Perturbation from symmetry and multiplicity of solutions for elliptic problems with subcritical exponential growth in $\mathbb{R}^2$

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We consider the following boundary value problem

\[
\begin{align*}
-\Delta u &= g(x,u) + f(x,u) \quad x \in \Omega \\
u &= 0 \quad x \in \partial \Omega
\end{align*}
\]

where $g(x,-\xi) = -g(x,\xi)$ and $g$ has subcritical exponential growth in $\mathbb{R}^2$. Using the method developed by Bolle, we prove that this problem has infinitely many solutions under suitable conditions on the growth of $g(u)$ and $f(u)$.

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1 Introduction

In the last few years, many authors have widely investigated existence and multiplicity of solutions for semilinear elliptic problems with Dirichlet boundary conditions by using variational methods and topological arguments (see [St1] and references therein). In particular, the following model

\[
\begin{align*}
-\Delta u &= |u|^{p-2}u + f(x) \quad x \in \Omega \\
u &= 0 \quad x \in \partial \Omega
\end{align*}
\]

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has been extensively studied, where $\Omega$ is an open bounded domain of $\mathbb{R}^N$, $N \geq 3$, $f \in L^2(\Omega)$ and $2 < p < 2N/N - 2$. If $f \equiv 0$, equation (1) possesses natural symmetry, which guarantees the existence of an unbounded sequence of critical values for the symmetric functional associated to the problem. On the contrary, if $f \neq 0$ the problem loses its $\mathbb{Z}_2$ symmetry and a natural question is whether the infinite number of solutions is preserved or not under perturbation of the odd equation; a partial answer was independently obtained by Rabinowitz [Ra], Bahri-Berestycki [BB] and Struwe [St2], who showed in important works that the multiplicity structure can be maintained also in the perturbed case, restricting the growth range of the nonlinearity with suitable bounds depending on $N$. The main idea is to think of the non-symmetric functional $I$ under study as a perturbation of its symmetric part $I_0$ and then to estimate how the growth of rate of the critical levels of $I_0$ is affected by perturbation from symmetry $I - I_0$.

More recently, a new type of perturbation from symmetry has been considered, resulting from second order systems with non-homogeneous boundary conditions: if $f = 0$ in (1) but $u|_{\partial \Omega} = u_0 \neq 0$, the symmetry is again broken and the perturbation - due to the non-homogeneous boundary condition - is of higher order. The standard perturbative method can be applied but yields the result for even smaller range of $p$ values. It was to deal with this type of perturbation that Bolle [Bo] developed his new approach: this new method deals with $I$ as the end-point of a continuous path of functionals $I_\theta$, $\theta \in [0, 1]$, which starts at the symmetric functional $I_0$. Bolle’s abstract theorem states, roughly speaking, that the preservation of the min-max critical levels along the path of functionals $I_\theta$ depends only on the velocities of deformation $\frac{d}{d\theta}I_\theta(u)$ at the critical points $u$ of $I_0$. This fact often allows to obtain better estimates at such points since they obey certain conservation laws, being solutions of the corresponding Euler-Lagrange equations. Bolle, Ghoussoub and Tehrani [BGT] tested this approach on several other problems, including the non-homogeneous problem

$$\begin{cases} -\Delta u = |u|^{p-2}u & x \in \Omega \\ u = u_0 & x \in \partial \Omega, \end{cases}$$

proving the existence of infinitely many solutions for a larger range, namely for $1 < p < (N + 1)/(N - 1)$. Later, Chambers and Ghoussoub [CG] have applied Bolle’s approach to establish a general multiplicity result
for problems with broken symmetry, where the forcing term $f$ depends also on $u$; they have been able to prove that the infinite sequence of critical values is preserved if $p$ belongs to a range of values depending also on the growth of $f(u)$: roughly speaking, the lower is the growth of the perturbation $f(u)$, the better is the result obtained.

