1 Introduction

This paper is concerned with approximating certain singular schemes by smooth stacks, and applications of having such an approximation. More precisely, if $X$ is a scheme over a field $k$, we are interested in finding a smooth Artin stack $\mathcal{X}$ over $k$ and a morphism $f : \mathcal{X} \to X$ over $k$ which realizes $X$ as the good moduli space (in the sense of [Al]) of $\mathcal{X}$. Moreover, we would like the stacky structure of $\mathcal{X}$ to be supported on the singular locus of $X$. That is, we would like the base change of $f$ to $X^{sm}$ to be an isomorphism. It is a well-known result that this is possible when $X$ has quotient singularities prime to the characteristic of $k$ (see for example [Vi, 2.9] or [FMN, Rmk 4.9]). In this case $\mathcal{X}$ is Deligne-Mumford and $f$ realizes $X$ as the coarse space of $\mathcal{X}$. If the singularities of $X$ are worse than quotient singularities, we can no longer hope to find a Deligne-Mumford stack $\mathcal{X}$ as above. Typically Artin stacks do not have coarse spaces and the appropriate notion that replaces coarse space is that of good moduli space.

Inspired by the work of Iwanari [Iw], we take a different approach toward the problem of approximating $X$ by a smooth algebraic stack. Namely, we restrict attention to a class of schemes which carry more structure in hopes of being able to both construct our desired stack as above and say more about that stack than we could for an arbitrary scheme. This richer class of schemes we look at is that of fs log smooth log schemes $\mathcal{X}$ over $k$, where Spec$k$ is given the trivial log structure (or equivalently, the class of toroidal embeddings which are not necessarily strict). Our main approximation theorem is then:

**Theorem 3.2.** Let $k$ be field and $X$ be an fs log scheme which is log smooth over $S = \text{Spec}k$, where $S$ is given the trivial log structure. Then there exists a smooth, log smooth log Artin stack $\mathcal{X}$ over $S$ and a morphism $f : \mathcal{X} \to X$ over $S$ which realizes $X$ as the good moduli space of $\mathcal{X}$. Moreover, the base change of $f$ to the smooth locus of $X$ is an isomorphism.

This is a generalization of [Iw Thm 3.3] where the result is proved for $X$ all of whose charts are given by simplicial toric varieties. Our method of proof is a direct generalization of Iwanari’s. In particular, our stack $\mathcal{X}$ has a moduli interpretation (in terms of log geometry) and agrees with the stack Iwanari constructs when $X$ is as in [Iw, Thm 3.3]. We also give a slight improvement of [Iw Thm 3.3(2)] in Remark 3.3.

We give two applications of Theorem 3.2, a generalization of the Chevalley-Shephard-Todd theorem ([Bo, §5 Thm 4]) to the case of diagonalizable group schemes and a generalization of the work of [BCS] on toric Deligne-Mumford stacks to the case of toric Artin stacks.

**Chevalley-Shephard-Todd Theorem for diagonalizable group schemes**

We recall that if $k$ is a field and $G$ is a finite (abstract) group which acts faithfully on a $k$-vector space $V$, then $g \in G$ is called a pseudo-reflection if $V^g$ is a hyperplane. The Chevalley-Shephard-Todd Theorem then states that if the order of $G$ is prime to the characteristic of $k$, then the invariants $k[V]^G$ is a polynomial ring if and only if $G$ is generated
by pseudo-reflections.

There is a strong connection between the Chevalley-Shephard-Todd theorem and the fact that every scheme with quotient singularities is the coarse space of a smooth Deligne-Mumford stack. In fact, we hope to convince the reader that (at least in the setting of linearly reductive group schemes) proving approximation theorems as above is roughly equivalent to proving Chevalley-Shephard-Todd type theorems. This philosophy is demonstrated, for example, in [Sa] where the Chevalley-Shephard-Todd theorem is generalized to the case of finite linearly reductive group schemes and from this it is shown that every scheme with linearly reductive singularities ([Sa, Def 5.1]) is the coarse space of a smooth tame Artin stack (in the sense of [AOV]). Here, we go the opposite direction; that is, we show that the proof of Theorem 3.2 can be reinterpreted as a Chevalley-Shephard-Todd theorem for diagonalizable group schemes (see Theorem 4.2 which is a bit too technical to state in the introduction).

We remark that in the process of proving Theorem 4.2, we give necessary and sufficient conditions for when \( \text{Spec}(k[V]^G) \) is a simplicial toric variety (see Theorem 4.5). We also note that Theorem 4.2 recovers [We, Thm 5.6] which gives the result when \( G \) is a torus.

**Toric Artin stacks**

If we require that \( X \) is a toric variety rather than just a toroidal embedding, then we can in fact produce (see Theorem 5.11) many smooth Artin stacks \( X \), other than the canonical stack of Theorem 3.2, all of which have \( X \) as a good moduli space and all of which have moduli interpretations in terms of log geometry (however, only the canonical stack is isomorphic to \( X \) over \( X^{sm} \)). Each of these Artin stacks has a dense open torus whose action on itself extends to an action on the stack. Thus, these stacks should all be thought of as toric Artin stacks. Along the lines of [BCS], we develop a theory of generalized stacky fans and recast the construction of the above toric Artin stacks in terms of these stacky fans (see Theorem 5.13). We imagine that along the lines of [FMN], there is a more intrinsic description of the toric Artin stacks we introduce in this paper, but we do not attempt to prove such a result here.

The key difference between our stacky fans and those of [BCS], is that we allow marked points which do not lie on an extremal ray. More precisely, we define a generalized stacky fan \( \Sigma \) to be the choice of a finitely-generated abelian group \( N \), a rational fan \( \Sigma \subset N \otimes \mathbb{Q} \), a choice of \( r \in \mathbb{N} \), and a morphism \( \beta : Z(1) \times \mathbb{Z}^r \rightarrow N \). We require that if \( \rho_i \in \Sigma(1) \), then \( \beta(\rho_i) \otimes 1 \) lie on the the ray \( \rho_i \), and we require that if \( e_j \in \mathbb{Z}^r \), then \( \beta(e_j) \otimes 1 \) lie in some cone \( \sigma_j \) of the fan. In particular, we allow \( \beta \) to be the zero map. We remark that the toric Artin stack associated to \( (N = 0, \Sigma = 0, \beta : \mathbb{Z}^n \rightarrow N) \) is Lafforgue’s toric Artin stack \( \mathbb{A}^n/G_m \). Therefore, the theory of toric Artin stacks we develop here helps to unite that of [BCS] with that of [La].

This paper is organized as follows. In Section 2, we define minimal free resolutions and qmfrs. In Section 3, we construct the canonical stack of Theorem 3.2 as the moduli space of qmfrs. We also construct several other smooth log smooth log Artin stacks having our given log scheme as a good moduli space. These stacks are constructed as moduli spaces parameterizing what we call admissible qfrs. In Section 4, we reinterpret the proof of Theorem 3.2 as a generalization of the Chevalley-Shephard-Todd theorem for diagonalizable group schemes. In Section 5, we restrict our attention to toric varieties (rather than arbitrary toroidal embeddings) and construct many more smooth log smooth log Artin stacks.
having $X$ as a good moduli space. This construction relies on log geometry; however, we then compare our construction with that of [BCS]. Namely, we generalize the notion of stacky fan given in [BCS] and reinterpret our log geometric construction in the language of generalized stacky fans.

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Notation and prerequisites. We assume the reader is familiar with log geometry as in [KKa]. Given an abelian group $A$ and scheme $X$, we denote by $D_X(A)$ the diagonalizable group scheme over $X$ associated to $A$. We often drop the subscript $X$ when it is understood from context.

2 Minimal Free Resolutions

In this section, we define the objects which will be parameterized by the canonical stack $X$ of Theorem 3.2. A morphism $f : P \to Q$ of monoids is called close if for all $q \in Q$ there is a positive integer $n$ such that $nq$ is in the image of $f$. We recall ([Iw, Def 2.5]) that an injective morphism $i : P \to F$ from a saturated simplicially toric sharp monoid to a free monoid is called a minimal free resolution if $i$ is close and if for all injective close morphisms $i' : P \to F'$ to a free monoid $F'$ of the same rank as $F$, there is a unique morphism $j : F \to F'$ such that $i' = ji$. The stack Iwanari constructs in [Iw, Thm 3.3] has a moduli interpretation in terms of minimal free resolutions, so our first step is to generalize this notion. The key is to replace his use of closeness in the above definition with that of exactness.

Definition 2.1. A morphism $f : P \to Q$ of integral monoids is exact if the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{f} & Q \\
P^{gp} & \xlongleftarrow{f^{gp}} & Q^{gp}
\end{array}
$$

is set-theoretically cartesian.

Note that if $f : P \to Q$ is sharp and exact, then it is automatically injective.

Let $P$ be a saturated toric sharp monoid and let $C(P)$ denote the rational cone of $P$ in $P^{gp}$. Let $d$ be the number of rays of the dual cone $C(P)^\vee$. For $1 \leq i \leq d$, we denote by $v_i$ the first lattice point on each of the rays of $C(P)^\vee$ and by $F(P)$ the free monoid on the $v_i$. We obtain a morphism $i : P \to F(P)$ defined by $p \mapsto (v_i(p))$.

Proposition 2.2. The morphism $i : P \to F(P)$ is exact. Moreover, for any exact morphism $i' : P \to F$ to a free monoid $F$ of the same rank as $F(P)$, there is a unique morphism $j : F(P) \to F$ such that $i' = ji$.

