ON DELAUNAY TRIANGULATIONS OF GROMOV SETS

CURTIS PRO AND FREDERICK WILHELM

ABSTRACT. Let $Y$ be a subset of a metric space $X$. We say that $Y$ is $\eta$–Gromov provided $Y$ is $\eta$–separated and not properly contained in any other $\eta$–separated subset of $X$. In this paper, we review a result of Chew which says that any $\eta$–Gromov subset of $\mathbb{R}^2$ admits a triangulation $T$ whose smallest angle is at least $\pi/6$ and whose edges have length between $\eta$ and $2\eta$. We then show that given any $k = 1, 2, 3, \ldots$, there is a subdivision $T_k$ of $T$ whose edges have length in $[\eta, \frac{2\eta}{10}]$ and whose minimum angle is also $\pi/6$.

These results are used in the proof of the following theorem in [10]: For any $k \in \mathbb{R}, v > 0$, and $D > 0$, the class of closed Riemannian $n$–manifolds with sectional curvature $\geq k$, volume $\geq v$, and diameter $\leq D$ contains at most finitely many diffeomorphism types. Additionally, these results imply that for any $\varepsilon > 0$, if $\eta > 0$ is sufficiently small, any $\eta$–Gromov subset of a compact Riemannian 2–manifold admits a geodesic triangulation $T$ for which all side lengths are in $[\eta(1-\varepsilon), 2\eta(1+\varepsilon)]$ and all angles are $\geq \pi/6 - \varepsilon$.

Let $\mathcal{M}_{K,V,D}^{K,V,D} (n)$ denote the class of closed Riemannian $n$–manifolds $M$ with

$$
\begin{align*}
    k & \leq \sec M \leq K, \\
    v & \leq \vol M \leq V, \text{ and} \\
    d & \leq \diam M \leq D,
\end{align*}
$$

where $\sec$ is sectional curvature, $\vol$ is volume, and $\diam$ is diameter.

Cheeger’s finiteness theorem says that $\mathcal{M}_{K,V,D}^{K,V,D} (n)$ contains only finitely many diffeomorphism types ([2], [3], [12], [14]). In [17], we prove the following, which generalizes Cheeger’s finiteness theorem and the result of Grove, Petersen and Wu from [10].

**Theorem A.** For any $k \in \mathbb{R}, v > 0, D > 0$, and $n \in \mathbb{N}$, the class of closed Riemannian $n$–manifolds $\mathcal{M}_{k,v,0}^{\infty,\infty,D} (n)$ contains at most finitely many diffeomorphism types.

Except in dimension 4, this result was established in the early 1990s via work of Grove–Petersen–Wu, Perelman, and Kirby-Siebenmann in [10], [13], [11], and [12]. For further details on finiteness theorems we refer the reader to [9]. Our argument in [17] only treats the case $n = 4$, and depends on the fact that there is a special family of simplicial complexes $\mathcal{T}$ in $\mathbb{R}^2$. This family has a uniform lower bound for all angles and certain subdivision and extension properties. The purpose of this paper is to establish the existence of such a family. More specifically, we show that there are nonempty examples of

**Definition B.** Let $\mathcal{F}$ be a family of triangulations of subsets of $\mathbb{R}^2$. Given $\theta_0, \sigma_0 > 0$, we say that $\mathcal{F}$ is $(\theta_0, \sigma_0)$–nondegenerate, extendable, and subdividable provided:

1. All angles of all triangles in all $\mathcal{T} \in \mathcal{F}$ are $\geq \theta_0$.
2. For all $\mathcal{T} \in \mathcal{F}$ all ratios of all edge lengths of $\mathcal{T}$ are bounded from above by $\sigma_0$.

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Figure 1. The triangulation of the four vertices on the left fails the circumdisk property and is not Delaunay. The opposite is true for the triangulation on the right.

3. For all $T \in F$ there is a $T_{\text{ext}} \in F$ so that $T_{\text{ext}}$ triangulates $\mathbb{R}^2$ and $T \subset T_{\text{ext}}$.

4. Given any $\varepsilon > 0$ and any $T \in F$ there is a subdivision $\tilde{T}$ of $T$ so that $\tilde{T} \in F$ and all edges of $\tilde{T}$ have length $< \varepsilon$.

Here we show

**Theorem C.** The family $F$ of $\left( \frac{\pi}{6}, 2 \right)$–nondegenerate, extendable, and subdividable triangulations of $\mathbb{R}^2$ is not empty.

There are numerous papers in the computer science and computational geometry literature that address the non-degeneracy and extension aspects of this theorem (see e.g. [4] and the references therein). Among these, Chew’s is the most useful for our purposes ([5]). It is based on Delaunay triangulations of what we have decided to call Gromov sets.

A triangulation $T$ of a point set of $\mathbb{R}^2$ is called **Delaunay** if and only if the circumdisk of each 2-simplex contains no vertices of $T$ in its interior (see e.g. [8], Chapter 6). Among all possible triangulations of a given point set, it is well known that a Delaunay triangulation maximizes the minimal angle (see e.g. Theorem 9.9 of [7]).

The notion of a **Gromov subset** of a metric space is motivated by the proof of Gromov’s precompactness theorem, the notion of totally bounded metric spaces, and Gromov, Perelman, and Burago’s notion of rough volume ([1],[15]).

**Definition.** Given a metric space $X$ and $\eta > 0$, we say that $Y \subset X$ is $\eta$–**Gromov** provided $Y$ is $\eta$–separated and maximal with respect to inclusion, that is, $Y$ is not properly contained in any $\eta$–separated subset of $X$.

**Definition** (Chew-Delaunay-Gromov Complexes). We call a simplicial complex $T$ in $\mathbb{R}^2$ an $\eta$–**CDG**, provided, $T$ is a subcomplex of a Delaunay triangulation $\hat{T}$ of an $\eta$–Gromov subset of $\mathbb{R}^2$. If $T = \hat{T}$, then $T$ is called a **maximal CDG**.

The algorithm on Pages 7–8 of [5] combined with the Theorem on Page 9 and the Corollary on Page 10 of [5] give the following.
Theorem D (Chew’s Angle Theorem). Let $\mathcal{T}$ be a CDG complex in $\mathbb{R}^2$. Then all angles of $\mathcal{T}$ are $\geq \frac{\pi}{6}$ and all edges of $\mathcal{T}$ are in $[\eta, 2\eta]$.

Any non-collinear collection of points in $\mathbb{R}^2$ admits a Delaunay triangulation ([8], Theorem 6.10), and, by definition, CDGs can always be extended to all of $\mathbb{R}^2$, so Chew’s Angle Theorem implies that the family of all CDGs satisfies Properties 1–3 of Definition B with $(\theta_0, \sigma_0) = \left(\frac{\pi}{6}, 2\right)$. Thus to prove Theorem C, it suffices to show that CDGs also satisfy Property 4 of Definition B.

