Second order QCD corrections to the nonleptonic decay of the $b$ quark in the slow charm limit

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Abstract

The perturbative QCD corrections to the nonleptonic decay rate of the $b$ quark are discussed. By considering the limit where the final charmed quarks are slow, it is argued that the coefficients of the $\alpha_s^2$ terms, corresponding to next-to-next-to-leading order in the standard renormalization group expansion in $\ln(m_W/m_b)$, are naturally large. The large coefficients arise from the final-state gluon exchange between quarks and are associated with the region of rather low momenta, which may further enhance the phenomenological significance of these terms.
1 Introduction

The problem of the nonleptonic decays of the $B$ mesons is well known and is well documented in the literature (see e.g. in [1, 2]): the experimentally measured semileptonic branching ratio $B(B \to l \nu X)$ is too low, and leaves unaccounted in the existing theoretical calculations an enhancement of about 20% or so of the nonleptonic decay rate of the $B$ mesons relative to their semileptonic decays. One effect: a relative enhancement by possibly as much as 30% of the sub-dominant decay $b \to c\bar{c}s$ due to the mass of the charm quark in the $O(\alpha_s)$ QCD correction (which effect is contained in the results of Ref.[3], and which has been rediscovered more recently[4, 5, 6]), is not entirely sufficient for the overall enhancement of the total nonleptonic rate and also does not look to be supported by the data. Indeed a solution of the problem of the semileptonic branching ratio of the $B$ mesons by an enhancement of the decay $b \to c\bar{c}s$ would require an average yield of charmed quarks per $B$ decay of about 1.3, whereas the latest experimental result from CLEO for this number is $n_c = 1.15 \pm 0.044$, which does not support a relative enhancement of this decay mode with respect to the dominant one $b \to c\bar{c}ud$. Thus the experimental situation suggests a further theoretical study of the nonleptonic $b$ decays beyond the thus far considered terms.

A theoretical study of the ratio of nonleptonic to semileptonic decay rates of the $B$ mesons involves two ingredients: perturbative QCD corrections to decays of a $b$ quark and the nonperturbative effects related to the fact that the $b$ quark decays being confined in a hadron. The latter effects, when considered within the expansion in the inverse mass of the $b$ quark, are estimated[1] to be quite small: of the order of few percent. Thus the required by the data enhancement of the nonleptonic decay is presumed to be associated with the perturbative QCD effects in the decay of the $b$ quark. The standard way of analyzing the perturbative radiative corrections in the nonleptonic decays is through the renormalization group (RG) summation of the leading log terms and the first next-to-leading terms in the parameter $L \equiv \ln(m_W/m_b)$. For the semileptonic decays the logarithmic dependence on $m_W/m_b$ is absent in all orders due to the weak current conservation at momenta larger than $m_b$, thus the correction is calculated by the standard perturbative technique, and a complete expression in the first order in $\alpha_s$ is available both for the total rate[3, 9] and for the lepton spectrum[10]. In reality however the parameter $L \approx 2.8$ is not large, and non-logarithmic terms may well compete with the logarithmic ones. This behavior is already seen from the known expression for the logarithmic terms: when expanded up to the order $\alpha_s^2$ the result
of Ref. [4] for the rate of decays with single final charmed quark takes the form

\[
\frac{\Gamma(b \to c\bar{u}d) + \Gamma(b \to c\bar{u}s)}{3 \Gamma(b \to c\bar{e}\nu)} = 1 + \frac{\alpha_s}{\pi} + \frac{\alpha_s^2}{\pi^2} \left[ 4L^2 + \left( \frac{7}{6} + \frac{2}{3}c(m_c^2/m_b^2) \right) L \right],
\]

(1)

where, in terms of notation of Ref. [4], \( c(a) = c_{22}(a) - c_{12}(a) \). The behavior of the function \( c(a) \) is known explicitly \([4]\) and is quite weak: \( c(0) = 19/2 \), \( c(1) = 6 \), and \( c(m_c^2/m_b^2) \approx 9.0 \) for the realistic mass ratio \( m_c/m_b \approx 0.3 \). One can see that the term with the single logarithm \( L \) contributes about two thirds of that with \( L^2 \) in the term quadratic in \( \alpha_s \). Under such circumstances the RG summation of the terms with powers of \( L \) does not look satisfactory for numerical estimates of the QCD effects, at least at the so far considered level of the first next-to-leading order terms.

