Peripheral Convex Expansions of Resonance Graphs

Zhongyuan Che

Received: 18 October 2019 / Accepted: 2 November 2020 / Published online: 9 November 2020
© Springer Nature B.V. 2020

Abstract
A perfect matching of a graph is a set of independent edges that covers all vertices of the graph. A bipartite graph is elementary if and only if it is connected and each edge is contained in a perfect matching. The resonance graph of a plane bipartite graph \( G \) is a graph whose vertices are perfect matchings of \( G \), and two perfect matchings are adjacent if edges contained in their union but not intersection form a cycle surrounding a finite face of \( G \). It is well known that the resonance graph of a plane elementary bipartite graph is a median graph. Median graphs form an important subclass of partial cubes and have a wide range of applications. The most important structural characterization of a median graph is the Mulder’s convex expansion theorem. Fibonacci cubes used in network designs are median graphs and can be obtained from an edge by a sequence of peripheral convex expansions. Plane bipartite graphs whose resonance graphs are Fibonacci cubes were studied by Klavžar et al. first for their applications in chemistry and completely characterized by Zhang et al. later. Motivated from their work, we characterize all plane bipartite graphs whose resonance graphs can be constructed from an edge by a sequence of peripheral convex expansions.

Keywords (Peripheral) convex expansion · Distributive lattice · Forcing face · Median graph · Plane elementary bipartite graph · Resonance graph · Reducible face · Z-transformation graph

1 Introduction
Let \( G \) be a connected graph with the vertex set \( V(G) \) and the edge set \( E(G) \). The length of a shortest path between two vertices \( u \) and \( v \) in \( G \) is denoted by \( d_G(u, v) \). The set of all vertices on shortest paths between \( u \) and \( v \) in \( G \) is called the interval between two vertices in \( G \), and denoted by \( I_G(u, v) \). A connected graph \( G \) is called a median graph if there is a unique vertex contained in the intersection \( I_G(u, v) \cap I_G(u, w) \cap I_G(v, w) \) for every triple vertices \( u, v, w \) of \( G \). Assume that \( H \) is a subgraph of \( G \). For any two vertices \( x \) and \( y \) of \( H \), if \( d_H(x, y) = d_G(x, y) \), then \( H \) is isometric in \( G \); if all shortest paths between \( x \) and \( y \) in \( G \) are contained in \( H \), then \( H \) is convex in \( G \). Median graphs form an important...
subclass of partial cubes, that is, isometric subgraphs of hypercubes [12], and arise naturally in the study of ordered sets and discrete distributive lattices as stated in [8]. They have been applied in various areas such as human genetics, phylogenetics, social choice theory, and computer science. There are several survey papers on median graphs, see [2, 14] and [16].

Let \( G_1 \) and \( G_2 \) be isometric subgraphs of \( G \) such that \( V(G) = V(G_1) \cup V(G_2) \) and \( V(G_1) \cap V(G_2) \) is not empty. Let \( G_1 \cap G_2 \) be the induced subgraph of \( G \) on the vertex set \( V(G_1) \cap V(G_2) \). Assume that \( G_1 \cap G_2 \) is a convex subgraph of \( G \), and there are no edges between \( V(G_1) \setminus V(G_1 \cap G_2) \) and \( V(G_2) \setminus V(G_1 \cap G_2) \). Take disjoint copies of \( G_1 \) and \( G_2 \) and connect every vertex of \( G_1 \cap G_2 \) in \( G_1 \) with the same vertex of \( G_1 \cap G_2 \) in \( G_2 \) with an edge. Then we obtain a graph which is called a convex expansion of \( G \). In particular, if \( G_1 = G \) and \( G_2 \) is convex in \( G \), then the above convex expansion is called a peripheral convex expansion of \( G \). The Mulder’s convex expansion theorem states that a graph is a median graph if and only if it can be obtained from the one vertex graph by a convex expansion procedure [12].

Perfect matchings of graphs in mathematics correspond to Kekulé structures of molecules in chemistry. Let \( M_1 \) and \( M_2 \) be two perfect matchings of a graph. Then a cycle of the graph is called \((M_1, M_2)\)-alternating if its edges are in \( M_1 \) and \( M_2 \) alternately. The symmetric difference of two sets \( S_1 \) and \( S_2 \), denoted by \( S_1 \oplus S_2 \), is the set of elements belonging to the union but not the intersection of \( S_1 \) and \( S_2 \). It is well known that the symmetric difference \( M_1 \oplus M_2 \) of two perfect matchings \( M_1 \) and \( M_2 \) of a graph is a set of vertex disjoint \((M_1, M_2)\)-alternating cycles of the graph [17].

Any plane graph divides the plane into a set of regions called faces. The boundary of a face \( s \) is the closed walk surrounding \( s \) and denoted by \( \partial s \). The boundary of the infinite face of a plane graph is also called the boundary of the plane graph. The resonance graph of a plane bipartite graph \( G \), denoted by \( Z(G) \), is a graph whose vertices are perfect matchings of \( G \) and two perfect matchings \( M_1 \) and \( M_2 \) are adjacent in \( Z(G) \) if \( M_1 \oplus M_2 = \partial s \) for some finite face \( s \) of \( G \). In this case, we say that the edge \( M_1M_2 \) has the face-label \( s \). The resonance graph was first introduced in chemistry to study the possible double bond structures of molecules. It was reintroduced by many researchers independently under various names such as the Z-transformation graph and the perfect matching graph. The resonance graph of a plane elementary bipartite graph is a median graph [22]. Readers are referred to the survey paper by Zhang [21] for properties and applications of resonance graphs.

Any plane elementary bipartite graph with two vertices is the one edge graph whose resonance graph is the one vertex graph. Let \( G \) be a plane elementary bipartite graph with more than two vertices. Then \( G \) is 2-connected [17] and the boundary of each face (including the infinite face) of \( G \) is an even cycle. A finite face of \( G \) is called a peripheral face if its boundary has some edges on the boundary of \( G \). Let \( s \) be a peripheral face of \( G \) such that the intersection of the boundary of \( s \) and the boundary of \( G \) is an odd length path \( P \). Let \( H \) be the subgraph of \( G \) obtained by removing all internal vertices and edges of \( P \). If \( H \) is also elementary, then \( s \) is called a reducible face of \( G \) [19].

