Geometric Batyrev–Manin–Peyre for equivariant compactifications of additive groups

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Abstract

Building on previous works by Bilu, Chambert-Loir and Loeser, we study the asymptotic behaviour of the moduli space of sections of a given family over a smooth projective curve, assuming that the generic fiber is an equivariant compactification of a finite dimensional vector space. Working in a suitable Grothendieck ring of varieties, we show that the class of these moduli spaces converges, modulo an adequate normalisation, to a non-zero effective element, when the class of the sections goes arbitrary far from the boundary of the dual of the effective cone. The limit can be interpreted as a motivic Euler product in the sense of Bilu’s thesis. This result provides a positive answer to a motivic version of the Batyrev–Manin–Peyre conjectures in this particular setting.

Keywords Moduli space of curves · Motivic Euler product · Geometric Batyrev–Manin–Peyre conjecture

Mathematics Subject Classification 14H10 · 14G10

1 Introduction

The link between the structure of higher dimensional algebraic varieties and the rational curves on such varieties has been fruitfully exploited during the last decades (Kollár 1996; Debarre 2001). In particular, this approach is relevant when one studies varieties admitting many rational curves, such as Fano varieties, by the work of Mori (1979). Covering families of curves are mobile and it is known that the cone of mobile curves is exactly the dual of the effective cone, by the theorem of Boucksom et al. (2012). Starting from this remark, a new direction consists in studying the asymptotic behaviour of the moduli space of curves on a variety, when the class of the curves
goes arbitrary far from the boundary of the dual of the effective cone. This approach finds another source of inspiration in the Batyrev–Manin–Peyre conjectures (Batyrev and Manin 1990; Peyre 1995) which originally concern the estimation of the number of rational points of bounded height on an algebraic variety defined over a number field, when the bound tends to infinity. It is then natural to consider analogous questions over function fields, which amounts to studying the morphisms from a given projective curve $C$ to a given *sufficiently nice* variety, or more generally the sections of a family over $C$. Roughly speaking, in this context the notion of height translates into the notion of degree of a curve, and the height zeta function one may introduce in the arithmetic setting becomes a motivic height zeta function; its coefficients are the classes, in a Grothendieck ring of varieties, of the moduli spaces of sections of a given degree. The properties of such a generating series have been studied by Bourqui (2009) in the case of toric varieties and by Chambert-Loir and Loeser (2016) and Bilu (2018) in the case of compactifications of vector spaces.

In this paper we refine the study of the case of compactifications of vector spaces. More precisely, we study the asymptotic behaviour of the moduli space of sections of a family over a projective curve whose generic fibre is a compactification of a power of the additive group. One can attach to such a section a class in the dual of the Picard group of the generic fibre, called the *multidegree* of the section, and consider the class of the moduli space of sections of a given multidegree in the Grothendieck ring of varieties. The main result of this article is the following: if we normalise this last class by the class of the affine space of dimension the anticanonical degree of the curves, then it converges, when the multidegree tends to infinity *in a proper way*, to an effective element of the Grothendieck ring of varieties. In comparison, previous results only provided information concerning the moduli space of sections with a given degree relatively to the generic log-anticanonical line bundle of the family. By writing *in a proper way*, we mean that the multidegree goes arbitrary far from the boundary of the dual of the effective cone. Furthermore, the limit of the normalised classes of moduli spaces can be written as a motivic Euler product in the sense of Bilu’s thesis (Bilu 2018). It has the expected form in comparison with the predictions of the Batyrev–Manin–Peyre conjectures in the arithmetic context (Peyre 1995) as well as in the geometric (or motivic) context (Bourqui 2009, 2010). Our result also deals with the case of sections whose restriction to a fixed open subset of the curve has its image contained in a partial compactification of the vector space, a condition which requires the introduction of additional finer numerical invariants in order to distinguish the relevant components of the moduli space.

Since we will use properties of a certain motivic height zeta function mentioned above, we adopt the framework of the last chapter of Bilu (2018) which generalises the situation studied by Chambert-Loir and Loeser in (2016). We consider a quasi-projective smooth connected curve $C_0$ defined over an algebraically closed field $k$ of characteristic zero and $C$ its smooth projective compactification, of genus $g$ and field of rational functions $F = k(C)$. The complement of $C_0$ in $C$ is a finite scheme $S$. We take $G$ to be the additive group scheme $G^n_a$ for a given positive integer $n$ and we consider a projective irreducible $k$-scheme $\mathcal{X}$ together with a non-constant morphism $\pi : \mathcal{X} \to C$ satisfying the following assumptions. We assume that the generic fibre $X = \mathcal{X}_F$ is a smooth equivariant compactification of $G_F$, meaning that $X$ is a smooth
projective scheme over $F$ containing a dense open subset isomorphic to $G_F$, such that the group law of $G_F$ extends to an action of $G_F$ on $X$. Let $\mathcal{U}$ be a Zariski open subset of $X$. Similarly, $U = \mathcal{U} F$ is assumed to be stable under the action of $G_F$. We denote by $D$ the complement of $U$ in $X$. Our goal is to study the asymptotic behaviour of sections of such a family $\pi : X \to \mathbb{C}$, in a sense we make more precise in what follows.

By (Hassett and Tschinkel 1999, Theorem 2.7), the boundary $X \setminus G_F$ is a divisor whose irreducible components $(D_\alpha)_{\alpha \in \mathcal{A}}$ freely generate the Picard group of $X$, as well as its effective cone. There exist integers $\rho_\alpha \geq 2$ such that an anticanonical divisor is given by $\sum_{\alpha} \rho_\alpha D_\alpha$; in particular, it is big. A log-anticanonical divisor with respect to $D$ is then $\sum_{\alpha \in \mathcal{A}} \rho'_\alpha D_\alpha$ where $\rho'_\alpha = \rho_\alpha - 1$ if $D_\alpha$ is an irreducible component of $D$ and $\rho'_\alpha = \rho_\alpha$ otherwise.

For every $\alpha$ in $\mathcal{A}$, there is a line bundle $L_\alpha$ on $X$ extending $D_\alpha$. Given a tuple of integers $n = (n_\alpha)_{\alpha \in \mathcal{A}}$, the moduli space $M_n$ is the space parametrizing sections $\sigma : C \to X$ such that:

- $\sigma$ maps the generic point $\eta_C$ of $C$ to a point of $G_F$;
- the image of $C_0$ by $\sigma$ is contained in $\mathcal{U}$;
- for all $\alpha$ in $\mathcal{A}$, $\deg (\sigma^* L_\alpha) = n_\alpha$.

The existence of these moduli spaces follows from (Chambert-Loir and Loeser 2016, Proposition 2.2.2). In order to avoid any local obstruction to the existence of such sections, we assume that local sections exist: we suppose that for every closed point $v$ of $C_0$ the intersection of $G(F_v)$ with $\mathcal{U}(O_v)$ is non-empty, where $F_v$ is the completion of $F$ at $v$ and $O_v$ is its ring of integers.

Let $\mathcal{M}_k$ be the localisation of the Grothendieck ring of varieties over $k$ at the class $L$ of the affine line, that is $\mathcal{M}_k = \text{KVar}_k[L^{-1}]$. Our focus will be on studying the asymptotic behaviour of the class $[M_n]$ in $\mathcal{M}_k$ endowed with the weight topology of Bilu (2018), following a question raised by Peyre (Peyre 2021, Question 5.4).

Any section $\sigma : C \to X$ gives rise to an element of $\text{Pic}(X)^\vee$. It is explicitly given by

$$n_\sigma : \sum_{\alpha \in \mathcal{A}} \lambda_\alpha [D_\alpha] \mapsto \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \deg (\sigma^* L_\alpha)$$

and this defines a pairing

$$\langle \cdot, \cdot \rangle : \text{Pic}(X) \otimes \text{Pic}(X)^\vee \to \mathbb{Z}.$$

Let $L'_\rho = \otimes_{\alpha \in \mathcal{A}} \rho'_\alpha L_\alpha$ be the generic log-anticanonical line bundle. By the previous pairing, any section $\sigma$ parametrized by $M_n$ satisfies $n_\sigma = n$ and we will write $\langle \rho', n \rangle$ for $\langle \sum_{\alpha} \rho'_\alpha [D_\alpha], n_\sigma \rangle$. For now we restrain ourselves to a particular case of our main result.

**Theorem** Let $\pi : X \to C$ be a family as above, with $\mathcal{U} = X$. Then

$$[M_n] L^{-(\rho, n)}$$
converges in the completion of $\mathcal{M}_k$ for the weight topology, when $\min(n_\alpha)_{\alpha \in \mathcal{O}} \to \infty$. Furthermore, the limit is a non-zero effective element which can be interpreted as a motivic Euler product in the sense of Bilu’s thesis (Bilu 2018).

The condition $\min(n_\alpha) \to \infty$ actually means that the class of the sections in the dual of the effective cone goes arbitrary far from its boundaries in $\text{Pic}(\mathcal{X})^\vee$ (this class of curves is the multidegree mentioned before). However, note that this condition is not strong enough to fully distinguish components of the moduli space of sections when $\mathcal{U}$ is not necessarily equal to $\mathcal{X}$: one has to introduce additional invariants, corresponding to local intersection degrees above $S$, and take them to be arbitrarily large as well. We refer to Notation 2.1 for the precise definition of such a motivic Euler product and to Proposition 5.12 and Theorem 5.15 for our most general result. For example, still assuming in this introduction $\mathcal{U} = \mathcal{X}$ and $C = C_0$, then the convergence can be understood as

$$\left[ M_n \right] L^{-\langle \rho, n \rangle} \to \frac{L^{(1-g)\dim(X)}}{(1 - L^{-1})^\text{rg}(\text{Pic}(X))} \prod_{v \in C} \mathcal{c}_v \quad \text{when } \min_{\alpha \in \mathcal{O}} (n_\alpha) \to \infty$$

where for all but a finite number of places $v$

$$\mathcal{c}_v = (1 - L^{-1})^\text{rg}(\text{Pic}(X)) \left[ \mathcal{O}_v \right] \frac{L^{\dim(X)}}{L^{\dim(X)}}.$$

We give in Remarks 4.6 and 5.13 a complete interpretation of those local factors in terms of motivic volume.

This work is based on the study of the multivariate motivic height zeta function

$$Z(T) = \sum_{n \in \mathbb{Z}^\mathcal{O}} \left[ M_n \right] T^n$$

where $T$ is the family of indeterminates $(T_\alpha)_{\alpha \in \mathcal{O}}$ and thus $T^n$ stands for the product $\prod_{\alpha \in \mathcal{O}} T_\alpha^{n_\alpha}$ for all $n \in \mathbb{Z}^\mathcal{O}$. It is proven in (Bilu 2018, Chapter 6) that this function can be written as a sum over a space of characters. From this decomposition, Bilu deduces a description of the first pole of the restriction of the motivic height zeta function to the line generated by the canonical line bundle. Here, we push her method one step further to be able to distinguish the components given by all possible multidegrees. In order to separate components arising from the integrality conditions, introducing new variables and refining several tools due to Bilu is found to be necessary.

In the first section of this paper we start by collecting some definitions and classical results about Grothendieck rings of varieties. We shortly recall what are symmetric products of classes in the Grothendieck ring of varieties, leading to Bilu’s definition of the motivic Euler product (Bilu 2018). For completeness, we also add a motivic Fourier transform together with a motivic Poisson formula to this list of reminders. The second section is devoted to a study of the convergence of such Euler products. We briefly recall the main properties of the weight on the Grothendieck ring of varieties, include a useful convergence criterion for Euler products and quickly study sums of
families in this context. In the third part of this work we present a bunch of convergence results in the particular setting of compactifications of vector spaces. The fourth and last section is the heart of the proof of our result. For clarity’s sake, we start with treating the case of the projective line and \( U = \mathbb{X} \), then generalise our argument to any smooth projective curve, and finally treat the general case \( U \subset \mathbb{X} \).

2 Preliminaries

2.1 Rings of varieties

Along this paper we work in Grothendieck rings of varieties, or in localisations and completions of such rings. Here we collect some definitions and properties. References for this paragraph are the paper of Denef and Loeser (1999), the second chapter of the book by Chambert-Loir et al. (2018) and the articles of Chambert-Loir and Loeser (2016) and Cluckers and Loeser (2010).

Let \( R \) be a Noetherian scheme. The Grothendieck group of varieties \( K\text{Var}_R \) is defined as the abelian group generated by all \( R \)-varieties (that is, \( R \)-schemes of finite presentation), with relations

\[
X - Y
\]

whenever \( X \) and \( Y \) are isomorphic \( R \)-varieties and

\[
X - Y - U
\]

whenever \( X \) is an \( R \)-variety, \( Y \) is a closed subscheme of \( X \) and \( U \) is its open complement in \( X \). The product \( [X][Y] = [X \times_R Y] \) defines a ring structure on \( K\text{Var}_R \) with unit element the class of \( R \) over itself with the identity structural map. The class of the affine line \( \mathbb{A}^1_R \) in \( K\text{Var}_R \) is denoted by \( L \) and the localised Grothendieck ring of varieties \( K\text{Var}_R[L^{-1}] \) by \( K\text{Var}_R \). Any constructible subset \( X \) of a \( R \)-variety admits a class \( [X] \) in such rings (see e.g. page 59 in Chambert-Loir et al. 2018). In our case, a constructible subset is a finite union of locally closed subsets.

The Grothendieck ring of varieties with exponentials \( K\text{ExpVar}_R \) is defined in a similar way. Its generators are pairs of \( R \)-varieties \( X \) together with morphisms \( f : X \to \mathbb{A}^1 = \text{Spec}(\mathbb{Z}[T]) \). Relations are the isomorphism relation

\[
(X, f) - (Y, f \circ u)
\]

whenever \( X, Y \) are \( R \)-varieties, \( f : X \to \mathbb{A}^1 \) a morphism and \( u : Y \to X \) an isomorphism of \( R \)-varieties; the scissors relation

\[
(X, f) - (Y, f|_Y) - (U, f|_U)
\]

whenever \( X \) is a \( R \)-variety, \( Y \) a closed subscheme of \( X \), \( U \) its complement in \( X \) and \( f : X \to \mathbb{A}^1 \) a morphism; and finally the relation
whenever $X$ is a $R$-variety and pr$_2$ is the second projection. If $X$ is a constructible set and $f : X \to A^1$ a piecewise morphism (given by the datum of morphisms $f_i : X_i \to A^1$ on a partition $(X_i)_{1 \leq i \leq m}$ of $X$ into locally closed subsets) then the class $[X, f]$ is well-defined. The class $[A^1_R, 0]$ is denoted $L$. Sending a class of a $R$-variety $X$ to the class $[X, 0]$ defines a morphism of Abelian groups $\iota : \text{KVar}_R \to \text{KExpVar}_R$ sending $L$ to $L$. The product

$$[X, f][Y, g] = [X \times_R Y, f \circ \text{pr}_1 + g \circ \text{pr}_2]$$

defines a ring structure on $\text{KExpVar}_R$, with unit element $[\text{Spec}(R), 0]$. Then the morphism $\iota$ is actually a morphism of rings. Localising at $L$, one gets a ring $\text{Exp}\mathcal{M}_R$ together with a ring morphism $\iota : \mathcal{M}_R \to \text{Exp}\mathcal{M}_R$.

Any morphism $u : R \to S$ of $k$-varieties induces a morphism of group $u! : \text{KExpVar}_R \to \text{KExpVar}_S$

sending any effective element $[X, f]_R$ to $[X, f]_S$, where we view $X$ as an $S$-variety through $u$. If $u$ is an immersion, $u!$ is a morphism of rings. The morphism $u$ induces as well a morphism of rings in the other direction

$$u^* : \text{KExpVar}_S \to \text{KExpVar}_R$$

sending any effective element $[X, f]_S$ to $[X \times_S R, f \circ \text{pr}_X]_R$. If $T$ is another $S$-variety, combining pull-backs and product, one obtains an exterior product

$$\boxtimes : \text{KExpVar}_R \times \text{KExpVar}_T \xrightarrow{\text{pr}_R^* \text{pr}_T^*} \text{KExpVar}_{R \times S T}.$$ 

We conclude this subsection by introducing an analogue of the exponential sums of characters over finite field. Assume for a while that $k$ is a finite field and $\psi : k \to \mathbb{C}^*$ is a non-trivial character. Then the exponential sum associated to a pair $[X, f]_k = [X, f]_{\text{Spec}(k)}$ is

$$\sum_{x \in X(k)} \psi(f(x)).$$

Let $S$ be a $k$-variety, together with a morphism $u : S \to A^1$, and $g : X \to S$ a variety over $S$ together with a morphism $f : X \to A^1$.
We write $\theta = [X, f]_S$. Then over a point $s$ we can introduce the sum over the fibre

$$\theta(s) = \sum_{x \in X(k) \atop g(x) = s} \psi(f(x)).$$

Using the additivity of $\psi$ we decompose fibre by fibre the sum

$$\sum_{s \in S(k)} \theta(s) \psi(u(s)) = \sum_{s \in S(k)} \sum_{x \in X(k) \atop g(x) = s} \psi(f(x)) \psi(u(s)) = \sum_{x \in X(k)} \psi(f(x) + u(g(x)))$$

which is the exponential sum associated to the pair $[X \times_S S, f + u \circ g]_S = [X, f]_S[S, u]_S$ viewed in $\text{KExpVar}_k$.

Thus in general we define the sum over rational points

$$\sum_{x \in S} \theta(s) \psi(u(s))$$

for any $\theta \in \mathcal{E}_{\text{exp}} \mathcal{M}_S$ and $u : S \to \mathbb{A}^1$ as the image by $\text{KExpVar}_S \to \text{KExpVar}_k$ of the class $\theta \cdot [S, u]_S$ (we may sometimes omit the exponential factor $\psi(u(s))$ when writing the sum). This class is explicitly given by $[X, f + u \circ g]$ when $\theta = [X, f]_S$ and $g : X \to S$ is a $S$-variety. This notation easily extends to the relative setting when $v : S \to T$ is a morphism of varieties over $k$: we write

$$v_! \theta = \sum_{s \in S/T} \theta(s) \psi(u(s))$$

where $v_! : \text{KExpVar}_S \to \text{KExpVar}_T$ is the morphism of groups induced by $v$.

### 2.2 Symmetric products

Let $R$ be a variety over a perfect field $k$, $X$ a variety over $R$ and $\mathcal{X} = (X_i)_{i \in I}$ be a family of quasi-projective varieties above $X$, where $I$ is an arbitrary set. Let $\pi = (n_i)_{i \in I} \in \mathbb{N}^{(I)}$ be a family of non-negative integers with finite support, which we call a partition. This terminology is justified by particular cases of set $I$: for example if $I = \mathbb{N}^r \setminus \{0\}$ for some given $r \in \mathbb{N}^*$, then a partition of a non-zero tuple of non-negative integers $n \in I$ is a family $(n_i)_{i \in I}$ such that $\sum_{i \in I} n_i = n$.

Given a partition $\pi$, one can define an $R$-variety $S^\pi(\mathcal{X}/R)$ called the ($\pi$-th) symmetric product of $\mathcal{X}$. Its geometric points can be understood as collections $(D_i)_{i \in I}$ of effective zero-cycles such that $D_i \in S^{n_i} X_i$ for all $i \in I$ and the union of all the
images in $X$ of the supports of the $D_i$'s consists of exactly $\sum_{i \in I} n_i$ distinct geometric points (Bilu 2018, §3.1 & §3.2). This is formally defined as follows: if $(\prod_{i \in I} X_i^{n_i})_*$ is the complement of the diagonal of $\prod_{i \in I} X_i^{n_i}$, in other words $(\prod_{i \in I} X_i^{n_i})_*$ is the set of points with pairwise distinct coordinates, then there is a Cartesian diagram

$$(\prod_{i \in I} X_i^{n_i})_* \hookrightarrow \prod_{i \in I} X_i^{n_i}$$

$$(\prod_{i \in I} X_i^{n_i})_* \hookrightarrow \prod_{i \in I} X_i^{n_i}$$

defining an open subset $(\prod_{i \in I} X_i^{n_i})_*$ of points of $\prod_{i \in I} X_i^{n_i}$ with pairwise distinct coordinates. Now one considers the natural action of the symmetric groups $S_{n_i}$ on each of the $X_i^{n_i}$ and defines the symmetric product

$$S^{\pi} (\mathcal{X}/R) = \left( \prod_{i \in I} X_i^{n_i} \right)_*/\prod_{i \in I} S_{n_i}$$

which is well-defined since $\mathcal{X}$ is a family of quasi-projective varieties over $X$. One can go one step further and define the symmetric product $S^{\pi} (\mathcal{X}'/R)$ of a family $(X_i)_{i \in I}$ of non-effective classes of $\text{KVar}_X$ as well as symmetric products of varieties with exponentials, in a way compatible with the definition of symmetric product of $X$-varieties (Bilu 2018, §3.5–3.7). If $n \in \mathbb{N} \setminus \{0\}$, then $S^n (\mathcal{X}/R)$ is by definition the disjoint union of all the $S^{\pi} (\mathcal{X}/R)$ for every partition $\pi$ of $n$. In case all the $X_i$ are equal to $X$, we obtain a decomposition of the classical symmetric power $S^n X$ of $X$ into locally closed subsets $S^n X$ (Bilu 2018, §3.1.1).

Let $\mathcal{X} = (a_i)_{i \in I}$ be a family of elements of $\text{KVar}_X$ and $\mathcal{Y} = (b_i)_{i \in I}$, $\mathcal{Z} = (c_i)_{i \in I}$ the families obtained by restriction respectively to a closed subscheme $Y \subset X$ and to its complement $U$. Then

$$S^{\pi} (\mathcal{X}) = \sum_{\pi' \leq \pi} S^{\pi'} (\mathcal{Y}) \boxtimes S^{\pi - \pi'} (\mathcal{Z})$$

in $\text{KVar}_{S^n X}$, where $\boxtimes$ is the exterior product morphism

$$\text{KVar}_{S^n Y} \times \text{KVar}_{S^n - \pi' U} \to \text{KVar}_{S^n Y \times S^n - \pi' U}$$

composed with the morphism induced by $S^n Y \times S^n - \pi' U \to S^n X$ on the rings of varieties (Bilu 2018, Corollary 3.5.2.5).

If $\mathcal{X}$ and $\mathcal{X}'$ are two families of quasiprojective varieties over $X$, it is clear from the definition that there is an isomorphism

$$S^{\pi} (\mathcal{X} \times_X \mathcal{X}') \simto S^{\pi} (\mathcal{X}) \times_{S^n X} S^{\pi} (\mathcal{X}') \quad (2.2.2)$$
2.3 Motivic Euler products

In this paragraph, $R$ is a variety and $A_R$ is one of the rings $K\text{Var}_R$, $K\text{ExpVar}_R$, $M_R$ or $\text{Exp} \cdot M_R$ previously defined.

Bilu introduced in the third chapter of her thesis (Bilu 2018, §3.8) the following notation.

**Notation 2.1** (Motivic Euler product) Consider a $R$-variety $X$ and $\mathcal{X} = (X_i)_{i \in I}$ a family of elements of $A_X$. Then the product

$$\prod_{x \in X/R} \left( 1 + \sum_{i \in I} X_{i,x} T_i \right)$$

is defined as a notation for the formal series

$$\sum_{\pi \in \mathbb{N}(I)} [S^\pi (\mathcal{X}/R)] T_\pi \in A_R[[T]]$$

where $T = (T_i)_{i \in I}$ is a family of indeterminates indexed by the set $I$ and $T_\pi$ stands for $\prod_{i \in I} T_i^{n_i}$ for every partition $\pi \in \mathbb{N}(I)$.

The following proposition shows that this object behaves, in a sense, *like a product*.

**Proposition 2.2** (§3.8.1 of Bilu 2018) Let $R$ be a variety, $X$ be a $R$-variety and $\mathcal{X} = (X_i)_{i \in I}$ be a family of elements of $A_X$.

- The Euler product notation is compatible with the $X = R$ case:

$$\prod_{x \in R/R} \left( 1 + \sum_{i \in I} X_{i,x} T_i \right) = 1 + \sum_{i \in I} X_i T_i.$$

- The Euler product notation is associative: if $X = U \cup Y$ with $Y$ a closed subscheme of $X$ and $U$ its complement, then

$$\prod_{x \in X/R} \left( 1 + \sum_{i \in I} X_{i,x} T_i \right) = \left( \prod_{u \in U/R} \left( 1 + \sum_{i \in I} X_{i,u} T_i \right) \right) \left( \prod_{y \in Y/R} \left( 1 + \sum_{i \in I} X_{i,y} T_i \right) \right)$$

when considering the Euler products of the restrictions $\mathcal{Y} = (X_i \times_X Y)_{i \in I}$ and $\mathcal{U} = (X_i \times_X U)_{i \in I}$. 

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As a consequence of the previous identity, the Euler product notation is compatible with finite products: if \( X \) is a finite disjoint union of varieties \( Y_1, \ldots, Y_m \) isomorphic to \( R \), and if \((Y_{i,j})_{i \in I}\) is the restriction of \( X \) to \( Y_j \) for \( j = 1, \ldots, m \), then

\[
\prod_{x \in X/R} \left( 1 + \sum_{i \in I} X_{i,x} T_i \right) = \prod_{j=1}^{m} \left( 1 + \sum_{i \in I} Y_{i,j} T_i \right).
\]

Another illustration of this behaviour is the following non-trivial property of multiplicativity, which is a particular case of (Bilu 2018, Proposition 3.9.2.4). Note that in the remainder of the article, we will mostly restrict ourselves to the case \( I = \mathbb{N}^r \setminus \{0\} \) and \( T_i = T^m \), where \( T_1, \ldots, T_r \) is a finite family of indeterminates and \( T^m = \prod_{j=1}^{r} T^{m_j} \) for every \( r \)-tuple \( m = (m_j)_{1 \leq j \leq r} \) of non-negative integers.

**Proposition 2.3** Let \( R \) be a variety, \( X \) a variety over \( R \), \( A = (A_m)_{m \in \mathbb{N}^r} \) a family of effective elements of \( A_R \), and \( B = (B_m)_{m \in \mathbb{N}^r} \) a family of elements of \( A_R \) such that \( A_0 = B_0 = 1 \). Then

\[
\prod_{x \in X/R} \left( \left( \sum_{m \in \mathbb{N}^r} A_{m,x} T^m \right) \left( \sum_{m \in \mathbb{N}^r} B_{m,x} T^m \right) \right) = \left( \prod_{x \in X/R} \left( \sum_{m \in \mathbb{N}^r} A_{m,x} T^m \right) \right) \left( \prod_{x \in X/R} \left( \sum_{m \in \mathbb{N}^r} B_{m,x} T^m \right) \right)
\]

in \( A_R[[T]] \).

### 2.4 Motivic Fourier transform and Poisson formula in families

In this subsection we recall the motivic analogue of a bunch of Fourier analysis tools used in the sixth chapter of Bilu (2018), where an easy-to-handle expression of the motivic height zeta function is obtained by using a motivic Poisson formula. We will explicitly need this construction at the end of our proof (Sect. 5.3), where we will have to check an uniform convergence. We follow Chambert-Loir and Loeser (2016) and the fifth chapter of Bilu (2018), which extends the scope of Hrushovski and Kazhdan’s motivic Poisson formula (Hrushovski and Kazhdan 2009).

