Unique Cartan decomposition for II$_1$ factors arising from arbitrary actions of hyperbolic groups

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Abstract

We prove that for any free ergodic probability measure preserving action $\Gamma \curvearrowright (X, \mu)$ of a non-elementary hyperbolic group, or a lattice in a rank one simple Lie group, the associated group measure space II$_1$ factor $L^\infty(X) \rtimes \Gamma$ has $L^\infty(X)$ as its unique Cartan subalgebra, up to unitary conjugacy.

1 Introduction and main results

A Cartan subalgebra $A$ in a (separable) II$_1$ factor $M$ is a maximal abelian $*$-subalgebra $A \subset M$ with normalizer $N_M(A) = \{ u \in \mathcal{U}(A) \mid uAu^* = A \}$ generating $M$. Its presence amounts to realizing $M$ as a generalized (twisted) version of the group measure space construction, for a measure preserving ergodic countable equivalence relation $\mathcal{R}$ on a probability space $X$ and a 2-cocycle $v$ for $\mathcal{R}$. Showing uniqueness (up to conjugacy by an automorphism) of Cartan subalgebras is important, because the classification of factors $M$ satisfying this property reduces to the classification of the associated pairs $(\mathcal{R}, v)$ ($[FM75]$). In particular, the classification of group measure space factors $M = L^\infty(X) \rtimes \Gamma$ with unique Cartan subalgebras, reduces to the classification up to orbit equivalence of the corresponding free ergodic probability measure preserving (pmp) actions $\Gamma \curvearrowright X$.

It has been known since [CFW81] that any two Cartan subalgebras of the hyperfinite II$_1$ factor $R$ are conjugated by an automorphism, and thus any 2-cocycle of any free ergodic pmp action of an amenable group vanishes (untwists) and any two ergodic actions of any two amenable groups are orbit equivalent. While in the nonamenable case examples of group measure space factors with two distinct Cartan subalgebras were already constructed in [CJ81], uniqueness results started to emerge in [Po01], where it was shown that all Cartan subalgebras $A \subset M$ that satisfy a certain rigidity property in a factor of the form $L^\infty(X) \rtimes F_n$, with $F_n$ being the free group on $2 \leq n \leq \infty$ generators, is unitarily conjugate to $L^\infty(X)$. This led to the conjecture that such a property could hold without any condition on the Cartan subalgebra. Further supporting evidence came with the work in [OP07], where it was shown that group measure space factors arising from profinite actions of $F_n$ have unique Cartan decomposition.

We solved this conjecture in [PV11], where we actually found a large class of groups $\Gamma$, containing $F_n$, with the property that the II$_1$ factor $L^\infty(X) \rtimes \Gamma$ associated with an arbitrary free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$ has $L^\infty(X)$ as its unique Cartan subalgebra up to unitary conjugacy, i.e. $\Gamma$ is $C$-rigid, in the sense of [PV11] Definition 1.4]. More precisely, we showed in [PV11] Theorem 1.2] that all weakly amenable groups that admit a proper 1-cocycle into a nonamenable representation are $C$-rigid. To prove this result, we first showed in [PV11].

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Theorem 5.1] (by only using the weak amenability of $\Gamma$!) that the normalizer of any Cartan subalgebra $A \subset L^\infty(X) \rtimes \Gamma$ has a special almost invariance property, that can be viewed as a relative version (w.r.t. $L^\infty(X)$) of the notion of weak compactness in [OP07, Definition 3.1]. The second part of the proof consisted in applying to this relative weak compactness the malleable deformation associated in [Si10] with a 1-cocycle into an orthogonal representation of $\Gamma$. As such, we derived that if $A$ is not unitarily conjugate to $L^\infty(X)$ then its normalizer generates an amenable subalgebra (thus contradicting the regularity of $A$).

The degree of generality of the results in [PV11] was thus limited by the assumption that $\Gamma$ admits a proper 1-cocycle into a nonamenable orthogonal representation $\eta$ on $K_R$, i.e., of a proper map $c : \Gamma \to K_R$ satisfying $c(gh) = c(g) + \eta_g c(h)$ for all $g, h \in \Gamma$.

In the particular case of profinite actions, this type of limitation had already been circumvented in [CS11] under the weaker assumption that the group $\Gamma$ belongs to their class $\mathcal{QH}_{\text{reg}}$, requiring that $\Gamma$ has an orthogonal representation $\eta$ on $K_R$ that is weakly contained in the left regular representation and that merely admits a proper map $c : \Gamma \to K_R$ coarsely satisfying the 1-cocycle relation, i.e. $\sup_{k \in \Gamma} \| \eta_g c(k) - c(gkh) \| < \infty$, $\forall g, h \in \Gamma$. Thus, it is shown in [CS11] that for all profinite free ergodic pmp actions of all weakly amenable, nonamenable groups in the class $\mathcal{QH}_{\text{reg}}$, the crossed product has a unique Cartan subalgebra up to unitary conjugacy. This result was then extended in [CSU11] to cover as well products of weakly amenable groups in $\mathcal{QH}_{\text{reg}}$.

As we will later explain, the class of exact groups in $\mathcal{QH}_{\text{reg}}$ coincides with the class of bi-exact groups in the sense of [Oz03] (see Definition 2.3 and Proposition 2.7 below). In this paper, which should be viewed as a follow-up to [PV11], we show that weakly amenable, nonamenable, bi-exact groups are in fact $C^*$-rigid, i.e., all their group measure space factors have unique Cartan subalgebra. To prove this result, we first use the relative weak compactness property (which was obtained in [PV11, Theorem 5.1] from the weak amenability assumption) and then apply the bi-exactness property, by using an argument inspired by the proof of [BO08, Theorem 15.1.5]. More precisely, we obtain the following general result.

**Theorem 1.1.** Let $\Gamma$ be a weakly amenable, nonamenable, bi-exact group, or let $\Gamma$ be a direct product of $1 \leq n < \infty$ such groups. If $\Gamma \curvearrowright (X, \mu)$ is an arbitrary free ergodic pmp action, then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$, up to unitary conjugacy.

In particular, all of the following groups are $C^*$-rigid.

1. non-elementary hyperbolic groups,
2. nonamenable discrete subgroups of a connected noncompact rank one simple Lie group with finite center,
3. limit groups in the sense of Sela,
4. direct products of $1 \leq n < \infty$ groups as in 1, 2 and 3.

One should point out that, although in our proof of Theorem 1.1 we use an approach based on bi-exactness rather than the $\mathcal{QH}_{\text{reg}}$ property, we owe much to ideas in [CS11], on how to go beyond groups admitting proper 1-cocycles. In fact, in a first version of this paper we gave a proof of Theorem 1.1 using the methods of [CS11], before we found the present much simpler and direct argument.

Recall from [PV11, Definition 1.4] the following definition.
**Definition 1.2.** We say that a countable group $\Gamma$ is $C$-rigid (Cartan-rigid) if for every free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$, the $\text{II}_1$ factor $L^\infty(X) \rtimes \Gamma$ has $L^\infty(X)$ as its unique Cartan subalgebra up to unitary conjugacy.

In view of [OP07, Proposition 4.12], we say that a countable group $\Gamma$ is $C_s$-rigid if for every free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$, the $\text{II}_1$ factor $\mathcal{M} = L^\infty(X) \rtimes \Gamma$ has the following property: every maximal abelian subalgebra $A \subset \mathcal{M}$ whose normalizer $N_{\mathcal{M}}(A)^\prime\prime$ is a finite index subfactor of $\mathcal{M}$, is unitarily conjugate to $L^\infty(X)$.

The groups $\Gamma$ in Theorem 1.1 are in fact $C_s$-rigid. Moreover the same holds for all groups that are measure equivalent with $\Gamma$ (see Definition 2.5).

**Theorem 1.3.** The following groups are $C$-rigid and $C_s$-rigid:

1. groups that are measure equivalent to a direct product of $1 \leq n < \infty$ weakly amenable, nonamenable, bi-exact groups,
2. discrete subgroups $\Gamma$ of a direct product $G = G_1 \times \ldots \times G_n$ of connected noncompact rank one simple Lie groups $G_i$ with finite center, such that the image of $\Gamma$ in $G_i$ has a nonamenable closure for all $1 \leq i \leq n$.

Following [OP07], a finite von Neumann algebra $\mathcal{M}$ is called strongly solid if the normalizer of any diffuse amenable subalgebra of $\mathcal{M}$ is still amenable. It is shown in [OP07] that the free group factors $L_{F_n}$ are strongly solid. As explained above, it was proven in [CS11] that in fact the group von Neumann algebras $L\Gamma$ of all hyperbolic groups $\Gamma$ are strongly solid. Crossed products $\mathcal{B} \rtimes \Gamma$ are typically not strongly solid, but we establish the following relative strong solidity property: for weakly amenable, bi-exact groups $\Gamma$, we prove the dichotomy that if a subalgebra $A$ of a crossed product $\mathcal{B} \rtimes \Gamma$ is amenable relative to $\mathcal{B}$ (see Section 2.3), then either $A$ embeds into $\mathcal{B}$ (in the sense of intertwining-by-bimodules, see Definition 2.1), or $A$ has a normalizer that remains amenable relative to $\mathcal{B}$.

**Theorem 1.4.** Let $\Gamma$ be a weakly amenable, bi-exact group and let $\Gamma \curvearrowright (\mathcal{B}, \tau)$ be an arbitrary trace preserving action on the tracial von Neumann algebra $(\mathcal{B}, \tau)$. Put $\mathcal{M} = \mathcal{B} \rtimes \Gamma$.

If $q \in \mathcal{M}$ is a projection and $A \subset q\mathcal{M}q$ is a von Neumann subalgebra that is amenable relative to $\mathcal{B}$ and put $P := N_{q\mathcal{M}q}(A)^\prime\prime$. Then $\mathcal{P} := N_{q\mathcal{M}q}(A)^\prime\prime$ remains amenable relative to $\mathcal{B}$, or $A \prec_{\mathcal{M}} \mathcal{B}$.

Both Theorem 1.1 and 1.4 will be deduced in Section 4 from our more technical Theorem 3.1 also yielding the following new class of tensor product $\text{II}_1$ factors without Cartan subalgebras, improving [OP07, Corollary 2].

**Theorem 1.5.** Let $\Gamma$ be a nonamenable, icc, weakly amenable, bi-exact group and let $\mathcal{N}$ be an arbitrary $\text{II}_1$ factor. Then $\mathcal{N} \otimes L\Gamma$ has no Cartan subalgebra.

A statement similar to 1.4 holds for direct product groups and goes as follows. We use the strong intertwining notation $\prec$ that is introduced in Definition 2.1 below.

**Theorem 1.6.** Let $\Gamma = \Gamma_1 \times \ldots \times \Gamma_n$ be the direct product of $n \geq 1$ weakly amenable, bi-exact groups $\Gamma_i$. Let $\Gamma \curvearrowright (\mathcal{B}, \tau)$ be an arbitrary trace preserving action on the tracial von Neumann algebra $(\mathcal{B}, \tau)$. Put $\mathcal{M} = \mathcal{B} \rtimes \Gamma$. Let $A \subset q\mathcal{M}q$ be a von Neumann subalgebra that is amenable relative to $\mathcal{B}$ and put $P := N_{q\mathcal{M}q}(A)^\prime\prime$.

Then there exist projections $p_0, \ldots, p_n \in \mathcal{Z}(P)$, some of which might be zero, such that $p_0 \vee \cdots \vee p_n = q$ and
• $Pp_0$ is amenable relative to $B$,

• for every $i=1,\ldots,n$ we have $A_i \prec_M B \rtimes \hat{\Gamma}_i$ where $\hat{\Gamma}_i$ is the direct product of all $\Gamma_j$, $j \neq i$.

Note that results of the same type as Theorems 1.4, resp. 1.6, were established in [CS11], resp. [CSU11], under the additional assumption that $A \subset qMq$ is a weakly compact embedding and that $A$ and $B$ are amenable von Neumann algebras.

Since for $\mathcal{C}$-rigid groups $\Gamma$, the classification of group measure space factors $L^\infty(X) \rtimes \Gamma$ reduces to the classification of the associated free ergodic pmp actions $\Gamma \curvearrowright (X,\mu)$ up to orbit equivalence (OE), Theorem 1.1 can be combined with existing OE rigidity results, in particular with the work of [MS02] on OE rigidity for direct products of hyperbolic groups. This leads to the following result. We refer to Section 6 for terminology and to [PV11, Section 12] for further applications in $W^*$-superrigidity.

**Theorem 1.7.** Let $\Gamma = \Gamma_1 \times \Gamma_2$ be the direct product of two non-elementary hyperbolic groups. Assume that $\Gamma \curvearrowright (X,\mu)$ is a free ergodic pmp action that is aperiodic and irreducible.

