Orthogonality catastrophe and decoherence of a confined Bose-Einstein condensate at finite temperature

A. B. Kuklov\textsuperscript{1} and Joseph L. Birman\textsuperscript{2}

\textsuperscript{1} Department of Engineering Science and Physics, The College of Staten Island, CUNY, Staten Island, NY 10314
\textsuperscript{2} Department of Physics, City College, CUNY, New York, NY 10031

(Received 6 September 2000; published 10 November 2000)

We discuss mechanisms of decoherence of a confined Bose-Einstein condensate at finite temperatures under the explicit condition of conservation of the total number of bosons $N$ in the trap. A criterion for the irreversible decay of the condensate two-time correlator is formulated in terms of the Orthogonality Catastrophe (OC) for the exact $N$-body eigenstates, so that no irreversible decay occurs without the OC. If an infinite external bath contacts a finite condensate, the OC should practically always occur as long as the bath degrees of freedom are interacting with each other. We find that, if no external bath is present and the role of the bath is played by the normal component, no irreversible decay occurs. We discuss the role of the effect of level repulsion in eliminating the OC. At finite temperatures, the time-correlations of the condensate isolated from the environment are dominated by reversible dephasing which results from the thermal ensemble averaging over realizations of the normal component. Accordingly, the correlator exhibits gaussian decay with certain decay time $\tau_d$ which depends on temperature as well as on intensity of the shot noise determined by the statistical uncertainty in the number of bosons $N$ in the trap.

PACS: 03.75.Fi, 05.30.Jp

1. INTRODUCTION

Creation of trapped atomic Bose-Einstein condensates \textsuperscript{1} has made possible experimental study of the fundamental concepts which so far were tested mostly in ”thought experiments”. Many intriguing questions are associated with the finiteness of the number of bosons $N$ forming the condensate. As discussed by Leggett and Sols \textsuperscript{2,3}, depending on the environment, the Josephson phase of a finite capacity Josephson junction may exhibit either a ballistic or a diffusive spread in time. For example, if the condensate randomly exchanges bosons with some environment, phase diffusion becomes completely irreversible \textsuperscript{4}.

A dephasing of the wave function of the confined condensate - the so called phase diffusion – has been discussed in Refs. \textsuperscript{1} \textsuperscript{5} in recent times. The nature of this effect is closely related to the concept of broken gauge symmetry associated with the formation of the condensate wave function $\langle \Psi(x,t) \rangle \neq 0$, where $\Psi(x,t)$ is the Bose field operator, and the averaging is performed over the initial state taken as the coherent state. At zero temperature, the phase diffusion refers to the decay of $\langle \Psi(x,t) \rangle$ due to the two-body interaction \textsuperscript{4,6}. This effect is purely quantum, and occurs without any bath. Such a decay was predicted to be reversible in time, so that spontaneous revivals of the wave function should occur \textsuperscript{7,8}. The phase diffusion effect in an atomic condensate which is completely separated from the environment was considered by Graham \textsuperscript{7,8}, and it has been suggested that $\langle \Psi(x,t) \rangle$ decays irreversibly due to the particle exchange between the condensate and the normal component confined in the same trap.

The experimental study of the temporal phase correlations has been conducted by the JILA group \textsuperscript{9}. It has been found that no detectable decay of the phase exists on the time scale of the experiment $\leq 100$ms. The question was then posed in \textsuperscript{10} why the phase correlations are so robust despite an apparent fast relaxation of other degrees of freedom such as, e.g., the relative motion of the two condensates.

In the present work, we address the question \textsuperscript{10} of the phase correlations. However, this question is not analyzed in the context of the problem of decoherence of $\langle \Psi(x,t) \rangle$. Strictly speaking, as long as the operator $\Psi(x,t)$ describes conserved particles, the mean $\langle \Psi(x,t) \rangle$ is not physically observable. We analyze a physically measurable quantity which is the two-time (and space) correlator

$$\rho(x,x',t,t') = \langle \Psi(\mathbf{x},t)\Psi(\mathbf{x}',t') \rangle. \tag{1.1}$$

This correlator can be measured by, e.g., scattering of fast atoms \textsuperscript{10}. We formulate a condition for the irreversible decay of the correlator \textsuperscript{11} in terms of the Orthogonality Catastrophe (OC) for the projected $N$-body eigenstates (defined below).

We consider a situation when the one-particle density matrix (OPDM) $\rho(x,x',t) = \rho(x,x',t,t)$ exhibits equilibrium off-diagonal long range order (ODLRO) \textsuperscript{11} (see also the discussion in Ref. \textsuperscript{12} on how the ODLRO should be defined in the atomic traps). We ask how the existence of the ODLRO affects the temporal behavior of the correlator \textsuperscript{11}.
as \( t - t' \to +\infty \). It is worth stressing that this question should not be identified with the problem of the decay of the broken \( U(1) \) symmetry state, because such a state is not a prerequisite for ODLRO \cite{13}. This difference becomes especially important while studying the long time correlations in large (but finite) \( N \) condensates. As was found by Wright et al. \cite{14}, the interaction between bosons introduces some break time \( \sim \sqrt{N} \), so that employing the classical field \( \langle \Psi(x, t) \rangle \) as a proper variable obeying the Gross-Pitaevskii equation at times longer than this break time becomes no longer valid. A similar situation occurs in the description of the normal excitations, when the mean field approach breaks down at long times \cite{15,16}, with the break time being \( \sim \sqrt{N} \) as well. The limitations of the Gross-Pitaevskii equation at long times have also been pointed out by Castin and Dum \cite{17}. If one is interested in the long time evolution, the correlator \( \rho \) must be analyzed.

In this paper we are interested in the long time correlations which, however, can be quite measurable in the current traps. We calculate the condensate part of the correlator \( \rho \) in equilibrium at \( T \neq 0 \) and show that, as long as no external bath is present, the time dependence of the correlator \( \rho \) can be represented as a result of the thermal averaging of the non-decaying exponents where each one can be viewed as representing a specific realization of the chemical potential (defined below) for the exact \( N \)-body eigenenergies. This averaging results in the dephasing of the correlator. The time \( \tau_d \) of this dephasing is determined by the thermal fluctuations of the normal component for fixed \( N \). In any realistic experiment, the value of \( N \) varies statistically from run to run, and produces shot noise, also contributing to the dephasing rate \( \tau_d^{-1} \). The equilibrium condensate evolution can be viewed as though it occurs in the frozen environment created by the normal component. The dephasing, then, results from the ensemble averaging over the possible realizations of the normal component and \( N \).

In the following we will formulate a necessary condition for the dissipative decay of the correlator \( \rho \) in terms of the Orthogonality Catastrophe for the exact \( N \)-body eigenstates.

### II. ORTHOGONALITY CATASTROPHE AS A NECESSARY CONDITION FOR THE DISSIPATIVE DECAY OF THE CONDENSATE TWO-TIME CORRELATOR AT \( T \neq 0 \)

The correlator \( \rho \) carries the most complete information about confined bosons. The averaging \( \langle \cdots \rangle \) in Eq.\( (1.1) \) is performed over the initial state (or states). In what follows we will assume that the system of \( N \) bosons is in the thermal state, so that the averaging is performed over the thermal ensemble of the exact \( N \)-body eigenstates \( |m, N \rangle \) and the eigenenergies \( E_m(N) \), where \( m \) denotes a set of the quantum numbers specifying the eigenstate with given \( N \) of the many body Hamiltonian \( H \). In other words, if the ground state \( |0, N \rangle \) corresponds to the pure condensate (characterized by the energy \( E_0(N) \)), the states with \( m \neq 0 \) describe the condensate and the normal component in the trap, with \( m \) carrying the meaning of the set of all possible quantum numbers of the normal excitations. Should some external bath be present in addition to the normal component, providing (coining the terminology \cite{9}) the intrinsic bath, these numbers characterize both the normal component and the bath. In the thermal equilibrium Eq.\( (1.1) \) is reduced to \( \rho(x, x', t, t') = \rho(x, x', t - t', 0) \). Accordingly, the OPDM becomes time independent \( \rho(x, x', t) = \rho(x, x', 0, 0) \). In order to simplify notation, we will set \( t' = 0 \).

As a matter of fact, for temperatures \( T \) below and not very close to the Bose-condensation temperature \( T_c \), the behavior of \( \rho \) is dominated by the condensate part. The condensate part \( \rho_0(x, x', t, 0) \) of \( \rho(x, x', t, 0) \) is defined in terms of some macroscopically populated eigenstate \( \varphi_0(x) \) of the OPDM. In the case of the weakly interacting Bose gas, \( \rho_0(x, x', t, 0) \) can be selected by employing the following standard representation

\[
\Psi(x, t) = a_0(t) \varphi_0(x) + \psi(x, t),
\]

where \( a_0(t) \) removes one boson from the condensate, and \( \psi \) accounts for the non-condensed bosons. Thus,

\[
\rho_0(x, x', t, 0) = \varphi_0^*(x) \varphi_0(x') a_0^\dagger(t) a_0(0) + \rho'_0(x, x', t),
\]

where \( \rho'_0(x, x', t) \) accounts for the non-condensed bosons. Thus, \( \rho_0(x, x', t, 0) = \varphi_0^*(x) \varphi_0(x') a_0^\dagger(t) a_0(0) \).

In what follows we will omit the coordinate dependencies of \( \rho_0(x, x', t) \), and will call the correlator \( \rho_0(t) = \langle a_0^\dagger(t) a_0(0) \rangle \) as the condensate (time) correlator. Accordingly, the condensate OPDM becomes \( \rho_0 = \rho_0(t = 0) \). Apparently, \( \rho_0 \) should be identified with the mean population \( N_0 \) of the condensate.

We assume that the equilibrium condensate can be well described by a macroscopic population of only one single-particle state \( \varphi_0 \). There are situations when this assumption is not valid. These are related to the low D geometries in which the condensate may exist as a quasi-condensate \cite{15}, or is thermally smeared over many states characterized by different winding of the phase \( \Omega \). In this paper we do not consider such situations. Thus, our analysis can be applied to the 3D case only. Furthermore, if there is the spin degeneracy, the so called fragmented condensate can be formed \cite{20}. In this case, many states can also be macroscopically populated. Here we do not consider such a situation as well.

Employing the standard definitions of the Heisenberg operators as well as of the thermal mean, one finds

\[
\rho_0(t) = \langle a_0^\dagger(t) a_0(0) \rangle = \sum_{m,m'} P_m e^{i(E_m(N) - E_{m'}(N-1))t} \langle |a_0(0)\rangle_{m', m} \rangle^2,
\]

where \( P_m \) is the probability of finding the state \( |m, N \rangle \).
where the Boltzmann factor is $p_m = Z^{-1}(N, \beta) \exp(-\beta E_m(N))$, and $Z(N, \beta)$ denotes the canonical partition function as a function of the total number of bosons $N$ and of the temperature $T = 1/\beta$; the notation

$$ (a_0)_{m', m} = \langle N - 1, m' | a_0 | m, N \rangle, \tag{2.3} $$

has been introduced. Eqs. (2.2), (2.3) are exact. Below, we will represent $\rho_0(t)$ as a cumulant expansion which is actually the $1/N$ expansion in the exponent, and we will find the leading term of this expansion.

We note that several papers have been devoted to developing an approach for treating $N$ bosons under the explicit condition of conservation of $N$. In this section, we introduce a natural phase-space for the $N$-conserving approaches. When $N$ is conserved, it is convenient to express the matrix elements (2.3) in such a way that $N$ becomes a parameter. First, we represent the exact eigenstate $|m, N\rangle$ of $N$ bosons as an expansion

$$ |m, N\rangle = \sum_{N_1, N_2, \ldots} C_{N_1, N_2, \ldots}(m, N)|N_0, N_1, N_2, \ldots\rangle, \quad N_0 = N - (N_1 + N_2 + \ldots), \tag{2.4} $$

in the Fock space $|N_0, N_1, N_2, \ldots\rangle$ of the population numbers $N_0, N_1, N_2, \ldots$ of some set of the single particle states $\varphi_0(x), \varphi_1(x), \varphi_2(x), \ldots$, respectively, given that $\varphi_0(x)$ is the only macroscopically populated state. Eq. (2.4) takes into account explicitly that the total number of bosons $N$ is conserved; $C_{N_1, N_2, \ldots}(m, N)$ denotes the coefficients of the expansion. These coefficients form the Fock representation of the states with given $N$. We call $C_{N_1, N_2, \ldots}(m, N)$ the projected states, and introduce a short notation $|m, N\rangle$ for them. Accordingly, the product of two projected states is defined as $\langle \tilde{N}', m' | \tilde{m}, N\rangle = \sum_{N_1, N_2, \ldots} C_{N_1, N_2, \ldots}(m', N')C_{N_1, N_2, \ldots}(m, N)$. Thus, for the same $N$, the product of any two projected states coincides with the product of the corresponding eigenstates (2.4), that is, $\langle \tilde{N}, m' | \tilde{m}, N\rangle = \langle N, m' | m, N\rangle$. Therefore, the orthogonality condition $\langle N, m' | m, N\rangle = \delta_{m'm}$ holds.

We note that, in general, no orthogonality exists between two projected states with different $N$. In other words, $\langle N, m | m', N'\rangle = \sum_{N_1, N_2, \ldots} C_{N_1, N_2, \ldots}(m, N)C_{N_1, N_2, \ldots}(m', N') = 0$ for $N' \neq N$. This should be contrasted with the trivial orthogonality $\langle N, m | m', N'\rangle = 0$ for $N' = N$ of the eigenstates (2.4) insured by the orthogonality of the two Fock subspaces with different $N$. In order to clarify the meaning of the projected states $|m, N\rangle$, it is straightforward to recall the radial part $R_{n, L}(r)$ of the full wavefunction $R_{n, L}(r)Y_{L, \ell}(\theta, \phi)$ of a particle moving in a central potential. Then, conserving $N$ is analogous to conserving angular momentum $L$. While $R_{n, L}(r)$ and $R_{n', L}(r)$ characterized by different radial numbers $n$ are orthogonal for given $L$, no orthogonality exists between $R_{n, L}(r)$ and $R_{n', L}(r)$ for $L \neq L'$.

