A SHORT PROOF THAT MINIMAL SETS OF PLANAR ORDINARY 
DIFFERENTIAL EQUATIONS ARE TRIVIAL

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Abstract. We present a short proof, relying on the divergence theorem, verifying that minimal sets in the plane are trivial.

1. Introduction

We consider the ordinary differential equation in the plane defined by

\begin{equation}
\frac{dx}{dt} = f(x),
\end{equation}

where \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is a locally Lipschitz function. (Although, we only use uniqueness with respect to initial conditions of (1.1) and the continuity of \( f \).)

A minimal set is a nonempty closed invariant set, which is minimal with respect to inclusions. A trivial minimal set is a set that is the image if either a stationary solution or a periodic solution.

We present a new short proof of the following well known result.

**Theorem 1.** Any minimal set of (1.1) either corresponds to a stationary solution or to the image of a periodic solution, namely, all minimal sets are trivial.

The text-book proof of this theorem relays on the Poincaré–Bendixson theorem, and employs dynamical arguments. The proof presented in this paper relays on a different argument, relying on a property of the velocity of Jordan curves. This idea was introduced in [1] and further developed in [2, 3].

2. Proof of Main Result

In the proof of the main result we use the following notation. The 2-dimensional euclidean space is denoted by \( \mathbb{R}^2 \), and the norm of a vector \( y \in \mathbb{R}^2 \) is denoted by \( |y| \). The open ball in \( \mathbb{R}^2 \), centered at \( y \) with radius \( r \), is denoted by \( B(y, r) \). The closure of an open set \( O \subset \mathbb{R}^2 \) is denoted by \( O \), its boundary by \( \partial O \), and its exterior normal and tangent vector at the point \( y \in \partial O \) are denoted by \( N_{\partial O}(y) \) and \( T_{\partial O}(y) \), respectively.

We shall use the following results that are well known in the smooth case.

**Lemma 2.** Let \( O \subset \mathbb{R}^2 \) be a bounded open set with rectifiable boundary. Then

\[ v = \int_{\partial O} N_{\partial O}(y) \, dy = 0 \in \mathbb{R}^2. \]

**Proof.** Assume in contradiction that \( v \neq 0 \), and set \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( g \equiv \frac{v}{|v|} \). The divergence theorem for sets of finite perimeter (see, e.g., [4]) implies that

\[ |v| = \frac{v}{|v|} \cdot v = \left| \int_{\partial O} g(y) N_{\partial O}(y) \, dy \right| = \left| \int_{\partial O} \nabla \cdot g(y) \, dy \right| = 0, \]

in contradiction. \( \Box \)

The following lemma appears in [5].
Lemma 3. Suppose $I \subset \mathbb{R}$ is a bounded interval and $g : I \to \mathbb{R}$ is a Lipschitz function. Then for almost every $r \in \mathbb{R}$ the set $g^{-1}(r) = \{ t \in I | g(t) = r \}$ is finite.

To prove the main theorem, let us now fix a minimal set $\Omega \subset \mathbb{R}^2$ and a solution $x^*(\cdot)$ of (1.1), defined on $[0, \infty)$, with trajectory contained in $\Omega$.

We shall also use the following well known fact.

Lemma 4. For every $y_0 \in \Omega$ and $\delta > 0$ there exists $t > s$ such that $|x^*(t) - y_0| < \delta$.

Proof. Otherwise, suppose that the lemma does not hold for some $y_0, \delta$ and $s$. Then the curve $y^*(t) = x^*(s + t)$ is a solution of (1.1) with trajectory contained in $\Omega \setminus B(y_0, \delta)$ for a suitable $\delta > 0$, in contradiction to the minimality of $\Omega$.

If $\Omega$ is not a singleton we choose $D > 0$ such that $\Omega \setminus B(x^*(0), 3D) \neq \emptyset$ and apply the following construction:

Construction 5. Set $\delta_0 = D$ and $t_0$ as the first time point where $x^*(\cdot)$ meets $\partial B(x^*(0), \delta_0)$. For $i = 1, 2, \ldots$ do the following:

1. Choose $\delta_i < \delta_{i-1}/2$ small enough, such that $|x^*(0) - x^*(t)| > \delta_i$ for all $t \in [t_i, t_{i-1}]$.
2. Set $t_i$ as the first time point after $t_0$ where the curve $x^*(\cdot)$ meets $\partial B(x^*(0), \delta_i)$. (Here we use Lemma 2).
3. Starting from $x^*(t_i)$ follow the line connecting it to $x^*(0)$, until first meeting a point in $x^*(\{0, t_0\})$. Let $x^*(s_i)$ be this point.
4. Let $\gamma_i$ be the parametrized Jordan curve obtained by following the curve $x^*(\cdot)$ in the interval $[s_i, t_i]$ and then the line connecting its endpoints, with velocity of norm 1.

