Set-invariance-based interpretations for the $L_1$ performance of nonlinear systems with non-unique solutions

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Abstract
This article is concerned with tackling the $L_1$ performance analysis problem of continuous and piecewise continuous nonlinear systems with non-unique solutions by using the involved arguments of set-invariance principles. More precisely, this article derives a sufficient condition for the $L_1$ performance of continuous nonlinear systems in terms of the invariant set. With respect to the case such that solving a nonlinear differential equation is difficult and thus an employment of the invariant set-based sufficient condition is a non-trivial task, we also derive another sufficient condition through the extended invariance domain approach. Because this extended approach characterizes set-invariance properties in terms of the corresponding vector field and an extended version of contingent cones, the $L_1$ performance analysis problem could be solved without considering both the explicit solutions for the differential equation and the relevant solution uniqueness. These arguments associated with the $L_1$ performance of continuous systems are further extended to the involved case of piecewise continuous nonlinear systems, and we establish parallel results relevant to the set-invariance principles obtained for the continuous nonlinear systems. Finally, numerical examples are provided to demonstrate the effectiveness as well as the applicability of the overall results derived in this article.

KEYWORDS
external contingent cone, invariance domain, invariant set, $L_1$ performance, piecewise continuous nonlinear system, set-invariance

1 | INTRODUCTION

Based on the fact that the $L_\infty$-induced norm of linear time-invariant (LTI) systems for the single-input/single-output (SISO) case coincides with the $L_1$ norm of the impulse response of the systems, the problem of dealing with this induced norm has been called the $L_1$ problem.1-3 Here, it is quite difficult to compute the $L_\infty$-induced norm explicitly even for LTI systems, and thus several approximation methods have been developed in References 4-7 for LTI continuous-time and sampled-data systems, respectively, while this induced norm can be exactly obtained for the specific case of linear positive systems without any approximation as discussed in References 8 and 9. To put it another way, an upper bound and a lower bound on the $L_\infty$-induced norms of general LTI continuous-time and sampled-data systems can be obtained in References 4-7 within any degree of accuracy.
On the other hand, the $L_1$ analysis problem might be extended for the case of nonlinear systems, but it cannot be formulated in an essentially equivalent fashion to LTI systems. This is because it is intrinsically impossible to define an analytic representation of the $L_\infty$-induced norm for general nonlinear systems. In connection with this, the $L_1$ performance of nonlinear systems is introduced in Reference 10 as an alternative for the $L_\infty$-induced norm. More precisely, a nonlinear system is regarded as satisfying the $L_1$ performance if the $L_\infty$ norm of the output is not larger than 1 for all input bounded by 1 with respect to the $L_\infty$ norm. In connection with this, the idea of set invariance principles used in Reference 11 for discrete-time systems is extended to the $L_1$ analysis problem of continuous-time nonlinear systems.

To put it another way, the set invariance principles (or equivalently, forward invariance) of dynamical systems imply the property that makes any state variable (or solution) $x(t)$ ($t \geq 0$) with $x(0) \in K$ always remain in $K$ for all $t \geq 0$. If this property holds for at least one solution $x(t)$ ($t \geq 0$) corresponding to an arbitrary initial condition $x(0) \in K$, then we call it the weak invariance or the viability property. In contrast, if this property holds for every solution $x(t)$ ($t \geq 0$) with respect to an initial condition $x(0) \in K$, we call it the strong invariance. For discrete-time systems, these two invariance properties coincide with each other because the solution for the corresponding difference equation is uniquely determined through a sort of algebraic computations. Motivated by this advantage, there have been a number of studies on the invariance property of discrete time systems. On the other hand, in the case of continuous-time systems, the weak and strong invariances do not coincide with each other as discussed in Reference 12 since nonlinear differential equations do not have unique solutions in general. In connection with this, aiming at employing some useful properties in the strong invariance, the arguments in the previous studies are confined themselves to (non)linear differential equations with unique solutions. The strong invariance has also shed a new light on the study of estimating reachable sets for linear delay systems and linear positive systems. In a similar fashion with respect to the employment of the strong invariance, the necessary and sufficient condition for the $L_1$ performance of continuous-time nonlinear systems in the aforementioned study is also established by assuming the solution uniqueness of the corresponding differential equations.

However, if we are confined ourselves to nonlinear systems with unique solutions and take the strong invariance for solving the $L_1$ problem of nonlinear systems, there could exist some significant problems as follows. First of all, it is intrinsically impossible to derive solutions for the $L_1$ performance analysis of the general case with non-unique solutions such as discontinuous dynamical systems and differential inclusion systems. Regarding the $L_1$ controller synthesis as an extended issue, it is also unclear how we can derive an adequate controller even for continuous nonlinear systems if the solution uniqueness is still assumed. More precisely, the synthesis procedure of the $L_1$ controller for continuous nonlinear systems is considered only in Reference 10 for the best knowledge of the authors, and the closed-loop systems obtained by connecting a continuous nonlinear plant with the $L_1$ controller are shown in that study to satisfy the $L_1$ performance if they have unique solutions. However, it could not be guaranteed in Reference 10 that the obtained $L_1$ controllers always lead to the solution uniqueness with respect to the corresponding closed-loop systems.

To resolve these issues, this article aims at establishing sophisticated arguments on the $L_1$ performance analysis problem of nonlinear systems, regardless of the solution uniqueness of the systems. As a preliminary step to tackle the $L_1$ performance analysis problem for continuous nonlinear systems, we first derive a sufficient condition in terms of invariant sets. To consider the case such that employing the invariant set-based sufficient condition is a non-trivial task due to difficulties in obtaining the corresponding solutions, we next establish another sufficient condition applicable to wider class of nonlinear systems. More precisely, similarly to the conventional invariance domain-based arguments, in which the relevant set-invariance properties are characterized in terms of vector fields and contingent cones, we provide the so-called extended invariance domain arguments through the external contingent cone approach. This external contingent cone can be interpreted as a generalized version of the contingent cone, and the rationale behind taking the former (i.e., the external contingent cone) not the latter (i.e., the contingent cone) is for wider applicability of the arguments with respect to the solution uniqueness; the arguments in References 10 and 12 are confined themselves to the case of nonlinear differential equations with unique solutions. To summarize, in contrast to the invariant set-based arguments and the conventional invariance domain-based arguments, the extended invariance domain is equipped with the external contingent cone and allows us to characterize the $L_1$ performance of nonlinear systems with non-unique solutions in terms of the vector fields.

Stimulated by the success in the $L_1$ performance analysis of continuous nonlinear systems with non-unique solutions, we are also concerned with a more involved problem of the $L_1$ analysis for piecewise continuous nonlinear systems. Note that the solutions of such systems should be intrinsically described by Filippov’s differential equations and the solution uniqueness cannot be assumed in general. Similarly to the case of continuous nonlinear systems,
sufficient conditions for the $L_1$ performance of piecewise continuous nonlinear systems are derived in terms of invariant sets and extended invariance domains, through a sophisticated analysis of the arguments on differential inclusions in Reference 28.

The contributions of the results derived in this article beyond the existing studies associated with the set invariance principles might be briefly summarized as follows. Even though the classical invariance theorem in Reference 14 (i.e., Theorem 2 in p. 202) enables us to obtain the strong invariance by using only subtangential conditions, it is confined itself to the nonlinear systems such that the vector fields satisfy Lipschitz continuity (see also Reference 30 for details). In contrast to the existing method assuming the Lipschitz continuity, the class of nonlinear differential equations taken in this article is not limited to that with Lipschitz continuous vector fields and contains those of non-unique solutions. Furthermore, the ideas of Lyapunov functions and barrier certificates have been recently developed in References 20, 30, and 31 to establish set invariance results in an equivalent fashion to those in this article, but they always require an involved task to constructing adequate scalar candidates for the Lyapunov functions and/or barrier certificates. In a comparison with these ideas, this article can be interpreted as giving a more intuitive and systematic method for solving the $L_1$ analysis problem of nonlinear systems since it is not required in this article to develop such scalar candidates.