To provide approaches to an open question raised in the survey paper [21] on characterizing Z-transformation graphs (or, resonance graphs), we first gave a convex expansion structure of the resonance graph \( Z(G) \) with respect to a reducible face of \( G \) in [3]. We further showed that the resonance graph of a 2-connected outerplane bipartite graph \( G' \) can be obtained from an edge by a sequence of peripheral convex expansions with respect to a reducible face decomposition of \( G' \) in [4]. Recently, we proved that \( Z(G) \) is cube-free if and only if it can be obtained from an edge by a sequence of convex path expansions with respect.
to a reducible face decomposition of $G$ in [5]. As a corollary, a structure characterization was obtained for a plane elementary bipartite graph whose resonance graph is cube-free.

The Fibonacci cube $\Gamma_n$, where $n \geq 1$, is a subgraph of the $n$-dimensional hypercube induced by the vertices without two consecutive ones. The Fibonacci cube was suggested in [11] as a new interconnection topology for parallel computing, and it has been used to emulate hypercube algorithms efficiently in network designs. Fibonacci cubes are median graphs and can be obtained from an edge by a sequence of peripheral convex expansions [13]. Klavžar and Žigert [15] proved that Fibonacci cubes are the resonance graphs of zigzag hexagonal chains. Zhang, Ou and Yao [23] characterized all plane bipartite graphs whose resonance graphs are only Fibonacci cubes.

In this paper, we show that a plane elementary bipartite graph $G$ whose resonance graph can be obtained from an edge by a sequence of peripheral convex expansions with respect to a reducible face decomposition of $G$ if and only if the subgraph of $G$ obtained by deleting all vertices on the boundary is either empty or has a unique perfect matching. This not only generalizes our previous result on the peripheral convex expansion structure of the resonance graph of a 2-connected outerplane bipartite graph [4], but also extends the characterizations of plane bipartite graphs [15] and [23] whose resonance graphs are only Fibonacci cubes.

2 Preliminaries

Let $M$ be a perfect matching of a graph $G$. An $M$-alternating path (resp., $M$-alternating cycle) is a path (resp., cycle) of $G$ whose edges are alternately in $M$ and not in $M$. A path $P$ of $G$ is called weakly $M$-augmenting if either $P$ is a single edge and not contained in $M$, or $P$ is an $M$-alternating path such that its two end edges are not contained in $M$. A cycle of $G$ is said to be a nice cycle if it is $M$-alternating for some perfect matching $M$ of $G$. A face (including the infinite face) of a plane bipartite graph is called $M$-resonant if its boundary is an $M$-alternating cycle for a perfect matching $M$ of the graph. A bipartite graph is elementary if and only if it is connected and each edge is contained in a perfect matching of the graph [17]. A plane bipartite graph is called weakly elementary if and only if for each nice cycle $C$ of the graph, the subgraph consisting of $C$ together with its interior is elementary. Every plane elementary bipartite graph is weakly elementary [25].

A plane bipartite graph with more than two vertices is elementary if and only if it has a reducible face decomposition [25]. A plane bipartite graph $G(=G_n)$ has a reducible face decomposition $G_i(1 \leq i \leq n)$ associated with a sequence of finite faces $s_i(1 \leq i \leq n)$ and a sequence of odd length paths $P_i(2 \leq i \leq n)$ if it can be constructed as follows. Start from $G_1$ which is an even cycle surrounding a finite face $s_1$ of $G$. Assume that $G_{i-1}$ has been constructed. Then $G_i$ can be obtained from $G_{i-1}$ by adding an odd length path $P_i$ in the exterior of $G_{i-1}$ such that only two end vertices of $P_i$ are contained in $G_{i-1}$, and $P_i$ and a part of the boundary of $G_{i-1}$ form an even cycle surrounding a finite face $s_i$ of $G$ for all $2 \leq i \leq n$. From the above definition, we can see that each subgraph $G_i$ of $G$ is a plane elementary bipartite graph for $1 \leq i \leq n$, and each $s_i$ is a reducible face of $G_i$ for $2 \leq i \leq n$.

Let $S$ be a poset with a partial order $\leq$. Assume that $x$ and $y$ are two elements of $S$. If $x \leq y$ and $x \neq y$, then we write $x < y$. We say that $y$ covers $x$ if $x < y$ and there is no element $z \in S$ such that $x < z < y$. The Hasse diagram of $S$ is a digraph with the vertex set $S$ such that $\overrightarrow{xy}$ is a directed edge from $y$ to $x$ if and only if $y$ covers $x$. A lattice is a poset with two operations $\lor$ and $\land$ such that $x \lor y$ is the unique least upper bound of $x$ and $y$, $x \land y$ is the unique greatest lower bound of $x$ and $y$. A Hasse diagram is a digraph with the vertex set $S$ such that $\overrightarrow{xy}$ is a directed edge from $y$ to $x$ if and only if $y$ covers $x$. A lattice is a poset with two operations $\lor$ and $\land$ such that $x \lor y$ is the unique least upper bound of $x$ and $y$, $x \land y$ is the unique greatest lower bound of $x$ and $y$.
and $x \land y$ is the unique greatest lower bound of $x$ and $y$. A lattice is distributive if its two operations $\lor$ and $\land$ admit distributive laws.

Assume that $G$ is a plane bipartite graph whose vertices are colored black and white such that adjacent vertices receive different colors. Let $M$ be a perfect matching of $G$. An $M$-alternating cycle $C$ of $G$ is $M$-proper (resp., $M$-improper) if every edge of $C$ belonging to $M$ goes from white to black vertices (resp., from black to white vertices) along the clockwise orientation of $C$. A plane bipartite graph $G$ with a perfect matching has a unique perfect matching $M_0$ (resp., $M_1$) such that $G$ has no proper $M_0$-alternating cycles (resp., no improper $M_1$-alternating cycles) [24]. The $Z$-transformation digraph (or, resonance digraph), denoted by $\bar{Z}(G)$, is the digraph obtained from $Z(G)$ by adding a direction for each edge so that $M_1M_2$ is a directed edge from $M_1$ to $M_2$ if $M_1 \oplus M_2$ is a proper $M_1$-alternating (or, an improper $M_2$-alternating) cycle surrounding a finite face of $G$. Let $\mathcal{M}(G)$ be the set of all perfect matchings of $G$. Then a partial order $\leq$ can be defined on $\mathcal{M}(G)$ such that $M' \leq M$ if there is a directed path from $M$ to $M'$ in $\bar{Z}(G)$.

