Behaviour of $L_q$ norms of the sinc$_p$ function

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Abstract

An integral inequality due to Ball involves the $L_q$ norm of the sinc$_p$ function; the dependence of this norm on $q$ as $q \to \infty$ is now understood. By use of recent inequalities involving $p$–trigonometric functions ($1 < p < \infty$) we obtain asymptotic information about the analogue of Ball’s integral when sin is replaced by sin$_p$.

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1 Introduction

In [1], K. Ball proved that every section of the unit cube in $\mathbb{R}^n$ by an $(n - 1)$–dimensional subspace has $(n - 1)$–volume at most $\sqrt{\pi}$, which is attained if and only if this section contains an $(n - 2)$–dimensional face of the cube. To show this, Ball made essential use of the inequality

$$\sqrt{q} \int_{-\infty}^{\infty} \left| \frac{\sin x}{x} \right|^q \, dx \leq \sqrt{2} \pi, \quad q \geq 2,$$

in which equality holds if and only if $q = 2$.

It is now known (see [3]) that

$$\lim_{q \to \infty} \sqrt{q} \int_{-\infty}^{\infty} \left| \frac{\sin x}{x} \right|^q \, dx = \sqrt{\frac{3\pi}{2}}.$$

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Moreover, the asymptotic properties of the $q$-norm of the sinc function were studied. In fact, more precise results of an asymptotic nature of the integral in Ball's integral inequality are now known (see [8] for more details).

Stimulated by applications to such differential operators as the $p$-Laplacian, there is now a large amount of recent work concerning generalisations of the sine function (and other trigonometric functions): see, for example, [5]. This encourages us to look for an extension of (1) to a more general setting. With this in mind, for $q \in (1, \infty)$, define

$$I_p(q) := q^{1/p} \int_0^\infty \left| \frac{\sin_p x}{x} \right|^q \, dx$$

where $p \in (1, \infty)$ and $\sin_p$ is the generalised sine function. This function is defined to be the inverse of the function $F_p : [0, 1] \rightarrow [0, \frac{\pi}{2}]$ given by

$$F_p(y) := \int_0^y (1 - t^p)^{-\frac{1}{p}} \, dt,$$

where $\pi := 2F_p(1) = \frac{2\pi}{p \sin(\frac{\pi}{p})}$; it is increasing on $[0, \frac{\pi}{2}]$ and is extended to the whole of $\mathbb{R}$ to be a $2\pi_p$-periodic function (still denoted by $\sin_p$) by means of the rules

$$\sin_p(-x) = -\sin_p(x) \quad \text{and} \quad \sin_p\left(\frac{\pi_p}{2} - x\right) = \sin_p\left(\frac{\pi_p}{2} + x\right).$$

The choice $p = 2$ corresponds to the standard trigonometric setting: $\sin_2 \equiv \sin$, $\pi_2 = \pi$. Moreover, $\pi_p$ is a decreasing function in $p \in (1, \infty)$ such that

$$\begin{cases} \pi_p \rightarrow \infty & \text{when } p \rightarrow 1^+ \\ \pi_p \rightarrow 2 & \text{when } p \rightarrow \infty.\end{cases}$$

The main purpose of this paper is to show that for each $p \in (1, \infty)$ there is an analogue of (1), namely

$$\lim_{q \rightarrow \infty} q^{1/p} \int_0^\infty \left| \frac{\sin_p x}{x} \right|^q \, dx = p^{-1}(p+1)^{1/p} \Gamma(1/p).$$

This is achieved by appropriate use of certain recently-obtained inequalities concerning $\sin_p$. Moreover, information is obtained about the asymptotic behaviour of the above integral as $q \rightarrow \infty$; this complements that known when $p = 2$.

2 Properties of the $\text{sinc}_p$ function

Given $p \in (1, \infty)$, the function $\text{sinc}_p$ is defined by

$$\text{sinc}_p x = \begin{cases} \frac{\sin_p x}{x}, & x \in \mathbb{R} \setminus \{0\}, \\ 1, & x = 0.\end{cases}$$

It is even and its roots are the points $n\pi_p$ with $n \in \mathbb{Z} \setminus \{0\}$. Since $|\sin_p x| \leq 1$ for all $x \in \mathbb{R}$, $\lim_{|x| \rightarrow \infty} \text{sinc}_p x = 0$. 

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Lemma 2.1. (a) \(|\text{sinc}_p x| \leq 1\) for all \(x \in \mathbb{R}\).

