Einstein clusters as models of inhomogeneous spacetimes

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Abstract

We study the effect of small-scale inhomogeneities for Einstein clusters. We construct a spherically symmetric stationary spacetime with small-scale radial inhomogeneities and propose the Gedankenexperiment. An hypothetical observer at the center constructs, using limited observational knowledge, a simplified homogeneous model of the configuration. An idealization introduces tensions and side effects. The inhomogeneous spacetime and the effective homogeneous spacetime are given by exact solutions to Einstein equations. They provide a simple toy-model for studies of the effect of small-scale inhomogeneities in general relativity.

1 Introduction

The concept of idealization is one of basic tools of modern physics. Macroscopic physical systems could be modelled only if unimportant details are neglected. Unfortunately, it is not always easy to decide which elements in the construction of the model are essential and which are not. It is believed that decisive role is played by observational or experimental falsification. Again, this is not always straightforward. The most famous example is the model of our Universe. Its foundations have been proposed hundred years ago. This extremely simple model, which extrapolated by many orders of
magnitude our faith in applicability of general relativity, turned out to be very successful. A hundred years later the model is alive and able to accommodate enormous flux of observational data provided by advances of modern technology. But, what some people see as a pure success for others is a failure: 96% of the energy content of the model has not been previously known and is seen only via gravitational interactions. This apparent contradiction motivated broad studies of validity of a basic assumption of the model — exact spatial homogeneity. In our article, we take on this topic. Our approach is restricted to a simple exact toy-model and, as such, it is only indirectly relevant for cosmology (for other studies based on exact solutions see also

\cite{2, 12, 13, 14, 10, 11})

\emph{Einstein cluster} is a class of solutions to Einstein equations which was discovered by Albert Einstein in 1939 \cite{5}. It provides an effective description of a cloud of massive particles moving in randomly inclined circular geodesics under the collective gravitational field of all the masses (see figure \ref{fig:1}). The spacetime is spherically symmetric and stationary. The radial pressure vanishes because the whole system is centrifugally supported. Einstein clusters have been studied extensively in literature (see \cite{7} and references there). In the astrophysical context, they have been proposed as models of galactic dark matter haloes \cite{3}.

The vanishing of radial pressure allows to construct stationary spacetimes with small-scale inhomogeneities without introducing unphysical equation of state. This property suits our purpose superbly.

The problem of small-scale inhomogeneities may be split into two topics: the effect of inhomogeneities on geodesics (light, gravitational waves, test bodies) which alters interpretation of our observations and the so-called backreaction effect which alters the structure of spacetime in a sense which will be explained below.

The backreaction problem is usually formulated as a \emph{fitting problem} \cite{6}. In this approach, one asks how to fit an idealized solution to a realistic (‘lumpy’) spacetime. The aim of this approach is to find covariant procedure which uniquely assigns the best effective spacetime to a realistic one. However, it is more common in a down-to-earth scientific work to assume an effective model \emph{a priori}. A physicist who want to describe the complicated system usually neglects ‘details’ and propose a simplified model. This model is later being tested in experiments or against observations. In cosmology, spatial isotropy and homogeneity of the universe was a natural first guess. These assumptions led to our standard cosmological model $\Lambda CDM$. This
Figure 1: The solution called ‘Einstein cluster’ describes a spacetime filled with a high number of massive particles moving in their own gravitational field on randomly inclined and directed circular orbits. (The Cartesian coordinates $x$, $y$, $z$ were scaled for simplicity.)

model, with free parameters estimated by astronomers, constitutes the ‘effective spacetime’. Therefore, instead of looking for a fitting procedure one may formulate backreaction problem in an alternative way and ask what kind of errors has been introduced by idealization.

In this alternative approach, the effective spacetime is known from the beginning. An idealised geometry does not fit to the matter content exactly and a discrepancy between the left hand side (geometry) and the right hand side of Einstein equations (the energy-matter content) arises. If one assumes that Einstein equations hold, then additional or missing terms are incorrectly interpreted as a contribution to the energy-momentum tensor.
These artificial terms are known as a backreaction tensor. Since $\Lambda CDM$ energy-momentum tensor is dominated by dark matter and dark energy — the forms of energy and matter detected so far only through their gravitational interactions, then the backreaction effect has a potential to clarify our understanding of the Universe.

