Extended Importance Sampling for Reliability Analysis under Evidence Theory

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Abstract. In early engineering practice, the lack of data and information makes uncertainty difficult to deal with. However, evidence theory has been proposed to handle uncertainty with limited information as an alternative way to traditional probability theory. In this contribution, a simulation-based approach, called ‘Extended importance sampling’, is proposed based on evidence theory to handle problems with epistemic uncertainty. The proposed approach stems from the traditional importance sampling for reliability analysis under probability theory, and is developed to handle the problem with epistemic uncertainty. It first introduces a nominal instrumental probability density function (PDF) for every epistemic uncertainty variable, and thus an ‘equivalent’ reliability problem under probability theory is obtained. Then the samples of these variables are generated in a way of importance sampling. Based on these samples, the plausibility and belief (upper and lower bounds of probability) can be estimated. It is more efficient than direct Monte Carlo simulation. Numerical and engineering examples are given to illustrate the efficiency and feasible of the proposed approach.

1. Introduction

The aim of conducting structural reliability analysis and optimization design is to make the optimum choice between the economy and reliability, considering various inevitable uncertainty factors of the structure, like boundary conditions, material properties, random load and so on. Most uncertainties can be divided into two kinds, objective uncertainty and epistemic uncertainty [1]. Objective uncertainty can be studied under probability model after a large number of data statistics. Thus reliability analysis [2][3] and optimization design [4] methods for random uncertainty has been well developed. Epistemic uncertainty is due to the lack of data and information, this kind of problems, it is usually more difficult to handle. There is a few theories can be used to study epistemic uncertainty based on non-probabilistic description, like evidence theory (Dempster-Shafer theory)[5]-[7], possibility theory[8][9], fuzzy theory[10], convex models [11][12] and so on.

Evidence theory can be used to effectively deal with uncertainty information, like interval, fuzziness, random and so on. In addition, reliability problem based on probability theory can be a particular case of evidence theory. Thus, evidence theory is an effective way to handle discontinuous variables. However, due to the cost of calculation [13], research achievement in the area of reliability analysis under evidence theory is rare, although some progress has been made in this research orientation. Bae et al. [13][14]built multi-point approximate surface based on a certain point at the limit state surface, in order to solve reliability analysis under the epistemic uncertainty. Du [15] et al proposed a new reliability-analysis method that probability and evidence theory are combined. Jiang et al. [16] proposed a structural reliability method using evidence theory by introducing a non-probabilistic reliability index approach. Nevertheless, high efficient method for reliability analysis still need to be developed.
2. Reliability analysis based on evidence theory

For simplicity, a two-dimensional case is used to describe the propagation process of evidence theory

\[ y = g(x) = g(x_1, x_2) \]  

(1)

where \( x = [x_1, x_2]^T \), \( x_1 \in A, x_2 \in B \) are two independent variables of the problem( \( A, B \) are two sets); \( y \) is the output; \( g(x) \) is the limit state function. Let \( m(x_1), m(x_2) \) be the basic probability assignment (BPA) of variables \( x_1 \) and \( x_2 \), respectively. The combined BPAs of both \( x_1 \) and \( x_2 \) can be calculated by Dempster’s rule. Defining a vector \( c_{ij} = [x_{ij}, x_{ij}] \) where \( i \) and \( j \) indicate focal elements, the BPA of \( C \) can be defined as:

\[ C = A \times B = \{ c_{ij} = [x_{ij}, x_{ij}], x_{ij} \in A, x_{ij} \in B \} \]  

(2)

Considering the independence of \( x_1 \) and \( x_2 \), the BPA of \( C \) can be calculated by:

\[ m(c_{ij}) = m(x_{ij})m(x_{ij}) \]  

(3)

In reliability analysis, suppose the safe domain \( G \) is defined as

\[ G = \{ y : y = g(x_1, x_2) > 0, c_{ij} = [x_{ij}, x_{ij}] \subseteq C \} \]  

(4)

Then \( Bel(G) \) and \( Pl(G) \) can be calculated by:

\[ Bel(G) = \sum_{c_{ij} \subseteq G} m(c_{ij}) \]  

(5)

\[ Pl(G) = \sum_{c_{ij} \subseteq G} m(c_{ij}) \]  

(6)

where \( c \subseteq G \) indicates that the joint focal element \( c_{ij} \) must be entirely within the domain \( G \), while \( c_{ij} \cap G \neq 0 \) means that \( c_{ij} \) entirely or partly within \( G \). However, it’s difficult to deal with \( c_{ij} \subseteq G \) and \( c_{ij} \cap G \neq 0 \). In order to determine whether \( c_{ij} \) satisfies \( c_{ij} \subseteq G \) or \( c_{ij} \cap G \neq 0 \), the following equation should be calculated:

\[ [y_{min}, y_{max}] = [\min_{x_{ij}} g(x), \max_{x_{ij}} g(x)] \]  

