Entanglement changing power of two-qubit unitary operations

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Abstract

We consider a two-qubit unitary operation along with arbitrary local unitary operations acts on a two-qubit pure state, whose entanglement is $C_0$. We give the conditions that the final state can be maximally entangled and be non-entangled. When the final state can not be maximally entangled, we give the maximal entanglement $C_{\text{max}}$ it can reach. When the final state can not be non-entangled, we give the minimal entanglement $C_{\text{min}}$ it can reach. We think $C_{\text{max}}$ and $C_{\text{min}}$ represent the entanglement changing power of two-qubit unitary operations. According to this power we define an order of gates.

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I. INTRODUCTION

Entanglement is a fundamental resource in quantum information theory. It is used in quantum key distribution \[1\], dense coding \[2\], teleportation \[3\] and so on. Since entanglement is such a valuable resource, many efforts are devoted to generate it \[4, 5\] and quantify it \[6\]. Recently some researchers begin to investigate nonlocal operations \[7, 8, 9, 10\]. On the one hand, nonlocal operations can generate entanglement. On the other hand, entanglement can be used to implement nonlocal operations if local operations and classical communication are permitted \[11, 12, 13\].

Entanglement can be generated by nonlocal operations, so we should ask entangling capacity of nonlocal operations, nonlocal Hamiltonians or unitary gates. Some results have been derived \[8, 14, 15\], especially Kraus and Cirac \[16\] calculate the maximal final entanglement after a two-qubit gate along with arbitrary local unitary operations acted on an initial non-entangled state. Leifer et al. \[17\] consider a similar question. Their efforts are devoted to maximize the entanglement of the final state minus the entanglement of the initial state. They think this quantity represents the entanglement generating ability of a nonlocal gate. In this paper we consider a general question. Suppose the entanglement of the initial pure state is given, denoted by \(C_0\), we want to know the reachable maximal entanglement \(C_{\text{max}}\) of the final state after a two-qubit gate acted on the initial state, where local unitary operations can be freely used. Obviously Kraus and Cirac \[16\] solved the question where \(C_0\) is zero. We solve the question for a general \(C_0\). In this paper we also calculate the the minimal reachable entanglement \(C_{\text{min}}\) of the final state for a general \(C_0\). The minimal entanglement can be zero if measurements are permitted, but we still consider it mathematical interest. We think \(C_{\text{max}}\) and \(C_{\text{min}}\) represent the entanglement changing power of a two-qubit gate. The entanglement of the final state can be any value between them due to continuity.

The structure of the paper is as follows. In Sec. II we introduce concurrence \[18\] and canonical decomposition of two-qubit gates \[16\]. We use concurrence to quantify two-qubit entanglement and use canonical decomposition to classify two-qubit gates. In Sec. III we calculate the minimal final entanglement after a two-qubit gate acted on a maximal entangled state. This result will be used to judge whether the maximal final entanglement \(C_{\text{max}}\) can be 1 or not for a general \(C_0\). In Sec. IV we calculate the maximal and minimal entanglement of the final state for a general initial state. Finally we conclude this paper in Sec. V and
II. CONCURRENCE AND CANONICAL DECOMPOSITION

Concurrence \[18\] is defined to quantify entanglement of formation of mixed two-qubit states. For pure states it has a simple form. We write two-qubit states in magic basis \[|\Psi\rangle = \sum_{k=1}^{4} b_k |\Phi_k\rangle,\] then the concurrence \[C(|\Psi\rangle) = |\sum_{k=1}^{4} b_k^2|,\] where \(|\{\Phi_k\}|_{k=1}^{4}\) is defined as follows,

\[
|\Phi_1\rangle = \frac{-i}{\sqrt{2}} (|00\rangle - |11\rangle),
\]
\[
|\Phi_2\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle),
\]
\[
|\Phi_3\rangle = \frac{-i}{\sqrt{2}} (|01\rangle + |10\rangle),
\]
\[
|\Phi_4\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle).
\]

The concurrence \(C\) is zero iff the two-qubit state is a product state. When the state is maximally entangled the concurrence is 1, which requires the coefficients \(\{b_k\}_{k=1}^{4}\) are real, except for a global phase.

