HIGHER LEVEL AFFINE CRYSTALS AND YOUNG WALLS

SEOK-JIN KANG* AND HYEONMI LEE**

Abstract. Using combinatorics of Young walls, we give a new realization of arbitrary level irreducible highest weight crystals $B(\lambda)$ for quantum affine algebras of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$. The irreducible highest weight crystals are realized as the affine crystals consisting of reduced proper Young walls. The notion of slices and splitting of blocks plays a crucial role in the construction of crystals.

1. Introduction

The crystal bases, introduced by Kashiwara in [11], have many nice combinatorial features reflecting the internal structure of integrable modules of quantum groups. Moreover, it is known that the crystal bases are preserved under the direct sum decomposition and have extremely simple behavior with respect to taking the tensor product. Hence it is a very natural and important problem to find explicit realizations of crystal bases for irreducible highest weight modules over quantum groups. (See [2], for example, and the references there in.)

In [6], Kang introduced the notion of Young walls as a new combinatorial scheme for realizing crystal bases for quantum affine algebras. The Young walls consists of colored blocks with various shapes, and can be viewed as generalizations of colored Young diagrams.

For classical quantum affine algebras of type $A_n^{(1)}$, $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$, the rules and patterns for building Young walls and the action of Kashiwara operators are given explicitly in terms of combinatorics of Young walls, which defines an affine crystal structure on the set of proper Young walls. In particular, the level-1 irreducible highest weight crystals are realized as the affine crystals consisting of reduced proper Young walls.

However, in [6], the problem of Young wall realization of crystal bases were left open for quantum affine algebras of type $C_n^{(1)}$ $(n \geq 2)$, because this case is more difficult to deal with than the other classical quantum affine algebras. The main difficulty lies in the fact that the level-1 perfect crystal for this case are intrinsically of level-2.

This difficulty was resolved in [3] by introducing the notion of slices and splitting blocks, which plays a crucial role in constructing the desired realization of crystal bases for quantum affine algebras of type $C_n^{(1)}$: the level-1 irreducible highest weights crystals are realized as the affine crystals consisting of reduced proper Young walls.

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In this paper, we develop the combinatorics of higher level Young walls for classical quantum affine algebras of type \( A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, A^{(2)}_{2n-1}, A^{(2)}_{2n}, \) and \( D^{(2)}_{n+1} \). As in [3], we first introduce the notion of slices and splitting blocks, and give a new realization of higher level perfect crystals as the equivalence classes of slices. We then proceed to define the notion of higher level Young walls, proper Young walls, reduced proper Young walls, and ground-state walls, etc. Finally, we prove that the arbitrary level irreducible highest weight crystals are realized as the affine crystals consisting of reduced proper Young walls.

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## 2. Quantum affine algebras and perfect crystals

In this section, we fix the notations for quantum affine algebras and review some of the basic properties of perfect crystals. Let \((A, P^\vee, \Pi^\vee, P, \Pi)\) be an affine Cartan datum, where

- \(I = \{0, 1, \ldots, n\}\) is the index set for the simple roots,
- \(A = (a_{ij})_{i,j \in I}\) is an affine generalized Cartan matrix of type \( A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, A^{(2)}_{2n-1}, A^{(2)}_{2n}, D^{(2)}_{n+1}\),
- \(P^\vee = \left( \bigoplus_{i \in I} \mathbb{Z} h_i \right) \oplus \mathbb{Z} d\) is the dual weight lattice,
- \(\Pi^\vee = \{ h_i | i \in I\}\) is the set of simple coroots,
- \(\mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} P^\vee\) is the Cartan subalgebra,
- \(P = \{ \lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subset \mathbb{Z} \}\) is the weight lattice,
- \(\Pi = \{ \alpha_i | i \in I\}\) is the set of simple roots.

We denote the null root by \(\delta\) and the fundamental weights by \(\Lambda_i (i = 0, 1, \ldots, n)\), so that we have

\[
P = \left( \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \right) \oplus \frac{1}{d_0} \mathbb{Z} \delta,
\]

Here, \(d_0\) is the coefficient of \(\alpha_0\) in the null root \(\delta = d_0 \alpha_0 + \cdots + d_n \alpha_n\). We also denote by \(P^+ = \{ \lambda \in P | \lambda(h_i) \in \mathbb{Z}_{\ge 0} \text{ for all } i \in I \}\) the set of affine dominant integral weights.

Let \(U_q(\mathfrak{g})\) be the quantum affine algebra associated with the affine Cartan datum \((A, P^\vee, \Pi^\vee, P, \Pi)\), and let \(e_i, f_i, K_i^{\pm 1}, q^d (i \in I)\) be the generators of \(U_q(\mathfrak{g})\). Let \(U'_q(\mathfrak{g})\) be the subalgebra of \(U_q(\mathfrak{g})\) generated by \(e_i, f_i, K_i^{\pm 1} (i \in I)\). Then \(U'_q(\mathfrak{g})\) can be regarded as the quantum group associated with the classical Cartan datum \((A, P^\vee, \Pi^\vee, P, \Pi)\), where \(P^\vee = \bigoplus_{i \in I} \mathbb{Z} h_i\) is the classical dual weight lattice and \(\mathcal{P} = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i\) is the classical weight lattice.

**Definition 2.1.** An affine crystal (resp. classical crystal) is a set \(B\) together with the maps \(w_t : B \to P\) (resp. \(w_t : B \to \mathcal{P}\)), \(\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}\), \(\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}\) satisfying the following conditions: for all \(i \in I\) and \(b \in B\),

1. \(\varphi_i(b) = \varepsilon_i(b) + (h_i, \text{wt}(b))\);
2. \(\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i\) if \(\tilde{e}_i b \in B\);
3. \(\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i\) if \(\tilde{f}_i b \in B\);
(4) if $\tilde{e}_ib \in B$, then
$$\varepsilon_i(\tilde{e}_ib) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_ib) = \varphi_i(b) + 1;$$

(5) if $\tilde{f}_ib \in B$, then
$$\varepsilon_i(\tilde{f}_ib) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_ib) = \varphi_i(b) - 1;$$

(6) $\tilde{f}_ib = b'$ if and only if $b = \tilde{e}_ib'$ for all $i \in I$ and $b, b' \in B$;

(7) if $\varepsilon_i(b) = -\infty$, then $\tilde{e}_ib = \tilde{f}_ib = 0$.

For example, for an affine dominant integral weight $\lambda \in P^+$, the crystal graph $B(\lambda)$ of the irreducible highest weight module $V(\lambda)$ is an affine crystal, which will be called the *irreducible highest weight crystal*. An affine crystal (resp. classical crystal) will also be called a $U_q(g)$-crystal (resp. $U'_q(g)$-crystal). We will often denote by $\text{wt}$ the weight function of a classical crystal.

**Definition 2.2.** Let $B_1$ and $B_2$ be (affine or classical) crystals. A **crystal morphism** $\psi: B_1 \rightarrow B_2$ is a map $\psi: B \cup \{0\} \rightarrow B \cup \{0\}$ satisfying the following conditions:

1. $\psi(0) = 0$;
2. if $b \in B_1$ and $\psi(b) \in B_2$, then $\text{wt}(\psi(b)) = \text{wt}(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b)$;
3. if $b, b' \in B_1, \psi(b), \psi(b') \in B_2$ and $\tilde{f}_ib = b'$, then $\tilde{f}_i\psi(b) = \psi(b'), \quad \psi(b) = \tilde{e}_i\psi(b').$

Let $B$ be a classical crystal. For $b \in B$, we define
$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i, \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i.$$

**Definition 2.3.** For each positive integer $l > 0$, a finite classical crystal $B$ is called a **perfect crystal** of level-$l$ if

1. there is a finite dimensional $U'_q(g)$-module with a crystal basis whose crystal graph is isomorphic to $B$,
2. $B \otimes B$ is connected,
3. there exists some $\lambda_0 \in \bar{P}$ such that $\text{wt}(B) \subset \lambda_0 + \frac{1}{d_0} \sum_{i \neq 0} \mathbb{Z}_{\leq 0}\alpha_i, \quad \#(B_{\lambda_0}) = 1,$

4. for any $b \in B$, we have $\langle c, \varepsilon(b) \rangle \geq l$,
5. for each $\lambda \in \bar{P}^+$ with $\langle c, \lambda \rangle = l$, there exist unique vectors $b^\lambda \in B$ and $b_{\lambda} \in B$ such that $\varepsilon(b^\lambda) = \lambda, \quad \varphi(b_{\lambda}) = \lambda$.

Here, $d_0$ is the coefficient of $\alpha_0$ in the null root $\delta$. We recall the following fundamental crystal isomorphism theorem proved in [8].

**Proposition 2.4.** [8] Let $B$ be a perfect crystal of level-$l > 0$. Then for any dominant integral weight $\lambda \in \bar{P}^+$ of level-$l$, there exists a crystal isomorphism
$$\Psi: B(\lambda) \rightarrow B(\varepsilon(b_{\lambda})) \otimes B \quad \text{given by} \quad u_{\lambda} \mapsto u_{\varepsilon(b_{\lambda})} \otimes b_{\lambda},$$

where $b_{\lambda}$ is the unique element in $B$ such that $\varphi(b_{\lambda}) = \lambda$ and $u_{\lambda}$ (resp. $u_{\varepsilon(b_{\lambda})}$) is the highest weight vector of $B(\lambda)$ (resp. $B(\varepsilon(b_{\lambda}))$).
For \( k \geq 0 \), set

\[
\lambda_0 = \lambda, \quad \lambda_{k+1} = \varepsilon(b_{\lambda_k}), \quad \text{and} \quad b_0 = b_\lambda, \quad b_{k+1} = b_{\lambda_{k+1}}.
\]

By taking the composition of crystal isomorphism given in Proposition 2.4, we get a crystal isomorphism

\[
\Psi_{k} : B(\lambda) \overset{\sim}{\longrightarrow} B(\lambda_k) \otimes B^\otimes k
\]

given by

\[
u_{\lambda} \longmapsto u_{\lambda_k} \otimes b_{k-1} \otimes \cdots \otimes b_1 \otimes b_0.
\]

The sequence

\[
p_{\lambda} = (b_k)_{k=0}^{\infty} = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0
\]

is called the **ground-state path** of weight \( \lambda \). A **\( \lambda \)-path** in \( B \) is a sequence

\[
p = (p(k))_{k=0}^{\infty} = \cdots \otimes p(k+1) \otimes p(k) \otimes \cdots \otimes p(1) \otimes p(0)
\]

in \( B \) such that \( p(k) = b_k \) for all \( k \geq 0 \). Let \( \mathcal{P}(\lambda) \) denote the set of all \( \lambda \)-paths. Then we can define a classical crystal structure on \( \mathcal{P}(\lambda) \) by the tensor product rule, which gives the **path realization** of the irreducible highest weight crystal \( B(\lambda) \).

**Proposition 2.5.** [8] There exists an isomorphism of classical crystals

\[
\Psi : B(\lambda) \overset{\sim}{\longrightarrow} \mathcal{P}(\lambda) \quad \text{given by} \quad \nu_{\lambda} \longmapsto p_{\lambda}.
\]

For each of classical quantum affine algebras, it was shown in [7, 9] that there exists a coherent family of perfect crystals \( \{B^{(l)} | l \in \mathbb{Z}_{\geq 0}\} \). In the following, we will give an explicit description of these perfect crystals.

