Nakayama automorphism of quasi-commutative skew PBW extensions over AS-regular algebras

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Abstract

Graded quasi-commutative skew PBW extensions are isomorphic to graded iterated Ore extensions of endomorphism type, whence graded quasi-commutative skew PBW extensions with coefficients in AS-regular algebras are skew Calabi-Yau and the Nakayama automorphism exists for these extensions. With this in mind, in this paper we give a description of Nakayama automorphism for these non-commutative algebras using the Nakayama automorphism of the ring of the coefficients.

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1 Introduction

Skew PBW extensions and quasi-commutative skew PBW extensions were defined in [7]. In [26] and [1], respectively, the automorphisms and derivations of skew PBW extensions were studied. Another properties of skew PBW extensions have been studied (see for example [1], [4], [9], [10], [12], [13], [14], [15], [16], [17], [18], [19], [21], [22], [23], [24] and [25]). It is known that quasi-commutative skew PBW extensions are isomorphic to iterated Ore extensions of endomorphism type ([10], Theorem 2.3). For $\mathbb{K}$ a field, in [21], the first author defined graded skew PBW extensions and showed that if $R$ is a finite presented Koszul $\mathbb{K}$-algebra, then every graded skew PBW extension of $R$ is Koszul (note that every graded iterated Ore extension of an Artin-Schelter Regular algebra, AS-regular for short, is AS-regular, see [11]). Since every graded quasi-commutative skew PBW extension is isomorphic to a graded iterated Ore extension of endomorphism type (see Proposition 2.11), we have that if $A$ is a graded quasi-commutative skew PBW extension of an AS-regular algebra $R$, then $A$ is AS-regular (see

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Proposition 3.1. Now, for $B$ a connected algebra, $B$ is AS-regular if and only if $B$ is skew Calabi-Yau (see [20]), and hence the Nakayama automorphism of AS-regular algebras exists. Therefore, if $R$ is an AS-regular algebra with Nakayama automorphism $\nu$, then the Nakayama automorphism $\mu$ of a graded quasi-commutative skew PBW extension $A$ exists, and we compute it using the Nakayama automorphism $\nu$ together some especial automorphisms of $R$ and $A$ (see Theorem 4.11). Note that a skew Calabi-Yau algebra is Calabi-Yau if and only if its Nakayama automorphism is inner.

2 Graded skew PBW extensions

**Definition 2.1** ([7], Definition 1). Let $R$ and $A$ be rings. We say that $A$ is a *skew PBW extension over $R$*, if the following conditions hold:

(i) $R \subseteq A$;

(ii) there exist elements $x_1, \ldots, x_n \in A$ such that $A$ is a left free $R$-module, with basis the basic elements $\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$. In this case, it is said also that $A$ is a left polynomial ring over $R$ with respect to $\{x_1, \ldots, x_n\}$ and $\text{Mon}(A)$ is the set of standard monomials of $A$. Moreover, $x_1^{n_1} \cdots x_n^{n_n} := 1 \in \text{Mon}(A)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For any natural elements $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n.$$  \hfill (2.1)

Under these conditions, we will write $A := \sigma(R)\langle x_1, \ldots, x_n \rangle$.

**Remark 2.2.** Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension with endomorphisms $\sigma_i$, $1 \leq i \leq n$, as in the Proposition 2.3. We establish the following notation (see [7], Definition 6): $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$; $\sigma^n := (\sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n})$; $|\alpha| := \alpha_1 + \cdots + \alpha_n$; if $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$; for $X = x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.

We have the following properties whose proofs can be found in [7], Remark 2 and Theorem 7.

(i) Since $\text{Mon}(A)$ is a left $R$-basis of $A$, the elements $c_{i,r}$ and $c_{i,j}$ in Definition 2.1 are unique. In Definition 2.1 (iv), $c_{i,i} = 1$. This follows from $x_i^2 - c_{i,i} x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_j \in R$, which implies $1 - c_{i,i} = 0 = s_j$, for $0 \leq j \leq n$.

(ii) Each element $f \in A \setminus \{0\}$ has a unique representation as $f = c_1 X_1 + \cdots + c_t X_t$, with $c_i \in R \setminus \{0\}$ and $X_i \in \text{Mon}(A)$, for $1 \leq i \leq t$.

(iii) For every $x^\alpha \in \text{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$ and $p_{\alpha,r} \in A$ such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$, if $p_{\alpha,r} \neq 0$.

(iv) For every $x^\alpha$, $x^\beta \in \text{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^{\alpha} x^{\beta} = c_{\alpha,\beta} x^{\alpha + \beta} + p_{\alpha,\beta}$ where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$, if $p_{\alpha,\beta} \neq 0$. 


Proposition 2.3 ([7], Proposition 3). Let $A$ be a skew PBW extension of $R$. For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and a $\sigma_i$-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, $r \in R$.

The notation $\sigma(R)(x_1, \ldots, x_n)$ and the name of the skew PBW extensions are due to the Proposition 2.3. In the following definition we recall some sub-classes of skew PBW extensions. Examples of these sub-classes of algebras can be found in [25].

Definition 2.4. Let $A$ be a skew PBW extension of $R$, $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and $\Delta := \{\delta_1, \ldots, \delta_n\}$, where $\sigma_i$ and $\delta_i$ $(1 \leq i \leq n)$ are as in Proposition 2.3

(a) $A$ is called pre-commutative, if the conditions (iv) in Definition 2.1 are replaced by:

For any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R x_1 + \cdots + R x_n$.

(b) $A$ is called quasi-commutative, if the conditions (iii) and (iv) in Definition 2.1 are replaced by

(iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r = c_{i,r} x_i,$$ 

(2.2)

(iv') for any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j.$$ 

(2.3)

(c) $A$ is called bijective, if $\sigma_i$ is bijective for each $\sigma_i \in \Sigma$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

(d) If $\sigma_i = id_R$, for every $\sigma_i \in \Sigma$, we say that $A$ is a skew PBW extension of derivation type.

