Inverse Spectral Problems for Collapsing Manifolds I: Uniqueness and Stability

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Abstract

We consider the geometric inverse problem of determining a closed Riemannian manifold from measurements of the heat kernel in an open subset of the manifold. In this paper we analyze the stability of this problem in the class of \( n \)-dimensional Riemannian manifolds with bounded diameter and sectional curvature. It is well-known that a sequence in this class of manifolds can collapse to a lower dimensional stratified space when the injectivity radius of the sequence of manifolds goes to zero. We prove the uniqueness of the inverse problem on the limiting spaces of the collapsing manifolds. As a result, we obtain stability results for the inverse problem in the class of manifolds with bounded diameter and sectional curvature.

1 Introduction

1.1 Inverse problems for collapsing manifolds

Let \((M, h)\) be a connected, closed, smooth Riemannian manifold of dimension \( n \) with Riemannian metric \( h \). Given \( p \in M \), we say that the triple \((M, p, h)\) is a pointed Riemannian manifold. We denote by \( dV_h \) the Riemannian volume element of \((M, h)\), and by \( d\mu_M \) the normalized measure,

\[
d\mu_M = \frac{1}{\text{Vol}(M)} dV_h,
\]

where \( \text{Vol}(M) \) is the Riemannian volume of \((M, h)\). We consider the heat kernel \( H(x, y, t) \) on \((M, h)\) associated to the Laplacian operator,

\[
\left( \frac{\partial}{\partial t} + \Delta_M \right) H(\cdot, y, t) = 0 \quad \text{on } M \times \mathbb{R}_+, \quad H(\cdot, y, 0) = \delta_y,
\]

where \( \delta_y \) is the normalized Dirac delta-distribution, i.e., \( \int_M \delta_y(x) \varphi(x) d\mu_M(x) = \varphi(y) \) for all \( \varphi \in C^\infty(M) \), and \( \Delta_M \) is the (nonnegative definite) Laplace-Beltrami operator on \((M, h)\), which has the following form in local coordinates,

\[
\Delta_M u = -|h|^{-\frac{1}{2}} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left( |h|^{\frac{1}{2}} h^{j,k} \frac{\partial}{\partial x_k} u \right), \quad |h| = \det (h_{jk}).
\]
In the following, we assume that we are given the values of the heat kernel at points \( \{ z_\alpha : \alpha = 1, 2, \ldots \} \), which form a dense set in a ball \( B = B_M(p, r) \) of \( (M, p, h) \) having center at \( p \) and radius \( r > 0 \). We define pointwise heat data, \( \text{PHD} \), that is the ordered sequence
\[
\text{PHD} = (H_{\alpha,\beta,\ell})_{\alpha,\beta,\ell=1}^{\infty}, \quad H_{\alpha,\beta,\ell} := H(z_\alpha, z_\beta, t_\ell),
\]
where \( \{ t_\ell : \ell = 1, 2, \ldots \} \) is a dense set in \( \mathbb{R}^+ \). Let us emphasize that the mutual relations of the measurement points \( z_\alpha \), e.g. the distances between these points, and the topology of the ball \( B \) are not \textit{a priori} known.

We consider the following generalization of Gel’fand’s inverse problem [31].

\textbf{Problem 1.1.} Suppose that the pointwise heat data (1.4) of two connected, closed, smooth, pointed Riemannian manifolds \( (M, p) \) and \( (M', p') \) coincide, i.e., for some \( r > 0 \),
\[
H(z_\alpha, z_\beta, t_\ell) = H'(z'_\alpha, z'_\beta, t_\ell), \quad \text{for all } z_\alpha \in B_M(p, r), \ z'_\alpha \in B_{M'}(p', r), \ t_\ell \in \mathbb{R}^+.
\]
Are the two manifolds isometric?

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Figure 1: Applications in manifold learning for almost collapsed manifolds. \textbf{Left:} Data points \( X = \{ x_j \in \mathbb{R}^3 : \ j = 1, 2, \ldots, 2048 \} \) sampled from a 2-dimensional torus embedded in \( \mathbb{R}^3 \). The points are sampled along a helical curve. The distances between the points \( x_j \) are defined using the flat metric of the torus \( M_{R,r} = S^1_R \times S^1_r \), with the larger radius \( R = 10 \) and the smaller radius \( r = \frac{1}{2} \). When \( r \) is small, the torus \( M_{R,r} \) is ‘almost collapsed’ to the circle \( S^1_R \). Note that the points could also be sampled randomly, but we have used points sampled randomly from a helical curve on \( M_{R,r} \) to make the visualization clearer. \textbf{Right:} The image of the 2-dimensional eigenfunction map \( \Phi^{(2,2)}(X) \) associated to the eigenfunctions for an approximation of the heat kernel of \( M_{R,r} \). The image \( \Phi^{(2,2)}(X) \subset \mathbb{R}^2 \) is close to a circle, that is, it is an approximation of the limiting space \( S^1_R \), see [1,5]. We study the local version of Gel’fand’s problem 1.1 where the heat kernel \( H(x_j, x_j', t_\ell) \) are given at points \( x_j \) that do not fill the whole manifold \( M = M_{R,r} \), but only fill a possibly small metric ball \( B \subset M \), for example, only the red part of the helical curve on the left picture. The data missing from the points \( x \in M \setminus B \) are compensated by measuring the heat kernel at several times \( t_\ell > 0 \), see Theorem 1.4. For details of the figure, see the part II of this paper [48, Section 7].

Gel’fand’s problem 1.1 is encountered in imaging sciences and inverse problems for partial differential equations, and in dimensionality reduction, especially in the study of
diffusion maps \[20, 21\] and manifold learning \[24, 66\]. In this paper we are particularly concerned with the situation where the spaces have collapsing structures. One such example is manifold learning, also called dimensionality reduction in data science \[66\], where the data sampled from a submanifold of a high-dimensional space need to be approximated by a submanifold of a low-dimensional space. In manifold learning one is given a point cloud, that is, a finite subset \(X = \{x_j\}_{j=1}^N \subset \mathbb{R}^n\) of \(n\) points which lie close to a \(d\)-dimensional submanifold \(M_0\) in an \(n\)-dimensional Euclidean space, where \(n\) is much larger than \(d\) \[24\]. Then, the goal in learning is to find a manifold \(M \subset \mathbb{R}^m\) that is diffeomorphic to \(M_0\) so that \(d < m < n\). In a more general problem, called the geometric Whitney problem \[22\], one is given a possibly discrete metric space \((Y,d_Y)\) and the task is to construct a smooth Riemannian manifold \(M\) that is close to \(Y\) in the Gromov-Hausdorff sense. An extensively studied method in manifold learning is the diffusion maps (or the spectral embedding) introduced by Coifman and Lafon in \[20, 21\]. In this method, one considers the data points \(X = \{x_j\}_{j=1}^N\), which are sampled from some manifold \(M_0\) and are given with distances \(d_X(x_j, x_j')\), and computes a kernel function \(h_t : X \times X \to \mathbb{R}\) that approximates the heat kernel of the manifold \(M_0\) at a time \(t > 0\). Then, one continues the algorithm by computing the first \(J\) eigenfunctions \(\phi_j(x)\) of the integral operator defined by the kernel \(h_t(x,y), x,y \in X\), and uses the eigenfunction map, defined as

\[
\Phi^{(K,J)} : X \to \mathbb{R}^{J},
\phi^{(K,J)}(x) = (\phi_K(x), \phi_{K+1}(x), \ldots, \phi_{K+J-1}(x)),
\]

(1.5)

to construct the set \(\Phi^{(K,J)}(X) \subset \mathbb{R}^{J}\) that approximates the Riemannian manifold \(M_0\), see Figure 1 for a toy example. In this paper we consider a similar problem of reconstructing an approximation of the Riemannian manifold \(M_0\) in the case when we have restricted data, that is, one has only local data \(h_{t_\ell}(x_k, x_k') = H(x_k, x_k', t_\ell)\), where the sample points \(\{x_k : k = 1, 2, \ldots, K\}\) are an \(\varepsilon\)-dense set of points in a possibly small metric ball \(B(x_0, r)\) of \(M_0\), not an \(\varepsilon\)-dense set of points in the whole manifold \(M_0\). We also define and analyze a local version of the eigenfunction map \(1.5\), see Definition 6.1 later. The collapsing of manifolds studied in this paper are closely related to multi-scale models in manifold learning where the intrinsic dimension of the data set is modelled by a function that depends on the scale, see \[69\]. For example, this occurs when the cryogenic electron-microscopy \[37\] is applied to image the structure of a large molecule which is connected to a small part of the molecule with a connection that allows a rotation (roughly speaking, the molecule has a moving ‘tail’). In this case, the problem of imaging the molecule is to find a manifold diffeomorphic to, e.g. \(SO(3) \times S^1(\epsilon)\) which almost collapses to \(SO(3)\) of lower dimension, where \(S^1(\epsilon)\) is a circle of small radius \(\epsilon\).

Problem 1.1 for collapsing manifolds is also encountered in mathematical physics, for example in a predecessor of the string theory, the Kaluza-Klein theory, in which Einstein equations are considered on \(M_\epsilon = \mathbb{R}^4 \times S^1(\epsilon)\), and when \(\epsilon \to 0\), the Einstein equations on \(M_\epsilon\) converge to equations on \(\mathbb{R}^4\) containing both the standard Einstein equations and Maxwell’s equations. In this case, the observations analogous to the data \(1.4\) give information of the fine structure of the almost collapsed metric. These applications for
collapsing manifolds are considered in the part II of this paper [48, Section 7].

When the positions of \( z_\alpha \in B \) are known, the positive answer to Problem 1.1 was essentially given in [7, 41, 45], using the boundary control method pioneered by Belishev [6] on domains of \( \mathbb{R}^n \) and Tataru’s unique continuation theorem [63], see also [1, 13, 50].

Generalizations and alternative methods to solve the problem have been studied in [3, 17, 42, 47, 49], and the related determination of smooth structure was recently studied in [22, 23]. On a given domain of the Euclidean space, the problem can be reduced to inverse coefficient problems for elliptic equations which were solved in [62]. We are concerned with the stability of the inverse problem.

**Problem 1.2.** Does the solution of the Inverse Problem 1.1 depend continuously on the given data?

Gel’fand’s inverse problem is ill-posed in the sense of Hadamard, as one can make large changes to the geometry without affecting the local measurements much. Moreover, in the study of inverse coefficient problems in Euclidean spaces, allowing non-smooth coefficients in operators can lead to counterexamples where even the change of topology is not observed in the measurements [35]. This phenomenon, in turn, has led to engineering applications in the emerging field of transformation optics and invisibility cloaking [34] in Calderón’s inverse problem, see [5, 44, 52]. This also makes clear the necessity of studying the question of uniqueness in the limiting non-smooth case, in order to understand the stability of the solution of Gel’fand’s inverse problem for smooth manifolds.

To stabilize the inverse problem, it is natural to impose a priori bounds in an invariant form on geometric parameters such as dimension, diameter, sectional curvature and injectivity radius. Within such a class of manifolds, an abstract continuity result for the stability of the inverse problem and a log-log type of stability estimate were proved in [3, 13, 16]. In this class of manifolds, imposing a lower bound on the injectivity radius is essential as the limiting space is always a Riemannian manifold of the same dimension. Stronger Hölder type of stability estimates can be obtained in [1, 3, 58, 59], see also [37, 60, 61, 70] for results on closely related tomography problems, if additional geometric assumptions are assumed, e.g., if the metric is close to being simple.

However, the situation gets much more complicated in general, as it is well-known that a sequence of smooth \( n \)-dimensional manifolds without a lower bound on the injectivity radius can collapse to a space of lower dimension and the limiting space is not necessarily a manifold, see e.g. [36]. It might be possible that a similar invisibility phenomenon could occur in geometric inverse problems: the information on microstructures can vanish in the collapse of dimension and one can determine only some effective properties of the metric. In this paper we study the question of what global information on collapsing manifolds can be determined from the local measurements, and prove the stability of the Inverse Problem 1.1 in the class of manifolds with bounded diameter and sectional curvature.

In this paper, we work with the class \( \mathcal{RM}_p = \mathcal{RM}_p(n, \Lambda, D) \) of connected, closed, smooth, pointed Riemannian manifolds \( (M, p, \mu_M), \ p \in M \), satisfying

\[
\dim(M) = n, \quad |R(M)| \leq \Lambda^2, \quad \text{diam}(M) \leq D, \tag{1.6}
\]
where $R(M), \text{diam}(M)$ are the sectional curvature and diameter of $M$, and $\mu_M$ is the normalized Riemannian measure $\mu_M$. We also use the notation $\mathcal{M}(n, \Lambda, D)$ for the same class of manifolds when there is no need to specify a point $p$.

When $(X, d)$ is a metric space, we denote by $d_X(x, y)$ the distance between $x$ and $y$ on $X$. When $A \subset B \subset X$, we say that $A$ is a $\delta$-net in $B$ if for any $y \in B$, there is $x \in A$ such that $d_X(x, y) < \delta$. Denote by $A^\varepsilon$ the $\varepsilon$-neighborhood of $A$, that is, $A^\varepsilon := \{x \in X : d_X(x, A) < \varepsilon\}$. We say that $\psi : X \to X'$ is an $\varepsilon$-Gromov-Hausdorff approximation (or $\varepsilon$-approximation in short) if

$$|d_{X'}(\psi(x), \psi(y)) - d_X(x, y)| < \varepsilon, \quad \text{and} \quad \psi(X) \text{ is an $\varepsilon$-net in } X'. \quad (1.7)$$

A metric-measure space $(X, d, \mu)$ is a metric space $(X, d)$ endowed with a Borel measure $\mu$. Below we often denote $(X, d, \mu)$ and $(X', d', \mu')$ just by $(X, \mu)$ and $(X', \mu')$, respectively. We use the following measured Gromov-Hausdorff “distance” (which is equivalent to the Prokhorov metric on the space of probability measures on a fixed space, see [12]).

**Definition 1.3.** Let $(X, \mu)$ and $(X', \mu')$ be metric-measure spaces such that $X$ and $X'$ are compact. The measured Gromov-Hausdorff distance $d_{mGH}((X, \mu), (X', \mu'))$ is defined as the infimum of those $\varepsilon > 0$ such that there are measurable $\varepsilon$-approximations $\psi : X \to X'$ and $\psi' : X' \to X$ satisfying

$$\mu(\psi^{-1}(A')) < \mu'((A')^\varepsilon) + \varepsilon, \quad \mu'((\psi')^{-1}(A)) < \mu(A^\varepsilon) + \varepsilon, \quad (1.8)$$

for all Borel sets $A \subset X$ and $A' \subset X'$, where $A^\varepsilon$ is the $\varepsilon$-neighborhood of $A$.

For pointed compact metric-measure spaces $(X, p, \mu)$ and $(X', p', \mu')$, the pointed measured Gromov-Hausdorff distance $d_{pmGH}$ is defined as the infimum of those $\varepsilon > 0$ such that there are measurable $\varepsilon$-approximations $\psi : X \to X'$ and $\psi' : X' \to X$ satisfying

$$d_{X'}(\psi(p), p') < \varepsilon, \quad d_X(\psi'(p'), p) < \varepsilon. \quad (1.9)$$

Let $\overline{\mathcal{M}}_p(n, \Lambda, D)$ denote the completion of the set $\mathcal{M}_p(n, \Lambda, D)$, that is, the space of equivalence classes of the Cauchy sequences, with respect to the pointed measured Gromov-Hausdorff distance $d_{pmGH}$. Usually we say that $\overline{\mathcal{M}}_p(n, \Lambda, D)$ is the closure of $\mathcal{M}_p(n, \Lambda, D)$.

It is well-known that a sequence of $n$-dimensional manifolds in the class $\mathcal{M}_p(n, \Lambda, D)$ can collapse to a lower dimensional space when the injectivity radius of the sequence of manifolds goes to zero. The structure of the metric-measure spaces $(X, \mu_X) \in \overline{\mathcal{M}}_p(n, \Lambda, D)$ was studied in [25–29]. The space $X$ has the stratification

$$X = S_0(X) \supset S_1(X) \supset \cdots \supset S_d(X), \quad (1.10)$$

with the following property: if $S_j(X) \setminus S_{j+1}(X)$ is non-empty, then it is a $(d - j)$-dimensional Riemannian manifold of class $C^{1, \alpha}$ for any $0 < \alpha < 1$, where $d = \text{dim}(X) \leq n$. The regular part of $X$ is the set $X^{\text{reg}} := S_0(X) \setminus S_1(X)$ which is an open $d$-dimensional manifold of class $C^{1, \alpha}$, and the singular set is its complement, $X^{\text{sing}} := X \setminus X^{\text{reg}}$. 

In particular, the singular set $X^{\text{sing}}$ has dimension at most $d-1$. In this paper we improve on the regularity of the limiting spaces. We show that on $X^{\text{reg}}$, the metric is locally determined by $C^2(X^{\text{reg}})$-smooth Riemannian tensor $h_X$ and the measure $\mu_X$ is absolutely continuous with respect to the Riemannian measure $dV_{h_X}$ with Radon-Nikodym derivative $\rho_X \in C^2_*(X^{\text{reg}})$. Here, $C^2_*(X^{\text{reg}})$ is the Zygmund space having the relation $C^{1,1}(X^{\text{reg}}) \subset C^2_*(X^{\text{reg}}) \subset C^{1,\alpha}(X^{\text{reg}})$ for any $0 < \alpha < 1$. Moreover, $X^{\text{reg}}$ is convex and Laplacian, denoted by $\Delta_X$. More details on the structure of collapsing are reviewed in Section 2.

As shown in [26], the Dirichlet’s quadratic form $A[u,v] = \langle du, dv \rangle_{L^2(X^{\text{reg}}, d\mu_X)}$, $u,v \in C^{0,1}(X)$, defines a self-adjoint operator $\Delta_X$ on $(X, \mu_X)$, which we call the weighted Laplacian, denoted by $\Delta_X$. In local coordinates on $X^{\text{reg}}$, it has the form

$$\Delta_X u = -\frac{1}{\rho_X |h_X|^\frac{1}{2}} \sum_{j,k=1}^d \frac{\partial}{\partial x^j} \left( \rho_X |h_X|^\frac{1}{2} h_X^{jk} \frac{\partial}{\partial x^k} u \right), \quad |h_X| = \det((h_X)_{jk}). \quad \text{(1.11)}$$

The associated semigroup $e^{-t\Delta_X}$ has the Schwartz kernel $H_X(x,y,t)$ that we call the heat kernel associated to $(X, \mu_X)$. Denote by $\lambda_j$ the $j$-th eigenvalue of the weighted Laplacian $\Delta_X$ and by $\phi_j$ the corresponding orthonormalized eigenfunction in $L^2(X, \mu_X)$. Fukaya proved in [26] that the $j$-th eigenvalue, for any $j$, is a continuous function on $\mathcal{M}_p(n, \Lambda, D)$ with respect to the measured Gromov-Hausdorff topology.

### 1.2 Main results

Our first main result is the following uniqueness theorem for the inverse problem for collapsing manifolds.

**Theorem 1.4.** Let $r, \Lambda, D > 0$, $n \in \mathbb{Z}_+$. Denote by $\mathcal{M}_p(n, \Lambda, D)$ the closure of the class of connected closed smooth pointed Riemannian manifolds defined by the conditions (1.6) in the pointed measured Gromov-Hausdorff topology. Let $(X, p, \mu), (X', p', \mu') \in \mathcal{M}_p(n, \Lambda, D)$. Let $\{z_\alpha\}_{\alpha=0}^\infty \subset X$, $\{z'_\alpha\}_{\alpha=0}^\infty \subset X'$ be dense sequences in the ball $B_X(p, r)$, $B_{X'}(p', r)$ with $z_0 = p$, $z'_0 = p'$, and $\{t_\ell\}_{\ell=1}^{\infty}$ be a dense set in $\mathbb{R}_+$. Suppose that

$$H(z_\alpha, z_\beta, t_\ell) = H'(z'_\alpha, z'_\beta, t_\ell), \quad \text{for all } \alpha, \beta \in \mathbb{N}, \ell \in \mathbb{Z}_+, \quad \text{(1.12)}$$

where $H, H'$ are the heat kernels on $X, X'$. Then there exists an isometry $F : X \to X'$ satisfying $F(p) = p'$ and $\mu = F^*\mu'$.

In particular, the spaces $X$ and $X'$ necessarily have the same dimension. Note that the assumption on the positions of $p, p'$ relative to the dense sequences $\{z_\alpha\}, \{z'_\alpha\}$ is necessary: otherwise, it is not possible to determine the location of $p, p'$ when $r > \text{diam}(X)$. Theorem 1.4 implies our second main result: the stability theorem for the Inverse Problem 1.1.
Let \( r, \Lambda, D > 0, n \in \mathbb{Z}_+ \). Then, there exists an increasing, continuous function \( \omega(s) = \omega_{(r, n, \Lambda, D)}(s) \) that depends on \( r, n, \Lambda, D \), which maps \( \omega : [0, 1) \to [0, \infty) \) with \( \omega(0) = 0 \), such that the following holds.

Let \((X, p, \mu), (X', p', \mu') \in \mathbb{MR}_p(n, \Lambda, D)\). Let \( \{z_\alpha\}_{\alpha=0}^N \subset X \), \( \{z'_\alpha\}_{\alpha=0}^N \subset X' \) be \( \delta \)-nets in the ball \( B_X(p, r), B_{X'}(p', r) \) with \( z_0 = p, z'_0 = p' \), and \( \{t_\ell\}_{\ell=1}^L \) be a \( \delta \)-net in \((\delta, \delta^{-1}) \subset \mathbb{R}_+\), where \( N, L \in \mathbb{Z}_+ \). Suppose that

\[
|H(z_\alpha, z_\beta, t_\ell) - H'(z'_\alpha, z'_\beta, t_\ell)| < \delta, \quad \text{for } 0 \leq \alpha, \beta \leq N, 1 \leq \ell \leq L, \tag{1.13}
\]

where \( H, H' \) are the heat kernels on \( X, X' \). Then

\[
d_{pmGH}((X, p, \mu), (X', p', \mu')) < \omega(\delta). \tag{1.14}
\]

In particular, in the case of the spaces \( X, X' \) being \( n \)-dimensional manifolds, Theorem 1.5 removes the injectivity radius assumption for the stability result in \([3]\).

If \((X, p, \mu) \in \mathbb{MR}_p(n, \Lambda, D)\) is \textit{a priori} an orbifold, i.e., \( \dim(X) = n - 1 \), it is possible to obtain an explicit form of the modulus of continuity \( \omega \), given the interior spectral data \( \{\lambda_j, \phi_j|_{B_X(p, r)}\} \) for the weighted Laplacian \( \Delta_X \) on a ball \( B_X(p, r) \subset X^{reg} \), provided that the metric tensor on the regular part \( X^{reg} \) is of class \( C^4 \). To this end, let us denote by \( \mathbb{MR}_p(n, \Lambda, D, \Lambda_3) \) the class of connected closed smooth pointed Riemannian manifolds \((M, p, \mu_M) \in \mathbb{MR}_p(n, \Lambda, D)\) satisfying additionally \( \|\nabla R(M)\| \leq \Lambda_3 \) for \( i = 1, 2, 3 \), and denote its closure by \( \overline{\mathbb{MR}}_p(n, \Lambda, D, \Lambda_3) \).

**Theorem 1.6.** Let \((X, p, \mu) \in \overline{\mathbb{MR}}_p(n, \Lambda, D, \Lambda_3)\) with \( p \in X^{reg} \). Suppose \( \dim(X) = n - 1 \) and \( \text{Vol}_{n-1}(X) \geq v_0 \). Let \( \sigma \in (0, 1) \), and \( r > 0 \) such that the ball \( B_X(p, r) \subset X^{reg} \). Then there exists \( \tilde{\delta} = \tilde{\delta}(X, r, \sigma) > 0 \), such that the finite interior spectral data \( \{\lambda_j, \phi_j|_{B_X(p, r)}\}_{j=1}^{\delta^{-1}} \) for \( \delta < \tilde{\delta} \) determine a finite metric space \( \tilde{X} \) such that

\[
d_{GH}(X \setminus S_{\sigma, \tilde{\delta}}, \tilde{X}) < C_1(X, \sigma)\left( \log \log |\log \delta| \right)^{-C_2},
\]

where \( S_{\sigma, \delta} \) is a subset of the \( C_0\sigma^{1/4} \)-neighborhood of the singular set of \( X \), and \( X \setminus S_{\sigma, \delta} \) is equipped with the restriction of the metric of \( X \). The constant \( C_1(X, \sigma) \) depends on \( X, \sigma, r, n, \Lambda, \Lambda_3, D, v_0 \), and \( C_1(X, \sigma) \to \infty \) as \( \sigma \to 0 \). The constant \( C_2 \) depends on \( n, \) and \( C_3 \) depends on \( n, \Lambda, D \).

