A Full Review of the Theory of Electromagnetism

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Abstract

We will provide detailed arguments showing that the set of Maxwell equations, and the corresponding wave equations, do not properly describe the evolution of electromagnetic wave-fronts. We propose a nonlinear corrected version that is proven to be far more appropriate for the modellization of electromagnetic phenomena. The suitability of this approach will soon be evident to the reader, through a sequence of astonishing congruences, making the model as elegant as Maxwell’s, but with increased chances of development. Actually, the new set of equations will allow us to explain many open questions, and find links between electromagnetism and other theories that have been searched for a long time, or not even imagined.

1 Short introduction

The theory of electromagnetism, in the form conceived by J.C. Maxwell, can boast 130 years of honored service. It survived the severest tests, proving itself to be, for completeness and elegance, among the most solid theories. Very few would doubt its validity, to the extent that they may be more inclined to modify the point of view of other theories, rather than question the Maxwell equations. The trust in the model has been strong enough to obscure a certain number of “minor” incongruities and to incite the search for justifications in the development of other theories.
Nevertheless, even if the time-honored equations excellently solve complex problems, they are not able to simulate the simplest things. They are not capable for instance of describing what a solitary signal-packet is, one of the most elementary electromagnetic phenomena. Alternative models have been proposed with the aim of including solitons, but they did not succeed in gaining a long-lasting relevance, because they were based on deliberate adjustments, that, accommodating specific aspects on one hand, were causing the model to lose general properties on the other.

The development of modern field theory, which was very prosperous in the years 1930-1960, has magnified the role of the equations, giving them a universal validation in the relativistic framework. This progress came to a stop, leaving however the impression of being not too far from the goal of compenetrating electromagnetism and gravitation theory.

We are going to make some statements that many readers will certainly consider heretic. We think that the various anomalies, which are present in the model, are not incidental, but consequences of a still insufficient theoretical description of electromagnetic phenomena. Actually, it is our opinion that the flaws are more severe than expected, and therefore, such a fundamental “brick” of Physics needs extensive revision. The review process must be so deep that the entire setting necessitates re-planning from the beginning. On the other hand, if it were just a matter of small adaptations, this revision would have already been made a long time ago.

We shall start to analyse some substantial facts, that at a practical level may be considered marginal, with the aim to evidentiate contradictions. We solve these problems by suitably redesigning the Maxwell equations. This will allow for the construction of a new model, solving all the inconsistencies and achieving the scope of a better understanding of electromagnetic phenomena. In a very natural way, the new approach also leaves the door more than open, to those links and generalizations that were expected to come from the Maxwell equations, but which, although vaguely insinuated, could never be realized in practice.

None of the gracefulness that characterizes the Maxwell model will be lost. The reader who has the patience to follow our arguments through to the end, will discover that all the pieces find their exact place in a global scheme, with due elegance and harmony. We do not wish to say more in this short introduction. The model will be developed step by step, up to its final form, in order to let the reader appreciate the phases of its maturation. The mathematical tools used are classical, and maybe dated. On the other hand, our intention is to examine what would have happened to the evolution of
Physics, if our model was taken into consideration, in place of the Maxwell equations. We will elaborate and clarify many important concepts, leaving the path well clear for future developments, not considered here due to lack of time.

2 Criticism of the theory of electromagnetism

In this section, we make some fine considerations regarding the evolution of electromagnetic waves, and the way they are modelled by the Maxwell equations. We start by pointing out deficiencies mainly at the level of mathematical elegance. These will reveal other more severe incoherences. In the end, even taking into account the correctness, up to a certain degree of approximation, of the physical approach, our judgement will be rather negative. As a matter of fact, in section 3, with the aim of finding a remedy to the problems that have emerged, substantial revision will be proposed.

From now on, until section 11, we assume that we are in void three-dimensional space. As usual, the constant $c$ indicates the speed of light. In this case, the classical Maxwell equations are:

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \text{curl} \mathbf{B}$$ (2.1)

$$\text{div} \mathbf{E} = 0$$ (2.2)

$$\frac{\partial \mathbf{B}}{\partial t} = -\text{curl} \mathbf{E}$$ (2.3)

$$\text{div} \mathbf{B} = 0$$ (2.4)

where the vector field $\mathbf{E}$ is dimensionally equivalent to an acceleration multiplied by a mass and divided by an electrical charge; while $\mathbf{B}$ is a frequency multiplied by a mass and divided by a charge.

The above equations are supposed to be satisfied point-wise at any instant of time. Their solutions are assumed to be smooth enough to allow differential calculus. Therefore, discontinuous or singular solutions are not allowed. The equations (2.2) and (2.4) could be considered unnecessary, since they are easily deduced from (2.1) and (2.3) respectively, after applying the divergence operator. Later on, for the reasons that we are going to explain, we will question the validity of (2.2) and (2.4). As a consequence, the entire formulation will lose its credibility.
As far as the evolution of an electromagnetic plane wave (with infinite extent and linearly polarized) is concerned, we have no objections to make. In Cartesian coordinates, a monochromatic wave of this type, moving along the direction of the z-axis, is written as:

\[ E = (c \sin \omega (t - z/c), 0, 0) \quad B = (0, \sin \omega (t - z/c), 0) \quad (2.5) \]

In this case, the Maxwell equations are all satisfied point-wise.

The next step is to examine the case of a spherical wave, which is far more delicate. The wave could be generated by an oscillating dipole of negligible size. However, the way the wave is produced and supplied is not of interest to us at the moment, being more concerned with analyzing the geometrical aspects of its evolution at a distance from the source.

Let us denote by \( P = E \times B \) the Poynting vector. It is customary to assume that \( E \) and \( B \) are orthogonal, and that the wave-front propagates at constant speed \( c \), through spherical concentric surfaces. One may argue that perfect spherical waves do not exist in nature. Nevertheless, for the sake of simplicity, we maintain this hypothesis which can be removed later, without modifying the essence of our reasoning.

We are basically confronted with two possibilities. In the first one, the Poynting vector follows exactly the radial direction. This means that \( E \) and \( B \) locally belong to the tangent plane to the wave-front. In such a circumstance, as detailed below, we are able to show that (2.1) and (2.3) cannot be both satisfied everywhere. More precisely, it is known that (2.1) and (2.3) are true up to an error that decays quadratically with the distance from the source. Since the intensity of a spherical electromagnetic wave only decays linearly in amplitude, the above mentioned inaccuracy has no influence on practical applications. However, we record a first negative mark.

The second possibility is that, in order to satisfy all the set of Maxwell equations, we loose the orthogonality of the Poynting vector with respect to the wave-front surface. This is a more unpleasant situation, considering that the Poynting vector represents the direction of propagation of the energy flow. The lack of orthogonality between the wave-front tangent plane and the direction of propagation violates the Huygens principle (recall that we are in vacuum), leading to a deformation of the front itself. As we will check later, this results in relevant defects in the development of the wave-shape.

Let us study the problem more in detail, by taking into account the transformation in spherical coordinates:

\[ (x, y, z) = (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) \quad (2.6) \]
with \(0 \leq \theta < 2\pi\), \(0 \leq \phi \leq \pi\) and \(r\) large enough. We look for vector fields having the following form:

\[
\mathbf{B} = (0, 0, u) \quad \mathbf{E} = (v, w, 0) \quad (2.7)
\]

where \(u, v, w\) are functions of the variables \(t, r\) and \(\phi\) (no dependency on \(\theta\) is assumed). In (2.7), the first component of the vectors is referred to the variable \(r\), the second one to \(\phi\), and the third one to \(\theta\). The unknowns in the system of Maxwell equations reduce from six to three. Choosing a more general form for the fields only complicates the computations, without adding anything to the substance.

We start by observing that equation (2.4) is immediately satisfied. Moreover:

\[
\text{curl} \mathbf{B} = \left( \frac{u \cos \phi}{r \sin \phi}, \frac{u \phi}{r}, -\left( \frac{u}{r} + u_r \right), 0 \right) \quad (2.8)
\]

\[
\text{div} \mathbf{E} = v_r + \frac{2v}{r} + \frac{w \cos \phi}{r \sin \phi} + \frac{w \phi}{r} \quad (2.9)
\]

\[
\text{curl} \mathbf{E} = \left( 0, 0, w_r + \frac{w}{r} - \frac{v \phi}{r} \right) \quad (2.10)
\]

Therefore, the equations in spherical coordinates become:

\[
u_t = -\left( w_r + \frac{w}{r} \right) + \frac{w \phi}{r} \quad (2.11)
\]

\[
v_t = \frac{c^2}{r} \left( \frac{u \cos \phi}{\sin \phi} + u \right) \quad (2.12)
\]

\[
w_t = -c^2 \left( \frac{u}{r} + u_r \right) \quad (2.13)
\]

To avoid discontinuities, we must introduce the following boundary constraints:

\[
\begin{align*}
\mathbf{B}(t, r, 0) &= \mathbf{B}(t, r, \pi) = 0 & \mathbf{E}(t, r, 0) &= \mathbf{E}(t, r, \pi) = 0 \quad (2.14) \\
\frac{\partial v}{\partial \phi}(t, r, 0) &= \frac{\partial v}{\partial \phi}(t, r, \pi) = 0 \quad (2.15)
\end{align*}
\]

In the case of the pure radiation field of an oscillating dipole, when \(r\) is sufficiently large, one usually sets \(v = 0\) and \(w = cu\). Within this hypothesis,
the two equations (2.11) and (2.13) are equivalent. They bring us to the
general solution:

\[ w(t, r, \phi) = c u(t, r, \phi) = \frac{c}{r} f(\phi) g(t - r/c) \quad (2.16) \]

where \( f \) (with \( f(0) = f(\pi) = 0 \)) and \( g \) are arbitrary functions (the only
restrictions apply to their regularity). Among these solutions there is the
one corresponding to \( f(\phi) = \sin \phi \), which is often present in classical texts
(see for instance [2], p.284), being the one with more physical relevance.
Nevertheless, we unfortunately note that equation (2.12) is compatible with
\( v = 0 \) only when:

\[
B = \left( 0, 0, \frac{1}{r \sin \phi} g(t - r/c) \right) \quad (2.17)
\]

\[
E = \left( 0, \frac{c}{r \sin \phi} g(t - r/c), 0 \right) \quad (2.18)
\]

which manifest singularities at the points corresponding to \( \phi = 0 \) and \( \phi = \pi \).
In general, we have the following statement:

\[
\text{div} E = \frac{1}{r^2} g(t - r/c) \left( \frac{\cos \phi}{\sin \phi} f(\phi) + f'(\phi) \right) = 0 \quad \Leftrightarrow \quad f(\phi) = \frac{1}{\sin \phi} \quad (2.19)
\]

Note that such a strong singularity at the poles cannot be removed only by
requiring the wave-front not to be perfectly spherical.

We observe that \( f \) can be taken in such a way that \( \text{div} E \) is vanishing at
the poles (for example \( f(\phi) = (\sin \phi)^2 \)), but not in the proximity of them.
In addition, we observe that, if \( f \) is regular with \( f(0) = f(\pi) = 0 \), for any
fixed \( r \), the points in which the divergence of \( E \) does not vanish belong to
a bidimensional set whose measure is different from zero. For instance, if
\( f(\phi) = \sin \phi \), we find out that \( \text{div} E \) is proportional to \( \cos \phi \), so that this
set consists of all points of the sphere of radius \( r \), with the exception of the
equator. It is certainly true that even if the divergence is not zero, it
is negligible when designing, for instance, a device like an antenna. This
argument, however, is not going to be valid here, since we would like to
carry out an in depth analysis of what is really happening in the evolution
of an electromagnetic wave, compared to what the Maxwell theory is able
to predict.

Let us now follow a different path and try to find other solutions, of the
form given in (2.7), satisfying the set of all Maxwell equations (including
\( \text{div} E = 0 \)). If we do not want \( f \) to be singular somewhere, we have to accept
that \( v \) is different from zero. This means that \( E \) has a radial component,
so that the Poynting vector cannot be perfectly radial. We have to better check what happens in this last case.

It is well-known that the Maxwell equations lead to:

\[
\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \Delta \mathbf{E} \quad \text{and} \quad \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = \Delta \mathbf{B}
\] (2.20)

The above are usually called “wave equations”, but, shortly, we will see that this name is not appropriate. The terminology is correct only if the fields involved are scalar. By deriving (2.11) with respect to time and using (2.12) and (2.13), we arrive at the equation:

\[
\frac{1}{c^2} u_{tt} = \frac{1}{r^2} (r^2 u_r)_r + \frac{1}{r^2 \sin \phi} (u \sin \phi)_\phi
\] (2.21)

corresponding to the third component of the second equation in (2.20) in spherical coordinates.

It is worthwhile noting that (2.21) is not the wave equation for the scalar field \(u\) in spherical coordinates, due to the fact that in this framework the Laplacian of a vector field is not the Laplacian of its coordinates (even if only one of them is different from zero). The wave equation for \(u\) reads as follows:

\[
\frac{1}{c^2} u_{tt} = \frac{1}{r^2} (r^2 u_r)_r + \frac{1}{r^2 \sin \phi} (u \sin \phi)_\phi = \Delta u
\] (2.22)

This is not a trivial warning, since many texts in electromagnetism erroneously confuse (2.22) with (2.21). Implicitly, we made the same mistake before, when looking for d’Alembert type solutions of the form (2.16), generating, for this reason, solutions not compatible with all the Maxwell equations.

By separation of variables, for any \(k \geq 1\) and any \(n \geq 1\), we discover that (2.21) admits the following basis of solutions:

\[
\begin{align*}
& r^{-\frac{1}{2}} \cos(ckt) J_{n+\frac{1}{2}}(kr) \sin \phi P'_n(\cos \phi) \\
& r^{-\frac{1}{2}} \sin(ckt) J_{n+\frac{1}{2}}(kr) \sin \phi P'_n(\cos \phi) \\
& r^{-\frac{1}{2}} \cos(ckt) Y_{n+\frac{1}{2}}(kr) \sin \phi P'_n(\cos \phi) \\
& r^{-\frac{1}{2}} \sin(ckt) Y_{n+\frac{1}{2}}(kr) \sin \phi P'_n(\cos \phi)
\end{align*}
\] (2.23)

where \(J_{n+\frac{1}{2}}\) and \(Y_{n+\frac{1}{2}}\) are Bessel functions of first and second kind respectively, while \(P'_n\) is the \(n\)-th Legendre polynomial.
A classical reference for Bessel functions is [11]. It is important to note that the solutions given in [11] at page 127, for the scalar wave equation in spherical coordinates, differ from the ones shown in (2.23). The reason is that the functions in [11] (having $P_n(cos\phi /sin\phi P'_n(cos\phi)$ in place of $sin\phi P'_n(cos\phi)$) are those solving (2.22), which is not the vector version of the wave equation, as we already mentioned.

For example, if $n = 1$ we have (see [11], p.54):

$$J_{3/2}(kr) = \sqrt{2\pi kr / (sin kr - cos kr)}$$

$$Y_{3/2}(kr) = \sqrt{2\pi kr / (cos kr + sin kr)}$$

$$P'_1(cos\phi) = 1$$ (2.24)

In order to understand what the solutions in (2.24) look like, it is standard to introduce some approximation. Thus, for $n = 1$ and $r$ large, by taking the combination $r^{-1/2}(sin(ckt)J_{3/2}(kr) + cos(ckt)Y_{3/2}(kr)) sin\phi$, up to multiplicative constants, it is possible to get asymptotically the monocromatic solution $u = r^{-1} sin\phi sin k(ct-r)$ (compare to (2.16)), up to an error which decays quadratically with $r$. Once again, one ends up with something similar to a travelling wave, although some cheating has been necessary (that is equivalent, in the end, to replacing once again (2.22) by (2.21)).

On the other hand, suppose that $u$ is evaluated exactly as linear combination of the functions in (2.23). Then, one recovers $v e w$ by (2.12) and (2.13), through time integration. Successively, it is possible to compute the Poynting vector:

$$P = -c^2 (u \int (u_r + u_r) dt, \frac{u}{r} \int (u \cos \phi / \sin \phi + u_\phi) dt, 0)$$ (2.25)

which has, as expected, a non radial component. Now, let us fix $r$ and study the behavior, by varying $\phi$, of the two components of $P$. In particular, we are interested to see what happens near the poles ($\phi = 0$ or $\phi = \pi$). We start by noting that, for any $n \geq 1$, the term $P'_n(cos\phi)$ tends towards a finite limit for $\phi \to 0$ or $\phi \to \pi$ (recall that $P'_n(\pm1) = \frac{1}{2}(\pm1)^{n+1}n(n+1)$). Therefore, according to (2.23), the first component in (2.25) behaves as $(sin\phi)^2$ near the poles. It is a matter of using known properties of Legendre polynomials, in particular the differential equation:

$$(sin\phi)^2 P''_n(cos\phi) - 2 cos\phi P'_n(cos\phi) + n(n+1)P_n(cos\phi) = 0$$ (2.26)

to check that the second component in (2.25) behaves as $sin\phi$ near the poles.
We are ready to draw some preliminary conclusions. Let us note that finally \( \text{div}\mathbf{E} = 0 \), hence all the Maxwell equations are satisfied. As already remarked, it has been necessary to keep the nonradial component of \( \mathbf{P} \). Surprisingly, for any fixed \( r \), such a nonradial component prevails on the radial one, when approaching the poles. This implies that the shape of the wave-fronts does not resemble a sphere, but rather a kind of doughnut with the central hole reduced to a single point. The parts of the fronts corresponding to the internal side of the doughnut, progressively stratify along the \( z \)-axis. We do not see a chance of recognising any sort of Huygens principle here. This is not what we would call a travelling wave. It may be argued that this behavior is due to the influence of the source located at \( r = 0 \). But, if we stop the source, the wave-fronts already produced continue to develop. If their motion is ruled by the Huygens principle, the hole should fill up quickly, and each front should transform to something rounded which is almost a perfect sphere. The problem is that, during this smoothing process, the vector fields \( \mathbf{E} \) and \( \mathbf{B} \) are not compatible with both the constraints \( \text{div}\mathbf{E} = 0 \) and \( \text{div}\mathbf{B} = 0 \). The clue is that “wave equations” in vector form have nothing to do with real waves.

Some mild analogy between the Maxwell equations and the eikonal equation, governing the movement of the fronts, was devised a long time ago. The equivalence is valid within the limits of geometrical optics (see [5], p.110). In spite of this, examining the behavior of the fronts, our impression is that their natural evolution is in conflict with all restrictions imposed by Maxwell equations. This statement will be clearer as we proceed with our study. The right connections with the eikonal equation will be defined in section 10.

The different situations analyzed up to now are summarized in figure 1, which should clarify our point of view: either we keep the singularity at the poles (manifested by infinite amplitude of the fields or strong geometrical distortion), or we allow the divergence of the electric field to be different from zero. To sustain this proposition, let us collect other elements.

Some confusion usually arises when one tries to simulate the evolution of a “fragment” of wave. We examine the case of the plane wave given in (2.5). For any fixed \( z \), we can cut out a region \( \Omega \) in the plane determined by the variables \( x \) and \( y \), and follow its evolution in time. For simplicity, \( \Omega \) can be the square \([0,1] \times [0,1]\). Inside \( \Omega \) we assume that the electromagnetic fields evolve following (2.5), in full agreement with Maxwell equations. Outside \( \Omega \), the fields \( \mathbf{E} \) and \( \mathbf{B} \) are supposed to vanish. The question is understanding what happens at the boundary \( \partial\Omega \) of \( \Omega \). It is not difficult to realize that, on the sides \( \{0\} \times [0,1[ \) and \( \{1\} \times ]0,1[ \), \( \text{curl}\mathbf{B} \) and \( \text{div}\mathbf{E} \) become singular,
Figure 1: Qualitative behavior of the field $\mathbf{E}$ as a function of the angle $\phi$. Case 1: $\mathbf{E} = (0, r^{-1}(\sin \phi)g(t - r/c), 0)$, the wavefronts are perfect spheres, but $\text{div}\mathbf{E} \neq 0$. Case 2: $\mathbf{E} = (0, (r\sin \phi)^{-1}g(t - r/c), 0)$ the condition $\text{div}\mathbf{E} = 0$ is satisfied, but there are singularities at $\phi = 0$ and $\phi = \pi$; Case 3: the corresponding Poynting vector is given in (2.25), the divergence of the electric field is vanishing, but the wave-fronts are far from being spherical surfaces.

producing concentrated distributions. Similarly, on the two sides $]0, 1[\times\{0\}$ and $]0, 1[\times\{1\}$, the quantities $\text{curl}\mathbf{E}$ and $\text{div}\mathbf{B}$ present singularities.

Some readers may complain because discontinuities of the fields may not exist in nature. Commonly, the right way to proceed is to consider a thin layer around $\partial\Omega$, where the solution given by (2.5) smoothly decays to zero. Then, one lets the width of the layer tend towards zero. This in general allows us to determine special relations to be satisfied on $\partial\Omega$ (in place of the Maxwell equations, which are meaningless there). Unfortunately, the procedure presents some drawbacks. Let us first assume that the wave-fronts shift along the $z$-axis maintaining their squared shape. We also assume that the fields $\mathbf{E}$ and $\mathbf{B}$ are orthogonal and smoothly decaying to zero in a neighbourhood of $\partial\Omega$ (like for instance in figure 2). Our conjecture is that
there exists at least one point where Maxwell equations are not all satisfied, because $\text{div}\mathbf{E}$ and $\text{div}\mathbf{B}$ cannot both be zero at the same time. Actually, examining figure 2, we discover that there are infinite points where either $\text{div}\mathbf{E} \neq 0$ or $\text{div}\mathbf{B} \neq 0$. These points form a set whose area is different from zero. We are free to try other configurations by modifying the orientation of the vector fields at each point near $\partial \Omega$, but we always arrive at the same conclusion: some rule of Physics breaks down when approaching $\partial \Omega$. Now, the question is: if we do not know what the governing rules are in the layer around $\partial \Omega$, how can we go to the limit for the size of the layer tending to zero?

