Galois actions by finite quantum groupoids

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Abstract

Proposing a certain category of bialgebroid maps we show that the balanced depth 2 extensions appear as they were the finitary Galois extensions in the context of quantum groupoid actions, i.e., actions by finite bialgebroids, weak bialgebras or weak Hopf algebras. We comment on deformation of weak bialgebras, on half grouplike elements, on uniqueness of weak Hopf algebra reconstructions and discuss the example of separable field extensions.

For extensions of rings, algebras or $C^*$-algebras the notion of depth 2, introduced originally for von Neumann factors by A. Ocneanu, has many features that make it the analogue of Galois extension of fields. The extension $N \subset M$ of $k$-algebras is called of depth 2 if the canonical $N$-$M$ bimodule $X = \mathcal{N}M_M$ and $M$-$N$ bimodule $\mathcal{X} = M\mathcal{N}$ satisfy: $X \otimes \mathcal{X} \otimes X$ is a direct summand in a finite direct sum of copies of $X$ and $\mathcal{X} \otimes X \otimes \mathcal{X}$ is a direct summand in a finite direct sum of $\mathcal{X}$'s. The right module $M_N$ is called balanced if $\operatorname{End} \mathcal{E} M \cong N$ where $\mathcal{E} = \operatorname{End} M_N$. For any balanced depth 2 extension the endomorphism ring $A = \operatorname{End} \mathcal{N}M_N$ carries a bialgebroid structure which is finite projective over the centralizer, or relative commutant, $R = C_M(N)$ both as a left and as a right module. Moreover the canonical

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action of $A$ on $M$ makes $M$ to be a left $A$-module algebroid with invariant subalgebra equal to $N$ \([7]\). This generalizes the result of Nikshych and Vainerman in subfactor theory \([12]\) because finite index depth 2 extensions of von Neumann or $C^*$-algebras are always balanced depth 2 extensions. Moreover they are Frobenius extensions which causes the appearence of antipodes, hence leading to weak $C^*$-Hopf algebra structure \([1]\) on $A$.

Our main purpose in this paper will be to study the uniqueness problem of the bialgebroid $A$. Using a kind of category for bialgebroids we show that $A$ satisfies a universal property analogous to the one of the Galois group of a field extension. While the natural action of $A$ on $M$ generalizes Hopf-Galois extensions of Kreimer and Takeuchi \([8]\), Hopf-Galois extensions do not have the universal property w.r.t the category of Hopf algebras as the example of Greither and Pareigis \([4]\) has shown. Our proposal of a Galois bialgebroid $\text{Gal}(M/N)$ of an algebra extension in Section \([2]\) is close in spirit to Pareigis’ Quantum Automorphism Group \([14]\) but technically not a mature one.

In case of depth 2 Frobenius extensions we present an optimistic interpretation of the non-uniqueness of its associated weak Hopf algebra. The natural object to which a unique (measured) weak Hopf algebra can be associated is a Frobenius system \([3]\), i.e., a Frobenius extension $N \subset M$ together with a Frobenius homomorphism $\phi: M_N \to N_N$. Let me recall \([2]\) for the definition of weak bialgebra (WBA) which is one of the main theme in this paper. Let $K$ be a field. A finite dimensional $K$-space $A$ together with a $K$-algebra structure $\langle A, m, u \rangle$ and a $K$-coalgebra structure $\langle A, \Delta, \varepsilon \rangle$ is called a weak bialgebra if

1. $\Delta$ is multiplicative / $m$ is comultiplicative, i.e., as maps $A \otimes A \to A \otimes A$,

$$\Delta \circ m = (m \otimes m) \circ (\text{id} \otimes \Sigma \otimes \text{id}) \circ (\Delta \otimes \Delta)$$

where $\Sigma: A \otimes A \to A \otimes A$ denotes the flip map $x \otimes y \mapsto y \otimes x$,

2. $\varepsilon$ is weakly multiplicative, i.e., as maps $A \otimes A \otimes A \to K$,

$$(\varepsilon \otimes \varepsilon) \circ (m \otimes m) \circ (\text{id} \otimes \Delta \otimes \text{id}) = \varepsilon \circ m \circ (m \otimes \text{id})$$

$$(\varepsilon \otimes \varepsilon) \circ (m \otimes m) \circ (\text{id} \otimes \Delta_{\text{op}} \otimes \text{id}) = \varepsilon \circ m \circ (m \otimes \text{id})$$

where $\Delta_{\text{op}} := \Sigma \circ \Delta$ is opposite comultiplication,

3. $u$ is weakly comultiplicative, i.e., as maps $K \to A \otimes A \otimes A$,

$$(\text{id} \otimes m \otimes \text{id}) \circ (\Delta \otimes \Delta) \circ (u \otimes u) = (\Delta \otimes \text{id}) \circ \Delta \circ u$$

$$(\text{id} \otimes m_{\text{op}} \otimes \text{id}) \circ (\Delta \otimes \Delta) \circ (u \otimes u) = (\Delta \otimes \text{id}) \circ \Delta \circ u$$

where $m_{\text{op}} := m \circ \Sigma$ is opposite multiplication.
Weak bialgebras reduce to ordinary bialgebras iff $\Delta$ is unital. Weak bialgebras have canonical subalgebras $A^L$ and $A^R$ that are spanned by the right leg and left leg of $\Delta(1)$, respectively. $A^L$ belongs to the relative commutant of $A^R$ and there is a canonical antiisomorphism $A^L \to A^R$. The subalgebras $A^L$ and $A^R$ are separable $K$-algebras.

Takeuchi’s $\times_R$-bialgebras [19] or, what is the same [3, 22] bialgebroids [9] are defined over a ground ring $R$ which is not supposed to be separable but plays the role of $A^L$ (or of $A^R$). Indeed weak bialgebras are just the bialgebroids over separable base [16].

The weak bialgebras as well as the bialgebroids we speak about here are finite dimensional over the ground field and finitely generated projective over the base ring, respectively. Briefly saying they are finite quantum groupoids.

1 Weak bialgebras versus bialgebroids

A category of bialgebroids is introduced the arrows of which are called bialgebroid maps. They intend to be special arrows in a possible larger category. They have a special trend to point from bialgebras to bialgebroids but not vice versa. The category of maps of left/right bialgebroids will be denoted by $\text{Bia}_l$, $\text{Bia}_r$, respectively. Using the forgetful functor from weak bialgebras to bialgebroids we obtain weak morphisms of weak bialgebras as lifts of bialgebroid maps. We comment on deformed versions of WBA’s and on half grouplike elements.

1.1 The category of bialgebroid maps

Let $k$ be a commutative ring. All objects and maps below will be $k$-algebras and $k$-algebra maps, respectively. Thus our base category is $k$-$\text{Alg}$.

1.1.1 The objects

Following the original definition [19, 3] its reformulations in [22, 18], and the terminology of [7] we say that $A = (A, R, s, t, \gamma, \pi)$ is a left bialgebroid if

- $R \xrightarrow{s} A \xleftarrow{t} R^{\text{op}}$ are $k$-algebra homomorphisms such that $s(r)t(r) = t'(r)s'(r')$ for $r, r' \in R$. Then $A$ is made into an $R$-$R$-bimodule by setting $r \cdot a \cdot r' := s(r)t(r')a$.

