Multiplicity of Radially Symmetric Small Energy Solutions for Quasilinear Elliptic Equations Involving Nonhomogeneous Operators

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Abstract: We investigate the multiplicity of radially symmetric solutions for the quasilinear elliptic equation of Kirchhoff type. This paper is devoted to the study of the $L^\infty$-bound of solutions to the problem above by applying De Giorgi’s iteration method and the localization method. Employing this, we provide the existence of multiple small energy radially symmetric solutions whose $L^\infty$-norms converge to zero. We utilize the modified functional method and the dual fountain theorem as the main tools.

Keywords: radial solution; quasilinear elliptic equations; De Giorgi iteration; modified functional methods; dual fountain theorem

MSC: 35J50; 35J62; 46E30; 46E35

1. Introduction

This paper is devoted to the study of the multiplicity of radially symmetric solutions for the following quasilinear elliptic equation with general nonlinearities in Orlicz-Sobolev spaces:

\[
-\mathcal{M}\left(\int_{\mathbb{R}^N} \phi\left(|\nabla u|^2\right) dx\right) \text{div}\left(\phi'(|\nabla u|^2) \nabla u\right) + |u|^{\alpha-2} u = \lambda h(x,u),
\]

in $\mathbb{R}^N$,\hspace{1cm} (1)

where $N \geq 2$, $1 < p < q < N$, $1 < \alpha \leq p^* q'/p'$, $\alpha < q$, $\phi(t)$ behaves like $t^{q/2}$ for small $t$ and $t^{p'/2}$ for large $t$, and $p'$ and $q'$ are the conjugate exponents of $p$ and $q$. $h: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $h(\cdot, u)$ is radial, and the Kirchhoff function $\mathcal{M}: \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:

(M1) $\mathcal{M} \in C(\mathbb{R}_+)$ satisfies $\inf_{t \in \mathbb{R}_+} \mathcal{M}(t) \geq m_0 > 0$, where $m_0$ is a constant.
(M2) There exists $\theta \in [1, \frac{q}{p'})$ such that $\theta \mathcal{M}(t) = \theta \int_0^t \mathcal{M}(\tau)d\tau \geq \mathcal{M}(t)t$ for any $t \geq 0$.

A typical example for $\mathcal{M}$ is given by

\[\mathcal{M}(t) = b_0 + b_1 t^n,\]

where $n > 0$, $b_0 > 0$, and $b_1 \geq 0$. This operator appears in the Kirchhoff equation that occurs in nonlinear vibrations. We refer to [1–4] and the references therein for more physical motivation for the Kirchhoff problem. Also the function $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfies the following properties:

(A1) $\phi(0) = 0$;
(A2) There exist $\tilde{c} > 0$ and $\tilde{C} > 0$ such that

$$
\begin{align*}
&\tilde{c} t^2 \leq \phi(t) \leq \tilde{C} t^2, & t \geq 1, \\
&\tilde{c} t^q \leq \phi(t) \leq \tilde{C} t^q, & 0 \leq t \leq 1;
\end{align*}
$$

(A3) There exists $0 < \mu < \frac{q}{2q}$ such that

$$
\phi'(t) t \leq \frac{s \mu}{2} \phi(t)
$$

for all $t \geq 0$, and $s$ is given in (H2) below.

(A4) The map $t \to \phi(t^2)$ is strictly convex.

When $M(t) = 1$ and $\phi(t) = 2[(1 + t)^{\frac{1}{2}} - 1]$, if the term $|u|^\alpha - 2u$ is missing, our problem corresponds to

$$
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \lambda h(x, u),
$$

known as the prescribed mean curvature equation.

Recently, considerable attention has been focused on the study of certain nonlinear equations, including nonhomogeneous operators of the type

$$
-\text{div}(\phi'(|\nabla u|^2)\nabla u), \quad \phi \in C^1(\mathbb{R}^+, \mathbb{R}^+).
$$

The interest in such kinds of operators has widely developed in light of the pure or applied mathematical theory to some concrete phenomena, such as nonlinear elasticity, fluid mechanics, plasticity theory, biophysics problems, and plasma physics; see [5–9] and the references therein.

In the case of nonlinear quasilinear elliptic problems, a functional setting is the use of the classical Sobolev spaces to deal with the problem variationally. In contrast, the investigation on problems with nonhomogeneous differential operators is based on the theory of Orlicz-Sobolev spaces. In this regard, variational problems for elliptic equations of this type have been extensively studied in recent years; for instance, see [1,2,10–19] and their references.

In particular, A. Azzollini et al. [20,21] recently dealt with the existence of non-negative radially symmetric solutions to nonlinear problems associated with a new class of differential operators in an Orlicz-Sobolev space when $\phi$ has a different behavior near zero and at infinity; for instance,

$$
\phi(t) = \frac{2}{p} \left[ \left(1 + t^2 \right)^{\frac{p}{2}} - 1 \right], \quad 1 < p < q < N, \quad t \in \mathbb{R}^+.
$$

A different approach has been used in [22] where a suitable formulation of the problem is given in the Banach space $W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. As remarked in [21], the theory of classical Sobolev spaces cannot be immediately used, because they considered the different growth of the principal part $\phi$ on the whole space $\mathbb{R}^N$. Hence, in order to obtain the existence results, they took an adequate functional framework based on the paper [23] into account. This approach to the sum of Lebesgue spaces is new and different from the preceding related works [2,10–14,17–19], even if the functional setting is considered in an Orlicz-Sobolev space. Inspired by the results in [20,21], N. Chorfi and V. D. Rădulescu [24] established the existence of at least one nontrivial solution for the quasilinear Schrödinger equation by using the mountain pass theorem which is originally suggested by the paper [25]. Very recently, under various conditions on the nonlinear term $h$, the authors in [26] studied the multiplicity of radially symmetric large- or small-energy solutions for problem (1) by employing the fountain theorem and the dual fountain theorem in [27], respectively.

The main purpose of the present paper is to provide the existence of a sequence of infinitely many radially symmetric solutions whose $L^\infty$-norms converge to zero when the nonlinear term $h(\cdot, t)$ is odd in $t$ for a small $t$, and no conditions on $h(\cdot, t)$ exist at infinity. This approach, initially based on the work of Z.-Q. Wang [28], utilizes the modified functional method and global variational formulation.
in [29] as the main tools. We also refer to the papers [28,30–35]. However, we design our consequence under a different approach from the previous works. To be precise, we point out that in contrast to aforementioned papers [28,30–32,34,35], which investigate the existence of such a sequence of solutions belonging to the $L^\infty$ space, we take the dual fountain theorem in place of global variational formulation in [29] into consideration. As we know, multiplicity results that apply the dual fountain theorem to derive the existence of small energy solutions for elliptic equations of variational type do not ensure the boundedness of solutions; see [36,37]. In this light, we firstly show the global uniform boundedness for weak solutions of problem (1). Unfortunately, to the best of our knowledge, there are no results about $L^\infty$-bound for weak solutions of the problem above. To overcome this difficulty, we use De Giorgi’s iteration method and a truncated energy technique, originally given in [38], as the main tools for obtaining this result. These arguments together with the modified functional method and the dual fountain theorem, allow us to prove the existence of multiple small-energy radially symmetric solutions converging to zero in $L^\infty$ space. To our knowledge, the present paper is the first to study the existence and regularity type results of this type for our problem with the Kirchhoff function $M$ (even if $M \equiv 1$) and different growth conditions on $\phi$.

The remainder of this paper is organized as follows. We firstly review some well-known facts for the sum of Lebesgue spaces and Orlicz-Sobolev spaces. Based on this background, we present the $L^\infty$-bound of solutions to the problem (1) by applying De Giorgi’s iteration method and the localization method, and finally we provide a sequence of infinitely many small energy radially symmetric solutions whose $L^\infty$-norms converge to zero.

