Topological Excitation in an Antiferromagnetic Bose-Einstein Condensate

Yi-shi Duan, Jun-ping Wang∗, Xin Liu and Peng-ming Zhang
Institute of theoretical physics, Lanzhou University, Lanzhou 730000, P. R. China

February 12, 2003

Abstract

Two kinds of topological excitations, skyrmions and monopoles, are studied. It is revealed that these two types of excitations originate from different order parameter fields that reflect different spatial distributions of the Neel vector $\vec{m}$. Critical phenomena of the generation and annihilation of skyrmion-antiskyrmion pairs are also discussed.

1 Introduction

The realization of a spinor Bose-Einstein condensate (BEC) [1, 2, 3] offers the opportunity to find richer topological excitations in such dilute atomic gases. The spinor condensate is described by a macroscopic wavefunction, $\Psi = \sqrt{n}\zeta$, where $n$ is the total density of the gas, $\zeta$ is a normalized spinor that determines the average local spin through the relation $\langle S \rangle = \zeta^* S_{ab} \zeta_b$, and $S$ represents the usual spin matrices obeying the commutation relations $[S_\alpha, S_\beta] = i\epsilon_{\alpha\beta\gamma} S_\gamma$ [4]. In the case of spin-1 bosons, consideration of two ground states is necessary, because the effective interaction between two spins can be either ferromagnetic or antiferromagnetic [5, 6]. Also, there is a fundamental difference between the possible topological excitations of the two states [4]. In the antiferromagnetic case, the ground state energy is minimized for $\langle S \rangle = 0$, and the spinor $\zeta$ is given by

$$\zeta = \frac{e^{i\vartheta}}{\sqrt{2}} \begin{pmatrix} -m^x + im^y \\ \sqrt{2}m^z \\ m^x + im^y \end{pmatrix},$$

(1)

∗Corresponding author; e-mail: wangjp0101@st.lzu.edu.cn
where $\vartheta$ is the superfluid phase and $\vec{m}$ is the unit vector field known as the Néel vector [7]. It can be seen from (1) that the parameter space for the spinor $\zeta$ is $U(1) \times S^2$, because both its overall phase and the spin quantization axis can be chosen freely. Our interest is in the topological excitations that are not analogous to the vortex excitations in the scalar condensate and find $\pi_2(U(1) \times S^2) = e \oplus \pi_2(S^2) = \mathbb{Z}$. The meaning of this result is twofold. It not only reveals the existence of a monopole excitation in a three-dimensional antiferromagnetic BEc [7] but also implies the existence of a skyrmion excitation in the two-dimensional case [8, 9]. Both types of excitations have been studied in other fields of physics. Skyrmions appear in planar condensed matter systems that can be described by the nonlinear sigma model (NSM) in the continuum limit, including high temperature superconductors and systems exhibiting the quantum Hall effect [10]. Monopoles appear in gauge field theory [11]. However, the creation of such topological structures in BEc presents an exciting opportunity to study the properties of skyrmions and monopoles in exquisite detail, both theoretically and experimentally.

In this work, we study these two kinds of topological excitations, skyrmions and monopoles, in an antiferromagnetic spin-1 BEc in the context of the $\phi$-mapping topological current theory [12, 13, 14, 15]. Because these two kinds of excitations originate from different spatial distributions of the Néel vector field $\vec{m}$, it is necessary to introduce additional vector order parameter fields in order to study these two excitations individually. It is shown that quantities such as position, density and velocity of the two excitations expressed in terms of the corresponding order parameter fields can be rigorously determined with the $\phi$-mapping theory. Critical phenomena of the generation and annihilation of skyrmion-antiskyrmion pairs are also discussed.

## 2 Skyrmion excitation

In this section we discuss the skyrmion excitation of a two-dimensional antiferromagnetic BEc. The vector $\vec{m}$ sweeps the unit sphere $S^2$ an integral number of times inside the core of a skyrmion and is uniform outside the core. In analogy to previous work on skyrmions [16], we introduce a topological three-current as

$$J_{\mu}^s = \frac{1}{8\pi} \varepsilon^{\mu\nu\lambda} \varepsilon_{abc} m^a \partial_\nu m^b m^c \quad (\mu, \nu, \lambda = 0, 1, 2, \quad a, b, c = 1, 2, 3) \quad (2)$$

to describe the skyrmion excitations of the condensate, and its time component is defined as the density of the total skyrmion charges, i.e. $J_0^s = \rho_s$. It is easy to prove that this current can be expressed as