Another possibility is that the wave-fronts, due to the strong variation of the fields near the boundary of $\Omega$, are forced to bend a little. The electromagnetic fields are no longer on a plane, so we could probably find out the way to enforce all the Maxwell equations. However, this implies that the Poynting vectors are not parallel to the $z$-axis anymore. Thus, the shape of $\Omega$ is going to be further modified during the evolution. A little diffusion is bearable, yet our impression is that the wave-fronts would rapidly change their form. The more they bend, the faster they produce other distortions. This is in contrast for instance with the fact that neat electromagnetic signals, of arbitrary transversal shape, reach our instruments after travelling for years between galaxies. The only acceptable rule is that all the Poynting vectors must stay orthogonal to the fronts and parallel to the actual direction of movement; if this does not happen the wave quickly deteriorates, fading completely.

To prove what we claimed before, we show using very standard arguments that it is not possible to construct solutions to Maxwell equations, having finite energy and travelling unperturbed at constant speed along a straight-line. We assume that the speed is $c$ and the straight-line is the $z$-axis. Without loss of generality, such a signal-packet is supposed to be of the following type:

\[
\mathbf{E} = \left( E_1(x, y), E_2(x, y), E_3(x, y) \right) g(t - z/c) \\
\mathbf{B} = \left( B_1(x, y), B_2(x, y), B_3(x, y) \right) g(t - z/c)
\]

where $g$ is a bounded function and all the components $E_1, E_2, E_3, B_1, B_2, B_3$ are zero outside a bidimensional set $\Omega$. It is not difficult to check that (2.1) and (2.3) only hold when $\mathbf{E}$ and $\mathbf{B}$ are identically zero. Actually, it is straightforward to discover that $E_3$ and $B_3$ must be constant (and the sole
constant allowed is zero). Then, one finds out that $E_1, E_2, B_1, B_2$ must be harmonic functions in $\Omega$. Since they have to vanish at the boundary, they must vanish everywhere.

Due to the above mentioned reasons, solitonic solutions are not described by the classical theory of electromagnetism. Efforts have been made in the past to generalize the Maxwell model, in a nonlinear way, in order to include solitons. Just to mention an example, the Born-Infeld theory (see [4]) predicts the existence of finite-energy soliton-like solutions (that have been successively called BIons). These last equations have no relation with the ones we are going to develop in this paper. However, they point out the necessity of looking for nonlinear versions of the model. We will come back

Figure 2: Example of electromagnetic field smoothly reducing to zero at the boundary of the square. The Poynting vector is orthogonal to the page at each point. The Maxwell equations are satisfied in the central part. Instead, approaching the boundary, the divergence of the fields turns out to be different from zero.
to the subject of solitary waves in section 5.

In many applications, a standard approach is to reconstruct the bidimensional profile of the fields inside $\Omega$ with the help of a truncated Fourier series. This is accomplished by a complete orthogonal set of plane waves, each one carrying a suitable eigenfunction in the variables $x$ and $y$. We must pay attention, however, to the fact that these eigenfunctions are of the periodic type. Therefore, they reproduce the same profile, not only inside $\Omega$, but in a lattice of infinite contiguous domains. In this way, the represented solution turns out to have infinite energy. Considering only one of these profiles, thus forcing to zero the solution outside $\Omega$, unavoidably brings us again to a violation of the Maxwell equations near the boundary of $\Omega$. Some clarifying comments on this issue can be found in [8], p.42.

We recognize that the techniques based on Fourier expansions provide excellent results in many practical circumstances, as for example the study of diffraction. Nevertheless, in this last case and in the ones treated before, it is necessary to adapt the solutions, introducing some approximation, if we want them to correspond to the real phenomenon. Indeed, these adjustments are within the so-called limits of the model. Hence, we could just stop our analysis here, with the trivial (well-known) conclusion that the Maxwell model is not perfect. We believe instead that the discrepancies pointed out are not just imperfections, but symptoms of a more profound pathology affecting the theory of classical electromagnetism.

What we learn in these pages is that there are plenty of simple and interesting phenomenon, which are inadequately explained by the Maxwell model, because the equations impose too many restrictions. Consequently, the idea we shall follow in the next section is of weakening the equations, with the aim of widening the range of solutions.

### 3 Modified Maxwell equations

The demolition process is finished, now it is time to rebuild. To begin, we propose the following model:

\[
\frac{\partial \mathbf{E}}{\partial t} = c^2 \text{curl} \mathbf{B} - c (\text{div} \mathbf{E}) \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E} \times \mathbf{B}|} \tag{3.1}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = -\text{curl} \mathbf{E} \tag{3.2}
\]
\[
\text{div} \mathbf{B} = 0 \tag{3.3}
\]

that will be further adjusted in the subsequent sections. The norm \(|\cdot|\) is the usual one in \(\mathbb{R}^3\), i.e.: \(|(x, y, z)| = \sqrt{x^2 + y^2 + z^2}\). We define \(\mathbf{J} = \mathbf{P}/|\mathbf{P}| = (\mathbf{E} \times \mathbf{B})/|\mathbf{E} \times \mathbf{B}|\). Note that, when \(\mathbf{P} = 0\), the direction of \(\mathbf{J}\) is not determined (see also the comments at the end of section 7). The vector \(\mathbf{J}\) is supposed to be adimensional (or, equivalently, \(\mathbf{P}/|\mathbf{P}|\) is multiplied by a constant, equal to 1, whose dimension is the inverse of the dimension of \(\mathbf{P}\)). Consequently, \(c\mathbf{J}\) is a velocity vector.

As the reader may notice, the “awkward” relation \(\text{div} \mathbf{E} = 0\) has been eliminated. It is also evident that in all the points in which \(\text{div} \mathbf{E} = 0\), we find again the classical Maxwell system. This states that the solutions of (2.1)-(2.2)-(2.3)-(2.4) are also solutions of (3.1)-(3.2)-(3.3). Therefore, the replacement of (2.1)-(2.2) by (3.1) brings us to the property we wanted, that is the enlargement of the range of solutions.

Afterwards, we have to understand and justify what is happening from the point of view of Physics. Let us recall that we are in empty space, and that there are no electrical charges or masses anywhere. We are only examining the behavior of waves. In spite of that, we pretend that there may be regions where \(\text{div} \mathbf{E} \neq 0\). The situation is not alarming, since we checked that the condition \(\text{div} \mathbf{E} \neq 0\) is quite frequent in the study of waves. Anyway, such a hypothesis is acceptable, as long as it is coherent with the basic laws of Physics. First of all, we observe that the equation (3.1) has been obtained by adding a nonlinear term to (2.1). The term has strong analogy with the corresponding one of classical electromagnetism, appearing on the right-hand side of (2.1) as a consequence of the Ampère law, and due to the presence of moving charges. In fact, by setting \(\rho = \text{div} \mathbf{E}\), the vector \(\rho c \mathbf{J}\) can be assimilated, up to dimensional constants, to an electric current density. Thus, even if in our case there are no real charged particles, we have to deal with a continuous time-varying medium, consisting of infinitesimal electrical charges, living with the electromagnetic wave during its evolution. Moreover, the added term does not compromise the theoretical study of a functioning device like an antenna, since, at a certain distance, the quantity \(\text{div} \mathbf{E}\) is negligible.

By taking the divergence of (3.1), we get a very important relation:

\[
\frac{\partial \rho}{\partial t} = -c \text{div}(\rho \mathbf{J}) \tag{3.4}
\]

which is, actually, the continuity equation for the density \(\rho = \text{div} \mathbf{E}\). The equation (3.4) testifies to the presence of a transport, at the speed of light,
along the direction determined by $J$. Hence, something is flowing together with the electromagnetic fields; something that later, in sections 9 and 10, will be compared to a true mechanical fluid. On the other hand, this was also the interpretation at the end of the 19th century, before the theory of fields was rigorously developed. The fluid changes in density, but preserves its quantity, as stated by the continuity equation. It is extremely significant to remark that this property comes directly from (3.1), so it is not an additional hypothesis. In section 12, based on the density $\rho$, we will construct a mass tensor that, due to (3.1), can be perfectly combined with the standard electromagnetic energy tensor. The skilful reader has already understood that this will allow us to find the link between electromagnetic and gravitational fields.

We are now going to collect other properties about the new set of equations. Considering that $E \times B$ is orthogonal to both $E$ and $B$, a classical result is obtainable:

\[
\frac{1}{2} \frac{\partial}{\partial t} (|E|^2 + c^2 |B|^2) = c^2 (\text{curl} B \cdot E - \text{curl} E \cdot B) = -c^2 \text{div} P \quad (3.5)
\]

where the quantity $|E|^2 + c^2 |B|^2$, up to a multiplicative dimensional constant, is related to the energy of the electromagnetic field. Thus, the nonlinear term in (3.1) is not disturbing at this level, and the Poynting vector $P$ preserves its meaning.

By noting that $J \cdot J = 1$ and that $E \cdot J = 0$, we get another interesting relation:

\[
E \cdot \frac{\partial J}{\partial t} = \frac{\partial (E \cdot J)}{\partial t} - \frac{\partial E}{\partial t} \cdot J = -c^2 (\text{curl} B) \cdot J + c \text{ div} E \quad (3.6)
\]

Finally, one has:

\[
\frac{\partial^2 B}{\partial t^2} = -\text{curl} \frac{\partial E}{\partial t} = -c^2 \text{curl}(\text{curl} B) + c \text{curl}(\rho J)
\]

\[
= -c^2 \nabla(\text{div} B) + c^2 \Delta B + c \text{curl}(\rho J) \quad (3.7)
\]

from which we deduce the following second-order vector equation with a nonlinear forcing term:

\[
\frac{\partial^2 B}{\partial t^2} = c^2 \Delta B + c \text{curl}(\rho J) \quad (3.8)
\]

that generalizes the second equation in (2.20). We are sorry to announce that the “wave” equations for the fields $E$ and $B$ are no longer true. On the
other hand, it has emerged in section 2 that, in vector form, they are only a source of a lot of trouble.

In the classical Maxwell equations the role of the field $\mathbf{E}$ can be interchanged with that of field $c\mathbf{B}$. This is not true for the new formulation. We will later see, in section 9, how to solve this problem. For the moment, we keep working with (3.1)-(3.2)-(3.3), just because the theory will be more easy. In the coming sections 4 and 5, we will see how elegantly it is possible to solve the problems raised in section 2.

4 Perfect spherical waves

In the case of a plane wave of infinite extension, for both the Maxwell model and the new one, we are able to enforce the condition $\text{div}\mathbf{E} = 0$ and realize the orthogonality of the Poynting vectors with respect to the propagation fronts. Concerning a “fragment” of plane wave, the classical method runs into problems. However, with the new approach the situation radically improves. Let us see why.

With the same assumptions of section 2, let $\Omega$ be the square $[0,1] \times [0,1]$. We already noted that, on the sides $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$, the quantities $\text{div}\mathbf{E}$ and $\text{curl}\mathbf{B}$ become infinite. Nevertheless, when substituted into equation (3.1), they come to a difference of the type $+\infty - \infty$. The two singular terms reciprocally cancel out, leaving a finite quantity, so that the equation has a chance to be satisfied. To show this, we can create a layer around the boundary of $\Omega$. Then, without developing singularities, we pass to the limit for the width of the layer tending to zero. The trick now works, because, in contrast to the classical Maxwell case, equation (3.1) can be satisfied exactly in all the points, since it is compatible with the condition $\text{div}\mathbf{E} \neq 0$. In the limit process we can also guarantee that the Poynting vectors remain parallel to the $z$-axis. Therefore, the fragment does not change its transversal shape. Explicit computations will be carried out in section 5, in the case in which $\Omega$ is a circle.

For example, the situation represented in figure 2 is perfectly allowed for by our equations, except near the lower and the upper sides. Actually, on the sides $]0,1[ \times \{0\}$ and $]0,1[ \times \{1\}$, given that $\text{div}\mathbf{B}$ and $\text{curl}\mathbf{E}$ are singular, we still have problems (clearly equation (3.2) and (3.3) are not true). Similar problems are encountered by modifying the polarization of the wave. These troubles will be solved in section 9, by unifying (3.2) and (3.3) in a single
equation similar to (3.1), in such a way that the roles of \(E\) and \(cB\) are interchangeable.

The case of a spherical wave is very interesting. Let us consider the transformation of coordinates given in (2.6). Let us also suppose that the fields are given as in (2.7), with \(u, v, w\) not depending on \(\theta\). We have:

\[
\mathbf{E} \times \mathbf{B} = (uw, -uv, 0) \quad (4.1)
\]

\[
\mathbf{J} = \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E} \times \mathbf{B}|} = \frac{s(u)}{\sqrt{v^2 + w^2}}(w, -v, 0) \quad (4.2)
\]

where \(s(u) = u/|u|\) is the sign of \(u\).

The new equations in spherical coordinates become:

\[
u_t = - \left( w_r + \frac{w}{r} \right) + \frac{v_\phi}{r} \quad (4.3)
\]

\[
v_t = \frac{c^2}{r} \left( \frac{u \cos \phi}{\sin \phi} + u_\phi \right) - c s(u) \frac{v_r + \frac{2v}{r} + \frac{1}{r} w_\phi + \frac{\cos \phi}{r \sin \phi} w}{\sqrt{v^2 + w^2}} \quad (4.4)
\]

\[
w_t = - c^2 \left( \frac{u}{r} + u_r \right) + c s(u) \frac{v_r + \frac{2v}{r} + \frac{1}{r} w_\phi + \frac{\cos \phi}{r \sin \phi} w}{\sqrt{v^2 + w^2}} \quad (4.5)
\]

These expressions may seem rather complicated, but there is nothing to be afraid of.

To avoid discontinuities, we also impose the boundary conditions (2.14) and (2.15). Now, by choosing \(v = 0\) and \(w = cu\), one obtains:

\[
\frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E} \times \mathbf{B}|} \text{div} \mathbf{E} = \left( \frac{cu \cos \phi}{r \sin \phi} + \frac{c}{r} u_\phi, 0, 0 \right) \quad (4.6)
\]

Therefore, one gets:

\[
u_t = - \left( w_r + \frac{w}{r} \right) \quad (4.7)
\]

\[
v_t = 0 \quad (4.8)
\]

\[
w_t = - c^2 \left( u_r + \frac{u}{r} \right) \quad (4.9)
\]
Once again, the first and the last equations are equivalent, providing the general solution (2.16). Anyway, this time, thanks to the nonlinear corrective term of (3.1), the second equation is compatible with \( v = 0 \). We are not obliged to choose \( f(\phi) = 1/\sin \phi \), in order to enforce the condition \( \text{div} \mathbf{E} = 0 \), because this constraint is no longer required. The conclusion is that perfect spherical waves are admissible with the new model. The functions \( f, e, g \) may be truly arbitrary (the only restriction is \( f(0) = f(\pi) = 0 \)). Continuing with our analysis, we will construct later infinite other solutions which are unobtainable with the classical Maxwell model.

In the perfect spherical case, the Poynting vector \( \mathbf{P} = (cu^2, 0, 0) \) only has the radial component different from zero. As expected, this component has a constant sign (even if \( u \) and \( w \) oscillate). Since the set of equations is of a hyperbolic type, we can introduce the characteristic curves. In the example of the spherical wave, such curves are semi straight-lines emanating from the point \( r = 0 \), and the vector \( \mathbf{J} = \mathbf{P}/|\mathbf{P}| = (1, 0, 0) \) is aligned with them. The nonlinear term introduced in (3.1) does not adversely affect the behavior of the wave, because, with \( v = 0 \) and \( w = cu \), the corresponding equations (4.7) and (4.9) are linear. Therefore, the superposition principle is still valid. Any piece of information, present at the boundary of the sphere of radius \( r > 0 \), propagates radially at the speed of light, without being disturbed (except by the natural decay in intensity). The nonlinear effects of the model are latent. They show up when we try to force, with some external solicitations, the Poynting vector not to follow the characteristic lines. This circumstance will be taken into account in sections 7 and 8.

In a very mild form, we can state that the divergence vanishes, by observing that, for any \( T \):

\[
\int_{T}^{T+2\pi/\omega} \text{div} \mathbf{E} \, dt = 0
\]

that is \( \text{div} \mathbf{E} \) has zero average when integrated over a period of time. Nevertheless, in section 13, we will get an astonishing result. We will see that an electromagnetic wave produces, during its passage, a modification of space-time. In the new geometry, the 4-divergence of the electric field is zero. This could make it difficult, or even impossible, to set up an experiment that emphasizes the condition \( \text{div} \mathbf{E} \neq 0 \) at some point. The measure could be affected by the modified space-time geometry in such a way the condition cannot be revealed.

Among the stationary solutions we find:

\[
u(t, r, \phi) = \frac{K_1}{r \sin \phi} \quad v(t, r, \phi) = \frac{K_2}{r^2} \quad w(t, r, \phi) = \frac{K_3}{r \sin \phi}
\]
as well as:

\[ u(t, r, \phi) = \frac{K_1}{r \sin \phi} \quad v(t, r, \phi) = K_2 \cos \phi \quad w(t, r, \phi) = -K_2 \sin \phi \]  

(4.12)

with \( K_1, K_2, K_3 \) arbitrary constants. In particular, we recognize the classical stationary electric field:

\[ \mathbf{E} = (K_2r^{-2}, 0, 0) \]

whose divergence is zero for any \( r > 0 \). Unfortunately, most of these solutions show singularities.

Due to the nonlinearity of the equations, the study of the interference of waves looks quite complicated. As long as the waves are such that \( \text{div} \mathbf{B} = 0 \) and \( \text{div} \mathbf{E} = 0 \) (as in the plane case) there are no problems, since the nonlinear terms do not actually activate. For waves of different shape, the situation may be truly intricate. In first approximation, the nonlinear effects should attenuate faster than the amplitude of the waves. Thus, at a certain distance, these anomalies may not normally be observed. Although we do not wish to discuss it here, the subject is of crucial relevance and deserves to be studied in more detail.

## 5 Travelling signal-packets

In this section, it is convenient for us to express our new set of equations in cylindrical coordinates. After taking \((x, y, z) = (r \cos \theta, r \cos \theta, z)\), we assume that the fields are of the form \( \mathbf{B} = (0, 0, u) \), \( \mathbf{E} = (v, w, 0) \), where the first component is referred to the variable \( r \), the second one to the variable \( z \) and the third one to the variable \( \theta \). Moreover, for simplicity, the functions \( u, v \) and \( w \) will not depend on \( \theta \). In cylindrical coordinates, the counterparts of equations (4.3)-(4.4)-(4.5) are:

\[
\begin{align*}
    u_t &= -u_r + v_z \\
    v_t &= c^2 u_z - c s(u) \frac{v_r + \frac{v}{r} + w_z}{\sqrt{v^2 + w^2}} w \\
    w_t &= -c^2 \left( \frac{u}{r} + u_r \right) + c s(u) \frac{v_r + \frac{v}{r} + w_z}{\sqrt{v^2 + w^2}} v
\end{align*}
\]  

(5.1) (5.2) (5.3)
By setting $v = 0$, we can easily find solutions when $u$ and $w$ do not depend on $z$. In this case one has $\text{div} B = 0$ and $\text{div} E = 0$. From (5.1) and (5.3) it is easy to get the equations:

$$
\begin{align*}
  u_{tt} &= c^2 \left( \frac{u}{r} + u_r \right)_r \\
  w_{tt} &= c^2 \left( \frac{w}{r} + w_r \right)_r
\end{align*}
$$

(5.4)

whose solutions are related to Bessel functions.

Anyhow, extremely interesting solutions in cylindrical coordinates turn out to be the following ones:

$$
\begin{align*}
  u(t, r, z) &= g(t \pm z/c)f(r) \\
  v(t, r, z) &= \pm c g(t \pm z/c)f(r) \\
  w(t, r, z) &= 0
\end{align*}
$$

(5.5)

Note that the divergence of $E$ is equal to $\rho = v_r + r^{-1}v$, so that it is different from zero, unless $f$ is proportional to $1/r$. The functions $f$ and $g$ can be arbitrary (to guarantee the continuity of the vector fields, we only impose $f(0) = 0$). The relations in (5.5) give raise to electromagnetic waves shifting at the speed of light along the $z$-axis. If $f$ and $g$ vanish outside a finite measure interval, for any fixed time $t$ the wave is constrained inside a bounded cylinder. The packet travels unperturbed for an indefinite amount of time. The corresponding field $E$ is perfectly radial and the vector $J$ is parallel to the $z$-axis.