2. Preliminaries and Main Result

In this section, we briefly list some definitions and essential properties of the sum of Lebesgue spaces and Orlicz-Sobolev space. For a deeper treatment of these spaces, we refer to [10,39]. For simplicity, $C$ is used to represent a generic constant, which may change from line to line unless otherwise noted.

**Definition 1.** Let $1 < p < q < N$. We denote by $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N, \mathbb{R})$ in the norm
\[
\|u\|_{L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)} = \inf \{ \|v\|_{L^p(\mathbb{R}^N)} + \|w\|_{L^q(\mathbb{R}^N)} \mid v \in L^p(\mathbb{R}^N), w \in L^q(\mathbb{R}^N), u = v + w \}.
\]
We set $\|u\|_{L^{p,q}(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)}$.

Now, we define the Orlicz-Sobolev space for our analysis (see [23]).

**Definition 2.** For $\alpha > 1$, $W$ is denoted by the completion of $C_0^\infty(\mathbb{R}^N, \mathbb{R})$ in the norm
\[
\|u\|_W = \|u\|_{L^\alpha(\mathbb{R}^N)} + \|\nabla u\|_{L^{p,q}(\mathbb{R}^N)}.
\]
Following [21], we note that $(W, \|\cdot\|_W)$ is a reflexive Banach space and recall the following embedding result.

**Lemma 1.** The space $W$ is continuously embedded into $L^{p'}(\mathbb{R}^N)$ for every $1 < \alpha \leq \frac{p'-q'}{p'}$, where $p'$ and $q'$ are the conjugate exponents of $p$ and $q$, respectively. In addition, $W$ is continuously embedded into $L^t(\mathbb{R}^N)$ for any $1 \leq t \leq p^*$.

Following [23], let us recall the following Hölder type inequality.

**Lemma 2.** For any $u \in W$ and $\psi \in L^{p'}(\mathbb{R}^N) \cap L^{q'}(\mathbb{R}^N)$, we have
\[
\int_{\mathbb{R}^N} |u\psi| d\mu \leq \|u\|_W \max \{ \|\psi\|_{L^{p'}(\mathbb{R}^N)}, \|\psi\|_{L^{q'}(\mathbb{R}^N)} \}.
\]
To obtain some compactness results in whole space, we study radially symmetric functions in $W$.

**Definition 3.** We define 
\[
(C^\infty_c(\mathbb{R}^N, \mathbb{R}))_{\text{rad}} = \{ v \in C^\infty_c(\mathbb{R}^N, \mathbb{R}) \mid v \text{ is radially symmetric} \},
\]
and let $W_r$ be the completion of $(C^\infty_c(\mathbb{R}^N, \mathbb{R}))_{\text{rad}}$ in the norm $\| \cdot \|$, namely 
\[
W_r = \left( C^\infty_c(\mathbb{R}^N, \mathbb{R}) \right)_{\text{rad}}^{\| \cdot \|}.
\]

**Remark 1.** Generally, it is not trivial to determine whether $W_r$ coincides with the set of radial functions of $W$. On the other hand, if $1 < p < q < N$, $1 < \alpha \leq p^* \frac{d'}{p}$ and $q < p^*$, then, arguing as in the proof of Theorem 2.8 in [21], we can show that the two sets are equal.

The compact embedding result is given in [21] as follows.

**Lemma 3.** For every $\alpha \in (1, p^* \frac{d'}{p}]$, the space $W_r$ is compactly embedded into $L^\alpha(\mathbb{R}^N)$ with $\alpha < \gamma < p^*$.

The estimate for radial symmetry functions is as follows; see [21] (Lemma 2.13):

**Lemma 4.** If $1 < p < q < N$, then there exists $\kappa > 0$ such that for every $u \in W_r$,
\[
|u(x)| \leq \frac{\kappa}{|x|^{\frac{N-1}{2}}\|\nabla u\|_{L^p(\mathbb{R}^N)}} \quad \text{for } |x| \geq 1.
\]

Throughout this paper, we assume that the conditions (M1), (M2), and (A1)–(A4) are fulfilled.

**Definition 4.** We say that $u \in W_r$ is a weak solution of problem (1) if 
\[
M \left( \int_{\mathbb{R}^N} \phi(|\nabla u|^2) \, dx \right) \int_{\mathbb{R}^N} \phi'(|\nabla u|^2) \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^N} |u|^{a-2}uv \, dx = \lambda \int_{\mathbb{R}^N} h(x, u)v \, dx,
\]
for any $v \in W_r$.

We define $H(x, t) = \int_0^t h(x, s) \, ds$ and then suppose that

(H1) $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and $h(\cdot, t)$ is radial.

(H2) There exist nonnegative functions $\rho \in L^\infty(\mathbb{R}^N)$ and $\sigma \in L^\infty(\mathbb{R}^N)$ such that 
\[
|h(x, t)| \leq \rho(x) + \sigma(x) |t|^{\delta-1}
\]
for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ and $\alpha < s < p^*$.

(H3) There exists a constant $s_0 > 0$ such that $pH(x, t) - h(x, t)t > 0$ for $0 < |t| < s_0$ and for any $x \in \mathbb{R}^N$, where $H(x, t) = \int_t^{s_0} h(x, s) \, ds$.

(H4) $\lim_{|t| \to 0} \frac{h(x, t)}{|t|^{\delta-1}} = +\infty$ uniformly for all $x \in \mathbb{R}^N$.

Next, we define the functional $I_\lambda : W_r \to \mathbb{R}$ by 
\[
I_\lambda(u) = \frac{1}{2}M \left( \int_{\mathbb{R}^N} \phi(|\nabla u|^2) \, dx \right) + \frac{1}{\alpha} \int_{\mathbb{R}^N} |u|^\alpha \, dx - \lambda \int_{\mathbb{R}^N} H(x, u) \, dx.
\]
Here, we define the functional $\Psi : W_r \to \mathbb{R}$ as 
\[
\Psi(u) = \int_{\mathbb{R}^N} H(x, u) \, dx.
\]
Then, we can easily check that $\Psi \in C^1(W_r, \mathbb{R})$ and its Fréchet derivative is

$$
\langle \Psi'(u), v \rangle = \int_{\mathbb{R}^N} h(x, u)v\,dx
$$

for any $u, v \in W_r$. Then it follows that the functional $I_\lambda \in C^1(W_r, \mathbb{R})$ and its Fréchet derivative is

$$
\langle I_\lambda'(u), v \rangle = M \left( \int_{\mathbb{R}^N} \phi(|\nabla u|^2)\,dx \right) \int_{\mathbb{R}^N} \phi'(|\nabla u|^2)\nabla u \cdot \nabla v\,dx + \int_{\mathbb{R}^N} |u|^{n-2}uv\,dx - \lambda \int_{\mathbb{R}^N} h(x, u)v\,dx
$$

for any $u, v \in W_r$.

In the next lemma, we provide a list of useful properties for the sum of weighted Lebesgue spaces (see [21] (Proposition 2.2)).

**Lemma 5.** Let $\Omega \subset \mathbb{R}^N$, $u \in L^p(\Omega) + L^q(\Omega)$ and $\Lambda_u = \{ x \in \Omega : |u(x)| > 1 \}$. We have

(i) for $\Omega' \subset \Omega$, if $|\Omega'| < \infty$, then $u \in L^p(\Omega')$;

(ii) for $\Omega' \subset \Omega$ if $u \in L^\infty(\Omega')$, then $u \in L^q(\Omega')$;

(iii) $|\Lambda_u| < \infty$;

(iv) $u \in L^p(\Lambda_u) \cap L^q(\Lambda_u')$;

(v) $L^p(\Omega) + L^q(\Omega)$ is reflexive;

(vi) $\|u\|_{L^p(\Omega)+L^q(\Omega)} \leq \max\{\|u\|_{L^p(\Omega_u)}, \|u\|_{L^q(\Lambda_u')}\}$;

(vii) if $\Omega' \subset \Omega$, then $\|u\|_{L^p(\Omega')+L^q(\Omega')} \leq \|u\|_{L^p(\Omega)+L^q(\Omega)} + \|u\|_{L^p(\Omega\setminus\Omega')} + \|u\|_{L^q(\Omega\setminus\Omega')}$.\n
From now on, we present the $L^\infty$-bound of solutions to the problem (1). In order to employ the De Giorgi technique, we need the following Vital Lemma. The proof is given in the paper [38] (Lemma 2.2).