In this paper we deal with an analogous of problem (1) in dimension $N = 2$. Let $\Omega$ be an open bounded subset of $\mathbb{R}^2$ with smooth boundary $\partial \Omega$; we are concerned with existence and multiplicity results for nonlinear elliptic equations of the type

$$
\begin{cases}
-\Delta u = g(x, u) + f(x, u) & x \in \Omega \\
u = 0 & x \in \partial \Omega
\end{cases}
$$

where $g(x, -\xi) = -g(x, \xi)$ and $g$ has subcritical growth in $\mathbb{R}^2$. When $N = 2$, the notion of criticality, that is, the maximal growth on $u$ which allows to treat problem (2) variationally, is motivated by the so called Trudinger-Moser inequality [Tr], [M], which says that for $\alpha \leq 4\pi$

$$
\sup_{\|u\|_{H^1_0} \leq 1} \int_B e^{\alpha u^2} \leq c(\alpha) |B| \leq c(4\pi) |B| = C_{TM} |B|
$$

where $|B|$ denotes the Lebesgue measure of $B$ and $C_{TM}$ is a constant which does not depend on $u$; hence, the maximal growth permitted to study problem (2) variationally is of exponential type. Motivated by the Trudinger-Moser inequality, we say that $g$ has subcritical growth at $+\infty$ if for all $\alpha > 0$

$$
\lim_{\xi \to +\infty} \frac{|g(\xi)|}{e^{\alpha t^2}} = 0,
$$

and $g$ has critical growth at $+\infty$ if there exists $\alpha_0 > 0$ such that

$$
\lim_{\xi \to +\infty} \frac{|g(\xi)|}{e^{\alpha t^2}} = 0 \quad \forall \alpha > \alpha_0; \quad \lim_{\xi \to +\infty} \frac{|g(\xi)|}{e^{\alpha t^2}} = +\infty \quad \forall \alpha < \alpha_0.
$$

We will consider only the subcritical case. To our knowledge, the problem of perturbation from symmetry for equation with exponential growth in bounded domain of $\mathbb{R}^2$ has been approached only by Sugimura [Sug], who proved that the infinite number of solutions is preserved if the nonlinear term has an exponential growth of the kind $e^{\xi q}$, $0 < q < 1/2$, and the forcing term
\( f = f(x) \) does not depend on \( u \). In this paper we approach the problem \( (2) \) using Bolle’s method: following the idea in [CG], we are able to extend the result of Sugimura to perturbed problem with forcing term depending also on \( u \). As just remarked, maximal growth allowed depends now on the growth of \( f(u) \): in particular we prove the existence of infinite solutions for \( (2) \) if, roughly speaking, \( g(u) \sim e^{\xi q}, 0 < q \leq 1 \) and \( f(u) \) satisfies suitable growth’s conditions. Our result includes the one obtained by Sugimura as a special case: we remark that in this case we are able to include also the case \( q = 1/2 \), which was not considered in [Sug]. Whether the result would still hold for all \( q \) up to the Trudinger-Moser critical exponent \( q = 2 \) is still open.

2 Preservation of critical levels under deformation of an even functional

In this section we recall Bolle’s method for dealing with problems with deformation from symmetry (see e.g. [BGT], [CG]). Consider two continuous functions \( \rho_1, \rho_2 : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) which are Lipschitz continuous relative to the second variable. Assume \( \rho_1 \leq \rho_2 \) and denote by \( \psi_1, \psi_2 \) the scalar fields associated to them, defined on \([0, 1] \times \mathbb{R}\) by

\[
\begin{aligned}
\psi_1(0, s) &= s \\
\frac{\partial}{\partial \theta} \psi_i(\theta, s) &= \rho_i(\theta, \psi_i(\theta, s)).
\end{aligned}
\]

Note that \( \psi_1 \) and \( \psi_2 \) are continuous and that for all \( \theta \in [0, 1] \), \( \psi_1(\theta, \cdot) \) and \( \psi_2(\theta, \cdot) \) are non decreasing on \( \mathbb{R} \); moreover, since \( \rho_1 \leq \rho_2 \) we have \( \psi_1 \leq \psi_2 \).