The results presented in the article [8] suggest that that this potential has not been realised in nature: under appropriate mathematical conditions the backreaction tensor is traceless, thus it may mimic radiation, but it cannot mimic cosmological constant nor cold dark matter. In the context of the $\Lambda CDM$ model this implies that backreaction effect introduces a minor correction and it is definitely not the ‘order of magnitude effect’ (which is needed to explain cosmological observations without cosmological constant or other forms of dark energy).

One may, at least formally, find relevant examples of spacetimes with small-scale inhomogeneities, such that the formalism [8] cannot be directly applied to them, e.g. a vacuum cosmological model with all the mass concentrated in a statistically homogeneously distributed black holes [1]. Moreover, even if the backreaction vanishes, the effect of small-scale inhomogeneities still may alter interpretation of our observations, as will be illustrated by our example.

The aim of the article is to conduct the Gedankenexperiment. We construct an exact solution to Einstein equations which contains small-scale inhomogeneities. We show that in our model the backreaction vanishes (in the sense of the Green-Wald framework). Moreover, we present a heuristic analysis which implies that the backreaction vanishes in all possible models constructed within Einstein cluster class. Next, we adopt a point of view of an astrophysicist who would like to model our inhomogeneous spacetime by available idealised exact solutions. We argue that astronomical observations interpreted within simplified model would lead to the misinterpretation of the energy content of the model. Our analysis is restricted to the particular class of solutions to Einstein equations, but it illustrates what the effect of small-scale inhomogeneities could be in principle.
2 Setting

Any spherically symmetric stationary spacetime could be written in the following form

\[ g = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) , \]  

(1)

where \( \nu, \lambda \) are functions of \( r \) only. Two of the Killing fields could be immediately read out from the form of the metric: \( \partial_t, \partial_\varphi \). For the centrifugally supported cloud of massive particles (the so-called Einstein cluster) the energy–momentum tensor non-vanishing components are

\[ T^t_t = -\rho, \quad T^\theta_\theta = T^\varphi_\varphi = p , \]  

(2)

where \( \rho = \rho(r) \) is the energy density and \( p = p(r) \) is a tangential pressure. The Einstein equations imply

\[ \lambda = \ln(1 + r\nu'), \quad p = \frac{r\nu'}{4} \rho , \]  

(3)

and

\[ rv'' + r(v')^2 + 2v' = 8\pi r(1 + rv')^2 \rho . \]  

(4)

In order to find a particular solution one may set \( \rho(r) \), solve \( (4) \) for \( \nu(r) \) and calculate \( \lambda(r) \) from \( (3) \). The standard pseudopotential analysis reveals \( [7] \) that the radial stability conditions have forms

\[ 0 < rv'/2 < 1 , \]  

(5)

\[ rv'' - r(v')^2 + 3v' > 0 . \]  

(5)

The equation \( (4) \) if written in terms of auxiliary function \( \lambda = \ln \zeta \) [using \( (3) \)] reduces to the Bernoulli differential equation

\[ \zeta' + P\zeta = Q\zeta^2 , \]  

(6)

where \( P = -1/r, \) \( Q = -1/r + 8\pi r\rho \). The substitution \( \zeta \rightarrow 1/\mu \) leads to a linear equation of the form

\[ -\mu' + P\mu = Q , \]  

(7)

which has a solution

\[ \mu = 1 - \frac{8\pi}{r} \int \rho r^2 dr , \]  

(8)
where an integration constant is fixed by regularity at the center (it depends on the form of $\rho$). Therefore, for a given density profile solution to Einstein equations is given by

$$\nu = \int \frac{dr}{r} \left( \frac{1}{\mu} - 1 \right) ,$$  \hspace{1cm} (9)

$$\lambda = \ln \frac{1}{\mu} ,$$  \hspace{1cm} (10)

where $\mu$ is given by (8).