**Lemma 6.** Let $\{Z_n\}_{n=1}^\infty$ be a sequence of positive numbers, satisfying the recursion inequality

$$
Z_{n+1} \leq cb^n Z_{n+\delta}, \quad n = 0, 1, 2, \ldots
$$

for some $b > 1, c > 0$ and $\delta > 0$. If $Z_0 \leq \min\{1, c(-1)/\delta b(-1)/\delta^2\}$ then $Z_n \leq 1$ for some $n \in \mathbb{N} \cup \{0\}$. Moreover,

$$
Z_n \leq \min\left\{1, c(-1)/\delta b(-1)/\delta^2 b(-n)/\delta\right\}
$$

for any $n \geq n_0$, where $n_0$ is the smallest $n \in \mathbb{N} \cup \{0\}$ satisfying $Z_n \leq 1$. In particular, $Z_n \to 0$ as $n \to \infty$.

Next, we prove the following consequence, which is a regularity type result via De Giorgi technique and the localization method.

**Proposition 1.** Assume that (H1)–(H2) hold. If $u$ is a weak solution of the problem (1), then $u \in L^\infty(\mathbb{R}^N)$ and there exist positive constants $C, \eta$ independent of $u$ such that

$$
\|u\|_{L^\infty(\mathbb{R}^N)} \leq C\|u\|^{\eta}_{L^1(\mathbb{R}^N)}.
$$

**Proof.** Let $A_k = \{ x \in \mathbb{R}^N : u(x) > k \}, \bar{A}_k = \{ x \in \mathbb{R}^N : -u(x) > k \}$ for $k > 0$. Note that $|A_k|$ and $|\bar{A}_k|$ are finite for any $k \in \mathbb{N}$. Taking a test function $v = (u - k)_+ \in W$ from Definition 4 we obtain

$$
M \left( \int_{\mathbb{R}^N} \phi(|\nabla u|^2)\,dx \right) \int_{\mathbb{R}^N} \phi'(|\nabla u|^2)\nabla u \cdot \nabla v\,dx + \int_{\mathbb{R}^N} |u(x)|^{n-2}uv\,dx = \lambda \int_{\mathbb{R}^N} h(x, u)v\,dx.
$$

Equivalently,

$$
M \left( \int_{\mathbb{R}^N} \phi(|\nabla u|^2)\,dx \right) \int_{A_k} \phi'(|\nabla u|^2)\,dx + \int_{\bar{A}_k} |u|^{n-2}u(u - k)\,dx = \lambda \int_{A_k} h(x, u)(u - k)\,dx.
$$
Note that, by (A3) and (A4),
\[ \phi(t) < 2\phi'(t) t \leq s\mu\phi(t) \]
for all \( t > 0 \). Hence, since \( u \geq u - k > 0 \) on \( A_k \), by the definition of the function \( M \), assumptions (A2) and (H2), we note that
\[
\frac{c m_0}{2} \max \left\{ \int_{\mathbb{R}^N} |\nabla u|^q dx, \int_{\mathbb{R}^N} |\nabla u|^p dx \right\} \\
\leq -\int_{A_k} |u|^q - u(u - k) dx + \lambda \int_{A_k} (\rho(x) + \sigma(x)|u|^{q-1})(u - k) dx \\
\leq \lambda \int_{A_k} (\rho(x) + \sigma(x)|u|^{q-1}) u dx \\
\leq \lambda \|\rho\|_{L^\infty(\mathbb{R}^N)} \int_{A_k} u dx + \lambda \|\sigma\|_{L^\infty(\mathbb{R}^N)} \int_{A_k} u^q dx \\
\leq \lambda (1 + k^{q-s}) (\|\rho\|_{L^\infty(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}) \int_{A_k} u^q dx.
\]
(2)

Put \( k_n := k_s(2 - 1/2^n), n = 0, 1, 2, \cdots \), with \( k_s > 0 \) specified later and
\[
Z_n := \int_{A_{k_n}} (u - k_n)^s dx.
\]
Since \( k_s \leq k_n \leq k_{n+1} \leq 2k_s \) for all \( n \in \mathbb{N} \), we have
\[
\int_{A_{k_n}} (u - k_n)^s dx \geq \int_{A_{k_{n+1}}} u^s \left( 1 - \frac{k_n}{k_{n+1}} \right)^s dx \geq \int_{A_{k_{n+1}}} u^s \frac{u^s}{2^s(n+2)} dx
\]
and therefore
\[
Z_n \geq \int_{A_{k_{n+1}}} \frac{u^s}{2^s(n+2)} dx.
\]
(3)

Thus
\[
\int_{A_{k_{n+1}}} u^s dx \leq c_1^{s+2} Z_n,
\]
where \( c_1 = 2^s > 1 \). It follows from (2) and (3) that
\[
\frac{c m_0}{2} \max \left\{ \int_{A_{k_{n+1}}} |\nabla (u - k_{n+1})|^q dx, \int_{A_{k_{n+1}}} |\nabla (u - k_{n+1})|^p dx \right\} \\
\leq \lambda (1 + k_s^{q-s}) (\|\rho\|_{L^\infty(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}) c_1^{s+2} Z_n.
\]
(4)

For the Lebesgue measure of \( A_{k_{n+1}} \), we deduced that
\[
|A_{k_{n+1}}| \leq \int_{A_{k_{n+1}}} \left( \frac{u - k_n}{k_{n+1} - k_n} \right)^s dx = \int_{A_{k_{n+1}}} \left( \frac{2^{n+1}}{k_s} \right)^s (u - k_n)^s dx.
\]
So
\[
|A_{k_{n+1}}| \leq \frac{c_1^{s+1}}{k_s^s} Z_n.
\]
(5)

Observe that
\[
\max \left\{ \int_{A_{k_{n+1}}} |\nabla (u - k_{n+1})|^q dx, \int_{A_{k_{n+1}}} |\nabla (u - k_{n+1})|^p dx \right\} \\
\geq \max \left\{ \int_{A_{k_{n+1}}} |\nabla (u - k_{n+1})|^q dx, \int_{A_{k_{n+1}}} |\nabla (u - k_{n+1})|^p dx \right\} - |A_{k_{n+1}}|.
\]
Note that $1 + k_n^{-s} \leq 2(1 + k_n^{-s})$. Then, it follows from the above inequality, relations (4) and (5) that we obtain

$$\tilde{C} = \min\left\{ \frac{\tilde{\varepsilon}m_0}{2} \max\left\{ \int_{\mathcal{N}(u-k_n+1) \cap \mathcal{A}_{k_n+1}} |\nabla(u-k_n)|^q \, dx, \int_{\mathcal{N}(u-k_n+1) \cap \mathcal{A}_{k_n+1}} |\nabla(u-k_n+1)|^p \, dx \right\}, \frac{\lambda(1 + k_n^{-s})}{\|p\|_{L^\infty(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}} e^{1+2Z_n} + |\mathcal{A}_{k_n+1}| \right\}
$$

$$\leq \lambda(1 + k_n^{-s}) (\|p\|_{L^\infty(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}) e^{1+2Z_n} + |\mathcal{A}_{k_n+1}|$$

$$= 2(1 + k_n^{-s}) (\lambda \tilde{C} e^{1} + \varepsilon_1) e^{\varepsilon_1} Z_n$$

where $\tilde{C} := \|p\|_{L^\infty(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}$ and $e_2 := 2(1 + k_n^{-s}) (\lambda \tilde{C} e^{1} + \varepsilon_1)$. Define

$$\mathfrak{s} := \begin{cases} \frac{s+p^*}{2} & \text{if } p^* < \infty, \\ s + 1 & \text{if } p^* = \infty. \end{cases}$$

