Field-emission from parabolic tips: current distributions, the net current and effective emission area

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Field emission from nano-structured emitters primarily takes place from the tips. Using recent results on the variation of enhancement factor around the apex (Biswas et al, Ultramicroscopy 185, 1-4 (2018)), analytical expressions for surface distribution of net emitted electrons as well as the total and normal energy distributions are derived in terms of the apex radius \( R_a \) and the local electric field at the apex \( E_a \). Formulae for the net emitted current and effective emission area in terms of these quantities are also obtained.

I. INTRODUCTION

Field emission is increasingly a preferred source of electrons in several applications including vacuum microwave and terahertz devices, microscopy, lithography and space and medical applications. Theoretical prediction of the field-emission current largely relies on the Fowler-Nordheim (FN) model which assumes a planar emission surface in addition to the free-electron model. The Murphy-Good adaptation of the Fowler-Nordheim law serves as the basis of modern analytical field emission formulae. It includes the image-charge correction and makes allowance for curved emitters by replacing the electric field on the planar emitter surface by the (enhanced) local electric field \( E(r) \) at a point \( r \) on the curved emitter surface. This quasi-planar formalism serves as a reasonable approximation to the current density for even high aspect-ratio curved field emitter when the local field at the emitter surface is not too low. The bottleneck in most theoretical predictions lies in not knowing much about the enhanced electric field \( E(r) \) or its local variation. In the absence of such information, it is impossible to predict the net emission current or the distribution of electrons (that are emitted from a single emitter comprising of the apex and its neighbourhood) with respect to launch angle, the total and normal energy.

Our aim here is to provide a partial rectification of this lacuna for the class of smooth axially symmetric curved emitters placed on a planar surface in situations where the asymptotic (away from the emitter apex) electric field \( E_0 \) is uniform and aligned parallel to the emitter axis. Under such circumstances, a recent result shows that the local field

\[
E(r) = E_a \cos \hat{\theta}
\]

where \( E_a = \gamma_a E_0 \) is the field at the emitter apex,

\[
\cos \hat{\theta} = \frac{z/h}{\sqrt{(z/h)^2 + (\rho/R_a)^2}}
\]  

and \( \gamma_a \) is the field enhancement factor at the apex. In Eq. 2, \( R_a \) is the apex radius of curvature and \( h \) is the height of the emitter-apex measured from the planar surface. It is assumed that the curved protrusion acting as a field emitter is aligned in the direction of the externally applied field (the \( z \)-axis) and has a smooth tip described by \( z = g(\rho) \) where \( \rho^2 = x^2 + y^2 \) for a point on the emitter surface. Thus a Taylor expansion at the apex yields

\[
z = h + \rho \frac{dg}{d\rho} \bigg|_{\rho=0} + \frac{\rho^2}{2} \frac{d^2g}{d\rho^2} \bigg|_{\rho=0} + \ldots \simeq h - \frac{\rho^2}{2R_a}
\]  

since \( (dg/d\rho) |_{\rho=0} = 0 \). Thus, the tip is locally parabolic. Using Eq. 1 and assuming that \( E_a \) is known, an ideal theoretical prediction for field-emission current from a single vertically aligned emitter can in principle be made. This also allows us to address the issue of distribution of electrons from a single emitter. A joint distribution of the net emitted electrons with respect to generalization angle \( \theta \), the total energy \( E_T \) and the normal energy \( E_N \) can thus be written down immediately using Eqs. 1 and 4 while distributions with respect to individual quantities can be arrived at by integrating over the other two quantities. For instance, we denote by the quantity \( f_S(\hat{\theta}) \) the current emitted from the surface at generalized angle \( \hat{\theta} \) between \( \hat{\theta} \) and \( \hat{\theta} + d\hat{\theta} \). We refer to this as the surface-angular current distribution and \( f_S(\hat{\theta}) \) as the corresponding density. While the tip or apex (\( \rho = 0 \)) has the maximum field enhancement, the emitted current from its immediate neighbourhood is clearly negligible as the emission area is small. On the other hand, as \( \theta \) increases, the enhancement factor \( \gamma \) decreases while the size of the area element increases. Thus the surface-angular distribution of emitted electrons must have a peak close to the apex. We show here that this distribution is universal in the sense that all emitters with a given apex curvature and local apex field have identical distribution with respect to \( \theta \). We also establish that \( \hat{\theta} \) is in fact the launch angle \( \theta_L \) for sharp emitters and thus has a physical significance.

