HERMITE-HADAMARD TYPE INEQUALITIES FOR GA-s-CONVEX FUNCTIONS

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Abstract. In this paper, the author introduces the concepts of the GA-s-convex functions in the first sense and second sense and establishes some integral inequalities of Hermite-Hadamard type related to the GA-s-convex functions.

1. Introduction

In this section, we firstly list several definitions and some known results. The following concept was introduced by Orlicz in [11]:

**Definition 1.** Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ where $\mathbb{R}_+ = [0, \infty)$, is said to be $s$-convex in the first sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in I$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. We denote this class of real functions by $K_{1s}$. In [4], Hudzik and Maligranda considered the following class of functions:

**Definition 2.** A function $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ where $\mathbb{R}_+ = [0, \infty)$, is said to be $s$-convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in I$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $s$ fixed in $(0, 1]$. They denoted this by $K_{2s}$.

It can be easily seen that for $s = 1$, $s$-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In [2], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the $s$-convex functions.

**Theorem 1.** Suppose that $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an $s$-convex function in the second sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L[a, b]$, then the following inequalities hold

$$2^{s-1} \frac{a+b}{2} \leq \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1},$$

the constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.1).

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and results:

Inequalities for special means and they used the following lemma to prove their properties for GA-convex functions and applied these inequalities to construct several be the geometric, logarithmic, identric, arithmetic and p-logarithmic means of \(x, y\).

Let \(x, y \in I\) and \(t \in [0, 1]\), where \(x^ty^{1-t}\) and \((1-t)f(x) + tf(y)\) are respectively called the weighted geometric mean of two positive numbers \(x\) and \(y\) and the weighted arithmetic mean of \(f(x)\) and \(f(y)\).

For \(b > a > 0\), let \(G(a, b) = \sqrt{ab}\), \(L(a, b) = (b - a) / (\ln b - \ln a)\), \(I(a, b) = (1/e) \left( b/a^q \right)^{1/(b-a)}\), \(A(a, b) = a^{1/b} b^{1/a}\), and \(L_p(a, b) = \left( \frac{(a^{p+1} - b^{p+1})}{(p+1)(b-a)} \right)^{1/p}\), \(p \in \mathbb{R} \setminus \{ -1, 0 \}\), be the geometric, logarithmic, identric, arithmetic and \(p\)-logarithmic means of \(a\) and \(b\), respectively. Then

\[
\min \{ a, b \} < G(a, b) < L(a, b) < I(a, b) < A(a, b) < \max \{ a, b \}.
\]

In [13], Zhang et al. established some Hermite-Hadamard type integral inequalities for GA-convex functions and applied these inequalities to construct several inequalities for special means and they used the following lemma to prove their results:

**Lemma 1.** Let \(f : I \subseteq \mathbb{R}_+ \to \mathbb{R}\) be a differentiable mapping on \(I^o\), and \(a, b \in I^o\), with \(a < b\). If \(f' \in L[a, b]\), then

\[
bf(b) - af(a) - \int_a^b f(x)dx = (\ln b - \ln a) \int_0^1 b^{2t}a^{2(1-t)} f'(b^{t}a^{1-t}) dt.
\]

Also, the main inequalities in [13] are pointed out as follows:

**Theorem 2.** Let \(f : I \subseteq \mathbb{R}_+ \to \mathbb{R}\) be differentiable on \(I^o\), and \(a, b \in I\) with \(a < b\) and \(f' \in L[a, b]\). If \(|f'|^q\) is GA-convex on \([a, b]\) for \(q \geq 1\), then

\[
\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq \frac{([b - a] A(a, b))^{1 - 1/q}}{2^{1/q}} \times \left\{ [L(a^2, b^2) - a^2] |f'(a)|^q + [b^2 - L(a^2, b^2)] |f'(b)|^q \right\}^{1/q}.
\]

**Theorem 3.** Let \(f : I \subseteq \mathbb{R}_+ \to \mathbb{R}\) be differentiable on \(I^o\), and \(a, b \in I\) with \(a < b\) and \(f' \in L[a, b]\). If \(|f'|^q\) is GA-convex on \([a, b]\) for \(q > 1\), then

\[
\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq (\ln b - \ln a) \times \left[ L(a^{2q/(q-1)}, b^{2q/(q-1)}) - a^{2q/(q-1)} \right]^{1 - 1/q} \times \left[ A(|f'(a)|^q, |f'(b)|^q) \right]^{1/q}.
\]