Proof. The exactness of $i$ follows easily from the discussion on the top of page 12 of [Fu]. Arbitrarily choosing an isomorphism of $F$ with $F(P)$, we can assume $F(P) = F$. Let
\( i' = (\varphi_i) \). Exactness of \( i' \) shows that \( p \in P^{gp} \) is in \( P \) if and only if \( \varphi_i^{gp}(p) \geq 0 \) for all \( i \). Since \( p \) is in \( P \) if and only if \( f(p) \geq 0 \) for all \( f \in C(P)^{\vee} \), we see

\[
\text{Cone}(\varphi_i) = C(P)^{\vee} = \text{Cone}(v_i).
\]

Since \( C(P)^{\vee} \) has \( d \) rays, it follows that every \( \varphi_i \) lies on a ray. Composing \( i' \) by a uniquely determined permutation, we can assume that \( \varphi_i \) lies on ray generated by \( v_i \). Since \( \varphi_i \) takes integer values on \( P \), we see that \( \varphi_i \) is a lattice point of \( C(P)^{\vee} \). Since \( v_i \) is defined to be the first lattice point on the ray defined by \( v_i \), we have \( \varphi_i = n_i v_i \) for uniquely determined \( n_i \in \mathbb{N} \). Hence, multiplication by \( (n_1, \ldots, n_d) \) is our desired \( j \).

In light of this proposition, we make the following definition.

**Definition 2.3.** Let \( P \) be a saturated toric sharp monoid and let \( F(P) \) have rank \( d \). A morphism \( i : P \to F \) to a free monoid of rank \( d \) is a **minimal free resolution** if it is exact and if for every exact morphism \( i' : P \to F' \) to a free monoid \( F' \) of rank \( d \), there is a unique morphism \( j : F \to F' \) such that \( i' = ji \).

If all of the charts of \( X \) are given by simplicial toric varieties (as in the case Iwanari considers), then in constructing the canonical stack \( X \), one need only consider minimal free resolutions. As we see shortly, in the non-simplicial case, however, certain quotients of minimal free resolutions naturally arise:

**Definition 2.4.** Let \( P \) be a saturated toric sharp monoid and let \( F(P) \) have rank \( d \). A morphism \( i' : P \to F' \) to a free monoid is **qmfr** if it is of the form

\[
P \xrightarrow{i} F \xrightarrow{\pi} F/H,
\]

where \( i \) is a minimal free resolution, \( H \) is a face of \( F \) such that \( i(P) \cap H = 0 \), and \( \pi \) is the natural projection.

If \( P \) is a saturated simplicially toric sharp monoid, then the rank of \( F(P) \) is equal to the rank of \( P^{gp} \). If \( i : P \to F \) is a morphism to a free monoid of rank equal to \( P^{gp} \), then \( i \) is exact if and only if it is injective and closed. Hence, \( i \) is a minimal free resolution in the sense of [Iw] Def 2.5 if and only if it is a minimal free resolution in the sense of Definition 2.3 if and only if it is qmfr. As in [Iw] Def 2.11, we define a minimal free resolution morphism of log schemes.

**Definition 2.5.** A morphism \( f : (Y, \mathcal{M}_Y) \to (X, \mathcal{M}_X) \) of fs log schemes is a **minimal free resolution**, resp. is qmfr if for all geometric points \( y \) of \( Y \), the induced morphism

\[
\mathcal{M}_{X,f(y)} \to \mathcal{M}_{Y,y}
\]

is a minimal free resolution, resp. is qmfr.

**Proposition 2.6.** Let \( P \) be a saturated toric sharp monoid with minimal free resolution \( i : P \to F \). If \( P_0 \) is a face of \( P \) and \( F_0 \) is the face of \( F \) generated by \( P_0 \), then the induced morphism \( P/P_0 \to F/F_0 \) is a minimal free resolution.

**Proof.** One easily checks that \( P/P_0 \) is a saturated toric sharp monoid. Let \( \pi : P \to P/P_0 \) be the natural morphism. Let \( \{v_i\}_{i=1}^d \) be the extremal rays of \( C(P)^{\vee} \). If \( v_i(P_0) = 0 \), then we obtain a well-defined morphism \( w_i : P/P_0 \to \mathbb{N} \) given by \( w_i(p) = v_i(p) \). Note that \( w_i \) is an extremal ray since if \( \psi + \psi' = w_i \), then \( \psi \pi + \psi' \pi = v_i \). It follows that \( \psi \pi = av_i \) for some \( a \in \mathbb{Q}_{\geq 0} \), and so \( \psi = aw_i \).
We claim that the $w_i$ generate $C(P/P_0)^\vee$. Let $\psi \in C(P/P_0)^\vee$. Then $\psi \pi = \sum_j a_j v_j$ for some $a_j \in \mathbb{Q}_{\geq 0}$. If there is some $p_0 \in P_0$ such that $v_k(p_0) \neq 0$, then since 

$$0 = \psi \pi(p_0) = \sum_j a_j v_j(p_0),$$

we see $a_k v_k(p_0) = 0$ and so $a_k = 0$. Therefore, the $w_i$ generate $C(P/P_0)^\vee$.

To complete the proof of the proposition, we need only show that $e_i \notin F_0$ if and only if $v_i(p_0) = 0$. If $v_i(p_0) \neq 0$ for some $p_0 \in P_0$, then 

$$e_i + ((v_i(p_0) - 1)e_i + \sum_{j \neq i} v_j(p_0)e_j)$$

is in the image of $P_0$ and so $e_i \in F_0$. Conversely, if $e_i \in F_0$, then there exists some $p_0 \in P_0$ and $b_j \in \mathbb{N}$ such that 

$$e_i + \sum_j b_j e_j = \sum_j v_j(p_0)e_j.$$ 

As a result, $v_i(p_0) \neq 0$. \hfill \Box

We remark that if $H$ is any face of $F$ and $P_0 = H \cap P$, then $P/P_0 \to F/H$ is not in general a minimal free resolution. For example, let $P$ be the submonoid of $\mathbb{N}^4$ generated by $x = (1, 0, 0, 1), y = (0, 1, 1, 0), z = (1, 0, 1, 0),$ and $w = (0, 1, 0, 1)$. Note that the only relation among the generators is $x + y = z + w$. This is a saturated sharp non-simplicial toric monoid. Its minimal free resolution is given by the embedding into $F = \mathbb{N}^4$ that is used to define it.

If $H$ is any of the four faces of $F$ which are generated by a single element, then $P_0 := H \cap P = 0$. So, $P = P/P_0 \to F/H \simeq \mathbb{N}^3$ cannot be a minimal free resolution since the rank of $F/H$ is too small. Nonetheless, this morphism is qmfr. This general phenomenon is the content of the following proposition which generalizes [Iw, Prop 2.12].

**Proposition 2.7.** Let $P$ be a saturated toric sharp monoid and let $i : P \to F$ be its minimal free resolution. If $R$ is a ring, then the induced morphism $f : \text{Spec } R[F] \to \text{Spec } R[P]$ on log schemes is qmfr.

**Proof.** Let $i$ be a geometric point of $\text{Spec } R[F]$ and let $p$ be the corresponding prime ideal of $R[F]$. Let $H$ be the face of $F$ consisting of elements which map to units under $F \to R[F] \to R[F]_p$. Then $\tilde{M}_{P,f(i)} \to \tilde{M}_{F,i}$ is given by the natural map $\eta : P/P_0 \to F/H$, where $P_0 = H \cap P$. If we let $F_0$ be the face of $F$ generated by $P_0$, we see $F_0 \subset H$ and so $\eta$ factors as

$$P/P_0 \xrightarrow{\pi} F/F_0 \to F/H.$$ 

By Proposition 2.6 we see that $\pi$ is a minimal free resolution. Since $H/F_0 \cap P/P_0 = 0$, we see then that $\eta$ is qmfr. \hfill \Box

We now prove an analogue of [Iw, Prop 2.17].

**Proposition 2.8.** Let $P$ be a saturated toric sharp monoid and $i : P \to F$ an injective morphism to a free monoid $F$. Let $R$ be a ring and let $(f, h) : (T, \mathcal{M}_T) \to \text{Spec } R[P]$ be a morphism of fine log schemes. If we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_P & \xrightarrow{i} & \mathcal{M}_F \\
\downarrow & & \downarrow \\
\tilde{h}_s & & \tilde{M}_s \\
\end{array}
$$

where $\tilde{h}_s$ is a morphism of fine log schemes.
and α étale locally lifts to a chart, then there is an fppf neighborhood of s and a chart ε : F → M making the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{i} & F \\
\downarrow & & \downarrow \alpha \\
f^*M_P & \xrightarrow{h} & M
\end{array}
\]

commute.

Proof. The proof is the same as that of [Iw, Prop 2.17], except here the ranks of \( P^{gp} \) and \( F^{gp} \) are no longer the same. Letting the ranks be \( r \) and \( d \), respectively, but otherwise keeping Iwanari’s notation, we can choose isomorphisms \( \phi : P^{gp} \to \mathbb{Z}^r \) and \( \psi : F^{gp} \to \mathbb{Z}^d \) so that \( a \) is given by \( e_i \mapsto \lambda_i e_i \) for \( 1 \leq i \leq r \). Note that the \( \lambda_i \) are positive integers since \( P^{gp} \to F^{gp} \) is injective. Letting \( O' = O_{T,s}[T_1, \ldots, T_r]/(T_i^{\lambda_i} - u_i) \), we define \( \eta : F^{gp} \to M^{gp}_i \) as Iwanari does on the \( e_i \) for \( 1 \leq i \leq r \), and for \( r < i \leq d \), we send the \( e_i \) to 0. \( \square \)

3 The Approximation Theorem

Throughout this section \( k \) is a field and \( S = \text{Spec } k \) has the trivial log structure. Given \( X \) a log scheme over \( S \), we define a fibered category \( \mathcal{X} \) over \( X \)-schemes as follows. Objects are qmfr morphisms \( (T, \mathcal{N}) \to (X, \mathcal{M}_X) \), where \( \mathcal{N} \) is a fine log structure on \( T \), and morphisms are maps of \( (X, \mathcal{M}_X) \)-log schemes \( h : (T, \mathcal{N}) \to (T', \mathcal{N}') \) with \( h^*\mathcal{N}' \to \mathcal{N} \) an isomorphism. Then \( \mathcal{X} \) is a stack on the étale site of \( X \), and in fact also on the fppf site by [Ol1, Thm A.1].

Proposition 3.1. Let \( P \) be a saturated toric sharp monoid with minimal free resolution \( i : P \to F \). Let \( R \) be a ring and \( G \) be the group scheme \( \text{Spec } R[F^{gp}/P^{gp}] \). If \( X = \text{Spec } R[P] \), then \( \mathcal{X} \) is isomorphic to \( \mathcal{Y} := [\text{Spec } R[F]/G] \) over \( \text{Spec } R[P] \).