- $\gamma: A \to A \otimes_R A$ and $\pi: A \to R$ are $R$-$R$-bimodule maps such that the triple $(A, \gamma, \pi)$ is a comonoid in the category $R \mathcal{M}_R$. 

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- $\gamma$ is a ring homomorphism into the Takeuchi $\times_R$-product $A \times_R A$, i.e.,

$$\gamma(a)(t(r) \otimes 1) = \gamma(a)(1 \otimes s(r)) \quad (1.1)$$

$$\gamma(a)\gamma(b) = \gamma(ab) \quad (1.2)$$

$$\gamma(1) = 1 \otimes 1 \quad (1.3)$$

for all $a, b \in A$ and $r \in R$.

- $\pi$ is compatible with the algebra structure, i.e.,

$$\pi(as(\pi(b))) = \pi(ab) = \pi(at(\pi(b))), \quad a, b \in A \quad (1.4)$$

$$\pi(1) = 1_R. \quad (1.5)$$

### 1.1.2 The arrows

For two left bialgebroids $A$ and $B$ a pair $\langle \phi, \omega \rangle$ of algebra homomorphisms $\phi: A \to B$ and $\omega: R_A \to R_B$ is called a **map of left bialgebroids** if

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\uparrow{s_A} & & \uparrow{s_B} \\
R_A & \xrightarrow{\omega} & R_B
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\uparrow{s_A} & & \uparrow{s_B} \\
R_A & \xrightarrow{\omega} & R_B
\end{array}
\]

are commutative diagrams in $k$-Alg and

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & \Phi_\omega(B) \\
\downarrow{\pi_A} & & \downarrow{\pi_B} \\
R_A & \xrightarrow{\omega} & R_B
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & \Phi_\omega(B) \\
\downarrow{\pi_A} & & \downarrow{\pi_B} \\
R_A & \xrightarrow{\omega} & R_B
\end{array}
\]

are commutative diagrams in $R_A \mathcal{M}_{R_A}$. Here $\langle \Phi^\omega, \tau^\omega, \omega \rangle$ is the monoidal forgetful functor $R_B \mathcal{M}_{R_B} \to R_A \mathcal{M}_{R_A}$ associated to the algebra homomorphism
That is to say for $\omega: R \to S$

$$\Phi_\omega: \quad S M_S \to R M_R \quad \text{the forgetful functor} \quad (1.9)$$

$$\tau_{X,Y}^\omega: \Phi_\omega(X) \otimes_R \Phi_\omega(Y) \to \Phi_\omega(X \otimes_S Y) \quad \text{the canonical bimodule epimorphism} \quad (1.10)$$

$$\zeta_\omega: \quad R R_R \to R S_R \quad \omega \text{ as a bimodule map.} (1.11)$$

Notice that $\omega$ is uniquely determined by $\varphi$ as $\omega = \pi_B \circ \varphi \circ s_A$.

In order to see that the above properties of $\varphi$ are preserved under composition of such maps, so we indeed have a category, one uses functoriality of $\omega \mapsto \Phi^\omega$, i.e., that in fact the monoidal forgetful functor is the arrow map of a functor $\Phi: k-\text{Alg}^{op} \to \text{MonCat}$. The object map of this functor is

$$R \mapsto \langle R M_R, \otimes_R, R R_R \rangle$$

Then functoriality means the identities

$$\Phi_\omega \circ \Phi_\sigma = \Phi_{\sigma \circ \omega}$$

$$(\Phi_\omega \circ \tau_\sigma) \circ (\tau_\omega \circ (\Phi_\sigma \times \Phi_\sigma)) = \tau_{\sigma \circ \omega}$$

$$\Phi_\omega(\zeta_\sigma) \circ \zeta_\omega = \zeta_{\sigma \circ \omega}$$

for $R \xrightarrow{\omega} S \xrightarrow{\sigma} T$ in $k-\text{Alg}$.

In this way we have constructed a category $\text{Bia}_l$ of left bialgebroids over the base category $k-\text{Alg}$, i.e., the objects are left bialgebroids in $k-\text{Alg}$. In $\text{Bia}_l$ there is no fixed base ring and there are arrows between bialgebroids over different base rings. In particular there are bialgebroid maps from ordinary bialgebras to bialgebroids.

In a similar way one defines the category $\text{Bia}_r$ of right bialgebroids and right bialgebroid maps the details of which we omit.

With the terminology "maps of bialgebroids" we intend to leave place for more general arrows between bialgebroids. Certain bimodules with a coproduct, so bimodule coalgebras, are natural candidates, they allow to formulate Morita equivalence [7] when only the forgetful functor $\mathcal{M}_A \to R M_R$ of the bialgebroid is considered as relevant. However, in the Galois problem of non-commutative rings maps of bialgebroids do play a role as we shall see in Section 3. In other words, $\text{Bia}_l$ is large enough to contain maps from group algebras or bialgebras to bialgebroids but also small enough to contain only very restrictive isomorphisms.
1.2 From weak bialgebras to bialgebroids

Let $\underline{W} = \langle W, \Delta, \varepsilon \rangle$ be a WBA over $K$. Its left and right subalgebras are defined by

\begin{align}
L & = \{ \varepsilon(1_{(1)}w)1_{(2)} | w \in W \} \\
R & = \{ 1_{(1)}\varepsilon(w1_{(2)}) | w \in W \}
\end{align}

and are the images of the maps

\begin{align}
\pi^L : W \to L, & \quad w \mapsto \varepsilon(1_{(1)}w)1_{(2)} \\
\pi^R : W \to R, & \quad w \mapsto 1_{(1)}\varepsilon(w1_{(2)})
\end{align}

For the basic properties of these maps see [2].

Now we introduce data for bialgebroids as follows. Let

\begin{align}
s^L : & \quad L \to W \\
s^R : & \quad R \to W
\end{align}

just the inclusion maps, hence algebra homomorphisms. Let

\begin{align}
t^L : & \quad L^{\text{op}} \to W, \quad l \mapsto 1_{(1)}\varepsilon(1_{(2)}l) \\
t^R : & \quad R^{\text{op}} \to W, \quad r \mapsto \varepsilon(r1_{(1)})1_{(2)}
\end{align}

which are also algebra maps (if antipode exists they are the restrictions of $S^{-1}$). The ranges of $s^L$ and $t^L$ are $L$ and $R$, so they commute. Similarly for $s^R$ and $t^R$. This allows us to introduce bimodule structures $LW_L$ and $RW_R$, respectively, via the formulae

\begin{align}
l \cdot w \cdot l' := s^L(l)t^L(l')w, & \quad l, l' \in L, \ w \in W \\
r \cdot w \cdot r' := wt^R(r)s^R(r'), & \quad r, r' \in R, \ w \in W .
\end{align}

Finally, let

\begin{align}
\tau^L : & \quad W \otimes W \to W \otimes_L W \\
\tau^R : & \quad W \otimes W \to W \otimes_R W
\end{align}

be the canonical epimorphisms associated to the units $K \to L, K \to R$, respectively.

