Now let

\[ \Lambda^{(k)} := \sum_{j=m_k-1+1}^{m_k-1+|I_k|} \gamma_j, \quad O^{(k)} := \sum_{j=m_k-1+1}^{m_k-1+|I_k|} \epsilon_j. \]
where
\[ \gamma_j = \mathbb{E}f^2(S_j - \Delta_{k-1}) - \mathbb{E}f^2(S_j) \]
\[ \varepsilon_j = \mathbb{E}f^2(S_j) - \mathbb{E}f^2(F_{\{S_j\}}(\{S_j\})) . \]

Repeating the argument above for \( f^2 - \|f\| \) we get
\[ \sum_{j=m_{k-1}+1}^{m_k + |I_k|} \mathbb{E}f^2(S_j - \Delta_{k-1}) = \|f\|^2[n^\alpha] + \Lambda^{(k)} + O^{(k)} \]
and
\[ |\Lambda^{(k)}| \leq Cn^\alpha e^{-\lambda[n^\beta]} , \]
\[ |O^{(k)}| \leq Cn^\alpha e^{-\lambda[n^\beta]} . \]

We now turn to the cross terms. Define \( T_{\ell}^q = X_{q+1} + \cdots + X_{q+\ell} \) and split the product expectation \( \mathbb{E}f(S_q - \Delta_{k-1})f(S_{q+\ell} - \Delta_{k-1}) \) into a sum of terms:
\[ e_q := \mathbb{E}f(S_q - \Delta_{k-1})f(S_{q+\rho} - \Delta_{k-1}) - \mathbb{E}f(S_q)f(S_{q+\rho} - \Delta_{k-1}) , \]
\[ g_q := \mathbb{E}f(S_q)f(S_{q+\rho} - \Delta_{k-1}) - \mathbb{E}f(S_q)f(S_{q+\rho}) , \]
\[ h_q := \mathbb{E}f(S_q)f(S_{q+\rho}) - \mathbb{E}f(F_{\{S_q\}}(\{S_q\}))f(S_{q+\rho}) , \]
\[ i_q := \mathbb{E}f(F_{\{S_q\}}(\{S_q\}))f(S_q + T_{\rho}^q) - \mathbb{E}f(F_{\{S_q\}}(\{S_q\}))f(F_{\{S_q\}}(\{S_q\}) + T_{\rho}^q) , \]
\[ c_{\rho}^q := \mathbb{E}f(F_{\{S_q\}}(\{S_q\}))f(F_{\{S_q\}}(\{S_q\}) + T_{\rho}^q) . \]

Here \( F_{\{S_q\}}(\{S_q\}) \) is a uniformly distributed random variable independent of \( T_{\ell}^q \) and thus letting \( U \) denote a uniform \((0,1)\) random variable independent of \((X_j)_{j \geq 1}\) we have
\[ c_{\rho}^q = c_{\rho} = \mathbb{E}f(U)f(U + S_{\rho}) \]

is a \( q \)-independent quantity. Exactly as before
\[ |e_q| \leq Ce^{-\lambda[n^\beta]} , \quad |g_q| \leq Ce^{-\lambda[n^\beta]} , \quad |h_q| \leq Ce^{-\lambda q} , \quad |i_q| \leq Ce^{-\lambda q} . \]

Thus letting
\[ E^{(k)} := 2 \sum_{\rho=1}^{[I_k] - 1} \sum_{q=m_{k-1}+1}^{m_{k-1} + |I_k| - \rho} e_q , \quad G^{(k)} := 2 \sum_{\rho=1}^{[I_k] - 1} \sum_{q=m_{k-1}+1}^{m_{k-1} + |I_k| - \rho} g_q , \]
\[ H^{(k)} := 2 \sum_{\rho=1}^{[I_k] - 1} \sum_{q=m_{k-1}+1}^{m_{k-1} + |I_k| - \rho} h_q , \quad F^{(k)} := 2 \sum_{\rho=1}^{[I_k] - 1} \sum_{q=m_{k-1}+1}^{m_{k-1} + |I_k| - \rho} i_q . \]
we have

\[ |E^{(k)}| \leq Cn^{2\alpha}e^{-\lambda|n^\beta|}, \quad |G^{(k)}| \leq Cn^{2\alpha}e^{-\lambda|n^\beta|}, \]

\[ |H^{(k)}| \leq Cn^{2\alpha}e^{-\lambda|n^\beta|}, \quad |I^{(k)}| \leq Cn^{2\alpha}e^{-\lambda|n^\beta|}. \]

