Research Article

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Analytical properties of the two-variables Jacobi matrix polynomials with applications

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Abstract: In the current study, we introduce the two-variable analogue of Jacobi matrix polynomials. Some properties of these polynomials such as generating matrix functions, a Rodrigue-type formula and recurrence relations are also derived. Furthermore, some relationships and applications are reported.

Keywords: Jacobi matrix polynomials, generating matrix relations, recurrence formulas

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1 Introduction

The generating function of the classical Jacobi polynomials is given by (cf., e.g., [1,2])

\[ \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(z) t^n = 2^{\alpha+\beta} q^{-1}(1-t+q)^{-\alpha}(1+t+q)^{-\beta}, \]

(1.1)

where

\[ P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n! \Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)} \left( \frac{z-1}{2} \right)^m, \]

\[ q = q(z, t) = (1 - 2zt + t^2)^{\frac{1}{2}} \] and \( \Gamma(.) \) is the usual gamma function.

These polynomials are generalizations of several families of orthogonal polynomials like the Legendre, Chebyshev and Gegenbauer (ultraspherical) polynomials. In addition, the classical orthogonal polynomials of Jacobi have played important roles in many different applications of mathematics, physics and engineering sciences (see, e.g., [1–6]).

In contrast, the special functions and polynomials with matrix parameters have many applications in various areas of mathematical analysis, physics, probability theory, statistics and engineering (see, [7–16]). One particular special matrix polynomial which frequently appears in the recent investigations is the Jacobi matrix polynomial (JMP) that has been introduced in [17–19].

The aim of the present work is to study two-variable analogue of Jacobi matrix polynomials (2VAJMP) \( J_n^{(E,F,z,w)} \) and their properties, which have been proposed on the pattern for two-variables Konhauser matrix polynomials [20], two-variable Shivley’s matrix polynomials [21], two-variable Laguerre matrix polynomials [22], two-variable Hermite generalized matrix polynomials [23], two-variable Gegenbauer matrix...
polynomials [24] and the second kind Chebyshev matrix polynomials with two variables [25]. The current work is assumed to be extensions to the matrix setting of the results of [26].

The paper is organized as follows. In Section 2, we summarize definition and previous results to be used in the following sections. Section 3 contains the definition of the 2VAJMP $J_f(E, F, z, w)$, for parameter matrices $E$ and $F$ associated with some generating matrix relations. A Rodrigue-type formula and recurrence relations for 2VAJMP $J_n(E, F, z, w)$ are archived in Section 4. Finally, we give some relationships and applications in Section 5.

2 Preliminaries

In this section, we recall some definitions and facts whose more detailed accounts and applications can be found in [7,16].

If $E$ is a matrix in the complex space $\mathbb{C}^{d \times d}$ ($d \in \mathbb{N}$), its spectrum $\sigma(E)$ denotes the set of all the eigenvalues of $E$ and $\mu(E) = \max \{\Re(z) : z \in \sigma(E)\}$, $\nu(E) = \min \{\Re(z) : z \in \sigma(E)\}$. The square matrix $E$ is said to be positive stable if and only if $\nu(E) > 0$. $I$ and $0$ stand for the identity matrix and the null matrix in $\mathbb{C}^{d \times d}$, respectively.

If $\Phi(z)$ and $\Psi(z)$ are holomorphic functions of the complex variable $z$, which are defined in an open set $\Omega$ of the complex plane, and $E$ is a matrix in $\mathbb{C}^{d \times d}$ such that $\sigma(E) \subset \Omega$, then from the properties of the matrix functional calculus, it follows that

$$\Phi(E) \Psi(E) = \Psi(E) \Phi(E).$$

Hence, if $F$ in $\mathbb{C}^{d \times d}$ is a matrix for which $\sigma(F) \subset \Omega$ and also if $E F = F E$, then

$$\Phi(E) \Psi(F) = \Psi(F) \Phi(E).$$

By application of the matrix functional calculus, for $E$ in $\mathbb{C}^{d \times d}$, then from [7], the Pochhammer symbol or shifted factorial defined by

$$(E)_n = \begin{cases} E^{(n)} & n \geq 1, \\ I, & n = 0, \end{cases}$$

with the condition

$$E + n I \quad \text{is invertible for all integers } n \geq 0.$$  

**Definition 2.1.** [27] If $E$ is a matrix in $\mathbb{C}^{d \times d}$, such that $\Re(z) > 0$ for all eigenvalues $z$ of $E$, then $\Gamma(E)$ is well defined as

$$\Gamma(E) = \int_0^\infty \tau^{E-I} e^{-\tau} d\tau; \quad \tau^{E-I} = \exp((E-I) \ln \tau).$$