**Lemma 1.1** Let $\underline{W} = \langle W, \Delta, \varepsilon \rangle$ be a WBA/K. Then the maps

\begin{align}
\gamma^L := \tau^L \circ \Delta : & \quad W \to W \otimes_L W \\
\gamma^R := \tau^R \circ \Delta : & \quad W \to W \otimes_R W
\end{align}
are such that
\[
\beta_l(W) := \langle W, L, s^L, t^L, \gamma^L, \pi^L \rangle \text{ is a left bialgebroid} \tag{1.26}
\]
\[
\beta_r(W) := \langle W, R, s^R, t^R, \gamma^R, \pi^R \rangle \text{ is a right bialgebroid.} \tag{1.27}
\]

**Proof** First notice that \( \pi^L \circ s^L = \text{id}_L \) and \( \pi^L \circ t^L = \text{id}_L \) by \[2\] (2.3a), (2.25a). Also using \[2\] Lemma 2.5
\[
\pi^L(l \cdot w \cdot l') = \pi^L(l^t(l')w) = l\pi^L(w)(\pi^L(t^L(l')) = l\pi^L(w)\pi^L(t^L(l'))
\]

therefore \( \pi^L \) is an \( L-L \) bimodule map. Now \( \langle L W, \gamma^L, \pi^L \rangle \) is a comonoid in \( L M_L \) because \( \tau^L \) is natural and \( \tau^L(w(1))w(2) = w \) and \( t^L \circ \pi^L(w(2))w(1) = w \) are general WBA identities. Coassociativity follows using that \( \tau^L \) extends to a natural transformation \([1.10]\), namely, \( \tau^L = \tau^{uL}_{W,W} \) where \( u_L: K \to L \) is the unit of the \( K \)-algebra \( L \), and the latter satisfies the hexagon diagram of a (lax) monoidal functor. In order to show that the image of \( \gamma^L \) is in \( W \times_L W \) it suffices to refer to the old WBA identity \( 1(1) \otimes 1(2) = 1(1) t^L(l) \otimes 1(2) \) (cf. \[2\] (2.31a)). Multiplicativity of \( \gamma^L \) then follows from multiplicativity of \( \Delta \). It remains to show the counit properties \([1.4]\) but they are just the identities \([2\] (2.5a),(2.25a)]. Passing to the opposite-coopposite WBA one obtains the statement for \( \beta_r \).

The \( \beta_l \) and \( \beta_r \) defined by the Lemma are expected to be the object maps of two functors
\[
\text{Bial} \leftarrow WBA \xrightarrow{\beta_l} \text{Bial} \xrightarrow{\beta_r} \text{Bial} \tag{1.28}
\]
the arrows of WBA, however, will be discussed later.

### 1.3 From bialgebroids over separable base to weak bialgebras

Since the left/right subalgebras of a WBA are always separable, we start from a left bialgebroid \( B = \langle B, R, s, t, \gamma, \pi \rangle \) in which \( R \) is a separable \( K \)-algebra. That is to say there exists an element \( e = \sum_i e_i \otimes e^i \in R \otimes K R \) such that
\[
\sum_i e_i e^i = 1_R \quad \text{and} \quad \sum_i r e_i \otimes e^i = \sum_i e_i \otimes e^i r \quad r \in R. \tag{1.29}
\]

Such a separability idempotent provides a splitting map for the canonical epimorphism \( \tau \), namely
\[
\sigma: B \otimes_R B \to B \otimes_K B, \quad b \otimes b' \mapsto \sum_i b \cdot e_i \otimes e^i \cdot b' \tag{1.30}
\]
This formula is the same for right bialgebroids. For left bialgebroids we can write also
\[ \sigma(b \otimes_R b') = \sum_i t(e_i)b \otimes s(e^i)b'. \] (1.31)

But there is more than separability of \( R \) in a WBA. There is also a separability structure for \( R \). For any weak bialgebra with left subalgebra \( R \) the restriction of the counit \( \psi := \varepsilon|_R \) is a nondegenerate functional of index one. This means that \( \psi \) distinguishes a special separability idempotent, namely \( e = S(1_{(1)}) \otimes 1_{(2)} \) which is the quasibasis of \( \psi \). Comparing this with the above expression for \( \sigma \) one recognizes that \( \sigma \) is multiplication from the left by \( \Delta(1) \) on any element from the inverse image \( \tau^{-1}(\{b \otimes_R b'\}) \).

**Lemma 1.2** Given a pair \( \langle B, \psi \rangle \), where \( B \) is a left or right bialgebroid over \( R \) and \( \psi: R \to K \) is a nondegenerate functional of index 1, define
\[ \Delta := \sigma \circ \gamma: B \to B \otimes_K B \] (1.32)
\[ \varepsilon := \psi \circ \pi: B \to K \] (1.33)
where \( \sigma \) is the splitting map of the canonical epimorphism \( B \otimes_K B \to B \otimes_R B \) that is associated to the quasibasis \( e \) of \( \psi \) as in (1.30). Then the triple \( \langle B, \Delta, \varepsilon \rangle \) is a WBA over \( K \).

**Proof:** Mutatis mutandis, the proof has already been given in [7, Proposition 9.4] and in [16, Theorem 5.5].

The above Lemma characterizes the fibres of the functor \( \beta_l \) in the following sense. The WBA’s \( W \) with a fixed underlying (let’s say left) bialgebroid \( \beta_l(W) = B \) are in one-to-one correspondence with separability structures \( \langle R, \psi, e \rangle \) on \( R \).

### 1.4 Strict and weak morphisms of weak bialgebras

**Definition 1.3** Let \( \langle W, \Delta, \varepsilon \rangle \) and \( \langle W', \Delta', \varepsilon' \rangle \) be weak bialgebras over \( K \). Then a \( K \)-linear map \( f: W \to W' \) is called a

- **strict morphism** of weak bialgebras if \( f \) is an algebra map and a coalgebra map;
- **weak left morphism** of weak bialgebras if \( f: \beta_l(W) \to \beta_l(W') \) is a map of the underlying left bialgebroids;
- **weak right morphism** of weak bialgebras if \( f: \beta_r(W) \to \beta_r(W') \) is a map of the underlying right bialgebroids.
A strict morphism \( f \) not only preserves the left and right subalgebras, \( f(L) \subset L' \) and \( f(R) \subset R' \), but, because \( (f \otimes f) \circ \Delta(1) = \Delta'(1') \), establishes isomorphisms \( L \xrightarrow{\sim} L' \) and \( R \xrightarrow{\sim} R' \). Therefore strict morphisms exist between two WBA’s only if they have isomorphic left, resp. right subalgebras. This is definitely too strong since the original philosophy of [1] was to “blow up” Hopf algebras in order they could afford non-integral categorical dimensions, but the amount of the blowing up, i.e., the size of the left/right subalgebras should be considered as a gauge degree of freedom. Using weak morphisms we pursue this idea to some extent.

Since weak morphisms are just lifts of the rather involved bialgebroid maps into the WBA framework, they are useful only if they can be recognized directly without reference to bialgebroids. Therefore we make the

**Proposition 1.4** For weak bialgebras \( W \) and \( W' \) a \( K \)-linear map \( f: W \rightarrow W' \) is a weak left morphism iff

1. \( f \) is a \( K \)-algebra map,
2. \( f(R) \subset R' \),
3. \( \pi^L \circ f = f \circ \pi^L \),
4. and \( \Delta'(1') \cdot (f \otimes f)(\Delta(w)) = \Delta'(f(w)), \ w \in W \).

It is a weak right morphism iff

1. \( f \) is a \( K \)-algebra map,
2. \( f(L) \subset L' \),
3. \( \pi^R \circ f = f \circ \pi^R \),
4. and \( (f \otimes f)(\Delta(w)) \cdot \Delta'(1') = \Delta'(f(w)), \ w \in W \).