This solution may be matched to the Schwarzschild exterior. The active gravitational mass inside of the sphere with an area radius $r$ is given by [7]

$$M(r) = 4\pi \int_0^r \rho(\hat{r}) \hat{r}^2 d\hat{r} = \frac{r^2}{2} \frac{\nu'}{1 + r\nu'} .$$  \hspace{1cm} (11)

### 3 Small-scale inhomogeneities

Let $\rho$ contain ‘a high frequency component’ such that

$$\rho(r) = \rho_B(r) + \rho_l(r) ,$$  \hspace{1cm} (12)

where $\rho_l$ is an oscillating function and $l$ is a constant (small $l$ corresponds to high frequency oscillations). Using equations (8), (9), (10) one may calculate metric functions $\nu(r)$, $\lambda(r)$ that correspond to $\rho(r)$ given by formula above. In this way, we construct a spacetime with small-scale inhomogeneities.

It is not aim of this paper to model any realistic astrophysical system, but in order to gain physical intuition one may pretend that our inhomogeneous spacetime describes the galactic halo. For simplicity, we choose $\rho(r)$ in such a way that it oscillate about a constant density $\rho_0$ for a class of stationary observers. Moreover, our system constitute a finite configuration: at some radius $r = R$ it is matched to the Schwarzschild solution.

We assume that a hypothetical astrophysicist living at the center of the system does not know $\rho(r)$ precisely, but knows that $\rho(r)$ is ‘approximately’ constant and that the configuration is finite. Both facts would become basic assumptions of his idealised model: the Einstein cluster with a constant energy density and anisotropic pressure, from now on called the model A.

It follows from the Birkhoff theorem that vacuum spacetime outside a spherically symmetric configuration is given by the Schwarzschild metric.
Thus, any effective spacetime must be also matched to the Schwarzschild solution. Observations of trajectories of satellite stars and dwarf galaxies, in such a hypothetical configuration, would allow to estimate gravitational mass of the system $M$. We assume that this parameter is known. Moreover, we assume that the astrophysicist observes most distant stars (at the matching surface $r = R$) and known their blueshift $z$.

To sum up, assumptions and hypothetical observational results which are made/known to our astrophysicist:

- the spacetime is stationary,
- the spacetime is spherically symmetric,
- observer is at the center,
- the matter is distributed uniformly on average $\rho(r) = \text{const}$,
- the configuration is finite (vacuum outside),
- the gravitational mass $M$ of the configuration is known (based on observations of satellite dwarf galaxies and orbits of stars encircling the halo),
- the blueshift $z$ of most distant stars in the halo is known (we assume that $z$ has been corrected for a perpendicular Doppler shift),
- the state of art observations are not good enough to resolve individual inhomogeneities (their density profiles, etc.) — the observer may detect only the cumulative effects.

It will be more instructive for a reader to start with the description of a constant density Einstein cluster (our effective and background spacetime — our approach does not distinguish between these two concepts).

4 Model A: constant density Einstein cluster

A constant density profile $\rho(r) = \rho_A$ and the equation (7) give (see also [4])

$$\mu_0 = 1 - a_A r^2,$$

(13)
where \( a_A = 8\pi \rho_A / 3 \) is a constant and where an additive constant was chosen to satisfy regularity at the center \( r = 0 \). We have from (9), (10)

\[
\nu_A = -\ln \sqrt{1 - a_A r^2} + 3 \ln \sqrt{1 - a_A R_A^2} , \quad (14)
\]

\[
\lambda_A = -\ln (1 - a_A r^2) , \quad (15)
\]

where without loss of generality we have chosen the additive constant \( 3 \ln \sqrt{1 - a_A R_A^2} \) in \( \nu_0 \) and where \( R_A \) is a new constant \( 0 < R_A < 1 / \sqrt{a_A} \).

