Quantum entanglement as information theoretic resource

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Abstract: We address the following criterion for quantifying the quantum information resources: classically simulable vs. classically non-simulable information processing. This approach gives rise to existence of a deeper level of quantum information processing—which we refer to as "quantum communication channel". We particularly show, that following the recipes of the standard theory of entanglement measures does not necessarily give rise to un-locking the quantum communication channel, which is naturally quantified by Bell inequalities.

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**Introduction.** - Quantum entanglement is recognized as a resource for useful quantum information/computation (henceforth: QIC) processing [1-6]. Consequently, quantifying entanglement is among the central issues of QIC theory. The standard theory of entanglement measures (cf., e.g., [1] for a review) offers the recipes for improving entanglement (and/or purity) of a quantum state (system) while bearing the following point as a background: "the system is more entangled if it allows for better performance of some task (impossible without entanglement)" [1]. However, it is natural (for many reasons) to expect that certain additional criteria/requirements concerning the state preparation/manipulation might improve the performance of some task. E.g. a more elaborate *information theoretic* analysis might shed some new light in this concern; particularly, the analysis devoted to the efficient (classically non-simulable) tasks is not yet fully developed in the standard theory [1]. To this end, it is useful to recall: distinguishing the classically non-simulable yet quantum mechanically achievable processing is probably the main motivation for developing QIC theory [7].

In this paper, we point out existence of the more fundamental information processing level that we refer to as "entanglement as information resource". Actually, we move the focus of the task of quantifying entanglement to the following criterion/divide: *classically simulable* vs. *classically non-simulable* information processing. This new approach directly addresses the issue raised by Feynman [7] which is still un-resolved by the standard theory of entanglement measures. With this, hopefully, we sharpen the issue of necessity of entanglement for the efficient (classically non-simulable) information processing. Finally, recognizing "entanglement as information resource" as the "quantum communication channel (QCC)", we may phrase our main result as follows: manipulating entangled states according to the recipes of the standard theory does not necessarily imply opening (un-locking) QCC; in general, certain *additional operations* are required for opening QCC, which as a quantum information resource can be quantified by Bell-inequalities (henceforth: BI) [8, 9].

**Quantum entanglement relativity.** - By definition, a bipartite system is in entangled state if the state can not be written in the *separable form* (with the straightforward generalization to the mixed entangled states) [6, 10, 11]:

$$|\psi_1\rangle|\chi_2\rangle,$$

where $|\psi_1\rangle, |\chi_2\rangle$ are the subsystems’ states, and we omit the symbol of the tensor product. Whilst this definition is as simple as clear, *operationally to
justify entanglement is a bit subtle task. E.g., if the $S$-factor of CHSH inequality [9] satisfies $S > 2$, one may conclude that the system is in entangled state. However, the result $S \leq 2$ does not necessarily imply that the state is of the separable form eq. (1). This is what we call "entanglement relativity". Namely, as it is well-known since the pioneering paper of Bell [8], the trick is properly to choose the observables to be measured in respect (i.e. relative) to the quantum state to be tested on entanglement.

By "entanglement relativity", we do not mean that quantum entanglement is a relative concept–according to (1), the state is either an entangled, or a separable state. What we have in mind is the fact that, operationally, quantum entanglement need not reveal itself. An example given in Appendix A clarifies our notions.

Interestingly enough, this relativity of entanglement perfectly fits with the Copenhagen interpretation of quantum mechanics [12]: depending on a physical situation (here: of measurement), a quantum state either reveals, or does not reveal entanglement, which is here quantified by $CC$. To this end, it is worth emphasizing: the operational estimation of entanglement we deal with is only weakly linked to the problem of deciding (by measurement) whether an unknown state is entangled or not. The issue we have in mind is a bit more subtle. Actually, and bearing in mind the Principle of Complementarity [12], we want to emphasize that a quantum system (in known or unknown entangled state) need not reveal entanglement, very much like a quantum system need not reveal its e.g. corpuscular nature (behaviour).

Extending this reasoning to the QIC protocols gives rise to some interesting observations.

