Complex anti-self-dual instantons and Cayley submanifolds

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1 Introduction

Let $M$ be a manifold of dimension 8, and let $\Omega$ be a 4-form which defines an almost $\text{Spin}(7)$-structure on $M$. An $\Omega$-anti-self-dual instanton is a connection $A$ on a vector bundle over $M$ such that the curvature $F_A$ satisfies

$$F_A + *(\Omega \wedge F_A) = 0. \quad (1)$$

If $M$ is an almost Calabi-Yau manifold, then the 4-form $\Omega$ can be written as

$$\Omega = 4 \text{Re}(\theta) + \frac{1}{2} \omega^2,$$

where $\omega \in \Omega^{1,1}(M)$ denotes the symplectic form and $\theta \in \Omega^{0,4}(M)$ is the complex volume form. The complex volume form induces an anti-linear involution $*_{\theta} : \Omega^{0,2}(M) \to \Omega^{0,2}(M)$. Then the anti-self-duality equation (1) is equivalent to

$$F_A^{1,1} \cdot \omega = 0 \quad (2)$$

and

$$(1 + *_{\theta}) F_A^{0,2} = 0. \quad (3)$$

The space of 2-forms splits as a direct sum

$$\Lambda^2 TM = \Lambda^2_+ TM \oplus \Lambda^2_- TM, \quad (4)$$

where

$$\Lambda^2_+ TM = \{ \varphi \in \Lambda^2 M : 3\varphi - *(\Omega \wedge \varphi) = 0 \} \quad (5)$$

and

$$\Lambda^2_- TM = \{ \varphi \in \Lambda^2 M : \varphi + *(\Omega \wedge \varphi) = 0 \}. \quad (6)$$
Note that $\Lambda^2_+ M$ is a vector space of dimension 7 and $\Lambda^2_-(M)$ is a vector space of dimension 21. Let $P_+$ and $P_-$ be the projections associated to the splitting (4). This implies

$$P_+ \varphi = \frac{1}{4} (\varphi + *(\Omega \wedge \varphi))$$

and

$$P_- \varphi = \frac{1}{4} (3 \varphi - *(\Omega \wedge \varphi)).$$

We denote by $\Omega^2_+ (M)$ the space of sections of the vector bundle $\Lambda^2_+ TM$. Similarly, $\Omega^2_- (M)$ is the space of sections of the vector bundle $\Lambda^2_- TM$.

If $\Omega$ is closed, then the anti-self-duality equation (1) implies the Yang-Mills equation $D^* A F_A = 0$.

The equations (1),(2) generalize the anti-self-dual equations in dimension 4 (see e.g. [7, 23]), and have been studied by various authors, including S. K. Donaldson and R. P. Thomas [8, 26], L. Baulieu, H. Kanno, and I. M. Singer [3], J. Chen [6], and G. Tian [27]. These submanifolds are also of considerable interest in mathematical physics.

G. Tian constructed a compactification of the moduli space of $\Omega$-anti-self-dual instantons over $M$. He proved that every sequence $A_k$ of $\Omega$-anti-self-dual instantons over $M$ has a subsequence, still denoted by $A_k$, such that

$$\lim_{k \to \infty} \int_M c_2(A_k) \wedge \psi = \int_M c_2(A_{\infty}) \wedge \psi + \int_S \Theta \psi,$$

where $c_2$ denotes the 4-form representing the second Chern class of the bundle, and $\psi$ is a smooth 4-form on $M$. Furthermore, $A_{\infty}$ is a $\Omega$-anti-self-dual instanton which is smooth outside a set of vanishing $H^4$-measure. Furthermore, $S$ is a Cayley submanifold, i.e. a submanifold calibrated by the 4-form $\Omega$. Cayley submanifolds were studied by R. Harvey and H. B. Lawson [9]. There is a rich class of examples. For instance, this class contains as limiting cases the holomorphic subvarieties and the special Lagrangian submanifolds of $M$. Special Lagrangian submanifolds have been studied extensively, see e.g. [10]. Cayley submanifolds play a role in high-energy physics, see for example [4].

Our aim in this paper is to construct smooth complex anti-self-dual instantons such that the energy density $|F_A|^2$ is concentrated near a given Cayley submanifold $S$. 
In the first step, we construct a suitable family of approximate solutions. To this end, we assume that the normal bundle $NS$ can be endowed with a complex structure $J$ and a complex volume form $\omega$. Each approximate solution is described by a set $(v, \lambda, J, \omega)$, where $v$ is a section of the normal bundle of $S$, $\lambda$ is a positive function on $S$, and $(J, \omega)$ is a $SU(2)$-structure on $NS$. The covariant derivative of the pair $(J, \omega)$ can be described by a 1-form $\theta$ with values in the Lie algebra $\Lambda^2 NS$.

The covariant derivative of the 4-form $\Omega$ can be written in the form

$$\nabla_X \Omega = \sum_{k=1}^{8} i_{e_k} \alpha(X) \wedge i_{e_k} \Omega,$$

where $\alpha$ is a 1-form with values in $\Lambda^2 TM$.

We consider the elliptic complex

$$0 \rightarrow \Omega^0(M) \rightarrow \Omega^1(M) \rightarrow \Omega^2_+(M) \rightarrow 0.$$

The first and the second cohomology groups associated to this elliptic complex are $H^0(M)$ and $H^1(M)$. The third cohomology group is denoted by $H^2_+(M)$.

**Theorem 1.1.** Suppose that $H^2_+(M) = 0$. Then, for each $\varepsilon > 0$, there exists a mapping $\Xi_\varepsilon$ which assigns to each set of glueing data $(v, \lambda, J, \omega) \in C^{2,\gamma}(S)$ a section of the vector bundle $V \oplus W$ of class $C^{\gamma}(S)$ such that the following holds.

(i) If $(v, \lambda, J, \omega)$ is a set of glueing data such that

$$\|v\|_{C^{1,\gamma}(S)} \leq K,$$
$$\|\lambda\|_{C^{1,\gamma}(S)} \leq K, \quad \inf \lambda \geq 1,$$
$$\|(J, \omega)\|_{C^{1,\gamma}(S)} \leq K,$$

then we have the estimate

$$\left\| \Xi_\varepsilon(v, \lambda, J, \omega) \right\|_{C^{1,\gamma}(S)} \leq C \varepsilon^{\frac{1}{32}}.$$
(ii) If $\Xi_\varepsilon(v, \lambda, J, \omega) = 0$, then the approximate solution $A$ corresponding to $(v, \lambda, J, \omega)$ can be deformed to a nearby connection $\tilde{A}$ satisfying $F_{\tilde{A}} + *(\Omega \wedge F_{\tilde{A}}) = 0$.

In Section 2, we study the mapping properties of a model operator on $\mathbb{R}^8$.

In Section 3, we construct a family of approximate solutions of the Yang-Mills equations. More precisely, given any set of gluing data $(v, \lambda, J, \omega)$ satisfying
\[
\|v\|_{C^{1,\gamma}(S)} \leq K, \\
\|\lambda\|_{C^{1,\gamma}(S)} \leq K, \quad \inf \lambda \geq 1, \\
\|(J, \omega)\|_{C^{1,\gamma}(S)} \leq K,
\]
we construct a connection $A$ such that
\[
\|F_A + *(\Omega \wedge F_A)\|_{C^{\gamma}_3(M)} \leq C \varepsilon^2.
\]
Here, the weighted Hölder space $C^{\gamma}_\nu(M)$ is defined as
\[
\|u\|_{C^{\gamma}_\nu(M)} = \sup (\varepsilon + \text{dist}(p, S))^{\nu} |u(p)| \\
\quad + \sup_{4\text{dist}(p_1, p_2) \leq \varepsilon + \text{dist}(p_1, S) + \text{dist}(p_2, S)} (\varepsilon + \text{dist}(p_1, S) + \text{dist}(p_2, S))^{\nu + \gamma} \frac{|u(p_1) - u(p_2)|}{\text{dist}(p_1, p_2)^\gamma}.
\]

In Section 4, we derive estimates for the linearized operator which are independent of $\varepsilon$.

In Section 5, we apply the contraction mapping principle to deform the approximate solution $A$ to a nearby connection $\tilde{A} = A + a$ such that
\[
(I - P)(F_{\tilde{A}} + *(\Omega \wedge F_{\tilde{A}})) = 0,
\]
where $(I - P)$ is the fibrewise projection from $C^{\gamma}_\nu(M)$ to the subspace $G^{\gamma}_\nu(M)$.

In particular, if the balancing condition
\[
P(F_{\tilde{A}} + *(\Omega \wedge F_{\tilde{A}})) = 0
\]
is satisfied, then $\tilde{A}$ is an $\Omega$-anti-self-dual instanton.

In Section 6, we calculate the leading term in the asymptotic expansion of
\[
P(F_{\tilde{A}} + *(\Omega \wedge F_{\tilde{A}})) = 0.
\]
This concludes the proof of Theorem 1.1.

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2 The model problem on $\mathbb{R}^8$

The $\text{Spin}(7)$-structure on $\mathbb{R}^8$ is given by

$$\Omega = -e_1 \wedge e_2 \wedge e_1^+ \wedge e_2^+ - e_1 \wedge e_2 \wedge e_3^+ \wedge e_4^+ - e_3 \wedge e_4 \wedge e_1^+ \wedge e_2^+
- e_3 \wedge e_4 \wedge e_3^+ \wedge e_4^+ + e_1 \wedge e_3 \wedge e_2^+ \wedge e_4^+ - e_1 \wedge e_3 \wedge e_1^+ \wedge e_3^+
- e_2 \wedge e_4 \wedge e_2^+ \wedge e_3^+ + e_2 \wedge e_4 \wedge e_1^+ \wedge e_3^+ - e_1 \wedge e_4 \wedge e_2^+ \wedge e_3^+
- e_1 \wedge e_4 \wedge e_1^+ \wedge e_4^+ - e_2 \wedge e_3 \wedge e_2^+ \wedge e_3^+ - e_2 \wedge e_3 \wedge e_1^+ \wedge e_4^+
+ e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_1^+ \wedge e_2^+ \wedge e_3^+ \wedge e_4^+.$$

Hence, the 2-forms

$$e_1 \wedge e_2 + e_3 \wedge e_4 - e_1^+ \wedge e_2^+ - e_3^+ \wedge e_4^+,\,$$
$$e_1 \wedge e_3 - e_2 \wedge e_4 - e_1^+ \wedge e_3^+ + e_2^+ \wedge e_4^+,
$$
$$e_1 \wedge e_4 + e_2 \wedge e_3 - e_1^+ \wedge e_4^+ - e_2^+ \wedge e_3^,+\,$$
$$e_1 \wedge e_1^+ + e_2 \wedge e_2^+ + e_3 \wedge e_3^+ + e_4 \wedge e_4^+,$$
$$e_1 \wedge e_2^+ - e_2 \wedge e_1^+ - e_3 \wedge e_4^+ + e_4 \wedge e_3^+,$$
$$e_1 \wedge e_3^+ + e_2 \wedge e_4^+ - e_3 \wedge e_1^+ - e_4 \wedge e_2^+,\,$$
$$e_1 \wedge e_4^+ - e_2 \wedge e_3^+ + e_3 \wedge e_2^+ - e_4 \wedge e_1^+$$

form a basis for $\Lambda^2_+ \mathbb{R}^8$.

