An Observer’s View on Relativity
Space-Time Splitting and Newtonian Limit

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Maik Reddiger

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Gutachter: Prof. Dr. Yuri B. Suris
Prof. Dr. Horst-Heino von Borzeszkowski

Betreuer: Dr. Wolfgang Hasse
dedicated to
free thought, equality and mutual respect,
the very foundations of a free society
Abstract

We motivate and construct a mathematical theory for the separation of space and time in general relativity. The formalism only requires a single observer and an optional choice of reference frame at each instant. As the splitting is done via the observer’s past light cone, it is both closer to the experimental situation and mathematically less restrictive than the splitting via observer vector fields or spacelike hypersurfaces. Indeed, the theory can in principle be applied to all spacetimes and adapted to other ‘metric’ theories of gravity. Instructive examples are developed along with the general theory. In particular, we obtain an alternative description for accelerated frames of reference in Minkowski spacetime.

Further, we use the splitting formalism to motivate a new mathematical approach to the Newtonian limit of the motion of mass points. This employs a general formula for their observed motion, distinguishing between ‘actual’ forces (i.e. those detectable via an accelerometer) and pseudo-forces. Via this formula we show that for inertial frames of reference in Minkowski spacetime the essential laws of non-gravitational Newtonian mechanics can be derived.

Physically relevant, related, open problems are indicated throughout the text. These include the proof, that the Newtonian limit gives rise to the central pseudo-forces known from Newtonian mechanics (‘constant gravity’, Euler, Coriolis and centrifugal force) for non-inertial frames of reference in Minkowski spacetime, as well as the derivation of Newton’s law of gravitation in the Schwarzschild spacetime under said limit.

Keywords: space-time splitting - Newtonian limit - relativistic kinematics - frame of reference - gravitational lensing
Zusammenfassung

Wir motivieren und entwickeln eine mathematische Theorie zur Aufteilung von Raum und Zeit in der Allgemeinen Relativitätstheorie. Der Formalismus benötigt lediglich einen einzelnen Beobachter und eine optionale Wahl eines Bezugssystems zu jedem Zeitpunkt. Da die Trennung über den Vergangenheitssichtkegel des Beobachters erfolgt, ist sie sowohl näher an der experimentellen Situation als auch mathematisch weniger restriktiv als die Teilung mittels Beobachtervektorfeldern oder raumartigen Hyperflächen. Tatsächlich ist die Theorie im Prinzip auf alle Raumzeiten anwendbar und kann an andere 'metrische' Gravitationstheorien angepasst werden. Instruktive Beispiele werden zusammen mit der allgemeinen Theorie entwickelt. Insbesondere erhalten wir eine alternative Beschreibung beschleunigter Bezugssysteme in der Minkowski-Raumzeit.

Weiter benutzen wir den Trennungsformalismus, um einen neuen mathematischen Zugang zum Newtonschen Grenzfall der Bewegung von Massepunkten zu begründen. Dies wird über den Gebrauch einer allgemeinen Formel zu ihrer beobachteten Bewegung erreicht, welche 'echte' Kräfte (jene, die sich mit einem Beschleunigungsmesser nachweisen lassen) und Scheinkräfte voneinander unterscheidet. Mit Hilfe dieser Formel zeigen wir, dass sich für inertiale Bezugssysteme in der Minkowski-Raumzeit die wesentlichen Gesetze der gravitationsfreien Newtonschen Mechanik herleiten lassen.

Physikalisch bedeutsame, verwandte, offene Probleme werden im Text angeschnitten. Beispiele dafür sind zum einen der Beweis, dass der Newtonsche Grenzfall tatsächlich zu den aus der Newtonschen Mechanik bekannten zentralen Scheinkräften ('konstante Gravitation', Euler-, Coriolis- und Zentrifugalkraft) für nicht-inertiale Bezugssysteme in der Minkowski-Raumzeit führt, und zum anderen die Herleitung von Newtons Gravitationsgesetz in der Schwarzschild-Raumzeit unter diesem Grenzfall.

Schlüsselwörter: Raum-Zeit-Trennung - Newtonscher Grenzfall - Relativistische Kinematik - Bezugssystem - Gravitationslinsen
# Contents

1 Introduction .................................................. 4

2 Mathematical Preliminaries .................................. 10
   2.1 Pullback Bundles ........................................ 10
   2.2 (First Order) $G$-structures ............................ 11
      2.2.1 Mathematical Definitions ............................ 11
      2.2.2 Lorentzian Structures .............................. 15
      2.2.3 Lorentzian Orientations ............................ 19
   2.3 Connections on the Tangent Bundle ..................... 30
      2.3.1 Ehresmann Connections .............................. 31
      2.3.2 Covariant Derivatives & Connectors ................ 31
   2.4 Jacobi Fields and the Lorentzian Exponential ........... 35

3 The Splitting Construction ................................ 41
   3.1 General Considerations ................................ 41
   3.2 Heuristic Motivation of the Space-Time Splitting ...... 50
   3.3 Static Splitting .......................................... 60
   3.4 Kinematic Splitting ..................................... 68
      3.4.1 Kinematic Observer Mapping ......................... 68
      3.4.2 Moving Frames of Reference ........................ 75
      3.4.3 Observer Spacetime and relative Motion ............ 80

4 The Newtonian Limit ........................................ 86
   4.1 General Newtonian Limit ................................ 87
   4.2 Newtonian Limit in Special Relativity .................. 93

References ...................................................... 98

Acknowledgements ............................................. 104
1 Introduction

With the 1687 publication of his ‘Philosophiae Naturalis Principia Mathematica’, Newton not only gave birth to modern mathematical physics, but also carved in stone a scientific view of the world: Space is Euclidean and absolute, time is eternal in both directions and everywhere the same. They are separate, metaphysical entities, providing the stage for all physical occurrences.

Up to the advent of non-Euclidean geometry in the first half of the nineteenth century with the works by Lobachevski, Bolyai, Gauß, Riemann and others, the inherent truth of the Newtonian paradigm had been out of the question. Once, it became apparent, however, that Euclidean geometry was not the only one, it could no longer be asserted a priori that it represented the true geometry of physical space. Among the first to realize this was F. K. Schweikart, who, contrary to what one might expect, was neither a mathematician, nor a physicist, but a professor of law at the university of Marburg [17, p. 147]. Yet it did not take long for some of the founding fathers of non-Euclidean geometry to follow suit. In the 1820s Gauß famously decided to settle the issue by measuring the inner angles of a triangle formed by the mountains Brocken, Hoher Hagen and Inselberg in Germany, but, after taking their sum, could not detect any clear deviation from the anticipated 180 degrees [17, p. 147; 87]. Had his measurements been more precise, he might have dealt a serious blow to the view of the world at the time.

The Newtonian paradigm remained largely unchallenged for almost another 90 years, until the young Einstein published his article ‘On the Electrodynamics of Moving Bodies’ [11] in 1905. With his foundational work in what is today known as the special theory of relativity, he took the courage to scientifically discard the notion of universal simultaneity, as well as to turn length into a relative concept. The mathematical axiomatization of his ‘theory of relativity’ was carried out by his teacher and mathematician Hermann Minkowski [11]. Minkowski realized that the new physics called for a unification of space and time into a single concept, named spacetime, and therefore completed the new view of the world.

It took only 10 years until Einstein again publicly defied the prevailing conception of space and time. His struggle to include gravitation into the picture forced him to impugn the Euclidean nature of space, thereby reviving a centuries-old discussion. The successful fusion of non-Euclidean geometry with Minkowski’s spacetime concept in his ‘general theory of relativity’ ultimately lead to a second revolution in our collective understanding of space and time.

Still, the more general spacetime concept brought forth new questions. The proximity of the special theory of relativity to Newtonian mechanics did not lead to any serious issues regarding the relation between spacetime and the subjectively more familiar notions of space and time. As soon as Einstein modeled gravity as a consequence of the curvature of spacetime, however, he was unable to clearly identify space and time by themselves in the theory. While it was apparent to him that he had buried the Euclidean conception of
space [12, p. 69sq.] and that time was only a meaningful concept for individual clocks, he
needed to employ a combination of heuristic reasoning and approximations to separate
the two again [12, p. 96sq.]. Even though his reasoning proved itself to be sufficient for
the establishment of his new theory of gravity, it left a vacancy to be filled by future
generations of physicists and mathematicians. The main question awaiting to be answered
was: “Does there exist a physically well-motivated, mathematically rigorous separation of
space and time in general relativity?”

If one surveys the contemporary literature regarding those so called space-time splittings,
one finds that mathematically well-defined constructions do indeed exist. A list of references
[43] was compiled by Bini and Jantzen, but it is not exhaustive, of course. Excluding
the rather ad hoc coordinate-based methods, the two most common splitting formalisms
involve a choice of global timelike vector field on the spacetime, taken to be a Lorentzian
manifold for now. In the so called ‘threading approach’, originally due to Landau and
Lifshitz [21, §10-4] (see also [78] and [52]), this choice is explicit, while in the ‘slicing
approach’, being part of the Arnowitt-Deser-Misner formalism [23, p. 419sqq. & §21.7]
(short: ADM formalism), the choice is implicit. Philosophically, the approaches are rather
similar and can be put in mathematical agreement, provided certain topological conditions
on the spacetime, as well as integrability conditions of the vector field, are met. We refer
to the articles [78, 79] by Jantzen, Carini and Bini for an introduction to the different
approaches, including a historical review. For a more geometric approach, the reader may
also find the article [62] by van Elst and Uggla, as well as the book by O’Neill [24, p.
358sqq.] beneficial.

While from a mathematical perspective the aforementioned approaches are in principle
unproblematic, we believe the issue is not yet settled. Physically, space-time splittings
ought to directly relate to our individual experience of space and time. Therefore, they
should be carried out for individual curves representing physical motion, not via a timelike
vector field representing infinitely many such curves. So the choice of a timelike vector
field is ultimately arbitrary and estranged from the experimental situation. In addition,
focusing on what is observed leads us to reject the general philosophy that spacelike
submanifolds ought to be identified as physical space. We believe that this philosophically
flawed approach both in the ‘threading’ and the ‘slicing’ ansatz is the origin of the rather
restrictive conditions required for a full splitting to be carried out. For instance, in the
plane wave spacetimes one of the major conditions is not met (see e.g. the article by
Perlick [96, §5.11] and the original one due to Penrose [98]). So does it not make sense to
speak of space and time individually here?

Based on an reexamination of the underlying philosophy and its relation to what is
actually observed, an alternative approach is proposed here. Our original motivation for
it came from reading the diploma and PhD theses [14, 15] by Wolfgang Hasse, as well
as from the attendance of lectures in the philosophy of space and time, given by Dennis
Dieks at the University of Utrecht. Later we discovered that the relativistic separation
of space and time via observer mappings, as they are named here, is not unheard of in
the literature. However, we are not aware of any source, where the observer mapping has
explicitly been named as a tool for doing so. As far as we know, the first instance, where
the observer mapping implicitly appears, is a 1938 article [100, p. 128] due to G. Temple,
who was one of Eddington’s students [82, p. 390]. It may also be found in a 1959 article
[89] by Mast and Strathe, which has been a valuable reference to us. Further analyses
have been carried out, e.g. by Kristian and Sachs [84] and Ellis et al. [58]. In addition, the
observer mapping is closely related to the physical phenomenon of gravitational lensing. So in this context, the works by Perlick [94,96], Ehlers [54], as well as Ellis, Basset and Dunsby [59] should be mentioned.

In this work we attempt to answer two main questions:

1) How does our individual perception of the separateness of space and time relate to the spacetime concept on a physical and mathematical level?

2) In what sense is relativity theory a generalization of Newtonian mechanics?

The second question needs to be raised, because the concepts of space and time, as defined by the splitting, need to reduce to the Newtonian ones in an approximation. In the literature, this approximation is referred to as the Newtonian limit. Indeed, the mathematical construction (and the theory as a whole) is only physically tenable, if it can be shown that the Newtonian limit exists under assumptions that are compatible with the domain of validity of the Newtonian theory. We refer to page 86 for a more detailed discussion.

Accordingly, the structure of this thesis follows the main questions: First we give a review of the required mathematical machinery and then employ it for the construction of the splitting formalism in the subsequent chapter. The final chapter discusses the Newtonian limit in this context and shows that it indeed exists for the special theory of relativity. We invite the reader to skip the technical chapter 2 on first reading and refer to it when necessary.

Contrary to what one might expect initially, the existence of the Newtonian limit in the special theory of relativity already constitutes a non-trivial test of the splitting construction. There are two other cases, where the Newtonian limit needs to be shown to exist, but their treatment here would blow the size of this thesis out of proportion. Nevertheless, they are certainly the most crucial tests of the construction and are thus to be considered important open problems. They are elaborated upon in section 4.1.

If the theory withstands these attempts of falsification, then it may be applied, for instance, to elaborate on philosophical issues of relativity theory, the subject of ‘gravitoelectromagnetism’ and to attack the question whether general relativity is really unable to account for the internal motion of spiral galaxies (being part of the so called ‘dark matter problem’ [101]). Of course, this would also necessitate the development or application of a variety of approximation formalisms, as well as a formulation of the theory, which is more suitable for direct application by physicists.

We close this introduction with a few important remarks.

**Type of work:** This is a thesis in mathematical physics. However, the term mathematical physics is not as well-defined as one might expect - indeed, there appear to be two polar views of the field: One may be named ‘physical mathematics’, where one aims to solve purely mathematical problems, that are either directly or indirectly related to physics. The second approach aims to contribute to the clarification - or even correction - of physical theories and the related solution of physical problems by means of rigorous mathematics. As such, it necessarily requires a certain philosophical understanding of the physical situation at hand and is a supplement as well as a direct competitor [64] to the field of theoretical physics. In our mind, both approaches to mathematical physics are interdependent, often hard to separate and fertilize each other. In fact, mathematics and
physics share a common origin in natural philosophy, so it should not come as a surprise that the fields have become so intertwined as to give rise to an own discipline.

The reader might have guessed already that we take the second approach in this work. Therefore, the aim is, at least within the bounds of the theory, to make statements on physical reality, not to elaborate on the underlying mathematical machinery - unless required to achieve this goal. Though we follow the mathematical tradition in giving rigorous definitions and proving theorems, the main emphasis is placed in answering the two questions stated above. So not the most general version of a theorem is stated and proved, but only the one of interest to the model. Moreover, we have tried hard to keep the mathematics separate from the physics, but our empirical data strongly suggests that this is an impossible task for a mathematical physicist of the second type. We have found a middle way in not using undefined terminology in theorems and definitions and, whenever we do so elsewhere in an attempt to reason heuristically, we have usually indicated this with words like ‘heuristically’, ‘roughly’, ‘intuitively’, etc. In cases where we forgot to do this, the context should tell the reader whether the reasoning is mathematical or philosophical in its nature. Consequently, in between theorems and definitions as well as within remarks and examples, the reader may find undefined terminology and heuristic reasoning. While this is almost a crime for mathematicians, it is absolutely necessary when discussing the physical and philosophical aspects of the matter. Again, we discuss phenomena in the physical world and not just purely mathematical structures. These phenomena are of interest precisely because we physicists find them on the interface between the known and the unknown. Therefore, in this research it is a necessary state of affairs that concepts are only rigorously defined once they have passed well into the realms of the known and other new concepts, that have not yet passed this boundary, may show themselves to be ill motivated or even nonsensical after further progress has been made. Nonetheless, founding on top of the known can lead deep into the realms of the unknown. Precisely this is attempted in this work and as such, it is a work in natural philosophy, as much as it is a work in mathematics and physics.

**Terminology:** The choice of terminology in the field of mathematical and theoretical physics is a problem of its own: As opposed to pure mathematics, where the terminology is often chosen on categorical grounds, there is often a philosophical concept attached to the words, extending beyond the mathematical definition. This inevitably leads to a conflict: On one hand, one would like to capture the physical idea adequately and on the other hand one would like to express this idea as precisely as possible in mathematical terms.

This in turn leads to the problem, that varying the mathematical ansatz for tackling the physical issue leaves many fundamental physical ideas invariant, while requiring a change of their mathematical definitions. This justifies some choice of terminology here, which differs partially from the one used in the physical literature. Whenever such a deviation occurs, we guarantee that it is not without warrant and it is done, because we believe that it captures the underlying physical idea better. A particular example is the word ‘frame of reference’ to be discussed later.

In addition, we chose to keep the mathematical terminology, where the introduction of terminology due to physicists is redundant. For example, in the general relativity literature one often reads the word ‘tetrad’, which is simply a choice of basis in the tangent space of
1 Introduction

interest. This is a purely mathematical object and mathematicians have already devised the word ‘frame’ for it, hence there is no need for the word ‘tetrad’. We believe that the introduction of redundant mathematical terminology in physics contributes to the mutual separation of the fields, which, in our opinion, is harmful to both and should thus be avoided. Of course, if the word carries an additional physical meaning, the matter is different.

Choice of examples: The reader will observe, that the examples we chose in the context of observer mappings are mostly given in flat spacetimes. The reason is not that the construction does not work in the general setting, but due to the fact that examples in curved spacetimes are computationally very challenging and we wished to put the emphasis on the abstract, general theory, rather than on computations.

Prerequisites: As for all works in advanced mathematics, the reader needs a fair amount of background knowledge in order to be able to fully comprehend this thesis. Accordingly, she or he should be familiar with the major concepts of differential geometry (e.g. manifolds, tensor fields, pseudo-Riemannian metrics, Lie groups, covariant derivatives, fiber bundles, ...), as well as the basic structure of relativity theory (e.g. its physical motivation, what a spacetime roughly is, equations of free fall motion, the Einstein equation). For the former the books by Lee [22], and Rudolph and Schmidt [31, Chap. 1 to 4] provide a good, rigorous introduction. More advanced differential geometric topics are reviewed in chapter 2, so appropriate references are given there. For an introduction to relativity theory, we recommend the books by Carroll [10, Chap. 1 to 5] and Wald [39, Part I]. Moreover, the book by O’Neill [24] should be mentioned as an excellent work in mathematical relativity. Knowledge of special relativity is explicitly not required. On the contrary, it is conceptually simpler to view special relativity through the lens of general relativity as, from our experience, the theory can be very deceiving otherwise.

Conventions: We write definitions in italic in the hope that it will help the reader with a distaste in successive reading to easily spot terminology. Also, we sometimes write words in round brackets to prevent misunderstanding or to emphasize that the words can be omitted. For instance, as we are working in the category of smooth manifolds, there is no need to explicitly state that every mapping should be smooth, but sometimes we nonetheless write ‘(smooth) mapping’ instead of just ‘mapping’. The set of natural numbers $\mathbb{N}$ starts with 1, not 0. $c$ is always the speed of light in vacuum, unless it appears as an index. Concerning fiber bundles, if it is not clear that it has a global section, then sections are assumed to be local in general. Usually, however, there should be no ambiguity as to what is meant: We explicitly write $s \in \Gamma^\infty (\mathcal{U}, \mathcal{E})$ to say that $s$ is a (smooth, local) section of the fiber bundle $\mathcal{E}$ over some $\mathcal{M}$ with domain $\mathcal{U} \subseteq \mathcal{M}$. For trivial bundles we like to drop the distinction between sections and maps into the fiber, though there is technically a difference. That is, if $\mathcal{E} = \mathcal{M} \times \mathcal{F}$ is trivial, then we sloppily write $s \in C^\infty (\mathcal{U}, \mathcal{F})$ instead of $s \in \Gamma^\infty (\mathcal{U}, \mathcal{M} \times \mathcal{F})$. Moreover, it should be said that if we use the word ‘natural’, it is meant in the physicist’s vague sense of the word, so a priori there is no inherent mathematical meaning to it.

Notation: A joke among mathematicians says ‘differential geometry is what stays invariant under change of notation’ and there is definitely some truth to it. Our notation is a mixture of personal taste and the one used in the book by Rudolph and Schmidt [31]. In
fact, most of the notation is explained in the text or can be inferred from the context. Nevertheless, we shall make some basic remarks.

We use curly letters like $M, N, Q, U$ for manifolds, except for the classical Lie groups, which get their common declaration $GL, O, SO$ and so on. For a (smooth) map $\varphi$ between manifolds, $\varphi_*$ denotes its differential/pushforward and $\varphi^*$ the respective pullback. If $V$ is a subset of the domain $\text{dom} \varphi$, then $\varphi|_V$ is the restricted map. The upright letter ‘$d$’ is reserved for the Cartan derivative and ordinary derivatives like $d/dt$. $\text{pr}_a$ always denotes projection onto the $a$th factor of some product of sets. The letter $1$ is always some kind of identity. $\mathbb{R}_+$ is the open interval $(0, \infty)$, $\mathbb{R}^n$ the $n$-fold product of the reals $\mathbb{R}$ with $i$th standard basis vector $e_i$, and $(\mathbb{R}^n)^*$ denotes the dual (vector) space with $j$th cobasis vector $e^j$. Bars under letters always indicate an inverse, e.g. the (algebraic) inverse of the matrix $A$ is $\bar{A}$. Where appropriate, we use Einstein summation convention with $i,j,k,l,\ldots$ going from 0 to the end and $a,b,c,\ldots$ starting at 1 instead. When we write matrices, they are to be understood as homomorphisms of vector spaces given in a particular basis. So if $A: V \rightarrow W$ is a linear map with $\dim V = m$ and $\dim W = n$, then

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix}$$

gives the components $A_{ab}$ of $A$ with respect to named bases in $V$ and $W$. The components of the identity on a vector space with respect to a given basis are $\delta_{ab}$.

The dot $\cdot$ denotes contraction with the next sensible adjacent entry from the left or right. For instance, when $\theta$ is a 1-form and $X$ is a vector field over the same manifold, then the contraction is

$$\theta (X) = \theta \cdot X = X \cdot \theta .$$

The left hand views $\theta$ as a covector field, the middle is just $\theta_i X^i$ in some coordinates and the right hand side can be read as $X^i \theta_i$. The same formula holds, for example, if we let $\theta$ be an $\mathcal{E}$-valued 1-form, where $\mathcal{E}$ is (real) vector bundle over $Q$ being neither the cotangent bundle $T^*Q$ nor any tensor bundle ‘built’ from $T^*Q$. Otherwise, the first expression would not be defined; the second and third would differ in general. The notation has the advantage that it correctly views contraction as a generalization of ordinary matrix multiplication. Also note that $\varphi_* \cdot X = \varphi_* X$, so we sometimes view $\varphi_*$ point-wise as a matrix.

**Contact:** Queries regarding this work (e.g. comments, errors, remarks or questions) are received with gratitude and should be submitted to the author’s email address: maik.reddiger@zoho.com
2 Mathematical Preliminaries

In this chapter we review some mathematical results needed for the rigorous formulation of the theory of relativity, including the construction and analysis of the space-time splitting. With the exception of section 2.2.3, the discussion here only serves as a reminder and is not intended to be an in-depth review. For the latter, including detailed proofs, the reader is advised to consult the references provided in the respective sections. Nonetheless, we have attempted to create a coherent overview of those mathematical results and do prove some propositions, where we have not found a treatment in the literature suitable for our purposes or consider it pedagogically worthwhile.

In the first section 2.1 we quickly recall how to ‘pull back’ smooth fiber bundles, a concept that appears both implicitly and explicitly throughout this thesis. Afterwards we give a general treatment of (first order) $G$-structures. Two examples of particular interest to relativity theory are discussed in the subsequent two subsections, namely those of Lorentzian metrics as well as Lorentzian orientations. Their combination will give rise to the mathematical definition of spacetime in the next chapter. Section 2.3 introduces the notion of connector and tangent bundle connection, first from the more general point of view of Ehresmann connections and then becoming more specific in subsection 2.3.2. We close the chapter with a treatment of Jacobi fields and their relation to the Lorentzian exponential map in section 2.4.

2.1 Pullback Bundles

Due to the omnipresence of pullback bundles in this thesis, we shall give a brief review.

Let $\pi: E \to N$ be a (smooth) fiber bundle and $\xi: M \to N$ be a (smooth) mapping between manifolds. Then the set

$$\xi^* E := \{(x, T) \in M \times E | \xi(x) = \pi(T)\}, \quad (2.1.1)$$

has a unique manifold structure (cf. [22, p. 13] for a definition), such that it is a (smooth) fiber bundle over $N$ with the same typical fiber as $E$ and $\pi := \text{pr}_1|_{\xi^* E}: \xi^* E \to M$ is its smooth projection (cf. [2, Satz 2.2; Satz 2.1]). We call $\xi^* E$ the pullback bundle of $E$ by $\xi$. Indeed, it is equipped with the subspace topology and is thus an embedded submanifold of the product manifold $M \times E$ (for a proof adapt [31, Prop. 2.6.1] to the more general case of fiber bundles). We now follow the book by Sachs and Wu [33, §2.0.1], in defining $T$ (in $E$) over $\xi$ to be an element of $E$ with $\pi(T) \in \xi(M)$, i.e. $T$ is just the projection onto the second factor of an element $(x, T) \in \xi^* E$. In this spirit, if $U \subseteq M$ is open and $(., T): U \to \xi^* E$ is a (smooth, local) section of $\xi^* E$, then the mapping $T: U \to E$ is called a (smooth, local) section (of $E$) over $\xi$. In the special case of $\xi$ being a curve, we also call $T$ a (smooth, local) section (of $E$) along $\xi$. Note that we always assume domains of curves to be open and connected subsets of $\mathbb{R}$, i.e. open intervals.
2.2 (First Order) \(\mathcal{G}\)-structures

The use of these concepts is necessary as the map \(\xi\) need not be injective, so intuitively we wish to vary \(T\) on \(\mathcal{N}\), rather than on \(\mathcal{M}\), in order to allow for several \(T\)s at the same point in the image \(\xi(\mathcal{M})\). For instance, a case of particular interest to us is the one where \(\mathcal{E} = T\mathcal{N}\), \(\mathcal{M} \subseteq \mathbb{R}\) is an (open) interval and \(\xi: \mathcal{M} \rightarrow \mathcal{N}\) is a smooth curve. Intuitively, a mapping \(T: \mathcal{M} \rightarrow T\mathcal{N}\) attaches for every parameter \(s \in \mathcal{M}\) a vector \(T_s \in T_{\xi(s)}\mathcal{N}\) to the curve \(\xi\) in \(\mathcal{N}\). Should the curve intersect itself once at \(q \in \mathcal{N}\), then there will be two usually different vectors at \(q\). As vector fields are particular instances of sections of fiber bundles, we then call \(T\) a vector field over \(\xi\) or a vector field along \(\xi\). Analogous terminology is used for differential forms, tensor fields, frame fields and so on.

Finally, we remark that fiber bundles over open subsets of \(\mathbb{R}^n\) with \(n \in \mathbb{N}\) are always trivial and thus admit global sections.

2.2 (First Order) \(\mathcal{G}\)-structures

The notion of \(\mathcal{G}\)-structures allows for an (almost) unified view on geometric structures in differential geometry and hence they are also of interest to the (mathematical) relativist. The fundamental idea is that many geometric structures on a manifold are constructed from objects in multi-linear algebra (e.g. tensors), which can be brought to a ‘standard form’ by an appropriate choice of basis, unique up to the linear action of a ‘symmetry group’ \(\mathcal{G}\) onto that basis. An analogue is then constructed locally on a manifold by choosing ’appropriate’ local frame fields and the structure is ’globalized’ by a partition of unity argument. However, for non-parallelizable manifolds there may be topological obstructions to the existence of the geometric structure as it is not always possible to choose the frame fields appropriately. In subsection 2.2.1 we will give a general definition of (first order) \(\mathcal{G}\)-structures, as well as dual \(\mathcal{G}\)-structures, and make some general statements. In subsections 2.2.2 and 2.2.3 we consider two particular examples illustrating the relation between the mathematical definition of \(\mathcal{G}\)-structure and ‘actual’ geometric structures on a manifold.

Standard references for \(\mathcal{G}\)-structures are for instance [37, Chap. VII] and [18]. We will not recall the mathematical notion and properties of principal bundles here as this is not the topic of this thesis and it is a standard topic in the differential geometry literature. For the German-speaking reader the book by Baum [2] provides a good reference, for the English-speaking one we recommend the books by Poor [26] and Rudolph and Schmidt [32]. Principal bundles are also necessary for the mathematical formulation of so-called gauge theories in particle physics [2,32,45].

2.2.1 Mathematical Definitions

Recall that the frame bundle \(\text{Fr} (\mathcal{E})\) of a (smooth, real) vector bundle \(\mathcal{E}\) over a (smooth) manifold \(\mathcal{Q}\) with typical fiber \(\mathcal{V}\) (i.e. a real vector space) is a proper subset of \(\mathcal{E} \otimes \mathcal{V}^*\) and canonically a principal \(\text{GL} (\mathcal{V})\)-bundle\(^1\) over \(\mathcal{Q}\) (cf. [26, §1.45e]), where

\[
\text{GL} (\mathcal{V}) := \{ A \in \text{End} (\mathcal{V}) = \mathcal{V} \otimes \mathcal{V}^* \mid \exists \bar{A} \equiv A^{-1} : A \cdot A = A \cdot A = 1_{\mathcal{V}} \}.
\]

\(^1\)For \(\dim \mathcal{V} = n\) one often reads that \(\text{Fr} (\mathcal{E})\) ought to be a \(\text{GL} (\mathbb{R}^n)\)-bundle over \(\mathcal{Q}\). While this is certainly a correct point of view (as all non-trivial finite dimensional, real vector spaces are isomorphic to \(\mathbb{R}^n\) for some \(n \in \mathbb{N}\)), considering \(\text{Fr} (\mathcal{E})\) as a \(\text{GL} (\mathcal{V})\)-bundle is more ‘natural’ - at least for most tensor bundles \(\mathcal{E}\). As an example, consider the endomorphism bundle \(T\mathcal{Q} \otimes T^* \mathcal{Q}\).
In the context of $G$-structures, we are interested in the (tangent) frame bundle $\text{Fr}(TQ)$ - with $\mathbb{R}^n$ being its typical fiber - and the co(tangent)frame bundle $\text{Fr}(T^*Q)$ - with typical fiber $(\mathbb{R}^n)^*$ - of an $n$-manifold $Q$. These two principal bundles are dual in the sense that for every frame $X \in \text{Fr}(TQ)$ there exists a unique coframe $\bar{X} \in \text{Fr}(T^*Q)$ with the property that

$$X \cdot \bar{X} = \mathbb{I}_{TQ}, \quad \bar{X} \cdot X = \mathbb{I}_{\mathbb{R}^n}. \quad (2.2.1)$$

So the notation $\bar{X} = X^{-1}$ is admissible, if taken in the algebraic sense. Moreover, both $\text{Fr}(TQ)$ and $\text{Fr}(T^*Q)$ can be viewed as principal $\text{GL}_n$-bundles, where $\text{GL}_n := \text{GL}(\mathbb{R}^n)$.