Theorem 1.6 will be proved in the part II of this paper \([48]\).

### 1.3 Plan of the exposition

This paper is organized as follows. Section 2 is of an expository nature, where we review, in a slightly modified form appropriate for our purposes, Fukaya’s results on the measured Gromov-Hausdorff convergence of Riemannian manifolds and provide some further results in this direction. In Section 3, we show that the density function \( \rho_X \) for \((X, \mu_X) \in \mathbb{MR}_p(n, \Lambda, D)\) is of \( C^4 \) on \( X^{reg} \). This improves on the earlier results in \([26, 39, 40]\). Note that our proof differs from that in the above papers as it is based
on the analysis of smoothness of the transformation groups into the scale of Zygmund-type functions. In turn, this requires an extension of the classical Montgomery-Zippin results, which we will discuss in the part II of this paper \[48\]. Section 4 is of an auxiliary nature. Here we prove various results concerning the behavior of the spectrum, eigenfunctions and heat kernels on $\mathfrak{M}_p(n, \Lambda, D)$, obtaining uniform estimates for the eigenfunctions and heat kernels, and prove the pointwise convergence of the heat kernels with respect to the pointed measured Gromov-Hausdorff convergence. Some auxiliary results dealing with the relations between the Laplacian on manifolds and their orthonormal frame bundles, as well as the corresponding structures on $\mathfrak{M}_p(n, \Lambda, D)$, are discussed in Appendix A. In Section 6 by extending the geometric boundary control method to the collapsing manifolds, we show that the heat kernel data for any $(X, \mu_X) \in \mathfrak{M}_p(n, \Lambda, D)$ uniquely determine its metric-measure structure (Theorem 1.4). At last, Section 7 is devoted to the proof of the stability result, Theorem 1.5.

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2 Basic results in the theory of collapsing

Given $n \in \mathbb{N}_+$ and $\Lambda, D \in \mathbb{R}_+$, the notation $\mathfrak{M}(n, \Lambda, D)$ stands for the class of $n$-dimensional connected, closed, smooth Riemannian manifolds $(M, h)$ satisfying

$$\dim(M) = n, \quad |R(M)| \leq \Lambda^2, \quad \text{diam}(M) \leq D,$$

where $R(M)$ and $\text{diam}(M)$ are the sectional curvature and diameter of $M$. The structure of collapsing in the moduli space $\mathfrak{M}(n, \Lambda, D)$, with respect to the Gromov-Hausdorff distance, was extensively studied by Fukaya \([26, 27, 29]\). In this section we review some of the results.

2.1 Properties of the limit spaces

2.1.1 Basic example. It is known (e.g. \[36\]) that $\mathfrak{M}(n, \Lambda, D)$ is precompact in the Gromov-Hausdorff topology. Therefore we assume that a sequence $(M_i, h_i) \in \mathfrak{M}(n, \Lambda, D)$, $i = 1, 2, \ldots$, converges to some compact metric space $X$ with respect to the Gromov-Hausdorff distance. The space $X \in \mathfrak{M}(n, \Lambda, D)$ can be locally described as orbit spaces. It has the stratification

$$X = S_0(X) \supset S_1(X) \supset \cdots \supset S_d(X),$$

(2.1)
such that, if $S_j(X) \setminus S_{j+1}(X)$ is non-empty, it is a $(d - j)$-dimensional Riemannian manifold of class $C^{1,\alpha}$ for any $0 < \alpha < 1$, where $d := \dim(X) \leq n$. Actually $S_j(X) \setminus S_{j+1}(X)$ is defined as the set of all point $p \in X$ such that the tangent cone $K_p(X)$ of $X$ at $p$ (see e.g. [14, 15]) is isometric to a product of the form $\mathbb{R}^{d-j} \times Y_j$, where $Y$ has no such a nontrivial $\mathbb{R}$-factor.

We denote by

$$X^{\text{sing}} := S_1(X), \quad X^{\text{reg}} := X \setminus S(X)$$

the singular and the regular set of $X$, respectively. By definition, the regular set $X^{\text{reg}}$ is an open $d$-dimensional Riemannian manifold with $d = \dim(X)$, and the singular set $X^{\text{sing}}$ has dimension at most $d - 1$. When $\dim(X) \leq n - 1$, we say that $M_i$ collapses to $X$, and that the collapsing is $m$-dimensional for $m = n - \dim(X)$.

Let us briefly explain the structure of 1-dimensional collapse in the following example.

**Example 2.1.** For $\epsilon > 0$, let $M_\epsilon$ be a 3-dimensional Riemannian manifold of the following form. We start with a product manifold $S^2 \times [0, \epsilon]$, where $S^2 \subset \mathbb{R}^3$ is the 2-dimensional unit sphere with the canonical metric. Consider the action of the finite group $\mathbb{Z}_m, m \geq 2$, on $S^2$ by a rotation of angle $2\pi/m$ along the $z$-axis. Then we define $M_\epsilon$ by identifying points $(x, 0) \in S^2 \times \{0\}$ with $(e^{2\pi i / m} \cdot x, \epsilon) \in S^2 \times \{\epsilon\}$ in the manifold $S^2 \times [0, \epsilon]$, where $e^{2\pi i / m} \cdot x$ stands for the rotation by the angle $2\pi/m$. Observe that this gives rise to closed vertical geodesics of length $m\epsilon$ except for the points corresponding to the north and south poles. When $\epsilon \to 0$, $M_\epsilon$ collapses to a 2-dimensional space $X = S^2 / \mathbb{Z}_m$, a Riemannian orbifold as illustrated on the right. This orbifold $X$ has conic singular points only at the north and south poles. For analytically-oriented readers, we cite to [48, Section 7.2] for the details and additional pictures on this example.

An example of the limit spaces above is so-called good orbifolds (e.g. [67]), that is,

$$X = N/G,$$

where $N$ is a Riemannian manifold and $G$ is a discrete group acting properly and isometrically on $N$. For another example, one could consider $N = S^2 \times S^1$, and $\mathbb{Z}_m$ is the finite group acting on $S^2 \subset \mathbb{R}^3$ by a $2\pi/m$-rotation along the $z$-axis as in Example 2.1. Then the quotient space $N/\mathbb{Z}_m = (S^2/\mathbb{Z}_m) \times S^1$ is a good orbifold, where the singular set of the orbifold is $\{\text{north pole, south pole}\} \times S^1$ and the complement is the regular set.

**2.1.2 Local pseudogroup construction.** In this subsection, we consider the geometric properties of the collapsing manifolds and the properties of the limit space $X$. An analytically-oriented reader may consider the orbifold space described in Example 2.1 as a basic example of the limit of collapsing manifolds, and skip in the first reading the general local construction in this subsection.
Let \((M_i, h_i) \in \mathfrak{M}(n, \Lambda, D)\) be a sequence of closed Riemannian manifolds converging to a compact metric space \(X\) in the Gromov-Hausdorff topology. Fix any point \(p \in X\) and put \(p_i := \psi_i(p)\), where \(\psi_i : X \to M_i\) is an \(\varepsilon_i\)-approximation as defined in [1.7] with \(\lim_{i \to \infty} \varepsilon_i = 0\). Let \(B \subset \mathbb{R}^n\) be the open ball around the origin \(O\) in \(\mathbb{R}^n\) of radius \(\pi/\Lambda\), and let \(\exp_i : B \to M_i\) be the composition of the exponential map \(\exp_{p_i} : T_{p_i}(M_i) \to M_i\) and a Li isometric embedding \(B \to B(O, \pi/\Lambda) \subset T_{p_i}(M_i)\). Since \(|R(M_i)| \leq \Lambda^2\), the exponential map on \(B(O, \pi/\Lambda)\) has maximal rank, and we have the pull-back metric \(\tilde{h}_i := \exp_i^*(h_i)\) on \(B \subset \mathbb{R}^n\). Moreover, the injectivity radius of \((B, \tilde{h}_i)\) is uniformly bounded from below, e.g. [36] Lemma 8.19]. Therefore, we may assume that \((B, \tilde{h}_i)\) converges to a \(C^2\)-metric \((B, \tilde{h}_0)\) in the \(C^{1,\alpha}\)-topology, for any \(0 < \alpha < 1\) (see [29, 33, 54]).

Let \(G_i\) denote the set of all isometric embeddings \(\gamma : (B', \tilde{h}_i) \to (B, \tilde{h}_i)\) such that \(\exp_i \circ \gamma = \exp_i\) on \(B'\), where \(B' := B(O, \frac{\pi}{2\Lambda}) \subset \mathbb{R}^n\). Then \(G_i\) forms a local pseudogroup, see e.g. [29, Section 7]. Passing to a subsequence, we may assume that \(G_i\) converges to a local pseudogroup \(G\) consisting of isometric embeddings \(\gamma : (B', \tilde{h}_0) \to (B, \tilde{h}_0)\), and that \((B, \tilde{h}_i, G_i)\) converges to \((B, \tilde{h}_0, G)\) in the equivariant Gromov-Hausdorff topology. This means that there exist \(\varepsilon_i\)-approximations \(\phi_i : (B, \tilde{h}_i) \to (B, \tilde{h}_0)\), \(\psi_i : (B, \tilde{h}_0) \to (B, \tilde{h}_i)\) with \(\lim_{i \to \infty} \varepsilon_i = 0\), and maps \(\rho_i : G_i \to G\), \(\sigma_i : G \to G_i\) such that for every \(x, y \in B\) and \(\gamma_i \in G_i\), \(\gamma \in G\), the following hold
\[
d_0((\phi_i(\gamma_i(x)), \rho_i(\gamma_i))(\phi_i(x))) < \varepsilon_i, \quad d_i((\psi_i(\gamma(x)), \sigma_i(\gamma))(\psi_i(x))) < \varepsilon_i, \tag{2.3}
\]
whenever they make sense, where \(d_0, d_i\) denote the distance function on \((B, \tilde{h}_0)\), \((B', \tilde{h}_i)\), respectively. Roughly speaking, this means that the pseudogroup action of \(G_i\) on \((B, \tilde{h}_i)\) is close to that of \(G\) on \((B, \tilde{h}_0)\). In particular, the quotient space \((B', \tilde{h}_i)/G_i = B(p_i, \frac{\pi}{2\Lambda}) \subset M_i\) converges to \((B', \tilde{h}_0)/G,\) which implies that
\[
(B', \tilde{h}_0)/G = B(p, \frac{\pi}{2\Lambda}) \subset X. \tag{2.4}
\]
(See [30] for further details on basic properties of the equivariant Gromov-Hausdorff convergence.)

**2.1.3 Geometric structure of the limit space.** The following lemma is known, where (1), (2) are due to Fukaya [27] (and [3]), and (3) is the consequence of [55].

**Lemma 2.2.** Let \(X\) be any element of \(\mathfrak{M}(n, \Lambda, D)\) which is not a point. Then

(1) \(X^{reg}\) is a \(C^2\)-Riemannian manifold, that is, there exist coordinate charts on \(X^{reg}\) for which the transition maps are of class \(C^3\), and the metric tensor in these charts is of class \(C^2\).

(2) For any compact subset \(K \subset X^{reg}\), there exists a positive number \(i_K > 0\) such that \(\text{inj}(p) \geq i_K\) for all \(p \in K\), where \(\text{inj}(p)\) denotes the injectivity radius at \(p\).

(3) \(X^{reg}\) is convex in \(X\), that is, every minimizing geodesic joining two points in \(X^{reg}\) is contained in \(X^{reg}\).
For reader’s convenience, we shall sketch the outline of the proof of Lemma 2.2(1) and (2) below. Let \( X \) be the limit of \((M_i, h^\varepsilon_i) \in \mathfrak{M}(n, \Lambda, D)\). By [27], there exist Riemannian metrics \( h^\varepsilon_i \) on \( M_i \) such that, as \( \varepsilon \to 0 \), \( h^\varepsilon_i \to h_i \) in the \( C^{1,\alpha} \)-topology, for any \( 0 < \alpha < 1 \), and

(i) \((M_i, h^\varepsilon_i)\) converges to \( X^\varepsilon \), as \( i \to \infty \), with respect to the Gromov-Hausdorff distance;

(ii) the regular part \((X^\varepsilon)^{reg}\) of \( X^\varepsilon \) is a Riemannian manifold of class \( C^\infty \);

(iii) \( X^\varepsilon \) is \( \varepsilon \)-isometric to \( X \). Namely, there exists a bi-Lipschitz map \( f^\varepsilon : X \to X^\varepsilon \) satisfying

\[
\left| d(f^\varepsilon(x), f^\varepsilon(y)) - d(x, y) \right| < \varepsilon.
\]

More precisely, the norm of the \( k \)-th covariant derivatives of the curvature tensor \( R_{h^\varepsilon} \) of \((M_i, h^\varepsilon_i)\) is uniformly bounded \( \|\nabla^k R_{h^\varepsilon}\| \leq C(n, \Lambda, k, \varepsilon) \) for any fixed \( k \) and \( \varepsilon \) (see [9]). The space \( X^\varepsilon \) here is called a smooth element in [27].

Since \( X \) and \( X^\varepsilon \) have orbit-type singularities, the above (iii) implies that \( f^\varepsilon(S_j(X)) = S_j(X^\varepsilon) \) for small \( \varepsilon \), where \( S_j(X) \) are the stratification of \( X \) in (2.1). In particular, \( f^\varepsilon(X^{reg}) = (X^\varepsilon)^{reg} \).

For a small \( \delta > 0 \), let \( V \) be the \( \delta \)-neighborhood of \( K \) in \( X \) such that \( \overline{V} \subset X^{reg} \). Set \( V^\varepsilon := f^\varepsilon(V) \). Then,

\[
-\Lambda^2 \leq R(V^\varepsilon)|_{V^\varepsilon} \leq C(n, \Lambda), \tag{2.5}
\]

see [27] Theorem 0.9. Note that, by (iii), \( \text{diam}(V^\varepsilon) \leq C, \text{Vol}(V^\varepsilon) \geq C \) for some uniform constant \( C = C(K) > 0 \), which is independent of \( \varepsilon \). Thus, (2.5) together with Cheeger’s theorem [19] implies that there is a positive number \( i_K < \delta \) independent of \( \varepsilon \) such that

\[
\text{inj}_{X^\varepsilon}(p) \geq i_K, \tag{2.6}
\]

for all \( p \in K^\varepsilon := f^\varepsilon(K) \). Now a standard argument using the Cheeger-Gromov compactness applied to the convergence \( V^\varepsilon \to V \) together with Alexandrov geometry proves (2). It follows from (2.5), (2.6) and [3] that the metric of \( V \) and hence of \( X^{reg} \) is of class \( C_2^\varepsilon \). Then the \( C_2^\varepsilon \)-smoothness of the transition maps between harmonic coordinate charts is due to [2] [3].

Let us recall here the definition of the Zygmund spaces, e.g. [68, Chapter 2.7]. The Zygmund space \( C_s^k(B) \) on a ball \( B \subset \mathbb{R}^n \) coincides with the Hölder space \( C^s(B) \) for \( s \in \mathbb{R}_+ \setminus \mathbb{Z}_+ \), and forms a complex interpolation scale (for an overview on the interpolation theory of function spaces, see [11]). For \( s \in \mathbb{Z}_+ \), \( C_s^k(B) \) is defined as the space of functions in \( C^{s-1}(B) \) satisfying that the following norm is finite,

\[
\|f\|_{C_s^k(B)} := \|f\|_{C^{s-1}(B)} + \sum_{|\alpha| = s-1} \sup_{x \neq y \in B} \frac{|D^\alpha f(x) - 2D^\alpha f(\frac{x+y}{2}) + D^\alpha f(y)|}{|x-y|}, \quad s \in \mathbb{Z}_+.
\]

(2.7)

We point out that when \( k \) is a positive integer and \( 0 < \alpha < 1 \), the Hölder spaces \( C_2^{k,\alpha}(B) \) and the Zygmund spaces \( C_s^k(B) \) have the relation \( C^{k-1.1}(B) \subset C_s^k(B) \subset C^{k-1.\alpha}(B) \).

The geometric structure of the limit space \( X \) can be described in the following ways.
Theorem 2.3 ([27]). For every $p \in X$, let $G$ be local pseudogroup defined as above, and set $\ell := n - \dim(G \cdot O)$, where $G \cdot O$ denotes the orbit $G(O)$.

(1) There exist a neighborhood $U$ of $p$, a compact Lie group $G_p$ and a faithful representation of $G_p$ into the orthogonal group $O(\ell)$, a $G_p$-invariant smooth metric on a neighborhood $V$ of $O$ in $\mathbb{R}^\ell$ such that $U$ is bi-Lipschitz homeomorphic to $V/G_p$.

(2) There exists a $C^2_*$-Riemannian manifold $Y$ with $\dim(Y) = \dim(X) + \dim(O(n))$ on which $O(n)$ acts as isometries in such a way that

(a) $X$ is isometric to $Y/O(n)$. Let $\pi : Y \to X$ be the projection;

(b) For every $p \in X$ and $\bar{p} \in \pi^{-1}(p)$, the isotropy group

$$H_p := \{ g \in O(n) \mid g(\bar{p}) = \bar{p} \}$$

is isomorphic to $G_p$, where $G_p$ is as in (1).

We will make an extensive use of Theorem 2.3(2) in this paper, and the fact of $Y$ being a Riemannian manifold is crucial for our purposes. Here we briefly explain the construction of $Y$. Let $X$ be the limit of $M_i \in \mathcal{M}(n, \Lambda, D)$. Let $F M_i$ denote the orthonormal frame bundle of $M_i$ endowed with the natural Riemannian metric, which has uniformly bounded sectional curvature and diameter. Note that $O(n)$ isometrically acts on $F M_i$, and $M_i = F M_i/O(n)$. Passing to a subsequence, we may assume that $(F M_i, O(n))$ converges to $(Y, O(n))$ in the equivariant GH-topology. Then it follows that $X = Y/O(n)$.

To show that $Y$ is a Riemannian manifold, let $B' \subset B \subset \mathbb{R}^n$, $(B, \tilde{h}, G_i)$ and $(B, \tilde{h}_0, G)$ be as described at the beginning of this section, so that $(B', \tilde{h})/G_i = B(p_i, \frac{\pi}{2\Lambda})$ and $(B', \tilde{h}_0)/G = B(p, \frac{\pi}{2\Lambda})$. The pseudogroup action of $G_i$ on $(B', \tilde{h}_i)$ induces an isometric pseudogroup action, denoted by $\tilde{G}_i$, on the frame bundle $F(B', \tilde{h}_i)$ of $(B', \tilde{h}_i)$ defined by differential. Therefore, $F(B', \tilde{h})/\tilde{G}_i = F B(p_i, \frac{\pi}{2\Lambda})$. Passing to a subsequence, we may assume that $(F(B', \tilde{h}), \tilde{G}_i)$ converges to $(F(B', \tilde{h}_0), \tilde{G})$ in the equivariant GH-topology, where $\tilde{G}$ denotes the isometric pseudogroup action on $F(B', \tilde{h}_0)$ induced from that of $G$ on $(B', \tilde{h}_0)$. Since the action of $G$ on $(B', \tilde{h}_0)$ is isometric, the action of $\tilde{G}$ on $F(B', \tilde{h}_0)$ is free. Therefore, $F(B', \tilde{h}_0)/\tilde{G}$ is a Riemannian manifold, and so is $Y$.

Theorem 2.4 ([27], Theorem 10.1). Suppose a sequence $X_i$ in $\mathcal{M}(n, \Lambda, D)$ converges to $X$ with respect to the Gromov-Hausdorff distance. Then there are $O(n)$-Riemannian manifolds $Y_i$ and $Y$ of class $C^2_*$ and $O(n)$-maps $\bar{f}_i : Y_i \to Y$ and maps $f_i : X_i \to X$ such that

(1) $X_i = Y_i/O(n)$, $X = Y/O(n)$. Let $\pi_i : Y_i \to X_i$, $\pi : Y \to X$ be the projections;

(2) $\bar{f}_i$ are $\varepsilon_i$-Riemannian submersions as well as $\varepsilon_i$-approximations, where $\lim_{i \to \infty} \varepsilon_i = 0$.

Namely, $\bar{f}_i$ satisfies

$$e^{-\varepsilon_i} < \frac{|d\bar{f}_i(\xi)|}{|\xi|} < e^{\varepsilon_i},$$

for all tangent vectors $\xi$ orthogonal to fibers of $\bar{f}_i$;
\[(3) \ f_i \circ \pi_i = \pi \circ \tilde{f}_i; \]

(4) for every \( y \in Y \), the isotropy subgroup \( \{ g \in O(n) \mid g(y) = y \} \) is isomorphic to \( G_{\pi(y)} \), where \( G_{\pi(y)} \) is as in Theorem 2.3.

We shall call the maps \( f_i : X_i \to X \) regular \( \varepsilon_i \)-approximations for simplicity.

Note that \( p \) is an orbifold point if and only if \( G_p \simeq H_\beta \) is finite. This actually occurs for every \( p \in X \) in the case of collapsing being one dimensional.

**Corollary 2.5** ([29], Proposition 11.5). If \( \dim X = n - 1 \), then \( X \) is an orbifold.

### 2.2 Measured Gromov-Hausdorff distance

Let \( \mu_i \) and \( \mu \) be probability Borel measures on compact metric spaces \( X_i \) and \( X \). Fukaya defined the notion of measured Gromov-Hausdorff convergence \( (X_i, \mu_i) \to (X, \mu) \), or weak convergence in short. By definition, this is the case when there are measurable \( \varepsilon_i \)-approximations \( \psi_i : X_i \to X \) with \( \lim_{i \to \infty} \varepsilon_i = 0 \) such that the pushforward measure \( (\psi_i)_*\mu_i \) weakly converges to \( \mu \) in the usual sense: namely,

\[
\int_{X_i} f \circ \psi_i \, d\mu_i \to \int_X f \, d\mu, \quad \text{as } i \to \infty,
\]

(2.9)

for any \( f \in C(X) \), where \( C(X) \) denotes the space of continuous functions on \( X \). In this subsection, we provide basic properties of the measured Gromov-Hausdorff distance \( d_{mGH}((X, \mu), (X', \mu')) \) defined in Definition 1.3 between compact metric measure spaces.

**Lemma 2.6.** The measured Gromov-Hausdorff “distance” satisfies an almost triangle inequality:

\[
d_{mGH}((X_1, \mu_1), (X_2, \mu_2)) \\
\leq 2 \left( d_{mGH}((X_1, \mu_1), (X_3, \mu_3)) + d_{mGH}((X_3, \mu_3), (X_2, \mu_2)) \right).
\]

**Proof.** Let \( d_{ij} := d_{mGH}((X_i, \mu_i), (X_j, \mu_j)) \), and \( d := d_{13} + d_{32} \). By definition, for any \( \varepsilon > 0 \) and for any \( i, j \in \{1, 3\} \) or \( i, j \in \{3, 2\} \), there are \((d_{ij} + \varepsilon/2)\)-approximation \( \psi_{ij} : X_i \to X_j \), satisfying

\[
\mu_i(\psi_{ij}^{-1}(A_j)) < \mu_j(A_j^{d_{ij}+\varepsilon/2}) + d_{ij} + \varepsilon/2,
\]

for any Borel set \( A_j \subset X_j \). Define \( \psi_{12} : X_1 \to X_2 \) by \( \psi_{12} := \psi_{32} \circ \psi_{13} \), which is a \( 2(d + \varepsilon) \)-approximation. Then for any Borel set \( A_2 \subset X_2 \),

\[
\mu_1(\psi_{12}^{-1}(A_2)) = \mu_1(\psi_{32}^{-1}(A_2)) < \mu_3((\psi_{32}^{-1}(A_2))^{d_{13}+\varepsilon/2}) + d_{13} + \varepsilon/2 \]

\[
< \mu_3((A_2)^{d_{13}+d_{32}+\varepsilon/2}) + d_{13} + \varepsilon/2 \]

\[
< \mu_2((A_2)^{d+d_{32}+\varepsilon}) + d + \varepsilon.
\]

Similarly, for \( \psi_{21} := \psi_{31} \circ \psi_{23} \) we have \( \mu_2(\psi_{21}^{-1}(A_1)) < \mu_1((A_1)^{d+d_{31}+\varepsilon}) + 1 + \varepsilon \), for any Borel set \( A_1 \subset X_1 \), and, therefore, the almost triangle inequality follows. \( \Box \)
Lemma 2.7. $d_{mGH}((X, \mu), (X', \mu')) = 0$ if and only if there exists an isometry $\psi : X \to X'$ such that $\psi_*(\mu) = \mu'$.

