

DIAGONALLY SCALED MEMORYLESS QUASI–NEWTON METHODS WITH APPLICATION TO COMPRESSED SENSING

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Abstract. Memoryless quasi–Newton updating formulas of BFGS (Broyden–Fletcher–Goldfarb–Shanno) and DFP (Davidon–Fletcher–Powell) are scaled using well-structured diagonal matrices. In the scaling approach, diagonal elements as well as eigenvalues of the scaled memoryless quasi–Newton updating formulas play significant roles. Convergence analysis of the given diagonally scaled quasi–Newton methods is discussed. At last, performance of the methods is numerically tested on a set of CUTEr problems as well as the compressed sensing problem.

1. Introduction. Quasi–Newton (QN) algorithms are among the popular techniques for solving unconstrained optimization problems (often) with smooth real-valued cost functions [44]. Especially, QN methods are efficient because of acceptably approximating (inverse) Hessian of the cost function and then, explicitly using the approximation to determine the search direction. Also, under standard situations, the methods generate descent search directions and have reasonable global as well as local superlinear convergence features [47].

Starting from some positive definite (PD) matrices, in the QN methods successive approximations of the (inverse) Hessian are updated to satisfy the secant (QN) equation [47]. To make the generated matrix approximations well-conditioned, the scaled QN updates have been developed based on eigenvalue analyses [1,40,41]. Moreover, to adjust the methods for large-scale problems, memoryless QN techniques have been proposed in which the search directions are computed by performing a few vector inner products [11].

As known, BFGS and DFP methods are regarded as the most popular and efficient QN methods. Recently, the methods have been much heeded in practical applications such as image processing [37, 48], time series prediction [49], neural networks training [17, 27], document categorization [21], managing demands in the water distribution networks [53], machine learning [4], robotics [50], solving systems of nonlinear equations [3,23,55], curve fitting by B-splines [26], matrix approximation in Frobenius norm [42], computing the matrix geometric mean [52] and estimating unitary symmetric eigenvalues of the complex tensors [18]. The methods

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have been also well-combined with the classical optimization tools such as conjugate gradient methods [8,16,22,35,36] as well as the metaheuristic algorithms [17,38,48]. Real-world applications of the QN algorithms motivated the researchers to improve efficiency of the methods. Such attempts can be mainly categorized into modifying the secant equation to approximate the curvature more accurately [2,20,29,45,46,51,54] or to achieve convergence without convexity suppositions [33,34], improving the scaling scheme [9,12–14], and structuring the updating formulas to solve special problems such as nonlinear least-squares [5,24]. Here, we deal with scaling the memoryless QN updating formulas by special diagonal matrices. Remaining parts of this study include stating necessary preliminaries in Section 2, describing our diagonally scaled memoryless QN methods together with analyzing their global convergence in Section 3, testing practical efficiency of the given methods in Section 4 and finally concluding in Section 5.

2. Preliminaries. As line search-based strategies for solving the unconstrained optimization problem

\[ \min_{x \in \mathbb{R}^n} f(x), \]

QN iterations are generally in the form of

\[ x_{k+1} = x_k + s_k, \quad k = 0, 1, \ldots, \]

starting from some \( x_0 \in \mathbb{R}^n \), where \( s_k = \alpha_k d_k \) in which \( \alpha_k > 0 \) is a step length determined by a line search (LS) along the descent direction \( d_k \) [44]. In the QN methods, we have

\[ d_0 = -g_0, \quad d_k = -H_k g_k, \quad k = 0, 1, \ldots, \]

where \( g_k = \nabla f(x_k) \) and \( H_k \in \mathbb{R}^{n \times n} \) is a PD approximation of \( \nabla^2 f(x_k)^{-1} \). Here, we use the popular Wolfe LS conditions [44], i.e.

\[ f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k g_k^T d_k, \]

\[ \nabla f(x_k + \alpha_k d_k)^T d_k \geq \varrho g_k^T d_k, \]

with \( 0 < \delta < \varrho < 1 \).