Let \( E \) be a Hilbert space (with scalar product \( \langle \cdot, \cdot \rangle \) and associated norm \( \| \cdot \| \)) and consider a \( \mathcal{C}^2 \) functional \( I_0 \) on \( E \); let \( I \) be a \( \mathcal{C}^2 \) functional : \([0, 1] \times E \rightarrow \mathbb{R}\). For subsets \( U \subset V \) of \( E \), denote by

\[
c_U(\theta) = \sup_U I_\theta, \quad c_{V,U}(\theta) = \inf_{g \in S_{V,U}} \sup_{g(U)} I_\theta
\]

where

\[
S_{V,U} = \{ g \in \mathcal{C}(V, E) : g(v) = v \text{ for } v \in U \text{ and } g(v) = v \text{ for } \| v \| > R, \text{ for some } R > 0 \}.
\]

We make the following assumptions ( we shall use the abbreviation \( I_\theta \) for \( I(\theta, \cdot) \)): 

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(H1) $I$ satisfies a kind of Palais-Smale condition: for every sequence $(\theta_n, v_n)$ in $[0, 1] \times E$ such that $I(\theta_n, v_n)$ is bounded and $\lim_{n \to \infty} \|I'_{\theta,n}(v_n)\| = 0$ there is a subsequence converging in $[0, 1] \times E$. (The limit $(\theta, v)$ then satisfies $I'_{\theta}(v) = 0$).

(H2) For all $b > 0$ there is a constant $C_1(b)$ such that:

$$|I_{\theta}(v)| < b \implies \left| \frac{\partial I}{\partial \theta}(\theta, v) \right| \leq C_1(b)(\|I'_{\theta}(v)\| + 1)(\|v\| + 1).$$

(H3) For all critical points $v$ of $I_{\theta}$,

$$\rho_1(\theta, I_{\theta}(v)) \leq \frac{\partial I}{\partial \theta}(\theta, v) \leq \rho_2(\theta, I_{\theta}(v))$$

(H4) There are two closed subsets of $E$, $B$ and $A \subset B$ such that

(i) $I_0$ has an upper-bound on $B$ and for some $\theta_0 \in [0, 1]$  
$$\lim_{|v| \to +\infty, v \in B, \theta \in [0, \theta_0]} \sup I_{\theta}(v) = -\infty.$$  

(ii) $c = c_{B,A}(0) > b = c_A(0)$

In the sequel we will say that $I_0$ has a min-max configuration $(c, b)$ if it satisfies hypothesis (H4).

(H4') $I_0$ is even and for any finite dimensional subspace $W$ of $E$ and any $\theta$ we have

$$\lim_{\|w\| \to \infty, w \in W} \sup_{\beta \in [0, \theta]} I(\beta, w) = -\infty.$$  

Observe that we are assuming implicitly in the above definition that the starting functional $I_0$ satisfies the Palais-Smale condition $(H_1)$ and $(H_4)$ for $\theta = 0$. Set

$$\overline{\rho}_1(\theta, t) = \sup_{\beta \in [0, \theta]} \rho_1(\beta, t), \quad \overline{\rho}_2(\theta, t) = \sup_{\beta \in [0, \theta]} \rho_2(\beta, t).$$

The main idea of Bolle’s result is the following: If one assumes a min-max critical level for the initial functional $I_0$, then the deformation velocities $\rho_1$
and $\rho_2$ will determine whether this critical level persists along the path. We are now ready to present a reformulation of Bolle’s result due to Chambers and Ghoussoub (see [CG] for further references and [Bo] for the original result)

**Theorem 2.1.** Let $I_0$ be a $C^2$-functional on $E$ with a min-max configuration $c_{B,A}(0) > c_A(0)$, as defined in (H4); let $\rho_1 \leq \rho_2$ be two velocity fields and $\psi_1$, $\psi_2$ be the corresponding scalar flows. If $\psi_1(\theta_0, c_{B,A}(0)) > \psi_2(\theta_0, c_A(0))$ for some $\theta_0 \in [0, 1]$ then for any path of functionals $I : [0, 1] \times E \to \mathbb{R}$ satisfying (H1), (H2), (H3) and (H4) the functional $I_{\theta_0}$ has a critical point at a level $\overline{c}$ such that:

$$\psi_1(\theta_0, c_{B,A}(0)) \leq \overline{c} \leq \psi_2(\theta_0, c_{B,A}(0)).$$

Note that if $c = c_{B,A}(0) > b = c_A(0)$, as assumed in (ii) of (H4), it is standard to show that the functional $I_0$ ha a critical point at level $c$: Boll’s theorem assures that this min-max critical level is preserved along any path of functionals satisfying the above hypotheses if $\psi_1(\theta_0, c_{B,A}(0)) > \psi_2(\theta_0, c_A(0))$.

