Stroh formalism in analysis of skew-symmetric and symmetric weight functions for interfacial cracks

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Abstract
The focus of the paper is on the analysis of skew-symmetric weight functions for interfacial cracks in two-dimensional anisotropic solids. It is shown that the Stroh formalism proves to be an efficient tool for this challenging task. Conventionally, the weight functions, both symmetric and skew-symmetric, can be identified as non-trivial singular solutions of a homogeneous boundary-value problem for a solid with a crack. For a semi-infinite crack, the problem can be reduced to solving a matrix Wiener–Hopf functional equation. Instead, the Stroh matrix representation of displacements and tractions, combined with a Riemann–Hilbert formulation, is used to obtain an algebraic eigenvalue problem, which is solved in a closed form. The proposed general method is applied to the case of a quasi-static semi-infinite crack propagating between two dissimilar orthotropic media: explicit expressions for the weight functions are evaluated and then used in the computation of the complex stress intensity factor corresponding to a general distribution of forces acting on the crack faces.

Keywords
Interfacial crack, Riemann–Hilbert problem, Stroh formalism, weight functions, stress intensity factor

1. Introduction
Evaluation of the coefficients in the asymptotic representations of the displacements and stress fields represents an important issue for vector problems of crack propagation in elastic materials [1, 2]. The explicit derivation of weight functions is fundamental for the evaluation of stress intensity factors corresponding to a general distribution of forces acting on the crack faces, as well as for the calculation of higher order coefficients in the asymptotic expressions of the fields. The latter are used in asymptotic models of incremental crack growth, and hence are essential for evaluation of the crack path and the analysis of fracture stability. The weight functions for several types of cracks in homogeneous elastic media, both in two and three dimensions, have been defined in the work by Bueckner [3,4].

In this paper, the term ‘symmetric’ load is associated with forces of the same magnitude applied on both crack faces in opposite directions, while the load generated by forces acting on both crack faces in the same direction is called ‘skew-symmetric’. For cracks in homogeneous elastic materials, in the two-dimensional setting, the skew-symmetric loading does not contribute to stress intensity factors, whereas it becomes relevant and...
must be accounted for in three-dimensional solids [3, 5]. The situation is different when the crack is placed at the interface between two dissimilar elastic materials: even for two-dimensional problems, the skew-symmetric loads generate a non-zero contribution to stress intensity factors [1, 6, 7]. In particular, for mode III interfacial cracks, the stress components do not oscillate, but a non-vanishing skew-symmetric component of the weight function still has to be accounted for refs. [7, 8]. In the case of isotropic media, the weight functions for semi-infinite cracks can be defined as singular non-trivial solutions of a homogeneous boundary-value problem with zero tractions on the crack faces but unbounded elastic energy [9]. For interfacial cracks between dissimilar isotropic elastic media, the weight functions are well discussed in the literature [6–8]. The problem is generally reduced to a functional equation of Wiener–Hopf type, and its solution gives the symmetric weight function matrix [10], while the skew-symmetric component is obtained by constructing the corresponding full-field singular solution of the elasticity boundary-value problem discussed in Piccolroaz et al. [7]. For interfacial cracks between anisotropic dissimilar elastic media, although symmetric weight functions have been derived by Gao [11, 12] and Ma and Chen [13], the results for skew-symmetric weight functions are not readily available.

In this paper, we illustrate a general procedure for calculating symmetric and skew-symmetric weight functions for semi-infinite two-dimensional interfacial cracks in anisotropic elastic bi-materials. It is shown that the challenging analysis of the matrix functional Wiener–Hopf equation can be replaced by solving a matrix eigenvalue problem deduced via an equivalent formulation, based on the Stroh representation of the displacement and stress fields [14].

Section 4 illustrates the proposed method for the case of a semi-infinite two-dimensional stationary crack between two dissimilar orthotropic materials under plane stress deformation. Similar to ref. [15], Stroh representations for displacements and stresses corresponding to this problem are explicitly calculated and used for deriving symmetric and skew-symmetric weight functions. In Section 5, both symmetric and skew-symmetric weight functions are utilised together with Betti’s formula, in order to evaluate the complex stress intensity factor for an interfacial crack in an orthotropic bi-material subject to non-symmetric loading. In a particular case of isotropic media, the obtained result is consistent with the stress intensity factor derived for a non-symmetric distribution of forces by Piccolroaz et al. [7].

Finally, in the appendix, the evaluated skew-symmetric weight functions are compared to those calculated using the full field singular solution of the plane elasticity problem, following the approach illustrated in ref. [7]. Perfect agreement between the expressions derived using two alternative formulations holds.

2. Interfacial cracks: preliminary results

Here we introduce the main notations and the mathematical framework of the model. Consider a quasi-static advance of a semi-infinite plane crack between two dissimilar anisotropic elastic materials under asymmetric loading applied to the crack faces; the geometry of the system is shown in Figure 1.

The loading is defined via tractions acting on the crack faces. Considering a Cartesian coordinate system with the origin at the crack tip (see Figure 1), traction components behind the crack tip are then defined as

\[
\sigma_{2j}^+(x_1,0^+) = p_j^+(x_1) \quad \text{for} \quad x_1 < 0, \quad j = 1, 2,
\]

where \(p_j^\pm(x_1)\) are given functions.

Since the load is assumed to be self-balanced, its resultant force and moment vectors are equal to zero. Moreover, we assume that the forces are applied outside a neighbourhood of the crack tip and vanish at infinity. The body forces are assumed to be zero. The symmetric and skew-symmetric parts of the loading are given by

\[
[p_j](x_1) = \frac{1}{2} \left( p_j^+(x_1) + p_j^-(x_1) \right), \quad [p_j^\pm](x_1) = p_j^\pm(x_1) - p_j^\mp(x_1), \quad j = 1, 2.
\]

The solutions in the form of functions which vanish at infinity and possess finite elastic energy are sought. Expressions for the stress field and displacements for a semi-infinite interfacial crack in anisotropic bi-materials have been obtained by means of Stroh formalism [14] by Suo [15]. This approach to the physical problem of the crack, which is used for the derivation of the weight functions in the present paper, is reported in Section 2.1.

In Section 2.2, the weight functions are defined as non-trivial singular solutions of a homogeneous elasticity problem for an interfacial crack with zero tractions on the crack faces but unbounded elastic energy, following the approach of Piccolroaz et al. [7].
Section 2.3 reports the application of the Betti formula to the physical fields and weight functions and the derivation of the fundamental identity in the Fourier space, discussed in detail refs. [9, 16, 17]. This integral identity will be used in the paper for the evaluation of the coefficients in the asymptotic representations of the stress field near the crack tip by means of a procedure based on the weight functions theory.

2.1. Stroh formalism in the analysis of interfacial cracks

Physical displacements and stress fields for an interfacial crack between two different anisotropic materials can be derived by means of the Lekhnitskii or Stroh approaches [14, 18]. Following the paper by Suo [15], we introduce the stress vectors $\mathbf{t}_j = (\sigma_{1j}, \sigma_{2j})^T$ together with the displacement $\mathbf{u} = (u_1, u_2)^T$. The constitutive relations for both elastic media occupying the upper and lower half-planes can be written using the Stroh formulation [14]

$$
\mathbf{t}_1 = \mathbf{Qu}_1 + \mathbf{Ru}_2, \quad (3)
$$

$$
\mathbf{t}_2 = \mathbf{R}^T\mathbf{u}_1 + \mathbf{T}\mathbf{u}_2, \quad (4)
$$

where the matrices $\mathbf{Q}, \mathbf{R}$ and $\mathbf{T}$ depend on the material constants. A semi-infinite static crack is placed at the interface between the two materials as illustrated in Figure 1. The derivative of the displacement $\mathbf{u}_1(x_1, x_2)$ and the traction $\mathbf{t}(x_1, x_2) = \mathbf{t}_2(x_1, x_2)$ can be written as

$$
\mathbf{t}(x_1, x_2) = \mathbf{B}g(z) + \mathbf{B}g(z) \quad (5)
$$

and

$$
\mathbf{u}_1(x_1, x_2) = \mathbf{A}g(z) + \mathbf{A}g(z), \quad (6)
$$

where $\mathbf{A}$ and $\mathbf{B}$ are constant matrices, $g(z)$ is an analytic vector function with components $g_j(x_1 + \mu_j x_2)$, and $\mu_j$ are complex numbers with positive imaginary parts. Using the analytic continuation argument [15], we introduce a single complex variable $z = x_1 + \mu x_2$ with $\text{Im} \mu > 0$. The connection between the elements of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{Q}, \mathbf{R}, \mathbf{T}$ is given by the following relations (see Ting [19], pp. 170, 171)

$$
(Q + (R + R)\mu + T\mu^2)\mathbf{A} = \mathbf{0}, \quad (7)
$$

$$
\mathbf{B} = (R + \mu T)\mathbf{A}. \quad (8)
$$

Thus, each column of $\mathbf{A}$ is a non-trivial solution of the eigensystem (7), while the eigenvalue $\mu$ is the root of the characteristic equation

$$
|Q + (R + R)\mu + T\mu^2| = 0. \quad (9)
$$

In turn, for each of the two phases, we introduce the Hermitian matrix $\mathbf{Y} = i\mathbf{AB}^{-1}$, which will be used in the text below. Further in the text, we shall use the superscripts (1) and (2) to denote the quantities related to the upper and lower half-planes, respectively.