**Definition 2.2.** [7] Suppose that $N_1$, $N_2$ and $N_3$ are matrices in $\mathbb{C}^{d \times d}$, such that $N_3$ satisfies the condition (2.4). Then, the hypergeometric matrix function $\gin_2 F_1(N_1, N_2; N_3; z)$ is given by

$$\gin_2 F_1(N_1, N_2; N_3; z) = \sum_{n=0}^\infty (N_1)_n (N_2)_n (N_3)_n^{-1} \frac{z^n}{n!}.$$  

**Definition 2.3.** [7,17,18] Let $E$ and $F$ be positive stable matrices in $\mathbb{C}^{d \times d}$, then the JMP $\gin_2 F_1^E (z)$ is defined by

$$\gin_2 F_1^E (z) = \frac{(E + I)_n}{n!} - n I, \quad \frac{E + F + (n + 1) I}{E + I} \left[ 1 - \frac{z}{2} \right].$$
Definition 2.4. [7,11] Let $N_1, N_2, N_3$ and $N_4$ be commutative matrices in $\mathbb{C}^{d \times d}$ with $N_3 + nI$ and $N_4 + nI$ invertible for all integers $n \geq 0$. Then the four Appell hypergeometric matrix function $F_4[N_1, N_2, N_3, N_4; z, w]$ is defined in the following form:

$$F_4[N_1, N_2, N_3, N_4; z, w] = \sum_{\mu, \nu = 0}^{\infty} \frac{(N_1)_{\mu+1}(N_2)_{\nu+1}(N_3)_{\mu}\mu^{-1}(N_4)_{\nu}\nu^{-1}}{\mu!\nu!}z^\mu w^\nu, \quad (|z|^2 + |w|^2 < 1).$$ (2.8)

In [28], it was shown that JMPs are generated by

$$\sum_{n=0}^{\infty} P_{n,E}F(z)t^n = F_3[I + F, I + E, I + F; \frac{(z - 1)t}{2}; \frac{(z + 1)t}{2}],$$ (2.9)

where $E, F \in \mathbb{C}^{d \times d}$ with $E + nI$ and $F + nI$ invertible for every integer $n \geq 0$ and $EF = FE$.

3 Some generating relations for 2VAJMP

Let $E$ and $F$ be matrices in the complex space $\mathbb{C}^{d \times d}$, satisfying the conditions

$$\text{Re}(z) > -1 \quad \forall z \in \sigma(E), \quad \text{Re}(z) > -1 \quad \forall z \in \sigma(F) \quad \text{and} \quad EF = FE.$$ (3.1)

In view of (1.1), we define 2VAJMP with matrix generating form:

$$\sum_{n=0}^{\infty} \mathcal{J}_n(E, F, z, w)t^n = 2^{E+F}R^{-1}(1 + \sqrt{w}t + R)^{-E}(1 - \sqrt{w}t + R)^{-F},$$ (3.2)

where $\mathcal{R} = \mathcal{R}(z, w, t) = (1 - 2zt + wt^2)\frac{1}{2}$.

In this section, we obtain power series and matrix generating relations for 2VAJMP $\mathcal{J}_n(E, F, z, w)$.

