The Alon-Tarsi number of Halin graphs

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Abstract

The Alon-Tarsi number of a graph $G$ is the smallest $k$ for which there is an orientation $D$ of $G$ with max outdegree $k - 1$ such that the number of Eulerian subgraphs of $G$ with an even number of edges differs from the number of Eulerian subgraphs with an odd number of edges. In this paper, we obtain the Alon-Tarsi number of a Halin graph equals 4 when it is a wheel of even order and 3 otherwise.

Keywords: Alon-Tarsi number; list chromatic number; chromatic number; Halin graph

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1 Introduction

All graphs considered in this article are finite, and all graphs are either simple graphs or simple directed graphs. A $k$-list assignment of a graph $G$ is a mapping $L$ which assigns to each vertex $v$ of $G$ a set $L(v)$ of $k$ permissible colors. Given a $k$-list assignment $L$ of $G$, an $L$-coloring of $G$ is a mapping $\phi$ which assigns to each vertex $v$ a color $\phi(v) \in L(v)$ such that $\phi(u) \neq \phi(v)$ for every edge $e = uv$ of $G$. A graph $G$ is $k$-choosable if $G$ has an $L$-coloring for every $k$-list assignment $L$. The choice number of a graph $G$ is the least positive integer $k$ such that $G$ is $k$-choosable, denoted by $ch(G)$.

In the classic article[1], Alon and Tarsi have obtained the upper bound for the choice number of graphs by applying algebraic techniques, which was later called the Alon-Tarsi number of $G$, and denoted by $AT(G)$ (See e.g. Jensen and Toft (1995) [2]). They have transformed the computation of the Alon-Tarsi number of $G$ from algebraic manipulations to the analysis of the structural properties of $G$. Their characterization can be restated as follows. The Alon-Tarsi number of $G$, $AT(G)$, is the smallest $k$ for which there is an orientation $D$ of $G$ with max
outdegree $k - 1$ such that the number of odd Eulerian subgraphs of $G$ is not the same as the number of even Eulerian subgraphs of $G$.

As pointed out by Hefetz [3], the Alon-Tarsi number has some different features and we are interested in studying $AT(G)$ as an independent graph invariant. Let $\chi_p(G)$ be the paint number of $G$ [4]. In [5], U. Schauz generalizes the result of Alon and Tarsi [1]: $ch(G) \leq \chi_p(G) \leq AT(G)$ for any graph $G$ and the equalities are not hold in general. Nevertheless, it is also known that the upper bounds of the choice number and the Alon-Tarsi number are the same for several graph classes. For example, In [6], Thomassen has shown that the choice number of any planar graph is at most 5, and it was proved by Schauz in [7] that every planar graph $G$ satisfies $\chi_p(G) \leq 5$. As a strengthening of the results of Thomassen and Schauz, X. Zhu proves that every planar graph $G$ has $AT(G) \leq 5$ by Alon-Tarsi polynomial method and $AT$-orientation method [8]. It is of interest to find graph $G$ for which these parameters are equal.

Furthermore, Grytczuk and X. Zhu have used polynomial method to prove that every planar graph $G$ has a matching $M$ such that $AT(G - M) \leq 4$ in [9], it implies a positive answer to the more difficult question — whether every planar graph is 1-defective 4-choosable [10]. Let $T_{m,n} = C_m \Box C_n$ be a toroidal grid, the first author et al. use the same method to show that the Alon-Tarsi number of $T_{m,n}$ equals 4 when $m, n$ are both odd and 3 otherwise in [11]. T. Abe et al. prove that for a $K_5$-minor-free graph $G$, $AT(G) \leq 5$ [12], which generalizes the result of X. Zhu [8].

A Halin graph $H = T \cup C_n$ is a plane graph, where $T$ is a tree with no vertex of degree two and at least one vertex of degree three or more, and $C_n$ is a cycle connecting the pendant vertices of $T$ in the cyclic order determined by the drawing of $T$. Vertices of $C_n$ and $H - C_n$ are referred to as outer and inner vertices of $H$, respectively. In particular, a wheel graph is a Halin graph which contains only one inner vertex. In a wheel graph, if we delete an edge of $C_n$, the rest of the graph is called a fan.

The chromatic number and choice number of Halin graphs are determined in [13] and [14], respectively. In this article, we obtain the exact values of Alon-Tarsi number of Halin graphs by constructing an $AT$-orientation method.

