Robust Estimation in Generalised Linear Models : The Density Power Divergence Approach *

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Abstract
The generalised linear model (GLM) is a very important tool for analysing real data in biology, sociology, agriculture, engineering and many other application domain where the relationship between the response and explanatory variables may not be linear or the distributions may not be normal in all the cases. However, quite often such real data contain a significant number of outliers in relation to the standard parametric model used in the analysis; in such cases the classical maximum likelihood estimator may fail to produce reasonable estimators and related inference could be unreliable. In this paper, we develop a robust estimation procedure for the generalised linear models that can generate robust estimators with little loss in efficiency. We will also explore two particular cases of the generalised linear model in details — Poisson regression for count data and logistic regression for binary data — which are widely applied in real life experiments. We will also illustrate the performance of the proposed estimators through several interesting data examples.

Keywords: Density power divergence; Generalised linear model; Logistic regression; Poisson regression; Robustness.

1 Introduction

Many real life problems require suitable techniques to describe some response data through a set of related explanatory variables. Parametric regression helps the experimenter to model such scenarios by means of some pre-specified functional relationship between response and explanatory variables described through a set of real parameters. The most widely used regression model is linear regression for continuous responses that depends on the covariates linearly. In practice, though, there are lots of different types of response data like count data, binary response data and others which arise frequently in real life experiments such as clinical trials, medical surveys, designed experiments etc. The generalised linear model is the general tool that can be used with all such types of response variables. This generalised linear model allows the experimenter to model the response variables by any distribution within a large family of distributions, namely the exponential family, and the expected response by any (suitably smooth) function of the explanatory variables, with the only restriction that this function should depend on the explanatory variables linearly. As a special case, it also includes the ordinary linear regression problem; the study of the generalised linear model helps us to deal with a very large superfamily of parametric regression problems.

The classical procedure to estimate the parameters of the generalised linear regression model is the maximum likelihood estimation method generating most efficient estimators. This theoretical advantage of the maximum likelihood estimator is, however, tempered by its known lack of robustness to outliers

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and model misspecification. In many real life experiments, outliers show up as a matter of routine which influence the maximum likelihood estimators and often produce nonsensical results. So, there is a real need for developing robust estimation procedures for the generalised linear regression model. Although there is a crowded field of robust estimators in the ordinary linear regression problem, there exist only a few for the generalised linear regression case. Cantoni and Ronchetti (2001) and Hosseinian (2009) present some such approaches; most of these approaches consider the explanatory variables to be stochastic. In this paper, we will develop an estimation procedure for the generalised linear model from a design perspective, where we will assume that the explanatory variables are fixed and given to us; each response is independent and follows the same distribution specified by the generalised linear model, but having different distributional parameters depending on the values of the corresponding explanatory variables. The idea is motivated by the work of Ghosh and Basu (2013a) where a robust minimum divergence estimation procedure was developed under the general set-up of independent but non-homogeneous observations using the density power divergence. This work considered the case of simple linear regression in great detail and illustrated the properties of the corresponding estimates of regression parameters demonstrating that many of them are highly robust with a very little loss in efficiency. Here, we will follow a similar approach to develop the minimum density power divergence estimators of the parameters of generalised linear model, which will be extremely robust in presence of the influential observations and also have comparable high efficiency.

The rest of the paper is organized as follows. In Section 2, we will briefly describe the generalised regression model and develop the corresponding minimum density power estimators of parameters. We also prove the asymptotic properties and present the influence function analysis of the proposed minimum density power divergence estimator in case of generalised linear regression model in this section. We will then explore the special case of Poisson regression for count data and logistic regression for binary data in Sections 3 and 4 respectively. Subsequently, we will present a short discussion on a data-driven choice of the tuning parameter \( \alpha \) in Section 5. Section 6 contains the application of the proposed minimum density power divergence estimation to some interesting real life data sets that illustrate the performance of the proposed methodology under several differing scenarios. Finally the paper ends with some concluding remarks in Section 7.

2 The Minimum Density Power Divergence Estimator in Generalised Linear Models

2.1 The Generalised Linear Model (GLM)

Generalised linear models are indeed generalizations of the normal linear regression model where the response variables \( Y_i \) are independent and assumed to follow the general exponential family of distributions having density

\[
f(y_i; \theta_i, \phi) = \exp \left\{ \frac{y_i \theta_i - b(\theta_i)}{a(\phi)} + c(y_i, \phi) \right\},
\]

where the canonical parameter \( \theta_i \) is a measure of location depending on the predictor \( x_i \) and \( \phi \) is the nuisance scale parameter. The mean \( \mu_i \) of \( Y_i \) is linked to the explanatory variables \( x_i \) through the relation \( g(\mu_i) = \eta_i = x_i^T \beta \), where \( g \) is called the link function and \( \eta_i = x_i^T \beta \) is known as the linear predictor. Thus \( g(\mu_i) \), rather than \( \mu_i \), itself is linearly related to the explanatory variables. The link function \( g \) is assumed to be monotone and differentiable. It is easy to see from the theory of exponential family of distributions that the canonical parameter \( \theta_i \) is related to the mean \( \mu_i \) by \( \mu_i = b'(\theta_i) \) and also the variance of \( y_i \) is given by \( V(y_i) = b''(\theta_i)a(\phi) \), where \( b'(\cdot) \) and \( b''(\cdot) \) represent the first two derivatives of \( b(\cdot) \) with respect to its argument. Thus our main parameter of interest in the generalised linear model becomes the regression coefficient \( \beta \) and \( \phi \) acts as the nuisance parameter which is involved only in the error variance. Clearly generalised linear model allows us to choose several possible densities \( f \) from the exponential family
including Normal, Binomial, Poisson, Exponential, Negative Binomial etc. and the link function \( g \) to form a wide variety of regression models based on the patterns of the available data. Thus we can suitably model a large number of different types of data.

Note that choosing \( f \) to be the normal density and \( g \) as the identity link function the generalised linear model reduces to usual normal linear regression model. Further, choosing \( f \) as the Poisson density and \( g \) as the log link \( g(\mu) = \log(\mu) \), we get the Poisson regression case useful in modeling data and cases of over-dispersion. Also for the binomial \( f \), choosing Logit link function \( g(\mu) = \log(\mu(1 - \mu)) \) or Probit link function \( g(\mu) = \Phi^{-1}(\mu) \), we get the logistic and Probit regression models respectively which are useful in modeling the binary response variables. Similarly, many other useful models can be generated from the generalised linear model.

### 2.2 The Minimum Density Power Divergence Estimator and its Estimating Equation

We will define the minimum density power divergence estimators for the generalised linear model with general density \( f \) and link function \( g \) so that we can estimate the regression coefficients for any regression model as a special case of it by substituting the form of \( f \) and \( g \). In the later sections, we will consider some of these special cases in detail. Since the form of the density power divergence is now well known in the literature we do not repeat it here. A description of the form of this divergence may be found, for example, in Basu et al. (1998) and Ghosh and Basu (2013a), among others.

Let us assume that we have a data set \( (y_i; x_i); i = 1, \ldots, n \) from the generalised linear model with density \( f \) given by equation (1) and a general link function \( g(\mu_i) = \eta_i = x_i^T \beta \). We will further assume that the independent variables \( x_i \)'s are given and fixed so that we are indeed considering a fixed carrier generalised linear model. Then we have the set-up of independent but non-homogeneous observations, where \( y_1, \ldots, y_n \) are independent and \( y_i \) has density \( f_i(.; (\beta, \phi)) = f(y_i; \theta_i, \phi) \) for all \( i = 1, \ldots, n \). Hence we can use the approach of Ghosh and Basu (2013a), where the minimum density power divergence estimator for the independent but non-homogeneous observations was defined. Following this approach, the minimum density power divergence estimator of \( (\beta, \phi) \) has to be obtained by minimizing

\[
H_n(\beta, \phi) = \frac{1}{n} \sum_{i=1}^{n} V_i(Y_i; (\beta, \phi)),
\]

where

\[
V_i(Y_i; (\beta, \phi)) = \int f_i(y_i; (\beta, \phi))^{1+\alpha} dy - \left( 1 + \frac{1}{\alpha} \right) f_i(y_i; (\beta, \phi))^\alpha.
\]

Note that, in the usual generalised linear model estimation, we use a robust estimate of scale parameter \( \phi \) and then estimate the regression parameter \( \beta \). However, in the proposed minimum density power divergence estimation, we can simultaneously estimate \( \beta \) and \( \phi \) robustly by just minimizing \( H_n(\beta, \phi) \) with respect to both the parameters. The estimating equation of the parameters are then given by \( \frac{1}{n} \sum_{i=1}^{n} \nabla V_i(Y_i; (\beta, \phi)) = 0 \) or,

\[
\frac{1 + \alpha}{n} \sum_{i=1}^{n} \left[ \int u_i(y_i; (\beta, \phi)) f_i(y_i; (\beta, \phi))^{1+\alpha} dy - u_i(Y_i; (\beta, \phi)) f_i(Y_i; (\beta, \phi))^\alpha \right] = 0.
\] (2)

where \( u_i(y; (\beta, \phi)) = \nabla \log(f_i(y; (\beta, \phi)) \); \( \nabla \) represents the derivative with respect to \( (\beta, \phi) \). Let \( \nabla_\beta \) and \( \nabla_\phi \) denote the individual derivatives with respect to \( \beta \) and \( \phi \) respectively. Then, a simple calculation shows that

\[
\nabla_\beta \log(f_i(y_i; (\beta, \phi)) = \frac{(y_i - \mu_i)}{\text{Var}(y_i) g'(\mu_i)} x_i = K_1(y_i; (\beta, \phi)) x_i,
\] (3)

and

\[
\nabla_\phi \log(f_i(y_i; (\beta, \phi)) = \frac{(y_i - b(\theta_i))}{\phi^2} a'(\phi) + \frac{\partial}{\partial \phi} c(y_i, \phi) = K_2(y_i; (\beta, \phi)),
\] (4)
where $K_{1i}$ and $K_{2i}$ are the indicated functions. Thus our estimating equations become

\[
\sum_{i=1}^{n} x_i \left[ \int K_{1i}(y; (\beta, \phi)) f_i(y; (\beta, \phi))^{1+\alpha} dy - K_{1i}(y_i; (\beta, \phi)) f_i(y_i; (\beta, \phi))^{1+\alpha} \right] = 0, \tag{5}
\]

\[
\sum_{i=1}^{n} \left[ \int K_{2i}(y; (\beta, \phi)) f_i(y; (\beta, \phi))^{1+\alpha} dy - K_{2i}(y_i; (\beta, \phi)) f_i(y_i; (\beta, \phi))^{1+\alpha} \right] = 0. \tag{6}
\]

