Explicit Fermionic Tree Expansions

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Abstract We express connected Fermionic Green’s functions in terms of two totally explicit tree formulas. The simplest and most symmetric formula, the Brydges-Kennedy formula is compatible with Gram’s inequality. The second one, the rooted formula of [AR1], respects even better the antisymmetric structure of determinants, and allows the direct comparison of rows and columns which correspond to the mathematical implementation in Grassmann integrals of the Pauli principle. To illustrate the power of these formulas, we give a “three lines proof” that the radius of convergence of the Gross-Neveu theory with cutoff is independent of the number of colors, using either one or the other of these formulas.

I) Introduction

Perturbation theory for fermion systems is often said to converge, whether for boson systems it is said to diverge. But what does this mean exactly? Unnormalized Fermionic perturbation series with cutoffs are not only convergent but entire, whether Bosonic perturbation series with cutoffs have zero radius of convergence. But this is of little use since in actual computations, especially renormalization group computations, one needs to compute connected functions. It was not obvious until recently how to compute these connected functions in a way which preserves the algebraic cancellations of the Pauli principle to obtain convergent power series with accurate and easy estimates on the convergence radius, without using the heavy techniques of cluster and Mayer expansions.

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The first solution to this problem which does not require cluster and Mayer expansions is in [FMRT] where some inductive way of computing such Fermionic connected functions is given and applied to the uniform convergence of two-dimensional many fermion systems in a slice. However in the corresponding computation the expansions steps were intertwined with constructions of layers of some tree, and the final outcome was therefore not totally explicit yet.

In [AR1] we developed two explicit tree formulas which are especially suited for cluster and Mayer expansions. Shortly thereafter we made the remark that the second formula leads to a completely explicit version of the inductive cluster expansion used in [FMRT] and [FKLT]. More recently we realized that the first formula is also compatible with Gram’s inequality, and can therefore also be used in fermionic expansions. It is possible to develop a continuous renormalization group formalism for Fermions using this formula [DR], a problem which has attracted attention recently [S].

We also mention that an other (closely related) version of this kind of expansions has been introduced recently, using Gram’s inequality [FKT]. It relies on a resolvent expansion rather than an explicit tree formula.

II. Tree formulas and Grassmann integrals

In this section we recall the notations and the “Forest Formulas” of [AR1] and apply them to Grassmann (fermionic) integrals.

Let \( n \geq 1 \) be an integer, \( I_n = \{1, \ldots, n\} \), \( P_n = \{ \{i, j\} / i, j \in I_n, i \neq j \} \) (the set of unordered pairs in \( I_n \)). An element \( l \) of \( P_n \) will be called a link, a subset of \( P_n \), a graph. A graph \( \mathcal{F} = \{l_1, \ldots, l_\tau\} \) containing no loops, i.e. no subset \( \{\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_k, i_1\}\} \) with \( k \geq 3 \) elements, is called a forest.

A Forest formula is a Taylor formula with integral remainder, which expands a quantity such as \( \exp\left( \sum_{l \in P_n} u_l \right) \) to search for the explicit presence or absence of links \( u_l \). Taylor formulas with remainders in general are provided with a “stopping rule” and forests formulas stop at the level of connected sets. This means that two points which are joined by a link are treated as a single block. (More sophisticated formulas with higher stopping rules are useful for higher particle irreducibility analysis, or renormalization group computations but are no longer forests formulas in the strict sense [AR2]). Any such forest formula contains a “weakening factor” \( w \) for the links which remain underived. Many different such forests formulas exist, with different rules for \( w \). Two of them were identified as the most natural ones in [AR1], corresponding to two different logics: in one of them, the Brydges-Kennedy formula, the forest grows in the most symmetric and random way, and in the other, the “rooted formula” it grows layers by layers from a preferred root.

A forest is a union of disconnected trees \( \mathcal{T} \), the supports of which are disjoint subsets of \( I_n \) called the connected components or clusters of \( \mathcal{F} \).