Let $G$ be a plane (weakly) elementary bipartite graph. Then its resonance graph $Z(G)$ is a median graph [22], and $\mathcal{M}(G)$ is a finite distributive lattice with the minimum $M_0$ and the maximum $M_1$, and the Hasse diagram of $\mathcal{M}(G)$ is isomorphic to $\bar{Z}(G)$ [18]. The height of $\mathcal{M}(G)$ is defined as the length of a directed path from $M_1$ to $M_0$ in $\bar{Z}(G)$, we write $\height(\mathcal{M}(G))$ briefly. By the Jordan-Dedekind theorem [1] in finite distributive lattice, all directed paths (if exist) between any pair of vertices of its Hasse diagram have the same length and are shortest paths between them. So, $\height(\mathcal{M}(G))$ is the length of a shortest path between $M_1$ and $M_0$ in $Z(G)$ and can be written as $\height(\mathcal{M}(G)) = d_{Z(G)}(M_1, M_0)$.

Let $G$ be a plane elementary bipartite graph and $\mathcal{M}(G)$ be the set of all perfect matchings of $G$. Assume that $s$ is a reducible face of $G$. By definition [19], the intersection of the boundary of $s$ and the boundary of $G$ is an odd length path $P$. By Proposition 4.1 in [3], $P$ is $M$-altering for any perfect matching $M$ of $G$, and $\mathcal{M}(G) = \mathcal{M}(G; P^-) \cup \mathcal{M}(G; P^+)$, where $\mathcal{M}(G; P^-)$ (resp., $\mathcal{M}(G; P^+)$) is the set of perfect matchings $M$ of $G$ such that $P$ is weakly $M$-augmenting (resp., $P$ is not weakly $M$-augmenting). Furthermore, $\mathcal{M}(G; P^-)$ and $\mathcal{M}(G; P^+)$ can be partitioned as

$$
\mathcal{M}(G; P^-) = \mathcal{M}(G; P^-, \partial s) \cup \mathcal{M}(G; P^-, \overline{\partial s})
$$

$$
\mathcal{M}(G; P^+) = \mathcal{M}(G; P^+, \partial s) \cup \mathcal{M}(G; P^+, \overline{\partial s})
$$

where $\mathcal{M}(G; P^-, \partial s)$ (resp., $\mathcal{M}(G; P^-, \overline{\partial s})$) is the set of perfect matchings $M$ in $\mathcal{M}(G; P^-)$ such that $s$ is $M$-resonant (resp., not $M$-resonant), and $\mathcal{M}(G; P^+, \partial s)$ (resp., $\mathcal{M}(G; P^+, \overline{\partial s})$) is the set of perfect matchings $M$ in $\mathcal{M}(G; P^+)$ such that $s$ is $M$-resonant (resp., not $M$-resonant).

Two edges $uv$ and $xy$ of a graph $G$ are said to be in relation $\Theta$, denoted by $uv\Theta xy$, if $d_G(u, x) + d_G(v, y) \neq d_G(u, y) + d_G(v, x)$. Relation $\Theta$ divides the edge set of a median graph into $\Theta$-classes as an equivalent relation [12]. Based on the above partitions of $\mathcal{M}(G)$, a convex expansion structure of the resonance graph $Z(G)$ was given in [3] using the tool of $\Theta$-relation.

**Assumption 1** Assume that $G$ is a plane elementary bipartite graph and $s$ is a reducible face of $G$. Let $P$ be the odd length path that is the intersection of the boundary of $s$ and the boundary of $G$. Let $H$ be the subgraph of $G$ obtained by removing all internal vertices and
edges of $P$. Assume that $Z(G)$ and $Z(H)$ are resonance graphs of $G$ and $H$ respectively. Let $F$ be the set of all edges in $Z(G)$ with the face-label $s$.

Let $\langle W \rangle$ represent the induced subgraph of a graph $G$ on $W \subseteq V(G)$.

**Theorem 1** [3] Let Assumption 1 hold true. Then $F$ is a $\Theta$-class of $Z(G)$ and $Z(G) - F$ has exactly two components $\langle M(G; P^-) \rangle$ and $\langle M(G; P^+) \rangle$. Furthermore,

(i) $F$ is a matching defining an isomorphism between $\langle M(G; P^-, \partial s) \rangle$ and $\langle M(G; P^+, \partial s) \rangle$;

(ii) $\langle M(G; P^-) \rangle$ is convex in $\langle M(G; P^-) \rangle$, $\langle M(G; P^+, \partial s) \rangle$ is convex in $\langle M(G; P^+) \rangle$;

(iii) $\langle M(G; P^-) \rangle$ and $\langle M(G; P^+) \rangle$ are median graphs, where $\langle M(G; P^-) \rangle \cong Z(H)$.

In particular, $Z(G)$ can be obtained from $Z(H)$ by a peripheral convex expansion if and only if $\langle M(G; P^+) \rangle = \langle M(G; P^+, \partial s) \rangle$.

**Proposition 1** Let Assumption 1 hold true. Then both subgraphs $\langle M(G; P^-) \rangle$ and $\langle M(G; P^+) \rangle$ are convex in $Z(G)$.

**Proof** Let $xy$ be an edge of $Z(G)$ with the face-label $s$ where $x \in M(G; P^-, \partial s)$ and $y \in M(G; P^+, \partial s)$. Let $F_{xy} = \{e \in E(Z(G)) \mid e \Theta xy\}$ be the set of all edges of $Z(G)$ that are in relation $\Theta$ with the edge $xy$. Then by Proposition 4.4 in [3], $F = F_{xy}$. Note that $P$ is weakly $x$-augmenting. Then by Proposition 4.5 in [3], $M(G; P^-)$ is the set of perfect matchings $M$ of $G$ such that $M$ is closer to $x$ than to $y$ in $Z(G)$, and $M(G; P^+)$ is the set of perfect matchings $M$ of $G$ such that $M$ is closer to $y$ than to $x$ in $Z(G)$. Recall that $Z(G)$ is a median graph and so is a partial cube. Theorem 11.8 in [12] states that a connected graph is a partial cube if and only if it is bipartite and for each edge $xy$ of the graph, the induced subgraph on the set vertices that are closer to $x$ (resp., closer to $y$ than to $x$) in the graph is a convex subgraph. Therefore, both $\langle M(G; P^-) \rangle$ and $\langle M(G; P^+) \rangle$ are convex in $Z(G)$. \qed

**Proposition 2** Let Assumption 1 hold true. Then $\langle M(G; P^-) \rangle$, $\langle M(G; P^+) \rangle$, $\langle M(G; P^-, \partial s) \rangle$ and $\langle M(G; P^+, \partial s) \rangle$ are distributive sublattices of $\langle M(G) \rangle$.

