A CHARACTERIZATION OF KAZHDAN–LUSZTIG RIGHT CELLS CONTAINING SMOOTH ELEMENTS

ZHANQIANG BAI AND ZHENG-AN CHEN

Abstract. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. Its Weyl group is the symmetric group $S_n$. In this paper, we want to describe some Kazhdan–Lusztig right cells containing smooth elements, which is closely related to the study of associated varieties of highest weight modules of $\mathfrak{sl}(n, \mathbb{C})$. Firstly, we give a complete classification of the KL right cells containing only smooth elements. Then we give a sufficient condition for a KL right cell containing only non-smooth elements by using invariant subsequences and a sufficient condition for a KL right cell containing some smooth elements. Finally, we give an efficient algorithm to find out all the smooth elements in a given KL right cell.

Keywords: Young tableau; Pattern avoidance; Kazhdan–Lusztig right cell.

Contents

1. Introduction 1
2. Preliminaries 3
2.1. Associated variety 3
2.2. Pattern avoidance 4
2.3. Hecke algebra and cells 4
2.4. Robinson–Schensted insertion algorithm 4
3. Smooth KL right cells 6
4. Some examples of KL right cells with non-smooth elements 11
4.1. Invariant subsequences with the pattern 3412 11
4.2. Invariant subsequences with the pattern 4231 12
5. Young tableaux with two columns 13
6. Right cells containing some smooth elements 15
Acknowledgments 18
References 18

1. Introduction

In their famous paper [KL79], Kazhdan and Lusztig introduced the concepts of right, left and two-sided cells in order to study representations of the Hecke algebras associated to a Coxeter group $W$. Now these concepts are studied by many people from representation theory and combinatorics.

Let $G = SL(n, \mathbb{C})$ be the special linear group. Let $\mathfrak{g}$ be its simple complex Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra. Let $\Phi^+ \subset \Phi$ be the set of positive

2010 Mathematics Subject Classification. Primary 20B30; Secondary 05E10.
roots determined by a Borel subalgebra $b$ of $g$. Denote by $\Pi$ the set of simple roots in $\Phi^+$. We fix a Borel subgroup $B \subset G$ corresponding to $b$. We have a triangular decomposition $g = n \oplus h \oplus n^-$. The Weyl group $W$ of $g$ is $S_n$.

For $\lambda \in h^*$, the Verma module $M(\lambda)$ is defined by

$$M(\lambda) = U(g) \otimes U(b) C_{\lambda - \rho},$$

where $C_{\lambda - \rho}$ is a one-dimensional $b$-module with weight $\lambda - \rho$ and $\rho$ is half the sum of positive roots. Denote by $L(\lambda)$ the simple quotient of $M(\lambda)$.

We use $L_w$ to denote the simple highest weight $g$-module of highest weight $-w\rho - \rho$ with $w \in W$. Joseph [J84] proved that the associated variety $V(L_w)$ is a union of orbital varieties defined as follows. Let $O \subseteq g$ be a nilpotent $G$-orbit. The irreducible components of $O \cap n$ are called orbital varieties of $O$. They all take the form $V(w) = B(n \cap wn)$ for some $w \in W$. Melnikov [FM13, M04a, M04b, M06] did a lot of work for the properties of orbital varieties of type $A$. The associated variety $V(L_w)$ is called irreducible if and only if it contains only one orbital variety. For a long time, people conjectured that the associated variety of any highest weight module $L_w$ is irreducible in the case of type $A$ (see [BB85] and [M93]). However, Williamson [W15] showed that there exist counter-examples in 2014. So the structure of $V(L_w)$ or $V(L(\lambda))$ is still mysterious in type $A$.

We refer to [KL79] or §2.3 for the definition of Kazhdan–Lusztig right (resp. left and two-sided) cell equivalence relation and use $\sim^R$ (resp. $\sim^L$ and $\sim^{LR}$) to denote the right (resp. left and two-sided) cell equivalence relation.

Let $X_w = BwB/B$ be the Schubert variety indexed by $w$ in the flag manifold $G/B$. Lakshmibai and Sandhya [LS90] determined the smoothness of Schubert varieties for type $A$ by using pattern avoidance. It is known that for type $A$, a Schubert variety is smooth if and only if certain Kazhdan–Lusztig polynomials are trivial, see for example [CK03].

From Sagan [S01] or Bai–Xie [BX19, Lemma 4.1], we know that there is a bijection between the KL right cells in the symmetric group $S_n$ and the Young tableaux through the famous Robinson–Schensted insertion algorithm. We use $P(w)$ to denote the corresponding Young tableau for any $w \in S_n$.

From [J84], we know that the associated variety $V(L_w)$ is constant on each KL right cell. It is also known that if the associated variety $V(L_w)$ is reducible, the Schubert variety $X_w$ will be singular, see for example [BB85, Corollary 4.3.2]. So there is a relationship between the reducibility of associated varieties and non-smoothness (or smoothness) of Schubert varieties.

In this paper, we consider the following problem: For which Kazhdan–Lusztig right cell $C^R_w$, the Schubert variety $X_w$ is smooth for every $w \in C^R_w$? Equivalently, the Kazhdan–Lusztig polynomial $P_{e,w}(q) = 1$ for all $w$ in these KL right cells $C^R_w$.

For other KL right cells, we will give an algorithm to determine that it contains smooth elements or not. We have incorporated this in a simple program. The program takes some $w$ as the input and returns the set of smooth elements. It is available at

https://github.com/zhengan-chen/Young_tableaux.
From our program, we can easily find many elements in $S_n$ such that the associated variety of $L_w$ is irreducible. See Corollary 6.6.

This paper is organized as follows. In §2, we prepare some necessary preliminaries on associated varieties, pattern avoidance and KL right cells. In §3, we will give a complete classification of KL right cells containing only smooth elements, see Theorem 3.5. Then in §4, we describe some special KL right cells containing only non-smooth elements. In §5, we give a characterization for some special KL right cells (containing smooth elements) corresponding to some Young tableaux with two columns. In §6, we give an algorithm to find out all the smooth elements in a given KL right cell.

2. Preliminaries

In this section, we give some brief preliminaries on associated varieties of highest weight modules, pattern avoidance and the Robinson–Schensted insertion algorithm.

2.1. Associated variety. Let $\mathfrak{g}$ be a simple complex Lie algebra. Let $M$ be a finite generated $U(\mathfrak{g})$-module. Fix a finite dimensional generating subspace $M_0$ of $M$. Let $U_n(\mathfrak{g})$ be the standard filtration of $U(\mathfrak{g})$. Set $M_n = U_n(\mathfrak{g}) \cdot M_0$ and $\text{gr}(M) = \bigoplus_{n=0}^\infty \text{gr}_n M$, where $\text{gr}_n M = M_n/M_{n-1}$. Thus $\text{gr}(M)$ is a graded module of $\text{gr}(U(\mathfrak{g})) \cong S(\mathfrak{g})$.