The Brydges-Kennedy formula is then
Brydges-Kennedy Forest Formula

\[
\exp\left(\sum_{l \in \mathcal{P}_n} u_l \right) = \sum_{\mathcal{S} = \{i_1, \ldots, i_\tau\}, \text{u-forest}} \left( \prod_{\nu=1}^{\tau} \int_0^1 dw_{l_\nu} \right) \left( \prod_{\nu=1}^{\tau} u_{l_\nu} \right) \exp\left( \sum_{l \in \mathcal{P}_n} w_{l,\mathcal{S},BK}^{\mathcal{S},BK}(w), u_l \right),
\]

(II.1)

where the summation extends over all possible lengths \( \tau \) of \( \mathcal{S} \), including \( \tau = 0 \) hence the empty forest. To each link of \( \mathcal{S} \) is attached a variable of integration \( w_l \); and \( w_{\{ij\},BK}^{\mathcal{S},BK}(w) = \inf\{w_l, l \in L_{\mathcal{S}}\{ij\}\} \) where \( L_{\mathcal{S}}\{ij\} \) is the unique path in the forest \( \mathcal{S} \) connecting \( i \) to \( j \). If no such path exists, by convention \( w_{\{ij\},BK}^{\mathcal{S},BK}(w) = 0 \).

The rooted formula is absolutely identical, but with a different rule for the weakening factor \( w \), now called \( w_{\{ij\}}^{\mathcal{S},R}(w) \). It is a less symmetrical formula since we have to give a rule for choosing a root in each cluster. For each non empty subset or cluster \( C \) of \( I_n \), choose \( r_C \), for instance the least element in the natural ordering of \( I_n = \{1, \ldots, n\} \), to be the root of all the trees with support \( C \) that appear in the following expansion. Now if \( i \) is in some tree \( \mathcal{S} \) with support \( C \) we call the height of \( i \) the number of links in the unique path of the tree \( \mathcal{S} \) that goes from \( i \) to the root \( r_C \). We denote it by \( l^\mathcal{S}(i) \). The set of points \( i \) with a fixed height \( k \) is called the \( k \)-th layer of the tree. The Rooted Forest Formula is then:

Rooted Forest Formula

\[
\exp\left(\sum_{l \in \mathcal{P}_n} u_l \right) = \sum_{\mathcal{S} = \{i_1, \ldots, i_\tau\}, \text{u-forest}} \left( \prod_{\nu=1}^{\tau} \int_0^1 dw_{l_\nu} \right) \left( \prod_{\nu=1}^{\tau} u_{l_\nu} \right) \exp\left( \sum_{l \in \mathcal{P}_n} w_{l,\mathcal{S},R}^{\mathcal{S},R}(w), u_l \right),
\]

(II.2)

where the only difference with (II.1) lies in the definition of \( w_{\{ij\}}^{\mathcal{S},R}(w) \) which is different from the one of \( w_{\{ij\},BK}^{\mathcal{S},BK}(w) = 0 \). We define still \( w_{\{ij\}}^{\mathcal{S},R}(w) = 0 \) if \( i \) and \( j \) are not connected by the \( \mathcal{S} \). If \( i \) and \( j \) fall in the support \( C \) of the same tree \( \mathcal{S} \) of \( \mathcal{S} \) then

\[
\begin{align*}
w_{\{ij\}}^{\mathcal{S},R}(w) &= 0 & & \text{if } |l^\mathcal{S}(i) - l^\mathcal{S}(j)| \geq 2 \quad (i \text{ and } j \text{ in distant layers}) \\
w_{\{ij\}}^{\mathcal{S},R}(w) &= 1 & & \text{if } l^\mathcal{S}(i) = l^\mathcal{S}(j) \quad (i \text{ and } j \text{ in the same layer}) \\
w_{\{ij\}}^{\mathcal{S},R}(w) &= w_{\{ii'\}} & & \text{if } l^\mathcal{S}(i) - 1 = l^\mathcal{S}(j) = l^\mathcal{S}(i'), \text{ and } \{ii'\} \in \mathcal{S}. \quad (i \text{ and } j \text{ in neighboring layers, } i' \text{ is then unique and is called the ancestor of } i \text{ in } \mathcal{S}).
\end{align*}
\]

Proofs of these formulas are given in [AR1].