Proof. Let \( h : \mathcal{Y} \to \text{Spec } R[P] \) and \( \pi : \text{Spec } R[F] \to \mathcal{Y} \) be the natural morphisms. By [Ol1 Prop 5.20], the stack \( \mathcal{Y} \) has the following moduli interpretation. The fiber over \( f : T \to \text{Spec } R[P] \) is the groupoid of triples \( (\mathcal{N}, \eta, \gamma) \), where \( \mathcal{N} \) is a fine log structure on \( T \), and morphisms are maps of \( (X, \mathcal{M}_X) \)-log schemes \( h : (T, \mathcal{N}) \to (T', \mathcal{N}') \) with \( h^*\mathcal{N}' \to \mathcal{N} \) an isomorphism which étale locally lifts to a chart, and where \( \eta : f^*\mathcal{M}_P \to \mathcal{N} \) is a morphism of log structures such that

\[
\begin{array}{ccc}
P & \xrightarrow{i} & F \\
\downarrow & & \downarrow \gamma \\
f^{-1}\mathcal{M}_P & \xrightarrow{\eta} & \mathcal{N}
\end{array}
\]

commutes. We claim that \( \eta \) is qmfr. Let \( g : T \to \mathcal{Y} \) be the morphism representing \( (\mathcal{N}, \eta, \gamma) \) and let \( \bar{t} \) be a geometric point of \( T \). Since smooth morphisms étale locally have sections by [EGA4 17.16.3], we have a dotted arrow making the diagram

\[
\begin{array}{ccc}
\text{Spec } R[F] & \xrightarrow{\pi} & \mathcal{Y} \\
\downarrow & & \downarrow \pi \\
\bar{t} & \xrightarrow{f} & \text{Spec } R[P] \\
\end{array}
\]

\[ g \]

\[ h \]

\[ \text{Spec } R[F] \]

\[ \pi \]

\[ \bar{t} \]
commute. Recall from the proof of [Ol11, Prop 5.20] that \( \eta \) is simply the pullback under \( g \) of the natural morphism \( h^* M_P \to M_Y \). Therefore, \( \tilde{\eta} \) is the morphism
\[
\tilde{M}_{P,h^*\tilde{\eta}} = (\pi^* h^* M_P)_\tilde{t} \to (\pi^* M_Y)_\tilde{t} = \tilde{M}_{F,\tilde{t}},
\]
which is qmfr by Proposition 2.7.

We have, then, a morphism \( \Phi : Y \to X \) of stacks which forgets \( \gamma \). To prove full faithfulness of \( \Phi \), we must show that if
\[
(\eta_i : f^* M_P \to N_i, \gamma_i : F \to N_i)
\]
are objects of \( Y \) for \( i = 1, 2 \), then any isomorphism of log structures \( \xi : N_1 \to N_2 \) such that \( \xi \eta_1 = \eta_2 \) automatically satisfies \( \xi \gamma_1 = \gamma_2 \). The equality \( \xi \gamma_1 = \gamma_2 \) can be checked on stalks. Let \( t \in T \). Since the \( \gamma_i \) étale locally lift to charts \( \epsilon_i : F \to N_i \), we see that \( N_{i,t} \simeq \mathbb{N}^r \) for some \( r \) and that \( (\epsilon_i)_t \) is a projection followed by a permutation of coordinates. We have therefore reduced to proving that if
\[
(\phi_j, \psi_j) : F \to \mathbb{N}^r
\]
are morphisms given by projecting and permuting coordinates, and if
\[
(\phi_j(p)) = (\psi_j(p))
\]
for all \( p \in P \), then \( \phi_j = \psi_j \) for all \( j \). Post-composing \( (\phi_j) \) and \( (\psi_j) \) by the projection to the \( j \)th factor for some fixed \( j \), we may assume that \( r = 1 \). That is, we have reduced to the statement that if \( j \neq j' \), then there is some \( p \in P \) such that \( v_j(p) \neq v_{j'}(p) \), which is clearly true.

We now prove essential surjectivity of \( \Phi \). Let \( f : (T,N) \to \text{Spec } R[P] \) be qmfr. By full faithfulness, we need only show that \( f \) is fppf locally in the image of \( \Phi \). Let \( \tilde{t} \) be a geometric point of \( T \). Then \( \tilde{M}_{P,f(\tilde{t})} = P/P_0 \) for some face \( P_0 \) of \( P \). If \( P_0 \) is the face of \( F \) generated by \( P_0 \), then by Proposition 2.6, the natural morphism \( P/P_0 \to F/F_0 \) is a minimal free resolution. Since \( \tilde{f}_t : \tilde{M}_{P,f(\tilde{t})} \to \tilde{N}_t \) is qmfr by assumption, it has the form
\[
P/P_0 \to F/F_0 \to (F/F_0)/F_1
\]
for some face \( F_1 \) of \( F/F_0 \) such that \( F_1 \cap P/P_0 = 0 \). Letting \( H \) be the face of \( F \) such that \( H/F_0 = F_1 \), we see that
\[
P \to f^{-1}\tilde{M}_{P,\tilde{t}} \to \tilde{N}_t
\]
factors as
\[
P \to F \to F/H.
\]
Proposition 2.8 then shows that fppf locally, \( f \) is in the image of \( \Phi \).

**Theorem 3.2.** Let \( X \) be an fs log scheme which is log smooth over \( S \). Then there exists a smooth, log smooth log Artin stack \( X \) over \( S \) and a morphism \( f : X \to X \) over \( S \) which realizes \( X \) as the good moduli space of \( X \). Moreover, the base change of \( f \) to the smooth locus of \( X \) is an isomorphism.

**Proof.** By [FKa, Thm 4.8], since \( X \) is log smooth over \( S \), we have an étale cover \( h : Y \to X \) and a smooth strict morphism \( g : Y \to Z \), where \( Z = \text{Spec } k[P] \) with \( P \) a saturated toric
sharp monoid. Let \( i : P \to F \) be the minimal free resolution. Then Proposition 3.1 shows that \( X \) is étale locally \([U/G]\), where

\[
U = Y \times_Z \text{Spec } k[F]
\]

and \( G = \text{Spec } k[F^{gp}/P^{gp}] \). Hence, \( X \) is a smooth Artin stack. Moreover, the log structure on \([\text{Spec } k[F]/G]\) induces a log structure on \( X \) which makes it log smooth over \( S \). We see that \( X \to X \) is a good moduli space by [AI, Ex 8.3]. Lastly, the base change of this map to \( X^{sm} \) is an isomorphism since \( \mathcal{M}_{X,\bar{x}} \) is free if and only if \( x \in X^{sm} \) by [Iw, Lemma 3.5], which shows that if \((f, h) : (T, N) \to (X^{sm}, \mathcal{M}_{X|X^{sm}})\) is qmfr, then \( h \) is an isomorphism.

**Remark 3.3.** We can improve slightly on [Iw, Thm 3.3(2)]. If \( X \) is a good toroidal embedding ([Iw, 1.2]), then \( X \) is a tame Artin stack in the sense of [AOV]. It follows from [Sa, Lemma 5.5] that [Iw, Thm 3.3(2)] still holds for such \( X \). That is, we do not need to assume that \( X \) is a tame toroidal embedding ([Iw, 1.2]).

We end this section by showing that Iwanari’s stack of admissible free resolutions can also be generalized to the case when the charts of \( X \) are given by toric monoids which are not necessarily simplicial. Throughout the rest of this section, \( k \) is a field, \( S = \text{Spec } k \) has the trivial log structure, and \( X \) is a log scheme which is log smooth over \( S \).

**Definition 3.4.** If \( P \) is a saturated toric sharp monoid, \( i : P \to F \) is its minimal free resolution, and \( b_i \) are positive integers for every irreducible element \( v_i \) of \( F \), then \( P \to F' \) is an admissible free resolution of type \((b_i)\) if it is isomorphic to

\[
P \overset{i'} \to F \overset{(b_i)} \to F.
\]

We say that \( P \to F' \) is admissibly qfr of type \((b_i)\) if it is isomorphic to

\[
P \overset{i'} \to F \to F/H,
\]

where \( i' \) is an admissible free resolution of type \((b_i)\), and where \( H \) is a face of \( F \) such that \( H \cap i'(P) = 0 \).

Note that if \( P \) is simplicial, then \( i' : P \to F' \) is an admissible free resolution of type \((b_i)\) in the sense of [Iw, Def 2.5] if and only if it is in the sense of Definition 3.4 if and only if it is admissibly qfr of type \((b_i)\).

To define the corresponding notions for morphisms of log schemes, we first generalize [Iw, Prop 3.1].

**Proposition 3.5.** For every geometric point \( \bar{x} \) of \( X \), there is a canonical bijection between the irreducible elements of the minimal free resolution of \( \mathcal{M}_{X,\bar{x}} \) and the irreducible components of \( X - X^{triv} \) on which \( \bar{x} \) lies.

**Proof.** As the proof of [Iw, Prop 3.1] shows, we need only address the case when \( X = \text{Spec } k[P] \), where \( P \) is a saturated toric sharp monoid. We can further assume that \( \bar{x} \) maps to the torus-invariant point, so that \( \mathcal{M}_{X,\bar{x}} = P \). Then the irreducible components of \( X - X^{triv} \) are the torus-invariant divisors of \( X \), and we see that \( \bar{x} \) lies on all of them. The torus-invariant divisors are in canonical bijection with the extremal rays of \( C(P)^{\vee} \), which are precisely the irreducible elements of the minimal free resolution of \( P \). \( \square \)
**Definition 3.6.** Let $b_i$ be a positive integer for every irreducible component $D_i$ of $X - X^{triv}$. For every geometric point $\bar{x}$ of $X$, let $I(\bar{x})$ be the set of irreducible components of $X - X^{triv}$ on which $\bar{x}$ lies. Then a morphism $f : (Y, \mathcal{M}_Y) \to (X, \mathcal{M}_X)$ from a fine log scheme is an admissible free resolution of type $(b_i)$, resp. is admissibly qfr of type $(b_i)$ if for all geometric points $\bar{y}$ of $Y$, the induced morphism

$$\tilde{\mathcal{M}}_{X,f(\bar{y})} \to \tilde{\mathcal{M}}_{Y,\bar{y}}$$

is an admissible free resolution of type $(b_i)_{i \in I(f(\bar{y}))}$, resp. is admissibly qfr of type $(b_i)_{i \in I(f(\bar{y}))}$.