**Proof** By [18], for any plane (weakly) elementary bipartite graph, the set of all its perfect matchings forms a finite distributive lattice with the Hasse diagram isomorphic to its resonance digraph.

Since $H$ is a plane elementary bipartite graph, $\langle M(H) \rangle$ forms a finite distributive lattice with the Hasse diagram isomorphic to the resonance digraph $\overline{Z}(H)$. By Theorem 1, $\langle M(G; P^-) \rangle \cong Z(H) = \langle M(H) \rangle$. Therefore, $\langle M(G; P^-) \rangle$ is a distributive sublattice of $\langle M(G) \rangle$.

Remove two end vertices of $P$ and their incident edges. If there is a pendent edge, then remove both end vertices of the pendent edge and incident edges. Continue this way, if all vertices of $G$ can be removed, then $\langle M(G; P^+) \rangle$ is the single vertex graph. Otherwise, we obtain a nontrivial plane bipartite graph $H' \subset G$ without pendent edges. Then $\langle M(G; P^+) \rangle \cong Z(H') = \langle M(H') \rangle$. By Theorem 1, $\langle M(G; P^+) \rangle$ is a median graph and so is connected. It follows that $Z(H')$ is connected. By [10] and [26], the resonance graph of
a plane bipartite graph is connected if and only if the plane bipartite graph is weakly elementary. Hence, $H'$ is a plane weakly elementary bipartite graph. It follows that $\mathcal{M}(H')$ forms a finite distributive lattice with the Hasse diagram isomorphic to the resonance digraph $\overrightarrow{Z}(H')$. Therefore, $\mathcal{M}(G; P^+)$ is a distributive sublattice of $\mathcal{M}(G)$.

By Theorem 1, we have that $\langle \mathcal{M}(G; P^-, \partial s) \rangle$ is convex in $\langle \mathcal{M}(G; P^-) \rangle$, and $\langle \mathcal{M}(G; P^+, \partial s) \rangle$ is convex in $\langle \mathcal{M}(G; P^+) \rangle$, where $\langle \mathcal{M}(G; P^-) \rangle$ and $\langle \mathcal{M}(G; P^+) \rangle$ are median graphs. By definition, $\langle \mathcal{M}(G; P^-, \partial s) \rangle$ (resp., $\langle \mathcal{M}(G; P^+, \partial s) \rangle$) is a connected graph. Similarly to the argument in the above observation of edge directions, we can show that $\langle \mathcal{M}(G; P^-, \partial s) \rangle$ (resp., $\langle \mathcal{M}(G; P^+, \partial s) \rangle$) is isomorphic to the resonance graph of a plane weakly elementary bipartite graph $H_1 \subset G$ (resp., $H_2 \subset G$). Therefore, both $\mathcal{M}(G; P^-, \partial s)$ and $\mathcal{M}(G; P^+, \partial s)$ are distributive sublattices of $\mathcal{M}(G)$.

\section{3 Main Results}

**Lemma 1** Let Assumption 1 hold true. Assume that $M_0$ and $M_1$ are the minimum and maximum of the finite distributive lattice $\mathcal{M}(G)$ on the set of all perfect matchings of $G$ respectively. Then $Z(G)$ can be obtained from $Z(H)$ by a peripheral convex expansion if and only if exactly one of $M_0$ and $M_1$ is contained in $\mathcal{M}(G; P^+, \partial s)$.

\textit{Proof} By Theorem 1, $Z(G)$ can be obtained from $Z(H)$ by a peripheral convex expansion if and only if $\mathcal{M}(G; P^+, \partial s) = \mathcal{M}(G; P^+)$. It remains to show that $\mathcal{M}(G; P^+, \partial s) = \mathcal{M}(G; P^+)$ if and only if exactly one of $M_0$ and $M_1$ is contained in $\mathcal{M}(G; P^+, \partial s)$.

By the definition of $\overrightarrow{Z}(G)$, we can see that all edges in $F$ are directed from one of $\mathcal{M}(G; P^-, \partial s)$ and $\mathcal{M}(G; P^+, \partial s)$ to the other in $\overrightarrow{Z}(G)$. We distinguish two cases based on the above observation of edge directions.

Case 1. All edges in $F$ are directed from $\mathcal{M}(G; P^-, \partial s)$ to $\mathcal{M}(G; P^+, \partial s)$ in $\overrightarrow{Z}(G)$. By Theorem 1, $Z(G) - F$ has exactly two components $\langle \mathcal{M}(G; P^-) \rangle$ and $\langle \mathcal{M}(G; P^+) \rangle$. Then $M_1 \in \mathcal{M}(G; P^-)$ and $M_0 \in \mathcal{M}(G; P^+)$.

**Claim** $\mathcal{M}(G; P^+, \partial s) = \mathcal{M}(G; P^+)$ if and only if $M_0 \in \mathcal{M}(G; P^+, \partial s)$.