(e) If $\delta_i = 0$, for every $\delta_i \in \Delta$, we say that $A$ is a skew PBW extension of endomorphism type.

(f) Any element $r$ of $R$ such that $\sigma_i(r) = r$ and $\delta_i(r) = 0$ for all $1 \leq i \leq n$, will be called a constant. $A$ is called constant if every element of $R$ is constant.

(g) $A$ is called semi-commutative if $A$ is quasi-commutative and constant.

As we said in the Introduction, the letter $K$ denotes a field, and if it is not said otherwise, every algebra is a $K$-algebra. The symbol $\mathbb{N}$ will be used to denote the set of natural numbers including zero.

The next proposition was proved by the first author in [21].

Proposition 2.5. Let $R = \bigoplus_{m \geq 0} R_m$ be a $\mathbb{N}$-graded algebra and let $A = \sigma(R)(x_1, \ldots, x_n)$ be a bijective skew PBW extension of $R$ satisfying the following two conditions:

(i) $\sigma_i$ is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded $\sigma_i$-derivation for all $1 \leq i \leq n$, where $\sigma_i$ and $\delta_i$ are as in Proposition 2.3.

(ii) $x_j x_i - c_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, as in (2.1) and $c_{i,j} \in R_0$. 

3
For $p \geq 0$, let $A_p$ the $\mathbb{K}$-space generated by the set
$$\left\{ r_t x^\alpha \mid t + |\alpha| = p, \ r_t \in R_t \text{ and } x^\alpha \in \text{Mon}(A) \right\}.$$  

Then $A$ is a $\mathbb{N}$-graded algebra with graduation $A = \bigoplus_{p \geq 0} A_p$.

Proof. It is clear that $1 = x^0_1 \cdots x^0_n \in A_0$. Let $f \in A \setminus \{0\}$, then by Remark 2.2-(ii), $f$ has a unique representation as $f = r_1 X_1 + \cdots + r_s X_s$, with $r_i \in R \setminus \{0\}$ and $X_i := x^\alpha_1 \cdots x^\alpha_n \in \text{Mon}(A)$ for $1 \leq i \leq s$. Let $r_t = r_{t_1} + \cdots + r_{t_m}$ with the unique representation of $r_t$ in homogeneous elements of $R$. Then $f = (r_{t_1} \cdots + r_{t_{q+1}}) x_{ij}^{\sigma_1} \cdots x_{ij}^{\sigma_n} + \cdots + (r_{s_{q+1}} \cdots + r_{s_q}) x_{ij}^{\sigma_1} \cdots x_{ij}^{\sigma_n} = r_{t_1} x_{ij}^{\sigma_1} \cdots x_{ij}^{\sigma_n} + \cdots + r_{t_{q+1}} x_{ij}^{\sigma_1} \cdots x_{ij}^{\sigma_n} + \cdots + r_{s_q} x_{ij}^{\sigma_1} \cdots x_{ij}^{\sigma_n}$ is the unique representation of $f$ in homogeneous elements of $A$. Therefore $A$ is a direct sum of the family $\{A_p\}_{p \geq 0}$ of subspaces of $A$.

Now, let $x \in A_p A_q$. Without loss of generality we can assume that $x = (r_{t_1} x^\alpha)(r_{t_2} x^\beta)$ with $r_t \in R_t, r_s \in R_s, x^\alpha, x^\beta \in \text{Mon}(A)$, $t + |\alpha| = p$ and $s + |\beta| = q$. By Remark 2.2-(iii), we have that for $r_s$ and $x^\alpha$ there exist unique elements $\sigma^\alpha(r_s) \in R \setminus \{0\}$ and $p_{a, \beta} \in A$ such that
$$x = r_t r_{t_1} x^\alpha + p_{a, \beta} x^\beta = r_t r_{t_1} x^\alpha + r_t p_{a, \beta} x^\beta,$$
where $p_{a, \beta} = 0$ or $\deg(p_{a, \beta}) < |\alpha|$ if $p_{a, \beta} \neq 0$. Now, by Remark 2.2-(iv), we have that for $x^\alpha$, $x^\beta$ there exist unique elements $c_{a, \beta} \in R$ and $p_{a, \beta} \in A$ such that
$$x = r_t r_{t_1} c_{a, \beta} x^\alpha + p_{a, \beta} x^\beta = r_t r_{t_1} c_{a, \beta} x^\alpha + r_t p_{a, \beta} x^\beta,$$
where $c_{a, \beta}$ is left invertible, $p_{a, \beta} = 0$ or $\deg(p_{a, \beta}) < |\alpha + \beta|$ if $p_{a, \beta} \neq 0$. We note that:

1. Since $\sigma_i$ is graded for $1 \leq i \leq n$, then $\sigma^\alpha_i$ is graded and therefore $\sigma^\alpha$ is graded. Then
$$r_{t_1} := \sigma^\alpha(r_s) \in R_s$$
and $\sigma^\alpha_i(r_s) \in R_s + \alpha_i$, for $1 \leq i \leq n$ and $\alpha_i \geq 0$.

2. If $W[i \sigma^\alpha_i - \nu]$ represents the sum of the possible words that can be constructed with the alphabet composed of $\nu$ times the symbol $\delta_i$ and $\alpha_i - \nu$ times the symbol $\sigma_i$, then
$$x^\alpha_i r_s = \sum_{\nu=0}^{\alpha_i} W[i \sigma^\alpha_i - \nu](r_s) x^\alpha_i - \nu \in A_{s + \alpha_i},$$
since each summand in the above expression is in $A_{s + \alpha_i}$.

