Real-analytic, volume-preserving actions of lattices on 4-manifolds

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\textbf{Abstract}

We prove that if $\Gamma$ is a lattice of $\mathbb{Q}$-rank at least 7 in a simple linear Lie group, then any real-analytic, volume-preserving action of $\Gamma$ on a closed 4-manifold of nonzero Euler characteristic factors through a finite group action. \textit{To cite this article: B. Farb, P.B. Shalen, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1011–1014}. \textcopyright 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

\textbf{1. Results}

Zimmer conjectured in \cite{Zimmer} (see also \cite{McMullen}) that the standard action of $\text{SL}(n, \mathbb{Z})$ on the $n$-torus is minimal in the following sense:

\textbf{Conjecture 1.1}. – Any smooth, volume-preserving action of any finite-index subgroup $\Gamma < \text{SL}(n, \mathbb{Z})$ on a closed $r$-manifold factors through a finite group action if $n > r$.

While Conjecture 1.1 has been proved for actions which also preserve an extra geometric structure such as a pseudo-Riemannian metric (see, e.g., \cite{Zimmer}), almost nothing is known in the general case. For $r = 2$ and $n > 4$, the conjecture was proved for real-analytic actions in \cite{McMullen} and \cite{Benoit}. Quite recently, Polterovich \cite{Polterovich} has brought ideas from symplectic topology to the problem, using these to give a proof of Conjecture 1.1 for orientable surfaces of genus $> 1$; his methods actually prove Conjecture 1.1 for the torus as well (see \cite{Polterovich}). For $r = 3$, Conjecture 1.1 is known only in some special cases where $\Gamma$ contains some torsion and the action is real-analytic (see \cite{Benoit}).

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The main result of this note, Theorem 1.2 below, implies that Conjecture 1.1 is true in the case where \( r = 4, n \geq 8, M \) has nonzero Euler characteristic, and the action is real-analytic. To state the general version of the theorem, we follow the conventions used by Witte in [12]. Consider a nonuniform lattice \( \Gamma \) in a simple linear Lie group \( G \) with \( \mathbb{R}\text{-}\text{rank}(G) \geq 2 \). Then \( G \) may be given the structure of an algebraic group over \( \mathbb{Q} \) in such a way that \( \Gamma \) is commensurate with the group of \( \mathbb{Z} \)-points in \( G \). After passing to a torsion-free subgroup of finite index, one deduces this from Margulis’s Arithmeticity Theorem and Remark 6.17 of [12]. We then define the \( \mathbb{Q} \)-rank of \( \Gamma \) to be the \( \mathbb{Q} \) rank of \( G \) with this \( \mathbb{Q} \)-structure; it follows from Theorem 2.10 of [12] that this notion of \( \mathbb{Q} \)-rank is well-defined.

**Theorem 1.2.** – Let \( \Gamma \) be a lattice of \( \mathbb{Q} \)-rank \( \geq 7 \) in a simple linear Lie group \( G \). Then any real-analytic, volume-preserving action of \( \Gamma \) on a closed 4-manifold of nonzero Euler characteristic factors through a finite group action.

The main ingredient in the proof of Theorem 1.2 is Theorem 7.1 of [2] on real-analytic actions which preserve a volume form. This theorem, which is the most difficult result in [2], gives an invariant of finite group action.

2. Proof of Theorem 1.2

Before giving the proof of Theorem 1.2, we will need two algebraic properties of lattices with large \( \mathbb{Q} \)-rank.

**Proposition 2.1.** – Let \( \Gamma \) be a lattice of \( \mathbb{Q} \)-rank \( d \) in a simple linear Lie group \( G \). Then the following hold:

1. If \( d \geq 7 \) then \( \Gamma \) contains commuting subgroups \( A \) and \( B \) which are respectively isomorphic to lattices of \( \mathbb{Q} \)-rank \( 2 \) and \( d - 3 \) in simple linear Lie groups.
2. If \( d \geq 4 \) then \( \Gamma \) contains a torsion-free nilpotent subgroup which is not metabelian.

