Characterization of the Peak Value Behavior of the Hilbert Transform of Bounded Bandlimited Signals

H. Boche and U. J. Mönich

Lehrstuhl für Theoretische Informationstechnik, Technische Universität München, Germany
boche@tum.de

Research Laboratory of Electronics, Massachusetts Institute of Technology, USA
moenich@mit.edu

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Abstract—The peak value of a signal is a characteristic that has to be controlled in many applications. In this paper we analyze the peak value of the Hilbert transform for the space $B^\infty_{\pi}$ of bounded bandlimited signals. It is known that for this space the Hilbert transform cannot be calculated by the common principal value integral, because there are signals for which it diverges everywhere. Although the classical definition fails for $B^\infty_{\pi}$, there is a more general definition of the Hilbert transform, which is based on the abstract $H^1$-BMO($\mathbb{R}$) duality. It was recently shown in [1] that, in addition to this abstract definition, there exists an explicit formula for calculating the Hilbert transform. Based on this formula we study properties of the Hilbert transform for the space $B^\infty_{\pi}$ of bounded bandlimited signals. We analyze its asymptotic growth behavior, and thereby solve the peak value problem of the Hilbert transform for this space. Further, we obtain results for the growth behavior of the Hilbert transform for the subspace $B^\infty_{\pi,0}$ of bounded bandlimited signals that vanish at infinity. By studying the properties of the Hilbert transform, we continue the work [2].

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1. INTRODUCTION

The peak value is a basic characteristic of signals. In many applications it is crucial to control the peak value. For example, in wireless communication systems high peak-to-average power ratios (PAPRs) are problematic because high peak values can overload power amplifiers, which in turn leads to undesirable out-of-band radiation [3–5]. In this paper we analyze the asymptotic growth behavior of the Hilbert transform for the space of bounded bandlimited signals, and thereby solve the peak value problem of the Hilbert transform for this space.

The Hilbert transform is an important operation in numerous fields, in particular in communication theory and signal processing. For example, the “analytic signal” [6], which was used by Dennis Gabor in his “Theory of Communication” [7], is based on the Hilbert transform. Further concepts and theories in which the Hilbert transform is an integral part are the instantaneous amplitude, phase, and frequency of a signal [6,8–13] and the theory of modulation [6,14–16].

In an analytic signal $\psi(t) = u(t) + iv(t)$, the imaginary part $v$ is the Hilbert transform of the real part $u$, i.e., $v = Hu$. Based on the analytic signal it is possible to define the instantaneous
amplitude and frequency of a signal [8,9]. The instantaneous amplitude $A_u(t)$ of a signal $u$ is then defined by

$$A_u(t) := \sqrt{u^2(t) + v^2(t)},$$

the instantaneous phase $\varphi_u(t)$ by

$$\varphi_u(t) = \arctan \left( \frac{v(t)}{u(t)} \right),$$

and the instantaneous frequency, which is the derivative of the instantaneous phase, by

$$\varphi'_u(t) = \frac{u'(t)v(t) - v'(t)u(t)}{u^2(t) + v^2(t)}.$$

Although there are other possibilities to define the instantaneous amplitude and frequency [9,17], it was shown in [9] that the only definition that satisfies certain physical requirements is the definition based on the Hilbert transform and analytic signal. In [12,13] interesting approaches are developed to find generalizations of the amplitude-phase representation to nonsmooth functions. In these papers, application of the Hilbert transform and the use of techniques from the theory of analytic functions are central. Our approach in this paper is different, because we consider smooth signals, more precisely, the practically important class of bandlimited signals.

A further interesting application of the Hilbert transform is presented in [18], where the classical Hardy spaces are characterized as $L^p(\mathbb{R})$ functions with nonnegative spectrum, and an $L^p(\mathbb{R})$ extension of the Bedrosian theorem is developed.

Classically, the Hilbert transform of a smooth signal $f$ with compact support is defined as the principal value integral

$$\langle H f \rangle(t) = \frac{1}{\pi} \mathrm{V.P.} \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} \, d\tau = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|t - \tau| \leq \frac{1}{\varepsilon}} \frac{f(\tau)}{t - \tau} \, d\tau$$

$$= \frac{1}{\pi} \lim_{\varepsilon \to 0} \left( \int_{t - \varepsilon}^{t + \varepsilon} \frac{f(\tau)}{t - \tau} \, d\tau + \int_{t + \varepsilon}^{t} \frac{f(\tau)}{t - \tau} \, d\tau \right). \quad (1)$$

The above integral (1) can be used to define the Hilbert transform for more general spaces only if the integral converges for all signals from this space. There are cases where the integral converges only for almost all $t$, but where the Hilbert transform can be defined in the $L^p$-sense. However, the convergence of the integral is delicate and has to be checked from case to case. For bounded bandpass signals, the Hilbert transform exists and is bounded. If $f$ is a bandpass signal the distributional Fourier transform of which vanishes outside $[-\pi, -\varepsilon \pi] \cup [\varepsilon \pi, \pi]$, $0 < \varepsilon < 1$, then $f$ has a bounded Hilbert transform satisfying

$$\|Hf\|_\infty \leq \left( C_1 + \frac{2}{\pi} \log \left( \frac{1}{\varepsilon} \right) \right) \|f\|_\infty,$$

where $C_1 < 4/\pi$ is a constant [16,19]. That is, the upper bound on the peak value of the Hilbert transform diverges as $\varepsilon$ tends to zero. Probably, observations of this kind led to the conclusion “that an arbitrary bounded bandlimited function does not have a Hilbert transform...” [16]. Such a nonexistence of the Hilbert transform for certain bounded bandlimited signals would have far-reaching consequences.
In this paper we use a new representation of the Hilbert transform for bounded bandlimited signals, which was recently found in [1]. With this representation, we can explicitly calculate the Hilbert transform of such signals using a mixed signal system. Based on this new mixed signal representation, we are able to characterize the peak value behavior of the Hilbert transform and understand the problems in the evaluation of the standard Hilbert transform integral that probably led to the above-cited statement about the nonexistence of the Hilbert transform for arbitrary bounded bandlimited functions. Our approach is restricted to bandlimited signals. In the literature, other methods for the calculation of the Hilbert transform and treatment of related applications have been developed for nonsmooth functions. For example, approaches in [12,13,18] use Hardy spaces and techniques from complex integration.

The paper is structured as follows. In Section 2 we introduce some notation. In Sections 3 and 4 we define the Hilbert transform for general bounded bandlimited signals and present a new constructive formula for its calculation. The material in these two sections is a summary of the most important facts from [1] which are necessary for the further understanding of this paper. However, proofs are omitted, because they can be found in [1]. In Section 5 the peak value problem of the Hilbert transform is solved, and in Section 6 further results about the peak value of the Hilbert transform for the important subspace of bounded bandlimited signals that vanish at infinity are presented. In Section 7 a sufficient condition for boundedness of the Hilbert transform is derived, and in Section 8 an example of a bounded bandlimited signal that vanishes at infinity with unbounded Hilbert transform is given. Finally, in Section 9 we characterize a subset of the bounded bandlimited signals for which the common Hilbert transform integral (1) converges.

2. NOTATION

Let \( \hat{f} \) denote the Fourier transform of a function \( f \). Let \( L^p(\mathbb{R}), 1 \leq p < \infty \), denote the space of all \( p \)-th-power Lebesgue integrable functions on \( \mathbb{R} \), with the usual norm \( \| \cdot \|_p \), and let \( L^\infty(\mathbb{R}) \) denote the space of all functions for which the essential supremum norm \( \| \cdot \|_\infty \) is finite. A function that is defined and holomorphic over the whole complex plane is called an entire function. For \( 0 < \sigma < \infty \), let \( B_\sigma \) be the set of all entire functions \( f \) with the property that for all \( \varepsilon > 0 \) there exists a constant \( C(\varepsilon) \) with \( |f(z)| \leq C(\varepsilon) \exp((\sigma + \varepsilon)|z|) \) for all \( z \in \mathbb{C} \). The Bernstein space \( B_\sigma^p, 1 \leq p \leq \infty \), consists of all functions in \( B_\sigma \) whose restriction to the real line is in \( L^p(\mathbb{R}) \). The norm for \( B_\sigma^p \) is given by the \( L^p \)-norm on the real line, i.e., \( \| \cdot \|_{B_\sigma^p} = \| \cdot \|_p \). A signal in \( B_\sigma^p, 1 \leq p \leq \infty \), is called bandlimited to \( \sigma \), and \( B_\sigma^\infty \) is the space of bandlimited signals that are bounded on the real axis. We call a signal in \( B_\sigma^\infty \) a bounded bandlimited signal. By the Paley–Wiener–Schwartz theorem [20], the Fourier transform of a signal bandlimited to \( \sigma \) is supported in \([\sigma, \sigma]\). For \( 1 \leq p \leq 2 \), the Fourier transform is defined in the classical, and for \( p > 2 \), in the distributional sense.

3. OPERATOR Q

Consider the linear time-invariant (LTI) system \( Q = DH \), which consists of the concatenation of the Hilbert transform \( H \) and the differential operator \( D \), as an operator acting on \( B_\pi^2 \). Since both operators \( H : B_\pi^2 \to B_\pi^2 \) and \( D : B_\pi^2 \to B_\pi^2 \) are stable LTI systems, \( Q : B_\pi^2 \to B_\pi^2 \), as a concatenation of two stable LTI systems, is a stable LTI system. The system \( Q : B_\pi^2 \to B_\pi^2 \) has a frequency domain representation

\[
(Qf)(t) = (DHf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}_Q(\omega) \hat{f}(\omega) e^{i\omega t} d\omega,
\]

where

\[
\hat{h}_Q(\omega) = \begin{cases} 
|\omega|, & |\omega| \leq \pi \\
0, & |\omega| > \pi.
\end{cases}
\]
It is easy to show (for details, see [1]) that the system $Q: \mathcal{B}_0^2 \rightarrow \mathcal{B}_0^2$ has also a mixed signal representation

$$(Qf)(t) = \sum_{k=-\infty}^{\infty} a_{-k} f(t-k),$$

where the coefficients $a_k$, $k \in \mathbb{Z}$, are given by

$$a_k = \begin{cases} \frac{\pi}{2}, & k = 0, \\ \frac{(-1)^k - 1}{\pi k^2}, & k \neq 0. \end{cases}$$

We call this representation a mixed signal representation, because for a fixed $t \in \mathbb{R}$ we need signal values on the discrete grid $\{t-k\}_{k \in \mathbb{Z}}$ in order to calculate $(Qf)(t)$. However, for different $t \in \mathbb{R}$ we need other signal values in general. As $t$ ranges over $[0, 1]$, we need all the signal values $f(\tau)$, $\tau \in \mathbb{R}$.