The proof of Theorem A also exploits complexes that are close to, but not quite, CDGs. In particular, we will study Delaunay triangulations of sets $S_i$ with the same combinatorial structure as a fixed Delaunay triangulation on a set $S$, provided $S_i$ and $S$ are sufficiently close. One complication is that a fixed subset $S \subset \mathbb{R}^2$ can have more than one Delaunay triangulation, hence subsets $S_i$ arbitrarily close to $S$ can have Delaunay triangulations combinatorially distinct from a prescribed triangulation of $S$. The following two definitions are part of our strategy to account for these issues.

Definition E. ([8], page 74) For $S \subset \mathbb{R}^2$ we say a segment $e$ between two points of $S$ is in the Delaunay graph of $S$ if and only if $e$ is an edge of every Delaunay triangulation of $S$.

Definition F. Given a discrete $S \subset \mathbb{R}^2$, $\xi > 0$, and a segment $ab$ between two points $a$ and $b$ of $S$, we say that $ab$ is $\xi$–stable provided the following holds. For any embedding $\iota : S \rightarrow \mathbb{R}^2$ so that $|\iota - \text{id}_S| < \xi$, the segment $\iota(a)\iota(b)$ is in the Delaunay graph of $\iota(S)$.

The fact that CDGs satisfy Property 4 of Definition B is a consequence of the following result.

Theorem G. Let $\mathcal{T}$ be an $\eta$–CDG. There is a subdivision $\tilde{\mathcal{T}}$ of $\mathcal{T}$ that is an $\frac{\eta}{10}$–CDG. Moreover, each edge $\tilde{e}$ of $\tilde{\mathcal{T}}$ that is a subedge of an edge $e$ of $\mathcal{T}$ satisfies one of the following conditions:

1. If $\tilde{e}$ does not contain a vertex of $e$, then $e$ is $\left(\frac{1}{100}, \frac{\eta}{10}\right)$–stable.
2. If $\tilde{e}$ contains a vertex of $e$, then

$$\text{length}(\tilde{e}) = \frac{\text{length}(e)}{10}.$$  

In particular, each triangle $\tilde{\Delta}$ of $\tilde{\mathcal{T}}$ that has a vertex in $\mathcal{T}$ is similar to the triangle $\Delta$ of $\mathcal{T}$ with $\tilde{\Delta} \subset \Delta$.

In Section 1, we review some basic facts about Delaunay triangulations. In Section 2, we review the proof of Theorem D. In Section 3, we prove Theorem G, and in Section 4 we explore various deformations of Theorems D and G that we will need to prove Theorem A. Throughout the paper, we let $\mathcal{T}_k$ denote the set of $k$–simplices of a simplicial complex $\mathcal{T}$. We let $|\mathcal{T}|$ be the polyhedron determined by $\mathcal{T}$. 
1. Review of Delaunay Triangulations

Definition 1.1. ([8], Definition 6.8) The circumdisk of a triangle in \( \mathbb{R}^2 \) is the unique disk whose boundary circle passes through the three vertices of the triangle (see Figure 1).

A triangulation \( \mathcal{T} \) of a discrete point set \( P \subset \mathbb{R}^2 \) is called Delaunay if and only if every circumdisk of every triangle in \( \mathcal{T} \) contains no points of \( P \) in its interior.

The existence of Delaunay triangulations of discrete point sets in \( \mathbb{R}^2 \) is guaranteed by the following theorem.

Theorem 1.2. ([8], Theorem 6.10, Lemma 6.16, Figure 6.6c) Every discrete point set \( V \subset \mathbb{R}^2 \) has a Delaunay triangulation, provided \( V \) does not lie on a line. The Delaunay triangulation of \( V \) is unique provided no four points of \( V \) lie on a circle. On the other hand, if four points of \( V \) lie on a circle, then \( V \) has more than one Delaunay triangulation.

The proof takes any triangulation \( \mathcal{T} \) of \( V \) and then performs a sequence of edge replacements to \( \mathcal{T} \) described as follows (see Figure 1). If \( \Delta_1 = \Delta_{ab}p \) and \( \Delta_2 = \Delta_{ax}b \) are two triangles of \( \mathcal{T} \) that share a common edge \( e = ab \), the following test is applied to \( e \) to determine if it should be replaced:

**Lawson Flip Test:** Let \( D_{\Delta_1} \) be the circumdisk of \( \Delta_1 \).
- If \( x \in \text{int}D_{\Delta_1} \), then replace \( e = ab \) with \( \tilde{e} = px \).
- If \( x \notin D_{\Delta_1} \), then do not replace \( e = ab \).
- If \( x \in \partial D_{\Delta_1} \), then either \( e = ab \) or \( \tilde{e} = px \) are acceptable.

This algorithm produces a Delaunay triangulation of \( \mathcal{T} \). In fact,

**Lemma 1.3.** (See e.g., Proposition 6.13 in [8].) Let \( e \) be an edge of a triangulation \( \mathcal{T} \) in \( \mathbb{R}^2 \). If \( e \) passes the Lawson flip test, then \( e \) is an edge of a Delaunay triangulation of \( \mathcal{T}_0 \).

For our purposes, it will be convenient to use an alternative to the Lawson Flip Test which we call the

**Angle Flip Test:** Let \( Q = apbx \) be the quadrilateral formed by \( \Delta_1 \), and \( \Delta_2 \). Let \( \angle a, \angle p, \angle b, \) and \( \angle x \), denote the angles of \( Q \) at \( a, p, b, \) and \( x \), respectively and note \( \angle a + \angle p + \angle b + \angle x = 2\pi \).
- If \( \angle p + \angle x > \pi \), then replace \( e = ab \) with \( \tilde{e} = px \).
- If \( \angle p + \angle x < \pi \), then do not replace \( e = ab \).
- If \( \angle p + \angle x = \angle a + \angle b = \pi \), then either \( e = ab \) or \( \tilde{e} = px \) is acceptable.

To see that these tests are equivalent recall

**Theorem 1.4.** (Thale’s Theorem, page 194, [7]) Let \( C \) be a circle in \( \mathbb{R}^2 \) that contains the points \( a, p, b, \) and \( q \). Suppose that the four points \( b, q, r, \) and \( s, \) lie on the same side of \( ap \) and that \( s \) is outside of \( C \) and that \( r \) is inside of \( C \). Then

\[
\angle (a, r, p) > \angle (a, b, p) = \angle (a, q, p) > \angle (a, s, p) .
\]

Let \( Q = apbx \) be a quadrilateral in \( \mathbb{R}^2 \) whose vertices \( a, p, b, x \) are listed in counterclockwise order. If \( Q \) is a rectangle then all four vertices lie on a circle \( C \), and

\[
\angle a + \angle b = \angle p + \angle x = \pi .
\]
If \( Q \) is perturbed in a way so that the points \( a \) and \( b \) are fixed and \( x, p \in C \), then by Euclid’s Central Angle Theorem, the angles \( \angle x \) and \( \angle p \) remain constant. So (1.4.1) holds if and only if \( Q \) is inscribed in a circle \( C \).

Again fix three points \( a, p, \) and \( b \) on \( C \). If \( x \) is outside of \( C \), then by Thale’s theorem \( \angle x \) is smaller than it is in (1.4.1), so \( \angle p + \angle x < \pi \). Conversely, if \( x \) is inside \( C \), \( \angle p + \angle x > \pi \).