**Theorem 4.** Let \(f : I \subseteq \mathbb{R}_+ \to \mathbb{R}\) be differentiable on \(I^o\), and \(a, b \in I\) with \(a < b\) and \(f' \in L[a, b]\). If \(|f'|^q\) is GA-convex on \([a, b]\) for \(q > 1\) and \(2q > p > 0\), then

\[
\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq \frac{((\ln b - \ln a))^{1 - 1/q}}{p^{1/q}}.
\]
Lemma 2. Let Hadamard-like type inequalities for the geometrically convex functions.

Definition 4. Let $H$ be the set of GA-convex (concave) functions.

Theorem 5. For GA-convex (concave) functions:

In [13], Zhang et al. established the following Hermite-Hadamard type inequality for GA-convex (concave) functions:

**Theorem 5.** If $b > a > 0$ and $f : [a, b] \to \mathbb{R}$ is a differentiable GA-convex (concave) function then

$$f(I(a, b)) \leq \left( \frac{1}{b-a} \int_a^b f(x) dx \right) \leq \left( \frac{b - L(a, b)}{b-a} f(b) + \frac{L(a, b) - a}{b-a} f(a) \right).$$

In [14], the author proved the following identity and established some new Hermite-Hadamard-like type inequalities for the geometrically convex functions.

**Lemma 2.** Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable mapping on $I$, and $a, b \in I$, with $a < b$. If $f' \in L[a, b]$, then

$$f^{(2)} \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx = \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx$$

In this paper, we will give concepts $s$-GA-convex functions in the first and second sense and establish some new integral inequalities of Hermite-Hadamard-like type for these classes of functions by using Lemma 2.

2. Definitions of GA-$s$-convex functions in the first and second sense

Now it is time to introduce two concepts, GA-$s$-convex functions in the first and second sense.

**Definition 4.** Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ is said to be a GA-$s$-convex (concave) function in the first sense on $I$ if

$$f(x^t y^{1-t}) \leq (\geq) \, t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

**Definition 5 (Definition 2.1).** Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ is said to be a GA-$s$-convex (concave) function in the second sense on $I$ if

$$f(x^t y^{1-t}) \leq (\geq) \, t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. 
It is clear that when \( s = 1 \), GA-\( s \)-convex functions in the first and second sense become GA-convex functions.

3. INEQUALITIES FOR GA-\( s \)-CONVEX FUNCTIONS IN THE FIRST AND SECOND SENSE

Now we are in a position to establish some inequalities of Hermite–Hadamard type for GA-\( s \)-convex functions in the first and second sense

**Theorem 6.** Let \( 0 < s \leq 1 \). Suppose that \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) is GA-\( s \)-convex function in the first sense and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \), then one has the inequalities:

\[
(3.1) \quad f \left( \sqrt{ab} \right) \leq \frac{1}{\ln b - \ln a} \int_{a}^{b} \frac{f(x)}{x} \, dx \leq \frac{f(a) + sf(b)}{s+1}
\]

**Proof.** As \( f \) is GA-\( s \)-convex function in the first sense, we have, for all \( x, y \in I \)

\[
(3.2) \quad f \left( \sqrt{xy} \right) \leq \frac{1}{2^s} f(x) + \left( 1 - \frac{1}{2^s} \right) f(y).
\]

Now, let \( x = a^{1-t}b^t \) and \( y = a^tb^{1-t} \) with \( t \in [0, 1] \). Then we get by (3.2) that:

\[
f \left( \sqrt{ab} \right) \leq \frac{1}{2^s} f(a^{1-t}b^t) + \left( 1 - \frac{1}{2^s} \right) f(a^tb^{1-t})
\]

for all \( t \in [0, 1] \). Integrating this inequality on \([0, 1] \), we deduce the first part of (3.1).

