Zygmund’s Type Inequality to the Polar Derivative of A Polynomial

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Abstract. In this paper we improve a result recently proved by Irshad et al. [On the Inequalities Concerning to the Polar Derivative of a Polynomial with Restricted Zeroes, Thai Journal of Mathematics, 2014 (Article in Press)] and also extend Zygmund’s inequality to the polar derivative of a polynomial.

Introduction
Let $P(z)$ be a polynomial of degree $n$, then
\[
\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|
\] (1)

inequality (1) is a well known result of S. Bernstein [1]. Equality holds in (1) if and only if $P(z)$ has all its zeros at the origin.

Inequality (1) was extended to $L_p$-norm $p \geq 1$ by Zygmund [2], who proved that if $P(z)$ is a polynomial of degree $n$, then
\[
\left\{ \frac{1}{2\pi} \int_0^{2\pi} |P'(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}
\] (2)

Equality holds in (2) for $P(z) = \alpha z^n$, $|\alpha| \neq 0$. If we let $p \to \infty$ in (2), we get inequality (1).

Let $\alpha$ be a complex number. If $P(z)$ is a polynomial of degree $n$, then the polar derivative of $P(z)$ with respect to the point $\alpha$, denoted by $D_\alpha P(z)$, is defined by
\[
D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)
\]
clearly $D_\alpha P(z)$ is a polynomial of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that
\[
\lim_{\alpha \to \infty} \left[ \frac{D_\alpha P(z)}{\alpha} \right] = P'(z).
\] (3)

As an extension of (1) to the polar derivative, Aziz and Shah [3], have shown that if $P(z)$ is a polynomial of degree $n$, then for every real or complex number $\alpha$ with $|\alpha| > 1$ and for $|z| = 1$,
\[
|D_\alpha P(z)| \leq n|\alpha| \max_{|z|=1} |P(z)|
\] (4)

As a generalization of (2) to the polar derivative Aziz et al. [4], proved the following result.
**Theorem A** If \( P(z) \) is a polynomial of degree \( n \), then for every complex number \( \alpha \) with \(|\alpha| \geq 1\) and \( p \geq 1 \)

\[
\left\{ \int_0^{2\pi} |D_{\alpha} P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n(|\alpha| + 1) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}
\]

(5)

For the Class of polynomials having no zeros in \(|z| < 1\), inequality (2) was improved by D-Bruijn [5] that if \( P(z) \neq 0 \) in \(|z| < 1\), then for \( p \geq 1 \)

\[
\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n C_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}
\]

(6)

where

\[
c_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + e^{i\theta} \right|^p d\theta \right\}^{-\frac{1}{p}}
\]

(7)

As an extension to the polar derivative. A. Aziz and N. Rather [6], proved the following generalization of (5). In fact they proved.

**Theorem B** If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in \(|z| < 1\), then for every complex number \( \alpha \) with \(|\alpha| \geq 1\) and \( p \geq 1 \)

\[
\left\{ \int_0^{2\pi} |D_{\alpha} P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n C_p \left\{ \int_0^{2\pi} |D_{\alpha} P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}
\]

(8)

where \( C_p \) is defined by (7).

Recently, Irshad et al. [7] proved the following result.

**Theorem C** If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in \(|z| < K \leq 1\), then for every \( \alpha, \beta \in C \) with \(|\alpha| \geq K\), \(|\beta| \leq 1\) and \( p \geq 1 \)

\[
\left\{ \int_0^{2\pi} \left| e^{i\theta} D_{\alpha} P(e^{i\theta}) + n \frac{|\alpha| - K}{K + 1} \beta P(e^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}} \leq n \left( 1 + |\alpha| + 2 \frac{(|\alpha| - K)}{K + 1} |\beta| \right) C_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}
\]

(9)

where \( C_p \) is defined by (7).

