A set of $q$-coherent states for the Rogers–Szegő oscillator

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Abstract
We discuss a model of a $q$-harmonic oscillator based on Rogers–Szegő functions. We combine these functions with a class of $q$-analogs of complex Hermite polynomials to construct a new set of coherent states depending on a nonnegative integer parameter $m$. Our construction leads to a new $q$-deformation of the $m$-true-polyanalytic Bargmann transform whose range defines a generalization of the Arik–Coon space. We also give an explicit formula for the reproducing kernel of this space. The obtained results may be exploited to define a $q$-deformation of the Ginibre-$m$-type process on the complex plane.

Keywords q-deformed 2D complex Hermite polynomials · $q$-coherent states · Rogers-Szego oscillator · $q$-deformed Bargmann transform · Generalized Arik-Coon spaces · Reproducing kernels

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1 Introduction

Coherent states (CS) are a type of quantum states which were first introduced by Schrödinger [1] when he described certain states of the harmonic oscillator (HO). Later, Glauber [2] used these states for his quantum mechanical description of coherent laser light and coined the term CS. Like canonical CS, they satisfy a set of relevant
properties and they too have found wide applications in different branches of physics such as quantum optics, statistical mechanics, nuclear physics and condensed matter physics [3]. One of many definitions of CS is a special superposition with the form

\[ \Psi_z := (e^{z\bar{z}})^{-1/2} \sum_{j=0}^{+\infty} \frac{\bar{z}^j}{\sqrt{j!}} \phi_j, \]  

(1.1)

where the \( \phi_j \)'s span a Hilbert space \( \mathcal{H} \) usually called Fock space.

The terminology of generalized CS (GCS) was first appeared and studied in \([4,5]\) in connection with states discussed in \([6]\). These states are usually associated with potential algebras other than the oscillator one \([3,7,8]\). An important example is provided by the \( q \)-deformed CS \((q-CS)\) for brevity related to deformations of boson operators \([9–11]\). \( q-CS \) are usually constructed in a way that they reduce to their standard counterparts as \( q \to 1 \) and are special case of a more general class, the so-called \( f \)-deformed CS \([12]\). Among \( q-CS \), there are those associated with the relation

\[ a_q a_q^\dagger - qa_q^\dagger a_q = 1 \]  

(1.2)

with \( 0 < q < 1 \), where \( a_q \) are often termed maths-type \( q \)-bosons operators \([13,14]\) because the basic numbers and special functions attached to them have been extensively studied in mathematics \([15]\). As a consequence, the corresponding wavefunctions of these \( q \)-boson operators were found in terms of many orthogonal \( q \)-polynomials \([16–18]\).

The \( q-CS \) may be defined \([9]\) through a \( q \)-analog of the expansion (1.1) by a superposition of a set of number states \( \phi_j^{(q)} \) spanning a Hilbert space \( \mathcal{H}_q \) which stands for a \( q \)-analog of the Fock space \( \mathcal{H} \) as:

\[ \Psi_z^q := (e_q(z\bar{z}))^{-1/2} \sum_{j=0}^{+\infty} \frac{\bar{z}^j}{\sqrt{[j]_q!}} \phi_j^{(q)}, \]  

(1.3)

for \( z \in \mathbb{C}_q := \{ \zeta \in \mathbb{C}, \ (1-q)\zeta \bar{\zeta} < 1 \} \) where \( [j]_q = \frac{1-q^{-j}}{1-q} \to j \) as \( q \to 1 \),

\[ [j]_q! = \frac{(q; q)_j}{(1-q)_j} \]  

(1.4)

denotes the \( q \)-factorial of \( j \) and

\[ e_q(\xi) := \sum_{n \geq 0} \frac{\xi^n}{[n]_q!} = \frac{1}{((1-q)\xi; q)_\infty}, \quad |\xi| < \frac{1}{1-q} \]  

(1.5)

being a \( q \)-exponential function satisfying \( e_q(\xi) \to e^\xi \) as \( q \to 1 \). For details on basic notations of \( q \)-calculus, we refer to \([15,20,29]\).
In this paper, we adopt the Hilbertian probabilistic formalism [19] in order to generalize the expression in (1.3) with respect to a fixed integer parameter $m \in \mathbb{N}$. Precisely, we here introduce the following superposition of the $\Phi_j^{(q)}$’s:

$$
\Psi_{z}^{q,m} := \left( N_{q,m}(z\bar{z}) \right)^{-\frac{1}{2}} \sum_{j=0}^{+\infty} \Phi_{j}^{q,m}(z) \Phi_{j}^{(q)}, \tag{1.6}
$$

where $N_{q,m}(z\bar{z})$ is a normalization factor (given by (5.2) below) and

$$
\Phi_{j}^{q,m}(z) := \frac{q^{(m/j)} \sqrt{1 - q^{m-j}} |z|^{m-j} e^{-i(m-j) \arg(z)}}{((q; q)_{m \vee j})^{-1} (-1)^{m \wedge j} (q; q)_{m-j} \sqrt{q^{m+j} (q; q)_{m} (q; q)_{j}}}
\times P_{m \wedge j} \left( (1 - q)z\bar{z}; q^{m-j} |q| \right) \tag{1.7}
$$

are coefficients defined in terms of Wall polynomials [20, p. 109]:

$$
P_n(x; a|q) \equiv 2 \phi_1 \left( q^{-n}, 0; aq |q, qx \right). \tag{1.8}
$$

Here, $m \vee j = \max(m, j)$, $m \wedge j = \min(m, j)$ and $2\phi_1$ is the basic hypergeometric series [20, p. 12]. In the case, $m = 0$, $\Phi_{j}^{q,0}(z)$ reduce to the coefficient $z^j / \sqrt{[j]_q}!$. For general $m \in \mathbb{N}$, the coefficients (1.7) are a slight modification of a class of 2D orthogonal $q$-polynomials, denoted $H_{j,m}(z, \bar{z}|q)$, which are $q$-deformation [21] of the well-known 2D complex Hermite polynomials [22] $H_{j,m}(z, \bar{z})$ whose Rodriguez-type formula reads $H_{j,m}(z, \bar{z}) = (-1)^{j+m} e^{z\bar{z}} \partial_z^j \partial_{\bar{z}}^m e^{-z\bar{z}}$. This last expression turns out to be the diagonal representation (or upper symbol [23] with respect to CS (1.1)) of the operator $(a^*)^j a^m$ whose expectation values are needed [24] in the study of squeezing properties involving the Heisenberg uncertainty relation. Here, $a$ and $a^*$ are the classical annihilation and creation operators. In other words, the polynomial $H_{j,m}(z, \bar{z})$ also represents a classical observable on the phase space or a Glauber–Sudarshan $P$-function for $(a^*)^j a^m$. This means that the G$q$-CS $\Psi_{z}^{q,m}$ in (1.6) we are introducing may play an analog role in the Berezin $de$-quantization procedure of $\left( a^* \right)^j a^m$, see [25] for a similar discussion related the $q$-Heisenberg–Weyl group.