Given $r_0 > 0$, suppose that $f$ is zero for $r > r_0$. Suppose also that $f$, in a neighborhood of $r_0$ has a sharp gradient. It is evident that the vector $c^2 \text{curl} B - c(\text{div} E) J = (c^2 u_z, 0, 0) = (\pm cg'f, 0, 0)$ remains bounded even if we let the derivative of $f$ tend to $\infty$ at $r_0$. Therefore, as we anticipated at the beginning of section 4, we can give a meaning to equation (3.1), even if $f$ is a discontinuous function in $r_0$.

We can get a transport equation for the unknown $\rho = \text{div} E$ by using the equation (3.3), i.e.:

$$
\frac{\partial \rho}{\partial t} = - c \text{div}(\rho J) = - c \rho \text{div} J - c \nabla \rho \cdot J
$$

(5.6)

which, thanks to the fact that $J$ is a constant field, takes the simplified form of:

$$
\frac{\partial \rho}{\partial t} = \pm c \frac{\partial \rho}{\partial z}
$$

(5.7)

with the sign depending on the orientation of $J$. 

20
We recall that the Maxwell equations do not allow for the existence of solitary waves, as the ones we have introduced right now. Therefore, here we obtained another important result.

The energy $E$ of these solitary waves is obtained by integrating the energy density, given by: $\frac{1}{2}\varepsilon_0(|E|^2 + c^2|B|^2)$. Thus, one gets:

$$E = 2\pi\varepsilon_0c^2\int_0^{\infty} f^2(r)rdr \int_{-\infty}^{+\infty} g^2(\xi)d\xi$$

(5.8)

Suppose that, at an initial time $t_0$, the electromagnetic fields are assigned compatibly with (5.5). The vector $J$ turns out to be automatically determined, then the wave is forced to move in the direction of $J$ at speed $c$. There are no stationary solutions, unless $g$ is constant. But, in this last case, due to (5.8), the energy is not going to be finite. The wave-packets take their energy far away, with no dissipation, until they react with other waves or more complicated structures (like, for instance, particles).

Let us study more closely the expressions given in (5.5). Assume that, for $r \geq 0$, the function $f$ is non negative, and that $f(0) = 0$. If the function $g$ has a constant sign, we distinguish between two cases, depending if the sign is positive or negative (see figure 3). The sign determines the “orientation” of the vector $\text{curl} B$ (note however that $\text{curl} B$ is not exactly parallel to the $z$-axis, despite what is shown in figure 3). Then, we have subcases, depending whether $E$ is directed toward the $z$-axis or not. In conclusion, two possible types of solitary waves can occur, depending on the orientation of the electric field (external or internal). These will be denoted by $\gamma^+$ and $\gamma^−$, respectively. In figure 3, $J$ indicates the direction of motion. Of course, $g$ could also have a non-constant sign. In this case, the corresponding wave can be seen as a sequence of waves of type $\gamma^+$ and $\gamma^−$, shifting one after the other.

In vacuum, the electromagnetic fields at rest, are assumed to be identically zero. During the passage of a soliton, the calm is momentarily broken only at the points “touched” by the wave. The information shifts, but does not irradiate. Thus, also if the wave-packet displays a negative or positive sign, depending on the orientation of $E$, this is not in relation to what is usually called electric charge. Hence, as long as the cylinders containing two different solitons do not collide, they can cross very near without influencing each other. On the contrary, we expect some scattering phenomena, through a mechanism that will be studied in section 8.

If we place a mirror parallel to the $z$-axis, at some distance from it, the reflected image of the travelling wave-packet will be the same wave-packet shifting in the opposite direction (because $\text{curl} B$ changes sign, while
Figure 3: Behavior of the electric field for the two possible wave-packets (section for a fixed angle $\theta$): $\gamma^-$ shifting from left to right, $\gamma^+$ shifting from right to left, $\gamma^-$ shifting from right to left, $\gamma^+$ shifting from left to right.

$E$ maintains its orientation). This disagrees with our common sense. In other words, equation (3.1) does not preserve plane symmetries. The same is true for the Maxwell equations. In both cases we have no elements to decide the correct sign of the vector product $\times$ (left-hand or right-hand). As a matter of fact, without modifying the equations, a change of the sign of $\times$ can be compensated for by a change of the sign of the electric (or the magnetic) field. To learn more about this problem we need to study the interactions between waves and matter. Hence, for the moment, we have no sufficient information to understand from which part of the mirror is our universe. An answer to these crucial questions will be given in section 8.

In cylindrical coordinates, we can find many other solitonic solutions.
Here is another example:

\[
E = (\pm cu, 0, \mp cv) \quad B = (v, 0, u)
\]

(5.9)

with \(u = f_1(r, \theta)g(t \pm z/c)\) and \(v = f_2(r, \theta)g(t \pm z/c)\). In order to fulfill the condition \(\text{div}\mathbf{B} = 0\), it is necessary to impose:

\[
\frac{\partial f_1}{\partial \theta} + \frac{\partial (rf_2)}{\partial r} = 0
\]

(5.10)

Note that \(\mathbf{E} \cdot \mathbf{B} = 0\) and \(\mathbf{J} = (0, \mp 1, 0)\). Except for the condition (5.10), the functions \(f_1\) and \(f_2\) are arbitrary, so that the new solutions are very general. Actually, they include the previous ones (take \(f_1 = f\) and \(f_2 = 0\)). In section 9, we will remove the condition \(\text{div}\mathbf{B} = 0\), allowing for the existence of even more solutions. We can force \(f_1\) and \(f_2\) to vanish outside a bidimensional domain \(\Omega\). Again, this plane front, modulated by the function \(g\), travels along the \(z\)-axis at the speed of light.

We may reasonably expect that these solitary solutions are modified, or even destroyed, when they encounter another external electromagnetic field. In fact, the equations being nonlinear, the superposition principle does not hold, in particular if the motion is disturbed in a way that is in contrast to the natural evolution along the characteristic curves. As far as we know, there are no documented experiments evidencing these facts. In section 11, we instead examine the behavior of solitary waves under the action of gravitational fields.

An electromagnetic radiation can be suitably considered as an envelope of solitary waves, travelling in the same direction. If, transversally, these solitons are of infinitesimal size, they can be compared to “light rays”. This observation clarifies how a wave can be viewed at the same time as a whole electromagnetic phenomenon and as a bundle of infinite microscopic rays.

And then there are photons. They are also pure electromagnetic manifestations, but, unfortunately, they are not modelled by the Maxwell equations. Modern atomic and subatomic physics would not exist without photons, yet they find no place in the classical theory of electromagnetism. This is an unpleasant gap. Although physicists are acquainted with this dualism, the general framework remains blurred. From our new standpoint, we contend that the photons observed in nature have very good chances to be modelled by the equations introduced here. As a matter of fact, we have enough freedom to be able to build solutions (no matter how complicated) resembling real photons. We can assign a frequency to them, longitudinally or transversally. Then, we know that they must move at the speed of light and can
have finite energy, given by the energy of their electromagnetic vector fields. They can be “positive” or “negative” without being electrical charges. Even with no mass at rest, their motion can be distorted by gravitational fields (see section 11). A concept of spin can be also introduced (see section 15). If what we are proposing here is a new functioning model for electromagnetism (and we will collect other evidence supporting this hypothesis), then it explains why photons can be self-contained elementary entities and electromagnetic emissions at the same time. In this case, a first link between classical and quantum physics is set forth.

6 Lagrangian formulation

In order to recover the equations (3.1)-(3.2)-(3.3) from the principle of least action, we follow the same path bringing to the classical Maxwell equations. Thus, we introduce the scalar potential Φ and the vector potential \( A = (A_1, A_2, A_3) \), such that:

\[
\mathbf{B} = \frac{1}{c} \text{curl} \mathbf{A} \quad \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi \tag{6.1}
\]

From the above definitions we easily get the equations: \( \text{div} \mathbf{B} = 0 \) and \( \frac{\partial}{\partial t} \mathbf{B} = -\text{curl} \mathbf{E} \). The third equation (3.1) is going to be deduced from the minimization of a suitable action function.

Let us first note that, by taking the potential \( \Phi \) equal to zero and setting \( \xi = t - (xJ_1 + yJ_2 + zJ_3)/c \), one obtains:

\[
\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial \xi} \quad c\mathbf{B} = \nabla \xi \times \frac{\partial \mathbf{A}}{\partial \xi} = -\mathbf{J} \times \frac{1}{c} \frac{\partial \mathbf{A}}{\partial \xi} \tag{6.2}
\]

This allows us to infer that \( \mathbf{B} \) is orthogonal to \( \mathbf{E} \) and that \( |\mathbf{E}| = c|\mathbf{B}| \) (see also [10], p.126).

Successively, for \( i \) and \( k \) between 0 and 3, we introduce the electromagnetic tensor:

\[
F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \tag{6.3}
\]

where \( A_0 = \Phi \) and \( (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z) \). Explicitly, we have:

\[
F_{ik} = \begin{pmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & -cB_3 & cB_2 \\
E_2 & cB_3 & 0 & -cB_1 \\
E_3 & -cB_2 & cB_1 & 0
\end{pmatrix} \tag{6.4}
\]
with $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$. Replacing $\mathbf{E}$ by $-\mathbf{E}$, one gets instead the contravariant tensor $F^{ik}$:

$$
F^{ik} = \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & -cB_3 & cB_2 \\
-E_2 & cB_3 & 0 & -cB_1 \\
-E_3 & -cB_2 & cB_1 & 0
\end{pmatrix}
$$

(6.5)

Therefore, up to multiplicative constants, the action turns out to be (see for instance [3], p.596):

$$
S = - \int F^{ik} F_{ik} dx_0 dx_1 dx_2 dx_3 = c \int F^{ik} F_{ik} dx dy dz dt
$$

(6.6)

where, summing-up over repeated indices, the Lagrangian is $L = F^{ik} F_{ik} = 2(c^2 |\mathbf{B}|^2 - |\mathbf{E}|^2)$. As customary, the variations are functions $\delta A_i$, having compact support both in space and time (between two fixed instants).

With well-known results, one obtains:

$$
\delta S = -4 \int \frac{\partial F^{ik}}{\partial x_k} \delta A_i \ dx_0 dx_1 dx_2 dx_3
$$

(6.7)

Imposing $\delta S = 0$, the corresponding Euler equations are exactly the standard Maxwell equations. As a matter of fact, due to the arbitrariness of the variations $\delta A_i$, one recovers: $- \frac{\partial}{\partial x_k} F^{ik} = 0$ (for $i = 0, 1, 2, 3$), that is equivalent to write $\text{div} \mathbf{E} = 0$ and $\frac{\partial}{\partial t} \mathbf{E} = c^2 \text{curl} \mathbf{B}$.

Let us now introduce a novelty. We require that the variations $\delta A_i$ are subjected to a certain constraint, so that the conclusions are going to be different. Actually, we impose the condition:

$$
\delta \Phi - \mathbf{J} \cdot \delta \mathbf{A} = \delta A_0 - J_1 \delta A_1 - J_2 \delta A_2 - J_3 \delta A_3 = 0
$$

(6.8)

where $\mathbf{J}$ is the vector $(\mathbf{E} \times \mathbf{B})/|\mathbf{E} \times \mathbf{B}|$, already defined in section 3. The relation (6.8) says for instance that, if the vector variation $(\delta A_1, \delta A_2, \delta A_3)$ locally belongs to the tangent plane generated by $\mathbf{E}$ and $\mathbf{B}$, then the variation $\delta A_0$ is zero. Although for the moment we can only provide vague explanations, we assert that (6.8) is the germ of the Huygens principle. The picture will become more focused as we proceed with our analysis.

Consequently, we discover that the 4-vector $- \frac{\partial}{\partial x_k} F^{ik}$ is not identically vanishing, but, due to (6.8), it must have a component along the 4-vector $(1, -\mathbf{J})$. This leads to:

$$
\text{div} \mathbf{E} = \lambda
$$

(6.9)
\[
\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - c \text{ curl} \mathbf{B} = -\lambda \mathbf{J} \quad (6.10)
\]

where the parameter \(\lambda\) is a Lagrange multiplier. By eliminating \(\lambda\) we easily arrive at equation (3.1). Thanks to (6.8), the set of possible variations is smaller than the set in which we impose no restrictions at all. Therefore, the equation \(\delta S = 0\) is now less stringent. As we already know, this shows that (3.1) admits a space of solutions which is larger than the one corresponding to (2.1) and (2.2) together.

Using the electromagnetic tensor, the equation (3.1) can be written as:

\[
-c \left( \frac{\partial F^{ik}}{\partial x_k} + \frac{\partial F^{0k}}{\partial x_k} J_i \right) = 0 \quad \text{for } i = 1, 2, 3 \quad (6.11)
\]

By defining \(J_0 = -1\), the above relation is also trivially satisfied for \(i = 0\).

In section 11, within the framework of general relativity, we will be able to write (6.11) in invariant tensor form.

Let us try to understand the reason for the constraint (6.8). As far as the evolution of a plane or a spherical wave is concerned (and, surely, in more general cases), it is easy to check that:

\[
\mathbf{A} = \Phi \mathbf{J} \quad (6.12)
\]

A straightforward way to get (6.12) is from explicit solutions. For example, in spherical coordinates, we can use (2.16) in order to find:

\[
\mathbf{E} = \left( 0, \frac{1}{r} f(\phi) g(t - r/c), 0 \right) \quad \mathbf{B} = \left( 0, 0, \frac{1}{cr} f(\phi) g(t - r/c) \right)
\]

\[
\mathbf{J} = (1, 0, 0) \quad \mathbf{A} = \left( -F(\phi) g(t - r/c), 0, 0 \right) \quad \Phi = -F(\phi) g(t - r/c) \quad (6.13)
\]

where \(F\) is such that \(F' = f\). The relation (6.12) is also true in the case of solitons. In fact, in cylindrical coordinates, thanks to (5.5) one has:

\[
\mathbf{E} = \left( \pm cf(r) g(t \pm z/c), 0, 0 \right) \quad \mathbf{B} = \left( 0, 0, f(r) g(t \pm z/c) \right)
\]

\[
\mathbf{J} = (0, \mp 1, 0) \quad \mathbf{A} = \left( 0, cF(r) g(t \mp z/c), 0 \right) \quad \Phi = \mp cF(r) g(t \mp z/c) \quad (6.14)
\]

where \(F\) is such that \(F' = f\) and \(f(0) = 0\). Note that, in general, \(\Phi\) and \(A\) are not uniquely determined. However, there exists at least one choice of \(\Phi\) and \(\mathbf{A}\) such that (6.12) is satisfied.
Now, the equation \( \mathbf{A} = \Phi \mathbf{J} \) implies \( |\Phi| = |\mathbf{A}| \), or equivalently: \( \Phi^2 - |\mathbf{A}|^2 = 0 \). Taking the variation of the last relation brings us to the constraint:

\[
2 \left( \Phi \delta \Phi - \mathbf{A} \cdot \delta \mathbf{A} \right) = 2\Phi \left( \delta \Phi - \mathbf{J} \cdot \delta \mathbf{A} \right) = 0 \quad (6.15)
\]

which is the same as in (6.8). Another way to get (6.8) is by directly evaluating the variation of (6.12):

\[
\delta \mathbf{A} = \delta (\Phi \mathbf{J}) = \delta \Phi \mathbf{J} + \Phi \delta \mathbf{J} \Rightarrow \mathbf{J} \cdot \delta \mathbf{A} = \delta \Phi \quad (6.16)
\]

where we used that \( |\mathbf{J}| = 1 \) and \( \delta \mathbf{J} \cdot \mathbf{J} = 0 \) (a normalized vector is orthogonal to its variation).

Obviously, the vector relation \( \mathbf{A} = \Phi \mathbf{J} \) implies the scalar relation:

\[
\Phi = \mathbf{J} \cdot \mathbf{A} \quad (6.17)
\]

obtainable after scalar multiplication by \( \mathbf{J} \) and by observing that \( |\mathbf{J}| = 1 \). Another way of expressing (6.17) is to require that the scalar product between the 4-vectors \((\Phi, \mathbf{A})\) and \((1, -\mathbf{J})\) is zero. This makes (6.17) an invariant in the context of general relativity.

In conclusion, the equation (3.1) can be recovered from the constrained minimization of the action function associated with the classical Lagrangian. The constraint originates from (6.12) which says, in particular, that \( \mathbf{A} \) is lined up with \( \mathbf{J} \). As will be better explained in section 10, such a condition characterizes the propagation of waves, whose evolution is ruled by the Huygens principle. From now on, these will be called “free waves”. In sections 7 and 8, we will see that not all waves are of this type.

An interesting relation, that is directly obtained by checking (6.13) or (6.14), is the following one:

\[
\mathbf{E} + c \mathbf{J} \times \mathbf{B} = 0 \quad (6.18)
\]

The above equation is extremely important, since it represents another characterization of free waves, perhaps simpler than (6.12). Indeed, it is the analogous of the Lorentz law for moving electric charges (recall that \( c\mathbf{J} \) is a velocity), even if here there are no particles around. As it will be explained in the coming sections, equation (6.18) tells us that the mechanical forces acting on a free wave are zero. Therefore, the wave actually moves without external disturbances and in agreement with the Huygens principle. We will prove all these statements in sections 9 and 10.
The covariant version of (6.18) is:

\[-cB + J \times E = 0\]  \hspace{1cm} (6.19)

obtained by vector multiplication of (6.18) by J. Both (6.18) and (6.19) can be trivially deduced from the orthogonality of E with respect to B, and by the equality |E| = c|B| (see also the beginning of this section). Therefore, they have quite a general validity.

Before ending this section, we will collect a few more properties. We begin by considering the following writing:

\[|E|^2 - c^2|B|^2 = E \cdot \left( -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \Phi \right) - c B \cdot \text{curl}A\]

\[= -\frac{1}{c} \frac{\partial}{\partial t} (E \cdot A) + \frac{1}{c} \frac{\partial E}{\partial t} \cdot A - E \cdot \nabla \Phi - c A \cdot \text{curl}B - c \text{div}(A \times B)\]

\[= -\frac{1}{c} \frac{\partial}{\partial t} (E \cdot A) + \frac{1}{c} \left( \frac{\partial E}{\partial t} - c^2 \text{curl}B \right) \cdot A - \text{div}(\Phi E) + \Phi \text{div}E - c \text{div}(A \times B)\]  \hspace{1cm} (6.20)

where we used the definitions in (6.1) and known formulas of vector calculus. Introducing the constraint A = \Phi J (hence also \Phi = J \cdot A) we have that A is orthogonal to E, because so it is J. Thus, (6.20) can be simplified and becomes:

\[|E|^2 - c^2|B|^2 = \text{div}(\Phi (E + c J \times B)) + \frac{1}{c} \left( \frac{\partial E}{\partial t} - c^2 \text{curl}B + c \text{div}(\Phi E)J \right) \cdot A\]  \hspace{1cm} (6.21)

Then, it is interesting to point out that, when both (6.18) and (3.1) are satisfied, the Lagrangian vanishes. On the contrary, when E is orthogonal to B and |E| = c|B| (so that (6.18) holds), the relation (6.21) reveals that imposing equation (3.1) is a natural requisite.

Finally, due to (6.1), equation (3.1) entails:

\[-\frac{1}{c} \frac{\partial^2 A}{\partial t^2} - \nabla \frac{\partial \Phi}{\partial t} = c \text{curl(curl}A) + c \left( \frac{1}{c} \text{div} \frac{\partial A}{\partial t} + \Delta \Phi \right) J\]

\[= -c \Delta A + c \nabla (\text{div}A) + \left( \frac{\partial (\text{div}A)}{\partial t} + c \Delta \Phi \right) J\]  \hspace{1cm} (6.22)
We also assume the following Lorentz condition:
\[
\text{div} \mathbf{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0 \quad (6.23)
\]
which is known to be an invariant expression in general relativity. With the help of (6.23), from (6.22) we get:
\[
\frac{\partial^2 \mathbf{A}}{\partial t^2} - c^2 \Delta \mathbf{A} = \left( \frac{\partial^2 \Phi}{\partial t^2} - c^2 \Delta \Phi \right) \mathbf{J} \quad (6.24)
\]
which is in perfect agreement with the continuity equation (3.4), once we define
\[
\rho = c^{-2} \frac{\partial^2 \Phi}{\partial t^2} - \Delta \Phi. \quad \text{Note however that (3.4) is true independently of (6.23) and (6.24). In the classical Maxwell case, the right and the left terms of (6.24) both vanish, providing, together with (2.20), additional “wave” equations. This is not necessarily true in our case.}

7 Encounter of a wave with an obstacle

In this section and in the following one, we study what happens to a free wave when it meets an obstacle that we define of “mechanical type”. The phenomenon can be extremely complicated, therefore the analysis will be carried out on very simple cases. For the moment, here we just discuss some basic facts, trying to catch the underlying ideas. In section 9, we formalize the problem better, by writing down the final equations.

We first take into account an example concerning the reflection of an electromagnetic radiation. We assume to have a monocromatic plane wave, linearly polarized, which is totally reflected by a perfectly-conducting metallic wall. Hence, we suppose that there is no refraction at all. In Cartesian coordinates, the wall corresponds to the plane \(y = 0\). Initially, the wave-front propagates forming an angle \(\zeta \neq 0\) with the \(y\)-axis. Referring to figure 4, the incident wave is described by the fields:
\[
\mathbf{E}^{(i)} = \left(0, \ -c \sin \zeta, \ c \cos \zeta \right) \sin \omega (t - (y \cos \zeta + z \sin \zeta)/c)
\]
\[
\mathbf{B}^{(i)} = \left(1, \ 0, \ 0 \right) \sin \omega (t - (y \cos \zeta + z \sin \zeta)/c) \quad (7.1)
\]

The reflected electric field \(\mathbf{E}^{(r)}\) is such that, for \(y = 0\), the component parallel to the obstacle, of the vector \(\mathbf{E}^{(r)} + \mathbf{E}^{(i)}\), vanishes (see for instance
Figure 4: Reflection of a plane wave when the magnetic field is normal to the plane of incidence \( x = 0 \). The vectors \( \mathbf{B}^{(i)} \) and \( \mathbf{B}^{(r)} \) are therefore orthogonal to the page.