1. \( A_n^{(1)} (n \geq 1) \)

   \[
   B^{(l)} = \left\{(x_0, x_1, \ldots, x_n) \mid x_i \in \mathbb{Z}_{\geq 0}, \sum_{i=0}^{n} x_i = l\right\}.
   \]

2. \( B_n^{(1)} (n \geq 3) \)

   \[
   B^{(l)} = \left\{ (x_1, \ldots, x_n | x_0 | x_1, \ldots, x_1) \mid x_0 = 0 \text{ or } 1, \, x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \, x_0 + \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \right\}.
   \]

3. \( C_n^{(1)} (n \geq 2) \)

   \[
   B^{(l)} = \left\{ (x_1, \ldots, x_n | x_1, \ldots, x_1) \mid x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \, 2l \geq \sum_{i=1}^{n} (x_i + \bar{x}_i) \in 2\mathbb{Z}\right\}.
   \]

4. \( A_{2n-1}^{(2)} (n \geq 3) \)

   \[
   B^{(l)} = \left\{ (x_1, \ldots, x_n | x_1, \ldots, x_1) \mid x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \, \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \right\}.
   \]

5. \( A_{2n}^{(2)} (n \geq 1) \)

   \[
   B^{(l)} = \left\{ (x_1, \ldots, x_n | x_1, \ldots, x_1) \mid x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \, \sum_{i=1}^{n} (x_i + \bar{x}_i) \leq l \right\}.
   \]

6. \( D_{n+1}^{(2)} (n \geq 2) \)

   \[
   B^{(l)} = \left\{ (x_1, \ldots, x_n | x_0 | x_1, \ldots, x_1) \mid x_0 = 0 \text{ or } 1, \, x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \, x_0 + \sum_{i=1}^{n} (x_i + \bar{x}_i) \leq l \right\}.
   \]
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For the reader's convenience, we also give an explicit description of the maps \( \tilde{e}_i, \tilde{f}_i, \varepsilon_i \) and \( \varphi_i \) \( (i \in I) \) for this coherent family of perfect crystals over the quantum affine algebras of type \( B_n^{(1)} \). For the other quantum affine algebras, see [7, 13].

Let \( b = (x_1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_1) \in \mathcal{B}^{(1)} \) so that \( x_0 = 0 \) or \( x_0 = 1 \), \( x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0} \), and \( x_0 + \sum x_i + \sum \bar{x}_i = l \).

For \( i = 0 \) and \( i = n \), the Kashiwara operators are given by

\[
\begin{align*}
\tilde{e}_0 b &= \begin{cases} 
(x_1, x_2 - 1, x_3, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_2, \bar{x}_1 + 1) & \text{if } x_2 > \bar{x}_2, \\
(x_1 - 1, x_2, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_3, \bar{x}_2 + 1, \bar{x}_1) & \text{if } x_2 \leq \bar{x}_2,
\end{cases} \\
\tilde{f}_0 b &= \begin{cases} 
(x_1, x_2 + 1, x_3, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_2, \bar{x}_1 - 1) & \text{if } x_2 \geq \bar{x}_2, \\
(x_1 + 1, x_2, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_3, \bar{x}_2 - 1, \bar{x}_1) & \text{if } x_2 < \bar{x}_2,
\end{cases} \\
\tilde{e}_n b &= \begin{cases} 
(x_1, \ldots, x_{n-1}, x_0 + 1 | x_0 - 1 | \bar{x}_n, \ldots, \bar{x}_1) & \text{if } x_0 = 1, \\
(x_1, \ldots, x_n | x_0 + 1 | \bar{x}_n - 1, \bar{x}_{n-1}, \ldots, \bar{x}_1) & \text{if } x_0 = 0,
\end{cases} \\
\tilde{f}_n b &= \begin{cases} 
(x_1, \ldots, x_{n-1}, x_n - 1 | x_0 + 1 | \bar{x}_n, \ldots, \bar{x}_1) & \text{if } x_0 = 0, \\
(x_1, \ldots, x_n | x_0 - 1 | \bar{x}_n + 1, \bar{x}_{n-1}, \ldots, \bar{x}_1) & \text{if } x_0 = 1.
\end{cases}
\]

For \( i = 1, \ldots, n - 1 \), we have

\[
\begin{align*}
\tilde{e}_i b &= \begin{cases} 
(x_1, \ldots, x_i + 1, x_{i+1} - 1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_1) & \text{if } x_{i+1} > \bar{x}_{i+1}, \\
(x_1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_{i+1} + 1, \bar{x}_i - 1, \ldots, \bar{x}_1) & \text{if } x_{i+1} \leq \bar{x}_{i+1},
\end{cases} \\
\tilde{f}_i b &= \begin{cases} 
(x_1, \ldots, x_i - 1, x_{i+1} + 1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_1) & \text{if } x_{i+1} \geq \bar{x}_{i+1}, \\
(x_1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_{i+1} - 1, \bar{x}_i + 1, \ldots, \bar{x}_1) & \text{if } x_{i+1} < \bar{x}_{i+1}.
\end{cases}
\]

The remaining maps are given below.

\[
\begin{align*}
\varphi_0 (b) &= \bar{x}_1 + (\bar{x}_2 - x_2)_+, \\
\varphi_i (b) &= x_i + (\bar{x}_{i+1} - x_{i+1})_+ \quad (i = 1, \ldots, n - 1), \\
\varphi_n (b) &= 2x_n + x_0, \\
\varepsilon_0 (b) &= x_1 + (x_2 - \bar{x}_2)_+, \\
\varepsilon_i (b) &= \bar{x}_i + (x_{i+1} - \bar{x}_{i+1})_+ \quad (i = 1, \ldots, n - 1), \\
\varepsilon_n (b) &= 2\bar{x}_n + x_0,
\end{align*}
\]

\[
\text{wt}(b) = \sum_{i=0}^{n} (\varphi_i(b) - \varepsilon_i(b)) \Lambda_i,
\]

where we use the notation \( (x)_+ = \max(0, x) \).

3. Slices and splitting of blocks

In this section, we introduce the notion of slices and splitting blocks, and define a classical crystal structure on the set \( C^{(l)} \) of equivalence classes of slices. In the next section, we will show that the classical crystal \( C^{(l)} \) is isomorphic to the level-\( l \) perfect crystal \( B^{(l)} \).
To build a slice, we use the following colored blocks of three different types.

- \( \begin{array}{c} \text{i} \\ \text{i} \\ \text{i} \end{array} \): half-unit height, unit width, unit depth.
- \( \begin{array}{c} \text{i} \\ \text{i} \\ \text{i} \end{array} \): unit height, unit width, half-unit depth.
- \( \begin{array}{c} \text{i} \\ \text{i} \\ \text{i} \end{array} \): unit height, unit width, unit depth.

The coloring of a block will be given differently according to the types of blocks and the types of quantum affine algebras. For simplicity, we will use the following notations:

- \( \begin{array}{c} \text{i} \\ \text{i} \end{array} \) \( \leftrightarrow \) \( \begin{array}{c} \text{j} \\ \text{j} \end{array} \)
- \( \begin{array}{c} \text{i} \\ \text{i} \end{array} \) \( \leftrightarrow \) \( \begin{array}{c} \text{j} \\ \text{j} \end{array} \)
- \( \begin{array}{c} \text{i} \\ \text{i} \end{array} \) \( \leftrightarrow \) \( \begin{array}{c} \text{j} \\ \text{j} \end{array} \)

The thin rectangle at the top of each notation is a reminder that the block is stacked in a wall of unit depth. The dark shading shows that it is of full unit depth. Unshaded ones show that the unit depth has not been filled completely.

The following example is for a set of blocks stacked in a wall of unit thickness.

**Definition 3.1.** If \( g \neq C_n^{(1)} \), we define a level-1 slice to be a set of finitely many blocks of given type stacked in one column of unit depth following the pattern given below. In stacking the blocks, no block should be placed on top of a column of half-unit depth. If \( g = C_n^{(1)} \), such a set of blocks will be called a level-\( \frac{1}{2} \) slice.

(1) \( A_n^{(1)} (n \geq 1) \)

\( \begin{array}{c} \text{i} \\ \text{i} \end{array} \) \( (i = 0, \cdots, n) \)
(2) $B_n^{(1)}(n \geq 3)$ and $A_{2n-1}^{(2)}(n \geq 3)$

(3) $C_n^{(1)}(n \geq 2)$ and $A_{2n}^{(2)}(n \geq 1)$
As we can see in the figure, the blocks are stacked in a repeating pattern, and, roughly speaking, this pattern is symmetric with respect to the \( n \)-block except for the \( A_n^{(1)} \) type. We say that an \( i \)-block is a covering \( i \)-block (resp. supporting \( i \)-block) if it is closer to the \( n \)-block that sits below (resp. above) it than to the \( n \)-block that sits above (resp. below) it. An \( i \)-block that appears only once in each cycle is regarded as both a supporting block and a covering block. An \( i \)-slot is the top of a level-1 (or level-1/2) slice where one may add an \( i \)-block. The notion of covering \( i \)-slot or supporting \( i \)-slot is self-explanatory.

We define a \( \delta \)-column to be a set of blocks appearing in a cycle of the stacking pattern. For a level-1 (or level-1/2) slice \( C \), we define \( C + \delta \) (resp. \( C - \delta \)) to be the level-1 (or level-1/2) slice obtained from \( C \) by adding (resp. removing) a \( \delta \)-column.

**Example 3.2.** If \( g = B_3^{(1)} \), we have

\[
\begin{array}{cccc}
2 & & \n & \\
1 & 0 & & \\
2 & & & \\
\end{array}
\]

\[
\begin{array}{cccc}
2 & & \n & \\
1 & 0 & & \\
2 & & & \\
\end{array}
\]

**Definition 3.3.**

1. A level-\( l \) slice for \( g \neq C_n^{(1)} \) is an ordered \( l \)-tuple \( C = (c_1, \ldots, c_l) \) of level-1 slices satisfying the following conditions :
2. \( A_n^{(1)} \), \( B_n^{(1)} \), and \( A_{2n}^{(2)} \)
   - \( c_1 \subset c_2 \subset \cdots \subset c_l \subset c_1 + \delta \);
(b) $A_{2n-1}^{(2)}$
- $c_1 \subset c_2 \subset \cdots \subset c_l \subset c_1 + \delta$,
- it contains an even number of $n$-blocks;

(c) $D_{n+1}^{(2)}$
- $c_1 \subset c_2 \subset \cdots \subset c_l \subset c_1 + \delta$,
- at most one of the top blocks is a supporting $n$-block.

(2) A **level-1 slice** for $g = C_n^{(1)}$ is an ordered 2l-tuple $C = (c_1, \ldots, c_{2l})$ of level-$\frac{1}{2}$ slices such that
- $c_1 \subset c_2 \subset \cdots \subset c_{2l} \subset c_1 + \delta$,
- it contains an even number of 0-blocks.

For affine types that allow more than one stacking pattern, only one should be used in a level-l slice. Each level-1 (or level-$\frac{1}{2}$) slice $c_i$ in $C$ is called the $i$-th layer of $C$. For each affine type, the set of all level-l slices, that uses the same stacking pattern, is denoted by $S^{(l)}$.

**Remark 3.4.** According to the above definition, for affine types that allow more than one stacking pattern, more than one set is being denoted by $S^{(l)}$. But the contents of this section shall show that this will cause no confusion.

We will often just say slice for level-l slice. A level-l slice can be viewed as the set of $l$ columns with the $i$-th layer placed in front of the $(i+1)$-th layer. For simplicity, we will often use the front-and-top view when representing a slice. We will also use side views whenever it is necessary. We explain the two methods used in drawing a slice with the following example for $B_3^{(1)}$-type.

\[
\begin{align*}
(c_1 = \begin{array}{c}
\begin{array}{c}
5
\end{array}
\end{array} & \quad c_2 = \begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array} & \quad c_3 = \begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array} & \quad c_4 = \begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array} & \quad c_5 = \begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array} & \quad c_6 = \begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
4
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{align*}
\]

Next, we will explain the notion of **splitting an $i$-block** in a level-l slice. Let $C$ be a level-l slice and fix an $i$ which may be chosen as follows:
- $i = 01, 2, \ldots, n-1$ for $g = B_n^{(1)}$, $A_{2n-1}^{(2)}$.
- $i = 1, \ldots, n$ for $g = C_n^{(1)}$, $A_{2n}^{(2)}$.
- $i = 1, \ldots, n-1$ for $g = D_{n+1}^{(2)}$. 
Here, the choice $i = 01$ is not a typographical error. The 01-block is the unit cube obtained by gluing a 0-block and a 1-block together. Note that in any fixed slice, there can be at most two heights in which a covering or supporting $i$-block may appear as the top block of a layer. Similarly, there can be at most two heights in which a supporting or covering $i$-slot may appear.