(b) The function \(\text{sinc}_p\) is strictly decreasing on the interval \((0, \frac{\pi_p}{2})\).

Proof. \(\Box\) Observe that for \(x \in [0, \frac{\pi_p}{2}]\) we have the \(p\)-analogue of the classical Jordan inequality \([4, \text{Proposition 2.3}]\),

\[
\frac{2}{\pi_p} \leq \text{sinc}_p x < 1 \quad \forall x \in \left[0, \frac{\pi_p}{2}\right].
\]

On the other hand, for \(x \in \left(\frac{\pi_p}{2}, \infty\right)\) and since \(\pi_p \in (2, \infty)\) for all \(p \in (1, \infty)\) we conclude that

\[
|\text{sinc}_p x| = \left|\frac{\sin_p x}{x}\right| < 1.
\]

Since \(\text{sinc}_p\) is an even function, \((a)\) is complete.

\(\Box\) Observe that

\[
\frac{d}{dx} \text{sinc}_p x = \frac{\cos_p x}{x^2}(x - \tan_p x)
\]

and \(\cos_p x > 0\) for any \(x \in (0, \frac{\pi_p}{2})\). Let \(g(x) = x - \tan_p x\). Then, \(g'(x) = -\tan_p^p x < 0\) for all \(x \in (0, \frac{\pi_p}{2})\) and we conclude that \(g(x)\) is strictly decreasing in this interval. Then \(g(x) < g(0) = 0\) for \(x \in (0, \frac{\pi_p}{2})\). The result follows. \(\Box\)

3 The \(p\)-version of Ball’s integral inequality

For \(p, q \in (1, \infty)\) define

\[
I_p(q) := q^{1/p} \int_0^\infty \left|\frac{\sin_p x}{x}\right|^q dx.
\]

Here we establish the existence of \(\lim_{q \to \infty} I_p(q)\). We use Laplace’s method for integrals which suggests approximating the integrand in a neighbourhood by simpler functions for which the integral can be evaluated after proving that the integral on the complementing interval is very small when \(q\) is large enough.

Lemma 3.1. Let \(p, q \in (1, \infty)\). For any real \(\alpha > 0\), we have

\[
\left.\int_{\alpha}^\infty \left|\frac{\sin_p x}{x}\right|^q dx \right| \\
\quad \geq \left\{ \begin{array}{ll}
\frac{1}{q-1} \alpha^{1-q} & \text{for } \alpha \geq 1, \\
\left(\frac{\sin_p \alpha}{\alpha}\right)^q (1-\alpha) + \frac{1}{q-1} & \text{for } \alpha < 1.
\end{array} \right.
\]

Moreover,

\[
\lim_{q \to \infty} q^{1/p} \int_{\alpha}^\infty \left|\frac{\sin_p x}{x}\right|^q dx = 0, \quad \forall \alpha > 0.
\]
Proof. We first discuss the case $\alpha \in [1, \infty)$.

\[
\int_{\alpha}^{\infty} \left| \frac{\sin p x}{x} \right|^q \, dx = \lim_{\beta \to \infty} \int_{\alpha}^{\beta} \left| \frac{\sin p x}{x} \right|^q \, dx \\
\leq \lim_{\beta \to \infty} \int_{\alpha}^{\beta} x^{-q} \, dx \\
= \lim_{\beta \to \infty} \frac{1}{q} \left[ \frac{1}{\beta^{q-1}} - \frac{1}{\alpha^{q-1}} \right] = \frac{1}{q - 1} \alpha^{1-q}.
\]

Then,

\[
\lim_{q \to \infty} q^{1/p} \int_{\alpha}^{\infty} \left| \frac{\sin p x}{x} \right|^q \, dx \leq \lim_{q \to \infty} \frac{q^{1/p}}{q - 1} = 0.
\]

For $\alpha \in (0, 1)$, we have

\[
\int_{\alpha}^{\infty} \left| \frac{\sin p x}{x} \right|^q \, dx = \int_{\alpha}^{1} \left| \frac{\sin p x}{x} \right|^q \, dx + \int_{1}^{\infty} \left| \frac{\sin p x}{x} \right|^q \, dx.
\]

From the previous case we have,

\[
q^{1/p} \int_{1}^{\infty} \left| \frac{\sin p x}{x} \right|^q \, dx \leq \frac{q^{1/p}}{q - 1},
\]

which approaches zero as $q \to \infty$.