Secondly, we observe that for all \( t \in [0, 1] \)

\[f(a^tb^{1-t}) \leq t^sf(a) + (1-t^s)f(b).\]

Integrating this inequality on \([0, 1] \), we get

\[
\int_{0}^{1} f(a^tb^{1-t}) \, dt \leq \frac{f(a) + sf(b)}{s+1}.
\]

As the change of variable \( x = a^tb^{1-t} \) gives us that

\[
\int_{0}^{1} f(a^tb^{1-t}) \, dt = \frac{1}{\ln b - \ln a} \int_{a}^{b} \frac{f(x)}{x} \, dx,
\]

the second inequality in (3.1) is proved. \( \square \)

Similarly to Theorem 6, we will give the following theorem for GA-\( s \)-convex function in the second sense:

**Theorem 7.** Suppose that \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) is GA-\( s \)-convex function in the second sense and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \), then one has the inequalities:

\[
(3.3) \quad 2^{s-1} f \left( \sqrt{ab} \right) \leq \frac{1}{\ln b - \ln a} \int_{a}^{b} \frac{f(x)}{x} \, dx \leq \frac{f(a) + f(b)}{s+1}
\]
Proof. As $f$ is GA-$s$-convex function in the second sense, we have, for all $x, y \in I$

\[(3.4) \quad f(\sqrt{xy}) \leq \frac{f(x) + f(y)}{2^s}.\]

Now, let $x = a^{1-t}b^t$ and $y = a^t b^{1-t}$ with $t \in [0, 1]$. Then we get by (3.4) that:

\[f(\sqrt{ab}) \leq \frac{f(a^{1-t}b^t) + f(a^t b^{1-t})}{2^s}\]

for all $t \in [0, 1]$. Integrating this inequality on $[0, 1]$, we deduce the first part of (3.3).

Secondly, we observe that for all $t \in [0, 1]$

\[f(a^t b^{1-t}) \leq t^s f(a) + (1-t)^s f(b).\]

Integrating this inequality on $[0, 1]$, we get

\[\int_0^1 f(a^t b^{1-t}) dt \leq \frac{f(a) + f(b)}{s + 1}.\]

As the change of variable $x = a^t b^{1-t}$ gives us that

\[\int_0^1 f(a^t b^{1-t}) dt = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx,\]

the second inequality in (3.3) is proved. \qed

Remark 1. The constant $k = 1/(s+1)$ for $s \in (0, 1]$ is the best possible in the second inequality in (3.3). Indeed, as the mapping $f : [a, b] \to [a, b]$ given $f(x) = s + 1$, $0 < a < b$, is GA-$s$-convex in the second sense and

\[\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx = s + 1 = \frac{f(a) + f(b)}{s + 1}\]

Theorem 8. Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be differentiable on $I^\circ$, and $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$.

a) If $|f'|^q$ is GA-$s$-convex function in the second sense on $[a, b]$ for $q \geq 1$ and $s \in (0, 1]$, then

\[(3.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right|\]

\[\leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{2 - \frac{1}{q}} \left[ a \{c_1(s, q) |f'(a)|^q + c_2(s, q) |f'(b)|^q \}^{\frac{1}{q}} + b \{c_3(s, q) |f'(b)|^q + c_4(s, q) |f'(a)|^q \}^{\frac{1}{q}} \right]\]

\[(3.6) \quad \left| f(\sqrt{ab}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right|\]
\[ \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{3 - \frac{1}{q}} \left[ a \left\{ c_1(s, q/2) |f'(a)|^q + c_2(s, q/2) \left| f' \left( \sqrt[3]{ab} \right) \right|^q \right\} \right] ^{\frac{1}{q}} + b \left\{ c_3(s, q/2) |f'(b)|^q + c_4(s, q/2) \left| f' \left( \sqrt[3]{ab} \right) \right|^q \right\} \]

where

\[ c_1(s, q) = \int_0^1 t (1 - t)^s \left( \frac{b}{a} \right)^{qt} dt, \quad c_2(s, q) = \int_0^1 t^{s+1} \left( \frac{b}{a} \right)^{qt} dt, \]

\[ c_3(s, q) = \int_0^1 t (1 - t)^s \left( \frac{a}{b} \right)^{qt} dt, \quad c_4(s, q) = \int_0^1 t^{s+1} \left( \frac{a}{b} \right)^{qt} dt, \]

