Spectral expansion for finite temperature two-point functions and clustering

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Received 8 October 2012
Accepted 11 November 2012
Published 6 December 2012

Online at stacks.iop.org/JSTAT/2012/P12002
doi:10.1088/1742-5468/2012/12/P12002

Abstract. Recently, the spectral expansion of finite temperature two-point functions in integrable quantum field theories was constructed using a finite volume regularization technique and the application of multidimensional residues. In the present work, the original calculation is revisited. By clarifying some details in the residue evaluations, we find and correct some inaccuracies of the previous result. The final result for contributions involving no more than two particles in the intermediate states is presented. The result is verified by proving a symmetry property which follows from the general structure of the spectral expansion, and also by numerical comparison to the discrete finite volume spectral sum. A further consistency check is performed by showing that the expansion satisfies the cluster property up to the order of the evaluation.

Keywords: correlation functions, form factors, integrable quantum field theory

ArXiv ePrint: 1210.0331
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1. Introduction

Correlation functions play a central role in the formulation of many-body quantum systems. Integrable models present a unique opportunity to study strongly correlated quantum systems in situations where conventional methods break down. Recent experimental advances resulted in renewed interest in integrable models, since it is now possible to realize certain models with the help of optical and magnetic traps [1]–[4] or in low dimensional magnets [5, 6].

In a recent paper [7] finite temperature (i.e. thermal) two-point correlation functions were constructed using the exact form factors in 1 + 1-dimensional integrable models. In an integrable quantum field theory, the basic object is the factorized $S$-matrix [8, 9]. The matrix elements of the local operators (form factors) satisfy a certain set of equations (the form factor bootstrap equations), which follow from general field theoretical arguments supplemented with the special analytic properties of the $S$-matrix [10]–[13]. Solving these equations gives the form factor functions, which can then be used to construct correlation functions by expanding in the basis formed by the infinite volume asymptotic scattering states.

The form factor expansion of zero-temperature correlations in integrable QFT is very well understood. In general, the series has very good convergence properties in massive models and can be evaluated numerically to any desired precision [12, 14]. However, the problem of thermal correlation functions is much more complicated and has been the subject of active research in the last two decades [15]–[24]. The form factor construction of the spectral series is plagued with problems due to the presence of disconnected terms in the expansion, which lead to formally divergent expressions. Following Balog, it can be shown that the divergent parts cancel with contributions from the partition function [25]. Nevertheless, it is a completely nontrivial task to obtain the correct finite answer. Leclair and Mussardo conjectured an answer for the spectral expansion for one-point and two-point functions in terms of form factors dressed by appropriate occupation number factors containing the pseudo-energy function from the thermodynamical Bethe ansatz [16]. Their proposal for the two-point function was questioned by Saleur [17]; on the other hand, in the same paper he also gave a proof of the Leclair–Mussardo formula for one-point functions provided the operator considered is the density of some local conserved charge. By comparison to an alternative proposal [19], it was also shown that the results obtained by naive regularization are ambiguous [20].

The idea behind our approach is to use a finite volume setting to regularize the divergences. In [26, 27] this was applied to one-point functions giving a confirmation of the Leclair–Mussardo formula to third order; later, a derivation to all orders was also obtained [28]. The crucial point is that finite volume is not an ad hoc, but a physical
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regulator (since physically realizable systems are always of finite size), therefore one is virtually guaranteed to obtain the correct result when taking the infinite volume limit. The natural small parameter for the finite temperature expansion is the Boltzmann factor $e^{-m/T}$ where $m$ is the mass gap (which is assumed to be nonzero). The result is an integral series, where the $N$th term represents $N$-particle processes over the Fock vacuum. The contributions with a low number of particles can be interpreted as disconnected terms of matrix elements calculated in a thermal state with a large number of particles [23, 28]. In this sense the approach is similar to the one used in the algebraic Bethe ansatz [29, 30].

Besides correlation functions, the finite volume regularization can also be applied to numerous other problems. The finite volume form factor approach was extended to boundary operators as well [31], and was used to compute finite temperature one-point functions of boundary operators [32]. Another application of the bulk finite volume form factors is the construction of one-point functions of bulk operators on a finite interval [33]. It was also used [34] to construct the form factor perturbation expansion in non-integrable field theories (originally proposed by Delfino et al [35]) beyond the leading order. It also turned out that this approach can be applied to quenches in field theory [33, 36, 37].

Regarding thermal two-point functions, the finite volume regularization method was first applied by Essler and Konik [38, 39]; however, their methods do not have any obvious extension to higher order. Despite this shortcoming, their results are very useful as shown by their relevance to inelastic neutron scattering experiments [6]. An independent early calculation of the one-particle–one-particle contribution can also be found in [40].

In [7], we developed a systematic method to compute the finite temperature form factor expansion to arbitrary orders. It turns out that the machinery of multidimensional residues provides an appropriate formalism to evaluate higher order corrections systematically, and this was demonstrated to all orders which involve only intermediate states with at most two particles. To verify the result, we applied two consistency checks. The first of them was that the correlator should have a finite limit when the volume is taken to infinity, therefore all terms containing positive powers of the volume had to cancel, which was indeed true. The second one took into account that for some contributions there are two independent ways to arrive at the answer, and agreement between them also provides validating evidence. However, for the term $D_{22}$, which contains the contributions when both intermediate states involved in the spectral sum contain two particles, the second one is not available and the first is insufficient to check the structure of the result in detail.

Therefore, we decided to provide numerical evidence, especially since the analytic manipulations themselves are rather tedious and complicated, with many possible sources of mistakes. As it turned out, the result for $D_{22}$ reported in [7] is unfortunately incorrect. By further investigation, it turned out that some fine details of the residue calculation needed to be carried out more carefully.

Here we report the correct version of the computation and its result, and present the final formula for the finite temperature two-point function including all contributions with at most two-particle intermediate states. To be confident in our results, we perform several checks. First we check that the result for $D_{22}$ satisfies a particular symmetry property following from the general form of the spectral expansion. Then we apply a detailed numerical verification of our analytic manipulations, and also verify that the final result satisfies the physically required cluster property.

doi:10.1088/1742-5468/2012/12/P12002
2. Finite volume regularization

2.1. The thermal two-point function

A field theory with finite temperature $T$ can be defined using a compact Euclidean (Matsubara) time $t$:

$$t \equiv t + R \quad \text{where } R = 1/T.$$  

We are interested in the two-point function in 1 + 1-dimensional field theories:

$$\langle O_1(x,t)O_2(0) \rangle^R = \frac{\text{Tr}(e^{-R \mathcal{H}} O_1(x,t)O_2(0))}{\text{Tr}(e^{-R \mathcal{H}})}. \quad (2.2)$$

A naive spectral sum leads to an ill defined expression due to the presence of disconnected contributions (cf, e.g., the discussion in [7]). However, one can put the system in a finite spatial volume $L$ with periodic boundary conditions

$$x \equiv x + L \quad \text{(2.3)}$$

so that

$$\langle O_1(x,t)O_2(0) \rangle^R_L = \frac{\text{Tr}_L(e^{-R \mathcal{H}_L} O_1(x,t)O_2(0))}{\text{Tr}_L(e^{-R \mathcal{H}_L})} \quad (2.4)$$

where $\text{Tr}_L$ denotes the trace over the finite volume states and $\mathcal{H}_L$ is the Hamiltonian in volume $L$. This expression can be expanded inserting two complete sets of states

$$\text{Tr}_L(e^{-R \mathcal{H}_L} O_1(x,t)O_2(0)) = \sum_{m,n} e^{-RE_{m}(L)} \langle n|O_1(x,t)|m \rangle_L \langle m|O_2(0)|n \rangle_L \quad (2.5)$$

where the matrix elements of local operators are also taken in the finite volume system. To evaluate it, we need an expression for form factors in finite volume.

2.2. The form factor bootstrap

In a 1 + 1-dimensional field theory, the energy and the momentum of an on-shell particle is parametrized by the rapidity variable as $E = m \cosh \theta$ and $p = m \sinh \theta$. For the sake of simplicity, let us suppose that the spectrum of the model consists of a single-particle
mass $m$. Incoming and outgoing asymptotic states are defined as

$$|\theta_1, \ldots, \theta_n\rangle = \begin{cases} |\theta_1, \ldots, \theta_n\rangle^\text{in}: & \theta_1 > \theta_2 > \cdots > \theta_n \\ |\theta_1, \ldots, \theta_n\rangle^\text{out}: & \theta_1 < \theta_2 < \cdots < \theta_n. \end{cases}$$

Integrability leads to factorized scattering, which can be summarized by the relation

$$|\theta_1, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_n\rangle = S(\theta_k - \theta_{k+1})|\theta_1, \ldots, \theta_{k+1}, \theta_k, \ldots, \theta_n\rangle$$

where $S$ denotes the two-particle amplitude; from this any multi-particle scattering process can be obtained by reordering the particles. States are normalized as

$$\langle \theta'|\theta \rangle = 2\pi \delta(\theta' - \theta).$$

The form factors of a local operator $\mathcal{O}(t, x)$ are defined as

$$F^\mathcal{O}_{mn}(\theta_1, \ldots, \theta_m|\theta_1, \ldots, \theta_n) = \langle \theta_1, \ldots, \theta_m|\mathcal{O}(0, 0)|\theta_1, \ldots, \theta_n\rangle.$$ (2.7)

With the help of the crossing relations

$$F^\mathcal{O}_{mn}(\theta_1, \ldots, \theta_m|\theta_1, \ldots, \theta_n) = F^\mathcal{O}_{m-1n+1}(\theta'_1, \ldots, \theta'_{m-1}|\theta'_m + i\pi, \theta_1, \ldots, \theta_n)$$

$$+ \sum_{k=1}^{n} 2\pi \delta(\theta'_m - \theta_k) \prod_{l=1}^{k-1} S(\theta_l - \theta_k)$$

$$\times F^\mathcal{O}_{m-1n-1}(\theta'_1, \ldots, \theta'_{m-1}|\theta_1, \ldots, \theta_{k-1}, \theta_{k+1}, \ldots, \theta_n)$$ (2.8)

all form factors can be expressed in terms of the elementary form factors

$$F^\mathcal{O}_n(\theta_1, \ldots, \theta_n) = \langle 0|\mathcal{O}(0, 0)|\theta_1, \ldots, \theta_n\rangle$$

which satisfy the form factor bootstrap equations [10, 41, 11]

Lorentz symmetry: $F^\mathcal{O}_n(\theta_1 + \Lambda, \theta_2 + \Lambda, \ldots, \theta_n + \Lambda) = \exp(s_\mathcal{O}\Lambda) F^\mathcal{O}_n(\theta_1, \theta_2, \ldots, \theta_n)$ (2.9)

Exchange: $F^\mathcal{O}_n(\theta_1, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_n) = S(\theta_k - \theta_{k+1}) F^\mathcal{O}_n(\theta_1, \ldots, \theta_{k+1}, \theta_k, \ldots, \theta_n)$ (2.10)

Cyclic property: $F^\mathcal{O}_n(\theta_1 + 2i\pi, \theta_2, \ldots, \theta_n) = F^\mathcal{O}_n(\theta_2, \ldots, \theta_n, \theta_1)$ (2.11)

Kinematical poles: $-i\text{Res}_{\theta = \theta'} F^\mathcal{O}_{n+2}(\theta + i\pi, \theta', \theta_1, \ldots, \theta_n)$

$$= \left(1 - \prod_{k=1}^{n} S(\theta' - \theta_k)\right) F^\mathcal{O}_n(\theta_1, \ldots, \theta_n)$$ (2.12)

where $s_\mathcal{O}$ denotes the Lorentz spin of the operator $\mathcal{O}$. There is also a further equation related to bound states, which we do not need in the following.

### 2.3. Form factors in finite volume

A formalism that gives the exact quantum form factors to all orders in $L^{-1}$ was introduced in [26, 27]. The finite volume multi-particle states can be denoted

$$|\{I_1, \ldots, I_n\}\rangle_L$$

where $I_i$ is an operator in the $i$th level of the lattice.
where the $I_k$ are momentum quantum numbers, ordered as $I_1 \geq \cdots \geq I_n$ by convention. The corresponding energy levels are determined by the Bethe–Yang equations
\[ e^{imL \sinh \hat{\theta}_k} \prod_{l \neq k} S(\hat{\theta}_k - \hat{\theta}_l) = 1. \]

Defining the two-particle phase shift $\delta(\theta)$ by the relation
\[ S(\theta) = -e^{i\delta(\theta)} \]
the derivative of $\delta$ will be denoted by
\[ \varphi(\theta) = \frac{d\delta(\theta)}{d\theta}. \]
Due to unitarity, $\delta$ is an odd and $\varphi$ is an even function. We can write
\[ Q_k(\hat{\theta}_1, \ldots, \hat{\theta}_n) = mL \sinh \hat{\theta}_k + \sum_{l \neq k} \delta(\hat{\theta}_k - \hat{\theta}_l) = 2\pi I_k, \quad k = 1, \ldots, n \]
where the quantum numbers $I_k$ take integer/half-integer values for odd/even numbers of particles respectively. Equations (2.15) must be solved with respect to the particle rapidities $\hat{\theta}_k$, where the energy (relative to the finite volume vacuum state) can be computed as
\[ \sum_{k=1}^{n} m \cosh \hat{\theta}_k \]
up to corrections which decay exponentially with $L$. The density of $n$-particle states in rapidity space can be calculated as
\[ \rho(\theta_1, \ldots, \theta_n) = \det \mathcal{J}^{(n)}, \quad \mathcal{J}_{kl}^{(n)} = \frac{\partial Q_k(\theta_1, \ldots, \theta_n)}{\partial \theta_l}, \quad k, l = 1, \ldots, n. \]

The finite volume behavior of local matrix elements can be given as [26]
\[ \langle \{I'_1, \ldots, I'_m\} | O(0,0) | \{I_1, \ldots, I_n\} \rangle_L = \frac{F_{m+n}^{\mathcal{O}}(\hat{\theta}'_m + i\pi, \ldots, \hat{\theta}'_1 + i\pi, \hat{\theta}_1, \ldots, \hat{\theta}_n)}{\sqrt{\rho(\hat{\theta}_1, \ldots, \hat{\theta}_n) \rho(\hat{\theta}'_1, \ldots, \hat{\theta}'_m)}} + O(e^{-\mu L}) \]
where $\hat{\theta}_k (\hat{\theta}'_k)$ are the solutions of the Bethe–Yang equation (2.15) corresponding to the state with the specified quantum numbers $I_1, \ldots, I_n$ $(I'_1, \ldots, I'_m)$ at the given volume $L$. The above relation is valid provided there are no disconnected terms (the left and the right states do not contain particles with the same rapidity), i.e. the sets $\{\hat{\theta}_1, \ldots, \hat{\theta}_n\}$ and $\{\hat{\theta}'_1, \ldots, \hat{\theta}'_m\}$ are disjoint.

It is easy to see that in the presence of nontrivial scattering there are only two cases when exact equality of (at least some of) the rapidities can occur [27].

(1) The two states are identical, i.e. $n = m$ and
\[ \{I'_1, \ldots, I'_m\} = \{I_1, \ldots, I_n\} \]
in which case the corresponding diagonal matrix element can be written as a sum over all bipartite divisions of the set of the $n$ particles involved (including the trivial ones...
where $A$ is the empty set or the complete set $\{1, \ldots, n\}$

$$\langle \{I_1 \cdots I_n\} | O | \{I_1 \cdots I_n\}\rangle_L = \sum_{A \subseteq \{1, \ldots, n\}} \mathcal{F}(A)_L \rho(\{1, \ldots, n\} \setminus A)_L + O(e^{-\mu L}) \quad (2.19)$$

where

$$\rho(\{k_1, \ldots, k_r\})_L = \rho(\tilde{\theta}_{k_1}, \ldots, \tilde{\theta}_{k_r})$$

is the $r$-particle Bethe–Yang Jacobi determinant (2.17) involving only the $r$-element subset $1 \leq k_1 < \cdots < k_r \leq n$ of the $n$ particles, and

$$\mathcal{F}(\{k_1, \ldots, k_r\})_L = F^n_{2r}(\tilde{\theta}_{k_1}, \ldots, \tilde{\theta}_{k_r})$$

$$F^n_{2l}(\theta_1, \ldots, \theta_l) = \lim_{\epsilon \to 0} F^\omega_{2l}(\theta_1 + i\pi + \epsilon, \ldots, \theta_l + i\pi + \epsilon, \theta_1, \ldots, \theta_l)$$

is the so-called symmetric evaluation of diagonal multi-particle matrix elements.

