THE SEARCH FOR THE ORIGINS OF M-THEORY:
LOOP QUANTUM MECHANICS, LOOPS/STRINGS
AND BULK/BOUNDARY DUALITIES

Carlos Castro*

* Center for Theoretical Studies of Physical Systems, Clark Atlanta University,
Atlanta, Georgia 30314 USA

ABSTRACT

The construction of a covariant Loop Wave functional equation in a 4D spacetime is attained by introducing a generalized eleven dimensional categorical C-space comprised of $8 \times 8$ antisymmetric matrices. The latter matrices encode the generalized coordinates of the histories of points, loops and surfaces combined. Spacetime Topology change and the Holographic principle are natural consequences of imposing the principle of covariance in C-space. The Planck length is introduced as a necessary rescaling parameter to establish the correspondence limit with the physics of point-histories in ordinary Minkowski space, in the limit $l_P \to 0$. Spacetime quantization should appear in discrete units of Planck length, area, volume .....All this seems to suggest that the generalized principle of covariance, representing invariance of proper area intervals in C-space, under matrix-coordinate transformations, could be relevant in discovering the underlying principle behind the origins of M theory. We construct an ansatz for the $SU(\infty)$ Yang-Mills vacuum wavefunctional as a solution of the Schroedinger Loop Wave equation associated with the Loop Quantum Mechanical formulation of the Eguchi-Schild String . The Strings/Loops (SU(\infty) gauge field) correspondence implements one form of the Bulk/Boundary duality conjecture in this case.

*Electronic mail address: castro@hubble.cau.edu
1. INTRODUCTION

Recently, a very interesting relation between string quantization based on the Schild string path integral and the Nambu-Goto string path integral was established at the semi-classical level by a saddle point evaluation method. Quantum mechanics of Matrices, M(atrix) models theory have been conjectured as a leading candidate to understand nonperturbative string theory: M theory.

We will follow another approach related to ordinary Matrix Quantum Mechanics and study Quantum Mechanics but in Loop spaces. The Eguchi quantization of the Schild string is essentially a sort of quantum mechanics formulated in the space of loops. The authors have constructed a Loop Wave equation associated with the Eguchi-Schild string quantization in the Schroedinger form, where the area spanned by the evolution of a closed spatial string (loop) served as the role of the “time parameter” and the holographic shadows of the loop-shapes, onto the spacetime coordinate planes, \( \sigma^{\mu\nu}(C) \), played the role of space coordinates.

It was argued by that the large scale properties of the string condensate, within the framework of Loop Quantum Mechanics, are responsible for the effective Riemannian geometry of spacetime at large distances. On the other hand, near Planck scales, the condensate “evaporates” and what is left behind is a “vacuum” characterized by an effective fractal geometry.

What is required now is to covariantize the Schroedinger Loop wave equation by including the areal time \( A \) and the “spatial coordinates” \( \sigma^{\mu\nu}(C) \) into one single footing. We will see that the construction of a covariant Loop Wave functional equation in a 4D spacetime, a Klein-Gordon Loop Wave equation, is attained by introducing a generalized eleven dimensional categorical C-space comprised of \( 8 \times 8 \) antisymmetric matrices. The latter matrices encode the generalized coordinates of the histories of points, loops and surfaces combined. The points correspond to the center of mass coordinates of a loop/surface. The areal time \( A \) and \( \sigma^{\mu\nu}(C) \) will also be a part of those \( 8 \times 8 \) antisymmetric matrices. We find that Spacetime Topology change and the Holographic principle are natural consequences of imposing the principle of covariance in C-space. The Planck length must be introduced as a necessary rescaling parameter to establish the correspondence limit with the physics of point-histories in ordinary Minkowski space, in the limit \( l_P \to 0 \); i.e when the loops and
surfaces shrink to a point, the field theory limit.

In section II we will review very briefly the Loop Quantum Mechanics of the Schild-string and write down the Schroedinger Loop Wave equation. In III we construct boundary wavefunctional solutions to the Loop Wave equation by evaluating the phase path integral over open surfaces with boundary C. In IV we propose an ansatz for the SU(∞) Yang-Mills vacuum wavefunctional as a solution of the Schroedinger Loop Wave equation associated with the Loop Quantum Mechanical formulation of the Eguchi-Schild String. This String/Loop (SU(∞) gauge field) correspondence implements one form of the Bulk/Boundary duality conjecture in this case.

Finally in V we write down the Covariant Loop Wave Functional equation in the Klein Gordon form (a classical master field theory) and we argue how Moyal-Fedosov deformation quantization, if, and only if, it is applicable, may be suited to quantize the classical master field theory. We will find that the search for a covariant Loop Wave equation where areal time A and holographic shadows, \( \sigma^{\mu} \), are on the same footing, requires to embed the ordinary string theory in spacetime into the categorical C-space comprised of point, loop and surface histories. All this seems to suggest that the generalized principle of covariance, representing invariance of proper area intervals in this C-space (whose elements are antisymmetric matrices), under matrix-coordinate transformations, could be relevant in discovering the underlying principle behind the origins of M theory.

2. LOOP QUANTUM MECHANICS

We shall present a very cursory description of Eguchi’s areal quantization scheme of the Schild string action. For further details we refer to [2]. The starting Lagrangian density is:

\[
L = \frac{1}{4} \{X^\mu, X^\nu\} \{X_\mu, X_\nu\}. \quad X^\mu(\sigma^0, \sigma^1). \tag{2.1}
\]

\( X^\mu(\sigma^0, \sigma^1) \) are the embedding string coordinates in spacetime and the brackets represent Poisson brackets w.r.t the \( \sigma^0, \sigma^1 \) world sheet coordinates. The Schild action is just the areal squared of the world sheet instead of the area interval described by the Dirac-Nambu-Goto actions. The corresponding Schild action is only invariant under area-preserving transformations.
Eguchi’s areal quantization scheme is based on keeping the area of the string histories fixed in the path integral and then taking the averages over the string tension values. The Nambu-Goto approach, on the other hand, keeps the tension fixed while averaging over the world sheet areas. If one wishes to maintain the analog of quantum mechanics of point particles and that of loops, the Eguchi quantization scheme of the Schild action requires the following Correspondence principle between area-momentum variables and time (areal time) $A$:

$$P_{\mu\nu}(s) \rightarrow \frac{1}{\sqrt{x'(s)^2}} \frac{\delta}{\delta x'^{\mu}(s) \delta x'^{\nu}(s)} \quad H \rightarrow -i \frac{\partial}{\partial A} \quad \hbar = 1. \quad (2.2)$$

The Loop Schroedinger-like Wave equation $H\Psi = -i(\partial/\partial A)\Psi$ is obtained once we extract the areal time independence, $\Psi = e^{-iEA}\Psi$.

$$\frac{1}{\lambda_C} \int_0^1 ds \sqrt{x'(s)^2} \frac{-\delta^2}{4m^2 \delta x'^{\mu}(s) \delta x'^{\nu}(s)} \Psi[\sigma^{\mu\nu}(C)] = E\Psi[\sigma^{\mu\nu}(C)]. \quad (2.3)$$

where the string wave functional $\Psi[\sigma^{\mu\nu}(C)]$ is the amplitude to find the loop $C$ with area-elements $\sigma^{\mu\nu}$ (holographic shadows) as the only boundary of a two-surface of internal area $A$ in a given quantum state $\Psi$.

Plane wave solutions and Gaussian wavepackets (superposition of fundamental plane wave solutions) were constructed by freezing all the modes, associated with the arbitrary loop shapes, except the constant momentum modes on the boundary. The plane wave solutions were of the type:

$$\Psi[\sigma^{\mu\nu}(C)] \sim exp \left[ iE - i \int_C x'^{\mu}(s) dx'^{\nu}(s) P_{\mu\nu} \right]. \quad (2.4)$$

A loop space momentum/string shape-uncertainty principle was given as:

$$\Delta \sigma^{\mu\nu} \Delta P_{\mu\nu} \geq 1. \quad (2.5)$$

in suitable units.

Having presented this very brief review of the Schroedinger Loop Wave equation we shall elaborate further details in the next sections and construct more general solutions than the plane wave case by evaluating the Schild string phase space path integral in III. In IV we propose our ansatz for the $SU(\infty)$ YM vacuum wavefunctional as another
candidate solution to the Schrödinger Loop Wave equation. And in the final section we present the covariant Loop Wave functional equation and its relation to the plausible origins of $M$ theory.

3. BOUNDARY WAVE FUNCTIONAL EQUATIONS

We shall construct a boundary wavefunctional solution to the Loop Schrödinger-like wave equation (besides the standard plane wave solution) which is associated with a Riemann surface of arbitrary topology and with one boundary. For simplicity we shall start with a Riemann surface of spherical topology, with the boundary, $C$, parametrized by the curve: $x^\mu(s)$. Writing the Schild action in Hamiltonian form allows one to define a boundary wavefunctional solution to the Loop Schrödinger-like wave equation, by:

$$
\Psi[\sigma^{\mu\nu}(C)] \equiv \int_C [DX^\mu(\sigma)] \int [DP_{\mu a}(s)] \int_{\Pi_{\mu a}|\partial \Sigma = P_{\mu a}} [D\Pi_{\mu a}(\sigma)]
exp \left[ i \int \frac{1}{2} \Pi_{\mu a} d\Sigma^{\mu\nu}(\sigma) - i \int d^2 \sigma \frac{\Pi_{\mu a}^2}{4m^2} \right]
$$

The physical meaning of $\Psi[\sigma^{\mu\nu}(C)]$ is the probability amplitude of finding any given loop $C$ with area-components (holographic shadows onto the spacetime coordinate planes) $\sigma^{\mu\nu}$ in a given quantum state $\Psi$.

One is performing the canonical path integral, firstly, by summing over all bulk momentum configurations of the string world sheet with the restriction that on the boundary they equal the pre-assigned value of the boundary momentum, $P^{\mu\nu}(s)$, and afterwards, one performs the summation over all values of the boundary momentum, $P^{\mu\nu}(s)$. It is important to emphasize, that one is also summing over all string world sheet embeddings, maps from a world sheet with a disk topology to a target spacetime, $X^\mu(\sigma)$ such that maps $X^\mu$ restricted to boundary of the disk are equal to $x^\mu(s)$. The latter are just the spacetime parametrization coordinates of any free loop $C$; i.e the boundary shapes, $C$, are completely arbitrary.

Furthermore, since we are using the Schild action which is not fully reparametrization invariant (only under area-preserving diffs), we are not performing the functional integral over a family of world sheet auxiliary metrics. In we carried out such functional integral.

The world sheet conjugate area-momentum is defined as the pullback of the target spacetime area conjugate momentum:
\[ \Pi^a_{\mu} \equiv \Pi_{\mu \nu} \epsilon^{ab} \partial_b X^\nu (\sigma^a). \]  

(3.2)

and:

\[ \Pi_{\mu a} \Pi^\mu_a = \Pi_{\mu \nu} \partial_b X^\nu \Pi^{\mu \rho} \partial^a X_\rho = \Pi_{\mu \nu} \Pi^{\mu \rho} \delta_\rho^\nu = \Pi_{\mu \nu} \Pi^{\mu \nu}. \]  

(3.3)

which was used in the definition of the Schild Hamiltonian. Where the bulk momentum is restricted to obey the boundary condition:

\[ \Pi_{\mu \nu} (\sigma^a)|_{\partial \Sigma} = P_{\mu \nu} (s). \quad \Pi_{\mu a}|_{\partial \Sigma} = P_{\mu a} (s) \]  

(3.4)

The area components enclosed by the arbitrary boundary curve \( x^\mu (s) \) is given by Stokes theorem:

\[ \sigma^{\mu \nu} (C) = \frac{1}{2} \oint_C (x^\mu dx^\nu - x^\nu dx^\mu) \]  

(3.5)

An integration by parts of the integral

\[ i \int d^2 \sigma \Pi_{\mu \nu} d\Sigma^{\mu \nu} (\sigma) \]  

(3.6a)

yields

\[ i \int_C P_{\mu \nu} d\sigma^{\mu \nu} (s) - i \int d^2 \sigma X^\mu \{ \Pi_{\mu \nu}, X^\nu \} \]  

(3.6b)

where the Poisson bracket is taken with respect to the world sheet coordinates \( \sigma^0, \sigma^1 \).