To any such algebraic formula, there is an associated “Taylor” tree formula, in which \( u_l \) is interpreted not as a real number but as a differential operator \( u_l = \frac{\partial}{\partial x_l} \) [AR1]. Let us recall these associated Taylor formulas. Let \( \mathcal{S} \) be the space of smooth functions from \( \mathbb{R}^{P_n} \) to an arbitrary Banach space \( \mathcal{V} \). An element of \( \mathbb{R}^{P_n} \) will be generally denoted by \( x = (x_l)_{l \in \mathcal{P}_n} \). The vector with all entries equal to 1 will be denoted by \( \mathbb{1} \). Applied to an element \( H \) of \( \mathcal{S} \), the Taylor Rooted Forest formula takes the form:
The Taylor forest formulas

\[
H(1) = \sum_{\text{u-forest}} \left( \prod_{l \in \delta} \int_0^1 dw_l \right) \left( \prod_{l \in \delta} \frac{\partial}{\partial x_l} \right) H \left( X^{BK} \text{ or } R(w) \right). \tag{II.3}
\]

Here \( X^{BK} \text{ or } R(w) \) is the vector \( (x_l)_{l \in P_n} \) of \( \mathbb{R}^n \) defined by \( x_l = w_l^{BK} \text{ or } R(w) \), which is the value at which we evaluate the derivative of \( H \). The symbol \( BK \text{ or } R \) means that the formula is true using either the Brydges-Kennedy or the rooted weakening factor \( w \).

Again the proof of (II.3) is in [AR1] (but we recall that it is a rather trivial consequence of (II.1-2) applied to \( u_l = \frac{\partial}{\partial x_l} \)).

Any forest is a union of connected trees. Therefore any forest formula has an associated tree formula for its connected components. And therefore, at least formally, any forest formula solves the problem of computing normalized correlation functions. Indeed applying the forest formula to the functional integral for the unnormalized functions, the connected functions are simply given by the connected pieces of the forest formula, hence by the corresponding tree formula. It is in this sense that forest formulas exactly solve the well known snag that makes connected functions difficult to compute. This snag is that since typically there are many trees in a graph, one does not know “which one to choose” when one tries to compute connected functions in the (desirable) form of tree sums. Any forest formula gives a particular answer to that problem. It tells us exactly by how much our pondered “tree choice” has “weakened” the remaining loop lines: just by the weakening factor \( w \).

To illustrate this by the simplest possible example, let us consider the second order connected graphs of the pressure for the \( \phi^4 \) theory in dimension 0 with propagator 1, which simply counts multiplicities of connected graphs. There are two second order connected graphs, pictured in Fig 1. To count contractions correctly it is convenient to label all fields or half lines. The first graph \( G_1 \) corresponds to \( 2.C_4^2 \cdot C_4^2 = 72 \) contractions, and has therefore weight 72. The second graph has weight \( 4! = 24 \). Now to select a tree in this case is to select the trivial trees \( T \) and there are \( 4^2 = 16 \) such trees. If the same measure was used for the contractions of the three loop lines, we would find a weight \( 16.C_3^2 \cdot C_3^2 = 144 \) for \( G_1 \), and a weight \( 16.3! = 96 \) for \( G_2 \). This illustrates the overcounting problem of choosing a tree, and we see that this overcounting varies with the graphs. Now
we see how the weakening of the loop lines corrects precisely this defect: with
the weakening parameters \( w^{BK} \) or \( w^R \), the graph \( G_1 \) is weakened by a factor
\( \int_0^1 wdw = 1/2 \) and the graph \( G_2 \) is weakened by a factor \( \int_0^1 w^3 dw = 1/4 \) so that
correct weights are recovered! The reader is invited to try higher order cases to
see the distinction between \( w^{BK} \) and \( w^R \) appear.

It remains to see which formulas can be used in the constructive sense
and for which theories. It is well known that the Brydges-Kennedy Formula
\([BK][AR1]\), when applied to interpolate a symmetric positive matrix, preserves
positivity. Indeed \( w_{ij}^{BK}(w) \) completed as a symmetric matrix by the definition
\( w_{ii}^{BK}(w) = 1 \forall i \) is a positive matrix, since it is a convex combination of block
matrices with 1 everywhere \([AR1]\). Therefore if \( C \) is a positive matrix (like a
bosonic covariance) the matrix \( w_{ij}^{BK}(w)C(x_i, x_j) \), being the Hadamard product
of two positive matrices, remains positive for any values of the tree parameters \( w \).
This means that it is suited to the construction of bosonic models (\([B][AR2]\) and
references therein). But this positivity property also allows Gram’s estimates for
fermionic theories:

Lemma

Let \( A = (a_{\alpha\beta})_{\alpha,\beta} \) be a Gram matrix: \( a_{\alpha\beta} = \langle f_\alpha, g_\beta \rangle \) for some inner
product \( \langle ., . \rangle \). Suppose each of the indices \( \alpha \) and \( \beta \) is of the form \((i, \sigma)\) where
the first index \( i \), \( 1 \leq i \leq n \), has the same range as the indices of the positive matrix
\( w_{ij}^{BK}(w) \), and \( \sigma \) runs through some other index set \( \Sigma \).