The mixed signal representation (3) will be important for Section 4, where the Hilbert transform is extended to $\mathcal{B}_0^\infty$.

**Extension of $Q$ to $\mathcal{B}_0^\infty$.** So far, we have considered the LTI system $Q$ acting on signals in $\mathcal{B}_0^2$ only. Next, we extend $Q$ to a bounded operator $Q^E$ acting on the larger space $\mathcal{B}_0^\infty$ of bandlimited signals that are bounded on the real axis. For the operator $Q: \mathcal{B}_0^2 \rightarrow \mathcal{B}_0^2$ we had representations (2) and (3). However, the frequency domain representation, which involves the Fourier transform of the signal, makes no sense for signals in $\mathcal{B}_0^\infty$. The next theorem shows that the mixed signal representation (3) is still meaningful for signals in $\mathcal{B}_0^\infty$, because it is also a valid representation of the extension $Q^E$.

**Theorem 1.** The mapping

$$Q^E f = \sum_{k=-\infty}^{\infty} a_{-k} f(\cdot - k),$$

where the coefficients $a_k$ are defined as in (4), defines a bounded linear operator $Q^E: \mathcal{B}_0^\infty \rightarrow \mathcal{B}_0^\infty$ with norm $\|Q^E\| = \pi$ which coincides with $Q$ on $\mathcal{B}_0^2$, i.e., which satisfies $Q^E f = Q f$ for all $f \in \mathcal{B}_0^2$.

A proof of Theorem 1 can be found in [1].

### 4. HILBERT TRANSFORM FOR $\mathcal{B}_0^\infty$

Despite the convergence problems for the principal value integral, there is a way to define the Hilbert transform for signals in $\mathcal{B}_0^\infty$. This definition uses Fefferman’s duality theorem, which states that the dual space of $\mathcal{H}^1$ is $\text{BMO}(\mathbb{R})$ [21]. In addition to this rather abstract definition, we will also give a constructive procedure for the calculation of the Hilbert transform. We briefly review some definitions.

**Definition 1.** The space $\mathcal{H}^1$ denotes the Hardy space of all signals $f \in L^1(\mathbb{R})$ for which $Hf \in L^1(\mathbb{R})$. It is a Banach space endowed with the norm $\|f\|_{\mathcal{H}^1} := \|f\|_{L^1(\mathbb{R})} + \|Hf\|_{L^1(\mathbb{R})}$.

**Definition 2.** A function $f: \mathbb{R} \rightarrow \mathbb{C}$ is said to belong to $\text{BMO}(\mathbb{R})$ if it is locally in $L^1(\mathbb{R})$ and $\frac{1}{\mu(I)} \int_I |f(t) - m_I(f)| \, dt \leq C_2$ for all bounded intervals $I$, where $m_I(f) := \frac{1}{\mu(I)} \int_I f(t) \, dt$ and the constant $C_2$ is independent of $I$. By $\mu$ we denote the Lebesgue measure.

For our further examinations, we need an important fact that the dual space of $\mathcal{H}^1$ is $\text{BMO}(\mathbb{R})$ [22, p. 245]. In order to state this duality, we use the space $\mathcal{H}^1_D = \mathcal{H}^1 \cap \mathcal{S}$, which is dense in $\mathcal{H}^1$. By $\mathcal{S}$ we denote the usual Schwartz space of functions $\varphi: \mathbb{R} \rightarrow \mathbb{C}$ that have continuous derivatives of all orders and satisfy $\sup_{t \in \mathbb{R}} |t^a \varphi^{(b)}(t)| < \infty$ for all $a, b \in \mathbb{N} \cup \{0\}$.
Theorem 2 (Fefferman). Assume that \( f \in \text{BMO}(\mathbb{R}) \). Then the linear functional \( \mathcal{H}_D^1 \rightarrow \mathbb{C} \),
\( \varphi \mapsto \int_{-\infty}^{\infty} f(t) \varphi(t) \, dt \), has a bounded extension to \( \mathcal{H}^1 \). Conversely, every continuous linear functional \( L \) on \( \mathcal{H}^1 \) is created in this way by a function \( f \in \text{BMO}(\mathbb{R}) \), which is unique up to an additive constant.

The function \( f \in \text{BMO}(\mathbb{R}) \) in Theorem 2 is only unique up to an additive constant, because \( \varphi \in \mathcal{H}^1 \) implies \( \int_{-\infty}^{\infty} \varphi(t) \, dt = 0 \). Therefore, it will be beneficial to identify two functions in \( \text{BMO}(\mathbb{R}) \) that differ only by a constant. We do this by introducing an equivalence relation \( \sim \) on \( \text{BMO}(\mathbb{R}) \). We write \( f \sim g \) if and only if \( f(t) = g(t) + C_{\text{BMO}} \) for almost all \( t \in \mathbb{R} \), where \( C_{\text{BMO}} \) is a constant. By \( [f] \) we denote the equivalence class \( [f] = \{ g \in \text{BMO}(\mathbb{R}) : g \sim f \} \), and \( \text{BMO}(\mathbb{R})/\mathbb{C} \) is the set of all equivalence classes in \( \text{BMO}(\mathbb{R}) \).

A possible extension of the Hilbert transform, which is based on the \( \mathcal{H}^1\)-BMO(\( \mathbb{R} \)) duality, is given in the next definition \[23\].

**Definition 3.** We define the Hilbert transform \( \mathcal{H} f \) of a function \( f \in \text{L}^\infty(\mathbb{R}) \) to be the function in \( \text{BMO}(\mathbb{R})/\mathbb{C} \) that generates the linear continuous functional

\[
\langle \mathcal{H} f, \varphi \rangle = \int_{-\infty}^{\infty} f(t) (H \varphi)(t) \, dt, \quad \varphi \in \mathcal{H}^1.
\]

Note that this definition is very abstract, because it gives no information on how to calculate the Hilbert transform \( \mathcal{H} f \). However, in \[24\] it was shown that for bounded signals that are additionally bandlimited, i.e., for signals \( f \in \mathcal{B}_\infty^2 \), it is possible to explicitly calculate the Hilbert transform \( \mathcal{H} f \).

Next, we will give this formula, which is based on the \( Q^E \) operator from Section 3.

Since \( Q^E f \) is continuous, for every \( f \in \mathcal{B}_\infty^2 \), the operator \( J \) given by

\[
( Jf)(t) = \int_{0}^{t} (Q^E f)(\tau) \, d\tau, \quad t \in \mathbb{R},
\]

(6)

is well defined. Since the operator \( Q: \mathcal{B}_\pi^2 \rightarrow \mathcal{B}_\pi^2 \), as an operator on \( \mathcal{B}_\pi^2 \), was defined to be the concatenation of the Hilbert transform \( H \) and the differential operator \( D \), it is clear that, for \( g \in \mathcal{B}_\pi^2 \), the integral of \( Qg \) as in (6) gives—up to a constant—the Hilbert transform \( Hg \) of \( g \). Note that for \( g \in \mathcal{B}_\pi^2 \) we have \( Hg \in \mathcal{B}_\pi^2 \), which implies that \( Hg \) is continuously differentiable. Hence, the fundamental theorem of calculus can be applied in the next equation without problems. For \( g \in \mathcal{B}_\pi^2 \) we have

\[
( Jg)(t) = \int_{0}^{t} (Q^E g)(\tau) \, d\tau = \int_{0}^{t} (Qg)(\tau) \, d\tau = \int_{0}^{t} (DHg)(\tau) \, d\tau = (Hg)(t) - (Hg)(0),
\]

(7)

i.e., for every signal \( g \in \mathcal{B}_\pi^2 \), we have \( (Hg)(t) = (Jg)(t) + C_3(g) \), \( t \in \mathbb{R} \), where \( C_3(g) \) is a constant that depends on \( g \).

Based on this observation, one could conjecture that, for signals \( f \in \mathcal{B}_\infty^2 \), the integral \( Jf \) is somehow connected to the Hilbert transform \( \mathcal{H} f \) of \( f \). In \[1\] it was shown that such a connection exists in the sense that \( Jf \) is a representative of the equivalence class \( \mathcal{H} f \).

**Theorem 3.** Let \( f \in \mathcal{B}_\pi^2 \). Then we have \( \mathcal{H} f = [Jf] \).

Note that according to Definition 3, the Hilbert transform \( \mathcal{H} f \) of a signal \( f \in \mathcal{B}_\pi^2 \) is only defined up to an arbitrary additive constant. This is a consequence of the \( \mathcal{H}^1\)-BMO(\( \mathbb{R} \)) duality, which was employed for the definition. However, the mapping \( J \) does not have this ambiguity; it maps every input signal \( f \in \mathcal{B}_\pi^2 \) uniquely to an output signal \( Jf \in \text{BMO}(\mathbb{R}) \).
Theorem 3 is very useful, because it enables us to compute the Hilbert transform of bounded bandlimited signals in $B^\infty_\pi$ by using constructive formula (6) instead of using abstract Definition 3. This result is also a key for solving the peak value problem of the Hilbert transform.

Remark 1. Formula (6) for the calculation of the Hilbert transform is based on the mixed signal representation of the operator $Q^E$. It is an interesting question whether the Hilbert transform of $\mathrm{sign} \in B^\infty_\pi$ can be calculated by using only samples of a signal. In [25] we have shown that a Nyquist-rate-sampling-based representation of the Hilbert transform that is based on the Shannon sampling series is not possible even for the subspace $PW^1_\pi$ of $B^\infty_\pi$. We conjecture that this negative result holds even in more generality, as long as no oversampling is used. However, if oversampling is used, then a sampling-based representation of the Hilbert transform is possible for $B^\infty_\pi$.

An important fact about the Hilbert transform of bounded bandlimited signals is stated in the next theorem.

Theorem 4. Let $f \in B^\infty_\pi$. Then we have $Hf \in B_\pi$.