Let now $E$ be a Cayley subspace of $\mathbb{R}^8$, i.e. a subspace calibrated by $\Omega$. The group $\text{Spin}(7)$ acts transitively on the set of Cayley subspaces and leaves the 4-form $\Omega$ invariant (cf. [7]). Hence, we may assume without loss of generality that $E$ is spanned by $\{e_i^+ : 1 \leq i \leq 4\}$ and $E^\perp$ is spanned by $\{e_i^: : 1 \leq i \leq 4\}$.

We define two vector spaces $V \subset E \otimes E^\perp$ and $W \subset E \otimes E^\perp \otimes E^\perp$ over the submanifold $S$. The following elements form a basis for $V$:

$$e_1 \otimes e_1^+ + e_2 \otimes e_2^+ + e_3 \otimes e_3^+ + e_4 \otimes e_4^+,\,$$
$$e_1 \otimes e_2^+ - e_2 \otimes e_1^+ - e_3 \otimes e_4^+ + e_4 \otimes e_3^+,$$
$$e_1 \otimes e_3^+ + e_2 \otimes e_4^+ - e_3 \otimes e_1^+ - e_4 \otimes e_2^+,$$
$$e_1 \otimes e_4^+ - e_2 \otimes e_3^+ + e_3 \otimes e_2^+ - e_4 \otimes e_1^+.$$

The following elements form a basis for $W$:

$$e_1 \otimes (e_1^+ \otimes e_1^+ + e_2^+ \otimes e_2^+ + e_3^+ \otimes e_3^+ + e_4^+ \otimes e_4^+)
+ e_2 \otimes (e_2^+ \otimes e_1^+ - e_1^+ \otimes e_2^+ + e_4^+ \otimes e_3^+ - e_3^+ \otimes e_4^+),\,$$
$$e_3 \otimes (e_3^+ \otimes e_1^+ - e_4^+ \otimes e_2^+ - e_1^+ \otimes e_3^+ + e_2^+ \otimes e_4^+),\,$$
$$e_4 \otimes (e_4^+ \otimes e_1^+ + e_3^+ \otimes e_2^+ - e_2^+ \otimes e_3^+ - e_1^+ \otimes e_4^+).$$
agrees with the basic instanton along $E$

The adjoint operator $L$
The linearized operator $B$

$B e = L e - e e^\perp$

\begin{align*}
e_1 \otimes (e_1^+ + e_4^+ - e_1^+ e_1^+ - e_4^+ e_2^+ + e_4^+ e_3^+ - e_3^+ e_4^+) \\
- e_2 \otimes (e_4^+ + e_4^+ e_2^+ + e_3^+ e_3^+ e_4^+ e_4^+) \\
- e_3 \otimes (e_1^+ e_1^+ + e_3^+ e_2^+ - e_2^+ e_3^+ - e_1^+ e_4^+) \\
+ e_4 \otimes (e_3^+ e_1^+ - e_1^+ e_2^+ - e_1^+ e_3^+ + e_2^+ e_4^+) \\

&= e_1 \otimes (e_3^+ e_1^+ - e_4^+ e_2^+ - e_1^+ e_3^+ + e_2^+ e_4^+) \\
+ e_2 \otimes (e_4^+ e_1^+ + e_3^+ e_2^+ - e_2^+ e_3^+ - e_1^+ e_4^+) \\
- e_3 \otimes (e_1^+ e_1^+ + e_3^+ e_2^+ + e_3^+ e_3^+ + e_4^+ e_1^+) \\
- e_4 \otimes (e_2^+ e_1^+ - e_1^+ e_2^+ + e_4^+ e_3^+ - e_3^+ e_4^+),
\end{align*}

\begin{align*}
e_1 \otimes (e_4^+ e_1^+ + e_3^+ e_2^+ - e_1^+ e_3^+ + e_2^+ e_4^+) \\
- e_2 \otimes (e_4^+ e_1^+ - e_4^+ e_2^+ - e_1^+ e_3^+ + e_2^+ e_4^+) \\
+ e_3 \otimes (e_2^+ e_1^+ - e_1^+ e_2^+ + e_4^+ e_3^+ - e_3^+ e_4^+) \\
- e_4 \otimes (e_1^+ e_1^+ + e_2^+ e_2^+ + e_3^+ e_3^+ + e_4^+ e_4^+).
\end{align*}

Let $B$ be a connection which is invariant under translations along $E$ and agrees with the basic instanton along $E^\perp$. More precisely, we define

\begin{align*}
B(e_1^+) &= \frac{-y_2 i - y_3 j - y_4 \ell}{\varepsilon^2 + |y|^2} \\
B(e_2^+) &= \frac{y_1 i - y_4 j + y_3 \ell}{\varepsilon^2 + |y|^2} \\
B(e_3^+) &= \frac{y_1 i + y_4 j - y_2 \ell}{\varepsilon^2 + |y|^2} \\
B(e_4^+) &= \frac{-y_3 i + y_2 j + y_1 \ell}{\varepsilon^2 + |y|^2},
\end{align*}

where

\begin{align*}
i(e_1^+) &= -e_2^+, \quad i(e_2^+) = e_1^+, \quad i(e_3^+) = e_4^+, \quad i(e_4^+) = -e_3^+, \\
j(e_1^+) &= -e_3^+, \quad j(e_2^+) = -e_4^+, \quad j(e_3^+) = e_1^+, \quad j(e_4^+) = e_2^+, \\
\kappa(e_1^+) &= -e_1^+, \quad \kappa(e_2^+) = e_3^+, \quad \kappa(e_3^+) = -e_2^+, \quad \kappa(e_4^+) = e_1^+.
\end{align*}

Furthermore, $B(e_i) = 0$ for $1 \leq i \leq 4$.

The linearized operator $L_B : \Omega^1(\mathbb{R}^8) \rightarrow \Omega^2_+(\mathbb{R}^8)$ is given by

$L_B a = 2 P a D_B a.$

The adjoint operator $L_B^* : \Omega^2_+(\mathbb{R}^8) \rightarrow \Omega^1(\mathbb{R}^8)$ is given by

$L_B^* \varphi = 2 D_B^* \varphi.$
We define the weighted Hölder space $C^\gamma_\nu(R^8)$ by
\[
\|u\|_{C^\gamma_\nu(R^8)} = \sup (\varepsilon + |y|)^\nu |u(x,y)| + \sup_{4(|x_1-x_2|+|y_1-y_2|) \leq \varepsilon + |y_1|+|y_2|} (\varepsilon + |y_1| + |y_2|)^{\nu+\gamma} \frac{|u(x_1,y_1)-u(x_2,y_2)|}{(|x_1-x_2|+|y_1-y_2|)^\gamma}.
\]

More generally, we define
\[
\|u\|_{C^k,\gamma_\nu(R^8)} = \sum_{l=0}^k \|\nabla^l u\|_{C^\gamma_\nu(R^8)}.\]

Let $G^{\nu,\gamma}(R^8)$ be the set of all $\varphi \in \Omega^2_+(R^8)$ such that $\varphi \in C^{k,\gamma_\nu}(R^8)$, and
\[
\int_{x+E^\perp} \sum_{i,j=1}^4 (\varepsilon s_{ik} + t_{ikl} y_l) \langle \varphi(e_i, e_\perp^j), F_B(e^\perp_k, e^\perp_j) \rangle = 0
\]
for all $x \in E$, $s \in V$, and $t \in W$.

We first derive a Weitzenböck formula for the operator $L_B L^*_B : \Omega^2_+(R^8) \to \Omega^2_+(R^8)$. We shall need two algebraic facts which can be verified by direct calculation. For simplicity, let $e_5 = e^\perp_1$, $e_6 = e^\perp_2$, $e_7 = e^\perp_3$, $e_8 = e^\perp_4$.

**Lemma 2.1.** For every $\varphi \in \Lambda^2_+ \mathbb{R}^8$, we have
\[
2 P_+ (e_k \wedge (i_{e_1} \varphi) + e_l \wedge (i_{e_k} \varphi)) = \delta_{kl} \varphi.
\]

**Lemma 2.2.** For every $\varphi \in \Lambda^2_+ \mathbb{R}^8$, we have
\[
\sum_{k,l=1}^8 e_k \wedge [F_B(e_k, e_l), i_{e_l} \varphi] \in \Lambda^2_+ \mathbb{R}^8.
\]

**Proposition 2.3.** The operator $L_B L^*_B$ satisfies the Weitzenböck formula
\[
L_B L^*_B \varphi = \nabla_B^2 \varphi - 2 \sum_{k,l=1}^4 e^\perp_k \wedge [F_B(e^\perp_k, e^\perp_l), i_{e^\perp_l} \varphi]
\]
for every $\varphi \in \Omega^2_+(R^8)$. 

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Proof. For every \( \varphi \in \Omega^2_+ (\mathbb{R}^8) \), we obtain

\[
4 D_B D_B^* \varphi = -4 \sum_{k,l=1}^8 e_k \wedge (i_{e_l} D_B e_k D_B e_l \varphi)
\]

\[
= -2 \sum_{k,l=1}^8 (e_k \wedge (i_{e_l} D_B e_k D_B e_l \varphi) + e_l \wedge (i_{e_k} D_B e_k D_B e_l \varphi))
\]

\[
- 2 \sum_{k,l=1}^8 e_k \wedge (i_{e_l} (D_B e_k D_B e_l \varphi - D_B e_l D_B e_k \varphi))
\]

\[
= -2 \sum_{k,l=1}^8 (e_k \wedge (i_{e_l} D_B e_k D_B e_l \varphi) + e_l \wedge (i_{e_k} D_B e_k D_B e_l \varphi))
\]

\[
- 2 \sum_{k,l=1}^8 e_k \wedge [F_B(e_k, e_l), i_{e_l} \varphi].
\]

Since \( \varphi \in \Omega^2_+ (\mathbb{R}^8) \) and \( \Omega \) is parallel, it follows that \( D_B e_k D_B e_l \varphi \in \Omega^2_+ (\mathbb{R}^8) \). Using Lemma 2.1, we obtain

\[
2 P_+(e_k \wedge (i_{e_l} D_B e_k D_B e_l \varphi) + e_l \wedge (i_{e_k} D_B e_k D_B e_l \varphi)) = \delta_{kl} D_B e_k D_B e_l \varphi.
\]

From this it follows that

\[
4 P_+ D_B D_B^* \varphi = \nabla_B^* \nabla_B \varphi - 2 \sum_{k,l=1}^8 P_+(e_k \wedge [F_B(e_k, e_l), i_{e_l} \varphi]).
\]

Moreover, Lemma 2.2 implies that

\[
\sum_{k,l=1}^8 e_k \wedge [F_B(e_k, e_l), i_{e_l} \varphi] \in \Omega^2_+ (\mathbb{R}^8).
\]

Thus, we conclude that

\[
4 P_+ D_B D_B^* \varphi = \nabla_B^* \nabla_B \varphi - 2 \sum_{k,l=1}^8 e_k \wedge [F_B(e_k, e_l), i_{e_l} \varphi].
\]

This proves the assertion.