However, if we want to ignore the nuisance parameter $\phi$, as per the usual practice, and estimate $\beta$ taking $\phi$ fixed (or, substituted suitably), it is enough to consider only estimating equation (5). Further, for $\alpha = 0$, we have $\int \frac{(y_i - \mu_i)}{\text{Var}(y_i)} g'(\mu_i) x_i f_i(y_i; (\beta, \phi))^{1+\alpha} dy = 0$ and hence the estimating equations for $\beta$ (ignoring $\phi$) simplify to

\[
\sum_{i=1}^{n} \frac{(Y_i - \mu_i)}{\text{Var}(Y_i)g'(\mu_i)} x_i = 0.
\]

Note that this is just the maximum likelihood estimating equation and also is the same as the ordinary least squares (OLS) estimating equation for $\beta$ assuming $\phi$ to be fixed. Thus the minimum density power divergence estimator of $\beta$ with $\alpha = 0$ equals the maximum likelihood estimator as well as the ordinary least squares estimate of $\beta$. That is the minimum density power divergence estimator proposed here is just a natural robust generalization of the maximum likelihood estimator.

Also it is interesting to note that, if our density $f$ is such that $\int f(y; \theta_1, \phi)^{1+\alpha} dy$ is independent of the location parameter $\theta_1$, like the normal density, then we have $\int \frac{(y_i - \mu_i)}{\text{Var}(y_i)} g'(\mu_i) x_i f_i(y_i; (\beta, \phi))^{1+\alpha} dy = 0$ and hence the estimating equation (5) simplifies to

\[
\sum_{i=1}^{n} \frac{(Y_i - \mu_i)}{\text{Var}(Y_i)g'(\mu_i)} x_i f_i(Y_i; (\beta, \phi))^{\alpha} = 0. \tag{7}
\]

### 2.3 Asymptotic Properties

We will now derive the joint asymptotic distribution of the minimum density power divergence estimator $(\hat{\beta}, \hat{\phi})$ of the parameters $(\beta, \phi)$ obtained by solving the estimating equations (5) and (6). For simplicity, we will assume that the true data generating distribution also belongs to the model density with parameters $(\beta^0, \phi^0)$. Define, for $i = 1, \ldots, n$,

\[
\gamma_{1i} = \gamma_{1i}^{1+\alpha}(\beta, \phi) = \int K_{1i}(y; (\beta, \phi)) f_i(y; (\beta, \phi))^{1+\alpha} dy,
\]

\[
\gamma_{2i} = \gamma_{2i}^{1+\alpha}(\beta, \phi) = \int K_{2i}(y; (\beta, \phi)) f_i(y; (\beta, \phi))^{1+\alpha} dy,
\]

\[
\gamma_{jki} = \gamma_{jki}^{1+\alpha}(\beta, \phi) = \int K_{ji}(y; (\beta, \phi)) K_{ki}(y; (\beta, \phi)) f_i(y; (\beta, \phi))^{1+\alpha} dy, \quad j, k = 1, 2,
\]

so that

\[
N_i^{1+\alpha}(\beta, \phi) = \int u_i(y; (\beta, \phi)) f_i(y; (\beta, \phi))^{1+\alpha} dy = \begin{pmatrix} \gamma_{1i} x_i \\ \gamma_{2i} \end{pmatrix}, \tag{8}
\]

\[
M_i^{1+\alpha}(\beta, \phi) = \int u_i(y; (\beta, \phi)) u_i(y; (\beta, \phi))^T f_i(y; (\beta, \phi))^{1+\alpha} dy = \begin{pmatrix} \gamma_{11} x_i x_i^T \\ \gamma_{12} x_i \\ \gamma_{21} x_i \\ \gamma_{22} \end{pmatrix}. \tag{9}
\]

Now, put $\Gamma_j^{(\alpha)} = \text{Diag}(\gamma_{ji})_{i=1, \ldots, n}$ and $\Gamma_{jk}^{(\alpha)} = \text{Diag}(\gamma_{jki})_{i=1, \ldots, n}$ for $j, k = 1, 2$ and $X^T = [x_1, \ldots, x_n]$. Then we have

\[
\Psi_n(\beta, \phi) = \frac{1}{n} \sum_{i=1}^{n} M_i^{1+\alpha}(\beta, \phi) = \frac{1}{n} \begin{pmatrix} X^T \Gamma_1^{(\alpha)} X & X^T \Gamma_2^{(\alpha)} \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ X^T \Gamma_1^{(\alpha)} \mathbf{1} \end{pmatrix}; \tag{10}
\]
\[
\Omega_n(\beta, \phi) = \frac{1}{n} \sum_{i=1}^{n} \left[ M_i^{1+2\alpha}(\beta, \phi) - N_i^{1+\alpha}(\beta, \phi)(N_i^{1+\alpha}(\beta, \phi))^T \right] \tag{11}
\]
\[
= \frac{1}{n} \left( X^T \Gamma_1^{2\alpha} - \Gamma_1^{\alpha} T_1^{\alpha} X \right) \left( 1^T \Gamma_2^{2\alpha} - \Gamma_1^{\alpha} T_2^{\alpha} \right) \left( 1^T \Gamma_2^{1\alpha} - \Gamma_2^{\alpha} T_2^{\alpha} \right) 1 \tag{12}
\]

Then, the asymptotic distribution of \((\hat{\beta}, \hat{\phi})\) follows from a simple modification of Theorem 3.1 of Ghosh and Basu (2013a), provided the Assumptions (A1) to (A7) hold in case of the generalised linear models. Note that, Assumptions (A1) to (A3) hold directly from the properties of the exponential family of distributions.

**Theorem 2.1** Under Assumptions (A1)-(A7) of Ghosh and Basu (2013a), there exists a consistent sequence \((\hat{\beta}_n, \hat{\phi}_n)\) of roots to the minimum density power divergence estimating equations \(\hat{\beta}\) and \(\hat{\phi}\). Also, the asymptotic distribution of \(\Omega_n^{-\frac{1}{2}} \Psi_n \sqrt{n}(\hat{\beta}_n, \hat{\phi}_n) \) is \((p+1)\)-dimensional normal with mean 0 and variance \(I_{p+1}\), the identity matrix of dimension \(p+1\) where \(\Psi_n = \Psi_n(\beta^*, \phi^*)\) and \(\Omega_n = \Omega_n(\beta^*, \phi^*)\).

It follows from above theorem that the reciprocal of the matrix \(\Psi_n^{-1}\Omega_n\Psi_n^{-1}\) gives a estimate of the asymptotic efficiency of the minimum density power divergence estimators \((\hat{\beta}_n, \hat{\phi}_n)\). Though this depends on the sample size \(n\) and the given covariates \(x_i\)’s, it will give a reasonable estimate of the asymptotic efficiency for large \(n\). We will examine the performance of this measure for several particular problems in the subsequent sections.

Further, note that the asymptotic covariance of the estimators \(\hat{\beta}_n\) and \(\hat{\phi}_n\) are not in general 0 and hence these estimators are not asymptotically independent for all the generalised regression models. However, for some particular cases including the normal linear regression case, they turn out to be independent. One possible set of sufficient conditions for their independence are \(\gamma_{1i2i}^{1+2\alpha} = 0\) and \(\gamma_{1i}^{1+\alpha}\gamma_{2i}^{1+\alpha} = 0\) for all \(i\). These conditions hold for the normal linear regression case.