The Poisson bracket in the second integrand of eq-(3-6b ) can be written in terms of the canonical conjugate variables : \( X^\mu, \Pi_{\mu a} \) because:

\[ X^\mu \{ \Pi_{\mu \nu}, X^\nu \} = X^\mu \epsilon^{ab} \partial_a \Pi_{\mu \nu} (\sigma) \partial_b X^\nu (\sigma) = X^\mu \epsilon^{ab} \partial_a [ \Pi_{\mu \nu} (\sigma) \partial_b X^\nu (\sigma) ] = X^\mu \partial_a \Pi^a_{\mu} \]  

(3.7)

due to the condition:

\[ \epsilon^{ab} \partial_a \partial_b X^\nu = 0. \]  

(3.8)

and after using the definition of the world sheet canonical momentum; i.e the pullback of \( \Pi_{\mu \nu} \) to the world sheet. Notice that the integrand given by eq-(3-7 ) which is to appear in
the exponential weight of the path integral depends solely on $X^\mu$ and $\Pi_{\mu a}$. Upon doing firstly the functional integration with respect to all the string embeddings $X^\mu$, of a world sheet of a disk topology into spacetime, one will produce a delta functional constraint which is nothing but the classical Schild string equations of motion:

$$\delta(K_\mu). \quad K_\mu = \{\{X_\mu, X_\nu\}, X^\nu\} = \{\Pi_{\mu \nu}, X^\nu\} = \partial_a \Pi_{\mu a}^0 = 0. \quad (3.9)$$

where we have made use of eqs-(3-1, 3-6b, 3-7, 3.8) . The on-shell equations of motion are then equivalent to a divergence-free condition on the area-momentum variables:

$$\partial_a \Pi_{\mu a} = 0. \quad (3.10)$$

Therefore, the $X^\mu$ path integral imposes the *bulk worldsheet* classical Schild string equations of motion. This is tantamount of minimizing the proper world sheet area or *areal* time. A congruence (or family) of classical string configurations is usually parametrized by the boundary data: $x^\mu(s);P_{\mu a}(s)$. This is the string version of the classical free particle motion. One requires to specify the initial position and velocity of the particle to determine the trajectory.

We shall study now the *quantum* boundary string dynamics *induced* by the classical world sheet dynamics. Later, we will extend this procedure to a membrane and all $p$-branes. The boundary wavefunctional becomes:

$$\Psi[\sigma^{\mu \nu}(C)] = \int [DP_{\mu a}(s)] \exp \left[ i \oint_C \frac{1}{2} P_{\mu \nu} d\sigma^{\mu \nu}(s) \right] \int_{\Pi_{\mu a}|_{\partial\Sigma=P_{\mu a}}} [D\Pi_{\mu a}^{class}(\sigma)] \exp \left[ -i \int d^2 \sigma \frac{(\Pi_{\mu a}^{class})^2}{4m^2} \right] \quad (3.11)$$

The other alternative way to proceed, is firstly by *covariantizing* the Schild action by introducing auxiliary two-dim metrics, $g^{ab}$. We explicitly performed such *phase* space path integral in [1] . It was also required to find the moduli space of solutions of the classical Schild string equations of motion, $\partial_a \Pi_{\mu a} = 0$, and then to integrate over the moduli. Since in two dimensions, a vector is dual to a scalar field, the solutions for $\Pi_{\mu a}$
could be expressed in terms of a set of scalar fields and, in this fashion, we made contact with a bulk Conformal Field Theory. A further functional integral with respect to the auxiliary two-metrics, $g_{ab}$, was also necessary. We were able to compute explicitly the bulk momentum path integral and to establish a duality relation between loop states (living on $C$) and string states (living on the bulk) through a functional loop transform.

From this last expression (3-11) we can recognize immediately that the boundary area-wavefunctional $\Psi[\sigma^{\mu\nu}(C)]$ is nothing but the loop transform of the boundary momentum wavefunctional which is defined as:

$$\tilde{\Psi}[P_{\mu a}] \equiv \int_{P_{\mu a}} [D\Pi_{\mu a}(\sigma)]_{\text{class}} \exp \left[ -i \int_{\Sigma} d^2 \sigma \left( \frac{\Pi^{\text{class}}_{\mu a}}{4m^2} \right)^2 \right]. \quad (3-12)$$

This bulk momentum path integral defined by eq-(3.12), was explicitly evaluated in [1], when we computed the covariantized Schild action path integral in phase space. We found that the phase space path integral factorizes into a product of an Eguchi wavefunctional, encoding the boundary dynamics, and a bulk path integral term of the Polyakov type, with induced scalar curvature and cosmological constant on the bulk and an induced boundary cosmological constant and extrinsic curvature on the boundary. The final result of the functional loop transform that maps $\tilde{\Psi}[P_{\mu a}]$ into $\Psi[\sigma^{\mu\nu}(C)]$ is [1]:

$$\Psi[\sigma^{\mu\nu}(C)] = \int_0^\infty dA \ e^{iA} \Psi^{\text{Eguchi}}[\sigma^{\mu\nu}(C); A] \ Z_{\text{Bulk}}^A[\sigma^{\mu\nu}(C)] \quad (3-13a)$$

and the loop functional was expressed in terms of the holographic coordinates $\sigma^{\mu\nu}(C)$ and the loop coordinates $x^\mu(s)$ as follows:

$$\Psi[x^\mu(s), \sigma^{\mu\nu}(C)] \sim e^{iS_{\text{eff}}[C]} \Psi[\sigma^{\mu\nu}(C)]. \quad (3.13b)$$

Notice that $\Psi[x^\mu(s), \sigma^{\mu\nu}(C)]$ is a functional of two arguments and this differs from the standard construction of the $\Psi[C]$ in the literature. $S_{\text{eff}}[C]$ is the boundary effective action induced by the quantum fluctuations of the string world sheet. It is a local quantity written in terms of the counterterms needed to cancel the boundary ultraviolet divergent terms. $\Lambda$ is the world sheet cosmological constant. $A$ is the world sheet area. Such path integral required to evaluate the covariantized Schild action,

$$\int d^2 \sigma \sqrt{h} \left( \frac{\Pi^{\text{class}}_{\mu a} \Pi^{\text{class}}_{\nu a}}{4m^2} \right)^2. \quad (3.14)$$
over each classical momentum trajectory parametrized by the pre-assigned boundary momentum, \( P_{\mu a}(s) \), and summing over all possible values of the bulk area of the world sheet and auxiliary \( h_{mn} \) metrics. In this fashion one eliminates any dependence on the bulk world sheet area. At the end, in order to get \( \Psi[\sigma^{\mu \nu}(C)] \), one must sum over all the congruence of classical trajectories by integrating over all values of the boundary momentum and auxiliary metrics (einbeins) induced on the boundary. To regularize ultraviolet infinities requires the introduction of the curvature scalar and cosmological constant on the bulk and the extrinsic curvature and boundary cosmological constant.

Hence, the boundary area-wavefunctional \( \Psi[\sigma^{\mu \nu}(C)] \) associated with the area-components enclosed by a closed loop \( C \) is nothing but the analog of the Wilson loop transform of the boundary momentum wave functional:

\[
\Psi[\sigma^{\mu \nu}(C)] \equiv \int [DP_{\mu a}(s)] \exp \left[ i \oint_C \frac{1}{2} P_{\mu \nu}(s) d\sigma^{\mu \nu}(s) \right] \tilde{\Psi}[P_{\mu a}(s)].
\] (3.15)

and vice versa, the inverse Fourier/Loop transform:

\[
\tilde{\Psi}[P_{\mu a}(s)] \equiv \int [D\sigma^{\mu \nu}] \exp \left[ -i \oint_C \frac{1}{2} P_{\mu \nu}(s) d\sigma^{\mu \nu}(s) \right] \Psi[\sigma^{\mu \nu}].
\] (3.16)

yields the boundary momentum wavefunctional \( \tilde{\Psi}[P_{\mu a}(s)] \) in terms of the boundary area-wavefunctional \( \Psi[\sigma^{\mu \nu}(C)] \) with:

\[
P_{\mu a}(s) \equiv \Pi_{\mu a}[\partial \Sigma] = (\Pi_{\mu \nu} \partial_a X^\nu) |_{\partial \Sigma}.
\] (3.17)

It is important to have the precise computation of the \( \tilde{\Psi}[P_{\mu a}(s)] \) given by eq-(3.12) in order to find \( \Psi[\sigma^{\mu \nu}(C)] \) as its Fourier/functional loop transform. In general, there is an ambiguity in assigning in a unique fashion a string shape \( C \) to a given set of area-components (Plucker coordinates), \( \sigma^{\mu \nu}(C) \). There could be two or more loop-shapes that have the same shadows or coordinates, \( \sigma^{\mu \nu} \). The Plucker conditions in \( D = 4 \) read:

\[
\epsilon^{\alpha \beta \mu \nu} \sigma_{\alpha \beta} \sigma_{\mu \nu} = 0.
\] (3.18)

to reassure us that there is a one-to-one correspondence between the loop \( C \) and its Plucker (area components) coordinates. The classical Schild action is invariant under temporal area-preserving diffeomorphisms (it is not fully reparametrization invariant), and one
must not confuse this invariance with that of spatial area-preserving diffs: invariance of \((\sigma^{\mu\nu})^2\). One would expect the invariance of the boundary wave functional, under the action of (temporal) area-preserving diffs, to be preserved in the quantum theory. The issue of area-preserving-diffs anomalies will be the subject of future study.

It is straightforward to verify that \(\Psi[\sigma^{\mu\nu}(C)]\) defined by eq-(3-15), is a solution of the Loop Schroedinger-like Wave equation \(H\Psi = -i(\partial/\partial A)\Psi\) once we extract the areal time independence, \(\Psi = e^{-iE/A}\Psi\).

\[
\frac{1}{l_C} \int_0^1 ds \sqrt{x'(s)^2 - \delta^2} \frac{P^{\mu a}(s)P^a_{\mu}(s)}{4m^2} \Psi[\sigma^{\mu\nu}(C)] = \mathcal{E}[\sigma^{\mu\nu}(C)].
\]

(3.19)

where the on-shell dispersion relation for the Eguchi-Schild string is:

\[
\mathcal{E} = \frac{1}{l_C} \int_0^1 ds \sqrt{x'(s)^2} \frac{P^{\mu a}(s)P^a_{\mu}(s)}{4m^2}.
\]

(3.20)

with \(l_C\) being the reparametrization invariant length of the loop, \(C\).

The expectation value of the Hamiltonian Operator, in the state \(\Psi[\sigma^{\mu\nu}(C)]\), after inserting the expression (3-15), and using the definition of \(\delta[P^{\mu a} - P'^{\mu a}]\), is simply:

\[
< \mathcal{H} >_\Psi = \frac{1}{l_C} \int_0^1 ds \sqrt{x'(s)^2} \int [D\sigma^{\mu\nu}(s)]\Psi^*[\sigma^{\mu\nu}(C)](\frac{-\delta^2}{4m^2\delta\sigma^{\mu\nu}\delta\sigma_{\mu\nu}})\Psi[\sigma^{\mu\nu}(C)] =
\]

\[
\frac{1}{l_C} \int_0^1 ds \sqrt{x'(s)^2} \left\{ \int [DP^{\mu a}(s)]\Psi^*[P^{\mu a}(s)] \frac{P^{\mu a}(s)P^{\mu a}(s)}{4m^2} \Psi[P^{\mu a}(s)] \right\} =
\]

\[
\frac{1}{l_C} \int_0^1 ds \sqrt{x'(s)^2} < P^{\mu a}(s)P^{\mu a}(s) >_{\Psi(P)} = \mathcal{E}.
\]

(3.21)

where the quantity under the brackets in eq-(3.21) is the defining expression for the averages of \(P^{\mu a}(s)P^{\mu a}(s) / 4m^2\) over the boundary momentum quantum states. As expected we recover the on-shell dispersion relation for the Eguchi-Schild string if, and only if,

\[
< \frac{P^{\mu a}(s)P^{\mu a}(s)}{4m^2} >_{\Psi(P)} = \frac{P^{\text{class}}_{\mu a}(s)P^{\text{class}}_{\mu a}(s)}{4m^2}.
\]

(3.22)

This is precisely what is expected for a free string; i.e eigenstates of the boundary momentum operator, \(\hat{P}_{\mu a}(s)\). It is important also to impose the normalization condition:
\[
\int [D P_{\mu a}(s)] \bar{\Psi} [P_{\mu a}(s)] \Psi [P_{\mu a}(s)] = 1.
\] (3.23)

Concluding, \( \Psi[\sigma^{\mu \nu}(C)] \) solves the Loop Schroedinger-like wave equation and the expectation value of the Hamiltonian operator is indeed equal to those values of \( \mathcal{E} \) which are consistent with the Eguchi-Schild string on-shell dispersion relation.