**Proof:** It suffices to prove the statement for left morphisms. Since \( s^L \) is just the injection \( L \subset W \) and so is \( s^L' \), the first diagram in (1.6) is equivalent to \( f(L) \subset L' \), which in turn is a consequence of (1.7), which is nothing but condition 1 above. Having condition 3 anyway the second diagram of (1.6) is equivalent to the condition 2 because if 2 holds then \( f \circ \tau^L = \tau^L \circ \pi^L \circ f \circ \tau^L = \tau^L \circ f \circ \pi^L \circ \tau^L = \tau^L \circ f \) and backwards is obvious. This proves that the three diagrams of (1.6) and (1.7) are equivalent to 1, 2, and 3. Assuming this we can equip \( W' \) with \( L-L \) bimodule structure \( W' = \Phi_\omega(L \cdot W'_L) \) and lift \( f \) to an \( L-L \) bimodule map \( \hat{f} \). Then (1.8) takes the form

\[
\tau^\omega \circ (\hat{f} \otimes_L \hat{f}) \circ \gamma = \gamma' \circ \hat{f}.
\]
Let $\tau'$ be the canonical epi for $W'$ and $\sigma'$ be its splitting map associated to the quasibasis $e'$ of $\varepsilon'|_L$. Let $\hat{\tau}$ be the canonical epi for $\hat{W}'$ and $\hat{\sigma}$ its splitting map that is associated to $e$, or to $(f \otimes f)(e)$ in some (bad) sense. Then we have $\tau' \circ \hat{\sigma} = \hat{\tau}'$ and using the observation we made just before Lemma 1.2 we can write for all $w'_1 \otimes w'_2 \in \hat{W}' \otimes L \hat{W}'$ that

$$\sigma' \circ \tau' (w'_1 \otimes w'_2) = \sigma' \circ \tau' \circ \hat{\sigma}(w'_1 \otimes w'_2) = 1'_1 f(1(1)) w'_1 \otimes 1'_2 f(1(2)) w'_2.$$ 

Now acting by $\sigma'$ on (1.34) we obtain 4 and acting by $\tau'$ on 4 we obtain (1.34).

For weak Hopf algebras one defines weak left/right morphisms as those of its underlying weak bialgebra, disregarding whether they preserve antipodes or not.

The category of bialgebras, as well as the category of Hopf algebras, are full subcategories in each one of $\mathbf{Bial}, \mathbf{WBA}$ and $\mathbf{Bia}_r$.

**Example 1.5** Let $H$ be a Hopf algebra over $K$. Define its blowing up as the algebra $W := H \otimes M_n(K)$ with comultiplication $\Delta(h \otimes e_{ij}) := (h_{(1)} \otimes e_{ij}) \otimes (h_{(2)} \otimes e_{ij})$. Then $W$ becomes a weak Hopf algebra. Its left and right subalgebras coincide and equal to the diagonal matrices with entries from $K$. The diagonal embedding of $H$, $f : H \to W$, $f(h) = h \otimes I_n$, is clearly an algebra map. It is not a coalgebra map however, but we have

$$\Delta(f(h)) = \sum_i (h_{(1)} \otimes e_{ii}) \otimes (h_{(2)} \otimes e_{ii}) = \Delta(1_W)(f \otimes f)(\Delta_H(h)) = (f \otimes f)(\Delta_H(h)) \Delta(1_W)$$

$$\pi^l(f(h)) = f(1_H)\varepsilon_H(h) = \pi^H(f(h)).$$

Now using the above Proposition it is plain that $f$ is weak left and right morphism of weak bialgebras and there is no strict morphism from $H$ to $W$ unless $n = 1$.

### 1.5 Weak automorphisms, twists and half grouplike elements

Let $f : W \to W'$ be a weak left morphism of WBA’s. Then $\varepsilon'(f(w)) = \varepsilon'(\pi^L(f(w))) = \varepsilon'(f(\pi^L(w)) = \varepsilon(1(1)w)\varepsilon'(f(1(2)))$ therefore

$$\varepsilon'(f(w)) = \varepsilon(uw) , \quad w \in W , \quad \text{where } u = \varepsilon'(f(1(1))) 1(2) \in L \quad (1.35)$$

where, in order to get $u \in L$, we also made the $R \to L$ transformation $t^L(1) \varepsilon'(f(1(1))) = \varepsilon'(f(\pi^L(1))) 1(2) = u$. Especially we have $\varepsilon'(f(l)) =$
ε(ul) for l ∈ L. So assuming that f|L is an isomorphism onto L′ we have u as a Radon-Nikodym derivative of a nondegenerate functional w.r.t another, hence invertible. Comparing their quasibases we obtain the equality
\[ \omega^{-1}(\pi^L(1'(1))) \otimes \omega^{-1}(1'(2)) = \pi^L(1(1)) \otimes u^{-1}1(2) \]  
(1.36)
as elements of L ⊗ L. Applying f ⊗ f and using that π^L restricts to an isomorphism R → L (the would-be antipode), we get
\[ 1'(1) \otimes 1'(2) = f(1(1)) \otimes f(u^{-1})f(1(2)) . \]  
(1.37)
Inserting this result to the 4th property of weak left morphisms in Proposition 1.4 one immediately arrives to the

**Lemma 1.6** Let f: W → W′ be a weak left morphism of WBA’s such that its restriction ω: L → L′ is an isomorphism. Then there is an invertible element u ∈ L such that
\[ \Delta'(f(w)) = (f \otimes f)((1 \otimes u^{-1})\Delta(w)) \]  
(1.38)
\[ \varepsilon'(f(w)) = \varepsilon(uw) \]  
(1.39)
for all w ∈ W.

This result holds in particular if f is an isomorphism. As a matter of fact property 3 in Proposition 1.4 is invariant under changing f to f^{-1}. For completeness we remark that the forgetful functor Bia_l → k-Alg reflects isomorphisms. That is to say, if a weak left morphism is invertible as an algebra map then its inverse is a weak left morphism. So the Lemma holds for f = id_W. In this case u describes a deformation in the sense of [11, Remark 3.7]. Such (left) deformed WBA’s have identical underlying left bialgebroids, so deformations should be interpreted as weak left automorphisms. Although the deformation changes the Nakayama automorphism of the counit, there may be no deformation at all which produces a tracial ε′, unless the base L possesses a nondegenerate trace of index 1. For example if L is split semisimple then the only such trace is the regular trace. Since the Radon-Nikodym derivative of the regular trace w.r.t. ε|L is 1(1)_2S(1(1)), tracial deformation exists iff 1(1)_2S(1(1)) is invertible. In characteristic zero this is always the case, otherwise there are counter examples [21].