Thus we have proven

**Proposition 1.5.** Let \( e \) be an edge of a triangulation \( T \) in \( \mathbb{R}^2 \). \( e \) passes the Lawson flip test if and only if \( e \) passes the angle flip test.

**Definition 1.6.** An edge \( e \) of a triangulation \( T \) will be called a Delaunay edge provided \( e \) is not replaced by the Angle Flip Test. A triangle \( \Delta \) in \( T \) is called a Delaunay triangle provided each of its edges are Delaunay.

### 2. Chew’s Angle Theorem

To prove Theorem D, it suffices to consider the case when \( T \) is a maximal CDG. This, together with the fact that all edges of an \( \eta \)-CDG have length \( \geq \eta \), means that Theorem D follows from the following result.

**Theorem 2.1.** Let \( T_0 \) be a subset of \( \mathbb{R}^2 \) that is \( \eta \)-dense, and let \( T \) be a Delaunay triangulation of \( T_0 \). Then all side lengths of \( T \) are \( \leq 2\eta \), and any triangle of \( T \) whose side lengths are \( \geq \eta \) has angles \( \geq \frac{\pi}{6} \).

We begin the proof with

**Proposition 2.2.** Let \( T_0 \) be a subset of \( \mathbb{R}^2 \) that is \( \eta \)-dense, and let \( T \) be a Delaunay triangulation of \( T_0 \). If \( \Delta \) is a triangle in \( T \), and \( D_\Delta \) is its circumdisk, then the radius of \( D_\Delta \) is \( \leq \eta \).

**Proof.** Let \( v \) be a vertex of \( \Delta \). Let \( r \) be the radius of \( D_\Delta \), and assume that \( r > \eta \). Let \( x \) be the center of \( D_\Delta \). Since \( T \) is Delaunay, it has no vertices in the interior of \( D_\Delta \). Hence the distance from \( x \) to all vertices of \( T \) is \( > \eta \). This contradicts the hypothesis that the vertices of \( T \) are \( \eta \)-dense. So the radius \( r \) of \( D_\Delta \) is \( \leq \eta \), as claimed. \( \square \)

We are ready to prove the first statement of Theorem 2.1, which is the content of the next result.

**Proposition 2.3.** Let \( T_0 \) be a subset of \( \mathbb{R}^2 \) that is \( \eta \)-dense, and let \( T \) be a Delaunay triangulation of \( T_0 \). Then all side lengths of \( T \) are \( \leq 2\eta \).

In particular, if \( T \) is a maximal \( \eta \)-CDG, then all edges of \( T \) have length in the interval \([\eta, 2\eta]\).

**Proof.** Assume \( e \) is an edge of \( \Delta \). Since \( D_\Delta \) has radius \( \leq \eta \), and \( e \) is a chord of \( D_\Delta \), by the triangle inequality, \( |e| \leq 2\eta \).

If \( T \) is a maximal \( \eta \)-CDG, then the vertices of \( T \) are \( \eta \)-separated, we also have \( \eta \leq |e| \). \( \square \)

To complete the proof of Theorem 2.1, we show
Proposition 2.4. Let $T_0$ be a subset of $\mathbb{R}^2$ that is $\eta$–dense, and let $T$ be a Delaunay triangulation of $T_0$. If $\Delta = \Delta abc$ is a triangle in $T$ whose edge lengths are in $[\eta, 2\eta]$, then all angles of $\Delta$ are $\geq \pi/6$.

Proof. Let $D_\Delta$ be the circumdisk of $\Delta$ and let $x$ be the center of $D_\Delta$. Since the edges of $\Delta$ are in the interval $[\eta, 2\eta]$ and the radius $r$ of $D_\Delta$ is $\leq \eta$, for any two vertices, say $a,b$, of $\Delta$, we have

$$\angle axb \geq \frac{\pi}{3}.$$ 

By Euclid’s Central Angle Theorem, $\angle acb \geq \pi/6$. \hfill \square

Before leaving the topic of Chew’s Angle Theorem, we record the following two results on the geometry of the triangles of CDGs that we use in the sequel.

Lemma 2.5. Let $\Delta = \Delta (a,b,c)$ be a triangle with all side lengths $\geq \eta$ and all angles $\geq \pi/6$. If $l_{ab}$ denotes the line through $a$ and $b$, then

$$\text{dist} (c, l_{ab}) \geq \frac{\eta}{2}.$$ 

If equality occurs, then $\angle (a, c, b) = \frac{\pi}{3}$ and $|ca| = |cb| = \eta$.

Proof. At least one of $\angle a$ or $\angle b$ is acute. If for instance $\angle a$ is acute, then

$$\text{dist} (c, l_{ab}) = |ac| \sin \angle a \geq \frac{\eta}{2},$$

as claimed. Notice that equality forces $|ac| = \eta$ and $\angle a = \frac{\pi}{3}$, and repeating this argument with $\angle b$ shows that in the equality case, $|bc| = \eta$ and $\angle b = \frac{\pi}{6}$. \hfill \square

The following is an immediate corollary.

Corollary 2.6. Let $T$ be a simplicial complex in $\mathbb{R}^2$ with side lengths in $[\eta, 2\eta)$ and angles $\geq \frac{\pi}{6}$. Then for any $v \in T_0$,

$$B \left( v, \frac{\eta}{2} \right) \cap |T| \subseteq \bigcup_{s \in T \mid v \in |s|} |s|,$$

where

$$B \left( v, \frac{\eta}{2} \right) := \left\{ x \in \mathbb{R}^2 \mid \text{dist} (x, v) < \frac{\eta}{2} \right\}.$$ 

3. Almost Legal Subdivisions

In this section, we prove Theorem G. The strategy is to subdivide the 1–skeleton and then to extend to the interior of the original 2–skeleton. Subdividing the 1–skeleton turns out to be the bigger challenge, so for most of the section, we focus on subdividing the following type of graph.

Definition 3.1. We call a graph $\eta$–geometric provided its vertices are $\eta$–separated and its edge lengths are in the interval $[\eta, 2\eta]$. 
To force our subdivided edges to be part of the new, finer Delaunay triangulation, in most cases, we arrange that they be in the Delaunay graph. To do this, we start by recalling

**Lemma 3.2.** (see Lemma 6.16 on page 74 of [8]) For $S \subset \mathbb{R}^2$, a segment $e$ between two points $a, b \in S$ is in the Delaunay graph of $S$ if and only if the closed disk $D_e$ with diameter $e$ contains no point of $S \setminus \{a, b\}$.

Using this we will show

**Proposition 3.3.** For $\eta > 0$, let $S$ be an $\eta$–Gromov subset of $\mathbb{R}^2$. If $a, b \in S$ satisfy $\text{dist}(a, b) < \sqrt{2}\eta$, then the segment $ab$ between $a$ and $b$ is in the Delaunay graph of $S$.

