Semi-Local Analysis and Real Life Applications of Higher-Order Iterative Schemes for Nonlinear Systems

Ramandeep Behl 1, Sonia Bhalla 2, Ioannis K. Argyros 3,* and Sanjeev Kumar 4

1 Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia; ramanbehl87@yahoo.in
2 Department of Mathematics, Chandigarh University, Gharuan 140413, Mohali, Punjab, India; soniamaths5@gmail.com
3 Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
4 School of Mathematics, Thapar Institute of Engineering and Technology University, Patiala 147004, Punjab, India; sanjeevbakshi1@gmail.com

* Correspondence: ioannisa@cameron.edu

Received: 25 November 2019; Accepted: 31 December 2019; Published: 6 January 2020

Abstract: Our aim is to improve the applicability of the family suggested by Bhalla et al. (Computational and Applied Mathematics, 2018) for the approximation of solutions of nonlinear systems. Semi-local convergence relies on conditions with first order derivatives and Lipschitz constants in contrast to other works requiring higher order derivatives not appearing in these schemes. Hence, the usage of these schemes is improved. Moreover, a variety of real world problems, namely, Bratu’s 1D, Bratu’s 2D and Fisher’s problems, are applied in order to inspect the utilization of the family and to test the theoretical results by adopting variable precision arithmetics in Mathematica 10. On account of these examples, it is concluded that the family is more efficient and shows better performance as compared to the existing one.

Keywords: computational efficiency; system of nonlinear equations; semi-local convergence analysis; Steffensen’s method

MSC: 65G99; 65H10; 65J15; 65Y20; 47J25

1. Introduction

A lot of problems related to obtaining the solutions of nonlinear systems are brought forward in many applied sciences and engineering applications. Generally, the zeros of a nonlinear system cannot be expressed in closed form. Therefore, iterative methods for obtaining the approximating solutions of systems of nonlinear equations are the most frequently used techniques. This topic has always been of paramount importance in mathematics for approximating the roots of nonlinear system of the form

\[ \Gamma(\vartheta) = 0, \]  

where \( \Gamma(\vartheta) = (f_1(\vartheta), f_2(\vartheta), \ldots, f_n(\vartheta))^T \), \( \vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_n)^T \) and \( \Gamma : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a sufficiently differentiable vector function. One of the most basic and common among iterative methods for approximating the solutions of nonlinear system is Newton’s method [1–6], defined as follows:

\[ \vartheta^{\sigma+1} = \vartheta^\sigma - \Gamma'(\vartheta^\sigma)^{-1}\Gamma(\vartheta^\sigma), \quad \sigma = 0, 1, 2, \ldots, \]  

where \( \{\Gamma'(\vartheta^\sigma)\}^{-1} \) is the inverse of the first-order Fréchet derivative of \( \Gamma(\vartheta) \) and has a quadratic convergence. Some higher-order Newton-like schemes can be found in [7–16]. However, these
schemes need the usage of \( \{ \Gamma'(\theta)^{-1} \} \). On the other hand, this usage is very expensive and/or it demands a huge amount of time for them to be given, computed or calculated. That is why Traub [17] established a second-order convergent scheme, which is defined as follows:

\[
\theta^{p+1} = \theta^p - [\Lambda^p, \theta^p; \Gamma]^{-1}\Gamma(\theta^p),
\]

where \( \Lambda^p = \theta^p + \beta(\theta^p), \beta \in \mathbb{R} - \{0\} \). The finite difference \([\Lambda^p, \theta^p; \Gamma]\) (see [18,19]) is given by

\[
[\Lambda^p, \theta^p; \Gamma]_{ij} = \frac{f_i(\lambda^p_i, ..., \lambda^p_{i-1}, \theta^p_i, \theta^p_{i+1}, ..., \theta^p_n) - f_i(\lambda^p_i, ..., \lambda^p_{i-1}, \theta^p_{j+1}, ..., \theta^p_n)}{\lambda^p_j - \theta^p_j},
\]

where \( \theta^p = (\theta^p_1, ..., \theta^p_{n-1}, \theta^p_n), \Lambda^p = (\lambda^p_1, ..., \lambda^p_{n-1}, \lambda^p_n); 1 \leq i, j \leq n \). For \( \beta = 1 \), the Traub’s scheme reduces to Steffensen’s scheme [20].

Bhalla et al. [21] have also presented the following scheme:

\[
\begin{align*}
\psi^p_1 &= \theta^{p+1} = \theta^p - [\theta^p + \Gamma, \theta^p - \Gamma; \Gamma]^{-1}\Gamma(\theta^p), \\
\psi^p_2 &= \psi^p_1 - \eta \Gamma(\psi^p_1), \\
\psi^p_3 &= \psi^p_2 - \eta \Gamma(\psi^p_2), \\
\vdots \\
\psi^p_{i-2} &= \psi^p_{i-3} - \eta \Gamma(\psi^p_{i-3}), \\
\psi^p_{i-1} &= \psi^p_{i-2} - \eta \Gamma(\psi^p_{i-2}), \\
\psi^p_i &= \psi^p_{i-1} - \eta \Gamma(\psi^p_{i-1}).
\end{align*}
\]

where

\[
\eta = \tau^{-1}\left(- (a_2^2 - 2a_1a_2)\psi^p_1, \theta^p, \Gamma + (a_2^2 + a_2^2 - 3a_1a_2)\psi^p_1 + \Gamma, \theta^p - \Gamma; \Gamma\right),
\]

\[
\tau = (2a_1a_2 - a_2^2)\psi^p_1, \theta^p, \Gamma + a_1a_2\psi^p_1, \theta^p, \Gamma - (2a_1a_2 - a_2^2)\psi^p_1 + \Gamma, \theta^p - \Gamma; \Gamma\right) \mbox{ and } t = 2, 3, 4, \ldots.
\]

Here \( a_1 \) and \( a_2 \) are non-real parameters provided both parameters do not equal zero simultaneously. In particular, for global convergence schemes for algebraic systems can be found in [22–24]. However, here we work on Banach spaces to determine only a locally unique solution.