Using the Hölder inequality and Lemma 1, we get

$$\int_{\mathcal{A}_{k_n+1}} (u-k_n+1)^{s} \, dx$$

$$\leq \left( \int_{\mathbb{R}^N} (u-k_n+1)^{s} \, dx \right)^{\mathfrak{s}/\mathfrak{s}} |\mathcal{A}_{k_n+1}|^{1-s/\mathfrak{s}}$$

$$\leq \|u-k_n+1\|_{L^\mathfrak{s}(\mathbb{R}^N)} |\mathcal{A}_{k_n+1}|^{1-s/\mathfrak{s}}$$

$$\leq \mathcal{C} \left( \|u-k_n+1\|_{L^\mathfrak{s}(\mathbb{R}^N)} + |\nabla(u-k_n+1)|_{L^\mathfrak{s}(\mathbb{R}^N)} \right)^{\frac{\mathfrak{s}}{\mathfrak{s}+1}} |\mathcal{A}_{k_n+1}|^{1-s/\mathfrak{s}}$$

where $\tau$ is either $p$ or $q$. Meanwhile, we have

$$\int_{\mathbb{R}^N} |u-k_n+1|^{\mathfrak{s}} \, dx \leq \int_{\mathcal{A}_{k_n+1}} u^{\mathfrak{s}} \, dx \leq e^{1+2Z_n}$$

and from (6) we estimate

$$\|\nabla(u-k_n+1)\|_{L^\mathfrak{s}(\mathbb{R}^N)}$$

$$\leq \max\left\{ \int_{\mathcal{N}(u-k_n+1) \cap \mathcal{A}_{k_n+1}} |\nabla(u-k_n+1)|^q \, dx, \int_{\mathcal{N}(u-k_n+1) \cap \mathcal{A}_{k_n+1}} |\nabla(u-k_n+1)|^p \, dx \right\}$$

$$\leq \frac{2e_2 e^{1}}{\tilde{c}_m} Z_n.$$
In other words,
\[ Z_{n+1} \leq \left( k_*^{-s(1-\frac{s}{2})} + k_*^{-s(1-\frac{s}{2})} \right) e_1^{1+\delta} \int_{\mathbb{R}^N} u_n^{1+\delta}, \tag{10} \]
where \( e_3 = C \left( e_1^2 + \frac{4(A_1 + e_1)}{c_{\text{me}}^2} \right)^{\frac{s}{\min\{a, e\}}} \) and \( \delta = \frac{s}{\min\{a, e\}} - \frac{s}{2}. \) This implies
\[ Z_{n+1} \leq e_3 \left( k_*^{-\gamma_1} + k_*^{-\gamma_2} \right) b^n Z_n^{1+\delta}, \quad n \in \mathbb{N} \cup \{0\}, \tag{11} \]
where
\[ 0 < \gamma_1 := s \left( 1 - \frac{s}{2} \right) < \gamma_2 := s \left( 1 - \frac{s}{2} + \frac{s}{\min\{a, \tau\}} \right) \quad \text{and} \quad b := e_1^{1+\delta}. \]

Applying Lemma 6 with (10), we obtain that
\[ Z_n = \int_{\mathbb{R}^N} (u - k_n)^{1+\delta} dx \to 0 \quad \text{as} \quad n \to \infty, \tag{12} \]
provided that
\[ Z_0 \leq \min \left\{ 1, e_3^{-\frac{1}{2}} \left( k_*^{-\gamma_1} + k_*^{-\gamma_2} \right)^{-\frac{1}{2}} b^{-\frac{1}{2\tau}} \right\}. \]

We note that for \( k \) large enough, it is \( Z_0 \leq 1 \) since \( A_{k_*} \to 0 \) as \( k_* \to \infty. \) Moreover, observe that
\[ Z_0 = \int_{A_{k_*}} (u - k_n)^{1+\delta} dx \leq \int_{\mathbb{R}^N} u_n^{1+\delta} dx. \tag{13} \]

Meanwhile,
\[ \int_{\mathbb{R}^N} u_n^{1+\delta} dx \leq e_3^{-\frac{1}{2}} \left( k_*^{-\gamma_1} + k_*^{-\gamma_2} \right)^{-\frac{1}{2}} b^{-\frac{1}{2\tau}} \]
is equivalent to
\[ k_*^{-\gamma_1} + k_*^{-\gamma_2} \leq e_3^{-1} b^{-\frac{1}{2\tau}} \left( \int_{\mathbb{R}^N} u_n^{1+\delta} dx \right)^{-\delta}. \tag{14} \]

Moreover,
\[ \left\{ \begin{array}{l}
2k_*^{-\gamma_1} \leq e_3^{-1} b^{-\frac{1}{2\tau}} \left( \int_{\mathbb{R}^N} u_n^{1+\delta} dx \right)^{-\delta} \\
2k_*^{-\gamma_2} \leq e_3^{-1} b^{-\frac{1}{2\tau}} \left( \int_{\mathbb{R}^N} u_n^{1+\delta} dx \right)^{-\delta}
\end{array} \right. \]
is equivalent to
\[ \left\{ \begin{array}{l}
k_* \geq \left( 2e_3 \right)^{\frac{1}{\gamma_1}} b^{\frac{1}{2\tau}} \left( \int_{\mathbb{R}^N} u_n^{1+\delta} dx \right)^{\frac{\gamma_1}{\gamma_1}} \\
k_* \geq \left( 2e_3 \right)^{\frac{1}{\gamma_2}} b^{\frac{1}{2\tau}} \left( \int_{\mathbb{R}^N} u_n^{1+\delta} dx \right)^{\frac{\gamma_2}{\gamma_2}}
\end{array} \right. \]

Hence, by choosing
\[ k_* = \max \left\{ \left( 2e_3 \right)^{\frac{1}{\gamma_1}} b^{\frac{1}{2\tau}} \left( \int_{\mathbb{R}^N} u_n^{1+\delta} dx \right)^{\frac{\gamma_1}{\gamma_1}}, \left( 2e_3 \right)^{\frac{1}{\gamma_2}} b^{\frac{1}{2\tau}} \left( \int_{\mathbb{R}^N} u_n^{1+\delta} dx \right)^{\frac{\gamma_2}{\gamma_2}} \right\}, \tag{15} \]
we obtain the inequality (14). Combining this and (13), we deduce the relation (12). Since \( k_n \uparrow 2k_* \), the relation (12) and the Lebesgue dominated convergence theorem infer that
\[ \int_{\mathbb{R}^N} (u - 2k_*)^\delta dx = 0. \]
Therefore, \((u - 2k_\ast)_+ = 0\) almost everywhere in \(\mathbb{R}^N\) and hence \(\text{ess sup}_{\mathbb{R}^N} u \leq 2k_\ast\). By replacing \(u\) with \(-u\) and \(A_k\) with \(A_k\), we have analogously that \(u\) is bounded from below. Therefore

\[
\|u\|_{L^\infty(\mathbb{R}^N)} \leq C \max \left\{ \left( \int_{\mathbb{R}^N} |u|^p \, dx \right)^{\frac{1}{p}}, \left( \int_{\mathbb{R}^N} |u|^q \, dx \right)^{\frac{1}{q}} \right\},
\]

where \(C\) is a positive constant independent of \(u\). This completes the proof. \(\square\)

We are ready to assert our main result for the existence of a sequence of multiple radially symmetric small energy solutions whose \(L^\infty\)-norms converge to zero. As seen before, the primary tools are the modified functional method and the dual fountain theorem.

**Remark 2.** Define a cut-off function \(\chi \in C^1(\mathbb{R}, \mathbb{R})\) satisfying \(\chi(t) = 1\) for \(|t| \leq t_0\), \(\chi(t) = 0\) for \(|t| \geq 2t_0\), \(|\chi'(t)| \leq 2/t_0\), and \(\chi'(t) \leq 0\). So, we set

\[
\tilde{H}(x, t) = \chi(t)H(x, t) + (1 - \chi(t))\xi |t|^\alpha \quad \text{and} \quad \tilde{h}(x, t) = \frac{\partial}{\partial t} \tilde{H}(x, t),
\]

where \(\xi\) is a positive constant.