**Remark 2.2.1**

Viewing $\text{Fr}(T^*Q)$ as a principal $\text{GL}(\mathbb{R}^n)$-bundle rather than a principal $\text{GL}((\mathbb{R}^n)^*)$-bundle is more convenient from the point of view of $G$-structures. The reason for this will become apparent later when we introduce dual $G$-structures. Mathematically, this corresponds to replacing the right Lie group action

$$\text{Fr}(T^*Q) \times \text{GL}((\mathbb{R}^n)^*) : (\bar{X}, A) \to \bar{X} \cdot A$$

with

$$\text{Fr}(T^*Q) \times \text{GL}(\mathbb{R}^n) : (\bar{X}, A) \to (X \cdot A) = A \cdot X = X \cdot A^T,$$

which is also a right Lie group action. ♦

Assume now we are given a Lie subgroup $(G, \rho)$ of $\text{GL}_n$. Hence

$$\rho : G \to \text{GL}_n$$

is a (smooth) faithful representation. In general, $G$ does not need to be a subset of $\text{GL}_n$, nor do we require $\rho$ to be open onto its image, i.e. Lie subgroups do not need to be embedded. Then mathematically, a (first order) $G$-structure $P$ on $Q$ is a $G$-reduction of the frame bundle of $Q$, where $P \subseteq \text{Fr}(TQ)$ and the (right) action of $G$ on $P$ is induced by the canonical action of $G$ on $\text{Fr}(TQ)$ via the representation $\rho$. More explicitly, we have a (smooth) principal $G$-bundle $(P, \pi, Q, G)$ with right action

$$\alpha : P \times G \to P : (X, A) \to \alpha(X, A) = X \cdot \rho(A)$$

and the reduction mapping is the inclusion $\iota : P \hookrightarrow \text{Fr}(TQ)$.

**Remark 2.2.2**

i) The action of $G$ on the fibers of $P$ is simply transitive, i.e. for all $q \in Q$ and $X,Y \in \pi^{-1}([q])$ there exists a unique $A \in G$ such that $Y = X \cdot \rho(A)$.

ii) Point i) implies that if $X, X'$ are (smooth, local) sections of $P$ over $\mathcal{U}, \mathcal{U}'$, respectively, with $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, then there exists a unique (smooth) function

$$A : \mathcal{U} \cap \mathcal{U}' \to G$$

such that $X' = X \cdot \rho(A)$ over $\mathcal{U} \cap \mathcal{U}'$. 

12
iii) Since $G$ is only a Lie subgroup of $GL_n$, the pair $(P, \iota)$ is a submanifold of $Fr(TQ)$, but it is not necessarily embedded. However, if $(G, \rho)$ is an embedded Lie subgroup of $GL_n$, then $(P, \iota)$ is also an embedded submanifold. The proof of these statements (cf. [2, p. 66]) employs the fact that locally over some open $U \subseteq Q$ the mapping $\iota|_{\pi^{-1}(U)}$ can be viewed as the mapping

$$U \times G \to U \times GL_n: (q, A) \to (q, \rho(A)).$$

We continue by recalling an important theorem for the theory of $G$-structures.

**Theorem 2.2.3**

Let $(H, \rho)$ be a Lie subgroup of $G$ and let $(P, \pi, Q, G)$ be a (smooth) principal $G$-bundle over $Q$ with (smooth) right action $\alpha$.

If $P' \subseteq P$ satisfies

i) $\alpha(P', \rho(A)) = P'$ for all $A \in H$,

ii) for every $q \in Q$ and $X, Y \in P' \cap \pi^{-1}\{\{q\}\}$ with $Y = \alpha(X, A)$ we have $A \in \rho(H)$, and

iii) for every $q \in Q$ there exists an open neighborhood $U$ of $q$ and a smooth, local section $X: U \to P$ such that the image $X_U$ lies in $P'$,

then there exists a unique (smooth) manifold structure on $P'$ such that $(P', \pi|_{P'}, Q, H)$ with the action

$$P' \times H \to P': (X, A) \to \alpha(X, \rho(A))$$

is a principal $H$-bundle over $Q$. With respect to this manifold structure $P' \hookrightarrow P$ is an $H$-reduction and smooth submanifold of $P$.

**Proof** A proof can be found in the book by Baum [2, Satz 2.14]. Uniqueness of the manifold structure follows from the fact that [2, Satz 2.1] was used in the proof.  

Note that if $H$ is embedded, so is $P'$ (cf. Remark 2.2.2/iii)). Moreover, the manifold structure on $P'$ is independent of the choice of local sections $X$ as for every $A \in C^\infty(U, H)$ the section $\alpha(X, A)$ gives rise to the same manifold structure.

Theorem 2.2.3 is used within the theory of $G$-structures to turn subsets $P$ of the frame bundle $Fr(TQ)$ into $G$-structures, once one has found an (open, countable) cover

$$\{U_\beta \subseteq Q| \beta \in I\}$$

with (smooth, local) frame fields $X \in \Gamma^\infty(U_\beta, Fr(TQ))$ taking values in $P$ for every $\beta \in I$ and an appropriate ‘symmetry group’ $G$ such that conditions i) and ii) are satisfied. To avoid confusion, we emphasize that $G$ is the ‘smaller group’ here.

The theorem can also be used in showing that a $G$-structure $P$ on $Q$ gives rise to a reduction $P^*$ of the coframe bundle $Fr(T^*Q)$ via equation (2.2.1).

This works as follows. For every $X \in P \subseteq Fr(TQ)$ we consider the coframe $X$ and observe that for every $A \in G$

$$X \cdot X = X \cdot \rho(A) \cdot \rho(A) \cdot X = 1_{TQ}.$$  

(2.2.3)

Thus we define the set

$$P^* := \{\theta \in Fr(T^*Q)| \exists X \in P: \theta = X\}$$

(2.2.4)
and then equation (2.2.3) states that the action of $\mathcal{G}$ on $\mathcal{P}$ induces an action of $\mathcal{G}$ on $\mathcal{P}^*$. This is in fact the one induced by the (smooth) right action of $\text{GL}_n$ on $\text{Fr}(T^*Q)$

$$\tilde{\alpha}: \text{Fr}(T^*Q) \times \text{GL}_n \to \text{Fr}(T^*Q): (X,A) \mapsto \tilde{\alpha}(X,A) := A \cdot X$$ (2.2.5)

and the dual representation of $\rho: \mathcal{G} \to \text{GL}_n$ as given by

$$\mathcal{G} \to \text{GL}((\mathbb{R}^n)^*): A \mapsto (\rho(A))^T.$$ (2.2.6)

**Corollary 2.2.4 (Dual $\mathcal{G}$-structures)**

Let $(\mathcal{P}, \pi, Q, \mathcal{G})$ be a $\mathcal{G}$-structure and $\mathcal{P}^*$ be a subset of the coframe bundle $\tilde{\pi}: \text{Fr}(T^*Q) \to Q$, as defined in (2.2.4). Then there exists a unique manifold structure on $\mathcal{P}^*$, such that $(\mathcal{P}^*, \tilde{\pi}|_{\mathcal{P}^*}, Q, \mathcal{G})$ together with the action

$$\mathcal{P}^* \times \mathcal{G} \to \mathcal{P}^*: (X,A) \mapsto \rho(A) \cdot X = X \cdot (\rho(A))^T,$$

as induced by the action (2.2.5) and the dual representation (2.2.6) of $\rho$, is a principal $\mathcal{G}$-bundle. With respect to this manifold structure $\mathcal{P}^* \hookrightarrow \text{Fr}(T^*Q)$ is a $\mathcal{G}$-reduction and smooth submanifold of $\text{Fr}(T^*Q)$. $\diamond$

**Proof** As noted above, this is a corollary of Theorem 2.2.3, so we need to check the assumptions. Since $\mathcal{P}$ is a $\mathcal{G}$-structure, $(\mathcal{G}, \rho)$ is a Lie subgroup of $\text{GL}_n$. By Remark 2.2.1, the latter is the structure group of the principal bundle $\text{Fr}(T^*Q)$.

“i)”: We have $\mathcal{P} \cdot \rho(A) = \mathcal{P}$ for all $A \in \mathcal{G}$ and formally $\mathcal{P} = \mathcal{P}^*$ by definition (2.2.4) of $\mathcal{P}^*$. Upon inversion

$$X \cdot \rho(A) \to X \cdot \rho(A) = \rho(A) \cdot X$$

for $X \in \text{Fr}(T^*Q), A \in \mathcal{G}$, it follows

$$\rho(A) \cdot \mathcal{P}^* = \mathcal{P} \cdot \rho(A) = \mathcal{P}^*.$$

“ii)”: Again invert and use simple transitivity of the $\mathcal{G}$-action on $\mathcal{P}$, see Remark 2.2.2/i).

“iii)”: This is true for $\mathcal{P}$ and upon inversion of the frame field, it is true for $\mathcal{P}^*$. $\blacksquare$

We call the principal $\mathcal{G}$-bundle $\mathcal{P}^*$ the **dual $\mathcal{G}$-structure to $\mathcal{P}$**. The dual point of view is often needed to understand the relation between $\mathcal{G}$-structures and ‘actual’ geometric structures on the base manifold.

**Remark 2.2.5**

i) Recall that if $\{(U_{\alpha}, \phi_{\alpha})|\alpha \in I\}$ is a system of local principal bundle trivializations of $\mathcal{P}$ (cf. [32, p. 5]), then second countability of $Q$ implies that we can always choose the set $I$ to be countable. We call such a countable cover $\{U_{\alpha}|\alpha \in I\}$ a **trivializing cover**. Now recall that there is a one-to-one correspondence between the diffeomorphisms $\phi_{\alpha}$ and smooth local sections

$$\bar{\alpha}: U_{\alpha} \to \mathcal{P}: q \mapsto \bar{\alpha}_q := \phi_{\alpha}^{-1}(q,1_{\mathcal{G}}).$$

Due to Remark 2.2.2/ii) on page 12, for all $\alpha, \beta \in I$ with $U_{\alpha} \cap U_{\beta} \neq \emptyset$ there exist (smooth) transition functions

$$\alpha\beta: U_{\alpha} \cap U_{\beta} \to \mathcal{G}: q \mapsto A_q,$$
uniquely defined by the equation

$$\beta X = \alpha X \cdot \rho(A).$$

Conversely, if \(\{U_\alpha | \alpha \in I\}\) is an (open, countable) cover of \(Q\) with corresponding frame fields \(\{X_\alpha\}_{\alpha \in I}\) satisfying the above equation for some \(A \in C^\infty(U_\alpha \cap U_\beta, G)\) with \(\alpha, \beta \in I\) such that \(U_\alpha \cap U_\beta \neq \emptyset\), then we can use this to put a unique topology and smooth structure on the union of the images

\[ P := \bigcup_{\alpha \in I} (X_{U_\alpha} \cdot \rho(G)) \]

such that \(P\) is a \(G\)-structure (cf. [2, Satz 2.31]). This yields an alternative proof of Corollary 2.2.4.

In many cases the existence of (smooth) frame fields with transition functions \(A_{\alpha\beta}\), taking values in \(\rho(G)\) and satisfying the cocycle condition

$$A_{\alpha\beta} \cdot A_{\beta\gamma} = A_{\alpha\gamma}$$

(2.2.8)

for all \(\alpha, \beta, \gamma \in I\) with non-empty \(U_\alpha \cap U_\beta \cap U_\gamma\), gives a simpler proof that \(P\) is a \(G\)-structure, than the use of Theorem 2.2.3. See, for instance, [32, Prop. 1.1.10] for a formal proof.

ii) In general, if \(P\) is a \(G\)-structure on an \(n\)-manifold \(Q\) and \((G, \chi)\) is also a Lie subgroup of the Lie subgroup \((G', \rho')\) of \(\text{GL}_n\) with \(\rho = \rho' \circ \chi\), then the argument in i) shows that this gives rise to a \(G'\)-structure \(P'\), where the action of \(G' \) on \(P'\) is induced by \(\rho'\). \(P\) is then called a \(G'\)-extension of \(P\). For more on extensions in German see [2, §2.5].

As indicated in the beginning, the mathematical definition of \(G\)-structures is not sufficient to capture the philosophical concept. If one speaks of \(G\)-structures, the choice of \(G\) and \(\rho\) is not arbitrary, but is thought of as a ‘symmetry group’ of an object in linear algebra on \(\mathbb{R}^n\). This object is called the ‘linear model’. Among many possible choices, this can be a vector subspace of \(\mathbb{R}^n\), an orientation on \(\mathbb{R}^n\), a particular tensor or a scalar product. Geometric structures on \(Q\) are then constructed from frame and coframe fields taking values in \(P\) and its dual \(P^*\), respectively, hence the need for Corollary 2.2.4. As it is difficult to give a general definition of how that construction of geometric structures on \(Q\) works precisely and as this would be an unnecessary abstraction for us, we will consider two particular cases of interest in the following two sections. As we will observe in the next chapter, both of these cases are constitutive for the mathematical theory of relativity.

### 2.2.2 Lorentzian Structures

Though we assume that the reader is familiar with the topic of Lorentzian metrics, their discussion in this section is of use both for fixing conventions and for understanding the philosophical idea of \(G\)-structures in terms of a familiar example. Specifically, we show how a Lorentzian metric is constructed from a particular \(G\)-structure on a manifold \(Q\).
In accordance with the statements made in the beginning of section 2.2, we first consider a particular object in linear algebra and its ‘symmetry group’ \( G \) that leaves the object invariant.

Recall that a \((\text{real})\) Lorentz vector space is a pair \((V, g)\), where \( V \) is a finite dimensional \((\text{real})\) vector space equipped with a Lorentz product \( g \). The latter is a bilinear form on \( V \), with the property that there exists a basis such that it takes the form

\[
g = \begin{pmatrix}
1 & & \\
-1 & \ddots & \\
& \ddots & -1
\end{pmatrix}
\] (2.2.9)

in said basis. Obviously, this is only possible if the dimension of \( V \) is at least 2. Since (2.2.9) constitutes a diagonalization, \( g \) must be non-degenerate, but not positive definite. A non-zero vector \( v \in V \) is called timelike if \( g(v, v) > 0 \), lightlike if \( g(v, v) = 0 \) and spacelike\(^2\) if \( g(v, v) < 0 \). It is also convenient to call a vector \( v \neq 0 \) causal, if it is either time- or lightlike. The property of being space-, time- or lightlike is sometimes referred to as the ‘causal character’ of the vector.

The standard example of a Lorentz vector space is Minkowski space \((\mathbb{R}^{n+1}, \eta)\), where \( n \in \mathbb{N} \) and \( \eta \) has components as in (2.2.9) with respect to the canonical basis on \( \mathbb{R}^{n+1} \). Trivially, all Lorentz vector spaces of the same dimension are linearly isomorphic to it via a suitable choice of basis, so we can restrict our attention to this instance. The ‘symmetry group’ \( G \) of Minkowski space is the Lorentz group \( O_{1,n} \equiv O(\mathbb{R}^{n+1}, \eta) \), defined by

\[
O_{1,n} := \{ \Lambda \in \text{GL}_{n+1} \mid \Lambda^T \cdot \eta \cdot \Lambda = \eta \}.
\] (2.2.10)

This group \( G = O_{1,n} \) acts on \( \mathbb{R}^{n+1} \) via the standard representation

\[
\rho: O_{1,n} \hookrightarrow \text{GL}_{n+1}: \Lambda \mapsto \Lambda,
\]

which is just an inclusion map. In fact, \( O_{1,n} \) admits a unique manifold structure such that \( (O_{1,n}, \rho) \) becomes an embedded Lie subgroup of \( \text{GL}_{n+1} \). The proof that it admits such a structure is a standard application of Cartan’s theorem [22, Thm. 20.12] and works in full analogy to the one of Lemma 2.2.8 in the next subsection. Uniqueness of the manifold structure follows from the fact that the manifold structure of embedded submanifolds is always unique (cf. [22, Prop. 5.18]). With respect to this topology, \( O_{1,n} \) has 4 components [24, Cor. 9.7]. We denote the identity component, which is also a Lie group, by \( \text{Lor}_{n+1} \). This is characterized by

\[
\text{Lor}_{n+1} = \left\{ \Lambda = \begin{pmatrix}
\Lambda^0 & v^T \\
u & A_S
\end{pmatrix} \in O_{1,n} \mid \Lambda^0 > 0, \det A_S > 0 \text{ and } u, v \in \mathbb{R}^n \right\}
\] (2.2.11)

(cf. [24, p. 237 sq.]). The other 3 components are obtained by multiplication with the matrices

\[
\begin{pmatrix}
-1 & & \\
1 & \ddots & \\
& \ddots & -1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & \pm1 & \\
& \ddots & \\
& & \pm1
\end{pmatrix}
\] (2.2.12)

\(^2\)In the book by O’Neill [24], which is in a sense constitutive for the mathematical theory of relativity, the zero vector is taken to be spacelike, but we believe this to be an unnatural convention and hence do not follow it.
with an odd number of minus signs. These matrices are called the *time inversion matrix* and *space inversion matrices*, respectively. Obviously, $O_{1,n}$ and $\text{Lor}_{n+1}$ have the same Lie algebra $\mathfrak{lor}_{n+1} = T_1 \text{Lor}_{n+1}$. To express it, we first note that, as commonly done in mathematics, we canonically identify the Lie algebra of the general linear group $\mathfrak{gl}_k \equiv \mathfrak{gl} \left( \mathbb{R}^k \right) \approx T_1 \text{GL} \left( \mathbb{R}^k \right)$ in $k$ dimensions with $\text{End} \left( \mathbb{R}^k \right) = \mathbb{R}^k \otimes \left( \mathbb{R}^k \right)^*$ and hence we may write $\mathfrak{lor}_{n+1} \subset \mathfrak{gl}_{n+1} = \text{End} \left( \mathbb{R}^{n+1} \right)$. Under this identification, which is basis independent, the exponential map of the Lie group $O_{1,n}$ is just the matrix exponential and hence

$$\mathfrak{lor}_{n+1} = \{ \lambda \in \mathfrak{gl}_{n+1} \mid \lambda^T \cdot \eta + \eta \cdot \lambda = 0 \} .$$

(2.2.13)

Now we use this ‘linear model’ $\left( \mathbb{R}^{n+1}, \eta \right)$ with ‘symmetry group’ $O_{1,n}$ to construct a geometric structure on an $(n+1)$-manifold $Q$ admitting an $O_{1,n}$-structure. An $O_{1,n}$-structure $(P, \pi, Q, O_{1,n})$ is also known as a *Lorentzian structure on $Q$*. The idea is that in the linear case, that is for an $(n+1)$-dimensional Lorentz vector space $(V, g)$, a Lorentz product $g \in V^* \otimes V^*$ can always be written as

$$g = X^T \cdot \eta \cdot X$$

(2.2.14)

where $X \in V \otimes \left( \mathbb{R}^{n+1} \right)^*$ is a basis. This basis is unique up to the action of the group $O_{1,n}$ on $X$ from the right. To construct a Lorentzian metric we think of $V$ as the tangent space $T_q Q$ at a point $q \in Q$ and then let $X$ ‘vary smoothly’ with $q$. If the manifold $Q$ is parallelizable, the tangent frame bundle $\text{Fr} \left( TQ \right)$ and hence every $\mathcal{G}$-structure is trivial, so in this case we simply choose a global frame field $X : Q \to P$ or equivalently a global coframe field $\bar{X} : Q \to P^*$ dual to $X$ (see section 2.2.1) and define a Lorentzian metric $g$ on $Q$ via (2.2.14).

If $\text{Fr} \left( TQ \right)$ is not trivial, then neither is the $O_{1,n}$-structure $P$ nor its dual $P^*$, but we can still find a (countable) trivializing (open) cover $\{ U_\alpha \mid \alpha \in I \}$ with (smooth, local) frame fields

$$\bar{X} : U_\alpha \to P \subset \text{Fr} \left( TQ \right)$$

for every $\alpha \in I$. Moreover, for $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$ we obtain smooth transition functions

$$\bar{\Lambda} : U_\alpha \cap U_\beta \to O_{1,n-1} : q \to \bar{\Lambda}_q^{-1}$$

via $\bar{X} = X \cdot \bar{\Lambda}$ on $U_\alpha \cap U_\beta$. If we again use the ansatz of defining $g$ via (2.2.14), we can use the freedom in the choice of basis, the local frame fields $\bar{X}$ and a partition of unity to construct the Lorentzian metric $g$. More explicitly, we first define a (smooth) partition of unity subordinate to the cover $\{ U_\alpha \}_{\alpha \in I}$:

$$\rho_\alpha : U_\alpha \to [0,1] \quad \forall \alpha \in I , \quad 1 = \sum_{\alpha \in I} \rho_\alpha ,$$

and then construct the global tensor field

$$g := \sum_{\alpha \in I} \rho_\alpha \bar{X}_\alpha^T \cdot \eta \cdot \bar{X}_\alpha .$$

(2.2.15)
For any $\alpha \in I$ and $q \in \mathcal{U}_\alpha$, we calculate:

$$g_q = \sum_{\beta \in I} \rho_\beta (q) \ X_q^T \cdot \eta \cdot \bar{X}_q$$

$$= \sum_{\beta \in I} \rho_\beta (q) \ (\bar{X} \cdot \Lambda^\beta)_q \cdot \eta \cdot (\bar{X} \cdot \Lambda^\beta)_q$$

$$= \sum_{\beta \in I} \rho_\beta (q) \ X_q^T \cdot (\bar{\Lambda}_q \cdot \eta \cdot \bar{\Lambda}_q) \cdot X_q$$

$$= \sum_{\beta \in I} \rho_\beta (q) \ X_q^T \cdot \bar{\Lambda}_q \cdot \eta \cdot \bar{X}_q$$

$$= \frac{\bar{X}^T}{X_q} \cdot \eta \cdot \bar{X}_q.$$  \hspace{1cm} (2.2.16)

This calculation shows that definition (2.2.15) is independent of the choice of $\{\rho_\alpha\}_{\alpha \in I}$. By a similar argument, it is independent of the choice of frame fields taking values in $\mathcal{P}$.

Now we simply define a Lorentzian metric $g$ to be a smooth covariant 2-tensor field such that for every $q \in \mathcal{Q}$ there exists an open neighborhood $\mathcal{U}$ of $q$ and a local frame field $X \in \Gamma^\infty (\mathcal{U}, \text{Fr}(T\mathcal{Q}))$ with coframe field $\bar{X} \in \Gamma^\infty (\mathcal{U}, \text{Fr}(T\mathcal{Q}))$ dual to $X$ such that

$$g|_\mathcal{U} = X^T \cdot \eta \cdot X.$$  

As we have shown with (2.2.16), the tensor field defined by (2.2.15) is a Lorentzian metric and conversely every Lorentzian metric can be written in this way. We conclude that the existence and choice of a Lorentzian metric on a manifold $\mathcal{Q}$ is equivalent to the existence and choice of an $O_{1,n}$-structure. As stated in the end of section 2.2.1, this is not by accident, but illustrates the general scheme of (first order) $G$-structures. Similarly one can construct Riemannian metrics, volume forms, symplectic forms, etc. from appropriate reductions of the frame bundle and vice versa.

**Remark 2.2.6 (Implicit geometric structures)**

Extensions of $G$-structures as described in Remark 2.2.5 (ii) yield new geometric structures on $\mathcal{Q}$, once one has identified the corresponding 'linear model' for which the 'larger group' $G'$ acts as a 'symmetry group'. If one can argue that the extension is 'natural', these new geometric structures can in turn be used to understand and formulate physical laws without needing to introduce additional postulates or mathematical assumptions.

For instance, assume we are given a $\text{Lor}_{n+1}$-structure on an $(n + 1)$-manifold $\mathcal{Q}$. Obviously $\text{Lor}_{n+1}$ is a Lie subgroup of $O_{1,n}$, hence we can use the $\text{Lor}_{n+1}$-structure to construct a Lorentzian metric on $\mathcal{Q}$ via the procedure described in this section. Moreover, $\text{Lor}_{n+1}$ is also a Lie subgroup of the special linear group

$$\text{SL}_{n+1} \equiv \text{SL} (\mathbb{R}^{n+1}) := \{ A \in \text{GL}_{n+1} | \det A = 1 \},$$

which is the symmetry group of the determinant map. So if $X = X_i \otimes e^i$ is a basis in $\mathbb{R}^{n+1}$, then for every $A \in \text{SL}_{n+1}$

$$(X \cdot A)^0 \wedge \cdots \wedge (X \cdot A)^n = \det (A) \ X^0 \wedge \cdots \wedge X^n = X^0 \wedge \cdots \wedge X^n.$$
In full analogy to the procedure of constructing a Lorentz metric $g$, we can use the above calculation to construct an $(n+1)$-form on $Q$ from sections of the $\text{Lor}_{n+1}$-structure. The resulting $(n+1)$-form is in fact the canonical volume form with respect to $g$.

In chapter 3 we will see that our definition of spacetime indeed yields a $\text{Lor}_{n+1}$-structure, namely the so called ‘frame of reference bundle’. Hence the mathematical theory of extensions of $\mathcal{G}$-structures gives rise to a natural, physical notion of spacetime volume.

To conclude this section, we add that an $\text{O}_{1,n}$-structure $\mathcal{P}$ on an $n$-manifold $Q$ has a natural interpretation in terms of the metric $g$, as constructed above. If $X \in \Gamma^\infty (\mathcal{U}, \mathcal{P})$ is a local frame field over $U \subseteq Q$, then we can calculate the component functions of $g$ with respect to $X$:

$$g_{ij} := g(X_i, X_j) = X_i \cdot (X^T \cdot \eta \cdot X) \cdot X_j$$

$$= X_i \cdot (\eta_{kl} X^k \otimes X^l) \cdot X_j = \eta_{kl} \delta_i^k \delta_j^l$$

$$= \eta_{ij}.$$ 

In other words, for each $q \in U$ the frame $X_q$ is an orthonormal basis in $T_q Q$ with respect to $g_q$. Hence we call $X$ an orthonormal frame field (with respect to $g$) and $\mathcal{P} = \text{OFr} (Q, g)$ the orthonormal frame bundle (on $Q$ with respect to $g$).

### 2.2.3 Lorentzian Orientations

In this section we discuss another example of a $\mathcal{G}$-structure on a manifold $Q$, namely that of so called ‘Lorentzian orientations’. The example is in fact a class of examples encompassing space, time, as well as spacetime orientations on manifolds. Their construction again follows the general recipe: First find the linear model, second find its symmetry group and third use a trivializing cover with corresponding frame fields and a partition of unity to construct the geometric structure from the linear model. However, Lorentzian orientations differ from the example in the preceding subsection in the sense that the linear model is not a tensor on $\mathbb{R}^n$ and hence we will not obtain a tensor field on $Q$ as a result.

As noted before, the discussion here is more explicit than in the other sections, since we have not been able to find a suitable reference. Many textbooks, including the one by O’Neill [24, Chap. 9] treat the issue of Lorentzian orientations implicitly by assuming that a Lorentzian metric is given. The treatment here in terms of open Lie subgroups of the indefinite conformal group is more conceptual and more general. Nonetheless, we do recommend O’Neill’s treatment as a reference. For more information on conformal geometry in the context of general relativity, we refer to [39, Appendix D; 85].

The linear model we wish to consider is an adaptation of the concept of orientation to a Lorentz vector space $(\mathcal{V}, g)$, which we will later call a ‘(linear) Lorentzian orientation’. First recall that, if $\mathcal{V}$ is a vector space of dimension $n \in \mathbb{N}$, an orientation $O$ on $\mathcal{V}$ is a choice of basis $X = X_i \otimes \mathfrak{g}$ modulo the canonical action of the (Lie) group

$$\text{GL}_n^+ := \text{GL}^+ (\mathbb{R}^n) := \{ A \in \text{GL} (\mathbb{R}^n) | \det A > 0 \}.$$ 

from the right. As a group orbit

$$O := X \cdot \text{GL}_n^+ = \{ Y \in \mathcal{V} \otimes (\mathbb{R}^n)^* | \exists A \in \text{GL}_n^+: Y = X \cdot A \}.$$

19
of a basis $X$, an orientation defines a subset of the set of bases of $\mathcal{V}$. A basis is called right-handed, if it is an element of that set and left-handed, if it is not. The terminology stems from the 'right-hand rule'.

However, if we have a Lorentz vector space $(\mathcal{V}, g)$ and want to find an ‘adapted’ notion of orientation on it, it is natural to require this notion of orientation to preserve the causal character of at least some elements $Z \in \mathcal{V}$, i.e. we ask for time-, space- or lightlike vectors (or some combination thereof) to stay time-, space- or lightlike under the action of the subgroup $G \subseteq \text{GL}(\mathcal{V})$ from the right. The next theorem characterizes this group $G$.

**Theorem 2.2.7**

Let $\mathcal{V}$ be a vector space with Lorentz products $g, g'$. Then the following are equivalent:

i) A vector is timelike with respect to $g$ if and only if it is timelike with respect to $g'$.

ii) A vector is spacelike with respect to $g$ if and only if it is spacelike with respect to $g'$.

iii) A vector is lightlike with respect to $g$ if and only if it is lightlike with respect to $g'$.

iv) There exists a constant $\lambda \in \mathbb{R}_+$ such that $g' = \lambda g$.