Similarly, the total-energy distribution \( f_T(E_T) \) and normal-energy distribution \( f_N(E_N) \) of emitted electron current are also of interest. These are distinct from earlier theoretical studies that deal with the energy distribution of current density at a single point on the emitter surface. The two sets of distributions are thus very different but can be related by a surface integration over the emitter surface using the variation in field enhancement factor in the neighbourhood of the apex.
thus study distributions of the total emitted current from an emitter rather than the emitted current density at a point on the emitter.

In the following, we shall establish analytical formulae for total emitted current from a parabolic tip as well as the various distributions mentioned above in terms of \(E_a\), \(R_a\) and \(h\). While \(R_a\) and \(h\) are experimentally measurable quantities, we shall assume the local field at the apex \(E_a\), to be known as well\([12]\).

### II. THE SURFACE (ANGULAR) DISTRIBUTION OF EMITTED ELECTRONS

We shall first address the question of surface distribution of emitted electrons. It is assumed that the tip is smooth having a radius of curvature \(R_a > 20\) nm\([13,15]\).

In the regime, \(R_a > 20\) nm, and for local field strengths less than \(10\) V/nm, the zero-temperature Murphy-Good formulation\([10]\) of Fowler-Nordheim (FN) type field emission formula can be used for the local electron current density, \(J(r)\), at a point \(r\) on the emitter surface. It can be expressed as\[1\]

\[
J(r) = \frac{1}{t_F(r)} \frac{A_{\text{FN}}}{\phi} (E(r))^2 e^{-B_{\text{FN}}V_F(r)\phi^{3/2}/E(r)}. \tag{4}
\]

In the above, the free electron model is assumed and barrier lowering due to the image potential is incorporated. Here, \(A_{\text{FN}} \approx 1.541434\) \(\mu\)A eV\(^{-1}\) and \(B_{\text{FN}} \approx 6.830890\) eV\(^{-3/2}\) nm\(^{-1}\) are the conventional FN constants, \(\phi\) is the work function while \(V_F \approx 1 - f + \frac{1}{2k}f\ln f\) and \(t_F \approx 1 + f/9 - \frac{1}{18k}\ln f\) in \(f\) and \(t_F\) are correction factors due to the image potential with \(f = f(r) \approx c_S^2 E(r)/\phi^2\). The local field \(E(r)\) is the magnitude of the local electric field at the emitter surface while \(c_S\) is the Schotky constant with \(c_S^2 = 1.439965\) eV\(^2\) V\(^{-1}\) nm. Note that Eq. (4) does not depend on the Fermi energy as it relies on a WKB formula for the transmission coefficient. This induces errors that have been extensively tabulated by Mayer\[16\] for different work function, applied field and Fermi energy. We shall ignore these subtleties here and merely note that a correction factor to Eq. (4) can be used when absolute or unnormalized quantities are evaluated.

The current emitted from the strip on the emitter surface between radius \(\rho\) and \(\rho + d\rho\) can be expressed as

\[
f(r)dr = J(r)2\pi\rho \sqrt{1 + (dz/dr)^2} d\rho \tag{5}
\]

where \(r = (\rho, z, \phi)\). For axially symmetric vertically aligned emitters, \(z = h - \rho^2/(2R_a)\) near the tip. Thus,

\[
f(r)dr = J(r)2\pi\rho \sqrt{1 + (\rho/R_a)^2} d\rho. \tag{6}
\]

The transformation from \(d\rho\) to \(d\theta\) where \(\tan \theta = \rho/z\) should in principle give the angular distribution of emitted electrons. However, it is clear that such a distribution cannot be universal and will depend on the emitter height and radius of curvature. A transformation to normalized co-ordinates \(\tilde{\rho} = \rho/R_a\) and \(\tilde{z} = z/h\) is however helpful. With \(\tan \tilde{\theta} = \tilde{\rho}/\tilde{z}\), the surface angular distribution is expressed as

\[
f_S(\tilde{\theta})d\tilde{\theta} = J(r)2\pi\tilde{\rho} \sqrt{1 + \tilde{\rho}^2} \frac{d\rho}{d\tilde{\theta}} \tag{7}
\]