Theorem 4, a proof of which can be found in [1], shows that the Hilbert transform of a bounded bandlimited signal is again bandlimited.

5. PEAK VALUE PROBLEM

The peak value of signals is important for many applications, e.g., for hardware design in mobile communications [3, 4]. In the peak value problem we are interested in $\sup_{t \leq T} |f(t)|$, i.e., in the peak value of a signal $f$ on the interval $[-T, T]$. Next, we study the Hilbert transform of signals in $B^\infty_\pi$, in particular its growth behavior on the real axis, and thereby solve the peak value problem for the Hilbert transform.

For all $f \in B^\infty_\pi$, we have an upper bound

$$
|(Hf)(t)| \leq \int_0^t |(Q^E f)(\tau)| \, d\tau \leq \|Q^E f\|_\infty |t| \leq \pi \|f\|_\infty |t|, \quad t \in \mathbb{R},
$$

(8)

which shows that the asymptotic growth of the Hilbert transform $Hf$ of signals $f \in B^\infty_\pi$ is at most linear. More precisely, for all $f \in B^\infty_\pi$ there exists a signal $g \in \text{BMO}(\mathbb{R})$ such that $Hf = [g]$ and $g(t) = O(t)$.

On the other hand, using identity (6), it has been shown in [1] that for the $B^\infty_\pi$-signal

$$
f_1(t) = \frac{2}{\pi} \int_0^\pi \frac{\sin(\omega t)}{\omega} \, d\omega
$$

(9)

we have

$$
|(Hf_1)(t)| \geq \frac{2}{\pi} \left( \log(|t|) - \frac{\pi^2}{4} - 1 - \frac{1}{\pi} \right)
$$

(10)

for all $t \in \mathbb{R}$ with $|t| \geq 1$. The signals $f_1$ and $Hf_1$ are visualized in Fig. 1 and Fig. 2, respectively. Thus, there are signals $f \in B^\infty_\pi$ such that the growth of the Hilbert transform $Hf$ is logarithmic, in the sense that there exists a signal $g \in \text{BMO}(\mathbb{R})$ such that $Hf = [g]$ and $g(t) = \Omega(\log(t))$.

From this the question arises of whether the asymptotically logarithmic growth is actually the maximum possible growth, i.e., whether the upper bound (8) can be improved. The next theorem gives a positive answer.

Theorem 5. There exist two positive constants $C_4$ and $C_5$ such that for all $f \in B^\infty_\pi$ and all $t \in \mathbb{R}$ we have

$$
|(Hf)(t)| \leq C_4 \log(1 + |t|) \|f\|_\infty + C_5 \|f\|_\infty.
$$
For the proof we need the following lemma, a proof of which can be found in [1].

Lemma 1. Let \( f \in \mathcal{B}_\pi^\infty \) and, for \( 0 < \varepsilon < 1 \),

\[
f_\varepsilon(t) = f((1 - \varepsilon)t) \frac{\sin(\varepsilon \pi t)}{\varepsilon \pi t}, \quad t \in \mathbb{R}.
\]

Then we have \((\mathcal{J}f)(t) = \lim_{\varepsilon \to 0} (\mathcal{J}f_\varepsilon)(t)\) for all \( t \in \mathbb{R} \).

Now, we are in a position to prove Theorem 5.

Proof of Theorem 5. Let \( f \in \mathcal{B}_\pi^\infty \) be arbitrary but fixed. For \( 0 < \varepsilon < 1 \) consider the functions \( f_\varepsilon \) that were defined in (11). We have \( f_\varepsilon \in \mathcal{B}_\pi^2 \) and \( \|f_\varepsilon\|_\infty \leq \|f\|_\infty \) for all \( 0 < \varepsilon < 1 \), as well as \( \lim_{\varepsilon \to 0} f_\varepsilon(t) = f(t) \) for all \( t \in \mathbb{R} \), where the convergence is locally uniform. Lemma 1 is a key observation. Due to representation (6) and properties of the operator \( Q \), we can work with \( \mathcal{B}_\pi^2 \)-functions in the following. Next, we analyze

\[
(\mathcal{J}f_\varepsilon)(t) = \int_0^t (Qf_\varepsilon)(\tau) d\tau.
\]
We have to distinguish two cases: $|t| < 2$ and $|t| \geq 2$. For $|t| < 2$ we have
\[
\left| \int_0^t (Qf_\varepsilon)(\tau) \, d\tau \right| \leq \|Qf_\varepsilon\|_\infty |t| \leq 2\pi \|f\|_\infty,
\] (12)
where we used $\|Qf_\varepsilon\|_\infty = \|Q^E f_\varepsilon\|_\infty \leq \|Q^E f_\varepsilon\|_\infty \leq \pi \|f\|_\infty$ in the second inequality. Now, we come to the second case $|t| \geq 2$. We can restrict ourselves to the case $t \geq 2$, because the case $t \leq -2$ is treated analogously. Let $t \geq 2$ be arbitrary but fixed. Using (2), i.e., the frequency domain representation of $Q$, we obtain
\[
\int_0^t (Qf_\varepsilon)(\tau) \, d\tau = \int_0^t \frac{1}{2\pi} \int_{-\pi}^{\pi} |\omega| \tilde{f}_\varepsilon(\omega) e^{i\omega \tau} \, d\omega \, d\tau
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\omega| \tilde{f}_\varepsilon(\omega) \int_0^t e^{i\omega \tau} \, d\tau \, d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\omega| \hat{\tilde{f}}_\varepsilon(\omega) \frac{e^{i\omega t} - 1}{i\omega} \, d\omega.
\] (13)
The order of integration was exchanged according to Fubini’s theorem, which can be applied because
\[
\int_0^t \frac{1}{2\pi} \int_{-\pi}^{\pi} |\omega| |\hat{f}_\varepsilon(\omega)| \, d\omega \, d\tau \leq |t| \pi \|f\|_{L_2^B} < \infty.
\]
Furthermore, we have
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |\omega| \hat{f}_\varepsilon(\omega) \frac{e^{i\omega t} - 1}{i\omega} \, d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} -i \text{sgn}(\omega) \varphi(\omega) \hat{f}_\varepsilon(\omega) e^{i\omega t} \, d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} -i \text{sgn}(\omega) \varphi(\omega) \hat{f}_\varepsilon(\omega) \, d\omega,
\] (14)
where the function
\[
\varphi(\omega) = \begin{cases} 1, & |\omega| \leq \pi, \\ 2 - |\omega|/\pi, & \pi < |\omega| < 2\pi, \\ 0, & |\omega| \geq 2\pi, \end{cases}
\]
was inserted without altering the integrals, because $\varphi(\omega) = 1$ for $\omega \in [-\pi, \pi]$. Using the abbreviation $\hat{u}(\omega) = -i \text{sgn}(\omega) \varphi(\omega)$ and applying the generalized Parseval equality, we obtain from (13) and (14) that
\[
\int_0^t (Qf_\varepsilon)(\tau) \, d\tau = \int_{-\infty}^{\infty} f_\varepsilon(\tau) u(t - \tau) \, d\tau - \int_{-\infty}^{\infty} f_\varepsilon(\tau) u(-\tau) \, d\tau
\]
\[
= \int_{-\infty}^{\infty} f_\varepsilon(\tau) u(t - \tau) \, d\tau + \int_{-\infty}^{\infty} f_\varepsilon(\tau) u(\tau) \, d\tau,
\] (15)
where $u$ is given by
\[
u(\tau) = \frac{1}{\pi \tau} + \frac{\sin(\pi \tau) - \sin(2\pi \tau)}{(\pi \tau)^2}, \quad \tau \in \mathbb{R}.
\]
Dividing the integration range of the first and second integral in (15) into three parts gives

\[
\int_0^t (Qf_\varepsilon)(\tau) \, d\tau = \int_{|\tau| \leq 1} f_\varepsilon(\tau)u(t-\tau) \, d\tau + \int_{|\tau-t| \leq 1} \ldots \, d\tau + \int_{|\tau| \geq 1} \ldots \, d\tau 
\]

\[
= (A_1) + (A_2) + (A_3)
\]

For \((A_1)\) we have

\[
|(A_1)| = \left| \int_{|\tau| \leq 1} f_\varepsilon(\tau)u(t-\tau) \, d\tau \right| \leq \int_{|\tau| \leq 1} |f_\varepsilon(\tau)||u(t-\tau)| \, d\tau \leq 2\|f_\varepsilon\|_\infty \|u\|_\infty.
\]

The same calculation shows that \(|(A_2)| \leq 2\|f_\varepsilon\|_\infty \|u\|_\infty\), \(|(B_1)| \leq 2\|f_\varepsilon\|_\infty \|u\|_\infty\), and \(|(B_2)| \leq 2\|f_\varepsilon\|_\infty \|u\|_\infty\). It remains to analyze \((A_3) + (B_3)\). We have

\[
|(A_3) + (B_3)| = \left| \int_{|\tau-t| \geq 1} f_\varepsilon(\tau)u(t-\tau) + u(\tau) \, d\tau \right|
\]

\[
\leq \|f_\varepsilon\|_\infty \left( \int_{|\tau-t| \geq 1} \frac{1}{\pi(t-\tau)} + \frac{1}{\pi t} \, d\tau + \int_{|\tau-t| \geq 1} \left( \frac{2}{\pi(t-\tau)^2} + \frac{2}{\pi t^2} \right) \, d\tau \right)
\]

\[
\leq \|f_\varepsilon\|_\infty \left( \frac{1}{\pi} \int_{|\tau-t| \geq 1} \frac{|t|}{|t-\tau||\tau|} \, d\tau + \frac{8}{\pi} \right),
\]

because

\[
\int_{|\tau-t| \geq 1} \left( \frac{2}{\pi(t-\tau)^2} + \frac{2}{\pi t^2} \right) \, d\tau \leq \frac{8}{\pi}.
\]