**Proof** Trivially, iv) implies the other statements. Thus it is sufficient to prove “iii) $\implies$ iv)”, “i) $\implies$ iii)” and “ii) $\implies$ iii)”.

“iii) $\implies$ iv)” The main idea is as follows: If we are given a Lorentz product $g$, then for any timelike $X$ and any spacelike $Y$ in $\mathcal{V}$ we can use the parameter $s \in \mathbb{R}$ to ‘move’ $X + sY$ into and out of the light cone

$$c(g) := \{Z \in \mathcal{V} | Z \neq 0 \text{ and } g(Z, Z) = 0\}.$$  

This idea is due to Hawking and Ellis [16, p. 61] and was later taken up by Dajczer and Nomizu [49] to prove a related result.

Let $g, g'$ be Lorentz products on $\mathcal{V}$, $X \in \mathcal{V}$ be timelike, $Y \in \mathcal{V}$ be spacelike with respect to $g$ and $s \in \mathbb{R}$. Define $f: \mathbb{R} \to \mathbb{R}$ via

$$f(s) := g(X + sY, X + sY) = g(X, X) + 2s g(X, Y) + s^2 g(Y, Y),$$

which has zeros $s_+, s_-$ satisfying

$$s_{\pm} = -\frac{g(X, Y)}{g(Y, Y)} \pm \sqrt{\left(\frac{g(X, Y)}{g(Y, Y)}\right)^2 - \frac{g(X, X)}{g(Y, Y)}}$$

and $s_- < 0 < s_+$, since $-g(X, X)/g(Y, Y) > 0$. Analogously, we define $f'$ for the Lorentz product $g'$.

As $X + s_\pm Y$ are lightlike with respect to $g$, they are lightlike with respect to $g'$. Hence $f'(s_\pm) = 0$ and therefore the zeros of $f'$ are $s'_\pm = s_\pm$. Thus

$$s_+ s_- = \frac{g(X, X)}{g(Y, Y)} = s'_+ s'_- = \frac{g'(X, X)}{g'(Y, Y)},$$

20
which is equivalent to
\[ \frac{g'(X)}{g(X)} = \frac{g'(Y)}{g(Y)} =: \lambda \in \mathbb{R}_+ \]
for all \( g \)-timelike \( X \) and \( g \)-spacelike \( Y \). Since a vector \( Z \in \mathcal{V} \) is either space-, time-, lightlike or trivial, we obtain:
\[ g'(Z, Z) = \lambda g(Z, Z). \]

The polarization identity, given by
\[ g(X, Y) = \frac{1}{2} (g(X + Y, X + Y) - g(X, X) - g(Y, Y)) \tag{2.2.17} \]
for all \( X, Y \in \mathcal{V} \), finishes the proof.

\( \text{“i) } \Rightarrow \text{ iii)”} \): The idea of proof is to reconstruct the light cone by taking sequences of timelike vectors converging to lightlike ones. So we show that, under some ‘natural’ choice of topology on \( \mathcal{V} \), the light cone \( \mathcal{C}(g) \) with respect to \( g \) is the boundary \( \partial t \) of the set of timelike vectors \( \mathcal{I} \) minus the zero vector. Since the set of timelike vectors coincide for both \( g \) and \( g' \), the boundary must coincide and so the assertion follows.

\( \text{“} \mathcal{C}(g) \subseteq (\partial t \setminus \{0\}) \text{”} \): Equip \( \mathcal{V} \) with a norm \( \| \cdot \| \). Let \( X \) be \( g \)-lightlike and choose a \( g \)-timelike \( Z \) such that \( g(X, Z) > 0 \). For \( k \in \mathbb{N} \) define a sequence via
\[ X_k := X + \frac{1}{k} Z \]
and observe that each element is timelike. Since
\[ \|X - X_k\| = \frac{1}{k} \|Z\| \]
the sequence converges.

\( \text{“} \mathcal{C}(g) \supseteq (\partial t \setminus \{0\}) \text{”} \): With respect to the topology induced by the above norm, the quadratic form
\[ p: \mathcal{V} \to \mathbb{R}: Z \mapsto p(Z) := g(Z, Z) \]
is continuous (i.e. bounded with respect to the operator norm). Thus the set of \( g \)-spacelike vectors \( p^{-1}((-\infty, 0)) \) is open. Since the boundary \( \partial t \) is the closure \( \bar{t} \) without the interior \( t \), any point \( X \) in \( \partial t \setminus \{0\} \) is the limit of some sequence \( \{X_k\}_{k \in \mathbb{N}} \) in \( t \). Now \( X \notin t \), so by continuity
\[ \lim_{k \to \infty} p(X_k) = p(X) \neq 0. \]
If \( p(X) < 0 \), then it is spacelike, but the set of spacelike vectors is open, so \( X \) cannot be a limit point of \( \{X_k\}_{k \in \mathbb{N}} \subset t \). Thus \( p(X) = 0 \).

Repeating the argument for the set of \( g' \)-timelike vectors \( t' \), we conclude
\[ \mathcal{C}(g) = (\partial t \setminus \{0\}) = (\partial t' \setminus \{0\}) = \mathcal{C}(g'). \]

\( \text{“ii) } \Rightarrow \text{ iii)”} \): The proof is entirely analogous to the previous one. \( \blacksquare \)

Theorem 2.2.7 shows that asking either for the set of time-, space- or lightlike vectors to be preserved under the linear action of a group \( \mathcal{G} \) implies that the group has to preserve the
2 Mathematical Preliminaries

causal character of every vector in the Lorentz vector space \((V, g)\). By Theorem 2.2.7/iv), this group \(G\) is given by
\[
CO(V, g) := \{ A \in \text{GL}(V) \mid \exists \lambda > 0: A^T \cdot g \cdot A = \lambda g \},
\]
(2.2.18)
canonically equipped with the restricted multiplication and inversion mappings of \(\text{GL}(V)\). The proof that \(CO(V, g)\) is indeed a subgroup of \(\text{GL}(V)\) is elementary. We call \(CO(V, g)\) the (linear) conformal group of \((V, g)\). Note that the notation \(CO\) for this group follows the one used in the book by Kobayashi [18, Ex. 2.6] and appears to be standard. For \(n \in \mathbb{N}\) we write \(CO_{1,n} := CO(\mathbb{R}^{n+1}, \eta)\). As every Lorentz vector space \((V, g)\) is linearly isomorphic to Minkowski space of the same dimension \(n + 1\), the algebraic group \(CO(V, g)\) is isomorphic to the algebraic group \(CO_{1,n}\). The corresponding group isomorphism is obtained by a choice of basis \(X_i = X_i \otimes e^i\) in \(V\), which needs to be orthonormal with respect to \(g\) up to a real factor. More precisely, if \(X\) is a basis of \(V\) with \(\lambda g = X^T \cdot \eta \cdot X\) for some \(\lambda \in \mathbb{R}_+\), then the map
\[
CO_{1,n} \rightarrow CO(V, g): A \rightarrow X \cdot A \cdot X
\]
is an isomorphism of groups. Conversely, if we are just given a vector space \(V\) together with a basis \(X\), we may define
\[
O := X \cdot CO_{1,n} = \left\{ Y \in V \otimes (\mathbb{R}^{n+1})^* \mid \exists A \in CO_{1,n}: Y = X \cdot A \right\},
\]
i.e. \(O\) is the \(CO_{1,n}\)-orbit of \(X\). Then every element \(Y \in O\) and hence the set \(O\) itself uniquely defines a Lorentz product \(g\) on \(V\) up to a positive factor. We call an \((n+1)\)-dimensional vector space \(V\) equipped with an orbit \(O\) of \(CO_{1,n}\) a causal vector space (of signature \((1, n)\)) and \(O\) a (linear) causal structure. In this setting, it is then natural to define a non-zero vector \(Z \in V\) to be space-, time- or lightlike, if it is space-, time- or lightlike with respect to some and hence every \(g := X^T \cdot \eta \cdot X\) induced by \(X \in O\). Thus a causal structure is enough to define the causal character of any element of \(V\), i.e. we do not need a particular Lorentz product. By Theorem 2.2.7, it is clear that this is the most general setting in which it makes sense to speak of the causal character of vectors. Yet even in this case, we have a notion of (hyperbolic) angle (cf. [24, Chap. 5 Lem. 30]) between any two non-lightlike vectors \(Z, Z' \in V \setminus \{0\}\), as for all \(\lambda \in \mathbb{R}_+\) it holds that
\[
\frac{g(Z, Z')}{\sqrt{|g(Z, Z)|} \sqrt{|g(Z', Z')|}} = \frac{\lambda g(Z, Z')}{\sqrt{|g(Z, Z)|} \sqrt{|g(Z', Z')|}}.
\]
Moreover, orthogonality is well-defined for any \(Z, Z' \in V\). This angle-preserving property is characteristic of the conformal group in the sense that it can also be used as its definition.

In a causal vector space \((V, O)\) one can classify linear subspaces \(W \subseteq V\) in accordance with the causal character of the vectors they contain. Adapting the definition from O’Neill for Lorentz vector spaces [24, pp. 141 sqq.] to the causal case (and our sign convention), we call \(W\) spacelike, if the restricted product \(g|_W\) induced by some \(X \in O\) is negative definite. An equivalent condition is that \((W, -g|_W)\) is an inner product space. A subspace \(W\) is called timelike, if \((W, g|_W)\) is a Lorentz vector space, and it is called lightlike, if \(g|_W\) is degenerate.\(^3\) Again, those definitions do not depend on the choice of \(X \in O\).

\(^3\)It would be more natural to call a subspace \(W\) timelike, if \(g|_W\) is positive definite and Lorentzian, if \(g|_W\) is a Lorentz product on \(W\). We shall submit to O’Neill’s convention here, but the point deserves to be made.
terminology is motivated by the fact that, if $Z \in \mathcal{V}$ is time-, space- or lightlike, then the subspace span \{Z\} = \mathbb{R}Z is time-, space- or lightlike, respectively. Using these definitions, one can show (cf. [24, p. 141]) that a subspace $\mathcal{W}$ is spacelike if and only if its orthogonal subspace $\mathcal{W}^\perp = \{Z \in \mathcal{V} | \forall Y \in \mathcal{W}: g(Y, Z) = 0\}$ is timelike and a subspace is lightlike if and only if its orthogonal subspace is lightlike. Note that any lightlike vector is orthogonal to itself and hence care must be taken with the terminology ‘orthogonal complement’.

Before we turn to the issue of how $CO_{1,n}$ gives a natural notion of ‘Lorentzian orientation’ on $\mathbb{R}^{n+1}$ and hence more general vector spaces $\mathcal{V}$, we shall prove that $CO_{1,n}$ is in fact a Lie group and have a closer look at its properties.

**Lemma 2.2.8**

For every $n \in \mathbb{N}$ there exists a unique manifold structure on $CO_{1,n}$ such that, together with this manifold structure, it is an embedded Lie subgroup of $GL_{n+1}$. Moreover, the map

$$CO_{1,n} \to \mathbb{R}_+ \times O_{1,n}: A \to \left( \sqrt[n+1]{|\det A|}, A/\sqrt[n+1]{|\det A|} \right)$$  (2.2.20)

is a Lie group isomorphism.

**Proof** One of the standard methods to obtain a manifold structure on an algebraic subgroup is to show that (2.2.18) gives a ‘closed condition’ and to use Cartan’s theorem [22, Thm. 20.12]. Uniqueness of the manifold structure then follows from topological embeddedness [22, Prop. 5.31].

For any $A \in CO_{1,n}$ we have

$$\det (\eta \cdot A^T \cdot \eta \cdot A) = (\det (A))^2 = \det (\lambda I) = \lambda^{n+1}$$

with $\lambda$ as in (2.2.18). As $\det$ never vanishes and is continuous on $GL_{n+1}$ the function

$$\xi: GL_{n+1} \to (\mathbb{R}^{n+1})^* \otimes (\mathbb{R}^{n+1}) : A \to A^T \cdot \eta \cdot A \sqrt[n+1]{|\det A|^2}$$

is continuous and hence $CO_{1,n} = \xi^{-1}(\{\eta\})$ is closed. Again, [22, Thm. 20.12] together with [22, Prop. 5.31] yields the first assertion.

For the second assertion, we note that $\mathbb{R}_+ \times O_{1,n}$ is the product of the Lie groups $O_{1,n}$ and $\mathbb{R}_+$ together with ordinary multiplication, hence $\mathbb{R}_+ \times O_{1,n}$ is canonically a Lie group. Indeed, for every $A \in CO_{1,n}$ we can write

$$A = \sqrt[2n+1]{\lambda} \frac{A}{\sqrt[2n+1]{\det A}} = \sqrt[n+1]{\det A} \frac{A}{\sqrt[n+1]{\det A}}.$$

The first factor is a positive number, the second one is an element of $O_{1,n}$ by definition of $\lambda$ in (2.2.18). As the factorization is unique, (2.2.20) defines a bijection. It is a group homomorphism, since $\det$ and taking roots of positive numbers are group homomorphisms. As taking absolute values of non-zero reals and roots of positive numbers is smooth, the map (2.2.20) is smooth. The result now follows from the fact that bijective, smooth group homomorphisms between Lie groups are Lie group isomorphisms, see e.g. [22, Cor. 7.6].
As a corollary of Lemma 2.2.8, we find that $O_{1,n}$ is canonically an embedded Lie subgroup of $CO_{1,n}$. Moreover, as $CO_{1,n}$ is diffeomorphic to $\mathbb{R}_+ \times O_{1,n}$ via (2.2.20), $\mathbb{R}_+$ is connected and $O_{1,n}$ has 4 components, so does $CO_{1,n}$. In particular, its identity component, denoted by $CL_{or_{n+1}}$, is diffeomorphic to $\mathbb{R}_+ \times Lor_{n+1}$.

Now, to define Lorentzian orientations, we consider the analogy to ordinary orientations on a vector space $V$. The general linear group $GL_n$ has two connected components, but only the component $GL_n^+$ is a (Lie) subgroup of $GL_n$. This is the identity component and the one used to define an orientation. The other component is obtained by multiplication with the reflection matrix $-1 \in GL_n$. Carrying this line of thought over to the conformal group $CO_{1,n}$, we are lead to the conclusion that there should be four kinds of Lorentzian orientations, since there are four open submanifolds of $CO_{1,n}$, besides $CO_{1,n}$ itself, that are also Lie subgroups: The identity component $CL_{or_{n+1}}$, the Lie group generated by $CL_{or_{n+1}}$ together with time inversion, the Lie group generated by $CL_{or_{n+1}}$ together with space inversion as well as the Lie group generated by $CL_{or_{n+1}}$ together with

$$
\begin{pmatrix}
-1 & -1 \\
-1 & 1 \\
 & \ddots \\
 & & 1
\end{pmatrix},
$$

where the $+1$s are dropped for $n+1 = 2$. We thus rigorously define a (linear) Lorentzian orientation on a vector space $V$ to be an orbit of one of these four groups in the set of bases of $V$ under the canonical action from the right. In the first case we call the Lorentzian orientation a (linear) spacetime orientation, in the second case a (linear) space orientation, in the third case a (linear) time orientation and in the last case a (linear) causal orientation. The latter name is derived from the fact that the group is $CO_{1,n} \cap GL_{n+1}^+ = \{ A \in CO_{1,n} | \det A > 0 \}$, so its orbit in $V$ is just an ordinary orientation respecting the causal structure. Clearly, every spacetime orientation induces a unique space- and a unique time-orientation, but causal orientations induce neither. A vector space $V$ is said to be spacetime-, space-, time- or causally oriented, if it is equipped with the respective orientation $O$. In each case it is Lorentz-oriented. Note that in a Lorentz-oriented vector space $(V, O)$, the ‘causal character’ of any non-zero element and the notion of orthogonality is well-defined, since $O$ is contained in an orbit of $CO_{1,n}$ and hence $(V, O \cdot CO_{1,n})$ is a causal vector space.

Of course, we would like to use Lorentzian orientations on an $(n + 1)$-dimensional vector space $V$ to classify bases or vectors, but this classification differs from ordinary orientations on $V$ and is physically motivated. For instance, one does not just define a basis $X$ of $V$ to be time-oriented, if it is an element of $O$. This would be a natural definition from a mathematical perspective, but physically, we want time-orientations to define whether a single vector points into the ‘past’ or the ‘future’. So instead, for a time-oriented vector space $(V, O)$, we define a timelike vector $Z \in V$ to be future-directed, if there exists a basis $X \in O$ such that $Z = X_0$. Else we call $Z$ past-directed. If a timelike vector is known to be future-directed, there exists a convenient way to check whether another timelike vector is also future-directed.
2.2 (First Order) $G$-structures

Proposition 2.2.9
Let $(V, O)$ be a time-oriented vector space and let $Z \in V$ be future directed timelike. Then a timelike $Z' \in V$ is future directed if and only if

$$g(Z, Z') > 0$$

with respect to some Lorentz product $g$ induced by a basis in $O$. \hfill \Box

Proof
Denote by $n+1$ the dimension of $V$. Obviously the above condition is independent of the choice of $g$.

$\implies$: If $Z, Z'$ are future directed timelike, then by definition there exist respective bases $X, X' \in O$ such that $Z = X_0$ and $Z' = X_0'$. Since $X, X' \in O$, there exists a time orientation preserving $A \in C_{O^-}^1$ such that $X' = X \cdot A$. Now let $g$ be the Lorentz product induced by $X$, then

$$g(Z, Z') = g(X_0, X_i A^i_0) = \eta_{0i} A^i_0 = A^0_0.$$  

By Lemma 2.2.8, $A$ is the product of a positive number $\lambda$, an element $\Lambda \in \text{Lor}_{n+1}$ and possibly a space-inversion matrix (cf. (2.2.12) on page 16 for a definition). Since $\Lambda^0_0 > 0$ by the characterization (2.2.11) of $\text{Lor}_{n+1}$ and neither $\lambda$ nor the space inversion matrix change the sign of $A^0_0$, the assertion is true.

$\impliedby$: Let $Z'$ be a timelike vector with strictly positive $g(Z', Z)$. Again, choose a basis $X \in O$ such that $Z = X_0$ and assume without loss of generality that $g$ is induced by $X$. Since $Z$ and $Z'$ are both timelike, there exists a basis $X'$ in the $C_{O^-}^1$-orbit of $X$ such that $Z' = X_0'$. In other words, we have an $A \in C_{O^-}^1$ such that $X' = X \cdot A$. Repeating the calculation above we find $A^0_0 > 0$ and therefore $X' \in O$. \hfill \blacksquare

Regarding space orientations $O$, we would like to obtain an orientation on a spacelike hyperplane $W$, i.e. a linear $n$-dimensional subspace of $V$ with positive definite $-g|_W$, where $g$ is induced by some $X \in O$. We emphasize that a spacelike hyperplane is always spanned by $n$ linearly independent spacelike vectors, but not every hyperplane spanned by $n$ linearly independent spacelike vectors is spacelike. Yet for a timelike vector the orthogonal subspace is always a spacelike hyperplane (cf. [24, Chap. 5 Lem. 26]), so, given a spacelike subspace $W$, we may write

$$W^\perp = \text{span} \{Z\} = \mathbb{R}Z$$

for some timelike $Z \in V$. We may therefore call an ordered set of $n$ linearly independent, spacelike vectors $\{Y_1, \ldots, Y_n\}$ in a space-oriented vector space $(V, O)$ (space-)right handed, if there exists an $X \in O$ such that

$$W := \text{span} \{Y_1, \ldots, Y_n\} = \text{span} \{X_1, \ldots, X_n\}$$  \hspace{1cm} (2.2.22)

and $Y_a \otimes e^a$ is right-handed with respect to the ordinary orientation induced by $X_a \otimes e^a$ on $W$. An ordered set of $n$ spacelike vectors $\{Y_1, \ldots, Y_n\}$ spanning a spacelike hyperplane is (space-)left handed, if it is not (space-)right handed.

For a spacetime-oriented vector space $(V, O)$, we call a basis $Y$ of $V$ spacetime-oriented, if $Y_0$ is timelike and future-directed, and $Y_a \otimes e^a$ are (space-)right handed. In this sense, it can be said that a spacetime-orientation on a vector space $V$ is a space- together with a
time-orientation giving rise to the same causal structure. In contrast, causal orientations are simply ordinary orientations on a causal vector space.

There also exists an extension of time- and space orientations to lightlike vectors and lightlike hyperplanes in vector spaces $V$ carrying the respective Lorentzian orientation $O$. As these will be employed in chapter 3, we shall give a brief discussion. First observe that for time- or spacelike $Z \in V$, we can define the parallel projection (endomorphism) with respect to $Z$:

$$
\pi^\parallel := \frac{Z \otimes Z \cdot g}{g(Z, Z)}.
$$

(2.2.23)

Indeed, (2.2.23) is independent of the choice of $g$ as induced by $X \in O$. The orthogonal projection (endomorphism) $\pi^\perp$ is then defined via $1 = \pi^\parallel + \pi^\perp$. As projections, those endomorphisms satisfy

$$
\pi^\parallel \cdot \pi^\parallel = \pi^\parallel, \quad \text{and} \quad \pi^\perp \cdot \pi^\perp = \pi^\perp.
$$

The image of $\pi^\parallel$ is $\mathbb{R}Z$ and the image of $\pi^\perp$ is $(\mathbb{R}Z)^\perp$, for the kernels the situation is reversed. As noted before, if $Z$ is timelike, then $\mathbb{R}Z$ is timelike and $(\mathbb{R}Z)^\perp$ is spacelike. Since all non-zero vectors in a spacelike subspace are spacelike, it follows that lightlike $K \in V$ satisfy $K \notin \pi^\perp(V) = \ker \pi^\parallel$ and therefore $\pi^\parallel \cdot K \neq 0$. Since all non-zero vectors in $\pi^\parallel(V) = \mathbb{R}Z$ are timelike, $\pi^\perp \cdot K$ is timelike. So if $O$ is a time-orientation on $V$, we may define lightlike $K \in V$ to be future-directed, if for some timelike vector $Z \in V$, the vector $\pi^\parallel \cdot K$ is future-directed. Else $K$ is past-directed. For the case of a space-oriented vector space $(V, O)$, assume $K_1, \ldots, K_n \in V$ span a lightlike hyperplane $W$ in $V$. Now take some $X \in O$, consider the corresponding spacelike hyperplane

$$
W' := \text{span} \{X_1, \ldots, X_n\}
$$

and the parallel projection $\pi^\parallel$ with respect to $X_0$. Since $W$ is lightlike, it does not contain any timelike vector [24, Chap. 5 Lem. 28], so each $K_a$ with $a \in \{1, \ldots, n\}$ is either light- or spacelike. In either case, $K_a \notin \pi^\parallel(V) = \ker \pi^\parallel$ and thus $\pi^\perp \cdot K_a \neq 0$ for each $a$. Thus the restriction $\pi^\perp|_W: W \to W'$ is a vector space isomorphism and hence $\pi^\perp \cdot K_1, \ldots, \pi^\perp \cdot K_n$ form a basis of $W'$. It is thus natural to define an ordered tuple of vectors $\{K_1, \ldots, K_n\}$ spanning a lightlike hyperplane to be (space-)right handed, if for some $X \in O$ the ordered set $\{\pi^\perp \cdot K_1, \ldots, \pi^\perp \cdot K_n\}$ yields a space-right-handed basis on $(\mathbb{R}X_0)^\perp$. Else it is (space-)left handed. Formally, we still need to show that the definitions are independent of the choice of $Z$ and $X$.

**Theorem 2.2.10 (Lightlike time & space orientations are well-defined)**

i) Let $(V, O)$ be a time-oriented vector space and let $K \in V$ be lightlike. Then $K$ is future-directed with respect to a timelike vector $Z \in V$ if and only if it is future-directed with respect to any other timelike $Z' \in V$.

ii) Let $(V, O)$ be a space-oriented vector space and let $K_1, \ldots, K_n \in V$ span a lightlike hyperplane. Then $K_1, \ldots, K_n$ are right-handed with respect to $X \in O$ if and only if they are right-handed with respect to any other $X' \in O$.

\[\Diamond\]

**Proof** It is enough to show one direction in each case. We set $n + 1 = \dim V$.

"i)" The idea is to decompose $K$ and $Z'$ into ‘spatial’ and ‘temporal’ parts with respect
to $Z$ and then use the Cauchy-Schwarz inequality. We borrowed it from a related proof by O’Neill [24, Lem. 5.29].

Choose some $X \in O$ such that $Z = X_0$ and let $g$ be the induced Lorentz product.

We introduce some notation: Any $Y \in \mathcal{V}$ may be decomposed into a timelike and spacelike part via

$$Y = Y^0 X_0 + \vec{Y}$$

where $\vec{Y}$ is a linear combination of the $X_a$s with $a \in \{1, \ldots, n\}$. Applying the same decomposition on $Y' \in \mathcal{V}$, we write

$$g(Y, Y') = Y^0 Y'^0 - \vec{Y} \cdot \vec{Y}'$$

with $\cdot$ denoting the standard inner product in the spacelike hyperplane spanned by the $X_a$s. We write the length of $\vec{Y}$ with respect to this inner product as $|\vec{Y}| := \sqrt{\vec{Y} \cdot \vec{Y}}$. Now employ this notation for $K$ and $Z'$.

Since $K$ is future directed with respect to $X_0$, Proposition 2.2.9 implies that $K^0 > 0$. Lightlikeness of $K$ yields

$$K^0 = |\vec{K}|.$$

Similarly Proposition 2.2.9 and timelikeness of $Z'$ implies that

$$Z'^0 > |\vec{Z}'|.$$

Now by definition of the projection $K^\|\$ of $K$ onto the subspace $\mathbb{R}Z'$ we have

$$g(K^\|, Z') = g(K, Z') = K^0 Z'^0 - \vec{K} \cdot \vec{Z}'$$

Again by Proposition 2.2.9, $K$ is future directed with respect to $Z'$ if and only if the expression is strictly positive. So the Cauchy-Schwartz inequality applied to

$$\vec{K} \cdot \vec{Z} \leq |\vec{K}| |\vec{Z}'| < K^0 Z'^0$$

yields the assertion.

“ii)”: The idea of proof is to continuously transform the projections of the $K_a$s onto the orthogonal subspaces into each other and observe that no reflection can occur in this process.

The basis $X$ induces a spacetime orientation $O'$ on $\mathcal{V}$. Since time inversion leaves the space orientation invariant, we may assume that $X' = X \cdot A$. Now, since $\text{CLor}_{n+1}$ is (path-)connected, there exists a continuous function

$$B : [0, 1] \to \text{CLor}_{n+1} : s \to B(s)$$

such that $B(0) = 1$ and $B(1) = A$. Define now the ‘intermediate’ bases $Y' : [0, 1] \to O'$ via $Y'(s) := X \cdot B(s)$ for $s \in [0, 1]$. This gives rise to corresponding orthogonal projections

$$\pi^\perp : [0, 1] \to \text{End} \mathcal{V} : s \to \pi^\perp (s)$$

with respect to $Y_0$. These can be written as

$$\pi^\perp = Y_a \otimes Y^a,$$  \hspace{1cm} (2.2.24)
where we sum over \( a \in \{1, \ldots, n\} \). Now collect the \( K_a \)'s into

\[
K := K_a \otimes e^a
\]

in order to define

\[
K^\perp (s) := \pi^\perp (s) \cdot K = K^\perp_a (s) \otimes e^a
\]

for each \( s \in [0, 1] \). By assumption, \( K^\perp (0) \) is right handed with respect to \( X = Y (0) \) in the sense that there exists a \( C \in \text{GL}_n^+ \) such that

\[
K^\perp (0) = X \cdot \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}.
\]

This motivates us to define the (matrix) function \( D \) via

\[
K^\perp = Y \cdot \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix},
\]

which necessarily exists as a transformation in the \( Y_1, \ldots, Y_n \) hyperplane. Use (2.2.24) to show this algebraically. Due to

\[
\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} = \bar{Y} \cdot \pi^\perp \cdot K,
\]

\( D \) is continuous and invertible, i.e. we may write \( D: [0, 1] \to \text{GL}_n \). Since \( D (0) = C \in \text{GL}_n^+ \), \( \text{GL}_n^+ \) is connected in \( \text{GL}_n \), and \( D \) is continuous, we have \( D (1) \in \text{GL}_n^+ \). This proves the assertion.

If \((V, g)\) is a Lorentzian vector space that is also Lorentz oriented via \( O \), then we call \( O \) compatible with \( g \), if the causal character of vectors with respect to \( g \) and \( O \) coincide. That is, if \( g' \) is one of the Lorentz products induced by \( O \), then \( O \) is compatible with \( g \), if and only if one of the equivalent conditions of Theorem 2.2.7 holds. So a Lorentz-oriented Lorentz vector space \((V, g, O)\) is a Lorentz vector space \((V, g)\) together with a Lorentzian orientation \( O \) compatible with \( g \). Space oriented, time oriented, spacetime oriented and oriented Lorentz vector spaces are defined accordingly, hence ignoring the possibility that Lorentzian orientations and Lorentz products need not be compatible. If \((V, g, O)\) is a time-oriented Lorentz vector space and \( c \in \mathbb{R}_+ \) is a particular distinguished number, e.g. 1 or the speed of light (in vacuum), then a vector \( Z \in V \) is called an observer vector, if \( Z \) is future directed timelike and \( g (Z, Z) = c^2 \). The motivation for this definition will become apparent in chapter 3. At this point we are finished with the discussion of the linear model.