Let us consider as a motivational example the nonlinear mixed Hammerstein integral type appearing in numerous studies [9–12] given by

\[
x(s) = c^1_0 K(s, x) \left( x(\xi)^2 + x(\xi)^2 \right) d\xi, \quad 0 \leq s \leq 1,
\]

where

\[
K(s, x) = \begin{cases}
\xi(1-s), & \xi \leq s \\
(1-s)\xi, & s \leq \xi.
\end{cases}
\]

Solving Equation (7) is equivalent to solving \( Q(\theta) = 0 \), with \( Q : D \subseteq C[0, 1] \to C[0, 1] \) given as

\[
\left(Q(\theta)\right)(s) = x(s) - \int_0^1 K(s, \xi) \left( x(\xi)^2 + x(\xi)^2 \right) d\xi.
\]

It follows that

\[
Q'(\theta)y(s) = y(s) - \int_0^1 K(s, \xi) \left( \frac{y(s)^2}{3} + x(\xi)^2 \right) d\xi.
\]
The local convergence analysis given in [21] uses Taylor expansions and derivatives up to the eighth order. Consequently, these results cannot be used on $Q(\theta) = 0$. These types of suppositions confine the usage of Scheme (3) and related schemes. Motivated by these problems, we develop a semi-local analysis for Scheme (5) but with suppositions only for the first Fréchet derivative as well as divided differences of order one (see, in particular, the continuation of this example in Example 1). Equation (10) is used to determine how close the initial guess should be to the solution to assure convergence of the method. Moreover, the $(H)$ conditions that follow indicate the constraints on the involved operators. Notice also that the work in [22–24] cannot be used to solve the motivational example.

2. Semi-Local Convergence

To introduce the semi-local analysis of Scheme (5), consider $\Gamma : D \subset B \to B$ to be a Fréchet-differentiable operator and $B$ to be a Banach space. Let $a, a_2 \in \mathbb{R}$ (or $\mathbb{C}$), $\gamma_0 \geq 0$, $d > 0$, $d_0 > 0$, $d_1 \geq 0$ and $d_2 > 0$ be parameters. Moreover, let $w_0 : [0, \infty)^2 \to [0, \infty)$, $w : [0, \infty)^2 \to [0, \infty)$ be continuous and nondecreasing functions in both variables. Define $r$, $b$ and $b_i$, $i = 1, 2, 3, 4, 5$ as

\[
\gamma = \frac{\gamma_0}{1 - w_0(d_1, d_1)},
\]

\[
b_1 = -a_2 + a_1 a_2,
\]

\[
b_2 = 2a_1^2 + a_2^2 - 3a_1 a_2,
\]

\[
b_3 = -a_1 a_2,
\]

\[
b_4 = 2a_1 a_2 - a_2^2,
\]

\[
b_5 = a_1^2 + a_2^2 - 3a_1 a_2,
\]

and

\[
b = b_1 + b_2.
\]

Suppose that

\[
b \neq 0.
\]

Define functions $v_0$, $v_1$, $v_2$, $v_3$ and $v$ on the interval $[0, \infty)$ in the following ways

\[
v_0(t) = |b|^{-1} \left| b_1 w_0(t + d_1, t + d_1) + b_2 w_0(t, t) + \left( \frac{|b_1| + d_2 |b_1| d_0 d}{1 - w_0(t + d_1, t + d_1)} \right) \right|,
\]

\[
v_1(t) = \left( \frac{|b_1| d}{1 - w_0(t + d_1, t + d_1)} + |b_2| \right) \frac{w(\gamma + d_1, d_1)}{b(1 - v_0(t))},
\]

\[
v_2(t) = \frac{|b_2| d}{1 - w_0(t + d_1, t + d_1)} + |b_2 + 1|,
\]

\[
v_3(t) = \frac{d_0 d_2}{b(1 - v_0(t))(1 - v_2(t))} \left( \frac{|b_4| d}{1 - w_0(t + d_1, t + d_1)} + |b_2| \right) \left( \frac{d(2|b_4| + |b_3|)}{1 - w_0(t + d_1, t + d_1)} + 2|b_2| + |b_1| \right),
\]

and

\[
v(t) = \frac{\gamma}{1 - v_3(t)} - t.
\]

Suppose function $v$ has a smallest zero $R > 0$ with

\[
w_0(R + d_1, R + d_1) < 1,
\]

\[
v_0(R) < 1,
\]

\[
v_1(R) < 1,
\]

\[
v_2(R) < 1
\]
Theorem 1. Let us assume that the

Proof. We first show that Scheme (5) is well defined and remains in \( \bar{H} \) and condition in \( (\ref{eq:5}) \) by adopting the preceding notations.

Equations (9)–(15) that the "\( v \)" functions are well defined. Next, we demonstrate the convergence of Equation (5) by adopting the preceding notations.

Let the \((H)\) conditions be used in the semi-local convergence in the following way

\( (H_1) \) Let \( \Gamma : D \subset B \to B \) be a continuous operator that is Fréchet differentiable at some \( \theta^0 \in D \) with \( \Gamma'(\theta^0)^{-1} \in L(B) \).

\( (H_2) \) There exists a divided difference of order one \( [\cdot, \cdot; \Gamma] : D \times D \to L(B) \).

\( (H_3) \) There exist \( \gamma_0 \geq 0, d_0 > 0, d_1 \geq 0, d_2 > 0 \) and \( d > 0 \) such that for each \( \theta, \Lambda \in D \)

\[
\begin{align*}
&\left\| \Gamma'(\theta^0)^{-1}\Gamma(\theta^0) \right\| \leq \gamma_0, \\
&\left\| [\theta, \Lambda; \Gamma] \right\| \leq d_0, \\
&\left\| \Gamma(\theta) \right\| \leq d_1, \\
&\left\| \Gamma'(\theta^0)^{-1} \right\| \leq d_2 \\
\end{align*}
\]

and

\[
\left\| \Gamma'(\theta^0)^{-1}[\theta, \Lambda; \Gamma] \right\| \leq d.
\]

\( (H_4) \) There exist functions \( w_0 : [0, \infty) \times [0, \infty) \to [0, \infty), w : [0, \infty) \times [0, \infty) \to [0, \infty) \) continuous and nondecreasing with \( \theta, \Lambda, Z \in D \)

\[
\begin{align*}
&\left\| \Gamma'(\theta^0)^{-1}\left( [\theta, \Lambda; \Gamma] - \Gamma'(\theta^0) \right) \right\| \leq w_0\left( \left\| \theta - \theta^0 \right\|, \left\| \Lambda - \theta^0 \right\| \right) \\
&\left\| \left( \left[ \theta, \Lambda; \Gamma - [Z, W; \Gamma] \right) \right\| \right\| \leq w\left( \left\| \theta - Z \right\|, \left\| \Lambda - W \right\| \right)
\end{align*}
\]

1. Function \( v \) defined by Equation (10) has the smallest positive zero denoted by \( R \).
2. Conditions (9) and (14)–(15) hold.
3. \( U(\theta^0, R + d_1) \subseteq D \).

