NEW MODULAR REPRESENTATIONS AND FUSION ALGEBRAS FROM QUANTIZED \( SL(2,R) \) CHERN-SIMONS THEORIES

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ABSTRACT

We consider the quantum-mechanical algebra of observables generated by canonical quantization of \( SL(2,R) \) Chern-Simons theory with rational charge on a space manifold with torus topology. We produce modular representations generalizing the representations associated to the \( SU(2) \) WZW models and we exhibit the explicit polynomial representations of the corresponding fusion algebras. The relation to Kac-Wakimoto characters of highest weight \( \hat{sl}(2) \) representations with rational level is illustrated.
1. INTRODUCTION

Three-dimensional topological Chern-Simons gauge theory with gauge group $G$ is described by the following action [1]:

$$ S = \frac{k}{2\pi} \int_{M_3} \eta_{ab} A^a dA^b + \frac{2}{3} f_{abc} A^a A^b A^c $$

where $f_{abc}$ are the structure constants of the Lie algebra of $G$, $\eta_{ab}$ is the invariant Killing metric on it, and $A^a$ are the gauge field 1-forms.

Chern-Simons theory with $G$ compact was solved non-perturbatively by means of “holomorphic” canonical quantization methods which uncovered its relation to the two-dimensional Wess-Zumino-Witten model on the group manifold $G$ [2]-[4]. In this talk we will discuss the much less well understood Chern-Simons theory with non-compact gauge group $G = SL(2,\mathbb{R})$ [5]-[8]. Since the meaning of the $SL(2,\mathbb{R})$ WZW model as a conformal quantum field theory is far from clear [9], the relation of $SL(2,\mathbb{R})$ Chern-Simons theory to two-dimensional conformal field theory, if it exists, must be of a novel type. In what follows we will describe various features (such as modular properties and fusion algebras) of the yet unknown two-dimensional counterpart of $SL(2,\mathbb{R})$ Chern-Simons theory.

Let us first briefly review the main aspects of the holomorphic canonical quantization of the action (1) to point out why it does not extend to theories with non-compact gauge groups. In the Hamiltonian formalism, the three-dimensional space-time manifold $M_3$ is the product $\Sigma \times \mathbb{R}^1$ of a two-dimensional compact surface $\Sigma$ and of the time axis $\mathbb{R}^1$. Going to the $A_0 = 0$ gauge, one obtains a free gauge-fixed action

$$ S = \frac{k}{2\pi} \int dt \int_{\Sigma} \epsilon^{ij} \eta_{ab} A^a_i \dot{A}^b_j d^2x $$

where $\epsilon^{ij}$ is the anti-symmetric tensor on the two-dimensional space manifold $\Sigma$. The constraint

$$ \epsilon^{ij} F^a_{ij} = 0 $$

associated to the gauge-fixing encodes the non-linearity of the theory. The “Gauss law” (3) states that the classical physical phase space $\mathcal{M}$ is the space of flat $G$-connections on the two-dimensional surface $\Sigma$.

In the holomorphic quantization, one selects a complex structure on $\Sigma$ which determines a complex structure on the space of $G$-connections on $\Sigma$. States, in the “quantize-first” approach, are then described by holomorphic wave-functionals $\Psi(A^a_z)$ which depend
only on the holomorphic components of the gauge field 1-forms \( A^a = A^a_z dz + A^a_{\bar{z}} d\bar{z} \) and which are normalizable in the scalar product

\[
<\Psi_1, \Psi_2> = \int [DA_z DA_{\bar{z}}] e^{-\frac{k}{\pi} \int_\Sigma \eta_{ab} A^a_z A^b_{\bar{z}} \Psi_1(A_{\bar{z}}) \Psi_2(A_z)}.
\]  

(4)

Physical states \( \Psi(A^a_z) \) are normalizable solutions of the functional equation which is the quantum version of the “Gauss law” constraint (3):

\[
D_{ab}^z \frac{\delta}{\delta A^b_z} \Psi = \frac{k}{2\pi} \bar{\partial} A^a_z \Psi.
\]

(5)

Eqs.(5) are identical to the Ward Identities for the generating functional of current correlators of the two-dimensional WZW model on the Riemann surface \( \Sigma \). This shows that for \( G \) compact, the vector space of physical states of the Chern-Simons theory (4) is isomorphic to the space of current blocks of the two-dimensional \( \hat{G} \) current algebra. The central task of canonical quantization of Chern-Simons theory is to prove that the unitary structure (4) on the vector space of quantum states of the Chern-Simons theory is in fact the same as the natural unitary structure on the current blocks of the WZW theory for which modular transformations are represented by unitary matrices. For \( G \) compact, this has been proven explicitly for \( \Sigma \) of genus zero and one [4],[10].

When \( G \) is non-compact, \( \eta_{ab} \) is not positive-definite, and the scalar product (4) is not, even formally, well defined. In this case, the holomorphic gauge-invariant polarization \( \Psi(A^a_z) \) does not define a genuine positive-definite Kähler structure on the space of \( G \) connections on \( \Sigma \). If \( G \) is a non-compact but complex group, one can find a family of real gauge-invariant polarizations which is preserved by the reparametrizations of \( \Sigma \) [11]. The existence of a reparametrization invariant family of gauge-invariant polarizations is the essential condition that makes it possible to discuss topological invariance at quantum level.

