Disordered 2d quasiparticles in class D: Dirac fermions with random mass, and dirty superconductors

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Abstract

Disordered noninteracting quasiparticles that are governed by a Majorana-type Hamiltonian – prominent examples are dirty superconductors with broken time-reversal and spin-rotation symmetry, or the fermionic representation of the 2d Ising model with fluctuating bond strengths – are called class D. In two dimensions, weakly disordered systems of this kind may possess a metallic phase beyond the insulating phases expected for strong disorder. We show that the 2d metal phase emanates from the free Majorana fermion point, in the direction of the RG trajectory of a perturbed WZW model. To establish this result, we develop a supersymmetric extension of the method of nonabelian bosonization. On the metallic side of the metal-insulator transition, the density of states becomes nonvanishing at zero energy, by a mechanism akin to dynamical mass generation. This feature is explored in a model of \( N \) species of disordered Dirac fermions, via the mapping on a nonlinear sigma model, which encapsulates a \( \mathbb{Z}_2 \) spin degree of freedom. We compute the density of states in a finite system, and obtain agreement with the random-matrix prediction for class \( D \), in the ergodic limit. Vortex disorder, which is a relevant perturbation at the free-fermion point, changes the density of states at low energy and suppresses the local \( \mathbb{Z}_2 \) degree of freedom, thereby leading to a different symmetry class, \( BD \).

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1 Introduction

As is well-known, the two-dimensional Ising model has a magnetic phase transition, the critical behavior of which is governed by the relativistic field theory of a massless Majorana fermion or, on squaring the partition function, a massless Dirac fermion. A theme of some debate over the last fifteen years has been the effect of disorder on this phase transition. Disorder, when introduced in the form of spatial inhomogeneities in the bond strengths, is known to add a random mass term to the Majorana Lagrangian. Perturbative renormalization group calculations by Dotsenko and Dotsenko [1] showed that randomness in the mass is a marginally irrelevant perturbation leading to logarithmic corrections to the pure Ising critical behavior. In particular, the singularity in the specific heat of the Ising model persists (in a weakened form) in the presence of disorder. This conclusion was confirmed by Shankar [2] and Ludwig [3], who also calculated the effect of disorder on the moments of the spin-spin correlation function.

The picture emerging from this work is that the 2d Ising model with weakly disordered bond strengths undergoes a phase transition controlled by the pure Ising fixed point, and the only effect the disorder has are logarithms. Although this picture came to be widely accepted, a constant challenge has been the work of Ziegler [4, 5, 6], who claims that the thermodynamic singularities of the Ising model are rounded off by the disorder. Specifically, he argues that the vanishing density of states (DoS) of the relativistic fermion at zero energy becomes finite as a result of nonperturbative effects. To motivate this scenario, he refers to a model of disordered Dirac fermions on the lattice, for which the DoS at $E = 0$ can be proved to be strictly positive.

Ziegler's field-theoretic calculation imitates the standard treatment of disordered metals, and is based on a Hubbard-Stratonovich transformation introducing a composite field $Q$, followed by a saddle-point approximation. Such a strategy is in principle not unreasonable. Indeed, in the case of disordered metals the effective theory for $Q \sim \psi \bar{\psi}$, where $\psi$ is the basic electron field, not only allows the systematic study of quantum interference corrections to metallic behavior, but also captures the nonperturbative physics of localization in low dimension. By analogy, one expects that in the present case, too, an advantage might be gained by transforming from Majorana fields $\psi$ to composite fields $Q \sim \psi \bar{\psi}$.

An important lesson learned from disordered metals is that, when using the field-theoretic formulation, we must exercise particular care to correctly
implement the symmetries of the disordered Hamiltonian. For example, any inaccuracy in the treatment of time-reversal invariance (when present) will spoil the weak localization effects due to the cooperon mode. What, then, are the symmetries of the disordered Majorana theory? The situation is most easily explained if we switch from two real (Majorana) fermions to the equivalent representation by one complex (Dirac) fermion, and is as follows. The first-quantized Hamiltonian for a two-dimensional Dirac fermion with random mass \( m(x) \),

\[
H = \begin{pmatrix}
  m(x) & -i\partial_1 - \partial_2 \\
  -i\partial_1 + \partial_2 & -m(x)
\end{pmatrix},
\]

(1)

has a symmetry of the “particle-hole” type:

\[
H = -\sigma_1 H^T \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

According to the general classification of disordered single-particle systems \[7, 8\], this symmetry places the Hamiltonian (1) in class \( D \). From the analysis of Ref. [8], one infers that the supersymmetric field-theory representation for random-mass Dirac fermions has a global invariance under the orthosymplectic Lie supergroup \( \text{OSp}(2n|2n) \) if the energy vanishes and the disorder average of a product of \( n \) Green functions is to be calculated.

Actually, the existence of this orthosymplectic symmetry was first pointed out by D. Bernard \[9\] in his Cargèse lectures on the application of conformal field-theory techniques to two-dimensional disordered systems. (Random Dirac fermions have also been treated by CFT techniques in \[10\].)

It is clear that the existence of such a symmetry will have an important bearing on the low-energy physics. For one thing, it was shown in \[4\] that the Gaussian random-matrix ensemble for class \( D \) has a \textit{positive} density of states at zero energy. (This can be understood from the fact \[4\] that the eigenvalues of a Gaussian random matrix in class \( D \) behave as a noninteracting gas of harmonically confined fermions on the half-line, with Neumann

\footnote{The equation (2) fixes an orthogonal Lie algebra in even dimension, \( \text{so}(2N) \), which is denoted by \( D_N \) in Cartan’s table. Hence the name “\( D \)”.}

\footnote{Throughout this paper, supergroups such as \( \text{GL}(n|n) \) and \( \text{OSp}(2n|2n) \) are understood to be defined over the complex number field, \( \mathbb{C} \), unless specified otherwise. In places where this fact is of particular importance, we will switch to the notation \( \text{GL}_C(n|n) \) and \( \text{OSp}_C(2n|2n) \).}
boundary conditions at the origin.) For another, the density of states enjoys
the status of an “order parameter” in the field-theoretic formalism. It is well
known that, when the condensation of an order parameter is associated with
a spontaneous breaking of global symmetries, there appear massless modes
due to Goldstone’s theorem. In the present context, the saddle-point value of
the $Q$-field (the order parameter) breaks $\text{OSp}(2n|2n)$ symmetry, and Gold-
stone’s theorem hence forces the existence of a supersymmetric $\text{OSp}(2n|2n)$
multiplet of Goldstone modes. (As usual, these modes have a physical in-
terpretation as diffusion-like modes. They are characteristic of class $D$, and
were called the “spin-singlet $D$-type diffuson” in [7].) It is a perplexing fact
that these Goldstone modes appear nowhere in Ziegler’s work. Nonetheless,
they exist, and because they do, the fate of the system in the thermody-
namic limit cannot be decided on the basis of a plain saddle-point analysis,
but is determined by the notoriously subtle problem of interacting Goldstone
modes in two dimensions. Solving this problem requires the use of the renor-
malization group. Thus we are led to ask: once we have augmented Ziegler’s
approach by incorporating the orthosymplectic symmetry of the disordered
Majorana theory, what is the prediction for the local density of states at $E = 0$?
Does it vanish in the thermodynamic limit, or does it not?

Further motivation for the present paper comes from several issues beyond
the density-of-states controversy. According to the general scheme of [7], an-
other realization of symmetry class $D$ is by the low-energy quasiparticles
of disordered superconductors with broken time-reversal and spin-rotation
invariance. Physically, such a situation may occur in spin-triplet supercon-
ductors, or in spin-singlet superconductors with spin-orbit scattering, when
time-reversal symmetry is broken spontaneously or by a magnetic field. Al-
ternatively, both time-reversal and spin-rotation invariance can be broken by
randomly adding classical Heisenberg impurity spins. With neither the elec-
tric charge nor the spin of a single quasiparticle being conserved in class $D$,
the only constant of the motion is the energy. Hence, quasiparticle diffusion
and localization in a superconductor of class $D$ has to be probed via thermal
transport (or transport of energy).

The qualitative physics of class-$D$ quasiparticles was the subject of a
recent paper by Senthil and Fisher [11]. These authors drew a schematic
phase diagram for the two-dimensional case in particular. There are three
phases, namely thermal insulator, thermal quantum Hall fluid, and thermal
metal, with three distinct phase boundaries, which meet in a multicritical
point, $\mathcal{M}^*$. The structure of the phase diagram is determined on very basic
grounds, by localization of all states for strong disorder (giving the insulating phases), the renormalization group flow of a weakly coupled nonlinear sigma model (giving the metal), and the topological distinction between the insulator and the quantum Hall fluid with edge currents.

In a more speculative attempt, Senthil and Fisher went on to try and match the phases of disordered 2d quasiparticles in class $D$ with the phase diagram of the 2d random-bond Ising model. Such an identification is prompted by the representation of the 2d Ising model by a Majorana fermion. However, there appeared to be a difficulty: the three stable phases of symmetry class $D$ in two dimensions seem to have only two counterparts, namely the ferromagnet and the paramagnet, in the Ising model. Where is the missing third phase? To resolve this puzzle, Senthil and Fisher suggested two alternative scenarios. In brief, the first one identifies the multicritical point $M^*$ with a point on the Nishimori line [12], while the second one interprets the missing phase as a spin glass phase. (The latter, however, is thought to exist only at zero temperature in $d = 2$.) Neither scenario looks convincing, which will motivate us to take a fresh look at the puzzle.

The latest contribution to the subject was made by Read and Green [13], who expounded the fact that another physical realization of the massless Majorana theory exists (in mean-field approximation) at the transition between paired quantum Hall states, and in chiral $p$-wave superconductors at the transition between the topologically distinct phases of weak and strong pairing. Concerning the role of disorder, they suggested to extend the definition of class $D$ so as to include vortices. Moreover, they argued that randomly placed vortices are a strongly relevant perturbation at the free-fermion point. The question, however, exactly what the renormalization group flows to, was not answered conclusively. Based on symmetry grounds only, Read and Green wrote down a nonlinear sigma model which differs from the one we obtain for class $D$ in that the target space in the fermion-fermion (FF) sector, $O(2n)/U(n)$, is replaced by its connected component $SO(2n)/U(n)$. Thus we may ask: what happened to the $\mathbb{Z}_2$ degree of freedom that distinguishes between $O(2n)$ and $SO(2n)$? Giving an answer to this subtle and yet pertinent question provides the final motivation for writing the present paper.

In view of the length of the paper, we now summarize our main results.
1.1 Overview

For generality and better perspective, we study a family of models with $N \geq 1$ species of Dirac fermions, replacing the random mass $m(x)$ for a single species by a mass matrix $M_{kl}(x)$ for $N$ species. A perturbative renormalization group calculation shows that in addition to the random-mass coupling, a few other couplings are generated by the RG flow for $N > 1$. By extending the initial formulation of the model, we incorporate the most important one of these. We then use standard technology to approximate the disordered Dirac theory by an effective action for nonlinear fields $Q : \mathbb{R}^2 \to X$. As expected from [8], the target space $X$ is a Riemannian symmetric superspace of type $C|D_{III}$. Its bosonic base (or “body”) is a product $M_B \times M_F$ where $M_B = \text{Sp}(2n, \mathbb{R})/U(n)$ is a noncompact symmetric space of type $C_I$, and $M_F = \text{O}(2n)/U(n)$ is a compact symmetric space of type $D_{III}$. Note that by $\text{O}(2n)$ we do mean the full orthogonal group, which consists of two connected components (the orthogonal matrices with determinant $+1$ or $-1$). Thus the present target space is distinguished by the striking feature of having two disjoint components, and there exists the possibility, not previously encountered in this context, of forming $\mathbb{Z}_2$ domain walls in the $Q$-field theory.

The effective action for the nonlinear field $Q$ is the logarithm of a superdeterminant, resulting from integration over the Dirac fields. One now wants to expand the effective action in gradients of $Q$, to produce a nonlinear sigma model. It turns out that performing this expansion requires a certain amount of care – the mathematical subtlety involved is known by the name of chiral anomaly. To do the expansion correctly, we have to resort to a variant of nonabelian bosonization, which is the celebrated statement [14] that the free theory of $n$ Majorana fermions has an equivalent representation by a level-one $\text{O}(n)$ Wess-Zumino-Novikov-Witten (WZW) model. Supersymmetry extends this to an equivalence between $2n+2n$ Majorana fields (fermions and bosonic ghosts), and a level-one WZW model of fields taking values in a Riemannian symmetric superspace of type $C|D$, based on $\mathcal{M}_B \times \mathcal{M}_F$ where $\mathcal{M}_B = \text{Sp}(2n, \mathbb{C})/\text{Sp}(2n)$ and $\mathcal{M}_F = \text{O}(2n)$.

Using a supersymmetric extension of the bosonization rules for the chiral densities $\psi_{\pm} \bar{\psi}_{\mp}$, we are then able to compute the gradient expansion of the effective action. The Lagrangian of the $C|D_{III}$ nonlinear sigma model thus obtained contains a topological term, or winding number term, with topological coupling $\theta$. Such a term is permitted by symmetry, since the massless Dirac Hamiltonian, $H_0$, depends on the choice of orientation of $\mathbb{R}^2$.  


or, in other words, reversing the orientation by a parity transformation, say by exchanging the two coordinates \(x_1\) and \(x_2\) of \(\mathbb{R}^2\), takes \(H_0\) into an inequivalent Hamiltonian. (Note that, quite generally, Dirac operators exist on spin manifolds, which are manifolds carrying a spin structure and thus an orientation.) The topological term is trivial for \(n = 1\), but nontrivial for \(n \geq 2\), since

\[
\Pi_2(O(2n)/U(n)) = \begin{cases} 
0 & \text{for } n = 1, \\
\mathbb{Z} & \text{for } n \geq 2.
\end{cases}
\]

By reduction of the multi-valued action of the \(C|D\) WZW model, we show that the topological angle has the value \(\theta = N\pi\).

The passage from random-mass Dirac fermions to the nonlinear sigma model is under good control, \(i.e.\) the terms omitted are small, for \(N \gg 1\). The limit of a large number \(N\) of species is of course unrealistic. Fortunately, we can also control the case \(N = 2\), if the kinetic energy is anisotropic, with the first species being more mobile in the \(x_1\) direction and the second more mobile in the \(x_2\) direction. A closely related situation is relevant for the application of our results to disordered \(d\)-wave superconductors with a subcritical concentration of localized impurity spins \([15]\). For \(N \gg 1\), or \(N = 2\) with large anisotropy, the nonlinear sigma model is at weak coupling, and the one-loop beta function predicts renormalization group flow to a Gaussian fixed point describing a perfect metal. The local density of states at \(E = 0\) in this case diverges logarithmically in the thermodynamic limit. We also compute the density of states for a finite system in the ergodic regime, and obtain agreement with the random-matrix prediction of \([7]\).

In contrast, for \(N = 1\) the mapping on the nonlinear sigma model is far from being controlled. Even if we trust the mapping, the model is strongly coupled, and no safe statements can be easily made from it. We have to concede that our method fails in that case, and the problem is better analysed through its original formulation in terms of Dirac fields. Our conclusion thus is the one commonly accepted: random mass is a marginally irrelevant perturbation, and the theory in the infrared flows to the free-fermion point, where the density of states at zero energy vanishes.

There now seems to be a conflict, but only superficially so, with Ziegler’s rigorous proof of a positive lower bound for the density of states. The apparent discrepancy is resolved by observing that Ziegler works with a naive lattice discretization of the Dirac operator, thereby imposing an additional lattice symmetry on his model, as a result of which the Hilbert space de-
composes into two decoupled sectors. In each of these, the Hamiltonian can be shown to be a pi-flux model (details of the argument will be published in a separate comment), which belongs to the time-reversal invariant Wigner-Dyson class A1 and, in fact, is known \cite{16, 17} to have a finite density of states at zero energy. Hence Ziegler’s rigorous result, albeit correct, is a statement about a different symmetry class, and does not falsify our results. We may safely ignore his proof in all that follows.

There is another point of possible contention that deserves to be made clear. Recall that our large-$N$ saddle-point analysis yields a target manifold consisting of two connected components. Thus there is a $\mathbb{Z}_2$ degree of freedom in the effective theory, and we anticipate the existence of $\mathbb{Z}_2$ domains and domain walls, across which the nonlinear sigma model field jumps from one connected component of the target manifold to the other. (Note that in our computation of the one-loop beta function of the CI|DIII nonlinear sigma model, the nonperturbative effect of these domain walls is neglected.) The same $\mathbb{Z}_2$ degree of freedom arises for small $N$, when the method of nonabelian bosonization is used. Indeed, nonabelian bosonization according to Witten \cite{14} transforms $n$ Majorana fermions into a WZW model over the full orthogonal group $O(n)$, not just its identity component $SO(n)$. The necessity to work with $O(n)$, which consists of two connected components, is readily seen by looking at the special case $n = 1$: a single Majorana fermion is not an empty theory, as would be the case with the trivial group $SO(1)$, but is equivalent to a theory of local $O(1) \equiv \mathbb{Z}_2$ degrees of freedom, namely the 2d Ising model. In a sense, then, passing from random-mass fermions to the effective description by a nonlinear sigma model, reintroduces the local Ising degrees of freedom underlying the free fermion.

Having established the general formalism and its validity, we turn to a specific application: disordered non-chiral $d$-wave superconductors with broken time-reversal and spin-rotation symmetry. Following a standard procedure \cite{18, 19, 20}, we linearize the dispersion relation around four low-energy points (or “nodes”) in wave vector space, and then add generic disorder (in the form of a random scalar potential, random complex order parameter, and random spin-orbit scattering) to place the model in class $D$. The number of species per node is $N = 2$ due to spin. By drawing on our previous results, we map the low-energy physics of the quasiparticles on a nonlinear sigma model. Doing this for every node separately, we obtain kinetic energy terms that are spatially anisotropic, and a topological angle which is positive for two of the nodes and negative for the other two. The inclusion of quasipar-
article scattering between the nodes merges the four anisotropic models into a single isotropic theory. In the last step, all the topological terms cancel, as is required for a non-chiral superconductor invariant under reflections of space.

Next, we recall the 2d phase diagram proposed by Senthil and Fisher [11], and address the puzzle why the two stable phases of the 2d random-bond Ising model seem to have three counterparts in class D. To resolve this mismatch, one needs to understand the nature of the multicritical point $\mathcal{M}^*$ and the renormalization group flow in its vicinity. Our resolution of the puzzle is very simple: the “multicritical” point $\mathcal{M}^*$, where the three phases of class $D$ meet, is to be identified with the free-fermion point! The evidence in favor of such an identification is twofold. Firstly, by an argument involving nonabelian bosonization, we show that the free-fermion point sits right on the boundary of the 2d metal phase. Secondly, we argue that (barring vortex disorder) the thermal metal phase of class $D$ can only be realized in the enlarged parameter space available for more than one species ($N > 1$), and is absent in the fundamental case $N = 1$. This makes it natural to invert some of the flows assumed in [11] and arrive at a plausible 2d phase diagram, where the paramagnetic and ferromagnetic states of the Ising model match the thermal insulator and thermal quantum Hall fluid phases of class $D$ for $N = 1$, and there is no redundant third phase.

Another suggestion by Senthil and Fisher [11] says that, from the perspective of the nonlinear sigma model, the existence of the insulating phases, which are distinguished by a quantized thermal Hall conductance, can be attributed to the presence of the topological term in the action functional. One might then think that this very term is what is driving the phase transition from the thermal insulator to the thermal quantum Hall fluid. That scenario is widely accepted for a close cousin, namely Pruisken’s nonlinear sigma model [21] of the integer quantum Hall effect. However, as was stated earlier, the topological term of the $CI|DIII$ nonlinear sigma model is trivial for $n = 1$. Since the phase transition occurs for all $n$, including $n = 1$, it is hard to see how the topological term could be its driving agent. A better scenario is to attribute the phase transition to the $Z_2$ degree of freedom of the $CI|DIII$ nonlinear sigma model.