In this paper we prove the following more general result which also generalize Theorem B and yields a number of known polynomial inequalities.
Theorem 1. If \( P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}, \) \( 1 \leq \mu \leq n \) be a polynomial of degree \( n \) which does not vanish in \( |z| < K \leq 1 \), then for every \( \alpha, \beta \in C \) with \( |\alpha| \geq K, |\beta| \leq 1 \) and \( p \geq 1 \)

\[
\left\{ \int_{0}^{2\pi} e^{i\theta} D_{\alpha} P(e^{i\theta}) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta}) \right)^p d\theta \right\}^{\frac{1}{p}} \leq n \left( 1 + |\alpha| + 2 \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} |\beta| \right) \right) C_p \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}
\]

(10)

where \( C_p \) is defined by (7), or equivalently

\[
\left\| e^{i\theta} D_{\alpha} P(e^{i\theta}) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta}) \right) \right\|_p \leq n \left( 1 + |\alpha| + 2 \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} |\beta| \right) \right) \left\| \frac{P(e^{i\theta})}{1 + e^{i\theta}} \right\|_p
\]

(11)

Remark. If we choose \( \mu = 1 \) in (10), we get Theorem C and if we choose \( \beta = 0 \) and \( K = 1 \) in (10), we get Theorem B.

\section*{Lemmas}

For the proof of this theorem, we need the following lemmas. The first lemma is due to Gulshan Singh et al. [8].

\begin{lemma}
Let \( P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}, \) \( 1 \leq \mu \leq n \) be a polynomial of degree having all its zeros in the disk \( |z| \leq K, K \leq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq K, K \leq 1 \) and for \( |z| = 1 \)

\[ |D_{\alpha} P(z)| \geq n \left( \frac{|\alpha| - K^\mu}{K^\mu} \right) |P(z)| \]

\end{lemma}

\begin{lemma}
Let \( Q(z) \) be a polynomial of degree \( n \) having all its zeros in \( |z| < K, K \leq 1 \) and \( P(z) \) is a polynomial of degree not exceeding that of \( Q(z) \). If \( |P(z)| \leq |Q(z)| \) for \( |z| = K \leq 1 \), then for every \( \alpha, \beta \in C \) with \( |\alpha| \geq K, |\beta| \leq 1 \)

\[ |zD_{\alpha} P(z) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(z) \right)| \leq |zD_{\alpha} Q(z) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta Q(z) \right)| \]

Proof. Since \( |\lambda P(z)| \leq |P(z)| \leq |Q(z)| \), for \( \lambda < 1 \) and \( |z| = K \), then for Rouche’s Theorem \( Q(z) - \lambda P(z) \) and \( Q(z) \) have the same number of zeros in \( |z| < K \). On the other hand by inequality \( |P(z)| \leq |Q(z)| \) for \( |z| = K \), any zero of \( Q(z) \), that lies on \( |z| = K \), in the zero of \( P(z) \). Therefore, \( Q(z) - \lambda P(z) \) has all its zero in the closed disk \( |z| \leq K \). Hence by Lemma 1 for all real or complex numbers \( \alpha \) with \( |\alpha| \geq K \) and \( |z| = 1 \), we have

\[ |zD_{\alpha}(Q(z) - \lambda P(z))| \geq n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) |Q(z) - \lambda P(z)| \]

(12)

Now consider a similar argument that for any value of \( \beta \) with \( |\beta| < 1 \), we have

\[ |zD_{\alpha}(Q(z) - \lambda P(z))| \geq n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) |Q(z) - \lambda P(z)| \]
\[ \frac{n|\beta| \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) |Q(z) - \lambda P(z)|}{K^\mu + 1} \]  

(13)

where \( |z| = 1 \), resulting in

\[ T(z) = |zD_{\alpha}Q(z) - \lambda zD_{\alpha}P(z)| + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} \{Q(z) - \lambda P(z)\} \neq 0 \]  

(14)

where \( |z| = 1 \).

That is

\[ T(z) = \left| zD_{\alpha}Q(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z) - \lambda \left| zD_{\alpha}P(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z) \right| \right| \neq 0 \]  

(15)

for \( |z| = 1 \).

We also conclude that

\[ \left| zD_{\alpha}Q(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z) \right| \geq \left| zD_{\alpha}P(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z) \right| \]  

(16)

for \( |z| = 1 \).