If $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ for any free mildly mixing pmp action $\Lambda \curvearrowright (Y,\eta)$, then $\Gamma \cong \Lambda$ and the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are conjugate.

**2 Preliminaries**

Throughout this article we call *tracial von Neumann algebra* $(M,\tau)$, any von Neumann algebra $M$ equipped with a faithful normal tracial state $\tau$.

**2.1 Intertwining by bimodules**

We recall from [Po03, Theorem 2.1 and Corollary 2.3] the theory of *intertwining-by-bimodules*, summarized in the following definition.

**Definition 2.1.** Let $(M,\tau)$ be a tracial von Neumann algebra and $P,Q \subset M$ possibly non-unital von Neumann subalgebras. We write $P \prec_M Q$ when one of the following equivalent conditions is satisfied.

- There exist projections $p \in P$, $q \in Q$, a normal $*$-homomorphism $\varphi : pPp \to qQq$ and a nonzero partial isometry $v \in pMq$ such that $xv = v\varphi(x)$ for all $x \in pPp$.

- It is impossible to find a net of unitaries $u_n \in \mathcal{U}(P)$ satisfying $\|E_Q(xu_n y^*)\|_2 \to 0$ for all $x,y \in 1QM1P$.

We write $P \prec_M^f Q$ if $Pp \prec_M Q$ for every projection $p \in P' \cap 1_PM1P$.

**2.2 Jones’ basic construction**

Let $(M,\tau)$ be a tracial von Neumann algebra and $B \subset M$ a von Neumann subalgebra. Jones’ basic construction $\langle M,e_B \rangle$ is defined as the von Neumann algebra acting on $L^2(M)$ generated by $M$ and the orthogonal projection $e_B$ of $L^2(M)$ onto $L^2(B)$. Recall that $\langle M,e_B \rangle$ coincides with the commutant of the right $B$-action on $L^2(M)$. 

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2.3 Relative amenability

Recall that a functional $\Omega$ on a von Neumann algebra $\mathcal{N}$ with subalgebra $P \subset \mathcal{N}$ is called $P$-central if $\Omega(xS) = \Omega(Sx)$ for all $x \in P$, $S \in \mathcal{N}$.

Let $(M, \tau)$ be a tracial von Neumann algebra, $p \in M$ a projection and $P \subset pMp, B \subset M$ von Neumann subalgebras. Following [OP07, Section 2.2] we say that $P$ is amenable relative to $B$ if the von Neumann algebra $p(M, e_B)p$ admits a $P$-central positive functional whose restriction to $pMp$ coincides with $\tau$. We need the following variant of [PVII, Corollary 2.6].

**Lemma 2.2.** Let $(M, \tau)$ be a tracial von Neumann algebra. Assume that $P_1 \subset P_2 \subset M$ and $Q \subset M$ are von Neumann subalgebras such that $P_1 \subset P_2$ is a finite index subfactor.

If $p_1 \in P_1' \cap M$ is a nonzero projection such that $P_1p_1$ is amenable relative to $Q$ and if $p_2$ denotes the smallest projection in $P_2' \cap M$ that dominates $p_1$, then $P_2p_2$ is amenable relative to $Q$.

**Proof.** Take a Pimsner-Popa basis (see [PP84, Proposition 1.3]) for the finite index subfactor $P_1 \subset P_2 :$ we find elements $v_1, \ldots, v_n \in P_2$ and a projection $q \in M_n(\mathbb{C}) \otimes P_1$ such that the map $U : q(\mathbb{C}^n \otimes L^2(P_1)) \to L^2(P_2) : U(q(e_i \otimes x)) = v_ix$ for all $i = 1, \ldots, n$, $x \in P_1$, is a unitary operator. Define the normal $*$-homomorphism $\varphi : P_2 \to q(M_n(\mathbb{C}) \otimes P_1)q$ such that $U(\varphi(x)\xi) = xU(\xi)$ for all $x \in P_2$ and $\xi \in q(\mathbb{C}^n \otimes L^2(P_1))$. Defining $V \in M_{1,n}(\mathbb{C}) \otimes P_2$ given by $V = \sum_{i=1}^n e_{i1} \otimes v_i$, we get that $xV = V\varphi(x)$ for all $x \in P_2$.

Write $T := \sum_{i=1}^n v_ip_1v_i^*$. A direct computation shows that $T$ is a positive element in $P_2' \cap M$. The support projection of $T$ equals the projection onto the closed linear span of $\{v_ip_1x \mid i = 1, \ldots, n, x \in M\}$. Since $p_1$ commutes with $P_1$ and since the linear span of $v_ip_1$ equals $P_2$, it follows that the support projection of $T$ equals the projection onto the closed linear span of $P_2p_1M$. Thus, the support projection of $T$ equals $p_2$.

Since $P_1p_1$ is amenable relative to $Q$, we get a $P_1p_1$-central positive functional $\Omega_1$ on $p_1(M, e_Q)p_1$ such that $\Omega_1(x) = \tau(x)$ for all $x \in p_1Mp_1$. Define the positive functional $\Omega_2$ on $p_2(M, e_Q)p_2$ given by

$$\Omega_2(S) = \sum_{i=1}^n \Omega_1(p_1v_i^*Sv_ip_1).$$

A direct computation shows that $\Omega_2$ is $P_2p_2$-central. Also, for all $x \in p_2Mp_2$, we have that $\Omega_2(x) = \tau(xT)$. Since $T \in P_2' \cap M$ and since the support projection of $T$ equals $p_2$, we can take a sequence of positive elements $T_n \in P_2' \cap M$ such that $T_nT = TT_n \leq p_2$ and $T_nT \to p_2$ strongly. If we choose the positive functional $\Omega$ on $p_2(M, e_Q)p_2$ as a weak$^*$-limit point of the sequence of positive functionals $S \mapsto \Omega_2(T_n^{1/2}ST_n^{1/2})$, it follows that $\Omega$ is a $P_2p_2$-central positive functional on $p_2(M, e_Q)p_2$ with $\Omega(x) = \tau(x)$ for all $x \in p_2Mp_2$. Hence, $P_2p_2$ is amenable relative to $Q$. 

2.4 Bi-exactness and the classes $\mathcal{QH}_{\text{reg}}$ and $\mathcal{S}$

**Definition 2.3.** Let $\Gamma$ be a countable group.

- ([CHSS]) The group $\Gamma$ is called weakly amenable if there exists a sequence of finitely supported functions $f_n : \Gamma \to \mathbb{C}$ tending to 1 pointwise and satisfying $\sup_n \|f_n\|_{cb} < \infty$. Here $\|f\|_{cb}$ is the Herz-Schur norm, i.e. the $\text{cb}$-norm of the linear map $L(\Gamma) \to L(\Gamma) : u_g \mapsto f(g)u_g$. 

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The group $\Gamma$ is called bi-exact if $\Gamma$ is exact and if there exists a map $\mu : \Gamma \rightarrow \text{Prob}\Gamma$ satisfying
\[
\lim_{k \to \infty} \| \mu(gkh) - g \cdot \mu(k) \|_1 = 0 \quad \text{for all } g, h \in \Gamma.
\] (2.1)

Collecting several results from the literature, we get the following large classes of weakly amenable, bi-exact groups.

**Lemma 2.4.** The following groups are weakly amenable and bi-exact:

- (\[Oz03, Oz07\]) hyperbolic groups,
- (\[CH88, Sk88\]) discrete subgroups of a connected noncompact rank one simple Lie group with finite center,
- (\[Oz05, Oz12\]) limit groups in the sense of Sela.

Finally, by \[Su09, Oz10\], the family of weakly amenable, bi-exact groups is stable under measure equivalence and under the passage to measure equivalence subgroups (see Definition 2.5).

Before proving Lemma 2.4, recall from \[Oz04, Section 4\] that the class $S$ is defined as the class of countable groups $\Gamma$ for which the action of $\Gamma \times \Gamma$ by left right multiplication on the Stone-Čech remainder $\partial^\beta(\Gamma)$ of $\Gamma$ is topologically amenable. By \[BO08, Proposition 15.2.3\], a countable group $\Gamma$ belongs to the class $S$ if and only if $\Gamma$ admits a compactification $\Gamma \subset X$ such that

- the left multiplication action of $\Gamma$ extends to an action of $\Gamma$ by homeomorphisms of $X$ that is topologically amenable;
- the right multiplication action of $\Gamma$ extends to an action of $\Gamma$ by homeomorphisms of $X$ that are equal to the identity on $X - \Gamma$.

By \[BO08, Proposition 15.2.3\], a countable group $\Gamma$ belongs to the class $S$ if and only if $\Gamma$ is bi-exact in the sense of Definition 2.3.

**Definition 2.5.** A countable group $\Gamma$ is called a measure equivalence subgroup of a countable group $\Lambda$ if $\Gamma \times \Lambda$ admits a measure preserving action on a, typically infinite, standard measure space $(\Omega, m)$ such that both the actions $\Gamma \curvearrowright \Omega$ and $\Lambda \curvearrowright \Omega$ are free and admit a fundamental domain, with the fundamental domain of $\Lambda \curvearrowright \Omega$ having finite measure.

If the action $\Gamma \times \Lambda \curvearrowright \Omega$ can be chosen in such a way that also the fundamental domain of $\Gamma \curvearrowright \Omega$ has finite measure, the groups $\Gamma$ and $\Lambda$ are called measure equivalent.

**Proof of Lemma 2.4.** As explained above, by \[BO08, Proposition 15.2.3\], the class $S$ equals the class of bi-exact groups.

The action of a word hyperbolic group on its Gromov boundary is topologically amenable (see e.g. \[BO08, Theorem 5.3.15\]) and hence all hyperbolic groups belong to the class $S$. By \[Oz07\], hyperbolic groups are weakly amenable.

Let $\Gamma$ be a discrete subgroup of a connected noncompact rank one simple Lie group with finite center. By \[CH88\], the group $\Gamma$ is weakly amenable. In particular, $\Gamma$ is exact. It then follows from \[Sk88, Proof of Theorem 4.4\] and \[BO08, Lemma 15.1.4\] that $\Gamma$ belongs to the class $S$. 


Let $\Gamma$ be a limit group in the sense of Sela. By \[Da02\] Theorem 0.3, the group $\Gamma$ is hyperbolic relative to a family of cyclic subgroups. So, by \[Oz05\] Proposition 12, the group $\Gamma$ belongs to the class $S$. The following argument of \[Oz12\] shows that $\Gamma$ is weakly amenable. The group $\Gamma$ is a subgroup of an ultraproduct of free groups. Since all free groups are a subgroup of $\text{SL}(2,\mathbb{Z})$, we can view $\Gamma$ as a subgroup of $\text{SL}(2,\mathbb{Z})^\omega$, for some free ultrafilter $\omega$ on $\mathbb{N}$. Denoting by $K$ the ultrapower field $K := \mathbb{Q}^\omega$, we see that $\Gamma < \text{SL}(2, K)$. In \[GHW04\] Theorem 4 it is shown that all countable subgroups of $\text{SL}(2, K)$ have the Haagerup approximation property. The same argument actually shows that they are as well weakly amenable.

By \[Sa09\] Theorem 3.1, the class $S$ is stable under the passage to ME-subgroups, and in particular under measure equivalence. Finally, it was proven in \[Oz10\] End of Section 2 that weak amenability is stable under the passage to ME-subgroups.

When $\mathcal{G}$ is a family of subgroups of $\Gamma$, a subset $F \subset \Gamma$ is said to be small relative to $\mathcal{G}$ if $F$ is contained in the union of finitely many subsets of the form $g\Sigma h$ with $g, h \in \Gamma$ and $\Sigma \in \mathcal{G}$. We always tacitly assume that $\mathcal{G}$ contains the trivial subgroup $\{e\}$, so that finite subsets of $\Gamma$ always are small relative to $\mathcal{G}$. When $K$ is a normed space and $f : \Gamma \to K$, we say that

- $\lim_{k \to \infty} f(k) = 0$ if for every $\varepsilon > 0$, the set $\{k \in \Gamma \mid \|f(k)\| > \varepsilon\}$ is small relative to $\mathcal{G}$;
- $f : \Gamma \to K$ is proper relative to $\mathcal{G}$, if for every $\kappa > 0$ the set $\{k \in \Gamma \mid \|f(k)\| < \kappa\}$ is small relative to $\mathcal{G}$.

We denote by $\text{Prob}\,\Gamma$ the set of probability measures on a (countable) group $\Gamma$. We identify $\text{Prob}\,\Gamma$ with the natural convex subset of $\ell^1(\Gamma)$ and use the 1-norm on $\ell^1(\Gamma)$. If $g \in \Gamma$ and $\mu \in \text{Prob}\,\Gamma$, we denote by $g \cdot \mu$ the left translation of $\mu$ by $g$.