Definition 2.1. The associated variety of $M$ is defined by

$$V(M) := \{ X \in \mathfrak{g}^* | f(X) = 0 \text{ for all } f \in \text{Ann}_{S(\mathfrak{g})}(\text{gr} M) \}.$$ 

The above definition is independent of the choice of $M_0$ (e.g., [NOT01]).

Let $G$ be a connected semisimple finite dimensional complex algebraic group with Lie algebra $\mathfrak{g}$. We fix some triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Let $O$ be a nilpotent $G$ orbit. The irreducible components of $\overline{O \cap \mathfrak{n}}$ are called orbital varieties associated to $O$. Usually, an orbital variety has the following form

$$V(w) = \overline{B(n \cap w(n))}$$

for some $w$ in the Weyl group $W$ of $\mathfrak{g}$, where $B$ is the Borel subgroup of $G$ corresponding the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$.

We have the following propositions.

Proposition 2.2 ([J84]). Let $L(\lambda)$ be a highest weight module of a simple Lie algebra $\mathfrak{g}$ with highest weight $\lambda - \rho$. Then its associated variety $V(L(\lambda))$ equals the union of some orbital varieties associated with the nilpotent coadjoint orbit $O_{\text{Ann}(L(\lambda))}$ in $\mathfrak{g}^*$.

The associated variety $V(L(\lambda))$ is called irreducible if it equals one orbital variety.

Proposition 2.3 ([BB85, Corollary 4.3.2]). If the Schubert variety $X_w = \overline{BwB/B}$ is smooth, we will have $V(L_w) = V(w)$. 
2.2. Pattern avoidance. By the definition, an element \( w \in S_n \) is a permutation of the set \( \{1, 2, ..., n\} \). In general, we use \( w = (w_1, ..., w_n) \) to denote this permutation, where \( w_i = w(i) \).

Firstly we have the following definition.

**Definition 2.4.** The element \( w = (w_1, ..., w_n) \in S_n \) contains the pattern 3412 (resp. 4231) if there exist integers \( 1 \leq i < j < k < l \leq n \) such that \( w_k < w_l < w_i < w_j \) (resp. \( w_k < w_j < w_k < w_i \)). If there is no such integers, we say \( w \) avoids the pattern 3412 and 4231.

We have the following criterion for smoothness of Schubert varieties.

**Proposition 2.5 ([LS90]).** For \( g = sl(n, \mathbb{C}) \) and \( W = S_n \), the Schubert variety \( X_w = BwB/B \) is smooth if and only if \( w \) avoids the two patterns 3412 and 4231.

In general, \( w \) is called a smooth element when \( X_w \) is smooth.

2.3. Hecke algebra and cells. Recall that the Weyl group \( W \) is a Coxeter group generated by \( S = \{s_\alpha \mid \alpha \in \Delta\} \). Let \( \ell(\cdot) \) be the length function on \( W \). Given an indeterminate \( v \), the Hecke algebra \( \mathcal{H} \) over \( A := \mathbb{Z}[q, q^{-1}] \) is generated by \( T_w, w \in W \) with relations

\[
T_wT_{w'} = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w'),
\]

and \((T_s + q^{-1})(T_s - q) = 0\) for any \( s \in S \).

The unique elements \( C_w \) such that

\[
\overline{C_w} = C_w, \quad C_w \equiv T_w \mod \mathcal{H}_{<0}
\]

are known as the Kazhdan–Lusztig (KL) basis of \( \mathcal{H} \), where \( -: \mathcal{H} \to \mathcal{H} \) is the bar involution such that \( \overline{q} = q^{-1}, \overline{T_w} = T_{w^{-1}}^{-1} \), and \( \mathcal{H}_{<0} = \bigoplus_{w \in W} A_{<0}T_w \) with \( A_{<0} = q^{-1}\mathbb{Z}[q^{-1}] \).

If \( C_y \) occurs in the expansion of \( hC_w \) (resp. \( C_w h \)) with respect to the KL-basis for some \( h \in \mathcal{H} \), then we write \( y \leftarrow_L w \) (resp. \( y \leftarrow_R w \)). Extend \( \leftarrow_L \) (resp. \( \leftarrow_R \)) to a preorder \( \leq_L \) (resp. \( \leq_R \)) on \( W \). For \( x, w \in W \), write \( x \leq_{LR} w \) if there exists \( x = w_1, \ldots, w_n = w \) such that for every \( 1 \leq i < n \) we have either \( w_i \leq_L w_{i+1} \) or \( w_i \leq_R w_{i+1} \). Let \( \sim, \sim_L, \sim_R \) be the equivalence relations associated with \( \leq_L, \sim, \sim_{LR} \) (for example, \( x \sim w \) if and only if \( x \leq_{LR} w \) and \( w \leq_{LR} x \)). The equivalence classes on \( W \) for \( \sim_L, \sim_R, \sim_{LR} \) are called left cells, right cells and two-sided cells respectively.

**Proposition 2.6 ([SS82, II. 9.8]).** Suppose \( x, y \in S_n \). Then we have \( \mathcal{V}(x) = \mathcal{V}(y) \) if and only if \( x \sim y \).

**Proposition 2.7 ([J84, Lemma 6.6]; [BB85, Corollary 6.3]).** \( \mathcal{V}(L_w) \) is constant on each KL right cell.

2.4. Robinson–Schensted insertion algorithm. In this subsection, we recall the famous Robinson–Schensted insertion algorithm. Some details can be found in [A98] and [S01].
Definition 2.8 (Robinson–Schensted insertion algorithm). For an element \( w \in S_n \), we write \( w = (w_1, \ldots, w_n) \). We associate to \( w \) a Young tableau \( P(w) \) as follows. Let \( P_0 \) be an empty Young tableau. Assume that we have constructed Young tableau \( P_k \) associated to \( (w_1, \ldots, w_k) \), \( 0 \leq k < n \). Then \( P_{k+1} \) is obtained by adding \( w_{k+1} \) to \( P_k \) as follows. Firstly we add \( w_{k+1} \) to the first row of \( P_k \) by replacing the leftmost entry \( x \) in the first row which is strictly bigger than \( w_{k+1} \). (If there is no such an entry \( x \), we just add a box with entry \( w_{k+1} \) to the right side of the first row, and end this process). Then add \( x \) to the next row as the same way of adding \( w_{k+1} \) to the first row. Finally we put \( P(w) = P_n \). Let \( Q(w) \) be the recording tableau such that 1, 2, ..., \( n \) are placed in the \( Q \)'s so that shape of \( P_k \) equal to the shape of \( Q_k \) for all \( 1 \leq k \leq n \). Thus \( Q(w) = Q_n \) and \( \text{sh}(P(w)) = \text{sh}(Q(w)) \).

