On the variety generated by completions of representable relation algebras

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Abstract. Maddux recently defined the variety $V$ generated by the completions of representable relation algebras. In this note, we observe that $V$ is canonical, answering Maddux’s problem 1.1(3), and show that the variety of representable relation algebras is not finitely axiomatisable over $V$.

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1. Introduction

In a recent paper [9, 1.10], Maddux defined the variety $V = \text{HSP} \ RRA^c$, where $RRA$ is the variety of representable relation algebras and $RRA^c = \{ \mathfrak{A}^c : \mathfrak{A} \in RRA \}$. Here, $\mathfrak{A}^c$ denotes the completion of the relation algebra $\mathfrak{A}$. For details of these notions and further ones used below, see the very attractive introduction to [9].

In [9, problem 1.1(3)], Maddux asked whether $V$ is closed under canonical extensions.

Theorem 1.1. $V$ and $S \ RRA^c$ are closed under canonical extensions.

Proof. By [4, theorem 3.8], if $K$ is a class of relation algebras that is closed under ultraproducts, and $K^c = \{ \mathfrak{A}^c : \mathfrak{A} \in K \}$, then $S K^c$ and $\text{HSP} \ K^c$ are both closed under canonical extensions. Theorem 1.1 follows by taking $K = RRA$, which is a variety and so closed under ultraproducts. \qed

$RA$ denotes the class of all relation algebras. Using the known facts that $RA$ is closed under completions [10] but $RRA$ is not [7], Maddux noted that

$$RRA \subsetneq V \subseteq RA,$$
and he showed that \( V \) contains a number of non-representable ‘Monk algebras’, so that the gap between \( \text{RRA} \) and \( V \) is substantial. In [9, problem 1.1(1)], he asked whether \( V = \text{RA} \). This was answered negatively by Andréka and Németi [1], where it is shown that in fact there are continuum-many varieties lying between \( V \) and \( \text{RA} \). That might suggest that \( V \) is ‘nearer’ to \( \text{RRA} \) than to \( \text{RA} \), but as ‘evidence’ in the other direction, we show below that \( \text{RRA} \) is not finitely axiomatisable over \( V \).

2. \( \text{RRA} \) is not finitely axiomatisable over \( V \)

It suffices to show that \( \text{RRA} \) contains an ultraproduct of algebras in \( V \setminus \text{RRA} \).

To this end, we use a construction from [6] of relation algebras from graphs.

2.1. Graphs

Graphs here are undirected and loop-free. Let \( G \) be a graph whose set of nodes is \( N \), say. Recall that a cycle of length \( l \geq 3 \) in \( G \) is a subset \( \{ v_0, \ldots, v_{l-1} \} \subseteq N \) of size \( l \) such that \( (v_i, v_{(i+1) \mod l}) \) is an edge of \( G \) for each \( i < l \). A subset \( X \subseteq N \) is said to be independent if no pair of nodes in \( X \) is an edge of \( G \). The chromatic number \( \chi(G) \) of \( G \) is the least natural number \( n \) such that \( N \) is the union of \( n \) (possibly empty) independent sets, and \( \infty \) if there is no such \( n \). It is well known (see, e.g., [2, 1.6.1]) that \( \chi(G) \leq 2 \) iff \( G \) has no cycles of odd length. We let + and \( \sum \) denote disjoint union of graphs. Then if \( G_i \) \( (i \in I) \) are graphs, \( \chi(\sum_{i \in I} G_i) \) is the least upper bound (possibly \( \infty \)) of \( \{ \chi(G_i) : i \in I \} \).

Let \( n \) be a positive integer. Let \( K_n \) be a complete graph with precisely \( n \) nodes. Clearly, \( \chi(K_n) = n \). Let \( E_n \) be a graph with \( \chi(E_n) \geq n \) and with no cycles of length at most \( n \)—finite examples were constructed by Erdős [3].

We use these graphs to construct some infinite graphs \( (G^k_n, G^k, G^\omega) \), and compute their chromatic numbers. First define

\[
G^0_n = \sum_{m \geq n} E_m,  
\]

\[
G^k_n = G^0_n + K_k \quad \text{if } 0 < k < \omega.  
\]

Since \( \{ \chi(E_m) : m \geq n \} \) is unbounded, each \( G^k_n \) has chromatic number \( \infty \).

Next, fix a non-principal ultrafilter \( D \) over \( \omega \setminus 1 \). Define the ultraproduct

\[
G^k = \prod_D \{ G^k_n : 0 < n < \omega \}, \quad \text{for each } k < \omega.  
\]

First consider the case when \( k = 0 \). Observe that \( G^0_n \) in (2.1) has no cycles of length \( \leq n \). Now for each \( l \geq 3 \), the property of having no cycles of length \( l \) can be expressed by a first-order sentence, and is true for all but finitely many \( G^0_n \). So by Loš’s theorem, \( G^0 \) in (2.3) has no cycles of length \( l \). This holds for each \( l \), so in fact \( G^0 \) has no cycles at all, and hence \( \chi(G^0) \leq 2 \).