Let us introduce the vector function $\mathbf{h}(z)$, such that

$$
\mathbf{h}(z) = \begin{cases} 
\frac{\mathbf{B}g(z)}{2}, & \text{Im} \, z \geq 0, \\
-\frac{\mathbf{B}g(z)}{2}, & \text{Im} \, z < 0.
\end{cases}
$$

Figure 1. Semi-infinite crack on a bi-material interface.
The expression (5), in the limit as \( x_2 \to 0^\pm \), leads to a non-homogeneous Riemann–Hilbert problem

\[
h^+(x_1) - h^-(x_1) = \tau(x_1), \quad x_1 \in \mathbb{R}. \tag{10}
\]

A crack advancing along the negative semi-axis \( x_1 < 0 \) is considered, the traction-free condition is imposed for \( x_1 < 0 \), while the continuity of the tractions is assumed at the interface ahead of the crack. The following equations are satisfied on the real axis [15]

\[
h^+(x_1) + \overline{H}^{-1}Hh^-(x_1) = \tau(x_1) \quad \text{for} \quad x_1 > 0, \tag{11}
\]

\[
h^+(x_1) + \overline{H}^{-1}Hh^-(x_1) = 0 \quad \text{for} \quad x_1 < 0. \tag{12}
\]

The detailed analysis of this problem is included in refs. [15, 20], and it shows that the stress and displacement fields do not have oscillations near the crack tip for the case when the matrix \( H = \mathbf{Y}^{(1)} + \mathbf{Y}^{(2)} \) is real, otherwise the stresses and displacements are characterised by the oscillatory behaviour near the crack tip. The branch cut for the function \( h(z) \) is assumed to be along the crack line \( x_1 < 0 \). Assuming that the stresses vanish at infinity, the solution of the homogeneous Riemann–Hilbert problem (12) can be written in the form

\[
Hw = e^{2i\varepsilon}Hw. \tag{13}
\]

The traction \( \tau \) ahead of the crack has the form

\[
\tau(x_1) = \frac{1}{\sqrt{2\pi x_1}} \text{Re} \left( K x_1^\varepsilon w \right), \tag{14}
\]

where \( K = K_I + iK_{II} \) is the complex stress intensity factor which includes both mode I and mode II contributions to the traction, \( \varepsilon \) is the bi-material parameter, and \( w \) is the eigenvector of (13).

For the case of a plane stress load, it has been shown in Suo [15] and Suo et al. [20] that the displacement jump \( [u] \) across a semi-infinite crack propagating along the negative semi-axis \( x_1 < 0 \) is given by

\[
[u](x_1) = \left( \frac{2(-x_1)}{\pi} \right)^{1/2} \frac{(H + H')}{\cosh \pi \varepsilon} \text{Re} \left( K(-x_1)^\varepsilon w \right). \tag{15}
\]

### 2.2. Weight functions: definition

Following the theory developed by Willis and Movchan [9], we define a vector function \( \mathbf{U} = (U_1, U_2)^T \) as a singular solution of the elasticity problem with zero tractions on the crack faces where the crack is placed along the positive semi-axis \( x_1 > 0 \). The traces of these functions on the plane containing the crack are known as the weight functions, and notations \([U]\) and \((U)\) will be used in the present paper to denote symmetric and skew-symmetric weight functions, respectively,

\[
[U](x_1) = U(x_1, x_2 = 0^+) - U(x_1, x_2 = 0^-), \tag{16}
\]

\[
(U)(x_1) = \frac{1}{2} (U(x_1, x_2 = 0^+) + U(x_1, x_2 = 0^-)). \tag{17}
\]

The traction vector \( \mathbf{S} = (\Sigma_1, \Sigma_2)^T \) associated with the singular solutions is continuous on the plane containing the crack and vanishes for \( x_1 > 0 \) (homogeneous boundary conditions are imposed). In practice, for singular solutions we impose the traction-free condition for \( x_1 > 0 \) and the continuity of traction at the interface for \( x_1 < 0 \)

\[
h^+(x_1) + \overline{H}^{-1}Hh^-(x_1) = 0 \quad \text{for} \quad x_1 > 0, \tag{18}
\]

\[
h^+(x_1) + \overline{H}^{-1}Hh^-(x_1) = \Sigma(x_1) \quad \text{for} \quad x_1 < 0. \tag{19}
\]
For the singular solution $\mathbf{h}(x_1)$, the branch cut is along the line $x_1 > 0$. Using the representation $\mathbf{h}(z) = vz^{-\frac{3}{2}+i\epsilon}$, we reduce (19) to the eigenvalue problem

$$\overline{\mathbf{H}} \mathbf{v} = e^{-2i\epsilon} \mathbf{H} \mathbf{v}.$$  \hspace{1cm} (20)

The bi-material matrix $\mathbf{H}$ is Hermitian, and the eigenvector corresponding to the parameter $-\varepsilon$ is $\mathbf{v} = \overline{\mathbf{w}}$ [15]. The singular traction $\mathbf{S}$ is sought in the form

$$\mathbf{S} = \frac{(-x_1)^{-\frac{3}{2}}}{\sqrt{2\pi}} \text{Re} \left( C \mathbf{w}(-x_1)^{i\epsilon} \right),$$  \hspace{1cm} (21)

where $C = C_I + iC_{II}$ is a complex constant representing both mode I and mode II contributions to the traction (21). The shear and normal opening modes for plane stress and plane strain problems are coupled. Two linearly independent vectors $\mathbf{S}$ and associated weight functions can be identified similar to refs. [7, 16].

### 2.3. Fundamental Betti identity

As mentioned above, $\mathbf{U}$ is discontinuous along the positive semi-axis $x_1 > 0$, whereas $\mathbf{u}$ is discontinuous along the negative semi-axis $x_1 < 0$. The field $\mathbf{S}$ is zero for $x_1 > 0$, whereas $\mathbf{T}$ is zero for $x_1 < 0$. For general non-symmetric loading applied to the crack faces, the physical traction acting at the interface on the entire $x_1$-axis can be written as

$$\sigma(x_1, 0+) = \mathbf{p}^+(x_1) + \mathbf{T}(x_1), \quad \sigma(x_1, 0-) = \mathbf{p}^-(x_1) + \mathbf{T}(x_1).$$  \hspace{1cm} (22)

According to the approach illustrated [9, 17], in order to derive explicit formulae for calculating the coefficients in the asymptotic representations of the stress fields near the crack tip, we apply Betti’s formula to the physical fields and to the weight functions. The following relations are obtained

$$\int_{x_2=0^+} \left\{ \mathbf{U}^T(x_1' - x_1, 0+) \mathcal{R} \sigma(x_1, 0+) - \mathbf{S}^T(x_1' - x_1, 0+) \mathcal{R} \mathbf{u}(x_1, 0+) \right\} dx_1 = 0,$$  \hspace{1cm} (23)

$$\int_{x_2=0^-} \left\{ \mathbf{U}^T(x_1' - x_1, 0-) \mathcal{R} \sigma(x_1, 0-) - \mathbf{S}^T(x_1' - x_1, 0-) \mathcal{R} \mathbf{u}(x_1, 0-) \right\} dx_1 = 0,$$  \hspace{1cm} (24)

where $\mathcal{R}$ is the rotation matrix,

$$\mathcal{R} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Subtracting (24) from (23) and using (22), we derive the following integral identity

$$\int_{x_2=0} \left\{ \mathbf{U}^T(x_1' - x_1, 0+) \mathcal{R} \mathbf{p}^+(x_1) + \mathbf{U}^T(x_1' - x_1, 0+) \mathcal{R} \mathbf{T}(x_1) - \mathbf{U}^T(x_1' - x_1, 0-) \mathcal{R} \mathbf{p}^-(x_1) - \mathbf{U}^T(x_1' - x_1, 0-) \mathcal{R} \mathbf{T}(x_1) \right\} dx_1 = 0.$$

Referring to (16) and (17), we deduce

$$\int_{x_2=0} \left\{ \mathbf{U}^T(x_1' - x_1, 0+) \mathcal{R} \mathbf{T}(x_1) - \mathbf{S}^T(x_1' - x_1, 0+) \mathcal{R} \mathbf{u}(x_1, 0+) \right\} dx_1$$

$$= - \int_{x_2=0} \left\{ \mathbf{U}^T(x_1' - x_1, 0+) \mathcal{R} \mathbf{p}^+(x_1) + \mathbf{S}^T(x_1' - x_1, 0+) \mathcal{R} \mathbf{p}^-(x_1) \right\} dx_1.$$  \hspace{1cm} (25)

We define the Fourier transform of the skew-symmetric weight function and associated traction $\mathbf{S}$ with respect to $x_1$ as

$$\hat{\mathbf{U}}^+(\xi) = \int_0^{+\infty} \mathbf{U}(x_1)e^{i\xi x_1} dx_1, \quad \hat{\mathbf{S}}^+(\xi) = \int_0^{+\infty} \mathbf{S}(x_1)e^{i\xi x_1} dx_1,$$  \hspace{1cm} (26)
where the superscript $^+$ indicates that $[\hat{U}]^+(\xi)$ is analytic in the upper half-plane $(\text{Im}\xi > 0)$ and the superscript $^-$ indicates that $\hat{\Sigma}^- (\xi)$ is analytic in the lower half-plane $(\text{Im}\xi < 0)$. The physical traction and the jump function are defined in such a way that the transforms $\hat{T}^+(\xi)$ and $[\hat{u}]^-(\xi)$ are analytic in the upper and lower half-planes, respectively. Applying the Fourier transform to (25), we deduce

$$[\hat{U}]^{+T} \mathcal{R} \hat{T}^+ - \hat{\Sigma}^{-T} \mathcal{R} [\hat{u}]^- = -[\hat{U}]^{+T} \mathcal{R} (p) - (\hat{U})^{+T} \mathcal{R} [\hat{p}], \quad \xi \in \mathbb{R}. \quad (27)$$

This identity, obtained by Willis and Movchan [9] and Piccolroaz et al. [17], relates transforms of the physical solutions to transforms of the weight functions, and it is used for evaluation of the stress identity factors.