### 2.4 Influence Function

To illustrate the robustness properties of the proposed estimation methodology for the generalised regression model, we will now consider the influence function of the minimum density power divergence estimator of the parameter \(\theta = (\beta, \phi)\). For this we need to consider them in terms of a statistical functional at the true data generating distributions. Let \(T_\beta(G_1, \ldots, G_n)\) and \(T_\alpha(G_1, \ldots, G_n)\) denote the minimum density power divergence functional for the parameters \(\beta\) and \(\phi\) respectively. Let \(T_\alpha(G_1, \ldots, G_n) = (T_\beta(G_1, \ldots, G_n)^T, T_\alpha(G_1, \ldots, G_n)^T)^T\), which is defined by

\[
\frac{1}{n} \sum_{i=1}^{n} d_\alpha(g_i(\cdot), f_i(\cdot; T_\alpha(G_1, \ldots, G_n))) = \min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} d_\alpha(g_i(\cdot), f_i(\cdot; \theta)),
\]

where \(g_i\) is the probability density function corresponding to \(G_i\). We consider the contaminated density \(g_{i,\epsilon} = (1 - \epsilon)g_i + \epsilon \delta_{t_i}\), where \(t_i\) is the point of contamination and \(G_{i,\epsilon}\) denotes the corresponding distribution function for all \(i = 1, \ldots, n\). Let \(T_\alpha(G_1, \ldots, G_{n-1}, G_{i_0,\epsilon}, G_{n})\) be the minimum density power divergence functional with contamination only in the \(i_0^{th}\) direction. Then a fairly straightforward (albeit lengthy and tedious) calculation shows that the influence function of \(T_\alpha\) for contamination at the direction \(i_0\) will be

\[
IE_{i_0}(t_{i_0}, T_\alpha, G_1, \ldots, G_n) = \Psi_n^{-\frac{1}{2}} \left[ f_{i_0}(t_{i_0}; (\beta, \phi)) x_{i_0} - N_{i_0}^{1+\alpha} \right] \tag{13}
\]
\[
= \Psi_n^{-\frac{1}{2}} \left( f_{i_0}(t_{i_0}; (\beta, \phi)) x_{i_0} - \gamma_{1i_0} x_{i_0} \right) . \tag{14}
\]
Note that for any fixed sample size \( n \) and any given (finite) values of \( X_i \)'s, if \( \Psi_n \) and \( \gamma_{ji0} \)'s are assumed to be bounded, the influence function of the minimum density power divergence estimator of the parameters \((\beta, \phi)\) will be bounded with respect to the contamination in any direction \( i_0 \) provided the terms \( f_i(t_i; (\beta, \phi))^{\alpha} K_j(t_i; (\beta, \phi)) \) are bounded for all \( i \) and \( j = 1, 2 \). Under assumptions (A1) to (A7) of Ghosh and Basu (2013a), the values of \( \Psi_n \) and \( \gamma_{ji0} \)'s are necessarily bounded. This can be easily seen to hold for the majority of generalised linear models with \( \alpha > 0 \) because of the exponential nature of the density function and the polynomial nature of the functions \( K_{ji}(t_i; (\beta, \phi)) \). This demonstrates the robust nature of the minimum density power divergence estimator in most generalised linear models with \( \alpha > 0 \). However, for \( \alpha = 0 \) the term \( f_i(t_i; (\beta, \phi))^{\alpha} K_j(t_i; (\beta, \phi)) = K_j(t_i; (\beta, \phi)) \) is clearly unbounded implying the non-robust nature of the maximum likelihood estimator in the case of any generalised linear model.

As in Ghosh and Basu (2013a), in this context also we can define some measures of sensitivity based on the influence function. The (unstandardized) **gross-error sensitivity** of the functional \( T_n \) at the true distributions \( G_1, \ldots, G_n \), considering contamination only in the \( i_0 \)th direction may be defined as

\[
\gamma_{i_0}^g(T_\alpha, G_1, \ldots, G_n) = \sup_{t_{i_0}} \{|\mathcal{I}F_{i_0}(t_{i_0}, T_\alpha, G_1, \ldots, G_n)|\} 
\]

(15)

\[
= \frac{1}{n} \sup_{t_{i_0}} \{[f_i(t_{i_0}; \theta)^{\alpha} u_i(t_{i_0}; \theta) - \xi_i] \mathbf{T} \Psi_n^{-2} [f_i(t_{i_0}; \theta)^{\alpha} u_i(t_{i_0}; \theta) - \xi_i]^T \}. 
\]

(16)

Since it is not invariant to scale transformations of the individual parameter components, we will consider the **Self-Standardized Sensitivity**. For contamination in only the \( i_0 \)th direction, this may be defined as

\[
\gamma_{i_0}^s(T_\alpha, G_1, \ldots, G_n) = \sup_{t_{i_0}} \{[f_i(t_{i_0}; \theta)^{\alpha} u_i(t_{i_0}; \theta) - \xi_i] \mathbf{T} [\Psi_n^{-1} \Omega_n \Psi_n^{-1}]^{-1} \mathcal{I}F_{i_0}(t_{i_0}, T_\alpha, G_1, \ldots, G_n) \}^{\frac{1}{2}},
\]

(17)

\[
= \frac{1}{n} \sup_{t_{i_0}} \{[f_i(t_{i_0}; \theta)^{\alpha} u_i(t_{i_0}; \theta) - \xi_i] \mathbf{T} \Omega_n^{-1} [f_i(t_{i_0}; \theta)^{\alpha} u_i(t_{i_0}; \theta) - \xi_i] \}^{\frac{1}{2}}.
\]

(18)

As discussed before, in the particular case when \( \gamma_{12i}^{1+2\alpha} = 0 \) and \( \gamma_{1i}^{1+\alpha}, \gamma_{2i}^{1+\alpha} = 0 \) for all \( i \) (like the normal linear regression case), the minimum density power divergence estimator of \( \beta \) and \( \phi \) become asymptotically independent and we can also separate out the influence function for the minimum density power divergence estimator of \( \beta \) and \( \phi \). Due to the special form of the matrix \( \Psi_n \) in this case, these two influence functions simplify to

\[
\mathcal{I}F_{i_0}(t_{i_0}, T_\alpha^\beta, G_1, \ldots, G_n) = (X^T \Gamma_{11}^{(\alpha)} X)^{-1} x_{i_0} [f_{i_0}(t_{i_0}; (\beta, \phi))^{\alpha} K_{1i_0}(t_{i_0}; (\beta, \phi)) - \gamma_{1i_0}],
\]

and

\[
\mathcal{I}F_{i_0}(t_{i_0}, T_\alpha^\phi, G_1, \ldots, G_n) = (1^T \Gamma_{22}^{(\alpha)} 1)^{-1} [f_{i_0}(t_{i_0}; (\beta, \phi))^{\alpha} K_{2i_0}(t_{i_0}; (\beta, \phi)) - \gamma_{2i_0}],
\]

respectively. Then the (unstandardized) gross-error sensitivities of \( \beta \) and \( \phi \) are given respectively by

\[
\gamma_{i_0}^{\alpha}(T_\alpha^\beta, G_1, \ldots, G_n) = \sqrt{x_{i_0}^T (X^T \Gamma_{11}^{(\alpha)} X)^{-1} x_{i_0}} \sup_{t_{i_0}} |f_{i_0}(t_{i_0}; (\beta, \phi))^{\alpha} K_{1i_0}(t_{i_0}; (\beta, \phi)) - \gamma_{1i_0}|,
\]

and

\[
\gamma_{i_0}^{\alpha}(T_\alpha^\phi, G_1, \ldots, G_n) = (1^T \Gamma_{22}^{(\alpha)} 1)^{-1/2} |f_{i_0}(t_{i_0}; (\beta, \phi))^{\alpha} K_{2i_0}(t_{i_0}; (\beta, \phi)) - \gamma_{2i_0}|.
\]

The corresponding Self-Standardized Sensitivities also have the same form as above with \( \Gamma_{11}^{(\alpha)} \) and \( \Gamma_{22}^{(\alpha)} \) replaced by \( \Gamma_{11}^{(\alpha)} - \Gamma_{11}^{(\alpha)2} \) and \( \Gamma_{22}^{(\alpha)} - \Gamma_{22}^{(\alpha)2} \) respectively. Clearly these sensitivities are infinite for \( \alpha = 0 \) and are generally finite for most of the generalised linear model with \( \alpha > 0 \).

In the following we will derive the explicit form of the influence functions and sensitivity measures for some special cases of the generalised linear model.
3 Special Case (I) : Poisson Regression for Count Data

Samples that have the count data structure are usually modeled by the Poisson distribution. And the most useful regression tool for count data is the Poisson regression model where, given the values of explanatory variables, dependent variables independently follow the Poisson distribution but with different mean parameters depending on the corresponding values of the explanatory variable. More precisely, let \((y_1, x_1), \cdots, (y_n, x_n)\) be the sample observations from the Poisson regression model. We will assume that the values \(x_i\) of the explanatory variable are fixed known values. Then, in the Poisson regression model, the count variable \(y_i\) is assumed to be independent and have Poisson distributions with

\[
E(y_i | x_i) = e^{(x_i^T \beta)}
\]

and we want to estimate the parameter \(\beta\) efficiently and robustly.

3.1 The minimum density power divergence estimator for Poisson Regression

Poisson regression is indeed a special case of generalised linear models with known shape parameter \(\phi = 1\) and \(\theta_i = x_i^T \beta\), \(b(\theta_i) = e^{\theta_i}\) and \(c(y_i) = -\log(y_i!)\). Since here the mean is \(\mu_i = e^{(x_i^T \beta)}\), the link function \(g\) is the natural logarithm function and the variance of \(y_i\) is also \(e^{(x_i^T \beta)}\). Thus, we can estimate the unknown parameter \(\beta\) using our minimum density power divergence estimation procedure as described earlier. Using the above notation and the form of the Poisson distribution, we get

\[
N_i^{1+\alpha}(\beta) = \sum_{y=0}^{\infty} \left( y - e^{(x_i^T \beta)} \right) x_i f_i(y; \beta)^{1+\alpha} = \gamma_{11} x_i,
\]

where \(f_i(y; \beta)\) is the probability mass function of the Poisson distribution with mean \(e^{(x_i^T \beta)}\). Then for \(\alpha \geq 0\), the minimum density power divergence estimating equation is given by

\[
\sum_{i=1}^{n} \left[ \gamma_{11}(\beta) - \left( y_i - e^{(x_i^T \beta)} \right) f_i(y; \beta)^{\alpha} \right] x_i = 0.
\]

In particular, for \(\alpha = 0\) above estimating equation simplifies to the maximum likelihood estimating equation given by

\[
\sum_{i=1}^{n} \left( y_i - e^{(x_i^T \beta)} \right) x_i = 0.
\]

However, for \(\alpha > 0\), there is no simplified form for \(\gamma_{11}\) and \(\gamma_{111}\) so that we need to compute these quantities numerically and then numerically solve the estimating equation \((21)\) with respect to \(\beta\).