For a membrane of spherical topology spanning a world tube \( S^2 \times R \) the higher-loop analog of the loop wave equation is:

\[
\frac{1}{A} \int d^2 \sigma \sqrt{\sigma} \frac{1}{2 \cdot 3 ! m^3} \left( \frac{-\delta^2}{\delta \sigma^{\mu \nu}(S^2) \delta \sigma^{\mu \nu}(S^2)} \right) \Psi[\sigma^{\mu \nu}(S^2)] = \mathcal{E} \Psi[\sigma^{\mu \nu}(S^2)].
\] (3.24)

where \( \sigma^{\mu \nu}(S^2) \) will be the volume components enclosed by the spherical membrane (bubble) \( S^2 \); i.e the projection (shadows) of the volume onto the target spacetime coordinate planes. This higher-loop boundary wave equation corresponds to the quantum dynamics of a Euclidean world sheet or bubble, the sphere \( S^2 \); i.e the boundary of the world tube of the membrane. The \( \mathcal{E} \) eigenvalue has units of membrane tension : energy per unit area (mass\(^3\)). The \( A \) represents the reparametrization invariant area of the bubble (sphere). The action for the membrane is in this case the generalization of the Schild action, a volume squared

\[
\frac{1}{l^3} \int d^3 \sigma \frac{\partial(X^\mu, X^\nu, X^\rho)^2}{\partial(\sigma^0, \sigma^1, \sigma^2)}.
\] (3.25)

The Mechanics associated with p-branes admits the so-called Nambu-Poisson Hamiltonian Mechanics where the ordinary Poisson bracket of two quantities \( \{A, B\} \) is replaced by a multi-bracket : \( \{A, B, C, \ldots\} \) which is essentially the volume form/Jacobian. The subject of Higher-Dimensional Loop Spaces and their algebras is a very complex one. We refer to \[34\] for an introduction.

4. Wavefunctionals of Loops and \( SU(\infty) \) Gauge Theories

In this section we shall construct loop wavefunctionals, \( \Psi[C] \) associated with a given loop and must not be confused with area-wavefunctionals of the previous section : \( \Psi[\sigma^{\mu \nu}(C)] \) associated with a given loop \( C \) with area-components \( \sigma^{\mu \nu}(C) \).

Based on the known observation \[28, \bar{3}] that classical “vacuum” configurations of a \( SU(\infty) \) YM theory (space time independent gauge field configurations) are given by
classical solutions of the Eguchi-Schild string, after the \( A^\mu \leftrightarrow X^\mu(\sigma^0, \sigma^1) \) correspondence is made, we shall construct “vacuum” loop functionals \( \Psi_{\text{vac}}[\gamma] \) using the standard Wilson loop transform associated with the \( SU(\infty) \) YM theory; i.e gauge theory of area-preserving diffs. By “vacuum” one means classical solutions to the \( SU(\infty) \) YM equations of motion for the special case when the \( A^\mu(x^\mu; \sigma^a) \) fields are spacetime independent:

\[
D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \{A_\mu, F^{\mu\nu}\} = 0 \Rightarrow \{A_\mu, \{A^\mu, A^{\nu}\}\} = 0 \leftrightarrow \{X_\mu, \{X^\mu, X^{\nu}\}\} = 0. \quad (4.1)
\]

after the gauge field/string correspondence is made one recovers the classical Schild string equations of motion. From now on we shall use the term “vacuum” to represent the classical solutions of the Schild string and must not be confused with the true vacuum of the theory; i.e the state which has true zero expectation values for all physical observables:

\[
< F^{\mu\nu}_{\text{vac}} > = 0 \text{ and zero energy, } < H > = < F^{\mu\nu}_{\text{vac}} > = 0.
\]

The Wilson loop holonomy operator associated with a \( SU(\infty) \) gauge field, \( A^\mu(x^\mu, \sigma^0, \sigma^1) \) is defined:

\[
W[A^\mu, \gamma] = Tr P \exp \left[ i \oint_\gamma A_\mu dx^\mu \right]_{N \to \infty} = \int d^2 \sigma exp \left[ i \oint_\gamma A_\mu dx^\mu \right] = < \gamma | A^\mu > \quad (4.2)
\]

and its complex conjugate associated with a different contour \( C \) is:

\[
W^*[A^\mu, C] = Tr P \exp \left[ -i \oint_C A_\mu dx^\mu \right]_{N \to \infty} = \int d^2 \sigma exp \left[ -i \oint_C A_\mu dx^\mu \right] = < A^\mu | x^\mu(s) > . \quad (4.3)
\]

we choose the contours \( \gamma, C \) to be different for reasons which will become clear later. For the time being we shall not be concerned with path orderings nor traces. After all, the \( SU(\infty) \) gauge field is now an ordinary number and not a matrix. The trace in the \( SU(\infty) \) YM case is replaced by an integral w.r.t the internal space “color” \( \sigma^a \) coordinates.

There is a signature subtlety if one wishes to identify the \( \sigma \) internal color coordinates with the string worldsheet ones. The former live in a Euclidean internal world whereas the latter in a Minkowskian spacetime. However, if one performs a dimensional reduction of the \( A^\mu \) field to one temporal dimension ( Matrix Models \[3\]) one makes contact with a true physical membrane coordinate : \( A^\mu(t, \sigma^0, \sigma^1) \leftrightarrow X^\mu(t, \sigma^0, \sigma^1) \) because now we
have the correct signature for the world volume of the membrane. A timelike slice of the membrane reproduces a string worldsheet whereas a spatial slice a Euclidean worldsheet. The signature subtleties have been recently studied by [48] and [53].

We turn attention to what we consider one of the important aspects in the essence of duality: \( q \leftrightarrow p \) in the standard Hamiltonian dynamics. It has been a long-sought goal to construct a formulation of extended objects where dualities are already manifest. A formulation of all bosonic p-branes as Composite Antisymmetric Tensor Gauge theories of the volume-preserving diffs where \( S, T \) duality were incorporated from the start was achieved in [11]. For example, in the standard canonical quantization of electromagnetism, \( \vec{A}, \vec{E} = F_{\alpha i} \) are a canonical pair of conjugate variables. In the Schild formulation one has: \( X^\mu, \Pi_{\mu a} \) as a canonical pair which have a correspondence with the \( SU(\infty) \) \( c \)-number variables: \( A^\mu, F_{\mu \nu} \), respectively. The \( c \)-number valued field strength is:

\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \{A_\mu, A_\nu\}_{PB}. \tag{4.4}
\]

the Poisson brackets are taken w.r.t the internal color indices: \( \sigma^0, \sigma^1 \). Moyal deformations are very natural, for recent results on this we refer to [14], [15], [10]. For (vacuum solutions) spacetime independent field configurations one can make direct contact with the Schild string string since the classical YM Lagrangian becomes now equivalent to the Schild one:

\[
F_{\mu \nu}^2 = \{A_\mu, A_\nu\}_{PB} \{A^\mu, A^\nu\}_{PB} \leftrightarrow \{X_\mu, X_\nu\}_{PB} \{X^\mu, X^\nu\}_{PB} \tag{4.5}
\]

Spacetime independent field configurations yield a trivial holonomy factor (since the vector sum of all the tangents along a closed loop is zero) leaving only the internal color space area operator: \( W[A, \gamma] = Area[\int d^2 \sigma] \) as the holonomy. If one wishes to identify the colour coordinates with the world-sheet ones, colour confinement would imply a finite colour-area: a compact world sheet with boundary \( C \) that will be mapped into the Wilson loop \( \gamma \) living in the target spacetime. It is important to remark that if one evaluates the loop derivative on the Wilson loop \( W[A^\mu, \gamma] \) in the vacuum case, one must firstly take the derivatives and afterwards set the values of \( A^\mu = A^\mu(\sigma^a) \) yielding:
\[
\frac{\delta}{\delta x^\mu(s)} W[A^\mu, \gamma] = x'^\nu F_{\mu\nu}(x^\mu(s)) W[A^\mu, \gamma], \quad x'^\nu(s) = \frac{\partial x^\nu(s)}{\partial s}. \quad (4.6)
\]

otherwise one will trivially get zero since the spacetime loop derivative does not affect the area in the internal colour space.

These area and volume operators are very important in Loop Quantum Gravity especially in regards to the fact that these operators have discrete eigenvalues: spacetime is quantized in multiples of Planck areas and Planck volumes. This will not be surprising if the group of area and volume preserving diffs are truly relevant symmetry groups in nature. However, it is important to realize that area in color space is not the same as area in spacetime. Strings and membranes can be interpreted as gauge theories of the area-preserving diffs algebra where the effective dimensions will be \(D + 2\), two internal dimensions (the infinite color space) in addition to the usual ones. If one identifies the internal color directions, \(\sigma^0, \sigma^1\), with an Euclidean worldsheet one makes contact with the Euclideanized string for those spacetime independent gauge field configurations of the \(SU(\infty)\) gauge field. In the case that \(A^\mu(x^\mu; \sigma^a)\) is dimensionally reduced to one temporal dimension one will then make contact with the membrane (this time with the correct signature); i.e with M(atrix) Models.

Therefore, the \(SU(\infty)\) gauge field/Schild string correspondence is then:

\[
W[A^\mu, \gamma] = \int d^2\sigma \exp \left[ i \oint_C A_\mu dx^\mu \right] \leftrightarrow \exp \left[ i \int \Pi_{\mu\nu} d\sigma^{\mu\nu} \right] = \exp \left[ i \oint_C P_{\mu\nu}(s) d\sigma^{\mu\nu}(s) - i \int d^2 \sigma X^\mu \{ \Pi_{\mu\nu}, X^\nu \} \right]. \quad (4.7)
\]

after an integration by parts. The Eguchi-Schild string equations of motion set to zero the last term of (4-7) and one gets (at the classical vacuum level) the correspondence:

\[
W[A^\mu, \gamma] = \text{Area} \int d^2\sigma \leftrightarrow \exp \left[ i \oint_C P_{\mu\nu}(s) d\sigma^{\mu\nu}(s) \right]. \quad (4.8)
\]

The correspondence between the loop transform of \(\Psi[A^\mu]\) and that of the loop momentum boundary wavefunctional of the Eguchi-Schild string is:

\[
\Psi[\gamma] \leftrightarrow \Psi[\sigma^{\mu\nu}(C)]
\]

\[
\int [DA^\mu] W[A^\mu, \gamma] \Psi[A^\mu] \leftrightarrow \int [DP_{\mu\nu}(s)] \exp \left[ i \oint_C P_{\mu\nu}(s) d\sigma^{\mu\nu}(s) \right] \Psi[P_{\mu\nu}(s)]. \quad (4.9)
\]
We emphasize once more that $\Psi[\gamma]$ and $\Psi[\sigma^{\mu\nu}(C)]$ are not the same objects even in the case that $\gamma = C$. There is a correspondence between them. $\Psi[\gamma]$ should obey a suitable $SU(\infty)$ loop equation and $\Psi[\sigma^{\mu\nu}(C)]$ obeys the Loop Schroedinger-like Wave equation discussed in the previous section. We propose that $\Psi[\gamma]$ (for the vacuum) should obey the following Loop Wave equation 2:

$$\frac{1}{l_C} \int_{0}^{1} \frac{ds}{\sqrt{\rho'(s)^2}} \left( \frac{1}{4m^2} \frac{\delta^2}{\delta x^\mu(s) \delta x_\mu(s)} \right) \Psi_{vac}[\gamma] = E \Psi_{vac}[\gamma].$$

(4.10)

Notice that this Loop wave equation clearly differs from the $SU(N)$ loop wave equation in the literature 15. In the $N \to \infty$ limit, the standard loop equation would require a wavefunctional of an infinite number of loops $\Psi[\gamma_1, \gamma_2, \ldots]$ . Whereas, the Loop equation above (4.10) requires one loop only. It can be generalized to many loops (closed strings) when interactions are included. What allows us to avoid the problem of using an infinite number of loops is precisely when we impose the spacetime independent (vacuum) $SU(\infty)$ YM gauge field configurations / Eguchi-Schild string correspondence.

Such gauge field/string correspondence has been discussed by many authors in particular by 22 pertaining strings, loops, knots, quantum gravity and BF theories and also by 24. The former authors have shown that the large $N$ limit of QCD in $D = 2$ corresponds to a string theory, although it was never established which particular string theory it actually referred to. The action for the string theory that reproduces the partition function of the large $N$ QCD in $D = 2$ was never constructed. The quantum YM partition function could be matched with an infinite sum of branched string coverings (with singularities) of a family of Riemann surfaces of arbitrary genus, $g$ to a compact Riemann surface of fixed genus $G$ and fixed area. The latter is the compact Riemann surface where the original $SU(\infty)$ YM field theory lived in.

Hence, we propose that the Wilson loop transform of the vacuum $SU(\infty)$ YM wavefunctional, $\Psi_{vac}[A^\mu] \to \Psi_{vac}[\gamma]$ is a solution of the Schild string Loop Wave equation described by eq-(4.10). Furthermore, due to the $A^\mu \leftrightarrow X^\mu$ correspondence, the $\Psi_{vac}[\gamma]$ can also be obtained by performing the Schild string path integral over a Riemann surface of spherical topology with one boundary equal to the closed loop $\gamma$. 


\[
\int_{SU(\infty)} [DA^\mu] W[A^\mu, \gamma] \Psi_{\text{vac}}[A^\mu] = \Psi_{\text{vac}}[\gamma] = \int_{X^\mu[\partial \Sigma = \gamma]} [DX^\mu(\sigma)] e^{iS_{\text{Schild}}[X^\mu]}.
\] (4.11)

the family of maps \(X^\mu(\sigma)\) from a Riemann surface of a disk topology to spacetime is restricted to obey a boundary condition: the surface must have as boundary the loop \(\gamma\).