Definition 2.6 ([BO08, Definition 15.1.2]). A countable group $\Gamma$ with a family of subgroups $\mathcal{G}$ is said to be bi-exact relative to $\mathcal{G}$ if $\Gamma$ is exact and if there exists a map $\mu : \Gamma \to \text{Prob}\,\Gamma$ such that

$$\lim_{k \to \infty} \|\mu(gkh) - g \cdot \mu(k)\|_1 = 0 \quad \text{for all} \quad g, h \in \Gamma.$$  

(2.2)

By definition, a group is bi-exact if and only if it is bi-exact relative to $\{\{e\}\}$.

As observed by Ozawa, a group is bi-exact if and only if it is exact and belongs to the class $\mathcal{QH}_{\text{reg}}$ of \[CS11\]. We actually have the following more general result, parts of which were already proven in \[CS11\, CSU11\]. For the sake of completeness, we give a detailed proof, using the methods of \[BO08\, Chapter 15\]. Note however that we do not use this result in the rest of the paper.

Proposition 2.7. Let $\Gamma$ be a countable group and $\mathcal{G}$ a family of subgroups of $\Gamma$ with $\{e\} \in \mathcal{G}$. The following statements are equivalent.

1. There exists a map $\mu : \Gamma \to \text{Prob}\,\Gamma$ satisfying (2.2) in Definition 2.6.

2. There exist a map $c : \Gamma \to \ell^2(\Gamma)$ that is proper relative to $\mathcal{G}$ and that satisfies

$$\sup_{k \in \Gamma} \|c(gkh) - \lambda_g c(k)\|_2 < \infty \quad \text{for all} \quad g, h \in \Gamma.$$  

(2.3)
3. There exists an orthogonal representation \( \eta : \Gamma \rightarrow \mathcal{O}(K_{\mathbb{R}}) \) that is weakly contained in the regular representation and a map \( c : \Gamma \rightarrow K_{\mathbb{R}} \) that is proper relative to \( \mathcal{G} \) and satisfies
\[
\sup_{k \in \Gamma} \|c(gh) - \eta_0 c(k)\| < \infty \quad \text{for all } g, h \in \Gamma. \tag{2.4}
\]

In particular, \( \Gamma \) is bi-exact relative to \( \mathcal{G} \) if and only if \( \Gamma \) is exact and \( \Gamma \) satisfies the above equivalent conditions.

Note that the class \( \mathcal{QH}_{\mathrm{reg}} \) of \( \text{[CS11]} \) is defined as the class of groups \( \Gamma \) that satisfy 3 w.r.t. \( \mathcal{G} = \{ \{ e \} \} \). So, we indeed have that \( \mathcal{S} = \mathcal{QH}_{\mathrm{reg}} \cap \{ \text{exact} \} \).

**Proof.** 1 \( \Rightarrow \) 2. Assume that \( \mu : \Gamma \rightarrow \text{Prob} \Gamma \) satisfies (2.2). Define \( \zeta : \Gamma \rightarrow \ell^2_\mathbb{R}(\Gamma) \) given by \( \zeta(k) := \mu(k)^{1/2} \). Note that \( \|\zeta(k)\|_2 = 1 \) for all \( k \in \Gamma \) and that
\[
\lim_{k \to \infty/\mathcal{G}} \|\zeta(gh) - \lambda g \zeta(k)\|_2 \leq \lim_{k \to \infty/\mathcal{G}} \|\mu(gh) - g \cdot \mu(k)\|_1^{1/2} = 0 \quad \text{for all } g, h \in \Gamma. \tag{2.5}
\]

Let \( \{ e \} = E_0 \subset E_1 \subset E_2 \subset \cdots \) be finite subsets of \( \Gamma \) such that \( E_n^{-1} = E_n \) for all \( n \) and \( \bigcup_{n=0}^{\infty} E_n = \Gamma \). Inductively define the subsets \( F_n \subset \Gamma \) given by \( F_0 = \{ e \} \) and, for all \( n \geq 1 \),
\[
F_n := E_n F_{n-1} E_n \cup \bigcup_{g, h \in E_n} \{ k \in \Gamma \mid \|\zeta(gh) - \lambda g \zeta(k)\|_2 > \frac{1}{n} \}.
\]

By construction, the sets \( F_n \) are small relative to \( \mathcal{G} \). Also, the subsets \( F_n \) are increasing and their union equals \( \Gamma \), because \( E_n \subset F_n \). So we can uniquely define the map
\[
c : \Gamma \rightarrow \ell^2_\mathbb{R}(\Gamma) : c(k) = \begin{cases} 0 & \text{if } k \in F_1, \\
_\zeta(k) & \text{if } n \geq 1 \text{ and } k \in F_{n+1} - F_n. \end{cases}
\]

Whenever \( k \in \Gamma - F_n \), we have \( \|c(k)\|_2 \geq n \). So, \( c \) is proper relative to \( \mathcal{G} \). We prove that \( c \) satisfies (2.3). So fix \( g, h \in \Gamma \). Take \( m \geq 1 \) such that \( g, h \in E_m \). It suffices to prove that
\[
\|c(gh) - \lambda g c(k)\|_2 \leq 2m \quad \text{for all } k \in \Gamma. \tag{2.6}
\]

We first prove (2.6) if \( k \in \Gamma - F_m \). So, \( k \in F_{n+1} - F_n \) for some \( n \geq m \). Hence \( c(k) = n \zeta(k) \). Since \( g, h \in E_m \subset E_n = E_n^{-1} \), we also get that \( gkh \in F_{n+2} - F_{n-1} \). So \( c(gkh) \) can be \( n+1 \) times, or \( n \) times, or \( n - 1 \) times \( \zeta(k) \). In all cases \( \|c(gkh) - n \zeta(gkh)\|_2 \leq 1 \). Since \( k \notin F_n \) and \( g, h \in E_n \subset E_m \), we have \( \|\zeta(gkh) - \lambda g \zeta(k)\|_2 \leq 1/n \). Multiplying by \( n \), we get that
\[
\|c(gkh) - \lambda g c(k)\|_2 \leq \|c(gkh) - n \zeta(gkh)\|_2 + n \|\zeta(gkh) - \lambda g \zeta(k)\|_2 \leq 2 \leq 2m.
\]

So (2.6) is proven for all \( k \in \Gamma - F_m \). If \( k \in F_m \), we have \( \|c(k)\|_2 \leq m - 1 \). Since \( g, h \in E_m \), we also have \( gkh \in F_{m+1} \) and hence \( \|c(gkh)\|_2 \leq m \). Combining both we get that
\[
\|c(gkh) - \lambda g c(k)\|_2 \leq \|c(gkh)\|_2 + \|c(k)\|_2 \leq 2m - 1 < 2m.
\]

So (2.6) is proven and hence 2 holds.

2 \( \Rightarrow \) 3 is trivial by taking \( \eta \) to be the regular representation.

3 \( \Rightarrow \) 1 For a finite group \( \Gamma \) all statements in the proposition are trivially true (and rather silly).

So we assume that \( \Gamma \) is a countably infinite group satisfying 3 and we prove that \( \Gamma \) satisfies 1. Take \( \eta : \Gamma \rightarrow \mathcal{O}(K_{\mathbb{R}}) \) and \( c : \Gamma \rightarrow K_{\mathbb{R}} \) as in 3. Replacing \( K_{\mathbb{R}} \) by the closed linear span of
\{\eta_g c(k) \mid g, k \in \Gamma\}
we may assume that \(K_{\mathbb{R}}\) is separable. Let \(\zeta_0 \in K_{\mathbb{R}}\) be an arbitrary unit vector and define
\[
\zeta : \Gamma \to K_{\mathbb{R}} : \zeta(k) = \begin{cases} ||c(k)||^{-1} c(k) & \text{if } c(k) \neq 0, \\ \zeta_0 & \text{if } c(k) = 0. \end{cases}
\]
By construction \(||\zeta(k)|| = 1\) for all \(k \in \Gamma\). Since for all nonzero vectors \(\xi\) and \(\xi'\) in a Hilbert space, one has
\[
\left\| \frac{\xi}{\|\xi\|} - \frac{\xi'}{\|\xi'\|} \right\| \leq 2 \frac{\|\xi - \xi'\|}{\|\xi'\|},
\]
the properness of \(c\) relative to \(\mathcal{G}\) together with (2.4) implies that
\[
\lim_{k \to \infty} \|\zeta(gkh) - \eta_g \zeta(k)\| = 0 \quad \text{for all } g, h \in \Gamma.
\]
Denote by \(K\) the complexification of \(K_{\mathbb{R}}\) and still denote by \(\eta : \Gamma \to \mathcal{U}(K)\) the complexified representation. Denote \(\hat{K} = K \otimes \ell^2(\Gamma)\) and \(\hat{\Gamma} = \Gamma \times \Gamma\). Consider the unitary representation \(\tilde{\eta} : \hat{\Gamma} \to \mathcal{U}(\hat{K})\) given by \(\tilde{\eta}(g,h) = \eta_g \otimes \lambda_g \rho_h\). Since \(\eta\) is weakly contained in the regular representation of \(\Gamma\), we get that \(\tilde{\eta}\) is weakly contained in the representation \((g, h) \mapsto \lambda_g \otimes \lambda_g \rho_h\) that in turn is unitarily equivalent with the regular representation of \(\hat{\Gamma}\). So we get a unital \(*\)-homomorphism \(\theta : C^*_\text{red}(\hat{\Gamma}) \to B(\hat{K})\) satisfying \(\theta(\lambda_s) = \tilde{\eta}_s\) for all \(s \in \hat{\Gamma}\). Since \(\hat{\Gamma}\) is an infinite group, \(C^*_\text{red}(\hat{\Gamma}) \cap \mathcal{K}(\ell^2(\hat{\Gamma})) = \{0\}\). So by Voiculescu’s theorem (see e.g. [Da96, Corollary II.5.5]), there exists a unitary operator \(V : \ell^2(\hat{\Gamma}) \oplus \hat{K} \to \ell^2(\hat{\Gamma})\) such that
\[
aV - V(a \oplus \theta(a)) \quad \text{is a compact operator for all } a \in C^*_\text{red}(\hat{\Gamma}).
\]
For \(k \in \Gamma\), denote by \(\delta_k \in \ell^2(\Gamma)\) the canonical basis vector. Define
\[
\zeta' : \Gamma \to \ell^2(\hat{\Gamma}) : \zeta'(k) = V(0 \oplus (\zeta(k) \otimes \delta_k)) \quad \text{for all } k \in \Gamma.
\]
By construction \(\|\zeta'(k)\|_2 = 1\) for all \(k \in \Gamma\). We claim that
\[
\lim_{k \to \infty} \|\zeta'(gkh) - (\lambda_g \otimes \lambda_h)\zeta'(k)\| = 0 \quad \text{for all } g, h \in \Gamma.
\]
To prove this claim, fix \(g, h \in \Gamma\). Put
\[
T := \lambda_{(g,h)} V - V(\lambda_{(g,h)} \oplus \tilde{\eta}_{(g,h)}).
\]
From (2.8), we know that \(T\) is a compact operator. If \(k \to \infty/\mathcal{G}\), certainly \(k \to \infty\) and hence \(\zeta(k) \otimes \delta_k\) tends to 0 weakly. Since \(T\) is compact, it follows that
\[
\lim_{k \to \infty/\mathcal{G}} \|T(0 \oplus (\zeta(k) \otimes \delta_k))\|_2 = 0.
\]
This precisely means that
\[
\lim_{k \to \infty/\mathcal{G}} \|V(\eta_g c(k) \otimes (\delta_{gkh}^{-1})) - V(0 \oplus (\eta_g c(k) \otimes (\delta_{gkh}^{-1})))\|_2 = 0.
\]
From (2.7), it follows that
\[
\lim_{k \to \infty/\mathcal{G}} \|\eta_g c(k) \otimes (\delta_{gkh}^{-1}) - (\zeta(gkh^{-1}) \otimes \delta_{gkh}^{-1})\| = 0.
\]
In combination with (2.10), we exactly get the claim (2.9).
To every unit vector \( \zeta \in \ell^2(\tilde{\Gamma}) \), we associate the probability measure \( T(\zeta) \in \text{Prob} \Gamma \subset \ell^1(\Gamma) \) given by

\[
(T(\zeta))(s) = \sum_{t \in \Gamma} |\zeta(s, t)|^2 \quad \text{for all } s \in \Gamma.
\]

Clearly, \( T((\lambda_g \otimes \lambda_h)\zeta) = g \cdot T(\zeta) \) and, using the Cauchy-Schwarz inequality, \( \|T(\zeta_1) - T(\zeta_2)\|_1 \leq 2\|\zeta_1 - \zeta_2\|_2 \) for all unit vectors \( \zeta_1, \zeta_2 \in \ell^2(\tilde{\Gamma}) \). Defining \( \mu : \Gamma \to \text{Prob} \Gamma \) by \( \mu(k) = T(\zeta'(k)) \) for all \( k \in \Gamma \), we then get that (2.9) implies (2.2). So we have proven that (1) holds.

