Extreme wave events for a nonlinear Schrödinger equation with linear damping and Gaussian driving

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We perform a numerical study of the initial-boundary value problem, with vanishing boundary conditions, of a driven nonlinear Schrödinger equation (NLS) with linear damping and a Gaussian driver. We identify Peregrine-like rogue waveforms, excited by algebraically decaying initial data. The observed extreme events emerge on top of a decaying support. Depending on the spatial/temporal scales of the driver, the transient dynamics – prior the eventual decay of the solutions – resembles the one in the semiclassical limit of the integrable NLS, or may, e.g., lead to large-amplitude breather-like patterns. The effects of the damping strength and driving amplitude, in suppressing or enhancing respectively the dynamics, are numerically analyzed.

I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation

$$i u_t + \frac{1}{2} u_{xx} + |u|^2 u = 0,$$

is one of the universal nonlinear evolution equations associated with the theory of solitons and integrable systems, with applications ranging from deep water waves to optics. Its dissipative counterparts, incorporating gain, loss, external driving, or combinations thereof, may exhibit (and potentially be attracted to) low-dimensional dynamical features, such as metastability, periodic and quasi-periodic orbits, even low-dimensional complex dynamics. In particular, the linearly damped and driven NLS equation, expressed in dimensionless form:

$$i u_t + \frac{1}{2} u_{xx} + |u|^2 u = f - i \gamma u,$$

is one of the prototypical partial differential equations exhibiting spatiotemporal chaotic behavior. Here, $f = f(x, t)$ stands for the driving (or forcing) of the system, while $-\gamma u$ accounts for linear damping of strength $\gamma > 0$; thus, Eq. (2) defines a non-autonomous perturbation of the integrable NLS ($\gamma = 0$, $f = 0$), which may give rise to the above complex dynamics.

Generically, this dynamics is captured by the global attractor of the associated infinite dimensional dynamical system to Eq. (2), for its existence, finite dimensionality and regularity, see.

The present study on Eq. (2) is motivated by an important feature of the integrable NLS limit, (1): the existence of rational solutions, in the form of the Peregrine rogue wave (PRW), as well as space- or time-periodic solutions, referred to as the Akhmediev and Kuznetsov–Ma (KMb) breathers. In recent years, the study of rational solutions is of wide interest, due to their ar...
these curves bound the region where the space-time oscillations occur.

For Eq. (2), we consider the simplest example of quadratically decaying initial data:

\[ u_0(x) = \frac{1}{1 + x^2}. \]  

(4)

Studying the effect of the damping strength \( \gamma \) and driving amplitude \( \Gamma \), we find that the above dynamical effect weakens for increasing values of \( \gamma \), while it emerges above a threshold value of \( \Gamma \), where the support sustaining the spatiotemporal oscillations becomes observable.

Our main findings can be summarized as follows. First, semi-classical type dynamics studied in Refs. [45, 46] persist – for certain spatiotemporal scales – in the presence of damping and external driving, even far from the integrable limit. Second, as the temporal localization width of the forcing is increasing, the transient dynamics prior to decay, is manifested by the formation of almost time-periodic breathing modes of (extremely, in some cases) large amplitude. We also establish a strong connection between the spatial width of the driving and the form/width of the emerging spatiotemporal patterns.

The paper is structured as follows. In Section II, we report the results of our numerical simulations. In Section III, we summarize and discuss the implications of our results with an eye towards future work currently in progress.

II. NUMERICAL INVESTIGATIONS

In this section, we present results of direct numerical simulations for Eq. (2), with the initial and boundary conditions discussed above. We assume that the driving, has the form of a Gaussian function, centered at \((x = 0, t = 0)\), having spreads \(\sigma_x, \sigma_t > 0\), with respect to the space and time variable, respectively:

\[ f(x, t) = g(x, t) + ih(x, t), \]

with \( g(x, t) = h(x, t) = \Gamma \exp \left( -\frac{x^2}{2\sigma_x^2} - \frac{t^2}{2\sigma_t^2} \right). \]  

(5)

The function \( f \), serves as a simple phenomenological example, of a spatiotemporal, exponentially localized driver, of amplitude \( \Gamma > 0 \).

