Approximation of a random process with variable smoothness

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Abstract

We consider the rate of piecewise constant approximation to a locally stationary process \( X(t), t \in [0, 1] \), having a variable smoothness index \( \alpha(t) \). Assuming that \( \alpha(\cdot) \) attains its unique minimum at zero and satisfies

\[
\alpha(t) = \alpha_0 + bt^\gamma + o(t^\gamma) \quad \text{as} \quad t \to 0,
\]

we propose a method for construction of observation points (composite dilated design) such that the integrated mean square error

\[
\int_0^1 \mathbb{E}\{(X(t) - X_n(t))^2\} \, dt \sim \frac{K}{n^{\alpha_0(\log n)^{(\alpha_0+1)/\gamma}}} \quad \text{as} \quad n \to \infty,
\]

where a piecewise constant approximation \( X_n \) is based on \( N(n) \sim n \) observations of \( X \). Further, we prove that the suggested approximation rate is optimal, and then show how to find an optimal constant \( K \).

Keywords: locally stationary processes, multifractional Brownian motion, piecewise constant approximation.

1 Introduction

Probabilistic models based on the locally stationary processes with variable smoothness became recently an object of interest and a convenient tool for applications in various areas (such as modelling of internet traffic or artificial landscapes, finance, geophysics, biomedicine, etc) due to their flexibility. The most known representative random process of this class is a multifractional Brownian motion (mBm) independently introduced in [6] and [22]. A more general class of \( \alpha(t) \)-locally stationary Gaussian processes with a variable smoothness index \( \alpha(t), t \in [0, 1] \), was elaborated in [15]. We refer to [3] for a survey and to [4, 5, 17, 27] for studies of particular aspects of mBm.

Whenever we need to model such processes with a given accuracy, the approximation (time discretization) accuracy has to be evaluated.

More specifically, consider a random process \( X(t), t \in [0, 1] \), with finite second moment and variable quadratic mean smoothness (see precise definition [1])
below). The process $X$ is observed at $N = N(n)$ points and a piecewise constant approximation $X_n$ is built upon these observations. The approximation performance on the entire interval is measured by integrated mean square error (IMSE) $\int_0^1 \mathbb{E}\{(X(t) - X_n(t))^2\} dt$. We construct a sequence of sampling designs (i.e., sets of observation points) taking into account the varying smoothness of $X$ such that on a class of processes, the IMSE decreases faster when compared to conventional regular sampling designs (see, e.g., [26]) or to quasi-regular designs, [2], used for approximation of locally stationary random processes and random processes with an isolated singularity point, respectively.

The approximation results obtained in this paper can be used in various problems in signal processing, e.g., in optimization of compressing digitized signals, (see, e.g., [12]), in numerical analysis of random functions (see, e.g., [7, 13, 14]), in simulation studies with controlled accuracy for functionals on realizations of random processes (see, e.g., [1, 16]). It is known that piecewise constant approximation gives an optimal rate for certain class of continuous random processes satisfying Hölder condition (see, e.g., [11, 26]). In this paper we develop a technique improving this rate for a certain class of locally stationary processes with variable smoothness. The developed technique can be generalized for more advanced approximation methods (e.g., Hermite splines) and various classes of random processes and fields.

Some related approximation results for continuous and smooth random functions can be found in [18, 19, 25]. The book [23] contains a very detailed survey of various random function approximation problems.

The paper is organized as follows. In Section 2 we specify the problem setting. We recall a notion of locally stationary process, introduce a class of piecewise constant approximation processes, and define integrated mean square error (IMSE) as a measure of approximation accuracy. Furthermore, we introduce a special method of composite dilated sampling designs that suggests how to distribute the observation points sufficiently densely located near the point of the lowest smoothness. The implementation of this design depends on some functional and numerical parameters, and we set up a certain number of mild assumptions about these parameters. In Section 3, our main results are stated. Namely, for a locally stationary process with known smoothness, we consider the piecewise constant interpolation related to dilated sampling designs (adjusted to smoothness parameters) and find the asymptotic behavior of its approximation error. In the second part of that section, the approximation for conventional regular and some quasi-regular sampling designs are studied. In Section 4, the results and conjectures related to optimality of our bounds are discussed. Section 5 contains the proofs of the statements from Section 3.
2 Variable smoothness random processes and approximation methods. Basic notation

2.1 Approximation problem setting

Let $X = X(t), t \in [0, 1]$, be an $\alpha(\cdot)$-locally stationary random process, i.e., $\mathbb{E}\{X(t)^2\} < \infty$ and

$$
\lim_{s \to 0} \frac{||X(t+s) - X(t)||^2}{|s|^\alpha(t)} = c(t) \text{ uniformly in } t \in [0, 1],
$$

where $||Y|| := (\mathbb{E}Y^2)^{1/2}, \alpha(\cdot), c(\cdot) \in C([0, 1])$ and $2 \geq \alpha(t) > 0, c(t) > 0$.

We assume that the following conditions hold for the function $\alpha(\cdot)$ describing smoothness of $X$:

(C1) $\alpha(\cdot)$ attains its global minimum $\alpha_0 := \alpha(0)$ at the unique point $t_0 = 0$.

(C2) there exist $b, \gamma > 0$ such that $\alpha(t) = \alpha_0 + bt^\gamma + o(t^\gamma)$ as $t \to 0$.

The choice $t_0 = 0$ in (C1) is of course just a matter of notation convenience. The results are essentially the same for any location of the unique minimum of $\alpha(\cdot)$.

Let $X$ be sampled at the distinct design points $T_n = (t_0(n), \ldots, t_N(n))$ (also referred as knots), where $0 = t_0(n) < t_1(n) < \cdots < t_N(n) = 1, N = N(n)$.

We suppress the auxiliary integer argument $n$ for design points $t_j = t_j(n)$ and for number of points $N = N(n)$ when doing so causes no confusion.