The string path integral can be evaluated \textit{perturbatively} (approximately). However, one will not recapture the nonperturbative string physics in that fashion. String perturbative corrections are obtained as usual by summing over all surfaces of arbitrary genus. The path integral can be evaluated in different steps: we fix first the genus and area, then sum over all areas and afterwards over all genus. Auxiliary metrics are introduced in the \textit{covariantized} Schild action. The latter is on-shell equivalent to the Nambu-Dirac-Goto and Polyakov actions. A sum over all metrics is then performed and in this way the path integral is computed modulo the action of the diffs and Weyl group. At the end one ends up with integrals in the Moduli space \(\mathcal{M}_{h,1}\) of Riemann surfaces with \(h\) handles and one puncture (associated with one boundary). The invariant measure in the moduli space is obtained by means of the \(b, c\) ghost contributions (Faddev-Popov determinants).

Noncritical string dimensions require the Liouville field.

Inserting \(\Psi_{\text{vac}}[\gamma]\) provided by the standard Wilson loop transform, left hand side of (4.11), into the Loop Wave equation (4.10) yields for the expectation values of the YM Hamiltonian (that corresponds to the Schild Hamiltonian) after using eq-(4.5):

\[
\frac{1}{l_c} \int_0^1 ds \sqrt{x'(s)^2 + \frac{1}{4m^2}} \int [DA^\mu] \Psi_{\text{vac}}^*[A^\mu] \left(g_{YM}^2 \frac{F_{\mu\nu}^2[x^\mu(s)]}{g_{YM}^2} \right) \Psi_{\text{vac}}[A^\mu] = \mathcal{E}_{\text{vac}}.
\] (4.12)

and one obtains that the \textit{vacuum} expectation value of the \(SU(\infty)\) YM energy density, \((F_{\mu\nu}^2[x^\mu(s)]/g_{YM}^2)\), restricted to the values on the loop \(x^\mu(s)\), is precisely proportional to the Schild string tension. This is due to the fact that \(\mathcal{E}\) has units of energy per unit length which is a tension. We have introduced the YM coupling, \(g_{YM}\) in the form \(g_{YM}^2(F_{\mu\nu}^2/g_{YM}^2)\) to match units appropriately. The latter term has units of \textit{length}^{-4} as it should in order that the l.h.s of (4.12) has the units of a string tension. For recent work on the QCD vacuum wavefunctional and the large \(N\) ’t Hooft’s expansion in relation to strings see \cite{38, 37}. 

16
Inserting $\Psi_{\text{vac}}[\gamma]$ provided by the right hand side of (4-11) yields a solution of the Loop Wave equation after one identifies the $\Psi_{\text{vac}}[\gamma]$ (up to a normalization constant $N$ of units $L^{-1/2}$ to absorb the units coming from the $X^\mu$ integral) with the boundary effective action induced by the bulk quantum dynamics:

$$\Psi_{\text{vac}}[\gamma] = e^{iS_{\text{eff}}[x^\mu(s)]} \equiv \int_{X^\mu[\partial \Sigma = \gamma]} [DX^\mu(\sigma)] e^{iS_{\text{Schild}}[X^\mu]}.$$  \hspace{1cm} (4.13)

In ordinary $SU(2)$ YM gauge theories in $D = 4$, the Chern-Simmons wavefunctional $\Psi[A] = e^{iS_{\text{CS}}[A]}$ (the boundary term) is used to evaluate the loop transform from $\Psi[A] \rightarrow \Psi[\gamma]$. In the $SU(\infty)$ case we shall use the boundary effective action induced by bulk Schild string quantum dynamics.

Plugging the above solution (4-13), into the Loop Wave equation yields for the expectation value of the Schild Hamiltonian density:

$$\frac{1}{l_C} \int_0^1 \frac{ds}{\sqrt{x'(s)^2}} \frac{1}{4m^2} \int [Dx^\mu(s)]N^2 \left[ \frac{\delta^2 S_{\text{eff}}[x^\mu(s)]}{\delta x^\mu \delta x_\mu} + \left( \frac{\delta S_{\text{eff}}[x^\mu(s)]}{\delta x^\mu} \right)^2 \right] = \mathcal{E}_{\text{vac}}.$$  \hspace{1cm} (4.14)

The $\left( \frac{\delta S_{\text{eff}}[x^\mu(s)]}{\delta x^\mu} \right)^2$ are the standard momentum squared terms (to order $\hbar^0$), kinetic terms of the Hamilton-Jacobi equation and the $\frac{\delta^2 S_{\text{eff}}[x^\mu(s)]}{\delta x^\mu \delta x_\mu}$ are the so-called “quantum potential” terms (to order $\hbar$). We can see that the units match appropriately.

Given a $\Psi_{\text{vac}}[\gamma]$ associated with the $SU(\infty)$ YM theory that obeys the Loop wave equation, the next question to ask: is what happens for states other than the vacuum? Clearly one cannot hope to use the Eguchi-Schild correspondence any longer. We used, originally, the Schild string as a guiding principle but one should not expect the Schild string to be the actual theory behind the full fledged $SU(\infty)$ YM theory, especially in $D \geq 2$. At the classical level there was a simple correspondence between the Schild vacua and those vacua of the $SU(\infty)$ YM theory (space-time independent field configurations). However, this is a very restricted case. The question is again: what string theory, if any, can one use to construct wavefunctionals other than the vacuum in any dimension?

Since $W_\infty$ is an area-preserving difs algebra, containing the Virasoro algebra, which appears naturally in the physics of membranes, we believe that the sort of string theory one is looking for could be a $W_\infty$ string theory; i.e an Extended Higher Conformal Spin Field Theory representing $W_\infty$ gravity coupled to $W_\infty$ Conformal Matter plus an
infinite tower of Liouville fields and ghosts in the noncritical $W_\infty$ string case. It has been formally shown in $^{11}$ that $D = 27/11$ are the appropriate target spacetime dimensions for the bosonic/supersymmetric non-critical $W_\infty$ string, if the theory is devoid of BRST anomalies. It is interesting to see that these are precisely the membrane/supermembrane spacetime dimensions that are devoid of Lorentz anomalies $^{48}$. The self-dual sector of the membrane/supermembrane spectrum contains non-critical $W_\infty$ strings/superstrings.

Higher Spin Gauge theories have been revisited very recently by $^{8,9}$ in connection to Higher Spin Gauge interactions of Massive fields in Anti-de-Sitter space in $D = 3$ and $N = 8$ Higher Spin Supergravities. It has been argued that Higher Spin Gauge theories may actually be the underlying theory behind the bulk dynamics of Anti-de-Sitter space in any dimensions. For a review of the historical precedents of Maldacena’s conjecture $^{16}$ we refer to Duff $^{17}$.

In the case that the boundary loops do not coincide, $\gamma$ is not equal to $C$, we propose the following vacuum loop wavefunctional as the solution to the Loop Wave equation. In $^{1}$ we constructed the area-wavefunctional $\Psi[\sigma^{\mu\nu}(\gamma)]$ directly from the computation of covariantized Schild string phase space path integral. The latter is not the same as the $\Psi[\gamma]$ defined as:

$$\Psi_{\text{vac}}[\gamma] = \langle \gamma | \Psi'_{\Sigma} \rangle = \int [DA^\mu] \langle \gamma | A^\mu \rangle < A^\mu | \Psi'_{\Sigma} > =$$

$$\int [DA^\mu] W[A^\mu, \gamma] \int [Dx^\mu(s)] < A^\mu | x^\mu > \int_0^\infty dA e^{iEA(A(\gamma,C))} x^\mu | \Psi_{\Sigma} > =$$

$$\int [DA^\mu] W[A^\mu, \gamma] \int [Dx^\mu(s)] W^*[A^\mu, C] \int_0^\infty dA e^{iEA(A(\gamma,C))} \Psi_{\Sigma}[x^\mu(s)]. \quad (4.15)$$

where: $A = A(\gamma,C)$ is the proper area whose boundaries are $\gamma$, $C$; i.e. Areal time spanned between $\gamma$, $C$. We have used the relation $|\Psi'_{\Sigma} > = e^{i\hat{H}A}|\Psi_{\Sigma} > = e^{iEA}|\Psi_{\Sigma} >$ since the Schild Hamiltonian operator is the evolution operator of the initial quantum eigenstate $|\Psi_{\Sigma} >$ (assigned to the boundary $C$), to the final quantum eigenstate $|\Psi'_{\Sigma} >$ assigned to the boundary $\gamma$, in a given areal time: $A(\gamma,C)$. At the end one must sum over all possible values of the proper area spanning between $\gamma$ and $C$.

The wavefunctional assigned to the $C$ boundary is once again:

$$\Psi_{\Sigma}[x^\mu(s)] \equiv \int_{x^\mu(s)} [DX^\mu(\sigma)] \exp \left[ iS_{\text{Schild}}[X^\mu(\sigma)] \right] = e^{iS_{\text{eff}}[x^\mu(s)]}. \quad (4.16)$$
The path integral of the bulk field theory is evaluated on all the open surfaces that have the loop $C$ as the boundary: $X^\mu(\sigma)|_{\partial \Sigma} = x^\mu(s)$. The wavefunctional $\Psi_\Sigma[x^\mu(s)]$ agrees exactly with the one described by the operator formalism of Riemann surfaces. The Schild action functional, $S_{\text{Schild}}[X^\mu]$, will reassure us that $\Psi_{\text{vac}}[\gamma]$ will indeed be a solution of the Loop wave equation (4-10) if, and only if, $\Psi_\Sigma[x^\mu(s)]$ obeys the Loop Wave equation as it should, since it was constructed by performing the Schild-string path integral. Therefore, the two wavefunctionals must be related through the Schild-string kernel:

$$\Psi_{\text{vac}}[\gamma] = \int [DA^\mu] \int^\infty_0 dA \ K_{\text{Schild}}[C, \gamma, A(\gamma, C)] \Psi_\Sigma[x^\mu(s)]. \quad (4.17)$$

with $A[\gamma, C]$ being the proper area (areal time) spanned between the initial loop $C$ and the final loop $\gamma$. In the limiting case that the two loops coincide ($A = 0$) the Schild loop propagator turns into a delta functional $\delta[\gamma - C]$ and one gets from eqs- (4-11, 4-17):

$$\Psi_\Sigma[x^\mu(s)] = \Psi_{\text{vac}}[\gamma] = \int [DA^\mu] W[A^\mu, C] \Psi_{\text{vac}}[A^\mu] \quad (4.18)$$

which is indeed the loop transform of the vacuum wavefunctional of the $SU(\infty)$ YM theory. And the latter, in turn, is indirectly given in terms of the bulk field theory by using, precisely, the inverse loop transform of the boundary wavefunctional associated with the conformal field theory living on the bulk; i.e the Riemann surface (string world sheet) with boundary $\gamma = C$:

$$\Psi_{\text{vac}}[A^\mu] = \int [DX^\mu] W^*[A^\mu, \gamma] \Psi_{\text{vac}}[\gamma]. \quad (4.19)$$

Concluding: It is precisely when one sets $\Psi_{\text{vac}}[\gamma] = \Psi_\Sigma[x^\mu(s) = C]$, that we have the following string (bulk)$/SU(\infty)$ gauge field (boundary) duality for the vacuum wavefunctionals:

$$\Psi_{\text{vac}}[\gamma] = \int [DA^\mu] W[A^\mu, \gamma] \Psi_{\text{vac}}[A^\mu] = \int_{x^\mu(s) = \gamma} [DX^\mu(\sigma)] \exp [iS_{\text{Schild}}[X^\mu(\sigma)]] \quad (4.20)$$

the left hand side is just the Wilson loop transform of the $SU(\infty)$ YM vacuum wavefunctional whereas the right hand side is the induced boundary effective action defined above.
by performing the path integral of the Schild string bulk field theory with the standard boundary restriction on the string embeddings: \( X^\mu |_{\partial \Sigma} = x^\mu(s) = \gamma \).

For states other than the vacuum we postulate that:

\[
\Psi[\gamma] = \int [DA^\mu] W[A^\mu, \gamma] \Psi[A^\mu] = \int_{x^\mu(s) = \gamma} [DX^\mu(\sigma)] \exp [i S_{\text{bulk}}[X^\mu(\sigma)]] . \tag{4.21}
\]

The crucial question is once more: What is the theory behind \( S_{\text{Bulk}}[X^\mu] \) in the more general case besides the vacuum? Is it a Higher Spin Gauge Theory; i.e a \( W_{\infty} \) gauge theory? A \( W_{\infty} \) string theory? As stated earlier, we used, originally, the Schild string as a guiding principle but one should not expect the Schild string to be the actual theory behind the action functional \( S_{\text{Bulk}}[X^\mu] \). We believe that \( W_{\infty} \) geometry, higher spin gauge theories, could be the theory behind the \( S_{\text{Bulk}}[X^\mu] \). \( W_{\infty} \) algebras and their Moyal and \( q \) deformations are the natural candidates to construct the Quantum Gauge Theories of the area-preserving diffs group.