The remaining of this article is organized as follows. The mathematical preliminaries used in this article are given in Section 2. The problem definition associated with the $L_1$ performance analysis for continuous nonlinear systems as well as the corresponding results are given in Section 3. The parallel extension of the results for continuous nonlinear systems to piecewise continuous nonlinear systems is discussed in Section 4. Some numerical examples are provided in Section 5 to verify the effectiveness as well as the validity of the results obtained in this article.

2 | MATHEMATICAL PRELIMINARIES

We use the notations $\mathbb{R}^\nu$ and $\mathbb{N}$ to denote the sets of $\nu$-dimensional real vectors and positive integers, respectively, while the notations $\mathbb{R}_+$, $\mathbb{R}_-$, and $\mathbb{N}_0$ are used to imply the sets of non-negative real numbers, non-positive real numbers and $\mathbb{N} \cup \{0\}$, respectively. The $\infty$-norm of a finite-dimensional vector is denoted by $| \cdot |_\infty$, that is,

$$|v|_\infty := \max_i |v_i|.$$ 

Equipped with this notation, we use for $r > 0$ with $x \in \mathbb{R}^n$ the notations $B_r(x)$ and $\overline{B}_r(x)$ to denote the open and closed balls defined, respectively, as

$$B_r(x) = \{ v \in \mathbb{R}^n \mid |v - x|_\infty < r \}, \quad \overline{B}_r(x) = \{ v \in \mathbb{R}^n \mid |v - x|_\infty \leq r \}.$$

For the notational simplicity, the notation $\mathbb{B}^\nu_0$ is taken to mean the closed unit ball $\overline{B}(0)$ with $0 \in \mathbb{R}^n$. The space of $\nu$-dimensional Lebesgue measurable functions is denoted by $L_\nu^\infty$, such that the corresponding $L_\infty$ norm is well-defined and bounded, that is,

$$||f(\cdot)||_\infty := \text{ess sup}_{t \geq 0} |f(t)|_\infty < \infty.$$ 

We also denote the space of functions $f \in L_\nu^\infty$ with $||f||_\infty \leq 1$ by $W_\nu$, for the notational simplicity, that is,

$$W_\nu := \{ f \in L_\infty^\nu \mid ||f||_\infty \leq 1 \}.$$

For a nonempty subset $A$ of $\mathbb{R}^n$, the notations $\text{Int}(A)$, $\overline{A}$ and $\partial A$ denote the sets of interior points, closure points and boundary points of $A$, respectively. Subsequently, for two nonempty subsets $A, B$ of $\mathbb{R}^n$, the Minkowski sum of $A, B$ and scalar multiplication of $A$ are denoted, respectively, by $A \oplus B$ and $cA$, that is,

$$A \oplus B := \{ a + b \in \mathbb{R}^n \mid \forall a \in A, \forall b \in B \}, \quad cA = \{ ca \in \mathbb{R}^n \mid \forall a \in A \}.$$
If \( F : X \rightarrow Y \) is a set-valued function with \( X = \mathbb{R}^n \) and \( Y = \mathbb{R}^m \), \( F \) is called *upper-semicontinuous* at \( x_0 \in X \) if for any open set \( O \) in \( Y \) containing \( F(x_0) \) the following relation holds:

\[
\exists V \subseteq X \text{ such that } x_0 \in V \text{ with } F(V) \subseteq O.
\]

This set-valued function \( F \) is also called *Lipschitz continuous* at \( x_0 \in X \) if there exists a neighborhood \( O \) of \( x_0 \) and a constant \( \lambda > 0 \) such that

\[
F(x_1) \subseteq F(x_2) + \lambda |x_1 - x_2|_\infty \mathbb{B} \mathbb{R}^m, \quad \forall x_1, x_2 \in O.
\]

This definition is applied to a single-valued function in an equivalent fashion.

Let \( K \) be a nonempty subset of the normed vector space \( (\mathbb{R}^n, | \cdot |_\infty) \). Then, the contingent cone \( T_K(x) \) of \( K \) for some \( x \in K \) is defined as

\[
T_K(x) := \left\{ v \in X \mid \liminf_{h \to 0^+} \frac{d_K(x + hv) - d_K(x)}{h} \leq 0 \right\}
\]

with the distance function defined as \( d_K(x) := \inf_{y \in K} |x - y|_\infty \). It should be remarked that the contingent cone \( T_K(x) \) can be defined only on \( x \in K \) and \( K \) is not required to be convex. If \( K \) is convex, \( T_K(x) \) coincides with the (well-known) tangent cone \( S_K(x) \) of \( K \) at \( x \) defined as

\[
S_K(x) := \bigcup_{h > 0} \left( K \oplus \{-x\} \right).
\]

Note that \( T_K(x) \neq S_K(x) \) in general for a non-convex \( K \).

In contrast, the external contingent cone \( \hat{T}_K(x) \) is defined for any point of \( x \in \mathbb{R}^n \) as

\[
\hat{T}_K(x) := \left\{ v \in X \mid \liminf_{h \to 0^+} \frac{d_K(x + hv) - d_K(x)}{h} \leq 0 \right\}.
\]

To put it another way, \( \hat{T}_K(x) \) obviously allows us to take \( x \) even in the outside of \( \overline{K} \) and is an extended version of \( T_K(x) \) in the sense that \( \hat{T}_K(x) = T_K(x) \) for all \( x \in \overline{K} \). For convenience of the understanding of the relation between these cones, they are depicted in Figure 1 for both the cases of convex and non-convex \( K \); \( \hat{T}_K(x) \) (as well as \( T_K(x) \)) may not coincide with \( S_K(x) \) for a non-convex \( K \). Furthermore, we take the notation \( T_K(x) \) to denote the external contingent cone when it is not required to take \( x \) in the outside of \( \overline{K} \) for the notational simplicity, which will be clear from the context.

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**Figure 1** The contingent cone \( T_K(x) \), external contingent cone \( \hat{T}_K(x) \), and tangent cone \( S_K(x) \) with respect to \( x_1 \in K \) and \( x_2 \in X - K \) for both the cases of (A) convex \( K \) and (B) non-convex \( K \).
3 | THE L1 PERFORMANCE ANALYSIS FOR CONTINUOUS NONLINEAR SYSTEMS

Let us consider the following continuous nonlinear system

\[
\Sigma_c : \begin{cases} 
\dot{x}(t) = f(x(t)) + g(x(t))w(t), \\
z(t) = h(x(t)) + k(x(t))w(t),
\end{cases}
\] (1)

where \(x(t) \in \mathbb{R}^n\) is the state, \(w(t) \in \mathbb{R}^p\) is the exogenous input, and \(z(t) \in \mathbb{R}^m\) is the output. We further assume that \(w \in W_p\) and the coefficient functions \(f(\cdot), g(\cdot), h(\cdot), k(\cdot)\) are continuous on \(\mathbb{R}^n\).

For a given \(w \in W_p\), a solution of the differential equation \(x(t) = f(x(t)) + g(x(t))w(t)\) in (1) can be described by using Carathéodory’s arguments.\(^{33}\) More precisely, \(x\) is called a solution of the differential equation on \([0, \delta]\) with the initial condition \(x(0) = x_0\) when \(x(t)\) is an absolutely continuous function in \(t\) such that the equality

\[
x(t) = x_0 + \int_0^t f(x(r)) + g(x(r))w(r) \, dr
\] (2)

holds for almost everywhere \(t \in [0, \delta]\). Even though such a solution \(x(\cdot)\) is uniquely determined if both \(f(x)\) and \(g(x)\) are (locally) Lipschitz continuous in \(x\), this section is concerned with the more general case of non-unique solutions with respect to \(x(\cdot)\) and thus both the coefficient functions (i.e., \(f(x)\) and \(g(x)\)) are not confined themselves to be (locally) Lipschitz continuous throughout the article.