Assume now that the Hilbert space is decomposed as $E = \bigcup_{k=0}^{\infty} E_k$ with $E_0 = E_-$ being a finite dimensional subspace and $(E_k)_{k=1}^{\infty}$ is an increasing sequence of subspaces of $E$ such that $E_k = E_{k-1} \oplus \mathbb{R}e_k$; let us set

$$\mathcal{G} = \{g \in C(E, E) : g \text{ is odd and for a fixed } R > 0 \ g(v) = v \text{ for } \|v\| \geq R\}$$

and

$$c_k = \inf_{g \in \mathcal{G}} \sup_{v \in g(E_k)} I_0(v).$$

In this framework, the following abstract result can be proved (see [CG]).

**Lemma 2.2.** Let $\rho_1 \leq \rho_2$ be two velocity fields and let $\psi_1, \psi_2$ be the corresponding scalar flows. Let $I_0$ be an even $C^2$ functional on $E$ and consider the levels $c_k$ associated to $I_0$ defined by (3). Then there is $C > 0$, depending only on $\rho_1, \rho_2$ such that for every $k \in \mathbb{N}$ and every $\theta \in [0, 1]$:

(i) either $\psi_2(\theta, c_k) < \psi_1(\theta, c_k)$, or

(ii) $c_{k+1} - c_k \leq C(\overline{\rho}_1(\theta, c_{k+1}) + \overline{\rho}_2(\theta, c_k) + 1)$. 

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Moreover, in case (i) there exists a level $\ell_k(\theta)$, only depending on $I_0$ and $\rho_1, \rho_2$, such that for any good path of functionals $I$ satisfying hypotheses (H1), (H2), (H3) and (H4') there exists a critical level $\overline{c}_k(\theta)$ for $I_0$ with $\psi_2(\theta, c_k) < \psi_1(\theta, c_{k+1}) \leq \overline{c}_k(\theta) \leq \ell_k(\theta)$.

We are now ready to prove the main result of this paper

**Theorem 2.3.** Let $\rho_1 \leq \rho_2$ be two velocity fields and let $\psi_1, \psi_2$ be the corresponding scalar flows. Assume now that the Hilbert space is decomposed as $E = \bigcup_{k=0}^{\infty} E_k$ with $E_0 = E_\ell$ being a finite dimensional subspace and $(E_k)_{k=1}^{\infty}$ is an increasing sequence of subspaces of $E$ such that $E_k = E_{k-1} \oplus \mathbb{R}e_k$. Let $I_0$ be an even $C^2$ functional on $E$ and consider the levels $c_k$ associated to $I_0$ defined by (3). We have:

(i) if $\psi_1(\theta, c_k) \uparrow +\infty$ as $k \to \infty$, then for every $N \in \mathbb{N}$ there exists a $\theta_N \in (0, 1]$, depending only on $I_0$ and $\rho_1, \rho_2$, such that for any good path of functionals $I : [0, 1] \times E \to \mathbb{R}$ satisfying (H1), (H2), (H3) and (H4') the functional $I_0$ has at least $N$ distinct critical levels, for any $\theta \in [0, \theta_N]$;

(ii) if $c_k \geq B_1 + B_2k(\ln k)\overline{\beta}$ where $\overline{\beta} > 0$, $B_1 \in \mathbb{R}$, $B_2 > 0$ and if $\overline{v}_i(\theta, t) \leq A_1 + A_2(\ln (|t| + 1))^{\overline{\tau}}(\ln \ln (|t| + 1))^{-1}$ where $\overline{\tau} \geq 0$ and $A_1, A_2 \geq 0$, then $I_1$ has an unbounded sequence of critical levels provided $\overline{\beta} \geq \overline{\tau}$.

**Proof of Theorem 2.3** Theorem 2.3 is a consequence of Lemma 2.2.

(i) Our aim is to prove that for any $N \in \mathbb{N}$ and for any good path of functional $I_0$, there is a $\theta_N \in (0, 1]$ such that the functionals $I_0$ have $N$ distinct critical levels $d_1(\theta) < d_2(\theta) < \ldots < d_N(\theta)$, for every $\theta \in [0, \theta_N]$. We obtain the desired sequence of $N$ critical levels by induction. Let $C > 0$ denote the constant appearing in Lemma 2.2 define

$$\eta_k = \inf \{ \theta \in [0, 1] : c_{k+1} - c_k \leq C\theta[\overline{v}_1(\theta, c_{k+1}) + \overline{v}_2(\theta, c_k) + 1] \}.$$ 