To conclude, string/gauge field duality is tantamount of a bulk/boundary duality for vacuum \( SU(\infty) \) gauge field configurations when \( \gamma = C \). In the more general case that \( \gamma \) does not coincide with \( C \) the Schild loop propagator \( K[C, \gamma, A] \) is the kernel operator that evolves the initial state \( \Psi_{\Sigma}[C] \), in a given areal time \( A(\gamma, C) \) between the two loops \( \gamma, C \), to a new quantum state \( \Psi_{\text{vac}}[\gamma] \). This is equivalent to attaching a plumbing fixture or throat, typical of Closed String Field Theory, to the original Riemann surface (disk topology) at the boundary \( C \) extending it all the way to the final boundary \( \gamma \). And then, propagating (sliding) the initial loop \( C \) along the throat all the way to the final loop \( \gamma \). As it propagates, the shape of the initial loop \( C \) changes ("elastic") to match \( \gamma \) at the other end. And, finally, one performs the sum over all values of areal time (area) \( A \) as shown in (4-15).

This is consistent with the fact that if \( \Psi_{\Sigma}[C] , \Psi[\gamma] \) are both solutions of the loop wave equation, then the standard operator form of the Loop wave equation gives:

\[
\hat{H} \Psi > = E \Psi > = e^{i\hat{H}A(\gamma, C)} \Psi > = e^{i\mathcal{E}A(\gamma, C)} \Psi > = \Psi' > . \tag{4.22}
\]

It is clear that when \( \gamma \rightarrow C \) the area (areal time) tends to zero and both wavefunctionals will coincide as expected. Finally, the induced boundary effective action was required to be identified with the phase of the boundary vacuum wavefunctional \( \Psi_{\text{vac}} = \)
\[ \Psi_{\Sigma} [C] = N \exp \left[ i S_{\text{eff}} [x^\mu (s)] \right] \] if this boundary wavefunctional is a true solution of the Vacuum Boundary Loop Wave equation.

To finalize this section, we emphasize once more that one must differentiate between the three different representations associated with the quantum state \( |\Psi > \). The \( \Psi_{\text{vac}} [\sigma^{\mu\nu} (\gamma)] \), is the area-wavefunctional that was evaluated explicitly in (3.16), and is related to the standard loop functional \( \Psi_{\text{vac}} [\gamma] \) as follows:

\[ \Psi [\sigma^{\mu\nu} (\gamma)] = \langle \sigma^{\mu\nu} | \Psi \rangle = \int [D\gamma] \langle \sigma^{\mu\nu} | \gamma \rangle \langle \gamma | \Psi \rangle = \int [D\gamma] \langle \sigma^{\mu\nu} | \gamma \rangle \Psi [\gamma]. \quad (4.23) \]

The quantity \( \langle \sigma^{\mu\nu} | \gamma \rangle \) represents the probability amplitude that the loop \( \gamma \) has for Plucker area-components, the values of \( \sigma^{\mu\nu} \). As usual, \( \Psi [\gamma] \) is the loop wavefunctional discussed previously in full detail; i.e the Wilson loop transform associated with the \( SU(\infty) \) YM vacuum functional \( \Psi_{\text{vac}} [A^\mu] \) or defined via the string path integral with boundary \( \gamma \).

The Fourier conjugate state given earlier by the functional loop transform in eqs- (3.15,3.16) can also be represented as:

\[ \tilde{\Psi}_{\text{vac}} [P_{\mu a} (s)] = \langle P_{\mu a} | \Psi \rangle = \int [D\gamma] \langle P_{\mu a} | \gamma \rangle \langle \gamma | \Psi \rangle = \int [D\gamma] \langle P_{\mu a} | \gamma \rangle \Psi_{\text{vac}} [\gamma]. \quad (4.24) \]

and coincides with the expression (3.12). \( \tilde{\Psi} [P_{\mu a} (s)] \).

The quantity \( \langle P_{\mu a} | \gamma \rangle \) represents now the probability amplitude that the given loop \( \gamma \) carries an area-momentum of \( P_{\mu a} (s) \).

Concluding, finally, the three descriptions:

\[ \Psi_{\text{vac}} [\gamma], \; \Psi_{\text{vac}} [\sigma^{\mu\nu} (\gamma)], \; \tilde{\Psi}_{\text{vac}} [P_{\mu a} (s)]. \quad (4 - 25) \]

are nothing but just three different representations of the same quantum state, \( |\Psi >_{\text{vac}} \) of the \( SU(\infty) \) YM theory: the Wilson loop, the area and the momentum representations, respectively. Since the area-wavefunctional representation \( \Psi_{\text{vac}} [\sigma^{\mu\nu} (\gamma)] \) was given explicitly by (3.16) one can then evaluate, in principle, the functional maps which take one representation into the other.

**5 The Master Covariant Loop Wavefunctional Equation and M Theory**

The loop wave equation that has been discussed so far are of the Schroedinger type. This is not covariant since \( A \) (areal time) and \( \sigma^{\mu\nu} \) (the analog space like coordinates)
are on a different footing. What we need now is the analog of the Klein Gordon equation. Pavsic has written a covariant functional Schroedinger equation in the Fock-Schwinger proper time formalism by adding two Lagrange multipliers (the price of having an unconstrained formulation) to the standard action for the p-branes. We shall follow an alternative approach by embedding the theory of strings(p-branes) into a larger space that we call C-space. We find that the principle of covariance in such a C-space forces upon us a drastic new view of spacetime and extended objects and, consequently, an extension/modification of the ordinary QFT that we are accustomed to. In particular, modifications of the Heisenberg uncertainty principle and the notion of an observer-dependent Hilbert space.

The Klein-Gordon-like Loop Master field $\Phi[X^{MN}]$ will depend now on an antisymmetric matrix (to be defined shortly) living in an enlarged space that we shall call C-space (some sort of superspace not to be confused with the one in supersymmetry). The elements of this C-space, a Category in the modern language, are comprised of a collection of histories of point-particles, closed lines (loops), bubbles (world-sheet surfaces), lumps, p-branes, of arbitrary topologies, D-branes..... The bubbles may have also a boundary $\gamma$: i.e a disk topology with any number of handles attached to it. The lumps (world volume of a membrane) may have two-dimensional boundaries and so forth and so forth. For simplicity and without loss of generality we shall study the simple case of points, loops and bubbles only.

The interacting master field theory associated with $\Phi[X^{MN}]$ (corresponding to a given quantum state $|\Phi>$) will involve:

(i) bubbles of non-zero areal time $A$ (of arbitrary topologies) emerging from the vacuum (zero origin of coordinate-matrices in this C-space); i.e a virtual closed string history. These bubbles will also carry with them their center of mass coordinates.

(ii) An open surface with boundary $\gamma$ and proper areal time $A$, with boundary holographic coordinates $\sigma^{\mu\nu}(\gamma)$ (shadows onto the spacetime coordinate planes); i.e it will involve disks of areal time $A$, with boundary $\gamma$ with any number of handles attached to it emerging from the vacuum. These open surfaces will also carry with them their center of mass coordinates (inside).

(iii) loops emerging out of the vacuum, with shape $\gamma$, with a given center of mass $x^\mu_{CM}$
in spacetime; i.e a sort of “loop-instanton” in areal time $A$.

(iv) null world sheets of areal time $A = 0$; i.e null/tensionless closed string-histories with null boundary shape $\gamma_{null}$. These null world sheets will also carry with them their center of mass coordinates: a null line in spacetime.

(v) It will, in addition, involve point particle-world lines; i.e the zero modes or center of mass coordinates of the loops and open/closed surfaces.

(vi) the master field will in general create an “object” that encompasses all of the above for the most general value of the matrix $X^{MN}$ (defined shortly).

In the case of a membrane one requires a rank three antisymmetric tensor: $X^{MNP}$ and for a $p$-branes a rank $p+1$ antisymmetric tensor. All these histories are observer dependent in $C$-space. For example, an observer moving very fast ( “infinite-momentum “ frame) in this $C$-space will see the original loops and bubbles (observed by the first static observer in $C$-space), ”contract” to a point and to a line, respectively. The former single point must not be confused with the zero-matrix origin of coordinates, the vacuum. Therefore, “Special Relativity “ in this $C$-space will transform the above objects into one-another and combinations thereof. In this fashion the categorical aspects of such space will be manifested; i.e topology change in particular.

There is an infinite number of reference frames linked to each observer. As each single one of these infinite observers explores, from his/her own reference frame, each event in $C$-space labeled by a “point”, an antisymmetric matrix $X^{MN}$, he/she will see many histories. Depending on the values of $X^{MN}$: he/she will see a point-history; a loop-history or open surface with boundary $\gamma$; a bubble (a virtual string history); a virtual loop (a “loop-instanton” in areal time $A$ which is not the same as the instanton in coordinate imaginary time) .... Another observer will see a different scenario from a different frame of reference.

$C$-space is the arena for all these histories or events in the language of John Wheeler. An observer in $C$-space witnesses the “history of all possible histories” associated with points, loops, bubbles, depending on where he/she is “sitting”. It is in this $C$-space, a category comprised of point, loops and bubbles, where one quantizes spacetime.

Recent recents based on the Loop Quantum Gravity formalism, have found that the length, area, volume operators have discrete eigenvalues $z$. This seems to suggest that the true quanta of spacetime are now strings of Planck length, loops of Planck-area, bubbles
of Planck volume, etc....This also appears to be consitent with the ideas that there seems to be an *alleged* minimum distance, minimum area, minimum volume,...in Nature implied by the stringy-uncertainty principle. There is no such a thing as a point in this $C$-space. The only dimensionless “point” is the vacuum (zero matrix) which is the zero event (the origin) from which all histories emerge/spring from. It must not be confused with the center of mass coordinates of the loops, bubbles!

Evidence accumulates today which seems to suggest that that spacetime is *fuzzy* at Planck scales: one cannot resolve a point with an infinite accuracy. This is what the modified stringy-uncertainty principle and Noncommutative geometry have taught us [51], [7], [24], [25]. There is an alleged minimum resolution-distance in Nature [1], and consequently, a new principle of Relativity: The theory of Scale Relativity proposed by Nottale where the Planck scale is the minimum *resolution* attained in Nature. These spacetime resolutions are *not statistical* uncertainties of the spacetime coordinates. They are the natural *extra resolutions* one has to introduce in a truly Fractal Space Time. The latter space requires both coordinates and resolutions. It is roughly speaking like a sort of generalized phase space $(q, p)$ of double dimensionality than the configuration, $X^\mu$ spacetime. Increasing evidence seems to point that the Universe appears to be fractal at very small and very large scales.

We must not confuse the string length scale $l_s$ with the Planck length, $l_p$ and *resolutions* of spacetime intervals, that a physical apparatus can resolve, with with the actual value of the spacetime interval. For example, a closed loop of radius $l_p$ will have for Euclidean perimeter a length $l_s = 2\pi l_p$ and area $\pi l_s^2 > \pi l_p^2$. D-branes, can probe distances smaller than $l_s$ [2]. Scale Relativity suggests that it takes an *infinite* amount of energy to attain Planck scale *resolutions*. Like ordinary motion Special Relativity sets a maximum speed in Nature, the speed of light, in Scale Relativity there is a minimum resolution distance. If a massive object were to attain such speed of light it would consume an *infinite* amount of energy. A hypothetical infinite Lorentz contraction (null like “observer”) of a line segment to zero does not imply that the *resolution* at which one observes the length of zero is also equal to zero! For example, the value of $q = 0$ in phase space does not imply that the value of $p = 0$. Scale Relativity in a fractal spacetime imposes upon us the need to enlarge our ordinary concept of a smooth spacetime to include extra independent degrees
of freedom: the resolutions.

Ordinary geometrical concepts do not apply at those (fractal) scales where the Hausdorff dimensions can attain an infinite value. A phase transition occurs where one has pure topology: there is no notion of distance, below the Planck scale: \( < g_{\mu\nu} > = 0 \). It is the result of the breakdown of this topological symmetric phase, at the Planck scale and beyond, when the notion of metric emerges: \( < g_{\mu\nu} > \neq 0 \) and, consequently, a distance, light-cone, particle propagation, dynamics, is possible etc.

Furthermore, what is more intriguing, if one chooses \( D = 4 \) as the spacetime dimension where all these events take place, and one considers trivial topologies corresponding to stringy-tree level amplitudes (spherical bubbles, disks with no handles, ...) we can show that the dimension of this C-space associated with the space of antisymmetric matrices, \( X^{MN} \), will be precisely 11 dimensional. We have at this stringy-tree level point a \( D = 4 \) spacetime unified description of points, closed lines (loops) and closed/open surfaces of trivial topology, in an 11 dimensional C-space.