Now we introduce canonical decomposition of two-qubit unitary operations \[16\]. Any unitary operation acting on two qubits has 15 parameters but it can be locally equivalent to an operation which only has 3 parameters. According to the canonical decomposition given by Kraus and Cirac \[16\], we can decompose \(U_{AB} = (U_A \otimes U_B) U_d (V_A \otimes V_B)\), where \(U_A, U_B, V_A\) and \(V_B\) are local unitary operations and \(U_d\) has a special form

\[
U_d = \exp \left( i \sum_{j=1}^{3} \alpha_j \sigma_j^A \otimes \sigma_j^B \right),
\]

where \(\pi/4 \geq \alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0\) and \(\sigma_{1,2,3}\) are Pauli matrix. Because local unitary operations do not change the entanglement, we only discuss the entanglement changing power of \(U_d\) instead of \(U_{AB}\) in the following. In fact we can always take \(\alpha_3 \geq 0\) when we discuss entanglement changing power \[16, 17\]. And because entanglement is invariant under conjugation, the entanglement changing power of \(U_d\) is the same as \(U_d^* (U_d^\dagger)\). This means if
$U_d$ can change the states of entanglement $C_1$ to the states of entanglement $C_2$, conversely it can change the states of entanglement $C_2$ to the states of entanglement $C_1$. This result will be used in the following. A very important character of $U_d$ is that the magic basis states are its eigenstates, $U_d |\Phi_j\rangle = e^{i\lambda_j} |\Phi_j\rangle$, where

$$\lambda_1 = -\alpha_1 + \alpha_2 + \alpha_3,$$

$$\lambda_2 = +\alpha_1 - \alpha_2 + \alpha_3,$$

$$\lambda_3 = +\alpha_1 + \alpha_2 - \alpha_3,$$

$$\lambda_4 = -\alpha_1 - \alpha_2 - \alpha_3.$$

### III. THE CASE THAT THE INITIAL STATE IS MAXIMALLY ENTANGLED

In this section we consider the situation where $C_0 = 1$. We want to know the maximal and minimal entanglement of the final state after a two-qubit unitary gate acted on an initial maximally entangled state. Because the Bell states are eigenstates of the unitary operation $U_d$, we can easily find that the maximal entanglement of the final state $C_{\text{max}}$ is 1. To find the minimal entanglement of the final state, we write the initial state in the magic state, $|\Psi_0\rangle = \sum_{j=1}^{4} b_j |\Phi_j\rangle$. The coefficients $\{b_j\}_{j=1}^{4}$ are real and $\sum_{j=1}^{4} b_j^2 = 1$. The final state is

$$|\Psi\rangle = U_d |\Psi_0\rangle = \sum_{j=1}^{4} b_j e^{i\lambda_j} |\Phi_j\rangle. \quad (10)$$

We want to minimize the concurrence $C$ of the final state $|\Psi\rangle$. We define a Lagrangian function

$$L = C^2 - \mu \left( \sum_{k=1}^{4} b_k^2 - 1 \right), \quad (11)$$

where $\mu$ is a Lagrangian multiplier, which is real. Differentiating gives

$$\frac{\partial L}{\partial b_j} = 2b_j e^{2i\lambda_j} \left( \sum_{l=1}^{4} b_l^2 e^{-2i\lambda_l} \right) + 2b_j e^{-2i\lambda_j} \left( \sum_{k=1}^{4} b_k^2 e^{2i\lambda_k} \right) - 2\mu b_j = 0, \quad (12)$$

4
multiplying $b_j$ and summing over $j$ gives
\[ \mu = 2C^2. \]  
(13)

We write \((\sum_{k=1}^{4} b_k^2 e^{2i\lambda_k}) = Ce^{i\eta}\), then from Eq. (12) we get
\[ b_j C \cos (2\lambda_j - \eta) = b_j C^2. \]  
(14)