The eigenproblem $H|m, N\rangle = E_m(N)|m, N\rangle$ is equivalent to finding the coefficients $C_{N_1, N_2, \ldots}(m, N)$. Accordingly, this problem can equivalently be reformulated as $H(N)|m, N\rangle = E_m(N)|m, N\rangle$ in terms of the projected states, which are the eigenstates of the corresponding projected Hamiltonian $H(N)$ with the same eigenenergies $E_m(N)$ for given $N$. Resorting back to the analogy with the motion in a central potential, $H(N)$ can be mnemonically viewed as the "radial" part of the total Hamiltonian. The projected states take care explicitly of the conservation of $N$. As a matter of fact, the projected states form a basis for previously used $N$-conserving approaches (21, 22). Employing them, it is possible, in principle, to construct explicitly the projected Hamiltonian $H(N) = \sum_m |m, N\rangle E_m(N)|\tilde{N}, m\rangle$ in terms of the projected states. It should be noted that, despite being possible in principle, representing the projected states $|m, N\rangle$ in the original Fock basis by the coefficients $C_{N_1, N_2, \ldots}(m, N)$ is impractical as long as the interparticle interaction is finite. The most productive way of constructing the projected states is in terms of the pair operators suggested in Refs. (21, 22). We, however, will first analyze general properties of the projected states without constructing them explicitly (Sec.III). For this purpose, any complete representation is suitable. Then, we will demonstrate the meaning of the projected states by employing an exactly solvable model of the bath of non-interacting oscillators (Sec.IVA). The role of interactions between the oscillators will be analyzed in sections IVB, IV.C. We will also show (Sec.V) that the traditional hydrodynamical approach (23, 24) provides a natural formalism for constructing the projected states.

The matrix elements (2.3) can be expressed in terms of the overlaps of the projected states with $N$ differing by 1. Indeed, a substitution of Eq. (2.4) into Eq. (2.3) yields

$$ (a_0)_{m', m} = \sum_{N_1, N_2, \ldots} \sqrt{N_0} C_{N_1, N_2, \ldots}^*(m', N - 1)C_{N_1, N_2, \ldots}(m, N), \tag{2.5} $$

where $N_0$ is given in Eq. (2.4). We consider a situation when the mean $\langle N_0\rangle$ of the condensate population $N_0$ for given $N$ is macroscopically large. As was shown by Giorgini et al. in Ref. (25), the relative mean square fluctuation $\delta N_0/\langle N_0\rangle$ of $N_0$ vanishes in the canonical ensemble for $N \gg 1$. Thus, in the calculation of the matrix elements (2.5) one can
perform an expansion with respect to \( \delta N_0/N_0 \ll 1 \), and retain only the zeroth order term \( \sqrt{N_0} \), so that within the accuracy \( o(\delta N_0/N_0) \) Eq. (2.3) gives

\[
(a_0)_{m',m} = \sqrt{N_0} \chi_{m',m},
\]

where we have introduced the overlap of the projected states

\[
\chi_{m',m} = \sum_{N_1,N_2,...} C_{N_1,N_2...}(m',N-1)C_{N_1,N_2...}(m,N) = (N-1,m'|m,N)
\]

with \( N \) differing by 1. In what follows we will assume that \( N_0 \approx N \). It is useful to mention some obvious properties of the overlap (2.7). Employing the orthonormality condition for the projected states with given \( N \), one finds

\[
\sum_{m'} \chi_{m',m'} \chi_{m,m'} = \delta_{m,m'}.
\]

In Beliaev’s work [29] it has been shown that, for the exact ground state \((0,0)\), one can write \( (a_0)_{m',0} = \sqrt{N_0}(\delta_{m',0} + o(1/N)) \). In other words, creation or destruction of one boson in the condensate does not lead to creation of excitations within the accuracy \( o(1/N) \) (no external bath was considered in Ref. [24]). Beliaev’s result can be reformulated in terms of the overlap (2.7) as \( \chi_{0,0} = 1 + o(1/N) \). This immediately yields Eq. (2.2) as \( \rho_0(t) = N_0 \exp[i(E_0(N) - E_0(N-1))t] + o(1) \) at \( T = 0 \), which implies no decay as long as \( N \) is fixed.

At finite temperatures \( T \neq 0 \), Beliaev’s result is traditionally extended [27] as follows \( (a_0)_{m',m} \approx \sqrt{N_0} \delta_{m',m} \). In terms of the overlap (2.7), this is identical to

\[
\chi_{m',m} = \delta_{m',m} + o(1/N).
\]

We introduce the notation \( \rho_0^{(d)} = \sum_m p_m |(a_0)_{m,m}|^2 \). Obviously, \( \rho_0^{(d)} \), which plays an important role in the following analysis, is the diagonal part (with respect to the excitation label) of the condensate OPDM \( \rho_0 = \sum_{m,m'} p_m |(a_0)_{m,m'}|^2 = \sqrt{N_0} \). Employing Eq. (2.8), it is convenient to represent \( \rho_0^{(d)} \) as

\[
\rho_0^{(d)} = \sqrt{N_0} \chi^2, \quad \chi = \sqrt{\sum_m p_m |\chi_{m,m}|^2},
\]

where we have introduced the mean overlap \( \chi \). A direct consequence of (2.9) is

\[
\chi = 1 + o(1/N).
\]

The physical interpretation of Eq. (2.11) is that, while entering or exiting the condensate, a boson does not significantly disturb the excitations reservoir.

Employing Eqs. (2.6), (2.9) in Eq. (2.2), we obtain the condensate correlator (2.2) as \( \rho_0(t) = \rho_0^{(d)}(t) + o(1) \), where we have introduced the diagonal part (with respect to the excitations) \( \rho_0^{(d)}(t) \) of the correlator (2.3). Given the condition (2.3), it is

\[
\rho_0^{(d)}(t) = \sqrt{N_0} \sum_m p_m e^{i\mu_m t} + o(1),
\]

\[
\mu_m = E_m(N) - E_m(N-1) = \frac{\partial E_m(N)}{\partial N} + o(1/N).
\]

Following the standard definition [27], the quantity \( \mu_m \) introduced in Eq. (2.12) will be called the chemical potential of the \( m \)-th exact \( N \)-body eigenstate, so that the dephasing can be viewed as occurring due to the canonical ensemble fluctuations of the so defined chemical potential.

Obviously, \( \rho_0^{(d)}(t) \approx \sqrt{N_0} \approx N \) for times short enough, so that no dephasing takes place. It is worth emphasizing that, while no decay is observed for each particular exponent representing the \( m \)-th eigenstate, the summation in Eq. (2.12) will normally produce a dephasing at times longer than some dephasing time \( \tau_d \). We also mention that a statistical uncertainty inevitably present in the initial value of \( N \), namely: the shot noise (from realization to realization), will result in the dephasing as well [28]. The evolution represented by (2.12) can be viewed as reversible dephasing in a
sense that no normal excitations are disturbed by such an evolution (see also in Ref. [29]). We, however, note that this definition does not necessarily imply that the evolution can always be reversed in time practically.

An alternative to the situation described above is that the mean overlap introduced in Eq.(2.10) becomes \( \chi = o(1/N) \). It is important to emphasize that such a situation does not conflict with the existence of ODLRO. Let us discuss this. Apparently, if \( \chi = o(1/N) \), the diagonal part \( \rho_{0}^{0} (2.10) \), instead of being \( \approx N \), becomes

\[
\rho_{0}^{d} = N_{0} \chi^{2} = o(1).
\]  

On the other hand, the condensate OPDM \( \rho_{0} = \sum_{m,m'} p_{m} \langle \chi_{m,m'} |^{2} = N_{0} \approx N \) due to the condition (2.8). In fact, the summation here runs essentially over \( m' \neq m \), that is, the sum is collected from the amplitudes of the processes when one boson, enters or exits the condensate, significantly disturbs the bath. Thus, it is conceivable that the ODLRO, which can be expressed as \( \rho_{0} \sim N \), can coexist with the situation represented by Eq.(2.13). We will refer to such a situation as the Orthogonality Catastrophe (OC) [31] in the system consisting of the Bose-Einstein condensate and the bath, which can be either extrinsic [1] or intrinsic [1].

We point out that, should the OC occur, the dynamics of the correlator (2.2) would be totally determined by \( \chi_{m',m} \) with \( m \neq m' \), that is by the processes of creation and destruction as well as of scattering of the normal excitations. Consequently, the condensate correlator would exhibit a dissipative decay as long as the excitations have finite life-time due to the many-body interactions.

The notion of the OC [30] was introduced by P.W. Anderson with respect to the Fermi-edge singularity where creation or annihilation of a single hole dramatically changes all states of the Fermi sea, so that the overlap of the states (projected on the Fermi-sea) which differ in the number of holes by 1 is essentially zero in the thermodynamical limit. The OC in the bosonic system is defined above in terms of the mean square overlap \( \chi \) of the projected states which differ only in the total number of bosons \( N \) by 1. Thus, in this case the role of the hole is played by 1 boson removed from (or added to) the condensate, and the role of the Fermi sea is played by the ensemble of the normal excitations (or by the external bath).

Summarizing, if \( \chi \to 0 \), the OC occurs, and the condensate exhibits the irreversible decay. If \( \chi = 1 + o(1/N) \), no OC occurs, and the condensate correlator (2.2) is essentially described by Eq.(2.12), which, however, may exhibit a dephasing as a result of the thermal averaging. Below we will derive a general expression for \( \chi \) for the case when \( N \) is macroscopically large. We will also obtain an expression for the correlator (2.2) in the main 1/N limit.

### III. OVERLAP AND THE SPECTRAL FUNCTION OF THE BATH

If the total number of bosons \( N \) is conserved, the projected Hamiltonian \( H(N) \) can be constructed. In this Hamiltonian, \( N \) plays a role of the parameter. Consequently, the mean overlap \( \chi \) between the projected states with different \( N \) (see Eqs.(2.7), (2.10)) can be found by employing the perturbation expansion with respect to

\[
H' = H(N) - H(N - 1) = \frac{\partial H(N)}{\partial N} + ....
\]  

Each additional derivative \( \partial.../\partial N \) in the expansion (3.1) introduces a factor \( \sim 1/N \). Therefore, in the limit \( N \gg 1 \), it is enough to consider the first term only in (3.1). Correspondingly, calculations of the overlap are based on the 1/N expansion in Eq.(3.1) [31]. Below we follow the approach first introduced by Feynman and Vernon in Ref. [32] for describing weak interaction with the bath. As pointed out in [32], it is sufficient to consider the effect of the bath in the second order with respect to this interaction, and the result must be considered as a first term of the expansion of the exponent. In our case the weakness is insured by the 1/N expansion. In other words, we look for the eigenstates \( |m, N - 1\rangle \) of \( H(N - 1) \) assuming that the eigenstates \( |m, N\rangle \) and the eigenenergies \( E_{m}(N) \) of \( H(N) \) are known. Accordingly, in the lowest order with respect to \( H' \) (3.1) we find for (2.7)

\[
\chi_{m,m} = 1 - \frac{1}{2} \sum_{n \neq m} \frac{|(H')_{mn}|^{2}}{\omega_{mn}^{2}}, \quad \chi_{m,m'} = \frac{(H')_{mm'}}{\omega_{mm'}}
\]  

where the following notations

\[
(H')_{mn} = \langle \hat{\pi}_{0}, m|H'|n, \hat{\pi}_{0}\rangle, \quad \omega_{mn} = E_{m}(N) - E_{m}(N),
\]  

have been introduced, and the eigenstates \( |n, \hat{\pi}_{0}\rangle \) as well as the eigenenergies \( E_{m}(N) \) are exact with respect to the Hamiltonian \( H(N) \).
As pointed out by Feynman and Vernon [32], if there is an extremely large number of bath degrees of freedom, the exponentiation of \( [3.2] \) must be done after the averaging. This is a consequence of the central-limit theorem [22]. Thus, averaging \( |\chi_{m,m}|^2 \) in Eq. (3.2) over the ensemble and then exponentiating, we find the mean overlap

\[
\chi = \exp \left( -\int_0^\infty d\omega \frac{J(\omega)}{2\omega^2} \right)
\]

(3.4)

where the spectral weight \( J(\omega) \) is defined as

\[
J(\omega) = \sum_{m,n} p_m |(\delta H)_{mn}|^2 \delta(\omega - \omega_{mn}),
\]

(3.5)

and we have introduced the operator \( \delta H \) whose matrix elements are

\[
(\delta H)_{mn} = (H')_{mn} - (H')_{mm} \delta_{mn}.
\]

(3.6)

We note that the accuracy of the result (3.4) is well controlled by the \( 1/N \) expansion (3.1).

The OC occurs if the integral in (3.4) diverges at the low frequency limit \( \omega \to 0 \). The borderline for this divergence is the so-called ohmic dissipation [33], characterized by the limiting behavior \( J(\omega) \sim \omega^s \) \((s = 1)\) as \( \omega \to 0 \). For \( s > 1 \), the integral in Eq. (3.4) is finite, which insures that no OC occurs. In the case when the bath spectral function is ohmic (and subohmic) the OC will occur, so that the condensate correlator will irreversibly decay at large times.

Let us assume that no OC occurs, that is, \( \chi \approx 1 \) in Eq. (3.4). Then, the correlator (2.2) is represented by Eq. (2.12). Its structure can be interpreted in terms of the thermodynamic fluctuations of the chemical potential \( \mu_m \). Employing the central limit theorem, we find

\[
\rho_0^{(d)}(t) = N_0 e^{-(t/\tau_d)^2} + o(1)
\]

(3.7)

where the mean value of \( \mu_m \) and the dephasing time \( \tau_d \) are given as

\[
\overline{\mu} = \sum_m p_m (H')_{mm}, \quad \tau_d^{-2} = \frac{1}{2} \sum_m p_m ((H')_{mm} - \overline{\mu})^2,
\]

(3.8)

and the definition of \( \mu_m \) in Eq. (2.13), which can be written as \( \mu_m = \partial E_m(N)/\partial N + o(1/N) \), as well as the identity \( \partial E_m(N)/\partial N = (H')_{mm} \), with \( H' \) defined by (3.1), have been employed.