Proof. Suppose $d_{mGH}((X, \mu), (X', \mu')) = 0$. By definition, there are $\varepsilon_i$-approximations $\psi_i : X \to X'$ with $\lim \varepsilon_i = 0$ such that

$$\mu(\psi_i^{-1}(A')) < \mu'((A')^{\varepsilon_i}) + \varepsilon_i,$$

for every closed subset $A' \subset X'$. As $X, X'$ are compact, we may assume, using (1.7), that $\psi_i$ uniformly converges to an isometry $\psi : X \to X'$. Since $\psi^{-1}(A') \subset (\psi_i^{-1}(A'))^{\varepsilon_i}$ for some $\delta_i \to 0$, it follows that

$$\mu(\psi^{-1}(A')) \leq \mu(((\psi_i^{-1}(A'))^{\delta_i}) \leq \mu'(((A')^{\delta_i+2\varepsilon_i}) + \varepsilon_i.$$

Letting $i \to \infty$, we obtain $\mu(\psi^{-1}(A')) \leq \mu'(A')$. Taking complement, we have $\mu(\psi^{-1}(U')) \geq \mu'(U')$ for any open set $U' \subset X'$. It follows that $\mu(\psi^{-1}(A')) \geq \mu'(A')$. Letting $\varepsilon \to 0$, we obtain $\mu(\psi^{-1}(A')) \geq \mu'(A')$. Thus, we have $\mu(\psi^{-1}(A')) = \mu'(A')$ for every closed set $A'$ and hence for every Borel subset $A'$. This completes the proof of the lemma. \hfill \square

Proposition 2.8. Let $\mu_i$ and $\mu$ be probability Borel measures on compact metric spaces $X_i$ and $X$. A sequence $(X_i, \mu_i)$ weakly converges to $(X, \mu)$ if and only if

$$\lim_{i \to \infty} d_{mGH}((X_i, \mu_i), (X, \mu)) = 0.$$

Proof. Take $\varepsilon_i$-approximations $\psi_i : X_i \to X$ with $\lim \varepsilon_i = 0$ such that $\int X_i f \circ \psi_i d\mu_i \to \int X f d\mu$ for every $f \in C(X)$. First, using the weak convergence $(\psi_i)_* \mu_i \to \mu$, we show by contradiction that, for any Borel set $A \subset X$,

$$((\psi_i)_* \mu_i)(A) < \mu(A^{\varepsilon_i}) + \varepsilon_i' \quad \quad \quad \quad \quad (2.10)$$

$$\mu(A) < ((\psi_i)_* \mu_i)(A^{\varepsilon_i'}) + \varepsilon_i' \quad \quad \quad \quad \quad (2.11)$$

for some $\varepsilon_i' \to 0$. Suppose (2.10) does not hold. Then there are Borel sets $A_i$ of $X$ such that

$$((\psi_i)_* \mu_i)(A_i) \geq \mu(A_i^e) + c \quad \quad \quad \quad \quad (2.12)$$

for some constant $c > 0$ independent of $i$. We may assume that $A_i$ converges to a closed set $A$ with respect to the Hausdorff distance in $X$. Take $\varepsilon_2 > \varepsilon_1 > 0$ with $A_i \subset A^{\varepsilon_1} \subset A^{\varepsilon_2} \subset A^e_1$ for sufficiently large $i$. Choose $f \in C(X)$ such that $0 \leq f \leq 1$, $f = 1$ on $A^{\varepsilon_1}$, and $	ext{supp}(f) \subset A^{\varepsilon_2}$. Then

$$\mu(A_i^e) \geq \mu(A_i^{\varepsilon_2}) \geq \int X f d\mu = \lim_{i \to \infty} \int_{X_i} f \circ \psi_i d\mu_i \geq \limsup_{i \to \infty} ((\psi_i)_* \mu_i)(A_i^{\varepsilon_1}) \geq \limsup_{i \to \infty} ((\psi_i)_* \mu_i)(A_i).$$

This is a contradiction to (2.12).
Next suppose (2.11) does not hold. Then for some Borel sets $A_i$ of $X$ we have

$$\mu(A_i) \geq ((\psi_i)_*\mu_i)(A_i^c) + c,$$

for some constant $c > 0$ independent of $i$. Let $A_i \subset A^{c1} \subset A^{c2} \subset A_i^c$ and $f \in C(X)$ be given as above. Then,

$$\liminf_{i \to \infty}((\psi_i)_*\mu_i)(A_i^c) \geq \liminf_{i \to \infty}((\psi_i)_*\mu_i)(A^{c2}) \geq \lim_{i \to \infty} \int_{X_i} f \circ \psi_i \, d\mu_i = \int_X f \, d\mu \geq \mu(A^{c1}) \geq \mu(A_i).$$

This is a contradiction to (2.13).

Let $\psi'_i : X \to X_i$ be any measurable $\varepsilon_i$-approximation such that $d_i(\psi'_i \circ \psi_i(x), x_i) < \varepsilon_i$ for every $x_i \in X_i$, and $d(\psi_i \circ \psi'_i(x), x) < \varepsilon_i$ for every $x \in X$, see e.g. [26, Lemma 2.5]. Using (2.11), we obtain, for any Borel $A_i \subset X_i$,

$$\mu((\psi'_i)^{-1}(A_i)) < \mu_i((\psi'_i)^{-1}((\psi'_i)^{-1}(A_i)^c)) + \varepsilon'_i < \mu_i(A_i^{2\varepsilon_i + \varepsilon'_i}) + \varepsilon'_i.$$

Together with (2.10), we have $d_mGH((X_i, \mu_i), (X, \mu)) < 2\varepsilon_i + \varepsilon'_i$.

Finally we shall prove the converse. Since $\mu_i, \mu$ are probability measures, shifting $f \in C(X)$, $f \mapsto f - \min(f)$, and normalising it, $f \mapsto f/\max(f)$, we may assume $0 \leq f \leq 1$. Take a large positive integer $k$, and set $A_j := \{x \in X \mid f(x) \geq j/k\}$ for $0 \leq j \leq k$. It is straightforward to see that, for any Borel measure $\mu$ on $X$,

$$\frac{1}{k} \sum_{j=1}^{k} \mu(A_j) \leq \int_X f \, d\mu \leq \frac{1}{k} + \frac{1}{k} \sum_{j=1}^{k} \mu(A_j).$$

(2.14)

Now take $\varepsilon_i$-approximation $\psi_i : X_i \to X$ with $\varepsilon_i \to 0$ such that $((\psi_i)_*\mu_i)(A) < \mu(A^{c1}) + \varepsilon_i$ for every closed set $A \subset X$, where $\varepsilon_i \to 0$. Letting $i \to \infty$, we have $\limsup_{i \to \infty}((\psi_i)_*\mu_i)(A) \leq \mu(A)$. Therefore, in the above situation, we obtain $\limsup_{i \to \infty}((\psi_i)_*\mu_i)(A) \leq \mu(A)$ for each $0 \leq j \leq k$. It follows from (2.14) that $\limsup_{i \to \infty} \int_X f \, d((\psi_i)_*\mu_i) \leq \frac{1}{k} + \int_X f \, d\mu$, and letting $k \to \infty$,

$$\limsup_{i \to \infty} \int_X f \, d((\psi_i)_*\mu_i) \leq \int_X f \, d\mu.$$ 

Replacing $f$ by $1 - f$, we get $\liminf_{i \to \infty} \int_X f \, d((\psi_i)_*\mu_i) \geq \int_X f \, d\mu$, and

$$\lim_{i \to \infty} \int_X f \, d((\psi_i)_*\mu_i) = \int_X f \, d\mu,$$

as required. ∎
3 Smoothness of the density functions

In this section, we consider the class $\mathcal{M}_p(n, \Lambda, D)$ of pointed closed Riemannian manifolds $(M, p, \mu_M)$ with $M \in \mathcal{M}(n, \Lambda, D)$ and the normalized measure $\mu_M = dV_M/\text{Vol}(M)$. By Fukaya [26], we have the precompactness of $\mathcal{M}_p(n, \Lambda, D)$ with respect to the pointed measured Gromov-Hausdorff topology.

Let us consider a sequence $M_i \in \mathcal{M}(n, \Lambda, D)$ converging to $X \in \mathcal{M}(n, \Lambda, D)$. Let $\varphi_i : M_i \to X$ be a measurable $\varepsilon_i$-approximation with $\lim_{i \to \infty} \varepsilon_i = 0$. Passing to a subsequence, we may assume that $(M_i, \mu_{M_i})$ converges to some $(X, \mu)$ in the pointed measured Gromov-Hausdorff topology. Here $\mu$ is some probability measure of $X$, see [26 Section 3]. More precisely, the pushforward measure $(\varphi_i)_*(\mu_i)$ weak* sub-converges to $\mu$.

The following lemma is known in [26]. Recall the notations (2.2) that $S(X) = X^{\text{sing}}$ denotes the singular set of $X$ and $X^{\text{reg}}$ denotes the regular part of $X$. We say a point $x \in X$ is an orbifold point if the group $G_x$ stated in Theorem 2.3(1) is finite.

**Lemma 3.1** ([26]). Let $\hat{X}^{\text{sing}}$ be the set of all points of $X^{\text{sing}}$ which are not orbifold points. Then

1. $\mu(X^{\text{sing}}) = 0$;

2. there exists a continuous density function $\rho_X$ on $X$ with $\mu = \rho_X \mu_X$, where $\mu_X$ denotes the normalized Riemannian volume element of $X^{\text{reg}}$, such that $\hat{X}^{\text{sing}} = \{ x \in X^{\text{sing}} \mid \rho_X(x) = 0 \}$.

In this section, we discuss some properties of $\rho_X$ concerning the smoothness, and prove the following:

**Lemma 3.2.** $\rho_X$ is of class $C^2_*$ on the regular part $X^{\text{reg}}$.

Concerning Lemma 3.2, Kasue [39, 40] proved that $\rho_X$ is of class $C^{1, \alpha}$ for any $0 < \alpha < 1$. The method in [39, 40] used smooth approximations of the metric of $M_i$ as in the proof of Lemma 2.2. Our method discussed below is more direct and contains an extension of Montgomery and Zippin’s result on the smoothness of isometric group actions, see [18], although we follow a basic line in [26].

First we consider the case when $X^{\text{sing}}$ is empty, namely the case when $X$ is a Riemannian manifold. In this case, we can approximate $\varphi_i$ by an almost Riemannian submersion $f_i : M_i \to X$ such that (see [26 Section 3])

1. the pushforward measure $(f_i)_*(\mu_i)$ weak* converges to $\mu$,.

2. $\text{Vol}(f_i^{-1}(q))/\text{Vol}(M_i)$ converges to $\rho_X(q)/\text{Vol}(X)$ in the $C^0$-topology.

Fix $q_0 \in X$ and put $q_i := \psi_i(q_0)$, where $\psi_i : X \to M_i$ is an $\varepsilon_i$-approximation such that $d_X(f_i \circ \psi_i(x), x) < \varepsilon_i$. Let $B' \subset B \subset \mathbb{R}^n$, $(B', \bar{h}_i, G_i)$ and $(B, \bar{h}_0, G)$ be as in Section 2 so that $(B', \bar{h}_i)/G_i = B(q_i, \frac{\pi}{2\Lambda})$ and $(B', \bar{h}_0)/G = B(q_0, \frac{\pi}{2\Lambda})$. Let $\pi_i : B' \to B(q_i, \frac{\pi}{2\Lambda}) \subset M_i$, $\pi : B' \to B(q_0, \frac{\pi}{2\Lambda}) \subset X$,
be the natural projections.

We now need the following result on the smoothness of isometric group actions.

**Theorem 3.3 ([48]).** Let $G$ be a Lie group, and $M$ be a Riemannian manifold of class $C^k$ with $k \geq 1$. Suppose that the action of $G$ on $M$ is isometric. Then the $G$-action on $M$,

$$G \times M \to M, \ (g, x) \to gx,$$

is of class $C^{k+1}_*$, where we consider the analytic structure on $G$.

It is proved in Calabi-Hartman [18] and Shefel [56] that the transformation $g : M \to M$ defined by each $g \in G$ is of class $C^{k, \alpha}$. Theorem 3.3 is a generalization of Montgomery-Zippin [51, p. 212], where it is stated that the $G$-action on $M$ is of class $C^k$. The proof of Theorem 3.3 can be found in [48, Appendix A].

Note that in our situation, the pseudo-group $G$ can be extended to a nilpotent Lie group ([27]). It follows from Theorem 3.3 that the pseudo-group action of $G$ on $B$ is of class $C^3_*$.

Let $d = \dim(X), m := \dim(G)$ with $n = d + m$, and take a $d$-dimensional $C^\infty$-submanifold $Q$ of $B$ which transversally meets the orbit $G \cdot O$ at the origin $O$. Let $s : U_0 \to Q$ be a smooth coordinate chart of $Q$ around 0, where $U_0$ is an open subset of $\mathbb{R}^d$. From Theorem 3.3 together with the inverse function theorem, taking $Q$ smaller if necessary, we may assume that, for some neighborhood $U$ of $O$ in $B$ and for a neighborhood $G^*$ of the identity in $G$, the mapping

$$G^* \times U_0 \to U, \ (g, x) \to g(s(x)), \ g \in G^*, \ x \in U_0,$$

(3.1)

gives $C^*_*$-coordinates in $U$.

We can consider every element $V$ of the Lie algebra $\mathfrak{g}$ of $G$ as a Killing field on $B'$ by setting

$$V(x) := \frac{d}{dt} (\exp tV \cdot x)\big|_{t=0},$$

where $x$ denotes points in $U$. Then, for any $x \in U$, there is a unique $g \in G^*$ and $x \in U_0$ with $x = g(s(x))$. We define

$$\bar{\rho}(x) = [Ad_g(V_1)(x) \wedge \cdots \wedge Ad_g(V_m)(x)],$$

where $V_1, \ldots, V_m$ be a basis of $\mathfrak{g}$, and the norm is taken with respect to $\bar{h}_0$. Here the adjoint representation $Ad : G \to GL(\mathfrak{g})$ is defined by

$$Ad_g(V) := \frac{d}{dt} (g \cdot \exp tV \cdot g^{-1})\big|_{t=0}.$$  

Obviously $\bar{\rho}$ is $G$-invariant. We show that $\bar{\rho}$ is of class $C^2_*$. Recall that the correspondence $x = g(s(x)) \to (g, x)$ is of class $C^*_*$ from the inverse function theorem. It follows that the map $(t, x) \to g(\exp tV_i)g^{-1}(x)$ is of class $C^2_*$. Thus $x \to Ad_g(V_i)(x)$ is of class $C^2_*$, and so is $\bar{\rho}(x)$.
Since $\bar{\rho}$ is $G$-invariant, there is a function $\rho$ defined on a neighborhood $\pi(U)$ of the point $q_0$ such that $\bar{\rho} = \rho \circ \pi$. Then $\rho$ is of class $C^2_\ast$.

We shall prove that $\rho_X$ is of class $C^2_\ast$ by showing that $\rho_X(x)/\rho(x)$ is constant for $x \in \pi(U)$. Basically we follow the argument in [28]. Let

$$G'_i := \{ g \in G_i | d_{(B, \tilde{h}_i)}(g(O), O) < 1/2 \},$$

$$G' := \{ g \in G | d_{(B, \tilde{h}_0)}(g(O), O) < 1/2 \}.$$

Let us consider a left-invariant Riemannian metric on $G$. First we need to show that

$$\frac{\text{Vol}(G'(s(x)))}{\rho(x)} = \text{const},$$

on a neighborhood of $q_0$. Define $F^x : G' \to G'(s(x))$ by $F^x(g) = g(s(x))$. Let $V_1, \ldots, V_m$ be an orthonormal basis of $g$. For any $g \in G'$, we have

$$F^x(V_i(g)) = \left. \frac{d}{dt}(g \exp tV_i(s(x))) \right|_{t=0} = Ad_g(V_i)(F^x(g)).$$

Therefore,

$$\text{Vol}(G'(s(x))) = \int_{G'} |Ad_g(V_1) \wedge \cdots \wedge Ad_g(V_m)|(g(s(x)))$$

$$= \int_{G'} \bar{\rho}(g(s(x)) = \rho(x)\text{Vol}(G').$$

For the rest of the argument, we can go through along the same line as in [28], which we outline below for reader’s convenience.

Set

$$E_i(x, \delta) := \{ y \in B | \text{there exists } g \in G'_i \text{ such that } d_{\tilde{h}_i}(y, g(s(x))) < \delta \},$$

$$E_0(x, \delta) := \{ y \in B | \text{there exists } g \in G' \text{ such that } d_{\tilde{h}_0}(y, g(s(x))) < \delta \}.$$

Then one can check

$$\lim_{i \to \infty} \sup_{x \in \pi(U)} \frac{\text{Vol}(E_i(x, \delta))}{\text{Vol}(E_0(x, \delta))} = 1, \quad (3.2)$$

$$\lim_{\delta \to 0} \frac{\text{Vol}(E_0(x, \delta))}{\delta^d} = \omega_d \text{Vol}(G'(s(x))), \quad (3.3)$$

where $\omega_d$ denotes the volume of unit ball in $\mathbb{R}^d$. This implies

$$\lim_{\delta \to 0} \frac{\text{Vol}(E_0(x, \delta))}{\text{Vol}(E_0(x', \delta))} \frac{\rho(x')}{\rho(x)} = 1,$$

for all $x, x' \in W$. One can prove that there exists $c > 0$ independent of $x$ such that

$$\lim_{\delta \to 0} \frac{\text{Vol}(E_i(x, \delta))}{\text{Vol}(G'_i) \delta^n \text{Vol}(f^{-1}_i(x))} = c. \quad (3.4)$$
These yield
\[ \frac{\text{Vol}(f_i^{-1}(x))}{\text{Vol}(f_i^{-1}(x'))}\frac{\rho(x')}{\rho(x)} = 1. \]

Since
\[ \lim_{i \to \infty} \frac{\text{Vol}(f_i^{-1}(x))}{\text{Vol}(M_i)} = \frac{\rho_X(x)}{\text{Vol}(X)}, \]
we conclude
\[ \frac{\rho(x')}{\rho(x)}\frac{\rho_X(x)}{\rho_X(x')} = 1, \]
which shows that \( \rho_X \) is of class \( C^2 \).

Next consider the general case when \( X \) is not a Riemannian manifold. Since the above argument is local, it follows that \( \rho_X \) is of class \( C^2 \) on \( X^{\text{reg}} \). This completes the proof of Lemma 3.2. QED

Later on, we will also need the properties of the class of the orthonormal frame bundles \( F_M \) over Riemannian manifolds \( (M, p, \mu) \in \mathfrak{M}_p(n, \Lambda, D) \). The frame bundles \( F_M \) are equipped with the Riemannian metric \( h \), inherited from \( (M, h) \) and the corresponding probability measure \( \tilde{\mu} \), see (3.6). We denote this class by \( \mathfrak{MM}(n, \Lambda, D) \). By Theorem 2.3 and O’Neill’s formula [53], \( \dim(F_M) = n + \dim(O(n)) \), \( \text{diam}(F_M) \leq D_F \), \( |\text{sec}(F_M, h_{F_M})| \leq \Lambda_F^2 \), for \( F_M \in \mathfrak{MM}(n, \Lambda, D) \).\( (3.5) \)

The closure \( \overline{\mathfrak{MM}}(n, \Lambda, D) \) with respect to the measured Gromov-Hausdorff topology gives rise to \( C^3 \)-smooth manifolds \( Y \) equipped with \( C^2 \)-smooth Riemannian metric \( h_Y \), which appears in Theorem 2.3(2) and in the proof of Lemma 3.1. The \( C^3 \)-smooth structure of \( Y \) is defined by the limit of the transition maps for harmonic coordinates on a sequence \( F_{M_i} \) converging to \( Y \), see [2, 3, 33, 54]. Hence \( Y \) has a compatible \( C^\infty \)-smooth structure contained in the \( C^3 \)-smooth structure. Analyzing the proof of Lemma 3.1 we have

**Corollary 3.4.** Let \( Y \in \overline{\mathfrak{MM}}(n, \Lambda, D) \). Then \( Y \) is a smooth manifold with \( C^2 \)-smooth Riemannian metric \( h_Y \) and strictly positive density function \( \rho_Y \in C^2(Y) \). Moreover, \( O(n) \) acts by isometries on \( Y \) and \( \rho_Y \) is \( O(n) \)-invariant.

Let \( M_i \) be a sequence in \( \mathfrak{M}(n, \Lambda, D) \) converging to \( X \in \mathfrak{M}(n, \Lambda, D) \), and let \( F_{M_i} \) denote the orthonormal frame bundle of \( M_i \) endowed with the natural Riemannian metric, which has uniformly bounded sectional curvature and diameter. Put
\[ d\tilde{\mu}_i = \frac{dV_{F_{M_i}}}{\text{Vol}(F_{M_i})}, \]
where \( dV_{F_{M_i}} \) is the natural volume element on \( F_{M_i} \) with \( O(n) \) acting isometrically on \( F_{M_i} \). Passing to a subsequence, we may assume that \( (F_{M_i}, \tilde{\mu}_i, O(n)) \) converges to \( (Y, \tilde{\mu}, O(n)) \) in the equivariant measured Gromov-Hausdorff topology, where \( Y \) is a...
Riemannian manifold and \( \tilde{\mu} \) is a probability measure on \( Y \) invariant under the \( O(n) \)-action. The notion of equivariant measured Gromov-Hausdorff topology is defined in a way similar to that of measured Gromov-Hausdorff topology, where a measurable \( \epsilon_i \)-approximation maps \( \tilde{\psi}_i : FM_i \to Y \) with the property \( (\tilde{\psi}_i)_* (\tilde{\mu}_i) \to \tilde{\mu} \) is required to satisfy conditions similar to (2.3). By Theorem 2.4, there are \( \epsilon_i \)-regular maps \( \tilde{f}_i : FM_i \to Y \) and \( f_i : M_i \to X \) such that \( f_i \circ \tilde{\pi}_i = \pi \circ \tilde{f}_i \). We may replace \( \tilde{\psi}_i \) by \( \tilde{f}_i \).

The probability measure \( \tilde{\mu} \) on \( Y \) can be written as

\[
d\tilde{\mu} = \rho_Y \frac{dV_Y}{\text{Vol}(Y)},
\]

with a strictly positive \( O(n) \)-invariant density function \( \rho_Y \) by Corollary 3.4. It follows that there is a strictly positive function \( \nu_X \) on \( X \) with \( \nu_X \circ \pi = \rho_Y \), where \( \pi : Y \to X \) is the projection. Since \( d_X(x, x') = d_Y(\pi^{-1}(x), \pi^{-1}(x')) \), then \( \nu_X \) is Lipschitz.

The projection \( \pi_i : FM_i \to M_i \) is a Riemannian submersion with totally geodesic fibers isometric to \( O(n) \). Thus,

\[
d\tilde{\mu}_i = \frac{dV_{FM_i}}{\text{Vol}(O(n)) \times \text{Vol}(M_i)}.
\]

Then it follows that \( (\pi_i)_* (\tilde{\mu}_i) = \mu_i \). For any continuous function \( f \) on \( X \), by Fubini’s theorem, we have

\[
\int_M f(x) \rho_X(x) \frac{dV_X}{\text{Vol}(X)} = \int_Y f \left( \frac{dV_Y}{\text{Vol}(Y)} \right) \rho_X(x) \frac{dV_X}{\text{Vol}(X)} = \int_M (f \circ f_i) \frac{d\mu_M}{\text{Vol}(M)}
\]

\[
= \lim_{i \to \infty} \int_{FM_i} (f \circ f_i \circ \pi_i) d\tilde{\mu}_i = \lim_{i \to \infty} \int_{FM_i} (f \circ \pi \circ \tilde{f}_i) d\tilde{\mu}_i
\]

\[
= \int_Y (f \circ \pi) d\tilde{\mu} = \int_Y (f \circ \pi) \rho_Y \frac{dV_Y}{\text{Vol}(Y)}
\]

\[
= \frac{1}{\text{Vol}(Y)} \int_X \left( \int_{\pi^{-1}(x)} \rho_Y(y) \text{dH}^\ell_{\pi^{-1}(x)}(y) \right) f(x) dV_X(x)
\]

where \( \text{H}^\ell_{\pi^{-1}(x)} \) is the \( \ell \)-dimensional Hausdorff measure of \( \pi^{-1}(x) \) with \( \ell = \text{dim}(O(n)) \). Then it follows

\[
\pi_*(\tilde{\mu}) = \mu
\]

and the function

\[
\rho_X(x) = \nu_X(x) \frac{\text{Vol}(\pi^{-1}(x))}{\text{Vol}(Y)}
\]

is the required density function on \( X \) with \( d\mu_X = \rho_X dV_X / \text{Vol}(X) \), as stated in Lemma 3.1.2.
4 On properties of eigenfunctions

In this section, we consider some properties of the eigenpairs \( \{\lambda_j, \phi_j\}_{j=0}^{\infty} \) of the weighted Laplace operator \( \Delta_X \), \((X, \rho, \mu_X) \in \mathfrak{M}M_p(n, \Lambda, D) \) and \( \{\tilde{\lambda}_j, \tilde{\phi}_j\}_{j=0}^{\infty} \) for \( \Delta_Y \), \((Y, \mu_Y) \in \mathfrak{M}M_p(n, \Lambda, D) \). The basic properties of these operators are given in Appendix A.