Here, we introduce an explicit realization of $q$-creation and $q$-annihilation operators associated with a Rogers–Szegő oscillator whose eigenstates are chosen to be our vectors $\Phi_j^{(q)}$. Next, we give a closed form for the superposition (1.6) and the associated CS transform (CST) $B_{m}^{(q)}$ which may be viewed as a new $q$-deformation of the $m$-true polyanalytic Bargmann transform [26,27]. As a consequence, the range of $B_{m}^{(q)}$ defines a generalization of the well-known Arik–Coon space [9], whose reproducing kernel will be given explicitly. The latter one may be used to define a $q$-deformation of the Ginibre-$m$-type process on the complex plane [28].
The paper is organized as follows. In Sect. 2, we give a brief review of the standard coherent state formalism. In Sect. 3, we discuss the Hilbert space carrying the coefficients needed in the superposition defining our Gq-CS. Section 4 is devoted to discuss a model of a q-deformed HO based on Rogers–Szegő functions with a realization of q-creation and q-annihilation operators on $L^2(\mathbb{R})$. In Sect. 5, we introduce a new set of Gq-CS for the Rogers–Szegő oscillator and we give an explicit formula for the associated CST whose range provides us with a generalization of the Arik–Coon space. For the latter one, we also obtain explicitly the reproducing kernel. Most technical proofs and calculations are postponed in Appendices.

2 A CS formalism

Here, we adopt the prototypical model of CS presented in [19, pp. 72–77] and described as follows. Let $X$ be a set equipped with a measure $d\mu$ and $L^2(X)$ the Hilbert space of $d\mu$-square integrable functions $f(x)$ on $X$. Let $A^2 \subset L^2(X)$ be a closed subspace with an orthonormal basis $\{\Phi_j\}_{j=0}^\infty$ such that

$$\mathcal{N}(x) := \sum_{j \geq 0} |\Phi_j(x)|^2 < +\infty, \quad x \in X. \quad (2.1)$$

Let $\mathcal{H}$ be another (functional) Hilbert space with the same dimension as $A^2$ and $\{\phi_j\}_{j=0}^\infty$ is a given orthonormal basis of $\mathcal{H}$. Then, consider the family of states $\{\Psi_x\}_{x \in X}$ in $\mathcal{H}$, through the following linear superpositions

$$\Psi_x := (\mathcal{N}(x))^{-1/2} \sum_{j \geq 0} \Phi_j(x)\phi_j. \quad (2.2)$$

These CS obey the normalization condition

$$\langle \Psi_x | \Psi_x \rangle_{\mathcal{H}} = 1 \quad (2.3)$$

and the following resolution of the identity operator on $\mathcal{H}$

$$1_{\mathcal{H}} = \int_X |\Psi_x \rangle \langle \Psi_x | \mathcal{N}(x) d\mu(x) \quad (2.4)$$

which is expressed in terms of Dirac’s bra–ket notation $|\Psi_x \rangle \langle \Psi_x |$ meaning the rank one operator defined by $\varphi \mapsto \langle \Psi_x | \varphi \rangle_{\mathcal{H}} \cdot \Psi_x$, $\varphi \in \mathcal{H}$. The choice of the Hilbert space $\mathcal{H}$ defines in fact a quantization of the space $X$ by the coherent states in (2.2), via the inclusion map $X \ni x \mapsto \Psi_x \in \mathcal{H}$ and the property (2.4) is crucial in setting the bridge between the classical and the quantum worlds. The CST associated with the set $\Psi_x$ is the map $B : \mathcal{H} \longrightarrow A^2$ defined for every $x \in X$ by

$$B[\phi](x) := (\mathcal{N}(x))^{1/2} \langle \phi, \Psi_x \rangle_{\mathcal{H}}. \quad (2.5)$$
From the resolution of the identity (2.4) and for \( \phi, \psi \in H \), we have \( \langle \phi, \psi \rangle_H = \langle B[\phi], B[\psi] \rangle_{L^2(X)} \), meaning that \( B \) is an isometric map.

The formula (2.2) can be considered as a generalization of the series expansion of the canonical CS in (1.1) with \( \{\phi_j\}_{j=0}^\infty \) being an orthonormal basis of the Hilbert space \( H := L^2(\mathbb{R}) \), consisting of eigenstates of the quantum HO given by

\[
\phi_j(\xi) = \left( \sqrt{\pi} 2^j j! \right)^{-1/2} H_j(\xi)e^{-\frac{1}{2} \xi^2}, \quad \xi \in \mathbb{R}
\]  

where

\[
H_j(\xi) := j! \sum_{k=0}^{\lfloor j/2 \rfloor} \frac{(-1)^k}{k!(j-2k)!} (2\xi)^{j-2k}
\]  

is the Hermite polynomial of degree \( j \) [20, p. 59]. Here, the space \( \mathcal{A}^2 \) is the Bargmann–Fock space \( \mathcal{F}(\mathbb{C}) \) of entire complex-valued functions which are \( \pi^{-1}e^{-z\bar{z}}d\lambda \)-square integrable, \( \mathcal{N}(z) = e^{z\bar{z}}, \ z \in \mathbb{C} \) and \( d\lambda \) denotes the Lebesgue measure on \( \mathbb{C} \cong \mathbb{R}^2 \).

In this case, the associated CST \( B_0 : L^2(\mathbb{R}) \to \mathcal{F}(\mathbb{C}) \), defined for any function \( f \in L^2(\mathbb{R}) \) by

\[
B_0[f](z) := \pi^{-\frac{1}{4}} \int_\mathbb{R} e^{-\frac{1}{2} \xi^2 + \sqrt{2}z\xi - \frac{1}{2} z^2} f(\xi) d\xi, \quad z \in \mathbb{C}
\]

turns out to be the well-known Bargmann transform [30, p. 12].

### 3 The Hilbert space \( \mathcal{F}_q \)

We observe that the coefficients

\[
\Phi^q_j(z) = \frac{z^j}{\sqrt{(j)_q!}}, \quad j = 0, 1, 2, \ldots,
\]

occurring in the number state expansion of \( \Psi^q_j \) in (1.3) constitute a particular case of the complex-valued functions \( \Phi^q_m(z) \) in (1.7). Indeed, for \( m = 0 \) we have that \( \Phi^q_0(z) = \Phi^q_m(z) \). For \( m \in \mathbb{N} \), these coefficients are a slight modification of the 2D complex orthogonal polynomials [21, p. 6783]:

\[
H_{m,j}(z, \zeta|q) := \sum_{k=0}^{m\wedge j} \binom{m}{k}_q \binom{j}{k}_q (-1)^k q^{\frac{k(k+1)}{2}} (q; q)_{k} z^{m-k} \zeta^{j-k}, \ z, \zeta \in \mathbb{C}.
\]

Precisely,

\[
\Phi^q_m(z) := \frac{H_{m,j}(\sqrt{1-\beta^2} z, \sqrt{1-\beta^2} \zeta|q)}{\sqrt{q^m (q; q)_j (q; q)_m}}.
\]
Note, also, that by using (1.4), one can check that \( \Phi_{j}^{q,m}(z) \to (m!j!)^{-1/2} H_{m,j}(z, \bar{z}) \) as \( q \to 1 \), where

\[
H_{m,j}(z, \xi) = \sum_{k=0}^{m+j} (-1)^k k! \binom{m}{k} \binom{j}{k} z^{m-k} \bar{\xi}^{j-k}
\]  

(3.4)
denote the 2D complex Hermite polynomials introduced by Itô [22] in the context of complex Markov processes.

