The Two-Spectra Inverse Problem for Semi-Infinite Jacobi Matrices in The Limit-Circle Case

Luis O. Silva and Ricardo Weder

Departamento de Métodos Matemáticos y Numéricos
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas
Universidad Nacional Autónoma de México
C.P. 04510, México D.F.
silva@leibniz.iimas.unam.mx
weder@servidor.unam.mx
September 13, 2008

Abstract

We present a technique for reconstructing a semi-infinite Jacobi operator in the limit circle case from the spectra of two different self-adjoint extensions. Moreover, we give necessary and sufficient conditions for two real sequences to be the spectra of two different self-adjoint extensions of a Jacobi operator in the limit circle case.

*Mathematics Subject Classification(2000): 47B36, 49N45, 81Q10, 47A75, 47B37, 47B39
†Keywords: Jacobi matrices; Two-spectra inverse problem; Limit circle case
‡Research partially supported by CONACYT under Project P42553F.
§Fellow Sistema Nacional de Investigadores.
1. Introduction

Let $l_{fin}(\mathbb{N})$ be the linear space of sequences with a finite number of non-zero elements. In the Hilbert space $l_2(\mathbb{N})$, consider the operator $J$ defined for every $f = \{f_k\}_{k=1}^\infty$ in $l_{fin}(\mathbb{N})$ by means of the recurrence relation

$$
(Jf)_k := b_{k-1}f_{k-1} + q_kf_k + b_kf_{k+1}, \quad k \in \mathbb{N} \setminus \{1\},
$$

$$
(Jf)_1 := q_1f_1 + b_1f_2,
$$

where, for $n \in \mathbb{N}$, $b_n$ is positive and $q_n$ is real. Clearly, $J$ is symmetric since it is densely defined and Hermitian due to (1.1) and (1.2). Thus $J$ is closable and henceforth we shall consider the closure of $J$ and denote it by the same letter.

We have defined the operator $J$ so that the semi-infinite Jacobi matrix

$$
\begin{pmatrix}
q_1 & b_1 & 0 & 0 & \cdots \\
b_1 & q_2 & b_2 & 0 & \cdots \\
0 & b_2 & q_3 & b_3 & \\
0 & 0 & b_3 & q_4 & \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}
$$

(1.3)

is its matrix representation with respect to the canonical basis $\{e_n\}_{n=1}^\infty$ in $l_2(\mathbb{N})$ (see [3, Sec. 47] for the definition of the matrix representation of an unbounded symmetric operator). Indeed, $J$ is the minimal closed symmetric operator satisfying

$$
(Je_n, e_n) = q_n, \quad (Je_n, e_{n+1}) = (Je_{n+1}, e_n) = b_n, \quad \forall n \in \mathbb{N}.
$$

We shall refer to $J$ as the Jacobi operator and to (1.3) as its associated matrix. It is well known that $J$ can have either $(1,1)$ or $(0,0)$ as its deficiency indices [2, Sec. 1.2 Chap. 4], [28, Cor. 2.9]. By our definition, $J$ is closed, so the case $(0,0)$ corresponds to $J = J^*$, while $(1,1)$ implies that $J$ is a non-trivial restriction of $J^*$. The latter operator is always defined on the maximal domain in which the action of the matrix (1.3) make sense [3, Sec. 47].

Throughout this paper we assume that $J$ has deficiency indices $(1,1)$. Jacobi operators of this kind are referred as being in the limit circle case and the moment problem associated with the Jacobi matrix is said to be indeterminate [2, 28]. In the limit circle case, all self-adjoint extensions of a Jacobi operator have discrete spectrum [28, Thm. 4.11]. The set of all self-adjoint extensions of a Jacobi operator can be characterized as a one parameter family of operators (see below).

In this paper we show that a Jacobi matrix can be recovered uniquely from the spectra of two different self-adjoint extensions of the Jacobi operator $J$ corresponding to that matrix. These spectra also determine the parameters that define the self-adjoint extensions of $J$ for which they are the spectra. Our proof is constructive and it gives a method for the unique reconstruction. The uniqueness of this reconstruction has been already announced in [12] without proof.
We also give necessary and sufficient conditions for two sequences to be the spectra of two self-adjoint extensions of a Jacobi operator in the limit circle case. This is a complete characterization of the spectral data for the two-spectra inverse problem of a Jacobi operator in the limit circle case.

It is worth remarking that the formulation of the inverse spectral problem that we consider here differs from the usual setting for this kind of problems. Usually one reconstructs a certain self-adjoint operator from the spectra of two different rank-one self-adjoint perturbations of the operator to be reconstructed. This is the case of recovering the potential of a Schrödinger differential expression in $L^2(0, \infty)$, being regular at the origin and limit point at $\infty$, from the spectra of two operators defined by the differential expression with two different self-adjoint boundary conditions at the origin [1, 5, 9, 14, 16, 20, 22, 25, 26]. The analogous inverse problem of recovering a Jacobi matrix from the spectra of rank one self-adjoint perturbations was studied in [8, 11, 18, 19, 29, 32]. A complete characterization of the spectral data for this two-spectra inverse problem is given in [27]. Note that in the formulation of the inverse problem studied in the present paper the aim is to recover a non-self-adjoint operator from the spectra of its self-adjoint extensions, as well as the parameters that characterize the self-adjoint extensions. Results for the setting studied here have been obtained in [13, 21] for Sturm-Liouville operators and in [12] for Jacobi matrices.

Let us make a final remark concerning the two different inverse spectral problems mentioned above. Self-adjoint extensions of symmetric operators with deficiency indices $(1,1)$ can be treated within the rank-one perturbation theory (cf. [6, Sec. 1.1–1.3] and, in particular, [6, Thm. 1.3.3]). Thus, both settings may be regarded as particular cases of a general two-spectra inverse problem. A consideration similar to this is behind the treatment of inverse problems in [10]. For Jacobi operators, however, the type of rank-one perturbations in the referred formulations of the inverse spectral problem are different [6]. This difference is evident when comparing the results here with the ones in [27].

2. Preliminaries

The spectral analysis of $J$ may be carried out by studying the following second order difference system

$$b_{n-1}f_{n-1} + q_nf_n + b_nf_{n+1} = \zeta f_n, \quad n > 1, \quad \zeta \in \mathbb{C},$$

(2.1)

with the “boundary condition”

$$q_1f_1 + b_1f_2 = \zeta f_1.$$  

(2.2)

If one sets $f_1 = 1$, then $f_2$ is completely determined by (2.2). Having $f_1$ and $f_2$, equation (2.1) gives all the other elements of a sequence $\{f_n\}_{n=1}^{\infty}$ that formally satisfies (2.1) and (2.2). Clearly, $f_n$ is a polynomial of $\zeta$ of degree $n - 1$, so we
denote \( f_n =: P_{n-1}(\zeta) \). The polynomials \( P_n(\zeta), n = 0, 1, 2, \ldots \), are referred to as the polynomials of the first kind associated with the matrix \((1.3)\) [2, Sec. 2.1 Chap. 1].

The sequence \( P(\zeta) := \{P_{k-1}(\zeta)\}_{k=1}^\infty \) is not in \( l_{fin}(\mathbb{N}) \), but it may happen that

\[
\sum_{k=0}^{\infty} |P_k(\zeta)|^2 < \infty ,
\]

in which case \( P(\zeta) \in \text{Ker}(J^* - \zeta I) \).

The polynomials of the second kind \( Q(\zeta) := \{Q_{k-1}(\zeta)\}_{k=1}^\infty \) associated with the matrix \((1.3)\) are defined as the solutions of

\[
b_{n-1} f_{n-1} + q_n f_n + b_n f_{n+1} = \zeta f_n , \quad n \in \mathbb{N} \setminus \{1\} ,
\]

under the assumption that \( f_1 = 0 \) and \( f_2 = b_1^{-1} \). Then

\[
Q_{n-1}(\zeta) := f_n , \quad \forall n \in \mathbb{N} .
\]

\( Q_n(\zeta) \) is a polynomial of degree \( n - 1 \).

As pointed out in the introduction, \( J \) has either deficiency indices \((1,1)\) or \((0,0)\) [2, Sec. 1.2 Chap. 4] and [28, Cor. 2.9]. These cases correspond to the limit circle and limit point case, respectively. In terms of the polynomials of the first kind, \( J \) has deficiency indices \((0,0)\) if for one \( \zeta \in \mathbb{C} \setminus \mathbb{R} \) the series in (2.3) diverges. In the limit circle case (2.3) holds for every \( \zeta \in \mathbb{C} \) [2, Thm. 1.3.2], [28, Thm. 3] and, therefore, \( P(\zeta) \) is always in \( \text{Ker}(J^* - \zeta I) \). Another peculiarity of the limit circle case is that every self-adjoint extension of \( J \) has purely discrete spectrum [28, Thm. 4.11]. Moreover, the resolvent of every self-adjoint extension is a Hilbert-Schmidt operator [30, Lem. 2.19].

In what follows we always consider \( J \) to have deficiency indices \((1,1)\). Additionally, all self-adjoint extensions of \( J \) are assumed to be restrictions of \( J^* \). When dealing with all self-adjoint extensions of \( J \), including those which imply an extension of the original Hilbert space, the self-adjoint restrictions of \( J^* \) are called von Neumann self-adjoint extensions of \( J \) (cf. [3, Appendix I], [28, Sec. 6]).

There is also a well known result for \( J \) in the limit circle case, namely, that \( J \) is simple [2, Thm. 4.2.4]. In its turn this imply that the eigenvalues of any self-adjoint extension of \( J \) have multiplicity one [3, Thm.3 Sec. 81].