#### 2.4.1 Local motivic Schwartz–Bruhat functions

The first building block of a motivic Poisson formula is a motivic analogue of classical Schwartz–Bruhat functions on the non-archimedean local field \( K = k((t)) \). One should keep in mind that this field can be thought about as the completion of the field of rational functions of a curve, for a valuation given by a closed point. The letter \( t \) denotes an uniformiser of this completion; then the ring of integers of \( K \) is \( \mathcal{O}_K = k[[t]] \).

Recall that a classical Schwartz–Bruhat function \( \varphi \) on a locally compact, non-archimedean local field \( L \) is a locally constant and compactly supported function on
If \( \sigma \) is an uniformiser for \( L \) and \( \mathcal{O}_L \) is the ring of integers of \( L \), then there exist integers \( M \leq N \) such that \( \varphi \) is zero outside \( \sigma^M \mathcal{O}_L \) and invariant modulo \( \sigma^N \mathcal{O}_L \). The pair \((M, N)\) is called the level of \( \varphi \) and the function \( \varphi \) on \( L \) can be seen as a function on the quotient \( \sigma^M \mathcal{O}_L / \sigma^N \mathcal{O}_L \).

The motivic analogue of such a function, as it has been introduced by Hrushovski and Kazhdan in (2009), takes values in the Grothendieck ring for all \( N \) at a closed point \( v \) (with fibre \( \mathcal{O}_v \)).

Let \( \mathbf{A}^n \) be integers. A (local) motivic Schwartz-Bruhat function of level \((M, N)\) is a function \( t^M \mathcal{O}_K / t^N \mathcal{O}_K \to \mathcal{E}x. \mathcal{M}_k \). Through the identification

\[
t^M \mathcal{O}_K / t^N \mathcal{O}_K \to A_k^{M-N}
\]

which endows \( t^M \mathcal{O}_K / t^N \mathcal{O}_K \) with the structure of a \( k \)-variety, such a motivic Schwartz-Bruhat function on \( K \) is basically seen as an element of \( \mathcal{E}x. \mathcal{M}_{A_k^{N-M}} \). In what follows we will freely use the notation

\[
A_k^{n(M,N)} = A_k^{n(N-M)}.
\]

for any \( M \leq N \) and non-negative integer \( n \). The previous identification naturally extends to \( K^n \), leading to the following definition.

**Definition 2.4** Let \( M \leq N \) and \( n \geq 1 \) be integers. A (local) motivic Schwartz-Bruhat function of level \((M, N)\) on \( K^n \) is an element of \( \mathcal{E}x. \mathcal{M}_{A_k^{n(M,N)}} \).

The *extension-by-zero* morphism \( t^M \mathcal{O}_K / t^N \mathcal{O}_K \to t^{M-1} \mathcal{O}_K / t^N \mathcal{O}_K \) and the *cutting* morphism \( t^M \mathcal{O}_K / t^{N+1} \mathcal{O}_K \to t^M \mathcal{O}_K / t^N \mathcal{O}_K \) induce respectively a closed immersion \( A_k^{(M,N)} \to A_k^{(M-1,N)} \) and a trivial fibration \( A_k^{(M,N+1)} \to A_k^{(M,N)} \) (with fibre \( A_k^1 \)). Such morphisms in turn provide homomorphisms at the level of Grothendieck rings with exponentials, turning \( \mathcal{E}x. \mathcal{M}_{A_k^{n(M,N)}} \) into a direct system, the direct limit of which is by definition the set of all motivic Schwartz-Bruhat functions on \( K^n \).

Fourier kernel and transform. Recall that we see \( K = K(t) \) as the completion \( F_v \) at a closed point \( v \) of the field of fractions \( F = k(C) \) of a smooth projective curve over \( k \), together with the choice of an uniformiser \( t \). If one fixes a non-zero rational differential form \( \omega \in \Omega_{K/k} \), then one obtains a non-zero \( k \)-linear map \( r_v : K \to k \) defined by sending any element \( a \in K \) to the residue at \( v \) of the rational form \( a\omega \):

\[
r_v(a) = \text{res}_v(a\omega).
\]

There exists a smallest integer \( v \) such that \( r_v \) vanishes on \( t^v \mathcal{O}_K \), given by the order of the pole of \( \omega \) at \( v \) (the *conductor* of \( r_v \)). In particular, \( r_v \) is invariant modulo \( t^N \mathcal{O}_K \) for all \( N \geq v \) and can be seen as a linear function \( r_{(M,N)} : A_k^{(M,N)} \to A_k^1 \) for every \( M \leq N \) such that \( N \geq v \).
The product $K \times K \to K$ restricts to a morphism $A^k_{(M,N)} \times A^k_{(M',N')} \to A^k_{(M+M',N'')}$, for every $M \leq N$, $M' \leq N'$ and $N'' = \min(M' + N, M + N')$. If $N, N', N'' \geq v$ it can be composed with $r_v^{(M+M',N'')}$ and one obtains an element

$$r_v : A^k_{(M,N)} \times A^k_{(M',N')} \to A^k_{(M+M',N'')}$$

of $\mathcal{E}xp.\mathcal{M}_{A^k_{(M,N)} \times k A^k_{(M',N')}}$, called the Fourier kernel on $K$ and written $e(xy)$. This notation is the analogue of the exponential factor $e^{2i\pi xy}$ of the integrand in classical Fourier analysis.

More generally, the pairing $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ on $K^n$ provides a morphism

$$A^n_{k(M,N)} \times A^n_{k(M',N')} (\cdot, \cdot) \to A^n_{k(M+M',N'')}$$

which can be composed with $r_v^{(M+M',N'')}$ to give the Fourier kernel $e(\langle x, y \rangle)$ on $K^n$.

The Fourier transform of a motivic Schwart-Bruhat function $\varphi \in \mathcal{E}xp.\mathcal{M}_{A^k_{n(v-N,y-M)}}$ of level $(M, N)$ on $K^n$ is defined as

$$\mathcal{F} \varphi(y) = \int_{K^n} \varphi(x) e(\langle x, y \rangle) dx$$

(2.4.3)

which is a notation for the class of $\mathcal{E}xp.\mathcal{M}_{A^k_{n(v-N,y-M)}}$ given by

$$\mathcal{F} \varphi = L^{-N} \varphi \times A^k_{n(M,N)} [A^k_{n(M,N)} \times_k A^k_{n(v-N,y-M)}, r]$$

where $r = r^{(M+v-N,N+y-M)} \circ (\cdot, \cdot)$. The formal variable $y$ is thus living in $A^k_{n(v-N,y-M)}$. We can be more explicit when $\varphi = [U, f]$ with $g : U \to A^k_{n(M,N)}$ and $f : U \to A^1$; in this case $\mathcal{F} \varphi$ is the class

$$\mathcal{F} \varphi = L^{-N} [U \times_k A^k_{n(v-N,y-M)}], f \circ pr_1 + r^{(M+v-N,N+y-M)}((g \circ pr_1, pr_2))]$$

in $\mathcal{E}xp.\mathcal{M}_{A^k_{n(v-N,y-M)}}$.

$$
\begin{array}{cccccc}
U \times_k A^k_{n(v-N,y-M)} & \sim & U \times_k A^k_{n(M,N)} \times_k A^k_{n(v-N,y-M)} & \to & U & \to A^1 \\
& & ↓ & & ↓^f & ↓^g \\
& & A^k_{n(M,N)} \times_k A^k_{n(v-N,y-M)} & \to & A^k_{n(M,N)} & \to \text{Spec}(k) \\
& & ↓ & & ↓ & \\
& & A^k_{n(v-N,y-M)} & \to & \text{Spec}(k) & \\
\end{array}
$$

Notation 2.4.3 can be seen as a variant of the exponential sum notation of § 2.1.
From local functions to global ones and summation over rational points. Such a construction easily extends to finite products \( \prod_{s \in S} F_s \) of completions of \( F = k(C) \) at a finite number of closed points \( s \in S \) of the curve \( C \). The motivic Schwartz-Bruhat functions of level \( (M_s, N_s)_{s \in S} \) are the elements of the ring

\[
\mathcal{E}xp \prod_{s \in S} \text{Res}_{k(s)/k} A^n_{k(s)}^{n(M_s, N_s)}
\]

where \( \text{Res}_{k(s)/k} \) denotes the Weil restriction functor. The set of global motivic Schwartz-Bruhat functions on \( K^n \) is defined as a direct limit over the set \( S \) of closed points of \( C \) and one can easily extend the previous Fourier kernel and transform to this setting; we will not be more explicit about this construction for now, since it will be generalised in the next paragraph.

Nevertheless it might be enlightening for the reader to define the sum over rational points of a global motivic Schwartz-Bruhat in this simple setting. Let \( \varphi \) be a global Schwartz-Bruhat function of level \( (M_s, N_s)_{s \in S} \) where \( S \subset C \) is a finite set of closed points of \( C \). We choose an uniformiser \( t_s \) of \( F_s \) for every \( s \in S \). Consider the divisor \( D = -\sum_{s \in S} M_s[s] \) on \( C \) and remark that the embeddings \( F \hookrightarrow F_s \) map the Riemann-Roch space

\[
L(D) = \Gamma(C, \mathcal{O}_C(D))
\]

(the space of global sections of the invertible sheaf \( \mathcal{O}_C(D) \)) to \( t_s^{M_s} \mathcal{O}_s \) for every \( s \in S \). This mapping provides a morphism of varieties \( \theta : L(D)^n \to \text{Res}_{k(s)/k} A^n_{k(s)}^{n(M_s, N_s)} \).

The sum over rational points of \( A^n \)

\[
\sum_{x \in k(C)^n} \varphi(x) = \sum_{x \in L(D)^n} \theta^* \varphi
\]

is by definition the image in \( \mathcal{E}xp \mathcal{M}_k \) of the pull-back \( \theta^* \varphi \), that is, the image of \( \varphi \) through the map

\[
\mathcal{E}xp \prod_{s \in S} \text{Res}_{k(s)/k} A^n_{k(s)}^{n(M_s, N_s)} \xrightarrow{\theta^*} \mathcal{E}xp L(D)^n \xrightarrow{} \mathcal{E}xp \mathcal{M}_k
\]

which can be seen as another variant of the exponential sum notation of Sect. 2.1. In this context, one gets a motivic Poisson formula

\[
\sum_{x \in k(C)^n} \varphi(x) = L(1-g)^n \sum_{y \in k(C)^n} F \varphi(y)
\]

(see e.g. (Chambert-Loir and Loeser 2016, Theorem 1.3.10) for a proof).
2.4.2 Constructible families of Schwartz-Bruhat functions

Until the end of this subsection we follow (Bilu 2018, Chapter 5). We will say that a function \( f : X \to \mathbb{Z} \) on a \( k \)-variety is \textit{constructible} if the inverse image of any integer by \( f \) is a constructible subset of \( X \).

Again \( C \) is a smooth projective connected curve over an algebraically closed field \( k \) of characteristic zero. Let \( M, N : C \to \mathbb{Z} \) be constructible functions such that \( M \leq N \). Such functions are constant respectively equal to \( M_0 \) and \( N_0 \) on a dense open subset \( U \) of the curve \( C \). Then \( A_C^{(M,N)} \) stands for the \( C \)-variety isomorphic to \( U \times A_k^{(M_0,N_0)} \) over \( U \) and with fibre over \( u \not\in U \) equal to \( A_k^{(M_0,N_0)} \). Let \( n \) be a positive integer. One can as well define the \( n \)-th product over \( C \)

\[
A_C^{(M,N)} = A_C \times_C ... \times_C A_C^{(M,N)}
\]

\( n \) times.

Until the end of this section we fix two constructible functions \( a, b : C \to \mathbb{Z} \) such that \( a \leq 0 \leq b \) and we assume that there exists a dense open subset \( U \) of \( C \) such that \( a|_U = b|_U = 0 \). Let \( \Sigma \) be the complement of \( U \) in \( C \), it is a finite set of points. Let \( M = (M_i) \) and \( N = (N_i) \) be families of non-negatives integers indexed by \( \mathbb{N}^p \), such that \( M_0 = N_0 = 0 \). This data provides a family

\[
\mathcal{A}_n(a, b, M, N) = S^n \left( \left( A_C^{(n(a-M_i,b+N_i))} \right)_{i \in \mathbb{N}^p} \right)
\]

for any \( p \)-tuple \( n \) of non-negative integers, which is a variety over \( S^n C \). Let us quickly describe the fibre of \( \mathcal{A}_n(a, b, M, N) \to S^n C \) over a schematic point \( D \in S^n C \). The latter can be decomposed formally as

\[
D = \sum_{v \in C} \tau_v v = \sum_{v \in U} \tau_v v + \sum_{v \in \Sigma} \tau_v v = D_U + D_{\Sigma}
\]

where the \( \tau_v \) are \( p \)-tubles of non-negative integers (almost all equal to \( 0 \in \mathbb{N}^p \)),

\[
D_U = \sum_{v \in U} \tau_v v \in S^n U
\]

for some partition \( \pi = (n^U_i)_{i \in \mathbb{N}^p} \) of some \( n^U \leq n \) and \( D_{\Sigma} = \sum_{v \in \Sigma} \tau_v v \). Then, the fibre over \( D \) can be seen as the domain of definition of a Schwartz-Bruhat function \( \Phi_D \), up to a finite extension of the residue field \( \kappa(D) \) of \( D \) (seen as a point of \( S^n C \)) (Bilu 2018, §5.3.2): it is of the form

\[
\prod_{i \in \mathbb{N}^p} A_{\kappa(D)}^{n^U_i \times n(M_i+N_i)} \prod_{v \in \Sigma} A_{\kappa(D)}^{n(a_{v-M_{i_v}},b_v+N_{i_v})}.
\]

(2.4.4)
Note that the first product is actually finite, since \((n_i^U)_{i \in \mathbb{N}^r}\) has finite support as a partition.

For this reason, elements of \(\mathcal{E}xp.M_{\phi_{\Phi}}(a,b,M,N)\) are called constructible families of Schwartz-Bruhat functions of level \(n\) (Bilu 2018, §5.3.4.1). Two special cases arise. On one hand, if all the integers \(N_i\) are zero, the family is said to be uniformly smooth. On the other hand, if all the integers \(M_i\) are zero, the family is said to be uniformly compactly supported (Bilu 2018, §5.3.5.1).

### 2.4.3 Motivic Fourier transform in families

It is possible to define the Fourier transform of a family \(\Phi \in \mathcal{E}xp.M_{\phi_{\Phi}}(a,b,M,N)\) of Schwartz-Bruhat functions, independently of the choice of \(b\) and \(N\). Let us fix a non-zero rational differential form \(\omega \in \Omega_k(C)/k\) and define \(v\) by \(v_\omega = -\text{ord}_p \omega\) for every closed point \(v \in C\). This allows one to define a constructible Fourier kernel morphism

\[
\mathcal{A}_C^{\mathcal{N}(a-M_1,b+N_1)} \times_C \mathcal{A}_C^{n(b-N_1,v-a+M_1)} \to A^1
\]

which by (2.2.2) induces morphisms on the symmetric products

\[
r_n : \mathcal{A}_n(a, b, M, N) \times_{S^n C} \mathcal{A}_n(v - b, v - a, N, M) \to A^1
\]

hence an element of \(\mathcal{E}xp.M_{\phi_{\Phi}}(a,b,M,N) \times_{S^n C} \mathcal{A}_n(v - b, v - a, N, M)\) for any \(n \in \mathbb{N}^r\) (Bilu 2018, §5.4.2). Then one can show that the class \([S^n((\mathcal{A}_C^{n(b,N_i)})_{i \in \mathbb{N}^r})]\) has an inverse in \(\mathcal{E}xp.M_{S^n C}\). We finally define a motivic Fourier transform

\[
\mathcal{F} : \mathcal{E}xp.M_{\phi_{\Phi}}(a,b,M,N) \to \mathcal{E}xp.M_{\phi_{\Phi}}(v-b,v-a,N,M)
\]

given for any \(\Phi = [V, f] \in \mathcal{E}xp.M_{\phi_{\Phi}}(a,b,M,N)\), with \(g : V \to \mathcal{A}_n(a, b, M, N)\) the structure morphism, by the class in \(\mathcal{E}xp.M_{\phi_{\Phi}}(v-b,v-a,N,M)\)

\[
[V \times_{S^n C} \mathcal{A}_n(v - b, v - a, N, M), f \circ \text{pr}_1 + r_n(g \circ \text{pr}_1 \cdot \text{pr}_2)[S^n((\mathcal{A}_C^{n(b,N_i)})_{i \in \mathbb{N}^r})]^{-1}
\]

where \(\cdot\) denotes the product in the ring \(\mathcal{E}xp.M_{\phi_{\Phi}}(a,b,M,N) \times_{S^n C} \mathcal{A}_n(v-b,v-a,N,M)\) (Bilu 2018, §5.4.3). This Fourier transform is independent of the choice of \(b\) and \(N\), and is compatible with symmetric products (Bilu 2018, Proposition 5.4.4.2). It generalises the local and global Fourier transforms of a Schwartz-Bruhat function we presented in the first paragraph of this subsection.

### 2.4.4 Local Poisson formula

Let \(D'_U\) be a pull-back of \(D_U \in S^\pi U\) via the quotient map

\[
\left( \prod_{l \in \mathbb{N}^r \setminus \{0\}} U^{m_l} \right)_s \to S^\pi U
\]
defining $S^n U$, and $v_{i,j}$ the projection of $D'_U$ on the $j$-th copy of $U$ in $U^{m_i}$. A different choice of $D'_U$ only permutes the order of the $v_{i,j}$'s. Let $E_D$ be the effective $\kappa(D)$-zero-cycle on the curve $C_{\kappa(D)}$ defined by

$$E_D = \sum_{i \in \mathbb{N}_p \setminus \{0\}} M_i(v_{i,1} + \cdots + v_{i,m_i}) - \sum_{v \in \Sigma} (a_v - M_v)v.$$ 

The divisor $E_D$ does not depend on the choice of $D'_U$ since it is invariant under the action of $\prod_{i \in \mathbb{N}_p \setminus \{0\}} S_{m_i}$ on the product $\prod_{i \in \mathbb{N}_p \setminus \{0\}} U^{m_i}$ (Bilu 2018, Remark 5.5.1.1). It can be rewritten as

$$\sum_{v \in C} (M_v - a_v)v$$

since $a$ is zero on $U$. One then defines a constructible morphism of $\kappa(D)$-varieties

$$\theta_D : L_{\kappa(D)}(E_D)^a \to \mathcal{H}_n(a, b, M, N)_D$$

where $L_{\kappa(D)}(E_D)$ is the Riemann-Roch space over $E_D$

$$L_{\kappa(D)}(E_D) = \Gamma(C_{\kappa(D)}, \mathcal{O}_{C_{\kappa(D)}}(E_D)).$$

Pointwise, the morphism $\theta_D$ sends a function $f \in L_{\kappa(D)}(E_D)$ to its $v$-adic expansions in the ranges given by the exponents defining $\mathcal{H}_n(a, b, M, N)_D$ in 2.4.4. For example, its image in

$$\prod_{v \in \Sigma} \left( m_v^{a_v-M_v} \mathcal{O}_v/m_v^{b_v+N_v} \mathcal{O}_v \right)$$

may be understood as the coefficients of its $v$-adic expansion in the range $a_v - M_v, ..., b_v + N_v - 1$ (see Bilu 2018, page 164) for an explicit definition).

The summation over rational points of $\Phi_D$ is by definition the class in $\mathcal{E}xp.\mathcal{M}_{\kappa(D)}$ of

$$\sum_{x \in \kappa(D)(C)^a} \Phi_D(x) = \sum_{x \in L_{\kappa(D)}(E_D)^a} (\theta^*_D \Phi_D)(x).$$

Here the sum of the right side is the exponential sum notation (2.1.1) corresponding to the morphism

$$\mathcal{E}xp.\mathcal{M}_{L_{\kappa(D)}(E_D)^a} \to \mathcal{E}xp.\mathcal{M}_{\kappa(D)}$$

induced by the projection on $\kappa(D)$.

For a proof of the following, see (Chambert-Loir and Loeser 2016, Theorem 1.3.10) and (Bilu 2018, Lemma 5.5.1.4).
Proposition 2.5 (Poisson formula) Let $D$ be a schematic point of $S^n C$ and $\Phi_D$ an element of the fibre $\mathcal{E}xp\mathcal{M}_{\mathcal{A}_n(a,b,M,N)}$. Then

$$\sum_{x \in \kappa(D)(C)^n} \Phi_D(x) = L(1-g)^n \sum_{x \in \kappa(D)(C)^n} \mathcal{F}\Phi_D(x).$$

2.4.5 Poisson formula in families

If the family $\Phi_1$ is uniformly compactly supported, the zero cycle $E_D$ is actually equal to the $k$-zero-cycle $D_a = -\sum v_a v$ for any schematic point $D \in S^n C$. Its Riemann-Roch space over $k$ is by definition

$$L(D_a) = \{ f \in k(C) \mid f = 0 \text{ or } \operatorname{div}(f) + D_a \geq 0 \}.$$

By flat base change, there is a $\kappa(D)$-linear canonical isomorphism $L_{\kappa(D)}(D_a) \cong L(D_a) \otimes_k \kappa(D)$. Then one can prove the existence of a constructible morphism

$$\theta_n : L(D_a) \times S^n C \rightarrow A_n(a, b, 0, N)$$

over $S^n C$, whose restriction to the fibres above a schematic point $D \in S^n C$ induces $\theta_D$ (Bilu 2018, §5.5.2.1). Given an uniformly compactly supported family $\Phi \in \mathcal{E}xp\mathcal{M}_{\mathcal{A}_n(a,b,0,N)}$, the image in $\mathcal{E}xp\mathcal{M}_{S^n C}$ of the pullback $\theta_n^* \Phi$ is called the uniform summation over rational points. It is denoted by

$$\left( \sum_{x \in \kappa(D)(C)^n} \Phi_D(x) \right)_{D \in S^n C}.$$

This is an example of what Bilu calls an uniformly summable family, that is, a constructible family of functions $\Phi \in \mathcal{A}_n(a, b, M, N)$ such that there exists a function $\Sigma \in \mathcal{E}xp\mathcal{M}_{S^n C}$ on $S^n C$ whose pullback $D^* \Sigma$ in $\mathcal{E}xp\mathcal{M}_{\kappa(D)}$ is the sum over rational points $\sum_{x \in \kappa(D)(C)^n} \Phi_D(x)$, for any $D \in S^n(C)$.

Now one may remark that there is a commutative diagram of group morphisms

$$\begin{array}{ccc}
\mathcal{E}xp\mathcal{M}_{L(D_a)^n} \times S^n C & \rightarrow & \mathcal{E}xp\mathcal{M}_{S^n C} \\
\downarrow & & \downarrow \\
\mathcal{E}xp\mathcal{M}_{L(D_a)^n} & \rightarrow & \mathcal{E}xp\mathcal{M}_{k}
\end{array}$$

which means that it is possible to permute the sums as follows:

$$\sum_{D \in S^n C} \sum_{x \in \kappa(D)(C)^n} \Phi_D(x) = \sum_{x \in k(C)^n} \sum_{D \in S^n C} \Phi_D(x).$$
Proposition 2.6 (Poisson formula in families) If $\Phi$ is an uniformly smooth family of Schwartz-Bruhat functions, then its Fourier transform $\mathcal{F}\Phi$ is uniformly compactly supported and one has

$$
\sum_{D \in S^n C} \sum_{x \in \kappa(D)(C)^n} \phi_D(x) = L^{(1-g)n} \sum_{y \in k(C)^n} \sum_{D \in S^n C} \mathcal{F}\phi_D(y).
$$

in $\mathcal{E}xp.M_k$.

2.5 Arc schemes and integrals of motivic residual functions

In this paragraph we collect a few facts about jet schemes, arc schemes and a particular case of motivic measure on such spaces. A concise reference for this paragraph could be Blickle (2005) or Craw (2004). A more exhaustive one is the third chapter of Chambert-Loir et al. (2018).

2.5.1 Jet schemes, arc schemes and integration

The field $k$ is an algebraically closed field of characteristic zero and $\mathcal{Y}$ is a flat scheme of finite type over $k[[t]]$, which we assume to be equidimensional of relative dimension $n$. A simple example is given by $\mathcal{Y} = Y \times_{\text{Spec} k} \text{Spec}(k[[t]])$ for some $k$-variety $Y$. For any non-negative integer $m$, the jet-scheme $\mathcal{L}_m(\mathcal{Y})$ of order $m$ of $\mathcal{Y}$ is the $k$-variety representing the functor

$$A \mapsto \text{Hom}_k(\text{Spec}(A[[t]]/(t^{m+1})), \mathcal{Y})$$

on the category of $k$-algebras. There are canonical affine projection morphisms

$$p_m^{m+1} : \mathcal{L}_{m+1}(\mathcal{Y}) \to \mathcal{L}_m(\mathcal{Y})$$

and one can consider the proscheme over $k$

$$\mathcal{L}(\mathcal{Y}) = \lim_{\leftarrow m} \mathcal{L}_m(\mathcal{Y})$$

of arcs on $\mathcal{Y}$. This projective limit carries canonical projections $p_m : \mathcal{L}(\mathcal{Y}) \to \mathcal{L}_m(\mathcal{Y})$.

The projections $p_m^{m+1}$ induce ring morphisms

$$\left(p_m^{m+1}\right)^* : \mathcal{E}xp.M_{\mathcal{L}_m(\mathcal{Y})} \to \mathcal{E}xp.M_{\mathcal{L}_{m+1}(\mathcal{Y})}$$

between the corresponding localised Grothendieck rings with exponential. The ring of motivic residual functions on $\mathcal{L}(\mathcal{Y})$ is defined as the direct limit $\lim_{\rightarrow m} \mathcal{E}xp.M_{\mathcal{L}_m(\mathcal{Y})}$.
Definition 2.7 Let $h$ be a motivic residual function, which we assume to be of the form $[H, f]_{\mathcal{L}_m(\mathcal{Y})}$ with $H$ a variety over $\mathcal{L}_m(\mathcal{Y})$ for some $m$ and $f : H \to \mathbb{A}^1$ a morphism. The integral of $h$ over $\mathcal{L}(\mathcal{Y})$ is the element of $\mathcal{E}xp.M_k$

\[ \int_{\mathcal{L}(\mathcal{Y})} h(x)dx = L^{-(m+1)n}[H, f]_k. \]

Since the projection $\mathcal{L}_m'(\mathcal{Y}) \to \mathcal{L}_m(\mathcal{Y})$ is a locally trivial fibration with fibres isomorphic to $\mathbb{A}^{(m'-m)n}$ for any $m' \geq m$ (Chambert-Loir et al. 2018, Proposition 3.7.5), this definition does not depend on the choice of $m$.

2.5.2 Motivic volumes

An useful particular case of such integrals is given by the characteristic function of a constructible subset $W = p^{-(m-1)}(W_m)$ of $\mathcal{L}(\mathcal{Y})$. The integral of $1_W$ over $\mathcal{L}(\mathcal{Y})$ is by definition the volume $\text{vol}(W)$ of $W$. There are three particular volumes of interest for our purpose. The first one is the volume of the whole arc space $\mathcal{L}(\mathcal{Y}) = p^{-1}(\mathcal{Y}_k)$, which is

$$\text{vol}(\mathcal{L}(\mathcal{Y})) = L^{-n}[\mathcal{Y}_k, 0].$$

Then the volume of the subspace $W = p^{-1}([0]) = \mathcal{L}(\mathbb{A}^1, 0)$ of arcs in $\mathbb{A}^1$ with origin at 0 is

$$\text{vol}(\mathcal{L}(\mathbb{A}^1, 0)) = L^{-1}[[0], 0] = L^{-1}.$$

Finally, an arc on $\mathbb{A}^1$ of order zero at $\{0\}$ in an arc whose image in $\mathbb{A}^1$ do not belongs to $\{0\}$, thus the corresponding volume is $1 - L^{-1}$. More generally, the set of arcs of order $m \in \mathbb{N}$ at $\{0\}$ has volume $(1 - L^{-1})L^{-m}$.

