Robust semiparametric inference for polytomous logistic regression with complex survey design

Elena Castilla · Abhik Ghosh · Nirian Martin · Leandro Pardo

Abstract
Analyzing polytomous response from a complex survey scheme, like stratified or cluster sampling is very crucial in several socio-economics applications. We present a class of minimum quasi weighted density power divergence estimators for the polytomous logistic regression model with such a complex survey. This family of semiparametric estimators is a robust generalization of the maximum quasi weighted likelihood estimator exploiting the advantages of the popular density power divergence measure. Accordingly robust estimators for the design effects are also derived. Using the new estimators, robust testing of general linear hypotheses on the regression coefficients are proposed. Their asymptotic distributions and robustness properties are theoretically studied and also empirically validated through a numerical example and an extensive Monte Carlo study.

Keywords
Cluster sampling · Design effect · Minimum quasi weighted DPD estimator · Polytomous logistic regression model · Pseudo minimum phi-divergence estimator · Quasi-likelihood · Robustness

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Nirian Martin
nimartin@ucm.es

1 Interdisciplinary Mathematics Institute and Department of Statistics and O.R, Complutense University of Madrid, 28040 Madrid, Spain

2 Interdisciplinary Statistical Research Unit, Indian Statistical Institute, 700108 Kolkata, India

3 Interdisciplinary Mathematics Institute and Department of Financial, Actuarial Economics and Statistics, Complutense University of Madrid, 28003 Madrid, Spain
1 Introduction

In several real-life applications, we come across data collected through a complex survey scheme, like stratified or cluster sampling procedure, etc., rather than the simple random sampling. Such situations commonly arise in large scale data collection, for example, within several states of a country or even among different countries. Suitable statistical methods are required to analyze these data by taking care of their stratified structure; this is because there often exist several inter and intra-class correlations within such stratification and ignoring them might lead to erroneous inference. Further, in such a complex survey, if stratified observations are collected on some categorical responses having two or more mutually exclusive unordered categories along with some related covariates, thorough inference about their relationship is of up-most interest for insight generation and policy making. Polytomous logistic regression (PLR) model is a useful and popular tool for this purpose of modeling categorical responses with associated covariates. However, most classical literature on PLR deals exclusively with the cases of simple random sampling scheme (e.g. McCullagh 1980; Lesaffre and Albert 1989; Agresti 2002; Gupta et al. 2006, 2008). The application of the PLR model under complex survey settings can be found, e.g., in Binder (1983), Roberts et al. (1987), Morel (1989), Morel and Neerchal (2012) and Castilla et al. (2018); except the last one, they are all based on the quasi maximum likelihood approach.

Even though the maximum quasi weighted likelihood estimator is the base of most existing literature on logistic models under complex survey designs, this is known to be highly non-robust in the presence of outliers in the data. In practice, with such a complex survey design, it is quite natural to have some outlying observations that make the likelihood based inference highly unstable (Chambers 1986; Tambay 1988). So, we often may need to make additional efforts to find and discard the outliers from the data before their analyses. A robust method providing stable solution even in the presence of outliers will be really helpful and more efficient in practice. The cited work by Castilla et al. (2018) has developed an alternative minimum divergence estimator based on $\phi$-divergences (Pardo 2005), but the important issue of robustness is still ignored there.

In this work, based on a minimum quasi weighted divergence approach, we develop a robust estimator under the PLR model with a complex survey. In particular, we exploit the nice properties of the density power divergence (DPD) of Basu et al. (1998). This measure has become very popular in recent literature for yielding highly robust and efficient estimators under various statistical models; see, for example, Ghosh and Basu (2013, 2016, 2018) and, in particular, the recent paper by Castilla et al. (2019). We also derive the asymptotic distributions of the proposed DPD based estimator for the PLR model under complex surveys as well as the underlying design effects. Based on our new estimators, we also develop a new family of Wald-type tests for testing general linear hypotheses about the parameters of the PLR along with its asymptotic properties. The important robustness properties of the proposed estimators and Wald-type tests are studied theoretically through the influence function analysis and also numerically through extensive simulation and real data applications.
At this point, it is important to clarify the innovations in our present contribution here over the previous studies by Castilla et al. (2018, 2019) on the PLR model. Castilla et al. (2018) proposed pseudo minimum phi-divergence estimators for PLR models with complex sample design, focusing mainly on the efficiency of the resulting estimators. In particular, the Cressie–Read divergence estimator was found to yield good efficiency even for small sample sizes. However, neither the robustness issue, nor the hypothesis testing problems were considered in Castilla et al. (2018). In order to get robust estimators for the PLR model, the DPD measures were considered in Castilla et al. (2019) both to develop robust estimators and associated Wald-type tests but exclusively under the simple random sampling scheme. In this paper we extend the robust inference procedures and results presented in Castilla et al. (2019) for the complex sample designs, with the aim of combining the two important (but often neglected) features of inference, namely recognition of the sampling design and robustness against outliers. This required non-trivial extensions for both the development of estimation procedure as well as the theoretical derivations due to the dependence structures of stratified complex sampling, which are successfully solved in our present work. Additionally, we also empirically compare the proposed estimators with those considered in Castilla et al. (2018) to show the benefit of the present proposal to address the robustness issues under the complex survey settings. Its relations to other existing robust methods for finite populations are also discussed towards the end of the paper.

The rest of the paper is organized as follows. We first start with the mathematical description of the PLR model with complex survey set-up and a brief discussion is given about the maximum quasi weighted likelihood estimator of the underlying parameters in Sect. 2. Thereafter, we introduce the class of new robust parameter estimates for the PLR model with complex survey by minimizing a suitably defined DPD measure in Sect. 3; the asymptotic distributional results are also described there. Robust Wald-type test procedures based on our new estimators are discussed in Sect. 4. In Sect. 5, we theoretically study the robustness properties through the influence function analysis. After presenting two illustrative real data examples in Sect. 6, an extensive simulation study is presented in Sect. 7 to investigate the performance of our proposed estimators and tests. The paper ends with some concluding remarks and discussions in Sect. 8. For brevity in presentation, proofs of all the results, as well as some of the theoretical and empirical results, are presented in the Online Supplementary Material.

2 Maximum quasi weighted likelihood estimator

Let us assume that the whole population is partitioned into \( H \) distinct strata and the data consist of \( n_h \) clusters in stratum \( h \) for each \( h = 1, \ldots, H \). Further, for each cluster \( i = 1, \ldots, n_h \) in the stratum \( h \), we have observed the values of a categorical response variable \( (Y) \) for \( m_{hi} \) units. Assuming \( Y \) has \((d + 1)\) categories, we denote these observed responses by \((d + 1)\)-dimensional classification vectors

\[
y_{hij} = \left( y_{hij1}, \ldots, y_{hij,d+1} \right)^T, \quad h = 1, \ldots, H, \quad i = 1, \ldots, n_h, \quad j = 1, \ldots, m_{hi},
\]

with \( y_{hijr} = 1 \) and \( y_{hijl} = 0 \) for \( l \in \{1, \ldots, d + 1\} - \{r\} \) if the \( j \)th unit selected from the \( i \)th cluster of the \( h \)th stratum falls in the \( r \)th category. We also have data on
(k + 1) explanatory variables which are common for all individuals in the ith cluster of the hth stratum (very common with dummy or qualitative explanatory variables) to be denoted as $x_{hi} = (x_{hi0}, x_{hi1}, \ldots, x_{hik})^T$; the first one $x_{hi0} = 1$ is associated with the intercept. Let us denote the sampling weight from the ith cluster of the hth stratum by $w_{hi}$. For each $i$, $h$ and $j$, the expectation of the rth element of the random variable $Y_{hij} = (Y_{hij1}, \ldots, Y_{hij,d+1})^T$, corresponding to the realization $y_{hij}$, is given by the PLR model

$$
 \pi_{hir}(\beta) = E[Y_{hijr}|x_{hi}] = \Pr(Y_{hijr} = 1|x_{hi})
$$

$$
= \begin{cases} 
\frac{\exp(x_{hi}^T \beta_r)}{1 + \sum_{l=1}^{d} \exp(x_{hi}^T \beta_l)}, & r = 1, \ldots, d \\
\frac{1}{1 + \sum_{l=1}^{d} \exp(x_{hi}^T \beta_l)}, & r = d + 1
\end{cases}, (2)
$$

with $\beta_r = (\beta_{r0}, \beta_{r1}, \ldots, \beta_{rk})^T \in \mathbb{R}^{k+1}$, $r = 1, \ldots, d$. Note that, the expectation of $Y_{hij}$ does not depend on the unit number $j$ (homogeneity), which is not a strong assumption as we generally have random sampling with the clusters in each stratum.

Let $\pi_{hi}(\beta)$ denote the $(d + 1)$-dimensional probability vector with the elements given in (2), i.e.,

$$
\pi_{hi}(\beta) = (\pi_{hi1}(\beta), \ldots, \pi_{hi,d+1}(\beta))^T. (3)
$$

Then, the associated parameter space for (2) is given by $\Theta = \{\beta = (\beta_1^T, \ldots, \beta_d^T)^T, \beta_r = (\beta_{r0}, \ldots, \beta_{rk})^T \in \mathbb{R}^{k+1}, \, r = 1, \ldots, d\} = \mathbb{R}^{d(k+1)}$.

Originally defined by Wedderburn (1974), the quasi-likelihood method is a widely used method for modeling data exhibiting overdispersion. The quasi-loglikelihood function is constructed without a complete distributional knowledge, only through the mean and the variance of the sampling units, all the clusters in all the strata. That is, there are $n = \sum_{h=1}^{H} n_h$ independent and non-homogenous clusters in the PLR model with complex sampling within each cluster, with the first two moments obtained from (2). It is a semiparametric method, since it only specifies the first two multivariate moments of $Y_{hij}$. Let $f_\beta(y_{hi}|x_{hi})$ be the probability mass function of $Y_{hij}$ when $\pi_{hi}(\beta)$ is modeled by the PLR model with complex sampling. Since the support of $Y_{hij}$, say $Y_{hi}$, is the set of column vectors of identity matrix $I_{d+1}$, it holds

$$
f_\beta(y_{hi}|x_{hi}) = \pi_{hi}^T(\beta) y_{hi} = \sum_{l=1}^{d} \pi_{hil}(\beta) y_{hl},
$$

$$
\log f_\beta(y_{hi}|x_{hi}) = \log (\pi_{hi}^T(\beta) y_{hi}) = \sum_{l=1}^{d} \log \pi_{hil}(\beta) y_{hl} = \log \pi_{hi}^T(\beta) y_{hi},
$$

where we use the notation $\log \pi_{hi}(\beta) = (\log \pi_{hi1}(\beta), \ldots, \log \pi_{hi,d+1}(\beta))^T$.