Furthermore,

\[ |I_{k}^{\rho}| = \sum_{\rho=1}^{I_{k}^{\rho}} c_{\rho} \sum_{q=m_{k-1}+1}^{\infty} q \sum_{\rho=1}^{I_{k}^{\rho}} c_{\rho} = \sum_{\rho=1}^{I_{k}^{\rho}} c_{\rho} - \sum_{\rho=\rho_{k}}^{\infty} c_{\rho} - \sum_{\rho=1}^{I_{k}^{\rho}} \rho c_{\rho}. \]

Putting all this together one can see that

\[ \sum_{k=1}^{t} \text{Var} \left( T_{k}^{(f)} \right) \sim nA^{(f)}. \]

**Lemma 3.** \( A^{(f-t-s)} \leq B(t-s)\log((t-s)^{-1}) \) for some suitably chosen constant \( B \).

**Proof.** Clearly

\[ A^{(f-t-s)} = \left( (t-s) - (t-s)^2 \right) + 2 \sum_{\rho=1}^{\infty} E_f t^{-s} (U) f t^{-s} (U + S_{\rho} x). \]

Define a sequence of uniformly distributed random variables \( U_{\rho} \) by

\[ U_{\rho} := \{ U + S_{\rho} x \}. \]

By the definition of \( f \)

\[ |E_f t^{-s} (U) f t^{-s} (U + S_{\rho} x) | = |Cov \{ I \{ U \in (s,t] \}, I \{ U_{\rho} \in (s,t] \} \} | \leq t-s \]

by simple algebra. On the other hand, the machinery we used in the proof of Lemma 2 yields the estimate

\[ |E_f t^{-s} (U) f t^{-s} (U + S_{\rho} x) | \leq C e^{-\lambda \rho}. \]

Then:

\[ A^{(f-t-s)} \leq (t-s) + 2 \sum_{\rho=1}^{\infty} \min \left( t-s, C e^{-\lambda \rho} \right) \]
\[ \leq (t-s) + 2 \sum_{\rho=1}^{\omega} (t-s) + 2 \sum_{\rho > \omega} C e^{-\lambda \rho} \]

and the result follows by choosing \( \omega = \log(t-s)^{-1} \) (whenever \( \log(t-s)^{-1} > 1 \)). \( \square \)

We now turn to the proof of our theorem. Using Borel–Cantelli lemma, one can see that statement (2) of our Theorem is equivalent to

\[
\sup_{s \in [0,1]} \left| n^{-1/2} \sum_{k=1}^{\ell} T_k^{(f_s)} + n^{-1/2} \sum_{k=1}^{\ell} T_k^{* (f_s)} - K_n(s) \right| = O(n^{-\gamma}) \quad \text{a.s.} \quad (5)
\]

Let \( \delta_n = n^{-\varepsilon} \) for some \( \varepsilon > 0 \) to be specified later, and define points \( z_j = j \cdot \delta_n, \) \( 0 \leq j \leq r_n \), where \( r_n \) is the largest integer with \( r_n \delta_n \leq 1 \). Define sequences of random vectors by

- (a) \( X_j := (I \{\{S_j x - \Delta_{k-1} \leq z_1\}, \ldots, I \{\{S_j x - \Delta_{k-1} \leq z_n\}) \),
- (b) \( Y_j := (I \{\{S_j x - \Pi_{k-1} \leq z_1\}, \ldots, I \{\{S_j x - \Pi_{k-1} \leq z_n\}) \),
- (c) \( \xi_k := \sum_{j \in I_k} (X_j - E X_j) \),
- (d) \( \eta_k := \sum_{j \in J_k} (Y_j - E Y_j) \).

We apply Lemma 4 of Berthet and Mason (see (1)) to \( (\xi_k)_{k \in \mathbb{N}} \). It is evident from the construction that, after enlarging the probability space, there exists a sequence \( (\xi_k^*)_{k \in \mathbb{N}} \) of independent, zero-mean Gaussian random vectors with the same covariance structure as the sequence \( (\xi_k)_{k \in \mathbb{N}} \) such that

\[
\mathbb{P} \left\{ \left| n^{-1/2} \sum_{j=1}^{\ell} (\xi_j - \xi_j^*) \right| > v \right\} \leq C_1 r_n^2 \exp \left( -C_2 v / n^{\frac{\varepsilon}{2} + \alpha} \right),
\]

where \( r_n, \varepsilon \) and \( \alpha \) are as described previously and \( C_1 \) and \( C_2 \) are constants. For \( s \in [z_{p-1}, z_p], 1 \leq p \leq r_n \) we define

\[
K_n(s) = n^{-1/2} \sum_{k=1}^{\ell} \xi_k^* (p) \quad \text{and} \quad K_n^*(s) = n^{-1/2} \sum_{k=1}^{\ell} \eta_k^* (p),
\]