When \( G = SL(2, R) \) a reparametrization invariant family of gauge-invariant (positive-definite) polarizations does not exist [11]. This is the main difficulty in quantizing the theory in a topological invariant way, or, equivalently, in proving that the mapping class group is implemented unitarily on the Hilbert space of quantum states. The difficulty is analogous to the one that is met when quantizing the Heisenberg algebra \([x_\mu, p_\nu] = i\eta_{\mu\nu}\) with a Lorentzian type of metric \( \eta_{\mu\nu} \). Choosing a real polarization, one obtains states represented by wave-functions \( \psi(x_\mu) \), and \( p_\mu = -i\partial_\mu \). This gives a perfectly unitary (but
not highest-weight) representation of the Heisenberg algebra. This representation is, however, not equivalent to the (heighest-weight but not positive-definite) Fock representation defined by the creation and annihilation operators $a_μ^+ = x_μ + ip_μ, a_μ = x_μ - ip_μ$. The point is that though dependence on the polarization is a ‘fact of life’ of the quantization process, it potentially jeopardizes quantum topological invariance of the Chern-Simons theory.

Because of this fundamental difficulty which affects any attempt to “first-quantize” the full space of two-dimensional $SL(2, R)$ connections on $Σ$ and to impose the “Gauss-law” (3) as an operatorial constraint on the physical states, we will work in the so-called “constraint-first” approach [2],[10] in which one quantizes directly the physical classical phase space $M$. Since $M$ is finite-dimensional, the canonical quantization problem actually has a finite number of degrees of freedom. However, the topology of $M$ is, for a generic $Σ$, quite intricate. For this reason, we will restrict ourselves to the case when $Σ$ has the torus topology; such a limitation has been sufficient to unravel the underlying two-dimensional current algebra structure in the case of the compact gauge group $SU(2)$ [10].

When $Σ$ is a torus, the problem of quantizing $M$ is reduced to the problem of quantizing the moduli space of flat-connections of an abelian gauge group [4]. This makes the computation for genus one drastically simpler than for higher genus, where non-abelian Chern-Simons theory appears to be vastly more complex than abelian. On the other hand, the factorization properties of 2-dimensional conformal field theories suggest that the torus topology already contains most, if not all, of the complexities of higher genus. The solution of this apparent paradox is that $M$ for a torus is almost the space of flat connections of an abelian group, but not quite: it is the space of abelian flat connections modulo the action of a discrete group whose fixed points give rise to orbifold singularities. It is only here that the quantization of non-abelian Chern-Simons theory with compact gauge group for genus one differs from the computationally trivial abelian case. The projection associated to such a discrete group is responsible for the emergence of a non-abelian structure for the $SL(2, R)$ Chern-Simons theory as well.
2. QUANTIZATION OF $\mathcal{M}$

The difficulties of the “quantize-first” approach for $SL(2, R)$ gauge group have their counterpart in the “constrain-first” method in the singular geometry of $\mathcal{M}$. Corresponding to the three types of inequivalent Cartan subgroups of $SL(2, R)$, there exist three different “branches” of $\mathcal{M}$ which have a non-vanishing intersection: $\mathcal{M} = \bigcup_{i=1,2,3} \mathcal{M}_i$, with $i = 1, 2, 3$. It is interesting to notice that similarly “branched” phase spaces appear also in the context of two-dimensional topological theories based on the gauged $SL(2, R)$ WZW model and related to solvable string theories [12].

Flat $SL(2, R)$ connections on a torus correspond to pairs $(g_1, g_2)$ of commuting $SL(2, R)$ elements modulo overall conjugation in $SL(2, R)$. The elements $g_1$ and $g_2$ represent the holonomies of the flat connections around the two non-trivial cycles of the torus.

The $\mathcal{M}_1$ branch is made of flat connections whose holonomies can be simultaneously brought by conjugation into the compact $U(1)$ subgroup of $SL(2, R)$. Therefore, $\mathcal{M}_1 \approx T^{(1)}$, the two dimensional torus.

$\mathcal{M}_2$ is the branch of flat connections whose holonomies, when represented by $2 \times 2$ real unimodular matrices, can be conjugated into a diagonal form: i.e. $g_i = \begin{pmatrix} \pm e^{x_i} & 0 \\ 0 & \pm e^{-x_i} \end{pmatrix}$, with $i = 1, 2$. However, one can still conjugate diagonal holonomies by an element of the gauge group which permutes the eigenvalues, mapping $x_i$ onto $-x_i$. Therefore, $\mathcal{M}_2$ consists of four copies of $R^{(2)}/Z_2$ whose origins are attached to the four points of $\mathcal{M}_1$ which correspond to flat connections with holonomies in the center of the gauge group $SL(2, R)$.

Finally, $\mathcal{M}_3$ is the branch of flat connections with holonomies which can be conjugated into an upper triangular form with units on the diagonal. Conjugation allows one to rescale the (non-vanishing) elements in the upper right corner by an arbitrary positive number. Thus, $\mathcal{M}_3 \approx S^1$, the real circle. Being odd-dimensional, $S^1$ cannot be a genuine non-degenerate symplectic space. In fact, when pushed down to $\mathcal{M}_3$, the symplectic form on the space of flat connections coming from the Chern-Simons action vanishes identically. $\mathcal{M}_3$ represents a “null” direction for the symplectic form of the $SL(2, R)$ Chern-Simons theory, reflecting the indefiniteness of the $SL(2, R)$ Killing form. Since $\mathcal{M}_3$ is a disconnected piece of the total phase space $\mathcal{M}$, it is consistent to consider the problem of quantizing $\mathcal{M}_1 \cup \mathcal{M}_2$ independently of $\mathcal{M}_3$.