It is important to be aware that the correct loglikelihood should be $\ell(\beta) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} \log f_\beta(y_{hi}|x_{hi})$, where $y_{hi} = (y_{hi1}, \ldots, y_{himhi})^T$. Under homogeneity assumption within the clusters, since $f_\beta(y_{hi}|x_{hi})$ is unknown, the quasi loglikelihood, $\ell(\beta)$, considers an approximation of the likelihood within each cluster to be the one associated with i.i.d. multivariate observations, i.e.
\[
\log f_{\beta}(y_{hi} | x_{hi}) \overset{\text{def}}{=} \sum_{j=1}^{m_{hi}} \log f_{\beta}(y_{hi j} | x_{hi}),
\]

(4)

and thus

\[
\ell(\beta) = \sum_{h=1}^{H} \sum_{i=1}^{n_{h}} \log \pi_{hi}^{T}(\beta) y_{hi j} = \sum_{h=1}^{H} \sum_{i=1}^{n_{h}} \log \pi_{hi}^{T}(\beta) \hat{y}_{hi},
\]

where \( \hat{y}_{hi} = 1_{m_{hi}}^{T} y_{hi} = \sum_{j=1}^{m_{hi}} y_{hi j} \) and \( \hat{y}_{hi} = (\hat{y}_{hi 1}, \ldots, \hat{y}_{hi,d+1})^{T} \) denotes the realization value of counts in the \( i \)th cluster of the \( h \)th stratum.

An important feature of the quasi loglikelihood is that the marginal distributions of \( Y_{hi j} \) are completely known but the components of \( Y_{hi} \), jointly, might be correlated. This means that distribution of their total, \( \hat{Y}_{hi} \), might be also unknown, but the expectation is obtained as the total of the expectation of \( Y_{hi j}, j = 1, \ldots, m_{hi} \). The most common assumption is to consider that \( \hat{Y}_{hi} \) has a multinomial sampling scheme, which means that \( Y_{hi j}, j = 1, \ldots, m_{hi} \) are independent random variables and

\[
\Sigma_{hi} = \Sigma_{hi}(\beta) = m_{hi} \Delta(\pi_{hi}(\beta)),
\]

(5)

where \( \Delta(\pi_{hi}(\beta)) = \text{diag}(\pi_{hi}(\beta)) - \pi_{hi}(\beta) \pi_{hi}^{T}(\beta) \). Since (4) is not an approximation, the term “quasi” should be dropped. A weaker assumption is to consider, with an overdispersion parameter \( \nu_{hi} = 1 + \rho_{hi}^{2}(m_{hi} - 1) \), that \( \hat{Y}_{hi} \) has a multinomial sampling scheme, which means that \( Y_{hi j}, j = 1, \ldots, m_{hi} \) are correlated random variables (\( \text{Cor}(Y_{hia}, Y_{hib}) = \rho_{hi}^{2}, a \neq b, a, b \in \{1, \ldots, m_{hi}\} \)) and

\[
\Sigma_{hi} = \Sigma_{hi}(\nu_{hi}, \beta) = \nu_{hi} m_{hi} \Delta(\pi_{hi}(\beta)),
\]

(6)

but the distribution of \( \hat{Y}_{hi} \) is in principle unspecified. Distributions such as Dirichlet Multinomial, Random Clumped and \( m \)-inflated could belong to this family of unspecified distributions (see Morel and Neerchal (2012), Raim et al. (2015) and Alonso-Revenga et al. (2017) for details). The weakest assumption is to consider that \( \hat{Y}_{hi} \) has an unknown distribution, with \( Y_{hi j}, j = 1, \ldots, m_{hi} \) being possibly correlated but with no specific pattern. It is worth of mentioning that \( \nu_{hi} \) plays here a role of nuisance parameter and it is possible to consider a model with additional nuisance parameters, which are more complex than (6) but simpler than the option of completely unknown distribution for \( \hat{Y}_{hi} \) (see Morel and Koehler (1995) for details).

Taking into account weights for each cluster, \( w = (w_{11}, \ldots, w_{Hn_{h}})^{T} \), the quasi weighted loglikelihood is defined as

\[
\ell(\beta, w) = \sum_{h=1}^{H} \sum_{i=1}^{n_{h}} w_{hi} \log \pi_{hi}^{T}(\beta) \hat{y}_{hi}.
\]

(7)
The maximum quasi weighted likelihood estimator of \( \beta \), say \( \hat{\beta}_P \), is obtained by maximizing \( \ell(\beta, w) \), given in (7), with respect to \( \beta \). The corresponding estimating equation is then given by

\[
\frac{H}{\sum_{h=1}^{H} \sum_{i=1}^{n_h}} \sum_{h=1}^{n_h} w_{hi} \frac{\partial \pi^T_{hi}(\beta)}{\partial \beta} \text{diag}^{-1}(\pi_{hi}(\beta)) \left[ \hat{Y}_{hi} - m_{hi} \pi_{hi}(\beta) \right] = 0_{d(k+1)},
\]

with

\[
\frac{\partial \pi^T_{hi}(\beta)}{\partial \beta} = \Delta^*(\pi_{hi}(\beta)) \otimes x_{hi}, \quad \Delta^*(\pi_{hi}(\beta)) = (I_d, 0_d) \Delta(\pi_{hi}(\beta)).
\]

The system of Eq. (8) can be written as \( u(\beta) = 0_{d(k+1)} \), where

\[
u(\beta) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} u_{hi}(\beta, x_{hi}), \quad u(\beta, x_{hi}) = w_{hi} \left[ \hat{Y}_{hi} - m_{hi} \pi_{hi}(\beta) \right] \otimes x_{hi},
\]

with superscript * denoting the vector (matrix) obtained by deleting the last row from the initial vector (matrix); thus \( \pi^*_{hi}(\beta) = (\pi_{h1}(\beta), \ldots, \pi_{hid}(\beta))^T \) and \( \hat{Y}^*_{hi} = (\hat{Y}_{h1}, \ldots, \hat{Y}_{hid})^T \). For the derivation of (9) from (8) see the Supplementary Material; additional details can be found in Morel (1989).

3 The minimum quasi weighted density power divergence estimators

Let \( f_\beta(y_{hij} | x_{hi}) \) be the probability mass function of \( Y_{hij} | x_{hi} \) as defined in the previous section, \( g(y_{hij} | x_{hi}) \) denote the unknown true probability mass function of \( Y_{hij} | x_{hi} \) and \( \mathbb{Y}_{hi} \) is their support. The DPD for tuning parameter \( \lambda > 0 \) based on the probability mass functions of a single observation of the sample, between \( f_\beta(y_{hij} | x_{hi}) \) and \( g(y_{hij} | x_{hi}) \), is given by

\[
d_\lambda^*(g(y_{hij} | x_{hi}), f_\beta(y_{hij} | x_{hi}))
= \int_{\mathbb{Y}_{hi}} \left( f_\beta^{\lambda+1}(y | x_{hi}) - \frac{\lambda + 1}{\lambda} f_\beta^{\lambda}(y | x_{hi}) g(y | x_{hi}) + \frac{1}{\lambda} g^{\lambda+1}(y | x_{hi}) \right) dy
= \int_{\mathbb{Y}_{hi}} f_\beta^{\lambda}(y | x_{hi}) dF(y | x_{hi}) - \frac{\lambda + 1}{\lambda} \int_{\mathbb{Y}_{hi}} f_\beta^{\lambda}(y | x_{hi}) dG(y | x_{hi}) + K
= d_\lambda^*(g(y_{hij} | x_{hi}), f_\beta(y_{hij} | x_{hi})) + K,
\]

with \( K \) being a constant not dependent on \( \beta \). \( F(y | x_{hi}) \) and \( G(y | x_{hi}) \) the distribution functions in correspondence with the densities \( f(y | x_{hi}) \) and \( g(y | x_{hi}) \) respectively, and

\[
d_\lambda^*(g(y_{hij} | x_{hi}), f_\beta(y_{hij} | x_{hi})) = E[f_\beta^{\lambda}(Y_{hij} | x_{hi})] \frac{\lambda + 1}{\lambda} \int_{\mathbb{Y}_{hi}} f_\beta^{\lambda}(y | x_{hi}) dG(y | x_{hi})
\]
the kernel of \( d_\lambda (g(y_{hij} | x_{hi}), f_\beta (y_{hij} | x_{hi})) \). In practice, since \( G \) is unknown, it must be estimated from the sample which is, in this framework, a single individual so that

\[
d_\lambda^* (\hat{g}(y_{hij} | x_{hi}), f_\beta (y_{hij} | x_{hi})) = E [f_\beta^*(Y_{hij} | x_{hi})] - \frac{\lambda + 1}{\lambda} f_\beta^*(y_{hij} | x_{hi}).
\]

Based on Ghosh and Basu (2013), the “kernel of the ordinary DPD” between the probability mass functions \( \hat{g}(y_{hij} | x_{hi}) \) and \( f_\beta (y_{hij} | x_{hi}) \) for the whole sample is defined as a total discrepancy given by

\[
d_\lambda^* (\hat{g}, f_\beta) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} \sum_{j=1}^{m_{hi}} \left( E \left[ f_\beta^*(Y_{hij} | x_{hi}) \right] - \frac{\lambda + 1}{\lambda} f_\beta^*(y_{hij} | x_{hi}) \right), \quad \text{for } \lambda > 0.
\]

Since \( Y_{hi} \) is the set of column vectors of identity matrix \( I_{d+1} \), it holds

\[
E \left[ f_\beta^*(Y_{hij} | x_{hi}) \right] = E \left[ \pi_{hi}^{\lambda, T} (\beta) Y_{hij} \right] = \sum_{l=1}^{d+1} \pi_{hili}^{\lambda+1} (\beta).
\]

In the present paper, for the first time, we define the “kernel of the quasi weighted DPD” between the probability mass functions \( \hat{g}(y_{hij} | x_{hi}) \) and \( f_\beta (y_{hij} | x_{hi}) \) for the whole sample as a weighted sum

\[
d_\lambda^* (\hat{g}, f_\beta, w) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} \sum_{j=1}^{m_{hi}} w_{hi} \left( E \left[ f_\beta^*(Y_{hij} | x_{hi}) \right] - \frac{\lambda + 1}{\lambda} f_\beta^*(y_{hij} | x_{hi}) \right), \quad \text{for } \lambda > 0,
\]

whose expression for the PLR model with complex sampling is given by

\[
d_\lambda^* (\hat{g}, f_\beta, w) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} \pi_{hi}^{\lambda, T} (\beta) \left( m_{hi} \pi_{hi} (\beta) - \frac{\lambda + 1}{\lambda} \hat{y}_{hi} \right), \quad \text{for } \lambda > 0.
\]

Based on (10) the minimum quasi weighted DPD estimator is formally defined as follows.