We use \( p(w) = [p_1, \ldots, p_k] \) to denote the shape of \( P(w) \), where \( p_i \) is the number of boxes in the \( i \)-th row of \( P(w) \). So \( [p_1, \ldots, p_k] \) is a partition of \( n \), denoted by \( p(w) \vdash n \).

Proposition 2.9 ([S01, Theorem 3.1.1 & Theorem 3.6.6]). The map \( w \rightarrow (P(w), Q(w)) \) is a bijection between elements of \( S_n \) and pairs of standard tableaux of the same shape \( p(w) \vdash n \). We also have \( P(w^{-1}) = Q(w) \) and \( Q(w^{-1}) = P(w) \) for any \( w \in S_n \).

Proposition 2.10 ([A98] or [KL79]). For \( g = s(n, \mathbb{C}) \) and \( W = S_n \), two elements \( x \) and \( y \) in \( S_n \) are in the same KL right cell if and only if \( P(x) = P(y) \).

Definition 2.11. Suppose \( x < y < z \). Then \( w, \pi \in S_n \) differ by a Knuth relation of the first kind, if

\[
w = (w_1, \ldots, y, x, z, \ldots, x_n) \quad \text{and} \quad \pi = (x_1, \ldots, y, z, x, \ldots, x_n) \quad \text{or vice versa.}
\]

They differ by a Knuth relation of the second kind, if

\[
w = (w_1, \ldots, x, z, y, \ldots, x_n) \quad \text{and} \quad \pi = (x_1, \ldots, z, x, y, \ldots, x_n) \quad \text{or vice versa.}
\]

The two elements are Knuth equivalent, written \( w \sim^K \pi \), if there is a sequence of elements in \( S_n \) such that \( w = \pi_1, \ldots, \pi_N = \pi \) such that \( \pi_i \) and \( \pi_{i+1} \) differ by a Knuth relation of the first kind or second kind for all \( 1 \leq i \leq N - 1 \).

Proposition 2.12 ([K70]). For two elements \( x \) and \( y \) in \( S_n \), we have \( x \sim^K y \) if and only if \( P(x) = P(y) \).

Definition 2.13. If \( P \) is a Young tableau, then the row word of \( P \) is the permutation

\[
\pi_P = R_N R_{N-1} \cdots R_1,
\]

where \( R_i \) is the \( i \)-th row of \( P \).

The column word of \( P \) is the permutation

\[
\pi^t_P = C^t_1 C^t_2 \cdots C^t_k,
\]

where \( C_i \) is the \( i \)-th column of \( P \) and \( C_i^t \) is the transpose of \( C_i \) and is starting from the bottom of \( C_i \).
Example 2.14. Suppose
\[
P(w) = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 \end{pmatrix},
\]
then the corresponding row word is \( \pi_P = (2,4,1,3,5) \) and column word is \( \pi^c_P = (2,1,4,3,5) \).

The following proposition is very useful in the proof our main results.

Proposition 2.15 (\cite{S01}, Theorem 3.5.3). Consider \( \pi \in S_n \). The length of the longest increasing subsequence of \( \pi \) is the length of the first row of \( P(\pi) \). The length of the longest decreasing subsequence of \( \pi \) is the length of the first column of \( P(\pi) \).

3. Smooth KL right cells

In this section, we will determine the KL right cells where all elements are smooth. Our result is based on a simple observation.

Lemma 3.1. Let \( \pi \in S_n \), for every \( i \), the relative position of \( i \) and \( i+1 \) remains unchanged under the action of the Knuth relation.

Proof. Based on the definition of the Knuth relation, if we want to change the position of two elements, we need the help of the middle number. But for \( i \) and \( i+1 \), there is not such middle number. So the lemma is correct. \( \square \)

Before we give the proof of our result, we discuss some special cases.

Proposition 3.2. Let \( P \) be a Young tableau. The shape of \( P \) is \([2,1,\ldots,1] = [2,1^{n-2}]\). Let \( \pi_P = (n,\ldots,k+1,k-1,\ldots,2,1,k) \) be the row word of \( P \). Then the right cell corresponding to the permutation \( \pi_P \) will be one of the two followings:

1. All elements in the right cell corresponding to \( \pi_P = (n,\ldots,3,1,2) \) or \( \pi_P = (n-1,\ldots,2,1,n) \) will be smooth;
2. The right cell contains smooth and non-smooth elements for \( 2 < k < n \).

Proof. We use \([n,m]\) to denote the set \( \{n, n+1, \ldots, m\} \) for \( n < m \). We use “\( k \)” to denote the \( k \)-th largest element in the pattern 4231 and the pattern 3412 for a given permutation \( \pi \in S_n \).

For case one, we can easily find that the pattern 3412 will not appear. For the pattern 4231, if we consider \( \pi_P = (n,\ldots,3,1,2) \), you will find that we can only take “\( 2 \)” from \([2,n]\). But Lemma 3.1 implies possible “\( 3 \)” is before “\( 2 \)”, which implies that the pattern 4231 will not appear. If we consider \( \pi_P = (n-1,\ldots,2,1,n) \), the discussion is similar.

For the case two, we find that there are no pattern 3412 and 4231 in the initial permutation. But after some Knuth relation, the pattern 4231 will appear:
\[
\pi_P = (n-1,\ldots,k+1,k-1,\ldots,2,1,k) \xrightarrow{K} (n-1,\ldots,k+1,k-1,\ldots,2,k,1)
\]
since \( 2 < k < n \). Now, \((k+1,k-1,k,1)\) will satisfy the pattern 4231. \( \square \)
Proposition 3.3. Let $P$ be a Young tableau. The shape of $P$ is $[n - 1, 1]$. Then the elements in the right cells are all smooth.

Proof. Let $\pi_P = (k, 1, 2, \ldots, k - 1, k + 1, \ldots, n)$ for some $2 \leq k \leq n$.

For the pattern 4231, note that it has a subsequence $(4, 2, 1)$. So by Proposition 2.15, the length of the first column of $P(\pi_P)$ is larger than 2, which is a contradiction.

For the pattern 3412, if “3” is taken from $[1, k - 1]$, then we can’t find “12”. If “3” is taken from $[k, n]$, we will take “12” from $[1, k - 1]$. But now the length of the longest increasing subsequence is less than before, which is a contradiction.