3.2 Properties of the minimum density power divergence estimator

The asymptotic properties of the minimum density power divergence estimator of \(\beta\) in this special case follows directly from the Theorem \(2.1\) of the previous section. Under the notation of Section \(2.3\), we have

\[
\Psi_n(\beta) = \frac{1}{n} \left( X^T \Gamma^{(\alpha)}_{11} X \right), \quad \Omega_n(\beta) = \frac{1}{n} \left( X^T [\Gamma^{(2\alpha)}_{11} - \Gamma^{(\alpha)^2}_{11}] X \right).
\]

Thus we have
Corollary 3.1 Under Assumptions (A1)-(A7) of Ghosh and Basu (2013a), there exists a consistent sequence \( \hat{\beta}_n = \hat{\beta}_n^{(a)} \) of roots to the minimum density power divergence estimating equations (21) at tuning parameter \( \alpha \). Also, the asymptotic distribution of \( \left( X^T [\Gamma_{11}^{(2\alpha)}(\beta^g) - \Gamma_1^{(\alpha)}(\beta^g)] X \right)^{-\frac{1}{2}} \left( X^T \Gamma_{11}^{(\alpha)}(\beta^g) X \right) (\hat{\beta}_n - \beta^g) \) is \( p \)-dimensional normal with mean 0 and variance \( I_p \).

Thus the asymptotic efficiency of the different minimum density power divergence estimator \( \hat{\beta}_n = \hat{\beta}_n^{(a)} \) of \( \beta \) can be measured based on the asymptotic variance

\[
AV_\alpha(\beta^g) = \left( X^T \Gamma_{11}^{(\alpha)}(\beta^g) X \right)^{-1} \left( X^T [\Gamma_{11}^{(2\alpha)}(\beta^g) - \Gamma_1^{(\alpha)}(\beta^g)] X \right) \left( X^T \Gamma_{11}^{(\alpha)}(\beta^g) X \right)^{-1},
\]

which can be consistently estimated by replacing \( \beta^g \) with \( \hat{\beta}_n \) in its expression, i.e., \( \widehat{AV}_n = AV_\alpha(\hat{\beta}_n) \). Thus an estimate of the relative efficiency of the different minimum density power divergence estimators of the \( i \)th component of the parameter vector \( \beta \) with respect to its maximum likelihood estimator (or the ordinary least squares estimator) is given by

\[
\widehat{RE}_{i,\alpha} = \frac{i^{th} \text{ diagonal entry of } \widehat{AV}_0}{i^{th} \text{ diagonal entry of } AV_\alpha} \times 100.
\]

Clearly, the above estimate of the relative efficiency depends on the sample size \( n \) and the choice of the given explanatory variables \( x_i \)’s. But it can be shown that the consistency of the estimator \( \hat{\beta}_n \) implies that the above measure gives us a consistent estimator of the asymptotic relative efficiency if the \( x_i \)’s are chosen suitably. For example we must have \( X^T X \) to be bounded. We have presented the empirical value of this measure of relative efficiency for different sample sizes \( n = 50 \) and \( n = 100 \) respectively under several different cases in Tables 1 and 2. We have reported six cases which are defined based on the true values of the regression coefficients \( \beta = (\beta_0, \beta_1, \ldots, \beta_p) \) and the given values of the explanatory variables \( x_i \) (i = 1, \ldots, n) as follows:

- **Case I**: \( p = 2; \beta = (1, 1) \) and \( x_i = (1, \sqrt{i}) \).
- **Case II**: \( p = 2; \beta = (1, 0.5) \) and \( x_i = (1, \sqrt{i}) \).
- **Case III**: \( p = 2; \beta = (1, 1) \) and \( x_i = (1, \frac{1}{i}) \).
- **Case IV**: \( p = 2; \beta = (1, 0.5) \) and \( x_i = (1, \frac{1}{i}) \).
- **Case V**: \( p = 3; \beta = (1, 1, 1) \) and \( x_i = (1, \sqrt{i}, \frac{1}{\sqrt{i}}) \).
- **Case VI**: \( p = 3; \beta = (2, 1, 0.5) \) and \( x_i = (1, \sqrt{i}, \frac{1}{\sqrt{i}}) \).

All the simulations are done based on 1000 replications. It is clear from the tables that the loss of efficiency is very negligible for the minimum density power divergence estimator with small positive \( \alpha \) under each of the cases considered here. Even for large positive \( \alpha \) near 0.5 also, we can get quite high efficiency if \( x_i \)’s are relatively small.

Next, in order to see the robustness of the minimum density power divergence estimator in case of the Poisson regression model, we will use the results from the Section 2.4. The influence function of the minimum density power divergence estimator in the direction \( t_0 \) simplifies to

\[
IF_{t_0}(t_0, T_\alpha, G_1, \ldots, G_n) = (X^T \Gamma_{11}^{(\alpha)} X)^{-1} x_{t_0} \left[ t_{t_0} - e^{(x^T x_0^\alpha)} \frac{e^{a t_{t_0} (x^T x_0^\alpha)}}{(t_{t_0})^\alpha} e^{a e^{(x^T x_0^\alpha)} - \gamma_{1 t_0}} \right] .
\]
Table 1: The estimated Relative efficiencies of the MDPDE for various values of the tuning parameter $\alpha$ under different cases of Poisson regression with sample size $n = 50$

| Case | Coefficients | $\alpha = 0$ | $\alpha = 0.01$ | $\alpha = 0.1$ | $\alpha = 0.25$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 1$ |
|------|--------------|--------------|-----------------|----------------|-----------------|----------------|----------------|----------------|----------------|
| I    | $\beta_0$    | 100.0        | 100.0           | 98.3           | 91.2            | 80.9           | 73.5           | 58.1           | 37.9           |
|      | $\beta_1$    | 100.0        | 100.0           | 98.3           | 91.1            | 80.7           | 73.1           | 57.2           | 36.4           |
| II   | $\beta_0$    | 100.0        | 99.9           | 98.5           | 93.2            | 85.9           | 80.7           | 70.5           | 56.5           |
|      | $\beta_1$    | 100.0        | 99.8           | 98.4           | 93.0            | 85.7           | 80.5           | 70.0           | 55.5           |
| III  | $\beta_0$    | 100.0        | 100.0          | 98.8           | 94.5            | 89.9           | 85.1           | 77.6           | 67.7           |
|      | $\beta_1$    | 100.0        | 100.0          | 98.8           | 93.7            | 88.4           | 84.3           | 76.0           | 64.8           |
| IV   | $\beta_0$    | 100.0        | 100.0          | 98.7           | 94.4            | 88.9           | 85.1           | 77.6           | 68.0           |
|      | $\beta_1$    | 100.0        | 100.0          | 98.9           | 94.3            | 88.4           | 84.6           | 76.6           | 66.3           |
|      | $\beta_2$    | 100.0        | 100.0          | 98.6           | 94.3            | 88.1           | 83.8           | 75.8           | 65.0           |
| V    | $\beta_0$    | 100.0        | 100.0          | 98.9           | 94.4            | 88.7           | 84.9           | 77.4           | 67.5           |
|      | $\beta_1$    | 100.0        | 100.0          | 98.9           | 94.3            | 88.4           | 84.3           | 76.1           | 65.7           |
|      | $\beta_2$    | 100.0        | 100.0          | 98.7           | 94.2            | 88.1           | 84.0           | 76.0           | 65.5           |
| VI   | $\beta_0$    | 100.0        | 100.0          | 98.6           | 94.3            | 88.1           | 83.8           | 75.8           | 65.0           |
|      | $\beta_1$    | 100.0        | 100.0          | 98.6           | 94.3            | 88.1           | 83.7           | 75.7           | 64.8           |