We record the following lemma from [BO08].

**Lemma 2.8 ([BO08, Lemma 15.3.3]).** Let \( \Gamma_1, \ldots, \Gamma_n \) be countable groups with families of subgroups \( G_1, \ldots, G_n \). Denote by \( \Gamma = \Gamma_1 \times \cdots \times \Gamma_n \) the direct product group. If every \( \Gamma_i \) is bi-exact relative to \( G_i \), then \( \Gamma \) is bi-exact relative to

\[
G = \bigcup_{i=1}^n \left\{ \Lambda \times \prod_{j \neq i} \Gamma_j \ \big| \ \Lambda \in G_i \right\}.
\]

**Proof.** Clearly \( \Gamma \) is an exact group. Take maps \( \mu_i : \Gamma_i \to \text{Prob} \Gamma_i \) satisfying (2.2) in Definition 2.6. Then the map

\[
\mu : \Gamma \to \text{Prob} \Gamma : \mu(g_1, \ldots, g_n) = \mu_1(g_1) \times \cdots \times \mu_n(g_n)
\]

satisfies the same condition. \( \square \)

### 3 Key theorem

We prove the following key theorem from which all other results in the paper will be deduced.

**Theorem 3.1.** Let \( \Gamma \) be a weakly amenable group that is bi-exact relative to a family \( G \) of subgroups of \( \Gamma \). Assume that \( \Gamma \rhd (B, \tau) \) is any trace preserving action on an arbitrary tracial von Neumann algebra \( (B, \tau) \). Put \( M = B \rtimes \Gamma \). Let \( q \in M \) be a projection and \( A \subset qMq \) a von Neumann subalgebra that is amenable relative to \( B \). Denote by \( P = N_{qMq}(A)^\prime\prime \) the normalizer of \( A \) inside \( qMq \). Then at least one of the following statements holds.

- \( P \) is amenable relative to \( B \).
- There exists a \( \Sigma \in G \) such that \( A \prec_M B \rtimes \Sigma \).

#### 3.1 It suffices to prove Theorem 3.1 for the trivial action

**Proposition 3.2.** If Theorem 3.1 holds for the trivial action on arbitrary tracial von Neumann algebras, then Theorem 3.1 also holds for arbitrary trace preserving actions.

**Proof.** Assume that Theorem 3.1 holds for the trivial action on arbitrary tracial von Neumann algebras. Let \( \Gamma \rhd (B, \tau) \) be any trace preserving action. Put \( M = B \rtimes \Gamma \). Let \( q \in M \) be a projection and \( A \subset qMq \) a von Neumann subalgebra that is amenable relative to \( B \). Denote by \( P = N_{qMq}(A)^\prime\prime \) the normalizer of \( A \) inside \( qMq \).
Define \( \mathcal{M} := M \otimes \mathbb{L} \Gamma \) which we view as the crossed product of \( \Gamma \) with the trivial action on \( M \).

Consider the trace preserving embedding
\[
\Delta : M \to \mathcal{M} : \Delta(bu_g) = bu_g \otimes u_g \quad \text{for all } b \in B, g \in \Gamma .
\]

Put \( \tilde{q} = \Delta(q) \) and \( A := \Delta(A) \). Denote by \( P := N_{\tilde{q} M \tilde{q}}(A)'' \) the normalizer of \( A \) inside \( \tilde{q} M \tilde{q} \).

Note that \( \Delta(P) \subset P \).

As explained in the first paragraphs of the proof of [PV11, Lemma 4.1], we have that \( A \) is amenable relative to \( M \otimes 1 \). Since Theorem 3.1 holds for the trivial action of \( \Gamma \) on \( M \), at least one of the following statements holds.

- \( \Delta(P) \) is amenable relative to \( M \otimes 1 \).
- There exists a \( \Sigma \in \mathcal{G} \) such that \( A \preceq M \otimes L \Sigma \).

If \( A \preceq M \otimes L \Sigma \), it is easy to check that \( A \preceq M \otimes \Sigma \) so that the second statement in the formulation of Theorem 3.1 holds.

Next assume that \( \Delta(P) \) is amenable relative to \( M \otimes 1 \). So we have a \( \Delta(P) \)-central positive functional \( \tilde{\Omega} \) on \( \tilde{q} \langle M, e \rangle \tilde{q} \) satisfying \( (\tilde{\Omega} \circ \Delta)|_{qMq} = \tau|_{qMq} \). Since \( E_{M \otimes 1} \circ \Delta = \Delta \circ E_B \), the embedding \( \Delta : M \to M \) can be extended to an embedding
\[
\Psi : \langle M, e \rangle \to \langle \mathcal{M}, e \rangle : \Psi(e_B) = e_{M \otimes 1} \quad \text{and} \quad \Psi(x) = \Delta(x) \quad \text{for all } x \in M .
\]

It follows that \( \tilde{\Omega} \circ \Psi \) is a \( \mathcal{P} \)-central positive functional on \( q\langle M, e \rangle q \) satisfying \( \Omega|_{qMq} = \tau|_{qMq} \). Hence, \( P \) is amenable relative to \( B \) and the first statement in the formulation of Theorem 3.1 holds.

### 3.2 Setup and notations for the proof of Theorem 3.1

By Proposition 3.2 we may assume that \( \Gamma \acts (B, \tau) \) is the trivial action. We put \( M := B \otimes \mathbb{L} \Gamma \).

For simplicity of notation we assume that \( q = 1 \). As in [PV11, Remark 6.3], this notational simplification is only cosmetic and does not hide any essential parts of the argument.

So we are given a von Neumann subalgebra \( A \subset M \) that is amenable relative to \( B \). We denote by \( P = N_{A}(A)'' \) its normalizer. Following [PV11, Theorem 5.1], we define \( \mathcal{N} \) as the von Neumann algebra generated by \( B \) and \( P^{\text{op}} \) on the Hilbert space \( L^2(M) \otimes_A L^2(P) \). Put \( \mathcal{N} := N \otimes \mathbb{L} \Gamma \) and define the tautological embeddings
\[
\pi : M \to \mathcal{N} : \pi(b \otimes u_g) = b \otimes u_g \quad \text{and} \quad \theta : P^{\text{op}} \to \mathcal{N} : \theta(y^{\text{op}}) = y^{\text{op}} \otimes 1
\]
for all \( b \in B, g \in \Gamma \) and \( y \in P \). Note that \( \pi(M) \) and \( \theta(P^{\text{op}}) \) commute and that together they generate \( \mathcal{N} \).

By [PV11, Theorem 5.1] we find a net of normal states \( \omega_n \in \mathcal{N}^* \) satisfying the following properties.

- \( \omega_n(\pi(x)) \to \tau(x) \) for all \( x \in M \),
- \( \omega_n(\pi(a)\theta(\pi)) \to 1 \) for all \( a \in \mathcal{U}(A) \),
- \( \| \omega_n \circ \text{Ad}(\pi(u)\theta(\pi)) - \omega_n \| \to 0 \) for all \( u \in N_M(A) \).
We fix a standard Hilbert space $H$ for $N$ and we always view $N$ as acting on $H$. This standard Hilbert space comes with the canonical anti-unitary involution $J$. Being the tensor product of $N$ and $L(\Gamma)$, the von Neumann algebra $\mathcal{N}$ is standardly represented on $\mathcal{H} := H \otimes \ell^2(\Gamma)$ by the formula

$$(x \otimes u_g) \cdot (\xi \otimes \delta_h) = x\xi \otimes \delta_{gh} \quad \text{for all } x \in N, \ g, h \in \Gamma, \ \xi \in H.$$ 

The corresponding anti-unitary involution $J : \mathcal{H} \to \mathcal{H}$ is given by $J(\xi \otimes \delta_g) = J\xi \otimes \delta_{g^{-1}}$. So the von Neumann algebras $\pi(M)$, $\mathcal{J}\pi(M)\mathcal{J}$, $\theta(P^{\text{op}})$ and $\mathcal{J}\theta(P^{\text{op}})\mathcal{J}$ all act on $\mathcal{H}$ and mutually commute.

Denote by $\xi_n \in \mathcal{H}$ the canonical positive unit vectors that implement the normal states $\omega_n$ on $\mathcal{N}$. Whenever $u \in N_M(A)$ it follows from [Ta03, Theorem IX.1.2.(iii)] that the vector$$
\pi(u) \theta(\pi) \mathcal{J} \pi(u) \theta(\pi) \xi_n
$$
is the canonical positive vector that implements $\omega_n \circ \text{Ad}(\pi(u^*)\theta(u^{\text{op}}))$. Using the Powers-Størmer inequality (see e.g. [Ta03, Theorem IX.1.2.(iv)]), the properties of $(\omega_n)$ can now be rewritten as follows in terms of the net $(\xi_n)$.

$$
(\pi(x)\xi_n, \xi_n) = \omega_n(\pi(x)) \to \tau(x) \quad \text{for all } x \in M,  \quad (3.1)
$$

$$
\|\pi(a)\theta(\pi)\xi_n - \xi_n\| \to 0 \quad \text{for all } a \in \mathcal{U}(A), \quad (3.2)
$$

$$
\|\pi(u)\theta(\pi)\mathcal{J} \pi(u)\theta(\pi) \xi_n - \xi_n\| \to 0 \quad \text{for all } u \in N_M(A). \quad (3.3)
$$

Since $\Gamma$ is bi-exact relative to $\mathcal{G}$, Definition 2.6 provides a map $\mu : \Gamma \to \text{Prob}\Gamma$ such that

$$
\lim_{k \to \infty/\mathcal{G}} \|\mu(gkh) - g \cdot \mu(k)\|_1 = 0 \quad \text{for all } g, h \in \Gamma.
$$

Define $\zeta : \Gamma \to \ell^2(\Gamma) : \zeta(k) = \mu(k)^{1/2}$. Note that

$$
\|\zeta(k)\|_2 = 1 \quad \text{for all } k \in \Gamma \quad \text{and} \quad \lim_{k \to \infty/\mathcal{G}} \|\zeta(gkh^{-1}) - \lambda_g \zeta(k)\|_2 = 0 \quad \text{for all } g, h \in \Gamma. \quad (3.4)
$$

Define the isometry

$$
V : \ell^2(\Gamma) \to \ell^2(\Gamma) \otimes \ell^2(\Gamma) : V \delta_k = \zeta(k) \otimes \delta_k \quad \text{for all } k \in \Gamma.
$$

We denote by $\mathcal{S}$ the directed set of subsets of $\Gamma$ that are small relative to $\mathcal{G}$. For every subset $\mathcal{F} \subset \Gamma$, we denote by $P_{\mathcal{F}}$ the orthogonal projection of $\ell^2(\Gamma)$ onto $\ell^2(\mathcal{F})$. Then (3.4) can be rewritten as

$$
\lim_{\mathcal{F} \in \mathcal{S}} \|((\lambda_g \otimes \lambda_h)\rho_h) V - V \lambda_g \rho_h) P_{\Gamma - \mathcal{F}}\| = 0 \quad \text{for all } g, h \in \Gamma.
$$

The representation $(g, h) \mapsto \lambda_g \otimes \lambda_h \rho_h$ of $\Gamma \times \Gamma$ is unitarily conjugate to the regular representation $(g, h) \mapsto \lambda_g \otimes \lambda_h$ through the unitary $U \in \mathcal{U}(\ell^2(\Gamma) \otimes \ell^2(\Gamma))$ given by $U(\delta_k \otimes \delta_r) = \delta_k \otimes \delta_{r^{-1}k}$. We define the isometry

$$
W : \ell^2(\Gamma) \to \ell^2(\Gamma) \otimes \ell^2(\Gamma) : W = UV.
$$

We then get

$$
\lim_{\mathcal{F} \in \mathcal{S}} \|((\lambda_g \otimes \lambda_h)W - W \lambda_g \rho_h) P_{\Gamma - \mathcal{F}}\| = 0 \quad \text{for all } g, h \in \Gamma. \quad (3.5)
$$

