A General formulation of the Moyal and Voros products and its physical interpretation

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Abstract

A unifying perspective on the Moyal and Voros products and their physical meanings has been recently presented in the literature, where the Voros formulation admits a consistent physical interpretation. We define a star product $\star$, in terms of an antisymmetric fixed matrix $\Theta$, and an arbitrary symmetric matrix $\Phi$, that is a generalization of the Moyal and the Voros products. We discuss the quantum mechanics and the physical meaning of the generalized star product.

Keywords: Noncommutative geometry; star product.
1 Introduction

Quantum field theories on noncommutative spaces are an important area of research in high energy physics due to the fact that they can be used as a tool to detect aspects of Planck scale physics, where one expects the spacetime to show noncommutative behavior, their emergence in string theory and also as a tool to regularize quantum field theories [1]. Studying quantum field on noncommutative spaces leads to better understanding of the structure and the setup of quantum field theory itself [2]. The starting point for a large part of what is now called the noncommutative geometry is the commutator

\[ x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}, \] (1)

implemented via the Moyal product often written in the asymptotic form

\[ f(x) \star_M g(x) = f(x)e^{i\theta^{ij}\partial_i\partial_j}g(x), \] (2)

where \( \theta^{ij} \) is a constant antisymmetric 2-tensor. It is a noncommutative, associative product introduced originally in quantum mechanics. It comes from a Weyl map between functions and operators. The commutation relation (1) has been introduced in the spacetime context by Doplicher, Fredenhagen and Roberts [3]. The Moyal product is not the only product which gives the above commutation relation. There is also the Voros product. In fact, it has been shown that the two products can be cast in the same general framework in the sense that they are both coming from a “Weyl map”. More precisely, it has been shown that the Moyal product comes from a map, called the Weyl map, which associates operators to functions with symmetric ordering, while the Voros one comes from a similar map, a weighted Weyl map, which associates operators to functions with normal ordering [4]. The Moyal and the Voros formulations of noncommutative field theory has been a point of controversy in the past. This issue has been recently addressed in the context of noncommutative non relativistic quantum mechanics [5]. In particular, it has been shown that the two formulations simply correspond to two different representations associated with two different choices of basis on the quantum Hilbert space. The connection between the Voros and Moyal Weyl products has been shown in Ref.[6] and their equivalence is well known in the sense of Kontsevich [7]. In the present paper, we define a star product in 2 + 1 dimensional space-time, in terms of an antisymmetric fixed matrix \( \Theta \) and an arbitrary symmetric matrix \( \Phi \). This definition generalize the formulation of the Moyal and the Voros star products. Our motivation is to explore the possible unification of the physical meaning of these star products. In section 2, we define the generalized star product followed by the quantum mechanics associated to this star product in section 3.
2 Generalized star product

We consider the (2+1) dimensional space-time operators where the operators \( \{ x_\mu \} \) satisfy the commutation relations
\[
[x_\mu, x_\nu] = 0, \quad \mu = 0, 1, 2, \quad \nu = 0, 1, 2.
\]

We define, new operators
\[
\hat{x}_\mu = x_\mu + \frac{i}{2} \Theta_{\mu\alpha} \partial_\alpha + \frac{i}{2} \Phi_{\mu\alpha} \partial_\alpha,
\]
where the matrix \( \Theta \) is fixed antisymmetric and the matrix \( \Phi \), an arbitrary symmetric matrix defined respectively as follows
\[
\Theta = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \theta \\
0 & -\theta & 0
\end{pmatrix}, \quad \Phi = \begin{pmatrix}
0 & 0 & 0 \\
0 & \varphi_{11} & \varphi_{12} \\
0 & \varphi_{12} & \varphi_{22}
\end{pmatrix},
\]
with the time taken to be an ordinary \( c \)-number. The operators defined in equation (4) satisfy the commutation relations
\[
[\hat{x}_i, \hat{x}_j] = i\theta \epsilon_{ij}, \quad i = 1, 2, \quad j = 1, 2 \quad \text{and} \quad [\hat{x}_0, \hat{x}_i] = 0, \quad i = 1, 2.
\]