With this definition in place, for any choice $(b_i)$ of positive integers indexed by the irreducible components of $X - X^{triv}$, let $\mathcal{X}(b_i)$ be the fibered category over $X$-schemes whose objects are morphisms $(T, \mathcal{N}) \to (X, \mathcal{M}_X)$ which are admissibly qfr of type $(b_i)$, and whose morphisms are maps of $(X, \mathcal{M}_X)$-log schemes $h : (T, \mathcal{N}) \to (T', \mathcal{N}')$ with $h^* \mathcal{N}' \to \mathcal{N}$ an isomorphism. As before, this fibered category is a stack on the fppf site by [Ol1, Thm A.1].

The proofs of Propositions 2.7 and 3.1 apply word for word after replacing “minimal free resolution” by “admissible free resolution of type $(b_i)$”, and “qmfr” by “admissibly qfr of type $(b_i)$” to show the following two propositions:

**Proposition 3.7.** Let $P$ be a saturated toric sharp monoid and $i : P \to F$ its minimal free resolution. Let $b_i$ be a positive integer for every irreducible element $v_i$ of $F$. If $i' : P \to F'$ is an admissible free resolution of type $(b_i)$ and $X = \text{Spec } k[P]$, then the induced morphism

$$f : X \to \text{Spec } k[F']$$

of log schemes is admissibly qfr of type $(b_i)$; here we are using Proposition 3.5 to identify the irreducible components of $X - X^{triv}$ and the irreducible elements of $F$.

**Proposition 3.8.** Let $P$ be a saturated toric sharp monoid and $i : P \to F$ its minimal free resolution. Let $b_i$ be a positive integer for every irreducible element $v_i$ of $F$. If $i' : P \to F'$ is an admissible free resolution of type $(b_i)$ and $X = \text{Spec } k[P]$, then

$$\mathcal{X}(b_i) \simeq [\text{Spec } k[F']/D(F^{gp}/i'(P^{gp})]]$$

over $X$.

Using Proposition 3.8 we prove an analogue of Theorem 3.2. Note that if some $b_i > 1$, then a morphism $F \to F'$ which is admissibly qfr of type $(b_i)$ has automorphisms. As a result, the stacks $\mathcal{X}(b_i)$ are not isomorphic to $X$ over $X^{sm}$; they are, however, isomorphic to $X$ over $X^{triv}$.

**Theorem 3.9.** Let $b_i$ be a positive integer for every irreducible component $D_i$ of $X - X^{triv}$. Then $\mathcal{X}(b_i)$ is a smooth, log smooth Artin stack over $S$. The natural morphism $\mathcal{X}(b_i) \to X$ is a good moduli space and the base change of this morphism to $X^{triv}$ is an isomorphism.

**Proof.** The proof is the same as that of Theorem 3.2. We address only the last assertion. If $(f, h) : (T, \mathcal{N}) \to (X^{triv}, \mathcal{M}_X|_{X^{triv}})$ is admissibly qfr of type $(b_i)$, then $\bar{h} = 0$ is an isomorphism, and so $\bar{h}$ is strict.

The $\mathcal{X}(b_i)$ are all root stacks over the stack $\mathcal{X} = \mathcal{X}(1)$ in Theorem 3.2. As we will see in Section 5 if we restrict $X$ to being a toric variety, rather than an arbitrary log smooth log scheme, then we can construct many other smooth log smooth stacks having $X$ as a good moduli space.
4 Polynomial Invariants of Diagonalizable Group Schemes

Throughout this section, \( k \) is a field and \( A \) is a finitely-generated abelian group. We let \( G = D(A) \) be the diagonalizable group scheme over \( k \) associated to \( A \) and we fix a faithful action of \( G \) on a finite-dimensional \( k \)-vector space \( V \). Our goal in this section is to give necessary and sufficient conditions for when the invariants \( k[V]^G \) is a polynomial algebra over \( k \). In the process of working toward this goal, we give necessary and sufficient conditions for when \( \text{Spec}(k[V]^G) \) is a simplicial toric variety (see Theorem 4.5).

Since \( G \) is diagonalizable, there is a free monoid \( F' \) such that \( k[V] = k[F'] \) and such that the action of \( G \) on \( k[V] \) is induced from a morphism of monoids \( \pi : F' \to A \). Then there is an exact morphism \( i' : P \to F' \) such that \( A = F'^{gp}/i'(P^{gp}) \) and the induced morphism \( k[P] \to k[F']^G \) is an isomorphism.

We state the main theorem of this section after first giving a definition which is derived from [We, Thm 4.1 (4)].

**Definition 4.1.** Given a torus \( T = \text{Spec} k[M] \) over \( k \) and a faithful action of \( T \) on a finite-dimensional \( k \)-vector space \( W \), we say the action has MSOP if there are weights \( m_1, \ldots, m_k \in M \) such that the \( m_i \otimes 1 \) is a basis for \( M \otimes \mathbb{Z} \mathbb{Q} \) and all other weights lie in the non-positive orthant of the \( m_i \). We say that \( m_1, \ldots, m_k \) is an MSOP basis.

**Theorem 4.2.** If the induced action of the torus of \( G \) on \( V \) does not have MSOP, then \( k[V]^G \) is not polynomial. If the action does have MSOP, then fix \( m_1, \ldots, m_k \) as in Definition 4.1 and let \( F'_0 \) be the face generated by the elements of \( F' \) which map to the positive orthant of \( m_1, \ldots, m_k \). If \( B \) denotes the image of \( F'_0^{gp} \) in \( A \) under \( \pi^{gp} \), then \( A/B \) is a finite abelian group and the induced action of \( D(A/B) \) on \( k[F'/F'_0] \) has the property that

\[
k[V]^G = k[F'/F'_0]^{D(A/B)};
\]

therefore \( k[V]^G \) is polynomial if and only if the action of \( D(A/B) \) on \( k[F'/F'_0] \) is generated by pseudo-reflections, as defined in [Sa, Def 1.2].

We begin with some observations. Let \( i : P \to F \) be the minimal free resolution and let the ranks of \( F \) and \( F' \) be \( d \) and \( d' \), respectively. Choosing an isomorphism \( F' \cong \mathbb{N}^{d'} \), we see that \( i'(p) = (w_j(p))_j \), where the \( w_j \) are morphisms from \( P \) to \( \mathbb{N} \). As in the proof of Proposition 2.2 exactness of \( i' \) shows that \( \text{Cone}(w_j) = C(P)^\vee \) and so rechoosing our isomorphism of \( F' \) with \( \mathbb{N}^{d'} \) if necessary, we can assume \( w_j = c_j v_j \) for \( 1 \leq j \leq d \) and positive integers \( c_j \). We see then that \( d \leq d' \). For \( d < j \leq d' \), we have

\[
w_j = \sum_{i=1}^{d} c_{ij} v_i
\]

for \( c_{ij} \in \mathbb{Q}_{\geq 0} \). We can therefore find a positive integer \( n \) and isomorphisms \( F \cong \mathbb{N}^d \) and \( F' \cong \mathbb{N}^{d'} \) such that the diagram

\[
\begin{array}{ccc}
F' & \overset{n}{\longrightarrow} & F' \\
\downarrow{i'} & & \downarrow{\psi} \\
P & \overset{i}{\longrightarrow} & F \\
\end{array}
\]

commutes, where \( b_i = c_i n \) and \( b_{ij} = c_{ij} n \), and \( \psi \) is given by the \( d' \times d \) matrix whose top \( d \times d \) block is \( \text{diag}(b_i) \) and whose bottom block has entries \( b_{ij} \). For the rest of this section,
we identify $F$ with $\mathbb{N}^d$ and $F'$ with $\mathbb{N}^{d'}$ via the above isomorphisms.

We now show the necessity of the MSOP condition in Theorem 4.2.

**Proposition 4.3.** If $k[V]^G$ is a polynomial algebra, then the action of the torus of $G$ on $V$ has MSOP.

**Proof.** If $k[V]^G$ is polynomial, then $P$ is free. So, $i$ is an isomorphism and we can take $n = 1$. If $C$ denotes the torsion-free part of $A$, then the torus $T$ of $G$ is $D(C)$ and the action of $T$ on $k[F']$ is induced from the morphism $\sigma : F' \rightarrow A \rightarrow C$. From the explicit description of $\psi$, we see $\sigma(e_i) \otimes 1$ for $i > d$ form a basis for $C \otimes \mathbb{Q}$ and the $\sigma(e_j)$ for $j \leq d$ lie in the non-positive orthant of the $\sigma(e_i)$ for $i > d$. Hence, the action of $T$ has MSOP. 

**Lemma 4.4.** The action of the torus of $G$ on $V$ has MSOP if and only if there is a subset $S$ of $\{1, 2, \ldots, d'\}$ and a basis $\{p_j\}_{j \notin S}$ of $P^{gp} \otimes \mathbb{Q}$ with each $p_j$ satisfying $w_j(p_j) > 0$ and $w_k(p_j) = 0$ for $j \neq k \notin S$. There is a canonical bijection

$$S \mapsto \{\bar{e}_i \otimes 1 \mid i \in S\}$$

between such subsets and the set of MSOP bases, where $\bar{e}_i$ denotes the image in $A$ of the standard generators $e_i \in \mathbb{N}^{d'} \simeq F'$.