**Proof.** By Lemma 3.2, $ab$ is in the Delaunay graph of $S$ if the closed disk $D_{ab}$ with diameter $e$ contains no point of $S \setminus \{a, b\}$. If $\text{dist}(a, b) < \sqrt{2}\eta$, then no point in $D_{ab}$ is further than $\eta$ from $\{a, b\}$. Since $S$ is an $\eta$–Gromov set, no point in $D_{ab}$ can be in $S \setminus \{a, b\}$. □

Motivated by Proposition 3.3 and Definition F, we make the following definition, wherein the constant $\frac{1}{10}$ could be any small, fixed positive number.

**Definition 3.4.** We call an edge of an $\eta$–geometric graph $\eta$–legal if its length is strictly less than $(\sqrt{2} - \frac{1}{10})\eta$. An $\eta$–geometric graph is called legal if and only if all of its edges are $\eta$–legal.

**Proposition 3.5.** If $e$ is a line segment with $\eta \leq \text{length}(e) \leq 2\eta$, then there is a subdivision of $e$ into a legal $\frac{\eta}{10}$–geometric graph.

**Proof.** We will subdivide $e$ into subedges of equal length. The number of subedges will be between 10 and 19 and is the following function of the length of $e$.

Let $n_\eta : [\eta, 2\eta] \to \{10, 11, \ldots, 19\}$ be the step function that takes the 10 disjoint subintervals

$$
\left[\eta, \frac{11\eta}{10}\right), \left[\frac{11\eta}{10}, \frac{12\eta}{10}\right), \ldots, \left[\frac{19\eta}{10}, 2\eta\right]
$$

successively to $\{10, 11, \ldots, 19\}$. Thus for $k = 0, 1, \ldots, 9$,

$$
n_\eta(s) = \begin{cases} 10 + k & \text{if } s \in \left[\frac{10+k\eta}{10}, \frac{11+k\eta}{10}\right) \\ 19 & \text{if } s = 2\eta. \end{cases}
$$

Now subdivide the edge $e$ into $n_\eta(\text{length}(e))$ subedges of equal length. The length of each subedge is then

$$
f(\text{length}(e)) := \frac{\text{length}(e)}{n_\eta(\text{length}(e))}.
$$

Notice that the restriction of $f$ to each interval $[\frac{10+k\eta}{10}, \frac{11+k\eta}{10})$ is increasing. At the endpoints we have

$$
f\left(\frac{(10+k)\eta}{10}\right) = \frac{\eta}{10} \quad \text{and} \quad \lim_{s \to \left(\frac{11+k}{10}\right)} f(s) = \frac{(11+k)\eta}{(10+k)10} \leq \frac{11\eta}{10}.
$$
Therefore, for all \( s \in [\eta, 2\eta] \),
\[
\frac{\eta}{10} \leq f(s) \leq \frac{11\eta}{10} < \left( \sqrt{2} - \frac{1}{10} \right) \frac{\eta}{10},
\]
and the resulting subdivision is a legal \( \frac{n}{10} \)-geometric graph. \( \square \)

For graphs in \( \mathbb{R} \) this proposition gives us

**Lemma 3.6.** Let \( \mathcal{T} \) be an \( \eta \)-geometric graph in \( \mathbb{R} \). There is a subdivision \( \tilde{\mathcal{T}} \) of \( \mathcal{T} \) that is legal and \( \frac{n}{10} \)-geometric.

If instead we assume that \( \mathcal{T} \) is an \( \eta \)-geometric graph in \( \mathbb{R}^2 \), applying Proposition 3.5 to each edge does not in general lead to an \( \frac{n}{10} \)-separated configuration. The problem arises when the angle between adjacent edges is small. Even Chew’s estimate that all angles of CDGs are \( \geq \frac{\pi}{6} \), is insufficient to resolve this issue. So to prove Theorem G we use the following slight modification of the concept of legal subdivisions.

**Definition 3.7.** Let \( \mathcal{T} \) be an \( \eta \)-geometric graph in \( \mathbb{R}^2 \). A subdivision \( \tilde{\mathcal{T}} \) of \( \mathcal{T} \) is called \( \frac{n}{10} \)-almost legal provided \( \tilde{\mathcal{T}} \) is \( \frac{n}{10} \)-geometric and all edges of \( \tilde{\mathcal{T}} \) are \( \frac{n}{10} \)-legal except possibly for edges with a bounding vertex in \( \mathcal{T}_0 \). We further require that each edge \( \tilde{e} \) with a bounding vertex in \( \mathcal{T}_0 \) satisfies
\[
\text{length (} \tilde{e} \text{)} = \frac{1}{10} \text{length (} e \text{)},
\]
where \( e \) is the edge of \( \mathcal{T} \) that contains \( \tilde{e} \).

To prove Theorem G it suffices to consider the case when \( \mathcal{T} \) is a maximal \( \eta \)-CDG. Moreover, it follows from Proposition 3.3 that a legal edge of an \( \frac{n}{10} \)-CDG is \( \left( \frac{1}{100}, \frac{n}{10} \right) \)-stable. Thus Theorem G follows from

**Theorem 3.8.** Every maximal \( \eta \)-CDG has a subdivision that is an almost legal \( \frac{n}{10} \)-CDG.

To begin the proof, we will subdivide our 1–skeleton so that it is \( \frac{n}{10} \)-almost legal. The following result asserts that this is possible.

**Lemma 3.9.** Let \( \mathcal{T} \) be an \( \eta \)-geometric graph in \( \mathbb{R}^2 \). There is a subdivision \( \hat{\mathcal{T}} \) of \( \mathcal{T} \) which is \( \frac{n}{10} \)-almost legal.

**Proof.** For each edge \( e \) in \( \mathcal{T} \), divide \( e \) into 3 subedges so that the two subedges that contain a vertex of \( e \) have length \( \text{length}(e)/10 \). With a modification of the step function \( n_\eta \) used in Proposition 3.5, produce a legal subdivision of the remaining interior edge. \( \square \)

**Proof of Theorem 3.8.** By Lemma 3.9 there is an almost legal subdivision \( \mathcal{G} \) of the 1–skeleton \( \mathcal{T}_0 \cup \mathcal{T}_1 \) of \( \mathcal{T} \). Let \( \tilde{\mathcal{T}}_0 \) be an extension of \( \mathcal{G}_0 \) to an \( \frac{n}{10} \)-Gromov subset of \( \mathbb{R}^2 \), and let \( \tilde{\mathcal{T}} \) be a Delaunay triangulation of \( \tilde{\mathcal{T}}_0 \). We claim that \( \tilde{\mathcal{T}} \) can be chosen to be a subdivision of \( \mathcal{T} \). This is equivalent to asserting that \( \tilde{\mathcal{T}} \) can be chosen so that its collection of edges includes the edges of \( \mathcal{G} \), that is, \( \mathcal{G}_1 \subset \tilde{\mathcal{T}}_1 \). To see this for \( \tilde{e} \in \mathcal{G}_1 \) we let \( e \) be the unique edge of \( \mathcal{T} \) so that \( \tilde{e} \subset e \). By construction, if \( \tilde{e} \) does not contain a vertex of \( e \), then \( \tilde{e} \) is legal and therefore \( \tilde{e} \subset \tilde{\mathcal{T}}_1 \). If on the other hand, \( \tilde{e} \) does contain a vertex \( v \) of \( e \), then it is precisely \( \frac{1}{10} \) the length of \( e \). It follows that \( \tilde{e} \) is contained in two triangles \( \tilde{\Delta}_1, \tilde{\Delta}_2 \) that are similar to the two
triangles $\Delta_1, \Delta_2$ of $\mathcal{T}$ that contain $e$. Since $e$ is an edge of the Delaunay triangulation $\mathcal{T}$, it follows from the angle flip test that we may choose $\mathcal{T}$ so that $\bar{e} \in \mathcal{F}_1$. Thus $\mathcal{T}$ is an $\frac{\eta}{10}$–CDG that is an almost legal subdivision of $\mathcal{T}$. □

4. Deforming CDGs

In essence, the proof of Theorem A exploits Theorems C, D, and G together with the principle that Riemannian manifolds are infinitesimally euclidean. Since Theorem A deals with infinitely many Riemannian manifolds simultaneously, turning this principle into a rigorous proof requires some careful analytic arguments on how these results deform. The purpose of this section is to carry out this analysis.

