Relations among Open-loop Control Ability, Control Strategy Space and Closed-loop Performance for Linear Discrete-time Systems

Mingwang Zhao

Information Science and Engineering School, Wuhan University of Science and Technology, Wuhan, Hubei, 430081, China
Tel.: +86-27-68863897
Work supported by the National Natural Science Foundation of China (Grant No. 61273005)

Abstract
In this article, the definition on the control ability, and the relation between the open-loop control ability and the closed-loop performance are studied systematically. Firstly, to define and compare the state control ability, the normalization of the input variables and state variables in the different control plants or one controlled plant with the different system parameters are discussed. With the help of the normalization, the control ability with the unit input constraint (input amplitude limited) can be defined. Finally, a theorem on the relations among the open-loop control ability, the control strategy space (i.e., the solution space of the input variable for control problems), and the closed-loop time performance is purposed and proven. Based on that, the conclusion that it is necessary to optimize the control ability for the practical engineering problems can be got.

Keywords: control ability, controllability region, closed-loop performance, time-optimal control, discrete-time systems, state controllability

1. Introduction
Putting forward the concept and criterion on the state controllability of the dynamical systems in 1960’s by R. Kalman, et al., [10] initiated a new era for control theory. As we known, the concept can reveal deeply the possibility controlling the state variable by the input variables and impels
us to understand and control well the dynamical systems. Therefore, the concept became one of the most important concept to support the 60 years development of the control theory.

It is a pity that the controllability concept is a qualitative concept with two-value logic and the dynamical systems is distinguished as only two classes of systems, controllable systems or uncontrollable systems, according to the corresponding controllability criterion. The concept and criterion could not tell us the control ability and control efficiency of the input variable to the state variables, and the quantitative concept and analysis method on the control ability and control efficiency are failed to establish. In fact, the quantitative concept and analysis method are very important for the control theory and engineering and many engineering problems are dying to these concept and method. For example, evaluating the control ability and control efficiency of the controlled plants can help us to solve the following important problems:

1) how to choose the controlled plant (e.g., DC or AC motor?), to choose the input variables (e.g., power supply of main circuit or excitation circuit for DC motor), to place the location of the actuators in larger mechanical system (e.g., mechanical cantilever, bridge, solar panels, etc), for maximum the control ability of the open-loop systems.

2) how to design and optimize the structure and technical parameters of the open-loop plants to get the more control ability and then to make designing and implementing the closed-loop controller easily.

3) how to determine the leader, the sub-leaders, and the connections between the nodes in the networked control systems and formation system.

4) how to determine reasonably the expected target state or state trace, the control horizon and the optimization horizon for these optimal control problems, adaptive control problems, predictive control problems, and the receding-horizon control (RHC) problems.

To summarize above, defining, quantifying and optimizing the control ability are with the very greater signification for the control theory and engineering.

In this paper, the definition on the control ability is studied systematically. Firstly, to define and compare the state control ability, the normalization of the input variables and state variables in the different control plants or one controlled plant with the different system parameters are discussed. With the help of the normalization, the control ability with the unit input variables can be defined. Finally, a theorem on the relations among the open-
loop control ability, the control strategy space (i.e., the solution space of the input variable for control problems), and the closed-loop time performance is purposed and proven. Based on that, the conclusion that it is necessary to optimize the control ability for the practical engineering problems can be got.

2. Normalization of the Variables for Comparing Control Ability

In this paper, the linear discrete-time systems (LDTs) is as a sample form studying the state control ability and the analysis method and the results can be generalized conveniently to other classes of dynamical systems. In general, the LDTs can be formulated as follows:

\[
x_{k+1} = Ax_k + Bu_k, \quad x_k \in \mathbb{R}^n, \quad u_k \in \mathbb{R}^r,
\]

where \(x_k\) and \(u_k\) are the state variable and input variable, respectively, and matrices \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times r}\) are the state matrix and input matrix, respectively, in the system models \([9], [3]\). To investigate the controllability of the linear dynamic systems (1), the controllability matrix and the controllability Grammian matrix can be defined as follows

\[
P_N = \begin{bmatrix} B, AB, \ldots, A^{N-1}B \end{bmatrix}
\]

\[
G_N = \sum_{i=0}^{N-1} A^i B (A^i B)^T
\]

That the rank of the matrix \(P_N/G_N\) is \(n\), the dimension of the state space the systems \([11]\) is the well-known criterion on the state controllability.