So the pattern 4231 and 3412 can not appear in any element of any KL right cell corresponding to some $\pi_P = (k, 1, 2, \ldots, k - 1, k + 1, \ldots, n)$. □

Remark 3.4. From the above two propositions, we can see that we cannot simply use symmetry to make inferences. The nature of $P$ and $P^t$ may be completely different.

Now we give the theorem:

Theorem 3.5. Let $n \in \mathbb{Z}_{>0}$. Suppose $P$ is a standard Young tableau with $n$ boxes. Then its corresponding KL right cell consists entirely of smooth elements if and only if $P$ is one of the followings:

1. When the shape of $P$ is $[n]$ or $[1, \ldots, 1] = [1^n]$, we will have:

   $$P = \begin{array}{c}
   1 \\
   2 \\
   \vdots \\
   n
   \end{array}$$

    or

   $$P = \begin{array}{c}
   1 \\
   2 \\
   \vdots \\
   n
   \end{array}$$

2. When the shape of $P$ is $[2, 1, \ldots, 1] = [2, 1^{n-1}]$ or $[n - 1, 1]$, we will have:

   $$P = \begin{array}{c}
   1 \\
   2 \\
   \vdots \\
   n
   \end{array}$$

   or

   $$P = \begin{array}{c}
   1 \\
   2 \\
   \vdots \\
   n - 1
   \end{array}$$

   or

   $$P = \begin{array}{c}
   1 \\
   2 \\
   \vdots \\
   n
   \end{array}$$

   or

   $$P = \begin{array}{c}
   1 \\
   2 \\
   \vdots \\
   n
   \end{array}$$
When the shape of $P$ is $[k,1,\ldots,1] = [k,1^{n-k}]$ for some $2 < k < n-1$ and the length of the first column is larger than 2, we will have:

$$P = \begin{array}{cccc}
1 & i+1 & \cdots & n \\
2 & \\
\vdots & \\
i & \\
\end{array}$$

or

$$P = \begin{array}{cccc}
1 & 2 & \cdots & i \\
i+1 & \\
\vdots & \\
n & \\
\end{array}$$

or

$$P = \begin{array}{cccc}
1 & 2 & \cdots & i \\
i+1 & \\
\vdots & \\
k-1 & \\
\end{array}$$

(4) When the shape of $P$ is $[n-2,2]$, we will have:

$$P = \begin{array}{cccc}
1 & 3 & \cdots & k-1 & k+1 & \cdots & n \\
2 & k & \\
\end{array}$$

Proof. (1). It is obvious that the elements of right cells corresponding to $(1,2,\ldots,n)$ and $(n,n-1,\ldots,1)$ are smooth, since we can’t find any Knuth relations in those permutations.

(2). First, we consider Young tableaux with only one row and one column whose length is larger than one. Since we have discussed about the case (2) in Proposition 3.2, we only focus on case (3).

(3). Let $\pi_P$ be like $(\ldots,a,1,b,\ldots)$. Now there is only one Knuth relation in $\pi_P$, which is $(a,1,b)$.

If $a < b$, we will have $\pi_P \sim (\ldots,a,b,1,\ldots)$. If the leftmost element, denoted as $c$, of $\pi_P$ is larger than $b$, $cab1$ will be the pattern 4231. So we assume that $c < b$, $\pi_P$ is like $(i,i-1,\ldots,2,1,i+1,\ldots,n)$. We claim that the elements of the right cell will avoid the pattern 3412 and 4231.

First, we consider the pattern 3412. If “3” is taken from $[3,i]$. By Lemma 3.1, we note “4” can only be taken from $[i+1,n]$ and “1”, “2” can only be taken from $[1,3-1]$. However, by Lemma 3.1, “2” must be before “1”. It is absurd. If “3” is taken from $[i+1,n]$, the discussion is the same. When we consider the pattern 4231, we find that “2” must be taken from $[1,i]$ and “3” must be taken from $[i+1,n]$. Using Lemma 3.1, we note that “4” must
be after "3", which is a contradiction. So all elements of this right cell will avoid the pattern 3412 and 4231.

If \( a > b \), we note that the lengths of the first row and first column are both larger than 2 and let \( \pi_P = (\ldots, a, 1, b, d, \ldots) \).

If \( a < d \), we can use the Knuth relation to change \( \pi_P \) into \( (\ldots, 1, a, d, b, \ldots) \). If the leftmost element of the changed permutation is larger than \( d \), then we have the pattern 4231. So we assume the leftmost element of the changed permutation is smaller than \( d \), then

\[
\pi_P^K \sim \pi'_P = (i, \ldots, 3, 1, 2, i + 1, \ldots, n).
\]

First we claim that there is no pattern like 3412. Observing \( \pi'_P \), we find that the longest increasing subsequence is \( (1, 2, i + 1, \ldots, n) \). But if we have the pattern 3412, we find that "1", "2" must be 1, 2 and "4" must be taken from \([i + 1, n]\). But when the pattern exists, the length of the longest increasing subsequence of the changed permutation is less than \( n - i + 2 \), which is a contraction. As for the pattern 4231, we can easily find a contradiction in the relative position between "2" and "3".

If \( a > d \) and \( a \) is larger than the last element of \( \pi_P \), we may assume \( \pi_P^K \sim \pi''_P = (n, \ldots, i + 1, 1, 2, \ldots, i) \). We claim that the elements of the right cell are all smooth. For the pattern 3412, if we take "3" from \([i + 1, n]\), we cannot find "4". If we take "3" from \([1, i]\), we cannot find "2". As for the pattern 4231, we know that "4" can only be taken from \([i + 1, n]\). Then if "2" is also taken from \([i + 1, n]\), we can't find "3". If "2" is taken from \([1, i]\), we can't find "1".

If there exists an element in the first row of \( P \) larger than \( a \), denoted as \( k \), and the leftmost element \( c \) of \( \pi_P \) is larger than \( k \). Then we can write

\[
\pi_P = (c, \ldots, a, 1, 2, 3, \ldots, a - 1, k, \ldots).
\]

Using the Knuth relations, we have:

\[
(c, \ldots, a, 1, 2, 3, \ldots, a - 1, k, \ldots) \sim (c, \ldots, 1, a, 2, 3, \ldots, a - 1, k, \ldots) \\
\sim (c, \ldots, 1, 2, a, 3, \ldots, a - 1, k, \ldots) \\
\sim (c, \ldots, 1, 2, 3, \ldots, a, a - 1, k, \ldots) \\
\sim (c, \ldots, 1, 2, 3, \ldots, a, k, a - 1 \ldots).
\]

We find that \((c, a, k, a - 1)\) has the pattern 4231, which implies that the right cell has a non-smooth element.