**Proposition 2.4.** Suppose that \( \psi \in G^{\gamma}_{3+\nu}(\mathbb{R}^8) \) has compact support. Then there exists some \( \varphi \in G^{2,\gamma}_{1+\nu}(\mathbb{R}^8) \) such that

\[
\| \varphi \|_{C^{2,\gamma}_{1+\nu}(\mathbb{R}^8)} \leq C \| \psi \|_{C^{\gamma}_{3+\nu}(\mathbb{R}^8)}
\]

and

\[
L_B L_B^* \varphi = \psi.
\]
Proof. Since $\psi \in \Omega^2_+(\mathbb{R}^8)$, we may write
\[
\psi = (e_1 \wedge e_2 + e_3 \wedge e_4 - e_1^+ \wedge e_2^+ - e_3^+ \wedge e_4^+) \otimes g_2 \\
+ (e_1 \wedge e_3 - e_2 \wedge e_4 - e_1^+ \wedge e_3^+ + e_2^+ \wedge e_4^+) \otimes g_3 \\
+ (e_1 \wedge e_4 + e_2 \wedge e_3 - e_1^+ \wedge e_4^+ - e_2^+ \wedge e_3^+) \otimes g_4 \\
+ (e_1 \wedge e_1^+ + e_2 \wedge e_2^+ + e_3 \wedge e_3^+ + e_4 \wedge e_4^+) \otimes g_5^+ \\
+ (e_1 \wedge e_2^+ - e_2 \wedge e_1^+ - e_3 \wedge e_4^+ + e_4 \wedge e_3^+) \otimes g_6^+ \\
+ (e_1 \wedge e_3^+ + e_2 \wedge e_4^+ - e_3 \wedge e_1^+ - e_4 \wedge e_2^+) \otimes g_7^+ \\
+ (e_1 \wedge e_4^+ - e_2 \wedge e_3^+ + e_3 \wedge e_2^+ - e_4 \wedge e_1^+) \otimes g_8^+,
\]
where $g_j, g_j^+ \in C^7_{3+\nu}(\mathbb{R}^8)$. Furthermore, since $\psi \in G^7_{3+\nu}(\mathbb{R}^8)$, we deduce that
\[
\int_{x+\mathbb{E}^+} \sum_{j=1}^4 \langle g_j^+, F_B(X, e_j^+) \rangle = 0
\]
for all $x \in \mathbb{E}$ and all vector fields of the form
\[
X = \varepsilon w_k e_k^+ + \mu y_k e_k^+ + r_{kl} y_l e_k^+.
\]
Using Corollary 3.6 in \[5\], we can find $f_j, f_j^+ \in C^2_{3+\nu}(\mathbb{R}^8)$ such that
\[
\nabla_B \nabla_B f_j = g_j
\]
and
\[
\nabla_B^* \nabla_B f_j^+ - 2 \sum_{k=1}^4 [F_B(e_k^+, e_k^+), f_k^+] = g_j^+
\]
and
\[
\int_{x+\mathbb{E}^+} \sum_{j=1}^4 \langle f_j^+, F_B(X, e_j^+) \rangle = 0
\]
for all $x \in \mathbb{E}$ and all vector fields of the form
\[
X = \varepsilon w_k e_k^+ + \mu y_k e_k^+ + r_{kl} y_l e_k^+.
\]
We now define
\[
\varphi = (e_1 \wedge e_2 + e_3 \wedge e_4 - e_1^+ \wedge e_2^+ - e_3^+ \wedge e_4^+) \otimes f_2 \\
+ (e_1 \wedge e_3 - e_2 \wedge e_4 - e_1^+ \wedge e_3^+ + e_2^+ \wedge e_4^+) \otimes f_3 \\
+ (e_1 \wedge e_4 + e_2 \wedge e_3 - e_1^+ \wedge e_4^+ - e_2^+ \wedge e_3^+) \otimes f_4 \\
+ (e_1 \wedge e_1^+ + e_2 \wedge e_2^+ + e_3 \wedge e_3^+ + e_4 \wedge e_4^+) \otimes f_5^+ \\
+ (e_1 \wedge e_2^+ - e_2 \wedge e_1^+ - e_3 \wedge e_4^+ + e_4 \wedge e_3^+) \otimes f_6^+ \\
+ (e_1 \wedge e_3^+ + e_2 \wedge e_4^+ - e_3 \wedge e_1^+ - e_4 \wedge e_2^+) \otimes f_7^+ \\
+ (e_1 \wedge e_4^+ - e_2 \wedge e_3^+ + e_3 \wedge e_2^+ - e_4 \wedge e_1^+) \otimes f_8^+.
\]
Then \( \varphi \in G^{2,\gamma}_{1+\nu}(\mathbb{R}^8) \), and

\[ L_B L^*_B \varphi = \psi \]

by Proposition 2.3. This proves the assertion.

**Corollary 2.5.** Suppose that \( \psi \in G^{2,\gamma}_{3+\nu}(\mathbb{R}^8) \) has compact support. Then there exists a 1-form \( a \in C^{1,\gamma}_{2+\nu}(\mathbb{R}^8) \) such that

\[ \| a \|_{C^{1,\gamma}_{2+\nu}(\mathbb{R}^8)} \leq C \| \psi \|_{C^{3,\gamma}_{3+\nu}(\mathbb{R}^8)} \]

and

\[ L_B a = \psi. \]

**Proof.** By Proposition 2.4, there exists some \( \varphi \in G^{2,\gamma}_{1+\nu}(\mathbb{R}^8) \) such that

\[ \| \varphi \|_{C^{2,\gamma}_{1+\nu}(\mathbb{R}^8)} \leq C \| \psi \|_{C^{3,\gamma}_{3+\nu}(\mathbb{R}^8)} \]

and

\[ L_B L^*_B \varphi = \psi. \]

Let \( a = L^*_B \varphi \). Then \( a \in C^{1,\gamma}_{2+\nu}(\mathbb{R}^8) \) satisfies

\[ \| a \|_{C^{1,\gamma}_{2+\nu}(\mathbb{R}^8)} \leq C \| \psi \|_{C^{3,\gamma}_{3+\nu}(\mathbb{R}^8)} \]

and

\[ L_B a = \psi. \]

This proves the assertion.

**Proposition 2.6.** Let \( 0 < \nu < 1 \). Suppose that \( \psi \in G^{3,\gamma}_{3+\nu}(\mathbb{R}^8) \) is supported in the set \( \{(x, y) \in \mathbb{R}^8 : |x| \leq \delta, |y| \leq 2\delta^4\} \). Then there exists a 1-form \( a \in C^{1,\gamma}_{2+\nu}(\mathbb{R}^8) \) such that \( a \) is supported in \( \{(x, y) \in \mathbb{R}^8 : |x| \leq 2\delta, |y| \leq 2\delta^2\} \),

\[ \| a \|_{C^{1,\gamma}_{2+\nu}(\mathbb{R}^8)} \leq C \| \psi \|_{C^{3,\gamma}_{3+\nu}(\mathbb{R}^8)} \]

and

\[ \| L_B a - \psi \|_{C^{3,\gamma}_{3+\nu}(\{(x, y) \in \mathbb{R}^8 : |y| \leq 2\delta^4\})} \leq C \delta \| \psi \|_{C^{3,\gamma}_{3+\nu}(\mathbb{R}^8)}, \]

and

\[ \| L_B a - \psi \|_{C^{3,\gamma}_{3+\nu}(\mathbb{R}^8)} \leq C | \log \delta |^{-1} \| \psi \|_{C^{3,\gamma}_{3+\nu}(\mathbb{R}^8)}. \]
Proof. By Corollary 2.5, there exists a 1-form \( a \in C^{1,\gamma}_{2+\nu}(\mathbb{R}^8) \) such that
\[
\|a\|_{C^{1,\gamma}_{2+\nu}(\mathbb{R}^8)} \leq C \|\psi\|_{C^{\gamma,\nu}_{3+\nu}(\mathbb{R}^8)}
\]
and
\[
\mathbb{L}_B a = \psi.
\]
Let \( \zeta \) be a cut-off function on \( E \) such that \( \zeta(x) = 1 \) for \( |x| \leq \delta \), \( \zeta(x) = 0 \) for \( |x| \geq 2\delta \), and
\[
\sup \delta |\nabla \zeta| \leq C.
\]
Furthermore, let \( \eta \) be a cut-off function on \( E^\perp \) satisfying \( \eta(y) = 1 \) for \( |y| \leq 2\delta^4 \), \( \eta(y) = 0 \) for \( |y| \geq 2\delta^2 \), and
\[
\sup \frac{|y|}{|\nabla \eta|} \leq C |\log \delta|^{-1}.
\]
Then we have the estimates
\[
\|\eta \zeta a\|_{C^{1,\gamma}_{2+\nu}(\mathbb{R}^8)} \leq C \|\psi\|_{C^{\gamma,\nu}_{3+\nu}(\mathbb{R}^8)}
\]
and
\[
\|\mathbb{L}_B(\zeta a) - \psi\|_{C^{\gamma}_{3+\nu}(\{(x,y)\in\mathbb{R}^8:|y|\leq 2\delta^4\})}
\]
\[=
\|\mathbb{L}_B(\zeta a) - \zeta \mathbb{L}_B a\|_{C^{\gamma}_{3+\nu}(\{(x,y)\in\mathbb{R}^8:|y|\leq 2\delta^4\})}
\]
\[\leq C \delta \|a\|_{C^{\gamma}_{3+\nu}(\mathbb{R}^8)}
\]
\[\leq C \delta \|\psi\|_{C^{\gamma}_{3+\nu}(\mathbb{R}^8)}
\]
and
\[
\|\mathbb{L}_B(\eta \zeta a) - \psi\|_{C^{\gamma}_{3+\nu}(\mathbb{R}^8)}
\]
\[=
\|\mathbb{L}_B(\eta \zeta a) - \eta \zeta \mathbb{L}_B a\|_{C^{\gamma}_{3+\nu}(\mathbb{R}^8)}
\]
\[\leq C \|\log \delta|^{-1} \|a\|_{C^{\gamma}_{3+\nu}(\mathbb{R}^8)}
\]
\[\leq C \|\log \delta|^{-1} \|\psi\|_{C^{\gamma}_{3+\nu}(\mathbb{R}^8)}.
\]
From this the assertion follows.

3 Construction of the approximate solutions

In this section, we outline the construction of a certain class of approximate solutions. To this end, we assume that the normal bundle \( NS \) can be endowed with a \( SU(2) \)-structure \( (J, \omega) \). Here, \( J \) is a complex structure and \( \omega \)
is a complex volume form on $NS$.

Let $\nabla' = \nabla + \theta$ be a connection on the normal bundle $NS$ such that $\theta$ is a 1-form with values in the Lie algebra $\Lambda^2 NS$ and $(J, \omega)$ is parallel with respect to the connection $\nabla'$. The 1-form $\theta$ is uniquely determined by the covariant derivative of the pair $(J, \omega)$ with respect to the Levi-Civita connection $\nabla$. Since $(J, \omega)$ is parallel with respect to $\nabla'$, the connection induced by $\nabla'$ on the bundle $\Lambda^2 NS$ is flat.

The connection $\nabla'$ induces a splitting of the tangent space $TNS$ into horizontal and vertical subspaces. Let $\{e'_i : 1 \leq i \leq 4\}$ be an orthonormal basis for the horizontal subspace with respect to $\nabla'$, and let $\{e^+_j : 1 \leq j \leq 4\}$ be a $SU(2)$ basis for the vertical subspace.

In the first step, we define a connection on the pull-back bundle $\pi^* NS$ of the normal bundle under the natural projection $\pi : NS \to S$. Since we may identify a neighborhood of $S$ in $M$ with a neighborhood of the zero section in $NS$, this gives a connection on a small neighborhood of $S$ in $M$. In the second step, we show that this connection can be extended to the whole of $M$ using suitable cut-off functions.