There are no “rigorous” ways to quantize a phase space consisting of different branches with a non-zero intersection. The strategy adopted both in [8] and [12] is to consider the smooth, non-compact manifold $\mathcal{M}/\mathcal{N}$ obtained by deleting the intersection $\mathcal{N} \equiv \mathcal{M}_1 \cap \mathcal{M}_2$.
of the two branches of $M$. $M/N$ consists of disconnected smooth components $M_1/N$ and $M_2/N$, which, upon quantization, give rise to Hilbert spaces of wave functions $H_{M_1/N}$ and $H_{M_2/N}$. It seems reasonable to think of a wave function on the union $M_1 \cup M_2$ as a pair $(\psi_1, \psi_2)$ of wave functions, with $\psi_1 \in H_{M_1/N}$ and $\psi_2 \in H_{M_2/N}$, “agreeing” in some sense on the intersection $N$. The proposal of [8] is that $\psi_1$ and $\psi_2$, when represented by holomorphic functions, should have the same behaviour around the points in $N$. This implies that the pair $(\psi_1, \psi_2)$ should be determined uniquely by $\psi_1$ and that most of the states $\psi_2$ in the infinite-dimensional $H_{M_2/N}$ should be discarded. The conclusion of the analysis in [8] is that the quantization of $M$ produces a Hilbert space $H_{CS}$ which is a subspace of $H_{M_1/N}$ with definite parity under the conjugation operator $C$,

$$C : (g_1, g_2) \to (g_1^{-1}, g_2^{-1}).$$  

(6)

The space $H_{M_1/N}$ coming from the quantization of $M_1/N$ is the representation space of the ’t Hooft algebra [13]:

$$AB = \mu BA,$$  

(7)

where $\mu$ is a phase related to the coupling constant appearing in the $SL(2, R)$ Chern-Simons action [1] through the equation:

$$\mu = e^{i\pi k}.$$  

(8)

The quantum operators $A$ and $B$ are the quantum versions of the classical holonomies $g_1$ and $g_2$. Unitary, irreducible representations of (7) are finite-dimensional when $k$ is rational:

$$2k = 2s/r = p/q,$$  

(9)

with $s, r$ and $p, q$ coprime integers.

Since we are interested in investigating the connection between $SL(2, R)$ Chern-Simons theory and two-dimensional rational conformal field theories, we will restrict ourselves to $k$ rational as in Eq.(9), and we will denote the corresponding ’t Hooft algebra by $O_{p/q}$.

Modular transformations are external automorphisms of the algebra $O_{p/q}$:

$$S : \begin{cases} A \to B^{-1} \\ B \to A \end{cases} \quad T : \begin{cases} A \to A \\ B \to \mu^{-1/2}AB \end{cases} \quad C : \begin{cases} A \to A^{-1} \\ B \to B^{-1} \end{cases}.$$  

(10)

One can verify that $S, T, C$ satisfy the modular group relations, $S^2 = C$ and $(ST)^3 = 1$ and that the conjugation operator $C$ commutes with the modular group generators, $SC = CS$, $TC = CT$. 

6
The requirement that the automorphisms $S, T, C$ be represented unitarily on the representation space $\mathcal{H}_{M_1/N}$ of the 't Hooft algebra (7) selects a unique (up to equivalence) unitary, irreducible representation of $O_{p/q}$ with dimension $p$:

\begin{align}
(A)_{MN} &= (-1)^{pqN}\delta_{N,M} \\
(B)_{MN} &= (-1)^{pq\delta_{M,N+1}}, \quad M, N = 0, 1, \ldots, p - 1. \tag{11}
\end{align}

The corresponding unitary representation of the modular group is:

\begin{align}
(S)_{MN} &= \frac{1}{\sqrt{p}}e^{2\pi i \frac{q}{p} N^2} \\
(T)_{MN} &= (-1)^{Npq}e^{2\pi i \frac{q}{p} N^2 - 2\pi i \theta(q;p)/3}\delta_{N,M} \tag{12} \\
(C)_{MN} &= \delta_{N+M,0} \quad M, N = 0, 1, \ldots, p - 1,
\end{align}

where the phase $\theta(q;p)$ is determined by the $SL(2,\mathbb{Z})$ relation $(ST)^3 = 1$ and can be written as a generalized Gauss sum:

\begin{equation}
\exp(2\pi i \theta(q;p)) = \frac{1}{\sqrt{p}} \sum_{n=0}^{p-1} (-1)^{pq} e^{2\pi i \frac{q}{p} n^2}. \tag{13}
\end{equation}

An explicit formula for $\theta(q;p)$ has been found in [8].

The modular invariant representations (11) of the 't Hooft algebra (7) admit concrete realizations in terms of holomorphic functions only if $2k$ is integer (i.e., if $q = 1$). When $q \neq 1$, a holomorphic realization of (11) involves rather $q$-multiplets of holomorphic functions [4],[7]. Geometrically this can be understood by noting that the compact $M_1$ is quantizable, in the sense of geometric quantization [15], only if $2k$ is an integer. However, the non-compact $M_1/N$ is quantizable for any real $k$ since holomorphic wave functions might have non-trivial monodromies around loops surrounding points of $N$. For $2k = p/q$ rational, monodromies are represented by $q \times q$ matrices, and we can think of a holomorphic wave function on $M_1/N$ with non-trivial monodromy around the points of $N$ as a $q$-multiplet of wave functions holomorphic on $M_1$ [8].