In papers \([15], [5], [13],\) and \([8]\), the determinant value \(\text{det} (G_N)\) and the minimum eigenvalue \(\lambda_{\text{min}} (G_N)\) of the controllability Grammian matrix \(G_N\) can be used to quantify the control ability of the input variable to the state space, and then be chosen as the objective function for optimizing and promoting the control ability of the linear dynamical systems. Due to lack of the analytical computing of the determinant \(\text{det} (G_N)\) and eigenvalue \(\lambda_{\text{min}} (G_N)\), these optimizing problems for the control ability are solved very difficulty, and few achievements about that were made. Out of the need of the practical control engineering, quantifying and optimizing the control ability are key problems in control theory and engineering fields.

To study rationally the control ability of the input variables to the state variables in two different dynamical systems, it is necessary to normalize the
input variable and state variables. Based on the normalization, the control ability can be defined and discussed in detail.

2.1. Normalization of the Input Variables and State Variables

In two different practical controlled plants, the physical dimensions, scales, value ranges of the input variables and state variables are different. Comparing rationally the control ability of these different practical plants, or these different input variables, firstly, the input variables and the state variable must be normalized according to the practical control problems. For example, to compare the control ability between the two different input variables in one controlled plant or two different controlled plants, the physical dimensions, scales, value ranges are necessary to be adjusted as proper compatible values with some rationalness. Similarly, to compare the controlled ability between the two different state variables, the dimensions, scales, ranges are also necessary to be adjusted as proper compatible values. Next, two examples are discussed for showing these adjustment and normalization.

1) If only one input variable can be used to be designed the speed controller of a practical DC motor, which voltage variable, the power supply of the main circuit or excitation circuit, is chosen as that for the maximum control ability? The value ranges of these voltage variables and the ratios between the voltage variables and the speed variable of the motor must be adjusted to be with uniformity. Based on this, comparing with the different input variables is with rationalness and signification.

2) Which motor, DC or AC motor, can be determined to be used to the some electric speed control system for the maximum control ability? The value ranges of the input variables and the state variables, the ratios between the input variables and the speed variable, and the power of the electric energy of the two motors, must be adjusted to be with uniformity. Based on this, comparing with the different controlled plants is with rationalness and signification.

2.2. Normalization of the amplitude and Energy of the Input Variables

The so-called state control ability is indeed the ability controlling the state variables by the input variables. The basis for comparison is the normalization of the input variables. In fact, the input variables of the most practical controlled plants are with some constraints, are bounded, or with saturation element [1], [6], [7]. For example, the power supply voltage variables as the input variable for the DC or AC motor are bounded, and the fuel carried or
the total energy wasted in the rocket is with some constraints. Therefore, based on these bounded values and constraints, the input variables can be normalized.

In control theory and engineering field, the most common bounded and constraint cases of the input variable vector \( u_k \) can be summarized as the two following cases

\[
U_{a,p} = \left\{ u_k : \| u_k \|_p = \left( \sum_{i=0}^{r} |u_{k,i}|^p dt \right)^{1/p} \leq U \right\} \tag{4}
\]

\[
U_{t,p} = \left\{ U_N : \| U_N \|_p = \left( \sum_{k=0}^{N-1} \| u_k \|_p^p dt \right)^{1/p} \leq U \right\} \tag{5}
\]

where \( r \) and \( u_{k,i} \) are the input variable numbers and the \( i \)-th input variable of the multi-input systems, respectively; \( U_N = [u_{0,T}^T, u_{1,T}^T, \ldots, u_{N-1,T}^T]^T \). The constraints with \( p = 1, 2, 3 \) are respectively the amplitude, fuel, and energy bounded. The condition (4) is for bounding the input variable in the \( k \)-time sampling, and the condition (5) is for constraining the total waste of the input in a control period \([0, N]\). In fact, for the single-input systems, the constraints (4) are a same constraint as

\[
|u_k| \leq U, \ \forall k \geq 0 \tag{6}
\]

In practical control engineering problems, the most common constraints are as follows

\[
\| u_k \|_{\infty} \leq U, \ \forall k \geq 0 \tag{7}
\]

\[
\| U_N \|_1 \leq U \tag{8}
\]

\[
\| U_N \|_2 \leq U \tag{9}
\]

These 3 constraints are bounded on the amplitude, total fuel, and total energy of the input variables respectively. Studying the control ability in this paper will be carried out for these 3 constraints. For comparing conveniently the control ability, the bounded value \( U \) is chosen as 1, and then these 3 constraints can be called as the unit input constraint, unit total fuel constraint, and unit total energy constraint.
3. The Definitions of the Controllability Region