The glueing data consist of a set $(v, \lambda, J, \omega)$, where $v$ is a section of the normal bundle $NS$, $\lambda$ is a positive function on $S$, and $(J, \omega)$ is a $SU(2)$ structure on the normal bundle $NS$. Let $\{i, j, k\}$ be a basis for the Lie algebra $su(NS)$ such that

\[
\begin{align*}
  i(e^+_1) &= -e^+_2, & i(e^+_2) &= e^+_1, & i(e^+_3) &= e^+_4, & i(e^+_4) &= -e^+_3, \\
  j(e^+_1) &= -e^+_3, & j(e^+_2) &= -e^+_4, & j(e^+_3) &= e^+_1, & j(e^+_4) &= e^+_2, \\
  k(e^+_1) &= -e^+_4, & k(e^+_2) &= e^+_3, & k(e^+_3) &= -e^+_2, & k(e^+_4) &= e^+_1.
\end{align*}
\]

We consider a connection of the form $D_A = \nabla' + A$. The vertical components of $A$ are defined by

\[
\begin{align*}
  A(e^+_1) &= \frac{-(y - ev)_2 i - (y - ev)_3 j - (y - ev)_4 k}{\varepsilon^2 \lambda^2 + |y - ev|^2}, \\
  A(e^+_2) &= \frac{(y - ev)_1 i - (y - ev)_4 j + (y - ev)_3 k}{\varepsilon^2 \lambda^2 + |y - ev|^2}, \\
  A(e^+_3) &= \frac{(y - ev)_4 i + (y - ev)_1 j - (y - ev)_2 k}{\varepsilon^2 \lambda^2 + |y - ev|^2}, \\
  A(e^+_4) &= \frac{-(y - ev)_3 i + (y - ev)_2 j + (y - ev)_1 k}{\varepsilon^2 \lambda^2 + |y - ev|^2}.
\end{align*}
\]

Since the basic instanton on $\mathbb{R}^4$ is $SU(2)$-equivariant, this definition is independent of the choice of $SU(2)$-frame $\{e^+_j : 1 \leq j \leq 4\}$. Furthermore, the
horizontal components of \( A \) are defined by
\[
A(e_i') = -\epsilon \nabla_i' v_k A(e_k^\perp) - \lambda^{-1} \nabla_i \lambda \left( (y - \epsilon v)_k \right) A(e_k^\perp)
\]
for \( 1 \leq i \leq 4 \).

**Lemma 3.1.** The curvature of \( A \) is given by
\[
F_A(e_i', e_j^\perp) = -\left( \epsilon \nabla_i' v_k + \lambda^{-1} \nabla_i \lambda \left( (y - \epsilon v)_k \right) \right) F_A(e_k^\perp, e_j^\perp)
\]
and
\[
F_A(e_i', e_j') = \left( \epsilon \nabla_i' v_k + \lambda^{-1} \nabla_i \lambda \left( (y - \epsilon v)_k \right) \right)
\cdot \left( \epsilon \nabla_j' v_l + \lambda^{-1} \nabla_j \lambda \left( (y - \epsilon v)_l \right) \right) F_A(e_k^\perp, e_l^\perp)
\]
\[
+ C_{ij} + A(C_{ij} (y - \epsilon v)),
\]
where \( C_{ij} \in \Lambda^2_{NS} \) is the curvature of the connection \( \nabla' \).

If \( \{e_i : 1 \leq i \leq 4 \} \) is an orthonormal basis for the horizontal subspace with respect to the Levi-Civita connection \( \nabla \), then we obtain the following result:

**Lemma 3.2.** The curvature of \( A \) satisfies
\[
F_A(e_i, e_j^\perp) = -\left( \epsilon \nabla_i v_k + \lambda^{-1} \nabla_i \lambda \left( (y - \epsilon v)_k \right) + \theta_{i,kl} (y - \epsilon v)_l \right) F_A(e_k^\perp, e_j^\perp)
\]
and
\[
F_A(e_i, e_j') = \left( \epsilon \nabla_i v_k + \lambda^{-1} \nabla_i \lambda \left( (y - \epsilon v)_k \right) + \theta_{i,km} (y - \epsilon v)_m \right)
\cdot \left( \epsilon \nabla_j v_l + \lambda^{-1} \nabla_j \lambda \left( (y - \epsilon v)_l \right) + \theta_{j,ln} (y - \epsilon v)_n \right) F_A(e_k^\perp, e_l^\perp)
\]
\[
+ C_{ij} + A(C_{ij} (y - \epsilon v)),
\]
where \( C_{ij} \in \Lambda^2_{NS} \) is the curvature of \( \nabla' \).

**Lemma 3.3.** Suppose that \( \mu \) is constant and \( r \) is a section of the vector bundle \( \Lambda^2_{NS} \) such that \( \nabla' r = 0 \). Let
\[
u = (\epsilon^2 \lambda^2 + |y - \epsilon v|^2)^{-\frac{1}{2}} \left( \mu (y - \epsilon v)_k + r_{kl} (y - \epsilon v)_l \right) e_k^\perp.
\]
Then the covariant derivative of \( \nu \) satisfies the estimate
\[
\|D_A \nu\|_{C^2(\mathcal{M})} \leq C \epsilon^2.
\]
Hence, as we move away from the submanifold $S$, the connection $A$ approaches a flat connection. Therefore, we can extend $A$ trivially to $M$.

Our aim is to derive estimates for $F_A + *(\Omega \wedge F_A)$ in $C^*_3(M)$. To this end, we assume that the glueing data $(v, \lambda, J, \omega)$ satisfy the estimates

\[ \|v\|_{C^{1,5}(M)(S)} \leq K, \]
\[ \|\lambda\|_{C^{1,5}(M)(S)} \leq K, \quad \inf \lambda \geq 1, \]
\[ \|(J, \omega)\|_{C^{1,5}(M)(S)} \leq K \]

for some $K > 0$. All implicit constants will depend on $K$.

**Proposition 3.4.** If the set $(v, \lambda, J, \omega)$ is admissible, then we have the estimate

\[ \|F_A + *(\Omega \wedge F_A)\|_{C^*_3(M)} \leq C \varepsilon^2. \]

**Proof.** Let $\Omega_0$ be a 4-form which defines an almost $Spin(7)$-structure on $M$ such that $\Omega(x) = \Omega_0(x)$ for all $x \in S$ and $\nabla_X \Omega(x) = 0$ for all $x \in S$ and $X \in NS_x$. Then we have the estimate

\[
\begin{align*}
\|F_A + *(\Omega_0 \wedge F_A)\|_{C^*_3(M)} & \leq \|F_A(e_1, e_2) + F_A(e_3, e_4) - F_A(e_1^+, e_2^+) - F_A(e_3^+, e_4^+)\|_{C^*_3(M)} \\
& + \|F_A(e_1, e_3) - F_A(e_2, e_4) - F_A(e_1^+, e_3^+) + F_A(e_2^+, e_4^+)\|_{C^*_3(M)} \\
& + \|F_A(e_1, e_4) + F_A(e_2, e_3) - F_A(e_1^+, e_4^+) - F_A(e_2^+, e_3^+)\|_{C^*_3(M)} \\
& + \|F_A(e_1, e_1^+) + F_A(e_2, e_2^+) + F_A(e_3, e_3^+) + F_A(e_4, e_4^+)\|_{C^*_3(M)} \\
& + \|F_A(e_1, e_2^+ - F_A(e_2, e_1^+) - F_A(e_3, e_4^+ + F_A(e_4, e_3^+)\|_{C^*_3(M)} \\
& + \|F_A(e_1, e_3^+) + F_A(e_2, e_4^+) - F_A(e_3, e_1^+) - F_A(e_4, e_2^+)\|_{C^*_3(M)} \\
& + \|F_A(e_1, e_4^+) - F_A(e_2, e_3^+) + F_A(e_3, e_2^+) - F_A(e_4, e_1^+)\|_{C^*_3(M)} \\
& \leq C \varepsilon^2.
\end{align*}
\]

Using the identities

\[
\begin{align*}
F_A(e_1^+, e_2^+) + F_A(e_3^+, e_4^+) &= 0 \\
F_A(e_1^+, e_3^+) + F_A(e_4^+, e_2^+) &= 0 \\
F_A(e_1^+, e_4^+) + F_A(e_2^+, e_3^+) &= 0,
\end{align*}
\]

we obtain

\[ \|F_A + *(\Omega_0 \wedge F_A)\|_{C^*_3(M)} \leq C \varepsilon^2. \]

Since $\Omega = \Omega_0 + O(|y|)$, we conclude that

\[ \|F_A + *(\Omega \wedge F_A)\|_{C^*_3(M)} \leq \|F_A + *(\Omega_0 \wedge F_A)\|_{C^*_3(M)} + \|F_A\|_{C^*_3(M)} \leq C \varepsilon^2. \]
4 Estimates for the linearized operator in weighted H"{o}lder spaces

Our aim in this section is to analyze the mapping properties of the linearized operator \( L_A : \Omega^1(M) \to \Omega^2_+(M) \).

As in Section 2, we define two vector bundles \( V \subset TS \otimes NS \) and \( W \subset TS \otimes NS \otimes NS \) over the submanifold \( S \). Both vector bundles have rank 4. The following elements form a basis for \( V \):

\[
e_1 \otimes e_1^+ + e_2 \otimes e_2^+ + e_3 \otimes e_3^+ + e_4 \otimes e_4^+, \\
e_1 \otimes e_2^+ - e_2 \otimes e_1^+ - e_3 \otimes e_4^+ + e_4 \otimes e_3^+, \\
e_1 \otimes e_3^+ + e_2 \otimes e_4^+ - e_3 \otimes e_1^+ - e_4 \otimes e_2^+, \\
e_1 \otimes e_4^+ - e_2 \otimes e_3^+ + e_3 \otimes e_2^+ - e_4 \otimes e_1^+.
\]