We will write down one Master Covariant Loop Wave Functional Equation describing all the quantum dynamics of both points, closed lines and closed/open surfaces combined. We will also have a holographic description of the quantum dynamics in this extended loop space (of points, loops and bubbles) given by their projections/shadows (Plucker-coordinates) from the C-space (essentially an extended loop space) onto the \( D = 4 \) spacetime. We have a C-space whose C-events (point, loop, bubble histories) are labeled by Noncommutative antisymmetric matrices. And finally, smooth topology change appears naturally as one moves from “point to point” in this C-space.

After this introduction we proceed to write down a Covariant Loop Wave Functional Master equation for the master field \( \Phi \). Firstly, one needs to coordinatize the areal time, the \( \sigma^{\mu\nu}(\gamma) \) and their center of mass coordinates into one single object (the analog of a four vector). Therefore, we incorporate the \( A, \sigma^{\mu\nu} \) and center of mass coordinates coordinates into one \( (2D) \times (2D) \) antisymmetric matrix whose block components are:

\[
X^{MN} \equiv \{ X^{ab} = -X^{ba}, \ a, b = 0, 2, 3, ...D - 1, \ X^{\mu\nu} = -X^{\nu\mu}, \ X^{a\mu} = -X^{\mu a} \}. \quad (5 - 1)
\]

writing explicitly this matrix:
\[ X^{MN} = \begin{pmatrix} X^{ab} & X^{a\mu} \\ X^{\mu a} & \sigma^{\mu\nu} \end{pmatrix} \] (5 - 2)

The spatial loop holographic components are the usual \( X^{\mu\nu} \equiv \sigma^{\mu\nu}(\gamma) \). They make up a matrix lying on the south east block corner. The \( D \times D \) matrix \( X^{ab} = -X^{ba} \), lying on the north-west block corner of \( X^{MN} \), has for the two nonzero components in the two off-diagonal corners the values of \( A, -A \) respectively. \( A \) in the upper right-hand corner and \( -A \) in the lower left-hand corner, respectively. Finally, the two matrices \( X^{a\mu} = -X^{\mu a} \), north-east and south-west blocks, are the remaining ones required to fit the center of mass coordinates \( x_0, x_1, ..., x_{D-1} \). We do this by assigning the (non-zero) diagonal components of \( X^{a\mu} \) the values of \( x_0, x_1, ..., x_{D-1} \) and for \( X^{\mu a} \) the values of \( -x_0, -x_1, -x_2, ..., -x_{D-1} \) to assure that the \( X^{MN} \) is indeed antisymmetric.

We notice that the elements of the matrix have dimensions of area and length. It proves convenient to rescale all the elements of the matrix in such a way to have a matrix of area dimensions. One should then leave fixed the areal time \( A \) and the holographic shadows, \( \sigma^{\mu\nu} \) and rescale the center of mass coordinates by a factor of \( l_P \) (Planck’s length). This will be important when we construct a metric in \( C \)-space that will define an interval of dimensions of area. It will also be very important to establish the correspondence principle. We will see that in the limit of \( l_P \to 0 \) everything collapses to a “point” (leaving only the commuting matrices \( X^{a\mu} \) containing the center of mass coordinates on their diagonal) and we recover the ordinary Minkowski spacetime interval from this limiting process of the interval in \( C \)-space.

The antisymmetric matrix \( X^{MN} \) encodes the histories of points (center of mass), loops and bubbles (with/without boundary) in one single footing. Also the formation of null loops of zero proper areal time, \( A \); i.e. a tensionless string appear here as well. Tensionless strings must be present in this covariant formalism.

For example, In \( D = 4 \) for trivial topologies (tree level stringy amplitudes), we have an \( 8 \times 8 \) antisymmetric matrix. There are \( \frac{43}{2} = 6 \) independent components for the holographic shadows of the loop. There are four components for the center of mass coordinates and one component for the areal time \( A \). The total number of independent matrices is 11. Any \( 8 \times 8 \) antisymmetric \( X^{MN} \) can be expanded into a basis of 11 \( 8 \times 8 \)
antisymmetric matrices, \( E_i^{MN} \) as \( \sum a_i E_i^{MN} \). \( i = 1, 2, 3...11 \).

One may be inclined to add extra matrices, \( X_h^{MN} \), to the elements of the \( C \)-space, to account for the self-interactions, like stringy-loop interactions, by including surfaces of arbitrary number of handles all the way to infinity and, in general, nodal singularities (pinching a handle to a point). An augmented moduli space.

However, interactions in string theory are mediated by topology. The four string tree level \( S \) matrix amplitude can be computed as the vacuum correlation function of four Vertex operators inserted at four punctures: on-shell emission or absorption of particles by the interpolating world sheet having as boundaries the locations of the strings. The latter can be conformally mapped to a world sheet with four punctures. Stringy-loop perturbative corrections are obtained by adding an arbitrary number of handles to the world sheet. In the string field theory language, for example, the three string vertex interaction requires to use \( \Psi^* \Psi^* \Psi \) string-field interactions.

Since the interacting field theory of the quantum master field \( \Phi[ X^{MN} ] \) should already encode the introduction of all sort of stringy-tree level interactions and stringy-loop corrections, it won't be necessary to add extra degrees of freedom to the \( C \)-space by using a set of \( X_h^{MN} \), for \( h = 0, 1, 2, .... \infty \). The interacting quantum master field theory will take into account all of that from the start. It will create an arbitrary number of handles, boundaries, holes,.....

For example, it will involve two bubbles interacting at one point and then merging into a third bubble or vice versa: the breaking of one big bubble into two smaller bubbles, at one point. These "elastic" bubbles of Planck volume will be our quanta of volume in spacetime; the "elastic" closed loops of Planck area will be our areal-quanta of spacetime.....two fundamental bubble-units (two separate volume quanta) can merge into one bigger bubble of twice the Planck volume; \( n \) bubble units (\( n \) quanta) can merge into one bigger bubble of \( n \) times the Planck volume...and vice versa: one big bubble of \( n \) times the Planck volume can break into \( n \) separate smaller bubbles of one unit of Planck volume. Similar arguments apply to the quanta of Planck area: two closed loops interact at one point and merge into a third loop of twice the area. Or vice versa: the big loop of two quanta of area breaks at one point into two smaller loops of one quanta of area.

As in ordinary quantum field theories, the master field propagator is obtained from
the kinetic terms in the action and the form of vertices are determined through the types of interactions one constructs for the master field. In the true categorical sense, the interacting quantum master field theory will play the role of a categorical functor that will map two incoming closed strings (of one areal quanta) into a third string (of two areal quanta); a torus (of one volume quanta) into a sphere (of one volume quanta); a two-handled Riemann surface (two units of volume quanta) into two torus (of one unit of volume quanta), etc... It is in this way how topology change will be captured, in a smooth fashion from the categorical point of view only, by the interacting quantum dynamics of the master quantum field $\hat{\Phi} [X^{MN}]$ in the C-space. However, the topology change it is not smooth from the ordinary spacetime point of view! The torus-sphere transition by pinching the handle of a torus to a nodal point is not a smooth transition in spacetime. We must emphasize the the Master QFT we are constructing is in C-space and not in spacetime!

Segal has advocated this Categorical picture to reformulate the physicist’s notion of 2D Conformal Field Theory in the Mathematical language of functors and categories. For example, the three-string vertex interaction admits a very natural functorial interpretation as the functor that maps two incoming loops to a third outgoing loop. The functorial map plays exactly the same role as the interpolating string world sheet (of arbitrary topology) having for boundaries the three loops with appropriate orientations. The quantum Master field $\hat{\Phi}$ will play an analogous role as Segal’s functorial map. It will generate surfaces of arbitrary interpolating topologies among any number of incoming/outgoing strings (the boundaries). Crane has proposed also to view quantum gravity as categorical (algebraic) process. Smolin has developed Penrose’s Spin-Networks program as an algebraic approach to quantum gravity and also a more recent holographic formulation of quantum general relativity in terms of states and operators defined on finite boundaries. For another interesting approach one has the work of based on Spin-Foam models and the Discrete and Combinatorial Geometric approach of Matroids.

Open strings and D-branes will be incorporated later. Timelike Wilson loops represent quark-antiquark pair creation and annihilation. These quarks/antiquarks are attached to the end points of two open strings. One moving forward in time and the other backwards in time. Hence, time-like Wilson lines correspond to mapping the boundaries of
an open string world-sheet into the quark-antiquark worldlines. Null-like loops represent tensionless strings and spatial loops the usual ones. These were mentioned above.

The zero origin of matrix-coordinates represents the vacuum; i.e the zero matrix. It can be “translated” to another point in C-space by the action of a “constant-vector translation” as it occurs with ordinary Minkowski spacetime under the action of the Poincare translation operators. It is with respect to this zero origin that we label our coordinates, $X^{MN}$.

For example, a spatial loop can be created out of the vacuum (zero matrix) and, in a given areal time, $A$, can appear at a given region, $\gamma$, of the usual target spacetime onto which we embed the loop. This history emerging from the vacuum is represented by the matrix $X^{MN}$ (relative to the zero matrix). Another picture is that the spatial loop could shrink to zero size at the end of the road: we will have a closed surface or bubble of areal time $A$ (a virtual closed string). Another possibility is to have the spatial loop shrink to zero from the very beginning and what we have is the world-line of a point particle (center-of-mass motion) from the origin to another point; etc...

If we view the origin (the vacuum) of coordinates attached to one particular observer (like any spacetime coordinate assigned to any spacetime event) as the zero matrix, under the analog of a Lorentz relativistic transformation it will be transformed also into the zero matrix. An additional “Poincare translation” will shift the latter zero matrix to a non-zero (constant) matrix that we may label as the “transformed vacuum” under the full Poincare-group-analog (Lorentz plus translations). Therefore, the notion of the so-called vacuum (origin of coordinates) is observer dependent for this general case. When one speaks of a “loop” as an element of a Loop space, one must talk of loops based at a fixed point. In a similar fashion, here one must specify, first, the point/origin to which one assigns the beginning/birth of the history of a bubble, loop, point....

For example, viewing a virtual closed string as the spatial loop history emerging from the vacuum (origin of coordinates) and ending at another point (shrinking the closed string to zero) in a given areal time $A$, as a bubble (baby universe) history, from the point of view of two different observers in C-space, one has two different pictures:

The action of the Poincare-group-analog, will transform one bubble (universe) history as seen by the first observer, into a new bubble (universe) history as seen by the second
observer. And the latter history, can be seen as the one which has been created out of a “translated vacuum”. Translated with respect to the old vacuum (from which the first bubble-universe history emerged from). If the second observer is moving “very fast” with respect to the first observer, he/she will see a different virtual string history: a different bubble that appears now “contracted”. For him/her now the bubble looks now more like a line (virtual point particle) instead of a closed surface (virtual closed string). We believe that this could be related to the plausible origins of Duality in String theory and one of the underlying principles behind M theory:

Covariance in C-space. It is the generalized principle of covariance in this categorical C-space comprised of point histories, loop histories, bubble histories,... it is this generalized principle of covariance (area-preserving diffs in C-space) apply to the categorical C-space, comprised of a whole collection of extended-loop spaces: points, loops, bubbles, lumps,..., that we believe may hold important clues in the underlying principle behind "MFS..." theory. It is relativity to its fullest potential: the relativity of histories. Each observer sees a different history/herstory accordingly to his/her motion in C-space. There is no preferred referential frame of reference. All are equivalent. To one observer one history/herstory looks like that of a bubble; to another it looks like the one of a line; to another like the one of a point.....all p-branes can be encompassed in this fashion into one single footing in one categorical space C-space. In the case that we have only points, loops, bubbles, their holographic shadows of an 11-dimensional C-world cast upon the ordinary 4D spacetime is what we observe. Plato's old view of reality.

Summarizing, there are an infinite number of histories associated with one, and only one loop, $\gamma$, since each history may be viewed by an infinite number of observers. Each history carries its own vacuum (attached to each observer). The family of vacua are related by “translations” in C-space. There is no preferred universal inertial frame of reference in C-space, and for this reason, there is no universal vacuum. In loop space there is no preferred base point.

We are ready now to implement this generalized relativity principle to the set of point, loop and bubble histories. The analog of an invariant proper time interval -in a fixed reference frame with a given zero origin of matrix coordinates -between two matrices or histories emerging from the vacuum, $X_{MN}(\tau);X^{LP}(\tau)$ (“points”), whose evolution is
parametrized by $\tau$ is:

$$(d\tau)^2 = [dX_{MN}(\tau)] \mathcal{G}_{LP}^{MN}[X^{AB}(\tau)] [dX^{LP}(\tau)].$$

(5 - 3)

where $\mathcal{G}_{LP}^{MN}[X^{AB}(\tau)]$ is a C-Metric in the space of matrices $X^{MN}$. For the time being let us assume the C-space is “Flat”. The introduction of “parallel transport”, “curvature”, and the “principle of equivalence” will be the subject of another work.

