ON THE FUJITA EXPONENT FOR A NONLINEAR PARABOLIC EQUATION WITH A FORCING TERM

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Abstract. The purpose of this work is to analyze the blow-up of solutions of the nonlinear parabolic equation

\[ u_t - \Delta u = |x|^\alpha |u|^p + a(t)w(x) \quad \text{for } (t, x) \in (0, \infty) \times \mathbb{R}^N, \]

where \( p > 1, \alpha \in \mathbb{R} \) and \( a, w \) are suitable given functions. We improve earlier results [11, 17] by considering a wide class of functions \( a(t) \).

1. Introduction

It is well known that many diffusion processes with reaction can be described by the following equation

\[ u_t - \Delta u = f(t, x, u), \quad (1.1) \]

where \( u(t, x) \) stands for the mass concentration in chemical reaction processes or temperature in heat conduction and \( f(t, x, u) \) is the rate of change due to reaction. See [22, 27, 28]. Equation (1.1) arises in many physical phenomena and biological species theories, such as the concentration of diffusion of some fluid, the density of some biological species, and heat conduction phenomena, see [9, 6, 27, 19, 20] and references therein.

Our aim in this paper is to investigate the finite time blow-up of solutions to (1.1) with the special non-linearity

\[ f(t, x, u) = |x|^\alpha |u|^p + a(t)w(x), \quad (1.2) \]

where \( w : \mathbb{R}^N \to \mathbb{R} \) is a continuous and globally integrable function, \( a : (0, \infty) \to [0, \infty) \) is a continuous locally integrable function, and \( \alpha \in \mathbb{R}, p > 1 \).

Actually, we consider Cauchy problem associated to (1.1)-(1.2)

\[
\begin{align*}
\left\{
\begin{array}{ll}
u_t = \Delta u + |x|^\alpha |u|^p + a(t)w(x) & \quad \text{in } S := (0, \infty) \times \mathbb{R}^N, \\
u(0, x) = u_0(x) & \quad \text{in } \mathbb{R}^N.
\end{array}
\right.
\end{align*}
\]

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In the case $a \equiv 0$ or $w \equiv 0$, problem (1.3) reduces to
\[
\begin{align*}
\left\{ \begin{array}{ll}
  u_t = \Delta u + |x|^\alpha |u|^p & \text{in } S, \\
  u(0,x) = u_0(x) & \text{in } \mathbb{R}^N.
\end{array} \right.
\end{align*}
\]
(1.4)

For nonnegative initial data $u_0$, the solution of (1.4) blows up in finite time if $u_0$ is sufficiently large. For arbitrary initial data $u_0 \geq 0$, $u_0 \neq 0$, the blow-up occurs if and only if $\alpha > -2$ and $p \leq p_F$, where
\[
p_F = 1 + \frac{2 + \alpha}{N}.
\]

This result was proved by Fujita in [5] for $\alpha = 0$, $p \neq 1 + \frac{2}{N}$, and by Hayakawa in [8] for $\alpha = 0$, $p = 1 + \frac{2}{N}$. Later, Qi in [26] was able to prove similar results for a wide class of parabolic problems including in particular (1.4). The number $p_F$ is called the critical Fujita exponent.

Note that the case $\alpha = 0$, $a \equiv 1$ was investigated in [3]. It was shown, among other results, that (1.3) has no global solutions provided that $p < \frac{N}{N-2}$ and $\int_{\mathbb{R}^N} w(x) \, dx > 0$. In [11], the authors consider (1.3) with $\alpha = 0$ and $a(t) = t^\sigma$ where $\sigma > -1$. They showed that the critical exponent is given by
\[
p^*(\sigma) = \left\{ \begin{array}{ll}
  \frac{N-2\sigma}{N-2\sigma-2} & \text{if } -1 < \sigma < 0, \\
  \infty & \text{if } \sigma > 0.
\end{array} \right.
\]