**Definition 3.5.**

(1) $g = B^{(1)}_n, A^{(2)}_{2n-1}$.
Suppose that there is a layer whose top is a supporting $i$-block and another layer whose top is a covering $i$-slot. Recall that there can be at most two heights for such layers. Among these layers, we choose the supporting $i$-block lying in the fore-front layer (i.e., the one with the smallest layer index) among the ones with the higher height, and the covering $i$-slot lying in the very back layer (i.e., the one with the largest layer index) among the ones with the lower height. To **split an $i$-block** means to break off the top half of the chosen supporting $i$-block and to place it in the chosen covering $i$-slot.

(2) $g = C^{(1)}_n, A^{(2)}_n, D^{(2)}_{n+1}$.
Suppose that there is a layer whose top is a covering $i$-block and another layer whose top is a supporting $i$-slot. Among these layers, we choose the covering $i$-block lying in the fore-front layer among the ones with the higher height, and the supporting $i$-slot lying in the very back layer among the ones with the lower height. To **split an $i$-block** means to break off the top half of the chosen covering $i$-block and to place it in the chosen supporting $i$-slot.

**Remark 3.6.** There is no notion of splitting for $g = A^{(1)}_n$.
Dotted lines shall be used to denote broken blocks, as seen in the next example.

**Example 3.7.** We illustrate the notion of splitting blocks for $g = B^{(1)}_3$.

(1) splitting an 01-block

(2) splitting a 2-block, twice
Remark 3.8. For a level-$l$ slice $C$, the result obtained after splitting all possible $i$-blocks is not a level-$l$ slice. We will call it the $i$-split form of $C$. The split form of $C$ is defined to be the result obtained after splitting all possible $i$-blocks for all $i$.

We now define the action of Kashiwara operators on the set $S(l)$ of level-$l$ slices. Let $C$ be a level-$l$ slice and fix an index $i \in I$.

(1) $\mathfrak{g} = A_n^{(1)}$

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by the rules $ea) - eb)$ and $fa) - fb)$, respectively.

ea) If there is no layer in $C$ whose top is an $i$-block, we define $\tilde{e}_i C = 0$.

eb) If $C$ contains some layers whose top is an $i$-block, we remove an $i$-block from the (top of) fore-front layer among the ones with the higher height.

fa) If there is no layer whose top is an $i$-slot, we define $\tilde{f}_i C = 0$.

fb) If $C$ contains some layer whose top is an $i$-slot, then we add an $i$-block on top of the very back layer among the ones with the lower height.

(2) $\mathfrak{g} = B_n^{(1)}$

For $i \neq 0, 1, 2$, let $C'$ be the $(i - 1)$-split form of $C$. For $i = 2$, let $C'$ be the 01-split form of $C$, and for $i = 0, 1$, let $C' = C$. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by the rules $ea) - ec)$ and $fa) - fc)$, respectively.

ea) If there is no layer in $C'$ whose top is an $i$-block, we define $\tilde{e}_i C = 0$.

eb) If $C'$ contains some layer whose top is an $i$-block and all of these $i$-blocks are supporting blocks, then remove an $i$-block from the fore-front layer among the ones with the higher height.

ecc) If $C'$ contains some layer whose top is an $i$-block and some of these $i$-blocks are covering blocks, then single out the layers with the higher height among the ones containing covering $i$-blocks. We remove an $i$-block from the fore-front layer among the chosen ones.

fa) If there is no layer in $C'$ whose top is an $i$-slot, we define $\tilde{f}_i C = 0$.

fb) If $C'$ contains some layer whose top is an $i$-slot and all of these $i$-slots are covering slots, then add an $i$-block on top of the very back layer among the ones with the lower height.

fc) If $C'$ contains some layer whose top is an $i$-slot and some of these $i$-slots are supporting slots, then single out the layers with the lower height among the ones containing supporting $i$-slots. We add an $i$-block on top of the very back layer among the chosen ones.

(3) $\mathfrak{g} = A_{2n-1}^{(2)}$

For $i = 0, 1, \ldots, n - 1$, we use the rules for $\mathfrak{g} = B_n^{(1)}$. For $i = n$, let $C'$ be the $(n - 1)$-split form of $C$. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by the rules $ea) - ed)$ and $fa) - fd)$, respectively.

ea) If there is no layer in $C'$ whose top is an $n$-block, we define $\tilde{e}_i C = 0$.

eb) If $C'$ contains some layer whose top is an $n$-block and all of these $n$-blocks are supporting blocks, then the number of $n$-blocks must be even. Single out the layers with the higher height among the ones containing $n$-blocks.

- If there is only one such layer, remove an $n$-block from that layer and another $n$-block from the fore-front layer among the remaining ones.
with \( n \)-blocks.
- If there are more than one such layers, remove two \( n \)-blocks from the two front layers (i.e., one \( n \)-block from each layer) among the chosen ones.

**cc)** If \( C' \) contains some layer whose top is a covering \( n \)-block and there is only one such layer, first of all, remove an \( n \)-blocks from that layer. Then the top of the layer from which the block was removed will be a supporting \( n \)-block. So this intermediate result contains at least one supporting \( n \)-block. Single out the layers with the higher height among the ones containing supporting \( n \)-blocks. We remove an \( n \)-block from the fore-front layer among the singled out layers.

**cd)** If \( C' \) contains more than one layers whose top is a covering \( n \)-block, then single out the layers with the higher height among the ones containing covering \( n \)-blocks.
- If there is only one such layer, remove an \( n \)-block from that layer and remove another \( n \)-block from the fore-front layer among the remaining ones with covering \( n \)-blocks.
- If there are more than one such layers, remove two \( n \)-blocks from the two front layers among the chosen ones.

\( fea)\) If there is no layer in \( C' \) whose top is an \( n \)-slot, we define \( \tilde{f}_i(C) = 0. \)

\( feb)\) If \( C' \) contains some layer whose top is an \( n \)-slot and all of these \( n \)-slots are covering slots, then the number of \( n \)-slots must be even. Single out the layers with the lower height among the ones containing \( n \)-slots.
- If there is only one such layer, add an \( n \)-block on top of that layer and another \( n \)-block on top of the very back layer among the remaining ones with \( n \)-slots.
- If there are more than one such layers, add two \( n \)-blocks on top of the two back layers (i.e., one \( n \)-block on top of each layer) among the chosen ones.

\( fec)\) If \( C' \) contains some layer whose top is a supporting \( n \)-slot and there is only one such layer, first of all, add an \( n \)-block on top of that layer. Then the slot on top of the block just added will be a covering \( n \)-slot. So this intermediate result contains at least one covering \( n \)-slot. Single out the layers with the lower height among the ones containing covering \( n \)-slots. We add an \( n \)-block to the very back of the singled out layers.

\( fed)\) If \( C' \) contains more than one layers whose top is a supporting \( n \)-slot, then single out the layers with the lower height among the ones containing supporting \( n \)-slots.
- If there is only one such layer, add an \( n \)-block on top of that layer and add another \( n \)-block on top of the very back layer among the remaining ones with supporting \( n \)-slots.
- If there are more than one such layers, add two \( n \)-blocks on top of the two very back layers among the chosen ones.

\( 4) \ g = A^{(2)}_{2n} \)

For \( i \neq n \), let \( C' \) be the \((i + 1)\)-split form of \( C \). For \( i = n \), let \( C' = C \). Then we use the rules for \( g = B^{(1)}_n \) with the following substitution of words:
• covering $\mapsto$ supporting,
• supporting $\mapsto$ covering.

(5) $g = C^{(1)}_n$
For $i = 1, \ldots, n$, we will use the rules for $A^{(2)}_{2n}$. For $i = 0$, let $C'$ be the 1-split form of $C$, and use the rules for $i = n$ over $A^{(2)}_{2n-1}$ with the following substitution of the words:
• $n$-block $\mapsto$ 0-block,
• $n$-slot $\mapsto$ 0-slot,
• covering $\mapsto$ supporting,
• supporting $\mapsto$ covering.

(6) $g = D^{(2)}_{n+1}$
For $i \neq n-1, n$, let $C'$ be the $(i+1)$-split form of $C$. For $i = n-1, n$, let $C' = C$. Then we use the rules for $g = A^{(2)}_{2n}$.

Let $C = (c_1, \ldots, c_l)$ (resp. $(c_1, \ldots, c_2l)$) for $C^{(1)}_n$ be a level-$l$ slice. We define the slices $C \pm \delta$ by
\begin{align*}
C + \delta &= (c_2, \ldots, c_l, c_l + \delta) \text{ (resp. } (c_2, \ldots, c_2l, c_l + \delta) \text{ for } C^{(1)}_n), \\
C - \delta &= (c_l - \delta, c_1, \ldots, c_l - 1) \text{ (resp. } (c_2l - \delta, c_1, \ldots, c_2l - 1) \text{ for } C^{(1)}_n).
\end{align*}
We say that two slices $C$ and $C'$ are related, denoted by $C \sim C'$, if one of the two slices may be obtained from the other by adding finitely many $\delta$'s. Let
\[ C^{(l)} = \mathcal{S}^{(l)} / \sim \]
be the set of equivalence classes of level-$l$ slices. We will use the same symbol $C$ for the equivalence class containing the level-$l$ slice $C$. By abuse of terminology, the equivalence class containing a slice $C$ will be often referred to as the slice $C$. Note that the map $C \mapsto C + \delta$ commutes with the action of Kashiwara operators. Hence they are well-defined on $C^{(l)}$. We define
\begin{align*}
\varepsilon_i(C) &= \max\{n \mid \tilde{e}_n^i C \in C^{(l)}\}, \\
\varphi_i(C) &= \max\{n \mid \tilde{f}_n^i C \in C^{(l)}\}, \\
\wt(C) &= \sum_i (\varphi_i(C) - \varepsilon_i(C)) \Lambda_i.
\end{align*}
Then it is lengthy but straightforward to prove the following proposition.

**Proposition 3.9.** The Kashiwara operators, together with the maps $\varepsilon_i, \varphi_i$ ($i \in I$), $\wt$, define a $U'_q(\mathfrak{g})$-crystal structure on the set $C^{(l)}$.

### 4. NEW REALIZATION OF PERFECT CRYSTALS

In this section, we will show that the $U'_q(\mathfrak{g})$-crystal $C^{(l)}$ gives a new realization of the level-$l$ perfect crystal $B^{(l)}$ described in Section 2. We first define a canonical map $\psi : B^{(l)} \to C^{(l)}$ as follows.
Recall that every element $b \in B^{(l)}$ has the form $b = (x_0, x_1, \ldots, x_n)$ with $x_i \in \mathbb{Z}_{\geq 0}$, $\sum x_i = l$. For ease of writing, we shall temporarily use the following notation:

$$u_i = \begin{bmatrix} i \end{bmatrix} \quad \text{for each } i \in I.$$  

The image of the above $b$ under the map $\psi$ is defined to the equivalence class of a level-$l$ slice obtained by pasting together $x_i$-many $u_i$’s for each $i$. Note that this equivalence class does not depend on the way we have pasted the $u_i$’s together, as long as the pasted result forms a slice.

We will use the notation $\psi(b) = [x_0, x_1, \ldots, x_n]$, as needed.

Let $b = (x_1, \ldots, x_n|x_0, \bar{x}_n, \ldots, \bar{x}_1) \in B^{(l)}$ over $A^{(2)}_{2n}$ (resp. $C^{(1)}_n$) with $x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}$, $\sum (x_i + \bar{x}_i) = k \leq l$ (resp. $\sum (x_i + \bar{x}_i) = 2k \leq 2l$). We shall temporarily use the following notation:

$$w_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_i = \begin{bmatrix} i-1 \\ i-2 \end{bmatrix} \quad \text{for } i = 2, \ldots, n,$$

$$v_n = \begin{bmatrix} n \end{bmatrix}$$

$$v_i = \begin{bmatrix} i \\ i+1 \end{bmatrix} \quad \text{for } i = 1, \ldots, n-1.$$  

The image of $b$ under the map $\psi$ is defined to the equivalence class of a level-$l$ slice obtained by pasting together $(l-k)$ (resp. $2(l-k)$) -many $w_0$’s, $x_i$-many $u_i$’s, and $\bar{x}_i$-many $v_i$’s, for each $i$.