As known, the BFGS and DFP updating formulas [44] are given by

\[ H_{k+1}^{\text{BFGS}} = H_k - \frac{s_k y_k^T H_k + H_k y_k s_k^T}{s_k^T y_k} + \left( 1 + \frac{y_k^T H_k y_k}{s_k^T y_k} \right) \frac{s_k s_k^T}{s_k^T y_k}, \]

and

\[ H_{k+1}^{\text{DFP}} = H_k + \frac{s_k s_k^T}{s_k^T y_k} - \frac{H_k y_k y_k^T H_k}{y_k^T H_k y_k}, \]

where \( y_k = g_{k+1} - g_k \). Replacing \( H_k \) by \( \vartheta_k I \) where \( \vartheta_k > 0 \) is called the scaling parameter [41], scaled memoryless BFGS and DFP updating formulas for the inverse Hessian, here respectively named by SMBFGS and SMDFP, are given by

\[ H_{k+1}^{\text{SMBFGS}} = \vartheta_k I - \vartheta_k \frac{s_k y_k^T + y_k s_k^T}{s_k^T y_k} + \left( 1 + \frac{||y_k||^2}{s_k^T y_k} \right) \frac{s_k s_k^T}{s_k^T y_k}, \]

and

\[ H_{k+1}^{\text{SMDFP}} = \vartheta_k I + \frac{s_k s_k^T}{s_k^T y_k} - \vartheta_k \frac{y_k y_k^T}{y_k^T y_k}, \]
for all $k \geq 0$, in which $\| \cdot \|$ denotes the $\ell_2$ norm and (often) $H_0 = I$ [11]. It has been believed that the two most effective formulas for $\vartheta_k$ are

$$
\vartheta_k^{OL} = \frac{\|s_k\|^2}{s_k^T y_k},
$$

and

$$
\vartheta_k^{OS} = \frac{s_k^T y_k}{\|y_k\|^2},
$$

as suggested in [19, 40, 41]. Note that it is not necessary to assign memory for storing the (matrices) $H_k^{SMBFGS}$ or $H_k^{SMDFP}$, being a memory-intensive practice which also slows the implementations. Instead, as already mentioned, search directions can be effectively determined by performing a few inner products. It can be seen that from (4) we have $s_k^T y_k > 0$, preserving heredity of positive definiteness in QN updating formulas (5)–(8) [44]. In convergence analyses, we consider the following standard assumption on the cost function.

**Assumption 2.1.** For an arbitrary $x_0 \in \mathbb{R}^n$, $L = \{ x : f(x) \leq f(x_0) \}$ is a bounded set and in some neighborhood $U$ of $L$, $\nabla f(x)$ is Lipschitz continuous; that is, there exists a constant $L > 0$ such that

$$
\| \nabla f(x) - \nabla f(\tilde{x}) \| \leq L \| x - \tilde{x} \|, \quad \text{for all } x, \tilde{x} \in U.
$$

Based on Assumption 2.1, positive constants $\Omega$ and $B$ exist such that

$$
\| \nabla f(x) \| \leq \Omega, \quad \text{for all } x \in L,
$$

and

$$
\| s_k \| \leq B, \quad \text{for all } k \geq 0.
$$

Also, when the directions are descent, from (3) we have $\{x_k\}_{k \geq 0} \subset L$. We say that directions $\{d_k\}_{k \geq 0}$ satisfy the sufficient descent condition when

$$
d_k^T g_k \leq -\varsigma \| g_k \|^2, \quad \text{for all } k \geq 0,
$$

where $\varsigma > 0$ is a constant. Our convergence analyses are structured based on the following essential result.

**Lemma 2.1.** [43] Suppose that Assumption 2.1 holds. Consider any iterative method in the form (1), where $d_k$ and $\alpha_k$ satisfy the sufficient descent condition (14) and the Wolfe conditions (3) and (4), respectively. If

$$
\sum_{k=0}^{\infty} \frac{1}{\|d_k\|^2} = \infty,
$$

then the method converges globally in the sense that

$$
\liminf_{k \to \infty} \| g_k \| = 0.
$$

3. **Diagonally scaled memoryless quasi–Newton updates.** Here, we extend the Barzilai–Borwein [19] approach of approximating the (inverse) Hessian by a positive scalar based on a least-squares scheme, to suggest diagonal approximations of the (inverse) Hessian. In this context, we use the diagonal elements as well as eigenvalues of the updating formulas (7) and (8). Next, we expatiate our diagonally scaled memoryless QN methods in detail and discuss their global convergence.
3.1. Diagonally scaled memoryless BFGS updating formulas. Here, firstly we propose a simple diagonal approximation for the inverse Hessian $\hat{H}_{k+1}^{\text{DSMBFGS}}$ as a solution of the following minimization problem:

$$\min_{D \in \mathbb{D}} ||H_{k+1}^{\text{DSMBFGS}} - D||_F^2,$$

in which $||.||_F$ denotes the Frobenius norm and $\mathbb{D}$ stands for the set of all diagonal matrices in $\mathbb{R}^{n \times n}$. Solving (16), we get the diagonal matrix $\hat{D}_k$,