An alternative way of evaluating the QCD corrections would be to use the straight expansion in the QCD coupling at least to the order \( \alpha_s^2 \), which amounts to a calculation of the non-logarithmic term in the ratio ([1]) in the order \( \alpha_s^2 \). In terms of the RG summation this corresponds to calculating the next-to-next-to-leading terms, which can be used for constructing the corresponding full RG expression in that high order. Unfortunately an explicit calculation of the \( \alpha_s^2 \) terms poses a quite complicated technical problem, which however, perhaps, can eventually be solved. The purpose of this paper is to present arguments that the non-logarithmic terms in the order \( \alpha_s^2 \) may naturally be large and in fact may compete by their numerical significance with the known logarithmic terms. In other words the natural parameter for these terms is \( \alpha_s^2 \) in the case of \( b \to c\bar{u}(s) \) decays and even \( \pi^2\alpha_s^2 \) in the case of the decay \( b \to c\bar{c}s \) rather than the \( (\alpha_s/\pi)^2 \) that is usual for the radiative effects. To see this the limit of small velocity of the \( c \) quark in the \( b \to c\bar{u}(s) \) decays and of small velocity of both charmed quarks in the decay \( b \to c\bar{c}s \) is considered (slow charm limit). The realistic kinematics of the \( b \) quark decay is known \([11]\) to be not too far from this limit, however it is far enough to prevent from obtaining a reliable quantitative estimates in the discussed problem of the real \( b \) quark decays\(^1\). The reason for the large coefficients, arising in the limit of slow charm is the Coulomb-like gluonic interaction of the quarks in the final state. The presence of this interaction makes the nonleptonic \( b \) quark decays fundamentally different from the semileptonic ones at least in the slow charm limit. The QCD corrections to the semileptonic decay are finite in this limit and are determined by exchange of virtual gluons.

\(^1\)It can be also noticed that for the decay \( b \to c\bar{c}s \) the inverse of the velocity \( v \) of either of the charmed quarks in the center of mass of the \( c\bar{c} \) system, averaged over the phase space in the decay, is \( <v^{-1}> \approx 2 \). Thus \( v^{-1} \) is not a much worse parameter than \( L \). However, as will be seen in the real kinematics, an expansion in \( v \) of the \( \alpha_s^2 \) terms lacks convergence similarly to the expansion in \( L \).
with the scale of momenta between $m_c$ and $m_b$ \[11\]. On the contrary, the QCD corrections to the nonleptonic decay rate are singular in the limit of zero velocity of the final charmed quark already in the coefficients of the $\alpha_s^2$ terms.

The set of essential graphs, which need to be calculated for a complete $O(\alpha_s^2)$ QCD analysis of the corrections to nonleptonic decays of $b$, is somewhat simplified\[5\] for the ratio $\Gamma_{nl}/\Gamma_{sl}$, as will be discussed in Sect.2. This simplification allows to ignore the non-Abelian nature of the QCD interaction and to relate the problem to the known solution of a Coulomb problem for non-relativistic as well as for relativistic fermions. In Sect.3 the relativistic case is discussed, relevant for the decay $b \rightarrow c\bar{u}d(s)$ and in Sect.4 the decay $b \rightarrow c\bar{c}s$ is considered, where both the relativistic and non-relativistic Coulomb dynamics comes into play. In Sect.5 the results of this paper are summarized and discussed.

