EXPRESSION OF A TENSOR COMMUTATION MATRIX IN TERMS OF THE GENERALIZED GELL-MANN MATRICES

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Abstract

We have expressed the tensor commutation matrix \( n \otimes n \) as linear combination of the tensor products of the generalized Gell-Mann matrices. The tensor commutation matrices \( 3 \otimes 2 \) and \( 2 \otimes 3 \) have been expressed in terms of the classical Gell-Mann matrices and the Pauli matrices.

Introduction

When we had worked on RAOELINA ANDRIAMBOLOLONA idea on the using tensor product in Dirac equation [1], [2] we had met the unitary matrix

\[
U_{2\otimes 2} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

This matrix is frequently found in quantum information theory [3], [4], [5] where one write, by using the Pauli matrices [3], [4], [5]

\[
U_{2\otimes 2} = \frac{1}{2} I_2 \otimes I_2 + \frac{1}{2} \sum_{i=1}^{3} \sigma_i \otimes \sigma_i
\]

with \( I_2 \) the \( 2 \times 2 \) unit matrix. We call this matrix a tensor commutation matrix \( 2 \otimes 2 \). The tensor commutation matrix \( 3 \otimes 3 \) is expressed by using the Gell-Mann matrices under the following form [6]
\[
U_{3\otimes 3} = \frac{1}{3} I_3 \otimes I_3 + \frac{1}{2} \sum_{i=1}^{8} \lambda_i \otimes \lambda_i
\]

We have to talk a bit about different types of matrices because in the generalization of the above formulas we will consider the commutation matrix as a matrix of fourth order tensor and in expressing the commutation matrices \(U_{3\otimes 2}, U_{2\otimes 3}\), at the last section, a commutation matrix will be considered as matrix of second order tensor.

\(\mathcal{M}_{m\times n} (\mathbb{C})\) denotes the set of \(m \times n\) matrices whose elements are complex numbers.

1 Tensor product of matrices

1.1 Matrices

If the elements of a matrix are considered as the components of a second order tensor, we adopt the habitual notation for a matrix, without parentheses inside, whereas if the elements of the matrix are, for instance, considered as the components of sixth order tensor, three times covariant and three times contravariant, then we represent the matrix of the following way, for example

\[
M = \begin{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
&
\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix}
&
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
&
\begin{pmatrix}
7 & 8 \\
6 & 8
\end{pmatrix}
\\
\begin{pmatrix}
3 & 2 \\
1 & 1
\end{pmatrix}
&
\begin{pmatrix}
3 & 2 \\
0 & 0
\end{pmatrix}
&
\begin{pmatrix}
3 & 2 \\
1 & 1
\end{pmatrix}
&
\begin{pmatrix}
9 & 0 \\
8 & 0
\end{pmatrix}
\\
\begin{pmatrix}
4 & 5 \\
1 & 1
\end{pmatrix}
&
\begin{pmatrix}
4 & 5 \\
0 & 0
\end{pmatrix}
&
\begin{pmatrix}
4 & 5 \\
1 & 1
\end{pmatrix}
&
\begin{pmatrix}
7 & 8 \\
6 & 6
\end{pmatrix}
\\
\begin{pmatrix}
1 & 6 \\
3 & 4
\end{pmatrix}
&
\begin{pmatrix}
1 & 6 \\
4 & 5
\end{pmatrix}
&
\begin{pmatrix}
1 & 6 \\
3 & 4
\end{pmatrix}
&
\begin{pmatrix}
5 & 4 \\
4 & 4
\end{pmatrix}
\\
\begin{pmatrix}
7 & 8 \\
5 & 6
\end{pmatrix}
&
\begin{pmatrix}
7 & 8 \\
6 & 6
\end{pmatrix}
&
\begin{pmatrix}
7 & 8 \\
5 & 6
\end{pmatrix}
&
\begin{pmatrix}
5 & 6 \\
4 & 4
\end{pmatrix}
\end{pmatrix}
\]
\[ M = \left( M_{ij}^{i_1j_1i_2j_2j_3} \right) \]