Finally, we delve into the quasiparticle physics of chiral $p$-wave superconductors, to shed more light on the effect of vortex disorder in those systems. Read and Green [13] have argued, both intuitively and formally, that randomly placed vortices are a relevant perturbation at the free-fermion point of a superconductor with (mean-field) order parameter symmetry $p_x+iy$. Un-
like the perturbation by a random mass matrix, which can become relevant only in the extended parameter space available for \( N \geq 2 \), vortex disorder turns out to be relevant even in the fundamental case of spinless particles \((N = 1)\). Read and Green further propose that vortices drive the superconductor from the free-fermion point into a thermal metal phase, the effective field theory for which lacks, as it stands, the \( \mathbb{Z}_2 \) (or Ising spin) degree of freedom of the nonlinear sigma model we obtain for random-mass fermions. We believe that their proposal is correct, and in order to provide supporting evidence for it, we outline an indirect argument, passing through a variant of the Chalker-Coddington network model with local O(1) invariance. (We are unable at present to give a direct field-theoretic proof, as we do not know how to incorporate vortex disorder into the nonabelian bosonization scheme.)

What we are learning, then, is that vortex disorder exerts a drastic influence on the effective field theory: while there exists a local \( \mathbb{Z}_2 \) degree of freedom (and domain walls in it) when vortices are absent, this degree of freedom is suppressed when vortices are inserted. Thus the cases with and without vortices map on different field theories. Actually, there must exist a whole one-parameter family of such theories, as the suppression proceeds continuously. Motivated by random-matrix limits, we propose to refer by the (hybrid) name \( BD \) to the generic symmetry class including a variable amount of vortex disorder. Class \( D \) is then viewed as a subclass of the generic class \( BD \). Of course, the field-theoretic distinction between classes does not fail to leave its imprint on the quasiparticle physics. We expect that when all quasiparticle states are localized, \( i.e. \) in the thermal insulator and quantum Hall fluid phases, the local density of states at zero energy vanishes in the absence of vortices (class \( D \)), but becomes finite when an extensive number of vortices is inserted (class \( BD \)).

In contrast, the physical distinction between systems with and without vortices is of relatively minor consequence in the metallic regime of extended states. There, and for the case of a finite system and energies \( E \) below the Thouless energy \( E_{\text{Th}} \), the nonlinear sigma model can be evaluated in zero-mode approximation. For the total density of states in the two cases at hand, we obtain:

\[
\rho_D(E) = \nu + \frac{\sin(2\pi \nu E)}{2\pi E}, \quad \rho_{BD}(E) = \nu + \frac{1}{2}\delta(E),
\]

where the scale of variation is set by \( \nu \), the inverse of the level spacing for energies much greater than the mean level spacing (but still less than
$E_{Th}$). The first expression results on integrating over the full target manifold (with $\mathbb{Z}_2$ present), the second from the reduced target (with $\mathbb{Z}_2$ completely suppressed). These results coincide and, as we shall see, not accidentally so, with the large-$N$ limits of the density of states for the Haar random-matrix ensembles on $SO(2N)$ and $O(2N)$, respectively.

The material of the paper is arranged as follows. In Sections 2–3, we introduce a Hamiltonian for $N$ species of Dirac fermions with random mass, and set up a field-theoretic representation of its Green functions by a supersymmetric Gaussian functional integral. Renormalization of the disorder-averaged field theory generates a total of four marginal perturbations, the flow equations for which are worked out in Section 4. Guided by this, we settle for a specific choice of Lagrangian and proceed to map it on the $Cl/DIII$ nonlinear sigma model, in the limit $N \gg 1$. The composite field $Q$ is introduced and subjected to a saddle-point approximation in Section 5, while the structure of the saddle-point manifold is elucidated in Section 6. Section 7 is the longest of the paper. There, we expand the effective action for $Q$ in gradients, using a supersymmetric extension of the method of nonabelian bosonization, which is developed in two major subsections. In Section 8, we renormalize the nonlinear sigma model (neglecting the $\mathbb{Z}_2$ degree of freedom) and show that its coupling flows to zero, resulting in a perfect metal with infinite thermal conductivity. We compute the density of states for a finite system, as well as in the thermodynamic limit. In Section 9, we clarify the relation between the model considered and disordered $d$-wave superconductors. Section 10 assembles various arguments that focus on the role of the free-fermion point inside class $D$, and culminate in the proposal of a local phase diagram. Finally, in Section 11 we comment on the role of vortices and the significant modification they cause in the nonlinear sigma model.

To avoid confusion, we emphasize that the present paper will always be concerned with the physics of Majorana fermions. Nevertheless, the words “Dirac fermions” or “Dirac theory” will be frequently encountered, the reason being that we carry out the disorder average by using the supersymmetry method, and a single Majorana fermion does not have a bosonic analog. The standard trick to get around this problem is to “square the partition function”, making two Majorana fermions out of one, and then combine the two to form a Dirac fermion. The latter can be augmented by a bosonic $b$-$c$ ghost system to make the theory supersymmetric and cancel all vacuum graphs.
2 Dirac Hamiltonian with random mass

The first-quantized Hamiltonian \( H \) for \( N \) species of Dirac fermions with random mass in two dimensions is written as

\[
H = \begin{pmatrix}
M(x) & -2i\partial \\
-2i\bar{\partial} & -M^T(x)
\end{pmatrix}.
\]  

(3)

Here \( M = (M_{kl}) \) is an \( N \times N \) mass matrix, and we use the convention \( \partial = \frac{1}{2}(\bar{\partial}_1 - i\bar{\partial}_2) \), \( \bar{\partial} = \frac{1}{2}(\bar{\partial}_1 + i\bar{\partial}_2) \). Aside from being Hermitian (\( H^\dagger = H \)), this Hamiltonian enjoys an important symmetry property:

\[
H = -\sigma_1 H^T \sigma_1 ,
\]

which will be called particle-hole symmetry, or the symmetry of class \( D \). The superscript \( T \) denotes joint transposition in particle-hole and position space (note \( \bar{\partial}^T = -\partial \)). The particle-hole symmetry of \( H \) is dictated by the Lagrangian for Majorana fermions,

\[
L_M = i\bar{\psi}_l \partial \bar{\psi}_l + i\psi_l \bar{\partial} \psi_l + \bar{\psi}_k M_{kl} \psi_l .
\]

(Summation over repeated indices is understood.) An immediate consequence of the particle-hole symmetry is that the eigenvalues of \( H \) occur in pairs with opposite sign \( \pm E \). Hence, the density of states is symmetric with respect to the point \( E = 0 \). By the same token, there exists a relation between retarded and advanced Green functions \( G^\pm(E) = (E \pm i0 - H)^{-1} \):

\[
G^-(E) = -\sigma_1 G^+(E)^T \sigma_1 .
\]

When the mass matrix is a constant multiple of unity, \( M(x) = m \times 1_N \), one easily computes the local density of states:

\[
\nu(E) = (\text{area})^{-1} \pi^{-1} \text{Im} \ Tr \ (E - i0 - H)^{-1} \\
= \frac{2N}{\pi} \lim_{\epsilon \to 0^+} \text{Im} \int \frac{d^2k}{(2\pi)^2} \frac{E - i\epsilon}{(E - i\epsilon)^2 - k^2 - m^2} \\
= \frac{N|E|}{2\pi} \theta(E^2 - m^2) ,
\]

which vanishes linearly at \( E = 0 \) in the limit of zero mass.
In what follows, we take the entries of the mass matrix to be Gaussian distributed random variables with zero mean and second moments given by

$$\langle M_{ij}(x)M_{kl}(y) \rangle = (2g_M/N) \delta_{il} \delta_{jk} \delta(x-y).$$

The density of states on average over the fluctuating mass matrix can be obtained by computing the average Green function. This will be done by an adaptation of Efetov’s supersymmetry method [22]. Although that method is in principle quite standard, we will see that its application to the present situation features some peculiarities which are well worth explaining.

3 Supersymmetric integral representation

We are going to employ a supersymmetric integral representation to set up the calculation of the disorder averaged Green functions. Although we concentrate on the case of a single Green function \(n = 1\) for notational simplicity, the generalization to arbitrary \(n > 1\) will be immediate [8]. We introduce \(2N\) supermultiplets \(\phi_k\) and \(\bar{\phi}_l\) \((k, l = 1, ..., N)\), each containing two fermionic components \(\psi_{\pm}\) and two bosonic superpartners \(b\) and \(c\). Each supermultiplet is arranged in the form

$$\phi = \begin{pmatrix} \psi_- \\ \psi_+ \\ c \\ b \end{pmatrix}, \quad \bar{\phi} = \begin{pmatrix} \bar{\psi}_- \\ \bar{\psi}_+ \\ \bar{c} \\ \bar{b} \end{pmatrix},$$

where the index \(l = 1, ..., N\) was omitted. The orthosymplectic transpose \(\phi^t\) of \(\phi\) is defined by

$$\phi^t \equiv (\psi_+, \psi_-, b, -c),$$

and \(\bar{\phi}^t\) (defined similarly) is the orthosymplectic transpose of \(\bar{\phi}\). Note that this choice of transpose determines an inner product which is skew:

$$\bar{\phi}^t \phi = \bar{\psi}_+ \psi_- + \bar{\psi}_- \psi_+ + \bar{b}c - \bar{c}b = -\phi^t \bar{\phi}. \quad (6)$$

Moreover, the orthosymplectic transpose on \(\phi\) defines a compatible transpose of \(4 \times 4\) supermatrices \(T\) by the requirement \((T\phi)^t \equiv \phi^t T^t\).

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3We use the conformal field theory convention of denoting antiholomorphic fields by an overbar. To avoid confusion where it might otherwise arise, complex conjugation is denoted by an asterisk \(*\) instead of the overbar.
The field-theory Lagrangian for $N$ species is taken to be

$$L_0 = \bar{\phi}_l \partial_\phi \phi_l + \phi_l^\dagger \partial_\phi \bar{\phi}_l + i \bar{\phi}_k M_{kl} \phi_l - i E \bar{\phi}_l \Sigma_3 \phi_l,$$

(7)

where

$$\Sigma_3 = 1_{\text{susy}} \otimes \sigma_3 = \text{diag}(+1, -1, +1, -1),$$

and summation over repeated $k, l$ indices is understood. If $j$ and $\bar{j}$ are source fields, the Green functions of the Dirac Hamiltonian (3) are generated by the functional

$$Z[j] = \left< \int \mathcal{D}\bar{\phi} \mathcal{D}\phi \exp - \int d^2x \left( L_0 + \bar{\phi}_l j_l + \phi_l^\dagger \bar{j}_l \right) \right>,$$

as is easily verified from the fact that $L_0$ is the quadratic form constructed by sandwiching the operator

$$1_{\text{susy}} \otimes i(H - E) = 1_{\text{susy}} \otimes \begin{pmatrix} iM - iE & 2\partial \\ 2\bar{\partial} & -iM^T - iE \end{pmatrix}$$

between

$$\left( \bar{\psi}_+, \bar{b}, \psi_+, b \right) \quad \text{and} \quad \begin{pmatrix} \psi_- \\ c \\ \bar{\psi}_- \end{pmatrix},$$

and integrating by parts to symmetrize the terms with derivatives. In order for the Gaussian functional integral over $\phi, \bar{\phi}$ to make sense, the bosonic fields must be related to each other by complex conjugation $*$:

$$b_l^* = \bar{c}_l, \quad c_l^* = \bar{b}_l.$$ 

(8)

Given this convention, the functional integral is purely oscillatory for $E \in \mathbb{R}$, and converges for energies in the upper half plane ($\text{Im} E > 0$). As stated in the introduction, the Lagrangian $L_0$ in the limit $E \to 0$ acquires an invariance under global transformations,

$$\phi_l(x) \mapsto T \cdot \phi_l(x), \quad \bar{\phi}_l(x) \mapsto T \cdot \bar{\phi}_l(x),$$

$$\phi_l^\dagger(x) \mapsto \phi_l^\dagger(x) \cdot T^\dagger, \quad \bar{\phi}_l^\dagger(x) \mapsto \bar{\phi}_l^\dagger(x) \cdot T^\dagger,$$

for elements $T \in \text{OSp}(2|2)$, i.e. if $T$ obeys $T^\dagger T = 1$, with $T^\dagger$ being the orthosymplectic supermatrix transpose defined above.
On performing the disorder average specified by (4), the Lagrangian for the case $E = 0$ becomes

$$L_1 = \bar{\phi}_t^i \partial \bar{\phi}_t^i + \phi_t^i \bar{\partial} \phi_t^i + (g_M/N) \Phi^{(1)}, \quad \Phi^{(1)} = \bar{\phi}_k^i \phi_l^i \phi_k^i.$$  

By simple power counting, the operator $\Phi^{(1)}$ is a perturbation of the massless Dirac theory which is marginal in the renormalization group sense. $\Phi^{(1)}$ may be marginally relevant or marginally irrelevant. The question of which is the case, is decided by calculating the short-distance expansion of the operator product of $\Phi^{(1)}$ with itself.

### 4 Marginal perturbations

It turns out that the operator product expansion of $\Phi^{(1)}$ with itself generates three additional operators:

$$\begin{align*}
\Phi^{(2)} &= \bar{\phi}_k^i \phi_l^i \phi_k^i, \\
\Phi^{(3)} &= \bar{\phi}_k^i \phi_l^i \phi_k^i, \\
\Phi^{(4)} &= \phi_l^i \phi_k^i \phi_k^i,
\end{align*}$$

for $N > 1$. Each of these is marginal and invariant under $OSp(2|2)$, and the set $\Phi^{(1)}, ..., \Phi^{(4)}$ exhausts the set of marginal perturbations permitted by that symmetry. Note that the situation simplifies for $N = 1$, where $\Phi^{(1)} \equiv \Phi^{(3)} \equiv \Phi^{(4)}$, and $\Phi^{(2)} \equiv 0$ from (3).

Given that the operators $\Phi^{(\alpha)}$ ($\alpha = 2, 3, 4$) are generated by the renormalization group flow anyway, the natural procedure is to include them in the bare theory. Our goal thus is to renormalize the extended Lagrangian

$$L_2 = \bar{\phi}_t^i \partial \bar{\phi}_t^i + \phi_t^i \bar{\partial} \phi_t^i + N^{-1} \sum_{\alpha=1}^{4} g_{\alpha} \Phi^{(\alpha)}.$$  

To do so, we use the fact [23, 24] that, if the leading short-distance singularity of the operator product expansion (OPE) is

$$\Phi^{(\alpha)}(x) \Phi^{(\beta)}(x') = |x - x'|^{-2} \sum_{\gamma} C_{\gamma}^{\alpha \beta} \Phi^{(\gamma)}(x') + ... ,$$

the one-loop beta functions determining the RG flow with increasing cutoff length scale $\ell$, are given by

$$\frac{2}{\pi} \hat{g}_{\gamma} \equiv \frac{d g_{\gamma}}{d \ln \ell} = \frac{-\pi}{N} \sum_{\alpha \beta} C_{\gamma}^{\alpha \beta} g_{\alpha} g_{\beta}.$$
The expansion of the operator product $\Phi^{(\alpha)}(x)\Phi^{(\beta)}(x')$ is completely determined by the OPEs for holomorphic ($z = x_1 + ix_2$) and antiholomorphic ($z^* = x_1 - ix_2$) fields. These are

$$\phi_k^a(z)\phi_l^b(w) \sim \frac{1}{2\pi} \frac{\delta_{kl} \tau^{ab}}{z - w}, \quad \bar{\phi}_k^a(z^*)\bar{\phi}_l^b(w^*) \sim \frac{1}{2\pi} \frac{\delta_{kl} \tau^{ab}}{z^* - w^*},$$  \hfill (11)

where

$$\tau = \sigma_1 \otimes E_{FF} + i\sigma_2 \otimes E_{BB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

determines the orthosymplectic structure. Using these formulas, a lengthy but straightforward calculation yields

$$\dot{g}_1 = g_1g_2 - N^{-1}(g_1g_2 + g_1g_4 - g_2g_3 + g_3g_4),$$
$$\dot{g}_2 = \frac{1}{2}(g_1^2 + g_2^2) + N^{-1}(g_1g_3 + g_1g_4 - g_2^2 - g_3g_4),$$
$$\dot{g}_3 = -(g_1 + g_2)g_3 - N^{-1}(g_1^2 - g_1g_3 + g_1g_4 + g_2g_3 + g_3^2 + g_3g_4),$$
$$\dot{g}_4 = -N^{-1}(g_1g_2 + g_1g_3 + g_2g_3 + g_4^2).$$  \hfill (12)

For $N = 1$ these equations simplify greatly. In that case, $\Phi^{(2)} \equiv 0$, so the coupling $g_2$ must be dropped, and since $\Phi^{(1)} = \Phi^{(3)} = \Phi^{(4)}$, the only coupling to renormalize is $g_M \equiv g_1 + g_3 + g_4$. By linearly combining the above equations, we get

$$\frac{dg_M}{d\ln \ell} = -\frac{2g_M^2}{\pi},$$  \hfill (13)

which reproduces the well-known result \cite{1} that random mass is a marginally irrelevant perturbation (for $N = 1$).

Let us quickly review how to deduce from this result the behavior of the local density of states $\nu(E, g_M)$ at small energies $E$ and weak coupling $g_M$. We start from the formula

$$\nu(E, g_M) = \pi^{-1}\text{Re} \left\langle (\psi_- \bar{\psi}_+ + \bar{\psi}_- \psi_+)(0) \right\rangle,$$

where the functional average is defined w.r.t. the Lagrangian (9) for $N = 1$. The OPE between $\Phi^{(1)}$ and the operator that couples to $E$, $\Phi^{(0)} \equiv \bar{\phi}^3 \Sigma^3 \phi$, reads $\Phi^{(1)}(x)\Phi^{(0)}(0) = -(2\pi^2)^{-1}\Phi^{(0)}(0)/|x|^2 + \ldots$. It then follows, by the same principle we just used for the coupling $g_M$, that the operator $\Phi^{(0)}$ has
scaling dimension $\gamma(g_M) = 1 - g_M/\pi + \mathcal{O}(g_M^2)$, and the RG flow equation for $E$ is $dE/d\ln \ell = (2 - \gamma(g_M))E$. By making in the functional integral an RG transformation which changes the cutoff from $\ell_0$ to $\ell$, we obtain

$$\nu(E(\ell_0), g_M(\ell_0); \ell_0) = \nu(E(\ell), g_M(\ell); \ell) \exp - \int_{\ell_0}^{\ell} \gamma(g_M(\ell'))d\ln \ell' .$$

We know from (13) that the coupling $g_M(\ell)$ flows to zero with increasing $\ell$. We also know the density of states of the pure system to be independent of the cutoff: $\nu(E, 0; \ell) = |E|/2\pi$. Hence, by integrating the flow equations for $g_M$ and $E$ up to the infrared cutoff, which for small enough energies is given by the system size $L$, we obtain

$$\nu(E, g_M; \ell_0) = \frac{|E|}{2\pi} (1 + (2g_M/\pi) \ln(L/\ell_0)) .$$

We see that, for $N = 1$, the density of states at $E = 0$ remains zero, and randomness in the mass only changes the slope of $\nu \sim |E|$ by a logarithm.

For $N > 1$ the flow equations for the couplings $g_1, ..., g_4$ are not very transparent, and the precise form of the RG flow remains unclear in general. However, for $N \gg 1$ the equations reduce to

$$\dot{g}_1 \pm \dot{g}_2 = \pm \frac{1}{2}(g_1 \pm g_2)^2 , \quad \dot{g}_3 = -(g_1 + g_2)g_3 , \quad \dot{g}_4 = 0 .$$

Clearly, the parameters $g_3$ and $g_4$, if initially set to zero, remain zero under the flow. The relevant couplings are $g_1$ and $g_2$, which are seen to be attracted to the line $g = g_1 = g_2$, with $g$ satisfying the equation

$$\frac{dg}{d\ln \ell} = \beta(g) = + \frac{2g^2}{\pi} . \quad (14)$$

We observe that the sign on the right-hand side has been reversed as compared to $N = 1$, and the coupling $g$ is now relevant.

