GLOBAL SOLUTION TO THE CAUCHY PROBLEM ON A UNIVERSE FIREWORKS MODEL

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Abstract. We prove existence and uniqueness of the global solution to the Cauchy problem on a universe fireworks model with finite total mass at the initial state when the ratio of the mass surviving the explosion, the probability of the explosion of fragments and the probability function of the velocity change of a surviving particle satisfy the corresponding physical conditions. Although the nonrelativistic Boltzmann-like equation modeling the universe fireworks is mathematically easy, this paper leads rather theoretically to an understanding of how to construct contractive mappings in a Banach space for the proof of the existence and uniqueness by means of methods taken from the famous work by DiPerna & Lions about the Boltzmann equation. We also show both the regularity and the time-asymptotic behavior of solution to the Cauchy problem.

1. Introduction

We are concerned with existence and uniqueness of the global solution to the Cauchy problem on a universe fireworks model (hereafter, UFM) without gravity described by the following nonrelativistic Boltzmann-like equation [12]

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = \eta(t, x, \xi) \int_{\mathbb{R}^3} \Gamma(t, x, \xi_1) P(-\xi \cdot \xi_1) f_1 d\xi_1 - \Gamma(t, x, \xi) f$$

(1.1)

where $f \equiv f(t, x, \xi)$ is the total mass distribution with respect to time $t$, space $x$ and velocity $\xi$ during the explosion of a huge cloud of matter; $f_1 \equiv f(t, x, \xi_1)$; $\eta \equiv \eta(t, x, \xi)$ is the mass ratio which is a measure of the mass surviving the explosion; $\Gamma \equiv \Gamma(t, x, \xi)$ is the explosion probability per unit time for a fragment; $P \equiv P(-\xi \cdot \xi_1)$ is the probability density so that $P(-\xi \cdot \xi_1) d\xi_1$ is the probability that a surviving particle will change its velocity $\xi$ to $\xi_1 \in d\xi_1$ during explosion.

Assume that $\Gamma, \eta$ and $P$ satisfy the following physical conditions

$$0 \leq \eta \leq 1, \ 0 \leq \Gamma \leq 1, \ 0 \leq P,$$

(1.2)

$$\int_{\mathbb{R}^3} P(-\xi \cdot \xi_1) d\xi_1 = 1 \text{ for } \forall \xi \in \mathbb{R}^3,$$

(1.3)

and that the initial total mass distribution $f_0 \equiv f_0(x, \xi)$ satisfies

$$0 \leq f_0(x, \xi) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3),$$

(1.4)

i.e., having finite total mass at the initial state.

In this paper we shall prove existence and uniqueness of the global solution to the Cauchy problem on the nonrelativistic Boltzmann-like equation (1.1) of the distribution $f$ with finite total mass at the initial state when the ratio $\eta$, the probability $\Gamma$ and the probability density $P$ satisfy the corresponding physical conditions (1.2) and (1.3). We shall also show both the regularity and the time-asymptotic behavior of solution to this problem.

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The nonrelativistic Boltzmann-like equation is mathematically almost trivial since it is a linear transport equation with an integrable kernel and bounded multiplicative coefficients. However, this paper leads rather theoretically to an understanding of how to construct contractive mappings in a Banach space by means of methods taken from the famous work by DiPerna & Lions [3] about the Boltzmann equation.

The fireworks model is one of the metagalaxy models which are used to give a physical explanation of the expansion. Alfvén [1] suggested the first fireworks model for the evolution of the universe as an alternative cosmology to big bang and described a possible mechanism for this model in an original work in 1983, but many details remain unexplained because of the extremely complicated physics involved [12]. Some assumptions on the mechanism of the explosion, such as that matter and antimatter are mixed, are invalid in a metagalaxy model given by Alfvén & Klein [2] for the highest redshift observed. Also, the highest redshift observed cannot be accounted for even if it is assumed that there is an upper limit to the velocities that could be reached by one explosion of about \(0.7c\) (\(c\) is the light velocity) [9]. A special relativistic model was investigated by Laurent & Carlqvist [10] but the evolution of the distributions in configuration space was not extended to general relativity by use of their approach. Due to these, Widlund [12] gave a fireworks model (defined by equation (1.1)) without prescribing the details of the explosion mechanism and developed the model using a fully general relativistic treatment of the fireworks. The nonrelativistic fireworks model, which is given by equation (1.1), is a special case of the nonrelativistic limit to the general relativistic model considered by Widlund [12]. Widlund has explained the observed redshifts only from the start of the study of an approximate equation of equation (1.1) because it is a differential equation and has an exact solution for many weak explosions. Our results mentioned above are not only a kind of reasoning to adopt the research method of replacing equation (1.1) by an approximate equation as Widlund has done, but additionally make it possible to do some research work about the redshifts directly from the start of the solution to equation (1.1).