Let us now introduce a convenient way of parametrizing the self-adjoint extensions of \( J \) in the non-self-adjoint case. We first define the Wronskian associated with \( J \) for any pair of sequences \( \varphi = \{\varphi_k\}_{k=1}^\infty \) and \( \psi = \{\psi_k\}_{k=1}^\infty \) in \( l_2(\mathbb{N}) \) as follows

\[
W_k(\varphi, \psi) := b_k(\varphi_k \psi_{k+1} - \psi_k \varphi_{k+1}) , \quad k \in \mathbb{N} .
\]

Now, consider the sequences \( v(g) = \{v_k(g)\}_{k=1}^\infty \) such that, for \( k \in \mathbb{N} \),

\[
v_k(g) := P_{k-1}(0) + g Q_{k-1}(0) , \quad g \in \mathbb{R} ,
\]

(2.4)
\begin{align*}
v_k(\infty) := Q_{k-1}(0) \, . \\
\end{align*}

All the self-adjoint extensions $J(g)$ of the non-self-adjoint operator $J$ are restrictions of $J^*$ to the set \[30\text{ Lem. 2.20}\]

\[D_g := \{ f = \{ f_k \}_{k=1}^{\infty} \in \text{Dom}(J^*) : \lim_{n \to \infty} W_n(v(g), f) = 0 \} , \quad g \in \mathbb{R} \cup \{ \infty \} \, . \] (2.6)

Different values of $g$ imply different self-adjoint extensions, so $J(g)$ is a self-adjoint extension of $J$ uniquely determined by $g$ \[30\text{ Lem. 2.20}\]. Observe that the domains $D_g$ are defined by a boundary condition at infinity given by $g$. We also remark that given two sequences $\varphi$ and $\psi$ in $\text{Dom}(J^*)$ the following limit always exists \[30\text{ Sec. 2.6}\]

\[\lim_{n \to \infty} W_n(\varphi, \psi) =: W_\infty(\varphi, \psi) \, . \]

It follows from \[28\text{ Thm. 3}\] that, in the limit circle case, $P(\zeta)$ and $Q(\zeta)$ are in $\text{Dom}(J^*)$ for every $\zeta \in \mathbb{C}$.

From what has just been said, one can consider the functions (see also \[2\text{ Sec. 2.4 Chap. 1, Sec. 4.2 Chap. 2}\])

\begin{align*}
W_\infty(P(0), P(\zeta)) &=: D(\zeta) \, , \\
W_\infty(Q(0), P(\zeta)) &=: B(\zeta) \, . \quad (2.7)
\end{align*}

The notation for these limits has not been chosen arbitrarily; they are the elements of the second row of the Nevanlinna matrix associated with the matrix (1.3) and they are usually denoted by these letters \[2\text{ Sec. 4.2 Chap. 2}, \[28\text{ Eq. 4.17}\].

It is well known that the functions $D(\zeta)$ and $B(\zeta)$ are entire of at most minimal type of order one \[2\text{ Thm. 2.4.3}, \[28\text{ Thm. 4.8}\], that is, for each $\epsilon > 0$ there exist constants $C_1(\epsilon), C_2(\epsilon)$ such that

\[|D(\zeta)| \leq C_1(\epsilon)e^{\epsilon|\zeta|}, \quad |B(\zeta)| \leq C_2(\epsilon)e^{\epsilon|\zeta|}.\]

If $P(\zeta)$ is in $D_g$ the following holds

\[0 = W_\infty(v(g), P(\zeta)) = \begin{cases} D(\zeta) + gB(\zeta) & \text{if } g \in \mathbb{R} \\ B(\zeta) & \text{if } g = \infty \end{cases} \, . \]

Thus, the zeros of the function

\[ \mathcal{R}_g(\zeta) := \begin{cases} D(\zeta) + gB(\zeta) & \text{if } g \in \mathbb{R} \\ B(\zeta) & \text{if } g = \infty \end{cases} \] (2.8)

constitute the spectrum of the self-adjoint extension $J(g)$ of $J$.

A Jacobi matrix of the form (1.3) determines, in a unique way, the sequence $P(t), t \in \mathbb{R}$. This sequence is orthonormal in any space $L^2(\mathbb{R}, d\rho)$, where $\rho$ is a
solution of the moment problem associated with the Jacobi matrix \([1.3] \text{ Sec. 2.1 Chap. 2}\). The elements of the sequence \(\{P_n(t)\}_{n=0}^\infty\) form a basis in \(L^2(\mathbb{R}, d\rho)\) if \(\rho\) is an N-extremal solution of the moment problem \([2, \text{ Def. 2.3.3}]\) or, in other words, if \(\rho\) can be written as

\[
\rho(t) = \langle E(t)e_1, e_1 \rangle, \quad t \in \mathbb{R},
\]

where \(E(t)\) is the spectral resolution of the identity for some von Neumann self-adjoint extension of the Jacobi operator \(J\) associated with (1.3) \([2, \text{ Thm. 2.3.3, Thm. 4.1.4}]\).

Let \(\rho\) be given by (2.9), then we can consider the linear isometric operator \(U\) which maps the canonical basis \(\{e_n\}_{n=1}^\infty\) in \(l^2(\mathbb{N})\) into the orthonormal basis \(\{P_n(t)\}_{n=0}^\infty\) in \(L^2(\mathbb{R}, d\rho)\) as follows

\[
Ue_n = P_{n-1}, \quad n \in \mathbb{N}.
\]

By linearity, one extends \(U\) to the span of \(\{e_n\}_{n=1}^\infty\) and by continuity, to all \(l^2(\mathbb{N})\). Clearly, the range of \(U\) is all \(L^2(\mathbb{R}, d\rho)\). The Jacobi operator \(J\) given by the matrix (1.3) is transformed by \(U\) into the operator of multiplication by the independent variable in \(L^2(\mathbb{R}, d\rho)\) if \(J = J^*\), and into a symmetric restriction of the operator of multiplication if \(J \neq J^*\). Following the terminology used in [10], we call the operator \(UJU^{-1}\) in \(L^2(\mathbb{R}, d\rho)\) the canonical representation of \(J\).

By virtue of the discreteness of \(\text{Sp}(J(g))\) in the limit circle case (here and in the sequel, \(\text{Sp}(A)\) stands for the spectrum of operator \(A\)), the function \(\rho_g\) given by (2.9), with \(E(t)\) being the resolution of the identity of \(J(g)\), can be written as follows

\[
\rho_g(t) = \sum_{\lambda_k \leq t} a(\lambda_k)^{-1}, \quad \{\lambda_k\}_k = \text{Sp}(J(g)),
\]

where the positive constant \(a(\lambda_k)\) is the so-called normalizing constant of \(J(g)\) corresponding to \(\lambda_k\). In the limit circle case it is easy to obtain the following formula for the normalizing constants \([2, \text{ Sec. 4.1 Chap. 3}], [28, \text{ Thm. 4.11}]\)

\[
a(\lambda_k) = \|P(\lambda_k)\|^2_{l^2(\mathbb{N})}, \quad \{\lambda_k\}_k = \text{Sp}(J(g)).
\]

Formula (2.11), which gives the jump of the spectral function at \(\lambda_k\), also holds true in the limit point case, when \(\lambda_k\) is an eigenvalue of \(J\) \([7, \text{ Thm. 1.17 Chap. 7}]\).

It turns out that the spectral function \(\rho_g\) uniquely determines \(J(g)\). There are two ways of recovering the matrix from the spectral function. One method, developed in [15] (see also [29]), makes use of the asymptotic behaviour of the Weyl \(m\)-function

\[
m_g(\zeta) := \int_{\mathbb{R}} \frac{\rho_g(t)}{t - \zeta} \, dt
\]

and the Ricatti equation \([15, \text{ Eq. 2.15}], [29, \text{ Eq. 2.23}]\),

\[
b_n^2 m_g^{(n)}(\zeta) = q_n - \zeta - \frac{1}{m_g^{(n-1)}(\zeta)}, \quad n \in \mathbb{N},
\]

(2.12)
where $m_{g}^{(n)}(\zeta)$ is the Weyl $m$-function of the Jacobi operator associated with the matrix $[1,3]$ with the first $n$ columns and $n$ rows removed.

The other method for the reconstruction of the matrix is more straightforward (see [7, Sec. 1.5 Chap. 7 and, particularly, Thm. 1.11]). The starting point is the sequence \( \{t^k\}_{k=0}^{\infty}, \ t \in \mathbb{R} \). From what we discussed above, all the elements of the sequence \( \{t^k\}_{k=0}^{\infty} \) are in $L_2(\mathbb{R},d\rho_g)$ and one can apply, in this Hilbert space, the Gram-Schmidt procedure of orthonormalization to the sequence \( \{t^k\}_{k=0}^{\infty} \). One, thus, obtains a sequence of polynomials \( \{P_k(t)\}_{k=0}^{\infty} \) normalized and orthogonal in $L_2(\mathbb{R},d\rho_g)$. These polynomials satisfy a three term recurrence equation [7, Sec. 1.5 Chap. 7], [28, Sec. 1]

\[
    tP_{k-1}(t) = b_{k-1}P_{k-2}(t) + q_kP_{k-1}(t) + b_kP_k(t), \quad k \in \mathbb{N} \setminus \{1\}, \quad (2.13)
\]

\[
    tP_0(t) = q_0P_0(t) + b_1P_1(t), \quad (2.14)
\]

where all the coefficients $b_k$ ($k \in \mathbb{N}$) turn out to be positive and $q_k$ ($k \in \mathbb{N}$) are real numbers. The system (2.13) and (2.14) defines a matrix which is the matrix representation of $J$.

After obtaining the matrix associated with $J$, if it turns out to be non-self-adjoint, one can easily obtain the boundary condition at infinity which defines the domain of $J(g)$. The recipe is based on the fact that the spectra of different self-adjoint extensions are disjoint [2, Sec. 2.4 Chap. 4]. Take an eigenvalue, $\lambda$, of $J(g)$, i.e., $\lambda$ is a point of discontinuity of $\rho_g$ or a pole of $m_g$. Since the corresponding eigenvector $P(\lambda) = \{P_k(\lambda)\}_{k=0}^{\infty}$ is in $\text{Dom}(J(g))$, it must be that

\[
    W_\infty(v(g), P(\lambda)) = 0.
\]

This implies that either $W_\infty(Q(0), P(\lambda)) = 0$, which means that $g = \infty$, or

\[
    g = -\frac{W_\infty(P(0), P(\lambda))}{W_\infty(Q(0), P(\lambda))}.
\]

We conclude this section with a remark on the notation. In (2.11) we represented the discrete set $\text{Sp}(J(g))$ by the sequence \( \{\lambda_k\}_{k} \). It was understood that $k$ ran through a countable set, so we did not indicated this. Sometimes in what follows we also use this notation for sequences. Let us denote by $K$ the countable set through which the subscript $k$ runs when the sequence \( \{\eta_k\}_{k} \) is given. We shall say that the series $\sum_k \eta_k$ converges to a number $c$ if for any sequence of sets \( \{K_j\}_{j=1}^{\infty} \), with $K_j \subset K_{j+1} \subset K$, such that $\bigcup_{j=1}^{\infty} K_j = K$, the sequences \( \{\sum_{k \in K_j} \eta_k\}_{j=1}^{\infty} \) tends to $c$ whenever $j \to \infty$.