$$\hat{D}_k = \text{diag}(\hat{D}_{k11}, \hat{D}_{k22}, ..., \hat{D}_{knn}),$$

with

$$\hat{D}_{kii} = \vartheta_k - 2\vartheta_k \frac{\langle s_k, y_k \rangle}{s_k^T y_k} \left(1 + \vartheta_k \frac{||y_k||^2}{s_k^T y_k} \right) \frac{s_k^2}{s_k^T y_k}, \quad i = 1, 2, ..., n,$$

being the diagonal elements of $H_{k+1}^{\text{DSMBFGS}}$. So, $\hat{D}_{kii} > 0$, $i = 1, 2, ..., n$, and as a result, $\hat{D}_k$ is a PD matrix. Now, replacing $H_k$ by $\hat{D}_k$ in (5), our first diagonally scaled memoryless BFGS updating formula is obtained as follows:

$$H_{k+1}^{\text{DSMBFGS1}} = \hat{D}_k - \frac{y_k^T \hat{D}_k y_k}{s_k^T y_k} + \left(1 + \frac{y_k^T \hat{D}_k y_k}{s_k^T y_k} \right) s_k s_k^T.$$

To suggest another diagonal approximation for the inverse Hessian, we take into account the eigenvalues of $H_{k+1}^{\text{DSMBFGS}}$, playing an important role in numerical behavior of the method [44]. As discussed in [11], $n - 2$ eigenvalues of $H_{k+1}^{\text{DSMBFGS}}$ are equal to $\vartheta_k$ and the two others, namely $\hat{\rho}_k^+$ and $\hat{\rho}_k^-$, can be computed as follows:

$$\hat{\rho}_k^+ = \frac{1}{2} \left(1 + \vartheta_k \frac{||y_k||^2}{s_k^T y_k} \right) \frac{||s_k||^2}{s_k^T y_k},$$

$$\hat{\rho}_k^- = \frac{1}{2} \sqrt{\left(1 + \vartheta_k \frac{||y_k||^2}{s_k^T y_k} \right)^2 \frac{||s_k||^4}{(s_k^T y_k)^2} - 4\vartheta_k \frac{||s_k||^2}{s_k^T y_k}},$$

for which $0 < \hat{\rho}_k^- \leq \vartheta_k \leq \hat{\rho}_k^+$. Now, we consider a diagonal matrix $\hat{\Lambda}_k$ with eigenvalues of $H_{k+1}^{\text{DSMBFGS}}$ as its diagonal entries. To find a proper arrangement for the eigenvalues on the diagonal positions of $\hat{\Lambda}_k$, we pay attention to diagonal structure of $\hat{D}_k$. That is, letting $i_{\text{max}} = \arg \max_{1 \leq i \leq n} \hat{D}_{kii}$ and $i_{\text{min}} = \arg \min_{1 \leq i \leq n} \hat{D}_{kii}$, for $i = 1, 2, ..., n$, we set

$$\hat{\Lambda}_{kii} = \begin{cases} \hat{\rho}_k^+, & \text{if } i = i_{\text{max}}, \\ \hat{\rho}_k^-, & \text{if } i = i_{\text{min}}, \\ \vartheta_k, & \text{otherwise}. \end{cases}$$

So, the largest and the smallest eigenvalues of $H_{k+1}^{\text{DSMBFGS}}$ are placed in the diagonal positions of $\hat{\Lambda}_k$ respectively corresponding to the largest and the smallest diagonal elements of $\hat{D}_k$. Now, replacing $H_k$ by $\hat{\Lambda}_k$ in (5), our second diagonally scaled memoryless BFGS updating formula is obtained as follows:

$$H_{k+1}^{\text{DSMBFGS2}} = \hat{\Lambda}_k - \frac{y_k^T \hat{\Lambda}_k y_k}{s_k^T y_k} + \left(1 + \frac{y_k^T \hat{\Lambda}_k y_k}{s_k^T y_k} \right) s_k s_k^T.$$