As the Riemannian metric \( h_Y \) and the density function \( \rho_Y \) on \( Y \) are of class \( C^2 \) by Corollary 3.4, it is natural to work in the \( C^3 \)-smooth structure of \( Y \) and invariantly define the Sobolev spaces \( W^{k,q}(Y) \), \( k \in \{0, 1, 2\} \), \( 1 \leq q < \infty \), and Hölder spaces \( C^{l,\alpha}(Y) \), \( l \in \{0, 1, 2\} \), \( 0 \leq \alpha < 1 \), and the Zygmund space \( C^3 (Y) \).

Moreover, due to Theorem 3.3 with \( M = Y, G = O(n) \), we can define \( O(n) \)-invariant subspaces \( W^{k,q}_O(Y) \subset W^{k,q}(Y) \), \( C^{l,\alpha}_O(Y) \subset C^{l,\alpha}(Y) \) and \( C^3_+ O(Y) \subset C^3(Y) \), which consist of functions invariant with respect to \( O(n) \)-action, that is, functions \( f \in W^{k,q}_O(Y) \) (resp. \( C^{l,\alpha}_O(Y) \), \( C^3_+ O(Y) \)) satisfying \( f(y) = f(o(y)) \) for all \( y \in Y \) and \( o \in O(n) \).

Introduce the operator
\[
P_O : L^2(Y) \to L^2(Y), \quad (P_O u^*)(y) = \int_{O(n)y} u^*(o(y)) d_n \mathcal{H}^\ell_{O(n)y}, \quad (4.1)
\]
where \( d_n \mathcal{H}^\ell_{O(n)y} := \frac{1}{\text{vol}(O(n)y)} d\mathcal{H}^\ell_{O(n)y} \) and \( \ell = \dim(O(n) \cdot y) = \dim(O(n)) \), where \( d\mathcal{H}^\ell_{O(n)y} \) is the \( \ell \)-dimensional Hausdorff measure on the orbit \( O(n) \cdot y \) of \( y \). In the future, the subindex \( O \) indicates \( O(n) \)-invariance of functions.

**Lemma 4.1.** The operator \( P_O \) in (4.1) is an orthogonal projector in \( L^2(Y, \mu_Y) \). Moreover, for \( l \in \{0, 1, 2\} \), \( 1 \leq q < \infty \),
\[
P_O : L^q(Y) \to L^q(Y), \quad P_O : W^2,q(Y) \to W^2,q(Y), \quad P_O : C^{l,1}(Y) \to C^{l,1}(Y), \quad P_O : C^3(Y) \to C^3_+ O(Y), \quad (4.2)
\]
are bounded. In addition, for \( 1 \leq q < \infty \), \( k \in \{0, 1, 2\} \), \( W^{k,q}_O(Y), C^3_+ O(Y) \) are dense in \( L^q(Y, \mu_Y) \).

**Proof.** Let \( u^*, v^* \in L^2(Y) \). Then,
\[
(P_O u^*, v^*)_{L^2(Y)} = \int_Y \left( \int_{O(n)y} u^*(o(y)) d_n \mathcal{H}^\ell_{O(n)y} \right) v^*(y) d\mu_Y(y)
\]
\[
= \int_Y \left( \int_{O(n)y} v^*(o^{-1}(y)) d_n \mathcal{H}^\ell_{O(n)y} \right) u^*(y) d\mu_Y(y) = (u^*, P_O v^*)_{L^2(Y)}.
\]
Here we have made the substitution \( \tilde{y} = o y \), and used the invariance of \( d\mathcal{H}^\ell_{O(n)y} \) with respect to \( O(n) \) and the fact that \( \mu_Y \) is \( O(n) \)-invariant.

Next we prove (4.2) for \( L^q \). We have
\[
\| P_O u^* \|^q_{L^q(Y)} = \int_Y \left( \int_{O(n)y} u^*(o(y)) d_n \mathcal{H}^\ell_{O(n)y} \right)^q d\mu_Y(y)
\]
\[
\leq \int_Y \int_{O(n)y} |u^*(o(y))|^q d_n \mathcal{H}^\ell_{O(n)y} d\mu_Y(y) = \| u^* \|^q_{L^q(Y)}
\]

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where we used Hölder’s inequality, invariance of $d_n \mathcal{H}^k_{O(n)y}$ and $\mu_Y$ with respect to $O(n)$, and $\int_{O(n)y} d_n \mathcal{H}^k_{O(n)y} = 1$.

As, by [46] Thm. 9.4.1, $(I + \Delta_Y)^{-1} : L^q(Y) \rightarrow W^{2,q}(Y)$, is an isomorphism and $\mathbb{P}_O$ commutes with $\Delta_Y$, we obtain (4.2) for $W^{2,q}(Y)$.

Since $O(n)$ acts by isometries, (4.2) on $C^{0,1}$ and $C^3_*$ follows from definition (4.1). As by [3] Prop. 2.4.1 $(I + \Delta_Y)^{-1} : C^3_*(Y) \rightarrow C^3_*(Y)$ is an isomorphism, we obtain (4.2) on $C^3_*(Y)$.

To obtain the density of $C^3_{s, O}(Y)$ and therefore, $W^{k,q}_O(Y)$ in $L^q_O(Y)$, we approximate $u^* \in L^q_O(Y)$ by $C^3_*$-functions $\tilde{u}_k$ and consider $u_k^* = \mathbb{P}_O \tilde{u}_k^* \in C^3_{s, O}(Y)$. Since $u^*, u_k^*$ are $O(n)$-invariant, then $u^*(y) - u_k^*(y) = \int_{O(n)y} (u^* - \tilde{u}_k^*) (o(y)) d_n \mathcal{H}^k_{O(n)y}$ and by Hölder’s inequality,

$$\|u^* - u_k^*\|_{L^q(Y)} \leq \|u^* - \tilde{u}_k^*\|_{L^q(Y)}.$$ 

Hence $u_k^*$ approximates $u^*$ in $L^q$.

Let $(X, \mu_X) \in \mathcal{MM}_p(n, \Lambda, D)$ and $(Y, \mu_Y) \in \mathcal{MM}_p(n, \Lambda, D)$ such that $X = Y/O(n)$, as constructed in Theorem 2.3. Let $\pi : Y \rightarrow X$ be the projection. As $X^{reg}$ is a $C^3_*$-smooth manifold (see [3]), we define function spaces $W^{k,q}_{loc}(X^{reg})$, $C^{r,\alpha}(X^{reg})$ and $C^3_{s, reg}(X^{reg})$.

**Lemma 4.2.** (1) Let $(Y, \mu_Y) \in \mathcal{MM}_p(n, \Lambda, D)$. Then the eigenfunction $\tilde{\phi}_j \in C^3_*(Y)$.

(2) Let $(X, \mu_X) \in \mathcal{MM}_p(n, \Lambda, D)$. Then the eigenfunction

$$\phi_j \in C^{0,1}(X) \bigcap C^3_{s, reg}(X^{reg})$$

**Proof.** (1) Since $Y$ is a $C^3_*$-manifold with $C^2_*$ metric tensor, the $C^3_*$-smoothness of $\tilde{\phi}_j$ follows from the standard interpolation arguments by the interior Schauder estimates, see e.g. [32] Thm. 6.17. Since $h_Y$ and $\rho_Y$ are $C^2_*$-smooth, these arguments can be adjusted to $C^3_*$-case by means of [3] Prop. 2.4.1, see also [65] Thm. 14.4.2-3 or [68].

(2) Denote by $\Delta^O_Y$ the $O(n)$-invariant part of $\Delta_Y$ on $L^2_O(Y)$, namely, $\Delta^O_Y := \Delta_Y \mathbb{P}_O$ where $\mathbb{P}_O : L^2(Y) \rightarrow L^2_O(Y)$ is the projection onto the $O(n)$-invariant subspace of $L^2(Y)$. By [26] Lemma 7.1, see also Appendix A,

$$\text{spec}(\Delta^O_Y) = \text{spec}(\Delta_Y), \quad \phi^O_j = \pi^* \phi_j,$$

where $\lambda^O_j, \phi^O_j$ are the eigenvalues and eigenfunctions of $\Delta^O_Y$. Then,

$$|\phi_j(x) - \phi_j(y)| = |\phi^O_j(x^*) - \phi^O_j(y^*)| \leq \|\phi^O_j\|_{C^0(Y)} d_X(x, y),$$

where $x, y \in X$ and $x^* \in \pi^{-1}(x)$, $y^* \in \pi^{-1}(y)$, satisfy $d_X(x, y) = d_Y(x^*, y^*)$. The $C^3$ inclusion in (4.3) follows from (4.4) and claim (1).

Denote by $Z_X, Z_Y, Z^O_Y$ the linear subspaces of finite linear combinations of the eigenfunctions $\{\phi_j\}_{j=0}^\infty$, $\{\phi^O_j\}_{j=0}^\infty$, and $\{\phi^O_j\}_{j=0}^\infty$, respectively.
Proposition 4.3.  
1. $Z_Y, Z_Y^O$ are dense in $W^{k,q}(Y), W^{k,q}_O(Y)$, for any $k \in \{0, 1, 2\}$, $1 \leq q < \infty$, $C^{l, \alpha}(Y)$, $C^{l, \alpha}_O(Y)$, $l \in \{0, 1, 2\}$, $0 \leq \alpha < 1$, and $C^3_\ast(Y)$, $C^3_{\ast, O}(Y)$, respectively.

2. $Z_X$ is dense in $W^{k,q}_\text{loc}(X^{\text{reg}})$, $C^{l, \alpha}(X^{\text{reg}})$, $C^3_\ast(X^{\text{reg}})$ and $C^{0,1}(X)$.

Proof. 1. We start with $W^{2,q}$. Using the density of $W^{k,q}$ in $W^{k,q'}$, if $q > q'$, we consider only the case $2 \leq q < \infty$. As, by [6, Thm. 9.1],

$$(I + \Delta_Y)^{-1} : L^q(Y) \to W^{2,q}(Y),$$

is an isomorphism, it is enough to show that $Z_Y$ is dense in $L^q$. Clearly, this is true for $q = 2$ and we will use boot-strap arguments to show this for any $q > 2$.

Indeed, assume that $Z_Y$ is dense in $L^q$ for some $q \geq 2$. Then, as $(I + \Delta_Y)^{-1}(Z_Y) = Z_Y$, $Z_Y$ is dense in $W^{2,q}$. Using Sobolev's embedding and the fact that $C^3_\ast(Y)$ is dense in $L^q(Y)$, $q < \infty$, $Z_Y$ is dense in $W^{2,q}$, if $\frac{1}{q} > \frac{1}{q'} - \frac{1}{\dim(Y)}$. Iterating the arguments, we obtain the density of $Z_Y$ in $W^{k,q}(Y)$, $k = 0, 1, 2$, $q \in [2, \infty)$.

Observe now that, due to $W^{k,q} \subset C^{0,1}$ for $q > \dim(Y)/2$, $Z_Y$ is dense in $C^{0,1}$. Then the interior Schauder regularity method, adjusted to Zygmund classes, see e.g. [3, Prop. 2.4.1], [65, Thm. 14.4.2-3] or [68], shows that $(I + \Delta_Y)^{-1} : C^1_\ast(Y) \to C^3_\ast(Y)$ is an isomorphism, completing the proof of (1) for $Z_Y$.

Turning to the case of $Z_Y^O$, we just note that

$$(I + \Delta_Y)^{-1} : L^q_O(Y) \to W^{2,q}_O(Y), \quad (I + \Delta_Y)^{-1} : C^1_{\ast, O}(Y) \to C^3_{\ast, O}(Y),$$

are isomorphisms. Since $Z_Y^O$ is dense in $L^2_O(Y)$, repeating the above arguments gives the desired result.

2. Note that

$$C^{0,1}_O(Y) = \pi^*(C^{0,1}(X)), \quad (4.6)$$

see Appendix A or [26] Section 7, where the classes $C^1$ can be easily substituted by classes $C^{0,1}$. This, together with (4.4), (4.5) and the density of $Z_Y^O$ in $C^{0,1}(Y)$ provides the density of $Z_X$ in $C^{0,1}(X)$.

Next, to prove the density of $Z_X|_{X^{\text{reg}}}$ in $W^{k,q}_\text{loc}(X^{\text{reg}})$, $k \in \{0, 1, 2\}$, $1 \leq q < \infty$, or in $C^{l, \alpha}(X^{\text{reg}})$, $l \in \{0, 1, 2\}$, $0 \leq \alpha < 1$, it is sufficient to show that $Z_X|_{X^{\text{reg}}}$ is dense in $C^3_\ast(X^{\text{reg}})$. To this end, let $K$ be a compact set in $X^{\text{reg}}$. To prove that last statement, it is sufficient to show that any $u \in C^3_\ast(X^{\text{reg}})$, supp$(u) \subset K$, can be approximated in $C^3_\ast(K)$ by a sequence of functions $z_m|_K$, $z_m \in Z_X$. To this end, consider

$$u^* = \pi^*(u) \in C^3_{\ast, O}(Y).$$

Since $O(n)$ acts isometrically on $\pi^{-1}(K)$, it follows from Theorem 3.3 that

$$\pi_* : C^3_{\ast, O}(\pi^{-1}(K)) \to C^3_{\ast}(K) \quad (4.7)$$

is a bounded operator.
By part (1), there are \( \{z_m^O\}_{m=1}^\infty, z_m^O \in Z_Y^O \) which approximate \( u^* \) in \( C^3_*(Y) \). Thus, \( \{z_m^{\pi^{-1}(K)}\}_{m=1}^\infty \subset Z_X \) approximates \( u^*|_{\pi^{-1}(K)} \) in \( C^3_*(\pi^{-1}(K)) \) and, by \((4.7)\),
\[
\lim_{m \to \infty} \pi_*(z_m^O)|_K = \pi_*(u^*)|_K = u|_K \quad \text{in} \quad C^3_*(K).
\]

\(\square\)

**Lemma 4.4.** (i) The map \( \Phi : X \to \mathbb{R}^N \) defined by
\[
\Phi(x) := (\phi_j(x))_{j=0}^\infty
\]
is injective, that is, if \( x \neq y \), there is an index \( j \in \mathbb{N} \) such that \( \phi_j(x) \neq \phi_j(y) \).

(ii) For any \( x_0 \in X^{\text{reg}} \), there is a neighborhood \( U \subset X^{\text{reg}} \) of \( x_0 \) and indices \( j = (j_1, \ldots, j_d) \), where \( d = \dim(X) \), such that
\[
\Phi_j : U \to \mathbb{R}^d, \quad \Phi_j(x) = (\phi_{jk}(x))_{k=1}^d, \quad x \in U,
\]
are \( C^3_* \)-smooth coordinates in \( U \).

**Proof.** Suppose that there exists \( x \neq y \) such that \( \phi_j(x) = \phi_j(y) \) for all \( j \in \mathbb{N} \). Then \( f(x) = f(y) \) for all \( f \in Z_X \). Due to the density of \( Z_X \) in \( C^{0,1}(X) \) (Proposition 4.3), this would imply that there is no Lipschitz function taking different values at \( x \) and \( y \), which is clearly a contradiction e.g., \( f(z) = d_X(z, y)/d_X(x, y) \). The second claim is due to \([3, \text{Lemma 4.2.1}]\).

By Lemma 4.4 we can use the following test to identify \( i := (i_1, \ldots, i_d) \in \mathbb{N}^d \) such that \( \Phi_i = (\phi_{i_1}, \ldots, \phi_{i_d}) \) form a coordinate system.

**Corollary 4.5.** Let \( x_0 \in X^{\text{reg}} \) and, for \( i := (i_1, \ldots, i_d) \in \mathbb{N}^d \), a neighborhood \( W_i \subset \mathbb{R}^d \) of \( \Phi_i(x_0) \), such that the function \( \phi_l \circ \Phi_i^{-1} : W_i \to \mathbb{R} \) is \( C^3_* \)-smooth, for any \( l \in \{i_1, \ldots, i_d\} \). Then there is a neighborhood \( U \) of \( x_0 \) such that \( \Phi_i : U \to \Phi_i(U) \) is a \( C^3_* \)-smooth diffeomorphism.

The above results show that the eigenfunctions \( \{\phi_j(x)\}_{j=0}^\infty \) given on open set \( \Omega^{\text{reg}} \subset X^{\text{reg}} \) determine the topological and \( C^3_* \)-differentiable structure of \( \Omega^{\text{reg}} \). Next we consider the metric tensor \( h \) and the density function \( \rho \) on \( \Omega^{\text{reg}} \).

**Lemma 4.6.** Let \( \Omega^{\text{reg}} \subset X^{\text{reg}} \) be an open set. The set \( \Omega^{\text{reg}} \) and the eigenpairs \( \{\lambda_j, \phi_j|_{\Omega^{\text{reg}}}\}_{j=0}^\infty \) of \( \Delta_X \) uniquely determine the metric \( h|_{\Omega^{\text{reg}}} \) and a function \( \tilde{\rho} \) on \( \Omega^{\text{reg}} \), such that \( \tilde{\rho}(x) = \tilde{c} \rho(x) \), where \( \tilde{c} > 0 \) is a constant.

**Proof.** Let \( x_0 \in \Omega^{\text{reg}} \) and \( x(x) = (x^1, \ldots, x^d), x(x_0) = 0 \) be \( C^3_* \)-smooth coordinates in a neighborhood \( U \subset \Omega^{\text{reg}} \) of \( x_0 \). Let \( \chi(x) \) be a \( C^3_* \)-smooth function with \( \text{supp}(\chi) \subset U \) and \( \chi(x) = 1 \) in a neighborhood \( V \) of 0. Let
\[
\chi_i(x) = x^i \chi(x), \quad 1 \leq i \leq d, \quad \chi_{j,k}(x) = x^j x^k \chi(x), \quad 1 \leq j \leq k \leq d.
\]
By Proposition 4.3(2), there are \( z_{i}^\ell, z_{j,k}^\ell \in Z_X \) such that, in \( C^3_*(U) \),
\[
\chi_i = \lim_{\ell \to \infty} z_{i}^\ell, \quad \chi_{j,k} = \lim_{\ell \to \infty} z_{j,k}^\ell.
\]
Therefore, for sufficiently large $N$ and $y \in V$, the vectors $\Psi_y(\phi_m)$, $m = 0, 1, \ldots, N$, span the space $\mathbb{R}^{d(d+3)/2}$, where

$$
\Psi_y[f] = \left( (\frac{\partial f}{\partial x^i}(y))_{1 \leq i \leq d}, (\frac{\partial^2 f}{\partial x^i \partial x^j}(y))_{1 \leq j \leq k \leq d} \right).
$$

Consider the equations $\Delta_X \phi_m = \lambda_m \phi_m$ at an arbitrary point $y \in V$ in coordinates $x$,

$$
\Delta_X \phi_m(y) = -h^{jk}(y) \frac{\partial^2 \phi_m}{\partial x^j \partial x^k}(y) - a^j(y) \frac{\partial \phi_m}{\partial x^j}(y) = \lambda_m \phi_m(y), \quad m = 0, 1, \ldots, N, \quad (4.9)
$$

where

$$
a^j = \frac{1}{\sqrt{h}_p} \frac{\partial}{\partial x^k} \left( \sqrt{h} h^{jk} \rho \right) = \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^k} \left( \sqrt{h} h^{jk} \right) + h^{jk} \frac{\partial}{\partial x^k} \log(\rho).
$$

Regarding (4.9) as a system of $li$ equations, the vectors $\Psi_y(\phi_m)$, $m = 0, 1, \ldots, N$, uniquely determine the coefficients $h^{jk}(y)$ and $a^j(y)$, which determines $h^{jk}(y)$ and, up to a multiplicative constant, $\rho(y)$. \hfill \Box

From now on, we choose and fix some positive function $\tilde{\rho}$ so that

$$
\int_{\Omega_0} \tilde{\rho}(x) dV_X = 1, \quad \tilde{\rho} = \tilde{c} \rho, \; \tilde{c} > 0. \quad (4.10)
$$

Now we move to the consideration of the pointwise heat data, PHD, see (1.2)-(1.4). We denote by $L(\lambda_j)$ the eigenspace of $\Delta_X$, corresponding to the (multiple) eigenvalue $\lambda_j$.

**Lemma 4.7.** Let $PHD = (H(z_\alpha, z_\beta; t_\ell))_{\alpha, \beta \in \mathbb{N}, \ell \in \mathbb{Z}_+}$ be the pointwise heat data of $\Delta_X$ with $\{z_\alpha\}_{\alpha \in \mathbb{N}}$ dense in an open set $\Omega_0 \subset X$. Then the data uniquely determine the closure $\overline{\Omega_0}$, the eigenvalues $\lambda_j$ and, up to an orthogonal transformation in $L(\lambda_j)$, the orthonormalized eigenfunctions $\phi_j|_{\Omega_0}$.

**Proof.** Due to the analyticity of $H(x, y; t)$ with respect to $t$, Re($t$) > 0, $H_{\alpha, \beta, \ell}$, $\ell \in \mathbb{Z}_+$, determine $H_{\alpha, \beta}(t) \in C(0, \infty)$, where $H_{\alpha, \beta}(t) = H(z_\alpha, z_\beta; t)$. As follows from Weyl’s asymptotics for eigenfunctions (Lemma 5.3) and together with the Lipschitz regularity estimate (Lemma 5.5), for a given $X$,

$$
\lambda_j \geq c \lambda_j^{2/n}, \quad \|\phi_j\|_{C^{0,1}(X)} \leq C(1 + \lambda_j^2)^s, \quad (4.11)
$$

for some uniform constants $c, s, C$. This implies that, in the standard representation,

$$
H(x, y; t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x) \phi_j(y), \quad (4.12)
$$

where the series in the right-hand side converges in $C^{0,1}(X \times X \times (0, \infty))$. Then the Laplace transform $\tilde{H}_{\alpha, \beta}(\omega)$ of $H_{\alpha, \beta}(t)$ has a meromorphic continuation to the whole plane $\mathbb{C}$. It has simple poles at $\omega = -\lambda_j$, $j \in \mathbb{N}$, with corresponding residues,

$$
\text{Res}_{\omega=-\lambda_j} (\tilde{H}_{\alpha, \beta}(\omega)) = \sum_{\lambda_{j'} = \lambda_j} \phi_j(z_\alpha) \phi_j(z_\beta). \quad (4.13)
$$
Using the results in [41, Lemma 4.9], these residues determine, up to an orthogonal
li transformation in the orthogonal group $O(m_j)$, where $m_j$ is the multiplicity of the
eigenvalue $\lambda_j$, the values $\phi_j(z_\alpha)$.

The map $\Phi : \overline{\Omega}_0 \to \mathbb{R}^N$ defined in (4.8) is injective and continuous with respect to
the product topology in $\mathbb{R}^N$. As $\overline{\Omega}_0$ is compact, $\Phi$ is a homeomorphism onto its image
so that $\phi_j(z_\alpha)$, $j, \alpha \in \mathbb{N}$, determine $\Phi(\overline{\Omega}_0)$.

It follows from the the proof of Lemma 4.7 that

**Corollary 4.8.** Let $X \in \mathcal{M}_p(n, \Lambda, D)$. Assume that $H_X(x, z, t) = H_X(x', z, t)$ for all
$z \in U$, $t \in (a, b) \subset \mathbb{R}_+$, where $U \subset X$ is open. Then $x = x'$.

**Remark 4.9.** The pointwise heat data on any open set $\Omega \subset X$ determine the dimension
d = dim($X$). Indeed, $d$ is the minimal number so that, in some open set $\Omega' \subset \Omega$, there are
$d$ eigenfunctions $\phi_{j(1)}, \ldots, \phi_{j(d)}$, which form $C^0_\alpha$-coordinates in $\Omega'$.

5 Continuity of the direct map

The principal goal of this section is to show that, in $\mathcal{M}_p(n, \Lambda, D)$, the pointed measured
Gromov-Hausdorff convergence implies the convergence of the heat kernel associated
to the weighted Laplacians. We define the topology in the set of the heat kernels,
$H_X(\cdot, \cdot, t)$, $X \in \mathcal{M}_p(n, \Lambda, D)$ in the following way.

**Definition 5.1.** Let $H_X(x, y, t)$, $x, y \in X$, $t > 0$, and $H_X'(x', y', t)$, $x', y' \in X'$, $t > 0$, be
the heat kernels for spaces $(X, p, \mu_X)$, $(X', p', \mu_X') \in \mathcal{M}_p(n, \Lambda, D)$. We say $H_X$, $H_X'$,
are $\varepsilon$-close, if for every $r > 0$, there exist $\varepsilon$-nets $\{x_i\}_{i=1}^{I(\varepsilon)}$, $\{x_i'\}_{i=1}^{I(\varepsilon)}$
in $B_X(p, r)$, $B_X'(p', r)$ satisfying $|d_X(x_i, x_j) - d_X'(x_i', x_j')| < \varepsilon$, and an $\varepsilon$-net $\{t_\ell\}_{\ell=1}^{L(\varepsilon)}$
in $(\varepsilon, \varepsilon^{-1})$, such that

$$|H_X(x_i, x_j, t_\ell) - H_X'(x_i', x_j', t_\ell)| < \varepsilon, \quad \text{for all} \quad 1 \leq i, j \leq I(\varepsilon), 1 \leq \ell \leq L(\varepsilon). \quad (5.1)$$

In this section we prove the following result.