2.5.3 Weights and volumes

The weight function of Bilu (2018) we will introduce in the next section satisfies the following property (Bilu 2018, Remark 6.3.1.3): for any constructible subset $W$ of $\mathcal{L}(\mathcal{Y})$ and motivic residual function $h$ we have the inequality

$$w\left(\int_{\mathcal{L}(\mathcal{Y})} 1_W(x)h(x)dx\right) \leq w(\text{vol}(W)).$$

3 Convergence of motivic Euler products

The aim of this section is to recall the main properties of the weight function on the Grothendieck ring $\mathcal{E}xp.M_X$ introduced by Bilu (Bilu 2018, Chapter 4) and to give an effective criterion, Proposition 3.6, ensuring the weight-linear convergence, in the corresponding completed ring, of multivariate motivic Euler products. We also include
Lemma 3.4 and Lemma 3.5, two lemmas dealing respectively with multiplicativity of weight-linear convergence and what one might call negligible convolution products.

3.1 Weight filtration

We refer to Lemmas 4.5.1.3, 4.6.2.1, 4.6.3.1 and 4.6.3.4 of Bilu (2018) for a proof of the following.

Proposition 3.1 Let $X$ be a complex variety. There exists a weight function

$$w_X : \mathcal{E}xp.\mathcal{M}_X \to \mathbb{Z} \cup \{-\infty\}$$

satisfying the following properties.

1. $w_X(0) = -\infty$
2. $w_X(a + a') \leq \max(w_X(a), w_X(a'))$ with equality if $w_X(a) \neq w_X(a')$, for any $a, a' \in \mathcal{E}xp.\mathcal{M}_X$.
3. If $\mathcal{A} = (a_i)_{i \in I}$ is a family of $\mathcal{E}xp.\mathcal{M}_X$ indexed by a set $I$ and $\pi = (n_i)_{i \in I} \in \mathbb{N}^I$ then

$$w_{\pi S \mathcal{X}}(S^\pi \mathcal{A}) \leq \sum_{i \in I} n_i w_X(a_i).$$

4. If $Y \to X$ is a variety over $X$ then

$$w_X(Y) = 2 \dim_X(Y) + \dim(X).$$

5. If $p : Y \to X$ and $q : Z \to X$ are smooth morphisms with fibres of dimension $d \geq 0$ and $Y, Z$ irreducible, then

$$w_X([Y \xrightarrow{p} X] - [Z \xrightarrow{q} X]) \leq 2d + \dim X - 1.$$

If $X = \text{Spec}(\mathbb{C})$, the weight function $w_X$ will be simply written $w$. It induces a filtration $(W_{\leq n} \mathcal{E}xp.\mathcal{M}_X)_{n \in \mathbb{Z}}$ on $\mathcal{E}xp.\mathcal{M}_X$ given by

$$W_{\leq n} \mathcal{E}xp.\mathcal{M}_X = \{a \in \mathcal{E}xp.\mathcal{M}_X | w_X(a) \leq n\}$$

for all $n \in \mathbb{Z}$.

Definition 3.2 The completion of $\mathcal{E}xp.\mathcal{M}_X$ with respect to the weight topology is

$$\mathcal{E}xp.\mathcal{M}_X = \lim_{\leftarrow n} (\mathcal{E}xp.\mathcal{M}_X / W_{\leq n} \mathcal{E}xp.\mathcal{M}_X).$$
3.2 Convergence criteria

Definition 3.3 Let \( F(T) = \sum_{i \geq 0} X_i T^i \) be a formal power series with coefficients in \( \mathcal{E}_{\text{xp}M_X} \). The radius of convergence of \( F \) is defined by

\[
\sigma_F = \limsup_{i \geq 1} \frac{w_X(X_i)}{2i}.
\]

We say that \( F \) converges for \( |T| < L^{-r} \) if \( r \geq \sigma_F \).

This terminology can be explained by the fact that if \( F \) converges for \( |T| < L^{-r} \) then for all \( a \in \mathcal{E}_{\text{xp}M_C} \) with weight \( w(a) < -2r \) the evaluation \( F(a) \) exists as an element of \( \widehat{\mathcal{E}}_{\text{xp}M_X} \). Indeed, by (Bilu 2018, Lemma 4.5.1.3) one has

\[
w_X(X_i \boxtimes a^i) \leq i(2r + w(a))
\]

for all \( i \) sufficiently large, so \( F(a) = \sum_{i \geq 0} X_i \boxtimes a^i \) can be thought as an element of \( \widehat{\mathcal{E}}_{\text{xp}M_X} \) by considering partial sums with respect to the weight (here \( \boxtimes \) denotes the exterior product defined in Sect. 2.1).

In what follows we will say that a formal series \( \sum_{i \geq 0} c_i T^i \) with terms in \( \mathcal{E}_{\text{xp}M_X} \) converges \textit{weight-linearly} at \( a \in \mathcal{E}_{\text{xp}M_C} \) if there exists \( \delta > 0 \) such that \( w_X(c_i a^i) < -\delta i \) for \( i \) sufficiently large. We endow \( \mathbb{N}^r \) with its poset structure

\[
m' \leq m \text{ if and only if } m'_\alpha \leq m_\alpha \text{ for all } \alpha
\]

and extend this definition to families indexed by \( \mathbb{N}^r \) by considering indices such that

\[
\langle \rho, m \rangle = \sum_{\alpha=1}^r \rho_\alpha m_\alpha = i
\]

for a fixed \( \rho \in \mathbb{N}^r \setminus \{0\} \). More precisely, a formal series \( \sum_{m \in \mathbb{N}^r} c_m T^m \) is said to converge \textit{weight-linearly with respect to} \( \rho \), which we may sometimes abbreviate by saying \( \rho \)-weight-linearly, at \( a = (a_1, ..., a_r) \in (\mathcal{E}_{\text{xp}M_C})^r \) if there exists a real number \( \delta > 0 \) such that for \( i \) sufficiently large one has

\[
w_X(c_m a^m) < -\delta i.
\]

for every \( m \in \mathbb{N}^r \) such that \( \langle \rho, m \rangle = i \). This property only depends on the support

\[
\text{Supp}(\rho) = \{1 \leq \alpha \leq r \mid \rho_\alpha > 0\}
\]

of \( \rho \). Indeed, if \( \rho_1, \rho_2 \in \mathbb{N}^r \) have same support, then

\[
\min_{\alpha \in \text{Supp}(\rho_1)} (\rho_{1,\alpha} / \rho_{2,\alpha})(\rho_1, m) \leq \langle \rho_2, m \rangle \leq \max_{\alpha \in \text{Supp}(\rho_1)} (\rho_{1,\alpha} / \rho_{2,\alpha})(\rho_1, m)
\]

for all \( m \in \mathbb{N}^r \).
In particular, for such a $\rho$-weight-linearly convergent series we can licitly consider the sum

$$\sum_{m \in \mathbb{N}} c_m a^m$$

in $\mathcal{M}_X$ as soon as $\sum_{m \in \mathbb{N}} c_m T^m$ belongs to the subring of formal series with indeterminates $T_\alpha$ for $\alpha \in \text{Supp}(\rho)$.

Then the value of the sum does not depend on the order of summation, in particular the partial sum

$$\sum_{m' \leq m} c_m a^{m'}$$

has a limit in $\mathcal{M}_X$ when $\langle \rho, m \rangle$ tends to infinity.

In the fourth section of this paper, we will have to deal with weight-linearly convergent series with respect to multiple $\rho$’s having different supports, each of these series depending only on the indeterminates $T_\alpha$ for $\alpha \in \text{Supp}(\rho)$. In this situation, simultaneous convergence for the various $\rho$’s will be obtained by asking $\min_{1 \leq \alpha \leq r} (m_\alpha)$ to be arbitrarily large.

**Lemma 3.4** Fix $\rho \in \mathbb{Z}_{\geq 1}$ and let $P(T) = \sum_{m \in \mathbb{N}} p_m T^m$ and $Q(T) = \sum_{m \in \mathbb{N}} q_m T^m$ be two formal series with coefficients in $\mathcal{M}_X$. Assume that they converge $\rho$-weight-linearly at $T_\alpha = L^{-\rho_\alpha}$, which means that $\sum_{m \in \mathbb{N}} p_m L^{-(\rho, m)}$ and $\sum_{m \in \mathbb{N}} q_m L^{-(\rho, m)}$ are $\rho$-weight-linearly convergent. Then the product $R(T) = P(T)Q(T)$ converges $\rho$-weight-linearly at $T_\alpha = L^{-\rho_\alpha}$.

**Proof** The $m$-th coefficient of $R(T)$ is given by

$$r_m = \sum_{m' \leq m} p_{m'} q_{m - m'}.$$ 

Let $\delta, \delta' > 0$ and $i_0, i_0' \in \mathbb{N}$ be such that

$$w_X \left( p_m L^{-(\rho, m)} \right) < -\delta \langle \rho, m \rangle$$

for any $m$ such that $\langle \rho, m \rangle \geq i_0$;

$$w_X \left( q_m L^{-(\rho, m)} \right) < -\delta' \langle \rho, m \rangle$$

for any $m$ such that $\langle \rho, m \rangle \geq i_0'$.

Then by Proposition 3.1

$$w_X(r_m L^{-(\rho, m)}) \leq \max_{m' \leq m} \left( w_X \left( p_{m'} L^{-(\rho, m')} \right) + w_X \left( q_{m-m'} L^{-(\rho, m-m')} \right) \right).$$

There is only a finite number of indices $m'$ such that $\langle \rho, m' \rangle < i_0$. Fix such an $m'$. Then for $m$ such that $\langle \rho, m \rangle$ is sufficiently large
\[ w_X \left( p_m L^{-\langle \rho, m' \rangle} \right) + w_X \left( q_{m-m'} L^{-\langle \rho, m-m' \rangle} \right) < -\frac{\delta'}{2} \langle \rho, m \rangle. \]

The case when \( \langle \rho, m - m' \rangle < i'_0 \) is symmetric, and the last case \( \langle \rho, m' \rangle \geq i_0 \) and \( \langle \rho, m - m' \rangle \geq i'_0 \) is left to the reader. This shows the existence of \( j_0 \in \mathbb{N} \) such that

\[ w_X \left( c_{m} L^{-\langle \rho, m \rangle} \right) < -\min(\delta, \delta') \langle \rho, m \rangle \]

for any \( m \) such that \( \langle m, \rho \rangle \geq j_0 \). Thus \( R(T) \) converges \( \rho \)-weight-linearly at \( T_\alpha = L^{-\rho_\alpha} \).

\[ \sum_{m \in \mathbb{N}^r} c_m T^m \]

be a convergent sum at \( a \) with coefficients in \( \mathcal{E}xp.MX \) and \( \varepsilon \in \{0, 1\}^r \setminus \{0\} \). For any \( m = (m_i)_{i=1}^r \in \mathbb{N}^r \) we write \( \langle \varepsilon, m \rangle = \sum_{i=1}^r \varepsilon_i m_i \).

Then

\[ \sum_{m' \leq m} c_{m'} a^{m'} L^{-\langle \varepsilon, m-m' \rangle} \]

tends to zero in \( \mathcal{E}xp.MX \) when \( \min_{1 \leq i \leq r} (m_i) \to \infty \).

If moreover \( \sum_{m \in \mathbb{N}^r} c_m T^m \) is weight linearly-convergent at \( a \) with respect to some given \( \rho \in \mathbb{Z}_{\geq 1}^r \), then there exist \( \delta > 0 \) and \( i_0 \in \mathbb{N} \) such that

\[ w_X \left( \sum_{m' \leq m} c_{m'} a^{m'} L^{-\langle \varepsilon, m-m' \rangle} \right) < -\delta \langle \varepsilon, m \rangle \]

whenever \( \langle \varepsilon, m \rangle \geq i_0 \).

**Proof** In the proof we write \( c_m \) for \( c_m a^m \). Fix an \( A \in \mathbb{Z} \). There exists \( m_0 \in \mathbb{N}^r \) such that

\[ w_X(c_m) < A \text{ for all } m \text{ such that } m \notin m_0. \]

Set \( \omega = \max(w_X(c_m \mid m \leq m_0)) \). On one hand if \( m' \leq m_0 \) one has

\[ w_X \left( c_{m'} L^{-\langle \varepsilon, m-m' \rangle} \right) \leq \omega - \langle \varepsilon, m - m' \rangle \leq \omega - \langle \varepsilon, m - m_0 \rangle < A \]

if \( \min(m_i) \) is large enough. On the other hand \( m' \notin m_0 \) implies \( w_X \left( c_{m'} L^{-\langle \varepsilon, m-m' \rangle} \right) < A \). Thus applying Proposition 3.1 one gets

\[ w_X \left( \sum_{m' \leq m} c_{m'} L^{-\langle \varepsilon, m-m' \rangle} \right) < A \]

for any \( m \) sufficiently large. This proves the lemma.
Now we assume that there exist $i_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$w_X(c_m) < -\delta \langle m, \rho \rangle$$

whenever $\langle m, \rho \rangle \geq i_0$. Remark that since we assumed that $\rho \in \mathbb{Z}_r^{r \geq 1}$ and $\varepsilon \in \{0, 1\}^r$, we have

$$\langle \rho, m \rangle \geq \langle \varepsilon, m \rangle$$

for all $m \in \mathbb{N}^r$. On one hand, if $\langle m', \rho \rangle < i_0$, then

$$w_X \left( c_{m'} L^{-\langle \varepsilon, m-m' \rangle} \right) \leq \omega' - \langle \varepsilon, m \rangle + i_0 < -\frac{1}{2} \langle \varepsilon, m \rangle$$

whenever $\langle \varepsilon, m \rangle > 2(\omega' + i_0)$ and $\omega' = \max(w_X(c_m \mid \langle m, \rho \rangle < i_0))$. On the other hand, if $\langle m', \rho \rangle \geq i_0$ then $w_X(c_{m'}) < -\delta \langle m', \rho \rangle$. In particular, we have this last inequality if $\langle \varepsilon, m' \rangle \geq i_0$. Finally by Proposition 3.1

$$w_X \left( \sum_{m' \leq m} c_{m'} L^{-\langle \varepsilon, m-m' \rangle} \right) < -\min \left( \frac{1}{2} \langle \varepsilon, m \rangle, \delta \langle \rho, m \rangle \right) \leq -\min \left( \frac{1}{2}, \delta \right) \langle \varepsilon, m \rangle$$

for all $m$ such that $\langle \varepsilon, m \rangle \geq 2(\omega' + i_0) + 1$.  

Now we state and prove a weight-linear convergence criterion for Euler products. This result and its proof are a slight extension of Proposition 4.7.2.1 in Bilu (2018) to the framework of multivariate series.

**Proposition 3.6** Fix a complex variety $X$, an integer $r \geq 1$ and an $r$-tuple $\rho \in \mathbb{Z}_r^{r \geq 1}$. Let $\mathcal{X}$ be a family $(X_m)_{m \in \mathbb{N}^r \setminus \{0\}}$ of elements of $\exp \mathcal{M}_X$. Assume that there exist an integer $M \geq 0$ and real numbers $\varepsilon > 0, \alpha < 1$ and $\beta$ such that

- $w_X(X_m) \leq \langle \rho, m \rangle - \frac{1}{2} - \varepsilon) w(X)$ whenever $1 \leq \langle \rho, m \rangle \leq M$;
- $w_X(X_m) \leq \alpha \langle \rho, m \rangle + \beta - \frac{1}{2} \varepsilon w(X)$ whenever $\langle \rho, m \rangle > M$.

Let $b_m$ be the coefficient of multidegree $m \in \mathbb{N}^r$ of the Euler product

$$\prod_{x \in X} \left( 1 + \sum_{m \in \mathbb{N}^r \setminus \{0\}} X_{m,x} T^m \right) \in \exp \mathcal{M}_{\mathbb{C}[[T]]}.$$  

Then there exists $\delta \in ]0, \frac{1}{2}]$ and $\delta' > 0$ such that

$$w \left( b_m a_1^{\rho_1 m_1} \cdots a_r^{\rho_r m_r} \right) \leq -\delta' \langle \rho, m \rangle$$
for every $m \in \mathbb{N}^r \setminus \{0\}$ and $a_1, \ldots, a_r \in \exp M$ such that $w(a_i) < -w(X) \left(1 - \delta + \frac{\beta}{M+1}\right)$ for all $1 \leq i \leq r$. In particular, the Euler product

$$
\prod_{x \in X} \left(1 + \sum_{m \in \mathbb{N}^r \setminus \{0\}} X_m \langle \rho, m \rangle\right) \in \exp M[[T]]
$$

converges for $|T| < L^{-\frac{w(X)}{2} \left(1-\delta + \frac{\beta}{M+1}\right)}$ and takes non-zero values for $|T| \leq L^{-\frac{w(X)}{2} \left(1-\eta + \frac{\beta}{M+1}\right)}$ for any $0 \leq \eta < \delta$.

**Proof** By definition of the motivic Euler product notation, for every $m \in \mathbb{N}^r$ the coefficient $b_m$ is given by the sum

$$
b_m = \sum_{\pi \text{ partition of } m} [S^\pi \mathcal{X}].
$$

Since $w(b_m)$ is smaller or equal to $\max\{w(S^\pi \mathcal{X}) | \pi \text{ partition of } m\}$, it is enough to find a uniform bound for the weight of $S^\pi \mathcal{X}$. So we fix $\pi = (n_i)_{i \in \mathbb{N}^r \setminus \{0\}}$ a partition of $m$, that is such that $\sum_{i \in \mathbb{N}^r \setminus \{0\}} n_i i = m$. Then

$$
w(S^\pi \mathcal{X}) \leq w_S X (S^\pi \mathcal{X}) + \dim(S^\pi X)
$$

$$
\leq \sum_{i \in \mathbb{N}^r \setminus \{0\}} n_i w_X (X_i) + \frac{1}{2} \sum_{i \in \mathbb{N}^r \setminus \{0\}} n_i w(X)
$$

$$
\leq \sum_{j \geq 1} \sum_{i \in \mathbb{N}^r \setminus \{0\}} n_i w_X (X_i) + \frac{1}{2} \sum_{j \geq 1} \sum_{i \in \mathbb{N}^r \setminus \{0\}} n_i w(X)
$$

$$
\leq \sum_{j=1}^M \left( \sum_{i \in \mathbb{N}^r \setminus \{0\}} n_i (j - \varepsilon) w(X) \right) + \sum_{j=M+1} \left( \sum_{i \in \mathbb{N}^r \setminus \{0\}} (\alpha j + \beta) n_i w(X) \right)
$$

$$
\leq \sum_{j=1}^M \left( \sum_{i \in \mathbb{N}^r \setminus \{0\}} n_i \left(1 - \varepsilon \frac{M}{j}\right) w(X) \right) + \sum_{j=M+1} \left( \sum_{i \in \mathbb{N}^r \setminus \{0\}} (\alpha + \frac{\beta}{M+1}) n_i w(X) \right)
$$
\[
= \left(1 - \frac{\varepsilon}{M}\right) \sum_{j=1}^{M} \left( \sum_{i \in \mathbb{N} \setminus \{0\}} n_{i} j w(X) \right)
+ \left(\alpha + \frac{\beta}{M + 1}\right) \sum_{j \geq M + 1} \left( \sum_{i \in \mathbb{N} \setminus \{0\}} n_{i} j w(X) \right)
\leq \left(1 - \delta + \frac{\beta}{M + 1}\right) \langle \rho, \mathbf{m} \rangle w(X)
\]

where \(\delta \in [0, \frac{1}{2}]\) is given by \(1 - \delta = \max \left(1 - \frac{\varepsilon}{M}, \alpha\right)\) if \(M \neq 0\) and \(\delta = 1 - \alpha\) otherwise. This proves the first part of the proposition.

To conclude the proof, remark that if \(w(a_{k}) < -w(X) \left(1 - \eta_{k} + \frac{\beta}{M + 1}\right)\) for some \(0 \leq \eta_{k} < \delta\), for all integer \(k\) in \(\{1, ..., r\}\), then the weight of \(S^{\pi} \mathfrak{X} a_{1}^{\rho_{1} m_{1}} \cdots a_{r}^{\rho_{r} m_{r}}\) is negative for any \(\mathbf{m}\) non-zero and any partition of \(\mathbf{m}\). This argument shows that the product takes a non-zero value at \(a\): it is equal to 1 plus some terms of negative weight. \(\square\)

### 4 Compactifications of additive groups

#### 4.1 Back to our setting

We recall in this paragraph a few facts and notation coming from the setting adopted in Bilu (2018) and Chambert-Loir and Loeser (2016).

From now on, we consider the situation presented in the introduction: briefly, our data consists in a Zariski open subset \(\mathfrak{X}\) of a projective irreducible scheme \(\mathfrak{X}\) over an algebraically closed field \(k\) of characteristic zero, together with a non-constant proper morphism

\[\pi : \mathfrak{X} \to C\]

whose generic fibre \(X = \mathfrak{X}_{F}\) is a smooth equivariant compactification of \(G_{F} = G^{n}_{a, F}\). Let \((D_{a})_{a \in \mathfrak{A}}\) be the family of irreducible components of the boundary divisor \(X \setminus G_{F}\). Their linear classes freely generate the Picard group of \(X\), as well as its effective cone. An anticanonical divisor of \(X\) is given by

\[\sum_{\alpha} \rho_{a} D_{\alpha}\]

for a certain family \(\rho_{a} \geq 2\) of integers. In particular, it is big.

Let \(\mathfrak{D}_{a}\) be the Zariski closure of \(D_{a}\) in \(\mathfrak{X}\), for all \(a \in \mathfrak{A}\). Using resolution of singularities in characteristic zero, one can assume that \(\mathfrak{X}\) is a good model, that is to say, \(\mathfrak{X}\) is smooth over \(k\) and the sum of the non-smooth fibres of \(\mathfrak{X}\) and of the
\( D_\alpha \) is a divisor with strict normal crossings. We may assume that \( \mathcal{X} \setminus \mathcal{U} \) is a divisor with strict normal crossings as well. These assumptions will not change our counting problem (see §3.2 and Lemma 3.4.1 of Chambert-Loir and Loeser 2016).

The restriction \( U = \mathcal{U}_F \) is assumed to be a partial compactification of \( G_F \). The boundary \( D = X \setminus U \) is a divisor which can be written \( D = \sum_{\alpha \in \mathcal{A}_D} D_\alpha \) for a subset \( \mathcal{A}_D \) of \( \mathcal{A} \). A log-anticanonical divisor with respect to \( D \) is then \( \sum_{\alpha \in \mathcal{A}} \rho_\alpha D_\alpha \) where \( \rho_\alpha = \rho_\alpha - 1 \) if \( D_\alpha \) is an irreducible component of \( D \) and \( \rho_\alpha = \rho_\alpha \) otherwise. We set

\[
\mathcal{A}_U = \mathcal{A} \setminus \mathcal{A}_D.
\]

We fix as well a dense open subset \( C_0 \subset C \) and assume that \( \mathcal{U}(\mathcal{O}_v) \neq \emptyset \) whenever \( v \in C_0(k) \). For any \( k \)-point \( v \) of \( C \), let \( \mathcal{B}_v \) be the set of irreducible components of \( \pi^{-1}(v) \). Given an irreducible component \( \beta \in \mathcal{B}_v \), let us denote by \( E_\beta \) the corresponding component and \( \mu_\beta \) its multiplicity in the special fibre of \( \mathcal{X} \) at \( v \) (that is, the length of the \( \mathcal{O}_v \)-module of regular functions on \( E_\beta \) in \( \mathcal{X}_v = \mathcal{X} \times_{\mathcal{O}_v} \text{Spec}(k) \)). Let \( \mathcal{B} = \bigcup_{v \in C(k)} \mathcal{B}_v \) be the union of all the \( \mathcal{B}_v \)'s and \( \mathcal{B}_1 \) the subset of components of multiplicity one. Over any place \( v \) this subset restricts to a subset \( \mathcal{B}_{1,v} = \mathcal{B}_1 \cap \mathcal{B}_v \) of \( \mathcal{B}_v \). If \( v \in C_0(k) \), we write \( \mathcal{B}^U \) for the subset of \( \mathcal{B}_{1,v} \) of vertical components intersecting \( \mathcal{U} \). This definition makes sense since we assumed that \( \mathcal{X} \setminus \mathcal{U} \) is a divisor: it is the union of the \( \mathcal{D}_\alpha \) for \( \alpha \in \mathcal{A}_D \) together with a finite number of vertical divisors. It is convenient to set \( \mathcal{B}_s^U = \mathcal{B}_{1,s} \) if \( s \in S \).

For every \( \alpha \) in \( \mathcal{A} \), we fix a line bundle \( \mathcal{L}_\alpha \) on \( \mathcal{X} \) extending \( D_\alpha \). There is a finitely supported family of integers \( (e^\beta_\alpha) \) indexed by \( \mathcal{A} \) and \( \mathcal{B} \) such that

\[
\mathcal{L}_\alpha = D_\alpha + \sum_{\beta \in \mathcal{B}} e^\beta_\alpha E_\beta
\]

for any \( \alpha \) in \( \mathcal{A} \), as well as a finitely supported family of integers \( (\rho^\beta) \) indexed by \( \mathcal{B} \) such that

\[
- \text{div}(\omega_X) = \sum_{\alpha \in \mathcal{A}} \rho_\alpha D_\alpha + \sum_{\beta \in \mathcal{B}} \rho^\beta E_\beta
\]

with \( \omega_X \) the unique (up to a multiplicative constant) \( G_F \)-invariant rational differential form on \( X \), which is understood here as a rational section of the relative canonical bundle of \( \mathcal{X} \) over \( C \). By Lemmas 3.3.3 and 3.3.4 in Chambert-Loir and Loeser (2016), there exists an open dense subset \( C_1 \subset C_0 \) such that \( |\mathcal{B}_v| = 1 \) for every \( v \in C_1(k) \) (so that \( \mathcal{B}_v = \mathcal{B}_{1,v} = \mathcal{B}^U \)) above \( C_1 \), since we assumed \( \mathcal{U}(\mathcal{O}_v) \neq \emptyset \). Furthermore one can assume \( e^\beta_\alpha = \rho^\beta v = 0 \) for all \( v \in C_1(k) \) and \( \mathcal{D}_\alpha \times_C C_1 \to C_1 \) smooth for all \( \alpha \in \mathcal{A}_U \). This open subset \( C_1 \) can be understood as the set of places of good reduction.