Similar to the definition of the state controllability region (a.k.a. “recover region”) for the input-saturated linear systems in papers [1], [6], [7], the state controllability regions of the LDTSs with the input constraints are defined as follows

$$R_a(N) = \left\{ x : x = \sum_{k=0}^{N-1} A^{-k-1} B u_k \quad \forall u_k \in U_s \right\}$$  \hspace{1cm} (10)

where * indicate the case of the input constraints in Eqs. (4) and (5). For the 3 most common constraints in Eqs. (7), (8), and (9), the controllability regions, i.e., $R_{a,\infty}(N)$, $R_{t,1}(N)$, and $R_{t,2}(N)$, are the biggest range of the controllable state with the unit input constraint, unit total fuel constraint, and unit total energy constraint, respectively.

The state controllability region $R_a(N)$ can be regarded as a convex geometry in $n$-dimensional space. Region $R_{a,\infty}(N)$ is a parallel polyhedron and can be regarded as a zonotope [6], [7], [16], region $R_{t,2}(N)$ is an ellipsoid (i.e., so called ”controllability ellipsoid” ) [4], [11], [14], [12], [2], but region $R_{t,1}(N)$ is a rhombohedral. In the following sections, the controllability with the 3 controllability regions are defined and analysis.

4. Control Ability under the Unit Input Constraint

4.1. The properties of the controllability Region $R_{a,\infty}(N)$

As stated in papers [6], [7], [16], the state controllability region $R_{a,\infty}(N)$ under the unit input constraint is a convex geometry, can be regard as a parallel polyhedron or a special zonotope. In fact, the region $R_{a,\infty}(N)$ is surrounded by a series of vertices, edges, 2-dimensional faces, 3-dimensional faces, .... All vertices, edges, $i$-dimensional faces ($i = 2, n-1$) construct the bound of the region $R_{a,\infty}(N)$.

Some properties about the vertices, shape and size of the region can be summarized as follows [6], [7], [16].

**Property 1.** The all vertice of the controllability region $R_{a,\infty}(N)$ can be computed as follows

$$\text{Ver} \left( R_{a,\infty}(N) \right) \left\{ x \mid x = \sum_{i=0}^{N-1} \text{sgn} \left( d^T A^{-i} B \right) A^{-i} B, \forall d \in \mathbb{R}^n \right\}$$ \hspace{1cm} (11)
Based on the vertex produced by Eq. (11), the all edges and $i$-dimensinal faces ($i = 2, n-1$) can be produced recursively.

**Property 2.** If the LDTSs (1) is the full controllable, when $N \geq n$ the controllability region $R_{a,\infty}(N)$ is a $n$-dimensional geometry, and then for any $N_1 < N_2$, we have

$$R_{a,\infty}(N_1) \subset R_{a,\infty}(N_2) \text{ and } \partial R_{a,\infty}(N_1) \cap \partial R_{a,\infty}(N_2) = \phi$$

(12)

that is, the geometry $R_{a,\infty}$ is strictly monotonic expansion, where $\partial R$ is the bound of the geometry $R$.

If the systems is not the full controllable, when $N \geq n$ the region $R_{a,\infty}(N)$ is a $n_c$-dimensional geometry, and then for any $N_1 < N_2$, we have

$$R_{a,\infty}(N_1) \subseteq R_{a,\infty}(N_2) \text{ and } \partial R_{a,\infty}(N_1) \cap \partial R_{a,\infty}(N_2) \neq \phi$$

(13)

that is, the geometry $R_{a,\infty}$ is monotonic expansion, where $n_c$ the controllability index, that is, $n_c = \text{rank}P_N$.

In the next discussion, the systems are assumed always as a controllable systems and the geometry $R_{a,\infty}$ is a $n$-dimensional zonotope.

4.2. The Definition of the Control Ability

As we know, the bigger of the controllability region $R_{a,\infty}$ is, the more the controllable states in the state space are, and then the stronger the control ability of the dynamical systems can be regarded as. In the next subsection, it will proven that for the control problem stabilizing an initial state to the original of the state space, the bigger of the controllability region, the bigger the solution space of the input variable is, and then the better the some closed-loop control performance is. Thus, the size of the controllability region $R_{a,\infty}$ is used to define and describe the control ability.