For the remaining integral, the corresponding interval $(\alpha, 1)$ is a subset of $(0, \pi/2)$ since $\pi_p \in (2, \infty)$. According to Lemma [2.1]

\[
0 < \frac{\sin p x}{x} < \frac{\sin \alpha}{\alpha} < 1, \quad \forall x \in (\alpha, 1).
\]

Then,

\[
\int_{\alpha}^{\infty} \left| \frac{\sin p x}{x} \right|^q \, dx < \left( \frac{\sin \alpha}{\alpha} \right)^q (1 - \alpha) + \frac{1}{q - 1}.
\]

Using Lemma [2.1 (a)], we conclude that

\[
\lim_{q \to \infty} q^{1/p} \int_{\alpha}^{\infty} \left| \frac{\sin p x}{x} \right|^q \, dx \leq \lim_{q \to \infty} q^{1/p} \left[ \left( \frac{\sin \alpha}{\alpha} \right)^q (1 - \alpha) + \frac{1}{q - 1} \right] = 0.
\]

Our main result is the following theorem:

**Theorem 3.1.** Let $p, q \in (1, \infty)$. Then

\[
\lim_{q \to \infty} I_p(q) = \frac{1}{p} \left( \frac{1}{p} \right) (p(p + 1))^{1/p}.
\]
Proof. From \[6, 3.251, \text{p. 324}\], for \(\mu > 0, \nu > 0, \lambda > 0\) we have

\[(6) \quad \int_0^1 x^{\mu-1}(1-x)^{\nu-1}dx = \frac{1}{\lambda} B \left(\frac{\mu}{\lambda}, \nu\right).\]

From \[2, \text{Theorem 1.1 (1)}\], for \(0 < x < \left[1 - \left(\frac{2}{\pi p}\right)^{(p+1)}\right]^{1/p}\) we have

\[(7) \quad \frac{x}{\sin_p x} > (1 - x^p)^{\frac{1}{p(p+1)}}.\]

Note that for all \(p \in (1, \infty)\) we have

\[0 < \left[1 - \left(\frac{2}{\pi p}\right)^{(p+1)}\right]^{1/p} < 1.\]

Let

\[\alpha_1 := \sin_p^{-1} \left(1 - \left(\frac{2}{\pi p}\right)^{(p+1)}\right) \in \left(0, \frac{\pi}{2}\right).\]

Changing the variable to \(y = \sin_p x\) and using the inequality in (7) we see that

\[
q^{1/p} \int_0^\infty \left|\frac{\sin_p x}{x}\right|^q \frac{dx}{x} > q^{1/p} \int_0^{\alpha_1} \left(\sin_p x\right)^q \frac{dx}{x} > q^{1/p} \int_0^{\alpha_1} \left(1 - y^p\right)^{-1/p} dy
\]

\[
> q^{1/p} \int_0^{\alpha_1} (1 - y^p)^{\frac{1}{p(p+1)}}(1 - y^p)^{-1/p} dy
\]

\[=: J_1(p, q) - J_2(p, q).\]

From (6) with \(\mu = 1, \lambda = p\) and \(v = \frac{q}{p(p+1)} - \frac{1}{p} + 1\) we get

\[\lim_{q \to \infty} J_1(p, q) = \lim_{q \to \infty} q^{1/p} \frac{1}{p} \Gamma \left(\frac{1}{p}\right) \Gamma \left(\frac{q}{p(p+1)} - \frac{1}{p} + 1\right) \Gamma \left(\frac{q}{p(p+1)} + 1\right)\]

\[(8) \quad = \frac{1}{p} \Gamma \left(\frac{1}{p}\right) (p(p+1))^{1/p}.\]

The last equality is due to the fact that

\[(9) \quad \frac{\Gamma(q + a)}{\Gamma(q + b)} \rightarrow q^{a-b} \quad \text{as} \quad q \to \infty,\]
which follows from Stirling’s formula: see also [7, Problem 2, p.45]. Moreover,

\[
\lim_{q \to \infty} J_2(p, q) = \lim_{q \to \infty} q^{1/p} \int_{\sin_p \alpha_1}^{1} (1 - y^p)^{1/(p+1)} (1 - y^p)^{-1/p} \, dy \\
\leq \lim_{q \to \infty} q^{1/p} \int_{\sin_p \alpha_1}^{1} (1 - y^p)^{-1/p} \, dy \\
\leq \lim_{q \to \infty} q^{1/p} \int_{0}^{1} (1 - y^p)^{-1/p} \, dy \\
= \frac{\pi}{2} q^{1/p} (1 - \sin_p \alpha_1)^{-1/p+1} = 0.
\]