Given any closed point \( v \in C(k) \), for every subset \( A \) of \( \mathcal{A} \) and every irreducible component \( \beta \) of multiplicity one above \( v \), let \( \Delta_v(A, \beta) \) be the set of points of the special fibre \( \mathcal{X}_v \) of \( \mathcal{X} \to C \) over \( v \) belonging exclusively to \( \mathcal{D}_\alpha \), for every \( \alpha \in A \), and to \( E_\beta \):
\[ \Delta_v(A, \beta) = \left( \bigcap_{\alpha \in A} D_{\alpha, v} \cap E_{\beta} \right) \setminus \left( \bigcup_{\alpha \notin A} D_{\alpha, v} \cup E_{\beta'} \right). \]

We denote by \( \Omega_v(A, \beta) \) the preimage of \( \Delta_v(A, \beta) \) in the arc space \( \mathcal{L}(\mathcal{X}_O_v) \) through the projection \( \mathcal{L}(\mathcal{X}_O_v) \to \mathcal{L}_0(\mathcal{X}_O_v) = \mathcal{D}_v \). Lemma 5.2.6 in Chambert-Loir and Loeser (2016) (recalled as Lemma 6.3.3.2 in Bilu 2018) gives the existence of an isomorphism

\[ \Theta : \Delta_v(A, \beta) \times \mathcal{L}(\mathbf{A}^1, 0)^A \times \mathcal{L}(\mathbf{A}^1, 0)^{n-|A|} \longrightarrow \Omega_v(A, \beta) \]

\[ x = (x_\Delta, (x_\alpha)_{\alpha \in A}, (y_\beta)) \longmapsto \Theta(x) \]

which preserves the motivic measures and such that \( \text{ord}_0(x_\alpha) = \text{ord}_{\mathcal{D}_\alpha}(\Theta(x)) \) whenever \( \alpha \in A \) and \( \text{ord}_{\mathcal{D}_\alpha}(\Theta(x)) = 0 \) otherwise. For further details, we refer to (Chambert-Loir and Loeser 2016, §2.4 & §6.2).

To conclude this section, one may note that our problem is defined by a finite number of polynomial equations over \( k \). Fixing an embedding of \( k \) into \( \mathbb{C} \), we can assume that everything is defined over the field of complex numbers (using the fact that the definition of the moduli spaces of sections is functorial). Moreover, the assumption \( \mathcal{U}(\mathcal{O}_v) \neq \emptyset \) for all \( v \in C_0 \) implies that for such \( v \), at least one of the \( \Delta_v(A, \beta) \) has a \( k \)-point, thus has a \( \mathbb{C} \)-point (Bilu 2018, §6.4.4).

### 4.2 Expression of the multivariate Zeta series

For the sake of completeness, we start by recalling the method leading to the decomposition of the motivic Zeta function, as it is developed in Chambert-Loir and Loeser (2016) and Bilu (2018). A motivation for doing that is giving the reader a precise meaning of the summation symbols involved here: our asymptotic study will require to permute a sum over rational points and a limit in the completed ring \( \hat{\text{exp}}\mathbb{M}_k \).

Given a \( k \)-point \( v \in C(k) \), it is possible to define local intersection degrees \( (g, D_\alpha)_v \) and \( (g, E_\beta)_v \) for all \( g \in G(F_v) \), \( \alpha \in \mathcal{A} \) and \( \beta \in \mathcal{B}_v \), in the following manner (Chambert-Loir and Loeser 2016, §3.3). Let \( g : \text{Spec}(F_v) \to G_F \) be such a point. Since \( \mathcal{X} \to C \) is proper, by the valuative criterion of properness this map extends to a map \( \tilde{g} : \text{Spec}(O_v) \to \mathcal{X} \).

\[
\begin{array}{ccc}
\text{Spec}(F_v) & \xrightarrow{g} & G_F \\
\downarrow & & \downarrow \pi \\
\text{Spec}(O_v) & \xrightarrow{\tilde{g}} & C
\end{array}
\]

The non-negative local intersection number \( (g, D_\alpha)_v \) is given by the effective Cartier divisor on \( \text{Spec}(O_v) \) obtained by pulling-back \( D_\alpha \):

\[ \tilde{g}^* D_\alpha = (g, D_\alpha)_v[v]. \]
The number \((g, E_\beta)_v\) is defined by pulling-back \(E_\beta\) on \(\text{Spec}(O_v)\). Such invariants satisfy the following two properties:

- they are compatible with the global degree of the section \(\sigma_g : C \to \mathcal{X}\) canonically extending \(g \in G(F)\), with respect to \(D_\alpha\) for any \(\alpha\):
  \[
  \deg_C(\sigma_g^*(D_\alpha)) = \sum_{v \in C(k)} (g, D_\alpha)_v;
  \]

- the \(F_v\)-point \(g\) intersects exactly one vertical component of multiplicity one: there exists an unique \(\beta \in \mathcal{B}_v\) such that \((g, E_\beta)_v = 1\) with \(\mu_\beta = 1\) and \(\beta' \neq \beta\) implies \((g, E_{\beta'})_v = 0\).

Then one defines the sets

\[
G(m, \beta)_v = \{ g \in G(F_v) \mid (g, E_\beta)_v = 1 \text{ and } (g, D_\alpha)_v = m_\alpha \text{ for all } \alpha \in \mathcal{A}\}
\]

for all \(m \in \mathbb{N}_\mathcal{A}\) and \(\beta \in \mathcal{B}_v\). These sets provide a decomposition of \(G(F_v)\) into disjoint bounded definable subsets (Chambert-Loir and Loeser 2016, Lemma 3.3.2).

A pair \((m, \beta) \in \mathbb{N}_\mathcal{A} \times \mathcal{B}_v\) is said to be \(v\)-integral if

- either \(v \in C_0\), \(\beta \in \mathcal{B}_v^{\mathcal{A}}\) and \(m_\alpha = 0\) for every \(\alpha \in \mathcal{A}_D\);
- or \(v \in C \setminus C_0\).

One then introduces the corresponding sets

\[
H(m, \beta)_v = G(m, \beta)_v \text{ if and only if } (m, \beta) \text{ is } v\text{-integral, and } \emptyset \text{ otherwise.}
\]

for any place \(v \in C(k)\) and any pair \((m, \beta)\). These definitions provide adelic sets

\[
H(m, \beta) = \prod_{v \in C(k)} H(m_v, \beta_v)_v \subset G(m, \beta) = \prod_{v \in C(k)} G(m_v, \beta_v)_v \subset G(\mathbb{A}_k(C))
\]

for any \(m = (m_v) \in S^n C\) and \(\beta = (\beta_v) \in \prod_v \mathcal{B}_v\). By propositions 6.2.3.3 and 6.2.4.2 of Bilu (2018), there exist

- almost zero functions \(s, s' : C \to \mathbb{Z}\) (playing the roles of \(a\) and \(b\) in Sect. 2.4.5);
- an unbounded family \(N = (N_m)_{m \in \mathbb{N}_\mathcal{A}}\) such that \(N_0 = 0\);
- for all \(m \in \mathbb{N}_\mathcal{A}\) and \(\beta \in \prod_v \mathcal{B}_v\), a constructible subset \(H_{m, \beta} \subset \mathbb{A}_C^n(s' - N_m, s)\);

such that for every \(v \in C(k)\), the fibre of \(H_{m, \beta}\) at \(v\) is \(H(m, \beta)_v\). By taking symmetric products, this data provides constructible subsets

\[
S^n((H_{m, \beta})_{m \in \mathbb{N}_\mathcal{A}}) \subset \mathcal{X}_n(s', s, N, 0)
\]
for any \( n \in \mathbb{N}^{\mathcal{A}} \), which themselves define uniformly smooth constructible families of Schwartz-Bruhat functions (of any level \( n \)), denoted by

\[
(\mathcal{F}(1_{H(m, \beta)}))_{m \in \mathcal{S}^n C} \in \mathcal{E}xp.\mathcal{M}_{\mathcal{A}}(s', s, n, 0).
\]

Such functions correspond to the characteristic functions of \( H(m, \beta) \), for fixed \( \beta \) and varying \( m \in \mathcal{S}^n C \). Therefore, by taking Fourier transforms, one gets uniformly compactly supported family

\[
(\mathcal{F}(1_{H(m, \beta)}))_{m \in \mathcal{S}^n C} \in \mathcal{E}xp.\mathcal{M}_{\mathcal{A}}(v-s, v-s', 0, N)
\]

where \( v \) is the conductor of a rational differential form \( \omega \in \Omega_k(C)/k \).

The moduli space \( M_n \) of sections \( \sigma : C \to \mathcal{X}^r \) such that:

- \( \sigma \) maps the generic point \( \eta_C \) of \( C \) to a point of \( G_F \);
- the image of \( C_0 \) by \( \sigma \) is contained in \( \mathcal{U} \);
- for all \( \alpha \) in \( \mathcal{A} \), \( \deg (\sigma^* \mathcal{L}_\alpha) = n_\alpha \),

can be rewritten as a disjoint union of constructible subsets \( M^\beta_n \), where for any \( \beta \in \prod_{v \in C(k)} \mathcal{B}_v^{\mathcal{A}} \), the subset \( M^\beta_n \) is the set of sections \( \sigma \in M_n \) such that \( (\sigma(\eta_C), E_{\beta_v})_v = 1 \) for all \( v \in C(k) \). In what follows,

\[
e^{\beta_v} = (e^{\beta_v})_{\alpha \in \mathcal{A}} \quad e^\beta = \sum_v e^{\beta_v} \quad n^\beta = n - e^\beta \in \mathbb{N}^{\mathcal{A}}
\]

hence elements of \( M^\beta_n \) are sections \( \sigma \in M_n \) such that \( (\deg(\sigma^* \mathcal{L}_\alpha))_{\alpha \in \mathcal{A}} = n^\beta \). Since \( \mathcal{B}_v^{\mathcal{A}} \) is the set of vertical components of multiplicity one lying above \( S \) or contained in \( \mathcal{U} \), one has \( e^{\beta_v} \in \mathbb{N}^{\mathcal{A} \setminus \mathcal{U}} \) whenever \( v \in C_0(k) \). Moreover, if \( v \in C_1(k) \) then the condition \( (\sigma(\eta_C), E_{\beta_v})_v = 1 \) is automatically satisfied since \( |\mathcal{B}_v| = 1 \). Thus the partition of \( M_n \) we are describing here is actually finite, since it only depends on the local intersection numbers with respect to a finite number of vertical divisors.

By (Bilu (2018), Lemma 6.2.6.1), if \( M^\beta_n \) is non-empty there is a morphism \( M^\beta_n \to S^{n^\beta} C \) of constructible sets sending a section \( \sigma_{\mathcal{D}} \) to the tuple of zero-cycles \( \sum_{v \in C(k)} ((g, \mathcal{D}_v)_{\alpha \in \mathcal{A}} [v]) \). The coarse height motivic zeta function \( Z(T) \) can be rewritten

\[
Z(T) = \sum_{\beta \in \prod_v \mathcal{B}_v^{\mathcal{A}}} \sum_{n^\beta \in \mathbb{N}^{\mathcal{A}}} [M^\beta_{n^\beta + e^\beta}] T^{n^\beta + e^\beta} = \sum_{\beta \in \prod_v \mathcal{B}_v^{\mathcal{A}}} T^{e^\beta} Z^\beta(T).
\]

where \( Z^\beta(T) = \sum_{n^\beta \in \mathbb{N}^{\mathcal{A}}} [M^\beta_{n^\beta + e^\beta}] T^{n^\beta} \) for every \( \beta \in \prod_v \mathcal{B}_v^{\mathcal{A}} \) (note again that this last product is finite).

When one considers \( S \)-integral points and use this function, it appears that it is not precise enough: poles of multiple order appear, which means that we miss relevant invariants which correspond to distinct components of the moduli space of sections. In order to deal with it, remark that a section \( \sigma : C \to \mathcal{X}^r \) intersects the divisors \( \mathcal{D}_\alpha \)

\( \mathcal{S} \) Springer
for $\alpha \in \mathcal{A}_D$ only above the finite set of closed points $S = C \setminus C_0$ and each point $s \in S$ gives an invariant

$$((\sigma(\eta_C), \mathcal{D}_\alpha)_s)_{\alpha \in \mathcal{A}_D}.$$  

This leads us to define for any $(n, \beta)$ and $m_S = (m_s)_{s \in S} \in \mathbb{N}^{\mathcal{A}_D \times S}$ the subset $M^\beta_{n, m_S}$ of sections $\sigma \in M^\beta_n$ such that $((\sigma(\eta_C), \mathcal{D}_\alpha)_s)_s \in S^\beta_C = C \setminus C_0$ and each point $s \in S$ gives an invariant $((\sigma(\eta_C), \mathcal{D}_\alpha)_s)_s \in \mathbb{N}^{\mathcal{A}_D \times S}$. This gives a decomposition of $M^\beta_n$ into a finite disjoint union of definable subsets. If $n^\beta = (n^\beta_U, n^\beta_D)$ is an element of $\mathbb{N}^{\mathcal{A}_U} \times \mathbb{N}^{\mathcal{A}_D}$, this subset can be identified with the fibre of $M^\beta_n \to S^{n^\beta}_C$ over the points $m \in S^{n^\beta}_C = S^{n^\beta_U}_C \times S^{n^\beta_D}_C$ whose image by the projection $S^{n^\beta}_C \to S^{n^\beta_D}_C$ has support in $S = C \setminus C_0$ and is given by $m_S$.

**Definition 4.1** Let $U = (U_0, (U_s)_{s \in S})$ be a family of indeterminates indexed by the set $\mathcal{A}_U \sqcup (\mathcal{A}_D \times S)$ and let

$$e^\beta = \left(\sum_{v \in C_0} e^{\beta_v} + \sum_{s \in S} e^{\beta_s}_{\mathcal{A}_U}, (e^{\beta_s}_{\mathcal{A}_D})_{s \in S}\right)$$

for all $\beta \in \prod_v \mathcal{B}_v^{\mathcal{A}_U}$.

We define the **refined motivic height zeta function** $\mathcal{Z}(U)$ by

$$\mathcal{Z}(U) = \sum_{\beta \in \prod_v \mathcal{B}_v^{\mathcal{A}_U}} U^{e^\beta} \mathcal{Z}^\beta(U)$$

(4.2.7)

where

$$\mathcal{Z}^\beta(U) = \sum_{n^\beta_U \in \mathbb{N}^{\mathcal{A}_U}} \sum_{m_S \in \mathbb{N}^{\mathcal{A}_D \times S}} [M^\beta_{n^\beta_U + e^\beta, m_S}]U^{n^\beta_U + m_S}$$

for every $\beta \in \prod_v \mathcal{B}_v^{\mathcal{A}_U}$.

**Remark 4.2** By definition, the specialisation of $\mathcal{Z}(U)$ obtained by replacing respectively $U_0 = (U_\alpha)_{\alpha \in \mathcal{A}_U}$ by $(T_\alpha)_{\alpha \in \mathcal{A}_U}$ and $U_s = (U_{\alpha, s})_{\alpha \in \mathcal{A}_D}$ by $(T_\alpha)_{\alpha \in \mathcal{A}_D}$, for every $s \in S$, is the coarse zeta function $Z(T)$. Furthermore, if $C_0 = C$ and $\mathcal{U} = \mathcal{A}$ then this refined function $\mathcal{Z}$ coincides with the coarse one $Z$, since in that case $S = \emptyset$ and $\mathcal{A} = \mathcal{A}_U$. A similar remark can be done for $\mathcal{Z}^\beta$ and $Z^\beta$.

The map sending a section $\sigma : C \to X$ to $\sigma(\eta_C) \in X(F)$ induces an exact correspondence between

$$\mathcal{Z}^\beta(U) = \sum_{n^\beta_U \in \mathbb{N}^{\mathcal{A}_U}} \sum_{m_S \in \mathbb{N}^{\mathcal{A}_D \times S}} [M^\beta_{n^\beta_U + e^\beta, m_S}]U^{n^\beta_U + m_S}$$

for every $\beta \in \prod_v \mathcal{B}_v^{\mathcal{A}_U}$.
sections \( \sigma \in M^n_\beta \) such that \( \sum_{v \in C(k)} ((g, D) v)_{\alpha \in \mathcal{A}}[v] = m \in S^{n_\beta} C \)

and

\[
\text{elements of } G(F) \cap H(m, \beta).
\]

Remark that if \( H(m, \beta) \) is non-empty then the image of \( m \) by the projection \( S^{n_\beta} C \to S^{n_\beta} C \) has support in \( S \).

By definition of the summation over rational points (2.4.5), one has the equalities in \( \mathcal{M}_{\kappa(m)} \)

\[
\sum_{x \in k(m)(C)} 1_{G(F) \cap H(m, \beta)}(x) = [G(F) \cap H(m, \beta)] = \left( M^n_\beta \right)_m
\]

where \( \left( M^n_\beta \right)_m \) is the class of the fibre of \( M^n_\beta \to S^{n_\beta} C \) over the schematic point \( m \in S^{n_\beta} C \). Applying the motivic Poisson formula for families (see Sect. 2.4.5) and following Bilu’s computations, we obtain for the coarse subspaces

\[
\left[ M^n_\beta \right] = \sum_{m \in S^{n_\beta}(C)} [G(F) \cap H(m, \beta)] = L^{(1-g)n} \sum_{\xi \in k(C)^n} \sum_{m \in S^{n_\beta}(C)} \mathcal{F}(1_{H(m, \beta)})(\xi)
\]

and for the refined subspaces

\[
\left[ M^n_{m, m_\beta} \right] = \sum_{m \in S^{n_\beta}(C)} [G(F) \cap H(m + m_\beta, \beta)] = L^{(1-g)n} \sum_{\xi \in k(C)^n} \sum_{m \in S^{n_\beta}(C)} \mathcal{F}(1_{H(m + m_\beta, \beta)})(\xi).
\]

Since \( (\mathcal{F}(1_{H(m, \beta)}))_{m \in S^{n_\beta} C} = S^n((\mathcal{F}(1_{H(m, \beta)}))_{m \in N^{n_\beta}}) \) (see Bilu 2018, Proposition 5.4.4.2), by definition of the motivic Euler product notation one can write

\[
Z^n_\beta(T) = L^{(1-g)n} \sum_{\xi \in k(C)^n} \sum_{m \in S^{n_\beta} C} \mathcal{F}(1_{H(m, \beta)})(\xi) T^m = L^{(1-g)n} \sum_{\xi \in k(C)^n} \prod_{v \in C} \left( \sum_{m_v \in N^{n_\beta_v}} \mathcal{F}(1_{H(m_v, \beta_v)})(\xi_v) T^{m_v} \right)
\]

\[
= L^{(1-g)n} \sum_{\xi \in k(C)^n} \prod_{v \in C} Z^n_\beta(T, \xi)
\]
and since the Euler product is compatible with finite products, we have

\[ Z(\mathbf{U}) = L^{(1-g)n} \sum_{\mathbf{m} \in \mathbb{N}_{D, \mathbf{C}}} \prod_{\mathbf{v} \in \mathbb{C}_{0}} \left( \sum_{m_{v} \in \mathbb{N}_{D, \mathbf{U}}} \mathcal{F}(\mathbf{1}_{H(m_{v}, \beta_{v})})(\xi_{v}) \mathbf{U}^{m_{v}} \right) \]

This last decomposition is consistent with the one of \( Z^{\beta}(\mathbf{T}) \) if one applies the specialisation of Remark 4.2. Combined with (4.2.7) it finally gives

\[ Z(\mathbf{U}) = L^{(1-g)n} \sum_{\mathbf{m} \in \mathbb{N}_{D, \mathbf{C}}} \prod_{\mathbf{v} \in \mathbb{C}_{0}} \left( \sum_{m_{v} \in \mathbb{N}_{D, \mathbf{U}}} \mathcal{F}(\mathbf{1}_{H(m_{v}, \beta_{v})})(\xi_{v}) \mathbf{U}^{m_{v}} \right) \]

In this expression, the summation over \( k(\mathbf{C})^{n} \) is actually a summation over \( L(\tilde{E})^{n} \), where \( L(\tilde{E}) \) is the Riemann-Roch space of the \( k \)-divisor

\[ \tilde{E} = -\sum_{v} (\nu_{v} - s_{v})[v], \quad (4.2.8) \]

\( s : C \to \mathbb{Z} \) being the almost zero function of (Bilu 2018, Proposition 6.2.3.3) and \( \nu \) being the order function of \( \omega \in \Omega_{k(\mathbf{C})/k} \) defined at the beginning of Sect. 2.4.3.

Hence the multivariate zeta functions \( Z(\mathbf{U}) \) and \( Z(\mathbf{T}) \) can be written as

\[ Z(\mathbf{U}) = L^{(1-g)n} \sum_{\mathbf{m} \in \mathbb{N}_{D, \mathbf{C}}} \prod_{\mathbf{v} \in \mathbb{C}_{0}} \left( \sum_{m_{v} \in \mathbb{N}_{D, \mathbf{U}}} \mathbf{U}^{m_{v}} \mathcal{Z}(\mathbf{U}, \xi_{v}) \right) \]

where \( V \) is the finite dimensional \( k \)-vector space \( V = L(\tilde{E})^{n} \) and each \( Z(v, \xi) \) (respectively \( Z(T, \xi) \)) can be expressed as a motivic Euler product with local factors

\[ Z(v, \xi) = \sum_{\beta_{v} \in \mathcal{R}_{v}} \mathbf{U}^{\beta_{v}} Z(v_{v}, \xi). \]
(resp. \(Z_v(T, \xi)\)) for any place \(v \in C\). For any \(\xi\) in \(V\), we will first study the asymptotic behaviour of the \(m\)-th coefficient of the Euler product

\[
\prod_{v \in C} \mathcal{Z}_v(U, \xi)
\]

when \(\min_{(\alpha, s) \in \mathcal{A} \times (\{0\} \cup S)} (m_{\alpha, s})\) tends to infinity. When we restrict this product to \(C_0\), its local factors coincide with the ones of the coarse version \(Z(T, \xi)\). Therefore it is natural for us to identify them and keep the notations of Bilu (2018); Chambert-Loir and Loeser (2016) for places of \(C_0\).

Note that this notation is slightly abusive: it actually means that we restrict a certain relative motivic Euler product to a constructible subset of \(V\) containing \(\xi\). Indeed, recall that we use the commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{E}\text{xp}.\mathcal{M}_V \times \mathcal{S}^{n\mathbb{C}} & \longrightarrow & \mathcal{E}\text{xp}.\mathcal{M}_{S^{n\mathbb{C}}} \\
\downarrow & & \downarrow \\
\mathcal{E}\text{xp}.\mathcal{M}_V & \longrightarrow & \mathcal{E}\text{xp}.\mathcal{M}_k
\end{array}
\]

while summing over \(\xi\) and \(m\) the families \((\theta_n^* \mathcal{F}(\mathbf{1}_{H(m, \beta)}))_{m \in S^{n\mathbb{C}}} \in \mathcal{E}\text{xp}.\mathcal{M}_V \times \mathcal{S}^{n\mathbb{C}}\). In particular, one can understand

\[
\prod_{v \in C} \left( \sum_{m_v \in \mathbb{N}^d} \mathcal{F}(\mathbf{1}_{H(m_v, \beta_v)_{v}}) T_{m_v}^n \right) = \sum_{n \in \mathbb{N}^d} \sum_{m \in S^{n(C)}} \mathcal{F}(\mathbf{1}_{H(m, \beta)}) T^n
\]

as a relative motivic Euler product with coefficients in \(\mathcal{E}\text{xp}.\mathcal{M}_V\) (in which we omit \(\theta_n^*\)). Its convergence will be studied with respect to a finite constructible partition of \(V\) (described in Sect. 5.3). Since the motivic sum

\[
\sum_{\xi \in V} : \mathcal{E}\text{xp}.\mathcal{M}_V \rightarrow \mathcal{E}\text{xp}.\mathcal{M}_k
\]

is a morphism, it is compatible with such a cutting process. In the end, the remaining task will be to justify that one can actually permute the sum over rational points \(\sum_{\xi \in k(C)^n}\) and the limit of the coefficients of \(\mathcal{Z}(U, \xi)\).

It is shown in Chambert-Loir and Loeser (2016) that the local factor of \(Z(T)\) can be rewritten as a motivic integral over the arc spaces \(\Omega_v(A, \beta)\) defined in Sect. 4.1. When \(v \in C_0\), this procedure gives

\[
Z_v(T, \xi) = \sum_{\beta_v \in \mathcal{B}_v^{\mathbb{C}}} T^{e_{\beta_v}} Z_{v}^{\beta_v}(T, \xi)
= \sum_{\mathcal{A} \subset \mathcal{A}_U} T^{e_{\beta_v}} L^{\beta_v} \int_{\Omega_{v}(A(\beta_v))} \prod_{\alpha \in A} (L^{\rho_{\alpha}} T_{\alpha})^{\text{ord}_v(x_{\alpha})} e(\langle x, \xi \rangle) dx
\]

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Remark 4.3 Remark that for any character $\xi$ and place $v \in C_0(k)$ we have a decomposition

$$Z_v^\beta(T, \xi) = \sum_{A \subset \mathcal{U}} T_{v, A}^\beta \mathcal{Z}_{v, A}(T, \xi)$$

where $Z_v^\beta(T, \xi)$ only depends on the indeterminates $(T^{\alpha})_{\alpha \in \mathcal{A}_U}$.

This remark remains valid for places $s$ of $S$, if one just replaces $\mathcal{A}_U$ by $\mathcal{A}$ in the decomposition above, and adapting this decomposition to $\mathcal{Z}(U, \xi)$ is straightforward.

4.3 Convergence of Euler products in our setting

As the reader familiar with Tauberian theorem in complex analysis might guess, the asymptotic behaviour of the coefficients of $Z(T)$ is closely linked to the poles of the series. The precise study of these poles has been done by Bilu in (Bilu 2018, Chapter 6). In this subsection we will both recall and adapt the relevant results for our purpose; in particular, we precisely check weight-linear convergence of these products, which is locally uniform with respect to a stratification of the space of characters. This information will be crucial both for the control of various error terms appearing through the last section of this paper and for the final step of our proof.

Since the motivic Euler product notation is compatible with finite products, it will be enough to prove convergence over $C_0$ (and even $C_1$). As we already pointed out, in this situation the local factors of the coarse zeta function and refined zeta function coincide. Therefore the content of this paragraph stays valid if one replaces everywhere $Z_v(T, \xi)$ by $\mathcal{Z}_v(U, \xi)$ and the indeterminates $T^{\alpha}$ by $U^{\alpha}$, $\alpha \in \mathcal{A}_U$, in Notation 4.4 and 4.7 below.

4.3.1 Trivial character

In this paragraph we study the main term of the motivic Zeta function.
Notation 4.4 For any place $v$ of $C$ we set $$F_v(T, 0) = Z_v(T, 0) \prod_{\alpha \in \mathcal{A}_U} (1 - L^\rho_{\alpha}^{-1} T_\alpha).$$

One can directly compute the local factors corresponding to the trivial character (Chambert-Loir and Loeser 2016, §6). First assume that $v$ is a place of $C_0$. Then, $$Z_v(T, 0) = \sum_{\beta \in \mathcal{B}_v} T^{e^\beta_v} Z_v^{\beta_v}(T, 0)$$
$$= \sum_{\beta \in \mathcal{B}_v} T^{e^\beta_v} L^\rho_{\beta} \sum_{A \subset \mathcal{A}_U} [\Delta_u(A, \beta)] L^{-n + |A|} (1 - L^{-1})^{|A|}$$
$$\times \prod_{\alpha \in A} \frac{L^\rho_{\alpha}^{-1} T_\alpha}{1 - L^\rho_{\alpha}^{-1} T_\alpha}.$$ 

In case $v$ is a place of the dense subset $C_1 \subset C_0$, since $\mathcal{B}_v = \{\beta_v\}$ and both $e^\beta_v$ and $\rho_{\beta}$ equal zero, this expression becomes slightly nicer:
$$Z_v(T, 0) = Z_v^{\beta_v}(T, 0) = \sum_{A \subset \mathcal{A}_U} [\Delta_u(A)] L^{-n + |A|} (1 - L^{-1})^{|A|} \prod_{\alpha \in A} \frac{L^\rho_{\alpha}^{-1} T_\alpha}{1 - L^\rho_{\alpha}^{-1} T_\alpha}.$$ 

Concerning places of $S = C \setminus C_0$, the local factor becomes
$$Z_v(T, 0) = \sum_{\beta \in \mathcal{B}_v} T^{e^\beta_v} L^\rho_{\beta} [\Delta_u(A, \beta)] L^{-n + |A|} (1 - L^{-1})^{|A|} \prod_{\alpha \in A} \frac{L^\rho_{\alpha}^{-1} T_\alpha}{1 - L^\rho_{\alpha}^{-1} T_\alpha}.$$

We will use these expressions to prove the following proposition.