One issue is that the boundary of an $\eta$–CDG need not be stable in the sense of Definition F. To remedy this, in Subsection 4.1, we show that every $\eta$–CDG, $\mathcal{T}$, has extension $\mathcal{T}_{st}$ whose boundary is stable and is also not too far from $\mathcal{T}$. More precisely, $|\mathcal{T}_{st}| \subset B(|\mathcal{T}|, 2\eta)$. In Subsection 4.7, we define a deformation of the concept of a CDG that we call an almost CDG. Most of the key properties of almost CDGs are proven in Subsection 4.7. In particular we generalize Theorem D. In Subsection 4.19, we complete this process by explaining how to subdivide almost CDGs in a manner that generalizes Theorem G.

4.1. The $\xi$–stable Graph. In this subsection, we show we can always choose an extension $\mathcal{T}$ of an $\eta$–CDG $\mathcal{T}$ so that the boundary edges of $\mathcal{T}_{st}$ are stable in the sense of Definition F. More specifically, we will prove

**Proposition 4.2.** There is an $\xi > 0$ with the following property. Let $\mathcal{T}$ be an $\eta$–CDG which is a subcomplex of the maximal $\eta$–CDG, $\mathcal{T}_{\text{max}}$. There is an $\eta$–CDG, $\mathcal{T}_{st}$, so that

$$\mathcal{T} \subset \mathcal{T}_{st} \subset \mathcal{T}_{\text{max}},$$

$$|\mathcal{T}_{st}| \subset B(|\mathcal{T}|, 2\eta),$$

and the boundary of $\mathcal{T}_{st}$ consists of $\xi\eta$–stable edges.

First notice that Proposition 3.3 gives us

**Corollary 4.3.** Let $\mathcal{T}$ be an $\eta$–CDG. For all sufficiently small $\varepsilon > 0$, an edge $e$ of $\mathcal{T}$ is $\varepsilon\eta$-stable provided

$$\text{length}(e) < \eta \left( \sqrt{2} - 10\varepsilon \right).$$

Next we note that definition of Delaunay triangulation gives us

**Corollary 4.4.** Let $\mathcal{T}_0$ be a discrete point set in $\mathbb{R}^2$. For $b, d \in \mathcal{T}_0$ suppose that $bd$ is an edge of a Delaunay triangulation of $\mathcal{T}_0$. Then $bd$ is not in the Delaunay graph of $\mathcal{T}_0$ if and only if $bd$ is a diagonal of a quadrilateral $Q$ of $\mathcal{T}$ that is inscribed in a disk $D$ and int $D \cap \mathcal{T}_0 = \emptyset$. (Cf. Corollary 6.17 in [8].)

Combining this with the definition of $\xi$-stable, gives us the following result, wherein we use the term $\xi$-unstable to denote an edge $e$ that is in some Delaunay triangulation of $\mathcal{T}_0$ but is not $\xi$–stable.
Corollary 4.5. There is a $\xi > 0$ with the following property. Let $T$ be a maximal $\eta$-CDG and let $bd \in T_1$ be the diagonal of the quadrilateral $Q$ of $T$. Then $bd \in T_1$ is $\eta\xi$--unstable if and only if there is a disk $D$ that is related to the vertex set $Q_0$ of $Q$ as follows:

$$Q_0 = D \cap \mathcal{T}_0 \text{ and } Q_0 \subset B(\partial D, \eta\xi),$$

where $B(\partial D, \eta\xi)$ denotes the open $\eta\xi$--ball around $\partial D$.

It is possible that the boundary edges of $Q$ are unstable, but the manner in which this can happen is constrained by the rigidity of CDGs via the following result.

**Proposition 4.6.** For every $\zeta > 0$ there is a $\xi > 0$ with the following property. Let $T$ be a maximal $\eta$-CDG. If an edge $bd \in T_1$ is $\eta\xi$--unstable, then all of the following hold.

1. There is an $n$--gon $P$ and a disk $D$ so that the vertex set $P_0$ of $P$ is related to the disk $D$ as follows:

$$\{b, d\} \subset P_0 = D \cap \mathcal{T}_0, \text{ and } P_0 \subset B(\partial D, \eta\zeta).$$

Moreover, $n = 4, 5, \text{ or } 6$.

2. The vertices $b, d$ are non-adjacent vertices of $P$.

3. The boundary edges of $P$ are $\eta\xi$-stable.

**Proof.** From the previous result, we have that there is $\xi > 0$ so that if $bd \in T_1$ is $\eta\xi$--unstable, then there is a disk $D$ and a quadrilateral $Q := Q(a, b, c, d)$ that satisfy Conclusions 1 and 2 with $\zeta = \xi$. If the boundary edges of $Q$ are $\eta\xi$-stable, we are done. Otherwise, one of the boundary edges, say $ab$ of $Q$ is $\eta\xi$-unstable. By the previous corollary, $ab$ is the diagonal of a quadrilateral $\tilde{Q}$ whose vertex set contains $a, b, d$, and one additional point $p$. Moreover, there is a disk $\tilde{D}$ that is related to $\tilde{Q}$ as in (4.5.1). Since

$$\{a, b, d\} \subset B(\partial D, \eta\xi) \text{ and } \{a, b, d\} \subset B(\partial \tilde{D}, \eta\xi),$$

$$\text{dist}_{\text{Haus}}(D, \tilde{D}) < \tau(\eta\xi),$$

where $\tau : \mathbb{R} \to \mathbb{R}_+$ is some function that satisfies $\lim_{t \to 0} \tau(t) = 0$.

Thus the pentagon $P(a, p, b, c, d)$ and the disk $D$ satisfy (4.6.1) with $\zeta := \tau(\xi)$. Again we are done if the boundary edges of $P$ are $\eta\xi$-stable.