where, using \(\sin \tilde{\theta} = \tilde{\rho}/\sqrt{\tilde{\rho}^2 + \tilde{\rho}^2}\), we have

\[
\cos \tilde{\theta} d\tilde{\theta} = \frac{d\rho}{\tilde{R}_a} \sqrt{1 - \tilde{\rho}^2} \left[1 - \frac{zR_a}{\tilde{h}^2}\right] \tag{8}
\]

\[
\approx \frac{d\rho}{\tilde{R}_a} \sqrt{1 + \tilde{\rho}^2} \cos \tilde{\theta} \tag{9}
\]

where we have neglected \(zR_a/\tilde{h}^2\) for sharp emitters \((h >> R_a)\). Thus

\[
\frac{d\rho}{d\tilde{\theta}} \approx \tilde{R}_a \sqrt{1 + \tilde{\rho}^2} \cos \tilde{\theta} \tag{10}
\]

since for sharp emitters, emission is limited to regions for which \(z \approx h\) (or \(\tilde{z} \approx 1\)). Finally,

\[
f_S(\theta)d\theta = 2\pi R_a^2 J(r) \frac{\hat{\rho}}{\sqrt{1 + \hat{\rho}^2}} \frac{(1 + \hat{\rho}^2)^{3/2}}{\cos \theta} d\hat{\theta} \tag{12}
\]

\[
\approx 2\pi R_a^2 J(r) \frac{\sin \hat{\theta}}{\cos^3 \hat{\theta}} d\hat{\theta} \tag{13}
\]

where

\[
J(r) = \frac{1}{t_F(\theta)^2} A_{\text{FN}} E_a^2 \cos^2 \theta e^{-B_{\text{FN}}V_F(\theta)\phi^{3/2}/(E_a \cos \theta)}. \tag{14}
\]

Note that \(t_F \approx 1 + f/9 - \frac{1}{18}\tilde{f}\ln f\) and \(V_F \approx 1 - f - \frac{1}{9}\tilde{f}\ln f\) also depend on \(\tilde{\theta}\) through \(f = c_S^2 E_a \cos \tilde{\theta}/\phi^2\). Putting everything together, we have

\[
f_S(\theta)d\theta = 2\pi R_a^2 \frac{\sin \hat{\theta}}{\cos^3 \hat{\theta}} A_{\text{FN}} E_a^2 \frac{E_a}{t_F(\theta)} e^{-B_{\text{FN}}V_F(\theta)\phi^{3/2}/(R_a \cos \theta)} d\hat{\theta}. \tag{15}
\]

Eq. (15) forms the central result of this section. It shows that in normalized co-ordinates, the angular distribution of emitted electrons is universal for sharp emitter tips for a given apex local field \((E_a)\) and radius of curvature \((R_a)\).

Fig. [I] shows a typical (un-normalized) angular density of currents for an emitter with apex radius \(R_a = 0.26\) \(\mu\)m and height 1500 \(\mu\)m, having an enhancement...
factor $\gamma_a = 617$. The work function $\phi = 4.5$ eV and $E_F = 8.5$ eV. The solid squares are “exact” results and have been obtained using a numerical scheme for the transmission coefficient instead of the WKB formula that is used in Eq. (14). Consequently, the exact values are multiplied by a correction factor $\lambda$ to match the prediction of Eq. (15). For $E_0 = 3, 6$ and 9 MV/m the value of $\lambda$ is 1.273, 1.15 and 1.18 respectively. Note that Fig. 1 indicates that emission away from the apex (larger $\theta$) becomes significant at higher applied fields.

The angular variation is thus reproduced exactly by Eq. (15) Keeping in mind the correction factor, the angular variation formula may be expressed as

$$f_S(\theta) d\theta = \lambda 2\pi R_a^2 \frac{\sin \hat{\theta}}{\cos^2 \theta} \frac{A_{FN}}{\phi} \frac{E_a^2}{t_F^2(\theta)} e^{-\frac{B_{FN} v_F(\theta)\sin \theta}{E_a}} d\theta$$

(16)

where $\lambda$ accounts for the discrepancy between the exact current density and Eq. (14). Note that when $f_S(\theta)$ is normalized for a probabilistic interpretation, the factor $\lambda$ is inconsequential.