Quantum teleportation: an analysis. - In this section, we strongly emphasize: quantifying the information resources cannot rely solely on investigating the quantum system’s states. Rather, the information theoretic analysis of a process should reveal the information resources. The following analysis of quantum teleportation clarifies these notions.

Quantum teleportation is probably the most investigated QIC protocol [3, 6, 13]. In the original paper [3], necessity of an entangled state in the protocol has been pointed out. However, as we point out in the sequel, there are still certain subtleties in this regard.

As it is well-known, quantum teleportation can be described by the stabilizer formalism [6, 14]. This formalism, in turn, can be efficiently simulated on a classical computer–a consequence of the profound Knill-Gottesman theorem [14]. That is, while teleportation requires entangled systems [3], the certain-states teleportation still can be classically simulated–which is fundamental for our considerations. Actually, as it is known since Bell [8], every
classical situation (here: the classical-computer simulation of the stabilizer formalism) can be described by a Local Hidden Variables (LHV) model, which, in turn, can be ascribed $S \leq 2$ [9]. At first sight, this may seem controversial: physically, there is entanglement (described by the stabilizer formalism) in the system implementing teleportation, yet the simulation of teleportation (the classical-computer simulation of the stabilizer formalism) is describable by $S \leq 2$. However, in analogy with "entanglement relativity", we may suppose that, operationally, entanglement need not reveal itself. On the other side, most quantum states can not be teleported by the use of the stabilizer formalism [6, 14]. For such states, it is expectable informaticly not to bear any LHV model thus eventually giving rise to $S > 2$.

Needless to say, physical processes are not identical with their computer-simulation counterparts. It is therefore not for surprise if the physical contents of a process do not reveal the information theoretic contents (e.g. information resources) of the process. Consequently, investigating the information resources of a process should primarily rely on investigating the information contents of a simulation of the process, as we have essentially learned from Feynman [7]. In the above analysis: quantum teleportation is a "physical process" employing entangled states (systems), while the possible classical-computer simulations of teleportation bear an LHV model. Certainly, quantum teleportation as a physical process can not be ascribed any LHV model, while certain simulations of teleportation can be ascribed an LHV model.

These observations force us to conclude that we should distinguish between "entanglement of a system (of quantum hardware)" and the "entanglement as information resource". While former refers to the physical systems (implementing the information processing), the later refers to the information contents of the processing on the deeper information theoretic level that reveal the information resources. That is, operating with an entangled system does not necessarily mean that, on the more fundamental level of information processing, entanglement as information resource has fully been employed in the execution of the processing.

Now, the two tasks are in order. First, we should more closely relate our findings to the standard theory of entanglement measures. Second, we should offer some quantification of "entanglement as information resource".

Quantum information resources. - Entangled quantum hardware (entangled systems; entangled states) seems to be necessary for the performance of (most of) the typical QIC tasks. On this level of the information processing, entangled states appear as a quantum information resource. This is exactly the issue of the standard theory of entangled measures, which relies on the definition of entanglement eq. (1). Physically, this resource may
be recognized as *quantum non-separability* [11]. To this end, relying to the definition eq. (1) is as simple as clear a criterion for non-separability, while quantifying non-separability by BI raises some questions. Actually, violation of BI (in quantum measurement) is linked with quantum nonlocality [6, 8, 9], which, in turn, does not necessarily apply to all the kinds of the entangled (non-separable) states—as it seems to be the case with the bound entangled states.

However, our analysis of quantum teleportation points out existence of the deeper level of information processing, which employs entangled systems. In this regard, implicit to the contents of the preceding section is the following criterion/divide:

\[ \text{classically simulable vs. classically non-simulable} \]  \hspace{1cm} (2)

**information processing.** Eq. (2) refers to the possible *simulations* of QIC tasks and therefore (cf. the preceding section) is suitable for deciding whether or not ”entanglement as information resource” has been employed in the execution of the information processing, still bearing the obvious measure (not yet in the mathematical sense):

\[ S \leq 2 \quad \text{vs.} \quad S > 2 \]  \hspace{1cm} (3)

in the order respective to (2); \( S \leq 2 \) reveals un-use while \( S > 2 \) reveals the possible use of ”entanglement as information resource”.