We note that the gaussian form (3.3) has its limitation time \( \tau' \), so that (3.7) is valid for \( t \ll \tau' \). It is important to realize that \( \tau' \gg \tau_d \) in the macroscopic system, so that when \( t \) is larger than \( \tau_d \) by only few times the correlator (3.3) is essentially zero. Therefore, the gaussian approximation (3.7) is practically exact (if no spontaneous revivals of the kind \( 0 \) are to be expected to occur at times \( \gg \tau_d \)). We will discuss this \( \tau' \) in detail later.

Note that the overlap (3.4) can be interpreted in a different manner. Indeed, transforming Eqs. (3.3), (3.4) into the time representation and introducing the spectral function \( G(t) \) in the time domain as the inverse Fourier transform of \( J(\omega) \), one finds

\[
\chi = \exp \left( -\int_0^\infty dt \int_0^t dt' G(t') \right), \quad G(t) = \langle \delta H(t') \delta H(0) \rangle,
\]

(3.9)

where the brackets denote the thermodynamic averaging over the canonical ensemble with given \( N \); and \( \delta H(t) = \exp[iH(N)t]|\delta H| \exp[-iH(N)t] \). Thus, the OC is tightly connected with the long time behavior of the correlator \( G(t) \). If \( G(t) \) does not exhibit long-time correlations, the conjecture of the presence of the OC implies that \( G(t) \) contains the white noise part, that is, \( G(t) = G_0(t) + \Gamma(t') \), where \( \Gamma > 0 \) is some effective decay constant, and \( G_0(t) \) stands for the part which satisfies \( \int_0^\infty dt G_0(t') = 0 \). The decay constant \( \Gamma \) determines the time scale \( t_{OC} = 1/\Gamma \) on which the OC develops. Evidently, the validity of the traditional treatment of the bosonic system is limited by times \( t \ll t_{OC} \) (or by the excitation frequencies \( \omega \gg t_{OC}^{-1} \)) under the assumption that the condensate correlator \( \langle a_0(t)a_0(0) \rangle = \exp(-t\overline{\mu}) \)

(2.4).

It is also possible to find the correlator (2.2) as a 1/N cumulant expansion, that is, 1/N expansion in the exponent. In the main 1/N order, we find

\[
\rho_0(t) = N_0 e^{\overline{\mu}-(t/\tau_d)^2} \chi(t); \quad \chi(t) = \exp \left( -\int_0^t dt_1 \int_0^{t_1} dt' G(t') \right),
\]

(3.10)
where we have employed Eqs. (3.2), (2.10) in Eq. (2.2), and, then, performed the exponentiation up to $o(1/N)$; the quantities $\tau_d$ and $G(t)$ are defined by Eqs. (3.8), (3.9). Here we have introduced $\chi(t)$ which can be interpreted as a time dependent overlap, so that $\chi = \chi(t = \infty)$.

We emphasize that the gaussian factor $\exp(-t^2/\tau_d^2)$ in the form (3.10) is not due to the short time approximation of the correlator $G(t)$. As the expressions (3.8) and (3.9) indicate, the dephasing rate $\tau_d^{-1}$ is determined by the diagonal elements of the derivative $H'(\omega)$ of the projected Hamiltonian $H(N)$, while the correlator $G(t)$ (3.9) is built on essentially off-diagonal elements of $H'$.

The physical interpretation of (3.10) is the following. The condensate correlator $\rho_0(t)$ is the ensemble mean of the correlators $\langle N, m | a_i^\dagger(t) a_0(0) | m, N \rangle$ in the given exact $N$-body eigenstates. These partial correlators contain the non-decaying parts $\exp(i \mu_m t)$, characterized by the chemical potential $\mu_m$ (defined in Eq. (2.13)). The correlators $\langle N, m | a_i^\dagger(t) a_0(0) | m, N \rangle$ may also contain the decaying parts $\exp(-\Gamma_m t)$, characterized by some decay constants $\Gamma_m > 0$. In general, there is no direct relation between the parameters $\Gamma_m$ and $\mu_m$, because $\mu_m$ is determined by the diagonal part of the operator $H'(\omega)$ (3.1), while $\Gamma_m$ are given by the off-diagonal elements of this operator. The thermal averaging of the partial correlators should not change the exponential decay much, so that it can be characterized by a single exponent $\sim \exp(-\Gamma t)$ with some mean decay constant $\Gamma$, which can be related to $G(t)$ in (3.9) as $\Gamma \approx \int_0^\infty dt G(t)$ at times longer than some typical correlation time of $G(t)$. It is important that the thermal averaging produces the gaussian time dependence in Eq. (3.10) due to the fluctuations of $\mu_m$ in the non-decaying factors $\exp(i \mu_m t)$. Thus, in general, one may expect the correlator (3.10) to acquire the form

$$\rho_0(t) = \tau_0 \exp(-t^2/\tau_d^2 - i \mu t),$$  

(3.11)
at times longer than some typical correlation time of $G(t)$. We note that considering thermal (ensemble) fluctuations of $\mu_m$ makes sense, if the dephasing time $\tau_d$ is shorter than (or at least comparable to) the OC time $t_{OC} = 1/\Gamma$.

As we will see later, perturbative calculations yield $t_{OC} \sim N$, while $\tau_d \sim \sqrt{N}$ in the case when no external bath is present. In this case, the OC becomes irrelevant for all practical purposes because $\rho_0(t)$ of Eq. (3.10) can be well approximated by the form (3.1) as long as $t < t_{OC}$, and it will decay to almost zero at $\tau_d < t < t_{OC}$. Furthermore, we will discuss that, in the isolated trap, the calculated $t_{OC}$ turns out to be longer than an inverse of the typical value $\Delta$ of the matrix elements of the interaction part of the many-body Hamiltonian. This implies that the effect of the level repulsion should modify significantly the correlator $G(t)$ at times $t \gtrsim \Delta^{-1}$. As will be shown, the effect of the level repulsion eliminates completely the OC in the isolated condensate, so that the evolution is dominated solely by the gaussian decay (3.1). Thus, while the necessary condition for irreversible decay of the condensate correlator (2.2) is the OC represented as $\chi = \chi(t = \infty) \to 0$, a sufficient condition for the irreversible decay is $t_{OC} \leq \tau_d$, provided the time scale on which the effect of the level repulsion may become significant is much longer than $t_{OC}$. This situation cannot be achieved in the isolated trap, and some external bath characterized by its own corresponding spectral function is required in order to create the OC. In the following discussion we will show that, as long as an infinite interacting bath contacts a finite condensate, the OC should always develop.

IV. ORTHOGONALITY CATASTROPHE IN THE BOSONIC TRAP IN THE PRESENCE OF AN EXTERNAL BATH

First, we consider the simplest model which exhibits the OC. In this model we neglect the normal excitations of the condensate, and take into account excitations of some external bath, represented by an infinite set of the non-interacting linear oscillators (32-33). In this case, the projected states as well as the spectral function $J(\omega)$ can be constructed explicitly. We will also analyze the role of the interaction in the bath, and will demonstrate that the OC should occur in the interacting infinite bath even though no OC was found without such an interaction.

A. Bath of non-interacting oscillators

We neglect now excitations inside the trap and assume a presence of some external bath which, however, does not exchange particles with the trap. For illustrative purposes, the trap is represented by a single level containing all the bosons. The bath is described by an infinite set of linear non-interacting oscillators (32-33). This toy model is a simplified version of the model suggested by Anglin (35). Thus, we have the Hamiltonian

$$H_0 = (\epsilon_0 + Q)a_0^\dagger a_0 + \sum_a \left[ \frac{p_a^2}{2} + \frac{\omega_q^2 q_a^2}{2} \right], \quad Q = \sum_a \left( g_a q_a + \frac{g_a^* q_a^2}{2} \right),$$

(4.1)
where $\epsilon_0$ stands for the energy of a single level accommodating $N$ bosons; the summation is performed over the non-interacting oscillators, with $\omega_a$ and $g_a$, $g_a'$ being the frequency of the $a$th oscillator and the interaction constants, respectively.

The advantage of the model (4.1) is that the eigenfunctions and the eigenenergies can be found explicitly. Furthermore, the *projected states* can be constructed explicitly as well. The eigenstates and the eigenenergies of the Hamiltonian (4.1) are

\[
|m, N\rangle = |N\rangle |m, N\rangle, \quad |m, N\rangle = \prod_a \Phi_{n_a}(q_a - g_a \Omega_a^{-2}N, \Omega_a),
\]

\[
E_m(N) = \sum_a \Omega_a(n_a + \frac{1}{2}) - \frac{1}{2} \left( \sum_a \frac{g_a'^2}{\Omega_a^2} \right) N^2,
\]

where $|N\rangle$ stands for the Fock state of $N$ bosons occupying the level $\epsilon_0$; $\Omega_a = \Omega_a(N) = \sqrt{\omega_a + g_a N}$, and we have explicitly indicated the dependence of $\Phi_{n_a}$ on $q_a$ and $\Omega_a$. The *projected states* $|m, N\rangle$, introduced in Sec.II, are actually the bath states which depend on $N$ as a parameter. The state $|m, N\rangle$ is represented here in terms of the oscillators eigenfunctions $\Phi_{n_a}(q_a, \Omega_a)$, with $m$ carrying the meaning of the set of all the integer quantum numbers $\{n_a\}$. The *projected* Hamiltonian $H(N)$ is, then, obtained from (4.1) by simply replacing $a_0\cdot a_0$ by $N$. Accordingly, the derivative (3.11), which determines the spectral weight and the chemical potential becomes

\[
H' = H(N) - H(N - 1) = \epsilon_0 + \sum_a \left( g_a q_a + \frac{g_a'}{2} g_a^2 \right).
\]

As discussed above in Sec.III, the diagonal elements of this derivative determine the chemical potential in the given eigenstate as $\mu_m = (H')_{mm} + o(1/N)$ or

\[
\mu_{\{n_a\}} = \epsilon_0 + \sum_a \frac{g_a'^2}{2\Omega_a} (n_a + \frac{1}{2}) - N \sum_a \frac{g_a'^2}{\Omega_a^2},
\]

where the label $m$ now carries the meaning of the occupation numbers $\{n_a\}$ of the oscillators. The averaging of

\[\exp(-it\mu_{\{q_a\}}) \sim \prod_a \exp(itg_a'^2n_a/2\Omega_a)\]

over $\{n_a\}$ produces the gaussian decay with the dephasing rate $\tau_d^{-1}$ following from Eq.(3.8) as

\[
\tau_d^{-2} = \sum_a \frac{g_a'^2}{\Omega_a} (n_a + \frac{1}{2}) + \frac{1}{8}.
\]

Here $\overline{n}_a = (\exp(\beta\omega_a) - 1)^{-1}$ stands for the mean thermal occupation of the $a$-th oscillator. It is important to note that in the canonical ensemble, the term $\sim g_a^2$ in Eq.(4.3) does not produce any dephasing. If, however, $N$ fluctuates from realization to realization (shot noise) and does not change during each realization, the dephasing rate (4.5) will make a contribution given by the variance $\Delta N$ of $N$, so that the result (4.6) should be modified as

\[
\tau_d^{-2} \rightarrow \tau_d^{-2} + (\Delta N)^2 \sum_a g_a'^2 \Omega_a^{-2}/2.
\]

The off-diagonal elements $(H')_{mn}$ define the spectral weight $J(\omega)$ which may lead to irreversible decay. Which one of these two effects – either the reversible dephasing or the irreversible decay – dominates depends on the model parameters $g_a$, $g_a'$ as well as on the density of states of the bath.

Let us calculate the overlap (2.7) directly in order to see how the central limit arguments work for an infinite bath. This overlap is $\langle N - 1, m|n, N \rangle = \prod_a \int dq_a \Phi_{n_a}^*(q_a - g_a \Omega_a^{-2}(N-1)(N-1), \Omega_a(N-1)) \Phi_{n_a}(q_a - g_a \Omega_a^{-2}N, \Omega_a(N))$, and it can be found explicitly. There is, however, no need to calculate it exactly because the coefficients $g_a$, $g_a'$ must be scaled with the number of the effective degrees of freedom of the bath. Let us assume that the bath volume $V_b \rightarrow \infty$ is proportional to this number, and the volume occupied by $N$ bosons is fixed. Then, the summation $\sum_a$ introduces a factor $V_b$. In order to maintain the energy an extensive quantity, we impose $g_a' \sim 1/V_b$, and $g_a \sim 1/\sqrt{V_b}$. This implies that the overlap, which can be represented as $\langle N - 1, m|n, N \rangle = \exp\{\sum_a \ln[\int dq_a \Phi_{n_a}^*(q_a - g_a \Omega_a^{-2}(N-1)(N-1), \Omega_a(N-1)) \Phi_{n_a}(q_a - g_a \Omega_a^{-2}N, \Omega_a(N))]}$, can be expanded in the inverse powers of $V_b$ in the exponent. Accordingly, the only terms surviving the limit $V_b \rightarrow \infty$ are linear in $g_a'$ and quadratic in $g_a$. This is a reiteration of the central-limit theorem (32). Thus, the mean overlap $\langle 2.7 \rangle$ acquires the form (3.3), with the spectral weight

\[
J(\omega) = \sum_a \frac{g_a'^2}{\omega_a} (\overline{n}_a + \frac{1}{2}) \delta(\omega - \omega_a).
\]

\[\text{Page 8}\]
Of course, the same expression (4.7) follows from Eqs. (3.3), (3.6) where (4.4) must be employed in the limit $V_b \to \infty$.

It is important that the dephasing rate (4.6) vanishes as $\tau_d^{-1} \sim 1/\sqrt{V_b} \to 0$ in the limit $V_b \to \infty$ (if no shot noise is taken into account). The occurrence of the OC, which is expressed as the condition (2.13), depends on the low frequency behavior of (4.7). If, e.g., $J(0) \neq 0$, the integral (3.4) will strongly diverge implying $\tau \to 0$. Thus, the OC is equivalent to the irreversible dissipation for the considered model. Because, should the OC occur, its time scale of the OC can be estimated as $t_{OC} \approx 1/J(0)$ does not contain any positive power of the bath volume $V_b$, so that $t_{OC} \ll \tau_d$ in the limit considered. This corresponds to the irreversible loss of the phase memory on the time scale which is finite in the thermodynamical limit. A concrete realization of this situation, requiring an external bath, will be discussed elsewhere.