As mentioned earlier the C-metric interval $\tau$ has area dimensions. Let us now rescale the interval $(d\tau)^2$ by a factor of $l_P^{-2}$ leaving us with a quantity of dimensions of $(length)^2$, which is the dimensions of the standard spacetime interval in Minkowski spacetime. Upon absorbing the factor of $l_P^{-2}$ into the $dX_{MN}dX^{LP}$ matrices as $[l_P^{-1}dX_{MN}][l_P^{-1}dX^{LP}]$ one can rescale each one of our original matrices (of area dimensions) by a factor of $l_P^{-1}$. The correspondence limit is defined as (i) $l_P \rightarrow 0$ and (ii) $A \rightarrow 0$ and $\sigma^{\mu\nu} \rightarrow 0$. One should end then with a physics involving points only; the bubbles and loops shrink to a point: the center of mass coordinates. We see that in this correspondence limit, the rescaled matrices $l_P^{-1}X_{MN}$ tend to a matrix whose only components will be those of the $X^{a\mu}$ containing the center of mass coordinates along their diagonals. This is fairly clear because in the limit:

$l_P^{-1}A = 0$; $l_P^{-1}\sigma^{\mu\nu} = 0$ (Number of Planck cells is kept finite) and $l_P^{-1} \cdot (l_P \cdot x_{CM}) = x_{CM}$.

The latter limit is what one expects since we had rescaled earlier the $x_{CM}$ by a factor of $l_P$ so that all elements of our matrices have the same area-dimensions. The quanta of areas come in discrete multiples of the Planck area.

Therefore, in the above correspondence limit (i) $l_P \rightarrow 0$; (ii) $A \rightarrow 0$ and $\sigma^{\mu\nu} \rightarrow 0$ (the Number of Planck cells is finite) one ends up with a standard spacetime interval of dimensions of $(length)^2$ in Minkowski space involving point-histories only. The anti-commuting matrices “collapse” to pure diagonal commuting ones. Taking the $l_P \rightarrow 0$ is tantamount of going from noncommutative (fuzzy) geometry to a commutative one (well defined points). It is the analog of taking the zero slope $\alpha' = 0$ limit in string theory, the field theory limit. Quantum effects in perturbative string theory are seen as a topological expansion in the genus of the Riemann surface. Since we wish to capture the full Nonperturbative physics of string theory we must follow a different approach. At the end of this section we will propose the Moyal-Fedosov deformation quantization technique to
construct the putative Nonperturbative interacting quantum field theory for the master field in C-space. This is still very preliminary.

One plausible outcome of this formalism is that the fundamental origins of gravity from string theory would no longer appear to be such a mystery: Gravity in spacetime emerges from the string as a result of the geometrization of the “parent” C-space onto which we embed the string theory itself. Furthermore, the initial spacetime with coordinates $X^\mu$, could in principle be any background spacetime. So this formulation will also be truly background independent. The simple fact that the spin two graviton appears in the perturbative string spectrum about a flat spacetime background is no satisfactory explanation of why gravity emerges from string theory. The background independent formulation is still missing. We hope that this approach based on loop spaces may shed some light into this problem.

$\tau$ is a parameter or proper-time clock that is intrinsic to the C-space manifold of antisymmetric matrices (evolution of the events that keep track of the history of histories emerging from the vacuum). There are now many “times” . The invariant proper time $\tau$ (intrinsic clock of the C-space) defined above in eq-(3-26). The string world sheet areal time $A$. The world sheet coordinate time $\sigma^0$. And the target spacetime coordinate $X^0$.

The analog of a “Lorentz” invariant interval is:

$$ (d\tau')^2 = (d\tau)^2 = dX_{MN}(\tau) \mathcal{G}_{LP}^{MN}[X^{AB}(\tau)] dX^{LP}(\tau) = $$

$$ dX'_{M'N'}(\tau') \mathcal{G}_{L'P'}^{M'N'}[X'^{A'B'}(\tau')] dX'^{L'P'}(\tau'). \quad (5-4) $$

and the corresponding finite “Lorentz” transformations implemented by the matrices $\Lambda^{MN}_{M'N'}$ is:

$$ X'_{M'N'} = X_{MN}^{\Lambda^{MN}_{M'N'}}(\beta_i). \quad X'^{L'P'} = (\Lambda^{-1})^{L'P'}_{LP}(\beta_i)X^{LP}. \quad (5-5) $$

where $\beta_i$ are the “boosts, angles..” parameters associated with each one of the generators of transformation group. As said above, the “Poincare-translations” will shift the vacua (origin of matrix-coordinates).

The invariant metric, $\mathcal{G} = \mathcal{G}'$ must transform as:

32
\[ G^{M'N'}_{L'P'} = (\Lambda)_{M N}^{-1} (\beta_i) G^{M N}_{L P} \Lambda_{L'P'}^P (\beta_i) = G^{M'N'}_{L'P'}. \] (5-6)

Notice that the proper-time interval in C-space has units of area. This will be important as discussed shortly. If one wishes to include membranes, three-branes,...p-branes into the C-space, one will need antisymmetric objects of any \( p+1 \) rank : \( X^{MNP} \). As we have said earlier, a new formulation of all (bosonic) p-branes as Composite Antisymmetric Tensor Field Theories of the volume-preserving diffs group, was given in [33]. The advantage of this formulation is that one has \( S \) and \( T \) duality already built in from the very beginning. There is no need to conjecture it.

The metric must be antisymmetric under the exchange of \( M, N \) and under the exchange of \( L, P \) and also symmetric under an exchange of the two pairs \((MN) \leftrightarrow (LP)\). The symmetry group that leaves invariant the “distance” between two antisymmetric matrices ( “area” between two histories) is the Lorentz-group-analog that characterizes the notion of relativistic invariance in the space of \( X^{MN} \). This C-space is endowed with a Lorentz-invariant-analog of the metric \( G^{MN}_{LP} \).

The main question now is : What is this symmetry acting in C-space that replaces ordinary spacetime Lorentz invariance ? It is the symmetry acting on the C-space which is the generalization of the area-preserving diffeomorphisms group. However these “Areas” are areas in C-space not in spacetime. It is more closely related to a topological symmetry instead of a spacetime symmetry [24, 25].

The action for these matrices in C-space should be the analog for the point particle :

\[ S = \mathcal{M}^2 \tau = \mathcal{M}^2 \int \sqrt{dX_{MN}(\tau)G^{MN}_{LP}[X^{AB}(\tau)]} dX^{LP}(\tau). \] (5-7)

The analog of invariant proper mass is now a tension-like quantity , \( \mathcal{M}^2 \). The massless case corresponds to tensionless strings ( null cone analog). \( \tau > 0; \tau < 0 \) will be timelike or spacelike analog respectively. The temporal evolution wave equation is then :

\[ \left( \frac{\partial}{\partial \tau} \right) \frac{\partial \Psi[X^{MN}, \tau]}{\partial \tau} = \frac{1}{\mathcal{M}^2} \frac{\delta^2 \Psi[X^{MN}, \tau]}{\delta X^{MN} \delta X_{MN}}. \] (5-8)

Setting as usual :
\[
\Psi[X^{MN}, \tau] = \Phi[X^{MN}] e^{-iS} = \Phi[X^{MN}] e^{-i\mathcal{M}^2 \tau}.
\] (5 - 9)

and inserting it into the temporal evolution equation yields the Klein-Gordon-like Covariant Loop Wave Master equation:

\[
\frac{\delta^2 \Phi[X^{MN}]}{\delta X^{MN} \delta X_{MN}} - \mathcal{M}^4 \Phi[X^{MN}] = 0.
\] (5 - 10)

where now the areal times, \(A\) and spatial areas (coordinates) \(\sigma^{\mu\nu} (\gamma)\) are treated on the same footing. Extra variables \(X^{\mu a} = -X_{a\mu}\) must also be added which represent the center of mass motion coordinates. Eq-(5.10) can be generalized to all \(p\)-branes by simply working with antisymmetric tensors of rank \(p + 1\) and inserting the \(p\)-brane tension squared for the “mass” squared terms.

If one chooses a complex valued field, \(\Phi[X^{MN}]\), one could reinterpret it as the field that creates loops, bubbles, point-particles and its complex conjugate, the field that will create anti-loops, anti-bubbles, anti-particles. The massless master field case, \(\mathcal{M} = 0\) corresponds to the whole array of processes like the ceation/annihilation of null loops (boundaries of the null world sheet of tensionless closed-strings), null bubbles, null lines for massless point-particles.... The Master Loop Wave Equation equation can be obtained from a free field theory action

\[
S[\Phi] = \frac{1}{2} \int [DX^{MN}] \frac{\delta \Phi^\dagger[X^{MN}]}{\delta X^{MN}} \frac{\delta \Phi[X^{MN}]}{\delta X_{MN}} + \mathcal{M}^4 \Phi^\dagger[X^{MN}] \Phi[X^{MN}].
\] (5 - 11)

Interactions of the type \(\Phi^3, \Phi^4\) ... can be also be added, similar to those in Closed String Field Theory \(^{44}\). These terms will provide the interacting field theory of the master field. Self interactions naturally introduce in string field theory the creation of handles. This corroborates that the full interacting field theory of the master field will be responsible for the creation of bubble-histories of arbitrary nontrivial topologies with/without boundaries.

As said above, it is for this reason that one does not need to add extra matrix coordinates to the \(C\)-space. The interacting field theory will reproduce all these terms.

To sum up: we are proposing a plausible Master scalar field theory in \(C\)-space that upon quantization ( to be discussed shortly) describes the Quantum Non-perturbative
dynamics of points, loops and bubbles encoded into a single antisymmetric matrix, $X^{MN}$.

In $D = 4$, for trivial topologies (stringy-tree level), this C-space is a Noncommutative 11-dimensional space of $8 \times 8$ antisymmetric matrices; it implements the holographic principle (projection of the C-shadows onto the spacetime coordinate planes) and its quantum dynamics is governed by a covariant loop wavefunctional equation of the Klein-Gordon type in the free field case. The interacting master field theory will automatically introduce objects of arbitrary topologies. And finally, the correspondence limit, $l_p \to 0, ...$ reproduces the standard spacetime proper interval after rescaling the area interval by $l_p^{-2}$, absorbing $l_p$ powers into the matrices $X^{MN}$ and taking the $l_p = 0$ limit.

“Square-roots” of the latter free field master equation will reproduce, a la Dirac, a “spinorial” Master wave equation. Doing the same square-root procedure for the “non-relativistic” Schroedinger Loop Wave equation will yield the analog of Supersymmetric Quantum Mechanics. This is no surprise: the M(atrix) models, infinite momentum frame of M-theory, are essentially a Supersymmetric Gauge Quantum Mechanical Model of the area-preserving diffs group $SU(\infty)$.

The fact that area-preserving diffs in this C-space of $2D \times 2D$ antisymmetric matrices (for trivial topologies) is the corresponding symmetry algebra (the relativistic Lorentz analog), suggests that the symplectic group (and its supergroup version which contains the conformal group) must play a fundamental role. Symplectic geometry and Moyal-Fedosov Deformation Quantization were used in $\mathfrak{M}$ to describe $W_\infty$ Geometry. The space of $2D \times 2D$ antisymmetric matrices is a generalized symplectic space. The generalized area is associated with the generalized symplectic form. It seems that it all boils down to symplectic geometry in higher-dimensional loop spaces. The holographic shadows of these higher-dimensional loop spaces (endowed with a symplectic structure) are the point-histories, the loop-histories, the bubble-histories,... the lump-histories, ...extended object-histories that we see in spacetime.

How will one implement, if possible, a Moyal-Fedosov Deformation Quantization program to the interacting quantum master field theory action? Assuming that one has an invertible non-degenerate symplectic structure which allows to compute “Poisson brackets” in loop spaces, that the notion of phase space is well defined, ...... The formal steps would be:
(i) Given an operator-valued field \( \hat{\Phi}[X^{MN}] \), living in the C-space of \( X^{MN} \), and belonging to a Hilbert space of square integrable functions of \( Q^{MN} \), that we call \( \mathcal{H} \), and whose phase space, \( \Gamma \), is represented by the \( Q^{MN}, P^{MN} \) “coordinates”, the formal Weyl-Wigner-Moyal symbol map \( \mathcal{W} \) takes:

\[
\mathcal{W} : \hat{\Phi}[X^{MN}] \rightarrow \Phi[X^{MN}, Q^{PM}, P^{MN}].
\]

It maps operator-valued objects of the Hilbert space, \( \mathcal{H} \), into smooth valued functions in phase space: the symbols of operators.

And:

\[
\mathcal{W}(\hat{\Phi}[X^{MN}]\hat{\Phi}[X^{MN}]) \rightarrow \Phi[X^{MN}, Q^{MN}, P^{MN}] \ast \Phi[X^{MN}, Q^{MN}, P^{MN}].
\]

The symbol of the “Weyl ordered” product of two operators is mapped into the Moyal product of their symbols. The Moyal product is taken w.r.t the \( Q, P \) phase space coordinates.