2. Existence and Uniqueness

One of our results mentioned in the previous section can be described as follows.

**Theorem 1.** Equation (1.1) with physical conditions (1.2) and (1.3) has a unique nonnegative global distributional solution \(f(t,x,\xi)\) through a total mass distribution \(f_0(x,\xi)\) satisfying a finite total mass condition (1.4) at the initial state, and this solution \(f(t,x,\xi)\) belongs to the Banach space \(L^\infty((0,+,\infty);L^1(R^3 \times R^3))\).

Theorem 1 shows existence and uniqueness of solution to the Cauchy problem on UFM without gravity no matter how long the fireworks era lasts.

The proof of Theorem 1 is as follows. We first define a mild solution to equation (1.1) as DiPerna & Lions did in [3]. Then, we establish a Banach space and a mapping of this space into itself which uniformly decreases distances, and shows that this mapping has a unique fixed point which is a unique global mild solution to equation (1.1). This is an application of the well-known Banach fixed point theorem. Finally, according to the relations between mild solution and distributional solution (e.g., [3], [5]), we know that this mild solution is also a global distributional solution.

**Definition 1.** Let \(f = f(t,x,\xi)\) be a nonnegative function which belongs to \(L^1_{loc}((0,T)\times R^3 \times R^3)\), and assume that for almost all \((x,\xi) \in R^3 \times R^3\),

\[
Q^\#(t,x,\xi) \in L^1(0,T_1) \quad (\forall T_1 \in (0,T))
\]

and

\[
f^\#(t,x,\xi) - f^\#(s,x,\xi) = \int_s^t Q^\#(\sigma,x,\xi)d\sigma \quad (\forall 0 \leq s < t \leq T),
\]

where \(Q^\#(t,x,\xi)\) is the convolution of the kernel \(Q(t,x,\xi,\eta)\) with the distribution \(f(t,x,\xi)\).
where \( h^\# \) denotes, for any measurable function \( h \) on \((0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3\), the following restriction to characteristics:

\[
h^\#(t, x, \xi) = h(t, x + t\xi, \xi).\]

If \( 0 < T < +\infty \), \( f \) is called a local-in-time mild solution to the equation

\[
\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = Q \tag{2.3}
\]

in the time interval \([0, T]\). If \( T = +\infty \), \( f \) is called a global mild solution to equation \((2.3)\).

By Definition 1 we have the following result:

**Lemma 1.** Suppose that \( f, h \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3) \), and that for almost all \( x, \xi \in \mathbb{R}^3 \), \( f^\# \) is absolutely continuous with respect to \( t \), \( h^\# \in L^1_{\text{loc}}(\mathbb{R}) \). Then \( f \) is a distributional solution to equation \((2.3)\) if and only if \( f \) is a mild solution to equation \((2.3)\).

This result is similar to that given by DiPerna & Lions \cite{3} for the Boltzmann equation. To show Theorem 1 we also have to define a mapping as follows.

**Definition 2.** A mapping \( J \) is defined as follows: for any \( f \in \mathcal{F} \) and \( t \in [0, T] \),

\[
J(f)(t, x, \xi) = f_0(x - t\xi, \xi) - \int_0^t \Gamma(\sigma, x + (\sigma - t)\xi, \xi) f(\sigma, x + (\sigma - t)\xi, \xi)d\sigma
\]

\[
+ \int_0^t \int_{\mathbb{R}^3} \eta(\sigma, x + (\sigma - t)\xi, \xi) \Gamma(\sigma, x + (\sigma - t)\xi, \xi_1) P(-\xi_1 \cdot \xi) f(\sigma, x + (\sigma - t)\xi_1, \xi_1)d\xi_1 d\sigma \tag{2.4}
\]

where \( \mathcal{F} = \{ f \equiv f(t, x, \xi) : f \in L^\infty((0, T); L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \text{ for all } T \in [0, +\infty) \} \).