The holomorphic representation of $O_{p/q}$ is better understood by considering the following isomorphisms of 't Hooft algebras

\[ O_{pq} \approx O_{q/p} \times O_{p/q}, \tag{14} \]
with \( \mathcal{O}_{q/p} \) and \( \mathcal{O}_{p/q} \) commuting among themselves. In fact, denoting by \( A \) and \( B \), \( \tilde{A} \) and \( \tilde{B} \), \( \hat{A} \) and \( \hat{B} \), the generators respectively of the algebras \( \mathcal{O}_{p/q}, \mathcal{O}_{q/p} \) and \( \mathcal{O}_{pq} \), the following relations hold:

\[
\begin{align*}
A &= \hat{A}^q, \quad B = \hat{B}^q \\
\tilde{A} &= \hat{A}^p, \quad \tilde{B} = \hat{B}^p \\
\hat{A} &= \tilde{A}^{\bar{m}} \tilde{A}^\bar{n}, \quad \hat{B} = \tilde{B}^{\bar{m}} \tilde{B}^\bar{n},
\end{align*}
\]

where \( \bar{m} \) and \( \bar{n} \) are integers determined by the conditions:

\[
1 = \bar{m}p + \bar{n}q \\
0 \leq \bar{m} \leq q - 1, \quad 0 \leq \bar{n} \leq p - 1.
\]

The modular invariant representation of \( \mathcal{O}_{pq} \) can be realized on holomorphic theta functions. When \( pq \) is even, the holomorphic realization of (11) is:

\[
\Psi_\lambda(\tau; z) = \theta_{\lambda, pq/2}(\tau; z), \quad \lambda = 0, 1, \ldots, pq - 1,
\]

where the \( \theta_{n,m}(\tau; z) \) (\( n \) integer modulo \( 2m \)) are level \( m \) \( SU(2) \) theta functions [16]:

\[
\theta_{n,m}(\tau; z) = \sum_{j \in \mathbb{Z}} e^{2\pi im\tau(j + \frac{n}{2m})^2 + 2\pi imz(j + \frac{n}{2m})}.
\]

If \( pq \) is odd, the holomorphic, modular invariant realization of (11) is instead:

\[
\Psi_\lambda(\tau; z) = (-1)^\lambda \left( \theta_{2\lambda + pq, 2pq}(\tau; z/2) - \theta_{2\lambda - pq, 2pq}(\tau; z/2) \right).
\]

Because of the algebra decomposition (14), the representations (17) and (18) decompose into \( q \) copies of the representation (11) of \( \mathcal{O}_{p/q} \). Defining the indices \( N \) and \( \alpha \) through

\[
\lambda = qN + p\alpha, \quad 0 \leq N \leq p - 1, \quad 0 \leq \alpha \leq q - 1,
\]

one obtains the \( q \)-components holomorphic representation of \( \mathcal{O}_{p/q} \):

\[
(\Psi_N(\tau; z))^\alpha = \theta_{qN + p\alpha, pq/2}(\tau; z/q)
\]

if \( pq \) is even, and

\[
(\Psi_N(\tau; z))^\alpha = (-1)^\lambda \left( \theta_{q(2N + p) + 2p\alpha, 2pq}(\tau; z/2q) - \theta_{q(2N - p) + 2p\alpha, 2pq}(\tau; z/2q) \right)
\]

if \( pq \) is odd.
The algebra decomposition (14) implies as well that the group of external automorphisms of $O_{pq}$ also factorizes into two copies of the modular group commuting among themselves and acting independently on $O_{q/p}$ and $O_{p/q}$. In particular, the center $C_{pq}$ of the group of external automorphisms factorizes: $C_{pq} = C_{q/p} \times C_{p/q}$. Thus, the conjugation operator $C_{pq} \in C_{pq}$ of the algebra $O_{pq}$, which in the representation (17),(18) acts as follows

$$C_{pq} : \lambda \to -\lambda,$$

satisfies the equation:

$$C_{pq} = C_{q/p} C_{p/q} = C_{p/q} C_{q/p},$$

where $C_{p/q}$ and $C_{q/p}$ are the conjugation operators of the algebras $O_{p/q}$ and $O_{q/p}$ with action given by

$$C_{p/q} : \lambda \to \bar{\lambda} \equiv -qN + p\alpha$$

$$C_{q/p} : \lambda \to -\bar{\lambda}.$$

Since both $C_{p/q}$ and $C_{q/p}$ are in the center $C_{pq}$, it is possible to project the holomorphic representation (17), (or (18)) onto subrepresentations with definite values of $C_{p/q}$ and/or $C_{q/p}$, each one carrying a unitary representation of the modular group. It is somewhat remarkable that in this way one obtains the characters of the $A$ and $D$ diagonal series of integrable representations of $SU(2)$ current algebra, the Kac-Wakimoto characters of admissible representations of $SL(2, R)$ current algebra with fractional level, and the Rocha-Caridi characters of the completely degenerate representations of the discrete Virasoro series.