Theorem 3.1. Let $E$ and $F$ be positive stable matrices in $\mathbb{C}^{d \times d}$ such that $E + nI$ and $F + nI$ are invertible for all integers $n \geq 0$. Then $\mathcal{J}_n(E, F, z, w)$ takes the following explicit forms:

$$\mathcal{J}_n(E, F, z, w) = \sum_{s=0}^{n} \frac{(E + I)_n(E + I)_{n-s}}{s!(n-s)!}(E + I)_n^{-1}[(E + I)_{n-s}]^{-1}\left(\frac{Z - \sqrt{w}}{2}\right)\left(\frac{Z + \sqrt{w}}{2}\right)^{n-s}$$ (3.3)

or

$$\mathcal{J}_n(E, F, z, w) = \sum_{s=0}^{n} \frac{(E + I)_n(E + F + I)_{n-s}}{s!(n-s)!}(E + F + I)_n^{-1}[(E + F + I)_{n-s}]^{-1}\left(\frac{Z - \sqrt{w}}{2}\right)^{n-s}.$$ (3.4)

Proof. Using the relation (2.9) with the following result (see [29]):

$$F_3[P, Q; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}] = (1 - xy)^{-1}(1 - x)^{p}(1 - y)^{q},$$ (3.5)

where $F_3$ is defined in (2.8) with $P$ and $Q$ positive stable matrices in $\mathbb{C}^{d \times d}$.

If we take $P = I + F$, $Q = I + E$ and

$$\begin{cases} -x &= t(z - 1) \\ (1-x)(1-y) &= \frac{2}{t} \\ -y &= t(z + 1) \\ (1-x)(1-y) &= \frac{2}{t}, \end{cases}$$ (3.6)
From (3.2), let us consider

\[
\begin{align*}
\mathcal{R} &= (1 - 2zt + wt^2)^{1/2}, \\
x &= 1 - \frac{2}{1 + \sqrt{\mathcal{W}}t + \mathcal{R}}, \\
y &= 1 - \frac{2}{1 - \sqrt{\mathcal{W}}t + \mathcal{R}}.
\end{align*}
\] (3.7)

Using (3.7), it follows that

\[
\frac{-x}{(1 - x)(1 - y)} = \frac{1}{1 - y} \left( 1 - \frac{1}{1 - x} \right) = \frac{1 - \sqrt{\mathcal{W}}t + \mathcal{R}}{2} \left( 1 - \frac{1 + \sqrt{\mathcal{W}}t + \mathcal{R}}{2} \right) = \frac{(1 - \sqrt{\mathcal{W}}t + \mathcal{R})(1 - \sqrt{\mathcal{W}}t - \mathcal{R})}{4} = \frac{(1 - \sqrt{\mathcal{W}}t)^2 - \mathcal{R}^2}{4} = \frac{t(z - \sqrt{\mathcal{W}})}{2}. 
\] (3.8)

Similarly,

\[
\frac{-y}{(1 - x)(1 - y)} = \frac{t(z + \sqrt{\mathcal{W}})}{2}. 
\] (3.9)

Now, according to (2.9), (3.8) and (3.9), we have

\[
\sum_{n=0}^{\infty} f_d(E, F, z, w) t^n = F_{k} \left[ I + F, I + E, t ; \frac{1}{2}(z - \sqrt{\mathcal{W}}), \frac{1}{2}(z + \sqrt{\mathcal{W}}) \right].
\] (3.10)

Therefore, equation (3.10) can be rewritten as

\[
\sum_{n=0}^{\infty} f_d(E, F, z, w) t^n = \sum_{n,s=0}^{\infty} \frac{(E + I)_{n+s} (F + I)_{n+s}}{s! n!} [(E + I)_{s}]^{-1} [(F + I)_{n}]^{-1} \frac{1}{2}(z - \sqrt{\mathcal{W}})^s \frac{1}{2}(z + \sqrt{\mathcal{W}})^n t^n,
\] (3.11)

which coincides with our assertion (3.3). □

**Remark 3.1.** It may be noted that for \( w = 1 \), (2VAJMP) \( f_d(E, F, z, w) \) are reduced to JMPs of one variable \( F_{\nu}^{(E,F)}(z) \) (see [17]).