Proof of Claim. Necessity is trivial. Only need to show Sufficiency. Assume that $M_0 \in \mathcal{M}(G; P^+, \partial s)$. Suppose that $\mathcal{M}(G; P^+, \partial s) \neq \mathcal{M}(G; P^+).$ Then there is an $x \in \mathcal{M}(G; P^+) \setminus \mathcal{M}(G; P^+, \partial s)$. Let $\overrightarrow{Q}$ be a directed path from $M_1$ passing through $x$ to $M_0$ in $\overrightarrow{Z}(G).$ Let $Q$ be the corresponding path of $\overrightarrow{Q}$ in $Z(G).$ By Theorem 1, $Q$ contains an edge $M_1' M_2' \in F$ satisfying $M_1' \in \mathcal{M}(G; P^-, \partial s)$ and $M_2' \in \mathcal{M}(G; P^+, \partial s)$. We can write

$$\overrightarrow{Q} = [Q_{M_1, M_1'}] \cup [Q_{M_1', M_2'}] \cup [Q_{M_2', x, M_0}],$$

where $[Q_{M_1, M_1'}]$ is a directed path from $M_1$ to $M_1'$, and $[Q_{M_2', x, M_0}]$ is a directed path from $M_2'$ passing through $x$ to $M_0$. By the Jordan-Dedekind theorem [1] in finite distributive lattice, all directed paths (if exist) between any pair of vertices of its Hasse diagram have the same length and are shortest paths between them. Note that $\langle \mathcal{M}(G; P^+, \partial s) \rangle$ is convex in $\langle \mathcal{M}(G; P^+) \rangle$ by Theorem 1; and $\langle \mathcal{M}(G; P^+) \rangle$ is convex in $Z(G)$ by Proposition 1. Therefore, $\langle \mathcal{M}(G; P^+, \partial s) \rangle$ is convex in $Z(G)$. Let $Q_{[M_2', x, M_0]}$ be the corresponding path of $[Q_{M_2', x, M_0}]$ in $Z(G)$. It follows that $Q_{[M_2', x, M_0]} \subset \langle \mathcal{M}(G; P^+, \partial s) \rangle$ since any shortest
path between $M'_2 \in \mathcal{M}(G; P^+, \partial s)$ and $M_0 \in \mathcal{M}(G; P^+, \partial s)$ in $Z(G)$ is contained in $(\mathcal{M}(G; P^+, \partial s))$. This is a contradiction since $\overrightarrow{Q}_{[M'_2, M_0]}$ contains an $x \in \mathcal{M}(G; P^+) \setminus \mathcal{M}(G; P^+, \partial s)$. Therefore, $\mathcal{M}(G; P^+, \partial s) = \mathcal{M}(G; P^+)$. This ends the proof of Claim.

Case 2. All edges of $F$ are directed from $\mathcal{M}(G; P^+, \partial s)$ to $\mathcal{M}(G; P^-, \partial s)$ in $\overrightarrow{Z}(G)$. Then $M_0 \in \mathcal{M}(G; P^-)$ and $M_1 \in \mathcal{M}(G; P^+)$. Similarly to the Claim in Case 1, we can show that $\mathcal{M}(G; P^+, \partial s) = \mathcal{M}(G; P^+)$ if and only if $M_1 \in \mathcal{M}(G; P^+, \partial s)$.

This ends the proof of the lemma. 

\textbf{Lemma 2} Let Assumption 1 hold true. Assume that $\mathcal{M}(G)$ and $\mathcal{M}(H)$ are the finite distributive lattices on the set of all perfect matchings of $G$ and $H$ respectively. Then $\text{height}(\mathcal{M}(G)) \geq \text{height}(\mathcal{M}(H)) + 1$ and the equality holds if and only if $Z(G)$ can be obtained from $Z(H)$ by a peripheral convex expansion.

\textbf{Proof} By Theorem 1 and the fact that all edges in $F$ are directed from one of $\mathcal{M}(G; P^-, \partial s)$ and $\mathcal{M}(G; P^+, \partial s)$ to the other in $\overrightarrow{Z}(G)$, we can see that $\text{height}(\mathcal{M}(G)) \geq \text{height}(\mathcal{M}(H)) + 1$. It remains to show that the equality holds if and only if $Z(G)$ can be obtained from $Z(H)$ by a peripheral convex expansion.

Sufficiency. Assume that $Z(G)$ can be obtained from $Z(H)$ by a peripheral convex expansion. By Lemma 1, without loss of generality, we can assume that $M_0 \in \mathcal{M}(G; P^+, \partial s)$. Then $M_1$ is contained in $\mathcal{M}(G; P^-)$. Let $M_0 = M_0 \oplus \partial s$ and $M_0|_H$ be the restriction of $M_0$ on $H$. Then $M_0|_H$ is a perfect matching of $H$. Note that all $M_0|_H$-alternating cycles of $H$ are $M_0|_H$-improper. By [18], $M_0|_H$ is the minimum element of $\mathcal{M}(H)$. It is easy to see that $M_0|_H$ can be extended uniquely to the perfect matching $M_0$ of $G$ in $\mathcal{M}(G; P^-)$. By Proposition 2, $\mathcal{M}(G; P^-)$ is a distributive sublattice of $\mathcal{M}(G)$. By Theorem 1, $(\mathcal{M}(G; P^-)) \cong Z(H) = (\mathcal{M}(H))$. Then $M_0$ is the minimum element of the sublattice $\mathcal{M}(G; P^-)$.

Let $\overrightarrow{Q}_{[M_1, M_0]}$ be a directed path from the maximum $M_1$ to the minimum $M_0$ in the Hasse diagram of the sublattice $\mathcal{M}(G; P^-)$. Then $|\overrightarrow{Q}_{[M_1, M_0]}| = \text{height}(\mathcal{M}(G; P^-)) = \text{height}(\mathcal{M}(H))$. Note that $\overrightarrow{Q}_{[M_1, M_0]}$ is a directed path in $\overrightarrow{Z}(G)$. By the construction of $M_0$, we can see that $x$ is proper $M_0$-resonant and $M_0 \oplus M_0 = \partial s$. Then $\overrightarrow{M_1M_0}$ is a directed edge in $\overrightarrow{Z}(G)$. It follows that $\overrightarrow{Q} = \overrightarrow{Q}_{[M_1, M_0]} \cup \overrightarrow{M_0M_0}$ is a directed path from $M_1$ to $M_0$ in $\overrightarrow{Z}(G)$.

Therefore, $\text{height}(\mathcal{M}(G)) = |\overrightarrow{Q}| = |\overrightarrow{Q}_{[M_1, M_0]}| + |\overrightarrow{M_0M_0}| = \text{height}(\mathcal{M}(H)) + 1$.