Knowing this, the notions of Lorentzian orientations \( O \) on manifolds \( Q \) follow immediately from the definition of a \( G \)-structure for the respective open Lie subgroups \( \mathcal{G} \) of \( \text{CO}_{1,n} \). In this manner we obtain space-, time-, spacetime- and causal orientations on manifolds. Since the fiber \( O_q \) of \( O \) at each \( q \in Q \) is a linear Lorentzian orientation on the tangent space \( T_q Q \), a Lorentzian orientation on a manifold yields a classification of tangent vectors into timelike, spacelike, future directed, etc. depending on the respective group \( \mathcal{G} \). Common to all these particular cases is that they induce a causal structure, that is a \( \text{CO}_{1,n} \)-structure on \( Q \). The following lemma gives a condition for the existence thereof and hence a necessary condition for the existence of Lorentzian orientations.
Lemma 2.2.11 (Existence of causal structures on manifolds)

Let $Q$ be a smooth $(n+1)$-manifold. Then $Q$ admits a $CO_{1,n}$-structure if and only if it admits an $O_{1,n}$-structure. ◊

Proof “$\iff$”: This is trivial, since $O_{1,n} \subset CO_{1,n}$ is a Lie subgroup.

"$\implies$": We need to show the existence of an $O_{1,n}$-reduction $\mathcal{P}'$ of some $CO_{1,n}$-structure $\mathcal{P} \subset Fr(TQ)$. By Remark 2.2.5i) on page 14, we can construct $\mathcal{P}$ by taking appropriate local sections and making sure that the transition functions take values in $O_{1,n}$. So we choose a trivializing cover $\{U_\alpha|\alpha \in I\}$ with frame fields $\alpha X \in \Gamma^\infty(U_\alpha,\mathcal{P})$ and transition functions $\alpha \beta A \in C^\infty(U_\alpha \cap U_\beta,CO_{1,n})$ for all $\alpha, \beta \in I$ with non-empty $U_\alpha \cap U_\beta$. From the Lie group isomorphism between $CO_{1,n}$ and $\mathbb{R}^+ \times O_{1,n}$ (cf. (2.2.20) on page 23), we are motivated to define

$$\Lambda := \det A^{-1/(n+1)} \alpha \beta^\alpha$$

taking values in $O_{1,n}$. Employing the $\alpha X$ as sections of $\mathcal{P}'$ and checking the cocycle condition (2.2.8) for the $\Lambda$ yields the result. ■

Since the choice of transition functions for the $CO_{1,n}$-structure in the above proof was arbitrary, a causal structure on $Q$ does in general not induce a unique Lorentzian metric $g$. Yet as in the linear case, any causal structure uniquely determines the metric up to a strictly positive factor, i.e. if $\mathcal{P}, \mathcal{P}'$ are $O_{1,n}$-reductions of a causal structure on an $(n+1)$-manifold $Q$ with respective Lorentzian metrics $g$ and $g'$, then there exists a strictly positive function $f \in C^\infty(Q,\mathbb{R}^+)$ such that

$$g' = fg.$$

The transformation from $g$ to $g'$ is called a conformal transformation and then $g, g'$ are called conformally equivalent. It is elementary to show that this is indeed an equivalence relation on the set of Lorentzian metrics on a manifold $Q$. Conversely, if a Lorentzian metric $g$ on $Q$ is given up to conformal equivalence, this uniquely determines a causal structure on $Q$. If we consider Lorentzian orientations instead, then a Lorentzian metric alone is not enough to reconstruct the orientation. However, we may ask for a given Lorentzian orientation $\mathcal{O}$ to be compatible with a Lorentzian metric $g$ by requiring that some (and hence every) metric induced by $\mathcal{O}$ is conformally equivalent to $g$. So we define a time, space or spacetime oriented Lorentzian manifold to be a tuple $(Q,g,\mathcal{O})$ such that $(Q,g)$ is a Lorentzian manifold and $\mathcal{O}$ is a compatible time, space or spacetime orientation on $Q$, respectively. Similarly, an oriented Lorentzian manifold can be viewed as a Lorentzian manifold equipped with a compatible causal orientation.

Spacetime oriented Lorentzian manifolds are the main objects of interest in chapter 3, so we complete this section with a statement on their existence.

Proposition 2.2.12 (Existence of spacetime orientations)

Let $(Q,g)$ be a Lorentzian manifold. Then it admits a compatible spacetime orientation if and only if it is orientable and there exists a global time-like vector field. ◊
2 Mathematical Preliminaries

Proof \( \Rightarrow \): Since \( \text{CLor}_{n+1} \subset \text{GL}^+_{n+1} \) for each \( n \in \mathbb{N} \), the spacetime orientation \( O \) on the \((n+1)\)-manifold \( Q \) can be extended to an ordinary orientation on \( Q \). For the existence of the vector field we note that \( O \) yields a time-orientation on \( Q \) and then a folklore theorem in relativity states that the existence of a time orientation is equivalent to the existence of a global timelike vector field (see e.g. [24, Lem. 5.32]). The proof thereof is essentially a partition of unity argument employing the fact that the sum of two future-directed timelike vector fields is again future-directed timelike.

\( \Leftarrow \): Without loss of generality, we may assume that the global timelike vector field \( Z \) is normalized with respect to \( g \). Now choose an at most countable trivializing cover \( \{U_\alpha | \alpha \in I\} \) such that \( X_\alpha \) are smooth orthonormal frame fields over each \( U_\alpha \). By applying appropriate Lorentz transformations, we may assume

\[
Z|_{U_\alpha} = X_0
\]

for each \( \alpha \in I \). Moreover, if \( X_\alpha \) is not right handed, we may multiply by \(-1 \in O_{1,n}\) to make it right handed. Now, for each \( \alpha, \beta \in I \) with non-empty \( U_\alpha \cap U_\beta \) there exists an \( \alpha \beta \) \( \Lambda \in C^\infty (U_\alpha \cap U_\beta, O_{1,n}) \) such that

\[
X_\beta = X_\alpha \cdot \alpha \beta \Lambda .
\]

Since each \( X_\alpha \) is right-handed, \( \det \Lambda > 0 \). By (2.2.25), \( \Lambda'_{0} = \delta_{0}^{0} \) and as it maps into \( O_{1,n} \), we also have \( \Lambda'_{0} = \delta_{0}^{0} \). Thus \( \Lambda \) takes values in the product Lie group \( \{1\} \times \text{SO}_n \). As this is a subgroup of \( \text{CLor}_{n+1} \), extension yields the assertion. \( \blacksquare \)

Indeed, the proof shows that a spacetime oriented Lorentzian \((n+1)\)-manifold admits a \( \{1\} \times \text{SO}_n \)-reduction of its frame bundle. It needs to be stressed, however, that this reduction is highly non-unique and thus, at least without further physical motivation of the choice of global vector field, a mere mathematical tool without intrinsic meaning. Moreover, we may combine Proposition 2.2.12 with Lemma 2.2.11 to conclude that a manifold can be equipped with a Lorentzian metric and a compatible spacetime orientation if and only if it is orientable and admits a global nowhere-vanishing vector field.\(^4\)

2.3 Connections on the Tangent Bundle

In this section we discuss the geometry of tangent bundles equipped with a covariant derivative. We approach the subject by first recapitulating the theory of Ehresmann connections on fiber bundles and then relating this general point of view to covariant derivatives on the tangent bundle. By taking this top-down perspective, we intend to convey an adequate understanding of the notion of connectors. This in turn is a prerequisite for understanding the detailed geometry of the space-time splitting in terms of Jacobi fields. Appropriate references are given in the respective subsections.

\(^4\)In the reverse implication we employed the fact, that the existence of a nowhere-vanishing vector field \( X \) on \( Q \) implies the existence of a Lorentzian metric \( g \). To show this, choose a Riemannian metric \( h \) on \( Q \) and assume for convenience that \( h(X, X) = 1 \). Then \( g := 2h \cdot X \otimes X \cdot h - h \) is a Lorentzian metric with respect to which \( X \) is timelike and all vectors in \( \ker (X \cdot h) \) are orthogonal to \( X \).
2.3 Connections on the Tangent Bundle

2.3.1 Ehresmann Connections

As stated above, we recollect the main theory of Ehresmann connections on fiber bundles here. A more in-depth treatment of connections and the closely related concept of parallel transport can be found in the German book by Baum [2] as well as the English ones by Poor [26] and Rudolph et al [32].

So let $\mathcal{E}$ be a fiber bundle over a manifold $Q$ with bundle projection $\pi: \mathcal{E} \to Q$. Then for every $q \in Q$ the fiber $\pi^{-1}\{q\}$ is an embedded submanifold of $\mathcal{E}$ (diffeomorphic to the typical fiber) due to the regular value theorem. Moreover, the fact that $\pi$ is a submersion also implies that the kernel of $\pi_*$, denoted by $\ker \pi_*$, defines a (smooth, regular geometric) distribution on the manifold $\mathcal{E}$. $\mathcal{V} := \ker \pi_*$ is called the vertical distribution. In accordance, we call a vector $X \in T\mathcal{E}$ vertical, if it is tangent to $\mathcal{V}$. Again by the regular value theorem, vertical vectors in $T\mathcal{E}$ are precisely those that are tangent to the fiber over their base point in $Q$. Comparing dimensions, this shows that the fibers are the integral manifolds of the vertical distribution, i.e. $\ker \pi_*$ is integrable.

One can now ask for a complementary distribution on $\mathcal{E}$, i.e. a smooth distribution $\mathcal{H}$ on $\mathcal{E}$ such that

$$T\mathcal{E} = \mathcal{V} \oplus \mathcal{H},$$

where $\oplus$ is the Whitney sum. The Whitney sum amounts to taking the fiber-wise direct sum of vector spaces and equipping the resulting set with a ‘natural’ manifold structure. A distribution $\mathcal{H}$ satisfying (2.3.1) is called an Ehresmann connection or, equivalently, a horizontal distribution. They are highly non-unique and in the generic case not integrable. Obviously, we call a vector horizontal, if it is tangent to $\mathcal{H}$.

If we wish to take a less abstract perspective, an Ehresmann connection $\mathcal{H}$ can be constructed from a vertical projection (endomorphism field) $\pi^\mathcal{V}$. This is a (smooth) tensor field on $\mathcal{E}$ taking values in $\text{End} (T\mathcal{E}) = T\mathcal{E} \otimes T^*\mathcal{E}$ and satisfying $\pi^\mathcal{V} \cdot \pi^\mathcal{V} = \pi^\mathcal{V}$ as well as $\pi^\mathcal{V} \cdot X \in \mathcal{V}$ for every $X \in T\mathcal{E}$. The horizontal projection (endomorphism field) $\pi^\mathcal{H}$ is then just

$$\pi^\mathcal{H} := 1 - \pi^\mathcal{V}. $$

Since $\pi^\mathcal{V}$ has constant rank, so does $\pi^\mathcal{H}$ and therefore the equation

$$\mathcal{H} := \pi^\mathcal{H} (T\mathcal{E}) = \ker \pi^\mathcal{V}$$

defines an Ehresmann connection on $\mathcal{E}$.

We thus conclude that connections exist on general fiber bundles $\mathcal{E}$. However, if the fiber bundle itself has a ‘symmetry’, one usually requires the induced ‘infinitesimal symmetry’ to carry over to $\mathcal{H}$. We give meaning to this statement in the following section.

2.3.2 Covariant Derivatives & Connectors

As the tangent bundle $TQ$ of a manifold $Q$ is a fiber bundle, we may equip it with an Ehresmann connection. Our choice is, however, not arbitrary, but we wish to relate it to the concept of covariant derivative in this section. Note that the following treatment can be generalized to arbitrary vector bundles in a straightforward manner, but we are primarily interested in the specific case of the tangent bundle. We recommend the books by Poor [26, 2.49 ff.], Burns and Gidea [8, §5.8] as well as the one by the group of French mathematicians under the pseudonym 'Arthur Besse' [4, 1.59 ff.] as references.
for connectors and vector bundle connections. The first book [8] provides a coherent
motivation, the second one [26] gives a good abstract definition and embeds it into the
general theory, while the third one [4] shows the relation to sprays and the symplectic
point of view.

To start off, consider a curve \( X: \mathcal{I} \to TQ \) in the tangent bundle of an \( n \)-manifold \( Q \),
equipped with a covariant derivative \( \nabla \), and let \( \pi: TQ \to Q \) be the bundle projection.
Intuitively, \( X \) can be thought of as the curve

\[
\gamma := \pi \circ X: \mathcal{I} \to Q: \tau \mapsto \gamma (\tau) = \pi (X_\tau)
\]

with a vector \( X_\tau \) attached to it for every \( \tau \). Recalling the terminology introduced in
section 2.1, \( X \) is a vector field along \( \gamma \). Note that \( X_\tau \) does not need to be tangent to \( \gamma \). If
\( X \) is parallel transported along \( \gamma \), then by definition it satisfies

\[
\left( \frac{\nabla X}{d\tau} \right)_\tau = 0 \tag{2.3.2}
\]

for every \( \tau \in \mathcal{I} \). Moreover, if the image of \( \gamma \) is contained in the domain \( U \) of a coordinate
map \( \kappa \), then equation (2.3.2) reads in those coordinates

\[
\dot{v}^k (\tau) + \Gamma^k_{ij} (\gamma (\tau)) \dot{\kappa}^i (\tau) v^j (\tau) = 0 . \tag{2.3.3}
\]

Here the \( \Gamma^k_{ij} \)'s are the connection coefficients with indices \( i, j, k \in \{1, \ldots, n\} \), the \( \kappa 's \) and
\( v^k \)'s are the components of \( X \) with respect to the coordinate map \( \kappa \) and the respective
coordinate vector field \( \partial/\partial \kappa^k \), and the dot denotes differentiation of the components with
respect to \( \tau \). Employing an analogy in Newtonian physics, equation (2.3.3) shows that
we can tell whether a vector \( X_\tau \) is (infinitesimally) parallel at time \( \tau \) by knowing the
position \( \kappa (\tau) \) and velocity \( \dot{\kappa} (\tau) \) of \( \gamma \) as well as the vector \( v (\tau) \in \mathbb{R}^n \) and its rate of change
\( \dot{v} (\tau) \in \mathbb{R}^n \) at time \( \tau \).

Now recall that the chart \((U, \kappa)\) on \( Q \) induces the canonical bundle chart \((\pi^{-1}(U),\)
\((\kappa \circ \pi, v))\) on \( TQ \) as follows: If \( Y = Y^i \partial_i |_q \) is a vector in \( \pi^{-1}(U) \subseteq TQ \) with base
point \( \pi (Y) = q \in U \), then \( (\kappa \circ \pi, v) (Y) = (\kappa (q), v^i e_i) \in \mathbb{R}^n \). Note that we do not
notationally distinguish between \( \kappa \) and \( \kappa \circ \pi \) in order to avoid cluttery formulas. It is also
common to write \( \dot{\kappa} \) instead of \( \dot{v} \) for obvious reasons.

If we repeat this procedure for the double tangent bundle \( \mathcal{T}TQ \to TQ \), we get canonical bundle coordinates \((\kappa, v, \dot{\kappa}, \dot{v})\) on \( \mathcal{T}TQ \) induced by the
chart \((U, \kappa)\). With respect to these coordinates and for every \( \tau \in \mathcal{I} \) the vector \( X_\tau := X_\pi ((\partial/\partial \tau)\kappa) \in \mathcal{T}TQ \) takes precisely the form \((\kappa (\tau), v (\tau), \dot{\kappa} (\tau), \dot{v} (\tau))\) as before. This
is true for any canonical bundle chart induced by a chart on \( U \subseteq Q \). So from an invariant
perspective, the vector \( X_\tau \) in the double tangent bundle \( \mathcal{T}TQ \) suffices to check whether
\( X_\tau \) is (infinitesimally) parallel at time \( \tau \). Namely, we evaluate \( X_\tau \) in arbitrary canonical
bundle coordinates and check whether it satisfies

\[
\dot{v}^k + \Gamma^k_{ij} (q) \dot{\kappa}^i v^j = 0 , \tag{2.3.4}
\]

where \( q := \pi \circ \pi' (X_\tau) \). As every \( Z \in \mathcal{T}TQ \) can be considered the tangent vector of
such a curve \( X \) in \( TQ \) at some \( \tau \), we obtain a condition for elements of \( \mathcal{T}TQ \) to be
'(infinitesimally) parallel'.
2.3 Connections on the Tangent Bundle

In fact, we have implicitly constructed a smooth map \( K: T^2 Q \to T Q \), which is locally given by
\[
(\kappa, v, \dot{\kappa}, \dot{v}) \mapsto \left( \kappa, \left( \dot{v}^k + \Gamma^k_{ij}(q) \kappa^i \dot{v}^j \right) e_k \right)
\]
with \( q := \pi \circ \pi'(Z) \). Moreover, for arbitrary curves \( X: \mathcal{I} \to T Q \) and all \( \tau \in \mathcal{I} \) it satisfies
\[
K(\dot{X}_\tau) = \left( \nabla X \frac{d\tau}{d\tau} \right)_\tau.
\]
Condition (2.3.6) gives an indirect global definition of \( K \) and conversely, if such a \( K \) is given, this uniquely determines a covariant derivative or, equivalently, a Koszul connection \( \nabla \) on \( T Q \). The converse follows from the fact that \( X \) is arbitrary. So we find that \( K \) defined via (2.3.6) is a uniquely defined (smooth) \( T Q \)-valued 1-form on \( T Q \), locally given by
\[
K = \Gamma^k_{ij} v^i \frac{\partial}{\partial \kappa^k} \otimes d\kappa^i + \frac{\partial}{\partial \kappa^i} \otimes dv^i
\]
in accordance with (2.3.5). We call \( K \in \Omega^1(T Q, T Q) \) as defined by (2.3.6) a connector (on the tangent bundle \( T Q \) induced by \( \nabla \)). If \( \nabla \) is the Levi-Civita connection, then \( K \) is called the Levi-Civita connector.

From the global (2.3.6) or the local expression (2.3.7) of \( K \), we directly observe that it has constant rank \( n \). Hence its kernel determines a smooth rank \( n \) distribution \( \ker K \) on \( T Q \). From the local expression of
\[
\pi_* = \frac{\partial}{\partial \kappa^i} \otimes d\kappa^i
\]
we deduce that a vertical vector \( Y \in \pi'^{-1}\left( \pi^{-1}(U) \right) \subset T T Q \) can be written as
\[
Y = Y^i \frac{\partial}{\partial v^i} \bigg|_{\pi'(Y)}
\]
with \( Y^i := \dot{v}^i(Y) \) and \( \kappa^i := \kappa^i(Y) = 0 \) for each \( i \in \{1, \ldots, n\} \). As canonical bundle charts exist everywhere on \( T T Q \), a vector \( Y \in V \subset T T Q \) is vertical and satisfies \( K(Y) = 0 \) if and only if \( Y = 0 \). This proves that \( \mathcal{H} := \ker K \) defines a horizontal distribution on \( T Q \).

In canonical bundle coordinates horizontal vectors \( Y \in \pi'^{-1}\left( \pi^{-1}(U) \right) \) with \( \pi'(Y) = y \) and \( \pi(y) = q \) take the form
\[
Y = \dot{Y}^i \frac{\partial}{\partial \kappa^k} \bigg|_y - \Gamma^k_{ij}(q) \dot{\kappa}^i y^j \frac{\partial}{\partial \kappa^k} \bigg|_y,
\]
(2.3.8)
and thus precisely those vectors $Y \in TTQ$ are horizontal, that are \(\text{'(infinitesimally) parallel'}\) in the aforementioned sense of satisfying (2.3.4). This is the reason why $\mathcal{H}$ is called a connection. Moreover, since $\ker K = \ker \pi^V = \mathcal{H}$ we can explicitly determine the local expression of the vertical projection in canonical bundle coordinates

$$\pi^V = \Gamma^k_{ij} v^j \frac{\partial}{\partial v^k} \otimes dk^i + \frac{\partial}{\partial v^i} \otimes dv^i$$

(2.3.9)

and thus the local expression for the horizontal projection reads

$$\pi^H = 1 - \pi^V = \frac{\partial}{\partial \kappa^i} \otimes d\kappa^i - \Gamma^k_{ij} v^j \frac{\partial}{\partial v^k} \otimes d\kappa^i.$$  

(2.3.10)

We would now like to consider the reverse construction. So assume we are given a horizontal distribution $H$ in the sense of section 2.3.1 on page 31 with vertical projection $\pi^V$ on the tangent bundle. We would like $\mathcal{H}$ to determine a unique $1$-form $K \in \Omega^1(TQ, TQ)$, which should be a 'connector' in the sense that we can use it to construct a Koszul connection on $TQ$ via equation (2.3.6). In general an element $K \in \Omega^1(TQ, TQ)$ can be locally written as

$$K = K^i_j \frac{\partial}{\partial \kappa^i} \otimes d\kappa^j + v^i K^i_j \frac{\partial}{\partial \kappa^i} \otimes dv^j$$

(2.3.11)

with smooth functions $K^i_j, v^i K^i_j$ on $\pi^{-1}(U)$ and indices $i, j \in \{1, \ldots, n\}$. Obviously, we would like to find invariant conditions such that $K$ takes the form (2.3.7) – which is equivalent to $\pi^V$ taking the form (2.3.9) (as $\ker \pi^V = \ker K$) and then we just set $\mathcal{H} = \ker \pi^V = \ker K$. Those conditions on $K$ might then in turn lead to restrictions on $H$, i.e. we only accept those Ehresmann connections $\mathcal{H}$ on $TQ$ that can be used to construct a sensible $K$.

To find such invariant conditions on $K$, consider the multiplication map

$$M : \mathbb{R} \times TQ \to TQ : (\lambda, X) \to M_\lambda (X) := \lambda X$$

(2.3.12)

and the vertical lift $\tilde{X}_Y$ of $X \in TQ$ at $Y \in \pi^{-1}(\{\pi(X)\})$, as defined by

$$\tilde{X}_Y (f) := \left. \frac{\partial}{\partial s} \right|_0 f(Y + sX)$$

(2.3.13)

for every $f \in C^\infty(TQ, \mathbb{R})$. It is straightforward to show that (2.3.13) defines a vector field $\tilde{X}$ over the fiber $\pi^{-1}(\{\pi(X)\})$. In particular $\tilde{X}$ is vertical at each point, hence the name vertical vector field. Fixing $\lambda \in \mathbb{R}$ and looking at $M_\lambda$ in canonical bundle coordinates induced by the chart $(U, \kappa)$ on $Q$, it is locally given by

$$\left. (\kappa, v) \mapsto (\kappa, \lambda v) \right.$$ 

and thus over $\pi^{-1}(U)$

$$(M_\lambda)_* = \frac{\partial}{\partial \kappa^i} \otimes \kappa^i + \lambda \frac{\partial}{\partial v^i} \otimes v^i.$$ 

Combining this with our desired expression (2.3.7) for $K$, we find $M^*_\lambda K = \lambda K$. Similarly, we find for $q \in U, X = X^i \left. (\partial/\partial \kappa^i) \right|_q \in T_qQ$

$$\tilde{X} = X^i \frac{\partial}{\partial v^i},$$

34
and combining this again with (2.3.7), we get $K(\tilde{X}) = X$. Thus to get our desired expression (2.3.7) from the general one (2.3.11), $K$ necessarily has to satisfy

$$M^*_\lambda K = \lambda K \quad \forall \lambda \in \mathbb{R}, \text{ and } K(\tilde{X}) = X \quad \forall X \in TQ.$$  \hspace{1cm} (2.3.14)

In fact, starting from (2.3.11) the second condition yields $K'_{ij} = \delta^i_j$ and the first condition implies that each $\tilde{K}^i_j$ is a first degree homogeneous polynomial in the components of $v$. Hence these two conditions specify $K$ modulo the (consistent) choice of the functions $t^k_{ij}$ in each chart. As the second condition is only relevant for vertical vectors, it does not put any restriction on $\mathcal{H} = \ker K$. Regarding the first one, we observe that for every horizontal $Y \in TTQ$ and $\lambda \in \mathbb{R}$

$$M^*_\lambda K (Y) = K ((M_\lambda)_* Y) = \lambda K (Y) = 0,$$

hence $(M_\lambda)_* Y \in \mathcal{H}$ and thus $(M_\lambda)_* \mathcal{H} = \mathcal{H}$. This is the sought-after condition. In particular, for $X \in TQ$ we can define $\mathcal{H}_X := \mathcal{H} \cap T_X TQ$ and then

$$(M_\lambda)_* \mathcal{H}_X = \mathcal{H}_\lambda X$$  \hspace{1cm} (2.3.15)

for every $\lambda \in \mathbb{R}$.

We have thus motivated a natural definition of the word ‘tangent bundle connection’ and of ‘connectors’ induced by them.

**Definition 2.3.1 (Tangent bundle connection)**

Let $\mathcal{H}$ be an Ehresmann connection on the tangent bundle $\pi: TQ \to Q$ of a manifold $Q$. Then $\mathcal{H}$ is called a **tangent bundle connection**, if (2.3.15) holds for every $X \in TQ$ and $\lambda \in \mathbb{R}$. An element $K \in \Omega^1 (TQ, TQ)$ is called a **connector (on the tangent bundle)**, if it satisfies (2.3.14). A connector is said to be **induced by a tangent bundle connection** $\mathcal{H}$, if $\mathcal{H} = \ker K$.

If we replace $TQ$ by a general (real) vector bundle $E$ in Definition 2.3.1, we obtain a definition of vector bundle connections and connectors on vector bundles. As noted before, one commonly requires Ehresmann connections on particular fiber bundles to satisfy an additional ‘infinitesimal symmetry condition’. For the tangent bundle and analogously for general vector bundles this condition is given by (2.3.15). Colloquially speaking, the condition guarantees that the Ehresmann connection respects the vector bundle structure of $TQ$.

Summing up, a Koszul connection $\nabla$ on $TQ$ gives rise to a connector $K$, whose kernel defines a tangent bundle connection $\mathcal{H}$. Conversely, a tangent bundle connection $\mathcal{H}$ can be considered as a choice of vertical projection $\pi^V$, which directly induces a corresponding connector $K$ on $TQ$. Locally this works via (2.3.9) and (2.3.7). The connector then gives rise to a Koszul connection $\nabla$ on $TQ$ via (2.3.6) and a corresponding notion of parallel transport.

### 2.4 Jacobi Fields and the Lorentzian Exponential

In the following we review the notion of geodesic variation, its relation to Jacobi fields and the exponential map within Lorentzian geometry. This will be essential in our construction.
2 Mathematical Preliminaries

later. For a discussion of Jacobi fields in Lorentzian geometry going beyond the material presented here, we refer to the books by Beem, Ehrlich and Easley [3, Chap. 10], O’Neill [24, Chap. 8 & 10], and Hawking and Ellis [16, Chap. 4 & 8]. A coherent introduction to the topic in the context of Riemannian geometry can also be found in the book by Burns and Gidea [8, §4.5, §5.5, §5.7, §5.9], as well as the one by Sakai [34, Chap. III & IV].

Throughout this section \((Q, g)\) is a (smooth) Lorentzian manifold. Although the discussion applies more generally to pseudo-Riemannian manifolds, we are only interested in the Lorentzian case.

Recall that for any Koszul connection \(\nabla\) on the tangent bundle, we call a (smooth) curve \(\gamma: I \to Q\) an autoparallel, if its tangent vector is parallel to itself, i.e.

\[
\left(\frac{\nabla \dot{\gamma}}{dt}\right)_r = 0 \quad \forall r \in I.
\]

(2.4.1)

In the Lorentzian case, we commonly take \(\nabla\) to be the Levi-Civita connection with respect to \(g\) and, neglecting conceptual subtleties, we call autoparallels geodesics. Intuitively, they are straight lines in a curved geometry. Metricity of \(\nabla\) implies that for geodesics \(\gamma\) the function \(g(\dot{\gamma}, \dot{\gamma})\) is a constant. By definition, a curve \(\gamma\) in \(Q\) is time-, light- or spacelike, if each tangent vector \(\dot{\gamma}_r\) is time-, light- or spacelike, respectively. So non-trivial geodesics are either time-, light- or spacelike.

Often we are not just interested in one single geodesic, but also in its behavior with respect to a parameter: If \(\gamma: I \to Q\) is a curve and for some \(\epsilon \in \mathbb{R}_+\) the (smooth) map

\[
\theta: (-\epsilon, \epsilon) \times I \to Q: (s, r) \to \theta_s(r)
\]

(2.4.2)
satisfies \(\theta_0 = \gamma\), then we call \(\theta\) a variation of \(\gamma\). If in addition \(\gamma\) itself and \(\theta_s\) are geodesics for all \(s \in (-\epsilon, \epsilon)\), we call \(\theta\) a geodesic variation of \(\gamma\). More generally, one might want to let the domain of \(\theta\) be an open, connected subset of \(\mathbb{R}^2\), but, as the definition (2.4.2) is fairly standard in the literature and this is only a brief review, we shall not consider this case.

Given a geodesic variation \(\theta\) of (a geodesic) \(\gamma\), we are naturally led to consider two particular tangent vectors at the point \(\gamma(r) = \theta_0(r)\), namely \(\theta_s(\partial/\partial r)_{(0, r)}\) and \(\theta_s(\partial/\partial s)_{(s, 0)}\). The former gives the tangent vector \(\dot{\gamma}_r\) at parameter value \(r\) and the second one gives its ‘infinitesimal displacement’ with respect to the variation. Recalling the terminology from section 2.1, the smooth mapping

\[
J: I \to TQ: r \to J_r := \left(\theta_s \frac{\partial}{\partial s}\right)_{(0, r)}
\]

is vector field along \(\gamma\). If we define the Riemann tensor field \(\mathcal{R}\) via

\[
\mathcal{R}(X, Y) Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z
\]

for \(X, Y, Z \in \mathfrak{X}(Q)\), then one can show (cf. [8, Thm. 5.5.3; 24, Chap. 8, Lem. 3]) that \(J\) satisfies

\[
\frac{\nabla^2 J}{dt^2} + \mathcal{R}(J, \dot{\gamma}) \dot{\gamma} = 0.
\]

(2.4.3)

Equation (2.4.3) is known as the Jacobi equation and vector fields \(J\) over a geodesic \(\gamma\) satisfying it are known as Jacobi fields. As we have chosen the domain \(\text{dom} \theta\) to be in
2.4 Jacobi Fields and the Lorentzian Exponential

accordance with (2.4.2), not every Jacobi field along $\gamma$ gives rise to a geodesic variation of $\gamma$, but the statement is true for finite intervals $I$. We refer to [8, p. 207] for a counterexample and remark that equation (2.4.7) below gives an explicit formula for this variation. Since (2.4.3) is a second order, linear ordinary differential equation, knowing $J_{r_0}$ and its derivative $(\nabla J/dr)_{r_0}$ at some $r_0 \in I$ entirely determines the Jacobi field $J$ (cf. [24, Chap. 8, Lem. 5]). Moreover, knowing these tangent vectors we can easily compute $g(J, \dot{\gamma})$ without solving the Jacobi equation.