$$J_0^s = \rho_s.$$
Here $A_\mu$ is the Wu-Yang potential \[17\]

$$A_\mu = \vec{e}_1 \cdot \partial_\mu \vec{e}_2,$$

where $\vec{e}_1$ and $\vec{e}_2$ are two unit vectors normal to $\vec{m}$, and $(\vec{e}_1, \vec{e}_2, \vec{m})$ forms an orthogonal frame: $\vec{e}_1 \times \vec{e}_2 = \vec{m}$, $\vec{e}_1 \cdot \vec{e}_2 = 0$. Now, consider a two-component vector field $\vec{\xi} = (\xi^1, \xi^2)$ residing in the plane formed by $\vec{e}_1$ and $\vec{e}_2$:

$$e_1^i = \xi^i / \|\xi\|, \quad e_2^i = \varepsilon_{ij} \xi^j / \|\xi\|. \quad (\|\xi\| = \sqrt{\xi^j \xi^j}, \ i, j = 1, 2)$$

Then we have

$$A_\mu = \varepsilon_{ij} \frac{\xi^i}{\|\xi\|} \frac{\partial}{\partial_\mu} \frac{\xi^j}{\|\xi\|}. \quad (4)$$

With Eq. (4), the current (3) is obtained as

$$J_\mu^s = \frac{1}{8\pi} \varepsilon^{\mu \nu \lambda} (\partial_\nu A_\lambda - \partial_\lambda A_\nu). \quad (5)$$

Equation (5) reveals the inherent conservation structure of the skyrmion current:

$$\partial_\mu J_\mu^s = 0. \quad (6)$$

Within the $\phi$-mapping theory [12, 13], the expression (5) is a topological current that can be rewritten in the compact form

$$J_\mu^s = \frac{1}{2} \delta^2 \left( \xi \right) D^\mu \left( \frac{\xi}{x} \right), \quad (7)$$

where $D^\mu (\xi/x)$ is the vector Jacobian of $\vec{\xi}$,

$$\varepsilon^{ij} D^\mu \left( \frac{\xi}{x} \right) = \varepsilon^{\mu \nu \lambda} \partial_\nu \xi^i \partial_\lambda \xi^j, \quad (8)$$

and $D^0 (\xi/x)$ is the usual two-dimensional Jacobian determinant,

$$D^0 \left( \frac{\xi}{x} \right) = D \left( \frac{\xi}{x} \right) = \frac{\partial (\xi^1, \xi^2)}{\partial (x^1, x^2)}.$$
\[ \xi^1(x^1, x^2, t) = 0, \quad \xi^2(x^1, x^2, t) = 0. \] (9)

Hence it is necessary to study the solutions of Eq. (9) to determine the nonzero solutions of \( J_s^\mu \). If \( D(\xi/x) = D^0(\xi/x) \neq 0 \), the solutions of Eq. (9) are

\[ x^1 = x^1_l(t), \quad x^2 = x^2_l(t), \quad (l = 1, 2, \cdots, N) \] (10)

which represent \( N \) isolated zero points \( \vec{z}_l(t) \) \( (l = 1, 2, \cdots, N) \) in space-time. These zero points are the skyrmion excitations. The motion of the \( l \)th skyrmion is determined by the \( l \)th world line \( \vec{z}_l(t) \).

With the \( \delta \)-function theory \([20]\), it can be proved that

\[ \delta^2(\vec{\xi}) = \sum_{l=1}^{N} \frac{\beta_l}{|D(\xi/x)|_{x_l}} \delta^2(\vec{r} - \vec{z}_l(t)), \]

where the positive integer \( \beta_l \) is called the Hopf index of the map \( x \to \vec{\xi} \). The meaning of \( \beta_l \) is that when the point \( \vec{r} \) covers the neighborhood of the zeros \( \vec{z}_l \) once, the vector field \( \vec{\xi} \) covers the corresponding region \( \beta_l \) times. Also, with the definition of vector Jacobians (8) and using the implicit function theorem \([15]\), the general velocity of the \( l \)th skyrmion can be expressed as

\[ V^\mu_l = \frac{dz^\mu_l}{dt} = \frac{D^\mu(\xi/x)}{D(\xi/x)} \big|_{\vec{z}_l}, \quad V^0_l = 1. \] (11)