From the point of view of the quantization of $\mathcal{M}$, the relevant projection is the one onto the subspace $\mathcal{H}_{\mathcal{M}_1/N}^{-} (\mathcal{H}_{\mathcal{M}_1/N}^{+})$ of $\mathcal{H}_{\mathcal{M}_1/N}$ with $C_{p/q} = -1$ ($C_{p/q} = 1$). For reasons which are still rather mysterious [4],[10], when $pq$ is even (odd) the projection onto $\mathcal{H}_{\mathcal{M}_1/N}^{+}$ ($\mathcal{H}_{\mathcal{M}_1/N}^{-}$) does not lead to characters related to two-dimensional conformal field theories. Therefore, in what follows, we will take as Chern-Simons space $\mathcal{H}_{CS}$ the subspace $\mathcal{H}_{\mathcal{M}_1/N}^{-}$, if $pq$ is even, and $\mathcal{H}_{\mathcal{M}_1/N}^{+}$, if $pq$ is odd.

The $C_{p/q}$-odd (or even) combinations of the multi-component wave functions (21) spanning $\mathcal{H}_{CS}$ turn out to be (the numerators of) the Kac-Wakimoto characters $\chi_{j(n,k):m}(z; \tau)$ [7],[8] of the irreducible, highest weight representations of $SL(2, R)$ current algebra with level $m \equiv t/u$ and spin $j(n,k) = 1/2(n - k(m + 2))$, with $n = 1, 2, ..., 2u + t - 1$ and $k = 0, 1, ..., t - 1$. 9
When \( p \) is even and \( q \) odd, the explicit relation between Kac-Wakimoto characters and the \( C_{p/q} \)-odd combinations \( \Psi_N^-(\tau; z) \) of the Chern-Simons multi-component wave functions (21) is:

\[
\mid N \equiv (\Psi_N^-)_{\alpha} \Pi(\tau; z) = \begin{cases} 
\chi_j(N,2\alpha);m(\tau; z) & \text{if } \alpha \leq \frac{q-1}{2} \\
-\chi_j(p/2-N,2\alpha-q);m(\tau; z) & \text{if } \alpha \geq \frac{q+1}{2},
\end{cases}
\]

(25)

where \( N = 1, \ldots, p/2 - 1 \), and the level \( m \) of the current algebra is related to the Chern-Simons coupling constant \( k \) through the equation

\[ m + 2 = k. \]

(26)

\( \Pi(\tau; z) \) is the Kac-Wakimoto denominator,

\[
\Pi(\tau; z) = \theta_{1,2}(\tau, z) - \theta_{-1,2}(\tau, z),
\]

(27)

which is holomorphic and non-vanishing on \( M_1/N \). Therefore, the wave functions \( \Psi_N^-(\tau; z) \) and the wave functions

\[
\Psi'_N(\tau; z) = \frac{\Psi_N^-(\tau; z)}{\Pi(\tau; z)}
\]

appearing in (25), describe equivalent wave functions on \( M_1/N \), related to each other by a Kähler transformation.

If \( p \) is odd and \( q \) even, Eqs. (25) and (26) are replaced by

\[
\mid N \equiv (\Psi_N^-)_{\alpha} \Pi(\tau; z) = \begin{cases} 
\chi_j(2N,\alpha);m(\tau; z) & \text{if } \alpha \leq q/2 - 1 \\
-\chi_j(p-2N,\alpha-q/2);m(\tau; z) & \text{if } \alpha \geq q/2,
\end{cases}
\]

(28)

with \( N = 1, \ldots, \frac{p-1}{2} \) and the level \( m \) given by

\[ m + 2 = 4k. \]

(29)

Finally, if both \( p \) and \( q \) are odd, Eq. (29) is true and the relation between \( C_{p/q} \)-even wave functions and characters becomes:

\[
\mid N \equiv (-1)^{\lambda} (\Psi_N^+)_{\alpha} \Pi(\tau; z) = \chi_j(p+2N,\alpha);m(\tau; z/2) + \chi_j(p-2N,\alpha);m(\tau; z/2),
\]

(30)

with \( N = 0, \ldots, \frac{p-1}{2} \).
Eqs. (25)-(30) generalize the relationship between Chern-Simons theories and two-dimensional current algebra to the case when the coupling constant $2k = p/q$ is fractional. To each Chern-Simons state $|N>$ there correspond not just a single current “block” as in the integer ($q = 1$) case, but a $q$-multiplet of Kac-Wakimoto characters. For example, when $p$ is even, the holomorphic representation of the Chern-Simons state $|N>$ is given by the multiplet \{ $\chi_j(N,0)$, $-\chi_j(p/2-N,1)$, $\chi_j(N,2)$, $\ldots$, $\chi_j(N,q-1)$ \}.