(2) Both states are parity symmetric states in the spin zero sector, i.e.

$$\{I_1, \ldots, I_n\} \equiv \{-I_n, \ldots, -I_1\}$$

$$\{I'_1, \ldots, I'_m\} \equiv \{-I'_m, \ldots, -I'_1\}.$$

Furthermore, both states must contain one (or possibly more, in a theory with more than one species) particle of zero quantum number. Writing $m = 2k + 1$ and $n = 2l + 1$ and defining

$$\mathcal{F}_{k,l}(\theta'_1, \ldots, \theta'_l | \theta_1, \ldots, \theta_l) \lim_{\epsilon \to 0} F^{2k+2l+2}(i\pi + \theta'_1 + \epsilon, \ldots, i\pi + \theta'_l + \epsilon, i\pi - \theta'_k + \epsilon, \ldots, i\pi - \theta'_1 + \epsilon, i\pi + \epsilon, 0, \theta_1, \ldots, \theta_l, -\theta_1, \ldots, -\theta_1) \quad (2.20)$$

the formula for the finite volume matrix element takes the form

$$\langle \{I'_1, \ldots, I'_k, 0, -I'_k, \ldots, -I'_1\} | O | \{I_1, \ldots, I_l, 0, -I_l, \ldots, -I_1\}\rangle_L$$

$$= (\rho_{2k+1}(\tilde{\theta}'_1, \ldots, \tilde{\theta}'_l, 0, -\tilde{\theta}'_k, \ldots, -\tilde{\theta}'_1))^{-1/2}$$

$$\times \rho_{2l+1}(\tilde{\theta}_1, \ldots, \tilde{\theta}_l, 0, -\tilde{\theta}_k, \ldots, -\tilde{\theta}_1)$$

$$\times [\mathcal{F}_{k,l}(\theta'_1, \ldots, \theta'_l | \theta_1, \ldots, \theta_l) + mL F^{2k+2l}(i\pi + \tilde{\theta}'_1, \ldots, i\pi + \tilde{\theta}'_k, \ldots, i\pi + \theta'_1, \ldots, i\pi - \theta'_k, \ldots, i\pi - \theta'_1, \ldots, \theta_1, -\theta_1, \ldots, -\theta_1)] + O(e^{-\mu L}) \quad (2.21)$$

2.4. The form factor expansion using finite volume regularization

Using the finite volume description introduced in section 2.3 we can write

$$\langle O_1(x,t) O_2(0) \rangle^R_L = \frac{1}{Z} \sum_{N,M} C_{NM} \quad (2.22)$$

where

$$C_{NM} = \sum_{I_1 \cdots I_N} \sum_{J_1 \cdots J_M} \langle \{I_1 \cdots I_N\} | O_1(0) | \{J_1 \cdots J_M\}\rangle_L$$

$$\times \langle \{J_1 \cdots J_M\} | O_2(0) | \{I_1 \cdots I_N\}\rangle_L e^{(P_1-P_2)x} e^{-E_1(R-t)} e^{-E_2 t} \quad (2.23)$$

and $E_{1,2}$ and $P_{1,2}$ are the total energies and momenta of the multi-particle states $\{|I_1 \cdots I_N\}_L$ and $\{|J_1 \cdots J_M\}_L$. The task is to calculate the sum in finite volume and then take the limit $L \to \infty$. 

doi:10.1088/1742-5468/2012/12/P12002
First we classify the contributions into different multi-particle orders following the procedure in [39, 7]. Introducing two auxiliary variables $u$ and $v$ (at the end both will be set to unity),

$$\langle \mathcal{O}_1(x,t) \mathcal{O}_2(0) \rangle^R_L = \frac{1}{Z} \sum_{N,M} u^N v^M C_{NM}. \quad (2.24)$$

Similarly for the partition function

$$Z = \sum_N (uv)^N Z_N$$

with $Z_N$ denoting the $N$-particle contribution. The inverse of the partition function is expanded as

$$Z^{-1} = \sum_N (uv)^N \tilde{Z}_N$$

where

$$\tilde{Z}_0 = 1, \quad \tilde{Z}_1 = -Z_1, \quad \tilde{Z}_2 = Z_1^2 - Z_2.$$

Putting this together we can rewrite the expansion as

$$\langle \mathcal{O}_1(x,t) \mathcal{O}_2(0) \rangle^R_L = \sum u^N v^N \tilde{D}_{NM} \quad (2.25)$$

with

$$\tilde{D}_{NM} = \sum_l C_{N-l,M-l} \tilde{Z}_l. \quad (2.26)$$

The first few nontrivial terms are given by

$$\tilde{D}_{1M} = C_{1M} - Z_1 C_{0,M-1}$$
$$\tilde{D}_{2M} = C_{2M} - Z_1 C_{1,M-1} + (Z_1^2 - Z_2) C_{0,M-2}. \quad (2.27)$$

In this way we produce a double series expansions in powers of the variables $e^{-mt}$ and $e^{-m(R-t)}$. Since these variables are independent, each quantity $\tilde{D}_{NM}$ must have a well defined $L \to \infty$ limit, which we denote as

$$D_{NM} = \lim_{L \to \infty} \tilde{D}_{NM} \quad (2.28)$$

and we obtain that

$$\langle \mathcal{O}_1(x,t) \mathcal{O}_2(0) \rangle^R = \lim_{L \to \infty} \langle \mathcal{O}_1(x,t) \mathcal{O}_2(0) \rangle^R_L = \sum_{N,M} D_{NM}. \quad (2.29)$$

A similar reordering was also used for the expansion of the one-point function in powers of $e^{-mR}$ [27], and for the boundary one-point function in [32]. It is evident from (2.23) that the $D_{NM}$ with $N > M$ can be obtained from those with $N < M$ after a trivial exchange of $t$ with $R-t$, $x$ with $-x$ and $\mathcal{O}_1$ with $\mathcal{O}_2$.

3. The spectral expansion for finite temperature correlators

To evaluate the finite temperature two-point function, it is necessary to evaluate the summation over two sets of intermediate states. For a given $C_{NM}$ this involves an $N$ and
an $M$ particle state. One can start with any of these; to simplify the calculations, it is best to start with the one containing the smallest number or particles, and do the other later. On the other hand, doing the calculation in the reverse order allows one to cross-check the result [7].

To evaluate the first summation, a systematic method was given in [7] based on a multidimensional residue method. Once this is done, all the singularities from the form factors are tamed, and the second summation can be performed by a simple transition from the discrete sum to an integral using the density of states. Then, after assembling $\tilde{D}_{NM}$ using the lower $C_{NM}'$ coefficients as in (2.26), and taking the limit $L \to \infty$ the final formula for the contribution $D_{NM}$ can be obtained. Another quick validity check of the calculation is provided by the existence of the infinite volume limit.

3.1. Converting sums to contour integrals

For sums over one-particle states $|\{I\}_L\rangle$ with quantum number $I \in \mathbb{Z}$ we can substitute

$$\sum_I \to \sum_I \int_{C_I} \frac{d\theta}{2\pi} \frac{\rho_1(\theta)}{e^{Q_1(\theta)} - 1}$$

where

$$Q_1(\theta) = mL \sinh \theta \quad \rho_1(\theta) = Q_1'(\theta) = mL \cosh \theta$$

and $C_I$ are small closed curves surrounding the solution of

$$Q_1(\theta) = 2\pi I$$

in the complex $\theta$ plane.

For two-particle sums over two-particle states $|\{I_1, I_2\}_L\rangle$ with quantum numbers $I_1, I_2 \in \mathbb{Z} + \frac{1}{2}$ we can use the multidimensional generalization of the residue theorem to write

$$\sum_{I_1>I_2} \to \sum_{I_1>I_2} \int_{C_{I_1,I_2}} \int_{C_{I_1,I_2}} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{\rho_2(\theta_1, \theta_2)}{2(e^{Q_1(\theta_1, \theta_2)} + 1)(e^{Q_2(\theta_1, \theta_2)} + 1)}$$

where $C_{I_1,I_2}$ is a multi-contour (a direct product of two curves in the variables $\theta_1$ and $\theta_2$) surrounding the solution of

$$Q_1(\theta_1, \theta_2) = mL \sinh \theta_1 + \delta(\theta_1 - \theta_2) = 2\pi I_1$$
$$Q_2(\theta_1, \theta_2) = mL \sinh \theta_2 + \delta(\theta_2 - \theta_1) = 2\pi I_2$$

where due to the definition (2.13) $I_1$ and $I_2$ take half-integer values, and

$$\rho_2(\theta_1, \theta_2) = \det \left( \begin{array}{cc} \frac{\partial Q_1}{\partial \theta_1} & \frac{\partial Q_1}{\partial \theta_2} \\ \frac{\partial Q_2}{\partial \theta_1} & \frac{\partial Q_2}{\partial \theta_2} \end{array} \right) = m^2 L^2 \cosh \theta_1 \cosh \theta_2 + mL(\cosh \theta_1 + \cosh \theta_2) \varphi(\theta_1 - \theta_2).$$

Since form factors vanish when any two of their arguments coincide, we can extend the sum by adding the diagonal:

$$\sum_{I_1>I_2} \to \frac{1}{2} \left( \sum_{I_1,I_2} - \sum_{I_1=I_2} \right)$$

doi:10.1088/1742-5468/2012/12/P12002
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Figure 1. Contour deformation procedure. The black dot shows a singularity not enclosed inside the contours following from the spectral sum.

In the next step, the contours are joined together and opened into straight lines, to a product contour whose components in each variable enclose the real axis. However, this can only be done by including other poles (apart from the ones needed for the state summations) in the interior, which come from singularities of the $Q$-dependent denominators and of the form factors. These must be classified and subtracted. This procedure was discussed in some detail in [7], and for one complex variable it is illustrated in Figure 1 (for more complex variable it must be performed in each variables separately). We shall only outline it for the case of the $D_{22}$ contribution, because of the corrections we make to the previous calculation performed in that paper.

3.2. The $D_{22}$ contribution revisited

The $D_{22}$ contribution is given by

$$D_{22} = \lim_{L \to \infty} [C_{22} - Z_1 C_{11} + (Z_1^2 - Z_2) C_{00}] = \lim_{L \to \infty} [C_{22} - Z_1 \tilde{D}_{11} - Z_2 C_{00}]$$

where

$$C_{22} = \sum_{I_1 > I_2} \sum_{J_1 > J_2} \langle \{I_1, I_2\}|O_1(0)\{J_1, J_2\}\rangle_L \langle \{J_1, J_2\}|O_2(0)\{I_1, I_2\}\rangle_L K^{(R)}_{t, x}(\tilde{\vartheta}_1, \tilde{\vartheta}_2|\tilde{\vartheta}_1', \tilde{\vartheta}_2')$$

with the notation

$$K^{(R)}_{t, x}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2') = e^{imx(\sinh \vartheta_1 + \sinh \vartheta_2 - \sinh \vartheta_1' - \sinh \vartheta_2')}$$

$$= e^{-m(R-t)(\cosh \vartheta_1 + \cosh \vartheta_2)} e^{-m(R-t)(\cosh \vartheta_1' + \cosh \vartheta_2')}$$

(3.1)

and where [7]

$$Z_1 = mL \int \frac{d\vartheta_1}{2\pi} \cosh \vartheta_1 e^{-mR \cosh \vartheta_1}$$

$$Z_2 = \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \rho_2(\vartheta_1, \vartheta_2) e^{-mR(\cosh \vartheta_1 + \cosh \vartheta_2)} - \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \rho_1(\vartheta_1) e^{-2mR \cosh \vartheta_1}$$

$$C_{00} = \langle O_1 \rangle \langle O_2 \rangle$$

$$D_{11} = \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} F_2^{O_1}(\vartheta_1 + i\pi, \vartheta_2) F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1)$$

$$\times e^{imx(\sinh \vartheta_1 - \sinh \vartheta_2)} e^{-m(R-t) \cosh \vartheta_1} e^{-mt \cosh \vartheta_2}$$

$$+ [\langle O_1 \rangle F_2^{O_2} + \langle O_2 \rangle F_2^{O_1}] \int \frac{d\vartheta_1}{2\pi} e^{-mR \cosh \vartheta_1}.$$
The rapidities are quantized by
\[ Q_1(\vartheta_1, \vartheta_2) = mL \sinh \vartheta_1 + \delta(\vartheta_1 - \vartheta_2) = 2\pi I_1 \]
\[ Q_2(\vartheta_1, \vartheta_2) = mL \sinh \vartheta_2 + \delta(\vartheta_2 - \vartheta_1) = 2\pi I_2 \] (3.2)
and
\[ Q'_1(\vartheta'_1, \vartheta'_2) = mL \sinh \vartheta'_1 + \delta(\vartheta'_1 - \vartheta'_2) = 2\pi J_1 \]
\[ Q'_2(\vartheta'_1, \vartheta'_2) = mL \sinh \vartheta'_2 + \delta(\vartheta'_2 - \vartheta'_1) = 2\pi J_2. \]

We perform the \( J_1, J_2 \)-sum first and separate it into a diagonal and an off-diagonal piece:
\[ \sum_{J_1 > J_2} = (\{J_1, J_2\} = \{I_1, I_2\} \text{ term}) + \sum_{J_1 > J_2}' \]
because the finite volume form factor expressions are different for the two types of contribution. In the second term, the prime indicates that the diagonal contributions are excluded.