2 Feynman graphs for $\Gamma_{nl}/\Gamma_{sl}$

The structure of the graphs, which need to be calculated in order to find the QCD corrections to $\Gamma_{nl}/\Gamma_{sl}$ up to $O(\alpha_s^2)$, becomes transparent if one represents the decay rate as the imaginary part of the $b$ quark self-energy. Then the zeroth order nonleptonic decay rate is given by the graph of Fig.1, while the $O(\alpha_s)$ and the $O(\alpha_s^2)$ corrections are represented by the classes of graphs shown respectively in Figures 2 and 3\[2\]. It is convenient for the purpose of discussion to call the quark line $b \rightarrow c \rightarrow b$ in those graphs as the heavy quark line, and the loop, made of $\bar{u}d(s)$ or $\bar{c}s$ quarks as the light quark loop. The graphs, where the gluons are attached only to the heavy quark line (like those shown in Fig.2a and Fig.3a), are identical to the same graphs for the semileptonic decays and thus they cancel in the ratio $\Gamma_{nl}/\Gamma_{sl}$. Therefore the corrections to this ratio should contain either gluon exchange within the light quark loop, or a gluon exchange between the loop and the heavy quark line, or both. Notice however that a color trace is taken over the light quark loop, thus the graphs where an overall color goes into the loop do not contribute to the decay rate (examples of such graphs are in Fig.2c and also in Fig.4). Therefore in the first order in $\alpha_s$ only the graph of Fig.2b is left with the gluon exchange within the loop. For light quarks this exchange is very well known to result in the factor $(1 + \alpha_s/\pi)$, which enters eq.(1), while for the $\bar{c}s$ loop the charm quark mass enhances the correction, so that for realistic quark masses the correction factor is $1 + \delta \alpha_s/\pi$ with $\delta \approx 4.5$.\[2\]

\[2\]We neglect here the penguin type contributions, originating from the process $b \rightarrow c\bar{c}s$, since these are known and small\[8\].
In the order $\alpha_s^2$ the graphs for the corrections to the ratio $\Gamma_{nl}/\Gamma_{sl}$ fall into two categories: QCD corrections in the order $\alpha_s^2$ entirely contained within the loop and a two-gluon exchange between the heavy quark line and the loop. Due to the color trace over the loop there is no interference in this order of the gluon exchange between the heavy quark line and the loop and either the corrections inside the loop or on the heavy quark line. Also the gluon self-energy, or non-Abelian splitting of the gluon into two for the gluon exchanged between the heavy line and the loop does not contribute due to the color trace. Thus the only exchange graphs that are relevant are those with two gluons, where both gluons originate on the heavy quark line and end on the quarks within the loop. An example of such graph is shown in Fig.3c.

The second order QCD corrections contained within the loop are well known for the massless quarks\cite{12,13}. They replace the first-order factor $(1 + \alpha_s/\pi)$ by

$$1 + \frac{\alpha_s^{\text{MS}}(q)}{\pi} + \left[ \left( \frac{41}{8} - \frac{11}{3} \zeta(3) \right) C_A - \frac{1}{8} C_F + \left( -\frac{11}{12} + \frac{2}{3} \zeta(3) \right) n_f \right] \left( \frac{\alpha_s^{\text{MS}}(q)}{\pi} \right)^2 \approx$$

$$1 + \frac{\alpha_s^{\text{MS}}(q)}{\pi} + 1.64 \left( \frac{\alpha_s^{\text{MS}}(q)}{\pi} \right)^2$$

with the number of light quark flavors $n_f = 3$ and $q^2$ being the square of the invariant mass of the quarks in the loop. Thus in this case the effect of the second order basically reduces to specifying the normalization scale for the coupling $\alpha_s$ in the first-order term and does not contain any spectacular features beyond that. For the $\bar{c}s$ loop, where the mass of the charmed quark cannot be neglected, the corresponding result for the QCD correction in order $\alpha_s^2$ is not known.