Finally, the metric reads

\[
g_A = -\frac{\sqrt{1 - a_A R_A^2}}{\sqrt{1 - a_A r^2}} dt^2 + \frac{1}{1 - a_A r^2} dr^2 + r^2 d\Omega^2 , \quad (16)
\]

The metric is regular and of the Lorentzian signature for \( 0 \leq r < 1 / \sqrt{a_A} \). The Ricci and Kretschmann scalars blows up at \( r = 1 / \sqrt{a_A} \), so there is a curvature singularity. From now on we assume that \( 0 \leq r \leq R_A < 1 / \sqrt{a_A} \). For \( r = R_A \) the spacetime is matched to the vacuum exterior Schwarzschild solution — the active gravitational mass inside of the sphere with an area radius \( r \) is given by (11). For a constant density profile \( M(r) = a_A r^3 / 2 \). The radial stability conditions (5) reduce to \( 0 < 3 a_A r^2 < 2 \) and \( a_A r^2 < 4 / 3 \) which gives additional restriction on the matching hypersurface \( r = R_A \).

5 Inhomogeneous spacetime

A toy-model studied in this paper is constructed as follows. We assume that \( \rho(r) = \rho_l(r) = 2 \rho_0 \cos^2(2\pi r / l + \pi / 4) \), where \( \rho_0 \) and \( l \) are constant. The parameter \( \rho_0 \) is an average density as measured by stationary observers in our coordinate system. We introduce auxiliary constant \( a \) such that \( \rho_0 = \frac{3a}{8\pi} \).

The frequency of density perturbations is fixed by \( l \) (we assume that this parameter is small \( l \ll 1 \) which corresponds to high-frequency oscillations). Moreover, we assume that

\[
R = nl , \quad (17)
\]

for some integer number \( n \). This condition together with the choice of phase \( \pi / 4 \) eliminates local variations of the gravitational potential — the energy density at the center (at position of the observer) and at the matching surface \( R \) corresponds to its average value \( \rho_0 \). We split \( \mu \) into two parts: one which
does not depend on $l$ and the second one which is $O(l)$: $\mu = \mu_0 + \mu_l$. Using \cite{8} we find

\begin{align*}
\mu_0 &= 1 - ar^2, \\
\mu_l &= \frac{3al}{32\pi^3} \left[ -\frac{l^2}{r} + 4\pi l \sin\left(\frac{4\pi r}{l}\right) + \left(\frac{l^2}{r} - 8\pi^2 r\right) \cos\left(\frac{4\pi r}{l}\right) \right],
\end{align*}

where an additive constant was chosen to satisfy regularity at the center. We have $\nu = \nu_0 + \nu_l$, where from \cite{9}

\begin{align*}
\nu_0 &= -\ln \sqrt{1 - ar^2} + 3 \ln \sqrt{1 - aR^2}, \\
\nu_l &= -\int \frac{dr}{r} \frac{\mu_l}{\mu_0 + \mu_l}, \\
\lambda &= \ln \left(\mu_0 + \mu_1\right).
\end{align*}

Therefore, we have $g_{tt} = -e^\nu$, $g_{rr} = 1/(\mu_0 + \mu_l)$ and

\begin{equation}
g = \frac{-\sqrt{1 - aR^2}}{\sqrt{1 - ar^2}} e^\nu dt^2 + \frac{1}{1 - ar^2 + \mu_l} dr^2 + r^2 d\Omega^2. \tag{23}
\end{equation}

One may show that the Ricci and Kretschmann scalars blows up at $\mu = 0$, so there is a curvature singularity. From now on we assume that that $0 \leq r \leq R < 1/\sqrt{a}$. The term $\mu_l$ which is proportional to $l$ can be made arbitrary small, so the metric is regular and of the Lorentzian signature in $\mathbb{R} \times (0, R) \times S^2$. For $r = R$ the spacetime is matched to the vacuum exterior Schwarzschild solution — the active gravitational mass inside of the sphere with the area radius $r$ is given by \cite{11}.

The radial stability conditions \cite{5} have complicated form for this solution, but once the parameters of the model are fixed, one may verify by direct calculation that they hold.

### 5.1 Green-Wald framework

Our model corresponds to a one-parameter family of solutions to Einstein equations (with $l$ being a free parameter). One may verify by inspection that it satisfies all assumptions of the Green-Wald framework \cite{8} with the background spacetime $g^{(0)}$ which corresponds to $g_A$ with $a_A \to a$, $R_A \to R$, $\rho_A \to \rho_0$. 