[2], p.270). Concerning the magnetic field at \( y = 0 \), we have: \( \mathbf{B}^{(r)} = \mathbf{B}^{(i)} \).

Then, one easily gets:

\[
\mathbf{E}^{(r)} = \left( 0, -c \sin \zeta, -c \cos \zeta \right) \sin \omega(t + (y \cos \zeta - z \sin \zeta)/c)
\]

\[
\mathbf{B}^{(r)} = \left( 1, 0, 0 \right) \sin \omega(t + (y \cos \zeta - z \sin \zeta)/c) \tag{7.2}
\]

To justify the imposition of the boundary conditions on \( y = 0 \), it is customary to assume the existence of instantaneous electric currents on the conducting surface, which force the tangential electric field to zero (see [5], p.558, and [9], p.335).

After reflection, the wave is very similar to the incident one, with the exception that \( \mathbf{J}^{(i)} = (0, \cos \zeta, \sin \zeta) \) has changed in \( \mathbf{J}^{(r)} = (0, -\cos \zeta, \sin \zeta) \).

Note that the relation (6.18) is valid both for the incident and the reflected waves. Nevertheless, during the impact, in which an instantaneous flip of the signs occurs, the wave-front at \( y = 0 \) does not show a behavior consistent with the one corresponding to a free wave. Each wave-front, when reaching the wall, evolves for a single moment without respecting the eikonal equation. Of course, what we are considering here is just a mathematical idealization. More realistically, the wall is made of matter, so it would be
more correct to check what happens to the wave when it interacts with the atoms of the wall. Anyway, we do not think it is useful to study this more complicated situation, since it only modifies the form but not the substance of facts. We believe that the main idea has already been outlined: when encountering an obstacle a free wave may lose its likeness and become, for a small amount of time, a “constrained wave”. The reaction of the obstacle can be so strong that, as in the case of the reflecting wall, the wave-fronts are forced to modify abruptly the direction of their movement. What we would like to do in the following pages, is to understand what happens at those instants.

We recall that, for $y = 0$, the vector $\mathbf{E}^{(r)} + \mathbf{E}^{(i)}$ does not have the same length of $\mathbf{E}^{(i)}$ before the impact (or $\mathbf{E}^{(r)}$ after the impact). Therefore, some electromagnetic energy turns out to be missing. We conjecture that it has been spent, with the help of the mechanical constraint, to allow the variation of the direction of the wave-front of an angle $\pi - 2\zeta$. For a moment, this energy has gone elsewhere, compensated by something which is not of an electromagnetic kind. We would like to find out what is. To this end, let us

Figure 5: Reflection of a plane wave when the electric field is normal to the plane of incidence $x = 0$. The vectors $\mathbf{E}^{(i)}$ and $\mathbf{E}^{(r)}$ are therefore orthogonal to the page.
introduce a new vector field:

\[
G = \begin{cases} 
0 & \text{if } y \neq 0 \\
\frac{c^2 \lim_{y \to 0} J^{(r)} - J^{(i)}}{y} & \text{if } y = 0
\end{cases}
\]  

(7.3)

Note that, dimensionally, the vector \( G \) corresponds to an acceleration. In the case under study, at \( y = 0 \), \( G \) is oriented as the vector \((0, -1, 0)\), with an infinite magnitude. If we imagine the wave as a bundle of rays (we saw in section 5 that the two things are strictly related), then \( G \) provides a measure of their curvature. When we are dealing with a free wave, the rays proceed along straight-lines. Corresponding to this situation, we have \( G = 0 \). When the rays hit the wall, then \( G \) becomes different from zero. In the particular case we are examining, \( G \) is a singularly distributed field, but if we allow the rays to change their trajectories smoothly, then \( G \) is going to be finite. This would give the idea of a centripetal time-varying acceleration, responsible for the rotation of the rays and the corresponding wave-fronts.

In the coming sections, we will make evident that a nonvanishing vector \( G \) appears each time a wave evolves without following the Huygens principle, as a consequence of external perturbations. Some suitable way of estimating the magnitude \( G \) should allow us to compensate the missing electromagnetic energy. By the way, the theory is not going to be easy. In section 13, we will discover that, in order to change the trajectories of the rays, it is necessary to pass through a modification of the space-time structure. Thus, we cannot be more precise, until we are ready to carry out our analysis in the framework of general relativity. Before that, we have to work a little more on the definition (7.3). This will be done in section 9. For the moment, let us put together other basic facts.

We now vary the polarization of the incident wave of 90 degrees (see figure 5). For \( y = 0 \), the component, orthogonal to the reflecting wall, of the magnetic field must be zero. At the same time, the whole electric field vanishes (since, for \( y \) approaching to zero, the field \( E^{(r)} \) is opposite to \( E^{(i)} \)). Therefore, more electromagnetic energy is missing when the wave hits the wall. Actually, in this case, the effects of the obstacle are stronger: together with the deviation of the wave-front, we also observe a torsion that modifies the polarization by 180 degrees. Such a torsion process is instantaneuos, but qualitatively similar to that corresponding to a circularly-polarized plane wave, like the one for example expressed by the following fields:

\[
E = (c \cos \omega (t - y/c), 0, c \sin \omega (t - y/c))
\]
where the polarization constantly changes at finite speed. We note that this wave is also a solution to Maxwell equations and is more “energetic” than the corresponding one with constant polarization.

If, together with reflection, we also have refraction within a medium of different density, the study is more involved. When reaching the plane of reflection the rays bifurcate. In our opinion, this is due to the arbitrariness of the vector $J$ at the time of the impact (for example because $P$ is zero). The incident wave splits into two solutions (the reflected and the refracted waves), both compatible with the same boundary conditions on the plane $y = 0$, brought by the incoming solution. We do not necessarily have bifurcation each time that $P$ is zero (this actually happens very frequently), but only when, in addition to this, suitable uncommon conditions are verified.

Based on the above observations, we conclude this section with a little philosophical dissertation. The equations (3.1)-(3.2)-(3.3) are of local and deterministic type. Hence, for given initial data, the solution is uniquely determined. Nevertheless, there may be circumstances in which, following the evolution of a certain solution, several other branches of solutions of the equation (3.1) may be admissible. As we noticed, this could be true because the evolution of the vector $J$ turns out to be compatible with different options. Thus, the original solution splits, and this event is also deterministic. As a matter of fact, when the right conditions are fulfilled, an incident wave bifurcates, giving raise to a reflected and a refracted wave. There is no uncertainty: both the solutions are systematically observed. However, this situation becomes extremely unstable when reversing time. We are indeed allowed to think that, marching backward in time, the inverse of a bifurcation phenomenon could occur. In this case, as a film runs in reverse, two waves would meet in perfect phase and melt, producing a single wave. Nevertheless, this event has zero probability of happening. A little perturbation is sufficient to modify completely the evolution of the two waves, with no chances of seeing their fusion.

In conclusion, our equations are of hyperbolic type, deterministic and energy preserving. Nevertheless the particular nature of the nonlinear term can be a source of instabilities when reversing the sign of time. The consequence is that some original information can get lost, and there is no practical way to recover it by following the reverse path. We ask ourselves if this could be an explanation (at least in part) for the second law of thermodynamics.
8 Diffraction phenomena

We continue our qualitative analysis on the interaction of waves with matter. The second example that we take into account is the developing of diffraction, where an electromagnetic wave encounters an aperture. Once again, for simplicity, the device used to generate the phenomenon is a pure mathematical abstraction. So, let us suppose that a plane monochromatic wave propagates in the direction of the $y$-axis (with $y$ increasing) and hits a perfect-conducting wall at $y = 0$. This time, however, there is a passage through the strip $0 \leq z \leq a$, for some positive width $a$.

For $y < 0$, the incident wave is polarized as in (7.1) with $\zeta = 0$. As a consequence, we have:

$$
E = (0, 0, c \sin \omega(t - y/c)) \quad B = (\sin \omega(t - y/c), 0, 0)
$$

$$
E \times B = P = (0, c|\sin \omega(t - y/c)|^2, 0)
$$

$$
J = \frac{P}{|P|} = (0, 1, 0) \quad \text{div} E = 0 \quad \text{div} B = 0 \quad (8.1)
$$

For $z < 0$ and $z > a$ the wave is reflected. Actually, it is not a perfect reflection, since it is affected by some perturbations originating at the boundaries of the aperture. Here, we neglect this aspect and focus our attention on the study of the sources of these disturbances.

We assume that along the two straight-lines $y = 0, z = 0$ and $y = 0, z = a$, instantaneous electric currents push the electric field to zero, so that the condition $E = 0$ is enforced (see [5], p. 559). At the instant in which the wave reaches the obstacle, the electric field develops discontinuities. Therefore, its divergence is a concentrated Dirac distribution along the above mentioned straight-lines. Thus, for $y = 0$, it is easy to realize that:

$$
\text{div} E = c \left[ \delta_0(z) - \delta_a(z) \right] \sin \omega t \quad (8.2)
$$

On the other hand, for $0 < z < a$, we can expect that:

$$
\lim_{y \to 0^+} P = (0, c(\sin \omega t)^2, 0) \quad (8.3)
$$

because these are points in which the wave does not hit the obstacle.

Next, let us examine more in detail what happens on the straight-lines $y = 0, z = 0$ and $y = 0, z = a$. Using equation (8.1) and neglecting the term curl$B$, the remaining nonlinear term brings an instantaneous rotation of the electric field. As a matter of fact, let us define in Cartesian coordinates
$\mathbf{E} = (0, v, w)$ and $\mathbf{B} = (u, 0, 0)$, where $u$, $v$ and $w$ do not depend on $x$. When $y < 0$, we trivially have $v = 0$. In terms of the new unknowns, the equation (3.1) takes the form:

$$
v_t = u_z - \Xi w \quad \quad w_t = -u_y + \Xi v \quad \quad (8.4)
$$

where $\Xi = s(u)c(\text{div}\mathbf{E})/\sqrt{v^2 + w^2} = s(u)c(v_y + w_z)/\sqrt{v^2 + w^2}$ and $s(u)$ is the sign of $u$.

We have $u_z = 0$. Neglecting $u_y$ (which is bounded), the system (8.4) is equivalent to a rotation with angular speed $\Xi$ (note that, for $z = 0$ and $z = a$, $\Xi$ is infinite because so it is $w_z$). The rotation is anti-clockwise at the points $(x, 0, 0)$. It is clockwise at the points $(x, 0, a)$. This is true whatever the sign of the electric field (note that $\mathbf{E}$ and $\mathbf{B}$ change sign together and $\mathbf{J}$ always maintains the same orientation). Even if the rotation is at infinite speed, the rotation angle is finite and depends on the magnitude of the fields. In practice, by equation (3.1), a sudden change of the quantity $\text{div}\mathbf{E}$ is balanced by a modification of $\frac{\partial}{\partial t}\mathbf{E}$, forcing the field to vary in the direction $\pm\mathbf{J}$.

This behavior is not at all in agreement with what is usually observed. The diffraction is a diffusive phenomenon. Therefore, referring to figure 6, we should expect a clockwise rotation at $(x, 0, 0)$ and an anti-clockwise rotation at $(x, 0, a)$. We can correct this inconsistency by changing the sign to the vector product $\times$, which amounts, in other terms, to inverting the sign of the electric field. The conclusion we can draw is quite astonishing. According to our equations, the standard right-handed product $\times$ is not suitable for describing a natural event like diffraction. Answering the question raised in section 5, the correct side of the mirror is the one where $\times$ is left-handed. This means that, if we do not wish to modify the model equations, we need to switch the polarity of the electric fields, in such a way an electron turns out to be positive and a proton negative. We will return to this subject in section 15. The asymmetry of our universe and, consequently, the determination of its parity, is a problem that emerged about 50 years ago. The effects of this dichotomy were predicted by Lee and Yang (see for instance [12], p.534), but the reasons for preferring left or right have still to be found. If the above arguments are free from errors, then here may lie the solution to the problem.

In a more realistic situation, the change of direction of the Poynting vector field is not instantaneous, but develops smoothly. It is very important to point out that the diffraction comes as a consequence of the nonlinearity of the model equation (3.1), and not from the imposition of some artificial
boundary conditions at the points \((x, 0, 0)\) and \((x, 0, a)\), as supposed by other theories (see for instance [5], chapter XI). The successive evolution of the wave after the obstacle follows the Huygens principle. In fact, for \(y > 0\), the wave is free. It displays a slight diffusion due to the rearrangement of the electromagnetic fields, described before. It is well-known that the behavior depends on the ratio between the width \(a\) and the frequency \(\omega/2\pi\). We do not investigate this aspect, since it has been intensively studied in the past. Here, we were only concerned with detecting the mechanism that leads to the deflection of the rays, when they meet the border of the aperture. The same mechanism takes place in the scattering of two solitons (see section 5), when their electromagnetic fields collide, with reciprocal influence.

Some quantitative information can be recovered by examining equation (3.5). We know that the Poynting vector \(P\) presents a natural pulsating variation along the direction of propagation of the wave. But, during the encounter with the obstacle, we have to add another variation, due to the instantaneous transversal change to the flux of energy. As we can see from figure 6, during the impact, the vector field \(P\) shows some dispersion and its
divergence suddenly grows. Thus, for just a moment, the quantity \(-c^2 \text{div} \mathbf{P}\) registers a negative peek. To restore the normal energy flux in (3.5), some corrective term, taking care of the “reaction” of the obstacle, should be added. In fact, as in the previous section, the change of curvature of the rays is accompanied by the creation of a new vector field \( \mathbf{G} \), concentrated on the obstacle. We better formalize this idea in section 9.

Anyhow, in spite of the good achievements, we still have some problems to fix. Suppose that the incident wave is polarized in a different way, for instance by applying a rotation of 90 degrees:

\[
\mathbf{E} = (c \sin \omega(t - y/c), 0, 0) \quad \mathbf{B} = (0, 0, -\sin \omega(t - y/c)) \quad (8.5)
\]

On the contact with the straight-lines \(y = 0, z = 0\) and \(y = 0, z = a\), we should now observe a prompt change of the magnetic field \( \mathbf{B} \), along the direction \(z\). This event is not modelled by our equations, since we need to suppose that \(\text{div} \mathbf{B}\) can be different from zero. In order to proceed, it is necessary to correct the model in such a way that the fields \( \mathbf{E} \) and \(c\mathbf{B}\) have the same role, as in the classical Maxwell equations. We also discuss this in the coming section.

9 Adding the mechanical terms

Based on some problems emerged in the previous sections, we make a first adaptation of the set of equations (3.1)-(3.2)-(3.3), with the aim of bringing to the same level the two fields \( \mathbf{E} \) and \(c\mathbf{B}\). Thus, the new formulation is:

\[
\frac{\partial \mathbf{E}}{\partial t} = c^2 \text{curl} \mathbf{B} - c \text{div} \mathbf{E} \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E} \times \mathbf{B}|} \quad (9.1)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = -\text{curl} \mathbf{E} - c \text{div} \mathbf{B} \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E} \times \mathbf{B}|} \quad (9.2)
\]

Now, (9.1)-(9.2) do not change if we replace \( \mathbf{E} \) by \(c\mathbf{B}\) and \(c\mathbf{B}\) by \(-\mathbf{E}\), as in the Maxwell equations. This makes the model invariant under change of polarization. Clearly, if the divergence of \( \mathbf{B} \) vanishes, we come back to equations (3.2)-(3.3). The possibility for \(\text{div} \mathbf{B}\) to be different from zero, does not imply the existence of magnetic monopoles, as the condition \(\text{div} \mathbf{E} \neq 0\) does not imply the existence of electrical charges. However, the issue is delicate, and will be further discussed at the end of section 14 and in section 15.
The modified version (9.1)-(9.2) allows for an even greater space of solutions. The spherical waves analyzed in section 4 can be now constructed with the electric field following the parallels, and the magnetic field following the meridians. In this case, we have $\text{div} E = 0$. Different other intermediate polarizations, which may also vary in time, can be considered as well.

Finally, we can also get solutions as the one shown in figure 2. It is sufficient to set:

$$E = (c f(x, y) g(t - z/c), 0, 0)$$
$$B = (0, f(x, y) g(t - z/c), 0)$$
$$J = (0, 0, 1)$$  \hspace{1em} (9.3)$$

where $f$ is an arbitrary function, decaying to zero near the boundary of a bidimensional domain $\Omega$. The function $g$ can also be arbitrary. If $g$ vanishes outside a finite interval, then the solution in (9.3) is a travelling soliton, as the ones considered in section 5. The difference with respect to section 5 is that those solutions satisfied the condition $\text{div} B = 0$. Hence, in cylindrical coordinates, the solutions given in (5.9) satisfy (9.1)-(9.2), without the need to enforce relation (5.10).

However, we are not completely satisfied yet. There is too much symmetry now, while we know that, in most natural phenomena, the role of fields $E$ and $cB$ is very well differentiated. Actually, the difference is detectable when a wave interacts with matter. We can take for example the case of the wire-grid polarizers, where an incident wave hits a grate of parallel metallic wires. If the wave is polarized with the electric field orthogonal to the wires, then it passes the obstacle almost undisturbed (if its wave-length is much smaller than the distance between two wires of the grate). If the electric field has a component along the direction of the wire, then the wave changes the polarization of a certain angle.

Insisting on a similar example, we can go back to the end of section 8. We are now able to study the diffraction of the wave given in (8.5), where the electric field is parallel to the $x$-axis. Using the equations (9.1)-(9.2), we come to the same conclusions obtained for the wave given in (8.1), polarized in another way. But this result is incorrect, because in the case of the wave (8.5), together with the diffraction of the rays, there should be a change of the polarization after passing the obstacle, which is not modelled by the equations (9.1)-(9.2), and which is not present in the case of the wave polarized as in (8.1).

In addition to the above observations, we also note in the reflection-refraction phenomenon, that the way the incident wave is polarized affects
the final result. Thus, it is necessary to further improve the model. For this purpose it will be useful the material collected in sections 6, 7 and 8.

We need to introduce new vector fields (not of electromagnetic type), which are activated each time a free wave becomes a constrained wave. Let us begin by defining a velocity vector field $\mathbf{V}$. We will ask all the vectors to be of constant norm, in particular: $|\mathbf{V}| = c$, where $c$ is the speed of light. Therefore, what really matters is the orientation of the vectors. The idea is that $\mathbf{V}$ is the tangent vector field to a bundle of light rays. An example is given by the vector field $c \mathbf{J}$, introduced in section 3, representing the direction of propagation of a wave-front.

Afterwords, we propose the following system of time-dependent partial differential equations, with three unknown vector fields:

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \text{curl} \mathbf{B} - (\text{div} \mathbf{E}) \mathbf{V} \quad (9.4)$$

$$\frac{\partial \mathbf{B}}{\partial t} = - \text{curl} \mathbf{E} - (\text{div} \mathbf{B}) \mathbf{V} \quad (9.5)$$

$$\frac{\partial \mathbf{V}}{\partial t} = - (\mathbf{V} \cdot \nabla) \mathbf{V} + \mu (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \quad (9.6)$$

The constant $\mu$ is dimensionally equivalent to an electric charge divided by a mass. Finally, we add the condition previously anticipated:

$$|\mathbf{V}| = c \quad (9.7)$$

Concerning the choice of the norm in (9.7), the discussion is postponed to section 15.

It is customary, in fluid mechanics, to introduce the material (or substantial) derivative:

$$\mathbf{G} = \frac{D \mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \quad (9.8)$$

where $\mathbf{G}$ is an acceleration field. Hence, the equation (9.6) is equivalently written as:

$$\frac{D \mathbf{V}}{Dt} = \mu (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \quad (9.9)$$

Geometrically, the vector $\frac{D \mathbf{V}}{Dt}$ provides a measure of the curvature of the stream-lines, which in this case are identified with the rays (recall (7.3)). As will become clearer, the knowledge of the vector field $\mathbf{V}$ is secondary with respect to the determination of its variation $\mathbf{G}$.
Again, we assume to be in vacuum, with no particles of any kind around. In spite of that, the equation (9.9) has strong similarity with the Lorentz law for a density of charge moving at the speed of light. Actually, all the ingredients are there. Multiplying by a mass, the left-hand side in (9.9) is a force: its component along the direction of motion turns out to be proportional to the electric force field, while the orthogonal component is proportional to the magnetic force field. As we can see, the symmetry is broken, so that the electric and the magnetic fields cannot be interchanged anymore. But, this only happens in the case of constrained waves ($G \neq 0$). For free waves, we recall that the relation (6.18), corresponding to $V = cJ$ and $G = 0$, is true. Moreover, replacing $E$ by $cB$ and $cB$ by $-E$, we obtain the relation (6.19), which is also true. Therefore, all the free waves, no matter what kind of polarization they have, are included in the new model. The interesting part is to study the behavior of constrained waves. We will discuss some general properties in section 10.