Lemma 2.4.1

Let $(Q, g)$ be a Lorentzian manifold and let $J$ be a Jacobi field over the geodesic $\gamma: I \rightarrow Q$ with $r_0 \in I$ and $\gamma(r_0) = q$. Then

$$g_{\gamma(r)}(J_r, \dot{\gamma}_r) = g_q\left(\left(\frac{\nabla J}{dr}\right)_{r_0}, \dot{\gamma}_{r_0}\right) (r - r_0) + g_q(J_{r_0}, \dot{\gamma}_{r_0})$$

(2.4.4)

for all $r \in I$.

**Proof.** This is an adaptation of Lemma 10.9 found in the book by Beem et al. [3]. We consider the left hand side of (2.4.4) and derive twice with respect to $r$, keeping in mind the metricity of $g$ and the fact that $\gamma$ is a geodesic. Then using the Jacobi equation and the symmetry properties of $R$ (cf. [24, Chap. 3, Prop. 36(2)]), we get

$$\frac{\partial^2}{\partial r^2} (g_{\gamma(r)}(J_r, \dot{\gamma}_r)) = g_{\gamma(r)}\left(\left(\frac{\nabla^2 J}{dr^2}\right)_r, \dot{\gamma}_r\right) = -g\left(R_{\gamma(r)}(J_r, \dot{\gamma}_r, \dot{\gamma}_r)\right) \equiv 0.$$ Integrating twice yields (2.4.4). 

As the Jacobi equation (2.4.3) depends on the Riemann tensor field, Jacobi fields yield implicit information on the curvature of the Lorentzian manifold $Q$. Indeed, there is an elaborate theory on the precise nature of this relation. We again refer to the books by Beem et al. [3, Chap. 10] and by O’Neill [24, Chap. 8] for further reading. The booklet by Penrose [25] should also be mentioned here.

Of particular interest in this theory are so-called conjugate points. Two points $q, q'$ on the image of a non-trivial geodesic $\gamma: I \rightarrow Q$ are called conjugate along $\gamma$, if there exists a non-trivial Jacobi field $J$ and $r_1, r_2 \in I$ with $r_1 \neq r_2$ such that $J$ vanishes at $r_1$ and $r_2$. In that case, $r_1, r_2$ are called conjugate values along $\gamma$. More generally, $q, q' \in Q$ are called conjugate points, if there exists a non-trivial geodesic $\gamma$ such that $q, q'$ are conjugate along $\gamma$. It is possible for a point to be conjugate to itself. Intuitively, conjugate points are intersection points of nearby geodesics starting at the same initial point. Mathematically, a second geodesic intersecting the original one twice need not exist.

We may use Lemma 2.4.1 to determine some properties of the Jacobi fields giving rise to conjugate points.

**Corollary 2.4.2**

Let $(Q, g)$ be a Lorentzian manifold and $\gamma$ be a geodesic. If $q$ is conjugate to $q'$ along $\gamma$, then the corresponding Jacobi field $J$ satisfies

$$J \perp \dot{\gamma}, \quad \frac{\nabla J}{dr} \perp \dot{\gamma}.$$
2 Mathematical Preliminaries

Proof Without loss of generality, assume $\gamma(0) = q$ and $\gamma(r') = q'$ with $r' > 0$. Now use (2.4.4) for $J_0 = 0$ and $J_{r'} = 0$ to determine the slope and summand, then derive once. ■

It is noteworthy that for lightlike geodesics a Jacobi field $J$ can be both parallel and orthogonal to $\dot{\gamma}$. However, if $J_r$ is timelike at any point $r \in \mathcal{I}$, then it is not orthogonal to $\dot{\gamma}_r$ and hence Corollary 2.4.2 states that it cannot vanish at two separate parameter values $r_1, r_2 \in \mathcal{I}$.

Jacobi fields frequently occur in the context of the exponential map

$$\exp: \text{dom} \exp \to \mathcal{Q}: Z \to \exp(Z) := \gamma_Z \quad (1)$$

with (dom exp) being the maximal set in $TQ$ such that the autoparallel $\tilde{\gamma}_Z$ with $\dot{\gamma}_0 = Z$ is defined. By re-parametrization, one shows that $\exp(sZ) = \gamma_Z(s)$ for all $s \in \mathbb{R}$ such that $sZ \in \text{dom} \exp$. It is often convenient to restrict exp to the fiber $T_q \mathcal{Q}$ at $q \in \mathcal{Q}$. Correspondingly, we call $\exp_q := \exp|_{T_q \mathcal{Q}}$ the exponential map at $q$. Recall now the definition of the vertical lift $\tilde{X}_Y$ of a vector $X \in T_q \mathcal{Q}$ at $Y \in T_q \mathcal{Q}$ (cf. (2.3.13) on page 34) and define the (smooth) addition map $P_Y$ by $Y$ on $T_q \mathcal{Q}$ via

$$P_Y(X) := X + Y,$$

which has inverse $P_{-Y}$. Then for any $f \in C^\infty(T_q \mathcal{Q}, \mathbb{R})$ we compute

$$((\exp \circ P_{-Y})_* \tilde{X}_Y)(f) = \tilde{X}_Y(f \circ \exp \circ P_{-Y}) = \frac{\partial}{\partial s} (f \circ \exp \circ P_{-Y})(Y + sX) \bigg|_{s=0} = \frac{\partial}{\partial s} f \circ \tilde{\gamma}_X(s) \bigg|_{s=0} = X(f).$$

This proves that the vertical lift at $Y$ is a (smooth) linear isomorphism and yields a direct inverse in terms of the exponential map, which is independent of the particular choice of $g$. Moreover, it shows that $(\exp_q)_*$ has full rank at 0 and hence (cf. [22, Prop. 4.1 & Thm. 4.5]) it is a diffeomorphism from an open neighborhood of 0 onto its image in $\mathcal{Q}$. Thus any coordinates around 0 in the tangent space $T_q \mathcal{Q}$ can be viewed as coordinates on $\mathcal{Q}$. An important instance are normal coordinates at $q$, which are given by linear coordinates on $T_q \mathcal{Q}$ with respect to an orthonormal basis. Normal coordinates are useful for ‘approximating the manifold around $q$’. For a more precise statement, including an explicit definition of normal coordinates, we refer to the book by O’Neill [24, Chap. 3, Prop. 33]. One may also adapt the treatment in reference [34, Chap. II, Prop. 3.1] to the Lorentzian case. Later we will use the existence of normal coordinates in the heuristic construction of the space-time splitting.

We continue with an analysis of the exponential map.

Proposition 2.4.3 (Domain of exponential map)

Let $(\mathcal{Q}, g)$ be a Lorentzian manifold with exponential exp as defined by the Levi-Civita connection. Then the domain $\text{dom} \exp$ is open in $T\mathcal{Q}$ and for every $q \in \mathcal{Q}$ the domain $\text{dom} \exp_q$ is open in $T_q \mathcal{Q}$. Moreover, $\text{dom} \exp_q$ is star-shaped around 0. ♦

Proof The full proof can be found in the book by O’Neill [24, Chap. 5, Cor. 4]. The idea is that exp can be defined in terms of a flow on the tangent bundle, called the geodesic

38
flow, and flow domains are open. Star-shapedness follows from the fact that \( \exp_q(sZ) \) is defined \( \forall s \in [0,1] \) whenever \( \exp_q(Z) \) is defined. ■

The relation to Jacobi fields arises when one asks for the differential of the exponential map and is given by the following theorem. We refer to section 2.3.2 for a discussion of the concept of connectors.

**Theorem 2.4.4 (Differential of exponential map)**

Let \((Q, g)\) be a Lorentzian manifold with exponential \(\exp\) and connector \(K\), as defined by the Levi-Civita connection. Denote by \(\pi: TQ \to Q, \tilde{\pi}: TTQ \to TQ\) the respective bundle projections.

Then for all \(Z \in TTQ\) such that \(\tilde{\pi}(Z) \in \text{dom} \exp\), we have

\[
\exp_* Z = J_1, \tag{2.4.6}
\]

where \(J: r \to J_r\) is the unique Jacobi field along the geodesic \(r \to \exp(r \tilde{\pi}(Z))\) with \(J_0 = \pi_* Z\) and \((\nabla J/ dr)_0 = K(Z)\). ♦

**Proof** A special case of this theorem can be found in the book by O’Neill [24, Chap. 8, Prop. 6]. The full theorem can be found in the book by Sakai [34, Lem. 2.2 & Lem. 4.3] and Burns et al [8, 5.9.2]. We shall give an intrinsic proof here.

Since \(Z \in TTQ\), there exists an \(\epsilon > 0\) and a smooth curve

\[ Y: (-\epsilon, \epsilon) \to TQ: s \to Y_s \]

with \(\dot{Y}_0 = Z\) projecting to the curve \(\gamma := \pi \circ Y\) on \(Q\). Therefore the map

\[ \theta: (-\epsilon, \epsilon) \times I \to Q: (s,r) \to \exp (rY_s) \tag{2.4.7} \]

with

\[ I := \{ r \in \mathbb{R} | \forall s \in (-\epsilon, \epsilon): rY_s \in \text{dom} \exp \} \neq \emptyset \]

is a geodesic variation of \(r \to \exp (r \tilde{\pi}(Z))\). Hence \(J_r := \theta_s (\partial/ ds)_{(0,r)}\) defines a Jacobi field \(J\). Recalling the multiplication map \(M\) (cf. (2.3.12) on page 34), we see that \(M_1 = 1_{TQ}\) and thus

\[ J_1 = (\exp \circ M_1 \circ Y)_* \frac{\partial}{\partial s} \bigg|_0 = \exp_* Y_0 = \exp_* \dot{Y}_0 \]

yields (2.4.6).

It remains to express \(J\) in terms of \(Z\). For any \(f \in C^\infty(Q, \mathbb{R})\) we calculate

\[ J_0 (f) = \frac{\partial}{\partial s} \bigg|_0 f \circ \theta_s (0) = \frac{\partial}{\partial s} \bigg|_0 f \circ \exp_{\gamma(s)} (0) = \dot{\gamma}_0 (f), \]

so \(J_0 = \dot{\gamma}_0 = (\pi \circ Y)_0 = \pi_* \dot{Y}_0 = \pi_* Z\). On the other hand

\[ \left( \pi_* J_0 \right) (f) = \frac{\partial}{\partial r} \bigg|_0 f \circ \pi \circ J_r = \frac{\partial}{\partial r} \bigg|_0 f \circ \exp (rY_0) = Y_0 (f), \]

so \(\pi_* J_0 = Y_0 = \tilde{\pi}(Z)\). To get the vertical parts, we calculate

\[ \dot{J}_0 (df) = \frac{\partial}{\partial r} \bigg|_0 df (J_r) = \frac{\partial}{\partial r} \bigg|_0 J_r (f) \]

yields (2.4.6).
2 Mathematical Preliminaries

\[ \frac{\partial}{\partial r} \bigg|_0 \frac{\partial}{\partial s} \bigg|_0 f \circ \theta_s (r) = \frac{\partial}{\partial s} \bigg|_0 \frac{\partial}{\partial r} \bigg|_0 f \circ \theta_s (r) \]

\[ = \frac{\partial}{\partial s} \bigg|_0 Y_s (f) = \frac{\partial}{\partial s} \bigg|_0 df (Y_s) = \dot{Y}_0 (df) , \]

hence Fl \( \dot{J}_0 \) = \( \dot{Y}_0 \). Finally, since the Levi-Civita connector is torsion-free,

\[ K (Z) = K \left( \dot{Y}_0 \right) = K \left( \text{Fl} \left( \dot{J}_0 \right) \right) = K \left( \dot{J}_0 \right) = \left( \nabla J \bigg|_0 \right) . \]

By Theorem 2.4.4, two points \( q \) and \( q' \) in \( Q \) are conjugate to each other if and only if there exists a vertical \( Z \in T T Q \) with base point \( Y \in T_q Q \) such that

\[ q' = \exp Y \text{ and } Z \in \ker \exp_* \].

In other words, the set of critical points of the exponential at \( q \)

\[ \text{crit exp}_q := \left\{ Y \in \text{dom exp}_q \subseteq T_q Q \mid \ker \left( (\exp_q)_* \big|_Y \right) \neq \{0\} \right\} \]

is given by

\[ \text{crit exp}_q = \left\{ Y \in \text{dom exp}_q \mid \exp_q (Y) \text{ is conjugate to } q \right\} . \quad (2.4.8) \]

For this reason, we call \( \text{crit exp}_q \) the \textit{conjugate locus at } \( q \). In the Riemannian case an analysis of this set has been carried out by Warner [104]. In the Lorentzian case one needs to distinguish between the \textit{time-, light-} and \textit{spacelike conjugate locus}, which is defined as the intersection of the conjugate locus at \( q \) with the respective subsets of \( T_q Q \). The following Lemma indicates their structure.

**Lemma 2.4.5 (Causal conjugate values are isolated)**

Let \((Q, g)\) be a Lorentzian manifold equipped with the Levi-Civita connection.

Then conjugate values along any time- or lightlike geodesic \( \gamma: \mathcal{I} \to Q \) are \textit{isolated}, i.e. for any \( r \) in the set of conjugate values \( S_0 \subset \mathcal{I} \) to some \( r_0 \in \mathcal{I} \) there exists an open neighborhood \( \mathcal{J} \) of \( r \) with \( \mathcal{J} \cap S_0 = \{r\} \). \( \Diamond \)

**Proof** The statement is a corollary of the fact that along any finite causal geodesic, the number of conjugate values with respect to a given value is finite. The proof thereof is part of so called Morse index theory and rather elaborate. It can be found in the book by Beem et al: See [3, Lem. 10.26 & Thm. 10.27] for the timelike case and [3, Prop. 10.76 & Thm. 10.77] for the lightlike case.

If \( S_0 \) is empty, we are done. So take any \( r_1 \in S_0 \) conjugate to \( r_0 \). By the above statement, on any finite open neighborhood \( \mathcal{J}' \) of \( r_1 \) there exist at most finitely many points in \( S_0 \cap \mathcal{J}' \). If there is none, set \( \mathcal{J} = \mathcal{J}' \). If there is at least one, Hausdorffness of \( \mathbb{R} \) (and thus of \( \mathcal{J}' \)) says that \( r_1 \) and any \( r_2 \in S_0 \cap \mathcal{J}' \) admit mutually disjoint open neighborhoods. Since there are only finitely many such \( r_2 \), the assertion follows. \( \blacksquare \)

Regarding the adaption of the above statement to spacelike geodesics, Helfer has constructed a counterexample in reference [74].
3 The Splitting Construction

The main objective of this chapter is to philosophically motivate and mathematically define the construction of splitting relativistic spacetimes into their spatial and temporal components. We give consistency proofs and examples along with the general theory.

In the first section, we give a definition of the word ‘spacetime’, introduce some elementary concepts required for the mathematical theory of relativity and provide some physically relevant examples. Section 3.2 is devoted to the heuristic motivation of the splitting construction. The reader only interested in the mathematical machinery is invited to skip this section, but the underlying philosophy is intended to convince the reader that the construction is ‘natural’ rather than ad hoc. Afterwards, we erect the mathematical theory in two steps: First, the ‘static splitting’ is considered in section 3.3. It derives its name from the fact that there is no time evolution in this setting. The second step is done in section 3.4 with the ‘kinematic splitting’, which allows for time evolution and thus constitutes an actual ‘space-time splitting’. Mathematically, the static splitting lays the foundation for the kinematic one, so we recommend to read them in this order.

3.1 General Considerations

As a brief introduction to the mathematical theory of relativity, this section provides a mathematical definition and motivation of the relativistic concept of spacetime along with the two physically most important examples. We also introduce observers, light cones and frames of reference. Apart from their general relevance within the theory of relativity, those are needed for the space-time splitting developed in the following sections.

The mathematical concept of spacetime admits a condensed definition, if we employ our findings from section 2.2.2 on Lorentzian structures, as well as the ones from section 2.2.3 on Lorentzian orientations.

Definition 3.1.1 (Spacetime)

A spacetime is a spacetime oriented Lorentzian manifold \((\mathcal{Q}, g, \mathcal{O})\) equipped with the Levi-Civita connection.

It should be stressed that the spacetime orientation \(\mathcal{O}\) is implicitly assumed to be compatible with the metric \(g\), i.e. they give rise to the same causal structure on \(\mathcal{Q}\). For simplicity, we often call \(\mathcal{Q}\) the spacetime rather than \((\mathcal{Q}, g, \mathcal{O})\). Furthermore, Definition 3.1.1 should be read from a categorical perspective in the sense that we do not distinguish between isomorphic spacetimes. Clearly, two (smooth) spacetimes \((\mathcal{Q}, g, \mathcal{O})\) and \((\mathcal{Q}', g', \mathcal{O}')\) are called isomorphic, if there exists a (smooth, global) diffeomorphism \(\varphi: \mathcal{Q} \rightarrow \mathcal{Q}'\) such that

\[\varphi^* g' = g\quad \text{and} \quad \varphi_* \mathcal{O} = \mathcal{O}'\].

41
The first condition means that $\varphi$ is an isometry and the second one says that it is spacetime-orientation preserving. We remark that for a frame $X$ on $Q$ and a diffeomorphism $\varphi: Q \to Q'$, $\varphi_*X$ is always a frame on $Q'$, so the second condition is well-defined.

The choice of Definition 3.1.1 is the middle path between physical sensibility and mathematical generality. In the following we shall give some justification to this claim along with a brief physical motivation. A complete physical justification of the mathematical concept of spacetime, if even possible, would derail the content of this work, so we limit ourselves to a few remarks regarding the main points. Nonetheless we wish to indicate the underlying principles and ideas that lead to this mathematical formulation. We refer to the books by Carroll [10], Kriele [19] and Wald [39] for similar motivations.

**Dimension:** It is an experiment we leave to the reader that, at least within humanly accessible realms, space is 3-dimensional and time is 1-dimensional. So if we wish to mathematically represent space and time as one ‘object’ called spacetime, the definition should respect this empirical fact. Obviously, in general relativity this is done by assuming the manifold $Q$ to be 4-dimensional.

Despite this, we have chosen not to fix the dimension of $Q$ in the definition, since it is mathematically inconvenient, the dimension is not important in any of the general definitions or proofs here and it is sometimes useful to consider lower dimensional ‘toy models’. Funnily enough, dimension 4 also has some particular mathematical significance, as it is the only dimension $n \in \mathbb{N}$ for which $\mathbb{R}^n$, equipped with the standard topology, does not have a unique smooth structure (up to diffeomorphism, cf. [22, p. 39sq.]). Though we do not use this here, it is certainly noteworthy. We refer to the books by Scorpan [36] and Asselmeyer-Maluga & Brans [1] regarding this fact.

**Manifold:** The choice to model space and time on a manifold is a mathematical expression of the old principle “Natura non facit saltus”, roughly translating into “nature does not make jumps”. This refers to both the need to model space and time on a continuum of points as well as the assumption that the dimension of space and time is fixed. While it appears to have become fashionable in these days to doubt this principle (see e.g. [9, 40]), there is, at least to our knowledge, no evidence to the contrary (see e.g. [66]). For instance, the discrete spectral lines of light emitted by atoms do provide conclusive evidence in favor of the theory of quantum mechanics, but it is a fallacy to read this as evidence in favor of the discreteness of space and time itself. Indeed, quantum mechanics itself is formulated on a so called Newtonian spacetime (see e.g. [99, §2]), so it can hardly be taken as a justification for discarding the principle (along with the manifold model).

**Lorentzian metric:** A manifold alone cannot be taken as a physical model of space and time, as it lacks an appropriate notion of ‘distance’. The choice to use a Lorentzian metric for this purpose primarily evolved out of a mathematical generalization of the special theory of relativity. In special relativity, one equips the spacetime manifold $Q = \mathbb{R}^4$ with a flat Lorentzian metric $g$, but Einstein realized that the assumption of flatness is ad hoc and the phenomenon of gravity could be explained by dropping it. So instead he had to formulate a law for the curvature of $g$, which is today known as the Einstein (field) equation. The flat case could then be recovered locally by considering it as the ‘tangent space approximation’ of the ‘true geometry’ via Lorentzian normal coordinates. Of course, if compared with Newtonian physics, this constitutes a serious weakening of the former.
principles that space is Euclidean and the rate of time is the same everywhere, regardless
of how the two are implemented into the theory in detail.

**Spacetime orientation:** Once the manifold $Q$ and the metric $g$ is chosen, it is still not
possible to distinguish between past and future in time or between right-handed and
left-handed in space. As indicated in section 2.2.3, a spacetime orientation $O$ is to be used
to make mathematical sense of both of these concepts in each tangent space of $Q$. Due
to the characterization of the conformal group in Theorem 2.2.7, spacetime orientations
are the most general $G$-structures able to give rise to time and space orientations in a
philosophical sense and which also respect the causal structure induced by $g$. While
relativistic spacetimes are commonly assumed to be time-oriented in the physics literature,
space orientations are often not included in the definition. Yet a space orientation is
indispensable to differentiate between a relativistic model and its ‘spatial mirror image’. It
therefore needs to be included in the general definition of ‘spacetime’. Later in this
section we will see that space orientations are also required in order to give a mathematical
definition of the physical concept ‘frame of reference’.

**Levi-Civita connection:** As in Riemannian geometry, the metric $g$ naturally gives rise to
the class of metric connections, that is the set of those covariant derivatives $\nabla$ for which $g$
is covariantly constant:

$$\nabla g = 0.$$

For given initial conditions, (maximal) autoparallels with respect to any of these $\nabla$
coincide, so we may philosophically understand these autoparallels to be intrinsic to the
Lorentzian manifold $(Q, g)$. Again in full analogy to the Riemannian case, $\nabla$ is uniquely
defined by the choice of a torsion tensor field $T$, given by

$$T(Y, Z) = \nabla_Y Z - \nabla_Z Y - [Y, Z]$$

for vector fields $Y, Z$ on $Q$. While the choice of $T$ has no influence on the shape of
autoparallels, it is important for parallel transport of tangent vectors along curves. Thus
the choice of the Levi-Civita connection for $\nabla$, i.e. $T = 0$, constitutes an additional
assumption on the geometry of the spacetime and cannot be taken for granted. Physically,
non-vanishing torsion would lead to a spinning of (infinitesimal) rigid bodies in free fall
with initially vanishing angular momentum (as measured from the center of mass) – to
our knowledge this has not been observed. Nevertheless the case of $T \neq 0$ is considered,
for instance, within so called Einstein-Cartan theory. See e.g. the article by Hehl, von der
Heyde and Kerlick [73] for a discussion thereof.

**Connectedness:** One may define spacetimes to be connected, but we decide not to do
so. This is not due to a need to consider ‘multiple universes’, which would be physically
inaccessible even if they existed, but rather for practical, modeling reasons.

For instance, if $f$ is a smooth, real-valued function on a connected spacetime $Q$ and we
wish to consider the function $1/f$, then

$$Q' := f^{-1}(\mathbb{R} \setminus \{0\})$$

is an open submanifold of $Q$ and $1/f$ is smooth on $Q'$. If we do not require spacetimes to
be connected, then $Q'$ is also canonically a spacetime.
Inextendibility: It is sensible to ask for a certain ‘maximality condition’ on the connected components of the spacetime. This is a condition that can be defined for any Lorentzian manifold: An extension of a connected Lorentzian manifold \((\mathcal{Q}, g)\) is a connected Lorentzian manifold \((\mathcal{Q}', g')\) together with a (smooth) mapping \(\varphi: \mathcal{Q} \to \mathcal{Q}'\) such that \((\mathcal{Q}, \varphi)\) is an open submanifold of \(\mathcal{Q}'\) with \(\varphi^* g' = g\). A connected Lorentzian manifold \((\mathcal{Q}, g)\) is said to be inextendible, if there does not exist an extension with \(\varphi(\mathcal{Q}) \subset \mathcal{Q}'\) (cf. [16, p. 85 sq.]; [24, Def. 5.44]). A general Lorentzian manifold is inextendible, if each of its connected components is inextendible (as a connected Lorentzian manifold equipped with the restricted metric).

Clearly, the terminology carries over to spacetimes. As in the case of connectedness, modeling arguments speak against defining spacetimes to be inextendible.

Causality: The last requirement one might want to add to the definition of a spacetime is a so called ‘causality condition’. They are usually topological (hence global) restrictions on the manifold \(\mathcal{Q}\) and derive their name from the fact that, among others, they have implications for the possible trajectories of point masses, i.e. for (future directed) timelike curves on \(\mathcal{Q}\). As there are quite a few possible choices, we mention only one example and refer to the books by O’Neill [24, Chap. 14], Beem at al [3, §3.2 & §3.3], Penrose [25], as well as the article by Minguzzi and Sanchez [92] for further reading.

One of the weakest causality conditions is non-viciousness. By definition, a spacetime is called vicious, if there exists a periodic timelike curve, i.e. one for which the image is compact in \(\mathcal{Q}\). If such a spacetime were considered a serious physical model, it would imply the possibility of time travel into the past as well as time repeating itself over and over. The name derives itself from the phrase ‘vicious circle’ and it is arguably quite judgmental terminology. In the author’s opinion, the physicist’s judgment on the acceptability of these models depends more on cultural background than scientific evidence. Independent of where one stands on this issue, we are not aware of any good reason for not calling these models spacetimes.

We continue by giving two physically relevant examples of spacetimes.

Example 3.1.2 (Minkowski spacetime)
Consider the manifold \(\mathbb{R}^4\) with standard topology and smooth structure. In canonical global coordinates \(y = (y^0, \vec{y})\), we may define the Lorentzian metric

\[
g := \eta_{ij} \, dy^i \otimes dy^j
\]

with respect to which \(\partial\) is an orthonormal frame field. As a global frame field, this gives rise to a C\text{Lor}_4-reduction \(\mathcal{O}\) of the frame bundle \(\text{Fr}(T\mathbb{R}^4)\) (cf. Remark 2.2.5i) on page 14) and thus a spacetime orientation. Hence \((\mathbb{R}^4, g, \mathcal{O})\) is a spacetime, known as Minkowski spacetime. It is obviously flat and connected. In addition, open submanifolds of Minkowski spacetime can be canonically turned into flat spacetimes, but these are not necessarily connected.

Minkowski spacetime is the mathematical setting of the special theory of relativity. We refer to [11] for the original articles on the theory due to Einstein and Minkowski.

The following lemma is of use in defining spacetime orientations, if one has an ‘almost global’ orthonormal frame field on a parallelizable Lorentzian manifold.
Lemma 3.1.3
Let \((Q, g)\) be a parallelizable Lorentzian manifold, \(U \subseteq Q\) be open and let \(X: U \to \text{OFr}(Q, g)\) be a local, orthonormal frame field, such that the closure (\(\text{clos}\,U\)) of \(U\) is \(Q\). Then there exists a unique spacetime orientation \(O\) on \(Q\) such that \(X\) is a local section of \(O\) and \((Q, g, O)\) is a spacetime.

\(\diamond\)

**Proof** Since the \((n + 1)\)-manifold \(Q\) is parallelizable and \(g\) is Lorentzian, there exists a global, orthonormal frame field \(Y\) and a \(\Lambda \in C^\infty(\text{clos}\,U, \text{Lor}_{n+1})\) such that \(Y = X \cdot \Lambda\) on \(U\). Extension of this \(\text{Lor}_{n+1}\)-structure to \(\text{CLor}_{n+1}\) proves existence (see Remark 2.2.5/ii) on page 15 and Lemma 2.2.11 on page 29).

To get uniqueness, consider a second such frame field \(Y'\) with \(Y' = X \cdot \Lambda'\) on \(U\). Then \(Y' = Y \cdot \Lambda \cdot \Lambda'\) on \(U\). On the other hand, there must exist an \(A \in C^\infty(Q, O_{1, n})\) with \(Y' = Y \cdot A\) and \(A = \Lambda \cdot \Lambda'\) on \(U\). Choosing a sequence \((q_i)_{i \in \mathbb{N}}\) in \(U\) converging to \(q \in Q \setminus U\), we obtain

\[
\lim_{i \to \infty} (\Lambda \cdot \Lambda')_q = \lim_{i \to \infty} A_{q_i} = A_q.
\]

As a connected component, \(\text{Lor}_{n+1}\) is closed in \(O_{1, n}\) and hence \(A_q \in \text{Lor}_{n+1} \subset \text{CLor}_{n+1}\). As the sequence was arbitrary, \(Y\) and \(Y'\) induce the same spacetime orientation \(O\).

We apply Lemma 3.1.3 in the next example of a spacetime.

**Example 3.1.4 (Exterior Schwarzschild spacetime)**
Let \(R \in \mathbb{R}_+\), consider \(\mathbb{R} \times (R, \infty)\) equipped with the Lorentzian metric \(g'\), given by

\[
g'_{(ct, r)} := \left(1 - \frac{R}{r}\right) d(ct) \otimes d(ct) - \left(1 - \frac{R}{r}\right)^{-1} dr \otimes dr
\]

in canonical global coordinates \((ct, r)\), as well as the 2-sphere \(S^2 \subset \mathbb{R}^3\) equipped with the standard pseudo-Riemannian metric \(g''\). We define a new Lorentzian manifold \((Q, g)\) by taking the standard Riemannian metric \((S^2, g'')\) (cf. [24, Chap. 3, Lem. 5; 33, Ex. 1.4.2]). \(g\) is called the Schwarzschild metric and \(R\) is called the Schwarzschild radius.