The corresponding piecewise constant approximation is defined by

$$
X_n(t) := X(t_{j-1}), \quad t_{j-1} \leq t < t_j, \ j = 1, \ldots, N.
$$

In this article, we consider the accuracy of approximation to $X$ by $X_n$ with respect to the integrated mean square error (IMSE)

$$
e_n^2 = ||X - X_n||_2^2 := \int_0^1 ||X(t) - X_n(t)||^2 dt.
$$

We describe now a construction of sampling designs $\{T_n\}$ providing the fastest decay of $e_n^2$.

2.2 Sampling design construction

The construction idea is as follows. In order to achieve a rate-optimal approximation of $X$ by $X_n$, we introduce a sequence of dilated sampling designs $\{T_n\}$.

Recall first that any probability density $f(t), t \in [0, 1]$, generates a sequence of associated conventional sampling designs, (cf., e.g., [24], [7], [26]) defined by

$$
\int_0^{t_j} f(t) dt = \frac{j}{n}, \quad j = 0, \ldots, n,
$$

i.e., the corresponding sampling points are $(j/n)$-percentiles of the density $f(\cdot)$, say, a sampling density.

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Let $p(\cdot)$ be a probability density on $\mathbb{R}_+ := [0, \infty)$. In our problem, it turns out to be useful to *dilate* the design density by replacing it with a *dilated sampling density*

$$p_n(t) := d_n p(d_n t), \quad t \in [0, 1],$$

where $d_n \nearrow \infty$ is a *dilation coefficient*. Note, that formally $p_n(\cdot)$ is not a probability density, but

$$\int_0^1 p_n(t) dt = \int_0^{d_n} p(u) du \to 1 \text{ as } n \to \infty.$$

The idea of dilation is obvious: we wish to put more knots near the point of the worst smoothness. The delation coefficient should be chosen accordingly to the smoothness behavior at this critical point. In our case, (C2) requires the choice

$$d_n := (\log n)^{1/\gamma}$$

that will be maintained in the sequel. As in (2), we define the knots by

$$\int_0^{t_j} p_n(t) dt = \frac{j}{n}. \quad (4)$$

Further optimization of the approximation accuracy bound requires one more adjustment: it turns out to be useful to choose the knots $t_j$ as in (4) using different densities in a neighborhood of the critical point and outside of it. We call composite such sampling design constructions operating differently on two disjoint domains.

Now we pass to the rigorous description of our sampling designs. Let $p(u)$ and $\tilde{p}(u), u \in [0, \infty)$, be two probability densities. Let the dilated densities $p_n(\cdot)$ be defined as in (3). Similarly, $\tilde{p}_n(t) := d_n \tilde{p}(d_n t)$.

For $0 < \rho < 1$, we define the *composite dilated* $(p, \rho, \tilde{p})$-designs $T_n$ by choosing $t_j$ according to (4) for

$$0 \leq j \leq J(p, \rho, n) := n \int_0^\rho p_n(t) dt = n \int_0^{d_n \rho} p(u) du \leq n.$$

Notice that for these knots, we have $0 \leq t_j \leq \rho$. Furthermore, we fill the interval $(\rho, 1]$ with analogous knots $t_i$ using density $\tilde{p}(\cdot)$,

$$\int_0^{t_i} \tilde{p}_n(t) dt = \frac{j}{n}, \quad (5)$$

where

$$J(\tilde{p}, \rho, n) < j \leq J(\tilde{p}, 1, n),$$

$$i = j + J(p, \rho, n) - J(\tilde{p}, \rho, n).$$

For these knots we clearly have $\rho < t_i \leq 1$. Note, it follows by definition that

$$J(p, \rho, n) \sim n \int_0^{d_n \rho} p(u) du \sim n \quad \text{as } n \to \infty,$$
and similarly, in the interval $[\rho, 1]$, the number of points does not exceed
\[ n - J(\tilde{p}, \rho, n) \sim n \int_{\rho u_0}^{\infty} \tilde{p}(u) du = o(n) \quad \text{as } n \to \infty, \]
that is the total number of sampling points satisfies
\[ N(n) \sim J(p, \rho, n) \sim n \quad \text{as } n \to \infty. \quad (6) \]

In the sequel, we will use $(p, \rho, \tilde{p})$-designs satisfying the following additional assumptions on $p(\cdot), \rho$, and $\tilde{p}(\cdot)$:

(A1) The design density $p(\cdot)$ is bounded, non-increasing, and
\[ p(u) \geq q_1 \exp\{-q_2 u^\gamma\}, \quad u \geq 0, \quad q_1 > 0, \quad \frac{b}{a_0} > q_2 > 0. \quad (7) \]

(A2) We assume that $\tilde{p}$ is regularly varying at infinity with some index $r \leq -1$. This means that for all $\lambda > 0$,
\[ \frac{\tilde{p}(\lambda u)}{\tilde{p}(u)} \to \lambda^r \quad \text{as } u \to +\infty. \quad (8) \]
In this case, we write $\tilde{p}(\cdot) \in \mathcal{R}_r(+\infty)$.

(A3) Finally, we assume that the parameter $\rho$ is small enough. Namely, applying $q_2 < b/a_0$ and using (C2) we may choose $\rho$ satisfying
\[ q_2 \sup_{0 \leq t \leq \rho} \alpha(t) < \inf_{0 \leq t \leq \rho} \frac{\alpha(t) - \alpha(0)}{t^\gamma}, \quad (9) \]
and
\[ q_2 \rho^\gamma < 1. \quad (10) \]
For example, let $\alpha(t) = 1 + t^\gamma$. Then (C1), (C2) hold and (A3) corresponds to $\rho < (1/q_2 - 1)^{1/\gamma}$, where $0 < q_2 < 1$.

Regularly varying densities satisfy (7) for large $u$, thus we could simplify the design construction by letting $p = \tilde{p}$. However, this simplified choice does not provide an optimal constant $K$ in the main approximation error asymptotics below.

3 Main results

3.1 Dilated approximation designs

In the following theorem, we give the principal result of the paper and consider IMSE $e_n^2$ of approximation to $X$ by $X_n$ for the proposed sequence of composite dilated sampling designs $T_n, n \geq 1$. It follows from (A1) that the following constant is finite,
\[ K = K(c, \alpha, (p, \rho, \tilde{p})) := \frac{c_0}{\alpha_0 + 1} \int_{0}^{\infty} p(u)^{-\alpha_0 - b/u \gamma} du < \infty, \]
where $c_0 := c(0)$. 