Let the Lebesgue measure be used for all the integrals in this paper. Since the Lebesgue measure defined in \( \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \) is \( \sigma \)-finite, by use of the Fubini-Tonelli Theorem and the physical condition \((1.3)\), we know that \( f, f^\#, \int_{\mathbb{R}^3} P(-\xi \cdot \xi_1) f_1 d\xi_1, [\int_{\mathbb{R}^3} P(-\xi \cdot \xi_1) f_1 d\xi_1]^\# \in L^1_{\text{loc}}(0, +\infty) \) for almost every \((x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3\), i.e., the product \( P(-\xi \cdot \xi_1) f_1 \) in equation \((2.3)\) is integrable with respect to \( \xi_1 \). Thus Definition 2 is valid.

Take \( T_0 \) as a definite time the fireworks era lasts. Then, if \( 0 \leq T_0 < \frac{1}{2} \), equation \((1.1)\) has a local-in-time mild solution in the time interval \([0, T_0]\), or say, the total mass distribution of the universe fireworks is uniquely determined by the initial total mass data during the fireworks era. Indeed, by Definition 2 and with the help of the physical conditions \((1.2)\) and \((1.3)\), we can easily prove that \( J(f) \in \mathcal{F} \) for all \( f \in \mathcal{F} \), and that the following inequality holds:

\[
\max_{0 \leq t \leq T_0} \| J(f) - J(h) \|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \leq 2T_0 \max_{0 \leq t \leq T_0} \| f - h \|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \tag{2.5}
\]

for all \( f, h \in \mathcal{F} \). The inequality \((2.5)\) shows that \( J \) is a contractive mapping from \( \mathcal{F} \) into itself with the same norm as in \( C((0, T_0); L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \). Therefore there exists a unique element \( h_1 \in \mathcal{F} \) such that \( J(h_1) = h_1 \) for a.e. \((t, x, \xi) \in (0, T_0) \times \mathbb{R}^3 \times \mathbb{R}^3\). In order to prove that \( h_1 \) is a local-in-time mild solution to equation \((1.1)\), it is enough to show that \( h_1 \) is nonnegative for almost every \((t, x, \xi) \in (0, T_0) \times \mathbb{R}^3 \times \mathbb{R}^3\).

Let us take another mapping \( J_1 \) as follows:

\[
J_1(f) = \max(0, J(f)) \quad \text{for all } f \in \mathcal{F}^+ = \{ f : f \in \mathcal{F} \text{ and } f \geq 0 \}. \]

Obviously, \( \mathcal{F}^+ \) is a subset of \( \mathcal{F} \). Similarly, we can easily show that the mapping \( J_1 \) maps \( \mathcal{F}^+ \) itself and is uniformly contractive with the same norm as mentioned above. Then there exists a unique element \( f_0 \in \mathcal{F}^+ \) such that \( J_1(f_1) = f_1 \) for almost every \((t, x, \xi) \in [0, T_0] \times \mathbb{R}^3 \times \mathbb{R}^3\). Thus, if \( f_1 = J(f_1) \) for almost every \((t, x, \xi) \in [0, T_0] \times \mathbb{R}^3 \times \mathbb{R}^3\), \( f_1 \) is a local-in-time mild solution to equation \((1.1)\) through \( f_0 \) in the time interval \([0, T_0]\), and \( f_1 \equiv \xi \). We will below show that \( J(f_1) \equiv f_1 \), or equivalently, \( J(f_1)^\# \equiv f_1^\# \). In fact, by equation \((2.4)\), we know that \( J(f_1)^\# \) is absolutely continuous with respect to \( t \in [0, T_0] \) for almost every \((x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3\). We may assume without loss of generality that \( J(f_1)^\#(t, x, \xi) \) is continuous for all \((t, x, \xi) \). To prove that \( J(f_1)^\# = f_1^\# \), it suffices to prove that \( J(f_1)^\# \geq 0 \).
If $J(f_1)(t_0, x_0, \xi_0) < 0$ for some point $(t_0, x_0, \xi_0) \in [0, T_0] \times \mathbb{R}^3 \times \mathbb{R}^3$, we can find some value $t_1 \in [0, T_0]$ such that $J(f_1)(t, x_0, \xi_0) < 0$ for all $t \in [t_1, t_0]$, so that $f_1(t, x_0, \xi_0) = 0$ for all $t \in [t_1, t_0]$. By equation (2.4), we can know that