### 3.2.1. The diagonal piece.
This calculation is exactly the same as in [7], so we only highlight the main steps. We start from
\[ C_{22}^{\text{diag}} = \sum_{I_1 > I_2} \langle I_1, I_2|O_1(0)|I_1, I_2\rangle \langle I_1, I_2|O_2(0)|I_1, I_2\rangle K_i^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1, \vartheta_2) \]
where
\[ \langle I_1, I_2|O(0)|I_1, I_2\rangle = \frac{F_4^O(\vartheta_1, \vartheta_2) + (\rho_1(\vartheta_1) + \rho_1(\vartheta_2)) F_2^O + \rho_2(\vartheta_1, \vartheta_2)\langle O\rangle}{\rho_2(\vartheta_1, \vartheta_2)} \]
with
\[ F_4^O(\vartheta) = F_2^O(i\pi, 0) \]
\[ F_4^O(\vartheta_1, \vartheta_2) = \lim_{\varepsilon \to 0} F_4^O(\vartheta_1 + i\pi + \varepsilon, \vartheta_2 + i\pi + \varepsilon, \vartheta_2, \vartheta_1). \]

Writing the sum in terms of contour integrals, after opening the contours and performing the large \( L \) limit the diagonal contribution becomes
\[
C_{22}^{\text{diag}} = \frac{1}{2} \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \left[ F_{4s}^{O_1}(\vartheta_1, \vartheta_2) \langle O_2 \rangle \right.
\]
\[ + F_{4s}^{O_2}(\vartheta_1, \vartheta_2) \langle O_1 \rangle \right] e^{-mR(\cosh \vartheta_1 + \cosh \vartheta_2)} \]
\[ + \frac{1}{2} \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \left[ \cosh \vartheta_1 + \cosh \vartheta_2 \right]^2 F_{2s}^{O_1}(i\pi, 0) F_2^{O_2}(i\pi, 0) \]
\[ \times e^{-mR(\cosh \vartheta_1 + \cosh \vartheta_2)} \]
\[ - \int \frac{d\vartheta_1}{2\pi} F_{2s}^{O_1}(i\pi, 0) \langle O_2 \rangle + F_{2s}^{O_2}(i\pi, 0) \langle O_1 \rangle \right] e^{-mR \cosh \vartheta_1} \]
\[ + \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} mL \cosh \vartheta_1 [F_{2s}^{O_1}(i\pi, 0) \langle O_2 \rangle + F_{2s}^{O_2}(i\pi, 0) \langle O_1 \rangle] \]
\[ \times e^{-mR(\cosh \vartheta_1 + \cosh \vartheta_2)} + Z_2 C_{00}. \] (3.3)
3.2.2. The non-diagonal part. In the non-diagonal part, one can use

\[ \langle I_1, I_2 \mid O(0) \mid J_1, J_2 \rangle = \frac{F_4^O(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_2, \vartheta'_1)}{\sqrt{\rho_2(\vartheta_1, \vartheta_2) \rho_2(\vartheta'_1, \vartheta'_2)}} \]

to write

\[ C_{22}^{\text{nondiag}} = \sum_{I_1 > I_2, J_1 > J_2} \sum' \left\{ \frac{F_4^{O_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_2, \vartheta'_1) F_4^{O_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2, \vartheta'_1)}{\rho_2(\vartheta_1, \vartheta_2) \rho_2(\vartheta'_1, \vartheta'_2)} \right\} \times K_{I,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2) \]

\[ = \sum_{I_1 > I_2} \frac{\tilde{C}_{22}}{\rho_2(\vartheta_1, \vartheta_2)} \]

where

\[ \tilde{C}_{22} = \sum'_{J_1 > J_2} \oint \oint \frac{d\vartheta'_1 d\vartheta'_2}{2\pi i 2\pi i} \left\{ \frac{F_4^{O_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_2, \vartheta'_1) F_4^{O_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2, \vartheta'_1)}{[\omega^{Q_1}(\vartheta'_1, \vartheta'_2) + 1][\omega^{Q_2}(\vartheta'_1, \vartheta'_2) + 1]} \right\} \times K_{I,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2) \]

and the prime denotes the omission of the \{J_1, J_2\} = \{I_1, I_2\} term. We can substitute

\[ \sum_{J_1 > J_2} \rightarrow \frac{1}{2} \sum'_{J_1, J_2} \]

since the form factors vanish when any two of their rapidity arguments are identical.

Now we open the contours to encircle the real axis in \( \vartheta'_1 \) and \( \vartheta'_2 \). However, this brings more singularities inside the contour whose contribution must then be subtracted. These can be classified as follows.

1. **Spurious QQ-poles.** There are two such terms, which come from including the poles with \( J_{1,2} = I_{1,2} \) or \( J_{1,2} = I_{2,1} \). Their contribution vanishes for \( L \to \infty \) [7]: hence the term ‘spurious’. However, they must be included in the numerical tests, therefore we provide their form in equation (B.9). Note that the form factors are not singular in this case, although their limits in such points are direction dependent.

2. **QF-poles.** In this case one of the integrations has a pole from a form factor, and the other one from a Q-term:

   - **QFI:** \( \vartheta'_1 = \vartheta_1 \) and \( Q'_2(\vartheta_1, \vartheta'_2) = 2\pi J_2 \)
   - **QFII:** \( \vartheta'_1 = \vartheta_2 \) and \( Q'_2(\vartheta_2, \vartheta'_2) = 2\pi J_2 \)
   - **QFIII:** \( \vartheta'_2 = \vartheta_1 \) and \( Q'_1(\vartheta'_1, \vartheta_1) = 2\pi J_1 \)
   - **QFIV:** \( \vartheta'_2 = \vartheta_2 \) and \( Q'_1(\vartheta'_1, \vartheta_2) = 2\pi J_1 \).

3. **FF-poles.** In this case poles in both integrals come from form factors:

   - **FFI:** \( \vartheta'_1 = \vartheta'_2 = \vartheta_1 \)
   - **FFII:** \( \vartheta'_1 = \vartheta'_2 = \vartheta_2 \).

\[ \text{doi:10.1088/1742-5468/2012/12/P12002} \]
The poles of the form factors can be separated by introducing the regular connected part $F_{4rc}$:

$$F_{4}^{C_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_2, \vartheta'_1) = \frac{A}{\vartheta_2 - \vartheta'_1} + \frac{B}{\vartheta_2 - \vartheta'_2} + \frac{C}{\vartheta_1 - \vartheta'_1} + \frac{D}{\vartheta_1 - \vartheta'_2}$$

$$+ \frac{F_{4rc}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1, \vartheta'_2)}{\vartheta_1 - \vartheta'_1} + \frac{G}{\vartheta_2 - \vartheta'_1} + \frac{H}{\vartheta_2 - \vartheta'_2}$$

$$F_{4}^{C_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_1, \vartheta'_2) = \frac{E}{\vartheta_1 - \vartheta'_2} + \frac{F}{\vartheta_1 - \vartheta'_1} + \frac{G}{\vartheta_2 - \vartheta'_2} + \frac{H}{\vartheta_2 - \vartheta'_1}$$

$$+ \frac{F_{4rc}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2, \vartheta'_1)}{\vartheta_2 - \vartheta'_2} + \frac{G}{\vartheta_1 - \vartheta'_1} + \frac{H}{\vartheta_1 - \vartheta'_1}$$

(3.4)

where

$$A = i(S(\vartheta_2 - \vartheta_1) - S(\vartheta'_2 - \vartheta'_1))F_{4}^{C_1}(\vartheta_1 + i\pi, \vartheta'_2)$$

$$B = i(S(\vartheta'_2 - \vartheta'_1)S(\vartheta_2 - \vartheta_1) - 1)F_{4}^{C_1}(\vartheta_1 + i\pi, \vartheta'_1)$$

$$C = i(1 - S(\vartheta_2 - \vartheta_1)S(\vartheta'_2 - \vartheta'_1))F_{4}^{C_1}(\vartheta_2 + i\pi, \vartheta'_2)$$

$$D = i(S(\vartheta'_2 - \vartheta'_1)S(\vartheta_2 - \vartheta_1))F_{4}^{C_1}(\vartheta_1 + i\pi, \vartheta'_1)$$

$$E = i(S(\vartheta'_1 - \vartheta'_2)S(\vartheta_2 - \vartheta_1) - 1)F_{4}^{C_2}(\vartheta_2 + i\pi, \vartheta'_1)$$

$$F = i(S(\vartheta'_1 - \vartheta'_2)S(\vartheta_1 - \vartheta_2) - 1)F_{4}^{C_2}(\vartheta_1 + i\pi, \vartheta'_1)$$

$$G = i(1 - S(\vartheta_1 - \vartheta_2)S(\vartheta'_1 - \vartheta'_2))F_{4}^{C_2}(\vartheta_1 + i\pi, \vartheta'_1)$$

$$H = i(S(\vartheta'_1 - \vartheta'_2) - S(\vartheta_1 - \vartheta_2)F_{4}^{C_2}(\vartheta_1 + i\pi, \vartheta'_2).$$

Using the above notation, the pole terms resulting from the form factors can be obtained:

$$F_{4}^{C_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_2, \vartheta'_1)F_{4}^{C_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2, \vartheta'_1)$$

$$= F_{4rc}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_2, \vartheta'_1)F_{4rc}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2, \vartheta'_1)
+ \frac{F_{4rc}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_1, \vartheta'_2)}{\vartheta_1 - \vartheta'_2} + \frac{F_{4rc}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_1, \vartheta'_2)}{\vartheta_2 - \vartheta'_1}
+ \frac{F_{4rc}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_1, \vartheta'_2)}{\vartheta_2 - \vartheta'_2}$$

$$+ \frac{A}{\vartheta_1 - \vartheta'_1} + \frac{B}{\vartheta_2 - \vartheta'_2} + \frac{C}{\vartheta_1 - \vartheta'_2} + \frac{D}{\vartheta_1 - \vartheta'_2}$$

$$+ \frac{A}{\vartheta_1 - \vartheta'_2} + \frac{B}{\vartheta_2 - \vartheta'_1} + \frac{C}{\vartheta_2 - \vartheta'_2} + \frac{D}{\vartheta_2 - \vartheta'_1}$$

(3.5)

from which one can identify the terms giving $QF$ and $FF$ type singularities. For the residue calculation, the formulas of appendix A can be used. This results in certain differences from the result derived in [7], where too simplistic evaluation of residues resulted in some inaccuracies in the end result.

Once the residues are calculated, in the case of the $QF$ terms a further summation remains, which must be converted into an integral. It has the general form (here written for the case QFI)

$$\sum_{j_2 \neq j_2} G(\vartheta_1, \vartheta_2, \vartheta'_2) \left( \frac{\partial QF(\vartheta_1, \vartheta'_2) / \partial \vartheta'_2}{QF(\vartheta_1, \vartheta'_2)} \right)_{QF=2\pi j_2}$$
where $\vartheta'_2$ is a solution to
\[ Q'_2(\vartheta_1, \vartheta'_2) = 2\pi J_2 \]
and the case $J_2 = I_2$ was omitted since it is a spurious $QQ$ singularity. One can convert the $J_2$ summation into integrals using the residue formula
\[ - \sum_{J_2 \neq I_2} \oint_{C_{J_2}} \frac{d\vartheta'_2}{2\pi} \frac{G(\vartheta_1, \vartheta_2, \vartheta'_2)}{e^{Q_2(\vartheta_1, \vartheta'_2)} + 1} \]
Opening the contours and taking care to eliminate the contributions resulting from possible poles of the function $G$ lying on the real $\vartheta'_2$ axis,
\[ = - \oint_{C_{\vartheta'_2}} \frac{d\vartheta'_2}{2\pi} \frac{G(\vartheta_1, \vartheta_2, \vartheta'_2)}{e^{Q_2(\vartheta_1, \vartheta'_2)} + 1} - \frac{G(\vartheta_1, \vartheta_2, \vartheta_2)}{(\partial Q_2(\vartheta_1, \vartheta'_2)/\partial \vartheta'_2)|_{\vartheta'_2=\vartheta_2}} \]
\[ \sum_{\text{poles of } G} \oint_{C_{\vartheta'_2}} \frac{d\vartheta'_2}{2\pi} \frac{G(\vartheta_1, \vartheta_2, \vartheta'_2)}{e^{Q_2(\vartheta_1, \vartheta'_2)} + 1} \] (3.6)
where the second term corrects for the subtraction of the $J_2 = I_2$ case and $\vartheta'_2$ denotes the location of the poles of $G$. The notation $\Rightarrow$ corresponds to the straight line contours enclosing the real axis as illustrated in figure 1. The full results of the residue calculations are given in appendix B.

The $J_2 = I_2$ term typically is of order $O(1/L)$, except for second order pole contributions. This results in the following contribution to the $QF$ terms:
\[ F_2^{Q_1}(i\pi, 0)F_2^{Q_2}(i\pi, 0)K_{1,t}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_2, \vartheta_1) \]
\[ \times \left( \frac{mL \cosh \vartheta_1 - \varphi (\vartheta_1 - \vartheta_2)}{mL \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} + \frac{mL \cosh \vartheta_2 - \varphi (\vartheta_2 - \vartheta_1)}{mL \cosh \vartheta_1 + \varphi (\vartheta_1 - \vartheta_2)} \right) \] (3.7)
which is included in QF6 in (B.7). This term was omitted by the calculation performed in [7]; its presence is critical for the cluster property.

3.2.3. Performing the $I_1, I_2$ sum and the large volume limit. We can write
\[ C_{22}^{\text{mon}} = \sum_{I_1 > I_2} \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_2)}{\rho_2(\vartheta_1, \vartheta_2)} = \frac{1}{2} \left( \sum_{I_1 = I_2} - \sum_{I_1 = I_2} \right) \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_2)}{\rho_2(\vartheta_1, \vartheta_2)} \]
\[ = \frac{1}{2} \sum_{I_1 = I_2} \oint_{C_{I_1} \times C_{I_2}} \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_2)}{e^{Q_2(\vartheta_1, \vartheta_2)} + 1} \frac{\rho_2(\vartheta_1)}{e^{Q_1(\vartheta_1, \vartheta_2)} + 1} \]
\[ + \frac{1}{2} \sum_{I_1 = I_2} \oint_{C_{I_1}} \frac{d\vartheta_1}{2\pi} \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_1)}{e^{Q_1(\vartheta_1, \vartheta_1)} + 1} \frac{\rho_1(\vartheta_1)}{\rho_2(\vartheta_1, \vartheta_1)}. \]
Since $\tilde{C}_{22}$ does not have any pole we open the contours in the usual way, enclosing the real axis as illustrated in figure 1. For $L \rightarrow \infty$ it is necessary to examine the behavior of

doi:10.1088/1742-5468/2012/12/P12002
the $Q$-functions:
\[
iQ_1(\vartheta_1 + i\varepsilon_1, \vartheta_2 + i\varepsilon_2) = imL \sinh(\vartheta_1 + i\varepsilon_1) + i\delta(\vartheta_1 + i\varepsilon_1 - \vartheta_2 - i\varepsilon_2)
= imL \sinh \vartheta_1 \cos \varepsilon_1 - mL \cosh \vartheta_1 \sin \varepsilon_1 + i\delta(\vartheta_1 + i\varepsilon_1 - \vartheta_2 - i\varepsilon_2)
\]
and similarly for $Q_2$ and $Q_{1,2}'$. This results in the following limits:
\[
\lim_{L \to \infty} \frac{1}{e^{iQ_1(\vartheta_1 + i\varepsilon_1, \vartheta_2 + i\varepsilon_2)} + 1} = \begin{cases} 
1, & \varepsilon_i \in [0, \pi] + 2n\pi \\
0, & \varepsilon_i \in [\pi, 2\pi] + 2n\pi
\end{cases}
\]
\[
\lim_{L \to \infty} \frac{1}{e^{iQ_1(\vartheta_1 + i\varepsilon_1, \vartheta_2 + \varepsilon_1)} + 1} = \begin{cases} 
1, & \varepsilon_1 \in [0, \pi] + 2n\pi \\
0, & \varepsilon_1 \in [\pi, 2\pi] + 2n\pi
\end{cases}
\]
\[
\lim_{L \to \infty} \frac{1}{e^{iQ_1'(\vartheta_1' + i\varepsilon_1, \vartheta_2' + i\varepsilon_2)} + 1} = \begin{cases} 
1, & \varepsilon_i \in [0, \pi] + 2n\pi \\
0, & \varepsilon_i \in [\pi, 2\pi] + 2n\pi
\end{cases}
\]
Therefore only the upper contours need to be kept, since all other terms vanish exponentially for large $L$:
\[
\frac{1}{2} \int_0^{2\pi} \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \tilde{C}_{22}(\vartheta_1 + i\varepsilon, \vartheta_2 + i\varepsilon) = \frac{1}{2} \int_0^{2\pi} \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \tilde{C}_{22}(\vartheta_1 + i\varepsilon, \vartheta_1 + i\varepsilon) \frac{\rho_1(\vartheta_1 + i\varepsilon)}{\rho_2(\vartheta_1 + i\varepsilon, \vartheta_1 + i\varepsilon)}.
\]
In addition, the integrals can be shifted to the real axis. However, this leads to singularities in the contribution such as $QF_5$ (B.6) due to the term containing
\[
\frac{K_{4s}(\vartheta_1, \vartheta_2, \vartheta_1', \vartheta_2')} {\left[ e^{iQ_1'(\vartheta_1', \vartheta_1)} + 1 \right] (\vartheta_1 - \vartheta_1')}
\]
which can be treated using the identity
\[
\frac{1}{x + i\varepsilon} = \mathcal{P} \frac{1}{x} + i\pi\delta(x).
\]
\section{3.3. End result for $D_{22}$}

