On $m$-th Root Finsler Metrics

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Abstract

In this paper, we characterize locally dually flat and Antonelli $m$-th root Finsler metrics. Then, we show that every $m$-th root Finsler metric of isotropic mean Berwald curvature reduces to a weakly Berwald metric.

Keywords: Antonelli metric, locally dually flat metric, isotropic mean Berwald metric.

1 Introduction

The theory of $m$-th root metrics has been developed by H. Shimada [14], and applied to Biology as an ecological metric [3]. It is regarded as a direct generalization of Riemannian metric in the sense that the second root metric is a Riemannian metric. The third and fourth root metrics are called the cubic metric and quartic metric, respectively.

Recently studies show that the theory of $m$-th root Finsler metrics plays a very important role in physics, theory of space-time structure, gravitation, general relativity and seismic ray theory [4][7][10][11][12]. For quartic metrics, a study of the geodesics and of the related geometrical objects is made by S. Lebedev [8]. Also, Einstein equations for some relativistic models relying on such metrics are studied by V. Balan and N. Brinzei in two papers [5], [6]. Tensorial connections for such spaces have been recently studied by L. Tamassy [13]. B. Li and Z. Shen study locally projectively flat fourth root metrics under some irreducibility condition [9]. Y. Yu and Y. You show that an $m$-th root Einstein Finsler metric is Ricci-flat [17].

Let $(M, F)$ be a Finsler manifold of dimension $n$, $TM$ its tangent bundle and $(x^i, y^i)$ the coordinates in a local chart on $TM$. Let $F$ be a scalar function on $TM$ defined by $F = \sqrt[2m]{A}$, where $A$ is given by

$$A := a_{i_1 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m},$$

with $a_{i_1 \ldots i_m}$ symmetric in all its indices [14]. Then $F$ is called an $m$-th root Finsler metric.

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Let $F$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Put

$$A_i = \frac{\partial A}{\partial y^i}, \quad \text{and} \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}. $$

Suppose that the matrix $(A_{ij})$ defines a positive definite tensor and $(A^{ij})$ denotes its inverse. Then the following hold

$$g_{ij} = A^{2m} - 2m A_{ij} + (2 - m) A_i A_j, \quad (2)$$

$$g^{ij} = A^{-2m} [m A^{ij} + \frac{m - 2}{m - 1} y^i y^j], \quad (3)$$

$$y^i A_i = m A, \quad y^i A_{ij} = (m - 1) A_j, \quad y_i = \frac{1}{m} A^{\frac{m}{2} - 1} A_i, \quad (4)$$

$$A^{ij} A_{jk} = \delta_i^k, \quad A^{ij} A_i = \frac{1}{m - 1} y^j, \quad A_i A_j A^{ij} = \frac{m}{m - 1} A, \quad (5)$$

$$A_0 := A x^m y^m, \quad A_{0i} := A x^m y^i. \quad (6)$$

A Finsler metric $F = F(x, y)$ on a manifold $M$ is said to be locally dually flat, if at any point there is a standard coordinate system $(x^i, y^i)$ in $T M$ such that $[F^2]_{x^i y^i y^k} = 2[F^2]_{x^i}$. In this case, the coordinate $(x^i)$ is called an adapted local coordinate system \[13\]. Here, we characterize locally dually flat $m$-th root Finsler metrics.

**Theorem 1.1.** Let $F$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Then, $F$ is a locally dually flat metric if and only if the following holds

$$A_{x^i} = \frac{1}{2A} \left\{ \left( \frac{2}{m} - 1 \right) A_i A_0 + AA_{0i} \right\}. \quad (7)$$

Moreover, suppose that $A$ is irreducible. Then $F$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_i(x^i) y^i$ on $U$ such that the following holds

$$A_{x^i} = \frac{1}{3m} \left\{ 2 \theta A_i + m A \theta_i \right\}. \quad (8)$$

A Finsler metric $F$ is called an Antonelli metric, if there is a local coordinate system in $T M$, in which the spray coefficients are dependent on direction alone. In this case, the spray coefficients of $F$ are given by $G^i = \frac{1}{2} \Gamma^i_{jk} (y^j y^k)$.\[13\]

Antonelli metrics were first introduced by P. L. Antonelli for some studies in Biology and Ecology \[3\]. Antonelli calls them $y$-Berwald metrics. This class of metrics arises in time sequencing change models in the evolution of colonial systems. Here, we characterize $m$-th root Antonelli metrics. More precisely, we prove the following.