These considerations motivate us to adopt the modified Lagrangian

$$L_3 = \bar{\phi}_1 \partial \phi_1 + \bar{\phi}_2 \partial \phi_2 + (g/N) \left( \Phi^{(1)} + \Phi^{(2)} \right) . \quad (15)$$

At the level of the Dirac Hamiltonian (3), the inclusion of the operator $\Phi^{(2)}$ corresponds to adding a term

$$H \to H + \begin{pmatrix} 0 & \Delta \\ \Delta^t & 0 \end{pmatrix} ,$$
where $\Delta$ is a skew-symmetric $N \times N$ matrix whose entries are Gaussian distributed random variables with zero mean and second moments
\[
\langle \Delta_{ij}(x)\Delta_{kl}(y) \rangle = (2g/N)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \delta(x - y).
\]
Such a term is permitted by the fundamental particle-hole symmetry (2) of class $D$. Note that for $N = 1$, skew-symmetry forces $\Delta$ to vanish identically. (Equivalently, $\Phi^{(2)} = 0$ for $N = 1$, and $L_3$ coincides with the original Lagrangian.) For the application to disordered superconductors (Section 4), we put $N = 2$. The indices $i, j, k, l$ then label the spin degrees of freedom of the electron, while the random functions $M(x)$ and $\Delta(x)$ acquire a physical meaning as random scalar potential, random spin-orbit scattering, and the random part of the superconducting order parameter (or, to be precise, linear combinations thereof).

Before embarking on our project of analysing $L_3$, we wish to mention three other special choices of the couplings, where we can easily understand the nature of the RG flow. The first case is $g_1 = g_3 = 0, N > 2$. By tracing the couplings back to a Hermitian random Hamiltonian, we find the constraint $g_1 \pm g_4 \geq 0$. Thus, setting $g_1 = 0$ forces $g_4 = 0$. Inspection shows that the random Hamiltonian in this limit has a higher degree of symmetry:
\[
H = -\sigma_1 \otimes 1_N H^T(\sigma_1 \otimes 1_N) + (\sigma_2 \otimes 1_N) H^T(\sigma_2 \otimes 1_N),
\]
where the second equality is a kind of time-reversal invariance, placing $H$ in class $DIII$ [7]. The only coupling that remains is $g_2$, which flows according to $\dot{g}_2 \propto (N - 2)g_2^2$ and thus is relevant for $N > 2$. The methods developed in the present paper are readily adapted to deal with this case.

The second special case is obtained by setting $N = 2$ in the previous one. The underlying random Hamiltonian then is of the form
\[
H = \begin{pmatrix} 0 & -2i\bar{\partial} + a\sigma_2 \\ -2i\partial + a\bar{\sigma}_2 & 0 \end{pmatrix}.
\]
Because this commutes with $\text{diag}(\sigma_2, \sigma_2)$, the Hilbert space decomposes into two sectors invariant under the action of $H$. (For $N > 2$, the Hilbert space does not decompose.) The Hamiltonian restricted to either sector is a Dirac fermion in a random abelian vector potential $\pm a(x)$, which is class $AIII$ [4, 8], and explains [23] why the single coupling $g_2$ is exactly marginal for $N = 2$. 

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The extra time-reversal symmetry here does no more than connect the two restrictions of $H$ by a Lie algebra homomorphism. Apart from causing every energy eigenvalue to be doubly degenerate, this is of no consequence, and the symmetry class is $A_{III}$ instead of $D_{III}$.

There exists a third special case where the flow equations close on a single coupling. This is $g_2 = g_3 = 0 = g_1 + g_4$ with $g_1 > 0$, and again $N = 2$. Here the underlying random Hamiltonian is of the form

$$H = \begin{pmatrix} V\sigma_2 & -2i\partial \\ -2i\bar{\partial} & V\sigma_2 \end{pmatrix}.$$  

By a simple conjugation $H' = UHU^{-1}$, this can be transformed into $H' = \text{diag}(H_+, H_-)$ where

$$H_\pm = \begin{pmatrix} \pm V & -2i\partial \\ -2i\bar{\partial} & \pm V \end{pmatrix}$$

are two copies of a Dirac fermion with random scalar potential $V(x)$. In the transformed basis, each copy satisfies

$$H_\pm = \sigma_2 H^T_\pm \sigma_2,$$

which is the defining equation of the “symplectic” Wigner-Dyson class $A_{II}$. The extra particle-hole symmetry (class $D$) here does no more than relate the first copy to the second copy by a Lie algebra homomorphism. It is therefore redundant, and the symmetry class remains $A_{II}$.

By numerically solving the flow equations (12), we find that the line $g_2 = g_3 = 0 = g_1 + g_4$ has a large basin of attraction for $N = 2$. In particular, it attracts the initial condition $g_1 = g_2, g_3 = g_4 = 0$ in that case.

According to [8], the classes $D$, $D_{III}$ and $A_{II}$ exhaust the set of symmetry classes whose field-theory representation is acted upon by the present form of $\text{OSp}(2n|2n)$ (preserving a symplectic structure for bosons and an orthogonal structure for fermions). We therefore expect no further accidental closures to occur.

## 5 Composite field and diagonal saddles

As we saw, for $N \gg 1$ the coupling $g = g_1 = g_2$ is certain to increase under renormalization, leading to a strong-coupling problem in the infrared.
To handle this problem, it is mandatory to switch to a dual formulation in terms of a Hubbard-Stratonovich field \( Q \sim \phi_l^\dagger \phi_l + \bar{\phi}_l \phi_l \).

The composite field \( Q \) is brought on stage by doing two Gaussian integrals in a row. To prepare these steps, we reorganize the Lagrangian (15) as follows:

\[
L_3 = \bar{\phi}_l \partial \bar{\phi}_l + \phi_l \partial \phi_l + \frac{g}{2N} \text{STr} \left( \phi_k \bar{\phi}_k^\dagger + \bar{\phi}_k \phi_k^\dagger \right) \left( \phi_l \bar{\phi}_l^\dagger + \bar{\phi}_l \phi_l^\dagger \right).
\]

The quartic interaction can now be decoupled in a first step by introducing an auxiliary \( 4 \times 4 \) supermatrix field \( Q \):

\[
L_4 = \bar{\phi}_l \partial \bar{\phi}_l + \phi_l \partial \phi_l + \text{STr} \left( \phi_l \bar{\phi}_l^\dagger + \bar{\phi}_l \phi_l^\dagger \right) - \frac{N}{2g} \text{STr} Q^2 - \frac{N}{2g} \text{STr} Q^2.
\]

In the second step, one does the Gaussian integral over the Dirac field, resulting in the action functional

\[
S_5[Q] = -\frac{N}{2g} \int d^2 x \text{STr} Q^2 + \frac{N}{2} \text{STr} \ln \left( \frac{\Sigma_3 \phi_l^\dagger}{\partial} Q \right) + \frac{N}{2} \text{STr} \ln \left( \frac{\Sigma_3 \phi_l^\dagger}{\partial} Q \Sigma_3 \right),
\]

where \( \text{STr} \) combines the operations of taking the supertrace \( \text{STr} \) and integrating over position space. The factors \( \Sigma_3 \) under the logarithm appear after the transformation \( \phi_l \to \Sigma_3 \phi_l \) and \( \phi_l^\dagger \to \phi_l^\dagger \Sigma_3 \), correcting for the fact that \( \bar{\phi} \) is not the complex conjugate of \( \phi^\dagger \); see the definition (13) and the relations (8). The above result is for \( E = 0 \). To restore the dependence on energy, we need to shift \( Q \to Q - (iE/2) \Sigma_3 \) under the argument of the logarithm. Note that all manipulations done so far were exact.

As a result of the transformation to \( Q \), the parameter \( N \) now appears as a factor in the exponent, while \( Q \) itself is ignorant of the number of species. For large \( N \), this suggests treating the \( Q \) integral in saddle-point approximation, which is what we are going to do next. (Note, however, that the saddle-point approximation is uncontrolled for \( N = 1 \).) The saddle-point equation reads

\[
Q(x)/g = \frac{1}{2} \left( \frac{\partial}{\partial} \right)^{-1} (Q - \partial Q^{-1} \partial)^{-1} \langle x | Q(x) \rangle. \]

We look for a spatially homogeneous solution of the form \( Q(x) = \mu \Sigma_3 \). The saddle-point equation then reduces to

\[
g^{-1} = \int \frac{d^2 k}{(2\pi)^2} \left( \mu^2 + k^2 / 4 \right)^{-1}.
\]
Cutting off the integral in the ultraviolet by $|k| < 1/\ell_0$ yields the equation

$$\pi/g = \ln \left( 1 + (2\mu\ell_0)^{-2} \right),$$

and by inversion,

$$\mu = \left(2\ell_0\right)^{-1}/\sqrt{e^{\pi/g} - 1}. \quad (18)$$

For weak disorder ($g \ll 1$), this is well approximated by $\mu \approx (2\ell_0)^{-1}e^{-\pi/2g}$. From formula (14) we then infer that $\mu$ obeys the renormalization group equation

$$\left( \frac{d}{d \ln \ell} + \beta(g) \frac{d}{dg} \right) \mu(\ell, g) = 0,$$

and thus has the meaning of a dynamically generated mass. In other words, the mass scale $\mu$ remains invariant:

$$\mu(\ell_0, g(\ell_0)) = \mu(\ell, g(\ell)), \quad (19)$$

under an RG transformation $\ell_0 \mapsto \ell$. This means that the Dirac field undergoes the phenomenon of dimensional transmutation: while the bilinear $\bar{\phi}_l\phi_l$ has dimension one at the free-fermion point, its dimension approaches zero on large scales. At the same time, $Q \sim \phi_l\bar{\phi}_l + \bar{\phi}_l\phi_l$ acquires a nonvanishing expectation value.

In principle, all diagonal matrices $Q$ with entries $\pm \mu$ are candidate solutions of the saddle-point equation. It may happen, however, that some of these do not lie on the integration domain, or cannot be reached by analytic continuation from that domain. Therefore, in order to be able to select the proper solutions, we have to be more explicit about which integration domain to choose for the superfields $\phi_l$, $\bar{\phi}_l$ and $Q$.

### 6 Saddle-point manifold

The issue of how to choose correctly the integration domain for the superfields $\phi$, $\bar{\phi}$ and for the supermatrix $Q$ was carefully addressed in [8]. Drawing on (but not repeating) the detailed analysis of that reference, we argue as follows.

Let $\text{Im}E > 0$ for definiteness. Then, in order to have convergent integrals over the bosonic ghosts of the Dirac version of the theory, we must impose the conditions (8) or, equivalently, $\phi^\dagger_{l,A} = \phi^\dagger_{l,A}\sigma_3$. Note that there is no need to require such a relation among the Grassmann fields, and we do not impose any such requirement.
Next one easily verifies that the bilinear $A \equiv \phi_l \bar{\phi}_t^\dagger + \bar{\phi}_l \phi_t^\dagger$ in the Lagrangian $L_4$ is odd under the orthosymplectic transpose: $A^t = -A$, so $A \in \mathfrak{osp}(2|2)$, by definition of the orthosymplectic Lie algebra. Via its coupling to $A$ in $L_4$, the supermatrix $Q$ inherits the same property:

$$Q = -Q^t \in \mathfrak{osp}(2|2).$$

Now, in the process of decoupling the interaction $\Phi^{(1)} + \Phi^{(2)}$, severe convergence problems arise in the boson-boson (BB) block denoted by $Q_{BB}$. However, close inspection shows that the integrals over $Q_{BB}$ can be arranged to exist (without ruining the convergence of the integrals over the bosonic Dirac ghosts $b, c$) by choosing the following parametrization:

$$Q_{BB} = Y + \gamma e^X \Sigma_3 e^{-X},$$

where $X, Y \in \mathfrak{osp}(2|2)$ are odd resp. even with respect to conjugation by $\Sigma_3$:

$$\Sigma_3 X \Sigma_3 = -X, \quad \Sigma_3 Y \Sigma_3 = Y,$$

and the even (or bosonic) parts $X_0, Y_0$ of $X, Y$ are subject to

$$X_0 = X_0^\dagger, \quad Y_0 = -Y_0^\dagger.$$

The parameter $\gamma > 0$ is in principle arbitrary but, anticipating its saddle-point value, we set it to $\gamma = \mu$. This deals with the BB sector. The situation in the FF sector is more benign. There, convergence of the integrals over $Q_{FF}$ is simply achieved by requiring $Q_{FF} = Q_{FF}^\dagger$.

Now we can compare the above parametrization for $Q$ to the diagonal solutions of the saddle-point equation. The structure of the solution in the BB block is uniquely fixed by the convergence requirements to be

$$Q_{0, BB} = \mu \sigma_3.$$

Other diagonal solutions cannot be reached by deformation of the integration manifold without crossing some singularities of the integrand.

In contrast, in the FF sector analyticity provides no selection rule on the saddle points; there, the integrand does not possess any poles, but only has zeroes as a function of $Q_{FF}$, so that the path of integration can be analytically continued to cross any saddle point in that sector. However, in the weak-coupling (or large $N$) limit, some of the saddles contribute in a negligible
manner. The dominant contributions come from those that minimize the super-dimension of the transverse manifold. This extremality condition is fulfilled when the positive and negative eigenvalues of $Q_{0,FF}$ are equal in number. In the present case, this criterion leaves two possibilities:

$$Q_{0,FF} = \pm \mu \sigma_3 .$$

In summary, we retain as dominant (diagonal) saddle points:

$$Q_0 = \mu (\pm E_{FF} + E_{BB}) \otimes \sigma_3 .$$ (20)

Recall now the existence of a global $G \equiv \text{OSp}(2|2)$ symmetry (at $E = 0$). The symmetry group acts on the supermatrix $Q$ by conjugation: $Q(x) \mapsto TQ(x)T^{-1} (T \in G)$. Clearly, such transformations leave the action functional $S_5[Q]$ in (17) invariant. As a result, the saddles of $S_5[Q]$ are degenerate: given any solution $Q_0$ of the saddle-point equation, we get an orbit of solutions by acting on $Q_0$ with the symmetry group $G$. Because the stability condition $hQ_0h^{-1} = Q_0$ divides out a subgroup $H = \text{GL}(1|1)$, called the stabilizer of $Q_0$, the orbit of $G$ on $Q_0$ is a coset space $G/H = \text{OSp}(2|2)/\text{GL}(1|1)$. Notice that this is the orbit of the complex symmetry group.

To carry out the saddle-point integral, we need to restrict the bosonic degrees of freedom of $G/H$ to a real submanifold. We now briefly describe this submanifold, setting the Grassmann variables temporarily to zero. The bosonic part of the complex supergroup $\text{OSp}(2|2)$ is $\text{O}(2,\mathbb{C}) \times \text{Sp}(2,\mathbb{C})$. In the BB sector, the symmetry group is $\text{Sp}(2,\mathbb{C})$, which arises as the group of transformations $T$,

$$\begin{pmatrix} b \\ c \end{pmatrix} \mapsto T \cdot \begin{pmatrix} b \\ c \end{pmatrix} , \quad \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} \mapsto T \cdot \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} ,$$

leaving invariant the symplectic form

$$\bar{b}c - \bar{c}b = (\bar{b}, \bar{c}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} .$$

Those symplectic transformations of $(b, c)$ and $(\bar{b}, \bar{c})$ that preserve the reality conditions (8), form a real subgroup $\text{Sp}(2,\mathbb{R})$. Division by the stabilizer, which is isomorphic to $\text{U}(1) \in H$, then yields the BB manifold $M_B = \text{Sp}(2,\mathbb{R})/\text{U}(1) \simeq H^2$. This is a two-hyperboloid. (For general $n$, we get $\text{Sp}(2n,\mathbb{R})/\text{U}(n)$, which is a noncompact symmetric space of type CI.)
In the FF sector, the symmetry group acts by \(O(2, \mathbb{C})\), which is understood to be the invariance group of the symmetric form (or “orthogonal” structure)

\[
\bar{\psi}_+ \psi_- + \bar{\psi}_- \psi_+ = (\bar{\psi}_+, \bar{\psi}_-) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.
\]

We have emphasized the crucial fact that we never impose any reality conditions on the fermions. What determines then the real FF subgroup? The answer is that reality here enters through the Hermiticity constraint \(Q_{\text{FF}} = Q_{\text{FF}}^\dagger\) (needed for convergence), which selects from the complex symmetry group \(O(2, \mathbb{C})\) a real subgroup \(O(2)\), the usual orthogonal group in two dimensions. Dividing by the stabilizer \(U(1)\), we obtain \(M_F = O(2)/U(1) \simeq \mathbb{Z}_2\). (For general \(n\), we get \(O(2n)/U(n)\), a compact symmetric space of type \(D\).) Thus the base of the saddle-point manifold is \(M_B \times M_F = H^2 \times \mathbb{Z}_2\). On reinstating the Grassmann variables, we arrive at a saddle-point supermanifold, \(X_1\), which is a Riemannian symmetric superspace of type \(CI|D\) III \[8\].

A distinctive feature of \(X_1\) is that, instead of being connected, it consists of two disjoint pieces. This fact was already discovered in \[8\], and we are now going to elaborate briefly. The special orthogonal group \(SO(2)\) acts on two Majorana fermions \((\xi, \eta)\) as

\[
\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix},
\]

while the corresponding \(SO(2)\) action on the Dirac fermion \(\psi_\pm = \xi \pm i\eta\) is

\[
\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.
\]

This does not exhaust the symmetries of the FF sector. The Majorana fermion is known to possess an additional discrete symmetry, which is reflection \(\xi \leftrightarrow \eta\) (an element of \(O(2)\) with determinant minus one), or in Dirac language:

\[
\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = -\sigma_2 \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.
\]

Consider now the action of these symmetry transformations on the saddle point \(Q_{0,\text{FF}} = \sigma_3\) in the FF sector. The action of \(SO(2)\) simply fixes the saddle point: \(e^{i\theta}\sigma_3e^{-i\theta}\sigma_3 = \sigma_3\), and thus is trivial, while the discrete \(O(2)\) transformation by \(\sigma_2\) reverses the sign: \(\sigma_3 \mapsto \sigma_2\sigma_3\sigma_2 = -\sigma_3\). The latter then explains the origin of \(\mathbb{Z}_2\) in the symmetric superspace \(X_1\).
In summary, all of the saddle points of $S_5[Q]$ lie on a single orbit of the bosonic symmetry group $O(2) \times \text{Sp}(2, \mathbb{R}) \subset \text{OSp}(2|2)$, but the orbit is not connected, as $O(2)$ comes in two pieces. We mention in passing that this peculiar feature of universality class $D$ also occurs in class $DIII$ [8].

Although we have focused on the formalism for a single Green function ($n = 1$), all of our considerations readily extend to the case $n \geq 1$. Supercmatrices simply become longer or bigger. The symmetry group inflates to $G = \text{OSp}(2n|2n)$, the matrix $\Sigma_3$ tensors to $\Sigma_3 = 1_{\text{susy}} \otimes \sigma_3 \otimes 1_n$, and the stabilizer becomes $H = \text{GL}(n|n)$. The order parameter space for a general value of $n$ is denoted by $X_n$, and has bosonic submanifold

$$M_B \times M_F = (\text{Sp}(2n, \mathbb{R})/U(n)) \times (O(2n)/U(n)) .$$

Again this consists of two disjoint pieces, corresponding to the two connected components of $O(2n)$ (with determinant plus or minus one). In the following section, we shall work with the extension to general $n$.

As an important corollary to the discovery of a $\mathbb{Z}_2$ degree of freedom in the saddle-point manifold, we anticipate the existence of $\mathbb{Z}_2$ domains and domain walls, across which $Q$ jumps from one connected component of the saddle-point manifold to the other. We expect this Ising-like degree of freedom to be very important for the phenomenology of the insulating phases of class $D$ (Section 11), where $Q$ fluctuates strongly. On the other hand, when the field is stiff, domain walls are costly in energy, and therefore they should be of minor relevance in the metallic limit we shall focus on in Sections 7–9.

7 Gradient expansion

To summarize the current state of affairs, the problem at hand has a global $G = \text{OSp}(2n|2n)$ symmetry, which is broken to $\text{GL}(n|n)$ by making a naive saddle-point approximation. Whether the symmetry is truly broken or not, will have to be decided at a later stage. (The Mermin-Wagner-Coleman theorem, stating that continuous symmetries of compact type cannot be broken spontaneously in two dimensions, does not apply in the present case, as the saddle-point manifold is a noncompact superspace. Hence, the question whether symmetry breaking occurs or not remains open for now.) Our saddle-point analysis has identified an order parameter $Q = \mu q$, where $q \equiv T \Sigma_3 T^{-1}$ takes values in a symmetric superspace $X_n$. The low-energy configurations
of the action $S_5[Q]$ in (17) are given by slowly varying fields

$$q(x) = T(x)\Sigma_3 T(x)^{-1} \quad (T(x) \in G).$$

These are the Goldstone modes of the broken $G$ symmetry. Note that $q$ lies (on an adjoint orbit of $G$) in $\text{osp}(2n|2n)$, and satisfies the constraint $q^2 = 1$.