Note that the word ‘radius’ is potentially misleading. Choosing spherical coordinates \((\theta, \phi)\) on \(S^2 \subset \mathbb{R}^3\), we obtain Schwarzschild coordinates \(\kappa := (ct, r, \theta, \phi)\):

\[
\mathbb{R} \times (R, \infty) \times (S^2 \setminus \{\bar{y} \in \mathbb{R}^3 | y^1 = 0, y^2 \geq 0\}) \to \mathbb{R} \times (R, \infty) \times (0, \pi) \times (0, 2\pi)
\]

\[
(ct, r, \bar{y}) \to (ct, r, \theta(\bar{y}), \phi(\bar{y}))
\]

on \(Q\), and in these coordinates the metric reads

\[
g_{(ct, r, \theta, \phi)} = \begin{pmatrix}
(1 - \frac{R}{r}) & - (1 - \frac{R}{r})^{-1} & r^2 & -r^2 \sin^2 \theta \\
- (1 - \frac{R}{r})^{-1} & -r^2 & -r^2 \sin^2 \theta \\
r^2 & -r^2 \sin^2 \theta & 1 & \frac{1}{r} \\
-\frac{1}{r} & \frac{1}{r \sin \theta} & \frac{1}{r} & 1
\end{pmatrix}.
\]

To obtain a compatible spacetime orientation, we use \(\kappa\) to define the orthonormal frame field \(X\) on \(U := \text{dom} \, \kappa \subset Q\) via

\[
(X)_{\kappa} := \begin{pmatrix}
(1 - \frac{R}{r})^{-1/2} & \frac{1}{r} \\
(1 - \frac{R}{r})^{1/2} & \frac{1}{r \sin \theta}
\end{pmatrix}.
\]
3 The Splitting Construction

Contrary to Example 3.1.2, this is not a global frame field and, since $S^2$ is not parallelizable, it cannot be smoothly extended to one. However, $(R, \infty) \times S^2$ is diffeomorphic to $\{ \vec{y} \in R^3 | ||\vec{y}|| > R \}$ via $(r, \vec{y}) \rightarrow r \vec{y}$ and hence parallelizable. Thus $Q$ is parallelizable and since $\text{clos} \mathcal{U} = Q$, we may apply Lemma 3.1.3 to obtain a compatible spacetime orientation induced by $X$.

We call $(Q, g, \mathcal{O})$ the exterior Schwarzschild spacetime. It was discovered independently by Schwarzschild and Droste as a solution of Einstein’s vacuum (field) equation in the year 1916. Being a ‘vacuum solution’ means that its Ricci tensor field $R$ vanishes, i.e. it is Ricci-flat. See [96, p. 55] for a historical overview including references to the original works. Mathematically, it is interesting since the Lie group $R \times SO_3$ acts on it canonically by spacetime-orientation preserving isometries, i.e. by spacetime automorphisms.

In the remaining part of this section, we consider additional mathematical structures on spacetimes. These structures are relevant for the theory of relativity and our splitting construction.

First we recall the definition of observers and discuss their physical relevance. Within this thesis, the constant $c \in R_+$ is always the speed of light (in vacuum).

**Definition 3.1.5 (Observer)**

Let $(Q, g, \mathcal{O})$ be a spacetime.

A (smooth) observer is a (smooth) curve $\gamma: I \rightarrow Q: \tau \rightarrow \gamma(\tau)$ for which each tangent vector $\dot{\gamma}_\tau := \gamma^* (\partial/\partial \tau)_+$ is an observer vector. That is, $\gamma$ is a timelike, future-directed curve satisfying

$$g(\dot{\gamma}_\tau, \dot{\gamma}_\tau) = c^2.$$  

(3.1.2)

for all $\tau \in I$.

In the theory of relativity, future directed, timelike curves are of particular importance, as they describe physical motion of point masses. The normalization condition (3.1.2) fixes the parametrization: We say that $\gamma$ is proper time parametrized. This choice of parametrization is taken in accordance with the so called ‘clock hypothesis’. As general relativity is currently a well-established theory of nature, it is a principle, rather than a postulate or hypothesis.

**Principle 1 (Clock Principle)**

The time difference measured by an (ideal) clock moving along a future-directed timelike curve in spacetime is given by its Lorentzian arc-length divided by the speed of light.

$$\Delta \tau := \frac{1}{c} \int_{s_1}^{s_2} \sqrt{g(\dot{\gamma}_s^f, \dot{\gamma}_s^f)} \, ds$$  

(3.1.4)
3.1 General Considerations

According to the clock. Indeed, it is often convenient and always physically correct to picture observers \( \gamma \) (in the sense of Definition 3.1.5) as moving, point-like, ideal clocks. The force applied to such a clock is given by the general-relativistic generalization of Newton’s second law:

\[
m \frac{\nabla \dot{\gamma}}{d\tau} = F,
\]

(3.1.5)

where \( m \in \mathbb{R}_+ \) is the inertial mass and the force \( F \) is a (necessarily spacelike) vector field over \( \gamma \). We strongly emphasize that gravity is not a force. Hence the relativistic generalization of Newton’s first law states that point masses move geodesically in the absence of forces, i.e. the \textit{(proper/absolute) acceleration} \( \nabla \ddot{\gamma}/d\tau \) vanishes entirely or equivalently, \( \gamma \) is \textit{unaccelerated}. In the presence of forces \( F \neq 0 \), however, \( \gamma \) is \textit{accelerated}. In both cases, the function

\[
a := \sqrt{-g \left( \frac{\nabla \dot{\gamma}}{d\tau}, \frac{\nabla \dot{\gamma}}{d\tau} \right)}
\]

(3.1.6)

is called the \textit{absolute value of the (proper/absolute) acceleration}, accordingly. We shall consider two examples of observers.

**Example 3.1.6 (Observers in Minkowski spacetime)**

In the following, let \((\mathbb{R}^4, g, Q)\) be 4-dimensional Minkowski spacetime from Example 3.1.2.

i) The observer \( \gamma: \mathbb{R} \to \mathbb{R}^4: \tau \to \gamma(\tau) \), defined by

\[
\gamma(\tau) = (c\tau, 0, 0, 0),
\]

(3.1.7a)

is prototypical and often implicitly assumed to be given in discussions on special relativity. As all Christoffel symbols vanish in standard coordinates, it is unaccelerated.

ii) If we look for \textit{constantly accelerated observers}, then \( \gamma: \tau \to \gamma(\tau) \) must satisfy:

\[
0 \neq -a^2 = \eta_{ij} \dddot{\gamma}^i \dddot{\gamma}^j = (\dddot{\gamma}^0)^2 - (\dddot{\gamma}^1)^2 = \text{const. },
\]

where the dot denotes the derivative with respect to the parameter \( \tau \). Setting \( \gamma^2(\tau) = \gamma^3(\tau) \equiv 0 \), we obtain

\[
-a^2 = (\dddot{\gamma}^0)^2 - (\dddot{\gamma}^1)^2 \quad \text{and} \quad c^2 = (\dot{\gamma}^0)^2 - (\dot{\gamma}^1)^2.
\]

We solve the latter equation for \( \dot{\gamma}^0 \), observe that \( \dot{\gamma}^0 > 0 \), since \( \gamma \) is future-directed, and then plug the derivative into the former equation. After some rearrangement we get

\[
c^2 + (\dot{\gamma}^1)^2 = \left( \frac{c^2 + 1}{a} \right)^2,
\]

which may be transformed to a first-order equation by defining \( u := \dot{\gamma}^1/c \). After re-parametrization via \( s := a\tau/c \), we obtain

\[
1 + u^2 = \left( \frac{du}{ds} \right)^2.
\]
By comparing this with the identity
\[ 1 + \sinh^2 s = \cosh^2 s, \]
we get \( u(s) = \sinh(s + \text{const.}) \). Thus for the initial condition \( \dot{\gamma}(0) = 0 \), the tangent vector of \( \gamma \) at \( \tau \) reads:
\[ \dot{\gamma}_\tau = c \cosh \left( \frac{a\tau}{c} \right) \frac{\partial}{\partial y^0}\gamma(\tau) + c \sinh \left( \frac{a\tau}{c} \right) \frac{\partial}{\partial y^1}\gamma(\tau). \quad (3.1.7b) \]
So for \( \gamma(0) = 0 \), we conclude
\[ \gamma(\tau) = \left( \frac{c^2}{a} \sinh \left( \frac{a\tau}{c} \right), \frac{c^2}{a} \left( \cosh \left( \frac{a\tau}{c} \right) - 1 \right), 0, 0 \right), \quad (3.1.7c) \]
which is defined for all \( \tau \in \mathbb{R} \).

The need to solve non-linear differential equations even for physically simple situations is the norm in general-relativity, not the exception.

Light cones are the next mathematical structures we consider here.

**Definition 3.1.7 (Light cones)**

Let \( (Q, g, O) \) be a spacetime and let \( q \) be a point on \( Q \).

The **tangent light cone** \( c_q \) at \( q \) is the set of lightlike vectors in \( T_q Q \). The **future tangent light cone** \( c_q^+ \) at \( q \) is the set of future-directed lightlike tangent vectors in \( T_q Q \). Analogously, we define the **past tangent light cone** \( c_q^- \) at \( q \). If \( \exp: \text{dom} \exp \to Q \) is the exponential map induced by the Levi-Civita connection, we define the **light cone** \( C_q \) at \( q \) to be the image of \( \text{dom} \exp \cap c_q \) under \( \exp \). Similarly,

\[ C_q^+ := \exp \left( \text{dom} \exp \cap c_q^+ \right) \]
is the **future light cone** at \( q \) and

\[ C_q^- := \exp \left( \text{dom} \exp \cap c_q^- \right) \]
is the **past light cone** at \( q \).

The light cones in the tangent spaces are manifolds in a natural way.

**Proposition 3.1.8 (Tangent light cone as manifold)**

Let \( (Q, g, O) \) be a spacetime of dimension \( n + 1 \) and let \( q \in Q \). Then there is a unique manifold structure on the tangent light cone \( c_q \), such that it is an embedded submanifold of \( T_q Q \). With respect to this manifold structure, \( c_q \) splits into two connected components \( c_q^+ \) and \( c_q^- \). Moreover, for an orthonormal frame \( X \in \mathcal{O}Fr(Q, g) \) at \( q \), the maps
\[ \bar{\mathbf{x}}_\pm: c_q^+ \to \mathbb{R}^n \setminus \{0\} : K \to (X^1 \cdot K, \ldots, X^n \cdot K) \quad (3.1.8a) \]
define (compatible) coordinates on \( c_q^+ \) and \( c_q^- \), respectively, having inverses
\[ (\bar{\mathbf{x}}_\pm)^{-1} : \mathbb{R}^n \setminus \{0\} \to c_q^\pm : \mathbf{y} \to \pm |\mathbf{y}| X_0 + y^a X_a \quad (3.1.8b) \]
with \( a \in \{1, \ldots, n\} \).

48
3.1 General Considerations

Proof

As an open submanifold, $T_q Q \setminus \{0\}$ is embedded in $T_q Q$. Now consider

$$p: T_q Q \setminus \{0\} \to \mathbb{R}: Y \to g_q(Y,Y),$$

(3.1.9a)

which is smooth and $p^{-1}(\{0\}) = \mathcal{C}_q$. Since the vertical lift is a linear isomorphism and $p$ takes values in $\mathbb{R}$, we may compute the differential $p_*$ for $Y, Z \in T_q Q \setminus \{0\}$ via:

$$p_* \tilde{Z}_Y = \frac{\partial}{\partial s} \bigg|_0 p(Y + sZ) = 2g_q(Z,Y).$$

(3.1.9b)

Since $g_q$ is non-degenerate, $(p_*)^* Y$ is non-degenerate for each $Y \in T_q Q \setminus \{0\}$ and hence a submersion on $\mathcal{C}_q$. Thus by the regular value theorem, $\mathcal{C}_q$ (together with the inclusion) is an embedded submanifold of $T_q Q$.

To get the components, choose any future-directed timelike $Z \in T_q Q$ and define the continuous function

$$p': \mathcal{C}_q \to \mathbb{R}: K \to g_q(Z,K).$$

(3.1.9c)

As $p'(\mathcal{C}_q^+) = (0, \infty)$, $p'(\mathcal{C}_q^-) = (-\infty, 0)$, $\mathcal{C}_q^+ \cap \mathcal{C}_q^-$ are mutually disjoint.

Connectedness of $\mathcal{C}_q^\pm$ follows immediately, if we can prove that (3.1.8a) are indeed coordinates. If we consider $\tilde{x}_\pm$ as maps from $T_q Q$ to $\mathbb{R}^n$, then smoothness is trivial. Since $\mathcal{C}_q^\pm$ are open submanifolds of the submanifold $\mathcal{C}_q$, the restriction of this map is also smooth. The image under $\tilde{x}_\pm$ is indeed contained in $\mathbb{R}^n \setminus \{0\}$, since $K^0 X_0$ is not lightlike for any $K^0 \in \mathbb{R}$. We now consider the map (3.1.8b). Recalling that $X$ is orthonormal, one verifies with $g_q$ and $p'$ that (3.1.8b) indeed maps into $\mathcal{C}_q^\pm$. (3.1.8b) is also smooth, since taking absolute values for $\tilde{y} \in \mathbb{R}^n \setminus \{0\}$ is smooth, and multiplication is smooth on $T_q Q$ and hence on $\mathcal{C}_q^\pm$. Composing the maps in each direction, we get the identity. Hence $\tilde{x}_\pm$ are diffeomorphisms with inverses (3.1.8b).

Clearly, the coordinates $\tilde{x}_\pm$ depend highly on the choice of $X \in \text{OFr}(Q, g)$. From a mathematical perspective, one would prefer to take those frames $X$, that are not just orthonormal but also spacetime-oriented, i.e. $X \in \mathcal{O} \cap \text{OFr}(Q, g)$.

This leads us to the third set of important mathematical structures on spacetimes.

Definition 3.1.9 (Frame of Reference)

Let $(Q, g, \mathcal{O})$ be a spacetime of dimension $n + 1$.

The frame of reference bundle\(^5\) (over the spacetime $(Q, g, \mathcal{O})$) or reference frame bundle (over $(Q, g, \mathcal{O})$) is the tuple $(\mathcal{P}, \tilde{\pi}, Q, \text{Lor}_{n+1})$, where $\mathcal{P}$ is the set

$$\mathcal{P} := \mathcal{O} \cap (\text{OFr}(Q, g)),$$

(3.1.10a)

$$\tilde{\pi} := \pi|_\mathcal{P}: \mathcal{P} \to Q$$

is the restriction of $\pi: \text{Fr}(TQ) \to Q$ to $\mathcal{P}$, and $\text{Lor}_{n+1}$ acts canonically on $\mathcal{P}$ from the right via

$$\mathcal{P} \times \text{Lor}_{n+1} \to \mathcal{P} : (X, \Lambda) \to X \cdot \Lambda.$$

(3.1.10b)

A frame of reference is an element of $\mathcal{P}$ and a frame of reference at $q \in Q$ is an element of $\mathcal{P}_q := \tilde{\pi}^{-1}(q)$.

---

\(^5\)German: “Bezugssystembündel”
The vector $X_0$ of a frame of reference $X \in \mathcal{P}$ may be identified as the tangent vector (divided by $c$) of some observer $\gamma$ at time $\tau$ on $Q$. The vectors $X_1, X_2, X_3$ represent the orientation in ‘space’ of a physical observer at time $\tau$ moving along $\gamma$.

**Remark 3.1.10**

To our knowledge, the identification of particular orthonormal frames $X$ with physical frames of reference is due to Walker [103]. While he assumed $X_0$ to be future directed, he did not explicitly assume $X_1, X_2, X_3$ to be right-handed. As far as we know, the use of these frames of reference to define coordinates on the tangent past light cone (as in (3.1.8a) on page 48) is due to Mast and Strathdee [89].

We continue by showing that the frame of reference bundle is an embedded $\text{Lor}_{n+1}$-structure on $Q$.

**Theorem 3.1.11**

Let $(\mathcal{P}, \tilde{\pi}, Q, \text{Lor}_{n+1})$ be the frame of reference bundle over a spacetime $(Q, g, \mathcal{O})$. Then there is a unique manifold structure on $P$ such that $(\mathcal{P}, \tilde{\pi}, Q, \text{Lor}_{n+1})$ with the group action (3.1.10b) is a principal $\text{Lor}_{n+1}$-bundle. With respect to this manifold structure, $P$ is an embedded submanifold both of the spacetime orientation $\mathcal{O}$ and the orthonormal frame bundle $\text{OFr}(Q, g)$.

**Proof** As in the proof of Proposition 2.2.12 on page 29, let $\{U_\alpha | \alpha \in I\}$ be a trivializing cover of $Q$ with respective (smooth, local) orthonormal frame fields $\tilde{X}_\alpha$. If necessary, apply the time inversion matrix and a space inversion matrix to make them spacetime-oriented. Then the $\tilde{X}_\alpha$s map into $P$ and the resulting transition matrices are elements of $\text{Lor}_{n+1}$. So by Remark 2.2.5i) on page 14, $P$ indeed carries a unique manifold structure that turns it into a $\text{Lor}_{n+1}$-structure. Since $\text{Lor}_{n+1}$ is embedded both in $\text{CLor}_{n+1}$ and $\text{O}_{1,n}$, an analogous reasoning to the one in Remark 2.2.2iii) implies that $P$ is embedded both in $\mathcal{O}$ and $\text{OFr}(Q, g)$. We also refer to the book by Baum [2, p. 66] regarding the latter argument.

We close this section with the remark that the frame of reference bundle is a very ‘natural’ mathematical object. Indeed, if we only consider the bundle $P$, then we can recover both the metric $g$ and the spacetime orientation $\mathcal{O}$ on $Q$ by extension. This in turn leads to the point of view that, at least mathematically, a spacetime is a $\text{Lor}_{n+1}$-structure $P$ on a manifold $Q$ equipped with the Levi-Civita connection (considered either as a tangent bundle connection on $TQ$ or as a particular Ehresmann connection on $P$). This approach is geometrically more coherent, but physically less accessible.

### 3.2 Heuristic Motivation of the Space-Time Splitting

In the following we motivate the mathematical formalism of separating space and time in general relativity. We restrict ourselves to presenting the underlying philosophy, the detailed mathematical implementation is postponed to the two consequent sections. Contrary to other such splitting formalisms, we will discover that no further restrictions on the spacetime are required apart from the ones already imposed by the mathematical definition.
3.2 Heuristic Motivation of the Space-Time Splitting

It needs to be said in advance, that the mathematical machinery, as outlined in sections 3.3 and 3.4, is self-contained and can be applied without taking notice of the underlying philosophy as presented here - provided the mathematical quantities are interpreted appropriately. Yet we would not have been able to construct the splitting formalism without these philosophical considerations and thus believe them to be as much part of the theory as the mathematical formalism. In our mind, a physical theory is ideally build upon an ontology consisting of principles and postulates, not just mathematics. The relation between the mathematical formalism and the measured quantities is then derived from this ontology. We refer to the book by Frisch [13] for an in-depth discussion of this approach in the context of the theory of electrodynamics. Of course, regardless of one’s view towards the role of metaphysics in physics, any theory of nature needs to be assessed by the quantity and quality of its empirical predictions.

The main objective of this section is to provide an answer to the following questions: How does the mathematical formalism of relativistic spacetimes relate to our individual experience of the separateness of time and space? In other words, what is time and what is space in general relativity?

Indeed, we have already touched upon the role of time in the theory: On page 46 in the previous section, we stated the clock principle: The motion of physical objects in spacetime is represented by future directed timelike curves and the time measured by a clock moving along such a curve is the proper time (3.1.4). In order to give a definite answer to the question of what constitutes time in relativity, it is sufficient to realize that proper time is the only physically measurable time and all other concepts of time are derived thereof. We invite the reader to ponder this claim for some time. Once we accept it, we must conclude that Principle 1 reduces the problem of separating space and time to giving meaning to the physical concept of space in general relativity.

In the literature on general relativity, there exist several mutually conflicting answers to the question what ‘space’ is in the theory. In the following, we list what we believe to be the three most common misconceptions of space in general relativity.

**Local rest spaces:** Given a point \( q \) on the spacetime \( Q \) and an observer vector \( Z \) in the tangent space \( T_q Q \), the orthogonal complement \( (\mathbb{R}Z)^\perp \) with respect to \( g_q \) is sometimes referred to as the ‘local rest space’ with respect to \( Z \) (see e.g. [33, §2.1.4]). Philosophically, the vector \( Z \) should be viewed as a tangent vector of an observer in the spacetime.

While this conception of space has the advantage of being well-defined and relating directly to observers, it is not clear how it is connected to the dynamics on \( Q \) and observed quantities like relative position, velocity, etc. One may apply the exponential map to \( (\mathbb{R}Z)^\perp \) in order to relate it to \( Q \), but, as there can be no causal interaction between these points and the observer, this image is unrelated to what we experience as ‘space’.

**Spacelike hypersurfaces:** By definition, a spacelike hypersurface \( S \), if given as a subset of a spacetime \( Q \), is a submanifold of \( Q \) such that for every \( q \in S \) the tangent space \( T_q S \) is a spacelike hyperplane in the Lorentz vector space \( (T_q Q, g_q) \). As noted in the introduction, this is the view of ‘space’ taken in the so called ADM formalism (cf. [23, p. 419sqq. & §21.7]).

Trivially, spacelike hyperplanes are highly non-unique and thus one needs to motivate why and how one chooses one or more particular ones as ‘space’. There do exist approaches to this issue, but the fundamental problem remains that this conception of space is
substantially unrelated to our individual experience. We believe that, as in the previous case, the researchers have been mislead by the word ‘spacelike’. In addition, identifying physical ‘space’ as spacelike hypersurfaces seems only natural to us if we take an outside perspective towards the theory - an approach that is subtly, yet warningly reminiscent of Newtonian thinking.

**Coordinate hyperplanes:** In the practical application of the theory, spacetimes are usually given implicitly by writing down the component functions of the metric in a single chart. In addition, the coordinates often carry suggestive names such as $t$ and $x$, which can be misleading to those physicists lacking a formal training in differential geometry and being accustomed to Newtonian thinking. Those are then tempted to identify ($t = \text{const.}$)-hyperplanes as space.

In contrast, it is evident to the physically versed geometer that coordinates may only carry direct physical meaning, if they are ‘adapted’ to the underlying geometric structures. This means that the structures take a special form in those coordinates, as is, for instance, the case for normal coordinates with respect to a metric. In most cases, however, coordinates are meaningless ways of labeling points. The fact that geometric structures have usually been implicitly given in Newtonian physics has certainly assisted the fallacy of identifying ($t = \text{const.}$)-hyperplanes with physical space. To put it bluntly, calling a coordinate $t$ does not make it a physical measure of time just like calling a coordinate $x$ does not make it a physical measure of spatial distance.

Therefore, to understand what space is in general relativity, we need to let go of Newtonian concepts and grasp the role played by the different geometric structures.

The abandonment of Newtonian thinking is eased by the appreciation of the fact that the Newtonian way of seeing the world has already proven itself to be an inadequate, approximative at best, path of gaining insight into the inner workings of nature. So the function of Einstein’s theory in this inquiry needs to be openly exposed: It is nothing short of a revolutionary, scientific alternative to outdated metaphysical ways of thinking about space and time.

Hence, to make a step in understanding the role played by the geometric structures, we need to view the physical world from within the theory. If we ask for our individual experience of physical space, it has to be defined mathematically with respect to a single observer $\gamma: I \to Q$ at fixed time $\tau \in I$. That is, we ask what a single observer would identify as space at a fixed time.

A natural answer to this question is that space is what an observer sees at an instant. However, this ‘definition’ is more subtle than it may appear at first sight. In general relativity, light is commonly modeled with a closed 2-form on the spacetime, known as the Faraday-form or electromagnetic field. If the Einstein equation is correct, then the presence of this electromagnetic field has a direct effect on the spacetime geometry and vice versa. But even if one ignores this issue, it requires some more assumptions and a lengthy heuristic argument (cf. [23, §22.5; 38, §1.8]) to arrive at the so-called geometric optics approximation [96, p. 7sq.], where light rays may be identified with lightlike geodesics. In addition, it is conceptually problematic to implicitly presuppose the existence of light in an identification of ‘space’ within the mathematical framework, since ‘space’ should also be present in the absence of light. Hence the concept of space ought not to be based on the concept of light, despite the observation that our subjective experience of space
always involves light. For these reasons, we avoid the subject of light here altogether with the well-meant advice that the specific physical situation has to be analyzed when light is part of the model.

Fortunately, we may circumvent the issue by identifying space with the set of all points in the spacetime that can causally interact with the observer at an instant, i.e. those points that can be linked to the observer’s position via a future directed lightlike geodesic. This is physically vague, yet functional, as in the geometric optics approximation those are precisely the points \( q \in Q \) that can send light to the observer \( \gamma \) at that instant \( \tau \) of his or her proper time. Thus we identify the ‘space’ for an observer at time \( \tau \) with the past light cone \( C^-_{\gamma(\tau)} \) of the point \( \gamma(\tau) \). So for convenience, from now on we say that an observer \( \gamma \) sees a point \( q \in Q \) at time \( \tau \in \text{dom} \gamma \), if \( q \) is in the past light cone \( C^-_{\gamma(\tau)} \) of \( \gamma(\tau) \) - well-knowing that this physically refers to potential causal interactions, rather than the emission and reception of light. This subtlety makes it possible in principle to apply the theory of space-time splitting as presented here also in conjunction with other models of light and beyond the geometric optics approximation.

However, the identification of space at an instant \( \tau \) with the past light cone \( C^-_{\gamma(\tau)} \) leads to another problem: How do we define a natural distance function on \( C^-_{\gamma(\tau)} \)? Clearly, we would like this distance function to be closely related to the empirically measured distance we observe in the presence of light and where the geometric optics approximation is admissible. Yet, mathematically, lightlike geodesics have vanishing length, so it is inadmissible to directly use the Lorentzian distance to measure lengths on the past light cone.

The bad news is, that the careful motivation of this distance function is quite elaborate and will occupy the rest of this section. The good news is, that once this is done, figuratively speaking, everything else falls into place and we will then have obtained a well motivated, and, as we believe, physically correct splitting of spacetime into space and time.

Let us specify what precisely we mean with the word ‘empirically measured distance’. If we idealize a physical observer at an instant to be point-like, as it is commonly done in the theory of relativity, then we can abstractly think of its spatial alignment to be given by three vectors: The first one pointing in the forward direction, the second one pointing to the left and the third one pointing upwards in accordance with the right-hand rule. This is the original concept behind the words ‘frame of reference’, i.e. a point in ‘space’ to which three mutually orthogonal (as perceived by the physical observer), right-handed vectors are attached. Even if the ‘surrounding geometry’ is very complicated, the physical observer may assign to everything it sees (in the aforementioned sense) a polar angle \( \phi \) and an azimuthal angle \( \theta \) with respect to its frame of reference. The measurement of these angles is done with an (idealized, infinitely small) goniometer. Now, if we were to find an appropriate notion of radial distance \( r \), then we could use the definition of spherical coordinates in \( \mathbb{R}^3 \) to assign to each observed point a value \( \bar{x} \in \mathbb{R}^3 \) with respect to the frame of reference. Then the distances between two points \( \bar{x}, \bar{x}' \in \mathbb{R}^3 \) would be given by the usual Euclidean distance formula:

\[
\text{dist} (\bar{x}, \bar{x}') := |\bar{x} - \bar{x}'| . \tag{3.2.1}
\]

Please note that this expression is invariant under rotation, i.e. it does not change under the action

\[
\text{SO}_3 \times \mathbb{R}^3 \to \mathbb{R}^3 : (A, \bar{x}) \to A \cdot \bar{x} .
\]
One may object to our discussion that we have implicitly assumed the observed geometry as Euclidean at the onset and hence this is what we ultimately arrive at. This criticism is justified, but again the issue is more subtle: How a physical observer measures distance is in fact convention, i.e. up to its subjective choice, but it is objective, what is ultimately measured empirically in accordance with this convention. Hence the task of finding the physically correct distance function on the past light cone $C^-\gamma(\tau)$ at time $\tau$ is linking this choice of viewing the world from the (perhaps due to our cultural imprint) subjectively chosen perspective of Euclidean geometry to the actual spacetime geometry. In fact, the conventionality of the metrization of physical space was already emphasized by Poincaré, Reichenbach and Grünbaum (cf. [17, p. 207 sqq.;27]).

It is therefore our task to establish this link between the chosen Euclidean geometry and the actual geometry of spacetime. A priori it may depend on the physical observer’s state of motion, so we require an appropriate postulate. The following one is due to Mashoon [90,91], which he termed “the hypothesis of locality”. While we do not follow his wording and do not agree with his implementation, the postulate does appear to be implicit in the general theory of relativity.

**Postulate 1 (Mashoon)**
The spatial distances an observer measures at an instant are independent both of its absolute acceleration and rotation.

The word ‘absolute’ is used to distinguish the acceleration of the observer and the rotation of its frame of reference from relative acceleration and rotation, which are unrelated concepts. The presence of absolute acceleration and rotation can be empirically verified on in principle arbitrarily small spatiotemporal scales, as measured by the observer, with an accelerometer and gyroscope, respectively. That is, ideal accelerometers and ideal gyroscopes are pointlike and it makes sense to speak of them for a physical observer without knowing how the observer’s geometry relates to the spacetime geometry. Postulate 1 now states that independent of whether the accelerometer and gyroscope show the presence of acceleration or rotation, the distances, as measured by that observer at that instant, remain unaffected. In particular, for the definition of spatial distances at $\gamma(\tau)$ and on arbitrarily large length scales, we may assume the observer’s frame of reference at the event is inertial, i.e. non-accelerating and non-rotating. Moreover, scientific realism dictates that the distances are also independent of its spatial orientation. Thus in the mathematical theory of relativity, the definition of a distance function for an observer $\gamma$ on the past light cone at proper time $\tau \in \text{dom} \gamma$ may only depend on the point $\gamma(\tau)$ and the observer’s so-called 4-velocity $\dot{\gamma}_\tau$ at the point. Hence we do not need any additional mathematical structures for a so-called spacetime splitting. As observers exist on arbitrary spacetimes, no further constraints need to be put on it and we are not faced with the uncomfortable question of why we have chosen this particular mathematical structure among the possibly infinitely many possibilities.