3. From condition (ii), we have that for $1 \leq i < j < k \leq n$, $x x_j x_i = c x_j x_i + r_0 x_j + r_1 x_j x_i + \cdots + r_{n_i} x_j x_i$. Then, for $1 \leq i < j < k \leq n$, we have that
$$x_k x_j x_i = x_k (c_{ij} x_i x_j + r_0 x_j + r_1 x_j x_i + \cdots + r_{n_j} x_i)$$
$$= (\sigma_k(c_{ij}) x_k x_j + \delta_k(c_{ij}) x_j x_i + (\sigma_k(r_0) x_j + \delta_k(r_0))) + (\sigma_k(r_1) x_k x_j + \delta_k(r_1) x_j x_i) + \cdots + (\sigma_k(r_{n_j}) x_k x_i + \delta_k(r_{n_j}) x_i).$$
are graded skew PBW extensions. If we assume that Proposition 2.7.

\[ x \in \text{Mon}(A) \] and \( rx \in A_3 \), we have that \( x_k x_j x_i \in A_3 \). Following this procedure we get in general that \( x_i, x_{i_2} \cdots x_{i_m} \in A_m \) for \( 1 \leq i_k \leq n, 1 \leq k \leq m, m \geq 1 \).

4. In a similar way and following the proof of Theorem 7, in [7], we obtain that \( p_{0, r} \in A_{[\alpha]+s} \) and \( p_{0, \beta} \in A_{[\alpha]+[\beta]} \). Then \( rt_r s_n c_{r, \beta} x_{\alpha+\beta} \in A_{t+s+[\alpha]+[\beta]} \), \( r_t r_s p_{0, \beta} \in A_{t+s+[\alpha]+[\beta]} \) and \( r_t p_{0, r} x_{\beta} \in A_{t+[\alpha]+s+[\beta]} \), i.e., \( x \in A_{p+q} \).

\[ \square \]

**Definition 2.6** ([21], Definition 2.6). Let \( A = \sigma(R)\langle x_1, \ldots, x_n \rangle \) be a bijective skew PBW extension of a \( \mathbb{N} \)-graded algebra \( R = \bigoplus_{n \geq 0} R_n \). We say that \( A \) is a **graded skew PBW extension** if \( A \) satisfies the conditions (i) and (ii) in Proposition 2.5.

Note that the family of graded iterated Ore extensions is strictly contained in the family of graded skew PBW extensions (see [21], Remark 2.11).

**Proposition 2.7.** Quasi-commutative skew PBW extensions with the trivial graduation of \( R \) are graded skew PBW extensions. If we assume that \( R \) has a different graduation to the trivial graduation, then \( A \) is a graded skew PBW extension if and only if \( \sigma_i \) is graded and \( c_{i, j} \in R_0 \), for \( 1 \leq i, j \leq n \).

**Proof.** Let \( R = R_0 \) and \( r \in R = R_0 \). From (2.2) we have that \( x_i r = c_{i, r} x_i = \sigma_i(r)x_i \). So, \( \sigma_i(r) = c_{i, r} \in R = R_0 \) and \( \delta_i = 0 \), for \( 1 \leq i \leq n \). Therefore \( \sigma_i \) is a graded ring homomorphism and \( \delta_i : R(-1) \to R \) is a graded \( \sigma_i \)-derivation for all \( 1 \leq i \leq n \). From (2.3) we have that \( x_j x_i - c_{i, j} x_i x_j = 0 \in R_2 + R_1 x_1 + \cdots + R_1 x_n \) and \( c_{i, j} \in R = R_0 \). If \( R \) has a nontrivial graduation, then the result is obtained from de relations (2.2), (2.3) and Definition 2.6. \( \square \)

**Examples 2.8.** We present some examples of graded quasi-commutative skew PBW extensions.

1. The **Sklyanin algebra** is the algebra \( S = \mathbb{K}\langle x, y, z \rangle / \langle ayx + bxy + cz^2, axz + bzx + cy^2, azy + byz + cz^2 \rangle \), where \( a, b, c \in \mathbb{K} \). If \( c \neq 0 \) then \( S \) is not a skew PBW extension. If \( c = 0 \) and \( a, b \neq 0 \) then in \( S \): \( xy = \frac{b}{a} xy \); \( zx = \frac{c}{b} zx \) and \( zy = \frac{b}{a} yz \), therefore \( S \cong \sigma(\mathbb{K})\langle x, y, z \rangle \) is a skew PBW extension of \( \mathbb{K} \), and we call this algebra a **particular Sklyanin algebra**. The particular Sklyanin algebra is graded quasi-commutative skew PBW extension of \( \mathbb{K} \).
2. Let \( h \in \mathbb{K} \). The algebra of shift operators is defined by \( S_h := \mathbb{K}[t][x_h; \sigma_h] \), where \( \sigma_h(p(t)) := p(t - h) \). Notice that \( x_h t = (t - h)x_h \) and for \( p(t) \in \mathbb{K}[t] \) we have \( x_h p(t) = p(t - ih)x_h \). Thus, \( S_h \cong \sigma(\mathbb{K}[t])[x_h] \) is a graded quasi-commutative skew PBW extension of \( \mathbb{K}[t] \), where \( \mathbb{K}[t] \) is endowed with trivial graduation.

3. For a fixed \( q \in \mathbb{K} - \{0\} \), the algebra of linear partial q-dilation operators \( H \), with polynomial coefficients, is the free algebra \( \mathbb{K}(t_1, \ldots, t_n, H_1^{(q)}, \ldots, H_m^{(q)}) \), \( n \geq m \), subject to the relations:

\[
\begin{align*}
    t_j t_i &= t_i t_j, \quad 1 \leq i < j \leq n; \\
    H_i^{(q)} t_i &= q t_i H_i^{(q)}, \quad 1 \leq i \leq m; \\
    H_j^{(q)} t_i &= t_i H_j^{(q)}, \quad i \neq j; \\
    H_j^{(q)} H_i^{(q)} &= H_i^{(q)} H_j^{(q)}, \quad 1 \leq i < j \leq m.
\end{align*}
\]

The algebra \( H \) is a graded quasi-commutative skew PBW extension of \( \mathbb{K}[t_1, \ldots, t_n] \), where \( \mathbb{K}[t_1, \ldots, t_n] \) is endowed with usual graduation.