If (16) is not true, then there is a point \( z = z_0 \) with \( |z_0| = 1 \), such that

\[ \left| z_0D_{\alpha}Q(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z_0) \right| < \left| z_0D_{\alpha}P(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z_0) \right| \]  

(17)

Take

\[ \lambda = \frac{z_0D_{\alpha}Q(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z_0)}{z_0D_{\alpha}P(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z_0)} \]  

(18)

then \( |\lambda| < 1 \) with this choice, we have from (15), \( T(z_0) = 0 \) for \( |z_0| = 1 \). But this contradicts the fact that \( T(z) \neq 0 \) for \( |z| = 1 \). For \( \beta \) with \( |\beta| = 1 \), (16) follows by continuity.

This completes the proof.

The next lemma is due to Aziz and Rather [4].

**Lemma 3.** If \( P(z) \) is a polynomial of degree \( n \) such that \( P(0) \neq 0 \) and \( Q(z) = z^n p \left( \frac{1}{z} \right) \), then for every \( p \geq 0 \) and \( \phi \) real

\[ \int_0^{2\pi} \int_0^{2\pi} |Q'(e^{i\theta}) + e^{i\theta} P'(e^{i\theta})|^p d\theta d\phi \leq n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \]

**Proof of the theorem**

**Proof of the Theorem.** Let \( P(z) \) be a polynomial of degree \( n \) which does not vanish in \( |z| < K, K \leq 1 \). By Lemma 2 for complex numbers \( \alpha, \beta \) with \( |\alpha| \geq K, |\beta| \leq 1 \), we have

\[ \left| zD_{\alpha}P(z) + n \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(z) \right| \leq \left| zD_{\alpha}Q(z) + n \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta Q(z) \right| \]  