Since $\psi_1(\theta, c_k) \uparrow +\infty$ as $k \to +\infty$ by assumption, the sequence $c_k$ is unbounded and for any $M > 0$ there is a $K_M > 0$ such that

$$\psi_1(\theta, c_{k+1}) > M \quad \text{for all} \quad \theta \in [0, 1] \quad \text{and} \quad k > K_M;$$

let us fix $M_1 = 1$, $k_1 = K_{M_1} + 1$ and $\theta_1 = \eta_{k_1}/2$. By definition of $\theta_1$, for all $\theta \in [0, \theta_1]$ the alternative (ii) of Lemma 2.2 is not valid; therefore (i) holds,
so that for any \( \theta \in [0, \theta_1] \) and any path of functionals \( I : [0, 1] \times E \to \mathbb{R} \) satisfying (H1), (H2), (H3) and (H4'), the functionals \( I_\theta \) have critical values \( d_1(\theta) \) with

\[
1 = M_1 < \psi_2(\theta, c_{k_1}) < \psi_1(\theta, c_{k_1+1}) < d_1(\theta) \leq \ell_{k_1}(\theta).
\]

Let now \( N \in \mathbb{N} \) and suppose that there is a \( \theta_{N-1} \in (0, 1] \) such that for any path of functionals (satisfying hypotheses of Lemma 2.2) the functionals \( I_\theta \) have critical values \( d_1(\theta) < d_2(\theta) < \ldots < d_{N-1}(\theta) \leq \ell_{k_{N-1}}(\theta) \). Let

\[
M_N > \sup_{\theta \in [0, \theta_{N-1}]} \ell_{k_{N-1}}(\theta)
\]

and let \( K_N \in \mathbb{N} \) such that

\[
\psi_1(\theta, c_{k+1}) > M_N \quad \text{for all} \quad k > K_N, \quad \theta \in [0, 1],
\]

which exists by assumption; define \( k_N = K_N + 1 \) and \( \theta_N = \eta_{k_N}/2 \). Again, it is clear that for any \( \theta \in [0, \theta_N] \) and any good family of functionals \( I : [0, 1] \times E \to \mathbb{R} \), the functionals \( I_\theta \) have critical values \( d_N(\theta) \) with

\[
M_N < \psi_2(\theta, c_{k_N}) < \psi_1(\theta, c_{k_N+1}) < d_N(\theta) \leq \ell_{k_N}(\theta);
\]

but, by definition, \( M_N > \sup_{\theta \in [0, \theta_{N-1}]} \ell_{k_{N-1}}(\theta) \); hence, by hypothesis of induction, we can conclude that the functionals \( I_\theta \) have \( N \) distinct critical values satisfying \( d_1(\theta) < d_2(\theta) < \ldots < d_N(\theta) \), that is \((i)\).

\((ii)\) Let us suppose by contradiction that the functional \( I_1 \) has only finitely many critical levels. Since \( \rho_i \) is Lipschitz continuous in the second variable, then there are \( L_i > 0 \) such that \( |\psi_i - s| \leq \theta(\mathcal{P}_i(\theta, s) + L_i) \); hence

\[
\psi_1(1, s) \geq s - \mathcal{P}_1(1, s) - L_1 \geq s - A_3 - A_2 \frac{(\ln(|s| + 1))^{\alpha}}{\ln \ln(|s| + 1)}.
\]

(4)

Since \( c_k \) is unbounded, \((4)\) implies that also \( \psi_1(c_{k+1}) \) is unbounded; therefore, if we suppose that \( I_1 \) has only finitely many solutions, the alternative \((i)\) of Lemma 2.2 cannot hold, so that \((ii)\) must be true (with \( \theta = 1 \)). Then we have

\[
c_{k+1} - c_k \leq C(\mathcal{P}_1(1, c_{k+1}) + \mathcal{P}_2(1, c_k) + 1)
\]

\[
\leq C(A_1 + A_2 \frac{(\ln(|c_{k+1}| + 1))^{\alpha}}{\ln \ln(|c_{k+1}| + 1)}) + A_1 + A_2 \frac{(\ln(|c_k| + 1))^{\alpha}}{\ln \ln(|c_k| + 1)} + 1
\]