Let \( B = (b_{\alpha\beta})_{\alpha,\beta} \) be the matrix with entries \( b_{(i,\sigma)(j,\tau)} = w_{ij}^{BK}(w), \langle
f_{(i,\sigma)}, g_{(j,\tau)} \rangle \) and let \( (b_{\alpha\beta})_{\alpha,\beta \in B} \) be some square matrix extracted from \( B \), then
for any \( w \) we have the Gram inequality:

\[
|\det(b_{\alpha\beta})_{\alpha,\beta \in B}| \leq \prod_{\alpha \in A} ||f_\alpha|| \prod_{\beta \in B} ||g_\beta|| \ . \quad (II.4)
\]

Proof: Indeed we can take the symmetric square root \( v \) of the positive matrix
\( w_{ij}^{BK} \) so that \( w_{ij}^{BK} = \sum_{k=1}^n v_{ik}v_{kj} \). Let us denote the components of the vectors
\( f \) and \( g \), in an orthonormal basis for the scalar product \( \langle ., . \rangle \) with \( q \) elements,
by \( f_{(i,\sigma)}^m \) and \( g_{(j,\tau)}^m \), \( 1 \leq m \leq q \). (Indeed even if the initial Hilbert space is infinite
dimensional, the problem is obviously restricted to the finite dimensional subspace
generated by the finite set of vectors \( f \) and \( g \)). We then define the tensorized
vectors \( F_{(i,\sigma)} \) and \( G_{(j,\tau)} \) with components \( F_{(i,\sigma)}^k = v_{ik}f_{(i,\sigma)}^m \) and \( G_{(j,\tau)}^k = v_{jk}g_{(j,\tau)}^m \)
where \( 1 \leq k \leq n \) and \( 1 \leq m \leq q \). Now considering the tensor scalar product
\( \langle ., . \rangle_T \) we have

\[
\langle F_{(i,\sigma)}, G_{(j,\tau)} \rangle_T = \sum_{k=1}^n \sum_{m=1}^q v_{ik}v_{jk}f_{(i,\sigma)}^m g_{(j,\tau)}^m = b_{(i,\sigma)(j,\tau)} \ . \quad (II.5)
\]

By Gram’s inequality using the \( \langle ., . \rangle_T \) scalar product we get

\[
|\det(b_{\alpha\beta})_{\alpha \in A, \beta \in B}| \leq \prod_{\alpha \in A} ||f_\alpha||_T \prod_{\beta \in B} ||g_\beta||_T \quad (II.6)
\]
\[ \sum_{k=1}^{n} \sum_{m=1}^{q} (F_{km})^2 = \sum_{k=1}^{n} \sum_{m=1}^{q} v_{ik}^2 (f_{i})^2 = \sum_{i=1}^{n} \sum_{m=1}^{q} (f_{im})^2 = w_{ii} \sum_{m=1}^{q} (f_{i})^2, \]

since \( w_{ii} = 1 \) for any \( i, 1 \leq i \leq n \).

Let us now apply the Brydges-Kennedy formula to the computation of the connected functions of a Fermionic theory. The corresponding Grassmann integral is:

\[
\frac{1}{Z} \int d\mu \bar{C}(\psi, \bar{\psi}) P(\bar{\psi}_a, \psi_a) e^{S(\bar{\psi}_a, \psi_a)} \tag{II.7}
\]

where \( C \) is the covariance or propagator, \( P \) is a particular monomial (set of external fields) and \( S \) is some general action. We take as simplest example the massive Gross-Neveu model with cutoff, for which the action is local and quartic in a certain number of Fermionic fields. These fields are two-components because of the spin. In a finite box \( \Lambda \) the action is

\[
S_\Lambda = \frac{\lambda}{N} \int_\Lambda dx (\sum_a \bar{\psi}_a(x) \psi_a(x))^2 \tag{II.8}
\]

where \( a \) runs over some finite set of \( N \) “color” indices. \( \lambda \) is the coupling constant. The covariance \( C \) is massive and has an ultraviolet cutoff, hence in Fourier space it is for instance \( \eta(p)/(p^2 + m) \) where \( \eta \) is a cutoff function on large momenta. We only need to know that \( C \) is diagonal in color space \( C(x, a; y, a') = 0 \) for \( a \neq a' \), and that it can be decomposed as

\[
C(x, y) = \int_{\mathbb{R}^d} D(x, t) E(t, y) dt \tag{II.9a}
\]

with

\[
|C(x, y)| \leq K \frac{1}{(1 + |x - y|)^p} \tag{II.9b}
\]

\[
\int_{\mathbb{R}^d} |D(x, t)|^2 dt \leq K ; \int_{\mathbb{R}^d} |E(x, t)|^2 dt \leq K \tag{II.9c}
\]

for some constant \( K \).