Similarly, the following elements form a basis for \( W \):

\[
e_1 \otimes (e_1^+ \otimes e_1^+ + e_2^+ \otimes e_2^+ + e_3^+ \otimes e_3^+ + e_4^+ \otimes e_4^+) \\
+ e_2 \otimes (e_2^+ \otimes e_1^+ - e_1^+ \otimes e_2^+ + e_4^+ \otimes e_3^+ - e_3^+ \otimes e_4^+) \\
+ e_3 \otimes (e_3^+ \otimes e_1^+ - e_4^+ \otimes e_2^+ - e_2^+ \otimes e_3^+ + e_3^+ \otimes e_4^+) \\
+ e_4 \otimes (e_4^+ \otimes e_1^+ + e_3^+ \otimes e_2^+ - e_2^+ \otimes e_3^+ - e_1^+ \otimes e_4^+) ,
\]

\[
e_1 \otimes (e_2^+ \otimes e_1^+ - e_1^+ \otimes e_2^+ + e_4^+ \otimes e_3^+ - e_3^+ \otimes e_4^+ ) \\
- e_2 \otimes (e_1^+ \otimes e_1^+ + e_2^+ \otimes e_2^+ + e_3^+ \otimes e_3^+ + e_4^+ \otimes e_4^+) \\
- e_3 \otimes (e_1^+ \otimes e_1^+ + e_3^+ \otimes e_2^+ - e_2^+ \otimes e_3^+ - e_1^+ \otimes e_4^+) \\
+ e_4 \otimes (e_3^+ \otimes e_1^+ - e_4^+ \otimes e_2^+ - e_1^+ \otimes e_3^+ + e_2^+ \otimes e_4^+) ,
\]

\[
e_1 \otimes (e_3^+ \otimes e_1^+ - e_4^+ \otimes e_2^+ - e_1^+ \otimes e_3^+ + e_2^+ \otimes e_4^+) \\
+ e_2 \otimes (e_4^+ \otimes e_1^+ + e_3^+ \otimes e_2^+ - e_2^+ \otimes e_3^+ - e_1^+ \otimes e_4^+) \\
- e_3 \otimes (e_1^+ \otimes e_1^+ + e_2^+ \otimes e_2^+ + e_3^+ \otimes e_3^+ + e_4^+ \otimes e_4^+) \\
- e_4 \otimes (e_2^+ \otimes e_1^+ - e_1^+ \otimes e_2^+ + e_4^+ \otimes e_3^+ - e_3^+ \otimes e_4^+) ,
\]

\[
e_1 \otimes (e_4^+ \otimes e_1^+ + e_3^+ \otimes e_2^+ - e_2^+ \otimes e_3^+ - e_1^+ \otimes e_4^+) \\
- e_2 \otimes (e_3^+ \otimes e_1^+ - e_4^+ \otimes e_2^+ - e_1^+ \otimes e_3^+ + e_2^+ \otimes e_4^+) \\
+ e_3 \otimes (e_2^+ \otimes e_1^+ - e_1^+ \otimes e_2^+ + e_4^+ \otimes e_3^+ - e_3^+ \otimes e_4^+) \\
- e_4 \otimes (e_1^+ \otimes e_1^+ + e_2^+ \otimes e_2^+ + e_3^+ \otimes e_3^+ + e_4^+ \otimes e_4^+) .
\]
Proposition 4.1. Suppose that $\psi \in C^\gamma_{3+\nu}(M)$ is supported in the set $\{ p \in M : \text{dist}(p, S) \leq 2\delta^4 \}$ and satisfies

$$\int_{NS} \sum_{i,j=1}^{4} \left( \varepsilon s_{ik} + t_{ikl} (y - \varepsilon v) t \right) \langle \psi(e_i, e_j), F_A(e_k, e_l) \rangle = 0$$

for all $x \in S$, $s \in V_x$, and $t \in W_x$. Then there exists an $1$-form $a \in C^1_{2+\nu}(M)$ which is supported in the region $\{ p \in M : \text{dist}(p, S) \leq 2\delta^2 \}$ such that

$$\| a \|_{C^1_{2+\nu}(M)} \leq C \| \psi \|_{C^\gamma_{3+\nu}(M)}$$

and

$$\| L_Aa - \psi \|_{C^\gamma_{3+\nu}(\{ p \in M : \text{dist}(p, S) \leq 2\delta^4 \})} \leq C \delta \| \psi \|_{C^\gamma_{3+\nu}(M)},$$

and

$$\| L_Aa - \psi \|_{C^\gamma_{3+\nu}(M)} \leq C | \log \delta |^{-1} \| \psi \|_{C^\gamma_{3+\nu}(M)}.$$

Proof. Let $\{ \zeta^{(j)} : 1 \leq j \leq j_0 \}$ be a partition of unity on $S$ such that each function $\zeta^{(j)}$ is supported in a ball $B\delta(p_j)$, and

$$| \{ 1 \leq j \leq j_0 : x \in B_{4\delta}(p_j) \} | \leq C$$

for all $x \in S$ and some uniform constant $C$. For each $1 \leq j \leq j_0$, there exists a $1$-form $a^{(j)} \in C^1_{2+\nu}(M)$ which is supported in the region $\{ (x, y) \in NS : x \in B_{2\delta}(p_j), |y| \leq 2\delta^2 \}$ such that

$$\| a^{(j)} \|_{C^1_{2+\nu}(M)} \leq C \| \zeta^{(j)} \psi \|_{C^\gamma_{3+\nu}(M)}$$

and

$$\| L_Aa^{(j)} - \zeta^{(j)} \psi \|_{C^\gamma_{3+\nu}(\{ p \in M : \text{dist}(p, S) \leq 2\delta^4 \})} \leq C \delta \| \zeta^{(j)} \psi \|_{C^\gamma_{3+\nu}(M)},$$

and

$$\| L_Aa^{(j)} - \zeta^{(j)} \psi \|_{C^\gamma_{3+\nu}(M)} \leq C | \log \delta |^{-1} \| \zeta^{(j)} \psi \|_{C^\gamma_{3+\nu}(M)}.$$

We now define

$$a = \sum_{j=1}^{j_0} a^{(j)}.$$ 

Then we have the estimates

$$\| a \|_{C^1_{2+\nu}(M)} \leq C \sup_{1 \leq j \leq j_0} \| a^{(j)} \|_{C^1_{2+\nu}(M)}$$

$$\leq C \sup_{1 \leq j \leq j_0} \| \zeta^{(j)} \psi \|_{C^\gamma_{3+\nu}(M)}$$

$$\leq C \| \psi \|_{C^\gamma_{3+\nu}(M)}.$$
\[
\|L\partial a - \psi\|_{C_{3+\nu}^\gamma(M)} \leq C \delta \sup_{1 \leq j \leq j_0} \|\zeta(j)\psi\|_{C_{3+\nu}^\gamma(M)} \\
\leq C \delta \|\psi\|_{C_{3+\nu}^\gamma(M)},
\]

This proves the assertion.

**Proposition 4.2.** For every \(\Omega\)-self-dual 2-form \(\psi \in C_{3+\nu}^\gamma(M)\), there exists a 1-form \(a \in C_{2+\nu}^1(M)\) such that

\[
\|a\|_{C_{2+\nu}^1(M)} \leq C \|\psi\|_{C_{3+\nu}^\gamma(M)}
\]

and

\[P_+ da = \psi.\]

**Proof.** We consider the elliptic operator \(P_+ dd^* : \Omega^2_+ (M) \to \Omega^2_+ (M)\). Its kernel is given by

\[
\ker(P_+ dd^* : \Omega^2_+ (M) \to \Omega^2_+ (M)) = H^2_+ (M).
\]

Since the cohomology group \(H^2_+ (M)\) vanishes, the operator \(P_+ dd^* : \Omega^2_+ (M) \to \Omega^2_+ (M)\) is invertible. Consequently, there exists a \(\Omega\)-self-dual 2-form \(\varphi\) such that

\[P_+ dd^* \varphi = \psi.\]

We claim that

\[
\|\varphi\|_{C_{3+\nu}^\gamma(M)} \leq C \|P_+ dd^* \varphi\|_{C_{3+\nu}^\gamma(M)}.
\]

By Schauder estimates, it suffices to show that

\[
\sup (\varepsilon + \operatorname{dist}(p, S))^{1+\nu} |\varphi| \leq C \sup (\varepsilon + \operatorname{dist}(p, S))^{3+\nu} |P_+ dd^* \varphi|.
\]

If this estimate fails, then there exists a sequence of positive real numbers \(\varepsilon_j\) and a sequence of \(\Omega\)-self-dual 2-forms \(\varphi^{(j)} \in C_{3+\nu}^\gamma(M)\) such that

\[
\sup (\varepsilon_j + \operatorname{dist}(p, S))^{1+\nu} |\varphi^{(j)}| = 1
\]
and
\[ \sup (\varepsilon_j + \text{dist}(p, S))^{3+\nu} |P_+ dd^* \varphi^{(j)}| \to 0. \]

Then there exists a sequence of points \( p_j \in M \) such that
\[ \sup (\varepsilon_j + \text{dist}(p_j, S))^{1+\nu} |\varphi^{(j)}(p_j)| \geq \frac{1}{2}. \]

There are two possibilities:

(i) Suppose that \( \text{dist}(p_j, S) \) is bounded from below. After passing to a subsequence, we may assume that the sequence \( \varphi^{(j)} \) converges to a \( \Omega \)-self-dual 2-form \( \varphi \in \Omega_+^2(M) \) such that
\[ \sup \text{dist}(p, S)^{1+\nu} |\varphi| \geq 1 \]
and
\[ P_+ dd^* \varphi = 0. \]

From this it follows that \( \varphi \) is smooth. Since the operator \( P_+ dd^* : \Omega_+^2(M) \to \Omega_+^2(M) \) has trivial kernel, it follows that \( \varphi = 0 \). This is a contradiction.

(ii) We now assume that \( \text{dist}(p_j, S) \to 0 \). After rescaling and taking the limit, we obtain a \( \Omega \)-self-dual 2-form \( \tilde{\varphi} \in \Omega_+^2(\mathbb{R}^8) \) such that
\[ \sup |y|^{1+\nu} |\tilde{\varphi}| \leq 1 \]
and
\[ P_+ dd^* \tilde{\varphi} = 0. \]

Thus, we conclude that \( \tilde{\varphi} = 0 \). This is a contradiction.

This implies
\[ \|\varphi\|_{C^{2+\gamma}_{3+\nu}(M)} \leq C \|\psi\|_{C_{3+\nu}(M)}. \]

Letting \( a = d^* \varphi \), the assertion follows.

**Proposition 4.3.** Suppose that \( \psi \in C_{3+\nu}^1(M) \) is supported in the region \( \{p \in M : \text{dist}(p, S) \geq \delta^4 \} \). Then there exists a 1-form \( a \in C^{1+\gamma}_{2+\nu}(M) \) which is supported in the region \( \{p \in M : \text{dist}(p, S) \geq \delta^8 \} \) such that
\[ \|a\|_{C^{1+\gamma}_{2+\nu}(M)} \leq C \|\psi\|_{C_{3+\nu}(M)} \]
and
\[ \| LAa - \psi\|_{C_{3+\nu}^1(M)} \leq C \left( |\log \delta|^{-1} + \delta^{-16} \varepsilon^2 \right) \|\psi\|_{C_{3+\nu}^1(M)}. \]
Proof. By Proposition 4.2, exists a 1-form $a$ such that
\[ \|a\|_{C_{2+\nu}^1(M)} \leq C \|\psi\|_{C_{3+\nu}^\gamma(M)} \]
and
\[ 2P_+ da = \psi. \]
Let $\eta$ be a cut-off function such that $\eta(p) = 0$ for $\text{dist}(p, S) \leq \delta^8$, $\eta(p) = 1$ for $\text{dist}(p, S) \geq \delta^4$ and
\[ \sup \text{dist}(p, S) |\nabla \eta| \leq C |\log \delta|^{-1}. \]
Then the 1-form $\eta a$ is supported in the region \{ $p \in M : \text{dist}(p, S) \geq \delta^8$ \} and satisfies
\[ \|L_A(\eta a) - \psi\|_{C_{3+\nu}^\gamma(M)} \leq 2 \|P_+ D_A(\eta a) - \eta P_+ da\|_{C_{3+\nu}^\gamma(M)} \]
\[ \leq 2 \|P_+ D_A(\eta a) - P_+ d(\eta a)\|_{C_{3+\nu}^\gamma(M)} + 2 \|P_+ d(\eta a) - \eta P_+ da\|_{C_{3+\nu}^\gamma(M)} \]
\[ \leq C \delta^{-16} \varepsilon^2 \|a\|_{C_{3+\nu}^\gamma(M)} + C |\log \delta|^{-1} \|a\|_{C_{2+\nu}^\gamma(M)} \]
\[ \leq C \delta^{-16} \varepsilon^2 \|\psi\|_{C_{3+\nu}^\gamma(M)} + C |\log \delta|^{-1} \|\psi\|_{C_{3+\nu}^\gamma(M)}. \]
This proves the assertion.