Recently in [17], the author shows that blow-up depends on the behavior of $a$ at infinity by considering the case
\[
a(t) \sim \left\{ \begin{array}{ll}
  a_0 t^\sigma & \text{as } t \to 0, \\
  a_\infty t^m & \text{as } t \to \infty,
\end{array} \right.
\]
where $a_0, a_\infty > 0$, $\sigma > -1$ and $m \in \mathbb{R}$. It was shown that the critical exponent is given by
\[
p^*(m,\alpha) = \left\{ \begin{array}{ll}
  \frac{N-2m+\alpha}{N-2m-2} & \text{if } m \leq 0, \alpha > -2, \\
  \infty & \text{if } m > 0, \alpha > -2,
\end{array} \right.
\]
provided that $\int_{\mathbb{R}^N} w(x) \, dx > 0$. In addition, some local and global existence results was obtained in [17].

Our main goal here is to consider a wide class of functions $a$ and obtain new blow-up results that cover earlier ones. Let us mention that the fractional counterpart was considered in [18].

There exist several well-known methods of the study of the blow-up effect, which have their specific domain of applicability to corresponding problems of
mathematical physics. For a survey of blow-up results for solutions of first-order nonlinear evolution inequalities and related Cauchy problems we refer to [3, 4, 6, 12, 13, 16, 14, 15, 24, 25] and the references therein. See also the books [9, 22, 29, 27].

**Definition 1.1.** A function \( u = u(t,x) \) is said to be a solution of (1.3) on \( \mathbb{S} \) if

(i) \( u \) is bounded and continuous in \([0,T] \times \mathbb{R}^N\) for every \( T \in (0,\infty) \).

(ii) The initial condition is satisfied in the classical sense.

(iii) The partial differential equation is satisfied in the distribution sense in \( \mathbb{S} \).

To state our main results, we define the function

\[
A(t) = \frac{1}{t} \int_0^t a(s) \, ds,
\]

and assume that

\[
\lim_{t \to \infty} A(t) = \ell \in [0, \infty].
\]

We also define

\[
p_*(N, \alpha) = 1 + \frac{\alpha}{N}, \quad p^*(N, \alpha) = \frac{N + \alpha}{N - 2} \left( p^*(N, \alpha) = \infty \quad \text{if} \quad N = 1, 2 \right).
\]

**Theorem 1.1.** Suppose \( 0 < \ell < \infty, p_*(N, \alpha) < p < p^*(N, \alpha), \) and \( \int_{\mathbb{R}^N} w(x) \, dx > 0 \). Then (1.3) has no solutions on \( \mathbb{S} \) in the sense of Definition 1.1.

**Remark 1.1.** Clearly, the condition \( p > p_*(N, \alpha) \) is relevant only for \( \alpha > 0 \).

**Examples 1.1.** Some examples of functions \( a \) satisfying (1.5) with \( \ell \in (0, \infty) \) are listed below:

(i) **Constant at infinity:** \( a(t) = a_\infty + o(1) \) as \( t \to \infty \) where \( a_\infty > 0 \).

(ii) **Oscillating:** \( a(t) = \cos^2(t) \) or \( a(t) = \sin^2(t) \). Other combinations are allowed.

(iii) **Periodic:** More generally, if \( a \) is \( \vartheta \)-periodic then by (A.3) \( a \) satisfies (1.5) with

\[
\ell = \frac{1}{\vartheta} \int_0^\vartheta a(s) \, ds.
\]

**Theorem 1.2.** Suppose \( \ell = \infty, p > p_*(N, \alpha) \), and \( \int_{\mathbb{R}^N} w(x) \, dx > 0 \). Then (1.3) has no solutions on \( \mathbb{S} \) in the sense of Definition 1.1.

**Examples 1.2.** Some examples of functions \( a \) satisfying (1.5) with \( \ell = \infty \) are listed below:
(i) **Power functions**: \( a(t) = a_\infty t^m + o(1) \) as \( t \to \infty \) where \( a_\infty, m > 0 \).

(ii) **Power-Log functions**: \( a(t) = a_\infty t^m (\ln t)^q + o(1) \) as \( t \to \infty \) where \( a_\infty > 0 \) and either \( m > 0, q \in \mathbb{R} \) or \( m = 0, q > 0 \).