### 2.4. Asymptotic representations of the fields and stress intensity factors

As $x_1 \to 0$, the physical traction (14) and the displacement jump across the crack face (15) can be written in a matrix form as

$$\tau(x_1) = \frac{x_1^{-1/2}}{2\sqrt{2\pi}} \mathcal{T}(x_1) \mathbf{K} + \frac{x_1^{1/2}}{2\sqrt{2\pi}} \mathcal{T}(x_1) \mathbf{J} + \frac{x_1^{3/2}}{2\sqrt{2\pi}} \mathcal{T}(x_1) \mathbf{L} + O(x_1^{5/2}),$$

$$[\mathbf{u}](x_1) = \frac{(-x_1)^{1/2}}{\sqrt{2\pi}} \mathcal{U}(x_1) \mathbf{K} + \frac{(-x_1)^{3/2}}{\sqrt{2\pi}} \mathcal{U}(x_1) \mathbf{J} + \frac{(-x_1)^{5/2}}{\sqrt{2\pi}} \mathcal{U}(x_1) \mathbf{L} + O((-x_1)^{7/2}),$$

where $\mathbf{K} = (K, \overline{K}), \mathbf{J} = (J, \overline{J}), \mathbf{L} = (L, \overline{L})$, and $J$ and $L$ are the higher order coefficients, defined in the same way as the stress intensity factor: $J = J_1 + J_{II}, L = L_1 + J_{II}$. Matrices $\mathcal{T}(x_1)$ and $\mathcal{U}(x_1)$ are given by

$$\mathcal{T}(x_1) = 2 (\mathbf{w}_1^x, \mathbf{w}_1^\nu), \quad \mathcal{U}(x_1) = \frac{2(H + \overline{H})}{\cosh \pi \varepsilon} \left( \frac{\mathbf{w}(-x_1)^{i\nu} \mathbf{w}(-x_1)^{-i\nu}}{1 + 2i\varepsilon}, \frac{\mathbf{w}(-x_1)^{i\nu} \mathbf{w}(-x_1)^{-i\nu}}{1 - 2i\varepsilon} \right).$$

In the next section, the general expressions for the Fourier transforms of symmetric and skew-symmetric weight functions are derived using the Stroh formulation as defined in Section 2.1, while in Section 4, these expressions will be specialised to the case of a crack between two orthotropic materials [15] and the general formula obtained by Piccolroaz et al. [7] will be applied to calculate the stress intensity factor in this case.

### 3. Symmetric and skew-symmetric weight functions for anisotropic bi-materials

Here we derive general expressions for symmetric and skew-symmetric weight functions defined in Section 2.2 as the jump $[\mathbf{U}]$ and the average $\langle \mathbf{U} \rangle$ of a singular solution for a semi-infinite interfacial crack with traction-free conditions.

According to the Plemelj formula, the solution of the Riemann-Hilbert problem (10) $h(z)$ can be written as

$$h(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\tau(\eta)}{\eta - z} \, d\eta,$$

where the integral is understood in the principal value sense. Taking the Fourier transform with respect to $x_1$, for $x_2 = 0 \pm$ on the interface, we obtain

$$\hat{h}(\xi, 0 \pm) = \hat{h}(\pm)(\xi) = \pm \int_{-\infty}^{\infty} e^{i\eta \xi} H(\pm \xi) \tau(\eta) \, d\eta = \pm H(\mp \xi) \hat{T}^+(\xi), \quad \xi \in \mathbb{R},$$

where $H$ is the Heaviside function, and the traction $\tau$ is defined in such a way that its transform is analytic in the upper half-plane. To derive the above representation, we used the following relations

$$\int_{-\infty}^{\infty} \frac{e^{\xi(x_1 \pm 0)}}{\eta - (x_1 \pm i0)} \, dx_1 = \pm 2\pi i e^{i\eta \xi} H(\mp \xi).$$
The Fourier transforms of \( g \) and \( \bar{g} \) at the interface are then obtained in the form
\[
\hat{g}(\xi) = B^{-1} \hat{\mathbf{h}}^+(\xi) = H(-\xi)B^{-1} \hat{\mathbf{r}}^+(\xi), \quad \xi \in \mathbb{R},
\]
and
\[
\hat{\bar{g}}(\xi) = -\overline{B}^{-1} \hat{\mathbf{h}}^-(\xi) = H(\xi)\overline{B}^{-1} \hat{\mathbf{r}}^+(\xi), \quad \xi \in \mathbb{R}.
\]
By applying the Fourier transform to (6), for the derivative of the physical displacements, we deduce
\[
-i\xi \hat{\mathbf{u}}(\xi, 0+) = \mathbf{A} \hat{g}(\xi) + \overline{\mathbf{A}} \hat{\bar{g}}(\xi), \quad \xi \in \mathbb{R}.
\]
The Fourier transform of the displacements on the boundary of the upper half-plane is given by
\[
\hat{\mathbf{u}}(\xi, 0+) = \frac{i}{\xi} \left\{ H(-\xi)\mathbf{A}B^{-1} + H(\xi)\overline{\mathbf{A}}\overline{B}^{-1} \right\} \hat{\mathbf{r}}^+(\xi) = \frac{i}{\xi} \left\{ H(-\xi)\mathbf{Y}^{(1)} - H(\xi)\overline{\mathbf{Y}}^{(1)} \right\} \hat{\mathbf{r}}^+(\xi).
\]
By expressing the Heaviside function in the form
\[
H(\pm \xi) = \frac{1}{2} \left( 1 \pm \text{sign} \xi \right),
\]
we obtain
\[
\hat{\mathbf{u}}(\xi, 0+) = \left\{ \frac{1}{2\xi}(\mathbf{Y}^{(1)} - \overline{\mathbf{Y}}^{(1)}) - \frac{1}{2|\xi|}(\mathbf{Y}^{(1)} + \overline{\mathbf{Y}}^{(1)}) \right\} \hat{\mathbf{r}}^+(\xi), \quad \xi \in \mathbb{R}. \tag{30}
\]
Following the same pattern of the derivation as above, we derive the Fourier transform of the physical displacement on the boundary of the lower half-plane
\[
\hat{\mathbf{u}}(\xi, 0-) = \left\{ \frac{1}{2\xi}(\mathbf{Y}^{(2)} - \overline{\mathbf{Y}}^{(2)}) + \frac{1}{2|\xi|}(\mathbf{Y}^{(2)} + \overline{\mathbf{Y}}^{(2)}) \right\} \hat{\mathbf{r}}^+(\xi), \quad \xi \in \mathbb{R}. \tag{31}
\]
Next, we replace \( \mathbf{u} \) in the above text by the singular solution \( \mathbf{U} \) for a semi-infinite interfacial crack. Correspondingly, the vector of tractions \( \mathbf{\tau} \) is replaced by \( \mathbf{\Sigma} \), which is the traction vector corresponding to the singular solution \( \mathbf{U} \). The Fourier transforms of the singular displacements on the boundary can then be derived
\[
\hat{\mathbf{U}}(\xi, 0+) = \left\{ \frac{1}{2\xi}(\mathbf{Y}^{(1)} - \overline{\mathbf{Y}}^{(1)}) - \frac{1}{2|\xi|}(\mathbf{Y}^{(1)} + \overline{\mathbf{Y}}^{(1)}) \right\} \hat{\mathbf{\Sigma}}^{-}(\xi), \quad \xi \in \mathbb{R}, \tag{32}
\]
\[
\hat{\mathbf{U}}(\xi, 0-) = \left\{ \frac{1}{2\xi}(\mathbf{Y}^{(2)} - \overline{\mathbf{Y}}^{(2)}) + \frac{1}{2|\xi|}(\mathbf{Y}^{(2)} + \overline{\mathbf{Y}}^{(2)}) \right\} \hat{\mathbf{\Sigma}}^{-}(\xi), \quad \xi \in \mathbb{R}. \tag{33}
\]
According to the definitions (16) and (17), we take the difference and the average of (32) and (33), and we derive that
\[
[\hat{\mathbf{U}}]^+(\xi) = \frac{1}{|\xi|} \left\{ i \text{sign}(\xi) \text{Im}(\mathbf{Y}^{(1)} - \overline{\mathbf{Y}}^{(2)}) - \text{Re}(\mathbf{Y}^{(1)} + \overline{\mathbf{Y}}^{(2)}) \right\} \hat{\mathbf{\Sigma}}^{-}(\xi) \tag{34}
\]
and
\[
(\hat{\mathbf{U}})(\xi) = \frac{1}{2|\xi|} \left\{ i \text{sign}(\xi) \text{Im}(\mathbf{Y}^{(1)} + \overline{\mathbf{Y}}^{(2)}) - \text{Re}(\mathbf{Y}^{(1)} - \overline{\mathbf{Y}}^{(2)}) \right\} \hat{\mathbf{\Sigma}}^{-}(\xi), \quad \xi \in \mathbb{R}. \tag{35}
\]
We note that (34) is the functional equation of the Wiener–Hopf type, similar to the one studied in refs. [7, 16, 17] for the case of isotropic media (see the appendix).