\[ i_1 i_2 i_3 = 111, 112, 121, 122, 211, 212, 221, 222, 311, 312, 321, 322 \]

row indices

\[ j_1 j_2 j_3 = 111, 112, 121, 122, 211, 212, 221, 222 \]

column indices

The first indices \( i_1 \) and \( j_1 \) are the indices of the outside parenthesis which we call the first order parenthesis; the second indices \( i_2 \) and \( j_2 \) are the indices of the next parentheses which we call the second order parentheses; the third indices \( i_3 \) and \( j_3 \) are the indices of the most interior parentheses, of this example, which we call third order parentheses. So, for instance, \( M_{121}^{321} = 5 \).

If we delete the third order parenthesis, then the elements of the matrix \( M \) are considered as the components of a forth order tensor, twice contravariant and twice covariant.

A matrix is a diagonal matrix if deleting the interior parentheses we have a habitual diagonal matrix.

A matrix is a symmetric (resp. antisymmetric) matrix if deleting the interior parentheses we have a habitual symmetric (resp. antisymmetric) matrix.

We identify one matrix to another matrix if after deleting the interior parentheses they are the same matrix.

### 1.2 Tensor product of matrices

**Definition 1.1.** Consider \( A = \left( A_{ij}^k \right) \in \mathcal{M}_{m \times n}(\mathbb{K}), \quad B = \left( B_{ij}^k \right) \in \mathcal{M}_{p \times r}(\mathbb{K}) \).

The matrix defined by

\[
A \otimes B = \begin{pmatrix}
A_1^1 B & \ldots & A_1^j B & \ldots & A_1^m B \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
A_i^1 B & \ldots & A_i^j B & \ldots & A_i^m B \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
A_n^1 B & \ldots & A_n^j B & \ldots & A_n^m B
\end{pmatrix}
\]

is called the tensor product of the matrix \( A \) by the matrix \( B \).

\[
A \otimes B \in \mathcal{M}_{mp \times nr}(\mathbb{K})
\]
\[ A \otimes B = (C_{j_1, j_2}^{i_1, i_2}) = (A_{j_1}^{i_1} B_{j_2}^{i_2}) \]

(cf. for example [3]) where, 
\( i_1 i_2 \) are row indices 
\( j_1 j_2 \) are column indices.

## 2 Generalized Gell-Mann matrices

Let us fix \( n \in \mathbb{N}, n \geq 2 \) for all continuation. The generalized Gell-Mann matrices or \( n \times n \)-Gell-Mann matrices are the traceless hermitian \( n \times n \) matrices \( \Lambda_1, \Lambda_2, \ldots, \Lambda_{n^2-1} \) which satisfy the relation \( Tr (\Lambda_i, \Lambda_j) = 2 \delta_{ij} \), for all \( i, j \in \{1, 2, \ldots, n^2 - 1\} \), where \( \delta_{ij} = \delta^{ij} = \delta_i^j \) the Kronecker symbol [7].

However, for the demonstration of the Theorem 3.2 below, denote, for \( 1 \leq i < j \leq n \), the \( C^2 \) \( n \times n \)-Gell-Mann matrices which are symmetric with all elements 0 except the \( i \)-th row \( j \)-th column and the \( j \)-th row \( i \)-th column which are equal to 1, by \( \Lambda^{(ij)} \); the \( C^2 \) \( n \times n \)-Gell-Mann matrices which are antisymmetric with all elements are 0 except the \( i \)-th row \( j \)-th column which is equal \(-i\) and the \( j \)-th row \( i \)-th column which is equal to \( i \), by \( \Lambda^{(ij)} \) and by \( \Lambda^{(d)}, 1 \leq d \leq n - 1 \), the following \((n - 1)\) \( n \times n \)-Gell-Mann matrices which are diagonal:

\[
\Lambda^{(1)} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0
\end{pmatrix}, \quad \Lambda^{(2)} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & -2 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0
\end{pmatrix},
\]
\[ \Lambda^{(n-1)} = \frac{1}{\sqrt{C_n}} \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ & \\
& & \ddots & \vdots \\
& & & 1 \\
0 & \ldots & & -(n-1)
\end{pmatrix} \]

For \( n = 2 \) we have the Pauli matrices.