If, however, the bath spectral function does not lead to the OC (e.g., $J(0) \sim \omega^2$ as $\omega \to 0$), the time $t_{OC}$ is formally infinite, and the only effect is the dephasing with $\tau_d \sim \sqrt{V_b}$. We note that the last situation resembles most closely the situation in the isolated trap, where the role of the bath is played by the normal component confined in the same trap.

Concluding this section, let us determine the validity of the gaussian approximation (3.7) (or (3.10)) with respect to the given model in the case $V_b \to \infty$. Keeping in mind that $g'_a \sim 1/V_b$, as discussed above, the form (3.7) (or (3.10)) represents the first term in the exponent which is $\sim 1/V_b$. Accordingly, it is valid as long as the next term is much less than 1. The next term is $\sim \sum_a g''_a t^3 \sim t^3/V_b^2$. It becomes of the order of one, when $t \geq \tau' \sim V_b^{2/3}$. Keeping in mind that $\tau_d \sim \sqrt{V_b}$ (see (4.6)), we obtain that $\tau' \approx V_b^{1/6} \gg \tau_d$. Thus, as long as the system is macroscopic, the approximation (3.7) (or (3.10)) is practically exact, as it was discussed in Sec.III (see below Eq. (3.8)).

**B. The OC induced by an external infinite bath of interacting degrees of freedom.** The strong OC.

Now we consider the role of the interaction in the bath in producing the OC in the case when no OC existed without interaction. Referring to the previous model, let us assume that (4.7) is such that no OC occurs. What would happen, if the interaction between $q_a$ is turned on? Such a question is relevant for the confined condensate. As it will be shown below, the intrinsic bath of the non-interacting normal excitations in the trap does not produce the OC. Thus, it is important to understand if the OC would emerge when the interaction between the normal excitations is taken into account.

Let us assume that the total Hamiltonian is $H = H_0 + H_{int}$, where $H_0$ is the “free” part, and $H_{int}$ stands for some weak interaction. For example, the free part can be given by $H_0 = \frac{1}{2} \delta H_{mn}$, and the interaction is of the form $H_{int} = \sum_{abc} g_{abc} q_a q_b q_c$, with $g_{abc}$ being the corresponding interaction constants. In fact, a concrete form of the interaction does not matter much. We choose the third-power non-linearity as an example which resembles most closely the situation in the actual trap, where the most important interaction vertex contains three lines.

The emergence of the OC in the presence of the interaction can be viewed as an appearance of the exponential decay in the correlator Eq. (3.10) leading to the suppression of the mean overlap $\langle 0 | \rho | 0 \rangle$. This happens if the interaction induces the decay of $G(t)$ entering Eqs. (3.10), (3.3), so that, e.g., $G(t) \approx \exp(-\gamma t)G(0)$ with some $\gamma > 0$. Then, the correlator $\rho_0(t)$ (3.10) acquires the factor $\exp(-\Gamma t)$ with $\Gamma = G(0)/\gamma$ at times $t \gg 1/\gamma$. The gaussian factor $\exp(-t^2/\tau_d^2)$ in Eq. (3.3), which is due to the ensemble fluctuations of the chemical potential, is not sensitive to presence of the weak interaction because the chemical potential may gain only small corrections in the case $g_{abc} \to 0$. Thus, the gaussian factor will not be significantly modified by the interaction. Accordingly, the form (3.11) should now describe the condensate correlator.

We take another look at the emergence of the OC. In zeroth order with respect to $H_{int}$, there is usually a high degree of degeneracy between various multi-mode excitations. This degeneracy is represented by the condition $\omega_{mn}^{(0)} = E_{mn}^{(0)}(N) - E_{0}^{(0)}(N) = 0$ for some set $n \neq m$, where the energy levels $E_{n}^{(0)}(N)$ are found for $H_0$. Here and below, the superscript (0) denotes a quantity calculated in the zeroth order with respect to $H_{int}$. Let us first consider the result of the perturbation approach. If the interaction $H_{int}$ is considered in the lowest order, the matrix elements $(\delta H)_{mn}^{(0)}$ between some degenerate states may become finite with some statistically significant weight. Then, if the energy levels are taken unrenormalized, the summation close to the degeneracy point will make $J(0) \neq 0$ in Eq. (3.3). This yields the OC because the integral (3.4) diverges as $\int_0^\infty d\omega J(0) \to \infty$, implying $\tau \to 0$. Accordingly, the time scale of the OC can be estimated as $t_{OC} \approx J^{-1}(0) = \gamma/G(0)$.

It should be stressed, however, that the role of the interaction $H_{int}$ is two-fold. On the one hand, it opens transitions in $\delta H$ (3.0) between the degenerate states, and, on the other hand, it removes the degeneracy by introducing the level repulsion. While the first effect can be treated within the perturbation expansion, the second one is essentially non-perturbative. These two effects act in the ”opposite directions”. The first one tends to make $J(0) \neq 0$ even though originally it might be that $J(0) = 0$. The level repulsion removes the degeneracy, and should result in $J(0) = 0$. Thus,
whether the OC does or does not occur depends on how the level repulsion modifies the spectral weight in the limit \( \omega \to 0 \).

The level repulsion is presently extensively discussed by the Random Matrix Theory (RMT) \cite{33}. It has been shown \cite{33} that, due to the level repulsion, the probability \( P(\omega) = \langle \delta(\omega - \omega_{mn}) \rangle \) to find two eigenenergies \( E_m \) and \( E_n \) separated by the energy "distance" \( \omega \) exhibits the universal behavior \( P(\omega) \sim \omega^{s} \) for \( \omega \to 0 \), with \( s = 1, 2, 4 \) depending on the general symmetry structure of the theory \cite{34}. Recently, in Ref. \cite{37} this feature has been explored to predict a dramatic change in the dissipation in a chaotic system in the presence of a magnetic field.

We, however, note that it is possible to give simple physical arguments which justify the irrelevance of the level repulsion in the case when the external bath is infinite, while the condensate is finite. Indeed, the effect of the level repulsion becomes significant when the energy difference between two states is comparable (or less) with a typical value of the interaction matrix elements \( (H_{\text{int}})^{(0)}_{mn} \) linking these states. Then, some energy splitting \( \Delta \sim |(H_{\text{int}})^{(0)}_{mn}| \) will occur. Otherwise, no level repulsion should be taken into account. In an infinite bath characterized by the volume \( V_{b} \to \infty \), the largest matrix elements must be scaled as \( (H_{\text{int}})^{(0)}_{mn} \sim V_{b}^{-\alpha} \), where \( \alpha > 0 \) is some power determined from the requirement that the energy of the bath is an extensive quantity. Thus, \( \Delta \to V_{b}^{-\alpha} \to 0 \), and the level repulsion can be practically neglected because the physically relevant quantities, which describe the condensate, cannot depend on \( V_{b} \to \infty \). Specifically, the time \( t_{\text{OC}} \), discussed in Sec.III, should be finite in this limit. In this case, no level repulsion should be taken into account, and the overlap \( \chi(t) \) \eqref{overlap} becomes strongly suppressed as \( \chi(t) \sim \exp(-t/t_{\text{OC}}) \to 0 \) as long as \( t_{\text{OC}} < t \ll \Delta^{-1} \). We will refer to such a situation as the strong OC.

In the case of the strong OC, the spectral function can be modeled as \( J(\omega) = J_0 = \text{const} \) up to some high energy cut-off \( \omega_c \). The constant \( J_0 \) can be restored from the normalization condition \( \int d\omega J(\omega) = \langle (\delta H)^2 \rangle \) obtained from Eq.\eqref{overlap}. Thus, \( J_0 \approx \omega_c^{-1} \langle (\delta H)^2 \rangle \), and, in accordance with the previous discussion, the time scale \( t_{\text{OC}} \approx 1/J_0 \), during which the OC actually occurs can be found as

\[
t_{\text{OC}} = \frac{\omega_c}{\langle (\delta H)^2 \rangle} \ll \Delta^{-1},
\]

where the mean can be calculated in the zeroth order with respect to \( H_{\text{int}} \).

Thus, a contact of a finite condensate with an infinite interacting bath should practically always result in the strong OC.

### C. Absence of the strong OC in the condensate contacting no external bath. The Weak OC.

Now let us consider the case of the intrinsic bath, that is, the normal component confined together with the condensate (for a moment, we neglect the possibility that the dephasing effect may suppress the correlator \eqref{overlap} much before the OC occurs). In this case, the level repulsion plays an important role.

In general, if the operator \( \delta H \) entering Eq.\eqref{overlap} and defined by Eqs.\eqref{level_repulsion}, \eqref{overlap} is not simply \( \sim H_{\text{int}} \), it is conceivable that \( \delta H \) links those degenerate states (with statistical significance) which are not mixed by \( H_{\text{int}} \). In such case, these states remain degenerate (up to the corresponding order). Hence, the level repulsion would become insignificant, and the OC may emerge. We note, however, that such a case looks very artificial, when no external bath is present, because \( \delta H \) is simply the derivative \eqref{overlap} of the projected Hamiltonian. In the weakly interacting systems, the projected Hamiltonian can always be expanded in powers of \( N \). This automatically implies that \( \delta H \) is statistically dependent on \( H(N) \). In other words, those states which are linked by \( \delta H \) with statistical significance are already mixed by the corresponding term in \( H(N) \), and their energies are thereby repelled from each other.

As it will be seen in the next section, the operator \( \delta H \), which determines the spectral weight \eqref{overlap}, is actually given as \( N^{-1}H_{\text{int}} \), where \( H_{\text{int}} \) describes the interaction between the normal excitations. Thus, \( H_{\text{int}} \) is responsible for turning on of the transitions between the degenerate states as well as for the level repulsion. The above arguments, which justified the irrelevance of the level repulsion when the external bath is present, are not valid anymore for the intrinsic bath because the typical splitting \( \Delta \) introduced above (see Sec.IV.B) is scaled now with \( N \) instead of the infinite volume of the external bath. The value of \( \Delta \sim |(H_{\text{int}})^{(0)}_{mn}| \) stems from the nature of interaction between the excitations. The processes responsible for this interaction are the pair scattering so that one member of the pair either enters or leaves the condensate (for references see \cite{24} as well as the following discussion in Sec.V). Thus, \( \Delta \sim 1/\sqrt{N} \) (or \( \sim 1/\sqrt{N} \) for fixed density).

It is important to note that the OC time follows from Eq.\eqref{overlap} as \( t_{\text{OC}} \sim N \), because now \( \delta H = \delta H_{\text{int}}/N \) is nothing else but the fluctuation of the interaction energy per one particle. Accordingly, the system is now in the limit \( t_{\text{OC}} \gg \Delta^{-1} \), which is the opposite to the condition for the strong OC. This implies, that the matrix elements \( (\delta H)_{mn} \) as well as the excitation energies \( \omega_{mn} \) in \eqref{overlap} experience strong renormalization caused by the level repulsion.
The above arguments imply that the mean overlap $\langle \beta | \alpha \rangle$ (or $\langle \beta | \alpha \rangle$) in the isolated trap may be sensitive only to the frequencies where the level repulsion takes place $\omega \leq \Delta$. This contrasts with the previous case of the external bath, in which the OC occurs as the strong OC, that is, far from the region where the level repulsion may become significant. Such a situation, when the OC occurs due to the integration in Eq.(3.4) over the region $\omega \leq \Delta$, will be referred to as the weak OC.

The actual time scale $t_{OC}$ of such weak OC should always be long on any practical experimental time scale. For example, if the level repulsion modifies $J(\omega)$ from $J_0 \neq 0$ to $J(\omega) \sim \omega$, the divergence of the integral in (3.4) is logarithmic. This immediately yields $t_{OC} \sim \exp(...N) \gg \Delta$, implying that $t_{OC}$ is practically infinity as long as $N \gg 1$. Nevertheless, let us discuss a possibility of the weak OC in the isolated trap.

As a matter of fact, $J(\omega)$ in (3.4) should approach zero not slower than $\sim \omega^2$ as $\omega \to 0$, because the matrix elements $(\delta H)_{mn}$ in (3.3) are strongly renormalized by the effect of the level repulsion. Indeed, if $|\tilde{n}\rangle$ is a subspace of the degenerate states corresponding to some energy $E_m^{(0)}(N)$ (with respect to the free Hamiltonian $H_0$), the new states and energies can be found by means of solving the corresponding secular problem which includes this subspace only. Thus, the matrix $(H_0)_{\tilde{n},\tilde{n}'} + (H_{int})^{(0)}_{\tilde{n},\tilde{n}'}$ must be diagonalized in the subspace of $|\tilde{n}\rangle$. The first part of this matrix is $E_m^{(0)}(N)\delta_{\tilde{n},\tilde{n}'}$ due to the degeneracy. Thus, the new spectrum and the new states follow from the diagonalization of $(H_{int})^{(0)}_{\tilde{n},\tilde{n}'}$ alone. Accordingly, the renormalized $(H_{int})^{(0)}_{\tilde{n},\tilde{n}'}$ is diagonal at the degeneracy. In fact, close to the degeneracy, $(H_0)_{mn} - E_m^{(0)}\delta_{mn}$ plays a role of the perturbation with respect to $(H_{int})^{(0)}_{mn}$. This implies that the renormalized $(\delta H)_{mn} \approx N^{-1}(H_{int})_{mn} \to 0$ as $\omega_{mn} \to 0$. The matrix elements must be smooth in the parameters, therefore $\omega_{mn}^{(0)} \to 0$ provides a natural scale for the off-diagonal values of the renormalized matrix elements $(\delta H)_{mn}$. Hence, the spectral weight $\langle \beta | \alpha \rangle$ which is $\sim |(\delta H)_{mn}|^2$ gains an extra factor $\omega^2$ in the limit $\omega \to 0$. This implies that the frequency integration in (3.4) over the region $\omega < \Delta$ does not yield any divergence. In Appendix B, we will estimate the mean overlap (3.4) by employing the two-level approximation for treating the level repulsion. The conclusion is that this overlap $\chi = 1 + \alpha(1/\sqrt{N})$.