The quantity $f_S(\theta) d\theta$ describes the distribution of electrons launched from the surface of a parabolic emitter. Clearly, there is little emission along the axis and the position where the peak occurs shifts away from the axis as the local apex field is increased.

Note that the distribution can be equivalently expressed in terms of $\rho$ and for sharp parabolic emitters $(R_a/\rho << 1)$, $\rho \simeq R_a \tan \hat{\theta}$. Using this relation between $\rho$ and $\tan \theta$, it is easy to show that the angle $\theta_L$ that the normal (at any point $\rho$ on the parabolic surface $z = h - \rho^2/2R_a$) makes with the emitter axis, is such that $\tan \theta_L = \rho / R_a \simeq \tan \hat{\theta}$. Thus, $\theta_L \simeq \hat{\theta}$ so that $f_S(\theta)$ also describes the distribution of launch angles ($\theta_L$) of electrons from the surface of a parabolic emitter.

### III. CURRENT FROM A SINGLE EMITTER

The current emitted by a single sharp emitter tip can be calculated by integrating Eq. (15) over $\hat{\theta}$. Writing $A = 2\pi R_a^2$ and collecting the constants (non-$\hat{\theta}$ dependent terms) together as

$$C = \lambda A \frac{A_{FN}}{\phi} E_a^2$$

(17)

the current from the emitter can be expressed as

$$I = C \int_0^{\hat{\theta}_{max}} \frac{\sin \hat{\theta}}{\cos^2 \theta} \frac{1}{t_F^2(\theta)} e^{-B_{v_F}(\theta) / \cos \hat{\theta}} d\hat{\theta}$$

(18)

where $B = \frac{B_{FN} \phi^{3/2}}{E_a}$. Note that for sharp emitters, $\hat{\theta}_s \simeq \pi/4$ corresponds to $\rho = R_a$ while $\hat{\theta} = \pi/3$ corresponds to $\rho \simeq 2R_a$. It is seen that the quadratic approximation is generally valid upto $\hat{\theta} = \pi/3$. Also, since the current beyond $\hat{\theta} = \pi/3$ is extremely small, the upper limit of integration can for all practical purposes be taken as $\hat{\theta}_{max} = \pi/3$. Thus,

$$I = C \int_0^{\pi/3} \frac{\sin \hat{\theta}}{\cos^2 \theta} \frac{1}{t_F^2(\theta)} e^{-B_{v_F}(\theta) / \cos \hat{\theta}} d\hat{\theta}$$

(19)

$$= C \int_1^2 \frac{1}{t_F^2(u)} e^{-B_{v_F}(u)u} du$$

(20)

where we have used the substitution $u = 1 / \cos \hat{\theta}$. Since most of the emission occurs near the apex, it is profitable to use the substitution $u = 1 + x$. Thus,

$$I = C \int_0^1 \frac{1}{t_F^2(x)} e^{-B_{v_F}(x)(1+x)} dx$$

(21)

where

$$v_F(x) = 1 - \frac{f_0}{1+x} + \frac{1}{6} \frac{f_0}{1+x} \ln \left( \frac{f_0}{1+x} \right)$$

(22)

$$t_F(x) = 1 + \frac{f_0}{9} \frac{1}{1+x} - \frac{1}{18} \frac{f_0}{1+x} \ln \left( \frac{f_0}{1+x} \right)$$

(23)

$$f_0 = \frac{c^2 E_a}{\phi^2}.$$  

(24)

An expansion of $v_F(x)(1+x)$ and $1/t_F^2(x)$ in powers of $x$ is helpful in carrying out the integration in Eq. (21) since the dominant contribution is close to $x = 0$. Retaining the first two terms yields,
where

\[ D_0 = \nu_0 = 1 - f_0 + \frac{1}{6} f_0 \ln(f_0) \]
\[ D_1 = 1 - \frac{1}{6} f_0 \]
\[ F_0 = \frac{1}{\nu_0^2} = \frac{1}{1 + \frac{f_0^2}{9} - \frac{f_0}{18} \ln(f_0)}^2 \]
\[ F_1 = \frac{1}{9} \left( \frac{f_0 - f_0 \ln(f_0)}{1 + \frac{f_0^2}{9} - \frac{f_0}{18} \ln(f_0)} \right)^3. \]