9
We define $h(l) = g(l) - g^{(0)}$. The non-zero components of $h_{\alpha\beta}$ for small $l$ are
\[ h_{tt} \approx \frac{\nu_l}{\sqrt{1 - ar^2}}, \quad h_{rr} \approx -\frac{\mu_l}{(1 - ar^2)^2}. \] (24)

It follows from the equations (19), (21) that $\mu_l$ and $\nu_l$ and the first derivatives of $\nu_l$ vanish in the high frequency limit $l \to 0$ (or $n \to \infty$). The derivative $\partial_r \mu_l$ is not pointwise convergent, but it remains bounded. We have $\lim_{l \to 0} h_{\alpha\beta} = 0$ as expected. Although $\lim_{l \to 0} (\nabla_\delta h_{\alpha\beta} \nabla_\gamma h_{\kappa\iota})$ does not vanish for $\delta = \alpha = \beta = \gamma = \kappa = \iota = r$, the backreaction tensor is zero (w-lim denotes a weak limit as defined in [8] and a connection is associated with the spacetime $g^{(0)}$).

In summary, the one-parameter family of spacetimes (23) has a high frequency limit $\lim_{l \to 0} g = g_A$. It satisfies assumptions of the Green–Wald framework [8]. Although one component of $\nabla_\delta h_{\alpha\beta}$ is not pointwise convergent, the backreaction tensor vanishes.

Vanishing of backreaction gives rise to another interesting question: Does there exist one-parameter families of solutions within Einstein cluster class [different choices of $\rho(r)$] with non-trivial backreaction in the Green-Wald framework? We think that the answer to this question is no. We justify it as follows.

The possible source of backreaction is a nonlinear term $(\nu')^2$ in (4). In order to be a source of the backreaction it would have to be non-zero in the high-frequency limit — it should be at least $O(l^0)$. However, if $\nu'$ does not vanish for $l \to 0$, then it follows from (3) that $\lambda$ is not pointwise convergent which contradicts one of the Green-Wald assumptions about behavior of $h_{\alpha\beta}$ as $l \to 0$. Taking the high-frequency limit is a covariant procedure provided that the background (effective) spacetime is fixed. Therefore, all one parameter families of Einstein clusters to which the Green-Wald framework may be applied have vanishing backreaction.

6 Effective spacetime

Our inhomogeneous spacetime is defined by three parameters
- an average energy density $\rho_0$,
- a size — an area radius $R$,
- a frequency of inhomogeneities $l$ or a number $n$ of inhomogeneous regions (such that $nl = R$).
These parameters are fixed. The effective model $A$ is defined by analogous two parameters: $\rho_A$, $R_A$. We assume that the available observational data allow to determine gravitational mass of the system $M$ and gravitational blueshift $z$ of stars at the boundary of the configuration. The observer is at the center of configuration.

From (11), we have for the inhomogeneous spacetime

$$M = M(R) = \left(\frac{4}{3}\pi + \frac{1}{n}\right)R^3 \rho_0 .$$

(25)

Since the spacetime is spherically symmetric and stationary the blueshift $z$ is given by

$$1 + z = \sqrt{\frac{g_{tt}(r = 0)}{g_{tt}(r = R)}} = e^{-\nu(R)/2} ,$$

(26)

where $\nu(R) = \nu_0(R) + \nu_l(R)$ must be computed numerically from (20), (21).

For the effective spacetime $g_A$ given by (16) the mass $M$ and the blueshift $z$ may be calculated as follows. Let $a_A = 2M/R_A^3$, then at some $r = R_A$ the metric $g_A$ will match to the Schwarzschild solution. Since we have also $a_A = 8\pi/3\rho_A$, then

$$M = \frac{4}{3}\pi R_A^3 \rho_A .$$

(27)

The blueshift is

$$1 + z = \left(1 - \frac{2M}{R_A}\right)^{\frac{1}{4}} .$$

(28)