As for the remaining integral, we have

\[
\int_{|\tau-t| \geq 1} \frac{|t|}{|t-\tau||\tau|} \, d\tau = \int_{-\infty}^{-1} \frac{t}{(\tau-t)\tau} \, d\tau + \int_{1}^{t-1} \frac{t}{(\tau-t)\tau} \, d\tau + \int_{t+1}^{\infty} \frac{t}{(\tau-t)\tau} \, d\tau
\]

\[
= \int_{1}^{t-1} \frac{t}{(\tau-t)\tau} \, d\tau + 2 \int_{t+1}^{\infty} \frac{t}{(\tau-t)\tau} \, d\tau.
\]
Since
\[ \int_{t+1}^{\infty} \frac{t}{(\tau - t)\tau} d\tau = \lim_{M \to \infty} \int_{t+1}^{M} \left( \frac{1}{\tau - t} - \frac{1}{\tau} \right) d\tau = \lim_{M \to \infty} \left( \log \left( \frac{M - t}{M} \right) + \log(t + 1) \right) = \log(t + 1) \]
and
\[ \int_{1}^{t-1} \frac{t}{(t - \tau)\tau} d\tau = \int_{1}^{t-1} \frac{1}{t - \tau} + \frac{1}{\tau} d\tau = 2\log(t - 1), \]
we obtain
\[ |(A_3) + (B_3)| \leq \|f_\varepsilon\|_\infty \frac{4}{\pi} (\log(1 + t) + 2). \]

Combining the partial results gives
\[ \left| \int_{0}^{t} (Qf_\varepsilon)(\tau) d\tau \right| \leq 8\|f_\varepsilon\|_\infty \|u\|_\infty + \|f_\varepsilon\|_\infty \frac{4}{\pi} (\log(1 + t) + 2) \]
\[ = C_4 \log(1 + t)\|f_\varepsilon\|_\infty + C_6 \|f_\varepsilon\|_\infty \]
\[ \leq C_4 \log(1 + t)\|f\|_\infty + C_6 \|f\|_\infty. \] (16)

Finally, the assertion follows from Lemma 1, (12), and (16). △

Remark 2. The growth result in Theorem 5 implies that
\[ \int_{-\infty}^{\infty} \left| (Qf_\varepsilon)(t) \right|^{2} \frac{1}{(1 + t^2)^{\alpha}} dt < \infty \]
for all \( \alpha > 1/2 \). This shows that, for all \( f \in \mathcal{B}_\infty^\infty \), the Hilbert transform \( \mathcal{H}f \) is in the Zakai class, in the sense that there exists a signal \( g \) in the Zakai class satisfying \( \mathcal{H}f = [g] \).

A direct consequence of Theorem 5 is the following corollary concerning the peak value problem of the Hilbert transform.

Corollary 1. For all \( f \in \mathcal{B}_\infty^\infty \) and all \( g \in \text{BMO}(\mathbb{R}) \) satisfying \( \mathcal{H}f = [g] \) there exists a constant \( C_7 = C_7(g) \) such that for all \( T > 2 \) we have
\[ \max_{|t| \leq T} |g(t)| \leq C_7 \log(1 + T). \]

Remark 3. The signal \( f_1 \) is also a good example where commonly used formal substitution rules fail. The Hilbert transform of \( f_1 \) cannot be obtained by replacing \( \sin \) with \( -\cos \), because the resulting integral
\[ \frac{2}{\pi} \int_{0}^{\pi} \frac{-\cos(\omega t)}{\omega} d\omega \]
diverges for all \( t \in \mathbb{R} \).

6. ASYMPTOTIC BEHAVIOR FOR \( \mathcal{B}_\infty^\infty_{\pi,0} \)

Next, we present two further results about the peak value of the Hilbert transform for signals in the space \( \mathcal{B}_\infty^\infty_{\pi,0} \), which is the subspace consisting of \( \mathcal{B}_\infty^\infty \)-signals \( f \) that vanish on the real axis at infinity, i.e., satisfy \( \lim_{|t| \to \infty} |f(t)| = 0 \).
Theorem 6. For all \( f \in \mathcal{B}_{\pi,0}^\infty \) we have

\[
\lim_{T \to \infty} \frac{1}{\log(1+T)} \max_{|t| \leq T} |(\mathcal{H}f)(t)| = 0.
\]

Proof. Let \( f \in \mathcal{B}_{\pi,0}^\infty \) and \( \varepsilon > 0 \) be arbitrary but fixed. Since \( \mathcal{B}_{\pi}^2 \) is dense in \( \mathcal{B}_{\pi,0}^\infty \), there exists a function \( g \in \mathcal{B}_{\pi}^2 \) such that \( \|f - g\|_\infty < \varepsilon \). Thus, for all \( t \in \mathbb{R} \), we have

\[
|(\mathcal{H}f)(t)| = |(\mathcal{H}f)(t) - (\mathcal{H}g)(t) + (\mathcal{H}g)(t)|
\leq |(\mathcal{H}(f - g))(t)| + |(\mathcal{H}g)(t)|
\leq C_4 \log(1 + |t|)\|f - g\|_\infty + C_5\|f - g\|_\infty + |(Hg)(t)| + |(Hg)(0)|
\leq C_4 \log(1 + |t|)\varepsilon + C_5\varepsilon + 2\|Hg\|_\infty,
\]
where we used Theorem 5 and equation (7) in the second to last line. It follows, for \( T > 0 \), that

\[
\frac{1}{\log(1+T)} \max_{|t| \leq T} |(\mathcal{H}f)(t)| \leq C_4\varepsilon + \frac{C_5\varepsilon}{\log(1+T)} + \frac{2\|Hg\|_\infty}{\log(1+T)}.
\]

Choosing \( T_0 = \exp(\max\{2\|Hg\|_\infty, 1\}/\varepsilon) - 1 \), we obtain

\[
\frac{1}{\log(1+T)} \max_{|t| \leq T} |(\mathcal{H}f)(t)| \leq (C_4 + C_5 + 1)\varepsilon
\]
for all \( T \geq T_0 \). Since \( \varepsilon > 0 \) was arbitrary, the proof is complete. \( \triangle \)

Due to Theorem 3, we immediately obtain the following corollary about the asymptotic growth behavior of the Hilbert transform \( \mathcal{H}f \) for \( f \in \mathcal{B}_{\pi,0}^\infty \).

Corollary 2. For all \( f \in \mathcal{B}_{\pi,0}^\infty \) we have

\[
\lim_{T \to \infty} \frac{1}{\log(1+T)} \max_{|t| \leq T} |(\mathcal{H}f)(t)| = 0.
\]

A further result about the asymptotic growth is the following.

Theorem 7. Let \( \varphi \) be an arbitrary positive function with \( \lim_{t \to \infty} \varphi(t) = 0 \). Then there exists a signal \( f_2 \in \mathcal{B}_{\pi,0}^\infty \) such that

\[
\limsup_{T \to \infty} \frac{1}{\varphi(T) \log(1+T)} \max_{|t| \leq T} |(\mathcal{H}f_2)(t)| = \infty.
\]

Proof. For \( t \geq 1 \) consider the family of bounded linear functionals \( U_t : \mathcal{B}_{\pi,0}^\infty \to \mathbb{C} \) defined by

\[
U_t f = \frac{(\mathcal{H}f)(t)}{\varphi(t) \log(1+t)}.
\]

Further, for \( 0 < \varepsilon < 1 \), let

\[
f_{1,\varepsilon}(t) = \frac{f_1((1-\varepsilon)t) \sin(\varepsilon\pi t)}{||f_1||_\infty \varepsilon \pi t},
\]

where \( f_1 \) is the function defined in (9). Then we have \( ||f_{1,\varepsilon}||_\infty \leq 1 \), and it follows that

\[
||U_t|| = \sup_{f \in \mathcal{B}_{\pi,0}^\infty \|f\|_\infty \leq 1} |U_t f| \geq |U_t f_{1,\varepsilon}|.
\]
Since
\[ \lim_{\varepsilon \to 0} |U_{t_f}(t)| = \frac{\lim_{\varepsilon \to 0} |(\mathcal{I} f_1, \varepsilon)(t)|}{\varphi(t) \log(1 + t)} = \frac{|(\mathcal{I} f_1)(t)|}{\|f_1\|_\infty \varphi(t) \log(1 + t)} \geq \frac{2 \left( \log(t) - \frac{\pi^2}{4} - 1 - \frac{1}{\pi} \right)}{\|f_1\|_\infty \varphi(t) \log(1 + t)}, \]

where we used Lemma 1 in the second equality and (10) in the last inequality, we obtain
\[ \|U_t\| \geq \frac{2}{\pi \|f\|_\infty \varphi(t)} \left( \frac{\log(t)}{\log(1 + t)} - \left( \frac{\pi^2}{4} + 1 + \frac{1}{\pi} \right) \frac{1}{\log(1 + t)} \right). \]

From this we see that \( \lim_{t \to \infty} \|U_t\| = \infty. \) Thus, the Banach–Steinhaus Theorem [26, p. 68] implies that there exists a signal \( f_2 \in \mathcal{B}^{\infty}_{n,0} \) such that
\[ \limsup_{t \to \infty} |U_{t_f}f_2| = \limsup_{t \to \infty} \frac{|(\mathcal{I} f)(t)|}{\varphi(t) \log(1 + t)} = \infty, \]

which completes the proof. \( \triangle \)

Again, we obtain, as a corollary, an analogous result for the asymptotic growth of the Hilbert transform.

**Corollary 3.** Let \( \varphi \) be an arbitrary positive function with \( \lim_{t \to \infty} \varphi(t) = 0. \) Then there exists a signal \( f_2 \in \mathcal{B}^{\infty}_{n,0} \) such that
\[ \limsup_{T \to \infty} \frac{1}{\varphi(T) \log(1 + T)} \max_{|t| \leq T} |(\mathcal{H} f_2)(t)| = \infty. \]  

(18)

Corollaries 2 and 3 show that, for signals in the space \( \mathcal{B}^{\infty}_{n,0} \), the peak value of the Hilbert transform grows not as fast as \( \log(1 + T) \) but not “substantially” slower.

### 7. CONDITION FOR BOUNDEDNESS OF THE HILBERT TRANSFORM

Thanks to Theorem 3, we can use simple formula (6) to compute the Hilbert transform of bounded bandlimited signal. In Section 5 we have seen that there exists a signal \( f_1 \in \mathcal{B}^{\infty}_{n} \) such that \( \mathcal{I} f_1 \) is unbounded on the real axis. Thus, the Hilbert transform of a bounded bandlimited signal is again a bandlimited but not necessarily a bounded signal.