Let \( U(\gamma, \rho), U(\gamma, \rho) \) be, respectively, open and closed balls in \( B \) centered at \( \gamma \in B \) and of radius \( \rho > 0 \). Next, let the local convergence analysis of Scheme (5) be under the \((H)\) conditions as follows:

**Theorem 1.** Let us assume that the \((H)\) conditions are satisfied. Then, the sequence achieved by Scheme (5) is valid, remains in \( U(\theta^0, R) \) and converges to the required solution \( \theta^* \) of expression \( \Gamma(\theta) = 0 \).

**Proof.** We first show that Scheme (5) is well defined and remains in \( U(\theta^0, R) \). Using the third condition in \((H_3)\) and \((H_7)\), we have

\[
\begin{align*}
\left\| \theta + \Gamma(\theta) - \theta^0 \right\| &\leq \left\| \Gamma(\theta) \right\| + \left\| \theta - \theta^0 \right\| \leq d_1 + R, \\
\text{and} \ \\
\left\| \theta + \Gamma(\theta) - \theta^0 \right\| &\leq \left\| \theta - \theta^0 \right\| + \left\| \Gamma(\theta) \right\| \leq d_1 + R, \ \text{for each} \ \theta \in U(\theta^0, R + d_1) \subseteq D.
\end{align*}
\]

Hence, we have

\[
\theta + \Gamma(\theta), \theta - \Gamma(\theta) \in U(\theta^0, R + d_1).
\]

In particular

\[
\theta^0 + \Gamma(\theta^0), \theta^0 - \Gamma(\theta^0) \in U(\theta^0, R + d_1).
\]
By the first condition in \((H_4)\) and Equation (11), we yield
\[
\left\| \Gamma' (\vartheta^0)^{-1} \left( [\vartheta + \Gamma (\vartheta), \vartheta - \Gamma (\vartheta); \Gamma] - \Gamma' (\vartheta^0) \right) \right\|
\leq w_0 \left( \left\| \vartheta + \Gamma (\vartheta) - \vartheta^0 \right\|, \left\| \vartheta - \Gamma (\vartheta) - \vartheta^0 \right\| \right)
\leq w_0 \left( \left\| \vartheta - \vartheta^0 \right\| + \left\| \Gamma (\vartheta) \right\|, \left\| \vartheta - \vartheta^0 \right\| + \left\| \Gamma (\vartheta) \right\| \right)
= w_0 (R + d_1, R + d_1) < 1. 
\] (17)

This takes from Equation (17) and the Banach Lemma for inverse operators \([10,11]\) so that \([\vartheta + \Gamma (\vartheta), \vartheta - \Gamma (\vartheta); \Gamma]^{-1} \in L(\vartheta)\) and
\[
\left\| [\vartheta + \Gamma (\vartheta), \vartheta - \Gamma (\vartheta); \Gamma]^{-1} \Gamma' (\vartheta^0) \right\| 
\leq \frac{1}{1 - w_0 (\left\| \vartheta - \vartheta^0 \right\| + \left\| \Gamma (\vartheta) \right\|, \left\| \vartheta - \vartheta^0 \right\| + \left\| \Gamma (\vartheta) \right\|)} 
\leq \frac{1}{1 - w_0 (R + d_1, R + d_1)}. 
\] (18)

We also have that by Equation (18) for \(\vartheta = \vartheta^0\) and the first sub-step of Scheme (5), \(\psi_1^0\) is well defined. Then, we have by the first condition in \((H_3)\) and Equation (18) the estimate
\[
\left\| \psi_1^0 - \vartheta^0 \right\| = \left\| \vartheta^0 - \vartheta^0 - \left[ \vartheta^0 + \Gamma (\vartheta^0), \vartheta^0 - \Gamma (\vartheta^0); \Gamma \right]^{-1} \Gamma' (\vartheta^0) \right\| 
\leq \left\| \vartheta^0 + \Gamma (\vartheta^0), \vartheta^0 - \Gamma (\vartheta^0); \Gamma \right\]^{-1} \Gamma' (\vartheta^0) \left\| \Gamma' (\vartheta^0)^{-1} \Gamma (\vartheta^0) \right\|
\leq \frac{\gamma_0}{1 - w_0 (d_1, d_1)} := \gamma. 
\] (19)

We can write
\[
\Gamma (\psi_1^0) = \Gamma (\psi_0^0) - \Gamma (\vartheta^0) - \left[ \vartheta^0 + \Gamma (\vartheta^0), \vartheta^0 - \Gamma (\vartheta^0); \Gamma \right] (\psi_1^0 - \vartheta^0)
= \left( [\psi_0^0, \vartheta^0; \Gamma] - \left[ \vartheta^0 + \Gamma (\vartheta^0), \vartheta^0 - \Gamma (\vartheta^0); \Gamma \right] \right) (\psi_1^0 - \vartheta^0), 
\] (20)

since
\[
\left[ \vartheta, \Lambda; \Gamma \right] (\vartheta - \Lambda) = \Gamma (\vartheta) - \Gamma (\Lambda).
\] (21)

By the second conditions in \((H_3)\), \((H_4)\) and Equation (20), we obtain
\[
\left\| \Gamma' (\vartheta^0)^{-1} \Gamma (\psi_1^0) \right\| = 
\left\| \Gamma' (\vartheta^0)^{-1} \left( [\psi_0^0, \vartheta^0; \Gamma] - \left[ \vartheta^0 + \Gamma (\vartheta^0), \vartheta^0 - \Gamma (\vartheta^0); \Gamma \right] \right) (\psi_1^0 - \vartheta^0) \right\|
\leq w \left( \left\| \psi_0^0 - \vartheta^0 - \Gamma (\vartheta^0) \right\|, \left\| \vartheta^0 - \vartheta^0 + \Gamma (\vartheta^0) \right\| \right) \left\| \psi_1^0 - \vartheta^0 \right\|
\leq w \left( \left\| \psi_1^0 - \vartheta^0 \right\| + \left\| \Gamma (\vartheta^0) \right\|, \left\| \vartheta^0 - \vartheta^0 \right\| + \left\| \Gamma (\vartheta^0) \right\| \right) \left\| \psi_1^0 - \vartheta^0 \right\|
= w (\gamma + d_1, d_1) \left\| \psi_1^0 - \vartheta^0 \right\|. 
\] (22)