Before going ahead, we feel that some clarification is necessary concerning the meaning of the word “mass”, used, perhaps improperly, several times in the paper. In our discussion, there are no masses in classical sense, since there are no elementary particles of any sort. Nevertheless, we needed to make distinction, in terms of dimensionality, between electromagnetic and mechanical (later they will be called gravitational) fields. This responsibility has been given to the constant $\mu$, which provides the dimensional link between the two “flavors”. Although other names could have been appropriate to this purpose, the choice of the term “mass” is not incidental, since, as we proceed with our arguments, it will come out to be consistent with the standard setting.

10 Properties of the new set of equations

The new system of equations (9.4)-(9.5)-(9.6) is able to describe electromagnetic phenomena where the wave-front, locally evolving in the direction determined by $V$, could be subjected to transversal perturbations modifying the trajectories of the rays. The propagation of the wave is governed by the first two equations. Through a feed-back process, the third equation, from the current knowledge of the local electromagnetic fields, allows for the determination of $\frac{D}{Dt}V$, setting up the new direction of motion. This coupling is possible because we have been able to include the vector $V$ in the description of the electromagnetic part, in the same way the term $cJ$ was added to
Thus, we got a remarkable result: a link between electromagnetic and mechanical forces. Using the standard Maxwell equations such a connection could never be established.

Let us continue with our analysis. From known formulas of vector calculus, we first deduce that:

\[
\frac{DV}{Dt} \cdot V = \frac{\partial V}{\partial t} \cdot V + \left( \frac{1}{2} \nabla |V|^2 - V \times \text{curl}V \right) \cdot V = 0 \tag{10.1}
\]

where we used that \( \nabla |V|^2 = 0 \) and that \( \frac{\partial}{\partial t} V \) is orthogonal to \( V \), since \( V \) has constant norm.

We recall that, by definition, \( \mathbf{E} \cdot \mathbf{J} = 0 \). Similarly, by (10.1) and by scalar multiplication of (9.9) by \( V \), one easily gets:

\[
\mathbf{E} \cdot V = 0 \tag{10.2}
\]

Although one has \( \mathbf{B} \cdot \mathbf{J} = 0 \), nothing can be deduced however for the scalar product \( \mathbf{B} \cdot V \).

By vector multiplication of (9.9) by \( V \), we get:

\[
V \times \frac{DV}{Dt} = \mu \left( V \times \mathbf{E} - c^2 \mathbf{B} + (\mathbf{V} \cdot \mathbf{B}) \mathbf{V} \right) \tag{10.3}
\]

that generalizes (6.19). Finally, by scalar multiplication of (9.4) by \( \mathbf{E} \) and (9.5) by \( \mathbf{B} \), one obtains:

\[
\frac{1}{2} \frac{\partial}{\partial t}(|\mathbf{E}|^2 + c^2 |\mathbf{B}|^2) = -c^2 \text{div}(\mathbf{E} \times \mathbf{B}) - c^2 (\mathbf{V} \cdot \mathbf{B}) \text{div} \mathbf{B} \tag{10.4}
\]

which is the counterpart of (3.5).

Referring to figure 7, let \( \mathbf{J}^{(t)} \) be the normalized Poynting vector at time \( t \) and \( \mathbf{J}^{(t+\delta t)} \) the one at time \( t + \delta t \). Let \( \mathbf{V} \) be the vector at time \( t \), obtained by backward parallel transport along the stream-lines of the vector \( \mathbf{J}^{(t+\delta t)} \).

Then, we have:

\[
\frac{DV}{Dt} = \lim_{\delta t \to 0} \frac{V - c \mathbf{J}^{(t)}}{\delta t} \tag{10.5}
\]

We recall that the same was done in section 7 in order to define the vector \( \mathbf{G} \) (see (7.3)). Therefore, for small time variations \( \delta t \), we are allowed to write:

\[
\mathbf{V} \approx c \mathbf{J} + \mathbf{G} \delta t = c \mathbf{J} + \mu(\mathbf{E} + c \mathbf{J} \times \mathbf{B}) \delta t \tag{10.6}
\]

with \( \mathbf{J} = (\mathbf{E} \times \mathbf{B})/|\mathbf{E} \times \mathbf{B}| \). If \( \mathbf{E} \cdot \mathbf{B} = 0 \), the relation (10.6) can be rewritten as:

\[
\mathbf{V} \approx c \mathbf{J} + \mu \frac{\mathbf{E}}{|\mathbf{E}|} \left( |\mathbf{E}| - c |\mathbf{B}| \right) \delta t \tag{10.7}
\]
after noting that: \((\mathbf{E} \times \mathbf{B}) \times \mathbf{B} = (\mathbf{E} \cdot \mathbf{B})\mathbf{B} - |\mathbf{B}|^2 \mathbf{E} = -|\mathbf{B}|^2 \mathbf{E}\). This shows that it is sufficient to have \(|\mathbf{E}| \neq c|\mathbf{B}|\), in order to activate the transversal field \(\mathbf{G}\).

We can compare the evolution of an electromagnetic phenomenon to that of an inviscid fluid, whose mass density, up to dimensional constants, is given by \(\rho = \text{div}\mathbf{E}\). Note, however, that a real “mass” does not exist. Note also that \(\rho\) can also attain negative values. We do not define the density \(\rho = \text{div}\mathbf{B}\) for reasons that will be detailed at the end of section 14. The following continuity equation holds (see also (3.4)):

\[
\frac{\partial \rho}{\partial t} = - \text{div}(\rho \mathbf{V}) \tag{10.8}
\]

obtainable by taking the divergence of (9.4). The equation (10.8) can be also written as:

\[
\frac{D\rho}{Dt} = - \rho \text{ div}\mathbf{V} \tag{10.9}
\]

For a plane wave (or soliton) having \(\rho \neq 0\), we obtain \(\text{div}\mathbf{V} = 0\) as well as \(\mathbf{G} = 0\). Then, (10.9) tells us that the fluid shifts, without modifications, along the direction determined by \(\mathbf{V}\). The fluid travels at constant speed.
c, showing rarefactions and compressions. More properly, it evolves like an incompressible fluid, but with density not equally distributed in space. Regarding a spherical wave, one has $\text{div} \mathbf{V} > 0$ and $\mathbf{G} = 0$. As expected, this implies that the mass density diminishes (in absolute value), while time passes, since it spreads on spheres of growing area. In both examples (the plane and the spherical) we have $\text{curl} \mathbf{V} = 0$. In other words, the fluid is irrotational.

Let us suppose that $\mathbf{V}$ is a gradient, i.e.: $\mathbf{V} = \nabla \Psi$, where $\Psi$ is a scalar potential not depending on time. Then, the corresponding fluid is irrotational. Thanks to (9.7), we trivially have:

$$|\nabla \Psi| = c \quad (10.10)$$

which is the eikonal equation. Then, we observe that (10.10) and the relation $\frac{\partial \Psi}{\partial t} = 0$, imply:

$$\frac{D\mathbf{V}}{Dt} = \nabla \left( \frac{\partial \Psi}{\partial t} + \frac{1}{2} |\nabla \Psi|^2 \right) = 0 \quad (10.11)$$

This confirms a remarkable result: the eikonal equation (hence, the evolution of the wave-fronts based on the Huygens principle) is perfectly compatible with the condition $\mathbf{G} = 0$. This analytic property, obtained without approximations, goes beyond the famous limits of geometrical optics. So, here, with a very elementary proof, we obtained another important result.

In the equation (9.9), the term $\mathbf{V} \times \mathbf{B} = -\mathbf{B} \times \mathbf{V} = \mathcal{T}(\mathbf{V})$ can be viewed as a suitable stress tensor $\mathcal{T}$ applied to the vector normal to the front of propagation (see for instance [1], p.10). As we already know, forced variations of the electric field produce changes in the motion of the fronts. If these are combined with forced variations of the magnetic field, a torsion is also introduced, which modifies the polarization of the wave. We guess, that, when the external perturbations end, the electromagnetic fields return to their natural equilibrium in which $|\mathbf{E}| = c|\mathbf{B}|$ and (6.18) is satisfied, so that the fluid takes again an irrotational motion. From the examples discussed in sections 7 and 8, this behavior corresponds to what is commonly observed in nature, and certainly comes from the minimization of some Lagrangian. At the moment, we do not however have theoretical explanation for this conjecture.

Furthermore, we note that stationary electric fields, for example with $\mathbf{B} = 0$, are no longer solutions. We can check this by examining relations (9.9) and (10.2). They force the velocity field $\mathbf{V}$ to turn itself around ($\mathbf{V}$
deviates in the direction of \( \mathbf{E} \), but \( \mathbf{E} \) remains orthogonal to it). More generally, equation (9.6) requires the solutions to be in continuous evolution. We contend that the new system of equations does not admit stationary solutions having finite energy. We made the same consideration in section 5, in the particular case of solitons. Nevertheless, there could be nonstationary solutions localized in space. We can imagine for instance the case of two (or more) solitons, in such a way that they are constrained, by influencing their electromagnetic fields each other, to revolve around a common center. We still do not have all the elements to study these phenomena, which, as we will see in the following pages, need the environment of general relativity to be stated properly. Some remarks about the case of two rotating solitons will be given in section 15.

We are unable at the moment to obtain the equations (9.4)–(9.5)–(9.6) from the minimization of a suitable action function as we did in section 6 (concerning (9.6) alone, something in this direction will be obtained in the next section). One may consider the usual Lagrangian

\[
L = 2(c^2|\mathbf{B}|^2 - |\mathbf{E}|^2)
\]

and the generalization of the relation (6.12), i.e.:

\[
\mathbf{A} = \frac{1}{c} \Phi \mathbf{V} \quad \text{(10.12)}
\]

Then, we could minimize the same action function given in (6.6) using the constraint (10.12). Nevertheless, we would not obtain the desired result, since (10.12) is too restrictive. In this way, we only get a set of equations describing free waves. As a matter of fact, we can prove that, if \( \mathbf{V} \) has the same direction as \( \mathbf{A} \), then one automatically has \( \mathbf{G} = 0 \). Assuming \( \text{div}\mathbf{B} = 0 \), this check can be done as follows. Considering (6.1) and (10.12), we have:

\[
\frac{D}{Dt} \left( \frac{c}{\Phi} \mathbf{A} + \frac{\mu}{c} \mathbf{A} \right) = \frac{D\mathbf{V}}{Dt} + \frac{\mu}{c} \frac{\partial \mathbf{A}}{\partial t} + \frac{\mu}{c} (\mathbf{V} \cdot \nabla) \mathbf{A}
\]

\[
= \frac{D\mathbf{V}}{Dt} - \mu \mathbf{E} - \mu \nabla \Phi + \frac{\mu}{\Phi} (\mathbf{A} \cdot \nabla) \mathbf{A}
\]

\[
= \frac{D\mathbf{V}}{Dt} - \mu \mathbf{E} + \frac{\mu}{\Phi} \left( -\frac{1}{2} \nabla \Phi^2 + (\mathbf{A} \cdot \nabla) \mathbf{A} \right)
\]

\[
= \frac{D\mathbf{V}}{Dt} - \mu \mathbf{E} + \frac{\mu}{\Phi} \left( -\frac{1}{2} \nabla |\mathbf{A}|^2 + (\mathbf{A} \cdot \nabla) \mathbf{A} \right)
\]

\[
= \frac{D\mathbf{V}}{Dt} - \mu \left( \mathbf{E} + \frac{1}{\Phi} \mathbf{A} \times \text{curl} \mathbf{A} \right) = \frac{D\mathbf{V}}{Dt} - \mu (\mathbf{E} + \mathbf{V} \times \mathbf{B}) = 0 \quad \text{(10.13)}
\]
where we used that $\Phi^2 = |A|^2$. The last equality is true thanks to (9.9). Then, noting that $(c/\Phi + \mu/c)A = (1 + \mu\Phi/c^2)V$, the relation (10.13) leads to:
\[
\frac{D}{Dt} \left[ \left(1 + \frac{\mu\Phi}{c^2}\right)V \right] = \frac{\mu}{c^2} \frac{D\Phi}{Dt} V + \left(1 + \frac{\mu\Phi}{c^2}\right) \frac{DV}{Dt} = 0 \quad (10.14)
\]

By scalar multiplication by $V$, due to (10.1), we recover:
\[
\frac{\mu}{c^2} \frac{D\Phi}{Dt} |V|^2 + \left(1 + \frac{\mu\Phi}{c^2}\right) \frac{DV}{Dt} \cdot V = \mu \frac{D\Phi}{Dt} = 0 \quad (10.15)
\]

Thus, $\Phi$ turns out to be constant along the stream-lines. For this reason, from (10.13), also $A$ is constant along the stream-lines. Therefore, one has $\frac{D}{Dt} V = G = 0$. We also conclude that, when the rays bend ($G \neq 0$), then the vector $A$ cannot be aligned in the direction of motion.

We mentioned in the previous sections that the mechanical effects are implicitly included in the term $c^2 \text{div} P$, where it is necessary to distinguish between the contribution due to the variation of the Poynting vector along the actual direction of propagation of the front, and the transversal contribution (which is zero when $G = 0$). Differentiating with respect to time the expression $J = P/|P|$, we get:
\[
G = \frac{1}{|P|} \left( \frac{\partial P}{\partial t} \right) - \frac{P \cdot \frac{\partial}{\partial t} P}{|P|^2 P} \quad (10.16)
\]

In particular, by scalar multiplication of $G$ by $P$, (10.16) shows the orthogonality relation $G \cdot P = 0$. Furthermore, from (3.5), the energy can be described as a work by integrating $-2c^2 \text{div} P$ with respect to time. This yields:
\[
-2c^2 \int \text{div} P \, dt = -2 \int_\Gamma \text{div} P \cdot V \cdot ds \quad (10.17)
\]

where we set $ds = V dt$, which implies $V \cdot ds = |V|^2 dt = c^2 dt$. The last integration is made along the curve $\Gamma$ representing the path of the light ray.

We end this section by illustrating another interesting relation. Let us define as usual $\rho = \text{div} E$. Afterwords, let us assume that $\rho \neq 0$ and define $\bar{\omega} = F/\rho$, where $F = \text{curl} V + \mu B$. Then, along the stream-lines we have:
\[
\frac{D\bar{\omega}}{Dt} = (\bar{\omega} \cdot \nabla) V \quad (10.18)
\]

Note that $\bar{\omega}$ is dimensionally equivalent to a time multiplied by a charge and divided by a mass. The equation (10.18) recalls the analogous one for
isentropic flows, which is introduced in fluid dynamics by defining $\tilde{\omega}$ as the curl of the velocity field divided by the mass density (see [6], p.24). Using (6.1), the field $\tilde{\omega}$ also takes the following form:

$$\tilde{\omega} = \frac{\text{curl}(\mathbf{V} + \mu \mathbf{A}/c)}{-\frac{1}{c}\frac{\partial}{\partial t}\text{div}\mathbf{A} - \Delta \Phi}$$  (10.19)

The equation (10.18) can be proven as follows:

$$\rho \left[ \frac{D\tilde{\omega}}{Dt} - (\tilde{\omega} \cdot \nabla)\mathbf{V} \right] = \rho \left[ \frac{1}{\rho} \frac{DF}{Dt} - \frac{1}{\rho^2} \frac{D\rho}{Dt} \mathbf{F} \right] - (\mathbf{F} \cdot \nabla)\mathbf{V}$$

$$= \text{curl} \left( \frac{\partial \mathbf{V}}{\partial t} \right) + \mu \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{F} + \mathbf{F} \text{div}\mathbf{V} - (\mathbf{F} \cdot \nabla)\mathbf{V}$$

$$= \text{curl} \left[ - (\mathbf{V} \cdot \nabla)\mathbf{V} + \mu(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \right] + \mu(-\text{curl}\mathbf{E} - \mathbf{V} \text{divB})$$

$$+ (\mathbf{V} \cdot \nabla)\mathbf{F} + \mathbf{F} \text{div}\mathbf{V} - (\mathbf{F} \cdot \nabla)\mathbf{V}$$

$$= \left[ \text{curl}(\mathbf{V} \times \mathbf{F}) - \mathbf{V} \text{div}\mathbf{F} + (\mathbf{V} \cdot \nabla)\mathbf{F} + \mathbf{F} \text{div}\mathbf{V} - (\mathbf{F} \cdot \nabla)\mathbf{V} \right]$$

$$- \left[ \text{curl}(\mathbf{V} \cdot \nabla)\mathbf{V} + \text{curl}(\mathbf{V} \times \text{curl}\mathbf{V}) \right]$$

$$= -\text{curl}(\mathbf{V} \cdot \nabla)\mathbf{V} + (\mathbf{V} \times \text{curl}\mathbf{V}) = -\frac{1}{2} \text{curl}(\nabla|\mathbf{V}|^2) = 0$$  (10.20)

where, in the order, we used (10.9), (9.6), (9.5), some well-known calculus properties and the fact that $\nabla|\mathbf{V}|^2 = 0$.

In the case of plane solitary waves, we have $\text{curl}\mathbf{V} = 0$, hence $\tilde{\omega} = \mu\mathbf{B}/\rho$ (when $\rho \neq 0$). Therefore, $\tilde{\omega}$ remains orthogonal to $\mathbf{V}$, so that the relation (10.18) becomes $\frac{D\tilde{\omega}}{Dt} = 0$. Then, the quantity $\tilde{\omega}$ shifts, remaining constant along the stream-lines determined by the velocity field $\mathbf{V}$ (which are straight-lines in this case).

11 Towards general relativity

Our first step, in this section, is to recover the equation (9.9) through the minimization of a suitable Lagrangian. To this end we work in space-time using 4-vectors. Let us start by defining $(x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$
and \((e_0, e_1, e_2, e_3) = (1, -1, -1, -1)\). Then, for the vector \((V_0, V_1, V_2, V_3) = (V_0, \mathbf{V})\), one has:
\[
\sum_{i=0}^{3} e_i V_i^2 = V_0^2 - |\mathbf{V}|^2
\]  
(11.1)

As in section 6, we assume that \(\text{div}\mathbf{B} = 0\) and introduce the potentials \(\Phi\) and \(\mathbf{A}\) by \((6.1)\). Let also be \((A_0, A_1, A_2, A_3) = (\Phi, \mathbf{A})\). Up to multiplicative constants, we can define a Lagrangian in the following way (see also \([10], p.50\)):
\[
L = c \sqrt{V_0^2 - |\mathbf{V}|^2} + \mu \left( \Phi - \frac{1}{c} \mathbf{A} \cdot \mathbf{V} \right)
\]  
(11.2)

The quantities \(V_i, i = 0, 1, 2, 3\), are the independent variables, while the potentials depend on \(x_i, i = 0, 1, 2, 3\). By setting \(V_0 = c\), the term in parenthesis of \((11.2)\) can be written as: \(c^{-1} \sum_{i=0}^{3} e_i A_i V_i\). For the moment, we do not impose the condition \((9.7)\), implying that the sum in \((11.1)\) is zero.

Suppose that we are moving along a stream-line (or curved ray), between two instants of time \(t_1\) and \(t_2\), then the action function takes the form:
\[
S = -\int_{t_1}^{t_2} L \, dt
\]  
(11.3)

Its minimization brings to the Euler-Lagrange equation (see \([9], p.577\)):
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{V}_i} \right) = \frac{\partial L}{\partial x_i} \quad i = 1, 2, 3
\]  
(11.4)

where we observed that \(\left( \frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right) = \left( \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)\). In particular, for \(i = 1, 2, 3\), we have:
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial V_i} \right) = \frac{d}{dt} \left( \frac{-c V_i}{\sqrt{V_0^2 - |\mathbf{V}|^2}} - \frac{\mu}{c} A_i \right)
\]  
(11.5)

and
\[
\frac{\partial L}{\partial x_i} = \frac{\mu}{c} \frac{\partial \Phi}{\partial x_i} - \frac{\mu}{c} \mathbf{V} \cdot \frac{\partial \mathbf{A}}{\partial x_i}
\]  
(11.6)

where, in \((11.5)\), the substantial derivative \(\frac{D}{Dt} V_i\) gives the variation, along the stream-lines, of the coordinates of the velocity field, parametrized with respect to the arc-length: \(s = |c|^{-1} \int_{t_1}^{t} \sqrt{V_0^2 - |\mathbf{V}|^2} \, d\xi\).
If we now define $\frac{dx_k}{dt} = V_k$, for $k = 1, 2, 3$, thanks to (6.3) and (6.5), we get:

$$\frac{DV_i}{Dt} = \mu \left( \frac{\partial \Phi}{\partial x_i} - \frac{1}{c} \frac{\partial A_i}{\partial t} \right) - \frac{\mu}{c} \mathbf{V} \cdot \left( \nabla A_i + \frac{\partial \mathbf{A}}{\partial x_i} \right) = -\frac{\mu}{c} F^{ik} V_k$$  \hspace{1cm} (11.7)

When $V_0 = c$, the last term in (11.7) is equal to the $i$-th component of the vector $\mu (\mathbf{E} + \mathbf{V} \times \mathbf{B})$. This implies the equation (9.9).