Finally, unknown parameters of the model $A$ (the effective spacetime), namely, $\rho_A$, $R_A$ in terms of ‘observational parameters’ $M$, $z$ and parameters of the inhomogeneous spacetime $\rho_0$, $R$, $n$ are given by

$$\rho_A = \frac{3}{32\pi} \frac{((-z)(2 + z)(2 + z)z + 2)^3}{M^2} = \frac{3}{32\pi} \frac{(1 - e^{-2\nu(R)})^3}{[(4/3\pi + 1/n)R^3 \rho_0]^2} ,$$

$$R_A = -\frac{2M}{-z(2 + z)(2 + z)z + 2} = 2\frac{(4/3\pi + 1/n)R^3 \rho_0}{1 - e^{-2\nu(R)}} .$$

(29)

It follows from the radial stability inequalities (5) that the most compact stable/metastable configurations [7] in the homogeneous case correspond to $R_A = 6M$, $R_A = 3M$, respectively. Using (29) one may show that the blueshifts for these configurations are given by $z = -1 + (2/3)^{1/4}$, $z = -1 +$
1/3^{1/4}. We start our analysis with the inhomogeneous spacetime with $z \approx -1 + 1/3^{1/4} \approx -0.24016$, thus a relativistic system.

For $R = 3M$ and $n = 100$ we get from (26) $z \approx -0.23926$. Using observational data in the form of $M$ and $z$ within the homogeneous model the observer at the center will estimate $\rho_A$ to a different value than physical $\rho_0$. Therefore, other measurements of the energy density, i.e., from radiation that originates in decay of dark matter particles of the galactic halo (assuming that this will be known one day) will lead to a disagreement with $\rho_A$. The inhomogeneity effect is presented in figure 2. The effect of inhomogeneities for tens of inhomogeneous regions is of order of a few percent. As the number of inhomogeneities grows and the amplitude decreases the effect vanishes (the limit $l \to 0$ or $n \to +\infty$) in accordance with the analysis within the Green-Wald framework (the high-frequency limit). The density contrast remains constant in this limit. What is interesting, the effect of inhomogeneities slightly increases with the ratio $R/M$ (the area radius over the gravitational mass of the system) — see figure 3. Since there is no backreaction in the sense of the Green-Wald framework, the misinterpretation of the energy content is of trivial nature. It reduces to misinterpretation of the parameters of the model. New form of the energy content cannot appear here because the

Figure 2: The discrepancy between the average energy density $\rho_0$ in the inhomogeneous model and the estimated energy density in the homogeneous model $\rho_A$ for $R = 3M$ as a function of a number of inhomogeneous regions.
effective spacetime belong to the same class of solutions to Einstein equations as the original one.

It was not an aim of our paper to model a realistic astrophysical system, but we find it instructive to calculate the effect of inhomogeneities for parameters corresponding to the dark matter halo of our Milky Way. We assume that in geometrized units the mass is \( M = 10^{12} M_\odot = 1.477 \times 10^{15} m \) and the radius \( R = 400000 ly = 3.784 \times 10^{21} m \) which gives \( R/M = 2.563 \times 10^6 \). The Schwarzschild radius is one order smaller than stellar distances \( 2M = 0.312 ly \). The energy density for the system compressed million times to the minimal configuration \( R = 3M \) would be \( 5.45 \times 10^{-6} kg/m^3 \) which qualifies as a high vacuum for Earth standards. If the local clustering scale is assumed to be \( l \approx 1 kpc \) (the size of satellite dwarf galaxies), then \( n \approx 40 \). For these parameters the inhomogeneity effect is small \( (\rho_0 - \rho_A)/\rho_0 \approx 1\% \).

7 Summary

We have constructed the spherically symmetric stationary Einstein cluster with small-scale radial inhomogeneities and applied the Green-Wald frame-
work to show that there is no backreaction. Next, we have conducted the
Gedankenexperiment: an observer at the center of this configuration modelled
surrounding spacetime by an effective solution — an homogeneous Einstein
cluster. The parameters of this effective spacetime are based on two straight-
forward astronomical ‘observations’: the gravitational mass of the system $M$
and the blueshift of stars at its outer boundary. The idealization of the inho-
mogeneous spacetime resulted in the misinterpretation of the energy content.
The effective energy density is lower than the original average energy density.
The effect is not bigger than a few percent and, as expected, it vanishes in
the limit in which the size of inhomogeneous regions are reduced, but the
density contrast is kept constant.

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