Proceeding to the discussion of the graphs with two-gluon exchange between the heavy quark line and the light quark loop, we first notice that such graphs have no non-Abelian structure and their contribution is exactly the same as it would be in an Abelian theory. The only feature of the color structure of QCD that is relevant is the color factor originating from the color traces. One can readily see that to correctly reproduce the effect of the two-gluon exchange between the heavy line and the loop relative to the zeroth order contribution of the diagram of Fig.1, one can replace the QCD interaction by an Abelian theory with the coupling $\pi$ related to $\alpha_s$ as $\pi^2 = 2\alpha_s/9$. (This takes into account the color factor of 3 present in the graph of Fig.1).

A part of the contribution of these graphs is contained in the $\alpha_s^2$ term in eq.(1), that is the part, which is due to the exchange of virtual gluons with momenta between $m_b$ and $m_W$.\[4\]
Here we will be concerned with the opposite part of the spectrum, i.e. with gluons with momenta much less than $m_c$. In general, the imaginary part of graphs of the type shown in Fig.3c is contributed by exchange of virtual gluons and by radiation and absorption of real transversal gluons. However, for a slow charmed quark its radiation and absorption arises only in the order $v^2$ of the expansion in its velocity. Thus, the leading terms are determined only by the exchange of virtual gluons. Furthermore, it is only the spin-independent (for heavy quark) exchange of Coulomb gluons that contributes in the leading order for a slow quark. Therefore the problem reduces to the one-particle problem of motion of a quark in an Abelian Coulomb-like field of a heavy quark. For the non-relativistic motion of the two charmed quarks in the decay $b \rightarrow c\bar{c}s$ this is the standard Quantum Mechanical problem, while for the light quarks both in this decay and in $b \rightarrow c\bar{d}(s)$ the problem reduces to the known solution of the Dirac equation in a Coulomb field (see e.g. in the textbook [14]). In the former case the singularity in the velocity of the charmed quarks has the well known form $\pi^2 \alpha_s^2/v^2$, while in the latter case a logarithmic singularity arises due to the corresponding singular behavior of the relativistic Coulomb wave function at the origin in the order $\alpha_s^2$.

3 The decay $b \rightarrow c\bar{d}(s)$.

We first consider the decay $b \rightarrow c\bar{d}$, where both quarks recoiling against the charmed quark are genuinely light (the decay $b \rightarrow c\bar{u}s$ is obviously similar in as much as the masses of both the $d$ and $s$ quarks can be neglected). As described above, this decay is considered in the limit of slow charm, i.e. in the limit $\Delta \equiv m_b - m_c \ll m_b, m_c$. In this limit the final charmed quark is considered to be at rest, and, as it will be shown, the $\alpha_s^2$ term in the correction to the decay rate develops a coefficient proportional to $\ln(m_c/\Delta)$. Certainly, the decay rate itself goes to zero in the limit $\Delta \rightarrow 0$ due to the overall $\Delta^5$ factor, and it is only the ratio of the $\alpha_s^2$ correction term to the zeroth order decay rate that displays a logarithmic singularity in this limit. Also as explained above, in order to find the soft gluon contribution in the two-gluon exchange graphs the QCD interaction can be replaced by an Abelian one with the coupling $\alpha_s^2 = 2\alpha_s^2/9$. The interaction of the final light quarks with the charmed quark, acting as a Coulomb center, is taken into account by replacing the plane wave functions for the light quarks by their wave functions in the Coulomb field. Thus we briefly discuss an adaptation of the known textbook results [14] to the case considered here. Before proceeding

\[3\] Notice, that the graphs where a gluon is emitted and absorbed by a light quark are not of the type of Fig.3c. In the graphs of this type at least one emission or absorption has to be by a heavy quark.
to this discussion, let us note that in the calculation of the Coulomb wave functions of
the light quarks the charmed quark can be considered as a static center located at \( r = 0 \),
only for the wave functions at the distances \( r \gg r_0 \approx m_c^{-1} \), since at shorter distances
the momenta of the exchanged gluons are not small in comparison with the mass of the charmed
quark, and its recoil cannot be neglected. Thus we will use the light quark wave functions
with the logarithmic accuracy down to these distances in the weak interaction Hamiltonian
proportional to \( (u^\dagger(r_0) (1 - \gamma_5) d(r_0)) \).

