On the maximal solution of the conjugate discrete-time algebraic Riccati equation

Hung-Yuan Fan  
Department of Mathematics, National Taiwan Normal University, Taipei 116325, Taiwan.

Chun-Yueh Chiang*  
Center for General Education, National Formosa University, Huwei 632, Taiwan.

Abstract

In this paper we consider a class of conjugate discrete-time Riccati equations, arising originally from the linear quadratic regulation problem for discrete-time antilinear systems. Under some mild assumptions and the framework of the fixed-point iteration, a constructive proof is given for the existence of the maximal solution to the conjugate discrete-time Riccati equation, in which the control weighting matrix is nonsingular and its constant term is Hermitian. Moreover, starting with a suitable initial matrix, we also show that the nonincreasing sequence generated by the fixed-point iteration converges at least linearly to the maximal solution of the Riccati equation. An example is given to demonstrate the correctness of our main theorem and provide considerable insights into the study of another meaningful solutions.

Keywords: conjugate discrete-time algebraic Riccati equation, maximal solution, fixed-point iteration, conjugate Stein matrix equation, LQR control problem, antilinear system

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1. Introduction

In this paper we are mainly concerned with the existence of the maximal solution to the conjugate discrete-time algebraic Riccati equation (CDARE)

\[ X = R(X) := A^HXA - A^HXB(R + B^HB)^{-1}B^HXA + H, \]  

or its equivalent expression

\[ X = A^H(I + G)^{-1}A + H, \]

where \( A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times m}, R \in \mathbb{H}_m \) is nonsingular, \( H \in \mathbb{H}_n, I \) is the identity matrix of compatible size and \( G := BR^{-1}B^H \), respectively. Here, \( \mathbb{H}_\ell \) denotes the set of all \( \ell \times \ell \) Hermitian matrices. We use the notation \( \overline{M} := MM^* \) throughout the paper, where \( \overline{M} \) is the matrix obtained by taking the complex conjugate of each entry of a matrix \( M \). The solution \( X \in \mathbb{H}_n \) of the CDARE (1a), with \( R_X := R + B^HXB \) being positive definite, is considered in this paper.

For any \( M, N \in \mathbb{H}_n \), the positive definite and positive semidefinite matrices are denoted by \( M > 0 \) and \( M \geq 0 \), respectively. Moreover, we usually denote by \( M \geq N \) (or \( M \leq N \)) if \( M - N \geq 0 \) (or \( N - M \geq 0 \)) in the context. For the sake of simplicity, the spectrum and spectral radius of \( A \in \mathbb{C}^{n \times n} \) are denoted by \( \sigma(A) \)

*Corresponding author  
Email addresses: hyfan@ntnu.edu.tw (Hung-Yuan Fan), chiang@nfu.edu.tw (Chun-Yueh Chiang)

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and $\rho(A)$, respectively. We say that the CDARE (1) has a maximal solution $X_M \in \mathbb{H}_n$ if $X_M \geq X$ for any solution $X \in \mathbb{H}_n$ of the CDARE. Thus, it follows from the definition of the maximality that $X_M$ is unique if it exists. A matrix operator $f : \mathbb{H}_n \to \mathbb{H}_n$ is order preserving (resp. reversing) on $\mathbb{H}_n$ if $f(A) \geq f(B)$ (resp. $f(A) \leq f(B)$) when $A \geq B$ and $A, B \in \mathbb{H}_n$.

A class of CDAREs (1) arises from the linear quadratic regulation (LQR) optimal control problem for the discrete-time antilinear system of the state space representation

$$x_{k+1} = Ax_k + Bu_k, \quad k \geq 0,$$

(2)

where $x_k \in \mathbb{C}^n$ is the state response and $u_k \in \mathbb{C}^n$ is the control input. The main goal of this control problem is to find a state feedback control $u_k = -Fx_k$ such that the performance index

$$J(u_k, x_0) := \sum_{k=0}^{\infty} (x_k^H H x_k + u_k^H R u_k)$$

is minimized with $H \geq 0$ and $R > 0$. If the antilinear system (2) is controllable, it is shown in Theorem 12.7 of [2] that the optimal state feedback controller is $u_k^* = -R_k^{-1} B^H X_k x_k$ for $k \geq 0$ and thus the minimum value of $J(u_k^*, x_0) = x_0^H X_* x_0$ is achieved, where $X_* \geq 0$ is the unique solution of the CDARE (1), which is also called the discrete-time algebraic anti-Riccati equation [3, 4].