This is not completely surprising. Though the Kac-Wakimoto characters share several properties of the characters of two-dimensional conformal field theories (like modular invariance, unicity of the vacuum, etc.), they cannot possibly come from a conventional conformal field theory since the associated fusion rules, as computed from the Verlinde formula [19], may be negative. On the other hand, as we will show in the next section, there is a well-defined Verlinde algebra, with positive integer fusion rules, acting on the Chern-Simons states $|N>$. This is compatible with the relation that we found between the states $|N>$ and the Kac-Wakimoto characters, because of the minus signs appearing in Eqs. (25), (28), (30). In fact, Eqs. (25)-(30) suggest that the objects relevant to the two-dimensional counterpart of $SL(2,R)$ Chern-Simons theory are the current algebra “super-characters”

$$\tilde{\chi}_{j(n,k)} = (-1)^k \chi_{j(n,k)} \quad (31)$$

if $u$ is odd, and

$$\tilde{\chi}_{j(n,k)} = (-1)^{n+1} \chi_{j(n,k)} \quad (32)$$

if $u$ is even. The fusion rules for the $\tilde{\chi}_{j(n,k)}$’s computed from the Verlinde formula are positive, as we will check in the next section. The fact that the Kac-Wakimoto fusion rules can be made simultaneously all positive by the redefinition (31), (32) seems to indicate that the two-dimensional theory underlying the $SL(2,R)$ Chern-Simons theory (and possibly representing a suitable definition of the $SL(2,R)$ WZW model) assigns non-trivial “ghost-parities” ($(-1)^k$ or $(-1)^{n+1}$ for $u$ odd or even) to the “primaries” $\tilde{\chi}_{j(n,k)}$. It is intriguing that two-dimensional gravity, believed to be related to $SL(2,R)$ Chern-Simons theory on completely different grounds [5], exhibits a similar property [20].
3. VERLINDE ALGEBRAS

The algebra of the observables of Chern-Simons theory with compact gauge group is the Verlinde algebra [19] of the underlying two-dimensional conformal field theory, and the maximally commuting subalgebra of the Chern-Simons observables is the fusion algebra of the conformal theory. It is interesting, therefore, to construct for the $SL(2,R)$ Chern-Simons theory the corresponding objects whose two-dimensional counterparts are not yet understood. From the previous discussion it follows that the $SL(2,R)$ Chern-Simons algebra of observables is the image of $C(O_{p/q})$, the $C$-invariant subalgebra of the ’t Hooft algebra $O_{p/q}$, in the representations (25)-(30).

The Verlinde basis of an algebra of observables might be defined as a basis \(\{\Phi_n(a), \Phi_n(b)\}\) with the properties

\[
\Phi_n(b) = S^{-1}\Phi_n(a)S
\]

\[
\Phi_n(a)\Phi_m(a) = \sum_k N_{nm}^k \Phi_k(a)
\]

\[
\Phi_0(a) = \Phi_0(b) = Id,
\]

where \(N_{nm}^k\) are positive integers. Moreover, there should be a basis \(\{v_m\}\) of eigenvectors of \(\Phi_n(a)\),

\[
\Phi_n(a)v_m = \lambda_n^{(m)}v_m,
\]

satisfying the equation:

\[
\{v_n = \Phi_n(b)v_0\}.
\]

The existence of such a basis is not guaranteed in general, but appears to be a specific property of Chern-Simons observables algebras, like $O_{p/q}$ and its $C$-invariant subalgebras. The Verlinde basis for $O_{p/q}$ is well known:

\[
\{\Phi_n(a) = A^n, \Phi_n(b) = B^n, \ n = 0, 1, ..., p - 1\}.
\]

A basis for $C(O_{p/q})$ is obviously

\[
\{\Phi_n(a) = A^n + A^{-n}, \Phi_n(b) = B^n + B^{-n}, \ n = 0, 1, ..., [p/2]\},
\]

where \([p/2]\) is the integer part of \(p/2\). This basis is not, however, a Verlinde basis since the associated fusion rules \(N_{nm}^k\) may be negative. Moreover, the image of $C(O_{p/q})$ in the representations (25)-(30) is \([p-1]\)-dimensional if \(pq\) is even, but \(\frac{p+1}{2}\)-dimensional if \(pq\) is
odd. To obtain the Verlinde basis, it is useful to recall Verlinde’s observation \cite{19} that the eigenvalues $\lambda_n^{(m)}$ of $\Phi_n(a)$ are related to the modular matrix $S$ in the basis $\{v_n\}$ through the equation

$$\lambda_n^{(m)} = \frac{S_{mn}}{S_{m0}}. \quad (34)$$

When $p$ is even, the holomorphic basis of $\mathcal{H}_{CS}$ in Eq.(25) defines a Verlinde basis if one takes $v_m = |m+1>$, with $m = 0, 1, ..., p/2 - 2$. In this basis, the modular matrix is

$$S_{mn} = \frac{2}{\sqrt{p}} \sin 2\pi \frac{q(n+1)(m+1)}{p}, \quad (35)$$

and thus:

$$\Phi_n(a)|M > = \frac{\sin 2\pi q(n+1)M}{\sin 2\pi qM} |M >. \quad (36)$$

Defining $A = e^{i\theta}$ and $x = 2 \cos \theta$, we obtain

$$\Phi_n(a) = \frac{\sin(n+1)\theta}{\sin \theta} = P_n(x), \quad (37)$$

where $P_n(x)$ are the Chebyshev polynomials of the second kind. It follows from $\cos \theta|M > = \cos 2\pi qM/p |M >$, that

$$P_{p/2-1}(x) = 0. \quad (38)$$

Therefore, for $p$ even, the Verlinde algebra is the same as the Verlinde algebra of $SU(2)$ current algebra of level $p/2 - 2$ \cite{21} and is, in fact, independent of the (odd) number $q$.