Next, 3 equivalent definitions on the stronger control ability between the two differential controlled plants, or of one controlled plants with two sets of systems parameters are purposed as follows.

**Definition 1.** For the given controllability regions $R_{a,\infty}^{(1)}(N)$ and $R_{c,\infty}^{(2)}(N)$ of the two LDTSs $\Sigma_1\Sigma_2$, if the given $x_0 \in \partial R_{a,\infty}^{(1)}(N_1) \cap \partial R_{c,\infty}^{(2)}(N_2)$ and $N_1 \leq N_2$, the control ability of the systems $\Sigma_1$ at state $x_0$ is stronger than the systems $\Sigma_2$. If the control ability of the systems $\Sigma_1$ at all state in $\partial R_{a,\infty}^{(1)}(N_1) \cap \partial R_{c,\infty}^{(2)}(N_2)$ is stronger than the systems $\Sigma_1$, the control ability of the systems $\Sigma_1$ is fully stronger than the systems $\Sigma_2$. 

7
Definition 2. For any finite number $N$, if the two controllability regions $R_{a,\infty}^{(1)}(N)$ and $R_{c,\infty}^{(2)}(N)$ of the LDTSs $\Sigma_1\Sigma_2$ satisfy

$$R_{a,\infty}^{(1)}(N) \supset R_{a,\infty}^{(2)}(N) \text{ and } \partial R_{a,\infty}^{(1)}(N) \cap \partial R_{a,\infty}^{(2)}(N) = \emptyset, \quad (14)$$

the control ability of the systems $\Sigma_1$ is fully stronger than the systems $\Sigma_2$. if $R_{a,\infty}^{(1)}(N)$ and $R_{c,\infty}^{(2)}(N)$ for any $N$ satisfy

$$R_{a,\infty}^{(1)}(N) \supseteq R_{a,\infty}^{(2)}(N) \text{ and } \partial R_{a,\infty}^{(1)}(N) \cap \partial R_{a,\infty}^{(2)}(N) \neq \emptyset, \quad (15)$$

the control ability of the systems $\Sigma_1$ is not weaker than the systems $\Sigma_2$.

According to the above definitions, the control ability of the dynamical systems can be compared for many control engineering problems as stated above.

5. Theorem on Relation between the Open-loop Control Ability and the Closed-loop Performances

In this section the control ability for the unit input constraint is discussed in detail and the results can be generalized convenient to the other constraints of the input variables. Before the discussion, a property about the time-optimal property is stated as follows.

Property 3. If the LDTSs $\Pi$ is the full controllable, if the initial state $x_0$ satisfy

$$x_0 \in R_{a,\infty}(N) R_{a,\infty}(N - 1) \quad (16)$$

the time waste of the time-optimal control problem for stabilizing the state $x_0$ to the original of the state space under the input amplitude constraint, is $N$, that is, the fewest control sampling number is $N$.

Based on the definition of the control ability and above properties, a theorem on the relations among the open-loop control ability, the solution space of the input variables, and the Closed-loop Performances are purposed and proven as follows.
Theorem 1. It is assumed that two linear discrete-time systems \( \Sigma_1 \) and \( \Sigma_2 \) are controllable, and their controllability regions are \( R^{(1)}_{a,\infty}(N) \) for \( \Sigma_1 \) and \( R^{(2)}_{a,\infty}(N) \) for \( \Sigma_2 \) respectively. If we have

\[
R^{(1)}_{a,\infty}(i) \subseteq R^{(2)}_{a,\infty}(i), \quad \forall i \leq N,
\]

for the control problem stabilizing the state \( x_0 \in R^{(1)}_{a,\infty}(N) \cap R^{(2)}_{a,\infty}(N) \) to the origin of the state space, the following conclusions hold under the input amplitude constraint.

1) The time waste of the time-optimal control for the system \( \Sigma_2 \) is not more than that of \( \Sigma_1 \), that is, there exist some control strategies with the less control time and the faster response speed for the system \( \Sigma_2 \).

2) There exist more control strategies for the system \( \Sigma_2 \), that is, the bigger the controllability region is, the bigger the solution space of the input variable for the control problems, the easier designing and implementing the control are.