If the leftmost element is smaller than \( k \). We can write

\[
\pi_P = (k - 1, \ldots, i + 1, 1, 2, 3, \ldots, i, k, \ldots).
\]

For the pattern 3412, if "3" is taken from \([1, i]\), we can't find "1" and "2". If "3" is taken from \([i + 1, k - 1]\), "4" must be taken from \([k, n]\), then "1" and "2" must be taken from \([1, i]\). But the length of the longest increasing subsequence will decrease, which is a contradiction. If "3" is taken from \([k, n]\), the contradiction is the same.

For the pattern 4231, if we take "4" from \([i + 1, k - 1]\), then if we take "2" from \([i + 1, k - 1]\), "3" will have no choice. If we take "2" from \([1, i]\),
“1” will have no choice. If we take “4” from \([1, i]\), “1” and “2” will have no choices. If we take “4” from \([k, n]\), we will find that “3” will have no choice.

(4) Now we consider Young tableaux with multiple rows and columns whose lengths are larger than one. We find that if the given Young tableau \(P\) has more than three rows whose lengths are larger than one, the pattern 4231 will appear. And if the length of the second row is larger than 2, the pattern 3412 will also appear. Also the first element of the second row, denoted it by \(b\), must be 2. Otherwise, we will find \(b, c, 1, a\) will satisfy the pattern 3412. So the Young tableau corresponding to \(\pi\) and avoiding the pattern 3412 and 4231 must be like:

\[
P(w) = \begin{array}{cccc}
1 & a & \cdots \\
2 & c \\
\vdots
\end{array}
\]

If there exists some element larger than \(c\) in the first column, \(\pi\) will have the pattern 4231. So we can assume the elements in the first column are smaller than \(c\).

If \(P\) has only two rows, the corresponding tableau will be like:

\[
P = \begin{array}{ccc}
1 & 3 & \cdots \\
2 & k
\end{array}
\]

and \(\pi_P = (2, k, 1, 3, \ldots)\). For the pattern 3412, if we take “4” from \([3, k - 1]\), we can’t find the correct “2”. If we take “4” from \([k + 1, n]\), denoted it as \(l\), then “2” must be taken from \([3, k - 1]\), denoted it as \(j\). Now the length of the longest increasing subsequence is actually less than before, which is a contradiction. For the pattern 4231, since the length of the first column is 2, there will be no longer decreasing subsequence. So it avoids the pattern 4231.

The last case we need to consider is:

\[
P = \begin{array}{cccc}
1 & k + 1 & \cdots & n \\
2 & j \\
\vdots \\
k
\end{array}
\]

Now \(\pi_P = (k, k - 1, \ldots, 3, 2, j, 1, k + 1, \ldots, j - 1, j + 1, \ldots, n)\). We can easily find that

\[
\pi_P = (k, k - 1, \ldots, 3, 2, j, 1, k + 1, \ldots, j - 1, j + 1, \ldots, n)
\]

\[
\sim (k, k - 1, \ldots, 3, j, 2, 1, k + 1, \ldots, j - 1, j + 1, \ldots, n)
\]

\[
\sim (k, k - 1, \ldots, 3, j, 2, k + 1, 1, \ldots, j - 1, j + 1, \ldots, n).
\]

So \((j, 2, k + 1, 1)\) satisfies the pattern 4231. Thus \(P\) contains some non-smooth element. \(\square\)
We call a KL right cell in the above theorem a smooth KL right cell. If we use \( C_R(P) \) to denote the corresponding KL right cell for a given Young tableau \( P \), we will have the following.

**Corollary 3.6.** A KL right cell \( C_R(P) \) is a smooth cell if and only if its column word \( \pi_P \) is one of the following elements:

- \( x_1 = (1, 2, ..., n) \);
- \( x_2 = (n, n - 1, ..., 1) \);
- \( x_3 = (n, n - 1, ..., 3, 1, 2) \);
- \( x_4 = (n - 1, n - 2, ..., 1, n) \);
- \( y_k = (k, 1, 2, ..., k - 1, k + 1, ..., n), 2 \leq k \leq n \);
- \( z_i = (i, i - 1, ..., 1, i + 1, i + 2, ..., n), 3 \leq i \leq n - 2 \);
- \( z_i' = (n, n - 1, ..., i + 1, 1, 2, 3, ..., i), 3 \leq i \leq n - 2 \);
- \( s_{ik} = (n, n - 1, ..., i + 1, 1, 2, 3, ..., k), 1 \leq i < k \leq n - 2 \);
- \( t_k = (2, 1, k, 3, 4, ..., k - 1, k + 1, ..., n), 4 \leq k \leq n \).

4. Some examples of KL right cells with non-smooth elements

By Lemma 3.1, we can find some special KL right cells with all elements being non-smooth. In general, we prove that the permutations corresponding to these tableaux have some invariant subsequences under the action of the Knuth relations to illustrate their non-smoothness.

4.1. Invariant subsequences with the pattern 3412.

**Proposition 4.1.** For \( k \geq 3 \), we have the fact that all elements of the following KL right cells are non-smooth:

\[
P = \begin{array}{cccc}
1 & 2 & \cdots & k \\
k + 1 & k + 2 & \cdots & 2k \\
\end{array}
\]

and

\[
P' = \begin{array}{cccc}
1 & 2 & \cdots & k & k + 1 \\
k + 2 & k + 3 & \cdots & 2k + 1 \\
\end{array}
\].

Actually, \((k + 1, k + 2, k - 1, k)\) and \((k + 2, k + 3, k, k + 1)\) are respectively the invariant subsequences.

**Proof.** We only prove the first case and the second one is exactly the same. By Lemma 3.1, \( k + 1 \) is always before \( k \), \( k + 2 \) is always after \( k + 1 \) and \( k - 1 \) is before \( k \). If the relative position of \((k + 1, k + 2, k - 1, k)\) will not change under the action of the Knuth relations, we can prove the proposition. If not, the only possibility is that \( k + 2 \) is after \( k - 1 \) under some actions of the Knuth relations. But now there will exist an increasing subsequence \((1, 2, \ldots, k - 1, k + 2, \ldots, 2k)\) by Lemma 3.1. The length of the subsequence is \(2k - 2\). But we know that the longest increasing subsequence is the length of the first row by Proposition 2.15. So we have \(2k - 2 \leq k\), which means \( k \leq 2 \). This is a contradiction since \( k \geq 3 \). \(\square\)
For the following two cases, we can similarly prove that all elements of the given KL right cells are non-smooth:

\[
P = \begin{array}{cccc}
1 & 2 & \cdots & k \\
& k+1 & & 2k \\
& & k+2 & \cdots & 2k \\
\end{array}
\]

and

\[
P = \begin{array}{cccc}
1 & 2 & \cdots & k \\
& k+1 & & 2k \\
& & k+2 & \cdots & 2k' \\
& & & 2k+1 \\
\end{array}
\]

Now we give a sufficient condition that all elements in a given KL right cell are non-smooth.