b) If \(|f'|^q\) is GA-s-convex function in the first sense on \([a, b]\) for \(q \geq 1\) and \(s \in (0, 1]\), then

\[ \left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{2 - \frac{1}{q}} \left[ a \left\{ c_5(s, q) |f'(a)|^q + c_2(s, q) \left| f' \left( \sqrt[3]{ab} \right) \right|^q \right\} \right] ^{\frac{1}{q}} + b \left\{ c_6(s, q) |f'(b)|^q + c_4(s, q) \left| f' \left( \sqrt[3]{ab} \right) \right|^q \right\} \]

\[ \left| \frac{f \left( \sqrt[3]{ab} \right)}{x} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{3 - \frac{1}{q}} \left[ a \left\{ c_5(s, q/2) |f'(a)|^q + c_2(s, q/2) \left| f' \left( \sqrt[3]{ab} \right) \right|^q \right\} \right] ^{\frac{1}{q}} + b \left\{ c_6(s, q/2) |f'(b)|^q + c_4(s, q/2) \left| f' \left( \sqrt[3]{ab} \right) \right|^q \right\} \]

where

\[ c_5(s, q) = \int_0^1 t (1 - t)^s \left( \frac{b}{a} \right)^{qt} dt, \quad c_6(s, q) = \int_0^1 t (1 - t)^s \left( \frac{a}{b} \right)^{qt} dt. \]

Proof. a) (1) Since \(|f'|^q\) is GA-s-convex function in the second sense on \([a, b]\), from lemma and power mean inequality, we have

\[ \left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \leq \ln \left( \frac{b}{a} \right) 2 \left[ a \int_0^1 \left( \frac{b}{a} \right)^t |f'(a^{1-t}b^t)| dt + b \int_0^1 \left( \frac{a}{b} \right)^t |f'(b^{1-t}a^t)| dt \right] \]
\[ \begin{align*}
\leq & \frac{a \ln \left( \frac{b}{a} \right)}{2} \left( \int_0^1 \frac{1}{t} \left( \int_0^1 \left( \int_0^t \frac{1}{a} q^t f' \left( a^{1-t} b^t \right) \right) \right) \right)^{\frac{1}{q}} \\
& + \frac{b \ln \left( \frac{b}{a} \right)}{2} \left( \int_0^1 \frac{1}{t} \left( \int_0^1 \left( \int_0^t \frac{1}{a} q^t f' \left( b^{1-t} a^t \right) \right) \right) \right)^{\frac{1}{q}} \\
\end{align*} \]

(3.11)

\[ \begin{align*}
& \leq \frac{a \ln \left( \frac{b}{a} \right)}{2} \left( \int_0^1 \frac{1}{t} \left( \int_0^1 \left( \int_0^t \frac{1}{a} q^t \left( (1-t)^s f' \left( a \right) \right) + t^s f' \left( b \right) \right) \right) \right)^{\frac{1}{q}} \\
& + \frac{b \ln \left( \frac{b}{a} \right)}{2} \left( \int_0^1 \frac{1}{t} \left( \int_0^1 \left( \int_0^t \frac{1}{a} q^t \left( (1-t)^s f' \left( b \right) \right) + t^s f' \left( a \right) \right) \right) \right)^{\frac{1}{q}} \\
& \leq a \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{2-\frac{1}{q}} \left\{ c_1(s, q) \left| f' \left( a \right) \right|^q + c_2(s, q) \left| f' \left( b \right) \right|^q \right\}^{\frac{1}{q}} \\
& + b \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{2-\frac{1}{q}} \left\{ c_3(s, q) \left| f' \left( b \right) \right|^q + c_4(s, q) \left| f' \left( a \right) \right|^q \right\}^{\frac{1}{q}}.
\end{align*} \]