Now consider inner weak left automorphisms associated to left grouplike elements. For a bialgebroid (B, L, s, t, γ, π) an element g ∈ B is grouplike if g is invertible and γ(g) = g ⊗ g. For a WBA W define
\[ \mathcal{G}^L(W) := \{ g \in W \mid g \text{ is grouplike in } \beta_{l}(W) \} \]  
(1.40)
\[ \mathcal{G}^R(W) := \{ g \in W \mid g \text{ is grouplike in } \beta_{r}(W) \} \]  
(1.41)
the sets of left/right grouplike elements. This is of course equivalent to saying e.g. that $g$ is left grouplike if it is invertible and $\Delta(g) = \Delta(1)(g \otimes g)$. In the next computations we assume that $g, h \in G^L(W)$ and $w \in W$ is arbitrary.

$$\pi^L(g) = \varepsilon(1(1)g)1(2) = \varepsilon(g(1))g(2)g^{-1} = gg^{-1} = 1 \quad (1.42)$$

$$\Delta(g^{-1}) = \Delta(g^{-1})\Delta(1) = \Delta(g^{-1})\Delta(g)(g^{-1} \otimes g^{-1})$$

$$= \Delta(1)((g^{-1} \otimes g^{-1}) \quad (1.43)$$

$$\Delta(gh) = \Delta(g)\Delta(1)(h \otimes h) = \Delta(h \otimes h) = \Delta(1)(gh \otimes gh) \quad (1.44)$$

$$\pi^L(gwg^{-1}) = \pi^L(gw\pi^L(g^{-1})) = \pi^L(gw) = \varepsilon(1(1)g\pi^L(w))1(2)$$

$$= \varepsilon(g(1)\pi^L(w))g(2)g^{-1} = g\pi^L(w)g^{-1} \quad (1.45)$$

$$t^L \circ \pi^L(gwg^{-1}) = 1(1)\varepsilon(1(2)gwg^{-1}) = g(1)g^{-1} \varepsilon(g(2)\pi^L(w))$$

$$= g(t^L \circ \pi^L(w))g^{-1} \quad (1.46)$$

$$\Delta(1)(gw(1)g^{-1} \otimes gw(2)g^{-1}) = g(1)w(1)g^{-1} \otimes g(2)w(2)g^{-1}$$

$$= g(1)w(1)g^{-1}_{(1)} \otimes g(2)w(2)g^{-1}_{(2)} = \Delta(gwg^{-1}) \quad (1.47)$$

This shows that $G^L$ is a group and for all $g \in G^L$ the inner automorphism $w \mapsto gwg^{-1}$ is a weak left automorphism $W \to W$. It does not leave the counit invariant but

$$\varepsilon(gwg^{-1}) = \varepsilon(uw) \quad w \in W, \quad \text{where } u = \varepsilon(g1(1))1(2) \quad (1.48)$$

implying for their quasibasis the relation

$$g^{-1}\pi^L(1(1))g \otimes g^{-1}1(2)g = \pi^L(1(1)) \otimes u^{-1}1(2).$$

Applying $t^L \otimes id$ and using that $\text{Ad}_{g^{-1}}$ commutes with $\pi^L$ and $t^L \circ \pi^L$ one obtains

$$g^{-1}1(1)g \otimes g^{-1}1(2)g = 1(2) \otimes u^{-1}1(2)$$

therefore

$$\Delta(g) = (g \otimes gw^{-1})\Delta(1) = (g\pi^L(u^{-1}) \otimes g)\Delta(1). \quad (1.49)$$

Since $\pi^R$ on $L$ is an algebra antiisomorphism and $\pi^R(u) = 1(1)\varepsilon(gs^{-1}(1(2))) = \pi^R(g)$, the above definition of $G^L$ is equivalent to the one given by Vecsényes [21].

2 Galois quantum groupoids

In this section we argue that the balanced depth 2 extensions [4] of rings or $k$-algebras are the proper analogues of the Galois extensions of fields (i.e.,
normal and separable field extensions) because they have finite quantum automorphism groups (cf. [14]) with invariant subalgebra just \( N \) and which are characterized by a universal property, hence unique. The role of finite groups are played by bialgebroids, i.e., \( \times_R \)-bialgebras, that are finitely generated projective over their base as a left and as a right module. They will be called finite bialgebroids. They are presumably also Hopf algebroids but the antipode raises several questions, so we skip their discussion altogether. The difference between Galois bialgebroid and Galois WBA will be found in the difference between depth 2 Frobenius extensions and depth 2 extensions with a Frobenius structure.

2.1 Quantum automorphisms

Recall the definition of left module algebroids over a left bialgebroid \( A \) in [7]. They are the monoids in the category of left \( A \)-modules.

**Definition 2.1** Let \( N \rightarrow M \) be an extension of \( k \)-algebras. Define the category \( \text{Aut}(M/N) \) as follows. Its objects are the pairs \( \langle B, \alpha_B \rangle \) where \( B \) is a finite left bialgebroid in \( k \text{-Alg} \) and \( \alpha_B: B \otimes M \rightarrow M \) is a left \( B \)-module algebroid action such that \( N \) is contained in the invariant subalgebra \( M^B \). The arrows from \( \langle B, \alpha_B \rangle \) to \( \langle C, \alpha_C \rangle \) are the maps \( f: B \rightarrow C \) of left bialgebroids such that

\[
\alpha_C \circ (f \otimes \text{id}_M) = \alpha_B.
\]

A terminal object in \( \text{Aut}(M/N) \) is called a **universal action** on the extension \( M/N \). The bialgebroid \( A \), unique up to isomorphism in \( \text{Bia}_k \), in a universal action is called the **Galois quantum groupoid** of the extension \( M/N \) and is denoted by \( \text{Gal}(M/N) \). If the invariant subalgebra of the universal action is equal to \( N \) the extension \( N \rightarrow M \) is called a **Galois extension**.

If \( F \subset E \) is a field extension then every finite group action on \( E \) which leaves \( F \) pointwise fixed factors uniquely through the Galois group \( \text{Gal}(E/F) \). This trivial fact is generalized by the above definition. Also, if the fixed points of \( \text{Gal}(E/F) \) coincide with the elements of \( F \) the extension is normal and separable, i.e., Galois, by Artin’s Theorem.

Note that the real beauty of universal monoids of [14] has not been used in the above definition. One could consider much more general arrows \( \alpha: B \otimes M \rightarrow M \) than just actions.

If \( H \) is a Hopf algebra, f.g.p. over \( k \) then Kreimer and Takeuchi defines an \( H \)-Galois extension to be a ring extension \( N \subset M \) such that
• there is a left $H$-module algebra action $\alpha: H \otimes M \to M$,

• $N = M^H$, the invariant subring,

• $M_N$ is finitely generated projective and

• the map

$$\phi: M \otimes H \to \text{End}M_N, \quad m \otimes h \mapsto \{m' \mapsto m\alpha(h \otimes m')\} \quad (2.1)$$

is an isomorphism.

(More precisely, this is a reformulation by Ulbrich [2].)

Hopf-Galois extensions in this sense, however, do not have the universal property with respect to the category of Hopf algebras. As it was pointed out by Greither and Pareigis in [4] there are separable field extensions which are $H$-Galois for two different Hopf algebras. We come back to this example in the last section.

### 2.2 Universal bialgebroid actions

The advantage of using bialgebroids is that there is a very general class of ring extensions for which a universal bialgebroid action exists. These are the depth 2 extensions $N \subset M$ for which $M_N$ is balanced. They include all ring extensions that are $H$-Galois for some Hopf algebra, as it was shown by Kadison recently [3], but many more. The universal bialgebroid of a depth 2 balanced ring extension is a canonical structure on the endomorphism ring $A = \text{End}_N M_N$ and it has been constructed in [3] although its universality was not formulated there. Below we shall give a proof for the special case of separable centralizer which leads us to weak bialgebra actions as follows.

