On the Existence of Solutions to the
Hasegawa-Mima Equation in Periodic Sobolev Spaces

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Abstract

In this paper, we represent the highly non-linear Hasegawa-Mima PDE model as a coupled system of linear Elliptic-Hyperbolic PDEs, for which we apply the Petrov-Galerkin method to obtain a sequence of fixed-point approximate solutions that converge weakly to a solution that satisfies periodic boundary conditions. We obtain existence results under weak assumptions on the initial data, such as $u_0 \in H^3_P$ with $-\Delta u_0 + u_0 \in H^1_P \cap L^\infty(\Omega)$. Throughout the paper, we give a brief introduction to Periodic Sobolev Spaces and prove relevant results. In the sequel, we establish a mathematical model that is suitable for Finite Element discretization.

Keywords: Plasma Confinement, Drift Waves, Hasegawa-Mima, Periodic Sobolev Spaces, Petrov-Galerkin Approximations.

1 Introduction

Magnetic plasma confinement is one of the most promising ways in future energy production. To understand the phenomena related to energy production through plasma confinement, several mathematical models can be found in literature [1, 2, 3, 4], of which the simplest and powerful 2D turbulent system model is the Hasegawa-Mima equation that models the time evolution of drift waves in magnetically-confined plasma. It was originally derived by Akira Hasegawa and Kunioki Mima during late 70s [2, 3], but can [5, 6] be extended and put as

$$-\Delta u_t + u_t = \{u, \Delta u\} + ku_y$$

where $\{u, v\} = u_x v_y - u_y v_x$ is the Poisson bracket, $u(x, y, t)$ describes the electrostatic potential, $k = \partial_x \ln \frac{n_0}{\omega_{ci}^2}$ is a constant depending on the background particle density $n_0$ and the ion cyclotron frequency $\omega_{ci}$, which in turn depends on the initial magnetic field. So, $k = 0$ refers to homogeneous plasma, and $k \neq 0$ refers to non-homogeneous plasma. As a cultural note, equation (1) is also referred as the Charney-Hasegawa-Mima equation in geophysical context that models the time-evolution of Rossby waves in the atmosphere [5].

Due to the highly nonlinear nature of $\{u, \Delta u\} = u_x \Delta u_y - u_y \Delta u_x$, it is difficult to define a mapping whose fixed-point is a solution directly to (1). For this reason in 2004, L. Paumond [7] perturbed (1) into a Cauchy problem with the strongly elliptic operator $\epsilon \Delta (\Delta - I)$ acting on $u$ and used analytic semigroup methods to prove both the existence of a unique local strong solution for initial data $u_0 := u(0) \in H^m(\mathbb{R}^2)$ with $m \geq 4$, and the existence of a global weak solution for initial data $u_0 \in H^2(\mathbb{R}^2)$.

In 2016, H. Karakazian [8] perturbed (1) into a Cauchy problem with the strongly elliptic operator $\epsilon(\Delta(\Delta - I) + 2I)$ acting on $u$ and proved the local existence of a unique strong solution in Periodic Sobolev Spaces, that is, with some periodic boundary conditions, on a square domain $\Omega$ for $u_0 \in H^m_\Omega$ with $m \geq 4$ (see Definition

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Nevertheless, both methods above never provided a numerical approach for the purpose of simulating the Hasegawa-Mima model, whether on $\mathbb{R}^2$ or on $\Omega$.

We present the Hasegawa-Mima problem on $\Omega$, in its basic classical form, as follows.

**Problem 1.1.** Given a temporal bound $T > 0$, an open domain $\Omega = (0, L) \times (0, L)$ with boundary $\Gamma$, and an initial data $u_0$ defined on $\Omega$, seek $u : \Omega \times [0, T] \rightarrow \mathbb{R}$ such that

\[
\begin{aligned}
(u - \Delta u)_t &= \{u, \Delta u\} + ku_y & \text{on } \Omega \times (0, T) \\
\text{PBCs on } u, u_x, u_y & & \text{on } \Gamma \times (0, T) \\
\begin{pmatrix} u(x, y, 0) \\ u(x, y, 0) \end{pmatrix} &= u_0(x, y) & \text{on } \Omega
\end{aligned}
\]

where PBCs stands for periodic boundary conditions.

Not only to lift the theoretic difficulty in handling the Poisson bracket, but also to provide a numerical Finite Element discretization for finding a solution, which builds upon on a previously done Finite Difference simulation by F. Hariri in 2010 [6], we reformulate this highly non-linear Hasegawa-Mima problem as a coupled Elliptic-Hyperbolic system of linear PDEs as follows:

Letting $w = -\Delta u + u$, the PDE in (HM) becomes

\[
w_t = \{u, u - w\} + ku_y = \{w, u\} + ku_y = w_xu_y - w_yu_x + ku_y
\]

or equivalently

\[
w_t + \nabla (\vec{V}(u) \cdot w) = ku_y
\]

where $\vec{V}(u) = -u_y\vec{j} + u_x\vec{i}$ is a divergence-free vector field. That is, we consider

**Problem 1.2.** Given a temporal bound $T > 0$, an open domain $\Omega = (0, L) \times (0, L)$ with boundary $\Gamma$, and an initial data $u_0$ defined on $\Omega$, seek $\{u, w\} : \Omega \times [0, T] \rightarrow \mathbb{R}^2$ such that

\[
\begin{aligned}
-\Delta u + u &= w & \text{on } \Omega \times (0, T) \\
\begin{pmatrix} w_t + \nabla (\vec{V}(u) \cdot w) \\ w_t + \nabla (\vec{V}(u) \cdot w) \end{pmatrix} &= ku_y & \text{on } \Omega \times (0, T) \\
\text{PBCs on } u, u_x, u_y, w & & \text{on } \Gamma \times (0, T) \\
\begin{pmatrix} u(x, y, 0) \\ u(x, y, 0) \end{pmatrix} &= u_0(x, y) & \text{on } \Omega \\
\begin{pmatrix} w(x, y, 0) \\ w(x, y, 0) \end{pmatrix} &= w_0(x, y) := -\Delta u_0 + u_0 & \text{on } \Omega
\end{aligned}
\]

In section 2 we state our main theoretical and Finite Element discretization numerical results regarding Problem 1.2 in the context of two of its equivalent semi-variational formulations, the first one being for initial data $u_0 \in H^2(\Omega)$, and second being for initial data $u_0 \in H^1_p(\Omega)$ with $-\Delta u_0 + u_0 \in H^1_{\text{PBC}}(\Omega) \cap L^\infty(\Omega)$, where $H^1_{\text{PBC}}(\Omega)$ are Periodic Sobolev Spaces briefly introduced in section 3. In sections 4 and 5 we study each of the elliptic and the hyperbolic PDEs independently. In section 6 we consider a sequence of fixed-point problems on particular subsets of $C(0, T; E_N)$, where $E_N$’s are finite-dimensional subspaces of the Periodic Sobolev Space

\[
H^1_{\text{PBC}}(\Omega) := \{ v \in H^1(\Omega) \mid \text{PBCs on } v \text{ on } \Gamma \}
\]

and obtain a uniformly bounded sequence of pairs $\{u_N, w_N\}$ of approximate solutions to each of the semi-variational formulations described in section 2. In section 7 we extract a weakly convergent subsequence to construct a candidate local solution $\{u, w\}$. Finally in section 8 we prove our main results.

In order to state and prove our main results, we put Problem 1.2 in its semi-variational formulations as a consequence of the following Lemma.

**Lemma 1.3.** Let $u \in H^2_{\text{PBC}}(\Omega) := \{ v \in H^2(\Omega) \mid v, v_x, v_y \in H^1_{\text{PBC}}(\Omega) \}$. Then
\[ (i) \quad \left\langle \vec{V}(u) \cdot \nabla w, \varphi \right\rangle_{L^2} = - \left\langle \vec{V}(u) \cdot \nabla \varphi, w \right\rangle_{L^2} \quad \forall w, \varphi \in H^1_P(\Omega) \] (7)

\[ (ii) \quad \left\langle \vec{V}(u) \cdot \nabla \varphi, \varphi \right\rangle_{L^2} = 0 \quad \forall \varphi \in H^1_P(\Omega) \] (8)

provided that \( \vec{V}(u) \cdot \nabla w \) and \( \vec{V}(u) \cdot \nabla \varphi \) are in \( L^2(\Omega) \).

\textbf{Proof.} Integrating by parts, we get

\[ \left\langle \vec{V}(u) \cdot \nabla w, \varphi \right\rangle_{L^2} = \int_{\Gamma} w \varphi \vec{V}(u) \cdot \nu \, ds - \left\langle \vec{V}(u) \cdot \nabla \varphi, w \right\rangle_{L^2} \] (9)

where the boundary integral vanishes due to the periodicity of \( u_x, u_y, w \) and \( \varphi \) on \( \Gamma \), which proves (i). As a corollary, we get

\[ \left\langle \vec{V}(u) \cdot \nabla \varphi, \varphi \right\rangle_{L^2} = - \left\langle \vec{V}(u) \cdot \nabla \varphi, w \right\rangle_{L^2} \] (10)

which proves (ii).