$$0 > J(f_1)(t, x_0, \xi_0) \geq J(f_1)(t_1, x_0, \xi_0) \quad (t_0 > t > t_1).$$

Repeating the above analysis, we can conclude that $J(f_1)(0, x_0, \xi_0) < 0$, i.e., $f_0(x_0, \xi_0) < 0$, which is a contradiction. Therefore (1.3) has a global mild solution through the total mass distribution and with the help of the physical conditions (1.2) and (1.3), it is also easy to see that for all $f, h \in F$ and with the help of the physical conditions (1.2) and (1.3), it is also easy to see that $\|J(f)\|_a \leq A$,

$$\|J(f) - J(h)\|_a \leq \frac{2}{a}\|f - h\|_a.$$  (2.7)

for all $f, h \in F_a$. We know from the two inequalities (2.6) and (2.7) that $J$ is a contractive mapping of $F_a$ into itself with the norm $\| \cdot \|_a$ in the time interval $[0, +\infty)$ for any given $a > 2$. Thus, choosing a real $a_0$ such that $a_0 > 2$ and taking $B = F_{a_0}$, we can have a unique element $h_0 \in B$ such that $J(h_0) = h_0$ in $[0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3$. As stated above about the fixed point nonnegativity in the $0 \leq T_0 < \frac{T}{2}$ case, we can easily show that $h_0$ is nonnegative for almost every $(t, x, \xi) \in [0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3$. Therefore equation (1.1) with the physical conditions (1.2) and (1.3) has a global mild solution through the total mass distribution $f_0$ satisfying the finite total mass condition (1.4) at the initial state. Finally, by Lemma 1, Theorem 1 holds true.

We can also construct another contractive mapping to prove Theorem 1. Let us write $F(t) = J_\#(\sigma, x, \xi) d\sigma$. Then, using the same device as given by DiPerna & Lions for the Boltzmann equation, we have

**Lemma 2.** $f = f(t, x, \xi)$ is a mild solution to equation (1.1) if and only if $f = f(t, x, \xi)$ satisfies

$$f(t, x, \xi) - f(s, x, \xi) e^{-\int_{s}^{t} F(\tau)d\tau} = \int_{s}^{t} \int_{\mathbb{R}^3} \eta(\sigma, x, \xi, \xi_1) \Gamma(\sigma, x, \xi_1) P(-\xi \cdot \xi_1) f(\sigma, x, \xi_1) e^{-\int_{\sigma}^{t} F(\tau)d\tau} d\xi_1 d\sigma$$

for almost all $(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$ and all $0 < s < t < +\infty$.

**Definition 3.** A mapping $J^+$ is defined as follows: $\forall f \in F^+$ and $\forall t \in [0, T]$, $J^+(f)(t, x, \xi) = e^{-\int_{0}^{t} \Gamma(\sigma, x, (\sigma-t)\xi, \xi)d\sigma} \int_{0}^{t} \Gamma(\sigma, x, (\sigma-t)\xi, \xi)d\sigma$, $J^+(f)(t, x, \xi) = e^{-\int_{0}^{t} \Gamma(\sigma, x, (\sigma-t)\xi, \xi)d\sigma} \int_{0}^{t} \Gamma(\sigma, x, (\sigma-t)\xi, \xi)d\sigma$.