**Main Theorem.** For a Halin graph $H$, we have

$$AT(H) = \begin{cases} 
4, & \text{if } H \text{ is a wheel of even order;} \\
3, & \text{otherwise.}
\end{cases}$$

2 Preliminaries

**Definition 2.1.** [1] A subdigraph $H$ of a directed graph $D$ is called Eulerian if $V(H) = V(G)$ and the indegree $d^-_H(v)$ of every vertex $v$ of $H$ in $H$ is equal to its outdegree $d^+_H(v)$. Note that $H$ might
not be connected. For a digraph $D$, we denote by $\mathcal{E}(D)$ the family of Eulerian subdigraphs of $D$. $H$ is even if it has an even number of edges, otherwise, it is odd. Let $\mathcal{E}_e(D)$ and $\mathcal{E}_o(D)$ denote the numbers of even and odd Eulerian subgraphs of $D$, respectively. Let $\text{diff}(D) = |\mathcal{E}_e(D)| - |\mathcal{E}_o(D)|$. We say that $D$ is Alon-Tarsi if $\text{diff}(D) \neq 0$. If an orientation $D$ of $G$ yields an Alon-Tarsi digraph, then we say $D$ is an Alon-Tarsi orientation (or an AT-orientation, for short) of $G$.

Generally, it is difficult to determine whether an orientation $D$ of a graph $G$ is an AT-orientation. Nevertheless, in some cases this problem is very simple. Observe that every digraph $D$ has at least one even Eulerian subdigraph, namely, the empty subgraph. If $D$ has no odd directed cycle, then $D$ has no odd Eulerian subdigraph, so $D$ is an AT-orientation.

An acyclic orientation of an undirected graph is an assignment of a direction to each edge (an orientation) that does not form any directed cycle and therefore makes it into a directed acyclic graph. If $D$ is an acyclic orientation of $G$, then $D$ has no odd Eulerian subdigraph, so $D$ is an AT-orientation. We denote the maximum outdegree of an acyclic orientation $D$ by $d_a$. By the definition of $\text{AT}(G)$ we have the following:

**Lemma 2.1.** If a graph $G$ has an acyclic orientation $D$ with maximum outdegree $d_a$, then $\text{AT}(G) \leq d_a + 1$.

**Definition 2.2.** [15] A graph $G$ is $k$-degenerate if there exists an ordering $v_1, \ldots, v_n$ of vertices of $G$ such that for $i = 1, \ldots, n$, the vertex $v_i$ has at most $k$ neighbors among $v_1, v_2, \ldots, v_{i-1}$.

**Lemma 2.2.** If a graph $G$ is $k$-degenerate, then $\text{AT}(G) \leq k + 1$.

**Proof.** Suppose that $G$ has degeneracy $k$ and $\sigma$ is a vertex ordering which witnesses this. By orienting each edge toward its endpoint that appears earlier in the vertex ordering, we can get an acyclic orientation with maximum outdegree $k$. By Lemma 2.1, $\text{AT}(G) \leq k + 1$. 

Let $H = T \cup C_n$, where $C_n$ is the cycle $v_1v_2\ldots v_nv_1$. Then every vertex of $V(C_n)$ is adjacent to exactly one vertex in $V(H) \setminus V(C_n)$, and every edge of $E(C_n)$ is adjacent to exactly two edge in $E(H) \setminus E(C_n)$. An inner vertex $u$ of a Halin graph $H$ is called special if it is a neighbor of a unique inner vertex. Let $v_1, v_2, \ldots, v_k$ be the neighbors of $u$ on $C_n$. If a Halin graph $H$ is not a wheel, then $\{u, v_1, v_2, \ldots, v_k\}$ induces a fan and $H$ contains at least two special inner vertices [13].

### 3 Proof of the main theorem

The proof will be completed by a sequence of lemmas.

**Lemma 3.1.** [14] Every Halin graph is 3-degenerate.
By Lemma 3.1 and Lemma 2.2, we have

**Lemma 3.2.** $\text{AT}(H) \leq 4$ for each Halin graph $H$.

**Lemma 3.3.** If $H$ is a wheel of even order, then $\text{AT}(H) = 4$.

**Proof.** By Lemma 3.2, we know that $\text{AT}(H) \leq 4$. $H$ has chromatic number 4, so $\text{AT}(H) \geq 4$. Hence $\text{AT}(H) = 4$.

![Fig.1 The graph $H = C_8 \cup u$.](image)

**Lemma 3.4.** Let $H$ be a Halin graph with an even outer cycle, then $\text{AT}(H) = 3$.

**Proof.** Assume $H = T \cup C_n$, $n$ is even. We know that $\text{AT}(H) \geq 3$ since $\chi(H) = 3$. It remains to show that $\text{AT}(H) \leq 3$. Consider the following two cases.