The terms divergent as $L \to \infty$ drop out when including the contribution $-Z_1 \tilde{D}_{111} - Z_2 C_{00}$. We can also combine some terms by introducing the function
\[
F_{4s}^C(\vartheta_1 + i\varepsilon, \vartheta_2 + i\varepsilon, \vartheta_1', \vartheta_2') = \frac{i(S(\vartheta_1 - \vartheta_2) - S(\vartheta_1' - \vartheta_2'))}{\vartheta_1 - \vartheta_2} F_2^C(\vartheta_2 + i\varepsilon, \vartheta_1')
+ \frac{i(1 - S(\vartheta_1 - \vartheta_2)S(\vartheta_1' - \vartheta_2'))}{\vartheta_2 - \vartheta_2'} F_2^C(\vartheta_1 + i\varepsilon, \vartheta_1)
+ \frac{i(S(\vartheta_1' - \vartheta_2') - S(\vartheta_1 - \vartheta_2))}{\vartheta_2' - \vartheta_1' F_2^C(\vartheta_1 + i\varepsilon, \vartheta_2')
+ \frac{i(1 - S(\vartheta_1 - \vartheta_2)S(\vartheta_1' - \vartheta_2'))}{\vartheta_2 - \vartheta_2'} F_2^C(\vartheta_1 + i\varepsilon, \vartheta_2')
\]
The end result is
\[
D_{22} = \frac{1}{4} \int_0^{2\pi} \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} F_{4s}^C(\vartheta_1 + i\varepsilon, \vartheta_1 + i\varepsilon, \vartheta_1' + i\varepsilon, \vartheta_2 + i\varepsilon)
\times F_{4s}^C(\vartheta_1 + i\varepsilon, \vartheta_1 + i\varepsilon, \vartheta_1' + i\varepsilon, \vartheta_2' + i\varepsilon) K_{1,2}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1' + i\varepsilon, \vartheta_2' + i\varepsilon).
\]
present in the diagonal contribution (3.3); the dependence on
\[
\begin{align*}
\left[ \cosh \vartheta_1 + \cosh \vartheta_2 \right]^2 \\
\cosh \vartheta_1 \cosh \vartheta_2
\end{align*}
\]

is simplified by the inclusion of the contribution (3.7), coming from the second order pole terms collected in QF6 (B.7). In the large \(L\) limit, the terms depending on the \(\cosh\) ratios cancel, leaving us with the last underlined piece in (3.10). As mentioned before, one of the mistakes made in the evaluation of \(D_{22}\) in [7] was the omission of this piece.

\[
doi:10.1088/1742-5468/2012/12/P12002
\]
3.4. The full two-point function up to $D_{22}$

For completeness, we also give here the lower contributions to the two-point function. These are exactly the same as in [40, 7], so we do not give the derivations here. The calculations are almost trivial with the exception of $D_{12}$, where one can use either the derivations presented in [7], or follow the steps outlined above, with slight modifications. The terms $D_{NM}$ with $N \leq M \leq 2$ are

\[
\begin{align*}
D_{00} &= \langle O_1 \rangle \langle O_2 \rangle \\
D_{01} &= \int \frac{d\theta_1}{2\pi} F_1^{O_1} F_1^{O_2} e^{-imx \sinh \vartheta_1 - mt \cosh \vartheta_1} \\
D_{02} &= \frac{1}{2} \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} F_1^{O_1} (\vartheta_1, \vartheta_2) F_2^{O_2} (\vartheta_2, \vartheta_1) e^{-imx (\sinh \vartheta_1 + \sinh \vartheta_2) - mt (\cosh \vartheta_1 + \cosh \vartheta_2)} \\
D_{11} &= \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} F_1^{O_1} (\vartheta_1 + i\pi, \vartheta_2) F_2^{O_2} (\vartheta_2 + i\pi, \vartheta_1) e^{imx (\sinh \vartheta_1 - \sinh \vartheta_2)} e^{-m(R-t) \cosh \vartheta_1} e^{-mt \cosh \vartheta_2} \\
&\quad + \left[ \langle O_1 \rangle F_{2s}^{O_2} + \langle O_2 \rangle F_{2s}^{O_1} \right] \int \frac{d\theta_1}{2\pi} e^{-mR \cosh \vartheta_1} \\
D_{12} &= \frac{1}{2} \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} F_1^{O_1} (\vartheta_1 + i(\pi + \varepsilon), \vartheta_1', \vartheta_2') F_2^{O_2} (\vartheta_1 + i(\pi + \varepsilon), \vartheta_2, \vartheta_1') \\
&\quad \times K^{(R)}_{t,x} (\vartheta_1 + i\varepsilon | \vartheta_1', \vartheta_2') \\
&\quad + \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} [F_1^{O_1} F_{3^{sc}}^{O_2} (\vartheta_1' + i\pi | \vartheta_1', \vartheta_2') + F_1^{O_2} F_{3^{sc}}^{O_1} (\vartheta_1' + i\pi | \vartheta_1', \vartheta_2')] \\
&\quad \times K^{(R)}_{t,x} (\vartheta_1' | \vartheta_1', \vartheta_2') \\
&\quad + \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} F_1^{O_1} F_1^{O_2} K^{(R)}_{t,x} (\vartheta_1' | \vartheta_1', \vartheta_2') (S (\vartheta_1' - \vartheta_2') - 1) \\
&\quad \times (mx \cosh \vartheta_1' + im (R-t) \sinh \vartheta_1') \\
&\quad - \int \frac{d\theta_1}{2\pi} F_1^{O_1} F_1^{O_2} K^{(R)}_{t,x} (\vartheta_1' | \vartheta_1', \vartheta_1')
\end{align*}
\]

and $D_{22}$ is given in (3.10). In $D_{12}$ we defined the regular connected form factor function $F_{3^{sc}}$ via the following separation of the kinematical pole terms:

\[
F_3^O (\vartheta_1 + i\pi, \vartheta_1', \vartheta_2') = \frac{i (1 - S(\vartheta_1' - \vartheta_2')) F_1^O}{\vartheta_1 - \vartheta_1'} + \frac{i (S(\vartheta_1' - \vartheta_2') - 1) F_2^O}{\vartheta_1 - \vartheta_2'} + F_{3^{sc}}^O (\vartheta_1 + i\pi | \vartheta_1', \vartheta_2')
\]

and used the abbreviation

\[
K_{t,x}^{(R)} (\vartheta_1 | \vartheta_1', \vartheta_2') = e^{imx (\sinh \vartheta_1 - \sinh \vartheta_1' - \sinh \vartheta_2')} e^{-m(R-t) \cosh \vartheta_1} e^{-mt (\cosh \vartheta_1' + \cosh \vartheta_2')}.
\]

The other contributions $D_{NM}$ for $M > N$ can be obtained from $D_{MN}$ by exchanging $O_1$ with $O_2$ and replacing $t \rightarrow R - t, x \rightarrow -x.$

doi:10.1088/1742-5468/2012/12/P12002
3.5. The symmetry of the $D_{22}$ term

The relation between the coefficients $D_{NM}$ and $D_{MN}$ stated above, when applied to $D_{22}$, leads to the property that $D_{22}$ must be symmetric under the following transformation:

\[
\begin{align*}
t &\rightarrow R - t \\
O^1 &\leftrightarrow O^2 \\
x &\rightarrow -x.
\end{align*}
\]

This is the same as requiring that the result should be independent of which two-particle summation is performed first. However, when implementing such a transformation in (3.10), the signs of the $\epsilon$ terms change, and therefore the contours must be pulled back to their original positions. The contour deformation encounters all the singularities on the real axis that were treated previously in this section, so the appropriate residue contributions must be computed. This computation is relegated to appendix C, where it is demonstrated that the required symmetry property indeed holds, providing the first nontrivial test of the result (3.10).

4. Numerical verification of the analytic results

The goal of this section is to validate the $D_{22}$ formula numerically. For this purpose we evaluated directly the sum for the two-particle states and compare it with the result of the contour integrals. For calculations we used the sinh–Gordon model with the Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{m^2}{g^2} \cosh g \Phi. \tag{4.1}
\]

The model contains one massive particle, and its two-particle scattering matrix is simple but nontrivial:

\[
S(\theta) = \frac{\sinh \theta - i \sin(\pi B/2)}{\sinh \theta + i \sin(\pi B/2)}
\]

where

\[
B = \frac{2g^2}{8\pi + g^2}.
\]

The nontrivial $S$-matrix is important, since our formula contains the scattering matrix and its derivative in an essential way, which we would like to verify. For the fields in the correlator, we chose the exponential operators

\[
e^{k g \Phi} \tag{4.2}
\]

normalized to have vacuum expectation value unity, since their form factors are explicitly known [42, 43]:

\[
F_n^{(k)}(\theta_1, \theta_2, \ldots, \theta_n) = H_n \frac{P_n^{(k)}(x_1, x_2, \ldots, x_n)}{\prod_{i<j} (x_i + x_j)} \prod_{i<j} f(\theta_i - \theta_j) \tag{4.3}
\]

\[x_i = e^{\theta_i}.
\]
where the polynomials $P_n^{(k)}$ are given by

\[ P_1^{(k)} = [k] \]
\[ P_n^{(k)} = [k] \det M^{(n)}(k) \quad n > 1 \]
\[ M_{ij}^{(n)}(k) = [i - j + k] \sigma_{2i-j}^{(n)}(x_1, x_2, \ldots, x_n) \quad i, j = 1, \ldots, n - 1 \]

with

\[ H_n = \left( \frac{4 \sin \pi B/2}{f(i \pi)} \right)^{n/2} \quad \text{and} \quad [n] = \frac{\sin(n \pi B/2)}{\sin(\pi B/2)}. \]

Furthermore, $\sigma_{l}^{(n)}$ denotes the elementary symmetric polynomials of $n$ variables defined by

\[ \prod_{i=1}^{n} (x + x_i) = \sum_{l=1}^{n} x^{n-l} \sigma_{l}^{(n)}(x_1, \ldots, x_n) \]
\[ \sigma_{l}^{(n)} \equiv 0 \quad \text{if} \ l < 0 \text{ or } l > n \]

and the minimal two-particle form factor is given by

\[ f(\theta) = \mathcal{N} \exp \left[ 8 \int_{0}^{\infty} \frac{dx}{x} \sin^2 \left( \frac{x(i \pi - \theta)}{2\pi} \right) \frac{\sinh(xB/4) \sinh(1 - (B/2))(x/2) \sinh(x/2)}{\sinh^2 x} \right] \]  \hspace{1cm} (4.4)

where

\[ \mathcal{N} = \exp \left[ -4 \int_{0}^{\infty} \frac{dx}{x} \frac{\sinh(xB/4) \sinh(1 - (B/2))(x/2) \sinh(x/2)}{\sinh^2 x} \right]. \]  \hspace{1cm} (4.5)

### 4.1. Evaluating the two-particle sum

Numerical evaluation of the sum is only possible at finite volume. The factors $K_{t,x}^{(R)}$ decrease exponentially at large rapidities, so it is possible to choose a rapidity cutoff and restrict the summation up to the corresponding Bethe–Yang quantum number. However, for large volume this quantum number cutoff is still too big and it is practically impossible to evaluate the four-particle sum. The compromise is to evaluate only the inner two-particle sum with fixed outer rapidities at moderate volume. This is enough to check the validity of all the nontrivial contour deformations and residue manipulation in the $D_{22}$ calculation. We can write

\[ C_{22}^{\text{nondiag}} = \sum_{I_1, I_2} \tilde{C}_{22}(\vartheta_1, \vartheta_2) / \rho_2(\vartheta_1, \vartheta_2) \]

with

\[ \tilde{C}_{22}(\vartheta_1, \vartheta_2) = \sum_{J_1, J_2} \mathcal{F}_1^{C_{1}}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1', \vartheta_2') F_1^{Q_{2}}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_1', \vartheta_2') \rho_2(\vartheta_1', \vartheta_2') \]
\[ \times K_{t,x}^{(R)}(\vartheta_1, \vartheta_2, \vartheta_1', \vartheta_2') \]

and evaluate $\tilde{C}_{22}$ for some given value of $\vartheta_{1,2}$, corresponding to a solution of the Bethe–Yang equations (3.2) with some quantum numbers $\{I_1, I_2\}$. 

doi:10.1088/1742-5468/2012/12/P12002
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The parameters for the evaluation can be chosen to help with the convergence of the summation, while ensuring that the structure of the expression tested remains general. The exponential operators (4.2) in the sinh–Gordon model can be parametrized by the number $k$ that we chose for our evaluations as $k_1 = 2$ for $O_1$ and $k_2 = 4$ for $O_2$. The essential structure of the formula does not depend on this choice. The space–time parameters and the temperature were chosen as $m_x = 0.0$, $m_t = 0.4$, and $m_R = 0.8$. Setting $m_x$ to zero does not hide any important structure of the equation, but makes the expression real, and this helps in comparing the results with the contour integrals. The sum was evaluated with several values for the volume, the sinh–Gordon coupling constant and the outer rapidities:

$$m_L = (10, 15, 20, 25, 30)$$
$$B = (0.1, 0.2, 0.3, 0.4, 0.55, 0.7, 0.9)$$

$$\{I_1, I_2\} \in \left\{ \left\{ \frac{5}{2}, \frac{1}{2} \right\}, \left\{ \frac{11}{2}, -\frac{5}{2} \right\}, \left\{ -\frac{5}{2}, -\frac{21}{2} \right\}, \left\{ \frac{7}{2}, -\frac{7}{2} \right\}, \left\{ 1, -\frac{1}{2} \right\} \right\}.$$  

The rapidity cutoff for the quantum numbers included in the sum was chosen as $\vartheta = 3.0, 4.0, 5.0, 6.0$, and the numerical results showed that for the value $\vartheta = 6.0$ the discrete sum was evaluated within a relative error of less than $10^{-14}$.

4.2. Evaluating the contour integrals

To compare the results of the contour integrals with the direct sum, the calculation must be performed at the same volumes. In this regime the exponential and power corrections in volume $L$ are not negligible, so they must be taken into account. Exponential corrections come from the integration on the contours going under the real axis, while power corrections come from the total derivative contribution in the second order pole calculation and from the $\{J_1, J_2\} = \{I_1, I_2\}$ point in the pole and the double integral contributions. The explicit formulas can be found in appendix B.

The integration contours run below and above the real axis, and it is important to find a choice that is optimal for numerical evaluation. The form factors and hence the integrands have poles on the real axis, so it would be better to integrate as far from the real axis as possible. However, for imaginary parts of rapidities larger than $(\pi/2)$ the factor $K_{t,x}^{(R)}(\vartheta_1, \vartheta_2, \vartheta_1', \vartheta_2')$ becomes oscillating and exponentially growing in the rapidity parameters instead of decaying. Another issue is that the form factors and scattering matrices are also evaluated at rapidities that lie out of the physical strip. In the sinh–Gordon model the scattering matrix and hence the minimal form factor have poles out of the physical strip, with imaginary positions that are proportional to the coupling parameter $B$ [42]. Therefore, the contour must be chosen to lie between these poles on the one hand and the poles on the real axis on the other hand. At the same time it must run as far away from all singularities as possible, and also be closer to the real axis than $(\pi/2)$. For small $B$ this leaves little space for the contours, so they run relatively close to the poles, resulting in a larger error in the numerical integration. The integration itself was performed using Mathematica\textsuperscript{4} and the Cuba library for multidimensional numerical integrations [44].