This decomposition amounts roughly to defining square roots of the covariance in momentum space and prove their square integrability. It is usually easy for any reasonable cutoff model. For instance if \( \eta \) is a positive function, we can define \( D \) in Fourier space as \( \eta^{1/2}(p)/(p^2 + m)^{1/4} \) and \( E \) as \( (-p^2 + m)^{1/2}(p)/(p^2 + m^2)^{3/4} \).

It is the Fermionic covariance or propagator \( C = \langle \bar{\psi}_a(x) \bar{\psi}_a(y) \rangle \) which is interpolated with the forest formulas. We obtain, for instance for the pressure that the formal power series in the coupling constant is a sum over trees on \( \{1, \ldots, n\} \), with 1, a distinguished vertex sitting at the origin to break translation invariance (similar formulas with external fields of course exist for the connected functions).

**Fermionic Tree Expansion**

\[
p = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log Z(\Lambda) = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \left( \int d\mu \bar{C}(\psi, \bar{\psi}) e^{S_\Lambda(\bar{\psi}_a, \psi_a)} \right)
\]
\[
= \sum_{n=0}^{\infty} \frac{\lambda^n}{N^n n!} \sum_{a_1, \ldots, a_n, b_1, \ldots, b_n=1}^{N} \sum_{\Omega} \sum_{\xi} \sum_{\psi} \epsilon(\xi, \Omega) \prod_{l \in \xi} \int_0^1 dw_l \]
\[
\int_{\mathbb{R}^n} dx_1 \ldots dx_n \delta(x_1 = 0) \prod_{l \in \xi} (C(x_{i(l)}, x_{j(l)}) \delta_l) \times \det(b_{\alpha\beta})_{\alpha, \beta \in B} . \tag{II.10}
\]

The sum over the \(a_i\)'s and \(b_i\)'s are over the colors of the fields and antifields of the vertices obtained by expanding the interaction and of the form:
\[
\tilde{\psi}_{a_i}(x_i) \psi_{a_i}(x_i) \bar{\psi}_{b_i}(x_i) \psi_{b_i}(x_i) \tag{II.11}
\]
with \(1 \leq i \leq n\). The sum over \(\xi\) is over all trees which connect together the \(n\) vertices at \(x_1, \ldots, x_n\). The sum over \(\Omega\) is over the compatible ways of realising the bonds \(l = \{i, j\} \in \xi\) as contractions of a \(\psi\) and \(\bar{\psi}\) between the vertices \(i\) and \(j\) (compatible means that we do not contract twice the same field or antifield). A priori there are 8 choices of contraction for each \(l = \{i, j\}\), including the choice in direction for the arrow. \(\epsilon(\xi, \Omega)\) is a sign we will express later. For any \(l \in \xi\), \(i(l) \in \{1, \ldots, n\}\) labels the vertex where the field, contracted by the procedure \(\Omega\) concerning the link \(l\), was chosen. Likewise \(j(l)\) is the label for the vertex containing the contracted antifield. \(\delta_l\) is 1 if the colors (among \(a_1, \ldots, a_n, b_1, \ldots, b_n\)) of the field and antifield contracted by \(l\) are the same and else is 0. Finally the matrix \((b_{\alpha\beta})_{\alpha, \beta}\) is defined in the following manner.

The row indices \(\alpha\) label the \(2n\) fields produced by the \(n\) vertices, so that \(\alpha = (i, \sigma)\) with \(1 \leq i \leq n\) and \(\sigma\) takes two values 1 or 2 to indicate whether the field is the second or the fourth factor in (II.11) respectively. The column indices \(\beta\) label in the same way the \(2n\) antifields, so that \(\beta = (j, \tau)\) with \(1 \leq j \leq n\) and \(\tau = 1\) or 2 according to whether the antifield is the first or the third factor in (II.11) respectively. The \(\alpha\)'s and \(\beta\)'s are ordered lexicographically. We denote by \(c(i, \sigma)\) the color of the field labeled by \((i, \sigma)\) that is \(a_i\), if \(\sigma = 1\), and \(b_i\) if \(\sigma = 2\). We introduce the similar notation \(\bar{c}(j, \tau)\) for the color of an antifield. Now
\[
b_{(i, \sigma)(j, \tau)} = w^{\xi, BK}_{ij}(w) C(x_i, x_j) \delta(c(i, \sigma), \bar{c}(j, \tau)) . \tag{II.12}
\]