Furthermore, we observe that the functions \( \Phi_{j}^{q,m}(z) \) constitute an orthonormal system in the Hilbert space \( \mathfrak{H}_{q} := L^2(\mathbb{C}, d\mu_q(z)) \) of square integrable functions on \( \mathbb{C} := \{ z \in \mathbb{C}, (1-q)|\xi| \leq q^m \} \) with respect to the measure

\[
d\mu_q(z) = \frac{d\theta}{2\pi} \sum_{l=0}^{\infty} \frac{q^l(q; q)_l}{(q; q)_l} \delta \left( r - \frac{q^{\frac{1}{2}l}}{\sqrt{1-q}} \right),
\]  

(3.5)

\( z = re^{i\theta}, \ r \in \mathbb{R}^+, \ \theta \in [0, 2\pi) \) and \( \delta(r - \bullet) \) is the Dirac mass function at the point \( \bullet \). Indeed, we may assume that \( m \geq j \) because of the symmetry \( H_{s,r}(w, z|q) = H_{s,r}(w, z|q) \), then we use the expression (1.7) to obtain

\[
\langle \Phi_{j}^{q,m}, \Phi_{k}^{q,m} \rangle_{\mathfrak{H}_{q}} = \int_{\mathbb{C}_{q,m}} \Phi_{j}^{q,m}(z) \overline{\Phi_{k}^{q,m}(z)} d\mu_q(z)
\]

\[
= (-1)^{j+k} \frac{(q; q)_m q^{\frac{l}{2}} \sqrt{1-q}^{m-j}}{(q; q)_{m-j}(q; q)_m \sqrt{1-q}^{m-k}} \frac{(q; q)_m q^{\frac{l}{2}} \sqrt{1-q}^{m-k}}{(q; q)_{m-k}(q; q)_k} \int_{0}^{2\pi} e^{i(j-k)} \frac{d\theta}{2\pi}
\]

\[
\times \sum_{l=0}^{\infty} \frac{q^l(q; q)_l}{(q; q)_l} P_j(r^2, q^{m-j}|q) P_k(r^2, q^{m-k}|q) r^{2(m-j-k)} \delta \left( r - \frac{q^{\frac{1}{2}l}}{\sqrt{1-q}} \right)
\]

\[
= (-1)^{j+k} \frac{(q; q)_m q^{\frac{l}{2}+\frac{l}{2}}}{(q; q)_{j}(q; q)_{m-j}(q; q)_{m-k}} \frac{\delta_{j,k}}{\delta_{j,k}} \sum_{l=0}^{\infty} \frac{q^l(q; q)_l}{(q; q)_l} \delta_{j,k}
\]

\[
\times P_j(q^{m-j}|q) P_k(q^{m-j}|q) = \delta_{j,k}.
\]  

(3.6)

Here, we have used the orthogonality relations of Wall polynomials [20, p. 107]:

\[
\sum_{l=0}^{\infty} \frac{(\tau q)^l}{(q; q)_l} P_s(q^l; \tau|q) P_n(q^l; \tau|q) = \frac{(\tau q)^n}{(q; q)_n} \delta_{s,n}, \quad 0 < \tau < q^{-1}
\]  

(3.7)

for parameters \( \tau = q^{m-j}, s = j \) and \( n = k \).

Finally, we denote by \( \mathcal{A}_{q}^{2}(\mathbb{C}) \) the completed space of holomorphic functions on \( \mathbb{C}_q = \mathbb{C}_{q,0} \), equipped with the scalar product

\[
\langle \varphi, \phi \rangle = \int_{\mathbb{C}_q} \varphi(z) \overline{\phi(z)} d\mu_q(z).
\]  

(3.8)
The element $\Phi^{q,0}_j(z) = ([j]_q!)^{-1/2} z^j$ form an orthonormal basis of the space $A^2_q(\mathbb{C})$ which coincides with Arik–Coon space [9] whose reproducing kernel is given by

$$K_q(z, w) := e_q(z \bar{w}), \quad z, w \in \mathbb{C}_q. \quad (3.9)$$

Note that by letting $q \to 1$, the measure $d\mu_q$ reduces to the Gaussian measure $\pi^{-1} e^{-z \bar{z}} d\lambda$ on $\mathbb{C}$.

### 4 A Rogers–Szegő Hamiltonian

Following [31], we consider a $q$-deformed creation and annihilation operators as follows:

$$B^*_q = \frac{e^{i\kappa x}}{i \sqrt{1-q}} \left( e^{i\kappa x} - q^{3/4} e^{i\kappa \partial_x} \right) \quad (4.1)$$

$$B_q = - \frac{e^{-i\kappa x}}{i \sqrt{1-q}} \left( e^{-i\kappa x} - q^{1/4} e^{i\kappa \partial_x} \right). \quad (4.2)$$

Here, $\kappa$ is a deformation parameter related to a finite-difference method with respect to $x$ and $q = e^{-2\kappa^2} > 0$. Note that for $a \in \mathbb{C}$ the operator $e^{a \partial_x}$ acts on a function $f(x)$ as $e^{a \partial_x} [f](x) = f(x + a)$. Moreover, it is not difficult to verify that operators (4.1)–(4.2) satisfy the $q$-commutation relation

$$[B_q, B^*_q] = B_q B^*_q - q B^*_q B_q = 1. \quad (4.3)$$

Similarly to the case of the quantum HO, the $q$-deformed HO is described by the Hamiltonian

$$H_q = \frac{1}{2} \hbar \omega \left( B_q B^*_q + B^*_q B_q \right) \quad (4.4)$$

where $\omega$ is the oscillator frequency and $\hbar$ denotes the Planck’s constant. Explicitly,

$$H_q = \frac{1}{2(q-1)} \hbar \omega \left( -2 + (q^{1/4} + q^{5/4}) e^{i\kappa x} e^{i\kappa \partial_x} + (q^{-1/4} + q^{-3/4}) e^{-i\kappa x} e^{i\kappa \partial_x} + (q^{3/2} + q^{1/2}) e^{2i\kappa \partial_x} \right). \quad (4.5)$$

For the sake of simplicity, we will take $\hbar = \omega = 1$. The eigenstates of the Hamiltonian in (4.5) are given by

$$\phi^{RS}_j(x) := \frac{(i \sqrt{q})^j}{\pi^{1/2} \sqrt{(q^j; q)_j}} H_j \left( -e^{2i\kappa x}, q \right) e^{-\frac{1}{2} x^2}, \quad x \in \mathbb{R} \quad (4.6)$$

in terms of the Rogers–Szegő polynomials [29, p. 455]

$$H_n(\xi; q) := \sum_{k=0}^n \binom{n}{k}_q (q^{-1/2} \xi)^k, \quad \xi \in \mathbb{C}. \quad (4.7)$$
It was proved [31, p. 612] that the functions $\varphi_{RS_j}$ satisfy the orthonormality relations on the full real line, i.e.,

$$\int_{\mathbb{R}} \varphi_{RS_j}(x) \varphi_{RS_k}(x) dx = \delta_{jk}.$$  

(4.8)

Furthermore, each of these functions can also be reproduced by iterating $j$ times the action of the $q$-creation operator $B_q^\ast$ on the Gaussian function as

$$\varphi_{RS_j}(x) = \left( B_q^\ast \right)^j \left( \frac{1}{\pi^2} e^{-\frac{1}{2} x^2} \right)$$

(4.9)

and one can check that the operators $B_q^\ast$ and $B_q$ act on them as

$$B_q^\ast \varphi_{RS_j}(x) = \varphi_{RS_{j+1}}(x), \quad B_q \varphi_{RS_j}(x) = [j]_q \varphi_{RS_{j-1}}(x).$$

(4.10)

Then, it follows that the Hamiltonian (4.4) is diagonal on these states and has the eigenvalues

$$\varepsilon_j^q = \frac{1}{2} \left( [j+1]_q + [j]_q \right)$$

(4.11)

as $q$-deformed energy levels.