If $C$ is equal to 0, this means $U_d$ can change the maximal entangled states to the product states. Actually this question has been solved by Kraus and Cirac [16] though they considered a different question. We write the result here: if
\[ \alpha_1 + \alpha_2 \geq \pi/4 \text{ and } \alpha_2 + \alpha_3 \leq \pi/4, \]  
(15)

then the two-qubit unitary operation $U_d$ can change maximally entangled pure qubit states to product states along with local unitary operations. In the following we focus on the cases where $C$ is not 0. Now the Eq. (14) becomes
\[ b_j \cos (2\lambda_j - \eta) = b_j C. \]  
(16)

One possible solution of the Eq. (16) is $b_j = 0$. But there must have some nonzero coefficients. If there are only one nonzero coefficient, the initial state is the eigenstate of $U_d$ and the final state is also a maximally entanglement state. So there are at least two nonzero coefficients. Suppose $b_k \neq 0$ and $b_l \neq 0$, then we have
\[ \cos (2\lambda_k - \eta) = \cos (2\lambda_l - \eta) = C. \]  
(17)

This means
\[ \lambda_k - \lambda_l = n\pi \text{ or } \lambda_k + \lambda_l - \eta = n\pi, \]  
(18)

where $n$ is an integer. Suppose no parameters $\lambda_k$ are equal. From the value range of $\alpha_{1,2,3}$ we can find that $\lambda_k - \lambda_l = n\pi$ is impossible. If there is another coefficient $b_m$ is also nonzero, it will satisfy $\lambda_k + \lambda_m - \eta = n'\pi$ for some integer $n'$. Then we can easily find that $\lambda_l - \lambda_m = (n - n')\pi$, but this is impossible. So there are only one pair of $\lambda_k$'s satisfy the Eq. (18). That is to say there are only two coefficients which are nonzero and we denote them $b_k$ and $b_l$. Now our purpose is to minimize the concurrence $C = |b_k^2 e^{2i\lambda_k} + b_l^2 e^{2i\lambda_l}|$ of the final state under the condition $b_k^2 + b_l^2 = 1$. Because
\[ C^2 = b_k^4 + b_l^4 + 2b_k^2 b_l^2 \cos (2\lambda_k - 2\lambda_l) \]  
(19)
\[ \geq |\cos (\lambda_k - \lambda_l)|^2, \]
so the minimal entanglement of the final state is $C_{\text{min}} = \min_{k,l} |\cos(\lambda_k - \lambda_l)|$, which is achieved when $b_k^2 = b_l^2 = 1/2$. This minimal entanglement $C_{\text{min}}$ is calculated when we suppose no parameters $\lambda_k$ are equal. Suppose the point $(\alpha_{10}, \alpha_{20}, \alpha_{30})$ in parameter space makes some parameters $\lambda_k$ equal, but for arbitrary small positive number $\xi$, there always exists some point $(\alpha'_1, \alpha'_2, \alpha'_3)$ which can not make any two parameters $\lambda_k$ equal, where $|\alpha'_1 - \alpha_{10}| + |\alpha'_2 - \alpha_{20}| + |\alpha'_3 - \alpha_{30}| < \xi$. So this constraint can be removed by continuity.

The result in this section can be applied in gate simulation. A maximal entangled state can be used to implement deterministic controlled unitary operations if local operations and classical communication (LOCC) are permitted [11, 12]. If a two-qubit unitary gate can change some product initial state into a maximal entangled state, then it can be used to simulate controlled unitary operations under LOCC. If the non-local operation can not change some product state into a maximal entangled one, we can let it act on an initially entangled state to get a maximally entangled one. Then what is the minimum entanglement of the initial state? It is $C_{\text{min}}$ we calculate in this section. We emphasize that if we can use ancillas the situation will be different. For example, swap gate can not change a nonmaximally entangled state into a maximal one without ancillas, but it can produce two maximally entangled states from product states when ancillas are permitted.