As a preliminary step to tackle the \(L_1\) performance analysis problem, we next consider the input-to-state stability of \(\Sigma_c\) as follows.

**Assumption 1.** Assume that the continuous nonlinear system \(\Sigma_c\) (described by (1)) is input-to-state stable. To put it another way, there exists a constant \(L > 0\) for any \(w(\cdot) \in W_p\) such that the associated Caratheodory’s solution \(x(\cdot)\) with the initial condition \(x(0) = 0\) defined as (2) satisfies \(\|x\|_\infty \leq L\).

Here, it would be worthwhile to note that similarly to the case of linear systems\(^{34,35}\) in which the asymptotic stability should be satisfied for some system norms to be well-defined and bounded, the input-to-state stability has been first established for dealing with any system gain in nonlinear systems.\(^{36,37}\) We next introduce the definition of the \(L_1\) performance as follows.

**Definition 1 (\(L_1\) performance of \(\Sigma_c\)).** If any solution \(x(t)\) of the differential equation in (1) with the initial condition \(x(0) = 0\) for an arbitrary \(w \in W_p\) can be actually defined on the interval \([0, \infty)\) and every corresponding output \(z(t)\) satisfies \(\|z\|_\infty \leq 1\), then the continuous nonlinear system \(\Sigma_c\) is called to satisfy the \(L_1\) performance.

**Remark 1.** It should be remarked that the \(L_1\) performance is not essentially confined itself to the specific case with respect to the unit magnitude of both \(w\) and \(z\) (i.e., \(\|w\|_\infty \leq 1\) and \(\|z\|_\infty \leq 1\)), but could be readily extended to the general case such that \(\|z\|_\infty \leq \beta\) is considered for all \(\|w\|_\infty < \alpha\) for some \(\alpha, \beta > 0\) by replacing \(g(x), h(x),\) and \(k(x)\) with \(a \cdot g(x), h(x) / \beta,\) and \((a / \beta) \cdot k(x)\), respectively.

**Remark 2.** If we are interested in SISO LTI systems as a specific case of nonlinear systems (i.e., \(f(x) \equiv A, g(x) \equiv B, h(x) \equiv C,\) and \(k(x) \equiv D\) for some constant matrices \(A, B, C,\) and \(D\)), the \(L_1\) performance considered in Definition 1 is essentially the same as that the \(L_1\) norm of the impulse response is less than or equal to 1.

The \(L_1\) performance is obviously established even for the case of non-unique solutions in Definition 1, while that in the previous study\(^{10}\) is considered on assuming Lipschitz continuity of \(f(x)\) and \(g(x)\) as well as the solution uniqueness. In this sense, the \(L_1\) performance tackled in this article could be regarded as an extended and generalized version of that in Reference 10.

On the other hand, one might naturally regard that direct applications of the \(L_1\) performance to general nonlinear systems are readily possible, but it is often quite difficult to solve the nonlinear differential equation in (1). In this sense, it could be also quite meaningful to formulate the analysis problem with respect to the \(L_1\) performance in Definition 1 independent of the analytic solutions of the differential equations in (1), and we consider the following problem statement.
Problem 1: \((L_1\) performance analysis for \(\Sigma_c\)). Characterize the \(L_1\) performance for \(\Sigma_c\) without directly solving the differential equation in (1) but using its coefficient functions \(f(\cdot), g(\cdot), h(\cdot),\) and \(k(\cdot)\).

In connection with providing a solution procedure to Problem 1 in terms of set invariance principles,\(^{13,28}\) we first introduce the definition of invariant set, which plays an important role in establishing a sufficient condition for the \(L_1\) performance of \(\Sigma_c\).

**Definition 2** (Invariant set of \(\Sigma_c\)). Suppose that \(K\) is a nonempty subset of \(\mathbb{R}^n\). Then, \(K\) is called an invariant set of \(\Sigma_c\) if every solution \(x(t)\) of the differential equation in (1) with an arbitrary initial condition \(x(0) \in K\) for any \(w \in W_p\) satisfies \(x(t) \in K\) for all \(t \in [0, \delta)\) on which the solution can be defined.

With the employment of invariant set for tackling Problem 1 in mind, we are led to the following lemma, which is associated with the solution existence of the differential equation in (1).

**Lemma 1** (Existence of solutions for \(\Sigma_c\)). Let \(K\) be a compact subset of \(\mathbb{R}^n\). Then, there exists a positive constant \(c\) such that the differential equation in (1) has a solution on the interval \([0, c)\) with an arbitrary initial condition \(x(0) \in K\) for any \(w \in W_p\).

**Proof.** For an arbitrary \(a > 0\), the compactness assumption on \(K\) together with the continuity property of \(f\) and \(g\) implies that there exist positive constants \(M_f\) and \(M_g\) such that

\[
M_f = \sup_{x \in K} |f(x)|_\infty, \quad M_g = \sup_{x \in K} |g(x)|_\infty. \tag{3}
\]

By taking \(\varphi(t) := M_f + M_g |w(t)|_\infty\) and \(c := a/(M_f + M_g)\), it immediately follows that

\[
\int_0^c \varphi(t) \leq a. \tag{4}
\]

This together with the solution existence theorem in Reference 29 ensures that there exists a solution \(x(t)\) of the differential equation in (1) on the interval \([0, c)\). This together with the fact that \(c\) is independent of the choice of the initial condition \(x(0) \in K\) completes the proof.

It would be worthwhile to note that the constant \(c\) in Lemma 1 is independent of the choice of the initial time \(t_0\), and this fact allows us to arrive at the following result, which is relevant to forward completeness of solutions of the differential equation in (1).

**Proposition 1** (Forward completeness of \(\Sigma_c\)). If \(K\) is a compact invariant set of \(\Sigma_c\), then every solution \(x(t)\) of the differential equation in (1) with an arbitrary initial condition \(x(0) \in K\) for any \(w \in W_p\) can be actually defined on the interval \([0, \infty)\) and satisfies \(x(t) \in K\) for all \(t \geq 0\).

**Proof.** From Lemma 1, there exists a constant \(c\) such that solutions of the differential equation in (1) can be defined on \([0, c)\) with an arbitrary initial condition \(x(0) \in K\) for any \(w \in W_p\). Hence, it is obvious from Definition 2 together with the compactness assumption on \(K\) that there exists a solution \(\tilde{x}_0(t)\) of the differential equation such that \(\tilde{x}_0(t) \in K\) \(\forall t \in [0, c)\) and the limit \(\lim_{t \to c^-} x(t)\) is also contained in \(K\). Applying this procedure repeatedly with the initial condition \(\tilde{x}_{i+1}(0) := \tilde{x}_i(c) \ (\forall i \in \mathbb{N}_0)\) leads to the function sequence \(\{\tilde{x}_i\}_{i=0}^\infty\) such that \(\tilde{x}_i(t) \in K\) \(\forall t \in [0, c], \forall i \in \mathbb{N}_0\). Because \(K\) is an invariant set of \(\Sigma_c\), every \(\tilde{x}_i(\cdot)\) is a solution of the differential equation. Then, \(x\) defined as \(x(0 + t) := \tilde{x}_i(t)\) \(\forall t \in [0, c), i \in \mathbb{N}_0\) satisfies that \(x(t) \in K\) \(\forall t \geq 0\) and is obviously a solution of the differential equation over \([0, \infty)\). This completes the proof.

**Remark 3.** If the compactness property is not assumed on \(K\) in Proposition 1, there could exist a case such that the assertion in the proposition does not hold. For example, the solution of the differential equation \(\frac{dx(t)}{dt} = x^2(t), \ x(0) > 0\) tends to \(\infty\) in a finite time.