Assume, for the time being, that the pair \( \{u, w\} \in (H^3(\Omega) \cap H^1_P(\Omega)) \times H^1_P(\Omega) \). Using the invertibility (Theorem 4.1) of \( I - \Delta : H^2_P(\Omega) \rightarrow L^2(\Omega) \), we write the hyperbolic equation (4) as

\[ u_t = -(I - \Delta)^{-1}[\vec{V}(u) \cdot \nabla w] + k(I - \Delta)^{-1}u_y \quad \text{on } \Omega \times (0, T) \] (11)

Taking its \( L^2 \) inner-product with \( \varphi \in L^2(0, T; L^2(\Omega)) \), and integrating it over \([0, T]\), we obtain

\[ \int_0^T \langle u_t, \varphi \rangle_{L^2} \, dt = - \int_0^T \left\langle (I - \Delta)^{-1}[\vec{V}(u) \cdot \nabla w], \varphi \right\rangle_{L^2} \, dt + \int_0^T \left\langle k(I - \Delta)^{-1}u_y, \varphi \right\rangle_{L^2} \, dt \] (12)

which, by the fact that \( (I - \Delta)^{-1} \) is self-adjoint and Lemma 4.3, leads us to the first semi-variational formulation:

\[ (VF \ 1) \quad \begin{cases} -\Delta u + u = w \\ \int_0^T \langle u_t, \varphi \rangle_{L^2} \, dt = \int_0^T \left\langle \vec{V}(u) \cdot \nabla (I - \Delta)^{-1} \varphi, w \right\rangle_{L^2} \, dt \\ + \int_0^T \left\langle k(I - \Delta)^{-1}u_y, \varphi \right\rangle_{L^2} \, dt \quad \forall \varphi \in L^2(0, T; L^2(\Omega)) \end{cases} \quad \text{a.e. on } \Omega \times [0, T] \]

\[ u(0) = u_0 \]

\[ w(0) = w_0 \]

that is compatible with \( \{u, w\} \in H^2_P(\Omega) \times L^2(\Omega) \), and can be used it for initial data \( u_0 \in H^2(\Omega) \) and \( u_0 \in H^1_P(\Omega) \). Note that since \( \vec{V}(u) \) and \( \nabla (I - \Delta)^{-1} \varphi \) are in \( H^1_P(\Omega) \), then \( \vec{V}(u) \cdot \nabla (I - \Delta)^{-1} \varphi \) will be in \( L^2(\Omega) \).

On the other hand, one could have directly taken the \( L^2 \) inner-product of the hyperbolic equation (4) with \( \varphi \in L^2(0, T; H^1_P(\Omega)) \) and integrated it over \([0, T]\) to obtain a second equivalent semi-variational formulation

\[ (VF \ 2) \quad \begin{cases} -\Delta u + u = w \\ \int_0^T \langle w_t, \varphi \rangle_{L^2} \, dt = \int_0^T \left\langle \vec{V}(u) \cdot \nabla \varphi, w \right\rangle_{L^2} \, dt + \int_0^T \left\langle ku_y, \varphi \right\rangle_{L^2} \, dt \quad \forall \varphi \in L^2(0, T; H^1_P(\Omega)) \end{cases} \quad \text{a.e. on } \Omega \times [0, T] \]

\[ u(0) = u_0 \]

\[ w(0) = w_0 \]

which we will use for the case when the initial data \( u_0 \in H^3_P(\Omega) \) with \( -\Delta u_0 + u_0 \in H^1_P(\Omega) \cap L^\infty(\Omega) \).
2 Main Results

Theorem 2.1 (Local weak solution for Problem 1.2). For each \( u_0 \in H^3_P(\Omega) \) with \( w_0 := -\Delta u_0 + u_0 \in H^1_P(\Omega) \cap \mathcal{L}^\infty(\Omega) \), setting
\[
T = \frac{1}{2C_E(|k| + 8 \|w_0\|_{\mathcal{L}^\infty} + 4\sqrt{|k|} \|LC\|_{\mathcal{L}^\infty}\|w_0\|_{\mathcal{L}^1})} > 0
\]
where \( C_E > 0 \) is the elliptic regularity constant and \( C_\infty > 0 \) is the constant from the continuous embedding of \( H^2(\Omega) \) into \( \mathcal{L}^\infty(\Omega) \), there exists a pair
\[
\{u, w\} \in L^2(0, T; H^3_P(\Omega)) \times L^2(0, T; H^1_P(\Omega)) \quad \text{with} \quad \{u_t, w_t\} \in L^2(0, T; H^1_P(\Omega)) \times L^2(0, T; L^2(\Omega))
\]
that satisfies (VF 2). In fact, \( \{u, w\} \) is in \( C([0, T]; H^3_P(\Omega)) \times C([0, T]; L^2(\Omega)) \) by the Sobolev Embedding Theorem (see Proposition 2.46(vii) of [9]).

Corollary 2.2 (Local strong solution for Problem 1.2). In the context above, the pair \( \{u, w\} \) is a unique strong local solution to
\[
\begin{cases}
-\Delta u + u = w & \text{a.e. on } \Omega, \forall t \in [0, T] \\
w_t + \vec{V}(u) \cdot \nabla w = ku_y & \text{a.e. on } \Omega \times (0, T) \\
PBCs \text{ on } u, u_x, u_y, \text{ and } w & \text{a.e on } \Gamma \times (0, T) \\
u(0) = u_0 & \text{a.e on } \Omega \\
w(0) = w_0 & \text{a.e on } \Omega
\end{cases}
\]

Corollary 2.3 (Finite Element Discretization). One can simulate the solution \( \{u, v\} \) from the following computational formulation
\[
\begin{cases}
-\Delta u + u = w & \text{a.e. on } \Omega \times [0, T] \\
\langle w(t_{i+1}), \psi \rangle_{L^2} = \langle w(t_i), \psi \rangle_{L^2} + \int_{t_i}^{t_{i+1}} \langle \vec{V}(u) \cdot \nabla \psi, w \rangle_{L^2} dt + \int_{t_i}^{t_{i+1}} \langle ku_y, \psi \rangle_{L^2} dt & \forall \psi \in H^1_P(\Omega) \\
PBCs \text{ on } u, u_x, u_y, \text{ and } w & \text{a.e on } \Gamma \times (0, T) \\
w(t_0) = u_0 & \text{a.e on } \Omega \\
w(t_0) = w_0 & \text{a.e on } \Omega
\end{cases}
\]
for \( t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < T \).

Proposition 2.4. Let \( 0 < T < (C_E|k| + 1)^{-1} \). Then for every \( u_0 \in H^2(\Omega) \), there exists a sequence of approximate solution pairs
\[
\{u_N, w_N\} \in C^1(0, T; E_N) \times C^1(0, T; E_N)
\]
satisfying
\[
\begin{cases}
-\Delta u_N + u_N = w_N & \text{on } \Omega \times [0, T] \\
\langle w_N, v \rangle_{L^2} = \langle \vec{V}(u) \cdot v, w_N \rangle_{L^2} + k \langle u_y, v \rangle_{L^2} & \forall v \in E_N \\
u(0) = \text{proj}_{E_N} u_0 & \text{on } \Omega \\
w(0) := -\Delta u(0) + u(0) & \text{on } \Omega
\end{cases}
\]
such that
\[
\begin{cases}
u_N \rightarrow u \text{ weakly in } L^2(0, T; H^2_P(\Omega)) \\
w_N \rightarrow w \text{ weakly in } L^2(0, T; L^2(\Omega)) \\
u_N' \rightarrow u_t \text{ weakly in } L^2(0, T; L^2(\Omega))
\end{cases}
\]
Conjecture 2.5 (Global weak solution for (VF2)). Let $0 < T < (C_E|k| + 1)^{-1}$. Then for every $u_0 \in H^2(\Omega)$, there exists a pair 

$$\{u, w\} \in L^2(0, T; H^p(\Omega)) \times L^2(0, T; L^2(\Omega)) \text{ with } u_t \in L^2(0, T; L^2(\Omega))$$

that satisfies (VF 1). In this case, $u \in C(0, T; L^2(\Omega))$ by the Sobolev Embedding Theorem. Further, since $T$ doesn’t depend on the initial data $u_0$, the solution $u$ can be continued globally to all of $[0, \infty)$.

The difficulty of proving this conjecture, even when $u_0$ is assumed to be in $H^2_P(\Omega)$ with $-\Delta u_0 + u_0 \in L^\infty(\Omega)$, lies in the absence of a uniform bound on $\|w_N\|_{L^\infty}$, otherwise the proof will be very similar to that of Theorem 2.4.

3 Periodic Sobolev Spaces

To handle PBCs properly, we work in the context of Periodic Sobolev Spaces that were first introduced in [8]. For the sake of completeness, we give a brief introduction and construct them as follows.

Definition 3.1. Let $k$ be a non-negative integer. We say that a real-valued function $u$ satisfies the periodic boundary conditions PBC$^k$ of order $k$ if and only if

$$(\text{PBC}^k) \begin{cases}
\frac{\partial^k}{\partial x^k} u(0, y) = \frac{\partial^k}{\partial x^k} u(L, y) & \forall y \in (0, L) \\
\frac{\partial^k}{\partial y^k} u(x, 0) = \frac{\partial^k}{\partial y^k} u(x, L) & \forall x \in (0, L)
\end{cases}$$

Imposing such a PBC$^k$, with $0 \leq k \leq m - 1$ on a function in a Sobolev space $W^{m,p}(\Omega)$ can be done through the Trace Theorem. As a result, we have

Definition 3.2. Let $m$ be a positive integer. We define the periodic Sobolev space of order $m$ as

$$W^{m,p}_P(\Omega) = \left\{ v \in W^{m,p}(\Omega) : \text{Tr}(v) \text{ satisfies PBC}^k \text{ a.e. } \forall k = 0, 1, \cdots m - 1 \right\}$$

Also for convenience, we set

(i) $W^{0,p}_P(\Omega) = L^p(\Omega)$.

(ii) $H^{0}_P(\Omega) = W^{m,2}_P(\Omega)$, so that $H^0_P(\Omega) = L^2(\Omega)$.