We will use the notation $\psi(b) = [t_0|x_1, \ldots, x_n|x_0, \bar{x}_n, \ldots, \bar{x}_1]$, as needed. Here, $t_0 = l - k$ (resp. $2(l - k)$).

Let $b = (x_1, \ldots, x_n|x_0, \bar{x}_n, \ldots, \bar{x}_1) \in B^{(l)}$ over $D^{(2)}_{n+1}$ with $x_0 = 0$ or 1, $x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}$, $x_0 + \sum (x_i + \bar{x}_i) = k \leq l$. We shall temporarily use the following notation:

$$w_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
The image of \( b \) under the map \( \psi \) is defined to the equivalence class of a level-\( l \) slice obtained by pasting together \( x_0 \)-many \( u_0 \), \( (l - k) \)-many \( w_0 \)'s, \( x_i \)-many \( u_i \)'s, and \( \bar{x}_i \)-many \( v_i \)'s, for each \( i \).

We will use the notation \( \psi(b) = [t_0 | x_1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_1] \), as needed. Here, \( t_0 = l - k \).

\[ (4) \quad g = B_n^{(1)} \]

Let \( b = (x_1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_1) \in B_n^{(l)} \) over \( B_n^{(1)} \) with \( x_0 = 0 \) or 1, \( x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0} \), \( x_0 + \sum(x_i + \bar{x}_i) = l \). And set

\[ x'_1 = (x_1 - \bar{x}_1)_+ , \]
\[ x'_2 = (x_2 - \bar{x}_2)_+ + \min\{x_1, \bar{x}_1\} , \]
\[ : \]
\[ x'_n = (x_n - \bar{x}_n)_+ + \min\{x_{n-1}, \bar{x}_{n-1}\} , \]
\[ x'_0 = x_0 + 2\min\{x_n, \bar{x}_n\}, \]
\[ \bar{x}'_n = (\bar{x}_n - x_n)_+ + \min\{x_{n-1}, \bar{x}_{n-1}\} , \]
\[ : \]
\[ \bar{x}'_2 = (\bar{x}_2 - x_2)_+ + \min\{x_1, \bar{x}_1\}, \]
\[ \bar{x}'_1 = (\bar{x}_1 - x_1)_+ . \]

Here, we used the notation \((x)_+ = \max(0, x)\). We shall temporarily use the following notation :

\[ u_1 = \begin{cases} 0 & \text{or} \quad 0 \\ \begin{array}{c} 1 \\ \vdots \\ i-1 \end{array} \end{cases} \]
\[ u_i = \begin{cases} \begin{array}{c} 1 \\ \vdots \\ i-1 \end{array} \\ i-2 \end{cases} \quad \text{for } i = 2, \ldots, n, \]
\[ u_0 = \begin{cases} \begin{array}{c} 1 \\ \vdots \\ n \end{array} \end{cases} \]
\[ v_n = \begin{cases} \begin{array}{c} 1 \\ \vdots \\ n \end{array} \end{cases} \]
\[ v_i = \begin{cases} \begin{array}{c} 1 \\ \vdots \\ i+1 \end{array} \end{cases} \quad \text{for } i = 1, \ldots, n - 1. \]
The image of $b$ under the map $\psi$ is defined to the equivalence class of a level-$l$ slice obtained by pasting together $x'_0$-many $u_0$’s, $x'_i$-many $u_i$’s, and $\bar{x}'_i$-many $v_i$’s, for each $i$.

We will use the notation $\psi(b) = [x'_{\bar{1}}, \ldots, x'_{\bar{n}}|$ $x'_{\bar{0}}$, $\ldots$, $x'_{\bar{n}}]$, as needed.

Let $b = (x_{\bar{1}}, \ldots, x_{\bar{n}}|\bar{x}_{\bar{n}}, \ldots, \bar{x}_{\bar{1}}) \in \mathcal{B}^{(l)}$ over $A_{2n-1}^{(2)}$ with $x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}$, $\sum(x_i + \bar{x}_i) = l$. And set

$$x'_1 = (x_1 - \bar{x}_1)_+, \quad x'_2 = (x_2 - \bar{x}_2)_+ + \min\{x_1, \bar{x}_1\}, \quad \ldots ,$$

$$x'_n = (x_n - \bar{x}_n)_+ + \min\{x_{n-1}, \bar{x}_{n-1}\}, \quad x'_0 = 2\min\{x_n, \bar{x}_n\}, \quad \bar{x}'_n = (\bar{x}_n - x_n)_+ + \min\{x_{n-1}, \bar{x}_{n-1}\}, \quad \ldots ,$$

$$\bar{x}'_2 = (\bar{x}_2 - x_2)_+ + \min\{x_1, \bar{x}_1\}, \quad \bar{x}'_1 = (\bar{x}_1 - x_1)_+.$$

We shall temporarily use the following notation:

$u_1 = \begin{cases} 0 \\ \# \end{cases}$ or $\begin{cases} \# \\ 0 \end{cases}$

$u_i = \begin{cases} i-1 \\ \# \end{cases}$ for $i = 2, \ldots, n$,

$w_0 = \begin{cases} n-1 \\ \# \end{cases}$

$v_n = \begin{cases} 1 \\ \# \end{cases}$

$v_i = \begin{cases} i+1 \\ \# \end{cases}$ for $i = 2, \ldots, n$,

$v_1 = \begin{cases} 1 \\ \# \end{cases}$ or $\begin{cases} \# \\ 1 \end{cases}$
The image of $b$ under the map $\psi$ is defined to the equivalence class of a level-$l$ slice obtained by pasting together $x_0'$-many $w_0$'s, $x_i'$-many $u_i$'s, and $\bar{x}_i'$-many $v_i$'s, for each $i$.

We will use the notation $\psi(b) = [x_1', \ldots, x_n'|x_0'|\bar{x}_n', \ldots, \bar{x}_1']$, as needed.

When $g = A_{(1)}^n, A_{(2)}^{(2)}, C_{(1)}^n, D_{(2)}^{(2)}$, it is easy to see that the map $\psi$ is a bijection.

In the case $g = B_{(1)}^n$, we define a new map $\phi : C^{(l)} \to B^{(l)}$ to see that the map $\psi$ is a bijection. Since any element of $C^{(l)}$ may be obtained by pasting together some number of $u_0$, $u_i$, and $v_i$, we may denote an arbitrary element of $C^{(l)}$ by $C = [y_1, \ldots, y_n|y_0|\bar{y}_n, \ldots, \bar{y}_1]$, where $y_1\bar{y}_1 = 0$, $y_0, y_i, \bar{y}_i \in \mathbb{Z}_{\geq 0}$, and $y_0 + \sum_{i=1}^{n} (y_i + \bar{y}_i) = l$. For such an element $C$, we define $\phi(C) = (x_1, \ldots, x_n|x_0|\bar{x}_n, \ldots, \bar{x}_1)$, where

\[
\begin{align*}
x_1 &= y_1 + \min\{y_2, \bar{y}_2\}, \\
x_2 &= (0, y_2 - \bar{y}_2)_+ + \min\{y_3, \bar{y}_3\}, \\
&\vdots \\
x_{n-1} &= (0, y_{n-1} - \bar{y}_{n-1})_+ + \min\{y_n, \bar{y}_n\}, \\
x_n &= (0, y_n - \bar{y}_n)_+ + \left\lfloor \frac{y_0}{2} \right\rfloor, \\
x_0 &= y_0 - 2\left\lfloor \frac{y_0}{2} \right\rfloor, \\
\bar{x}_n &= (0, \bar{y}_n - y_n)_+ + \left\lfloor \frac{y_0}{2} \right\rfloor, \\
\bar{x}_{n-1} &= (0, \bar{y}_{n-1} - y_{n-1})_+ + \min\{y_n, \bar{y}_n\}, \\
&\vdots \\
\bar{x}_2 &= (0, \bar{y}_2 - y_2)_+ + \min\{y_3, \bar{y}_3\}, \\
\bar{x}_1 &= \bar{y}_1 + \min\{y_2, \bar{y}_2\}.
\end{align*}
\]

Similarly, in the case $g = A_{2n-1}^{(2)}$, any element $C$ of $C^{(l)}$ can be denoted by $C = [y_1, \ldots, y_n|y_0|\bar{y}_n, \ldots, \bar{y}_1]$, where $y_1\bar{y}_1 = 0$, $y_0 \in 2\mathbb{Z}_{\geq 0}$, $y_i, \bar{y}_i \in \mathbb{Z}_{\geq 0}$, and $y_0 + \sum_{i=1}^{n} (y_i + \bar{y}_i) = l$. For such an element $C$, we define $\phi(C) = (x_1, \ldots, x_n|x_0|\bar{x}_n, \ldots, \bar{x}_1)$, where

\[
\begin{align*}
x_1 &= y_1 + \min\{y_2, \bar{y}_2\}, \\
x_2 &= (0, y_2 - \bar{y}_2)_+ + \min\{y_3, \bar{y}_3\}, \\
&\vdots \\
x_{n-1} &= (0, y_{n-1} - \bar{y}_{n-1})_+ + \min\{y_n, \bar{y}_n\}, \\
x_n &= (0, y_n - \bar{y}_n)_+ + \left\lfloor \frac{y_0}{2} \right\rfloor, \\
x_0 &= y_0 - 2\left\lfloor \frac{y_0}{2} \right\rfloor, \\
\bar{x}_n &= (0, \bar{y}_n - y_n)_+ + \left\lfloor \frac{y_0}{2} \right\rfloor, \\
\bar{x}_{n-1} &= (0, \bar{y}_{n-1} - y_{n-1})_+ + \min\{y_n, \bar{y}_n\}, \\
&\vdots \\
\bar{x}_2 &= (0, \bar{y}_2 - y_2)_+ + \min\{y_3, \bar{y}_3\}, \\
\bar{x}_1 &= \bar{y}_1 + \min\{y_2, \bar{y}_2\}.
\end{align*}
\]
\[ \sum_{i=1}^{n} (y_i + \bar{y}_i) = l. \]

We define \( \phi(C) = (x_1, \ldots, x_n | \bar{x}_n, \ldots, \bar{x}_1) \), where

\[
\begin{align*}
x_1 &= y_1 + \min\{y_2, \bar{y}_2\}, \\
x_2 &= (0, y_2 - \bar{y}_2)_+ + \min\{y_3, \bar{y}_3\}, \\
&\quad \vdots \\
x_{n-1} &= (0, y_{n-1} - \bar{y}_{n-1})_+ + \min\{y_n, \bar{y}_n\}, \\
x_n &= (0, y_n - \bar{y}_n)_+ + \frac{y_0}{2}, \\
\bar{x}_n &= (0, \bar{y}_n - y_n)_+ + \frac{y_0}{2}, \\
\bar{x}_{n-1} &= (0, \bar{y}_{n-1} - y_{n-1})_+ + \min\{y_n, \bar{y}_n\}, \\
&\quad \vdots \\
\bar{x}_2 &= (0, \bar{y}_2 - y_2)_+ + \min\{y_3, \bar{y}_3\}, \\
\bar{x}_1 &= \bar{y}_1 + \min\{y_2, \bar{y}_2\}.
\end{align*}
\]

It is now easy to verify that the map \( \phi \) is well-defined and that it is actually the inverse of \( \psi \).