**Case 1.** $H$ is a wheel with even outer vertices.

Since $H$ is a wheel, $H$ contains exactly one interior vertex $u$, $d(u) = n$, and $d(v_i) = 3$ for $i = 1, 2, \ldots, n$. Let $D$ be an orientation of $H$ in which the edges of $H$ are oriented in such a way by orientating the outer cycle $C_n$ in clockwise and orientating edge $v_i u$ as $(v_i, u), i = 1, 2, \ldots, n$ [See Figure 1]. $D$ has no odd directed cycle, so $D$ has no odd Eulerian subgraph and hence $D$ is an $\text{AT}$-orientation with maximum outdegree 2. Therefore $\text{AT}(H) \leq 3$.

**Case 2.** $H$ is not a wheel with even outer vertices.

Similarly, orient $C_n$ in clockwise. For $T$, let $X$ be the set of all the inner vertices that are adjacent to $V(C_n)$. All the arcs between $V(C_n)$ and $X$ are oriented from $V(C_n)$ to $X$. The unoriented edges of $T$ induce a subgraph, denoted by $T_1$. It is easy to see that $T_1$ has at least two leaves. Let $L_1$ and $X_1$ be the set of all the leaves of $T_1$ and all the vertices of $T_1$ that are adjacent to the leaves, respectively. All the arcs between $L_1$ and $X_1$ are oriented from $L_1$ to $X_1$. The unoriented edges of $T_1$ induce a subgraph, denoted by $T_2$. Let $L_2$ and $X_2$ be the set of all the leaves of $T_2$ and all the vertices of $T_2$ that are adjacent to the leaves, respectively. All the arcs between $L_2$ and $X_2$ are oriented from $L_2$ to $X_2$.  


Fig.2  $H$ is not a wheel and have even outer vertices.

Repeat this process until all edges of $H$ are oriented or only one edge left unoriented, then the edge is oriented arbitrarily. Obviously $D$ has no odd directed cycle, so $D$ has no odd Eulerian subdigraph [See Figure 2]. It is easy to see that the followings hold: (1) $D$ is an $AT$-orientation, (2) $d^+(v) \leq 1$ for each inner vertex $v$, (3) $d^+(v) = 2$ for each outer vertex $v$. Hence $AT(H) \leq 3$.

Lemma 3.5. [16] Assume that $D$ is a digraph and $V(D) = X_1 \cup X_2$. For $i = 1, 2$, let $D_i = D[X_i]$ be the subdigraph of $D$ induced by $X_i$. If all the arcs between $X_1$ and $X_2$ are from $X_1$ to $X_2$, then $D$ is Alon-Tarsi if and only if $D_1, D_2$ are both Alon-Tarsi.

Lemma 3.6. Let $H$ be a Halin graph with an odd outer cycle but not a wheel. Then $AT(H) = 3$.

Proof. Suppose that $H = T \cup C_n$, where $n$ is odd. $H$ has chromatic number 3, so we know that $AT(H) \geq 3$. It remains to show that $AT(H) \leq 3$. Similar to above lemma, we consider two cases.

Case 1. $H$ has a special inner vertex $u$ which is adjacent to an odd number of vertices of $C_n$.

Let $v_1, v_2, \ldots, v_k$ be the neighbors of $u$ on $C_n$ and $k$ is odd. Then $\{u, v_1, v_2, \ldots, v_k\}$ induces a fan, denoted by $G_1$. Assume $G_2 = H - V(G_1)$. The subgraph $G_1$ has an orientation $D_1$ as the following way: orient the edge $v_i v_{i+1}$ as $(v_i, v_{i+1})$, for $1 \leq i \leq k - 1$, the edge $uw_1$ as $(u, v_1)$, and the edge $v_j u$ as $(v_j, u)$ for $j = 2, 3, \ldots, k$. Observe that $uw_1 v_2 \ldots v_i u$ is an odd directed cycle when $i$ is even, and $uv_1 v_2 \ldots v_i u$ is an even directed cycle when $i$ is odd, for $2 \leq i \leq k$. Hence $D_1$ contains $k - 1$ directed cycles. Specifically $D_1$ contains $\frac{k-1}{2}$ odd directed cycles and $\frac{k-1}{2}$ even directed cycles. It is easy to see that the arc $(u, v_1)$ is contained in all directed cycles. Since Eulerian subdigraph is the arc disjoint union of directed cycles and empty subdigraph is an even Eulerian subdigraph, $D_1$ has $\frac{k-1}{2}$ odd Eulerian subgraphs and $\frac{k-1}{2}$ even Eulerian subdigraphs. $\text{diff}(D_1) = |E_e(D_1)| - |E_o(D_1)| = 1 \neq 0$. Therefore $D_1$ is an $AT$-orientation of $G_1$.