(10)

Then from (8) and (10),

\[
\liminf_{q \to \infty} q^{1/p} \int_{0}^{\infty} |\sin_p x|^q \, dx \geq \frac{1}{p} \Gamma \left( \frac{1}{p} \right) (p(p + 1))^{1/p}.
\]

(11)

On the other hand, from [6, 3.251, p.325], for \(0 < \mu < p \nu, b > 0 \) and \( p > 0 \) we have

\[
\int_{0}^{\infty} x^{\mu-1}(1 + bx^p)^{-\nu} \, dx = \frac{1}{p} b^{\mu/p} B \left( \frac{\mu}{p}, \nu - \frac{\mu}{p} \right).
\]

(12)

From [2, Theorem 1.1 (1)], for \(x \in (0, 1)\) we have

\[
\frac{x}{\sin_p x} < \left( 1 + \frac{x^p}{p(p + 1)} \right)^{-1}.
\]

(13)

Now let \(\alpha_2 \in (0, \pi_p/2]\). Changing the variable to \(y = \sin_p x\) and using the inequality in (13) we obtain

\[
\int_{0}^{\alpha_2} \left( \frac{\sin_p x}{x} \right)^q \, dx = \int_{0}^{\sin_p \alpha_2} \left( \frac{y}{\sin_p^{-1} y} \right)^q (1 - y^p)^{-1/p} \, dy \\
< \int_{0}^{\sin_p \alpha_2} \left( 1 + \frac{y^p}{p(p + 1)} \right)^{-q} (1 - y^p)^{-1/p} \, dy \\
< (1 - \sin_p \alpha_2)^{-1/p} \int_{0}^{\sin_p a_2} \left( 1 + \frac{y^p}{p(p + 1)} \right)^{-q} \, dy \\
= : (1 - \sin_p \alpha_2)^{-1/p} J_3(p, q).
\]

(14)

From (12), with \(\mu = 1, b = (p(p + 1))^{-1}\) and \(\nu = q \in (1/p, \infty)\), we obtain

\[
J_3(p, q) < \frac{1}{p} (p(p + 1))^{1/p} B \left( \frac{1}{p}, q - \frac{1}{p} \right) \\
= \frac{1}{p} (p(p + 1))^{1/p} \Gamma \left( \frac{1}{p} \right) \frac{\Gamma \left( q - \frac{1}{p} \right)}{\Gamma(q)}.
\]

(15)
With the help of (9) it follows that
\[
\limsup_{q \to \infty} q^{1/p} \int_0^{x_2} \left| \frac{\sin_p x}{x} \right|^q \, dx \leq (1 - \sin^p x_2)^{-1/p} \frac{1}{p} (p(p + 1))^{1/p} \Gamma \left( \frac{1}{p} \right).
\]
Letting \(x_2 \to 0^+\) we conclude that,
\[
\limsup_{q \to \infty} q^{1/p} \int_0^{x_2} \left( \frac{\sin_p x}{x} \right)^q \, dx \leq \frac{1}{p} (p(p + 1))^{1/p} \Gamma \left( \frac{1}{p} \right).
\]
From Lemma 3.1 we conclude that
\[
\limsup_{q \to \infty} q^{1/p} \int_\alpha^\infty \left| \frac{\sin_p x}{x} \right|^q \, dx \leq \frac{1}{p} (p(p + 1))^{1/p} \Gamma \left( \frac{1}{p} \right).
\]
(16)
From (11) and (16) we deduce that
\[
\lim_{q \to \infty} I_p(q) = \frac{1}{p} \Gamma \left( \frac{1}{p} \right) (p(p + 1))^{1/p}.
\]
\
\[
\text{4 Asymptotic Expansion
}
\]
Here we investigate the asymptotic expansion of the \(p\)-Ball integral \(I_p(q)\) by performing explicit calculations leading to a precise knowledge of the first two coefficients of the expansion. The study involved provides another proof of Theorem 3.1, the technique used is an adaptation of that developed in [8].