Our next goal is to derive the low-energy effective action for the Goldstone modes $q(x)$. (We shall neglect field fluctuations transverse to the saddle-point manifold, since these are massive. We shall also assume $q(x)$ to be smooth, which means we will ignore domain walls in the $\mathbb{Z}_2$ degree of freedom.) On general grounds, the effective action must be of the form

$$S_{\text{eff}}[q] = -\frac{1}{16\pi f} \int d^2 x \text{STr } \partial_\mu q \partial_\mu q + \frac{\theta}{32\pi} \int d^2 x \epsilon_{\mu\nu} \text{STr } q \partial_\mu q \partial_\nu q. \quad (21)$$

The two terms displayed are the only ones that contain no more than two derivatives, respect rotational invariance of position space, and are compatible with global $G$ symmetry. (Later we will add a symmetry-breaking term for finite energy $E \neq 0$.) The second term is a topological or winding number term. It arises by pulling back the closed two-form $\text{STr } q d^2 q \land dq$ (called the Kähler form) of $X_n$, and is nontrivial since

$$\Pi_2(X_n) = \Pi_2(M_B \times M_F) = \Pi_2(O(2n)/U(n)) = \Pi_1(U(n)) = \mathbb{Z}$$

for $n > 1$. (In contrast, the $n = 1$ winding number $\Pi_2(O(2)/U(1)) = 0$ is trivial. It may seem curious that there exists such a marked difference between $n = 1$ and $n > 1$, as the topological term is supposed [11] to play an important role in controlling the phase transition between the two insulating phases of class $D$ in two dimensions. We will suggest an explanation later, when we discuss the phase diagram.) Quantitatively put, for any closed surface $\Sigma$ we have

$$\frac{1}{32\pi i} \int_\Sigma d^2 x \epsilon_{\mu\nu} \text{STr } q \partial_\mu q \partial_\nu q \in \mathbb{Z}.$$

For $n = 2$ the correct normalization factor $(32\pi i)^{-1}$ can be figured out by direct calculation, using $O(4)/U(2) \simeq S^2 \times \mathbb{Z}_2$. The generalization to $n > 2$ is dictated by the connection with the multi-valued action $\Gamma[M]$ in equation (24) below. We note that the topological term is odd under parity $x_1 \leftrightarrow x_2$. Nevertheless, its presence is permitted, as a choice of pure Dirac Hamiltonian
\[ H_0 = \sigma_1 p_1 + \sigma_2 p_2, \] as compared to \[ H_0 = \sigma_2 p_1 + \sigma_1 p_2, \] implies a choice of orientation of position space.

Our task now is to work out what the values of the couplings \( f \) and \( \theta \) are. This is easily done for \( f \), by straightforward gradient expansion of

\[ S_5[q] = \frac{N}{2} \ln \text{SDet} \left( \begin{array}{cc} \mu \Sigma_3 q & \partial \\ \mu q \Sigma_3 & \partial \end{array} \right). \] (22)

On the other hand, the value of the topological angle \( \theta \) is a subtle issue. This is not easily found by gradient expansion. The reason is that topological excitations such as instantons, whose topological charges are what is counted by the winding number term, necessarily involve large variations of the field \( q \), thereby defying simple considerations based on Taylor expansion of the action functional. One viable option would be to evaluate both \( S_{\text{eff}}[q] \) and \( S_5[q] \) on a well-chosen configuration \( q(x) \), say an instanton solution (such solutions exist for \( n > 1 \)) and equate the two answers to determine \( \theta \). Unfortunately, the evaluation of \( S_5[q] \) on an instanton is neither easy nor rewarding (which is to say that it does not lead to any significant payoff beyond solving the technical problem at hand). A powerful standard trick for computing determinants of Dirac operators is to take the logarithmic derivative with respect to a parameter \( s \), and integrate over \( s \) at the end. Such a strategy is doomed to fail here, as differentiating a winding number necessarily produces zero (and, moreover, topologically distinct sectors cannot be continuously connected with one another).

The most elegant and painless scheme for extracting the low-energy effective action from \( S_5[q] \) proceeds via the method of nonabelian bosonization. Following a celebrated paper by Witten [14], this method has become a standard tool for dealing with perturbations of the free fermion theory. Unfortunately, in the supersymmetric context of Dirac fermions augmented by a \( b-c \) ghost system, the method is less established. Therefore, in the subsections that are appended below, we will make a digression to explain the needed extension of nonabelian bosonization, using the language of functional integrals. For now, we describe how the method is used to achieve our goal of expanding \( S_5[q] \) in gradients.

We start by undoing the integration over the fields \( \phi, \bar{\phi} \):

\[ e^{-S_5[q]/N} = \int \mathcal{D}\phi \, \mathcal{D}\bar{\phi} \, \exp \left( \int d^2 x \left( \bar{\phi}^i \partial \bar{\phi}^i + \phi^i \partial \phi^i + \mu \bar{\phi}^i q \phi + \mu \phi^i q \bar{\phi} \right) \right). \]

The notation is the same as before, except that we have dropped the summation over the species index \((l = 1, ..., N)\) and divided \( S_5 \) by \( N \) accordingly.
The next step is to bosonize, using the principle of Bose-Fermi equivalence in two dimensions. Witten’s nonabelian version asserts that the free theory of $2n$ Majorana or $n$ Dirac fermions has an equivalent representation by a Wess-Zumino-Novikov-Witten (WZW) model with target $O(2n)$ resp. $U(n)$ at level $k = 1$. As will be shown in Subsections 7.1 and 7.2 below, this equivalence can be generalized to allow for the presence of a $b$-$c$ ghost system on the free-fermion side. The equivalent “bosonized” theory turns out to be what might loosely, but only loosely, be called an “OSp($2n|2n$)” WZW model. We will give the precise definitions later. Here we simply state that the WZW field, which we denote by $M$, takes values in a subspace of the complex supergroup OSp($2n|2n$), and the action functional of the WZW model has the usual form,

$$W[M] = \frac{1}{16\pi} \int d^2 x \text{Str} \left( (M^{-1} \partial_\mu M)^2 + i\Gamma[M] \right).$$

(23)

Following Witten, the multi-valued functional $\Gamma[M]$ is expressed by assuming some extension $\tilde{M}$ of $M$ to a 3-ball $B$ that has position space $\Sigma$ for its boundary ($\partial B = \Sigma$):

$$\Gamma[M] = \int_B \text{Str} \left( (\tilde{M}^{-1} d\tilde{M})^3 \right)$$

$$= \int_B d^3 x \epsilon_{\mu\nu\lambda} \text{Str} \tilde{M}^{-1} \partial_\mu \tilde{M} \tilde{M}^{-1} \partial_\nu \tilde{M} \tilde{M}^{-1} \partial_\lambda \tilde{M}.$$  

(24)

Of course, in order for this to work we must assume the position space $\Sigma$ to be a surface without boundary, say a two-sphere or a two-torus.

The nonabelian bosonization rules tell us to replace the free Dirac theory plus $b$-$c$ ghost system by the WZW model with action $W[M]$, and the bilinears $\bar{\phi}\phi\Sigma_3$ and $\Sigma_3\bar{\phi}\phi$ by $\ell^{-1}M$ resp. $\ell^{-1}M^{-1}$. (The factor $\ell^{-1}$ is a large mass scale which enters for dimensional reasons and depends on the regularization scheme.) Doing so, and setting $T = e^X$ with $X = -\Sigma_3X\Sigma_3$, so that $q\Sigma_3 = T\Sigma_3T^{-1}\Sigma_3 = e^{2X} = T^2$, we obtain

$$e^{-S_5/N} = \int \mathcal{D}M \exp \left( -W[M] - \frac{\mu}{\ell} \int d^2 x \text{Str}(MT^{-2} + T^2M^{-1}) \right).$$

Next, we remove the factor $T^2 = e^X$ from the second term in the exponent, by changing integration variables from $M$ to $M' = MT^{-2}$:

$$e^{-S_5/N} = \int \mathcal{D}M' \exp \left( -W[M'T^2] - \frac{\mu}{\ell} \int d^2 x \text{Str}(M' + M'^{-1}) \right).$$
(Note that, since the adjoint orbit of $T$ on $\Sigma_3$ consists of two disjoint components, this change of variables can only be valid if $M$ runs through two components, too.) $T^2$ now appears in the argument of the WZW functional $W$, while the other term has become a plain mass term.

The last step is to argue that for small length $\ell$, or large mass $\mu$, the WZW field $M'$ fluctuates only weakly around $M' = 1$. Indeed, the potential $\text{STr}(M + M^{-1})$ has an absolute minimum at $M = 1$, as follows from the definition of the target space as a Riemannian symmetric superspace, see equation (29) below. The leading approximation then is to set $M'$ simply to unity, which yields

$$S_5[q]_{q = T \Sigma_3 T^{-1}} = NW[T^2].$$

Calculating the first correction due to fluctuations of $M'$ around 1, one finds a term with four gradients, multiplied by the square of the length scale $\sqrt{\ell/\mu}$. This four-gradient term becomes small, and the approximation of setting $M' = 1$ is therefore justified, on length scales larger than $\sqrt{\ell/\mu}$. Note that if the saddle-point approximation is applied after renormalization has brought the initially small value of $g$ to about unity, the two length scales $\ell$ and $\mu^{-1} \approx 2\ell \sqrt{g/\pi}$ are parametrically the same. It is then this length scale $\mu^{-1} \sim \ell$ which will set the short-distance cutoff of the nonlinear sigma model.

By inserting $M = T^2 = e^{2X}$ with $X = -\Sigma_3 X \Sigma_3$ into the first term of $W[M]$ and comparing with $S_{\text{eff}}[q]$ for $q = T \Sigma_3 T^{-1} = e^{2X} \Sigma_3$, we readily infer $f = 1/N$. This result for $f$ is easy to verify by direct gradient expansion of $S_5$, without passing through the WZW model. A more accurate calculation, taking into account the finite value of $\mu/\ell$, gives

$$f = \frac{1}{N(1 - e^{-\pi/g})}.$$  \hfill (25)

Now we tackle the delicate task of calculating the coupling $\theta$. Since we expect the topological term of the nonlinear sigma model to arise from the topological term $\Gamma[M]$ of the WZW model, we substitute $M = T^2$ into $\Gamma[M]$. Doing so, we naively get zero – the technical reason is that $X_n$ does not support such a term – but only naively so. The point to observe is that writing $\Gamma[M]$ in the form (24) requires a smooth extension of $M$ to the ball $\mathcal{B}$. If we insist on $M \Sigma_3 = T^2 \Sigma_3 = q$ taking values in $X_n$, such an extension does not exist, by the topological obstruction $\Pi_2(X_n) \neq 0$ (for $n > 1$).

We can avoid the obstruction by allowing $q$ to vary over a larger set, say the complex group $\text{OSp}(2n|2n)$. Then, if $0 \leq s \leq 1$ is a radial coordinate for
the ball $B$, we can take the extension to be
\[ \tilde{M}(x, s) = T(x) \exp(\pm is\pi \Sigma_3/2) T(x)^{-1}(\mp i\Sigma_3). \]

It is seen that for $s = 1$ we have $\tilde{M}(x, 1) = T(x)\Sigma_3 T(x)^{-1} \Sigma_3 = q(x)\Sigma_3$, while for $s = 0$ we get $\tilde{M}(x, 0) = \mp i\Sigma_3$, independent of $x$. By inserting this extension into the expression (24) for $\Gamma[M]$, and converting the integral over $B$ into an integral over $\Sigma = \partial B$ using Stokes’ theorem, we find
\[ \frac{i}{24\pi} \Gamma[q\Sigma_3] = \pm \frac{1}{32} \int_\Sigma d^2x \epsilon_{\mu\nu} \text{STr} q \partial_\mu q \partial_\nu q . \]

Given the relation $S_5[q] = NW[q\Sigma_3]$ and the formula for $W[M]$, comparison of this result with the nonlinear sigma model action (21) yields
\[ \theta = \pm N\pi . \quad (26) \]

The sign ambiguity arises from the multi-valuedness of $\Gamma[M]$. It has no consequence, since the physics of the nonlinear sigma model is periodic in the topological angle $\theta$, as long as the position space $\Sigma$ has no boundary.

For a complete description, what remains to be done is to augment the effective action with a symmetry-breaking term for finite energy $E \neq 0$. By shifting $q \mapsto q - i(E/2\mu)\Sigma_3$ in (22), and expanding to linear order in $E$, we obtain the full effective action
\[ S_E[q] = S_{\text{eff}}[q] - \frac{i\mu NE}{2g} \int d^2x \text{STr} q \Sigma_3 . \quad (27) \]

Finally, a standard calculation starting from the Lagrangian (7) yields the following expression for the local density of states:
\[ \nu(E; x) = \frac{\mu N}{2\pi g} \text{Re} \int \mathcal{D}q \text{Tr} q_{\Phi}(x) \sigma_3 e^{-S_E[q]} , \quad (28) \]

where the functional integral has to be computed with a positive imaginary part $\text{Im}E > 0$, in the limit $\text{Im}E \to 0$.

The above derivation of the couplings $f$ and $\theta$ was only of formal validity, as the precise definition of the WZW model was omitted and no justification of the bosonization rules was given. In the following two subsections, we are going to fill in the gaps. Readers not interested in these details are urged to proceed directly to Section 8.
7.1 WZW model of type $C|D$

In the present subsection we will construct the target space of the WZW model, based on the general notion of Riemannian symmetric superspace. Although this construction is by no means new, it is not as well known as it should be. Many influential authors have based their considerations on WZW models over the real supergroups $\text{GL}_R(n|n)$ or $\text{OSp}_R(2n|2n)$, or on the unitary supergroups $U(n|n)$. While such practice may have become standard, and is admittedly rather convenient and quite satisfactory for a number of formal calculations in current algebra, it does not take the WZW model seriously as a functional integral, and we are not going to adopt it. Bosonic (i.e. non-super) nonlinear sigma models, including the class of bosonic WZW models, are defined as functional integrals of maps from a Riemann surface into a target space with Riemannian structure. If our purpose is to extend this definition to the super world in a mathematically sound way, we have to address the fact that the natural or invariant geometry on a supergroup invariably is non-Riemann.

To appreciate the difficulty in some detail, consider $U(n|n)$ for example. The even part of the Lie superalgebra of $U(n|n)$ is the direct sum of two copies of $u(n)$, which is spanned by the anti-Hermitian $n \times n$ matrices. Thus, bosonic tangent vectors at the unit element of $U(n|n)$ are pairs $(A, B)$ with $A^\dagger = -A$ and $B^\dagger = -B$. The natural $U(n|n)$ invariant quadratic form (or metric) evaluated on $(A, B)$ is the supertrace $\text{Tr} A^\dagger A - \text{Tr} B^\dagger B$, which has indefinite sign. As a result, the action functional of the principal chiral nonlinear sigma model with target supermanifold $U(n|n)$ is bounded neither from below nor from above. On field configurations that fluctuate rapidly in space, the action becomes arbitrarily large, and since the action can be either positive or negative, the functional integral is unstable with respect to fluctuations for any choice of sign of the coupling constant. Therefore, unless some additional procedure (such as analytic continuation from $A^\dagger A - B^\dagger B$ to $A^\dagger A + B^\dagger B$) is specified, the theory with target $U(n|n)$ does not exist.

Such reasoning is not restricted to $U(n|n)$ but holds for the other cases as well. Indeed, the Cartan-Killing form of any Lie superalgebra is a supertrace, i.e. a difference between two traces. For the supergroups listed above, this implies that any rank-two (super)symmetric tensor $\kappa$ invariant under the left and right group actions is necessarily non-Riemann, by which we mean that the metric tensor obtained by restriction of $\kappa$ to the bosonic support has indefinite signature. For this universal reason, supergroups are ruled out as
a target spaces for nonlinear sigma models, at least in the literal sense.

We understand that the Lagrangians of the “supergroup” WZW models which exist in the literature, are used mainly for bookkeeping purposes, or as a device to generate equations of motion or other identities, such as the Knizhnik-Zamolodchikov equation, that do not depend on working with a Riemannian geometry. In the present paper, we wish to put the WZW model (or some descendant theory) to a more severe test: we intend to integrate carefully over the global zero modes (treating the theory in zero-mode approximation), so as to establish the connection with exact random-matrix limits that have previously been obtained. For that nonperturbative purpose, we must construct a functional integral that exists as such.

While the basic difficulty with supergroup targets is easily recognized, it is not so easy to circumvent. For example, switching from $U(n|n)$ to its noncompact analog $\text{GL}_C(n|n)/U(n|n)$ does not improve the situation. The latter space is still non-Riemann, and the nonlinear sigma models over it make sense only in combination with some procedure of analytic continuation [26].

It turns out that the difficulty cannot be overcome by using only the toolshed of standard supermanifold theory. What is required is a novel concept, namely that of cs-manifold[4], which transcends the traditional categories of real-analytic and complex-analytic supermanifolds. In short, cs-manifolds are super bundles supported by a real-analytic manifold, with a fibre which is a Grassmann algebra over $\mathbb{C}$ carrying no operation of complex conjugation or adjoint. In plain physics language, the “bosons are real while the fermions are complex”. It is not hard to see that this is just the right kind of mathematical setting to use for the construction of nonlinear sigma models with superspace target. Indeed, what we want is a target space with a Riemannian metric on its bosonic base manifold, giving an action functional bounded from below. A Riemannian metric distinguishes the real numbers from the imaginary numbers, in the sense that the tangent vectors with positive length form a vector space over $\mathbb{R}$ (and not over $i\mathbb{R}$ or $\mathbb{C}$). This is what is meant by saying that the “bosons are real”. On the other hand, fermionic “integration” according to Berezin is nothing but differentiation, and the definition of Berezin’s integral requires no notion of reality, so it is most natural to “leave the fermions complex” [28]. Our intention as statistical physicists is to do integrals (computing disorder averages of Green functions), and nothing

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[4] This terminology is due to Bernstein [27]. The letter “c” stands for complex, and “s” for super.
but integrals. Hence, we may take the extreme point of view that complex conjugation, while indispensable for the bosons, is a forbidden operation on fermions. This is the point of view advocated in [8], and it is the one we adopt here. Note that this means that we bar supergroups such as $U(n|n)$, the definition of which requires the use of an adjoint for the fermions.

Having prepared the stage with these remarks, we now turn to the description of the target space of the WZW model. Our starting point is the complex supergroup $G \equiv \text{OSp}_C(2n|2n)$. Its elements, which we denote by $M$, satisfy the equation $M^{-1} = M^t$, where the orthosymplectic transpose $M^t$ was defined by (5) and $(M\phi)^t = \phi^t M^t$. The group $G$ comes with a rank-two supersymmetric tensor $\kappa = -\text{STr} dM^{-1}dM = \text{STr} (M^{-1}dM)^2$, which has the distinctive property of being invariant under left and right translations $M \mapsto g_L M g_R^{-1}$. Let $G_0 = O(2n, \mathbb{C}) \times \text{Sp}(2n, \mathbb{C})$ denote the ordinary (or bosonic) subgroup of $G$. Since $G_0$ is a group of matrices with complex entries, the geometry induced on $G_0$ by restriction of $\kappa$ is non-Riemann.

What we need to do now is to specify a submanifold of $G_0$ on which the geometry induced by $\kappa$ is Riemann. This is done as follows. From $O(2n, \mathbb{C})$ (the FF-sector) we select a submanifold $M_F$ isomorphic to the compact orthogonal group $O(2n)$. Near the group unit, $M_F$ is parametrized by $e^Y$ where $Y = -Y^\dagger = -\sigma_1 Y^T \sigma_1$. The group $O(2n)$ acts on $M_F$ independently on the left and right by $e^Y \mapsto O_L e^Y O_R^{-1}$, and this action is transitive, which is to say that all elements of $M_F$ are translates of unity. In the BB-sector we proceed differently, by selecting the intersection, $M_B$, of $\text{Sp}(2n, \mathbb{C})$ with the set of positive Hermitian matrices. The elements of $M_B$ can be written as $gg^\dagger$ with $g \in \text{Sp}(2n, \mathbb{C})$. Since forming the product $gg^\dagger$ divides out the maximal compact subgroup $\text{Sp}(2n)$ of $\text{Sp}(2n, \mathbb{C})$ on the right, the set $M_B$ is isomorphic to $\text{Sp}(2n, \mathbb{C})/\text{Sp}(2n)$. Using the exponential map, $M_B$ can be parametrized by $e^X$ where $X = +X^\dagger = -\sigma_2 X^T \sigma_2$. Note that the product of two positive Hermitian matrices is not positive Hermitian in general, so $M_B$ is not a group. Rather, $M_B$ is a noncompact symmetric space, and the complex group $\text{Sp}(2n, \mathbb{C})$ acts on $M_B$ transitively by $e^X \mapsto g e^X g^\dagger$.