Proof of Theorem 1. First, according to the definition of the controllability region, for the state controllable systems \( \Sigma_1 \) and \( \Sigma_2 \), we have,

\[
R^{(j)}_{a,\infty}(i) \subset R^{(j)}_{a,\infty}(i), \quad j = 1, 2; \quad i = 1, 2, \ldots, N - 1
\]  

(18)

So, by Eq.(17) and Eq.(18), we know, for any state \( x_0 \in R^{(1)}_{a,\infty}(N) \setminus R^{(1)}_{a,\infty}(1) \), there must exist two finite positive number \( k_1 \) and \( k_2 (k_2 \leq k_1 \leq N) \) satisfied

\[
x_0 \in \left\{ R^{(1)}_{a,\infty}(k_1) \setminus R^{(1)}_{a,\infty}(k_1 - 1) \right\} \cap R^{(2)}_{a,\infty}(k_2)
\]

(19)

\[
R^{(1)}_{a,\infty}(k_1) \subseteq R^{(2)}_{a,\infty}(k_2)
\]

(20)

Therefore, for controlling the system state variable from the given \( x_0 \) to the origin, the fewest sampling steps must be \( k_1 \) for the system \( \Sigma_1 \), but must less than or equal to \( k_2 \) for the system \( \Sigma_2 \). So, for the any state \( x_0 \in R^{(1)}_{a,\infty}(N) \setminus R^{(1)}_{a,\infty}(1) \), for \( k_2 \leq k_1 \), the time waste of the time-optimal control for the system \( \Sigma_2 \) is not more than that of the system \( \Sigma_1 \).

In addition, for any state \( x_0 \in R^{(1)}_{a,\infty}(1) \), the fewest control times (sampling steps) are 1 for both of the two systems, that is, the time waste of the time-optimal control for the system \( \Sigma_2 \) is not more than that of the system \( \Sigma_1 \).

In summary, for any state \( x_0 \in R^{(1)}_{a,\infty}(N) \cap R^{(2)}_{a,\infty}(N) \), the conclusion 1) in Theorem 1 holds.
(2) Denoting the input sequence and its solution space for controlling the state $x_0$ to the origin for systems $\Sigma_i$ as $u_{0,N-1}^{(i)}(x_0)$ and $U_N^{(i)}(x_0)$, respectively. Without loss of the generality, it is assumed that $r = 1$, that is, the systems are single-input systems. So, by Eq. (17) and Eq. (18), we know, for any state $x_0 \in R_0^{(1)}(N)$, there must exist two finite positive number $k_1$ and $k_2$ ($k_2 < k_1 \leq N$) satisfied one of the following conditions

\begin{align}
\text{a) } x_0 & \in \partial R_0^{(1)}(k_1) \cap (R_0^{(2)}(k_1) \setminus \bar{R}_0^{(2)}(k_1 - 1)) \\
\text{b) } x_0 & \in (\partial R_0^{(1)}(k_1) \setminus \bar{R}_0^{(1)}(k_1 - 1)) \cap (R_0^{(2)}(k_1) \setminus \bar{R}_0^{(2)}(k_1 - 1)) \\
\text{c) } x_0 & \in (\partial R_0^{(1)}(k_1) \setminus \bar{R}_0^{(1)}(k_1 - 1)) \cap (\partial R_0^{(2)}(k_2) \setminus \bar{R}_0^{(2)}(k_2 - 1)) \\
\text{d) } x_0 & \in \partial R_0^{(1)}(1)
\end{align}

where $\bar{R} = R \setminus \partial R$. For controlling the state $x_0$ to the origin, the input sequence $u_{0,N-1}^{(i)}(x_0)$ must satisfies the following state equation.

$$x_0 = [A^{-1}B, A^{-2}B, \ldots, A^{-N}B] P_N \times u_{0,N-1}^{(i)}(x_0)$$

Then, corresponding to the above 4 conditions, the dimension of the solution space $U_N^{(i)}(x_0)$ of the Eq. (25) are as follows

\begin{align}
1) \dim U_N^{(1)}(x_0) & = N - k_1 \leq \dim U_N^{(2)}(x_0) \\
2) \dim U_N^{(1)}(x_0) & = \dim U_N^{(2)}(x_0) = N - k_1 + 1 \\
3) \dim U_N^{(1)}(x_0) & = N - k_1 + 1 < N - k_2 + 1 = \dim U_N^{(2)}(x_0) \\
4) \dim U_N^{(1)}(x_0) & = \dim U_N^{(2)}(x_0) = N - 1
\end{align}

Hence, we have

$$\dim U_N^{(1)}(x_0) \leq \dim U_N^{(2)}(x_0)$$

So, considered that the higher the space dimension is and the more the number of the states in the state space is, for any state $x_0 \in R_0^{(2)}(N)$, the solution space of the system $\Sigma_2$ is larger than that of the system $\Sigma_1$, and then the systems $\Sigma_2$ for controlling the state to $x_0$ to the original will be with more control strategies than the system $\Sigma_1$. 