Recently, some accelerated iterations have been proposed for solving the unique positive definite solution of the CDARE (1) under positive definiteness assumptions with $G > 0$ and $H > 0$, see, e.g., [2, 3, 5] and the references therein. In addition, this numerical technique has also been utilized to some real-life applications recently, see, e.g., [2, 4, 5].

In this work the concept of the maximal solution to discrete-time algebraic Riccati equations (DAREs) presented in Theorem 13.1.1 of [1] will be extended for the CDARE (1) with $G, H \in \mathbb{H}_n$, and hopefully, it would play an important role in the LQR control problem for discrete-time antilinear systems. The paper is organized as follows. In Section 2 we introduce some useful notations and auxiliary lemmas that will be used in our main result. In Section 3 based on the framework of the fixed-point iteration (FPI), the monotonicity property of a sequence generated by the FPI and the existence of the maximal solution to CDAREs (1) will be addressed in Theorem 3.1 under reasonable hypotheses in (3). We will give an example of the scalar CDARE (1) to illustrate the correctness of Theorem 3.1 in Section 4. Finally, we conclude the paper in Section 5.

2. Preliminaries

In this section we introduce some notations and auxiliary lemmas that will be used below. First, let the conjugate Stein matrix operator $C_A : \mathbb{H}_n \to \mathbb{H}_n$ associated with a matrix $A \in \mathbb{C}^{n \times n}$ be defined by

$$C_A(X) := X - A^H \overline{X} A,$$

(3)

for any $X \in \mathbb{H}_n$. In general, the operator $C_A$ is neither order preserving nor order reversing. However, under the assumption that $\rho(\overline{AA}) < 1$, its inverse operator $C_A^{-1}$ is order preserving, since $C_A^{-1}(X) = \sum_{k=0}^{\infty} ((\overline{AA})^k)^H (X + A^H \overline{X} A)^k \geq \sum_{k=0}^{\infty} ((\overline{AA})^k)^H (Y + A^H \overline{Y} A)^k = C_A^{-1}(Y)$ for $X, Y \in \mathbb{H}_n$, with $X \geq Y$.

It will be clear later on that the matrix operator $\mathcal{R} : \text{dom}(\mathcal{R}) \to \mathbb{H}_n$ defined by (13) plays an important role in Theorem 3.1 below, where $\text{dom}(\mathcal{R}) := \{X \in \mathbb{H}_n \mid \det(R_X) \neq 0\}$. Moreover, the following sets

$$\mathcal{R}_\leq := \{X \in \text{dom}(\mathcal{R}) \mid X \leq \mathcal{R}(X)\}, \quad \mathcal{R}_\geq := \{X \in \text{dom}(\mathcal{R}) \mid X \geq \mathcal{R}(X)\},$$

(4)

will be used throughout the paper. Let $F_X = R_X^{-1} B^H \overline{X} A$ and $T_X = A - B F_X$ for any solution $X \in \mathbb{H}_n$ of the CDARE (1a). Note that $T_X = (I + C_X)^{-1} A$ and $\mathcal{R}(X) = A^H \overline{X} T_X + H = T_X^H \overline{X} A + H$, respectively.

The following lemma characterizes a useful identity depending on the matrix operator $\mathcal{R}(\cdot)$ and its associated conjugate Stein operator.
Lemma 2.1. For any $X, Y \in \text{dom}(\mathcal{R})$, the following matrix identity
\[ X - \mathcal{R}(X) = C_{X^*}(X) - H_Y + K(Y, X) \] (5)
holds, where $K(Y, X) := (F_Y - F_X)^T R_X (F_Y - F_X)$ and $H_Y := H + K(Y, 0) = H + F_Y^T R_Y$. 

Proof. The proof is based on a direct calculation and the details are left to the reader. □

Let $T := \{ X \in \mathbb{H}_n | \rho(T_X) < 1 \}$ and $\mathbb{P} := \{ X \in \mathbb{H}_n | R_X > 0 \}$. A sufficient condition is provided in the following result to guarantee the stability property of a sequence generated by the fixed-point iteration, which will be described in the proof of Theorem 3.1 later.

Lemma 2.2. Assume that $X_\infty \in T$. If $X \in \mathbb{P}$ and $Y \geq 0$ both satisfy the matrix inequality
\[ C_{T_X}(Y) \geq K(X_\infty, X), \] (6)
then $X \in T$, i.e., $\rho(T_X) < 1$.