Thus,

\[ I = C \int_0^1 (F_0 + F_1 x)e^{-B(D_0 + D_1 x)}dx \]
\[ \simeq C e^{-BD_0} \left[ \frac{F_0}{BD_1} + \frac{F_1}{(BD_1)^2} - \frac{e^{-BD_1}}{BD_1} \left( \frac{F_0}{F_0 + \frac{F_1}{F_0} BD_1} \right) \right] \]
\[ = 2\pi R_a^2 J_{\text{apex}} \mathcal{G} \]

is the total current emitted by a single emitter which can be expressed in terms of the apex current density \( J_{\text{apex}} \), the area \( A = 2\pi R_a^2 \) of a hemisphere of radius \( R_a \) and the area factor \( \mathcal{G} \) where

\[ \mathcal{G} = \frac{1}{BD_1} + \frac{F_1}{F_0 (BD_1)^2} - \frac{e^{-BD_1}}{BD_1} \left( \frac{F_0}{F_0 + \frac{F_1}{F_0} BD_1} \right). \]

as predicted by Eq. 14 Fig. 2 shows a plot of the area factor \( \mathcal{G} \) for local apex fields in the range 2-10 V/nm using Eq. 32 alongside the prediction using Eq. 14 and the exact result using numerically determined transmission coefficient. The agreement is excellent except at higher apex fields where the analytical result (based on Eq. 14) is closer to the exact result. Note that the factor \( \lambda \) is immaterial as far as the area factor is concerned.

For materials with workfunction in the range 4-5 eV and apex fields in the range 1-10 V/nm, the term \( e^{-BD_1}/(BD_1) \) is negligible. A simple and reasonably accurate formula for the area factor is thus

\[ \mathcal{G} \simeq \frac{1}{BD_1} + \frac{F_1}{F_0 (BD_1)^2} \]
\[ = \frac{E_a}{B_{\text{FN}} \phi^{3/2}} \left( 1 - \frac{f_0}{6} \right) \left[ 1 + \frac{F_1}{F_0} \frac{1}{F_0 (BD_1)} \right], \]

where the second term in the square bracket is a small correction even at high apex fields and may be neglected. For small \( E_a \) therefore, \( \mathcal{G} \sim E_a^{19} \).

IV. THE TOTAL ENERGY DISTRIBUTION

The joint distribution of emitted electrons \( f_J \) with respect to the quantities \( \theta \), the normal energy \( E_N \) and the total energy \( E_T \) can be expressed as

\[ f_J(\theta, E_N, E_T)d\theta dE_N dE_T = \left[ A \frac{\sin \theta}{\cos^4 \theta} n(E_T) D(E_N, \theta) \right] d\theta dE_N dE_T. \]

where the supply function density

\[ n(E_T) = \frac{2em}{(2\pi)^2 h^3} f_{FD}(E_T, T) \]

and \( f_{FD}(E_T, T) \) is the Fermi-Dirac distribution at a temperature \( T \). By integrating over any two of these quantities, the distribution over a third quantity can be determined. It can also be used to arrive at joint distribution of any two of \( (\theta, E_N, E_T) \) by integrating over the third. In the rest of this paper, we shall be interested in the distribution of emitted electrons with respect to the total and normal energy . For a given normal energy, the total energy ranges from \( E_N \) to infinity (or \( E_F \) at zero temperature) while the normal energy component which determines the transmission coefficient, varies from 0 to the total energy \( E_T \). The total energy distribution of the current can thus be expressed as

\[ f_T(E_T)dE_T = \left[ A \int_{\cos \theta}^{\sin \theta} \frac{d\theta}{\cos^4 \theta} f_{FD}(E_T, T) \right] dE_T. \]

FIG. 2. The variation of the area factor \( \mathcal{G} \) with the apex field \( E_a \) using the exact transmission coefficient (triangle), using Eq. 14 for the current density (squares) and Eq. 32 (continuous line).