**Remark 3.2.** From (3.3) and (3.4), we have the matrix representation for (2VAJMP) \( f_d(E, F, z, w) \) in terms of matrix hypergeometric series in the following forms:

\[
f_d(E, F, z, w) = \frac{(A + I)_{n}}{n!} (z + \sqrt{\mathcal{W}})^n \, _2F_1 \left[ -nI, -(F + nI) ; I + E \right],
\] (3.12)

and

\[
f_d(E, F, z, w) = \frac{(E + I)_{n}}{n!} \, _2F_1 \left[ -nI, E + F + I(n + 1) ; \frac{\sqrt{\mathcal{W}} - z}{2} \right].
\] (3.13)

**Remark 3.3.** For \( E = F = 0 \), (3.12) reduces to Legendre polynomials of two complex variables \( P_{\nu}(z, w) \) in the following form:

\[
P_{\nu}(z, w) = \left( \frac{z + \sqrt{\mathcal{W}}}{2} \right)^n \, _2F_1 \left[ -nI, -nI ; \frac{z - \sqrt{\mathcal{W}}}{2} \right].
\] (3.14)
Theorem 3.2. Let $E, F \in \mathbb{C}^{d \times d}$ be positive stable matrices, with $|z| < 1$, $\left| \frac{z - \sqrt{w}}{1 - t} \right| < 1$. The generating matrix function of $J(E, F, z, w)$ is as follows:

$$
\sum_{\nu=0}^{\infty} (E + F + I)_{\nu} [(E + 1)_{\nu}]^{-1} J(E, F, z, w) t^\nu
$$

$$
= (1 - t)^{-I(E + F, F)} F_2 \left[ \frac{1}{2} (E + F + I), \frac{1}{2} (2 I + E + F); \frac{2t (z - \sqrt{w})}{(1 - t)^2} \right]
$$

(3.15)

Proof. For convenience, suppose that the left-hand side of (3.15) is denoted by $\Xi$. From relation (3.3) to $\Xi$, we obtain

$$
\Xi = \sum_{\nu=0}^{\infty} (E + F + I)_{\nu} [(E + 1)_{\nu}]^{-1} J(E, F, z, w) t^\nu
$$

$$
\Xi = \sum_{\nu=0}^{\infty} \sum_{k=0}^{\nu+1} \frac{(E + F + I)_{\nu+k}}{k! (\nu - k)!} [(E + 1)_{k}]^{-1} \left( \frac{z}{2} - \frac{\sqrt{w}}{2} \right)^k t^\nu
$$

$$
\Xi = \sum_{\nu=0}^{\infty} \sum_{k=0}^{\nu+1} \frac{(E + F + I)_{\nu+k}}{k! (\nu - k)!} [(E + 1)_{k}]^{-1} \left( \frac{z}{2} - \frac{\sqrt{w}}{2} \right)^k t^\nu
$$

$$
\Xi = \sum_{k=0}^{\infty} k! 2^k \nu! [(E + F + I)_{k}]^{-1} \left( \frac{z}{2} - \frac{\sqrt{w}}{2} \right)^k t^\nu
$$

Using the identity

$$
(E)_{2k} = 2^{2k} \left( \frac{E}{2} \right)_k \left( \frac{E + I}{2} \right)_k
$$

evidently leads us to the required result in (3.15).

Remark 3.4. Setting $w = 1$ in (3.15), we get the result of [28] given by

$$
\sum_{n=0}^{\infty} (E + F + I)_n \theta_n^{(E, F)}(z) [(E + 1)_n]^{-1} t^n = (1 - t)^{-I(E + F, F)} F_2 \left[ \frac{E + F + I}{2}, \frac{E + F + 2I}{2}; \frac{2t (z - \sqrt{w})}{(1 - t)^2} \right]
$$

Theorem 3.3. Let $E$ and $F$ be positive stable matrices in $\mathbb{C}^{d \times d}$ such that $E$ and $F$ satisfy the spectral condition (2.4) with $\left| \frac{t}{2} (z - \sqrt{w}) \right| < 1$ and $\left| \frac{t}{2} (z + \sqrt{w}) \right| < 1$, the following Bateman’s generating matrix function holds true:

$$
\sum_{\nu=0}^{\infty} J(E, F, z, w) [(E + 1)_{\nu}]^{-1} [(F + 1)_{\nu}]^{-1} t^\nu = o_{F_1} \left[ - \frac{E + I}{2} (z - \sqrt{w}) \right] \times o_{F_1} \left[ - \frac{F + I}{2} (z + \sqrt{w}) \right]
$$