Proposition 4.5 The product
$$\prod_{v \in C_0} F_v(T, 0)$$
is $\rho$-weight-linearly convergent at $T_\alpha = L^{-\rho_{\alpha}}$ and the resulting sum is a non-zero effective element of $\hat{M}_k$.

Proof The proof is essentially the same as the one of Proposition 6.3.5.2 of Bilu (2018). We write down the details here for the sake of completeness. In particular, it is important to check that we are in the situation of Proposition 3.6, in order to obtain linear convergence with respect to $\rho$.

It is enough to prove convergence for places of the dense open subset $C_1$ of $C_0$. Furthermore one can assume $\mathcal{A}_U = \mathcal{A}$. For a place $v \in C_1$ the local factor $F_v(T, 0) = \prod_{\alpha \in \mathcal{A}} (1 - L^\rho_{\alpha}^{-1} T_\alpha) Z_v(T, 0)$ is given by
\[ F_v(T, 0) = \sum_{A \subset \mathcal{A}} [\Delta_v(A)] L^{-n+|A|} (1 - L^{-1})^{|A|} \prod_{\alpha \in A} L^{\rho_{\alpha} - 1} T_{\alpha} \prod_{\alpha \in \mathcal{A} \setminus A} (1 - L^{\rho_{\alpha} - 1} T_{\alpha}) \]

If we expand the product over \( \mathcal{A} \setminus A \) for any \( A \subset \mathcal{A} \) we get

\[
\prod_{\alpha' \in \mathcal{A} \setminus A} (1 - L^{\rho_{\alpha'} - 1} T_{\alpha'}) = \sum_{B \subset \mathcal{A} \setminus A} (-1)^{|B|} \prod_{\alpha' \in B} L^{\rho_{\alpha'} - 1} T_{\alpha'}
\]

and

\[
F_v(T, 0) = \sum_{A \subset \mathcal{A}} \sum_{B \subset \mathcal{A} \setminus A} (-1)^{|B|} [\Delta_v(A)] L^{-n+|A|} (1 - L^{-1})^{|A|} \prod_{\alpha \in A} L^{\rho_{\alpha} - 1} T_{\alpha} \prod_{\alpha' \in B} L^{\rho_{\alpha'} - 1} T_{\alpha'}.
\]

Remark that \([\Delta_v(\emptyset)] = [G^g_{\alpha}] = L^n\). Then the previous sums can be decomposed with respect to the cardinalities of the sets \( A \) and \( B \):

\[
F_v(T, 0) = 1 - \sum_{\alpha' \in \mathcal{A}} L^{\rho_{\alpha'} - 1} T_{\alpha'} + \sum_{\alpha' \in B} (-1)^{|B|} \prod_{\alpha' \in B} L^{\rho_{\alpha'} - 1} T_{\alpha'} \quad (A = \emptyset)
\]

\[
+ \sum_{\alpha \in \mathcal{A}} [\Delta_v(\{\alpha\})] L^{-n+1} (1 - L^{-1}) L^{\rho_{\alpha} - 1} T_{\alpha} \quad (|A| = 1, B = \emptyset)
\]

\[
+ \sum_{\alpha \in \mathcal{A}} \sum_{B \subset \mathcal{A} \setminus \{\alpha\}} (-1)^{|B|} [\Delta_v(\{\alpha\})] L^{-n+1} (1 - L^{-1}) L^{\rho_{\alpha} - 1} T_{\alpha}
\]

\[
\times \prod_{\alpha' \in B} L^{\rho_{\alpha'} - 1} T_{\alpha'} \quad (|A| = 1, B \neq \emptyset)
\]

\[
+ \sum_{A \subset \mathcal{A}} \sum_{B \subset \mathcal{A} \setminus A} (-1)^{|B|} [\Delta_v(A)] L^{-n+|A|} (1 - L^{-1})^{|A|} \prod_{\alpha \in A} L^{\rho_{\alpha} - 1} T_{\alpha} \prod_{\alpha' \in B} L^{\rho_{\alpha'} - 1} T_{\alpha'} \quad (|A| \geq 2)
\]

The definition of \( \Delta_v(A) \) gives the equality of classes

\[
[\mathcal{A}_{\alpha,v}] = [\Delta_v(\{\alpha\})] + \sum_{A' \subset \mathcal{A} \setminus \{\alpha\}} [\Delta_v(A' \cup \{\alpha\})]
\]

for every \( \alpha \in \mathcal{A} \). Finally

\[
F_v(T, 0) = 1 + \sum_{\alpha \in \mathcal{A}} ([\mathcal{A}_{\alpha,v}] - L^{-n}) L^{-n} L^{\rho_{\alpha} - 1} T_{\alpha} + P_v(T)
\]
where $P_v(T)$ is the polynomial

$$
P_v(T) = \sum_{B \subset \mathcal{A}, |B| \geq 2} (-1)^{|B|} \prod_{\alpha' \in B} L^{\rho_{\alpha'} - 1} T_{\alpha'}$$

$$- \sum_{\alpha \in \mathcal{A}} \sum_{A' \subset \mathcal{A} \setminus \{\alpha\}, A' \neq \emptyset} [\Delta_v(A' \cup \{\alpha\})] L^{-n+1} (1 - L^{-1}) L^{\rho_{\alpha} - 1} T_{\alpha}$$

$$+ \sum_{A \subset \mathcal{A}} \sum_{B \subset \mathcal{A} \setminus A, |A| = 1 \Rightarrow B \neq \emptyset} (-1)^{|B|} [\Delta_v(A)] L^{-n+|A|} (1 - L^{-1})^{|A|} \prod_{\alpha \in A} L^{\rho_{\alpha} - 1} T_{\alpha}$$

$$\times \prod_{\alpha' \in B} L^{\rho_{\alpha'} - 1} T_{\alpha'}.$$

Let us analyse the dimensions of the coefficients of this polynomial term by term, and compare them to the multidegrees of the corresponding monomials. In the sum of the first line, since $|B| \geq 2$, the corresponding coefficient has dimension at most $-2 + \sum_{\alpha \in B} \rho_{\alpha}$. Here the corresponding monomial is $T^m$ with $m_\alpha = 1$ if $\alpha \in B$ and 0 otherwise; thus $\sum_{\alpha \in B} \rho_{\alpha} = \langle \rho, m \rangle$. Concerning the sum of the second line, since we assumed that $X$ is a good model, the dimension of $\Delta_v(A' \cup \{\alpha\})$ is at most $n - 2$ and the whole coefficient has dimension at most $\rho_{\alpha} - 2$. This remark still applies to the coefficients of the third and last sum when $|A| \geq 2$: they have dimension at most $-2 + \sum_{\alpha \in A \cup B} \rho_{\alpha}$. It corresponds to the monomial $T^m$ with $m_\alpha = 1$ if $\alpha \in A \cup B$ and 0 otherwise; thus $\langle \rho, m \rangle = \sum_{\alpha \in A \cup B} \rho_{\alpha}$. Finally, in the case $|A| = 1$, then $|A \cup B| \geq 2$ and the relation

$$\prod_{\alpha \in A \cup B} L^{\rho_{\alpha} - 1} = L^{\sum_{\alpha \in A \cup B} \rho_{\alpha} - |A \cup B|}$$

ensures that the dimension of the corresponding coefficient is again at most $-2 + \sum_{\alpha \in A \cup B} \rho_{\alpha}$.

Remark that our computations do not depend on $v$ in the sense that there exist polynomials $F(T, 0)$ and $P(T)$ with coefficients in $\mathcal{E}_{xp, M_{C_1}}$, such that their pullbacks by $v$ are respectively $F_v$ and $P_v$. Therefore we will be able to use Proposition 3.6 with $X = C_1$, $\varepsilon = \frac{1}{2}$, $M = \deg(P((T^{\rho_{\alpha}})_{\alpha \in \mathcal{A}}))$ and $\beta = 0$ so that the first condition of Proposition 3.6 becomes in our case

$$w_{C_1}(c_m) \leq 2 \left(\langle \rho, m \rangle - 1 \right).$$

By the last property of the weight recalled in Proposition 3.1, we obtain the crucial argument of this proof, which is the inequality

$$w_{C_1} \left(\left[[\mathcal{I}_{\alpha, v}] - L^{n-1} \right] L^{-n} L^{\rho_{\alpha}}\right) \leq 2(n - 1) - 2n + 2\rho_{\alpha} = 2(\rho_{\alpha} - 1).$$
As Bilu points out in (Bilu 2018, Remarks 6.3.4.2-3), the bounds on dimensions obtained in the previous paragraph ensure that the coefficients of multidegree \( m \) of \( P_v(T) \) satisfy (4.3.9). Thus we can apply Proposition 3.6 and \( \rho \)-weight-linear convergence over \( C_1 \) at \( T_\alpha = L^{-\rho_\alpha} \) follows. Since the Euler product notation is compatible with finite products and since we already know that the local factors converge weight-linearly, applying Lemma 3.4 we deduce that the \( \rho \)-weight-linear convergence holds for the product over \( C_0 \).

**Remark 4.6** Since for any place \( v \in C_1 \) the fibre \( \mathcal{V}_v \) is the disjoint union of all the \( \Delta_v(A) \) for \( A \subset \mathcal{A}_1 \), and in general \( E_{\beta,v} \) is the disjoint union of all the \( \Delta_v(A, \beta) \), it is straightforward to check that the value at \( T_\alpha = L^{-\rho_\alpha} \) of the motivic Euler product

\[
\prod_{v \in C_0} \frac{\mathcal{O}^\rho_{\mathcal{V}_v} \left( T_v \right) \prod_{\alpha \in \mathcal{A}_1} \left( 1 - L^{\rho_\alpha - 1} T_\alpha \right) \left( (L^{-\rho_\alpha})_{\alpha \in \mathcal{A}} \right)}{(1 - L^{-1})^{rg(Pic(U))} \left[ \mathcal{V}_v \right] L^n} \times \prod_{v \in C_0 \setminus C_1} \left( 1 - L^{-1} \right)^{rg(Pic(U))} L^{\rho_\beta_v - \rho \cdot e^\beta} \left[ E_{\beta_v}^0 \right] L^n)
\]

The local term \( L^{\rho_\beta_v - \rho \cdot e^\beta} \left[ E_{\beta_v}^0 \right] L^n \) (which is just \( \left[ \mathcal{V}_v \right] L^n \) if \( v \in C_1 \)) can be interpreted as the motivic integral

\[
\int_{G(F_v, \beta_v)} L^{-g, \mathcal{L}_{\rho_\beta}} |\omega_X|
\]

over the adelic space \( G(F_v, \beta_v) = \{ g \in G(F_v) \mid (g, E_{\beta_v}) = 1 \} \). By (Chambert-Loir and Loeser 2016, Lemma 6.1.1) this integral can we rewritten as the motivic integral over arc spaces

\[
\int_{\mathcal{L}(\mathcal{X}_v, E_{\beta_v})} L^{-ord_{\mathcal{X}_{\rho}}(x)} L^{-ord_{w_X}(x)}
\]

\[
= \int_{\mathcal{L}(\mathcal{X}_v, E_{\beta_v})} L^{-\sum_{\alpha \in \mathcal{A}_1} \rho_\alpha (ord_v(x_\alpha) + e^\beta_\alpha)} L^{\rho_\beta_v + \sum_{\alpha \in \mathcal{A}_1} \rho_\alpha ord_v(x_\alpha)}
\]

\[
= \int_{\mathcal{L}(\mathcal{X}_v, E_{\beta_v})} L^{\rho_\beta_v - \rho \cdot e^\beta}
\]

\[
= L^{\rho_\beta_v - \rho \cdot e^\beta} \text{vol}(\mathcal{L}(\mathcal{X}_v, E_{\beta_v}))
\]

\[
= L^{\rho_\beta_v - \rho \cdot e^\beta} \left[ E_{\beta_v}^0 \right] L^n.
\]
This comes from the fact that $\mathcal{X}(\mathcal{O}_v) = X(F_v)$ so that one can view $G(F)$ as a subset of $\mathcal{X}(\mathcal{O}_v)$ and any Schwartz-Bruhat function on $G(F_v)$ as a motivic function on $\mathcal{X}(\mathcal{X})$, see (Chambert-Loir and Loeser 2016, §6.1).

4.3.2 Non-trivial characters

Given a place $v \in C$ and a non-trivial character $\xi \in G(F)\backslash$, the linear form $x \in G(F) \mapsto \langle x, \xi \rangle$ can be seen as a rational function $f_\xi$ on $X$ whose divisor of poles has support contained in the union of the $D_\alpha$. We denote by $d_\alpha(\xi)$ the order of the pole of $f_\xi$ with respect to $D_\alpha$ and define a subset of $\mathcal{A}$ by setting

$$\mathcal{A}_U^0(\xi) = \{\alpha \in \mathcal{A}_U \mid d_\alpha(\xi) = 0\}.$$ 

If $U \neq X$, this is automatically a proper subset of $\mathcal{A}$. Otherwise if $U = X$, since $X$ is projective and $\xi$ is non-trivial, this is a proper subset of $\mathcal{A}$.

**Notation 4.7** For any place $v$ and any non-trivial character $\xi$, we write

$$F_v(T, \xi_v) = Z_v(T, \xi_v) \prod_{\alpha \in \mathcal{A}_U^0(\xi)} \left(1 - L^{\rho_\alpha - 1} T_\alpha\right).$$

**Proposition 4.8** The product

$$\prod_{v \in C_0} F_v(T, \xi_v)$$

is weight-linearly convergent with respect to $\rho$ at $T_\alpha = L^{\rho_\alpha}$ and the resulting sum is a non-zero effective element of $\hat{\mathcal{M}}_k$.

The proof of this proposition we give here consists of a summary of the arguments of the proof of Proposition 6.3.5.3 in Bilu’s thesis Bilu (2018), since our main interest lies in obtaining weight-linear convergence of $\prod_{v \in C_0} F_v(T, \xi)$ at $T_\alpha = L^{\rho_\alpha}$.

**Proof of Proposition 4.8** Once again it is enough to check convergence on the dense subset $C_1 \subset C$. In order to apply Proposition 3.6, one has to bound the weight of the coefficients of $F_v(T, \xi)$. We start by doing this for $Z_v(T, \xi)$.

First, the divisor of $f_\xi$ can be written

$$\text{div}(f_\xi) = \Xi_\xi - \sum_{\alpha \in \mathcal{A}} d_\alpha(\xi) D_\alpha$$

where $\Xi_\xi$ is the Zariski-closure of $\{\langle x, \xi \rangle = 0\}$ in $X$. Over a place $v \in C_1$, one has $\mathcal{B}_v^0 = \{\beta_v\}$ and the local factor $Z_v(T, \xi)$ can be written $Z_v(T, \xi) = \sum_{A \subset \mathcal{A}_U} Z_{v, A}(T, \xi)$ with

$$Z_{v, A}(T, \xi) = \int_{\Omega_v(A, \beta_v)} \prod_{\alpha \in A} (L^{\rho_\alpha} T_\alpha)^{\text{ord}_x \alpha} e(f_\xi(x)) \, dx.$$
as we already pointed out in Remark 4.3. It is possible to compute or at least bound the weight of this integral. We refer to (Bilu 2018, §6.3.5.2) for the details of such computations; in this proof we will restrict ourselves to giving a list of results we need to apply Proposition 3.6. In particular we obtain weight-linear convergence of the Euler product \( \prod_{v \in C_1} F_v(T, \xi) \). In what follows \( \mathcal{X}_v \) is written \( \mathcal{X} \) for conciseness and the index \( v \) may be dropped.

If \( A = \emptyset \) then the integral above equals one. If \( A = \{ \alpha \} \) the intermediate step is to cut the integral into two pieces: one part corresponding to arcs with origin in the zero divisor \( \Xi_\xi \) of \( f_\xi \) and another part corresponding to arcs with origin outside \( \Xi_\xi \).

\[
Z_{v,\{\alpha\}}(T, \xi) = \int \mathcal{L}(\mathcal{X}, D_\alpha^o \setminus \Xi_\xi) \prod_{\alpha \in A} (L^{\rho_\alpha} T_\alpha)^{\text{ord}_x f_\xi(x)} dx \\
+ \int \mathcal{L}(\mathcal{X}, D_\alpha^o \cap \Xi_\xi) \prod_{\alpha \in A} (L^{\rho_\alpha} T_\alpha)^{\text{ord}_x f_\xi(x)} dx
\]

Then there are several cases to distinguish, according to the order of the pole of \( f_\xi \) at \( D_\alpha \). On one hand, concerning arcs with origins outside \( \Xi_\xi \), one has the following results.

- If \( d_\alpha(\xi) = 0 \) then
  \[
  \int \mathcal{L}(\mathcal{X}, D_\alpha^o \setminus \Xi_\xi) (L^{\rho_\alpha} T_\alpha)^{\text{ord}_x f_\xi(x)} dx \\
  = (1 - L^{-1}) \frac{L^{\rho_\alpha - 1} T_\alpha}{1 - L^{\rho_\alpha - 1} T_\alpha} [D_\alpha^o \setminus \Xi_\xi] L^{-n + 1}.
  \]

- If \( d_\alpha(\xi) = 1 \) then
  \[
  \int \mathcal{L}(\mathcal{X}, D_\alpha^o \setminus \Xi_\xi) (L^{\rho_\alpha} T_\alpha)^{\text{ord}_x f_\xi(x)} dx = -L^{-2} [D_\alpha^o \setminus \Xi_\xi] L^{1-n} L^{\rho_\alpha} T_\alpha.
  \]

Since \( \dim(D_\alpha^o \setminus \Xi_\xi) = n - 1 \), one has the following upper bound on the weight

\[
w_{C_1} \left( -L^{2} [D_\alpha^o \setminus \Xi_\xi] L^{1-n} L^{\rho_\alpha} \right) \leq 2(\rho_\alpha - 2) + 1 \leq 2\rho_\alpha - 1.
\]

- If \( d_\alpha(\xi) > 1 \) then this integral equals zero.

On the other hand, concerning arcs with origin in \( \Xi_\xi \), the corresponding integral can be rewritten

\[
\int \mathcal{L}(\mathcal{X}, D_\alpha^o \cap \Xi_\xi) (L^{\rho_\alpha} T_\alpha)^{\text{ord}_x f_\xi(x)} dx \\
= \sum_{m \geq 1} (L^{\rho_\alpha} T_\alpha)^m \int_{(D_\alpha^o \cap \Xi_\xi) \times \mathcal{L}(A_1)^n \times \mathcal{L}(A_1, 0)^{n-1}} \text{ord}_x = m \ e(f_\xi(t^m x, y)) dx dy.
\]
By (2.5.3), the weight of the coefficient of order $m$ is smaller than the weight of the motivic volume

$$\text{vol} \left( \left\{(w, x, y) \in (D^o_A \cap \mathcal{X}_\xi) \times \mathcal{L}(A) \times \mathcal{L}(A^1, 0)^{n-1} \mid \text{ord } x = m \right\} \right)$$

$$= \left[ D^o_A \cap \mathcal{X}_\xi \right] L^{-m} \left( 1 - L^{-1} \right) L^{-n+1}.$$  

One has the upper bound on dimensions

$$\text{dim} \left( L^{\rho_{\alpha}} \left[ D^o_A \cap \mathcal{X}_\xi \right] L^{-m} \left( 1 - L^{-1} \right) L^{-n+1} \right) \leq m \rho_{\alpha} + (n - 2) - m - (n - 1)$$

$$= m(\rho_{\alpha} - 1) - 1$$

and thus on weights

$$w_{C_1} \left( L^{\rho_{\alpha}} \left[ D^o_A \cap \mathcal{X}_\xi \right] L^{-m} \left( 1 - L^{-1} \right) L^{-n+1} \right) \leq 2(m(\rho_{\alpha} - 1) - 1) + \text{dim}(C_1).$$

Setting $c = \max_{\alpha \in \mathcal{A}_U} \left( 1 - \frac{1}{2\rho_{\alpha}} \right)$ as in Bilu (2018), one gets $2\rho_{\alpha} - 1 \leq 2\rho_{\alpha}c$ for every $\alpha \in \mathcal{A}_U$ and

$$w_{C_1} \left( L^{\rho_{\alpha}} \left[ D^o_A \cap \mathcal{X}_\xi \right] L^{-m} \left( 1 - L^{-1} \right) L^{-n+1} \right) \leq 2cm\rho_{\alpha} - 1.$$  

Now if $|A| \geq 2$ the idea is similar. The corresponding local term is

$$Z_{v, A} = \sum_{m \in \mathbb{N}_{\geq 0}} \prod_{\alpha \in A} (L^{\rho_{\alpha} T_{\alpha}})^{m_{\alpha}} \int_{\text{ord } x_{\alpha} = m_{\alpha}} e(f_{\xi}(x, y)) dx dy.$$  

The volume of the constructible subsets of $\Omega_v(A)$ over which integration is done is actually

$$\text{vol} \left( \left\{(w, (x_\alpha)_{\alpha \in A}, y) \in D^o_A \times \mathcal{L}(A^1)^A \times \mathcal{L}(A^1, 0)^{n-|A|} \mid \text{ord } x_{\alpha} = m_{\alpha} \text{ for every } \alpha \in A \right\} \right)$$

$$= \left[ D^o_A \right] \prod_{\alpha \in A} \left( L^{-m_{\alpha}} \left( 1 - L^{-1} \right) \right) L^{-n+|A|}.$$  

By definition of $D^o_A$, it has dimension at most $n - |A|$. Thus the dimension of the integral is at most $- \sum_{\alpha \in A} m_{\alpha}$ and, using again the inequality of § 2.5.3, the weight of the $m$-th coefficient of $Z_{v, A}(T, \xi)$ is bounded by

$$2 \sum_{\alpha \in A} m_{\alpha}(\rho_{\alpha} - 1) + \dim C_1 \leq 2c \sum_{\alpha \in A} m_{\alpha}\rho_{\alpha} - 1.$$
To summarize, we obtained

\[
Z_v(T, \xi) = 1 + \sum_{\alpha \in \mathcal{U}_0 \setminus \{\xi\}} Z_v, \alpha(T, \xi) + \sum_{|A| \geq 2} Z_v, A(T, \xi)
\]

\[
Z_v(T, \xi) = 1 + L^{-n} \left(1 - L^{-1}\right) \sum_{\alpha \in \mathcal{U}_0 \setminus \{\xi\}} [D^\alpha \setminus \Xi_{\xi}] \frac{L^{\rho_{\alpha} T_{\alpha}}}{1 - L^{\rho_{\alpha} - 1} T_{\alpha}}
\]

\[
- L^{-n-1} \sum_{\alpha \in \mathcal{U}_0 \setminus \{\xi\}, d_{\alpha}(\xi) = 1} [D^\alpha \setminus \Xi_{\xi}] L^{\rho_{\alpha} T_{\alpha}}
\]

\[\text{+ terms of weights bounded as in Proposition 3.6, locally uniformly in } \xi\]

The last step of the proof consists in multiplying by \(\prod_{\alpha \in \mathcal{U}_0 \setminus \{\xi\}} \left(1 - L^{\rho_{\alpha} - 1} T_{\alpha}\right)\). This operation does not affect the bounds we need in order to apply Proposition 3.6, since \(\rho_{\alpha} - 1 \leq c \rho_{\alpha}\), and allows one to control the diverging term

\[
\sum_{\alpha \in \mathcal{U}_0 \setminus \{\xi\}} [D^\alpha \setminus \Xi_{\xi}] \frac{L^{\rho_{\alpha} T_{\alpha}}}{1 - L^{\rho_{\alpha} - 1} T_{\alpha}}
\]

in the expression of \(Z_v(T, \xi)\). One gets

\[
F_v(T, \xi) = 1 - \sum_{\alpha \in \mathcal{U}_0 \setminus \{\xi\}} L^{\rho_{\alpha} - 1} T_{\alpha} + L^{1-n} \sum_{\alpha \in \mathcal{U}_0 \setminus \{\xi\}} [D^\alpha \setminus \Xi_{\xi}] L^{\rho_{\alpha} - 1} T_{\alpha} + P_v(T, \xi)
\]

\[
= 1 + \sum_{\alpha \in \mathcal{U}_0 \setminus \{\xi\}} \left([D^\alpha \setminus \Xi_{\xi}] - L^{-n-1}\right) L^{1-n} L^{\rho_{\alpha} - 1} T_{\alpha} + P_v(T, \xi)
\]

where \(P_v(T, \xi)\) is a Laurent series consisting of terms satisfying the bounds of Proposition 3.6 for

\[
c = \max_{\alpha \in \mathcal{U}_0} \left(1 - \frac{1}{2 \rho_{\alpha}}\right).
\]

Finally, by Proposition 3.1 one has for every \(\alpha \in \mathcal{U}_0 \setminus \{\xi\}\)

\[
w_{C_1} \left(\left([D^\alpha \setminus \Xi_{\xi}] - L^{-n-1}\right) L^{1-n} L^{\rho_{\alpha} - 1}\right) \leq 2(n - 1) - 2(n - 1) + 2(\rho_{\alpha} - 1)
\]

\[= 2(\rho_{\alpha} - 1).\]

Our analysis shows that we can apply Proposition 3.6 with \(X = C_1, \varepsilon = \frac{1}{2}, M\) arbitrary and \(\beta = 0\). The result follows.

**Remark 4.9** The previous proof shows that the weight-linear convergence of \(\prod_{\nu \in C_1} F_v(T, \xi)\) with respect to \(\rho\) is uniform on each set of a finite partition of \(V \setminus \{0\}\) given by
\[ V_{A_0^D, A_1^D, A_{\geq 2}^D} = \left( \bigcap_{\alpha \in A_0^D} d_{\alpha}^{-1}([0]) \right) \cap \left( \bigcap_{\alpha \in A_1^D} d_{\alpha}^{-1}([1]) \right) \cap \left( \bigcap_{\alpha \in A_{\geq 2}^D} d_{\alpha}^{-1}(\mathbb{N}_{\geq 2}) \right) \]

\[ A_0^D \sqcup A_1^D \sqcup A_{\geq 2}^D = \mathcal{A}_U \]

(recall that \( V \) is the \( n \)-th power of the Riemann-Roch space of the divisor \( \tilde{E} \) (4.2.8)).

5 Moduli spaces of curves: asymptotic behaviour

5.1 Simplified case: rational curves

For the sake of simplicity we start the proof of our result assuming that \( C \) is the projective line over \( k \). For any \( a \) and \( b \) in \( \mathbb{Z}^{\mathcal{A}} \) we write \( a \leq b \) if and only if \( b - a \) lies in \( \mathbb{N}^{\mathcal{A}} \).