Otherwise, we repeat the argument above and obtain a hexagon $H$ of $T$ whose vertex set includes $\{a, p, b, c, d\}$ and satisfies (4.6.1) for some disk $\tilde{D}$. In an $\eta$-CDG all edge lengths are $\geq \eta$ and all circumdisks have radius $< \eta$; so if an $n$--gon $P$ of an $\eta$--CDG satisfies (4.6.1) for some disk $D$, then $n \leq 6$, and all edge lengths of $P$ are nearly $\eta$. Thus by Corollary 4.3, there is an $\xi_6 > 0$ so that all edges of $P$ are $\xi_6\eta$--stable. \qed

**Proof of Proposition 4.2.** For $\xi \in \left(0, \frac{1}{1000}\right)$, let $\xi > 0$ be as in Proposition 4.6. Let $e$ be an $\eta\xi$--unstable boundary edge of $T$. By Proposition 4.6, there is an $n \in \{4, 5, 6\}$ so that $e$ connects two nonadjacent vertices of an $n$--gon $P$ of $T_{\text{max}}$. $P$ is almost inscribed in a disk, and the boundary edges of $P$ are $\eta\xi$-stable.

Our edge $e$ separates $|P|$ into two components. Call them $C_-$ and $C_+$, and let $C_-$ be the component that contains the face of $T$ that has $e$ in its boundary.
Figure 2. Assume the side lengths of these squares is just under 2. Then the red vertices are a 1–Gromov subset of the red square, and the red edges are a Delaunay triangulation of the red vertices. The blue vertices are an extension of the red ones to a 1–Gromov subset of the union of the two squares, but the dashed red edge fails the angle flip test and hence is not Delaunay for the extended configuration. The Delaunay triangulation of the extended set consists of all of the edges pictured except for the dashed red one.

Now form a new complex $T_e$ which is the union of $T$ with the simplices of $T_{max} \setminus T$ that are contained in $|P| \cap C_+$. Since $P$ is almost inscribed in a disk of radius $\eta$, and since $P$ is an $n$–gon with $n \in \{4, 5, 6\}$ and side lengths $\geq \eta$,

$$T_e \setminus T \subset B (|e|, 2\eta).$$

Thus

$$T_e \subset B (|T|, 2\eta).$$

Since the number of unstable boundary edges of $T_e$ is one less than the number of unstable boundary edges of $T$, repeating this process a finite number of times completes the proof. □

4.7. Almost CDGs. In this subsection we define almost CDGs and discuss their key properties. First, notice that $A \subset X$ is $\eta$–Gromov if and only if

$$B (A, \eta) = X,$$

and for distinct $a, b \in A$,

$$\text{dist} (a, b) \geq \eta.$$

So the following is a natural deformation of this condition.

**Definition 4.8.** Given $\eta, \delta > 0$, we say a subset $A$ of a metric space $X$ is an $(\eta, \delta \eta)$–Gromov set if and only if

$$B (A, \eta (1 + \delta)) = X,$$

and for distinct $a, b \in A$,

$$\text{dist} (a, b) \geq \eta (1 - \delta).$$

A naive definition of CDGs is that they are Delaunay triangulations of Gromov sets. The problem with this idea is that while Gromov sets can always be extended, their Delaunay triangulations cannot necessarily be extended (see Figure 2). Our actual definition of CDGs
skirts this issue by placing CDGs inside of maximal ones. In our application, almost CDGs will not be presented inside of maximal complexes, but fortunately the technicality presented in Figure 2 is local at heart, and the following definition exploits this fact.

**Definition.** Let $X$ be a metric space with $U, U_x \subset X$ open in $X$. Given $\eta > 0$ and $\delta \geq 0$, let $A$ and $A_x$ be $(\eta, \delta \eta)$–Gromov subsets of the closures of $U$ and $U_x$ respectively. We say that $A$ is a buffer of $A$ provided:

1. The closed ball $D(U, 6\eta) \subset U_x$.
2. $A = A_x \cap \text{closure}(U)$.

While a perturbation of a CDG can fail the angle flip test, it will nevertheless pass the following easier test.

**Definition 4.9.** Let $T$ be a simplicial complex in $\mathbb{R}^2$. Let $e$ be an edge of $T$ which bounds two faces. We say that $e$ passes the $\varepsilon$–angle flip test if and only if the sum of the angles opposite $e$ is $\leq \pi + \varepsilon$.

We are now ready for the definition of almost CDGs.

**Definition 4.10.** Given $\eta, \delta, \varepsilon > 0$, let $T_0$ be an $(\eta, \delta \eta)$–Gromov subset of the closure of an open set $U \subset \mathbb{R}^2$ with buffer $(T_0)_x$. A triangulation $T$ of $T_0$ is called an $(\eta, \delta \eta, \varepsilon)$–CDG provided $T$ is a subcomplex of a triangulation $T_x$ of $(T_0)_x$, and each edge of $T_x$ passes the $\varepsilon$–angle flip test.

Arguing as in the proof of Proposition 4.2, we get

**Corollary 4.11.** There are $\varepsilon, \xi, \delta > 0$ with the following property. If $T$ is an $(\eta, \delta \eta, \varepsilon)$–CDG with buffer $T_x$, then there is an $(\eta, \delta \eta, \varepsilon)$–CDG, $T_{st}$, so that

$$T \subset T_{st} \subset T_x,$$

$$\left| T_{st} \right| \subset B \left( \left| T \right|, 2\eta \right),$$

and the boundary of $T_{st}$ consists of $\xi \eta$–stable edges.

**Definition 4.12.** We say that an $(\eta, \delta \eta, \varepsilon)$–CDG is $\xi \eta$–stable provided each of its boundary edges $T$ is $\xi \eta$–stable.

While an $(\eta, \delta \eta, \varepsilon)$–CDG is not necessarily Delaunay, it is almost Delaunay in the following sense.

**Definition 4.13.** A triangulation $T$ of a point set of $\mathbb{R}^2$ is called $\varepsilon$–Delaunay if and only if for each 2–simplex $\Delta \in T$, all vertices of $T$ that are in $D_\Delta$ are within $\varepsilon$ of $\partial D_\Delta$.

The proof of Proposition 1.5 gives us

**Proposition 4.14.** Given $\kappa > 0$ and a sufficiently small $\delta > 0$, there is an $\varepsilon > 0$ so that every $(\eta, \delta \eta, \varepsilon)$–CDG is $\kappa$–Delaunay.

Arguing as in the proof of Chew’s Angle Theorem (D) we get

**Proposition 4.15.** Given $d, \theta > 0$, there are $\varepsilon, \delta > 0$ so that for every $(\eta, \delta \eta, \varepsilon)$–CDG,

1. The radius of every circumdisk of every two simplex of $T$ is $\leq \eta + d$. 
2. Every edge length of $\mathcal{T}$ is in the interval $[\eta - d, 2\eta + d]$.
3. All angles of $\mathcal{T}$ are $\geq \frac{\pi}{6} - \theta$.

Using Proposition 4.14 we get

**Proposition 4.16.** Given $\kappa > 0$, there are $\varepsilon, \delta > 0$ that satisfy the following. Given any $(\eta, \delta \eta, \varepsilon)\text{-CDG, } \mathcal{T}$, in $[0, \beta] \times [0, \beta] \subset \mathbb{R}^2$ and any finite set of points $V_0$ in $\mathbb{R}^2 \setminus B(\mathcal{T}_0, 3\eta)$, there is a $\kappa\text{-Delaunay triangulation of } \mathcal{T}_0 \cup V_0$ that contains $\mathcal{T}$.