The Dirac-Weyl equation for a massless two-component left-handed \( d \) quark spinor in the
repulsive Coulomb field \( U(r) = \frac{\alpha}{r} \) reads as
\[
(\varepsilon_d - U(r) - i \sigma \cdot \nabla) d(r) = 0 ,
\]
where \( \varepsilon_d \) is the energy of the \( d \) quark. A non-vanishing near the origin solution to this
equation arises only in the lowest partial wave, where the ansatz for the spinor structure of
the solution has the form
\[
d(r) = (f(r) + i g(r) \sigma \cdot n) w
\]
with \( w \) being a constant two-component spinor, and \( n = \frac{r}{r} \). Upon substitution of this
ansatz in eq.(3) the resulting equations for the functions \( f(r) \) and \( g(r) \) exactly coincide with
the textbook ones \([14]\) in the massless case \( m = 0 \) and the angular moment parameter \( \kappa = -1 \).
Thus the solution can be readily read off the textbook:
\[
\begin{align*}
& f \\
& g \end{align*} = 2^{\frac{2}{3}} e^{-\frac{\varepsilon_d}{2}} \frac{\Gamma(\gamma + 1 + i \alpha)}{\Gamma(2\gamma + 1)} \frac{2 p r}{r} \left\{ \right. \\
& \left. \text{Im} \left\{ e^{i(pr+\xi)} \frac{\Gamma(\gamma + 1 + i \alpha)}{\Gamma(2\gamma + 1)} \right\} \text{Re} \left\{ e^{i(pr+\xi)} \frac{\Gamma(\gamma + 1 + i \alpha)}{\Gamma(2\gamma + 1)} \right\} \right\} ,
\]
where \( \gamma = \sqrt{1 - \alpha^2} \), \( p = \varepsilon_d \), and the phase angle \( \xi \) is defined as \( \exp(-2i \xi) = -\gamma - i\alpha \). The
wave function of the right handed \( \bar{u} \) antiquark is obtained from this solution by a formal
replacement \( \overline{\alpha} \rightarrow -\overline{\alpha} \), \( g \rightarrow -g \).

In calculating the matrix element of the weak interaction Hamiltonian we are interested
in these functions at a distance \( r_0 \) such that \( pr_0 \ll 1 \), but still \( pr_0 \gg \overline{\alpha} \), since we are seeking
an expansion in powers of \( \overline{\alpha} \) and if the latter condition were not satisfied such an expansion
would be impossible. Due to the former condition the degenerate hypergeometric function
can be replaced by 1. Multiplying the solutions to the wave equations in the expression
\( (u^\dagger(r_0) d(r_0)) \), one finds the ratio of the Coulomb-corrected matrix element to the bare one
in the following form
\[
\frac{(u^\dagger(r_0) d(r_0))_{\text{Coulomb}}}{(u^\dagger(0) d(0))_{\text{free}}} = \left| \frac{2 \Gamma(\gamma + 1 + i \alpha)}{\Gamma(2\gamma + 1)} \right|^2 (4 \varepsilon_u \varepsilon_d r_0^2) \gamma^{-1} .
\]
One can readily see that the parameter for the expansion of this expression is $\alpha^2$ rather than $(\alpha/\pi)^2$. Under the assumptions made we are however bound to retain only the logarithmic part of this expression and only in the order $\alpha^2$. With the logarithmic accuracy one can set $r_0 \sim m_c^{-1}, \varepsilon_u \sim \varepsilon_d \sim \Delta$ thus finding for the correction to the rate of the decay, given by the square of the ratio in eq.(3), in the form of the factor

$$F_{ud} = \left(1 + \frac{4}{9} \alpha_s^2 \ln \frac{m_c}{\Delta}\right),$$

where the relation $\alpha^2 = 2\alpha_s^2/9$ is used. This formula is the final result for the logarithmic singularity of the $\alpha^2_s$ terms in the decay $b \to \bar{u}d(s)$ in the slow charm limit of $\Delta \to 0$.