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Theorem 1 Let $X(t), t \in [0,1]$, be an $\alpha(\cdot)$-locally stationary random process such that assumptions (C1), (C2) hold. Let $X_n$ be the piecewise constant approximations corresponding to composite dilated $(p, \rho, \tilde{p})$-designs $\{T_n\}$ satisfying (A1)-(A3). Then $N(n) \sim n$ and

$$
\|X - X_n\|^2 \sim \frac{K}{n^{\alpha_0(\log n)^{\alpha_0+1}/\gamma}} \sim \frac{K}{N^{\alpha_0(\log N)^{\alpha_0+1}/\gamma}} \text{ as } n \to \infty.
$$

Remark 1. Among the assumptions of Theorem 1, the monotonicity of $p(\cdot)$ is worth of a discussion. Of course, it agrees with the heuristics to put more knots at places where the smoothness of the process is worse. However, this assumption may be easily replaced by some mild regularity assumptions on $p(\cdot)$.

Remark 2. The following density $p^*(\cdot)$

$$
p^*(u) = Ce^{-bu^{\gamma}/(\alpha_0+1)}, \quad C = \frac{b^{1/\gamma}}{(\alpha_0+1)^{1/\gamma} \Gamma(1/\gamma+1)}.
$$

minimizes the constant $K$ in Theorem 1 and generates the asymptotically optimal sequence of designs $T_n^*$. For the optimal $T_n^*$,

$$
K^* := \frac{c_0}{\alpha_0+1} \left( \int_0^{\infty} e^{-bu^{\gamma}/(\alpha_0+1)} du \right)^{\alpha_0+1} = \frac{c_0}{\alpha_0+1} \left( \frac{(\alpha_0+1)^{1/\gamma} \Gamma(1/\gamma+1)}{b^{1/\gamma}} \right)^{\alpha_0+1},
$$

see, e.g., [26]. We stress that $p^*(\cdot)$ satisfies assumption (7) but it is not regularly varying. In other words, a simple design based on $\tilde{p} = p = p^*$ does not fit in theorem’s assumptions.

Remark 3. The idea of considering composite designs might seem to be overcomplicated from the first glance. However, in some sense it can not be avoided. The previous remark shows that if we want to get the optimal constant $K$, we must handle the exponentially decreasing densities. Assume that

$$
p(u) \leq q_1 \exp\{-q_2 u^{\gamma}\}. \quad (12)
$$

If we would simplify the design by defining $t_j(n)$ as in (4) for the entire interval, i.e., with $\rho = 1$, then we would have

$$
\int_0^{t_j} p_n(t) dt = \frac{j}{n},
$$

hence,

$$
\frac{1}{n} = \int_{t_j}^{t_{j+1}} p_n(t) dt = \int_{t_j}^{t_{j+1}} d_n p(d_n t) dt \leq d_n q_1 \int_{t_j}^{t_{j+1}} \exp\{-q_2 (d_n t)^{\gamma}\} dt
$$

$$
\leq d_n q_1 (t_{j+1} - t_j) \exp\{-q_2 (d_n t_j)^{\gamma}\}.
$$

Let $a \in (0,1)$ and $t_j \in [1-a,1]$. Then for the length of corresponding intervals, we have

$$
t_{j+1} - t_j \geq \frac{\exp\{q_2 (d_n t_j)^{\gamma}\}}{nd_n q_1} \geq \frac{\exp\{q_2 \log n (1-a)^{\gamma}\}}{nd_n q_1}.
$$
If \( q_2 > 1 \) and \( a \) is so small that \( q_2(1-a)^\gamma > 1 \), we readily obtain \( t_{j+1} - t_j > a \) for large \( n \) which is impossible. Therefore, for \( q_2 > 1 \) there are no sampling points \( t_j \) in \([1-a,1]\), i.e., clearly \( e_n^2 > C > 0 \) for any \( n \), i.e., IMSE does not tend to zero at all.

The confusion described above may really appear in practice because \( q_2 > 1 \) is compatible with assumption \( q_2 < b_0 \alpha_0 \) from (7) whenever \( b > \alpha_0 \).

Theorem 1 shows that for the design densities with regularly varying tails, we may define all knots by (4) without leaving empty intervals as above. However, we can not achieve the optimal constant \( K \) on this simpler way.

**Remark 4.** Actually, the choice of knots outside of \([0, \rho]\) is not relevant for approximation rate. One can replace the knots from (5) with a uniform grid of knots \( t_i = i n^{-\mu} \) with appropriate \( \mu < 1 \).

### 3.2 Regular sampling designs

The approximation algorithm investigated in Theorem 1 is based upon the assumption that we know the point where \( \alpha(\cdot) \) attains its minimum, as well as the index \( \gamma \) in (C2). If for the same process neither critical point nor index \( \gamma \) are known, a conventional regular design can be used.

Let a random process \( X(t), t \in [0,1] \), be an \( \alpha(\cdot) \)-locally stationary, i.e., (1) hold. Consider now sampling designs \( T_n = \{t_j(n), j = 0,1,\ldots,n\} \) generated by a regular positive continuous density \( p(t), t \in [0,1] \), (see, e.g., [24], [26]) through (13), i.e.,

\[
\int_0^{t_j} p(t) dt = \frac{j}{n}, \quad 0 \leq j \leq n. \tag{13}
\]

Let the constant
\[
K_1 := \frac{c_0}{\alpha_0 + 1} \frac{\Gamma(1/\gamma + 1)}{p_0^{\alpha_0 b/\gamma}}, \quad p_0 := p(0).
\]

**Theorem 2** Let \( X(t), t \in [0,1] \), be an \( \alpha(\cdot) \)-locally stationary random process and (C1), (C2) hold. Let \( X_n \) be the piecewise constant approximations corresponding to (regular) sampling designs \( \{T_n\} \) generated by \( p(\cdot) \). Then

\[
||X - X_n||_2^2 \sim \frac{K_1}{n^{\alpha_0(\log n)^{1/\gamma}}} \quad \text{as } n \to \infty.
\]