**Remark 4.** It is well known that there exist discontinuous signals the Hilbert transforms of which have singularities [27,28]. However, those signals are not bandlimited, and divergence effects are consequences of the nonsmoothness of the signals. This is in contrast to this paper, where we treat bandlimited, smooth signals.

For practical applications it is important to know when the Hilbert transform is bounded. Theorem 8 gives a necessary and sufficient condition for boundedness of the Hilbert transform. A proof of Theorem 8 is given in Appendix 1.

**Theorem 8.** Let \( f \in \mathcal{B}^{\infty}_{n} \) be real-valued. We have \( \mathcal{I} f \in \mathcal{B}^{\infty}_{n} \) if and only if there exists a constant \( C_8 \) such that
\[ \left| \frac{1}{\pi} \int_{\varepsilon \leq |t| - |\tau| \leq \frac{1}{2}} f(\tau) \frac{d\tau}{t - \tau} \right| \leq C_8 \]

for all \( 0 < \varepsilon < 1 \) and all \( t \in \mathbb{R} \).

Thanks to Theorem 8 we have a complete characterization of the signals in \( \mathcal{B}^{\infty}_{n} \) that have a bounded Hilbert transform.
8. A SIGNAL IN $\mathcal{B}_{\pi,0}^\infty$ WITH UNBOUNDED HILBERT TRANSFORM

In Section 5 we have seen that there exists a signal $f_1 \in \mathcal{B}_{\pi}^\infty$ whose Hilbert transform is unbounded. Next, we strengthen this result by showing that there even exists a signal $f_3 \in \mathcal{B}_{\pi,0}^\infty$ with unbounded Hilbert transform.

**Theorem 9.** There exists a signal $f_3 \in \mathcal{B}_{\pi,0}^\infty$ such that $\|Hf_3\|_\infty = \infty$.

We do not prove Theorem 9 here, but only sketch the idea of the proof. Theorem 9 can be proved by showing that the Hilbert transform of the signal

$$f_3(t) = \frac{2}{\pi} \int_0^\pi \frac{1}{\log \left(\frac{2\pi}{\omega}\right)} \frac{\sin(\omega t)}{\omega} \, d\omega,$$

Remark 5. Note that condition (19) does not imply that the principal value integral of the Hilbert transform converges, i.e., that the limit in (1) exists. Convergence of the principal value integral is the content of Theorem 10.

Remark 6. Theorem 8 shows that unbounded divergence of the principal value integral for a signal $f \in \mathcal{B}_{\pi}^\infty$ implies that the Hilbert transform of $f$ (which is nevertheless well-defined) is not a signal in $\mathcal{B}_{\pi}^\infty$.

**Fig. 3.** Plot of the signal $f_3$.

**Fig. 4.** Plot of the signal $\Im f_3$. 

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which is plotted in Fig. 3, is unbounded. Using integration by parts it is easy to show that \( f_3 \) satisfies \( \lim_{|t| \to \infty} f_3(t) = 0 \), i.e., that \( f_3 \) is a signal in \( B^\infty_{\pi,0} \). The plot of \( \Im f_3 \) in Fig. 4 indicated the unboundedness of the Hilbert transform of \( f_3 \). An actual proof of Theorem 9 can be done indirectly by using Theorem 8 and showing that

\[
\lim_{\varepsilon \to 0} \int_{\varepsilon \leq |\tau| \leq \frac{1}{\varepsilon}} \frac{f_3(\tau)}{\tau} d\tau = \infty.
\]

9. CONVERGENCE OF THE HILBERT TRANSFORM INTEGRAL

Theorem 8 characterizes when \( \Im f \) is bounded. It links boundedness of \( \Im f \) to boundedness of the principal value integral (1). In Theorem 10 we characterize a subset of the bounded bandlimited signals, for which the Hilbert transform integral (1) converges, and thus give a sufficient condition for being able to calculate the Hilbert transform by integral (1).

**Theorem 10.** Let \( f \in B^\infty_{\pi,0} \) be real-valued. If \( \Im f - C_I \in B^\infty_{\pi,0} \) for some constant \( C_I \), then we have

\[
\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon \leq |t-\tau| \leq \frac{1}{\varepsilon}} \frac{f(\tau)}{t-\tau} d\tau = (\Im f)(t) - C_I
\]

and

\[
\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon \leq |t-\tau| \leq \frac{1}{\varepsilon}} \frac{(\Im f)(\tau)}{t-\tau} d\tau = -f(t)
\]

for all \( t \in \mathbb{R} \).

A proof of Theorem 10 is given in Appendix 2.

10. CONCLUSION

In this paper we solved the peak value problem of the Hilbert transform for bounded bandlimited signals. By analyzing the problem we also clarified the causes which, in the classical literature, led to the misbelief that general bounded bandlimited signals do not have a Hilbert transform. For general bounded bandlimited signals, the Hilbert transform cannot be calculated by the principal value integral (1), and formal substitution rules, like the one where, in certain expressions, “\( \sin \)” is simply replaced by “\(-\cos\)” can no longer be used. Theory necessary to define the Hilbert transform for bounded signals is build based on the abstract and nonconstructive \( H^1 \)-BMO(\( \mathbb{R} \)) duality. Because the duality approach gives no constructive procedure for calculation of the Hilbert transform, its usefulness for practical applications is limited. However, in this paper we considered bounded signals that are additionally bandlimited and thus could use a simple formula, which was recently found, for calculating the Hilbert transform, avoiding the abstract duality theory. Based on this novel formula, we were able to solve the peak value problem of the Hilbert transform and provide growth estimates.

**APPENDIX 1: PROOF OF THEOREM 8**

For the proof of Theorem 8, we need the following lemma.

**Lemma 2.** Let \( f \in B^\infty_{\pi} \) and \( \Im f \in B^\infty_{\pi} \). Then, for \( F = f + i\Im f \), we have

\[
|F(t + iy)| \leq \|F\|_{\infty}
\]

for all \( t \in \mathbb{R} \) and \( y \geq 0 \).
Proof. Let \( t \in \mathbb{R} \) and \( y > 0 \) be arbitrary but fixed. Since \( f \in \mathcal{B}_\pi^\infty \) and \( \mathcal{J}f \in \mathcal{B}_\pi^\infty \), we have \( F \in \mathcal{B}_\pi^\infty \), and therefore the integral

\[
G(t, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\tau) \frac{y}{y^2 + (t - \tau)^2} \, d\tau
\]

is absolutely convergent.

Next, we show that

\[
G(t, y) = F(t + iy).
\] (20)

For \( 0 < \varepsilon < 1 \) consider the functions

\[
f_\varepsilon(z) = f((1 - \varepsilon)z) \frac{\sin(\varepsilon \pi z)}{\varepsilon \pi z}, \quad z \in \mathbb{C},
\]

which are in \( \mathcal{B}_\pi^2 \), and set

\[
F_\varepsilon(z) = f_\varepsilon(z) + i(\mathcal{J}f_\varepsilon)(z), \quad z \in \mathbb{C}.
\]

Thus, according to (7), we have

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} F_\varepsilon(\tau) \frac{y}{y^2 + (t - \tau)^2} \, d\tau
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} (f_\varepsilon(\tau) + i(Hf_\varepsilon)(\tau)) \frac{y}{y^2 + (t - \tau)^2} \, d\tau - \frac{1}{\pi} \int_{-\infty}^{\infty} i(Hf_\varepsilon)(0) \frac{y}{y^2 + (t - \tau)^2} \, d\tau. \quad (21)
\]

Since \( f_\varepsilon \in \mathcal{B}_\pi^2 \), we obtain

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} (f_\varepsilon(\tau) + i(Hf_\varepsilon)(\tau)) \frac{y}{y^2 + (t - \tau)^2} \, d\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\hat{f}_\varepsilon(\omega) + i(-i \text{sgn}(\omega))\hat{f}_\varepsilon(\omega))e^{-y|\omega|e^{i\omega t}} \, d\omega
\]

\[
= \frac{1}{2\pi} \int_{0}^{\pi} \hat{f}_\varepsilon(\omega) e^{i\omega(t+iy)} \, d\omega
\]

\[
= f_\varepsilon(t + iy) + i(Hf_\varepsilon)(t + iy)
\]

for the first integral on the right-hand side of (21). For the second integral we have

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} i(Hf_\varepsilon)(0) \frac{y}{y^2 + (t - \tau)^2} \, d\tau = i(Hf_\varepsilon)(0),
\]

because \( (Hf_\varepsilon)(0) \) is a constant and

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (t - \tau)^2} \, d\tau = 1.
\]

Thus, it follows that

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} F_\varepsilon(\tau) \frac{y}{y^2 + (t - \tau)^2} \, d\tau = f_\varepsilon(t + iy) + i((Hf_\varepsilon)(t + iy) - (Hf_\varepsilon)(0))
\]

\[
= f_\varepsilon(t + iy) + i(\mathcal{J}f_\varepsilon)(t + iy)
\]

\[
= F_\varepsilon(t + iy). \quad (22)
\]
Let $\delta > 0$ be arbitrary but fixed. Then there exists a $\tau_0 = \tau_0(\delta) > 0$ such that

$$\int_{-\infty}^{-\tau_0} \log(1 + |\tau|) \frac{y}{y^2 + (t - \tau)^2} d\tau < \delta$$  \hspace{1cm} (23)$$

and

$$\int_{\tau_0}^{\infty} \log(1 + \tau) \frac{y}{y^2 + (t - \tau)^2} d\tau < \delta.$$  

Further, it can be shown that there exists an $\varepsilon_0 = \varepsilon_0(\tau) > 0$ such that

$$\max_{|\tau| \leq \tau_0} |F_\varepsilon(\tau) - F(\tau)| < \delta$$  \hspace{1cm} (24)$$

for all $0 < \varepsilon \leq \varepsilon_0$. Using (22), we have

$$|F_\varepsilon(t + iy) - G(t, y)| = \left| \frac{1}{\pi} \int_{-\infty}^{\infty} (F_\varepsilon(\tau) - F(\tau)) \frac{y}{y^2 + (t - \tau)^2} d\tau \right|$$

$$\leq \frac{1}{\pi} \int_{-\infty}^{-\tau_0} |F_\varepsilon(\tau) - F(\tau)| \frac{y}{y^2 + (t - \tau)^2} d\tau$$

$$+ \frac{1}{\pi} \int_{\tau_0}^{\infty} |F_\varepsilon(\tau) - F(\tau)| \frac{y}{y^2 + (t - \tau)^2} d\tau$$

$$+ \frac{1}{\pi} \int_{\tau_0}^{\infty} |F_\varepsilon(\tau) - F(\tau)| \frac{y}{y^2 + (t - \tau)^2} d\tau.$$  \hspace{1cm} (25)$$