\textsuperscript{4} Wolfram Research, Mathematica, version 8.0, Champaign, IL, 2010.
Table 1. Relative error of the difference between the direct sum and the contour integral evaluation of $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$ with $\{I_1, I_2\} = \{\frac{\pi}{2}, \frac{\pi}{2}\}$.

| $B$ | 10   | 15   | 20   | 25   | 30   |
|-----|------|------|------|------|------|
| 0.1 | $8.56 \times 10^{-8}$ | $1.03 \times 10^{-7}$ | $4.93 \times 10^{-9}$ | $3.29 \times 10^{-7}$ | $1.31 \times 10^{-8}$ |
| 0.2 | $7.38 \times 10^{-10}$ | $3.53 \times 10^{-10}$ | $6.29 \times 10^{-11}$ | $9.74 \times 10^{-10}$ | $1.96 \times 10^{-10}$ |
| 0.3 | $1.74 \times 10^{-10}$ | $1.13 \times 10^{-10}$ | $1.25 \times 10^{-10}$ | $1.26 \times 10^{-10}$ | $1.26 \times 10^{-10}$ |
| 0.4 | $1.42 \times 10^{-10}$ | $1.42 \times 10^{-10}$ | $1.41 \times 10^{-10}$ | $1.4 \times 10^{-10}$ | $1.4 \times 10^{-10}$ |
| 0.55 | $1.41 \times 10^{-10}$ | $1.42 \times 10^{-10}$ | $1.42 \times 10^{-10}$ | $1.42 \times 10^{-10}$ | $1.42 \times 10^{-10}$ |
| 0.7 | $1.33 \times 10^{-10}$ | $1.34 \times 10^{-10}$ | $1.33 \times 10^{-10}$ | $1.33 \times 10^{-10}$ | $1.33 \times 10^{-10}$ |
| 0.9 | $1.3 \times 10^{-10}$ | $1.3 \times 10^{-10}$ | $1.3 \times 10^{-10}$ | $1.3 \times 10^{-10}$ | $1.3 \times 10^{-10}$ |

Table 2. Relative error of the difference between the direct sum and the contour integral evaluation of $\tilde{C}_{12}(\vartheta_1)$ with $I_1 = 17$.

| $B$ | 10   | 15   | 20   | 25   | 30   |
|-----|------|------|------|------|------|
| 0.1 | $2.03 \times 10^{-6}$ | $3.56 \times 10^{-6}$ | $3.41 \times 10^{-7}$ | $1.56 \times 10^{-7}$ | $8.81 \times 10^{-8}$ |
| 0.2 | $7.5 \times 10^{-9}$ | $5.03 \times 10^{-9}$ | $1.97 \times 10^{-10}$ | $1.52 \times 10^{-9}$ | $7.73 \times 10^{-10}$ |
| 0.3 | $1.48 \times 10^{-10}$ | $7.76 \times 10^{-11}$ | $3.1 \times 10^{-11}$ | $6.22 \times 10^{-11}$ | $3.13 \times 10^{-11}$ |
| 0.4 | $2.35 \times 10^{-11}$ | $5.29 \times 10^{-11}$ | $5.81 \times 10^{-11}$ | $6.77 \times 10^{-11}$ | $6.59 \times 10^{-11}$ |
| 0.55 | $5.87 \times 10^{-11}$ | $6.61 \times 10^{-11}$ | $6.87 \times 10^{-11}$ | $6.9 \times 10^{-11}$ | $6.96 \times 10^{-11}$ |
| 0.7 | $2.96 \times 10^{-11}$ | $5.32 \times 10^{-11}$ | $6.02 \times 10^{-11}$ | $6.25 \times 10^{-11}$ | $6.38 \times 10^{-11}$ |
| 0.9 | $6.01 \times 10^{-11}$ | $6.48 \times 10^{-11}$ | $6.52 \times 10^{-11}$ | $6.51 \times 10^{-11}$ | $6.52 \times 10^{-11}$ |

4.3. Comparing the results

Table 1 shows the relative deviation between the direct sum and the contour integral evaluation of $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$ with $\{I_1, I_2\} = \{\frac{\pi}{2}, \frac{\pi}{2}\}$. Note that the relative error decreases as $B$ grows, which can be understood from the conditions for the choice of the integration contour mentioned above. Based on the above understanding of the deviations in the relative errors for different parameters, and the fact that this pattern of dependence was the same for every value of $I_1, I_2$ we checked, it can be inferred that the difference of the sum and the contour integration is only due to the numerical errors of integration.

To provide a further support for this conclusion, the above numerical test was repeated for $C_{12}$. The formula of $C_{12}$ is derived in two independent ways in [7] (depending on whether the one-particle or the two-particle summation is performed first), and therefore its validity is quite certain even without a numerical test. As in the case of $C_{22}$, let us denote by $\tilde{C}_{12}(\vartheta_1)$ the result of performing the two-particle summation first with fixed rapidity of the one-particle state. Table 2 shows the relative deviation between the direct sum and the contour integral evaluation of $\tilde{C}_{12}(\vartheta_1)$ with $I_1 = 17$ as the Bethe–Yang quantum number of the one-particle state. The relative deviation has the same pattern as for $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$, and is essentially of the same magnitude. Therefore, this evaluation gives an independent support for the assertion that the deviations are caused by errors of numerical integration.
As the derivation of $D_{22}$ from $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$ is almost trivial, the above numerical tests also confirm the details of our analytic result for $D_{22}$.

5. The cluster property of the two-point function

Another important test of the results is provided by checking that the two-point function has the cluster property

$$\langle O_1(x, t)O_2(0, 0) \rangle^R \sim \langle O_1(0, 0) \rangle^R \langle O_2(0, 0) \rangle^R$$

when the spatial separation $x$ grows large. Using the expansion up to $D_{22}$ one can write

$$\langle O_1(x, t)O_2(0, 0) \rangle^R = \sum_{N, M} D_{NM}$$

$$= D_{00} + D_{01} + D_{10} + D_{11} + D_{02} + D_{20} + D_{12} + D_{21} + D_{22} + \cdots.$$ 

For $mx \gg 1$ the terms containing

$$e^{imx \left( \sum_k \sinh \vartheta_k - \sum_l \sinh \vartheta'_l \right)}$$

oscillate very fast, and therefore the support of the (multiple) rapidity integrals is restricted to the zero measure set

$$\sum_k \sinh \vartheta_k = \sum_l \sinh \vartheta'_l$$

and the integral vanishes. Although this argument looks simple, there is a possible problem: namely, the argument only works if the integrands of the $x$-dependent terms are all regular. A nontrivial example is the term

$$\frac{1}{2} \int \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \left\{ 2 \left[ F_{1}^{O_1} F_{2}^{O_2} \left( S(\vartheta'_1 - \vartheta'_2) - 1 \right) K_{t,x}^{(R)}(\vartheta'_1 | \vartheta'_1, \vartheta'_2) + (\vartheta'_1 \leftrightarrow \vartheta'_2) \right] \right\}$$

in the contribution $D_{12}$ (cf equation (3.11)), which is in fact regular at $\vartheta'_1 = \vartheta'_2$ when the two terms inside the braces are added together. In the case of the principal value integral in $D_{22}$ in equation (3.10), the regularity of the integrand is ensured by the principal value prescription itself.

As a result, one only needs to examine the terms that are $x$ independent. We denote these by putting a bar over the respective contribution $D_{NM}$, and they read

$$\bar{D}_{00} = \langle O_1 \rangle \langle O_2 \rangle$$

$$\bar{D}_{01} = \bar{D}_{10} = \bar{D}_{02} = \bar{D}_{20} = 0$$

$$\bar{D}_{11} = \left[ \langle O_1 \rangle F_{2}^{O_2} (i\pi, 0) + \langle O_2 \rangle F_{2}^{O_1} (i\pi, 0) \right] \int \frac{d\vartheta_1}{2\pi} e^{-mR \cosh \vartheta_1}$$

$$\bar{D}_{12} = \bar{D}_{21} = 0$$

$$\bar{D}_{22} = \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \left[ F_{4s}^{O_1} (\vartheta_1, \vartheta_2) \langle O_2 \rangle + F_{4s}^{O_2} (\vartheta_1, \vartheta_2) \langle O_1 \rangle \right] e^{-mR (\cosh \vartheta_1 + \cosh \vartheta_2)}$$

$$+ \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} F_{1}^{O_1} (i\pi, 0) F_{2}^{O_2} (i\pi, 0) e^{-mR (\cosh \vartheta_1 + \cosh \vartheta_2)}$$

$$- \int \frac{d\vartheta_1}{2\pi} \left[ F_{2}^{O_1} (i\pi, 0) \langle O_2 \rangle + F_{2}^{O_2} (i\pi, 0) \langle O_1 \rangle \right] e^{-2mR \cosh \vartheta_1}. $$
The one-point function up to two-particle order is \[27\]
\[
\langle O \rangle^R = \langle O \rangle + \int \frac{d\vartheta_1}{2\pi} F_2^O (i\pi, 0) e^{-mR\cosh \vartheta_1} - \int \frac{d\vartheta_1}{2\pi} F_2^O (i\pi, 0) e^{-2mR\cosh \vartheta_1} + \frac{1}{2} \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} F_{4s}^O (\vartheta_1, \vartheta_2) e^{-mR(\cosh \vartheta_1 + \cosh \vartheta_2)} + O \left( e^{-3mR} \right).
\]

As a result one obtains that
\[
\bar{D}_{00} + \bar{D}_{11} + \bar{D}_{22} = \langle O_1 \rangle^R \langle O_2 \rangle^R + O \left( e^{-3mR} \right)
\]
and therefore the cluster property is satisfied to the given order. Note that the same argument shows that the formula for \(D_{22}\) derived in \[7\] violates the cluster property, providing another argument that it needs to be corrected.

6. Conclusions

In this work we revisited the finite volume regularization of thermal correlators introduced in \[7\], which is based on the finite volume form factor formalism \[26, 27\]. We have shown that the original results for the two-particle–two-particle contribution \(D_{22}\) need to be slightly corrected, and presented a modified prescription for evaluation of residue contributions. As a result, we now have the expansion up to all terms involving intermediate states with no more than two particles. In addition, the result for the nontrivial terms \(D_{12}\) and \(D_{22}\) was cross-checked with a numerical evaluation of the finite volume summation over the intermediate multi-particle states. It was also established that the correlation function given by the final formulas \(3.10\) and \(3.11\) satisfies the cluster property. In addition, it was shown to possess a symmetry property which follows from the general structure of the spectral expansion.

The formalism presented here can be extended to compute any higher correction in the series. However, the calculation of \(D_{22}\) is already very tedious, and it is expected to become even longer for higher terms. In view of potential applications to condensed matter systems, it is likely that the present evaluation would suffice for most of the cases.

Another reason why the result evaluated up to \(D_{22}\) is interesting is that this is the part which generalizes to non-integrable field theories. Breaking integrability in general allows inelastic processes; however, below the inelastic threshold the finite volume levels can still be described using only the elastic phase shift and the same quantization conditions as in \(3.2\) \[45, 46\]. Therefore, the present computation can be extended self-consistently whenever the states dominating the spectral expansion are below the inelastic threshold.

On the other hand, it remains an open question whether the full expansion for the finite temperature two-point function in integrable models can be recast in a form similar to the expression conjectured by Leclair and Mussardo \[16\]. Such an expression would represent a partial re-summation of the series, expressing the correlator in terms of Fermi–Dirac distribution for the dressed particles corresponding to the representation of the system as a free gas of quasi-particles under the thermodynamic Bethe ansatz \[47\]. It is known that this re-summation is possible for the one-point function \[27, 28\], even for the case of operators located on a system boundary \[32\]. In that respect, the explicit dependence of \(D_{22}\) on the two-particle \(S\)-matrix (cf the first underlined term in equation \(3.10\)) does not bode well. The TBA equation contains only the derivative of the phase shift \(\varphi\), so
any dressing term from there is expected to depend only on $\varphi$. Some partial integration
tricks can be performed to shift this dependence around, but we have found no way of
eliminating it. On the other hand, it was noticed in [7] that the $D_{1n}$ contributions allow
some re-summation by dressing the contributions $D_{0n-1}$. There is a possibility that a
redefinition of the form factor terms could help, i.e. if one used some definition for the
desingularized form factors different from the $F_{rc}$ or $F_{ss}$. At present it is not known how to
accomplish this; however, there is still hope for recovering some expression similar to the
original Leclair–Mussardo conjecture. Evaluation of some higher order corrections could
shed light on the structure of the series, and the experience gained in the present work
opens the way to performing these calculations, armed with a numerical method to verify
the results of the complicated analytic manipulations. We hope to return to this line of
thought in the near future.

The present calculation was performed for a theory with a single massive particle.
Adding more particles to the spectrum is rather straightforward as long as the scattering
remains diagonal; it is only necessary to add particle species labels in appropriate places.
For non-diagonal scattering theories, recent progress has made almost all matrix elements
available in finite volume [48]–[50], except for matrix elements involving disconnected
pieces when the states involved in the matrix element are subject to non-diagonal
scattering. However, more recently we have solved the issue for two-particle states\footnote{In fact, the paper [51] presents a conjectured solution for any number of particles, but only the two-particle case is backed up by (very strong) numerical evidence.} [51],
which means that the series presented here can be evaluated for general integrable field
theories, including those with non-diagonal scattering such as the sine–Gordon or O(3) $\sigma$
models [8].

Acknowledgments

We are grateful to Balázs Pozsgay for useful discussions and valuable comments on the
paper. GT was partially supported by the Hungarian OTKA grants K75172 and K81461.

Appendix A. Residue evaluations

A.1. First order poles

In this case, the two-dimensional residue formula

$$\oint_{C_a} \oint_{C_b} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{g(z_1, z_2)}{f_1(z_1, z_2)f_2(z_1, z_2)} = \frac{g(a, b)}{\det (\partial f_i/\partial z_j) \bigg|_{(z_1, z_2)=(a, b)}}$$

can be applied directly as

$$\oint_{C_a} \oint_{C_J} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{g(z_1, z_2)}{[a - z_1][e^{iQ_2(z_1, z_2)} + 1]} = \frac{g(a, z_2^*)}{i(\partial Q_2(a, z_2)/\partial z_2) \bigg|_{z_2=z_2^*}}$$

where $(a - z_1)$ is the pole term coming from the appropriate form factor, and $z_2^*$ is the root of

$$e^{iQ_2(a, z_2^*)} + 1 = 0.$$
A.2. Second order poles

These can be evaluated by successive integration, performing first the integral over the second order pole coming from the form factor. The fundamental formula to use is

$$\text{Res}_{z=a} \frac{h(z)}{g(z)} = \frac{2h'(a)}{g''(a)} - \frac{2g''(a)}{3[g''(a)]^2} h(a)$$  \hspace{1cm} \text{(A.1)}$$

where for a second order pole $g(a) = g'(a) = 0$ but $g''(a) \neq 0$. The terms we need to evaluate have the form

$$\oint_{C_a} \oint_{C_j} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{g(z_1, z_2)}{(a - z_1)^2 (e^{Q_2(z_1, z_2)} + 1)}.$$ 

Performing the $z_1$ integral leads to

$$\oint_{C_j} \frac{dz_2}{2\pi i} \left\{ \frac{(\partial g(z_1, z_2)/\partial z_1)}{e^{Q_2(z_1, z_2) + 1}} - \frac{g(a, z_2) e^{Q_2(a, z_2)} i(\partial Q(z_1, z_2)/\partial z_1) |_{z_1 = a}}{e^{Q_2(z_1, z_2) + 1}} \right\}$$

$$= i \frac{(\partial g(z_1, z_2)/\partial z_1) |_{z_1 = a}}{e^{Q_2(z_1, z_2) + 1}} - \oint_{C_j} \frac{dz_2}{2\pi i} \frac{g(a, z_2) e^{Q_2(a, z_2)} i(\partial Q(z_1, z_2)/\partial z_1) |_{z_1 = a}}{e^{Q_2(z_1, z_2) + 1}}.$$ 

For the second integral, introducing the notation $l(z_1, z_2) = g(z_1, z_2) i(\partial Q_2(z_1, z_2)/\partial z_1)$ and using (A.1) with $e^{Q_2(a, z_2)} = -1$,

$$\oint_{C_j} \frac{dz_2}{2\pi i} \frac{l(a, z_2) e^{Q_2(a, z_2)} - 1}{e^{Q_2(z_1, z_2) + 1}} = \frac{\partial}{\partial z_2} \frac{l(a, z_2)}{e^{Q_2(a, z_2)/\partial z_2} |_{z_2 = z_2^*}}.$$ 

Putting it together,

$$\oint_{C_a \times C_j} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{g(z_1, z_2)}{(a - z_1)^2 (e^{Q_2(z_1, z_2)} + 1)} = \frac{1}{\partial Q_2(a, z_2)/\partial z_2} \left\{ \frac{g(a, z_2) i(\partial Q(z_1, z_2)/\partial z_1) |_{z_1 = a}}{\partial Q(a, z_2)/\partial z_2} \right\}_{z_2 = z_2^*}$$

where again $z_2^*$ is the root of $e^{Q_2(a, z_2)} + 1 = 0$.