**Proposition 4.2.** Let \( P \) be a Young tableau and \( l \) be the length of the first row. When \( P \) is like:

\[
P = \begin{array}{cccc}
1 & 2 & \cdots & k \\
& k+1 & & \cdots \\
& & k+2 & \cdots \\
& & & k+m \\
\end{array}
\]

and \( k+m-2 > l \), all elements of the KL right cell are non-smooth.

**Proof.** The subsequence \((k+1, k+2, k-1, k)\) in the row word of \( P \) makes the pattern 3412. If \( k+2 \) is after \( k-1 \) under some actions of the Knuth relations, there will exist an increasing subsequence \((1, 2, \ldots, k-1, k+2, \ldots, k+m)\) in the new permutation by Lemma 3.1. The length of this subsequence is \( k+m-2 \). But based on our assumption, it is larger than the length of the first row. This is a contradiction since Proposition 2.15. \( \square \)

Note that in the above proposition, we have \( k \geq m \).

### 4.2. Invariant subsequences with the pattern 4231

We find that some special permutations will have some invariant subsequences satisfying the pattern 4231.

**Proposition 4.3.** If the Young tableau \( P \) has only two columns and is like the following:

\[
P = \begin{array}{cccc}
& & \cdots \\
& & \cdots \\
k-2 & k-1 & & \\
k & k+1 & & \\
k+2 & k+3 & & \\
\end{array}
\]
then all elements of the right cell are non-smooth. Actually, the subsequence 
\((k + 2, k, k + 1, k - 1)\) always exists under the action of the Knuth relation, 
which satisfies the pattern 4231.

**Proof.** By Lemma 3.1, we find that \(k + 2\) is before \(k + 3\). If \(k\) is before \(k + 2\), 
the subsequence \((k, k + 2, k + 3)\) exists, which implies the length of the first 
row of \(P\) is larger than 2. This is a contradiction. If \(k - 1\) is before \(k + 1\), 
then the subsequence \((k - 2, k - 1, k + 1)\) exists, which is also a contradiction. 
So the subsequence \((k + 2, k, k + 1, k - 1)\) always exists under the actions of 
the Knuth relations. \(\square\)

**Remark 4.4.** The above discussion does not indicate that for all elements 
in a KL right cell with all elements being non-smooth, we can find an invariant 
subsequence which satisfies the pattern 3412 or 4231. Let \(\pi_P = (6, 7, 3, 4, 1, 2, 5)\). Under the actions of the Knuth relations, there exists one 
permutation avoiding the pattern 3412. For the pattern 4231, we find that 
\((6, 7, 3, 4, 1, 2, 5)\) and \((3, 6, 4, 7, 1, 5, 2)\) don't have common subsequences satisfying the pattern 4231.

5. Young tableaux with two columns

The Young tableaux with only two columns always appear in representation 
theory of Lie algebras and Lie groups, see for example [BX19, BXX23]. Now we pay 
attention on KL right cells corresponding to these special tableaux. Let the Young tableau \(P\) be as follows:

\[
P = \begin{array}{c}
1 & b \\
\vdots & \vdots \\
\vdots & c \\
a
\end{array}
\]

We can easily find that the permutation \((a, \ldots, 1, c, \ldots, b)\) is the column 
word corresponding to \(P\). Now we have the following proposition.

**Proposition 5.1.** Let \(P\) be a Young tableau with two columns. \(P\) is like 
the tableau above. If \(P\) satisfies one of the following conditions:

1. The numbers below \(b\) are all larger than \(a\).
2. There exist numbers below \(b\) smaller than \(a\). Let \(d\) be the largest one 
   and the elements of the first column which are smaller than \(d\) are 
   also smaller than \(b\).

then the KL right cell will have at least one smooth element.

**Proof.** Firstly we know that column word \((a, \ldots, 1, c, \ldots, b)\) avoids the pattern 
3412. So we focus on the pattern 4231. Let us assume that \(a\) is “4” and \(b\) is “1”. Now if the numbers below \(b\) are larger than \(a\), the pattern 4231 will 
not appear. If there exist numbers below \(b\) smaller than \(a\), we denote the 
largest one by \(d\). If the first column has an element \(e\) satisfying \(b < e < d\), 
the pattern 4231 will appear. Otherwise, the pattern 4231 will not appear. \(\square\)
For this point, we can further discuss the Young tableau of the following:

\[
P = \begin{array}{ccc}
1 & b \\
\vdots \\
d & c \\
\end{array}.
\]

First, we have the fact that the corresponding KL right cells have non-smooth elements since Theorem 3.5. From the column word expression, we have the followings:

1. If \( c > a \), the KL right cell will have smooth elements.
2. If \( c < a \) and the elements of the first column which are smaller than \( c \) are also smaller than \( b \), then the KL right cell will have smooth elements.

Now if the first column has elements which are smaller than \( c \) and larger than \( b \), we can solve one special case.

**Proposition 5.2.** Let the Young tableau \( P \) be as follows:

\[
P = \begin{array}{ccc}
1 & 2 \\
3 & c \\
\vdots \\
b \\
\end{array}.
\]

Suppose \( a > c \) and \( 3 < b < c \) in the first column. Then the KL right cell has both smooth and non-smooth elements.

**Proof.** Let \( b \) be the largest element in the first column which is smaller than \( c \). Then \( \pi_P = (a, \ldots, b, \ldots, 3, c, 1, 2) \). Using the Knuth relations, we will have:

\[
(a, \ldots, b, \ldots, 3, c, 1, 2) \overset{K}{\sim} (a, \ldots, b, c, \ldots, 3, 1, 2) \\
\overset{K}{\sim} (b, a, \ldots, c, \ldots, 3, 1, 2) \\
\overset{K}{\sim} (b, a, \ldots, c, \ldots, 1, 3, 2) \\
\overset{K}{\sim} \ldots \\
\overset{K}{\sim} (b, 1, a, \ldots, c, \ldots, 3, 2).
\]

Obviously the permutation \((b, 1, a, \ldots, c, \ldots, 3, 2)\) avoids the two patterns 3412 and 4231.