Then, the spatial and temporal components of the skyrmion current, \( J_{s1} \) and \( J_{s0} \), can be respectively written in the forms of the current and the density of a system of \( N \) classical point particles in \((2+1)\)-dimensional space-time with topological charge \( \frac{1}{2}W_l = \frac{1}{2}\beta_l \eta_l \):

\[ \vec{j}_s = \sum_{l=1}^{N} \frac{W_l}{2} V_l \delta^2(\vec{r} - \vec{z}_l(t)), \] (12)

\[ \rho_s = \frac{1}{2} \delta^2(\vec{\xi}) D \left( \frac{\xi}{x} \right) = \sum_{l=1}^{N} \frac{W_l}{2} \delta^2(\vec{r} - \vec{z}_l(t)). \]

Here \( W_l = \beta_l \eta_l \) is the winding number of the \( \vec{\xi} \) field at the zero point \( \vec{z}_l(t) \), and \( \eta_l = sgn(D(\xi/x) \big|_{\vec{z}_l}) = \pm 1 \) is the Brouwer degree \([18]\): \( \eta_l = +1 \) corresponds to skyrmion solutions, while \( \eta_l = -1 \) corresponds to anti-skyrmion solutions. It is clear that Eq. (12) describes the motion of the skyrmions in space-time. According to Eq. (6), the topological charges of these skyrmions are conserved.
\[
\frac{\partial \rho_s}{\partial t} + \nabla \cdot \vec{j}_s = 0. \tag{13}
\]

The total charge of the system is

\[
\int \rho_s d^2x = \sum_{i=1}^{N} \frac{W_i}{2}.
\]

Note that, because the charge of a skyrmion is integer valued, the winding number \(W_i\) must take even number values \([19]\).

### 3 Generation and annihilation of skyrmion-antiskyrmion pairs

The preceding analysis shows that the zeros of the vector field \(\vec{\xi}\) play an important role in describing the skyrmions of an antiferromagnetic BEC. The following analysis reveals a picture of great physical interest. Note that the solutions (10) of Eq. (9) are based on the condition \(D(\xi/x) = D^0(\xi/x) \neq 0\). When this condition does not hold, the result (10) will be altered \([14]\). We denote one such zero point as \((t^*, \vec{z}_l)\) (i.e., \(D(\xi/x)|_{(t^*, \vec{z}_l)} = 0\)) and assume that the Jacobian \(D^1\) satisfies

\[
D^1\left(\frac{\xi}{x}\right)|_{(t^*, \vec{z}_l)} \neq 0. \tag{14}
\]

From Eq. (11), it can be seen that at \((t^*, \vec{z}_l)\), the velocity of the skyrmion is infinite:

\[
\left.\frac{dx^1}{dt}\right|_{(t^*, \vec{z}_l)} = \frac{D^1(\xi/x)|_{(t^*, \vec{z}_l)}}{D(\xi/x)|_{(t^*, \vec{z}_l)}} = \infty. \tag{15}
\]

In order to explicitly analyze the behavior of the solutions of Eq. (9) near the point \((t^*, \vec{z}_l)\), we can use the Jacobian \(D^1(\xi/x)\) instead of \(D(\xi/x)\) for the purpose of using the implicit function theorem. Then we have a unique solution of Eq. (9) in the neighborhood of \((t^*, \vec{z}_l)\),

\[
t = t(x^1), \quad x^2 = x^2(x^1),
\]

with \(t^* = t(z^1_l)\). It can be seen from (15) that

\[
\left.\frac{dt}{dx^1}\right|_{(t^*, \vec{z}_l)} = 0.
\]

Then the Taylor expansion of \(t = t(x^1)\) at the point \((t^*, \vec{z}_l)\) is
\[ t - t^* = \frac{1}{2} \frac{d^2 t}{dx_1^2} |(t^*, \vec{z}_l)(x^1 - z^1_l)^2, \]  

which is a parabola in the \( x^1-t \) plane. From Eq. (16), we obtain two solutions, \( x^1_1(t) \) and \( x^1_2(t) \), which give two branch solutions (world lines of the skyrmions) of Eq. (9). If \( \frac{d^2 t}{dx_1^2} |(t^*, \vec{z}_l) > 0 \), we have a branch solution for \( t > t^* \); otherwise, we have branch solutions for \( t < t^* \). These two cases are related to the origin and the annihilation of a skyrmion-antiskyrmion pair. From (15), we see that the velocity of skyrmions is infinite when they are being annihilated or generated at limit points. The restriction on the conservation of the skyrmion current (13) implies that the topological charges of these two skyrmions must be opposite, i.e.,

\[ \frac{1}{2} \beta_1 \eta_1 = - \frac{1}{2} \beta_2 \eta_2, \]

which shows that \( \beta_1 = \beta_2 \) and \( \eta_1 = - \eta_2 \).