The smallest value of the half-length of the spatial computational interval \([-L, L] \) considered in the simulations is \( L = 250 \), so that effects that may be produced at the boundaries are negligible. The system is integrated by using both Runge–Kutta and pseudo-spectral schemes, for both Dirichlet and periodic boundary conditions on \([-L, L] \). Note that there were no differences between the dynamics produced by the two different methods.

a. Emergence of extreme wave events To characterize the observed spatiotemporally localized waveforms as extreme, we compare the numerical solution with time-translations of the analytical PRW solution of the NLS \([1] \) [17]:

\[ u_{ps}(x, t; \Gamma_0) = \sqrt{P_0} \left\{ 1 - \frac{4(1 + 2t P_0 (t - t_0))}{1 + 4P_0 x^2 + 4P_0^2 (t - t_0)^2} \right\} e^{i P_0 (t - t_0)}, \]  

(6)

where \( P_0 \) stands for the power of the continuous background. Figure 1 shows snapshots of the evolution of

FIG. 1: (Color Online) Snapshots of the evolution of the density \(|u(x, t)|^2\) [solid (blue) curves], for the initial condition \((4)\). Parameters: \( \gamma = 0.01, L = 250, \) Gaussian driving \((5)\), with \( \Gamma = 1, \sigma_x = 100 \) and \( \sigma_t = 0.5 \). The density of the numerical solution is compared against the density of the PRW \((6), u_{ps}(x, t; 1.74; 1.07) \) [dashed (red) curves].

the density \(|u(x, t)|^2\) for damping strength \( \gamma = 0.01 \), forcing amplitude \( \Gamma = 1 \) and spreads \( \sigma_x = 100, \sigma_t = 0.5 \). We observe that the initial condition \((4)\) evolves towards an extreme event reminiscent to a PRW. The numerical solution is plotted by the continuous (blue) curve, against the dashed (red) curve depicting the evolution of the PRW-profile \((6), u_{ps}(x, t; 1.74; 1.07) \). The maximum amplitude of the event is attained at \( t = t^* = 1.74 \). The snapshots, clearly support the identification of the emerging structure around the peak formation as one of the Peregrine family. The central localized part of the waveform in the numerical solution, exhibits an algebraic-in-time growth/decay rate, remarkably close to that of the PRW-soliton (except for the region far from the core). In particular, as shown in the left panel of the top row of Fig. 2 for \( t \in [1, 3] \), the peak amplitude evolution is nearly indistinguishable from that of the corresponding member of the PRW family. The
The bottom left panel of Fig. 2 offers another view of the extreme event occurring at \( t = 1.74 \), revealing a remarkable feature of the dynamics: the extreme event occurs on the top of an emergent decaying support of the numerical solution (as opposed to the uniform background of the exact PRW waveform), which appears due to the existence of the driving – see discussion below. The bottom right panel of Fig. 2 depicts a magnification of the support of peak amplitude \( h_s \) [continuous (red) vertical line] and half-width \( w_s \).

Beyond the formation of the PRW-extreme event, further interesting dynamical features arise at later times. This can be seen in Fig. 3 where the spatiotemporal evolution of the density, for \( x \in [-75, 75] \), \( t \in [0, 60] \) and parameters as in Figs. 1 and 2 is shown. Here, it should be pointed out that the observed dynamics – prior to the ultimate decay of the solution – is found to be partially reminiscent of that of the semi-classical NLS \((\ref{eq:SCLNS})\): this is due to the fact that the change of variables \( x \to x/\sigma_x \) and \( t \to t/\sigma_x \) (for \( \sigma_x = 100 \), i.e., \( \sigma_x^{-1} \approx \epsilon \ll 1 \)) in Eq. \((\ref{eq:PRW})\) renders its left-hand side identical to that of Eq. \((\ref{eq:SCLNS})\). Indeed, as shown in Fig. 3 and similarly to Refs. \([45, 46]\), where the semi-classical NLS was studied, distinct spatiotemporal regions are formed; these are separated by nonlinear caustics (dotted curves), which bound the pattern of the transient decaying spatiotemporal oscillations. On the other hand, there exist also nontrivial differences to what was observed in Refs. \([45, 46]\). In particular, “lattice” of extreme events – in the form of PRW structures – was found to occupy the region in between the caustics, at points corresponding to the poles of the tritronquée solution of the Painlevé-I equation. In our case, while in the region bounded by the caustics we observe the formation of PRW events as discussed above, yet the pattern formed appears to be more close to the PRW lattice of \([45, 46]\) in the early stage of the evolution, while, at its final stages it resembles more to the cnoidal waveforms of Ref. \([48]\) (see Fig. 3 of this work). In the latter setting, in the inner region, a spatially oscillatory behavior, described by slow modulations of the periodic traveling wave solutions of the NLS, was found.