The significance of Proposition 1 is to establish the forward completeness of \(\Sigma_c\) without assuming the solution uniqueness. Based on this proposition, we are led to the following result.
Theorem 1. Let us define

$$\Omega := \{ x \in \mathbb{R}^n \mid |h(x) + k(x)w|_\infty \leq 1, \ \forall w \in \mathbb{R}^p \},$$

where $\mathbb{R}^p$ is the p-dimensional closed unit ball and is introduced in Section 2. Then, $\Sigma_c$ satisfies the $L_1$ performance if there exists a compact invariant set $K$ such that

$$0 \in K \subseteq \Omega.$$

Proof. Since $K(\emptyset 0)$ is a compact invariant set of $\Sigma_c$, it immediately follows from Proposition 1 that any solution $x(t)$ of the differential equation in (1) with the initial condition $x(0) = 0$ can be actually defined on $[0, \infty)$ and remains in $K$ for all $t \in [0, \infty)$. This together with (6) obviously implies that

$$|z(t)|_\infty = |h(x(t)) + k(x(t))w(t)|_\infty \leq 1, \ \forall t \in [0, \infty).$$

This completes the proof. □

Remark 4. Unlike the case of positive systems, for which unbounded cones (e.g., the first orthants) should be inherently utilized to characterize the relevant set invariance properties, it should be required to take the compactness of $K$ in the $L_1$ analysis problem. More precisely, $x(\cdot)$ for an external input in $W_p$ is always remained in a compact set $B_t(0)$ from Assumption 1 and thus taking a sort of unbounded sets (such as the aforementioned cones) cannot lead to any quantitative characterization with respect to the $L_1$ performance. Furthermore, taking an unbounded invariant set $K$ might not ensure the forward completeness of $\Sigma_c$ as discussed in Remark 3. In this sense, the compact property of $K$ cannot be regarded as a conservative assumption for tackling the $L_1$ analysis problem.

Remark 5. One might argue that the assertions of Theorem 1 are somewhat conservative in a comparison with Proposition 2.6 in Reference 10, in which a necessary and sufficient condition for the $L_1$ performance analysis of $\Sigma_c$ is derived while Theorem 1 only provides the sufficient condition. However, it is intrinsically impossible to establish an exact necessity direction of Theorem 1; the converse of Theorem 1 is not true in general due to the absence of solution uniqueness.

Theorem 1 provides us the sufficient condition for the $L_1$ performance analysis of $\Sigma_c$, but it would be worthwhile to note that one of the most immediate methods for applying the arguments in Theorem 1 is to clarify an existence of the invariant set by obtaining analytic solutions of the differential equation. However, this is not a typical case, and thus it is required to establish another framework for the $L_1$ performance analysis. By noting the fact that this difficulty occurred in nonlinear systems with non-unique solutions might be alleviated by employing the invariance-domain-based arguments, as in Reference 10, we provide the so-called extended invariance domain approach equipped with external contingent cones, with which the $L_1$ performance for $\Sigma_c$ can be described in terms of its coefficient functions.

Definition 3 (Extended invariance domain of $\Sigma_c$). Let $K$ be a closed subset of $\mathbb{R}^n$. Then, the set $K$ is called an extended invariance domain of $\Sigma_c$ if there exists an open neighborhood $O$ of $K$ such that

$$f(x) + g(x)w \in T_K(x), \ \forall w \in \mathbb{R}^p, \ \forall x \in O.$$

Remark 6. It should be remarked that $T_K(x)$ used in (8) is an external contingent cone introduced in Section 2, and the employment of this notion allows us to take the dynamic behavior of the differential equation in (1) at the outside of $K$, although the conventional contingent cone $T_K(x)$ considered in References 10 and 12 takes only that at the inside of $K$.

In connection with the relation between the invariant set and the extended invariance domain, we next introduce the following lemma by slightly modifying invariance theorem in Reference 28 (but without affecting the essentials).

Lemma 2. Suppose that $K$ is a nonempty subset of $\mathbb{R}^n$ and $a(t, x)$ is a function defined on $\mathbb{R}_+ \times \mathbb{R}^n$. If $a(t, x)$ is continuous in $x \in \mathbb{R}^n$ and $O$ is an open neighborhood of $K$ such that

$$a(t, x) \in T_K(x), \ \forall t \geq 0, \ \forall x \in O$$

then any solution $x(t)$ of $\dot{x}(t) = a(t, x(t))$ defined on $[0, \delta)$ with an arbitrary initial condition $x(0) \in K$ satisfies $x(t) \in K$, $\forall t \in [0, \delta)$. 

Remark that taking the external contingent cone $T_K(x)$ in Lemma 2 allows us to consider the case of non-unique solutions with respect to $\dot{x}(t) = a(t, x(t))$. Based on Lemma 2, we can derive the following result.

**Proposition 2.** If $K$ is an extended invariance domain of $\Sigma_c$, then $K$ is also an invariant set of $\Sigma_c$.

**Proof.** For an arbitrary fixed $w \in W_p$, define $a(t, x)$ as

$$a(t, x) := f(x) + g(x)w(t). \quad (10)$$

Then, it readily follows from (8) that the relation (9) holds since $w(t)$ in (10) can be in fact replaced by $w \in \mathbb{R}^p$ for each fixed time $t \in \mathbb{R}_+$. By Lemma 2, every solution $x(t)$ of $\dot{x}(t) = a(t, x(t))$ with an arbitrary initial condition $x(0) \in K$ stays in $K$ for all $t$ in $[0, \delta)$ on which the solution can be defined. From this together with Definition 2, $K$ is obviously an invariant set of $\Sigma_c$.

**Remark 7.** It should be noted that the converse of Proposition 2 does not hold in general. This is because we are also concerned with the dynamic behavior of $\Sigma_c$ at the outside of $O$ (i.e., points in $O - K$). In contrast, the converse is shown in the existing study $^{10}$ to be correct under the assumption of solution uniqueness. Thus, it is essential to obtain only sufficient conditions rather than necessary and sufficient conditions for the $L_1$ performance analysis of $\Sigma_c$ in this article for dealing with non-unique solutions.

Finally, we can obtain from Theorem 1 and Proposition 2 the following theorem, which is the main result of this section.

**Theorem 2.** The continuous nonlinear system $\Sigma_c$ satisfies the $L_1$ performance if there exists a compact extended invariance domain $K$ such that $0 \in K \subseteq \Omega$, where $\Omega$ is defined as (5).

This theorem clearly gives us a solution to Problem 1, that is, a sufficient condition for the $L_1$ performance of $\Sigma_c$ in terms of its coefficient functions. Furthermore, Theorem 2 can be also interpreted as an extended version of Theorem 2.12 in Reference 10.

On the other hand, it would be worthwhile to remark that Lemma 2 can be extensively used in accordance with the treatment of differential inclusions. $^{28}$ This fact motivates us to consider a further extension of the results obtained in this section to piecewise continuous systems whose dynamics are described by differential inclusions. To put it another way, the arguments in Section 3 play a key role in establishing involved conditions with respect to the $L_1$ performance analysis of piecewise continuous systems, which will be dealt with in the subsequent section.

# 4 Extension to Piecewise Continuous Nonlinear Systems

Let us consider the following piecewise continuous nonlinear system

$$\Sigma_{pc} : \begin{cases} \dot{x}(t) = f_{pc}(x(t)) + g_{pc}(x(t))w(t), \\ \dot{z}(t) = h_{pc}(x(t)) + k_{pc}(x(t))w(t), \end{cases} \quad (11)$$

where $x(t)$, $w(t)$, and $z(t)$ are the same variables as those in (1) and the coefficient functions $f_{pc}(\cdot)$, $g_{pc}(\cdot)$, $h_{pc}(\cdot)$, and $k_{pc}(\cdot)$ are piecewise continuous and locally bounded functions. It is further assumed that there exist open subsets $D_1, D_2, \ldots, D_N$ of $\mathbb{R}^n$ such that

- $\bigcup_{i=1}^N \overline{D_i} = \mathbb{R}^n$.
- $D_i \cap D_j = \emptyset$ for all $i \neq j$.
- $M := \mathbb{R}^n - \bigcup_{i=1}^N D_i$ has Lebesgue measure zero.
- Each $D_i$ becomes domain of continuity for all the coefficient functions $f_{pc}(\cdot)$, $g_{pc}(\cdot)$, $h_{pc}(\cdot)$, and $k_{pc}(\cdot)$.