**Proof.** To prove the “if” direction we show that the $\bar{e}_i \otimes 1$ with $i \in S$ form an MSOP basis. Let $a_j = w_j(p_j)$ and let $a_{ij} = w_i(p_j)$ for $i \in S$. Since the $p_j$ form a basis for $P^{gp} \otimes \mathbb{Q}$, we see that

$$|S| = d' - \text{rank } P^{gp},$$

which is the rank of the torus of $G$. To prove that the $\bar{e}_i \otimes 1$ form a basis, we therefore need only show that they are linearly independent. If $\sum_{i \in S} c_i \bar{e}_i = 0$ for some $c_i \in \mathbb{Q}$, then

$$\sum_{i \in S} c_i e_i = i'(p)$$

in $F^{gp}$ for some $p \in P^{gp} \otimes \mathbb{Q}$. Writing $p = \sum_{j \notin S} c_j' p_j$ for $c_j' \in \mathbb{Q}$, and comparing the $e_j$ coordinates above, we see $c_j' a_j = 0$, which shows that $p = 0$, and so the $c_i = 0$ as well.

To show that the $\bar{e}_i \otimes 1$ form an MSOP basis, note that for $j \notin S$,

$$a_j e_j + \sum_{i \in S} a_{ij} e_i = \sum_{k=1}^{d'} w_k(p_j) e_k = i'(p_j).$$

Since the $a_{ij} \geq 0$, we see then that $\bar{e}_j \otimes 1$ is in the non-positive orthant of the $\bar{e}_i \otimes 1$ for $i \in S$.

We now prove the “only if” direction. If $j \notin S$, then $\bar{e}_j \otimes 1$ is in the non-positive orthant of the $\bar{e}_i \otimes 1$ for $i \in S$. We therefore have some positive integer $a_j$, non-negative integers $a_{ij}$, and some $p_j \in P^{gp}$ such that

$$a_j e_j = - \sum_{i \in S} a_{ij} e_i + \sum_{k=1}^{d'} w_k(p_j) e_k.$$ 

We see then that $w_j(p_j) = a_j$, $w_i(p_j) = a_{ij}$ if $i \in S$, and $w_k(p_j) = 0$ otherwise. Since $w_\ell(p_j)$ is non-negative for all $\ell$, exactness of $i'$ shows that $p_j \in P$. 

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We prove that the $p_j$ form a basis for $P^{gp} \otimes \mathbb{Q}$. Since the rank of $P^{gp}$ is $d' - |S|$, we need only show that the $p_j$ are linearly independent. If $\sum_{j \notin S} c_j p_j = 0$ for $c_j \in \mathbb{Q}$, then applying $w_{j_0}$ for $j_0 \notin S$ shows that $c_{j_0}a_{j_0} = 0$. Since $a_{j_0}$ is positive, we have $c_{j_0} = 0$, as desired.

The following theorem gives necessary and sufficient conditions for when $k[V]^G$ is simplicial (c.f. [We] Thm 4.1).

**Theorem 4.5.** The action of the torus of $G$ on $V$ has MSOP if and only if $P$ is simplicial. Furthermore, when these equivalent conditions are satisfied, if we let $S \subset \{1, 2, \ldots, d'\}$ correspond to our choice of MSOP basis as in Lemma 4.4, then the extremal rays of $C(P)^V$ are precisely the rays defined by the $w_j$ for $j \notin S$, each lying on a distinct extremal ray.

**Proof.** Suppose first that $P$ is simplicial. Then the minimal free resolution $i : P \to F$ is a close morphism (see p.2). For all $j \leq d$, let $p_j \in P$ such that $i(p_j) = \lambda_j e_j$ for some $\lambda_j \in \mathbb{N}$. Since $P$ is simplicial, we see that the $p_j$ form a basis for $P^{gp} \otimes \mathbb{Q}$, that $v_j(p_j) = \lambda_j > 0$, and that $v_k(p_j) = 0$ for $j \neq k \leq d$. Since $w_j$ is a multiple of $v_j$ for $j \leq d$, we see that Lemma 4.4 finishes the proof of the “if” direction.

Assume now that the action of the torus of $G$ on $V$ has MSOP, and let $S$ be as in Lemma 4.4. We prove the “only if” direction in the process of proving the second assertion of the theorem. We begin by showing that the $w_j$ for $j \notin S$ lie on extremal rays. Fix $j_0 \notin S$ and suppose that $w_{j_0}$ does not lie on an extremal ray. Then we have

$$w_{j_0} = \sum_{k=1}^{r} c'_k v_{i_k}$$

for positive rational numbers $c'_k$, distinct $i_k$, and $r \geq 2$. If $j \notin S$ and $j \neq j_0$, then $w_{j_0}(p_j) = 0$. Since the $p_j \in P$, we see the $v_{i_k}(p_j) \geq 0$, and so $v_{i_k}(p_j) = 0$. Since the $p_j$ for $j \notin S$ form a basis for $P^{gp} \otimes \mathbb{Q}$, we see that the $v_{i_k}$ are determined by their values on $p_{j_0}$. They therefore all define the same extremal ray, which is a contradiction.

To prove that the $w_j$ for $j \notin S$ lie on distinct extremal rays, assume that $w_k = cw_j$ for some positive rational number $c$. Then we see

$$0 = w_k(p_j) = cw_j(p_j) = ca_j > 0$$

which is a contradiction.

Lastly, we show that every extremal ray actually does occur as one of the $w_j$ for $j \notin S$. Note that this implies that $P$ is simplicial, for then

$$d = d' - |S| = \text{rank } P^{gp}.$$  

Since the $w_j$ for $j \notin S$ lie on distinct extremal rays, they define a basis for the $\mathbb{Q}$-vector space $\text{Hom}(P^{gp}, \mathbb{Z}) \otimes \mathbb{Q}$. Hence for $i \in S$, we have $c'_{ij} \in \mathbb{Q}$ such that

$$w_i = \sum_{j \notin S} c'_{ij} w_j.$$  

Since $0 \leq w_i(p_j) = c'_{ij} a_j$, we see that the $c'_{ij} \geq 0$. Therefore, if $w_i$ is an extremal ray, it must be one of the $w_j$ for $j \notin S$. This finishes the proof, as every extremal ray of $C(P)^V$ must occur as some $w_k$. 

\square
Proof of Theorem 4.2. The necessity of the MSOP condition is shown in Theorem 4.3 so we assume now that the action of the torus of $G$ on $V$ has MSOP. By Proposition 4.5 we can assume that the $\bar{e}_i \otimes 1$ for $i > d$ is an MSOP basis. Then $F'_0$ is the face of $F'$ generated by the $e_i$ for $i > d$. Note that $F'/F'_0$ is free of rank $d$ and that we have a commutative diagram

$$
\begin{array}{ccc}
F'/F'_0 & \to & F'/F'_0 \\
\pi' & \uparrow & \cong \\
F' & \to & \mathbb{N}^d \\
\psi' & \uparrow & \psi' \\
F & \to & \mathbb{N}^d
\end{array}
$$

where $\psi'(a_1, \ldots, a_n) = (a_1, \ldots, a_d)$. We see then that $\psi' \psi = \text{diag}(b_i)$ and is therefore exact. Since the multiplication by $n$ map from $F'/F'_0$ to itself is exact, $[Og]$ Prop I.4.1.3(2) shows that $\pi' i'$ is exact as well. Hence,

$$k[V]^G = k[P] = k[F'/F'_0]^{D((F'/F'_0)^{gp}/\pi' i'(P^{gp}))}.$$ 

Furthermore, since $P$ is simplicial by Theorem 4.5 and since $F'/F'_0$ is free of rank $d$, we see that $(F'/F'_0)^{gp}/\pi' i'(P^{gp})$ is a finite group. Note that by the definition of $B$, there exists a morphism $\varphi$ making the diagram

$$
\begin{array}{ccc}
F^{gp} & \to & F^{gp}/F'_0^{gp} \\
\pi' & \uparrow & \varphi \\
A & \to & A/B
\end{array}
$$

commute. One easily checks that the kernel of $\varphi$ is $\pi i'(P^{gp})$, and so $A/B$ is isomorphic to $(F'/F'_0)^{gp}/\pi' i'(P^{gp})$. This proves that $A/B$ is a finite group and that

$$k[V]^G = k[F'/F'_0]^{D(A/B)}$$

as desired. Lastly, since $D(A/B)$ is a finite diagonalizable group scheme, we see from Theorem 1.6 and Proposition 2.1 of [Sa] that $k[V]^G$ is polynomial if and only if the action of $D(A/B)$ on $k[F'/F'_0]$ is generated by pseudo-reflections.

5 Toric Artin Stacks

Throughout this section, let $k$ be a field and $S = \text{Spec } k$ have trivial log structure. In Subsection 5.1, we generalize the definition of stacky fan given in [BCS] and associate to a generalized stacky fan $\Sigma$ a smooth log smooth Artin stack $X(\Sigma)$ having $X(\Sigma)$ as a good moduli space. In Subsection 5.2, we show that if $N$ is torsion-free, then $X(\Sigma)$ has a natural moduli interpretation in terms of log geometry.

5.1 Generalized Stacky Fans

In keeping with the notation of [BCS], for this subsection only, we let $A^* = \text{Hom}(A, \mathbb{Z})$ for an abelian group $A$; recall that in all other parts of this paper, $P^*$ denotes the units of a monoid $P$. Given a finitely-generated abelian group $N$ and a rational fan $\Sigma \subset N \otimes \mathbb{Q}$, we let $\Sigma(1)$ be the set of rays of the fan. We denote by $d$ the rank of $N$ and $n$ the order of $\Sigma(1)$.
We introduce the following notion of a generalized stacky fan. We frequently drop the word “generalized” when referring to it.

**Definition 5.1.** A generalized stacky fan $\Sigma$ consists of a finitely-generated abelian group $N$, a rational fan $\Sigma \subset N \otimes \mathbb{Q}$, a choice of $r \in \mathbb{N}$, and a morphism $\beta : \mathbb{Z}\Sigma(1) \times \mathbb{Z}^r \rightarrow N$. We require that if $\rho_i \in \Sigma(1)$, then $\beta(\rho_i) \otimes 1$ lie on the the ray $\rho_i$, and we require that if $e_j \in \mathbb{Z}^r$, then $\beta(e_j) \otimes 1$ lie in some cone $\sigma_j$ of the fan. We often suppress $r$ and write $\Sigma = (N, \Sigma, \beta)$.