On the basis of the work in [28,30], we get the following two lemmas.

**Lemma 7.** Let the assumptions (H1)–(H3) hold. Then

\[
\mathcal{I}_\lambda(u) = 0 = \langle \mathcal{I}_\lambda'(u), u \rangle \quad \text{if and only if} \quad u = 0.
\]

**Lemma 8.** Assume that (H1)–(H4) hold. Then there exist \(t_0 > 0\) with \(t_0 < \min\{s_0, 1\}/2\) and \(\tilde{h} \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})\) such that \(\tilde{h}(x, t)\) is odd for \(t\), \(\tilde{H}(x, t) \geq 0\) and

\[
\tilde{H}(x, t) = 0 \quad \text{iff} \quad t \equiv 0 \quad \text{or} \quad |t| \geq 2t_0,
\]

where \(\tilde{H}(x, t) := \alpha \tilde{H}(x, t) - \tilde{h}(x, t)t\) and \(\frac{\partial}{\partial t} \tilde{H}(x, t) = \tilde{h}(x, t)\).

Let \(\mathcal{X}\) be a reflexive and separable Banach space. Then there are \(\{e_n\} \subseteq \mathcal{X}\) and \(\{f_n^*\} \subseteq \mathcal{X}^*\) such that

\[
\mathcal{X} = \text{span}\{e_n : n = 1, 2, \cdots\}, \quad \mathcal{X}^* = \text{span}\{f_n^* : n = 1, 2, \cdots\},
\]

and

\[
\langle f_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\]

Let us denote \(\mathcal{X}_k = \text{span}\{e_k\}, \mathcal{Y}_k = \bigoplus_{m=1}^k \mathcal{X}_m,\) and \(Z_k = \bigoplus_{m=k}^\infty \mathcal{X}_m\) for \(k \in \mathbb{N}\) (see [40] (p. 21)).

**Definition 5.** Let \(\mathcal{X}\) be a reflexive Banach space and \(I \in C^1(\mathcal{X}, \mathbb{R})\). For every \(c \in \mathbb{R}\), we say that \(I\) satisfies the \((PS)_c^\ast\)-condition (with respect to \(\mathcal{Y}_n\)) if any sequence \(\{v_n\}_{n \in \mathbb{N}} \subset \mathcal{X}\) for which \(v_n \in \mathcal{Y}_n\), for any \(n \in \mathbb{N}\),

\[
I(v_n) \to c \quad \text{and} \quad \|(I|_{\mathcal{Y}_n})'(v_n)\|_{\mathcal{X}^*} \to 0 \quad \text{as} \quad n \to \infty,
\]

contains a subsequence converging to a critical point of \(\mathcal{X}\).

**Proposition 2** (Dual Fountain Theorem [27] (Theorem 3.18)). Assume that \(\mathcal{X}\) is a reflexive Banach space, \(I \in C^1(\mathcal{X}, \mathbb{R})\) is an even functional. If there exists \(k_0 > 0\) such that, for each \(k \geq k_0\), there is \(\rho_k > \delta_k > 0\) such that
Theorem 1. Suppose that (H1)–(H4) hold. If \( h(x, t) \) is odd in \( t \) for a small \( t \), then the problem (1) has a sequence of nontrivial radially symmetric solutions \( \{ u_n \} \) in \( W_r \) such that \( I_{\lambda}(u_n) \to 0 \) and \( \| u_n \|_{L^\infty(\mathbb{R}^N)} \to 0 \) as \( n \to \infty \) for every

\[
\lambda \in \Gamma := \left( 0, \min \left\{ \frac{m_0}{2b}, \frac{1}{\bar{a}} \right\} \right) \min \left\{ \left( 1 + \zeta C \right)^{-1}, \left( 2C\| \rho \|_{L^\infty(\mathbb{R}^N)} + \frac{1}{s} \| \sigma \|_{L^\infty(\mathbb{R}^N)} + \zeta \right)^{-1} \right\}.
\]

Proof. Consider the modified energy functional \( \tilde{I}_\lambda : E \to \mathbb{R} \) given by

\[
\tilde{I}_\lambda(v) := \frac{1}{2} \mathcal{M} \left( \int_{\mathbb{R}^N} \phi(\| \nabla v \|^2) \, dx \right) + \frac{1}{\bar{a}} \int_{\mathbb{R}^N} |v|^\bar{a} \, dx - \lambda \tilde{\Psi}(v),
\]

and

\[
\tilde{\Psi}(v) = \int_{\Omega} \tilde{H}(x, v) \, dx, \quad v \in W_r.
\]

Then it is clear by Lemma 8 that \( \tilde{I}_\lambda \in C^1(W_r, \mathbb{R}) \) is an even functional. Now we will show that conditions (D1)–(D4) of Proposition 2 are satisfied.

(D1): From (H2), we have

\[
|H(x, t)| \leq \rho(x)t + \frac{1}{s} \sigma(x)|t|^s, \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}.
\]

For convenience, we denote \( \theta_k = \sup_{\| v \|_{W_r} = 1, v \in \mathcal{Z}_k} \| v \|_{L^\infty(\mathbb{R}^N)} \). From Lemma 5, the definition of \( \chi_r(\mathbb{R}^2) \) and (A2) it follows that

\[
\tilde{I}_\lambda(v) := \frac{1}{2} \mathcal{M} \left( \int_{\mathbb{R}^N} \phi(\| \nabla v \|^2) \, dx \right) + \frac{1}{\bar{a}} \int_{\mathbb{R}^N} |v|^\bar{a} \, dx - \lambda \tilde{\Psi}(v),
\]

\[
\geq \frac{1}{2} \mathcal{M} \left( \int_{\mathbb{R}^N} \phi(\| \nabla v \|^2) \, dx \right) + \frac{1}{\bar{a}} \int_{\mathbb{R}^N} |v|^\bar{a} \, dx - \lambda \int_{\mathbb{R}^N} \chi_r(v)H(x, v) \, dx - (1 - \chi_r(v))\zeta|v|^\bar{a} \, dx
\]

\[
\geq \frac{1}{2 \alpha} \mathcal{M} \left( \int_{\mathbb{R}^N} \phi(\| \nabla v \|^2) \, dx \right) + \frac{1}{\bar{a}} \int_{\mathbb{R}^N} |v|^\bar{a} \, dx - \lambda \int_{\mathbb{R}^N} \rho(x)|v| + \frac{1}{s} \sigma(x)|v|^s + \zeta|v|^\bar{a} \, dx
\]

\[
\geq \frac{m_0}{2 \alpha} \max \left\{ \int_{\mathcal{N}_r} |v|^\bar{a} \, dx + \frac{1}{\alpha} \| v \|_{L^\infty(\mathbb{R}^N)} \lambda \int_{\mathbb{R}^N} \rho(x)|v| + \frac{1}{s} \sigma(x)|v|^s + \zeta|v|^\bar{a} \, dx \right\}
\]

\[
\geq \frac{m_0}{2 \alpha} \| v \|_{L^\infty(\mathbb{R}^N)} + \frac{1}{\alpha} \| v \|_{L^\infty(\mathbb{R}^N)} \lambda \int_{\mathbb{R}^N} \rho(x)|v| + \frac{1}{s} \sigma(x)|v|^s + \zeta|v|^\bar{a} \, dx
\]

\[
\geq \min \left\{ \frac{m_0}{2 \alpha}, \frac{1}{\bar{a}} \right\} \| v \|_{L^\infty(\mathbb{R}^N)} + \frac{1}{s} \| v \|_{L^\infty(\mathbb{R}^N)} \left( \frac{1}{s} \| v \|_{L^\infty(\mathbb{R}^N)} + \frac{1}{s} \| v \|_{L^\infty(\mathbb{R}^N)} \right) - \lambda \zeta|v|^\bar{a} \, dx
\]

\[
\geq \min \left\{ \frac{m_0}{2 \alpha}, \frac{1}{\bar{a}} \right\} \| v \|_{L^\infty(\mathbb{R}^N)} + \frac{1}{s} \| v \|_{L^\infty(\mathbb{R}^N)} \left( 1 + \| v \|_{L^\infty(\mathbb{R}^N)} \right) - \lambda \zeta|v|^\bar{a} \, dx
\]

\[
\geq \left( \min \left\{ \frac{m_0}{2 \alpha}, \frac{1}{\bar{a}} \right\} - \lambda \zeta C \right) \| v \|_{L^\infty(\mathbb{R}^N)} - \lambda \| v \|_{L^\infty(\mathbb{R}^N)} \zeta|v|^\bar{a} \, dx
\]