Set \(A_\vartheta = b_4 [\vartheta + \Gamma (\vartheta^0), \vartheta^0 - \Gamma (\vartheta^0); \Gamma]^{-1} [\psi_1^0, \vartheta^0; \Gamma] + b_2 I\)
\[23\]
and
\[
B_\vartheta = b_4 [\vartheta + \Gamma (\vartheta^0), \vartheta^0 - \Gamma (\vartheta^0); \Gamma] + b_2 [\psi_1^0, \vartheta^0; \Gamma]
+ b_3 [\vartheta + \Gamma (\vartheta^0), \vartheta^0 - \Gamma (\vartheta^0); \Gamma]^{-1} [\psi_1^0, \vartheta^0; \Gamma] [\vartheta^0 + \Gamma (\vartheta^0), \vartheta^0 - \Gamma (\vartheta^0); \Gamma]
+ b_4 [\vartheta + \Gamma (\vartheta^0), \vartheta^0 - \Gamma (\vartheta^0); \Gamma]^{-1} [\psi_1^0, \vartheta^0; \Gamma]^2. 
\] (24)
We must show $A_0^{-1}, B_0^{-1} \in L(\theta)$. By Equations (14), (18) and (23) and the last condition in $(H_3)$, we get that
\[
\|A_0 + I\| \leq |b_4| |\|\theta^0 + \Gamma(\theta^0) - \Gamma(\theta^0); \Gamma^{-1} \Gamma'(\theta^0)\| + |b_2 + 1| \|I\| \\
\leq \frac{|b_4| d}{1 - w_0(R + d_1, R + d_1)} + |b_2 + 1| = v_2(R) < 1.
\] (25)

It follows from Equation (25) that
\[
\|A_0^{-1}\| \leq \frac{1}{1 - v_2(R)}.
\] (26)

Similarly, by Equations (9) and (12), and $(H_3)$ and $(H_4)$, we get that
\[
\left\| \left( b \Gamma'(\theta^0) \right)^{-1} \left( B_0 - (b_1 + b_2) \Gamma'(\theta^0) \right) \right\|
\leq |b|^{-1} \left[ |b_1| \left\| \Gamma'(\theta^0)^{-1} \left( [\theta^0 + \Gamma(\theta^0), \theta^0 - \Gamma(\theta^0); \Gamma - \Gamma'(\theta^0)] \right) \right\| \\
+ |b_2| \left\| \Gamma'(\theta^0)^{-1} \left( [\psi_1^0, \theta^0; \Gamma - \Gamma'(\theta^0)] \right) \right\| \\
+ |b_3| \left\| \Gamma'(\theta^0)^{-1} \left( [\psi_1^0, \theta^0; \Gamma - \Gamma'(\theta^0)] \right) \right\| \\
\times \left\| \left[ \theta^0 + \Gamma(\theta^0), \theta^0 - \Gamma(\theta^0); \Gamma - \Gamma'(\theta^0) \right] \right\|
\leq |b|^{-1} \left[ |b_1| w_0(\|\theta^0 - \theta^0 + \Gamma(\theta^0)\|, \|\theta^0 - \Gamma(\theta^0) - \theta^0\|) \\
+ |b_2| w_0(\|\psi_1^0 - \theta^0\|, \|\theta^0 - \theta^0\|) \\
+ \frac{|b_3| d_0 d_2 d_2}{1 - w_0(R + d_1, R + d_1)} + \frac{|b_4| d_0 d}{1 - w_0(R + d_1, R + d_1)} \right]
\leq |b|^{-1} \left[ |b_1| w_0(R + d_1, R + d_1) + |b_2| w_0(R, R) + \frac{d_0 d (|b_4| + d_2 |b_3|)}{1 - w_0(R + d_1, R + d_1)} \right]
\] (27)

Hence, by Equation (27), we get that
\[
\left\| B_0^{-1} \Gamma'(\theta^0) \right\| \leq \frac{1}{|b| (1 - v_0(R))}. \] (28)

Then, by Equations (13), (16), (25) and (28), we have that
\[
\|\psi_2^0 - \psi_1^0\| \leq \|A_0\| \left\| B_0^{-1} \Gamma'(\theta^0) \right\| \left\| \Gamma'(\theta^0)^{-1} \Gamma(\psi_1^0) \right\|
\times \left( \frac{|b_4| d}{1 - w_0(R + d_1, R + d_1)} + |b_2| \right) \frac{w(\gamma + d_1, d_1)}{|b| (1 - v_0(R))} \|\psi_1^0 - \psi_0^0\| \\
\leq v_1(R) \|\psi_1^0 - \psi_0^0\| \leq q \|\psi_1^0 - \psi_0^0\|.
\] (29)
Since \( \theta^0 = \psi^0_0 \), we have that

\[
\| \psi^0_2 - \psi^0_0 \| = \| \psi^0_2 - \psi^0_0 \| \leq \| \psi^0_2 - \psi^0_1 \| + \| \psi^1_0 - \theta^0 \| = \frac{1 - q^2}{1 - q} \gamma < \frac{\gamma}{1 - q} \leq R,
\]

so \( \psi^0_2 \in \mathcal{U}(\theta^0, R) \), and we yield

\[
\Gamma(\psi^0_2) = \Gamma(\psi^0_2) - \Gamma(\psi^0_1) - \eta^{-1}(\psi^0_2 - \psi^0_1) = A^{-1}_0 \left( A_0[\psi^0_2, \psi^0_1; \Gamma] - B_0 \right) (\psi^0_2 - \psi^0_1).
\]

By Equations (23), (24), (26) and (31), and \((H_3)\) and \((H_4)\), we have

\[
\| \Gamma'(\theta^0)^{-1} \Gamma(\psi^0_2) \| \leq \| \Gamma'(\theta^0)^{-1} \| \| A^{-1}_0 \| \left[ \| A_0[\psi^0_2, \psi^0_1; \Gamma] \| + \| B_0 \| \right] \| \psi^0_2 - \psi^0_1 \|
\]

\[
\leq \frac{d_2}{d_1} \left[ \frac{d_2|d_4|}{1 - w_0(R + d_3, R + d_1)} + d_0(|b_1| + |b_2|) + \frac{d_4|b_3|}{1 - w_0(R + d_1, R + d_1)} \right] \| \psi^0_2 - \psi^0_1 \|.
\]

Then, using Equations (15), (16), (28) and (32), we get that

\[
\| \psi^0_3 - \psi^0_2 \| \leq \| A_0 \| \left[ B_0^{-1} \Gamma'(\theta^0) \right] \| \Gamma'(\theta^0)^{-1} \Gamma(\psi^0_2) \|
\]

\[
\leq w_3(R) \| \psi^0_2 - \psi^0_1 \| \leq q \| \psi^0_2 - \psi^0_1 \|.
\]

We also have that

\[
\| \psi^0_3 - \theta^0 \| \leq \| \psi^0_3 - \psi^0_2 \| + \| \psi^0_2 - \psi^0_1 \| + \| \psi^0_1 - \theta^0 \|
\]

\[
\leq (1 + q + q^2) \| \psi^0_1 - \theta^0 \| < \frac{1 - q^3}{1 - q} \gamma \leq R.
\]