### 3. Unique reconstruction of the matrix

In this section we show that, given the spectra of two different self-adjoint extensions $J(g)$, $J(f)$ of the Jacobi operator $J$ in the limit circle case, one can always
recover the matrix, being the matrix representation of $J$ with respect to the canonical basis in $l^2(\mathbb{N})$, and the two parameters $g$, $f$ that define the self-adjoint extensions. It has already been announced [12, Thm. 1] that, when $g, f \in \mathbb{R}$ and $g \neq f$, the spectra $\text{Sp}(J(g))$ and $\text{Sp}(J(f))$ uniquely determine the matrix of $J$ and the numbers $g$ and $f$. A similar result, but in a more general setting can be found in [10, Thm. 7].

Consider the following expression which follows from the Christoffel-Darboux formula [2, Eq. 1.17]:

$$\sum_{k=0}^{n-1} P_k^2(\zeta) = b_n \left( P_{n-1}(\zeta)P_n'(\zeta) - P_n(\zeta)P'_{n-1}(\zeta) \right) = W_n(P(\zeta), P'(\zeta)).$$

It is easy to verify, taking into account the analogue of the Liouville-Ostrogradskii formula [2, Eq. 1.15], that

$$W_n(P(\zeta), P'(\zeta)) = W_n(P(0), P(\zeta))W_n(Q(0), P'(\zeta)) - W_n(Q(0), P(\zeta))W_n(P(0), P'(\zeta)).$$

Thus,

$$\sum_{k=0}^{\infty} P_k^2(\zeta) = W_\infty(P(\zeta), P'(\zeta)) = D(\zeta)B'(\zeta) - B(\zeta)D'(\zeta).$$

Indeed, due to the uniform convergence of the limits in (2.7) [2, Sec. 4.2 Chap. 2], the following is valid

$$B'(\zeta) = W_\infty(Q(0), P'(\zeta))$$

$$D'(\zeta) = W_\infty(P(0), P'(\zeta)).$$

Now, a straightforward computation yields ($f, g \in \mathbb{R}$, $f \neq g$)

$$\mathcal{R}_g(\zeta)\mathcal{R}_f'(\zeta) - \mathcal{R}_f'(\zeta)\mathcal{R}_g(\zeta) = (f - g) \left[ D(\zeta)B'(\zeta) - B(\zeta)D'(\zeta) \right].$$

On the other hand one clearly have

$$\mathcal{R}_g(\zeta)\mathcal{R}_\infty'(\zeta) - \mathcal{R}_\infty'(\zeta)\mathcal{R}_g(\zeta) = D(\zeta)B'(\zeta) - B(\zeta)D'(\zeta), \quad g \in \mathbb{R}.$$

Hence,

$$a(\zeta) := \sum_{k=0}^{\infty} P_k^2(\zeta) = \begin{cases} 
\mathcal{R}_g(\zeta)\mathcal{R}_f'(\zeta) - \mathcal{R}_f'(\zeta)\mathcal{R}_g(\zeta) & f \neq g, \quad f, g \in \mathbb{R} \\
\mathcal{R}_g(\zeta)\mathcal{R}_\infty'(\zeta) - \mathcal{R}_\infty'(\zeta)\mathcal{R}_g(\zeta) & g \in \mathbb{R}.
\end{cases} \quad (3.1)$$

It follows from (2.11) that the values of the function $a(\zeta)$ evaluated at the points of the spectrum of some self-adjoint extension of $J$ are the corresponding normalizing constants of that extension.
The analogue of (3.1) with \( f \neq g \) and \( f, g \in \mathbb{R} \), for the Schrödinger operator in \( L^2(0, \infty) \) being in the limit circle case is [13, Eq. 1.20]. Formula [13, Eq. 1.20] plays a central rôle in proving the unique reconstruction theorem for that operator [13, Thm. 1.1]. The discrete counterpart of [13, Thm. 1.1] is [12, Thm. 1]. It is worth mentioning that the reconstruction technique we present below is also based on (3.1).

It is well known that the spectra of any two different self-adjoint extensions of \( J \) are disjoint [2, Sec. 2.4 Chap. 4]. One can easily conclude this from (3.1). Moreover, the assertion of Proposition[1] stated below, holds true. The proof of this statement for regular simple symmetric operators can be found in [17, Prop. 3.4 Chap. 1]. We provide below a simple proof of the assertion for Jacobi matrices.

**Proposition 1.** The eigenvalues of two different self-adjoint extensions of a Jacobi operator interlace, that is, there is only one eigenvalue of a self-adjoint extension between two eigenvalues of any other self-adjoint extension.

**Proof.** The proof of this assertion follows from the expression (3.1). It is similar to the proof of [2, Thm. 1.2.2].

Note that (2.8) implies that the entire function \( R_g(\zeta), g \in \mathbb{R} \cup \{\infty\} \), is real, i.e., it takes real values when evaluated on the real line. Let \( \lambda_k < \lambda_{k+1} \) be two neighboring eigenvalues of the self-adjoint extension \( J(f) \) of \( J, f \in \mathbb{R} \cup \{\infty\} \). So \( \lambda_k, \lambda_{k+1} \) are zeros of \( R_f \) and by (3.1) these zeros are simple. Since \( R_f'(\lambda_k) \) and \( R_f'(\lambda_{k+1}) \) have different signs, it follows from (3.1) that \( R_g(\lambda_k) \) and \( R_g(\lambda_{k+1}) \) (\( g \in \mathbb{R} \cup \{\infty\}, g \neq f \)) have also opposite signs. From the continuity of \( R_g \) on the interval \([\lambda_k, \lambda_{k+1}]\), there is at least one zero of \( R_g \) in \((\lambda_k, \lambda_{k+1})\). Now, suppose that in this interval there is more than one zero of \( R_g \), so one can take two neighboring zeros of \( R_g \) in \((\lambda_k, \lambda_{k+1})\). By reproducing the argumentation above with \( g \) and \( f \) interchanged, one obtains that there is at least one zero of \( R_f \) somewhere in \((\lambda_k, \lambda_{k+1})\). This contradicts the assumption that \( \lambda_k \) and \( \lambda_{k+1} \) are neighbors.

The assertion of the following proposition is known long ago (see, for instance [21, Thm. 1]), nevertheless, for the reader’s convenience, we provide the proof. Note that a non-constant entire function of at most minimal type of order one must have zeros, otherwise, by Weierstrass theorem on the representation of entire functions by infinite products [23, Thm. 3 Chap. 1], it would be a function of at least normal type.

Before stating the proposition we remind the definition of convergence exponent of a sequence of complex numbers (see [23, Sec. 4 Chap. 1]). The convergence exponent \( \rho_1 \) of a sequence \( \{\nu_k\}_k \) of non-zero complex numbers accumulating only at infinity is given by

\[
\rho_1 := \inf \left\{ \gamma \in \mathbb{R} : \sum_k \frac{1}{|\nu_k|^{\gamma}} < \infty \right\}.
\]

(3.2)

We also remark that, as it is customary, whenever we say that an infinite product is convergent we mean that at most a finite number of factors may be zero and
the partial product formed by the non-vanishing factors tends to a number different from zero [1, Sec. 2.2 Chap. 5].

**Proposition 2.** Let \( f(\zeta) \) be an entire function of at most minimal type of order one with an infinite number of zeros. Let the elements of the sequence \( \{\nu_k\}_k \), which accumulate only at infinity, be the non-zero roots of \( f \), and let \( m \in \mathbb{N} \cup \{0\} \) be the order of the zero of \( f \) at the origin. Then there exists a complex constant \( C \) such that

\[
f(\zeta) = C\zeta^m \lim_{r \to \infty} \prod_{|\nu_k| \leq r} \left( 1 - \frac{\zeta}{\nu_k} \right),
\]

where the limit converges uniformly on compacts of \( \mathbb{C} \).

**Proof.** The convergence exponent \( \rho_1 \) of the zeros of an arbitrary entire function does not exceed its order [23, Thm. 6 Chap. 1]. Then, for a function of at most minimal type of order one, \( \rho_1 \leq 1 \). According to Hadamard’s theorem [23, Thm. 13 Chap. 1], the expansion of \( f \) in an infinite product has either the form:

\[
f(\zeta) = \zeta^m e^{a\zeta+b} \lim_{r \to \infty} \prod_{|\nu_k| \leq r} G \left( \frac{\zeta}{\nu_k}; 0 \right), \quad a, b \in \mathbb{C}
\]

if the limit

\[
\lim_{r \to \infty} \sum_{|\nu_k| \leq r} \frac{1}{|\nu_k|} = 0
\]

converges, or

\[
f(\zeta) = \zeta^m e^{c\zeta+d} \lim_{r \to \infty} \prod_{|\nu_k| \leq r} G \left( \frac{\zeta}{\nu_k}; 1 \right), \quad c, d \in \mathbb{C}
\]

if (3.5) diverges. We have used here the Weierstrass primary factors \( G \) (for details see [23, Sec. 3 Chap. 1]). Let us suppose that the order is one and (3.5) diverges, then, in view of the fact that \( f \) is of minimal type, by a theorem due to Lindelöf [23, Thm. 15 a Chap. 1], we have in particular that

\[
\lim_{r \to \infty} \sum_{|\nu_k| \leq r} \nu_k^{-1} = -c.
\]

This implies the uniform convergence of the series \( \lim_{r \to \infty} \sum_{|\nu_k| \leq r} \frac{\zeta}{\nu_k} \) on compacts of \( \mathbb{C} \). In its turn, since \( \rho_1 = 1 \), this yields that \( \lim_{r \to \infty} \prod_{|\nu_k| \leq r} \left( 1 - \frac{\zeta}{\nu_k} \right) \) is uniformly convergent on any compact of \( \mathbb{C} \). Therefore,

\[
\lim_{r \to \infty} \prod_{|\nu_k| \leq r} G \left( \frac{\zeta}{\nu_k}; 1 \right) = e^{-c} \lim_{r \to \infty} \prod_{|\nu_k| \leq r} \left( 1 - \frac{\zeta}{\nu_k} \right).
\]

Thus, (3.6) can be written as (3.3).