**Theorem 4.1.** There exist constants \(\gamma_3, \gamma_4, \ldots\) such that for \(q\) large enough
\[
I_p(q) \sim (p(p + 1))^{1/p} \left( \frac{1}{p} \Gamma \left( \frac{1}{p} \right) + \frac{1}{p} \Gamma \left( \frac{1}{p} \right) \frac{(-p^2 + p + 1)(p + 1)}{2p(2p + 1)} \frac{1}{q} + \sum_{j=3}^{\infty} \Gamma \left( j + 1/p \right) \frac{\gamma_j}{q^j} \right).
\]

**Proof.** For \(\alpha \in (0, 1)\), let
\[
J(q, \alpha) := q^{1/p} \int_0^\alpha \left( \frac{\sin_p x}{x} \right)^q \, dx
\]
\[
= q^{1/p} \int_0^\alpha \exp \left( -q x^p \right) \left[ \exp \left( \frac{x^p}{p(p + 1)} \right) \frac{\sin_p x}{x} \right]^q \, dx.
\]
By Lemma 3.1
\[
q^{1/p} \int_\alpha^{\infty} \left| \frac{\sin_p x}{x} \right|^q \, dx \leq q^{1/p} \left( \frac{\sin_p \alpha}{\alpha} \right)^q (1 - \alpha) + q^{1/p} \frac{\alpha}{q - 1}.
\]
It is therefore enough to establish the existence of constants \( \gamma_3, \gamma_4, \ldots \) such that
\[
J(q, \alpha) \sim (p(p+1))^{1/p} \left( \frac{1}{p} \frac{\Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} \right)} + \frac{1}{p} \frac{\Gamma \left( \frac{1}{p} \right)}{\Gamma \left( \frac{1}{p} \right)} \frac{(-p^2 + p + 1)(p+1)}{2p(2p+1)} \frac{1}{q} + \sum_{j=3}^{\infty} \frac{\Gamma(j+1/p) \gamma_j}{q^j} \right).
\]
Changing the variable to \( u = \frac{x}{(p(p+1))^{1/p}} \) yields
\[
J(q, \alpha) = q^{1/p} (p(p+1))^{1/p} \int_{0}^{\infty} \exp(-q u^p) \left[ \exp(u^p) \sin \left((p(p+1))^{1/p} u \right) \right] \frac{\sin_p \left((p(p+1))^{1/p} u \right)}{(p(p+1))^{1/p} u} \] 
\[
\text{du.}
\]
For the exponential term we have
\[
\exp(u^p) = \sum_{j=0}^{\infty} \frac{u^{pj}}{j!}.
\]
While for \( \sin_p \left((p(p+1))^{1/p} u \right) \), we have from [4 (2.17)] the power series expansion of \( \sin_p^{-1} x \), and by the Lagrange reversion theorem this gives the existence of constants \( a_j \) such that
\[
\frac{\sin_p \left((p(p+1))^{1/p} u \right)}{(p(p+1))^{1/p} u} = \sum_{j=0}^{\infty} a_j (p(p+1))^j u^{pj};
\]
the series converges for sufficiently small \( u \). The coefficients of the first three terms of this expansion involve \( a_0 = 1, a_1 = \frac{-1}{p(p+1)} \) and \( a_2 = \frac{-p^2+2p+1}{2p^2(p+1)(2p+1)} \). The Cauchy product formula then gives,
\[
\exp(u^p) \sin_p \left((p(p+1))^{1/p} u \right) = 1 + \sum_{j=2}^{\infty} \sum_{l=0}^{j} \frac{a_{j-l}(p(p+1))^{j-l}}{l!} b_j u^{pj},
\]
where
\[
b_j = \frac{j}{j-l}(p(p+1))^{j-l}
\]
and the power series converges for sufficiently small \( u \).

We know that for small values of \( u \),
\[
\left| \exp(u^p) \sin_p \left((p(p+1))^{1/p} u \right) - 1 \right| = \sum_{j=2}^{\infty} b_j u^{pj} \leq \sum_{j=2}^{\infty} |b_j| u^{pj} < 1.
\]
Note that the power series \( \sum_{j=2}^{\infty} b_j u^{pj} \) is absolutely convergent for sufficiently small \( u \).