It is worth noting that the situation with the irreversible decay of the condensate correlator is very special with respect to the picture of the decay of the normal excitations. Indeed, the decay time of the long living normal excitations scales with the typical size $L$ of the system, which is $L \approx N^{1/3}$ in 3D. It is much shorter than $\Delta^{-1} \sim \sqrt{N}$. Thus, the normal excitations decay long before the level repulsion may produce any effect. In contrast, the OC time $t_{OC}$ calculated perturbatively with respect to $H_{int}$ is $t_{OC} \sim N$, which is much longer than $\Delta^{-1} \sim \sqrt{N}$. Thus, the effect of the level repulsion is crucial for the long time dynamics of the condensate.

The above analysis shows that neither the strong OC nor the weak OC should be anticipated in the isolated trap. The results discussed above are not sensitive to particular details of the model. Nevertheless, below we will estimate explicitly the overlap $\chi$ within the hydrodynamic approximation.

To conclude this section, we emphasize that the question of the existence of the OC in a confined condensate is, as a matter of fact, irrelevant as long as no external bath is present. Indeed, the earliest time when the OC may become important is $t_{OC}$. This time turns out to be $\sim N$ (see above). On the other hand, the dephasing time $\tau_d \sim \sqrt{N}$ (see Sec.VI). Therefore, the correlator (3.10) will decay long before the OC may become relevant. The mechanism of the decay of the condensate correlator should be distinguished from that for the normal excitations, which decay in 3D long before either the OC or the dephasing may take place.

**V. THE OVERLAP IN THE HYDRODYNAMIC APPROXIMATION**

The notion of the OC [30] is well defined in the thermodynamic limit $V \to \infty$ and $N \to \infty$. Strictly speaking, the projected Hamiltonian $H(N)$ should be constructed by employing the $N$-conserving approaches [21,22,17]. As was discussed above, the occurrence of the OC is determined by the low frequency (long time) response of the excitation ensemble. Thus, in order to construct this response, it suffices to consider the Hydrodynamic Approximation (HA) which is known as correctly reproducing the behavior of the cloud at low frequencies and momenta [23]. The HA takes into account correctly the conservation of the total number of particles $N$ as well as the vertex renormalization effects. A drawback of the HA is the occurrence of divergences at large frequencies. These are, however, non-physical and can be taken care of by introducing the upper cut-off at some energy $\omega_c$ of the order of the chemical potential, and at some momentum $k_c$ of the order of the inverse healing length. On the contrary, the divergences at low momenta and frequencies are real and determine the peculiar physics of the condensate [23,24]. Thus, in what follows we will employ the HA in order to construct the projected Hamiltonian $H(N)$. 

11
A. Dephasing due to non-interacting phonon gas

We start with the classical Lagrangian

\[ L = \int d^3x \left( i\hbar \frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \Psi - U(x) + \mu - \frac{g}{2} \Psi^* \Psi \right), \quad (5.1) \]

where the interaction term with \( g = \frac{4\pi \hbar^2}{m} a > 0 \) (\( a \) is the scattering length) is taken in the contact form, with \( m \) being the atomic mass and \( U(x) \) standing for the trapping potential. The chemical potential \( \mu \) is introduced to take into account the constraint

\[ \int d^3x \Psi^* \Psi = N. \quad (5.2) \]

In the HA, the ansatz

\[ \Psi = \sqrt{n} e^{i\theta}, \quad (5.3) \]

is employed in Eqs. (5.2), (5.1), where \( n, \theta \) stand for the density and the phase, respectively. This results in the Hamiltonian

\[ H = \int d^3x \left\{ \frac{\hbar^2}{2m} n(\nabla \theta)^2 + (U - \mu)n + \frac{g}{2} n^2 \right\}, \quad (5.4) \]

where the terms containing gradients of the density, which are not relevant for the long wave dynamics, have been omitted. We note that, if viewed together with the constraint (5.2) in which \( N \) is just a \( c \)-number (not the operator), Eq. (5.4) represents the classical version of the \textit{projected} Hamiltonian \( H(N) \). Keeping this in mind, in what follows we will use the notations \( H \) and \( H(N) \) interchangeably.

In order to construct the non-interacting Hamiltonian \( H_0 \), the density is represented as

\[ n(x) = n_0(x) + n'(x), \quad (5.5) \]

where \( n_0(x) \) minimizes \( H \) (5.4), and saturates the constraint (5.2). In the Thomas-Fermi approximation (see in [24])

\[ n_0(x) = (\mu - U(x))/g, \quad (5.6) \]

with the boundary condition \( n_0(x) = 0 \) for \( U(x) \geq \mu \). Accordingly, the constraint (5.2), which becomes

\[ \int d^3x n_0(x) = N, \quad (5.7) \]

yields the size of the condensate and the value of \( \mu \). Here we neglect the possibility of topological excitations in the condensate. Thus, no winding of the phase needs to be taken into account.

In Eq. (5.5), \( n'(x) \) denotes small perturbations (phonons) of the density which do not result in any change of \( N \). Thus, \( \int d^3x n'(x) = 0 \). Then, the Hamiltonian (5.4) takes the form \( H = H_0 + H_{int} \), where the quadratic part is

\[ H_0 = \int d^3x \left\{ \frac{\hbar^2}{2m} n_0(\nabla \theta)^2 + \frac{g}{2} n^2 + \frac{g}{2} n_0^2 \right\}, \quad (5.8) \]

provided (5.6) holds, and the interaction is defined as

\[ H_{int} = \int d^3x \frac{\hbar^2}{2m} n'(\nabla \theta)^2. \quad (5.9) \]

Subsequent quantization of the Lagrangian (5.1) shows that the density \( n'(x) \) and the phase \( \theta(x) \) are the conjugate variables. The phonon contributions can be described in terms of the Fourier harmonics \( \theta_k \) and \( n'_k \) of \( \theta(x) \) and \( n'(x) \), respectively ( \( \theta(x) = \sum_k \frac{1}{\sqrt{V}} \exp(ikx) \theta_k, \quad n'(x) = \sum_k \frac{1}{\sqrt{V}} \exp(ikx) n'_k \) ), satisfying the commutation relation

\[ [\theta_k, n'_k] = -i\delta_{k,k'}. \quad (5.10) \]
Thus, one can express \( n_k' = i \frac{\Delta}{\hbar} \). The corresponding Hamiltonian is actually the projected Hamiltonian \( H(N) \), in which \( N \) plays the role of a parameter through \( n_0 \) given by Eqs. (5.6), (5.7). The eigenstates of \( H(N) \) are the projected states \( |m, N\rangle \), which can be expressed as some functionals \( \Phi_m(\{\theta_k\}, n_0) \).

For sake of simplicity, we will consider a finite cubic box with the side \( L \), and will set the trapping potential to 0, with, e.g., periodic boundary conditions. Then, \( n_0(x) \) becomes just a constant \( n_0 = N/V, \ V = L^3 \), and \( H_0 \) in Eq. (5.8) can be represented as

\[
H_0 = \sum_k \left( \frac{g}{2} \overline{\partial^2 \theta_k} - \frac{\hbar^2 k^2}{2m} \theta_k \theta_{-k} \right) + \frac{gN^2}{2V^2}, \tag{5.11}
\]

where \( \theta_k = \theta^*_k \). Taking into account Eqs. (5.8), (5.9), \( H' \) in (5.13) is given as

\[
H' = \frac{\partial H(N)}{\partial N} = \sum_k \frac{\hbar^2 k^2}{2Vm} \theta_k \theta_{-k} + \frac{gN}{V}. \tag{5.12}
\]

Here the first term represents the global kinetic energy density (notice the factor \( V \) in the denominator) of the phonon gas. This term contributes to the dephasing rate \( \tau_\tau^{-1} \) at finite temperatures. The second term, which is the mean interaction energy per one boson, does not contribute to the dephasing in the canonical ensemble. However, in case the evolution of the correlator \( \langle \chi \rangle \) is considered for the case when the initial state is a mixture of states with different \( N \) (or the averaging is performed over different realizations of \( N \)), the last term in Eq. (5.12) may contribute to the dephasing rate \( \langle \omega \rangle \). We will discuss this later with respect to the effect of the shot noise inevitably present in the destructive measurements \( \delta \).

Let us neglect for a while the last terms in Eqs. (6.11), (6.12). Then, comparison of Eq. (6.12) with Eq. (4.14) indicates that the condensate and the normal component can be viewed as the case \( g_a = 0, \omega_a = 0, g'_a \neq 0 \) of the toy model considered in Sec.IVA. Specifically, in the absence of the interaction between phonons, the projected Hamiltonian \( H(N) \) obtained above in the HA is practically the projected Hamiltonian of the form \( (\frac{3}{2}) \), with \( g_a = 0 \) and \( g'_a = \sqrt{\frac{g_a}{V}} = m \). The spectral weight \( (\frac{3}{2}) \) then follows as \( J(\omega) \approx \sum (g'_a/\Omega)^2 \frac{\delta(\omega - \Omega_a)}{\sqrt{\omega^2/\Omega_a}} \sim \omega^2 T^2/N \) at large \( T \), where Eqs. (3.4). (5.12), (5.11) have been employed. Thus, no OC occurs due to the ideal phonon gas because \( J(\omega) \sim \omega^2/N \), so that the integral in the overlap \( (\frac{3}{2}) \) does not diverge for \( \omega \to 0 \). Furthermore, due to \( J(\omega) \sim 1/N \), the overlap \( \sqrt{\frac{\Omega}{\omega}} = 1 + o(1/N) \). Consequently, the correlator \( (\frac{\chi}{\chi^*}) \) is given by the form \( (\frac{\chi^*}{\chi}) \), where the dephasing rate given by Eq. (4.14) turns out to be \( \tau_d^{-1} \sim T/\sqrt{N} \) (in Sec.VI, we will present accurate calculations of \( \tau_d^{-1} \)). The following discussion will be devoted to proving that the above simple picture is not altered practically by the interaction between phonons.

**B. Interacting phonon gas**

Now let us consider the role of the interaction between the phonons. It is convenient to remove the dependence on \( n_0 \) from \( H_0 \) \( (\frac{5.11}) \) and transfer it to the interaction part \( (\frac{5.9}) \). We employ the scaling transformation \( \theta_k = \lambda^{-1} \theta_k^0 \), with \( \lambda = (n_0/n_r)^{1/4} \) where \( n_r \) stands for some reference density which will be kept constant with respect to changing \( N \). As a result of this transformation, \( H_0 \) \( (\frac{5.11}) \) changes to \( \lambda^2 H'_0 \) (we neglect the last term in Eq. \( (\frac{5.11}) \)), where \( H'_0 \) has the form \( (\frac{5.11}) \) in which \( \theta_k \) is replaced by \( \theta_k' \) and \( n_0 \) is replaced by \( n_r \). This scaling transformation will change the interaction part \( (\frac{5.9}) \) as \( H_{int} \to \lambda^{-1} H_{int} \). Thus, the total Hamiltonian can now be represented in the new variables as

\[
H = \left( \frac{n_0}{n_r} \right)^{1/2} (H'_0 + H'_{int}), \tag{5.13}
\]

\[
H'_0 = \sum_k \left( \frac{g}{2} \overline{\partial^2 \theta_k} - \frac{\hbar^2 k^2}{2m} \theta_k \theta_{-k} \right) + \frac{gN^2}{2V^2}, \tag{5.14}
\]

\[
H'_{int} = \left( \frac{n_0}{n_r} \right)^{-3/4} \frac{\hbar^2}{2m \sqrt{V}} \sum_{k,q} \bar{\theta}'_k \theta'_k \frac{\partial}{\partial \theta'_{-k-q}} \left( \bar{\theta}'_{-k} \right). \tag{5.15}
\]

In the representation \( (\frac{5.13}-\frac{5.15}) \), the matrix elements of the operator \( \delta H \), which determines the spectral weight in Eq. \( (\frac{5.5}) \) and which is given by Eqs. (3.6), (3.3), (5.1), become
\[(\delta H)_{mn} = -\frac{3}{4N}((H'_{\text{int}})_{mn} - \delta_{mn}(H'_{\text{int}})_{mm}). \] (5.16)

It is important to notice the factor \(1/N\) in this equation.

The meaning of the scaling transformation introduced above must be discussed. The overlap \(\Theta\) is calculated between two families of the eigenstates. Specifically, the change \(N \rightarrow N - 1\) produces the change of \(\eta_0 = N/V\) as \(n_0 \rightarrow n_0 - 1/V\). Thus, the overlap is to be found between the states \(\Phi_m(\{\theta_k\}, n_0 - V^{-1})\) and \(\Phi_m(\{\theta_k\}, n_0)\). We represent the overlap \(\Theta\) explicitly as an integral

\[\Theta = \int \delta \lambda \Phi_m(\{\theta_k\}, n_0 - V^{-1})\Phi_m(\{\theta_k\}, n_0)\]

over all the harmonics \(\{\theta_k\}\). The eigenstates \(\Phi_m(\{\theta_k\}, n_0)\) can be expressed in terms of the eigenstates \(\Phi'_m(\{\theta'_k\}, n_0)\) of the new Hamiltonian \(H'_0 + H'_{\text{int}}\) (5.13)-(5.15) (where \(\theta_k = \lambda^{-1}\theta'_k\) and \(\lambda = (n_0/n_0)^{1/4}\)). Specifically, \(\Phi_m(\{\theta_k\}, n_0) = \lambda^{M/2}\Phi'_m(\lambda\{\theta_k\}, n_0)\), where \(M\) stands for the dimension of \(D\theta_k = \prod_k d\theta_k\). Then, Eq.(5.17) takes the form

\[\Theta_m = \int D\theta_k \Phi'_m(\{\theta_k\}, n_0 - V^{-1})\Phi'_m(\{\theta_k\}, n_0),\] (5.18)

where \(\delta \lambda\) is the change of \(\lambda\) as \(n_0\) changes by \(1/V\). In fact, it is enough to consider only the term lowest in \(1/N\), so that \(\delta \lambda/\lambda = -(4N)^{-1}\). Expanding (5.18) in \(1/N\) and exponentiating the result within the accuracy \(N^{-2}\), we find the logarithm of the mean overlap as

\[\ln \Theta = -\frac{M}{4(4N)^2} - B_0(T) - B_1(T) - B_2(T)\] (5.19)

where the terms linear in \(1/N\) cancel automatically, and the terms \(B_0(T), B_1(T), B_2(T)\) represent the following contributions

\[B_0(T) = \frac{1}{2} \left(\frac{1}{4N}\right)^2 \sum m p_m \int D\theta_k \sum_{k,k'} \Phi'_m(\{\theta_k\}, n_0)\theta_k\theta_{k'} \frac{\partial^2}{\partial\theta_k\partial\theta_{k'}} \Phi'_m(\{\theta_k\}, n_0),\] (5.20)

\[B_1(T) = 4n_0 \left(\frac{1}{4N}\right)^2 \sum m p_m \int D\theta_k \sum_k \Phi'_m(\{\theta_k\}, n_0)\theta_k \frac{\partial^2}{\partial n_0\partial\theta_k} \Phi'_m(\{\theta_k\}, n_0),\] (5.21)

\[B_2(T) = -8n_0^2 \left(\frac{1}{4N}\right)^2 \sum m p_m \int D\theta_k \Phi'_m(\{\theta_k\}, n_0) \frac{\partial^2}{\partial n_0^2} \Phi'_m(\{\theta_k\}, n_0).\] (5.22)

The first term in (5.14) contains the formally divergent dimension \(M\). This can be represented as \(M = \sum_k k \sim V/\int d^3k \rightarrow \infty\). The divergence of the integral \(\int d^3k\) is a general high momenta divergence of the HA, and therefore it is non-physical. The integral must be cut off from above at the inverse healing length \(k_c\). Thus, \(M \sim N\), and the first term in (5.19) behaves as \(\sim 1/N \rightarrow 0\), and therefore it can be eliminated.