(iii) $W^{\infty,2}_P(\Omega) = \bigcap_{m=0}^{\infty} W^{m,2}_P(\Omega)$, so that $H^\infty_P(\Omega) = W^{\infty,2}_P(\Omega)$.

(iv) $C^{\infty}_P(\Omega) = C^{\infty}(\Omega) \cap H^\infty_P(\Omega)$.

One can also assume that $v \in W^{m,p}_P(\Omega)$ has a zero-average on $\Omega$, to be able to use Poincaré Inequality. In fact, if $v \in W^{1,p}_P(\Omega)$ has a zero-average then both $v_x$ and $v_y$ have as well. Further, this construction of Periodic Sobolev Spaces generalizes to any open box in $\mathbb{R}^N$.

Remark. $H^m_P(\Omega)$ is slightly general than $H^m(\mathbb{T})$ and it shouldn’t be confused with $H^m_{per}(\Omega)$ found in literature, because the latter assumes that $\text{Tr}(v)$ satisfies PBC$^0$ only. The advantage of $W^{m,p}_P(\Omega)$ is that boundary integrals vanish.

Proposition 3.3. The following assertions are true (at least for rectangular domains $\Omega \subset \mathbb{R}^2$):

(i) For every function $f \in W^{1,p}_P(\Omega) = \{ v \in W^{1,p} \mid \text{Tr}(v) \text{ satisfies PBC}^0 \}$ and 2D vector $\vec{g}$ whose components lie in $W^{1,p}_P(\Omega)$,

$$\int_{\Gamma} f \vec{g} \cdot \vec{\nu} \, ds = 0$$

(21)
There exists a Hilbert basis of Proposition 4.2.

Theorem 4.1. (In other words, the linear operator \( \xi \) and Proposition 3.3 (i), both (C) for some elliptic regularity constant \( \lambda \).)

Remark. Note that the temporal regularity of \( w \) carries onto \( u \) as sometimes we will ignore the temporal variable when talking about results related to the elliptic equation.

Proof. Lax-Milgram Theorem on \( H^1_p(\Omega) \), elliptic regularity and Proposition 3.6 of \[8\] establish the base cases for \( m = 0 \) and \( m = 1 \), respectively. Then induction, exactly done as in Proposition 3.7 of \[8\], concludes the proof.

In other words, the linear operator \( (I - \Delta) : H^m_p(\Omega) \rightarrow H^m_p(\Omega) \) is invertible. Also by Green’s formula and Proposition 3.3 (i), both \( (I - \Delta) \) and its inverse are self-adjoint on \( H_p^m(\Omega) \) for large enough enough \( m \). In particular,

\[
\langle u, v \rangle_{L^2} = \langle (I - \Delta)^{-1}u, (I - \Delta)v \rangle_{L^2} \quad \forall u \in L^2(\Omega), \forall v \in H^1_p(\Omega)
\]

Proposition 4.2. There exists a Hilbert basis of \( H^1_p(\Omega) \) composed of eigenvectors \( \{ \phi_j \}_{j=1}^{\infty} \) of \( I - \Delta \) with increasing eigenvalues \( \lambda_j \geq 1 \). Moreover \( \langle \phi_i, \phi_j \rangle_{L^2} = \delta_{ij}/\lambda_j \). In fact, \( \phi_j \)'s is the Fourier basis

\[
\phi_j = \frac{1}{\sqrt{L^2 + 4\pi^2 |\xi_j|^2}} e^{2\pi i x \xi_j/L} \quad \text{with eigenvalues } \lambda_j = 1 + \frac{4\pi^2 |\xi_j|^2}{L^2}
\]

where \( \xi_j = (\xi_{j,x}, \xi_{j,y}) \) is a bijection between \( \mathbb{Z}^+ \) and \( \mathbb{N} \times \mathbb{N} \) with non-decreasing \( |\xi_j|^2 = \mathcal{O}(j) \), and so for all \( j \in \mathbb{N}, \phi_j \in C_p^\infty(\Omega) \) with a zero average on \( \Omega \).

Proof. We have that the solution operator \( \mathcal{T} \) of the elliptic PDE

\[
\mathcal{T} : (I - \Delta)^{-1} : H^1_p(\Omega) \xrightarrow{\text{continuously}} H^1_p(\Omega) \xrightarrow{\text{compactly}} H^1_p(\Omega)
\]
is a compact self-adjoint operator. Thus by Theorem 6.11 of [10], there exists a Hilbert basis for $H^1_0(\Omega)$ consisting of eigenvectors $\{\phi_j\}_{j=1}^{\infty}$ of $(I - \Delta)^{-1}$ with eigenvalues $\{\eta_j\}_{j=1}^{\infty}$. Now since the eigenvalues of an invertible operator cannot be zero, then

$$(I - \Delta)^{-1}\phi_j = \eta_j\phi_j \iff (I - \Delta)\phi_j = \frac{1}{\eta_j}\phi_j$$

(29)

so that $\{\phi_j\}_{j=1}^{\infty}$ are eigenvectors of $I - \Delta$ with non-zero eigenvalues $\lambda_j := 1/\eta_j$, where

$$\lambda_j = \langle \lambda_j \phi_j, \phi_j \rangle_{H^1} = \langle (I - \Delta)\phi_j, \phi_j \rangle_{H^1} = \langle \phi_j, \phi_j \rangle_{H^1} + \langle -\Delta \phi_j, \phi_j \rangle_{H^1} = 1 + \|\nabla \phi_j\|_{L^2}^2 \geq 1$$

(30)

Finally,

$$\langle \phi_i, \phi_j \rangle_{L^2} = \frac{1}{\lambda_j}\langle \phi_i, (I - \Delta)\phi_j \rangle_{L^2} = \frac{1}{\lambda_j}\left(\langle \phi_j, \phi_j \rangle_{L^2} + \langle \nabla \phi_i, \nabla \phi_j \rangle_{L^2}\right) = \frac{1}{\lambda_j}\langle \phi_i, \phi_j \rangle_{H^1} = \delta_{ij}$$

(31)

The remaining results are due to simple computations. □

**Corollary 4.3.** $C^\infty_0(\Omega)$ is dense in $H^1_0(\Omega)$, and so in $L^2(\Omega)$ via convergence of Fourier Series of $L^2$ functions.

**Proposition 4.4.** If $\sum_{j=1}^{\infty} a_j (I - \Delta)\phi_j \in L^2(\Omega)$, then

$$(I - \Delta)\sum_{n=j}^{\infty} a_j\phi_j = \sum_{j=1}^{\infty} a_j(I - \Delta)\phi_j$$

(32)

**Proof.** By Theorem 4.1 there exists a unique $u \in H^1_0(\Omega)$ such that $(I - \Delta)u = \sum_{j=1}^{\infty} a_j(I - \Delta)\phi_j$. Taking the $L^2$-inner-product of this with $\phi_i$, we obtain

$$\langle (I - \Delta)u, \phi_i \rangle_{L^2} = \sum_{i=1}^{\infty} a_i \langle (I - \Delta)\phi_j, \phi_i \rangle_{L^2}$$

(33)

which is equivalent to

$$\langle u, \phi_i \rangle_{H^1} = \sum_{i=1}^{\infty} a_i \langle \phi_j, \phi_i \rangle_{H^1} = a_i$$

(34)

so that $u = \sum_{j=1}^{\infty} a_j \phi_j$, which completes the proof. □

**Corollary 4.5.** If $w = \sum_{j=1}^{\infty} c_j \phi_j \in L^2(\Omega)$, then

$$u = \mathcal{T}(w) = \sum_{j=1}^{\infty} \frac{c_j}{\lambda_j}\phi_j$$

(35)

**Proof.**

$$w = \sum_{j=1}^{\infty} c_j \phi_j = \sum_{j=1}^{\infty} c_j(I - \Delta)(I - \Delta)^{-1}\phi_j = (I - \Delta)\sum_{j=1}^{\infty} \frac{c_j}{\lambda_j}\phi_j$$

(36)

from which by uniqueness in Theorem 4.1 the conclusion follows. □

In the rest of the paper, we will consider $E_N := \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\}$ to be the $N$-dimensional subspace of $H^1_0(\Omega)$, and denote the orthogonal projection $\text{proj}_{E_N}$ by $P_N : H^1_0(\Omega) \rightarrow E_N$. In light of equation (28), it’s worth to mention that $\mathcal{T}_{|E_N} : E_N \rightarrow E_N$ is compact, as well as invertible, and is given by

$$\mathcal{T} \left(\sum_{j=1}^{N} c_j\phi_j\right) = \sum_{j=1}^{N} \frac{c_j}{\lambda_j}\phi_j$$

(37)

Moreover, we have
Proposition 4.6. \( \| \cdot \|_{H^1} \) and \( \| \cdot \|_{L^2} \) are equivalent norms on \( E_N \). In particular,

\[
\| v \|_{L^2} \leq \| v \|_{H^1} \leq \sqrt{\lambda_N} \| v \|_{L^2} \quad \forall v \in E_N
\]

(38)

Proof. Let \( v = \sum_{j=1}^{N} a_j \phi_j \in E_N \), then

\[
\| v \|_{H^1}^2 = \| v \|_{L^2}^2 + \| \nabla v \|_{L^2}^2 = \sum_{j=1}^{N} a_j^2 \| \phi_j \|_{L^2}^2 + \| \phi_j \|_{L^2}^2
\]

(39)

where

\[
\| \phi_j \|_{L^2}^2 = \frac{2 \pi i \xi_j}{L} \frac{2 \pi i \xi_j}{L} = 4 \pi^2 \xi_j^2 \| \phi_j \|_{L^2}^2
\]

(40)

and similarly

\[
\| \phi_j \|_{L^2}^2 = 4 \pi^2 \xi_j^2 \| \phi_j \|_{L^2}^2
\]

(41)

and thus

\[
\| v \|_{L^2}^2 \leq \| v \|_{H^1}^2 = \| v \|_{L^2}^2 + \sum_{j=1}^{N} a_j^2 \| \phi_j \|_{L^2}^2 \leq \left( 1 + \frac{4 \pi^2}{L^2} \| \xi_N \|_{L^2}^2 \right) \| v \|_{L^2}^2 = \lambda_N \| v \|_{L^2}^2
\]

(42)

from which the conclusion follows. \( \square \)

5 Existence of Approximate Solutions to the Hyperbolic Equation

In this section, we follow a semi-discrete Petrov-Galerkin method to construct (see Theorem 5.3) approximate \( C^1 \) solutions to the hyperbolic equation on finite-dimensional subspaces of \( H^1_0(\Omega) \). For that purpose, we make the following definition.