A.3. Useful formulas for practical evaluation

Using the form of the Bethe–Yang equations for $\vartheta_1, \vartheta_2$

$$-e^{iQ_1(\vartheta_1, \vartheta_2)} = e^{imL \sin \vartheta_1} S(\vartheta_1 - \vartheta_2) = 1$$

$$-e^{iQ_2(\vartheta_1, \vartheta_2)} = e^{imL \sin \vartheta_2} S(\vartheta_2 - \vartheta_1) = 1$$

we can easily substitute their solutions into the Bethe–Yang equations for $\vartheta_1', \vartheta_2'$

$$e^{iQ_1'(\vartheta_1', \vartheta_2')} = e^{imL \sin \vartheta_1} (-S(\vartheta_1 - \vartheta_2')) = -S(\vartheta_2 - \vartheta_1)S(\vartheta_1 - \vartheta_2')$$

$$e^{iQ_1'(\vartheta_2', \vartheta_1')} = e^{imL \sin \vartheta_2} (-S(\vartheta_2 - \vartheta_1')) = -S(\vartheta_1 - \vartheta_2)S(\vartheta_2 - \vartheta_1')$$

$$e^{iQ_2'(\vartheta_1', \vartheta_1')} = e^{imL \sin \vartheta_1} (-S(\vartheta_1 - \vartheta_1')) = -S(\vartheta_2 - \vartheta_1)S(\vartheta_1 - \vartheta_1')$$

$$e^{iQ_2'(\vartheta_2', \vartheta_2')} = e^{imL \sin \vartheta_2} (-S(\vartheta_2 - \vartheta_2')) = -S(\vartheta_1 - \vartheta_2)S(\vartheta_2 - \vartheta_2').$$
Denoting
\[ T(\vartheta_1, \vartheta_2, \vartheta_3) = 1 - S(\vartheta_1 - \vartheta_2)S(\vartheta_2 - \vartheta_3) \]
we can also write
\[ e^{iQ_1^1(\vartheta_1, \vartheta_2')} + 1 = T(\vartheta_2, \vartheta_1, \vartheta_2') \]
\[ e^{iQ_1^2(\vartheta_2, \vartheta_2')} + 1 = T(\vartheta_1, \vartheta_2, \vartheta_2') \]
\[ e^{iQ_2(\vartheta_2, \vartheta_1')} + 1 = T(\vartheta_2, \vartheta_1, \vartheta_1') \]
\[ e^{iQ_2(\vartheta_2, \vartheta_2')} + 1 = T(\vartheta_1, \vartheta_2, \vartheta_1') \]

**Appendix B. Pole terms for \( \tilde{C}_{22} \)**

Here we list the complete result for the pole term subtractions that appear in
\[ \tilde{C}_{22}(\vartheta_1, \vartheta_2) = \frac{1}{2} \oint \oint \frac{d\vartheta_1' d\vartheta_2' }{2\pi i^2} \left[ K^{(R)}_{1,x}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2') \right. \]
\[ \times \left. \frac{F_4^{O_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_2', \vartheta_1') F_4^{O_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_2', \vartheta_1')}{[e^{i\vartheta_1'(\vartheta_1, \vartheta_2')} + 1][e^{i\vartheta_1(\vartheta_1, \vartheta_2')} + 1]} \right] \]
\[ - \text{QF1}(\vartheta_1, \vartheta_2) - \text{QF2}(\vartheta_1, \vartheta_2) - \text{QF3}(\vartheta_1, \vartheta_2) - \text{QF4}(\vartheta_1, \vartheta_2) \]
\[- \text{QF5}(\vartheta_1, \vartheta_2) - \text{QF6}(\vartheta_1, \vartheta_2) - \text{FF}(\vartheta_1, \vartheta_2) - 2\text{SQQ}(\vartheta_1, \vartheta_2) \tag{B.1} \]

where the QF are the contributions from the QF singularities, FF comes from the FF singularities and SQQ is the spurious QQ singularity term. As in the main text, the notation \( \leftrightarrow \) corresponds to the straight line contours enclosing the real axis as illustrated in figure 1. The explicit forms of the individual singularity terms to (B.1) are as follows:

\[ \text{QF1}(\vartheta_1, \vartheta_2) = \oint \frac{d\vartheta_1'}{2\pi} \]
\[ \times \left( \frac{F_4^{O_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_1', \vartheta_1) F_2^{O_1}(\vartheta_2 + i\pi, \vartheta_2') K^{(R)}_{1,x}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)}{e^{i\vartheta_1'(\vartheta_1, \vartheta_2')} + 1} \right. \]
\[ + \{ \vartheta_1 \leftrightarrow \vartheta_2, \text{O1} \leftrightarrow \text{O2} \} \]
\[ + \left( \frac{F_4^{O_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_2, \vartheta_1) F_2^{O_1}(\vartheta_1, \vartheta_2') K^{(R)}_{1,x}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)}{mL \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} \right. \]
\[ + \{ \vartheta_1 \leftrightarrow \vartheta_2, \text{O1} \leftrightarrow \text{O2} \} \tag{B.2} \]

\[ \text{QF2}(\vartheta_1, \vartheta_2) = \oint \frac{d\vartheta_1'}{2\pi} \]
\[ \times \left( \frac{F_4^{O_1}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_1', \vartheta_1) F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_2') K^{(R)}_{1,x}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)}{e^{i\vartheta_1'(\vartheta_1, \vartheta_2')} + 1} \right. \]
\[ + \{ \vartheta_1 \leftrightarrow \vartheta_2, \text{O1} \leftrightarrow \text{O2} \} \]
+ \left( \frac{F_{4c}^O(\vartheta_1 + i\pi, \vartheta_2 + i\pi \mid \vartheta_2, \vartheta_1) F_{2}^{O_2}(i\pi, 0) K_{t,x}^{(O)}(\vartheta_1, \vartheta_2, \vartheta_2, \vartheta_1)}{m L \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} + \{ \vartheta_1 \leftrightarrow \vartheta_2, O \leftrightarrow O^2 \} \right)

\text{(B.3)}

\text{QF3}(\vartheta_1, \vartheta_2) = -i \oint \frac{d\vartheta'_{1}}{2\pi} \left\{ \frac{K_{t,x}^{(O)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1)}{[e^{Q_1(\vartheta', \vartheta_1)} + 1]} (\vartheta_2 - \vartheta_1) \right. \\
\times \left[ F_2^{O_2}(\vartheta_1 + i\pi, \vartheta_1') F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1') + \{ O_1 \leftrightarrow O_2 \} + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right] \\
- i \left\{ \frac{K_{t,x}^{(O)}(\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_2)}{m L \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} (\vartheta_2 - \vartheta_1) \right. \\
\times \left[ F_2^{O_2}(i\pi, 0) F_2^{O_2}(i\pi, 0) + \{ O_1 \leftrightarrow O_2 \} + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right] \left. \text{(B.4)} \right. \\
\text{QF4} (\vartheta_1, \vartheta_2) = -i \oint \frac{d\vartheta'_{1}}{2\pi} \left\{ \frac{K_{t,x}^{(O)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1)}{[e^{Q_1(\vartheta', \vartheta_1)} + 1]} (\vartheta_2 - \vartheta_1) \right. \\
\times \left[ F_2^{O_2}(i\pi, 0) F_2^{O_2}(i\pi, 0) + \{ O_1 \leftrightarrow O_2 \} + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right] \\
- \left\{ \frac{K_{t,x}^{(O)}(\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_2)}{m L \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} \right. \\
\times \left[ F_2^{O_2}(i\pi, 0) + \{ O_1 \leftrightarrow O_2 \} + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right] \left. \text{(B.5)} \right. \\
\text{QF5} (\vartheta_1, \vartheta_2) = -i \oint \frac{d\vartheta'_{1}}{2\pi} \left\{ \frac{K_{t,x}^{(O)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1)}{[e^{Q_1(\vartheta', \vartheta_1)} + 1]} (\vartheta_2 - \vartheta_1) \right. \\
\times \left[ F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1') F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1') + \{ O_1 \leftrightarrow O_2 \} + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right] \\
- i \left\{ \frac{K_{t,x}^{(O)}(\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_2)}{m L \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} \right. \\
\times \left[ F_2^{O_2}(i\pi, 0) F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1) + \{ O_1 \leftrightarrow O_2 \} + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right] \\
- \left\{ \frac{K_{t,x}^{(O)}(\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_2)}{m L \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} \right. \\
\times \left[ F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1) F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1) + \{ O_1 \leftrightarrow O_2 \} + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right] \left. \text{(B.6)} \right. \\
\text{QF6} (\vartheta_1, \vartheta_2) = -i \oint \frac{d\vartheta'_{1}}{2\pi} \left\{ \frac{F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1') F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1') K_{t,x}^{(O)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1)}{[e^{Q_1(\vartheta', \vartheta_1)} + 1]} \right. \\
\times \left[ T(\vartheta_1', \vartheta_1, \vartheta_2) \right] \left\{ -m x \cosh \vartheta_1 + i m t \sinh \vartheta_1 \right. \\
\times \left. -m L \cosh (\vartheta_1) + \varphi(\vartheta_1' - \vartheta_1) \right. \\
\times \left. S(\vartheta_1' - \vartheta_1) S(\vartheta_1 - \vartheta_2) \right\} + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right.$
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\[ + F^C_2(i\pi, 0) F^C_2(i\pi, 0) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_1) \]
\[ \times \left( \frac{mL \cosh \vartheta_1 - \varphi (\vartheta_1 - \vartheta_2)}{mL \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} + \frac{mL \cosh \vartheta_2 - \varphi (\vartheta_2 - \vartheta_1)}{mL \cosh \vartheta_1 + \varphi (\vartheta_1 - \vartheta_2)} \right) \]
\[ + \int \frac{d\vartheta_1'}{2\pi} \left( \frac{1}{[e^{Q_1(\vartheta_1', \vartheta_1)} + 1] + \frac{1}{[e^{Q_2(\vartheta_1', \vartheta_1)} + 1]} \right) \]  
\[ \times \frac{\partial}{\partial \vartheta_1'} \left[ K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1) T(\vartheta_1', \vartheta_1, \vartheta_2) \varphi (\vartheta_1' - \vartheta_1) \right] \]
\[ \times F^C_2(\vartheta_2 + i\pi, \vartheta_1') F^C_2(\vartheta_2 + i\pi, \vartheta_1') + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \]
\[ + K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta_2, \vartheta_1) F^C_2(i\pi, 0) F^C_2(i\pi, 0) \varphi (\vartheta_2 - \vartheta_1)^2 \]
\[ \times \left( \frac{1}{mL \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)^2} + \frac{1}{mL \cosh \vartheta_1 + \varphi (\vartheta_1 - \vartheta_2)^2} \right) \]  
\[ (B.7) \]

FF = \[ K^{(R)}_{t,x}(\vartheta_1, \vartheta_2, \vartheta_1, \vartheta_1) S(\vartheta_1 - \vartheta_2) F^C_2(\vartheta_2 + i\pi, \vartheta_1) F^C_2(\vartheta_2 + i\pi, \vartheta_1) + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \]  
\[ (B.8) \]

SQQ (\vartheta_1, \vartheta_2) = \[ \int_\mathcal{C}_{1,2} \int_\mathcal{C}_{1,2} \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} \]
\[ \times \left\{ \frac{F^C_4(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1', \vartheta_2') F^C_2(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_1', \vartheta_2')}{[e^{Q_1(\vartheta_1', \vartheta_2')} + 1][e^{Q_2(\vartheta_1', \vartheta_1')} + 1]} \right\} \]
\[ \times K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2') \]  
\[ (B.9) \]

We gave these contributions in their exact finite volume form (i.e. including the full volume dependence); although they simplify when taking the volume to infinity, and the SQQ term does not even contribute in this limit, all terms must be kept in order for the numerical verification of section 4 to work properly.

Appendix C. Symmetry of D_{22}

We want to prove that \( D_{22} \) is symmetric under

\[ t \rightarrow R - t \]
\[ \mathcal{O}^1 \leftrightarrow \mathcal{O}^2 \]
\[ x \rightarrow -x. \]  
\[ (C.1) \]

First of all, notice that the diagonal terms contain no \( t \) and \( x \) factors, and are manifestly symmetric under exchanging \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \), so it remains only to treat the non-diagonal part.

C.1. The four-integral term

First we treat the term in (3.10) that contains a fourfold integral. After the transformation we change the variables \( \vartheta_{1,2} \leftrightarrow \vartheta'_{1,2} \), and shift every contour with \(-2i\varepsilon\). This results in the...
contour now running under the real axis:

\[
\frac{1}{4} \int \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F^{(R)}_{t,x}(\vartheta_1, \vartheta_2) \left( \vartheta'_1 - i\varepsilon, \vartheta'_2 - i\varepsilon \right) \times F^{G_1}_4(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1 - i\varepsilon, \vartheta'_2 - i\varepsilon) \times F^{G_2}_4(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2 - i\varepsilon, \vartheta'_1 - i\varepsilon).
\]

By shifting the contour above the real axis we can transform this term back to its form in (3.10), but during this process we pick up some pole contributions from the poles in equation (3.5).