Note that because of sparse structures of the diagonal matrices $\hat{D}_k$ and $\hat{\Lambda}_k$, search directions of DSMBFGS1 and DSMBFGS2 can be computed by performing a few vector inner products, no requiring to store the matrices. Next, we study convergence of the methods DSMBFGS1 and DSMBFGS2. Henceforth, we suppose
that Assumption 2.1 holds, the line search fulfills (3) and (4), the scaling parameter \( \vartheta_k \) is bounded, that is,

\[
\vartheta_k \in [m, M],
\]

for some positive constants \( m \) and \( M \). Also, we assume that \( f \) is strongly convex on \( U \), namely,

\[
f(\tau x + (1 - \tau)\bar{x}) \leq \tau f(x) + (1 - \tau)f(\bar{x}) - \frac{1}{2}\nu\tau(1 - \tau)||x - \bar{x}||^2, \quad \forall x, \bar{x} \in U,
\]

for some constant \( \nu > 0 \), which considering Theorem 1.3.16 of [44] with smoothness supposition of \( f \) leads to

\[
s_k^T y_k \geq \nu||s_k||^2, \quad \text{for all } k \geq 0. \tag{24}
\]

**Theorem 3.1.** Suppose that there exist positive constants \( \eta_1 \) and \( \eta_2 \) such that \( \eta_1 \leq \hat{D}_{k,i} \leq \eta_2 \), for \( i = 1, 2, ..., n \) in (18). Then, the DSMBFGS1 method converges in the sense that (15) holds.

**Proof.** At first, let \( \theta_k^1 \) be the smallest eigenvalue of the inverse Hessian approximation \( H_k \). When \( H_k \) is a PD matrix, we can write \( \text{tr}(H_k^{-1}) \geq \frac{1}{\theta_k^1} \), and hence, from (2) we get

\[
g_k^T d_k = -g_k^T H_k g_k \leq -\theta_k^1||g_k||^2 \leq -\frac{1}{\text{tr}(H_k^{-1})}||g_k||^2,
\]

for all \( k \). Now, since \( g_k^T d_0 = -||g_0||^2 \), to ensure sufficient descent condition (14) it suffices to demonstrate \( \text{tr}(H_k^{-1}) \) is bounded above. To proceed, for DSMBFGS1 with the updating matrix (19), using the Sherman–Morrison formula [44] we have

\[
\text{tr}(H_{k+1}^{-1}) = \text{tr}\left(\hat{D}_{k+1}^{-1} + \frac{y_k y_k^T}{s_k^T y_k} - \frac{\hat{D}_k^{-1} s_k s_k^T \hat{D}_k^{-1}}{s_k^T \hat{D}_k^{-1} s_k}\right)
\]

\[
\leq \frac{n}{\min_{1 \leq i \leq n} \hat{D}_k(i,i)} + \frac{||y_k||^2}{s_k^T y_k} + \frac{||s_k||^2 \max_{1 \leq i \leq n} \hat{D}_k(i,i)}{\min_{1 \leq i \leq n} \hat{D}_k(i,i)}
\]

\[
\leq \frac{n}{\eta_1} + \frac{L^2}{\nu} + \frac{\eta_2}{\eta_1^2}, \tag{25}
\]

Now, to establish convergence of DSMBFGS1 in the sense of (15), from Lemma 2.1 it is enough to show \( \sum_{k=0}^{\infty} \frac{1}{||d_k||^2} \) is divergent which can be ensured when \( \{||d_k||\}_{k \geq 0} \) is bounded above. In this context, from (11), (19) and (24), we have

\[
||H_{k+1}^{\text{DSMBFGS1}}|| \leq ||\hat{D}_k|| + 2||\hat{D}_k|| ||s_k|| ||y_k|| + \left(1 + \frac{||\hat{D}_k|| ||y_k||^2}{s_k^T y_k}\right) ||s_k||^2
\]

\[
\leq \eta_2 + 2\eta_2 \frac{L}{\nu} + \frac{1}{\nu} \left(1 + \frac{\eta_2 L^2}{\nu}\right) := \Gamma_1. \tag{26}
\]

Hence, from (2) and (12), we get

\[
||d_{k+1}|| \leq ||H_{k+1}^{\text{DSMBFGS1}}|| ||g_{k+1}|| \leq \Gamma_1 \Omega,
\]

which completes the proof. \( \square \)
As stated in Theorem 3.1, \( \hat{D}_{k,i} \) \((i = 1, 2, \ldots, n)\) given by (18) is assumed to be uniformly bounded. In order to ensure boundedness of \( \hat{D}_{k,i} \), we can set \( \hat{D}_{k,i} \leftarrow \min\{\max\{\hat{D}_{k,i}, \eta_1\}, \eta_2\} \), for all \( i = 1, 2, \ldots, n \), where \( \eta_1 \) and \( \eta_2 \) respectively are enough small and enough large positive constants. To justify such truncation, note that when \( s_k \) tends to zero, from (18) and (23) we have \( \hat{D}_{k,i} \approx \hat{\vartheta}_k \geq m \), and in the case where \( s_k \) is defined by 

\[
\hat{\vartheta}_k = \vartheta_k \left( 1 - \frac{s_k y_k}{s_k^T y_k} \right)^2 + \frac{s_k^2}{s_k^T y_k^2} \geq \frac{\varepsilon^2}{L B^2}.
\]