**Remark 5.3.** For a given Young tableau \( P \), the KL right cell \( C_R(P) \) will contain smooth elements if the column word \( \pi_P^c \) or row word \( \pi_P \) avoids the two patterns 3412 and 4231. But it can happen that both \( \pi_P^c \) and \( \pi_P \) are not smooth when \( C_R(P) \) contains smooth elements. For example, in the two-sided cell corresponding to a given partition \( \mu \vdash n \), the longest element \( w_\mu \) in the parabolic subgroup \( W_\mu \) is smooth since it avoids the two patterns 3412 and 4231. \( w_\mu \) is the column word in the KL right cell containing \( w_\mu \).
For 

\[
P = \begin{array}{cc}
1 & 2 \\
3 & 5 \\
4 & \\
6 & \\
\end{array}
\]

its column word \( \pi^c_P = (6, 4, 3, 1, 5, 2) \) contains the pattern 4231 (since we can choose the subsequence \((6, 3, 5, 2)\)) and its row word \( \pi_P = (6, 4, 3, 5, 1, 2) \) contains the pattern 3412 (since we can choose the subsequence \((3, 5, 1, 2)\)). But from Proposition 5.2, we know that the KL right cell \( C_R(P) \) contains both smooth and non-smooth elements. For example \( w = (4, 1, 6, 5, 3, 2) \in C_R(P) \) is smooth.

6. Right cells containing some smooth elements

In this section, we want to give some algorithms to determine that a right cell contains a smooth element or not by using the Knuth relations.

To give our algorithm, we recall the famous hook formula, which was found by Frame, Robinson and Thrall [FRT54].

**Definition 6.1.** If \( v = (i, j) \) is a node in the diagram or Young tableau \( P \), then it has hook

\[
H_v = H_{i,j} = \{(i', j) \mid j' \geq j\} \cup \{(i, j') \mid i' \geq i\}
\]

with corresponding hooklength

\[
h_v = h_{i,j} = |H_{i,j}|.
\]

**Example 6.2.** Let

\[
P = \begin{array}{cc}
1 & 2 \\
3 & 5 \\
4 & \\
6 & \\
\end{array}
\]

then the dotted cells in

\[
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \\
\bullet & \\
\end{array}
\]

are the hook \( H_{2,1} \) with hooklength \( h_{2,1} = 4 \).

**Proposition 6.3** (Hook formula [FRT54]). Let \( P \) be a standard Young tableau and \( C \) be the corresponding Kazhdan–Lusztig right cell in the symmetric group \( S_n \). Then

\[
\#C = \frac{n!}{\prod_{(i,j) \in P} h_{i,j}}.
\]

There is another formula which is much older than the hook formula. In the following, we set \( 1/r! = 0 \) if \( r < 0 \) and \( 0! = 1 \).
Proposition 6.4 (Determinantal formula [S01]). Let $P$ be a standard Young tableau with shape $p = [p_1, \ldots, p_l]$ and $C$ be the corresponding Kazhdan–Lusztig right cell in the symmetric group $S_n$. Then

$$\#C = \frac{n!}{\prod (a_{ij})_{i\times j}},$$

where $(a_{ij})_{i\times j}$ is a $l \times l$ matrix with $a_{ij} = \frac{1}{(p_i+i+j)!}$.

Some more details about this formula can be found in Sagan [S01, §3.11].

Now recall that we have $x \sim y$ if and only if $P(x) = P(y)$ for any two elements $x$ and $y$ in $S_n$. Let $w \in S_n$ and $P(w)$ be the corresponding Young tableau which corresponds to a right cell $C^R(P(w))$. Then we have the following algorithm to count the number of smooth elements in the right cell $C^R(P(w))$:

1. If $w$ is smooth, then we put it in a set and denote it by $S_1$. If $w$ is not smooth, we write $w = (w_1, w_2, w_3, \ldots, w_n)$ and put it in another set and denote it by $N_1$. Thus we have $\#S_1 = 1$ or $\#N_1 = 1$;
2. From $w = (w_1, w_2, w_3, \ldots, w_n)$, we consider any triple $(w_i, w_{i+1}, w_{i+2})$ for $1 \leq i \leq n - 2$. If it is neither a decreasing nor an increasing sequence, we can use the Knuth relation to get a different element $w'_i$ such that $w'_i \sim w$. Let $G_1 := \{w'_i \mid 1 \leq i \leq n - 2\}$. Then we extract all smooth elements from $G_1$ and denote this new set by $S'_1$. Denote $N'_1 = G_1 - S'_1$. Then we define $S_2 = S'_1 \cup S_1$ and $N_2 = N'_1 \cup N_1$;
3. For each element in the set $G_1$, we repeat the procedure described in step 2. This will yield new sets of smooth and non-smooth elements, denoted as and get some new smooth elements and non-smooth elements, denoted as $S'_2$ and $N'_2$ respectively. Then we define $S_3 = S'_2 \cup S_2$ and $N_3 = N'_2 \cup N_2$;
4. We continue the above process. Since the right cell $C^R(P(w))$ has a finite number of elements, and every element in $C^R(P(w))$ is Knuth equivalent to $w$, the process will terminate after a finite number of steps. We denote the set of all smooth elements obtained throughout this process as $S$, and the set of all non-smooth elements as $N$. Then we have $\#S + \#N = \#C^R(P(w))$.

The above algorithm is called smooth elements finding algorithm (SEF algorithm for short).

Example 6.5. Let

$$P = \begin{array}{c|c}
1 & 2 \\
3 & 5 \\
4 & \\
6 & 
\end{array}$$

then the hooklengths are: $h_{11} = 5$, $h_{12} = 2$, $h_{21} = 4$, $h_{22} = 1$, $h_{31} = 2$, $h_{11} = 1$. Thus we have $\#C^R(P(w)) = \frac{6!}{5!2!1!1!} = 9$. Let $w = (6, 4, 3, 1, 5, 2)$ be the column word of $P$. Then $w$ contains the pattern 4231 (since we can choose the subsequence $(6, 3, 5, 2)$) and is not smooth. We consider the
triples \((w_i, w_{i+1}, w_{i+2})\) and find that there are only two triples: \((3, 1, 5)\) and \((1, 5, 2)\) from which we can use the Knuth relation to get new elements. Thus we get \(w'_3 = (6, 4, 3, 5, 1, 2) = w'_2\), which is not smooth. Then from \(w'_3\), we can use the triples: \((4, 3, 5), (3, 5, 1)\) and \((5, 1, 2)\). Thus we get a new element \((w'_3)_{2} = (6, 4, 5, 3, 1, 2)\). Here \((w'_3)_{1} = (w'_3)_{1} = (6, 4, 3, 1, 5, 2) = w\) is not a new element. Then from \((w'_3)_{2}\), we can use the triples: \((4, 6, 5, 3, 1, 2)\) and \((4, 1, 5, 3, 2)\). Thus we get \((w'_3)_{2} = (6, 4, 5, 3, 1, 2)\). Here \((w'_3)_{1} = (w'_3)_{1} = (6, 4, 3, 1, 5, 2) = w\) is not a new element. Then from \((w'_3)_{2}\), we can use the triples: \((4, 6, 5, 3, 1, 2)\) and \((4, 1, 5, 3, 2)\). Thus we get \((w'_3)_{2} = (6, 4, 5, 3, 1, 2)\). We find that there is only one smooth element among these 9 elements, i.e., \((4, 1, 6, 5, 3, 2)\). We draw the process as follows:

\[
\begin{align*}
(6,4,3,1,5,2) & \\
\downarrow & \\
(6,4,3,5,1,2) & \\
\downarrow & \\
(6,4,5,3,1,2) & \\
\downarrow & \\
(4,6,5,3,1,2) & (6,4,5,1,3,2) \\
\downarrow & \\
(4,6,5,1,3,2) & (6,4,1,5,3,2) \\
\downarrow & \\
(4,6,1,5,3,2) & \\
\downarrow & \\
(4,1,6,5,3,2) &
\end{align*}
\]

If we use our program “Young”, the input is \\{6 4 3 1 5 2\}. The output will be the following:

**P tableau:**
1 2
3 5
4
6

**Q tableau:**
1 5
2 6
3
4

**Hook lengths:**
5 2
4 1
2
Number of standard Young tableaux for shape (2, 2, 1, 1): 9
Number of smooth permutations: 1
Number of non-smooth permutations: 8
Number of smooth permutations: \{(4, 1, 6, 5, 3, 2)\}

Thus we have the same result but it is much faster to get these results.

**Corollary 6.6.** Let \(L_w\) be a highest weight module of \(\mathfrak{sl}(n, \mathbb{C})\). If we can find a smooth element in the KL right cell \(C^R(P(w))\) by using our SEF algorithm, we will have \(V(L_w) = V(w)\).

**Proof.** From Proposition 2.7, we know that \(V(L_w)\) is a constant in the KL right cell \(C^R(P(w))\). If we can find a smooth element \(x\) in the KL right cell \(C^R(P(w))\) by using our SEF algorithm, we will have \(V(L_x) = V(x)\) by Proposition 2.3. Thus by Proposition 2.7, we will have \(V(L_w) = V(L_x) = V(x)\). By Proposition 2.6, \(V(x) = V(w)\) since \(x \sim w\). Therefore, we have \(V(L_w) = V(x) = V(w)\).

\(\square\)

**Acknowledgments.** The first author is supported by NSFC Grant No. 12171344.

**References**

[A98] S. Ariki, Robinson-Schensted correspondence and left cells, in: Combinatorial Methods in Representation Theory, Kyoto, 1998, in: Adv. Stud. Pure Math., vol. 28, Kinokuniya, Tokyo, 2000, pp. 1-20. 4, 5

[BXX23] Z. Q. Bai, W. Xiao and X. Xie, Gelfand–Kirillov dimensions and associated varieties of highest weight modules, Int. Math. Res. Not. IMRN 2023, no.10, 8101-8142. 13

[BX19] Z. Q. Bai and X. Xie, Gelfand–Kirillov dimensions of highest weight Harish-Chandra modules for SU(p,q), Int. Math. Res. Not. IMRN 2019, no. 17, 4392-4418. 13

[BB85] W. Borho and J.-L. Brylinski, Differential operators on homogeneous spaces. III. Characteristic varieties of Harish-Chandra modules and of primitive ideals. Invent. Math. 80 (1985), no. 1, 1-68. 2, 3, 4

[CK03] J. B. Carrell and J. Kuttler, On the smooth points of \(T\)-stable varieties in \(G/B\) and the Peterson map. Invent. Math. 151 (2003), 353-370. 2

[FRT54] J. S. Frame, G. de B. Robinson and R. M. Thrall, The hook graphs of the symmetric groups, Canad. J. Math. 6 (1954), 316-324. 15

[FM13] L. Fresse and A. Melnikov, Smooth orbital varieties and orbital varieties with a dense \(B\)-orbit. Int. Math. Res. Not. IMRN 2013, no. 5, 1122-1203. 2

[J84] A. Joseph, On the variety of a highest weight module. J. Algebra 88 (1984), no. 1, 238-278. 2, 3, 4

[KL79] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras. Invent. Math. 53 (1979), no. 2, 165-184. 1, 2, 5

[K70] D. E. Knuth, Permutations, matrices, and generalized Young tableaux. Pacific J. Math. 34 (1970), 709-727. 5

[LS90] V. Lakshmibai and B. Sandhya, Criterion for smoothness of Schubert varieties in \(SL(n)/B\). Proc. Indian Acad. Sci. Math. Sci. 100 (1990), 45-52. 2, 4

[M93] A. Melnikov, Irreducibility of the associated varieties of simple highest weight modules in \(\mathfrak{sl}(n)\), C. R. Acad. Sci. Paris Sér. I Math. 316 (1993), no.1, 53-57.
[M04a] A. Melnikov, On orbital variety closures in \( \mathfrak{sl}_n \). I. Induced Duflo order. J. Algebra 271 (2004), no. 1, 179-233.

[M04b] A. Melnikov, On orbital variety closures in \( \mathfrak{sl}_n \). II. Descendants of a Richardson orbital variety. J. Algebra 271 (2004), no. 2, 698–724.

[M06] A. Melnikov, On orbital variety closures in \( \mathfrak{sl}_n \). III. Geometric properties. J. Algebra 305 (2006), no. 1, 68–97.

[NOT01] K. Nishiyama, H. Ochiai, and K. Taniguchi, Bernstein degree and associated cycles of Harish-Chandra modules - Hermitian case, Nilpotent Orbits, Associated Cycles and Whittaker Models for Highest Weight Representations (K. Nishiyama, H. Ochiai, K. Taniguchi, H. Yamashita, and S. Kato, eds.), Astérisque, vol. 273, Soc. Math. France, 2001, pp. 13-80.

[S01] B.E. Sagan, The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions, second edition, Graduate Texts in Mathematics, vol. 203, Springer-Verlag, New York, 2001.

[S82] N. Spaltenstein, Classes unipotentes et sous-groupes de Borel, Lecture Notes in Mathematics 946, Springer, 1982.

[W15] G. Williamson, A reducible characteristic variety in type A. Representations of reductive groups, 517-532, Progr. Math. 312, Birkhäuser/Springer, Cham, 2015.

(Bai) School of Mathematical Sciences, Soochow University, Suzhou 215006, P. R. China

Email address: zqbai@suda.edu.cn

(Chen) School of Mathematical Sciences, Shanghai Jiaotong University, Shanghai 200240, P. R. China

Email address: zhengan_chen@sjtu.edu.cn