Recall that by (Hassett and Tschinkel 1999, Theorem 2.7) there exists an \( r \)-tuple of integers \( (\rho_{\alpha})_{\alpha \in \mathcal{A}} \) such that \( \rho_{\alpha} \geq 2 \) for all index \( \alpha \) and

\[ \sum_{\alpha \in \mathcal{A}} \rho_{\alpha} D_{\alpha} \]

is an anticanonical divisor on \( X = \mathcal{Z}_F \). A log-divisor with respect to the boundary \( D \) is then given by

\[ \sum_{\alpha \in \mathcal{A}} \rho'_{\alpha} D_{\alpha} \]

where \( \rho'_{\alpha} = \rho_{\alpha} - 1 \) if \( \alpha \in \mathcal{A}_D \) and \( \rho'_{\alpha} = \rho_{\alpha} \) otherwise. As it has already been the case in the introduction, for any \( r \)-tuple \( \mathbf{m} = (m_{\alpha})_{\alpha \in \mathcal{A}} \) of integers, we will use freely the pairing

\[ \langle \rho', \mathbf{m} \rangle = \sum_{\alpha \in \mathcal{A}} \rho'_{\alpha} m_{\alpha} \]

as well as its obvious restriction \( \langle \rho', \mathbf{m} \rangle_{A} \) to any subset \( A \) of \( \mathcal{A} \). Recall that this quantity corresponds to a generic log-anticanonical degree.

5.1.1 Main term

First we study the contribution of the trivial character \( \xi = 0 \), which is expected to be the only one to contribute asymptotically after normalization by \( L^{\langle \rho', \mathbf{m} \rangle} \). For sake of simplicity we begin assuming \( \mathcal{W} = \mathcal{Z} \) so that \( \mathcal{A}_U = \mathcal{A} \) and the coarse and refined height zeta functions coincide.
For every place $v$ of $\mathbb{P}^1_k$ recall that we already defined a polynomial

$$F_v(T, 0) = Z_v(T, 0) \prod_{\alpha \in \mathcal{A}} \left(1 - L^{\rho_\alpha - 1} T\alpha\right)$$

with Notation 4.4. Thanks to Proposition 2.3 we are able to permute the products $\prod_{\alpha \in \mathcal{A}}$ and $\prod_{v \in \mathbb{P}^1_k}$ when considering $Z(T, 0)$. It means that

$$Z(T, 0) = \prod_{v \in \mathbb{P}^1_k} Z_v(T, 0) = \left(\prod_{v \in \mathbb{P}^1_k} F_v(T, 0)\right) \left(\prod_{\alpha \in \mathcal{A}} \prod_{v \in \mathbb{P}^1_k} \left(1 - L^{\rho_\alpha - 1} T\alpha\right)^{-1}\right).$$

By definition, the Kapranov Zeta function of $\mathbb{P}^1_k$ is the power series

$$Z_{\mathbb{P}^1_k}(T) = \prod_{v \in \mathbb{P}^1_k} (1 - T)^{-1}$$

which is given explicitly (see e.g. (Kapranov 2000, Theorem 1.1.9) or (Chambert-Loir et al. 2018, Chapter 7, Theorem 1.3.1)) by

$$Z_{\mathbb{P}^1_k}(T) = \frac{1}{(1 - T)(1 - LT)} = \frac{1}{1 - L} \left(\frac{1}{1 - T - \frac{L}{1 - LT}}\right).$$

In our case, this gives for every index $\alpha$

$$\prod_{v \in \mathbb{P}^1_k} \left(1 - L^{\rho_\alpha - 1} T\alpha\right)^{-1} = Z_{\mathbb{P}^1_k}(L^{\rho_\alpha - 1} T\alpha) = \frac{1}{1 - L} \left(\frac{1}{1 - L^{\rho_\alpha - 1} T\alpha} - \frac{L}{1 - L^{\rho_\alpha} T\alpha}\right).$$

Expanding the right side of this equality, one gets that the coefficient of $T^m\alpha$ for any non-negative integer $m$ is given by

$$\frac{L^{\rho_\alpha m}(L^{-m} - L)}{1 - L} = \frac{1}{1 - L^{-1}} \left(1 - L^{-m-1}\right) L^{\rho_\alpha m}.$$

Thus, loosely speaking, after normalisation by $L^{\rho_\alpha m}$, the contribution of $(1 - L^{\rho_\alpha - 1} T\alpha)^{-1}$ in this last formula tends to zero when $m$ tends to infinity. In order to find the expected limit of $L^{-(\rho, m)}[M_U, m]$ when $\min_{\alpha \in \mathcal{A}} (m_\alpha)$ tends to infinity, it may be natural to consider for a while the coefficients of the series

$$\bar{Z}(T) = \left(\prod_{\alpha \in \mathcal{A}} \left(\left(1 - L^{-1}\right) \left(1 - L^{\rho_\alpha} T\alpha\right)^{-1}\right)\right) \left(\prod_{v \in \mathbb{P}^1_k} F_v(T, 0)\right)$$

instead of considering those of $Z(T, 0)$.
For any \( r \)-tuple \( m = (m_\alpha)_{\alpha \in a} \) of integers, let us denote by \( a_m \) the coefficient of the monomial \( T^m \) in \( \mathbb{Z}(T) \). Denoting by \( b_n \) the coefficient of \( T^n \) in \( \prod_{v \in \mathcal{P}_k} F_v(T, 0) \) for any \( r \)-tuple \( n \) of integers, one gets the finite sum

\[
\sum_{n \in \mathbb{Z}^a} b_n \times L^{(\rho, n')}.
\]

After normalisation by \( L^{(\rho, m)} \), this is actually the \( m \)-th partial sum of the Euler motivic product \( \prod_{v \in \mathcal{P}_k} F_v \left( \left( L^{-\rho_\alpha} \right)_{\alpha \in a} \right, 0 \). Therefore by Proposition 4.5

\[
\tilde{a}_m L^{-(\rho, m)}
\]

converges in \( \hat{\mathcal{M}}_k \) to

\[
\frac{1}{(1 - L^{-1})^r} \prod_{v \in \mathcal{P}_k^1} F_v \left( \left( L^{-\rho_\alpha} \right)_{\alpha \in a}, 0 \right)
\]

when \( \min_{\alpha \in a} (m_\alpha) \) tends to infinity.

This heuristic argument given, now we have to justify that this is indeed the limit of \( L^{-(\rho, m)} [M_{U, m}] \) (up to a factor \( L^n \)) in \( \hat{\mathcal{M}}_k \) when \( \min_{\alpha \in a} (m_\alpha) \) becomes infinitely large. First we evaluate the error term we introduced, then we check that terms corresponding to non-trivial characters do not contribute to the limit. We postpone the summation over all characters to the last section of this paper, where it will be performed in full generality.

### 5.1.2 Contribution of the error term

In order to control the error term we implicitly introduced in the previous paragraph, we develop the denominator of \( Z(T, 0) \) as follows. We still assume \( U = \mathcal{X} \). Then

\[
G(T, 0) = \prod_{\alpha \in a} Z_{\mathbb{P}_k^1} \left( L^{\rho_\alpha - 1} T_\alpha \right)
\]

\[
= \prod_{\alpha \in a} \prod_{v \in \mathcal{P}_k^1} \left( 1 - L^{\rho_\alpha - 1} T_\alpha \right)^{-1}
\]

\[
= \prod_{\alpha \in a} \frac{1}{1 - L} \left( \frac{1}{1 - L^{\rho_\alpha - 1} T_\alpha} - \frac{L}{1 - L^{\rho_\alpha} T_\alpha} \right)
\]

\[
= \frac{1}{(1 - L)^r} \sum_{\varepsilon \in \{0, 1\}^a} (-L)^{|a| - |\varepsilon|} \prod_{\alpha \in a} \left( 1 - L^{\rho_\alpha - \varepsilon_\alpha} T_\alpha \right),
\]

where \( |\varepsilon| = \sum_{\alpha} \varepsilon_\alpha \). Let us introduce a few convenient notations. For any \( \varepsilon \in \{0, 1\}^a \) we define
so that \( G(T,0) \) admits a decomposition of the form

\[
G(T,0) = \sum_{\varepsilon \in \{0,1\}} \frac{(-L)^{r-|\varepsilon|}}{(1-L)^r} G_{\varepsilon}(T).
\]

Now one easily sees that the term of multidegree \( m \in \mathbb{Z}^d \) of \( G_{\varepsilon}(T) \prod_{v \in P_1} F_v(T,0) \) is the finite sum

\[
g_{\varepsilon}^m = \sum_{n \in \mathbb{Z}^d \atop n' \in \mathbb{N}^d \atop n+n'=m} b_n L^{(\rho-\varepsilon,n')} \]

from which we deduce

\[
g_{\varepsilon}^m L^{-(\rho,m)} = \sum_{n \in \mathbb{Z}^d \atop n' \in \mathbb{N}^d \atop n+n'=m} b_n L^{-(\rho,n)} L^{-(\varepsilon,n')}. \]

The case \( \varepsilon = 0 \) has been studied in the previous paragraph and it suffices to apply Lemma 3.5 to \( \sum b_n L^{-(\rho,n)} \) in case \( \varepsilon \neq 0 \). We obtain the following proposition.

**Proposition 5.1** \( (U = \mathcal{O}, C = C_0 = P_k^1, \xi = 0) \) There exists a decomposition

\[
Z(T,0) = \sum_{m \in \mathbb{Z}^d} a_m T^m = \sum_{\varepsilon \in \{0,1\} \setminus \{0\}} \sum_{m \in \mathbb{Z}^d} a^\varepsilon_m T^m
\]

and a real number \( \delta > 0 \) such that for all \( \varepsilon \in \{0,1\} \setminus \{0\} \)

\[
w\left(a^\varepsilon_m L^{-(\rho,m)}\right) < -\delta \langle \varepsilon, m \rangle
\]

while

\[
a^\varepsilon_m L^{-(\rho,m)} \xrightarrow{\rho\text{-weight-lin.}} \frac{1}{(1-L^{-1}) \mathrm{rg}(\mathrm{Pic}(X))} \prod_{v \in P_k^1} F_v \left((L^{-\rho_a})_{a \in \mathcal{A}}, 0\right)
\]

when \( \min_{a \in \mathcal{A}} (m_a) \) tends to infinity.

As an immediate corollary of this proposition, the terms \( a^\varepsilon_m L^{-(\rho,m)} \), for \( \varepsilon \neq 0 \), are negligible when \( \min_{a \in \mathcal{A}} (m_a) \) becomes arbitrarily large, in comparison with \( a^0_m L^{-(\rho,m)} \).
5.1.3 Non-trivial characters

We now study the asymptotic contribution of the coefficients of $Z(T, \xi)$ for $\xi \in G(F)^\vee$ non-trivial. We still assume $C = \mathbb{P}^1_k$.

Recall that given a place $v$ of $C$, the linear form $x \mapsto \langle x, \xi \rangle$ on $G_F$ can be seen as a rational function $f_\xi$ on $X$ with poles contained in $\bigcup_{\alpha \in \mathcal{A}} D_\alpha$. If $d_\alpha(\xi)$ denotes the order of the pole of $f_\xi$ with respect to $D_\alpha$, one can define a subset of $\mathcal{A}_U$ by setting

$$\mathcal{A}_U^0(\xi) = \left\{ \alpha \in \mathcal{A}_U \mid d_\alpha(\xi) = 0 \right\}.$$ 

The character $\xi$ being non-trivial, the cardinality of this set is strictly smaller than $|\mathcal{A}|$. Using Notation 4.7, the term of the motivic height Zeta function corresponding to the character $\xi$ is

$$Z(T, \xi) = \prod_{v \in \mathbb{P}^1_k} \left( F_v(T, \xi_v) \prod_{\alpha \in \mathcal{A}_U^0(\xi)} \left( 1 - L^{\rho_\alpha - 1} T_\alpha \right)^{-1} \right).$$

Again we restrict our analysis to the case $\mathcal{U} = \mathcal{X}$ so that $\mathcal{A} = \mathcal{A}_U$. The general case will be treated together with the case of a general curve. For any $\varepsilon \in \{0, 1\}$, we introduce

$$G_\varepsilon(T, \xi) = \prod_{\alpha \in \mathcal{A}_U^0(\xi)} \left( 1 - L^{\rho_\alpha - \varepsilon \alpha} T_\alpha \right)^{-1}.$$

Adapting the computation done in the previous paragraph, one gets

$$G(T, \xi) = \prod_{v \in \mathbb{P}^1_k} \left( \prod_{\alpha \in \mathcal{A}_U^0(\xi)} \left( 1 - L^{\rho_\alpha - \varepsilon \alpha} T_\alpha \right)^{-1} \right) = \sum_{\varepsilon \in \{0, 1\}} \frac{(-L)^{|\mathcal{A}_U^0(\xi)| - |\varepsilon|}}{(1 - L)^{|\mathcal{A}_U^0(\xi)|}} G_\varepsilon(T, \xi)$$

and the coefficient of multidegree $m \in \mathbb{N}^{\mathcal{A}}$ of the product

$$G_\varepsilon(T, \xi) \prod_{v \in \mathbb{P}^1_k} F_v(T, \xi_v)$$

is

$$g_{\varepsilon, m} = \sum_{n, n' \in \mathbb{Z}^{\rho_\mathcal{A}}}bf_{n, n'}^{(\rho - \varepsilon, n')^{\mathcal{A}_U^0(\xi)}}$$

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where $b_n^\xi$ is the coefficient of multidegree $n \in \mathcal{A}^{\text{eff}}$ of $\prod_{v \in P_k^1} F_v(T, \xi)$. Then we consider the following normalisation:

$$g_{\xi, m} L^{-\langle \rho, m \rangle} = \sum_{n \in \mathcal{A}^{\text{eff}}} b_n^\xi L^{-\langle \rho, n \rangle} L^{-\langle \rho, n' \rangle} L^{\langle \varepsilon, n \rangle} L^{\langle \varepsilon, n' \rangle} L^{\langle \varepsilon, 0 \rangle} L^{\langle \varepsilon, 1 \rangle}.$$

Applying Lemma 3.5 to this sum for every $\varepsilon$, we get the following proposition.

**Proposition 5.2** ($\mathcal{U} = \mathcal{X}, C = C_0 = P_k^1, \xi \neq 0$) There exists a decomposition

$$Z(T, \xi) = \sum_{\varepsilon \in \{0, 1\}} \sum_{m \in \mathcal{A}^{\text{eff}}} a_{m, \varepsilon} T^m$$

and a positive real number $\delta_\xi > 0$ (which only depends on the $d_\alpha(\xi)$) such that for every $\varepsilon \in \{0, 1\}$

$$w\left(a_{m, \varepsilon} L^{-\langle \rho, m \rangle}\right) < -\delta_\xi \left(\langle \rho, m \rangle_{\mathcal{A} \setminus \mathcal{A}_0(\xi)} + \langle \varepsilon, m \rangle_{\mathcal{A}_0(\xi)}\right)$$

when $\min(m_\alpha) \to \infty$.

Since $\mathcal{A}_U \setminus \mathcal{A}_U^0(\xi)$ is non-empty for every $\xi \neq 0$, the inequality of this last proposition means that the normalised terms $a_{m, \varepsilon} L^{-\langle \rho, m \rangle}$ coming form $Z(T, \xi)$ are negligible in comparison with the main term $a_m^0 L^{-\langle \rho, m \rangle}$ coming from $Z(T, 0)$. Furthermore we have a linear bound on the weight which is uniform on a finite partition of the space of characters by Remark 4.9.

### 5.2 The case of a general curve

From now on we assume that $C$ is a projective smooth irreducible curve of genus $g$ over $k$ having a $k$-rational point. In this case, for $m \geq 2g - 1$ the class of $\text{Sym}^m(C)$ in $K\text{Var}_k$ is $[\text{Pic}^0(C)] [\mathbb{P}^{m-\delta}_k]$ (it follows from Riemann–Roch and Serre’s duality, see (Chambert-Loir et al. 2018, Chapter 7, Example 1.1.10)) and the Kapranov zeta function of $C$ is still rational (see (Kapranov 2000, Theorem 1.1.9) or (Chambert-Loir et al. 2018, Chapter 7, Theorem 1.3.1)). We have

$$Z_C^{\text{Kap}}(T) = \sum_{m \in \mathbb{N}} [\text{Sym}^m(C)] T^m \sum_{m \geq 2g-1} \frac{L^{m-g+1} - 1}{L - 1} T^m$$
(where the first sum is empty if \( g = 0 \)). Consider for any \( \xi \in G(F)^{\vee} \)

\[
G(T, \xi) = \prod_{v \in C} \prod_{\alpha \in \mathcal{A}_U'(\xi)} (1 - L^{\rho_{\alpha}^{-1}T_{\alpha}})^{-1}
\]

(remark that if \( \xi = 0 \) then \( \mathcal{A}_U'(\xi) = \mathcal{A}_U \)). By Proposition 2.3 one has

\[
G(T, \xi) = \prod_{\alpha \in \mathcal{A}_U'(\xi)} Z_{Kap}^{C}(L^{\rho_{\alpha}^{-1}T_{\alpha}})
\]

In order to easily adapt the computations done before, we may assume \( g \geq 1 \), replace \( Z_{Kap}^{C} \) by

\[
\tilde{Z}_{Kap}^{C}(T) = \sum_{m \geq 0} [\text{Pic}^0(C)] L^{m-g+1} - 1 T^m
\]

in the expression of \( G(T, \xi) \) and then control the error term coming from this slight modification. If we do so, then we obtain

\[
Z(T, \xi) = G(T, \xi) \prod_{v \in C} F_v(T, \xi) = \tilde{G}(T, \xi) \prod_{v \in C} F_v(T, \xi) + H(T, \xi) \prod_{v \in C} F_v(T, \xi)
\]

where

\[
\tilde{G}(T, \xi) = \prod_{\alpha \in \mathcal{A}_U'(\xi)} \tilde{Z}_{Kap}^{C}(L^{\rho_{\alpha}^{-1}T_{\alpha}}) \tag{5.2.10}
\]

and

\[
H(T, \xi) = \prod_{\alpha \in \mathcal{A}_U'(\xi)} Z_{Kap}^{C}(L^{\rho_{\alpha}^{-1}T_{\alpha}}) - \prod_{\alpha \in \mathcal{A}_U'(\xi)} \tilde{Z}_{Kap}^{C}(L^{\rho_{\alpha}^{-1}T_{\alpha}}). \tag{5.2.11}
\]

We then put

\[
\tilde{Z}(T, \xi) = \tilde{G}(T, \xi) \prod_{v \in C} F_v(T, \xi). \tag{5.2.12}
\]

### 5.2.1 Main term

Basically, the contribution of \( \tilde{Z}(T, \xi) \) has already been treated in Sect. 5.1.2 for the case \( \xi = 0 \) and in Sect. 5.1.3 for non-trivial characters. The only difference with the
particular case of the projective line is a factor $[\text{Pic}^0(C)]L^{1-g}$. Indeed, starting from (5.2.10) one gets

$$ \tilde{G}(T, \xi) = \prod_{\alpha \in A_0} \frac{[\text{Pic}^0(C)]}{1 - L} \left( \frac{1}{1 - L^{\rho_\alpha - 1} T_\alpha} - \frac{L^{1-g}}{1 - L^{\rho_\alpha} T_\alpha} \right) $$

$$ = \frac{[\text{Pic}^0(C)]}{(1 - L)} \sum_{\xi \in \{0, 1\}^{0 \not U(\xi)}} \frac{(-L^{1-g})^{0 \not U(\xi)} [-\xi]}{\prod_{\alpha \in A_0} (1 - L^{\rho_\alpha} T_\alpha)} $$

$$ = \left( \frac{[\text{Pic}^0(C)]L^{1-g}}{L - 1} \right) \sum_{\xi \in \{0, 1\}^{0 \not U(\xi)}} (-1)^{|\xi|} \tilde{G}_\xi(T, \xi)L^{|\xi|(g-1)} $$

where

$$ \tilde{G}_\xi(T, \xi) = \prod_{\alpha \in A_0} (1 - L^{\rho_\alpha - \xi_\alpha} T_\alpha)^{-1} = \sum_{m \in \mathbb{Z}^{0 \not U(\xi)}} L^{(\rho - \xi, m)} \not U(\xi) \sum_{m \in \mathbb{Z}^{0 \not U(\xi)}} \tilde{a}_m \xi^m. $$

The following proposition summarizes what we obtain if we replace $G_\xi(T, \xi)$ by $\tilde{G}_\xi(T, \xi)$ in the previous paragraph. Again, it ensures the negligibility of the terms corresponding to non-trivial characters (up to the error term coming from $H(T, \xi)$, which is treated in the next paragraph).

**Proposition 5.3** ($\mathcal{U} = \mathcal{X}, C = C_0, \mathcal{A} = \mathcal{A}(U)$) For any character $\xi$, there exists a decomposition

$$ \tilde{Z}(T, \xi) = \sum_{m \in \mathbb{Z}^{\mathcal{A}} \not U(\xi)} \tilde{a}_m \xi^m = \sum_{\xi \in \{0, 1\}^{0 \not U(\xi)}} \sum_{m \in \mathbb{Z}^{\mathcal{A}}} \tilde{a}_m \xi^m \xi^m $$

and a real number $\delta_\xi > 0$ (depending only on the $d_\alpha(\xi)$) such that

- if $\xi = 0$
  $$ \tilde{a}_m L^{-\langle \rho, m \rangle} = \left( \frac{[\text{Pic}^0(C)]L^{1-g}}{L - 1} \right) \left| \not U \right| \sum_{m' \in \mathbb{Z}^{\mathcal{A}}, m' \leq m} b_{m'} L^{-\langle \rho, m' \rangle} $$
  $$ \longrightarrow \rho \text{-weight-lin.} \left( \frac{[\text{Pic}^0(C)]L^{1-g}}{L - 1} \right) \left| \not U \right| \prod_{v \in C_0} F_v((L^{-\rho_\alpha})_{\alpha \in \mathcal{A}} \not U, 0) $$

where $b_{m'}$ is the $m'$-th coefficient of $\prod_{v \in C_0} F_v(T, 0)$ and

$$ w(\tilde{a}_m^0 \xi L^{-\langle \rho, m \rangle}) < -\delta_0 \langle \epsilon, m \rangle \text{ for all } \epsilon \in \{0, 1\}^{0 \not U(\xi) \setminus \{0\}} $$

when $\min_{\alpha \in \mathcal{A}_0} (m_\alpha) \to \infty$. 

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• if $\xi \neq 0$

$$w \left( \tilde{a}_{m}^{\xi} L^{-\langle \rho, m \rangle} \right) < -\delta_{\xi} \left( \langle \rho, m \rangle_{\mathcal{A} \backslash \mathcal{A}^{0}_{U}(\xi)} + \langle \xi, m \rangle_{\mathcal{A}^{0}_{U}(\xi)} \right) \quad \text{for all } \varepsilon \in \{0, 1\}^{\mathcal{A}^{0}_{U}(\xi)}$$

when $\min(m_{\alpha}) \to \infty$.

5.2.2 Error term

Now we study the contribution of $H(T, \xi)$, still assuming $U = \mathcal{A}$. We rewrite this error term as follows, using the convenient notations $g = (g, \ldots, g)$ and $1 = (1, \ldots, 1)$.

In this paragraph $\xi$ is any character, including the trivial one; in this case $\mathcal{A}^{0}_{U}(0) = \mathcal{A}$.

$$H(T, \xi) = \sum_{\emptyset \neq A \subset \mathcal{A}^{0}_{U}(\xi)} \prod_{\alpha \in \mathcal{A}^{0}_{U}(\xi) \backslash A} \tilde{Z}_{C}^{Kap}(L^{\rho_{\alpha}-1}T_{\alpha})$$

$$\times \prod_{\alpha \in A} \left( Z_{C}^{Kap}(L^{\rho_{\alpha}-1}T_{\alpha}) - Z_{C}^{Kap}(L^{\rho_{\alpha}-1}T_{\alpha}) \right)$$

$$= \sum_{\emptyset \neq A \subset \mathcal{A}^{0}_{U}(\xi)} \left[ \text{Pic}^{0}(C) \right]^{\mathcal{A}^{0}_{U}(\xi) \backslash A} \left[ -L^{1-g} \right]^{\mathcal{A}^{0}_{U}(\xi)} \left[ -|\varepsilon| \right]^{\mathcal{A}^{0}_{U}(\xi)} \left( 1 - L \right)^{\mathcal{A}^{0}_{U}(\xi)} \left[ \text{Pic}^{0}(C) \right]^{\mathcal{A}^{0}_{U}(\xi)} \left[ L^{m_{\alpha}} - 1 \right]^{\mathcal{A}^{0}_{U}(\xi)} \left( 1 - L \right)^{\mathcal{A}^{0}_{U}(\xi)}$$

with for any $A \subset \mathcal{A}^{0}_{U}(\xi)$ non-empty and $\varepsilon \in \{0, 1\}^{\mathcal{A}^{0}_{U}(\xi) \backslash A \times \{1\}^{A}}$

$$H_{A, \varepsilon}(T, \xi) = \sum_{m \in \mathbb{N}^{\mathcal{A}^{0}_{U}(\xi)}} L^{\langle \rho - \varepsilon, m \rangle} T^{m} \prod_{\alpha \in A} \left( [\text{Sym}^{m_{\alpha}} C] - [\text{Pic}^{0}(C)] \frac{L^{m_{\alpha}} - g - 1}{L - 1} \right).$$

Thus studying the contribution of $H(T, \xi)$ amounts to studying the $m$-th coefficient of the product

$$H_{A, \varepsilon}(T, \xi) \prod_{v \in C} F_{v}(T, \xi)$$

for every $A \subset \mathcal{A}^{0}_{U}(\xi)$ non-empty and $\varepsilon \in \{0, 1\}^{\mathcal{A}^{0}_{U}(\xi) \backslash A \times \{1\}^{A}}$. In what follows we fix such an $A$. We know that the term of multi-degree $m$ of $H_{A, \varepsilon}(T)$ is zero whenever there is $\alpha \in A$ such that $m_{\alpha} \geq 2g - 1$. So the $m$-th term of the product $H_{A, \varepsilon}(T, \xi) \prod_{v \in C} F_{v}(T, \xi)$ is equal to
\[
\mathcal{e}_{A,\varepsilon} = L^{\langle \rho - \varepsilon, m \rangle_{\mathcal{A}, \mathcal{U}^0(\xi)}} \sum_{(n, n') \in \mathbb{Z}^0 \times \mathbb{N}^0 \atop n + n' = m \atop n'_A \leq 2(g_A - 1)} b_n L^{\langle \rho - \varepsilon, n \rangle_{\mathcal{A}, \mathcal{U}^0(\xi)}} \times \prod_{\alpha \in A} \left( [\text{Sym}^{n'_\alpha} C] - [\text{Pic}^0(C)] L^{n'_\alpha - g - 1} - 1 \right) \quad (5.2.13)
\]

where \( b_n \) is the \( n \)-th coefficient of \( \prod_{v \in C} F_v(T, \xi) \) and \( n'_A \) denotes the restriction to \( A \) of \( n' \in \mathbb{N}^0 \). Since \( A \neq \emptyset \), by Proposition 4.5 (for the trivial character), Proposition 4.8 (for non-trivial characters) and Lemma 3.5 we get the existence of a \( \delta > 0 \) (which only depends on the \( d_\alpha(\xi) \)) such that for any \( \varepsilon \in \{0, 1\} \)

\[
w \left( \sum_{(n, n') \in \mathbb{Z}^0 \times \mathbb{N}^0 \atop n + n' = m \atop n'_A \leq 2(g_A - 1)} b_n L^{\langle \rho - \varepsilon, n \rangle_{\mathcal{A}, \mathcal{U}^0(\xi)}} - \langle \rho, n \rangle_{\mathcal{A}, \mathcal{U}^0(\xi)} - \langle \rho, n \rangle_{\mathcal{A}, \mathcal{U}^0(\xi) \setminus \mathcal{A}} \right) < -\delta \left( \langle \varepsilon, m \rangle_{\mathcal{A}, \mathcal{U}^0(\xi)} + \langle \rho, m \rangle_{\mathcal{A}, \mathcal{U}^0(\xi)} \right) \quad (5.2.14)
\]

since for any \( n \in \mathbb{Z}^0 \) one has the decomposition

\[
\langle \rho - \varepsilon, n \rangle_{\mathcal{A}, \mathcal{U}^0(\xi)} = \langle \rho, n \rangle - \langle \varepsilon, n \rangle_{\mathcal{A}, \mathcal{U}^0(\xi)} - \langle \rho, n \rangle_{\mathcal{A}, \mathcal{U}^0(\xi) \setminus \mathcal{A}}.
\]

Remark that the product over \( A \) can only take a finite number of values, its weight is therefore bounded. Combining (5.2.13) and (5.2.14), we conclude that there exists \( \delta' > 0 \) such that whenever \( \langle \varepsilon, m \rangle_{\mathcal{A}, \mathcal{U}^0(\xi)} + \langle \rho, m \rangle_{\mathcal{A}, \mathcal{U}^0(\xi)} \) is large enough.