Finally each time a field \((i, \sigma)\) is contracted by \(\Omega\) the corresponding row is deleted from the \(2n \times 2n\) matrix \((b_{\alpha\beta})\). Likewise, for any contracted antifield the corresponding column is erased. \(A\) and \(B\) denote respectively the set of remaining rows and the set of remaining columns. The minor determinant featuring in formula (II.10) is now \(\det(b_{\alpha\beta})_{\alpha \in A, \beta \in B}\) which is \((n+1) \times (n+1)\). Indeed for each of the \(n - 1\) links of \(\xi\), a row and a column are erased.

Although it is not important for the bounds, we indicate the rule for computing the sign \(\epsilon(\xi, \Omega)\). Let \((1, 1) \leq \alpha_1 < \cdots < \alpha_{n-1} \leq (n, 2)\) and \((1, 1) \leq \beta_1 < \cdots < \beta_{n-1} \leq (n, 2)\) be the erased rows and columns respectively. Suppose that \(\alpha_{\pi_{r}}\) was contracted by \(\xi\) and \(\Omega\) together with \(\beta_{\pi_{r}}\), for \(1 \leq r \leq n - 1\). Clearly \(\pi\) is a permutation of \(\{1, \ldots, n - 1\}\). Then
\[
\epsilon(\xi, \Omega) = (-1)^{\sigma(\alpha_1) + \cdots + \sigma(\alpha_{n-1}) + \tau(\beta_1) + \cdots + \tau(\beta_{n-1})} \epsilon(\pi) \tag{II.13}
\]
where \(\epsilon(\pi)\) is the signature of \(\pi\) and for any \(\alpha = (i, \sigma)\) we introduced the notation \(i(\alpha) = i\) and \(\sigma(\alpha) = \sigma\) and similarly for any \(\beta = (j, \tau)\), we write \(j(\beta) = j\) and \(\tau(\beta) = \tau\).
Proof: The proof of the previous expansion although a bit tedious is straightforward. First expand \( S_A(\bar{\psi},\psi) \). Each term will involve a Berezin integral

\[
\int d\mu_C(\psi,\bar{\psi}) \prod_{i=1}^{n} (\bar{\psi}_{a_i}(x_i)\psi_{a_i}(x_i)\bar{\psi}_{b_i}(x_i)\psi_{b_i}(x_i))
\]  

(II.14)

which is a \( 2n \times 2n \) determinant like the one of the matrix \((b_{\alpha\beta})\). Then in order to use (II.3) we introduce for any pair \( \{i,j\} \in P_n \) a weakening factor \( x_{\{i,j\}} \) multiplying each of the eight entries involving both vertices \( i \) and \( j \). The output of (II.3) is a first formula when applied to \( Z(\Lambda) \). However the amplitudes corresponding to each connected tree in the forest factorize (actually, to check that, one needs to be careful and compute \( \epsilon(\Xi,\Omega) \)). Taking the sum over trees instead of forests simply computes \( \log Z(\Lambda) \) instead of \( Z(\Lambda) \). Finally \( \delta(x_1=0) \) accounts for the division by the volume \( |\Lambda| \) to get the pressure as an intensive quantity. \( \Box \)

Let us now use this fermionic tree formulas for proofs of convergence.

III) Convergence of the tree formulas

A typical constructive result for this Gross-Neveu model with cutoff is to prove:

Theorem

The pressure and the connected functions of the cut-off Gross-Neveu model are analytic in \( \lambda \) in a disk of radius \( R \) independent of \( N \).