It is almost obvious that the maps \( \overrightarrow{wt}, \varepsilon_i, \varphi_i \) are preserved under \( \psi \). It still remains to show that \( \psi \) commutes with the Kashiwara operators. However, it is a lengthy but a straightforward case-by-case verification. For instance, if \( g = B_n(1) \) and \( b = (x_1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_1) \in \mathcal{B}(l) \) with \( x_1 < \bar{x}_1, x_2 \geq \bar{x}_2 \), then we have

\[ \tilde{f}_0 b = (x_1, x_2 + 1, \ldots, x_n | x_0 | \bar{x}_n, \ldots, \bar{x}_2, \bar{x}_1 - 1), \]

and \( \psi(b) = [y_1, \ldots, y_n | y_0 | \bar{y}_n, \ldots, \bar{y}_1] \), where \( y_1 = 0, y_2 = x_2 - \bar{x}_2 + x_1, \ldots, \bar{y}_2 = x_1, \)
\( \bar{y}_1 = \bar{x}_1 - x_1 \). By applying \( \tilde{f}_0 \), we get \( \tilde{f}_0 \psi(b) = [y_1 + 1, \ldots, y_n | y_0 | \bar{y}_n, \ldots, \bar{y}_1 - 1] \), which is the same as \( \psi(\tilde{f}_0 b) \) as desired.

Therefore, we obtain a new realization of level-\( l \) perfect crystals as the sets of equivalence classes of level-\( l \) slices.

**Theorem 4.1.** The map \( \psi : \mathcal{B}(l) \to \mathcal{C}(l) \) defined above is an isomorphism of \( U'_q(g) \)-crystals.

**Example 4.2.** The following is a drawing of a portion of the level-3 perfect crystal for type \( B_3^{(1)} \). The elements have been represented by elements of \( \mathcal{C}(l) \).
5. Combinatorics of higher level Young walls

In this section, we introduce the notion of higher level Young walls. Roughly speaking, the level-$l$ Young walls are constructed by lining up level-$l$ slices defined in the previous section, and can be viewed as the $l$-tuples of level-1 Young walls introduced in [6] and [3]. The patterns for building Young walls are given below.

(1) $A_n^{(1)} (n \geq 1)$
(2) $B_n^{(1)}(n \geq 3)$ and $A_{2n-1}^{(2)}(n \geq 3)$

| 2 | 2 | 2 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 2 | 2 | 2 | 2 |
| : | : | : | : |
| n-1 | n-1 | n-1 | n-1 |
| n | n | n | n |
| n | n | n | n |
| n-1 | n-1 | n-1 | n-1 |
| : | : | : | : |
| 2 | 2 | 2 | 2 |
| 0 | 0 | 0 | 0 |

(3) $C_n^{(1)}(n \geq 2)$ and $A_{2n}^{(2)}(n \geq 1)$

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |
| : | : | : | : |
| n | n | n | n |
| : | : | : | : |
| 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

(4) $D_{n+1}^{(2)}(n \geq 2)$
Definition 5.1. A level-1 Young wall of type $A_{n}^{(1)}$, $B_{n}^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$ (or a level-$\frac{1}{2}$ Young wall of type $C_{n}^{(1)}$) is a set of blocks stacked in a wall of unit depth satisfying the following conditions:

1. The colored blocks are stacked in the pattern given above.
2. Except for the rightmost column, there is no free space to the right of any block.
3. No block can be placed on top of a column of half-unit depth.

Note that every column of a level-1 Young wall (resp. a level-$\frac{1}{2}$ Young wall for $C_{n}^{(1)}$) is a level-1 slice (resp. level-$\frac{1}{2}$ slice for $C_{n}^{(1)}$) defined in the previous section.

For a level-1 Young wall (or a level-$\frac{1}{2}$ Young wall for $C_{n}^{(1)}$) $Y$, we define $Y+\delta$ (resp. $Y-\delta$) to be the Young wall obtained by adding a $\delta$ to (resp. removing a $\delta$ from) each and every column of $Y$.

Definition 5.2.

1. A level-$l$ Young wall of type $A_{n}^{(1)}$, $B_{n}^{(1)}$, $A_{2n}^{(2)}$ is an ordered $l$-tuple of level-1 Young walls $Y = (Y_{1}, \ldots, Y_{l})$ such that
   - $Y_{1} \subset \cdots \subset Y_{l} \subset Y_{1} + \delta$.
2. A level-$l$ Young wall of type $A_{2n-1}^{(2)}$ is an ordered $l$-tuple of level-1 Young walls $Y = (Y_{1}, \ldots, Y_{l})$ such that
   - $Y_{1} \subset \cdots \subset Y_{l} \subset Y_{1} + \delta$,
   - each column contains an even number of $n$-blocks.
3. A level-$l$ Young wall of type $D_{n+1}^{(2)}$ is an ordered $l$-tuple of level-1 Young walls $Y = (Y_{1}, \ldots, Y_{l})$ such that
   - $Y_{1} \subset \cdots \subset Y_{l} \subset Y_{1} + \delta$,
   - in each column, at most one of the top blocks is a supporting $n$-block.
4. A level-$l$ Young wall of type $C_{n}^{(1)}$ is an ordered $2l$-tuple of level-$\frac{1}{2}$ Young walls $Y = (Y_{1}, \ldots, Y_{2l})$ such that
   - $Y_{1} \subset \cdots \subset Y_{2l-1} \subset Y_{2l} \subset Y_{1} + \delta$,
   - each column contains an even number of 0-blocks.

The level-1 Young wall (or level-$\frac{1}{2}$ Young wall for $C_{n}^{(1)}$) $Y_{i}$ in a level-$l$ Young wall $Y = (Y_{1}, \ldots, Y_{l})$ (or $Y = (Y_{1}, \ldots, Y_{2l})$ for $C_{n}^{(1)}$) is called the $i$-th layer of the Young wall $Y$. 

\[\begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
n-1 & n-1 & n-1 & n-1 \\
n & n & n & n \\
n-1 & n-1 & n-1 & n-1 \\
\vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}\]
We will write \( Y = (Y(k))_{k=0}^{\infty} \), where \( Y(k) \) denotes the \( k \)-th column of \( Y \) (reading from right to left). We will denote by \( Y_j(k) \) the \( k \)-th column of the \( j \)-th layer of \( Y \), which is, of course, the same as the \( j \)-th layer of the \( k \)-th column of \( Y \).

Let \( Y \) be a level-\( l \) Young wall and let \( Y' \) be the split form of \( Y \), which is obtained from \( Y \) by splitting every possible block in every column of \( Y \).

**Definition 5.3.**

1. A level-\( l \) Young wall \( Y \) is said to be **proper** if it satisfies the following conditions:
   - in the split form \( Y' \), except for the rightmost column, there is no free space to the right of any block (or broken halves of blocks);
   - for each layer in the split form \( Y' \), none of the columns which is of integer height and whose top is of unit depth have the same height.
2. A column in a level-\( l \) proper Young wall is said to **contain a removable** \( \delta \) if one may remove a \( \delta \) from that column to get a proper Young wall.
3. A level-\( l \) proper Young wall is said to be **reduced** if none of its columns contain a removable \( \delta \).

We will denote by \( Z \) and \( \mathcal{Y} \) the set of all proper Young walls and reduced proper Young walls, respectively.

Let \( Y \) be a level-\( l \) proper Young wall and let \( C \) be a column of \( Y \). For each \( i \in I \), we define \( \varepsilon_i(C) \) (resp. \( \varphi_i(C) \)) to be the largest integer \( k \geq 0 \) such that \( \varepsilon_i^k(C) \neq 0 \) (resp. \( \varphi_i^k(C) \neq 0 \)) and that \( k \) is the maximal number of times we may act \( \varepsilon_i \) (resp. \( \varphi_i \)) to the column \( C \) while the resulting wall remains a proper Young wall. We will often say that \( C \) is \( \varepsilon_i \)-times \( i \)-removable and \( \varphi_i \)-times \( i \)-admissible.

**Remark 5.4.** Here, we would like to give a warning to the readers. Viewed a level-\( l \) slice in the \( U_q^l(\mathfrak{g}) \)-crystal \( \mathcal{C}^{(0)} \), we have already assigned the nonnegative integers \( \varepsilon_i(C) \) and \( \varphi_i(C) \). However, these numbers are not necessarily the same as \( \varepsilon_i(C) \) and \( \varphi_i(C) \). Note that the number \( \varepsilon_i(C) \) (resp. \( \varphi_i(C) \)) depends on the column lying to the left (resp. right) of \( C \). Hence we have \( \varepsilon_i(C) \leq \varepsilon_i(C) \) and \( \varphi_i(C) \leq \varphi_i(C) \) for all \( i \in I \).

We now define the action of Kashiwara operators \( \varepsilon_i, \varphi_i \) \( (i \in I) \) on \( Y \) as follows.

1. For each column \( C \) of \( Y \), we write \( \varepsilon_i(C) \)-many 1’s followed by \( \varphi_i(C) \)-many 0’s under \( C \). This sequence is called the **\( i \)-signature of \( C \)**.
2. From this sequence of 1’s and 0’s, cancel out each \((0,1)\)-pair to obtain a finite sequence of 1’s followed by 0’s (reading from left to right). This sequence is called the **\( i \)-signature of \( Y \)**.
3. We define \( \varepsilon_i Y \) to be the proper Young wall obtained from \( Y \) by replacing the column \( C \) corresponding the rightmost 1 in the \( i \)-signature of \( Y \) with the column \( \varepsilon_i C \).
4. We define \( \varphi_i Y \) to be the proper Young wall obtained from \( Y \) by replacing the column \( C \) corresponding the leftmost 0 in the \( i \)-signature of \( Y \) with the column \( \varphi_i C \).
5. If there is no 1 (resp. 0) in the \( i \)-signature of \( Y \), we define \( \varepsilon_i Y = 0 \) (resp. \( \varphi_i Y = 0 \)).
We define the maps \( \overline{w_t} : \mathcal{Z} \rightarrow \bar{P} \), \( \varepsilon_i, \varphi_i : \mathcal{Z} \rightarrow \mathbb{Z} \) by

\[
\overline{w_t}(Y) = \sum_{i \in I} (\varphi_i(Y) - \varepsilon_i(Y))\Lambda_i,
\]

\( \varepsilon_i(Y) = \) the number of 1's in the \( i \)-signature of \( Y \),

\( \varphi_i(Y) = \) the number of 0's in the \( i \)-signature of \( Y \).

Then it is straightforward to verify that the following theorem holds.

**Theorem 5.5.** The set \( \mathcal{Z} \) of all level-\( l \) proper Young walls, together with the maps \( \overline{w_t}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i \) (\( i \in I \)), forms a \( U_q'(\mathfrak{g}) \)-crystal.

**6. Young wall realization of \( \mathcal{B}(\lambda) \)**

In this section, we give a new realization of higher level irreducible highest weight crystals in terms of reduced proper Young walls. Let \( \lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_n\Lambda_n \) be a dominant integral weight of level-\( l \). We define the **ground-state wall** \( Y_{\lambda} \) of weight \( \lambda \) as the following reduced proper Young walls.

(1) \( A^{(1)}_n \) \((n \geq 1)\)

(2) \( B^{(1)}_n \) \((n \geq 3)\)
The choice between the two depends on whether $a_1 \geq a_0$ or $a_1 \leq a_0$.

(3) $C_n^{(1)} (n \geq 2)$

(4) $A_{2n-1}^{(2)} (n \geq 3)$
The choice between the two depends on whether \( a_1 \geq a_0 \) or \( a_1 \leq a_0 \).

\[(5) \ A^{(2)}_{2n} \ (n \geq 1)\]

\[(6) \ D^{(2)}_{n+1} \ (n \geq 2)\]
A level-$l$ proper Young wall obtained by adding finitely many blocks to the ground-state wall $Y_{\lambda}$ is said to have been built on $Y_{\lambda}$. We denote by $Z(\lambda)$ (resp. $Y(\lambda)$) the set of all proper Young walls (resp. reduced proper Young walls) built on $Y_{\lambda}$.

For each $Y \in Z(\lambda)$, we define its affine weight to be

$$\text{wt}(Y) = \lambda - \sum_{i=0}^{n} k_i \alpha_i,$$

where $k_i$ is the number of $i$-blocks (or half the number when dealing with 0-blocks of $C_n^{(1)}$ type and $n$-blocks of $A_{2n-1}^{(2)}$ type) that have been added to $Y_{\lambda}$. Then we obtain:

**Proposition 6.1.** The set $Z(\lambda)$ of all level-$l$ proper Young walls built on $Y_{\lambda}$, together with the maps $\tilde{e}_i$, $\tilde{f}_i$, $\varepsilon_i$, $\varphi_i$ $(i \in I)$, wt, forms a $U_q(\mathfrak{g})$-crystal.