Using the continuity of tractions \( \mathbf{\Sigma} \) across the interface, together with (34) and (35), we express the Fourier transform of the skew-symmetric weight function \( \langle \mathbf{U} \rangle \) in the form
\[
(\hat{\mathbf{U}})(\xi) = \mathcal{A} [\hat{\mathbf{U}}]^+(\xi) + \frac{i}{\xi} \mathbf{B} \hat{\mathbf{\Sigma}}^{-}(\xi), \quad \xi \in \mathbb{R}, \tag{36}
\]
where the diagonal matrix $\mathbf{A}$ and the off-diagonal matrix $\mathbf{B}$ are

$$\mathbf{A} = \frac{1}{2} \text{Re}(\mathbf{Y}^{(1)} - \mathbf{Y}^{(2)}) \left( \text{Re}(\mathbf{Y}^{(1)} + \mathbf{Y}^{(2)}) \right)^{-1}$$

and

$$\mathbf{B} = \frac{1}{2} \text{Im}(\mathbf{Y}^{(1)} + \mathbf{Y}^{(2)}) - \mathbf{A} \text{Im}(\mathbf{Y}^{(1)} - \mathbf{Y}^{(2)}).$$

The formulae (34) and (35) give the expressions for the Fourier transforms of the symmetric and skew-symmetric weight functions for an interfacial crack in an anisotropic bi-material in terms of the transformed singular traction $\hat{\Sigma}$.

4. **Interfacial cracks in orthotropic bi-materials**

In this section, explicit expressions for the Fourier transforms of the symmetric and skew-symmetric weight functions for an interfacial crack in orthotropic bi-materials are derived using equations (34) and (35). Here we assume that the principal axes of the two orthotropic media coincide with the $x_1$ and $x_2$ axes.

4.1. **Stroh representations for orthotropic bi-materials**

For the case of orthotropic two-dimensional media, the matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ introduced in Section 2.1 are given by

$$\mathbf{Q} = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{66} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 0 & c_{12} \\ c_{66} & 0 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} c_{66} & 0 \\ 0 & c_{22} \end{pmatrix},$$

where $c_{ij}$ are the elements of the stiffness matrix, which can be expressed in terms of the elements of the compliance matrix $s_{ij}$ as

$$c_{11} = \frac{s_{22}}{s_{12}^2 - s_{11}s_{22}}, \quad c_{12} = \frac{s_{12}}{s_{12}^2 - s_{11}s_{22}}, \quad c_{22} = \frac{1}{s_{12}^2 - s_{11}s_{22}}, \quad c_{66} = \frac{s_{66}}{s_{66}}.$$

The characteristic equation (9) then becomes

$$s_{11}\mu^4 + (2s_{12} + s_{66})\mu^2 + s_{22} = 0.$$

The matrices $\mathbf{A}$ and $\mathbf{B}$, defined by the relations (5) and (6), are equivalent to those provided by alternative Lekhnitskii formulation [18], more precisely the Lekhnitskii approach gives a specially normalised eigenvector matrix $\mathbf{A}$, and the characteristic equation derived using the Lekhnitskii formalism is identical to (39) [15, 19]. The relationships between the two alternative formulations are derived and reported in detail in ref. [22], where the elements of $\mathbf{A}$ and $\mathbf{B}$ in the Stroh representation are given as functions of the coefficients proposed by Lekhnitskii. Here, we write the Stroh matrices $\mathbf{A}$ and $\mathbf{B}$ using the normalisation adopted in ref. [22]

$$\mathbf{A} = \begin{pmatrix} \frac{s_{11} \mu_1^2 + s_{12}}{\sqrt{\mu_1^2 (s_{22} - s_{11} \mu_1^2)}} & \frac{s_{11} \mu_1^2 + s_{12}}{\sqrt{\mu_1^2 (s_{22} - s_{11} \mu_1^2)}} \\ \frac{s_{12} \mu_2 + s_{22}}{\sqrt{s_{12}^2 + s_{22}^2}} & \frac{s_{12} \mu_2 + s_{22}}{\sqrt{s_{12}^2 + s_{22}^2}} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} -\mu_1 & -\mu_2 \\ \mu_2 & \mu_1 \end{pmatrix},$$

where $\mu_1$ and $\mu_2$ are the two roots of the characteristic equation with positive imaginary parts. The Hermitian matrix $\mathbf{Y}$ evaluated using (40) and (41) is

$$\mathbf{Y} = i\mathbf{A}\mathbf{B}^{-1} = \begin{pmatrix} s_{11} \text{Im}(\mu_1 + \mu_2) - i (s_{11} \mu_1 \mu_2 - s_{12}) \\ i (s_{11} \mu_1 \mu_2 - s_{12}) - s_{22} \text{Im} \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \end{pmatrix},$$

(42)
Following the notation introduced in Suo [15], we define two non-dimensional parameters measuring the material anisotropy

$$\lambda = \frac{s_{11}}{s_{22}}, \quad \rho = \frac{2s_{12} + s_{66}}{2\sqrt{s_{11}s_{22}}}.$$  

The condition

$$\lambda > 0, \quad -1 < \rho < +\infty,$$

guarantees the positive definiteness of the strain energy density. The characteristic equation (39) then becomes

$$\lambda \mu^4 + 2 \rho \lambda^{1/2} \mu^2 + 1 = 0. \quad (43)$$

The above equation possesses four roots that occur in two conjugate pairs. The roots $\mu_1$ and $\mu_2$, with positive imaginary parts, have the form [15]

$$\mu_1 = i\lambda^{-1/4}(n + m), \quad \mu_2 = i\lambda^{-1/4}(n - m) \quad \text{for} \quad 1 < \rho < +\infty,$$

$$\mu_1 = \lambda^{-1/4}(in + m), \quad \mu_2 = \lambda^{-1/4}(in - m) \quad \text{for} \quad -1 < \rho < 1,$$

$$\mu_1 = \mu_2 = i\lambda^{-1/4} \quad \text{for} \quad \rho = 1,$$

where

$$n = \sqrt{(1 + \rho)/2}, \quad m = \sqrt{|1 - \rho|/2}.$$

The matrix $Y$ in (42) then becomes [15]

$$Y = \begin{pmatrix} 2n\lambda^{1/4}(s_{11}s_{22})^{1/2} & i((s_{11}s_{22})^{1/2} + s_{12}) \\ -i((s_{11}s_{22})^{1/2} + s_{12}) & 2n\lambda^{-1/4}(s_{11}s_{22})^{1/2} \end{pmatrix}. \quad (44)$$

For two orthotropic materials with aligned principal axes, the bi-material matrix $H$ has the form [15]

$$H = Y^{(1)} + Y^{(2)} = \begin{pmatrix} H_{11} & -i\beta\sqrt{H_{11}H_{22}} \\ i\beta\sqrt{H_{11}H_{22}} & H_{22} \end{pmatrix}, \quad (45)$$

where

$$H_{11} = [2n\lambda^{1/4}\sqrt{s_{11}s_{22}}]^{(1)} + [2n\lambda^{1/4}\sqrt{s_{11}s_{22}}]^{(2)},$$

$$H_{22} = [2n\lambda^{-1/4}\sqrt{s_{11}s_{22}}]^{(1)} + [2n\lambda^{-1/4}\sqrt{s_{11}s_{22}}]^{(2)},$$

$$\beta\sqrt{H_{11}H_{22}} = [s_{12} + \sqrt{s_{11}s_{22}}]^{(2)} - [s_{12} + \sqrt{s_{11}s_{22}}]^{(1)}.$$

The quantity $\beta$ in (45) is the generalised Dundurs parameters [23], connected to the bi-material oscillatory index $\varepsilon$ by the relation

$$\varepsilon = \frac{1}{2\pi} \ln \left( \frac{1 - \beta}{1 + \beta} \right).$$

The normalised eigenvector $w$ of the eigensystem (13), associated with the displacement jump (15) and the traction (14) ahead of the crack, has the form [15]

$$w = \begin{pmatrix} -1/2, & 1/2 \sqrt{H_{11}} \end{pmatrix}^T. \quad (46)$$

### 4.2. Weight functions for an in-plane deformation

In order to derive explicit expressions for the Fourier transforms of the weight functions (34) and (35), we need to evaluate the Fourier transform of the singular traction $\Sigma$ given by (21). For a crack between two orthotropic media, the vector $w$ has the expression (46). Mode I and mode II are coupled, and are associated with one single
derive the Fourier transforms of the two independent symmetric and skew-symmetric weight functions

Substituting (49) and (50) into (34) and (35), and expressing them in terms of the elements of the matrix

where \( \text{Im} \) denotes the imaginary part of a complex number.