(3.16)

Proof. Starting with the explicit series

$$
\sum_{\nu=0}^{\infty} J(E, F, z, w) [(E + 1)_{\nu}]^{-1} [(F + 1)_{\nu}]^{-1} t^\nu
$$

$$
= \sum_{\nu=0}^{\infty} \sum_{k=0}^{\nu} \left[ \frac{1}{2} (z - \sqrt{w}) \right]^{k} \left[ \frac{1}{2} (z + \sqrt{w}) \right]^{\nu - k} [(E + 1)_{k}]^{-1} [(F + 1)_{\nu-k}]^{-1} t^\nu
$$

$$
= \sum_{\nu=0}^{\infty} \sum_{k=0}^{\nu} \left[ \frac{1}{2} (z - \sqrt{w}) \right]^{k} \left[ \frac{1}{2} (z + \sqrt{w}) \right]^{\nu - k} [(E + 1)_{k}]^{-1} [(F + 1)_{\nu-k}]^{-1} t^\nu
$$

$$
= o_{F_1} \left[ - \frac{E + I}{2} (z - \sqrt{w}) \right] \times o_{F_1} \left[ - \frac{F + I}{2} (z + \sqrt{w}) \right]
$$

which is precisely assertion (3.16) of Theorem 3.3.
Working on the same lines as in previous theorems, we state matrix version of Brafman’s generating function in the next theorem without proof.

Theorem 3.4. Let $D, E$ and $F$ be commutative matrices in $\mathbb{C}^{d \times d}$ such that $E$ and $F$ satisfy the spectral condition (2.4) with $\frac{1 - \sqrt{w - R}}{2} < 1$ and $\frac{1 + \sqrt{w - R}}{2} < 1$, the following Brafman’s generating matrix function holds true:

$$
\sum_{v=0}^{\infty} (D)_v (E + F - D + I)_v [(E + I)_v]^{-1} [(F + I)_v]^{-1} J_v (E, F, z, w) t^v
$$

$$
= 2 F_1 \left[ D, E + F - D + I ; \frac{1 - \sqrt{w t - R}}{2} \right] 2 F_1 \left[ D, E + F - D + I ; \frac{1 + \sqrt{w t - R}}{2} \right], \quad R = (1 - 2zt + wt^2)^{-1}.
$$

4 Rodrigues’ formula and recurrence relations

Two more basic properties of the 2VAJMP $J_d(E, F, z, w)$ are developed in this section, which enjoy a Rodrigues’ formula obtained from Theorem 4.1 and with the help of (3.3). Also, some various recurrence relations for the 2VAJMP are given.

4.1 Rodrigues’ formula

Theorem 4.1. Let $E$ and $F$ be matrices in $\mathbb{C}^{d \times d}$ that satisfy (3.1). Then the 2VAJMP $J_d(E, F, z, w)$ defined in (3.3) may be expressed as

$$
J_d(E, F, z, w) = \frac{(z - \sqrt{w})^E (z + \sqrt{w})^F}{2^n n!} D^n [z - \sqrt{w}]^{E^+ n} (z + \sqrt{w})^{F^+ n}], \quad D \equiv \frac{d}{d(z \pm \sqrt{w})}.
$$

Proof. We begin with the explicit expression (3.3) for $J_d(E, F, z, w)$:

$$
J_d(E, F, z, w) = \sum_{s=0}^{n} \frac{(E + I)_m (F + I)_n}{2^n s!(n - s)!} [(E + I)_s]^{-1} [(F + I)_n]^{-1} (z - \sqrt{w})^s (z + \sqrt{w})^{n - s}.
$$

We recall that (cf., e.g., [17,18])

$$
D^k z^{m+E} = (E + I)_m [(E + I)_{m-k}]^{-1} z^{(m-k)+E}, \quad D \equiv \frac{d}{dz}.
$$

In view of (4.3), we get

$$
D^k (z + \sqrt{w})^{F^+ m} = [(F + I)_{m-k}]^{-1} (F + I)_m (z + \sqrt{w})^{m-k+F} (4.4)
$$

and

$$
D^{m-k} (z - \sqrt{w})^{F^+ n} = [(F + I)_m]^{-1} (E + I)_m (z - \sqrt{w})^{E+kF} (4.5)
$$