**Remark 5.** If the point where \( \alpha(\cdot) \) attains its minimum, is known but \( \gamma \) is unknown, we may build the designs without dilating the density. Instead, one could use quasi-regular sampling designs generated by a possibly unbounded density \( p(t), t \in (0,1] \), at the singularity point \( t_0 = 0 \) (cf., [2]). For example, if \( p(\cdot) \) is a density on \((0,1]\) such that

\[
p(t) \sim A t^{-\kappa} \quad \text{as } t \searrow 0, \quad 0 < \kappa < 1,
\]

and \( t_j(n) \) are chosen through (13), then for a locally stationary process \( X \) satisfying (C1) and (C2), it is possible to show a slightly weaker asymptotics than that of Theorem 1

\[
e_n^2 \sim \frac{K_2}{n^{\alpha_0(\log n)^{(1+\kappa\alpha_0)/\gamma}}} \quad \text{as } n \to \infty,
\]
with $K_2 := c_0 A^{-\alpha_0} \Gamma(1/\gamma + 1)/((\alpha_0 + 1)b^{1/\gamma})$.

Of course, all above mentioned asymptotics differ only by a degree of logarithm while the polynomial rate is determined by the minimal regularity index $\alpha_0$.

4 Optimality

4.1 Optimality of the rate for piecewise constant approximations

We explain here that the approximation rate $l_n^{-1} = n^{-\alpha_0} d_n^{-(\alpha_0+1)}$ achieved in Theorem 1 is optimal in the class of piecewise constant approximations for every locally stationary process satisfying (C1) and (C2). For a sampling design $T_n$, let the mesh size $|T_n| := \max\{(t_j - t_{j-1}), j = 1, \ldots, n\}$.

**Proposition 1** Let $X_n$ be piecewise constant approximations to a locally stationary process $X$ satisfying (C1) and (C2) constructed according to designs $\{T_n\}$ such that $N_n \sim n$ and $|T_n| \to 0$ as $n \to \infty$. Then

$$\liminf_{n \to \infty} l_n e_n^2 > 0.$$  \hspace{1cm} (14)

4.2 Optimality of the rate in a class of linear methods

We explain here that the approximation rate $l_n^{-1}$ achieved in Theorem 1 is optimal not only in the class of piecewise constant approximations but in a much wider class of linear methods, – at least for some locally stationary processes satisfying (C1) and (C2). The corresponding setting is based on the notion of Gaussian approximation numbers, or $\ell$-numbers, that we recall here.

Gaussian approximation numbers of a Gaussian random vector $X$ taking values in a normed space $\mathcal{X}$ are defined by

$$\ell_n(X; \mathcal{X})^2 = \inf_{\xi_1, \ldots, \xi_{n-1}} \mathbb{E} \left\{ \left\| X - \sum_{j=1}^{n-1} \xi_j x_j \right\|_{\mathcal{X}}^2 \right\},$$  \hspace{1cm} (15)

where infimum is taken over all $x_j \in \mathcal{X}$ and all Gaussian vectors $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, see [20, 21]. If $\mathcal{X}$ is a Hilbert space, then

$$\ell_n(X; \mathcal{X})^2 = \sum_{j=n}^{\infty} \lambda_j,$$

where $\lambda_j$ is a decreasing sequence of eigenvalues of covariance operator of $X$.

Recall that a multifractional Brownian motion (mBm) with a variable smoothness index (or fractality function) $\alpha(\cdot) \in (0, 2)$ introduced in [6, 22] and studied in [3, 4, 5] is a Gaussian process defined through its white noise representation

$$X(t) = \int_{-\infty}^{\infty} e^{itu} - 1 \frac{1}{|u|^{(\alpha(t)+1)/2}} dW(u),$$
where $W(t), t \in \mathbb{R}$, is a conventional Brownian motion. Notice that mBm is a typical example of a locally stationary process whenever $\alpha(\cdot)$ is a continuous function.

In particular case of the constant fractality $\alpha(t) \equiv \alpha$, we obtain an ordinary fractional Brownian motion $B^\alpha, \alpha \in (0, 2)$. For $X = B^\alpha$ considered as an element of $\mathcal{X} = L_2[0, 1]$, the behavior of eigenvalues $\lambda_j$ is well known, cf. [9]. Namely,

$$\lambda_j \sim c_\alpha j^{-\alpha - 1} \quad \text{as } j \to \infty$$

with some $c_\alpha > 0$ continuously depending on $\alpha \in (0, 2)$. It follows that

$$\ell_n(B^\alpha; L_2[0, 1])^2 \sim \alpha^{-1} c_\alpha n^{-\alpha} \quad \text{as } n \to \infty.$$ 

Hence, for all $n \geq 1$,

$$\ell_n(B^\alpha; L_2[0, 1])^2 \geq C_\alpha n^{-\alpha}.$$ 

Furthermore, since $B^\alpha$ is a self-similar process, we can scale this estimate from $\mathcal{X} = L_2[0, 1]$ to $\mathcal{X} = L_2[0, r]$ with arbitrary $r > 0$. An easy computation shows that

$$\ell_n(B^\alpha; L_2[0, r])^2 = r^{\alpha+1} \ell_n(B^\alpha; L_2[0, 1])^2 \geq C_\alpha r^{\alpha+1} n^{-\alpha}.$$ 

Let us now consider a multifractional Brownian motion $X$ parameterized by a fractality function $\alpha(\cdot)$ satisfying (C2). For example, let

$$\alpha(t) := \alpha_0 + b t^\gamma, \quad 0 \leq t \leq 1,$$ 

with $\alpha_0, b > 0$ chosen so small that $\alpha_0 + b < 2$. This choice secures the necessary condition

$$0 < \alpha(t) < 2, \quad 0 \leq t \leq 1.$$ 

Then, letting $r = r_n := d_n^{-1}$, we have

$$\ell_n(X; L_2[0, 1])^2 \geq \ell_n(X; L_2[0, r_n])^2 \geq M \ell_n(B^{\alpha(r_n)}; L_2[0, r_n])^2 \geq M C_{\alpha(r_n)} r_n^{\alpha(r_n)+1} n^{-\alpha(r_n)} = M C_{\alpha(r_n)} d_n^{-\alpha(r_n)-1} n^{-\alpha(r_n)} \geq C l_n^{-1}(d_n n)^{\alpha_0-\alpha(r_n)} = C l_n^{-1}(d_n n)^{-b \gamma} \geq C l_n^{-1},$$

for some positive $M, C_{\alpha(r_n)}, C, \tilde{C}$. All bounds here are obvious except for the second inequality comparing approximation rate of multifractional Brownian motion with that of a conventional fractional Brownian motion.