**Theorem 5.2.** For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $X, X' \in \mathcal{M}_p(n, \Lambda, D)$
are $\varepsilon$-close then their heat kernels $H_X$ and $H_{X'}$ are $\varepsilon$-close.

Note that the choice of $\delta = \delta(\varepsilon)$ above is uniform on $\mathcal{M}_p(n, \Lambda, D)$. Theorem 5.2
is a direct consequence of Theorem 5.15 applying to the ball $B_X(p, r)$. The proof of
Theorem 5.15 based on [20], is rather long and will occupy the rest of this section.

5.1 Spectral estimates in $\mathcal{M}_p(n, \Lambda, D)$

First, we recall the Weyl-type estimate for the eigenvalues. To this end, recall the
counting function, $N_X(E)$ of $X$, which is the number of the eigenvalues of $-\Delta_X$, counted
with their multiplicities, that are not larger than $E$. 26
Lemma 5.3. There exist constants $c, C > 0$ such that, for any $X \in \mathfrak{M}_p(n, \Lambda, D)$,
\[
\lambda_j^X \geq c \, j^{2/n}, \ j = 0, 1, \ldots, \ N_X(E) \leq 1 + CE^{n/2}. \tag{5.2}
\]

Proof. By \cite{10}, there exists $c > 0$ such that the estimate \cite{5.2} is valid for all manifolds in $\mathfrak{M}(n, \Lambda, D)$. Since the measured GH-convergence implies the convergence of eigenvalues, see \cite{20}, the inequality \cite{5.2} for eigenvalues remains valid for any space in the closure $\overline{\mathfrak{M}}(n, \Lambda, D)$. The second inequality in \cite{5.2} follows from the first. \hfill \Box

We start with uniform spectral estimates in $\mathfrak{M}_p(n, \Lambda, D)$ and $\overline{\mathfrak{M}}_p(n, \Lambda, D)$. Since these estimates are independent of point $p$, we often omit the sub-index $p$. Our first goal are the Zygmund-type estimates of the eigenfunctions on the sequence $FM_k \in \mathfrak{M}$ collapsing to $Y \in \mathfrak{M}$. However, since the injectivity radii on $FM_k$ are not uniformly bounded from below, we work by lifting the eigenfunctions to the tangent space $T(FM_k)$ where the injectivity radii are uniformly bounded below, see Section 2.1.

We use the notation $\phi$ for the eigenfunctions on $X \in \mathfrak{M}$, and $\bar{\phi}$ for the eigenfunctions on $Y \in \overline{\mathfrak{M}}$.

Proposition 5.4. There are $C_F = C_F(n, \Lambda, D) > 0$ and $s_F = s_F(n, \Lambda, D) > 0$ such that for any $(FM, \mu_{FM}) \in \mathfrak{M}$, and $\bar{\phi}$ satisfying
\[
\Delta_Y \bar{\phi} = \lambda \bar{\phi}, \tag{5.3}
\]
we have
\[
\|\bar{\phi}\|_{C^{0,1}(FM)} \leq C_F(1 + \lambda^2)^{s_F/2}\|\phi\|_{L^2(FM, \mu_{FM})}. \tag{5.4}
\]

Proof. For $y \in FM$, let $\bar{B}_y(r) \subset T_y(FM)$ be a closed ball of radius $r < \pi/\Lambda_F$, where $\Lambda_F$ is the sectional curvature bound for $FM$, see \cite{3.5}. Let $\bar{h}$ denote the metric on $FM$, and let $\hat{h} := (\exp_y)^*\bar{h}$ denote the lifted metric on $\bar{B}_y(r)$. Recall that $\bar{B}_y(r)$ with the lifted metric $\hat{h}$ has injectivity radii uniformly bounded from below. Then by \cite{31}, there exist uniform constants $C_h, r_h > 0$ such that there are harmonic coordinates $\Psi : \bar{B}_y(r_h) \to \mathbb{R}^\ell$ with $\ell = \dim(FM)$, in which the metric tensor satisfies
\[
C_h^{-2}I \leq \Psi_*\hat{h} \leq C_h^2I, \quad \|\Psi_*\hat{h}\|_{C^2(\Psi(\bar{B}_y(r_h)))} \leq C_h. \tag{5.5}
\]
In the following, let $R_0 = r_h/2C_h$. Then the ball $B(R_0) \subset \mathbb{R}^\ell$ satisfies $B(R_0) \subset \Psi(\bar{B}(r_h/2))$. Below, we identify functions on $B(R_0) \subset \mathbb{R}^\ell$ and $\Psi^{-1}(B(R_0))$ via $\Psi$, and use the Lebesgue spaces $L^p(B(R_0))$ and Sobolev spaces $W^{k,p}(B(R_0))$ which are defined using the usual Lebesgue measure on $\mathbb{R}^\ell$. We note that as the metric $\hat{h}$ satisfies \cite{5.5}, the norms of these spaces are equivalent to the norms in the spaces defined using the metric $\bar{h}$ on $T_y(FM)$.

As $\bar{\phi}$ satisfies the eigenvalue equation, the lifted function $\bar{\phi} := (\exp_y)^*\bar{\phi}$ satisfies
\[
\Delta_{\hat{h}} \bar{\phi} = \lambda \bar{\phi} \quad \text{in } B(R_0). \tag{5.6}
\]
Let us consider radii $R_k = (1 - k/(8\ell))R_0$ and $q_k$ given by $1/q_k = 1/2 - k/(2\ell)$, $k = 0, 1, \ldots, \ell$. Then in the harmonic coordinates above, the standard local elliptic regularity estimates, see e.g. [26], yield that

$$\|\hat{\phi}\|_{W^{2,q_k}(B(R_k))} \leq c_{k,\ell}(1 + \lambda^2)^{1/2}\|\hat{\phi}\|_{W^{2,q_k-1}(B(R_{k-1}))},$$

(5.7)

where the constant $c_{k,\ell} > 0$ is uniform for $FM \in \mathcal{MM}$. Combining the estimates (5.7), for $k = 0, 1, \ldots, \ell$, together with the Sobolev embedding theorem, we obtain that

$$\|\hat{\phi}\|_{C^{0,1}(B(R_{k-2}/2))} \leq C(\ell)(1 + \lambda^2)^{(\ell)/2}\|\hat{\phi}\|_{L^2(B(R_{k-1}))}.$$ 

(5.8)

By [40] Lemma 1.2, the right-hand side $\|\hat{\phi}\|_{L^2(B(R_0))}$ of (5.8) is controlled by $\|\hat{\phi}\|_{L^2(FM,\mu_{FM})}$, namely

$$\|\hat{\phi}\|_{L^2(B(r),dV_h)}^2 \leq \frac{V_h(\hat{B}(4r))}{V_h(B(y,r))} \int_{B(y,r)} |\hat{\phi}|^2 dV_h \leq \frac{V_h(\hat{B}(4r))}{V_h(B(y,r))} \int_{FM} |\hat{\phi}|^2 d\mu_{FM}. $$

(5.9)

Recall that $d\mu_{FM} = dV_h/V_h(FM)$ is the normalized Riemannian volume. The volume $V_h(\hat{B}(4r))$ is bounded above due to (5.5), and $V_h(B(y,r))/V_h(FM)$ is bounded below due to the Bishop-Gromov volume comparison theorem.

At last, observe that for any Lipschitz function $u$ on $FM$, it satisfies that

$$\|u\|_{C^{0,1}(FM)} \leq \max_{y \in FM} \|\exp_y u\|_{C^{0,1}(B(R_{k}/2))}, $$

(5.10)

which concludes the proof. \hfill \Box

**Lemma 5.5.** There are $C_F > 0$ such that, for any $(X,\mu_X) \in \mathcal{MM}$ and $(Y,\mu_Y) \in \mathcal{MM}$, the estimate (5.4) remains valid for the eigenfunctions $\phi$ and $\bar{\phi}$, correspondingly.

**Proof.** It suffices to consider $Y \in \mathcal{MM} \setminus \mathcal{MM}$. Let $(Y,\mu_Y)$ be the limit of $(FM_k,\bar{\mu}_k) \in \mathcal{MM}$ in the measured GH topology and let

$$\varepsilon_k = d_{\text{min GH}}(Y,FM_k) \to 0, \text{ as } k \to \infty.$$ 

Denote by $\tilde{f}_k : FM_k \to Y$ a regular fibration of $Y$ which enjoys $\varepsilon$-approximation properties (1.7). Let $\tilde{\phi}$ be an eigenfunction of $-\Delta_Y$ with $\|\tilde{\phi}\|_{L^2(Y)} = 1$, corresponding to an eigenvalue $\lambda < E$. By [26], there exists $\delta_k > 0$ such that $\lim_{k \to \infty} \delta_k = 0$ and there are

$$\bar{\phi}_k \in \bar{L}_k(\lambda - \delta_k, \lambda + \delta_k), \quad \|\bar{\phi}_k\|_{L^2(FM_k)} = 1,$$

(5.11)

satisfying

$$\|\tilde{f}_k(\tilde{\phi}) - \bar{\phi}_k\|_{L^2(FM_k)} < \delta_k.$$ 

(5.12)

Here and later, for $a < b$, $a,b \notin \text{spec}(-\Delta_{FM_k})$, we denote by $\bar{L}_k(a,b) = \bar{L}_{FM_k}(a,b)$ the combination of the eigenfunctions of $-\Delta_{FM_k}$ corresponding to the eigenvalues in the interval $(a,b)$. 

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Let \( \tilde{f}'_k \) be a (non-necessary continuous) right inverse to \( \tilde{f}_k \), i.e., \( \tilde{f}_k \circ \tilde{f}'_k = \text{id}_Y \). Denote
\[
\tilde{\phi}'_k(y) = \tilde{\phi}_k(\tilde{f}'_k(y)), \quad y \in Y.
\] (5.13)

Observe that
\[
\| \tilde{f}'_k(\tilde{\phi}_k) - \tilde{\phi}_k \|_{L^\infty(FM_k)} \leq c(1 + E^2)^s/2 \delta_k,
\] (5.14)
where \( c = c(E), s = s(E) \) are uniform on \( \mathcal{M} \). Indeed, for any \( y_k, y'_k \in \tilde{f}'_k(\tilde{\phi}_k) \) formula (1.7) yields \( d_{FM_k}(y_k, y'_k) \leq c \delta_k \). This, together with the uniform Lipschitz continuity (5.4) of functions \( \tilde{\phi}_{FM_k} \), yields the inequality (5.14), which in particular gives
\[
\| \tilde{f}'_k(\tilde{\phi}_k) - \tilde{\phi}_k \|_{L^2(FM_k)} \leq c(1 + E^2)^s/2 \delta_k.
\]
Combining this with (5.12) yields
\[
\| \tilde{f}'_k(\tilde{\phi}_k - \tilde{\phi}) \|_{L^2(FM_k)} \leq c(1 + E^2)^s/2(\delta_k + \varepsilon_k).
\] (5.15)

On the other hand, there exists a subsequence, \( k = k(p) \), and a function \( \tilde{\phi}' \in C^{0,1}(Y) \) such that, for any \( y \in Y \),
\[
\lim_{k \to \infty} \tilde{\phi}'_k(y) = \tilde{\phi}'(y).
\]
Indeed, choosing a dense subset \( \{ y_p \}_{p=1}^\infty \subset Y \) and using the diagonalization procedure we find \( k = k(n), k(n) \to \infty \) as \( n \to \infty \) such that for all \( p \in \mathbb{Z}_+ \) there exist limits
\[
\tilde{\phi}'(y_p) = \lim_{n \to \infty} \tilde{\phi}'_{k(n)}(y_p).
\]

Using the uniform Lipschitz bound (5.4), the estimate \( d_{FM_k}(y_k, y'_k) \leq c \varepsilon_k \), for \( y_k, y'_k \in \tilde{f}'_k(\tilde{\phi}_k) \), and the definition of the GH distance, we see that
\[
|\tilde{\phi}'_{k(n)}(y) - \tilde{\phi}'_{k(n)}(y')| \leq c(1 + E^2)^s/2 \left( \varepsilon_{k(n)} + d_Y(y, y') \right), \quad y, y' \in Y.
\] (5.16)

Thus, we extend \( \tilde{\phi}' \) from \( \{ y_p : p \in \mathbb{Z}_+ \} \subset Y \) to the whole space \( Y \) so that
\[
|\tilde{\phi}'(y) - \tilde{\phi}'(y')| \leq c(1 + E^2)^s/2d_Y(y, y'), \quad \text{for all } y, y' \in Y.
\] (5.17)

This shows that \( \tilde{\phi}' \) satisfies the desired estimate and we need to show \( \tilde{\phi}' = \tilde{\phi} \).

From (5.16) and (5.17), one can show that the pointwise convergence of the subsequence \( \tilde{\phi}'_k \) to \( \tilde{\phi}' \) is uniform, i.e.,
\[
\lim_{k \to \infty} \| \tilde{\phi}'_k - \tilde{\phi}' \|_{L^\infty(Y)} = 0,
\] (5.18)
and consequently \( \lim_{k \to \infty} \| \tilde{f}'_k(\tilde{\phi}_k - \tilde{\phi}') \|_{L^\infty(FM_k)} = 0 \). Hence, together with (5.15), we have
\[
\lim_{k \to \infty} \| \tilde{f}'_k(\tilde{\phi}' - \tilde{\phi}) \|_{L^2(FM_k)} = 0,
\]
which gives \( \| \tilde{\phi}' - \tilde{\phi} \|_{L^2(Y)} = 0 \) by definition (2.9). Hence \( \tilde{\phi}' = \tilde{\phi} \) since they are both continuous.

At last, due to (A.6) and (4.5), the estimate for \( \tilde{\phi}' \) follows from that for \( \tilde{\phi} \).
Similarly, denote by $\tilde{L}_Y(a, b)$, for $Y \in FMM_p$, the li combination of the eigenfunctions of $-\Delta_Y$ corresponding to the eigenvalues in the interval $(a, b)$.

**Corollary 5.6.** Let $\tilde{u} \in \tilde{L}_Y(a, b)$, $a, b \notin \text{spec}(-\Delta_Y)$ or $u \in L_X(a, b)$, $a, b \notin \text{spec}(-\Delta_X)$. Then (5.4) remains valid for $\tilde{u}$, $u$ (with $a, b$ instead of $\lambda$).

### 5.2 Spectral convergence on $FMM_p$ and $\mathcal{M}_p$

Let $(X_k, \mu_k) \in \mathcal{M}_p$ be a sequence converging to $(X, \mu_X)$ in the measured Gromov-Hausdorff topology. Then as stated in Theorem 2.4 there exist Riemannian manifolds $(Y_k, \bar{\mu}_k) \in \mathcal{M}_p$ converging to $(Y, \bar{\mu}_Y)$ in the measured Gromov-Hausdorff topology, and regular $\varepsilon_k$-approximations $\tilde{f}_k : Y_k \to Y$, $f_k : X_k \to X$, such that $X_k = Y_k/O(n)$, $X = Y/O(n)$, and $\tilde{f}_k$ are $\varepsilon_k$-Riemannian submersions.

For $a < b$, $a, b \notin \text{spec}(-\Delta_Y)$, we denote by $\bar{L}_Y(a, b)$ the li combinations of the eigenfunctions of $-\Delta_Y$ corresponding to eigenvalues in the interval $(a, b)$, and define similarly $L_X(a, b)$ as the li combinations of the eigenfunctions of $-\Delta_X$. Using pullback notations, denote

$$\bar{L}_k^*(a, b) = \tilde{f}_k^*(\tilde{L}_Y(a, b)) \subset C^{0,1}(Y_k), \quad (5.19)$$

$$\bar{L}_k^*(a, b) = f_k^*(L_X(a, b)) \subset C^{0,1}(X_k). \quad (5.20)$$

Similarly, denote by $\bar{L}_k(a, b), L_k(a, b)$ the li combinations of the eigenfunctions of the Laplacian on $Y_k, X_k$, respectively.

Assuming $a, b \notin \text{spec}(-\Delta_Y)$. In the case of $Y_k = FM_k \in \mathcal{M}_p$, it follows from [26] that for large $k$, we have $a, b \notin \text{spec}(-\Delta_{FM_k})$, and

$$d_{L^2(FM_k)}(B(\bar{L}_k^*(a, b)), B(\bar{L}_k(a, b))) \to 0, \quad \text{as } k \to \infty, \quad (5.21)$$

with respect to the normalized Riemannian measure on $FM_k$, where $B(Z)$ stands for the unit ball in $Z$, and $d_{L^2(FM_k)}$ is the Hausdorff distance in $L^2(FM_k)$.

**Lemma 5.7.** In the case of $Y_k = FM_k \in \mathcal{M}_p$, the convergence in (5.21) is valid in $C^\alpha(FM_k)$, for any $0 \leq \alpha < 1$.

**Proof.** We start with the case $\alpha = 0$. Assume the convergence does not hold in $C^0(FM_k)$. Then there exists $\bar{\delta} > 0$ such that, for any $k$, either there exists $\tilde{u}_k \in B(\bar{L}_k(a, b))$ such that

$$\|\tilde{u}_k - \bar{v}_k\|_{C(FM_k)} > \bar{\delta}, \quad \text{for all } \bar{v}_k \in B(\bar{L}_k^*(a, b)), \quad (5.22)$$

or there exists $\bar{v}_k \in B(\bar{L}_k^*(a, b))$ such that

$$\|\bar{v}_k - \tilde{u}\|_{C(FM_k)} > \bar{\delta}, \quad \text{for all } \tilde{u} \in B(\bar{L}_k(a, b)).$$

Without loss of generality we consider the former situation.
On the other hand, by (5.21), for all \( k \) there exists \( \tilde{v}_k^* \in \mathcal{B}(\tilde{L}_k^*(a, b)) \) such that
\[
\| \tilde{u}_k - \tilde{v}_k^* \|_{L^2(FM_k)} \to 0, \quad \text{as } k \to \infty. \tag{5.23}
\]
However, by (5.22), there exists \( y_k \in FM_k \) such that \( |\tilde{u}_k(y_k) - \tilde{v}_k^*(y_k)| > \tilde{\delta} \). Since functions in \( \mathcal{B}(L_k(a, b)) \) and \( \mathcal{B}(\tilde{L}_k^*(a, b)) \) are uniformly bounded in \( C^{0,1} \)-norm due to (5.4) and (2.8), one can find a neighborhood \( \tilde{U}_k \) of \( y_k \) with uniform radius such that \( |\tilde{u}_k(y) - \tilde{v}_k^*(y)| > \tilde{\delta}/2 \) for all \( y \in \tilde{U}_k \). This implies that \( \| \tilde{u}_k - \tilde{v}_k^* \|_{L^2(FM_k)} \) is uniformly bounded away from 0, which is a contradiction to (5.23). This proves the convergence in \( C^0 \).

Due to (5.4) and (2.8),
\[
d_{C^{0,1}(FM_k)}\left( \mathcal{B}(\tilde{L}_k^*(a, b)), \mathcal{B}(\tilde{L}_k(a, b)) \right) \quad \text{is bounded. Then standard interpolation gives the convergence in } C^\alpha. \tag{5.24}
\]

The following lemma deals with the general case \( Y_k \in \mathcal{F} \mathcal{M}_p \).

**Lemma 5.8.** 1. Assume \( Y \in \mathcal{F} \mathcal{M}_p, a, b \notin \text{spec}(-\Delta_Y) \), and \( Y_k \in \mathcal{F} \mathcal{M}_p \) converge to \( Y \) in the measured GH-topology. Then for any \( 0 \leq \alpha < 1 \),
\[
d_{C^\alpha(Y_k)}\left( \mathcal{B}(\tilde{L}_k^*(a, b)), \mathcal{B}(\tilde{L}_k(a, b)) \right) \to 0, \quad \text{as } k \to \infty. \tag{5.25}
\]

2. Assume \( X \in \mathcal{F} \mathcal{M}_p, a, b \notin \text{spec}(-\Delta_X) \) and \( X_k \in \mathcal{F} \mathcal{M}_p \) converge to \( X \) in the measured GH-topology. Then for any \( 0 \leq \alpha < 1 \),
\[
d_{C^\alpha(X_k)}\left( \mathcal{B}(\tilde{L}_k^*(a, b)), \mathcal{B}(\tilde{L}_k(a, b)) \right) \to 0, \quad \text{as } k \to \infty. \tag{5.26}
\]

**Proof.** 1. As earlier, we start with the case \( \alpha = 0 \). Assuming the opposite, there is \( \tilde{\delta} > 0 \) such that
\[
d_{C(Y_k)}\left( \mathcal{B}(\tilde{L}_k^*(a, b)), \mathcal{B}(\tilde{L}_k(a, b)) \right) > \tilde{\delta}, \quad \text{for all } k. \tag{5.27}
\]
Approximate, in the measured GH-topology, \( (Y_k, \tilde{\mu}_Y^k) \) by \( (FM_k, \tilde{\mu}_k) \in \mathcal{F} \mathcal{M}_p \). Let \( \Delta_{kk} \) stand for the Laplacian on \( FM_k \). Denote by \( \tilde{L}_kk(a, b) \subset L^2(FM_k) \) the eigenspace of the Laplacian \( -\Delta_{kk} \) on \( FM_k \) corresponding to the eigenvalues in \( (a, b) \), and by
\[
\tilde{L}_kk(a, b) := f_{kk}^* \left( \tilde{L}_k(a, b) \right),
\]
where \( f_{kk} : FM_k \to Y_k \) is the regular approximation described in Theorem 2.4 with \( \bar{\varepsilon}_k \) in (2.8) changed into \( \varepsilon_{kk} \). Moreover, we can assume that
\[
d_{L^2(FM_k)}\left( \mathcal{B}(\tilde{L}_kk(a, b)), \mathcal{B}(\tilde{L}_k(a, b)) \right) < \frac{1}{k}. \tag{5.28}
\]
Let \( \bar{f}_k : Y_k \to Y \) be a \( \bar{\varepsilon}_k \)-Riemannian submersion, satisfying (2.8) and providing the measured GH-convergence of \( Y_k \) to \( Y \), which is guaranteed by Theorem 2.4. We can assume \( \bar{f}_k \) to be an \( O(n) \)-map.
Consider
\[ \tilde{f}_{kk} = \tilde{f}_k \circ f_{kk} : FM_k \to Y, \]
which provides a regular \( \tilde{\varepsilon}_{kk} \)-Riemannian submersion of \( FM_k \) to \( Y \), \( \tilde{\varepsilon}_{kk} \to 0 \), as \( k \to \infty \). Therefore,
\[
\lim_{k \to \infty} d_{L^2(FM_k)} \left( \mathcal{B}(\mathcal{L}_{kk}(a, b)), \mathcal{B}(\mathcal{L}_{kkk}^*(a, b)) \right) = 0,
\]
where \( \mathcal{L}_{kk}(a, b) \) is the eigenspace of \( -\Delta_{kk} \) on \( FM_k \) corresponding to the interval \( (a, b) \), and
\[
\mathcal{L}_{kkk}^*(a, b) := \tilde{f}_{kk}^*(\mathcal{L}_Y(a, b)) = f_{kk}^*(\mathcal{L}_k^*(a, b)).
\] (5.28)
This, together with (5.27) implies that
\[
\lim_{k \to \infty} d_{L^2(FM_k)} \left( \mathcal{B}(\mathcal{L}_{kk}(a, b)), \mathcal{B}(\mathcal{L}_{kkk}^*(a, b)) \right) = 0.
\]
On the other hand, (5.26) together with definition (5.28) yields that, for large \( k \),
\[
d_{C(FM_k)} \left( \mathcal{B}(\mathcal{L}_{kk}(a, b)), \mathcal{B}(\mathcal{L}_{kkk}^*(a, b)) \right) > \delta/2.
\]
Then a similar argument as the proof of Lemma 5.7 shows that the above two inequalities lead to a contradiction. This proves (5.24) for \( \alpha = 0 \).

To obtain the result for any \( 0 \leq \alpha < 1 \), we use again the fact that, due to Lemma 5.5 and (2.8), functions in \( \mathcal{B}(\mathcal{L}_{k}(a, b)) \) and \( \mathcal{B}(\mathcal{L}_{k}^*(a, b)) \) are uniformly bounded in \( C^{0,1}(Y_k) \). Then the convergence in \( C^\alpha \) follows from interpolation.