That is \( \psi^0_3 \in \mathcal{U}(\theta^0, R) \). By simply replacing \( \theta^0, \psi^0_0, \psi^0_2 \) by \( \psi^m_k, \psi^m_{k+1}, \sigma = 1, 2, 3, \ldots, m = 0, 1, 2, \ldots \) in the preceding estimates, we get that

\[
\| \psi^m_{k+1} - \psi^m_k \| \leq q \| \psi^m_{k+1} - \psi^m_k \|
\]

and

\[
\| \psi^m_{k+1} - \theta^0 \| \leq R,
\]

which complete the induction. However, then Scheme (5) is complete in \( B \) (Banach space) and by Equation (35) and \( \psi^m_{k+1} \in \mathcal{U}(\theta^0, R) \) converges to some \( \theta^* \in \mathcal{U}(\theta^0, R) \). Moreover, there exists (see Equations (14), (15) and (32)) \( \delta \geq 0 \) with

\[
\| \Gamma'(\theta^0)^{-1} \Gamma(\psi^i_p) \| \leq \delta \| \psi^i_p - \psi^i_{p-1} \|, \quad i = 1, 2, \ldots, p, \quad j = 0, 1, 2, \ldots.
\]

By letting \( j \to \infty \) in (37) we deduce that \( \Gamma(\theta^*) = 0 \). Hence, \( \psi^0_0 \in \mathcal{U}(\theta^0, R) \). \( \square \)

Next, a uniqueness result is presented.

**Proposition 1.** By adopting hypotheses \((H)\), we consider that there will be an \( R_1 \geq R \) such that

\[
w_0(R, R_1) < 1.
\]

Then, \( \theta^* \) is the unique solution of the expression \( \Gamma(\theta) = 0 \) in \( \mathcal{U}(\theta^0, R_1) \).
Proof. The existence of the solution $\dot{\theta}^*$ was established in Theorem 1. Let $S = [\dot{\theta}^*, \Lambda^*; \Gamma]$, where $\Lambda^* \in \tilde{U}(\dot{\theta}^*, R_1)$ with $\Gamma(\Lambda^*) = 0$. Then, using the first condition in $(H_4)$ and Equation (38), we get that

$$\left\|\Gamma'(\dot{\theta}^0)^{-1} \left( [\dot{\theta}^*, \Lambda^*; \Gamma] - \Gamma'(\dot{\theta}^0) \right) \right\| \leq w_0(\|\dot{\theta}^* - \dot{\theta}^0\|, \|\Lambda^* - \dot{\theta}^0\|) \leq w_0(R, R_1) < 1. \quad (39)$$

It follows from Equation (39) that $S^{-1} \in L(\dot{\theta})$. Then, by $\dot{\theta}^* = \Lambda^*$ is deduced from $0 = S(\dot{\theta}^*) - S(\Lambda^*) = S(\dot{\theta}^* - \Lambda^*)$. \hfill $\square$

3. Numerical Results

Several real life problems are provided to illustrate the convergence nature of the schemes shown in Scheme (5). Therefore, we consider all five higher-order methods out of our proposed scheme, namely $(\psi_2$ for $a_1 = \pm 10^{20} & \alpha_2 = \pm 10^{-1000})$, $(\psi_2$ for $a_1 = \pm \sqrt{3} & \alpha_2 = \pm 10^{-2000})$, $(\psi_3$ for $a_1 = \pm 10^{-1000})$, $(\psi_3$ for $a_1 = \pm \sqrt{3} & \alpha_2 = \pm 10^{-2000})$, $(\psi_4$ for $a_1 = \mathbb{R} - \{0\} & \alpha_2 = 0)$, $(\psi_4$ for $a_1 = \pm 10^{20} & \alpha_2 = \pm 10^{-1000})$ and $(\psi_4$ for $a_1 = \pm \sqrt{3} & \alpha_2 = \pm 10^{-2000})$, denoted by $(OM_14)$, $(OM_24)$, $(OM_38)$, $(OM_45)$, $(OM_58)$ and $(OM_68)$, respectively, to test the convergence criteria. Out these seven schemes first two are of order four; the third and fourth are of order six; and the last three are of order eight. We compare them with fourth-order schemes proposed by Sharma and Arora [25], out of them we consider Equations (2) and (3) denoted by $(SA_14)$ and $(SA_24)$, respectively.

In addition, we consider two higher-order methods of order four and six presented by Grau et al. [18], out of them we choose Equations (12) and (14), denoted by $(GM_14)$ and $(GM_26)$, respectively. Finally, we also compared them with seventh-order methods given by Wang and Zhang [26]; out them we consider Equations (56) and (57), denoted $WZ_17$ and $WZ_27$, respectively.

For fair contrast of our schemes, we have depicted the error among the exact and estimated zero $\|\dot{\theta}^m - \dot{\theta}^*\|$. COC stands for computational order of convergence, and CPU stands for time in Tables 1–4. We used the succeeding formulas [14]

$$\rho = \frac{\ln \|\dot{\theta}^{m+1} - \dot{\theta}^*\|}{\ln \|\dot{\theta}^m - \dot{\theta}^*\|}, \quad m = 1, 2, \ldots \quad (40)$$

or

$$\rho^* = \frac{\ln \|\dot{\theta}^{m+1} - \dot{\theta}^m\|}{\ln \|\dot{\theta}^m - \dot{\theta}^{m-1}\|}, \quad m = 2, 3, \ldots \quad (41)$$

COC and the approximate computational order of convergence (ACOC), respectively.

It is vital to note that $\rho$ or $\rho^*$ do not need any kind of higher-order derivatives to compute the error bounds. The notation $a_1$ (or $a_2$) employs $a_1 \times 10^{(\pm a_2)}$.

The computational works are performed with programming software Mathematica-10 (Wolfram Research, Champaign, IL, USA) [27] and the configurations of our computer are given below:

- A processor Intel(R) Core(TM) i5-3210M (Intel, Santa Clara, CA, USA)
- CPU @ 2.50 GHz (64-bit machine) (Intel, Santa Clara, CA, USA)
- Microsoft Windows 8 (Microsoft Corporation, Albuquerque, NM, USA).

We consider at least 1000 digits of mantissa in order to minimize the round-off errors. In addition, all the problems are first transformed into nonlinear systems of equations and then solved using the proposed iterative scheme.
Example 1. Returning back to Equation (7), for \( [\vartheta, \Lambda : \Gamma] = \int_{0}^{1} \Gamma'(\Lambda + v(\vartheta - \Lambda))dv \), we see that our results can apply, if we choose

\[
\vartheta_0(t) = \vartheta_1(t) = d_2 \left( \frac{5}{24} t^{2/3} + 2t \right), \quad \text{and} \quad \vartheta_0(s, t) = \vartheta(s, t) = \frac{1}{2} (\vartheta_0(s) + \vartheta_0(t)),
\]

for \( \vartheta^0 \) sufficiently close to the solution \( \vartheta^* = 0 \).