**Proof.** Choose $\delta, \varepsilon > 0$ so that Part 1 of Proposition 4.15 holds with $d = \eta/2$ and so that Proposition 4.14 implies $\mathcal{T}$ is $\kappa\text{-Delaunay.}$ Let $\Delta \in \mathcal{T}_2$ have circumdisk $D_\Delta$. Since $\mathcal{T}$ is a $\kappa\text{-Delaunay triangulation of } \mathcal{T}_0$, $D_\Delta \cap \mathcal{T}_0 \subset B(\partial D_\Delta, \kappa)$.

Since $V_0 \subset \mathbb{R}^2 \setminus B(\mathcal{T}_0, 3\eta)$ and the radius of $D_\Delta$ is less than $3\eta/2$, $D_\Delta \cap V_0 = \emptyset$.

Thus $D_\Delta \cap (\mathcal{T}_0 \cup V_0) \subset B(\partial D_\Delta, \kappa)$, and $\Delta$ is a $\kappa\text{-Delaunay triangle of } \mathcal{T}_0 \cup V_0$. \qed

We can now state the main result of the section, which is the following extension theorem for almost CDGs. It has the added feature that error estimates, $\varepsilon$ and $\delta$, for the new simplices can be chosen to be 0.

**Theorem 4.17.** For $\varepsilon$, $\xi$, and $\delta$ as in Corollary 4.11 and $\beta \geq 50\eta$, let $\mathcal{T}$ be a $\xi\eta\text{-stable } (\eta, \delta \eta, \varepsilon)\text{-CDG in } [0, \beta] \times [0, \beta] \subset \mathbb{R}^2$. There is a $\xi\eta\text{-stable } (\eta, \delta \eta, \varepsilon)\text{-CDG, } \tilde{\mathcal{T}}$, which extends $\mathcal{T}$ and has the following properties.

1. $[0, \beta] \times [0, \beta] \subset B\left(\tilde{\mathcal{T}}_0, \eta\right)$.
2. The buffer of $\tilde{\mathcal{T}}$ extends the buffer of $\mathcal{T}$, that is,
   \[(\mathcal{T}_x)_0 \subset \left(\tilde{\mathcal{T}}_x\right)_0.\] (4.17.1)
3. $\left(\tilde{\mathcal{T}}_x\right)_0$ is a subset of $[0, \beta + 6\eta] \times [0, \beta + 6\eta]$ so that
   \[
   \text{dist}\left(\left(\tilde{\mathcal{T}}_x\right)_0 \setminus (\mathcal{T}_x)_0, \mathcal{T}_0\right) \geq 3\eta, \] (4.17.2)
   \[
   [0, \beta + 6\eta] \times [0, \beta + 6\eta] \subset B\left(\left(\tilde{\mathcal{T}}_x\right)_0, \eta\right), \text{ and} \] (4.17.3)
   \[
   \text{dist } (v, w) \geq \eta, \] (4.17.4)
   for $v \in \left(\tilde{\mathcal{T}}_x\right)_0 \setminus (\mathcal{T}_x)_0$ and $w \in \left(\tilde{\mathcal{T}}_x\right)_0$.
4. $\tilde{\mathcal{T}}_x$ contains all legal subgraphs of $\mathcal{T}_x$.
5. Every edge of $\tilde{\mathcal{T}}$ that has a vertex in $\left(\tilde{\mathcal{T}}_0\right)_x \setminus \mathcal{T}_0$ is Delaunay in the sense that it passes the $\varepsilon\text{-angle flip test with } \varepsilon = 0$.
6. Every edge with a vertex in $\left(\tilde{\mathcal{T}}_0\right)_x \setminus (\mathcal{T}_0)_x$ has length $\leq 2\eta$. 

Proof. We construct \((\tilde{T}_0)_x\) by consecutively choosing points in \([0, \beta + 6\eta] \times [0, \beta + 6\eta] \setminus B(T_0, 3\eta)\) that are \(\eta\)-separated from each other and from \((T_0)_x\). By compactness the process ends in a finite number of steps. The final set, \((\tilde{T}_0)_x\) satisfies (4.17.1), (4.17.2), and (4.17.4) by construction. \((\tilde{T}_0)_x\) also satisfies (4.17.3), since otherwise the construction would have continued for at least one more step.

Let \(\tilde{T}_x\) be an \(\varepsilon\)-Delaunay triangulation of \((\tilde{T}_0)_x\) in the sense that every edge passes the \(\varepsilon\)-angle flip test. Since \((\tilde{T}_0)_x \setminus (T_0)_x \subset \mathbb{R}^2 \setminus B(T_0, 3\eta)\), it follows from Proposition 4.16 that we may choose the triangulation of \(\tilde{T}\) to be one that extends \(T\). By flipping the edges of \(\tilde{T}_x\) that are not Delaunay and have a vertex in \((\tilde{T}_0)_x \setminus (T_0)_x\), we force \(\tilde{T}_x\) to satisfy Conclusion 5. Since the boundary edges of \(T\) are \(\xi\eta\)-stable, we can do this without needing to flip these boundary edges and while preserving the condition that \(\tilde{T}_x\) is an \(\varepsilon\)-Delaunay triangulation of \((\tilde{T}_0)_x\). Since an edge passes the angle flip test if and only if it passes the Lawson flip test, an edge with a vertex in \((\tilde{T}_0)_x \setminus (T_0)_x\) is the diameter of a disk that has no vertices in its interior. Part 6 follows from this and (4.17.3).

Let 
\[
\tilde{T}_0^{\text{pre}} := (\tilde{T}_0)_x \cap \{[0, \beta + \eta] \times [0, \beta + \eta] \}. 
\]

Let \(\tilde{T}^{\text{pre}}\) be the subcomplex of \(\tilde{T}_x\) all of whose vertices are in \(\tilde{T}_0^{\text{pre}}\). By Corollary 4.11, there is a \(\xi\eta\)-stable \((\eta, \delta\eta, \varepsilon)\)-CDG, \(\tilde{T}\), so that
\[
T^{\text{pre}} \subset \tilde{T} \subset \tilde{T}_x \quad \text{and} \quad \left| \tilde{T} \right| \subset B\left(\left| T^{\text{pre}} \right|, 2\eta \right). 
\]

It follows from Proposition 3.3 and the definition of a legal graph (3.4) that \(\tilde{T}\) contains all legal subgraphs of \(\tilde{T}_x\). □

Informally, Parts 3 and 5 of the previous result say that \(\tilde{T}\) is an extension of \(T\) that corrects the two error estimates, \(\delta\) and \(\varepsilon\). Motivated by this, we propose

**Definition 4.18.** If \(T\) and \(\tilde{T}\) are related as in Theorem 4.17, then we will say that \(\tilde{T}\) is an error correcting extension of \(T\).

### 4.19. Almost Legal Subdivisions of Almost CDGs.

In this subsection, we show how the proof of Theorem 3.8 gives us almost legal subdivisions of almost CDGs.

