A GENERALIZATION THEOREM OF KATZ AND MOTIVIC INTEGRATION

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INTRODUCTION

In what follows, we are interested in an extension of a theorem of Nicholas Katz, which will be useful in studying the cohomology of generalized arc spaces develop by Hans Schoutens in [5] and [6]. As is well known, one is typically interested in the motivic volume of a definable subset of $X \times X \times \mathbb{Z}^n$ where $X$ is a scheme over $k((t))$ and $X$ the special fiber of $X$, cf., [2]. Schoutens has introduced the possibility of developing a motivic integration for limit points other than $k[[t]]$. In this note, we are concerned with a special type of limit point $k[[T]]$ where $T = (t_i)_{i \in \mathbb{N}}$.

1. TWO LEMMAS

We start with a few lemmas from commutative algebra which we will need.

1.1. Lemma. Let $R := k[[x_1, x_2, \ldots, x_n, \ldots]]$ be the $m$-adic completion of the polynomial ring $k[x_1, x_2, \ldots, x_n, \ldots]$ along the maximal ideal $m = (x_1, x_2, \ldots, x_n, \ldots)$. For all $n \in \mathbb{N}$, let $R_n := k[[x_1, \ldots, x_n]] \cong R/(x_{n+1}, x_{n+2}, \ldots)$ and let $R_n \to R_{n-1}$ be the homomorphism with kernel $(x_n)R_n$. Then there is an isomorphism

$$R \cong \lim_{\longrightarrow} R_n$$

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Moreover, $R$ is a local ring with maximal ideal $mR$.

Proof. It is straightforward to verify\textsuperscript{1} that $mR$ is the maximal ideal of $R$. For the other claim, we define a homomorphism from $R$ to $\lim_{\leftarrow n} R_n$ by $x_i \mapsto (y_j)_{j \in \mathbb{N}}$ where $y_j = 0$ if $j < i$ and $y_j = x_i$ if $i \geq j$. By definition of inverse limit, this map is injective. By the universal property of inverse limits, we conclude that it is surjective. \hfill $\square$

1.2. Remark. Note that this isomorphism takes place in the category of $k$-algebras and not the category of topological $k$-algebras.

Below we will state a version of Nakayama’s Lemma which will be important for our work below

1.3. Lemma. Let $R$ be any local ring (or, in particular, the one above), and let $M$ be a finitely generated $R$-module. Then there is a surjective homomorphism of $R$-modules

$$M/mM \otimes_{\mathbb{Z}} R \to M$$

Proof. This is a special case of Proposition 2.6 of [1] \hfill $\square$

2. The theorem

From now on, we assume that $k$ is of characteristic zero. What follows is a natural extension, mutatis mutadis, of a theorem contained in a paper of Katz, cf., Proposition 8.9 of [3]. The original argument is originally due to Cartier, whereas my contribution is to show that it works with an inverse system.

2.1. Theorem. Let $M$ be finite $R$-module with a connection $\nabla$ arising from the continuous $k$-derivations coming from $R$ to $M$. Then $M^\nabla$ is finitely generated and

$$M \cong M^\nabla \otimes_k R$$

Proof. For all $i \in \mathbb{N}$ we define

$$D_i = \nabla\left(\frac{\partial}{\partial x_i}\right)$$

and for each $j \in \mathbb{N}$ we define

$$D_i^{(j)} = \frac{1}{j!}(\nabla\left(\frac{\partial}{\partial x_i}\right))^j$$

For any $n$ and any $n$-tuple $J_n = (j_1, \ldots, j_n) \subset \mathbb{N}^n$, we define the following\textsuperscript{1}

\textsuperscript{1}You can do this by verifying that $m$ is the additive subgroup of non-units
\[ D^J_n = \prod_{i=1}^n D_{i}^{j_i} \quad x^J_n = \prod_{i=1}^n x_i^{j_i} \quad (-1)^J_n = \prod_{i=1}^n (-1)^{j_i} \]

Then, for each \( n \in \mathbb{N} \) we successfully define an (additive) endomorphism \( P_n \) by

\[ P_n : M \to M, \quad P_n = \sum_{J_n} (-1)^J_n x^J_n D^J_n \]

Now the action of \( R \) on \( M \) is actually the inverse limit homomorphisms \( \rho_n : R_n \to M \) of \( k \)-modules. In fact, \( P_n \) will be considered an additive endomorphism of \( M \) as an \( R_n \)-module (via the isomorphism established in Proposition 1). We define \( P : M \to M \) to be the inverse limit

\[ P = \lim_{\leftarrow} P_n \]

More explicitly, consider \( f \in R \), which by Proposition 1, can be identified with a sequence \((f_n)_{n \in \mathbb{N}} \) where \( f_n \in R_n \), then

\[ P(fm) = (P_n(f_n))P(m) \]

It is straightforward, that \( P_n(f_n) = f_n(0) \forall n \in \mathbb{N} \), from which it follows that for all \( f \in R \) and all \( m \in M \)

\[ P(fm) = f(0)P(m) \]

Therefore, the kernel of \( P \) contains \( mM \) where \( m \) is the maximal ideal of \( R \).