## 3 Tensor commutation matrices

For \( p, q \in \mathbb{N}, p \geq 2, q \geq 2 \), we call tensor commutation matrices \( p \otimes q \)
the permutation matrix \( U_{p \otimes q} \in M_{pq \times pq}(\mathbb{C}) \) formed by 0 and 1, verifying the
property

\[ U_{p \otimes q} (a \otimes b) = b \otimes a \]

for all \( a \in M_{p \times 1}(\mathbb{C}), b \in M_{q \times 1}(\mathbb{C}). \)

Considering \( U_{p \otimes q} \) as a matrix of a second order tensor, we can construct
it by using the following rule [6].

**Rule 3.1.** Let us start in putting 1 at first row and first column, after that
let us pass into second column in going down at the rate of \( p \) rows and put
1 at this place, then pass into third column in going down at the rate of \( p \)
rows and put 1, and so on until there is only for us \( p - 1 \) rows for going down
(then we have obtained as number of 1 : \( q \)). Then pass into the next column
which is the \((q + 1)\)-th column, put 1 at the second row of this column and
repeat the process until we have only \( p - 2 \) rows for going down (then we
have obtained as number of 1 : \( 2q \)). After that pass into the next column
which is the \((2q + 2)\)-th column, put 1 at the third row of this column and
repeat the process until we have only \( p - 3 \) rows for going down (then we
have obtained as number of 1 : \( 3q \)). Continuing in this way we will have that
the element at \( p \times q \)-th row and \( p \times q \)-th column is 1. The other elements are
0.

**Theorem 3.2.** We have
Consider at first, the $C^2_n$ symmetric $n \times n$-Gell-Mann matrices which can be written

$$
\Lambda^{(ij)} = \left( \Lambda^{(ij)} \right)^l_k \begin{array}{c} l \\
1 \leq l \leq n, 1 \leq k \leq n
\end{array}
= \left( \delta^{il}_k \delta^{lj}_k \right) \begin{array}{c} l \\
1 \leq l \leq n, 1 \leq k \leq n
\end{array} + \left( \delta^{il}_k \delta^{lj}_k \right) \begin{array}{c} l \\
1 \leq l \leq n, 1 \leq k \leq n
\end{array}
= \left( \delta^{il}_k + \delta^{lj}_k \right) \begin{array}{c} l \\
1 \leq l \leq n, 1 \leq k \leq n
\end{array}
$$

Then

$$
\Lambda^{(ij)} \otimes \Lambda^{(ij)} = \left( \Lambda^{(ij)} \otimes \Lambda^{(ij)} \right)^{l_1l_2}_{k_1k_2} = \left( \delta^{il_1}_k \delta^{lj}_k + \delta^{il_2}_k \delta^{lj_2}_k \right) \left( \delta^{il_2}_k \delta^{lj_2}_k + \delta^{il_1}_k \delta^{lj_1}_k \right)
$$

$l_1l_2$ row indices, $k_1k_2$ column indices.

That is

$$
\left( \Lambda^{(ij)} \otimes \Lambda^{(ij)} \right)^{l_1l_2}_{k_1k_2} = \delta^{il_1}_k \delta^{lj_1}_k \delta^{il_2}_k \delta^{lj_2}_k + \delta^{il_1}_k \delta^{lj_1}_k \delta^{il_2}_k \delta^{lj_2}_k + \delta^{il_1}_k \delta^{lj_1}_k \delta^{il_2}_k \delta^{lj_2}_k + \delta^{il_1}_k \delta^{lj_1}_k \delta^{il_2}_k \delta^{lj_2}_k
$$