C.1.1. First order pole terms containing \( F_{\text{arc}} \). Using equation (3.5) we can identify a contribution containing \( F_{\text{arc}} \). In this contribution, all poles are of first order, so one can apply the Cauchy formula directly. One set of such terms is given by

\[
\frac{1}{4} \int \int_{\mathbb{C}_-} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F^{G_1}_{\text{arc}}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta'_1, \vartheta'_2) \left[ \frac{E}{\vartheta_1 - \vartheta'_1} + \frac{F}{\vartheta_1 - \vartheta_1} + \frac{G}{\vartheta_2 - \vartheta'_2} + \frac{H}{\vartheta_2 - \vartheta_1} \right] \times K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2)
\]

\[ = - \frac{1}{4} \int \int_{\mathbb{C}_+} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F^{G_1}_{\text{arc}}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta'_1, \vartheta'_2) \left[ \frac{E}{\vartheta_1 - \vartheta'_2} + \frac{F}{\vartheta_1 - \vartheta_1} + \frac{G}{\vartheta_2 - \vartheta'_2} + \frac{H}{\vartheta_2 - \vartheta_1} \right] \times K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2)
\]

\[ + \frac{1}{4} \int \frac{d\vartheta'_1}{2\pi} F^{G_1}_{\text{arc}}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta'_1, \vartheta_1) F^{G_2}_{\text{arc}}(\vartheta_2 + i\pi, \vartheta'_1) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta_1) \times \left[ S(\vartheta_1 - \vartheta_2) - S(\vartheta_1 - \vartheta'_1) \right]
\]

\[ + \frac{1}{4} \int \frac{d\vartheta'_2}{2\pi} F^{G_1}_{\text{arc}}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta'_1, \vartheta_2) F^{G_2}_{\text{arc}}(\vartheta_2 + i\pi, \vartheta'_2) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta'_2) \times \left[ S(\vartheta'_2 - \vartheta_1) S(\vartheta_1 - \vartheta_2) - 1 \right]
\]

\[ + \frac{1}{4} \int \frac{d\vartheta'_1}{2\pi} F^{G_1}_{\text{arc}}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta'_1, \vartheta_2) F^{G_2}_{\text{arc}}(\vartheta_1 + i\pi, \vartheta'_1) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta_1) \times \left[ 1 - S(\vartheta_1 - \vartheta_2) S(\vartheta_1 - \vartheta'_1) \right]
\]

\[ + \frac{1}{4} \int \frac{d\vartheta'_2}{2\pi} F^{G_1}_{\text{arc}}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta'_1, \vartheta_2) F^{G_2}_{\text{arc}}(\vartheta_1 + i\pi, \vartheta'_2) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta'_2, \vartheta_2) \times \left[ S(\vartheta'_2 - \vartheta_2) - S(\vartheta_1 - \vartheta_2) \right]
\]

and a similar contribution from

\[
\frac{1}{4} \int \int_{\mathbb{C}_-} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F^{G_2}_{\text{arc}}(\vartheta_1 + i\pi, \vartheta_2 + i\pi | \vartheta'_1, \vartheta'_2) \left[ \frac{A}{\vartheta_2 - \vartheta'_1} + \frac{B}{\vartheta_2 - \vartheta_1} + \frac{C}{\vartheta_1 - \vartheta'_2} + \frac{D}{\vartheta_1 - \vartheta_2} \right] \times K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2)
\]

\[ = - \frac{1}{4} \int \int_{\mathbb{C}_+} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F^{G_2}_{\text{arc}}(\vartheta_1 + i\pi, \vartheta_2 + i\pi | \vartheta'_1, \vartheta'_2) \left[ \frac{A}{\vartheta'_1 - \vartheta_2} + \frac{B}{\vartheta'_2 - \vartheta_2} + \frac{C}{\vartheta'_1 - \vartheta_1} + \frac{D}{\vartheta'_2 - \vartheta_1} \right]
\]

\[ \times \int \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F^{(R)}_{t,x}(\vartheta_1, \vartheta_2) \left( \vartheta'_1 - i\varepsilon, \vartheta'_2 - i\varepsilon \right) \times F^{G_1}_4(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1 - i\varepsilon, \vartheta'_2 - i\varepsilon) \times F^{G_2}_4(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2 - i\varepsilon, \vartheta'_1 - i\varepsilon).
\]
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\[ \times K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \]
\[ + \frac{1}{4} \int \frac{d\vartheta'_1}{2\pi} \int \frac{d\vartheta'_2}{2\pi} F^{O_2}_{4\pi c}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta'_1, \vartheta'_2) F^{O_1}_{2\pi}(\vartheta_1 + i\pi, \vartheta'_1) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \]
\[ \times [S(\vartheta_2 - \vartheta_1) - S(\vartheta_2 - \vartheta'_2)] \]
\[ + \frac{1}{4} \int \frac{d\vartheta'_1}{2\pi} \int \frac{d\vartheta'_2}{2\pi} F^{O_2}_{4\pi c}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta'_1, \vartheta'_2) F^{O_1}_{2\pi}(\vartheta_1 + i\pi, \vartheta'_1) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \]
\[ \times [S(\vartheta'_2 - \vartheta_2) S(\vartheta_1 - \pi)] \]
\[ + \frac{1}{4} \int \frac{d\vartheta'_1}{2\pi} \int \frac{d\vartheta'_2}{2\pi} F^{O_2}_{4\pi c}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta'_1, \vartheta'_2) F^{O_1}_{2\pi}(\vartheta_2 + i\pi, \vartheta'_2) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \]
\[ \times [1 - S(\vartheta_2 - \vartheta_1) S(\vartheta'_1 - \vartheta'_2)] \]
\[ + \frac{1}{4} \int \frac{d\vartheta'_1}{2\pi} \int \frac{d\vartheta'_2}{2\pi} F^{O_2}_{4\pi c}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta'_1, \vartheta'_2) F^{O_1}_{2\pi}(\vartheta_2 + i\pi, \vartheta'_1) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \]
\[ \times [S(\vartheta'_1 - \vartheta_1) - S(\vartheta_2 - \vartheta_1)] \]

where \( C_{\pm} \) denote integration running above/below from the real axis, in the direction from left to right.

Since the exchange property (2.10) is valid for \( F_{4\pi c} \) in the first and last two variables, it can be used to simplify the total contribution to

\[ \frac{1}{2} \int \frac{d\vartheta'_1}{2\pi} \{ K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \} + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} - \frac{1}{2} \int \frac{d\vartheta'_1}{2\pi} \{ K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \} \]
\[ \times F^{O_2}_{2\pi}(\vartheta_2 + i\pi, \vartheta'_1) + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \]

\[ + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \}

**C.1.2. First order poles without \( F_{4\pi c} \).** The form of these terms is

\[ \frac{1}{4} \iiint_{c_{\pm}} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{d\vartheta'_3}{2\pi} K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \]
\[ \times \left[ \begin{array}{c}
AE + DH \\
(\vartheta_2 - \vartheta'_1)(1 - \vartheta'_2)
\end{array} \right] + \left[ \begin{array}{c}
AF + CH \\
(\vartheta_1 - \vartheta'_1)(1 - \vartheta'_2)
\end{array} \right] + \left[ \begin{array}{c}
AG + BH \\
(\vartheta_2 - \vartheta'_1)(2 - \vartheta'_2)
\end{array} \right] \]
\[ + \left[ \begin{array}{c}
BE + DG \\
(\vartheta_2 - \vartheta'_2)(1 - \vartheta'_1)
\end{array} \right] + \left[ \begin{array}{c}
BF + CG \\
(\vartheta_1 - \vartheta'_1)(2 - \vartheta'_2)
\end{array} \right] + \left[ \begin{array}{c}
CE + DF \\
(\vartheta_1 - \vartheta'_1)(\vartheta_2 - \vartheta'_2)
\end{array} \right] \]

Their evaluation is tedious, but straightforward. For example,

\[ AE + DH = -(S(\vartheta_2 - \vartheta_1) - S(\vartheta'_2 - \vartheta'_1))(S(\vartheta_1 - \vartheta_2) - S(\vartheta'_1 - \vartheta'_2)) \]
\[ \times \left\{ F^{O_2}_{2\pi}(\vartheta_1 + i\pi, \vartheta'_2) + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right\} \]

and one can evaluate the contributions resulting from the contour shift as

\[ \frac{1}{4} \iiint_{c_{\pm}} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{d\vartheta'_3}{2\pi} K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \]
\[ \times \frac{AE + DH}{(\vartheta_1 - \vartheta_2)(\vartheta'_2 - \vartheta_1)} \]
\[ = \frac{1}{4} \iiint_{c_{\pm}} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{d\vartheta'_3}{2\pi} K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \]
\[ \times \frac{AE + DH}{(\vartheta'_1 - \vartheta_2)(\vartheta'_2 - \vartheta_1)} \]
C.1.3. Second order poles. We get four contributions which contain a second order pole. All four are the same after relabeling the rapidities:

\[
\begin{align*}
+ \frac{1}{4} & \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \left( \frac{AH}{(\vartheta_2 - \vartheta'_1)^2} + \frac{BG}{(\vartheta_2 - \vartheta'_2)^2} + \frac{CF}{(\vartheta_1 - \vartheta'_1)^2} + \frac{DE}{(\vartheta_1 - \vartheta'_2)^2} \right) \\
& \times K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \\
= & \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{DE}{(\vartheta_1 - \vartheta'_2)^2} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \\
= & \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{DE}{(\vartheta_1 - \vartheta'_2)^2} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \\
+ & i \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{DE}{(\vartheta_1 - \vartheta'_2)^2} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \Bigg|_{\vartheta'_2 = \vartheta_1}.
\end{align*}
\]

After performing the differentiation, the final result is

\[
\int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{DE}{(\vartheta_1 - \vartheta'_2)^2} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \\
+ \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F_2^O(\vartheta_2 + i\pi, \vartheta'_1) F_2^O(\vartheta_2 + i\pi, \vartheta'_1) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) \\
\times \{ [(nx \cosh \vartheta_1 - i n t \sinh \vartheta_1] T(\vartheta_2, \vartheta_1, \vartheta'_1) T('1, \vartheta_1, \vartheta_2) \\
+ [S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta'_1) - S(\vartheta'_1 - \vartheta_1) S(\vartheta_1 - \vartheta_2)] \varphi(\vartheta_1 - \vartheta'_1) \}
\]

where the notation

\[ T(\vartheta_1, \vartheta_2, \vartheta_3) = 1 - S(\vartheta_1 - \vartheta_2) S(\vartheta_2 - \vartheta_3) \]

was used.
C.1.4. Putting together the results. Putting together the result for the fourfold-integral term, one obtains

\[
\frac{1}{4} \int \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} F_4^{\text{O}1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1' - i\epsilon, \vartheta_2' - i\epsilon) \\
	imes F_4^{\text{O}2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_2' - i\epsilon, \vartheta_1' - i\epsilon) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1' - i\epsilon, \vartheta_2' - i\epsilon) \\
= \frac{1}{4} \int \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} F_4^{\text{O}1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1' + i\epsilon, \vartheta_2' + i\epsilon) \\
	imes F_4^{\text{O}2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta_2' + i\epsilon, \vartheta_1' + i\epsilon) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1' + i\epsilon, \vartheta_2' + i\epsilon) \\
+ \int \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} F_2^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1) \\
\times \{ F_{4\text{ic}}^{(i\pi, 0)}(\vartheta_1, \epsilon, \vartheta_1', \vartheta_1) F_2^{\text{O}2}(\vartheta_2 + i\pi, \vartheta_1') + \{ \text{O}1 \leftrightarrow \text{O}2 \} \} \\
- \int \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} F_2^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1) \\
\times \{ F_{4\text{ic}}^{(i\pi, 0)}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1') F_2^{\text{O}2}(\vartheta_2 + i\pi, \vartheta_1') + \{ \text{O}1 \leftrightarrow \text{O}2 \} \} \\
- i \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \int \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} F_2^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1) \\
\times \{ F_2^{\text{O}1}(\vartheta_1 + i\pi, \vartheta_1) F_2^{\text{O}2}(\vartheta_2 + i\pi, \vartheta_1) + \{ \text{O}1 \leftrightarrow \text{O}2 \} \} \left[ S(\vartheta_1' - \vartheta_1) - S(\vartheta_2 - \vartheta_1) \right] \left[ S(\vartheta_1 - \vartheta_2) - S(\vartheta_1 - \vartheta_1') \right] \\
- i \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \int \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} F_2^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1) \\
\times \{ F_2^{\text{O}1}(\vartheta_2 + i\pi, \vartheta_1) F_2^{\text{O}2}(\vartheta_2 + i\pi, \vartheta_1) + \{ \text{O}1 \leftrightarrow \text{O}2 \} \} \left[ 1 - S(\vartheta_2 - \vartheta_1)S(\vartheta_1' - \vartheta_1) \right] \left[ S(\vartheta_1 - \vartheta_2') - S(\vartheta_1 - \vartheta_2) \right] \\
- i \int \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2) \\
\times \{ F_2^{\text{O}1}(\vartheta_1 + i\pi, \vartheta_1') F_2^{\text{O}2}(\vartheta_2 + i\pi, \vartheta_1') + \{ \text{O}1 \leftrightarrow \text{O}2 \} \} \left[ S(\vartheta_2 - \vartheta_1) - S(\vartheta_1 - \vartheta_1') \right] \left[ 1 - S(\vartheta_1' - \vartheta_1)S(\vartheta_2 - \vartheta_2) \right] \\
+ \int \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} F_2^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1) \\
\times \{ F_{4\text{ic}}^{(i\pi, 0)}(\vartheta_1, \epsilon, \vartheta_1', \vartheta_1) T(\vartheta_2, \vartheta_1, \vartheta_2') T(\vartheta_1', \vartheta_1, \vartheta_2) \\
+ S(\vartheta_2 - \vartheta_1)S(\vartheta_1 - \vartheta_1') - S(\vartheta_1' - \vartheta_1)S(\vartheta_2 - \vartheta_2) \} \varphi(\vartheta_1 - \vartheta_1').
\]

C.2. Other terms

The following two terms in (3.10)

\[
- \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} F_2^{\text{O}1}(\vartheta_2 + i\pi, \vartheta_1) F_2^{\text{O}2}(\vartheta_2 + i\pi, \vartheta_1) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_1) \\
- \int \frac{d\vartheta_1}{2\pi} \int \frac{d\vartheta_1'}{2\pi} F_2^{\text{O}1}(\vartheta_1 + i\pi, \vartheta_1') F_2^{\text{O}2}(\vartheta_1 + i\pi, \vartheta_1') K_{t,x}^{(R)}(\vartheta_1, \vartheta_1 | \vartheta_1, \vartheta_1')
\]

transform to each other under the symmetry (C.1).

\[\text{doi:10.1088/1742-5468/2012/12/P12002}\]
For the remaining terms in (3.10), after the transformation we redefine \( \vartheta_2 \leftrightarrow \vartheta'_1 \) to have the same \( K \) factor as before and get
\[
\frac{1}{4} \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \iint \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F_{4c}^{(R)}(\vartheta_1, \vartheta_2, \vartheta'_1, \vartheta'_2) \times \left( F_{4c}^{(R)}(\vartheta_1 + i\pi, \vartheta'_1 + i\pi|\vartheta_2, \vartheta_1) F_{2}^{(R)}(\vartheta_2 + i\pi, \vartheta'_2) + \{O^1 \leftrightarrow O^2\} \right)
\]
\[
- \left( F_{4c}^{(R)}(\vartheta_1 + i\pi, \vartheta_2) + \vartheta'_1 + i\pi, \vartheta'_2 \right) F_{2}^{(R)}(\vartheta_2 + i\pi, \vartheta'_2) + \{O_1 \leftrightarrow O_2\}
\]
\[
- \left( F_{4c}^{(R)}(\vartheta_1 + i\pi, \vartheta_2) + \vartheta'_1 + i\pi, \vartheta'_2 \right) F_{2}^{(R)}(\vartheta_2 + i\pi, \vartheta'_2) + \{O_1 \leftrightarrow O_2\}
\]
\[
- \left( F_{4c}^{(R)}(\vartheta_1 + i\pi, \vartheta_2) + \vartheta'_1 + i\pi, \vartheta'_2 \right) F_{2}^{(R)}(\vartheta_2 + i\pi, \vartheta'_2) + \{O_1 \leftrightarrow O_2\}
\]
\[
+ \left( \vartheta_2 + i\pi, \vartheta'_2 \right) \left[ -m \cosh \vartheta_1 + \text{im}(R - t) \sinh \vartheta_1 \right]
\]
\[
+ \varphi(\vartheta_2 - \vartheta_1) S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta'_1)
\]