Also, bearing in mind relations (11), (23) and (24), we can write 

\[
\hat{D}_{k,i} \leq \vartheta_k + 2\vartheta_k \frac{||y_k||}{s_k^T y_k} + \left( 1 + \frac{\vartheta_k}{s_k^T y_k} \right) \frac{||s_k||^2}{s_k^T y_k} \leq M + 2M \frac{L}{\nu} + \frac{1}{\nu} \left( 1 + \frac{ML^2}{\nu} \right),
\]

for all \( i = 1, 2, \ldots, n \). Moreover, boundedness of the two scaling parameters \( \vartheta_k^{OL} \) and \( \vartheta_k^{OS} \) respectively defined by (9) and (10) in the sense of (23) can be obtained from (11) and (24), as follows:

\[
\frac{\nu}{L^2} \leq \frac{s_k^T y_k}{||y_k||^2} \leq \frac{||s_k||^2}{s_k^T y_k} \leq \frac{1}{\nu}.
\]

Next, we deal with convergence of DSMBFGS2 without boundedness assumption of the diagonal elements.

**Theorem 3.2.** The DSMBFGS2 method converges in the sense that (15) holds.

**Proof.** The proof can be presented following carefully the proof of Theorem 3.1, namely, by finding proper upper bounds for both \( \text{tr}(H_{k+1}^{\text{DSMBFGS2}^{-1}}) \) and \( ||H_{k+1}^{\text{DSMBFGS2}}|| \). As known, \( \hat{\vartheta}_k = \frac{1}{\hat{\lambda}_k^+} \) where \( \hat{\lambda}_k^+ \) as the largest eigenvalue of \( H_{k+1}^{\text{DSMBFGS2}^{-1}} \) is defined by

\[
\hat{\lambda}_k^+ = \frac{1}{2} \left( \frac{1}{\vartheta_k} + \frac{||y_k||^2}{s_k^T y_k} \right) + \frac{1}{2} \sqrt{\left( \frac{1}{\vartheta_k} + \frac{||y_k||^2}{s_k^T y_k} \right)^2 - \frac{4}{\vartheta_k} \frac{s_k^T y_k}{||s_k||^2}}.
\]

Hence, from (11) and (23) we can write

\[
\hat{\lambda}_k^+ \leq \frac{1}{2} \left( \frac{1}{m} + \frac{L^2}{\nu} \right) + \frac{1}{2} \sqrt{\left( \frac{1}{m} + \frac{L^2}{\nu} \right)^2 + \frac{4L^2}{m} \frac{1}{\nu}} \leq \frac{1}{\zeta_1},
\]

yielding \( \hat{\vartheta}_k \geq \zeta_1 \). Also, from (20) we get

\[
\hat{\vartheta}_k^+ \leq \frac{1}{2\nu} \left( 1 + \frac{ML^2}{\nu} \right) + \frac{1}{2} \sqrt{\left( 1 + \frac{ML^2}{\nu} \right)^2 + \frac{4}{\nu} + \frac{M}{\nu} \leq \zeta_2}.
\]

Therefore, \( \zeta_1 \leq \hat{\lambda}_{k,i} \leq \zeta_2 \), for \( i = 1, 2, \ldots, n \) in (21) and so, upper bounds of \( \text{tr}(H_{k+1}^{\text{DSMBFGS2}^{-1}}) \) and \( ||H_{k+1}^{\text{DSMBFGS2}}|| \) can be respectively achieved similar to (25) and (26) with substituting \( \eta_1 \) by \( \zeta_1 \) and \( \eta_2 \) by \( \zeta_2 \). \( \square \)
3.2. Diagonally scaled memoryless DFP updating formulas. Here, we put forward diagonally scaled memoryless DFP updating formulas in light of similar approaches already presented for the diagonally scaled memoryless BFGS updating formulas. At first, by solving $\min_{D \in \mathcal{D}} \| H^\text{DSMDFP1}_{k+1} - D \|^2_F$, we get the first diagonal matrix as follows:

$$
\hat{D}_k = \text{diag}(\hat{D}_{k_{i1}}, \hat{D}_{k_{i2}}, \ldots, \hat{D}_{k_{nn}}),
$$

with

$$
\hat{D}_{k_{ii}} = \vartheta_k + \frac{s_{ki}^2}{s_{ki}^T y_k} - \vartheta_k \frac{y_k^T y_k}{y_k^T y_k}, \quad i = 1, 2, \ldots, n.
$$