Example 2. Bratu Problem:
Here, we assume the well known Bratu Problem [28], which is given by

\[
y'' + Ce^{y} = 0, \quad y(0) = y(1) = 0.
\]  
(42)

It has a wide area of application, for example, in radioactive heat transfer, thermal reaction, the fuel ignition model of thermal combustion, chemical reactor theory, the Chandrasekhar model of the expansion of the universe and nanotechnology [28–31].

The finite difference discretization is used to convert the above boundary value problem, Equation (42), into a non-linear system of size \( 40 \times 40 \) with step size \( h = 1/41 \). For second derivative central difference has been used and is as follows

\[
y''_{\sigma} = \frac{\lambda_{\sigma - 1} - 2\lambda_{\sigma} + \lambda_{\sigma + 1}}{h^2}, \quad \sigma = 1, 2, \ldots, 40.
\]

The computational comparison of the solution of this problem is shown in Table 1 and the graphical solution in Figure 1.

Example 3. Bratu Problem in 2D:
We choose a prominent 2D Bratu problem [32,33], which is defined by

\[
u_{xx} + u_{tt} + Ce^{u} = 0, \text{ on } \Omega : (x, t) \in 0 \leq x \leq 1, 0 \leq t \leq 1,
\]

along boundary hypothesis \( u = 0 \) on \( \Omega \).

Let us assume that \( \Gamma_{i,j} = u(x_i, t_j) \) is a numerical result over the grid points of the mesh. In addition, we consider that \( \tau_1 \) and \( \tau_2 \) are the number of steps in the direction of \( x \) and \( t \), respectively. Moreover, we choose that \( h \) and \( k \) are the respective step sizes in the direction of \( x \) and \( y \), respectively. In order to find the solution of partial differential equation (PDE) (43), we adopt the following approach

\[
u_{xx}(x_i, t_j) = \frac{\Gamma_{i+1,j} - 2\Gamma_{i,j} + \Gamma_{i-1,j}}{h^2}, \quad C = 0.1, \quad t \in [0, 1],
\]  
(44)

which further yields the succeeding system of nonlinear equation (SNE)

\[
\Gamma_{i,j+1} + \Gamma_{i,j-1} - \Gamma_{i,j} + \Gamma_{i+1,j} + \Gamma_{i-1,j} + h^2 C \exp \left( \Gamma_{i,j} \right) i = 1, 2, 3, \ldots, \tau_1, j = 1, 2, 3, \ldots, \tau_2.
\]  
(45)

By choosing \( \tau_1 = \tau_2 = 11 \), \( h = \frac{1}{11} \) and \( C = 0.1 \), we get a large SNE of order \( 100 \times 100 \). The starting point is

\[
\vartheta^0 = 0.1 (\sin(\pi h) \sin(\pi k), \sin(2\pi h) \sin(2\pi k), \ldots, \sin(10\pi h) \sin(10\pi k))^T
\]

and the results are depicted in Table 1. Further, the estimated solution has been plotted in Figure 2.
Example 4. Fisher’s Equation:

Here, we assume another typical non-linear Fisher’s equation [34], which is given by

\[
\begin{align*}
    u_t &= Du_{xx} + u(1-u) = 0, \\
    u(x,0) &= 1.5 + 0.5\cos(\pi x), 0 \leq x \leq 1, \\
    u_x(0,t) &= 0, \forall t \geq 0, \\
    u_x(1,t) &= 0, \forall t \geq 0, \\
\end{align*}
\]

(46)

where \( D \) is the diffusion coefficient. Let us assume that \( \Gamma_{i,j} = u(x_i, t_j) \) is a numerical result over the grid points of the mesh. In addition, we consider \( \tau_1 \) and \( \tau_2 \) to be the number of steps in the direction of \( x \) and \( t \), respectively. Moreover, we choose that \( h \) and \( k \) are the respective step sizes in the direction of \( x \) and \( y \), respectively. In order to find the solution of PDE (46), we adopt the following approach

\[
\begin{align*}
    u_{xx}(x_i, t_j) &= \frac{\Gamma_{i+1,j} - 2\Gamma_{i,j} + \Gamma_{i-1,j}}{h^2}, \\
    u_t(x_i, t_j) &= \frac{\Gamma_{i,j} - \Gamma_{i,j-1}}{k}, \\
    u_x(x_i, t_j) &= \frac{\Gamma_{i+1,j} - \Gamma_{i,j}}{(h)^2}, \\
\end{align*}
\]

(47)

which further yields the succeeding SNE

\[
\frac{\Gamma_{i,j} - \Gamma_{i,j-1}}{k} - \Gamma_{i,j}\left(1 - \Gamma_{i,j}\right) - H \frac{\Gamma_{i+1,j} - 2\Gamma_{i,j} + \Gamma_{i-1,j}}{h^2}, \quad i = 1, 2, 3, \ldots, \tau_1, j = 1, 2, 3, \ldots, \tau_2. \quad (48)
\]

By choosing \( \tau_1 = \tau_2 = 21, \quad h = \frac{1}{\tau_1} \) and \( k = \frac{1}{\tau_2} \), we get a large SNE of order \( 400 \times 400 \). The starting point is

\[
\theta^0 = (i/(\tau_1 - 1)^2)^T, \quad i = 1, 2, \ldots, \tau_1 - 1
\]

and results are mentioned in Table 3. The approximate solution has been plotted in Figure 3.

Example 5. Finally, we deal with the following SNE

\[
\Gamma(\theta) = \begin{cases} 
-1 + \theta_j^2 x_{j+1} = 0, & 1 \leq j \leq \sigma - 1, \\
-1 + \theta_1^2 \theta_1 = 0.
\end{cases}
\]

(49)

In order to access a giant system of nonlinear equations of order \( 100 \times 100 \), we pick \( \sigma = 100 \). In addition, we consider the following starting approximation for this problem:

\[
\theta^{(0)} = (1.13, 1.13 \cdots, 1.13(100 \text{ times}))^T,
\]

and converges to \( \xi^* = (1, 1, 1, \cdots, 1(100 \text{ times}))^T \). The attained computation outcomes are illustrated in Table 4.

Remark 1. It follows from Tables 1–4 that our methods have a very small error difference between the exact and approximated root as compared to the other mentioned methods. In addition, they have a more stable computational order of convergence and take less CPU time for better accuracy in the required zero.