4 Monopole excitation

There are monopole excitations in three-dimensional antiferromagnetic BEc. The vector \( \vec{m} \) sweeps \( S^2 \) an integral number of times when once traversing a path around the sphere in which a monopole is contained. According to the results of Ref. [7] and our previous work [21], we can deduce the monopole four-current

\[ J^\mu_m = \frac{1}{8\pi} \varepsilon^{\mu\nu\lambda\rho} \varepsilon_{abc} \partial_\nu m^a \partial_\lambda m^b \partial_\rho m^c. \quad (\mu, \nu, \lambda, \rho = 0, 1, 2, 3) \]  

It is clear that the current (17) is conserved, i.e. \( \partial_\mu J^\mu_m = 0 \). Using the \( \phi \)-mapping theory [13], it can be proved that the current (17) has the compact form

\[ J^\mu_m = \delta^3(\vec{\phi}) D^\mu \left( \frac{\phi}{x} \right), \]

where \( \vec{\phi} \) is a three-component vector field defined as

\[ \frac{\phi^a}{||\phi||} = m^a, \quad ||\phi|| = \sqrt{\phi^a \phi^a}, \quad (a = 1, 2, 3) \]

and \( D^\mu \left( \frac{\phi}{x} \right) \) is the vector Jacobian of \( \phi(x) \):

\[ \varepsilon^{abc} D^\mu \left( \frac{\phi}{x} \right) = \varepsilon^{\mu\nu\lambda\rho} \partial_\nu \phi^a \partial_\lambda \phi^b \partial_\rho \phi^c. \]
It can be seen from Eq. (18) that the monopole four-current $J_{m}^{\mu}$ is non-vanishing only at the zero points of $\vec{\phi}$:

$$\phi^{a} \left( x^{1}, x^{2}, x^{3}, t \right) = 0. \quad (a = 1, 2, 3)$$

(21)

The solutions of Eq. (21) can be generally expressed as

$$x^{1} = x_{i}^{1}(t), \quad x^{2} = x_{i}^{2}(t), \quad x^{3} = x_{i}^{3}(t), \quad (i = 1, 2, \cdots, K)$$

which represent the world lines of $K$ isolated zero points $\vec{z}_{i}(t)$ ($i = 1, 2, \cdots, K$). These zero points are the monopole excitations. The motion of the $i$th monopole is determined by the $i$th world line $\vec{z}_{i}(t)$.

The $\delta$-function theory [20] demonstrates the relation

$$\delta^{3}(\vec{\phi}) = \sum_{i=1}^{K} \frac{\beta_{i}}{|D(\phi/x)|_{\vec{z}_{i}}(\vec{r} - \vec{z}_{i}(t))},$$

where $\beta_{i}$ is the Hopf index of the map $x \rightarrow \vec{\phi}$. With the definition of the vector Jacobians (20), and using the implicit function theorem, we can obtain the general velocity of the $i$th monopole:

$$V_{i}^{\mu} = \frac{dx_{i}^{\mu}}{dt} = \frac{D^{\mu}(\phi/x)}{D(\phi/x)}|_{\vec{z}_{i}}, \quad V_{i}^{0} = 1.$$

Then, the monopole current $J_{m}^{\mu}$ can be written in the form of the current and the density of a system of $K$ classical point particles in (3+1)-dimensional space-time with topological charge $W_{i} = \beta_{i}\eta_{i}$:

$$\vec{J}_{m} = \sum_{i=1}^{K} W_{i} \vec{V}_{i} \delta^{3}(\vec{r} - \vec{z}_{i}(t)), \quad (22)$$

$$\rho_{m} = \delta^{3}(\vec{\phi}) D\left(\frac{\phi}{x}\right) = \sum_{i=1}^{K} W_{i} \delta^{3}(\vec{r} - \vec{z}_{i}(t)),$$

where $\eta_{i} = sgn(D(\phi/x)|_{\vec{z}_{i}}) = \pm 1$ is the Brouwer degree, and $W_{i} = \beta_{i}\eta_{i}$ is the winding number of $\vec{\phi}$ at the zero point $\vec{z}_{i}(t)$. It is clear that Eq. (22) describes the motion of the monopoles in space-time. Here, $\eta_{i} = +1$ corresponds to a monopole and $\eta_{i} = -1$ corresponds to an antimonopole.