**Definition 4.20.** Given \(\eta, \delta > 0\), we say that a graph \(T\) is \((\eta, \delta\eta)\)-geometric provided its vertices are \(\eta(1 - \delta)\)-separated and its edge lengths are all in \([\eta(1 - \delta), 2\eta(1 - \delta)]\).

**Definition 4.21.** Let \(T\) be an \((\eta, \delta\eta)\)-geometric graph. Given \(\eta, \delta > 0\), a subdivision \(\tilde{T}\) of \(T\) is called \((\frac{n}{10}, \frac{\delta n}{10})\)-almost legal if and only if \(\tilde{T}_0\) is \((\frac{n}{10}(1 - \delta))\)-separated and all edges
of $\tilde{T}$ have length in the interval $[\frac{n}{10}, \frac{n}{10}(\sqrt{2} - \frac{1}{10})]$, except possibly for edges with a bounding vertex in $T_0$. We further require that each edge $\tilde{e}$ with a bounding vertex in $T_0$ satisfies
\[
\text{length} (\tilde{e}) = \frac{1}{10} \text{length} (e),
\]
where $e$ is the edge of $T$ that contains $\tilde{e}$.

With a minor numerical adjustment, which we leave to the reader, the proof of Lemma 3.9 gives us

**Corollary 4.22.** Let $T$ be an $(\eta, \delta \eta)$–geometric graph in $\mathbb{R}^2$. If $\delta$ is sufficiently small, then there is a subdivision $\tilde{T}$ of $T$ which is $(\frac{n}{10}, \delta \frac{n}{10})$-almost legal.

Let $T$ be an $(\eta, \delta \eta, \varepsilon)$–CDG. Let $L(\text{sk}_1(T_x))$ be the almost legal subdivision of the 1-skeleton $\text{sk}_1(T)$ obtained by applying the previous corollary to $\text{sk}_1(T_x)$. As in the proof of Theorem 3.8, extend $L(\text{sk}_1(T_x))$ to a maximal subset $L(T_x)_0$ so that
\[
\text{dist} (v, w) \geq \frac{\eta}{10}
\]
for all $v \in L(T_x)_0$ and all $w \in L(T_x)_0 \setminus L(\text{sk}_1(T_x))_0$. Define $L(T)_0$ analogously, and let $L(T_x)$ and $L(T)$ be $\varepsilon$–Delaunay triangulations of $L(T_x)_0$ and $L(T)_0$, respectively. Arguing as in the proof of Theorem 3.8, we see that $L(T)$ is an $(\frac{n}{10}, \delta \frac{n}{10}, \varepsilon)$–CDG which is a subdivision of $T$.

This construction respects error correcting extensions. In fact,

**Theorem 4.23.** Let $T$ be an $(\eta, \delta \eta, \varepsilon)$–CDG in $[0, \beta] \times [0, \beta] \subset \mathbb{R}^2$. If $\tilde{T}$ is an error correcting extension of $T$, then there are subdivisions $L(T)$, $L(\tilde{T})$ of $T$ and $\tilde{T}$, respectively, so that

1. $L(T)$ and $L(\tilde{T})$ are $(\frac{n}{10}, \delta \frac{n}{10}, \varepsilon)$–CDGs.
2. $L(\tilde{T})$ is an error correcting of $L(T)$.
3. If $\tilde{G} \subset \tilde{T}_1$ is a legal subgraph of the 1-skeleton of $\tilde{T}$ and $L(\tilde{G})$ is a legal, $(\frac{n}{10}, \delta \frac{n}{10})$–geometric subdivision of $\tilde{G}$, then we can choose $L(\tilde{T})$ so that it contains $L(\tilde{G})$.

**Proof.** With one exception, everything follows from the construction. The exceptional property is that
\[
\text{dist} (v, w) \geq \frac{\eta}{10}
\]
for $v \in L(\tilde{T}_x)_0 \setminus L(T_x)_0$ and $w \in L(T_x)_0$. This, also, is immediate from the construction if either point is in $L(\text{sk}_1(\tilde{T}_x))_0$.

So suppose that $v \in L(\tilde{T}_x)_0 \setminus L(T_x)_0$ and $w \in L(T_x)_0$ satisfy
\[
\text{dist} (v, w) < \frac{\eta}{10},
\]
(4.23.1)
and neither point is in $\mathcal{L}\left(\text{sk}_1\left(\tilde{T}_x\right)\right)_0$. It follows that both $v$ and $w$ are $\eta/10$-separated from $\mathcal{L}\left(\text{sk}_1\left(\tilde{T}_x\right)\right)_0$. Applying Corollary 2.6 with $\eta/10$ playing the role of $\eta$, and using (4.23.1), we see that $v$ and $w$ must be opposite vertices of two triangles that share an edge. By Lemma 2.5, this edge must nearly have length $\sqrt{3}\eta/10$, and the angles at $v$ and $w$ of the respective triangles must nearly be $2\pi/3$. Since such an edge fails the angle flip test by a large margin, no such configuration can exist. □

The notion of error correcting extensions also makes sense for graphs in $\mathbb{R}$.

**Definition 4.24.** Let $\tilde{T}$ and $T$ be $(\eta, \delta\eta)$-geometric graphs with $T \subset \tilde{T}$ and $|T|, |\tilde{T}| \subset [0, \beta]$. We say that $\tilde{T}$ is an error correcting extension of $T$ provided $\left(\tilde{T}\right)_0$ is a maximal subset of $[0, \beta]$ so that $\text{dist}(v, w) \geq \eta$ for all $v \in \tilde{T}_0$ and all $w \in T_0$.

Via simpler arguments we get

**Corollary 4.25.** Let $\tilde{T}$ and $T$ be $(\eta, \delta\eta)$-geometric graphs with $|T|, |\tilde{T}| \subset [0, \beta]$. If $\tilde{T}$ is an error correcting extension of $T$, then we can choose the legal subdivisions $\mathcal{L}(T)$ and $\mathcal{L}(\tilde{T})$ of Lemma 3.6 so that $\mathcal{L}(\tilde{T})$ is an error correcting extension of $\mathcal{L}(T)$.

Chew’s angle theorem combined with the fact that Riemannian manifolds are infinitesimally euclidean and the definition of CDGs immediately yields

**Theorem 4.26.** Let $S$ be a compact Riemannian 2–manifold. For every $\varepsilon > 0$ there is an $\eta_0$ so that for all $\eta \in (0, \eta_0)$ every $\eta$–Gromov subset of $S$ admits a triangulation $T$ for which all side lengths are in $[\eta(1 - \varepsilon), 2\eta(1 + \varepsilon)]$ and all angles are $\geq \pi/6 - \varepsilon$.

The analog of this result for surfaces that are isometrically embedded in $\mathbb{R}^3$ is proven by Chew in [6].

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**Department of Mathematics, California State University, Stanislaus**  
*N-E-mail address: cpro@csustan.edu*

**Department of Mathematics, University of California, Riverside**  
*N-E-mail address: fred@math.ucr.edu*  
**URL:** https://sites.google.com/site/frederickwilhelmjr/home