Definition 5.1. A function \( w_N : [0, T] \rightarrow E_N \) is an approximate solution of the finite-dimensional hyperbolic PDE

\[
\begin{aligned}
(P) & \quad \begin{cases} 
  w'(t) + V(u(t)) \cdot \nabla w(t) = k u_y(t) \\
  w(0) = u_0 := -\Delta u_0 + u_0
  \end{cases} \\
\end{aligned}
\]

(43)

associated to a function \( u \in C([0, T], H^1_0(\Omega)) \) with \( u_0 := u(0) \in H^1_0(\Omega) \) iff all of the following conditions hold:

1. \( w_N \in C^1(0, T; E_N) := C^1([0, T], E_N) \cap C([0, T], E_N) \)
2. Semi-variational formulation

\[
\langle w'_N, v \rangle_{L^2} + \langle \nabla w_N, \nabla v \rangle_{L^2} = k \langle u_y, v \rangle_{L^2}
\]

(44)

or equivalently via Lemma [1.3.1],

\[
\langle w'_N, v \rangle_{L^2} = \langle \nabla w_N, \nabla v \rangle_{L^2} + k \langle u_y, v \rangle_{L^2}
\]

(45)

is satisfied for every \( v \in E_N \)

3. \( w_N(0) = (I - \Delta) P_N(u_0) \in E_N \). That is if \( u_0 = \sum_{j=1}^{\infty} a_j \phi_j \), then \( w_N(0) = \sum_{j=1}^{N} a_j \lambda_j \phi_j \).

In this case, we write \( w_N = P_N(u) \).

Proposition 5.2. In the context of the definition above,
(i) If \( u_0 \in H^1_0(\Omega) \), then \( w_N(0) = P_N((I - \Delta)u_0) = P_N(w_0) \).

(ii) For all \( N \in \mathbb{Z}^+ \), \( \|w_N(0)\|_{L^2} \leq \|w_0\|_{L^2} \).

(iii) If \( w_0 \in H^1_0(\Omega) \), then there exists \( M \in \mathbb{Z}^+ \) such that \( N \geq M \implies \|w_N(0)\|_{L^p} < (L^{2/p} + 1) \|w_0\|_{L^\infty} \).

Proof. For part (i), apply \( P_N \) to \((I - \Delta)u_0 = (I - \Delta) \sum_{j=1}^{\infty} a_j \phi_j = \sum_{j=1}^{\infty} a_j (I - \Delta) \phi_j = \sum_{j=1}^{\infty} a_j \lambda_j \phi_j \).

For part (ii), apply square root to

\[
\|w_N(0)\|_{L^2}^2 = \sum_{j=1}^{\infty} a_j \lambda_j a_i \phi_j \phi_i \|_{L^2} = \sum_{j=1}^{\infty} a_j^2 \lambda_j \leq \sum_{j=1}^{\infty} a_j^2 \lambda_j = \sum_{j=1}^{\infty} a_j \lambda_j \|_{L^2} \quad \text{for all } N \in \mathbb{Z}^+ \quad \text{implies} \quad \|w_N(0)\|_{L^2} \leq \|w_0\|_{L^2} \tag{46}
\]

where we have used Proposition 4.4.

Finally, for part (iii), by continuous embedding \( H^1(\Omega) \subset L^p(\Omega) \) for \( p \geq 2 \) (see Corollary 9.14 of [10]),

\[
\|w_N(0) - w_0\|_{L^p} \leq C \|w_N(0) - w_0\|_{H^1} \rightarrow 0 \tag{47}
\]

so that there exists \( M \in \mathbb{Z}^+ \) such that \( N \geq M \) implies \( \|w_N(0) - w_0\|_{L^p} < \|w_0\|_{L^\infty} \), and so

\[
\|w_N(0)\|_{L^p} < \|w_0\|_{L^p} + \|w_0\|_{L^\infty} < (L^{2/p} + 1) \|w_0\|_{L^\infty} \tag{48}
\]

\(\square\)

Notation. Let \( I \subset \mathbb{R} \) be an interval and \( E \) a Banach space. We will denote the norm of the Banach Space \( C(I,E) \) by

\[
\|v\|_{L^\infty(E)} = \sup_{t \in I} \|v(t)\|_E \quad \forall v \in C(I,E) \tag{49}
\]

Theorem 5.3. For every \( u \in C([0,T], H^1_0(\Omega)) \) and for every \( N \in \mathbb{Z}^+ \), there exists a unique approximate solution \( w_N = P_N(u) \) of (P) associated to \( u \). Further for \( N \geq M \) as in part (iii) of Proposition 5.2 we have

1. If \( u \in L^\infty(0,T; H^2_0(\Omega)) \), then

\[
\|w_N\|_{L^\infty(L^2)} \leq |k| T \|u_y\|_{L^\infty(L^2)} + \|w_N(0)\|_{L^2} \leq |k| T \|u\|_{L^\infty(H^1)} + \|w_0\|_{L^2} \tag{50}
\]

2. If \( u \in L^\infty(0,T; H^3_0(\Omega)) \) with \( w_0 := (I - \Delta)u_0 \in H^1_0(\Omega) \cap L^\infty(\Omega) \), then

(i)

\[
\|w_N\|_{L^\infty(L^\infty)} \leq |k| L T \|u_y\|_{L^\infty(L^\infty)} + \|w_N(0)\|_{L^\infty} \leq |k| L T C_\infty \|u\|_{L^\infty(H^3)} + 2 \|w_0\|_{L^\infty} \tag{51}
\]

(ii)

\[
\|w_N\|_{L^\infty(H^1)} \leq 8 |k| L C_\infty T^2 \|u_y\|_{L^\infty(H^3)}^2 + 2 T (|k| + 8 \|w_0\|_{L^\infty}) \|u\|_{L^\infty(H^3)} + 2 \|w_0\|_{H^1} \tag{52}
\]

(iii)

\[
\|w_N'\|_{L^\infty(L^2)} \leq 4 \|u\|_{L^\infty(W^{1,\infty})} \cdot \|w_N\|_{L^\infty(H^1)} + |k| \cdot \|u_y\|_{L^\infty(L^2)} \leq (4C_\infty \|w_N\|_{L^\infty(H^1)} + |k|) \|u\|_{L^\infty(H^3)} \tag{53}
\]

9
Proof of existence and uniqueness of $w_N$. Writing $w_N = \sum_{i=1}^{N} c_i(t) \phi_i$ and setting $v = \phi_j$ in (44) for $j = 1, \cdots, N$, we get

$$\frac{c_j(t)}{\lambda_j} + \sum_{i=1}^{N} \left\langle \nabla(u(t)) \cdot \nabla \phi_i, \phi_j \right\rangle_{L^2} c_i(t) = k \left\langle u_y(t), \phi_j \right\rangle_{L^2} \quad \forall j = 1, \cdots, N \tag{54}$$

Letting $A_{ij}(t) := \lambda_j \left\langle \nabla(u(t)) \cdot \nabla \phi_i, \phi_j \right\rangle_{L^2}$ and $B_j(t) := k \lambda_j \left\langle u_y(t), \phi_j \right\rangle_{L^2}$, this becomes

$$c_j'(t) + \sum_{i=1}^{N} A_{ij}(t)c_i(t) = B_j(t) \quad \forall j = 1, \cdots, N \tag{55}$$

which is the system of $N$ ODEs

$$C'(t) + A(t)C(t) = B(t) \tag{56}$$

Now since $A_{ij}(t)$ and $B_j(t)$ are defined and continuous on $[0, T]$, then by a usual Picard iteration (see Theorem V.7 of [11]), we get a unique solution $C \in C^1(0, T; \mathbb{R}^N)$, which establishes the existence of $w_N$. \qed

For ease of notation in the following proofs, we will write $w$ instead of $w_N$.