Proof. The inequality (6) implies
\[ S_{T_X}(Y) \geq K(X_\infty, X) + T_X^T K(X_\infty, X) T_X, \] (7)
where $S_{T_X}(Y) := Y - T_X R_Y T_X$. Assume that there exists a scalar $\lambda$ with $|\lambda| > 1$ such that $T_X x = \lambda x$ for some nonzero vector $x \in \mathbb{C}^n$. Then we see that
\[ 0 \geq (1 - |\lambda|^2)(x^T Y x) = x^T S_{T_X}(Y) x \geq x^T K(X_\infty, X) x + x^T T_X^T K(X_\infty, X) T_X x \geq 0. \]

Note that $\text{Ker}(K(X_\infty, X)) = \text{Ker}(F_X - F_X) \subseteq \text{Ker}(T_X - T_X)$ and $\text{Ker}(T_X^T K(X_\infty, X) T_X) \subseteq \text{Ker}(T_X X_X = T_X T_X x = \lambda x \in \mathbb{C}^n$, which contradicts to the assumption, and hence we conclude that $\rho(T_X) < 1$. □

Let $S_{\geq} := \bigcup_{X_\infty \in T} \{ X \in \mathbb{H}_n | C_{T_X}(X) \geq H_{X_\infty} \}$ if $T \neq \emptyset$. For any $X_\infty \in T$, notice that the operator $C_{T_X}$ is a bijection because $\rho(T_X) < 1$ and its inverse operator $C_{T_X}^{-1}$ is order preserving. Therefore, $S_{\geq}$ must be a nonempty subsets of $\mathbb{H}_n$ when $T \neq \emptyset$.

In order to deduce our main theorem presented in the next section, we assume that
\[ T \neq \emptyset \quad \text{and} \quad \mathcal{R}_\subset \cap \mathbb{P} \neq \emptyset \] (8)
in the following content of the paper. Starting with $X_0 \in S_{\geq}$, the monotonicity property of a sequence $\{X_k\}_{k=0}^{\infty}$ generated by the following FPI of the form
\[ X_{k+1} = \mathcal{R}(X_k), \quad k \geq 0, \] (9)
will be ensured by the following lemma.

Lemma 2.3. If $X_\infty \in S_{\geq}$, then the following statements hold:
(i) $X_\infty \geq X_0$ for any $X_0 \in \mathcal{R}_\subset \cap \mathbb{P}$.
(ii) $X_\infty - \mathcal{R}(X_\infty) \geq K(X_\infty, X_\infty)$ for some $X_\infty \in T$. Furthermore, we have $S_{\geq} \subseteq \mathcal{R}_\subset \cap \mathbb{P}$.

Proof. Since $X_\infty \in S_{\geq}$, there is a matrix $X_\infty \in T$ associated with $X_\infty$ such that $C_{T_X}(X_\infty) \geq H_X$.
(i) For any $X_0 \in \mathcal{R}_\subset \cap \mathbb{P}$, it follows from (5) that
\[ 0 \geq X_0 - \mathcal{R}(X_0) = C_{T_X}(X_0) - H_X + K(X_\infty, X_0). \]
Since $X_0 \in \mathbb{P}$, $K(X_\infty, X_0) \geq 0$ and hence we deduce that $H_X \geq C_{T_X}(X_0)$. Therefore,
\[ C_{T_X}(X_\infty - X_0) = C_{T_X}(X_\infty) - C_{T_X}(X_0) \geq H_X - C_{T_X}(X_0) \geq 0 \]
and then $X_\infty \geq X_0$. Furthermore, we also show that $S_{\geq} \subseteq \mathcal{R}_\subset \cap \mathbb{P}$, since $R_{X_\infty} \geq R_{X_\infty} > 0$.
(ii) From (5), note that $X_\infty - \mathcal{R}(X_\infty) = C_{T_X}(X_\infty) - H_X + K(X_\infty, X_\infty) \geq K(X_\infty, X_\infty) \geq 0$ because $X_\infty \in \mathbb{P}$, and we thus conclude that $S_{\geq} \subseteq \mathcal{R}_\subset \cap \mathbb{P}$. □
3. The maximal solution of the CDARE

In this section, the existence of the maximal solution to the CDARE \( (1) \) will be established iteratively, utilizing the framework of the FPI \( (1) \) with an appropriate initial matrix \( X_0 \in \mathbb{H}_n \), under the assumptions in \( (5) \). The main results of this paper are summarized in the following theorem.