(iii) **Oscillating**: \( a(t) = t^m \psi(t) + o(1) \) as \( t \to \infty \) with \( m > 0 \) and \( \psi \) is a continuous \( \vartheta \)-periodic function such that \( \int_0^\vartheta \psi(s) ds > 0 \). It follows by (A.3) that \( a \) satisfies (1.5) with \( \ell = \infty \).

As it will be clear in the proofs, our method doesn’t cover the case \( \ell = 0 \). To handle this case, let

\[
J = \left\{ q \in \mathbb{R}; \lim_{T \to \infty} T^q \int_{\frac{T}{2}}^{\frac{2T}{3}} a(t) dt = \infty \right\}.
\]

Note that the choice of \((\frac{T}{2}, \frac{2T}{3})\) is technical and is related to the test function method used here. It can be any interval \((\lambda T, \mu T)\) with \( 0 < \lambda < \mu < 1 \).

**Remark 1.2.** We may have \( J = \emptyset \) if we take, for example, \( a(t) = e^{-t} \).

The following description of the set \( J \) is straightforward.

**Proposition 1.1.** Suppose that \( J \neq \emptyset \). Then \( J = (q_0, \infty) \) where

\[
q_0 = \inf J \in [-\infty, \infty).
\]

**Remark 1.3.**

(i) If \( a(t) = \frac{1}{t} \) then \( J = (0, \infty) \).

(ii) If \( a(t) = e^t \) then \( J = \mathbb{R} \).

The following result covers the case \( \ell = 0 \) and can be seen as a general statement for blow-up.

**Theorem 1.3.** Suppose \( \int_{\mathbb{R}^N} w(x) dx > 0 \) and \( J \neq \emptyset \).

(i) If \( J = \mathbb{R} \) then (1.3) has no solutions on \( S \) in the sense of Definition 1.1 provided that \( p > p_*(N, \alpha) \).

(ii) If \( J = (q_0, \infty) \) with \( q_0 \in \mathbb{R} \) then (1.3) has no solutions on \( S \) in the sense of Definition 1.1 provided that

\[
\frac{2p + \alpha}{2(p - 1)} - \frac{N}{2} - 1 > q_0. \tag{1.7}
\]

**Remark 1.4.** This theorem covers and, in some cases, improves earlier results given in [3, 11, 17].
• For $a(t) \equiv 1$, we have $q_0 = -1$. The condition (1.7) with $\alpha = 0$ translate to $p < \frac{N}{N-2} (p < \infty$ for $N = 1, 2)$. Hence we recover [3, Theorem 2.1, Part (a)].

• Let $a(t) = t^\sigma$ with $\sigma > -1$ and $\alpha = 0$. The condition $\sigma > -1$ ensure the local integrability of $a$ on $(0, \infty)$. Clearly $q_0 = -1 - \sigma$. Owing to (1.7), we see that blow-up occurs for $\frac{2p}{p-1} > N - 2\sigma$. This leads to

$$p^*(\sigma) = \begin{cases} \frac{N-2\sigma}{N-2\sigma-2} & \text{if } -1 < \sigma < \frac{N}{2} - 1, \\ \infty & \text{if } \sigma \geq \frac{N}{2} - 1. \end{cases}$$

This covers and improves the result in [11].

• Suppose now that $a \in L^1_{loc}(0, \infty)$ and $a(t) \sim a_\infty t^m$ as $t \to \infty$ for some constants $a_\infty > 0$ and $m \in \mathbb{R}$. Clearly $q_0 = -1 - m$. Thanks to (1.7), we find the exponent

$$p^*(m, \alpha) = \begin{cases} \frac{N-2m+\alpha}{N-2m-2} & \text{if } m < \frac{N}{2} - 1, \\ \infty & \text{if } m \geq \frac{N}{2} - 1. \end{cases}$$

This improves [17, Theorem 3, Theorem 4].

The paper is organized as follows. In Section 2, we give the proofs of Theorems 1.1-1.2. The third section is devoted to the proof of Theorem 1.3. In the sequel, $C$ will be used to denote a constant which may vary from line to line.