Suppose now that the limit (3.5) converges. If the order of the function is less than one, then, by [23, Thm. 13 Chap. 1], one may write (3.4) as (3.3). If the order
of the function is one, by [23, Thm. 12, Thm. 15 b Chap. 1], one concludes again that (3.4) can be written as (3.3) (cf. Thm 15 in the Russian version of [23] or, alternatively, [24, Lect. 5]).

Let \( \{ \lambda_n(g) \} \) be the eigenvalues of \( J(g) \). In view of the fact that \( R_g(\zeta) \) is an entire function of at most minimal type of order one, by Proposition 2, one can always write

\[
\mathfrak{R}_g(\zeta) = C_g \lim_{r \to \infty} \prod_{\lambda_k(g) \leq r} \left( 1 - \frac{\zeta}{\lambda_k(g)} \right), \quad g \in \mathbb{R} \cup \{ \infty \}, \quad g \neq 0, \tag{3.7}
\]

and

\[
\mathfrak{R}_0(\zeta) = C_0 \zeta \lim_{r \to \infty} \prod_{0 < \lambda_k(0) \leq r} \left( 1 - \frac{\zeta}{\lambda_k(0)} \right), \tag{3.8}
\]

where \( C_g \) is some real constant and the limits converge uniformly on compacts of \( \mathbb{C} \).

When writing (3.7) and (3.8), we have taken into account, on the one hand, that \( R_0(0) = 0 \), which follows from (2.8) and the definition of the function \( D \), and on the other, that different self-adjoint extensions have disjoint spectra (see Section 2).

Now, let us consider the following expressions derived from the Green’s formula [2, Eqs. 1.23, 2.28]

\[
D(\zeta) = \zeta \sum_{k=0}^{\infty} P_k(0)P_k(\zeta), \tag{3.9}
\]

\[
B(\zeta) = -1 + \zeta \sum_{k=0}^{\infty} Q_k(0)P_k(\zeta). \tag{3.10}
\]

Again we verify from (3.9) that \( D(0) = 0 \), while from (3.10) we have \( B(0) = -1 \). Therefore \( \mathfrak{R}_g(0) = -g \) for every \( g \in \mathbb{R} \), and \( \mathfrak{R}_\infty(0) = -1 \). Thus, \( C_g = -g \) provided that \( g \in \mathbb{R} \) and \( g \neq 0 \), and \( C_\infty = -1 \).

**Theorem 1.** Let \( g, f \in \mathbb{R} \cup \{ \infty \} \) with \( g \neq f \). The spectra \( \{ \lambda_k(g) \} \), \( \{ \lambda_k(f) \} \) of two different self-adjoint extensions \( J(g) \), \( J(f) \) of a Jacobi operator \( J \) in the limit circle case uniquely determine the matrix associated with \( J \), and the numbers \( g \) and \( f \).

**Proof.** First consider the case when the sequences \( \{ \lambda_k(g) \} \) and \( \{ \lambda_k(f) \} \) do not contain any zero element. This means that \( g \) and \( f \) are both different from zero. Let us denote

\[
R_g(\zeta) := \lim_{r \to \infty} \prod_{\lambda_k(g) \leq r} \left( 1 - \frac{\zeta}{\lambda_k(g)} \right), \quad g \neq 0. \tag{3.11}
\]

By (3.1), we have

\[
a(\lambda_k(f)) = MR_g(\lambda_k(f))R'_f(\lambda_k(f)), \tag{3.12}
\]

\[
11
\]
where 

\[
M = \begin{cases} 
\frac{fg}{f-g} & \text{if } f, g \in \mathbb{R} \setminus \{0\}, f \neq g \\
g & \text{if } f = \infty \\
-f & \text{if } g = \infty 
\end{cases}
\]  

(3.13)

Now, since \(a(\lambda_k(f))\) are the normalizing constants of \(J(f)\) we must have

\[
1 = \sum_k \frac{1}{a(\lambda_k(f))} = \frac{1}{M} \sum_k \frac{1}{R_g(\lambda_k(f))R'_f(\lambda_k(f))}.
\]

Therefore

\[
M = \sum_k \frac{1}{R_g(\lambda_k(f))R'_f(\lambda_k(f))}.
\]  

(3.14)

Thus, \(M\) is completely determined by the sequences \(\{\lambda_k(g)\}_k\) and \(\{\lambda_k(f)\}_k\). Inserting the obtained value of \(M\) into (3.12) one obtains the normalizing constants. Having the normalizing constants allows us to construct the spectral measure for \(J(f)\). Then, by standard methods (see Section 2), one reconstructs the matrix associated with \(J\) and the boundary condition at infinity \(f\). From the value of \(M\) and \(f\) one obtains \(g\).

Now consider that one of the sequences \(\{\lambda_k(g)\}_k\) and \(\{\lambda_k(f)\}_k\) contains a zero element. For definiteness assume that \(0 \in \{\lambda_k(g)\}_k\). As has been established above, this implies that \(g = 0\). Denote

\[
R_0(\zeta) := \zeta \lim_{r \to \infty} \prod_{0 < |\lambda_k(0)| \leq r} \left(1 - \frac{\zeta}{\lambda_k(0)}\right).
\]

Then, due to (3.1) one has

\[
a(\lambda_k(f)) = -C_0 R_0(\lambda_k(f)) R'_f(\lambda_k(f)).
\]  

(3.15)

As in the previous case

\[
C_0 = - \sum_k \frac{1}{R_0(\lambda_k(f))R'_f(\lambda_k(f))}.
\]

Hence we have found the normalizing constants and therefore the spectral measure for \(J(f)\). From the spectral measure of \(J(f)\) one obtains the matrix and \(f\).

Remark 1. Note that the proof of Theorem 1 gives a reconstruction method of the Jacobi matrix. Although mentioned earlier, we also remark here that the assertion of Theorem 1 for the case of \(g, f \in \mathbb{R}\), was announced without proof in [12, Thm. 1].
4. Necessary and sufficient conditions

In this section we give a complete characterization of our two-spectra inverse problem. We remind the reader about the remark on the notation at the end of Section 2.

First we prove the following simple proposition related to the converse of Proposition 2.

**Proposition 3.** Let \( \{ \nu_k \}_k \) be an infinite sequence of non-vanishing complex numbers accumulating only at \( \infty \), and whose convergence exponent \( \rho_1 \) does not exceed one. Suppose that the infinite product

\[
\lim_{r \to \infty} \prod_{|\zeta| \leq r} \left( 1 - \frac{\zeta}{\nu_k} \right) = \lim_{r \to \infty} \prod_{|\zeta| \leq r} G \left( \frac{\zeta}{\nu_k}; 0 \right)
\]

(4.1)

converges uniformly on any compact of \( \mathbb{C} \). Then this product is an entire function of at most minimal type of order one if either (3.5) converges or if (3.5) diverges but the following holds

\[
\lim_{r \to \infty} \frac{n(r)}{r} = 0,
\]

(4.2)

where \( n(r) \) is the number of elements of \( \{ \nu_k \}_k \) in the circle \( |\zeta| < r \).

**Proof.** Clearly, by the conditions of the theorem, one can express (4.1) in terms of canonical products [23, Sec. 3 Chap. 1] either in the form

\[
\lim_{r \to \infty} \prod_{|\zeta| \leq r} \left( 1 - \frac{\zeta}{\nu_k} \right) = \lim_{r \to \infty} \prod_{|\zeta| \leq r} G \left( \frac{\zeta}{\nu_k}; 0 \right)
\]

(4.3)

whenever (3.5) converges, or in the form

\[
\lim_{r \to \infty} \prod_{|\zeta| \leq r} \left( 1 - \frac{\zeta}{\nu_k} \right) = e^{\zeta} \lim_{r \to \infty} \prod_{|\zeta| \leq r} G \left( \frac{\zeta}{\nu_k}; 1 \right)
\]

(4.4)

otherwise, where

\[
\lim_{r \to \infty} \sum_{|\zeta| \leq r} \nu_k^{-1} = -c
\]

(4.5)

In the case (4.3), in which the genus of the product is less than the convergence exponent of \( \{ \nu_k \}_k \), it is clear that (4.1) does not grow faster than an entire function of minimal type of order one. Indeed, by [23, Thm. 7 Chap. 1], the order of a canonical product is equal to the convergence exponent, so when \( \rho_1 < 1 \) the assertion is obvious. For \( \rho_1 = 1 \) the statement follows from [23, Thm. 15 b Chap. 1].

If we have the representation (4.4), then \( \rho_1 = 1 \). By [23, Thm. 7 Chap. 1], the canonical product has order one. Since the product of functions of the same order is of that same order, the order of (4.1) is one. Then, the assertion follows from [23]...
Before passing on to the main results of this section, we establish an auxiliary result which is related to part of the proof of Theorem 1 in the Addenda and Problems of [2, Chap. 4].

**Lemma 1.** Consider an infinite real sequence \( \{\kappa_j\}_j \) and a sequence \( \{\alpha_j\}_j \) of positive numbers such that

\[
\sum_j \frac{\kappa_j^{2m}}{\alpha_j} < \infty, \quad m = 0, 1, \ldots
\]

Let \( \mathcal{R} \) be an entire function of at most minimal type of order one whose zeros, \( \{\kappa_j\}_j \), are simple, and such that

\[
|\mathcal{R}(i\xi)| \to \infty, \quad \text{as} \quad \xi \to \pm \infty, \quad \xi \in \mathbb{R}. \tag{4.6}
\]

If

\[
\sum_j \frac{\alpha_j}{(1 + \kappa_j^2)[\mathcal{R}'(\kappa_j)]^2} < \infty,
\]

then

\[
\sum_j \frac{\kappa_j^m}{\mathcal{R}'(\kappa_j)} \quad (4.7)
\]

is absolutely convergent for \( m = 0, 1, \ldots \), and the absolutely convergent expansion

\[
\frac{1}{\mathcal{R}(\zeta)} = \sum_j \frac{1}{\mathcal{R}'(\kappa_j)(\zeta - \kappa_j)}
\]

holds true.