Concerning $k = 0$, if we fix $V_0$ to be constantly equal to $c$, one obtains $\frac{D}{Dt} V_0 = 0$. Therefore, we have:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial V_0} \right) = \frac{DV_0}{Dt} + \mu \frac{d\Phi}{dt} = \mu \frac{\partial \Phi}{\partial t} + \mu \mathbf{V} \cdot \nabla \Phi$$  \hspace{1cm} (11.8)

and

$$\frac{\partial L}{\partial t} = \mu \frac{\partial \Phi}{\partial t} - \frac{\mu}{c} \mathbf{V} \cdot \frac{\partial \mathbf{A}}{\partial t}$$  \hspace{1cm} (11.9)

Due to (11.4), by equating these two last expressions, we recover:

$$0 = -\mu \left( \nabla \Phi + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \cdot \mathbf{V} = \mu F^{0k} V_k = \mu \mathbf{E} \cdot \mathbf{V}$$  \hspace{1cm} (11.10)

which corresponds to (10.2). Considering (11.10), by scalar multiplication of (9.9) by $\mathbf{V}$, we deduce that the field minimizing the action satisfies $\mathbf{V} \cdot \frac{D}{Dt} \mathbf{V} = 0$. Hence, the norm $|\mathbf{V}|$ is constant. If such a constant is $c$, we finally obtain the relation (9.7), that says that the solutions evolve on the light-cone.

At this point, it should be noted that, by choosing $|\mathbf{V}|^2 = c^2$, the first part of the Lagrangian in (11.2) vanishes. This does not mean that it vanishes identically, but only in correspondence to the minimum. Instead, the second part of the Lagrangian is zero when $\mathbf{A} \cdot \mathbf{V} = c \Phi$, which is very similar to the condition (6.17), obtained from the constraint (6.12) (see also (10.12)). This coincidence is quite significant. Perhaps, in the future, it will suggest a way to build a Lagrangian for the entire set of equations (9.4)-(9.5)-(9.6).

By multiplying the equation (11.7) by $V_i$, $i = 1, 2, 3$, and the equation (11.10) by $V_0$, we get:

$$F^{ik} V_k V_i = 0$$  \hspace{1cm} (11.11)

where the sum is for $i$ and $k$ going from 0 to 3. This also trivially follows from the anti-symmetry of the tensor $F^{ik}$. The equation (6.11) is also written as:

$$\left( \frac{\partial F^{ik}}{\partial x_k} \right) V_0 - \left( \frac{\partial F^{0k}}{\partial x_k} \right) e_i V_i = 0 \quad \text{for } i = 0, 1, 2, 3$$  \hspace{1cm} (11.12)
Otherwise, the equations (2.3) and (2.4), can be recovered from the expression (see for instance [7], p.150):

\[ F_{ikj} = \frac{\partial F_{ik}}{\partial x_j} + \frac{\partial F_{kj}}{\partial x_i} + \frac{\partial F_{ji}}{\partial x_k} = 0 \quad (11.13) \]

where there is no sum on repeated indices. The rank-three tensor \( F_{ikj} \) is anti-symmetric and called the cyclic derivative of \( F_{ik} \). On the other hand, the equation (9.5) follows from the expression:

\[ V_0 \left( \frac{\partial F_{ik}}{\partial x_j} + \frac{\partial F_{kj}}{\partial x_i} + \frac{\partial F_{ji}}{\partial x_k} \right) = \pm e_m V_m \left( \frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} + \frac{\partial F_{12}}{\partial x_3} \right) \quad (11.14) \]

where the indices \( m, j, i, k \) (taken in this order) are all different. The sign \( \pm \) depends on the permutation (even or odd) of the indices (the sign is plus if \( m = 0, j = 1, i = 2, k = 3 \)). In (11.14) the term in parenthesis on the right-hand side is equal to \( \text{cdivB} \). In a more contracted form, the last equation reads as follows:

\[ V_0 F_{jik} = \pm e_m V_m F_{123} \quad (11.15) \]

In the results obtained above, we basically considered \( V \) as the velocity field of an infinitesimal particle moving at the speed of light. On the other hand, in a wave there are infinite contiguous trajectories. As a matter of fact, the evolution of a wave is a global phenomenon, that should be taken as a whole, and not studied independently along each stream-line. For such a more in depth analysis, we need to work in the context of general relativity. We are going to show that the passage of a wave modifies the space-time structure. For a free wave this does not affect the evolution of the wave itself (see section 13), but for constrained waves the change of the geometry influences their entire behavior. The analysis will allow us to find the coupling between the fields describing the wave and space-time geometry, hence the link between electromagnetic and gravitational phenomena.

We first need to introduce some classical definitions. Mainly, we adopt the notation used in [7]. The space-time geometry is locally determined by a symmetric bilinear form (the metric tensor), whose coefficients are denoted by \( g_{ij} \). Then, the Christoffel symbols are defined in the following way:

\[ \Gamma^i_{kj} = \frac{g^{im}}{2} \left( \frac{\partial g_{mk}}{\partial x_j} + \frac{\partial g_{mj}}{\partial x_k} - \frac{\partial g_{kj}}{\partial x_m} \right) \quad (11.16) \]
where we sum over the index \( m \). The coefficients \( g^{ij} \) are in such a way that:

\[
g_{im} g^{mj} = \delta_{ij} \quad (11.17)
\]

The coefficients \( g_{ij} \) are adimensional, while the Christoffel symbols are the inverse of a distance. If the space is “flat” (Euclidean or Minkowski space), all the Christoffel symbols vanish. In this case, one has \( g^{ik} = g_{ik} = \epsilon_i \delta_{ik} \).

As usual, we denote by \( g \) the determinant (which is negative) of the tensor \( g_{ik} \). A lemma due by Ricci (see [7], p.129) claims that the 4-divergence of the metric tensor is zero. In detail, one has:

\[
\nabla_k g^{ik} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} g^{ik})}{\partial x_k} + \Gamma^i_{jm} g^{jm} = 0 \quad (11.18)
\]

where \( \nabla_k \) is the covariant differentiation operator. The same is true for the coefficients \( g_{ik} \). Moreover, the coefficients \( g^{ik} \) are said to be harmonic when:

\[
\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} g^{ik})}{\partial x_k} = 0 \quad (11.19)
\]

Next, we define \( V^i = g^{im} V_m \). The values \( V_m \) are the entries of a velocity vector expressed in the coordinates system \( x_0, x_1, x_2, x_3 \). Let us set \( V^0 = c \). Then, the condition (9.7) is generalized in the following way:

\[
V^i V^i = g^{im} V_m V^m = 0 \quad (11.20)
\]

In this more general framework, the equations (11.7) and (11.10) are rewritten as:

\[
\frac{DV^i}{Dt} + \Gamma^i_{jk} V^j V^k = -\frac{\mu}{c} \Gamma^{im} V_m \quad \text{for } i = 0, 1, 2, 3 \quad (11.21)
\]

For \( i = 0, 1, 2, 3 \), we also define (see [7], p.217):

\[
G^i = \frac{DV^i}{Dt} + \Gamma^i_{jk} V^j V^k = V^m \nabla_m V^i \quad (11.22)
\]

From (11.21) we easily recover the orthogonality relations:

\[
G_i V^i = G^i V_i = 0 \quad (11.23)
\]

where \( G_i = g_{im} G^m \). Finally, let us define \( \mathbf{G} = (G_1, G_2, G_3) \), which is dimensionally equivalent to an acceleration.
In general relativity, the gravitational field is somehow identified with the tensor $g_{ij}$. Of course, the vector $G$ may vanish without having that the space is flat. Although $G$ does not fully characterize the properties of space-time, for us it will be the “real” gravitational field, that is the one we can measure in our everyday life. In the following, $G$ will be called the vector gravitational field.

The equation (11.21) enables us to understand how the trajectory of a “thin” solitary wave can be distorted when immerged in a given gravitational field (having $G \neq 0$). Being the soliton a free wave, the right-hand side of (11.21) vanishes (see (6.18)). Thus, its path follows a suitable geodesic in space-time, the shape of which is determined by the external gravitational field. This should correspond to some transversal bending in the direction locally individuated by the vector $G$. With this reasoning, we have to neglect a couple of facts, both due to the change of direction: the modification of the electromagnetic fields and the “gravitational reaction” of the soliton (a curving wave produces new gravitational field). As we argued in section 10, these should be minor effects, since the wave, for some principle of least action, tries to compensate the electromagnetic fields in order to enforce (6.18). From the point of view of the soliton, the path followed is straight, even if it actually travels on a curved geodesic. To get more reliable quantitative results, we must solve a quite complicated system of equations. Comparing the computed results with the experimental evidence, one could probably evaluate the constant $\mu$. This is however an exercise that we would prefer to avoid here. Also if some theoretical passages may be formally similar, the important clue is that there is no need to suppose that a soliton has an infinitesimal mass to justify that is attracted by a gravitational field.

After recalling that $F_{ik}$ is an anti-symmetric tensor, in general coordinates, the equation (11.12) becomes:

$$\frac{1}{\sqrt{-g}} \left( \frac{\partial (\sqrt{-g} F_{ik})}{\partial x_k} V^0 - \frac{\partial (\sqrt{-g} F_{0k}^i)}{\partial x_k} V^i \right) = 0 \quad i = 0, 1, 2, 3 \quad (11.24)$$

or, in more contracted form:

$$(\nabla_k F_{ik}) V^0 - (\nabla_k F_{0k}^i) V^i = 0 \quad i = 0, 1, 2, 3 \quad (11.25)$$

The equation (11.13) remains unchanged. However, it can be also written in the following way (see [7], p.133):

$$F_{ikj} = \nabla_j F_{ik} + \nabla_i F_{kj} + \nabla_k F_{ji} = 0 \quad (11.26)$$
Besides, equation (11.15) becomes:

\[ V^0 F_{jik} = \pm V^m F_{1j} \]  (11.27)

By taking the 4-divergence of the contravariant vector in (11.24) and considering once again that the tensor \( F_{ik} \) is anti-symmetric, we arrive at the continuity equation:

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (\sqrt{-g} \rho E V^i) = 0 \quad \text{with} \quad \rho E = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} (\sqrt{-g} F^{0k})
\]  (11.28)

We got an analogous result in section 3, by taking the standard divergence of the vector equation (3.1). The time derivative came from the term \( \text{div} \left( \frac{\partial}{\partial t} E \right) \) and the term \( \text{div} (\text{curl} B) \) was zero. All the pieces here combine in a completely different manner. Nevertheless, the final result is extraordinarily similar.

## 12 The energy tensor

Let us first work in Cartesian coordinates. We will define the symmetric electromagnetic stress tensor in the classical way (see [7], p.96), i.e.:

\[ U_{ik} = - \left( \sum_{j=0}^{3} e_j F_{ij} F_{kj} - \frac{1}{2} (c^2 |B|^2 - |E|^2) e_i \delta_{ik} \right) \]  (12.1)

We have \( U_{00} = \frac{1}{2} (|E|^2 + c^2 |B|^2) \) and \( \sum_{i=0}^{3} e_i U_{ii} = 0 \).

Its contravariant version is given by \( U^{ik} = e_k e_i U_{ik} \) and of course we have \( \sum_{i=0}^{3} e_i U^{ii} = 0 \). The explicit expression of the contravariant tensor is the following:

\[
\begin{pmatrix}
\frac{1}{2}(|E|^2 + c^2 |B|^2) & cB_2 E_3 - cE_2 B_3 & cE_1 B_3 - cB_1 E_3 & cB_1 E_2 - cE_1 B_2 \\
-cB_2 E_3 + cE_2 B_3 & -E_1^2 + c^2 B_3^2 + c^2 B_2^2 & -E_1 E_2 - c^2 B_1 B_2 & -E_1 E_3 - c^2 B_1 B_3 \\
-cE_1 B_3 + cB_1 E_3 & -E_1 E_2 - c^2 B_1 B_2 & -E_1^2 + c^2 B_3^2 + c^2 B_2^2 & -E_2 E_3 - c^2 B_2 B_3 \\
cB_1 E_2 - cE_1 B_2 & -E_1 E_3 - c^2 B_1 B_3 & -E_2 E_3 - c^2 B_2 B_3 & -E_1^2 + c^2 B_3^2 + c^2 B_2^2
\end{pmatrix}
\]

If (9.4) and (9.5) are satisfied, then an important property of this last tensor is that, in the case of free waves (hence in absence of mechanical
terms), its 4-divergence vanishes. Indeed, we have for $i = 0, 1, 2, 3$:

$$\frac{\partial U^{ik}}{\partial x_k} = 0 \quad (12.2)$$

provided (6.18) (or (6.19)) is satisfied. This implies that $\mathbf{E} \cdot \mathbf{B} = 0$ and $|\mathbf{E}| = c|\mathbf{B}|$. These hypotheses also imply that $|\mathbf{E} \times \mathbf{B}| = |\mathbf{E}||\mathbf{B}|$ and $\mathbf{V} \cdot \mathbf{B} = 0$.

Let us prove (12.2) starting from $i = 0$. Thanks to (3.5), one has:

$$\frac{\partial U^{0k}}{\partial x_k} = \frac{1}{2c} \frac{\partial}{\partial t} \left( |\mathbf{E}|^2 + c^2 |\mathbf{B}|^2 \right) + c \text{div}(\mathbf{E} \times \mathbf{B}) = 0 \quad (12.3)$$

As far as the other values of $i$ are concerned, let us begin to define:

$$\mathbf{N} = (N_1, N_2, N_3) = \frac{\partial \mathbf{E}}{\partial t} - c^2 \text{curl} \mathbf{B} + (\text{div} \mathbf{E}) \mathbf{V}$$

$$\mathbf{M} = (M_1, M_2, M_3) = \frac{\partial \mathbf{B}}{\partial t} + \text{curl} \mathbf{E} + (\text{div} \mathbf{B}) \mathbf{V} \quad (12.4)$$

Thus, if the equations (9.4) and (9.5) are true, then we get $\mathbf{N} = 0$ and $\mathbf{M} = 0$. We are ready to check (12.2) for $i = 1$ (the other cases are treated in a very similar way). We have:

$$\frac{\partial U^{1k}}{\partial x_k} = \frac{\partial}{\partial t} (B_2 E_3 - E_2 B_3) - \frac{\partial}{\partial x} (-E_1^2 + c^2 B_2^2 + c^2 B_3^2)$$

$$+ \frac{1}{2} \frac{\partial}{\partial x} \left( c^2 |\mathbf{B}|^2 - |\mathbf{E}|^2 \right) - \frac{\partial}{\partial y} (-E_1 E_2 - c^2 B_1 B_2) - \frac{\partial}{\partial z} (-E_1 E_3 - c^2 B_1 B_3)$$

$$= (M_2 E_3 - M_3 E_2 + N_3 B_2 - N_2 B_3) + (E_1 + V_2 B_3 - V_3 B_2) \text{div} \mathbf{E}$$

$$+ \left( c^2 B_1 + V_3 E_2 - V_2 E_3 \right) \text{div} \mathbf{B} = 0 \quad (12.5)$$

The last three terms in (12.5) are actually zero for the following reasons. In the first one we recognize the second and the third components of $\mathbf{N}$ and $\mathbf{M}$, which vanish, if we assume that the equations (9.4) and (9.5) are satisfied. The second one contains the first component of the vector $\mathbf{E} + \mathbf{V} \times \mathbf{B}$, which vanishes in the case of a free wave. Concerning the last term, the part in parenthesis is the first component of the vector $c^2 \mathbf{B} - \mathbf{V} \times \mathbf{E}$, which also vanishes (see (6.19)).

The property (12.2) is reported in many texts (see for instance [7], p.97). But, it is extremely important to observe that, in the case of Maxwell equations, the last two terms are zero because $\text{div} \mathbf{E} = 0$ and $\text{div} \mathbf{B} = 0$. This is the reason why we decided to double check equation (12.2), which turns out
to be fulfilled even when the divergence of the fields \( E \) and \( B \) is not zero (the assumption we supported throughout this paper). Therefore (12.2) holds under weaker hypotheses.

As expected, a converse statement also holds: assuming that (12.2) is true, then we can recover the equations (9.4) and (9.5). This amounts to differentiate the equation of energy conservation, in order to obtain the corresponding Euler equations. Arguing as we did to get (12.5), we arrive at:

\[
\left( \frac{\partial U^{1k}}{\partial x_k}, \frac{\partial U^{2k}}{\partial x_k}, \frac{\partial U^{3k}}{\partial x_k} \right) = (M \times E - N \times B) + (E + V \times B) \text{div}E + (c^2B - V \times E) \text{div}B
\] (12.6)

Assuming, as previously done, that we are dealing with a free wave, after eliminating in (12.6) the vanishing terms, we are left with \((M \times E - N \times B)\). Since, by hypothesis, the equation (12.2) is true, if the vector \( N \) is zero, then \( M \) must be also zero (likewise, if \( M \) is zero, then \( N \) is zero). Therefore, (9.4) is satisfied if and only if (9.5) is satisfied. This is the same situation encountered in the classical Maxwell equations, where \( \frac{\partial}{\partial t} B + \text{curl}E \) and \( \text{div}B \) both vanish if and only if \( \frac{\partial}{\partial t} E - c^2 \text{curl}B \) and \( \text{div}E \) are both zero. In the standard approach, the first pair of equations are satisfied by choosing the potentials \( A \) and \( \Phi \) as in (6.1). The second pair is obtained by means of variational type arguments.

Of course, we can find “intermediate” situations, by suitably redefining the two potentials. Let us take for example:

\[
\mathcal{B} = \text{curl}A \quad \quad \mathcal{E} = -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \Phi
\]

with \( \mathcal{E} = \frac{\lambda E + (1 - \lambda)cB}{\sqrt{\lambda^2 + (1 - \lambda)^2}} \) and \( \mathcal{B} = \frac{\lambda cB - (1 - \lambda)E}{\sqrt{\lambda^2 + (1 - \lambda)^2}} \) (12.7)

where \( \lambda \) is a real parameter. From the relations (12.7) we can explicitly compute the fields \( E \) and \( B \) in terms of \( \mathcal{E} \) and \( \mathcal{B} \). These also imply:

\[
\text{div}\mathcal{B} = 0 \quad \quad \frac{\partial \mathcal{B}}{\partial t} = -c \text{curl}\mathcal{E}
\] (12.8)

Then, it is a matter of minimizing the usual Lagrangian. At this point, introducing the constraint \( A \cdot V = \Phi \), one gets the equation:

\[
\frac{\partial \mathcal{E}}{\partial t} = c \text{curl}\mathcal{B} - V \text{div}\mathcal{E}
\] (12.9)
that, for $\lambda = 1$, is equivalent to equation (9.4). The equations in (12.8) are equivalent to require $(\lambda - 1)\mathbf{N} + \lambda\mathbf{M} = 0$, while the one in (12.9) brings to $\lambda\mathbf{N} + (1 - \lambda)\mathbf{M} = 0$.

It is to be noted that $\mathbf{V}$ has the same direction of $\mathbf{E} \times \mathbf{B}$, which is also like that of $\mathbf{E} \times \mathbf{B}$. So, from the energy tensor it is not possible to figure out what the parameter $\lambda$ is, as well as the polarization of the wave. This information has to be provided with the initial conditions. For instance, the wave in (12.1), circularly polarized, produces the same tensor $U^{ik}$ of a linearly polarized wave, moving in the same direction with twice the intensity. As a further consequence, we finally observe that $U^{ik}$ does not change if $\mathbf{E}$ takes the place of $-c\mathbf{B}$ and $c\mathbf{B}$ takes the place of $\mathbf{E}$. Such a permutation corresponds to the choice $\lambda = 0$.

We can now argue in a general framework. For a given metric tensor $g_{ik}$, the electromagnetic stress tensors must be modified in the following way (see [7], p.151):

$$ U^{ik} = - \left( g^{mj} F_{im} F_{kj} - \frac{1}{4} g_{ik} F_{mj} F^{mj} \right) $$

$$ U^{ik} = - \left( g_{mj} F^{im} F^{kj} - \frac{1}{4} g^{ik} F_{mj} F^{mj} \right) $$

(12.10)

where $F^{ik}$ is given in (6.4), while $F^{ik}$ comes from the relation:

$$ F^{ik} = g^{im} g^{kl} F_{ml} $$

(12.11)

Assuming to be as in the case of a free wave, the equation (12.2) has to be replaced by the following one:

$$ \nabla_k U^{ik} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} U^{ik})}{\partial x_k} + \Gamma^i_{mj} U^{mj} = 0 $$

(12.12)

The proof of (12.12) is given for instance in [9], p.606, for the classical Maxwell equations. This is also true in the case of our new set of equations (hence under weaker hypotheses). At the end of this section, we will evaluate the 4-divergence of the tensor $U^{ik}$ in general coordinates. Such generalizations are unavoidable since, even the simple case of a plane wave provokes a modification of the space-time geometry, requiring to work with tensors of the form (12.10). These aspects will be better studied in the next section.