The term \(B_0(T)\) represents a contribution which is finite in the absence of the interaction (5.15). Thus, it originates from the free phonons as discussed above in Sec.VA. Accordingly, elementary gaussian calculations with the free Hamiltonian (5.14) yield

\[B_0(T) = \frac{1}{2} \left(\frac{1}{4N}\right)^2 \sum k \frac{\exp(h\omega_k)}{(\exp(h\omega_k) - 1)^2},\] (5.23)

with \(h\omega_k \sim k\) being the spectrum of the normal excitations of the free Hamiltonian (5.14). The term (5.23) is \(\sim 1/N\), and it converges at small \(k\) in the 3D case. The interaction may slightly change the "free" form (5.23) without changing the main scaling dependence on \(N\). Thus, the term (5.23) can be eliminated from the consideration as well.

The term (5.21) describes a contribution which is non-singular at the degeneracy points \(\omega_{mn}\) (see the discussion in Sec.IV). The proof of this is presented in Appendix A. Thus, the term (5.21) contributes as \(\sim 1/N\) in the overlap (5.19). Consequently, we eliminate \(B_1(T)\) as well.
Finally, the term \( \chi \) is the one which is formally singular at the degeneracy points. Thus, one can represent \( \chi = \exp(-B_2(T)) \). It is possible to recognize (see the Appendix A) this form as \( \text{Eq. (5.4)} \) with the spectral weight

\[
J(\omega) = \left( \frac{3}{4N} \right)^2 \sum_{m,n \neq m} p_m |(H'_{\text{int}})_{mn}|^2 \delta(\omega - \omega_{mn}), \tag{5.24}
\]

where \( H'_{\text{int}} \) is given by \( \text{Eq. (5.15)} \). It is important to note the factor \( 1/N^2 \) in \( \text{Eq. (5.24)} \).

Above we have justified that the projected Hamiltonian \( H(N) \) as well as its space of states can be expressed in such a way that the dependence on \( N \) is included in the interaction part \( H'_{\text{int}} = H'_{\text{int}}(N) \) only. This implies that, should the OC occur, solely the interaction would be responsible for this. It is worth noting that in \( \text{Eq. (5.24)} \), the matrix elements are taken between exact eigenstates of the "primed" Hamiltonian \( H'_0 + H'_{\text{int}} \).

\( \text{Eq. (5.24)} \) contains usual HA high momenta divergences. Such divergences, which are non-physical, can be eliminated by the following procedure. Let us replace \( \chi \rightarrow \chi/\chi_{0,0} \), where \( \chi_{0,0} \) stands for the overlap of the projected ground states \( |0_N, \tilde{N} - 1 \rangle \) and \( |0_N, \tilde{N} \rangle \). It has been shown that such an overlap is essentially 1 (see the discussion below \( \text{Eq. (2.8)} \)). Thus, the renormalized overlap \( \chi \) becomes

\[
\chi = \exp\left(-\int_0^{\infty} \frac{d\omega}{2\omega^2} \frac{J(\omega, T) - J(\omega, 0)}{J(\omega, 0)}\right), \tag{5.25}
\]

where the temperature dependence is shown explicitly in \( J(\omega) = J(\omega, T) \). \( \text{Eqs. (5.24), (5.25)} \) still have divergences at high momenta, which, however, vanish at \( T = 0 \). These divergences must be cut off from above at the momentum \( k_c \approx \sqrt{\pi a_0} \).

Let us first make naive estimates of \( t_{\text{OC}} \) from \( \text{Eq. (4.8)} \). That is, we ignore the level repulsion effect, and consider the form \( \langle 4.8 \rangle \) in the lowest order with respect to the perturbation theory, where \( \delta H \) is taken from \( \text{Eq. (5.16)} \). Thus, we obtain

\[
t_{\text{OC}} \approx \frac{N^2 \omega_c}{\sum_{m,n \neq m} p_m |(H'_{\text{int}})_{mn}|^2}, \tag{5.26}
\]

where \( \text{Eq. (5.15)} \) should be employed. The denominator of \( \text{Eq. (5.26)} \) is the mean square total interaction energy between the phonons. Thus, it is scaled as \( \sim N \) as any extensive quantity, and \( \text{Eq. (5.26)} \) yields \( t_{\text{OC}} \sim N \). On the other hand, the typical matrix element \( \Delta \sim (H'_{\text{int}})_{mn}^{(0)} \) (see discussion in Sec.IVC), where \( H'_{\text{int}} \) is given by \( \text{Eq. (5.15)} \), is scaled as \( \sim 1/\sqrt{N} \). Thus, \( t_{\text{OC}} \gg \Delta^{-1} \), and the strong OC cannot occur (see discussion in Sec.IVC). We note that the dephasing time \( t_d \sim \sqrt{N} \) (see below \( \text{Eq. (6.12)} \)). Thus, regardless of whether the weak OC does or does not formally occur, the dephasing effect dominates the evolution \( \text{Eq. (5.16)} \) (because \( \tau_d \ll t_{\text{OC}} \) in the limit \( N \gg 1 \)). This implies that the question whether the OC actually occurs becomes irrelevant because the correlator \( \chi_{0,0} \) decays at times \( \geq \tau_d \) which are much shorter than the time \( t_{\text{OC}} \) of the possible OC. Nevertheless, in Appendix B, we have discussed the role of the level repulsion, and show that the OC occurs. Furthermore, in Appendix B, we will evaluate the mean overlap \( \chi \), and it will be shown that the only effect caused by the interaction between the phonons is that \( \chi = 1 + o(1/\sqrt{N}) \), while in the non-interacting phonon gas \( \chi = 1 + o(1/N) \) as it was concluded in Sec.VA.

Thus, in the case of no extrinsic bath, the decoherence of the condensate correlator is essentially given by \( \text{Eq. (5.7)} \). Below we will estimate the dephasing time for the box.

VI. THE DEPHASING TIME OF A CONFINED CONDENSATE

Here we will calculate the dephasing rate \( \tau_d^{-1} \) of the condensate time-correlator as described by \( \text{Eq. (3.7)} \). The chemical potential \( \mu_m = (H')_{mn} \), where \( H' \) is given by \( \text{Eq. (1.14)} \), consists of two parts \( \mu^{(0)} \) and \( \mu'_{mn} \). The term \( \mu^{(0)} = gN/V \) results from the last term in \( \text{Eq. (5.8), (5.11), (5.12)} \), and it is not associated with excitations. In the canonical ensemble, it does not produce any contribution to \( \tau_d^{-1} \) in \( \text{Eq. (3.8)} \). We note, however, that in the destructive measurements \( \Omega \), \( N \) will necessarily fluctuate from the realization to the realization producing the shot noise. Therefore, the term \( \mu^{(0)} \) will contribute into the dephasing rate \( \tau_d^{-1} \). To estimate this effect, we assume that the shot fluctuations are characterized by some known variance \( \Delta N \). This yields the dephasing rate due to the shot noise as

\[
(\tau_d^{(0)})^{-1} = \frac{g\Delta N}{\sqrt{2V}}. \tag{6.1}
\]
It is important to emphasize that the shot noise, which is due to the uncertainty of \( N \), does not disrupt the phase coherence in the each realization. It should not be identified with the situation when \( N \) is not conserved due to the unwanted escape (or deposition) of particles from (to) the trap during the evolution. The latter effect is a source of the white noise which erases any memory, and which leads to the exponential decay instead of the gaussian dephasing.

The term \( \mu'_m \) is due to the normal component. In order to obtain it, the spectrum of the elementary excitations \( \epsilon_k(N) \) must be found. According to the preceding analysis, the processes of scattering (elastic and inelastic) of the excitations do not contribute to the correlator \( \langle \ldots \rangle \) in the leading order \( N \gg 1 \). Thus, in order to find \( \tau_d \) in \( \text{(6.3)} \), the interaction between the excitations can be essentially ignored. This will give the energy \( E_m(N) \) as \( E_m(N) = \sum_k \epsilon_k(N)n_k \), where \( n_k \) are the corresponding population factors of the elementary excitations.

Strictly speaking, in order to find the effect (\( \tau_d^{-1} \neq 0 \)) for \( N \) finite, the \( N \)-conserving approaches \( \text{[21,22]} \) should be employed. Above we have employed the HA, which is the long wave limit of these approaches. The HA has been sufficient for estimating the mean overlap in order to justify the absence of the OC. However, here we are interested in accurate calculations of the dephasing time \( \tau_d \) from Eq.\( \text{(6.3)} \). These calculations contain the integration over the momenta which are divergent on the high end, and which therefore cannot be accurately eliminated within the HA approach. Thus, the methods \( \text{[21,22]} \) is the only known alternative. However, we note that in the limit \( N \gg 1 \), the differences vanish between the approaches \( \text{[21,22]} \) and the non-conserving \( N \) Bogolubov theory \( \text{[18]} \). This implies that, if we employ the approaches \( \text{[21,22]} \) in evaluating the elementary excitation spectrum, we will gain corrections of the order \( 1/N \) compared with the Bogolubov method. Keeping in mind that the result in \( \text{(6.3)} \) is itself \( \sim 1/N \), the corrections to the spectrum will produce terms \( \sim 1/N^2 \), which should be neglected.

Thus, in what follows we will employ the Bogolubov method \( \text{[23,24]} \), and take \( \epsilon_k \) as the standard Bogolubov spectrum (see \( \text{[23,24]} \)). We note that the form \( \text{(1.12)} \) indicates that, while changing \( N \) by 1, the number of excitations \( n_k \) must be maintained unchanged. Thus, neglecting terms \( o(1/N^2) \) in Eq.\( \text{(6.3)} \), we obtain

\[
\tau_d^{-2} = \frac{1}{2} \left( \sum_k \frac{\partial \epsilon_k}{\partial N} n_k - \left\langle \sum_k \frac{\partial \epsilon_k}{\partial N} n_k \right\rangle \right)^2. \tag{6.2}
\]

It is important to emphasize that the thermodynamic fluctuations, over which the averaging \( \langle \ldots \rangle \) is to be performed in Eq.\( \text{(6.2)} \), affect not only the term containing the population factors \( n_k \) explicitly. Indeed, as has been discussed by Giorgini et. al \( \text{[23]} \), the condensate fraction \( N_0 \) exhibits anomalous fluctuations \( \langle \delta N_0^2 \rangle \). Consequently, these fluctuations must affect the spectrum of the normal excitations \( \epsilon_k \) depending on \( N_0 \). Below we will see that the contribution due to these fluctuations can be dominant in Eq.\( \text{(6.2)} \) for large \( N \) and not very low \( T \). We represent \( \epsilon_k = \epsilon_k(N_0) = \epsilon_k(N_0 + \delta N_0) \) as an expansion in a smallness quantity \( \sqrt{\langle \delta N_0^2 \rangle}/N_0 \ll 1 \) around the mean value \( N_0 \) of \( N_0 \). This gives

\[
\frac{\partial \epsilon_k(N_0)}{\partial N} = \frac{\partial \epsilon_k(N_0)}{\partial N_0} + \frac{\partial \epsilon_k(N_0)}{\partial N_0} \frac{\partial \delta N_0}{\partial N} + \frac{\partial^2 \epsilon_k(N_0)}{\partial N_0 \partial N} \delta N_0 + o(\delta N_0^2/N_0^2). \tag{6.3}
\]

The fluctuation \( \delta N_0 \) is understood here as happening at fixed \( N \), thus one can employ \( \partial \delta N_0/\partial N = \delta \partial N_0/\partial N \). We note that \( \partial N_0/\partial N \) can be taken as \( \partial N_0/\partial N = 1 \), because the corrections due to the interaction contribute to the higher order in the gas parameter \( \xi \). Both this expansion and \( n_k = n_k + \delta n_k \), where \( n_k \) stands for the mean of \( n_k \), should be substituted into Eq.\( \text{(6.2)} \). In calculating the average, only the lowest terms in the fluctuations (\( \langle \delta n^2 \rangle \), \( \langle \delta N_0^2 \rangle \), etc.) must be considered. For large \( N \), the differences between canonical and grand canonical ensembles practically disapper with respect to the normal component. Thus, we employ \( (\langle n_k - \langle n_k \rangle \rangle)^2 = n_k(\bar{n}_k + 1) \), where

\[
\bar{n}_k = \frac{1}{\exp(\frac{\epsilon_k}{T})} - 1. \tag{6.4}
\]