Using (4.4) and (4.5) in (4.1), it follows that

$$
J_d(E, F, z, w) = \frac{(z - \sqrt{w})^E (z + \sqrt{w})^F}{2^n n!} \sum_{s=0}^{\infty} \frac{n^s}{s! (n - s)!} [D^n (z - \sqrt{w})^{E^+ n}] [D^s (z + \sqrt{w})^{F^+ n}]
$$

Applying the Leibnitz rule yields the desired result of Theorem 4.1.
4.2 Recurrence relations

Following some various matrix recurrence relations satisfied by 2VAJMP $J_n(E, F, z, w)$ in (3.2) as follows:

First, the 2VAJMP $J_n(E, F, z, w)$ satisfy the following total differential matrix recurrence relations:

$$
\left( z - \frac{\sqrt{w}}{\sqrt{w}} \right) [(E + F + nl)Df_n(E, F, z, w) + (E + In)f_n(E, F, z, w)] = (E + F + nl) [nJ_n(E, F, z, w) - (E + nl)f_{n-1}(E, F, z, w)], \quad D = \frac{d}{d\left( \frac{z}{\sqrt{w}} \right)}.
$$

(4.6)

$$
\left( z + \frac{\sqrt{w}}{\sqrt{w}} \right) [(E + F + nl)Df_n(E, F, z, w) + (F + In)f_n(E, F, z, w)] = (E + F + nl) [nJ_n(E, F, z, w) - (F + nl)f_{n-1}(E, F, z, w)],
$$

(4.7)

$$
\left( z - \frac{\sqrt{w}}{\sqrt{w}} \right) Df_n(E, F, z, w) - nf_{n-1}(E, F, z, w)
$$

$$
= -(E + I)_n[(E + F + I)_n]^{-1} \sum_{s=0}^{n-1} (E + F + 2kI)_n[(E + I)_n]^{-1}
$$

$$
\times \left[(E + F + nl)f_n(E, F, z, w) + 2\left( \frac{z - \sqrt{w}}{\sqrt{w}} \right)Df_n(E, F, z, w) \right],
$$

(4.8)

$$
\left( z - \frac{\sqrt{w}}{\sqrt{w}} \right) Df_n(E, F, z, w) - nf_{n-1}(E, F, z, w)
$$

$$
= (E + I)_n[(E + F + I)_n]^{-1} \sum_{s=0}^{n-1} (-1)^{n-s}(E + F + 2kI)_n[(E + I)_n]^{-1}f_n(E, F, z, w),
$$

(4.9)

$$
(E + F + 2nI) \left( \frac{z^2 - w}{w} \right) (E + F + 2nl)Df_n(E, F, z, w)
$$

$$
= n \left[ (F - E) \frac{\sqrt{w} + z(E + F + 2nl)}{\sqrt{w}} \right]f_n(E, F, z, w) - 2(E + nl)(F + nl)f_{n-1}(E, F, z, w),
$$

(4.10)

$$
2z(E + F + nl)Df_n(E, F, z, w) + [z(E - F) - (E + F + 2nl)\sqrt{w}]Df_{n-1}(E, F, z, w)
$$

$$
= \sqrt{w}(E + F + nl)2nf_n(E, F, z, w) - (E - F)f_{n-1}(E, F, z, w),
$$

(4.11)

$$
2\sqrt{w}(E + F + nl)Df_n(E, F, z, w) + [\sqrt{w}(E - F) - (E + F + 2nl)z]Df_{n-1}(E, F, z, w)
$$

$$
= \sqrt{w}(E + F + nl)(E + F + 2nl)f_{n-1}(E, F, z, w),
$$

(4.12)

and

$$
D^s f_n(E, F, z, w) = z^{-s}(E + F + (n + 1)l)_{s}f_{n-s}(E + sl, F + sl, z, w), \quad 0 < s \leq n,
$$

(4.13)

where $E$ and $F$ are matrices in $C^{d \times d}$ that satisfy conditions (3.1).