Choose \( \rho_k = \left( \frac{C}{s} \bar{a} \right)^{1/(\alpha-s)} \). Let \( v \in \mathcal{Z}_k \) with \( \| v \|_{W_r} = \rho_k > 1 \) for sufficiently large \( k \). Then, there exists \( k_0 \in \mathbb{N} \) such that

\[
\tilde{I}_\lambda(v) \geq \left( \min \left\{ \frac{m_0}{2 \alpha}, \frac{1}{\bar{a}} \right\} - \lambda \zeta C - \lambda \right) \left( \frac{C}{s} \bar{a} \right)^{1/(\alpha-s)} - \lambda \frac{C_2}{s} \geq 0.
\]
for all \( k \in \mathbb{N} \) with \( k \geq k_0 \), because
\[
\lim_{k \to \infty} \left( \min \left\{ \frac{m_0 \epsilon}{2B}, \frac{1}{\alpha} \right\} - \lambda \xi C - \lambda \right) \left( \frac{C \theta_k}{s} \right)^{\frac{\alpha}{n}} = \infty.
\]

Therefore, one has
\[
\inf \{ \tilde{I}_\lambda(v) : v \in \mathcal{Z}_k, \|v\|_W = \rho_k \} \geq 0.
\]

(D2): Observe that \( \| \cdot \|_{L^\infty(\mathbb{R}^N)}, \| \cdot \|_{L^p(\mathbb{R}^N)} \) and \( \| \cdot \|_W \) are equivalent on \( \mathcal{Y}_k \). Then there are positive constants \( \xi_{1,k} \) and \( \xi_{2,k} \) such that
\[
\xi_{1,k} \| \cdot \|_{L^\infty(\mathbb{R}^N)} \leq \| \cdot \|_W \leq \xi_{2,k} \| \cdot \|_{L^p(\mathbb{R}^N)}
\]
for any \( v \in \mathcal{Y}_k \). From (H3) and (H4), for any \( K > 0 \) there exists \( s_1 \in (0, s_0) \) such that
\[
H(x, t) \geq \frac{K C^a}{\alpha} |t|^a
\]
for almost all \( x \in \mathbb{R}^N \) and all \( |t| \leq s_1 \). Choose \( \delta_k := \min \{ \frac{1}{2}, s_1 \xi_{1,k} \} \) for all \( k \in \mathbb{N} \). Then we know that \( \|u\|_{L^\infty(\Omega)} \leq s_1 \) for \( u \in \mathcal{Y}_k \) with \( \|u\|_W = \delta_k \) and so \( \tilde{H}(x, u) = H(x, u) \) for \( \|u\|_{L^\infty(\mathbb{R}^N)} \leq s_1 \). Hence we derive by (16) that
\[
\tilde{I}_\lambda(v) \leq \frac{C M(1)}{2} \left[ \left( \int_{\Omega \subset \mathbb{R}^N} |\nabla v|^q \, dx \right)^{\frac{1}{q}} + \int_{\Omega \subset \mathbb{R}^N} |\nabla v|^p \, dx \right] + \frac{1}{\alpha} \int_{\mathbb{R}^N} |v|^\alpha \, dx - \frac{\lambda K C^a}{\alpha} \int_{\mathbb{R}^N} |v|^\alpha \, dx
\]
\[
\leq \frac{C M(1)}{2} \left( \|\nabla v\|_{L^q(\mathbb{R}^N)}^{\frac{1}{q}} + \|\nabla v\|_{L^p(\mathbb{R}^N)}^{\frac{1}{p}} \right) + \frac{1}{\alpha} \|v\|_{L^\infty(\mathbb{R}^N)}^{\alpha} - \lambda \frac{K C^a}{\alpha} \|v\|_{L^\infty(\mathbb{R}^N)}^{\alpha}
\]
\[
\leq \frac{C M(1)}{2} \left( \int_{\Omega \subset \mathbb{R}^N} |\nabla v|^q \, dx + \int_{\Omega \subset \mathbb{R}^N} |\nabla v|^p \, dx \right) + \frac{1}{\alpha} \|v\|_{L^\infty(\mathbb{R}^N)}^{\alpha} - \lambda \frac{K C^a}{\alpha} \|v\|_{L^\infty(\mathbb{R}^N)}^{\alpha}
\]
\[
\leq \max \left\{ \frac{C M(1)}{2}, \frac{1}{\alpha} \right\} \|v\|_W^{\alpha} - \lambda K C^a \|v\|_W^{\alpha}
\]
\[
\leq \max \left\{ \frac{C M(1)}{2}, \frac{1}{\alpha} \right\} \|v\|_W^{\alpha} - \lambda K C^a \|v\|_W^{\alpha}
\]
for any \( v \in \mathcal{Y}_k \) with \( \|v\|_W = \delta_k \). If we choose \( K > 0 \) large enough such that
\[
\max \left\{ \frac{C M(1)}{2}, \frac{1}{\alpha} \right\} - \frac{\lambda K C^a}{\alpha} < 0,
\]
we obtain that
\[
b_k = \max \{ \tilde{I}_\lambda(v) : v \in \mathcal{Y}_k, \|v\|_W = \delta_k \} < 0.
\]
If necessary, we can change \( k_0 \) to a larger value, so that \( \rho_k > \delta_k > 0 \) for all \( k \geq k_0 \).

(D3): Because \( \mathcal{Y}_k \cap \mathcal{Z}_k \neq \emptyset \) and 0 < \( \delta_k < \rho_k \), we have \( d_k \leq b_k < 0 \) for all \( k \geq k_0 \). Let us denote
\[
\tilde{\delta}_k = \sup \left\{ \int_{\mathbb{R}^N} \rho(x) |v(x)| \, dx : v \in \mathcal{Z}_k, \|v\|_W \leq 1 \right\}.
\]
Then, it is easy to verify that \( \tilde{\theta}_k \to 0 \) as \( k \to \infty \)(see [41]). For any \( v \in \mathcal{W}_k \) with \( \|v\|_{\mathcal{W}} = 1 \) and \( 0 < t < \rho_k \), we have

\[
\tilde{I}_\lambda(tv) = \frac{1}{2} \mathcal{M} \left( \int_{\mathbb{R}^N} \phi(|\nabla tv|^2) \, dx \right) + \frac{1}{\alpha} \int_{\mathbb{R}^N} |tv|^\alpha \, dx - \lambda \int_{\mathbb{R}^N} H(x, tv) \, dx \geq -\lambda \int_{\mathbb{R}^N} \rho(x) |tv| + \frac{\sigma(x)}{s} |tv|^s \, dx \geq -\lambda \rho_k \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{s} \theta_k - \lambda \rho_k \tilde{\theta}_k.
\]

Therefore, we deduce that for any \( (\theta_k) \), we have

\[
d_k \geq -\lambda \rho_k \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{s} \theta_k - \lambda \rho_k \tilde{\theta}_k.
\]

Because \( \theta_k \to 0 \) and \( \tilde{\theta}_k \to 0 \) as \( k \to \infty \), we conclude that \( \lim_{k \to \infty} d_k = 0 \).