Finally, let us mention that in the limit $q \to 1$ ($\kappa \to 0$), the operators (4.1)–(4.2) reduce to the well-known non-relativistic HO creation and annihilation operators. Also, from [31, p. 614] one has that

$$\lim_{q \to 1} \varphi_{RS_j}(x) = \left( \sqrt{\pi} 2^j j! \right)^{-1/2} H_j(x) e^{-\frac{1}{2} x^2}$$

(4.12)

where $H_j(\cdot)$ is the Hermite polynomial (2.7), meaning that the $\varphi_{RS_j}$'s stand for $q$-analogs of the HO wave functions which justify our choice in (4.6).

5 A new set of $Gq$-CS

For $m \in \mathbb{N}$ and $q \in ]0, 1[$, we define a new set of $Gq$-CS by the following superposition of the $\varphi_{RS_j}$'s:

$$\Psi^{q,m}_z := (N_{q,m}(z\bar{z}))^{-\frac{1}{2}} \sum_{j \geq 0} \Phi_{j}^{q,m}(z) \varphi_{RS_j}$$

(5.1)

where

$$N_{q,m}(z\bar{z}) = \frac{q^{-m}(q^{1-m}(1 - q)z\bar{z}; q)_m}{(q^{-m}(1 - q)z\bar{z}; q)_\infty}$$

(5.2)

is the normalization factor ensuring $\langle \Psi^{q,m}_z | \Psi^{q,m}_z \rangle_{L^2(\mathbb{R})} = 1$, which is well defined for $z \in \mathbb{C}_{q,m}$.

Using the normalization factor (5.2) together with the orthonormality relations (3.6) of the coefficients $\Phi_{j}^{q,m}(z)$ and the completeness of the basis $\varphi_{RS_j}$ in $L^2(\mathbb{R})$, one can
check that the states $\Psi^{q,m}_z$ provide us with a resolution of the identity operator on $L^2(\mathbb{R})$ as

$$1_{L^2(\mathbb{R})} = \int_{\mathbb{C}} N_{q,m}(z \bar{z}) d\mu_q(z) |\Psi^{q,m}_z\rangle \langle \Psi^{q,m}_z|.$$  \hspace{0.75cm} (5.3)

We now establish (see “Appendix A” for the proof) that the wave function of the CS (5.1) can be expressed as

$$\Psi^{q,m}_z(\xi) = (-1)^m \left( \frac{q^m e^{-\xi^2} (q^{-m}(1 - q)z\bar{z}; q)_\infty}{\sqrt{\pi}(q; q)_m(q^{1-m}(1 - q)z\bar{z}; q)_m} \right)^{1/2} \frac{q^{-m/4} e^{i m \xi \zeta}}{(iz\sqrt{1-q^{m-1}}, -iz\sqrt{1-q^{m-1}} e^{2i\xi}; q)_\infty}$$

$$\times Q_m \left( \frac{ie^{-i \xi \zeta} q^{1/4} - ie^{i \xi \zeta} q^{-1/4}}{2}; zq^{-1/4} \sqrt{1-q} e^{i \xi \zeta}, \bar{z}q^{1/4} \sqrt{1-q} e^{-i \xi \zeta} |q\rangle, \xi \in \mathbb{R}, \right.$$  \hspace{0.75cm} (5.4)

in terms of AL-Salam–Chihara polynomials $Q_m(\cdot, \cdot, |q)$, which are defined by [20, p. 80]:

$$Q_m(x; \alpha, \beta |q) = \frac{(\alpha \beta; q)_m}{\alpha^m} 3\phi_2 \left( \begin{array}{c} q^{-m}, \alpha u, \alpha u^{-1} \\ \alpha \beta, 0 \end{array} \right) |q\rangle, \quad x = \frac{1}{2}(u + u^{-1}).$$  \hspace{0.75cm} (5.5)

Further, by applying (2.5), the CST $B^{(q)}_m : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{C}, dm_q)$ defined by

$$B^{(q)}_m[f](z) = (N_{q,m}(z \bar{z}))^{1/2} \langle f, \Psi^{q,m}_z \rangle_{L^2(\mathbb{R})}, \quad z \in \mathbb{C},$$  \hspace{0.75cm} (5.6)

is an isometric map. Explicitly,

$$B^{(q)}_m[f](z) = \left( \frac{q^{-m/2}}{\sqrt{\pi}(q; q)_m} \right)^{1/2} \frac{(-1)^m}{(iz\sqrt{1-q^{m-1}}; q)_\infty} \int_{\mathbb{R}} e^{im \xi \zeta} e^{-\frac{1}{2} \xi^2}$$

$$\times Q_m \left( \frac{ie^{-i \xi \zeta} q^{1/4} - ie^{i \xi \zeta} q^{-1/4}}{2}; zq^{-1/4} \sqrt{1-q} e^{i \xi \zeta}, \bar{z}q^{1/4} \sqrt{1-q} e^{-i \xi \zeta} |q\rangle f(\xi) d\xi, \right. \hspace{0.75cm} (5.7)$$

for every $z \in \mathbb{C}$. Particularly, for $m = 0$, the wave function (5.4) has the form

$$\Psi^{q,0}_z(\xi) = (e_q(z \bar{z}))^{-1/2} \left( \frac{\pi^{-1/4} e^{-\frac{1}{2} \xi^2}}{(-iz\sqrt{1-q} e^{2i\xi \zeta}; q)_\infty (iz\sqrt{q(1-q)}; q)_\infty} \right), \quad \xi \in \mathbb{R}$$  \hspace{0.75cm} (5.8)
where \( z \in \mathbb{C}_q := \mathbb{C}_{q,0} \) which is the domain of convergence of the exponential function in \( e_q(\cdot) \). In this case, the transform (5.7) reduces to \( \mathcal{B}^{(q)}_0 : L^2(\mathbb{R}) \to \mathcal{A}^2_q(\mathbb{C}) \)

\[
\mathcal{B}^{(q)}_0[f](z) = \frac{\pi^{-\frac{1}{4}}}{(iz\sqrt{q(1-q)}; q)_\infty} \int_{\mathbb{R}} e^{-\frac{1}{2}z^2} e^{-\frac{1}{2}z^2 + \sqrt{q}iz\xi} f(\xi) d\xi
\]  

for every \( z \in \mathbb{C}_q \). Moreover, when \( q \to 1 \), \( \mathcal{B}^{(q)}_0 \) goes to the Bargmann transform (2.8).