As we will now pass to the quotient \( M/mM \), we mention that it is not hard to see that the inverse system defined in Proposition 1 and hence above satisfies the Mittag Leffler Condition. Therefore, there is an isomorphism

\[ M/mM \cong \lim_{\leftarrow} M/(x_1, \ldots, x_n)M \]

Note that, for all \( n \), \( P_n \) induces the identity on \( M/(x_1, \ldots, x_n)M \), and so \( P \) induces the identity on \( M/mM \)− i.e.,

\[ P(m) \equiv m \mod m \]

Therefore, the kernel of \( P \) is \( m \). In a similar fashion it is easy to check that \( P \) as the following properties

\[ P|_{M^\nabla} = id_{M^\nabla} \quad P(M) \subset M^\nabla \quad P^2 = P \]

Therefore, \( P \) induces an isomorphism vector spaces over \( k \)

\[ M/mM \cong M^\nabla \]
Therefore, $M^\nabla$ is a finite $R$-module. Using Nakayama’s Lemma (see Proposition 2 above), we have a surjective map

$$M^\nabla \otimes_k R \to M$$

Now, we will show that it is an isomorphism. Let $m_1, \ldots, m_l$ be $k$-linearly independent elements of $M^\nabla$ and let $f_1, \ldots, f_l$ be elements of $R$, we need to show

$$\sum_{k=1}^l f_k m_k \neq 0$$

In other words, writing $f_k$ as its corresponding sequence $(f_n^{(k)})$ in the inverse system, we need to show that for sufficiently large $n$

$$\sum_{k=1}^l f_n^{(k)} m_k \neq 0 \quad (*)$$

The only reason we to specify that $n$ be sufficiently large is to insure that there is an $n$ so that $f_n^{(k)} \neq 0$ for some $k$, which is clearly satisfied or else there is nothing to prove. Thus, we may assume there exists an $N$ such that for all $n > N$

$$f_n^{(1)} \neq 0$$

Then for all $n \geq N$ there exists a tuple $J_n = (j_1, \ldots, j_n)$ such that

$$\Pi_{\nu=1}^n \frac{1}{j_\nu!} \left( \frac{\partial}{\partial x_\nu} \right)^{j_\nu} (f_n^{(1)}(0)) \neq 0$$

Now, assume for the sake of contradiction that

$$\sum_{k=1}^l f_n^{(k)} m_k = 0$$

Applying $D^{J_n}$ to this equation, we get

$$0 = D^{J_n} \left( \sum_{k=1}^l f_n^{(k)} m_k \right) = \sum_{k=1}^l \Pi_{\nu=1}^n \frac{1}{j_\nu!} \left( \frac{\partial}{\partial x_\nu} \right)^{j_\nu} (f_n^{(i)}) m_k$$

This is a sum of the form

$$\sum_{k=1}^l g_k m_k = 0, \quad g_1(0) \neq 0, \quad g_k \in R_n$$

Applying $P$ to this sum, we obtain

$$\sum_{k=1}^l g_k(0) m_k = 0$$
which is impossible as \( g_1(0) \neq 0 \) and the \( m_1, \ldots, m_l \) are a \( k \)-linearly independent set. Therefore, this must be an isomorphism. \( \square \)

3. Application of result

To apply the above theorem, we take \( M = H_{DR}(X/S) \) to be finite sheaf of modules on \( S \), which is assured to us when we take \( f : X \to S \) to locally of finite type. We can define arc spaces by a universal property: we say that \( T \to X \) is the arc space of \( X \) along a scheme \( Z \), working in the category of \( k \)-schemes, if for every closed fat point \( \eta \) of \( T \) we have a unique morphism \( \eta \times_k Z \to X \) making the following diagram commute

\[
\begin{array}{ccc}
\eta & \xrightarrow{\cong} & \eta \times_k Z \\
\downarrow & & \downarrow \\
T & \to & X
\end{array}
\]

and which is unique in the sense that if \( T' \to X \) is any other such space, we have a unique map \( T' \to T \). When such a scheme exists, we write \( A_Z X \) for the arc space of \( X \) along \( Z \). This is a generalization of the notion of arc space found in \([6]\).

Using this description of \( A_Z \Spec(k) = Z \) to conclude that \( A_Z X \) is a scheme over \( Z \). In particular, if \( x \) is a limit point (the direct limit of an infinite sequence of fat points), then we have have the following relation

\[
H_{DR}(A_x X/x) \cong H_{DR}(X/k)
\]

when \( \nabla_x X \to x \) is smooth. This last condition implies, for suitable point systems, that \( X \) is rationally \( x \)-laxly stable – cf., \([4]\). Therefore, we would expect a further decomposition of \( H_{DR}(A_x X/x) \) which is captured motivically by the rational motivic measure as displayed loc. cit.

References

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