Necessity. Suppose that $Z(G)$ cannot be obtained from $Z(H)$ by a peripheral convex expansion. By Lemma 1, neither $M_0$ nor $M_1$ is contained in $\mathcal{M}(G; P^+, \partial s)$. Without loss of generality, we can assume that all edges in $F$ are directed from $\mathcal{M}(G; P^-, \partial s)$ to $\mathcal{M}(G; P^+, \partial s)$ in $\overrightarrow{Z}(G)$. Then $M_1 \in \mathcal{M}(G; P^-)$ and $M_0 \in \mathcal{M}(G; P^+) \setminus (\mathcal{M}(G; P^+, \partial s))$. Assume that $x$ is the minimum element of the sublattice $\mathcal{M}(G; P^-)$.

Let $\overrightarrow{Q}$ be a directed path from $M_1$ passing through $x$ to $M_0$ in $\overrightarrow{Z}(G)$. Let $Q$ be the corresponding path of $\overrightarrow{Q}$ in $Z(G)$. By Theorem 1, we can see that $Q$ contains a unique edge $e \in F$ since all edges in $F$ are directed from $\mathcal{M}(G; P^-, \partial s)$ to $\mathcal{M}(G; P^+, \partial s)$ in $\overrightarrow{Z}(G)$. By our assumption, $x$ is the minimum element of the sublattice $\mathcal{M}(G; P^-)$, it follows that $x$ must be one end vertex of the edge $e$ and $x \in \mathcal{M}(G; P^-, \partial s)$. Let $y$ be the other end vertex of the edge $e$. Then $y \in \mathcal{M}(G; P^+, \partial s)$. We can write $\overrightarrow{Q} = \overrightarrow{Q}_{[x,y]} \cup \overrightarrow{Q}_{[y,M_0]}$. 

where \( \hat{Q}_{[M_1,x]} \) is a directed path from the maximum \( M_1 \) to the minimum \( x \) in the Hasse diagram of the sublattice \( \mathcal{M}(G; P^-) \), and \( \hat{Q}_{[y,M_0]} \) is a directed path from \( y \) to \( M_0 \). Note that \( |\hat{Q}_{[M_1,x]}| = \text{height}(\mathcal{M}(G; P^-)) = \text{height}(\mathcal{M}(H)) \) and \( |\hat{Q}_{[y,M_0]}| > 0 \) since \( y \in \mathcal{M}(G; P^+, \partial s) \) and \( M_0 \in \mathcal{M}(G; P^+) \setminus \mathcal{M}(G; P^+, \partial s) \) are distinct vertices. Therefore, \[
\text{height}(\mathcal{M}(G)) = |\hat{Q}| = |\hat{Q}_{[M_1,x]}| + |\hat{Q}_{[y,M_0]}| > \text{height}(\mathcal{M}(H)) + 1.
\]

A face of a plane bipartite graph is called a forcing face if the boundary of the face is an even cycle and the subgraph obtained by removing all vertices of the face is either empty or has exactly one perfect matching. The concept of a forcing face was first introduced for finite faces in [6] and extended to all faces (including the infinite face) in [7].

**Theorem 2** Let \( G \) be a plane elementary bipartite graph with more than two vertices. Then its resonance graph \( Z(G) \) can be obtained from an edge by a sequence of peripheral convex expansions with respect to a reducible face decomposition of \( G \) if and only if the infinite face of \( G \) is forcing.

**Proof** Any plane elementary bipartite graph with more than two vertices has a reducible face decomposition. It is trivial when \( G \) is an even cycle as its resonance graph is an edge. Assume that \( G \) has more than one finite face. Let \( G_i (1 \leq i \leq n) \) be a reducible face decomposition of \( G (= G_0) \) associated with a sequence of finite faces \( s_i (1 \leq i \leq n) \). Let \( \mathcal{M}(G_i) \) be the finite distributive lattice on the set of all perfect matchings of \( G_i \) for all \( 1 \leq i \leq n \). Then \( \text{height}(\mathcal{M}(G_i)) = 1 \) since \( G_i \) is an even cycle surrounding a finite face \( s_1 \) of \( G \). By Lemma 2, for \( 2 \leq i \leq n \), \( \text{height}(\mathcal{M}(G_i)) \geq \text{height}(\mathcal{M}(G_{i-1})) + 1 \) and the equality holds if and only if \( Z(G_i) \) can be obtained from \( Z(G_{i-1}) \) by a peripheral convex expansion. It follows that \( Z(G) \) can be obtained from an edge by a sequence of peripheral convex expansions with respect to a reducible face decomposition of \( G \) if and only if \( \text{height}(\mathcal{M}(G)) = n \). It remains to show that \( \text{height}(\mathcal{M}(G)) = n \) if and only if the infinite face of \( G \) is forcing.

Assume that \( M_1 \) and \( M_0 \) are the maximum and the minimum of \( \mathcal{M}(G) \). Let \( \partial G \) denote the boundary of \( G \). By [19], \( \partial G \) is an even cycle that is both \( M_1 \)-alternating and \( M_0 \)-alternating. Hence, \( \partial G \) is an \( (M_1, M_0) \)-alternating cycle. Let \( \mathcal{F} \) be the set of all finite faces of \( G \). Let \( M \) be an arbitrary perfect matching in \( \mathcal{M}(G) \). Then \( M \oplus M_0 \) is a set of vertex disjoint \( (M, M_0) \)-alternating cycles in \( G \) [17].

Let \( f \in \mathcal{F} \). Define \( \phi_M(f) \) as the number of \((M, M_0)\)-alternating cycles in \( M \oplus M_0 \) containing \( f \) in their interiors [22]. Then \( \phi_{M_0}(f) = 0 \) and \( \phi_{M_1}(f) > 0 \) since \( \partial G \) is an \((M_1, M_0)\)-alternating cycle. Theorem 3.2 in [22] states that distance \( d_{Z(G)}(M, M') = \sum_{f \in \mathcal{F}} |\phi_M(f) - \phi_{M'}(f)| \) for any two vertices \( M \) and \( M' \) of \( Z(G) \). Hence, we have the following height formula for \( \mathcal{M}(G) \).

\[
\text{height}(\mathcal{M}(G)) = d_{Z(G)}(M_1, M_0) = \sum_{f \in \mathcal{F}} |\phi_{M_1}(f) - \phi_{M_0}(f)| = \sum_{f \in \mathcal{F}} \phi_{M_1}(f). \tag{1}
\]