We define a star product denoted by \( \ast \) as follows
\[
f(x) \ast g(x) = (f \ast g)(x) = f(x) \exp\left(\frac{i}{2} (\Phi + \Theta)_{\mu\nu} \partial_\mu \partial_\nu\right)g(x),
\]
that is associative but not commutative. From the general point of view, we define the commutator of two functions with respect to the product (7) as
\[
[f, g]_\ast = f \ast g - g \ast f,
\]
that is bilinear, antisymmetric, satisfies the Jacobi identity and the Leibniz rule \( 1 \). When \( \Phi \equiv 0 \), then the equation (7) is equivalent to the formulation of the Moyal product \( \ast_M \) and when
\[
\Phi \equiv \Phi_\theta = \begin{pmatrix}
0 & 0 & 0 \\
0 & -i\theta & 0 \\
0 & 0 & -i\theta
\end{pmatrix},
\]
the equation (7) is equivalent to the formulation of the Voros product \( \ast_V \). Since the matrix \( \Phi \) is arbitrary symmetric, it induces a family of star products that are all

\[1] [f, g \ast h]_\ast = [f, g]_\ast \ast h + g \ast [f, h]_\ast \]
equivalent to the Moyal product. For the proof, we consider on the space of functions on the Minkowski space-time, where the metric is of signature \((- , + , +)\), the map

\[ T = e^{\frac{i}{\hbar} x_{\mu} \partial_{\mu} \partial_{\nu}}, \quad \mu = 0 , 1, 2; \nu = 0, 1, 2 \]  

(10)

and it is equivalence if

\[ T( f \ast_M g) = T( f) \ast T( g). \]  

(11)

The convenient framework to show the equality (11) is the momentum space, where in the momentum representation

\[ f( x) = \int \frac{d^3 p}{(2\pi)^3} \tilde{f}( p) e^{i p \cdot x}, \]  

(12)

with \( p \cdot x = -p_0 x_0 + p_1 x_1 + p_2 x_2 \). We have

\[ ( f \ast_M g)( x) = \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \tilde{f}( p) \tilde{g}( q) e^{-\frac{i}{\hbar} \theta_{ij} p_i q_j} e^{i( p+q) \cdot x}, \]  

(13)

where

\[ \theta_{ij} = \theta \epsilon_{ij}, \quad i = 1, 2, \; j = 1, 2. \]  

(14)

It is straightforward to show that

\[ T( f \ast_M g)( x) = \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \tilde{f}( p) \tilde{g}( q) e^{-\frac{i}{\hbar} \theta_{ij} p_i q_j} \times e^{-\frac{i}{\hbar}( \phi_{11}( p_{1} + q_{1})^{2} + \phi_{22}( p_{2} + q_{2})^{2} + 2\phi_{12}( p_{1} + q_{1})( p_{2} + q_{2}))} e^{i( p+q) \cdot x}. \]  

(15)

Now let us compute

\[ T( f) \ast T( g) = e^{\frac{i}{\hbar} \phi_{ij} \partial_{i} \partial_{j}} \int \frac{d^3 p}{(2\pi)^3} \tilde{f}( p) e^{i p \cdot x} \ast e^{\frac{i}{\hbar} \phi_{ij} \partial_{i} \partial_{j}} \int \frac{d^3 q}{(2\pi)^3} \tilde{g}( q) e^{i q \cdot x} \times e^{i q \cdot x} e^{\frac{i}{2}( \Phi + \Theta)_{ij} \delta_{i} \delta_{j}} e^{i q \cdot x}. \]  

(16)

Since

\[ e^{i q \cdot x} e^{\frac{i}{2}( \Phi + \Theta)_{ij} \delta_{i} \delta_{j}} e^{i q \cdot x} = e^{-\frac{i}{2} \theta_{ij} p_i q_j} e^{-\frac{i}{2}( \phi_{11}( p_{1} + q_{1})^{2} + \phi_{22}( p_{2} + q_{2})^{2} + 2\phi_{12}( p_{1} + q_{1})( p_{2} + q_{2}))} e^{i( p+q) \cdot x}. \]  

(17)

