DIAGONAL RECURRENCE RELATIONS FOR THE STIRLING NUMBERS OF THE FIRST KIND

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Abstract. In the paper, the author presents diagonal recurrence relations for the Stirling numbers of the first kind. As by-products, the author also recovers three explicit formulas for special values of the Bell polynomials of the second kind.

1. Introduction

In combinatorics, the Bell polynomials of the second kind, or say, the partial Bell polynomials, denoted by $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ for $n \geq k \geq 0$, are defined by

\[
B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{1 \leq i \leq n, \ell \in \{0\} \cup \mathbb{N}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left( \frac{x_i}{i!} \right)^{\ell_i}.
\]

(1.1)

See [1, p. 134, Theorem A]. For more information on the Bell polynomials in general and Dyck paths in particular, please look at the papers [6] and [7] and plenty of references therein.

In mathematics, the Stirling numbers arise in a variety of combinatorics problems and were introduced by James Stirling in the eighteen century. There are two different kinds of the Stirling numbers. The Stirling numbers of the first kind $s(n,k)$, which are also called the signed the Stirling numbers of the first kind, may be generated by

\[
\frac{\ln(1+x))^k}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!}, \quad |x| < 1.
\]

(1.2)

The mathematical meaning of the unsigned Stirling numbers of the first kind $(-1)^{n-k}s(n,k)$ can be interpreted as the number of permutations of $\{1,2,\ldots,n\}$ with $k$ cycles.

Several “triangular”, “horizontal”, and “vertical” recurrence relations for the Stirling numbers of the first kind $s(n,k)$ were listed in [1, pp. 214–215, Theorems A, B, and C] as

\[
s(n,k) = s(n-1, k-1) - (n-1)s(n-1, k),
\]

(1.3)

\[(n-k)s(n,k) = \sum_{k+1 \leq \ell \leq n} (-1)^{\ell-k} \binom{\ell}{k-1} s(n, \ell),
\]

(1.4)

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Theorem 1.1. For three explicit formulas for special values of the Bell polynomials of the second kind $B_{n,k}$, reads that, for $1 \leq n \leq r$, the conventions that $s(n, k)$ may be described in terms of the Bell polynomials of the second kind $B_{n,k}$.

The main results may be formulated in the following theorem.

**Theorem 1.1.** For $n \geq k \geq 1$, we have

\[
B_{n,k} \left( \frac{1!}{2} \frac{2!}{3} \cdots \frac{(n-k+1)!}{n-k+2} \right) = (-1)^{n-k} \frac{1}{k!} \sum_{m=1}^{k} (-1)^{m} \frac{k}{m+n} s(n+m,m), \tag{1.8}
\]

\[
B_{n,k}(0,1,\ldots,(n-k)!) = (-1)^{n-k} \frac{n}{k} \sum_{m=0}^{k} (-1)^{m} \frac{k}{m+n-k} s(n-m,k-m), \tag{1.9}
\]

and

\[
s(n, k) = (-1)^{k} \sum_{m=1}^{n} (-1)^{m} \sum_{\ell=k-m}^{k-1} (-1)^{\ell} \binom{n}{\ell} \binom{\ell}{k-m} s(n-\ell,k-\ell), \tag{1.10}
\]

\[
= (-1)^{n-k} \sum_{\ell=0}^{k-1} (-1)^{\ell} \binom{n}{\ell} \binom{\ell-1}{k-n-1} s(n-\ell,k-\ell), \tag{1.11}
\]

where the conventions that \( \binom{0}{0} = 1 \), \( \binom{-1}{-1} = 1 \), and \( \binom{p}{q} = 0 \) for $p \geq 0 > q$ are adopted in (1.11).

2. **Proof of Theorem 1.1**

Recently, three integral representations for the Stirling numbers of the first kind $(-1)^{n-k}s(n,k)$ were discovered in [10]. The first one among them, [10, Theorem 2.1], reads that, for $1 \leq k \leq n$,

\[
s(n,k) = \binom{n}{k} \lim_{x \to 0} \frac{d^{n-k}}{dx^{n-k}} \left\{ \left[ \int_{0}^{\infty} \left( \int_{1/e}^{1} t^{\ell} d t \right)^{e-\ell} d u \right]^{k} \right\}. \tag{2.1}
\]

In combinatorial analysis, Faà di Bruno formula plays an important role and may be described in terms of the Bell polynomials of the second kind $B_{n,k}$ by

\[
\frac{d^{n}}{dt^{n}} f \circ h(t) = \sum_{k=1}^{n} f^{(k)}(h(t))B_{n,k}(h'(t), h''(t), \ldots, h^{(n-k+1)}(t)). \tag{2.2}
\]
See [1, p. 139, Theorem C]. The Bell polynomials of the second kind $B_{n,k}$ satisfy
\[
\sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right)^k, \quad (2.3)
\]
where $a$ and $b$ are any complex numbers. See [1, pp. 133 and 135–136].