This analysis proves that the slight modification we performed on the Kapranov Zeta function of \( C \) is painless in the case \( \mathcal{U} = \mathcal{X} \). It allows us to extend the result to any smooth projective irreducible curve of genus \( g \). The control of the error terms coming from \( H(T, \xi) \) is given by the following proposition.

**Proposition 5.4** (\( \mathcal{U} = \mathcal{X}, C = C_0 \)) Let \( \xi \) be any character. There exists a decomposition

\[
H(T, \xi) \prod_{v \in C_0} F_v(T, \xi) = \sum_{\emptyset \neq A \subset \mathcal{A}, \mathcal{U}^0(\xi)} \sum_{m \in \mathbb{Z}^0} b_{A,\varepsilon} m \mathcal{U}^0(\xi) \times \{1\}^A
\]
and a real number $\delta_{\xi} > 0$ (depending only on $\prod_{\nu \in C} F_{\nu}(T, \xi)$) such that for all non empty subset $A \subset \mathcal{A}_{1}^{\emptyset}(\xi)$ and $\varepsilon \in \{0, 1\} \mathcal{A}_{1}^{\emptyset}(\xi) \times \{1\}^{A}$

$$w \left( L_{\varepsilon}^{-\langle \rho, m \rangle} T_{m, \varepsilon} \right) < -\delta_{\xi} \left( \langle \varepsilon, m \rangle_{\mathcal{A}_{1}^{\emptyset}(\xi)} + \langle \rho, m \rangle \right)$$

when $\min(m_\alpha) \to \infty$.

5.2.3 $S$-integral points and the trivial character

We now start treating the general case $\mathcal{U} \subset X$. For clarity’s sake we will keep using the coarse height zeta function $Z(T)$ until it becomes necessary to switch to its refined version $\mathcal{Y}(U)$, that is, when we will integrate the Euler product over $S$ to the global one over $C$.

In order to treat the case of $S$-integral points, as in (Chambert-Loir and Loeser 2016, §2.3) we introduce a simplicial complex encoding the intersecting data of the boundary’s components. Let $Y$ be a smooth algebraic variety over a field $L$ and $\Delta$ a divisor on $Y$ with strict normal crossings. The Clemens complex $\text{Cl}(Y, \Delta)$ is the simplicial complex whose vertices are the irreducible components $\Delta_i \times \{1\}$ of $\Delta$. It has an edge between $\Delta_i$ and $\Delta_j$ for $i \neq j$ if and only if $\Delta_i \cap \Delta_j \neq \emptyset$. It has a two dimensional face, given by the vertices $\Delta_i, \Delta_j$ and $\Delta_k$, for $i, j, k$ pairwise distinct, if and only if $\Delta_i \cap \Delta_j \cap \Delta_k \neq \emptyset$, and so on for higher dimensional faces: in general, a subset $J \subset I$ corresponds to a face of dimension $|J| - 1$ of $\text{Cl}(Y, \Delta)$ if and only if $\Delta_J = \bigcap_{j \in J} \Delta_j \neq \emptyset$. Since we assumed that $Y$ is smooth and $\Delta$ has strict normal crossings, the intersections we are considering are smooth as well.

Then a maximal face of $\text{Cl}(Y, \Delta)$ is a simplex whose vertices are indexed by a subset $J \subset I$ such that $\Delta_J \neq \emptyset$ and $\Delta_J \cap \Delta_k = \emptyset$ for any $k \in I \setminus J$. In particular, the dimension of $\text{Cl}(Y, \Delta)$ is the maximal number of components of $\Delta$ with non-empty intersection minus one.

The $L$-Clemens complex $\text{Cl}_L(Y, \Delta)$ is defined in a similar way, by restriction to $L$-points. More precisely, $\text{Cl}_L(Y, \Delta)$ is the subcomplex of $\text{Cl}(Y, \Delta)$ consisting of simplices $\Delta_J \in \text{Cl}(Y, \Delta)$ such that $\Delta_J(L) \neq \emptyset$. The set of maximal faces of $\text{Cl}_L(Y, \Delta)$ is $\text{Cl}_L^{\text{max}}(Y, \Delta)$.

Let $s \in S$. By (Chambert-Loir and Loeser 2016, §6) and (Bilu 2018, §6.3.6) we have

$$Z_s^\beta(T, 0) = \sum_{A \subset \mathcal{A}_1} L^{\rho_\beta_{\mathcal{A} \setminus \beta}(A, \beta)}[\Delta_s(A, \beta)] L^{-n + |A|} (1 - L^{-1})^{|A|} \prod_{\alpha \in A} \frac{L_{\rho_\alpha} T_{\alpha}}{1 - L_{\rho_\alpha} T_{\alpha}}.$$ 

Our goal here is to isolate the contribution to the poles of the boundary $D$. To do so, following Bilu (2018); Chambert-Loir and Loeser (2016) we associate to any pair $(A, \beta)$ such that $\Delta_s(A \cap \mathcal{A}_D, \beta) \neq \emptyset$ a maximal subset $M_s$ of $\mathcal{A}_D$ such that $A \cap \mathcal{A}_D \subset M_s$ and $\Delta_s(M_s, \beta) \neq \emptyset$. For a general $A \subset \mathcal{A}_1$, there is no canonical choice of such a maximal subset: we arbitrary choose such a map. This will not be a

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1 This complex is called analytic Clemens complex in Chambert-Loir and Loeser (2016).
key issue, as we will see in Remarks 5.6 & 5.7 below. We assume furthermore that \( \Delta_s(M_s, \beta) \) has a \( F_s \)-point; this assumption is natural when one considers sections intersecting, above \( s \), the divisors \( D_\alpha \) given by \( M_s \subset A \), see Remark 5.14.

**Lemma 5.5** Such an \( M_s \subset A \) corresponds to a maximal face of the analytic Clemens complex \( \text{Cl}_s(X, D) = \text{Cl}_{F_s}(X_{F_s}, D_{F_s}) \).

**Proof** By definition, \( \Delta_s(A, \beta) \) is the set of points of the fibre \( X_s \) belonging exclusively to \( D_\alpha \) and \( E_\beta \), that is to say

\[
\Delta_s(A, \beta) = \left( \bigcap_{\alpha \in A} D_\alpha \right) \cap \left( \bigcup_{\beta \neq \beta'} \bigcup_{\alpha \in A} \bigcap_{s} E_\beta \right).
\]

For simplicity we assume that \( B_1, s = \{ \beta \} \).

We argue by contradiction: assume that \( M_s \) is not a maximal face of \( \text{Cl}_{F_s}(X_{F_s}, D_{F_s}) \).

It means that there exists a non-empty subset \( M'_s \subset A \) which contains \( M_s \) as a proper subset and such that

\[
\left( \bigcap_{\alpha \in M'_s} (D_\alpha)_{F_s} \right) (F_s) \neq \emptyset.
\]

We can assume furthermore that \( M'_s \) is maximal for this property. In other words \( M'_s \) is a maximal face of \( \text{Cl}_{F_s}(X_{F_s}, D_{F_s}) \) containing \( M_s \) as a proper subface. Remark that since \( X \setminus U = \bigcup_{\alpha \in A} D_\alpha \) and \( X_{F_s}(F_s) = X(F_s) \), together with the maximality of \( M'_s \), then

\[
\bigcap_{\alpha \in M'_s} D_\alpha \cap D_{\alpha'}(F_s) = \emptyset \quad (5.2.15)
\]

for every \( \alpha' \in A \setminus M'_s \).

Our argument relies on the fact that \( \mathcal{X} \to C \) is proper. Composing with the projection \( X_{F_s} \to \mathcal{X} \) and applying the valuative criterion of properness, this \( F_s \)-point \( x_s \) uniquely lifts to a \( O_s \)-point \( \tilde{x}_s \) of \( \mathcal{X} \).

Since for every \( \alpha \), \( D_\alpha \) is the closure in \( \mathcal{X} \) of \( D_\alpha \cdot \tilde{x}_s \) is actually a \( O_s \)-point of \( \bigcap_{\alpha \in M'_s} D_\alpha \). By reduction modulo the maximal ideal of \( O_s \), one gets a well-defined reduction \( x_s \) to the special fibre \( \mathcal{X}_s \).

\[
\mathcal{X}_{F_s} \to \mathcal{X} \to X \\
\text{Spec}(F_s) \to \text{Spec}(O_s) \to C
\]
This very last point $\bar{x}_s$ is actually a point of $\bigcap_{\alpha \in M_s} \mathcal{D}_\alpha$ which does not belong to any of the $\mathcal{D}_\alpha'$ for $\alpha' \in \mathcal{A}' \setminus M'_s$: since all the intersections we are considering are smooth, we use Hensel's lemma here. This contradicts the maximality of $M$, with respect to the hypothesis $\Delta_s(M, \beta) \neq \emptyset$, since we have found a $M'_s$ strictly containing $M$, and such that $\Delta_s(M'_s, \beta)$ is non-empty. Thus $M_s$ is a maximal face of $\text{Cl}_s(X, D)$. In the general case where the fibre above $s$ is not irreducible, then the above argument gives at least one irreducible component $\beta$ for which $\Delta_s(M'_s, \beta) \neq \emptyset$. The conclusion follows. □

**Remark 5.6** If $M_s$ is a maximal face of $\text{Cl}_s(X, D)$, then for every $A \subset \mathcal{A}'$ and $\beta \in \mathcal{B}_{1,s}$ the pair $(A \cup M_s, \beta)$ is sent to $M_s$.

Collecting the terms with respect to every such a maximal face $M_s$, one gets

$$Z_s^{\beta_s}(T, 0) = \sum_{M_s \in \text{Cl}_s^{\text{max}}(X, D)} \sum_{A \subset \mathcal{A}' \atop (A, \beta_s) \rightarrow M_s} L^{\rho_s} [\Delta_s(A, \beta)] L^{-n+|A|} (1 - L^{-1})^{|A|} \times \prod_{\alpha \in A \setminus M_s} \frac{L^{\rho_{\alpha^{-1}} - 1} T_\alpha}{1 - L^{\rho_{\alpha^{-1}} - 1} T_\alpha} \prod_{\alpha \in A \cap M_s} \frac{L^{\rho_{\alpha^{-1}} - 1} T_\alpha}{1 - L^{\rho_{\alpha^{-1}} - 1} T_\alpha} = \sum_{M_s \in \text{Cl}_s^{\text{max}}(X, D)} \frac{P_{M_s}^{\beta_s}(T)}{\prod_{\alpha \in \mathcal{A}' \setminus A} (1 - L^{\rho_{\alpha^{-1}} - 1} T_\alpha)} \prod_{\alpha \in \mathcal{A}' \setminus (A \cup M_s)} \frac{1}{1 - L^{\rho_{\alpha^{-1}} - 1} T_\alpha}.$$

where for all $M_s \in \text{Cl}_s^{\text{max}}(X, D)$

$$P_{M_s}^{\beta_s}(T) = \sum_{A \subset \mathcal{A}' \atop (A, \beta_s) \rightarrow M_s} L^{\rho_s} \left( \prod_{\alpha \in A} L^{\rho_{\alpha^{-1}} - 1} T_\alpha \right) \times [\Delta_s(A, \beta_s)] L^{-n} (L - 1)^{|A|} \prod_{\alpha \in (\mathcal{A}' \setminus A) \cup (M_s \setminus A)} (1 - L^{\rho_{\alpha^{-1}} - 1} T_\alpha).$$

**Remark 5.7** In the expression of $P_{M_s}^{\beta_s}(T)$, the polynomials $(1 - L^{\rho_{\alpha^{-1}} - 1} T_\alpha)$ for $\alpha \in M_s$ give vanishing factors at $T_\alpha = L^{-\rho_{\alpha'}}$. In particular, for any $A \subset \mathcal{A}'$ such that $A \cap \mathcal{A}_D \subset M_s$, we have

$$\prod_{\alpha \in (\mathcal{A}' \setminus A) \cup (M_s \setminus A)} (1 - L^{\rho_{\alpha^{-1}} - 1} T_\alpha)(L^{-\rho_{\alpha'}}) \neq 0$$

if and only if $A \cap \mathcal{A}_D = M_s$, and thus using Remark 5.6

$$P_{M_s}^{\beta_s}(L^{-\rho_{\alpha'}})_{\alpha \in \mathcal{A}'} = \sum_{A \subset \mathcal{A}' \atop \mathcal{A}' \cap \mathcal{A}_D = M_s} L^{\rho_s} L^{-|A \cap \mathcal{A}'|} [\Delta_s(A, \beta_s)] L^{-n} (L - 1)^{|A|} (1 - L^{-1})^{\mathcal{A}' \setminus A}. \times \sum_{A \subset \mathcal{A}' \atop \mathcal{A}' \cap \mathcal{A}_D = M_s} [\Delta_s(M_s \cup A, \beta_s)].$$

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By definition of $\Delta_s(M_s \cup A, \beta)$ one has

$$\sum_{A \subset \mathcal{A}_U} [\Delta_s(M_s \cup A, \beta)] = [E^o_\beta \cap \mathcal{M}_s]$$

where $E^o_\beta$ stands for $E_\beta \setminus \cup_{\beta' \neq \beta} E_{\beta'}$ and $\mathcal{M}_s$ is the intersection $\cap_{\alpha \in M_s} \mathcal{D}_{\alpha,s}$. Finally the value of $P^\beta_{M_s}$ at $(L - \rho^\alpha)_\alpha \in \mathcal{A}$ is

$$P^\beta_{M_s} \left( (L - \rho^\alpha)^{-1}_\alpha \right) = (L - 1)^{|M_s|} (1 - L^{-1})^{|\mathcal{A}_U|} L^{\rho_\beta_{\mathcal{M}_s}} [E^o_\beta \cap \mathcal{M}_s] L^{-n}.$$ 

Now we are able to consider the whole Euler product for $\xi = 0$ and evaluate its coefficients. For any $\beta \in \prod_v \mathcal{B}^\mathcal{W}_v$, which we call a choice of vertical components (recall that this product is actually finite, see Sect. 4.1 for the definition of $\mathcal{B}^\mathcal{W}_v$), let us refine Notation 4.4 by setting for every $v \in C(k)$

$$\mathcal{F}_v^\beta (U, 0) = \prod_{\alpha \in \mathcal{A}_U} (1 - L^{\rho^\alpha_{\mathcal{U}} - 1} U^\alpha) \mathcal{Z}_v^\beta (U, 0)$$

where $\mathcal{Z}_v^\beta (U, 0)$ is the local factor of the refined zeta function $\mathcal{Z}_v^\beta (U, 0)$, see Definition 4.1. This local factor only depends on the indeterminates $U_0$ when $v \in C_0(k)$ and on $(U_0, U_s)$ when $v = s \in S$. Similarly we define $\mathcal{Z}_v^\beta (U_0, U_s)$ starting from $P^\beta_{M_s}(T)$ for every $s \in S$ and we write $\mathcal{C}^{\text{max}}_s(X, D)$ for the product $\prod_{s \in S} \mathcal{C}^{\text{max}}_s(X, D)$. Then

$$\mathcal{Z}_v^\beta (U_0, 0) = \prod_{v \in C} \left( \prod_{\alpha \in \mathcal{A}_U} (1 - L^{\rho^\alpha_{\mathcal{U}} - 1} U^\alpha)^{-1} \mathcal{F}_v^\beta (U_0, 0) \right)$$

$$= \prod_{\alpha \in \mathcal{A}_U} Z_C(L^{\rho^\alpha_{\mathcal{U}} - 1} U^\alpha) \prod_{v \in C} \mathcal{F}_v^\beta (U_0, 0)$$

$$= \left( \prod_{\alpha \in \mathcal{A}_U} Z_C(L^{\rho^\alpha_{\mathcal{U}} - 1} U^\alpha) \prod_{v \in C_0} \mathcal{F}_v^\beta (U_0, 0) \right)$$

$$\times \prod_{s \in S} \left( \sum_{M_s \in \mathcal{C}^{\text{max}}_s(X, D)} \mathcal{Z}_v^\beta (U_0, U_s) \prod_{\alpha \in M_s} \frac{1}{1 - L^{\rho^\alpha_{U} - 1} U^{\alpha,s}} \right)$$

$$= \sum_{M \in \mathcal{C}^{\text{max}}_s(X, D)} \mathcal{Z}_v^\beta (M, 0)$$
where for every $M \in Cl^\text{max}_S (X, D) \subset \mathcal{P} (\mathcal{A}_D) ^S$

$$\mathcal{Z}_M (U, 0) = \left( \prod_{s \in S} \prod_{\alpha \in M_s} \frac{1}{1 - L^{L^{\rho_{\alpha}} - 1} U_{\alpha, s}} \right) \left( \prod_{\alpha \in \mathcal{A}_D} Z_C (L^{L^{\rho_{\alpha}} - 1} U_{\alpha}) \right)$$

$$\times \left( \prod_{v \in C_0} \mathcal{F}_v (U_0, 0) \right) \left( \prod_{s \in S} \mathcal{P}_{\mathcal{M}_s}^\beta (U_0, U_s) \right).$$

This series only depends on the indeterminates indexed by $\mathcal{A}_U$ or $M$. Let us introduce the series

$$\mathcal{G}_M (U, 0) = \prod_{\alpha \in \mathcal{A}_U} Z_C \left( L^{L^{\rho_{\alpha}} - 1} U_{\alpha} \right) \prod_{s \in S} \prod_{\alpha \in M_s} \left( 1 - L^{L^{\rho_{\alpha}} - 1} U_{\alpha, s} \right)$$

for all choice of maximal faces $M \in Cl^\text{max}_S (X, D)$. These notations allow us to rewrite $\mathcal{Z}_M (T, 0)$ as

$$\mathcal{Z}_M (U, 0) = \mathcal{G}_M (U, 0) \prod_{v \in C_0} \mathcal{F}_v (U_0, 0) \prod_{s \in S} \mathcal{P}_{\mathcal{M}_s}^\beta (U_0, U_s).$$

We may decompose $\mathcal{G}_M (U, 0)$ as

$$\mathcal{G}_M (U, 0) = \tilde{\mathcal{G}}_M (U, 0) + \mathcal{H}_M (U, 0)$$

where

$$\tilde{\mathcal{G}}_M (U, 0) = \tilde{G} (U_0, 0) \prod_{s \in S} \prod_{\alpha \in M_s} \left( 1 - L^{L^{\rho_{\alpha}} - 1} U_{\alpha, s} \right)^{-1}$$

$$\mathcal{H}_M (U, 0) = H (U_0, 0) \prod_{s \in S} \prod_{\alpha \in M_s} \left( 1 - L^{L^{\rho_{\alpha}} - 1} U_{\alpha, s} \right)^{-1}$$

with $\tilde{G} (T, 0)$ and $H (T, 0)$ defined in (5.2.10) and (5.2.11) (this definition is licit since $\tilde{G} (T, 0)$ and $H (T, 0)$ only depends on the indeterminates indexed by $\mathcal{A}_U$). This leads to a decomposition

$$\mathcal{Z}_M (U, 0) = \mathcal{Z}_M (U, 0) \mathcal{P}_{\mathcal{M}_s}^\beta (U_0, U_s) \left( \prod_{v \in C_0} \mathcal{F}_v (U_0, 0) \right) \left( \prod_{s \in S} \mathcal{P}_{\mathcal{M}_s}^\beta (U_0, U_s) \right)$$

where

$$\mathcal{Z}_M (U, 0) = \mathcal{Z}_M (U, 0) \mathcal{P}_{\mathcal{M}_s}^\beta (U_0, U_s) \left( \prod_{v \in C_0} \mathcal{F}_v (U_0, 0) \right) \left( \prod_{s \in S} \mathcal{P}_{\mathcal{M}_s}^\beta (U_0, U_s) \right).$$
Now we are ready to adapt the \( U = X \) case to the study of the coefficients of \( \mathcal{X}_M^B(U, 0) \) and \( \mathcal{X}_M^B(U, 0) \) for a fixed \( M \in \text{Cl}_S^{\text{max}}(X, D) \). We are going to show that asymptotically the main contribution in this decomposition of \( \mathcal{X}_M^B(U, 0) \) comes from \( \mathcal{X}_M^B(U, 0) \) while the error term coming from \( \mathcal{X}_M^B(U, 0) - \mathcal{X}_M^B(U, 0) \) is negligible.

An expression of \( \mathcal{X}_M(U, 0) \) is given by

\[
\mathcal{X}_M(U, 0) = \left( \frac{[\text{Pic}^0(C)]L^{1-g}}{L - 1} \right) \sum_{\varepsilon \in \{0, 1\}^{|\mathcal{A}_U|}} (-1)^{|\varepsilon|} \mathcal{X}_{M, \varepsilon}(U, 0)L^{|\varepsilon|(g-1)}
\]

where for all \( \varepsilon \in \{0, 1\}^{|\mathcal{A}_U|} \)

\[
\mathcal{X}_{M, \varepsilon}(U, 0) = \prod_{\alpha \in \mathcal{A}_U} \left( 1 - L^{\rho'_{\alpha} - \varepsilon_{\alpha}U_{\alpha}} \right) \prod_{s \in S} \prod_{\alpha \in M_s} \left( 1 - L^{\rho_{\alpha}U_{\alpha,s}} \right)^{-1}
\]

\[
= \sum_{m_0 \in \mathbb{N}^{|\mathcal{A}_U|}, (m_s) \in \prod_{s \in S} \mathbb{N}^{|M_s|}} L^{(\rho' - \varepsilon, m_0)_{\mathcal{A}_U} + \sum_{s \in S} \rho'_{s}m_s}_{M_s} U_{m_0 + (m_s)_{s \in S}}.
\]

The coefficient of order \( m \in \mathbb{N}^{|\mathcal{A}_U|} \times \mathbb{N}^{|M|} \subset \mathbb{N}^{|\mathcal{A}_U|} \times \mathbb{N}^{|D|} \times S \) of \( \mathcal{X}_M(U, 0) \) is equal to

\[
\sum_{n, n' \in \mathbb{N}^{|\mathcal{A}_U|} \times \mathbb{N}^{|M|}} c^\beta_M(n)L^{(\rho' - \varepsilon, n')_{\mathcal{A}_U} + nM} \tag{5.2.16}
\]

where \( c^\beta_M(n) \) is the \( n \)-th coefficient (with \( n \in \mathbb{N}^{|\mathcal{A}_U|} \times \mathbb{N}^{|M|} \)) of

\[
\prod_{v \in C_0} \mathcal{P}_v^\beta(U_0, 0) \prod_{s \in S} P^\beta_{M_s}(U_0, U_s)
\]

and \( \mathcal{A}_U \cup M \) is the disjoint union of \( \mathcal{A}_U \) and \( M \) in \( \mathcal{A}_U \cup \mathcal{A}_D^S \). After normalising by \( L^{(\rho', m)_{\mathcal{A}_U} + nM} \), the coefficient (5.2.16) for \( \varepsilon = 0 \) is the \( m' \)-th partial sum of the motivic Euler product

\[
\left( \prod_{v \in C_0} \mathcal{P}_v^\beta(U_0, 0) \right) \left( (L^{-\rho'_{\alpha}})_{\alpha \in \mathcal{A}_U} \times \prod_{s \in S} \left( P^\beta_{M_s}(U_0, U_s)((L^{-\rho'_{\alpha}})_{\alpha \in \mathcal{A}_U}, (L^{-\rho_{\alpha}})_{\alpha \in \mathcal{A}_D}) \right) \right)
\]
to which one can apply Proposition 4.5. If \( \varepsilon \neq 0 \), the only difference is again the factor \( L^{(\varepsilon, m - m')} \), which is controlled with Lemma 3.5 as in the case \( U = X \). Since by Lemma 3.4 we know that weight-linear convergence is stable by finite product, we deduce the following proposition.
Proposition 5.8 For all choice of maximal faces $M \in \text{Cl}^{\text{max}}_S(X, D)$ and vertical components $\beta \in \prod_v \mathcal{B}_v^\beta$, there exists a decomposition
\[
\tilde{\varphi}_M^\beta(U, 0) = \sum_{m \in \mathbb{N}^{\varrho(U)} \times N^M} \tilde{a}_m^\beta U^m = \sum_{\varepsilon \in \{0, 1\}^{\varrho(U)}} \sum_{m \in \mathbb{N}^{\varrho(U)} \times N^M} \tilde{a}_{m, \varepsilon}^\beta U^m
\]
as well as a real number $\delta > 0$ such that in $\tilde{\mathcal{M}}_k$
\[
\tilde{a}_{m, 0}^\beta L^{-(\rho', m)\cdot \varepsilon \cdot U \cdot M}
\]
converges $\rho'$-weight-linearly to the non-zero effective element
\[
\left(\frac{[\text{Pic}^0(C)] L^{1-g}}{L - 1}\right)^{\text{rg}(U)} \prod_{\nu \in C_1} \left(1 - L^{-1}\right)^{\text{rg}(U)} \left(\frac{[\mathcal{M}_\nu]}{L^n}\right)
\]
\times \prod_{\nu \in C_0 \setminus C_1} \left(1 - L^{-1}\right)^{\text{rg}(U)} \left(\frac{[E_{\beta_\nu}^\rho \cap \mathcal{M}_\nu]}{L^n}\right)
\]
\times \prod_{s \in S} \left(1 - L^{-1}\right)^{|M_s| + \text{rg}(U)} \left(\frac{[E_{\beta_s}^\rho \cap \mathcal{M}_s]}{L^n}\right)
\]
and for every $\varepsilon \in \{0, 1\}^{\varrho(U) \setminus \{0\}}$
\[
w \left(\tilde{a}_{m, \varepsilon}^\beta L^{-(\rho', m)\cdot \varepsilon \cdot U \cdot M}\right) < -\delta(\varepsilon, m)_{\varrho(U)}
\]
when $\min(m_\alpha) \to \infty$.