The equation (16) transforms to

\[ T( f) \ast T( g) = \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \tilde{f}( p) \tilde{g}( q) e^{-\frac{i}{\hbar} \theta_{ij} p_i q_j} \times e^{-\frac{i}{2}( \phi_{11}( p_{1} + q_{1})^{2} + \phi_{22}( p_{2} + q_{2})^{2} + 2\phi_{12}( p_{1} + q_{1})( p_{2} + q_{2}))} e^{i( p+q) \cdot x}. \]  

(18)
The following equality

\[ T(f \ast_M g) = T(f) \ast T(g), \]  

(19)

holds for any symmetric matrix \( \Phi \).

3 Quantum Mechanics with star product

In this section, we consider the formalism of noncommutative quantum mechanics as in Ref. [8], where noncommutative quantum mechanics is formulated as a quantum system on the Hilbert space of Hilbert-Schmidt operators acting on classical configuration space. Here, we consider \((2 + 1)\) dimensional space-time with only spacial noncommutativity. Restricting to two dimensions, the coordinates of the noncommutative configuration space satisfy the commutation relation

\[ [\hat{x}_1, \hat{x}_2] = i\theta. \]  

(20)

It is convenient to define the creation and annihilation operators

\[ b = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 + i\hat{x}_2); \quad b^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 - i\hat{x}_2), \]  

(21)

where \( \dagger \) is the hermitian conjugate notation, that satisfy the Fock algebra \([b, b^\dagger] = 1\).

The noncommutative configuration space is then isomorphic to boson Fock space

\[ \mathcal{H}_c = \text{span}\left\{ |n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle \right\}_{n=0}^{\infty}, \]  

(22)

where the span is taken over the field of complex numbers.

The quantum Hilbert space, is identified with the set of Hilbert Schmidt operators acting on noncommutative configuration space

\[ \mathcal{H}_q = \{ \psi(\hat{x}_1, \hat{x}_2) : \psi(\hat{x}_1, \hat{x}_2) \in \mathcal{B}(\mathcal{H}_c), \text{tr}_c[\psi^\dagger(\hat{x}_1, \hat{x}_2)\psi(\hat{x}_1, \hat{x}_2)] < \infty \}. \]  

(23)

Here \( \text{tr}_c \) denotes the trace over noncommutative configuration space and \( \mathcal{B}(\mathcal{H}_c) \) the set of bounded operators on \( \mathcal{H}_c \). This space has a natural inner product and norm

\[ (\phi(\hat{x}_1, \hat{x}_2), \psi(\hat{x}_1, \hat{x}_2)) = \text{tr}_c[\phi^\dagger(\hat{x}_1, \hat{x}_2)\psi(\hat{x}_1, \hat{x}_2)], \]  

(24)

and form a Hilbert Space. To distinguish the noncommutative configuration space, which is also a Hilbert space, from the quantum Hilbert space above, we use the notation \(|\cdot\rangle\) for elements of the noncommutative configuration space, while elements of the quantum Hilbert space are denoted by \(\psi(\hat{x}_1, \hat{x}_2) \equiv |\psi\rangle\). The elements of its dual (linear functionals) are as usual denoted by bras, \(\langle \psi |\), which maps elements of
\( \mathcal{H}_q \) onto complex numbers by \((\phi|\psi) = \langle \phi, \psi \rangle = tr_c[\phi^\dagger(\hat{x}_1, \hat{x}_2)\psi(\hat{x}_1, \hat{x}_2)]\). We reserve the notation \(\dagger\) for Hermitian conjugation on noncommutative configuration space and the notation \(\underline{\dagger}\) for Hermitian conjugation on quantum Hilbert space. The operators acting on the quantum Hilbert space are denoted by capital letters. The noncommutative Heisenberg algebra in two dimensions