**Proof.** The absolutely convergence of (4.7) follows from

\[
\sum_j \left| \frac{\kappa_j^m}{\mathcal{R}'(\kappa_j)} \right| = \sum_j \left| \frac{\sqrt{\alpha_j}}{\sqrt{1 + \kappa_j^2}[\mathcal{R}'(\kappa_j)]} \right| \left| \frac{\kappa_j^m}{\sqrt{1 + \kappa_j^2}} \right| \left| \frac{1}{\sqrt{\alpha_j}} \right| \leq \sqrt{\sum_j \frac{\alpha_j}{(1 + \kappa_j^2)[\mathcal{R}'(\kappa_j)]^2}} \left| \sum_j \frac{\kappa_j^{2m} + \kappa_j^{2m+2}}{\alpha_j} \right| < \infty
\]

Construct the function

\[
h(\zeta) := 1 - \mathcal{R}(\zeta) \sum_j \frac{1}{\mathcal{R}'(\kappa_j)(\zeta - \kappa_j)},
\]

where the series is absolutely convergent in compact subsets of \( \mathbb{C} \setminus \{\kappa_j\}_j \) because of (4.7). Clearly, \( h(\kappa_j) = 0 \) for any \( j \). Moreover, it turns out that \( h \) is an entire
function of at most minimal type of order one. To show this, first consider the case
when $\sum |\kappa_j|^{-1} < \infty$. Here, by what we have discussed in the proof of Proposition 2,
the function $R(\zeta)/(\zeta - \kappa_j)$ can be expressed by a canonical product of genus zero.
It follows from [23, Lem. 3 Chap. 1] (see also the proof of [23, Thm. 4 Chap. 1]) that,
on the one hand,
\[
\max_{|\zeta|=r} \left| \frac{R(\zeta)}{(\zeta - \kappa_j)} \right| < \exp \left( C(\alpha) r^\alpha \right), \quad \rho_1 < \alpha < 1,
\]
for any $r > 0$ provided that $\rho_1 < 1$. If, on the other hand, $\rho_1 = 1$, then for any
$\epsilon > 0$, there exists $R_0 > 0$ such that
\[
\max_{|\zeta|=r} \left| \frac{R(\zeta)}{(\zeta - \kappa_j)} \right| < \exp \left( \epsilon r \right) \tag{4.8}
\]
for all $r > R_0$. Hence, in any case, we have the uniform, with respect to $j$, asymptotic
estimation (4.8) when $\sum |\kappa_j|^{-1} < \infty$.

Suppose now that $\sum |\kappa_j|^{-1} = \infty$. In this case, as was shown in the proof of
Proposition 2,
\[
\frac{R}{(\zeta - \kappa_j)} = -\frac{1}{\kappa_j} \zeta^m e^{(c + \kappa_j^{-1}) \zeta + d} \lim_{r \to \infty} \prod_{\substack{|\kappa_k| \leq r \\kappa_k \neq j}} G\left( \frac{\zeta}{\kappa_k}; 1 \right),
\]
where
\[
\lim_{r \to \infty} \sum_{|\kappa_k| \leq r} \kappa_k^{-1} = -c. \tag{4.9}
\]
On the basis of the estimates found in the proof of [23, Thm. 15 Chap. 1] (see in
particular the inequality next to [23, Eq. 1.43]), one can find $R_1$, independent of $j$, such that
\[
\max_{|\zeta|=r} \left| \frac{R(\zeta)}{(\zeta - \kappa_j)} \right| < \exp \left[ r \left( c + \sum_{|\kappa_k| \leq r} \kappa_k^{-1} \right) + C \left( \limsup_{r \to \infty} \frac{n(r)}{r} + \epsilon \right) + O \left( \frac{1}{r} \right) \right]
\]
for all $r > R_1$ and $\epsilon > 0$ (see the definition of $n(r)$ in the statement of Proposition 3).
Note that if $|\kappa_j| \leq r$, then the above inequality follows directly from the inequality
next to [23, Eq. 1.43]. If $|\kappa_j| > r$, the same inequality holds due to
\[
\left| c + \kappa_j^{-1} + \sum_{|\kappa_k| \leq r} \kappa_k^{-1} \right| \leq \left| c + \sum_{|\kappa_k| \leq r} \kappa_k^{-1} \right| + \frac{1}{r}.
\]
Since $R$ does not grow faster than a function of minimal type of order one, by [23,
Thm. 15 a Chap. 1], one again verifies that, for any $\epsilon > 0$, (4.8) holds for all $r$ greater
than a certain $R_2$ depending only on the velocity of convergence in the limits (4.9).
and (4.2).

Thus, one concludes that, for any \( \epsilon > 0 \), there is \( R > 0 \) such that

\[
\max_{|\zeta| = r} \left| \Re(\zeta) \sum_j \frac{1}{\Re(\kappa_j)(\zeta - \kappa_j)} \right| \leq \sum_j \frac{1}{\Re(\kappa_j)} \max_{|\zeta| = r} \left| \Re(\zeta) \right| < \exp(\epsilon r)
\]

for all \( r > R \), which shows that \( h \) is an entire function of at most minimal type of order one.

Now, the function \( h/R \) is also an entire function of at most minimal type of order one [23, Cor. Sec. 9 Chap. 1]. By the hypothesis (4.6),

\[
\lim_{\xi \to \pm \infty} h(i\xi) R(i\xi) = 0,
\]

which implies that \( h/R \equiv 0 \) (see Corollary of [23, Sec. 14 Chap. 1]).

\[ \square \]

**Theorem 2.** Given two infinite disjoint sequences of real non-zero numbers \( \{\lambda_k\}_k \) and \( \{\mu_k\}_k \), with the point at infinity being their only accumulation point, and such that \( \lambda_k \neq \lambda_j, \mu_k \neq \mu_j \) with \( k \neq j \), there exist unique \( f, g \in \mathbb{R} \cup \{\infty\} \setminus \{0\} \), with \( f \neq g \), and a unique Jacobi operator \( J \neq J^* \) such that \( \{\lambda_k\}_k = \text{Sp}(J(f)) \) and \( \{\mu_k\}_k = \text{Sp}(J(g)) \) if and only if the following conditions are satisfied.

1. The convergence exponent of the sequences \( \{\lambda_k\}_k \) and \( \{\mu_k\}_k \) does not exceed one. Additionally, if

\[
\lim_{r \to \infty} \sum_{|\lambda_k| \leq r} \frac{1}{|\lambda_k|} = \infty, \quad \text{require that} \quad \lim_{r \to \infty} \frac{n_{\lambda}(r)}{r} = 0,
\]

and if

\[
\lim_{r \to \infty} \sum_{|\mu_k| \leq r} \frac{1}{|\mu_k|} = \infty, \quad \text{require that} \quad \lim_{r \to \infty} \frac{n_{\mu}(r)}{r} = 0,
\]

where \( n_{\lambda}(r) \) and \( n_{\mu}(r) \) are the number of elements of \( \{\lambda_k\}_k \) and \( \{\mu_k\}_k \), respectively, in the circle \( |\zeta| < r \).

2. The limits

\[
\lim_{r \to \infty} \prod_{|\mu_k| \leq r} \left( 1 - \frac{\zeta}{\mu_k} \right), \quad \lim_{r \to \infty} \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\zeta}{\lambda_k} \right)
\]

converge uniformly on compact subsets of \( \mathbb{C} \).

3. All numbers

\[
\frac{1}{\lambda_j} \lim_{r \to \infty} \left\{ \prod_{|\mu_k| \leq r} \left( 1 - \frac{\lambda_j}{\mu_k} \right) \prod_{|\lambda_k| \leq r, k \neq j} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \right\}
\]
have the same sign for every $j$. The same is true for the numbers

$$\frac{1}{\mu_j} \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \prod_{|\mu_k| \leq r \atop k \neq j} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}.$$  

4. For every $m = 0, 1, 2, \ldots$ the series below are convergent and the following equalities hold

$$\sum_j \frac{\lambda_j^{m+1}}{\lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \prod_{|\mu_k| \leq r \atop k \neq j} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}} = - \sum_j \frac{\mu_j^{m+1}}{\lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\mu_j}{\mu_k} \right) \prod_{|\mu_k| \leq r \atop k \neq j} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \right\}}.$$  

5. The series

$$\sum_j \lambda_j \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r \atop k \neq j} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \right\} \quad \text{and} \quad \sum_j \mu_j \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r \atop k \neq j} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}$$

diverge either to $-\infty$ or $+\infty$.

6. The series

$$\sum_j \frac{\lambda_j}{1 + \lambda_j^2} \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r \atop k \neq j} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \right\}, \quad \sum_j \frac{\mu_j}{1 + \mu_j^2} \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r \atop k \neq j} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}$$

are convergent.

**Proof.** We begin by proving that if $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ are, respectively, the spectra of the self-adjoint extensions $J(f)$ and $J(g)$ of a Jacobi operator $J$, then conditions 4 and 6 hold true.

The functions $\mathcal{R}_f$ and $\mathcal{R}_g$, given by (2.8), have the sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$, respectively, as their sets of zeros. These functions do not grow faster than an entire function of minimal type of order one. By Proposition 2 (see (3.7)), the limits

$$\lim_{r \to \infty} \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\zeta}{\lambda_k} \right), \quad \lim_{r \to \infty} \prod_{|\mu_k| \leq r} \left( 1 - \frac{\zeta}{\mu_k} \right)$$

are convergent.
converge uniformly on compacts of \( \mathbb{C} \). This is condition \( \text{2} \). Moreover, by \( (3.6) \) and \( [23, \text{Thm. 15 a Chap. 1}] \), condition \( \text{7} \) holds.