The $C^2_n$ antisymmetric $n \times n$-Gell-Mann matrices can be written

$$
\Lambda^{[ij]} = \left( \Lambda^{[ij]} \right)^l_k \begin{array}{c} l \\
1 \leq l \leq n, 1 \leq k \leq n
\end{array} = \left( -i \delta^{il}_k + i \delta^{lj}_k \right) \begin{array}{c} l \\
1 \leq l \leq n, 1 \leq k \leq n
\end{array}
$$

Then

$$
\Lambda^{[ij]} \otimes \Lambda^{[ij]} = \left( \Lambda^{[ij]} \otimes \Lambda^{[ij]} \right)^{l_1l_2}_{k_1k_2}
$$

$$
\left( \Lambda^{[ij]} \otimes \Lambda^{[ij]} \right)^{l_1l_2}_{k_1k_2} = -\delta^{il_1}_k \delta^{lj_1}_k \delta^{il_2}_k \delta^{lj_2}_k + \delta^{il_1}_k \delta^{lj_1}_k \delta^{il_2}_k \delta^{lj_2}_k + \delta^{il_1}_k \delta^{lj_1}_k \delta^{il_2}_k \delta^{lj_2}_k - \delta^{il_1}_k \delta^{lj_1}_k \delta^{il_2}_k \delta^{lj_2}_k
$$
and
\[
\sum_{1 \leq i < j \leq n} (\Lambda^{ij} \otimes \Lambda^{ij})_{k_1 k_2}^{l_1 l_2} + \sum_{1 \leq i < j \leq n} (\Lambda^{ij} \otimes \Lambda^{ij})_{k_1 k_2}^{l_1 l_2} = 2 \sum_{1 \leq i < j \leq n} (\delta_{i1}^d \delta_{k_1}^j \delta_{j2}^i k_2 + \delta_{j1}^d \delta_{k_1}^i \delta_{i2}^j k_2)
\]
\[
= 2 \sum_{i \neq j} \delta_{i1}^d \delta_{k_1}^j \delta_{j2}^i k_2
\]

the \(l_1 l_2\)-th row, \(k_1 k_2\)-th column of the matrix
\[
\sum_{1 \leq i < j \leq n} \Lambda^{ij} \otimes \Lambda^{ij} + \sum_{1 \leq i < j \leq n} \Lambda^{ij} \otimes \Lambda^{ij}.
\]

Now, consider the diagonal \(n \times n\)-Gell-Mann matrices. Let \(d \in \mathbb{N}, \ 1 \leq d \leq n - 1\),
\[
\Lambda^{(d)} = \frac{1}{\sqrt{C_{d+1}^2}} \left( \delta_{k}^d \sum_{p=1}^{d} \delta_{k}^p - d \delta_{k}^d \delta_{k}^{d+1} \right)
\]
and the \(l_1 l_2\)-th row, \(k_1 k_2\)-th of the matrix \(\Lambda^{(d)} \otimes \Lambda^{(d)}\) is
\[
(\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} = \frac{1}{C_{d+1}^2} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} \left( \sum_{q=1}^{d} \sum_{p=1}^{d} \delta_{k_1}^q \delta_{k_2}^p \right)
\]
\[- \frac{1}{C_{d+1}^2} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} \left( d \delta_{k_2}^{d+1} \sum_{p=1}^{d} \delta_{k_1}^p \right)
\]
\[- \frac{1}{C_{d+1}^2} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} \left( d \delta_{k_1}^{d+1} \sum_{p=1}^{d} \delta_{k_2}^p \right)
\]
\[+ \frac{1}{C_{d+1}^2} \delta_{k_1}^{l_1} \delta_{k_2}^{l_2} \left( d^2 \delta_{k_1}^{d+1} \delta_{k_2}^{d+1} \right)\]
\( \Lambda^{(d)} \otimes \Lambda^{(d)} \) is a diagonal matrix, so all that we have to do is to calculate the elements on the diagonal where \( l_1 = k_1 \) and \( l_2 = k_2 \). Then,