\[
[\hat{x}_1, \hat{x}_2] = i\theta, \ [\hat{x}_1, \hat{p}_1] = i\hbar, \ [\hat{x}_2, \hat{p}_2] = i\hbar, \ [\hat{p}_1, \hat{p}_2] = 0, \ [\hat{x}_1, \hat{p}_2] = 0, \ [\hat{x}_2, \hat{p}_1] = 0, \ (25)
\]
is now represented in terms of operators \(\hat{X}_1, \hat{X}_2\) and \(\hat{P}_1, \hat{P}_2\) acting on the quantum Hilbert space \(\mathbb{C}_2\) with the inner product \(\mathbb{C}_2\), which is the analog of the Schrödinger representation of the Heisenberg algebra. These operators are given by

\[
\hat{X}_1\psi(\hat{x}_1, \hat{x}_2) = \hat{x}_1\psi(\hat{x}_1, \hat{x}_2), \quad \hat{X}_2\psi(\hat{x}_1, \hat{x}_2) = \hat{x}_2\psi(\hat{x}_1, \hat{x}_2), \quad (26)
\]

\[
\hat{P}_1\psi(\hat{x}_1, \hat{x}_2) = \frac{\hbar}{\theta}\hat{\theta}[\hat{x}_2, \psi(\hat{x}_1, \hat{x}_2)], \quad \hat{P}_2\psi(\hat{x}_1, \hat{x}_2) = -\frac{\hbar}{\theta}[\hat{x}_1, \psi(\hat{x}_1, \hat{x}_2)]. \quad (27)
\]
The position operators act by left multiplication and the momentum acts adjointly. It is also useful to introduce the following quantum operators

\[
B = \frac{1}{\sqrt{2\theta}}(\hat{X}_1 + i\hat{X}_2), \quad B^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{X}_1 - i\hat{X}_2), \quad \hat{P} = \hat{P}_1 + i\hat{P}_2, \quad \hat{P}^\dagger = \hat{P}_1 - i\hat{P}_2. \quad (28)
\]

These operators act as following

\[
B\psi(\hat{x}_1, \hat{x}_2) = b\psi(\hat{x}_1, \hat{x}_2), \quad B^\dagger\psi(\hat{x}_1, \hat{x}_2) = b^\dagger\psi(\hat{x}_1, \hat{x}_2),
\]

\[
\hat{P}\psi(\hat{x}_1, \hat{x}_2) = -i\hbar\sqrt{\frac{2}{\theta}}[b, \psi(\hat{x}_1, \hat{x}_2)], \quad \hat{P}^\dagger\psi(\hat{x}_1, \hat{x}_2) = i\hbar\sqrt{\frac{2}{\theta}}[b^\dagger, \psi(\hat{x}_1, \hat{x}_2)]. \quad (29)
\]

Let us consider now the above formalism in a system of units such that \(\hbar = 1\). The momentum eigenstates \(|p\rangle\) are given by

\[
|p\rangle = \sqrt{\frac{\theta}{2\pi}}e^{ip\cdot\hat{x}}, \quad \hat{P}_i|p\rangle = p_i|p\rangle, \quad (30)
\]
and they satisfy the usual resolution of identity and orthogonality condition

\[
\int d^2p|p\rangle(p\langle p| = Id, \quad (p|p') = \delta(p_1 - p'_1)\delta(p_2 - p'_2). \quad (31)
\]

We consider now the following states as in \([5]\) obtained by expansion in terms of the momentum states as follows

\[
|x\rangle = \int \frac{d^2p}{2\pi} e^{-ip\cdot\hat{x}}|p\rangle. \quad (32)
\]
We have
\[
(p|x) = \frac{1}{2\pi} e^{-ip\cdot x}, \quad (x|x') = \delta(x_1 - x'_1)\delta(x_2 - x'_2),
\]
that means that the states \(|x\rangle\) are orthogonal. Do these states resolve the identity with respect to the star product \(\star\) as follows
\[
\int d^2 x |x\rangle \star (x) = Id?
\]
(34)

In order to respond to this question, we compute
\[
(p|\left(\int d^2 x|x\rangle \star (x)\right)|p') = e^{\frac{i}{2}(\phi_{11}p_1p'_1 + (\phi_{12} + \theta)p_1p'_2 + (\phi_{21} - \theta)p_2p'_1 + \phi_{22}p_2p'_2)} e^{\frac{i}{2}(p_1 + p'_1) + (p_2 - p'_2)} \delta(p_1 - p'_1) \delta(p_2 - p'_2).
\]
(35)