Clearly, for the realistic $b$ quark decay the equation (4) can serve only as an indicator of possible presence of large terms of order $\alpha^2_s$, rather than as a quantitative estimate of these terms, since in reality the charm in $b$ decay is not sufficiently slow for considering $\ln(m_c/\Delta)$ as a large parameter. It is interesting to note however, that the equation (3) can be used to find the exact expression for the $\alpha^2_s$ correction to a heavy quark decay in an entirely artificial kinematical situation, where both the $b$ and $c$ quarks are heavier than the $W$ boson: $\Delta \ll m_W \ll m_b, m_c$. In this limit the $W$ boson propagator provides a cutoff at short distances $r_0 \sim m_W^{-1}$, at which the recoil of the heavy quark can still be ignored, thus the one-particle wave functions of the form in eq.(5) can be used for a complete calculation of the matrix element of the weak Hamiltonian. This matrix element is proportional to

$$J = \int (u^+(r) d(r)) \exp\left(-\frac{m_W r}{r}\right) d^3 r.$$  

Calculating this integral with the solutions described by eq.(5), one readily finds for the square of the ratio of the Coulomb-corrected integral to the free one the following expression

$$\left|\frac{J(\varepsilon_u, \varepsilon_d)_{\text{Coulomb}}}{J(\varepsilon_u, \varepsilon_d)_{\text{free}}}\right|^2 = \Gamma(2\gamma)^2 \left|\frac{2 \Gamma(\gamma + 1 + i\alpha)}{\Gamma(2\gamma + 1)}\right|^4 \left(\frac{4 \varepsilon_u \varepsilon_d}{m_W^2}\right)^{2\gamma - 2}\left[1 + O\left(\frac{\Delta^2}{m_W^2}\right)\right].$$

Since the phase space integral with the free wave functions goes in the considered kinematical arrangement as $\varepsilon_u^2 \varepsilon_d^2 d\varepsilon_d$ with $\varepsilon_u = \Delta - \varepsilon_d$, one finds the Coulomb correction factor for the decay rate as

$$\mathcal{F} = \Gamma(2\gamma)^2 \frac{30 \Gamma(2\gamma + 1)^2}{\Gamma(4\gamma + 2)} \left|\frac{2 \Gamma(\gamma + 1 + i\alpha)}{\Gamma(2\gamma + 1)}\right|^4 \left(\frac{4\Delta^2}{m_W^2}\right)^{2\gamma - 2} \left[1 + O\left(\frac{\Delta^2}{m_W^2}\right)\right].$$

Actually the condition $\Delta \ll m_W$ can be dropped. It is assumed here in order to simplify the resulting expression, which anyway serves only for the purpose of illustration.
(the identity $\gamma^2 + \bar{\alpha}^2 = 1$ is used in the last transition). When expanded to the order $\pi^2$, this factor takes the form

$$F = 1 + \left( \frac{167}{30} - \frac{\pi^2}{3} + 2 \ln \frac{m_W}{2 \Delta} \right) \frac{2}{9} \alpha_s^2.$$  \hspace{1cm} (11)

This expression describes both the logarithmic term in the $\alpha_s^2$ correction and the non-logarithmic one, which is justified to be retained in the assumed artificial limit $\Delta \ll m_W \ll m_b, m_c$. When combined with the result in eq.(2) for the QCD corrections contained within the light quark loop, this would completely describe the QCD corrections up to the order $\alpha_s^2$ in this limit. It can be also mentioned that in this limit there of course are no terms with $\ln(m_W/m_b)$.