Applying the Fourier transform to (47) and (48) then yields

\[
\begin{align*}
\hat{\Sigma}^1_-(\xi) &= \frac{\xi^{1/2}}{1 + 4e^2} \left( -e_0 e^{-i\tau} \frac{-1}{c^+} - i\frac{\epsilon}{c^+} + \frac{\epsilon}{c^+} \right), \\
\hat{\Sigma}^2_-(\xi) &= \frac{\xi^{1/2}}{1 + 4e^2} \left( -e_0 e^{-i\tau} \frac{-1}{c^+} - i\frac{\epsilon}{c^+} + \frac{\epsilon}{c^+} \right),
\end{align*}
\]

where \( \text{Im} \xi_- < 0 \) and

\[
e_0 = e^{\frac{\pi}{2}}, \quad c^+ = \frac{(1 + i)\sqrt{\pi}}{2\Gamma \left( \frac{1}{2} + i\epsilon \right)}.
\]

Substituting (49) and (50) into (34) and (35), and expressing them in terms of the elements of the matrix \( \mathbf{H} \), we derive the Fourier transforms of the two independent symmetric and skew-symmetric weight functions

\[
\begin{align*}
\left( \hat{\mathbf{U}}_1^+ \quad \hat{\mathbf{U}}_2^+ \right) &= -\sqrt{\frac{H_{11}H_{22}}{H_{12}^2}} \left( \frac{H_{11}}{H_{22}} \right)^{i\beta \text{sign}(\xi)} \left( \frac{H_{12}}{H_{22}} \right)^{i\beta \text{sign}(\xi)} \left( \hat{\mathbf{S}}_1^- \quad \hat{\mathbf{S}}_2^- \right), \\
\left( \hat{\mathbf{U}}_1^- \quad \hat{\mathbf{U}}_2^- \right) &= -\sqrt{\frac{H_{11}H_{22}}{H_{12}^2}} \left( \frac{H_{11}}{H_{22}} \right)^{-i\gamma \text{sign}(\xi)} \left( \frac{H_{12}}{H_{22}} \right)^{-i\gamma \text{sign}(\xi)} \left( \hat{\mathbf{S}}_1^- \quad \hat{\mathbf{S}}_2^- \right),
\end{align*}
\]

where the Dundurs-like parameters \( \delta_1, \delta_2 \) and \( \gamma \) are defined as

\[
\begin{align*}
\delta_1 &= \frac{[2n\lambda^2(s_{11}s_{22})^\frac{1}{2}]^{(1)} - [2n\lambda^2(s_{11}s_{22})^\frac{1}{2}]^{(2)}}{H_{11}}, \\
\delta_2 &= \frac{[2n\lambda^2(s_{11}s_{22})^\frac{1}{2}]^{(1)} - [2n\lambda^2(s_{11}s_{22})^\frac{1}{2}]^{(2)}}{H_{22}}, \\
\gamma &= \frac{[(s_{11}s_{22})^\frac{1}{2} + s_{12}]^{(1)} + [(s_{11}s_{22})^\frac{1}{2} + s_{12}]^{(2)}}{\sqrt{H_{11}H_{22}}}.
\end{align*}
\]

The relation (51) is the Wiener–Hopf type equation. The skew-symmetric part of the weight function can then be decomposed using the relation (36). This yields

\[
\hat{\mathbf{U}} = \mathbf{A}\hat{\mathbf{U}}^+ + \frac{i}{\xi} \mathbf{B}\hat{\mathbf{S}}^-, \]

where \( \mathbf{A} \) and \( \mathbf{B} \) are matrices defined in terms of the weight function and the parameters of the problem.
where the matrices $\mathcal{A}$ and $\mathcal{B}$ for orthotropic bi-materials are given by

$$
\mathcal{A} = \frac{1}{2} \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}, \quad \mathcal{B} = \frac{\sqrt{H_{11}H_{22}}}{2} \begin{pmatrix} 0 & \gamma + \beta \delta_1 \\ -(\gamma + \beta \delta_2) & 0 \end{pmatrix}.
$$

The explicit expressions for the weight functions are obtained by the inversion of the derived transforms. The symmetric weight function $[U](x_1)$ is equal to zero for $x_1 < 0$, while for $x_1 > 0$, it is given by

$$
[U^1_1](x_1) = \frac{H_{11}x_1^{-\frac{1}{2}}}{2c^+c-\sqrt{2}\pi(1+4\varepsilon^2)} \left\{ (\beta - 1) \left( -\frac{1}{2} + i\varepsilon \right) x_1^{-ie} + (\beta + 1) \left( -\frac{1}{2} - i\varepsilon \right) x_1^{ie} \right\},
$$

$$
[U^1_2](x_1) = \frac{i\sqrt{H_{11}H_{22}}x_1^{-\frac{1}{2}}}{2c^+c-\sqrt{2}\pi(1+4\varepsilon^2)} \left\{ (\beta - 1) \left( -\frac{1}{2} + i\varepsilon \right) x_1^{-ie} - (\beta + 1) \left( -\frac{1}{2} - i\varepsilon \right) x_1^{ie} \right\},
$$

$$
[U^2_1](x_1) = -\frac{iH_{11}x_1^{-\frac{1}{2}}}{2c^+c-\sqrt{2}\pi(1+4\varepsilon^2)} \left\{ (\beta - 1) \left( -\frac{1}{2} + i\varepsilon \right) x_1^{-ie} - (\beta + 1) \left( -\frac{1}{2} - i\varepsilon \right) x_1^{ie} \right\},
$$

$$
[U^2_2](x_1) = \frac{\sqrt{H_{11}H_{22}}x_1^{-\frac{1}{2}}}{2c^+c-\sqrt{2}\pi(1+4\varepsilon^2)} \left\{ (\beta - 1) \left( -\frac{1}{2} + i\varepsilon \right) x_1^{-ie} + (\beta + 1) \left( -\frac{1}{2} - i\varepsilon \right) x_1^{ie} \right\}.
$$

(54)

The skew-symmetric weight function $\langle U \rangle(x_1)$ is equal to $\mathcal{A}[U](x)$ for $x_1 > 0$, while for $x_1 < 0$, it is given by

$$
\langle U^1_1 \rangle(x_1) = -\frac{iH_{11}(\gamma + \beta \delta_1)(-x_1)^{-\frac{1}{2}}}{4c^+c-\sqrt{2}\pi(1+4\varepsilon^2)} \left\{ \left( -\frac{1}{2} + i\varepsilon \right) \frac{(-x_1)^{-ie}}{e_0^2} + \left( -\frac{1}{2} - i\varepsilon \right) e_0^2 (-x_1)^{ie} \right\},
$$

$$
\langle U^2_1 \rangle(x_1) = \frac{\sqrt{H_{11}H_{22}}(\gamma + \beta \delta_2)(-x_1)^{-\frac{1}{2}}}{4c^+c-\sqrt{2}\pi(1+4\varepsilon^2)} \left\{ \left( -\frac{1}{2} + i\varepsilon \right) \frac{(-x_1)^{-ie}}{e_0^2} - \left( -\frac{1}{2} - i\varepsilon \right) e_0^2 (-x_1)^{ie} \right\},
$$

$$
\langle U^2_1 \rangle(x_1) = -\frac{H_{11}(\gamma + \beta \delta_1)(-x_1)^{-\frac{1}{2}}}{4c^+c-\sqrt{2}\pi(1+4\varepsilon^2)} \left\{ \left( -\frac{1}{2} + i\varepsilon \right) \frac{(-x_1)^{-ie}}{e_0^2} - \left( -\frac{1}{2} - i\varepsilon \right) e_0^2 (-x_1)^{ie} \right\},
$$

$$
\langle U^2_2 \rangle(x_1) = -\frac{i\sqrt{H_{11}H_{22}}(\gamma + \beta \delta_2)(-x_1)^{-\frac{1}{2}}}{4c^+c-\sqrt{2}\pi(1+4\varepsilon^2)} \left\{ \left( -\frac{1}{2} + i\varepsilon \right) \frac{(-x_1)^{-ie}}{e_0^2} + \left( -\frac{1}{2} - i\varepsilon \right) e_0^2 (-x_1)^{ie} \right\}.
$$

(55)

4.3. Stress intensity factors for orthotropic bi-materials

The symmetric and skew-symmetric weight functions are now used to evaluate the complex stress intensity factor due to an arbitrary system of forces. With the use of (28), for $x_2 \to 0+$, the traction becomes

$$
\tau(x_1) = \frac{x_1^{-\frac{1}{2}}}{2\sqrt{2}\pi} \mathcal{T}(x_1) \mathbf{K} + \frac{x_1^{\frac{1}{2}}}{2\sqrt{2}\pi} \mathcal{T}(x_1) \mathbf{J} + \frac{x_1^{\frac{3}{2}}}{2\sqrt{2}\pi} \mathcal{T}(x_1) \mathbf{L} + \mathcal{O}(x_1^{\frac{5}{2}}).
$$

(56)

For orthotropic bi-materials, the matrix $\mathcal{T}(x_1)$ is

$$
\mathcal{T}(x_1) = \begin{pmatrix}
-ix_1^{ie} & ix_1^{-ie} \\
\sqrt{H_{11}}x_1^{ie} & \sqrt{H_{11}}x_1^{-ie}
\end{pmatrix}.
$$