We now look at the product $M_B \times M_F$. This may seem like a somewhat unnatural hybrid to consider, since $M_F$ is group whereas $M_B$ is not. However, both $M_F$ and $M_B$ are Riemannian symmetric spaces, the former of compact and the latter of noncompact type, and the product $M_B \times M_F$ has the desired property of being a Riemannian submanifold of $G_0$. To establish the Riemannian property, we first inspect the tangent space at unity,
\( \mathcal{T}_1 (\mathcal{M}_B \times \mathcal{M}_F) \). Its elements are pairs \( A \oplus B = \frac{d}{ds} e^{s(A \oplus B)} \bigg|_{s=0} \), and evaluation of \( \kappa = \text{STr} (M^{-1} dM)^2 \) on these (at \( M = 1 \)) yields

\[
\text{STr} (A \oplus B)^2 = \text{Tr} A^2 - \text{Tr} B^2 = \text{Tr} A^\dagger A + \text{Tr} B^\dagger B \geq 0.
\]

From this we clearly see that \( \kappa \) is positive definite on \( \mathcal{T}_1 (\mathcal{M}_B \times \mathcal{M}_F) \). This property carries over to all of \( \mathcal{M}_B \times \mathcal{M}_F \), by the invariance of \( \kappa \) under the transitive action of the group \( \text{Sp}(2n, \mathbb{C}) \times (\text{O}(2n)_L \times \text{O}(2n)_R) \). Thus, \( \kappa \) restricts to a Riemannian structure on \( \mathcal{M}_B \times \mathcal{M}_F \) as claimed.

As an immediate consequence, we have that the numerical part of the function \( \text{STr} M = \text{STr} M^{-1} \) on \( \mathcal{M}_B \times \mathcal{M}_F \) is locally expressed by

\[
\text{STr} M_0 = \text{Tr} e^A - \text{Tr} e^B = \text{Tr} e^{-A} - \text{Tr} e^{-B} = \text{Tr} \cosh \sqrt{A^\dagger A} - \text{Tr} \cos \sqrt{B^\dagger B} .
\]

(29)

This function has an absolute minimum at \( M = 1 \).

In summary, the object at hand is highly structured, consisting of the complex supergroup \( G = \text{OSp}_C(2n|2n) \) with metric \( \kappa = \text{STr} (M^{-1} dM)^2 \), and a Riemannian submanifold \( \mathcal{M}_B \times \mathcal{M}_F \). The triple \( (G, \kappa, \mathcal{M}_B \times \mathcal{M}_F) \) is what is called \[8\] a Riemannian symmetric superspace of type \( C|D \). The name encodes the fact that the noncompact BB-sector is symplectic, or \( C \), while the compact FF-sector is orthogonal in even dimension, or \( D \).

The virtue of Riemannian symmetric superspaces, as opposed to supergroups, is that they make valid target spaces for nonlinear sigma models, as follows. Let the action functional be denoted by \( S \) and constructed in the usual way, \( i.e. \) if \( \kappa_{ij}(\varphi) d\varphi^i d\varphi^j \) is the expression of the metric in terms of target space supercoordinates \( \varphi^i \), which are bosonic for \( i = 1, \ldots, p \) and fermionic for \( i = p + 1, \ldots, p + q \), we have \( S = \int d^2 x \kappa_{ij}(\varphi) \partial_{\mu} \varphi^i \partial_{\mu} \varphi^j \). According to Berezin \[29\] superintegration is a two-step process. First we do the Fermi integral, which is to say we differentiate \( e^{-S} \) with respect to the fermionic fields. (For this no definition of an adjoint or complex conjugation of the fermions is needed.) This “integral” always exists – provided we take care to control the infinite number of integration variables – in the sense that the derivative of an analytic function always exists. The result of doing the Fermi integral is a functional, say \( e^{-S_0} D_0 \), on the bosonic fields. In the second step, we carry out the integral over the bosonic fields, which take values in a real target manifold \( (\mathcal{M}_B \times \mathcal{M}_F \text{ in the present case}) \). This functional integral exists (modulo the notorious need for regularization in the ultraviolet,

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regularization of zero modes etc.) because we have constructed the target as a Riemannian manifold, with a bosonic action $S_0$ bounded from below.

One might object that we are violating “supersymmetry” by using complex conjugation on the bosons (to fix the Riemannian submanifold $\mathcal{M}_B \times \mathcal{M}_F$), while barring the use of complex conjugation and any other adjoint on the fermions. This is not so. The role of supersymmetry here is to equip the theory with a BRST symmetry, thereby turning it into a kind of topological field theory and ensuring normalization of the partition function to unity. BRST symmetry does not require that we treat bosons and fermions in an egalitarian manner with respect to complex conjugation. The essence of the argument can be captured by looking at a simple zero-dimensional example.

For any analytic function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{C}$ with rapid decay at infinity and $f(0) = 1$, consider the superintegral

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} dx \, dy \, \partial_\xi \partial_\eta \, f(x^2 + y^2 + 2\xi \eta) = f(0) = 1,$$

where $x$ and $y$ (the “bosons”) are Cartesian coordinates of $\mathbb{R}^2$, and $\xi$ and $\eta$ (the “fermions”) are Grassmann variables. The integral is always equal to unity, and this holds true irrespective of whether any conjugation properties are imposed on $\xi$ and $\eta$ or not. (Thus we do not need to say that $\xi$ and $\eta$ are “real”, or $\eta$ is the “complex conjugate” of $\xi$, or anything of that sort.) The integral reduces to $f(0) = 1$ by dimensional reduction [30], as a consequence of invariance of the integrand under BRST transformations

$$\delta x = \varepsilon \xi, \quad \delta y = \bar{\varepsilon} \eta, \quad \delta \xi = -\bar{\varepsilon} y, \quad \delta \eta = \varepsilon x.$$

Since the generator of the BRST transformation vanishes only at the origin $x = y = 0$, the integrand on $\mathbb{R}^2 \setminus (0,0)$ can be written as the BRST derivative of another function, causing localization of the integral to the origin. The same localization principle applies in the functional setting.

One might ask again the basic question why we use the highly structured concept of Riemannian symmetric superspace instead of more conventional constructions (invoking an adjoint for both bosons and fermions). Isn’t there a simpler way of doing it? We have tried hard, but apparently the answer is no, if we insist on global supergroup symmetry and the stringent requirement of an invariant Riemannian structure on the bosonic base of the target space. There is one thing, however, that we could do in the way of simplification or economy of formulation, which would be to eliminate most of the complex
supergroup $G$ from the triple $(G, \kappa, \mathcal{M}_B \times \mathcal{M}_F)$, and retain just the Grassmann algebra fibres over the points of the base $\mathcal{M}_B \times \mathcal{M}_F$. The resulting object would be a cs-manifold in the terminology of Bernstein. We do not take this step here.

The WZW model whose existence we postulated earlier for the purpose of expanding $S_5[q]$ in gradients, is the functional integral of maps $M$ from position space into the Riemannian symmetric superspace of type $C|D$,

$$\left( \text{OSp}_C(2n|2n), \text{STr}(M^{-1}dM)^2, \mathcal{M}_B \times \mathcal{M}_F \right),$$

with the action functional given in (23). (To make the model completely well-defined, we add an infinitesimal mass term $\epsilon \int d^2x \text{STr}(M + M^{-1})$ to regularize the zero modes.) We propose to call it the $C|D$ WZW model at level $k = 1$. The simpler name “OSp(2n|2n)” WZW model would be inappropriate for two reasons. Firstly, it would belittle the above discussion emphasizing the necessity to use a Riemannian symmetric superspace instead of a supergroup target. Secondly, such a name could cause some confusion since there exists yet another “OSp(2n|2n)” WZW model, still at level $k = 1$, which differs from the present one by the exchange of the BB- and FF-sectors – in other words, the bosons carry an orthogonal structure while the fermions are symplectic. (The latter model, with $k = 2$, may have some relation to the fixed point governing the spin quantum Hall transition [31, 32, 33].)

There is one point that needs further attention. Although the existence and stability of the principal chiral nonlinear sigma model with target $C|D$ should now be clear, the situation for the WZW model is still precarious. The reason is the presence of the multi-valued term $\Gamma[M]$ in addition to the kinetic term induced by the metric tensor $\kappa$. While $i\Gamma[M]/24\pi$ is imaginary (and ambiguous by integer multiples of $2\pi i$) in the FF-sector, it is real (and single-valued) in the BB-sector. Indeed, if $A = A^\dagger$, $B = B^\dagger$ and $C = C^\dagger$ are elements of the tangent space of $\mathcal{M}_B$ at unity, the 3-linear form $i\text{Tr}(A[B, C])$, which induces the BB-part of $i\Gamma[M]$, takes values in $\mathbb{R}$. Moreover, this term reverses its sign under $(A, B, C) \mapsto (-A, -B, -C)$, so there exist field directions along which $e^{-i\Gamma[M]/24\pi}$ increases exponentially, thereby jeopardizing the existence of the functional integral with action $W[M]$. Fortunately, one can prove the following bound:

$$\left| \frac{\text{Re}}{24\pi} i\Gamma[M] \right| < \frac{\text{Re}}{16\pi} \int d^2x \text{STr}(M^{-1}\partial_\mu M)^2,$$  \hspace{1cm} (30)
which ensures the stability of the functional integral $\int e^{-W[M]}$. For the special case $\mathcal{M}_B = H^3$ (three-hyperboloid), which is isomorphic to $\mathcal{M}_B = \text{Sp}(2n, \mathbb{C})/\text{Sp}(2n)$ for $n = 1$, this bound was established in [34].

As a final remark, let us mention that the bound (30) is optimal. In other words, if we multiply the kinetic term of the WZW action $W[M]$ by a factor less than unity, and arbitrarily close to unity, there exist field configurations violating the bound, and the functional integral becomes unstable. This means that the $C|D$ WZW model resides exactly on the border of stability. An important consequence is that $W[M]$ tolerates the addition of a $J_LJ_R$ current-current perturbation only with a definite sign of the coupling.

### 7.2 Nonabelian bosonization

With a good functional integral in hand, we can now go ahead and establish a supersymmetric extension of nonabelian bosonization. We are going to show that the free Dirac theory plus $b$-$c$ ghost system is equivalent to the $C|D$ WZW model at level $k = 1$. To that end, we consider the partition function

$$Z_{\text{WZW}}[\bar{A}, A] = \int \mathcal{D}M \exp -W[M; \bar{A}, A],$$

where $W[M; \bar{A}, A]$ is a gauged WZW action:

$$W[M; \bar{A}, A] = W[M] - \frac{1}{2\pi} \int d^2x \text{STr} \left( M^{-1} \bar{\partial} MA + \bar{A} M \partial M^{-1} + \bar{A} M A M^{-1} - \bar{A} A \right),$$

and $\bar{A}$ and $A$ are external sources taking values in $\text{osp}(2n|2n)$. Our main tool for analysing the WZW model is the Polyakov-Wiegmann relation

$$W[gh^{-1}] = W[g] + W[h^{-1}] - \frac{1}{2\pi} \int d^2x \text{STr} \ g^{-1} \bar{\partial} g h^{-1} \partial h. \quad (31)$$

By using it, we easily derive the identity

$$W[g_L M g_R^{-1}] = W[M; g_L^{-1} \partial g_L, g_R^{-1} \partial g_R] + W[g_L g_R^{-1}],$$

and parametrizing the external sources by

$$\bar{A} = g_L^{-1} \partial g_L, \quad A = g_R^{-1} \partial g_R, \quad (32)$$
we can compute the partition function $Z_{WZW}[\bar{A}, A]$ exactly:

$$Z_{WZW}[\bar{A}, A] = \int D M e^{-W[g_L M g_R^{-1}]+W[g_L g_R^{-1}]} = \exp W[g_L g_R^{-1}].$$

To arrive at the last expression, we changed variables from $M$ to $M' = g_L M g_R^{-1}$ in the functional integral. Such a substitution is certainly valid at the infinitesimal level, i.e. as long as the Taylor expansion of $Z_{WZW}[\bar{A}, A]$ with respect to the sources $\bar{A}, A$ is truncated at finite order.

From the Polyakov-Wiegmann relation, one can also verify the invariance of $W[M]$ under local $OSp_C(2n|2n)_L \times OSp_C(2n|2n)_R$ transformations,

$$M(z, \bar{z}) \mapsto g_L(z) M(z, \bar{z}) g_R(\bar{z})^{-1},$$

which is characteristic of any WZW model and takes the form of an affine Lie symmetry in the quantum theory, which here is nonunitary. (Again, the invariance holds without doubt at the infinitesimal level, since we can always pass to the complexified tangent space with impunity.) By Noether’s theorem, the invariance entails two sets of conserved currents:

$$J = M \partial M^{-1}, \quad \bar{J} = M^{-1} \bar{\partial} M,$$

satisfying the equations of motion $\bar{\partial} J = \partial \bar{J} = 0$.

Consider now the free Dirac theory plus $b$-$c$ ghost system, with the partition function

$$Z_{Dirac}[\bar{A}, A] = \int D \phi D \bar{\phi} \exp - \int d^2 x \left( \bar{\phi}^i (\partial + A) \phi^i + \phi^i (\bar{\partial} + \bar{A}) \bar{\phi} \right)$$

$$= S \text{Det}^{-1/2} \begin{pmatrix} 0 & \partial + A \\ \bar{\partial} + \bar{A} & 0 \end{pmatrix}. $$

This functional (super)determinant is ill-defined and needs to be regularized in the infrared, say by adding to the Dirac Lagrangian a small mass term,

$$\epsilon \left( \bar{\phi} \Sigma_3 \phi + \phi \Sigma_3 \bar{\phi} \right) = 2\epsilon (|b|^2 + |c|^2 + ...).$$

Using standard heat kernel techniques [36], the regularized superdeterminant can then be computed to be the exponential of another WZW action:

$$Z_{Dirac}[\bar{A}, A] = S \text{Det}^{-1/2} \begin{pmatrix} \epsilon \Sigma_3 & \partial + A \\ \bar{\partial} + \bar{A} & \epsilon \Sigma_3 \end{pmatrix} = \exp W[g_L \Sigma_3 g_R^{-1} \Sigma_3],$$
where the sources $\bar{A}$ and $A$ were again assumed to be of the form (32). By comparing answers, we see that the partition functions of the $C|D$ WZW$_{k=1}$ model (derived from the complex supergroup OSp($2n|2n$)) and the Dirac+ ghost system (with $n$ species of particles) coincide:

$$Z_{\text{Dirac}}[\bar{A}, A] = Z_{\text{WZW}}[\Sigma_3 \bar{A} \Sigma_3, A],$$
onumber

on gauge source fields of the form (32). This equivalence furnishes the basis for nonabelian bosonization.

As an immediate consequence we get bosonization rules for the currents:

$$\phi \bar{\phi}^t \sim \frac{1}{2\pi} \partial M M^{-1}, \quad \Sigma_3 \bar{\phi}^3 \Sigma_3 \sim -\frac{1}{2\pi} M^{-1} \bar{\partial} M,$$

by comparing the $J \cdot A$ perturbations in the two theories. (Using these bosonization rules we can rewrite a single species of Dirac fermions with random mass as a WZW model with marginally irrelevant current-current interaction. No additional insight seems to be gained from that alternative representation.) However, in our computation of the gradient expansion of $S_5[q]$, Eq. (22), we employed the bosonization rules

$$\phi \bar{\phi}^t \Sigma_3 \sim (2\pi \ell)^{-1} M,$$  
$$\Sigma_3 \bar{\phi}^3 \Sigma_3 \sim (2\pi \ell)^{-1} M^{-1} \bar{\partial} M.$$  

To justify these, further considerations are necessary.

The $C|D$ WZW$_{k=1}$ model, as any WZW model, is solvable to a high extent, owing to the availability of exact operator product expansions between the conserved currents and the fundamental field $M$ (and all other primary fields). To write these down, we introduce the convenient notation $J^A = -\frac{1}{2} \text{STR} A M \partial M^{-1}$, where $A$ is any spatially constant element $A \in \text{osp}(2|2)$. Then, making in the functional integral $\langle M(w, \bar{w}) \rangle$ a change of variables $M \rightarrow e^{-\epsilon X} M$, where $X$ is holomorphic in a small neighborhood of the point $w$ and smoothly goes to zero outside, we obtain the operator product expansion

$$J^A(z) M(w, \bar{w}) = -\frac{A}{z - w} M(w, \bar{w}) + ... .$$  

For the antiholomorphic current defined by $\bar{J}^A(\bar{z}) = -\frac{1}{2} \text{STR} A M^{-1} \bar{\partial} M$, an analogous calculation gives

$$J^A(\bar{z}) M(w, \bar{w}) = M(w, \bar{w}) \frac{A}{\bar{z} - \bar{w}} + ... .$$  

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The OPE of the conserved currents among themselves can be obtained in a similar manner. For the holomorphic current we have

$$J^A(z)J^B(w) = -\frac{1}{2}\text{Str} AB \frac{1}{(z - w)^2} + \frac{J[A,B](w)}{z - w} + \ldots ,$$  \hspace{1cm} (37)

and the same formula (with \(z \to \bar{z}\) etc.) holds for the antiholomorphic one.

On general grounds, the holomorphic component \(T\) of the stress-energy tensor has the canonical Sugawara form

$$T(z)J^i(z) : J^j(z) : , \quad \kappa_{ij} = -\frac{1}{2}\text{Str} e^iM\partial M^{-1} ,$$

where \(\{e^i\}\) is a basis of the Lie superalgebra \(\text{osp}(2n, 2n)\). The metric tensor is expressed by \(\kappa_{ij} = -\text{Str} e^i e^j\), indices being lowered via \(\delta^i_j = \text{Str} e^i e^j\). Classical considerations (by the construction of \(T\) from the Lagrangian via Legendre transform) would suggest a constant of proportionality \(1/k\), where \(k = 1\) in the present case. However, the constant is renormalized by quantum fluctuations. Its correct value is deduced from the requirement

$$T(z)J^A(w) = \frac{J^A(w)}{(z - w)^2} + \ldots ,$$

expressing the fact that \(J\) is a holomorphic conserved current and therefore must have conformal dimensions \((\Delta, \bar{\Delta}) = (1, 0)\). Expanding the product \(T(z)J^A(w)\) with the help of (37) and the associativity of the operator product algebra, one finds

$$T(z) = \frac{\kappa_{ij}}{1 + h_*} : J^i(z)J^j(z) : ,$$  \hspace{1cm} (38)

where \(h_*\), called the dual Coxeter number, is the quadratic Casimir invariant evaluated in the adjoint representation:

$$\kappa_{ij} [e^i, [e^j, A]] = h_* A .$$

The superbracket \([\cdot, \cdot]\) here means the commutator or the anticommutator, as is appropriate. A standard calculation using the root system of \(\text{osp}(2n, 2n)\) yields

$$h_* = -2 ,$$

independent of \(n\). Note that \(h_*\) is negative! We will see shortly that this has to be so, in order for the dimensions of operators such as the fundamental field \(M\) to come out being positive. (We can also check the sign of \(h_*\) by interpreting the \(C|D\) WZW model with say, \(n = 1\), as the replica limit \(r \to 0\) of the familiar \(O(r)\) WZW model. As is well-known, the Lie algebra of \(O(r)\)
has dual Coxeter number $h_\ast = r - 2$, which indeed becomes $-2$ at $r = 0$.) The Fourier modes of $T(z)$,
\[ L_m = (2\pi i)^{-1} \oint T(z) z^{m+1} dz , \]
are the generators of a Virasoro algebra, with the central charge $c$ being zero by supersymmetry. Similar statements hold for the antiholomorphic component $\bar{T}(\bar{z})$ of the stress-energy tensor. The presence of these Virasoro algebras implies the conformal invariance of the WZW model. From equations (35)–(38) one sees that the leading singularities in the operator product expansion of the stress-energy tensor with the fundamental field $M$ are

\[ T(z) M(w, \bar{w}) = \frac{\Delta_M}{(z - w)^2} M(w, \bar{w}) + ... , \]
\[ \bar{T}(\bar{z}) M(w, \bar{w}) = \frac{\bar{\Delta}_M}{(\bar{z} - \bar{w})^2} M(w, \bar{w}) + ... . \]

where

\[ \Delta_M = \bar{\Delta}_M = \frac{C_M}{1 + h_\ast} , \]

and $C_M = \kappa_{ij} e^i e^j$ is the quadratic Casimir in the fundamental representation of $osp(2n|2n)$. The numbers $(\Delta_M, \bar{\Delta}_M)$ are the conformal dimensions of the field $M$. We can calculate $C_M$ by making an explicit choice of basis $\{e^i\}$. In this way we find $C_M = -1/2$, which yields a total dimension of one:

\[ \dim(M) = \Delta_M + \bar{\Delta}_M = 2 \times \frac{C_M}{1 + h_\ast} = 2 \times \frac{-1/2}{1 - 2} = 1 . \]

This means that the two-point function $\langle MM \rangle$ must vary as the inverse square of the distance. Global $G_L \times G_R$ invariance then constrains the OPE of $M$ with itself to be of the form

\[ \text{STr} \ AM(z, \bar{z}) \times \text{STr} \ BM^{-1}(w, \bar{w}) = -\frac{\ell^2}{(2\pi |z - w|)^2} \text{STr} \ AB + ... , \]

where $\ell$ is a length scale depending on the choice of UV regularization scheme. The supermatrices $A$ and $B$ are arbitrary elements of $osp(2n|2n)$.