We state this fact as a separate result.

**Proposition 2** Let $X(t), a \leq t \leq b$, be a multifractional Brownian motion corresponding to a continuous fractality function $\alpha : [a, b] \to (0, 2)$. Let $B^\beta$ be a fractional Brownian motion such that $\inf_{a \leq t \leq b} \alpha(t) \leq \beta < 2$. Then there exists $M = M(\alpha(\cdot), \beta) > 0$ such that

$$\ell_n(X; L_2[a, b]) \geq M \ell_n(B^\beta; L_2[a, b]), \quad n \geq 1.$$
The proof of this proposition requires the methods very different from those used in this article. We relegate it to another publication.

Our conclusion is that a multifractional Brownian motion with fractality function \( (16) \) provides an example of a locally stationary process satisfying assumptions \((C_1)\) and \((C_2)\) such that no linear approximation method provides a better approximation rate than \( l_n^{-1} \).

5 Proofs

Proof of Theorem 1

We represent the IMSE \( e_n^2 = \|X(t) - X_n(t)\|^2 \) as the following sum

\[
e_n^2 = \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \|X(t) - X_n(t)\|^2 dt = \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \|X(t) - X(t_{j-1})\|^2 dt =: \sum_{j=1}^{N} e_{n,j}^2.
\]

Next, for a large \( U > 0 \), let

\[
e_n^2 = \sum_{j=1}^{N} e_{n,j}^2 = S_1 + S_2 + S_3,
\]

where the sums \( S_1, S_2, S_3 \) include the terms \( e_{n,j}^2 \) such that \([t_{j-1}, t_j] \) belongs to \([0, U/d_n], [U/d_n, \rho], \) and \([\rho, 1]\), respectively. Let \( J_1 \) and \( J_2 \) denote the corresponding boundaries for index \( j \). Denote by

\[
l_n := n^{\alpha_0} d_n^{\alpha_0 + 1} = n^{\alpha_0} (\log n)^{(\alpha_0 + 1)/\gamma}.
\]

Recall that \( l_n^{-1} \) is the approximation rate announced in the theorem. We show that only \( S_1 \) is relevant to the asymptotics of \( e_n^2 \), namely, that \( l_n S_3 = o(1) \) as \( n \to \infty \), while

\[
\limsup_{n \to \infty} l_n S_2 = o(1) \quad \text{as } U \to \infty.
\]

Let \( w_j := t_j - t_{j-1}, u_j := d_n t_j \) be the normalized knots and denote by \( v_j := u_j - u_{j-1} = d_n w_j \) the corresponding dilated interval lengths.

It follows by the definition of \( \alpha(\cdot) \)-local stationarity \([11]\) that for large \( n \),

\[
e_{n,j}^2 = c(t_{j-1}) \int_{t_{j-1}}^{t_j} (t - t_{j-1})^{\alpha(t_{j-1})} dt (1 + r_{n,j})
= B_{j-1} (t_j - t_{j-1})^{\alpha(t_{j-1}) + 1} (1 + r_{n,j})
= B_{j-1} (v_j/d_n)^{\alpha(t_{j-1}) + 1} (1 + r_{n,j}),
\]

where \( |T_n| = \max_j w_j = o(1) \) and \( \max_j r_{n,j} = o(1) \) as \( n \to \infty \) and

\[
B_j := \frac{c(t_j)}{\alpha(t_j) + 1}, \quad j = 1, \ldots, N.
\]

First, we evaluate \( S_3 \). Recall that for \( j > J_2 \) we have \( \rho d_n \leq u_{j-1} < u_j \leq d_n \).

We use now the following property of regularly varying functions (see, e.g., \([10]\)):
convergence in \(K\) is uniform for all intervals \(0 < a \leq \lambda \leq b < \infty\). Using this uniformity we obtain, for some \(C_1 > 0\),

\[
\inf_{u_{j-1} \leq u \leq u_j} \tilde{p}(u) \geq \inf_{p_{d_n} \leq u \leq d_n} \tilde{p}(u) \geq C_1 \tilde{p}(d_n).
\]

It follows by (5) that

\[
\int_{u_{j-1}}^{u_j} \tilde{p}(u)du = \int_{t_{j-1}}^{t_j} \tilde{p}_n(t)dt = \frac{1}{n}.
\]

Hence, for some \(C_2 > 0\),

\[
v_j \leq \left( \inf_{u_{j-1} \leq u \leq u_j} \tilde{p}(u) \right)^{-1} \int_{u_{j-1}}^{u_j} \tilde{p}(u)du \leq \frac{1}{n} \left( \inf_{u_{j-1} \leq u \leq u_j} \tilde{p}(u) \right)^{-1}
\]

\[
\leq \frac{1}{C_1 n \tilde{p}(d_n)} \leq C_2 \frac{d_n^{\alpha_1+1}}{n}, \quad j = J_2 + 1, \ldots, N, \quad (20)
\]

and \(\max_{j > J_2} w_j = d_n^{\alpha_1}/n\). Recall that by assumption (C1),

\[
\alpha_1 := \inf_{t \in [\rho, 1]} \alpha(t) > \alpha_0.
\]

Therefore, for large \(n\), we get by (19) and (20), \(C_3, C_4 > 0\),

\[
S_3 \leq n \max_{j > J_2} c_{n,j}^2 \leq nC_3(v_j/d_n)^\alpha_1+1 \leq C_4 \frac{d_n^{\alpha_1+1}}{n^{\alpha_1}} = o(l_n^{-1}) \text{ as } n \to \infty. \quad (21)
\]