Proof of parts 1 and 2(i). Letting $2 \leq p < \infty$ and substituting $v = p |w(t)|^{p-2} w(t) \in E_N$ in the semi-variational formulation (44), we get

$$p \int_{\Omega} w' |w|^{p-2} w \, d\mu = -p \int_{\Omega} u_x w_y |w|^{p-2} w \, d\mu + p \int_{\Omega} u_y w_x |w|^{p-2} w \, d\mu + pk \int_{\Omega} u_y |w|^{p-2} w \, d\mu \tag{57}$$

which is equivalent to

$$\int_{\Omega} \frac{\partial}{\partial t} |w|^p \, d\mu = -\int_{\Omega} u_x \frac{\partial}{\partial y} |w|^p \, d\mu + \int_{\Omega} u_y \frac{\partial}{\partial x} |w|^p \, d\mu + pk \int_{\Omega} u_y |w|^{p-2} w \, d\mu \tag{58}$$

Now integrating the two middle terms by parts, where boundary integrals vanish due to periodicity of $u_x, u_y$ and $|w|^p$, we get

$$\frac{d}{dt} \|w\|^p_{L^p} = \int_{\Omega} u_{xy} |w|^p \, d\mu - \int_{\Omega} u_{yx} |w|^p \, d\mu + pk \int_{\Omega} u_y |w|^{p-2} w \, d\mu = pk \int_{\Omega} u_y |w|^{p-2} w \, d\mu \tag{59}$$

When $p = 2$, equation (59) becomes

$$\frac{d}{dt} \|w\|^2_{L^2} = 2k \int_{\Omega} u_y w \, d\mu \tag{60}$$

from which Cauchy-Schwarz inequality implies

$$\frac{d}{dt} \|w\|_{L^2} \leq |k| \cdot \|u_y\|_{L^2} \leq |k| \cdot \|u_y\|_{L^\infty(L^2)} \tag{61}$$

When $2 < p < \infty$, equation (59) becomes

$$p \|w\|^{p-1}_{L^p} \frac{d}{dt} \|w\|_{L^p} \leq p |k| \cdot \|u_y\|_{L^\infty} \|w\|^{p-1}_{L^p} \leq p |k| \cdot \|u_y\|_{L^\infty} L^{2/p} \|w\|^{p-1}_{L^p} \tag{62}$$

and so

$$\frac{d}{dt} \|w\|_{L^p} \leq |k| L \|u_y\|_{L^\infty} \leq |k| L \|u_y\|_{L^\infty(L^\infty)} \tag{63}$$
Finally, using Proposition 5.2 (iii) and letting $w$ where

$$\|w\|_{L^2} \leq |k| t \|u_y\|_{L^\infty(L^2)} + \|w(0)\|_{L^2}$$

(64)

and

$$\|w\|_{L^p} \leq |k| L^t \|u_y\|_{L^\infty(L^\infty)} + \|w(0)\|_{L^p}$$

(65)

where $w(0) = w_N(0) \in E_N \subset L^p(\Omega)$ for $2 \leq p \leq \infty$. Taking the sup of (64) over $[0, T]$, part 1 is proved. Finally, using Proposition 5.2 (iii) and letting $p \to \infty$ in (65), and then taking the sup over $[0, T]$, part 2(i) is proved.

**Proof of part 2(ii).** Substituting $v = -\Delta w(t) \in E_N$ in the semi-variational formulation (44), we get

$$\frac{1}{2} \frac{d}{dt} \|\nabla w\|_{L^2}^2 = -\langle \nabla (u_xw_y), \nabla w \rangle_{L^2} + \langle \nabla (u_yw_x), \nabla w \rangle_{L^2} + k \langle \nabla u_y, \nabla w \rangle_{L^2}$$

(66)

where the terms with $\frac{1}{2}$ in front cancel out, and the terms with form:

(a) $\|D^3 uDw, w\|_{L^2} \leq \|w\|_{H^1} \|\nabla w\|_{L^2} \|w\|_{L^\infty}$

(b) $\|D^2 uD^2 w, w\|_{L^2} \leq \|w\|_{L^\infty} \|D^2 u\| \|D^2 w\|_{L^2} = \|w\|_{L^\infty} \langle \text{sign}(D^2 w)\text{sign}(D^2 u)D^2 u, D^2 w \rangle_{L^2}$

$$= -\|w\|_{L^\infty} \langle D \text{sign}(D^2 w)\text{sign}(D^2 u)D^2 u, Dw \rangle_{L^2}$$

$$= -\|w\|_{L^\infty} \langle \text{sign}(D^2 w)\text{sign}(D^2 u)D^3 u, Dw \rangle_{L^2}$$

(67)

where $\text{sign}(v) \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and $D\text{sign}(v) = 0$ a.e. on $\Omega$ for all $v \in H^1_0$.

Hence $\|D^2 uD^2 w, w\|_{L^2} \leq \|w\|_{L^\infty} \|D^3 u, Dw\|_{L^2} \leq \|w\|_{L^\infty} \|u\|_{H^4} \|\nabla w\|_{L^2}$.

Thus after using Hölder’s and Triangle inequalities and canceling out the term $\|\nabla w\|_{L^2}$, we get

$$\frac{d}{dt} \|\nabla w\|_{L^2} = 8 \|w\|_{L^\infty} \|u\|_{H^3} + |k| \|\nabla u_y\|_{L^2} \leq (8 \|w\|_{L^\infty} |L^\infty| + |k|) \|u\|_{L^\infty(H^3)}$$

(68)

Now integrating over the temporal interval $[0, t]$, with $t \leq T$, and taking the sup over $[0, T]$, we obtain

$$\|\nabla w\|_{L^\infty(L^2)} \leq (8 \|w\|_{L^\infty(L^\infty)} + |k|) T \|u\|_{L^\infty(H^3)} + \|\nabla w(0)\|_{L^2}$$

(69)
Finally,
\[ ||w||_{L^\infty(H^1)} \leq ||w||_{L^\infty(L^2)} + ||\nabla w||_{L^\infty(L^2)} \]
\[ \leq |k|T ||u_y||_{L^\infty(L^2)} + ||w(0)||_{L^2} + (8 ||w||_{L^\infty(L^\infty)} + |k|)T ||u||_{L^\infty(H^3)} + ||\nabla w(0)||_{L^2} \]  
(70)
\[ \leq 2 |k|T ||u||_{L^\infty(H^3)} + 8 ||w||_{L^\infty(L^\infty)} T ||u||_{L^\infty(H^3)} + 2 ||w(0)||_{H^1} \]
from which the result follows by part 2(i).

**Proof of part 2(iii).** Substituting \( v = w'(t) \in E_N \) in the semi-variational formulation (44), we get
\[ \langle w', w' \rangle_{L^2} = - \langle \tilde{V}(u) \cdot \nabla w, w' \rangle_{L^2} + k \langle u_y, w' \rangle_{L^2} \]
(71)
so that by the triangle inequality
\[ ||w'||_{L^2}^2 \leq \int |u_y w_x w'| \ d\mu + \int |u_x w_y w'| \ d\mu + |k| \int |u_y w'| \ d\mu \]  
(72)
\[ ||w'||_{L^2}^2 \leq ||u_y||_{L^\infty} \int |w_x w'| \ d\mu + ||u_x||_{L^\infty} \int |w_y w'| \ d\mu + |k| \int |u_y w'| \ d\mu \]
(73)
which by Cauchy-Schwarz inequality implies
\[ ||w'||_{L^2}^2 \leq 2 ||w||_{W^{1,\infty}} \left( ||w_x||_{L^2} + ||w_y||_{L^2} \right) ||w'||_{L^2} + |k| \cdot ||u_y||_{L^2} ||w'||_{L^2} \]
(74)
from which the result follows.

For our convenience in the rest of the paper, we rename the subsequence \( \{w_{N_k}\} \) back as \( \{w_N\} \).

## 6 A Sequence of Fixed-Point Approximate Solutions \( \{u_N, w_N\} \)

In this section, for each \( N \in \mathbb{Z}^+ \), we find a fixed-point of the composition \( T \circ P_N \) of the solution operators \( T \) and \( P_N \) of the elliptic and the finite-dimensional hyperbolic PDEs, respectively, on \( C([0, T], E_N) \); thus obtaining a sequence of approximate, in light of Definition 5.1, solution pairs \( \{u_N, w_N\} \), where \( u_N = (I - \Delta)^{-1} w_N \). For each given regularity of initial data, we setup the fixed-point problem in the diagram below for particular sets \( \mathcal{X} \) and \( \mathcal{Y} \), however the proofs for existence of a fixed-point in each case are identical.

![Diagram](https://via.placeholder.com/150)

### 6.1 The case when initial data is \( u_0 \in H^2(\Omega) \)

We consider two constants
\[ 0 < T < (C_E |k| + 1)^{-1} \quad \text{and} \quad C_X = \frac{2C_E \|u_0\|_{H^2}}{1 - C_E |k| T} > 0 \]
(75)
and for each \( N \in \mathbb{Z}^+ \), from which \( T \) is independent of, we consider \( \mathcal{X} \) and \( \mathcal{Y} \) to be the following non-empty closed, bounded, convex sets
\[ \mathcal{X} = \left\{ u \in C([0, T], E_N) \mid ||u||_{L^\infty(H^2)} \leq C_X \right\} \]
(76)
\[ \mathcal{Y} = \left\{ w \in C([0, T], E_N) \mid ||w||_{L^\infty(L^2)} \leq C_X/C_E \right\} \]
(77)
Observe that each solution operator is well-defined due to Theorems 4.1 and 5.3 and the assertions:
(i) $\mathcal{P}(X) \subseteq \mathcal{Y}$ as through estimate \([50]\) we have
\[
\|\mathcal{P}(u)\|_{L^\infty(L^2)} \leq |k| T \|u\|_{L^\infty(H^1)} + \|u_0\|_{L^2} \\
\leq |k| T \|u\|_{L^\infty(H^2)} + \|\Delta u_0\|_{L^2} + \|u_0\|_{L^2} \\
\leq |k| |\mathcal{T}| + 2 \|u_0\|_{H^2} = C_X/C_E
\]
(ii) $\mathcal{T}(\mathcal{Y}) \subseteq \mathcal{X}$ as through estimate \([25]\) we have $\|\mathcal{T}(w)\|_{L^\infty(H^2)} \leq C_E \|w\|_{L^\infty(L^2)} \leq C_X$.