(2) Since \( |f'|^q \) is GA-s-convex function in the second sense on \([a, b]\), from lemma and power mean inequality, we have

\[ \left| f \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b f(x) \frac{dx}{x} \right| \leq \frac{\ln \left( \frac{b}{a} \right)}{4} \left[ a \int_0^1 \left( \frac{b}{a} \right)^{\frac{1}{q}} \left| f' \left( a^{1-t} (ab)^\frac{1}{q} \right) \right| dt + b \int_0^1 \left( \frac{a}{b} \right)^{\frac{1}{q}} \left| f' \left( b^{1-t} (ab)^\frac{1}{q} \right) \right| dt \right] \]

\[ \leq \frac{a \ln \left( \frac{b}{a} \right)}{4} \left( \int_0^1 \frac{1}{t} \left( \int_0^1 \left( \int_0^t \frac{1}{a} q^t \left( (1-t)^s f' \left( a \right) \right) + t^s f' \left( b \right) \right) \right) \right)^{\frac{1}{q}} \\
& + \frac{b \ln \left( \frac{b}{a} \right)}{4} \left( \int_0^1 \frac{1}{t} \left( \int_0^1 \left( \int_0^t \frac{1}{a} q^t \left( (1-t)^s f' \left( b \right) \right) + t^s f' \left( a \right) \right) \right) \right)^{\frac{1}{q}} \\
\end{align*} \]

(3.12)
inequality (3.11), we have

\[ \dot{\text{IMDAT}} \dot{\text{IS}} \dot{\text{CAN}} \]

\[ \left\lfloor \left\lfloor b \lfloor \left\lfloor c_5(s, q/2) |f'(b)|^q + c_6(s, q/2) \right\rfloor \right\rfloor \right\rfloor + 1 \]

b) (1) Since \( |f'|^q \) is GA-s-convex function in the first sense on \([a, b]\), from the inequality (3.11), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|
\]

\[
\leq a \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{1 - \frac{1}{q}} \left( \int_0^1 \left( \frac{t}{a} \right)^{qt} \left( (1 - t^q) |f'(a)|^q + t^q |f'(b)|^q \right) \, dt \right)^{\frac{1}{q}}
\]

\[
+ b \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{1 - \frac{1}{q}} \left( \int_0^1 \left( \frac{a}{b} \right)^{qt} \left( (1 - t^q) |f'(b)|^q + t^q |f'(a)|^q \right) \, dt \right)^{\frac{1}{q}}
\]

\[
\leq \left[ a \left\lfloor \left\lfloor c_5(s, q) |f'(a)|^q + c_2(s, q) |f'(b)|^q \right\rfloor \right\rfloor \right]^{\frac{1}{q}} + b \left[ c_6(s, q) |f'(b)|^q + c_4(s, q) |f'(a)|^q \right]^{\frac{1}{q}}.
\]

(2) Since \( |f'|^q \) is GA-s-convex function in the first sense on \([a, b]\), the inequality 3.9 is easily obtained by using the inequality (3.12).

If taking \( s = 1 \) in Theorem 8, we can derive the following inequalities for GA-convex.

**Corollary 1.** Let \( f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R} \) be differentiable on \( I^o \), and \( a, b \in I^o \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'|^q \) is GA-convex on \([a, b]\) for \( q \geq 1 \), then

\[
\left( 3.13 \right) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|
\]

\[
\leq \left[ a \left\lfloor \left\lfloor c_5(1, q) |f'(a)|^q + c_2(1, q) |f'(b)|^q \right\rfloor \right\rfloor \right]^{\frac{1}{q}} + b \left[ c_6(1, q) |f'(b)|^q + c_4(1, q) |f'(a)|^q \right]^{\frac{1}{q}}.
\]

\[
\left( 3.14 \right) \quad \left| f \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|
\]

\[
\leq \left[ a \left\lfloor \left\lfloor c_5(1, q/2) |f'(a)|^q + c_2(1, q/2) |f'(b)|^q \right\rfloor \right\rfloor \right]^{\frac{1}{q}} + b \left[ c_6(1, q/2) |f'(b)|^q + c_4(1, q/2) |f'(a)|^q \right]^{\frac{1}{q}}.
\]
If taking \( q = 1 \) in Theorem 9 we can derive the following corollary.