We now return to the Dirac+ghost system, and rewrite the OPE given in (11) in the invariant form

\[ \text{STr} \ A\phi(z)\phi^\dagger(\bar{z})\Sigma_3 \times \text{STr} \ B\Sigma_3\phi(\bar{w})\phi^\dagger(w) = -\frac{\text{STr} \ AB}{(2\pi |z - w|)^2} + ... . \]
(The presence of the factors $\Sigma_3$ leads to the overall minus sign on the right-hand side.) By comparing the two OPEs we see that they agree to leading order if we identify $M/\ell$ with $\phi \bar{\phi} \Sigma_3$, and $M^{-1}/\ell$ with $\Sigma_3 \bar{\phi} \phi^t$. Moreover, the OPEs between these fields and the respective currents agree to leading order in the two theories. By computing the three- and four-point functions one finds [37] that the agreement persists to higher order. This then justifies the bosonization rules [34]. Their status is the same as in the usual case [38].

In summary, we have established the $C|D$ WZW$_{k=1}$ model to be equivalent to the Dirac plus $b$-$c$ ghost system at the current algebra (or infinitesimal) level. What about aspects that transcend the infinitesimal level? This is not an empty question, as the Riemannian symmetric superspace of type $C|D$ with bosonic submanifold $\mathcal{M}_B \times \mathcal{M}_F = (\text{Sp}(2n,\mathbb{C})/\text{Sp}(2n)) \times \text{O}(2n)$ has two connected components, owing to the topological equivalence $\text{O}(2n) \cong \mathbb{Z}_2 \times \text{SO}(2n)$. Since the discreteness of $\mathbb{Z}_2$ allows the WZW field to break up into domains separated by domain walls, we are led to ask: exactly how is the functional integral to be defined? Do we have to sum over all possible numbers and positions of the domain walls, or do we not?

We believe that the answer to the question is yes, and the precise non-perturbative definition of the functional integral does have to involve $\mathbb{Z}_2$ as a local degree of freedom, for the following reason. $\mathbb{Z}_2$ already exists in the classical setting, and is not a peculiarity of the supersymmetric formalism. Indeed, according to Witten [14], the free 2d massless $n$-component Majorana fermion theory is equivalent to the level-one WZW model not over $\text{SO}(n)$ but over the disconnected group $\text{O}(n)$. The argument given in [14] proceeds by the comparison of current algebras, and does not address the issue of domains and domain walls in the $\mathbb{Z}_2$ degree of freedom of $\text{O}(n)$. However, by specializing to the case $n = 1$, we see that $\mathbb{Z}_2$ does need to be summed over locally, for a single Majorana fermion is not an empty theory. Rather, its partition function bosonizes to that of the 2d Ising model, which is a theory of local $\mathbb{Z}_2$ spin degrees of freedom. In the continuum limit near criticality, the latter partition function is computed by summing over all possible numbers of domain walls and their positions. By counting the number of degrees of freedom we see that the situation is the same for $n > 1$: the fermionic Hilbert space contains states that cannot be produced by acting with currents on the vacuum. To achieve equality of the partition functions, one needs $\mathbb{Z}_2$ in the WZW model, and $\mathbb{Z}_2$ must be summed over as a local spin degree of freedom. We particularly emphasize this point, although we will not pursue it seriously in the present paper.
Here ends our extensive excursion into nonabelian bosonization and the $C|D$ WZW model. This material was included to make the present paper self-contained.

8 Density of states

In Section 5 we saw that, when fluctuations are neglected, the composite field $Q$ assumes a nonzero saddle-point value $\mu$. Since $Q$ enters the theory by coupling to the Dirac bilinear $\phi_l \bar{\phi}_l^t + \bar{\phi}_l \phi_l^t$, the latter acquires a nonvanishing expectation value, too, and the density of states at zero energy becomes finite. This happens in spite of the fact that the Dirac theory is devoid of any scale, and is an instance of dynamical mass generation. Of course, a fixed value of the “order parameter” $Q$ breaks the $OSp(2n|2n)$ symmetry and leads to the existence of Goldstone modes, the low-energy effective field theory for which is the nonlinear sigma model (21).

What is the effect of order parameter fluctuations? The Mermin-Wagner-Coleman theorem states that continuous symmetries cannot be spontaneously broken in two dimensions. This statement applies to compact symmetries, or nonlinear sigma models with compact target spaces. These flow under renormalization to strong coupling, so that the field fluctuations grow large, and the potentially broken symmetry is restored in the infrared. However, for nonlinear sigma models with supersymmetry, or zero replica number, another scenario is possible, as has long been known from the example of time-reversal invariant disordered 2d electron systems with spin-orbit scattering (class AII). In that case, the beta function at weak coupling has the “wrong” sign, which physically corresponds to weak anti-localization, and the nonlinear sigma model undergoes logarithmic flow to a Gaussian fixed point describing a perfect metal. As was already mentioned in [39], the same happens in the present case. Let us review how to arrive at this result.

8.1 Renormalization group

Friedan [40] has shown in great generality that the one-loop beta function of a nonlinear model with target space metric $\kappa$ is determined by the Ricci curvature $R$:

$$\frac{d\kappa}{d \ln \ell} = -\frac{R}{2\pi} + ... .$$
The metric of the present target space is \( \kappa = -(8\pi f)^{-1}\text{STr}(dq)^2 \), and its Ricci curvature can be described as follows. (Note that the topological term of the nonlinear sigma model does not renormalize at weak coupling, and can safely be ignored. We shall also neglect the nonperturbative \( \mathbb{Z}_2 \) degree of freedom of the target space.) We parametrize \( q \) by the exponential map, \( q = e^X \Sigma_3 e^{-X} \) with \( \Sigma_3 X \Sigma_3 = -X \), and denote the commutator (or adjoint) action by \([X, \bullet] \equiv \text{ad}(X)\). Then the Ricci curvature is the second-rank invariant tensor determined by the quadratic form \[ R_0(X, X') = -\text{STr} \text{ad}(X)\text{ad}(X') \] on the tangent space at \( X = 0 \), or \( q = \Sigma_3 \). For any irreducible symmetric space, this tensor will always be a constant multiple of the metric tensor, as follows on general differential geometric grounds and is necessary for the nonlinear sigma model to be renormalizable. Moreover, the constant of proportionality is independent of \( n \) by supersymmetry, and we may calculate it by looking at the simplest case \( n = 1 \). To do so, one evaluates both tensors on some element of the tangent space, say \( H = h E_{BB} \otimes \sigma_1 \). The eigenvalues of the adjoint action are called roots. In the present case with \( n = 1 \), there exist two nonvanishing roots, a bosonic root \( 2h \) with multiplicity one, and a fermionic root \( h \) with multiplicity two. Hence, \[-R_0(H, H) = (2h)^2 - 2h^2 = 2h^2 = \text{STr} H^2 ,\] which gives \( R = \text{STr}(dq)^2/4 \). In terms of the coupling \( f \), the one-loop RG equation then reads \[
 \frac{df}{d\ln \ell} = -f^2 . \tag{39}\]

By integrating this equation from the cutoff scale \( \ell_0 \) to \( \ell \), with initial condition \( f_0 = f(\ell_0) = N^{-1}(1 - e^{-\pi/\xi})^{-1} , \) we find that the coupling \( f \) flows to zero as an inverse logarithm: \[
 f(\ell) = \frac{f_0}{1 + f_0 \ln(\ell/\ell_0)} . \tag{40}\]

This results in the system being a perfect “metal”, with a resistivity \( \sim f \) that goes to zero in the thermodynamic limit. By the same token, the broken
OSp(2n|2n) symmetry will not be restored, but remains truly broken, and the density of states at $E = 0$ is sure to be nonzero.

As was mentioned earlier, spontaneous breaking of continuous OSp(2n|2n) symmetry also occurs for 2d metals with spin-orbit scattering (class AII). There is, however, one important difference we wish to mention. For class AII a metallic phase does not exist in disordered wires (the quasi-1d limit), where weak antilocalization at short scales always crosses over to strong localization at large scales. In contrast, for class D anomalous “metallic” behavior already occurs in quasi-1d systems! By solving the CI|DIII nonlinear sigma model using its quantum Hamiltonian, one finds that the (thermal) conductance decays algebraically for wires of arbitrary length. The anomalous behavior can also be seen on a more elementary level from the maximum entropy transfer matrix ensemble, evolving according to the so-called DMPK equation, for class $D$.\footnote{We thank John Chalker for pointing this out. See also \cite{42}.}

### 8.2 RG for the density of states

We now embark on a quantitative calculation, based on \eqref{28}, of the local density of states in the metallic limit. The idea is to follow the flow of the renormalization group starting from the short-distance cutoff $\ell_0$ all the way up to the system size $L$, and then evaluate the functional integral \eqref{28} in zero-dimensional approximation. (The local $\mathbb{Z}_2$ degree of freedom of the nonlinear sigma model is expected to play no essential role in that process, and will neglected but for its global part.) The initial cutoff is taken to be the length associated with the dynamically generated mass, $\ell_0 \sim \mu^{-1}$, which sets the scale over which the “ballistic” free fermion theory crosses over to the nonlinear sigma model describing diffusion. (From the nonabelian bosonization argument given at the beginning of Section 7, we saw that the reduction to the nonlinear sigma model is valid on length scales larger than $\mu^{-1}$.)

To begin, we add a source term $\lambda \Tr q_{FF}(0)\sigma_3$ to the action functional and differentiate with respect to $\lambda$ at $\lambda = 0$. From \cite{41}, the parameter $\lambda$ obeys the renormalization group equation

$$
\frac{d\lambda}{d \ln \ell} = -fC_q \lambda,
$$

\footnote{We thank John Chalker for pointing this out. See also \cite{42}.}
where \( C_q \) is the quadratic Casimir invariant of \( \text{osp}(2n|2n) \), normalized according to the metric \( -\text{STr} \left( dq \right)^2/8 \) and evaluated on the representation the field \( q \) transforms under. This number turns out to be negative:

\[
C_q = -1,
\]

so that \( \lambda \) grows with increasing scale. In the thermodynamic limit, this will lead to a logarithmically divergent density of states, as follows.

Let the system be finite and of size \( L \). By integrating the flow equation for \( \lambda \) from the microscopic cutoff \( \ell_0 \) up to \( L \),

\[
\lambda(L) = \lambda(\ell_0) \exp \int_{\ell_0}^{L} f(\ell')d\ln \ell',
\]

and inserting the scale dependence (40) of the running coupling \( f \), we obtain

\[
\lambda(L)/\lambda(\ell_0) = 1 + f_0 \ln(L/\ell_0).
\]

To keep track of the renormalization of the operator \( \int d^2x \text{STr} q\Sigma_3 \) in the action functional \( S_E \), it is convenient to introduce a running energy parameter \( \epsilon(\ell) \), with initial value \( \epsilon_0 = \mu NE/2g \). This variable, as compared to \( \lambda \), carries two extra dimensions from \( d^2x \), and therefore evolves according to the equation

\[
\frac{d\epsilon}{d\ln \ell} = (2 + f) \epsilon,
\]

which integrates to

\[
\epsilon(L)/\epsilon_0 = (L/\ell_0)^2 \left( 1 + f_0 \ln(L/\ell_0) \right).
\]

Given (28), all this leads to a scaling relation:

\[
\nu(E, f_0)_L = (1 + f_0 \ln(L/\ell_0)) \nu(E\epsilon(L)/\epsilon_0, f(\ell))_{\ell_0}.
\]

(41)

Here \( \nu(E, f_0)_L \) is the local density of states of a system of size \( L \), energy \( E \), and nonlinear sigma model coupling \( f_0 \), with the short-distance cutoff being understood to be \( \ell_0 \). For \( L \gg \ell_0 \) and \( E = 0 \), we immediately obtain a logarithmic law,

\[
\nu(0, f_0)_L \sim f_0 \ln(L/\ell_0),
\]

which is a result given before by Senthil and Fisher [11].
Actually, we can work out a more precise answer. The function $\nu(.)_{\ell_0}$ on the right-hand side of (41) is meant to be evaluated for a system whose size $L$ equals the cutoff $\ell_0$. Under such circumstances, the functional integral can be calculated by retaining only the spatially homogeneous mode $q(x) = q_0$ (zero-mode approximation):

$$
\nu(E)_{\ell_0} = \frac{\mu N}{2\pi g} \text{Re} \int Dq_0 \text{Tr}(q_{FF} \sigma_3) \exp \left( \frac{i\mu NE\ell_0^2}{2g} \text{Str} q_0 \Sigma_3 \right), \quad (42)
$$

which no longer depends on the coupling $f$. The computation of this integral is the subject of the next section. Using the answer given below in (46), we have

$$
\nu(E, f_0)_L = \bar{\nu} + \frac{\sin(2\pi \nu L EL^2)}{2\pi EL^2}, \quad (43)
$$

where

$$
\bar{\nu} = \frac{\mu N}{\pi g} \left( 1 + f_0 \ln(L/\ell_0) \right).
$$

Since we have neglected the influence of finite $E$ on the RG flow, the validity of this result is restricted to small energies $E \ll E_{Th}$. By standard reasoning, the relevant energy scale is the Thouless energy $E_{Th} = D/L^2$, where $D = g/(2\pi \mu N f_0)$ has the meaning of a diffusion constant. In the opposite regime $E \gg E_{Th}$, direct use of one-loop perturbation theory yields

$$
\nu(E) = \bar{\nu} + \frac{1}{\pi} \text{Re} \int \frac{d^2k}{(2\pi)^2} \frac{1}{Dk^2 - 2iE}
$$

$$
= \bar{\nu} + \frac{1}{8\pi^2 D} \ln \left( 1 + \left( \frac{D}{2EL_0^2} \right)^2 \right), \quad (44)
$$

where the momentum integral was cut off by $|k| < 1/\ell_0$. We observe that the energy-dependent part of the density of states behaves as $\ln(1/E)$ on intermediate scales, and as $E^{-2}$ in the asymptotic regime of large $E$.

### 8.3 Random matrix limit

For the case $n = 1$, we are now going to compute the superintegral

$$
I(\epsilon) = \frac{1}{2} \int Dq \text{Tr}(q_{FF} \Sigma_3) \exp \frac{1}{4} i\epsilon \text{Str} q \Sigma_3,
$$

47
where $Dq$ is the invariant Berezin measure on the Riemannian symmetric superspace $X_1$ (Section 6) normalized by $\int Dq \exp \frac{1}{4}i\epsilon \text{STr} q\Sigma_3 = 1$. This integral appeared in (42), and can be viewed as the density of states in the limit of small system size (also referred to as the ergodic regime, or the universal random matrix limit). Although $I(\epsilon)$ can be shown to be semiclassically exact, which is to say that $I(\epsilon)$ is calculated exactly in saddle-point approximation (by a supersymmetric generalization of the Duistermaat-Heckman theorem [43]), we shall not make any use of this deep fact and proceed in a more pedestrian way.

The first step is to write down an explicit parametrization for the supermatrix $q$. To prepare that step, it is convenient to change the arrangement of the multiplets $\phi, \phi_t$ to

$$\phi = \begin{pmatrix} \psi_- \\ c \\ \psi_+ \end{pmatrix}, \quad \phi^t = (\psi_+, b, \psi_-, -c).$$

Next recall the definition of the orthosymplectic transpose of a supermatrix $X$ by $(X\phi)^t = \phi^t X^t$. Elements of the Lie algebra $osp(2|2)$ are solutions of the equation $X = -X^t$, and are easily verified to be of the form

$$X = \begin{pmatrix} a & \alpha & 0 & -\beta \\ \delta & d & \beta & b \\ 0 & \gamma & -a & -\delta \\ \gamma & c & \alpha & -d \end{pmatrix},$$

(45)

where Roman letters stand for commuting numbers, while Greek letters denote Grassmann variables.

Now, as was emphasized in Section 6, the superspace $X_1$ consists of two disjoint pieces. The first contains the diagonal matrix $\Sigma_3 \in osp(2|2)$, which rearranges to

$$\Sigma_3 = \text{diag}(1, 1, -1, -1).$$

On this piece of $X_1$, the supermatrix $q$ will be parametrized as

$$q^{(1)} = \Sigma_3 + 2 \begin{pmatrix} Z\tilde{Z}(1 - \tilde{Z}\tilde{Z})^{-1} & -Z(1 - \tilde{Z}\tilde{Z})^{-1} \\ \tilde{Z}(1 - Z\tilde{Z})^{-1} & -\tilde{Z}Z(1 - \tilde{Z}\tilde{Z})^{-1} \end{pmatrix}.$$

This can be seen to obey the nonlinear constraint $q^2 = 1$. From (45) and the fact that $q = T\Sigma_3 T^{-1}$ lies in $osp(2|2)$, the $2 \times 2$ supermatrices $Z$ and $\tilde{Z}$ have
to be of the form

\[ Z = \begin{pmatrix} 0 & -\beta \\ \beta & b \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 0 & \gamma \\ \gamma & c \end{pmatrix}. \]

Following Section 6, the bosonic base of \( X_1 \) is fixed by the conditions

\[ b = c^*, \quad |b|^2 < 1. \]

The second piece of \( X_1 \) contains the diagonal matrix \( \text{diag}(-1, 1, 1, -1) \), which can be represented as

\[ \text{diag}(-1, 1, 1, -1) = O \Sigma_3 O^{-1} \]

with

\[ O = E_{FF} \otimes \sigma_1 + E_{BB} \otimes 1_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{OSp}(2|2). \]

Since \( O \) lies in the orthosymplectic Lie supergroup, the \( q \) matrix for the second piece of \( X_1 \) can be obtained from the first one by conjugation with \( O \):

\[ q^{(2)} = T' O \Sigma_3 O^{-1} T'^{-1} = O T \Sigma_3 T^{-1} O^{-1} = O q^{(1)} O^{-1}. \]

The next step is express the integration measure \( Dq \) in terms of \( Z \) and \( \tilde{Z} \). One way of doing it is to use the coordinate expression of the metric \( \text{STr} (d(q)^2) \). A straightforward calculation yields

\[ -\text{STr} (d(q)^2)/8 = \text{STr} (1 - \tilde{Z}Z)^{-1} d\tilde{Z}(1 - Z \tilde{Z})^{-1} dZ. \]

This now gives rise to a Berezin measure in the standard way, cf. Appendix F of Ref. [44], where a similar calculation was described in full detail for symmetry class \( C \). One finds

\[ Dq = D(Z, \tilde{Z}) \text{SDet} (1 - Z \tilde{Z})^{-1}, \]

where \( D(Z, \tilde{Z}) \) is a flat Berezin measure. On setting \( b = re^{i\phi} \), we arrive at the coordinate expression

\[ Dq = (2\pi)^{-1} r dr \wedge d\phi \partial_{\beta} \partial_{\gamma} \circ (1 - r^2 - 2\beta \gamma)^{-1}, \]
which has been normalized so as to satisfy \( \int Dq \exp \frac{1}{4} i\epsilon \text{Str} q \Sigma_3 = 1 \).