Condition \( \text{2} \) implies the uniform convergence of the expression
\[
\frac{d}{d\zeta} \left[ \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\zeta}{\lambda_k} \right) \right], \quad \text{as } r \to \infty.
\]

This allows us to write
\[
R'_f(\lambda_j) = -\frac{1}{\lambda_j} \lim_{r \to \infty} \prod_{|\lambda_k| \leq r, \ k \neq j} \left( 1 - \frac{\lambda_j}{\lambda_k} \right), \quad (4.10)
\]
where \( R_f \) is defined by \( (3.11) \). Substituting \( (4.10) \) into \( (3.12) \) yields
\[
a(\lambda_j) = -\frac{M}{\lambda_j} \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \prod_{|\mu_k| \leq r, \ k \neq j} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}, \quad (4.11)
\]
where \( M \) is given by \( (3.13) \). Analogously,
\[
a(\mu_j) = \frac{M}{\mu_j} \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\mu_j}{\lambda_k} \right) \prod_{|\mu_k| \leq r, \ k \neq j} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}. \quad (4.12)
\]

On the basis of the positiveness of the normalizing constants, from \( (4.11) \) and \( (4.12) \), we obtain condition \( \text{3} \).

From what we discussed in Section \( \text{2} \) all the moments exist for the spectral functions of \( J(f) \) and \( J(g) \), which are, respectively,
\[
\sum_{\lambda_k \leq t} \frac{1}{a(\lambda_k)} \quad \text{and} \quad \sum_{\mu_k \leq t} \frac{1}{a(\mu_k)}.
\]
Hence the series in both sides of condition \( \text{4} \) are convergent for \( m \in \mathbb{N} \cup \{0\} \). Moreover, the spectral functions \( (4.13) \) are solutions of the same moment problem associated with \( J \) (see the paragraph surrounding \( (2.9) \)), therefore the equality of condition \( \text{4} \) holds.

Theorem 1 in the Addenda and Problems of \( [2, \text{Chap. 4}] \) tells us that
\[
\sum_j \frac{a(\lambda_j)}{[R'_f(\lambda_j)]^2} = +\infty \quad (4.14)
\]
is a necessary condition for the sequences \( \{\lambda_j\}_j \) and \( \{a(\lambda_j)\}_j \) to be the spectrum.
of \(J(f)\) and its corresponding normalizing constants. Thus, substituting (4.10) and (4.11) into (4.14), one establishes the divergence of the first series in (4.14). Similarly,

\[
\sum_j \frac{a(\mu_j)}{[R'_g(\mu_j)]^2} = +\infty
\]

must hold, which, by the analogue of (4.10) for \(R'_g(\mu_j)\) and (4.12), implies the divergence of the second series in (4.14).

By the same theorem in [2] mentioned above, and taking into account (4.10) and its analogue for \(R'_g(\mu_j)\), along with (4.11) and (4.12), one obtains the convergence of the series in (4.14).

Let us now prove that the conditions 1–6 are sufficient. Using condition 2 and the convergence of the series in the left hand side of (4.14) with \(m = 0\), we define the real constant

\[
\mathcal{M} := -\sum_j \lambda_j \lim_{r \to \infty} \left\{ \prod_{|\mu_k| \leq r} \left(1 - \frac{\lambda_j}{\mu_k}\right) \prod_{|\lambda_k| \leq r \neq j} \left(1 - \frac{\lambda_j}{\lambda_k}\right) \right\} \tag{4.15}
\]

and the sequence of numbers

\[
a_j := -\frac{\mathcal{M}}{\lambda_j} \lim_{r \to \infty} \left\{ \prod_{|\mu_k| \leq r} \left(1 - \frac{\lambda_j}{\mu_k}\right) \prod_{|\lambda_k| \leq r \neq j} \left(1 - \frac{\lambda_j}{\lambda_k}\right) \right\}. \tag{4.16}
\]

By (4.15), it follows that \(a_j > 0\) for all \(j\). Moreover, (4.15) and (4.16) imply that \(\sum_j a_j^{-1} = 1\).

With the aid of the sequences \(\{\lambda_k\}\) and \(\{a_k\}\) define the function \(\sigma : \mathbb{R} \to \mathbb{R}_+\) as follows

\[
\sigma(t) := \sum_{\lambda_k \leq t} a_k^{-1}. \tag{4.17}
\]

Consider the self-adjoint operator of multiplication \(A_\sigma\) by the independent variable in \(L^2(\mathbb{R}, d\sigma)\). We show below that this operator is the canonical representation (see Section 2) of a self-adjoint extension of a Jacobi matrix in the limit circle case. The proof of this fact is similar to the proof of Theorem 2 in Addenda and Problems of [2] Chap. 4. Note, however, that our conditions are slightly different.

Consider a function \(\theta_k(t) \in L^2(\mathbb{R}, d\sigma)\) such that

\[
\theta_k(\lambda_j) = \sqrt{a_k} \delta_{kj}. \tag{4.18}
\]

Clearly, \(\theta_k(t)\) is a normalized eigenvector of \(A_\sigma\) corresponding to \(\lambda_k\). Let \(\varphi(t) \in\)
\(L^2(\mathbb{R}, d\sigma)\) be such that
\[
\langle \varphi, \theta_j \rangle_{L^2(\mathbb{R}, d\sigma)} = \frac{\lambda_j \sqrt{a_j}}{(\lambda_j - i) \lim_{r \to \infty} \prod_{\lambda_k | \leq r, k \neq j} \left( 1 - \frac{\lambda_j}{\lambda_k} \right)},
\]  
where the convergence of the limit in the denominator is of course ensured by \ref{eq:4.16}. Taking into account \ref{eq:4.16}, it is clear that the convergence of the first series in \ref{eq:4.16} ensures that \(\varphi(t)\) is indeed an element of \(L^2(\mathbb{R}, d\sigma)\). Define
\[
D := \{ \xi \in L^2(\mathbb{R}, d\sigma) : \xi = (A_\sigma + iI)^{-1} \psi, \psi \in L^2(\mathbb{R}, d\sigma), \psi \perp \varphi \}.
\]  
Since \(D \subset \text{Dom}(A_\sigma)\), we can consider the restriction of \(A_\sigma\) to the linear set \(D\). Let us show that this restriction is a symmetric operator with deficiency indices \((1, 1)\).

Consider now the restriction of \(A_\sigma\) to the set \(D\), denoted henceforth by \(A_\sigma \mid_D\), and let us find the dimension of \(\ker((A_\sigma \mid_D)^* - iI)\) which is characterized as the set of all \(\omega \in L^2(\mathbb{R}, d\sigma)\) for which the equation
\[
\langle (A_\sigma + iI)\xi, \omega \rangle_{L^2(\mathbb{R}, d\sigma)} = 0
\]  
is satisfied for any \(\xi \in D\). It is not difficult to show that any such \(\omega\) can be written as follows
\[
\omega = \tilde{C} \varphi, \quad 0 \neq \tilde{C} \in \mathbb{C}.
\]  
Hence \(\dim \ker((A_\sigma \mid_D)^* - iI) = 1\). Analogously, it can be shown that the dimension of \(\ker((A_\sigma \mid_D)^* + iI)\) also equals one. Indeed, if \(\omega \in \ker((A_\sigma \mid_D)^* + iI)\) then, up to a complex constant \(\omega = (A_\sigma - iI)(A_\sigma + iI)^{-1}\varphi\).

Now we show that \(A_\sigma \mid_D\) is the canonical representation of a Jacobi operator in the limit circle case and \(A_\sigma\) is the canonical representation of a self-adjoint extension of this Jacobi operator. We proceed stepwise.

I. We orthonormalize the sequence of functions \(\{t^n\}_{n=0}^{\infty}\) with respect to the inner product of \(L^2(\mathbb{R}, d\sigma)\). Note that condition \ref{eq:4.16} guarantees that all elements of the sequence \(\{t^n\}_{n=0}^{\infty}\) are in \(L^2(\mathbb{R}, d\sigma)\). We obtain thus a sequence of polynomials \(\{P_n(t)\}_{n=1}^{\infty}\) which satisfy the three term recurrence equation (\ref{eq:2.13}) and (\ref{eq:2.14}), where all the coefficients \(b_k (k \in \mathbb{N})\) turn out to be positive and \(q_k (k \in \mathbb{N})\) are real numbers.
II. We verify that the polynomials are dense in $L^2(\mathbb{R}, d\sigma)$, so the sequence we have constructed is a basis in $L^2(\mathbb{R}, d\sigma)$. Note first that the function

\[ R = \lim_{r \to \infty} \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\zeta}{\lambda_k} \right), \]

is entire of at most minimal type of order one. Indeed, this follows from Proposition 3 in view of conditions 1 and 2. Now, for any element $\lambda_{k_0}$ of the sequence $\{\lambda_k\}_k$, we clearly have

\[ |R(i\xi)| \geq \left| 1 + \frac{\xi^2}{\lambda_{k_0}^2} \right|, \quad \xi \in \mathbb{R}. \]

This implies that $R$ satisfies (4.6). Hence the function $R$ and the sequences $\{\lambda_j\}_j, \{a_j\}_j$ satisfy the condition of Lemma 1. Thus, we have shown that

\[ \sum_j \frac{\lambda_j^m}{\prod_{|\lambda_k| \leq r} \left( 1 - \frac{\lambda_j}{\lambda_k} \right)} < \infty, \quad m = 1, 2, \ldots \]  

(4.21)

and

\[ \lim_{r \to \infty} \frac{1}{\prod_{|\lambda_k| \leq r} \left( 1 - \frac{\zeta}{\lambda_k} \right)} = -\sum_j \frac{\lambda_j}{\prod_{|\lambda_k| \leq r} \left( 1 - \frac{\lambda_j}{\lambda_k} \right)} \left( \zeta - \lambda_j \right). \]  

(4.22)

Taking into account Definitions 1 and 2 of the Addenda and Problems of [2, Chap. 4], one obtains from Corollary 2 of [2, Addenda and Problems Chap. 4], together with [2] and [4], that $\{\lambda_k\}$ is a canonical sequence of nodes and $\{a_k^{-1}\}$ the corresponding sequence of masses for the moment problem given by $\{s_k\}_{k=0}^\infty$ with

\[ s_k := -\frac{1}{M} \sum_j \lim_{r \to \infty} \left\{ \frac{\lambda_j^{k+1}}{\prod_{|\mu_k| \leq r} \left( 1 - \frac{\lambda_j}{\mu_k} \right) \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\lambda_j}{\lambda_k} \right)} \right\}. \]  

(4.23)

Hence, for this moment problem, $\sigma$ is a canonical solution [2, Def. 3.4.1]. By definition, a canonical solution is N-extremal and by [2, Thm. 2.3.3], the polynomials are dense in $L^2(\mathbb{R}, d\sigma)$.