2. Since \( \tilde{f}_k \) is an \( O(n) \)-map, (5.24) remains valid with \( O(n) \)-invariant spaces \( \mathcal{L}_{k, O}^*(a, b) \) and \( \mathcal{L}_{k, O}(a, b) \) instead of \( \mathcal{L}_k^*(a, b), \mathcal{L}_k(a, b) \). Denote \( \pi_k : Y_k \to X_k = Y_k/O(n) \). Then (5.24) for the \( O(n) \)-invariant functions, together with (A.3) and (A.4) proves (5.25). \( \square \)

**Remark 5.9.** If \( X = \lim_{p \to GH} X_k = \{ \text{point}, 1 \} \), then \( L^2(X) = \mathbb{R} \) and \( \Delta_X = 0 \). Thus, the only eigenvalue is \( \lambda_0 = 0 \) with the corresponding eigenfunction \( 1 \). Due to (5.2), (5.25) remains trivially valid with, when \( k \) is sufficiently large, with \( \mathcal{L}_k(a, b), \mathcal{L}_k^*(a, b) \) consisting of constant functions, if \( 0 \in (a, b) \), and of only 0-function if \( 0 \notin (a, b) \).

Denote by \( f_k' : X \to X_k \) the almost right inverse to \( f_k \), i.e., \( f_k, f_k' \) satisfy (1.7) (see e.g. [26, Lemma 2.5]), and for \( x_k \in X_k, x \in X \),
\[
d_k(x_k, f_k' \circ f_k(x_k)), d_X(x, f_k \circ f_k'(x)) \to 0, \quad \text{as } k \to \infty.
\] (5.29)

Then using (5.25) and the uniform \( C^{0,1} \)-boundedness of \( \mathcal{B}(\mathcal{L}_k(a, b)), \mathcal{B}(\mathcal{L}(a, b)) \), we obtain

**Corollary 5.10.** Let \( (X_k, \mu_k), (X, \mu_X) \in \mathcal{MM}_p \) be such that \( (X_k, \mu_k) \) converges to \( (X, \mu_X) \) in the measured GH-topology. Then there exists \( \sigma_k \to 0 \), as \( k \to \infty \), such that
\[
d_{L^\infty(X)}(\mathcal{B}(\mathcal{L}_k^*(a, b)), \mathcal{B}(\mathcal{L}(a, b))) < \sigma_k,
\] (5.30)
where
\[ \widetilde{L}^*_k(a, b) = (f^*_k)^*(\mathcal{L}_k(a, b)). \]

Moreover, there is \( c > 0 \), such that if \( d_X(x, x') < \sigma_k \) then for all \( u^*_k \in \mathcal{B}(\widetilde{L}^*_k(a, b)) \) we have
\[ |u^*_k(x) - u^*_k(x')| < c \sigma_k. \] \hspace{1cm} (5.31)

Proof. The first claim follows from the uniform \( C^{0,1} \)-estimate in \( \mathcal{L}_X(a, b) \) together with (5.25) for \( \alpha = 0 \) and (5.29). To prove (5.31) we also use the uniform \( C^{0,1} \)-estimate in \( \mathcal{L}_X(a, b) \) together with (5.30). \( \square \)

Our next goal is to obtain continuity of the eigenfunctions in \( L^\infty \)-norm, with respect to the measured GH distance which is uniform on \( \mathcal{M}_p \).

**Definition 5.11.** Let \( (a_\ell, b_\ell), \ell = 1, \ldots, L, \) be a finite collection of open intervals, and \( d_\ell, \ell = 1, \ldots, L, \) be positive numbers. Denote the set \( (a_\ell, b_\ell, d_\ell)_{\ell=1}^L \) by \( \mathcal{I} \). Then we define
\[ \mathcal{M}_\mathcal{I} = \left\{ X \in \mathcal{M}_p : d(\text{spec}(X), \{a_\ell\}) \geq d_\ell \text{ and } d(\text{spec}(X), \{b_\ell\}) \geq d_\ell \right\}. \]

Note that \( \mathcal{M}_\mathcal{I} \) is closed and thus compact with respect to the measured GH-topology.

**Corollary 5.12.** Let \( \mathcal{I} = (a_\ell, b_\ell, d_\ell)_{\ell=1}^L \) and \( \mathcal{J} = (a_\ell, b_\ell, d'_\ell)_{\ell=1}^L \), \( d_\ell > d'_\ell \). Then, for any \( \varepsilon > 0 \) there is \( \sigma = \sigma_{\mathcal{I}, \mathcal{J}}(\varepsilon) > 0 \) such that, if \( X \in \mathcal{M}_\mathcal{I} \) and \( d_{mGH}(X, X') < \sigma \), then \( X' \in \mathcal{M}_\mathcal{J} \) and \( X, X' \) satisfy
\[ \dim(\mathcal{L}_X(a_\ell, b_\ell)) = \dim(\mathcal{L}_{X'}(a_\ell, b_\ell)) := n(\ell). \]

Moreover, there are measurable maps \( f : X \to X', f' : X' \to X, \) which are \( \varepsilon \)-approximations satisfying (1.7) and (5.29), such that
\[ d_{L^\infty(X)} \left( \mathcal{B}(f^*(\mathcal{L}_{X'}(a_\ell, b_\ell))), \mathcal{B}(\mathcal{L}_X(a_\ell, b_\ell)) \right) < \varepsilon, \]
\[ d_{L^\infty(X')} \left( \mathcal{B}((f')^*(\mathcal{L}_X(a_\ell, b_\ell))), \mathcal{B}(\mathcal{L}_{X'}(a_\ell, b_\ell)) \right) < \varepsilon. \] \hspace{1cm} (5.32)

In addition, for any \( \ell = 1, \ldots, L, \) if \( \phi_{i, \ell}, i = 1, \ldots, n(\ell), \) and \( \phi'_{i, \ell}, i = 1, \ldots, n(\ell), \) form an orthonormal eigenfunction basis in \( \mathcal{L}_X(a_\ell, b_\ell), \mathcal{L}_{X'}(a_\ell, b_\ell), \) correspondingly, then there exist orthogonal matrices
\[ U_\ell = [u_{ij}]_{i,j=1}^{n(\ell)} \in O(n(\ell)), \quad U'_\ell = [u'_{ij}]_{i,j=1}^{n(\ell)} \in O(n(\ell)), \]

such that \( \phi^*_i \equiv \sum_{j=1}^{n(\ell)} u_{ij} f^*(\phi'_{j, \ell}) \) satisfy
\[ \| \phi_{i, \ell} - \phi^*_i \|_{L^\infty(X)} < \varepsilon, \quad i = 1, \ldots, n(\ell), \ell = 1, \ldots, L. \] \hspace{1cm} (5.33)

Similar result is valid for \((f')^*(\phi_{i, \ell})\) if we use \( U'_\ell \).
Proof. By compactness arguments, the first statement of the corollary follows immediately from Theorem 0.4(A) in [26]. To prove the second statement, in particular, (5.32), assume that there are $X_k$, $\hat{X}_k$ and $\delta_k \to 0$ satisfying $d_{mGH}(X_k, \hat{X}_k) < \delta_k$, but (5.32) is not valid. Without loss of generality, we can assume that $X_k$, $\hat{X}_k \to X$, with respect to the measured GH-topology, and $f_k : X_k \to \hat{X}_k$, $f_k' : \hat{X}_k \to X_k$ which provide $\delta_k$-approximations, are of the form

$$f_k = \hat{f}_{kk} \circ f_{kk} \quad f_k' = f_{kk} \circ \hat{f}_{kk},$$

where

$$f_{kk} : X \to X_k, \quad f_{kk}' : X_k \to X$$

and \(\hat{f}_{kk} : X \to \hat{X}_k, \hat{f}_{kk}' : \hat{X}_k \to X,\)

provide $\delta_k/4$-approximations, in the sense of (1.7), of $X, X_k$ and $\hat{X}_k, \hat{X}$, correspondingly. Note that, if $f_{kk}, f_{kk}'$ and $\hat{f}_{kk}, \hat{f}_{kk}'$ are $\delta_k/4$-almost inverse of each other in the sense of (5.29), then $f_k, f_k'$ are $\delta_k$-almost inverse.

By (5.30) and the triangle inequality, we have

$$d_{L^\infty(X)} \left(B(f_k^*(\mathcal{L}_k(a, b))) \right) \to 0, \quad k \to \infty,$$

where $\mathcal{L}_k(a, b))$ stand for $\mathcal{L}_{X_k}(a, b)$, $\hat{\mathcal{L}}_{\hat{X}_k}(a, b)$. Then, using the facts that

$$f_k^* = (\hat{f}_{kk} \circ f_{kk})^* = (f_{kk} \circ \hat{f}_{kk})^*,$$

and $f_{kk}, f_{kk}'$ are almost inverse of each other, and the $C^{0,1}$-boundedness of eigenfunctions, one can show that

$$d_{L^\infty(X_k)} \left(\mathcal{B}(\mathcal{L}_k(a, b)), \mathcal{B}(f_k^*(\hat{\mathcal{L}}_k(a, b))) \right) \to 0 \quad k \to \infty.$$

To prove the last statement of the corollary, recall that, due to the measure convergence in (1.8) and Lemma 5.5 $f^*(\phi^*_{j, \ell}), j = 1, \ldots, n(\ell),$ satisfy

$$\int_X f^*(\phi^*_{j, \ell}) f^*(\phi^*_{i, \ell}) d\mu_X = \delta_{ij}, \quad \text{as } d_{mGH}(X, X') \to 0.$$

Together with (5.32), this implies the existence of $U_\ell = U_\ell(X, X'), \ell = 1, \ldots, L,$ such that for $i = 1, \ldots, n(\ell),$

$$\|\phi_{i, \ell} - \phi^*_{i, \ell}\|_{L^\infty(X)} \to 0, \quad \text{as } d_{mGH}(X, X') \to 0,$$

which yields (5.33). \(\square\)

### 5.3 Heat kernel convergence

In this subsection, we consider the continuity of the heat kernel $H_X(x, y, t)$, with respect to the measured GH-convergence on $\mathfrak{M}_p(n, \Lambda, D)$. Recall that the heat kernel on $X$ can be written in terms of eigenfunctions and eigenvalues as

$$H_X(x, y, t) = \sum_{j=0}^\infty e^{-\lambda_j t} \phi_j(x)\phi_j(y), \quad (5.34)$$

where the eigenfunctions $\phi_j, j = 0, 1, \ldots,$ form an orthonormal basis in $L^2(X, \mu_X)$ and $\lambda_0 = 0$ and $\phi_0 = 1$. 

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Lemma 5.13. There exist constants $C > 0$ and $s_F > 0$, as introduced in \((5.4)\), such that, for any $E \geq 1$ and $X \in \mathbb{W}^p(n, \Lambda, D)$, we have

$$
|H_X(x, y, t) - \sum_{\lambda_j < E} e^{-\lambda_j t} \phi_j(x)\phi_j(y)| \leq Ct^{-(2s_F + \frac{p}{2} + 1)}e^{-\frac{1}{2}Et}, \quad t > 0. \quad (5.35)
$$

Proof. For any $x, y \in X$, using the $C^0$-norm estimate in Lemma 5.5 and (5.2), we have

$$
\left| \sum_{\lambda_j \geq E} e^{-\lambda_j t} \phi_j(x)\phi_j(y) \right| \leq \sum_{\lambda_j \geq E} e^{-\lambda_j t} |\phi_j(x)||\phi_j(y)|
$$

$$
\leq C^2 \sum_{\lambda_j \geq E} e^{-\lambda_j t}(1 + \lambda_j^{2s_F})
$$

$$
\leq C \sum_{\lambda_j \geq E, |\lambda_j - \lambda_k| \geq 1} e^{-\lambda_j t}\lambda_j^{2s_F + \frac{p}{2}},
$$

where $C_F, s_F$ are the same constants as in \((5.4)\). It is straightforward to verify that, for any $t > 0$,

$$
e^{-\frac{1}{2}t}\lambda_j^{2s_F + \frac{p}{2}} \leq C(n, s_F)t^{-(2s_F + \frac{p}{2})}, \quad \text{for any } \lambda \geq 0. \quad (5.36)
$$

Without loss of generality, we can take $E$ to be an integer. Then,

$$
\left| \sum_{\lambda_j \geq E} e^{-\lambda_j t} \phi_j(x)\phi_j(y) \right| \leq Ct^{-(2s_F + \frac{p}{2})} \sum_{\lambda_j \geq E, |\lambda_j - \lambda_k| \geq 1} e^{-\frac{1}{2}\lambda_j t}
$$

$$
\leq Ct^{-(2s_F + \frac{p}{2})} \sum_{j=E}^{\infty} e^{-\frac{1}{2}jt}
$$

$$
= Ct^{-(2s_F + \frac{p}{2})} \frac{e^{-\frac{1}{2}Et}}{1 - e^{-\frac{1}{2}t}} \leq Ct^{-(2s_F + \frac{p}{2} + 1)}e^{-\frac{1}{2}Et}.
$$

As a consequence, we obtain an estimate for the continuity of the heat kernel in a particular space $X \in \mathbb{W}^p(n, \Lambda, D)$.

Corollary 5.14. Let $X \in \mathbb{W}^p(n, \Lambda, D)$ and $0 < \hat{\varepsilon} < 1$. Then there exists a constant $C > 0$ such that, for any $x, y, x', y' \in X$ and $t, t' > \hat{\varepsilon}$, we have

$$
|H_X(x, y, t) - H_X(x', y', t')| < C\hat{\varepsilon}^{-(2s_F + \frac{p}{2} + 2)}(d_X(x, x') + d_X(y, y') + |t - t'|). \quad (5.37)
$$

Proof. For the part of the eigenfunction expansion of the heat kernel with $\lambda_j \geq 1$, one can follow the same argument in Lemma 5.13 with $E = 1$ using the Lipschitz norm estimate in Lemma 5.5. The additional order in $\hat{\varepsilon}^{-(2s_F + \frac{p}{2} + 2)}$ comes from the derivative of $e^{-\lambda_j t}$ with respect to $t$. The lower part of the eigenfunction expansion with $\lambda_j < 1$ is clearly Lipschitz with bounded Lipschitz constant by Lemma 5.5 and (5.2). \qed
We now prove the continuity of the heat kernel with respect to the measured Gromov-Hausdorff topology.

**Theorem 5.15.** For any \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) > 0 \) such that, if \( X, X' \in \mathcal{G} \) satisfy \( d_{mGH}(X, X') < \delta \), then there exist \( \varepsilon \)-nets \( \{x_i\}_{i=1}^{I(\varepsilon)} \subset X \) and \( \{x_i'\}_{i=1}^{I(\varepsilon)} \subset X' \) satisfying \( |d_X(x_i, x_k) - d_{X'}(x_i', x_k')| < \varepsilon \), and

\[
|H_X(x_i, x_k, t) - H_{X'}(x_i', x_k', t)| < \varepsilon, \quad \text{for } t \in (\varepsilon, \varepsilon^{-1}).
\] (5.38)

**Proof.** By Lemma 5.13, for sufficiently small \( \varepsilon \) and \( E = E(\varepsilon) = \varepsilon^{-2} \), the right-hand side of (5.38) is smaller than \( \varepsilon/4 \) for any \( X \in \mathcal{G} \) and any \( t > \varepsilon \).

By (5.2), for any \( X \in \mathcal{G} \), we can choose in the interval \((-1, \varepsilon^{-2} + 4\varepsilon^{\beta+n/2+1})\), where \( \beta > 0 \) is to be chosen later, the subintervals \( (a_{\ell}, b_{\ell}) \), \( \ell = 1, \ldots, L = L(X) \), satisfying the following properties:

(i) \( b_{\ell} - a_{\ell} \leq \varepsilon^{\beta} \),

(ii) \( a_{\ell+1} > b_{\ell} + 4\varepsilon^{\beta+n+1} \),

(iii) \( \text{spec}(-\Delta_X) \cap (-\infty, \varepsilon^{-2}) \subset \bigcup_{\ell=1}^{L} (a_{\ell}, b_{\ell}) \),

(iv) \( d(\text{spec}(-\Delta_X), a_{\ell}), d(\text{spec}(-\Delta_X), b_{\ell}) > 4\varepsilon^{\beta+n+1} \), \( \ell = 1, \ldots, L \).

Note that this system of subintervals, associated with \( X \), satisfies conditions of Definition 5.11 with \( \beta = 4\varepsilon^{\beta+n+1} \). Denote it by \( I = I(\varepsilon(X)) \) and consider \( \mathcal{G} \). Choose \( d' = 3\varepsilon^{\beta+n+1} \) to define the corresponding \( J(X) \). Then there is \( \tau = \tau(I(X), \varepsilon, \beta) \), such that, if

\[
X' \in U_{\tau,X}(\mathcal{G}_I) = \{ X' \in \mathcal{G} : d_{mGH}(\mathcal{G}_I, X') < \tau \},
\]

then \( X' \in \mathcal{G}_J(X) \). Clearly, \( \{U_{\tau,X} : X \in \mathcal{G} \} \) form an open covering of \( \mathcal{G} \).

Choose a finite subcovering, \( U_1, \ldots, U_N, N = N(\varepsilon) \). Observe that

\[
U_k \subset \mathcal{G}_J(X_k) \quad \text{for some } X = X_k.
\]

Then, due to Corollary 5.12, there is \( \sigma_k > 0 \) so that if \( d_{mGH}(X, X') < \sigma_k \), \( X \in U_k \), then equations (5.32), (5.33) are valid with \( \varepsilon = \varepsilon^n \) (and \( d' = 2\varepsilon^{\beta+n+1} \)). Since \( U_k = \mathcal{G} \), taking \( \sigma(\varepsilon) = \min \sigma_k \), we see that, for any \( X, X' \in \mathcal{G} \), if \( d_{mGH}(X, X') < \sigma(\varepsilon) \), equations (5.32), (5.33) are valid with \( \varepsilon = \varepsilon^n \). Moreover, there is \( k \) such that \( X, X' \in U_k \).

Consider

\[
\left| \sum_{\lambda_j < \varepsilon^{-2}} e^{-\lambda_j t} \phi_j(x) \phi_j(y) - \sum_{\lambda_j < \varepsilon^{-2}} e^{-\lambda_j' t} \phi_j'(f(x)) \phi_j'(f(y)) \right| \quad (5.39)
\]

and

\[
\left| \sum_{\ell=1}^{L} \left( \sum_{\lambda_j < \varepsilon} e^{-\lambda_j t} \phi_j(x) \phi_j(y) - \sum_{\lambda_j' < \varepsilon} e^{-\lambda_j' t} \phi_j'(f(x)) \phi_j'(f(y)) \right) \right|.
\]
where $f : X \to X'$, $f' : X' \to X$ are almost inverse $\sigma$-approximations. We analyze each term on the right side of (5.39) separately.

First, observe that for $\varepsilon < \varepsilon_1$, $\lambda_j, \lambda'_j \in (a_t, b_t)$,
\[
\max \{ |e^{-\lambda_j t} - e^{-a_t t}|, |e^{-\lambda'_j t} - e^{-a_t t}| \} \leq c(\beta)\varepsilon^{(\beta-1)} e^{-a_t t}.
\]
Using Lemma 5.5 together with (5.2), we see from the previous equation that replacing $\beta$ term on the right side of (5.39) separately.

Next, observe that
\[
\begin{align*}
| \sum_{t=1}^{L} \sum_{\ell=1}^{L} \left( e^{-\lambda_j t} - e^{-a_t t} \right) \phi_j(x)\phi_j(y) \\
- \sum_{t=1}^{L} \sum_{\ell=1}^{L} \left( e^{-\lambda'_j t} - e^{-a_t t} \right) \phi'_j(f(x))\phi'(f(y)) | < \frac{\varepsilon}{16}.
\end{align*}
\]
This equality, together with Lemma 5.5 and (1.7) with $\varepsilon = \varepsilon^\beta$ implies that, that there is $0 < \varepsilon < \varepsilon_1$, such that for $\varepsilon < \varepsilon_2$,
\[
|H_X(x, y, t) - H_{X'}(f(x), f(y), t)| \\
\leq \frac{1}{2} \varepsilon + \sum_{t=1}^{L} \sum_{i=1}^{n(t)} e^{-a_t t} | \phi_i(x)\phi_i(y) - \phi^*_i(x)\phi^*_i(y) |.
\]
Therefore, if equation (5.38) is valid with $\varepsilon = \varepsilon^\beta$, there is $\varepsilon_3 < \varepsilon_2$ such that for $\varepsilon < \varepsilon_3$,
\[
|H_X(x, y, t) - H_{X'}(f(x), f(y), t)| < \varepsilon, \quad \text{for } t \in (\varepsilon, \varepsilon^{-1}).
\]
At last, choosing $\{x_i\}_{i=1}^{I(\varepsilon)} \subset X$ to be an $\varepsilon/2$-net in $X$, we see from (1.7) that, for $\delta < \varepsilon(\varepsilon)$, the set $\{f(x_i)\}_{i=1}^{I(\varepsilon)} \subset X$ is $\varepsilon$-dense in $X'$. Due to the compactness of $\mathcal{MM}_p$.

6 From the local spectral data to the metric-measure structure

In this section, we consider the inverse spectral problem on $\mathcal{MM}_p$. 

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**Corollary 6.3.** Let \((X, \mu) \in \mathcal{M}_X\), and let \(\Omega \subset X\) be open and non-empty. The set 
\[
(\Omega, (\lambda_k)_{k=0}^{\infty}, (\phi_k|_{\Omega})_{k=0}^{\infty}),
\]
where \(\phi_k\) is orthonormalized in \(L^2(X, \mu)\), is called the local spectral data (LSD) for \(X\) on \(\Omega\).

Moreover, suppose that \((\Omega, (\lambda_k)_{k=1}^{\infty}, (\phi_k|_{\Omega})_{k=0}^{\infty})\) and \((\Omega', (\lambda_k)_{k=0}^{\infty}, (\phi_k|_{\Omega'})_{k=0}^{\infty})\) are the LSD of \((X, \mu)\) and \((X', \mu')\), respectively. We say that these data are equivalent by a map \(\Psi_{\Omega} : \Omega \rightarrow \Omega'\), if \(\Psi_{\Omega}\) is a homeomorphism, and
\[
\lambda_k = \lambda'_{k}, \quad \Psi_{\Omega}(\phi_k|_{\Omega'}) = \phi_k|_{\Omega}, \quad \text{for all } k \in \mathbb{N}.
\]

Our aim is to prove the uniqueness result.

**Theorem 6.2.** Assume that the LSD of \((X, \mu), (X', \mu') \in \mathcal{M}_X\) are equivalent by a homeomorphism \(\Psi_{\Omega} : \Omega \rightarrow \Omega'\). Then there exists a measure-preserving isometry \(\Psi : X \rightarrow X'\) such that \(\Psi|_{\Omega} = \Psi_{\Omega}\), i.e.,
\[
d_{X'}(\Psi(x), \Psi(y)) = d_X(x, y), \quad x, y \in X, \quad \mu = \Psi^*(\mu').
\]
Moreover, \(\Psi|_{X^{\text{reg}}} : X^{\text{reg}} \rightarrow (X')^{\text{reg}}\) is a \(C^3\)-Riemannian isometry.

**Corollary 6.3.** Let \((X, p, \mu), (X', p', \mu') \in \mathcal{M}_X\). Let \(z_\alpha, z'_\alpha, \alpha = 0, 1, \ldots,\) be dense in the ball \(B_X(p, r), B_{X'}(p', r)\) of some radius \(r > 0\) with \(z_0 = p, z'_0 = p'\), respectively, and \(t_\ell, \ell = 1, 2, \ldots,\) be dense in \((0, \infty)\). Assume that
\[
H_X(z_\alpha, z_\beta, t_\ell) = H_{X'}(z'_\alpha, z'_\beta, t_\ell), \quad \text{for all } \alpha, \beta \in \mathbb{N}, \ell \in \mathbb{Z}_+.
\]
Then the conclusion of Theorem 6.2 is valid and, in addition, \(\Psi(p) = p'\).

First, we prove the corollary using Theorem 6.2.