**Proof.** – Without loss of generality, we may assume that \( G \) is a \( \mathbb{Q} \)-algebraic group of \( \mathbb{Q} \)-rank \( d \) and that \( \Gamma \) is the group of \( \mathbb{Z} \)-points of \( G \).

The proof of the first statement is similar to that of Proposition 2.1 of [2]. Note that, after passing if necessary to a \( \mathbb{Q} \)-split subgroup of the algebraic \( \mathbb{Q} \)-group \( G \) whose root system is the reduced subsystem of the \( \mathbb{Q} \)-root system of \( G \), we may assume \( G \) is \( \mathbb{Q} \)-split.

Since \( G \) is \( \mathbb{Q} \)-simple, the \( \mathbb{Q} \)-root system \( \Phi \) of \( G \) is irreducible, and the Dynkin diagram determined by \( \Phi \) therefore appears in the list given in Section 11.4 of [7]. By going through this list, one sees that in every case where \( d \geq 7 \), one may “erase a vertex” of the diagram to obtain a graph with \( 2 \) components: one with two vertices and another which is a Dynkin diagram with at least \( d - 3 \) vertices. Let \( G_1 \) and \( G_2 \) be the root subgroups corresponding to these two components of the Dynkin diagram. Then the group of \( \mathbb{Q} \)-points of \( G_1 \) has \( \mathbb{Q} \)-rank at least \( 2 \), the group of \( \mathbb{Q} \)-points of \( G_2 \) has \( \mathbb{Q} \)-rank at least \( d - 3 \), and \( G_1 \) commutes with \( G_2 \).

Now \( \Gamma_i = \Gamma \cap G_i \) is the group of \( \mathbb{Z} \)-points of the algebraic \( \mathbb{Q} \)-group \( G_i \). The groups \( A = \Gamma_1 \) and \( B = \Gamma_2 \) have the required properties.

To prove the second statement, note that since \( G \) has \( \mathbb{Q} \)-rank \( \geq 4 \), we can find a connected, nilpotent Lie subgroup \( N \) of \( G \) which is defined over \( \mathbb{Q} \) and has derived length \( \geq 3 \), i.e., is not metabelian. As \( \Gamma \cap N \) is the group of \( \mathbb{Z} \)-points of the \( \mathbb{Q} \)-group \( N \), it is a lattice in \( N \), and in particular is Zariski-dense in \( N \). Hence \( \Gamma \cap N \) is nilpotent and has no metabelian subgroup of finite index. As \( \Gamma \cap N \) must have a torsion-free subgroup of finite index, the assertion follows. \( \square \)
We now turn to the proof of Theorem 1.2. We shall say that a group action $\rho: \Gamma \to \text{Diff}(M)$ is finite if $\rho$ has finite image, and infinite otherwise. We assume that the lattice $\Gamma$, of $\mathbb{Q}$-rank $d \geq 7$, admits an infinite, volume-preserving, real-analytic action on $M$, a 4-manifold of nonzero Euler characteristic; this would lead to a contradiction. By part (1) of Proposition 2.1, $\Gamma$ contains commuting subgroups $A$ and $B$ which are isomorphic to lattices of $\mathbb{Q}$-rank 2 and $d - 3 \geq 4$ respectively.

Let $\gamma_0$ be any infinite order element of $A$. By a theorem of Fuller [4], any homeomorphism of a closed manifold of nonzero Euler characteristic has a periodic point; the proof is an application of the Lefschetz fixed-point theorem and basic number theory. Hence some positive power $\gamma$ of $\gamma_0$ has a fixed point.