In some tables the values of error approximations \( \|\hat{\theta}^{(m)} - \hat{\xi}^*\| \) appear the same for different \( \alpha_1 \) and \( \alpha_2 \), but actually they are different; if we mention the errors in more significant digits in those tables then we can see the clear difference. However, due to limited page space only three significant digits are depicted for different \( \alpha_1 \) and \( \alpha_2 \).
Table 1. Comparison of distinct iterative schemes on Example 2.

| Scheme | ||\(\theta^{(1)} - \theta^*||\) | ||\(\theta^{(2)} - \theta^*||\) | ||\(\theta^{(3)} - \theta^*||\) | \(\rho\) | CPU time (s) |
|--------|-----------------|-----------------|-----------------|-----------------|--------|-----------------|
| SA1\(_4\) | 4.8 (−2) | 1.1 (−8) | 2.8 (−35) | 3.997 | 17.59 |
| SA2\(_4\) | 1.0 (−1) | 3.6 (−7) | 5.4 (−29) | 3.993 | 21.36 |
| GM1\(_4\) | 4.8 (−2) | 1.1 (−8) | 3.3 (−35) | 3.997 | 15.70 |
| OM1\(_4\) | 4.8 (−2) | 1.1 (−8) | 3.3 (−35) | 3.997 | 15.29 |
| OM2\(_4\) | 4.8 (−2) | 1.1 (−8) | 3.3 (−35) | 3.997 | 15.92 |
| GM2\(_6\) | 6.4 (−3) | 4.4 (−18) | 4.5 (−109) | 5.999 | 15.90 |
| OM3\(_6\) | 6.4 (−3) | 4.4 (−18) | 4.5 (−109) | 5.999 | 15.34 |
| OM4\(_6\) | 6.4 (−3) | 4.4 (−18) | 4.5 (−109) | 5.999 | 16.15 |
| WZ1\(_7\) | 6.8 (−5) | 1.2 (−35) | 5.7 (−251) | 7.000 | 29.50 |
| WZ2\(_7\) | 5.1 (−4) | 2.4 (−29) | 1.4 (−206) | 6.999 | 29.56 |
| OM5\(_8\) | 8.6 (−4) | 6.3 (−31) | 5.4 (−248) | 8.000 | 15.64 |
| OM6\(_8\) | 8.6 (−4) | 6.3 (−31) | 5.4 (−248) | 8.000 | 15.37 |
| OM7\(_8\) | 8.6 (−4) | 6.3 (−31) | 5.4 (−248) | 8.000 | 16.20 |

Figure 1. Approximated solution of Bratu problem with \(C = 3\) for \(t \in [0, 1]\).

Table 2. Comparison of distinct iterative schemes on 2D Bratu problem Example 3.

| Scheme | ||\(\theta^{(1)} - \theta^*||\) | ||\(\theta^{(2)} - \theta^*||\) | ||\(\theta^{(3)} - \theta^*||\) | \(\rho\) | CPU time (s) |
|--------|-----------------|-----------------|-----------------|-----------------|--------|-----------------|
| SA1\(_4\) | 2.8 (−10) | 2.8 (−47) | 2.6 (−195) | 3.999 | 241.63 |
| SA2\(_4\) | 2.8 (−10) | 2.9 (−47) | 3.1 (−195) | 3.999 | 303.07 |
| GM1\(_4\) | 4.2 (−10) | 2.1 (−46) | 1.6 (−191) | 3.999 | 224.73 |
| OM1\(_4\) | 4.2 (−10) | 2.1 (−46) | 1.6 (−191) | 3.999 | 218.45 |
| OM2\(_4\) | 4.2 (−10) | 2.1 (−46) | 1.6 (−191) | 3.999 | 238.30 |
| GM2\(_6\) | 2.0 (−15) | 8.2 (−102) | 5.0 (−620) | 5.999 | 217.65 |
| OM3\(_6\) | 2.0 (−15) | 8.2 (−102) | 5.0 (−620) | 5.999 | 210.22 |
| OM4\(_6\) | 2.0 (−15) | 8.2 (−102) | 5.0 (−620) | 5.999 | 220.96 |
| WZ1\(_7\) | 1.9 (−19) | 2.4 (−148) | 1.1 (−1050) | 6.999 | 1343.25 |
| WZ2\(_7\) | 1.9 (−19) | 2.6 (−148) | 1.8 (−1050) | 6.999 | 1310.85 |
| OM5\(_8\) | 9.2 (−21) | 1.6 (−178) | 1.2 (−1440) | 7.999 | 395.73 |
| OM6\(_8\) | 9.2 (−21) | 1.6 (−178) | 1.2 (−1440) | 7.999 | 431.84 |
| OM7\(_8\) | 9.2 (−21) | 1.6 (−178) | 1.2 (−1440) | 7.999 | 441.66 |
Figure 2. Approximated solution for 2D Bratu problem with $C = 0.1$ $t \in [0, 1]$.

Table 3. Comparison of distinct iterative schemes on Fisher’s Equation (4).

| Scheme | $\|v^{(1)} - v^*\|$ | $\|v^{(2)} - v^*\|$ | $\|v^{(3)} - v^*\|$ | $\rho$ | CPU time (s) |
|--------|------------------|------------------|------------------|--------|--------------|
| SA14   | div              | div              | div              | –      | –            |
| SA24   | div              | div              | div              | –      | –            |
| GM14   | 1.2              | 4.3(-6)          | 3.5(-28)         | 4.049  | 1127.58      |
| OM14   | 1.2              | 4.3(-6)          | 3.5(-28)         | 4.049  | 1409.30      |
| OM24   | 1.2              | 4.3(-6)          | 3.5(-28)         | 4.049  | 1405.62      |
| GM26   | 2.5(-1)          | 9.1(-14)         | 1.1(-88)         | 6.024  | 1376.54      |
| OM36   | 2.5(-1)          | 9.1(-14)         | 1.1(-88)         | 6.024  | 1383.58      |
| OM46   | 2.5(-1)          | 9.1(-14)         | 1.1(-88)         | 6.024  | 1189.68      |
| WZ17   | div              | div              | div              | –      | –            |
| WZ27   | div              | div              | div              | –      | –            |
| OM58   | 4.8(-2)          | 1.6(-24)         | 2.0(-204)        | 8.0091 | 1093.62      |
| OM68   | 4.8(-2)          | 1.6(-24)         | 2.0(-204)        | 8.0091 | 1356.34      |
| OM78   | 4.8(-2)          | 1.6(-24)         | 2.0(-204)        | 8.0091 | 1359.12      |
4. Concluding Remarks

In this work, semi-local convergence analysis of the family proposed by [21] has been proved by adopting Lipschitz constants and order one divided differences on a Banach space setting under weak conditions, so that we can expand the applicability of Scheme (5) and other related schemes. The use of this family on real life problems, namely Bratu’s 1D (SNE 40 × 40), Bratu’s 2D (SNE 100 × 100), Fisher’s problem (SNE 400 × 400) and polynomial equations (SNE 100 × 100), also confirms the applicability of this family.