**Corollary 2.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be differentiable on \( I^o \), and \( a, b \in I^o \) with \( a < b \) and \( f' \in L[a, b] \).

a) If \(|f'| \) is GA-s-convex function in the second sense on \([a, b], s \in (0, 1]\), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|
\]

\[
\leq \frac{\ln \left( \frac{b}{a} \right)}{2} \left| (ac_1(s, 1) + bc_4(s, 1)) |f'(a)| + (bc_3(s, 1) + ac_2(s, 1)) |f'(b)| \right|
\]

b) If \(|f'|^q \) is GA-s-convex function in the first sense on \([a, b], s \in (0, 1]\), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|
\]

\[
\leq \frac{\ln \left( \frac{b}{a} \right)}{4} \left| (ac_5(s, 1/2) + bc_4(s, 1)) |f'(a)| + (ac_2(s, 1) + bc_6(s, 1)) \left| f' \left( \sqrt{ab} \right) \right| \right|
\]

**Theorem 9.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be differentiable on \( I^o \), and \( a, b \in I^o \) with \( a < b \) and \( f' \in L[a, b] \).

a) If \(|f'|^q \) is GA-s-convex function in the second sense on \([a, b] \) for \( q > 1 \) and \( s \in (0, 1]\), then

\[
(3.15) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|
\]

where \( c_1, c_2, c_3, c_4 \) are defined by (3.7).

b) If \(|f'|^q \) is GA-s-convex function in the first sense on \([a, b], s \in (0, 1]\), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|
\]

\[
\leq \frac{\ln \left( \frac{b}{a} \right)}{4} \left| (ac_5(s, 1/2) + bc_4(s, 1)) \left| f' \left( \sqrt{ab} \right) \right| \right|
\]

where \( c_2, c_4, c_5, c_6 \) are defined by (3.7) and (3.10).
\[
\leq \frac{\ln \left( \frac{q}{2} \right)}{2} \left( \frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left[ a \left\{ c_7(s, q) |f'(a)|^q + c_8(s, q) |f'(b)|^q \right\}^\frac{1}{q} \\
+ b \left\{ c_9(s, q) |f'(b)|^q + c_{10}(s, q) |f'(a)|^q \right\}^\frac{1}{q} \right]
\]

(3.16)
\[
\left| f \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|
\]
\[
\leq \frac{\ln \left( \frac{q}{2} \right)}{4} \left( \frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left[ a \left\{ c_7(s, q/2) |f'(a)|^q + c_8(s, q/2) |f'(\sqrt{ab})|^q \right\}^\frac{1}{q} \\
+ b \left\{ c_9(s, q/2) |f'(b)|^q + c_{10}(s, q/2) |f'(\sqrt{ab})|^q \right\}^\frac{1}{q} \right]
\]

where
\[
c_7(s, q) = \int_0^1 (1-t)^s \left( \frac{b}{a} \right)^{qt} \, dt,
\]
\[
c_8(s, q) = \int_0^1 t^s \left( \frac{b}{a} \right)^{qt} \, dt,
\]
\[
c_9(s, q) = \int_0^1 (1-t)^s \left( \frac{a}{b} \right)^{qt} \, dt,
\]
\[
c_{10}(s, q) = \int_0^1 t^s \left( \frac{a}{b} \right)^{qt} \, dt,
\]
\[
b) If |f'|^q is GA-convex function in the first sense on [a, b] for q > 1 and s \in (0, 1], then
\]
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|
\]
\[
\leq \frac{\ln \left( \frac{q}{2} \right)}{2} \left( \frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left[ a \left\{ c_{11}(s, q) |f'(a)|^q + c_8(s, q) |f'(b)|^q \right\}^\frac{1}{q} \\
+ b \left\{ c_{12}(s, q) |f'(b)|^q + c_{10}(s, q) |f'(a)|^q \right\}^\frac{1}{q} \right]
\]

(3.18)
\[
\left| f \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right|
\]
\[
\leq \frac{\ln \left( \frac{q}{2} \right)}{4} \left( \frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left[ a \left\{ c_{11}(s, q/2) |f'(a)|^q + c_8(s, q/2) |f'(\sqrt{ab})|^q \right\}^\frac{1}{q} \\
+ b \left\{ c_{12}(s, q/2) |f'(b)|^q + c_{10}(s, q/2) |f'(\sqrt{ab})|^q \right\}^\frac{1}{q} \right],
\]

where
\[
c_{11}(s, q) = \int_0^1 (1-t)^s \left( \frac{b}{a} \right)^{qt} \, dt,
\]
\[
c_{12}(s, q) = \int_0^1 (1-t)^s \left( \frac{a}{b} \right)^{qt} \, dt.
\]
Proof. a) (1) Since $|f'|^q$ is GA-s-convex function in the second sense on $[a, b]$, from lemma 2 and Hölder inequality, we have