In addition, by letting \( q \to 1 \), \( \mathcal{B}^{(q)}_m \) goes to the generalized Bargmann transform \( \mathcal{B}_m : L^2(\mathbb{R}) \to \mathcal{A}^2_m(\mathbb{C}) \subset L^2(\mathbb{C}, \pi^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} d\lambda) \), defined by [26]:

\[
\mathcal{B}_m[f](z) = (-1)^m (2^m m! \sqrt{\pi})^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{1}{2}z^2 - \frac{1}{2}z^2 + \sqrt{q}iz\xi} H_m \left( \frac{\xi - \frac{z + \bar{z}}{2}}{\sqrt{q}} \right) f(\xi) d\xi
\]  

where \( H_m(\cdot) \) denotes the Hermite polynomial, see “Appendix B” for the proof. This transform, also called \( m \)-true-polyanalytic Bargmann transform, has found applications in time-frequency analysis [32] and discrete quantum dynamics [33]. For more details on (5.10), see [34] and reference therein. Here, the arrival space \( \mathcal{A}^2_m(\mathbb{C}) \) is the generalized Bargmann–Fock space whose reproducing kernel is given by [35]:

\[
K_m(z, w) = e^{z\bar{w}} L_m^0(\|z - w\|^2), \quad z, w \in \mathbb{C}.
\]  

Consequently, the range of the CST (5.7) leads us to define a generalization, with respect to \( m \in \mathbb{N} \), of the Arik–Coon space \( \mathcal{A}^2_q \) as \( \mathcal{A}^2_{q,m} = \mathcal{B}^{(q)}_m(L^2(\mathbb{R})) \) which also coincides with the closure in \( \mathfrak{H}_q \) of the linear span of \( \{\Phi^{j,m}_q\}_{j \geq 0} \). In “Appendix C”, we prove that the reproducing kernel of the space \( \mathcal{A}^2_{q,m} \) is given by

\[
K_{q,m}(z, w) = \frac{(q^{1-m}(1-q)z\bar{z}; q)_m}{q^m(q^{-m}(1-q)z\bar{w}; q)_\infty} \binom{3}{q, q^{-m+1}} \binom{q^{-m}}{(1-q)z\bar{z}} \binom{q; (1-q)w\bar{w}}{q^{-m+1}}
\]  

for every \( z, w \in \mathbb{C}_{q,m} \). In particular, the limit \( K_{q,m}(z, w) \to K_m(z, w) \) as \( q \to 1 \) holds true.

Finally, note that the expression (5.12) may also constitute a starting point to construct a \( q \)-deformation for the determinantal point process associated with an \( m \)th Euclidean Landau level or Ginibre-type point process in \( \mathbb{C} \) as discussed by Shirai [28].

**Remark 5.1** Note that the Stieltjes–Wigert polynomials [20, p. 116] defined by

\[
s_n(x; q) := \sum_{k=0}^n \binom{n}{k}_q q^{k^2} x^k
\]
are connected to Rogers–Szegő polynomials (4.7) by

\[ s_n(x; q^{-1}) = H_n(xq^{\frac{1}{2}-n}; q). \] (5.14)

This relation may also suggest the construction of another extension of \( q \)-deformed CS via the same procedure, which would be attached to a suitable Arik–Coon oscillator with \( q > 1 \) [36].

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**Appendix A: The wave function of the CS \( \Psi_{2,q}^{m,m} \)**

By using (5.1) and replacing \( \Phi_j^{m,q}(z) \) by their expressions in (1.7), we need to establish a closed form for the following series

\[
S^m = \sum_{j \geq 0} \frac{(-1)^{m-j}(q; q)_m q^{m(j/2)} \sqrt{1-q^{m-j}|z|^{m-j}} e^{-i(m-j)\theta}}{(q; q)_m \sqrt{q^{m_j}(q; q)_m(q; q)_j}} \times P_{m-j}
\]

This sum can be cast into two quantities as

\[
S^m_{(<\infty)} = \sum_{j=0}^{m-1} \frac{(-1)^j(q; q)_m q^{j(z/2)} \sqrt{1-q^jz^{m-j}}}{(q; q)_m \sqrt{q^{m_j}(q; q)_m(q; q)_j}} \times P_j (1-q)z \overline{z}; q^{m-j}|q) \psi_{RS}^j(\xi) \]

and

\[
S^m_{(\infty)} (z, \xi; q) = \frac{(-1)^m q^{m(z/2)} (z \sqrt{1-q})^{-m} e^{-\frac{1}{4}z^2}}{\pi^{\frac{1}{4}} (q; q)_m} \eta_{(\infty)}(m, z, \xi),
\] (5.15)

with

\[
\eta_{(\infty)}(m, z, \xi) = \sum_{j=0}^{\infty} \frac{(z \sqrt{1-q})^j}{\sqrt{q^{m_j}(q; q)_{j-m}}} P_m ((1-q)z \overline{z}; q^{j-m}|q) (i \sqrt{q})^j H_j \left(-e^{2ix}\xi; q\right).
\] (5.17)
Now, we apply the relation [37, p. 3]:

\[ P_n(x; q^{-N} | q) = x^N (-1)^{-N} q^{N(N+1-2n)/2} \left( \frac{q^{N+1}; q}{q^{1-N}; q} \right)_n P_{n-N}(x; q^{N} | q) \]  

(5.18)

for \( N = j - m, \ n = j \) and \( x = (1 - q)z \bar{z} \), to obtain that \( S_m^{\infty} = 0 \). For the infinite sum, we first rewrite the expression of the Wall polynomial as [20, p. 260]:

\[ P_n(x; a|q) = \frac{(x^{-1}; q)_n}{(aq; q)_n} (-x)^n q^{-\binom{n}{2}}_{2\phi_1} \left( \left( q^{-n}, 0 \right| q; aq^{n+1} \right), \]  

(5.19)

in which we apply the identity

\[ (a; q)_n = (a^{-1} q^{1-n}; q)_n (-a)^n q^{\binom{n}{2}} \]  

(5.20)

to the term \((x^{-1}; q)_n\). Therefore, (5.19) reduces to

\[ P_n(x; a|q) = \frac{(xq^{1-n}; q)_n}{(aq; q)_n} 2\phi_1 \left( \left( q^{-n}, 0 \right| q; aq^{n+1} \right). \]  

(5.21)

Setting \( n = m, \ x = (1 - q)z \bar{z} \) and \( a = q^{j-m} \) in (5.21), and appealing to the fact that

\[ (a; q)_{n+k} = (a; q)_n (aq^n; q)_k, \]  

(5.22)

the r.h.s of (5.17) takes the form

\[ \eta(\infty)(m, z, \xi) = (q^{1-m}\lambda; q)_m \sum_{j=0}^{\infty} \frac{y^j}{(q; q)_j} H_j(\alpha; q)_{2\phi_1} \left( \left( q^{-m}, 0 \right| \lambda q^{1-m} q; q^{1+j} \right) \]  