IV. ENTANGLEMENT CHANGING POWER

Assume that the entanglement of the initial pure two-qubit state is $C_0$. A nonlocal $U_d$ acts on this state and we want to know the possible maximal and minimal entanglement of the final state. Some results have been derived and we list them here:

When $C_0$ is $0$ [16], the minimal entanglement of the final state $C_{\text{min}}$ is $0$ [19]. The maximal entanglement of the final state $C_{\text{max}}$ is $1$ if $\alpha_1 + \alpha_2 \geq \pi/4$ and $\alpha_2 + \alpha_3 \leq \pi/4$, otherwise it is $\max_{k,l} |\sin(\lambda_k - \lambda_l)|$. This maximal entanglement of the final state has special use and we name it $C_{0\text{max}}$.

When $C_0$ is $1$, the maximal entanglement of the final state $C_{\text{max}}$ is $1$. The minimal entanglement of the final state $C_{\text{min}}$ is $0$ if $\alpha_1 + \alpha_2 \geq \pi/4$ and $\alpha_2 + \alpha_3 \leq \pi/4$, otherwise it is $\min_{k,l} |\cos(\lambda_k - \lambda_l)|$. This minimal entanglement of the final state also has special use and we name it $C_{1\text{min}}$.

For a general $C_0$, we first want to know whether the entanglement of the final state can
be 1 and 0. Now the question is easy to answer. If \( C_0 \geq C_{1\text{min}} \), then the entanglement of final state can be 1. If \( C_0 \leq C_{0\text{max}} \), then the entanglement of final state can be 0. So in the following we do not concern the question where the entanglement of final state is 1 or 0.

We write the initial state in the magic basis: \( |\Psi_0\rangle = \sum_{j=1}^{4} b_j |\Phi_j\rangle \). The coefficients satisfy two conditions: \( \sum_{j=1}^{4} |b_j|^2 = 1 \) and \( \left| \sum_{j=1}^{4} b_j^2 \right|^2 = C_0^2 \). We want to calculate the possible maximal and minimal entanglement of the final state, \( C = \left| \sum_{j=1}^{4} b_j^2 e^{2i\lambda_j} \right| \). We define a Lagrangian function

\[
L = \left| \sum_{j=1}^{4} b_j^2 e^{2i\lambda_j} \right|^2 - \mu_1 \left( \sum_{j=1}^{4} |b_j|^2 - 1 \right) - \mu_2 \left( \left| \sum_{j=1}^{4} b_j^2 \right|^2 - C_0^2 \right),
\]

where \( \mu_1 \) and \( \mu_2 \) are real. Differentiating gives

\[
2b_j e^{2i\lambda_j} \left( \sum_{j=1}^{4} (b_j^*)^2 e^{-2i\lambda_j} \right) - \mu_1 b_j^* - 2\mu_2 b_j \sum_{j=1}^{4} (b_j^*)^2 = 0.
\]

Multiplying \( b_j \) and summing over \( j \) gives

\[
\mu_1 = 2C^2 - 2\mu_2 C_0^2.
\]

We write \( \sum_{j=1}^{4} (b_j^*)^2 e^{-2i\lambda_j} = C e^{2i\eta} \) and \( \sum_{j=1}^{4} (b_j^*)^2 = C_0 e^{2i\epsilon} \). Substituting them into Eq. (21), we get

\[
2b_j C e^{2i(\lambda_j + \eta)} - \mu_1 b_j^* - 2\mu_2 b_j C_0 e^{2i\epsilon} = 0.
\]

One possible solution of the Eq. (23) is \( b_j = 0 \). To find nonzero \( b_j \), we write \( b_j = \beta_j e^{i\gamma_j} \). Then the Eq. (23) becomes

\[
C^2 - \mu_2 C_0^2 - e^{2i(\lambda_j + \eta + \gamma_j)} + \mu_2 C_0 e^{2i(\gamma_j + \epsilon)} = 0.
\]

If \( \mu_2 = 0 \), then \( C^2 - C e^{2i(\lambda_j + \eta + \gamma_j)} = 0 \). Because \( C \) is real, it will be 0 or 1. We have found the condition that the entanglement of the final state is 0 or 1. So we assume that \( \mu_2 \) is nonzero in the following. If there is only one nonzero coefficient, the initial and final state will both be maximally entangled, which is trivial. So there are at least two nonzero coefficients. Assume \( b_j \) and \( b_k \) are nonzero. Similar to Eq. (24), we have

\[
C^2 - \mu_2 C_0^2 - e^{2i(\lambda_k + \eta + \gamma_k)} + \mu_2 C_0 e^{2i(\gamma_k + \epsilon)} = 0.
\]