In the anthropomorphic terms, ”employing entanglement as information resource” may be described as ”opening (un-locking) the quantum communication channel”. Now, in analogy with ”entanglement relativity”, we may read eqs. (2), (3) as follows: the choice of *physical implementation* of a QIC protocol gives, in principle, rise either to non-opening \( (S \leq 2) \) or to un-locking \( (S > 2) \) the quantum communication channel (QCC). In other words: quantum correlations (non-separability) in a system do not *per se* constitute QCC—it is a matter of physical situation whether or not this *virtue of entangled systems* will be employed in the execution of the processing. Based on eq. (2), this observation distinguishes QCC as the possible basis for quantum protocols to beat the classical ones, in analogy with the recently observed necessity and sufficiency of violation of BI in the similar regard [15]. Needless to say, non-opening of the quantum channel refers to the QIC tasks that can be efficiently simulated on the ”classical hardware”.

Therefore, we may conclude that quantum entanglement bears (cf. below) at least a double role as information resource: (i) quantum non-separability of entangled systems, and (ii) the quantum communication channel—the later being quantifiable by BI (the r.h.s. of eq. (3)). While QCC apparently
requires non-separability, as we show in this paper, the later is not sufficient for opening QCC.

Another information resource, quantum non-locality, is linked with quantum non-separability—quantifying non-separability goes through quantifying non-locality by (non)validity of BI [16], of course except (probably) for the bound entangled states. Therefore, QCC (as we introduce it in Section 4) is not necessarily identical with the "quantum communication channel" of the standard theory which is sometimes identified with quantum non-locality. This is the reason we point out above the two, mutually distinguishable resources—quantum non-separability and QCC.

Discussion. - Following the recipes of the standard theory of entanglement measures, one can improve entanglement in a quantum system, yet without any guarantees about un-locking QCC as information resource. The distinction between quantum non-separability and QCC as the information resources is not quite surprising yet. To this end, we have the following lessons in mind. First, the classic lesson of the Complementarity Principle gives rise to the expectation that, operationally, entanglement need not reveal itself. Actually, in quantum measurement, entanglement reveals itself in the specific quantum measurement situations (cf. Section 2), while QCC reveals itself through the classically non-simulable information processing (cf. Sections 3, 4). Second, the true topic of QIC theory, as Feynman has pointed it out [7] (cf. also [6]), is the performance of the classically non-simulable tasks, which seems merely un-tackled by the standard theory of entanglement measures.

Our approach to the issue of entanglement measures mainly refers to QCC. It ultimately relies on the criterion (2), thus presenting a general yet simple approach in quantifying entanglement as information resource. By imposing the criterion (2), we tackle the truly fundamental issue of QIC theory: ‘whether or not entanglement appears ultimate to efficiency of certain QIC protocols/algorithms?’. Whilst the definite answer to this question is a remote goal of the theory yet, classifying QIC tasks in respect to (2) and (3) might help in setting the (e.g., empirically-based) recipe(s) for designing the efficient QIC processing. E.g. the classical simulability of the superdense coding, of the BB84 cryptographic protocol, as well as of the certain-states teleportation clearly stems non-opening of QCC in the course of the physical implementation of these protocols. It is therefore easy to speculate about the future theory of entanglement measures: the list of recipes from the standard theory (referring to non-separability and/or probably to non-locality) is extended by the recipes referring to the opening of QCC—in an attempt to perform the classically both inachievable and non-simulable information pro-
cessing. Along this line of reasoning, one may eventually clarify the relation between quantum non-locality and QCC as the information resources: e.g., if it appears that entangled states non-violating BI can not be used for the performance of the classically non-simulable tasks, it might be interpreted in favor of identifying the two kinds of the information resources—(ii) and (iii) in the above list of resources.