Next, we analyze the three integrals on the right-hand side of (25). In order to bound the first and third integral from above, we need an auxiliary result. According to Theorem 5 there exist two constants $C_4$ and $C_5$ such that

$$|(3f)(\tau)| \leq (C_5 + C_4 \log(1 + |\tau|)) \|f\|_\infty$$

for all $\tau \in \mathbb{R}$. Hence, we have

$$|F(\tau)| = |f(\tau) + i(3f)(\tau)| \leq |f(\tau)| + |(3f)(\tau)|$$

$$\leq \|f\|_\infty + (C_5 + C_4 \log(1 + |\tau|)) \|f\|_\infty$$

$$= (1 + C_5 + C_4 \log(1 + |\tau|)) \|f\|_\infty$$

and

$$|F_\varepsilon(\tau)| \leq (1 + C_5 + C_4 \log(1 + |\tau|)) \|f\|_\infty \leq (1 + C_5 + C_4 \log(1 + |\tau|)) \|f\|_\infty$$

for all $\tau \in \mathbb{R}$. This implies that

$$|F_\varepsilon(\tau) - F(\tau)| \leq 2(1 + C_5 + C_4 \log(1 + |\tau|)) \|f\|_\infty$$  \hspace{1cm} (26)$$

for all $\tau \in \mathbb{R}$. Thus, for the first integral on the right-hand side of (25), we obtain

$$\frac{1}{\pi} \int_{-\infty}^{-\tau_0} |F_\varepsilon(\tau) - F(\tau)| \frac{y}{y^2 + (t - \tau)^2} d\tau \leq C_0 \|f\|_\infty \int_{-\infty}^{-\tau_0} \log(1 + |\tau|) \frac{y}{y^2 + (t - \tau)^2} d\tau$$

$$< C_0 \|f\|_\infty \delta.$$  \hspace{1cm} (27)$$

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where we used (26) in the first inequality and (23) in the second, and $C_9$ is a constant. By the same arguments, we obtain

$$\frac{1}{\pi} \int_{\tau_0}^{\infty} |F_\varepsilon(\tau) - F(\tau)| \frac{y}{y^2 + (t - \tau)^2} d\tau < C_9 \|f\|_\infty \delta. \quad (28)$$

For the second integral on the right-hand side of (25), we have for all $0 < \varepsilon \leq \varepsilon_0$ that

$$\frac{1}{\pi} \int_{-\tau_0}^{\tau_0} |F_\varepsilon(\tau) - F(\tau)| \frac{y}{y^2 + (t - \tau)^2} d\tau \leq \max_{|\tau| \leq \tau_0} |F_\varepsilon(\tau) - F(\tau)| \frac{1}{\pi} \int_{-\tau_0}^{\tau_0} \frac{y}{y^2 + (t - \tau)^2} d\tau < \delta, \quad (29)$$

where we used (24) in the last inequality. Combining (25) and (27)–(29) implies that

$$|F_\varepsilon(t + iy) - G(t, y)| < (1 + 2C_9 \|f\|_\infty)\delta$$

for all $0 < \varepsilon \leq \varepsilon_0$, which shows that

$$\lim_{\varepsilon \to 0} F_\varepsilon(t + iy) = G(t, y).$$

Further, a short calculation gives

$$\lim_{\varepsilon \to 0} F_\varepsilon(t + iy) = F(t + iy).$$

Hence, we have (20), i.e.,

$$F(t + iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\tau) \frac{y}{y^2 + (t - \tau)^2} d\tau.$$

It follows that

$$|F(t + iy)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} |F(\tau)| \frac{y}{y^2 + (t - \tau)^2} d\tau \leq \|F\|_\infty \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (t - \tau)^2} d\tau = \|F\|_\infty,$$

which completes the proof. △

**Proof of Theorem 8.** We start with the proof of the “⇒” direction. Let $f \in B^\infty_\pi$ be real-valued, such that $3f \in B^\infty_\pi$. Further, let $\varepsilon$ with $0 < \varepsilon < 1$ and $t \in \mathbb{R}$ be arbitrary but fixed, and
consider the complex contour that is depicted in Fig. 5. Since \( F = f + iJf \) is an entire function, we have according to Cauchy’s integral theorem that
\[
\int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi = 0.
\]
Further, we have
\[
\int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi = \int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi + \int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi + \int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi.
\]
Thus, it follows that
\[
\int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi = -\int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi - \int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi.
\]
Next, we analyze the two integrals on the right-hand side of (30). For the first integral we have
\[
\int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi = \int_{-\pi}^{0} \frac{F(t + \varepsilon e^{i\varphi})}{\varepsilon e^{i\varphi}} i\varepsilon e^{i\varphi} \, d\varphi = i\int_{-\pi}^{0} F(t + \varepsilon e^{i\varphi}) \, d\varphi,
\]
and consequently
\[
\left| \int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi \right| \leq \pi \sup_{\text{Im}(z) \geq 0} F(z) \leq \pi \| F \|_\infty,
\]
where we used Lemma 2 in the last inequality. For the second integral, a similar calculation yields
\[
\left| \int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi \right| \leq \pi \| F \|_\infty.
\]
Combining (30), (32), and (33), we obtain
\[
\left| \frac{1}{\pi} \int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi \right| \leq 2 \| F \|_\infty.
\]
Since \(|\text{Re } z| \leq |z|\) for all \( z \in \mathbb{C} \) and \( f \) is real-valued, this implies that
\[
\left| \frac{1}{\pi} \int_{|\varepsilon - |t - \tau|| \leq \frac{1}{2}} \frac{f(\tau)}{t - \tau} \, d\tau \right| = \left| \frac{1}{\pi} \int_{\gamma} \frac{F(\xi)}{t - \xi} \, d\xi \right| \leq 2 \| F \|_\infty,
\]
which completes the proof of the “⇒” direction, because \( 0 < \varepsilon < 1 \) and \( t \in \mathbb{R} \) were arbitrary.

Next, we prove the “⇐” direction. Consider the operator \( J \) defined by
\[
(Jf)(t) = \lim_{\varepsilon \to 0} \left( (H_\varepsilon f)(t) - (H_\varepsilon f)(0) \right), \quad t \in \mathbb{R},
\]
where
\[
(H_\varepsilon f)(t) = \frac{1}{\pi} \int_{|\varepsilon - |t - \tau|| \leq \frac{1}{2}} \frac{f(\tau)}{t - \tau} \, d\tau.
\]
We first show that $J$ is a well-defined operator on $B_{\infty}^{\pi}$. Let $t > 0$ be arbitrary but fixed. Without loss of generality, we may assume that $t > 0$. The case $t < 0$ is treated analogously. Set $t_0 = t/2$.

For $\varepsilon$ satisfying $0 < \varepsilon < t_0$ and $1/\varepsilon > t + t_0$, we obtain, by splitting and rearranging integrals, that

$$((H_{\varepsilon}f)(t) - (H_{\varepsilon}f)(0))\pi = \int_{\varepsilon \leq |t - \tau| \leq t_0} \frac{f(\tau)}{t - \tau} d\tau + \int_{\varepsilon \leq |\tau| \leq t_0} \frac{f(\tau)}{\tau} d\tau$$

$$+ \int_{t - \tau}^{t + \tau} \frac{f(\tau)}{t - \tau} d\tau + \int_{t - t_0}^{t + t_0} \frac{f(\tau)}{\tau} d\tau$$

$$+ \int_{t - t_0}^{-t_0} \frac{f(\tau)}{(t - \tau)\tau} d\tau + \int_{t_0}^{t_0} \frac{f(\tau)t}{(t - \tau)\tau} d\tau. \quad (34)$$

For the first integral in (34) we have

$$\int_{\varepsilon \leq |t - \tau| \leq t_0} \frac{f(\tau)}{t - \tau} d\tau = \int_{\varepsilon \leq |t - \tau| \leq t_0} \frac{f(\tau) - f(t)}{t - \tau} d\tau. \quad (35)$$

Since $|f(\tau) - f(t)| \leq |t - \tau|\pi\|f\|_{\infty}$, we see that the integrand of the integral on the right-hand side of (35) is continuous on $[t - t_0, t + t_0]$. Thus, it follows that

$$\lim_{\varepsilon \to 0} \int_{\varepsilon \leq |t - \tau| \leq t_0} \frac{f(\tau)}{t - \tau} d\tau = \int_{t - t_0}^{t + t_0} \frac{f(\tau) - f(t)}{t - \tau} d\tau.$$ 

The same consideration for the second integral in (34) shows that

$$\lim_{\varepsilon \to 0} \int_{\varepsilon \leq |\tau| \leq t_0} \frac{f(\tau)}{\tau} d\tau = \int_{-t_0}^{t_0} \frac{f(\tau) - f(0)}{\tau} d\tau.$$ 

As for the third integral in (34), we have

$$\int_{\frac{1}{2}}^{t + \frac{1}{2}} \frac{f(\tau)}{t - \tau} d\tau \leq \frac{\|f\|_{\infty} t}{\varepsilon - t},$$

and consequently

$$\lim_{\varepsilon \to 0} \int_{\frac{1}{2}}^{t + \frac{1}{2}} \frac{f(\tau)}{t - \tau} d\tau = 0.$$ 

The same holds for the fourth integral

$$\lim_{\varepsilon \to 0} \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{f(\tau)}{t - \tau} d\tau = 0.$$ 