The Moyal Deformed Lagrangian density of the interacting Master field action reads:

\[
\mathcal{L}[\Phi(X, Q, P)] = \frac{1}{2} \frac{\delta \Phi[X, Q, P]}{\delta X^{MN}} \ast \frac{\delta \Phi[X, Q, P]}{\delta X^{MN}} + \frac{\mathcal{M}^4}{2} \Phi[X, Q, P] \ast \Phi[X, Q, P] + \frac{1}{3!} \Phi[X, Q, P] \ast \Phi[X, Q, P] \ast \Phi[X, Q, P] + \frac{1}{4!} \Phi[X, Q, P] \ast \Phi[X, Q, P] \ast \Phi[X, Q, P] \ast \Phi[X, Q, P] + \cdots + \frac{1}{n!} \Phi[X, Q, P] \ast \cdots \ast \Phi[X, Q, P].
\]

This very formal Moyal deformation quantization of the interacting quantum master field theory using noncommutative Moyal star products w.r.t the variables \( Q, P \), will provide, formally, the Moyal quantization of the master field theory living in the noncommutative space of antisymmetric matrices \( X^{MN} \). As far as we know, the problem of quantization of noncommutative geometries has not been solved. This is a fundamental question that we hope could be solved via a formalism similar to the one described by eq-(3.34).

The action eq-(3.34) resembles Witten’s open string field theory action using the BRST quantization formalism with ghosts, antighosts, and to Zwiebach closed string field theory action. The latter requires the use of the full machinery of the Batalin-Vilkoviski antibracket formulation, the quantum master field equation, homotopy Lie algebras, operads,...........

These results (3.34) are very preliminary. All we wish to point out is that if, and only if, deformation quantization techniques are indeed valid in this context, then they could
be very useful in quantizing (if possible) the interacting field theory of the Master field $\Phi^{[X^{MN}]}$, in these $C$-spaces. Such interacting Quantum Master Field theory will encode, in principle, the quantum dynamics of a unified field theory encompassing gravity. It will provide a quantization of a master field living in a Noncommutative Geometrical space of antisymmetric matrices.

Early constructions of generalized spaces to describe non-perturbative string physics include the Universal Moduli space construction of Friedan and Shenker based on holomorphic vector bundles and flat connections constructed on the moduli space of infinite genus surfaces with nodes. Sato’s Universal Grassmanian which has been central to the theory of integrable systems, where all Riemann surfaces with a puncture and a local coordinate system of all genera appear on the same footing. This is similar to our categorical construction with the difference that we include loop and point histories also into the picture. An approach closely related to the loop-space picture was analyzed by Bowick and Rajeev using the co-adjoint orbit method quantization of $\text{Diff}_{S^1} S^1$. The latter is a Kahler manifold and the cancellation of the tangent bundle (ghosts) curvature against the vector bundle curvature (matter) was equivalent to finding a globally defined vacuum in the space of orbits: the matter plus ghost central charges vanishes in $D = 26$. Co-adjoint orbit methods can also be used for the area-preserving diffs orbits of $W_\infty$. See references.

More results shall be presented elsewhere.

In short, the generalization of the principle of covariance under area-preserving/volume-preserving diffs applied to a categorical $C$-space of extended-loops (points, loops, bubbles,......histories); i.e the construction of a generalized symplectic geometry in such categorical $C$-space, seems to be, in principle, a plausible appropriate ground to build $M$ theory.

This has been imposed upon us by simply covariantizing the “non-relativistic” Loop Wave Functional Equation (Schroedinger) in order to obtain the Covariant Klein-Gordon-like Loop Wave equation for the single classical master field $\Phi^{[X^{MN}]}$. Moyal-Fedosov deformation quantization techniques may be formally (if possible) suited to quantize such classical master field theory in $C$-space. Spacetime Topology change, the Holographic principle are naturally incorporated in this formalism. The fact that the Planck length $l_P$
is required to match units in the definition of the matrices $X^{MN}$ (areal dimensions) and to establish the correspondence principle with ordinary Minkowski spacetime proper-time intervals of point events (center of mass coordinates), in the $l_P = 0$ limit, suggests that the quantization of spacetime should come in fundamental discrete units of Planck length, area, volume,.... A 4D spacetime requires an 11 dimensional $\mathbb{C}$-space of $8 \times 8$ antisymmetric noncommuting matrices (the only commuting matrices will be those containing the center of mass coordinates) for the special case that the categorical space contains solely points, loops and bubbles.

**ACKNOWLEDGMENT**

The author wishes to thank Euro Spallucci and Stefano Ansoldi for stimulating discussions. We also acknowledge enlightening conversations with Octavio Obregon in Guanajuato, Mexico at the early stages of this work and to Devashis Banerjee, Miguel Cardenas for lengthy conversations at the Abdus Salam ICTP in Trieste, Italy. We express our gratitude to the Dipt. di Fisica Teorica of the University of Trieste for their hospitality and support.
1 S. Ansoldi, C. Castro, E. Spallucci: “String Representation of Quantum Loops” hep-th/9809182.

2 S. Ansoldi, A. Aurilia, E. Spallucci: Phys. Rev D 56 (4) (1997) 2352. S. Ansoldi, A. Aurilia, E. Spallucci: “Loop Quantum Mechanics and the Fractal Structure of Spacetime” to appear in the J. Chaos, Solitons and Fractals. Special issue on Superstrings, M, F, S...theory; Jan. 1999.

3 E. D’Hoker, D. H. Phong: Rev. Mod. Phys. vol 60 (4) (1988) 917.

4 G. Segal: The Meaning of Conformal Field Theory. Oxford preprint.

5 T. Banks, W. Fischler, S. Shenker, L. Susskind: Phys. Rev. D 55 (1997) 5112.

6 L. Nottale: Fractal Spacetime and Microphysics: Towards the Theory of Scale Relativity. World Scientific, (1992).

7 C. Castro: Found. Phys. Letts. 10 (3) (1997).

C. Castro “Beyond Strings, Multiple Times and Gauge Theories of Area-Scaling Relativistic Transformations hep-th/9707171; to appear in the J. Chaos, Solitons and Fractals. Special issue on Superstrings, M, F, S...theory; Jan. 1999.

8 M. Vasiliev: Int. Jour. Mod. Phys. D 5 (1996) 763.

S.F Prokushkin, M. Vasiliev: “Higher Spin Gauge Interactions for Massive Matter Fields in 3D AntideSitter Space Time” hep-th/9806236.

9 E. Sezgin, P. Sundell: “Higher Spin N = 8 Supergravity” hep-th/9805127.

10 C. Castro: “W Geometry from Fedosov’s Deformation Quantization” hep-th/9802023; Pys. Lett B 413 (1997) 53.

11 C. Castro: “Jour. Chaos, Solitons and Fractals 7 (1996) 711; hep-th/9612160.

12 G. Chapline, K. Yamagishi: Class. Quant. Grav. 8 (1991) 427.

13 E. Nissimov, S. Pacheva: Pys. Lett B 288 (1992) 254. hep-th/9201070.

14 A. Kavalov, B. Sakita: Ann. Phys 255 (1997) 1.
15 G. Chapline, A. Granik: “Moyal Quantization, Holography, and the Quantum Geometry of Surfaces” hep-th/9808033. To appear in the J. Chaos, Solitons and Fractals. Special issue on Superstrings, M, F, S...theory; Jan. 1999.

16 J. Maldacena: “The large $N$ limit of superconformal Field Theories and Supergravity” hep-th/9711200. “Wilson Loops in large $N$ field Theories”. hep-th/9803002

17 M. Duff: “Anti-de Sitter Space, branes, singletons, superconformal field theories and all that” hep-th/9808101

18 R. Gambini, J. Pullin: “Loops, Knots, Gauge Theories and Quantum Gravity”. Cambridge University Press. 1997.

19 C. Rovelli: “Strings, loops and others: a critical survey of the present approaches to Quantum Gravity”. gr-qc/9803024. “Loop Quantum Gravity” gr-qc/9709008. C. Rovelli: “Relational Quantum Mechanics” quant-ph/9609002

20 L. Smolin: “Strings as perturbations of evolving spin networks”. hep-th/9801022

“The Future of Spin Networks”. gr-qc/9702030

21 L. Smolin: “A holographic formulation of quantum General Relativity” hep-th/9808191.

L. Smolin: “Towards a background independent approach to $M$ Theory”. hep-th/9808192. To appear in the Jour. of Chaos, Solitons and Fractals. Special issue on Superstrings, M, F, S...Theory; Jan. 1999.

22 J. Baez: “Strings, Loops, Knots and Gauge Fields” hep-th/9309067 in Knots and Quantum Gravity. Oxford University Press. Oxford 1994.

23 J. Baez “Spin Foam Models” gr-qc/9709052.

24 I. Oda: “Topological Symmetry, Background Independence and Matrix Models”. hep-th/9806096. To appear in the Jour. of Chaos, Solitons and Fractals. Special issue on Superstrings, M, F, S...Theory; Jan. 1999.
25. M. Li, T. Yoneya: “Short Distance Spacetime Structure, and Black Holes in String Theory: A short Review of the Present Status” [hep-th/9806210]. To appear in the Jour. of Chaos, Solitons and Fractals. Special issue on Superstrings, M, F, S,...Theory; Jan. 1999.

26. D. Gross, W. Taylor: Nuc. Phys. B 403 (1993) 395

27. J. Hoppe: “Constraints Theory and Surface Relativistic Dynamics” MIT PhD Thesis. 1982.

28. D.B. Fairlie, P. Fletcher, C. Zachos: Jour. Math. Phys 31 (1990) 1088.

29. P. Bouwknegt, K. Schoutens: Phys. Reports 223 (1993) 183-276.

30. J. Moyal: Proc. Cam. Phil. Soc 45 (1945) 99.

31. B. Fedosov: J. Diff. Geometry 40 (1994) 213.

32. M. Pavsic: “The Dirac-Nambu-Goto p-Branes as Particular Solutions to a Generalized Unconstrained Theory”. Ljubljana IJS-TP/96-10 preprint.

33. C. Castro: Int. Jour. Mod. Phys. A 13 (1998) 1263

34. M. Cederwall, G. Ferreti, B. Nilsson, A. Westerberg: “Higher Dimensional Loop Algebras, non-abelian extensions and p-branes” [hep-th/9401027].

35. L. Alvarez-Gaume, C. Gomez, G. Moore, C. Vafa: Nuc. Phys. B 303 (1988) 455. 
   L. Alvarez-Gaume, C. Gomez, P. Nelson, G. Sierra, C. Vafa “Nuc. Phys. B 311” (1988) 333

36. H. Sugawara: “F and M Theories as Gauge Theories of Area Preserving Algebra “ [hep-th/9708029]

37. Z. Kakushadeze: “String Expansion as ’t Hooft’s Expansion “ [hep-th/9808019]

38. W. Brown, J. Garrahan, I. Kogan, A. Kovner: “Instantons and the QCD Vacuum Wavefunctional “ [hep-ph/9808216].
A. Polyakov : Nuc. Phys. B 268 (1986) 406 H. Kleinert : Phys. Lett B 174 (1986) 335.

A. Polyakov : “ String Theory and Quark Confinement “ [hep-th/9711002].

A. Nieto : “Matroid Theory and Supergravity “ [hep-th/9807107].

E. Witten : Nuc. Phys B 268 (1986) 253. Nuc. Phys B 276 (1986) 291.

E. Witten : “Some comments on String Dynamics in various Dimensions “. [hep-th/9507121].

B. Zwiebach : Nucl. Phys. B 390 (1993) 33; [hep-th/9206084].

J. Ambjorn, J. Nielsen, J. Rolf, R. Loll : “ Euclidean and Lorentz Quantum Gravity :Lessons from D = 2. [hep-th/9806241]. To appear in the Jour. of Chaos, Solitons and Fractals. Special issue on Superstrings, M, F, S,...Theory; Jan. 1999.

C. Hull : “ Branes, Times and Dualities “ [hep-th/9808063].

L. Crane : Jour. Math. Phys. 36 (1995) 6180.

U. Marquard, M. Scholl : Phys. Lett B 227 (1989) 227. U. Marquard, R. Kaiser, M. Scholl : Phys. Lett B 227 (1989) 234.

C. Vafa : Nuc. Phys B 469 (1996) 403.

I. Bars, C. Kounnas : Phys. Rev. D 56 (1997) 3664.

For a nice review of Connes Noncommutative geometry with many references we refer to : G. Landi : “ An Introduction to Noncommutative Spaces and their Geometries “ Springer-Verlag Monographs in Physics, 1997.

M. Douglas, D. Kabat, P. Pouliot, S. Shenker : Nuc. Phys. B 485 (1997) 85.