III. We prove that the elements of the basis $\{P_{n-1}(t)\}_{n=1}^\infty$ are in $D$. From (4.21) and (4.22), by Lemma 1 of the Addenda and Problems of [2, Chap. 4], one has for $m = 1, 2, \ldots$

\[ \sum_j \frac{\lambda_j^m}{\prod_{|\lambda_k| \leq r} \left( 1 - \frac{\lambda_j}{\lambda_k} \right)} = 0. \]  

(4.24)
Then, if $S(t)$ is a polynomial

$$\langle (A_{\sigma} + iI)S, \varphi \rangle_{L^2(\mathbb{R}, d\sigma)} = \sum_j \frac{S(\lambda_j)\lambda_j}{\lim_{r \to \infty} \prod_{\lambda_k \leq r, k \neq j} \left(1 - \frac{\lambda_j}{\lambda_k}\right)} = 0.$$  

Whence it follows that $S \in D$.

Now, by (2.13), it is straightforward to show that $U^{-1}A_{\sigma} |_{D} U$ (see (2.10)) is a Jacobi operator in the limit circle case.

Denote by $J$ the Jacobi operator $U^{-1}A_{\sigma} |_{D} U$. Clearly, there is $f \in \mathbb{R} \cup \{\infty\} \setminus \{0\}$ such that the self-adjoint operator of multiplication in $L^2(\mathbb{R}, d\sigma)$ is the canonical representation of $J(f)$. $f$ cannot be zero since then $\{\lambda_k\}$ should contain the zero. We define

$$g := \begin{cases} M & \text{if } f = \infty \\ \infty & \text{if } f = -M \\ \frac{Mf}{f+M} & \text{in all other cases.} \end{cases}$$ \hspace{1cm} (4.24)

For the proof to be complete it remains to show that $\{\mu_k\}$ are the eigenvalues of $J(g)$. To this end we first show that $\{\mu_k\}$ are the eigenvalues of some self-adjoint extension of $J$. Let $\tilde{M} = -M$ and define

$$\tilde{a}_j = -\frac{\tilde{M}}{\mu_j} \lim_{r \to \infty} \left\{ \prod_{|\mu_k| \leq r} \left(1 - \frac{\mu_j}{\mu_k}\right) \prod_{|\mu_k| \leq r, k \neq j} \left(1 - \frac{\mu_j}{\mu_k}\right) \right\}.$$

From condition [4], it follows that $\tilde{a}_j > 0$ for any $j$ and $\sum_j \tilde{a}_j^{-1} = 1$, and that the function $\tilde{\sigma}(t) := \sum_{\mu_k \leq t} \tilde{a}_k^{-1}$ is a solution of the moment problem $\{s_k\}_{k=0}^{\infty}$ with $s_k$ given by (4.23). Moreover, taking into account conditions [7] and [8] one easily verifies as before that the sequences $\{\mu_j\}_j$ and $\{\tilde{a}_j\}_j$, and the function

$$\lim_{r \to \infty} \prod_{|\mu_k| \leq r} \left(1 - \frac{\zeta}{\mu_k}\right),$$

satisfy the conditions of Lemma [1]. Therefore, by Definitions 1, 2 and Corollary 2 of the Addenda and Problems of [2, Chap. 4], as well as conditions [2] and [8] it turns out that the sequence $\{\mu_j\}_j$ is a canonical sequence of nodes and $\{\tilde{a}_j\}_j$ the corresponding sequence of masses for the moment problem given by $\{s_k\}_{k=0}^{\infty}$ with $s_k$ satisfying (4.23). Hence $\tilde{\sigma}$ is a canonical solution of this moment problem. Denote by $J(\tilde{g})$ the self-adjoint extension of $J$ having $\tilde{\sigma}$ as its spectral function.

Let us consider now the functions $R_g$ and $R_{\tilde{g}}$ corresponding to $J(g)$ and $J(\tilde{g})$, respectively (see (3.11)). It is straightforward to verify that

$$R_g(\lambda_j) = \frac{a_j}{M R'_f(\lambda_j)} = R_{\tilde{g}}(\lambda_j),$$ \hspace{1cm} (4.25)
where the first equality follows from (3.12), while the second follows from (4.10) and (4.16). By (3.13), (3.14), and (4.24), one easily concludes from (4.25) that \( g = \tilde{g} \).

Remark 2. Note that, by Proposition 1, conditions 1–6 imply the interlacing of the sequences \( \{\lambda_k\}_k \) and \( \{\mu_k\}_k \).

In the previous theorem we required that the sequences \( \{\lambda_k\}_k \) and \( \{\mu_k\}_k \) do not contain the zero. This restriction was done for two reasons. First, we wanted to write conditions 1–6 without the introduction of new notation. Second, the proof would have become clumsier.

Theorem 3. Given two infinite disjoint sequences of real numbers \( \{\lambda_k\}_k \) and \( \{\mu_k\}_k \) with the point at infinity being their only accumulation point, such that \( \{\mu_k\} \ni 0 \) and \( \lambda_k \neq \lambda_j, \mu_k \neq \mu_j \) with \( k \neq j \), there exists a unique \( f \in (\mathbb{R} \setminus \{0\}) \cup \{\infty\} \) and a unique Jacobi operator \( J \neq J^* \) such that \( \{\lambda_k\}_k = \text{Sp}(J(f)) \) and \( \{\mu_k\}_k = \text{Sp}(J(0)) \) if and only if the following conditions are satisfied.

1'. The convergence exponent of the sequences \( \{\lambda_k\}_k \) and \( \{\mu_k\}_k \setminus \{0\} \) does not exceed one. Additionally, if

\[
\lim_{r \to \infty} \sum_{|\lambda_k| \leq r} \frac{1}{|\lambda_k|} = \infty, \quad \text{require that} \quad \lim_{r \to \infty} \frac{n_\lambda(r)}{r} = 0,
\]

and if

\[
\lim_{r \to \infty} \sum_{0 < |\mu_k| \leq r} \frac{1}{|\mu_k|} = \infty, \quad \text{require that} \quad \lim_{r \to \infty} \frac{n_\mu(r)}{r} = 0,
\]

where \( n_\lambda(r) \) and \( n_\mu(r) \) are the number of elements of \( \{\lambda_k\}_k \) and \( \{\mu_k\}_k \setminus \{0\} \), respectively, in the circle \( |\zeta| < r \).

2'. The limits

\[
\lim_{r \to \infty} \prod_{0 < |\mu_k| \leq r} \left(1 - \frac{\zeta}{\mu_k}\right), \quad \lim_{r \to \infty} \prod_{|\lambda_k| \leq r} \left(1 - \frac{\zeta}{\lambda_k}\right)
\]

converge uniformly on compact subsets of \( \mathbb{C} \).

3'. For all \( j \) the following inequality holds

\[
\lim_{r \to \infty} \left\{ \prod_{0 < |\mu_k| \leq r} \left(1 - \frac{\lambda_j}{\mu_k}\right) \prod_{|\lambda_k| \leq r \setminus k \neq j} \left(1 - \lambda_j \frac{\lambda_k}{\lambda_j}\right) \right\} > 0.
\]
Also, for every \( j \) with \( \mu_j \neq 0 \),
\[
\lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r, k \neq j} \left( 1 - \frac{\mu_j}{\lambda_k} \right) \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\} < 0.
\]

4’. The series below are convergent and the following equalities hold
\[
\sum_j \lambda_j^m \lim_{r \to \infty} \left\{ \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\lambda_j}{\mu_k} \right) \prod_{|\lambda_k| \leq r, k \neq j} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \right\}
= -\sum_j \mu_j^m \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r, k \neq j} \left( 1 - \frac{\mu_j}{\lambda_k} \right) \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}, \quad m = 1, 2, \ldots
\]
and
\[
\sum_j \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\mu_j}{\lambda_k} \right) \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}
= 1 - \sum_j \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r, k \neq j} \left( 1 - \frac{\mu_j}{\lambda_k} \right) \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}.
\]

5’. The series
\[
\sum_j \lambda_j^2 \lim_{r \to \infty} \left\{ \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\lambda_j}{\mu_k} \right) \right\} \quad \text{and} \quad \sum_j \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r, k \neq j} \left( 1 - \frac{\mu_j}{\lambda_k} \right) \right\}
\]
\[
\text{diverge either to } -\infty \text{ or } +\infty.
\]

6’. The series
\[
\sum_j \frac{\lambda_j^2}{1 + \lambda_j^2} \lim_{r \to \infty} \left\{ \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\lambda_j}{\mu_k} \right) \prod_{|\lambda_k| \leq r, k \neq j} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \right\}, \quad \sum_j \frac{1}{1 + \mu_j^2} \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r, k \neq j} \left( 1 - \frac{\mu_j}{\lambda_k} \right) \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}
\]
\[
\text{are convergent.}
\]

\textbf{Proof.} Let us show that 7’-10’ are necessary. First note that condition 7’ and 9’ follow from the fact that \( R_0 \) and \( R_f \) are entire functions of at most minimal type
of order one. Now, substitute (4.10) into (3.15) to obtain that

\[ a(\lambda_j) = C_0 \lim_{r \to \infty} \left\{ \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\lambda_j}{\mu_k} \right) \prod_{|\lambda_k| \leq r, k \neq j} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \right\}. \]  \hspace{1cm} (4.26)

Analogously, on the basis of (3.1) and taking into account that

\[ R'_0(\mu_j) = \begin{cases} - \lim_{r \to \infty} \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\mu}{\mu_k} \right) & \mu_j \neq 0 \\ 1 & \mu_j = 0 \end{cases}, \]  \hspace{1cm} (4.27)

one obtains

\[ a(\mu_j) = -C_0 \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\mu_j}{\lambda_k} \right) \prod_{0 < |\mu_k| \leq r, k \neq j} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}, \quad \mu_j \neq 0. \]  \hspace{1cm} (4.28)

and

\[ a(0) = C_0. \]  \hspace{1cm} (4.29)

Reasoning as in the proof of Theorem 2, conditions 3' and 4' follow from (4.26)–(4.29). Similarly, conditions 5' and 6' are obtained from (4.10), (4.14), and (4.26)–(4.29).