(5.23)

where \( y = iz\sqrt{q^{1-m}-1}, \ \alpha = -e^{2ix\xi} \) and \( \lambda = (1 - q)z \bar{z} \). By recalling the definition of the \( 2\phi_1 \) series and interchanging the order of summation, we get that

\[ \eta(\infty)(m, z, \xi) = (q^{1-m}\lambda; q)_m \sum_{k=0}^{\infty} \frac{(q^{-m}; q)_k}{(\lambda q^{m-1}; q)_k} \frac{q^k}{(q; q)_k} \sum_{j=0}^{\infty} \frac{(yq^k)^j}{(q; q)_j} H_j(\alpha; q). \]  

(5.24)

The last sum in (5.24) suggests us to make use of the generating function of the Rogers–Szegő polynomials [29, p. 456]:

\[ \sum_{j \geq 0} \frac{H_j(x; q)}{(q; q)_j} t^j = \frac{1}{(t; q)_{\infty}(q^{-1/2}xt; q)_{\infty}} \]  

(5.25)
for $t = yq^k$ and $x = \alpha$. Therefore, \((5.24)\) also reads

$$
\eta_{(\infty)}(m, z, \xi) = (q^{1-m} \lambda; q)_m \sum_{k=0}^{\infty} \frac{(q^{-m}; q)_k}{(\lambda q^{m-1}; q)_k} \frac{q^k}{(q; q)_k} \frac{1}{(yq^k; q)_\infty (q^{-1/2}ayq^k; q)_\infty}.
$$

(5.26)

By applying the identity

$$
(a; q)_s = \frac{(a; q)_\infty (aq^s; q)_\infty}{aq^s \neq q^{-n}, n \in \mathbb{N}},
$$

(5.27)

Equation \((5.26)\) transforms to

$$
\eta_{(\infty)}(m, z, \xi) = \frac{(q^{1-m} \lambda; q)_m}{(y, q^{-1/2}ay; q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-m}, q^{-1/2}ay, y; q)_k}{(\lambda q^{m-1}; q)_k} \frac{q^k}{(q; q)_k}
$$

(5.28)

where the last sum may be expressed in terms of a $3\phi_2$ series as

$$
\eta_{(\infty)}(m, z, \xi) = \frac{(q^{1-m} \lambda; q)_m}{(y, q^{-1/2}ay; q)_\infty} 3\phi_2 \left( \begin{array}{c} q^{-m}, y, q^{-1/2}ay \\ \frac{1}{\lambda q^{m-1}}, 0 \end{array} \right | q; q) \tag{5.29}
$$

with

$$
y = iz \sqrt{1 - q} q^{m-1} = q^{-1/4} e^{i \xi} z \sqrt{1 - q} q^{m-1} i q^{1/4} e^{-i \xi}
$$

and

$$
q^{-1/2}ay = -iz \sqrt{1 - q} q^{m-1} e^{2i \xi} = q^{-1/4} e^{i \xi} z \sqrt{1 - q} q^{m-1} i q^{1/4} e^{-i \xi}.
$$

Finally, recalling the definition of Al-Salam–Chihara polynomials \((5.5)\) and taking into account the previous prefactors, we arrive at the announced result \((5.4)\). □

**Appendix B: $B_{m}^{(q)}$ at the limit $q \to 1$**

We first use the identity \((1.4)\) to rewrite \((5.7)\) as

$$
B_{m}^{(q)}[f](z) = (-1)^m \left( q^{2m^2-m} 2^m [m]_q \sqrt{\pi} \right)^{-1/2} \int_{\mathbb{R}} \frac{e^{im \xi - \frac{1}{2} \xi^2}}{(iz \sqrt{1 - q} q^{m-1}, -iz \sqrt{1 - q} q^{m-1} e^{2i \xi}; q)_\infty} Q_m(\tau s; 2\tau \alpha, 2\tau \beta | q) \frac{Q_m(\tau s; 2\tau \alpha, 2\tau \beta | q)}{\tau^m} f(\xi) d\xi
$$

\[(5.30)\]
where

\[ s = \sqrt{\frac{q^{m-1} - i e^{-i \kappa \xi} q^{1/4} - i e^{i \kappa \xi} q^{-1/4}}{1 - q}} \],
\[ \alpha = q^{-1/4} \frac{z}{\sqrt{2}} e^{i \kappa \xi} , \quad \beta = q^{1/4} \frac{\bar{z}}{\sqrt{2}} e^{-i \kappa \xi} \quad \text{and} \quad \tau = \sqrt{\frac{1 - q}{2 q^{m-1}}} . \] (5.31)

Denoting

\[ G_q(z; \xi) := \frac{1}{iz \sqrt{\frac{1-q}{qm^2}}, -iz \sqrt{\frac{1-q}{qm} e^{2i \kappa \xi}; q}} \] (5.32)

and using (5.27), it follows that

\[
\Log G_q(z; \xi) = -\sum_{k \geq 0} \Log \left( 1 + i z q^{\frac{-m}{2}} \sqrt{1 - q} (e^{2i \kappa \xi} - \sqrt{q}) q^k + z^2 (1 - q) q^{\frac{1-m}{2}} e^{2i \kappa \xi} q^{2k} \right) = -iz q^{\frac{-m}{2}} (\sqrt{q} - e^{2i \kappa \xi}) \frac{1}{\sqrt{1-q}} - z^2 q^{\frac{1-m}{2}} e^{2i \kappa \xi} \frac{1}{1+q} + o(1 - q) \] (5.33)

where the quantities \( iz q^{\frac{m}{2}} (\sqrt{q} - e^{2i \kappa \xi}) \) and \( z^2 q^{\frac{1-m}{2}} e^{2i \kappa \xi} \frac{1}{1+q} \) go to \( \sqrt{2} \xi z \) and \( \frac{1}{2} z^2 \) as \( q \to 1 \) (or \( \kappa \to 0 \)), respectively. Therefore, \( G_q(z; \xi) \to \exp(\sqrt{2} \xi z - \frac{1}{2} z^2) \) as \( q \to 1 \).