Now consider the first two zones corresponding to \(S_1, S_2\). We have by definition

\[
\int_0^{u_j} p(u)du = \frac{j}{n}, \quad 0 \leq j < J.
\]

Since the function \(p_n(t), t \in [0, 1]\), is non-increasing, the sequence \(\{v_j\}\) is non-decreasing. In fact,

\[
\frac{1}{n} = \int_{u_{j-1}}^{u_j} p(u)du \in [p(u_j)v_j, p(u_{j-1})v_j],
\]

and therefore,

\[
\frac{1}{np(u_{j-1})} \leq v_j \leq \frac{1}{np(u_j)} \leq v_{j+1} \quad (22)
\]

and it follows by (A1) that \(\max_{j \leq J_2} w_j = o(1)\) as \(n \to \infty\). For \(j \leq J_2\), the bounds (19) and (22) yield for \(n\) large,

\[
e_{n,j}^2 = B_{j-1}(v_j/d_n)^{\alpha(t_{j-1})} \frac{v_j}{d_n}(1 + o(1))
\]

\[
\leq B_{j-1}(n d_n p(u_j))^{-\alpha(t_{j-1})} \frac{v_j}{d_n}(1 + o(1))
\]

\[
\leq B_{j-1}(n p(u_j))^{-\alpha(t_{j-1})} d_n^{\alpha_0-1} v_j(1 + o(1))
\]

\[
= B_{j-1} d_n^{\alpha(t_{j-1})-\alpha_0} p(u_j)^{-\alpha(t_{j-1})} v_j(1 + o(1)). \quad (23)
\]

From now on, we proceed differently in the first and in the second zone.
For the second zone, \( J_1 \leq j \leq J_2 \), we do not care about the constant by using
\[
B_j \leq B_* := \max_{0 \leq t \leq 1} \frac{c(t)}{\alpha(t) + 1}.
\] (24)
Next, (7) and (9) give
\[
p(u_j)^{-\alpha(t_{j-1})} \leq C \exp\{q_2 \alpha(t_{j-1})u_j^\gamma\} \leq C \exp\{\beta_1 u_j^\gamma\} \quad C > 0,
\] (25)
where \( \beta_1 := q_2 \sup_{0 \leq t \leq \rho} \alpha(t) \). On the other hand, we infer from (9) that
\[
n^{-(\alpha(t_{j-1}) - \alpha_0)} = n^{-\frac{\alpha(t_{j-1}) - \alpha_0}{t_{j-1}}} t_{j-1}^\gamma \leq n^{-\frac{\beta_2}{\rho(1)}} \exp\{-\beta_2 u_{j-1}^\gamma\},
\] (26)
where \( \beta_2 := \inf_{0 \leq t \leq \rho} (\alpha(t) - \alpha_0)/t^\gamma > 0 \) by (9).
Recall that by (10), we have \( 1 - q_2 \rho^\gamma > 0 \). Moreover, for \( U \leq u_j \leq \rho d_n \), we derive from (7) and (22)
\[
v_j \leq n^{-1} p(\rho d_n)^{-1} \leq C n^{-1} \exp\{q_2(\rho d_n)^\gamma\} = C n^{-1(1 - q_2 \rho \gamma)}, \quad C > 0,
\]
and it follows
\[
u_{j+1}^\gamma - v_j^\gamma = v_j^\gamma \left( \frac{u_{j+1}}{u_{j-1}} \right)^\gamma - 1 = O\left( d_n^\gamma n^{-1(1 - q_2 \rho \gamma)} \right) = o(1) \text{ as } n \to \infty
\] (27)
uniformly in \( J_1 \leq j \leq J_2 \).
Since \( \{v_j\} \) is non-decreasing, (27) implies an integral bound
\[
\exp\{ -\beta_2 u_{j-1}^\gamma \} \exp\{\beta_1 u_j^\gamma\} v_j = \exp\{\beta_2[u_{j+1}^\gamma - u_{j-1}^\gamma]\} \exp\{\beta_1 u_j^\gamma - \beta_2 u_{j+1}^\gamma\} v_j \leq C \inf_{u_j \leq u \leq u_{j+1}} \exp\{\beta_1 u^\gamma - \beta_2 u^\gamma\} v_{j+1}
\leq C \int_{u_j}^{u_{j+1}} e^{-(\beta_2 - \beta_1)u^\gamma} du, \quad C > 0.
\] (28)
By plugging (24), (25), and (28) into (23), and summing up the resulting bounds over \( J_1 < j \leq J_2 \), we obtain
\[
S_2 \leq B_* l_n^{-1} \int_U^\infty e^{-(\beta_2 - \beta_1)u^\gamma} du (1 + o(1)) \quad \text{as } n \to \infty.
\] (29)
Therefore, (18) is valid.
In the first zone, \( j \leq J_1 \), \( t_j \leq U/d_n \), the knots are uniformly small. Hence, \( B_{j-1} \) are uniformly close to \( B \) due to the continuity of the functions \( \alpha(\cdot) \) and \( c(\cdot) \). Moreover, by (C2) for any \( \varepsilon > 0 \), we have for all \( n \) large enough
\[
\alpha_0 + (b - \varepsilon) t_{j-1}^\gamma \leq \alpha(t_{j-1}) \leq \alpha_0 + (b + \varepsilon) t_{j-1}^\gamma, \quad j \leq J_1.
\] (30)
Hence (23) yields
\[
e_n^{2} \leq (B + \varepsilon) l_n^{-1} n^{-\alpha_0} p(u_j)^{-\alpha_0} p(u_j)^{-\alpha_0} v_j \leq (B + \varepsilon) l_n^{-1} n^{-\alpha_0} p(u_j)^{-\alpha_0} p(u_j)^{-\alpha_0} v_j.
\] (31)
Recall that by the definition of $d_n$, we have
\[ n^{-(b-\epsilon)(u_{j-1}/d_n)} = n^{-(b-\epsilon)u_{j-1}^\gamma/(\log n) } = \exp\{-(b-\epsilon)u_{j-1}^\gamma\} \]