5 Conclusion and discussion

In conclusion, two kinds of topological excitations, skyrmions and monopoles, in an antiferromagnetic BEc were studied in the context of the $\phi$-mapping
theory. Because these two kinds of excitations originate from different spatial distributions of the Néel vector field $\vec{m}$, two vector order parameter fields, $\vec{ξ}$ and $\vec{φ}$, defined in terms of $\vec{m}$ were introduced in order to study these two excitations separately. We found that these two kinds of excitations are generated from the zero points of the corresponding vector order parameter fields and that their topological charges are both characterized by the Hopf index and the Brouwer degree. We also found that quantities such as the density and velocity of these excitations can be rigorously determined using the $φ$-mapping theory. Further investigation of the topology of the two-component vector order parameter field $ξ$ revealed physical pictures of the generation and annihilation of skyrmion-antiskyrmion pairs. It was shown that the velocity of the skyrmion is infinite in such critical processes.

Unlike previous works, we concentrated mainly on topological properties rather than dynamical properties of skyrmions and monopoles. It was found the analysis of the topology of the corresponding vector order parameter fields of these excitations is sufficient to determine the topological characteristics as well as their kinematics. In addition, we have elucidated the inherent conservation structure of the skyrmion three-current and studied the critical phenomena of the generation and annihilation of skyrmion-antiskyrmion pairs on the basis of this structure. These results have general significance for subsequent investigation of skyrmions.

To study the stability of these two types of excitations, the fact that should be considered first is that they are different kinds of topological structures: monopoles originate from natural singularities of the Néel vector field $\vec{m}$ in three-dimensional space, while skyrmions are nonsingular excitations that originate from the nontrivial homotopy classes of mappings from the compactified $R^2 \sim S^2 \rightarrow S^2$ in two-dimensional space. Due to the singular nature of the spin texture of the monopole, the condensate density vanishes in the core, and the monopole turns out to be thermodynamically stable [7]. This is analogous to the case of vortex excitation in a scalar BEc. Contrastively, topology allows the spin texture of a skyrmion to be of arbitrary intrinsic size. As a result, the stability of a skyrmion is determined by energetic properties, not by topological properties [8].

Acknowledgements

This work was supported by the National Natural Science Foundation and the Doctor Education Fund of the Educational Department of China.
References

[1] C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell & C. E. Wieman, Phys. Rev. Lett. 78 (1997), 586.

[2] D. M. Stamper-Kurn et al., Phys. Rev. Lett. 80 (1998), 2027.

[3] M. Barrent, J. Sauer and M. S. Chapman, Phys. Rev. Lett. 87 (2001), 010404.

[4] U. Al Khawaja and H. T. C. Stoof, Phys. Rev. A 64 (2001), 0403612.

[5] T. Ohmi and K. Machida, J. Phys. Soc. Jpn. 67 (1998) 1822.

[6] T. L. Ho, Phys. Rev. Lett. 81 (1998), 742.

[7] H. T. C. Stoof, E. Vliegen and U. Al khawaja, Phys. Rev. Lett. 87 (2001), 120407.

[8] Hui Zhai, Wei-qiang Chen, Zhan Xu and Lee Chang, Phys. Rev. A 68 (2003), 043602.

[9] T. Mizushima, K. Machida and T. Kita, Phys. Rev. Lett. 89 (2002) 030401.

[10] D. H. Lee and C. L. Kane, Phys. Rev. Lett. 64 (1990), 1313.

[11] G. ’t Hooft, Nucl. Phys. B 79 (1974), 276.

[12] Y. S. Duan, H. Zhang and S. Li, Phys. Rev. B 58 (1998), 125.

[13] Y. S. Duan, T Xu and L. B. Fu, Prog. Theor. Phys. 101 (1999), 467.

[14] Y. S. Duan and H. Zhang, Phys. Rev. E 60 (1999), 2568.

[15] Y. S. Duan, S. Li and G. H. Yang, Nucl. Phys. B 514 (1998), 705.

[16] See, e.g., E. C. Marino, Phys. Rev. B 61 (2000), 1588.

[17] T. T. Wu and C. N. Yang, Phys. Rev. D 12 (1975), 3845; 14 (1975), 437.

[18] H. Hopf, Math. Ann. 96 (1929), 209.

[19] T. L. Ho, Phys. Rev. B 18 (1978), 1144.

[20] J. A. Schouten, *Tensor Analysis for Physicists* (Oxford, 1951).

[21] Y. S. Duan, Preprint SLAC-PUB-3301, 1984.