Hence, the dynamics observed in Fig. 3, although it appears to share characteristics of the behavior of Refs. \([45, 46, 48]\), nevertheless it is also distinct, due to the combined effect of loss and driving, as well as the form of the initial data. Concerning the latter, we remark that if the initial condition had the form of an exponentially localized pulse on top of a finite background (see also Refs. \([42, 49, 50]\)) then the numerical solution would follow closer the dynamics in Ref. \([48]\), albeit with some of the gain/loss-induced differences dis-
discussed above.

Considering the variation of the loss strength $\gamma$ (for fixed driving amplitude $\Gamma$) or $\Gamma$ (for fixed $\gamma$), we find the following. First, the increase of $\gamma$ results – as may be expected – in the decrease of the maximum amplitude of the first event $|u(0,t^*)|^2$, attained at $t = t^*$, and vice versa. On the other hand, the increase of $\Gamma$ (for fixed $\gamma$) results in the increase of $|u(0,t^*)|^2$, and vice versa.

We have also investigated the dependence of the height and the width of the emergent support of the extreme events on the magnitude of the driving: the relevant result is depicted in Fig. 4. In our computations we observe the existence of a threshold value $\Gamma^* = 0.2$ (for $\gamma = 0.01$), beyond which the emergence of the decaying support becomes noticeable (i.e., its amplitude is $\gtrsim 10^{-3}$), and increases with increasing $\Gamma$. This dependence is depicted in the top panel of Fig. 4. On the other hand, the bottom panel of Fig. 4 illustrates that the half-width $w_\gamma$ becomes comparable to that of the initial condition still for $\Gamma \geq \Gamma^*$; here, we have used $L = 500$. An interesting observation in the same panel is that the half width $w_\gamma$ approaches a constant value $w_\gamma^* \approx 250$ (i.e., saturates) for $\Gamma \geq 0.3$. The inset portrays a magnification of the support at $t = t^* = 1.74$, for $\Gamma = 1$, with localization width $[-w_\gamma^*, w_\gamma^*]$.

b. Dependence of the dynamics on $\sigma_x$ and $\sigma_t$. Another important feature of the system is that the driver’s spatial and temporal scales $\sigma_x$ and $\sigma_t$ affect the spatiotemporal localization of the numerical solution. This is verified in Fig. 5. There, the left column panels show contour plots of the spatiotemporal evolution of the density, for fixed $\sigma_t = 0.5$ and $\sigma_x = 50, 25, 2$ (top to bottom). Each of the right column panels depicts snapshots of the density, at $t = 1.7$, corresponding to the contour plots of the left column. Evidently, $\sigma_x$ controls the width of the decaying support of the emergent wave train. Particularly, we observe that, for $\sigma_x = 2$, the width of the support is considerably reduced and the solution is strongly localized around the center, exhibiting oscillations in time.

We also explore, in Fig. 6, the impact of increasing the value of $\sigma_t$ to $\sigma_t = 100$ (for sufficiently small $\sigma_x = 0.5$). The left panel shows the spatiotemporal evolution of the density of the numerical solution, for $t \in [0, 200]$. We can observe a dominating oscillation around the center and small-amplitude linear waves emanating from the main pattern. On the one hand, the large acting time of the forcing enhances time-periodic oscillations; on the other hand, the small value of $\sigma_x$ reduces considerably the effect of the formation of the decaying support and the localization width of the solution. The contour plots of the breathing oscillations are magnified, for $t \in [0, 100]$, in the upper right panel, as well as, for $t \in [100, 200]$, in the bottom right panel respectively; it is observed that these oscillations feature a significant lifetime prior to their eventual decay.

c. Dynamics in the case of purely spatial forcing. Having identified the role of the temporal localization of the driver above, we proceed by examining the case
FIG. 6: (Color Online) Left panel: Contour plot of the spatiotemporal evolution of the density of the initial condition \((4)\), for \(\gamma = 0.01, \Gamma = 1\), and for spreads of the forcing \(\sigma_x = 0.5, \sigma_t = 100\). Right panel: Contour plots magnifying the transient breathing localized mode observed in the left panel, for \(t \in [0, 100]\) (upper contour plot), and \(t \in [100, 200]\) (bottom contour plot).

of a driver showing solely spatial dependence, namely:

\[
f(x, t) = f(x) = g(x) + ih(x),
\]

where \(g(x) = h(x) = \Gamma \exp\left(-\frac{x^2}{2\sigma_x^2}\right)\).