C.3. Collecting the terms

Using the results so far, the transformed \( D_{22}^{\text{trans}} \) is the following:
\[
D_{22}^{\text{trans}} = \frac{1}{4} \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \iint \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F_{4c}^{(R)}(\vartheta_1, \vartheta_2, \vartheta'_1, \vartheta'_2)
\]
\[
\times \left( F_{4c}^{(R)}(\vartheta_1 + i\pi, \vartheta'_1 + i\pi|\vartheta_2, \vartheta_1) F_{2}^{(R)}(\vartheta_2 + i\pi, \vartheta'_2) + \{O^1 \leftrightarrow O^2\} \right)
\]
\[
- \left( F_{4c}^{(R)}(\vartheta_1 + i\pi, \vartheta_2) + \vartheta'_1 + i\pi, \vartheta'_2 \right) F_{2}^{(R)}(\vartheta_2 + i\pi, \vartheta'_2) + \{O_1 \leftrightarrow O_2\}
\]
\[
+ \{O_1 \leftrightarrow O_2\}
\]
\[
+ \{O_1 \leftrightarrow O_2\}
\]
\[
+ \{O_1 \leftrightarrow O_2\}
\]

where the \( F_{4c} \) terms are
\[
+ \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F_{4c}^{(R)}(\vartheta_1, \vartheta_2, \vartheta'_1, \vartheta'_2) \times \left( F_{4c}^{(R)}(\vartheta_1 + i\pi, \vartheta_2) + \vartheta'_1 + i\pi, \vartheta'_2 \right) F_{2}^{(R)}(\vartheta_2 + i\pi, \vartheta'_2) + \{O_1 \leftrightarrow O_2\}
\]
\[
+ \{O_1 \leftrightarrow O_2\}
\]
\[
+ \{O_1 \leftrightarrow O_2\}
\]

\[\text{doi:10.1088/1742-5468/2012/12/P12002}\]
the first order pole terms are

\[-i \int \int \frac{d\vartheta_1 d\vartheta_2}{2\pi} \int \frac{d\vartheta_1' K_{t,x}^{(R)}(\vartheta_1, \vartheta_2, \vartheta_1', \vartheta_1)}{2\pi} \left\{ F_2^{O_1}(i\pi, 0) F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1') + \{O^1 \leftrightarrow O^2\} \right\} \]

\[\quad + \{O^1 \leftrightarrow O^2\} \left\{ [S(\vartheta_1' - \vartheta_1) - S(\vartheta_2 - \vartheta_1)] [S(\vartheta_1 - \vartheta_2) - S(\vartheta_1 - \vartheta_1')] \right\}
\]

the second order pole terms are

\[+ \int \int \frac{d\vartheta_1 d\vartheta_2 d\vartheta_1'}{2\pi} \frac{d\vartheta_1'}{2\pi} F_2^{O_1}(\vartheta_2 + i\pi, \vartheta_1') F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1') K_{t,x}^{(R)}(\vartheta_1, \vartheta_2, \vartheta_1', \vartheta_1) \]

\[\times \{[\text{max} \cosh \vartheta_1 - \text{min} \sinh \vartheta_1] \left\langle T(\vartheta_2, \vartheta_1, \vartheta_1') T(\vartheta_1', \vartheta_1, \vartheta_2) + [S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta_1') - S(\vartheta_1' - \vartheta_1) S(\vartheta_1 - \vartheta_2)] \varphi(\vartheta_1 - \vartheta_1') \right\} \]

\[-i \int \int \frac{d\vartheta_1 d\vartheta_2 d\vartheta_1'}{2\pi} \frac{d\vartheta_1'}{2\pi} F_2^{O_1}(\vartheta_2 + i\pi, \vartheta_1') F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1') K_{t,x}^{(R)}(\vartheta_1, \vartheta_2, \vartheta_1', \vartheta_1) \]

\[\times \left\{ S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta_1') - S(\vartheta_1' - \vartheta_1) S(\vartheta_1 - \vartheta_2) \right\} \varphi(\vartheta_2 - \vartheta_1) \]

\[\times \left\langle T(\vartheta_2, \vartheta_1, \vartheta_1') \left\{ \text{max} \cosh \vartheta_1 + \text{min} (R - t) \sinh \vartheta_1 \right\} \right\} \]

\[\times \varphi(\vartheta_2 - \vartheta_1) S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta_1') \} \]

**C.3.1. F_{4rc} terms.** The last two F_{4rc} terms (C.2) cancel due to

\[F_{4rc}^{O_{1,2}}(\vartheta_1 + i\pi, \vartheta_1' + i\pi | \vartheta_2, \vartheta_1) - F_{4rc}^{O_{1,2}}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta_1, \vartheta_1') = 0\]
which can be easily proven from the definition (3.4); the remaining one is combined with the first order pole terms below into the $F_{4ss}$ contributions in (3.10).

C.3.2. First order pole terms. The first order pole terms (C.3) can be rearranged into the form

$$-i \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \phi \int \frac{d\vartheta'_1}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) \left\{ F_2^C(i\varpi, 0) F_2^C(\vartheta_2 + i\varpi, \vartheta'_1) + \{ O^1 \leftrightarrow O^2 \} \right\} [S(\vartheta'_1 - \vartheta_1)S(\vartheta_1 - \vartheta_2) - 1]$$

$$- i \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \phi \int \frac{d\vartheta'_1}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) \left\{ F_2^C(\vartheta_2 + i\varpi, \vartheta_1) F_2^C(\vartheta_2 + i\varpi, \vartheta'_1) + \{ O^1 \leftrightarrow O^2 \} \right\} [S(\vartheta'_1 - \vartheta_1) - S(\vartheta_1 - \vartheta_2)]$$

$$- i \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \phi \int \frac{d\vartheta'_1}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) \left\{ F_2^C(\vartheta_1 + i\varpi, \vartheta'_1) F_2^C(\vartheta_2 + i\varpi, \vartheta'_1) + \{ O^1 \leftrightarrow O^2 \} \right\} [S(\vartheta_1 - \vartheta_2) - S(\vartheta'_1 - \vartheta_1)]$$

using

$$S(\vartheta)S(-\vartheta) = 1.$$  

These terms are combined with the $F_{4rc}$ terms into the $F_{4ss}$ contributions in (3.10).

C.3.3. Second order pole terms. The second order pole terms (C.4) have a dependence on $R - t$ in the last line. One can carry out an integration by parts to transform it into a $t$-dependence as in the second line. Using

$$\frac{\partial K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1)}{\partial \vartheta_1} = -mR \sinh (\vartheta_1) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1)$$

we can make the transformation in the integrand

$$-iT(\vartheta_2, \vartheta_1, \vartheta'_1) mR \sinh \vartheta_1 K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) = iT(\vartheta_2, \vartheta_1, \vartheta'_1) \frac{\partial K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1)}{\partial \vartheta_1}$$

(partial integration) $\Rightarrow -iK_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) T(\vartheta_2, \vartheta_1, \vartheta'_1) \frac{\partial}{\partial \vartheta_1}$$

and so the contribution is transformed into

$$+ \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} F_2^C(\vartheta_2 + i\varpi, \vartheta'_1) F_2^C(\vartheta_2 + i\varpi, \vartheta'_1) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1)$$

$$\times \{ \text{mix cosh} \vartheta_1 - \text{int sinh} \vartheta_1 \} T(\vartheta'_1, \vartheta_1, \vartheta_2)$$

$$- S(\vartheta'_1 - \vartheta_1)S(\vartheta_1 - \vartheta_2) \varphi(\vartheta_1 - \vartheta'_1) \}.$$
With this the second order pole terms (C.4) reduce to the form of the second order pole term in (3.10).

C.4. End result

After putting together every term, using the definition of $F_{4ss}$ (3.9), and performing some simplifications, one obtains

$$D_{22}^{\text{trans}} = D_{22}$$

which is exactly what was to be proven.

References

[1] Petrov D, Gangardt D and Shlyapnikov G, Low-dimensional trapped gases, 2004 J. Physique IV 116 5 arXiv:cond-mat/0409230
[2] Moritz H, Stöferle T, Köhl M and Esslinger T, Exciting collective oscillations in a trapped 1D gas, 2003 Phys. Rev. Lett. 91 250402 arXiv:cond-mat/0307607
[3] Laburthe Tolra B, O’Hara K M, Huckans J H, Phillips W D, Rolston S L and Porto J V, Observation of reduced three-body recombination in a fermionized 1D Bose gas, 2004 Phys. Rev. Lett. 92 190401 arXiv:cond-mat/0312003
[4] van Amerongen A H, van Es J J P, Wicke P, Kheruntsyan K V and van Druten N J, Yang-Yang thermodynamics on an atom chip, 2008 Phys. Rev. Lett. 100 090402 arXiv:0709.1899 [cond-mat.other]
[5] Coldea R, Tennant D A, Wheeler E M, Wawrzynska E, Prabhakaran D, Telling M, Habicht K, Smeibidl P and Kieler K, Quantum criticality in an Ising chain: experimental evidence for emergent E-8 symmetry, 2010 Science 327 177
[6] Tennant D A, Lake B, James A J A, Essler F H L, Notbohm S, Mikeska H-J, Fielden J, Koegerler P, Canfield P C and Telling M T F, Anomalous dynamical line shapes in a quantum magnet at finite temperature, 2012 Phys. Rev. B 85 014402
[7] Pozsgay B and Takács G, Form factor expansion for thermal correlators, 2010 J. Stat. Mech. P11012 arXiv:1008.3810 [hep-th]
[8] Zamolodchikov A B and Zamolodchikov Al B, Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models, 1979 Ann. Phys. 120 253
[9] Mussardo G, Off critical statistical models: factorized scattering theories and bootstrap program, 1992 Phys. Rep. 218 215
[10] Karowski M and Weisz P, Exact form-factors in (1+1)-dimensional field theoretic models with soliton behavior, 1978 Nucl. Phys. B 139 455
[11] Smirnov F A, Form-factors in completely integrable models of quantum field theory, 1992 Adv. Ser. Math. Phys. 14 1
[12] Zamolodchikov A B, Two point correlation function in scaling Lee-Yang model, 1991 Nucl. Phys. B 348 619
[13] Delfino G, Simonetti P and Cardy J L, Asymptotic factorisation of form factors in two-dimensional quantum field theory, 1996 Phys. Lett. B 387 327 arXiv:hep-th/9607046
[14] Delfino G and Simonetti P, Correlation functions in the two-dimensional Ising model in a magnetic field at $T = T_c$, 1996 Phys. Lett. B 383 450 arXiv:hep-th/9605065
[15] Leclair A, Lesage F, Sachdev S and Saleur H, Finite temperature correlations in the one-dimensional quantum Ising model, 1996 Nucl. Phys. B 482 579 arXiv:cond-mat/9606104
[16] Leclair A and Mussardo G, Finite temperature correlation functions in integrable QFT, 1999 Nucl. Phys. B 552 624 arXiv:hep-th/9902075
[17] Saleur H, A comment on finite temperature correlations in integrable QFT, 2000 Nucl. Phys. B 567 602 arXiv:hep-th/990919
[18] Lukyanov S L, Finite temperature expectation values of local fields in the sinh-Gordon model, 2001 Nucl. Phys. B 612 391 arXiv:hep-th/0005027
Spectral expansion for finite temperature two-point functions and clustering

[19] Delfino G, One-point functions in integrable quantum field theory at finite temperature, 2001 J. Phys. A: Math. Gen. 34 L161 arXiv:hep-th/0101180
[20] Mussardo G, On the finite temperature formalism in integrable quantum field theories, 2001 J. Phys. A: Math. Gen. 34 7399 arXiv:hep-th/0103214
[21] Konik R, Haldane gapped spin chains: exact low temperature expansions of correlation functions, 2003 Phys. Rev. B 68 104435 arXiv:cond-mat/0105284
[22] Essler F H L and Konik R M, Applications of massive integrable quantum field theories to problems in condensed matter physics, 2004 arXiv:cond-mat/0412241
[23] Altshuler B L, Konik R M and Tavlevik A M, Finite temperature correlation functions in integrable models: derivation of the large distance and time asymptotics from the form factor expansion, 2006 Nucl. Phys. B 739 311 arXiv:cond-mat/0508618
[24] Doyon B, Finite temperature form factors: a review, 2007 SIGMA 3 011 arXiv:hep-th/0611066
[25] Balog J, Field theoretical derivation of the TBA integral equation, 1994 Nucl. Phys. B 419 480
[26] Pozsgay B and Takacs G, Form factors in finite volume I: form factor bootstrap and truncated conformal space, 2008 Nucl. Phys. B 788 167 arXiv:0706.1445 [hep-th]
[27] Pozsgay B and Takacs G, Form factors in finite volume II: disconnected terms and finite temperature correlators, 2008 Nucl. Phys. B 788 209 arXiv:0706.3605 [hep-th]
[28] Korepin V E, Mean values of local operators in highly excited Bethe states, 2011 J. Stat. Mech. P01011 arXiv:1009.4662 [hep-th]
[29] Korepin V E, Correlation functions of the one-dimensional bose gas in the repulsive case, 1984 Commun. Math. Phys. 94 93
[30] Izergin A G, Korepin V E and Reshetikhin N Y, Correlation functions in a one-dimensional bose gas, 1987 J. Phys. A: Math. Gen. 20 4799
[31] Kormos M and Pozsgay B, One-point functions in massive integrable QFT with boundaries, 2010 J. High Energy Phys. JHEP04(2010)112 arXiv:1002.2783 [hep-th]
[32] Takacs G, Finite temperature expectation values of boundary operators, 2008 Nucl. Phys. B 805 205 arXiv:0804.4096 [hep-th]
[33] Kormos M and Pozsgay B, One-point functions in massive integrable QFT with boundaries, 2010 J. High Energy Phys. JHEP04(2010)112 arXiv:1002.2783 [hep-th]
[34] Takacs G, Form factor perturbation theory from finite volume, 2010 Nucl. Phys. B 825 466 arXiv:0907.2109 [hep-th]
[35] Delfino G, Mussardo G and Simonetti P, Non-integrable quantum field theories as perturbations of certain integrable models, 1996 Nucl. Phys. B 473 469 arXiv:hep-th/9603011
[36] Schuricht D and Essler F H L, Dynamics in the Ising field theory after a quantum quench, 2012 J. Stat. Mech. P04017 arXiv:1203.5080 [cond-mat.str-el]
[37] Calabrese P, Essler F H L and Fagotti M, Quantum quench in the transverse field Ising chain I: time evolution of order parameter correlators, 2012 arXiv:1204.3911 [cond-mat.quant-gas]
[38] Essler F H L and Konik R M, Finite-temperature lineshapes in gapped quantum spin chains, 2008 Phys. Rev. B 78 100403 arXiv:0711.2524 [cond-mat.str-el]
[39] Essler F H L and Konik R M, Finite-temperature dynamical correlations in massive integrable quantum field theories, 2009 J. Stat. Mech. P09018 arXiv:0907.0779 [cond-mat.str-el]
[40] Peschel I, Finite temperature form factors and correlation functions at finite temperature, 2009 arXiv:0907.4306 [hep-th]
[41] Kirillov A N and Smirnov F A, A representation of the current algebra connected with the SU(2) invariant Thirring model, 1987 Phys. Lett. B 198 506
[42] Fring A, Mussardo G and Simonetti P, Form-factors for integrable Lagrangian field theories, the sinh–Gordon theory, 1993 Nucl. Phys. B 393 413 arXiv:hep-th/9211053 [hep-th]
[43] Koubek A and Mussardo G, On the operator content of the sinh–Gordon model, 1993 Phys. Lett. B 311 193 arXiv:hep-th/9306044 [hep-th]
[44] Hahn T, CUBA: a library for multidimensional numerical integration, 2005 Comput. Phys. Commun. 168 78 arXiv:hep-ph/0404043 [hep-ph]
[45] Luscher M, Volume dependence of the energy spectrum in massive quantum field theories. 2. Scattering states, 1986 Commun. Math. Phys. 105 153
[46] Luscher M, Two particle states on a torus and their relation to the scattering matrix, 1991 Nucl. Phys. B 354 351
Spectral expansion for finite temperature two-point functions and clustering

[47] Zamolodchikov A B, *Thermodynamic Bethe ansatz in relativistic models. Scaling three state potts and Lee-Yang models*, 1990 Nucl. Phys. B **342** 695

[48] Feher G and Takacs G, *Sine-Gordon form factors in finite volume*, 2011 Nucl. Phys. B **852** 441 arXiv:1106.1901 [hep-th]

[49] Feher G, Palmai T and Takacs G, *Sine-Gordon multi-soliton form factors in finite volume*, 2012 Phys. Rev. D **85** 085005 arXiv:1112.6322 [hep-th]

[50] Palmai T, *Regularization of multi-soliton form factors in sine-Gordon model*, 2012 Comput. Phys. Commun. **183** 1813 arXiv:1111.7086 [math-ph]

[51] Pálmai T and Takács G, *Diagonal multi-soliton matrix elements in finite volume*, 2012 arXiv:1209.6034 [hep-th]