**Theorem 3.1.** If the assumptions in \( (5) \) are fulfilled, then the maximal solution \( X_M \) of the CDARE \( (1) \) exists. Furthermore, the following statements hold:

(i) The sequence \( \{X_k\}_{k=0}^{\infty} \) generated by the FPI \( (9) \) with \( X_0 \in \mathbb{S}_+ \) is well-defined. Moreover, \( X_k \in \mathbb{S}_+ \cap \mathbb{T} \subseteq \mathbb{R}_+ \cap \mathbb{P} \cap \mathbb{T} \) for all \( k \geq 0 \).

(ii) \( X_k \geq X_{k+1} \geq X_P \) for all \( k \geq 0 \) and \( X_P \in \mathbb{R}_+ \cap \mathbb{P} \).

(iii) The sequence \( \{X_k\}_{k=0}^{\infty} \) converges at least linearly to \( X_M \), which is the maximal element of the set \( \mathbb{R}_+ \cap \mathbb{P} \) and satisfies \( \rho(T_{X_M}) \leq 1 \), with the rate of convergence

\[
\limsup_{k \to \infty} \sqrt[k]{\|X_k - X_M\|} \leq \rho(T_{X_M})
\]

whenever \( X_M \in \mathbb{T} \).

**Proof.** Let \( X_P \in \mathbb{R}_+ \cap \mathbb{P} \). The existence of \( X_M \) can be constructed by the following discussion.

(i) This result will be proven by induction. Since \( X_0 \in \mathbb{S}_+ \), it follows from Lemma 2.4 that \( X_0 \geq X_P \) and \( X_0 \in \mathbb{R}_+ \cap \mathbb{P} \). Thus, \( X_1 = \mathcal{R}(X_0) \) is well-defined with \( X_0 \geq X_1 \). Furthermore, we also have

\[
\begin{align*}
C_{T_{X_0}}(X_0 - X_P) &= C_{T_{X_0}}(X_0) - C_{T_{X_0}}(X_P) = HX_0 + X_0 - \mathcal{R}(X_0) - C_{T_{X_0}}(X_P) \\
&\geq HX_0 + K(X_0, X_0) - (H_{X_0} + X_P - \mathcal{R}(X_P)) = K(X_0, X_0),
\end{align*}
\]

for some \( X_T \in \mathbb{T} \). Then \( X_0 \in \mathbb{T} \) follows immediately from Lemma 2.2 since \( \rho(T_{X_T}) < 1 \). Assume that \( X_k \in \mathbb{S}_+ \cap \mathbb{T} \) for some positive integer \( k \). Again, it follows from Lemma 2.3 that \( X_k \geq X_P \), \( X_k \in \mathbb{R}_+ \cap \mathbb{P} \) and \( X_{k+1} = \mathcal{R}(X_k) \) is well-defined with \( X_k \geq X_{k+1} \). In addition, from Lemma 2.1 we have

\[
\begin{align*}
X_k - \mathcal{R}(X_k) &= C_{T_{X_k}}(X_k) - HX_k, \quad (10a) \\
X_{k+1} - \mathcal{R}(X_{k+1}) &= C_{T_{X_{k+1}}}(X_{k+1}) - HX_{k+1}, \quad (10b) \\
X_{k+1} - \mathcal{R}(X_{k+1}) &= \Delta_k + K(X_k, X_{k+1}), \quad (10c) \\
X_P - \mathcal{R}(X_P) &= (C_{T_{X_{k+1}}}(X_P) - HX_{k+1}) + K(X_{k+1}, X_P), \quad (10d)
\end{align*}
\]

where \( \Delta_k := C_{T_{X_k}}(X_{k+1}) - HX_k \). From \( 10a \), we see that

\[
\Delta_k = C_{T_{X_k}}(X_{k+1}) + X_k - X_{k+1} - C_{T_{X_k}}(X_k)
\]

\[
= -C_{T_{X_k}}(X_k - X_{k+1}) + X_k - X_{k+1} = T_k^H (X_k - X_{k+1}) T_k \geq 0.
\]