Now we can adapt the case $\mathcal{U} = \mathcal{B}$ to control the contribution of $\mathcal{H}_M(U, 0)$. One immediately gets a decomposition
\[
\mathcal{H}_M(U, 0) = \sum_{\emptyset \neq A \subset \varrho(U), \varepsilon \in \{0, 1\}^{\varrho(U) \setminus \{0\}} \times \{1\}^A} [\text{Pic}^0(C)]^{[\varrho(U) \setminus A]} \left(-L^{1-g}\right)^{[\varrho(U) \setminus \varepsilon]} \left(1 - L\right)^{[\varrho(U) \setminus A]} \mathcal{H}_M^{A, \varepsilon}(U, \xi)
\]
where for every $A \subset \varrho(U)$ non-empty and $\varepsilon \in \{0, 1\}^{\varrho(U) \setminus \{0\}} \times \{1\}^A$
\[
\mathcal{H}_M^{A, \varepsilon}(U, 0) = \sum_{m \in \mathbb{N}^{\varrho(U) \setminus A} \times N^M} L^{(\rho' - \varepsilon, m)\cdot \varepsilon \cdot U \cdot M}
\]
\times \prod_{\alpha \in A} \left([\text{Sym}^{m_\alpha} C] - [\text{Pic}^0(C)] \frac{L^{m_\alpha - g - 1} - 1}{L - 1}\right) U^m.
\]
Adapting the argument from the $\mathcal{U} = \mathcal{B}$ case, one deduces the following proposition.
Proposition 5.9 (U \subset X, C_0 \subset C, \xi = 0). For all choice of maximal faces \(M \in \text{Cl}^{\text{max}}_S(X, D)\), there is a decomposition

\[
\mathcal{H}_M(U, 0) \left( \prod_{v \in C_0} \mathcal{P}_v^\beta(U_0, 0) \right) \left( \prod_{s \in S} \mathcal{P}_{M_s}^\beta(U_0, U_s) \right) = \sum_{\emptyset \neq A \subset \mathcal{A}_U} \sum_{\varepsilon \in \{0, 1\}^{A \setminus \emptyset \times \{1\}^A}} \left( \prod_{s \in S} \mathcal{P}_v^\beta(U_0, U_s) \right)
\]

and a real number \(\delta > 0\) such that for all non-empty subset \(A \subset \mathcal{A}_U\) and \(\varepsilon \in \{0, 1\}^{A \setminus \emptyset \times \{1\}^A}\)

\[
w \left( \left( \prod_{M_0}^\beta A, s \right)_{\mathcal{A}_U} \right) < -\delta \left( \langle \varepsilon, m \rangle_{\mathcal{A}_U} + \langle \rho', m \rangle_{\mathcal{A}_U \cup M} \right)
\]

when \(\min(m_\alpha) \to \infty\).

5.2.4 S-integral points and non-trivial characters

Concerning places \(s \in S\), recall that the local factor of \(Z^\beta(T, \xi)\) is given by the formula

\[
Z^\beta_s(T, \xi) = \sum_{A \subset \mathcal{A}} L^\beta \int_{\Omega_s(A, \beta)} \prod_{\alpha \in A} (L^\beta_{\alpha, T_\alpha})^{d_A(x, \xi)} e(\langle x, \xi \rangle) dx.
\]

Let \(\text{Cl}_S(X, D)_0\) be the subcomplex of \(\text{Cl}_S(X, D)\) where we only keep vertices \(\alpha \in \mathcal{A}\) such that \(d_\alpha(\xi) = 0\). Then Proposition 5.3.1 of Chambert-Loir and Loeser (2016) can be rewritten as follows.

Proposition 5.10 Assume first that \(f_\xi\) extends to a regular map \(\mathcal{X}_{F_0} \to \mathbb{P}^1_{\mathbb{F}_v}\). Then there is a family \((P^\beta_{M_0}, M_0)_{M_0 \in \text{Cl}^{\text{max}}_S(X, D)_0}\) of polynomials with coefficients in \(\exp M_k\) such that

\[
Z^\beta_s(T, \xi_s) = \sum_{M_0 \in \text{Cl}^{\text{max}}_S(X, D)_0} \frac{P^\beta_{M_0}_s(T, \xi_s)}{\prod_{\alpha \in \mathcal{A}_0(\xi) \cup M_0}(1 - L^\rho_{\alpha, T_\alpha})}.
\]

If \(f_\xi\) does not extend to a regular map \(\mathcal{X}_{F_0} \to \mathbb{P}^1_{\mathbb{F}_v}\), then the argument of Chambert-Loir and Loeser in Chambert-Loir and Loeser (2016) consists in using a resolution of indeterminacies, that is a proper birational morphism \(\varphi : \mathcal{Y}_{\mathcal{O}_s} \to \mathcal{X}_{\mathcal{O}_s}\) such that \(\varphi^* f\) extends to a regular map from \(\mathcal{Y}_{\mathcal{O}_s}\) to \(\mathbb{P}^1_{\mathbb{F}_v}\). Moreover one can assume that the generic fibre \(\mathcal{Y}_{F_0} \to \mathcal{X}_{F_0}\) is invariant under the action of \(G_{F_0}\), and this can be done locally uniformly with respect to \(\xi\) (Chambert-Loir and Loeser 2016, Lemma 6.5.1). Let \(\mathcal{D}_{\alpha, s}\) be the strict transform of \(\mathcal{D}_{\alpha, s}\) through \(\varphi\) and \((\mathcal{E}_\gamma)_{\gamma \in \Gamma}\) the set of horizontal
exceptional divisors of $\sigma$. There exists a family $(m_{\gamma, \alpha})$ of nonnegative integers such that

$$
\pi^* (D_{\alpha,s})_{F_s} = (D'_{\alpha,s})_{F_s} + \sum_{\gamma \in \Gamma} m_{\gamma, \alpha} (E_{\gamma})_{F_s}
$$
as well as positive integers $\nu_{\gamma}$ such that

$$
K_{\mathcal{Y}_{F_s}/X_{F_s}} = \sum_{\gamma \in \Gamma} (\nu_{\gamma} - 1) (E_{\gamma})_{F_s}
$$

and $v$ a bounded constructible function on $\mathcal{L}(\mathcal{Y}_s)$ such that

$$
\text{ord}_{K_{\mathcal{Y}_{F_s}/X_{F_s}}} = \sum_{\gamma \in \Gamma} (\nu_{\gamma} - 1) \text{ord}_{E_{\gamma}} + v.
$$

Chambert-Loir and Loeser then use the change of variable formula (Cluckers and Loeser 2010, Theorem 13.2.2) and obtain an integral over $\mathcal{L}(\mathcal{Y}_s)$, which in our case gives

$$
Z^s_\xi (T, \xi_s) = \sum_{A \subset A} L^{p_A} \times \sum_{p \in Z_{\mathcal{Y}}} \left( L^{(\rho', p)} T^p \int_{W_p \cap \sigma^* \Omega_s (A, \beta)} L^{(\rho', \text{ord}_{\mathcal{Y}} (y))} T^{\text{ord}_{\mathcal{Y}} (y)} \times \prod_{\gamma \in \Gamma} \left( L^{v_{\gamma} - 1} L^{(\rho', m_{\gamma})} T^{m_{\gamma}} \right)^{\text{ord}_{E_{\gamma}} (y)} L^{v(y)} e^v (\sigma^* f_{\xi_s} (y)) dy \right)
$$

where $(W_p)$ is a finite partition of $\mathcal{L}(\mathcal{Y})$.

Then one applies Proposition 5.10 to $\sigma^* f_{\xi_s}$ and to the integral over $W_p \cap \sigma^* \Omega_s (A, \beta)$ to get a similar result: $Z_\xi (T, \xi_s)$ is rational with denominators given by products of polynomials of the form $(1 - L^{(\rho', a) T^a})$ for some $a \in N_{\mathcal{Y}}$. One has to justify that this procedure does not change the set $A$ and the relevant faces of Clemens complex. The argument is given in the very last paragraph of Chambert-Loir and Loeser (2016) and in (Chambert-Loir and Tschinkel 2012, §3.4): the $F_s$-Clemens complex $\text{Cl}_{F_s} (Y, Y \setminus \cup_{\alpha \in A} (D'_{\alpha,s}, F_s)$ has vertices coming from $X_s$, corresponding to the strict transforms $D'_\alpha$ for $\alpha \in A$, and vertices corresponding to the divisors $E_{\gamma}$ contracted by $\sigma$. The divisor of $\sigma^* f_{\xi_s}$ on $\mathcal{Y}_s$ takes the form

$$
\Sigma'_\xi = \sum_{\alpha \in A} d_\alpha (\xi) (D'_{\alpha,s})_{F_s} - \sum_{\gamma \in \Gamma} e_{\gamma} (E_{\gamma})_{F_s}
$$

where $\Sigma'_\xi$ is the strict transform of the hyperplane $\langle \xi, x \rangle = 0$. Since $Y_{F_s}$ is an equivariant compactification of $G_{F_s}$, the integers $e_{\gamma}$ are all non-negative. Furthermore
\( K_{\mathcal{F}_s} / \mathcal{F}_s \) is a linear combination of the \( (\delta^\gamma)_{\mathcal{F}_s} \) with non-negative coefficients. Consequently, looking back on the formula giving \( Z^\beta_s (T, \xi_s) \) above, one remarks that only the vertices coming from \( X_{\mathcal{F}_s} \) will contribute to the poles. Replacing \( \xi = 0 \) by \( \xi \neq 0, \mathcal{U} \) by \( \mathcal{U}^0(\xi) \) and \( \text{Cl}_s(X, D) \) by \( \text{Cl}_s(X, D)_0 \) everywhere in the previous paragraph, including in the definitions of \( \tilde{\mathcal{M}}^\beta_s, \tilde{\mathcal{H}}_s \) and \( \tilde{\mathcal{H}}_s \), one gets the following proposition. Since \( \text{Cl}_{s_{\max}}(X, D)_0 \) is a proper subset of \( \text{Cl}_{s_{\max}}(X, D) \) for every \( s \in S \) (Chambert-Loir and Tschinkel 2012, Lemma 3.5.4), this proposition implies that non-trivial characters do not contribute asymptotically, included in the case \( \mathcal{U} \neq \mathcal{X} \).

**Proposition 5.11** Let \( \xi \) be a non-trivial character. For all \( M^0_0 \in \text{Cl}_{s_{\max}}(X, D)_0 \subset \mathcal{P}(\mathcal{A}_D)^S \), there exist decompositions

\[
\tilde{\mathcal{M}}^\beta_{M^0}(U, \xi) = \sum_{m \in \mathbb{N}^A_U \times \mathbb{N}^{M^0_0}} \left( \tilde{a}^\beta_{M^0, \xi} \right)_m U^m \quad \text{and} \quad \tilde{\mathcal{H}}^\beta_{M^0}(U, \xi) = \sum_{\emptyset \neq A \subset \mathcal{A}^0_0(\xi)} \sum_{m \in \mathbb{N}^A_U \times \mathbb{N}^{M^0_0}} \left( \tilde{b}^\beta_{M^0, A, \xi} \right)_m U^m
\]

as well as a real number \( \delta_\xi > 0 \) such that for all \( \epsilon \in \{0, 1\}^{\mathcal{A}^0_0(\xi)} \)

\[
w\left( \left( \tilde{a}^\beta_{M^0_0, \xi, \epsilon} \right)_m U^{-\langle \rho', m \rangle_{\mathcal{A}_U \cup M^0}} \right) < -\delta_\xi \left( \langle \epsilon, m \rangle_{\mathcal{A}^0_0(\xi)} + \langle \rho', m \rangle \right)
\]

and for all \( \emptyset \neq A \subset \mathcal{A}_U \) and \( \epsilon \in \{0, 1\}^{\mathcal{A}^0_0(\xi) \setminus A} \times \{1\}^A \)

\[
w\left( \left( \tilde{b}^\beta_{M^0_0, A, \xi, \epsilon} \right)_m U^{-\langle \rho', m \rangle_{\mathcal{A}_U \cup M^0}} \right) < -\delta \left( \langle \epsilon, m \rangle_{\mathcal{A}^0_0(\xi)} + \langle \rho', m \rangle \right)
\]

when \( \min(m_\alpha) \to \infty \).

### 5.3 Summation over all the characters and convergence

In this subsection we perform the final step of our proof: we have to permute a motivic summation with taking a limit.

#### 5.3.1 Expression of \( [\mathcal{M}^{\beta}_{n, m_s}] \) in \( \mathcal{M}_k \)

The \( m \)-th coefficient of the Zeta function \( \mathcal{Z}^{\beta}(T) \) is by definition the class \( [\mathcal{M}^{\beta}_{n, m_s}] \) of the moduli space of section we are interested in. Recall that we have a decomposition

\[
\mathcal{Z}^{\beta}(U) = L^{n(1-g)} \left( \mathcal{Z}^{\beta}(U, 0) + \sum_{\xi \in V \setminus \{0\}} \mathcal{Z}^{\beta}(U, \xi) \right).
\]
where \( V = L(\tilde{E})^n \) is the \( n \)-th power of a finite dimensional Riemann-Roch space. The class of \( M_n \) in \( \mathcal{M}_k \) is given by

\[
\left[ M_{n,m_S}^\beta \right] = L^{(1-g)n} \sum_{\xi \in V} \sum_{\mu \in S^n U(C)} \theta_{n,\beta,}^* \mathcal{F}(1_{H(m+m_S,\beta)})(\xi)
\]

where the two sums here are motivic sums (given by the projections to \( S^n U(C) \) and then to \( k \), \( \tilde{E} \) is the divisor defined by equation (4.2.8) and

\[
\theta_{n,\beta} : V \times S^n U(C) \longrightarrow \mathcal{O}^{\beta}(v-s, v-s', N, 0)
\]

is the morphism introduced in Sect. 2.4.5. By definition,

\[
\sum_{\mu \in S^n U(C)} \theta_{n,\beta,}^* \mathcal{F}(1_{H(m+m_S,\beta)})(\xi) \in \mathcal{E}_{V \times M_V}
\]

is the coefficient of degree \((n_U, m_S)\) of

\[
\prod_{v \in C} \mathcal{O}_v^\beta(U, \xi).
\]

### 5.3.2 Uniform convergence

By Remark 4.9, we already know the existence of a finite constructible partition of \( V \setminus \{0\} \) over which the weight-linear convergence of \( \prod_{v \in C} F_v(T, \xi) \) with respect to \( \rho \) is uniform in \( \xi \). Concerning places in \( S \), Proposition 5.10 and (Chambert-Loir and Loeser 2016, Lemma 6.5.1) allows us to resolve indeterminacies of \( f_\xi \) uniformly on a partition (which is actually finite for the same reason). Taking a partition refining both previous partitions, we get a finite partition \( P \) of \( V \setminus \{0\} \) over which \( \xi \mapsto \mathcal{O}_U^0(\xi) \) and \( \xi \mapsto \mathcal{O}_U^1(\xi) \) are constant, as well as the corresponding degrees for \( \alpha \in \mathcal{D} \), which means that \( \rho' \)-weight-linear convergence is uniform over such a partition. This partition does not depend on a choice of \( \beta \). This provides a decomposition

\[
\left[ M_{n,m_S}^\beta \right] = L^{(1-g)n} \sum_{m \in S^n U(C)} \theta_{n,\beta,}^* \mathcal{F}(1_{H(m+m_S,\beta)})(0)
\]

\[
+ L^{(1-g)n} \sum_{P \text{ stratum}} \sum_{\xi \in P} \sum_{\mu \in S^n U(C)} \theta_{n,\beta,}^* \mathcal{F}(1_{H(m+m_S,\beta)})(\xi).
\]

All the results of §4.2 hold uniformly on each stratum \( P \), which means that we can pass from the decomposition of \( \prod_{v \in C} \mathcal{O}_v^\beta(U, \xi) \) given by Proposition 5.11 to a decomposition of

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\[
\sum_{\xi \in V} \prod_{v \in C} \mathcal{Z}_v^\beta(U, \xi)
\]
from which one deduces a decomposition of its coefficient
\[
\sum_{\xi \in P} \sum_{m \in S_\beta^U(C)} \theta_{n_{\beta}}(\mathcal{F}(1_H(m+m_\beta, \beta))(\xi))
\]
for every \((n_{\beta}, m_\beta) \in \mathbb{N}_{\beta}^D \times \mathbb{N}_{\beta}^D \times S\). Summing over the finite set of stratum \(P\) and adding the term coming from the trivial character, we get a decomposition of \([M_{n_{\beta}, m_\beta}]\).

### 5.3.3 Final result

We combine the uniform convergence argument above and the results of Propositions 5.8, 5.9 and 5.11, fixing a choice of vertical components \(\beta \in \prod_v \mathcal{B}_v^U\). Recall that the limit we obtained does not depend on the choice of the maps \((A, \beta) \mapsto M_s\) for every \(s \in S\).

**Proposition 5.12** Let \(M \in \text{Cl}_{S}^{\text{max}}(X, D)\) be any choice of maximal faces over the places \(s \in S\) and \(\beta \in \prod_v \mathcal{B}_v^U\) a choice of vertical components. When \(\min(m_{\alpha, s})\) becomes arbitrary large, the coefficients of
\[
\mathcal{L}_{M}^\beta(U) = L^{(1-g)n} \sum_{\xi \in V} \mathcal{L}_{M}^\beta(U, \xi),
\]
normalised by \(L^{(\rho_{\cdot, m})}\), converges in \(\widehat{\mathcal{M}}_k\) to a non-zero effective element of \(\widehat{\mathcal{M}}_k\) which can be written as the following motivic Euler product:
\[
L^{(1-g)n} \left( \frac{[\text{Pic}^0(C)]L^{1-g}}{L-1} \right)^{\text{rg}(\text{Pic}(U))} \times \prod_{v \in C_1} \left( 1 - L^{1-g} \frac{[\mathcal{Z}_v]}{L^n} \right) \times \prod_{v \in C_0 \setminus C_1} \left( 1 - L^{-1} \frac{\text{rg}(\text{Pic}(U))L^{\rho_{\beta_v}}[E_{\beta_v}]}{L^n} \right) \times \prod_{s \in S} \left( 1 - L^{-1} \frac{\text{rg}(\text{Pic}(U)) + |M_s|L^{\rho_{\beta_s}}[E_{\beta_s} \cap \mathcal{D}_{M_s}]}{L^n} \right).
\]

Furthermore, there exists a finite partition \((P)\) of \(V \setminus \{0\}\) such that the normalised coefficients of \(\mathcal{L}_{M}^\beta(U, \xi)\) tends to zero weight-linear-uniformly with respect to \(\rho\) on each stratum \(P\).
Remark 5.13 Over places $v \in C_0$, as we pointed out in Remark 4.6, the local term $L^{\rho \frac{v}{v} - \langle \rho, e^{\rho \frac{v}{v}} \rangle} \frac{[E^0_{\rho, v}]}{\Gamma_{\rho, v}}$ can be interpreted as the motivic integral

$$\int_{G(F_v, \beta_v)} L^{-(g, L_{\rho, v})}[\omega_X] = \int \mathcal{L}(\mathcal{F}_v, E^0_{\rho, v}) L^{-\operatorname{ord}_{\mathcal{F}_v}(x)} L^{-\operatorname{ord}_X(x)}.$$

This remark can be adapted for places $s \in S$ if one replaces $E^0_{\rho, v}$ by $E^0_{\rho, s} \cap \mathcal{D}_s$ and add a controlling term. Let $\epsilon$ be a non-negative integer, $\epsilon_\alpha = \epsilon$ if $\alpha \in M_s$ and zero otherwise, and

$$\mathcal{L}_{\rho, s} + \mathcal{L}_\rho = \mathcal{L}_{\rho, s} + \sum_{\alpha \in \mathcal{I}_s} \epsilon_\alpha \mathcal{L}_\alpha.$$

Then one considers for $\epsilon \geq 1$

$$\int \mathcal{L}(\mathcal{F}_s, E^0_{\rho, s} \cap \mathcal{D}_s) L^{-\operatorname{ord}_{\mathcal{F}_s}(x)} L^{-\operatorname{ord}_X(x)}$$

$$= \int \mathcal{L}(\mathcal{F}_s, E^0_{\rho, s} \cap \mathcal{D}_s) L^{-\sum_{\alpha \in \mathcal{I}_s \cup \mathcal{M}_s} (\rho_\alpha + \epsilon_\alpha)(\operatorname{ord}_{\mathcal{F}_s}(x) + \epsilon_\alpha)} L^{\rho \beta_s + \sum_{\alpha \in \mathcal{I}_s \cup \mathcal{M}_s} \rho_\alpha \operatorname{ord}_{\mathcal{F}_s}(x)}$$

$$= L^{\rho \beta_s - \langle \rho, e^{\rho \beta_s} \rangle} \int \mathcal{L}(\mathcal{F}_s, E^0_{\rho, s} \cap \mathcal{D}_s) L^{\sum_{\alpha \in \mathcal{M}_s} (1 - \epsilon_\alpha) \operatorname{ord}_{\mathcal{F}_s}(x)}$$

$$= L^{\rho \beta_s - \langle \rho, e^{\rho \beta_s} \rangle} \sum_{A \subset \mathcal{I}_s} \int \mathcal{L}_{\mathcal{O}_{s}(\mathcal{M}_s \cup \mathcal{A}, \beta)} L^{\sum_{\alpha \in \mathcal{M}_s} (1 - \epsilon_\alpha) \operatorname{ord}_{\mathcal{F}_s}(x)}.$$

This last family of integrals can be computing using the isomorphism

$$\Theta : \Delta_s(A, \beta) \times \mathcal{L}(A^1, 0)^A \times \mathcal{L}(A^1, 0)^{n-|A|} \rightarrow \Omega_{s}(A, \beta)$$

introduced at the beginning of Sect. 4, together with the motivic volumes computed in Sect. 2.5.2:

$$\int_{\mathcal{O}_{s}(A \cup \mathcal{M}_s, \beta)} L^{\sum_{\alpha \in \mathcal{M}_s} (1 - \epsilon_\alpha) \operatorname{ord}_{\mathcal{F}_s}(x)}$$

$$= \sum_{\mathcal{M}_s \times \mathcal{O}_{s}(A \cup \mathcal{M}_s, \beta)} \int_{\mathcal{O}_{s}(A \cup \mathcal{M}_s, \beta) \times \mathcal{L}(A^1, 0)^{A \cup \mathcal{M}_s} \times \mathcal{L}(A^1, 0)^{n-|A|-|\mathcal{M}_s|}} L^{(1 - \epsilon)|\mathcal{M}_s|}$$

$$= \sum_{\mathcal{M}_s \times \mathcal{O}_{s}(A \cup \mathcal{M}_s, \beta)} [\Delta_s(A \cup \mathcal{M}_s, \beta)] L^{(1 - \epsilon)|\mathcal{M}_s|} (1 - L^{-1}) |A \cup \mathcal{M}_s||L^{(n - n + |A| + |\mathcal{M}_s|)}$$

$$= \left[\Delta_s(A \cup \mathcal{M}_s, \beta)\right] \left(\frac{L^{-\epsilon}}{1 - L^{-\epsilon}}\right)^{|\mathcal{M}_s|} \left(\frac{L^{-1}}{1 - L^{-1}}\right)^{|A|} (1 - L^{-1}) |A \cup \mathcal{M}_s||L^{n - n + |A| + |\mathcal{M}_s|}$$

$$= \left(\frac{L^{-\epsilon}}{1 - L^{-\epsilon}}\right)^{|\mathcal{M}_s|} (1 - L^{-1}) |\Delta_s(A \cup \mathcal{M}_s, \beta)||L^{(n - n + |A| + |\mathcal{M}_s|)}.$$
This last equality shows that it makes sense to consider the value at $\epsilon = 0$ of
\[
(1 - L^{-\epsilon})^{[M_s]} \int \mathcal{L}(\mathcal{X}, E_{\beta_s}^0 \cap \mathcal{D}_{M_s}) L^{-\ord_{\rho' + \epsilon}(x)} L^{-\ord_{\omega_{\epsilon}}(x)}.
\]

Since $[E_{\beta_s}^0 \cap \mathcal{D}_{M_s}] = \sum \Delta_s (A \cup M_s, \beta)$, this residue is precisely the local term
\[
L^{\rho_s - (\rho', \epsilon^b)} (1 - L^{-1})^{[M_s]} [E_{\beta_s}^0 \cap \mathcal{D}_{M_s}] L^{|M_s| - n}.
\]

**Remark 5.14** A geometric interpretation of Proposition 5.12 is as follows.

For any $M \in \text{Cl}^\text{max}_S (X, D)$, let $M_{n, m_s}^{\beta, M} \subset M_{n, m_s}^{\beta}$ be the constructible moduli subspace of sections $\sigma \in M_{n, m_s}^{\beta}$ such that
\[
\forall s \in S \quad \forall \alpha \in M_s \quad (\sigma(\eta_C), \mathcal{D}_\alpha)_s > 0.
\]

Remark that by definition of the Clemens complexes, $M_{n, m_s}^{\beta, M}$ is non-empty if and only if the support of $m_s = ((\sigma(\eta_C), \mathcal{D}_\alpha)_s)_{s \in \mathcal{D}_P} \in \mathcal{N}^{\mathcal{D}_P}$ is exactly $M_s$ for all $s \in S$. The maximal number of $\mathcal{D}_\alpha$ a section can intersect above each point of $S$ is encoded in $\text{Cl}^\text{max}_S (X, D)$. This also means that when the multidegree of a section tends to infinity, there is an unique way to associate a maximal face to the section and $M_{n, m_s}^{\beta, M}$ is actually the right moduli space to consider. We showed that $[M_{n, m_s}^{\beta, M}]L^{-(n_U + m_s, \rho')}$ converges to the constant of Proposition 5.12. Since $[M_{n, m_s}^{\beta, M}]$ coincides with the coefficient of sufficiently high degree of the refined height zeta function $Z_M(U)$, this is the following statement and main result.

**Theorem 5.15** When the multidegree $(n_U^\beta, m_s) \in \mathcal{N}^{\mathcal{D}_P} \times \mathcal{N}^M$ goes arbitrary far from the boundaries of the corresponding cone, the normalised class
\[
[M_{n, m_s}^{\beta, M}]L^{-(\rho', n)}
\]
converges, in the completion of $\mathcal{M}_k$ for the weight topology, to the motivic Euler product
\[
L^{(1-g)n} \left( \frac{[\text{Pic}^0(C)]L^{1-g}}{L - 1} \right)^{rg(\text{Pic}(U))} \times \prod_{v \in C_1} \left( (1 - L^{-1})^{rg(\text{Pic}(U))} \frac{[\mathcal{D}_v]}{L^n} \right) \times \prod_{v \in C_0 \setminus C_1} \left( (1 - L^{-1})^{rg(\text{Pic}(U))} L^{\rho_v - (\rho', e_{\beta_v})} \frac{[E_{\beta_v}^0]}{L^n} \right) \times \prod_{s \in S} \left( (1 - L^{-1})^{rg(\text{Pic}(U)) + |M_s|} L^{\rho_s - (\rho', e_{\beta_s})} \frac{[E_{\beta_s}^0 \cap \mathcal{D}_{M_s}]L^{|M_s|}}{L^n} \right).
\]
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