6.2 The case when initial data is $u_0 \in H^1_0(\Omega)$ with $w_0 := -\Delta u_0 + u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$

We define the following quantities
\[
A = 8C_E |k| L C_\infty; \quad B = 2C_E(\|k\| + 8 \|w_0\|_{L^\infty}); \quad C = 2C_E \|w_0\|_{H^1};
\]
\[
T = \frac{1}{B + 2\sqrt{AC}}; \quad C_X = \begin{cases} \frac{C}{1 - BT} & \text{if } k = 0 \\ \frac{2AT}{B} & \text{if } k \neq 0 \end{cases}
\]
and for each $N \in \mathbb{Z}^+$, from which $T$ is independent of, we consider $\mathcal{X}$ and $\mathcal{Y}$ to be the following non-empty closed, bounded, convex sets
\[
\mathcal{X} = \left\{ u \in C([0,T],E_N) \mid \|u\|_{L^\infty(H^1)} \leq C_X \right\}
\]
\[
\mathcal{Y} = \left\{ w \in C([0,T],E_N) \mid \|w\|_{L^\infty(H^1)} \leq C_X/C_E \right\}
\]
Observe that each solution operator is well-defined due to Theorems \([4.1]\) and \([5.3]\) and the assertions:

(i) $\mathcal{P}(X) \subseteq \mathcal{Y}$ as through estimate \([52]\) if $k \neq 0$, we have
\[
\|\mathcal{P}(u)\|_{L^\infty(H^1)} \leq \frac{1}{C_E} \left( AT^2 C_X^2 + BT C_X + C \right) \\
\leq \frac{1}{C_E} \left[ AT^2 C_X^2 + (BT - 1) C_X + C \right] + C_X/C_E \\
\leq \frac{1}{C_E} \left[ C - \frac{(1 - BT)^2}{4AT^2} \right] + C_X/C_E = C_X/C_E
\]
where the last equality is due to $T = (B + 2\sqrt{AC})^{-1}$ being a solution to the quadratic $4ACT^2 - (1 - BT)^2 = 0$. On the other hand, if $k = 0$,
\[
\|\mathcal{P}(u)\|_{L^\infty(H^1)} \leq \frac{1}{C_E} \left( BT - 1 \right) + C \leq \frac{C}{1 - BT} + C_X/C_E = C_X/C_E
\]
(ii) $\mathcal{T}(\mathcal{Y}) \subseteq \mathcal{X}$ as through estimate \([25]\) we have $\|\mathcal{T}(w)\|_{L^\infty(H^1)} \leq C_E \|w\|_{L^\infty(L^1)} \leq C_X$.

6.3 Existence of a fixed-point $\{u_N, w_N\}$ of $\mathcal{T} \circ \mathcal{P}_N$

Theorem 6.1. For every $N \in \mathbb{Z}^+$, $\mathcal{T} \circ \mathcal{P}_N : \mathcal{X} \to \mathcal{X}$ is a continuous compact mapping, so that by Schauder Fixed-Point Theorem it has a fixed point $u_N$. Moreover via Theorems \([4.1]\) and \([5.3]\) each $u_N$ has an associate $w_N$, satisfying
\[
\begin{aligned}
\{u_N, w_N\} &\in C^1(0,T;E_N) \times C^1(0,T;E_N) \\
-\Delta u_N + u_N &= w_N \quad \text{on } \Omega \times [0,T] \\
\langle w'_N, v \rangle_{L^2} + \langle \nabla u_N, \nabla w_N, v \rangle_{L^2} &= k \langle u_N, v \rangle_{L^2} \quad \forall v \in E_N, \forall t \in (0,T) \\
u_N(0) &= \mathcal{P}_N(u_0) \quad \text{on } \Omega \\
w_N(0) &= (I - \Delta)\mathcal{P}_N(u_0) \quad \text{on } \Omega
\end{aligned}
\]
The proof of Theorem 6.1 heavily lies on the finite-dimensionality of $E_N$ as it rises a norm-equivalence between $\|\cdot\|_{H^1}$ and $\|\cdot\|_{L^2}$ on $E_N$ (see Proposition 4.6). At this point, we also let $C_G > 0$ to be the constant in the Gagliardo-Nirenberg inequality

$$\|v\|_{L^4} \leq C_G \|v\|_{L^2}^{1/2} \cdot \|v\|_{H^1}^{1/2} \leq C_G \|v\|_{H^1} \quad \forall v \in H^1(\Omega)$$  \hfill (84)

**Proof of Theorem 6.1.** Since $T$ is compact and continuous, it suffices to establish that $\mathcal{P}_N$ is continuous. So let $\epsilon > 0$ be given, set $\delta = 1/\sqrt{\lambda_N(2\lambda N C_G^2 C_N/C_E + |k| + 1)}$, and let $u_1, u_2 \in X$ be such that $\|u_1 - u_2\|_{L^\infty(\Omega)} \leq \delta$. Setting $w_1 = \mathcal{P}_N(u_1)$ and $w_2 = \mathcal{P}_N(u_2)$, the semi-variational formulation (44) gives us

$$\langle w_i', v \rangle_{L^2} = -\langle \{u_i, w_i\}, v \rangle_{L^2} + k \langle u_{i,y}, v \rangle_{L^2} \quad \forall v \in E_N, \text{ for } i = 1, 2$$  \hfill (85)

Now subtracting (85) with $i = 2$ from that of $i = 1$, we obtain

$$\langle w_1' - w_2', v \rangle_{L^2} = \{\langle u_2, w_2 \rangle - \{u_1, w_1\}, v \} \langle 1 \rangle_{L^2} + k \langle (u_1 - u_2)_y, v \rangle_{L^2} \quad \forall v \in E_N$$  \hfill (86)

Now setting $v = w_1 - w_2$, we get

$$\frac{1}{2} \frac{d}{dt} \|w_1 - w_2\|_{L^2}^2 = \langle \{u_2, w_2\} - \{u_1, w_1\}, w_1 - w_2 \rangle_{L^2} + k \langle (u_1 - u_2)_y, w_1 - w_2 \rangle_{L^2}$$

$$= \langle \{u_2, w_2 - w_1\} + \{u_2 - u_1, w_1\}, w_1 - w_2 \rangle_{L^2} + k \langle (u_1 - u_2)_y, w_1 - w_2 \rangle_{L^2}$$  \hfill (87)

$$= \langle \{u_2 - u_1, w_1\}, w_1 - w_2 \rangle_{L^2} + k \langle (u_1 - u_2)_y, w_1 - w_2 \rangle_{L^2}$$

$$\|w_1 - w_2\|_{L^2}^2 \frac{d}{dt} \|w_1 - w_2\|_{L^2} = \int_{\Omega} (u_2 - u_1)_x w_{1,y}(w_1 - w_2) \, d\mu$$

$$- \int_{\Omega} (u_2 - u_1)_y w_{1,x}(w_1 - w_2) \, d\mu + k \int_{\Omega} (u_1 - u_2)_y (w_1 - w_2) \, d\mu$$  \hfill (88)

Now by Hölder’s Inequality, after cancelling $\|w_1 - w_2\|_{L^2}$ from both sides, we obtain

$$\frac{d}{dt} \|w_1 - w_2\|_{L^2} \leq \|\{u_2 - u_1\}_x\|_{L^1} \|w_{1,y}\|_{L^1} + \|\{u_2 - u_1\}_y\|_{L^1} \|w_{1,x}\|_{L^1} \| \cdot \| + \|u_2 - u_1\|_{L^1} \|w_{1,x}\|_{L^1} \| \cdot \| + |k| \cdot \|u_1 - u_2\|_{L^2}$$

$$\leq C_G^2 I(t_0 - \pi)^H_{\|w_{1,y}\|_{H^1}} + C_G^2 I(t_0 - \pi)^H_{\|w_{1,x}\|_{H^1}} + |k| \cdot \|u_1 - u_2\|_{L^2}$$

$$\leq \lambda_N C_G^2 I(t_0 - \pi)^H_{\|w_{1,y}\|_{L^2} + \lambda_N C_G^2 I(t_0 - \pi)^H_{\|w_{1,x}\|_{L^2}} + |k| \cdot \|u_1 - u_2\|_{L^2}$$

$$\leq 2 \lambda_N C_G^2 I(t_0 - \pi)^H_{\|w_{1,y}\|_{L^2} + \|w_{1,x}\|_{L^2} + |k| \cdot \|u_1 - u_2\|_{H^1}}$$

$$\leq (2 \lambda_N C_G^2 C_N/C_E + |k|) \cdot \|u_2 - u_1\|_{L^\infty(H^1)}$$  \hfill (89)

which integrating over $[0, t]$, with $t \leq T$, and taking the sup over $[0, T]$ we get

$$\|w_1 - w_2\|_{L^\infty(L^\infty)} \leq T(2 \lambda_N C_G^2 C_N/C_E + |k|) \cdot \|w_1 - w_2\|_{L^\infty(H^1)} + \|w_1(0) - w_2(0)\|_{L^2}$$

$$\leq T(2 \lambda_N C_G^2 C_N/C_E + |k|) \delta < \epsilon/\sqrt{\lambda_N}$$  \hfill (90)

Hence

$$\|\mathcal{P}_N(u_1) - \mathcal{P}_N(u_2)\|_{L^\infty(H^1)} = \|w_1 - w_2\|_{L^\infty(H^1)} \leq \sqrt{\lambda_N} \|w_1 - w_2\|_{L^\infty(L^2)} < \epsilon$$  \hfill (91)

\[ \square \]