Eqs. 31 and 32 approximate the total current from an emitter and the effective emission area quite accurately
where \( \mathcal{E} \) and \( \mathcal{E}_N \) are the total and normal energy respectively, \( D(\mathcal{E}_N) \) is the transmission coefficient at a normal energy \( \mathcal{E}_N \):

\[
D(\mathcal{E}_N, \tilde{\theta}) = e^{-\nu_F(\cos \tilde{\theta}) \beta / \cos \tilde{\theta} e^{-(\mathcal{E}_F - \mathcal{E}_N) t_F(\cos \tilde{\theta}) / d_F \cos \tilde{\theta}}} (37)
\]

with \( 1/d_F = g_e \phi_1^{1/2}/E_a \) and \( g_e = \sqrt{8m/\hbar^2} \approx 10.24634 \text{ eV}^{-1/2} \text{nm}^{-1} \).

The integration over the normal energy \( \mathcal{E}_N \) can be carried over \((0, \mathcal{E}_T)\) to yield the joint distribution \( f(\tilde{\theta}, \mathcal{E}_T) \)

\[
f(\tilde{\theta}, \mathcal{E}_T) = K e^{-\nu_F(\cos \tilde{\theta}) \sin \tilde{\theta} e^{-(\mathcal{E}_F - \mathcal{E}_T) t_F/d_F \cos \tilde{\theta}}} (38)
\]

and on integrating \( f(\tilde{\theta}, \mathcal{E}_T) \) over \( \tilde{\theta} \), the total energy distribution of emitted electrons can be obtained as

\[
f_T(\mathcal{E}_T) = K \int_0^{\pi/3} d\tilde{\theta} e^{-\nu_F(\cos \tilde{\theta}) \sin \tilde{\theta} e^{-(\mathcal{E}_F - \mathcal{E}_T) t_F/d_F \cos \tilde{\theta}}} \cos \tilde{\theta} \ t_F (39)
\]

where \( K = K_0 f_F (\mathcal{E}_T, T) d_F, \ K_0 = \frac{2e^m}{(2\pi)^2 \hbar^2} 2\pi R_a^2 \) and \( \nu_F \) and \( t_F \) are functions of \( \cos \tilde{\theta} \).

\[
\nu_F(x)(1 + x) = D_0 + D_1 x \tag{41}
\]

\[
t_F(x)(1 + x) = G_0 + G_1 x \tag{42}
\]

\[
\frac{1 + x}{t_F(x)} = H_0 + H_1 x \tag{43}
\]

where \( D_0 \) and \( D_1 \) have been defined before and

\[
G_0 = t_0 = 1 + \frac{f_0}{9} - \frac{f_0}{18} \ln f_0 \tag{44}
\]

\[
G_1 = 1 + \frac{1}{18} \tag{45}
\]

\[
H_0 = \frac{1}{1 + \frac{f_0}{9} - \frac{f_0}{18} \ln f_0} = \frac{1}{G_0} \tag{46}
\]

\[
H_1 = \frac{1 - \frac{f_0}{6} - \frac{1}{18} f_0 \ln f_0}{(1 + \frac{f_0}{9} - \frac{1}{18} f_0 \ln f_0)^2}. \tag{47}
\]

The integrations can now be carried out so that the total energy probability density reduces to

\[
f_T(\mathcal{E}_T) = K e^{-\delta_0} \left[ (H_0 + \frac{H_1}{\delta_1}) \frac{1 - e^{-\delta_1}}{\delta_1} - \frac{H_1}{\delta_1} e^{-\delta_1} \right] \tag{48}
\]

where

\[
\delta_0 = BD_0 + (\mathcal{E}_F - \mathcal{E}_T)G_0/d_F \tag{49}
\]

\[
\delta_1 = BD_1 + (\mathcal{E}_F - \mathcal{E}_T)G_1/d_F. \tag{50}
\]

The total energy distribution can thus be expressed in terms of the apex electric field \( E_a \) and radius of curvature \( R_a \).

The total energy distribution of Eq. \( \text{[48]} \) is compared with the exact result obtained using numerically determined transmission coefficients in Fig. \( \text{[3]} \) at zero temperature. The density \( f_T(\mathcal{E}_T) \) is normalized in both cases so that the correction factor \( \lambda \) is immaterial. The agreement is excellent for the range of field strengths considered. Also, the longer tail at higher applied fields indicates larger contributions to the current from lower energy electrons. Thus, Eq. \( \text{[48]} \) serves as a good approximation for the total energy distribution and can be used to determine the peak position (also at non-zero temperature) with respect to the local field at the apex and the work function \( \text{[19]} \).