4. The quantum polynomial ring \( \mathcal{O}_n(\lambda_{ji}) \) is the algebra generated by \( x_1, \ldots, x_n \) subject to the relations: \( x_j x_i = \lambda_{ji} x_i x_j \), \( 1 \leq i < j \leq n \), \( \lambda_{ji} \in \mathbb{K} \setminus \{0\} \). Thus \( \mathcal{O}_n(\lambda_{ji}) \cong \sigma(\mathbb{K})\langle x_1, \ldots, x_n \rangle \cong \sigma(\mathbb{K}[x_1])\langle x_2, \ldots, x_n \rangle \).

5. Let \( n \geq 1 \) and \( q \) be a matrix \((q_{ij})_{n \times n}\) with entries in a field \( \mathbb{K} \) where \( q_{ii} = 1 \) and \( q_{ij}q_{ji} = 1 \) for all \( 1 \leq i, j \leq n \). Then multi-parameter quantum affine n-space \( \mathcal{O}_q(\mathbb{K}^n) \) is defined to be \( \mathbb{K} \)-algebra generated by \( x_1, \ldots, x_n \) with the relations \( x_j x_i = q_{ij} x_i x_j \), for all \( 1 \leq i, j \leq n \).

Examples of graded skew PBW extensions over commutative polynomial rings \( R \) which are not quasi-commutative, and where \( R \) has the usual graduation, can be found in [21].

**Remark 2.9** ([21], Remark 2.10). Let \( A = \sigma(R)\langle x_1, \ldots, x_n \rangle \) be a graded skew PBW extension. Then we have the following properties:

(i) \( A \) is a \( \mathbb{N} \)-graded algebra and \( A_0 = R_0 \).

(ii) \( R \) is connected if and only if \( A \) is connected.

(iii) If \( R \) is finitely generated then \( A \) is finitely generated.

(iv) For (i), (ii) and (iii) above, we have that if \( R \) is a finitely graded algebra then \( A \) is a finitely graded algebra.

(v) If \( R \) is locally finite, then \( A \) as \( \mathbb{K} \)-algebra is a locally finite.

(vi) \( A \) as \( R \)-module is locally finite.

(vii) If \( A \) is quasi-commutative and \( R \) is concentrate in degree 0, then \( A_0 = R \).

(viii) If \( R \) is a homogeneous quadratic algebra then \( A \) is a homogeneous quadratic algebra.

(ix) If \( R \) is finitely presented then \( A \) is finitely presented.
**Proposition 2.10** ([10], Theorem 2.3). Let $A$ be a quasi-commutative skew PBW extension of a ring $R$.

(i) $A$ is isomorphic to an iterated Ore extension of endomorphism type $R[z_1; \theta_1] \cdots [z_n; \theta_n]$, where $\theta_1 = \sigma_1$; $\theta_j : R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}] \to R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}]$ is such that $\theta_j(z_i) = c_{i,j}z_i$ $(c_{i,j} \in R$ as in (2.1)), $1 \leq i < j \leq n$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$.

(ii) If $A$ is bijective, then each $\theta_i$ in (i) is bijective.

The following proposition establishes the relation between graded skew PBW extensions and graded iterated Ore extensions.

**Proposition 2.11.** Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a graded quasi-commutative skew PBW extension of $R$. Then $A$ is isomorphic to a graded iterated Ore extension of endomorphism type.

**Proof.** By Proposition 2.10-(i) we have that $A$ is isomorphic to an iterated Ore extension of endomorphism type $R[z_1; \theta_1] \cdots [z_n; \theta_n]$, where $\theta_1 = \sigma_1$,

$$\theta_j : R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}] \to R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}]$$

is such that $\theta_j(z_i) = c_{i,j}z_i$ $(c_{i,j} \in R$ as in (2.1)), $1 \leq i < j \leq n$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$. Since $A$ is bijective then by Proposition 2.10-(ii) $\theta_i$ is bijective. Since $A$ is graded then $\sigma_i$ is graded and $c_{i,j} \in R_0$. Moreover, since $\theta_j(r) = \sigma_i(r)$, then $\theta_i$ is a graded automorphism for each $i$. Note that $z_i$ has graded 1 in $A$, for all $i$. Thus, $A \cong R[z_1; \theta_1] \cdots [z_n; \theta_n]$ is a graded iterated Ore extension. \qed

### 3 AS-Regular and Koszul algebras

Let $B = \mathbb{K} \oplus B_1 \oplus B_2 \oplus \cdots$ be a finitely presented graded algebra over $\mathbb{K}$. The algebra $B$ will be called $AS$-regular, if $B$ has the following properties:

(i) $B$ has finite global dimension $d$: every graded $B$-module has projective dimension $\leq d$.

(ii) $B$ has finite $GK$-dimension.

(iii) $B$ is Gorenstein, meaning that $\text{Ext}^i_B(\mathbb{K}, B) = 0$ if $i \neq d$, and $\text{Ext}^d_B(\mathbb{K}, B) \cong \mathbb{K}$

**Proposition 3.1** ([23], Theorem 17). Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a graded quasi-commutative skew PBW extension. If $R$ is AS-regular, then $A$ is AS-regular.

Let $B$ be a finitely graded algebra and let $M$, $N$ be $\mathbb{Z}$-graded $B$-modules. Then there is a natural inclusion $\underline{\text{Hom}}_B(M, N) \to \text{Hom}_B(M, N)$. If $M$ is an $B$-module finitely generated, then $\underline{\text{Hom}}_B(M, N) \cong \text{Hom}_B(M, N)$ and $\underline{\text{Ext}}^i_B(M, N) \cong \underline{\text{Ext}}^i_B(M, N)$. A graded algebra $B$ is quadratic if $B = T(V)/\langle R \rangle$ where $V$ is a finite dimensional $\mathbb{K}$-vector space concentrated in degree 1, $T(V)$ is the tensor algebra on $V$, with the induced grading and $\langle R \rangle$ is the ideal generated by a subspace $R \subseteq V \otimes V$. The dual of such a quadratic algebra is $B^! := T(V^*)/\langle R^! \rangle$, where $R^! = \{ \lambda \in V^* \otimes V^* \mid \lambda(r) = 0 \text{ for all } r \in R \}$. We identify $(V \otimes V)^*$ with $V^* \otimes V^*$ by defining $(\alpha \otimes \beta)(u \otimes v) := \alpha(u)\beta(v)$ for $\alpha, \beta \in V^*$ and $u, v \in V$. 