Since $p(\cdot)$ is non-increasing and $\{v_j\}$ is non-decreasing, we also have an integral bound
\[ \exp\{-(b-\epsilon)u_{j-1}^\gamma\} \cdot p(u_j)^{-\alpha_0} v_j \]
\[ = \exp\{(b-\epsilon)[u_{j+1}^\gamma - u_{j-1}^\gamma]\} \cdot p(u_j)^{-\alpha_0} \exp\{-(b-\epsilon)u_{j+1}^\gamma\} v_j \]
\[ \leq \exp\{(b-\epsilon)[u_{j+1}^\gamma - u_{j-1}^\gamma]\} \inf_{u_j \leq u \leq u_{j+1}} (p(u)^{-\alpha_0} e^{-\epsilon u^\gamma} v_j \]
\[ \leq \exp\{(b-\epsilon)[u_{j+1}^\gamma - u_{j-1}^\gamma]\} \int_{u_j}^{u_{j+1}} p(u)^{-\alpha_0} e^{-\epsilon u^\gamma} du. \quad (32) \]
Moreover, for $u_j \leq U$, we derive from (A1) and (22)
\[ v_j \leq n^{-1} p(U)^{-1}. \]
By using convexity and concavity of the power function for $\gamma \geq 1$ and $\gamma \leq 1$, respectively, we get
\[ u_{j+1}^\gamma - u_{j-1}^\gamma \leq \gamma U^{\gamma-1}(u_{j+1} - u_j) = \gamma U^{\gamma-1}(v_j + v_{j+1}) \]
\[ \leq 2\gamma U^{\gamma-1}v_{j+1} = O(1) \quad \text{as } n \to \infty \quad (\gamma \geq 1); \]
\[ u_{j+1}^\gamma - u_{j-1}^\gamma \leq (u_{j+1} - u_{j-1})^\gamma \]
\[ = (v_j + v_{j+1})^\gamma = O(1) \quad \text{as } n \to \infty \quad (\gamma \leq 1). \]
Therefore, the exponential factor in (32) turns out to be negligible.
Finally, for $u_j \leq U$, the property $d_n \to \infty$ yields
\[ p(u_j)^{-\alpha(t_j-\alpha_0)} \leq \max\{1, p(U)^{-\alpha_{\max} \leq t \leq U/d_n} (\alpha(t) - \alpha_0)\} = 1 + o(1). \quad (33) \]
By plugging (32) and (33) into (31), and summing up the resulting bounds over $j \leq J_1$, we obtain
\[ S_1 \leq (B + 2\epsilon) l_n^{-1} \int_0^\infty p(u)^{-\alpha_0} e^{-(b-\epsilon)u^\gamma} du \quad \text{as } n \to \infty. \]
Since $\epsilon$ can be chosen arbitrarily small, we arrive at
\[ \limsup_{n \to \infty} l_n S_1 \leq B \int_0^\infty p(u)^{-\alpha_0} e^{-bu^\gamma} du = K. \quad (34) \]
Combining (21), (29), and (31) gives the desired upper bound.
The lower bound is obtained along the same lines: we neglect $S_2$ and $S_3$, and evaluate $S_1$ starting again from (19). As in (23), we have
\[ e_{n,j}^2 = B_{j-1} (v_j/d_n)^{\alpha(t_j-\alpha_0)} v_j / d_n (1 + o(1)) \]
\[ \geq B_{j-1} (n d_n p(u_{j-1}))^{-\alpha(t_j-\alpha_0)} v_j / d_n (1 + o(1)) \]
\[ = B_{j-1} n^{-\alpha_0} n^{-\alpha(t_j-\alpha_0)} p(u_{j-1})^{-\alpha(t_j-\alpha_0)} d_n^{-\alpha_0} d_n^{-\alpha(t_j-\alpha_0)} v_j (1 + o(1)) \]
\[ = B_{j-1} l_n^{-1} n^{-\alpha(t_j-\alpha_0)} p(u_{j-1})^{-\alpha(t_j-\alpha_0)} d_n^{-\alpha(t_j-\alpha_0)} v_j (1 + o(1)). \quad (35) \]
Recall that for \( j \leq J_1 \), coefficients \( B_{j-1} \) are uniformly close to \( B \). Moreover, by using (30), we have for large \( n \),
\[
d_n^{\alpha_0 - \alpha(t_{j-1})} \geq d_n^{-(b+\epsilon)\alpha_0} \geq d_n^{-(b+\epsilon)(U/d_n)^\gamma} = 1 + o(1).
\]
Hence, (35) yields
\[
e_n^{2} \geq (B - \epsilon)l_n^{-1} n^{-(b+\epsilon)\alpha_0} p(u_{j-1})^{-\alpha_0} p(u_{j-1})^{-(\alpha(t_{j-1})-\alpha_0)} v_j
= (B - \epsilon)l_n^{-1} n^{-(b+\epsilon)(u_{j-1}/d_n)^\gamma} p(u_{j-1})^{-\alpha_0} p(u_{j-1})^{-(\alpha(t_{j-1})-\alpha_0)} v_j, \tag{36}
\]
where as before
\[
n^{-(b+\epsilon)(u_{j-1}/d_n)^\gamma} = n^{-(b+\epsilon)u_{j-1}/\log n} = \exp\{-\epsilon n u_{j-1}\}.
\]
Since \( p(\cdot) \) is non-increasing and \( \{v_j\} \) is non-decreasing, we also have an integral bound
\[
\exp\{-\epsilon n u_{j-1}\} v_j = \exp\{(b+\epsilon)\max_{j-1} u_j\} \exp\{-\epsilon n u_{j-1}\} \geq \exp\{(b+\epsilon)\max_{j-1} u_j\} \int_{u_{j-2}}^{u_{j-1}} p(u)^{-\alpha_0} e^{-(b+\epsilon)u} \, du. \tag{37}
\]
We have already seen that the exponential factor in (37) is negligible.