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which implies that there is a $k_1 > 0$ such that
\[ c_{k+1} \leq c_k + A_4 \frac{(\ln c_k)^\alpha}{\ln \ln c_k} \quad \text{for all } k > k_1. \tag{5} \]
Starting from this relation, one can obtain
\[ c_k < C_1 k \frac{(\ln k)^\alpha}{\ln \ln k} \quad \text{for all } k > k_1. \tag{6} \]
Estimate (6) has been proved by Sugimura [Sug], hence we will be brief; let us choose $C_1$ such that (6) is verified for $k = k_1$, and
\[ \frac{A_4}{C_1} (1 + \alpha + \ln C_1)^\alpha \leq 1; \]
assume now that for $k > k_1$ (6) is valid. Then, by the choice of $C_1$
\[ c_{k+1} \leq c_k + A_4 \frac{(\ln c_k)^\alpha}{\ln \ln c_k} \]
\[ \leq C_1 k \frac{(\ln k)^\alpha}{\ln \ln k} + A_4 \left[ \ln C_1 + \alpha \ln k + \ln \ln k - \ln \ln \ln k \right]^\alpha \]
\[ \leq C_1 \frac{(\ln (k + 1))^\alpha}{\ln \ln (k + 1)} \left[ k + \frac{A_4}{C_1} (\ln C_1 + 1 + \alpha)^\alpha \right] \]
\[ \leq C_1 (k + 1) \frac{(\ln (k + 1))^\alpha}{\ln \ln (k + 1)}, \]
that is (6) for $k + 1$.
Now, we recall that, by assumption, $c_k \geq B_1 + B_2 k (\log k)^\beta$, which is a contradiction under the further assumption that $\beta \geq \alpha$.

3 Perturbation of a symmetric elliptic problem with exponential growth

The aim of this section is to prove the existence of infinitely many solutions for the following perturbed elliptic problem
\[
\begin{cases}
-\Delta u = g(x, u) + f(\theta, x, u) & x \in \Omega \\
u = 0 & x \in \partial \Omega
\end{cases}
\tag{7}
\]
where \( g(x, \cdot) \) is odd, has exponential growth (as will be defined) and \( f \) is a perturbative term. Let us define \( G(x, \xi) = \int_0^\xi g(x, t) dt \) and \( F(x, \xi) = \int_0^\xi f(x, t) dx \). We make the following standard assumptions on the symmetric term \( g \) (see also [Sug]):

(g1) \( g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}) \);

(g2) there is a constant \( A_0 > 0 \) such that

\[
|g(x, \xi)| \leq A_0 e^{\phi(x)} \quad \text{for } (x, \xi) \in \overline{\Omega} \times \mathbb{R},
\]

where \( \phi : \mathbb{R} \to \mathbb{R} \) is a function satisfying \( \phi(\xi)\xi^{-2} \to 0 \) as \( |\xi| \to +\infty \);

(g3) there are constants \( \mu > 0 \) and \( r_0 \geq 0 \) such that \( 0 < G(x, \xi) \ln G(x, \xi) \leq \mu \xi g(x, \xi) \) for \( x \in \overline{\Omega} \) and \( |\xi| \geq r_0 \);

(g4) \( g(x, -\xi) = -g(x, \xi) \) for \( (x, \xi) \in \overline{\Omega} \times \mathbb{R} \);

(g5) there exist \( 0 < \alpha_1 \leq \alpha_2 < 1, A_1, A_2 > 0, \) and \( B_1, B_2 > 0 \) such that

\[
A_1 e^{|\xi|\alpha_1} - B_1 \leq G(x, \xi) \leq A_2 e^{|\xi|\alpha_2} + B_2 \quad \text{for } (x, \xi) \in \overline{\Omega} \times \mathbb{R},
\]

and we make the following assumptions on the perturbative term \( f \)

(f1) \( f \in C([0, 1] \times \overline{\Omega} \times \mathbb{R}, \mathbb{R}) \) and \( f(0, \cdot, \cdot) = 0 \);

(f2) there are \( 0 < \beta_1 < \alpha_1 \) and \( c_1, c_2 > 0 \) such that

\[
\left| \frac{\partial}{\partial \theta} F(\theta, x, \xi) \right| \leq c_1 e^{|\xi|\beta_1} + c_2
\]