Let
\[
h(x) = \int_{1/e}^{\infty} \left( \int_{1/e}^{1} t^{x-u-1} \frac{dt}{t} \right) e^{-u} \, du, \quad (2.6)
\]
It is clear that, for $\ell \in \mathbb{N},$
\[
h^{(\ell)}(x) = \int_{1/e}^{\infty} \left[ \int_{1/e}^{1} t^{x-u-1} (\ln t)^\ell d\ell \right] u^\ell e^{-u} \, du
\]
\[
\rightarrow \int_{1/e}^{\infty} \left[ \int_{1/e}^{1} \frac{(\ln t)^\ell}{t} d\ell \right] u^\ell e^{-u} \, du = \frac{(-1)^\ell}{\ell+1}
\]
as $x \to 0$. Applying in (2.2) $f(v) = v^k$ and the function (2.6) to compute (2.1) reveals
\[
s(n, k) = \binom{n}{k} \lim_{x \to 0} \sum_{m=1}^{n-k} f^{(m)}(h(x)) B_{n-k,m}(h'(x), \ldots, h^{(n-k-m+1)}(x))
\]
\[
= \begin{cases} 
\binom{n}{k} \lim_{x \to 0} \sum_{m=1}^{k} f^{(m)}(h(x)) B_{n-k,m}(h'(x), \ldots, h^{(n-k-m+1)}(x)), & n > 2k \\
\binom{n}{k} \lim_{x \to 0} \sum_{m=1}^{n-k} f^{(m)}(h(x)) B_{n-k,m}(h'(x), \ldots, h^{(n-k-m+1)}(x)), & k \leq n \leq 2k \\
\binom{n}{k} \sum_{m=1}^{k} f^{(m)}(h(0)) B_{n-k,m}(h'(0), \ldots, h^{(n-k-m+1)}(0)), & n > 2k \\
\binom{n}{k} \sum_{m=1}^{n-k} f^{(m)}(h(0)) B_{n-k,m}(h'(0), \ldots, h^{(n-k-m+1)}(0)), & k \leq n \leq 2k 
\end{cases}
\]
\[
= \begin{cases} 
\binom{n}{k} \sum_{m=1}^{k} \frac{k!}{(k-m)!} B_{n-k,m} \left( \frac{1}{2}, \ldots, \frac{(-1)^{n-k-m+1}(n-k-m+1)!}{n-k-m+2} \right), & n > 2k; \\
\binom{n}{k} \sum_{m=1}^{n-k} \frac{k!}{(k-m)!} B_{n-k,m} \left( \frac{1}{2}, \ldots, \frac{(-1)^{n-k-m+1}(n-k-m+1)!}{n-k-m+2} \right), & k \leq n \leq 2k.
\end{cases} \quad (2.7)
\]
Taking \(x_m = \frac{m^n}{m+1}\) in (2.3) and using (1.2) give

\[
\sum_{n=k}^{\infty} B_{n,k} \left( \frac{1!}{2}, \frac{2!}{3}, \ldots, \frac{(n-k+1)!}{n-k+2} \right) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{m=1}^{\infty} \frac{t^m}{m+1} \right)^k
\]

\[
= (-1)^k \frac{k!}{k!} \left[ \ln(1-t) + 1 \right]^k = (-1)^k \frac{k!}{k!} \sum_{i=0}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) \left[ \ln(1-t) \right]^i
\]

\[
= \frac{(-1)^k}{k!} \sum_{i=0}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) i! \sum_{\ell=0}^{\infty} (-1)^\ell s(\ell, i) \frac{t^\ell}{\ell!} = (-1)^k \sum_{i=0}^{k} \frac{1}{(k-i)!} \sum_{\ell=i}^{\infty} (-1)^\ell s(\ell, i) \frac{t^\ell}{\ell!}.
\]