Thus, it implies that \( X_{k+1} \in \mathbb{S}_+ \) because \( X_k \in \mathbb{T} \), and thus \( X_{k+1} \geq X_P \) follows from Lemma 2.3. Moreover, from \( 10b \), \( 10c \) and \( 10d \), the positive semidefinite matrix \( Y := X_{k+1} - X_P \) satisfies the conjugate Stein matrix inequality

\[
C_{T_{X_{k+1}}}(Y) = C_{T_{X_{k+1}}}(X_{k+1}) - C_{T_{X_{k+1}}}(X_P)
\]

\[
= (X_{k+1} - \mathcal{R}(X_{k+1}) + H_{X_{k+1}}) - (X_P - \mathcal{R}(X_P)) = K(X_{k+1}, X_P)
\]

\[
= \Delta_k + K(X_k, X_{k+1}) - (X_P - \mathcal{R}(X_P)) - H_{X_{k+1}} + K(X_{k+1}, X_P)
\]

\[
= \Delta_k + K(X_k, X_{k+1}) - (X_P - \mathcal{R}(X_P)) + K(X_{k+1}, X_P) \geq K(X_k, X_{k+1}),
\]

and hence \( X_{k+1} \in \mathbb{T} \) follows from Lemma 2.2 and \( X_k \in \mathbb{T} \). That is, we have deduced that \( X_{k+1} \in \mathbb{S}_+ \cap \mathbb{T} \) and this completes the inductive proof.
(ii) The statement follows immediately from the proof of (i) and Lemma 2.3.

(iii) Since it follows from the statement (ii) that \( \{X_k\}_{k=0}^\infty \) is a nonincreasing sequence of Hermitian matrices and bounded below by all elements of the set \( \mathcal{R} \subseteq \mathbb{P} \), \( X_M := \lim_{k \to \infty} X_k \) must be the maximal element of \( \mathcal{R} \subseteq \mathbb{P} \) such that \( X_M = \mathcal{R}(X_M) \) and \( \rho(T_{X_M}) \leq 1 \). Moreover, the proof of the rate of convergence for \( \{X_k\}_{k=0}^\infty \) follows from the Appendix A.1 of [2].

4. An example for the CDARE

In this section we shall give an example to illustrate the correctness of Theorem 3.1 presented in the previous section.

Example 4.1. We consider a scalar CDARE \([11]\) of the form

\[
x = |a|^2 \bar{x} - \frac{|a|^2 |b|^2}{r + |b|^2 \bar{x}} + h = \frac{|a|^2 \bar{x}}{1 + g \bar{x}} + h,
\]

where \( a, b \in \mathbb{C}, r, h \in \mathbb{R} \) with \( r + |b|^2 \bar{x} > 0 \) and \( g := |b|^2 / r \) with \( r \neq 0 \). Without loss of generality, we assume that \( |a| > 0, r > 0 \) and thus \( g > 0 \). From (11) and \( 1 + g \bar{x} > 0 \), we obtain

\[
g|\bar{x}|^2 + x - (|a|^2 + gh) \bar{x} - h = 0,
\]

which has two solutions \( x_M, x_m \in \mathbb{H}_1 = \mathbb{R} \) satisfying

\[
x_M := \frac{-(1 - |a|^2 + gh) + \sqrt{D}}{2g} \quad \text{and} \quad x_m := \frac{-(1 - |a|^2 - gh) - \sqrt{D}}{2g}
\]

if the discriminant \( D := (1 - |a|^2 - gh)^2 - 4gh \geq 0 \). Note that \( D \geq 0 \) if and only if \( h \geq h_M := \frac{-(1 - |a|)^2}{g} \) or \( h \leq h_m := \frac{-(1 + |a|)^2}{g} \).

Let \( \bar{x} := \bar{T}_x \cdot \frac{|a|^2}{1 + g|x|} \) for \( x = x_M \) or \( x = x_m \). Some facts are listed without proof as follows:

1. \( T = (-\infty, \frac{|a|^2 - 1}{g} \cup \frac{|a| - 1}{g}, \infty) \neq \emptyset \) and \( \mathcal{R} \subseteq \mathbb{P} = [x_m, x_M] \cap \left[ \frac{|a| - 1}{g}, \infty \right] \neq \emptyset \) if \( h \geq h_M \). Moreover, \( 1 + gx_M \geq 1 + gx_m > 0 \) and \( \mathcal{S}_2 \subseteq [x_M, \infty) \) if \( h \geq h_m \). When \( h \leq h_m \), we have \( 1 + gx_m < 1 + gx_M < 0 \) and \( \mathcal{N} \subseteq \mathbb{P} = \emptyset \).