where

$$F^+ = \left\{ f \equiv f(t, x, \xi) : f \geq 0 \text{ and } f \in L^\infty((0, T); L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \text{ for all } T \in [0, +\infty) \right\},$$

$$Q^+(t, x, \xi) = \int_{\mathbb{R}^3} \eta(t, x, \xi) \Gamma(t, x, \xi_1) P(-\xi \cdot \xi_1) f(t, x, \xi_1)d\xi_1.$$
Using Lemma 2 and Definition 3 we can give a different and brief proof of Theorem 1. Let us take $\mathcal{F}_a^+$ as follows:

$$
\mathcal{F}_a^+ = \left\{ f : \| f \|_a \equiv \sup_{0 \leq t < +\infty} \left\{ e^{-at} \int_{\mathbb{R}^3} |f(t, x, \xi)| dxd\xi \right\} \leq A, f \in \mathcal{F}^+ \right\}
$$

where $a > 0$, $A = a \| f_0 \|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)}$. Then $(\mathcal{F}_a^+, \| \cdot \|_a)$ is a complete metric space with the distance $\text{dist}(\cdot, \cdot) = \| \cdot - \cdot \|_a$. Under the physical conditions (1.2) and (1.3), it can be easily seen from Definition 3 that

$$
\| J^+(f) \|_a \leq A, \quad (2.9)
$$

and

$$
\| J^+(f) - J^+(h) \|_a \leq \frac{1}{a} \| f - h \|_a. \quad (2.10)
$$

for all $f, h \in \mathcal{F}_a^+$. By (2.9) and (2.10), it follows that $J^+$ is a contractive mapping from $\mathcal{F}_a^+$ into itself with the distance $\| \cdot - \cdot \|_a$ in the time interval $[0, +\infty)$ for any given $a > 1$. This implies that if we put $a > 1$, then there exists a unique element $g_0 \in \mathcal{F}_a^+$ such that $J^+(g_0) = g_0$. Hence, by Lemma 2, Theorem 1 holds true.

Similarly, we can further show the regularity of solution to the Cauchy problem, that is,

**Theorem 2.** Let $k$ be any fixed natural and $f$ a unique nonnegative global distributional solution under assumptions in Theorem 1. Support that $\eta$ and $\Gamma$ are in $C^k([0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$ and that $P \in C^k(\mathbb{R})$. If $f_0 \in C^k(\mathbb{R}^3 \times \mathbb{R}^3)$, then $f \in C^k([0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$.

In fact, under all the assumptions in Theorem 2 we can find that $J^+$ defined by (2.8) is still a contractive mapping from $\mathcal{F}_a^+ \cap C^k([0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$ into itself for any fixed $a > 1$, and then easily know that Theorem 2 holds.

**Remark 1.** In the relativistic case, the total rest mass distribution of UFM without gravity is uniquely determined by the initial total rest mass data as in the nonrelativistic case. Since the velocity $\xi$ in the relativistic case is not greater than the light velocity $c$ (for convenience, write $c = 1$), we have to choose a transformation: $\xi = \frac{p}{\sqrt{1+p^2}}$ where $p$ is a variable of momentum for the relativistic UFM, and regard the total rest mass distribution as that of the three variables of time $t$, space $x$ and momentum $p$. Thus we can easily show the results mentioned above as done in the nonrelativistic case, although the form of the transport operator $\frac{\partial}{\partial t} + \frac{p}{\sqrt{1+p^2}} \frac{\partial}{\partial x}$ of the relativistic Boltzmann-like equation (2) is different from that of the nonrelativistic Boltzmann-like equation (1.1). It is here worthwhile to mention that many results about the Cauchy problems relative to the relativistic Boltzmann or Enskog equation have been given in a number of papers (e.g., [4], [6], [7], [8]).

### 3. Asymptotic Behavior

In this section we shall study the time-asymptotic behavior of the solution to the Cauchy problem on UFM. We can find that this solution is time-asymptotically convergent to the free motion in $L^1$-norm under some suitable assumptions.