Throughout this subsection, we fix for every stacky fan $\Sigma = (N, \Sigma, \beta)$ an ordering on the rays of $\Sigma(1)$ so that $\beta$ is a map from $\mathbb{Z}^{n+r}$. In the next subsection, however, canonicity will be more important.

Note that in the above definition we do not require that the rays $\rho_i$ span $N \otimes \mathbb{Q}$ or that the $\beta(e_j)$ be distinct. Some of the $\beta(e_j)$ can even be zero, which as we will see, corresponds to the fact that the associated stack $X(\Sigma)$ contains a dense “Artin stacky torus”.

We remark that in [Ji], Jiang introduces a notion of extended stacky fans which is equivalent to our definition above, but the stacks he associates to them are all Deligne-Mumford. His goal in mind is not to construct toric Artin stacks, but rather to obtain presentations of toric Deligne-Mumford stacks other than the one given in [BCS].

We show now how to associate to a stacky fan $\Sigma = (N, \Sigma, \beta)$ an Artin stack $X(\Sigma)$, which we refer to as a toric Artin stack. We follow the procedure in [BCS]. As in [BCS, p. 195], we obtain an exact sequence

$$N^* \xrightarrow{\beta^*} (\mathbb{Z}^{n+r})^* \rightarrow H^1(\text{Cone}(\beta)^*) \rightarrow \text{Ext}^1_{\mathbb{Z}}(N, \mathbb{Z}) \rightarrow 0.$$  

Letting $DG(\beta) := H^1(\text{Cone}(\beta)^*)$, we define $\beta^\vee : (\mathbb{Z}^{n+r})^* \rightarrow DG(\beta)$ to be the connecting homomorphism above. More concretely, let

$$0 \rightarrow \mathbb{Z}^\ell \xrightarrow{Q} \mathbb{Z}^{d+\ell} \rightarrow N \rightarrow 0$$

be a projective resolution of $N$. If $B : \mathbb{Z}^{n+r} \rightarrow \mathbb{Z}^{d+\ell}$ is a lift of $\beta$, then

$$DG(\beta) = \text{coker}([BQ]^*)$$

and $\beta^\vee$ is the composite

$$(\mathbb{Z}^{n+r})^* \rightarrow (\mathbb{Z}^{d+r+\ell})^* \rightarrow DG(\beta).$$

The construction of $X(\Sigma)$ is then essentially the same as in [BCS, p.198]. Consider the ideal

$$J_\Sigma = \langle \prod_{\beta(e_i) \otimes 1 \notin \sigma} x_i \mid \sigma \in \Sigma \rangle$$

of $k[x_1, \ldots, x_{d+r}]$. Letting $G_\Sigma$ be the diagonalizable group scheme associated to $DG(\beta)$, then via $\beta^\vee$ we obtain a morphism $G_\Sigma \rightarrow G_m^{n+r}$ and hence an action of $G_\Sigma$ on $\mathbb{A}^{d+r} = \text{Spec} k[x_1, \ldots, x_{d+r}]$ via the action of $G_m^{n+r}$. Since $V(J_\Sigma)$ is a union of coordinate subspaces, we see that $Z_\Sigma := \mathbb{A}^{d+r} - V(J_\Sigma)$ is $G_\Sigma$-invariant. We obtain a log structure $M_{Z_\Sigma}$ on $Z_\Sigma$ by pulling back the canonical log structure on $\mathbb{A}^{d+r}$. Note that the $G_\Sigma$-action on $Z_\Sigma$ extends to the log scheme $(Z_\Sigma, M_{Z_\Sigma})$. We define

$$X(\Sigma) = [Z_\Sigma/G_\Sigma]$$

and obtain a log structure on $X(\Sigma)$ by descent theory. We see then that $X(\Sigma)$ is smooth and log smooth.
Example 5.2. If $\Sigma = (N,0,\beta : \mathbb{Z}^d \to 0)$ with $N$ torsion-free of rank $d$, then $X(\Sigma) = [\mathbb{A}^d/\mathbb{G}_m^d]$, which is a smooth toric Artin stack in the sense of Lafforgue [La, IV.1.a].

Remark 5.3. Given $N$ and a rational Artin fan $\Sigma$, we can define a canonical stacky fan $\Sigma^{can} = (N,\Sigma,\beta^{can} : \mathbb{Z}^n \to N)$ by letting $\beta^{can}(e_i)$ be the first lattice point on the $i^{th}$ ray. However, unlike in the theory of toric Deligne-Mumford stacks, given a stacky fan $\Sigma = (N,\Sigma,\beta)$, we do not necessarily have a morphism from $X(\Sigma)$ to $X(\Sigma^{can})$ as in [FMN]. It is sometimes necessary to take a root construction of $X(\Sigma)$ in order to get a map to $X(\Sigma^{can})$.

Note that given a stacky fan $\Sigma$, we can always write $N = N' \times N''$, where $N''$ is a free abelian group, the rays of $\Sigma$ span $N' \otimes \mathbb{Q}$, and the span of the rays of $\Sigma$ does not intersect $N''$. We then obtain another stacky fan $\Sigma' = (N',\Sigma,\beta)$ in the evident way, and the above construction shows that $X(\Sigma) = G^{d-n} \times X(\Sigma')$.

We work now toward showing that $X(\Sigma)$ is the good moduli space of $X(\Sigma)$. Given a stacky fan $\Sigma$ and $\sigma \in \Sigma$, along the lines of [Cox], we let

$$x^\sigma = \prod_{\beta(e_i) \otimes 1 \notin \sigma} x_i$$

and let $U_\sigma = \mathbb{A}^{d+r} - V(x^\sigma)$. Note that $U_\sigma$ is $G_\Sigma$-invariant and that $Z_\Sigma$ is the union of the $U_\sigma$.

Proposition 5.4. Let $\Sigma$ be a stacky fan and $\sigma \in \Sigma$. If $X_\sigma = \text{Spec} k[\sigma^\vee \cap M]$, then there is a natural map $[U_\sigma/G_\Sigma] \to X_\sigma$ which is a good moduli space.

Proof. We may assume $n = d$. Let $P_\sigma = \sigma^\vee \cap M$ and note that $U_\sigma = \text{Spec} k[P_\sigma]$, where

$$F_\sigma = \mathbb{N}[i\beta(e_i) \otimes 1 \notin \sigma] \times \mathbb{Z}^{[i\beta(e_i) \otimes 1 \notin \sigma]}.$$

Let $i_\sigma : P_\sigma \to F_\sigma$ be defined by $i_\sigma(p) = ((\beta(e_i) \otimes 1)(p))$. If $N$ is torsion-free, we can choose $\ell = 0$ and $B = \beta$ in the above construction of $X(\Sigma)$ so that the diagram

$$\begin{array}{ccc}
P_\sigma & \xrightarrow{i_\sigma} & F_\sigma^\text{opp} \\
\text{id} & \downarrow & \varphi \\
N^* & \xrightarrow{B^*} & (\mathbb{Z}^{n+r})^* \\
\end{array}$$

commutes; here $\varphi$ sends $e_i$ to the dual basis vector $e_i^\vee$. This shows that

$$k[P_\sigma] = k[F_\sigma]^{G_\Sigma}$$

and so the morphism $[U_\sigma/G_\Sigma] \to X_\sigma$ induced by $i_\sigma$ is a good moduli space.

We now handle the case when $N$ is not torsion-free. Let $N_{\text{tors}} \simeq \prod \mathbb{Z}/m_i\mathbb{Z}$, where the $m_i > 1$. Let $N' = N/N_{\text{tors}}$ and $\Sigma' = (N',\Sigma,\beta')$ where $\beta'$ is the composite of $\beta$ and the projection of $N$ to $N'$. We have then a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}^\ell & \xrightarrow{Q=0 \times \text{diag}(m_i)} & \mathbb{Z}^n + \mathbb{Z}^r \\
\downarrow & & \downarrow \text{id} \\
N' \times \mathbb{Z}^\ell & \xrightarrow{\pi} & N' \\
\downarrow \beta & & \downarrow \text{id} \\
\mathbb{Z}^{n+r} & \xrightarrow{\beta} & N & \xrightarrow{\pi} N'
\end{array}$$
where the columns are projective resolutions. Let $B' = \pi B$. We then obtain a commutative diagram

$$
\begin{array}{ccc}
(Z^n)^* & \xrightarrow{(B')^*} & (Z^{n+r})^* \xrightarrow{(\beta')^\vee} DG(\beta') \\
\downarrow & & \downarrow \eta \\
(N' \times Z^\ell)^* \xrightarrow{(BQ)^*} & (Z^{n+r+\ell})^* & \xrightarrow{DG(\beta)} \\
\end{array}
$$

The left and middle vertical arrows are injective. One easily checks that the left square is cartesian, and so $\eta$ is injective. This shows that the induced map

$$k[F_\sigma]^{G_{\Sigma'}} \to k[F_\sigma]^{G_{\Sigma}}$$

is an isomorphism, and hence, the composite

$$[U_\sigma/G_{\Sigma}] \to [U_\sigma/G_{\Sigma'}] \to X_\sigma$$

is a good moduli space as well.