Recall that the set $\mathcal{P}(\lambda)$ of all $\lambda$-paths gives a realization of the irreducible highest weight crystal $\mathcal{B}(\lambda)$ (see Section 2). We will show that there exists a $U_q(\mathfrak{g})$-crystal isomorphism $\Phi : Y(\lambda) \rightarrow \mathcal{P}(\lambda)$.

Let $Y = (Y(k))_{k=0}^{\infty}$ be a reduced proper Young wall built on $Y_{\lambda}$. Using the $U_q'(\mathfrak{g})$-crystal isomorphism $\psi : \mathcal{B}(l) \rightarrow \mathcal{C}(l)$ given in Theorem 4.1, we get a map $\Phi : Y(\lambda) \rightarrow \mathcal{P}(\lambda)$ which is defined by

$$\Phi(Y) = (\psi^{-1}(Y(k)))_{k=0}^{\infty}.$$

Note that the ground-state wall $Y_{\lambda}$ is mapped onto the ground-state path $p_{\lambda}$.

Conversely, to each $\lambda$-path $p = (p(k))_{k=0}^{\infty} \in \mathcal{P}(\lambda)$, we can associate a wall $Y = (Y(k))_{k=0}^{\infty}$ built on $Y_{\lambda}$ such that $\psi(p(k)) = Y(k)$ for all $k \geq 0$. By adding or removing an appropriate number of $\delta$'s (from left to right), one can easily see
that there exists a unique reduced proper Young wall \( Y = (Y(k))_{k=0}^\infty \) with this property, which shows that \( \Phi : \mathcal{Y}(\lambda) \to \mathcal{P}(\lambda) \) is a bijection.

Our main result is the following realization theorem.

**Theorem 6.2.** The bijection \( \Phi : \mathcal{Y}(\lambda) \to \mathcal{P}(\lambda) \) defined by (6.1) is a \( U_q(\mathfrak{g}) \)-crystal isomorphism. Therefore, we have a \( U_q(\mathfrak{g}) \)-crystal isomorphism

\[
\mathcal{Y}(\lambda) \xrightarrow{\sim} \mathcal{P}(\lambda) \xrightarrow{\sim} \mathcal{B}(\lambda).
\]

We shall focus our efforts on showing that the set \( \mathcal{Y}(\lambda) \) is a \( U_q(\mathfrak{g}) \)-subcrystal of \( \mathcal{Z}(\lambda) \) and that the map \( \Phi \) commutes with the Kashiwara operators \( \tilde{f}_i \) and \( \tilde{e}_i \). Other parts of the proof are similar or easy.

Now, recalling the fact that the map \( \Phi \) is determined by the \( U'_q(\mathfrak{g}) \)-crystal isomorphism \( \psi : \mathcal{B}^{(i)} \xrightarrow{\sim} \mathcal{C}^{(i)} \), we find that, to prove these two points, it suffices to prove the following two lemmas.

**Lemma 6.3.** Let \( Y = (Y(k))_{k=0}^\infty \) be a reduced proper Young wall in \( \mathcal{Y}(\lambda) \) and let \( p = \Phi(Y) \) be the corresponding \( \lambda \)-path in \( \mathcal{P}(\lambda) \). Then we have

1. The Kashiwara operator \( \tilde{e}_i \) (resp. \( \tilde{f}_i \)) acts on the \( j \)-th column of \( Y \) if and only if \( \tilde{e}_i \) (resp. \( \tilde{f}_i \)) acts on the \( j \)-th component of \( p \).
2. \( \tilde{e}_i Y = 0 \) (resp. \( \tilde{f}_i Y = 0 \)) if and only if \( \tilde{e}_i p = 0 \) (resp. \( \tilde{f}_i p = 0 \)).

**Lemma 6.4.** The action of Kashiwara operators on \( \mathcal{Y}(\lambda) \) satisfies the following properties:

\[
\tilde{e}_i \mathcal{Y}(\lambda) \subset \mathcal{Y}(\lambda) \cup \{0\}, \quad \tilde{f}_i \mathcal{Y}(\lambda) \subset \mathcal{Y}(\lambda) \cup \{0\} \quad \text{for all } i \in I.
\]

**Proof.** Suppose that there exists some \( Y \in \mathcal{Y}(\lambda) \) for which \( \tilde{f}_i Y \not\in \mathcal{Y}(\lambda) \cup \{0\} \). We assume that \( \tilde{f}_i \) has acted on the \( j \)-th column of \( Y \) and set \( p = \Phi(Y) \). Then, by Lemma 6.3 the action of \( \tilde{f}_i \) on \( p \) would also have been on the \( j \)-th tensor component of \( p \).

Since \( \tilde{f}_i Y \in \mathcal{Z}(\lambda) \), we may remove finitely many \( \delta \)'s from \( \tilde{f}_i Y \) to obtain a reduced proper Young wall \( Y' \). The number of \( \delta \)'s removed is nonzero since \( \tilde{f}_i Y \not\in \mathcal{Y}(\lambda) \). Since every columns of \( \tilde{f}_i Y \) is related to the corresponding column of \( Y' \) under the previously defined equivalence relation, we have

\[
\tilde{f}_i(p) = \Phi(Y').
\]

Let us apply \( \tilde{e}_i \) to both \( Y' \) and \( \tilde{f}_i(p) \). We have \( p = \tilde{e}_i(\tilde{f}_i(p)) \) and the action of \( \tilde{e}_i \) on \( \tilde{f}_i(p) \) would have been on the \( j \)-th tensor component. By Lemma 6.3, the action of \( \tilde{e}_i \) of \( Y' \) will also be on the \( j \)-th column. Hence the proper Young wall \( \tilde{e}_i Y' \) may be obtained from the reduced proper Young wall \( Y \) by removing finitely many \( \delta \)'s. We may now remove finitely many \( \delta \)'s from \( \tilde{e}_i Y' \) to obtain a reduced proper Young wall \( Y'' \) which also corresponds to \( p \) under the map \( \Phi \).

Recall that we started out with a reduced proper Young wall \( Y \), added an \( i \)-block to the \( j \)-th column of \( Y \) to obtain \( \tilde{f}_i Y \), removed finitely many \( \delta \)'s from \( \tilde{f}_i Y \) to obtain a reduced proper Young wall \( Y' \), removed an \( i \)-block from the \( j \)-th column of \( Y' \) to obtain \( \tilde{e}_i Y' \), and finally, removed finitely many \( \delta \)'s from \( \tilde{e}_i Y' \) to obtain a reduced proper Young wall \( Y'' \) which corresponds to \( p \) under the map \( \Phi \). Therefore, we have \( \Phi(Y') = p = \Phi(Y'') \), but \( Y \not= Y'' \), which is a contradiction. Hence \( \tilde{f}_i Y \) must be reduced.

Similarly, one can show that \( \tilde{e}_i \mathcal{Y}(\lambda) \subset \mathcal{Y}(\lambda) \cup \{0\} \) for all \( i \in I \), which completes the proof of our claim. \( \square \)
The rest of this section is devoted to proving Lemma 6.3. We first fix some notations. Let us consider two consecutive columns that form a part of a Young wall, or two consecutive perfect crystal elements from a path. We shall denote by left-\( \varphi \), the number of 0's to be written under the left column or crystal element when preparing for the Kashiwara operators. Similarly, left-\( \varepsilon \) denotes the number of 1's to be written under the left item. Right-\( \varphi \) and right-\( \varepsilon \) are defined in a similar manner. In the case of Young walls, left-\( \varepsilon \) depends on the column that sits to the left of the left column and right-\( \varphi \) depends on the column that comes to the right of the right column. So it is not possible to find the values left-\( \varepsilon \) and right-\( \varphi \) from the two Young wall columns. Hence, a straightforward comparison of the signatures between a Young wall and the corresponding path after cancellations of (0,1) pairs is not possible. But still, we can verify that what is left of the left-\( \varphi \) and right-\( \varepsilon \) signatures, after the (0,1)-pair cancellation, is the same for the path and the Young wall.

Proof of Lemma 6.3. We give a proof of our claim only for the \( B_n^{(1)} \) case. Other cases may be proved in a similar manner and are less complicated.

It suffices to check that, for all possible left-right pairs of perfect crystal elements and their corresponding Young wall columns, what remains after (0,1)-cancellation of left-\( \varphi \) and right-\( \varepsilon \) signatures agrees. (The right-most column may be dealt with in a similar way.)

We shall deal with the \( i = n \) case first, and the remaining cases will be covered in the Appendices.

Let us write the general level-\( l \) Young wall column(or, slice) in the following form.

\[
\begin{array}{cccccc}
\hline
n & n & n & n & \cdots & n \\
\hline
n & n & n & n & \cdots & n \\
\hline
a_1 & a_2 & a_3 & a_4 \\
\hline
l = \sum_{i=1}^{4} a_i
\end{array}
\]

Here, \( a_1, a_2, a_3 \) denote the number of layers having top blocks of the form given in the figure, and \( a_4 \) denotes the number of all other layers; that is, any layer having a top block that comes between the supporting (\( n-2 \))-block and the covering (\( n-1 \))-block (inclusive) is counted in \( a_4 \).

We break this into two cases and fix the notations for the two cases.

- \( a_1 \geq a_3 : \text{}(\pi)_{n-1} \)
- \( a_1 \leq a_3 : \text{}(n-1)_{\pi} \)

The same notation will be used to denote any other layer-rotations of these slices. Also the perfect crystal elements corresponding to these slices will be denoted by the same notations.

If we split every block possible from these slices, the result will be of the following two shapes.

- case \( (\pi)_{n-1} \) :
Now, we will place two slices side by side and also consider the corresponding pair of perfect crystal elements. When we use the above notations \((\pi) n_{-1}\) and \((n-1)_\pi\) for the right of the two slices, we will take the number of layers to be given by \(b_i\) instead of \(a_i\). For the left slices, we will use \(a_i\) as given in the figure.

Given any two slices, we may either add or remove finitely many \(\delta\)'s to or from either of the two slices so that the two may be considered as a part of a reduced proper Young wall. We shall take the following convention. First fix the right slice and next remove finitely many \(\delta\)'s to the left slice so that the two may be considered as a part of a reduced proper Young wall. To express exactly how many times this layer rotation (or \(\delta\)-removal) has taken place, we first need to designate a starting point for the two slices.

To make things easier later on, we choose the following starting shapes and relative heights for the two columns. The shape of right slice should be so that all layers with \(n\)-blocks at the top are placed at the rear, when every block possible in it has been split. Also the starting shape of the left slice should be so that all layers with \(n\)-slots at the top are placed at the rear, when every block possible in it has been split. Below is an example showing right slice \((\pi)_{n_{-1}}\) and left slice \((n-1)_\pi\). We have given the figures with every possible block split, for the readers’ convenience.
Finally, join the two slices so that the highest layer of the result forms a part of a level-$l$ reduced proper Young wall that has had all its blocks that may be split, split. This is taken to be the starting point.

Now, to bring this into a reduced proper form, we need to remove $\delta$’s from the left slice. We denote the number of $\delta$ removals needed by $k$.

Below, we list left-$\varphi$ and right-$\varepsilon$ values for each possible Young wall column pair. The line containing the bullet lists the two column types in left-right order. Each case is again separated into two cases according to the range $k$ falls into. These numbers have been directly observed from (mental) drawings of the two columns in reduced proper form.