(7)

If \( y_{min} \) and \( y_{max} \) are both greater than zero, then it means that at this focal element interval, the joint BPA \( m(c_{ij}) \) should be reckoned in both \( Bel(G) \) and \( Pl(G) \). However, if \( y_{min} \) is less than zero while \( y_{max} \) is greater than zero, \( m(c_{ij}) \) is only reckoned in \( Pl(G) \). If \( y_{min} \) and \( y_{max} \) are both less than zero, \( m(c_{ij}) \) should not reckoned in both \( Bel(G) \) and \( Pl(G) \). Through the above analysis, it can be seen that the calculation should be carried out at all focal element intervals. This is the difficulty of the reliability analysis under evidence theory. Approximate method [17]-[19] can be used to calculate the linear or approximate linear problem. However, for highly non-linear or complicated cases, simulation based method should be adopted.

3. Extended Importance sampling for reliability analysis under evidence theory

In order to achieve compute efficiency, traditional importance sampling method can be extended to solve the problem with epistemic uncertainty. The key of importance sampling for reliability analysis lies in the identifying of the important region and setting sampling centre on it. In this contribution, the ‘importance region’ is also identified which has larger joint BPA and is in the vicinity of limit state function. This can be done by introducing an ‘equivalent’ auxiliary sampling density according to the
joint BPA, and then solve the design point, and the region around the design point indicates the region contributing most to Bel and Pl. This is called ‘extended importance sampling’ in this paper, and it is illustrated as follows.

(1) Construct auxiliary sampling density
In order to apply traditional reliability analysis to solve problem with epistemic uncertainty, it is necessary to introduce an auxiliary sampling density (ASD) for each epistemic uncertainty variable. Generally, The ASD should be chosen according to the BPA of each variable. The principle of determine the ASD is that it reflects the amount of probability of BPA. For example, one optional way is given as

$$
BPA([x_a, x_b]) = p(x_b) - p(x_a)
$$

where $p(x)$ is the introduced ASD of variable $x$; $x_a$ and $x_b$ are interval endpoints. Eq.(8) guarantees that the ASD can reflect the probability assignment of the BPA.

(2) Solve the design point
The design point is solved in ‘equivalent’ reliability problem with variables are distributed as auxiliary sampling density $p(x)$. This can be done by using advance first order second moment method (AFOSM) or other optimization algorithm.

(3) Carry out importance sampling
Construct the importance sampling density $H(x)$ based on the design point, and generate the samples of variables according to $H(x)$.

(4) Calculate the $y_{\min}$ and $y_{\max}$ in each focal element
For every generated sample, say, $x^{(i)}$, first determine which focal element $c'$ it locates in, and then decide whether or not to calculate the limit state function $G$ by judging which case the focal element $c'$ is.

Case 1: $0 < y_{\min} < y_{\max}$, compute $y^{(i)} = g(x^{(i)})$, update the $y_{\min}$ and $y_{\max}$ in current focal element;

Case 2: $y_{\min} < 0 < y_{\max}$, no more compute;

Case 3: $y_{\min} < y_{\max} < 0$, compute $y^{(i)} = g(x^{(i)})$, update the $y_{\min}$ and $y_{\max}$ in current focal element.

Note that the $y_{\min}$ and $y_{\max}$ in current focal element is calculated according to the samples falling in this element.

(5) Estimate the Bel($G$) and Pl($G$)
By using eqs. (5) and (6), the measures Bel($G$) and Pl($G$) can be obtained. They can be viewed as the lower and upper bounds of the true reliability $P_r$, which satisfies

$$
Bel(G) \leq P_r \leq Pl(G)
$$

Note that if we use auxiliary sampling density $p(x)$ to directly generate samples in step (2) and (3), it can be seen as direct Monte Carlo simulation (MCS) method.

4. Examples
In this section, example is given to demonstrate the effectiveness of the proposed EIS method. Direct Monte Carlo and other methods are also used to compare with the proposed method.