Therefore by the Binomial expansion we get
\[
\left[ \exp(u^p) \sin_p \left((p(p+1))^{1/p} u \right) \right]^q = 1 + q \sum_{j=2}^{\infty} b_j u^{pj} + \frac{q(q-1)}{2} \left( \sum_{j=2}^{\infty} b_j u^{pj} \right)^2 + \cdots \frac{q(q-1)\cdots(q-m+1)}{m!} \left( \sum_{j=2}^{\infty} b_j u^{pj} \right)^m + \cdots
\]
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Since the right hand side of the Binomial expansion is bounded from above by
\[
1 + q \left[ \sum_{j=2}^{\infty} |b_j| u^p \right] + \frac{q(q-1)}{2} \left[ \sum_{j=2}^{\infty} |b_j| u^p \right]^2 \\
+ \cdots + \frac{q(q-1) \cdots (q-m+1)}{m!} \left[ \sum_{j=2}^{\infty} |b_j| u^p \right]^m \cdots = \left[ 1 + \sum_{j=2}^{\infty} |b_j| u^p \right]^q,
\]
we may rearrange terms and, for small enough \( u \), obtain (17). Hence,
\[
\exp \left( u^p \right) \sin \left( \frac{(p(p+1))^{1/p} \ u}{(p(p+1))^{1/p}} \right)^q = \sum_{j=0}^{\infty} c_j u^p,
\]
where \( c_0 = 1, c_1 = 0 \) and \( c_2 = q b_2 = \frac{q(-p^2+p+1)}{2(2p+1)} \). Observe that the other coefficients \( c_j = c_j(q) \) (\( j \geq 3 \)) can be obtained by the following rearrangements:
\[
q b_3 u^{3p} = c_3 u^{3p}, \quad \left( q b_4 + \frac{q(q-1)}{2} b_2^2 \right) u^{4p} = c_4 u^{4p}, \ldots
\]

Specialised to our case, [9, Theorem 8.1, p. 86] (with \( x = q, p(t) = u^p, q(t) = \exp(u^p) \frac{\sin_p((p(p+1))^{1/p} u)}{(p(p+1))^{1/p} u} \), \( s = p j, \lambda = 1 \) and \( \mu = p \)) establishes the existence of real constants \( \gamma_0, \gamma_1, \ldots \) such that
\[
J(q, \alpha) \sim q^{1/p} (p(p+1))^{1/p} \sum_{j=0}^{\infty} \frac{\Gamma(j+1/p)}{q^{1+1/p}} \frac{\gamma_j}{q^j}, \quad (q \to \infty)
\]
where
\[
\gamma_0 = \frac{1}{p}, \quad \gamma_1 = 0 \quad \text{and} \quad \gamma_2 = \frac{q(-p^2+p+1)}{2(2p+1)}.
\]

**Remark 4.1.** The asymptotic expansion in Theorem 4.1 complements that of [8] when \( p = 2 \); it involves the coefficients \( b_j \) of the expansion of \( \exp(u^p) \frac{\sin_p((p(p+1))^{1/p} u)}{(p(p+1))^{1/p} u} \) which depend on the constants \( a_j \) of the power series of the function \( \text{sinc}_p \). So far the first three terms in the expansion of \( \sin_p \) are known and no regular pattern has been obtained for the other subsequent terms. It remains to see whether or not higher-order terms in the expansion of \( I_p(q) \) can be determined.
5 Concluding remarks

In this section we present some results obtained from Theorem 3.1. The proofs are natural adaptations of those given in [3] and are therefore omitted.

For \( q \in (1, \infty) \) and \( n \in \mathbb{N} \cup \{0\} \), define

\[
\varphi_p(n, q) := \int_0^\infty \left( \ln \left| \frac{\sin_p x}{x} \right|^q \right) \left| \frac{\sin_p x}{x} \right| \, dx.
\]

Note that \( \varphi_p(0, q) := \varphi_p(q) = I_p(q) \).

A more general result of \( p \)-Ball integral inequality can also be achieved by induction for any non-negative integer \( n \).

Lemma 5.1. For \( n \in \mathbb{N} \cup \{0\} \) and \( p \in (1, \infty) \). Then

\[
\lim_{q \to \infty} q^{n+\frac{1}{p}} \varphi_p(n, q) = (-1)^n \frac{1}{p} \Gamma \left( \frac{1}{p} \right) (p(p + 1))^{1/p} \Gamma \left( n + \frac{1}{p} \right).
\]

The following gives the analyticity of the function \( \varphi_p(q) \) in a region containing \((1, \infty)\). The proof makes use of the \( L_q \)-Lebesgue integrability of the sinc\(_p\) functions when \( p, q \in (1, \infty) \).

Corollary 5.1. Let \( q \in (1, \infty) \). For \( 1 - q < z < q - 1 \),

\[
\varphi_p(q - z) = \sum_{n=0}^{\infty} (-1)^n \varphi_p(n, q) \frac{z^n}{n!},
\]

where \( \varphi_p^{(n)}(q) = \varphi_p(n, q) \).

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