**Definition 1** The minimum quasi weighted DPD estimator of \( \beta \), say \( \hat{\beta}_{\lambda, Q} \), is defined as

\[
\hat{\beta}_{\lambda, Q} = \arg \min_{\beta \in \mathbb{R}^{d(d+1)}} d_\lambda^* (\hat{g}, f_\beta, w).
\]

At the particular choice \( \lambda \to 0 \), the DPD measure, defined as the limit of (10) coincides (in limit) with the weighted quasi-loglikelihood, \( \ell(\beta, w) \), given in (7); thus the minimum quasi weighted DPD estimator of \( \beta \) at \( \lambda = 0 \) coincides with the maximum weighted quasi likelihood estimator. With the same philosophy, the following result generalizes \( u(\beta, x_{hi}) \), given in (9), which plays an important role for the derivation of the asymptotic distribution of \( \hat{\beta}_{\lambda, Q} \).
Theorem 2 The minimum quasi weighted DPD estimate of $\beta$, say $\hat{\beta}_{\lambda,Q}$, can be obtained by solving the system of equations $\lambda_\beta = 0_{d(k+1)}$, where

$$u_\lambda(\beta) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} u_\lambda(\beta, x_{hi}),$$  \hspace{1cm} (11)

$$u_\lambda(\beta, x_{hi}) = \left[ w_{hi} \Delta^*(\pi_{hi}(\beta)) \text{diag}^{\lambda-1}\{\pi_{hi}(\beta)\}\{\hat{y}_{hi} - m_{hi}\pi_{hi}(\beta)\}\right] \otimes x_{hi}. $$  \hspace{1cm} (12)

Let $U_\lambda(\beta, X)$ be a random variable generator of (12) associated with a generic random explanatory variable $X$, with no stratum and cluster assignment. In the following, we denote $U_\lambda(\beta, X)|X = x_{hi}$ by $U_\lambda(\beta, x_{hi})$ and note that

$$U_\lambda(\beta, x_{hi}) = \left[ w_{hi} \Delta^*(\pi_{hi}(\beta)) \text{diag}^{\lambda-1}\{\pi_{hi}(\beta)\}\{\hat{Y}_{hi} - m_{hi}\pi_{hi}(\beta)\}\right] \otimes x_{hi}.$$

The unbiasedness of the estimating equation given in Theorem 2 is a very important property in this article. It should be clarified that we refer to a stronger unbiasedness than the asymptotic one. This property is the key for constructing the asymptotic properties of the minimum quasi weighted DPD estimators of $\beta$, following a similar scheme as the maximum quasi weighted likelihood ones. This fact is not straightforward in advance since for other distance based estimators, such as the ones proposed in Castilla et al. (2018), does not happen.

3.1 Asymptotic distributions

The following results are generalizations associated with the PLR model with complex sampling and random explanatory variables. Without loss of generality, it is assumed that $\hat{P}(X = x_{hi}) = \frac{1}{n}$ is estimated from the sample of strata.

Theorem 3 For $\hat{\beta}_{\lambda,Q}$, the minimum quasi weighted DPD estimator of $\beta$ in the PLR model (2) under a complex survey, it holds

$$\sqrt{n}(\hat{\beta}_{\lambda,Q} - \beta_0) \xrightarrow{n \to \infty} \mathcal{N}\left(0_{d(k+1)}, \Psi^{-1}_\lambda(\beta_0) \Omega_\lambda(\beta_0) \Psi^{-1}_\lambda(\beta_0)\right),$$

where $\beta_0$ is the true parameter value and

$$\Omega_\lambda(\beta) = \lim_{n \to \infty} \Omega_{n,\lambda}(\beta) = \lim_{n \to \infty} \frac{1}{n} \sum_{h=1}^{H} \sum_{i=1}^{n_h} \Omega_{hi,\lambda}(\beta, \Sigma_{hi}),$$  \hspace{1cm} (13)

$$\Psi_\lambda(\beta) = \lim_{n \to \infty} \Psi_{n,\lambda}(\beta) = \lim_{n \to \infty} \frac{1}{n} \sum_{h=1}^{H} \sum_{i=1}^{n_h} \Psi_{hi,\lambda}(\beta),$$  \hspace{1cm} (14)
with \( \Sigma_{hi} = \text{Var}[\hat{Y}_{hi}] \) and

\[
\Omega_{hi, \lambda}(\beta, \Sigma_{hi}) = w_{hi}^2 \Delta^*(\pi_{hi}(\beta)) \text{diag}^{\lambda - 1}(\pi_{hi}(\beta)) \Sigma_{hi} \\
\times \text{diag}^{\lambda - 1}(\pi_{hi}(\beta)) \Delta^{*T}(\pi_{hi}(\beta)) \otimes x_{hi} x_{hi}^T, \\
\Psi_{hi, \lambda}(\beta) = \begin{cases} \\
\psi_{hi} m_{hi} \Delta^*(\pi_{hi}(\beta)) \text{diag}^{\lambda - 1}(\pi_{hi}(\beta)) \Delta^{*T}(\pi_{hi}(\beta)) \otimes x_{hi} x_{hi}^T, & \lambda > 0 \\
\psi_{hi} m_{hi} \Delta(\pi_{hi}(\beta)) \otimes x_{hi} x_{hi}^T, & \lambda = 0. 
\end{cases}
\]

Notice that the expression of \( \Psi_{\lambda=0}(\beta) \) is the same as the so called Fisher information matrix for multinomial sampling. In addition, if \( \hat{Y}_{hi} \) has an ordinary multinomial sampling scheme (without overdispersion), it holds \( \Omega_{\lambda=0}(\beta) = \Psi_{\lambda=0}(\beta) \) and \( \Psi_{\lambda=0}(\beta) \propto 0(\beta) \Omega_{\lambda=0}^{-1}(\beta) \) is the inverse of the Fisher information matrix.

Consistency is considered as a minimal requirement for an inference procedure. The following result is useful as a tool for estimating \( \Omega_{\lambda}(\beta) \) and \( \Psi_{\lambda}(\beta) \) consistently plugging a consistent estimator into \( \beta \), and hence also \( \Omega_{\lambda}^{-1}(\beta) \Psi_{\lambda}(\beta) \Omega_{\lambda}^{-1}(\beta) \) again as a (double) plug-in estimator.

**Corollary 4** The following ones are (weak) consistent estimators as \( n \) goes to infinity:

(a) \( \hat{\beta}_{\lambda, Q} \) is a consistent estimator of the true regression coefficient \( \beta_0 \).

(b) \( \Psi_{n, \lambda}(\hat{\beta}_{\lambda, Q}) = \frac{1}{n} \sum_{h=1}^H \sum_{i=1}^{n_h} \psi_{hi, \lambda}(\hat{\beta}_{\lambda, Q}) \) is a consistent estimator of \( \Psi_{\lambda}(\beta_0) \).

(c) \( \Omega_{n, \lambda}(\hat{\beta}_{\lambda, Q}, \{\hat{\Sigma}_{hi}\})_{h=1, \ldots, H; i=1, \ldots, n_h} = \frac{1}{n} \sum_{h=1}^H \sum_{i=1}^{n_h} \Omega_{hi, \lambda}(\hat{\beta}_{\lambda, Q}, \hat{\Sigma}_{hi}) \) with

\[
\Omega_{hi, \lambda}(\hat{\beta}_{\lambda, Q}, \hat{\Sigma}_{hi}) = w_{hi}^2 \Delta^*(\pi_{hi}(\hat{\beta}_{\lambda, Q})) \text{diag}^{\lambda - 1}(\pi_{hi}(\hat{\beta}_{\lambda, Q})) \hat{\Sigma}_{hi} \\
\times \text{diag}^{\lambda - 1}(\pi_{hi}(\hat{\beta}_{\lambda, Q})) \Delta^{*T}(\pi_{hi}(\hat{\beta}_{\lambda, Q})) \otimes x_{hi} x_{hi}^T,
\]

is a consistent estimator of \( \Omega_{\lambda}(\beta_0) \), whenever \( \Sigma_{hi} = \text{Var}[\hat{Y}_{hi}] \) is consistently estimated through \( \hat{\Sigma}_{hi} \) for all \((h, i)\), such that \( h = 1, \ldots, H; i = 1, \ldots, n_h \).

The following two cases of \( \Omega_{n, \lambda}(\hat{\beta}_{\lambda, Q}, \{\hat{\Sigma}_{hi}\})_{h=1, \ldots, H; i=1, \ldots, n_h} \) are taken into account for two cases of \( \Sigma_{hi} \) for which there exists a consistent estimator:

- if \( \hat{Y}_{hi} \) has an (ordinary) multinomial sampling scheme then

\[
\Omega_{n, \lambda}(\hat{\beta}_{\lambda, Q}) = \frac{1}{n} \sum_{h=1}^H \sum_{i=1}^{n_h} \Omega_{hi, \lambda}(\hat{\beta}_{\lambda, Q})
\]

where \( \Omega_{hi, \lambda}(\hat{\beta}_{\lambda, Q}) \) is \( \Omega_{hi, \lambda}(\hat{\beta}_{\lambda, Q}, \hat{\Sigma}_{hi}) \) in (16) with \( \hat{\Sigma}_{hi} = \Sigma_{hi}(\hat{\beta}_{\lambda, Q}) = m_{hi} \Delta(\pi_{hi}(\hat{\beta}_{\lambda, Q})) \);  

- if \( \hat{Y}_{hi} \) has an overdispersed multinomial sampling scheme then

\[
\Omega_{n, \lambda}(\hat{\beta}_{\lambda, Q}, \{\hat{\nu}_{hi}\})_{h=1, \ldots, H; i=1, \ldots, n_h} = \frac{1}{n} \sum_{h=1}^H \sum_{i=1}^{n_h} \Omega_{hi, \lambda}(\hat{\beta}_{\lambda, Q}, \hat{\nu}_{hi}),
\]
where \( \tilde{v}_{hi} \) is a consistent estimator of the overdispersion parameter \( \nu_{hi} \), which is established later in Corollary 7, and \( \Omega_{n,\lambda}(\tilde{v}_{hi}, \tilde{\beta}_{\lambda}, Q) \) is \( \Omega_{hi,\lambda}(\tilde{\beta}_{\lambda}, Q, \tilde{\Sigma}_{hi}) \) given in (16) with \( \tilde{\Sigma}_{hi} = \Sigma_{hi}(\tilde{v}_{hi}, \tilde{\beta}_{\lambda}, Q) = \tilde{v}_{hi} m_{hi} \Delta(\pi_{hi}(\beta_{\lambda}, Q)) \).