In the following, we will choose $\delta = \varepsilon^{\frac{1}{16}}$. Let $\kappa$ be a cut-off function such that $\kappa(p) = 1$ for $\text{dist}(p, S) \leq \varepsilon^{\frac{1}{4}}$ and $\kappa(p) = 0$ for $\text{dist}(p, S) \geq 2 \varepsilon^{\frac{1}{4}}$.

Let $\mathcal{G}_K^{k,\gamma}(M)$ be the set of all $\psi \in \Omega^2_+(M)$ such that $\psi \in \mathcal{C}_K^{k,\gamma}(M)$ and
\[ \int_{NSx} \kappa \sum_{i,j=1}^4 (\varepsilon s_{ik} + t_{ikl} (y - \varepsilon v)_l) \langle \psi(e_i, e^+_j), F_A(e_k^+, e_j^+) \rangle = 0 \]
for all $x \in S$, $s \in V_x$, and $t \in W_x$.

We denote by $I - P$ the fibrewise projection from $\mathcal{C}_K^\gamma(M)$ to the subspace $\mathcal{G}_0^0(M)$. Hence, if $\psi$ is an $\Omega$-self-dual 2-form, then the projection $P\psi$ is of the form
\[ P\psi(e_i, e^+_j) = \kappa (\varepsilon s_{ik} + t_{ikl} (y - \varepsilon v)_l) F_A(e_k^+, e_j^+) \]
for suitable $s \in V$ and $t \in W$. Let $\Pi$ be the linear operator which assigns to every $\Omega$-self-dual 2-form $\psi$ the pair
\[ \Pi \psi = (s, t) \in V \oplus W. \]
We shall need the following estimate for the operator norm of the projection operator $P$. 

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Proposition 4.4. For every $\Omega$-self-dual 2-form $\psi \in C^\gamma_{3+\nu}(M)$, we have the estimates
\[
\|\Pi \psi\|_{C^\gamma(S)} \leq C \epsilon^{-2-\nu-\gamma} \|\psi\|_{C^\gamma_{3+\nu}(M)}
\]
and
\[
\|\mathcal{P} \psi\|_{C^\gamma_{3+\nu}(M)} \leq C \epsilon^{-\nu-\gamma} \|\psi\|_{C^\gamma_{3+\nu}(M)}.
\]

Proof. This follows from [5], Proposition 5.4.

Proposition 4.5. For every $\psi \in G^\gamma_{3+\nu}(M)$ there exists a 1-form $a \in C^{1,\gamma}_{2+\nu}(M)$ such that
\[
\|a\|_{C^{1,\gamma}_{2+\nu}(M)} \leq C \|\psi\|_{C^\gamma_{3+\nu}(M)}
\]
and
\[
\|\mathcal{L}_A a - \psi\|_{C^\gamma_{3+\nu}(\{p \in M : \text{dist}(p, S) \leq \epsilon^{\frac{1}{2}}\})} \leq C \epsilon^{\frac{1}{16}} \|\psi\|_{C^\gamma_{3+\nu}(M)},
\]
and
\[
\|\mathcal{L}_A a - \psi\|_{C^\gamma_{3+\nu}(M)} \leq C |\log \epsilon|^{-1} \|\psi\|_{C^\gamma_{3+\nu}(M)}.
\]

Proof. Apply Proposition 4.1 to $\kappa \psi$ and Proposition 4.3 to $(1-\kappa) \psi$.

Proposition 4.6. For every $\psi \in G^\gamma_{3+\nu}(M)$ there exists a 1-form $a \in C^{1,\gamma}_{2+\nu}(M)$ such that
\[
\|a\|_{C^{1,\gamma}_{2+\nu}(M)} \leq C \|\psi\|_{C^\gamma_{3+\nu}(M)}
\]
and
\[
(I - \mathcal{P}) \mathcal{L}_A a = \psi.
\]
Furthermore, $a$ satisfies the estimate
\[
\|\Pi \mathcal{L}_A a\|_{C^\gamma(S)} \leq C \epsilon^{-2+\frac{\nu}{2}} \|\psi\|_{C^\gamma_{3+\nu}(M)}.
\]

Proof. By Proposition 4.5, there exists an operator $\mathcal{S} : G^\gamma_{3+\nu}(M) \to C^{1,\gamma}_{2+\nu}(M)$ such that
\[
\|\mathcal{S}\psi\|_{C^{1,\gamma}_{2+\nu}(M)} \leq C \|\psi\|_{C^\gamma_{3+\nu}(M)}
\]
and
\[
\|\mathcal{L}_A \mathcal{S}\psi - \psi\|_{C^\gamma_{3+\nu}(\{p \in M : \text{dist}(p, S) \leq \epsilon^{\frac{1}{2}}\})} \leq C \epsilon^{\frac{1}{16}} \|\psi\|_{C^\gamma_{3+\nu}(M)},
\]

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and
\[ \| L_A S \psi - \psi \|_{C_{3+\nu}^\gamma(M)} \leq C |\log \epsilon|^{-1} \| \psi \|_{C_{3+\nu}^\gamma(M)}. \]

This implies
\[ \| \Pi L_A S \psi \|_{C^\gamma(S)} = \| \Pi (L_A S \psi - \psi) \|_{C^\gamma(S)} \leq C \epsilon^{-2+\frac{1}{16}-\nu-\gamma} \| \psi \|_{C_{3+\nu}^\gamma(M)}. \]

From this it follows that
\[ \| (I - P) L_A S \psi - \psi \|_{C_{3+\nu}^\gamma(M)} \leq C |\log \epsilon|^{-1} \| \psi \|_{C_{3+\nu}^\gamma(M)}. \]

Therefore, the operator \((I - P) L_A S : \mathcal{G}_{3+\nu}^\gamma(M) \to \mathcal{G}_{3+\nu}^\gamma(M)\) is invertible.

Hence, if we define
\[ a = S [(I - P) L_A S]^{-1} \psi, \]
then \(a\) satisfies\n\[ \| a \|_{C_{2+\nu}^{1,\gamma}(M)} \leq C \| \psi \|_{C_{3+\nu}^\gamma(M)} \]

and
\[ (I - P) L_A a = \psi. \]

This proves the assertion.

\section{The nonlinear problem}

\textbf{Proposition 5.1.} For every approximate solution \(A\), there exists a nearby connection \(\tilde{A} = A + a\) such that
\[ \| a \|_{C_{3+\nu}^{1,\gamma}(M)} \leq C \epsilon^{2-\nu-\gamma} \]

and
\[ (I - P) (F_{\tilde{A}} + *(\Omega \wedge F_{\tilde{A}})) = 0. \]

Furthermore, \(a\) satisfies the estimate
\[ \| \Pi L_A a \|_{C^\gamma(S)} \leq C \epsilon^{\frac{1}{12}}. \]

\textbf{Proof.} The connection \(\tilde{A} = A + a\) satisfies
\[ F_{\tilde{A}} + *(\Omega \wedge F_{\tilde{A}}) = F_A + *(\Omega \wedge F_A) + D_A a + *(\Omega \wedge D_A a) + [a, a] + *(\Omega \wedge [a, a]). \]
This implies
\[ F_A + *(\Omega \wedge F_A) = F_A + *(\Omega \wedge F_A) + 2 \mathbb{L}_A a + [a, a] + *(\Omega \wedge [a, a]). \]

According to Proposition 4.6, there exists an operator \( G : \mathcal{G}_{3+\nu}(M) \to C_{2+\nu}(M) \) such that

\[
\|G\psi\|_{C_{2+\nu}^1(M)} \leq C \|\psi\|_{C_{3+\nu}^1(M)}
\]

and

\[
(I - \mathbb{P}) \mathbb{L}_A G = I.
\]

We now define a mapping \( \Phi : C_{2+\nu}^1(M) \to C_{2+\nu}^1(M) \) by

\[
\Phi(a) = -\frac{1}{2} G (I - \mathbb{P}) (F_A + *(\Omega \wedge F_A)) - \frac{1}{2} G (I - \mathbb{P}) ([a, a] + *(\Omega \wedge [a, a])).
\]

Then we have the estimate

\[
\|\Phi(a)\|_{C_{2+\nu}^1(M)} \leq C \|I - \mathbb{P}\| \|F_A + *(\Omega \wedge F_A)\|_{C_{3+\nu}^1(M)} + C \|I - \mathbb{P}\| \|[a, a] + *(\Omega \wedge [a, a])\|_{C_{3+\nu}^1(M)}
\]

\[
\leq C \varepsilon^{-\nu-\gamma} \|F_A + *(\Omega \wedge F_A)\|_{C_{3+\nu}^1(M)} + C \varepsilon^{-\nu-\gamma} \|[a, a]\|_{C_{3+\nu}^1(M)}
\]

\[
\leq C \varepsilon^{-\nu-\gamma} \|F_A + *(\Omega \wedge F_A)\|_{C_{3+\nu}^1(M)} + C \varepsilon^{-1-2\nu-\gamma} \|a\|^2_{C_{2+\nu}^1(M)}
\]

\[
\leq C \varepsilon^{2-\nu-\gamma}
\]

for all \( a \in C_{2+\nu}^1(M) \) satisfying

\[
\|a\|_{C_{2+\nu}^{1,\gamma}} \leq \varepsilon^{-\frac{\gamma}{4}}.
\]

Moreover, we have

\[
\|\Phi(a) - \Phi(a')\|_{C_{1+\nu}^2(M)} \leq C \varepsilon^{-\nu-\gamma} \|[a, a] - [a', a']\|_{C_{3+\nu}^1(M)}
\]

\[
\leq C \varepsilon^{\frac{3}{4}-2\nu-\gamma} \|a - a'\|_{C_{1+\nu}^2(M)}
\]

for all \( a, a' \in C_{2+\nu}^1(M) \) satisfying

\[
\|a\|_{C_{2+\nu}^{1,\gamma}} \leq \varepsilon^{-\frac{\gamma}{4}}, \quad \|a'\|_{C_{2+\nu}^{1,\gamma}} \leq \varepsilon^{-\frac{\gamma}{4}}.
\]

Hence, it follows from the contraction mapping principle that there exists a 1-form \( a \in C_{2+\nu}^{1,\gamma}(M) \) such that

\[
\|a\|_{C_{2+\nu}^{1,\gamma}(M)} \leq C \varepsilon^{2-\nu-\gamma}
\]
and
\[ \Phi(a) = a. \]

From this it follows that
\[ G(I - P)(F_A + *(\Omega \wedge F_A)) + 2a + G(I - P)([a, a] + *(\Omega \wedge [a, a])) = 0, \]
hence
\[ (I - P)(F_A + *(\Omega \wedge F_A)) + 2(I - P)\bar{L}_Aa + (I - P)([a, a] + *(\Omega \wedge [a, a])) = 0. \]
Thus, we conclude that
\[ (I - P)(F_{\tilde{A}} + *(\Omega \wedge F_{\tilde{A}})) = 0. \]

This proves the assertion.