7
Let $B = \mathbb{K} \oplus B_1 \oplus B_2 \oplus \cdots$ be a locally finite graded algebra and $E(B) = \bigoplus_{s,p} E^{s,p}(B) = \bigoplus_{s,p} \text{Ext}^{{s,p}}_{B}(\mathbb{K}, \mathbb{K})$ the associated bigraded Yoneda algebra, where $s$ is the cohomology degree and $-p$ is the internal degree inherited from the grading on $B$. Let $E^s(B) = \bigoplus_p E^{s,p}(B)$. $B$ is said to be Koszul if the following equivalent conditions hold:

(i) $\text{Ext}^{s,p}_{B}(\mathbb{K}, \mathbb{K}) = 0$ for $s \neq p$;

(ii) $B$ is one-generated and the algebra $\text{Ext}^*_{B}(\mathbb{K}, \mathbb{K})$ is generated by $\text{Ext}^1_{B}(\mathbb{K}, \mathbb{K})$, i.e., $E(B)$ is generated in the first cohomological degree;

(iii) The module $\mathbb{K}$ admits a linear free resolution, i.e., a resolution by free $B$-modules

$$\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{K} \to 0,$$

such that $P_i$ is generated in degree $i$.

(iv) $\text{Ext}^*_{B}(\mathbb{K}, \mathbb{K}) \cong B^!$ as graded $\mathbb{K}$-algebras.

The following theorem was proved by the first author in [21].

**Theorem 3.2.** Let $A = \sigma(R)(x_1, \ldots, x_n)$ be a graded skew PBW extension.

(i) The graded iterated Ore extension $A := R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is Koszul if and only if $R$ is Koszul ([21], Proposition 3.1).

(ii) If $A$ is quasi-commutative, then $R$ is Koszul if and only if $A$ is Koszul ([21], Proposition 3.3).

(iii) Let $R$ be a finitely presented algebra. If $R$ is a PBW algebra then $A$ is Koszul algebra ([21], Corollary 4.4).

(iv) If $R$ is a finitely presented Koszul algebra, then $A$ is Koszul ([21], Theorem 5.5).

4 Skew Calabi-Yau algebras and Nakayama automorphism

The enveloping algebra of a ring $B$ is defined as $B^e := B \otimes B^{op}$. We characterize the enveloping algebra of a skew PBW extension in [16]. If $M$ is an $B$-bimodule, then $M$ is an $B^e$ module with the action given by $(a \otimes b) \cdot m = amb$, for all $m \in M$, $a, b \in B$. Given automorphisms $\nu, \tau \in \text{Aut}(B)$, we can define the twisted $B^e$-module $\nu^* M^\tau$ with the rule $(a \otimes b) \cdot m = \nu(a)m\tau(b)$, for all $m \in M$, $a, b \in B$. When one or the other of $\nu, \tau$ is the identity map, we shall simply omit it, writing for example $M^\nu$ for $1^* M^\nu$.

**Proposition 4.1** ([6], Lemma 2.1). Let $\nu, \sigma$ and $\phi$ be automorphisms of $B$. Then

(i) The map $\nu^* B^\sigma \to \phi^* B^{\sigma\phi}$, $a \mapsto \phi(a)$ is an isomorphism of $B^e$-modules. In particular,

$$\nu^* B^\sigma \cong B^{\nu^{-1}\sigma} \cong \sigma^{-1}\nu B$$

and $B^\sigma \cong \sigma^{-1} B$.

(ii) $B \cong B^\sigma$ as $B^e$-modules if and only if $\sigma$ is an inner automorphism.
An algebra $B$ is said to be homologically smooth, if as an $B^e$-module, $B$ has a finitely generated projective resolution of finite length. The length of this resolution is known as the Hochschild dimension of $B$.

**Definition 4.2.** A graded algebra $B$ is said to be skew Calabi-Yau of dimension $d$ if

(i) $B$ is homologically smooth.

(ii) There exists an algebra automorphism $\nu$ of $B$ such that

$$\text{Ext}^i_{B^e}(B, B^e) \cong \begin{cases} 0, & i \neq d; \\ B^e(l), & i = d. \end{cases}$$

as $B^e$-modules, for some integer $l$. If $\nu$ is the identity, then $B$ is said to be Calabi-Yau.

Sometimes condition (ii) is called the skew Calabi-Yau condition.

The skew Calabi-Yau condition appears to have first been explicitly defined in [3] under the term rigid Gorenstein. The automorphism $\nu$ is called the Nakayama automorphism of $B$, and is unique up to inner automorphisms of $B$. As a consequence of Proposition 4.1, we have that a skew Calabi-Yau algebra is Calabi-Yau if and only if its Nakayama automorphism is inner. If $B$ is a Calabi-Yau algebra of dimension $d$, then the Hochschild dimension of $B$ (that is, the projective dimension of $A$ as an $A$-bimodule) is $d$ (see [2], Proposition 2.2). Moreover, the Hochschild dimension of $B$ coincides with the global dimension of $B$ (see [2], Remark 2.8).

**Proposition 4.3.** Let $B$ be a skew Calabi-Yau algebra with Nakayama automorphism $\nu$. Then $\nu$ is unique up to an inner automorphism, i.e., the Nakayama automorphism is determined up to multiplication by an inner automorphism of $B$.