Therefore, replacing $H_k$ by $\hat{D}_k$ in (6), our first diagonally scaled memoryless DFP updating formula is obtained as follows:

$$
H^\text{DSMDFP1}_{k+1} = \hat{D}_k + \frac{s_k s_k^T}{s_k^T y_k} - \frac{\hat{D}_k y_k y_k^T \hat{D}_k}{y_k^T y_k}.
$$

Similar to DSMBFGS1, to ensure boundedness of $\hat{D}_{k_{ii}}$, we can set $\hat{D}_{k_{ii}} \leftarrow \min \{\max\{\hat{D}_{k_{ii}}, \gamma_1\}, \gamma_2\}$, for all $i = 1, 2, \ldots, n$, where $\gamma_1$ and $\gamma_2$ respectively are enough small and enough large positive constants.

To determine the second diagonal matrix, we need to obtain eigenvalues of $H^\text{DSMDFP2}_{k+1}$. Following carefully the analysis carried out in [11], it can be seen that $H^\text{DSMDFP2}_{k+1}$ has $n - 2$ eigenvalues equal to $\vartheta_k$, and the two others, namely $\tilde{\rho}_k^+$ and $\tilde{\rho}_k^-$, are given by

$$
\tilde{\rho}_k^\pm = \frac{1}{2} \left( \vartheta_k + \frac{||s_k||^2}{s_k^T y_k} \right) \pm \frac{1}{2} \sqrt{\left( \vartheta_k + \frac{||s_k||^2}{s_k^T y_k} \right)^2 - 4 \vartheta_k \frac{s_k^T y_k}{||y_k||^2}}.
$$

Using Cauchy–Schwarz inequality, it can be proved that $\tilde{\rho}_k^- \leq \vartheta_k \leq \tilde{\rho}_k^+$. So, if $i_{\max} = \arg \max_{1 \leq i \leq n} \hat{D}_{k_{ii}}$ and $i_{\min} = \arg \min_{1 \leq i \leq n} \hat{D}_{k_{ii}}$, for $i = 1, 2, \ldots, n$ we set

$$
\tilde{\Lambda}_{k_{ii}} = \begin{cases} 
\tilde{\rho}_k^+, & \text{if } i = i_{\max}, \\
\tilde{\rho}_k^-, & \text{if } i = i_{\min}, \\
\vartheta_k, & \text{otherwise}.
\end{cases}
$$

Consequently, we can suggest the second diagonally scaled DFP updating formula as follows:

$$
H^\text{DSMDFP2}_{k+1} = \tilde{\Lambda}_k + \frac{s_k s_k^T}{s_k^T y_k} - \frac{\tilde{\Lambda}_k y_k y_k^T \tilde{\Lambda}_k}{y_k^T y_k}.
$$

**Theorem 3.3.** The DSMDFP1 and DSMDFP2 methods converge in the sense that (15) holds.

**Proof.** Taking Lemma 2.1 as well as proof of Theorem 3.1 into consideration, it is adequate to find upper bounds for the trace of the Hessian approximation as well as the norm of the inverse Hessian approximation. To this aim, considering the Sherman–Morrison formula we have

$$
H^\text{DSMDFP1}_{k+1}^{-1} = \tilde{D}_k^{-1} - \frac{\tilde{D}_k^{-1} s_k y_k^T + y_k s_k^T \tilde{D}_k^{-1}}{s_k^T y_k} + \left(1 + \frac{s_k^T \tilde{D}_k^{-1} s_k}{s_k^T y_k} \right) \frac{y_k y_k^T}{s_k^T y_k},
$$

which together with (11), (23) and (24) we get

$$
\text{tr}(H^\text{DSMDFP1}_{k+1}^{-1}) \leq \frac{n}{\min_{1 \leq i \leq n} \hat{D}_{k_{ii}}} + \frac{2}{\min_{1 \leq i \leq n} \tilde{D}_{k_{ii}}}.
$$
\[ \frac{L^2}{\nu} \left( 1 + \frac{1}{\nu \min_{1 \leq i \leq n} D_{k,i}} \right) \]
\[ \leq \frac{n + 2}{\gamma_1} + \frac{L^2}{\nu} \left( 1 + \frac{1}{\nu \gamma_1} \right). \]  
(32)