**Proof of Corollary 6.3.** By Lemma 4.7, the conditions of the corollary imply that \(\lambda_j = \lambda'_j, \phi_j(z_\alpha) = \phi'_j(z'_\alpha)\) for all \(j, \alpha \in \mathbb{N}\), where the last equality is considered modulus an orthogonal transformation in \(L(\lambda_j)\) and, without loss of generality, we assume this transformation to the identity. Let us denote \(B = B_X(p, r)\) and \(B' = B_{X'}(p', r)\). By taking closure in the product topology of \(\mathbb{R}^N\) of the set \(\Phi(z_\alpha) = \Phi'(z'_\alpha), \alpha = 0, 1, \ldots,\), where
\[
\Phi(x) = (\phi_j(x))_{j=0}^{\infty}, \quad \Phi'(x') = (\phi'_j(x'))_{j=0}^{\infty},
\]
we obtain the images \(\Phi(B) = \Phi'(B') \subset \mathbb{R}^N\). Since \(B, B'\) are compact, by Lemma 4.4, there exists a finite set \(J \subset \mathbb{N}\), such that \(\Phi_J : B \rightarrow \mathbb{R}^{|J|}, \Phi'_J : B' \rightarrow \mathbb{R}^{|J|}\), where \(\Phi_J(x) = (\phi_{j(1)}(x), \ldots, \phi_{j(|J|)}(x))\) and similarly for \(\Phi'_J\), are homeomorphisms onto their images. Then these images coincide and the map
\[
\Psi_B := (\Phi'_J)^{-1} \circ \Phi_J : \overline{B} \rightarrow \overline{B'}
\]
is a homemorphism, and the LSD are equivalent by \(\Psi_B\). Let us choose any open set \(\Omega \subset B\) such that the image \(\Psi_B(\Omega) \subset B'\). For example, the set \(\Omega\) can be chosen as
an open neighborhood of an interior point in $\Psi^{-1}_B(B_X'(p', r/2)) \subset B$. Then $\Psi_B|_\Omega$ is a homeomorphism from $\Omega$ to $\Psi_B(\Omega)$, and the LSD is equivalent on these sets by definition. Note that the sets $\Omega, \Psi_B(\Omega)$ are open with respect to the topology of $X, X'$, respectively. Hence Theorem 6.2 applies to $\Omega, \Psi_B(\Omega)$, and we obtain a measure-preserving isometry $\Psi : X \to X'$ such that $\Psi|_\Omega = \Psi_B|_\Omega$. In particular, $\Psi|_{\{z_0\}}$ is an isometry from the dense points $\{z_0\} \subset B$ to $\{z_0'\} \subset B'$, and thus $\Psi(p) = p'$.

QED

The proof of Theorem 6.2 is rather long and will consist of several parts.

### 6.1 Blagovestchenskii identity on $\Omega$

For $(X, \mu_X) \in \mathcal{M}_p$, consider the following initial-boundary value problem (IBVP) for the wave equation

\begin{equation}
(\partial_t^2 + \Delta_X) u(x, t) = H(x, t), \quad \text{in } X \times \mathbb{R}_+,
\end{equation}

\begin{equation}
u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1,
\end{equation}

with $u_0 \in H^1(X)$, $u_1 \in L^2(X)$, and $H \in L^2(\Omega \times \mathbb{R}_+)$ for an open set $\Omega \subset X$, where

$L^2(\Omega \times (0, T)) = \left\{ H \in L^2(X \times (0, T)) : H = 0 \text{ outside } \Omega \times (0, T) \right\}$.

We denote the unique solution of (6.2), when $u_0 = u_1 = 0$, by $u^H(x, t)$. Sometimes we denote it by $u^H(t) = u^H(\cdot, t)$. Recall that,

\begin{equation}
u^H(x, t) = \sum_{j=0}^{\infty} u_j^H(t) \phi_j(x), \quad u_j^H(t) = \langle u^H(t), \phi_j \rangle_{L^2(X, \mu_X)}.
\end{equation}

**Lemma 6.4.** Assume that we are given the local spectral data (LSD), that is, $\Omega \subset X^{\text{reg}}$, $(\lambda_j)_{j=0}^{\infty}$, $(\phi_j|_\Omega)_{j=0}^{\infty}$. Let $H \in L^2(\Omega \times \mathbb{R}_+)$. Then

\begin{equation}
u_j^H(t) = \bar{c}^{-1} \int_0^t \int_\Omega \sin(\sqrt{\lambda_j}(t - \tau)) \frac{H(x, \tau) \phi_j(x) \bar{\rho}(x)}{\sqrt{\lambda_j}} dV_X(x) d\tau,
\end{equation}

where the metric $h$ and $\bar{\rho}$ on $\Omega$ are determined by Lemma 4.6 and $\bar{c} > 0$ is some (unknown) constant, and $dV_X$ stands for the Riemannian volume element on the regular part $X^{\text{reg}}$. Hence, when we are given LSD on $\Omega$ and a function $H \in L^2(\Omega \times \mathbb{R}_+)$, we can determine the numbers $\bar{u}_j^H(t) = \bar{c} u_j^H(t)$.

**Proof.** Recall that $d\mu_X = \rho dV_X$ with the density function $\rho$. Using (6.2), the self-adjointness of $\Delta_X$ w.r.t. $d\mu_X$, we obtain

\begin{equation}
(\partial_t^2 + \lambda_j) u_j^H(t) = \int_X (\partial_t^2 + \lambda_j) u^H(x, t) \phi_j(x) d\mu_X
\end{equation}

\begin{equation}
= \int_X \phi_j(x) (-\Delta_X + \lambda_j) u^H(x, t) d\mu_X + \int_X H(x, t) \phi_j(x) d\mu_X
\end{equation}

\begin{equation}
= \int_X H(x, t) \phi_j(x) \rho(x) dV_X
\end{equation}

\begin{equation}
= \bar{c}^{-1} \int_\Omega H(x, t) \phi_j(x) \bar{\rho}(x) dV_X,
\end{equation}

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where \( \hat{\rho} = \check{\rho} \) and \( \check{c} \) defined in (4.10). Since \( u_j^H(\cdot,0) = \partial_t u_j^H(\cdot,0) = 0 \), equation (6.5) implies (6.4).

### 6.2 Approximate controllability

**Definition 6.5.** Let \( V \subset X \) be open, \( V \neq \emptyset \). The domain of influence of \( V \) in \( X \) at time \( T \), is the open set \( X(V,T) = \{ x \in X : d(x,V) < T \} \).

Below, we use the representation \( X = Y/O(n) \) for \( Y \) in \( \mathbb{R}^{2n} \), and denote by \( \pi : Y \to X \) the map from \( y \in Y \) to the corresponding orbit. We have the following generalization of finite speed propagation of waves.

**Lemma 6.6.** Let \( V \subset X \) be open and \( T > 0 \). Assume \( \text{supp}(u_0), \text{supp}(u_1) \subset V \) and \( \text{supp}(H) \subset V \times \mathbb{R}_+ \). Then \( \text{supp}(u(T)) \subset X(V,T) \).

**Proof.** Let \( \tilde{u}(y,t) \) be the solution of the wave equation on \( Y \),

\[
\begin{align*}
\left( \partial_t^2 + \Delta_Y \right) \tilde{u}(y,t) &= \tilde{H}(y,t), & y \in Y \times \mathbb{R}_+, \\
\tilde{u}(y,t)|_{t=0} &= \tilde{u}_0(y), & \partial_t \tilde{u}(y,t)|_{t=0} = \tilde{u}_1(y)
\end{align*}
\]

where \( \tilde{u}_0 = u_0 \circ \pi \) and \( \tilde{u}_1 = u_1 \circ \pi \) are functions on \( Y \) that are supported on \( \tilde{V} := \pi^{-1}(V) \subset Y \), and \( \tilde{H} = H \circ \pi \) is a function on \( Y \times \mathbb{R}_+ \) supported on \( \tilde{V} \times \mathbb{R}_+ \). Since \( \rho_Y, \ h_Y \in C^2(\tilde{Y}) \) by Corollary 3.4, a slight modification of the classical result shows that \( \text{supp}(\tilde{u}(\cdot,T)) \subset \tilde{Y}(\tilde{V},T) \), where \( \tilde{Y}(\tilde{V},T) \) is the domain of influence on \( Y \) corresponding to \( \tilde{V} \subset Y \). Observe that \( X(V,T) = \pi(Y(\tilde{V},T)) \).

Since \( \tilde{u}_0, \tilde{u}_1 \) and \( \tilde{H} \) are all \( O(n) \)-invariant, we see that the solution \( \tilde{u}(y,t) \) is also \( O(n) \)-invariant, see Appendix A. Thus, \( u(x,t) = \tilde{u}(y,t) \) for any \( y \in \pi^{-1}(x) \), and hence \( \text{supp}(u(T)) \subset \pi(Y(\tilde{V},T)) = X(V,T) \). \( \square \)

Next, we generalize Tataru’s controllability result.

**Theorem 6.7.** Let \( V \subset X \) be an open set. Then the set \( Z_X = \{ u^H(\cdot,T) : H \in C^2_s(X^{\text{reg}} \times (0,T)), \text{supp}(H) \subset V \times (0,T) \} \) is dense in \( L^2(X(V,T)) \).

**Proof.** Let \( \tilde{V} = \pi^{-1}(V) \) be an open subset of \( Y \). Since \( \rho_Y, \ h_Y \in C^2(\tilde{Y}) \) by Corollary 3.4, the classical Tataru’s unique continuation result, see [63], remains valid for the wave equation (6.6) on \( Y \). Let \( \tilde{u}(y,t) \) be the solution of (6.6) with \( \tilde{u}_0 = 0 \) and \( \tilde{u}_1 = 0 \). Using the standard duality arguments, e.g. [41] Sec. 2.5, this implies that the set \( Z_Y = \{ \tilde{u}(\cdot,T) : H \in C^2_s(Y \times (0,T)), \text{supp}(H) \subset \tilde{V} \times (0,T) \} \) is dense in \( L^2(\tilde{Y}(\tilde{V},T)) \). Hence by the proof of Lemma 6.6, the \( O(n) \)-invariant part \( Z_Y^O = \{ \tilde{u}(\cdot,T) : H \in C^2_{s,O}(Y \times (0,T)), \text{supp}(H) \subset \tilde{V} \times (0,T) \} \) is dense in \( L^2(\tilde{Y}(\tilde{V},T)) \), where \( L^2(\tilde{Y}(\tilde{V},T)) \) denotes the \( O(n) \)-invariant subspace of \( L^2(\tilde{Y}(\tilde{V},T)) \), see Appendix A. Together with (4.7), this implies that \( Z_X = \pi_*(Z_Y) \) is dense in \( L^2(X(V,T)) = \pi_*(L^2(\tilde{Y}(\tilde{V},T))) \). \( \square \)

**Lemma 6.8.** For any open set \( V \subset \Omega \subset X^{\text{reg}} \) and \( s > 0 \), there exist \( \{ F_k \}_{k=1}^\infty \subset L^2(V \times (0,s)) \) such that for \( a_k(x) = u^{F_k}(x,s) \),
where \( \chi \) is the indicator function of \( A \).

Then, as we can compute \( \bar{u}_j^{H_k}(s) \) and \( \bar{u}_j^{H_l}(s) \), and thus the non-negative quadratic form by

\[
Q[H_k, H_l] := \sum_{j=0}^{\infty} \bar{u}_j^{H_k}(s) \bar{u}_j^{H_l}(s) = \bar{c}^2(u^{H_k}(s), u^{H_l}(s))_{L^2(X, \mu_X)}. \tag{6.7}
\]

Applying the Gram-Smidt orthonormalization algorithm to the sequence \((H_k)_{k=1}^{\infty}\) with respect to the quadratic form \(Q[\cdot, \cdot]\), we obtain a sequence \((F_k)_{k=1}^{\infty}\) of functions \( F_k \in L^2(V \times (0, s)) \) such that \( Q[F_k, F_l] = \delta_{kl} \). Hence \( a_k = u^{F_k}(\cdot, s) \) are orthonormal and span a dense set in \( L^2(X(V, s), \bar{c}^2 \mu_X) \). \( \square \)

By (*), the functions

\[
\Phi_j(x) := \bar{c}^{-1} \phi_j(x)
\]

are an orthonormal basis of \( L^2(X, \bar{c}^2 \mu_X) \). For a measurable set \( A \subset X \), we denote

\[
(P_A v)(x) := \chi_A(x)v(x),
\]

where \( \chi_A \) is the indicator function of \( A \). By Lemma 6.8, we have

\[
(P_{X(V, s)} v)(x) = \chi_{X(V, s)}(x)v(x) = \sum_{k=1}^{\infty} \langle v, a_k \rangle_{L^2(X, \bar{c}^2 \mu_X)} a_k(x).
\]

Then,

\[
\langle P_{X(V, s)} \Phi_i, \Phi_j \rangle_{L^2(X, \bar{c}^2 \mu_X)} = \sum_{k=1}^{\infty} \langle \Phi_i, a_k \rangle_{L^2(X, \bar{c}^2 \mu_X)} a_k \Phi_j \rangle_{L^2(X, \bar{c}^2 \mu_X)}
\]

\[
= \sum_{k=1}^{\infty} \langle \Phi_i, a_k \rangle_{L^2(X, \bar{c}^2 \mu_X)} \langle a_k, \Phi_j \rangle_{L^2(X, \bar{c}^2 \mu_X)} = \sum_{k=1}^{\infty} \bar{u}_i^{F_k}(s)a_k \bar{u}_j^{F_k}(s).
\]

Thus, as we can compute \( \bar{u}_k^{F_k}(s) \) using Lemma 6.4, we see that if we are given LSD on \( \Omega \), we can compute the operator \( \mathcal{M}_{X(V, s)} \) for any \( V \subset \Omega \) corresponding to the orthogonal projector \( P_{X(V, s)} \), defined by

\[
\mathcal{M}_{X(V, s)} : \ell^2 \to \ell^2, \quad \mathcal{M}_{X(V, s)} = F_{\ell^2} \circ P_{X(V, s)} \circ F_{\ell^2}^{-1}, \tag{6.8}
\]

where \( F_{\ell^2} : L^2(X, \bar{c}^2 \mu_X) \to \ell^2 \) is the isometry

\[
F : v \mapsto \left( \int_X v(x) \Phi_j(x) \bar{c}^2 \mu_X \right)_j \in \ell^2.
\]
We note that the operator $\mathcal{M}_{X(V, s)} : \ell^2 \to \ell^2$ is independent of $\tilde{c}$, and it can be considered as an infinite matrix whose elements are

$$
\langle P_{X(V, s)} \phi_i, \phi_j \rangle_{L^2(X, e^2d\mu_X)} = \langle P_{X(V, s)} \phi_i, \phi_j \rangle_{L^2(X, d\mu_X)} = \int_{X(V, s)} \phi_i \phi_j \, d\mu_X. \quad (6.9)
$$

Let $V_l \subset \Omega$ be open sets and $s_l \geq 0$, $l = 1, 2, \ldots, 2L$. Denote

$$
I := \bigcap_{l=1}^L \left( X(V_{2l-1}, s_{2l-1}) \setminus X(V_{2l}, s_{2l}) \right).
$$

Then for given $V_l$ and $s_l$, LSD determine the operator

$$
\mathcal{M}_I = \mathcal{F}_c \circ P_l \circ \mathcal{F}_c^{-1}, \quad P_l = \prod_{l=1}^L P_{X(V_{2l-1}, s_{2l-1})}(I - P_{X(V_{2l}, s_{2l})}). \quad (6.11)
$$

The operator $\mathcal{M}_I$ is non-zero if and only if the set $I$ has non-zero measure.

**Lemma 6.9.** Let $(\Omega, (\lambda_j)_{j=0}^\infty, (\phi_j|\Omega)_{j=0}^\infty)$ and $(\Omega', (\lambda'_j)_{j=0}^\infty, (\phi'_j|\Omega')_{j=0}^\infty)$, where $\Omega \subset X^{\text{reg}}$, $\Omega' \subset (X')^{\text{reg}}$, be the LSD for $(X, \mu)$ and $(X', \mu')$, and assume that these data are equivalent by $\Psi: \Omega \to \Omega'$. Let $V_l \subset \Omega$, $V'_l := \Psi(V_l) \subset \Omega'$ be open sets, and $s_l \geq 0$, $l = 1, 2, \ldots, 2L$. Let $I \subset X$ be given by (6.10) and define

$$
I' := \bigcap_{l=1}^L \left( X'(V_{2l-1}, s_{2l-1}) \setminus X'(V_{2l}, s_{2l}) \right) \subset X'.
$$

Then $\mu(I) \neq 0$ if and only if $\mu'(I') \neq 0$, and

$$
\int I \phi_i(x) \phi_j(x) \, d\mu = \int I' \phi'_i(x') \phi'_j(x') \, d\mu', \quad \forall i, j \in \mathbb{N}. \quad (6.13)
$$

**Proof.** As the LSD on $X$ and $X'$ coincide, the operators $\mathcal{M}_{X(V_l, s_l)}$ and $\mathcal{M}_{X'(V'_l, s_l)}$ given by (6.8) coincide. Hence $\mathcal{M}_I$ is non-zero if and only if $\mathcal{M}_{I'}$ is non-zero, and the integrals (6.13) coincide by definition, see (6.9). \qed

### 6.3 Cut locus

All locally compact length spaces are geodesic spaces, that is, any pair of points can be joined by a distance-minimizing path. By [30], the Gromov-Hausdorff limit of a sequence of locally compact length spaces is again a locally compact length space, and hence, $X \subset \overline{\mathbb{R}}$ is a geodesic space.

**Definition 6.10.** We say that a path $\gamma : [0, \ell] \to X$ is a geodesic on $X$ if it is a locally distance-minimizing path and is parametrized with the arc length. The curve $\gamma \cap X^{\text{reg}}$ is a geodesic in the Riemannian sense on $X^{\text{reg}}$.

For $x \in X^{\text{reg}}$ and a unit vector $\xi \in S_x(X)$, we denote by $\gamma_{x, \xi}(t)$, $t \geq 0$ a geodesic of $X$ starting from $x$ in the direction $\xi$. We denote by $I_{x, \xi}$ the largest interval on which the geodesic $\gamma_{x, \xi}(t)$ can be defined.
Definition 6.11. For $x \in X^{\text{reg}}$ and $\xi \in S_x(X)$, let $i(x, \xi)$ be the supremum of those $t \in I_{x,\xi}$ that $d(x, \gamma_{x,\xi}(t)) = t$. We call $i(x, \xi)$ the cut locus distance function. The injectivity radius $i(x)$ at $x$ is defined as $i(x) = \inf_{\xi \in S_x(X)} i(x, \xi)$.

Lemma 6.12. For any $x \in X^{\text{reg}}$, we have $i(x) \leq d_X(x, X^{\text{sing}})$.

Proof. Let $z \in X^{\text{sing}}$ be a test point in $X^{\text{sing}}$ from $x$. If the claim is not valid, then we have $t_0 = d_X(x, z) < i(x)$. Then there exists a shortest $\gamma_{x,\xi}(t)$ joining $x$ and $z$ with $\gamma_{x,\xi}(t_0) = x'$, and the geodesic $\gamma_{x,\xi}([0, t_1])$ is distance-minimizing at least till $t_1 = i(x)$. However, due to [55 Thm. 1.1(A)], this implies that $\gamma_{x,\xi}([0, t_1]) \subset X^{\text{reg}}$, contradicting $z = \gamma_{x,\xi}(t_0) \in X^{\text{sing}}$.

Figure 2: The set $N = N(x, \xi; \rho, s, \varepsilon)$ where $y = \gamma_{x,\xi}(\rho)$.

Let $y = \gamma_{x,\xi}(\rho)$ with $0 < \rho < i(x)$ and $s, \varepsilon > 0$. We denote

$$N(x, \xi; \rho, s, \varepsilon) = B_{s+\varepsilon}(y) \setminus B_{\rho+s}(x), \quad (6.14)$$

where $B_r(x)$ denotes the open ball of radius $r$ centered at $x$, see Figure 2.

Lemma 6.13. Let $x \in X^{\text{reg}}$ and $0 < \rho < i(x)$.

(a) If $\rho + s < i(x)$, then $\bigcap_{\varepsilon>0} N(x, \xi; \rho, s, \varepsilon) = \{\gamma_{x,\xi}(\rho + s)\}$.

(b) If $\rho + s > i(x)$, then there exist $\xi \in S_x(X)$ and $\varepsilon > 0$ such that $N(x, \xi; \rho, s, \varepsilon) = \emptyset$.

(c) The injectivity radius satisfies

$$i(x) = \inf_{s>0} \left\{ \rho + s : \text{there exist } \xi \in S_x X, \varepsilon > 0 \text{ such that } \text{Vol}(N(x, \xi; \rho, s, \varepsilon)) = 0 \right\}.$$
Proof. (a) For any $\xi \in S_x(X)$ and sufficiently small $\varepsilon > 0$, $\gamma_{x,\xi}([0, \rho + s + \varepsilon]) \subset X$ is distance minimizing. This implies that $\gamma_{x,\xi}([\rho + s, \rho + s + \varepsilon]) \subset N(x, \xi; \rho, s, \varepsilon)$. Thus, $\gamma_{x,\xi}(\rho + s) \in \bigcap_{\varepsilon > 0} N(x, \xi; \rho, s, \varepsilon)$.

For any $q \in \bigcap_{\varepsilon > 0} N(x, \xi; \rho, s, \varepsilon)$, it satisfies that $d(y, q) < s + \varepsilon$ for any $\varepsilon > 0$ and $d(x, q) \geq \rho + s$. The former yields $d(y, q) \leq s$, which implies that

$$\rho + s \leq d(x, q) \leq d(x, y) + d(y, q) \leq \rho + s.$$  

Here we have used the condition that $\rho < \hat{i}(x)$ so that $d(x, y) = \rho$. The inequality above gives $d(x, q) = d(x, y) + d(y, q) = \rho + s$, which means that $q$ lies on the extension of the geodesic $\gamma_{x,\xi}$. Since $\rho + s < \hat{i}(x)$, we have $q = \gamma_{x,\xi}(\rho + s)$.

(b) Assume the claim is not true: $N(x, \xi; \rho, s, \varepsilon) \neq \emptyset$ for any $\xi \in S_x(X)$ and any $\varepsilon > 0$. Let $\xi_0 \in S_x(X)$ be a unit vector satisfying $\rho + s > \hat{i}(x, \xi_0)$. We pick a sequence $\varepsilon_i \to 0$ such that $N(x, \xi_0; \rho, s, \varepsilon_i) \neq \emptyset$ and pick a point $q_i \in N(x, \xi_0; \rho, s, \varepsilon_i)$. Then it satisfies that $d(y, q_i) < s + \varepsilon_i$ and $d(x, q_i) \geq \rho + s$. Taking a converging subsequence of $q_i \to q$, we have $d(y, q) \leq s$ and $d(x, q) \geq \rho + s$. Repeating the proof of (a) gives $d(x, q) = d(x, y) + d(y, q) = \rho + s$, which means that $q$ lies on the extension of the geodesic $\gamma_{x,\xi_0}(t)$, and the geodesic is minimizing until $t = \rho + s$, contradiction to $\rho + s > \hat{i}(x, \xi_0)$.

(c) Observe that $N(x, \xi; \rho, s, \varepsilon)$ contains an open ball of radius $\varepsilon/2$ centered at $\gamma_{x,\xi}(\rho + s + \varepsilon/2)$ for sufficiently small $\varepsilon > 0$ if $\rho + s < \hat{i}(x)$. Then the claim (c) is a consequence of claims (a) and (b). □

Next, we combine Lemma 6.9 and Lemma 6.13

Lemma 6.14. Let $(X, \mu)$, $(X', \mu') \in \mathcal{MR}$. Let $\Omega \subset X^{reg}$, $\Omega' \subset (X')^{reg}$, and $(\Omega, (\lambda_k)_{k=0}^\infty)$, $(\phi_k|\Omega)_{k=0}^\infty)$ and $(\Omega', (\lambda'_k)_{k=0}^\infty)$, $(\phi'_k|\Omega')_{k=0}^\infty)$ be the LSD of $X, X'$. Assume that the LSD are equivalent by a homeomorphism $\Psi_\Omega: \Omega \to \Omega'$. Then $\Psi_\Omega: \Omega \to \Omega'$ is a $C^3_\ast$-diffeomorphism.

Moreover, let $x \in \Omega$, $x' := \Psi_\Omega(x)$ and $\exp_\Omega: T_xX \to X$, $\exp_{x'}: T_{x'}X' \to X'$ be the exponential maps. Then $i(x) = i(x')$, and the map $E: B(x, i(x)) \to B'(x', i(x'))$ defined by

$$E(z) = \exp_{x'}\left(d\Psi_{|x}(\exp_{x'}^{-1}(z))\right) \quad (6.15)$$

satisfies $h = E^*(h')$, $\tilde{\rho} = E^*(\tilde{\rho'})$, and $\phi_j = \phi'_j \circ E$ on $B(x, i(x))$ for all $j \in \mathbb{N}$.

Proof. By Lemma 4.4 any $z \in \Omega$ has a neighborhood $V \subset \Omega$ such that there is an index $j = (j_1, \ldots, j_d)$ with $d = \dim(X)$, for which $\Phi_j: V \to \mathbb{R}^d$, $\Phi_j(x) = (\phi_j(x))_{j \in j}$, defines $C^3_\ast$-smooth coordinates. Let $z' = \Psi_\Omega(z)$ and $V' = \Psi_\Omega(V)$. Then, as $\phi'_j(\Psi_\Omega(x)) = \phi_j(x)$ for all $j$ by assumption, it follows from Lemma 4.5 and Remark 4.9 that the map $\Phi'_j: V' \to \mathbb{R}^d$ defines $C^3_\ast$-smooth coordinates for the same index $j$. Moreover, $\Psi_\Omega = (\Phi'_j)^{-1} \circ \Phi_j$ in $V$. Thus, $\Psi_\Omega: \Omega \to \Omega'$ is a $C^3_\ast$-diffeomorphism.