Finally, for \( u_j \leq U \), the property \( d_n \to \infty \) implies (cf., (33))
\[
p(u_{j-1})^{-(\alpha(t_{j-1})-\alpha_0)} \geq \min\{1, p(0)^{-\max_{0 \leq t \leq U/d_n}(\alpha(t)-\alpha_0)} \} = 1 + o(1). \tag{38}
\]
By plugging (37) and (38) into (36), and summing up the resulting bounds over \( j \leq J_1 \), we obtain
\[
S_1 \geq (B - 2\epsilon)l_n^{-1} \int_0^U p(u)^{-\alpha_0} e^{-(b+\epsilon)u} \, du \quad \text{as } n \to \infty.
\]

Since \( \epsilon \) can be chosen arbitrarily small, we arrive to
\[
\liminf_{n \to \infty} l_n S_1 \geq B \int_0^U p(u)^{-\alpha_0} e^{-bu} \, du.
\]
Finally,
\[
\liminf_{n \to \infty} l_n e_n^2 \geq \sup_{U > 0} \liminf_{n \to \infty} l_n S_1 \geq B \int_0^\infty p(u)^{-\alpha_0} e^{-bu} \, du = K. \tag{39}
\]
This is the desired lower bound.

Proof of Theorem 2

Applying the notation of Theorem 1, we have for an interval approximation error
\[
e_n^{2} = B_{j-1} w_j^{\alpha(t_{j-1})+1} (1 + r_{n,j}), \quad w_j = t_j - t_{j-1}, j = 1, \ldots, n,
\]
where \( \max_j r_{n,j} = o(1) \) as \( n \to \infty \). Now for a small enough \( \rho > 0 \), similarly to Theorem 1 we get
\[
\int_\rho^1 e_n(t)^2 dt \leq C/n^{\alpha_1}, \quad C > 0, \quad \alpha_1 := \inf_{t \in [\rho, 1]} \alpha(t) > \alpha_0,
\]
that is only \( e_{n,j} \) such that \([t_{j-1}, t_j] \subset [0, \rho] \) are relevant for the asymptotics, say, \( e_{n,j}, j = 1, \ldots, J = J(\rho, n) \). Let us denote the approximation rate \( L_n := n^{\alpha_0}(\log n)^{1/\gamma} \).

Next, for \( S_1 := \sum_{j}^J e_{n,j} \) and small enough \( \rho \), we have by continuity of the density \( p(\cdot) \) and the mean value theorem
\[
e_{n,j}^2 = B_{j-1} (np(\eta_j))^{-\alpha(t_j-1)} w_j (1 + o(1)) \]
\[
\leq \frac{B}{p(0)^{\alpha_0}} (1 + \epsilon) n^{-\alpha_0} \int_{t_{j-1}}^{t_j} e^{-(b-\epsilon) t \gamma} \log n \, dt (1 + o(1)) \]
\[
= L_n^{-1} \frac{B}{p(0)^{\alpha_0}} (1 + \epsilon) \int_{u_{j-2}}^{u_{j-1}} e^{-(b-\epsilon) u \gamma} \, du (1 + o(1)),
\]
where \( p_0 := p(0) \). Now by summing up, we obtain
\[
\limsup_{n \to \infty} L_n S_1 \leq \frac{B}{p(0)^{\alpha_0}} (1 + \epsilon) \int_0^\infty e^{-(b-\epsilon) u \gamma} \, du = \frac{B}{p(0)^{\alpha_0}} (1 + \epsilon) \frac{\Gamma(1/\gamma + 1)}{(b-\epsilon)^{1/\gamma}}
\]
Hence,
\[
\limsup_{n \to \infty} L_n e_n^2 = \limsup_{n \to \infty} L_n S_1 \leq \frac{B}{p_0^{\alpha_0}} (1 + \epsilon) \frac{\Gamma(1/\gamma + 1)}{(b-\epsilon)^{1/\gamma}}.
\]
Since \( \epsilon \) can be chosen arbitrary small, we get
\[
\limsup_{n \to \infty} L_n e_n^2 \leq \frac{B}{p_0^{\alpha_0}} \frac{\Gamma(1/\gamma + 1)}{b^{1/\gamma}} = K_1.
\]
The lower bound follows similarly. This completes the proof. \( \square \)

**Proof of Proposition 1.**

Let \( r_n := d_n^{-1} = (\log n)^{-1/\gamma} \) and \( J_n := \inf \{ j : t_j = t_j(n) \geq r_n \} \). Then (19) implies
\[
e_n^2 \geq \sum_{j=1}^{J_n} e_{n,j}^2 = \sum_{j=1}^{J_n} B_{j-1} w_j^{\alpha(t_j-1)+1} (1 + o(1)) = B \sum_{j=1}^{J_n} w_j^{a_n+1} (1 + o(1)),
\]
where \( a_n := \sup_{0 \leq t \leq r_n} \alpha(t) \) and \( w_j = t_j - t_{j-1} \). By using convexity of the power function \( w \to w^{a_n+1} \), we obtain
\[
\frac{1}{J_n} \sum_{j=1}^{J_n} w_j^{a_n+1} \geq \left( \frac{1}{J_n} \sum_{j=1}^{J_n} w_j \right)^{a_n+1} \geq \left( \frac{r_n}{J_n} \right)^{a_n+1},
\]
hence,
\[
\sum_{j=1}^{J_n} w_j^{a_n+1} \geq \frac{r_n^{a_n+1}}{J_n} \geq \frac{r_n^{a_n+1}}{N_n^{a_n}},
\]
whereas

\[ e_n^2 \geq B r_n^{\alpha_n+1} \frac{1}{N_n^{\alpha_n}} (1 + o(1)) = B \frac{1}{d_n^{\alpha_n+1}} \left( 1 + o(1) \right) \]

\[ = B \frac{1}{d_n^{\alpha_n+1}} \left( \frac{1}{n} \right)^{\alpha_n-\alpha_0} \left( \frac{n}{N_n} \right)^{\alpha_n} (1 + o(1)) \]

\[ = B l_n^{-1} \left( \frac{1}{d_n} \right)^{\alpha_n-\alpha_0} (1 + o(1)). \]

Recall that by (C2), \( a_n - \alpha_0 = O(r_n^\gamma) = O((\log n)^{-1}) \) and thus \( (14) \) follows. \( \square \)

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