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| 
\leq \frac{\ln (\frac{b}{a})}{2} \left[ a \int_0^1 t \left( \frac{b}{a} \right)^t |f'(a^{1-t}b^t)| dt + b \int_0^1 t \left( \frac{a}{b} \right)^t |f'(b^{1-t}a^t)| dt \right] 
\leq \frac{a \ln (\frac{b}{a})}{2} \left[ \int_0^1 \left( \frac{b}{a} \right)^{qt} (1 - t)^s |f'(a)|^q + t^s |f'(b)|^q \right]^{\frac{1}{q}} 
\leq \frac{b}{2} \ln (\frac{b}{a}) \left[ \int_0^1 \left( \frac{a}{b} \right)^{qt} (1 - t)^s |f'(b)|^q + t^s |f'(a)|^q \right]^{\frac{1}{q}} 
\leq \frac{\ln (\frac{b}{a})}{2} \left[ \int_0^1 \left( \frac{b}{a} \right)^{qt} (1 - t)^s |f'(a)|^q + t^s |f'(b)|^q \right]^{\frac{1}{q}} 
$$

(3.21)

(2) Since $|f'|^q$ is GA-s-convex function in the first sense on $[a, b]$, from lemma 2 and Hölder inequality, we have

$$
\left| f \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| 
\leq \frac{\ln b}{4} \left[ a \int_0^1 t \left( \frac{b}{a} \right)^{\frac{1}{2}} |f'(a^{1-t}(ab)^{\frac{1}{2}})| dt + b \int_0^1 t \left( \frac{a}{b} \right)^{\frac{1}{2}} |f'(b^{1-t}(ab)^{\frac{1}{2}})| dt \right] 
\leq \frac{a \ln b}{4} \left[ \int_0^1 \left( \frac{b}{a} \right)^{\frac{1}{4q}} (1 - t)^s |f'(a)|^q + t^s |f'(\sqrt{ab})|^q \right]^{\frac{1}{q}} 
\leq \frac{b \ln b}{4} \left[ \int_0^1 \left( \frac{a}{b} \right)^{\frac{1}{4q}} (1 - t)^s |f'(b)|^q + t^s |f'(\sqrt{ab})|^q \right]^{\frac{1}{q}} 
$$

(3.22)
inequality (3.21), we have

\[ \text{Corollary 3.} \]

and

\[ (3.9) \] is easily obtained by using the inequality (3.22).

(1)\quad Since \(|f'|^q|\) is GA-\(s\)-convex function in the first sense on \([a, b]\), from the inequality (3.21), we have

\[ \left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \]

\[ \leq \frac{a \ln \left( \frac{b}{a} \right)}{2} \left( \frac{q - 1}{2q - 1} \right)^{1 - \frac{1}{q}} \left( \int_0^1 \left( \frac{b(a)}{a(b)} \right)^q \left( (1 - t^q) |f'(a)|^q + t^q |f'(b)|^q \right) dt \right)^{\frac{1}{q}} 
\]

\[ + \frac{b \ln \left( \frac{b}{a} \right)}{2} \left( \frac{q - 1}{2q - 1} \right)^{1 - \frac{1}{q}} \left( \int_0^1 \left( \frac{a(b)}{b(a)} \right)^q \left( (1 - t^q) |f'(b)|^q + t^q |f'(a)|^q \right) dt \right)^{\frac{1}{q}} \]

\[ \leq \frac{\ln \left( \frac{b}{a} \right)}{2} \left( \frac{q - 1}{2q - 1} \right)^{1 - \frac{1}{q}} \left[ a \left\{ c_{11}(s, q) \right| f'(a) \right|^q + c_{8}(s, q) \right| f'(b) \right|^q \right]^{\frac{1}{q}} 
\]

\[ + b \left\{ c_{12}(s, q) \right| f'(b) \right|^q + c_{10}(s, q) \right| f'(a) \right|^q \right]^{\frac{1}{q}} \]

(2)\quad Since \(|f'|^q|\) is GA-\(s\)-convex function in the first sense on \([a, b]\), the inequality (3.9) is easily obtained by using the inequality (3.22).