\begin{proof}
We may assume $n = d$. For every $\sigma \in \Sigma$, let $P_\sigma, F_\sigma,$ and $i_\sigma$ be as in the proof of Proposition 5.4. We denote also by $i_\sigma$ the induced morphism $U_\sigma \to X_\sigma$. Note that if $\tau$ is face of $\sigma$, then the diagram

$$
\begin{array}{ccc}
U_\tau & \xrightarrow{i_\tau} & U_\sigma \\
\downarrow & & \downarrow i_\sigma \\
X_\tau & \xrightarrow{X_\tau} & X_\sigma \\
\end{array}
$$

commutes. We claim that it is, in fact, cartesian. To prove this, we show that if we have a commutative diagram of monoids

$$
\begin{array}{c}
Q \\
\downarrow \phi \\
\downarrow \psi \\
F_\tau \leftarrow F_\sigma \\
\downarrow i_\tau \leftarrow i_\sigma \\
P_\tau \leftarrow P_\sigma \\
\end{array}
$$

then there is a unique dotted arrow making the diagram commute. This is equivalent to showing that if $\beta(e_i) \otimes 1$ is in $\sigma$ but not in $\tau$, then $\phi(e_i)$ is a unit. By [Fu, \S 1.2 Prop 2], there is some $p \in P_\sigma$ such that $\tau = \sigma \cap p^\perp$ and $P_\tau = P_\sigma + \mathbb{N} \cdot (-p)$. Note then that $\psi(p)$ is a unit and that

$$\psi(p) = \phi i_\sigma(p) = \sum_i ((\beta(e_i) \otimes 1)(p)) \phi(e_i).$$

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\end{proof}

**Theorem 5.5.** If $\Sigma$ is a stacky fan, then there is a natural map $X(\Sigma) \to X(\Sigma)$ which is a good moduli space such that for all $\sigma \in \Sigma$,

$$
\begin{array}{ccc}
[U_\sigma/G_{\Sigma}] & \xrightarrow{} & X(\Sigma) \\
\downarrow & & \downarrow \ \\
X_\sigma & \xrightarrow{} & X(\Sigma) \\
\end{array}
$$

is cartesian; here $X_\sigma = \text{Spec } k[\sigma^\vee \cap M]$ and $[U_\sigma/G_{\Sigma}] \to X_\sigma$ is the morphism constructed in Proposition 5.4.
Let \( i \) be such that \( \beta(e_i) \oplus 1 \) is in \( \sigma \) but not in \( \tau \). Since it is in \( \sigma \), we see that \( (\beta(e_i) \oplus 1)(p) \geq 0 \). Since \( \beta(e_i) \oplus 1 \) is not in \( \tau \) and since \( \tau = \sigma \cap p^\perp \), we must have \( (\beta(e_i) \oplus 1)(p) > 0 \), and so \( \phi(e_i) \) is a unit, as desired.

We see then that
\[
\begin{array}{c}
[U_\tau/G\Sigma] \\
\downarrow \\
X_\tau
\end{array}
\begin{array}{c}
[U_\sigma/G\Sigma] \\
\downarrow \\
X_\sigma
\end{array}
\]

is cartesian. By Lemma 6.3 and Proposition 7.9 of [Al], it follows that there is a natural map \( X(\Sigma) \rightarrow X(\Sigma) \) which is a good moduli space whose base change to \( X_\sigma \) is as claimed.

\[\Box\]

5.2 Admissible qfrs and a Moduli Interpretation of \( X(\Sigma) \)

We begin this subsection by associating to a stacky fan \( \Sigma = (N, \Sigma, \beta) \) with \( N \) torsion-free, a smooth log smooth Artin stack \( X_\Sigma \) having \( X(\Sigma) \) as a good moduli space. The stack \( X_\Sigma \) is constructed as a moduli space along the same lines of the constructions in Theorems 3.2 and 4.3. We then show that \( X(\Sigma) \) is isomorphic to \( X_\Sigma \) as log stacks over \( X(\Sigma) \), thereby giving a moduli interpretation to \( X(\Sigma) \).

Note that if \( \Sigma = (N, \Sigma, \beta) \) is a stacky fan with \( N \) torsion-free, then giving the map \( \beta \) is equivalent to choosing a positive integer \( b_i \) for every \( \rho_i \in \Sigma(1) \) and choosing for every \( j \in \{1, 2, \ldots, r\} \) an element \( w_j \in N \) which lies in some cone \( \sigma_j \in \Sigma \). Given this equivalence, throughout this subsection, we denote stacky fans by \( \Sigma = (N, \Sigma; b_i; w_j) \).

Let us work toward defining the morphisms which \( X_\Sigma \) will parameterize.

**Definition 5.6.** If \( P \) is a saturated toric sharp monoid and \( i : P \rightarrow F \) its minimal free resolution, then we say a **datum** \( D \) for \( P \) is a choice of \( r \in \mathbb{N} \), a positive integer \( b_i \) for every irreducible element \( v_i \) of \( F \), and morphisms \( w_j : P \rightarrow \mathbb{N} \) for \( j \in \{1, 2, \ldots, r\} \). We frequently suppress \( r \) and write \( D = (b_i; w_j) \).

If \( \Sigma = (N, \Sigma; b_i; w_j) \) is a stacky fan with \( N \) torsion-free, then given a geometric point \( \bar{x} \) of \( X = X(\Sigma) \), let \( I(\bar{x}) \) be the set of irreducible components of \( X - X^{triv} \) on which \( \bar{x} \) lies and let
\[
F_{\bar{x}} = \alpha_{\bar{x}}^{-1}(\mathcal{O}_{X,\bar{x}}^*),
\]
where \( \alpha : \mathcal{M}_X \rightarrow \mathcal{O}_X \) is the structure morphism of the log structure \( \mathcal{M}_X \). If \( w_j(F_{\bar{x}}) = 0 \), then we obtain a morphism \( \bar{w}_j : \mathcal{M}_{X,\bar{x}} \rightarrow \mathbb{N} \). We can therefore define a datum for \( \mathcal{M}_{X,\bar{x}} \) by
\[
D_{\Sigma,\bar{x}} = (b_i \text{ s.t. } i \in I(\bar{x}); \bar{w}_j \text{ s.t. } w_j(F_{\bar{x}}) = 0);
\]
here we are using Proposition 3.5 to identify \( I(\bar{x}) \) with the set of irreducible elements of the minimal free resolution of \( \mathcal{M}_{X,\bar{x}} \).

**Definition 5.7.** If \( P \) is a saturated toric sharp monoid, \( i : P \rightarrow F \) its minimal free resolution, and \( D = (b_i; w_j) \) a datum for \( P \), then we say a morphism \( P \rightarrow F' \) is an **admissible free resolution of type** \( D \) if it is isomorphic to \( i' : P \rightarrow F \times \mathbb{N}^r \), where \( i' \) is such
that the diagram

\[
\begin{array}{c}
F \\
| \\
\downarrow \uparrow \\
P \\
| \\
\downarrow \uparrow \\
F \times \mathbb{N}^r \\
\end{array}
\]

commutes; the two arrows out of \(F \times \mathbb{N}^r\) are the natural projections. We say that \(P \to F'\) is \textit{admissibly qfr of type} \(D\) if it is isomorphic to

\[
P \to F' \to (F \times \mathbb{N}^r)/H,
\]

where \(i'\) is an admissible free resolution of type \(D\) and \(H \cap i'(P) = 0\).

**Definition 5.8.** If \(\Sigma\) is a stacky fan with \(N\) torsion-free and if \(X = X(\Sigma)\), then a morphism \(f : (Y, \mathcal{M}_Y) \to (X, \mathcal{M}_X)\) from a fine log scheme is \textit{admissibly qfr of type} \(\Sigma\) if for all geometric points \(\bar{y}\) of \(Y\), the induced morphism \(\mathcal{M}_{X,f(\bar{y})} \to \mathcal{M}_{Y,\bar{y}}\) is admissibly qfr of type \(D_{\Sigma,f(\bar{y})}\).

Note that if \(D\) is a datum for \(P\) in which \(r = 0\), then \(P \to F'\) is an admissible free resolution of type \(D\), resp. admissibly qfr of type \(D\) in the sense of Definition 5.7 if and only if it is in the sense of Definition 3.4. Similarly, if \(\Sigma\) is a stacky fan for which \(r = 0\) (i.e. a stacky fan in the sense of [BCS]) and for which \(N\) is torsion-free, then a morphism \(f : (Y, \mathcal{M}_Y) \to (X, \mathcal{M}_X)\) from a fine log scheme is admissibly qfr of type \(\Sigma\) if and only if it is admissibly qfr of type \((b)\) in the sense of Definition 3.6.

Given a stacky fan \(\Sigma\), let \(X = X(\Sigma)\). We define \(\mathfrak{X}_\Sigma\) as the fibered category over \(X\)-schemes whose objects are morphisms \((T, N') \to (X, \mathcal{M}_X)\) which are admissibly qfr of type \(\Sigma\), and whose morphisms are maps of \((X, \mathcal{M}_X)\)-log schemes \(h : (T, N') \to (T', N'')\) with \(h^*N' \to N\) an isomorphism. As before, this fibered category is a stack on the fppf site by [Ol1, Thm A.1].

The proof that these stacks are algebraic and have the properties mentioned earlier is similar to the proofs of Theorems 3.2 and 3.9, so we indicate only where changes are necessary.

**Proposition 5.9.** Let \(\Sigma\) be a stacky fan such that \(X := X(\Sigma) = \text{Spec} \, k[P]\) for some saturated toric sharp monoid \(P\). Let \(D = D_{\Sigma,0}\), where \(0 \in X\) is the point such that \(\tilde{\mathcal{M}}_{X,0} = P\). If \(i' : P \to F'\) is an admissible free resolution of type \(D\), then the induced morphism \(f : X \to \text{Spec} \, k[F']\) on log schemes is admissibly qfr of type \(\Sigma\).

**Proof.** Choosing an appropriate isomorphism, we can assume that \(F' = F \times \mathbb{N}^r\) and that \(i'\) is as in Definition 5.7. Let \(H'' = H \times H'\) be a face of \(F'\) and let \(P_0 = i'(P) \cap H''\). Let \(\bar{i}'\) be the resulting morphism which makes the diagram

\[
\begin{array}{c}
P \\
| \\
\downarrow \uparrow \\
P/P_0 \\
| \\
\downarrow \uparrow \\
F/H \times \mathbb{N}^r/H'
\end{array}
\]

commute. We must show that \(\bar{i}'\) is admissibly qfr of the appropriate type. Note first that if \(H'_0\) denotes the face of \(\mathbb{N}^r\) generated by the \(e_j\) with \(w_j(P_0) \neq 0\), then commutativity of...
the above diagram shows that $H' \supset H'_0$. Similarly, we see that $H \supset F_0$, where $F_0$ denotes the face of $F$ generated by $i(P_0)$. As a result, we have a commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{i'} & F' \\
\downarrow & & \downarrow \pi \\
P/P_0 & \xrightarrow{i''} & F/F_0 \times \mathbb{N}'/H'_0 \\
\end{array}
$$

where the bottom row composes to $i'$, and $\pi$ and $\pi'$ are the natural projections. Recall that by Proposition 2.6, the natural morphism from $P/P_0$ to $F/F_0$ is a minimal free resolution. Note now that $i''$ is given by $i''(\bar{p}) = (bi; vi(p); wj(p))$ for $i$ such that $vi(P_0) = 0$ and $j$ such that $wj(P_0) = 0$; that is, $i''$ is an admissible free resolution of the correct type.