2. Proofs of Theorem 1.1 and Theorem 1.2

As one will see, our proof borrows some arguments from [3]. Let $\varphi \in C^2(\mathbb{R}^N)$ be a smooth function such that:

$$\varphi = 1 \text{ in } B_1(0), \quad \varphi = 0 \text{ in } B_2^C(0) \text{ and } 0 \leq \varphi \leq 1 \text{ everywhere}, \quad (2.1)$$

$$\frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial(B_2(0) - B_1(0)), \quad (2.2)$$

$$|\Delta \varphi| \leq C_{\theta \varphi} \text{ in } B_2(0) - B_1(0) \text{ for all } \theta \in (0, 1), \quad (2.3)$$

where, for $r > 0$, $B_r(0)$ stands for the euclidean ball in $\mathbb{R}^N$ centered at 0 and with radius $r$. Define the functions
where \( \varphi_R(x) = \varphi(\frac{x}{R}) \), \( R > 0 \) and \( t_0 > 0 \). The following propositions summarize the main properties of functions \( F_R, G_R, H_R, K_R \) that will be crucial in our proofs.

**Proposition 2.1.** For \( R > 0 \) we have

\[
F_R'(t) \geq G_R(t) - CR^{-2 + \frac{N}{p} (p - p_*(N, \alpha))} \frac{1}{p} G_R^p(t) + a(t) \int_{\mathbb{R}^N} w(x) \varphi_R(x) \, dx, \tag{2.4}
\]

where \( p_*(N, \alpha) \) is given by (1.6).

**Proof.** Multiply the first equation in (1.3) by \( \varphi_R \) and making integration by parts, we get

\[
F_R'(t) = \int_{\mathbb{R}^N} u \Delta \varphi_R \, dx + \int_{\mathbb{R}^N} |x|^\alpha |u|^p \varphi_R \, dx + a(t) \int_{\mathbb{R}^N} w(x) \varphi_R \, dx.
\]

Using Hölder’s inequality, we infer

\[
\left| \int_{\mathbb{R}^N} u \Delta \varphi_R \, dx \right| \leq \left( \int_{\mathbb{R}^N} |x|^\alpha |u|^p \varphi_R \, dx \right)^\frac{1}{p} \left( \int_{R \leq |x| \leq 2R} |x|^{-\frac{na}{p}} \varphi_R^{-\frac{q}{p}} |\Delta \varphi_R|^q \, dx \right)^\frac{1}{q},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

From (2.3), one easily verifies that \( \varphi_R^{-\frac{2}{p}} |\Delta \varphi_R|^q \leq C R^{-2q} \). Hence

\[
\int_{R \leq |x| \leq 2R} |x|^{-\frac{na}{p}} \varphi_R^{-\frac{q}{p}} |\Delta \varphi_R|^q \, dx \leq C R^{-2q} R^{N - \frac{2a}{p}},
\]

which in turn completes the proof of (2.4). \( \square \)

**Proposition 2.2.** For \( R > 0 \), we have

\[
F_R'(t) \geq a(t) \int_{\mathbb{R}^N} w(x) \varphi_R(x) \, dx - CR^{\frac{p(N - 2) - (N + \alpha)}{p - 1}}. \tag{2.5}
\]
Proof. The proof of (2.5) immediately follows from (A.1) by choosing \( Z = G_R(t) \), \( \lambda = C R^{-2+\frac{2}{p}(p-p_*(N,\alpha))} \) and \( \theta = \frac{1}{p} \).