Second, the 2VAJMP $J_n(E, F, z, w)$ give the following pure matrix recurrence relation:

$$
2n(E + F + nl)(E + F + 2(n - 1)l)f_n(E, F, z, w)
$$

$$
= (E + F + (2n - 1)l) \left[ \frac{\sqrt{w}(E^2 - F^2) + z(E + F + 2nl)(E + F + 2l(n - 1))}{\sqrt{w}} \right]f_{n-1}(E, F, z, w)
$$

$$
- 2(E + (n - 1)l)(F + (n - 1)l)(E + F + 2nl)f_{n-1}(E, F, z, w),
$$

where $E$ and $F$ are matrices in $C^{d \times d}$ that satisfy conditions (3.1).

**Remark 4.1.** For $w = 1$, we obtain matrix recurrence relations of the Jacobi matrix polynomial $P_n^{(E,F)}(z)$. 

5 Applications

In this section, we obtain some other interesting results and applications involving $\mathcal{J}_d(E, F, z, w)$ by the formalism developed in the above sections.

(i) Following relationships can easily be obtained from (3.2) as follows:

$$
\sum_{n=0}^{\infty} \mathcal{J}_d(E, F, z, w) t^n = 2^{E+F} R^{-1}(1 + \sqrt{w} t + R)^F(1 - \sqrt{w} t + R)^{-E} \\
= 2^{E+F}(1 - 2zt + w^2)^{\frac{1}{2}}(1 + \sqrt{w} t + R)^F(1 - \sqrt{w} t + R)^{-E} \\
= \sum_{n=0}^{\infty} P_n(z, w) t^n \left( \frac{1 + \sqrt{w} t + R}{2} \right)^F \left( \frac{1 - \sqrt{w} t + R}{2} \right)^{-E},
$$

where $R = (1 - 2zt + w^2)^{\frac{1}{2}}$, thus, we have

$$
\mathcal{J}_d(E, F, z, w) = P_d(z, w) \left( \frac{1 + \sqrt{w} t + R}{2} \right)^F \left( \frac{1 - \sqrt{w} t + R}{2} \right)^{-E},
$$

(5.1)

where $P_d(z, w)$ is the two-variable Legendre polynomial, putting $w = 1$ in (5.1), we obtain

$$
P^{E,F}_{1,n}(z) = P_d(z) \left( \frac{1 + t + R}{2} \right)^F \left( \frac{1 - t + R}{2} \right)^{-E},
$$

where $P_d(z)$ is the Legendre polynomial of one veritable (see [1]).

(ii) In Theorem 3.1, if $x, y$ and $R$ are chosen as

$$
\begin{align*}
\{ \mathcal{R} &= \mathcal{R}(\eta, z, w, t) = (\eta - 2zt + w^2)^{\frac{1}{2}} \\
x &= 1 - \frac{2\sqrt{\eta}}{\sqrt{\eta} + \sqrt{w} t + \mathcal{R}} \\
y &= 1 - \frac{2\sqrt{\eta}}{\sqrt{\eta} - \sqrt{w} t + \mathcal{R}}.
\end{align*}
$$

(5.2)

Then, according to (2.9) with the same way as in Theorem 3.1, we get

$$
\sum_{n=0}^{\infty} \mathcal{J}_d(E, F, z, w) t^n = (2\sqrt{\eta})^{E+F} R^{-1}(\sqrt{\eta} + \sqrt{w} t + R)^F(\sqrt{\eta} - \sqrt{w} t + R)^{-E}.
$$

Also, we can write

$$
\sum_{n=0}^{\infty} \mathcal{J}_d(E, F, z, w) t^n = F_n \left[ I + F, I + E, \frac{1}{2} (z - \sqrt{\eta} \sqrt{w}), \frac{1}{2} (z + \sqrt{\eta} \sqrt{w}) \right].
$$

(5.3)

(iii) Taking

$$
\begin{align*}
\{ \mathcal{R} &= (1 - 2(z + \eta)t + w^2)^{\frac{1}{2}} \\
x &= 1 - \frac{2}{1 + \sqrt{w} t + \mathcal{R}} \\
y &= 1 - \frac{2}{1 - \sqrt{w} t + \mathcal{R}}.
\end{align*}
$$