Define the weakly dense $*$-subalgebra $M_0 \subset M$ given by $M_0 = B \otimes_{\text{alg}} \mathbb{C} \Gamma$. Then define the unital $*$-algebras

$$
\mathcal{D} := M \otimes_{\text{alg}} M^{\text{op}} \otimes_{\text{alg}} P^{\text{op}} \otimes_{\text{alg}} P \quad \text{with } \text{subalgebra} \quad \mathcal{D}_0 = M_0 \otimes_{\text{alg}} M_0^{\text{op}} \otimes_{\text{alg}} P^{\text{op}} \otimes_{\text{alg}} P.
$$
Define the unique $\ast$-homomorphisms

$$
\Psi: \mathcal{D} \to \mathcal{B}(H \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma)) \quad \text{and} \quad \Theta: \mathcal{D} \to \mathcal{B}(H \otimes \ell^2(\Gamma))
$$

that are separately normal in each of the tensor factors of $\mathcal{D} = M \otimes_{\text{alg}} M^{\text{op}} \otimes_{\text{alg}} P$ and that satisfy

$$
\Psi((b \otimes u_g) \otimes (c \otimes u_h))^{\text{op}} \otimes y^{\text{op}} \otimes z = b \ J c^* J \ y^{\text{op}} \ J \pi \ J x \ \lambda_g \otimes \lambda_{h-1} , \\
\Theta((b \otimes u_g) \otimes (c \otimes u_h))^{\text{op}} \otimes y^{\text{op}} \otimes z = \pi(b \otimes u_g) \ J \pi(c \otimes u_h)^* \ J \theta(y^{\text{op}}) \ J \theta(\pi) \ J ,
$$

for all $b, c \in B, g, h \in \Gamma$, and $y, z \in P$. Note that for a better understanding of the defining formulae of $\Psi$ and $\Theta$, one should identify $P$ with $(P^{\text{op}})^{\text{op}}$. Also note that by the definition of $\pi$ and $J$, we have

$$
\Theta((b \otimes u_g) \otimes (c \otimes u_h))^{\text{op}} \otimes y^{\text{op}} \otimes z = b \ J c^* J \ J y \ J z \ \lambda_g \rho_{h-1} ,
$$

for all $b, c \in B, g, h \in \Gamma$, and $y, z \in P$. By linearity, (3.5) thus implies that

$$
\lim_{F \in \mathcal{S}} \| (\Psi(S)(1 \otimes W) - (1 \otimes W)\Theta(S)) (1 \otimes P_{\mathcal{F} \setminus \mathcal{F}}) \| = 0 \quad \text{for all } S \in \mathcal{D}_0 . \quad (3.6)
$$

Since $W$ is an isometry, we get in particular that

$$
\lim_{F \in \mathcal{S}} \| \Theta(S) (1 \otimes P_{\mathcal{F} \setminus \mathcal{F}}) \| \leq \| \Psi(S) \| \quad \text{for all } S \in \mathcal{D}_0 . \quad (3.7)
$$

It is important to note that (3.6) and (3.7) only hold for $S \in \mathcal{D}_0$ and not necessarily for all $S \in \mathcal{D}$.

### 3.3 The proof of Theorem 3.1 splits up in two cases

We get the following dichotomy in terms of the net of unit vectors $(\xi_n)$ in $H \otimes \ell^2(\Gamma)$ that we introduced in the previous section.

**Case 1.** For every subset $\mathcal{F} \subset \Gamma$ that is small relative to $\mathcal{G}$, we have

$$
\lim_n \| (1 \otimes P_{\mathcal{F}})\xi_n \| = 0 .
$$

**Case 2.** There exists a subset $\mathcal{F} \subset \Gamma$ that is small relative to $\mathcal{G}$ and that satisfies

$$
\limsup_n \| (1 \otimes P_{\mathcal{F}})\xi_n \| > 0 .
$$

### 3.4 Proof of Theorem 3.1 in case 1

Choose a (typically non-normal) state $\Omega_1$ on $\mathcal{B}(H \otimes \ell^2(\Gamma))$ as a weak$^*$ limit point of the net of states $S \mapsto \langle S\xi_n, \xi_n \rangle$. From (3.1) and (3.3), we get that

$$
\Omega_1(\pi(x)) = \tau(x) \quad \text{and} \quad |\Omega_1(S \pi(x))| \leq \| S \| \| x \|_2 \quad \text{for all } x \in M, S \in \mathcal{B}(H \otimes \ell^2(\Gamma)) , \quad (3.8)
$$

$$
\Omega_1(\Theta(u \otimes \pi \otimes \pi \otimes u)) = 1 \quad \text{for all } u \in N_M(A) . \quad (3.9)
$$

Since the vectors $\xi_n$ are positive, we have $\mathcal{J}\xi_n = \xi_n$ for all $n$ and hence, $\Omega_1(\mathcal{J}S^*\mathcal{J}) = \Omega_1(S)$ for all $S \in \mathcal{B}(H \otimes \ell^2(\Gamma))$. Then (3.8) implies that

$$
|\Omega_1(S \mathcal{J} \pi(x) \mathcal{J})| \leq \| S \| \| x \|_2 \quad \text{for all } x \in M, S \in \mathcal{B}(H \otimes \ell^2(\Gamma)) . \quad (3.10)
$$
To prove this claim, note that (3.8) implies that

\[ \Omega_1(S) = \Omega_1(S(1 \otimes P_{1-\mathcal{F}})) \]

for all \( S \in \mathcal{B}(H \otimes \ell^2(\Gamma)) \) and all subsets \( \mathcal{F} \subset \Gamma \) that are small relative to \( \mathcal{G} \). In combination with (3.7), it follows that

\[
|\Omega_1(\Theta(S))| = \limsup_{\mathcal{F} \in \mathcal{S}} |\Omega_1(\Theta(S)(1 \otimes P_{1-\mathcal{F}}))| \leq \limsup_{\mathcal{F} \in \mathcal{S}} \|\Theta(S)(1 \otimes P_{1-\mathcal{F}})\| \leq \|\Psi(S)\| \quad \text{for all } S \in \mathcal{D}_0 . \tag{3.11}
\]

The main point of the proof will now be to prove the existence of \( \kappa > 0 \) such that

\[
|\Omega_1(\Theta(S))| \leq \kappa^2 \|\Psi(S)\| \quad \text{for all } S \in \mathcal{D} . \tag{3.12}
\]

Since \( \Gamma \) is weakly amenable, choose a sequence of finitely supported Herz-Schur multipliers \( f_i : \Gamma \to \mathbb{C} \) such that \( f_i \to 1 \) pointwise and \( \limsup_i \|f_i\|_{cb} = \kappa < \infty \). Denote by \( m_i : \mathbb{L}\Gamma \to \mathbb{L}\Gamma \) the normal completely bounded maps given by \( m_i(u_g) = f_i(g)u_g \) for all \( g \in \Gamma \). We define the corresponding normal completely bounded maps \( \varphi_i : M \to M \) and \( \tilde{\varphi}_i : M^{\text{op}} \to M^{\text{op}} \) given by

\[
\varphi_i(b \otimes u_g) = f_i(g)(b \otimes u_g) \quad \text{and} \quad \tilde{\varphi}_i(b \otimes u_g) = f_i(g)b \otimes u_g
\]

for all \( b \in B \) and \( g \in \Gamma \).

Observe that for all \( x \in M \), we have \( \lim_i \|x - \varphi_i(x)\|_2 = 0 \) and \( \lim_i \|x^{\text{op}} - \tilde{\varphi}_i(x^{\text{op}})\|_2 = 0 \). Since the functions \( f_i \) are finitely supported, we also note that for all \( S \in \mathcal{D} \), we have

\[
(\varphi_i \otimes \tilde{\varphi}_i \otimes \text{id} \otimes \text{id})(S) \in \mathcal{D}_0 .
\]

We claim that for all \( x_1, x_2 \in M \) and all \( y, z \in P \), we have

\[
\lim_i \Omega_1(\Theta(\varphi_i(x_1) \otimes \tilde{\varphi}_i(x_2^{\text{op}}) \otimes y^{\text{op}} \otimes z)) = \Omega_1(\Theta(x_1 \otimes x_2^{\text{op}} \otimes y^{\text{op}} \otimes z)) . \tag{3.13}
\]

To prove this claim, note that (3.8) implies that

\[
\limsup_i \left| \Omega_1(\Theta((\varphi_i(x_1) - x_1) \otimes \tilde{\varphi}_i(x_2^{\text{op}}) \otimes y^{\text{op}} \otimes z)) \right| \\
\leq \limsup_i \left\| \Theta(1 \otimes \tilde{\varphi}_i(x_2^{\text{op}}) \otimes y^{\text{op}} \otimes z) \right\| \|\varphi_i(x_1) - x_1\|_2 \\
\leq \limsup_i \left\| \tilde{\varphi}_i(x_2^{\text{op}}) \right\| \left\| y \right\| \left\| z \right\| \left\| \varphi_i(x_1) - x_1\right\|_2 \\
\leq \kappa \left\| x_2 \right\| \left\| y \right\| \left\| z \right\| \limsup_i \left\| \varphi_i(x_1) - x_1 \right\|_2 = 0 .
\]

Using (3.10), one similarly proves that

\[
\lim_i \Omega_1(\Theta(x_1 \otimes (\tilde{\varphi}_i(x_2^{\text{op}}) - x_2) \otimes y^{\text{op}} \otimes z)) = 0 .
\]

Summing up both, the claim (3.13) follows. By linearity, we get that

\[
\Omega_1(\Theta(S)) = \lim_i \Omega_1(\Theta((\varphi_i \otimes \tilde{\varphi}_i \otimes \text{id} \otimes \text{id})(S))) \quad \text{for all } S \in \mathcal{D} . \tag{3.14}
\]

We are now ready to prove (3.12). Observe that \( \Psi(\mathcal{D}) \subset \mathcal{B}(H) \boxplus \mathbb{L}\Gamma \boxplus \mathbb{L}\Gamma \) and that

\[
\Psi((\varphi_i \otimes \tilde{\varphi}_i \otimes \text{id} \otimes \text{id})(S)) = (\text{id} \otimes m_i \otimes m_i)(\Psi(S)) \quad \text{for all } S \in \mathcal{D} .
\]
In combination with \(\text{(3.14)}\) and \(\text{(3.11)}\), we get for all \(S \in \mathcal{D}\) that
\[
|\Omega_1(\Theta(S))| = \limsup_i |\Omega_1(\Theta((\varphi_i \otimes \tilde{\varphi}_i \otimes \text{id} \otimes \text{id})(S)))|
\leq \limsup_i \|\Psi((\varphi_i \otimes \tilde{\varphi}_i \otimes \text{id} \otimes \text{id})(S))\|
= \limsup_i \|\text{id} \otimes m_i \otimes m_i(\Psi(S))\|
\leq \limsup_i \|m_i\|_{\mathcal{B}}^2 \|\Psi(S)\| \leq \kappa^2 \|\Psi(S)\|.
\]
So, \(\text{(3.12)}\) is proven.

Define the unital C*-algebra \(\mathcal{Q} \subset \mathcal{B}(H \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma))\) as the norm closure of \(\Psi(\mathcal{D})\). Because of \(\text{(3.12)}\), there is a unique continuous functional \(\Omega_2 \in \mathcal{Q}^*\) such that \(\Omega_2(\Psi(S)) = \Omega_1(\Theta(S))\) for all \(S \in \mathcal{D}\). Since \(\Omega_1\) is positive, it follows that for all \(S \in \mathcal{D}\),
\[
\Omega_2(\Psi(S)^* \Psi(S)) = \|\Omega_1(\Theta(S^* S))\| = \|\Omega_1(\Theta(S)^* \Theta(S))\| \geq 0.
\]
By density, it follows that \(\Omega_2(T^* T) \geq 0\) for all \(T \in \mathcal{Q}\). So, \(\Omega_2\) is a positive functional on \(\mathcal{Q}\). Since \(\Omega_2(1) = 1\), we conclude that \(\Omega_2\) is a state on \(\mathcal{Q}\).