We are finally ready to compute \( I(\epsilon) \). Using

\[
\frac{1}{4} \text{Str} q^{(1)} \Sigma_3 = \frac{r^2}{1 - r^2} + \frac{2\beta \gamma}{(1 - r^2)^2},
\]

\[
\frac{1}{4} \text{Str} q^{(2)} \Sigma_3 = \frac{1}{1 - r^2} + \frac{2r^2 \beta \gamma}{(1 - r^2)^2},
\]

\[
\frac{1}{2} \text{Tr} q^{(1)}_{\text{FF}} \sigma_3 = 1 - \frac{2\beta \gamma}{1 - r^2} = -\frac{1}{2} \text{Tr} q^{(2)}_{\text{FF}} \sigma_3,
\]

and making the substitution \( t = (1 - r^2)^{-1} \), we obtain the following expression for the integral:

\[
I(\epsilon) = \int_{1}^{\infty} \frac{dt}{2t^2} \partial_{\beta} \partial_{\gamma} (t + 2t^2 \beta \gamma)(1 - 2t \beta \gamma) \times
\]
\[
\times \left( \exp \left( i\epsilon \left( t - 1 + 2t^2 \beta \gamma \right) \right) - \exp \left( i\epsilon \left( t + 2t(t - 1) \beta \gamma \right) \right) \right).
\]

where we have combined the contributions from the two pieces of \( X_1 \). The integral is now easily calculated to be

\[
I(\epsilon) = 1 - \frac{1}{i\epsilon} + \frac{e^{i\epsilon}}{i\epsilon}.
\]

(46)

On taking the real part, we get

\[
\text{Re} I(\epsilon) = 1 + \frac{\sin \epsilon}{\epsilon},
\]

in agreement with the known result \[7\] for the density of states (in scaled units) of the Gaussian random matrix ensemble for class \( D \).

Let us remark that exactly the same result for \( I(\epsilon) \) would have been obtained by treating the superintegral in saddle-point approximation, in the spirit of the Duistermaat-Heckman theorem. The constant part comes from a saddle point on the trivial component (with respect to \( Z_2 \)) of the target space, and the oscillatory part \( e^{i\epsilon} / i\epsilon \) from a saddle point on the nontrivial component.
9 Disordered \(d\)-wave superconductor

An important physical realization of symmetry class \(D\) is by the low-energy quasiparticles of dirty superconductors with broken spin-rotation and time-reversal symmetry. Although there is a number of interesting cases to consider [11, 13], we will here focus on a specific model of a disordered \(d\)-wave superconductor. The treatment in this section follows Ref. [45] to some extent. We shall work in two-dimensional space with Cartesian coordinates \(x, y\) and wave vector \(k = (k_x, k_y)\).

The effective quasiparticle Hamiltonian of the pure system is

\[
H_0 = \sum_{k,\sigma}(\varepsilon_k - \mu)c^\dagger_{k\sigma}c_{k\sigma} + \sum_k \Delta_k(c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow} + \text{h.c.}),
\]

where the single-particle energies \(\varepsilon_k\), shifted by the chemical potential \(\mu\), and the gap function \(\Delta_k\) are taken to be

\[
\varepsilon_k = -t(\cos k_x a + \cos k_y a),
\]

\[
\Delta_k = -\Delta_0(\cos k_x a - \cos k_y a),
\]

and \(a\) is the lattice constant of the tight-binding model that results on transforming to position space. The excitations of this Hamiltonian are gapped almost everywhere in the wave vector plane, the exceptional places being the four points given by \(|k_x| = |k_y| = a^{-1}\arccos(-\mu/2t)\), where \(\varepsilon_k - \mu = \Delta_k = 0\).

Since the low-temperature behavior of the system on large distance scales is expected to be dominated by the quasiparticles with low energy, it is natural to linearize \(H_0\) around the nodal points \(k_{x,y} = \pm a^{-1}\arccos(-\mu/2t)\), denoted in a self-explanatory notation by \((-\pm)\). By following a standard procedure [18, 19, 20], we obtain four Majorana Hamiltonians, one for each node. The one for \((++)\) reads

\[
H_0^{(+)} = \frac{1}{2}iv_1 \int d^2x \sum_a \left(f^\dagger_a \partial_1 f^\dagger_a + f_a \partial_1 f_a \right)
\]

\[
+ \frac{1}{2}iv_2 \int d^2x \sum_a \left(-i f^\dagger_a \partial_2 f^\dagger_a + i f_a \partial_2 f_a \right),
\]

with the two velocities given by \(v_1 = at\) and \(v_2 = a\Delta_0\), and we have introduced rotated coordinates by \(x_1 = \frac{1}{2}(x + y)\) and \(x_2 = \frac{1}{2}(x - y)\). The fermion operators \(f_a\) and \(f^\dagger_a\) are linear combinations of the operators \(c_\sigma\) and \(c^\dagger_\sigma\) in (47).
The Hamiltonian $H_0^{(+)}$ for the nodal point $k_x = -k_y = \pi/2a$ is obtained from $H_0^{(++)}$ by exchanging $\partial_1 \leftrightarrow \partial_2$. The remaining ones, $H_0^{(-+)}$ and $H_0^{(--)}$, are gotten from the previous two by reversing the overall sign.

We now introduce disorder into the problem. In the present context, disorder can be classified into two categories: intranode scattering, and internode scattering. We will deal with these scattering mechanisms separately. First we take into account the intranode scattering, and later we will include the internode scattering as a perturbation. Such a two-step procedure makes sense if the “backscattering” between nodes is weak compared to the “forward” scattering within a node.

In the family of symmetry classes of disordered superconductors, class $D$ is distinguished by the absence of any special symmetries such as time-reversal or spin-rotation invariance. Thus we are to add disorder of generic type. Without much loss, we can restrict the disorder to be local (involving no derivatives). To write down the most general, local, intranode scattering Hamiltonian, we again single out the node $(++)$ as an example. The expression for this Hamiltonian in terms of the operators $f_a, f_a^\dagger$ reads

$$H_1^{(++)} = \frac{1}{2} \int d^2 x \left( \sum_{ab} f_a^\dagger M_{ab} f_b + \Delta f_1^\dagger f_2^\dagger + \text{h.c.} \right),$$

where $\Delta$ now is a complex “random gap function”, and $M_{ab} = \bar{M}_{ba}$ is a $2 \times 2$ “mass matrix” fluctuating randomly in space. They are given as certain linear combinations of the perturbations (random scalar potential, random order parameter, etc.) that are added to the original pure Hamiltonian $H_0$.

Consider now the problem posed by the sum $H^{(++)} = H_0^{(++)} + H_1^{(++)}$. This is essentially the problem we solved in Sections 5–8. To make the correspondence precise, we choose anisotropic units of length in the $x_1$ and $x_2$ directions such that $v_1 = v_2 = 1$ (we will restore the proper length units shortly), and write

$$H^{(++)} = \frac{1}{4} \int d^2 x \left( f_a^\dagger f_a \right) \cdot \mathcal{H}^{(++)} \cdot \begin{pmatrix} f_a^\dagger \\ f_a \end{pmatrix} + \text{const},$$

$$\mathcal{H}^{(++)} = \begin{pmatrix} M_{11} & M_{12} & i\partial_1 + \partial_2 & \Delta \\ M_{21} & M_{22} & -\Delta & i\partial_1 + \partial_2 \\ i\partial_1 - \partial_2 & -\Delta & -M_{11} & -M_{21} \\ \Delta & i\partial_1 - \partial_2 & -M_{12} & -M_{22} \end{pmatrix}.$$

All information about the second-quantized Hamiltonian $H^{(++)}$ is encoded in the first-quantized Hamiltonian $\mathcal{H}^{(++)}$, and we can equivalently work with
the latter instead of the former. Doing so, and adopting white-noise disorder

\[
\langle M_{ab}(x)M_{cd}(x') \rangle = g \delta_{ac} \delta_{bd} \delta(x - x'),
\]

\[
\langle \Delta(x)\Delta(x') \rangle = g \delta(x - x'),
\]

with \( \langle M \rangle = \langle \Delta \rangle = 0 \), we arrive at the \( N = 2 \) version of the random Hamiltonian treated earlier. Drawing on the results of Section 4, we can immediately write down the low-energy effective field theory for this problem. (Of course, the anisotropic choice of length units has made the short-distance cutoff of the field theory anisotropic. This does not make a big difference, as the nonlinear sigma model couplings are essentially cutoff independent.)

On putting the proper units of length back in place, we get an anisotropic theory \( (q = q^{(++)}) \),

\[
S^{(++)} = -\frac{1}{8\pi} \int d^2x \text{STr} \left( \frac{v_1}{v_2} \partial_1 q \partial_1 q + \frac{v_2}{v_1} \partial_2 q \partial_2 q \right)
\]

\[
+ \frac{1}{16} \int d^2x \epsilon_{\mu\nu} \text{STr} q \partial_\mu q \partial_\nu q.
\]

Note that the topological term by its nature is ignorant of all length units and therefore cannot depend on the ratio \( v_1/v_2 \). Exactly the same effective action governs the field \( q^{(--)} \) for the node \( (--) \). The actions for the remaining two nodes are obtained by interchanging \( \partial_1 \leftrightarrow \partial_2 \), which has the particular consequence of reversing the sign of the topological coupling.

Finally, we turn on the scattering between nodes, which couples the four theories together. Here we can easily guess without any calculation what is going to happen. The fields \( q^{(st)} \) for the nodes \( (st) \) are the Goldstone modes of a broken orthosymplectic symmetry. In the absence of internode scattering, the four sectors described by the \( q^{(st)} \) are decoupled, and the symmetry group consists of four independent copies of OSp(2n|2n). Scattering between the nodes reduces the symmetry to the diagonal subgroup, which acts by the same factor in each copy. On these grounds, we expect that the effect of internode scattering is to “lock” the fields of the four nodes to each other: \( q^{(++)} = q^{(--)} = q^{(+\cdot)} = q^{(-\cdot)} \equiv q \), at large distance scales. The locked field \( q \) is the Goldstone field due to the remaining diagonal OSp(2n|2n) symmetry. The effective action for it will simply be a sum of actions:

\[
S_{\text{eff}} = S^{(++)} + S^{(--)} + S^{(+\cdot)} + S^{(-\cdot)}
\]

\[
= -\frac{v_1^2 + v_2^2}{4\pi v_1 v_2} \int d^2x \text{STr} \partial_\mu q \partial_\mu q.
\]
In this final expression the anisotropy of the single-node actions has cancelled, and $S_{\text{eff}}$ is isotropic. Also, the four topological terms have added up to zero. This was to be expected, since the presence of these terms violates parity, which is a good symmetry of the superconductor, unless an orientation is induced by a strong magnetic field or by an order parameter with nonzero chirality (such as $d_{x^2-y^2} + id_{xy}$).

Thus we have arrived at the nonlinear sigma model for class $D$. The model is at weak coupling if the ratio $v_1/v_2$ is large, which is the typical situation in experimental systems. The coupling $f^{-1} = 4(v_1^2 + v_2^2)/v_1v_2$ or rather, $\text{const} \times T f^{-1}$ where $T$ is the temperature, has an interpretation as the thermal conductivity of the disordered superconductor \cite{11}. (A nonzero topological angle $\theta$ would have corresponded to a thermal Hall conductivity.)

The renormalization group for the weakly coupled theory was worked out in Section 8. We found that the theory flows to a Gaussian fixed point describing a perfect (thermal) metal, and the local density of states in the thermodynamic limit diverges logarithmically at $E = 0$.

10 Free fermions and phase diagram

We have shown that a model of $N$ species of Dirac fermions in class $D$ supports a metallic phase in two dimensions. Our analysis has put the existence of that phase on solid ground for $N \gg 1$, or $N = 2$ with large and opposite anisotropies in the velocities. Here and in the next section, we wish to go further and address two questions that emerge from the recent literature \cite{12, 14, 13}: (i) Can the 2d metallic phase of class $D$ also be realized in the constrained parameter space of the fundamental case $N = 1$? (ii) What is the location of the free-fermion point relative to the metallic phase?

The answer to the second question is that the metallic phase has the free-fermion point sitting right on its boundary. We are going to demonstrate that this is so, by employing the supersymmetric extension of nonabelian bosonization developed in Section 7. As for the first question, we will see that the answer is no, but there exists a remarkable twist to the story.

We start with the second question. Recall the Lagrangian $L_2$ in (10), describing $N$ species of massless Dirac fermions plus $b$-$c$ ghost system, weakly perturbed by the operators $\Phi^{(\alpha)}$ with couplings $g_{\alpha}$ ($\alpha = 1, \ldots, 4$). As we have seen, a single species in the pure limit is equivalent to the $C|D \text{WZW}_{k=1}$ model with field $M$ and action functional $W[M]$. We now bosonize $L_2$, by
exploiting the equivalence for each species separately:

\[ \phi^t_l \bar{\phi}^t_l + \phi^t_l \bar{\phi}^t_l \rightarrow W[M_l] \quad (l = 1, \ldots, N). \]

This scheme can be justified by introducing auxiliary fields \( Q \) as in (16) to transform \( L_2 \) into a Lagrangian for \( N \) decoupled species. The perturbations \( \Phi^{(\alpha)} \) are then bosonized using the rules derived in Subsection 7.2, which results in

\[ S'_2 = \sum_{l=1}^{N} \left( W[M_l] - \frac{g_1 + g_3 + g_4}{(2\pi)^2 N} \int d^2x \text{Str} J_l \Sigma_3 \bar{J}_l \Sigma_3 \right) \]

\[ + \frac{1}{N} \sum_{k \neq l} \int d^2x \left( \frac{g_1}{\ell} \text{Str} M_k \Sigma_3 M_l \Sigma_3 + \frac{g_2}{\ell} \text{Str} M_k^{-1} M_l \right. \]

\[ \left. + \frac{g_3}{\ell^2} (\text{Str} M_k \Sigma_3) (\text{Str} M_l \Sigma_3) - \frac{g_4}{(2\pi)^2} \text{Str} J_k \Sigma_3 \bar{J}_l \Sigma_3 \right). \]

Here we isolated the part of the perturbation that acts within a single species \( l \), and bosonized it by reduction to the bosonization rule for the currents:

\[ (\phi^t_l \phi^t_l)^2 = -\text{Str}(\phi^t_l \phi^t_l)(\bar{\phi}^t_l \bar{\phi}^t_l) \rightarrow -(2\pi)^{-2} \text{Str} J_l \Sigma_3 \bar{J}_l \Sigma_3. \]

We observe that, since the fields \( J_l = M_l \partial M_l^{-1}, \bar{J}_l = M_l^{-1} \bar{\partial} M_l \) and \( M_l \) have conformal dimensions \((1, 0), (0, 1), \) and \((1/2, 1/2)\), respectively, bosonization has preserved the marginality of the perturbation. Of course, the couplings will not be truly marginal but will evolve under renormalization in a way that is determined by the operator product expansion among the various perturbations. Given that the perturbed \( C \mid D \) WZW \( k=1 \) model is a true image of the original theory, the flow equations are identical to the ones worked out earlier and given in (12). Note also that, by taking the beta functions from the Dirac representation, instead of recalculating them in the bosonized theory, we avoid the nontrivial question of how to renormalize the local \( \mathbb{Z}_2 \) degree of freedom of the perturbed WZW model.

To go further, we distinguish between cases. Let us first assume \( N > 2 \) and take the bare couplings to be \( g_1(\ell_0) = g_2(\ell_0), \) and \( g_3(\ell_0) = g_4(\ell_0) = 0. \) Numerical integration of the one-loop flow equations (12) then shows that the couplings \( g_1, g_2 \) grow \((g_1 \) more strongly than \( g_2)\), \( g_3 \) becomes nonzero and positive, and \( g_4 \) moves to negative values but respects the bound \( g_1 + g_4 \geq 0 \) dictated by Hermiticity of the random Hamiltonian. \( \) For \( N = 2, \) the flow
eventually is attracted to the line $g_1 + g_4 = g_2 = g_3 = 0$, which has a higher symmetry, namely that of class $A_{II}$; see Section 4. Thus the symmetry is dynamically enhanced in that case and the flow, though starting from a point in class $D$, terminates in class $A_{II}$. Hence we are facing a relevant perturbation that drives the system away from the free-fermion point. We make the reasonable assumption that the relevant nature of the flow persists beyond the one-loop approximation used in deriving the flow equations.

The fate of the theory on its way to strong coupling is quite transparent from the bosonized representation. First of all, the effect of the term $\text{STr} M^{-1}_k M_l$ with large coupling $g_2$ is to “lock” the fields, since the function $\text{STr} M^{-1}_k M_l$ has an absolute minimum at $M^{-1}_k M_l = 1$; recall expression (29). Second, in the locked configuration $M_k = M_l \equiv M$ the terms multiplying $g_1$ and $g_3$ become $\ell^{-2}(N - 1) \left( g_1 \text{STr}(M\Sigma_3)^2 + g_3 \text{STr}^2(M\Sigma_3) \right)$. By parametrizing $M = Te^YT$, with $Y = +\Sigma_3 Y \Sigma_3$ and $\Sigma_3 T \Sigma_3 = T^{-1}$, we can write them as

$$g_1 \text{STr}(M\Sigma_3)^2 + g_3 \text{STr}^2(M\Sigma_3) = g_1 \text{STr} e^{2Y} + g_3 \left( \text{STr} e^Y \Sigma_3 \right)^2.$$ 

This is a potential for $Y$ which, from (29), has an absolute minimum at $Y = 0$. (Here we again benefit from our careful construction of the target of the WZW model as a Riemannian symmetric superspace.) At strong coupling, we may set $Y = 0$. Doing so, and inserting $M_l = T^2$ into the expression for the bosonized action $S''_2$, we obtain

$$S''_2 = NW [T^2] + \frac{g_1 + g_3 + N g_4}{(2\pi)^2} \int d^2 x \text{STr} \partial T^2 \Sigma_3 \partial T^2 \Sigma_3 .$$ 

We finally set $q = T\Sigma_3 T^{-1}$ and identify $q$ with the field of the nonlinear sigma model. Then, by a slight extension of the calculation of Section 7, we find that $S''_2$ reduces to the effective action $S_{\text{eff}}[q]$ given in (21), with couplings

$$f = (N - (g_1 + g_3 + N g_4)/\pi)^{-1}, \quad \theta = \pm N\pi . \quad (48)$$

What will be the fate under renormalization of the nonlinear sigma model so obtained? Recall that for the choice of bare couplings we made, $g_1, g_2$ and $g_3$ flow to positive values whereas $g_4$ becomes negative. By linearly combining the basic flow equations (14) we find

$$\dot{g}_1 + \dot{g}_3 + N \dot{g}_4 = -\frac{(g_1 + g_4)^2}{N} - \frac{(g_3 + g_4)^2}{N} - (1 - 2/N) g_4^2 - 2(g_1 + g_2)g_3 .$$

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The right-hand side of this equation is negative, so the combination $g_1 + g_3 + N g_4$ has a negative rate of change, for all $N > 2$. (This happens in spite of the fact that the individual couplings increase in magnitude.) Hence, the nonlinear sigma model coupling in (13) is roughly of order $1/N$ and decreases under renormalization. It is therefore reasonable to expect that the model lies in the metallic phase, where the RG flow is attracted by the Gaussian fixed point $f = 0$. Our argument then says that the RG flow takes the Lagrangian $L_2$ with bare couplings $g_1 = g_2$ and $g_3 = g_4 = 0$ into the metallic phase. This remains true for arbitrarily small $g_1 = g_2$, because this coupling is (marginally) relevant. Thus, no matter how close to the free-fermion point we start, the RG flow will go to the metallic fixed point. (Recall that this is not true for $N = 2$ with equal velocities, in which case the flow is attracted toward symmetry class AII. However, we expect a finite anisotropy in the velocities to stabilize the flow inside class D.) In other words, for $N > 2$ we are sure of the existence of a metallic phase, and the free-fermion point sits right on the boundary of that phase, as claimed.