**Corollary 5.2.** If \( \tilde{A} \) satisfies
\[ \mathbb{P}(F_{\tilde{A}} + *(\Omega \wedge F_{\tilde{A}})) = 0, \]
then \( \tilde{A} \) is an \( \Omega \)-anti-self-dual instanton, i.e.
\[ F_{\tilde{A}} + *(\Omega \wedge F_{\tilde{A}}) = 0. \]

6 The balancing condition

By Corollary 5.2, the problem is reduced to finding a set of glueing data \((v, \lambda, J, \omega)\) such that
\[ \mathbb{P}(F_{\tilde{A}} + *(\Omega \wedge F_{\tilde{A}})) = 0. \]

Our aim in this section is to derive a formula for the error term
\[ \mathbb{P}(F_{\tilde{A}} + *(\Omega \wedge F_{\tilde{A}})). \]

**Proposition 6.1.** The curvature of \( A \) satisfies
\[ \Pi(F_A + *(\Omega_0 \wedge F_A)) = 4 \left( \text{proj}_V \left( \sum_{i,j=1}^{4} \nabla_i v_k e_i \otimes e_k^\perp \right) \right), \]
\[ \text{proj}_W \left( \sum_{i,k,l=1}^{4} (\lambda^{-1} \nabla_i \lambda \delta_{kl} + \theta_{i,kl}) e_i \otimes e_k^\perp \otimes e_l^\perp \right). \]
Proof. This is a consequence of the identity
\[ F_A(e_i, e_j^\perp) = - (\varepsilon \nabla_i v_k + \lambda^{-1} \nabla_i \lambda (y - \varepsilon v)_k + \theta_{i,kl} (y - \varepsilon v)_l) F_A(e_k^\perp, e_j^\perp). \]

The covariant derivative of \( \Omega \) can be described by a 1-form \( \alpha \) with values in \( \Lambda_2^+ TM \). For every vector field \( X \in TM \), we write
\[ \nabla_X \Omega = \sum_{k=1}^{8} i_{e_k} \alpha(X) \land i_{e_k} \Omega, \]
where \( \alpha(X) \in \Lambda_2^+ TM \). From this it follows that
\[ \Omega = \Omega_0 + \sum_{k=1}^{8} i_{e_k} \alpha(y) \land i_{e_k} \Omega_0 + O(|y|^2), \]
where \( \alpha(y) \in \Lambda_2^+ TM \).

**Proposition 6.2.** The curvature of \( A \) satisfies
\[ \left\| \Pi(F_A + *(\Omega \land F_A)) \right\| \leq C \varepsilon. \]

Proof. Using the identity
\[ \Omega - \Omega_0 - \sum_{k=1}^{8} i_{e_k} \alpha(y) \land i_{e_k} \Omega_0 = O(|y|^2), \]
we obtain
\[ \left\| \Omega \land F_A - \Omega_0 \land F_A - \sum_{k=1}^{8} i_{e_k} \alpha(y) \land i_{e_k} \Omega_0 \land F_A \right\|_{C^2(S)} \leq C \varepsilon^2. \]

This implies
\[ \left\| \Omega \land F_A - \Omega_0 \land F_A + \sum_{k=1}^{8} i_{e_k} \alpha(y) \land i_{e_k} (\Omega_0 \land F_A) \right. \]
\[ - \left. \Omega_0 \land \sum_{k=1}^{8} i_{e_k} \alpha(y) \land i_{e_k} F_A \right\|_{C^2(M)} \leq C \varepsilon^2, \]

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hence
\[ \left\| - \left( \Omega \wedge \sum_{k=1}^{8} i_{e_k} \alpha(y) \wedge i_{e_k} F_A \right) \right\|_{C^2(M)} ^* \leq C \varepsilon^2. \]

Therefore, we obtain
\[ \left\| (F_A + \ast(\Omega \wedge F_A)) - (F_A + \ast(\Omega_0 \wedge F_A)) + \sum_{k=1}^{8} i_{e_k} \alpha(y) \wedge i_{e_k} (F_A + \ast(\Omega_0 \wedge F_A)) \right\|_{C^2(M)} \leq C \varepsilon^2. \]

According to Proposition 3.4, we have
\[ \| F_A + \ast(\Omega_0 \wedge F_A) \|_{C^2(M)} \leq C \varepsilon^2, \]

hence
\[ \left\| \sum_{k=1}^{8} i_{e_k} \alpha(y) \wedge i_{e_k} (F_A + \ast(\Omega_0 \wedge F_A)) \right\|_{C^2(M)} \leq C \varepsilon^2. \]

Moreover, we have
\[ 3 \sum_{k=1}^{8} i_{e_k} \alpha(y) \wedge i_{e_k} F_A - \ast \left( \Omega_0 \wedge \sum_{k=1}^{8} i_{e_k} \alpha(y) \wedge i_{e_k} F_A \right) = 0. \]

Thus, we conclude that
\[ \left\| (F_A + \ast(\Omega \wedge F_A)) - (F_A + \ast(\Omega_0 \wedge F_A)) - 4 \sum_{k=1}^{8} i_{e_k} \alpha(y) \wedge i_{e_k} F_A \right\|_{C^2(M)} \leq C \varepsilon^2. \]

The assertion follows now from Proposition 6.1.

**Proposition 6.3.** The curvature of $\tilde{A}$ satisfies
\[ \left\| \Pi(F_{\tilde{A}} + \ast(\Omega \wedge F_{\tilde{A}})) \right\|_{C^2(S)} \leq C \varepsilon. \]
Proof. Using the estimate
\[ \|a\|_{C^1_{2\nu}(M)} \leq C \varepsilon^{2-\nu-\gamma}, \]
we obtain
\[ \|\Pi([a,a] + *(\Omega \wedge [a,a]))\|_{C^\gamma(S)} \leq C \varepsilon^{-2-\nu-\gamma} \|a\|_{C^1_{2\nu}(M)} \]
\[ \leq C \varepsilon^{-3-2\nu-\gamma} \|a\|_{C^1_{2\nu}(M)}^2 \]
\[ \leq C \varepsilon^{1-4\nu-3\gamma}. \]
Moreover, we have
\[ \|\Pi L_A a\|_{C^\gamma(S)} \leq C \varepsilon^{1-\frac{1}{2\nu}}. \]
Hence, the assertion follows from Proposition 6.2.

Proof of Theorem 1.1. Let
\[ \Xi_\varepsilon(v,\lambda,J,\omega) = \Pi(F_\tilde{A} + *(\Omega \wedge F_\tilde{A})). \]
The first part of Theorem 1.1 follows from Proposition 5.2, the second part from Proposition 6.3.

7 Discussion

In this final section, we show how the first order balancing condition derived in this paper is related to the second order balancing condition in [6]. To this end, we assume that \( \Omega \) is parallel. Then the Riemann curvature tensor of \( M \) belongs to \( \Lambda^2 TM \otimes \Lambda^2 TM \). Since \( S \) is a Cayley submanifold, the second fundamental form of \( S \) satisfies
\[ h(e_k,e_1,e_1^+) + h(e_k,e_2,e_2^+) + h(e_k,e_3,e_3^+) + h(e_k,e_4,e_4^+) = 0 \]
\[ h(e_k,e_1,e_2^+) - h(e_k,e_2,e_1^+) - h(e_k,e_3,e_4^+) + h(e_k,e_4,e_3^+) = 0 \]
\[ h(e_k,e_1,e_3^+) + h(e_k,e_2,e_4^+) - h(e_k,e_3,e_1^+) - h(e_k,e_4,e_2^+) = 0 \]
\[ h(e_k,e_1,e_4^+) - h(e_k,e_2,e_3^+) - h(e_k,e_3,e_2^+) + h(e_k,e_4,e_1^+) = 0. \]
We denote the curvature of the normal bundle \( NS \) by \( E \). Using the Gauss equations, we obtain
\[ E(e_i,e_j,e_k^+,e_l^+) = R(e_i,e_j,e_k^+,e_l^+) \]
\[ - \sum_{m=1}^{4} h(e_m,e_i,e_k^+) h(e_m,e_j,e_l^+) + h(e_m,e_i,e_l^+) h(e_m,e_j,e_k^+). \]
Since $\nabla \Omega = 0$, the first part of the balancing condition becomes

$$\begin{align*}
\nabla v_1 + \nabla v_2 + \nabla v_3 + \nabla v_4 &= 0 \\
\nabla v_2 - \nabla v_1 - \nabla v_4 + \nabla v_3 &= 0 \\
\nabla v_3 + \nabla v_4 - \nabla v_1 - \nabla v_2 &= 0 \\
\nabla v_4 - \nabla v_3 + \nabla v_2 - \nabla v_1 &= 0.
\end{align*}$$

This implies

$$\begin{align*}
0 &= \Delta v_1 \\
&+ \nabla_1 \nabla_2 v_2 - \nabla_1 \nabla_3 v_3 + \nabla_2 \nabla_3 v_2 - \nabla_3 \nabla_2 v_3 \\
&+ \nabla_1 \nabla_3 v_4 - \nabla_2 \nabla_4 v_3 + \nabla_3 \nabla_4 v_2 - \nabla_4 \nabla_3 v_4 \\
&+ \nabla_2 \nabla_4 v_3 - \nabla_1 \nabla_3 v_4 + \nabla_4 \nabla_3 v_2 - \nabla_3 \nabla_2 v_4 \\
0 &= \Delta v_2 \\
&- \nabla_1 \nabla_2 v_1 + \nabla_1 \nabla_3 v_2 - \nabla_1 \nabla_4 v_3 + \nabla_2 \nabla_4 v_3 \\
&- \nabla_1 \nabla_3 v_4 + \nabla_2 \nabla_4 v_3 + \nabla_4 \nabla_3 v_2 \\
&+ \nabla_2 \nabla_4 v_3 - \nabla_1 \nabla_3 v_4 + \nabla_4 \nabla_3 v_2 \\
0 &= \Delta v_3 \\
&+ \nabla_1 \nabla_2 v_4 - \nabla_2 \nabla_3 v_1 + \nabla_3 \nabla_4 v_3 \\
&- \nabla_1 \nabla_3 v_4 + \nabla_4 \nabla_2 v_3 + \nabla_3 \nabla_2 v_1 \\
0 &= \Delta v_4 \\
&- \nabla_1 \nabla_2 v_3 + \nabla_2 \nabla_3 v_1 - \nabla_2 \nabla_4 v_3 + \nabla_3 \nabla_2 v_1.
\end{align*}$$

From this it follows that

$$\begin{align*}
0 &= \Delta v_1 \\
&+ \left( E(e_1, e_2, e_1^1, e_2^1) + E(e_3, e_4, e_1^1, e_2^1) + E(e_1, e_3, e_1^1, e_3^1) + E(e_4, e_2, e_1^1, e_3^1) \\
&+ E(e_1, e_4, e_1^1, e_1^1) + E(e_2, e_3, e_1^1, e_4^1) \right) v_1 \\
&+ \left( E(e_1, e_3, e_2^1, e_3^1) + E(e_4, e_2, e_2^1, e_3^1) + E(e_1, e_4, e_2^1, e_4^1) + E(e_2, e_3, e_2^1, e_4^1) \right) v_2 \\
&+ \left( E(e_1, e_2, e_3^1, e_2^1) + E(e_3, e_4, e_3^1, e_2^1) + E(e_1, e_4, e_3^1, e_4^1) + E(e_2, e_3, e_3^1, e_4^1) \right) v_3 \\
&+ \left( E(e_1, e_2, e_1^1, e_2^1) + E(e_3, e_4, e_1^1, e_2^1) + E(e_1, e_3, e_1^1, e_3^1) + E(e_4, e_2, e_1^1, e_3^1) \right) v_4
\end{align*}$$
$0 = \Delta v_2$
\begin{align*}
0 &= \Delta v_2 \\
&\quad + (E(e_3, e_1, e_1^+_{e_1}, e_1^+_{e_3}) + E(e_2, e_4, e_1^+_{e_1}, e_1^+_{e_2}) + E(e_1, e_4, e_1^+_{e_1}, e_3^+_{e_3}) + E(e_2, e_3, e_1^+_{e_1}, e_3^+_{e_3})) v_1 \\
&\quad + (E(e_2, e_1, e_1^+_{e_2}, e_3^+_{e_3}) + E(e_4, e_3, e_1^+_{e_2}, e_1^+_{e_4}) + E(e_3, e_1, e_2^+_{e_2}, e_1^+_{e_4}) + E(e_2, e_4, e_2^+_{e_2}, e_4^+_{e_4})) v_2 \\
&\quad + (E(e_2, e_1, e_1^+_{e_3}, e_1^+_{e_3}) + E(e_4, e_3, e_3^+_{e_3}, e_1^+_{e_4}) + E(e_3, e_1, e_3^+_{e_3}, e_1^+_{e_4}) + E(e_2, e_4, e_3^+_{e_3}, e_1^+_{e_4})) v_3 \\
&\quad + (E(e_2, e_1, e_1^+_{e_4}, e_1^+_{e_4}) + E(e_4, e_3, e_4^+_{e_4}, e_1^+_{e_4}) + E(e_1, e_4, e_1^+_{e_4}, e_3^+_{e_3}) + E(e_2, e_3, e_1^+_{e_4}, e_3^+_{e_3})) v_4
\end{align*}