Subtract Eq. (25) from (24), we get

\[
(e^{2i(\lambda_j + \eta + \gamma_j)} - e^{2i(\lambda_k + \eta + \gamma_k)}) C = \mu_2 C_0 (e^{2i(\gamma_j + \epsilon)} - e^{2i(\gamma_k + \epsilon)}).
\]
Simplify Eq. (26), we get
\[
\sin (\lambda_j - \lambda_k + \gamma_j - \gamma_k) C = e^{i(2\epsilon - 2\eta - \lambda_j - \lambda_k)} \mu_2 C_0 \sin (\gamma_j - \gamma_k). \tag{27}
\]
We assume no \(\lambda'_j\)'s are equal. Because \(\mu_2 C_0 C\) is nonzero and \(\sin (\gamma_j - \gamma_k)\) can not be 0, from Eq. (27) we can get
\[
2\epsilon - 2\eta - \lambda_j - \lambda_k = n\pi, \; n \in \mathbb{Z}. \tag{28}
\]
Because no \(\lambda'_j\)'s are equal, there is only one pair of index \((j, k)\) satisfying Eq. (28). So there are only one pair of nonzero coefficients. Now Eq. (21) becomes
\[
2b_j e^{2i\lambda_j} \left( (b_j^*)^2 e^{-2i\lambda_j} + (b_k^*)^2 e^{-2i\lambda_k} \right) - \mu_1 b_j^* - 2\mu_2 b_j \left( (b_j^*)^2 + (b_k^*)^2 \right) = 0. \tag{29}
\]
Substituting \(b_j = \beta_j e^{i\gamma_j}\) and \(b_k = \beta_k e^{i\gamma_k}\) into Eq. (29), we have
\[
2\beta_j^2 + 2\beta_k^2 \cos (\alpha + \beta) - \mu_1 - 2\mu_2 \left( \beta_j^2 + \beta_k^2 \cos \alpha \right) = 0, \tag{30}
\]
\[
\sin (\alpha + \beta) = \mu_2 \sin \alpha, \tag{31}
\]
where \(\alpha = 2(\gamma_j - \gamma_k), \beta = 2(\lambda_j - \lambda_k)\). Similarly we have
\[
2\beta_k^2 + 2\beta_j^2 \cos (\alpha + \beta) - \mu_1 - 2\mu_2 \left( \beta_k^2 + \beta_j^2 \cos \alpha \right) = 0. \tag{32}
\]
Subtract Eq. (32) from (30), we get
\[
(\beta_j^2 - \beta_k^2) \left[ \sin^2 \left( \frac{\alpha + \beta}{2} \right) - \mu_2 \sin^2 \left( \frac{\alpha}{2} \right) \right] = 0. \tag{33}
\]
If \(\beta_j^2 \neq \beta_k^2\), then we have
\[
\left[ \sin^2 \left( \frac{\alpha + \beta}{2} \right) - \mu_2 \sin^2 \left( \frac{\alpha}{2} \right) \right] = 0. \tag{34}
\]
From Eq. (31) and (34), we have \(\beta/2 = n\pi\), where \(n\) is an integer. This result contradicts with our assumption that no \(\lambda'_j\)'s are equal. So we have \(\beta_j^2 = \beta_k^2 = 1/2\). Now we rewrite the concurrence of the initial state
\[
C_0^2 = |\beta_j e^{2i\gamma_j} + \beta_k e^{2i\gamma_k}|^2 = \cos^2 (\gamma_j - \gamma_k). \tag{35}
\]
So
\[
\gamma_j - \gamma_k = n\pi \pm \arccos C_0. \tag{36}
\]
The concurrence of the final state

\[ C^2 = \left| \beta_j^2 e^{2i(\gamma_j + \lambda_j)} + \beta_k^2 e^{2i(\gamma_k + \lambda_k)} \right|^2 \]

\[ = \cos^2 (\gamma_j - \gamma_k + \lambda_j - \lambda_k) \]

\[ = \cos^2 (\arccos C_0 \pm (\lambda_j - \lambda_k)) \].