**Remark 5** If \( \tilde{Y}_{hi} \) has a multinomial sampling scheme, Theorem 3 can be similarly formulated taking the assumption that \( \tilde{P}(X = x_{hi}) = \frac{1}{m} \), where \( m = \sum_{h=1}^{H} \sum_{i=1}^{n_h} m_{hi} \), for each individual of all clusters in the sample, and taking for the estimation equation \( m \) summands rather than \( n \), i.e. plugging \( \sum_{j=1}^{m_{hi}} y_{hij} = \tilde{y}_{hi} \) into (11) and considering the system of equations \( u_{\lambda}(\beta) = 0_{d(k+1)} \), where

\[
\begin{align*}
\boldsymbol{u}_{\lambda}(\beta) &= \sum_{h=1}^{H} \sum_{i=1}^{n_h} \sum_{j=1}^{m_{hi}} \boldsymbol{v}_{j,\lambda}(\beta, x_{hi}), \\
\boldsymbol{v}_{j,\lambda}(\beta, x_{hi}) &= \left[ w_{hi} \Delta(\pi_{hi}(\beta)) \text{diag}^{\lambda-1}(\pi_{hi}(\beta)) \{ \pi_{hi}(\beta) \} \right] \otimes x_{hi}.
\end{align*}
\]

Hence, it holds

\[
\sqrt{m}(\tilde{\beta}_{\lambda, Q} - \beta_0) \xrightarrow{m \to \infty} N\left( 0_{d(k+1)}, \Psi_{\lambda}^{-1}(\beta_0) \Omega_{\lambda}(\beta_0) \Psi_{\lambda}^{-1}(\beta_0) \right),
\]

with

\[
\begin{align*}
\Omega_{\lambda}(\beta) &= \lim_{m \to \infty} \begin{cases} 
\frac{1}{m} \sum_{h=1}^{H} \sum_{i=1}^{n_h} m_{hi} w_{hi}^{2} \Delta(\pi_{hi}(\beta)) \text{diag}^{\lambda-1}(\pi_{hi}(\beta)) \Delta(\pi_{hi}(\beta)) & \lambda > 0, \\
\frac{1}{m} \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi}^{2} m_{hi} \Delta(\pi_{hi}(\beta)) \otimes x_{hi} x_{hi}^T & \lambda = 0
\end{cases}, \\
\Psi_{\lambda}(\beta) &= \lim_{m \to \infty} \begin{cases} 
\frac{1}{m} \sum_{h=1}^{H} \sum_{i=1}^{n_h} m_{hi} w_{hi} \Delta(\pi_{hi}(\beta)) \text{diag}^{\lambda-1}(\pi_{hi}(\beta)) \Delta^{T}(\pi_{hi}(\beta)) \otimes x_{hi} x_{hi}^T & \lambda > 0, \\
\frac{1}{m} \sum_{h=1}^{H} \sum_{i=1}^{n_h} m_{hi} \Delta(\pi_{hi}(\beta)) \otimes x_{hi} x_{hi}^T & \lambda = 0.
\end{cases}
\end{align*}
\]

The formal proof is omitted, but the derivation of the expressions is almost the same considering \( U_{\lambda}(\beta, x_{hi}) = \sum_{j=1}^{m_{hi}} V_{j,\lambda}(\beta, x_{hi}) \), with \( V_{j,\lambda}(\beta, x_{hi}) \) i.i.d. random variables \( j = 1, \ldots, m_{hi} \). This idea matches the philosophy of the asymptotic result developed in Castilla et al. (2019), for \( H = 1 \) and \( w_{1i} = 1, i = 1, \ldots, m_{1i} \).

The following result is useful for any sample of polytomous logistic regression with complex sample design, more general in comparison with Corollary 4, since does not require getting any consistent estimators for \( \Sigma_{hi} \) in advance.
Theorem 6 The estimator $\widehat{\Omega}_{n,\lambda}(\widehat{\beta}_{\lambda}, Q) = \frac{1}{n} \sum_{h=1}^{H} \sum_{i=1}^{n_h} \widehat{\Omega}_{hi,\lambda}(\widehat{\beta}_{\lambda}, Q)$, with

$$
\widehat{\Omega}_{hi,\lambda}(\widehat{\beta}_{\lambda}, Q) = U_\lambda(\widehat{\beta}_{\lambda}, Q, x_{hi})U_\lambda^T(\widehat{\beta}_{\lambda}, Q, x_{hi})
= [w_{hi}^* \Delta^*(\pi_{hi}(\widehat{\beta}_{\lambda}, Q)) \text{diag}^{\lambda-1}(\pi_{hi}(\widehat{\beta}_{\lambda}, Q))] (\hat{Y}_{hi} - m_{hi} \pi_{hi}(\widehat{\beta}_{\lambda}, Q)) \times (\hat{Y}_{hi} - m_{hi} \pi_{hi}(\widehat{\beta}_{\lambda}, Q))^T \text{diag}^{\lambda-1}(\pi_{hi}(\widehat{\beta}_{\lambda}, Q)) \Delta^*(\pi_{hi}(\widehat{\beta}_{\lambda}, Q)) \otimes x_{hi}x_{hi}^T,
$$

is consistent for $\Omega_{\lambda}(\beta)$ as $n$ goes to infinity.

3.2 Estimates of the design effect

The following results presents two sets of estimators of design effects under the overdispersed multinomial sampling setting, as a corollary to Theorem 6.

Corollary 7 Let $\hat{Y}_{hi}$ be a random variable with overdispersed multinomial sampling scheme with a common overdispersion parameter $\nu$ and $m_{hi} = \bar{m}$,

$$
\Sigma_{hi} = \Sigma_{hi}(\nu, \beta) = \nu \bar{m} \Delta(\pi_{hi}(\beta)), \quad \nu = 1 + \rho^2(\bar{m} - 1),
$$

then, for $\nu$ and $\rho^2$:

(a) “robust and consistent estimators based on the estimating equation” are given by

$$
\tilde{\nu}_{n,\lambda}^{E} = \tilde{v}_{n,\lambda}^{E}(\widehat{\beta}_{\lambda}, Q) = \frac{1}{d(k + 1)} \text{trace} \left( \Omega_{n,\lambda}^{-1}(\widehat{\beta}_{\lambda}, Q) \widehat{\Omega}_{n,\lambda}(\widehat{\beta}_{\lambda}, Q) \right),
\tilde{\rho}_{n,\lambda}^{2E} = \frac{\tilde{v}_{n,\lambda}^{E}(\widehat{\beta}_{\lambda}, Q) - 1}{\bar{m} - 1},
$$

(17)

where $\Omega_{n,\lambda}(\widehat{\beta}_{\lambda}, Q) = \frac{1}{n} \sum_{h=1}^{H} \sum_{i=1}^{n_h} \Omega_{hi,\lambda}(\widehat{\beta}_{\lambda}, Q)$ is obtained from Theorem 6 and the inner matrices simplify for overdispersed multinomial sampling to have the form

$$
\Omega_{hi,\lambda}(\widehat{\beta}_{\lambda}, Q) = \bar{m}w_{hi}^2 \Delta^*(\pi_{hi}(\beta)) \text{diag}^{\lambda-1}(\pi_{hi}(\beta)) \Delta(\pi_{hi}(\beta)) \times \text{diag}^{\lambda-1}(\pi_{hi}(\beta)) \Delta^*(\pi_{hi}(\beta)) \otimes x_{hi}x_{hi}^T.
$$

(b) “robust and consistent estimators based on the method of moments” are given by

$$
\tilde{\nu}_{n,\lambda}^{M} = \tilde{v}_{n,\lambda}^{M}(\widehat{\beta}_{\lambda}, Q) = \frac{1}{nd} \sum_{h=1}^{H} \sum_{i=1}^{n_h} \sum_{j=1}^{d+1} (\hat{Y}_{hij} - \bar{m} \pi_{hij}(\widehat{\beta}_{\lambda}, Q))^2, \quad \tilde{\rho}_{n,\lambda}^{2M} = \frac{\tilde{v}_{n,\lambda}^{M}(\widehat{\beta}_{\lambda}, Q) - 1}{\bar{m} - 1},
$$

(18)

The preceding results of this whole section were established for $n = \sum_{h=1}^{H} n_h$ tending to infinity which implies, in practice, that $H$ is fixed and there exists
\[ \eta_h = \lim_{n \to \infty} \frac{n \eta_h}{n} \in (0, 1), \sum_{h=1}^{H} \eta_h = 1. \] So, it is also useful to consider alternative results assuming that \( \Pr(X_h = x_{hi}) = \frac{1}{\eta_h} \), which matches the philosophy of the asymptotic results developed in Castilla et al. (2018) for another family of divergences and the derivations of the results were quite different. Detailed results for our proposed estimator under the above assumption are provided in the Supplementary Material (Sect. 2).

### 4 Testing linear hypotheses for the PLR coefficients with complex survey design

Based on the asymptotic distribution of the minimum quasi weighted DPD estimator of \( \beta, \hat{\beta}_{\lambda, Q} \), presented in Theorem 3 for the PLR, we now introduce and study a family of Wald-type test statistics for testing

\[ H_0 : M^T \beta = l \text{ against } H_0 : M^T \beta \neq l \quad (19) \]

where \( M \) is a \( d(k+1) \times r \) full row-rank matrix with \( r \leq d(k+1) \) and \( l \) is an \( r \)-vector of constants; \( l = 0_r \) in many situations. For testing (19), we define a family of Wald-type test statistics based on minimum quasi DPD estimators as follows.

**Definition 8** For \( \hat{\beta}_{\lambda, Q} \), the minimum quasi weighted DPD estimator of \( \beta \), in the PLR model (2) under a complex survey the family of Wald-type test statistics for testing the null hypothesis of (19) is given by

\[ W_n(\hat{\beta}_{\lambda, Q}) = (M^T \hat{\beta}_{\lambda, Q} - l)^T (M^T \hat{Q}_{n, \lambda}(\hat{\beta}_{\lambda, Q})M)^{-1} (M^T \hat{\beta}_{\lambda, Q} - l), \quad (20) \]

where

\[ \hat{Q}_{n, \lambda}(\hat{\beta}_{\lambda, Q}) = \frac{1}{n} \Psi_{n, \lambda}(\hat{\beta}_{\lambda, Q}) \hat{\Omega}_{n, \lambda}(\hat{\beta}_{\lambda, Q}) \Psi_{n, \lambda}^{-1}(\hat{\beta}_{\lambda, Q}) \]

\[ = (n \Psi_{n, \lambda}(\hat{\beta}_{\lambda, Q}))^{-1} n \hat{\Omega}_{n, \lambda}(\hat{\beta}_{\lambda, Q}) (n \Psi_{n, \lambda}(\hat{\beta}_{\lambda, Q}))^{-1}, \]

with

\[ n \hat{\Omega}_{n, \lambda}(\hat{\beta}_{\lambda, Q}) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} \left[ w_{hi}^2 \Delta^*(\pi_{hi}(\hat{\beta}_{\lambda, Q})) \text{diag}^{\lambda-1}(\pi_{hi}(\hat{\beta}_{\lambda, Q})) \right] [\hat{Y}_{hi} - m_{hi}] \pi_{hi}(\hat{\beta}_{\lambda, Q})^T \text{diag}^{\lambda-1}(\pi_{hi}(\hat{\beta}_{\lambda, Q})) \Delta^{*T}(\pi_{hi}(\hat{\beta}_{\lambda, Q})) \otimes x_{hi} x_{hi}^T. \]

\[ n \Psi_{n, \lambda}(\hat{\beta}_{\lambda, Q}) = \begin{cases} \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} m_{hi} \Delta^*(\pi_{hi}(\hat{\beta}(\beta))) \text{diag}^{\lambda-1}(\pi_{hi}(\hat{\beta}(\beta))) \Delta^{*T}(\pi_{hi}(\hat{\beta}(\beta))) \otimes x_{hi} x_{hi}^T, & \lambda > 0 \\ \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} m_{hi} \Delta(\pi_{hi}(\beta)) \otimes x_{hi} x_{hi}^T, & \lambda = 0 \end{cases} \]

Obtained from estimating equations given in (8), for \( \lambda = 0, \hat{\beta}_{\lambda=0, Q} \) is the maximum quasi weighted likelihood estimator of \( \beta \). It is not difficult to see that \( \hat{Q}_{n, \lambda=0}(\hat{\beta}_{\lambda=0, Q}) \) is the Fisher information matrix and \( W_n(\hat{\beta}_{\lambda=0, Q}) \) the classical Wald test statistic.
Theorem 9  Under the null hypothesis given in (19) the asymptotic distribution of the Wald-type test statistics \( W_n(\hat{\beta}_{\lambda}, Q) \), defined in (20), is chi-square with \( r \) degrees of freedom.