We will also need the following two facts. One of the corollaries (see, e.g., Corollary II.7 of [12] or Theorem VIII.3.12 of [9]) of the Margulis Superrigidity theorem is that if $\Lambda$ is commensurable with the group of $\mathbb{Z}$-points a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group $G$ with $\mathbb{Q}$-rank($G) \geq 1$ and $\mathbb{R}$-rank($G) \geq 2$, then any representation of $\Lambda$ into a compact Lie group must have finite image. Since $\mathbb{R}$-rank($B) \geq \mathbb{Q}$-rank($B) \geq 4$, this fact together with the Superrigidity Theorem itself implies that any representation of $B$ into $\text{GL}(4, \mathbb{R})$ has finite image. Second, since $\Gamma$ is a lattice in a simple linear Lie group $G$ of $\mathbb{R}$-rank $\geq 2$, the Margulis Finiteness Theorem (see, e.g., Theorem 8.1 of [15]) gives that $\Gamma$ is almost simple in the sense that any normal subgroup of $\Gamma$ must be finite or of finite index.

The properties of $\Gamma$, $A$ and $B$ that we have stated show that they satisfy the hypotheses of Theorem 7.1 of [2] (with $n = 4$). For the reader’s convenience we recall the statement here.

**Theorem 7.1 of [2].** Let $\Gamma$ be an almost simple group. Suppose we are given an infinite, volume-preserving, real-analytic action of $\Gamma$ on a closed, connected $n$-manifold $M$. Suppose further that $\Gamma$ contains commuting subgroups $A$ and $B$ with the following properties:

- There exists an element $\gamma \in A$, noncentral in $\Gamma$, having a fixed point in $M$.
- $A$ is isomorphic to a lattice of $\mathbb{Q}$-rank $\geq 2$.
- $B$ is noncentral in $\Gamma$.
- Any representation of any finite-index subgroup of $B$ in $\text{GL}(n, \mathbb{R})$ has finite image.

Then there is a nonempty, connected, real-analytic submanifold $W \subset M$ of codimension at least 2 which is invariant under a finite-index subgroup $B'$ of $B$. Furthermore, the action of this subgroup on $W$ is infinite.

**Remark 2.2.** The action of $B'$ on the surface $W$ produced by this theorem is not necessarily area preserving.

We now conclude the proof of Theorem 1.2. Let $B'$ be the subgroup, and $W$ the submanifold, given by Theorem 7.1 of [2]. Then $B'$ is a lattice of $\mathbb{Q}$-rank at least 4, $W$ is a compact, connected manifold of dimension 0, 1, or 2, and the action of $B'$ on $W$ is infinite. If $\dim W = 0$ we have an immediate contradiction, since no group admits an infinite action on a point. If $\dim W = 1$ then we have a contradiction to Witte’s theorem [13] that a lattice of $\mathbb{Q}$-rank $\geq 2$ admits no infinite action on $S^1$. (For a generalization of Witte’s result, see Burger–Monod [1] or Ghys [6].) Now suppose that $\dim W = 2$, so that $W$ is a compact, connected surface. It follows from part (2) of Proposition 2.1 that $B'$ contains a torsion-free nilpotent subgroup $H$ which is not metabelian. But Rebelo [11] showed that any nilpotent group of real-analytic diffeomorphisms of a compact, connected surface must be metabelian. (Rebelo states his result only in the orientable case. However, an action of a nilpotent group on a non-orientable surface gives rise to an action of a $\mathbb{Z}/2\mathbb{Z}$-extension of that group on the orientable double cover; since a $\mathbb{Z}/2\mathbb{Z}$ extension of a nilpotent group is nilpotent, it follows that Rebelo’s result holds in the non-orientable case.) Hence the action of $H$ on $W$ is not effective. Since $H$ is torsion-free, there is an infinite-order element of $H \leq B'$ which acts trivially on $W$, so that the action of $B'$ on $W$ has infinite kernel. Since $B'$ is almost simple by the Margulis finiteness theorem, this kernel must have finite index in $B'$, so that the action of $B'$ on $W$ is finite, and we again have a contradiction. □

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