Table 2: The estimated Relative efficiencies of the MDPDE for various values of the tuning parameter $\alpha$ under different cases of Poisson regression with sample size $n = 100$

| Case | Coefficients | $\alpha = 0$ | $\alpha = 0.01$ | $\alpha = 0.1$ | $\alpha = 0.25$ | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 1$ |
|------|--------------|--------------|-----------------|----------------|-----------------|----------------|----------------|----------------|----------------|
| I    | $\beta_0$    | 100.0        | 100.0           | 98.2           | 89.8            | 77.3           | 67.7           | 48.0           | 24.1           |
|      | $\beta_1$    | 100.0        | 100.0           | 98.2           | 89.7            | 77.0           | 67.2           | 47.1           | 22.9           |
| II   | $\beta_0$    | 100.0        | 100.0           | 98.4           | 92.4            | 83.9           | 77.9           | 65.4           | 48.5           |
|      | $\beta_1$    | 100.0        | 100.0           | 98.4           | 92.3            | 83.7           | 77.5           | 64.7           | 47.2           |
| III  | $\beta_0$    | 100.0        | 100.0           | 98.7           | 94.4            | 88.9           | 85.1           | 77.8           | 67.9           |
|      | $\beta_1$    | 100.0        | 100.0           | 98.8           | 94.3            | 88.3           | 83.9           | 75.6           | 64.8           |
|      | $\beta_2$    | 100.0        | 100.0           | 99.4           | 93.8            | 89.0           | 84.8           | 76.9           | 66.7           |
| IV   | $\beta_0$    | 100.0        | 100.0           | 98.9           | 94.1            | 89.0           | 83.2           | 78.0           | 68.2           |
|      | $\beta_1$    | 100.0        | 100.0           | 99.4           | 93.8            | 89.0           | 84.8           | 76.9           | 66.7           |
|      | $\beta_2$    | 100.0        | 100.0           | 99.9           | 93.8            | 88.2           | 83.8           | 76.0           | 65.2           |
| V    | $\beta_0$    | 100.0        | 100.0           | 98.7           | 94.3            | 88.9           | 83.0           | 77.7           | 67.7           |
|      | $\beta_1$    | 100.0        | 99.9           | 98.6           | 93.8            | 88.2           | 83.9           | 76.2           | 65.6           |
|      | $\beta_2$    | 100.0        | 100.0           | 98.9           | 94.2            | 88.2           | 84.1           | 76.2           | 65.6           |
| VI   | $\beta_0$    | 100.0        | 100.0           | 99.2           | 94.2            | 88.3           | 84.2           | 76.0           | 65.9           |
|      | $\beta_1$    | 100.0        | 100.0           | 99.1           | 94.2            | 88.3           | 84.1           | 75.7           | 64.8           |

Clearly, whenever the inverse of the first matrix exists, this influence function is bounded in $t_{i0}$ for any $\alpha > 0$ implying the robustness of the minimum density power divergence estimator with $\alpha > 0$. However, at $\alpha = 0$, the influence function above further simplifies to

$$IF_{i0}(t_{i0}, T_0^0, G_1, \ldots, G_n) = (X^T \Gamma_{11}^{(0)} X)^{-1} x_{i0} (t_{i0} - e^{(x_{i0}^T \beta)})$$

which is linear and hence unbounded in $t_{i0}$. This indicates the non-robustness of the maximum likelihood estimator and equivalently ordinary least squares of the regression parameter in case of the Poisson regression model. Figures 1 and 2 show the Influence function of the minimum density power divergence estimator for different $\alpha$ under several specific Poisson regression models and for sample sizes 50 and 100 respectively. The redescending nature of the influence function with increasing $\alpha$ is quite clear in all the figures.
Figure 1: Plot of the influence function of MDPDE of slope parameter $\beta_1$ for different $\alpha$ and direction $i_0$ of contamination in case of three models [Model I: $x_i = (1, \sqrt{7})^T$, Model II: $x_i = (1, \frac{1}{7})^T$, Model III: $x_i = (1, \frac{1}{7}, \frac{1}{7})^T$ with $\beta_i = 1$ and $n = 50$]
Figure 2: Plot of the influence functions of MDPDE of slope parameter $\beta_1$ for different $\alpha$ and direction $i_0$ of contamination in case of three models [Model I: $x_i = (1, \sqrt{i})^T$, Model II: $x_i = (1, \frac{1}{\sqrt{i}})^T$, Model III: $x_i = (1, \frac{1}{\sqrt{i}}, \frac{1}{\sqrt{i}})^T$ with $\beta_i = 1$ and $n = 100$]
4 Special Case (II) : Logistic Regression for Binary Data

Another important special case of the generalised linear model is the logistic regression model which is used to model any categorical or binary dependent variables in terms of some explanatory variable. Given the values of the explanatory variables \(x_i\), the binary outcome variable \(y_i\) (or the binary transform of the categorical variables) are assumed to follow a Bernoulli trial with success probability \(\pi_i\) depending on the explanatory variable \(x_i\) (for each \(i = 1, \cdots, n\)). To ensure that the predicted values of \(\pi_i\) are in the interval \((0,1)\), in the logistic model it is assumed that

\[
\pi_i = \pi(x_i) = \frac{e^{x_i^T \beta}}{1 + e^{x_i^T \beta}}.
\]

We will now assume that the \(x_i\)'s are fixed and consider the logistic regression model from its design perspective to estimate \(\beta\) efficiently and robustly.

4.1 The minimum density power divergence estimator for the Logistic Regression

We can treat the logistic regression model as a particular case of the generalised linear model with known shape parameter \(\phi = 1\) and \(\theta_i = \eta_i = x_i^T \beta\), \(c(y_i) = 0\). The distribution of \(y_i\) is the Bernoulli distribution with mean \(\mu_i = \pi_i = \frac{e^{\eta_i}}{1+e^{\eta_i}}\), and \(\text{var}(y_i) = \pi_i(1-\pi_i) = \frac{e^{\eta_i}}{(1+e^{\eta_i})^2}\). Thus the link function \(g\) is the logit function and so we can use the minimum density power divergence estimation procedure discussed in Section 2 to estimate \(\beta\) robustly. Using the above notations and the form of the Bernoulli distribution, a simple calculation yields \(K_{11}(y_i; \beta) = (y_i - \mu_i)\) so that

\[
\gamma_{1i} = (1-\mu_i)(1+\alpha) - \mu_i(1-\mu_i)(1+\alpha) = \frac{e^{x_i^T \beta}(e^{\alpha x_i^T \beta} - 1)}{(1 + e^{x_i^T \beta})^{2+\alpha}},
\]

and

\[
\gamma_{11i} = (1-\mu_i)^2(1+\alpha) + \mu_i^2(1-\mu_i)(1+\alpha) = \frac{e^{x_i^T \beta}(e^{\alpha x_i^T \beta} + e^{\beta x_i^T \beta})}{(1 + e^{x_i^T \beta})^{3+\alpha}}. \tag{23}
\]

Then the minimum density power divergence estimating equation for \(\alpha \geq 0\) is given by,

\[
\sum_{i=1}^{n} \left[ \frac{e^{x_i^T \beta}(e^{\alpha x_i^T \beta} - 1)}{(1 + e^{x_i^T \beta})^{2+\alpha}} - \left( y_i - \frac{e^{x_i^T \beta}}{1 + e^{x_i^T \beta}} \right) \frac{e^{\alpha x_i^T \beta}y_i}{(1 + e^{x_i^T \beta})^{\alpha}} \right] x_i = 0, \tag{24}
\]

which can be further simplified to

\[
\sum_{i=1}^{n} (1 - 2y_i)e^{(x_i^T \beta)(1-y_i)} \left( \frac{e^{\alpha x_i^T \beta} + e^{x_i^T \beta}}{(1 + e^{x_i^T \beta})^{2+\alpha}} \right) x_i = 0. \tag{25}
\]

We can easily solve the above estimating equation with respect to \(\beta\) to compute the minimum density power divergence estimator for any \(\alpha \geq 0\). In particular, for \(\alpha = 0\), equation \((24)\) simplifies to

\[
\sum_{i=1}^{n} \left( y_i - \frac{e^{x_i^T \beta}}{1 + e^{x_i^T \beta}} \right) x_i = 0, \tag{26}
\]

which is the maximum likelihood estimating equation. Once again the minimum density power divergence estimator estimating equation is just a generalization of the maximum likelihood estimating equation.
4.2 Properties of minimum density power divergence estimator

We will now present the asymptotic distribution of the minimum density power divergence estimator of \( \beta \) in the logistic regression case as it follows from Theorem 2.1. In this special case, we have

\[
\hat{\beta} = \frac{1}{n} \sum_{i=1}^{n} e^{x_i^T \beta} \left( \frac{e^{\alpha(x_i^T \beta)} + e^{x_i^T \beta}}{(1 + e^{x_i^T \beta})^{\beta + \alpha}} \right) (x_i x_i^T)
\]

and

\[
\Omega_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} e^{x_i^T \beta} \left( \frac{e^{\alpha(x_i^T \beta)} + e^{x_i^T \beta}}{(1 + e^{x_i^T \beta})^{4 + 2\alpha}} \right) (x_i x_i^T).
\]

We then have the following result.

**Corollary 4.1** Under Assumptions (A1)-(A7) of Ghosh and Basu (2013a), there exists a consistent sequence \( \hat{\beta}_n = \hat{\beta}_n^{(\alpha)} \) of roots to the minimum density power divergence estimating equations (25) at the tuning parameter \( \alpha \). Also, the asymptotic distribution of

\[
\left( \sum_{i=1}^{n} e^{x_i^T \beta^y} \left( \frac{e^{\alpha(x_i^T \beta^y)} + e^{x_i^T \beta^y}}{(1 + e^{x_i^T \beta^y})^{\beta^y + \alpha}} \right) (x_i x_i^T) \right)^{-\frac{1}{2}} \times \left( \sum_{i=1}^{n} e^{x_i^T \beta^y} \left( \frac{e^{\alpha(x_i^T \beta^y)} + e^{x_i^T \beta^y}}{(1 + e^{x_i^T \beta^y})^{\beta^y + \alpha}} \right) (x_i x_i^T) \right) \left( \hat{\beta}_n - \beta^y \right)
\]

is \( p \)-dimensional normal with mean 0 and variance \( I_p \).