### 7 A Candidate Solution \{u, w\}

In this section, we construct a candidate solution to (VF2) and (VF 1), respectively, as the limit of some weakly convergent subsequence of the sequence obtained above.
7.1 The case when initial data is \( u_0 \in H^3_P(\Omega) \) with \( w_0 := -\Delta u_0 + u_0 \in H^1_P(\Omega) \cap L^\infty(\Omega) \)

We extract a subsequence of pairs \( \{u_N, w_N\} \) that converge weakly to the pair

\[
\{u, w\} \in L^2(0, T; H^3_P(\Omega)) \times L^2(0, T; H^1_P(\Omega))
\]

which we consider as a candidate solution to (VF 2). For this purpose, we recall that for any Hilbert space \( \mathcal{H} \), the Bochner space \( L^2(0, T; \mathcal{H}) \) is a Hilbert space with the inner product \( \int_0^T \langle \cdot, \cdot \rangle_{\mathcal{H}} \). Also, if a sequence \( \{v_n\} \) is uniformly bounded in \( L^2(0, T; \mathcal{H}) \), then by Theorem 3.18 of [10], it has a weakly convergent subsequence \( \{v_{n_k}\} \), which we will always rename it back as \( \{v_n\} \). That is, there exists \( v \in L^2(0, T; \mathcal{H}) \) such that

\[
\int_0^T \langle v_n, \varphi \rangle_{\mathcal{H}} \, dt \to \int_0^T \langle v, \varphi \rangle_{\mathcal{H}} \, dt \quad \forall \varphi \in L^2(0, T; \mathcal{H})^* \cong L^2(0, T; \mathcal{H})
\]  

(92)

Now since \( u_N \in \mathcal{X} \) with \( T < \infty \), then for \( m = 0, 1, 2, 3 \), \( \int_0^T \|u_N\|^2_{H^m} \) are uniformly bounded by \( TC_X^2 \). Hence, we can extract subsequences, one at a time for each of \( m = 3, 2, 1, 0 \) (in this order) to get a subsequence \( \{u_N\} \) which converges weakly to \( u \in L^2(0, T, H^3_P(\Omega)) \).

Note that in this process we have renamed each subsequence to its parent sequence and used the uniqueness of weak limits.

At this point, we consider the Sobolev-Bochner space

\[
\mathcal{W} = W^{1,2,2}(0, T; H^3_P(\Omega), H^2_P(\Omega)) := \{ u \in L^2(0, T; H^3_P(\Omega)) \mid u_t \in L^2(0, T; H^2_P(\Omega)) \}
\]

(93)

where \( u_t \) is the weak derivative of \( u \) with respect to \( t \). By Proposition 2.46 of [9], \( \mathcal{W} \) is a Hilbert space with the inner product

\[
\langle u, v \rangle_{\mathcal{W}} := \int_0^T \langle u, v \rangle_{H^3} \, ds + \int_0^T \langle u_t, v_t \rangle_{H^2} \, ds \quad \forall u, v \in \mathcal{W}
\]

(94)

and by Proposition 2.46 (vii) of [9], \( \mathcal{W} \subset C([0, T]; H^2_P(\Omega)) \). By \( u_N \) being in \( C_X \), uniqueness in Theorem 4.1 and estimate [53], the norm

\[
\|u_N\|_{\mathcal{W}} = \int_0^T \|u_N\|^2_{H^3} \, ds + \int_0^T \|u'_N\|^2_{H^2} \, ds
\]

\[
\leq TC_X + \int_0^T C_E \|u'_N\|^2_{L^2} \, ds
\]

\[
\leq TC_X^2 + TC_E(4C_X C_E + |k|)C_X^2
\]

(95)

is uniformly bounded, so that the subsequence \( \{u_N\} \) has a weakly convergent subsequence in \( \mathcal{W} \), which we rename it again as \( \{u_N\} \). That is, there exists \( u \in \mathcal{W} \) such that

\[
\langle u_N, \varphi \rangle_{\mathcal{W}} \to \langle u, \varphi \rangle_{\mathcal{W}} \quad \forall \varphi \in \mathcal{W}^* \cong \mathcal{W}
\]

(96)

Lemma 7.1. In the construction above, when the initial data is \( u_0 \in H^3_P(\Omega) \) with \( w_0 := -\Delta u_0 + u_0 \in H^1_P(\Omega) \cap L^\infty(\Omega) \), there exists a pair

\[
\{u, w\} \in L^2(0, T; H^3_P(\Omega)) \times L^2(0, T; H^1_P(\Omega)) \quad \text{with} \quad \{u_t, w_t\} \in L^2(0, T; H^2_P(\Omega)) \times L^2(0, T; L^2(\Omega))
\]

(97)

where \( w_t \) is the weak temporal derivative of \( w \), such that:

(a) \( u_N \rightharpoonup u \) weakly in \( L^2(0, T; H^3_P(\Omega)) \) for \( m = 0, 1, 2, 3 \).

(b) Let \( w = -\Delta u + u \in L^2(0, T; H^3_P(\Omega)) \), then \( w_N \rightharpoonup w \) weakly in \( L^2(0, T; H^m_P(\Omega)) \) for \( m = 0, 1 \).

(c) \( u'_N \rightharpoonup u_t \) weakly in \( L^2(0, T; H^3_P(\Omega)) \) for \( m = 0, 1, 2 \).
(d) \( w_N \to w_t \) weakly in \( L^2(0, T; L^2(\Omega)) \).

(e) \( u_N \to u \) strongly in \( L^2(0, T; H^1_0(\Omega)) \).

**Proof.** Part (a) is due to construction. Now let \( \varphi \in C^\infty_c(0, T; C^\infty_0(\Omega)) \). Then for \( m = 0, 1 \)
\[
\int_0^T \langle w_N - w, \varphi \rangle_{H^m} \, dt = \int_0^T \langle (I - \Delta)(u_N - u), \varphi \rangle_{H^m} \, dt = \int_0^T \langle u_N - u, (I - \Delta)\varphi \rangle_{H^m} \, dt \to 0 \tag{98}
\]
by part (a). Now, Fubini’s Theorem, integration by parts on \([0, T] \), and the definition of weak temporal derivatives imply
\[
\int_0^T \langle w'_N - w_t, \varphi \rangle_{H^1} \, dt = \int_0^T \langle w - w_N, \varphi_t \rangle_{H^1} \, dt = \int_0^T \langle u - u_N, (I - \Delta)\varphi_t \rangle_{H^1} \, dt \to 0 \tag{99}
\]
and for \( m = 0, 1, 2 \)
\[
\int_0^T \langle u'_N - u_t, \varphi \rangle_{H^m} \, dt = \int_0^T \langle u - u_N, \varphi_t \rangle_{H^m} \, dt \to 0 \tag{100}
\]
by part (a). Now by the density of \( C^\infty_c(0, T; C^\infty_0(\Omega)) \) in \( L^2(0, T; H^1_0(\Omega)) \), parts (b), (c), and (d) follow. Finally, since \( H^1_0(\Omega) \) can be compactly embedded into \( H^1_F(\Omega) \), part (e) follows immediately from Aubin-Lions Lemma (see Lemma 7.7 of \([12]\)) and parts (a) and (c).

**7.2 The case when initial data is \( u_0 \in H^2(\Omega) \)**

We extract a subsequence of pairs \( \{u_N, w_N\} \) that converge weakly to the pair
\[
\{u, w\} \in L^2(0, T, H^2_F(\Omega)) \times L^2(0, T, L^2(\Omega))
\]
which we consider as a candidate solution to (VF 1). We begin the uniform boundedness of \( \|u'_N\|_{L^\infty(L^2)} \).

**Lemma 7.2.** For every \( N \in \mathbb{Z}^+ \), the fixed point \( u_N \) satisfies
\[
\|u'_N\|_{L^\infty(L^2)} \leq 2C_G C_X^2 + |k| C_E C_X \tag{101}
\]

**Proof.** Substituting \( v = (I - \Delta)^{-1}u'_N \) in the semi-variational formulation in \((83)\) and using Lemma \([3](i)\), we get
\[
\langle w'_N, (I - \Delta)^{-1}u'_N \rangle_{L^2} = \langle \nabla(u_N) \cdot \nabla(I - \Delta)^{-1}u'_N, w_N \rangle_{L^2} + k \langle u_{N,y}, (I - \Delta)^{-1}u'_N \rangle_{L^2} \tag{102}
\]
which by self-adjointness of \((I - \Delta)^{-1}\) becomes
\[
\|u'_N\|_{L^2}^2 = \langle \frac{\partial}{\partial x}(u_N) \frac{\partial}{\partial y}((I - \Delta)^{-1}u'_N), w_N \rangle_{L^2} + k \langle (I - \Delta)^{-1}u_{N,y}, u'_N \rangle_{L^2} \leq \langle \|\frac{\partial}{\partial x}(u_N)\|_{L^4} \|\frac{\partial}{\partial y}((I - \Delta)^{-1}u'_N)\|_{L^4} \|\frac{\partial}{\partial x}(u_N)\|_{L^4} \|\frac{\partial}{\partial y}((I - \Delta)^{-1}u'_N)\|_{L^4} \|w_N\|_{L^2} + k \|\|((I - \Delta)^{-1}u_{N,y}, u'_N\|_{L^2} \leq 2C_G \|w_N\|_{H^2} \|\|((I - \Delta)^{-1}u'_N\|_{H^2} + |k| \|\|((I - \Delta)^{-1}u_N\|_{H^2} \|u'_N\|_{L^2} \leq 2C_E C_G \|w_N\|_{H^2} \|u'_N\|_{L^2} + |k| C_E \|u_N\|_{L^2} \|u'_N\|_{L^2} \leq (2C_G C_X^2 + |k| C_E C_X) \|u'_N\|_{L^2}
\]
from which the result follows.
Lemma 7.3. By a similar construction, when the initial data is considered to be $u_0 \in H^2(\Omega)$, there exists a pair
\[ \{u, w\} \in L^2(0, T; H^2_P(\Omega)) \times L^2(0, T; L^2(\Omega)) \] with $u_t \in L^2(0, T; L^2(\Omega))$ (104)
such that:

(a) $u_N \rightharpoonup u$ weakly in $L^2(0, T; H^m_P(\Omega))$ for $m = 0, 1, 2$

(b) Let $w = -\Delta u + u \in L^2(0, T; L^2(\Omega))$, then $w_N \rightharpoonup w$ weakly in $L^2(0, T; L^2(\Omega))$

(c) $u_t' N \rightharpoonup u_t$ weakly in $L^2(0, T; L^2(\Omega))$

(d) $u_N \rightarrow u$ strongly in $L^2(0, T; L^2(\Omega))$

Proof. Mimic the proof of Lemma 7.1.