Proposition 2.3. There exist \( R_0 > 0 \) and \( \delta > 0 \) such that, for all \( R \geq R_0 \), one has
\[
F'_R(t) \geq \delta a(t) - C R^{\frac{p(N-2)-(N+\alpha)}{p-1}}.
\] (2.6)

Proof. Taking into consideration \( w \in L^1 \), one obtains thanks to Lebesgue theorem
\[
\int_{\mathbb{R}^N} w(x) \varphi_R(x) \, dx \to \int_{\mathbb{R}^N} w(x) \, dx > 0 \text{ as } R \to \infty.
\] This obviously leads to (2.6). \( \Box \)

Proposition 2.4. Let \( R \geq R_0 \) where \( R_0 \) is as in Proposition 2.3. Suppose either
\[
\ell = \infty \quad \text{and} \quad p > p_*(N,\alpha),
\] (2.7)
or
\[
\ell \in (0, \infty) \quad \text{and} \quad p_*(N,\alpha) < p < p^*(N,\alpha).
\] (2.8)
Then
\[
\lim_{t \to \infty} F_R(t) = \infty.
\] (2.9)

Proof. Integrating (2.6) with respect to \( t \) yields
\[
F_R(t) \geq F_R(0) + t \left( \delta a(t) - C R^{\frac{p(N-2)-(N+\alpha)}{p-1}} \right).
\] (2.10)
Observe that
\[
\frac{p(N-2)-(N+\alpha)}{p-1} = \frac{N-2}{p-1} (p-p^*(N,\alpha)), \quad N \geq 3,
\] so that (2.9) follows from (2.10) under the assumption (2.8). The case (2.7) is easier. This finishes the proof of Proposition 2.4. \( \Box \)

Proposition 2.5. Let \( p > 1 + \frac{\alpha}{N} \) and \( R \geq R_0 \) where \( R_0 \) is as in Proposition 2.3. Then
\[
F_R(t) \leq H_R(t) \leq C R^{\frac{N(p-1)-\alpha}{p}} \left( G_R(t) \right)^{\frac{1}{p}}.
\] (2.11)
In particular
\[
\lim_{t \to \infty} G_R(t) = \infty.
\] (2.12)
Proof. Clearly $F_R(t) \leq H_R(t)$. Next, by invoking Hölder’s inequality together with (2.1), we get

$$
H_R(t) = \int_{\mathbb{R}^N} \left( |u(t,x)||x|^{\frac{\alpha}{p}} \varphi_R(x) \right) \left( |x|^{-\frac{\alpha}{p}} \varphi_R(x) \right) dx
\leq \left( \int_{\mathbb{R}^N} |x|^{-\frac{\alpha}{p-1}} \varphi_R(x) dx \right)^{1-\frac{1}{p}} \left( \int_{\mathbb{R}^N} |u(t,x)|^p |x|^\alpha \varphi_R(x) dx \right)^\frac{1}{p}
\leq \left( \int_{|x| \leq 2R} |x|^{-\frac{\alpha}{p-1}} dx \right)^{1-\frac{1}{p}} \left( \int_{\mathbb{R}^N} |u(t,x)|^p |x|^\alpha \varphi_R(x) dx \right)^\frac{1}{p}
\leq CR^{\frac{N(p-1)-\alpha}{p}} \left( G_1 R(t) \right)^{\frac{1}{p}}.
$$

The proof of Proposition 2.5 is now complete. \qed

**Proposition 2.6.** There exists $t_0 > 0$ such that, for all $t \geq t_0$ and $R \geq R_0$,

$$
F'_R(t) \geq \delta a(t) + \frac{1}{2} G_R(t), \tag{2.13}
$$

where $R_0$ is as in Proposition 2.3.

Proof. Owing to (2.4) and granted (2.12), one can write

$$
F'_R(t) \geq \delta a(t) + G_R(t) - CR^{2+\frac{N}{p}(p-p_*(N,\alpha))} G_R^{\frac{1}{p}}(t)
\geq \delta a(t) + G_R(t) \left[ 1 - CR^{2+\frac{N}{p}(p-p_*(N,\alpha))} G_R^{\frac{1}{p}-1}(t) \right]
\geq \delta a(t) + \frac{1}{2} G_R(t),
$$

for $t \geq t_0 > 0$ large enough. This finishes the proof. \qed

**Proposition 2.7.** For $t \geq t_0$ and $R \geq R_0$, we have

$$
K'_R(t) \geq CR^{\alpha-N(p-1)} \left( K_R(t) \right)^p. \tag{2.14}
$$

Proof. Using (2.13) and (2.11), we find that

$$
F'_R(t) \geq \delta a(t) + C R^{\alpha-N(p-1)} H_R^p(t). \tag{2.15}
$$

Upon integrating (2.15) on $[t_0, t]$, we obtain

$$
F_R(t) \geq F_R(t_0) + \delta \int_{t_0}^t a(s) ds + C R^{\alpha-N(p-1)} K_R(t) \geq C R^{\alpha-N(p-1)} K_R(t). \tag{2.16}
$$