Lemma 6. If $t_i \to t^*$ then $x^*(0) = x^*(t^*)$, and $x^*(\cdot)$ is a periodic solution with image $\Omega$.

Proof. According to our construction $|x^*(0) - x^*(t_i)| = \delta_i < 2^{-i}D$ for every $i$. Hence, by continuity $x^*(t^*) = x^*(0)$, and $x^*(\cdot)$ is periodic. By the minimality of $\Omega$, the image of $x^*(\cdot)$ is $\Omega$.

Proof of Theorem 1. Clearly, $\Omega$ is a singleton if and only if it contains a point $y \in \Omega$ such that $f(y) = 0$. In this case and when the condition of Lemma 3 holds, we are done. Thus, we assume that $f$ does not vanish in $\Omega$ and that $t_i \to \infty$.

Fix $y_0 \in \Omega$ such that $|y_0 - x^*(0)| > 2D$. Using Lemma 3 we fix an arbitrary small ball $B = B(y_0, r_0)$, such that $r_0 < D$, and that $\{ 0 \leq t \leq s \ | x^*(t) - y_0 | = r_0 \}$ is finite for every $s > 0$. Note that this implies that for every $i$ the Jordan curve $\gamma_i$ intersects $\partial B$ at a finite number of points, and that the portion of $\gamma_i$ in $B$ corresponds to the trajectory $x^*(\cdot)$.

For every $i$ we denote the interior of $\gamma_i$ by $O_i$, and, using the identity

$$
\partial (O_i \cap B) \subset (\partial O_i \cap B) \cup (O_i \cap \partial B) \cup (\partial O_i \cap \partial B),
$$

we obtain, by Lemma 2 that

$$
0 = \int_{\partial (O_i \cap B)} N_{\partial (O_i \cap B)}(y) \, dy = \int_{\partial O_i \cap B} N_{\partial O_i}(y) \, dy - \int_{O_i \cap \partial B} N_{\partial B}(y) \, dy,
$$

since $\partial O_i \cap \partial B$ has zero measure. This bounds

$$
(2.1) \quad \left| \int_{\partial O_i \cap B} N_{\partial O_i}(y) \, dy \right| = \left| \int_{O_i \cap \partial B} N_{\partial B}(y) \, dy \right| \leq \left| \int_{O_i \cap \partial B} |N_{\partial B}(y)| \, dy \right| \leq 2\pi r_0.
$$

For each $i$ the set $\partial O_i \cap B$ contains a finite number of arcs, and applying a change of variable it is easy to see that

$$
\int_{\partial O_i \cap B} N_{\partial O_i}(y) \, dy = P_i \int_{\partial O_i \cap B} T_{\partial O_i}(y) \, dy = P_i \int_{\{ t \leq t_i \ | x^*(t) \in B \}} \frac{d}{dt} x^*(t) \, dt,
$$

where $T_{\partial O_i}$ is chosen to agree with the direction of $\gamma_i$, and $P_i$ is a $\frac{\pi}{2}$-rotation matrix. Here we use the fact that the portion of $\gamma_i$ in $B$ corresponds to the original trajectory $x^*(\cdot)$. 

Combined with (2.1) we conclude that for every $i$

$$\left| \int_{\{t \leq t_i | x^*(t) \in B\}} f(x^*(t)) \, dt \right| \leq 2\pi r_0.$$  

The minimality of $\Omega$ and Lemma 4 implies that the set \{t | x^*(t) ∈ B\} has infinite measure. This implies that 0 is contained in the convex hull of $\{f(y) | y \in \overline{B}\}$. The radius $r_0$ can be chosen arbitrary small, thus, the continuity of $f$ implies that $f(y_0) = 0$, in contradiction.

References
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