This implies that

\[
B_{n,k} \left( \frac{1!}{2}, \frac{2!}{3}, \ldots, \frac{(n-k+1)!}{n-k+2} \right) = n! (-1)^k \sum_{i=0}^{k} \frac{(-1)^{n+i} s(n+i,i)}{(k-i)! (n+i)!}
\]

\[
= (-1)^{n-k} \frac{1}{k!} \sum_{i=0}^{k} \frac{\left( \begin{array}{c} k \\ i \end{array} \right)}{\left( \begin{array}{c} n+i \\ i \end{array} \right)} (-1)^i s(n+i,i).
\]

The formula (1.8) follows.
Substituting (1.8) into (2.5) leads to

\[
B_{n+k,k}(0, 1!, 2! \ldots, n!) = (-1)^{n-k} \left( \begin{array}{c} n+k \\ k \end{array} \right) \sum_{i=0}^{k} (-1)^i \frac{\left( \begin{array}{c} k \\ i \end{array} \right)}{\left( \begin{array}{c} n+i \\ i \end{array} \right)} s(n+i,i),
\]

which may be rearranged as (1.9).

By virtue of (2.4), we have

\[
B_{n-k,m} \left( \frac{1}{2}, \frac{2!}{3}, \ldots, \frac{(n-k-m+1)!}{n-k-m+2} \right) = (-1)^{n-k} B_{n-k,m} \left( \frac{1!}{2}, \frac{2!}{3}, \ldots, \frac{(n-k-m+1)!}{n-k-m+2} \right).
\]

Substituting (1.8) into (2.8), and then into (2.7), and simplifying find that

(1) when \(2k \geq n \geq k \geq 1\), we have

\[
s(n,k) = \sum_{m=1}^{n-k} \sum_{\ell=1}^{m} (-1)^{m+\ell} \binom{n}{k-\ell} \binom{k-\ell}{m-\ell} s(n-k+\ell, \ell);\quad (2.9)
\]

(2) when \(n > 2k > 0\), we have

\[
s(n,k) = \sum_{m=1}^{k} \sum_{\ell=1}^{m} (-1)^{m+\ell} \binom{n}{k-\ell} \binom{k-\ell}{m-\ell} s(n-k+\ell, \ell).\quad (2.10)
\]

Considering the convention that \(s(n,k) = 0\) for \(0 \leq n < k\), we can unify the above two formulas (2.9) and (2.10) into

\[
s(n,k) = \sum_{m=1}^{n} \sum_{\ell=1}^{m} (-1)^{m+\ell} \binom{n}{k-\ell} \binom{k-\ell}{k-m} s(n-k+\ell, \ell),\quad (2.11)
\]

which can be further formulated as (1.10).
Interchanging two sums in (1.10) and computing the inner sum yield

\[ s(n, k) = (-1)^k \sum_{\ell=1}^{n-k} \left( -1 \right)^\ell \binom{n}{\ell} \sum_{m=\ell-1}^{n-\ell} (-1)^m \binom{\ell}{k-m} s(n-\ell, k-\ell) \]

\[ = (-1)^{n-k} \sum_{\ell=k-n}^{k-1} \left( -1 \right)^\ell \binom{n}{\ell} \binom{\ell-1}{k-\ell-1} s(n-\ell, k-\ell) \]

which may be rearranged as (1.11). The proof of Theorem 1.1 is complete.

3. Remarks

Remark 3.1. The recurrence relations (1.10) and (1.11) are neither “triangular”, nor “vertical”, nor “horizontal” recurrence relations as listed in [1, pp. 214–215, Theorems A, B, and C], so we call them “diagonal” recurrence relations for the Stirling numbers of the first kind \( s(n, k) \).

Remark 3.2. The formula (1.10) is also true if changing the sum over \( m \) from 1 to \( k \) instead of from 1 to \( n \).

Remark 3.3. Corollary 2.3 in [9] states that the Stirling numbers of the first kind \( s(n, k) \) for \( 2 \leq k \leq n \) may be computed by

\[ s(n, k) = (-1)^{n-k} (n-1)! \sum_{\ell_1=1}^{n-k-1} \frac{1}{\ell_1} \sum_{\ell_2=\ell_1+1}^{n-k-2} \frac{1}{\ell_2} \cdots \sum_{\ell_{k-2}=\ell_{k-3}+1}^{n-k} \frac{1}{\ell_{k-2}} \sum_{\ell_{k-1}=1}^{\ell_{k-2}} \frac{1}{\ell_{k-1}} \]  

This formula may be reformulated as

\[ (-1)^n s(n, k) = \sum_{m=1}^{n-k} \frac{1}{m} \left[ (-1)^{m-(k-1)} s(m, k-1) \right] \]  

Remark 3.4. By applying the integral representation (2.1), some properties for the Stirling numbers of the first kind \( s(n, k) \), including the logarithmic convexity with respect to \( n \geq 0 \) of the sequence \( \left\{ \frac{|s(n+k,k)|}{\binom{n+k}{k}} \right\}_{n \geq 0} \) for any fixed \( k \in \mathbb{N} \), see [10, Corollary 5.1], were established in [10, Section 5].