What about \( \chi(G^k) \) for \( k > 0 \)? By standard ultraproduct considerations,

\[
G^k \cong G^0 + K_k,  
\]

so we further obtain \( \chi(G^k) = \max(\chi(G^0), \chi(K_k)) \leq \max(2, k) < \infty. \)
Finally let
\[ G^\omega = \prod_D \{G^k : 0 < k < \omega\}. \] (2.4)

For each \( m > 0 \), \( K_m \) embeds into \( G^k \) for every \( k \geq m \). It follows by Loś’s theorem that \( K_m \) embeds into \( G^\omega \), so plainly, \( \chi(G^\omega) \geq m \). This holds for every \( m \), so \( \chi(G^\omega) = \infty \).

2.2. Relation algebras from graphs

Let \( G \) be an infinite graph with set of nodes \( N \). We write \( N \times 3 \) for the set \( N \times \{0, 1, 2\} \), and \( G \times 3 \) for the graph whose set of nodes is \( N \times 3 \) and where \(((v, i), (w, j))\) is an edge of \( G \times 3 \) iff \( i \neq j \) or \((v, w)\) is an edge of \( G \). In simple words, \( G \times 3 \) consists of three disjoint copies of \( G \), with all possible edges added between the copies.

We now define a relation algebra atom structure \( \alpha(G) = (A, C, \cdot, I) \) isomorphic to one in [6, section 4] and [5, chapter 14]. We stipulate that \( A = (N \times 3) \cup \{1\} \), \( I = \{1\} \), \( \tilde{x} = x \) for every \( x \in A \), and for each \( x, y, z \in A \), \( C(x, y, z) \) holds iff

1. one of \( x, y, z \) is 1 and the other two are equal, or
2. \( \{x, y, z\} \subseteq N \times 3 \) and \( \{x, y, z\} \) is not independent (in the graph \( G \times 3 \)).

One can check that for any graphs \( G_n (0 < n < \omega) \),
\[ \alpha \left( \prod_D G_n \right) \cong \prod_D \alpha(G_n). \] (2.5)

We write \( \mathcal{A}(G) \) for the complex algebra of \( \alpha(G) \) (see [9, 1.13]). By [8, lemma 6.2], \( \mathcal{A}(G) \) is a relation algebra; of course it is atomic and its atom structure is \( \alpha(G) \). We will need the following fact about it, from [6, theorems 10–11] or [5, theorems 14.12–13] or (for \( \Leftarrow \)) [5, exercise 14.2(7)]. It uses our assumption that \( G \) is infinite.

**Fact 2.1.** \( \mathcal{A}(G) \in \text{RRA} \) if and only if \( \chi(G) = \infty \).

2.3. Why RRA is not finitely axiomatisable over \( V \)

Let \( k < \omega \). For \( 0 < n < \omega \), we saw that \( \chi(G^k_n) = \infty \), so \( \mathcal{A}(G^k_n) \in \text{RRA} \) by fact 2.1. Define the ultraproduct
\[ \mathcal{A}^k = \prod_D \{\mathcal{A}(G^k_n) : 0 < n < \omega\}. \] (2.6)

Then \( \mathcal{A}^k \in \text{RRA} \), since \( \text{RRA} \) is a variety and closed under ultraproducts. So by definition of \( V \),
\[ \mathcal{C}^k = (\mathcal{A}^k)^c \in V. \]

By Loś’s theorem, \( \mathcal{A}^k \) is atomic. So (e.g., [5, remark 2.67]) its completion \( \mathcal{C}^k \) is isomorphic to the complex algebra of the atom structure of \( \mathcal{A}^k \). This atom structure is \( \prod_D \alpha(G^k_n) \cong \alpha(\prod_D G^k_n) = \alpha(G^k) \) by (2.6, 2.5, 2.3). Hence, \( \mathcal{C}^k \cong \mathcal{A}(G^k) \). We saw that \( \chi(G^k) < \infty \), so by fact 2.1, \( \mathcal{C}^k \notin \text{RRA} \).
Finally let \( \mathcal{C} \) be the ultrapower \( \prod_{D} \{ \mathcal{C}_k : 0 < k < \omega \} \). As before, this is an atomic relation algebra with atom structure isomorphic to \( \prod_{D} \alpha(G^k) \cong \alpha(\prod_{D} G^k) = \alpha(G^\omega) \), so its completion \( \mathcal{C}^c \) is isomorphic to \( \mathfrak{A}(G^\omega) \). But \( \chi(G^\omega) = \infty \), so \( \mathfrak{A}(G^\omega) \in \text{RRA} \) by fact 2.1. Then \( \mathcal{C} \subseteq \mathcal{C}^c \subseteq \text{RRA} \), and as \( \text{RRA} \) is closed under subalgebras, we obtain \( \mathcal{C} \in \text{RRA} \).

We can now prove our main theorem.

**Theorem 2.2.** \( \text{RRA} \) is not finitely axiomatisable over \( V \).

**Proof.** We have shown that \( \mathcal{C}_k \in V \setminus \text{RRA} \) for \( k > 0 \), and \( \mathcal{C} = \prod_{D} \mathcal{C}_k \in \text{RRA} \). It follows by Loś’s theorem that \( \text{RRA} \) is not finitely axiomatisable over \( V \). \( \square \)

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