In contrast to the continuous nonlinear system $\Sigma_c$ discussed in the preceding section, the Carathéodory’s arguments $^{33}$ could not establish the existence of solutions for the differential equation $\dot{x}(t) = f_{pc}(x(t)) + g_{pc}(x(t))w(t)$ in (11) due to the discontinuity of the coefficient functions. To alleviate this difficulty, we describe the solutions of the differential equation...
based on Filippov’s arguments. More precisely, a function $x(t)$ is defined as a Filippov’s solution of the differential equation of $\Sigma_{pc}$ on $[0, \delta)$ if $x(t)$ is absolutely continuous in $t$ and satisfies the differential inclusion

$$x(t) \in F_w(t, x(t))$$  \hspace{1cm} (12)

for almost everywhere $t \in [0, \delta)$, with the set-valued function $F_w(t, x)$ given by

$$F_w(t, x) = \begin{cases} \{f_{pc}(x) + g_{pc}(x)w(t)\}, & \text{if } x \in C, \\ \overline{\{ \lim_{x^+ \to x^-} f_{pc}(x^+) + g_{pc}(x^+)w(t) \mid x^+ \in C \}}, & \text{otherwise,} \end{cases}$$  \hspace{1cm} (13)

where $C$ and $\overline{\circ}(-)$ mean the set of all continuous points of the functions $f_{pc}(-)$ and $g_{pc}(-)$ and the closed convex hull of $(-)$, respectively.

On the other hand, the vector field $F_w(t, x)$ is upper-semicontinuous with respect to $x$ but not Lipschitz continuous in general. This is because it is not required for the functions $f_{pc}(-)$ and $g_{pc}(-)$ to be (locally) Lipschitz continuous on their domains of continuity $D_i$ ($1 \leq i \leq N$), essentially equivalent to the case of continuous systems discussed in Section 3. This fact together with the arguments relevant to solution existence theorem in Reference 29 establishes the local existence of solutions of the differential equation in (11), but the solution uniqueness does not often hold. Similarly to the preliminary step to the $L_1$ performance analysis of continuous systems, we further consider the input-to-state stability of $\Sigma_{pc}$ as follows.

Assumption 2. The piecewise continuous system $\Sigma_{pc}$ (described by (11)) is assumed to be input-to-state stable. In other words, there exists a constant $L > 0$ for any $w(-) \in W_p$ such that the relevant Filippov solution $x(t)$ with the initial condition $x(0) = 0$ defined as (12) satisfies $\|x\|_\infty \leq L$.

Based on the Filippov’s arguments-based solution existence together with Assumption 2, we can naturally extend the $L_1$ performance as well as the corresponding analysis problem defined for the continuous nonlinear system $\Sigma_c$ (i.e., Definition 1 and Problem 1) to the piecewise continuous nonlinear system $\Sigma_{pc}$ as follows.

Definition 4 ($L_1$ performance of $\Sigma_{pc}$). If any Filippov’s solution $x(t)$ of the differential equation in (11) with the initial condition $x(0) = 0$ for an arbitrary $w \in W_p$ can be actually defined on $[0, \infty)$ and every corresponding output $z(t)$ satisfies $\|z\|_\infty \leq 1$, then the piecewise continuous system $\Sigma_{pc}$ is said to satisfy the $L_1$ performance.

Problem 2 ($L_1$ performance analysis for $\Sigma_{pc}$). Characterize the $L_1$ performance of $\Sigma_{pc}$ in terms of its coefficient functions $f_{pc}(-), g_{pc}(-), h_{pc}(-)$, and $k_{pc}(-)$ without directly solving the Filippov’s differential equation in (11).

It should be noted that taking Filippov’s solution makes the above definition and problem statement intrinsically different from those for the continuous nonlinear system $\Sigma_c$, in which the associated solutions are described in terms of Caratheodory’s arguments. It might be possible to regard the effects of discontinuity of $f_{pc}(-)$ and $g_{pc}(-)$ on the regulated output as those by exogenous inputs or smoothened continuous functions, rather than taking the involved arguments with Filippov’s solutions. However, such rough treatment of the discontinuity could not establish explicit characterizations with respect to the $L_1$ performance analysis for piecewise continuous systems, and the details will be discussed in Section 5.3. In this sense, we also consider the following definition of invariant set relevant to Filippov’s solutions in a modified fashion to that in Definition 2.

Definition 5 (Invariant set for $\Sigma_{pc}$). Let $K$ be a nonempty subset of $\mathbb{R}^n$. Then, $K$ is said to be an invariant set of $\Sigma_{pc}$ if every Filippov’s solution $x(t)$ of the differential equation in (11) with an arbitrary initial condition $x(0) \in K$ for any $w \in W_p$ satisfies $x(t) \in K$ for all $t \in [0, \delta)$ on which the solution $x(t)$ can be defined.

Then, the arguments on existence of solutions and forward completeness of the continuous nonlinear system $\Sigma_c$ (i.e., Lemma 1 and Proposition 1) can be naturally established for the piecewise continuous nonlinear system $\Sigma_{pc}$ with the considerations of Filippov’s solutions. To put it another way, we could arrive for the piecewise continuous nonlinear system $\Sigma_{pc}$ at the following results.

Lemma 3 (Existence of solutions for $\Sigma_{pc}$). Let $K$ be a compact subset of $\mathbb{R}^n$. Then, there exists a positive constant $c$ such that the differential equation in (11) has a Filippov’s solution on the interval $[0, c)$ with an arbitrary initial condition $x(0) \in K$ for any $w \in W_p$. 
Proof. For an arbitrary fixed \( a > 0 \) and \( w \in W_p \), let us take \( f_{pc}(x) + g_{pc}(x)w(t) \) as a (trivial) single-valued selection of \( F_w(t, x) \). Then, it immediately follows from the locally boundedness of \( f_{pc}(\cdot) \) and \( g_{pc}(\cdot) \) together with the compactness of \( K \) that this selection is bounded by a function \( \varphi(t) \) defined as

\[
\varphi(t) := M_f + M_g w(t),
\]

where

\[
M_f = \sup_{x \in K \cap \partial \mathbb{R}^n} |f_{pc}(x)|_\infty, \quad M_g = \sup_{x \in K \cap \partial \mathbb{R}^n} |g_{pc}(x)|_\infty.
\]

If we note the upper-semicontinuity of \( F_w(t, x) \), the remaining task is essentially the same as the proof of Lemma 1. \( \blacksquare \)

**Proposition 3** (Forward completeness of \( \Sigma_{pc} \)). If \( K \) is a compact invariant set of \( \Sigma_{pc} \), then every Filippov’s solution \( x(t) \) of the differential equation in (11) with an arbitrary initial condition \( x(0) \in K \) for any \( w \in W_p \) can be actually defined on the interval \([0, \infty)\) and satisfies \( x(t) \in K \) for all \( t \geq 0 \).

Even though we provide the details for the proof of Lemma 3 since taking a single-valued selection of \( F_w(t, x) \) is intrinsically different to the case of continuous nonlinear systems, we omit the proof of Proposition 3 because it is essentially equivalent to that of Proposition 1. From Proposition 3, we are led to the following theorem, which is parallel to Theorem 1.

**Theorem 3.** The system \( \Sigma_{pc} \) satisfies the \( L_1 \) performance if there exists a compact invariant set \( K \) such that \( 0 \in K \subseteq \Omega \), where \( \Omega \) is defined as (5) for \( h_{pc}(\cdot) \) and \( k_{pc}(\cdot) \).