To do so, we first have to show the following lemma:

**Lemma 3.** Let $\delta$ be a positive constant less than 1. In addition to all the assumptions in Theorem 1 if $\eta$ and $P$ satisfy the following inequality:

$$
\int_{\mathbb{R}^3} \eta(t, x, \xi) P(-\xi \xi_1) d\xi \leq \delta \quad (3.1)
$$

for all the three variables $(t, x, \xi_1)$, then the nonnegative solution $f$ to the Boltzmann-like equation (1.1) satisfies

$$
\frac{d}{dt} \int_{\mathbb{R}^3} f(t, x, \xi) dxd\xi + (1 - \delta) \int_{\mathbb{R}^3} \Gamma(t, x, \xi) f(t, x, \xi) dxd\xi \leq 0. \quad (3.2)
$$
Furthermore, we have
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f(s, x, \xi) dx d\xi \\
\leq (\delta - 1) \int_0^s \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Gamma(t, x, \xi) f(t, x, \xi) dtdx d\xi + \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x, \xi) dx d\xi
\]
for any \( s \in (0, +\infty) \).

**Proof.** Note that \( f^\#(t, x, \xi) \) is absolutely continuous with respect to the time \( t \) and that the solution \( f \) satisfies
\[
\frac{\partial}{\partial t} f^\#(t, x, \xi) = -\Gamma^\#(t, x, \xi) f^\#(t, x, \xi) \\
+ \eta^\#(t, x, \xi) \int_{\mathbb{R}^3} \Gamma(t, x + t\xi, \xi_1) P(-\xi \cdot \xi_1) f(t, x + t\xi, \xi_1) d\xi_1.
\]
Integrating equation (3.4) over the two variables \((x, \xi)\) and using assumptions (1.2) and (3.1), we can know that (3.2) holds true. Furthermore, with the help of (1.4), integrating (3.2) over the time \( t \) from zero to \( s \) gives (3.3).

**Remark 2.** The inequality (5.2) implies that the total mass in UFM does not increase as time increases.

Then, using Lemma 3, we can show that

**Theorem 3.** Under all the assumptions in Lemma 3, the solution \( f(t, x, \xi) \) to the Boltzmann-like equation (1.1) converges in \( L^1(\mathbb{R}^3 \times \mathbb{R}^3) \) to \( f_\infty(t, x, \xi) \) as \( t \to +\infty \), where
\[
f_\infty(t, x, \xi) = f_0(x - t\xi, \xi) - \int_0^{+\infty} \Gamma(\sigma, x + (\sigma - t)\xi, \xi) f(\sigma, x + (\sigma - t)\xi, \xi) d\sigma \\
+ \int_0^{+\infty} \int_{\mathbb{R}^3} \eta(\sigma, x + (\sigma - t)\xi, \xi) P(-\xi_1) \Gamma(\sigma, x + (\sigma - t)\xi, \xi_1) f(\sigma, x + (\sigma - t)\xi, \xi_1) d\sigma d\xi_1
\]
for any point \((t, x, \xi)\).

**Proof.** Note that \( f \) is a nonnegative solution to the Boltzmann-like equation (1.1). Then, by (3.5), we can easily know that
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |f(t, x, \xi) - f_\infty(t, x, \xi)| dx d\xi \leq \int_t^{+\infty} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Gamma(\sigma, x, \xi) f(\sigma, x, \xi) dx d\xi d\sigma \\
+ \int_t^{+\infty} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left[ \int_{\mathbb{R}^3} \eta(\sigma, x, \xi) P(-\xi_1) d\xi_1 \right] \Gamma(\sigma, x, \xi_1) f(\sigma, x, \xi_1) dx d\xi d\sigma.
\]
By (3.1), it then follows that
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |f(t, x, \xi) - f_\infty(t, x, \xi)| dx d\xi \leq (1 + \delta) \int_t^{+\infty} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Gamma(\sigma, x, \xi) f(\sigma, x, \xi) dx d\xi d\sigma.
\]
Since the inequality (3.3) in Lemma 3 implies that the integral on the right side of (3.7) is convergent to zero as \( t \to +\infty \), the left side of (3.7) converges to zero as time goes to infinity. This completes our proof of this theorem.

**Remark 3.** Since \( f_\infty(t, x, \xi) \) defined by (3.3) describes a free motion in UFM, Theorem 3 shows that the solution to the Cauchy problem on UFM is time-asymptotically convergent to the free motion in \( L^1 \)-norm under some assumptions. It is worth mentioning that this time-asymptotic convergence of solution still holds true for the Boltzmann or Enskog equation under some suitable assumptions (e.g., [11]).
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