So the maximal possible concurrence of the final state is

\[ C_{\text{max}} = \max_{j,k} \left| \cos (\arccos C_0 + (\lambda_j - \lambda_k)) \right| \].

(38)

The minimal possible concurrence of the final state is

\[ C_{\text{min}} = \min_{j,k} \left| \cos (\arccos C_0 + (\lambda_j - \lambda_k)) \right| \].

(39)

These results are derived when we assume no \( \lambda_k \)'s are equal. This constraint can be removed by continuity and the reason is the same as explained in Sec. III.

V. CONCLUSION

In this paper we discuss the entanglement changing power of two-qubit unitary operations without ancillas. A two-qubit unitary operation is characterized by \( \alpha_1, \alpha_2, \alpha_3 \), or \( \gamma_1, \gamma_2, \gamma_3, \) and \( \gamma_4 \). We consider a two-qubit unitary operation along with arbitrary local unitary operations act on a two-qubit pure state, whose entanglement is \( C_0 \). When the initial state is non-entangled (\( C_0 = 0 \)), Kraus and Cirac have calculated its reachable maximal entanglement and we name it \( C_{0\text{max}} \). When the initial state is maximally entangled, we calculate its reachable minimal entanglement and we name it \( C_{1\text{min}} \). Then we give the conditions that the final state can be maximally entangled and be non-entangled: when \( C_0 \geq C_{1\text{min}} \), the final state can be maximally entangled; when \( C_0 \leq C_{0\text{max}} \), the final state can be non-entangled. When the final state can not be maximally entangled, we give its reachable maximal entanglement in Eq. (38). When the final state can not be non-entangled, we give its reachable minimal entanglement in Eq. (39). We think \( C_{\text{max}} \) and \( C_{\text{min}} \) represent the entanglement changing power of two-qubit unitary operations. Now we write our results in an unified form, which is much easier to operate.

If \( \alpha_1 + \alpha_2 \geq \pi/4 \) and \( \alpha_2 + \alpha_3 \leq \pi/4 \), \( C_{\text{max}} = 1 \) and \( C_{\text{min}} = 0 \).
If $\alpha_1 + \alpha_2 < \pi/4$ and $\alpha_2 + \alpha_3 \leq \pi/4$,

$$C_{\text{max}} = \cos \left( \max \left[ \arccos C_0 - 2 (\alpha_1 + \alpha_2), 0 \right] \right)$$

and

$$C_{\text{min}} = \cos \left( \min \left[ \arccos C_0 + 2 (\alpha_1 + \alpha_2), \frac{\pi}{2} \right] \right).$$

If $\alpha_1 + \alpha_2 \geq \pi/4$ and $\alpha_2 + \alpha_3 > \pi/4$,

$$C_{\text{max}} = \cos \left( \max \left[ \arccos C_0 - 2 \left( \frac{\pi}{2} - \alpha_2 - \alpha_3 \right), 0 \right] \right)$$

and

$$C_{\text{min}} = \cos \left( \min \left[ \arccos C_0 + 2 \left( \frac{\pi}{2} - \alpha_2 - \alpha_3 \right), \frac{\pi}{2} \right] \right).$$

It can be easily found that for the same initial state entanglement, the set $[C_{\text{min}}, C_{\text{max}}]$ of one two-qubit gate is a subset of another’s, or vice versa, and this relation will not change for arbitrary $C_0$. So we can define an order of gates. We say two gates are equal if their sets $[C_{\text{min}}, C_{\text{max}}]$ are equal for any $C_0$. If the set $[C_{\text{min}}, C_{\text{max}}]$ of gate $U_1$ is a true subset of gate $U_2$’s for some $C_0$, we say $U_1 < U_2$. Because swap gate does not change entanglement when there is no ancilla, it is the smallest gate according to this order.

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