Then, as argued in Section 3.2 for the Poisson regression, the asymptotic efficiency of the different minimum density power divergence estimator \( \hat{\beta}_n = \hat{\beta}_n^{(\alpha)} \) of \( \beta \) for the logistic regression can also be measured in terms of its asymptotic variance

\[
AV_\alpha(\beta^y) = \left( \sum_{i=1}^{n} e^{x_i^T \beta^y} \left( \frac{e^{\alpha(x_i^T \beta^y)} + e^{x_i^T \beta^y}}{(1 + e^{x_i^T \beta^y})^{\beta^y + \alpha}} \right) (x_i x_i^T) \right)^{-1} \left( \sum_{i=1}^{n} e^{x_i^T \beta^y} \left( \frac{e^{\alpha(x_i^T \beta^y)} + e^{x_i^T \beta^y}}{(1 + e^{x_i^T \beta^y})^{\beta^y + \alpha}} \right) (x_i x_i^T) \right) \left( \sum_{i=1}^{n} e^{x_i^T \beta^y} \left( \frac{e^{\alpha(x_i^T \beta^y)} + e^{x_i^T \beta^y}}{(1 + e^{x_i^T \beta^y})^{\beta^y + \alpha}} \right) (x_i x_i^T) \right)^{-1}.
\]

This can be estimated consistently by \( \overline{AV}_\alpha = AV_\alpha(\hat{\beta}_n) \).

As in the Poisson regression case, here also we can compute the values of relative efficiencies of the minimum density power divergence estimators of the coefficients of the logistic regression model based on \( AV_\alpha \). This measure of relative efficiency clearly depends on the value of \( \beta \) and \( X_i \)'s. We present the empirical estimate of the relative efficiencies of the MDPDE in case of the logistic regression model in Tables 1 and 2 respectively for sample size \( n = 50 \) and \( n = 100 \). These are calculated based on a simulation study based on 1000 replications under several different cases of logistic regressions. These cases are defined based on the true values of the regression coefficients \( \beta = (\beta_0, \beta_1, \ldots, \beta_p) \) and the given values of the explanatory variables \( x_i \) \( (i = 1, \ldots, n) \) as follows:

- **Case I** : \( p = 2; \beta = (0.1, 0.1) \) and \( x_i = (1, \sqrt{i}) \).
- **Case II** : \( p = 2; \beta = (0.001, 0.0001) \) and \( x_i = (1, \sqrt{i}) \).
- **Case III** : \( p = 2; \beta = (1, 1) \) and \( x_i = (1, \frac{1}{i}) \).
- **Case IV** : \( p = 2; \beta = (0.1, 0.01) \) and \( x_i = (1, \frac{1}{i}) \).
Case V: \( p = 3; \beta = (0.1, 0.1, 0.1) \) and \( x_i = (1, \sqrt{i}, \frac{1}{i^2}) \).

Case VI: \( p = 3; \beta = (0.01, 0.001, 0.0001) \) and \( x_i = (1, \sqrt{i}, \frac{1}{i^2}) \).

It is clearly seen from the tables that for any value of the parameter and the explanatory variables, the loss of efficiency is negligible for small \( \alpha > 0 \). Further, if the values of \( x_i^T \beta \) is small, then we can get very quite high efficiency even for large positive \( \alpha \) near 0.5.

Table 3: The estimated Relative efficiencies of the MDPDE for various values of the tuning parameter \( \alpha \) under different cases of Logistic Regression with sample size \( n = 50 \)

| Case | Coefficients | \( \alpha = 0 \) | \( \alpha = 0.01 \) | \( \alpha = 0.1 \) | \( \alpha = 0.25 \) | \( \alpha = 0.4 \) | \( \alpha = 0.5 \) | \( \alpha = 0.7 \) | \( \alpha = 1 \) |
|------|--------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| I    | \( \beta_0 \) | 100.0          | 99.0           | 90.7           | 74.6           | 67.6           | 61.3           | 50.4           | 37.5           |
|      | \( \beta_1 \) | 100.0          | 99.2           | 92.7           | 79.6           | 73.8           | 68.4           | 58.7           | 46.7           |
| II   | \( \beta_0 \) | 100.0          | 99.3           | 93.3           | 81.2           | 75.8           | 70.7           | 61.5           | 50.0           |
|      | \( \beta_1 \) | 100.0          | 99.3           | 93.3           | 81.2           | 75.8           | 70.7           | 61.5           | 50.0           |
| III  | \( \beta_0 \) | 100.0          | 98.6           | 86.7           | 65.2           | 56.5           | 49.0           | 36.8           | 23.9           |
|      | \( \beta_1 \) | 100.0          | 98.1           | 82.8           | 56.9           | 47.2           | 39.2           | 27.1           | 15.6           |
| IV   | \( \beta_0 \) | 100.0          | 99.3           | 92.8           | 79.8           | 74.0           | 68.7           | 59.1           | 47.2           |
|      | \( \beta_1 \) | 100.0          | 99.2           | 92.4           | 79.0           | 73.0           | 67.5           | 57.7           | 45.6           |
| V    | \( \beta_0 \) | 100.0          | 99.3           | 92.7           | 79.8           | 74.0           | 68.6           | 59.0           | 47.1           |
|      | \( \beta_1 \) | 100.0          | 99.2           | 92.6           | 79.4           | 73.5           | 68.1           | 58.4           | 46.3           |
|      | \( \beta_2 \) | 100.0          | 99.2           | 92.4           | 78.9           | 72.9           | 67.4           | 57.3           | 45.4           |
| VI   | \( \beta_0 \) | 100.0          | 99.3           | 93.3           | 81.1           | 75.6           | 70.5           | 61.3           | 49.7           |
|      | \( \beta_1 \) | 100.0          | 99.3           | 93.3           | 81.1           | 75.6           | 70.5           | 61.3           | 49.7           |
|      | \( \beta_2 \) | 100.0          | 99.3           | 93.3           | 81.1           | 75.6           | 70.5           | 61.3           | 49.7           |

Table 4: The estimated Relative efficiencies of the MDPDE for various values of the tuning parameter \( \alpha \) under different cases of Logistic Regression with sample size \( n = 50 \)

| Case | Coefficients | \( \alpha = 0 \) | \( \alpha = 0.01 \) | \( \alpha = 0.1 \) | \( \alpha = 0.25 \) | \( \alpha = 0.4 \) | \( \alpha = 0.5 \) | \( \alpha = 0.7 \) | \( \alpha = 1 \) |
|------|--------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| I    | \( \beta_0 \) | 100.0          | 98.9           | 89.8           | 72.3           | 64.9           | 58.2           | 46.8           | 33.7           |
|      | \( \beta_1 \) | 100.0          | 99.2           | 92.6           | 79.5           | 73.6           | 68.2           | 58.5           | 46.5           |
| II   | \( \beta_0 \) | 100.0          | 99.3           | 93.3           | 81.2           | 75.8           | 70.7           | 61.5           | 50.0           |
|      | \( \beta_1 \) | 100.0          | 99.3           | 93.3           | 81.2           | 75.8           | 70.7           | 61.5           | 50.0           |
|      | \( \beta_2 \) | 100.0          | 98.1           | 82.9           | 57.2           | 47.5           | 39.7           | 27.4           | 15.9           |
| III  | \( \beta_0 \) | 100.0          | 99.3           | 92.8           | 79.9           | 74.1           | 68.8           | 59.2           | 47.3           |
|      | \( \beta_1 \) | 100.0          | 99.2           | 92.4           | 79.0           | 73.0           | 67.5           | 57.7           | 45.6           |
|      | \( \beta_2 \) | 100.0          | 99.2           | 92.4           | 78.9           | 72.9           | 67.3           | 57.5           | 45.3           |
| IV   | \( \beta_0 \) | 100.0          | 99.3           | 93.3           | 81.1           | 75.6           | 70.5           | 61.3           | 49.7           |
|      | \( \beta_1 \) | 100.0          | 99.3           | 93.3           | 81.1           | 75.6           | 70.5           | 61.3           | 49.7           |
| V    | \( \beta_0 \) | 100.0          | 99.3           | 93.3           | 81.1           | 75.6           | 70.5           | 61.3           | 49.7           |
5 A Data-driven Choice of the tuning parameter $\alpha$

The minimum density power divergence estimators depend on the choice of the tuning parameter $\alpha \geq 0$ defining the divergence. The properties of the minimum density power divergence estimator in the case of independent and identically distributed data have been extensively studied in the literature and it is well known that there is a trade off between efficiency and robustness for varying $\alpha$. Increasing $\alpha$ leads to greater robustness at the cost of efficiency. Ghosh and Basu (2013a, 2013b) also observed similar trade-offs for the linear regression case with fixed covariates. In the two previous sections, we have observed the same phenomenon in the context of the proposed minimum density power divergence estimator for the Poisson and the logistic regression models. Therefore, it is necessary to carefully choose the tuning parameter $\alpha$ while using the minimum density power divergence estimator in any of the generalised linear regression models. In this section, we will try to present a possible approach to choose the optimum value of $\alpha$ based on the observed data at hand.

In the context of the i.i.d. data problems, some data driven choices for selecting the optimum tuning parameter in the minimum density power divergence estimation context have been proposed by Hong and Kim (2001) and Warwick and Jones (2005). Ghosh and Basu (2013b) extended these approaches to the case of independent but non-homogeneous data and illustrated this approach for the case of linear regression. Kim (2001) and Warwick and Jones (2005). Ghosh and Basu (2013b) extended these approaches to the parameter in the minimum density power divergence estimation context have been proposed by Hong and Kim (2001) and Warwick and Jones (2005). In the two special cases of generalised linear model discussed above, namely the Poisson and the logistic regression models. Therefore, it is necessary to carefully choose the tuning parameter $\alpha$ while using the minimum density power divergence estimator in any of the generalised linear regression models.