4 The decay $b \to c\bar{c}s$

In the decay $b \to c\bar{c}s$ in the slow charm limit one has two slow quarks in the final state. An exchange of gluons between them results in the well-known non-relativistic Coulomb corrections, which have as their parameter $\pi\alpha_s/v$, where $v$ is the velocity of either of the charmed quarks in their center of mass frame: $v = \sqrt{1 - 4m_c^2/q^2}$ with $q^2$ being the invariant mass squared of the $c\bar{c}$ pair. Thus in the limit of small $v$ the dominant $\alpha_s^2$ term is given by the double Coulomb exchange between the $c$ and $\bar{c}$, and is described by the second term of the expansion of the Coulomb factor

$$\frac{\pi\alpha_s/v}{1 - \exp(-\pi\alpha_s/v)} = 1 + \frac{\pi\alpha}{v} + \frac{\pi^2\alpha^2}{12v^2} + \ldots$$  \hspace{1cm} (12)

The linear term in $\pi$ should be discarded, as explained in Sect.2, while the second term gives the correction factor

$$1 + \pi^2\alpha_s^2/(54v^2)$$  \hspace{1cm} (13)

for the rate (the relation $\alpha_s^2 = 2\alpha_s^2/9$ is again used here, which takes into account the color factors in the relative correction).

However, one in fact can estimate at least one more term in the expansion of the $\alpha_s^2$ correction in powers of $v$, and thus get some feeling of what $v$ is sufficiently small for applicability of the expansion. This is especially relevant, given rather small numerical coefficient in the leading correction term in eq.(13). The next term, linear in $1/v$ arises through the interference of the one Coulomb gluon exchange between $c$ and $\bar{c}$, described by the term $O(\pi)$ in eq.(12) with the less singular than $1/v$ part of the gluon exchange of either the $\bar{c}$
or $s$ quarks with the heavy quark line. For the $\bar{c}$ this corresponds to a hard gluon exchange with either the $c$ or the $b$ quark, while for the $s$ quark this corresponds to contribution of gluons with generally arbitrary momenta. The contribution of the soft gluons for the $s$ quark exchange with the heavy quark line is however not singular at all in $v$: the logarithmic singularity develops only starting from the order $\alpha'^2$, as described by the wave function in eq.(5). Therefore the linear in $1/v$ term has no additional logarithmic dependence on $v$.

The non-logarithmic coefficient can be estimated in the same approximation as in eq.(1) i.e. using $L$ as a parameter. In this approximation the exchange of the non-Coulomb gluon is dominated by the region of momenta between $m_b$ and $m_W$, and one finds that the interference of the hard gluon exchange with a Coulomb exchange within the $c\bar{c}$ pair adds a negative contribution to the correction factor (13):

$$-\frac{3}{2} \frac{\alpha^2}{v} s L = -\frac{2}{3} \frac{\alpha^2}{v} L.$$ 

Thus the final result for the $\alpha_s^2$ correction factor to the $b \to c\bar{c}s$ decay rate, unaccounted for by eq.(1), in the slow charm limit is estimated as

$$F_{\bar{c}s} = 1 + \left( \frac{\pi^2}{54 v^2} - \frac{2}{3v} (L + O(1)) \right) \alpha_s^2 . \quad (14)$$

The formula in eq.(14) describes the leading $\alpha_s^2$ terms in the part of the spectrum, where the $c\bar{c}$ pair is close to its threshold. In the total decay rate of the real $b$ quark the dominance of the singular in $v$ terms is however quite smeared. Indeed, for the average over the spectrum of the $b \to c\bar{c}s$ decay values of $v^{-2}$ and $v^{-1}$ one finds at $m_c/m_b \approx 0.3$:

$$\langle v^{-2} \rangle \approx 5.14 , \quad \langle v^{-1} \rangle \approx 1.97 . \quad (15)$$

Thus the linear in $1/v$ term in eq.(14) in fact dominates over the quadratic one and makes the overall coefficient of $\alpha_s^2$ negative and large: $F_{\bar{c}s} \approx 1 - 2.8 \alpha_s^2$. Certainly, under these circumstances one can not rely on the expansion in $v$ for the total rate and may only consider eq.(14) as an indication of presence of large contributions in the order $\alpha_s^2$.