Denote $X = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$, and let $X_1$ be the set of all the vertices in $V(G_2) - X$ that are adjacent to $X$. $G_2$ has an orientation $D_2$ that for each $k + 1 \leq i \leq n - 1$, orient edge $v_i v_{i+1}$ as $(v_i, v_{i+1})$. All the arcs between $X$ and $X_1$ are oriented from $X$ to $X_1$. The unoriented edges of $G_2$ induce a subgraph, denoted by $T_1$. Let $L_1$ and $X_2$ be the set of all the leaves of $T_1$ and all the
vertices of $T_1$ that are adjacent to the leaves, respectively. All the arcs between $L_1$ and $X_2$ are oriented from $L_1$ to $X_2$. The unoriented edges of $T_1$ induce a subgraph, denoted by $D$. Let $L_2$ and $X_3$ be the set of all the leaves of $T_2$ and all the vertices of $T_2$ that are adjacent to the leaves, respectively. All the arcs between $L_2$ and $X_3$ are oriented from $L_2$ to $X_3$. Repeat this process until all edges of $G_2$ are oriented or only one edge left unoriented, then the edge is oriented arbitrarily. It is obvious that $D_2$ is an acyclic orientation, hence $\text{diff}(D_2) = |E_e(D_2)| - |E_o(D_2)| \neq 0$, $D_2$ is an $AT$-orientation of $G_2$.

Let $v$ be the unique inner vertex which is adjacent to $u$, $u \in V(G_1)$ and $v \in V(G_2)$. Let $D$ be obtained from $D_1 \cup D_2$ by adding arcs $(u, v), (v_1, v_n)$ and $(v_k, v_{k+1})$. Such that all the arcs between $G_1$ and $G_2$ are oriented from $G_1$ to $G_2$ [See Figure 3]. By Lemma 3.5, $D$ is an $AT$-orientation. It is easy to see that $d^+_D(v) \leq 2$ for every $v \in V(G)$. Hence $AT(H) \leq 3$.

**Case 2.** All special inner vertices of $H$ are adjacent to an even number of vertices of $C_n$.

Let $u$ be a special inner vertex, $v_1, v_2, \ldots, v_k$ be the neighbors of $u$ on $C_n$. \{u, v_1, v_2, \ldots, v_k\} induces a fan, denoted by $G_1$. Let $G_2 = H - V(G_1)$. The subgraph $G_1$ has an orientation $D_1$ as the following way: orient the edge $v_{i-1}v_i$ as $(v_i, v_{i-1})$ for $2 \leq i \leq k$, and the edge $v_ju$ as $(v_j, u)$ for $j = 1, 2, \ldots k$. Observed that $D_1$ is an acyclic orientation, so $D_1$ is an $AT$-orientation of $G_1$.

In the tree $T$, any two vertices are connected by exactly one path, so there is a unique $xv_n$-path in $T$ connecting $x$ and $v_n$, where $x \in V(G_2) \setminus v_n$. $G_2$ has an orientation $D_2$ that for $k + 2 \leq i \leq n$, orient the edge $v_{i-1}v_i$ as $(v_i, v_{i-1})$, and let every $xv_n$-path be a directed path from $x$ to $v_n$. It is obvious that all the edges of $G_2$ are oriented. Denote $w$ is the unique vertex in $T$ which is adjacent to $v_n$, the arc $(w, v_n)$ is contained in all $xv_n$-directed paths. In $G_2$, all directed circles are made up of $v_iv_n$-directed path in $T$ and $v_nv_i$-directed path in $C_n$, for $k + 1 \leq i \leq n - 1$. $D_2$ has an even number of directed cycles. Empty subdigraph is an even Eulerian subdigraph of $D_2$, hence $|E(D_2)|$ is odd. $\text{diff}(D_2) = |E_e(D_2)| - |E_o(D_2)| \neq 0$, so $D_2$ is an $AT$-orientation of $G_2$.

Let $v$ be the unique inner vertex which is adjacent to $u$, $u \in V(G_1)$ and $v \in V(G_2)$. Let $D$ be obtained from $D_1 \cup D_2$ by adding arcs $(v, u), (v_n, v_1)$ and $(v_{k+1}, v_k)$. Obviously all the arcs between $G_1$ and $G_2$ are oriented from $G_2$ to $G_1$ [See Figure 4]. In a similar way as case 1, we can get $AT(H) \leq 3$. 

![Fig.3 The Halin graph H for n = 9 and k = 3.](image-url)
Corollary 3.7. For a Halin graph $H$, we have
\[
\chi(H) = ch(H) = \chi_P(H) = AT(H) = \left\{\begin{array}{ll}
4, & \text{if } H \text{ is a wheel of even order;} \\
3, & \text{otherwise.}
\end{array}\right.
\]

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