As in case of the area factor, the distribution \( f_T(\mathcal{E}_T) \) can be simplified considerably, for typical fields and work function of interest in field emission, without compromising much with its accuracy. A good first approximation that is reasonably accurate is

\[
f_T(\mathcal{E}_T) \approx K H_0 e^{-\delta_0 \delta_1} \tag{40}
\]

\[
= A \frac{2e^m}{\delta_1 (2\pi)^2 \hbar^3} \frac{d_F}{t_0} e^{-\frac{g_F e^a q_2}{2 \pi \hbar}} \nu_F e^{-(\mathcal{E}_F - \mathcal{E}_T) t_0/d_F} \tag{51}
\]
which can be compared with the total energy distribution of the current density at the apex

\[ f_T^{\text{apex}}(E_T) = \frac{2e\eta}{(2\pi)^{3/2}h^3} \frac{dF}{t_0} e^{-\frac{B_FN\phi^{3/2}}{E_a}m \epsilon^{-(E_F-E_T)t_0/d_F}}. \]

Since the two densities are unnormalized, the difference in this simplified picture lies essentially in the term. As a measure of the difference, the mean energy of the emitted electrons can be calculated. For the total energy distribution of the current density at the apex, we get the well known result

\[ \langle E_T \rangle = E_F - \frac{dF}{t_0} \Gamma(-1, \alpha) \]

where

\[ \alpha = \frac{B_FN\phi^{3/2}}{E_a}t_0(1 - f_0/6) + \frac{f_0/18}{1 + f_0/18}. \]

A comparison shows that the difference in mean electron energies is about 10% for fields exceeding 5 V/nm with \( \langle E_T \rangle \) being higher than \( \langle E_T^{\text{apex}} \rangle \).

V. THE NORMAL ENERGY DISTRIBUTION

The normal energy distribution can similarly be obtained by integrating \( f_J \) over \( \tilde{\theta} \) and \( E_T \). It is a quantity of interest in its own right and can be used in determining conditional distributions. It can be expressed as

\[ f_N(E_N)dE_N = [A \int d\tilde{\theta} \frac{\sin \tilde{\theta}}{\cos^4 \tilde{\theta}} D(E_N, \tilde{\theta}) \int_0^\infty n(E_T)dE_T]dE_N. \]

The integration over the total energy can be performed easily and with the approximation \( k_B T \ln (1 + e^{\frac{\epsilon - E_F}{k_B T}}) \approx E_F - E_N \), the expression for the normal energy probability density takes the form

\[ f_N(E_N) = K_0(E_F - E_N) \int_0^{\pi/3} \frac{\sin \tilde{\theta}}{\cos^4 \tilde{\theta}} D(E_N, \tilde{\theta})d\tilde{\theta} \]

where the transmission coefficient \( D(E_N) \) depends on \( \tilde{\theta} \). The joint distribution \( f(E_N, \tilde{\theta})dE_Nd\tilde{\theta} \) is thus

\[ f(E_N, \tilde{\theta}) = K_0(E_F - E_N) \frac{\sin \tilde{\theta}}{\cos^4 \tilde{\theta}} D(E_N, \tilde{\theta}) \]

where \( f(E_N, \tilde{\theta})dE_Nd\tilde{\theta} \) measures the current with normal energy between \( E_N \) and \( E_N + dE_N \) and (generalized) angle between \( \tilde{\theta} \) and \( \tilde{\theta} + d\tilde{\theta} \).

![Figure 4](image-url)

FIG. 4. The normalized normal-energy density \( f_N(E_N) \) of Eq. 61 for the system described in Fig. 1 at zero temperature for (a) \( E_a = 1.85 \) V/nm (square) (b) \( E_a = 3.70 \) V/nm (circle) and (c) \( E_a = 5.55 \) V/nm (triangle). The continuous lines are the corresponding exact results using numerically determined transmission coefficients.