Essentially equivalent to the case of continuous nonlinear systems as discussed in Remark 4, it is necessary from Assumption 2 and the assertion on forward completeness to establish Lemma 3, Proposition 3, and Theorem 3 by taking the compactness of \( K \). It should be further remarked that the converse of Theorem 3 cannot be established, similarly to the case of continuous nonlinear systems as mentioned in Remark 5. On the other hand, the proof of Theorem 3 is essentially the same as that of Theorem 1, and thus we omit the proof.

With an indirect solution procedure to Problem 2 in mind, we are next concerned with an extended invariance domain of \( \Sigma_{pc} \) as a modification of Definition 3. To this end, let us consider the time-independent vector field defined as

\[
F_w(x) = \bar{\partial}\{ \lim_{x^* \to x} f_{pc}(x^*) + g_{pc}(x)w \mid x^* \in C \}
\]

for any fixed \( w \in \mathbb{B} \mathbb{R}^p \). Then, we can obtain the following involved definition.

**Definition 6** (Extended invariance domain of \( \Sigma_{pc} \)). Let \( K \) be a closed subset of \( \mathbb{R}^n \). Then, the set \( K \) is defined as an extended invariance domain of \( \Sigma_{pc} \) if there exists a neighborhood \( O \) of \( K \) such that

\[
F_w(x) \subseteq \hat{T}_K(x), \quad \forall w \in \mathbb{B} \mathbb{R}^p, \quad \forall x \in O.
\]

Note that the set-valued vector field \( F_w(x) \) is employed for describing the extended invariance domain of the piecewise continuous nonlinear system \( \Sigma_{pc} \), in contrast to Definition 3. Regarding a relation between invariant set and extended invariance domain with respect to \( \Sigma_{pc} \), we also provide the following lemma by modifying Invariance Theorem for differential inclusions\(^\text{28}\) without affecting the essentials.

**Lemma 4.** Suppose that \( K \) is a nonempty subset of \( \mathbb{R}^n \) and \( F(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a locally bounded set-valued function with \( K \subseteq \text{dom}(F(t, \cdot)) \). If \( O \) is an open (relative to \( \text{dom}(F(t, \cdot)) \)) neighborhood of \( K \) such that

\[
F(t, x) \subseteq \hat{T}_K(x), \quad \forall t \geq 0, \quad \forall x \in O
\]

then any solution \( x(t) \) of \( \dot{x}(t) \in F(t, x(t)) \) defined on \([0, \delta)\) with arbitrary initial condition \( x(0) \in K \) satisfies \( x(t) \in K \) for all \( t \in [0, \delta) \).

By Lemma 4, we can derive the following result.

**Proposition 4.** If \( K \) is an extended invariance domain of \( \Sigma_{pc} \), then the set \( K \) is an invariant set of \( \Sigma_{pc} \).
Proof. For an arbitrary fixed \( w \in W_p \), we define the set-valued function \( F(t,x) \) as (13). Then, it immediately follows from Definition 6 and the essentially equivalent procedure for obtaining Proposition 2 that \( K \) is an invariant set of \( \Sigma_{pc} \).

Similar to the case of continuous nonlinear systems as discussed in Remark 7, the converse of Proposition 4 does not hold in general due to the consideration of dynamic behavior at outside points \( O - K \). Finally, we could obtain the following theorem from Theorem 3 and Proposition 4.

**Theorem 4.** The piecewise continuous nonlinear system \( \Sigma_{pc} \) satisfies the \( L_1 \) performance if there exists a compact extended invariance domain \( K \) such that \( 0 \in K \subseteq \Omega \), where \( \Omega \) is defined as (5).

This theorem corresponds to a solution to Problem 2, that is, a sufficient condition for the \( L_1 \) performance of \( \Sigma_{pc} \) in terms of its coefficient functions. Theorem 4 could be also interpreted as providing an extension of Theorem 2.12 in Reference 10 to the piecewise continuous system \( \Sigma_{pc} \).

### 5 Numerical Examples

This section aims at examining the effectiveness of Theorems 2 and 4 through some numerical examples associated with continuous nonlinear and piecewise continuous nonlinear systems, respectively. Furthermore, the significance of the rigorous treatment of piecewise continuity on the \( L_1 \) analysis problem is also verified through a numerical example in this section (i.e., Section 5.3).

#### 5.1 Case of continuous nonlinear systems with non-unique solutions

Let us consider the continuous nonlinear system

\[
\Sigma_c : \begin{cases} 
\dot{x}(t) = f(x) + w(t), \\
z(t) = \sqrt{x(t)}, 
\end{cases}
\]

(19)

where the coefficient function \( f(x) \) is defined as

\[
f(x) = \begin{cases} 
1, & \text{if } x < -1, \\
1 + \sqrt{x + 1}, & \text{if } -1 \leq x < 0, \\
2 - 3x^2, & \text{if } 0 \leq x.
\end{cases}
\]

(20)

We first obtain \( \Omega \) (defined as (5)) by

\[
\Omega = \{ x \in \mathbb{R} \mid z = |\sqrt{x}| \leq 1 \} = [-1, 1].
\]

(21)

In connection with the application of Theorem 2 to this system, it is required to establish that there exists a corresponding extended invariance domain \( K \) such that \( 0 \in K \subseteq \Omega \). In this regard, the remaining part of this subsection is devoted to showing that such a set \( K \) can be taken by \( K = [-1, 1] \).

Let us first note that \( K \) is an extended invariance domain, if there exists \( \epsilon > 0 \) such that

\[
f(x) + w \in \tilde{T}_K(x), \quad \forall w \in \mathbb{R}^1
\]

(22)

is satisfied for arbitrary \( x \in (-1 - \epsilon, 1 + \epsilon) \). Here, because the external contingent cone \( \tilde{T}_K(x) \) is given by

\[
\tilde{T}_K(x) = \begin{cases} 
\mathbb{R}^+, & \text{if } -1 - \epsilon \leq x \leq -1, \\
\mathbb{R}, & \text{if } -1 < x < 1, \\
\mathbb{R}^-, & \text{if } 1 \leq x \leq 1 + \epsilon.
\end{cases}
\]

(23)
It immediately follows from (20) together with (23) that the inclusion of (22) is satisfied for all \( x \in (-1 - \epsilon, 1 + \epsilon) \). Thus, it is obvious from Theorem 2 that the continuous nonlinear system given by (19) and (20) satisfies the \( L_1 \) performance.

On the other hand, it should be noted that this continuous nonlinear system is one of the examples to which the existing results of the \( L_1 \) performance analysis in Reference 10 cannot be applied. This is because the differential equation of (19) could have non-unique solutions in accordance with the choice of \( w(t) \). For example, if we take \( w(t) \equiv -1 \), then the differential equation of (19) has non-unique solutions described by

\[
\begin{align*}
x(t; t_0) &= \begin{cases} -1, & \text{if } 0 \leq t \leq t_0, \\
-1 + \frac{t-t_0^2}{4}, & \text{if } t > t_0,
\end{cases}
\end{align*}
\]

for arbitrary \( t_0 > 0 \) with the initial condition \( x(0) = -1 \). To put it another way, \( x(t; t_0) \) given by (24) becomes solutions of (19) for any \( t_0 > 0 \).

Such a non-uniqueness of solutions as in (24) could be interpreted as arising from the fact that the vector field \( f(x) \) of (20) does not satisfy Lipschitz continuity at \( x = -1 \). This situation is quite crucial for a controller synthesis because the closed-loop system obtained through a feedback connection between a nominal system and a controller cannot satisfy the Lipschitz property even if the vector field of the nominal system is Lipschitz continuous as discussed in Theorem 3.9 in Reference 10 and Definition 2.3 in Reference 12.

In connection with this, if the continuous nonlinear system described by (19) is interpreted as obtained by connecting the \( L_1 \) controller \( u(x) \) (which can be computed by using the arguments in Reference 10 and is given by)

\[
u(x) = \begin{cases} 2, & \text{if } x < -1, \\
2 + \sqrt{|x+1|}, & \text{if } -1 \leq x < 0, \\
3 - 3x^2, & \text{if } 0 \leq x,
\end{cases}
\]

then it is still unclear whether or not the closed-loop system satisfies the \( L_1 \) performance when we are confined ourselves to the existing arguments in Reference 10. Hence, this observation undoubtedly implies that the employment of extended invariance domain in this article allows us to significantly improve the existing results in Reference 10 in the sense that the acceptable class of the \( L_1 \) admissible controller would be widened.