If $p$ is odd and $q$ even, the basis defined above is not of Verlinde type since it leads to fusion rules which are not all positive. However, the basis

$$v_m = |\frac{p-1}{2} - m > \quad m = 0, 1, ..., \frac{p-3}{2}$$

is of Verlinde type, since from the Verlinde formula one obtains the equation

$$\Phi_n(a) = \frac{\sin(\frac{p-1}{2} - n)\theta}{\sin \frac{p-1}{2} \theta} = \frac{\sin(2n+1)\theta/2}{\sin \theta/2} = P_{2n}(y), \quad (39)$$

where $y = 2 \cos \theta/2$. Again, Eq.(11) implies the relation:

$$P_{p-1}(y) = 0. \quad (40)$$
The fusion algebra for $p$ odd is therefore independent of the (even) number $q$:

$$\Phi_n(a) \Phi_m(a) = \sum_{k=|m-n|}^{\text{min}\{m+n,p-2-m-n\}} \Phi_k(a).$$

(41)

Finally, let us consider the case when both $p$ and $q$ are odd. Because of the relations (30), we expect to obtain in this case the fusion algebra of the $D_{p+1}$ diagonal series. Let us show that the basis $v_m = \left\langle \frac{p-1}{2} - m \right\rangle$, with $m = 0, 1, ..., \frac{p-1}{2} \equiv \nu$ is of Verlinde type. From the expression for the $S$ matrix in this basis we get the eigenvalues $\lambda^{(m)}_n$ by means of the Verlinde formula:

$$\lambda^{(m)}_n = (-1)^n \frac{\cos \frac{2\pi q (\nu - n)(\nu - m)}{p}}{\cos \frac{2\pi q \nu (\nu - m)}{p}}.$$  

(42)

Defining $A = -e^{i\varphi}$, the eigenvalues of $\cos n\varphi$ are $\cos \frac{2\pi q n m}{p}$, as follows from the representation (11). Thus, the Verlinde operators are:

$$\Phi_n(a) = (-1)^n \frac{\cos (\nu - n)\varphi}{\cos \nu \varphi} = (-1)^n \frac{\cos (2n + 1)\varphi/2}{\cos \varphi/2}.$$  

(43)

By introducing the variable $z = -2 \sin \varphi/2$, one can rewrite the Verlinde operators once again in terms of Chebyshev polynomials

$$\Phi_n(a) = P_{2n}(z) \quad n = 0, 1, ..., \nu.$$  

(44)

However, the fusion algebra is not the same as for $q$ even (Eq.(14)) since the generator $P_{2\nu}(z)$ does not vanish on the $(\frac{p+1}{2})$-dimensional representation space $H_{CS}$. The polynomial relation among the $P_{2n}(z)$’s can be easily derived from Eq.(13):

$$P_{2\nu+2}(z) - P_{2\nu-2}(z) = 0.$$  

(45)

The ring defined by Eqs.(14) and (15) is indeed the fusion ring of the $D_{2\nu+2} = D_{p+1}$ series [22], with $\Phi_\nu(a) = \Phi^{(+)\nu} + \Phi^{(-)\nu}$ being the sum of the two “degenerate” blocks $\Phi^{(\pm)\nu}$ of the $D$ models, which cannot be distinguished, of course, by the Chern-Simons theory on the torus.

Having produced the Verlinde representation of the algebra of Chern-Simons observables $C(\mathcal{O}_{p/q})$, one can exploit the ’t Hooft algebras isomorphism of Eq.(14) to derive
explicit polynomial representations of the fusion rings of the Virasoro minimal models and
of the Kac-Wakimoto set of $SL(2, R)$ current algebra representations.

The minimal model $(r, s)$, with $r$ odd, is realized by taking the $C(O_{q/p}) \times C(O_{p/q})$
subalgebra of the ‘t Hooft algebra $O_{pq}$ acting on the $C_{q/p}$ and $C_{p/q}$ odd subrepresentations
of the representation $(17)$, with $p = 2s$ and $q = r$ [8]. It follows that the fusion algebra of
the minimal models is given by the products

$$P_M(x)P_{2N}(y)$$

of the Verlinde algebras (37) and (39) with $M = 0, 1, ..., s - 2$ and $N = 0, 1, ..., \frac{r-3}{2}$. The
relations (38),(40) become

$$P_{s-1}(x) = P_{r-1}(y) = 0.$$  

The identification with the standard primary fields of the minimal models $\phi_{(m, n)}$ labelled
by the Kac indices $m, n$ ($1 \leq m \leq s - 1, 1 \leq n \leq \frac{r-1}{2}$) is obtained by comparing the
Rocha-Caridi characters with the Chern-Simons holomorphic wave functions (25). One
obtains

$$\phi_{(m, n)} = \begin{cases} 
P_{m-1}(x)P_{n-1}(y) & \text{if } n \text{ odd} \\
P_{s-1-m}(x)P_{r-1-n}(y) & \text{if } n \text{ even} 
\end{cases}.$$  