**Proof.** Let $B$ be a skew Calabi-Yau algebra with Nakayama automorphism $\nu$ and let $\mu$ another Nakayama automorphism, i.e., $\text{Ext}^d_{B^e}(B, B^e) \cong B^e$, then $\text{Ext}^d_{B^e}(B, B^e) \cong B^e \cong B^e$ as $B^e$-modules. By Proposition 4.1-(i), $B \cong B^\nu^{-1}\mu$; by Proposition 4.1-(ii), $\nu^{-1}\mu$ is an inner automorphism of $B$. Let $\nu^{-1}\mu = \sigma$ where $\sigma$ is an inner automorphism of $B$, so $\mu = \nu\sigma$ for some inner automorphism $\sigma$ of $B$. \qed

**Proposition 4.4 ([20], Lemma 1.2).** Let $B$ be a connected graded algebra. Then $B$ is skew Calabi-Yau if and only if it is AS-regular.

**Proposition 4.5.** Let $R$ be a Koszul Artin-Schelter regular algebra of global dimension $d$ with Nakayama automorphism $\nu$.

(i) ([8], Theorem 3.3) The skew polynomial algebra $B = R[x; \nu]$ is a Calabi-Yau algebra of dimension $d + 1$.

(ii) ([28], Remark 3.13) There exists a unique skew polynomial extension $B$ such that $B$ is Calabi-Yau.

(iii) ([28], Theorem 3.16) If $\sigma$ is a graded algebra automorphism of $R$, then $B = R[x; \sigma]$ is Calabi-Yau if and only if $\sigma = \nu$. 

9
The following theorem can also be found in [23].

**Theorem 4.6.** Every graded quasi-commutative skew PBW extension \( A = \sigma(R)(x_1, \ldots, x_n) \) of a finitely presented skew Calabi-Yau algebra \( R \) of global dimension \( d \), is skew Calabi-Yau of global dimension \( d + n \). Moreover, if \( R \) is Koszul and \( \theta_i \) is the Nakayama automorphism of \( R[x_1; \theta_1] \cdots [x_{i-1}; \theta_{i-1}] \) for \( 1 \leq i \leq n \), then \( A \) is Calabi-Yau of dimension \( d + n \) (\( \theta_i \) as in Proposition 2.11-(ii), \( x_0 = 1 \)).

**Proof.** Since \( R \) is connected and skew Calabi-Yau, then by Proposition 4.4 we know that \( R \) is Artin-Schelter regular. From Proposition 3.1 we have that \( A \) is Artin-Schelter regular and, in particular, connected. Thus, using again Proposition 4.4, we have that \( A \) is a skew Calabi-Yau algebra. By the proof of Proposition 3.1 we have that the global dimension of \( A \) is \( d + n \).

For the second part, we know that graded Ore extensions of Koszul algebras are Koszul algebras and, as a particular case of Proposition 3.1, we have that a graded Ore extension of an Artin-Schelter regular algebra is an Artin-Schelter regular algebra. Now, by Proposition 2.11-(ii) we have that \( A \) is isomorphic to a graded iterated Ore extension \( R[x_1; \theta_1] \cdots [x_n; \theta_n] \).

It is known that if \( A \) is a Calabi-Yau algebra of dimension \( d \), then the global dimension of \( A \) is \( d \) (see for example [2], Remark 2.8). Then, using Proposition 4.5-(i) and applying induction on \( n \) we obtain that \( A \) is a Calabi-Yau algebra of dimension \( d + n \). \( \Box \)

**Corollary 4.7.** Let \( R \) be an Artin-Schelter regular algebra of global dimension \( d \). Then every graded quasi-commutative skew PBW extension \( A = \sigma(R)(x_1, \ldots, x_n) \) is skew Calabi-Yau of global dimension \( d + n \).

There are two notions of Nakayama automorphisms: one for skew Calabi-Yau algebras and one for Frobenius algebras. In this paper, we focus ourselves on skew Calabi-Yau algebras, or equivalently, AS-regular algebras in the connected graded case (Proposition 4.4). In this case, the two notions of Nakayama automorphisms will coincide in the sense of the Koszul duality (see [28], Proposition 1.4). Let \( B = T(V)/(R) \) be a Koszul algebra. For a graded automorphism \( \sigma \) of \( B \), we define a map \( \sigma^*: V^* \rightarrow V^* \) by \( \sigma^*(f)(x) = f(\sigma(x)) \), for each \( f \in V^* \) and \( x \in V \). Note that \( \sigma^* \) induces a graded automorphism of \( B^1 \) because \( \sigma \) is assumed to preserve the relation space \( R \). We still use the notation \( \sigma^* \) for this algebra automorphism (see [28]). Suppose that \( \{x_1, x_2, \ldots, x_n\} \) is a \( \mathbb{K} \)-linear basis of \( V \) and \( \{x_1^*, x_2^*, \ldots, x_n^*\} \) is the corresponding dual basis of \( V^* \). If \( \sigma(x_i) = \sum_j c_{ij} x_j \), for \( c_{ij} \in \mathbb{K} \), \( 1 \leq i, j \leq n \), then we have \( \sigma^*(x_i^*) = \sum_j c_{ji} x_j^* \). Moreover, for each \( i, j = 1, 2, \ldots, n \), we have \( \sigma^*(x_i^*)(x_j) = x_j^*(\sigma(x_i)) \). Let \( B \) be a Koszul AS-regular algebra of dimension \( d \). Then, the Nakayama automorphism \( \nu \) of \( B \) is equal to \( \epsilon^{d+1} \mu^* \), where \( \mu \) is the Nakayama automorphism of the Frobenius algebra \( B^1 \) and \( \epsilon \) is the automorphism of \( B \) defined by \( a \mapsto (-1)^{\deg a} a \), for each homogeneous element \( a \in B \) ([28], Proposition 1.4). Let \( B \) be a Koszul AS-regular algebra of global dimension \( d \), with Nakayama automorphism \( \nu \). Suppose that \( \sigma \) is a graded automorphism of \( B \) and \( \sigma^* \) is its corresponding dual graded automorphism of the dual algebra \( B^1 \). The **homological determinant**, denoted \( \text{hdet} \), is a homomorphism from the graded automorphism group \( \text{GrAut}(B) \) to the multiplicative group \( \mathbb{K} \setminus \{0\} \) such that \( \sigma^*(u) = (\text{hdet}\sigma)u \), for any \( u \in \text{Ext}^d_B(\mathbb{K}, \mathbb{K}) \) ([27], Proposition 1.11).