If $W$ is a weak bialgebra over $K$ and $B = \beta_l(W)$ its underlying left bialgebroid then the category of left $W$-modules and the category of left $B$-modules are monoidally equivalent [10], Proposition 5.3, in fact isomorphic. Therefore these categories have the same monoids. Therefore a module algebra over $W$ is the same as a module algebroid over $B$. This lends to a WBA action $\alpha: B \otimes M \to M$ the name **weak Galois action** if it has $N$ as its invariant subalgebra and if it has the universal property w.r.t. weak left morphisms of WBA’s.

**Theorem 2.2** Let $K$ be a field and $N \subset M$ a $K$ algebra extension such that

• $N \subset M$ is of depth 2,
• $M_N$ is balanced,
• $R := C_M(N)$ is a separable $K$-algebra.

Then the bialgebroid $A = \text{End}_N M_N$ constructed in [3] and acting on $M$ in the natural way is the Galois bialgebroid $\text{Gal}(M/N)$. That is to say, any weak bialgebra $(A, \Delta_A, \varepsilon_A)$ with underlying left bialgebroid $A$ has the following universal property. If $\alpha_W : W \otimes M \rightarrow M$ is a left module algebra action of a WBA $W$ such that $M^W \supset N$ then there is a unique weak left morphism $f : W \rightarrow A$ of weak bialgebras such that $\alpha_W \circ (f \otimes \text{id}_M) = \alpha_W$.

Proof: That the invariant subalgebra is $N$ was shown in [7]. To prove the universal property notice that for any weak bialgebra action on $M$

$$w \triangleright (nmn') = n(w \triangleright m)n', \quad w \in W, \; n, n' \in M^W, \; m \in M, \quad (2.2)$$

in particular for all $n, n' \in N$. Thus there is a unique algebra map $f : W \rightarrow A$ such that $f(w)(m) = w \triangleright m$.

Already this implies uniqueness so we are left to show that $f$ is a weak left morphism. We will use the criteria given in Proposition 1.4. At first compute the action of an $l \in W_L$.

$$l \triangleright m = (l_{(1)} \triangleright 1_M)(l_{(2)} \triangleright m) = 1 \triangleright ((l \triangleright 1_M)m) = (l \triangleright 1_M)m$$

Since $(l \triangleright 1_M)n = l \triangleright n = n(l \triangleright 1_M)$ for $n \in N$, $W^L \triangleright 1_M \subset C_M(N) = R$. Hence $f(W^L) \subset \lambda_M(R) = A^L$, where $\lambda_M(m)$ denotes left multiplication by $m$ on $M$. For $r \in W^R$ we have

$$r \triangleright m = (r_{(1)} \triangleright m)(r_{(2)} \triangleright 1_M) = 1 \triangleright (m(r \triangleright 1_M)) = m(r \triangleright 1_M) = m(\pi^L_W(r) \triangleright 1_M)$$

Since $A^R = \rho_M(R)$ where $\rho_M(m)$ is right multiplication by $m$ on $M$, we obtain $f(W^R) \subset \rho_M(R) = A^R$. Next recall that the counit of the left bialgebroid $A$ is $\pi^L_A(a) = \lambda_M(a(1_M))$. Therefore

$$\pi^L_A(f(w)) = \lambda_M(f(w)(1_M)) = \lambda_M(w \triangleright 1_M) = \lambda_M(\pi^L_W \triangleright 1_M) = f(\pi^L_W(w))$$

Turning to the last condition of Proposition 1.4 we recall [4], Prop. 3.9 stating that

$$A \otimes_R A \simto \text{Hom}_{N-N}(M \otimes_N M, M), \quad (a \otimes a')(m \otimes m') = a(m)a'(m').$$
Now consider the composite $K$-linear maps

$$
W \stackrel{\Delta}{\rightarrow} W \otimes W \stackrel{f \otimes f}{\rightarrow} A \otimes A \stackrel{\tau}{\rightarrow} A \otimes_R A
$$

and

$$
W \stackrel{f}{\rightarrow} A \stackrel{\gamma}{\rightarrow} A \otimes_R A
$$

which are equal because their images act on $M \otimes_N M$ in the same way due to the module algebra property. So, composing them with the splitting map $\sigma$ associated to $\Delta_A(1_A)$ and using $\sigma \circ \tau = \lambda_{A \otimes A}(\Delta_A(1_A))$ we obtain

$$
\Delta_A(1_A) \cdot (f \otimes f)(\Delta_W(w)) = \Delta_A(f(w)) , \quad w \in W .
$$

q.e.d.

2.3 Universal weak Hopf algebra actions

If we add to the conditions of Theorem 2.2 that $N \subset M$ is a Frobenius extension then it already implies that the WBA lift of its Galois bialgebroid is a WHA [7, Section 9]. This does not make the WHA unique, only up to weak left isomorphisms. This freedom of the WHA is precisely the freedom of choosing a Frobenius functional $\psi: R \rightarrow K$ of index 1. Therefore it is natural to associate WHA actions to Frobenius structures $\langle N \rightarrow M, \phi: N M_N \rightarrow N N_N \rangle$ rather than to just extensions $N \rightarrow M$.

If $\phi: N M_N \rightarrow N N_N$ is a Frobenius map, i.e., a bimodule map with quasibasis $\sum_i m_i \otimes m^i \in M \otimes_N M$, then its restriction to the centralizer $\phi|_R$ maps $R$ into the center $Z$ of $N$. Since $R$ is not only part of $M$ but belongs to the bialgebroid $A$ as well, it is very natural to build $\phi|_R$ into the data of the WHA as $\varepsilon|_R$. Strictly speaking, this is possible only if the center of $N$ is trivial. There is a tiny point here about the restriction. While in case of finite index $C^*$-algebra extensions one considers faithful conditional expectations $\phi$ which have faithful restrictions to the finite dimensional $R$, therefore $\phi|_R$ is a Frobenius map with invertible index, this is not automatic for general Frobenius algebra extensions.

**Theorem 2.3** Let $N \subset M$ be a depth 2 Frobenius extension of $K$-algebras with centralizer $R$ a separable $K$-algebra and with Center$N = K$. Assume $\phi: M \rightarrow N$ is a Frobenius map with its restriction $\phi|_R$ being an index 1 Frobenius map. Then there exists a unique weak Hopf algebra $A$ and a left module algebra action of $A$ on $M$ which satisfies the universal property of Theorem 2.2 and such that $\phi|_R = \varepsilon|_R$. 

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Proof. The antipode of a WHA is unique therefore uniqueness of \( A \) follows if we show that its WBA structure is unique. The latter is uniquely determined by its underlying left bialgebroid \( \beta_l(A) \) and by the restriction of its counit, \( \varepsilon|_R \). The former is uniquely determined by the universal property as \( \beta_l(A) = \text{Gal}(M/N) \) by Theorem 2.2 and the latter by the requirement \( \varepsilon|_R = \phi|_R \). This proves uniqueness. The existence part is an easy application of Theorem 9.5 of [7].

The question arises how to interpret \( \phi \) if only its restriction to the centralizer matters. Since \( \phi \) is an \( N\)-\( N \) bimodule map, it belongs to \( A \) as a nondegenerate left integral. Thus in fact the data of the Theorem determine a measurable quantum groupoid, i.e., a WHA with a distinguished nondegenerate integral.