### 5.2 Case of piecewise continuous nonlinear systems

Let us consider the helicopter system with a frictional force as shown in Figure 2. Assume that the dynamics of this system is given by

\[
\begin{align*}
\frac{d^2 \theta}{dt^2} &= \frac{1}{I_{xy}} \left( -mL_x g \cos(\theta) - mL_z g \sin(\theta) - F_{km} \text{sgn} \left( \frac{d\theta}{dt} \right) - F_{vm} \frac{d\theta}{dt} + u(t) \right) + w(t), \\
z &= \frac{1}{4} \max \left\{|\theta|, \left| \frac{d\theta}{dt} \right|\right\},
\end{align*}
\]

where \( I_{xy} \) is the moment of inertia, \(-mL_x g \cos(\theta)\) and \(-mL_z g \sin(\theta)\) are the torques generated from the gravitational force with respect to \( x \)-axis and \( z \)-axis, respectively, \(-F_{km} \text{sgn} \left( \frac{d\theta}{dt} \right)\) is the frictional force, \(-F_{vm} \frac{d\theta}{dt}\) is the damping force, and \( \text{sgn}(\cdot) \) denotes the signum function defined as

\[
\text{sgn}(x) : = \begin{cases} -1, & x < 0, \\
0, & x = 0, \\
1, & x > 0.
\end{cases}
\]

Furthermore, \( w \in W_1 \) is the external disturbance and \( u(t) \) is the control torque input.
In the following, we take $I_{yy} = 1$, $F_{km} = 1$, $F_{vm} = 5$, $g = 10$, $m = 0.1$, and $L_x = L_z = 10$ for the simplicity of the arguments. The control torque input $u(t)$ is given by

$$u(t) = 10 \cos(\theta) + 3 \text{sgn}\left(\frac{d\theta}{dt}\right),$$

(29)

where the first and second terms correspond to the gravity and friction compensations, respectively. The closed-loop system obtained by connecting (29) with (27) can be described by

$$\Sigma_{pc} : \begin{cases} \dot{x}(t) = f_{pc}(x(t), y(t)) + g_{pc}(x(t), y(t))\psi(t), \\ \dot{y}(t) = \frac{1}{4} \max\{|x(t)|, |y(t)|\} \\ z(t) = \frac{1}{4} \max\{|x(t)|, |y(t)|\}, \end{cases}$$

(30)

where

$$f_{pc}(x, y) = \begin{pmatrix} y \\ -10 \sin x - 5y + 2\text{sgn}(y) \end{pmatrix}, \quad g_{pc}(x, y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(31)

with $x(t) := \theta(t)$ and $y(t) := \frac{d\theta(t)}{dt}$.

With respect to the roles of each term in the control input, it could be first observed from (31) that the gravitational torque at the $x$-axis in (27) is completely compensated by the first term while the friction force is overly compensated. This over-compensation is regarded as a typical scheme for frictions in mechanical systems (and see Reference 40 for details) and makes the piecewise continuous system $\Sigma_{pc}$ (given by (30)) have non-unique solutions. The following arguments are devoted to establishing the corresponding $L_1$ performance analysis.

For this piecewise continuous nonlinear system $\Sigma_{pc}$, we first note that $\Omega$ (defined as (5)) is given by

$$\Omega = [-4, 4] \times [-4, 4] \subseteq \mathbb{R}^2.$$  

(32)

We next show that the set

$$K = \{(x, y) \in \mathbb{R}^2 \mid x \in [-2, -1], y \in [-2x - 2, 2x + 6]\} \cup \{(x, y) \in \mathbb{R}^2 \mid x \in [-1, 1], y \in [-2x - 2, -2x + 2]\} \cup \{(x, y) \in \mathbb{R}^2 \mid x \in [1, 2], y \in [2x - 6, -2x + 2]\}$$

(33)
becomes an extended invariance domain of the piecewise continuous nonlinear system $\Sigma_{pc}$ described by (30) such that $K \subseteq \Omega$.

To this end, it suffices from Definition 6 to show that there exists a small positive number $\epsilon > 0$ such that the inclusion

$$F_w(x, y) \subseteq T_K(x, y), \quad \forall (x, y) \in K$$

is satisfied, where $F_w(x, y)$ is described by

$$F_w(x, y) = \begin{cases} 
y + 10 \sin x - 5y + 2\text{sgn}(y) + w, & \text{if } y \neq 0, \\
\left\{ \left(y - 10 \sin x - 5y + \alpha + w \right) \right\} \mid \alpha \leq 2 \vrule, & \text{if } y = 0,
\end{cases}$$

and $K_\epsilon$ is defined as $K_\epsilon = K \oplus \epsilon B^2$. In other words, for an arbitrary $0 < \epsilon < 1/5$, we divide the cases of $(x, y) \in K_\epsilon$ into $(x, y) \in \text{Int}(K)$, $(x, y) \in \partial K$, $(x, y) \in (\mathbb{R}^2 - K) \cap K_\epsilon$, and establish the relation (34) for each case.

Here, we confine ourselves to the cases of $(x, y) \in \partial K$ and $(x, y) \in (\mathbb{R}^2 - K) \cap K_\epsilon$, since that of $(x, y) \in \text{Int}(K)$ can be easily established. Let us divide $\partial K$ and $\mathbb{R}^2 - K$, respectively, into

$$\partial K = \bigcup_{i=1}^{4} P_i, \quad \mathbb{R}^2 - K = \bigcup_{i=1}^{8} Q_i, \quad (36)$$

where each $P_i(i = 1, \ldots, 4)$ denotes each line segment of the boundary $\partial K$ and each $Q_i(i = 1, \ldots, 8)$ means each open subset of $\mathbb{R}^2 - K$ partitioned by the lines with slopes of 1 and $-1$, and they are depicted in Figure 3. Because both the vector field $f_{pc}(x, y)$ in (31) and the extended invariance domain shown in Figure 3 are symmetric around the origin in $x$-$y$ plane, it is not required to take into account of every region of $\partial K$ and $\mathbb{R}^2 - K$ with respect to dealing with (34), but it suffices to consider $(x, y) \in P_i$ and $(x, y) \in Q_j \cap K_\epsilon$ for $i = 1, 2$ and $j = 1, 2, 5, 8$. For such reduced cases, we can compute

\[ \text{Figure 3} \quad \text{The extended invariance domain } K \text{ for } \Sigma_{pc} \text{ together with the line segments } P_i (i = 1, 2, 3, 4) \text{ of the boundary } \partial K \text{ and the open subsets } Q_j (i = 1, 2, \ldots, 8) \text{ of } \mathbb{R}^2 - K \text{ partitioned by the lines with slopes 1 and } -1. \]
the external contingent cone $T_K(x, y)$ consisting of 2-dimensional vector $[u \ v]^T \in \mathbb{R}^2$ as follows.

$$T_K(x, y) = \begin{cases} 
\{2u + v \leq 0, 2u - v \leq 0\}, & \text{if } (x, y) \in \partial P_1 \cap \partial P_4, \\
\{2u + v \leq 0\}, & \text{if } (x, y) \in P_1 \cup Q_1, \\
\{2u + v \leq 0, -2u + v \leq 0\}, & \text{if } (x, y) \in \partial P_1 \cap \partial P_2, \\
\{-2u + v \leq 0\}, & \text{if } (x, y) \in P_2 \cup Q_2, \\
\{v \leq 0\}, & \text{if } (x, y) \in Q_5, \\
\{u \leq 0\}, & \text{if } (x, y) \in Q_8, \\
\{2u - v \leq 0, u \leq 0\}, & \text{if } (x, y) \in \partial Q_4 \cap \partial Q_8, \\
\{2u + v \leq 0, u \leq 0\}, & \text{if } (x, y) \in \partial Q_1 \cap \partial Q_8, \\
\{2u + v \leq 0, v \leq 0\}, & \text{if } (x, y) \in \partial Q_1 \cap \partial Q_5, \\
\{-2u + v \leq 0, v \leq 0\}, & \text{if } (x, y) \in \partial Q_2 \cap \partial Q_5. 
\end{cases} \quad (37)$$

Then, it readily follows from (35) and (37) that the inclusion (34) holds and thus $K$ defined as (33) is obviously an extended invariance domain of $\Sigma_{pc}$ given by (30). Hence, by Theorem 4, this piecewise continuous nonlinear system $\Sigma_{pc}$ satisfies the $L_1$ performance.