**Proposition 4.8** ([28], Proposition 3.15). Suppose that \( R \) is a Koszul AS-regular algebra with Nakayama automorphism \( \nu \), \( \sigma \) is a graded algebra automorphism of \( R \) and \( A = R[x; \sigma] \). The
Nakayama automorphism $\mu$ of $A$ is given by:

$$\mu(a) = \begin{cases} \sigma^{-1}\nu(a), & a \in R; \\ \text{hdet}(\sigma)a, & a = x. \end{cases}$$

**Theorem 4.9** ([11], Theorem 3.3). Let $K$ be a unital commutative ring and let $R$ be a projective $K$-algebra and $A = R[x; \sigma, \delta]$ be an Ore extension. Suppose that $R$ is a skew Calabi-Yau algebra of dimension $d$ with Nakayama automorphism $\nu$. Then $A$ is a skew Calabi-Yau of dimension $d+1$ and the Nakayama automorphism $\nu'$ of $A$ satisfies that $\nu'|_R = \sigma^{-1}\nu$ and $\nu'(x) = ux + b$ with $u, b \in A$ and $u$ invertible.

**Remark 4.10** ([11], Remark 3.4). $\nu'(x) = x + b$ if $\sigma = Id$, and $\nu'(x) = ux$ if $\delta = 0$.

Let $A = \mathbb{K}(x, y)/(yx - xy - x^2)$ be the Jordan plane, which is an AS-regular algebra of dimension 2. Note that $A = \mathbb{K}[x][y; \delta_1]$ with $\delta_1(x) = x^2$. It follows that $A$ is skew Calabi-Yau, with the Nakayama automorphism given by $\nu(x) = x$ and $\nu(y) = 2x + y$. Thus, $A$ is not Calabi-Yau. On one hand, $B = A[z; \nu]$ is an Ore extension of Jordan plane. Then $A$ is skew Calabi-Yau with the Nakayama automorphism $\nu'$ such that $\nu'(x) = x$ and $\nu'(y) = y$. On the other hand, $A = \mathbb{K}[x, z][y; \delta]$ where $\delta$ is given by $\delta(x) = x^2$ and $\delta(z) = -2xz$. So, $\nu'(z) = z$. It follows that $A$ is Calabi-Yau (see [11]).

**Theorem 4.11.** Let $R$ be an Artin-Schelter regular algebra with Nakayama automorphism $\nu$. Then the Nakayama automorphism $\mu$ of a graded quasi-commutative skew PBW extension $A = \sigma(R)[x_1, \ldots, x_n]$ is given by

$$\mu(r) = (\sigma_1 \cdots \sigma_n)^{-1}\nu(r), \text{ for } r \in R, \text{ and}$$

$$\mu(x_i) = u_i \prod_{j=i}^n c_{i,j}^{-1}x_i, \text{ for each } 1 \leq i \leq n,$$

where $\sigma_i$ is as in Proposition 2.3, $u_i, c_{i,j} \in \mathbb{K}\setminus\{0\}$, and the elements $c_{i,j}$ are as in Definition 2.1.

**Proof.** Note that $A$ is skew Calabi-Yau (see Corollary 4.7) and therefore the Nakayama automorphism of $A$ exists. By Proposition 2.11-(ii) and its proof we have that $A$ is isomorphic to a graded iterated Ore extension $R[x_1; \theta_1] \cdots [x_n; \theta_n]$, where $\theta_i$ is bijective; $\theta_1 = \sigma_1$;

$$\theta_j : R[x_1; \theta_1] \cdots [x_{j-1}; \theta_{j-1}] \to R[x_1; \theta_1] \cdots [x_{j-1}; \theta_{j-1}]$$

is such that $\theta_j(x_i) = c_{i,j}x_i$ ($c_{i,j} \in \mathbb{K}$ as in Definition 2.1), $1 \leq i < j \leq n$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$. Note that $\theta_j^{-1}(x_i) = c_{i,j}^{-1}x_i$. Now, since $R$ is connected then by Remark 2.9, $A$ is connected. So, the multiplicative group of $R$ and also the multiplicative group of $A$ is $\mathbb{K}\setminus\{0\}$, therefore the identity map is the only inner automorphism of $A$. Let $\mu_i$ the Nakayama automorphism of $R[x_1; \theta_1] \cdots [x_i; \theta_i]$.