Finally, making connection of our results with the standard theory (cf. [1] and references therein) is not quite straightforward a task. As yet, the relations of BI with the standard entanglement measures (e.g., with the "concurrence" [17, 18]) are only weakly established [18] and only weakly understood for the purposes of our considerations. Therefore, in this respect, there is some research work yet to be done.

**Conclusion.** - We point out and discuss the following quantum information resources linked with entanglement: (i) non-separability (of quantum systems), (ii) quantum non-locality, and (iii) quantum communication channel. The relation between the first two resources is the true issue of the standard theory of entanglement measures. The resource (iii) comes from the addressing the following criterion for quantum information resources: classically simulable vs. classically non-simulable information processing—which is un-resolved by the standard theory of entanglement measures. As we show, manipulating entangled systems (non-separability, and probably non-locality) does not necessarily mean that the quantum communication channel as an information resource has been employed in the execution of the information processing—which is naturally quantified by Bell inequalities.

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Let us consider the following Bell states [6]:

\[ |\psi^+\rangle = 2^{-1/2}(|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2), \quad |\phi^+\rangle = 2^{-1/2}(|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2) \]  
\( A.1 \)

By definition, these states are entangled pure states. Let us now introduce the following set of the observables to be measured on the composite (two-qubit) system:

\[ \hat{A}_1(\alpha) = 2^{-1}(|0\rangle_1 + \exp(i\alpha)|1\rangle_1)(|0\rangle_1 + \exp(-i\alpha)|1\rangle_1) - 2^{-1}(|0\rangle_1 + \exp(i(\alpha + \pi))|1\rangle_1)(|0\rangle_1 + \exp(-i(\alpha + \pi))|1\rangle_1), \]  
\( A.2 \)

\[ \hat{B}_2(\chi) = 2^{-1}(|1\rangle_2 + \exp(i\chi)|0\rangle_2)(|1\rangle_2 + \exp(-i\chi)|0\rangle_2) - 2^{-1}(|1\rangle_2 + \exp(i(\chi + \pi))|0\rangle_2)(|1\rangle_2 + \exp(-i(\chi + \pi))|0\rangle_2), \]  
\( A.3 \)

where the indices refer to the two qubits in the composite system, and \( \chi, \alpha \in [-\pi, \pi] \).

The standard S-factor of CHSH inequality [9] now reads:

\[ S = E(\alpha_1, \chi_1) - E(\alpha_1, \chi_2) + E(\alpha_2, \chi_1) + E(\alpha_2, \chi_2), \]  
\( A.4 \)

where \( E(\alpha_i, \chi_j) = \langle \phi | \hat{A}_1(\alpha_i) \otimes \hat{B}_2(\chi_j) | \phi \rangle, i, j = 1, 2. \)
In Fig. 1, we present the plot of the $S$-factor for $|\phi\rangle = |\Psi^+\rangle$, for which $E(\alpha, \chi) = \cos(\alpha + \chi)$ [19], for the fixed values $\alpha_1 = \pi/2$ and $\chi_1 = -\pi/4$.

The plot exhibits that the state $|\Psi^+\rangle$, which is by definition entangled state, does not reveal entanglement for the wrong choice of the observables (here: of the angles $\alpha, \chi$). And this is exactly what we mean by "entanglement relativity". The same conclusion applies to other choices of $\alpha_1$ and $\chi_1$, as well as for the state $|\phi^+\rangle$. 
FIGURE CAPTIONS

Fig. 1: The $S$-factor, eq. (A.4), for the state $|\Psi^+\rangle$ and for fixed values $\alpha_1 = \pi/2, \chi_1 = -\pi/4$. For the chosen (the maximal possible) value $S = 2^{3/2}$, the plot returns $\alpha_2 = 0$ and $\chi_2 = \pi/4$. For $S = 0$, the plot returns e.g. $\alpha_2 = 0, \chi_2 = -3\pi/4$. In general, the plot returns proportions of $\alpha_2, \chi_2$ for every fixed value of $S \in [2^{-3/2}, 2^{3/2}]$. 