\[
\sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{l_1 l_2}^{k_1 k_2} = \sum_{d=1}^{n-1} \frac{1}{C^2_{d+1}} \left( \sum_{q=1}^{d} \delta_{k_1}^{q} \right) \left( \sum_{p=1}^{d} \delta_{k_2}^{p} \right)
- \sum_{d=1}^{n-1} \frac{1}{C^2_{d+1}} d \delta_{k_2}^{d+1} \sum_{p=1}^{d} \delta_{k_1}^{p}
- \sum_{d=1}^{n-1} \frac{1}{C^2_{d+1}} d \delta_{k_1}^{d+1} \sum_{p=1}^{d} \delta_{k_2}^{p}
+ \sum_{d=1}^{n-1} \frac{1}{C^2_{d+1}} d^2 \delta_{k_1}^{d+1} \delta_{k_2}^{d+1}
\]

the \( l_1 l_2 \)-th row, \( k_1 k_2 \)-th column of the diagonal matrix \( \sum_{d=1}^{n-1} \Lambda^{(d)} \otimes \Lambda^{(d)} \) with \( l_1 = k_1 \) and \( l_2 = k_2 \).

Let us distinguish two cases.

1st case : \( k_1 \neq 1 \) or \( k_2 \neq 1 \)
   case 1 : \( k_1 \neq k_2 \)
   If \( k_1 < k_2 \),

\[
\sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{l_1 l_2}^{k_1 k_2} = \sum_{d=k_2}^{n-1} \frac{1}{C^2_{d+1}} - \frac{k_2 - 1}{C^2_{k_2}}
= 2 \left[ \sum_{d=k_2}^{n-1} \left( \frac{1}{d} - \frac{1}{d+1} \right) - \frac{1}{k_2} \right]
= \frac{2}{n}
\]
Similarly, if \( k_1 > k_2 \),
\[
\sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} = -\frac{2}{n}
\]

case 2: \( k_1 = k_2 \neq 1 \)
\[
\sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} = \frac{1}{C^2_{d+1}} + \frac{(k_2 - 1)^2}{C^2_{k_2}}
\]
\[
= \frac{2}{k_2} - \frac{2}{n} + \frac{(k_2 - 1)^2}{C^2_{k_2}}
\]
\[
= 2 - \frac{2}{n}
\]

2nd case : \( k_1 = k_2 = 1 \)
\[
\sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2} = \sum_{d=1}^{n-1} \frac{1}{C^2_{d+1}} = 2 - \frac{2}{n}
\]

We can condense these cases in one formula
\[
\left(\Lambda^{(d)} \otimes \Lambda^{(d)}\right)_{k_1 k_2}^{l_1 l_2} = -\frac{2}{n} \delta_{k_1}^{i_1} \delta_{k_2}^{i_2} + 2 \sum_{i=1}^{n} \delta^{i_1 i} \delta_{k_1}^{i_2} \delta_{k_2}^{i_1}
\]
which yields the diagonal of the diagonal matrix \( \sum_{d=1}^{n} \Lambda^{(d)} \otimes \Lambda^{(d)} \).

For all the \( n \times n \)- Gell-Mann matrices we have

\[
\sum_{1 \leq i < j \leq n} (\Lambda^{(ij)} \otimes \Lambda^{(ij)})_{k_1 k_2}^{l_1 l_2} + \sum_{1 \leq i < j \leq n} (\Lambda^{[ij]} \otimes \Lambda^{[ij]})_{k_1 k_2}^{l_1 l_2} + \sum_{d=1}^{n-1} (\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1 k_2}^{l_1 l_2}
\]
\[
= -\frac{2}{n} \delta_{k_1}^{i_1} \delta_{k_2}^{i_2} + 2 \sum_{i=1}^{n} \delta^{i_1 i} \delta_{k_1}^{i_2} \delta_{k_2}^{i_1} + 2 \sum_{i \neq j} \delta^{i_1 j} \delta_{k_1}^{i_2} \delta_{k_2}^{i_1}
\]
\[
= -\frac{2}{n} \delta_{k_1}^{i_1} \delta_{k_2}^{i_2} + 2 \sum_{j=1}^{n} \sum_{i=1}^{n} \delta^{i_1 j} \delta_{k_1}^{i_2} \delta_{k_2}^{i_1}
\]
\[
= -\frac{2}{n} \delta_{k_1}^{i_1} \delta_{k_2}^{i_2} + 2 \delta_{k_1}^{i_1} \delta_{k_2}^{i_2}
\]
for all \( l_1, l_2, k_1, k_2 \in \{1, 2, \ldots, n\} \).