By another side, we may apply the result [38, p. 3]:

\[
\lim_{q \to 1} \tau^{-m} Q_m(\tau s; 2 \tau \alpha, 2 \tau \beta | q) = H_m(s - \alpha - \beta) \] (5.34)

for the parameters \( s, \alpha, \beta \) and \( \tau \) as in (5.31), to arrive at

\[
\lim_{q \to 1} (-1)^m \left( q^{\frac{2m^2-m}{2} - 2m [m]_q ! \sqrt{\pi}} \right)^{-1/2} Q_m(\tau s; 2 \tau \alpha, 2 \tau \beta | q) = (-1)^m \left( 2m m ! \sqrt{\pi} \right)^{-1/2} H_m \left( \xi - \frac{z + \bar{z}}{\sqrt{2}} \right).
\]

Finally, summarizing the above calculations we complete the proof of the assertion (5.10). \( \square \)
Appendix C: The reproducing kernel $K_{q,m}(z, w)$ and its limit at $q \to 1$

Since the $\Phi_j^{m,q}$’s form an orthonormal basis of the space $\mathcal{A}_m^2(\mathbb{C})$, the corresponding reproducing kernel is given, according the general theory [40,41] by

$$K_{q,m}(z, w) := \sum_{j=0}^{\infty} \Phi_j^{q,m}(z) \overline{\Phi_j^{q,m}(w)}. \quad (5.35)$$

For that we replace the $\Phi_j^{q,m}$’s by their expressions in (1.7) and we split the sum (5.35) into two parts as

$$K_{q,m}(z, w) = S_{(\infty)}^m(z, w; q) + S_{(\infty)}^m(z, w; q) \quad (5.36)$$

where

$$S_{(\infty)}^m(z, w; q) = \sum_{j=0}^{m-1} \frac{(q, q; q)_m q^{2\binom{j}{2}} (1 - q)^{m-j} z^{m-j} w^{m-j}}{(q, q; q)_m (q; q)_m (q; q)_j}$$

$$\times P_j \left( (1 - q)z\bar{w}; q^{m-j} | q \right) P_j \left( (1 - q)w\bar{w}; q^{m-j} | q \right)$$

$$- \sum_{j=0}^{m-1} \frac{(q, q; q)_j q^{2\binom{m}{2}} (1 - q)^{j-m} z^{j-m} w^{j-m}}{(q, q; q)_j (q; q)_j (q; q)_m (q; q)_j}$$

$$\times P_m \left( (1 - q)z\bar{w}; q^{j-m} | q \right) P_m \left( (1 - q)w\bar{w}; q^{j-m} | q \right),$$

and

$$S_{(\infty)}^m(z, w; q) = \frac{q^{\binom{m}{2}}}{(q; q)_m} \left( \frac{1}{z} \right)^{\lambda}$$

$$\times \sum_{j=0}^{m-1} \frac{(q; q)_j}{(q; q)_j (q; q)_j} \left( \frac{\lambda}{q^m} \right)^j \frac{P_m \left( \alpha; q^{j-m} | q \right) P_m \left( \beta; q^{j-m} | q \right)}{P_m \left( \alpha; q^{-N} | q \right) P_m \left( \beta; q^{-N} | q \right)} \quad (5.37)$$

where $\lambda = (1 - q)z\bar{w}$, $\alpha = (1 - q)z\bar{w}$ and $\beta = (1 - q)w\bar{w}$. By making use of the relation [37, p. 3]:

$$P_n(x; q^{-N} | q) = x^N (1-x)^{-N} q^{N(N+1-2n)} \frac{(q^{N+1}; q)_n (q^{-N}; q)_n}{(q^{1-N}; q)_n} P_{n-N}(x; q^{-N} | q) \quad (5.38)$$

for parameters $N = j - m, n = j, x = \alpha$ and $x = \beta$, we obtain that $S_{(\infty)}^m(z, w; q) = 0$. For the infinite sum (5.37), we rewrite the Wall polynomial as [20, p. 260]:

$$P_n(x; a | q) = \frac{(x^{-1}; q)_n}{(aq; q)_n} (-x)^n q^{-\binom{n}{2}/2} \phi_1 \left( q^{-n}, 0 | xq^{-1-n} | aq^{n+1} \right) \quad (5.39)$$
with \( n = m \), \( x = (1 - q)\lambda \) and \( a = q^{j-m} \). Next, by using (5.20)–(5.22), respectively, Eq. (5.36) takes the form

\[
K_{q,m}(z, w) = \frac{q^{2\binom{m}{2}}}{(q; q)_m} \left( \frac{1}{\lambda} \right)^m (q^{1-m} \alpha; q)_m (q^{1-m} \beta; q)_m S_q^m(\alpha; \beta) \quad (5.40)
\]

where

\[
S_q^m(\alpha; \beta) := \sum_{j \geq 0} \frac{(q; q)_j}{(q; q)_j} \frac{1}{(q^{1-m}; q)_m^2} \left( \frac{\lambda}{q^m} \right)^j \times \phi_1 \left( q^{-m}, 0 | q^{j+1} \right) \phi_1 \left( q^{-m}, 0 | q; q^{j+1} \right). \quad (5.41)
\]

Using the identity \((q; q)_j^2 = (q; q)_j^2\), we may also write

\[
S_q^m(\alpha; \beta) = \sum_{j \geq 0} \frac{(q^{-m} \lambda)_j}{(q; q)_j} \sum_{k \geq 0} \frac{(q^{-m} ; q)_k}{(\alpha q^{1-m}; q)_k} \frac{(q^{j+1} \beta)_l}{(q; q)_l} \sum_{j \geq 0} \frac{(q^{-m} ; q)_l}{(q; q)_l} = \sum_{k,l \geq 0} \frac{(q^{-m} ; q)_k}{(\alpha q^{1-m}; q)_k} \frac{q^k}{(q; q)_k} \frac{(q^{-m} ; q)_l}{(\beta q^{1-m}; q)_l} \frac{q^l}{(q; q)_l} \sum_{j \geq 0} \frac{(q^{-m+k+l} \lambda)_j}{(q; q)_j}. \quad (5.42)
\]

Now, by applying the \( q \)-binomial theorem

\[
\sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} \xi^n = \frac{(a \xi; q)_\infty}{(\xi; q)_\infty}, \quad |\xi| < 1 \quad (5.43)
\]

for \( \xi = q^{-m+k+l} \lambda \) and \( a = 0 \), the last sum reads

\[
S_q^m(\alpha; \beta) = \sum_{k,l \geq 0} \frac{(q^{-m} ; q)_k}{(\alpha q^{1-m}; q)_k} \frac{q^k}{(q; q)_k} \frac{(q^{-m} ; q)_l}{(\beta q^{1-m}; q)_l} \frac{q^l}{(q; q)_l} \frac{1}{(q^{-m+k+l} \lambda; q)_\infty}. \quad (5.44)
\]

By applying the identity (5.27) to the factor \( \frac{1}{(q^{-m+k+l} \lambda; q)_\infty} \), Eq. (5.44) can be rewritten as

\[
S_q^m(\alpha; \beta) = \frac{1}{(q^{-m} \lambda; q)_\infty} \sum_{k \geq 0} \frac{(q^{-m} ; q)_k}{(\alpha q^{1-m}; q)_k} \frac{q^k}{(q; q)_k} \sum_{l \geq 0} \frac{(q^{-m} ; q)_l(q^{-m} \lambda; q)_{k+l}}{(\beta q^{1-m}; q)_l} \frac{q^l}{(q; q)_l} \quad (5.45)
\]
Next, by writing \( (q^{-m}; q)_l \alpha k = (q^{-m}; q)_k (q^{k-m}; q)_l \), Eq. (5.45) transforms to

\[
S_q^m (\alpha; \beta) = \frac{1}{(q^{-m}; q)_\infty} \sum_{k \geq 0} \frac{(q^{-m}; q_k)_l}{(\alpha q^{-1}; q)_k} \frac{q^k}{(q; q)_k} \sum_{l \geq 0} \frac{(q^{-m}; q_k)_l}{(\beta q^{-1}; q)_l} \frac{q^l}{(q; q)_l} \left( q^{-m}; q_k \right) \frac{2\phi_1 \left( \frac{(q^{-m}; q_k)_l}{(q; q)_l} \right)}{(q; q)_l}.
\]

in terms of the \( 2\phi_1 \) series. The latter one satisfies the identity [15, p. 10]:

\[
2\phi_1 \left( \frac{q^{-n}, b}{c} \right) = \frac{(b^{-1}; q)_n (c; q)_n}{(c; q)_n} b^n, \quad n = 0, 1, 2, \ldots
\]