Proof: The determinant \( \text{det}(b_{\alpha\beta})_{\alpha\in A,\beta\in B} \) is diagonal in the colors of the involved fields and antifields. Each of the block determinants falls in the category described by the lemma, and thus the complete determinant is bounded by \( K^{n+1} \) thanks to the decomposition (II.9a). The spatial integrals are bounded using (II.9b) for the propagators corresponding to the links in \( \Xi \), by \( K^{-1} \). The sum over \( \Omega \) is bounded by \( 8^{n-1} \) (the number of compatible contractions is actually equal to \( 8 \times 12^n \times 10^3 \times 6^2 \) where \( n_p \) is the number of vertices of the tree \( \Xi \) with coordination number \( p \)). The sum over colors is bounded by \( N^{n+1} \), indeed once \( \Xi \) and \( \Omega \) are known, the circulation of color indices is determined. The attribution of color indices costs \( N^2 \) at the first vertex and by induction a factor \( N \) for each of the remaining vertices of the tree. Indeed climbing inductively into the tree layer by layer, at every vertex there is one color already fixed by the line joining the vertex to the root, hence one remaining color to fix, except for the root, for which two colors have to be fixed. The number of \( \Xi \)'s is bounded by \( n^{n-2} \), by Cayley’s theorem. Finally the \( n \)-th term of the series is bounded by \( (|\lambda|^n/N^n n!)K^{n+1}K^{-1}8^{n-1}N^{n+1}n^{n-2} \) and the radius of convergence is thus at least \( 1/8eK^2 \). \( \Box \)

This is perhaps the shortest and most transparent proof of constructive theory yet! Remark in particular that it does not require any discretization of space, lattice of cubes, cluster or Mayer expansion, all features which are necessary for bosonic theories. This “three lines” treatment of the theory with cutoff can
presumably be extended into a “three pages” treatment of for instance the Gross-Neveu theory with renormalization in two dimensions [DR]. Such a treatment with no discretization of space, no discrete slicing of momenta, and continuous instead of discrete renormalization group equations makes the constructive version of these fermionic theories almost identical to their perturbative version (and no longer more difficult).

The theorem above is interesting not only for the analysis of the Gross-Neveu model but also for that of the two-dimensional interacting Fermions considered in [FMRT] or [FKLT]. In this latter case, the “colors” correspond to angular sectors on the Fermi sphere and the factor $1/N$ in the coupling is provided by power counting.

We now give a second and longer proof of the theorem using the rooted formula. This is worth the trouble since the weakening factor in the rooted formula completely factorizes out of the determinant. This second formula may therefore be useful in problems for which Gram’s inequality is not applicable and the method of “comparison of rows and columns” of [IM] or [FMRT] has to be used.

We return to formula (II.10) but use the rooted weakening factor. To bound the determinant, we need to introduce simply one further subtle modification. We need a further sum which decides for each field or anti field whether it contracts or not to the level in the tree immediately below it.

The only change in formula (II.10) is that the loop determinant has entries

$$b_{\alpha\beta} = u_w^{\mathcal{T},R}(w)C(x_{i(\alpha)},x_{j(\beta)})\delta(c(\alpha),\bar{c}(\beta))$$ (III.1)

with $\alpha \in A$, $\beta \in B$. Any of these entries we write as:

$$b_{\alpha\beta} = \sum_{u(\alpha)\in\{♯,♭,♮\}} \sum_{v(\beta)\in\{♯,♭,♮\}} b_{\alpha\beta}^{u(\alpha)v(\beta)}$$ (III.2)

with the convention that

$$b_{\alpha\beta}^{♭♭} = b_{\alpha\beta}\delta(l_{\mathcal{T}}(i(\alpha)),1,l_{\mathcal{T}}(j(\beta)))$$ (III.3a)

$$b_{\alpha\beta}^{♭♯} = b_{\alpha\beta}\delta(l_{\mathcal{T}}(i(\alpha)),1,l_{\mathcal{T}}(j(\beta)))$$ (III.3b)

$$b_{\alpha\beta}^{♯♭} = b_{\alpha\beta}\delta(l_{\mathcal{T}}(i(\alpha))-1,l_{\mathcal{T}}(j(\beta)))$$ (III.3c)

and $b_{\alpha\beta}^{u(\alpha)v(\beta)} = 0$ whenever $(u(\alpha),v(\beta)) = (♯,♭),(♭,♭),(♭,♯),(♯,♭),(♭,♭), (♭,♯)$ or $(♯,♯)$.