Besides, from (24) and (28) we can write
\[ \| H_{k+1}^{DSMDFP1} \| \leq \| \tilde{D}_k \| + \frac{\| s_k \|^2}{s_k^T y_k} + \frac{\| y_k \|^2}{y_k^T D_k y_k} \leq \gamma_2 + \frac{1}{\nu} + \frac{\gamma_2^2}{\gamma_1}. \]  
(33)

For DSMDFP2,  \( \rho_k^- = \frac{1}{\tilde{\lambda}_k^+} \) where  \( \tilde{\lambda}_k^+ \) is the largest eigenvalue of  \( H_{k+1}^{SMDFP} \) obtained from (20) by replacing  \( s_k \leftrightarrow y_k \) and  \( \vartheta_k \leftrightarrow \frac{1}{\vartheta_k} \). Hence, we get
\[ \tilde{\lambda}_k^+ \leq \frac{L^2}{2 \nu} \left( 1 + \frac{1}{\nu} \right) + \frac{1}{2} \sqrt{\left( 1 + \frac{1}{\nu} \right)^2 + \frac{4 L^4}{\nu^2 \mu} + \frac{4}{\nu \mu} : = \frac{1}{\xi_1}, \]
yielding  \( \tilde{\rho}_k \geq \xi_1 \). Moreover, from (29) we get
\[ \tilde{\rho}_k^+ \leq \frac{1}{2} \left( M + \frac{1}{\nu} \right) + \frac{1}{2} \sqrt{\left( M + \frac{1}{\nu} \right)^2 + \frac{4 M}{\nu} : = \xi_2. \]

Therefore, upper bounds of  \( \text{tr}(H_{k+1}^{DSMDFP2}^{-1}) \) and  \( \| H_{k+1}^{DSMDFP2} \| \) can be respectively achieved similar to (32) and (33) with substituting  \( \gamma_1 \) by  \( \xi_1 \) and  \( \gamma_2 \) by  \( \xi_2. \)

It is notable that one can make a modification on the updating formulas (19), (22), (28) and (31) based on the modified secant condition proposed by Li and Fukushima [33]. Then, global convergence of the modified methods can be established for general functions (without convexity assumption); see [10,15,56] for more details.

4. Numerical tests. Here, we study effect of the diagonally scaling scheme on the given LS-based methods. Test functions data (including 84 problems of CUTEr library [30]) and the software and hardware information have been provided in [6]. Also, to perform line search, strong Wolfe conditions have been used by Algorithm 3.5 of [39] with  \( \delta = 10^{-4} \) and  \( \varrho = 0.99 \), consisting of (3) and the inequality  \( |d_k^T g_k| \leq -\varrho d_k^T g_k \) instead of (4). The algorithms were terminated when  \( k > 10000 \) or  \( \| g_k \| < 10^{-6} (1 + |f_k|) \). In the experiments, we set  \( \vartheta_k = \vartheta_k^{OS} \) given by (10). Efficiency of the algorithms were compared applying the performance profile proposed by Dolan and Moré [25] (DM) on the total number of function and gradient evaluations (TNFGE) defined in [32], and the CPU time (CPUT). In the DM performance profile outputs, plots (A) and (B) respectively show the results of TNFGE and CPUT.

As a brief description on the DM performance profile, let  \( S \) be the set of methods,  \( P \) be the set of test problems and  \( t_{p,s} \) be TNFGE or CPUT needed to solve the problem  \( p \in P \) by the method  \( s \in S \). Then, the performance ratio is computed by
\[ r_{p,s} = \frac{t_{p,s}}{\min \{ t_{p,s}, s \in S \}}. \]
and the overall performance of algorithm $s$ is given by
\[ \rho_s(\omega) = \frac{1}{n_p} |\{p \in \mathcal{P} : r_{p,s} \leq \omega\}|. \]
In fact, $\rho_s(\omega)$ is the probability for algorithm $s$ that a performance ratio $r_{p,s}$ is within a factor $\omega \in \mathbb{R}$ of the best possible ratio. The function $\rho_s(\omega)$ is a distribution function for the performance ratio $r_{p,s}$.

At first, we evaluate performance of our scaling approaches on the steepest descent method. In this context, we compare spectral steepest descent (SSD) method with $d_k = -\vartheta_k g_k$ [19], by the diagonally scaled steepest descent methods DSSD1 with $d_k = -\hat{D}_k g_k$, DSSD2 with $d_k = -\hat{\Lambda}_k g_k$, DSSD3 with $d_k = -\hat{D}_k g_k$ and DSSD4 with $d_k = -\hat{\Lambda}_k g_k$. Corresponding diagonal matrices respectively have been defined by (17), (21), (27) and (30). Moreover, we compared the above methods with nonmonotone strong Wolfe line search in which (3) is replaced by the inequality (4) of [31] with the number of required recent iterates being equal to 5. The corresponding methods are respectively called NMSSD and NMDSSD$i$, for $i = 1, \ldots, 4$.