Denote \(\pi_0 : M \to \mathcal{B}(H \otimes \ell^2(\Gamma)) : \pi_0(b \otimes u_g) = b \otimes \lambda_g\) and note that \(\pi_0(x) \otimes 1 = \Psi(x \otimes 1 \otimes 1 \otimes 1)\) for all \(x \in M\). From \(\text{(3.8)}\) and \(\text{(3.9)}\), we get that
\[
\Omega_2(\pi_0(x) \otimes 1) = \tau(x), \quad \forall x \in M \quad \text{and} \quad \Omega_2(\Psi(u \otimes \pi \otimes \pi \otimes u)) = 1, \quad \forall u \in \mathcal{N}_M(A). \tag{3.15}
\]
By the Hahn-Banach theorem, we can extend \(\Omega_2\) to a functional on \(\mathcal{B}(H \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma))\) without increasing the norm of \(\Omega_2\). We still denote this extension by \(\Omega_2\). Since \(\|\Omega_2\| = 1 = \Omega_2(1)\), we get that the extended \(\Omega_2\) is still a state. Since the state \(\Omega_2\) equals 1 on the unitaries \(\Psi(u \otimes \pi \otimes \pi \otimes u), \quad u \in \mathcal{N}_M(A)\), we get that
\[
\Omega_2(S \Psi(u \otimes \pi \otimes \pi \otimes u)) = \Omega_2(S) = \Omega_2(\Psi(u \otimes \pi \otimes \pi \otimes u) S)
\quad \text{for all} \quad S \in \mathcal{B}(H \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma)) \quad \text{and} \quad u \in \mathcal{N}_M(A). \tag{3.16}
\]
Define the state \(\Omega \in B \overset{\otimes}{\mathcal{B}} \mathcal{B}(\ell^2(\Gamma))\) given by \(\Omega(S) = \Omega_2(S \otimes 1)\). Since \(\Psi(1 \otimes M^{\text{op}} \otimes P^{\text{op}} \otimes P)\) commutes with \(B \overset{\otimes}{\mathcal{B}} \mathcal{B}(\ell^2(\Gamma)) \otimes 1\), it follows from \(\text{(3.16)}\) that \(\Omega\) is a \(\pi_0(\mathcal{N}_M(A))\)-central state on \(B \overset{\otimes}{\mathcal{B}} \mathcal{B}(\ell^2(\Gamma))\). From \(\text{(3.15)}\), we get that \(\Omega(\pi_0(x)) = \tau(x)\) for all \(x \in M\).

We claim that \(\Omega\) is actually \(\pi_0(P)\)-central. Fix \(S \in B \overset{\otimes}{\mathcal{B}} \mathcal{B}(\ell^2(\Gamma))\). Since \(\Omega \circ \pi_0 = \tau\), it follows from the Cauchy-Schwarz inequality that
\[
|\Omega(S \pi_0(x))| \leq \|S\| \|x\|_2 \quad \text{and} \quad |\Omega(\pi_0(x) S)| \leq \|S\| \|x\|_2 \quad \text{for all} \quad x \in M.
\]
So, the set of \(x \in M\) satisfying \(\Omega(S \pi_0(x)) = \Omega(\pi_0(x) S)\) is a \(\| \cdot \|_2\)-closed vector subspace of \(M\). Since it contains \(\mathcal{N}_M(A)\), it also contains \(P = \mathcal{N}_M(A)^\prime\). This proves the claim that \(\Omega\) is a \(\pi_0(P)\)-central state.

The inclusion \(\pi_0 : M \to B \overset{\otimes}{\mathcal{B}} \mathcal{B}(\ell^2(\Gamma))\) is canonically isomorphic with the inclusion \(M \subset \langle M, e_{B \otimes 1} \rangle\). So, we have found a \(P\)-central state on \(\langle M, e_{B \otimes 1} \rangle\) whose restriction to \(M\) equals \(\tau\). This means that \(P\) is amenable relative to \(B \otimes 1\) and concludes the proof of Theorem \(\text{3.1}\) in case 1.
3.5 Proof of Theorem 3.1 in case 2

Take $\delta_1 > 0$ and take a subset $\mathcal{F} \subset \Gamma$ that is small relative to $\mathcal{G}$ and that satisfies
\[
\limsup_n \| (1 \otimes P_{\mathcal{F}}) \xi_n \| > \delta_1 .
\]

Since $\mathcal{F}$ is small relative to $\mathcal{G}$, we have that $\mathcal{F}$ is contained in the union of $m < \infty$ subsets of $\Gamma$ of the form $g_0 \Sigma_0 h_0$ with $g_0, h_0 \in \Gamma$ and $\Sigma_0 \in \mathcal{G}$. Putting $\delta = \delta_1 / m$, we find $g_0, h_0 \in \Gamma$ and $\Sigma_0 \in \mathcal{G}$ such that
\[
\limsup_n \| (1 \otimes P_{g_0 \Sigma_0 h_0}) \xi_n \| > \delta .
\]

Put $\Sigma = h_0^{-1} \Sigma_0 h_0$ and denote by $\mathcal{F}_0$ the singleton $\{ g_0 h_0 \}$. Replacing $(\xi_n)$ by a subnet it follows that $\mathcal{F}_0$ is a finite subset of $\Gamma$ satisfying
\[
\liminf_n \| (1 \otimes P_{\mathcal{F}_0}) \xi_n \| > \delta . \tag{3.17}
\]

We will show that $A \prec_M B \boxtimes \Sigma$, using an argument inspired by the proof of \cite{CSU11} Lemma 6.2. Since $\Sigma$ is a conjugate of $\Sigma_0$ it then also follows that $A \prec_M B \boxtimes \Sigma_0$. Since $\Sigma_0 \in \mathcal{G}$, this will conclude the proof of Theorem 3.1 in case 2.

Assume that $A \not\prec_M B \boxtimes \Sigma$. We will deduce below that for any finite subset $\mathcal{F}_0 \subset \Gamma$ and any $\delta > 0$ satisfying $\delta > \sqrt{2} \delta_1$, there exists a larger finite subset $\mathcal{F}_1 \subset \Gamma$ such that
\[
\liminf_n \| (1 \otimes P_{\mathcal{F}_1}) \xi_n \| > \sqrt{2} \delta . \tag{3.18}
\]

Take an integer $k$ such that $2^{k/2} \delta > 1$. Iterating the above procedure $k$ times, we find a finite subset $\mathcal{F}_k \subset \Gamma$ that satisfies the absurd statement
\[
1 = \lim_n \| \xi_n \| \geq \liminf_n \| (1 \otimes P_{\mathcal{F}_k}) \xi_n \| > 2^{k/2} \delta > 1 .
\]

So it remains to find a finite subset $\mathcal{F}_1 \subset \Gamma$ satisfying (3.18).

Following \cite{CSU11} Formula (6.9), we first claim that
\[
\limsup_n \| \pi(x) (1 \otimes P_{\mathcal{F}_0}) \xi_n \| \leq \| \mathcal{F}_0 \| \| x \|_2 \quad \text{for all } x \in M . \tag{3.19}
\]

To prove this claim it suffices to check that for all $g \in \Gamma$ and $x \in M$ we have
\[
\limsup_n \| \pi(x) (1 \otimes P_{g \Sigma}) \xi_n \| \leq \| x \|_2 . \tag{3.20}
\]

First observe that
\[
1 \otimes P_{g \Sigma} = \pi(1 \otimes u_g^*) (1 \otimes P_{\Sigma}) \pi(1 \otimes u_g^*) .
\]

It then follows that, writing $y = (1 \otimes u_g^*) x^* x (1 \otimes u_g)$, we have
\[
\| \pi(x) (1 \otimes P_{g \Sigma}) \xi_n \|^2 = \langle (1 \otimes P_{\Sigma}) \pi ((1 \otimes u_g^*) x^* x (1 \otimes u_g)) (1 \otimes P_{\Sigma}) \pi (1 \otimes u_g^*) \xi_n, (1 \otimes u_g^*) \xi_n \rangle = \langle (1 \otimes P_{\Sigma}) \pi (E_{B \boxtimes \Sigma}(y)) (1 \otimes u_g^*) \xi_n, (1 \otimes u_g^*) \xi_n \rangle = \| (1 \otimes P_{\Sigma}) \pi (E_{B \boxtimes \Sigma}(y)^{1/2} (1 \otimes u_g^*)) \xi_n \|^2 .
\]

Using (3.1), we conclude that
\[
\limsup_n \| \pi(x) (1 \otimes P_{g \Sigma}) \xi_n \|^2 \leq \| E_{B \boxtimes \Sigma}(y)^{1/2} (1 \otimes u_g^*) \|^2 = \tau(y) = \| x \|_2^2 .
\]
This establishes (3.20). Hence also the claim (3.19) follows.
Because of (3.17) we can take $\varepsilon > 0$ such that
\[
\limsup_n \|\xi_n - (1 \otimes P_{F_0 \Sigma})\xi_n\| < \sqrt{1 - \delta^2} - \varepsilon .
\] (3.21)
For every $x \in M$ we denote by $x = \sum_{g \in \Gamma} (x)_g \otimes u_g$, with $(x)_g \in B$, the Fourier decomposition of $x$. We claim that there exist $a \in \mathcal{U}(A)$ and $v \in B \otimes \text{alg} \Gamma$ such that
\[
\|a - v\| < \varepsilon \|F_0\|^{-1} \quad \text{and} \quad (v)_g = 0 \text{ for all } g \in F_0 \Sigma F_0^{-1} .
\] (3.22)
To prove this claim, first take $a \in \mathcal{U}(A)$ such that
\[
\|E_{B \otimes L \Sigma}((1 \otimes u_g^*)a(1 \otimes u_g))\| < \frac{\varepsilon}{3|F_0|^3} \quad \text{for all } g,h \in F_0 .
\]
This is possible by our assumption that $A \not\simeq B \otimes L \Sigma$. For any subset $F \subset \Gamma$ we also denote by $1 \otimes P_F$ the orthogonal projection of $L^2(M)$ onto the closure of span$\{b \otimes u_g \mid b \in B, g \in F\}$. So we have chosen the unitary $a \in \mathcal{U}(A)$ such that $\|(1 \otimes P_{F_0 \Sigma F_0^{-1}})(a)\| < \varepsilon/|F_0|^3$ for all $g\in F_0$. It follows that $\|(1 \otimes P_{F_0 \Sigma F_0^{-1}})(a)\| < \varepsilon/|F_0|^3$. Choose $a' \in B \otimes \text{alg} \Gamma$ such that $\|a - a'\| < \varepsilon/(|F_0|^3)$. It follows that $\|(1 \otimes P_{F_0 \Sigma F_0^{-1}})(a')\| < \varepsilon/(|F_0|^3)$. Defining $v := a' - (1 \otimes P_{F_0 \Sigma F_0^{-1}})(a')$, the elements $a \in \mathcal{U}(A)$ and $v \in B \otimes \text{alg} \Gamma$ satisfy claim (3.22).
From (3.2) we know that $\lim_n \|\xi_n - \pi(a^\ast)\theta(a^\ast)\xi_n\| = 0$. In combination with (3.21) it follows that
\[
\limsup_n \|\xi_n - \theta(\bar{a})\pi(a) (1 \otimes P_{F_0 \Sigma})\xi_n\| = \limsup_n \|\pi(a^\ast)\theta(a^\ast)\xi_n - (1 \otimes P_{F_0 \Sigma})\xi_n\|
\]
\[
= \limsup_n \|\xi_n - (1 \otimes P_{F_0 \Sigma})\xi_n\| < \sqrt{1 - \delta^2} - \varepsilon .
\] (3.23)
Since $\|a - v\| < \varepsilon/|F_0|$, it follows from (3.19) that
\[
\limsup_n \|\pi(a - v) (1 \otimes P_{F_0 \Sigma})\xi_n\| < \varepsilon .
\]
In combination with (3.23), we get that
\[
\limsup_n \|\xi_n - \theta(\bar{a})\pi(v) (1 \otimes P_{F_0 \Sigma})\xi_n\| < \sqrt{1 - \delta^2} .
\] (3.24)
Define the subset $S \subset \Gamma$ given by $S := \{g \in \Gamma \mid (v)_g \neq 0\}$. Since $v \in B \otimes \text{alg} \Gamma$, the set $S$ is finite. Since $(v)_g = 0$ for all $g \in F_0 \Sigma F_0^{-1}$, we get that $S \cap F_0 \Sigma F_0^{-1} = \emptyset$. This means that $S F_0 \Sigma \cap F_0 \Sigma = \emptyset$.
Note that $\theta(\bar{a}) = \bar{a} \otimes 1$ commutes with $1 \otimes P_{F_0 \Sigma}$. Hence
\[
\theta(\bar{a})\pi(v) (1 \otimes P_{F_0 \Sigma})\xi_n = \pi(v) (1 \otimes P_{F_0 \Sigma}) \theta(\bar{a}) \xi_n
\]
lies in the range of $1 \otimes P_{S F_0 \Sigma}$ for all $n$. It then follows from (3.24) that
\[
\limsup_n \|\xi_n - (1 \otimes P_{S F_0 \Sigma})\xi_n\| < \sqrt{1 - \delta^2} .
\]
This means that
\[
\liminf_n \|(1 \otimes P_{S F_0 \Sigma})\xi_n\| > \delta .
\]
We put $F_1 := S F_0 \cup F_0$. Since $S F_0 \Sigma$ is disjoint from $F_0 \Sigma$, the vectors $(1 \otimes P_{S F_0 \Sigma})\xi_n$ and $(1 \otimes P_{F_0 \Sigma})\xi_n$ are orthogonal. So in combination with (3.17), it follows that (3.18) holds. As explained right after (3.18), this concludes the proof of Theorem 3.1 in case 2.
4 Proofs of Theorems 1.1, 1.4, 1.5 and 1.6

Proof of Theorem 1.1. Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a direct product of $n \geq 1$ weakly amenable, nonamenable, bi-exact groups $\Gamma_i$. Take an arbitrary free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$. Write $B = L^\infty(X)$ and $M = B \rtimes \Gamma$. Let $A \subset M$ be a Cartan subalgebra. Because of [Po01 Theorem A.1], it suffices to prove that $A \vartriangleleft_M B$.