(5.4)

Then

$$
\frac{-x}{1 - x}(1 - y) = \frac{1}{1 - y} \left( 1 - \frac{1}{1 - x} \right) = 1 - \sqrt{w} t + \mathcal{R} \left( 1 - \frac{1}{1 - \sqrt{w} t + \mathcal{R}} \right).
$$
Hence,
\[
\frac{-x}{(1-x)(1-y)} = t\left( (z + \eta) - \sqrt{\omega} \right) \frac{1}{2}.
\]

Similarly,
\[
\frac{-y}{(1-x)(1-y)} = t\left( (z + \eta) + \sqrt{\omega} \right) \frac{1}{2}.
\] (5.5)

Using (5.4) and (5.5) in (2.9), we have
\[
\sum_{n=0}^{\infty} f_n(E, F, z + \eta, w) t^n = \left( 2^{E+F} R^{-1}(1 + \sqrt{\omega} t + R)^{-F}(1 - \sqrt{\omega} t + R)^{-E} \right)
\] (5.6)
or
\[
\sum_{n=0}^{\infty} f_n(E, F, z + \eta, w) t^n = F_1 \left[ \frac{I + F, I + E}{I + E, I + F}; \frac{t}{2}\left( (z + \eta) - \sqrt{\omega} \right), \frac{t}{2}\left( (z + \eta) + \sqrt{\omega} \right) \right].
\] (5.7)

(iv) Furthermore,
\[
\begin{aligned}
\mathcal{R} &= (1 - 2zt + (w + \eta)t^2)^{1/2}, \\
x &= 1 - \frac{1}{1 + \sqrt{(w + \eta)t + \mathcal{R}}}, \\
y &= 1 - \frac{2}{1 - \sqrt{(w + \eta)t + \mathcal{R}}},
\end{aligned}
\] (5.8)
give us
\[
\frac{-x}{(1-x)(1-y)} = \frac{1}{1-y} \left( 1 - \frac{1}{1-x} \right) = \frac{1 - \sqrt{(w + \eta)t + \mathcal{R}}}{2} \left( 1 - \frac{1 + \sqrt{(w + \eta)t + \mathcal{R}}}{2} \right) = t(z - \sqrt{(w + \eta)}) \frac{1}{2}.
\] (5.9)

Similarly,
\[
\frac{-y}{(1-x)(1-y)} = \frac{t(z + \sqrt{(w + \eta)})}{2}.
\] (5.10)

Applying (5.9) and (5.10) in (2.9), as follows that
\[
\sum_{n=0}^{\infty} f_n(E, F, z, (w + \eta)) t^n = \left( 2^{E+F} R^{-1}(1 + \sqrt{(w + \eta)t + \mathcal{R}})^{-F}(1 - \sqrt{(w + \eta)t + \mathcal{R}})^{-E} \right),
\] (5.11)
\[
\sum_{n=0}^{\infty} f_n(E, F, z, (w + \eta)) t^n = F_1 \left[ \frac{I + F, I + E}{I + E, I + F}; \frac{t}{2}\left( (z - \sqrt{(w + \eta)}) \right), \frac{t}{2}\left( (z + \sqrt{(w + \eta)}) \right) \right].
\] (5.12)

(v) Setting \( R = (1 - 2azt + beta wt^2)^{1/2}, \alpha, \beta \in C \), then from (2.9), we obtain
\[
\sum_{n=0}^{\infty} f_n(E, F, az, beta w) t^n = \left( 2^{E+F} R^{-1}(1 + \sqrt{(w + \eta)t + \mathcal{R}})^{-F}(1 - \sqrt{(w + \eta)t + \mathcal{R}})^{-E} \right),
\] (5.13)
\[
\sum_{n=0}^{\infty} f_n(E, F, az, beta w) t^n = F_1 \left[ \frac{I + F, I + E}{I + E, I + F}; \frac{t}{2}\left( (az - \sqrt{(w + \eta)}) \right), \frac{t}{2}\left( (az + \sqrt{(w + \eta)}) \right) \right].
\] (5.14)
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