- **$(\pi)_{n-1}$ $(\mathbb{P})_{n-1}$**
  
  rotation : $0 \leq k \leq a_2$
  
  left-$\varphi$ : $k$
  
  right-$\varepsilon$ : $b_2 + 2(b_1 - b_3) - (a_2 - k)$

  rotation : $a_2 \leq k$
  
  left-$\varphi$ : $a_2$
  
  right-$\varepsilon$ : $b_2 + 2(b_1 - b_3)$

- **$(\pi)_{n-1}$ $(n-1)\pi$**
  
  rotation : $0 \leq k \leq a_2$
  
  left-$\varphi$ : $k$
  
  right-$\varepsilon$ : $b_2 - (a_2 - k)$

  rotation : $a_2 \leq k$
  
  left-$\varphi$ : $a_2$
  
  right-$\varepsilon$ : $b_2$

- **$(n-1)\pi$ $(\pi)_{n-1}$**
  
  rotation : $0 \leq k \leq a_2 + (a_3 - a_1)$
  
  left-$\varphi$ : $k + (a_3 - a_1)$
  
  right-$\varepsilon$ : $b_2 + 2(b_1 - b_3) - (a_2 + (a_3 - a_1) - k)$

  rotation : $a_2 + (a_3 - a_1) \leq k$
  
  left-$\varphi$ : $a_2 + 2(a_3 - a_1)$
  
  right-$\varepsilon$ : $b_2 + 2(b_1 - b_3)$

- **$(n-1)\pi$ $(n-1)\pi$**
rotation : \(0 \leq k \leq a_2 + (a_3 - a_1)\)
left-\(\varphi\) : \(k + (a_3 - a_1)\)
right-\(\varepsilon\) : \(b_2 - (a_2 + (a_3 - a_1) - k)\)

rotation : \(a_2 + (a_3 - a_1) \leq k\)
left-\(\varphi\) : \(a_2 + 2(a_3 - a_1)\)
right-\(\varepsilon\) : \(b_2\)

Similarly, the following gives the signatures of the path description. The number in the list are the left-\(\varphi\) and right-\(\varepsilon\) values for the two corresponding crystal elements.

- \((\overline{\pi})_{n-1} (\overline{\pi})_{n-1}\)
  - left-\(\varphi\) : \(a_2\)
  - right-\(\varepsilon\) : \(b_2 + 2(b_1 - b_3)\)

- \((\pi)_{n-1} (\pi-1)\pi\)
  - left-\(\varphi\) : \(a_2\)
  - right-\(\varepsilon\) : \(b_2\)

- \((\pi-1)\pi (\pi)_{n-1}\)
  - left-\(\varphi\) : \(a_2 + 2(a_3 - a_1)\)
  - right-\(\varepsilon\) : \(b_2 + 2(b_1 - b_3)\)

- \((\pi-1)\pi (\pi-1)\pi\)
  - left-\(\varphi\) : \(a_2 + 2(a_3 - a_1)\)
  - right-\(\varepsilon\) : \(b_2\)

We can easily see that the signatures agree with those of the corresponding path description in all of the cases after \((0,1)\)-pair cancellations.

We have covered the remaining part of this proof for the \(B_3^{(1)}\) case in the Appendices.

We close this paper with an example of Young wall realization of irreducible highest weight crystals.

Example 6.5.

(1) The top part of the crystal graph \(\mathcal{Y}(3\Lambda_0)\) for \(U_q(B_3^{(1)})\) is given below.
(2) The next figure shows how the Kashiwara operators act on a reduced proper Young wall in $\mathcal{Y}(3\Lambda_0)$. 
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**Appendix A. case \( i = 0, 1 \)**

We are dealing with the \( B_n^{(1)} \) case with \( i = 0 \) or 1. We will write the general level-\( l \) Young wall column(or, slice) in the following forms and fix the notations.

Here, \( a_1, b_2 \) denote the number of layers having 0-blocks at the top, \( a_2, b_1 \) denote the number of layers having 1-blocks at the top, and \( a_3, b_3 \) denote the number of layers having covering 2-blocks at the top, as given in the figure. Also \( a_4, b_4 \) denote the number of all other layers, that is, any layer having a top block that comes between the covering 3-block and the supporting 2-block (inclusive) is counted in \( a_4, b_4 \).
Now, we break each of these into two cases and fix the notations for each of the cases.

- \( a_2 \leq a_1 \) and \( a_3 \leq a_2 \) : \( L(0)_1(01)_\varnothing \)
- \( a_2 \leq a_1 \) and \( a_2 \leq a_3 \) : \( L(0)_1(2)_01 \)
- \( a_1 \leq a_2 \) and \( a_3 \leq a_1 \) : \( L(1)_0(01)_\varnothing \)
- \( a_1 \leq a_2 \) and \( a_1 \leq a_3 \) : \( L(1)_0(2)_01 \)
- \( b_2 \leq b_1 \) and \( b_3 \leq b_2 \) : \( R(1)_0(10)_\varnothing \)
- \( b_2 \leq b_1 \) and \( b_2 \leq b_3 \) : \( R(1)_0(2)_10 \)
- \( b_1 \leq b_2 \) and \( b_3 \leq b_1 \) : \( R(0)_1(10)_\varnothing \)
- \( b_1 \leq b_2 \) and \( b_1 \leq b_3 \) : \( R(0)_1(2)_10 \)

The same notation will be used to denote any other layer-rotation of these slices and the perfect crystal elements corresponding to these slices.

If we split every block possible from these slices, the resulting slices will be of the following eight shapes.

**case \( L(0)_1(01)_\varnothing \):**

```
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 2 |   |   |   |   |
```

**case \( L(0)_1(2)_01 \):**

```
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 2 | 3 | 3 |   |   |
```

**case \( L(1)_0(01)_\varnothing \):**

```
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 2 |   |   |   |   |
```

**case \( L(1)_0(2)_01 \):**

```
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
|---|---|---|---|---|---|---|---|
| 3 | 3 |   |   |   |   |   |   |
```

**case \( R(1)_0(10)_\varnothing \):**

```
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 2 |   |   |   |   |
```

**case \( R(1)_0(2)_10 \):**

```
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
|---|---|---|---|---|---|---|---|
| 3 | 3 |   |   |   |   |   |   |
```
Now, we will place two slices side by side and also consider the corresponding pair of perfect crystal elements. For the left slice, we will use L(0)_{1}(01), L(0)_{1}(10), L(1)_{0}(10), or L(1)_{0}(10), and for the right column, use R(1)_{0}(01), R(1)_{0}(10), R(0)_{1}(10), or R(0)_{1}(10), as given in the above figures.

First, we fix the right slice and next remove finitely many δ's from the left slice. We give the following starting shapes and relative heights. Shape of the right slice is to be given in the form R(0)_{1} or R(1)_{0}. And the starting shape of the left slice should be so that when every possible block is split, all bottom halves of 01-blocks appear at the rear layers. The following is an example showing right slice R(1)_{0}(10) and left slice L(0)_{1}(10).

right:

left:
Finally, join the two slices in such a way that the highest layer of the result forms a part of a level-$l$ reduced proper Young wall that has had all its blocks that may be split, split.

Now, to bring this into a reduced proper form, we need to remove $\delta$’s from the left slice. We denote the number of $\delta$ removals needed by $k$.

Below, we list left-$\varphi$ and right-$\varepsilon$ values for only four of the sixteen possible cases of Young wall column pairs. The remaining cases are less complicated. The line containing the bullet lists the two column types in left-right order.

- $L(0)_1(\overline{2})_{01} \ R(1)_0(10)_{\overline{1}}$ or $L(0)_1(\overline{2})_{01} \ R(0)_1(10)_{\overline{1}}$

  (1) $a_3 - a_2 \leq b_1 - a_1$
  
  rotation : $0 \leq k \leq a_2$
  
  left-$\varphi$ : $k + a_3 - a_2$
  
  right-$\varepsilon$ : $k + b_2 - a_2$
  
  rotation : $a_2 \leq k$
  
  left-$\varphi$ : $a_3$
  
  right-$\varepsilon$ : $b_2$

(2) $b_1 - a_1 \leq a_3 - a_2$

  rotation : $0 \leq k \leq a_1 + a_3 - b_1$
  
  left-$\varphi$ : $k + b_1 - a_1$
  
  right-$\varepsilon$ : $k + b_2 + b_1 - a_1 - a_3$
  
  rotation : $a_1 + a_3 - b_1 \leq k$
  
  left-$\varphi$ : $a_3$
  
  right-$\varepsilon$ : $b_2$

- $L(1)_0(\overline{2})_{01} \ R(1)_0(10)_{\overline{1}}$ or $L(1)_0(\overline{2})_{01} \ R(0)_1(10)_{\overline{1}}$

  (1) $b_1 \leq a_3$
  
  rotation : $0 \leq k \leq a_2 + a_3 - b_1$
  
  left-$\varphi$ : $k + b_1 - a_1$
  
  right-$\varepsilon$ : $k + b_2 + b_1 - a_2 - a_3$
  
  rotation : $a_2 + a_3 - b_1 \leq k$
  
  left-$\varphi$ : $a_2 + a_3 - a_1$
  
  right-$\varepsilon$ : $b_2$

(2) $a_3 \leq b_1$

  rotation : $0 \leq k \leq a_2$
  
  left-$\varphi$ : $k + a_3 - a_1$
  
  right-$\varepsilon$ : $k + b_2 - a_2$
  
  rotation : $a_2 \leq k$
  
  left-$\varphi$ : $a_2 + a_3 - a_1$
  
  right-$\varepsilon$ : $b_2$
Similarly, the following gives the signatures of the path description. The number in the list are the left-$\varphi$ and right-$\varepsilon$ values for the two corresponding crystal elements.

- Any pair with $L(0)_1$ in the left column
  
  \[
  \begin{align*}
  \text{left-}\varphi & : a_3 \\
  \text{right-}\varepsilon & : b_2
  \end{align*}
  \]

- Any pair with $L(1)_0$ in the left column
  
  \[
  \begin{align*}
  \text{left-}\varphi & : a_3 + (a_2 - a_1) \\
  \text{right-}\varepsilon & : b_2
  \end{align*}
  \]

We can easily see that the signatures agree with those of the corresponding path description in all of the cases after $(0,1)$-pair cancellations.

Results for $i = 1$ can be obtained if we apply the following substitutions to the parts giving $\varphi$ and $\varepsilon$ values, appearing just above.

- $L(0)_1(01)_1 \leftrightarrow L(1)_0(01)_1$
- $L(0)_1(2)_01 \leftrightarrow L(1)_0(2)_10$
- $R(1)_0(10)_2 \leftrightarrow R(0)_1(10)_2$
- $R(1)_0(2)_10 \leftrightarrow R(0)_1(2)_10$
- $a_1 \leftrightarrow a_2$
- $b_1 \leftrightarrow b_2$

**Appendix B. case $2 \leq i \leq n - 1$**

Finally, we are dealing with the $2 \leq i \leq n - 1$ case. Let us write the general level-$l$ Young wall column in the following form.

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Here, $a_1, a_2, a_4, a_5$ denote the number of layers having top blocks of the form given in the figure. Also $a_3, a_6$ denote the number of all other layers, that is, any layer having a top block that comes between the supporting $(i-2)$-block and the covering $(i-1)$-block (inclusive) is counted in $a_3$ and any layer having a top block that comes between the covering $(i + 2)$-block and the supporting $(i + 1)$-block (inclusive) is counted in $a_6$.

We break this into four cases and fix notations for the cases.

- $a_1 \geq a_5$ and $a_2 \geq a_4 : (\frac{i}{i-1})_{i-1}$
- $a_1 \geq a_5$ and $a_2 \leq a_4 : (\frac{i}{i-1})_{i-1}$
- $a_1 \leq a_5$ and $a_2 \geq a_4 : (\frac{i+1}{i})_{i-1}$
- $a_1 \leq a_5$ and $a_2 \leq a_4 : (\frac{i+1}{i})_{i-1}$
If we split every block possible from these slices, the result will be of the following four shapes.

**case** $(\overline{i})_{\overline{i-1}}(i - 1)_i$ :

**case** $(\overline{i})_{\overline{i+1}}(i)_{i-1}$ :

**case** $(i+1)_{i+1}(i-1)_i$ :

**case** $(i+1)_{i+1}(i)_{i-1}$ :
Now, we will place two slices side by side and also consider the corresponding pair of perfect crystal elements. When we use the above notations for the right of the two slices, we will take the number of layers to be given by $b_i$ instead of $a_i$. For the left slice, we will use $a_i$ as given in the figure.