**Theorem 1.2.** Let $F$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Then, $F$ is an Antonelli metric if and only if there exist functions $\Gamma^i_{jk}$ dependent only on direction such that the following holds

$$A_{x^i} = [\Gamma^i_{jk} y^k + \frac{1}{2} \Gamma^i_{jk,l} y^j y^k] A_i, \quad (9)$$
where $\Gamma^i_{jk,l} = \frac{\partial^3 u_i}{\partial y^j \partial y^l \partial y^k}$.

A Finsler metric $F$ is called a Berwald metric if $G^i$ are quadratic in $y \in T_xM$ for any $x \in M$ or equivalently Berwald curvature vanishes. In this case, we have $G^i = \frac{1}{2} \Gamma^i_{jk}(x)y^j y^k$. Hence, Berwald metrics can be considered as $x$-Berwald metrics. The $E$-curvature is defined by the trace of the Berwald curvature. A Finsler metric $F$ is called of isotropic mean Berwald curvature if $E = \frac{n+1}{2} cF^{-1} h$, where $c = c(x)$ is a scalar function on $M$ and $h$ is the angular metric. If $c = 0$, then $F$ is called weakly Berwald metric.

**Theorem 1.3.** Let $F$ be an $m$-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$ with $n \geq 2$. Suppose that $F$ is of isotropic mean Berwald curvature. Then $F$ is a weakly Berwald metric.

### 2 Preliminaries

Let $M$ be an $n$-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, by $TM = \cup_{x \in M} T_xM$ the tangent bundle of $M$ and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle of $M$.

A Finsler metric on a manifold $M$ is a function $F : TM \to [0, \infty)$ with the following properties:

(i) $F$ is $C^\infty$ on $TM_0$;

(ii) $F(x, \lambda y) = \lambda F(x, y) \ \forall \lambda > 0, \ y \in TM$;

(iii) for each $y \in T_xM$, the following quadratic form $g_y$ on $T_xM$ is positive definite,

$$g_y(u, v) := \frac{1}{2} \left[ F^2(y + su + tv) \right]_{s, t = 0}, \ u, v \in T_xM.$$

In local coordinates $(x^i, y^i)$, the vector field $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a global vector field on $TM_0$, where $G^i = G^i(x, y)$ are local functions on $TM_0$ given by

$$G^i := \frac{1}{4} g^{ij} \left\{ \frac{\partial^2 F^2}{\partial x^k \partial y^i} y^k - \frac{\partial F^2}{\partial x^i} \right\}, \ y \in T_xM. \quad (10)$$

The vector field $G$ is called the associated spray to $(M, F)$ [10].

For $y \in T_xM_0$, define $B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $E_y : T_xM \otimes T_xM \to \mathbb{R}$ by $B_y(u, v, w) := B^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $E_y(u, v) := E^i_{jk}(y)u^j v^k$ where

$$B^i_{jkl} := \frac{\delta^3 u_i}{\delta y^j \delta y^k \delta y^l}, \quad E^i_{jk} := \frac{1}{2} B^{i}_{jkl}.$$

$B$ and $E$ are called Berwald curvature and mean Berwald curvature, respectively. Then $F$ is called a Berwald metric and weakly Berwald metric if $B = 0$ and $E = 0$, respectively [10].
A Finsler metric $F$ on an $n$-dimensional manifold $M$ is said to be isotropic mean Berwald metric or of isotropic $E$-curvature if
\[ E_{ij} = \frac{n+1}{2} cF^{-1}h_{ij}, \]
where $h_{ij} = g_{ij} - g_{ip}y^pg_{jq}y^q$ is the angular metric.

## 3 Proof of Theorem 1.1

Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory [1][2]. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which plays a very important role in studying flat Finsler information structure [13].