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some values of the tuning parameter $\alpha$ for which the MDPDE of the parameters become robust for the respective data set. Now, comparing it with the table of optimum $\alpha$, we see that the choice of pilot $\alpha$ is very crucial in order to get a optimum $\alpha$ leading to a robust estimator. In this regard, the choice $\alpha = 0.5$ as the pilot $\alpha$ is sufficient to give us the “good” optimum $\alpha$ values for each of the data set and leads to the robust estimators of the parameter in next stage. Thus, we suggest $\alpha = 0.5$ as a reasonable choice for “pilot” $\alpha$ for obtaining the MDPDE in generalized linear regression. Note that, this choice is also in-line with the work of Ghosh and Basu (2013b) where the same choice for the pilot $\alpha$ was proposed in the context of linear regression model.

6 Real Data Examples

In this section, we will explore the performance of the proposed minimum density power divergence estimators in Poisson and logistic regression models by applying it on some interesting real data-sets. These data-sets are obtained from several clinical trials or surveys which are very important in biological sciences to demonstrate the importance of the proposed methodology in that field of application.

6.1 Epilepsy Data

In this example we will consider an interesting data-set consisting of 59 epilepsy patients from Thall and Vail (1990). The data were obtained from a clinical trial carried out by Leppik et al. (1985) where the patients were treated by the anti-epileptic drug “prog-abide” or a placebo with randomized assignment. Then the total number of epilepsy attacks was noted which can be modeled by the Poisson distribution. Here, we will try to explain the total number of attacks by an appropriate set of explanatory variables through a Poisson regression model (Hosseinian, 2009). The variables considered in this regard are “Base”, the eight week baseline seizure rate prior to randomization in multiples of 4, “Age”, patients’ age in multiple of 10 years, and “Trt”, an binary indicator for the treatment-control group. Also, the interaction between treatment and baseline seizure rate is important in this case, because it represents either higher or lower seizure rate for the treatment group compared to the placebo group depending on the baseline count. In fact, the drug decreases the epilepsy only if the baseline count becomes sufficiently large in numbers with respect to some critical threshold.

The data were also analyzed by Hosseinian (2009) who compared the maximum likelihood estimator with the robust methodologies proposed by herself in the same paper and those by Cantoni and Ronchetti (2001). There it was observed that the data contain some outlying observations for which the interaction effect between treatment and baseline seizure rate turns out to be insignificant based on the maximum likelihood estimator whereas the robust estimators show this interaction to be significant. Here, we will apply our proposed robust minimum density power divergence estimators for this epilepsy data set and try to see if our proposed estimators are also robust enough to differentiate with maximum likelihood estimator for the interaction effect.

Table 5 presents the parameter estimates, their asymptotic standard errors and corresponding p-values based on the minimum density power divergence estimator with different $\alpha$. Clearly the estimators corresponding to $\alpha \geq 0.3$ are quite different from maximum likelihood estimator and for these estimators the interaction effect is also significant under the Poisson regression model. Indeed, these estimators are quite similar to the robust estimators considered in Hosseinian (2009) but have greater asymptotic efficiency.

6.2 Australia AIDS Data

We will consider an interesting data set on the number of AIDS patients in Australia by the date of diagnosis for successive quarters of 1984 to 1988 (Dobson, 2002). This is a part of a large survey by National Centre for HIV Epidemiology and Clinical Research and reported in 1994. Dobson (2002) modeled these count data
Table 5: The minimum density power divergence estimates, their standard errors and p-values for the Epilepsy Data

|          | $\alpha = 0$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 1$ |
|----------|--------------|----------------|----------------|----------------|----------------|-------------|
| Intercept| Estimate     | 1.9888         | 2.1089         | 1.9106         | 1.9691         | 2.0060      | 1.9653      |
|          | SE (×100)    | 13.6518        | 15.2509        | 12.6869        | 13.7081        | 14.9185     | 17.0043     |
|          | P-Value      | 0.0000         | 0.0000         | 0.0000         | 0.0000         | 0.0000      | 0.0000      |
| Trt      | Estimate     | -0.2375        | -0.3169        | -0.3871        | -0.3893        | -0.3516     | -0.3186     |
|          | SE (×100)    | 7.6816         | 8.6812         | 7.9139         | 8.4566         | 9.1111      | 10.1787     |
|          | P-Value      | 0.0030         | 0.0006         | 0.0000         | 0.0000         | 0.0003      | 0.0027      |
| Base     | Estimate     | 0.0858         | 0.0866         | 0.1689         | 0.1631         | 0.1622      | 0.1562      |
|          | SE (×100)    | 0.3698         | 0.4101         | 0.2778         | 0.3055         | 0.3359      | 0.3959      |
|          | P-Value      | 0.0000         | 0.0000         | 0.0000         | 0.0000         | 0.0000      | 0.0000      |
| Age      | Estimate     | 0.2308         | 0.1153         | 0.0408         | 0.0362         | 0.0242      | 0.0559      |
|          | SE (×100)    | 4.1498         | 4.7242         | 3.9374         | 4.2416         | 4.6138      | 5.2119      |
|          | P-Value      | 0.0000         | 0.0177         | 0.3045         | 0.3972         | 0.6017      | 0.2878      |
| Trt×Base | Estimate     | 0.0069         | 0.0107         | 0.0156         | 0.0165         | 0.0131      | 0.0098      |
|          | SE (×100)    | 0.4443         | 0.4893         | 0.3230         | 0.3537         | 0.3888      | 0.4600      |
|          | P-Value      | 0.1283         | 0.0323         | 0.0000         | 0.0000         | 0.0013      | 0.0373      |

Table 6: The minimum DPD estimates of the parameters for the Australia AIDS Data. The standard error of the estimates are given in the parenthesis.

|          | $\alpha = 0$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 1$ |
|----------|--------------|----------------|----------------|----------------|----------------|-------------|
| Intercept| Estimate     | 0.9953         | 1.0248         | 1.0196         | 1.0192         | 1.0151      | 0.8395      |
|          | SE (×100)    | 3.0554         | 3.0294         | 3.0266         | 3.0287         | 3.0239      | 3.1739      |
|          | P-Value      | 0.0115         | 0.0160         | 0.0180         | 0.0190         | 0.0190      | 0.0230      |
| Without  | log(time)    | 1.2431         | 1.1680         | 1.0630         | 1.2100         | 1.1587      | 1.0191      |
| outliers | SE (×100)    | 2.8404         | 2.9059         | 2.9955         | 2.8596         | 2.8993      | 3.0182      |
|          | P-Value      | 0.0140         | 0.0150         | 0.0160         | 0.0160         | 0.0180      | 0.0220      |
| With one | log(time)    | 1.7291         | 1.2360         | 1.2496         | 1.0812         | 1.1722      | 1.0758      |
| outlier  | SE (×100)    | 2.2968         | 2.8271         | 2.8153         | 2.9647         | 2.8805      | 2.9542      |
|          | P-Value      | 0.0140         | 0.0140         | 0.0150         | 0.0190         | 0.0180      | 0.0220      |

using a Poisson regression model with the logarithm of time as the explanatory variable. Indeed, there is no outlier in the data. We will first apply the proposed MDPDE for different $\alpha$ to illustrate its performance in the absence of outliers. The estimates obtained are given in Table 6 and are very close to the MLE (corresponding to $\alpha = 0$) although their standard error increases slightly with increasing $\alpha > 0$.

Next we will create some artificial outliers in this “good” data and check how the estimators are affected by those outliers for several $\alpha \geq 0$. We will consider two cases; — case (i) will have one outlier where we replace the first observation (at time 1) from 1 to 10, and case (ii) will have two outliers where we retain the outlier of case (i) and also replace the last observation (at time 20) from 159 to 15. The MDPDE for
different $\alpha \geq 0$ are shown in Table 6 for both these cases of outliers. It is clear from the table that the MDPDE corresponding to $\alpha > 0.3$ do not differ significantly after the insertion of the outliers; but the MLE (corresponding to $\alpha = 0$) changes drastically.

6.3 Leukemia Data

Our next data example consist of the observation on 33 leukemia patients obtained from Cook and Weisberg (1982, p. 193). Here, we will model the data by the logistic regression with a binary response variable taking value one for at least 52 survival of a leukemia patients and two covariates; — “WBC”, the white blood cell count (in multiple of $10^4$) and “AG”, a binary indicator of the presence of Acute Granuloma in the white blood cells (Hosseinian, 2009). It was identified by Cook and Weisberg (1982) that the 15th observation corresponds to a patients surviving for a long period despite of having large WBC counts of 100000 and so it generates an outlier in the data influencing the MLE.

Here, we estimate the coefficients of the fitted logistic regression model robustly avoiding the effect of the outlying observation in the data. For we apply the proposed MDPDE for several values of $\alpha$ for all the data including the outlier (15th observation) and also to the reduced data without the 15th observation. The estimated parameter values along with their asymptotic standard error are presented in Table 7 and 8 respectively. Comparing the two tables we can see that all the MDPDE corresponding to $\alpha \geq 0.3$ are almost unaffected by the outlier observation and gives similar results to the MLE after excluding outlier.