Setting \( n = m, b = q^{k-m} \), and \( c = \beta q^{-1} \) in (5.47) allows us to rewrite (5.46) as

\[
S_q^m (\alpha; \beta) = \frac{1}{(q^{-m}; q)_\infty} \sum_{k \geq 0} \frac{(q^{-m}; q_k)_l}{(\alpha q^{-1}; q)_k} \frac{q^k}{(q; q)_k} \sum_{l \geq 0} \frac{(q^{-m}; q_k)_l}{(\beta q^{-1}; q)_l} \frac{q^l}{(q; q)_l} \left( q^{-m}; q_k \right) \frac{2\phi_1 \left( \frac{(q^{-m}; q_k)_l}{(q; q)_l} \right)}{(q; q)_l}.
\]

Furthermore, applying the identity

\[
(a q^{-n}; q)_k = \frac{(a^{-1} q^{-1}; q)_n}{(a^{-1} q^{-1}; q)_n} (a; q)_k q^{-nk}, \quad a \neq 0
\]

to \( (q^{-k} \frac{b}{\alpha}; q)_m \), leads to

\[
S_q^m (\alpha; \beta) = \frac{(q^{-m}; q)_\infty}{(q^{-m}; q)_\infty} \sum_{k \geq 0} \frac{(q^{-m}; q_k)_l}{(\alpha q^{-1}; q)_k} \frac{q^k}{(q; q)_k} \sum_{l \geq 0} \frac{(q^{-m}; q_k)_l}{(\beta q^{-1}; q)_l} \frac{q^l}{(q; q)_l} \left( q^{-m}; q_k \right) \frac{2\phi_1 \left( \frac{(q^{-m}; q_k)_l}{(q; q)_l} \right)}{(q; q)_l}.
\]

We now summarize the above calculations as follows

\[
K_{q,m}(z, w) = \frac{q^{-m}(q^{-1}; q)_m(q^{-1}; q)_m}{(q^{-m}; q)_\infty(q; q)_m} \sum_{k \geq 0} \frac{(q^{-m}; q_k)_l}{(\alpha q^{-1}; q)_k} \frac{q^k}{(q; q)_k} \sum_{l \geq 0} \frac{(q^{-m}; q_k)_l}{(\beta q^{-1}; q)_l} \frac{q^l}{(q; q)_l} \left( q^{-m}; q_k \right) \frac{2\phi_1 \left( \frac{(q^{-m}; q_k)_l}{(q; q)_l} \right)}{(q; q)_l}.
\]

By making appeal to the finite Heine transformation [39, p. 2]:

\[
3\phi_2 \left( \frac{q^{-n}, \xi}{\gamma, q^{-1}; q} \right) = \frac{(\xi; q)_n}{(\tau; q)_n} 3\phi_2 \left( \frac{q^{-n}, \gamma/\beta, \xi}{\gamma, \xi \tau} \right)
\]

\[
\sum_{k \geq 0} \frac{(q^{-m}; q_k)_l}{(\alpha q^{-1}; q)_k} \frac{q^k}{(q; q)_k} \sum_{l \geq 0} \frac{(q^{-m}; q_k)_l}{(\beta q^{-1}; q)_l} \frac{q^l}{(q; q)_l} \left( q^{-m}; q_k \right) \frac{2\phi_1 \left( \frac{(q^{-m}; q_k)_l}{(q; q)_l} \right)}{(q; q)_l}.
\]
for the parameters $\xi = z/w$, $\beta = q^{-m}a$, $\gamma = q^{1-m}a$, $\tau = qw/z$, we obtain

$$K_{q,m}(z, w) = \frac{(q^{1-m}(1-q)z\bar{z}; q)_m}{q^m(q^{-m}(1-q)z\bar{w}; q)_\infty} \, _3\phi_2 \left( \begin{array}{c} q^{-m}, \bar{z}/w, q\bar{z}/w \; q \; q(1-q)w\bar{w} \end{array} \right).$$

(5.53)

For the limit of $K_{q,m}(z, w)$ as $q \to 1$, we may use (1.5) to see that

$$\lim_{q \to 1} \frac{(q^{1-m}(1-q)z\bar{z}; q)_m}{q^m(q^{-m}(1-q)z\bar{w}; q)_\infty} = \lim_{q \to 1} q^{-m}(q^{-1-m}(1-q)z\bar{z}; q)_m e_q(q^{-m}z\bar{w}) = \exp(z\bar{w}).$$  

(5.54)

By another side, we observe that $(q^{-m}; q)_k = 0$, $\forall k > m$, meaning that $\, _3\phi_2$ series in (5.12) terminates as

$$\sum_{k=0}^m \frac{(q^{-m}, z/w, q\bar{z}/\bar{w}; q)_k}{(q^{1-m}(1-q)z\bar{z}, q; q)_k} \frac{(q(1-q)w\bar{w})^k}{(q; q)_k}.$$  

(5.55)

Thus, from the identity

$$\begin{bmatrix} s \\ k \end{bmatrix}_q := (-1)^k q^{ks-\frac{k(k+1)}{2}} \frac{(q^{-s}; q)_k}{(q; q)_k}, \quad s \in \mathbb{C}$$

(5.56)

together with (1.4) we, successively, have

$$\lim_{q \to 1} \sum_{k=0}^m \frac{(q^{-m}, z/w, q\bar{z}/\bar{w}; q)_k}{(q^{1-m}(1-q)z\bar{z}, q; q)_k} \frac{(q(1-q)w\bar{w})^k}{(q; q)_k} = \sum_{k=0}^m \lim_{q \to 1} \left( \begin{array}{c} m \\ k \end{array} \right)_q (-1)^k q^{(\frac{k(k+1)}{2})} \frac{(z/w, q\bar{z}/\bar{w}; q)_k}{(q^{1-m}(1-q)z\bar{z}, q; q)_k} \frac{(1-q)^k}{(q; q)_k} q^k(w\bar{w})^k \right)$$

$$= \sum_{k=0}^m \lim_{q \to 1} \left( \begin{array}{c} m \\ k \end{array} \right)_q (-1)^k q^{(\frac{k(k+1)}{2})} \frac{(z/w, q\bar{z}/\bar{w}; q)_k}{(q^{1-m}(1-q)z\bar{z}, q; q)_k} \frac{q^k(w\bar{w})^k}{[k]q!} \right)$$

$$= \sum_{k=0}^m (-1)^k \left( \begin{array}{c} m \\ k \end{array} \right) \frac{|z-w|^{2k}}{k!}.$$  

(5.57)

Finally, by noticing that the last sum in (5.57) is the evaluation of the Laguerre polynomial $L_m^{(0)}$ at $|z-w|^2$ [20, p. 108], the proof is completed.  

$\square$

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