The proof of the sufficiency of 7'–9' repeats the argumentation of the corresponding part in the proof of Theorem 2. Here one should define

\[ C_0 := \sum_j \lim_{r \to \infty} \frac{1}{\prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\lambda_j}{\mu_k} \right) \prod_{|\lambda_k| \leq r, k \neq j} \left( 1 - \frac{\lambda_j}{\lambda_k} \right)} \]

and

\[ a_j := C_0 \lim_{r \to \infty} \left\{ \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\lambda_j}{\mu_k} \right) \prod_{|\lambda_k| \leq r, k \neq j} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \right\}. \]  \hspace{1cm} (4.30)

Further, one defines \( \sigma \) by (4.17) and consider the self-adjoint operator of multiplication \( A_\sigma \) by the independent variable in \( L^2(\mathbb{R}, d\sigma) \). The function \( \theta_k \) in \( L^2(\mathbb{R}, d\sigma) \) satisfying (4.18) is a normalized eigenvector of \( A_\sigma \) corresponding to \( \lambda_k \). We consider the function \( \varphi \) satisfying (4.19). It follows from (4.30) and 9' that \( \varphi \) is in \( L^2(\mathbb{R}, d\sigma) \). Define the linear set \( D \) by (4.20). In complete analogy with what we did before, we show that \( J := U^{-1} A_\sigma \mid_D U \) is a Jacobi operator in the limit circle case. Note that
the moments associated with this Jacobi operator, \( \{s_k\}_{k=0}^{\infty} \), are given by

\[
s_k := \frac{1}{C_0} \sum_j \lim_{r \to \infty} \left\{ \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\lambda_j}{\mu_k} \right) \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\lambda_j}{\lambda_k} \right) \right\}.
\]

(4.31)

As before, there is \( f \in \mathbb{R} \cup \{\infty\} \setminus \{0\} \) such that \( J(f) = U^{-1}A_0U \).

Let us now define

\[
\tilde{a}_j := \begin{cases} 
-C_0 \lim_{r \to \infty} \left\{ \prod_{|\lambda_k| \leq r} \left( 1 - \frac{\mu_j}{\lambda_k} \right) \prod_{0 < |\mu_k| \leq r} \left( 1 - \frac{\mu_j}{\mu_k} \right) \right\}, & \mu_j \neq 0 \\
C_0, & \mu_j = 0
\end{cases}
\]

Invoking the reasoning at the end of the proof of Theorem 2, one concludes from Definitions 1, 2 and Corollary 2 of the Addenda and Problems of [2, Chap. 4], as well as conditions 5’ and 6’, that \( \tilde{\sigma}(t) := \sum_{\mu_k \leq t} \tilde{a}_k^{-1} \) is a canonical solution of the moment problem \( \{s_k\}_{k=0}^{\infty} \) with \( s_k \) given by (4.31). Thus, \( \{\mu_k\}_k \) is the spectrum of a self-adjoint extension of \( J \). For what we discussed in Section 3 this self-adjoint extension should be \( J(0) \) since \( \{\mu_k\}_k \ni 0 \).

\[\square\]

Remark 3. Here we have the analogue of Remark 2.

Acknowledgments. We thank A. Osipov for drawing our attention to [13].

References

[1] Ahlfors, L. V.: Complex analysis: An introduction of the theory of analytic functions of one complex variable McGraw-Hill Book Co., New York, 1966.

[2] Akhiezer, N. I.: The classical moment problem and some related questions in analysis. Hafner Publishing Co., New York, 1965.

[3] Akhiezer, N. I. and Glazman, I. M.: Theory of linear operators in Hilbert space. Dover Publications Inc., New York, 1993.

[4] Aktosun, T. and Weder, R.: Inverse spectral-scattering problem with two sets of discrete spectra for the radial Schrödinger equation. Inverse Problems. 22 (2006) 89–114.

[5] Aktosun, T. and Weder, R.: The Borg-Marchenko theorem with a continuous spectrum. In Recent Advances in Differential Equations and Mathematical Physics, Contemp. Math. 412. Amer. Math. Soc., Providence, RI, 2006 15–30.

[6] Albeverio, S. and Kurasov, P.: Singular perturbations of differential operators. London Mathematical Society Lecture Note Series 271. Cambridge University Press, Cambridge, 2000.

26
[7] Berezans’kiĭ, J. M.: *Expansions in eigenfunctions of selfadjoint operators*. Translations of Mathematical Monographs 17. American Mathematical Society, Providence, R.I., 1968.

[8] Brown, B. M., Naboko, S. and Weikard, R.: The inverse resonance problem for Jacobi operators. *Bull. London Math. Soc.* 37 (2005) 727–737.

[9] Borg, G.: Uniqueness theorems in the spectral theory of $y'' + (\lambda - q(x))y = 0$. In: *Proc. 11th Scandinavian Congress of Mathematicians*. Johan Grundt Tanums Forlag, Oslo, 1952, pp. 276–287.

[10] Donoghue, W. F., Jr.: On the perturbation of spectra. *Comm. Pure Appl. Math.* 18 (1965), 559–579.

[11] Fu, L. and Hochstadt, H.: Inverse theorems for Jacobi matrices. *J. Math. Anal. Appl.* 47 (1974), 162–168.

[12] Gasymov, M. G. and Guseĭnov, G. S.: On inverse problems of spectral analysis for infinite Jacobi matrices in the limit-circle case. *Dokl. Akad. Nauk SSSR* 309(6) (1989), 1293–1296. In Russian. [Translation in *Soviet Math. Dokl.* 40(3) (1990), 627–630]

[13] Gasymov, M. G. and Guseĭnov, G. S.: Uniqueness theorems in inverse problems of spectral analysis for Sturm-Liouville operators in the case of the Weyl limit circle. *Differentsial’nye Uravneniya* 25(4) (1989), 588–599.[Translation in *Differential Equations* 25(4) (1989), 394–402]

[14] Gesztesy, F. and Simon, B.: Uniqueness theorems in inverse spectral theory for one-dimensional Schrödinger operators. *Trans. Amer. Math. Soc.* 348 (1996) 349–373.

[15] Gesztesy, F. and Simon, B.: $m$-functions and inverse spectral analysis for finite and semi-infinite Jacobi matrices. *J. Anal. Math.* 73 (1997), 267–297.

[16] Gesztesy, F. and Simon, B.: On local Borg-Marchenko uniqueness results. *Comm. Math. Phys.* 211 (2000) 273–287.

[17] Gorbachuk, M. L. and Gorbachuk, V. I.: *M. G. Krein’s lectures on entire operators*. Operator Theory: Advances and Applications, 97. Birkhäuser Verlag, Basel, 1997.

[18] Guseĭnov, G. Š.: The determination of the infinite Jacobi matrix from two spectra. *Mat. Zametki* 23(5) (1978), 709–720.

[19] Halilova, R. Z.: An inverse problem. *Izv. Akad. Nauk Azerbaidžan. SSR Ser. Fiz.-Tekhn. Mat. Nauk* 1967(3-4) (1967), 169–175. In Russian.

[20] Kreĭn, M. G.: Solution of the inverse Sturm-Liouville problem. *Doklady Akad. Nauk SSSR (N.S.)* 76 (1951), 21–24. In Russian.
[21] Kreĭn, M. G.: On the indeterminate case of the Sturm-Liouville boundary problem in the interval $(0, \infty)$. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16 (1952), 293–324. In Russian.

[22] Kreĭn, M.: On a method of effective solution of an inverse boundary problem. Doklady Akad. Nauk SSSR (N.S.) 94 (1954), 987–990. In Russian.

[23] Levin, B. Ja.: Distribution of zeros of entire functions. Translations of Mathematical Monographs 5. American Mathematical Society, Providence, R.I., 1980.

[24] Levin, B. Ja.: Lectures on entire functions. Translations of Mathematical Monographs 150. American Mathematical Society, Providence, R.I., 1996.

[25] Levitan, B. M. and Gasymov, M. G.: Determination of a differential equation by two spectra. Uspehi Mat. Nauk 19(2 (116)) (1964), 3–63.

[26] Marĉenko, V. A.: Some questions of the theory of one-dimensional linear differential operators of the second order. I. Trudy Moskov. Mat. Obšč. 1 (1952), 327–420. In Russian. [Translation in Am. Math. Soc. Transl. (ser. 2) 101 (1973) 1–104].

[27] Silva, L. O. and Weder, R.: On the two spectra inverse problem for semi-infinite Jacobi matrices. Math. Phys. Anal. Geom. 3(9) (2006), 263–290.

[28] Simon, B.: The classical moment problem as a self-adjoint finite difference operator. Adv. Math. 137(1) (1998), 82–203.

[29] Teschl, G.: Trace formulas and inverse spectral theory for Jacobi operators. Comm. Math. Phys. 196(1) (1998), 175–202.

[30] Teschl, G.: Jacobi operators and completely integrable nonlinear lattices. Mathematical Surveys and Monographs 72. American Mathematical Society, Providence, RI, 2000.

[31] Titchmarsh, E. C.: The theory of functions. Oxford Univ. Press, London, 1939.

[32] Weikard, R.: A local Borg-Marchenko theorem for difference equations with complex coefficients. In Partial differential equations and inverse problems, Contemp. Math. 362. Amer. Math. Soc., Providence, RI, 2004 403–410.