The variables $x = 2 \cos \theta$ and $y = 2 \cos \tilde{\theta}/2$ are not independent, since the operators
$A = e^{i \theta}$ and $\tilde{A} = e^{i \tilde{\theta}}$ are both related to the same operator $\hat{A} = e^{i \hat{\theta}}$ of the “parent” ‘t
Hooft algebra $O_{pq}$. Eq. (15) implies

$$x = 2 \cos r \hat{\theta}, \quad y = 2 \cos s \hat{\theta},$$

from which one derives the identity

$$P_{r-2}(y) = P_{s-2}(x).$$

This equation can be solved to eliminate $y$ from Eq.(17) and to write a representation
of the minimal models fusion algebra in terms of one-variable polynomials [22]. Substituting
$y = 2 \cos s \hat{\theta}$ in Eq.(17) with one of its eigenvalues, $y|1 \Rightarrow 2 \cos \frac{\pi}{r}|1 \Rightarrow \gamma|1 \rangle$, one
reproduces the representation found in [22]:

$$\phi_{(m, n)} = \begin{cases} 
P_{m-1}(x)P_{n-1}(\gamma) & \text{if } n \text{ odd} \\
P_{s-1-m}(x)P_{r-1-n}(\gamma) & \text{if } n \text{ even} \\
P_{s-1}(x)P_{n-1}(\gamma) & \text{if } n \text{ even} 
\end{cases}.$$
The result of [22], that the fusion rings of the $D$ series and of the minimal models can be represented in terms of one-variable polynomials (which are essentially Chebyshev polynomials), can be understood in the Chern-Simons framework as consequence of the fact that all of these models, together with the $A$ series models, are contained in the “parent” ’t Hooft algebra $O_{pq}$.

A polynomial representation of the fusion algebra defined by the Kac-Wakimoto representations is obtained in a similar way:

$$
\chi_j(n,k) = (-1)^k \begin{cases} 
P_{n-1}(x)y^{n/2} & \text{if } k \text{ even} \\
P_{2u+t-1-n}(x)y^{k+u/2} & \text{if } k \text{ odd}
\end{cases},
$$

with $n = 1, \ldots, 2u + t - 1$, $k = 0, 1, \ldots, u - 1$, and $u$ odd. A similar expression can be found for $u$ even. The polynomial relations defining the fusion ring are

$$P_{2u+t-1}(x) = 0, \quad y^u - 1 = 0. \quad (52)$$

It is easy to check that the Kac-Wakimoto operators $\tilde{\chi}_j(n,k) = (-1)^k \chi_j(n,k)$ do indeed generate a fusion algebra with positive fusion rules:

$$\tilde{\chi}_j(n,k)\tilde{\chi}_j(n',k') = \sum_{m=|n-n'|+1}^{\min\{n+n'-1,4u+2t-1-n-n'\}} \begin{cases} 
\tilde{\chi}_j(m,k+k') & \text{if } k + k' < u \\
\tilde{\chi}_j(2u+t-m,k+k'-u) & \text{if } k + k' \geq u
\end{cases}. \quad (53)$$

4. CONCLUSIONS

$SL(2, R)$ Chern-Simons theory with fractional charge motivates the study of the algebra $C(O_{p/q})$ whose representation theory gives rise to very simple and natural generalizations of the modular representations and fusion rings of the $SU(2)$ WZW models. And yet, the two-dimensional interpretation of these algebraic data appears to lie outside standard conformal field theory. It is intriguing, in particular, that the Kac-Wakimoto characters, which cannot have a conformal field theory interpretation because they have non-positive fusion rules, do appear in $SL(2, R)$ Chern-Simons theory, though their relation to Chern-Simons states is of a novel type and they are multiplied by exactly those minus signs which make their fusions positive.

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References

[1] E. Witten, Comm. Math. Phys. 121 (1988) 351.
[2] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, Nucl. Phys. B326 (1989) 108.
[3] M. Bos and V.P. Nair, Phys. Lett. B223 (1989) 61; Int. J. Mod. Phys. A5 (1990) 959.
[4] S. Axelrod, S. Della Pietra and E. Witten, J. Diff. Geom. 33 (1991) 787.
[5] H. Verlinde, Princeton preprint, PUTP-89/1140, unpublished.
[6] E. Verlinde and H. Verlinde, Princeton preprint, PUTP-89/1149, unpublished.
[7] C. Imbimbo, in: “String Theory and Quantum Gravity ‘91,” H. Verlinde, ed., World Scientific, Singapore (1992).
[8] C. Imbimbo, Nucl. Phys. B384 (1992) 484.
[9] P. Furlan, R. Paunov, A.Ch. Ganchev and V.B. Petkova, Phys. Lett. 276 (1991) 63; A.Ch. Ganchev and V.B. Petkova, Trieste preprint, SISSA-111/92/EP.
[10] C. Imbimbo, Phys. Lett. B258 (1991) 353.
[11] E. Witten, Comm. Math. Phys. 137 (1991) 29.
[12] E. Witten, Nucl. Phys. B371 (1991) 191.
[13] G. 't Hooft, Nucl. Phys. B138 (1978) 1.
[14] R. Jengo and K. Lechner, Phys. Rep. 213 (1992) 179.
[15] N. Woodhouse, “Geometric Quantization,” Oxford University Press, Oxford (1980).
[16] V.G. Kac, “Infinite Dimensional Lie Algebras,” Cambridge University Press, Cambridge (1985).
[17] V.G. Kac and M. Wakimoto, Proc. Nat. Acad. Sci. 85 (1988) 4956.
[18] S. Mukhi and S. Panda, Nucl. Phys. B338 (1990) 263.
[19] E. Verlinde, Nucl. Phys. B300 (1988) 360.
[20] B. Lian and G. Zuckerman, Phys. Lett. B254 (1991) 417.
[21] D. Gepner, Comm. Math. Phys. 141 (1991) 381.
[22] P. Di Francesco and J.-B. Zuber, Saclay preprint 92/138, hep-th/9211138.