Table 7: The minimum DPD estimates of the parameters for the Leukemia Data. The standard error of the estimates are given in the parenthesis.

| $\alpha$ | Intercept | AG   | WBC  |
|---------|-----------|------|------|
| 0       | -1.3059 (0.81) | 2.2613 (0.95) | -0.3181 (0.19) |
| 0.1     | -1.2426 (0.82) | 2.2058 (0.97) | -0.3405 (0.2) |
| 0.3     | 0.1017 (1.15) | 2.4381 (1.39) | -2.0017 (1.39) |
| 0.5     | 0.1386 (1.27) | 2.4574 (1.62) | -2.0246 (1.7) |
| 0.7     | 0.1376 (1.38) | 2.4512 (1.87) | -1.9844 (1.97) |
| 1       | 0.1442 (1.58) | 2.459 (2.35) | -1.9635 (2.47) |

Table 8: The minimum DPD estimates of the parameters for the Leukemia Data (without the 15th observation). The standard error of the estimates are given in the parenthesis.

| $\alpha$ | Intercept | AG   | WBC  |
|---------|-----------|------|------|
| 0       | 0.2152 (1.08) | 2.5582 (1.24) | -2.3609 (1.36) |
| 0.1     | 0.1868 (1.10) | 2.5261 (1.28) | -2.253 (0.14) |
| 0.3     | 0.1544 (1.17) | 2.4826 (1.43) | -2.111 (1.48) |
| 0.5     | 0.1407 (1.27) | 2.4592 (1.62) | -2.0286 (1.7) |
| 0.7     | 0.1374 (1.38) | 2.4516 (1.87) | -1.9846 (1.97) |
| 1       | 0.1436 (1.58) | 2.458 (2.35) | -1.9623 (2.46) |
6.4 Skin Data

Now, we will consider the popular skin data set of Finney (1947) on occurrence of “vaso constriction” in the skin of digits after air inspiration. The data set was obtained from a controlled study and analyzed by Pregibon (1982) and by Croux and Haesbroeck (2003). Again we will model this data by means of the logistic regression model where the binary response gives the occurrence of vaso constriction in the skin of digits after a single deep breath and the explanatory variables are the logarithm of the volume of air inspired (“log.Vol”) and the logarithm of the inspiration rate (“log.Rate”).

It can be seen that the 4th and 18th observations influence the MLE to make it difficult to partition the responses. However, after deleting these two observations the overlap between the two outcome of the response variable depends only through on observation giving the facility to partition the outcomes with only one error. So, here also we will apply the proposed MDPDE for different $\alpha$ on the whole data including the influential observations as well as on the outlier-deleted data after dropping observation 4 and 18. The results are presented in Table 9 and 10 respectively. Again, we can see from the tables that the MDPDEs corresponding to tuning parameter $\alpha \geq 0.3$ gives us robust estimators which remains unaffected by the two influential observations and generates the similar estimators as the most efficient MLE for the outlier free data.

Table 9: The minimum DPD estimates of the parameters for the Skin Data. The standard error of the estimates are given in the parenthesis.

| $\alpha$ | Intercept | log(Rate) | log(Vol) |
|----------|-----------|-----------|----------|
| 0        | -2.88 (1.32) | 4.56 (1.84) | 5.18 (1.86) |
| 0.1      | -3.14 (1.48) | 4.83 (2.07) | 5.46 (2.12) |
| 0.3      | -19.05 (12.89) | 24.89 (16.65) | 30.57 (21.68) |
| 0.5      | -21.05 (18.06) | 27.44 (23.42) | 34.13 (30.54) |
| 0.7      | -20.85 (21.35) | 27.2 (27.86) | 33.97 (36.27) |
| 1        | -23.77 (32.98) | 31.08 (43.45) | 39.34 (56.35) |

Table 10: The minimum DPD estimates of the parameters for the Skin Data (without the 4th and 18th observations). The standard error of the estimates are given in the parenthesis.

| $\alpha$ | Intercept | log(Rate) | log(Vol) |
|----------|-----------|-----------|----------|
| 0        | -24.58 (14.02) | 31.94 (17.76) | 39.55 (23.25) |
| 0.1      | -24.13 (14.89) | 31.36 (18.97) | 38.9 (24.81) |
| 0.3      | -22.01 (15.86) | 28.66 (20.42) | 35.57 (26.66) |
| 0.5      | -21.14 (18.16) | 27.55 (23.55) | 34.28 (30.71) |
| 0.7      | -20.86 (21.43) | 27.21 (27.94) | 33.97 (36.37) |
| 1        | -23.77 (32.98) | 31.08 (43.45) | 39.34 (56.35) |

6.5 Damaged Carrots Data

As an interesting data example leading to the logistic regression, we will consider the damaged carrots dataset of Phelps (1982). The data set was obtained from a soil experiment trial containing the proportion of insect damaged carrots with three blocks and eight dose levels of insecticide in the experiments and discussed by
Williams (1987). McCullagh and Nelder (1989) this data to illustrate the identification methods for isolated departures from the model through an outlier in the y-space present in the data (14th observation; dose level 6 and block 2). Later Cantoni and Ronchetti (2001) modeled this data by a binomial logistic model to illustrate the performance of their proposed robust estimators. However, it can be checked easily that the observation 14 is only the outlier in y-space and not a leverage point.

We will now apply the Minimum density power divergence estimation for several different $\alpha$ to see the performance of the proposed method in case of presence of outlier only in y-space. Table 11 presents the parameter estimates, their asymptotic standard error and corresponding p-value for different tuning parameter $\alpha$. The estimators corresponding to $\alpha \geq 0.3$ again turns out to be highly robust and also similar to the robust estimator obtained by Cantoni and Ronchetti (2001). Also, for these estimators the indicator of Block 1 turns out to be insignificant which became significant in case of the maximum likelihood estimator (corresponding to $\alpha = 0$) due to the outlying observation.

Table 11: The minimum density power divergence estimates, their standard errors and p-values for the Damage Carrots Data

|            | $\alpha = 0$ | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 1$ |
|------------|--------------|----------------|----------------|----------------|----------------|--------------|
| Intercept  | 1.4805       | 1.4880         | 1.4974         | 1.5157         | 1.5310         | 1.5569       |
| SE         | 0.6562       | 0.6648         | 0.6859         | 0.7118         | 0.7406         | 0.7868       |
| P-Value    |              |                |                |                |                |              |
| logdose    | -1.8175      | -1.8163        | -1.8102        | -1.8102        | -1.8102        | -1.8152      |
| SE         | 0.3439       | 0.3484         | 0.3601         | 0.3749         | 0.3917         | 0.4183       |
| P-Value    | 0.0000       | 0.0000         | 0.0000         | 0.0001         | 0.0001         | 0.0002       |
| Block1     | 0.5421       | 0.5330         | 0.5149         | 0.4969         | 0.4824         | 0.4654       |
| SE         | 0.2318       | 0.2338         | 0.2392         | 0.2462         | 0.2542         | 0.2668       |
| P-Value    | 0.0284       | 0.0322         | 0.0421         | 0.0554         | 0.0704         | 0.0945       |
| Block2     | 0.8430       | 0.8284         | 0.7973         | 0.7710         | 0.7483         | 0.7240       |
| SE         | 0.2260       | 0.2283         | 0.2344         | 0.2422         | 0.2510         | 0.2649       |
| P-Value    | 0.0011       | 0.0014         | 0.0025         | 0.0041         | 0.0067         | 0.0119       |

7 Conclusion

In this paper, we have proposed a new methodology for robust estimation in case of generalised linear models and considered two prominent special cases – Poisson regression and logistic regression. These models are very useful for analyzing count response and binary response respectively. We have established the robustness properties of the proposed method in terms of the influence function analysis and also applied it to several real data sets having different types of outliers. In these we have seen that the proposed estimators with moderate $\alpha > 0$ are highly robust in handling different kinds of outliers and generate robust estimates competitive to the existing ones in the literature. In fact the estimated standard errors of our parameter estimates are smaller than those of the existing robust estimates in most cases for the real data examples studied by us for all moderately small $\alpha \leq 0.5$. In some cases the same holds for $\alpha$ as high as 0.7. Thus we hope that the estimators discussed in this paper will help the researchers in several fields such as medical biology, epidemiology and controlled trial experiments in biological sciences to estimate the model parameters in generalised linear model (including Poisson and logistic regression) efficiently and robustly even in contaminated scenarios.
Table 12: The Optimum values of the tunning parameter $\alpha$ for different data sets obtained using several “Pilot” values of $\alpha$

| Data Sets       | Pilot $\alpha$ |
|-----------------|----------------|
|                 | 0   | 0.1 | 0.3 | 0.5 | 0.7 | 1   |
| AIDS Data       | 0   | 0.05| 0.05| 0.1 | 0.1 | 1   |
| (with one outlier) | 0   | 0.2 | 0.65| 0.35| 0.55| 0.45|
| AIDS Data       | 0   | 0.3 | 0.3 | 0.55| 0.5 | 0.55|
| (with two outliers) | 0   | 0.3 | 0.3 | 0.55| 0.5 | 0.55|
| Epilepsy Data   | 0   | 0.05| 0.35| 0.3 | 1   | 0.95|
| Leukemia Data   | 0   | 0.1 | 0.3 | 0.3 | 0.3 | 0.3 |
| (without outlier) | 0   | 0   | 0.1 | 0.1 | 0.1 | 0.1 |
| Skin Data       | 0   | 0.1 | 0.3 | 0.35| 0.35| 0.4 |
| (without outlier) | 0   | 0   | 0.25| 0.3 | 0.35| 0.05|
| Carrots Data    | 0   | 0.05| 0.3 | 0.55| 0.7 | 0.95|

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