Hence, by using (4.1)
\[
\sum_{i=1}^{n^2-1} \Lambda_i \otimes \Lambda_i = -\frac{2}{n} \mathbf{I}_n \otimes \mathbf{I}_n + 2 \mathbf{U}_n \otimes\mathbf{n}
\]
and the theorem is proved.

4 Expression of $U_{3\otimes 2}$ and $U_{2\otimes 3}$

In this section we derive formulas for $U_{3\otimes 2}$ and $U_{2\otimes 3}$, naturally in terms of the Pauli matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

and the Gell-Mann matrices

$$
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},
$$

$$
\lambda_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix},
$$

$$
\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
$$

For $r \in \mathbb{N}^*$, define $E_{ij}^{(r)}$ the elementary $r \times r$ matrix whose elements are zeros except the $i$-th row and $j$-th column which is equal to 1. We construct $U_{3\otimes 2}$ by using the Rule 3.1 and then we have

$$
U_{3\otimes 2} = E_{11}^{(6)} + E_{23}^{(6)} + E_{35}^{(6)} + E_{42}^{(6)} + E_{54}^{(6)} + E_{66}^{(6)}
$$

Take

$$
E_{11}^{(6)} = E_{11}^{(3)} \otimes E_{11}^{(2)}
$$

Let

$$
E_{11}^{(3)} = \alpha_0 I_3 + \alpha_3 \lambda_3 + \alpha_8 \lambda_8
$$

with $\alpha_0, \alpha_3, \alpha_8 \in \mathbb{C}$, then
\[\alpha_0 = \frac{1}{3}, \, \alpha_3 = \frac{1}{2}, \, \alpha_8 = \frac{\sqrt{3}}{6}\]

and

\[E_{11}^{(3)} = \frac{1}{3} I_3 + \frac{1}{2} \lambda_3 + \frac{\sqrt{3}}{6} \lambda_8\]

Let

\[E_{11}^{(2)} = \beta_0 I_2 + \beta_3 \sigma_3\]

with \(\beta_0, \beta_3 \in \mathbb{C}\), then

\[\beta_0 = \frac{1}{2}, \, \beta_3 = \frac{1}{2}\]

and

\[E_{11}^{(2)} = \frac{1}{2} I_2 + \frac{1}{2} \sigma_3\]

So we have

\[E_{11}^{(6)} = \left(\frac{1}{3} I_3 + \frac{1}{2} \lambda_3 + \frac{\sqrt{3}}{6} \lambda_8\right) \otimes \left(\frac{1}{2} I_2 + \frac{1}{2} \sigma_3\right)\]

By the similar way, we have

\[E_{23}^{(6)} = \left(\frac{1}{2} \lambda_1 + \frac{i}{2} \lambda_2\right) \otimes \left(\frac{1}{2} \sigma_1 - \frac{i}{2} \sigma_2\right)\]

\[E_{35}^{(6)} = \left(\frac{1}{2} \lambda_6 + \frac{i}{2} \lambda_7\right) \otimes \left(\frac{1}{2} I_2 + \frac{1}{2} \sigma_3\right)\]

\[E_{42}^{(6)} = \left(\frac{1}{2} \lambda_1 - \frac{i}{2} \lambda_2\right) \otimes \left(\frac{1}{2} I_2 - \frac{1}{2} \sigma_3\right)\]

\[E_{34}^{(6)} = \left(\frac{1}{2} \lambda_6 - \frac{i}{2} \lambda_7\right) \otimes \left(\frac{1}{2} \sigma_1 + \frac{i}{2} \sigma_2\right)\]