When the electromagnetic phenomenon is not a free wave, we cannot expect that (12.2) and its generalization (12.12) are verified. This means
that the system constituted by the sole electromagnetic part is not energy preserving. We know that, in this case, the energy balance has to take care of the mechanical effects. Thus, we study how to introduce them. We start by assuming that $\text{div} \mathbf{B} = 0$, leaving the discussion of the more general case to section 14. Then, let us define a mass tensor as follows:

$$M_{ik} = V_i V_k \text{div} \mathbf{E} \quad (12.13)$$

The contravariant version is given by $M^{ik} = \epsilon_i \epsilon_k M_{ik}$, which is explicitly written as:

$$M^{ik} = \rho E \begin{pmatrix}
V_0^2 & -V_0 V_1 & -V_0 V_2 & -V_0 V_3 \\
-V_0 V_1 & V_1^2 & V_1 V_2 & V_1 V_3 \\
-V_0 V_2 & V_1 V_2 & V_2^2 & V_2 V_3 \\
-V_0 V_3 & V_1 V_3 & V_2 V_3 & V_3^2
\end{pmatrix} \quad (12.14)$$

where $V_0 = c$ and $\rho E = \text{div} \mathbf{E}$ is a kind of mass density (dimensionally this is not correct, but this aspect will be altered later). We recall that $\rho E$ can also be negative. Let us check what happens to $\frac{\partial M^{ik}}{\partial x^k}$. For $i = 0$ we have:

$$\frac{\partial M^{0k}}{\partial x_k} = c \left( \frac{\partial \rho E}{\partial x_0} - \frac{\partial(\rho E V_1)}{\partial x_1} - \frac{\partial(\rho E V_2)}{\partial x_2} - \frac{\partial(\rho E V_3)}{\partial x_3} \right)$$

$$= c \left( \frac{\partial \rho E}{\partial t} + \text{div}(\rho E \mathbf{V}) \right) = 0 \quad (12.15)$$

This is true because of the continuity equation (10.8) with $\rho = \rho E$. For the other indices $i = 1, 2, 3$, we have:

$$\frac{\partial M^{ik}}{\partial x_k} = - \left( c \frac{\partial (\rho E V_i)}{\partial x_0} - \frac{\partial(\rho E V_i V_1)}{\partial x_1} - \frac{\partial(\rho E V_i V_2)}{\partial x_2} - \frac{\partial(\rho E V_i V_3)}{\partial x_3} \right)$$

$$= - \left( \frac{\partial (\rho E V_i)}{\partial t} + \text{div}(\rho E \mathbf{V}) \right) \quad = - V_i \left( \frac{\partial \rho E}{\partial t} + \text{div}(\rho E \mathbf{V}) \right)$$

$$- \rho E \left( \frac{\partial V_i}{\partial t} + (\mathbf{V} \cdot \nabla)V_i \right) = - \frac{D V_i}{D t} \text{div} \mathbf{E} \quad (12.16)$$

where we again used the continuity equation. We conclude for instance that, if the light rays are straight-lines (that is: $\mathbf{G} = \frac{D}{D t} \mathbf{V} = 0$), then one gets $\frac{\partial M^{ik}}{\partial x_k} = 0$, for $i = 0, 1, 2, 3$. 

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In non-Euclidean geometry, it is necessary to generalize the mass tensors in the following way:

\[ M_{ik} = \rho_E V_i V_k \quad \text{with} \quad \rho_E = \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} F^{0k})}{\partial x_k} \]  

(12.17)

With the help of the continuity equation (11.28) and the definition (11.22), it is easy to get, for \( i = 0, 1, 2, 3 \):

\[ \nabla_k M^{ik} = \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} M^{ik})}{\partial x_k} + \Gamma^{i}_{mj} M^{mj} \]

(12.18)

where the last equality is a consequence of (11.21). Hence, in a flat space \((G^i = 0)\), the 4-divergence of the mass tensor vanishes. Moreover, we observe that the mass tensor does not contain the pressure term (on the other hand, an equation of state is not defined).

In order to combine electromagnetic and mechanical effects, we sum up the corresponding tensors, by defining:

\[ T_{ik} = \frac{\mu}{c^4} \left( \mu U_{ik} + M_{ik} \right) \]

(12.19)

where the constant \(\mu\) is dimensionally equivalent to a charge divided by a mass. It follows that \(T_{ik}\) has the same dimension of a curvature, that is the inverse of the square of a distance. Now, in a flat space-time, even if we are not dealing with a free wave, we may write:

\[ \frac{\partial T^{ik}}{\partial x_k} = 0 \]  

(12.20)

As a matter of fact, due to (9.9), if the term \(\mu(E + V \times B) \text{div } E\) of the electromagnetic part does not vanish (see (12.6)), it is anyway compensated by the corresponding term \(-\text{div } E \frac{\partial}{\partial t} V\) of the mechanical part (see (12.16)).

In the general case, the relation (12.20) is substituted by:

\[ \nabla_k T^{ik} = \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} T^{ik})}{\partial x_k} + \Gamma^{i}_{mj} T^{mj} = 0 \]

(12.21)

Before ending this section, we would like to verify that (12.21) actually corresponds to the Euler equations. As a matter of fact, (12.21) is satisfied
when (11.25), (11.27) and (11.21) are true. We recall that these three last equations are the generalizations of (9.4), (9.5), (9.6), respectively. For the moment, we will only treat the case in which $\text{div} B = 0$, leaving the general discussion to section 14. Let us start by computing the 4-divergence of the tensor $U^{ik}$. First of all, we have:

$$\nabla_k U^{ik} = -\nabla_k (g_{mj} F^{im} F^{kj}) + \frac{1}{4} \nabla_k (g^{ik} F^{mj} F^{mj})$$

(12.22)

where we notice that $g_{mj} F^{im} = g^{im} F_{mj}$ (thanks to (12.11)), that $F^{kj} = -F^{jk}$ and that, due to (11.18), it is allowed to exchange the metric tensor with the covariant derivative (see also [7], p130). Going ahead, one gets:

$$\nabla_k U^{ik} = g^{im} (\nabla_k F^{jk}) F_{mj} + g^{im} (\nabla_k F_{mj}) F^{jk} + \frac{1}{2} g^{im} (\nabla_k F^{mj} F^{mj})$$

(12.23)

$$= c^{-1} (\nabla_k F^{0k}) F_{mj} g^{im} V^j + \frac{1}{2} g^{im} (\nabla_k F_{mj}) F^{jk}$$

$$+ \frac{1}{2} g^{im} (\nabla_k F_{mj} + \nabla_m F^{jk}) F^{jk}$$

The other passages have been obtained by a suitable renaming of the indices. Recalling the definition of $F_{ikj}$ given in (11.26), we have:

$$\nabla_k U^{ik} = \frac{\rho E}{c} F_{mj} g^{im} V^j + \frac{1}{2} g^{im} (\nabla_k F_{mj}) F^{jk}$$

$$+ \frac{1}{2} g^{im} F_{mjk} F^{jk} - \frac{1}{2} g^{im} (\nabla_j F_{km}) F^{jk}$$

$$= \frac{\rho E}{c} F^{im} V_m + \frac{1}{2} g^{im} F_{mjk} F^{jk}$$

(12.24)

In the last passage two terms have been deleted, since, after renaming the indices, they resulted in being equal and with opposite signs. The last term in (12.24) is zero because of (11.26) (remember that we are studying the case $\text{div} B = 0$, thus $F_{123} = 0$). Of course, the final result is zero when, for instance, $\rho E = 0$, as in the classical Maxwell case. But it is also zero when $F^{im} V_m = 0$, which corresponds to the case of a free electromagnetic wave. On the contrary, we need to consider the contribution of the mass tensor. If $F_{mjk} = 0$, taking into account the relations (12.19) and (12.18), one finally obtains:

$$\nabla_k T^{ik} = \frac{\mu}{c^4} \left( \mu \nabla_k U^{ik} + \nabla_k M^{ik} \right)$$

$$= \frac{\mu}{c^4} \left( \mu \frac{\rho E}{c} F^{im} V_m - \frac{\mu}{c} \rho E F^{ik} V_k \right) = 0$$

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13 Unified fields equations

In the previous section, we build the symmetric tensor $T_{ik}$ that includes both the energy contribution of an electromagnetic wave and that of mechanical type, taking into account possible deviations from the natural propagation path of the wave. The properties of $T_{ik}$ insure the preservation of energy and momentum. Hence, we can put $T_{ik}$ on the right-hand side of the Einstein equation:

$$R_{ik} - \frac{1}{2}g_{ik}R = \chi T_{ik} \quad (13.1)$$

in which we recognize the Ricci tensor:

$$R_{ik} = \frac{\partial \Gamma^m_{ik}}{\partial x_m} - \frac{\partial \Gamma^m_{im}}{\partial x_k} + \Gamma^j_{ik} \Gamma^m_{jm} - \Gamma^j_{im} \Gamma^m_{kj} \quad (13.2)$$

the scalar curvature:

$$R = g^{ik} R_{ik} \quad (13.3)$$

and an adimensional constant $\chi$. We recall that the Christoffel symbols are defined in (11.16). Let us note that, by (12.10) and (12.17), the metric tensor, which is now our unknown, also appears on the right-hand side of (13.1).

We soon examine the response of equation (13.1) to the passage of the most elementary plane wave. We take for instance the expression given in (2.5), where we have $E_1 = cB_2 = c \sin \omega (t - z/c)$, div$E = 0$ and div$B = 0$. We will verify that, even in this simple case, the space-time geometry, that comes from the solution of (13.1), is not Euclidean. In fact, we look for a metric tensor $g_{ik}$ of the following type:

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -p^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/p^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (13.4)$$

where $p$ is a function, to be determined, of the variable $\xi = t - z/c$. Somehow, we are expressing a preference for the direction of the $x$-axis, which is orientated with the electric field. The determinant $g$ of $g_{ik}$ is equal to $-p^2$. The corresponding Christoffel symbols are:

$$\Gamma^0_{11} = \frac{pp'}{c}, \quad \Gamma^3_{11} = -\frac{pp'}{c}, \quad \Gamma^1_{01} = \Gamma^1_{10} = \Gamma^1_{13} = \Gamma^1_{31} = \frac{p'}{cp} \quad (13.5)$$
where the prime denotes the derivative with respect to $\xi$. All the other symbols vanish. The non-zero coefficients of the Ricci tensor are instead:

\[ R_{00} = R_{03} = R_{30} = R_{33} = -\frac{p''}{c^2 p} \]  

(13.6)

The scalar curvature $R$ is zero.

Being zero the divergence of $E$, the mass tensors $M_{ik}$ and $M^{ik}$ vanish. Actually, one should check that $\rho_E = 0$. This is also true, as the comments at the end of this section illustrate. The tensors $U_{ik}$ and $U^{ik}$ have to be computed through (12.10). First of all, one has:

\[ F_{ik} = c \begin{pmatrix} 0 & -u & 0 & 0 \\ u & 0 & 0 & u \\ 0 & 0 & 0 & 0 \\ 0 & -u & 0 & 0 \end{pmatrix} \]  

(13.7)

\[ F^{ik} = c \begin{pmatrix} 0 & u/p^2 & 0 & 0 \\ -u/p^2 & 0 & 0 & u/p^2 \\ 0 & 0 & 0 & 0 \\ 0 & -u/p^2 & 0 & 0 \end{pmatrix} \]  

(13.8)

where $u = B_2 = E_1/c$. Note that $(V_0, V) = (c, 0, 0, c)$ and $(V^0, V^1, V^2, V^3) = (c, 0, 0, -c)$. Hence, one gets $F^{ik}V_k = 0$, from which we deduce that the wave is free, as is already known. Afterwards, we have:

\[ T_{ik} = \frac{\mu^2}{c^2} \begin{pmatrix} (u/p)^2 & 0 & 0 & (u/p)^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (u/p)^2 & 0 & 0 & (u/p)^2 \end{pmatrix} \]  

(13.9)

\[ T^{ik} = \frac{\mu^2}{c^2} \begin{pmatrix} (u/p)^2 & 0 & 0 & -(u/p)^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(u/p)^2 & 0 & 0 & (u/p)^2 \end{pmatrix} \]  

(13.10)

Thus, (13.1) and (13.10) bring us to the equation:

\[ -p''p = \frac{\mu^2 \chi}{u^2} \]  

(13.11)

For $u = \sin \omega(t - z/c)$, we finally obtain $p = (\mu \sqrt{\chi}/\omega) \sin \omega(t - z/c)$, which is the solution we were looking for. There are surely other geometries compatible with the same plane wave. Note that the one presented here satisfies the relation (11.19). We also observe that there are points in which the metric
becomes singular, that is, the determinant $g$ is zero. Another equivalent possibility is to exchange $g_{11}$ and $g_{22}$ in (13.4), and make $p$ oscillate with the magnetic field. Comments about this option will be given in the next section.

The solution just obtained can be assimilated to a transversal (perfectly plane) gravitational wave, travelling in phase with the electromagnetic one. It must also be noted that, even if the space is officially non Euclidean, the geodesics involved in the motion of the wave are straight-lines. The field $G$, defined by (11.22), is identically zero. This is in agreement with our viewpoint: the geometry may be deformed, but there is no creation of a real gravitational vector field.

Pure gravitational solutions resembling plane waves, were formerly detected in [3]. We have been able to get the above explicit (and very simple) solution because we were resolute enough to assume the dependence from the metric tensor of the right-hand side of the Einstein equation. As far as we could deduce from the current literature, in contrast to our general approach to the problem, it is customary to construct the electromagnetic energy tensor in vacuum (thus, in Minkowski space-time), also because such an assumption is supposed (erroneously) to simplify the computations. Then, one comes to a set of solutions, but, as we proved, this is not the correct setting. Note also that, commonly, gravitational waves are searched among the solutions of the linearized homogeneous Einstein equation, obtained after perturbation of the flat space-time.

We can recover the laws of motion by evaluating the 4-divergence of the tensor $T^{ik}$ in (13.10). The geometry is non Euclidean, therefore, the relation (12.20) has to be replaced by (12.21), where $\sqrt{-g} = |p|$. For $i = 0$, one has:

$$\frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} T^{0k})}{\partial x_k} + \Gamma^0_{mj} T^{mj} = \frac{\mu^2}{c^2 p} \left( \frac{1}{c} \frac{\partial (u^2/p)}{\partial t} + \frac{\partial (u^2/p)}{\partial z} \right)$$

$$= \frac{\mu^2}{c^2 p} \left[ \frac{1}{p} \left( \frac{1}{c} \frac{\partial u^2}{\partial t} + \frac{\partial u^2}{\partial z} \right) - \frac{u^2}{p^2} \left( \frac{1}{c} \frac{\partial p}{\partial t} + \frac{\partial p}{\partial z} \right) \right] \quad (13.12)$$

The situation is exactly the same for $i = 3$. The last term in (13.12) is zero, when for instance:

$$\frac{1}{c} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} = 0 \quad \frac{1}{c} \frac{\partial p}{\partial t} + \frac{\partial p}{\partial z} = 0 \quad (13.13)$$

Our plane electromagnetic-gravitational wave is actually the solution to both the above equations, once the proper initial conditions have been assigned.
Let us now discuss the example of a circularly-polarized plane wave:

\[
E = (c \cos \omega (t - z/c), \ c \sin \omega (t - z/c), 0)
\]

\[
B = (- \sin \omega (t - z/c), \ \cos \omega (t - z/c), 0)
\]  \ (13.14)

The classical divergence of the electric field vanishes, as well as that of the magnetic field. Then, let us take the following metric tensor:

\[
g_{ik} = \frac{\mu^2 \chi}{\omega^2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -[\cos \omega (t - z/c)]^2 & 0 & 0 \\
0 & 0 & -[\sin \omega (t - z/c)]^2 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]  \ (13.15)

in such a way that the coordinates \(x\) and \(y\) are synchronized with the electric field (the reasons for this choice will be explained at the end of section 14). In this case, the nonvanishing coefficients of the Ricci tensor are: \(R_{00} = R_{03} = R_{30} = R_{33} = 2\omega^2/c^2\). They coincide with the respective coefficients of the stress tensor: \(T_{00} = T_{03} = T_{30} = T_{33} = 2\omega^2/c^2\). Therefore, once again, the Einstein equation is verified. The wave is free and we have \(R = 0\) and \(G = 0\).

Slightly more complicated is the case of a plane wave where \(\text{div} E\) is non-zero. This happens for instance when \(u = B_2 = E_1/c = f(x) \sin \omega (t - z/c)\). As we know, the solution satisfies the equations (3.1), (3.2), (3.3), but not the classical Maxwell equations. We suggest looking for a metric tensor of the form:

\[
g_{ik} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -p^2 f^2 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]  \ (13.16)

where \(p\) is function of the variable \(\xi = t - z/c\). One has: \(\sqrt{-g} = |fp|\). The tensor \(F_{ik}\) is the same as in (13.7). Regarding the other tensors, we get:

\[
F^{ik} = c \begin{pmatrix}
0 & u/(pf)^2 & u/(pf)^2 & 0 \\
u/(pf)^2 & 0 & 0 & u/(pf)^2 \\
0 & 0 & 0 & 0 \\
0 & u/(pf)^2 & 0 & 0
\end{pmatrix}
\]  \ (13.17)

\[
T_{ik} = \frac{\mu^2}{c^2} \begin{pmatrix}
(u/(pf)^2 & 0 & 0 & (u/(pf)^2) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
(u/(pf)^2 & 0 & 0 & (u/(pf)^2)
\end{pmatrix}
\]  \ (13.18)
We must point out an extraordinary fact: in the new geometry, the 4-divergence of the electric field turns out to be zero. As a matter of fact, by noting that $u/f$ and $p$ do not depend on $x$, one has:

$$\rho_E = \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} F^{0k})}{\partial x_k}$$

$$= -\frac{1}{\left|fp\right|} \frac{\partial (|fp| F^{01})}{\partial x} = -\frac{c}{fp^2} \frac{\partial (u/f)}{\partial x} = 0 \quad (13.19)$$

Thus, the mass tensor is still vanishing. The Christoffel symbols are a little different from the ones in (13.5) (in particular $\Gamma^1_{11}$ is not zero), but the coefficients of the Ricci tensor are exactly equal to those given in (13.6). Then, the equation (13.11) must be modified as follows:

$$- p'' p = \mu^2 \chi \left( \frac{u}{f} \right)^2 \quad (13.20)$$

thereby admitting the same solution $p$ obtained in the case of the plane wave at uniform density. The laws of motion are the same as in (13.13). They tell us that $u$ and $p$ shift at the speed of light along the $z$-axis. They do not specify however the function $f$, which must be assigned through the initial conditions.

We are ready to illustrate the case of a spherical wave. With the same notation of sections 2 and 4, we set the coordinates in order to have: $(x_0, x_1, x_2, x_3) = (ct, -r, -\phi, -\theta)$. Let us assume that $E = (0, cu, 0)$ and $B = (0, 0, u)$, where $u = \frac{1}{r} f(\phi) \sin \omega(t - r/c)$. We recall that, to avoid singularities at the poles, the function $f$ is not allowed to be constant. So, this is similar to the case of a variable-density plane wave. We also have that $(V^0, V^1, V^2, V^3) = (c, -c, 0, 0)$. So, let us begin by giving the metric tensor:

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -p^2 f^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (13.21)$$

where $p$ is a function of the variable $\xi = t - r/c$. Note that here the case $g_{22} = -1$ corresponds to the standard spherical system of coordinates. For the electromagnetic tensors we get:

$$F_{ik} = c \begin{pmatrix} 0 & 0 & -ru & 0 \\ 0 & 0 & -ru & 0 \\ ru & ru & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
In order to evaluate $F_{ik}$, we started from (6.3), recalling that, by (6.13), one has $(\Phi, A) = (-F(\phi) \sin \omega (t - r/c), -F(\phi) \sin \omega (t - r/c), 0, 0)$, where $F$ is a primitive of $f$. The metric in (13.21) is the same as the one we would have obtained if we had worked with a plane electromagnetic wave. The fact that we are in spherical coordinates is actually contained in the electromagnetic tensors (in which we find $ru$ in place of $u$). Note that $(V_0, V_1, V_2, V_3) = (c, c, 0, 0)$, from which one obtains the relation $F^{ik} V_k = 0$, confirming that the wave is free. As far as energy is concerned, we get:

$$T_{ik} = \frac{\mu^2}{c^2} \begin{pmatrix} (ru/pf)^2 & (ru/pf)^2 & 0 & 0 \\ (ru/pf)^2 & (ru/pf)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, from the Einstein equation, we come to:

$$-p'' p = \mu^2 \chi \left( \frac{ru}{f} \right)^2$$

(13.22)

where we observe that the right-hand side only depends on the variable $\xi = t - r/c$. The equation (13.22) once again gives the solution $p = (\mu \sqrt{\chi}/\omega) \sin \omega (t - z/c)$.

Finally, by differentiating $T^{ik}$ (for $i = 0$ and $i = 1$), we find the Euler equation in spherical coordinates:

$$\frac{1}{c} \frac{\partial u}{\partial t} + \frac{1}{r \partial r} \frac{\partial (ru)}{\partial r} = 0$$

(13.23)

The function $u$ is the solution to (13.23), after assuming the appropriate initial conditions.

The results of this section, although only restricted to the analysis of free waves, bring to attention some important issues. Up to now, we have claimed that a good theory of electromagnetism was meaningful only by allowing $\text{div} E$ to be different from zero. Here instead we find that $\rho E = 0$. In our opinion, what is happening can be explained as follows. The spacetime “reacts” to the passage of a wave, by varying itself in syncronism, in order to make the 4-divergence of the electric field vanish. The perturbation of the geometry is however weak enough to maintain field curvature $G$ equal
to zero. The classical divergence \( \text{div}\mathbf{E} \) may instead attain arbitrary values. We ask ourselves if it is possible to set up an experiment showing that, at some point and at a certain time, one has \( \text{div}\mathbf{E} \neq 0 \). Perhaps, this is not possible since, due to the modification of the metric, the instruments are unavoidably affected by the deformation of time and distances (with respect to the Euclidean reference frame). Therefore, in place of \( \text{div}\mathbf{E} \), we could end up measuring \( \rho_E \). But the last quantity is always zero (at least for free waves). As a consequence, we conclude that some divergence vanishes, although is not the classical, but the relativistic one. According to this new interpretation of the facts, in some sense the Maxwell theory was correct.