The results in this case portray crucial differences in the dynamics when the limits of small and large \(\sigma_x\) are considered. The upper left and right panels of Fig. 7 depict the dynamics for \(\sigma_x = \sqrt{2}\). In the contour plot of the spatiotemporal evolution of the density, we observe the emergence of a triangular spatiotemporal region, occupied by a traveling wavetrain (see snapshots). This setting is strongly reminiscent of the defect scenario classified as flip-flop, emitting wavetrains alternately to the left and right – see Ref. [51]. The situation, however, drastically changes for large values of \(\sigma_x\), e.g., for \(\sigma_x = 20\). The corresponding snapshots reveal that the support exhibits interesting dynamics itself: it is decomposed to two symmetric outward moving parts, and a middle part, as detected in the snapshots at \(t = 2.5\) and \(t = 4\). Furthermore, the middle part expands in its own right, and is sustaining large-amplitude spatiotemporal oscillations, as seen in the contour plot for \(t \in [4, 8]\). Thus, the magnitude of \(\sigma_x\) drastically affects the nature of the resulting patterns in the case of purely spatial driving.

d. Comment on the dynamics of the integrable limit \(\gamma = 0, f = 0\) We complete this Section by briefly commenting on the dynamics of the integrable limit, \(\gamma = 0\) and \(f = 0\), with the initial condition

\[
u_0(x) = \frac{1}{1 + \frac{x^2}{\sigma_x^2}}.
\]

In such a situation, we have found (results not shown here) that the dynamics is reminiscent of that observed in Ref. [47]. In particular, for \(\sigma = 1\) (in this case, \((8)\) coincides with \((4)\)), the solution “locks” to a soliton of amplitude \(\approx 0.73\). As \(\sigma\) is increased, we progressively see a transition from oscillatory spatiotemporal patterns to a “Christmas tree” dynamical behavior, resembling the one in Fig. 3.

The above results are, in turn, reminiscent of the initial data width variation reported in Ref. [47] going from the solitonic limit of small width to the \textit{semiclassical limit} of large width (see the connection in [47] with the rigorous work of [45]). In addition, we should underline a significant difference between these emerging patterns and the case examples of our previous numerical experiments involving the external drive. In the case where the driving is present, the width of the emerging support in which the extreme event lies, is considerably wider, connected to the width of the driving. On the other hand, in the integrable case, due to the absence of such forcing, there is no emergent support and the background, in which the extreme events are hosted, is only dictated by the initial condition.

III. DISCUSSION AND CONCLUSIONS

In this work, direct numerical simulations revealed the excitation of extreme wave events for the linearly damped NLS equation supplemented with vanishing boundary conditions, in the presence of a spatiotemporally localized forcing. The driver assumed the form of a Gaussian function, and the dynamics emerged as a result of a quadratically decaying initial condition.

We found that when the spatial localization of the driving is considerably larger that its temporal localization, the excited Peregrine rogue waveforms appear on top of a decaying support, and the transient dy-
namics, prior to decay, resembles the one of the semi-classical NLS; nevertheless, the extreme events in our case do not appear as a lattice of Peregrine waveforms arising at points corresponding to the poles of the tritronquée solution of the Painlevé-I equation. The role of the driving/forcing strengths in suppressing/enhancing the dynamical features observed was also elucidated. Finally, the integrable limit for the same type of initial data was explored for completeness; in this case, it is solely the width of the initial data that determines the eventual fate of the algebraically localized initial condition.

Via comparison of the dynamics produced by varying the space and time localization width of the driver, we are led to expect PRW-type structures to form in the presence of effectively strong spatially extended exterior forces, acting for short times, while transient, large-amplitude, almost time-periodic breather-like structures emerge, when the acting time of the external driving is comparable to its spatial localization width. Connecting these findings further to the linearly damped and linearly forced NLS counterpart, we note that in the work of, an effectively strong linear driving which is acting for short times facilitates the occurrence of extreme wave conditions.

The results herein constitute a starting point for further studies, such as analytically estimating the decay rates of the numerical solutions, examining the effect of different external drivers and initial conditions possessing different types of decaying rates. In particular, exploring the “separatrix” between the decay rates that favor the scenario of vs. those that lead to the dynamics described in would be especially important to clarify. As an additional direction, the consideration of discrete counterparts to the presented phenomenology in the form of the discrete linearly damped and forced NLS equation will be of interest in their own right. Finally, some other interesting future directions would be to examine similar behaviors in systems of coupled NLSEs or in systems where nonlinearity and dispersion management is applied. Relevant works are in progress, and will be reported in future publications.

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