\[E_{66}^{(6)} = \left(\frac{1}{3} I_3 - \frac{\sqrt{3}}{3} \lambda_8\right) \otimes \left(\frac{1}{2} I_2 - \frac{1}{2} \sigma_3\right)\]
Hence

\[
U_{3\otimes 2} = \left( \frac{1}{3} I_3 + \frac{1}{2} \lambda_3 + \frac{\sqrt{3}}{6} \lambda_8 \right) \otimes \left( \frac{1}{2} I_2 + \frac{1}{2} \sigma_3 \right) \\
+ \left( \frac{1}{2} \lambda_1 + \frac{i}{2} \lambda_2 \right) \otimes \left( \frac{1}{2} \sigma_1 - \frac{i}{2} \sigma_2 \right) \\
+ \left( \frac{1}{2} \lambda_6 + \frac{i}{2} \lambda_7 \right) \otimes \left( \frac{1}{2} I_2 + \frac{1}{2} \sigma_3 \right) \\
+ \left( \frac{1}{2} \lambda_1 - \frac{i}{2} \lambda_2 \right) \otimes \left( \frac{1}{2} I_2 - \frac{1}{2} \sigma_3 \right) \\
+ \left( \frac{1}{2} \lambda_6 - \frac{i}{2} \lambda_7 \right) \otimes \left( \frac{1}{2} \sigma_1 + \frac{i}{2} \sigma_2 \right) \\
+ \left( \frac{1}{3} I_3 - \frac{\sqrt{3}}{3} \lambda_8 \right) \otimes \left( \frac{1}{2} I_2 - \frac{1}{2} \sigma_3 \right)
\]

From analogous way,

\[
U_{2\otimes 3} = \left( \frac{1}{2} I_2 + \frac{1}{2} \sigma_3 \right) \otimes \left( \frac{1}{3} I_3 + \frac{1}{2} \lambda_3 + \frac{\sqrt{3}}{6} \lambda_8 \right) \\
+ \left( \frac{1}{2} \sigma_1 + \frac{i}{2} \sigma_2 \right) \otimes \left( \frac{1}{2} \lambda_1 - \frac{i}{2} \lambda_2 \right) \\
+ \left( \frac{1}{2} I_2 + \frac{1}{2} \sigma_3 \right) \otimes \left( \frac{1}{2} \lambda_6 - \frac{i}{2} \lambda_7 \right) \\
+ \left( \frac{1}{2} I_2 - \frac{1}{2} \sigma_3 \right) \otimes \left( \frac{1}{2} \lambda_1 + \frac{i}{2} \lambda_2 \right) \\
+ \left( \frac{1}{2} \sigma_1 - \frac{i}{2} \sigma_2 \right) \otimes \left( \frac{1}{2} \lambda_6 + \frac{i}{2} \lambda_7 \right) \\
+ \left( \frac{1}{2} I_2 - \frac{1}{2} \sigma_3 \right) \otimes \left( \frac{1}{3} I_3 - \frac{\sqrt{3}}{3} \lambda_8 \right)
\]

We can develop these formulas in employing the distributivity of the tensor product.

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References

[1] RAKOTONIRINA.C, Thèse de Doctorat de Troisième Cycle de Physique Théorique, Université d’Antananarivo, Madagascar, (2003), unpublished.

[2] WANG.R.P, arXiv : [hep-ph/0107184].

[3] FUJII.K,arXiv : [quant-ph/0112090] prepared for 10th Numazu Meeting on Integral System, Noncommutative Geometry and Quantum theory, Numazu Shizuoka Japan, 7-9 Mai 2002.

[4] FADDEV.L.D, Int.J.Mod.Phys.A, Vol.10, No 13, May,1848 (1995).

[5] FRANK VERSTRAETE, Thèse de Doctorat, Katholieke Universiteit Leuven, (2002).

[6] RAKOTONIRINA.C, arXiv : [math.GM/0508053].

[7] NARISON.S, Spectral sum Rules, World Scientific Lecture Notes in Physics - Vol.26, 501, (1989).