By Theorem 4.9 and Remark 4.10 we have that the Nakayama automorphism $\mu_1$ of $R[x_1; \theta_1]$ is given by $\mu_1(r) = \sigma_1^{-1}\nu(r)$ for $r \in R$, and $\mu_1(x_1) = u_1x_1$ with $u_1 \in \mathbb{K}\setminus\{0\}$; the Nakayama automorphism $\mu_2$ of $R[x_1; \theta_1][x_2; \theta_2]$ is given by $\mu_2(r) = \sigma_2^{-1}\mu_1(r) = \sigma_2^{-1}\sigma_1^{-1}\nu(r)$, for $r \in R$;
Example 4.12. Let $A = \mathcal{O}_q(\mathbb{K}^n)$ be the quantum affine $n$-space. $A = \mathcal{O}_q(\mathbb{K}^n)$ is a graded quasi-commutative skew PBW extension of $\mathbb{K}[x_1]$ (see Example 2.8-5), with $\sigma_j(k) = k$ for $k \in \mathbb{K}$ and $\sigma_j(x_1) = q_j x_1$, $j \geq 2$. Therefore, according to Proposition 2.11 and its proof, $A$ is isomorphic to a graded iterated Ore extension $\mathbb{K}[x_1;x_2;\theta_2][x_2;\theta_3]$ for $1 < j < n$, where $\theta_j(k) = k$ for $k \in \mathbb{K}$ and $\theta_j(x_i) = q_j x_i$, for $1 \leq i < j \leq n$. Note that the Nakayama automorphism $\nu$ of $\mathbb{K}[x_1]$ is the identity map. Applying Theorem 4.11 we have that the Nakayama automorphism $\mu$ of $A$ is given by $\mu(k) = k$ for $k \in \mathbb{K}$, $\mu(x_1) = (\sigma_1 \cdots \sigma_n)^{-1} \cdot \mu(x_1) = (q_1 \cdots q_{n-1}) x_1 = (q_1 \cdots q_2) x_1$, and $\mu(x_i) = u_i q_{i(i+1)}^{-1} q_{i(i+2)}^{-1} \cdots q_{i-1}^{-1} x_i = u_i q_{i(i+1)} q_{i(i+2)} \cdots q_{i-1} x_i = u_i q_{i(i+1)} q_{i(i+2)} \cdots q_{i-1} x_i$, for each $2 \leq i \leq n$. Since $\mu$ is unique up to an inner automorphism (see Proposition 4.3) and the invertible elements in $\mathcal{O}_q(\mathbb{K}^n)$ are those nonzero scalars in $\mathbb{K}$, the identity map is the only inner automorphism of $\mathcal{O}_q(\mathbb{K}^n)$. Therefore, using the same reasoning of [11] in the proof of Proposition 4.1, we have that $u_i = q_1 q_2 \cdots q_{i-1}$. Then, $\mu(x_i) = (\prod_{j=1}^n x_i) x_i$, for $2 \leq i \leq n$.

Example 4.13. Let $R$ be an Artin-Schelter regular algebra of global dimension $d$, with Nakayama automorphism $\nu$. Let $A = R[x_1,\ldots,x_n;\sigma_1,\ldots,\sigma_n]$ be an iterated skew polynomial ring (see [5], page 23-24), with $\sigma_i$ graded. $A$ is a skew PBW extension of $R$ with relations $x_it = \sigma_i(t) x_i$ and $x_j x_i = x_i x_j$, for $r \in R$ and $1 \leq i,j \leq n$. As $R$ is graded and $c_{i,j} = 1 \in R_0$, then by Proposition 2.7 we have that $A$ is a graded quasi-commutative skew PBW extension of $R$. Therefore, $A$ is a skew Calabi-Yau algebra (Corollary 4.7). By Proposition 2.11, $R[x_1,\ldots,x_n;\sigma_1,\ldots,\sigma_n] \cong R[x_1;\theta_1] \cdots [x_n;\theta_n]$, where $\theta_j(r) = \sigma_j(r)$ and $\theta_j(x_i) = x_i$ for $i < j$. Applying Theorem 4.11 we have that the Nakayama automorphism $\mu$ of $A$ is given by $\mu(r) = (\sigma_1 \cdots \sigma_n)^{-1} \cdot \nu(r)$, if $r \in R$ and $\mu(x_i) = u_i \prod_{j=1}^n q_{i,j}^{-1} x_i = u_i x_i$, $u_i \in \mathbb{K} \setminus \{0\}$, $1 \leq i \leq n$.

Example 4.14. The algebra of linear partial $q$-dilation operators $H$ is a graded quasi-commutative skew PBW extension of $\mathbb{K}[t_1,\ldots,t_n]$ (see Example 2.8-3). According to Proposition 2.3, we have that $\sigma_j(t_i) = t_i$ for $i \neq j$, $1 \leq i \leq n$, $1 \leq j \leq m$; $\sigma_j(t_i) = q t_i$ for $i = j$, $1 \leq i,j \leq m$; $\delta_j = 0$, for $1 \leq j \leq m$. By Proposition 2.11, $H$ is isomorphic to a graded iterated Ore extension of endomorphism type $\mathbb{K}[t_1,\ldots,t_n][H^{(q)}_1;\theta_1] \cdots [H^{(q)}_{m-1};\theta_{m-1}] H^{(q)}_m;\theta_m]$, with $\theta_j(t_i) = t_i$ for $i \neq j$, $1 \leq i \leq n$, $1 \leq j \leq m$, $\theta_j(t_i) = q t_i$ for $i = j$, $1 \leq i,j \leq m$; $\theta_j(H^{(q)}_i) = H^{(q)}_i$ for $1 \leq i,j \leq m$. Since $\mathbb{K}[t_1,\ldots,t_n]$ is a Calabi-Yau algebra, then its Nakayama automorphism $\nu$ is the identity map. Applying Theorem 4.11, we have that the Nakayama automorphism $\mu$ of $H$ is given by

$$
\mu(t_i) = \sigma_m^{-1} \cdots \sigma_1^{-1}(t_i) = \begin{cases} 
\sigma_1^{-1}(t_i) = q^{-1} t_i, & 1 \leq i \leq m; \\
q t_i, & m < i \leq n,
\end{cases}
$$

$$
\mu(H^{(q)}_i) = H^{(q)}_j, \text{ for } 1 \leq j \leq m.
$$
Remark 4.15. If in Example 4.13 $R$ is also a Koszul algebra, then due to the Theorem 3.2-(ii), we have that $A$ is a Koszul algebra. The Nakayama automorphism $\mu$ of Example 4.13 is then $\mu(r) = (\sigma_1 \cdots \sigma_n)^{-1} \nu(r)$, if $r \in R$ and $\mu(x_i) = (\text{hdet}\sigma_i)x_i$, $1 \leq i \leq n$ (see [28], Theorem 4.6). Since $\mu$ is unique up to an inner automorphism (see Proposition 4.3) and the invertible elements $\text{in} \ R[x_1, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$ are those nonzero scalars in $K$, the identity map is the only inner automorphism of $R[x_1, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$. Therefore, $(\text{hdet}\sigma_i) = u_i$.

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