A Finsler metric $F = F(x, y)$ on a manifold $M$ is said to be locally dually flat, if at any point there is a standard coordinate system $(x^i, y^i)$ in $TM$ such that $L = F^2$ satisfies
\[ L_{x^iy^l}y^k = 2L_{x^l}. \]
In this case, the coordinate $(x^i)$ is called an adapted local coordinate system.

Every locally Minkowskian metric $F$ satisfies trivially the above equation, hence $F$ is locally dually flat.

In order to find explicit examples of locally dually flat metrics, we consider $m$-th root Finsler metrics.

**Proof of Theorem 1.1** Let $F$ be a locally dually flat metric
\[ (A^\frac{1}{m})_{x^iy^l}y^k = 2(A^\frac{1}{m})_{x^l}. \]
We have
\[ (A^\frac{1}{m})_{x^l} = \frac{2}{m} A^{\frac{2m}{2m}} A_{x^l}, \]
\[ (A^\frac{1}{m})_{x^ly^l} = \frac{2}{m} A^{\frac{2m}{2m}} \left[ \left( \frac{2}{m} - 1 \right) A_{x} A_{x^l} + A A_{x^l}y^l \right]. \]
By (11), (12) and (13), we have (7). The converse is trivial.

Now, suppose that $A$ is irreducible. One can rewrite (7) as follows
\[ A(2A_{x^l} - A_{0l}) = \left( \frac{2}{m} - 1 \right) A_{l} A_{0}. \]
Irreducibility of $A$ and $\deg(A_l) = m - 1$ imply that there exists a 1-form $\theta = \theta_l y^l$ on $U$ such that
\[ A_0 = \theta A. \]
Plugging (15) into (7), we get
\[ A_{0l} = A\theta_l + \theta A_l - A_{x^l}. \]
Substituting (15) and (16) into (7) yields (8). The converse is a direct computation. This completes the proof.

Corollary 3.1. Let $F = \sqrt{A}$ be a Riemannian metric on open subset $U \subset \mathbb{R}^n$. Then, $F$ is a locally dually flat metric if and only if there exists a 1-form $\theta$ on $U$ such that the following holds

$$A_{xl} = \frac{1}{3}(\theta A)_{yl}. \quad (17)$$

In this case, the spray coefficients of $F$ are given by

$$G^i = \frac{1}{12} \theta^i F^2 + \frac{1}{6} \theta^i y^i, \quad (18)$$

where $\theta^i = 2A^{ik} \theta_k$.

Proof. Putting $m = 2$ in (8), we get (17). Plugging $A = a_{ij} y^i y^j$ into (17), one can obtain

$$3 \frac{\partial a_{ij}}{\partial x^l} = \theta_l a_{ij} + \theta_i a_{lj} + \theta_j a_{il}. \quad (19)$$

Substituting (19) into (10), we get (18).

4 Proof of Theorem 1.2

In this section, we deal with Antonelli metrics. First, we remark the following.

Lemma 4.1. (17) Let $F$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Then the spray coefficients of $F$ are given by

$$G^i = \frac{1}{2}(A_{0j} - A_{x^j}) A^i_j. \quad (20)$$

Now, we are going to prove Theorem 1.2. First, we prove the following.

Lemma 4.2. Let $F$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. If $F$ is an Antonelli metric, then the following holds

$$A_{xl} = [\Gamma^i_{ik} y^k + \frac{1}{2} \Gamma^i_{jk,l} y^j y^k] A_i, \quad (21)$$