On the other hand, it would be also worthwhile to note that piecewise continuous nonlinear systems often have non-unique solutions, as mentioned in Sections 1 and 4. This is also true even for the piecewise continuous nonlinear system $\Sigma_{pc}$ dealt with in this article. For example, if we let $(x, y) = (0, 0)$ be an initial condition of the differential equation in (30) with $w = 0$, then the solution $(x(t), y(t))$ cannot be uniquely determined since

$$\lim_{y \to 0^+} f_{pc}(0, y) = 2, \quad \lim_{y \to 0^-} f_{pc}(0, y) = -2 \quad (38)$$

and this non-uniqueness could be interpreted as occurring from the repulsive mode\(^{41}\) of discontinuity for $f_{pc}(x, y)$; its direction is repulsive (divergent) along the discontinuity line $y = 0$. Thus, the effectiveness as well as applicability of Theorem 4 to piecewise continuous nonlinear systems with non-unique solutions is verified again through this numerical example.

### 5.3 The significance of the rigorous treatment of piecewise continuity in the $L_1$ performance analysis

This subsection discusses the significance of rigorous treatment of piecewise continuity in the $L_1$ performance analysis by comparing the two naive treatments that replace the discontinuities with an additive disturbance and a continuous smoothing function, respectively.

For the piecewise continuous system $\Sigma_1$ described by

$$\Sigma_1 : \begin{cases} \dot{x} = -x - \text{sgn}(x) + 1.5w, \\
z = x, \end{cases} \quad (39)$$

we consider replacing $\text{sgn}(\cdot)$ with an additive disturbance $w$ with $\|w\| \leq 1$ by noting the fact that $|\text{sgn}(x)| \leq 1$ for all $x \in \mathbb{R}$. This procedure makes us arrive at the continuous nonlinear system $\bar{\Sigma}_1$ given by

$$\bar{\Sigma}_1 : \begin{cases} \dot{x} = -x + 2.5w, \\
z = x. \end{cases} \quad (40)$$

Here, it readily follows from the essentially equivalent arguments in the preceding subsections that $K_1 := [-0.5, 0.5]$ is an invariant set of $\Sigma_1$ containing the origin $x = 0$. This together with the fact that $K_1 \subseteq \Omega = [-1, 1]$ ensures the $L_1$
performance of $\Sigma_1$. In contrast, the approximated system $\Sigma_1$ does not satisfy the $L_1$ performance since the solution $x(t) = z(t)$ of $\Sigma_1$ with $x(0) = 0$ and $w(t) = 1$ satisfies $x(t) \geq 1.25$, $\forall t \geq \ln 2$. This implies that replacing $\text{sng}(-)$ with a disturbance could lead to a different consequence with respect to the $L_1$ performance.

We next consider the piecewise continuous system $\Sigma_2$ described by

$$
\Sigma_2 : \begin{cases} 
\dot{x} = f_{pc}(x) + 0.1w, \\
z = x + 1,
\end{cases}
$$

(41)

where

$$
f_{pc}(x) := \begin{cases} 
-0.5 + 1.5\text{sng}(x), & \text{if } x \geq -0.5, \\
-4x - 4, & \text{otherwise}.
\end{cases}
$$

(42)

Similarly to the well-known sliding mode control, we replace $\text{sng}(x)$ with the so-called (continuous) saturation function with the thickness $\rho > 0$ defined as

$$
\text{sat}_\rho(x) := \begin{cases} 
\frac{x}{\rho}, & \text{if } |x| \leq \rho, \\
\text{sng}(x), & \text{otherwise}.
\end{cases}
$$

(43)

This makes the discontinuity of $\text{sng}(\cdot)$ at $x = 0$ to be smooth, without affecting the dynamics at the outside of a neighborhood of $x = 0$, that is, $[-\rho, \rho]$. This treatment allows us to obtain the smoothened system $\tilde{\Sigma}_2$ described by

$$
\tilde{\Sigma}_2 : \begin{cases} 
\dot{x} = \tilde{f}_{pc}(x) + 0.1w, \\
z = x + 1,
\end{cases}
$$

(44)

where

$$
\tilde{f}_{pc}(x) := \begin{cases} 
-0.5 + 1.5\text{sat}_\rho(x), & \text{if } x \geq -0.5, \\
-4x - 4, & \text{otherwise}.
\end{cases}
$$

(45)

Note that $\tilde{K}_2 = [-2, 0]$ becomes an extended invariance domain of the approximated system $\tilde{\Sigma}_2$ containing $x = 0$ for any sufficiently small $\rho > 0$. This together with the fact that $\Omega = [-2, 0]$ ensures the $L_1$ performance of $\tilde{\Sigma}_2$ for any $\rho > 0$. However, the original system $\Sigma_2$ does not satisfy the $L_1$ performance since one of its (Filippov) solution $x(t)$ with $x(0) = 0$ and $w(t) = 1$ is described by $x(t) \geq 1.1$ for any $t \geq 1$. Hence, $\|z\|_\infty (= \|x + 1\|_\infty) > 1$. This implies that the $L_1$ performance cannot be ensured for an original piecewise continuous nonlinear system even if its approximated system obtained through a continuous smoothing of a discontinuous vector field is shown to satisfy the $L_1$ performance no matter how the corresponding approximation error is small.

The above interpretations clearly establish the significance of the sophisticated treatments of piecewise continuity proposed in this article on the $L_1$ performance analysis problem.

6 | CONCLUSION

Beyond the $L_1$ performance analysis problem discussed in Reference 10, this article tackled an advanced issue on the $L_1$ performance analysis problem of continuous and piecewise continuous nonlinear systems without assuming solution uniqueness. More precisely, we first provided a sufficient condition for the $L_1$ performance in terms of the invariant set. To establish another framework for the case such that the invariant set-based arguments are not readily applicable due to difficulties of obtaining an analytic representation of the corresponding solutions, the so-called extended invariance domain equipped with external contingent cones was taken. More precisely, it is also not required to obtain explicit solutions in employing the conventional invariance domains but taking the outside dynamics of a given set with respect to the arguments based on the extended invariance domains (as well as external contingent cones) makes us being concerned with a
wider class of nonlinear systems with non-unique solutions. Furthermore, we established parallel results with respect to the $L_1$ performance analysis problem of piecewise continuous nonlinear systems by extending the set-invariance-based arguments used for continuous nonlinear systems. Numerical examples were provided to verify the effectiveness as well as the applicability of the results derived in this article, especially for the possibility of wider applications of the $L_1$ admissible controller synthesis discussed in Reference 10.

On the other hand, it would be worthwhile to note that a slightly modified version of the existing methods for computing invariant sets through the invariance kernel algorithms $^{28,42}$ and the barrier functions $^{20}$ is expected to be directly applied to the arguments developed in this article, although such practical issues on computing invariant sets and (extended) invariance domains do not lie in the interest of this article. Furthermore, the arguments developed in this article are expected to contribute to the $L_1$ controller synthesis for nonlinear systems with non-unique solutions and the $L_1$ performance analysis for positive systems $^{8,9}$ with more sophisticated results. These two issues are left as interesting future studies.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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