The Fourier transform of (56), as $\xi \to \infty$ and $\text{Im} \xi > 0$, is

$$
\hat{\tau}^+(\xi) = \frac{\xi^{-\frac{1}{2}}}{4} \hat{T}_1(\xi \cdot) \mathbf{K} + \frac{\xi^{-\frac{1}{2}}}{4\xi} \hat{T}_2(\xi \cdot) \mathbf{J} + \frac{\xi^{-\frac{1}{2}}}{4\xi^2} \hat{T}_3(\xi \cdot) \mathbf{L} + \mathcal{O}(\xi^{-\frac{7}{2}}),
$$

(57)
where

$$\hat{T}_1(\xi_i) = \begin{pmatrix}
\frac{e_0 \frac{\xi e^{i\beta}}{c}}{c^2 + e_0} & -\frac{e_0 \frac{\xi e^{i\beta}}{c}}{c^2}
\end{pmatrix}.
$$

Using the above expression, the explicit transforms of the traction \(\hat{\Sigma}^-\) and of the symmetric weight functions matrix \([\hat{U}]^+\) derived in the previous section and evaluating \([\hat{u}]^-\), we derive the following asymptotic expansions

$$[\hat{U}]^+\mathcal{R} \hat{t}^+ = \xi^{-1} \mathcal{M}_1 \mathbf{K} + \xi^{-2} \mathcal{M}_2 \mathbf{J} + \mathcal{O}(\xi^{-3}) \quad \text{for} \quad \text{Im} \, \xi > 0,$$

$$\hat{\Sigma}^- \mathcal{R} [\hat{u}]^- = \xi^{-1} \mathcal{M}_1 \mathbf{K} + \xi^{-2} \mathcal{M}_2 \mathbf{J} + \mathcal{O}(\xi^{-3}) \quad \text{for} \quad \text{Im} \, \xi < 0.$$  

The explicit form of the matrix \(\mathcal{M}_1\) is

$$\mathcal{M}_1 = -\frac{H_{11}}{4c^+ c^- (1 + 4e^4)} \begin{pmatrix}
-\frac{(\beta - 1)(1 - 2ie)}{e_0} & e_0^2 (\beta + 1)(1 + 2ie) \\
\frac{4(\beta - 1)(1 - 2ie)}{e_0} & i e_0^2 (\beta + 1)(1 + 2ie)
\end{pmatrix}. \quad (60)$$

Now we rewrite the Fourier transform of the Betti identity (27) in terms of a Riemann–Hilbert problem

$$\Psi^+(\xi) - \Psi^-(\xi) = -[\hat{U}]^+ T(\xi) \mathcal{R} [\hat{p}] (\xi) - \langle \hat{U} \rangle^+ T(\xi) \mathcal{R} [\hat{p}] [\xi], \quad \xi \in \mathbb{R},$$

where, according to the Plemelj formula, the functions \(\Psi^\pm(\xi)\) are given by

$$\Psi^\pm(\xi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Psi(\eta) \frac{d\eta}{\eta - \xi}, \quad (62)$$

with \(\text{Im} \, \xi > 0\) or \(\text{Im} \, \xi < 0\), respectively. Then, the solution of (61) is

$$[\hat{U}]^+ T \hat{t}^+ = \Psi^+, \quad \text{Im} \, \xi > 0,$$

$$\hat{\Sigma}^- \mathcal{R} [\hat{u}]^- = \Psi^-, \quad \text{Im} \, \xi < 0.$$  

From these expressions, the asymptotic estimates can be extracted. For \(\xi \to \infty\), we can expand Plemelj’s formula. Considering only the first term, we have

$$\Psi^\pm(\xi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Psi(\eta) \frac{d\eta}{\eta - \xi} = -\xi^{-1} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Psi(\eta) d\eta + \mathcal{O}(\xi^{-2}), \quad (63)$$

with \(\text{Im} \, \xi > 0\) or \(\text{Im} \, \xi < 0\), respectively. Substituting this expression and the expansions (58) and (59) into (61), comparing the corresponding terms, and considering only the first order, the following general formula for the complex stress intensity factor is obtained

$$\mathbf{K} = \frac{1}{2\pi i} \mathcal{M}_1^{-1} \int_{-\infty}^{\infty} \left\{ [\hat{U}]^+ T(\eta) \mathcal{R} [\hat{p}] (\eta) + \langle \hat{U} \rangle^+ T(\eta) \mathcal{R} [\hat{p}] [\eta] \right\} d\eta. \quad (64)$$

Since we have explicit expressions for matrix \(\mathcal{M}_1\) and for the Fourier transforms of both symmetric and skew-symmetric weight functions, using (64) we now can evaluate the complex stress intensity factor for an interfacial crack in an orthotropic bi-material under an arbitrary loading. In the next section, an illustrative example of computation of \(\mathbf{K}\) by means of (64) is reported.
5. An illustrative example

In this section, we present an illustrative example of computation of the complex stress intensity factor \( K \) for an interfacial crack loaded by a simple non-symmetric force system in an orthotropic bi-material. For the purpose of comparison, the considered force system is taken to be the same as in ref. [7]. The loading consists of a point force \( F \) applied to the upper crack face at a distance \( a \) behind the crack tip and two point forces \( F/2 \) applied to the lower crack face at distances \( a - b/2 \) and \( a + b/2 \), respectively.

The loading can be expressed in terms of the Dirac delta function

\[
p^+(x_1) = -F\delta(x_1 + a), \quad p^-(x_1) = -\frac{F}{2}\delta(x_1 + a + b) - \frac{F}{2}\delta(x_1 + a - b).
\]

The loading can be decomposed into the symmetric and skew-symmetric parts

\[
(p)(x_1) = -\frac{F}{2}\delta(x_1 + a) - \frac{F}{4}\delta(x_1 + a + b) - \frac{F}{4}\delta(x_1 + a - b),
\]

\[
[p](x_1) = -F\delta(x_1 + a) + \frac{F}{2}\delta(x_1 + a + b) + \frac{F}{2}\delta(x_1 + a - b),
\]

whose Fourier transforms are then given by

\[
(\hat{p})(\xi) = -\frac{F}{2}e^{-i\xi a} - \frac{F}{4}e^{-i\xi(a+b)} - \frac{F}{4}e^{-i\xi(a-b)},
\]

\[
[\hat{p}](\xi) = -Fe^{-i\xi a} + \frac{F}{2}e^{-i\xi(a+b)} + \frac{F}{2}e^{-i\xi(a-b)}.
\]

Using the above expressions and the explicit transforms (51) and (53) of the symmetric and skew-symmetric weight functions in orthotropic media, both symmetric and skew-symmetric parts of the complex stress intensity factor \( K = K^S + K^A \) corresponding to this load can then be evaluated using the integral formula (64)

\[
K^S = K_1^S + iK_2^S = \frac{G^2}{1 - \beta^2} \int_{H_{11}} \sqrt{\frac{H_{22}}{H_{11}}} \frac{2\pi}{\alpha} \frac{\alpha^{-\frac{1}{2}}}{\alpha^{-\frac{1}{2}}} \left\{ \frac{1}{2} + \frac{1}{4}(1-b/a)^{-\frac{1}{2}i\xi} + \frac{1}{4}(1+b/a)^{-\frac{1}{2}i\xi} \right\} d\xi,
\]

\[
K^A = K_1^A + iK_2^A = \frac{G^2}{2(1 - \beta^2)} \int_{H_{11}} \sqrt{\frac{H_{22}}{H_{11}}} \frac{2\pi}{\alpha} \frac{\alpha^{-\frac{1}{2}}}{\alpha^{-\frac{1}{2}}} \left\{ \frac{1}{2} - \frac{1}{4}(1-b/a)^{-\frac{1}{2}i\xi} - \frac{1}{4}(1+b/a)^{-\frac{1}{2}i\xi} \right\} d\xi.
\]

In order to study the behaviour of these symmetric and skew-symmetric contributions to the stress intensity factors as functions of \( b/a \), the following non-dimensional parameters have been used [15]

\[
\Phi = \left[ \frac{(s_{11}s_{22})^{\frac{1}{2}}}{(s_{11}s_{22})^{\frac{1}{2}}} \right]^{(2)}, \quad \Theta^{(1)} = \left[ \frac{s_{12}}{(s_{11}s_{22})^{\frac{1}{2}}} \right]^{(1)}, \quad \Theta^{(2)} = \left[ \frac{s_{12}}{(s_{11}s_{22})^{\frac{1}{2}}} \right]^{(2)}.
\]

We then express \( H_{11}, H_{22} \) and \( \delta_2 \) in terms of these parameters

\[
\frac{H_{22}}{H_{11}} = \frac{[2n\lambda^{-\frac{1}{2}}]^{(1)} + [2n\lambda^{-\frac{1}{2}}]^{(2)}}{[2n\lambda^{\frac{1}{2}}]^{(1)} + [2n\lambda^{\frac{1}{2}}]^{(2)}},
\]

\[
\delta_2 = \frac{[2n\lambda^{-\frac{1}{2}}]^{(1)} - [2n\lambda^{-\frac{1}{2}}]^{(2)}}{[2n\lambda^{-\frac{1}{2}}]^{(1)} + [2n\lambda^{-\frac{1}{2}}]^{(2)}},
\]

It is important to note that oscillations of the stress and displacement fields are excluded for \( \beta = 0, \varepsilon = 0 \) [15].