Setting the matrix \(\Phi \equiv 0\), then
\[
(p|\left(\int d^2 x|x\rangle \star_M (x)\right)|p') = e^{\frac{i}{2}(p_1p'_1 - p_2p'_2 + p_1p'_2 - p_2p'_1)} \delta(p_1 - p'_1) \delta(p_2 - p'_2)
\]
\[
= (p|p').
\]
(36)

In general,
\[
(p|\left(\int d^2 x|x\rangle \star (x)\right)|p') \neq (p|p').
\]
(37)

The states \(|x\rangle\) are then orthogonal and resolve the identity with respect to the Moyal product, then constitute a basis of the Hilbert space. Although this provides a consistent interpretational framework, the measurement of position needs more careful consideration as the position operators do not commute and thus a precise measurement of one of these observables leads to total uncertainty in the other. In order to preserve the notion of position in the sense of a particle being localized around a certain point, the best is to construct a minimum uncertainty state in noncommutative configuration space and use that to give meaning to the notion of position.

The minimum uncertainty states on noncommutative configuration space, which are isomorphic to the boson Fock space, are well known to be the normalized coherent states
\[
|z\rangle = e^{-z\bar{z}/2} e^{z\hat{b}^\dagger} |0\rangle,
\]
(38)
where \(z = \frac{1}{\sqrt{2\theta}}(x_1 + ix_2)\) is a dimensionless complex number that satisfies the relation \(b|z\rangle = z|z\rangle\). These states provide an overcomplete basis on the noncommutative configuration space. Corresponding to these states, we can construct a state (operator) in quantum Hilbert space as follows
\[
|z, \bar{z}\rangle = |z\rangle \langle z|,
\]
(39)
and these states satisfy
\[ B|z, \bar{z}\rangle = z|z, \bar{z}\rangle, \] (40)
and
\[ (z', \bar{z}'|z, \bar{z}\rangle = tr_c[(|z'\rangle\langle z'|)^\dagger(|z\rangle\langle z|)] = e^{-|z-z'|^2}. \] (41)

The star product defined in equation (7) can be expressed in terms of complex variables as
\[ \star \equiv e^{\frac{i}{\pi}(\varphi_{11} - \varphi_{22} + 2i\varphi_{12})\partial_z \partial_{\bar{z}} + (\varphi_{11} + \varphi_{22} - 2i\theta)\partial_z \partial_{\bar{z}} + (\varphi_{11} + \varphi_{22} + 2i\theta)\partial_z \partial_{\bar{z}} + (\varphi_{11} - \varphi_{22} - 2i\varphi_{12})\partial_z \partial_{\bar{z}}}. \] (42)

When \( \Phi \equiv 0 \), we recognize the form of the Moyal product
\[ f(z, \bar{z}) \star g(z, \bar{z}) = f(z, \bar{z})e^{\frac{i}{\pi}(\partial_z \partial_{\bar{z}} \phi)}g(z, \bar{z}) = f(z, \bar{z}) \star_M g(z, \bar{z}), \] (43)
and for non trivial matrix \( \Phi_\theta \) defined in equation (9), we recognize the form of the Voros product,
\[ f(z, \bar{z}) \star g(z, \bar{z}) = f(z, \bar{z})e^{\frac{i}{\pi}(\partial_z \partial_{\bar{z}} \phi)}g(z, \bar{z}) = f(z, \bar{z}) \star_V g(z, \bar{z}). \] (44)

The question is whether the states \(|z, \bar{z}\rangle\) resolve the identity
\[ \int \frac{dzd\bar{z}}{\pi}|z, \bar{z}\rangle \star (z, \bar{z}) = Id? \] (45)