Using the transformations \( u = 1/\cos \tilde{\theta} \) and \( v = 1 + x \) and the expansions for \( \nu_F(x)(1 + x) \) and \( t_F(x)(1 + x) \), Eq. 58 can be expressed as

\[ f_N(E_N) = K_0(E_F - E_N) \int_0^1 e^{-B(D_0 + D_1x)}(1 + x)^2 \times 
\]

\[ e^{-\frac{(E_F-E_N)}{\eta F}}(G_0 + G_1x)dE_N. \]

The integrations can be performed to yield

\[ f_N(E_N) = K_0 e^{-\delta_2(E_F - E_N)} \left( 1 - e^{-\delta_3} \right) \left[ \frac{2}{\delta_4} - \frac{3}{\delta_3} e^{-\delta_3} - \frac{2}{\delta_3} e^{-3\delta_3} \right] \]

where \( K_0 \) and

\[ \delta_2 = BD_0 + (E_F - E_N)G_0/d_F \]

\[ \delta_3 = BD_1 + (E_F - E_N)G_1/d_F. \]
Eq. [61] is found to be a good approximation of the normal-energy distribution of emission current as shown in Fig. [5].

As seen before, with an increase in applied field, the spread in the normal energy distribution increases and the peak shifts away from the Fermi energy. Eq. [61] can be used to determine other quantities of interest such as the peak position at non-zero temperature with respect to the local field at the apex and the work function.

The result may continue be of some significance when these conditions are somewhat relaxed.

An accurate use of these expressions for making useful predictions for experiments is however limited by our sketchy knowledge about the apex field enhancement factor, $\gamma_\alpha$, alternately the electric field at the apex, $E_\alpha$. A recent advancement in this direction[21] is likely to bring theoretical predictions closer to experimental results at least for single emitters.

VI. SUMMARY AND DISCUSSIONS

We have derived analytical expressions for total field-emission current, its distribution on the emitter surface in terms of generalized angle, $\theta$ (which coincides with the distribution of launch angles) as well as the distributions with respect to the total and normal energies. In the process, we also provide joint distributions of $(\hat{\theta}, E_T)$ and $(\hat{\theta}, E_N)$ and an expression for the effective emission area.

All of these are based on a recent result on the variation of field enhancement factor near the apex of a smooth emitter. Despite the approximations used, the expressions are in good agreement with exact results where the transmission coefficients are evaluated numerically.

The results presented here are strictly valid when the emitter has a uniform work function over the emission surface and when it is placed on a flat surface with its axis parallel to an asymptotic uniform electrostatic field. The result may continue be of some significance when these conditions are somewhat relaxed.

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\[ f_N(E_N) \approx k_0 e^{-\delta_3 (E_F - E_N)} \frac{1}{\delta_3} \]

\[ = \frac{k_0}{\delta_3} (E_F - E_N) e^{-\frac{B_{EN}\phi_3/2}{E_a}} \nu_0 e^{-\frac{E_F-E_N}{\nu_0}} \]

which can be compared with the normal energy distribution of the current density at the apex

\[ J_N^{\text{apex}}(E_N) = \frac{k_0}{A} (E_F - E_N) e^{-\frac{B_{EN}\phi_3/2}{E_a}} \nu_0 e^{-\frac{E_F-E_N}{\nu_0}}. \]

Since the densities are unnormalized, the difference essentially lies in the term

\[ \frac{1}{\delta_3} = \frac{B_{EN} \phi_3/2}{E_a} (1 - f_0) + \frac{(E_F - E_N)}{d_N} (1 + f_0/18). \]

As in case of the total energy, the mean normal energies can be calculated in order to compare the two distributions. Thus, we have

\[ \langle E_N^{\text{apex}} \rangle = E_F - 2 \frac{d_F}{t_0} \]

while

\[ \langle E_N \rangle = E_F - 2 \frac{d_F}{t_0} \frac{\Gamma(-2, \alpha)}{\Gamma(-1, \alpha)} \]

where $\alpha$ is defined in Eq. [56]. The difference for $E_\alpha > 5 \ \text{V/nm}$ is again about 10% with $\langle E_N \rangle$ higher than $\langle E_N^{\text{apex}} \rangle$. 

\[ \langle E_N \rangle = \frac{d_F}{t_0} \frac{\Gamma(-2, \alpha)}{\Gamma(-1, \alpha)} \]
The values of $\lambda$ follow the trend arrived at by Mayer\textsuperscript{16}. A recipe for determining the discrepancy factor between the transfer-matrix (TM) and Murphy-Good (MG) formalism at arbitrary $\phi$, $E_F$ and the local external field $E_a$ is given in Table 1 of Mayer\textsuperscript{16}.

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