(f3) there are \( b > 0 \) and \( \varphi(x) \in L^s(\Omega) \) for some \( s \geq 1 \) such that

\[
f(\theta, x, \xi) \leq be^{|\xi|\beta_1} + \varphi(x);
\]

(f4) there are constants \( p > 0, r > 0 \) and \( d_1, d_2 > 0 \) such that

\[
\left| \int_\Omega \frac{\partial}{\partial \theta} F(\theta, x, \xi) dx \right| \leq d_1 \| u \|_p^r + d_2
\]

whenever \( u \) is a solution of (7).
Using the above notation, we have

**Theorem 3.1.** Suppose that $g$ satisfies (g1)-(g5), and suppose that the perturbative term $f$ satisfies (f1)-(f4). Then we have:

(i) for every $N \in \mathbb{N}$ there exists a $\theta_N \in (0, 1]$ problem (7) has at least $N$ distinct solutions;

(ii) if in addition $2/\alpha_2 - 2 \geq r/\alpha_1$, then problem (7) has an infinite number of solutions for all $\theta \in [0, 1]$.

The above theorem is applicable to, e.g., the following problem:

\[
\begin{cases}
-\Delta u = ue^{u|u|^q} + \theta f(x)u^{r-1} & \text{on } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

where

\[0 < q < 1, \quad 0 \leq r \leq 2 - 2q\]

and $f \in L^s(\Omega)$ for some $s > \frac{r}{r-1}$. Theorem 3.1 includes as a special case the result obtained by Sugimura, which is now extended also to the exponent $q = 1/2$.

Let $E = H^1_0(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

\[\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2},\]

and define the path of functionals $I : [0, 1] \times E \to \mathbb{R}$ by

\[I_\theta(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - G(x, u) - F(\theta, x, u)\right) dx;\]

then $I_0$ is an even functional and the critical points of $I_\theta$ are solutions of problem (7). It is standard to verify that $I$ satisfies the Palais Smale condition $(H1)$ (see for example...). Lemma 3.2 assures that condition $(H2)$ is satisfied. It is also easy to show that condition $(H4')$ is satisfied in any finite dimensional subspace of $E$, thanks to the exponential growth of the nonlinear term $G(x, u)$.
For each \( k \), denote by \( E_k \) the subspace of \( E \) spanned by the first \( k \) eigenfunctions of \( \triangle \); then, let
\[
G = \{ g \in C(E, E) : g \text{ is odd and } g(x) = x \text{ for large } \|x\| \},
\]
and set \( c_k = \inf_{g \in G} \sup_{E_k} I_0 \). As before, \( c_k \) are critical levels of the even functional \( I_0 \). Lemma 3.3 below shows that \((H3)\) holds with
\[
\rho_1(\theta, s) = -C \frac{\ln(|s| + 1)}{\ln \ln(|s| + 1)},
\]
\[
\rho_2(\theta, s) = C \frac{\ln(|s| + 1)}{\ln \ln(|s| + 1)};
\]
on the other hand, it is shown by Sugimura \cite{Sug} that there are positive constants \( B_1, B_2 \) such that \( c_k \geq B_1 k (\ln k)^{\frac{2}{\alpha_1} - 2} - B_2 \). Therefore we can apply Theorem 2.3 \((ii)\) with \( \alpha = \frac{\alpha_1}{\alpha_1} \) and \( \beta = \frac{2}{\alpha_2} - 2 \) to obtain that \( I_1 \) has an infinite number of solutions when \( \frac{2}{\alpha_2} - 2 \geq \frac{2}{\alpha_1} \), which is the claim in Theorem 3.1.\((ii)\). Theorem 3.1.\((i)\) also follows directly from Theorem 2.3.\((i)\), since \( \psi_1(\theta, c_k) \uparrow +\infty \) as \( k \to \infty \) for the \( \rho_i \) defined above.