Recalling (2.11) yields (2.14) as desired. \qed
End of Proofs of Theorem 1.2 and Theorem 1.3. Suppose that assumptions of Theorem 1.2 (respectively Theorem 1.3) are fulfilled. Since the differential inequality (2.14) blows up in finite time, we deduce that the solution of (1.3) cannot be global in time. This completes the proofs. □

3. Proof of Theorem 1.3

We will focus in this section on blow-up results stated in Theorems 1.3. Suppose that the maximal solution \( u \) of (1.3) is global in time. In order to obtain a contradiction we use the so-called test function method [3, 11, 17, 21]. Pick two cut-off functions \( f, g \in C^\infty([0, \infty)) \) such that \( 0 \leq f, g \leq 1 \),

\[
f(\tau) = \begin{cases} 
1 & \text{if } 1/2 \leq \tau \leq 2/3, \\
0 & \text{if } \tau \in [0, 1/4] \cup [3/4, \infty), 
\end{cases}
\]

and

\[
g(\tau) = \begin{cases} 
1 & \text{if } 0 \leq \tau \leq 1, \\
0 & \text{if } \tau \geq 2.
\end{cases}
\]

For \( T, R > 0 \), we introduce \( \psi_{T,R}(t, x) = f_T(t) g_R(x) \), where

\[
f_T(t) = \left( f \left( \frac{t}{T} \right) \right)^{\frac{p}{p-1}},
\]

\[
g_R(x) = \left( g \left( \frac{|x|^2}{R^2} \right) \right)^{\frac{2p}{p-1}}.
\]

Multiplying both sides of the differential equation in (1.3) by \( \psi_{T,R} \) and integrating over \( (0, T) \times \mathbb{R}^N \) yields

\[
\int_0^T \int_{\mathbb{R}^N} |x|^{\alpha}|u|^p \psi_{T,R} \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} a(t) w(x) \psi_{T,R} \, dx \, dt + \int_{\mathbb{R}^N} u_0(x) \psi_{T,R}(0, x) \, dx \\
= - \int_0^T \int_{\mathbb{R}^N} u \Delta \psi_{T,R} \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} u \partial_t \psi_{T,R} \, dx \, dt.
\]

Granted to \( \psi_{T,R}(0, x) = 0 \), we obtain that

\[
(I) + (II) \leq (III) + (IV),
\]

where
\[(I) = \int_0^T \int_{\mathbb{R}^N} |x|^\alpha |u|^p f_T(t)g_R(x) \, dx \, dt, \quad (3.4)\]

\[(II) = \int_0^T \int_{\mathbb{R}^N} a(t)w(x) f_T(t)g_R(x) \, dx \, dt, \quad (3.5)\]

\[(III) = \int_0^T \int_{\mathbb{R}^N} |u| f_T(t)|\Delta g_R(x)| \, dx \, dt, \quad (3.6)\]

\[(IV) = \int_0^T \int_{\mathbb{R}^N} |u| g_R(x)|\partial_t f_T(t)| \, dx \, dt. \quad (3.7)\]

Next, applying Young’s inequality as in \[17\], we infer

\[(III) \leq \frac{1}{2} (I) + A(T,R), \quad (3.8)\]

\[(IV) \leq \frac{1}{2} (I) + B(T,R), \quad (3.9)\]

where

\[A(T,R) = C \int_0^T \int_{\mathbb{R}^N} |x|^{-\frac{\alpha}{p-1}} f_T(t)g_R^{-\frac{1}{p-1}}(x)|\Delta g_R(x)|^{\frac{p}{p-1}} \, dx \, dt, \quad (3.10)\]

\[B(T,R) = C \int_0^T \int_{\mathbb{R}^N} |x|^{-\frac{\alpha}{p-1}} f_T^{-\frac{1}{p-1}}(t)g_R(x)|\partial_t f_T(t)|^{\frac{p}{p-1}} \, dx \, dt, \quad (3.11)\]

where we have used (3.1), (3.2) and

\[|\Delta g_R(x)| \lesssim R^{-2} g_R^{1/p}(x). \quad (3.12)\]