Figures 1 and 2 illustrate the results. As shown, for both monotone and nonmonotone line searches $\hat{D}_k$-based scaling on the steepest descent method turns out to be more effective than the $\hat{\Lambda}_k$-based scaling, while both of them are generally preferable to the scalar scaling. Also, DSSD3 (NMDSSD3) and DSSD4 (NMDSSD4) are approximately competitive and both of them outperform SSD (NMSSD). It seems that computing the eigenvalues and then, finding max/min positions as done in (21) and (30) is to some extent time consuming. As another observation, the figures show that BFGS-based scaling methods are generally preferable to DFP-based scaling methods.

Next, we study computational performance of the SMBFGS, DSMBFGS1, DSMBFGS2, SMDFP, DSMDFP1 and DSMDFP2 methods with the directions (2) and the search direction matrix $H_{k+1}$ respectively given by (7), (19), (22), (8), (28) and (31). Results are shown by Figures 3 and 4. As seen in Figure 3, DSMBFGS1 is preferable to SMBFGS with respect to TNFGE while the methods are competitive with respect to CPUT. However, here the eigenvalue-based diagonal scaling method DSMBFGS2 performs poorly. From Figure 4 it can be observed that DSMDFP1 and DSMDFP2 are generally superior to SMDFP. So, diagonal scalings of the DFP updating formula are practically promising in contrast to the scalar scaling.
Besides, performance of DSMBFGS1 is compared by the limited memory BFGS (LMBFGS) method in the sense of Algorithm 5.7.1 of [44], with applying information of 5 recent iterations, the two-parameter memoryless BFGS (TPSMBFGS)
method proposed in [14] and the two efficient conjugate gradient methods obtained based on memoryless BFGS methods proposed in [16] (MLBFGSCG1) and [35] (MLBFGSCG2). Results are shown by Figure 5. As illustrated, the DSMBFGS1 is superior to the other methods.

In another part of our numerical experiments, we evaluate performance of the diagonally scaled memoryless BFGS methods for solving the compressed sensing problem which essentially deals with sparse solutions of an underdetermined system of linear equations. Details of the problem model as well as the smoothing technique can be found in [7]. The initial process starts from the origin.

To assess the restoration performance qualitatively, we report relative error (Rel-Err) [7] of the recovered signal (in percent). Figures 6–10 show the results for different choices of the sampling matrix [28]. In the figures, plots (A), (B), (C) and (D) respectively illustrate the successive function values, the successive relative errors, the original signal and the observation (noisy measurements). Also, recovered signals (red disks) by DSMBFGS1, DSMBFGS2 and SMBFGS in contrast to the original signal (blue peaks) have been respectively shown by plots (E), (F) and (G). As seen, approximate solutions obtained by DSMBFGS1 and DSMBFGS2 are competitive in the relative error point of view, and both are preferable to SMBFGS. Especially, although in the initial iterations DSMBFGS2 is not as good as DSMBFGS1 and SMBFGS, near the solution DSMBFGS2 behaves more promisingly.

5. Conclusions and future works. We have extended the Barzilai–Borwein approach of approximating the inverse Hessian by a positive scalar and developed several diagonal approximations for the matrix. In one of our approaches we used the diagonal elements of the scaled memoryless BFGS and DFP updating formulas while in the other we applied their eigenvalues. Convergence of the corresponding quasi–Newton methods has been studied. Using a class of problems of the CUTEr library, numerical tests have been implemented to investigate effectiveness of our diagonal scaling schemes. Results showed that the given methods are computationally promising. Further numerical experiments have been done on the well-known compressed sensing problem and the results showed that the proposed methods can be also capable to acceptably recover sparse solutions of the underdetermined systems.
Figure 6. Compressed sensing outputs for the Gaussian matrix
Figure 7. Compressed sensing outputs for the scaled Gaussian matrix
Figure 8. Compressed sensing outputs for the orthogonalized Gaussian matrix
Figure 9. Compressed sensing outputs for the Bernoulli matrix
Figure 10. Compressed sensing outputs for the Hadamard matrix
As a future work, one can investigate effective arrangements of the eigenvalues of the scaled memoryless quasi–Newton updating formulas on the corresponding diagonal approximations.

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