The proof is immediate using the asymptotic distribution of the minimum quasi weighted DPD estimator and taking into account the consistency of the matrix \( Q_\lambda(\hat{\beta}_{\lambda}, Q) \). Based on the previous theorem the null hypothesis in (19) will be rejected if

\[
W_n(\hat{\beta}_{\lambda}, Q) > \chi^2_{r, \alpha}.
\]

Further results in relation to the power function of this proposed Wald-type tests are given in the Supplementary Material (Sect. 3).

5 Robustness: influence function analyses

The influence function is a classical tool to measure robustness of an estimator (Hampel et al. 1986). However, the present set-up of complex survey is not as simple as the i.i.d. set-up; in fact the observations within a cluster of a stratum are i.i.d., the observations in different clusters and strata are independent but non-homogeneous. So we need to modify the definition of the influence function accordingly. Recently, Ghosh and Basu (2013, 2016, 2018) have discussed the extended definition of the influence function for the independent but non-homogeneous observations; we will extend their approach to define the influence function in the present case of PLR model under complex design.

5.1 Influence function of minimum quasi weighted DPD estimator

We first need to define the statistical functional corresponding to the minimum quasi weighted DPD estimator as the minimizer of the DPD between the true and model densities. Assume the set-up and notations of Sect. 3 with the DPD kernel being given by \( d^\lambda_\ast( g(y_{hi} | x_{hi}), f_{\beta}(y_{hi} | x_{hi}) ) \) for individual densities \( g(y_{hi} | x_{hi}) \) and \( f_{\beta}(y_{hi}) \). Then, following Ghosh and Basu (2013), the minimum quasi weighted DPD estimator functional is to be defined by the minimizer of the total weighted DPD measure given by

\[
d_\lambda( g, f_{\beta}, w ) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} d^\lambda_\ast( g(y_{hi} | x_{hi}), f_{\beta}(y_{hi} | x_{hi}) )
\]

\[
= \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} \left( m_{hi} \pi_{hi}^{\lambda, T}(\beta) \pi_{hi}(\beta) - \frac{\lambda + 1}{\lambda} \int_{\mathcal{Y}_{hi}} \pi_{hi}^{\lambda, T}(\beta) y dG( y | x_{hi}) \right), \quad \text{for } \lambda > 0.
\]

Note that, in Eq. (22), we have considered only the part (DPD kernel) of the divergence measure that we need to minimize in order to find the vector of the coefficients.
of the model. But, it is indeed gives equivalent definition as using the full DPD since the additional (third) term in the DPD measure is independent of the parameter vector.

Now, for notational compactness, let \( y \) be the vector of constants with components equal to \( y_{hi} \) in lexicographical order. And, let \( g \) and \( G \) be the function vectors with components equal to \( g(y) = (g_{1i}^T(y), \ldots, g_{HmH}^T(y))^T \) and \( G(y) = (G_{1i}^T(y), \ldots, G_{HmH}^T(y))^T \), respectively, such that

\[
g_{hi}(y) = (g_{hi1}(y), \ldots, g_{himhi}(y))^T = (g(y_{hi1}|x_{hi}), \ldots, g(y_{himhi}|x_{hi}))^T,
G_{hi}(y) = (G_{hi1}(y), \ldots, G_{himhi}(y))^T = (G(y_{hi1}|x_{hi}), \ldots, G(y_{himhi}|x_{hi}))^T.
\]

We could define similarly \( f_\beta(y) \) and \( F_\beta(y) \).

**Definition 10** We consider the PLR model with complex survey defined in (2). The minimum quasi weighted DPD estimator functional of \( \beta \), \( T_{\lambda, Q}(G) \), at \( G(y) \) is defined as

\[
T_{\lambda, Q}(G) = \arg \min_{\beta \in \Theta} d_\lambda(g, f_\beta, w),
\]

where \( d_\lambda(g, f_\beta, w) \) is as defined above in (22).

Note that, by the property of the DPD measures, it is immediate that the minimum quasi weighted DPD estimator functional \( T_{\lambda, Q}(G) \) is Fisher consistent at the assumed PLR model (2), i.e., \( T_{\lambda, Q}(F_\beta) = \beta \) for all values of \( \beta \). Also, following the proof of Theorem 2, one can see that the minimum quasi weighted DPD estimator functional \( T_{\lambda, Q}(G) \) can also be derived as a solution to the estimating equations \( u_\lambda(T_{\lambda, Q}(G)) = 0_{d(k+1)} \), where

\[
u_j(T_{\lambda, Q}(G), x_{hi}) = \left[ w_{hi} \Delta^x(\pi_{hi} (\beta)) \ diag^{\lambda-1}[\pi_{hi} (\beta)] \{y_{hij} - \pi_{hi} (\beta)\} \right] \otimes x_{hi}.
\]

Note that these estimating equations are unbiased at the model probability \( G = F_\beta \).

Now, in order to define the influence function of the minimum quasi weighted DPD estimator functional \( T_{\lambda, Q}(G) \), we note that the functional itself depends on the sample sizes and cluster weights and so its influence function would have the same dependence in analogue to the non-homogeneous case of Ghosh and Basu (2013, 2016, 2018). Also, note that the contamination can be in any one particular cluster within one stratum or simultaneously in many (or all) of them. For simplicity, let us first assume that the contamination is only in one cluster probability \( g_{h0i0j0} \) for some fixed unit at \( h_0, i_0 \) and \( j_0 \). Consider the contaminated probability \( g_{h0i0j0,t,\epsilon} = (1-\epsilon)g_{h0i0j0} + \epsilon \delta_t \), where \( \epsilon \) is the contamination proportion and \( \delta_t \) is the degenerate probability at the outlier point \( t = (t_1, \ldots, t_{d+1})^T \in \{0, 1\}^{d+1} \) with \( \sum_{s=1}^{d+1} t_s = 1 \). Denote the corresponding contaminated full probability vector as \( g_{h0i0j0,t,\epsilon} \) which is same as \( g \) except for \( g_{h0i0j0} \), \( j = 1, \ldots, m_{h0i0} \), being replaced by \( g_{h0i0j0,\epsilon} \) and let the corresponding contaminated
distribution vector be $G_{h_{0i0j0},t,\epsilon}$. Then, the corresponding influence function is defined as

$$IF(T_{\lambda, Q}, G, t, h_{0i0j0}) = \lim_{\epsilon \to 0^+} \frac{T_{\lambda, Q}(G_{h_{0i0j0},t,\epsilon}) - T_{\lambda, Q}(G)}{\epsilon} = \frac{\partial}{\partial \epsilon} T_{\lambda, Q}(G_{h_{0i0j0},t,\epsilon}) \bigg|_{\epsilon=0^+}.$$ 

Some comments regarding the choice of our contamination are in order here. Note that the outlier point $t$ in the contaminated density $g_{h_{0i0j0},t,\epsilon}$ can take only the values in a finite set, the unit vectors of dimension $(d + 1)$ rather than the infinite possible values towards infinity as considered in the usual definition of influence function. This is because the present PLR model densities have a finite support and hence it is intuitively not meaningful to take an outlier value outside the support of the model density (any such values can be easily discarded at the beginning). So, considering only the possible outliers within the support of the PLR model densities, the outlier points $t$ indeed produces the classification errors; if the only non-zero element (1) of $t$ is in the $r$th position, then the degenerate contamination distribution $\delta t$ always classify the response into the $r$th category irrespective of its original distribution $g_{h_{0i0j0}}$. Thus, our “outlier producing measure” $g_{h_{0i0j0},t,\epsilon}$ indeed provides 100% classification error if the outlier point $t$ yields its mass in a wrong category. Varying over all possible such outlier points $t$, we get all possible classification errors and our definition of influence function above then represents the bias in the estimator caused by infinitesimal amount of such classification errors. This view of considering outliers as classification errors in the PLR model is, in fact, in line with the general literature on robust analysis of categorical data, including different types of logistic regressions having finite supports for the model densities (Johnson 1985; Bianco and Yohai 1996; Croux and Haesbroeck 2003; Rousseeuw and Christmann 2003; Bondell 2008; Bianco and Martinez 2009; Basu et al. 2017; Ghosh and Basu 2018; Castilla et al. 2019). However, another standard aspects for robustness based on sampling on a finite space is the treatment using the so-called Pearson Residuals (Lindsay 1994; Jiménez and Shao 2001; Basu et al. 2011), and pertains to oversampling (outliers in this setting) and undersampling (inliers) from the target distribution, which are clearly associated also with the classification errors. Our contaminated density $g_{h_{0i0j0},t,\epsilon}$, although defined as a classical mixture, is now way conceptually different from this classical approach. As noted above, $g_{h_{0i0j0},t,\epsilon}$ indeed only produces classification error for different choices of $t$ considered, and hence has a direct connection to the concept of Pearson residual. When $t$ has its only non-zero element (1) in the $r$th position different from $\tilde{r}$, the category of maximum probability by for the true density $g_{h_{0i0j0}}$, the corresponding mixture contamination generates more observation in the $r$th category having low actual density (high Pearson residuals or oversampling) whereas it generates relatively less observations than what is expected under actual probability in the $\tilde{r}$th category (low Pearson residual or inliers). All these discussions clearly justifies the use of our contaminated densities $g_{h_{0i0j0},t,\epsilon}$ for investigating the robustness of the proposed inference procedures under the PLR model with complex survey.
Now, for the computation of the required influence function of our proposed estimator, we start by focusing on the estimating equation for $T_{\lambda,Q}$. Note that, $T_{\lambda,Q}(G_{h_0i_0j_0}, \epsilon)$ satisfies the equations

$$
u_{\lambda} \left( T_{\lambda,Q}(G_{h_0i_0j_0}, \epsilon) \right) = 0_{d(k+1)}$$

such that $u_{\lambda}(T_{\lambda,Q}(G_{h_0i_0j_0}, t, \epsilon)) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} \sum_{j=1}^{m_{hi}} u_{\lambda}(T_{\lambda,Q}(G_{h_0i_0j_0}, t, \epsilon), x_{hi})$.