Denote by $\mathring{\Gamma}_i$ the direct product of all $\Gamma_j$ with $j \neq i$. Put $M_i := B \rtimes \mathring{\Gamma}_i$. Fix $i = 1, \ldots, n$ and view $M$ as $M = M_i \rtimes \Gamma_i$. Because $\mathring{\Gamma}_i$ is nonamenable, we have that $M$ is not amenable relative to $M_i$. So, by Theorem 3.1, it suffices to prove that $A \vartriangleleft_M M_i$ for all $i = 1, \ldots, n$. Since we know that $M$ is a factor, that $A \subset M$ is regular and that $A \vartriangleleft_M M_i$ for all $i = 1, \ldots, n$, it follows from [Va10 Proposition 2.5 and Lemma 2.6] that $A \vartriangleleft_M B$.

By Lemma 2.4 the groups in 1, 2 and 3 are weakly amenable and bi-exact. Since they are also nonamenable, the theorem applies to these groups and their direct products.

Proof of Theorem 1.4. Take $M = B \rtimes \Gamma$ and $A \subset qMq$ as in the formulation of the theorem. Put $P = N_{qMq}(A)''$. By Theorem 3.1, we have that $P$ is amenable relative to $B$, or that $A \vartriangleleft_M B$.

Proof of Theorem 1.5. Put $M = N \bar{\otimes} L\Gamma$ and assume that $A \subset M$ is a Cartan subalgebra. Since $\Gamma$ is nonamenable, $M$ is not amenable relative to $N$. So Theorem 3.1 implies that $A \vartriangleleft_M N$. Taking relative commutants, it follows from [Va10 Lemma 3.5] that $L\Gamma \vartriangleleft_M M \cap A'$. Since $M \cap A' = A$ and since $\Gamma$ is nonamenable, there are no normal $*$-homomorphisms from $L\Gamma$ to an amplification of $A$. Hence the statement $L\Gamma \vartriangleleft_M A$ is absurd.

Proof of Theorem 1.6. Using e.g. [Va10 Proposition 2.5], we find projections $p_i \in Z(P)$ such that $Ap_i \vartriangleleft_M B \times \Gamma_i$ and $A(q - p_i) \not\subset B \times \Gamma_i$ for all $i = 1, \ldots, n$. Of course, some or even all of the $p_i$ could be zero. Define $p_0 = q - (p_1 \vee \cdots \vee p_n)$. We consider the subalgebra $Ap_0 \subset p_0 M p_0$ whose normalizer is given by $Pp_0$. By Lemma 2.8, the group $\Gamma$ is bi-exact relative to $G = \{\mathring{\Gamma}_1, \ldots, \mathring{\Gamma}_n\}$. By construction, $Ap_0 \not\subset B \times \mathring{\Gamma}_i$ for all $i = 1, \ldots, n$. So, by Theorem 3.1 we have that $Pp_0$ is amenable relative to $B$.

5 Proof of Theorem 1.3

We first prove the following more general result.

Theorem 5.1. Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ be a direct product of $1 \leq n < \infty$ weakly amenable, bi-exact groups. Assume that $\Gamma$ is a measure equivalence subgroup of $\Lambda$ through an action $\Gamma \times \Lambda \curvearrowright \Omega$ as in Definition 2.3. Then, at least one of the following statements holds.

(1) $\Gamma$ is $C$-rigid and $C'$-rigid.

(2) There exists an $i \in \{1, \ldots, n\}$, a nonnegligible $(\Gamma \times \Lambda)$-invariant subset $\Omega_0 \subset \Omega$ and a sequence of measurable maps $\xi_n : \Omega_0 \to \operatorname{Prob} \Lambda_i$ such that

$$\lim_{n} \|\xi_n((g, s) \cdot x) - s_i \cdot \xi_n(x)\|_1 = 0 \quad \text{for all } g \in \Gamma, s \in \Lambda \text{ and a.e. } x \in \Omega_0.$$ 

Moreover in the following two special cases, statement (2) has a simpler equivalent formulation.
(a) If also the fundamental domain of $\Gamma \curvearrowright \Omega$ has finite measure (i.e., $\Gamma$ and $\Lambda$ are measure equivalent), then statement (2) is equivalent with the amenability of one of the $\Lambda_i$.

(b) Assume that the $\Lambda_i < G_i$ are lattices in the second countable locally compact groups $G_i$ and assume that $\Gamma < G = G_1 \times \cdots \times G_n$ is a discrete subgroup. If we take $\Omega = G$ with the action of $\Gamma \times \Lambda \curvearrowright \Omega$ given by left right multiplication, then statement (2) is equivalent with the image of $\Gamma$ in one of the $G_i$ having an amenable closure.

Note that Theorem 1.3 is a direct consequence of Theorem 5.1. Case 1 of Theorem 1.3 follows immediately by using (a) in Theorem 5.1. To prove case 2 of Theorem 1.3, we choose lattices $\Lambda_i < G_i$. By Lemma 2.4 all $\Lambda_i$ are weakly amenable and bi-exact. So case 2 of Theorem 1.3 follows by using (b) in Theorem 5.1.

Proof of Theorem 5.1. Choose a free ergodic pmp action $\Gamma \curvearrowright X$. Put $M = L^\infty(X) \rtimes \Gamma$ and assume that $A \subset M$ is a maximal abelian von Neumann subalgebra whose normalizer $N_M(A)^\prime\prime$ is a finite index subfactor of $M$. From [Po01] Theorem A.1, we know that $A$ is unitarily conjugate with $L^\infty(X)$ if and only if $A \triangleleft_M L^\infty(X)$. We will prove that either $A \triangleleft_M L^\infty(X)$ or that (2) holds.

We denote the commuting actions $\Gamma \curvearrowright \Omega$ and $\Lambda \curvearrowright \Omega$ as actions on the left, resp. on the right, and denote these actions by a dot ·. Choose a fundamental domain $U \subset \Omega$ for the action $\Gamma \curvearrowright \Omega$ and choose a fundamental domain $\mathcal{V} \subset \Omega$ for the action $\Lambda \curvearrowright \Omega$. So, up to measure zero, we get partitions

$$\Omega = \bigsqcup_{g \in \Gamma} g \cdot \mathcal{V} \quad \text{and} \quad G = \bigsqcup_{s \in \Lambda} \mathcal{U} \cdot s.$$ 

Since $U$ is of finite measure, we may normalize $m$ such that $m(U) = 1$.

We identify $\Omega/\Lambda = \mathcal{U}$. Through this identification, the natural action $\Gamma \curvearrowright \Omega/\Lambda$ becomes a pmp action $\Gamma \curvearrowright \mathcal{U}$ that we denote by * to distinguish it from the action $\Gamma \curvearrowright \Omega$ denoted by ·. We then get the 1-cocycle $\omega : \Gamma \times \mathcal{U} \to \Lambda$ for the action $\Gamma \curvearrowright \mathcal{U}$ such that $g \cdot x = (g * x) \cdot \omega(g, x)$ for all $g \in \Gamma$ and a.e. $x \in \mathcal{U}$.

In particular, for $g \in \Gamma$ and a.e. $x \in \mathcal{U}$, we have $\omega(g, x) = s$ if and only if $g \cdot x \in \mathcal{U} \cdot s$.

Define the tracial von Neumann algebra $N := L^\infty(X \times \mathcal{U}) \rtimes \Gamma$, where $\Gamma$ acts on $X \times \mathcal{U}$ diagonally. We view $M$ as a von Neumann subalgebra of $N$ in the canonical way. For every $g \in \Gamma$, we denote by $V_g \in L^\infty(\mathcal{U}) \boxtimes \Lambda$ the unitary given by $V_g(x) = v_{\omega(g, x)}$. Here we use the notation $(v_s)_{s \in \Lambda}$ to denote the canonical unitaries in $\Lambda$. We then get a normal trace preserving *-homomorphism

$$\Delta : N \to N \boxtimes \Lambda : \Delta(a u_g) = (a u_g \otimes 1)V_g \quad \text{for all} \quad a \in L^\infty(X \times \mathcal{U}), g \in \Gamma.$$ 

We put $N_i := N \boxtimes \Lambda_i$ and identify $N \boxtimes \Lambda = N_1 \boxtimes \Lambda_1$. As such we view $N \boxtimes \Lambda$ as the crossed product of $N_1$ and $\Lambda_1$ w.r.t. the trivial action of $\Lambda_i$ on $N_i$. Since $\Lambda_i$ is weakly amenable and bi-exact, we will apply Theorem 3.1 to this crossed product.

Denoting $P = N_M(A)^\prime\prime$, we distinguish two cases.

Case 1. For all $i \in \{1, \ldots, n\}$ and all nonzero projections $p \in \Delta(P) \cap (N \boxtimes \Lambda)$, we have that $\Delta(P)p$ is not amenable relative to $N_i$.

Case 2. There exists an $i \in \{1, \ldots, n\}$ and a nonzero projection $p \in \Delta(P) \cap (N \boxtimes \Lambda)$ such that $\Delta(P)p$ is amenable relative to $N_i$. 

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We prove that in case 1, we have $A \prec_M L^\infty(X)$, while in case 2, there exists a sequence $\xi_n$ satisfying the conditions of statement (2).

**Proof of case 1.** It follows from Theorem 3.1 that $\Delta(A) \prec N$ for all $i \in \{1, \ldots, n\}$ and all nonzero projections $p \in \mathcal{P}(\Lambda') \cap (N \otimes \Lambda \Lambda)$. So, using e.g. [Va10, Proposition 2.6], we get that $\Delta(A) \prec \mathcal{I} N \otimes \Lambda(\Lambda_i)$ for all $i \in \{1, \ldots, n\}$. But then $\Delta(A) \prec \mathcal{I} N \otimes 1$ (see e.g. [Va10, Lemma 2.7]).

For every subset $\mathcal{F} \subset \Lambda$, we denote by $P_{\mathcal{F}}$ the orthogonal projection of $\ell^2(\Lambda)$ onto $\ell^2(\mathcal{F})$. We similarly denote, for every subset $\mathcal{F} \subset \Gamma$, by $P_{\mathcal{F}}$ the orthogonal projection of $\ell^2(M)$ onto the closed linear span of $\{bu_g \mid b \in L^\infty(X), g \in \mathcal{F}\}$. Choose $\varepsilon > 0$. We now prove that there exists a finite subset $\mathcal{F}' \subset \Gamma$ such that

$$
\|P_{\mathcal{F}'}(a)\|_2^2 > 1 - 2\varepsilon \quad \text{for all } a \in \mathcal{U}(A). \quad (5.1)
$$

Once (5.1) is proven, the required conclusion $A \prec_M L^\infty(X)$ follows from Definition 2.1.

Since $\Delta(A) \prec \mathcal{I} N \otimes 1$, we get from [Va10, Lemma 2.5], a finite subset $\mathcal{F} \subset \Lambda$ such that

$$
\|(1 \otimes P_{\mathcal{F}})\Delta(a)\|_2^2 > 1 - \varepsilon \quad \text{for all } a \in \mathcal{U}(A). \quad (5.2)
$$

For every $a \in M$, we denote by $a = \sum_{g \in \mathcal{F}}(a)g u_g$, with $(a)g \in L^\infty(X)$, the Fourier decomposition of $a$. A direct computation yields

$$
\|(1 \otimes P_{\mathcal{F}})\Delta(a)\|_2^2 = \sum_{g \in \Gamma} \|(a)g\|_2^2 \ m(\{x \in \mathcal{U} \mid \omega(g, x) \in \mathcal{F}\}). \quad (5.3)
$$

Note that $\omega(g, x) \in \mathcal{F}$ if and only if $g \cdot x \in \mathcal{U} \cdot \mathcal{F}$. Since $\mathcal{U} \subset \Omega$ has finite measure and $\Gamma \curvearrowright \Omega$ admits a fundamental domain, there exists a finite subset $\mathcal{F}' \subset \Gamma$ such that

$$
m(\{x \in \mathcal{U} \mid \omega(g, x) \in \mathcal{F}\}) < \varepsilon \quad \text{for all } g \in \Gamma - \mathcal{F}'.
$$

A combination of (5.3) and (5.2) then yields (5.1). This concludes the proof of case 1.