If taking \(s = 1\) in Theorem 9 we can derive the following inequalities for GA-

convex.

\[ \text{Corollary 3. Let } f : \mathbb{I} \subseteq (0, \infty) \to \mathbb{R} \text{ be differentiable on } \mathbb{I}^o, \text{ and } a, b \in \mathbb{I}^o \text{ with } a < b \text{ and } f' \in L [a, b]. \text{ If } |f'|^q \text{ is GA-convex function in the second sense on } [a, b] \text{ for } q > 1, \text{ then} \]

\[ \left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \]

\[ \leq \frac{\ln \left( \frac{b}{a} \right)}{2} \left( \frac{q - 1}{2q - 1} \right)^{1 - \frac{1}{q}} \left[ a \left\{ c_{7}(1, q) \right| f'(a) \right|^q + c_{8}(1, q) \right| f'(b) \right|^q \right]^{\frac{1}{q}} 
\]

\[ + b \left\{ c_{9}(1, q) \right| f'(b) \right|^q + c_{10}(1, q) \right| f'(a) \right|^q \right]^{\frac{1}{q}} \]

\[ \left| f\left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \]
\[\leq \frac{\ln \left( \frac{q}{a} \right)}{4} \left( \frac{q - 1}{2q - 1} \right)^{1 - \frac{q}{2}} \left[ a \left\{ c_7(1,q/2) |f'(a)|^q + c_8(1,q/2) \left| f' \left( \sqrt{ab} \right) \right|^q \right\}^{\frac{1}{q}} + b \left\{ c_9(1,q/2) |f'(b)|^q + c_{10}(1,q/2) \left| f' \left( \sqrt{ab} \right) \right|^q \right\}^{\frac{1}{q}} \right],\]

where \( c_7, c_8, c_9, c_{10} \) are defined by (3.17) and (3.20).

4. Application to Special Means

Proposition 1. Let \( 0 < a < b \) and \( q \geq 1 \). Then
\[
|A(a,b) - L(a,b)| \leq \left[ \ln \left( \frac{b}{a} \right) \right]^{1 - \frac{q}{2}} \left( \frac{1}{2} \right)^{2 - \frac{q}{2}} \left( \frac{1}{q} \right)^{\frac{1}{q}} \times \left\{ b^q - L(a^q,b^q) \right\}^{\frac{1}{q}} + \left\{ L(a^q,b^q) - a^q \right\}^{\frac{1}{q}} [1]
\]
\[
|G(a,b) - L(a,b)| \leq \left[ \ln \left( \frac{b}{a} \right) \right]^{1 - \frac{q}{2}} \left( \frac{1}{2} \right)^{2 - \frac{q}{2}} \left( \frac{2}{q} \right)^{\frac{1}{q}} \times \sqrt{a} \left\{ b^{q/2} - L(a^{q/2},b^{q/2}) \right\}^{\frac{1}{q}} + \sqrt{b} \left\{ L(a^{q/2},b^{q/2}) - a^{q/2} \right\}^{\frac{1}{q}} [2].
\]

Proof. The assertion follows from the inequalities (3.13) and (3.14) in Corollary 1 for \( f(x) = x, \ x > 0 \).

Proposition 2. Let \( 0 < a < b \leq 1 \) and \( q > 1 \). Then
\[
|A(a,b) - L(a,b)| \leq \ln \left( \frac{b}{a} \right) \left( \frac{q - 1}{2q - 1} \right)^{1 - \frac{q}{2}} L^{\frac{q}{2}}(a^q,b^q)
\]
\[
|G(a,b) - L(a,b)| \leq \frac{1}{2} \ln \left( \frac{b}{a} \right) \left( \frac{q - 1}{2q - 1} \right)^{1 - \frac{q}{2}} L^{\frac{q}{2}}(a^q,b^q) A \left( \sqrt{a}, \sqrt{b} \right).
\]

Proof. The assertion follows from the inequalities (3.23) and (3.24) in Corollary 3 for \( f(x) = x, \ x > 0 \).

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