The complex stress intensity factor has been computed for \( \rho^{(1)} = 0.74, \rho^{(2)} = 4.91, \lambda^{(1)} = 1, \lambda^{(2)} = 1/143, \Theta^{(1)} = 1/2, \Theta^{(2)} = 2, \Phi = 6.4 \) and five different values of the Dundurs oscillation parameter \( \beta = \{ -1/4, -1/2, 0, 1/4, 1/2 \} \); here, we assume that the material (1) is Aluminium and the material (2) is Boron. The
Figure 2. The symmetric and skew-symmetric stress intensity factors as functions of the ratio $b/a$ plotted for $\rho^{(1)} = 0.74, 1/\lambda^{(1)} = 1, \theta^{(1)} = 1/2, \rho^{(2)} = 4.91, 1/\lambda^{(2)} = 14.3, \theta^{(2)} = 2, \Phi = 6.4$ and different values of the Dundurs parameter $\beta$.

values of the stress intensity factors have been normalised by multiplying by $a^2 F$ and are plotted in Figure 2 as functions of the ratio $b/a$. The symmetric mode I and II stress intensity factors $K^S_I$ and $K^S_{II}$ are shown on the left part of the figure, while the skew-symmetric ones are on the right. As expected, both mode I and II skew-symmetric stress intensity factors $K^{A}_I$ and $K^{A}_{II}$ are zero when $b/a = 0$, since for $b = 0$, the load is symmetric. As we increase $b/a$, the skew-symmetric contribution of the loading become more apparent, and $K^{A}_I$ and $|K^{A}_{II}|$ correspondingly increase. In the case without oscillation (when $\beta = 0$), both mode II stress intensity factors $K^S_{II}$ and $K^{A}_{II}$ vanish.

Figure 3 shows the ratio $K^A_I/K^S_I$ as a function of $b/a$. We observe that as $b/a$ increases, $K^A_I$ may exceed 40% of $K^S_I$, and thus the contribution of the skew-symmetric part of the load is not negligible, and it needs to be taken into account in perturbative analysis of the fields in the vicinity of an interfacial crack between two dissimilar anisotropic elastic materials subject to non-symmetric load applied to the crack faces.

6. Conclusions

The developed general approach for the derivation of the symmetric and skew-symmetric weight functions for an interfacial plane crack between dissimilar anisotropic materials, based on the Stroh formulation, has been discussed in detail and tested by applying it to a plane stress problem for a crack placed at an interface between two orthotropic materials. The skew-symmetric weight functions obtained by means of the proposed method have been compared to those obtained for the same problem by constructing a singular solution of the elasticity problem in a half-plane [7], the comparison between the two different solutions is reported in the appendix.

Since the proposed Stroh representation is valid for stationary and steady-state elasticity problems in many anisotropic media [14, 15, 19], it can be used for evaluating explicit weight functions for plane interfacial cracks in several kind of materials (Stroh analysis has been proposed, for example, in piezoelectrics [20], quasicrystals [24, 25] and poroelastic media [26]). The derived weight functions can be applied to many important
Figure 3. The ratio $K_A^I/K_S^I$ as a function of $b/a$ computed for $\rho^{(1)} = 0.74, 1/\lambda^{(1)} = 1, \Theta^{(1)} = 1/2, \rho^{(2)} = 4.91, 1/\lambda^{(2)} = 14.3, \Theta^{(2)} = 2, \Phi = 6.4$ and different values of the Dundurs parameter $\beta = \{-1/4, -1/2, 0, 1/4, 1/2\}$.

applications such as perturbative expansions for growing cracks or wavy cracks problems [17], and in the computation of the stress intensity factor for a non-symmetric self-balanced load generated by a system of point-forces applied on the crack faces. An example of the stress intensity factor evaluation for an asymmetric loading has been reported in Section 5, both the symmetric and skew-symmetric weight functions, obtained for orthotropic bi-materials, have been used in the computations, and the results have shown that the contribution of the skew-symmetric part of the load is not negligible and must be taken into account when deriving asymptotic representations of the stresses near the crack tip; a similar tendency has already been observed by Piccolroaz et al. [7, 8] for the case of isotropic media.

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Conflict of interest

None declared.

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Appendix

In this appendix, the Fourier transforms of the singular solutions of the interfacial crack problem between two dissimilar orthotropic materials are derived by solving a boundary-value problem for a half-plane subjected to traction boundary conditions, following the procedure illustrated in ref. [7]. The derived expressions for the singular displacements and for the symmetric and skew-symmetric weight functions are compared with those obtained in Section 3 by means of the direct solution of the Riemann–Hilbert problem (9).

Initially, we consider the lower half-plane, denoted in the main text by the superscript (2). Introducing the Fourier transform of the stresses with respect to $\xi$, we consider the component $\hat{\sigma}_{22}$ as the primary unknown function, so that the plane stress elasticity problem for the orthotropic material (2) is reduced to the following ordinary differential equation

$$s_{11}^{(2)} \hat{\sigma}_{22}^{(m)} - \xi^2 \left( s_{66}^{(2)} + 2s_{12}^{(2)} \right) \hat{\sigma}_{22}^{(n)} + \xi^4 s_{22}^{(2)} \hat{\sigma}_{22}^{(s)} = 0,$$  \hspace{1cm} (68)

where a prime denotes the derivative with respect to $x_2$. The characteristic equation associated with (68) is

$$[\omega^{(2)}]^4 s_{11}^{(2)} - \xi^2 \left( s_{66}^{(2)} + 2s_{12}^{(2)} \right) [\omega^{(2)}]^2 + \xi^4 s_{22}^{(2)} = 0, \quad \xi \in \mathbb{R}.$$  

Introducing $\nu^{(2)} = \omega^{(2)}/|\xi|$ and using the notations of Section 4, this characteristic equation becomes

$$\lambda^{(2)} [\nu^{(2)}]^4 - 2\rho^{(2)} (\lambda^{(2)})^2 [\nu^{(2)}]^2 + 1 = 0.$$  \hspace{1cm} (69)

The above equation possesses four distinct roots ($\rho^{(2)} \neq 1$); the general solution of the plane stress elasticity problem in the lower half-plane is

$$\hat{\sigma}_{22}(\xi, x_2) = A_1^{(2)} e^{\xi |\nu^{(2)}| x_2} + A_2^{(2)} e^{\xi |\nu^{(2)}| x_2}, \quad \hat{\sigma}_{11}(\xi, x_2) = -\frac{1}{\xi^2} \hat{\sigma}_{22}^{(n)}, \quad \hat{\sigma}_{21}(\xi, x_2) = -\frac{i}{\xi} \hat{\sigma}_{22}^{(s)}.$$  \hspace{1cm} (70)
The Fourier transform of the displacement components are

\[ \hat{u}_1 = \frac{i}{\xi} \left( s^{(2)}_{11} \hat{\sigma}_{11} + s^{(2)}_{12} \hat{\sigma}_{22} \right), \quad \hat{u}_2 = \frac{1}{\xi^2} \left( s^{(2)}_{11} \hat{\sigma}_{11} + s^{(2)}_{12} \hat{\sigma}_{22} \right), \]  

(71)

where only the two eigenvalues with positive real parts for which the stresses vanish at infinity (i.e. \( \hat{\sigma}_{ij} \to 0 \) as \( x_2 \to -\infty \)) have been taken into account. Remembering the conditions for positive definiteness of the strain energy density mentioned in Section 4, the two eigenvalues with positive real part become

\[ v_1^{(2)} = \left[ \lambda - \frac{i}{2} (n + m) \right], \quad v_2^{(2)} = \left[ \lambda - \frac{i}{2} (n - m) \right], \quad \text{for} \quad 1 < \rho^{(2)} < +\infty, \]

\[ v_1^{(2)} = \left[ \lambda - \frac{i}{2} (n + im) \right], \quad v_2^{(2)} = \left[ \lambda - \frac{i}{2} (n - im) \right], \quad \text{for} \quad -1 < \rho^{(2)} < 1. \]

From this form, it is straightforward to note that these eigenvalues can be expressed as functions of Stroh eigenvalues

\[ v_1^{(2)} = -i\mu_1^{(2)}, \quad v_2^{(2)} = -i\mu_2^{(2)}. \]  

(72)

In order to derive the weight functions, we need to evaluate explicit expressions for the singular displacements utilising (71). The boundary conditions on the boundary \( x_2 = 0^- \) are defined as

\[ \hat{\sigma}_{22}(\xi, x_2 = 0^-) = \hat{\Sigma}_{22}^- (\xi), \quad \hat{\sigma}_{21}(\xi, x_2 = 0^-) = \hat{\Sigma}_{21}^- (\xi), \quad \xi \in \mathbb{R}, \]

where \( \Sigma_{21}, \Sigma_{22} \) are the components of the singular traction defined in Section 4 (see equations (47) and (48)). It then follows that

\[ \hat{\sigma}_{22}(\xi, x_2 = 0^-) = A_1^{(2)} + A_2^{(2)} = \hat{\Sigma}_{22}^- (\xi), \]

\[ \hat{\sigma}_{21}(\xi, x_2 = 0^-) = -i \text{sign}(\xi) \left( A_1^{(2)} v_1^{(2)} + A_2^{(2)} v_2^{(2)} \right) = \hat{\Sigma}_{21}^- (\xi), \]

and thus

\[ A_1^{(2)} = \frac{v_2^{(2)} \hat{\Sigma}_{22}^- - i \text{sign}(\xi) \hat{\Sigma}_{21}^-}{v_2^{(2)} - v_1^{(2)}}, \]  