There is another point that needs to be clarified. The problem is why the geometry changes depending on the electric field, and not the magnetic field, especially after we said that for free waves the two fields have the same role. Firstly, we note that, in all examples studied in this section, the condition \( \text{div}\mathbf{B} = 0 \) was always fulfilled. In addition, if similarly to \( \rho_E \), we define \( \rho_B \), we discover that this new quantity is also zero (see section 14). If we imagine the wave like a fluid in motion, then this condition says that there is no flow of some “magnetic density of matter”. In truth, it is reasonable to assume that a sole electromagnetic fluid exists (not two, a separate electrical one and a magnetic one). As will become clear in the next section, where we analyze the case \( \text{div}\mathbf{B} \neq 0 \), such a fluid turns out to pulsate along a specific tangential direction (in principle, not necessarily corresponding to that of the electric field). Exchanging cause with effect, in section 14 we will support the following statement: from the behavior of the natural events, we are inclined to name the direction of the electric field as being that identified by the transversal oscillations of the fluid in motion.

Far more complicated phenomena show up, when we suppose that the waves are no longer free (thus, \( \mathbf{G} \neq 0 \)). In this context, the real gravitational fields come into life. We do not have any specific examples to discuss, due to the difficulty of the problems involved. Some hints will be given in section 15.

### 14 The divergence of the magnetic field

In the previous sections, some situations were discussed under the hypothesis \( \text{div}\mathbf{B} = 0 \). Although our equations now have a general validity, the assumption is necessary, for instance when introducing the potentials \( \mathbf{A} \) and \( \Phi \). Regarding this condition, we would like to add further comments
in this section. It is standard to introduce a transformation that exchanges the role of the electric and magnetic fields. This can be done through the pseudo-tensor:

\[
\epsilon_{mjk} = \begin{cases} 
0 & \text{when at least two indices are equal} \\
1 & \text{if the indices form an even permutation} \\
-1 & \text{if the indices form an odd permutation}
\end{cases}
\]  

The parity of the permutations is counted starting from the set: \{0, 1, 2, 3\}. Then, we define:

\[
\epsilon^{mjk} = \epsilon_m e_j e_i e_k \epsilon_{mjk} = -\epsilon_{mjk}
\]  

We may now introduce the duals of the tensors (6.4) and (6.5) in the following way:

\[
\hat{F}_{mj} = \frac{1}{2} \epsilon_{mjk} F^{ik} \quad \hat{F}^{mj} = e_m e_j \hat{F}_{mj} = -\frac{1}{2} \epsilon^{mjk} F_{ik}
\]  

Therefore, we obtain for example:

\[
\hat{F}_{01} = F^{23} \quad \hat{F}_{02} = F^{31} \quad \hat{F}_{03} = F^{12} \quad \hat{F}_{23} = F^{01} \quad \hat{F}_{31} = F^{02} \quad \hat{F}_{12} = F^{03}
\]

The original tensors and their duals have the same structure, with the difference that \(E\) replaces \(-cB\) and \(cB\) replaces \(E\).

In a similar way, the dual of the anti-symmetric rank-three tensor \(F_{jik}\) (defined in (11.13)) is given by:

\[
\hat{F}^m = -\frac{1}{6} \epsilon^{mjk} F_{jik}
\]  

Hence, up to even permutations of the lower indices, one has:

\[
\hat{F}^0 = F_{123} \quad \hat{F}^1 = -F_{023} \quad \hat{F}^2 = F_{013} \quad \hat{F}^3 = -F_{012}
\]

In general coordinates, it is customary to define:

\[
\xi^{mjk} = \sqrt{-g} g^{mm'} g^{jj'} g^{ii'} g^{kk'} \epsilon_{m'j'i'k'} = -\frac{1}{\sqrt{-g}} \epsilon_{mjk}
\]

\[
\xi_{mjk} = \sqrt{-g} \epsilon_{mjk}
\]  

So that the duals in (14.3) and in (14.4) are generalized as follows:

\[
\hat{F}_{mj} = \frac{1}{2} \xi_{mjk} F^{ik} \quad \hat{F}^{mj} = -\frac{1}{2} \xi^{mjk} F_{ik} \quad \hat{F}^m = -\frac{1}{6} \xi^{mjk} F_{jik}
\]  

where \(F_{ik}\) is provided in (6.4) and \(F^{ik}\) can be found in (12.11).
Then, the following relation is known (see [7], p.134):

\[ \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} \hat{F}^{mj})}{\partial x_j} = \hat{F}^m \]  

(14.7)

where we supposed that \( \hat{F}^m \) is the dual of the cyclic derivative \( F_{jk} \) of the tensor \( F_{ik} \) (of which \( \hat{F}^{mj} \) is the dual).

Passing to the duals, the equation (11.27) becomes:

\[ V^0 \hat{F}^m = V^m \hat{F}^0. \]

Therefore, by (14.7) we get:

\[ \frac{1}{\sqrt{-g}} \left( \frac{\partial (\sqrt{-g} \hat{F}^{mj})}{\partial x_j} V^0 - \frac{\partial (\sqrt{-g} \hat{F}^{0j})}{\partial x_j} V^m \right) = 0 \quad m = 0, 1, 2, 3 \]  

(14.8)

which is the exact counterpart of (11.24). The equation (14.8) represents, in a general coordinates system, the equation (9.5), that is equivalent to (9.4), after taking \( E \) in place of \( -cB \) and \( cB \) in place of \( E \). From (14.8), we can recover the continuity equation:

\[ \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} \rho_B V^i)}{\partial x_i} = 0 \quad \text{with} \quad \rho_B = \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} \hat{F}^{0k})}{\partial x_k} \]  

(14.9)

It is worth noting that \( \rho_B \) has the same dimensions of \( \rho_E \). For example, according to (14.6), the dual of \( F_{ik} \) in (13.7) is:

\[ \hat{F}^{mj} = \frac{c}{\sqrt{-g}} \left( \begin{array}{cccc} 0 & 0 & -u & 0 \\ 0 & 0 & 0 & 0 \\ u & 0 & 0 & -u \\ 0 & 0 & u & 0 \end{array} \right) \]  

(14.10)

Since we supposed that \( u \) does not depend on \( y \), we obtain \( \rho_B = 0 \). Based on the metric given by (13.16) (where \( \sqrt{-g} = |fp| \)), we just checked that, together with the condition \( \text{div} B = 0 \), the 4-divergence of the magnetic field also vanishes.

Going back to the equation (12.24), this time we cannot assume that \( F_{mjk} = 0 \). On the other hand, we can use (14.7) and (14.8) with \( V^0 = c \), to get:

\[ \frac{1}{2} g^{im} F_{mj} F^{jk} = g^{lm} \hat{F}^l \hat{F}_m = g^{lm} \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} \hat{F}^{lj})}{\partial x_j} \hat{F}_m \]

\[ = \frac{\rho_B}{c} g^{lm} \hat{F}_m V^l = \frac{\rho_B}{c} \hat{F}^{il} g_{lm} V^l = \frac{\rho_B}{c} \hat{F}^{im} V_m \]  

(14.11)
The first passage follows on from a direct counting of the permutations of the indices, thanks to the definitions provided in (14.6). Substituting in (12.24), one finally gets:

$$\nabla_k U^{ik} = \frac{1}{c} \left( \rho_E F^{im} V_m + \rho_B \hat{F}^{im} V_m \right)$$

(14.12)

Let us observe that, for i = 0, we have $F^{0m} V_m = 0$ (due to (11.10)), while $\rho_B \hat{F}^{0m} V_m$ recalls the product $-c^2 (B \cdot V) \text{div} B$. Thus, the first line of (14.12) turns out to be equivalent to (10.4).

The equation (14.12) is the generalization of (12.6) with $N = 0$ and $M = 0$. In spite of its elegance, it is not very convincing, since it involves two mass densities (see also the comments at the end of section 13). Let us try to explain what is happening. Without going into technical detail, we may make some remarks. We first note that $\hat{\hat{F}}^{ik} = -F^{ik}$, which means that, after applying the dual twice, one gets the opposite of the original tensor. Then, for any real $\lambda$, we consider the two tensors:

$$F^{ik} = \frac{1}{\sqrt{\lambda^2 + (1-\lambda)^2}} \left[ \lambda F^{ik} + (1-\lambda) \hat{F}^{ik} \right]$$

$$\hat{F}^{ik} = \frac{1}{\sqrt{\lambda^2 + (1-\lambda)^2}} \left[ \lambda \hat{F}^{ik} - (1-\lambda) F^{ik} \right]$$

(14.13)

where the second one is the dual of the first one. As in (12.11) we have: $F^{ik} = g^{im} g^{kl} F_{ml}$. Moreover, we can check that the tensor $U_{ik}$ in (12.10) does not change if in place of $F^{ik}$ and $\hat{F}^{ik}$ we take $F^{ik}$ and $\hat{F}^{ik}$, respectively. Therefore, the electromagnetic stress tensor does not depend on $\lambda$, even if this parameter varies in space and time. Actually, we already observed in section 12 that the energy tensor does not recognize the polarization of the electromagnetic field.

At this point, we can introduce the two new densities (see also (12.74)): $\rho_E = \nabla_k F^{0k}$ and $\rho_B = \nabla_k \hat{F}^{0k}$, where $E = (\lambda E + (1-\lambda) cB) / \sqrt{\lambda^2 + (1-\lambda)^2}$ and $B = (\lambda c \hat{B} - (1-\lambda) E) / \sqrt{\lambda^2 + (1-\lambda)^2}$.

So, another equivalent way to write equation (14.12) is:

$$\nabla_k U^{ik} = \frac{1}{c} \left( \rho_E F^{im} V_m + \rho_B \hat{F}^{im} V_m \right)$$

(14.14)

For $\lambda = 1$ the two versions are actually the same. Now, by letting $\lambda$ to vary, suppose that it is possible to modify the polarization of the fields $E$ and $B$ at each point, in order to get $\rho_B = 0$. In this way, we are left with
a single density $\rho_E$, which is the one to be used in constructing the mass tensor $M_{ik} = \rho_E V_i V_k$.

Let us restate the situation in brief. Every non-trivial electromagnetic wave presents regions where the classical divergence of any of the two fields is non-zero. The electromagnetic energy tensor does not distinguish between the two types of fields (electric or magnetic). In the end, what matters is the intensity of the wave and the modality of propagation of its fronts, without paying attention to the way each front has been parametrized. We can associate a fluid in motion at the speed of light with the wave. Independently of the actual orientation of the fields $E$ and $B$, we can locally build two other fields $\mathcal{E}$ and $\mathcal{B}$, so that the first one oscillates together with the fluid and the second one satisfies $\text{div}\mathcal{B} = 0$. This fictitious change of polarization has no influence on the electromagnetic energy tensor. The 4-divergence of $\mathcal{E}$, when different from zero, represents the mass density of the fluid and it is used to construct the mass tensor. This last tensor is added to the electromagnetic energy one, to form the global energy tensor which is on the right-hand side of the Einstein equation. In principle, the fields $\mathcal{E}$ and $\mathcal{B}$ are not directly associated with $E$ and $B$. However, in the natural evolution of electromagnetic phenomena, the two entities usually coincide.

All the examples analyzed in the previous section satisfy $\rho_B = 0$ and $\rho_B = 0$, hence, they were already well suited to the case $\lambda = 1$, corresponding to $\mathcal{E} = E$ and $\mathcal{B} = cB$. In particular, the case in spherical coordinates simulates the real behavior of a wave generated by an infinitesimal electric dipole oscillating in a vertical direction. Somehow, the dipole imparts mechanical oscillations to the fluid, in the same direction as the electric field. Formally, we can now exchange the role of the fields $E$ and $cB$, by polarizing the spherical wave by 90 degrees. In this new situation, we have $\text{div}E = 0$, $\rho_E = 0$ and $\text{div}B \neq 0$. By choosing $\lambda = 0$, we realize the condition $\rho_B = 0$ and the fictitious fields $\mathcal{E}$ and $\mathcal{B}$ turn out to be anti-ruotated by 90 degrees. Therefore, there is no longer coincidence of $\mathcal{E}$ and $\mathcal{B}$ with the corresponding $E$ and $cB$. Nevertheless, a spherical wave having the second kind of polarization is difficult to observe in nature, since it should correspond to the one generated by an infinitesimal magnetic monopole.

It is certainly true that our equations are not capable of recognising the polarization of free waves. This is a property that comes with the initial conditions. However, free waves are created by some causes inherent to natural events, which have a strong influence in determining polarization. The problem resides at the origin, for example in the non existence of magnetic monopoles (we will have a short discussion about this in section 15). Recall
that, in equation (9.6), the electric and magnetic fields cannot be inter-
changed. Certainly, this equation influences the creation of a spherical wave
through the mechanical oscillations of an electric charge. The conclusion
is that, at least for free waves, we can expect $\lambda = 1$, which implies that
the direction of transversal propagation of the fluid is in accordance with
that of the electric field. More precisely, this can be taken as a definition
of electric field. Suppose that an external mechanical perturbation is applied
to a free wave having $\mathcal{E} = \mathbf{E}$, in a direction not aligned with that of field
$\mathbf{E}$, in such a way the direction of $\mathcal{E}$ changes. Then we may think that the
wave reacts by varying its polarization (see section 7, 8 and 9) in order to
correct its posture, bringing field $\mathbf{E}$ to once again coincide with field $\mathcal{E}$. In
other words, the electric field turns out to be identified with the one that
follows the transversal oscillations of the fluid, and such a definition matches
reality.

15 Other developments and conclusions

We start by recalling the primary results obtained by the paper. In section
9, we introduced the following equations:

$$
\frac{\partial \mathbf{E}}{\partial t} = c^2 \text{curl} \mathbf{B} - (\text{div} \mathbf{E}) \mathbf{V} \quad (15.1)
$$

$$
\frac{\partial \mathbf{B}}{\partial t} = - \text{curl} \mathbf{E} - (\text{div} \mathbf{B}) \mathbf{V} \quad (15.2)
$$

$$
\frac{\mathbf{DV}}{\mathbf{Dt}} = \mu (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \quad (15.3)
$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field and $\mathbf{V}$ is a velocity field
satisfying:

$$
|\mathbf{V}| = c \quad (15.4)
$$

The constant $\mu$ is a charge divided by a mass, and $c$ is the speed of light.

Then, in sections 11 and 12, we wrote the equations in covariant form.
In the same order they appear above, we have, for $i = 0, 1, 2, 3$:

$$
(\nabla_k \hat{F}^{ik})V^0 = (\nabla_k \hat{F}^{0k})V^i \quad (15.5)
$$

$$
(\nabla_m \hat{F}^{im})V^0 = (\nabla_m \hat{F}^{0m})V^i \quad (15.6)
$$

$$
\frac{DV^i}{Dt} + \Gamma^i_{jk} V^j V^k = - \frac{\mu}{c} F^{im} V_m \quad (15.7)
$$

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where $V^0 = c$, $F_{ik}$ is the electromagnetic tensor and $\hat{F}^{jm}$ its dual. In the general system of coordinates, the normalizing condition takes the form:

$$g_{im}V^i V^m = 0 \quad (15.8)$$

The metric tensor $g_{ik}$ is not given, but has to be determined through the Einstein equation:

$$R_{ik} - \frac{1}{2}g_{ik}R = \frac{\chi \mu}{c^4} \left(-\mu g^{mj} F_{im} F_{kj} + \frac{1}{4} \mu g_{ik} F_{mj} F^{mj} + V_i V_k \nabla_m F^{0m} \right) \quad (15.9)$$

where on the right-hand side we find a suitable energy tensor, obtainable with the rules provided in section 12 (in the construction of $\rho_E = \nabla_m F^{0m}$ remember to take into account the warnings at the end of section 14). Such a coupling corresponds to a quite complex system, able to describe space-time geometry in conjunction with electromagnetic phenomena.

Our set of equations contains the embryo of some of the main laws of Physics. In (15.1) and (15.2), we recognize the equations of electromagnetism, more or less with the same structure as the Maxwell equations. We discovered that, when $V$ is irrotational, then (15.4) is the eikonal equation, so that the Huygens principle is also latent. On the right-hand side of (15.1) we partly recognize the Ampère law. The equation (15.3) expresses the Lorentz law, anticipating the Newton law in the form of momentum equation for the dynamics of fluids. In fact, we claimed that the light rays can be assimilated to stream-lines of a certain fluid of density $\rho_E$. Moreover, we know that a continuity equation holds for $\rho_E$.

Throughout the paper, we assumed we were in a universe that we could call “pre-Coulombian”. As a matter of fact, we developed a theory of electromagnetism without introducing any charges, and we spoke about fluids without having any masses. The only elements at our disposal were the fields. Here comes the big question: can we now build matter from these fields respecting the rules that we wrote? In other terms: can an elementary particle be “solution” to our set of equations?

A particle is a quite complicated thing. It has charge, magnetic momentum, spin, mass. It evolves and interacts with other particles according to the rules of quantum mechanics. Can we contain all these factors in a solution localized in space? This problem was mentioned in section 10 where we discussed possible solutions, consisting of a stable system of two rotating solitons. Although the framework is still incomplete we collected some pieces of evidence, whose details will shortly be discussed below, that
support the possibility of creating particles from fields. We recall that other authors, through a qualitative analysis, followed a similar idea of building electrons from photons (see for example [13] and the references therein).

We can give a rough idea of how a “particle solution” looks by examining figure 7, that shows, projected on a plane, the rotation of the fronts around an axis. From a qualitative viewpoint, field $E$ oscillates radially, but, in the average, mainly pointing inward (or outward). This creates the polarity of the electrical charge. Field $B$ is orthogonal to the page. The rays form closed orbits and their vector curvature $G$ points toward the center, producing a non-vanishing gravitational field. If the sign of $E$ is changed, then $G$ again points toward the inside (gravity has only one polarity). The displacement of field $V$ matches the idea that something is “spinning”, and the associated electromagnetic fluid corresponds to a kind of vortex.

Still referring to figure 7, let us suppose that the particle is an electron. Then $E$ should be directed toward the center and, using the standard vector product $\times$, $B$ points downwards. Nevertheless, a negative charge rotating clockwise produces a spin angular momentum pointing downward and a magnetic field pointing upward, which is in contrast to what previously found. As we remarked in section 8, this happens because we do not use the suitable vector product $\times$. In fact, the correct one is the left-handed one. Since the magnetic dipole moment is independent of the sign of $\times$, the change of parity now confirms that $B$ points upward. If we want to maintain the same set of equations, we can solve the problem just by changing the sign of the electric field, so that the electron has a chance of existing only if the electric field vectors point outward. We can still call this particle an “electron” and give a negative sign to it, but we have to comply with the new rule stating that currents flow from a negative pole to a positive one.

Nevertheless, one can see that such a situation is still not compatible with equations (15.1)-(15.2)-(15.3)-(15.4). One of the reasons is that the outer orbits of the light rays are longer than the inner orbits, and this does not match the condition (15.4), telling us that the information propagates at constant speed. In order to have chances of finding solutions to the form described above, the use of the general relativity framework is unavoidable. The modification of space-time geometry allows for the preservation of the momentum of inertia (a typical mechanical concept), providing the “glue” that keeps the particle together. The rotating wave follows the geodesics in the new metric. At the same time, the curvature of such geodesics has to be compatible, through (15.7), with the electromagnetic setting. The geometry alters the relation between space and time in such a way that the rays, always
travelling at speed $c$, can accomplish paths of different length in the same amount of time. This recalls the problem of the rigid rotating disk in general relativity. It is clear that the particle solution (if it exists) involves the use of the whole set of equations. Therefore, its determination, even from the point of view of numerical computations, is a demanding problem. Finally, by heuristic arguments, one can recognise that a similar solution, where the magnetic field is exchanged with the electric one, should be forbidden by equation (15.7). This would imply the impossibility of building magnetic monopoles.

We finish the paper with some further speculations, not having enough theoretical background. One positive aspect is that particle solutions are expected to be extremely stable (an electron is quite a difficult object to destroy). Another aspect is that they are in some sense “unique” (there is only one type of electron or proton), and this property raises other questions. The equations (9.1) and (9.2) are “scalable”, by meaning that we can multiply the fields of a free wave by a constant, once more obtaining a solution. Thus, free waves may be of any size and intensity. But, if we take into account constrained waves, then this property is no longer true, since (15.4) is not a scalable equation. Together with $c$, $\mu$ and $\chi$ in (13.1), another constant is hidden in the set of equations (15.5)-(15.6)-(15.7)-(15.8), which is related to some “magnitude” of the geometry. This result does not penalize our theory. Actually it may give more strength to it. As a matter of fact, we cannot have electrons of any size! We have no elements for quantifying the values of the various parameters, unless we find the particle solution explicitly.

The last issue we discuss is the convenience of setting up experiments validating our theory. The problem is left to the experts. However, we think that many convincing arguments, also based on a multitude of practical observations, have already been collected, showing that our model is adequate. The real breakthrough would be in predicting the realization of an electromagnetic device, capable of producing gravitational field.

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