On the other hand, the situation for $N = 1$ is qualitatively different. In that case, the only type of local disorder available in class $D$ is randomness in the mass. If we set the statistical average of the random mass to zero, $m_0 \equiv \langle m(x) \rangle = 0$, there remains only a single coupling $g_M$, which is defined by (4). The bosonized action then reads simply

$$S'_2 = W[M] - \frac{g_M}{(2\pi)^2} \int d^2 x \text{STr} J \Sigma \bar{J} \Sigma_3.$$  

This is a WZW model perturbed by a current-current interaction. Computing the beta function directly from the OPEs for the WZW currents $J$ and $\bar{J}$, one finds that the perturbation is marginally irrelevant in the physical range $g_M > 0$, in agreement with the flow equation (13). (The perturbation would be marginally relevant for negative $g_M$. However, for that sign of the coupling the $C|D$ WZW model is unstable and does not exist; cf. the discussion at the very end of Subsection 7.2.) Thus, the RG flow is attracted by the WZW fixed point and the system in this sense is critical.

The discussion so far assumed $m_0 = 0$. When a nonzero average mass is introduced, the bosonized action $S'_2$ acquires an additional term

$$(i m_0/\ell) \int d^2 x \text{STr} M \Sigma_3.$$  

This term is relevant by power counting at the WZW fixed point, and puts the system off criticality. Physically speaking, a finite Dirac mass $m_0$ opens
a gap around $E = 0$ in the energy spectrum of the pure system, which causes localization of all low-energy states due to disorder (at least as long as $g_M$ is small enough). Therefore, the critical line segment $m_0 = 0$, $g_M > 0$ separates two insulating phases. As is known from [23], the distinction between the two phases is of a topological nature. If one of the two phases, say $m_0 < 0$, is a plain insulator, the other one ($m_0 > 0$) can be likened to a quantum Hall fluid, in the sense that the presence of a boundary gives rise to chiral edge excitations leading to a quantized Hall-type response. (In the language of the 2d Ising model with weakly disordered bond strengths, the two phases correspond to the paramagnetic and the ferromagnetic phase.) Thus we have two insulating phases, and there is no room for a metallic phase, at least not in the vicinity of the free-fermion point, for $N = 1$. On grounds of continuity, the local density of states at zero energy in these phases is expected to vanish, and we believe the local $\mathbb{Z}_2$ degree of freedom of the $C|D$ WZW model to play a crucial role in reproducing this feature in the field-theoretic formalism.

By combining the information given, we arrive at Figure 1, which draws a schematic picture of the phase diagram close to the free-fermion point. Three phases are seen to meet there: insulator, quantum Hall fluid, and metal. (Recall that in the physical application to superconductors, the terminology “metal” and “insulator” refers to thermal transport.) The two insulating phases are separated by a critical line, described by the field-theoretic model of Dirac fermions with random mass, or alternatively, upon nonabelian bosonization, by the $C|D$ WZW$_{k=1}$ model with a current-current interaction. The flow along the critical line terminates at the free-fermion point, which controls the critical behavior across the transition from the insulator to the quantum Hall fluid. The metallic phase exists for $N > 2$ species, or for $N = 2$ with anisotropy in the velocities, but is absent for $N = 1$. The field-theoretic model for this phase is the $CI|D$III nonlinear sigma model with couplings $f$ and $\theta$. It is expected [11] that the Hall response in this phase is not quantized but varies continuously with the angle $\theta$.

Two additional remarks are in order. First, a structurally similar phase diagram was suggested by Senthil and Fisher [11]. The main difference is that these authors did not identify the multicritical point of class $D$ as free fermions and, consequently, had part of the flow reversed. Second, recall from Section 7 that the topological term of the $CI|D$III nonlinear sigma model is trivial (at least in any continuum regularization of the field theory) for the case of one replica ($n = 1$), which contains all information about a single Green function at any energy $E$, and about a product of two Green functions.
functions at \( E = 0 \). It is hard to see how a term which is topologically trivial could drive a topological phase transition. This leaves two options: either the nonlinear sigma model (unlike Pruisken’s model for the integer quantum Hall effect) is not a valid description of the critical line separating the two insulating phases, or else the driving agent for the transition is the \( \mathbb{Z}_2 \) degree of freedom of the model. In either case, the sole function of the topological term is to give rise to the edge current that is expected to exist for a system with boundary in the metallic and quantum Hall fluid phases.

11 Vortices

This is not yet the end of the story. It has recently been argued by Read and Green [13] that vortex disorder has a significant effect on the phenomenology of quasiparticle transport and localization in class \( D \). Without repeating the detailed discussion of [13], we recall the following essential points.

Basic to our study is the spinor field \( \psi(x), \psi^\dagger(x) \) of a Majorana fermion...
with Hamiltonian
\[ H = \int d^2x \left( \psi\dagger \bar{i} \partial \psi\dagger + \psi \bar{i} \partial \psi + m \psi\dagger \psi \right), \]

which we here imagine to be the low-energy approximation to a mean-field Hamiltonian governing the time evolution of the quasiparticles in a chiral \( p \)-wave superconductor. Vortices are introduced by postulating a change of boundary condition: with (half-quantum) vortices present, it is no longer true that the Majorana spinor is a single-valued function of position. Rather, the spinor now reverses its sign on circling once around any one of the vortex singularities. (We assume the London limit where vortex cores are point-like objects.) The phase twist originates from gauging away the phase of the superconducting order parameter, which winds by \( 2\pi \) on going once around a vortex carrying half a magnetic flux quantum.

Thus the insertion of vortices amounts to the introduction of square-root singularities into the continuum Majorana field. Recall that the one-component Majorana theory in the massless limit \((m = 0)\) is equivalent to the critical 2d Ising model. Under this equivalence, the square-root singularities in the Majorana field correspond to spin fields (or Kadanoff-Ceba disorder fields in a dual picture) placed in the partition sum of the Ising model. By using the known \([46]\) conformal dimensions of the spin field at criticality, and transcribing them to the supersymmetric setting, Read and Green were able to conclude that the perturbation of the massless free Majorana theory by randomly placed vortices is strongly relevant.

The question then is: what does the relevant RG flow go to? Read and Green \([13]\) suggested that the flow leads into a metallic phase described by a nonlinear sigma model \textit{without the local \( \mathbb{Z}_2 \) degree of freedom} which we have identified as being characteristic of class \( D \). We have no direct proof of this conjecture at present, as we do not know how to incorporate square-root singularities into the supersymmetric extension of the nonabelian bosonization scheme. We can, however, give a partial verification as follows.

Throughout our discussion, we are going to assume that the random mass \( m(x) \) is a smooth function of \( x \). As a warm-up, let us start from the homogeneous limit \( m(x) = m_0 > 0 \) and insert a local inhomogeneity with the shape of a disk-like region \( \mathcal{D} \) where the mass switches to negative values. Thus, \( m(x) \) is positive outside of \( \mathcal{D} \), negative inside, and zero along the boundary \( \partial \mathcal{D} \). The zero-mass contour \( \partial \mathcal{D} \) is a kind of “inner edge”. As was nicely explained in \([24]\), moving a Majorana spinor at energy \( E = 0 \)
once around $\partial \mathcal{D}$ reproduces the spinor with a phase shift of $\pi$. Hence, Bohr-Sommerfeld quantization gives a bound state with low energy $E = \pi/L$ (actually, in the first-quantized formulation, a pair of bound states with energies $\pm E$), where $L$ is the circumference of the contour. The localization length of the bound state transverse to the contour is inversely proportional to the mass $m_0$.

At the critical point $m_0 = 0$, a randomly but smoothly varying mass function produces a percolating network of zero-mass contours, and thus a macroscopic number of low-energy states. To capture the quantum physics of this random system, we turn to a variant of the Chalker-Coddington network \cite{Chalker1988}, originally designed to model the quantum percolation transition between plateaus of the integer quantum Hall effect. The original model involves random U(1) phases on the links, in addition to scattering at the nodes, of a square network. Since we are interested in the limit of zero energy, where the states propagating along inner edges do not carry any phases, randomness in the link phases is forbidden in the present context. Thus we are led to study a network model without any random phases, just randomness in the probability for scattering to the right ($p_R$) or left ($p_L$). At the symmetric point of the model, where $p_L = p_R = 1/2$ on average, the continuum limit is known \cite{Chalker1990} to be a Dirac fermion (or, equivalently, two copies of a Majorana fermion) with random mass and $m_0 = 0$. Thus we have come full circle and are back to our starting point.

Consider now the effect of adding vortices to the system. As we recall, going once around a vortex in the continuum formulation twists the phase of the spinor wavefunction by minus one. For the case of an isolated zero-mass contour enclosing one vortex, the phase shift by $\pi$ has the important consequence of giving rise to a single zero-energy bound state or fermionic zero mode \cite{Wen1990}. In what follows, we shall be concerned with the effect of vortices on the critical system. This is captured in the network model by introducing “frustrated” plaquettes, by which we mean that the product of the link phase factors along any loop encircling such a plaquette equals minus one. Thus, randomly placed vortices in the continuum theory translate into randomly placed frustrated plaquettes in the network model. To create an isolated frustrated plaquette, we insert a semi-infinite string of minus signs on links, terminating at the plaquette. To create a large amount of frustration, we insert many strings. The maximal amount of disorder, corresponding to a high density of vortices, is realized by taking the link phases to be independent and identically distributed random variables drawn from the
uniform distribution on $\mathbb{Z}_2 = \{\pm 1\}$, which is the same as the orthogonal group in 1 dimension, $O(1)$. In this limit we arrive at the Chalker-Coddington network model with a single channel per link and $\mathbb{Z}_2$ invariant link disorder.

The simplest choice is to take the scattering probabilities to be nonrandom. Then, by the color-flavor transformation developed in [50] for $U(N)$ (which readily extends [51] to $O(N)$ for all $N$ including $N = 1$) or, alternatively, by Read’s second-quantized setup [52], the $\mathbb{Z}_2$ phase invariant network model at its left-right symmetric point readily maps on the $CI|DIII$ nonlinear sigma model with action (21), and couplings $f = 2/\pi$, $\theta = \pi$. However, there is one important difference from before, which is that the target space now has just one connected component. This can be understood by anticipating from Section 11.1 that $s \in \mathbb{Z}_2$ acts on the identity component by the trivial representation $|s| = 1$, and on the other component by the faithful representation $s = \pm 1$. The latter is wiped out by averaging over the uniform distribution on $\mathbb{Z}_2$. (It reappears when the distribution is made nonuniform.) Since the free-fermion point formally corresponds (by saddle-point approximation in the limit of small $g_M$ for $N = 1$; see Section 11) to the nonlinear sigma model with $f = 1$, and since the absence of a disconnected component of the target space does not change the perturbative RG beta function, the model with a weakened coupling $f = 2/\pi < 1$ is expected to flow to the metallic fixed point $f = 0$. This result is compatible with the expectation of Read and Green that the addition of vortex disorder at the free-fermion point results in metallic behavior. By computing the functional integral of the nonlinear sigma model in zero-mode approximation, we find that the density of states for a finite network in the ergodic regime is

$$\rho(E) = \nu + \frac{1}{2}\delta(E).$$

Note that this answer differs from the ergodic limit of the density of states for quasiparticles in class $D$; cf. Section 8 and equation (50) below.

Thus, vortices change the low-energy density of states in the metallic phase. The change is relatively minor there, being visible only on the microscopic scale of the mean level spacing $\nu^{-1}$. In the insulating phases, however, vortices are expected to have a much more pronounced effect. We have seen that, when vortices are absent, the local density of states at zero energy vanishes in that case. In contrast, the insertion of a finite density of vortices gives rise, by the above reasoning (for a slowly varying random-mass function), to an extensive number of quasiparticle states at very low energy.
In an insulating phase these do not communicate over large distances, and therefore they do not repel, leaving a finite density of states at \( E = 0 \). In the field-theoretic setting, we speculate that this comes about by the mechanism we identified above: *vortices suppress the local \( \mathbb{Z}_2 \) degree of freedom* of the nonlinear sigma model.

We believe these differences in local observables and their field-theoretic manifestation to be sufficiently significant to warrant a refinement in vocabulary. What we propose to say is that Majorana fermions subject to vortex disorder belong to a generic (hybrid) class \( BD \). This nomenclature is motivated by the following considerations.

### 11.1 Random-matrix limits

The fundamental point we are going to make, here, is that the Lie algebraic structure underlying the Majorana fermion provides us with more than one kind of universal level statistics, and this suggests a refinement of the symmetry classification scheme. We begin by recalling from [4] that the basic random-matrix model for class \( D \) is the Gaussian ensemble over \( D_N = \text{so}(2N) \), the Lie algebra of the orthogonal group in even dimension. A Hermitian matrix drawn at random from \( iD_N \) has \( N \) pairs of eigenvalues \( \pm E \), with the universal large-\( N \) limit of the density of states being

\[
\rho_D(E) = \nu + \frac{\sin(2\pi \nu E)}{2\pi E}.
\]

(50)

Note that this function is positive and smooth at \( E = 0 \).

Another Lie algebra, regarded as mathematically distinct from \( \text{so}(2N) \) (as the root systems differ in structure) is that of the orthogonal group in odd dimension, denoted by \( B_N = \text{so}(2N + 1) \) in Cartan’s notation. The distinction becomes especially evident in the level statistics: the density of states of the Gaussian ensemble over \( iB_N \) has the large-\( N \) limit

\[
\rho_B(E) = \nu - \frac{\sin(2\pi \nu E)}{2\pi E} + \delta(E).
\]

(51)

Here we see a singular term \( \delta(E) \), reflecting the fact that an antisymmetric matrix in odd dimension has one, and generically only one, eigenvalue at zero. (To understand this fact by way of example, recall that every proper rotation of Euclidian 3-space has one invariant vector, namely the axis of rotation.) Repulsion from that level is the cause of the quadratic law \( \rho_B(E) \sim E^2 \) near
\( E = 0 \). We mention in passing that the above law has also been found, in recent mathematical work [53], to be the universal limit of the density of zeros of suitably chosen families of algebraic \( L \)-functions.

What we have reviewed and illustrated, then, is the fact that the large family of “orthogonal” Lie algebras splits into two classes, conventionally denoted by \( D \) and \( B \). The latter class was mentioned in [4] but not pursued there for want of a good physical example. Recently, it has been proposed [54] that the universal random-matrix limit of the density of bound states of a disordered vortex in a chiral \( p \)-wave superconductor, is given by class \( B \). In earlier work [55, 44, 56], the same limit for a conventional \( s \)-wave superconductor had been shown to be governed by class \( D \).

What is the relevance of all this to systems in the thermodynamic limit, when an extensive number of vortices is present? To answer that question, notice first of all that the ergodic limit (49) of the density of states for the \( Z_2 \) phase invariant network model coincides with the arithmetic mean of the random-matrix answers for the classes \( B \) and \( D \). This is already a first indication that the network model should not be assigned to the symmetry class \( D \) (at least not in a narrow sense), but is better placed in a hybrid class, which we propose to call \( BD \). We are going to elaborate below.

For the moment, we shall explain why the coincidence of (49) with the arithmetic mean \( \frac{1}{2}(\rho_D + \rho_B) \) is not an accident. Let us interpret [57] the network model as a quantum dynamical system evolving in time by discrete steps of size \( \Delta t \). The discrete time-evolution operator \( U = \exp(-iH\Delta t) \) is a product of two unitary operators, one of which is diagonal (in the link basis) containing all the random \( Z_2 \) factors associated with the links, while the other one encodes the random tunneling at the nodes. The eigenvalues of \( U \) are written \( e^{-iE\Delta t} \), where \( E \) is called the quasi-energy. Let the network now be finite, with an even number \( 2N \) of links. Then \( U \), being a unitary matrix with real entries, is an element of the orthogonal group \( O(2N) \). Note that for the model with \( Z_2 \) phase invariant disorder, the probabilities for \( U \) to lie in the proper and improper parts of \( O(2N) \) are equal.

Thus the time-evolution operator of the network model is a (sparse) random matrix on \( O(2N) \), and in order to understand the ergodic low-energy limit of the density of quasi-energies \( E \), one needs to work out the random-matrix level statistics for that group. With some linear algebra, one can show that the level statistics for the two Haar ensembles on the proper and improper components of \( O(2N) \) coincide exactly with those of the classes \( D \) and \( B \), respectively. (A noteworthy feature here is that, since the nonreal
eigenvalues of an orthogonal matrix come in complex conjugate pairs, every $U$ in the improper component of $O(2N)$, where $\text{Det} U = -1$, must have an eigenvalue $-1$, which then is accompanied by a corresponding eigenvalue $+1$, giving the delta function in (51).) The general answer for the universal limit of the density of states is some linear combination of $\rho_D$ and $\rho_B$ as given by (50) and (51). When the weights assigned to the proper and improper components of $O(2N)$ are equal, the answer equals the arithmetic mean, which reproduces the result (49), obtained from the nonlinear sigma model.

Of course, the disordered quasiparticle systems we are interested in are typically very far from the random-matrix limit. One might therefore think that a nomenclature borrowed from that limit is too crude to capture the general situation. We insist, however, that the terminology we propose is fully supported from field theory, by the following argument.

The key is to understand how the expressions (50, 51) arise from the zero-dimensional limit of the field-theoretic representation by a nonlinear sigma model. As we recall, the target space of the field theory for class $D$ has two connected components, and the result (50) comes from summing over both of them with equal Boltzmann weights. By inspection, one finds that the expression for class $B$ results from a twisted target, where the Boltzmann weight of the component not containing the identity element carries a minus sign. It is useful to let the group $\mathbb{Z}_2$ act on the target space, so that twisting the Boltzmann weights is the same as acting with the nontrivial element of $\mathbb{Z}_2$. Now recall that the insertion of vortices amounts to creating frustrated plaquettes in the network model. Frustration of plaquettes, in turn, amounts to the presence of (strings of) links with minus signs. Any odd number of these will transfer the time-evolution operator $U$ from the proper to the improper part of the orthogonal group, and vice versa. The crucial proposition we are driving at is that $\mathbb{Z}_2$ vortex disorder acts as $\mathbb{Z}_2$ sign disorder on the target space of the nonlinear sigma model. (Indeed, as we saw, going from $D$ to $B$ in the random-matrix limit is like switching between the proper and improper components of $O(2N)$.) This proposition, albeit motivated by random-matrix considerations, does not depend on the random-matrix limit and we are therefore confident to predict its general validity. Thus, we expect that a square-root singularity (due to a half-quantum vortex) at a point $p$ in the continuum Majorana field acts as the nontrivial element of $\mathbb{Z}_2$ at $p$ on the target space of the CI|DIII nonlinear sigma model (whenever that model is a valid description); and, hence, average over randomly placed vortices suppresses the second target space component of that model.
This, ultimately, is the field-theoretic reason why vortices change the physics of disordered Majorana fermions, and why we believe that the subclass $D$ (without vortices) ought to be kept distinct from the more generic hybrid class $BD$.

12 Open questions

An extensive summary of our main results was already given in the introduction and will not be repeated here. Instead, we content ourselves with pointing out a few open questions we find interesting.

i) It has been conjectured \[54\] that the spectral statistics of the bound states of an isolated disordered vortex in a chiral $p$-wave superconductor, coincides with the random-matrix statistics of class $B$ in the universal limit. It would be desirable to try and prove that conjecture, by adapting the treatment of \[44\] to derive the nonlinear sigma model for class $B$ from a microscopic model.

ii) To elucidate the effect of vortex disorder on a system close to the free-fermion point, it would be useful to know how to correctly transform the vortex (or square-root) singularities in the Majorana field to the bosonized representation. Since the insertion of a square-root singularity changes the Neveu-Schwartz fermions of the radial quantization scheme into Ramond fermions, the question is how to modify the $C|D$ WZW$_{k=1}$ model to account for a degenerate (Ramond) vacuum.

iii) Although our work sheds much light on the 2d metallic phases of the classes $D$ and $BD$, the insulating phases remain poorly understood. There, we expect domain walls in the $\mathbb{Z}_2$ degree of freedom of the $CI|DI$ nonlinear sigma model to be important in determining the low-energy physics.

iv) According to \[58\], the 2d random-bond Ising model (with fluctuating signs, not bond strengths) can be cast in the form of a Chalker-Coddington network model with frustration (or vortices) on adjacent plaquettes. It would be interesting to understand better how this model fits into the expanded landscape of class $BD$ versus $D$. In particular, the nature of the multicritical Nishimori point remains to be clarified.

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