Remark 3.5. It is well known in combinatorics that

\[ B_{n,k}(1!, 2!, \ldots, (n-k+1)! \} = \binom{n}{k} \binom{n-1}{k-1} (n-k)! \]  

for \( n \geq k \geq 1 \). See [1, p. 135, Theorem B]. We now recover this identity alternatively.

In [12, Theorems 2.1 and 2.2], it was inductively obtained that, for \( i \in \mathbb{N} \) and \( t \neq 0 \),

\[ \frac{d^i e^{1/t}}{dt^i} = (-1)^i (1/t)^{i-1} \sum_{k=0}^{i-1} a_{i,k} k^k \]  

and

\[ \frac{d^i e^{-1/t}}{dt^i} = (-1)^i \sum_{k=0}^{i-1} (-1)^k a_{i,k} k^k \]  

where

\[ a_{i,k} = \binom{i}{k} \binom{i-1}{k} k! \]
for all $0 \leq k \leq i - 1$ and $a_{n,n-k}$ are Lah numbers $L(n,k)$. See also [11, Equations (1.3) and (1.4)]. For more information on Lah numbers $L(n,k)$, please refer to the recent references [2] and [5] and related reference therein.

By (2.2) and (2.4), it follows that, for $i \in \mathbb{N}$ and $t \neq 0$,
\[
\frac{d^i}{dt^i} e^{1/t} = e^{1/t} \sum_{k=1}^{i} B_{i,k} \left( -\frac{1}{t^2}, \frac{2!}{t^3}, \ldots, (-1)^{i-k+1} \frac{(i-k+1)!}{t^{i-k+2}} \right)
\]
\[
= (-1)^i e^{1/t} \sum_{k=1}^{i} \frac{1}{t^{i+k}} B_{i,k}(1!, 2!, \ldots, (i-k+1)!) \tag{3.7}
\]
and
\[
\frac{d^i}{dt^i} e^{-1/t} = e^{-1/t} \sum_{k=1}^{i} (-1)^k B_{i,k} \left( \frac{1!}{t^2}, \frac{2!}{t^3}, \ldots, (-1)^{i-k+1} \frac{(i-k+1)!}{t^{i-k+2}} \right)
\]
\[
= e^{-1/t} \sum_{k=1}^{i} \frac{1}{t^{i+k}} (-1)^{i+k} B_{i,k}(1!, 2!, \ldots, (i-k+1)!)
\]

Combining the formula (3.4) with (3.7) and the formula (3.5) with the above equation respectively show
\[
(-1)^i \frac{1}{t^{2i}} \sum_{k=0}^{i-1} a_{i,k} t^k = (-1)^i \sum_{k=1}^{i} \frac{1}{t^{i+k}} B_{i,k}(1!, 2!, \ldots, (i-k+1)!) \]
and
\[
\frac{1}{t^{2i}} \sum_{k=0}^{i-1} (-1)^k a_{i,k} t^k = \sum_{k=1}^{i} \frac{1}{t^{i+k}} (-1)^{i+k} B_{i,k}(1!, 2!, \ldots, (i-k+1)!)
\]
As a result,
\[
\sum_{k=1}^{i} a_{i,i-k} t^k = \sum_{k=1}^{i} B_{i,k}(1!, 2!, \ldots, (i-k+1)! t^k,
\]
which implies
\[
B_{n,k}(1!, 2!, \ldots, (n-k+1)! = a_{n,n-k}
\]
\[
= \binom{n}{n-k} \frac{(n-1)!}{(n-k)!} = \binom{n}{k} \binom{n-1}{k-1} (n-k)!, \quad (3.8)
\]
a recovery of the identity (3.3).

Remark 3.6. In [3] and [4] and related references therein, several special values of the Bell polynomials of the second kind $B_{n,k}$ are collected and applied.

Remark 3.7. The term $(-1)^{\ell-1}$ in (1.7) was misprinted as $(-1)^{n-1}$ in [1, p. 215, Theorem B].

Remark 3.8. This paper is a revised version of the preprint [8].
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