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It remains to analyze the seventh and eighth integral in (34). Since
\[
\int_{-\frac{t_0}{\varepsilon}}^{\frac{t_0}{\varepsilon}} \frac{\left| f(\tau) \right|}{(t-\tau)^{\varepsilon}} d\tau \leq \|f\|_{\infty} \int_{t_0}^{\frac{t_0}{\varepsilon}} \frac{t}{(t+\tau)^{\varepsilon}} d\tau
\]
\[
= \|f\|_{\infty} \int_{t_0}^{\frac{t_0}{\varepsilon}} \frac{1}{\tau} - \frac{1}{(t+\tau)^{\varepsilon}} d\tau
\]
\[
= \|f\|_{\infty} \left( \log(1-\varepsilon t) + \log \left( \frac{t+t_0}{t_0} \right) \right)
\]
\[
\leq \|f\|_{\infty} \log \left( \frac{t+t_0}{t_0} \right)
\]
\[
\leq C_{10},
\]
with a constant $C_{10} < \infty$ that is independent of $\varepsilon$, we see that the limit
\[
\lim_{\varepsilon \to 0} -\frac{t_0}{\varepsilon} \int_{t-\frac{1}{\varepsilon}}^{t} \frac{f(\tau)t}{(t-\tau)^{\varepsilon}} d\tau
\]
extists and is finite. By a similar calculation we see that
\[
\lim_{\varepsilon \to 0} \int_{t+\frac{1}{\varepsilon}}^{t_0} \frac{f(\tau)t}{(t-\tau)^{\varepsilon}} d\tau
\]
extists and is finite. Thus, the operator $J$ is well-defined on $B_{\infty}^{\pi}$ and we have
\[
(Jf)(t) = \frac{1}{\pi} \int_{t-t_0}^{t+t_0} \frac{f(\tau) - f(t)}{t-\tau} d\tau + \frac{1}{\pi} \int_{-\infty}^{t_0} \frac{f(\tau) - f(0)}{\tau} d\tau
\]
\[
+ \frac{1}{\pi} \int_{-\infty}^{t_0} \frac{f(\tau)}{\tau} d\tau + \frac{1}{\pi} \int_{t-t_0}^{t+t_0} \frac{f(\tau)}{\tau} d\tau
\]
\[
+ \frac{1}{\pi} \int_{-\infty}^{-t_0} \frac{f(\tau)t}{(t-\tau)^{\varepsilon}} d\tau + \frac{1}{\pi} \int_{t+t_0}^{\infty} \frac{f(\tau)t}{(t-\tau)^{\varepsilon}} d\tau
\]
(36)
for all $f \in B_{\infty}^{\pi}$ and $t \in \mathbb{R}$.

Next, let $f \in B_{\infty}^{\pi}$ be real-valued and arbitrary but fixed. For $n \in \mathbb{N}$, consider the $B_{\pi}^{2}$-functions
\[
f_n(\tau) = f \left( \left( 1 - \frac{1}{n} \right) \tau \right) \frac{\sin \left( \frac{1}{n} \pi \tau \right)}{\frac{1}{n} \pi \tau}, \quad \tau \in \mathbb{R}.
\]
We will show that $(Jf)(t) = \lim_{n \to \infty} (Jf_n)(t)$ for all $t \in \mathbb{R}$. Again, we may restrict ourselves to the case $t > 0$, because the case $t < 0$ is treated analogously. Therefore, let $t > 0$ be arbitrary but fixed. We need some additional preliminary considerations. First, note that $\|f_n\|_{\infty} \leq \|f\|_{\infty}$, $n \in \mathbb{N}$.
Moreover, $f_n$ converges locally uniformly to $f$, and $f'_n$ converges locally uniformly to $f'$. Let $\delta > 0$ be arbitrary but fixed. Then there exists a $\tau_0 = \tau_0(\delta) \geq t + t_0$ such that

$$
\int_{-\infty}^{-\tau_0} \frac{t}{(\tau - t)\tau} \, d\tau < \delta
$$

(37)

and

$$
\int_{\tau_0}^{\infty} \frac{t}{(\tau - t)\tau} \, d\tau < \delta.
$$

(38)

Due to the local uniform convergence of $f_n$, there exists a natural number $n_0 = n_0(\delta)$ such that

$$
\max_{\tau \in [-\tau_0, \tau_0]} |f(\tau) - f_n(\tau)| < \delta
$$

(39)

for all $n \geq n_0$. Since

$$
\frac{f(\tau) - f(\tau_1) - f_n(\tau) - f_n(\tau_1)}{\tau_1 - \tau} = \frac{1}{\tau_1 - \tau} \int_{\tau_1}^{\tau} f'(u) - f'_n(u) \, du,
$$

it follows that

$$
\left| \frac{f(\tau) - f(\tau_1) - f_n(\tau) - f_n(\tau_1)}{\tau_1 - \tau} \right| \leq \frac{1}{|\tau_1 - \tau|} \int_{\tau_1}^{\tau} |f'(u) - f'_n(u)| \, du
$$

$$
\leq \max_{u \in [\min\{\tau_1, \tau\}, \max\{\tau_1, \tau\}]} |f'(u) - f'_n(u)|
$$

for all $\tau, \tau_1 \in \mathbb{R}$. Hence, by the local uniform convergence of $f'_n$, it follows that there exists a natural number $n_1 = n_1(\delta)$ such that

$$
\left| \frac{f(\tau) - f(t) - f_n(\tau) - f_n(t)}{t - \tau} \right| < \delta
$$

(40)

and

$$
\left| \frac{f(\tau) - f(0) - f_n(\tau) - f_n(0)}{\tau} \right| < \delta
$$

(41)

for all $\tau \in [-t_0, t + t_0]$ and all $n \geq n_1$. Now, we have all preliminary considerations and can return to the proof. From (36) we obtain

$$
|(J(f - f_n))(t)| \leq \frac{1}{\pi} \int_{-t_0}^{t_0} \left| \frac{f(\tau) - f_n(\tau) - f(t) + f_n(t)}{t - \tau} \right| \, d\tau
$$

$$
+ \frac{1}{\pi} \int_{-t_0}^{t_0} \left| \frac{f(\tau) - f_n(\tau) - f(0) + f_n(0)}{\tau} \right| \, d\tau
$$

$$
+ \frac{1}{\pi} \int_{-t_0}^{t_0} \left| \frac{f(t) - f_n(t)}{t - \tau} \right| \, d\tau + \frac{1}{\pi} \int_{t - t_0}^{t + t_0} \left| \frac{f(\tau) - f_n(\tau)}{\tau} \right| \, d\tau
$$

$$
+ \frac{1}{\pi} \int_{-\infty}^{-t_0} \left| \frac{f(t) - f_n(t)}{(\tau - t)\tau} \right| \, d\tau + \frac{1}{\pi} \int_{t - t_0}^{t + t_0} \left| \frac{f(\tau) - f_n(\tau)}{(\tau - t)\tau} \right| \, d\tau.
$$

(42)
We treat the integrals in (42) separately. For the first and second integral we easily obtain
\[
\frac{1}{\pi} \int_{t-t_0}^{t+t_0} \left| \frac{f(\tau) - f_n(\tau) - f(t) + f_n(t)}{t - \tau} \right| d\tau < 2\delta t_0
\]
and
\[
\frac{1}{\pi} \int_{-t_0}^{t_0} \left| \frac{f(\tau) - f_n(\tau) - f(0) + f_n(0)}{\tau} \right| d\tau < 2\delta t_0
\]
for all \( n \geq n_1 \), by using (40) and (41), respectively. For the third integral we have
\[
\int_{-t_0}^{t_0} \left| \frac{f(\tau) - f_n(\tau)}{t - \tau} \right| d\tau \leq \max_{\tau \in [-t_0,t_0]} |f(\tau) - f_n(\tau)| \int_{-t_0}^{t_0} \frac{1}{t - \tau} d\tau < \delta C_{11}
\]
for all \( n \geq n_0 \), where we used (39) in the last inequality. Equally, we obtain
\[
\int_{t-t_0}^{t+t_0} \left| \frac{f(\tau) - f_n(\tau)}{t - \tau} \right| d\tau \leq \max_{\tau \in [t-t_0,t+t_0]} |f(\tau) - f_n(\tau)| \int_{t-t_0}^{t+t_0} \frac{1}{t - \tau} d\tau < \delta C_{12}
\]
for all \( n \geq n_0 \). The fifth integral can be upper bounded according to
\[
\int_{-\infty}^{t_0} \frac{|f(\tau) - f_n(\tau)| t}{(\tau - t)\tau} d\tau = \int_{-\infty}^{-t_0} \frac{|f(\tau) - f_n(\tau)| t}{(\tau - t)\tau} d\tau + \int_{-t_0}^{t_0} \frac{|f(\tau) - f_n(\tau)| t}{(\tau - t)\tau} d\tau \leq 2\|f\|_{\infty} \int_{-\infty}^{-t_0} \frac{t}{(\tau - t)\tau} d\tau + \max_{\tau \in [-t_0,-t_0]} |f(\tau) - f_n(\tau)| \int_{-\infty}^{t_0} \frac{t}{(\tau - t)\tau} d\tau < \delta(2\|f\|_{\infty} + C_{13})
\]
for all \( n \geq n_0 \), where we used (37) and (39) in the last inequality. For the last integral in (42), the same considerations together with (38) and (39) give
\[
\int_{t+t_0}^{\infty} \frac{|f(\tau) - f_n(\tau)| t}{(\tau - t)\tau} d\tau = \int_{t+t_0}^{t_0} \frac{|f(\tau) - f_n(\tau)| t}{(\tau - t)\tau} d\tau + \int_{t_0}^{\infty} \frac{|f(\tau) - f_n(\tau)| t}{(\tau - t)\tau} d\tau \leq \max_{\tau \in [t+t_0,t_0]} |f(\tau) - f_n(\tau)| \int_{t_0}^{\infty} \frac{t}{(\tau - t)\tau} d\tau + 2\|f\|_{\infty} \int_{t_0}^{\infty} \frac{t}{(\tau - t)\tau} d\tau < \delta(2\|f\|_{\infty} + C_{14})
\]
for all \( n \geq n_0 \). Combining all partial results yields
\[
|(Jf)(t) - (Jf_n)(t)| = |(J(f - f_n))(t)| < \delta(4t_0 + 4\|f\|_{\infty} + C_{11} + C_{12} + C_{13} + C_{14})
\]
for all \( n \geq \max\{n_0, n_1\} \). Since \( \delta > 0 \) was arbitrary, this shows that
\[
(Jf)(t) = \lim_{n \to \infty} (Jf_n)(t).
\]
(43)
Equality (43) is true for all \( t \in \mathbb{R} \), because \( t > 0 \) was arbitrary, and the case \( t < 0 \) is treated analogously. Thus, we have

\[
(\Im f)(t) = \lim_{n \to \infty} (\Im f_n)(t) = \lim_{n \to \infty} ((H f_n)(t) - (H f_n)(0))
\]

\[
= \lim_{n \to \infty} \lim_{\varepsilon \to 0} ((H \varepsilon f_n)(t) - (H \varepsilon f_n)(0)) = \lim_{n \to \infty} (J f_n)(t)
\]

\[
= (J f)(t) = \lim_{\varepsilon \to 0} ((H \varepsilon f)(t) - (H \varepsilon f)(0))
\]

for all \( t \in \mathbb{R} \), where we used Lemma 1 in the first equality and (43) in the fifth. It follows from the assumption of the theorem that

\[
|((\Im f)(t)| = \lim_{\varepsilon \to 0} \left| ((H \varepsilon f)(t) - (H \varepsilon f)(0)) \right| \leq \limsup_{\varepsilon \to 0} \left( |(H \varepsilon f)(t)| + |(H \varepsilon f)(0)| \right) \leq 2C8
\]

for all \( t \in \mathbb{R} \), which implies that \( \Im f \in \mathcal{B}_\pi^\infty \) because of Theorem 4. \( \triangleq \)

**APPENDIX 2: PROOF OF THEOREM 10**

For the proof of Theorem 10, we need the following lemma.