Generalizations to Center \( N = Z \) a separable \( K \)-algebra is possible. It requires to use a slight generalization of the notion of a WHA. It requires WHA’s not in \( k\text{-Alg} \) but in \( Z\mathcal{M}_Z \), cf. [18, Proposition 1.6].

In addition to the assumptions of Theorem 2.3 let us assume that \( \phi(1_M) \) is invertible or only assume that \( M/N \) is split. Then \( M_N \) is balanced therefore \( M/N \) is a Galois extension in the sense of Definition 2.1.

3 Separable field extensions are weak Hopf Galois

Let \( E|K \) be a separable field extension. Then the following results are standard.

1. \( E_K \) is finite dimensional, \( \dim E_K = n < \infty \).

2. Let \( \tau: E \rightarrow K \) be the trace associated to the regular \( E \)-module \( _EE \).

Then there exists \( x_i, y_i \in E \), \( i = 1, \ldots, n \) such that

\[
\sum_i x_i \tau(y_i x) = x, \quad \forall x \in E
\]

(3.1) \[
\sum_i x_i y_i = 1.
\]

(3.2)

3. The above set \( \{x_i, y_i\} \), called a quasibasis for \( \tau \), satisfies

\[
\sum_i x x_i \otimes y_i = \sum_i x_i \otimes y_i x, \quad \forall x \in E.
\]

4. There exists a \( \xi \in E \) which generates \( E \) as an \( K \)-algebra, i.e., \( E = K(\xi) \). (Primitive Element Theorem)
5. The non-zero $K$-algebra endomorphisms of $E$ are automorphisms. Their group $G$ forms an $K$-linearly independent set in the $K$-algebra of $K$-linear endomorphisms $\text{End}_K E$ of the $K$-module $E_K$. The $G$-invariants $F = E^G$ form a subfield of $E$ and $E|F$ is (classically) Galois with Galois group $G$. Hence $|G| = \dim_F E = n/m$ where $m = \dim_K F$.

3.1 The universal weak Hopf algebra of $E/K$

Define $A$ as the $K$-algebra $\text{End}_K E$ and its weak Hopf algebra structure by

$$\Delta_A(a) = \sum_i \sum_j x_i \tau(x_j \_ \_) \otimes y_j a(y_j \_ \_), \quad (3.3)$$

$$\varepsilon_A(a) = \tau(a(1)), \quad (3.4)$$

$$S_A(a) = \sum_i x_i \tau(a(y_i \_ \_)), \quad (3.5)$$

The WHA $A$ is a very special one:

1. The left and right subalgebras coincide with $E$. As a matter of fact, identifying $E$ with the subalgebra of $A$ of (left) multiplications on $E$

$$\pi^L(a) = a(1), \quad \pi^R(a) = \sum_i x_i \tau(a(y_i)))$$

thus $A^L = A^R = E$.

2. The antipode is involutive, $S_A^2 = \text{id}_A$. What is more, it is transposition w.r.t. the nondegenerate bilinear form on $E \otimes_K E$ given by $\tau$, i.e.,

$$\tau(xa(y)) = \tau(S_A(a)(x)y), \quad x, y \in E.$$

3. $A$ is cocommutative, $a_{(1)} \otimes a_{(2)} = a_{(2)} \otimes a_{(1)}$ as elements of $A \otimes_K A$, holds for all $a \in A$ as a consequence of commutativity of $E$ on the one hand and of the existence of an isomorphism $A \otimes_K A \cong \text{End}_K(E \otimes_K E)$, i.e., finite dimensionality of $E_K$ on the other hand.

4. Left integrals in $A$ are the endomorphisms $l$ of $E_K$ such that $l(E) \subset K$. Their general form is $l = \tau(r \_ \_)$ where $r \in E$. Normalized left integrals thus exist ($\Rightarrow A$ is a separable $K$-algebra) and the invariant subalgebra of $E$ is $K$.

5. $\tau$ is a 2-sided nondegenerate integral. If $n$ is invertible in $K$, especially in characteristic 0, then $\tau/n$ is a Haar integral in $A$. 

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6. The left grouplike elements of $A$ are precisely the algebra automorphisms of $E_K$. Thus the number $|G^L|$ of left grouplike elements is a divisor of $n$ and is equal to $n$ precisely if $E/K$ is classically Galois. In the latter case $A$ is a crossed product of $E$ with the group algebra $K G^L$.

7. The left $A$-module algebra $A^L$, i.e., the trivial $A$-module, coincides with $E$ with its canonical left $A$-module structure. That is to say,

$$a(x) = ax(1) = a_{(1)} x(1 \varepsilon_A(a_{(2)})) = a_{(1)} x S_A(a_{(2)})(1) = \pi^L(ax)(1)$$

for all $a \in A$ and $x \in E$.

8. The smash product $E \rtimes A$ is isomorphic to $A$ as $K$-algebras via the canonical map $x \rtimes a \mapsto \{y \mapsto xa(y)\}$.

### 3.2 Weak Hopf Galois extensions

As an immediate generalization of Hopf-Galois extensions one can make the

**Definition 3.1** Let $W$ be a weak Hopf algebra over $K$. A finite field extension $E/K$ is called $W$-Galois if there exists a weak Hopf module algebra action $\alpha: W \otimes_K E \to E$ such that the map

$$\Phi: E \otimes_L W \to \text{End}_K E, \quad x \otimes w \mapsto \{y \mapsto x \alpha(w \otimes y)\}$$

where $L$ stands for $W^L$, is an isomorphism.

Let $E/K$ be a finite extension which is $W$-Galois. Below we shall write $w \triangleright x$ for $\alpha(w \otimes x)$, $w \in W$, $x \in E$.

1. The map $\Phi$ in the above Definition is an $K$-algebra isomorphism from the smash product $E \rtimes W$ to $A$. As a matter of fact, the underlying $K$-space of the smash product is the tensor product $E \otimes_L W$ of $L$ modules where $E_L$ is defined by $x \cdot l := x(l \triangleright 1)$. So the definition of $W$-Galois extension just claims that the smash product is isomorphic to $A$ as an $K$-space. This map is an algebra map as it is obvious from the multiplication rule of the smash product. Let $\varphi: W \to A$ denote the restriction of this map.
2. The restriction of the $K$-algebra monomorphism $\varphi: W \to A$ to $L = W^L$ is $\varphi(l): x \mapsto (l \triangleright 1)x$. Therefore $\varphi(L) \subset E$ and therefore $\varphi$ identifies $L$ with an intermediate field $K \subset L \subset E$.

3. Let $r \in W^R$. Then $r \triangleright x = x(r \triangleright 1) = (S_W(r) \triangleright 1)x$. Therefore $\varphi(r) = \varphi(S_W(r)) \in L$ which, using injectivity of $\varphi$, implies that $W^L = W^R = L$ and $S_W$ acts as the identity on $L$.

4. It follows that $\Delta_W(1_W)$ is a separating idempotent for the separable algebra $L$ over $K$. By commutativity of $E$ it contains $\Delta_A(1_A)$ as a subprojection, i.e.,

$$(\varphi \otimes \varphi)(\Delta_W(1_W)) \Delta_A(1_A) = \Delta_A(1_A)(\varphi \otimes \varphi)(\Delta_W(1_W)) = \Delta_A(1_A)$$

5. $\varphi$ is a weak (two sided) morphism of weak bialgebras, i.e.,

$$\Delta_A(1_A)(\varphi \otimes \varphi)(\Delta_A(w)) = \Delta_A(\varphi(w)), \quad w \in W. \quad (3.6)$$

This can be seen as follows. Upon identifying $A \otimes_K A$ with $\text{End}_K(E \otimes_K E)$ the module algebra property of $W E$ boils down to

$$\tau \circ (\varphi \otimes \varphi)(\Delta_A(w)) = \tau \circ \Delta_A(\varphi(w)).$$

Composing both hand sides with the section $\sigma$ of $\tau$ which is given by $\sigma(1) = \Delta_A(1_A)$ we get precisely the required statement.