**Proof of case 2.** Since $P$ is a finite index subfactor of $M$, Lemma 2.2 provides a projection $q \geq p$ such that $q \in \mathcal{D}(\Lambda)' \cap (N \otimes \Lambda \Lambda)$ and such that $\Delta(M)q$ is amenable relative to $N$. Write $\mathcal{N} := (N \otimes \Lambda \Lambda, e_{\mathcal{N}})$. We get a $\Delta(M)q$-central positive functional $\Psi_1$ on $q N q$ such that $\Psi_1(x) = \tau(x)$ for all $x \in q (N \otimes \Lambda \Lambda)q$. We identify $\mathcal{N} = N \otimes \mathcal{B}(\ell^2(\Lambda_i))$. As such, we view $L^\infty(\mathcal{U} \times \Lambda_i) = L^\infty(\mathcal{U}) \otimes L^\infty(\Lambda_i)$ as a von Neumann subalgebra of $\mathcal{N}$. The unitaries $\Delta(u_g) \in \mathcal{N}$, $g \in \Gamma$, normalize $L^\infty(\mathcal{U} \times \Lambda_i) \subset \mathcal{N}$ and induce the action $\Gamma \curvearrowright \mathcal{U} \times \Lambda_i$ given by $g \cdot (x, s) = (g \cdot x, \omega(g, x) \cdot s)$. The formula $\Psi(F) = \Psi_1(q F q)$ then provides a nonzero positive $\Gamma$-invariant functional on $L^\infty(\mathcal{U} \times \Lambda_i)$ such that the restriction of $\Psi$ to $L^\infty(\mathcal{U})$ is normal and $\Gamma$-invariant.

Denote by $\mathcal{W} \subset \mathcal{U}$ the support of $\Psi_{|L^\infty(\mathcal{U})}$. Then, $\mathcal{W}$ is a nonnegligible $\Gamma$-invariant subset of $\mathcal{U}$. Modifying $\Psi$ by using the $\Gamma$-invariant Radon-Nikodym derivative between $\Psi_{|L^\infty(\mathcal{W})}$ and integration w.r.t. $m$, we may assume that $\Psi_{|L^\infty(\mathcal{W})}$ equals integration w.r.t. $m$. We approximate $\Psi \in L^\infty(\mathcal{W} \times \Lambda_i)^+$ in the weak* topology by a net of unit vectors in $L^1(\mathcal{W} \times \Lambda_i)^+$. We view the elements of $L^1(\mathcal{W} \times \Lambda_i)^+$ as measurable functions from $\mathcal{W}$ to $\ell^1(\Lambda_i)^+$. Passing to convex combinations, we then find a sequence of measurable maps $\eta_n : \mathcal{W} \to \ell^1(\Lambda_i)^+$ satisfying

$$
\lim_{n \to \infty} \int_{\mathcal{W}} \|\eta_n(x)\|_1 - 1 \ dm(x) = 0,
$$

$$
\lim_{n \to \infty} \int_{\mathcal{W}} \|\eta_n(g \cdot x) - \omega(g, x) \cdot \eta_n(x)\|_1 \ dm(x) = 0 \quad \text{for all } g \in \Gamma.
$$

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Passing to a subsequence, we get that \( \|\eta_n(x)\|_1 \to 1 \) for a.e. \( x \in \mathcal{W} \) and that \( \|\eta_n(g \ast x) - \omega(g,x)\|_1 \to 0 \) for all \( g \in \Gamma \) and a.e. \( x \in \mathcal{W} \). We define \( \psi_n(x) = ||\eta_n(x)||_1^{-1}\eta_n(x) \) and have found a sequence of measurable maps \( \psi_n : \mathcal{W} \to \text{Prob} \Lambda_i \) such that
\[
\lim_n \|\psi_n(g \ast x) - \omega(g,x)\|_1 = 0 \quad \text{for all } g \in \Gamma \text{ and a.e. } x \in \mathcal{W}.
\]

Define \( \Omega_0 := \mathcal{W} \cdot \Lambda \). Then, \( \Omega_0 \) is a nonnegligible \((\Gamma \times \Lambda)\)-invariant subset of \( \Omega \). Defining \( \xi_n : \Omega_0 \to \text{Prob} \Lambda_i \) given by \( \xi_n(x \cdot s) := s_i^{-1} \cdot \psi_n(x) \) for all \( x \in \mathcal{W} \) and \( s \in \Lambda \), it is easy to check that
\[
\lim_n \|\xi_n(g \cdot y \cdot r^{-1}) - r_1 \cdot \xi_n(y)\|_1 = 0 \quad \text{for all } g \in \Gamma, r \in \Lambda \text{ and a.e. } y \in \Omega_0.
\]
So, we have shown that (2) holds. This concludes the proof of case 2.

It remains to prove (a) and (b).

(a) If \( \Lambda_i \) is amenable, take a sequence \( \eta_n \in \text{Prob} \Lambda_i \) such that \( \lim_n \|s \cdot \eta_n - \eta_n\|_1 = 0 \) for all \( s \in \Lambda_i \). Defining \( \xi_n(x) = \eta_n \) for all \( x \in \Omega \), it follows that (2) holds. Conversely, assume that the sequence \( \xi_n : \Omega_0 \to \text{Prob} \Lambda_i \) satisfies (2) and that also \( m(\mathcal{V}) < \infty \). Identifying \( \Gamma \Lambda \Omega \) with \( \mathcal{V} \), we get a right action of \( \Lambda \) on \( \mathcal{V} \), denoted by \( \ast \), and a 1-cocycle \( \mu : \mathcal{V} \times \mathcal{V} \to \Gamma \) such that
\[
y \ast s = \mu(y,s) \ast (y \ast s) \quad \text{for a.e. } y \in \mathcal{V} \text{ and all } s \in \Lambda.
\]
Consider the action \( \Lambda \times \mathcal{V} \to \Lambda_i \) given by \( (y,t) = (y \ast s^{-1}, s_t) \). Since \( \Omega_0 \subset \Omega \) is nonnegligible and \((\Gamma \times \Lambda)\)-invariant, also \( \mathcal{V} \cap \Omega_0 \) is nonnegligible and invariant under the \( \ast \)-action of \( \Lambda \). The restriction of \( \xi_n \) to \( \mathcal{V} \cap \Omega_0 \) defines a sequence \( \xi_n \in L^1(\mathcal{V} \cap \Omega_0) \) with \( \|\xi_n\|_1 = m(\mathcal{V} \cap \Omega_0) \). Defining \( \xi_n \in L^1(\mathcal{V} \cap \Omega_0) \) with \( \|\xi_n\|_1 = m(\mathcal{V} \cap \Omega_0) \) for all \( n \), that satisfies \( \lim_n \|s \cdot \xi_n - \xi_n\|_1 = 0 \) for all \( s \in \Lambda \), by the dominated convergence theorem. The push forward of \( \xi_n \) along the factor map \( \mathcal{V} \times \Lambda_i \to \Lambda_i : (y,r) \mapsto r \), then provides, after rescaling, a sequence \( \eta_n \in \text{Prob} \Lambda_i \) such that \( \lim_n \|s \cdot \eta_n - \eta_n\|_1 = 0 \) for all \( s \in \Lambda_i \). So, \( \Lambda_i \) is amenable.

(b) Assume that \( \Omega = G = G_1 \times \cdots \times G_n \) as in statement (b). When \( x \in G \), we denote by \( x_i \) the \( i \)’th component of \( g \). First assume that we have a sequence \( \xi_n : \Omega_0 \to \text{Prob} \Lambda_i \) as in (2). We consider the action \( \Gamma \times \mathcal{U} \to \Lambda_i \) given by \( s \cdot (x,s) = (g \ast x, \omega(g,x) s) \). By the proof of (a), \( \mathcal{U} \cap \Omega_0 \) is nonnegligible, invariant under the \( \ast \)-action of \( \Gamma \) on \( \mathcal{U} \), and the restriction of \( \xi_n \) to \( \mathcal{U} \cap \Omega_0 \) defines a sequence \( \zeta_n \in L^1(\mathcal{U} \cap \Omega_0) \) with \( \|\zeta_n\|_1 = m(\mathcal{U} \cap \Omega_0) \). Defining \( \zeta_n \in L^1(\mathcal{U} \cap \Omega_0) \) with \( \|\zeta_n\|_1 = m(\mathcal{U} \cap \Omega_0) \) for all \( n \), that satisfies \( \lim_n \|g \cdot \zeta_n - \zeta_n\|_1 = 0 \) for all \( g \in \Gamma \). The push forward of \( \zeta_n \) along the factor map \( \mathcal{U} \to \Lambda_i : (x,s) \mapsto x_is \), then provides, after rescaling, a sequence of unit vectors \( \eta_n \in L^1(G_1) \) such that \( \lim_n \|g_1 \cdot \eta_n - \eta_n\|_1 = 0 \) for all \( g_1 \in G_i \). Denoting \( \pi_i : G \to G_i : \pi_i(g) = g_i \), it follows from Lemma 5.2 that \( \pi_i(\Gamma) \) has an amenable closure inside \( G_i \).

Conversely, assume that the closure \( H \) of \( \pi_i(\Gamma) \) in \( G_i \) is an amenable locally compact group. Then the action \( \Lambda_i \times H \backslash G_i \) is amenable in the sense of Zimmer. So, we find a sequence of measurable maps \( \psi_n : H \backslash G_i \to \text{Prob} \Lambda_i \) such that \( \lim_n ||\psi_n(s \cdot x) - s^{-1} \cdot \psi_n(x)||_1 = 0 \) for a.e. \( x \in H \backslash G_i \) and all \( s \in \Lambda_i \). We view \( \psi_n \) as a sequence of \( H \)-invariant measurable maps \( \psi_n : G_i \to \text{Prob} \Lambda_i \). Defining \( \Gamma : G \to \text{Prob} \Lambda_i : \xi_n(x) = \psi_n(x_i) \), statement (2) is satisfied.

For the convenience of the reader, we include a proof of the following standard, but slightly technical, lemma.

**Lemma 5.2.** Let \( G \) be a second countable locally compact group and \( H_0 < G \) any subgroup. Assume that \( L^\infty(G) \) admits a state that is invariant under left translation by all elements of \( H_0 \). Then the closure of \( H_0 \) is an amenable locally compact group.
Proof. Let $\omega$ be a left $H_0$-invariant state on $L^\infty(G)$. Denote by $H$ the closure of $H_0$ inside $G$. Since $G$ is second countable, we can take a Borel map $T : G \to H$ satisfying $T(hg) = hT(g)$ for all $h \in H$, $g \in G$ (see e.g., [Ke95, Theorem 12.17]). Consider the Banach space $LUC(H)$ of bounded left uniformly continuous functions on $H$. Then, $\mu : f \mapsto \omega(f \circ T)$ is a state on $LUC(H)$ that is invariant under left translation by $H_0$. By continuity, $\mu$ is invariant under left translation by $H$. So, $LUC(H)$ admits a left invariant mean. This implies that $H$ is amenable (see e.g. [BHVO8 Theorem G.3.1]).

6 An application to W*-rigidity

Recall that an ergodic pmp action $\Gamma \curvearrowright (X, \mu)$ is called aperiodic if all finite index subgroups of $\Gamma$ still act ergodically. Following [MS02, Definition 1.8] an ergodic pmp action $\Lambda \curvearrowright (Y, \eta)$ is called mildly mixing if there are no nontrivial recurrent subsets: if $A \subset Y$ is measurable and $\liminf_{g \to \infty} \eta(g \cdot A \Delta A) = 0$, then $\eta(A) = 0$ or $\eta(A) = 1$. Note that for a mildly mixing action $\Lambda \curvearrowright (Y, \eta)$ all infinite subgroups of $\Lambda$ act ergodically on $(Y, \eta)$. Finally, a pmp action $\Gamma_1 \times \Gamma_2 \curvearrowright (X, \mu)$ of a product group is called irreducible if both $\Gamma_1$ and $\Gamma_2$ act ergodically.

Proof of Theorem 1.7. Since $\Gamma$ is a product of hyperbolic groups, Theorem 1.1 applies. So the existence of an isomorphism $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ implies that $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ are orbit equivalent. Since non-elementary hyperbolic groups belong to the class $C_{reg}$ of Monod and Shalom, it follows from [MS02, Theorem 1.10] that the groups $\Gamma$ and $\Lambda$ must be isomorphic and that their actions must be conjugate.

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