Let $\rho < \min(i(x), i'(x'))$ be small so that $B_\rho(x) \subset \Omega$ and $B_\rho(x') \subset \Omega'$, and let $\xi \in S_x(X)$, $\xi' := d\Psi_{|x}(\xi) \in S_{x'}(X')$. By Lemma 6.9 for any $s, \varepsilon > 0$,

$$\mu(N(x, \xi; \rho, s, \varepsilon)) = 0 \quad \text{if and only if} \quad \mu'(N'(x', \xi'; \rho, s, \varepsilon)) = 0.$$
Therefore, due to Lemma 6.13, $i(x) = i(x')$.

Next, we again take $\xi \in S(x)$, $\xi' := d\Psi|_{\xi}(\xi)$ and $0 < s < i(x) - \rho$. Note that $B(x, i(x)) \subset X^{\text{reg}}$ and $B'(x', i(x')) \subset (X')^{\text{reg}}$ due to Lemma 6.12 and thus the exponential maps are defined in the Riemannian sense. Consider $x_0 = \gamma_{x',}\xi'(\rho + s)$, $x_0' = \gamma_{x',}\xi' (\rho + s) = E(x_0)$. Then by Lemma 6.9 we have

$$\phi_j(x_0) = \lim_{\varepsilon \to 0} \frac{1}{\mu(N)} \int_{N(x,|\xi|,\varepsilon)} \phi_j \phi_0 d\mu = \lim_{\varepsilon \to 0} \frac{1}{\mu'(N')} \int_{N'(x',|\xi'|,\varepsilon)} \phi_j' \phi_0' d\mu' = o_j(x_0'),$$

where we have used $\phi_0 = \phi_0' = 1$ since $\phi_j, \phi_j'$ are normalized w.r.t. $\mu, \mu'$. At last, using Lemma 4.6, we obtain that $h = E^*(h'), \rho = E^*(\rho')$ on $B(x, i(x))$.  

We are now ready to prove Theorem 6.2.

**Proof of Theorem 6.2.** Let $O$ be the set of all pairs, $(\Omega_i, \Omega'_i)$, of connected open subsets $\Omega_i \subset X^{\text{reg}}, \Omega'_i \subset (X')^{\text{reg}}$ containing $\Omega, \Omega'$ such that $\Psi_{\Omega}$ extends to a diffeomorphism $\Psi_1 : \Omega_1 \to \Omega'_1$ and satisfies $\Psi_1^* (h')|_{\Omega'_1} = h|_{\Omega_1}$, $\Psi_1^* (\rho')|_{\Omega'_1} = \rho|_{\Omega_1}$ and $\Psi_1^* (\phi_j'|_{\Omega'_1}) = \phi_j|_{\Omega_1}$ for every $j \in N$. Note that the map $\Psi_1$ is an isometry by the first condition, and by the last condition, the map $\Psi_1$ has to coincide with $(\Phi')^{-1} \circ \Phi|_{\Omega_1}$, where $\Phi : X \to \mathbb{R}^N$ and $\Phi' : X' \to \mathbb{R}^N$ are the maps defined in (6.1).

Let us consider $O$ as a partially ordered set, where the order is given by the inclusion $\subset$. Suppose that $I$ is an ordered index set and $(\Omega_i, \Omega'_i) \in O$, $i \in I$, are sets so that $\Omega_i \subset \Omega_l, \Omega'_i \subset \Omega'_l$ for $i < l$. Let $\Psi_i : \Omega_i \to \Omega'_i$ be a corresponding diffeomorphism. Then it follows from Lemma 4.4 that $\Psi_i|_{\Omega_l} = \Psi_i$ for $i < l$. Hence we see that $(\bigcup_{i \in I} \Omega_i, \bigcup_{i \in I} \Omega'_i) \in \mathcal{O}$. Then by Zorn Lemma, there exists a maximal element $(\Omega_{\text{max}}, \Omega'_{\text{max}})$ in $O$. Let $\Psi_{\text{max}} : \Omega_{\text{max}} \to \Omega'_{\text{max}}$ be a diffeomorphism (an isometry) corresponding to $(\Omega_{\text{max}}, \Omega'_{\text{max}}) \in O$.

**Lemma 6.15.** The maximal element satisfies $(\Omega_{\text{max}}, \Omega'_{\text{max}}) = (X^{\text{reg}}, (X')^{\text{reg}})$.

**Proof.** Assume that the claim is not valid. Then, without loss of generality, we can assume that $X^{\text{reg}} \setminus \Omega_{\text{max}} \neq \emptyset$. As $X^{\text{reg}}$ is connected, we see that there exists a point $z_0$ in $\partial \Omega_{\text{max}} \cap X^{\text{reg}}$. Let us choose a sequence of points $y_k \in \Omega_{\text{max}}$ such that $y_k \to z_0$ as $k \to \infty$ and let $y_k' = \Psi_{\text{max}}(y_k) \in \Omega'_{\text{max}}$. As $X'$ is compact, by choosing a subsequence, we can assume $y_k'$ converges to some $z_0' \in X'$. Note that at this moment we do not know whether $z_0' \in (X')^{\text{reg}}$.

Let $W = B(z_0, r_0) \subset X^{\text{reg}}$ be a neighborhood of $z_0$ such that $W \subset X^{\text{reg}}$. By Lemma 2.2 there exists a positive number $\iota_0 > 0$ such that $\text{inj}(p) \geq \iota_0$ for all $p \in \overline{W}$. Let $r_0 := \min\{\iota_0, r_0\}/2$.

Let us choose (and fix) $k$ such that $z_0 \in B(y_k, r_0) \subset X^{\text{reg}}$. Note that $i(y_k) \geq \iota_0$. Since $y_k \in \Omega_{\text{max}} \subset X^{\text{reg}}, y_k' \in \Omega'_{\text{max}} \subset (X')^{\text{reg}}$, there exists $0 < \rho < \min\{i(y_k), i(y_k')\}$ such that $B := B(y_k, \rho) \subset \Omega_{\text{max}}$ and $B' := B'(y_k', \rho) \subset \Omega'_{\text{max}}$. Then the LSD $(B, (\lambda_j)_j^\infty, (\phi_j|_{B})_j^\infty)$ and $(B', (\lambda'_j)_j^\infty, (\phi'_j|_{B'})_j^\infty)$ are equivalent by the isometry $\Psi_{\text{max}}|_{B}$. Using Lemma 6.14 we can determine $i(y_k) = i(y_k')$. Moreover, as $i(y_k) \geq \iota_0$, we can construct a diffeomorphism $E : B(y_k, r_0) \to B'(y_k', r_0)$ such that

$$\phi_j(x) = \phi'_j(E(x)),$$

for all $j \in \mathbb{N}$ and $x \in B(y_k, r_0)$.  

(6.16)
In particular, $B'(y_k', r_0) \in (X')^{reg}$. This implies that $E$ coincides with the map $(\Phi')^{-1} \circ \Phi|_{B(y_k, r_0)}$, where $\Phi, \Phi'$ are defined in (6.1). Note that $\Psi_{\max}$ is also the restriction of $(\Phi')^{-1} \circ \Phi$ onto $\Omega_{\max}$. Since $E : B(y_k, r_0) \to B'(y_k', r_0)$ and $\Psi_{\max} : \Omega_{\max} \to \Omega_{\max}'$ are surjections, we see that $\Psi_{\ext} := (\Phi')^{-1} \circ \Phi|_{\Omega_{\max} \cup B(y_k, r_0)}$ is a bijection from $\Omega_{\ext} = \Omega_{\max} \cup B(y_k, r_0)$ to $\Omega_{\ext}' = \Omega_{\max}' \cup B'(y_k', r_0)$. Then it follows that $\Psi_{\ext}$ is a diffeomorphism. In particular, we have $\Psi^*_{\ext}(\phi_k') = \phi_k$ on $\Omega_{\ext}$, and using Corollary 4.6 we see that $\Psi^*_{\ext}(h') = h$ and $\Psi^*_{\ext}(\rho') = \rho$ on $\Omega_{\ext}$. Thus, $(\Omega_{\ext}, \Omega_{\ext}') \in \mathcal{O}$.

At last, observe that $B(y_k, r_0) \setminus \Omega_{\max} \neq \emptyset$ since $z_0 \in B(y_k, r_0) \cap \partial \Omega_{\max} \neq \emptyset$. Hence $\Omega_{\ext} \subset \Omega_{\max}$, $\Omega_{\ext}' \subset \Omega_{\max}'$, which is in contradiction with the assumption that $(\Omega_{\max}, \Omega_{\max}')$ is the maximal element of $\mathcal{O}$. Thus, $(\Omega_{\max}, \Omega_{\max}') = (X^{reg}, (X')^{reg})$. \( \square \)

**End of Proof of Theorem 6.2.** Let $\Omega \subset X$ and $\Omega' \subset X'$ be the open sets in Theorem 6.2 possibly containing singular points. Since any homeomorphism between $\Omega$ and $\Omega'$ has to preserve the regular part, $\Psi|_{\Omega \cap X^{reg}}$ is a homeomorphism from $\Omega \cap X^{reg}$ to $\Omega' \cap (X')^{reg}$, and hence the LSD on $\Omega \cap X^{reg}, \Omega' \cap (X')^{reg}$ are equivalent. Thus, applying the arguments above yields that $X^{reg}$ and $(X')^{reg}$ are isometric via an isometry $\Psi_{\max}$. Since $X^{reg} = X, (X')^{reg} = X'$, extending $\Psi_{\max}$ by continuity to $X$, we obtain an isometry $\Psi : X \to X'$. By construction, $h = \Psi^*(h'), \rho = \Psi^*(\rho')$ on $X^{reg}$, and $\Psi|_{\Omega} = \Psi_{\Omega}$. Since $\mu(X) = \mu'(X') = 1$, the density functions satisfy $\rho = \Psi^*(\rho')$ on $X^{reg}$. Hence as $\mu(X^{sing}) = \mu'((X')^{sing}) = 0$, we see that $\mu = \Psi^*(\mu')$ on $X$. Moreover, the same considerations as in Lemma 6.14 with $\Omega = X^{reg}$, show that $\Psi|_{X^{reg}}$ is $C^3$-Riemannian isometry. \( \text{QED} \)

### 7 Stability of inverse problem

In this section we prove the main Theorem 1.5

**Proof of Theorem 1.5.** To prove the result it is enough to show that, for any $\varepsilon > 0$, there is $\delta > 0$, such that if the conditions of the theorem are satisfied with this $\delta$, then (1.14) is valid with $\varepsilon$ instead of $\omega^r(\delta)$.

Assume that, for some $\varepsilon > 0$, there exist pairs $(X_i, p_i, \mu_i), (X'_i, p'_i, \mu'_i) \in \mathcal{MM}_p$, which satisfy the conditions of the theorem with $i^{-(2k_F+n/2+1)}$ instead of $\delta$, but

$$d_{pmGH}(X_i, X'_i) \geq 6\varepsilon.$$  

Using regular approximations of $X_i, X'_i$ by pointed manifolds from $\mathcal{MM}_p$, there are pairs of pointed manifolds $(M_i, p_i, \mu_i), (M'_i, p'_i, \mu'_i) \in \mathcal{MM}_p$ which satisfy the conditions of the theorem with $1/i$ instead of $\delta$, but

$$d_{pmGH}(M_i, M'_i) \geq \varepsilon.$$  

(7.1)

Indeed, we can find $M_i, M'_i$ such that

$$d_{pmGH}(X_i, M_i), d_{pmGH}(X'_i, M'_i) < \varepsilon/2,$$  

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For any $\beta$ contradicting (7.1), which would prove the theorem. Let us show that regular approximations. In particular, for sufficiently large $t$ in the sense of the pointed measured GH convergence with $t$.

Going to a subsequence if necessary, we can assume that

\[ H_M(y_{i\alpha}^i, y_{i\alpha}^{i'}, t_m) - H_{X_i}(x_{i\beta}^i, x_{i\beta}^{i'}, t_m) < i^{-(2s_{p}+n/2+4)}, \]

(7.2)

For any $z_{i\alpha}^i$ involved in the definition of $d_{pmGH}(X_i, X'_i)$, take a point $x_{i\beta}^i$ which is $i^{-(2s_{p}+n/2+4)}$ close to $z_{i\alpha}^i$ and similarly for $z_{i\alpha}^{i'}$. Rename $y_{i\beta}^i, y_{i\alpha}^i, y_{i\alpha}^{i'}$ as $y_{i\alpha}^i, y_{i\alpha}^{i'}, \alpha = 1, \ldots, A(1/i)$. By the above inequality, for sufficiently large $i$, the points $y_{i\alpha}^i, y_{i\alpha}^{i'}$ form $2/i$-net on $M_i, M'_i$, correspondingly. Similarly, for any $t$ involved in the definition of $d_{pmGH}(X_i, X'_i)$ take a point $t$ above such that $|t - t_m| < i^{-(2s_{p}+n/2+4)}$. Compare $H_M(y_{i\alpha}^i, y_{i\alpha}^{i'}, t)$ and $H_M(y_{i\alpha}^i, y_{i\alpha}^{i'}, t), \alpha = 1, \ldots, A(1/i)$. Then, using (1.13) with $H_{X_i}, H_{X'_i}$ instead of $H, H'$, and (7.2), we see from (5.37) that

\[ |H_M(y_{i\alpha}^i, y_{i\alpha}^{i'}, t_m) - H_M(y_{i\alpha}^i, y_{i\alpha}^{i'}, t_{i'})| < i^{-(2s_{p}+n/2+4)} + 2Ci^{-2} < i^{-1}, \]

for sufficiently large $i$.

Going to a subsequence if necessary, we can assume that

\[ (M_i, p_i, \mu_i) \xrightarrow{f_i} (X, p, \mu_X), \quad (M'_i, p'_i, \mu'_i) \xrightarrow{f'_i} (X', p', \mu_X'), \]

(7.3)

in the sense of the pointed measured GH convergence with $f_i, f'_i$ being the corresponding regular approximations. In particular,

\[ B(p_i, r) \xrightarrow{f_i} B(p, r), \quad B'(p'_i, r) \xrightarrow{f'_i} B'(p', r). \]

(7.4)

Let us show that

\[ (X, p, \mu_X) \simeq (X', p', \mu_X'), \]

(7.5)

where $\simeq$ stands for the measure-preserving isometry. This would imply that

\[ d_{pmGH}((M_i, p_i, \mu_i), (M'_i, p'_i, \mu'_i)) \to 0, \quad i \to \infty, \]

contradicting (7.1), which would prove the theorem.

The rest of the proof is a proof of (7.5). Denote

\[ w_{i\alpha}^i := f_i(y_{i\alpha}^i) \in X, \quad w_{i\alpha}^{i'} := f'_i(y_{i\alpha}^{i'}) \in X', \quad i = 1, \ldots, \alpha = 1, \ldots, A(1/i). \]

(7.6)

Let $P$ be the set of the double sequences $q := \{i(k), \alpha(k)\}_{k=1}^{\infty}$, with $\{i(k)\}_{k=1}^{\infty}$ being a subsequence of $\{i\}_{i=1}^{\infty}$, such that for some point $w_q \in B(p, r), w_q' \in B'(p', r)$, we have

\[ w_{i(k)}^{\alpha(k)} \to w_q \quad \text{and} \quad w_{\alpha(k)}^{n(k)} \to w'_q, \quad \text{as} \quad k \to \infty. \]

(7.7)

Here we use $B(p, r), B'(p', r)$ to denote the closed ball in $X, X'$, respectively. Using the estimate (5.40) in the proof of Theorem 5.15 it follows from (1.13) that

\[ H(w_q, w_s, t) = H'(w_q', w_s', t), \quad \text{for all} \quad q, s \in P, \quad t > 0. \]

(7.8)
Lemma 7.1. Let $\mathbb{P}$ be the set of the convergent double sequences defined above. Then,

$$
\{w_q : q \in \mathbb{P}\} = B(p, r), \quad \{w_q' : q \in \mathbb{P}\} = B'(p', r).
$$

(7.9)

Proof. Due to (7.4) and the fact that $\{y^i_{\alpha}\}_{\alpha=1}^{A(1/i)}$ form an $1/i$-net in $B(p, r)$, the points $\{w^i_{\alpha}\}_{\alpha=1}^{A(1/i)}$ form a $\delta(i)$-net in $B(p, r)$, where $\delta(i) \to 0$ as $i \to \infty$. Analogously, the same property is valid for $\{w^i_{\alpha}\}_{\alpha=1}^{A(1/i)}$.

Therefore, for any $w \in B(p, r)$ there exists a sequence $\alpha(i)$, $i = 1, \ldots$, such that $w_{\alpha(i)} \to w$ as $i \to \infty$. Consider the corresponding points $w^i_{\alpha(i)}$ in $X'$. Since the closed ball $B'(p', r)$ is compact, there is a subsequence $\{i(k), \alpha(k) = \alpha(i(k))\}_{k=1}^{\infty}$ such that $w^i_{\alpha(i(k))}$ converge. Then $q = \{i(k), \alpha(k)\} \in \mathbb{P}$, and

$$
\begin{align*}
&\lim_{k \to \infty} w^i_{\alpha(i(k))} = w \quad \text{and} \quad \lim_{k \to \infty} w^i_{\alpha(i(k))} = w', \quad \text{as} \ k \to \infty.
\end{align*}
$$

This proves the first claim in (7.9). Switching the role of $X$ and $X'$ yields the second claim in (7.9).

Lemma 7.2. Suppose two double sequences $q, \tilde{q} \in \mathbb{P}$ satisfy $w_q = w_{\tilde{q}}$. Then $w_q' = w_{\tilde{q}}$.

Proof. From the condition of lemma and (7.8), we see that

$$
H(w_q, w_s, t) = H'(w_q', w_s', t), \quad H(w_q, w_s, t) = H'(w_q', w_s', t),
$$

for any $s \in \mathbb{P}, t > 0$. Therefore, by Lemma 7.1,

$$
H'(w_q', w_s', t) = H'(w_{\tilde{q}}', w_s', t),
$$

for any $w' \in B'(p', r), t > 0$. Then it follows from Corollary 4.8 that $w_q' = w_{\tilde{q}}'$.

Lemma 7.2 makes it possible to introduce an equivalence relation $\approx$ on $\mathbb{P}$:

$$
q \approx \tilde{q} \quad \text{iff} \quad w_q = w_{\tilde{q}} \quad \text{and} \quad w_q' = w_{\tilde{q}}'.
$$

Then the maps

$$
\mathcal{F} : \mathbb{P}/ \approx \to B(p, r), \quad \mathcal{F}(q) = w_q,
$$

$$
\mathcal{F}' : \mathbb{P}/ \approx \to B'(p', r), \quad \mathcal{F}'(q) = w_q',
$$

are bijections. Thus, the map $\Phi := \mathcal{F}' \circ \mathcal{F}^{-1} : B(p, r) \to B'(p', r)$ is a bijection. Moreover, due to (7.8),

$$
H(w, \tilde{w}, t) = H'(\Phi(w), \Phi(\tilde{w}), t), \quad \text{for all} \ w, \tilde{w} \in B(p, r), \quad t > 0.
$$

(7.10)

Observe that Corollary 6.3 remains valid if, instead of using the dense sequences $\{z_0\}_{\alpha \in \mathbb{N}}$, $\{z'_0\}_{\alpha \in \mathbb{N}}$, we use all points $w \in B(p, r)$ with the corresponding point $w'$ running over the whole $B'(p', r)$. Thus, using Corollary 6.3, $\Phi$ can be uniquely extended to a measure-preserving isometry $\Phi : (X, p, \mu_X) \to (X', p', \mu_{X'})$. Theorem 1.5 is proven. QED
A Operator-theoretical approach to $\Delta_X$.

In this section we redefine the weighted Laplacian operator $\Delta_X$ extending on Fukaya’s [26 Section 7]. Recall Theorem 2.3(2) and Corollary 3.4 that, for $(X, \mu_X) \in \text{Riem}(n, \Lambda, D)$, there exists a $C^2$ Riemannian manifold $Y$ on which $O(n)$ acts isometrically in such a way that $X = Y/O(n)$, and an $O(n)$-invariant probability measure $\mu_Y$ on $Y$ satisfying $\pi_*(\mu_Y) = \mu_X$, where $\pi : Y \to X$ is the natural projection. Moreover, by Theorem 3.3 (with $M = Y$ and $G = O(n)$), the $O(n)$-action on $Y$ is of class $C^3$:

$$F_O : O(n) \times Y \to Y, \quad F_O(o, y) = o(y), \quad F_O \in C^3(O(n) \times Y).$$  \hfill (A.1)

Using the operator (4.1), we decompose $L^2(\mu_Y)$ into

$$L^2(\mu_Y) = L^2_0(Y) \oplus L^2_\perp(Y),$$

where $L^2_0(Y), L^2_\perp(Y)$ are the invariant subspaces of $\Delta_Y$ so that

$$\Delta_Y = \Delta_O \oplus \Delta_\perp.$$

We denote by $\{\lambda^+_O, \phi^+_O\}$ and $\{\lambda^+_J, \phi^+_J\}$ the eigenpairs of $\Delta_O$ and $\Delta_\perp$.

Due to the fact that $\pi_*(\mu_Y) = \mu_X$ by (3.8), the map

$$\pi_* : L^2_0(Y, \mu_Y) \to L^2(X, \mu_X)$$

is an isometry. Thus, we can define a self-adjoint operator $A$ in $L^2(X, \mu_X)$ by

$$Au = \pi_* \circ \Delta_O \circ \pi^* u, \quad \mathcal{D}(A) = \pi_* (\mathcal{D}(\Delta_O)) = \pi_* \left( W^{2,2}_O(Y) \right).$$  \hfill (A.2)

On the other hand, modifying [26 Section 7], we define the Dirichlet form

$$a_O[u^*] = \int_Y |d\pi^*(y)|^2 h_Y d\mu_Y = \int_{\pi^{-1}(X^{reg})} |d\pi^*(y)|^2 d\mu_Y, \quad \mathcal{D}(a_O) = C^{0,1}_O(Y),$$

where we use the fact that $\mu_Y(\pi^{-1}(X^{reg})) = 1$. Using Kato’s theory of quadratic forms [43], $a_O$ is closable with $\mathcal{D}(a_O) = W^{1,2}_O(Y)$ and the associated self-adjoint operator is $\Delta_O$. Observe that

$$\pi_* : C^{0,1}_O(Y) \to C^{0,1}(X)$$  \hfill (A.3)

is an isometry, since

$$d_X(x, x') = d_Y(\pi^{-1}(x), \pi^{-1}(x')), \quad d_X(x, x') = d_Y(y, y'),$$  \hfill (A.4)

for some $y \in \pi^{-1}(x)$, $y' \in \pi^{-1}(x')$. For $u \in C^{0,1}(X)$, let us define

$$a_X[u] = \int_{X^{reg}} |d\pi^* u|^2 h_X d\mu_X = \int_{\pi^{-1}(X^{reg})} |d(\pi^* u)|^2 h_Y d\mu_Y = a_O[\pi^* u],$$  \hfill (A.5)
Then $a_X$ is closable with $\mathcal{D}(a_X) = \pi_*(W^{1,2}_O(Y))$. This defines a self-adjoint operator, $A'$ in $L^2(X, \mu_X)$. Using the distribution duality, we see that in local coordinates of $X^{reg}$, $A'$ is given by
\[
A'u(x) = -\frac{1}{\sqrt{h_X \rho_X}} \partial_i \left( \sqrt{h_X h_X^{ij} \rho_X} \partial_j u(x) \right).
\]
Thus, $\mathcal{D}(A') \subset W^{2,2}(X^{reg}, \mu_X)$ and we denote $A'$ by $\Delta_X$. Then it follows from (A.5) that
\[
\Delta_X = \pi_* \circ \Delta_Y \circ \pi^* u = A, \quad \mathcal{D}(\Delta_X) = \pi_* \left( W^{2,2}_O(Y) \right) \subset W^{2,2}(X^{reg}).
\]
In particular,
\[
\text{spec}(\Delta_X) = \text{spec}(\Delta_O) \subset \text{spec}(\Delta_Y).
\]
Therefore, the eigenpairs $\{\lambda_j, \phi_j\}$ of $\Delta_X$ satisfy
\[
\lambda_j = \lambda_j^O, \quad \phi_j = \pi_*(\phi_j^O).
\]

**Remark A.1.** If a sequence of manifolds $M_i$ collapse to a point $p$, $O(n)$ acts transitively on $Y$. Thus, $L^2_O(Y)$ consists only of constant functions. Therefore, $\text{spec}(\Delta_O) = \{0\}$. Similarly, $\Delta_X$ is the operator of multiplication by 0 in the space of $L^2$-functions on $\{p\}$, i.e. constants. The results of [26] remain valid for this case.

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