Now, differentiating it with respect to $\epsilon$, taking $\epsilon = 0^+$ and simplifying, we get the required influence function as given by

$$\mathcal{IF}(T_{\lambda,Q}, G, t, h_0i_0j_0) = \Psi_{n,\lambda}^{-1}(T_{\lambda,Q}(G)) \frac{1}{n} \left[ v_{j_0,\lambda}(T_{\lambda,Q}(G_{h_0i_0j_0}, t), x_{hi}) - v_{j_0,\lambda}(T_{\lambda,Q}(G), x_{hi}) \right],$$

where $v_{j_0,\lambda}(T_{\lambda,Q}(G_{h_0i_0j_0}, t), x_{hi}) = \left[ w_{hi} \Delta^*(\pi_{hi}(\beta)) \operatorname{diag}^{\lambda-1}\{\pi_{hi}(\beta)\}\{t_{hij} - \pi_{hi}(\beta)\} \right] \otimes x_{hi},$ and $A_{h_0i_0j_0,t}$ is the distribution function associated with the degenerate probability at the outlier point $t$ for some fixed unit at $h_0, i_0$ and $j_0$. Note that, at the model distribution $G = F_\beta$, we have

$$\mathcal{IF}(T_{\lambda,Q}, F_\beta, t, h_0i_0j_0) = \Psi_{n,\lambda}^{-1}(\beta) \frac{1}{n} \left[ w_{hi} \Delta^*(\pi_{hi}(\beta)) \operatorname{diag}^{\lambda-1}\{\pi_{hi}(\beta)\}\{t_{hij} - \pi_{hi}(\beta)\} \right] \otimes x_{hi},$$

Clearly, by the assumed form of the model distribution, this influence function is bounded for all $\lambda > 0$. So, the proposed minimum quasi weighted DPD estimators are expected to be robust for any $\lambda > 0$ but the corresponding maximum quasi weighted likelihood estimator (at $\lambda = 0$) is non-robust against data contamination having the maximum influence of contamination. Further, it can be seen that the extent of robustness increases, indicated by the decrease in the influence of contamination, with increasing values of $\lambda > 0$.

Similarly, one can show that, if there is contamination in selected individuals of some cluster within some stratum with indices belonging to $\Gamma \subseteq S$, such that $S = \{(h,i,j) : h = 1, \ldots, H; i = 1, \ldots, n_h; j = 1, \ldots, m_{hi}\}$, at the contamination points $t_{hij}$ with indices belonging to $\Gamma$, compiled in vector $t$ in lexicographical order, the corresponding influence function of the proposed minimum quasi weighted DPD estimator is given by

$$\mathcal{IF}(T_{\lambda,Q}, G, t, \Gamma) = \Psi_{n,\lambda}^{-1}(T_{\lambda,Q}(G)) \frac{1}{n} \sum_{(h,i,j) \in \Gamma} \left[ v_{j_0,\lambda}(T_{\lambda,Q}(A_{h_0i_0j_0,t}), x_{hi}) - v_{j_0,\lambda}(T_{\lambda,Q}(G), x_{hi}) \right].$$

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\[ \mathcal{IF}(T_{\lambda, Q}, F_{\beta}, t, \Gamma) = \Psi_{n, \lambda}^{-1}(\beta) \frac{1}{n} \sum_{(h,i,j) \in \Gamma} \left[ w_{hi} \Delta^\lambda(\pi_{hi}(\beta)) \text{diag}^{-1}(\pi_{hi}(\beta)) \{t_{hij} - \pi_{hi}(\beta)\} \right] \otimes x_{hi}. \]

The boundedness and robustness implications for these influence functions are exactly the same as before.

### 5.2 Influence function of the Wald-type tests in PLRM

We now study the robustness of the proposed Wald-type test through influence function of the corresponding test statistics in (20). Considering the notation and set-up of Sect. 2 and 3, we define

\[ \bar{Q}_{\lambda}(\beta) = \Psi_{\lambda}^{-1}(\beta) \Omega_{\lambda}(\beta) \Psi_{\lambda}^{-1}(\beta). \]

Then the Wald-type test functional

\[ W_n(T_{\lambda, Q}(G)) = n(M^T T_{\lambda, Q}(G)-I)^T \left( M^T \bar{Q}_{\lambda}(T_{\lambda, Q}(G)) M \right)^{-1} (M^T T_{\lambda, Q}(G)-I), \]

where \( T_{\lambda, Q} \) is the minimum quasi weighted DPD estimator functional as defined in the previous subsection. Also, let \( \beta_0 \) denote the true null parameter value for the hypothesis in (19).

Let us now derive the influence function of the test functional \( W_n(T_{\lambda, Q}(G)) \). As before, here also the contamination can be any particular observation of some cluster and strata (a given \( h_0, i_0, j_0 \) combination or in many (or all) of them compiled in the set \( \Gamma \). The influence function of general Wald-type tests under such non-homogeneous set-up has been extensively studied in Basu et al. (2018). Here, we follow the general theory of Basu et al. (2018) to conclude that the first order influence functions of \( W_n(T_{\lambda, Q}(G)) \), defined as the first order derivative of its value at the contaminated distribution with respect to \( \epsilon \) at \( \epsilon = 0 \), in either case of contamination become identically zero at the null distribution \( G = F_{\beta_0} \). Therefore, the first order influence function is not informative for Wald-type tests, and we need to investigate the second order influence function of \( W_n(T_{\lambda, Q}(G)) \).

The second order influence function, which measures the second order approximation to the asymptotic bias due to infinitesimal contamination, is defined as the second order derivative of the value of \( W_n(G_{h_0i_0j_0,t,\epsilon}) \) at \( \epsilon = 0 \). Again, following Basu et al. (2018), we can derive these second order influence functions of the Wald-type tests in either case of contaminations; at the null distribution \( G = F_{\beta_0} \), they are simplified to

\[ \mathcal{IF}^{(2)}(W_n(T_{\lambda, Q}(G)), F_{\beta_0}, t, h_0i_0j_0) \]

\[ = 2n \mathcal{IF}^{(2)}(T_{\lambda, Q}, F_{\beta_0}, t, h_0i_0j_0) M \left( M^T \bar{Q}_{\lambda}(\beta_0) M \right)^{-1} M^T \mathcal{IF}^{(2)}(T_{\lambda, Q}, F_{\beta_0}, t, h_0i_0j_0). \]
\[ \mathcal{IF}^{(2)}(W_n(T_\lambda, Q(\cdot), F_{\beta_0}, L, \Gamma)) = 2n \mathcal{IF}^T(T_\lambda, Q, F_{\beta_0}, L, \Gamma)M \left(M^T \tilde{Q}_\lambda(\beta_0)M\right)^{-1} M^T \mathcal{IF}(T_\lambda, Q, F_{\beta_0}, L, \Gamma) \]

for the two types of contamination as considered before in the case of estimator. Note that, the second order influence functions of the proposed Wald-type tests are a quadratic function of the corresponding influence functions of the minimum quasi weighted DPD estimator for any type of contamination. Therefore, the boundedness of the influence functions of minimum quasi weighted DPD estimator at \( \lambda > 0 \) also indicates the boundedness of the influence functions of the Wald-type test functional \( W_n(T_\lambda, Q(F_{\beta_0})) \) indicating their robustness against contamination in any cluster or stratum of the sample data.

### 6 Illustrative examples

#### 6.1 Education in Jordan

Let us use the 2017–2018 Jordan Population and Family Health Survey (JPFHS, Department of Statistics (DOS) and ICF 2019) in order to illustrate the method developed in previous sections. Information is obtained from Tables 3.2.1 and 3.2.2 of the cited report, where distribution of men and women by highest level of schooling attended or completed is recorded in three different regions (“Central”, “North” and “South”) which are considered as stratum. The categorical response variable is divided in five categories: “no education”, “some/completed elementary”, “some/completed preparatory”, “some/completed secondary” and “more than secondary”. The data is shown in Table 1. For each region, the percentage of men and women corresponding to each of the education categories is recorded. Note that the “Total” column in Table 1 is, indeed, the weighted total, computed in the JPFHS (Table 3.1) by taking into account the real population number of men and women in Central, South and North area of Jordan. We can observe how the percentage of female with any kind of education is higher in all the regions. Curiously, the percentage of female with the highest level of education is also higher in all the regions, specially in the South.

After computing the minimum quasi weighted DPD estimators for different tuning parameters \( \lambda \in \{0, 0.2, \ldots, 1\} \), we measure a pondered mean absolute error of the estimated probabilities as follows

\[
\hat{e}_\lambda = \frac{1}{m} \sum_{h=1}^{H} \sum_{i=1}^{n_h} m_{hi} \sum_{r=1}^{d+1} \frac{1}{d+1} \left| \pi_{hir}(\hat{\beta}_\lambda, Q) - \frac{y_{hir}}{m_{hi}} \right| , \tag{24}
\]

with, in this case, \( H = 3, n_h = 2 \) and \( d = 4 \). Results are shown in Table 2. While the errors are low with independence of the tuning parameter, these decrease as the value of the tuning parameter increases.
| Region | Sex    | No education | Some/comp. elementary | Some/comp. preparatory | Some/comp. secondary | More than secondary | Total |
|--------|--------|--------------|-----------------------|------------------------|----------------------|---------------------|-------|
| Central | Female | 1.7          | 6.4                   | 12.2                   | 42.2                 | 37.4                | 9171  |
|        | Male   | 1.4          | 5.9                   | 12.4                   | 45.6                 | 34.8                | 3560  |
| North  | Female | 2.3          | 8.8                   | 15.4                   | 42.3                 | 31.3                | 4119  |
|        | Male   | 1.4          | 7.4                   | 15.8                   | 46.3                 | 29                  | 1550  |
| South  | Female | 5.5          | 5.4                   | 9.3                    | 40.4                 | 39.4                | 1398  |
|        | Male   | 2.8          | 4.1                   | 11.6                   | 53.2                 | 28.4                | 513   |
Table 2  Estimated pondered mean absolute error, 2017–2018 JPFHS

| $\lambda$ | $\hat{e}_\lambda$ | $\lambda$ | $\hat{e}_\lambda$ |
|-----------|-------------------|-----------|-------------------|
| 0         | 0.01219           | 0.6       | 0.01135           |
| 0.1       | 0.01202           | 0.7       | 0.01122           |
| 0.2       | 0.01188           | 0.8       | 0.01110           |
| 0.3       | 0.01174           | 0.9       | 0.01099           |
| 0.4       | 0.01161           | 1         | 0.01088           |
| 0.5       | 0.01147           |           |                   |

Table 3  Body mass index (BMI) data set

| Age group (years) | Sex  | Body mass index (BMI) |
|-------------------|------|-----------------------|
|                   |      | Acceptable weight (18.5–24.9) | Overweight (25.0–29.9) | Obese (30 or higher) |
| 20–34             | Men  | 5438                  | 4790                  | 1470                  |
|                   | Women| 4910                  | 2878                  | 802                   |
| 35–44             | Men  | 2458                  | 3437                  | 1319                  |
|                   | Women| 3100                  | 1494                  | 1313                  |
| 45–64             | Men  | 1968                  | 3290                  | 1412                  |
|                   | Women| 1710                  | 1481                  | 1078                  |

6.2 BMI data set

Let us consider an illustrative real-life dataset on BMI which was previously studied in Castilla et al. (2018). This data set, obtained from CANSIM Canada’s database and presented in Table 3, shows the body mass indexes of population in Canada in the year 1994. Each person in the sample is divided by their body mass index category under the international standard: acceptable weight, overweight or obese. The data set consists on a stratified sample design with clusters nested on them, with the strata being three different age groups (20–34 years, 35–44 years and 45–64 years) and the genders (male or female) as the clusters. The qualitative explanatory variables are valid to distinguish the clusters within the strata. They are given by $x^T_{h1} = (1, 0)$, and $x^T_{h2} = (0, 1)$, $h = 1, \ldots, 5$ for men and women, respectively.