Of course, Postulate 1 does not say which angles and radial distance an observer should measure in $Q$. To answer this question, we will employ the Einstein equivalence principle, which directly gives us the angles $\theta$ and $\phi$, and to obtain the radial distances we employ an analogy in Riemannian geometry.

The following formulation of the Einstein equivalence principle\(^6\) has been taken from the book by Carroll [10, p. 50]. For an original discussion due to Einstein, see e.g. his lecture

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\(^6\)Dr. Hasse pointed out to us that there exist other formulations of the Einstein equivalence...
3.2 Heuristic Motivation of the Space-Time Splitting

notes [12]. It is also worth a note, that in the original anticipation of the theory Einstein identified what an observer ‘sees’ directly with coordinates in spacetime. However, since what an observer ‘sees’ is its past light cone and one can prove that in the presence of curvature this may intersect itself, this point of view is only justified on small enough spatiotemporal scales.

**Principle 2 (Einstein Equivalence Principle)**

It is impossible to detect the existence of a gravitational field by means of local experiments. Hence for inertial frames of reference within sufficiently small spatiotemporal scales, the laws of special relativity are approximately valid.

As observed by Mashoon, Principle 2 in conjunction with Postulate 1 imply that at an instant and on sufficiently small spatial scales, a rotating and accelerating frame of reference under the influence of gravitation measures approximately the same distances as one which is neither rotating, accelerating nor under the influence of gravitation. So intuitively, general relativistic distances are locally approximated by special relativistic ones. Mathematically, we approximate the sought-after distance function on $C_{\gamma(\tau)}$ for general observers $\gamma$ on $\mathcal{Q}$ at proper time $\tau$ with the special-relativistic distance function in the tangent past light cone $c_{\gamma(\tau)}$ at $\gamma(\tau)$, since a neighborhood of the origin in the tangent past light cone of $\gamma(\tau)$ approximates the past light cone at $\gamma(\tau)$ via the exponential map.

The latter statement is precisely the proposition [24, Chap. 3, Prop. 33] on the existence and properties of normal coordinates at the point $\gamma(\tau)$. Therefore, we may identify $T_{\gamma(\tau)}\mathcal{Q}$, along with the natural geometric structures, as dictated by this approximation, as Minkowski spacetime and $\dot{\gamma}_\tau$ as the 4-velocity vector of a special-relativistic observer at rest at the origin.

So let us recall how distances are measured in special relativity. Accordingly, let $(\mathbb{R}^4, g, \mathcal{O})$ be Minkowski spacetime (cf. Example 3.1.2 on page 44) and let $\gamma'$ be the standard observer from Example 3.1.6, given by

$$\gamma' : \mathbb{R} \to \mathbb{R}^4 : \tau \to \gamma(\tau) := (c\tau, 0, 0, 0) = c\tau e_0,$$

which is heuristically considered to be ‘at rest’ at the origin and sees physical occurrences in the spacetime. For two points $x = (ct, \vec{x}), x' = (ct', \vec{x}')$ in $\mathbb{R}^4$ the distance is then given by

$$\text{dist} (x, x') = |\vec{x} - \vec{x}'|,$$

in analogy to equation (3.2.1). Since we are only interested in the distances measured by the observer $\gamma'$ at one particular time, we may restrict the distance function (3.2.3) to the past light cone $C_{\gamma(0)} = C_0^-$ in $\mathbb{R}^4$ at time 0 and have thus obtained what we asked for.

So how precisely do we carry this construction over to the general relativistic case? Instead of considering an observer $\gamma$ in $\mathcal{Q}$, Postulate 1 allows us to simplify the situation by only considering an observer vector $c\mathcal{X}_0$ at $q \in \mathcal{Q}$ and interpreting it as the tangent vector of $\gamma$ at time $\tau = 0$. Since we would like to consider Minkowski spacetime as the ‘tangent cone approximation to the geometry on $\mathcal{Q}$’ at $q$, we need to construct an analogous distance function on $c_{\mathcal{X}_0}$. Obviously, $T_q\mathcal{Q}$ is equipped with the Lorentz product $g_q$ and for principle (see e.g. [86]). For our purposes we only require, that the measurement of spatial distances at an instant by an inertial, physical observer works in approximately the same way as in special relativity theory, provided the measured distances are ‘small enough’.
any \( Y, Z \in T_q \mathcal{Q} \) we can use the vertical lift \( \tilde{Z}_Y \in T_Y T_q \mathcal{Q} \) of \( Z \) at \( Y \) (see (2.3.13) on page 34) to define a Lorentzian metric \( \tilde{g} \) on \( T_q \mathcal{Q} \) via

\[
\tilde{g}_Y \left( \tilde{Z}_Y, \tilde{Z}'_Y \right) := g_q \left( Z, Z' \right)
\]

for any \( Y, Z, Z' \in T_q \mathcal{Q} \). Similarly, one may use the vertical lift together with the fiber \( O_q \) to define a spacetime orientation on \( T_q \mathcal{Q} \). This procedure turns \( T_q \mathcal{Q} \) into a spacetime categorically isomorphic to Minkowski spacetime, i.e. they are essentially the same mathematical object. Indeed, this procedure is precisely the one imposed by viewing \( T_q \mathcal{Q} \) as an approximation to the spacetime \( \mathcal{Q} \) at \( q \) via the exponential map. Hence the curve

\[
\gamma'' : \mathbb{R} \to T_q \mathcal{Q} : \tau \mapsto \gamma''(\tau) := \tau cX_0,
\]

can be identified with \( \gamma' \) above, and so \( cX_0 \in T_q \mathcal{Q} \) can be identified with \( c e_0 \in \mathbb{R}^4 \). Observing now that in Minkowski spacetime \( \mathbb{R}^4 \), we can reverse the procedure and carry the Lorentzian metric down to the standard Lorentz product \( \eta \) in the vector space \( \mathbb{R}^4 \), we may use \( \eta \) to identify the \( t = 0 \) hyperplane as \( (\mathbb{R} c e_0)^\perp \). The distance from equation (3.2.3) is then just the Euclidean distance of the orthogonal projections of \( x \) and \( x' \in \mathbb{R}^4 \) along \( c e_0 \), that is

\[
\text{dist} \left( x, x' \right) = \sqrt{ \delta \left( \bar{x} - \bar{x}', \bar{x} - \bar{x}' \right) }
= \sqrt{- \eta \left( (x - x')^\perp, (x - x')^\perp \right) }
= : \sqrt{- \eta^\perp (x - x', x - x') }.
\]

Therefore, the observer vector \( cX_0 \in T_q \mathcal{Q} \) defines a spacelike hyperplane \( (\mathbb{R} X_0)^\perp \) via the Lorentz product \( g_q \) and the distances on the tangent past light cone \( c_q^\perp \) are the ones induced by restricting the degenerate product \( -q^-_q \) to the cone. The construction is graphically depicted in figure 3.1. Explicitly, the mutual distances of any \( K, K' \in c_q^\perp \) are given by

\[
\text{dist} \left( K, K' \right) := \sqrt{-q^-_q \left( K - K', K - K' \right) }.
\]

(3.2.4)

Since the exponential is invertible on an open neighborhood of \( 0 \in T_q \mathcal{Q} \), we have thus obtained a distance for any two points \( q', q'' \) in the image of the restricted exponential \( \exp_q \):

\[
\text{dist} \left( q', q'' \right) \approx \text{dist} \left( \exp_q^{-1} \left( q' \right), \exp_q^{-1} \left( q'' \right) \right).
\]

The left hand side refers to the sought-after physical distance on \( c_q^- \) for an observer with tangent vector \( cX_0 \) at \( q \) and the right hand side refers to the just constructed distance on \( c_q^- \).

By assumption, the approximation gets better the closer \( \exp_q^{-1} (q') \) and \( \exp_q^{-1} (q'') \) are to the origin \( 0 \in T_q \mathcal{Q} \). Thus, if we are only interested in the mutual angles as seen by the observer \( \gamma \) at \( q \), we can make the approximation arbitrarily precise by decreasing the radial distance (in \( T_q \mathcal{Q} \) with respect to \( -q^-_q \) of \( \exp_q^{-1} (q') \) and \( \exp_q^{-1} (q'') \) to the origin. In the limit we get a precise value. Moreover, in the more general case where \( q' = \exp_q (K), q'' = \exp_q (K') \in c_q^- \) with \( K, K' \in c_q^- \) given, we can also decrease the radial distance of \( K \) and \( K' \) in \( c_q^- \) from the origin to get the mutual angles of arbitrary observed points \( q' \) and \( q'' \).
Figure 3.1: The picture indicates the measurement of distances on the past tangent light cone $c_q^-$ at a point $q$ in a three dimensional spacetime by embedding the tangent space $T_qQ$ into Euclidean 3-space. At the top we see the tangent vector of the observer at some fixed time, which gives rise to an orthogonal hyperplane. The cone $c_q^-$ is situated below. The surface of constant radial distance $r$ on $c_q^-$, as measured by the observer, is orthonally projected to a circle of radius $r$ on the hyperplane. The observer’s measurement of angles is analogous.
3 The Splitting Construction

It is, however, important to note that the fact that \( \mathcal{C}_q^- \) can intersect itself implies that an observer can see one and the same event \( q' \) from more than one direction. This corresponds to two different radial curves in \( c_q^- \) intersecting \( q' \) under the exponential map. Physically, the phenomenon is known as (strong) gravitational lensing and its existence has been empirically confirmed. We refer to the article by Perlick [96] for an introductory discussion of this phenomenon from the perspective of general relativity. This notion of angle has also been employed by Hasse in an analysis [69] of the observed size of astronomical objects in the geometric optics approximation.

Having determined how to measure angles with respect to a chosen frame of reference, we are left with identifying the radial distances. As this refers to the situation where the angles \( \phi, \theta \) are fixed, the radial distance has to be defined on past directed lightlike geodesics ‘starting’ at \( q \) and, as a distance between a point on the geodesic and the observer at \( q \), it should be monotonically increasing along the geodesic. Thus we actually ask for a particular (continuous) parametrization of each such geodesic. In addition, the equivalence principle requires that, at least ‘close’ to the observer, this parametrization is approximately given by

\[
\{ s \in \mathbb{R}_+ \mid sK \in \text{dom} \exp_q \} \to \mathbb{Q} : s \to \exp_q (sK)
\]

for a unit vector \( K \) in \( c_q^- \) (with respect to \( -g_q^\perp \)). The analogue in Riemannian geometry supports the conjecture that this \textit{affine parameter distance} is not merely an approximate, but an exact radial distance.

\textbf{Lemma 3.2.1 (Exponential preserves radial distances)}

Let \((\mathbb{Q}, g)\) be a Riemannian manifold with standard exponential \( \exp_q \) at \( q \). Further denote by \( \tilde{Z}_Y \in T_Y T_q \mathbb{Q} \) the vertical lift of \( Z \in T_q \mathbb{Q} \) at \( Y \in T_q \mathbb{Q} \) (cf. (2.3.13) on page 34). Then the equation

\[
\tilde{g}_Y \left( \tilde{Z}_Y, \tilde{Z}_Y' \right) := g_q \left( Z, Z' \right)
\]

for any \( Y, Z, Z' \in T_q \mathbb{Q} \) defines a Riemannian metric \( \tilde{g} \) on \( T_q \mathbb{Q} \). Moreover, for all radial curves

\[
\theta : (0, r) \to T_q \mathbb{Q} : s \to \theta (s) := sY
\]

with \( Y \in T_q \mathbb{Q} \) and \( r \in \mathbb{R}_+ \) such that \( \exp_q \circ \theta \) is defined, the corresponding Riemannian lengths of \( \theta \) and \( \exp_q \circ \theta \) coincide. In particular, if \( Y \) is a unit vector, the length of \( \exp_q \circ \theta \) is \( r \).

\textbf{Proof} \( \tilde{g} \) is a smooth Riemannian metric, since \((Y, Z) \to \tilde{Z}_Y \) is smooth and \( Z \to \tilde{Z}_Y \) is a linear isomorphism for all \( Y \in T_q \mathbb{Q} \). It is now sufficient to calculate

\[
\tilde{g}_{\theta(s)} \left( \dot{\theta}_s, \dot{\theta}_s \right) = \tilde{g}_{sY} \left( \tilde{Y}_{sY}, \tilde{Y}_{sY} \right) = g_q \left( Y, Y \right)
= g_{\exp_q(0Y)} \left( (\exp \circ \theta)_0, (\exp \circ \theta)_0 \right)
= g_{(\exp \circ \theta)(s)} \left( (\exp \circ \theta)_s, (\exp \circ \theta)_s \right),
\]

where the last equality follows from the fact that tangent vectors of geodesics have constant length.

\[\Box\]
Lemma 3.2.1 is a special case of the Gauß’ lemma (see e.g. [24, Lem. 5.1]). From a mathematical perspective, it is therefore natural to assume that the Lorentzian exponential preserves the radial distances on the past tangent light cone with respect to $-g_q^\perp$.

According to Perlick [96, p. 21], the affine parameter distance was discovered by Kermack, M’Crea and Whittacker [81]. Unfortunately, they did not give any physical interpretation of it.

**Remark 3.2.2 (Parallax Distance)**

There exist several common astronomical distance measures, we refer for instance to the articles by Hogg [77] and Perlick [96, §2.4].

Only few of them can serve as a parametrization of lightlike geodesics in the general case, but one might be tempted to employ the so called parallax distance for this purpose: If a physical observer sees an extended massive object and neither the observed angular size, shape nor location changes measurably, then the observer may accelerate without rotating in a direction orthogonal to the observed center of the object. After travelling a distance $s$ (as determined by the acceleration), this yields an angular displacement of the center by an angle $\alpha$ and thus by triangulation in the Euclidean plane, we may define the parallax distance to be

$$r = s \tan \alpha.$$  

In the process of taking the limit where $s$ tends to 0, $s$ becomes a better approximation to the affine parameter distance by the equivalence principle and $r$ should, at least intuitively, remain constant. Furthermore, by taking the limit, it might be possible to circumvent the problem that quite a few assumptions are necessary to make such an idealized situation mathematically and physically feasible.

However, in a personal correspondence W. Hasse provided a counterexample to the claim that the affine parameter distance and parallax distance coincide. We give a slight adaption of his argument here: Consider a spacetime region between the observer and the seen object and assume the parallax distance has been determined in accordance with the above procedure. Now apply a conformal transformation $g \rightarrow fg$ such that the strictly positive function $f$ is precisely one outside the region and greater than 1 inside the region. It can be shown (see e.g. [39, p. 446]) that this leaves lightlike geodesics invariant and leads to an increase in the affine parameter distance. Yet the invariance of lightlike geodesics under this transformation together with the fact that it acts neither on the observer nor the object implies that the parallax distance also stays invariant. Therefore the two distances are conceptually different.

In his article [96, p. 21] Perlick also claims that the affine parameter distance is “not an observable”. Indeed, the argument in Remark 3.2.2 supports his claim by indicating that the distance is not directly measurable. Nonetheless, we believe that the affine parameter distance is not devoid of physical meaning. In fact, we are not aware of any other (non-ad hoc) radial distance measure satisfying the aforementioned requirements and behaving sensibly under conformal transformations. Beyond these arguments, its mathematical ‘naturalness’ even suggests it to be a fundamental physical distance measure, despite the problem that its empirical measurement needs to be indirect in the presence of curvature. Its empirical evaluation depends strongly on the physical model and shall not concern us here.

References [57; 58; 60; 63; 69; 77; 84; 96, §2.4] provide further reading on the subject.
Returning to our original discussion, if the physical observer has determined the angles $\theta, \phi$ and radial distances $r$ of each event relative to its instantaneous frame of reference, it may employ the standard formula for spherical coordinates to relabel these in Cartesian coordinates $\vec{x} = (x^1, x^2, x^3)$ and the distance between these events is then simply given by the Euclidean distance in accordance with equation (3.2.1) on page 53. By doing this for every time $\tau$, the observer can assign to each observed event a point $(\tau, \vec{x})$ in its ‘observer spacetime’ and even assign distances to events at different times relative to its frame of reference. Therefore, the space-time splitting philosophically gives rise to a second ‘spacetime’ with its own geometric structures, which is of course not a spacetime in the mathematical sense, but de facto the spacetime of Newtonian mechanics: Its geometry is Euclidean and time may be treated as a simple parameter. As noted before, this is not by accident, but a result of convention. Ultimately, the separation of space and time in general relativity is the question of how the Newtonian conception of the physical world relates to the general relativistic one. Thus a space-time splitting construction is also a prerequisite for showing the precise mathematical relation between the theories including the so called Newtonian limit. For a discussion of the latter, we refer to chapter 4.

Summing up, we have identified the primary concept of time in general relativity to be the one measured by individual clocks along future directed timelike curves, in accordance with the clock principle. Therefore, we found that the concept of space within the theory also had to be defined with respect to individual observers and determined it to be the past light cone at the point where the physical observer is located at an instant of its time. We discussed that the choice of geometry with which the observer views ‘space’ is conventional, but the distances on the past light cone in accordance with that convention are not conventional. In an attempt to link the two, we applied Mashoon’s postulate and the Einstein equivalence principle to conclude that the measured angles with respect to an instantaneous frame of reference are determined solely by the observer’s tangent vector at the event. We then argued that the radial distance is obtained in a similar manner, even though it is not as empirically accessible as the measurement of angles. Finally, we added that this indeed yields a Euclidean conception of space, which can be recombined with the time dimension to give rise to a so called ‘observer spacetime’.

### 3.3 Static Splitting

After having laid out the philosophical foundation of the splitting formalism, we may now implement it mathematically. As for an observer $\gamma: I \rightarrow Q: \tau \rightarrow \gamma(\tau)$, the spatial distance function at time $\tau$ on the past light cone $C_{\gamma(\tau)}$ only depends on the tangent vector $\dot{\gamma}_\tau$ (Postulate 1), we may construct the splitting in two steps. First we consider the ‘static case’, where we are only given a point in the spacetime with ‘attached’ observer vector. This will be the content of this section. We then carry this construction over in the concluding section to the ‘dynamic case’, where we are actually given an observer $\gamma$ and hence ‘add the time dimension’.

As noted before, the mathematical machinery does formally not require a philosophical basis, provided the identification of the physical concepts with the mathematical ones is given. Nonetheless, we invite the reader to return to section 3.2 for a motivation of the specific definitions.

So in this section, let $(Q, g, O)$ be a spacetime, $q \in Q$ and let $cX_0$ be an observer vector
3.3 Static Splitting

The central object of the static splitting is the (static) observer mapping, which intuitively maps the world as the observer ‘sees’ it ‘to the world as it is’.\footnote{This is a phrase we borrowed from Perlick [96, p.10].}

**Definition 3.3.1 (Static observer mapping)**

Let \((Q, g, \mathcal{O})\) be a spacetime and \(q \in Q\). Define \(\mathcal{M}_q := c_q^- \cap \text{dom exp}_q\). Then the (static) observer mapping at \(q \in Q\) is

\[
\xi_q : \mathcal{M}_q \to Q : K \to \xi_q(K) := \exp K.
\]

(3.3.1)

As required, the image of \(\xi_q\) is the past light cone \(C_q^-\). It remains to show that \(\xi_q\) is smooth.

**Proposition 3.3.2 (Domain & smoothness of static observer mapping)**

Let \((Q, g, \mathcal{O})\) be a spacetime and \(\xi_q\) be the static observer mapping at \(q \in Q\). Then there is a unique manifold structure on the domain \(\mathcal{M}_q\) of \(\xi_q\), such that it is a smooth embedded submanifold in \(T_q Q\) of dimension \(n = \text{dim } Q - 1\). With respect to this manifold structure \(\xi_q\) is smooth. Moreover, \(\mathcal{M}_q \cup \{0\}\) is star-like about \(0\). To show this, recall \(p\) and \(p'\), as defined in (3.1.9a) and (3.1.9c) on page 49, and compute for all \(K \in c_q^- \cup \{0\}, \lambda \in [0, 1]\):

\[
p(\lambda K) = \lambda^2 p(K) = 0 \quad , \quad p'(\lambda K) = \lambda p'(K) \geq 0.
\]

Since the intersection of star-like sets about the same point is starlike and

\[
\mathcal{M}_q \cup \{0\} = (\text{dom exp}_q) \cap (c_q^- \cup \{0\}) ,
\]

\(\mathcal{M}_q \cup \{0\}\) is star-like about \(0\).

Finally, as \(\bar{x}_-\) are global coordinates on \(c_q^-\) and \(\mathcal{M}_q\) is open, the restriction \(\bar{x}\) defines global coordinates on \(\mathcal{M}_q\). \(\blacksquare\)

It follows from Proposition 3.3.2 that \(\mathcal{M}_q\) is diffeomorphic to an open subset of \(\mathbb{R}^n \setminus \{0\}\), that is star-like about \(0\) (if one adds the point) and hence connected. An example of such a domain is shown in figure 3.2. Note again that the image of the static observer mapping \(C_q^-\) is usually not a submanifold of \(Q\), since the conjugate locus at \(q\) may intersect \(\mathcal{M}_q\).
3 The Splitting Construction

Figure 3.2: The picture shows a typical domain of the static observer mapping in 3-spacetimes. We chose to transform the coordinates (3.3.2) to polar coordinates \((r, \phi)\). The origin is accentuated by the ring in the middle, the straight lines diverging away are \((\phi = \text{const.})\)-lines, the circles are \((r = \text{const.})\)-lines, both in equal spacings. The outer dotted line indicates the region where the mapping is undefined.

So far we have not used the observer vector \(cX_0\). The reason is that every physical observer positioned at the event \(q\) sees the same (tangent) past light cone. The 4-velocity \(cX_0\) at \(q\) is only needed to determine the distances the observer measures. We again refer to figure 3.1 on page 57. In accordance with equation (3.2.4) on page 56, this distance is given by

\[
\text{dist}: \quad \mathcal{M}_q \times \mathcal{M}_q \to [0, \infty) \quad (3.3.3)
\]

\[
(K, K') \mapsto \sqrt{-g_q \left((K - K')^\perp, (K - K')^\perp\right)},
\]

where \(\perp\) denotes the orthogonal projection along \(cX_0\). Moreover, if we normalize the vector to \(X_0\), we may ‘complete’ it to an orthonormal frame \(X\) at \(q\). According to (3.3.2) above, this in turn yields coordinates \(\vec{x}\) on \(\mathcal{M}_q\). Then from a straightforward computation we indeed find that

\[
\text{dist} (K, K') = |\vec{x}(K) - \vec{x}(K')|.
\]

This also gives a quick proof that \((\mathcal{M}_q, \text{dist})\) is a metric space, so dist is indeed a distance function in the mathematical sense. In addition, it proves that dist induces a flat Riemannian metric \(h\) on \(T\mathcal{M}_q\), given by

\[
h = \delta_{ab} \, dx^a \otimes dx^b. \quad (3.3.4)
\]

This metric only depends on \(g_q\) and \(cX_0\). In fact, we may express \(h\) invariantly by using the vertical lift: By a coordinate calculation one verifies directly that for every \(\tilde{Y}_K, \tilde{Z}_K \in T_K \mathcal{M}_q\):

\[
h \left(\tilde{Y}_K, \tilde{Z}_K\right) = -g_q^\perp (Y, Z), \quad (3.3.5)
\]
so formally $h = \widetilde{g_q^+}_{|T_qM_q \oplus T_qM_q}$.

Apart from the ability to measure distances, it should also be possible for the observer to distinguish the space it sees from its mirror image. So we have to find a ‘natural’ way to use the space orientation on $Q$ to define an orientation on $M_q$. First recall that we have applied the regular value theorem on the quadratic form $p$ for the proof that $c_q$ is a submanifold of $T_qQ$ (cf. Proposition 3.1.8 on page 48), so the tangent space $T_KM_q$ at each $K \in M_q$ is simply the kernel of $(p_*)_K$. Identifying the tangent spaces in $T_qQ$ with the space itself under the vertical lift, we obtain from the computed expression (3.1.9b) for $(p_*)_K$ that

$$T_KM_q = \ker (K \cdot g_q) = (\mathbb{R}K)^\perp,$$

(3.3.6) i.e. the tangent space $T_KM_q$ is then identified with the lightlike hyperplane $(\mathbb{R}K)^\perp \subset T_qQ$.

Now from our discussion on page 26 sqq. on orientations on lightlike hyperplanes, we know that the fiber $O_q$ defines a vector space orientation on $(\mathbb{R}K)^\perp$. Since the vertical lift is smooth, this construction indeed defines a smooth orientation on $M_q$ (in the sense of a $\text{GL}_n^+$-structure), which is what we asked for.

Summing up, we have a ‘space’ $M_q$ equipped with a smooth metric $h$ and a smooth orientation $O \subset \text{Fr}(T_qM_q)$. This construction is formalized in the next definition.

**Definition 3.3.3 (Observer space)**

Let $(Q, g, \mathcal{O})$ be a spacetime, $M_q$ be the domain of the static observer mapping at $q \in Q$ and $cX_0 \in T_qQ$ be an observer vector.

We define the (static) observer metric $h$ induced by $cX_0$ as $h = \widetilde{g_q^+}_{|T_qM_q \oplus T_qM_q}$, in accordance with (3.3.5). Furthermore, the observer space orientation $O$ (induced by $\mathcal{O}$) is the orientation on $M_q$ induced by the orientation $\mathcal{O}_q$ in each tangent space $(\mathbb{R}K)^\perp$ at $K \in M_q$ (under the identification (3.3.6)).

The tuple $(M_q, h, O)$ is called the observer space (at $q \in Q$ induced by $cX_0$).

The coordinates $\vec{x}$ induced by an orthonormal frame $X$ at $q$ allow us to identify $M_q$ with an open subset of Euclidean space $\mathbb{R}^n$. Mathematically speaking, every observer space is isomorphic to an open submanifold of Euclidean space of the same dimension in the category of oriented Riemannian manifolds.

One may now ask if and for which choices of $X \in \text{Ofr}(Q, g)$, the coordinates $\vec{x}$ on $M_q$ are adapted to the geometric structures on $M_q$, in the sense that in these coordinates the geometric structures take the standard form. From (3.3.5) and (3.3.4), we conclude that $X_0$ needs to be the zeroth frame vector of $X$, so we only need to identify those $X$ for which the vector fields $\partial/\partial x^1, \ldots, \partial/\partial x^n$ on $M_q$ are right-handed. Due to the fact that any frame of reference $X \in \mathcal{P}_q$ induces the linear spacetime orientation $\mathcal{O}_q$ on $T_qQ$ (see section 2.2.3), $\vec{x}$ are adapted coordinates precisely when $X$ is a frame of reference at $q$ and the observer metric is induced by $cX_0$.

**Definition 3.3.4 (Static observer coordinates)**

Let $(Q, g, \mathcal{O})$ be a spacetime and $(M_q, h, O)$ be the observer space at $q \in Q$ induced by the observer vector $Z \in T_qQ$.

If $X$ is a frame of reference at $q$ with $X_0 = Z/c$, then the coordinates $\vec{x}$ on $M_q$, as defined by (3.3.2), are called (static) observer coordinates (with respect to $X$).

The usage of the term ‘observer coordinates’ was inspired by G.F.R. Ellis, who considered the spherical coordinate analog of $\vec{x}$ and named these ‘observational coordinates’ (cf.
As indicated in the previous section, the choice of the frame of reference $X$ at $q$ has a physical meaning: It specifies the position, the 4-velocity and directions ‘forward’, ‘left’ and ‘up’ (in accordance with the right-hand rule) of the physical observer in spacetime and can thus be understood as an ‘infinitesimal rigid body’ at the event $q$. Observer coordinates with respect to $X$ then label each observed point in space in accordance with this choice.

With respect to these coordinates, the observer mapping is given by

$$\varphi_X(\vec{x}) := \exp_q( -|\vec{x}|X_0 + x^aX_a).$$

We call this map $\varphi_X := \xi_q \circ \vec{x}^{-1}$ the (static) observer mapping with respect to $X$. In fact, to compute the observer mapping at a point $q \in Q$ in practice, one takes a chart $(U, \kappa)$ around $q$, a frame of reference $X$ at $q$ and computes the past lightlike geodesics in the chart with initial conditions

$$\kappa_0 = \kappa(q) \quad \text{and} \quad \kappa^i_0 = -|\vec{x}|X^i_0 + x^aX^i_a$$

for $i \in \{0, \ldots, n\}$ and all possible $\vec{x} \in \mathbb{R}^n \setminus \{0\}$. To illustrate our construction, we give a simple example.