By Theorem 1, we have the following discussions:

(1) Not only the time waste can be reduced by promoting the control ability, but also other closed-loop performance related the control time waste can be improved.
(2) In fact, that the solution space of the input variable for the control problems is bigger implies that the control strategies in the solution space are with better robustness, and then the closed-loop control systems is also with better robustness.

Therefore, optimizing the open-loop control ability are with very greater signification for these practical control engineering problems and it’s very necessary to optimize the control ability for the practical engineering problems. To optimize the control ability, it is necessary to establish the quantify analysis and computing method for the control ability. Paper [16] prove an analytical computing equation for the volume of the controllability region and deconstruct the volume equation to construct the some analytical factors about the shape of the controllability region. Based on these analytical expressions of the volume and shape factor, the optimizing and promoting methods for the control ability can be set up conveniently.

6. Conclusions

In this article, the definition on the control ability, and the relation between the open-loop control ability and the closed-loop performance are studied systematically. Firstly, to define and compare the state control ability, the normalization of the input variables and state variables in the different control plants or one controlled plant with the different system parameters are discussed. With the help of the normalization, the control ability with the unit input constraint (input amplitude limited) can be defined. Finally, a theorem on the relations among the open-loop control ability, the control strategy space (i.e., the solution space of the input variable for control problems), and the closed-loop time performance is purposed and proven. Therefore, it is necessary to optimize the control ability for the practical engineering problems. Based on the results in paper [16], the optimizing and promoting methods for the control ability can be set up conveniently.

References

[1] D. Bernstein, A. Michel, A chronological bibliography on saturating actuators, Inter. J. of Robust and Nonlinear Control 5 (1995) 375–380.

[2] S.L. Canfield, R. Nkhumise, Controllability ellipse to evaluate performance of mobile manipulators for manufacturing tasks, Journal of Mechanisms and Robotics 9 (2017).
[3] C.T. Chen, Linear system theory and design, Oxford University Press, Inc. New York, NY, USA, 3rd edition, 1998.

[4] G.E. Dullerud, F. Paganini, A Course in Robust Control Theory: A Convex Approach, Springer, 2000.

[5] D. Georges, The use of observability and controllability gramians or functions for optimal sensor and actuator location in finite-dimensional systems, in: Proc. of IEEE Conf. on Decision and Control, New Orleans, LA, USA, p. 33193324.

[6] T. Hu, Z. Lin, Control systems with actuator saturation: analysis and design, Birkhäuser, Boston, 2001.

[7] T. Hu, Z. Lin, L. Qiu., An explicit description of the null controllable regions of linear systems with saturating actuators, Systems & Control Letters 47 (2002) 65–78.

[8] U. Ilkturk, Observability Methods in Sensor Scheduling, Ph.D. thesis, ARIZONA STATE UNIVERSITY, 2015.

[9] T. Kailath, Linear systems, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[10] R.E. Kalman, Y.C. Ho, K.S. Narendra, Controllability of dynamical systems, Contributions to Differential Equations 1 (1963) 189–213.

[11] A. Kurzhanskiy, P. Varaiya, Ellipsoidal techniques for reachability analysis of discrete-time linear systems, IEEE Trans. on Automatic Control 52 (2007) 26–38.

[12] R.M. Nkhumise, Controllability Ellipse – a Method to Eavluate Performance of Mobile Manipulators Applied to Welding, Master’s thesis, Tennessee Technological University, 2016.

[13] F. Pasqualetti, S. Zampieri, F. Bullo, Controllability metrics, limitations and algorithms for complex networks, IEEE Trans. on Control of Network Systems 1 (2014) 40–52.

[14] B. Polyak, S. Nazin, M. Khlebnikov, The invariant ellipsoid technique for analysis and design of linear control systems, in: Advances in Mechanics: Dynamics and Control: Proceedings of the 14th International
[15] W. VanderVelde, C. Carignan, A dynamic measure of controllability and observability for the placement of actuators and sensors on large space structures, Technical Report, NASA-CR-168520, SSL-2-82, 1982.

[16] M.W. Zhao, Exact volume of zonotopes generated by a matrix pair, arxiv.org:3123388 (2020).