We now introduce the momentum eigenstates
\[ |p\rangle = \frac{\theta}{2\pi} e^{i\sqrt{\frac{\pi}{2}}(pb + pb^\dagger)}, \quad \int d^2p |p\rangle(p) = Id, \quad \hat{P}_i |p\rangle = p_i |p\rangle, \] (46)

normalised such that \( \langle p|p'\rangle = \delta(p_1 - p'_1)\delta(p_2 - p'_2) \). The overlap of this basis with the momentum eigenstate is given by
\[ \langle z, \bar{z}|p\rangle = \frac{\theta}{2\pi} e^{-\frac{|p|^2}{4}} e^{i\sqrt{\frac{\pi}{2}}(p\bar{z} + \bar{p}z)}. \] (47)

We can check if equation (45) is satisfied with respect to the star product \( \star \) as follows
\[ \int \frac{dzd\bar{z}}{\pi}(p'|z, \bar{z}\rangle \star (z, \bar{z})|p\rangle = e^{\frac{i\pi}{\theta}(|p|^2 + |p'|^2)} e^{\frac{i}{\pi}(\varphi_{11} - \varphi_{22} + 2i\varphi_{12})p\bar{p} + (\varphi_{11} + \varphi_{22} - 2i\theta)p\bar{p}'} \]
\[ \times e^{\frac{i}{\pi}(\varphi_{11} + \varphi_{22} + 2i\theta)p\bar{p}' + (\varphi_{11} + \varphi_{22} - 2i\varphi_{12})p\bar{p}'} \delta(p_1 - p'_1)\delta(p_2 - p'_2). \] (48)
For the particular case of the matrix $\Phi_\theta$ defined in (9), the equation (48) becomes

$$\int \frac{dz \bar{z}}{\pi} \pi^{\theta}(|z, \bar{z}) \star_V (z, \bar{z}|p) = e^{-\theta(|p|^2+|p'|^2)} e^{\theta \delta(p_1-p'_1)\delta(p_2-p'_2)} = (p|p'),$$

that implies the resolution of the identity

$$\int \frac{dz \bar{z}}{\pi} \pi^{\theta}(|z, \bar{z}) \star_V (z, \bar{z}|p) = \text{Id.}$$

In general

$$\int \frac{dz \bar{z}}{\pi} (p'|z, \bar{z}) \star (z, \bar{z}|p) \neq (p'|p).$$

This means that the states $|z, \bar{z}\rangle$ do not resolve the identity operator with respect to the star product $\star$.

4 Conclusion

We have defined a star product $\star$ in terms of a fixed antisymmetric matrix $\Theta$ and an arbitrary symmetric matrix $\Phi$. We have shown that the Moyal and the Voros products are some particular cases of the star product $\star$. This formulation confirms their equivalence from a mathematical perspective [5]. As the matrix $\Phi$ is arbitrary, the star product $\star$ induces a family of star products with respect to $\Phi$ that are all equivalent to the Moyal product. In order to interpret the physical meaning of the star product, we set the problem in a completely general and abstract operator formulation of non-commutative quantum field theory and quantum mechanics. We expect to have some results that unify at least the physical meaning of the Moyal and the Voros products and then a complete generalization of the results in [5]. We set some physical states $|x\rangle$ as expansion of the momentum states and they do resolve the identity with respect to the Moyal product. In the coherent states framework, the states $|z, \bar{z}\rangle$ resolve the identity with respect to the Voros product. For both of the states $|x\rangle$ and $|z, \bar{z}\rangle$, we could not conclude the resolution of identity with respect to the star product $\star$. The physical interpretation of the star product $\star$ could not be pursued. Restricting to the Moyal and the Voros star products, in [5], it has been shown that only the Voros product can be interpreted as describing a maximally localized system. That is also reflected in the transition amplitudes that differ with only the Voros amplitude representing the physical amplitude. Similarly, in [9, 10], it has been shown that the low energy dynamics in the lowest Landau level approximation in relativistic quantum field theories in a magnetic field is described by the Voros rather than the Moyal product. It is known that a set of states that is overcomplete without having a resolution of identity is not practically useful. So for the star product $\star$ to be practically useful,
one has to redefine an overcomplete set of states that resolve the identity with respect to this star product, alternatively, one should set the matrix $\Theta$ and the matrix $\Phi$ in a more general way. This setting may turn out to be more complicated, but one may discover other star products equivalent to the Moyal product that unify the physical meaning of the Moyal and the Voros products.

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