$0 = \Delta v_3$
\begin{align*}
0 &= \Delta v_3 \\
&\quad + (E(e_1, e_2, e_1^+_{e_1}, e_4^+_{e_4}) + E(e_3, e_4, e_1^+_{e_1}, e_4^+_{e_4}) + E(e_4, e_1, e_1^+_{e_1}, e_2^+_{e_4}) + E(e_3, e_2, e_1^+_{e_1}, e_2^+_{e_4})) v_1 \\
&\quad + (E(e_1, e_2, e_2^+_{e_1}, e_4^+_{e_4}) + E(e_3, e_4, e_2^+_{e_1}, e_4^+_{e_4}) + E(e_3, e_1, e_2^+_{e_2}, e_4^+_{e_4}) + E(e_2, e_4, e_2^+_{e_2}, e_4^+_{e_4})) v_2 \\
&\quad + (E(e_1, e_2, e_3^+_{e_1}, e_4^+_{e_4}) + E(e_3, e_4, e_3^+_{e_1}, e_4^+_{e_4}) + E(e_3, e_1, e_3^+_{e_3}, e_4^+_{e_4}) + E(e_2, e_4, e_3^+_{e_3}, e_4^+_{e_4})) v_3 \\
&\quad + (E(e_1, e_2, e_4^+_{e_1}, e_4^+_{e_4}) + E(e_4, e_3, e_4^+_{e_1}, e_4^+_{e_4}) + E(e_1, e_3, e_4^+_{e_4}, e_2^+_{e_4}) + E(e_4, e_2, e_4^+_{e_4}, e_4^+_{e_4})) v_4
\end{align*}

$0 = \Delta v_4$
\begin{align*}
0 &= \Delta v_4 \\
&\quad + (E(e_2, e_1, e_1^+_{e_2}, e_3^+_{e_3}) + E(e_4, e_3, e_1^+_{e_2}, e_3^+_{e_3}) + E(e_1, e_3, e_1^+_{e_2}, e_3^+_{e_3}) + E(e_4, e_2, e_1^+_{e_2}, e_3^+_{e_3})) v_1 \\
&\quad + (E(e_2, e_1, e_2^+_{e_2}, e_3^+_{e_3}) + E(e_4, e_3, e_2^+_{e_2}, e_3^+_{e_3}) + E(e_4, e_1, e_2^+_{e_2}, e_3^+_{e_3}) + E(e_3, e_2, e_2^+_{e_2}, e_3^+_{e_3})) v_2 \\
&\quad + (E(e_2, e_1, e_3^+_{e_2}, e_3^+_{e_3}) + E(e_4, e_2, e_3^+_{e_2}, e_3^+_{e_3}) + E(e_4, e_1, e_3^+_{e_3}, e_3^+_{e_3}) + E(e_3, e_2, e_3^+_{e_3}, e_3^+_{e_3})) v_3 \\
&\quad + (E(e_2, e_1, e_4^+_{e_2}, e_3^+_{e_3}) + E(e_4, e_3, e_4^+_{e_2}, e_3^+_{e_3}) + E(e_1, e_3, e_4^+_{e_4}, e_3^+_{e_3}) + E(e_4, e_2, e_4^+_{e_4}, e_3^+_{e_3})) v_4
\end{align*}

Hence, we obtain
\begin{align*}
0 &= \Delta v_1 + \sum_{i,j,k=1}^{4} h(e_i, e_j, e_1^+_{e_1}) h(e_i, e_j, e_k^+_{e_k}) v_k + \sum_{i,k=1}^{4} R(e_i, e_1^+_{e_1}, e_k^+_{e_k}, e_i) v_k \\
0 &= \Delta v_2 + \sum_{i,j,k=1}^{4} h(e_i, e_1^+_{e_2}, e_1^+_{e_3}) h(e_i, e_j, e_1^+_{e_3}) v_k + \sum_{i,k=1}^{4} R(e_i, e_1^+_{e_2}, e_k^+_{e_k}, e_i) v_k \\
0 &= \Delta v_3 + \sum_{i,j,k=1}^{4} h(e_i, e_j, e_3^+_{e_3}) h(e_i, e_j, e_k^+_{e_k}) v_k + \sum_{i,k=1}^{4} R(e_i, e_3^+_{e_3}, e_k^+_{e_k}, e_i) v_k \\
0 &= \Delta v_4 + \sum_{i,j,k=1}^{4} h(e_i, e_1^+_{e_4}, e_1^+_{e_4}) h(e_i, e_j, e_1^+_{e_4}) v_k + \sum_{i,k=1}^{4} R(e_i, e_1^+_{e_4}, e_k^+_{e_k}, e_i) v_k.
\end{align*}
Furthermore, the second part of the balancing condition can be written in the form

\[ 2\lambda^{-1} \nabla_1 \lambda + (\theta_{2,21} + \theta_{2,43}) + (\theta_{3,31} + \theta_{3,24}) + (\theta_{4,41} + \theta_{4,32}) = 0 \]
\[ (\theta_{1,21} + \theta_{1,43}) - 2\lambda^{-1} \nabla_2 \lambda - (\theta_{3,41} + \theta_{3,32}) + (\theta_{4,31} + \theta_{4,24}) = 0 \]
\[ (\theta_{1,31} + \theta_{1,24}) + (\theta_{2,41} + \theta_{2,32}) - 2\lambda^{-1} \nabla_3 \lambda - (\theta_{4,21} + \theta_{4,43}) = 0 \]
\[ (\theta_{4,41} + \theta_{4,32}) - (\theta_{2,31} + \theta_{2,24}) + (\theta_{3,21} + \theta_{3,43}) - 2\lambda^{-1} \nabla_4 \lambda = 0. \]

This implies

\[ 0 = 2\lambda^{-1} \Delta \lambda - 2\lambda^{-2} |\nabla \lambda|^2 \]
\[ + \nabla_1 \theta_{2,21} - \nabla_2 \theta_{1,21} + \nabla_1 \theta_{2,43} - \nabla_2 \theta_{1,43} \]
\[ + \nabla_3 \theta_{4,21} - \nabla_4 \theta_{3,21} + \nabla_3 \theta_{4,43} - \nabla_4 \theta_{3,43} \]
\[ + \nabla_1 \theta_{3,31} - \nabla_3 \theta_{1,31} + \nabla_1 \theta_{3,24} - \nabla_3 \theta_{1,24} \]
\[ + \nabla_4 \theta_{2,31} - \nabla_2 \theta_{4,31} + \nabla_4 \theta_{2,24} - \nabla_2 \theta_{4,24} \]
\[ + \nabla_1 \theta_{4,41} - \nabla_4 \theta_{1,41} + \nabla_1 \theta_{4,32} - \nabla_4 \theta_{1,32} \]
\[ + \nabla_2 \theta_{3,41} - \nabla_3 \theta_{2,41} + \nabla_2 \theta_{3,32} - \nabla_3 \theta_{2,32}, \]

hence

\[ 0 = 2\lambda^{-1} \Delta \lambda - \frac{1}{2} |\theta|^2 \]
\[ + \nabla_1 \theta_{2,21} - \nabla_2 \theta_{1,21} + [\theta_1, \theta_2]_{21} + \nabla_1 \theta_{2,43} - \nabla_2 \theta_{1,43} + [\theta_1, \theta_2]_{43} \]
\[ + \nabla_3 \theta_{4,21} - \nabla_4 \theta_{3,21} + [\theta_3, \theta_4]_{21} + \nabla_3 \theta_{4,43} - \nabla_4 \theta_{3,43} + [\theta_3, \theta_4]_{43} \]
\[ + \nabla_1 \theta_{3,31} - \nabla_3 \theta_{1,31} + [\theta_1, \theta_3]_{31} + \nabla_1 \theta_{3,24} - \nabla_3 \theta_{1,24} + [\theta_1, \theta_3]_{24} \]
\[ + \nabla_4 \theta_{2,31} - \nabla_2 \theta_{4,31} + [\theta_4, \theta_2]_{31} + \nabla_4 \theta_{2,24} - \nabla_2 \theta_{4,24} + [\theta_4, \theta_2]_{24} \]
\[ + \nabla_1 \theta_{4,41} - \nabla_4 \theta_{1,41} + [\theta_1, \theta_4]_{41} + \nabla_1 \theta_{4,32} - \nabla_4 \theta_{1,32} + [\theta_1, \theta_4]_{32} \]
\[ + \nabla_2 \theta_{3,41} - \nabla_3 \theta_{2,41} + [\theta_2, \theta_3]_{41} + \nabla_2 \theta_{3,32} - \nabla_3 \theta_{2,32} + [\theta_2, \theta_3]_{32}. \]

From this it follows that

\[ 0 = 2\lambda^{-1} \Delta \lambda - \frac{1}{2} |\theta|^2 \]
\[ + E(e_1, e_2, e_1^+, e_2^+) + E(e_1, e_2, e_3, e_3^+) + E(e_3, e_4, e_1^+, e_2^+) \]
\[ + E(e_1, e_3, e_1^+, e_3^+) + E(e_1, e_3, e_4, e_4^+) + E(e_4, e_2, e_1^+, e_3^+) \]
\[ + E(e_1, e_4, e_1^+, e_4^+) + E(e_1, e_4, e_2, e_3^+) + E(e_2, e_3, e_1^+, e_4^+) \]
\[ + E(e_2, e_3, e_2^+, e_3^+). \]

Using the identities

\[ E(e_1, e_2, e_1^+, e_2^+) + E(e_3, e_4, e_1^+, e_2^+) + E(e_1, e_3, e_1^+, e_3^+) + E(e_4, e_2, e_1^+, e_3^+) \]
\[ + E(e_1, e_4, e_1^+, e_4^+) + E(e_2, e_3, e_1^+, e_4^+) \]
\[ = \sum_{i,j=1}^{4} h(e_i, e_j, e_1^+) h(e_i, e_j, e_1^+) + \sum_{i=1}^{4} R(e_i, e_1^+, e_1^+, e_i). \]
A similar calculation gives

Thus, we conclude that

we obtain

Thus, the first order balancing condition implies the second order balancing condition derived in [5].
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