To complete the proof, we must therefore show $(H/F_0 \times H'/H'_0) \cap i''(P/P_0) = 0$. This amounts to showing that if $bi; vi(p)vi \in H$ for all $i$ such that $vi(P_0) = 0$ and if $wj(p)e_j \in H'$ for all $wj(P_0) = 0$, then $bi; vi(p)vi \in H$ for all $i$ and $wj(p)e_j \in H'$ for all $j$. If $i$ is such that $vi(P_0) \neq 0$, then $vi \in F_0 \subset H$ and so $bi; vi(p)vi \in H$. Similarly, if $j$ is such that $wj(P_0) \neq 0$, then $e_j \in H'_0 \subset H'$ and so $wj(p)e_j \in H'$. This shows that the above intersection is trivial. □

The proof of Proposition 3.11 then yields:

**Proposition 5.10.** Let $\Sigma$ be a stacky fan such that $X := X(\Sigma) = \text{Spec} \, k[P]$ for some saturated toric sharp monoid $P$. Let $D = D_{\Sigma, 0}$, where $0 \in X$ is the point such that $\mathcal{M}_{X, 0} = P$. If $i' : P \to F'$ is an admissible free resolution of type $D$ and $G = D(F^{gp}/i'(P^{gp}))$, then $X_\Sigma$ is isomorphic to $[\text{Spec} \, k[F']/G]$ over $X$.

The first main theorem of this subsection is then:

**Theorem 5.11.** If $\Sigma = (N, \Sigma; b_i; w_j)$ is a stacky fan and $X = X(\Sigma)$, then $X_\Sigma$ is a smooth log smooth Artin stack over $\text{Spec} \, k$ having $X$ as a good moduli space. Moreover, we have a cartesian diagram

$$
\begin{array}{ccc}
T \times [\mathbb{A}^z/G_m^z] & \xrightarrow{\pi} & X_\Sigma \\
\downarrow & & \downarrow \\
T & \xrightarrow{\pi} & X
\end{array}
$$

where $T$ is the dense open torus of $X$, $\pi$ is the natural projection, and $z$ is the number of $w_j$ which are zero.

**Proof.** Zariski locally, $X$ is of the form $Y \times T'$, where $T'$ is a torus and $Y = \text{Spec} \, k[P]$ for some toric sharp monoid $P$. The proof of Theorem 3.1 then shows that $X_\Sigma$ is a smooth log smooth Artin stack and that the natural map $X_\Sigma \to X$ is a good moduli space. Now note that we have a cartesian diagram

$$
\begin{array}{ccc}
X_\Sigma' & \xrightarrow{\pi} & X_\Sigma \\
\downarrow & & \downarrow \\
T & \xrightarrow{\pi} & X \\
\end{array}
$$

where $\Sigma' = (N, 0; w_j \text{ s.t. } w_j = 0)$. Then by the first part of this proof, we see that $X_\Sigma' = T \times X_\Sigma''$, where $\Sigma'' = (0, 0; w_j \text{ s.t. } w_j = 0)$. Proposition 5.10 then shows that $X_\Sigma'' \simeq [\mathbb{A}^z/G_m^z]$, thereby completing the proof. □
We work now toward comparing the stacks $X_\Sigma$ and $X(\Sigma)$. If $\sigma \in \Sigma$, then let $P_\sigma$, $F_\sigma$, $i_\sigma$, $U_\sigma$, and $X_\sigma$ be as in the proof of Proposition 5.4.

**Proposition 5.12.** Let $\Sigma = (N, \Sigma; b_1; w_j)$ be a stacky fan in which $N$ is torsion-free. If $\sigma \in \Sigma$ is a maximal cone, then

$$[U_\sigma/G_\Sigma] \simeq X_\Sigma \times X(\Sigma) X_\sigma$$

as log stacks over $X_\sigma$.

**Proof.** We may assume that $n = d$, so that $P_\sigma$ is sharp. Let $D_\sigma = D_{\Sigma, \emptyset}$, where $0 \in X_\sigma$ is a point such that $N_{X_\sigma, 0} = P_\sigma$. Let $i'_\sigma : P_\sigma \to F'_\sigma$ be an admissible free resolution of type $D_\sigma$ as in Definition 5.7. Let $A = F^\gp_{\sigma}/i_\sigma(P^\gp_{\sigma})$, $A' = F'^\gp_{\sigma}/i'_\sigma(P'^\gp_{\sigma})$, and $G_\sigma = D(A')$. From the proof of Theorem 5.11, we have a cartesian diagram

$$\begin{array}{ccc}
\text{Spec } k[F'_\sigma/G_\sigma] & \longrightarrow & X_\Sigma \\
\downarrow & & \downarrow \\
X_\sigma & \longrightarrow & X(\Sigma)
\end{array}$$

where $\epsilon$ is induced from $i'_\sigma$. Let $I$ be the set of rays in $\sigma$ union the set of $w_j \in \sigma$. Let $J$ be the set of rays not in $\sigma$ union the set of $w_j \notin \sigma$. Then, we see that $F'_\sigma$ is a direct sum of copies of $N$ indexed by $I$. We have a commutative diagram

$$\begin{array}{ccc}
F_\sigma & \longrightarrow & F'_\sigma \\
\downarrow_{i_\sigma} & & \downarrow_{i'_\sigma} \\
P_\sigma & \longrightarrow & P'_\sigma
\end{array}$$

where $\pi$ is the natural projection.

To prove the proposition, we show that $U_\sigma$ and the pushout $G_\Sigma \times G_\sigma$ Spec $k[F'_\sigma]$ are isomorphic as schemes with $G_\Sigma$-action. By definition,

$$G_\Sigma \times G_\sigma \text{ Spec } k[F'_\sigma] = \text{ Spec } k[A \times N^I]^{A'}.$$

Since $i_\sigma$ and $i'_\sigma$ are injective, we see that the induced morphism

$$\mathbb{Z} = \ker(F^\gp_\sigma \to F'^\gp_\sigma) \to \ker(A \to A')$$

is an isomorphism. It follows that

$$k[A \times N^I]^{A'} \simeq k[\mathbb{Z}] \otimes_k k[A' \times N^I]^{A'} \simeq k[\mathbb{Z}] \otimes_k k[Q],$$

where $Q \subset A' \times N^I$ is the submonoid of elements of the form $(-q, q)$. Since the projection of $A' \times N^I$ to $N^I$ induces an isomorphism of $Q$ with $N^I$, we have an induced isomorphism

$$k[A \times N^I]^{A'} \simeq k[\mathbb{Z}] \otimes_k k[N^I].$$

Via the isomorphism $\varphi : \mathbb{Z} \times N^I \to F_\sigma$ sending $(a, b)$ to $(-a, b)$, and the automorphism of $G_\Sigma$ sending $g$ to its inverse, we obtain an isomorphism of $G_\Sigma \times G_\sigma$ Spec $k[F'_\sigma]$ and $U_\sigma$ respecting the $G_\Sigma$-actions. We see that this isomorphism respects the log structures as well. $\square$
Theorem 5.13. If \((\Sigma, N, \Sigma; b_i; w_j)\) is a stacky fan in which \(N\) is torsion-free, then \(X(\Sigma)\) and \(X_\Sigma\) are isomorphic as log stacks over \(X(\Sigma)\).

Proof. We show first that the composite \(Z_\Sigma \to X(\Sigma) \to X(\Sigma)\) is admissibly qfr of type \(\Sigma\). Note that this can be checked Zariski locally on \(X(\Sigma)\). By Theorem 5.5, we have a cartesian diagram

\[
\begin{array}{ccc}
U_\sigma & \rightarrow & Z \\
\downarrow^{g_\sigma} & & \downarrow^{h} \\
[U_\sigma/G_\Sigma] & \rightarrow & X(\Sigma) \\
\downarrow & & \downarrow \phi \\
X_\sigma & \rightarrow & X(\Sigma)
\end{array}
\]

for any cone \(\sigma \in \Sigma\). By Proposition 5.12 we see that if \(\sigma\) is maximal, then the morphism \(i_\sigma\) is admissibly qfr of type \(\Sigma\). Since the \(X_\sigma\) for \(\sigma\) maximal form a Zariski cover of \(X(\Sigma)\), we see that \(h\) is admissibly qfr of type \(\Sigma\).

We therefore have a strict morphism \(f : Z \to X_\Sigma\) over \(X(\Sigma)\). We have a \(G_\Sigma\)-action on \(Z\) over \(X(\Sigma)\). Since this action respects the log structure of \(Z\), we see that \(G_\Sigma\) acts on \(Z\) over \(X_\Sigma\).

We claim that \(f\) is a \(G_\Sigma\)-torsor. This can be checked Zariski locally on \(X(\Sigma)\). Since the above diagram is cartesian, we obtain a morphism

\[g'_\sigma : U_\sigma \to X_\Sigma \times_{X(\Sigma)} X_\sigma\]

over \(X_\sigma\). By the construction of \(f\), we see that for \(\sigma\) a maximal cone, \(\varphi_\sigma g'_\sigma = g_\sigma\), where \(\varphi_\sigma\) is the isomorphism from Proposition 5.12. It follows that \(g'_\sigma\) is a \(G_\Sigma\)-torsor, and since the \(X_\sigma\) for \(\sigma\) maximal form a Zariski cover of \(X(\Sigma)\), we see that \(f\) is as well. Hence,

\[X_\Sigma \simeq [Z_\Sigma/G_\Sigma] = X(\Sigma)\]

This is, moreover, an isomorphism of log stacks as the morphisms from \(Z_\Sigma\) to \(X_\Sigma\) and to \(X(\Sigma)\) are strict.

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