where $\Gamma^i_{jk,l} = \frac{\partial \Gamma^i_{jk}}{\partial y^l}.$

Proof. Let $F$ be an Antonelli metric metric. Then

$$G^i = \frac{1}{2} \Gamma^i_{jk} (y) y^j y^k. \quad (22)$$

Plugging (22) into (20), we get

$$\Gamma^i_{jk} y^j y^k = (A_{0j} - A_{x^j}) A^i_j. \quad (23)$$
Multiplying (22) with $A_{il}$ and $A_i$ implies that
\begin{align*}
\Gamma^i_{jk}y^jy^kA_{il} &= A_{0l} - A_{x^i}, \\
\Gamma^i_{jk}y^jy^kA_i &= A_0.
\end{align*}
(24) (25)
Differentiating (25) with respect to $y^l$, we have
\begin{align*}
[\Gamma^i_{jk,l}y^jy^k + 2\Gamma^i_{lk}y^k]A_i + \Gamma^i_{jk}y^jy^kA_{il} &= A_{0l} + A_{x^i}.
\end{align*}
(26)
Subtracting (26) from (24) yields
\begin{align*}
A_{x^i} &= [\Gamma^i_{jk}y^j + \frac{1}{2}\Gamma^i_{jk,l}y^jy^k]A_i.
\end{align*}
(27)
Then we get the proof.

**Lemma 4.3.** Let $F$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that $F$ satisfies the following equation
\begin{align*}
A_{x^i} &= [\Gamma^i_{jk}y^j + \frac{1}{2}\Gamma^i_{jk,l}y^jy^k]A_i,
\end{align*}
(28)
where $\Gamma^i_{jk}$ are functions only of direction. Then, $F$ is an Antonelli metric.

**Proof.** Now suppose that (28) holds. By differentiating of (28) with respect to $y^h$, we have
\begin{align*}
A_{x^i}y^h &= [\Gamma^i_{jk,h}y^k + \Gamma^i_{lh} + \frac{1}{2}\Gamma^i_{jk,l,h}y^jy^k + \Gamma^i_{hk,l}y^k]A_i \\
&\quad + [\frac{1}{2}\Gamma^i_{jk,l}y^jy^k + \Gamma^i_{lk}y^k]A_{ih}.
\end{align*}
(29)
Contracting (29) with $y^l$ and 0-homogeneity of functions $\Gamma^i_{lk}$ yield
\begin{align*}
A_{0h} &= [\frac{1}{2}\Gamma^i_{lk,h}y^k + \Gamma^i_{lh}y^l]A_i + \Gamma^i_{lk}y^k A_{ih}.
\end{align*}
(30)
Substituting (28) into (30) implies that
\begin{align*}
A_{0h} &= A_{x^i} + \Gamma^i_{lk}y^k y^l A_{ih}.
\end{align*}
(31)
Multiplying (31) with $A^{bij}$, we have
\begin{align*}
\Gamma^i_{lk}y^k y^l &= (A_{0h} - A_{x^i})A^{bij}.
\end{align*}
(32)
By (20) and (32), one can obtain that
\begin{align*}
G^j &= \frac{1}{2}\Gamma^i_{lk}y^k y^l.
\end{align*}
(33)
This means that $F$ is an Antonelli metric.

**Proof of Theorem 1.2** By Lemma 4.2 and Lemma 4.3 we get the proof of Theorem 1.2.
5 Proof of Theorem 1.3

Proof of Theorem 1.3: Let $F = \sqrt{A}$ be an $m$-th root Finsler metric and be of isotropic mean Berwald curvature, i.e.,

$$E_{ij} = \frac{n+1}{2} cF^{-1}h_{ij}, \quad (34)$$

where $c = c(x)$ is a scalar function on $M$. A direct computation implies that the angular metric $h_{ij} = g_{ij} - F^2g_{yi}g_{yj}$ are given by the following

$$h_{ij} = \frac{A^{\frac{n-2}{2}}}{m^2} [mA_{ij} + (1-m)A_iA_j]. \quad (35)$$

Plugging (35) into (34), we get

$$E_{ij} = \frac{(n+1)A^\frac{n-2}{2}}{2m^2A^2} c[mA_{ij} + (1-m)A_iA_j]. \quad (36)$$

By (20), one can see that $E_{ij}$ is rational with respect to $y$. Thus, (36) implies that $c = 0$ or

$$mA_{ij} + (1-m)A_iA_j = 0. \quad (37)$$

Suppose that $c \neq 0$. Contracting (37) with $A^j_k$ yields

$$mA\delta^k_i - A_iy^k = 0,$$

which implies that $mnA = mA$. This contradicts our assumption $n \geq 2$. Thus $c = 0$ and consequently $E_{ij} = 0$. \hfill \Box

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