Sufficiency. Assume that the infinite face of \( G \) is forcing. Then the subgraph \( G' \) of \( G \) obtained by removing all vertices on \( \partial G \) has a unique perfect matching. Recall that \( M_1 \oplus M_0 \) is a set of vertex disjoint \((M_1, M_0)\)-alternating cycles and \( \partial G \) is an \((M_1, M_0)\)-alternating cycle. Note that restrictions of \( M_1 \) and \( M_0 \) on the subgraph \( G' \) are perfect matchings of \( G' \) and must be identical. Then \( \partial G \) is the unique \((M_1, M_0)\)-alternating cycle in \( M_1 \oplus M_0 \). Then
\( \phi_{M_1}(f) = 1 \) for each \( f \in \mathcal{F} \). By the height formula (1), we have

\[
\text{height}(\mathcal{M}(G)) = \sum_{f \in \mathcal{F}} \phi_{M_1}(f) = \sum_{f \in \mathcal{F}} 1 = |\mathcal{F}| = n.
\]

Necessity. Assume that \( \text{height}(\mathcal{M}(G)) = n = |\mathcal{F}| \). By the height formula (1), we have \( \text{height}(\mathcal{M}(G)) = n = \sum_{f \in \mathcal{F}} \phi_{M_1}(f) \). It implies that \( \phi_{M_1}(f) = 1 \) for any \( f \in \mathcal{F} \) since \( \phi_{M_1}(f) > 0 \) for any \( f \in \mathcal{F} \). Suppose that the infinite face of \( G \) is not forcing. Then the subgraph \( G' \) of \( G \) obtained by removing all vertices on \( \partial G \) has at least two perfect matchings \( M_1' \) and \( M_2' \). Note that \( M_1' \oplus M_2' \) is a set of vertex disjoint \( (M_1', M_2') \)-alternating cycles. Choose an \( (M_1', M_2') \)-alternating cycle \( C \) such that \( C \) is not contained in the interior of any other \( (M_1', M_2') \)-alternating cycles in \( M_1' \oplus M_2' \). Without loss of generality, we can assume that \( C \) is proper \( M_1' \)-alternating and so improper \( M_2' \)-alternating. It is clear that \( M_1' \) (resp., \( M_2' \)) can be extended to a perfect matching \( M_1 \) (resp., \( M_2 \)) of \( G \) such that both \( C \) and \( \partial G \) are proper \( M_1 \)-alternating and improper \( M_2 \)-alternating, and \( C \) is not contained in the interior of any \( (M_1, M_2) \)-alternating cycles other than \( \partial G \) in \( M_1 \oplus M_2 \).

Let \( f \in \mathcal{F} \). Define \( \psi_{M_1,M_2}(f) \) as the number of proper \( M_1 \)-alternating cycles in \( M_1 \oplus M_2 \) with \( f \) in their interiors minus the number of improper \( M_1 \)-alternating cycles in \( M_1 \oplus M_2 \) with \( f \) in their interiors [22]. By the construction of \( M_1 \) and \( M_2 \), there exists an \( s \in \mathcal{F} \) contained in \( C \) such that \( \psi_{M_1,M_2}(s) \geq 2 \). By Lemma 2.4 in [22], \( \phi_{M_1}(s) - \phi_{M_2}(s) = \psi_{M_1,M_2}(s) \). Since both \( \phi_{M_1}(s) \geq 0 \) and \( \phi_{M_2}(s) \geq 0 \), it follows that \( \phi_{M_1}(s) \geq 2 \).

Assume that \( M \) covers \( M_1 \) in \( \mathcal{M}(G) \). Then there is a directed edge from \( M \) to \( M_1 \) in the resonance digraph \( \overrightarrow{Z}(G) \). So, \( M \oplus M_1 \) is the boundary of a finite face of \( G \). Lemma 2.5 in [22] states that \( M \) covers \( M_1 \) in \( \mathcal{M}(G) \) if and only if \( \phi_{M}(f) - \phi_{M_1}(f) = 1 \) when \( M \oplus M_1 = \partial f \) and \( \phi_{M}(f) - \phi_{M_1}(f) = 0 \) otherwise. This implies that \( \phi_{M_1}(s) \geq \phi_{M_1}(s) \geq 2 \) since \( M_1 \) is the maximum element of \( \mathcal{M}(G) \). On the other hand, we have shown that \( \phi_{M_1}(s) = 1 \). This is a contradiction. Therefore, the infinite face of \( G \) is forcing. \( \square \)
It is known that the Fibonacci cube $\Gamma_n$ can be constructed from an edge by a sequence of peripheral convex expansions [13]. Plane elementary bipartite graphs whose $Z$-transformation graphs are the Fibonacci cube $\Gamma_n$ are characterized as a class $\mathcal{F}_n$ of graphs in [23]. A plane elementary bipartite graph $G$ with more than two vertices is contained in $\mathcal{F}_n$ if it has a reducible face decomposition $G_i (1 \leq i \leq n)$ associated with a sequence of finite faces $s_i (1 \leq i \leq n)$ and a sequence of odd length paths $P_i (2 \leq i \leq n)$ under two extra conditions: (i) $P_i$ starts from the vertex $x_i$ and ends at the vertex $y_i$ along the clockwise orientation of the boundary of $s_i$ for $2 \leq i \leq n$ such that $x_{i-1}$ of $P_{i-1}$ and $x_i$ of $P_i$ have different colors for $3 \leq i \leq n$; (ii) the boundary of $s_i$ is formed by a part of $P_{i-1}$ and $P_i$ for $3 \leq i \leq n$. For example, all zigzag hexagonal chains and linear square chains are in class $\mathcal{F}_n$ [23]. By Theorem 2, if $G$ is contained in $\mathcal{F}_n$, then the infinite face of $G$ is forcing.

The following two corollaries can be obtained from Theorem 2 easily.

**Corollary 1** The infinite face of a plane elementary bipartite graph $G$ is forcing if and only if it has a reducible face decomposition $G_i (1 \leq i \leq n)$ such that the infinite face of each $G_i$ is forcing for $1 \leq i \leq n$.

**Corollary 2** Let $G$ be a plane elementary bipartite graph such that each finite face has a vertex on the boundary of $G$. Then $Z(G)$ can be obtained from an edge by a sequence of peripheral convex expansions with respect to a reducible face decomposition of $G$.