After computing the minimum quasi weighted DPD estimators for different tuning parameters $\lambda \in \{0, 0.2, \ldots, 1\}$, we measure again the pondered mean absolute error of the estimated probabilities defined in (24). Results are shown in Table 4, obtaining better results for some $\lambda > 0$.

To illustrate the robustness of minimum quasi weighted DPD estimators, we contaminate the BMI data by permuting the categories overweight and obese in the Men with age range 45–64 years. After obtaining the corresponding minimum quasi weighted DPD estimators estimates, we compute the mean absolute standardized deviations ($mssd$) between the estimated parameters and corresponding estimated
Table 4  Estimated pondered mean absolute error, BMI data

| λ  | $\hat{e}_\lambda$ | $\lambda$ | $\hat{e}_\lambda$ |
|----|------------------|----------|------------------|
| 0  | 0.05057          | 0.6      | 0.05054          |
| 0.1| 0.05051          | 0.7      | 0.05063          |
| 0.2| 0.05046          | 0.8      | 0.05072          |
| 0.3| 0.05040          | 0.9      | 0.0508           |
| 0.4| 0.05035          | 1        | 0.05089          |
| 0.5| 0.05045          |          |                  |

Table 5  Mean standardized deviations under contamination for BMI data

| λ  | 0  | 0.2 | 0.4 | 0.6 | 0.8 | 1  |
|----|----|-----|-----|-----|-----|----|
| Contamination of Men (45–64 years) |
| $\text{masd}(\hat{\beta}_\lambda^*, \hat{\beta}_\lambda)$ | 0.24396 | 0.23057 | 0.21731 | 0.20441 | 0.19187 | 0.17969 |
| $\text{masd}(\hat{\pi}_\lambda^*, \hat{\pi}_\lambda)$ | 0.10170 | 0.09700 | 0.09220 | 0.08750 | 0.08280 | 0.07810 |
| $\text{asd}(\hat{\rho}_\lambda^{2,E}, \hat{\rho}_\lambda^{2,E})$ | 0.64785 | 0.58974 | 0.52955 | 0.46868 | 0.40858 | 0.35044 |
| $\text{asd}(\hat{\rho}_\lambda^{2,M}, \hat{\rho}_\lambda^{2,M})$ | 0.60811 | 0.53418 | 0.46217 | 0.39339 | 0.32919 | 0.27036 |
| Contamination of Women (45–64 years) |
| $\text{masd}(\hat{\beta}_\lambda^*, \hat{\beta}_\lambda)$ | 0.10516 | 0.09484 | 0.08533 | 0.07665 | 0.0687 | 0.06148 |
| $\text{masd}(\hat{\pi}_\lambda^*, \hat{\pi}_\lambda)$ | 0.03250 | 0.03030 | 0.02810 | 0.02600 | 0.0240 | 0.02210 |
| $\text{asd}(\hat{\rho}_\lambda^{2,E}, \hat{\rho}_\lambda^{2,E})$ | 0.07038 | 0.05132 | 0.03358 | 0.01744 | 0.00321 | 0.00893 |
| $\text{asd}(\hat{\rho}_\lambda^{2,M}, \hat{\rho}_\lambda^{2,M})$ | 0.01300 | 0.00441 | 0.01928 | 0.03162 | 0.04137 | 0.04859 |

probabilities obtained for the modified and original data, as given by

\[
\text{masd}(\hat{\beta}_\lambda^*, \hat{\beta}_\lambda) = \frac{1}{4} \sum_{r=1}^{2} \sum_{s=1}^{2} \left| \frac{\hat{\beta}^*_{\lambda, rs} - \hat{\beta}_{\lambda, rs}}{\hat{\beta}_{\lambda, rs}} \right| \quad \text{and}\]

\[
\text{masd}(\hat{\pi}_\lambda^*, \hat{\pi}_\lambda) = \frac{1}{6} \sum_{r=1}^{3} \sum_{s=1}^{2} \left| \frac{\pi_{rs}(\hat{\beta}^*_\lambda) - \pi_{rs}(\hat{\beta}_\lambda)}{\pi_{rs}(\hat{\beta}_\lambda)} \right| ,
\]

where, with superscript * we denote the contaminated case. Assuming common overdispersion parameter, absolute standardized deviation (asd) of the two versions of the intra-cluster correlation estimator are also computed as in Corollary 7. Their values, as presented in Table 5, clearly show that the minimum quasi weighted DPD estimators become more robust as $\lambda$ increases. Similar results are obtained when permuting the categories for Women instead of Men, as can also be seen in Table 5.
7 Monte Carlo simulation study

7.1 Simulation scheme

We develop a complex design extension of the simulation scheme previously studied by Castilla et al. (2019), where a simple sample design was considered. Here, we consider $H$ strata consisting of $n_h = n$ clusters having $m_{hi} = m$ units each, are taken. Three overdispersed multinomial distributions: the random-clumped (RC), the $m$-inflated ($m$-I) and the Dirichlet Multinomial (DM) distributions (Alonso-Revenga et al. 2017) having the same parameters $\pi_{hi}(\beta_0)$ and $\rho$, are then considered for $\hat{Y}_{hi}$; which are characterized by $E[\hat{Y}_{hi}] = m\pi_{hi}(\beta_0)$ and $V[\hat{Y}_{hi}] = \nu_m \Delta(\pi_{hi}(\beta_0))$, where 

$$\nu_m = 1 + \rho^2(m - 1), \quad i = 1, \ldots, n; \quad h = 1, 2.$$

As in Castilla et al. (2019), we further assume that the outcome nominal variable $Y$ has $d + 1 = 3$ categories depending on $k = 2$ explanatory variables through the PLR model probabilities

$$\pi_{hir}(\beta) = \begin{cases} \frac{\exp\{x_{hi}^T\beta_r\}}{1 + \sum_{l=1}^d \exp\{x_{hi}^T\beta_l\}}, & r = 1, 2 \\ \frac{1 + \sum_{l=1}^d \exp\{x_{hi}^T\beta_l\}}{1 + \sum_{l=1}^d \exp\{x_{hi}^T\beta_l\}}, & r = 3 \end{cases},$$

with $\beta = (\beta_{01}, \beta_{11}, \beta_{21}, \beta_{02}, \beta_{12}, \beta_{22})^T = (0, -0.9, 0.1, 0.6, -1.2, 0.8)^T$ and $x_{hi} \sim \mathcal{N}(0, I)$ for all $i = 1, \ldots, n, h = 1, \ldots, H$.

Considering different values of the intra-cluster correlation parameter ($\rho^2$), the number of clusters in each stratum ($n$) and the clusters sizes ($m$), we simulate data from different scenarios.

1. Scenario 1: $H = 2$, $n \in \{10i\}_{i=4}^{15}$, $m = 21$, $\rho^2 = 0.25$, RC distribution.
2. Scenario 1b: $H = 2$, $n \in \{10i\}_{i=4}^{15}$, $m = 21$, $\rho^2 = 0.50$, RC distribution.
3. Scenario 2: $H = 2$, $n = 60$, $m \in \{10i\}_{i=2}^{12}$, $\rho^2 = 0.25$, RC m-I and DM distributions.
4. Scenario 3: $H = 2$, $n = 60$, $m = 21$, $\rho^2 \in \{0.1i\}_{i=0}^9$, RC, m-I and DM distributions.

In order to study the robustness issue, these simulations are repeated under contaminated data having 7% outliers. These outliers are generated by permuting the elements of the outcome variable, such that categories 1, 2, 3 are classified as categories 3, 1, 2 for the outlying observations; see Sect. 5.1 for justifications of such an outlier setting in the PLRMs.

7.2 Performance of minimum quasi weighted DPD estimators and $\rho^2$ estimates

For the above scenarios, we compute the minimum quasi weighted DPD estimator of $\beta$, for different tuning parameters $\lambda \in \{0, 0.2, 0.4, 0.6, 0.8\}$, and the corresponding
Fig. 1 Scenario 1: RMSEs of minimum quasi weighted DPD estimators of $\beta$ and $\rho^2$ by the equations method and the method of moments. Pure data (left) and contaminated data (right). RC distribution, $\rho^2 = 0.25$.

While classical maximum quasi weighted likelihood estimator presents the best behaviour under pure data, minimum quasi weighted DPD estimators with $\lambda > 0$ are much more robust. In particular, as $\lambda$ increases, the change on their behaviour is accentuated. This is independent to the sample size but more accentuated for moderate/large
Fig. 2 Scenario 1b: RMSEs of minimum quasi weighted DPD estimators of $\beta$ and $\rho^2$ by the equations method and the method of moments. Pure data (left) and contaminated data (right). RC distribution, $\rho^2 = 0.50$

sample sizes (Figs. 1, 2), the intra-cluster correlation and the distribution (Figs. 3, 4) considered.

Estimator of $\rho^2$ by the method of moments seems less precise than the method of estimating equations, with independence of the tuning parameter chosen (Figs. 1, 2). Best estimators of $\rho^2$ by the estimating equations are obtained from minimum quasi weighted DPD estimators with $\lambda > 0$, both for pure and contaminated data and for any of the distributions considered (Fig. 3). Error of the estimators of $\rho^2$ tends to be smaller with the DM and RC distributions in comparison with the mI distribution,

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Fig. 3 Scenario 2: RMSEs of $\rho^2$ by the equations method. Pure data (left) and contaminated data (right). RC (top), mI (middle) and DM (bottom) distribution, $\rho^2 = 0.25, n = 60$. 

as can be seen in Fig. 5, where density plots based on 1000 replications are shown for $\rho^2 = 0.5, n = 60, m = 21$ and tuning parameter $\lambda = 0.4$. These results on the design effect parameter are consistent with the previous work of Alonso-Revenga et al. (2017).