Arguing as in \[17\], we find that

\[A(T,R) \lesssim TR^{N-\frac{2\alpha}{p-1}}, \quad (3.13)\]

\[B(T,R) \lesssim T^{1-\frac{p}{p-1}} R^{N-\frac{\alpha}{p-1}}, \quad (3.14)\]

provided that \(p > 1 + \frac{\alpha}{N}\). Plugging all estimates above together, we find that

\[\int_0^T \int_{\mathbb{R}^N} a(t)w(x) \psi_{T,R}(t,x) \, dx \, dt \lesssim TR^{N-\frac{2p}{p-1} - \frac{\alpha}{p-1}} + T^{1-\frac{p}{p-1}} R^{N-\frac{\alpha}{p-1}}. \quad (3.15)\]
Now, since \( w \in L^1 \) and \( g_R(x) \to g(0) = 1 \) as \( R \to \infty \), we obtain by Lebesgue theorem that
\[
\int_{\mathbb{R}^N} w(x) g_R(x) \, dx \xrightarrow{R \to \infty} \int_{\mathbb{R}^N} w(x) \, dx > 0.
\]
It follows that there exists \( R_0 > 0 \) such that for all \( R \geq R_0 \) and \( T > 0 \), we have
\[
\left( \int_0^T a(t) \left( f \left( \frac{t}{T} \right) \right)^{\frac{p}{p-1}} \, dt \right) \left( \int_{\mathbb{R}^N} w(x) \, dx \right) \lesssim TR^{N-\frac{2p+\alpha}{p-1}} + T^{1-\frac{p}{p-1}} R^{N-\frac{\alpha}{p-1}}.
\]
Owing to (3.1), the above estimate translate to
\[
\int_{\mathbb{R}^N} w(x) \, dx \lesssim \frac{R^{N-\frac{2p+\alpha}{p-1}} + T^{-\frac{p}{p-1}} R^{N-\frac{\alpha}{p-1}}}{\frac{1}{T} \int_{\frac{T}{2}}^{\frac{3T}{2}} a(t) \, dt}.
\]
\( \text{End of the proof of Theorem 1.3.} \)

The following results are classical and the proofs are omitted here.

**Remark 3.1.** Note that we use in our proof (3.16) with \( R = \sqrt{T} \), that is (3.17), instead of (3.16). Nevertheless, (3.16) is useful to give an alternative proof of Theorems 1.1-1.2.

**Appendix A.**

**Lemma A.1.** Let \( \lambda > 0 \) and \( \theta \in (0, 1) \). Then
\[
\min_{Z \geq 0} (Z - \lambda Z^\theta) = (\theta - 1) \theta^{\frac{\theta}{1-\theta}} \lambda^{\frac{\theta}{1-\theta}}.
\]
Lemma A.2. Let $y : [t_0, T) \to (0, \infty)$ be a differentiable function such that

$$y'(t) \geq C y^p(t), \quad t_0 \leq t < T,$$

where $C > 0$ and $p > 1$. Then

$$T \leq t_0 + \frac{y^{1-p}(t_0)}{C(p-1)} < \infty.$$

Lemma A.3. Let $g : [a, b] \to \mathbb{C}$ be a $C^1$–function and $\psi : \mathbb{R} \to \mathbb{C}$ be a continuous periodic function with period $\vartheta > 0$. Then

$$\lim_{\lambda \to \infty} \int_a^b g(s) \psi(\lambda s) \, ds = \left( \frac{1}{\vartheta} \int_0^\vartheta \psi(s) \, ds \right) \left( \int_a^b g(s) \, ds \right).$$

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