8 Proof of the Main Theorem & It’s Corollaries

We begin by the following lemma.

Lemma 8.1. In the context of Lemma 7.1, for every $\varphi \in L^2(0, T, H^1_P(\Omega))$ we have

(i)
\[ \int_0^T \langle \tilde{V}(u_N) \cdot \nabla w_N, \varphi \rangle_{L^2} dt \rightarrow \int_0^T \langle \tilde{V}(u) \cdot \nabla w, \varphi \rangle_{L^2} dt \] (105)

(ii)
\[ \int_0^T \langle k \frac{\partial}{\partial y} u_N, \varphi \rangle_{L^2} dt \rightarrow \int_0^T \langle ku_y, \varphi \rangle_{L^2} dt \] (106)

Proof. Let $\varphi \in C_c^\infty(0, T, C^\infty(\Omega))$. Then using Triangle and Cauchy-Schwarz inequalities, and Lemma 1.3i), we get

\[
\left| \int_0^T \langle \tilde{V}(u_N) \cdot \nabla w_N, \varphi \rangle_{L^2} dt - \int_0^T \langle \tilde{V}(u) \cdot \nabla w, \varphi \rangle_{L^2} dt \right|
\]
\[
= \left| \int_0^T \langle \tilde{V}(u_N) \cdot \nabla \varphi, w_N \rangle_{L^2} dt - \int_0^T \langle \tilde{V}(u) \cdot \nabla \varphi, w \rangle_{L^2} dt \right|
\]
\[
\leq \int_0^T \left| \langle \tilde{V}(u_N - u) \cdot \nabla \varphi, w_N \rangle_{L^2} dt \right| + \int_0^T \left| \langle \tilde{V}(u) \cdot \nabla \varphi, w_N - w \rangle_{L^2} dt \right|
\]
\[
\leq \int_0^T \left| \frac{\partial}{\partial x}(u_N - u) \varphi_y, w_N \right|_{L^2} dt + \int_0^T \left| \frac{\partial}{\partial y}(u_N - u) \varphi_x, w_N \right|_{L^2} dt
\]
\[
+ \int_0^T \langle \tilde{V}(u) \cdot \nabla \varphi, w_N - w \rangle_{L^2} dt
\] (107)
so that 

\[
\|w(0) - w_0\|_{L^2} = \|(I - \Delta)^{-1}w_N(0) - u_0\|_{L^2} = \|P_N(u_0) - u_0\|_{L^2} \leq \|P_N(u_0) - u_0\|_{H^1} \rightarrow 0
\]

so that \(w(0) = u_0\) a.e. on \(\Omega\). Hence \(u(0) = u_0\) as elements of \(H^1_P(\Omega)\). Now by continuity of \(w\) at \(t = 0\) and definition,

\[
\|w(0) - w_0\|_{L^2} = \|(I - \Delta)(u(0) - u_0)\|_{L^2} \leq \|u(0) - u_0\|_{H^2} = 0
\]

so that \(w(0) = w_0\) a.e. on \(\Omega\). 

---

**Proof of Theorem 2.7** Through the fixed-point argument in Section 6 setting \(T = \frac{1}{\beta + 2\sqrt{\nu}}\) that is equivalent to equation (112), we obtain a sequence of fixed-points \(\{u_n, w_N\}\) which via Lemma 7.1 has a subsequence that converge weakly to a candidate solution pair \(\{u, w\}\) in \(L^2(0, T; H^1_P(\Omega))\) by construction, then \(-\Delta u + u = w\) a.e. on \(\Omega \times [0, T]\). Observe that through Lemmas 7.1(d) and 8.1 the semi-variational formulation converges to

\[
\int_0^T \langle w_t, \varphi \rangle_{L^2} \, dt = \int_0^T \langle \nabla w, \varphi \rangle_{L^2} \, dt + \int_0^T \langle k w_u, \varphi \rangle_{L^2} \, dt \quad \forall \varphi \in L^2(0, T; H^1_P(\Omega))
\]

Finally, since \(u_N \rightharpoonup u\) in \(L^2(0, T; H^1_P(\Omega))\) strongly, then there is a subsequence \(u_{N_m}(t) \rightharpoonup u(t)\) in \(H^1_P(\Omega)\) for a.e. \(t \in [0, T]\). In particular, there is a sequence \(\{t_m\}_{m=1}^\infty \subset [0, 1]\) converging to 0 such that for every \(m \in \mathbb{Z}^+, u_{N_k}(t_m) \rightharpoonup u(t_m)\) in \(H^1_P(\Omega)\). Now

\[
\|u(0) - u_0\|_{L^2} \leq \|u(0) - u(t_m)\|_{L^2} + \|u(t_m) - u_N(t_m)\|_{L^2} + \|u_N(t_m) - u_N(0)\|_{L^2} + \|u_N(0) - u_0\|_{L^2}
\]

where the last term

\[
\|u_N(0) - u_0\|_{L^2} = \|(I - \Delta)^{-1}w_N(0) - u_0\|_{L^2} = \|P_N(u_0) - u_0\|_{L^2} \leq \|P_N(u_0) - u_0\|_{H^1} \rightarrow 0
\]

so that \(u(0) = u_0\) a.e. on \(\Omega\). Hence \(u(0) = u_0\) as elements of \(H^1_P(\Omega)\). Now by continuity of \(w\) at \(t = 0\) and definition,
Proof of Corollary 2.2. The semi-variational formulation is equivalent to
\[
\int_0^T \left< w_t + \bar{V}(u) \cdot \nabla w - ku_y, \varphi \right>_{L^2} dt = 0 \quad \forall \varphi \in L^2(0, T; H^1_p(\Omega))
\] (114)
which in particular holds for \( \varphi \in C_{\infty}^c(0, T; C_{\infty}^\alpha(\Omega)) \), a dense subset of \( L^2(0, T; H^1_p(\Omega)) \). Thus
\[
w_t + \bar{V}(u) \cdot \nabla w = ku_y \quad \text{in} \ L^2(0, T; H^1_p(\Omega))
\] (115)
so that \( w_t + \bar{V}(u) \cdot \nabla w = ku_y \) a.e. on \( \Omega \times [0, T] \). Now to prove uniqueness, we suppose that the pairs \( \{u_1, w_1\} \) and \( \{u_2, w_2\} \) are a solution to (15). Then subtracting
\[
w_{i,t} = -\bar{V}(u_i) \cdot \nabla w_i + ku_{i,y}
\] (116)
for \( i = 2 \) from that of \( i = 1 \), taking the \( L^2 \)-innerproduct of the resulting equation with \( \varphi \in L^2(0, T; H^1_p(\Omega)) \), and using Lemma 1.3(i), we obtain
\[
\langle (w_1 - w_2)_t, \varphi \rangle_{L^2} = \langle \bar{V}(u_1) \cdot \nabla \varphi, w_1 \rangle_{L^2} - \langle \bar{V}(u_2) \cdot \nabla \varphi, w_2 \rangle_{L^2} + k \langle (u_1 - u_2)_y, \varphi \rangle_{L^2}
\] (117)
Now setting \( \varphi = w_1 - w_2 \) and using Lemma 1.3(ii), we get
\[
\frac{1}{2} \frac{d}{dt} \|w_1 - w_2\|^2_{L^2} = \langle k(u_1 - u_2)_y - \bar{V}(u_1 - u_2) \cdot \nabla w_1, w_1 - w_2 \rangle_{L^2}
\] (118)
where we have cancelled the term \( \|w_1 - w_2\|^2_{L^2} \). Now integrating over \([0, t]\) with \( t \leq T \) we get
\[
\|w_1(t) - w_2(t)\|_{L^2} \leq \int_0^t (|k| \cdot C_X + 2C_\infty C_X^2/C_E) \ d\tau + \|w_1(0) - w_2(0)\|_{L^2}
\] (119)
which by Gronwall’s inequality implies that \( \|w_1 - w_2\|_{L^2} \leq 0 \), so that \( w_1 = w_2 \) a.e on \( \Omega \). Consequently, \( u_1 = u_2 \) a.e on \( \Omega \).

Proof of Corollary 2.2. Consider the weight/test function(s) \( \psi \in H^1_p(\Omega) \) to be independent of \( t \). After repeating the process of choosing \( t_i, i+1 \in (t_i, T) \), considering the problem on \([t_i, t_{i+1}]\), integrating the left hand side of the semi-variational formulation with respect to \( t \) and using Lemma 1.3(i), the result will follow.

Proof of Proposition 2.4. Theorem 6.1 and Lemmas 7.3 and 1.3

\[\square\]
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