2. \( \bar{T}_{x_M} \bar{T}_{x_M} = 1 \). Furthermore, \( \bar{T}_{x_M} < 1 < \bar{T}_{x_m} \) if \( h > h_M \), \( \bar{T}_{x_M} = 1 = \bar{T}_{x_m} \) if \( h = h_M \) or \( h = h_m \), and \( \bar{T}_{x_M} > 1 > \bar{T}_{x_m} \) if \( h < h_m \).

3. For \( x_0 \in \mathcal{S}_2 \backslash \{x_M\} \), the equivalent expressions of the sequence \( \{x_k\}_{k=0}^\infty \) generated by the FPI \([5]\) can be rewritten in the form

\[
x_k = x_M + \frac{x_M - x_m}{s \bar{T}_{x_M} - 1} \quad \text{if} \quad 0 < \bar{T}_{x_M} < 1,
\]

\[
x_k = x_M + \frac{|a|(x_0 - x_M)}{g(x_0 - x_M) k + |a|} \quad \text{if} \quad \bar{T}_{x_M} = 1,
\]

for all \( k \geq 0 \), where \( s := \frac{\bar{x}_k - x_m}{x_M - x_m} \).

Based on these facts, the convergence behavior of the sequence presented in (12) is summarized below for different cases depending on \( h \).

(i) If \( h > h_M \) and \( 0 < \bar{T}_{x_M} < 1 \), then the assumptions in (5) hold and the sequence \( \{x_k\}_{k=0}^\infty \) in the expression (12a) converges decreasingly to \( x_M \), which is the maximal solution of the CDARE \([11]\). Furthermore, the convergence is linearly since \( \lim_{k \to \infty} \sqrt{|x_k - x_M|} = \bar{T}_M < 1 \). When \( x_0 \in (x_m, x_M) \), we see that \( s < 0 \) and thus \( \{x_k\}_{k=0}^\infty \) converges increasingly to \( x_M \) with linear convergence even though \( x_0 \in \mathcal{S}_2 \) is not true in this case.
(ii) If $h = h_M$, then $x_M = x_m$ with $\hat{t}_M = 1$ and the assumptions in (5) hold. In this case we see that the sequence $\{x_k\}_{k=0}^\infty$ in the expression (12b) converges decreasingly to $x_M$ with sublinear convergence, since $\lim_{k \to \infty} \frac{x_{k+1} - x_M}{x_k - x_M} = \hat{t}_M = 1$.

(iii) If $h < h_m$, then the assumptions in (5) do not hold and $\hat{t}_M > 1$. However, from (12a) it can be shown that $x_k \to x_M + \frac{x_m - x_M}{1 - \hat{t}_M} = x_m$ as $k \to \infty$. Namely, the sequence $\{x_k\}_{k=0}^\infty$ converges linearly to the minimal solution of the CDARE (11). Further investigations are left as a topic of our future work for this case.

5. Concluding remarks

In this paper we mainly deal with a class of conjugate discrete-time Riccati equations, arising originally from the LQR control problem for discrete-time antilinear systems. In this case, the design of the optimal controller usually depends on the existence of a unique positive semidefinite optimizing solution of CDAREs (1a) with $R > 0$ and $H \geq 0$, if the antilinear system is assumed to be controllable.

Analogous to Theorem 13.1.1 of [1] for standard DAREs, based on the framework of the fixed-point iteration, we have proved the existence of the maximal solution to the CDARE (1a), with nonsingular $R$ and $H \in \mathbb{H}_n$, under some weaker assumptions defined as in (8). Moreover, starting from $X_0 \in S_\geq$, the monotonicity of the sequence $\{X_k\}_{k=0}^\infty$ generated by the FPI (9) and the stability of its corresponding sequence $\{\hat{T}_X\}_{k=0}^\infty$ have also been deduced under the same assumptions, respectively. We believe that our theoretical results would be useful in the LQR control problem, or even the state-feedback stabilization problem, for discrete-time antilinear systems.

It is only to be expected that the existence of the minimal solution or other extremal solutions of the CDARE (1) will be investigated in the next work, and it also leads to our future work that how to apply the accelerated techniques presented in [2, 3] for solving the extremal solutions to the CDARE (1).

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