6. $W$ is cocommutative. As a matter of fact, module algebra property of $W E$ and commutativity of $E$ immediately imply that $(w_{(1)} \triangleright y)(w_{(2)} \triangleright x) = (w_{(1)} \triangleright x)(w_{(2)} \triangleright y)$ for $x, y \in E$ and $w \in W$. Therefore $\Phi((w_{(1)} \triangleright y) \times w_{(2)}) = \Phi((w_{(2)} \triangleright y) \times w_{(1)})$ and $\Phi$ being mono we have equality of the arguments in the smash product. Now the arguments are images under $v_y \otimes \text{id}_W$ of $w_{(1)} \otimes_L w_{(2)}$ and $w_{(2)} \otimes_L w_{(1)}$, respectively, where $v_y: W \to E, w \mapsto w \triangleright y$, is a right $L$-module map. Since the common kernel of maps $v_y \otimes \text{id}_W: W \otimes_L W \to E \otimes_L W, y \in E$, is the kernel of $\varphi \otimes \text{id}$, we conclude that

$$w_{(1)} \otimes_L w_{(2)} = w_{(2)} \otimes_L w_{(1)}, \quad w \in W.$$ 

Now use separability of $L/K$ and the isomorphism $W \otimes_L W \cong (W \otimes_K W) \Delta_W(1_W)$ to conclude that $w_{(1)} \otimes_K w_{(2)} = w_{(2)} \otimes_K w_{(1)}, w \in W$.

7. If $n \in W E$ is an invariant then $\varphi(n)$ commutes with $\varphi(W)$ and since $\text{End}_{E_K}$ is generated by $\varphi(W)$ and the commutative $E$, it belongs to $\text{Center}_{\text{End}_{E_K}} = K$. This proves that $E^W = K$. 

8. \( \pi^L_A(\varphi(w)) = w \triangleright 1 = \pi^L_W(w) \triangleright 1 = \pi^L_W(w) \), the last equation identifying \( W^L \) with \( L \subset E \). Therefore \( \varepsilon_A(\varphi(w)) = \varepsilon_A(\pi^L_A(\varphi(w))) = \varepsilon_A(\pi^L_W(w)) = \tau(\pi^L_W(w)) \). Since \( \varepsilon_W|_L \) is nondegenerate there exist \( u \in L \) such that \( \tau(l) = \varepsilon_W(ul) \) for \( l \in L \). It follows that
\[
\varepsilon_A(\varphi(w)) = \varepsilon_W(u\pi^L_W(w)) = \varepsilon_W(\pi^L_W(uw)) = \varepsilon_W(uw)
\]
thus
\[
\varepsilon_W(w) = \varepsilon_A(\varphi(u^{-1}w)), \quad w \in W.
\] (3.7)

In particular if \( E/K \) is \( H \)-Galois for some finite dimensional Hopf algebra \( H \) over \( K \) then \( H \) is embedded into \( A \) by a unique weak morphism of weak Hopf algebras, which is just the restriction of the Galois map. Moreover \( A \) is the crossed product of \( E \) with \( H \).

**Example 3.2 (Greither-Pareigis [4, 14])** Let \( K = \mathbb{Q} \) the rational field and \( E = \mathbb{Q}[x]/(x^4 - 2) \). Then \( E/K \) is separable but not normal. However it is \( H \)-Galois for two different Hopf algebras. One of these Hopf algebras is the commutative and co-commutative Hopf algebra \( H = \mathbb{Q}[c, s]/(c^2 + s^2 - 1, cs) \) with comultiplication \( \Delta(c) = c \otimes c - s \otimes s \), \( \Delta(s) = c \otimes s + s \otimes c \), counit \( \varepsilon(c) = 1 \), \( \varepsilon(s) = 0 \) and antipode \( S(c) = c, S(s) = -s \). The weak left embedding of \( H \) into \( A \) is clear from the presentation of \( A \) as
\[
A = \mathbb{Q} \text{-alg}(c, s, x| c^2 + s^2 - 1, cs, sc, cx - xs, sx + xc, x^4 - 2) \quad (3.8)
\]
The general WHA structure of (3.3) can be cast into the form
\[
\Delta(1) = \frac{1}{4} \otimes 1 + \frac{1}{8} \sum_{k=1}^{3} x^k \otimes x^{4-k} \quad (3.9)
\]
\[
\Delta(x) = \frac{1}{4} (x \otimes 1 + 1 \otimes x) + \frac{1}{8} (x^3 \otimes x^2 + x^2 \otimes x^3) \quad (3.10)
\]
\[
\Delta(c) = \Delta(1)(c \otimes c - s \otimes s) \quad (3.11)
\]
\[
\Delta(s) = \Delta(1)(c \otimes s + s \otimes c) \quad (3.12)
\]
\[
\varepsilon(c) = 4 \quad \varepsilon(s) = 0 \quad \varepsilon(x) = 0 \quad (3.13)
\]
\[
S(c) = c \quad S(s) = -s \quad S(x) = x \quad (3.14)
\]

### 3.3 Galois connection

Let \( E/K \) be separable and let \( A \) be its universal weak Hopf algebra. Define \( \text{Sub}_{\text{Alg}}(E) \) to be the set of subobjects of \( E \) in the category of \( K \)-algebras.
Also, let \( \text{Sub}_{WHA/K}(A) \) be the sub-WHA’s of \( A \). The latter means any \( K \)-subalgebra of \( A \) which is closed under comultiplication. That is to say we restrict ourselves to strict embeddings of WHA’s in the sense of Definition 1.3.

We can define two order reversing functions (contravariant functors between preorders)

\[
\text{Sub}_{WHA/K}(A) \xrightarrow{\text{Fix}} \text{Sub}_{Alg/K}(E) \xrightarrow{\text{Gal}} \text{Sub}_{WHA/K}(A)
\]
as follows.

\[
\text{Fix}(W) := \{ x \in E \mid a(x) = \pi^L(a)x, \forall a \in W \} \quad (3.15)
\]
\[
\text{Gal}(F) := \{ a \in A \mid a(xy) = a(x)y, \forall x \in E, y \in F \} \quad (3.16)
\]

Then we have the adjointness relations

\[
W \subset \text{Gal}(F) \iff F \subset \text{Fix}(W) \quad (3.17)
\]

Therefore the pair \((\text{Fix}, \text{Gal})\) is a Galois connection between sub-WHA’s of \( A = \text{End}_E K \) and intermediate fields \( E \supset F \supset K \). Moreover this is a half Galois correspondence since every intermediate field occurs as a fixed field of a sub-WHA,

\[
F = \text{Fix}(\text{Gal}(F)), \quad \forall F \in \text{Sub}_{Alg/K}(E). \quad (3.18)
\]

A full Galois correspondence would require further analysis of weak Hopf subalgebras and coideal subalgebras like in [13].

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