We give the following starting shapes and relative height. We fix the shape of the right slice to be one of the forms $(\frac{i+1}{i+1}(i-1)_{\overline{7}}, (\frac{i+1}{i+1}(i-1)_{\overline{7}})_1, (i+1)\frac{i}{i}(i-1)_{\overline{7}}, (i+1)\frac{i}{i}(i-1)_{\overline{7}})$. The starting shape of the left slice should be so that when every possible block is split, all bottom halves of $i$-blocks (stacked in supporting place) appear at the rear layers. The following is an example showing right slice $(\frac{i+1}{i+1}(i-1)_{\overline{7}})$ and left slice $(i+1)\frac{i}{i}(i-1)_{\overline{7}}$.
Finally, join the two slices in such a way that the highest layer of the result forms a part of a level-$l$ reduced proper Young wall that has had all its blocks that may be split, split.

Now, to bring this into a reduced proper form, we need to remove $\delta$’s from the left slice. We denote the number of $\delta$ removals needed by $k$.

Below, we list left-$\varphi$ and right-$\varepsilon$ values for only the most complicated case among the sixteen possible cases of Young wall column pairs. The remaining cases are less complicated. The line containing the bullet lists the two column types in left-right order. Unlike the $i = 0, 1$ or $i = n$ case, in this type, we know that there exist two groups of layers with $i$-slots and blocks at the top, that are apart from each other. But we shall determine $k$ based only on the shape of one of the two groups (the group of layers with the supporting part at the top). So, for some $k$, left-$\varphi$ and right-$\varepsilon$ values will have several possibilities.

\begin{itemize}
  \item $(i + 1)i(i-1)\Pi \Pi_1 i-1$
  
  (1) $a_5 - a_1 \leq a_2 - a_4$, $b_4 - b_2 \leq b_1 - b_5$, $a_5 \leq (b_4 - b_2) + b_5$ or $a_5 - a_1 \leq a_2 - a_4$, $b_4 - b_2 \geq b_1 - b_5$, $a_5 \leq b_1$

  rotation : $0 \leq k \leq (a_2 - a_4) - (a_5 - a_1)$
  left-$\varphi$ : $(a_5 - a_1) + k$
  right-$\varepsilon$ : $(b_4 - b_2) + b_1 - (a_1 + (a_2 - a_4) - k)$

  rotation : $(a_2 - a_4) - (a_5 - a_1) \leq k \leq a_1 + (a_2 - a_4)$
  left-$\varphi$ : $(a_2 - a_4)$
  right-$\varepsilon$ : $(b_4 - b_2) + b_1 - a_5$

  left-$\varphi$ : $(a_2 - a_4) + 1$
  right-$\varepsilon$ : $(b_4 - b_2) + b_1 - a_5 + 1$

  left-$\varphi$ : $(a_5 - a_1) + k$
  right-$\varepsilon$ : $(b_4 - b_2) + b_1 - (a_1 + (a_2 - a_4) - k)$

  rotation : $a_1 + (a_2 - a_4) \leq k$
  left-$\varphi$ : $(a_2 - a_4)$
  right-$\varepsilon$ : $(b_4 - b_2) + b_1 - a_5$

  left-$\varphi$ : $(a_2 - a_4) + 1$
  right-$\varepsilon$ : $(b_4 - b_2) + b_1 - a_5 + 1$

  left-$\varphi$ : $a_5 + (a_2 - a_4)$
  right-$\varepsilon$ : $(b_4 - b_2) + b_1$

  (2) $a_5 - a_1 \leq a_2 - a_4$, $b_4 - b_2 \leq b_1 - b_5$, $a_5 \geq (b_4 - b_2) + b_5$
rotation : \( 0 \leq k \leq (a_2 - a_4) + a_1 - (b_4 - b_2) - b_5 \)
left-\( \phi \) : \( (a_5 - a_1) + k \)
right-\( \varepsilon \) : \( (b_4 - b_2) + b_1 - (a_1 + (a_2 - a_4) - k) \)

rotation : \( (a_2 - a_4) + a_1 - (b_4 - b_2) - b_5 \leq k \leq a_1 + (a_2 - a_4) \)
left-\( \phi \) : \( (a_2 - a_4) + a_5 - (b_4 - b_2) - b_5 \)
right-\( \varepsilon \) : \( b_1 - b_5 \)
left-\( \phi \) : \( (a_2 - a_4) + a_5 - (b_4 - b_2) - b_5 + 1 \)
right-\( \varepsilon \) : \( b_1 - b_5 + 1 \)

\[ \vdots \]
left-\( \phi \) : \( (a_5 - a_1) + k \)
right-\( \varepsilon \) : \( (b_4 - b_2) + b_1 - (a_1 + (a_2 - a_4) - k) \)

rotation : \( a_1 + (a_2 - a_4) \leq k \)
left-\( \phi \) : \( (a_2 - a_4) + a_5 - (b_4 - b_2) - b_5 \)
right-\( \varepsilon \) : \( b_1 - b_5 \)
left-\( \phi \) : \( (a_2 - a_4) + a_5 - (b_4 - b_2) - b_5 + 1 \)
right-\( \varepsilon \) : \( b_1 - b_5 + 1 \)

\[ \vdots \]
left-\( \phi \) : \( a_5 + (a_2 - a_4) \)
right-\( \varepsilon \) : \( (b_4 - b_2) + b_1 \)

(3) \( a_5 - a_1 \leq a_2 - a_4, b_4 - b_2 \geq b_1 - b_5, a_5 \geq b_1 \)

rotation : \( 0 \leq k \leq (a_2 - a_4) + a_1 - b_1 \)
left-\( \phi \) : \( (a_5 - a_1) + k \)
right-\( \varepsilon \) : \( (b_4 - b_2) + b_1 - (a_1 + (a_2 - a_4) - k) \)

rotation : \( (a_2 - a_4) + a_1 - b_1 \leq k \leq a_1 + (a_2 - a_4) \)
left-\( \phi \) : \( (a_2 - a_4) + a_5 - b_1 \)
right-\( \varepsilon \) : \( b_4 - b_2 \)
left-\( \phi \) : \( (a_2 - a_4) + a_5 - b_1 + 1 \)
right-\( \varepsilon \) : \( b_4 - a_2 + 1 \)

\[ \vdots \]
left-\( \phi \) : \( (a_5 - a_1) + k \)
right-\( \varepsilon \) : \( (b_4 - b_2) + b_1 - (a_1 + (a_2 - a_4) - k) \)

rotation : \( a_1 + (a_2 - a_4) \leq k \)
left-\( \phi \) : \( (a_2 - a_4) + a_5 - b_1 \)
right-\( \varepsilon \) : \( b_4 - b_2 \)
left-\( \phi \) : \( (a_2 - a_4) + a_5 - b_1 + 1 \)
right-\( \varepsilon \) : \( b_4 - b_2 + 1 \)

\[ \vdots \]
left-\( \phi \) : \( a_5 + (a_2 - a_4) \)
right-\( \varepsilon \) : \( (b_4 - b_2) + b_1 \)

(4) \( a_5 - a_1 \geq a_2 - a_4, b_4 - b_2 \leq b_1 - b_5, a_1 + (a_2 - a_4) \leq (b_4 - b_2) + b_5 \) or \
\( a_5 - a_1 \geq a_2 - a_4, b_4 - b_2 \geq b_1 - b_5, a_1 + (a_2 - a_4) \leq b_1 \)
rotation : $0 \leq k \leq (a_5 - a_1) - (a_2 - a_4)$
left-$\varphi$ : $(a_2 - a_4) + k$
right-$\varepsilon$ : $(b_4 - b_2) + b_1 - (a_5 - k)$

rotation : $(a_5 - a_1) - (a_2 - a_4) \leq k \leq a_5$
left-$\varphi$ : $(a_5 - a_1)$
right-$\varepsilon$ : $(b_4 - b_2) + b_1 - (a_1 + (a_2 - a_4))$
left-$\varphi$ : $(a_5 - a_1) + 1$
right-$\varepsilon$ : $(b_4 - b_2) + b_1 - (a_1 + (a_2 - a_4)) + 1$

\vdots

left-$\varphi$ : $(a_2 - a_4) + k$
right-$\varepsilon$ : $(b_4 - b_2) + b_1 - a_5 + k$

rotation : $a_5 \leq k$
left-$\varphi$ : $(a_5 - a_1)$
right-$\varepsilon$ : $(b_4 - b_2) + b_1 - (a_1 + (a_2 - a_4))$
left-$\varphi$ : $(a_5 - a_1) + 1$
right-$\varepsilon$ : $(b_4 - b_2) + b_1 - (a_1 + (a_2 - a_4)) + 1$

\vdots

left-$\varphi$ : $(a_2 - a_4) + a_5$
right-$\varepsilon$ : $(b_4 - b_2) + b_1$

(5) $a_5 - a_1 \geq a_2 - a_4$, $b_4 - b_2 \leq b_1 - b_5$, $a_1 + (a_2 - a_4) \geq (b_4 - b_2) + b_5$

rotation : $0 \leq k \leq a_5 - b_5 - (b_4 - b_2)$
left-$\varphi$ : $(a_2 - a_4) + k$
right-$\varepsilon$ : $(b_4 - b_2) + b_1 - (a_5 - k)$

rotation : $a_5 - b_5 - (b_4 - b_2) \leq k \leq a_5$
left-$\varphi$ : $(a_2 - a_4) + a_5 - b_5 - (b_4 - b_2)$
right-$\varepsilon$ : $b_1 - b_5$
left-$\varphi$ : $(a_2 - a_4) + a_5 - b_5 - (b_4 - b_2) + 1$
right-$\varepsilon$ : $b_1 - b_5 + 1$

\vdots

left-$\varphi$ : $(a_2 - a_4) + k$
right-$\varepsilon$ : $(b_4 - b_2) + b_1 - a_5 + k$

rotation : $a_5 \leq k$
left-$\varphi$ : $(a_2 - a_4) + a_5 - b_5 - (b_4 - b_2)$
right-$\varepsilon$ : $b_1 - b_5$
left-$\varphi$ : $(a_2 - a_4) + a_5 - b_5 - (b_4 - b_2) + 1$
right-$\varepsilon$ : $b_1 - b_5 + 1$

\vdots

left-$\varphi$ : $(a_2 - a_4) + a_5$
right-$\varepsilon$ : $(b_4 - b_2) + b_1$

(6) $a_5 - a_1 \geq a_2 - a_4$, $b_4 - b_2 \geq b_1 - b_5$, $a_1 + (a_2 - a_4) \geq b_1$
Similarly, the following gives the signatures of the path description. The numbers in the list are the left-\(\varphi\) and right-\(\varepsilon\) values for the two corresponding crystal elements.

- Any pair with \((\overline{i})_{i+1}(\overline{i-1})\) or \((\overline{i})_{i+1}(\overline{i-1})\overline{i}\) in both the left and right columns
  
  \[
  \text{left-}\varphi : a_5 + (a_2 - a_4) \\
  \text{right-}\varepsilon : b_1 + (b_4 - b_2)
  \]

- Any pair with \((\overline{i})_{i+1}(\overline{i})_{i-1}\) or \((\overline{i})_{i+1}(\overline{i})_{i-1}\overline{i}\) in both the left and right columns
  
  \[
  \text{left-}\varphi : a_5 \\
  \text{right-}\varepsilon : b_1 + (b_4 - b_2)
  \]

- Any pair with \((\overline{i})_{i+1}(\overline{i-1})\overline{i}\) or \((\overline{i})_{i+1}(\overline{i-1})\overline{i}\overline{i}\) in the left column and \((\overline{i})_{i+1}(\overline{i})_{i-1}\) or \((\overline{i})_{i+1}(\overline{i})_{i-1}\overline{i}\) in the right column
  
  \[
  \text{left-}\varphi : a_5 + (a_2 - a_4) \\
  \text{right-}\varepsilon : b_1 + (b_4 - b_2)
  \]

We can easily see that the signatures agree with those of the corresponding path description in all of the cases after \((0,1)\)-pair cancellations.
This completes the proof of Lemma 5.5.

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