The plane elementary bipartite graph $G$ given in Figure 23 [9] obtained from a six crossing knot universe satisfies the property stated in Corollary 2. Its resonance graph $Z(G)$ can be obtained from an edge by a sequence of peripheral convex expansions with respect to a reducible face decomposition of $G$ associated with the face sequence $s_i (1 \leq i \leq 7)$.

The *induced graph* $\Theta(Z(G))$ on the $\Theta$-classes of the edge set of a resonance graph $Z(G)$ is the graph whose vertex set is the set of $\Theta$-classes, and two $\Theta$-classes $E_1$ and $E_2$ are adjacent if $Z(G)$ has two incident edges $e_1 \in E_1$ and $e_2 \in E_2$ such that $e_1$ and $e_2$ are not contained in a common 4-cycle of $Z(G)$. In [4], we showed that if $G$ is a 2-connected outerplane bipartite graph, then $Z(G)$ can be obtained from an edge by a sequence of peripheral convex expansions with respect to a reducible face decomposition of $G$. Moreover, $\Theta(Z(G))$ is a tree and isomorphic to the inner dual of $G$. This generalized the corresponding result in [20] for catacondensed hexagonal systems. Note that for a plane elementary bipartite graph $G$, if $Z(G)$ can be obtained from an edge by a sequence of peripheral convex...
expansions with respect to a reducible face decomposition of \( G \), then it is not necessarily true that \( \Theta(Z(G)) \) is isomorphic to the inner dual of \( G \). See Examples given in Figs. 1 and 2.

**Question 1** When a plane elementary bipartite graph \( G \) has the property that \( \Theta(Z(G)) \) is isomorphic to the inner dual of \( G \)?

**Acknowledgments** The research work is supported by the Research Development Grant (RDG) from Penn State University, Beaver Campus. The author would like to thank the referees for their helpful comments.

**Data Availability** Data sharing statement supplied by Springer: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

**References**

1. Birkhoff, G.: Lattice Theory. American Mathematical Society Colloquium Publications Vol. XXV, 3rd Edn.. American Mathematical Society, Providence (1967)
2. Bandelt, H.-J., Chepoi, V.: Metric graph theory and geometry: a survey. Contemp. Math. **453**, 49–86 (2008)
3. Che, Z.: Structural properties of resonance graphs of plane elementary bipartite graphs. Discret. Appl. Math. **247**, 102–110 (2018)
4. Che, Z.: Characterizations of the resonance graph of an outerplane bipartite graph. Discret. Appl. Math. **258**, 264–268 (2019)
5. Che, Z.: Cube-free resonance graphs. Discret. Appl. Math. **284**, 262–268 (2020)
6. Che, Z., Chen, Z.: Forcing faces in plane bipartite graphs. Discrete Math. **308**, 2427–2439 (2008)
7. Che, Z., Chen, Z.: Forcing faces in plane bipartite graphs (II). Discret. Appl. Math. **161**, 71–80 (2013)
8. Chung, F.R.K., Graham, R.L., Saks, M.E.: Dynamic search in graphs, Discrete Algorithms and Complexity, pp. 351–387. Kyoto (1986)
9. Cohen, M., Teicher, M.: Kauffman’s clock lattice as a graph of perfect matchings: a formula for its height. Electron. J. Combin. **21**, Paper 4.31, 39 (2014)
10. Fournier, J.C.: Combinatorics of perfect matchings in plane bipartite graphs and application to tilings, Theoret. Comput. Sci. **303**, 333–351 (2003)
11. Hsu, W.-J.: Fibonacci cubes - a new interconnection topology. IEEE Trans. Parallel Distrib. Syst. **4**, 3–12 (1993)
12. Hammack, R., Imrich, W., Klavžar, S. Handbook of Product Graphs, 2nd. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton (2011)
13. Klavžar, S.: On median nature and enumerative properties of Fibonacci-like cubes. Discrete Math. **299**, 145–153 (2005)
14. Knuth, D.E.: Median algebras and median graphs, The Art of Computer Programming, IV, Fascicle 0: Introduction to Combinatorial Algorithms and Boolean Functions, pp. 64–74. Addison-Wesley (2008)
15. Klavžar, S., Žigert, P.: Fibonacci cubes are the resonance graphs of fibonaccenes. Fibonacci Quart. **43**, 269–27 (2005)
16. Klavžar, S., Mulder, H.M.: Median graphs: characterizations, location theory and related structures. J. Combin. Math. Combin. Comput. **30**, 103–127 (1999)
17. Lovász, L., Plummer, M.D.: Matching Theory. North-Holland Publishing Co., Amsterdam (1986)
18. Lam, P.C.B., Zhang, H.: A distributive lattice on the set of perfect matchings of a plane bipartite graph. Order **20**, 13–29 (2003)
19. Taranenko, A., Vesel, A.: 1-Factors and characterization of reducible faces of plane elementary bipartite graphs. Discuss. Math. Graph Theory **32**, 289–297 (2012)
20. Vesel, A.: Characterization of resonance graphs of catacondensed hexagonal graphs. MATCH Commun. Math. Comput. Chem. **53**, 195–208 (2005)
21. Zhang, H.: Z-transformation graphs of perfect matchings of plane bipartite graphs: a survey. MATCH Commun. Math. Comput. Chem. **56**, 457–476 (2006)
22. Zhang, H., Lam, P.C.B., Shiu, W.C.: Resonance graphs and a binary coding for the 1-factors of benzenoid systems. SIAM J. Discret. Math. **22**, 971–984 (2008)
23. Zhang, H., Ou, L., Yao, H.: Fibonacci-like cubes as Z-transformation graphs. Discret. Math. 309, 1284–1293 (2009)
24. Zhang, H., Zhang, F.: The rotation graphs of perfect matchings of plane bipartite graphs. Discret. Appl. Math. 73, 5–12 (1997)
25. Zhang, H., Zhang, F.: Plane elementary bipartite graphs. Discret. Appl. Math. 105, 291–311 (2000)
26. Zhang, H., Zhang, F., Yao, H.: Z-transformation graphs of perfect matchings of plane bipartite graphs. Discrete Math. 276, 393–404 (2004)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.