Then we expand the determinant $det(b_{\alpha\beta})_{\alpha\in A,\beta\in B}$ by multilinearity with respect to rows, having for any $\alpha \in A$, a sum over $u(\alpha) \in \{♯,♭,♮\}$. Then we do the same thing for columns and we get a sum over $v(\beta) \in \{♯,♭,♮\}$, for any $\beta \in B$. We then obtain an expansion into $3^{2n+2}$ determinants, of the form $det(b_{\alpha\beta})_{\alpha\in A,\beta\in B}$. Let $f = \#\{\alpha \in A, u(\alpha) = ♭\}$ and $\tilde{f} = \#\{\beta \in B, v(\beta) = ♭\}$ be the number of flat fields and antifields. The number of sharp fields and antifields $s$, $\tilde{s}$, and of natural ones $n$, $\tilde{n}$ are defined similarly. For one of the determinants to be nonzero we must have $n = \tilde{n}$, $f = \tilde{s}$ and $s = \tilde{f}$. We can also group together
fields and antifields according to their level in the tree $\mathcal{T}$, so that we get a block diagonal determinant that can be factorized as:

$$
det(b^{w}_{\alpha\beta}v^{(\beta)})_{\alpha\in A,\beta\in B} = \pm \prod_{k=0}^{k_{max}} det(b^{\beta}_{\alpha\beta})_{\alpha\in A^1_k,\beta\in B^2_k}$$

$$
\times \prod_{k=0}^{k_{max}-1} det(b^{\beta}_{\alpha\beta})_{\alpha\in A^1_k,\beta\in B^2_{k+1}} \prod_{k=1}^{k_{max}} det(b^{\beta}_{\alpha\beta})_{\alpha\in A^1_k,\beta\in B^2_{k-1}} \tag{III.4}
$$

where $k_{max}$ is the maximal occupied level of the tree $\mathcal{T}$. $A^1_k$ is the number of $\alpha$’s in $A$ with $l_T(i(\alpha)) = k$ and $u(\alpha) = i$. $A^1_k$, $A^2_k$, $B^1_k$, $B^2_k$ and $B^3_k$ are defined in the same manner. We must have of course $\#(A^1_k) = \#(B^2_k)$, $\#(A^2_k) = \#(B^3_k)$, and $\#(A^2_k) = \#(B^3_{k-1})$ for any $k$, in order for the full determinant to be nonzero. Now by definition of $w_{ij}^T,R(w)$ we have

$$
det(b^{\beta}_{\alpha\beta})_{\alpha\in A^1_k,\beta\in B^2_k} = det\left( C(x_i(\alpha), x_j(\beta))\delta(c(\alpha),c(\beta)) \right)_{\alpha\in A^1_k,\beta\in B^2_k} \tag{III.5}
$$

Likewise

$$
det(b^{\beta}_{\alpha\beta})_{\alpha\in A^1_k,\beta\in B^2_{k+1}} = \prod_{\beta\in B^2_{k+1}} w_{j(\beta)j'(\beta)} \times det\left( C(x_i(\alpha), x_j(\beta))\delta(c(\alpha),c(\beta)) \right)_{\alpha\in A^1_k,\beta\in B^2_{k+1}} \tag{III.6}
$$

where $j'(\beta)$ denotes the ancestor of $j(\beta)$ in the rooted tree $\mathcal{T}$. Finally

$$
det(b^{\beta}_{\alpha\beta})_{\alpha\in A^1_k,\beta\in B^2_{k-1}} = \prod_{\alpha\in A^1_k} w_{i(\alpha)i'(\alpha)} \times det\left( C(x_i(\alpha), x_j(\beta))\delta(c(\alpha),c(\beta)) \right)_{\alpha\in A^1_k,\beta\in B^2_{k-1}} \tag{III.7}
$$

We see that the $w$’s factor out and are bounded by 1. The remaining determinants again can be factorized according to the colors $c((\alpha)$, and $c(\beta)$. Each determinant obtained that way is of the form $det(C(x_i(\alpha), x_j(\beta)))_{\alpha\beta}$.

At this stage we can complete the proof of the Theorem either using Gram’s inequality or the method of comparisons of rows and columns to bound these determinants.

Gram’s inequality is the fastest road. We use (II.9) to bound every $p$ by $p$ determinant by $K^p$. And the complete loop determinant is then bounded by

$$
|det(b_{\alpha\beta})_{\alpha\in A,\beta\in B}| \leq 3^{2n+2}K^{n+1} \tag{III.8}
$$

The end of the proof is as in the case of the Brydges-Kennedy interpolation. □
It is also clear on the form (III.3) that the method of comparisons of rows and columns of [IM], [FMRT] also works here. This method roughly corresponds to Taylor expanding around a middle point further and further when fields or antifields accumulate in any given cube of unit size of a lattice covering $\mathbb{R}^d$.

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