Notice that $\beta$ estimates are obtained through nlm() procedure with tolerance $10^{-6}$ in the software R, used for the whole Monte Carlo study. In Fig. 5 iterations needed to compute the corresponding minimum quasi weighted DPD estimators were also obtained, with not a significant difference among the different distributions.
In this Section we have illustrated how the quasi minimum quasi weighted DPD estimators present a much more robust behavior than the maximum quasi weighted likelihood estimator for the estimation of $\beta$ when a moderate/large sample size was considered. In the Sect. 7.3, we will empirically compare this behaviour with that presented with another family of divergence estimators.
7.3 Comparison of minimum quasi weighted DPD estimators with pseudo minimum phi-divergence estimators

In Castilla et al. (2018), the pseudo minimum phi-divergence estimators, were defined in the PLR model with complex survey. In particular, pseudo minimum Cressie–Read divergence estimators for positive tuning parameter $\lambda$ were studied. Results of the simulation study suggested these estimators as an interesting alternative to maximum quasi weighted likelihood estimator in terms of efficiency when a small sample size and a large intra cluster correlation was considered. Nevertheless, the robustness of these estimators was not studied.

In this section we want to make a general empirical comparison between minimum quasi weighted DPD estimator and pseudo minimum phi-divergence estimators (with positive and negative tuning parameters in the Cressie–Read subfamily) of $\beta$, in terms of efficiency and robustness. Although they were not studied in the cited paper of Castilla et al. (2018), phi-divergences with negative tuning parameter and, in particular, the Hellinger distance ($\lambda = -0.5$), are known to have good robustness properties in some models. See for example Beran (1977) Lindsay (1994) and Toma (2007). This seems to happen in the context of PLR model with complex survey too, when a moderate/large sample size is considered. Nevertheless, pseudo minimum phi-divergence estimators with positive tuning parameter present an important lack of robustness. This behaviour can be summarized in Fig. 6.

Let us now consider Scenario 3 of the previous section. A comparison between quasi minimum quasi weighted DPD estimators with $\lambda \in \{0.4, 0.8\}$, classical maximum
Fig. 6  Pseudo minimum phi-divergence estimators: best choice of the tuning parameter, $m = 21$ and $\rho^2 = 0.5$

Fig. 7  Comparison among pseudo minimum phi-divergence and minimum quasi weighted DPD estimators
Fig. 8 Estimated levels (top) and powers (bottom) under pure (left) and contaminated (right) data. Scenario 1

quasi weighted likelihood estimator, and pseudo minimum phi-divergence estimators with $\lambda \in \{-0.5, -0.3, 0.66\}$ is made (top of Fig. 7). Hellinger distance is shown to be, by far, the best choice when a contaminated scheme with low intra-cluster correlation is considered. With medium/high intra-correlation the behaviour turns to be the opposite. Minimum quasi weighted DPD estimators present a more stable behaviour, both in pure and contaminated schemes, with independence to the correlation parameter.

The same divergences are considered now in Scenario 1 for the estimation of the parameter $\rho^2$ (bottom of Fig. 7). The estimation of $\rho^2$ with pseudo minimum phi-divergence estimators is made by the method of Binder (Castilla et al. 2018) while the estimation of $\rho^2$ with minimum quasi weighted DPD estimators is made by the method of the estimating equations. Pseudo minimum phi-divergence estimators with negative tuning parameter are not competitive neither in the pure nor in the contaminated schemes. Pseudo minimum phi-divergence estimator with tuning parameter $\lambda = 2/3$ is a good alternative to the minimum quasi weighted DPD estimators in contaminated data. This estimator showed also a good behaviour in Castilla et al. (2018).
7.4 Performance of the Wald-type tests

With the same model as in Sect. 7.1 we now empirically study the robustness of the minimum quasi weighted DPD estimator based Wald-type tests for the PLR model. As in Castilla et al. (2018) we first study the level under the true null hypothesis $H_0: \beta_{02} = 0.6$. For studying the power robustness, the true data generating parameter value is considered as $\beta_{02} = 1.08$.

Under Scenario 1, and both for pure and contaminated data, we compute observed levels and powers (measured as the proportion of test statistics exceeding the corresponding chi-square critical value), as can be seen in Fig. 8. Under pure data, all levels present a very similar behaviour, while power attains their best value for classical maximum quasi weighted likelihood estimator. Under contamination, minimum quasi weighted DPD estimator based Wald-type tests for $\lambda > 0$ present a more robust behaviour than maximum quasi weighted likelihood estimator, in concordance with previous results.

7.5 Coupling empirical results with theory: effects of $\lambda$

Let us now reconcile our empirical results obtained from simulation with the theoretical properties of the proposed estimators and Wald-type tests and investigate the overall effects of the choice of the tuning parameter $\lambda$. Firstly, in terms of the robustness, performances of both the proposed DPD based estimators and associated Wald-type tests are (empirically) seen to improve at any finite sample size under contamination as the tuning parameter $\lambda$ increases; bias and MSE of the estimators decrease and the stability of the level and power of the tests increase with increasing values of $\lambda > 0$. The case $\lambda = 0$ corresponds to the usual likelihood based procedure and is the most non-robust one in the lot. These robustness patterns are exactly in-line with the asymptotic theory of influence function as derived in Sect. 5 where the they are justified by theoretically measuring the effects of outlier contamination (classification error) in the asymptotic bias for each $\lambda \geq 0$ and arguing their redescending pattern with increasing $\lambda$ values.

On the contrary, in terms of efficiency under pure data, the proposed estimators and tests are (empirically) seen to have slightly deteriorating patterns with increasing values of the tuning parameter $\lambda \geq 0$. The case $\lambda = 0$ is clearly the most efficient one, since it is indeed the likelihood based one. To reconcile the empirical patterns for other $\lambda > 0$ with their (asymptotic) theoretical properties, we consider the asymptotic variance of the proposed DPD estimators; this also controls the power of the proposed Wald-type tests (as shown in the Supplementary material). We compare the asymptotic variances over different $\lambda \geq 0$ by computing the asymptotic relative efficiency (ARE), computed with respect to the most efficient likelihood based estimator at $\lambda = 0$. The values of ARE under some special PLR models, as in our simulation set-ups, are given in Table 6 (and few more cases are given in the Supplementary Material, Sect. 4). These results, again being in line with our numerical findings, indicate there is a slight loss in efficiency (asymptotically as well as for any finite sample sizes) with increasing values of $\lambda$ when no outlier is present in the data. However, in most cases, this loss in
### Table 6

| $\rho^2$ | $\beta_{01}$ | $\beta_{11}$ | $\beta_{21}$ | $\beta_{02}$ | $\beta_{12}$ | $\beta_{22}$ |
|----------|--------------|--------------|--------------|--------------|--------------|--------------|
| $0$      | 1.0000 0.0000 | 1.0000 1.0000 | 1.0000 1.0000 | 1.0000 1.0000 | 1.0000 1.0000 |
| $0.2$    | 0.9951 0.9924 | 0.9920 0.9936 | 0.9904 0.9897 | 0.9624 0.9438 | 0.9557 0.9389 | 0.9430 0.9430 |
| $0.4$    | 0.9817 0.9715 | 0.9715 0.9773 | 0.9672 0.9677 | 0.9391 0.9071 | 0.9122 0.9313 | 0.9107 0.9204 |
| $0.6$    | 0.9624 0.9419 | 0.9438 0.9557 | 0.9389 0.9430 | 0.9391 0.9071 | 0.9122 0.9313 | 0.9107 0.9204 |
| $0.8$    | 0.9391 0.9071 | 0.9122 0.9313 | 0.9107 0.9204 | 0.9391 0.9071 | 0.9122 0.9313 | 0.9107 0.9204 |

**efficiency is not quite significant at smaller values of $\lambda > 0$ (sometimes even for the $\lambda$ values near 0.8 as well).** Therefore, consistently from both the empirical and theoretical investigations, it is clear that the tuning parameter $\lambda$ controls the trade-offs between the efficiency of our proposed inference procedures under pure data and their robustness against data contamination for the PLR models under complex sampling scheme. This is in line with the DPD based procedures for other parametric model and data set-ups (Basu et al. 2011; Ghosh and Basu 2015) From our extensive simulations, we can suggest that small values of $\lambda$ present, generally, the best trade-off between efficiency and robustness for our PLRMs. However, a data driven procedure for selecting an optimal value of $\lambda$ would be practically more useful which can be done by extending the algorithms of Warwick and Jones (2005) and Ghosh and Basu (2015); see also Castilla et al. (2019) where this algorithm is discussed for the PLRMs under simple random sampling. We hope investigate this issue for the PLR models under the complex survey setting in detail in our future works.

### 8 Concluding remarks

We have made a contribution to fill an important gap existing in models with complex survey, since we present a robust alternative to classical approaches, in the modeling of categorical responses with associated covariates through polytomous logistic regression models. Minimum quasi weighted DPD estimators are introduced and also
Wald-type family of tests based on them, for the problem of testing linear hypotheses on regression coefficients. The robustness of both estimators and tests are theoretically justified in terms of the influence function, which is shown to be bounded for the proposed new procedures. An extensive simulation study is also provided, to empirically illustrate their robustness. The results clearly show how the proposed minimum quasi weighted DPD estimators seem to be the best choice for dealing with the robustness issue for a moderate sample size under the complex survey designs.

This method may be of special interest for analyzing demographic and health surveys, such as the one presented in Sect. 6.1, as well as overall complex surveys for developing countries; the resulting robust inferential insights would be extremely beneficial to plan appropriate population and health programs and policies. One possible drawbacks of the proposed method from an application point of view is that, unlike classical PMLE-based methods, the proposal is not yet implemented in any ready-to-run software product (such as SAS, R, Python, etc.). By the moment, we provide an example R-code in the Supplementary Material (Sect. 5) to the readers who want to apply it to their own work. We hope to develop appropriate R package in future for wider range of practitioners to enhance the practical applications of the proposed methodologies.

Finally, as pointed out by the Associate Editor, a related interesting problem is the estimation of totals in a finite population containing outliers under complex sample designs. Briefly, let us consider a finite population $U$ of size $N$ and one is interested in estimating the total $T_y = \sum_{i=1}^{N} y_i$ of the response variable $y$, based on a sample of $n$ elements drawn from the population; see, for example, Chambers (1986); Beaumont and Rivest (2009) or Beaumont et al. (2013). Assuming that the values of the $N$ population units are generated by a linear model of unknown parameters $\beta$ and $\sigma$, Chambers (1986, eq. 1) considered a linear estimator of $T$ based on $\beta$ and $\sigma$. He proposed to use a robust estimator of $\beta$, since the classical the standard least squares estimator is well-known to be non-robust. Similarly, the estimator of $\beta$ could be any robust other estimator developed for infinite populations. Beaumont et al. (2013, Eq.11) also presented a linear estimator of $T$. Following this idea, if the values of the $N$ population units are generated by the assumed PLR model, we may have the possibility to get an estimator of $T$ adapting the linear estimator considered by Chambers (1986) and Beaumont et al. (2013) to this context, but additionally considering our proposed quasi minimum DPD estimator as the estimator of $\beta$. This would undoubtedly be a very interesting and practically important research; however, considering the length and the heavy contents of the present paper, we have left its detailed investigation for a future work.

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