(19)
For every real \( \phi \) and \( \xi \geq 1 \), we have
\[
|\xi + e^{i\phi}| \geq |1 + e^{i\phi}|
\]
which implies for any \( p \geq 0 \)
\[
\left\{ \int_0^{2\pi} |\xi + e^{i\phi}|^p d\phi \right\}^{1/p} \geq \left\{ \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \right\}^{1/p} \tag{20}
\]
If \( e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta}) \neq 0 \), we can take
\[
\xi = \frac{e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{|\alpha| - K^\mu}{K^\mu + 1} \beta Q(e^{i\theta})}{e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta})}
\]
where according to (19), \( |\xi| \geq 1 \). Now
\[
\int_0^{2\pi} \left| e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{|\alpha| - K^\mu}{K^\mu + 1} \beta Q(e^{i\theta}) \right|^p d\phi
\]
\[
+ e^{i\phi} \left[ e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta}) \right]^p d\phi
\]
\[
= \left| e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta}) \right|^p \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi
\]
\[
\geq \left| e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta}) \right|^p \int_0^{2\pi} |\xi + e^{i\phi}|^p d\phi \tag{21}
\]
This inequality is trivially true if
\[
e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta}) = 0
\]
Integrating both sides of (21) with respect to \( \theta \) from 0 to \( 2\pi \), we have
\[
\int_0^{2\pi} \int_0^{2\pi} \left| e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{|\alpha| - K^\mu}{K^\mu + 1} \beta Q(e^{i\theta}) \right|^p d\theta d\phi
\]
\[
+ e^{i\phi} \left[ e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta}) \right]^p d\theta d\phi
\]
\[
\geq \int_0^{2\pi} \left| e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta}) \right|^p d\theta \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \tag{22}
\]
Now for $0 \leq \theta < 2\pi$
\[
\left| e^{i\theta} D_\alpha Q(e^{i\theta}) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) \beta Q(e^{i\theta}) + e^{i\phi} \left[ e^{i\theta} D_\alpha P(e^{i\theta}) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) \beta P(e^{i\theta}) \right] \right|
\]
\[
= \left| e^{i\theta} \left( nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q'(e^{i\theta}) \right) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) \beta Q(e^{i\theta}) \right|
\]
\[
+ e^{i\phi} \left[ e^{i\theta} \left( nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}) \right) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) \beta P(e^{i\theta}) \right] \right| \quad (23)
\]
\[
= \left| e^{i\theta} \left( nQ(e^{i\theta}) - e^{i\theta} Q'(e^{i\theta}) \right) + \alpha e^{i\theta} Q'(e^{i\theta}) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) \beta Q(e^{i\theta}) \right|
\]
\[
+ e^{i\phi} \left[ e^{i\theta} \left( nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta}) \right) + \alpha e^{i\theta} P'(e^{i\theta}) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) \beta P(e^{i\theta}) \right] \right| \quad (24)
\]
Since $Q(z) = z^n P \left( \frac{1}{z} \right)$, we have $P(z) = z^n Q \left( \frac{1}{z} \right)$ and it can be easily verified that for $0 \leq \theta < 2\pi$
\[
nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta}) = e^{i(n-1)\theta} Q'(e^{i\theta})
\]
and
\[
nQ(e^{i\theta}) - e^{i\theta} Q'(e^{i\theta}) = e^{i(n-1)\theta} P'(e^{i\theta})
\]
From (24)
\[
\left| e^{i\theta} D_\alpha Q(e^{i\theta}) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) \beta Q(e^{i\theta}) + e^{i\phi} \left[ e^{i\theta} D_\alpha P(e^{i\theta}) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) \beta P(e^{i\theta}) \right] \right|
\]
\[
= \left| e^{i\theta} \left( e^{i(n-1)\theta} P'(e^{i\theta}) \right) \right| + \alpha e^{i\theta} Q'(e^{i\theta}) + e^{i\phi} \left[ e^{i\theta} P'(e^{i\theta}) \right] + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) \beta [Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta})] + e^{i\phi} e^{i(n-1)\theta} Q'(e^{i\theta}) \right| \quad (25)
\]
Therefore, (22) in conjunction with (25) gives
\[
\left\{ \begin{array}{l}
\int_0^{2\pi} \int_0^{2\pi} \left| e^{i\theta} e^{i(n-1)\theta} \left[ P'(e^{i\theta}) + e^{i\phi} Q'(e^{i\theta}) \right] + \alpha e^{i\phi} [Q'(e^{i\theta}) e^{i\phi} P'(e^{i\theta})] \\
+ n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) \beta [Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta})]\right| d\theta d\phi \right\} \frac{1}{p}
\]
\[
\geq \left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) \beta P(e^{i\theta}) \right| d\theta \int_0^{2\pi} \left| 1 + e^{i\phi} \right|^p d\phi \right\} \frac{1}{p} \quad (26)
\]
By Minkowski inequality, we have
\[
\left\{ \int_0^{2\pi} |e^{i\theta} D_\alpha P(e^{i\theta}) + n \left( \frac{\alpha - K^\mu}{K^\nu + 1} \right) \beta P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} 
\leq \left\{ \int_0^{2\pi} \left( \frac{\alpha}{K^\nu + 1} \right) \beta \int_0^{2\pi} |e^{i\theta} P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \{1 + |\alpha| \}
\]
\[
+ n \left( \frac{|\alpha| - K^\mu}{K^\nu + 1} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \{1 + |\alpha| \}
\]
By Lemma 3, we have
\[
\left\{ \int_0^{2\pi} |e^{i\theta} D_\alpha P(e^{i\theta}) + n \left( \frac{\alpha - K^\mu}{K^\nu + 1} \right) \beta P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} 
\leq 2n^p \left[ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right]^{\frac{1}{p}} \{1 + |\alpha| \}
\]
\[
+ 2n \left( \frac{|\alpha| - K^\mu}{K^\nu + 1} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \{1 + |\alpha| \}
\]
\[
= n(1 + |\alpha|) + 2n \left( \frac{|\alpha| - K^\mu}{K^\nu + 1} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \{1 + |\alpha| \}
\]
This implies
\[
\left\{ \int_0^{2\pi} |e^{i\theta} D_\alpha P(e^{i\theta}) + n \left( \frac{\alpha - K^\mu}{K^\nu + 1} \right) \beta P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} 
\leq n \left( 1 + |\alpha| + 2 \left( \frac{|\alpha| - K^\mu}{K^\nu + 1} \right) \right) C_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}
\]
where $C_p$ in defined by (7),
or equivalently,
\[
\left\| e^{i\theta} D_\alpha P(e^{i\theta}) + n \left( \frac{\alpha - K^\mu}{K^\nu + 1} \right) \beta P(e^{i\theta}) \right\| 
\leq n \left( 1 + |\alpha| + 2 \left( \frac{|\alpha| - K^\mu}{K^\nu + 1} \right) \right) \left\| P(e^{i\theta}) \right\|_p \left\| 1 + P(e^{i\theta}) \right\|_p
\]
Hence the result.
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