We neglect the terms of the order higher than \( 1/N \) in Eq.\( \text{(6.3)} \) (e.g., \( \langle \delta N_0 n_k \rangle \)). Close examination shows that each derivative of \( \epsilon_k \) in Eq.\( \text{(6.3)} \) introduces an additional power of the gas parameter after the momentum summation in Eq.\( \text{(6.2)} \). Thus, we retain only the first two terms in the expansion \( \text{(6.3)} \). To proceed, we employ the approach \( \text{[23,24]} \) in calculating the anomalous fluctuations. Thus, we choose the Bogolubov representation \( \text{[38,24]} \) of the normal component as

\[
\psi' = \frac{1}{\sqrt{N}} \sum_k (u_k \alpha_k + v_k \alpha_k^\dagger) e^{ikx}, \tag{6.5}
\]

\[
u_k^2 + \epsilon_k^2 = \frac{(\epsilon_k^2 + g^2 n_k^2)^{1/2}}{2\epsilon_k}, \tag{6.6}
\]
\[ u_k v_k = -\frac{g n_c}{2\epsilon_k} \quad (6.7) \]

where \( \alpha_k (\alpha_k^\dagger) \) destroys (creates) a quasiparticle with the momentum \( k \); \( n_c = N_0 / V \) stands for the condensate density, and the Bogolubov spectrum is

\[ \epsilon_k = \left[ \left( \frac{\hbar^2 k^2}{2m} + g n_c \right)^2 - g^2 n_c^2 \right]^{1/2}. \quad (6.8) \]

Fluctuations of the condensate population \( \delta N_0 \) are related to the fluctuations of the normal component by means of the relation \[25\]

\[ N_0 = N - \int dx \psi^\dagger \psi. \quad (6.9) \]

Accordingly, employing Eqs. (6.3)-(6.7) and \( n_k = \alpha_k^\dagger \alpha_k \) in Eq. (6.2), we obtain

\[ \tau_d^{-2} = \tau_1^{-2} + \tau_2^{-2}, \quad (6.10) \]

where the notations are

\[ \tau_1^{-2} = \frac{1}{2} \sum_k \left( \frac{\partial \epsilon_k (N_0)}{\partial N_0} \right)^2 n_k (n_k + 1), \quad (6.11) \]
\[ \tau_2^{-2} = \frac{1}{2} \left( \sum_k \frac{\partial \epsilon_k (N_0)}{\partial N_0} n_k \right)^2 \left( \frac{\partial N_0}{\partial N} \right)^2. \quad (6.12) \]

In the term (6.12) there is the factor \( \left\langle (\delta \partial N_0 / \partial N)^2 \right\rangle \) which can be found from Eq. (6.9). The change of \( N \) by 1 does not affect the excitations accordingly to the above discussion. Hence, the quasiparticle operators in Eq. (6.5) should be taken as independent of \( N \). Thus, we find from (6.9)

\[ \frac{\partial N_0}{\partial N} = 1 - \sum_k \left( \frac{\partial}{\partial N} (u_k^2) \alpha_k^\dagger \alpha_k + \frac{\partial}{\partial N} (v_k^2) \alpha_k \alpha_k^\dagger + \frac{\partial}{\partial N} (u_k v_k) (\alpha_k^\dagger \alpha_k^\dagger - \alpha_k \alpha_k - k) \right). \quad (6.13) \]

This representation should be employed in Eq. (6.12). As discussed in Ref. [25], the main contribution into \( \left\langle \delta N_0^2 \right\rangle \) comes from small momenta where \( u_k^2 \approx v_k^2 \sim \sqrt{N_0} \). Thus, Eq. (6.13) yields \( \partial N_0 / \partial N_0 \approx N_0 / 2 \). Accordingly, \( \left\langle (\delta \partial N_0 / \partial N)^2 \right\rangle \approx (\delta N_0^2) / 4 N_0^2 \). Employing this as well as the explicit form of \( \left\langle \delta N_0^2 \right\rangle \) for the box from Ref. [25], we obtain for the dephasing rate (6.10)

\[ \tau_d^{-1} = T \left[ \frac{1.8 \epsilon^{1/2}}{N} + \frac{6.0 \epsilon^{2/3}}{N^{2/3}} \left( \frac{T}{T_c} \right)^3 \right], \quad (6.14) \]

where the limit \( T \gg \mu \) is taken. The first term under the square root is due to the ensemble fluctuations of the population factors (see Eq. (6.11)). This contribution discussed by the authors in Ref. [23] is effectively gained at the momenta close to \( k_c \). These fluctuations do not practically contribute to the fluctuations of the condensate population \( N_0 \) [25]. The second term (see Eq. (6.14)) describes the contribution of the ensemble fluctuations of \( N_0 \) determined by the low momenta collective modes [25]. It can be seen, that the two terms under the square root can be dominant in different regions of \( N \) and \( \xi \).

The result (6.14) is obtained for the box. It is clear that, apart from the numerical coefficients, a similar expression can be derived for the actual oscillator trap. Eq. (6.14) can be employed for order of magnitude estimates for the conditions of the actual trap [9]. At large \( N \), the first term behaves as \( \sim N^{-4/5} \), and the second \( \sim N^{-2/5} \) [10]. Taking values typical for the experiment [1] \( N_0 = 5 \cdot 10^6 \), \( T \approx 50 \text{mK}, a \approx 5 \cdot 10^{-7} \text{cm}, n_c \approx 10^{14} \text{cm}^{-3} \), we find \( \xi = a^3 n_c \approx 2 \cdot 10^{-5} \), and \( T_c \approx 3.3 \hbar^2 n_0^{2/3} / m \approx 500 \text{mK} \). For these values, the second term under the square root in Eq. (6.14) is much smaller than the first one due to the temperature factor. In this case we find the dephasing time \( \tau_d \approx 1 \). If, however, temperature is increased up to \( T / T_c = 0.5 \), the second term becomes dominant. In this case, the dephasing time becomes \( \tau_d \leq 100 \text{ms} \), which is within the experimental range [9]. In this estimate we assumed no shot noise due to the uncertainty of the total value \( N \). If,
however, the variance $\Delta N$ in Eq.(4.1) is large enough, the total dephasing rate $\sqrt{\tau_d^{-2} + (\tau_d^{(0)})^{-2}}$ can vary over a large range.

We note, however, that the results obtained above should not be directly applied to the experiment \[3\] in which the evolution of the relative phase of the two-component condensate has been studied. In a following publication we will modify the above analysis in order to apply it to the experiment \[3\].

VII. DISCUSSION AND SUMMARY

In this work we have applied the concept of the Orthogonality Catastrophe to a bosonic ensemble in order to treat temporal correlations of a confined Bose-Einstein condensate, consisting of a finite and fixed (albeit large) number of bosons. The occurrence of the OC turns out to be a prerequisite to the irreversible decay of such correlations. We stress that by saying "the temporal correlations of the condensate" we mean the correlator which is associated with the evolution of the global phase of a single condensate, or the relative phase between two condensates which do not exchange particles. The above analysis does not apply to the decay of the normal excitations.

We have shown that the OC occurs if some infinite interacting bath contacts a finite condensate. On the contrary, if no external bath is present, the equilibrium evolution of the condensate correlator can be treated as though the normal component is not perturbed at all by such an evolution. The correlations are dominated by the irreversible dephasing caused by the ensemble averaging over the realizations of the normal component. The effect of the level repulsion is shown to be significant in suppressing the irreversible decay at times much longer than the dephasing time.

In general, relation between the three time scales – the time of reversible dephasing $\tau_d$ induced by the ensemble fluctuations of the chemical potential; the OC time $t_{\text{OC}}$ (Sec.III); and the time $\Delta^{-1}$, during which the effect of the level repulsion takes place - determines the nature of the condensate decoherence. In the isolated trap, the case $\tau_d \ll t_{\text{OC}}$ and $t_{\text{OC}} \gg \Delta^{-1}$ is realized. Thus, the decoherence is the reversible dephasing. In the presence of an external bath, $t_{\text{OC}} \ll \Delta^{-1}$, and the strong OC develops (the level repulsion is insignificant). In this case, the reversible dephasing can be observed if $\tau_d < t_{\text{OC}}$. If, however, $\tau_d > t_{\text{OC}}$, the decoherence of the equilibrium condensate correlator is completely irreversible. Realization of either case depends essentially on the nature of the external bath and its interaction with the condensate. The strong OC with short $t_{\text{OC}}$ deserves special attention because in this case the nature of the low energy excitations may change dramatically \[24\].

Our results should be compared with those obtained by Graham in Refs. \[7,8\]. We note that in the phenomenological approach of Refs. \[7,8\], the classical field $\langle \Psi(x, t) \rangle$ has been treated as a dynamical variable coupled to the reservoir of the normal excitations. A corresponding evolution equation with the white noise due to the normal component has been postulated. The conclusion has been given that a major reason for the decay of the field $\langle \Psi(x, t) \rangle$ is the irreversible developing at times longer than some decay time containing a factor larger than $\sqrt{N}$ (see, e.g., Eqs.(7.48) - (7.50) of Ref. \[8\]). In this regard we note that, the assumption that $\langle \Psi(x, t) \rangle \neq 0$ is a classical variable which satisfies classical equations of motion inevitably violates the conservation of $N$ on the level of an uncertainty $\approx \sqrt{N}$. Thus, the classical Bogolubov treatment is accurate within $1/\sqrt{N}$. This does not introduce any problem unless the long time limit is considered (times $\sim \sqrt{N}$ and longer) \[4,15–17\]. Thus, we consider the results of Refs. \[7,8\] as based on an unjustified use of the Gross-Pitaevskii equation as well as of the Bogolubov approximation in the domain where these approaches should not be employed. In contrast, our treatment takes into account the conservation of $N$ explicitly, and the Bogolubov approximation (in Sec.VI) was employed only for calculating the dephasing rate which itself has a prefactor $1/\sqrt{N}$, so that the corrections $\sim 1/N$ due to the Bogolubov approximation are insignificant.

In future work we will extend our results to the case of the specific situation realized by the JILA group \[9\]. We will also discuss how the reversibility of the dephasing can be revealed in the echo-type experiments \[29\]. It is also worth considering the possibility of creating a situation in which a confined equilibrium condensate contacts some external bath inducing fast irreversible decay of the condensate correlator \[33\].

ACKNOWLEDGMENTS

One of the authors (A.K.) is grateful to E. Cornell for useful discussion of the possibilities of observing the echo effect. A.K. is also thankful to D.Schmeltzer and A.Ruckenstein for reading the manuscript during its preparation and for stimulating conversations. We also acknowledge support from a CUNY collaborative grant.
APPENDIX A: TERMS $B_1(T)$ AND $B_2(T)$ IN Eqs. (5.21), (5.22)

The terms $B_1(T)$ (5.21) and $B_2(T)$ (5.22) can be written as

$$B_1(T) = -\frac{2n_0}{(4N)^2} \sum_m p_m \sum_k \langle \tilde{N}, m| (a_k^2 - a_k^{\dagger 2}) \frac{\partial}{\partial n_0} |\tilde{N}, m\rangle,$$

and

$$B_2(T) = -\frac{8n_0^2}{(4N)^2} \sum_m p_m \langle \tilde{N}, m| \frac{\partial^2}{\partial n_0^2} |\tilde{N}, m\rangle,$$

because $\Phi'_m(\{\theta_k\}, n_0)$ are the projected states $|\tilde{N}, m\rangle$. In Eq. (A1) we employed the standard representation

$$\theta_k = \frac{a_k + a_k^{\dagger}}{\sqrt{2z_k}}, \quad \frac{\partial}{\partial \theta_k} = (a_k - a_k^{\dagger}) \sqrt{z_k} \quad \text{and} \quad z_k = \sqrt{\frac{\hbar^2 k^2 n_r}{m g}}$$

in terms of the phonon annihilation and creation operators $a_k$ and $a_k^{\dagger}$, respectively. These operators diagonalize the free part (5.14) of the projected Hamiltonian (5.13)-(5.15), so that $H'_0 = \sum_k \epsilon_k a_k^{\dagger} a_k$, with the bare spectrum defined as $\epsilon_k = c_0 k, c_0 = \sqrt{\hbar^2 n_r g/m}$.

The matrix elements $\langle \tilde{N}, m| \frac{\partial}{\partial n_0} |\tilde{N}, m\rangle$ can be found by differentiating the relation $H(N)|\tilde{N}, m\rangle = E_m(N)|\tilde{N}, m\rangle$ with respect to $n_0$ (note that $n_0 = N/V$ is the only combination which contains the dependence on $N$). Then, we find

$$\langle \tilde{N}, m| \frac{\partial}{\partial n_0} |\tilde{N}, m\rangle = -\frac{3}{4n_0} \frac{(H'_\text{int})_{m'm}}{E_m(N) - E_{m'}(N)}$$

for $m \neq m'$ and zero otherwise. In here we employed an explicit form (5.15). We note that (A4) is exact. Then, (A1) can be represented as

$$B_1(T) = \frac{3}{2(4N)^2} \sum_k \sum_{m,m'} p_m (a_k^2 - a_k^{\dagger 2})_{mm'} \frac{(H'_\text{int})_{m'm}}{E_m(N) - E_{m'}(N)}.$$  

The expression (A5) can be evaluated by employing the perturbation theory with respect to $H'_\text{int}$ (5.15). We note that in the lowest order, $B_1(T) = 0$ because the zeroth order intermediate states $|\tilde{N}, m\rangle$ differ from the state $|\tilde{N}, m\rangle$ by two phonons, while $(H'_\text{int})_{m'm}$ (5.13) links those states which differ by either one or three phonons. The higher order terms do make $B_1(T) \neq 0$. However, the dominator $E_m(N) - E_{m'}(N)$ in this expression may lead only to the first order pole which does not lead to the divergence $\sim 1/\omega^2$ in the overlap (3.4). Thus, the term $B_1(T) \sim 1/N$.

The term $B_2(T)$ in (A2) can be rewritten by employing the relation $\langle \tilde{N}, m\rangle \frac{\partial^2}{\partial n_0^2} |\tilde{N}, m\rangle = -(\frac{\partial}{\partial n_0} |\tilde{N}, m\rangle) (\frac{\partial}{\partial n_0} |\tilde{N}, m\rangle)$ as well as Eq.(A4). Finally, we find

$$B_2(T) = \frac{1}{2} \left( \frac{3}{4N} \right)^2 \sum_{m,m'} p_m \frac{|(H'_\text{int})_{m'm}|^2}{(E_m(N) - E_{m'}(N))^2}.$$  

This expression coincides with $\frac{1}{2} \int d\omega J(\omega)/\omega^2$ where $J(\omega)$ is given by Eq. (5.24).