If the coefficients of the $\alpha_s^2$ terms may be as big as indicated by the present analysis, these terms may well compete with the first-order term in $\alpha_s$. Thus the $\alpha_s^2$ effects may essentially modify the conclusion based on the first order in $\alpha_s$ calculation about the relative enhancement of the $b \to c\bar{c}s$ decay.

It should be also noted in connection with the $b \to c\bar{c}s$ decay that the gluon exchange contained within the $c\bar{s}$ loop receives a logarithmic contribution near the threshold in the order $\alpha_s^2$ from the Coulomb exchange. This contribution also can be readily found from eq.(3). However in this case one has a competing logarithmic effect due to the so-called hybrid anomalous dimension of the current $(\bar{s}_L \gamma_\mu c)$ both in the leading order and in
the next-to-leading order. In any event, however, these terms are less singular in the slow charm limit than those in eq.(14).

## 5 Summary and discussion

Lacking a complete calculation of the second-order QCD corrections to the $b$ quark decays, we have to rely on parametric approximations for these corrections. Unfortunately, in the real situation there is no such parameter, which would be acceptably good. The widely used expansion of these corrections in $L = \ln(m_W/m_b) \approx 2.8$ by means of RG does not seem to work well, as illustrated by eq.(1). In this paper a different parameter is considered: the velocity of the charmed quarks in the final state of the decay. The terms which are parametrically large in the limit of large $1/v$ are in general independent of those, which dominate the expansion in $L$, thus they indicate the magnitude of the contributions lost in the so-far considered order of the expansion in $L$. Also in the decay $b \to c\bar{c}s$ the inverse velocity of the charmed quarks is almost as good (or as bad) a parameter as $L$. As is shown the leading in the slow charm limit the $\alpha_s^2$ corrections are associated with the interaction of the final state quarks with the Coulomb-like gluonic field of the slow quark. Therefore the parameter for these corrections turns out to be $\alpha_s^2$ in the case of the decay $b \to c\bar{u}d(s)$ with relativistic light quarks (eq.(1)) or $(\pi\alpha_s)^2$ for the case of the interaction between the non-relativistic $c$ and $\bar{c}$ in the decay $b \to c\bar{c}s$ (eq.(14)). This can be considered as a strong indication that the missing in eq.(1) terms of order $\alpha_s^2$ can have large coefficients, which may be considerably different in the $b \to c\bar{u}d(s)$ decay from those in the $b \to c\bar{c}s$ decay. In this case the missing contributions may well compete in their numerical significance with the terms present in eq.(1) and may considerably alter the theoretical predictions both for the semileptonic branching ratio and for the average charm yield in $B$ decays. Thus the main conclusion from the arguments presented in this paper is that it is rather premature to seek a contradiction of the theoretical predictions with the data on $B_{sl}$ and $n_c$ before a complete theoretical analysis of the $\alpha_s^2$ corrections becomes available.

This work is supported, in part, by the DOE grant DE-AC02-83ER40105.

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5 The known enhancement of this decay in the first order in $\alpha_s$ is partly due to the logarithmic rise of the $\alpha_s$ term towards the $\bar{c}s$ threshold.
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Figure 1: The lowest order graph, whose unitary cut describes the rate of the decay $b \rightarrow c\bar{q}_2q_1$. The small filled circles represent the W boson propagators.

Figure 2: Three types of graphs, whose unitary cuts describe the first QCD radiative corrections to the inclusive decay rate $b \rightarrow c\bar{q}_2q_1$. The dashed lines correspond to gluons. The gluon vertices can be anywhere on the $bc$ line (a), quark lines in the loop (b), or one vertex anywhere on the $bc$ line and the other vertex on either line in the loop (c). The graphs of the type c in fact do not contribute to the decay rate, as explained in the text.
Figure 3: Three types of graphs, whose unitary cuts contribute to $O(\alpha_s^2)$ QCD radiative corrections to the inclusive decay rate $b \to c\bar{q}_2q_1$.

Figure 4: One of the types of graphs of order $\alpha_s^2$, which do not contribute to the decay rate of $b \to c\bar{q}_2q_1$, because an overall color flows into the colorless quark loop.