**Lemma 3.** Let \( f \in \mathcal{B}_\pi^\infty \) such that \( \Im f - C_I \in \mathcal{B}_\pi^\infty \) for some constant \( C_I \), and let

\[
F^C_I(t + iy) = f(t + iy) + i((\Im f)(t) + iy) - C_I.
\]

Then for all \( \varepsilon > 0 \) there exists a natural number \( R_0 = R_0(\varepsilon) \) such that

\[
|F^C_I(t + iy)| < \varepsilon
\]

for all \( t \in \mathbb{R} \) and \( y \geq 0 \) satisfying \( \sqrt{t^2 + y^2} \geq R_0 \).

**Proof.** Consider the Möbius transformation

\[
\varphi(z) = \frac{z - i}{z + i},
\]

which maps the upper half plane to the unit disk. The inverse mapping is given by

\[
\varphi^{-1}(z) = i \frac{1 + z}{1 - z}.
\]

Since \( F^C_I \) is analytic in \( \mathbb{C} \) and \( |F^C_I(t + iy)| \leq \|F^C_I\|_\infty \) for all \( t \in \mathbb{R} \) and \( y \geq 0 \), according to Lemma 2 it follows that

\[
G(z) = F^C_I(\varphi^{-1}(z)) = F^C_I \left( i \frac{1 + z}{1 - z} \right)
\]

is analytic for \( |z| < 1 \) and that

\[
\sup_{|z| < 1} |G(z)| < \infty.
\]

Further, \( G \) is continuous on the unit circle, because \( F^C_I \) is continuous on the real axis,

\[
\lim_{\omega \downarrow 0} G(e^{i\omega}) = \lim_{t \to -\infty} F^C_I(t) = 0,
\]

and

\[
\lim_{\omega \uparrow 0} G(e^{i\omega}) = \lim_{t \to \infty} F^C_I(t) = 0.
\]
Hence, by [29, p. 340, Theorem 17.11], we have

\[ G(\rho e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{i\omega}) \frac{1 - \rho^2}{1 - 2\rho \cos(\omega - \theta) + \rho^2} d\omega \]

for all \( 0 \leq \rho < 1 \) and \(-\pi < \theta < \pi\).

Let \( \varepsilon > 0 \) be arbitrary but fixed. Equations (44) and (45) imply that there exits an \( \omega_0 = \omega_0(\varepsilon) \), \( 0 < \omega_0 < \pi \), such that

\[ |G(e^{i\omega})| < \frac{\varepsilon}{2} \] \hspace{1cm} (46)

for all \( |\omega| \leq \omega_0 \). Further, there exists a \( \rho_0 = \rho_0(\varepsilon) \), \( 0 < \rho_0 < 1 \), such that

\[ \frac{\|F_1^C\|_\infty (1 - \rho)}{\rho \left( 1 - \cos \left( \frac{\omega_0}{2} \right) \right)} < \frac{\varepsilon}{2} \] \hspace{1cm} (47)

for all \( \rho_0 \leq \rho < 1 \).

Next, let \( \rho \) satisfying \( \rho_0 \leq \rho < 1 \) and \( \theta \) satisfying \(-\omega_0/2 \leq \theta \leq \omega_0/2\) be arbitrary but fixed. Then, we have

\[ |G(\rho e^{i\theta})| \leq \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} |G(e^{i\omega})| \frac{1 - \rho^2}{1 - 2\rho \cos(\omega - \theta) + \rho^2} d\omega \]

\[ + \frac{1}{2\pi} \int_{\omega_0 \leq |\omega| \leq \pi} |G(e^{i\omega})| \frac{1 - \rho^2}{1 - 2\rho \cos(\omega - \theta) + \rho^2} d\omega \]

\[ < \frac{\varepsilon}{2} + \frac{\|F_1^C\|_\infty}{2\pi} \int_{\omega_0 \leq |\omega| \leq \pi} \frac{1 - \rho^2}{1 - 2\rho \cos(\omega - \theta) + \rho^2} d\omega, \]

where we used (46) and the fact [29, p. 233] that

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \rho^2}{1 - 2\rho \cos(\omega - \theta) + \rho^2} d\omega = 1. \]

Further, we have

\[ \frac{\|F_1^C\|_\infty}{2\pi} \int_{\omega_0 \leq |\omega| \leq \pi} \frac{1 - \rho^2}{1 - 2\rho \cos(\omega - \theta) + \rho^2} d\omega \leq \frac{\|F_1^C\|_\infty}{2\pi} \int_{\omega_0 \leq |\omega| \leq \pi} \frac{1 - \rho^2}{1 - 2\rho \cos \left( \frac{\omega_0}{2} \right) + \rho^2} d\omega \]

\[ < \frac{\|F_1^C\|_\infty (1 - \rho^2)}{1 - 2\rho \cos \left( \frac{\omega_0}{2} \right) + \rho^2} = \frac{\|F_1^C\|_\infty (1 - \rho)(1 + \rho)}{(1 - \rho^2) + 2\rho \left( 1 - \cos \left( \frac{\omega_0}{2} \right) \right)} < \frac{\|F_1^C\|_\infty (1 - \rho)}{\rho \left( 1 - \cos \left( \frac{\omega_0}{2} \right) \right)} < \frac{\varepsilon}{2}. \]
where we used (47) in the last inequality. Hence, it follows that \(|G(\rho e^{i\theta})| < \varepsilon\) for all \(\rho_0 \leq \rho < 1\) and \(-\omega_0/2 \leq \theta \leq \omega_0/2\). Let \(\mathcal{D} = \{\rho e^{i\theta} : \rho_0 \leq \rho < 1, -\omega_0/2 \leq \theta \leq \omega_0/2\}\). Thus, for \(z \in \varphi^{-1}(\mathcal{D})\), we have \(F(z) < \varepsilon\). The image of \(\mathcal{D}\) under the mapping \(\varphi^{-1}\) is depicted in Fig. 6. Finally, let \(R_0\) be the radius of the smallest circle around the origin whose restriction to the upper half plane lies completely in \(\varphi^{-1}(\mathcal{D})\). Then, we have \(|F(t + iy)| < \varepsilon\) for all \(t \in \mathbb{R}\) and \(y \geq 0\) satisfying \(\sqrt{t^2 + y^2} \geq R_0\). \(\triangle\)

Now we are in a position to prove Theorem 10.

Proof of Theorem 10. Let \(f \in \mathcal{B}_{\pi, 0}^{\infty}\) be real-valued and such that \(3f - C_1 \in \mathcal{B}_{\pi, 0}^{\infty}\) for some constant \(C_1\). Further, let \(t \in \mathbb{R}\) be arbitrary but fixed. Since \(F^t = f + i(3f - C_1) \in \mathcal{B}_\pi\) is an entire function, we can use the same argumentation as in the proof of Theorem 8 to obtain

\[
\int_{\mathbb{C}} \frac{F(t)}{t - \xi} \, d\xi = -\int_{\mathbb{C}} \frac{F(t)}{t - \xi} \, d\xi - \int_{\mathbb{C}} \frac{F(t)}{t - \xi} \, d\xi.
\]

From (31) we see that

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{C}} \frac{F(t)}{t - \xi} \, d\xi = \pi i F(t).
\]

Let \(\delta > 0\) be arbitrary but fixed. Then, according to Lemma 3, there exists a natural number \(R_0 = R_0(\delta)\) such that \(|F(t + iy)| < \delta\) for all \(t \in \mathbb{R}\) and \(y \geq 0\) satisfying \(\sqrt{t^2 + y^2} \geq R_0\). Let \(\varepsilon_0 = 1/(R_0 + |t|)\). Then it follows that \(|t + \frac{1}{\varepsilon} e^{i\varphi}| \geq R_0\) for all \(0 < \varepsilon \leq \varepsilon_0\) and consequently that \(|F(t + \frac{1}{\varepsilon} e^{i\varphi})| \leq \delta\) for all \(0 < \varepsilon \leq \varepsilon_0\) and \(0 \leq \varphi \leq \pi\). Thus, we have

\[
\left| \int_{\mathbb{C}} \frac{F(t)}{t - \xi} \, d\xi \right| \leq \pi \int_{\mathbb{C}} \left| F(t + \frac{1}{\varepsilon} e^{i\varphi}) \right| \, d\varphi \leq \pi \delta
\]

for all \(0 < \varepsilon \leq \varepsilon_0\), which shows that

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{C}} \frac{F(t)}{t - \xi} \, d\xi = 0.
\]

Hence, it follows that

\[
\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{C}} \frac{F(t)}{t - \xi} \, d\xi = -i F(t), \quad (48)
\]

which in turn implies that the real part of the left-hand side of (48) converges to the real part of the right-hand side of (48), i.e., that

\[
\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(t)}{t - \xi} \, d\xi = (3f)(t) - C_1,
\]

and that the imaginary part of the left-hand side of (48) converges to the imaginary part of the right-hand side of (48), i.e., that

\[
\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{C}} \frac{(3f)(t) - C_1}{t - \xi} \, d\xi = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{C}} \frac{(3f)(t)}{t - \xi} \, d\xi = -f(t). \quad \triangle
\]
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