APPENDIX B: LEVEL REPULSION AND ABSENCE OF THE OC IN AN ISOLATED TRAP

We evaluate the typical matrix element $\Delta \approx |(H'_\text{int})_{mn}|$, where the index $(0)$ indicates that the matrix elements are taken in zeroth order with respect to $H'_\text{int}$ (5.15). At high $T$ one can estimate $\theta_k \sim \sqrt{mT/\hbar^2 k^2 n_r}$ and $\partial\ldots/\partial \theta_k \sim \sqrt{T/g}$ from (5.14). Accordingly, we arrive at the estimate of $\Delta$ from Eqs. (5.13)-(5.13) as

$$\Delta \approx \frac{T}{\sqrt{N}} \left( \frac{T}{\mu} \right)^{1/2}. $$
This value sets up an energy scale below which the level repulsion becomes significant. Here the factor $1/\sqrt{N}$ indicates that no the strong OC occurs. Indeed, as discussed in Sec.IVB, the necessary condition for the strong OC is that the OC time $t_{OC}$ \[ \frac{\Delta}{c} \] (calculated by means of the perturbation expansion) is shorter than $\Delta^{-1}$. Eq.\[ \frac{\Delta}{c} \] yields $t_{OC} \sim N$, implying that for large enough $N$, the opposite limit holds $t_{OC} \gg \Delta^{-1}$. Hence, the OC in the isolated trap may occur only as the weak OC (see Sec.IVC).

The relevant frequency scale, at which the effect of the level repulsion dominates, is given by

$$\omega \leq \Delta.$$ \[ \text{(B2)} \]

At such frequencies the level repulsion effect strongly modifies the spectral weight \[ \frac{\Delta}{c} \] with respect to the perturbation expansion. As discussed in Sec.IVC, a strong renormalization of the matrix elements takes place. This effect is non-perturbative. This can be understood within the effective two-states model, in which a pair of the degenerate states $|m,N\rangle^{0}$ and $|n,N\rangle^{0}$ defined with respect to the free Hamiltonian $H'_{0}$ \[ \frac{\Delta}{c} \] is considered independently from the rest of the states.

In order to justify this two-state approximation we note that the degeneracy contributing to the spectral weight $J(\omega)$ \[ \frac{\Delta}{c} \] at $\omega \rightarrow 0$ in the lowest order of the perturbation expansion is associated with the processes of the decay and the recombination of two phonons. Given an initial phonon with some momentum $k$, the final states contain two phonons with the momenta $p$ and $q$, which obey the conservation condition $k = p + q$. The subspace of the final states, which may become degenerate with the initial state, is characterized by different $p$ and $q$. Only those final states, which are close enough in energy to the initial state, participate practically in the renormalization caused by the level repulsion. A natural criterion for the closeness is given by the typical value of the matrix element $\Delta$ \[ \frac{\Delta}{c} \]. Let us assume that some pair $p$ and $q$ satisfies this condition. Then, in a finite system, another closest final state will be characterized by the momenta differed by the amount $\sim 1/L \rightarrow 0$, where $L$ is a typical size of the system.

Accordingly, their energies should also differ by some amount $\sim 1/L \rightarrow 0$. The value of $\Delta$ \[ \frac{\Delta}{c} \] is scaled, however, as $\sim 1/L^{3/2} \ll 1/L$. Thus, for large enough system, such a closest state turns out to be out of the range of the level repulsion. These considerations justify that only pairs of the degenerate states participate practically in the level repulsion.

Then, the matrix element $(H'_{int})_{mn}^{0}$, which is defined with respect to the bare states $|m,N\rangle^{0}$ and $|n,N\rangle^{0}$, mixes these states (as long as $|\omega_{mn}| \leq \Delta$). The new states $|m,N\rangle$ and $|n,N\rangle$ are the linear combinations of the bare states $|m,N\rangle^{0}$ and $|n,N\rangle^{0}$. Then, the standard treatment of the two-level system close to the degeneracy shows that the renormalized matrix element $(H'_{int})_{mn}$ and the energy difference $\omega_{mn}$ become

$$(H'_{int})_{mn} = \frac{\omega_{mn}}{\omega_{mn}^{0}}(H'_{int})_{mn}^{0}, \quad \omega_{mn} = \pm \sqrt{|(H'_{int})_{mn}^{0}|^{2} + (\omega_{mn}^{0})^{2}}.$$ \[ \text{(B3)} \]

Thus, $(H'_{int})_{mn} \sim \omega_{mn}^{0} \rightarrow 0$ for $|\omega_{mn}^{0}| \ll \Delta$. This feature alone leads to a strong suppression of $J(\omega)$ \[ \frac{\Delta}{c} \] in the range \[ \frac{\Delta}{c} \].

Let us estimate the mean overlap $\overline{X} \frac{\Delta}{c}$ where the $J(\omega)$ is given by Eq.\[ \frac{\Delta}{c} \]. We introduce a distribution function of the bare matrix elements $|(H_{int})_{mn}^{0}|$ and of the excitation energies $|\omega_{mn}^{0}|$:

$$P(M,\omega_{0}) = \sum_{m, n \neq m} p_{m} \delta(M - |(H_{int})_{mn}^{0}|)\delta(\omega_{0} - |\omega_{mn}^{0}|),$$ \[ \text{(B4)} \]

so that the spectral weight \[ \frac{\Delta}{c} \] can be represented as

$$J(\omega) = \left( \frac{3}{4N} \right)^{2} \int_{0}^{\infty} dM \int_{0}^{\infty} d\omega_{0} P(M,\omega_{0}) \frac{\omega_{0}}{\omega} M^{2} \delta(\omega - \sqrt{M^{2} + \omega_{0}^{2}}),$$ \[ \text{(B5)} \]

where we have employed Eq.\[ \frac{\Delta}{c} \].

We assume that, if no level repulsion is taken into account (that is, the matrix elements as well as the excitation energies are taken unrenormalized), the spectral weight \[ \frac{\Delta}{c} \] can be well taken as a constant. This would imply the OC. A simplest form of $P(M,\omega_{0})$, which will produce such a spectral weight in Eq.\[ \frac{\Delta}{c} \], if the level repulsion were ignored, is

$$P(M,\omega_{0}) = P_{0} \exp\left(-\frac{M}{\Delta} - \frac{\omega_{0}}{\omega_{c}}\right),$$ \[ \text{(B6)} \]
where $P_0$ is some normalization constant which can be found from the sum rule

$$
\int d\omega J(\omega) = (3/4N)^2 \langle (H_{int}')^2 \rangle \tag{B7}
$$

following from Eq.(5.24). The form (B4) indicates that the distribution of the matrix elements and the excitation frequencies is uniform, as long as the matrix elements are less than some typical element $\Delta$ (B11) and the excitation frequencies are below the high energy cut-off $\omega_c$. Employing (B6), the unrenormalized spectral weight acquires the form

$$
J^{(0)}(\omega) = \left( \frac{3}{4N} \right)^2 P_0 \int_0^\infty dM \int_0^\infty d\omega_0 \ e^{-\frac{\omega_0}{\omega_c} - \frac{\omega}{\pi \Delta}} \ M^2 \delta(\omega - \omega_0) = 2 \left( \frac{3}{4N} \right)^2 \Delta^3 P_0 e^{-\frac{\omega}{\pi \Delta}}. \tag{B8}
$$

It satisfies $J(0) \neq 0$. In order to obtain $P_0$, it is enough to employ this unrenormalized form in Eq.(B7), because the renormalization (B5) becomes important only for $\omega \leq \Delta$, where $\Delta$ in Eq.(B1) is small as $\sim 1/\sqrt{N}$. Then, a substitution of (B8) into (B7) yields

$$
P_0 = \frac{1}{2\omega_c \Delta^3 \langle (H_{int}')^2 \rangle}. \tag{B9}
$$

This can now be employed in Eq.(B6), and then in Eq.(B7). This yields the renormalized $J(\omega)$ (B5) as

$$
J(\omega) = \left( \frac{3}{4N} \right)^2 \langle (H_{int}')^2 \rangle \left( \frac{\omega}{\Delta} \right)^3 \int_0^1 dx x^2 \sqrt{1 - x^2} e^{-\omega x / \Delta}, \tag{B10}
$$

where the condition $\omega \ll \omega_c$ is taken into account. Thus, $J(\omega) \sim \omega^3/N$ for $\omega \ll \Delta$, and $J(\omega) \approx \text{const} \sim 1/N$ for $\Delta \ll \omega \ll \omega_c$. Such a behavior implies that the OC is eliminated due to the level repulsion. This can be seen directly after employing Eq.(B10) in Eq.(5.23) for the mean overlap. We find

$$
\ln \chi = -\frac{9\pi \langle (H_{int}')^2 \rangle}{2^8 N^2 \omega_c \Delta}. \tag{B11}
$$

A quick glance on Eq.(B11) shows that $\ln \chi \sim -1/\sqrt{N} \to 0$. Indeed, $\langle (H_{int}')^2 \rangle$ is the mean square interaction energy $\sim N$. The value of $\Delta$ is $\sim 1/\sqrt{N}$ (B1). Thus, due to the $N^2$ in the denominator, the total factor $1/\sqrt{N}$ follows. Accordingly, $\chi = 1 + o(1/\sqrt{N})$ implying no the OC.

[1] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Science 269, 198 (1995); K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995); C.C. Bradley, C.A. Sackett, J.J. Tollett, and R.G. Hulet, Phys. Rev. Lett. 75, 1687 (1995).
[2] A.Leggett, F.Sols, Foundations of Phys. 21,353 (1991).
[3] F. Sol, Physica B 194-196 1389 (1994).
[4] E.M. Wright, D.F. Walls, J.C. Garrison, Phys. Rev. Lett. 77, 2158(1996).
[5] M. Lewenstein and L. You, Phys. Rev. Lett 77, 3489 (1996).
[6] J. Javanainen, M. Wilkens, Phys. Rev. Lett 78, 4675 (1997); A.J. Leggett, F. Sols, Phys. Rev. Lett 81, 1344 (1998)
[7] R. Graham, Phys. Rev. Lett 81, 5262 (1998).
[8] R. Graham, cond-mat/0005107.
[9] D.S. Hall, M.R. Matthews, C.E. Wieman, E.A. Cornell, Phys. Rev. Lett 81, 1543 (1998).
[10] A.B. Kuklov, B.V. Svistunov, Phys. Rev. 60, R769 (1999).
[11] C.N. Yang, Rev. Mod. Phys. 34, 694 (1962).
[12] C. J. Pethick, L.P. Pitaevskii, cond-mat/0004187 (2000).
[13] For example, consider a situation when $\tilde{N}$ is fixed, so that no broken gauge symmetry can strictly be introduced.
[14] such a time at zero temperature is approximately determined as $\sim (\tilde{N}/6NE_{in})$, where $\delta N$ stands for the variance of $N$ in the coherent state, and $E_{in}$ is a typical interaction energy per boson [4].
[15] A.B. Kuklov, N. Chencinski, A.M. Levine, W.M. Schreiber, and J.L. Birman, Phys.Rev. A 55, 488 (1997).
[16] L.P. Pitaevskii, Phys. Lett. A 229, 406 (1997).
[17] Y. Castin, R. Dum, Phys. Rev. A **57**, 3008 (1998).
[18] Yu. Kagan, V.A. Kashurnikov, A.V. Krasavin, N.V. Prokof’ev, B.V. Svistunov, Phys.Rev. A **61**, 043608 (2000).
[19] Yu. Kagan, N.V. Prokof’ev, B.V. Svistunov, Phys.Rev. A **61**, 045601 (2000).
[20] T.-L. Ho, S. K. Yip, Phys. Rev. Lett **84**, 4031 (2000).
[21] M. Girardeau, R. Arnowitt, Phys. Rev. **113**, 755 (1959); M.D. Girardeau, Phys. Rev. A **58**, 775 (1998);
[22] C.W. Gardiner, Phys.Rev. A **56**, 1414 (1997).
[23] S. Giorgini, L.P. Pitaevskii, S. Stringari, Phys. Rev. Lett. **46**, 6374 (1992).
[24] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, S. Stringari, Rev.Mod.Phys. **71**, 463 (1999).
[25] S. Giorgini, L.P. Pitaevskii, S. Stringari, Phys. Rev. Lett **80**, 5040 (1998).
[26] S.T. Beliaev, JETP **34**, 417 (1958).
[27] E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics, Part 2* (Pergamon, Oxford, 1981), Ch. III, Sec.26.
[28] Such a dephasing should be contrasted with the quantum phase diffusion [4] which refers to the decay of a single coherent state built on the infinite number of the Fock states.
[29] R. Walser, J. Cooper, M. Holland, cond-mat/0004257.
[30] G.D. Mahan, *Many-Particle Physics*, (Plenum Press, New York and London, 1993), Ch.8, Sec.8.3.
[31] Such 1/N expansion should not be identified with the 1/N expansion employed in Ref. [17].
[32] R.P. Feynman, F.L. Vernon, Ann. Phys. **24**, 118 (1963).
[33] A.J. Leggett, S. Chakravarty, A.T. Dorsey, M.P.A. Fisher, A. Garg, W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987).
[34] As pointed out by A. Ruckenstein, it would be interesting to find an example of the system where tOC is short enough to be comparable with typical inverse frequencies of the elementary excitations. In this case, the low energy elementary excitations become ill defined even in the case of the weakly interacting bosons.
[35] J. Anglin, Phys. Rev. Lett **76**, 6 (1997).
[36] M. L. Mehta, *Random Matrices*, Academic Press, Boston (1991); T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981).
[37] O.M. Auslaender, S. Fishman, Phys. Rev. Lett **84**, 1886 (2000).
[38] See in Ref. [2], Ch.III, Sec.25.
[39] A.B. Kuklov, J.L. Birman, DAMOP Meeting, ABS S4 7, June 14-17, 2000.
[40] G. Baym, C. Pethick, Phys. Rev. Lett **76**, 6 (1996);