(73)

\[ A_2^{(2)} = \frac{i \text{sign}(\xi) \hat{\Sigma}_{22}^- - v_2^{(2)} \hat{\Sigma}_{21}^-}{v_2^{(2)} - v_1^{(2)}}. \]  

(74)

The Fourier transforms of the singular displacements fields are then

\[ \hat{u}_1(\xi, x_2) = \frac{1}{\xi (v_2^{(2)} - v_1^{(2)})} \left\{ \text{sign}(\xi) \left[ s^{(2)}_{12} s^{(2)}_{11} \left[ v_1^{(2)} \right]^2 \right] \hat{\Sigma}_{11}^- + iv_2^{(2)} \left[ s^{(2)}_{12} - s^{(2)}_{11} \left[ v_1^{(2)} \right]^2 \right] \hat{\Sigma}_{22}^- \right\} e^{i|v_1^{(2)}| x_2} + \left[ \text{sign}(\xi) \left[ s^{(2)}_{11} \left[ v_2^{(2)} \right]^2 - s^{(2)}_{12} \right] \hat{\Sigma}_{11}^- + iv_1^{(2)} \left[ s^{(2)}_{11} \left[ v_2^{(2)} \right]^2 - s^{(2)}_{12} \right] \hat{\Sigma}_{22}^- \right] e^{i|v_2^{(2)}| x_2}, \]  

(75)

\[ \hat{u}_2(\xi, x_2) = \frac{1}{\xi (v_2^{(2)} - v_1^{(2)})} \left\{ i \left[ s^{(2)}_{11} \left[ v_1^{(2)} \right]^2 - s^{(2)}_{12} - s^{(2)}_{66} \right] v_1^{(2)} \hat{\Sigma}_{11}^- \right\} e^{i|v_1^{(2)}| x_2} + \left[ \text{sign}(\xi) \left[ s^{(2)}_{12} + s^{(2)}_{66} - s^{(2)}_{11} \left[ v_1^{(2)} \right]^2 \right] v_1^{(2)} \hat{\Sigma}_{22}^- \right] e^{i|v_1^{(2)}| x_2} + \left[ \text{sign}(\xi) \left[ s^{(2)}_{11} \left[ v_2^{(2)} \right]^2 - s^{(2)}_{12} - s^{(2)}_{66} \right] v_2^{(2)} \hat{\Sigma}_{22}^- \right] e^{i|v_2^{(2)}| x_2} + \left[ \text{sign}(\xi) \left[ s^{(2)}_{12} + s^{(2)}_{66} - s^{(2)}_{11} \left[ v_2^{(2)} \right]^2 \right] v_2^{(2)} \hat{\Sigma}_{22}^- \right] e^{i|v_2^{(2)}| x_2}. \]  

(76)
For the upper half-plane, we find the same expressions as in ref. [7], subject to replacing $|\xi|$ with $-|\xi|$ and the superscript (2) with (1). From equations (75) and (76) and their corresponding expressions in the upper half-plane, we can derive

$$
\hat{U}_1^+(\xi) = \hat{u}_1^+(\xi, x_2 = 0+) = \left( -\frac{[(v_2 + v_1) s_{11}]^{(1)}(\xi) I_1}{|\xi|}, i\frac{[s_{12} + s_{11} v_1 v_2]^{(1)}(\xi)}{|\xi|} \right) \hat{\Sigma}^-, \\
\hat{U}_2^+(\xi) = \hat{u}_2^+(\xi, x_2 = 0+) = \left( i\frac{[s_{11} + s_{66} - (v_2 + v_1 + v_1 v_2) s_{11}]^{(1)}(\xi)}{|\xi|}, -\frac{[s_{11} v_1 v_2 (v_2 + v_1)]^{(1)}(\xi)}{|\xi|} \right) \hat{\Sigma}^-,
$$

$$
\hat{U}_1^-(\xi) = \hat{u}_1^-(\xi, x_2 = 0-) = \left( \frac{[(v_2 + v_1) s_{11}]^{(2)}(\xi)}{|\xi|}, i\frac{[s_{12} + s_{11} v_1 v_2]^{(2)}(\xi)}{|\xi|} \right) \hat{\Sigma}^-, \\
\hat{U}_2^-(\xi) = \hat{u}_2^-(\xi, x_2 = 0-) = \left( i\frac{[s_{11} + s_{66} - (v_2 + v_1 + v_1 v_2) s_{11}]^{(2)}(\xi)}{|\xi|}, -\frac{[s_{11} v_1 v_2 (v_2 + v_1)]^{(2)}(\xi)}{|\xi|} \right) \hat{\Sigma}^-.
$$

Using the relation (72) for $\eta_1, \eta_2$ and the Stroh eigenvalues

$$v_1 = -i\mu_1, \quad v_2 = -i\mu_2,$

and considering the following relations

$$v_1^2 + v_2^2 = -(\mu_1^2 + \mu_2^2) = \frac{2s_{12} + s_{66}}{s_{11}}, \quad v_1^2 v_2^2 = \mu_1^2 \mu_2^2 = \frac{s_{22}}{s_{11}},
$$

we deduce the expressions for singular displacements along the axes of propagation of the crack ($x_2 = 0$) as

$$
\hat{U}_1^+(\xi) = \hat{u}_1^+(\xi, x_2 = 0+) = \left( -\frac{[\text{Im}(\mu_1 + \mu_2) s_{11}]^{(1)}(\xi)}{|\xi|}, -i\frac{[s_{11} \mu_1 \mu_2 - s_{12}]^{(1)}(\xi)}{\xi} \right) \hat{\Sigma}^-,
$$

$$
\hat{U}_2^+(\xi) = \hat{u}_2^+(\xi, x_2 = 0+) = \left( i\frac{[s_{11} \mu_1 \mu_2 - s_{12}]^{(1)}(\xi)}{\xi}, 1\frac{[\text{Im}(\mu_1 + \mu_2)]^{(1)}(\xi)}{\xi} \right) \hat{\Sigma}^-,
$$

$$
\hat{U}_1^-(\xi) = \hat{u}_1^-(\xi, x_2 = 0-) = \left( \frac{[\text{Im}(\mu_1 + \mu_2) s_{11}]^{(2)}(\xi)}{|\xi|}, -i\frac{[s_{11} \mu_1 \mu_2 - s_{12}]^{(2)}(\xi)}{\xi} \right) \hat{\Sigma}^-,
$$

$$
\hat{U}_2^-(\xi) = \hat{u}_2^-(\xi, x_2 = 0-) = \left( i\frac{[s_{11} \mu_1 \mu_2 - s_{12}]^{(2)}(\xi)}{\xi}, 1\frac{[\text{Im}(\mu_1 + \mu_2)]^{(2)}(\xi)}{\xi} \right) \hat{\Sigma}^-.
$$

These expressions can be written in the form similar to (30) and (31)

$$
\hat{U}^+(\xi) = \left\{ \frac{1}{2\xi} (Y^{(1)} - \bar{Y}^{(1)}) - \frac{1}{2|\xi|} (Y^{(1)} + \bar{Y}^{(1)}) \right\} \hat{\Sigma},
$$

$$
\hat{U}^-(\xi) = \left\{ \frac{1}{2\xi} (\bar{Y}^{(2)} - Y^{(2)}) + \frac{1}{2|\xi|} (\bar{Y}^{(2)} + Y^{(2)}) \right\} \hat{\Sigma},
$$

where the Hermitian matrices $Y^{(1)}$ and $Y^{(2)}$ are

$$
Y^{(1),(2)} = \begin{bmatrix}
[s_{11} \text{Im}(\mu_1 + \mu_2)]^{(1),(2)} & -i[s_{11} \mu_1 \mu_2 - s_{12}]^{(1),(2)} \\
i[s_{11} \mu_1 \mu_2 - s_{12}]^{(1),(2)} & s_{22} \text{Im}(\mu_1 + \mu_2)]^{(1),(2)}
\end{bmatrix}.
$$

According to (16) and (17), the Fourier transforms of the symmetric and skew-symmetric weight functions are defined, respectively, as the jump and the average of the singular displacements across the plane containing the crack

$$
[\hat{U}](\xi) = \frac{1}{|\xi|} \left\{ i \text{sign}(\xi) \text{Im}(Y^{(1)} - \bar{Y}^{(2)}) - \text{Re}(Y^{(1)} + \bar{Y}^{(2)}) \right\} \hat{\Sigma},
$$

$$
(\hat{U})(\xi) = \frac{1}{2|\xi|} \left\{ i \text{sign}(\xi) \text{Im}(Y^{(1)} + \bar{Y}^{(2)}) - \text{Re}(Y^{(1)} - \bar{Y}^{(2)}) \right\} \hat{\Sigma}.
$$

We finally recover the expressions (34) and (35), previously derived from the direct solution of the Riemann–Hilbert problem for an interfacial crack by means of the Stroh formalism.