**Lemma 3.2.** For all \( b > 0 \) there is a constant \( C_1(b) \) such that \(|I_0(v)| < b \) implies
\[
|\frac{\partial I}{\partial \theta}(\theta, v)| \leq C_1(b)(\|I_0(v)\| + 1)(\|v\| + 1).
\]

*Proof* Let \( b > 0 \) and suppose that \(|I_0(v)| < b\); then
\[
\int_\Omega e^{\|u\|^\alpha_1} dx \leq \int_\Omega e^{\|u\|^\alpha_1} dx + \int_\Omega F(\theta, x, u) dx \leq b + \frac{1}{2} \|u\|^2 \quad (9)
\]
Combining \((g3)\) with \((9)\) yields

\[
\begin{align*}
-\langle I'_\theta(u), u \rangle &= \int\! \left(-|\nabla u|^2 + g(x, u)u + f(\theta, x, u)u\right)dx \\
&\geq \int\! |\nabla u|^2dx + \int\! (g(x, u)u - 4G(x, u))dx \\
&\quad + \int\! (f(\theta, x, u)u - 4F(\theta, x, u))dx - 4I_\theta(u) \\
&\geq A_4\|u\|^2 - A_5 - A_6 \int\! e^{u_{|\beta|}}dx - 4b \\
&\geq A_7\|u\|^2 - A_6 \int\! e^{u_{|\beta|}}dx - A_8 \\
&\geq A_7\|u\|^2 - \varepsilon \int\! e^{u_{|\alpha|}}dx - C_\varepsilon \\
&\geq A_9\|u\|^2 - A_{10}.
\end{align*}
\]

Therefore,

\[
\|I'_\theta(u)\|\|u\| \geq -\langle I'_\theta(u), u \rangle \geq A_9\|u\|^2 - A_{10};
\]

(10) finally, combining \((f2)\) with \((10)\) and \((9)\)

\[
\begin{align*}
|\frac{\partial}{\partial \theta}I(\theta, u)| &= | \int\! \frac{\partial}{\partial \theta} F(\theta, x, u)dx | \\
&\leq c_1 \int\! e^{u_{|\beta|}}dx + c_2 |\Omega| \\
&\leq \varepsilon c_1 \int\! e^{u_{|\alpha|}}dx + c_\varepsilon \\
&\leq A_{11}\|u\|^2 + A_{12} \\
&\leq A_{13}\|I'_\theta(u)\|\|u\| + A_{14} \\
&\leq C(\|I'_\theta(u)\| + 1)(\|u\| + 1)
\end{align*}
\]

**Lemma 3.3.** There exists a constant \(C > 0\) such that if \(u \in H_0^1(\Omega)\) is a critical point of \(I_\theta\), then

\[
|\frac{\partial}{\partial \theta}(\theta, u)| \leq C\frac{\ln(\|I_\theta(u)\| + 1))^{\varepsilon_1}}{\ln \ln(\|I_\theta(u)\| + 1)}.
\]
Proof. Let us suppose that $I_\theta'(u) = 0$; then, combining $(g3)$, $(g5)$, $(f3)$ and Young’s inequality yields

$$I_\theta(u) = I_\theta(u) - \frac{1}{2} \langle I_\theta'(u), u \rangle$$

$$= \int_\Omega \left( \frac{1}{2} g(x,u) u - G(x,u) \right) dx + \int_\Omega \left( \frac{1}{2} f(\theta, x, u) u - F(\theta, x, u) \right) dx$$

$$\geq \left( \frac{1}{2\mu} - \varepsilon \right) \int_\Omega G(x,u) \ln G(x,u) dx - \int_\Omega \left( \frac{c_1}{2} e^{u|u|^\beta_1} + |u| e^{u|u|^\beta_1} \right) dx - C_\varepsilon$$

$$\geq C_1 \int_\Omega |u|^{\alpha_1} e^{u|u|^\alpha_1} dx - C_2$$

$$\geq C_3 \|u\|_{\frac{1}{p}, e^{u|u|^\alpha_1}} - C_4,$$

for any $p \geq 1$, where $C_3, C_4$ are positive constants depending only on $p$ and $\Omega$; therefore,

$$\|u\|_p \leq C_1(p) \left( \frac{\ln(|I_\theta(u)| + 1)^\frac{1}{\alpha_1}}{\ln(|I_\theta(u)| + 1)} + C_2(p) \right). \tag{11}$$

Finally, applying $(f4)$ and (11) we obtain

$$\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| = \left| \int_\Omega \frac{\partial}{\partial \theta} F(\theta, u) dx \right|$$

$$\leq d_1 \|u\|^r_p + d_2$$

$$\leq C_5 \left( \frac{\ln(|I_\theta(u)| + 1)^\frac{1}{\alpha_1}}{\ln(|I_\theta(u)| + 1)} + C_6, \right.$$

that is our thesis.

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