Author Contributions: R.B. and I.K.A.: Conceptualization; Methodology; Validation; Writing—Original Draft Preparation; Writing—Review & Editing. S.B. and S.K.: Validation; Review & Editing. All authors have read and agreed to the published version of the manuscript.
Funding: Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia, under Grant No. D-405-130-1441.

Acknowledgments: This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (D-405-130-1441). The authors, therefore, acknowledge with thanks DSR technical and financial support.

Conflicts of Interest: The authors declare no conflict of interest.

References
1. Ezquerro, J.A.; Grau-Sánchez, M.; Hernández-veron, M.A.; Noguera, M. A family of iterative methods that uses divided differences of first and second orders. *Numer. Algor.* **2015**, *70*, 571–589.
2. Grau-Sánchez, M.; Noguera, M. Frozen divided difference scheme for solving system of nonlinear equations. *J. Comput. Appl. Math.* **2011**, *235*, 1739–1743.
3. Ortega, J.M.; Rheinboldt, W.C. *Iterative Solution of Nonlinear Equations in Several Variables*; Academic Press: New York, NY, USA, 1970.
4. Ostrowski, A.M. *Solutions of Equations and System of Equations*; Academic Press: New York, NY, USA, 1960.
5. Petković, M.S. Remarks on On a general class of multipoint root finding methods of high computational efficiency. *SIAM J. Numer. Anal.* **2011**, *49*, 1317–1319.
6. Potra, F.A.; Pták, V. *Nondiscrete introduction and iterative process*; (Research Notes in Mathematics); Pitman Advanced Pub. Program: Roma, Italy, 1984.
7. Amat, S.; Busquier, S.; Plaza, S. Dynamics of the King and Jarratt iterations. *Aequationes Math.* **2005**, *69*, 212–223.
8. Amat, S.; Busquier, S.; Plaza, S. Chaotic dynamics of a third-order Newton-type method. *J. Math. Anal. Appl.* **2010**, *366*, 24–32.
9. Amat, S.; Hernández, M.A.; Romero, N. A modified Chebyshev’s iterative method with at least sixth order of convergence. *Appl. Math. Comput.* **2008**, *206*, 164–174.
10. Argyros, I.K. *Convergence and Application of Newton-Type Iterations*; Springer: New York, NY, USA, 2008.
11. Argyros, I.K.; Hilout, S. *Computational Methods in Nonlinear Analysis*; World Scientific Publishing Company: Hackensack, NJ, USA, 2013.
12. Argyros, I.K.; George, S.; Magrenan, A.A. Local convergence for multi-point-parametric Chebyshev-Halley-type methods of high convergence order. *J. Comput. Appl. Math.* **2015**, *282*, 215–224.
13. Behl, R.; Kanwar, V.; Sharma, K.K. Optimal equi-scaled families of Jarratt’s method. *Int. J. Comput. Math.* **2013**, *290*, 408–422.
14. Ezquerro, J.A.; Hernández, M.A. New iterations of R-order four with reduced computational cost. *BIT Numer. Math.* **2009**, *49*, 325–342.
15. Kelley, C.T. *Solving Nonlinear Equations with Newton’s Method*; SIAM: Philadelphia, PA, USA, 2003.
16. Wang, X.; Zhang, T.; Qian, W.; Teng, M. Seventh-order derivative-free iterative method for solving nonlinear systems. *Numer. Algor.* **2015**, *70*, 545–558.
17. Traub, J.F. *Iterative Method for the Solution of Equations*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1964.
18. Grau-Sánchez, M.; Noguera, M.; Amat, S. On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative methods. *J. Comput. Appl. Math.* **2013**, *237*, 363–372.
19. Grau-Sánchez, M.; Noguera, M.; Diaz-Barrero, J.L. On the local convergence of a family of two-step iterative methods for solving nonlinear equations. *J. Comput. Appl. Math.* **2014**, *255*, 753–764.
20. Steffensen, J.F. Remarks on iteration. *Skand. Aktuar. Tidskr* **1933**, *16*, 64–72.
21. Bhalla, S.; Kumar, S.; Argyros, I.K.; Behl, R. A family of higher order derivative free methods for nonlinear systems with local convergence analysis. *Comput. Appl. Math.* **2018**, *37*, 5807–5827.
22. Aizenshtein, M.; Bartoň, M.; Elber, G. Global solutions of well-constrained transcendental systems using expression trees and a single solution test. *Comput. Aided Geom. Des.* **2012**, *29*, 265–279.
23. Van Sosin, B.; Elber, G. Solving piecewise polynomial constraint systems with decomposition and a subdivision-based solver. *Comput. Aided Des.* **2017**, *90*, 37–47.
24. Bartoň, M. Solving polynomial systems using no-root elimination blending schemes. *Comput. Aided Des.* **2011**, *43*, 1870–1878.

25. Sharma, J.R.; Arora, H. An efficient derivative free iterative method for solving systems of nonlinear equations. *Appl. Anal. Discrete Math.* **2013**, *7*, 390–403.

26. Wang, X.; Zhang, T. A family of Steffensen-type methods with seventh-order convergence. *Numer. Algor.* **2013**, *62*, 429–444.

27. Wolfram, S. *The Mathematica Book*, 5th ed.; Wolfram Media: Champaign, IL, USA, 2003.

28. Gelfand, I.M. Some problems in the theory of quasi-linear equations. *Trans. Amer. Math. Soc. Ser.* **1963**, *2*, 295–381.

29. Jacobsen, J.; Schmitt, K. The Liouville Bratu Gelfand problem for radial operators. *J. Differ. Equ.* **2002**, *184*, 283–298.

30. Jalilian, R. Non-polynomial spline method for solving Bratu’s problem. *Comput. Phys. Commun.* **2010**, *181*, 1868–1872.

31. Wan, Y.Q.; Guo, Q.; Pan, N. Thermo-electro-hydrodynamic model for electrospinning process. *Int. J. Nonlinear Sci. Numer. Simul.* **2004**, *5*, 5–8.

32. Kapania, R.K. A pseudo-spectral solution of 2-parameter Bratu’s equation. *Comput. Mech.* **1990**, *6*, 55–63.

33. Simpson, R.B. A method for the numerical determination of bifurcation states of nonlinear systems of equations. *SIAM J. Numer. Anal.* **1975**, *12*, 439–451.

34. Sauer, T. *Numerical Analysis*, 2nd ed.; Pearson: London, UK, 2012.