where \( (\eta_k^*)_{k \in \mathbb{N}} \) are the exact analogue of \( (\xi_k^*)_{k \in \mathbb{N}} \) for the short blocks. By construction, these are Gaussian processes whose covariances we denote by \( \Gamma_n(s, t) \).
and $\Gamma_n^*(s, t)$, respectively. Assume that $1/2 - \gamma > 5\varepsilon/2 + \alpha$, $\alpha > \beta$. Then the Borel–Cantelli lemma gives

$$\max_{s \in H_n} \left| n^{-1/2} \sum_{k=1}^\ell T_k(f_s) + n^{-1/2} \sum_{k=1}^\ell T_k^*(f_s) - K_n(s) - n^{\beta-\gamma} K_n^*(s) \right| = O(n^{-\gamma}) \quad \text{a.s.}$$

where $H_n = \{z_0, \ldots, z_{r_n}\}$. Another Borel–Cantelli argument gives, under the additional assumption that $\alpha - \beta > 2\gamma$,

$$n^{\beta-\gamma} K_n^*(s) = o(n^{-\gamma}), \quad \text{a.s.}$$

Thus, under the conditions

$$1/2 - \gamma > 5\varepsilon/2 + \alpha, \quad \alpha > \beta, \quad \alpha - \beta > 2\gamma$$

we have

$$\max_{s \in H_n} \left| n^{-1/2} \sum_{k=1}^\ell T_k(f_s) + n^{-1/2} \sum_{k=1}^\ell T_k^*(f_s) - K_n(s) \right| = O(n^{-\gamma}) \quad \text{a.s.}$$

Borel–Cantelli lemma gives Next we show now that (7) holds with $\max_{s \in H_n}$ replaced by $\sup_{s \in [0, 1]}$. To this end, we have to control the fluctuations of the corresponding large and small block processes between points of $H_n$. By Bernstein’s inequality we have, for $s \in (z_p, z_{p+1}]$

$$\mathbb{P} \left( \left| \sum_{k=1}^\ell T_k^*(f_s) \right| > n^{1/2-\gamma} \right) \leq 2 \exp \left( -\frac{n^{1/2-\gamma}}{\sum_{k=1}^\ell \text{Var} T_k^*(f_s) + 2n^{\alpha-1/2-\gamma}} \right).$$

Using Lemmas 2 and 3 we deduce that the fluctuations are insignificant if $1/2 > \alpha + \gamma$ and $\gamma < \varepsilon/2$. Since $\alpha > \beta$ there will be no further conditions for controlling the smaller block fluctuations.

It remains to impose conditions implying the uniform convergence of the $\Gamma_n(s, t)$ and $\Gamma_n^*(s, t)$ to $\Gamma(s, t)$, over all $s, t \in [0, 1]$. To this end, using the same ideas as before and the fact that

$$\text{Cov}(K_n(s), K_n(t)) = \text{Cov}(K_n(z_{i+1}), K_n(z_{j+1}))$$
for all \( s \in (z_i, z_{i+1}], \ t \in (z_j, z_{j+1}] \)
we get
\[
|\Gamma_n(z_i, z_j) - \Gamma(z_i, z_j)| \leq C/n^\alpha.
\]
Moreover, using Lemma 3 we immediately deduce that
\[
\sup_{s,t\in[0,1]^2} |\Gamma_n(s, t) - \Gamma(s, t)| \leq C(\log n)/n^{\varepsilon}
\]
and that, under the condition \( \varepsilon > \alpha \), the second statement (2) of the Theorem holds as well.

Finally, we will find the largest value of \( \gamma \) for which our argument works. Clearly, we have to maximize \( \gamma \) subject to:
\[
\begin{align*}
1/2 - \gamma &> 5\varepsilon/2 - \alpha, \\
\alpha - \beta &> 2\gamma, \\
\gamma &< \varepsilon/2, \\
\alpha &> \beta, \\
\alpha &> 0, \\
\beta &> 0, \\
\gamma &> 0, \\
\varepsilon &> 0.
\end{align*}
\]
Then \( \gamma < 1/2 - \alpha - 5\varepsilon/2 < 1/2(1 - 7\alpha) \). Thus \( \gamma < \min(\alpha/2, 1/2(1 - 7\alpha)) \), the other constraints are not important. By plotting the corresponding feasible region we see that the largest possible value of \( \gamma \) corresponds to the intersection of the lines \( J = \alpha/2 \), \( J = 1/2(1 - 7\alpha) \), whence \( \alpha = 1/8 \). Thus \( \gamma < 1/2 \cdot 1/8 = 1/16 \) and the proof is complete.

References

[1] Berthet and Mason: Revisiting two strong approximation results of Dudley and Philipp, Institute of Mathematical Statistics-Monograph Series, 2006