\( \text{D4)} \): Let \( v \in \mathcal{W} \) and \( \|v\|_{\mathcal{W}} \geq 1 \). We set \( \Omega_1 := \{ x \in \mathbb{R}^N : |v(x)| \leq t_0 \} \), \( \Omega_2 := \{ x \in \mathbb{R}^N : t_0 \leq |v(x)| \leq 2t_0 \} \), and \( \Omega_3 := \{ x \in \mathbb{R}^N : 2t_0 \leq |v(x)| \} \), where \( t_0 \) is given in Lemma 8.

From the relation (2) and the conditions of \( \chi \), we have

\[
\tilde{I}_\lambda(v) = \frac{1}{2} \mathcal{M} \left( \int_{\mathbb{R}^N} \phi(|\nabla v|^2) \, dx \right) + \frac{1}{\alpha} \int_{\mathbb{R}^N} |v|^\alpha \, dx - \lambda \int_{\mathbb{R}^N} H(x, v) \, dx \geq \min \left\{ \frac{m_0 \xi}{2^\alpha} \frac{1}{\alpha}, \lambda \int_{\Omega_1} H(x, v) \, dx - \lambda \int_{\Omega_2} \chi(v) H(x, v) + (1 - \chi(v)) \xi |v|^\alpha \, dx - \lambda \int_{\Omega_3} \chi |v|^\alpha \, dx \right\} \geq \min \left\{ \frac{m_0 \xi}{2^\alpha} \frac{1}{\alpha}, \lambda \int_{\Omega_1 \cup \Omega_2} H(x, v) \, dx - \lambda \int_{\Omega_1} \chi |v|^\alpha \, dx \right\} \geq \min \left\{ \frac{m_0 \xi}{2^\alpha} \frac{1}{\alpha}, \lambda \int_{\Omega_1 \cup \Omega_2} \rho(x) |v| \, dx - \lambda \int_{\Omega_1 \cup \Omega_2} \sigma(x) |v|^s \, dx - \lambda \int_{\Omega_1 \cup \Omega_2} \chi |v|^\alpha \, dx \right\} \geq \min \left\{ \frac{m_0 \xi}{2^\alpha} \frac{1}{\alpha}, \lambda \int_{\Omega_1 \cup \Omega_2} \rho |v| \chi \, dx - \lambda \left( \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{s} + \xi \right) \int_{\mathbb{R}^N} |v|^\alpha \, dx \right\} \geq \min \left\{ \frac{m_0 \xi}{2^\alpha} \frac{1}{\alpha}, \lambda \int_{\Omega_1 \cup \Omega_2} \rho |v| \chi \, dx - \lambda \left( \frac{2c_0 \rho}{s} + \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{s} + \xi \right) \|v\|_{\mathcal{W}} \right\}.
\]

Therefore we deduce that for any \( \lambda \in \left( 0, \min \left\{ \frac{m_0 \xi}{2^\alpha} \frac{1}{\alpha}, \frac{2c_0 \rho}{s} + \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{s} + \xi \right\} \right)^{-1} \),

the functional \( \tilde{I}_\lambda \) is coercive, that is, \( \tilde{I}_\lambda(v) \to \infty \) as \( \|v\|_{\mathcal{W}} \to \infty \) and thus is bounded from below on \( \mathcal{W}_k \).

By the analogous arguments as those of Theorem 3 in [33] with Lemma 3, we infer that the functional \( \tilde{\Psi} : \mathcal{W} \to \mathcal{W}^*_s \), defined by

\[
\langle \tilde{\Psi}(v), \varphi \rangle = \int_{\mathbb{R}^N} \tilde{h}(x, v) \varphi \, dx \quad \text{for any} \quad \varphi \in \mathcal{W}^*_s,
\]

is compact in \( \mathcal{W}_k \). Then it follows from the coercivity of \( \tilde{I}_\lambda \) that the functional \( \tilde{I}_\lambda \) satisfies the (PS)-condition. Because \( \mathcal{W}_k \) is a reflexive Banach space, the proof is carried out by the same argument as in [41] (Lemma 3.12).

Consequently, all conditions of Proposition 2 are fulfilled, and hence for \( \lambda \in \Gamma \) we have a sequence \( c_n < 0 \) for \( \tilde{I}_\lambda \) satisfying \( c_n \to 0 \) when \( n \) goes to \( \infty \). Then for any \( v_n \in \mathcal{W} \) satisfying \( \tilde{I}_\lambda(v_n) = c_n \) and \( \tilde{I}'_\lambda(v_n) = 0 \), the sequence \( \{ v_n \} \) is a (PS)-sequence of \( \tilde{I}_\lambda(v) \) and \( \{ v_n \} \) admits a convergent subsequence. Thus, up to a subsequence, still denoted by \( \{ v_n \} \), one has \( v_n \to v \) in \( \mathcal{W}_k \) as \( n \to \infty \). Lemmas 7 and 8 imply that 0 is the only critical point with 0 energy and the subsequence \( \{ v_n \} \) has to converge.
to 0 in $W_t$ so $\|v_n\|_{L^r(R^N)} \to 0$ as $n \to \infty$ for any $t$ with $\alpha \leq t \leq p^*$. According to Proposition 1, any weak solution $v$ of our problem belongs to the space $L^\infty(\Omega)$ and there exist positive constants $C, \eta$ independent of $v$ such that

$$\|v\|_{L^\infty(R^N)} \leq C\|v\|_r^\eta.$$ 

From this fact, we know $\|v_n\|_{L^\infty(\Omega)} \to 0$ and thus by Lemma 8 again, we have $\|v_n\|_{L^\infty(\Omega)} \leq s_2$ for large $n$. Thus $\{v_n\}$ with large enough $n$ is a sequence of weak solutions of the problem (1). The proof is complete. 

3. Conclusions

In summary, we are concerned with the study of the existence, multiplicity and uniform estimates of infinitely many radially symmetric solutions for quasilinear elliptic equations whose $L^\infty$-norms converge to zero when the principal part $\phi$ has a different behavior near zero and at infinity. As we know, the dual fountain theorem is crucial to derive the existence of infinitely many small energy solutions for nonlinear elliptic equations of variational type. However the boundedness of solutions cannot be obtained from this variational method. Unfortunately, to the best of our knowledge, there are no results about $L^\infty$-bound for weak solutions of our problem with the Kirchhoff function $M$ (even if $M \equiv 1$). To overcome this difficulty, we use the De Giorgi’s iteration method and a truncated energy technique as the main tools for obtaining this result. This together with the modified functional method and the dual fountain theorem implies the existence of multiple small-energy radially symmetric solutions whose norms converge to zero in $L^\infty$ space. As far as we are aware, the present paper is the first to study the existence and regularity type results of this type for our problem with the different growth conditions on $\phi$.

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References

1. Chung, N.T. Existence of solutions for a class of Kirchhoff type problems in Orlicz-Sobolev spaces. Ann. Polon. Math. 2015, 113, 283–294. [CrossRef]
2. Figueiredo, G.; Santos, J.A. Existence of least energy nodal solution with two nodal domains for a generalized Kirchhoff problem in an Orlicz-Sobolev space. Math. Nachr. 2017, 290, 583–603. [CrossRef]
3. Kim, J.-M.; Kim, Y.-H.; Lee, J. Existence and multiplicity of solutions for Kirchhoff-Schrödinger type equations involving $p(x)$-Laplacian on the whole space. Nonlinear Anal. Real World Appl. 2019, 45, 620–649.
4. Pucci, P.; Xiang, M.; Zhang, B. Multiple solutions for nonhomogeneous Schrödinger–Kirchhoff type equations involving the fractional $p$-Laplacian in $\mathbb{R}^N$. Calc. Var. Partial Differ. Equ. 2015, 54, 2785–2806. [CrossRef]
5. Frehse, J.; Seregin, G.A. Regularity of solutions to variational problems of the deformation theory of plasticity with logarithmic hardening. Proc. St. Petersburg Math. Soc. 1998, 5, 184–222; English Translation: Am. Math. Soc. Trans. II 1999, 193, 127–152.
6. Fuchs, M.; Osmolovski, V. Variational integrals on Orlicz-Sobolev spaces. Z. Anal. Anwend. 1998, 17, 393–415. [CrossRef]
7. Fuchs, M.; Seregin, G. Variational methods for fluids of Prandtl–Eyring type and plastic materials with logarithmic hardening. Math. Methods Appl. Sci. 1999, 22, 317–351. [CrossRef]
8. Fukagai, N.; Narukawa, K. Nonlinear eigenvalue problem for a model equation of an elastic surface. Hiroshima Math. J. 1995, 25, 19–41. [CrossRef]
9. He, C.; Li, G. The existence of a nontrivial solution to the p & q-Laplace problem with nonlinearity asymptotic to \( u^{p-1} \) at infinity in \( \mathbb{R}^N \). *Nonlinear Anal.* **2008**, *68*, 1100–1119.