4.1. A ten-bar truss
Consider a ten-bar truss structure shown in Fig. 1. The length $L=360$ in, and the modulus of elasticity $E=15,000$ ksi. $A_j$ $(j = 1,2,...,10)$ denotes the cross-section area, and $A_j = 10$ in $(j = 7,8,9,10)$. The truss is subjected to three forces, $F_1=100kip, F_2=120kip$ and , $F_3=400kip$. The limit state function is given by

$$
g = d_y - \delta_z (A_1, A_2, A_3, A_4, A_5, A_6)
$$

where $d_y$ and $\delta_z$ are specified in current paper.
where \( d_y = 1.5 \) in is the allowable displacement; \( \delta_2 \) is the vertical displacement of joint 2 which is the function of \( A_1, \ldots, A_6 \) and

\[
\delta_2 = \frac{\sum_{j=1}^{6} N_j^0 A_j}{A} + \sqrt{2} \sum_{i=7}^{10} \frac{N_i^0 A_i}{E}
\]  

(11)

where \( N_j, j = 1, 2, \ldots, 10 \) are forces and can be obtained by solving the following equations.

\[
\begin{align*}
N_1 &= F_2 - \sqrt{2}/2 N_8, \quad N_2 = -\sqrt{2}/2 N_7, \quad N_3 = -F_1 - 2F_2 + F_3 - \sqrt{2}/2 N_8, \\
N_4 &= -F_2 + F_3 - \sqrt{2}/2 N_7, \quad N_5 = -F_2 - \sqrt{2}/2 N_8 - \sqrt{2}/2 N_10, \\
N_6 &= -\sqrt{2}/2 N_10, \quad N_7 = \sqrt{2}(F_1 + F_2) + N_8, \quad N_8 = (a_{12} b_1 a_{21} b_2) / (a_{11} a_{22} a_{12} a_{21}), \\
N_9 &= \sqrt{2} F_2 + N_{10}, \quad N_{10} = (a_{11} b_1 a_{21} b_2) / (a_{11} a_{22} a_{12} a_{21})
\end{align*}
\]  

(12)

And

\[
\begin{align*}
a_{11} &= \left(1/A_1 + 1/A_2, 1/A_3 + 1/A_4 + 2\sqrt{2}/A_5, 2\sqrt{2}/A_6\right) L/2E, \quad a_{12} = a_{21} = L/2A_6 E, \\
a_{22} &= \left(1/A_2 + 1/A_4 + 1/A_6 + 2\sqrt{2}/A_5, 2\sqrt{2}/A_6\right) L/2E, \\
b_1 &= (F_2 / A_3 - (F_1 + 2F_2 - F_3) / A_5 - F_2 / A_6 - 2\sqrt{2}(F_1 + F_3) / A_5) \sqrt{2} L / 2E, \\
b_2 &= (\sqrt{2}(F_3 - F_2) / A_4 - \sqrt{2} F_2 / A_7 - 4F_2 / A_9) L / 2E
\end{align*}
\]  

(13)

The cross-section areas \( A_1, \ldots, A_6 \) are treated as evidence variables, and the BPA structures are given in Table 1.

![Figure 1. Ten-bar structure](image)

| Interval | BPA |
|----------|-----|
| [2.0 6.0] | 0.09 |
| [6.0 10.0] | 0.41 |
| [10.0 14.0] | 0.41 |
| [14.0 18.0] | 0.09 |

The proposed EIS and direct MCS are used to calculate the \( Bel(G) \) and \( Pl(G) \). The computed results are summarized in Table 2. In order to compare the method more clearly, the mean values and standard deviations of \( Bel(G) \) and \( Pl(G) \) estimates are also given, which are obtained by carrying out the analysis method for a number of times, say 10, in this example. Also, the optimization methods,
i.e., Active Set and Genetic Algorithm, are applied to solve this problem. The results of different methods are listed in Table 2.

|                      | Bel(G) Mean | Standard deviation | Pl(G) Mean | Standard deviation | Averaged No. of calls |
|----------------------|-------------|--------------------|------------|--------------------|-----------------------|
| EIS                  | 0.6072      | 0.0082             | 0.9686     | 0.0018             | 10^4                  |
| MCS                  | 0.6701      | 0.0045             | 0.9722     | 0.0010             | 10^5                  |
| Active Set           | 0.6064      | ---                | 0.9926     | ---                | 4.68 x 10^5           |
| Genetic Algorithm    | 0.6059      | ---                | 0.9926     | ---                | 1.94 x 10^8           |

It can be seen from Table 2 that the results of different methods are approximately consistent with each other. Among which, the proposed obtains the results at the least computed cost which uses 10^4 function calls. By contrast, Direct MCS method uses 10^5 function calls. Note that as the number of focal elements in this example is 4^6= 4096, the optimization approach should be carried out at each focal element twice to find the minima and maximum value of the limit state function, thus the computational cost is very huge. As seen in Table 2, Active Set approach uses 4.68 x 10^5 function calls, while Genetic Algorithm uses more, 1.94 x 10^8. The proposed method can obtain the satisfied results at moderate computational cost, thus the feasibility and efficiency of the proposed method is shown.

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