10. Ait-Mahiout, K.; Alves, C.O. Existence and multiplicity of solutions for a class of quasilinear problems in Orlicz-Sobolev spaces. *Complex Var. Elliptic Equ.* **2017**, *62*, 767–785. [CrossRef]

11. Alves, C.O.; da Silva, A.R. Multiplicity and concentration of positive solutions for a class of quasilinear problems through Orlicz-Sobolev space. *J. Phys. Math.* **2016**, *57*, 1, 11502. [CrossRef]

12. Bonanno, G.; Bisci, G.M.; Rădulescu, V. Existence of three solutions for a non-homogeneous Neumann problem through Orlicz-Sobolev spaces. *Nonlinear Anal.* **2011**, *74*, 4785–4795. [CrossRef]

13. Clément, P.; García-Huidobro, M.; Manásevich, R.; Schmitt, K. Mountain pass type solutions for quasilinear elliptic equations. *Calc. Var. Partial Differ. Equ.* **2000**, *11*, 1, 33–62. [CrossRef]

14. Fang, F.; Tan, Z. Existence of three solutions for quasilinear elliptic equations: An Orlicz-Sobolev space setting. *Medit. J. Math.* **2004**, *1*, 241–267. [CrossRef]

15. Fang, F.; Tan, Z. Existence and multiplicity of solutions for a class of quasilinear elliptic equations: An Orlicz-Sobolev space setting. *J. Math. Anal. Appl.* **2012**, *389*, 420–428. [CrossRef]

16. Fang, F.; Tan, Z. Existence of three solutions for quasilinear elliptic equations: An Orlicz-Sobolev space setting. *Acta Math. Appl. Sin. Engl. Ser.* **2017**, *33*, 287–296. [CrossRef]

17. Fukagai, N.; Narukawa, K. On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems. *Ann. Mat. Pura Appl.* **2007**, *186*, 539–564. [CrossRef]

18. Gossez, J.P. A strongly nonlinear elliptic problem in Orlicz-Sobolev spaces. *Proc. Symp. Pure Math.* **1986**, *45*, 455–462.

19. Mihăilescu, M.; Rădulescu, V. Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: An Orlicz-Sobolev space setting. *J. Math. Anal. Appl.* **2007**, *330*, 416–432. [CrossRef]

20. Azzollini, A. Minimum action solutions for a quasilinear equation. *Calc. Var. Partial Differ. Equ.* **2014**, *49*, 197–213. [CrossRef]

21. Azzollini, A.; d’Avenia, P.; Pomponio, A. Quasilinear elliptic equations in \( \mathbb{R}^N \) via variational methods and Orlicz-Sobolev embeddings. *Calc. Var. Partial Differ. Equ.* **2014**, *49*, 197–213. [CrossRef]

22. Candela, A.M.; Salvatore, A. Existence of radial bounded solutions for some quasilinear elliptic equations in \( \mathbb{R}^N \) via variational methods and Orlicz-Sobolev embeddings. *Nonlinear Anal.* **2020**, *191*, 111625. [CrossRef]

23. Badiale, M.; Pisani, L.; Rolando, S. Sum of weighted Lebesgue spaces and nonlinear elliptic equations. *NoDEA Nonlinear Differ. Equ. Appl.* **2011**, *18*, 369–405. [CrossRef]

24. Chiorfi, N.; Rădulescu, V.D. Standing waves solutions of a quasilinear degenerate Schrödinger equation with unbounded potential. *Electron. J. Qual. Theory Differ. Equ.* **2016**, *2016*, 1–12. [CrossRef]

25. Ambrosetti, A.; Rabinowitz, P. Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **1973**, *14*, 349–381. [CrossRef]

26. Kim, J.-M.; Kim, Y.-H.; Lee, J. Radially symmetric solutions for quasilinear elliptic equations involving nonhomogeneous operators in Orlicz-Sobolev space setting. *Acta Math. Sci.* **2019**, submitted.

27. Willem, M. *Minimax Theorems*; Birkhauser: Basel, Switzerland, 1996.

28. Wang, Z.-Q. Nonlinear boundary value problems with concave nonlinearities near the origin. *Nonlinear Differ. Equ. Appl.* **2001**, *8*, 15–33. [CrossRef]

29. Heinz, H.P. Free Lusternik-Schnirelman theory and the bifurcation diagrams of certain singular nonlinear problems. *J. Differ. Equ.* **1987**, *66*, 263–300. [CrossRef]

30. Choi, E.B.; Kim, J.-M.; Kim, Y.-H. Infinitely many solutions for equations of \( p(x) \)-Laplace type with the nonlinear Neumann boundary condition. *Proc. Roy. Soc. Edinb. Sect. A* **2018**, *148*, 1–31. [CrossRef]

31. Guo, Z. Elliptic equations with indefinite concave nonlinearities near the origin. *J. Math. Anal. Appl.* **2010**, *367*, 273–277. [CrossRef]

32. Kim, Y.-H. Infinitely many small energy solutions for equations involving the fractional Laplacian in \( \mathbb{R}^N \). *J. Korean Math. Soc.* **2018**, *55*, 1269–1283.

33. Kim, J.-M.; Kim, Y.-H.; Lee, J. Multiplicity of small or large energy solutions for Kirchhoff-Schrödinger-type equations involving the fractional \( p \)-Laplacian in \( \mathbb{R}^N \). *Symmetry* **2018**, *10*, 436. [CrossRef]

34. Naimen, D. Existence of infinitely many solutions for nonlinear Neumann problems with indefinite coefficients. *Electron. J. Differ. Equ.* **2014**, *2014*, 1–12.

35. Tan, Z.; Fang, F. On superlinear \( p(x) \)-Laplacian problems without Ambrosetti and Rabinowitz condition. *Nonlinear Anal.* **2012**, *75*, 3902–3915. [CrossRef]
36. Dai, G.; Hao, R. Existence of solutions for a $p(x)$-Kirchhoff-type equation. *J. Math. Anal. Appl.* **2009**, *359*, 275–284. [CrossRef]

37. Teng, K. Multiple solutions for a class of fractional Schrödinger equations in $\mathbb{R}^N$. *Nonlinear Anal. Real World Appl.* **2015**, *21*, 76–86. [CrossRef]

38. Vergara, V.; Zacher, R. A priori bounds for degenerate and singular evolutionary partial integro-differential equations. *Nonlinear Anal.* **2010**, *73*, 3572–3585. [CrossRef]

39. Rao, M.M.; Ren, Z.D. *Theory of Orlicz Spaces*; Marcel Dekker, Inc.: New York, NY, USA, 1991.

40. Zhou, Y.; Wang, J.; Zhang, L. *Basic Theory of Fractional Differential Equations*, 2nd ed.; World Scientific Publishing Co. Pte. Ltd.: Singapore, 2017.

41. Miyagaki, O.H.; Hurtado, E.J.; Rodrigues, R.S. Existence and multiplicity of solutions for a class of elliptic equations without Ambrosetti-Rabinowitz type conditions. *J. Dyn. Diff. Equat.* **2018**, *30*, 405–432.

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