**Example 3.3.5 (Static observer mapping in Minkowski spacetime)**

Consider Minkowski spacetime $(\mathbb{R}, g, O)$, as defined in Example 3.1.2, a point $(ct_0, \vec{y}_0) = y_0 \in \mathbb{R}^4$ and a frame of reference $Y \in \mathcal{P}_{y_0}$. In canonical coordinates $y = (ct, \vec{y})$ on $\mathbb{R}^4$, the geodesic equation is simply $\ddot{y}^k = 0$. Hence for a tangent vector $K \in T_{y_0}\mathbb{R}^4$, we get

$$\left(\exp_{y_0}(K)\right)^k = (K^k s + y^k_0)\bigg|_{s=1} = K^k + y^k_0.$$

Thus, according to (3.3.9), the observer mapping with respect to $Y$ is

$$\varphi_Y(\vec{x}) = \begin{pmatrix} ct_0 - |\vec{x}|Y^0_0 + x^aY^0_a \\ y^1_0 - |\vec{x}|Y^1_0 + x^aY^1_a \\ y^2_0 - |\vec{x}|Y^2_0 + x^aY^2_a \\ y^3_0 - |\vec{x}|Y^3_0 + x^aY^3_a \end{pmatrix}.$$

Since $\partial$ is a global frame of reference field and $Y \in \mathcal{P}$, the matrix $\Lambda \in \text{End}(\mathbb{R}^4)$ with components $\Lambda^i_j \equiv Y^i_j$ is an element of Lor$_4$. Under a coordinate change via the Poincaré transformation $y' := \Lambda \cdot (y - y_0)$, the observer mapping takes the form

$$\varphi_{Y'}(\vec{x}) = \begin{pmatrix} -|\vec{x}| \\ \vec{x} \end{pmatrix}.$$

This shows that distances are indeed given by (3.2.3), as dictated by special relativity. Moreover, we see that $\varphi_{Y'}$ (and hence $\varphi_Y$) is invertible, if restricted to its image. In practice, we determine $\vec{x} = \vec{y}'$ and then check whether $ct' = -|\vec{y}'|$ is satisfied to guarantee that the observer actually sees the event. ♦

To compute the observer mapping in general spacetimes, we need to solve the geodesic equation for arbitrary initial conditions. Following Perlick (cf. [96]), Langrangian and Hamiltonian techniques may be applied to do so. See [96, §5] for specific examples and
further references. It should be noted that solving the geodesic equations analytically is usually a very difficult, if not impossible task.

A further difficulty is imposed by the fact that in practical applications it is often required to invert the observer mapping. An inverse is needed, since we would like to assign relative positions and mutual distances to different observed events on the past light cone, not on the tangent past light cone. The following lemma states that this is possible, if one only considers events ‘close enough’ to the observer.

**Lemma 3.3.6**
Let \((Q, g, O)\) be a spacetime and \(\xi_q : M_q \to Q\) be the static observer mapping. Then there exists an open neighborhood \(V\) of \(0 \in T_q Q\) such that the restriction of \(\xi_q\) to \(V \cap M_q\) is a diffeomorphism onto its image. \(\blacksquare\)

**Proof** As already shown, \(\exp_q\) has full rank at \(0\), hence there exists an open neighborhood \(V\) of \(0\), such that \(\exp_q|_V\) is a diffeomorphism onto its image. Again, since \(c_q^-\) is an embedded submanifold of \(T_q Q\) and \(V \cap c_q^- \neq \emptyset\), the latter is also an embedded submanifold of \(T_q Q\) having the same dimension as \(c_q^-\). Restricting \(\exp_q\) yields the result. \(\blacksquare\)

Of course, one would like to be able to (smoothly) invert \(\xi_q\) globally to define directions and distances on the whole of \(C_q^-\). Physically, the phenomenon of (strong) gravitational lensing implies that there cannot be a 1-1 map between ‘the world as the observer sees it’ and ‘the world as it is’ (cf. footnote 7 on page 61). Mathematically, this ‘non-invertibility’ of the observer mapping is due to two effects: First the conjugate locus in \(T_q Q\) may intersect the tangent past light cone, preventing the map from being an immersion, and second the light cone may intersect itself, preventing it from being injective. We refer to the article by Perlick [96, §2.6 & §2.7] for an introductory discussion on this issue.

References [54,59,71] provide further reading on the relativistic description of gravitational lensing.

Despite this problem, we can still prove a weak kind of invertibility for the static observer mapping. First we recollect some definitions.

**Definition 3.3.7 (Almost everywhere locally invertible maps)**
Let \(M, N\) be smooth manifolds.

i) A subset \(S\) of \(M\) is said to have \(\text{(Lebesque) measure zero}\), if it has Lebesque measure zero in each chart.

ii) A smooth map \(\varphi : M \to N\) is said to be \textit{almost everywhere locally invertible}, if there exists a set \(S\) of (Lebesque) measure zero in \(M\) such that for every \(m \in M \setminus S\) there is an open neighborhood \(V\) of \(m\) in \(M\) for which \(\varphi|_V\) is a diffeomorphism onto its image. \(\blacksquare\)

One can show (cf. [22, Lem. 6.6]) that for a subset \(S\) of a manifold to have Lebesque measure zero, it is sufficient to prove this for a collection of charts whose domains cover \(S\). Similarly, there exists an arguably simpler condition for a map to be almost everywhere locally invertible.

**Lemma 3.3.8 (Condition for local invertibility almost everywhere)**
A (smooth) map is almost everywhere locally invertible if and only if its set of critical points has measure zero and the dimension of the target manifold is greater than or equal to the dimension of the domain. \(\blacksquare\)
3 The Splitting Construction

Proof First observe that, if the dimension of the target manifold $N$ of any smooth map $\varphi: M \to N$ is less than the dimension of the domain manifold $M$, the map cannot be an immersion at any point $m \in M$ and thus cannot be a local diffeomorphism in a neighborhood of $m$. Consequently, we obtain this as a necessary condition for local invertibility.

Next we show that the set $S$ from above is the set of critical points. Let $m$ be a regular value of $\varphi$. Then there exists an open neighborhood $V$ of $m$ in $M$ such that $\varphi|_V$ is an immersion and by the constant rank theorem (see e.g. [22, Thm. 4.12]), we can choose $V$ such that $\varphi|_V$ is a diffeomorphism onto its image. Hence $S \subseteq \text{crit } \varphi$. On the other hand, if $m \in \text{crit } \varphi$, then $(\varphi_*)_m$ does not have full rank and hence there cannot exist an open neighborhood $V$ around $m$ on which $\varphi|_V$ is a diffeomorphism onto its image. So $\text{crit } \varphi \subseteq S$ and thus $\text{crit } \varphi = S$.

Finally, Lebesque-measurability of $\text{crit } \varphi$ is guaranteed by the fact that $\text{crit } \varphi$ is closed (cf. [22, Prop. 4.1]) and thus a Borel set. ■

Example 3.3.9 (Almost everywhere locally invertible map)
Consider the smooth function
$$\varphi: \mathbb{R} \to \mathbb{R}^2: s \to (s^2, s^2) .$$

Except at the origin $\{0\}$, it is an immersion everywhere. Thus the map is almost everywhere locally invertible and we may even analytically express its two (maximal) local inverses
$$\varphi^\pm: \{ (x^1, x^2) \in \mathbb{R}^2 | x^1 = x^2 > 0 \} \to \mathbb{R}_\pm$$
$$\quad : (x^1, x^2) \to \varphi^\pm (x^1, x^2) = \pm \sqrt{x^1} .$$

Of course, we intend to show that the static observer mapping is almost everywhere locally invertible. From our discussion of the Lorentzian exponential in section 2.4, we know that the set of critical points of the static observer mapping $\xi_q$ at $q \in \mathcal{Q}$ is the past lightlike conjugate locus, i.e. the intersection of the conjugate locus $\text{crit } \exp_q$ with the tangent past light cone $c^-_q$. So by Lemma 3.3.8, we require the following for the static observer mapping to be almost everywhere locally invertible.

Proposition 3.3.10 (Past lightlike conjugate locus has measure zero)
Let $(\mathcal{Q}, g, \mathcal{O})$ be a spacetime and $\xi_q$ be the static observer mapping at $q \in \mathcal{Q}$.
Then the past lightlike conjugate locus $\text{crit } \xi_q$ has measure zero in the domain $M_q$ of $\xi_q$. ◊

Proof The idea is to use the isolatedness of conjugate points along lightlike geodesics (cf. Lemma 2.4.5 on page 40) to show that the integral of the past lightlike conjugate locus $\text{crit } \xi_q$ vanishes in radial direction. This in turn implies that the entire volume of $\text{crit } \xi_q$ needs to vanish.

We shall first give meaning to the word ‘radial direction’: Take $n + 1$ to be the dimension of $\mathcal{Q}$. Since $M_q$ is an open submanifold of the past light cone $c^-_q$ and it is more convenient to work on the latter, we choose some timelike vector $Z$ in $T_q \mathcal{Q}$ and consider the map
$$\| \cdot \|: c^-_q \to \mathbb{R}: K \to \| K \| := \sqrt{-g_q(K^\perp, K^\perp)} .$$

Here $\perp$ denotes the orthogonal projection with respect to $Z$. For all $K \in c^-_q$ the vector $K^\perp$ is spacelike and $\| K \| > 0$, so $\| K \|$ may be interpreted as the ‘length’ of $K$. Using this
3.3 Static Splitting

length, identify the \((n - 1)\)-sphere \(S^{n-1}\) as a (possibly 0-dimensional) submanifold of \(\mathcal{Q}_q\). Observe that the map

\[
\mathcal{Q}_q \to \mathbb{R}_+ \times S^{n-1}: K \to \left(\|K\|, \frac{K}{\|K\|}\right)
\]

is a smooth bijection. Since the inverse is just multiplication by a strictly positive number, it is a diffeomorphism. So we have indeed obtained a splitting of the past light cone into ‘radial’ and ‘angular’ parts.

It remains to do the integration: Consider the induced Borel product measure \(B := B_{\mathbb{R}_+} \times B_{S^{n-1}}\) on \(\mathbb{R}_+ \times S^{n-1}\) as a measure on \(\mathcal{Q}_q\). The Borel measure coincides with the Lebesque measure on Borel sets, so a Borel set \(A\) on \(\mathcal{Q}_q\) has Lebesque measure zero if and only if it has measure zero with respect to the product measure \(B\). Again note that the set of critical points \(\text{crit} \xi_q\) is closed and thus Borel-measurable. Now, for every \(Y \in S^{n-1} \subset \mathcal{Q}_q\), the map

\[
\{r \in \mathbb{R}_+ | rY \in \mathcal{Q}\} \to \mathbb{Q}: r \to \exp_q (rY)
\]

is a geodesic, so we use the isolatedness of lightlike conjugate values (Lemma 2.4.5) to conclude that the set

\[
S_Y := \{r \in \mathbb{R}_+ | rY \in \text{crit} \xi_q\}
\]

is (at most) countable. Thus \(B_{\mathbb{R}_+} (S_Y) \equiv 0\). So if we denote by \(dY\) the ‘volume element’ for the measure \(B_{S^{n-1}}\) and apply ‘Fubini’s theorem in measure theory’ (cf. [6, Thm. 3.4.1]), we indeed find

\[
B (\text{crit} \xi_q) = \int_{S^{n-1}} B_{\mathbb{R}_+} (S_Y) \ dY = 0.
\]

Therefore the set, where the static observer mapping is not locally invertible, is ‘negligible’. For the sake of coherence and ease of referencing, we state the proven theorem below.

**Corollary 3.3.11 (Static observer mapping inversion theorem)**
The static observer mapping is almost everywhere locally invertible. The critical set is the past lightlike conjugate locus.

**Proof** Again, the second sentence follows from the characterization (2.4.8) of the critical points of the exponential on page 40. Now Lemma 3.3.8 in conjunction with Proposition 3.3.10 yields the assertion.

We conclude our treatment of the static splitting with a computation of the differential of \(\xi_q\) in terms of Jacobi fields.

**Proposition 3.3.12 (Differential of static observer mapping)**
Let \((\mathcal{Q}, g, \mathcal{O})\) be a spacetime of dimension \(n + 1\) and \(\varphi_X\) be the static observer mapping with respect to the frame of reference \(X\) at \(q\).

Then for all \(\vec{x} \in \text{dom} \varphi_X \subseteq \mathbb{R}^n \setminus \{0\}\) and \(a \in \{1, \ldots n\}\):

\[
\left(\left(\varphi_X\right)_* \frac{\partial}{\partial x^a}\right)_{\vec{x}} = J^a_{\vec{x}} (\vec{x}) ,
\]

(3.3.12a)
3 The Splitting Construction

where \( J^a (\vec{x}) : s \rightarrow J^a_s (\vec{x}) \) is the unique Jacobi field along the geodesic

\[
s \rightarrow \exp_q (s(-|\vec{x}| X_0 + x^a X_a))
\]

with \( J^a_0 (\vec{x}) = 0 \) and

\[
\left( \frac{\nabla J^a (\vec{x})}{ds} \right)_0 = -\delta_{ab} x^b |\vec{x}| X_0 + X_a.
\] (3.3.12b)

\[\Box\]

**Proof** We consider the curve \( x^a \rightarrow -|\vec{x}| X_0 + x^b X_b \) in \( TQ \), its tangent vector field \( Z : x^a \rightarrow Z_{x^a} \) and apply Theorem 2.4.4 from page 39. Since the curve stays within the fiber \( T_q Q \), we have \( J^a_0 (\vec{x}) \equiv 0 \). If \( K \) is the Levi-Civita connector, then

\[
\left( \frac{\nabla J^a (\vec{x})}{ds} \right)_0 = K (Z_{x^a}) = \frac{\nabla}{dx^a} (-|\vec{x}| X_0 + x^b X_b)
\]

yielding (3.3.12b).

Proposition 3.3.12 might be useful for finding an approximate expression of \( \xi^*_q g \) in observer coordinates. We refer to the book by Sakai [34, §3.1] for an analogous expression in Riemannian geometry, and to the article by Klein and Collas [83] for references to similar work already done in relativity theory. Also note that the differential of \( \xi_q \) in radial direction never vanishes, hence the kernel of \( (\xi_q)_* \) is at most \((n-1)\)-dimensional for \((n+1)\)-dimensional \( Q \). In particular, for 2-spacetimes the static observer mapping, if restricted to a small enough domain, is always a diffeomorphism onto its image.

3.4 Kinematic Splitting

Based on the findings of sections 3.1 and 3.3, we obtain a natural definition of a spacetime-splitting by ‘adding the time-dimension’. However, going over to the kinematic case requires additional considerations, as the kinematics on the spacetime needs to be related to the kinematics in the ‘observer spacetime’ and appropriately interpreted. To do this, we first define and analyze the kinematic observer mapping in section 3.4.1 and then consider so called moving frames of reference in section 3.4.2 in order to introduce so called ‘observer spacetimes’ thereafter. Observer spacetimes are needed to rigorously relate the dynamics and kinematics on the spacetime \( Q \) to the observed kinematics, as seen by an actual physical observer.

3.4.1 Kinematic Observer Mapping

We now adapt the static splitting to the kinematic case, i.e. where motion comes into play. In the spirit of Definition 3.3.1 and Proposition 3.3.2, we first define the kinematic observer mapping and then prove its smoothness. Afterwards, we treat the measurement of time in the formalism and the question whether the kinematic observer mapping can be inverted.

As already indicated, the kinematic splitting is obtained by taking an observer \( \gamma : I \rightarrow Q : \tau \rightarrow \gamma (\tau) \) and considering for each time \( \tau \in I \) the observer space \( M_{\gamma (\tau)} \) induced by \( \dot{\gamma}_\tau \), as well as the observer mapping \( \xi_{\gamma (\tau)} \).
3.4 Kinematic Splitting

Definition 3.4.1 (Kinematic observer mapping)
Let \((Q, g, O)\) be a spacetime and \(\gamma: \mathcal{I} \to Q\) be an observer. The set

\[ C^-_\gamma := \bigcup_{\tau \in \mathcal{I}} c^-_{\gamma(\tau)} = \bigcup_{\tau \in \mathcal{I}} \{\tau\} \times c^-_{\gamma(\tau)} \subset \gamma^* TQ \]

is called the past tangent light cone along \(\gamma\). For each \(\tau \in \mathcal{I}\) define

\[ \mathcal{M}^-_\tau := \mathcal{M}_{\gamma(\tau)} = c^-_{\gamma(\tau)} \cap \left(\text{dom exp}_{\gamma(\tau)}\right) , \]

and

\[ \mathcal{M}^- := \bigcup_{\tau \in \mathcal{I}} \mathcal{M}^-_\tau \subseteq C^-_\gamma . \]

Then the (kinematic) observer mapping (for \(\gamma\)) is

\[ \xi^-: \mathcal{M}^- \to Q: (\tau, K) \to \xi^- (\tau, K) := \xi_{\gamma(\tau)} (K) = \exp_{\gamma(\tau)} (K) . \quad (3.4.1) \]

The image of the kinematic observer mapping \(\xi^-\) is the union of the past light cones \(C^-_{\gamma(\tau)}\) over all \(\tau \in \mathcal{I}\). In general \(\xi^-\) is not surjective, which corresponds to the physical situation that the observer does not see the entire spacetime in the temporal interval \(\mathcal{I}\).

One may now object to this construction, that it does not yield a ‘full splitting’ of the spacetime. However, this criticism is physically unwarranted, as, following the discussion in section 3.2, the separation between space and time is only sensible for individual physical observers, and so it would be an inadmissible assumption to demand that it ‘sees’ the entire spacetime.

To prove that the map \(\xi^-\) is smooth we employ the following Lemma.

Lemma 3.4.2 (Global sections are closed maps)
Let \(M, E\) be manifolds and let \(\pi: E \to M\) be a surjective submersion. Then every global section \(s: M \to E\) is a closed map.

Proof Let \(V\) be closed in \(M\) and consider an arbitrary sequence \(\{y_i \in s(V) | i \in \mathbb{N}\}\) converging to \(y \in E\). Then \(x_i := \pi(y_i)\) with \(i \in \mathbb{N}\) defines a sequence in \(V\) with \(y_i = s(x_i)\). As \(V\) is closed in \(M\), its limit \(x = \pi(y)\) lies in \(V \subseteq \text{dom } s\). Continuity of \(s\) yields:

\[ y = \lim_{i \to \infty} s(x_i) = s(x) \in s(V) \subseteq E . \]

We remark that the statement is false for local sections, since their domains are not always closed in \(M\) and thus one may have \(\pi(y) \notin \text{dom } s\).

Proposition 3.4.3 (Domain & smoothness of kinem. observer mapping)
Let \((Q, g, O)\) be a spacetime of dimension \(n + 1\) and let \(\xi^-\) be the kinematic observer mapping for the observer \(\gamma: \mathcal{I} \to Q\).

Then there is a unique manifold structure on the domain \(\mathcal{M}^-\) of \(\xi^-\), such that it is an embedded submanifold of \(\gamma^* TQ\). With respect to this manifold structure, \(\mathcal{M}^-\) is connected and \(\xi^-\) is smooth. Moreover, if \(X: \mathcal{I} \to \text{Ofr}(Q, g): \tau \to (X)_\tau\) is an orthonormal frame field along \(\gamma\), then the map

\[ x: \mathcal{M}^- \to \bigcup_{\tau \in \mathcal{I}} \{\epsilon\} \times \mathbb{R}^- (\mathcal{M}^-_\tau) \subseteq (\epsilon \mathcal{I}) \times (\mathbb{R}^n \setminus \{0\}) \]
3 The Splitting Construction

\[ (\tau, K) \rightarrow x(\tau, K) = (c\tau, X^1_{\tau} \cdot K, \ldots, X^n_{\tau} \cdot K) \]  

(3.4.3)

defines global coordinates on the manifold \( M^\gamma \).

Please note that the factor of \( c \) in (3.4.3) is a convention, which guarantees that all coordinate values have the physical dimension of length.

**Proof** First we show that the tangent light cone along \( \gamma \)

\[ \mathcal{C}_\gamma = \bigsqcup_{\tau \in \mathcal{I}} \mathcal{C}_\gamma(\tau) \]

is an embedded submanifold of \( \gamma^* TQ \).

In full analogy to the proof of Proposition 3.1.8 on page 48, we exclude from \( \gamma^* TQ \) the image of \( \mathcal{I} \) under the zero-section along \( \gamma \) to obtain the set \( \mathcal{N} \). By Lemma 3.4.2, this image is closed, hence \( \mathcal{N} \) is a (non-empty) open submanifold of \( \gamma^* TQ \). Now consider the map

\[ p: \mathcal{N} \subset \mathcal{I} \times TQ \rightarrow \mathbb{R}: (\tau, Z) \rightarrow p(\tau, Z) := g_{\gamma(\tau)}(Z, Z) \]

and observe that \( p^{-1}(\{0\}) = \mathcal{C}_\gamma \subset \gamma^* TQ \). As in (3.1.9b), we want to compute the differential of \( p \) and apply the regular value theorem. So first we define the vertical lift on \( \gamma^* TQ \) via

\[ \bar{Z}_{(\tau,Y)}(f) := \frac{\partial}{\partial s} \bigg|_0 f(\tau, Y + sZ) \]

for each \((\tau, Y) \in \gamma^* TQ, Z \in T_{\gamma(\tau)}Q \) and \( f \in C^\infty(\gamma^* TQ, \mathbb{R}) \). Then for each \((\tau, K) \in \mathcal{C}_\gamma \) and \( Z \in T_{\gamma(\tau)}Q \) we find

\[ p_* \bar{Z}_{(\tau,K)} = 2g_{\gamma(\tau)}(K, Z). \]

Again, non-degeneracy of \( g \) and \( K \neq 0 \) implies that \( p \) is a submersion on \( \mathcal{C}_\gamma \). Thus \( \mathcal{C}_\gamma \) is an embedded submanifold of \( \mathcal{N} \) and of \( \gamma^* TQ \) of dimension \((1 + (n + 1)) - 1 = n + 1 \). As such, its manifold structure is unique.

To obtain \( \mathcal{C}_-^\gamma \), we proceed as in Proposition 3.1.8 and use the continuous function

\[ p': \mathcal{C}_\gamma \rightarrow \mathbb{R}: (\tau, K) \rightarrow p'(\tau, K) := g_{\gamma(\tau)}(\dot{\gamma}_{\tau}, K) \]

to show that it splits into \( \mathcal{C}_-^\gamma = p'((\infty, 0)) \) and \( \mathcal{C}_+^\gamma =: p'((0, \infty)) \).

Recalling the coordinates \( \vec{x}_- \), as given in (3.1.8a) on page 48, we find that

\[ \mathcal{C}_-^\gamma \rightarrow \mathcal{I} \times (\mathbb{R}^n \setminus \{0\}) : (\tau, K) \rightarrow (c\tau, \vec{x}_-(K)) \]

is also a global coordinate map, which is compatible with the smooth structure. According to Proposition 2.4.3, the set \( \text{dom exp} \) is open and thus \( M^\gamma = \mathcal{C}_-^\gamma \cap \text{dom exp} \neq \emptyset \) is an open submanifold of \( \mathcal{C}_-^\gamma \) with global coordinates \( x' \), as given by (3.4.3).

As each slice \( \{\tau\} \times M^\gamma_{\gamma(\tau)} \) is connected, so is \( M^\gamma \). As \( M^\gamma \) is a submanifold of \( \text{dom exp} \) and \( \text{exp} \) is smooth, the restriction \( \xi^\gamma \) is also smooth.  

\[ \blacksquare \]

Obviously, each slice \( \{\tau\} \times M^\gamma_\tau \) at proper time \( \tau \in \mathcal{I} \) is to be considered the observer space at \( \gamma(\tau) \) induced by the observer vector \( \dot{\gamma}_\tau \). Yet we will postpone the definition of an ‘observer spacetime’, carrying the physically appropriate geometric structures, to the next subsection. Roughly speaking, the reason is that it is not possible to assign mutual
3.4 Kinematic Splitting

Figure 3.3: In this image we see a portion of 3-dimensional Minkowski spacetime embedded in Euclidean 3-space. The accelerated observer $\gamma$ is indicated by a curved line going upwards. At three different instances $\tau_1, \tau_2, \tau_3$ of $\gamma$’s proper time, a part of the respective past light cone is shown. Since the spacetime is flat, the cones do not intersect themselves. Lines of constant radial distance, as measured by the observer, are schematically drawn in accordance with the direction of $\gamma$’s tangent vector. We remind the reader that orthogonality in Minkowski space differs from orthogonality in Euclidean space.
3 The Splitting Construction

distances to events seen at different times without the choice of a particular frame of reference at those times. Nonetheless, the crude mathematical construction is set up at this point and an example is depicted in figure 3.3.

The measurement of temporal distances does, however, not require any choice of reference frame. As the observer $\gamma$ is parametrized with respect to proper time, the time passed between two observed events $(\tau_1, K_1), (\tau_2, K_2) \in \mathcal{M}_\gamma$ is simply $|\tau_2 - \tau_1|$. In addition, the sign of $\tau_2 - \tau_1$ gives information on which observed event precedes the other one in an obvious way, so we also have a ‘temporal orientation’. Infinitesimally, both can be encoded in the 1-form $d\tau$, where $x^0 = c\tau$ denotes the zeroth coordinate function in (3.4.3). Hence a tangent vector $Y \in T\mathcal{M}_\gamma$ may be called future-directed relative to $\gamma$, if $d\tau(Y) > 0$ and past-directed relative to $\gamma$, if $d\tau(Y) < 0$. For $d\tau(Y) = 0$, it is spatial relative to $\gamma$. Since $\mathcal{M}_\gamma \subset I \times TQ$, this happens precisely when it is tangent to the respective past tangent light cone.

In principle, if $\varrho: J \rightarrow \mathcal{M}_\gamma: s \rightarrow \varrho(s) = (\tau(s), K_s)$ (3.4.4) is a curve in $\mathcal{M}_\gamma$ with $\dot{\tau} := \frac{d\tau}{ds} > 0$, we may measure the time passed between the endpoints by integration:

$$\int_{\varrho} d\tau = \int_{J} \varrho^* d\tau = \int_{J} \frac{d\tau}{ds} ds.$$ 

On categorical grounds, we call such a curve future-directed relative to $\gamma$. Analogously, we have past-directed and spatial curves relative to $\gamma$. A priori, it is not guaranteed, that a future-directed curve $\varrho$ relative to $\gamma$, is also future-directed on $Q$ under the kinematic observer mapping $\xi$. Indeed, it may happen that $\xi \circ \varrho$ is not even time- or lightlike. A counterexample is provided later in Example 3.4.20. The next theorem gives a direct relation between the time directions on $Q$ and $\mathcal{M}_\gamma$.

**Theorem 3.4.4 (Consistency of time directions)**

Let $(Q, g, \mathcal{O})$ be a spacetime and $\xi$ be the kinematic observer mapping for the observer $\gamma$. Further, let $pr_2: \gamma^* TQ \rightarrow TQ$ be the projection on the second factor.

i) If the curve $\varrho$, as in (3.4.4), is a future-directed curve relative to $\gamma$ such that $\xi \circ \varrho$ is either time- or lightlike on $Q$, then $\xi \circ \varrho$ is future-directed. In the former case, there exists a smooth (orientation-preserving) reparametrization such that the reparametrized curve is an observer.

ii) Conversely, if $\varrho': J \rightarrow Q$ is an observer on $Q$, such that there exists a smooth curve $\varrho$ with $\varrho' = \xi \circ \varrho$, then $\varrho$ is future-directed relative to $\gamma$.

iii) If instead $\varrho': J \rightarrow Q$ is a future-directed, lightlike curve, such that there exists a smooth curve $\varrho$ with $\varrho' = \xi \circ \varrho$, then for each $s \in J$ the tangent vector $\dot{\varrho}_s$ is either future-directed or spatial relative to $\gamma$. If it is spatial, then it is tangent to the geodesic $r \rightarrow \exp (rK_s)$ with $K_s := pr_2 \circ \varrho(s)$.

**Proof** “i) ” We borrowed the idea of proof from Perlick [94]. As $K_s := pr_2 \circ \varrho(s)$ is lightlike for each $s \in J$

$$0 = \frac{d}{ds} g(K, K) = 2 g \left( \frac{\nabla K}{ds}, K \right).$$
We now write $\xi^\gamma \circ \vartheta = \exp \circ K$ and thus $(\xi^\gamma \circ \vartheta) = \exp_s K$. For $s \in J$, apply Theorem 2.4.4 from page 39 to get $\exp_s K = J^s$ for the respective Jacobi field $J^s$ along the geodesic $\gamma: r \to \dot{\gamma}(r) = \exp(r K_s)$. Afterwards, we use Lemma 2.4.1 from page 37 to compute

$$g(\dot{\gamma}_1^s, J^s_m) = g\left(\frac{\dot{\gamma}_0^s}{\gamma_1^s}, \left(\frac{\nabla J^s_m}{dr}\right)_0\right) + g\left(\dot{\gamma}_0^s, J^s_m\right) = g\left(K^s, \left(\frac{\nabla K}{ds}\right)_s\right) + g\left(K^s, (\gamma \circ \tau)_s\right) = \dot{\tau}(s) g\left(K^s, \dot{\gamma}_{\tau(s)}\right),$$

(3.4.5a)

The expression is negative, since $\dot{\tau} > 0$, $\dot{\gamma}_{\tau(s)}$ is future-directed timelike and $K^s$ is past-directed lightlike. On the other hand $\frac{\dot{\gamma}_1^s}{\gamma_1^s}$ is the parallel transport $P^0_{\dot{\gamma}}(K^s)$ of $K^s$ to $\exp(K^s)$ along $\dot{\gamma}$, hence past-directed timelike. As $J_1 = (\xi^\gamma \circ \vartheta)$ is time- or lightlike, it must be future-directed. Finally, every smooth, future-directed, timelike curve $\vartheta': J \to Q$ can be reparametrized to a smooth observer curve by calculating

$$s'(s) = \frac{1}{c} \int_{\inf J}^s \sqrt{g\left(\dot{\vartheta}_s', \dot{\vartheta}_s'\right)} \, ds'$$

for each $s \in J$.

\textit{ii) ”:} Again we have for all $s \in J$

$$g\left(P^{0,1}_{\dot{\gamma}}(K^s), \dot{\vartheta}_s\right) = \dot{\tau}(s) g\left(K^s, \dot{\gamma}_{\tau(s)}\right),$$

(3.4.5b)

as in (3.4.5a) above. As $\dot{\vartheta}_s$ is future-directed timelike, it must ‘lie above’ the lightlike hyperplane $\left(\mathbb{R} P^{0,1}_{\dot{\gamma}}(K^s)\right)^\perp$, so the left hand side is negative. Since the second factor on the right hand side is also negative, we conclude $\dot{\tau} > 0$.

\textit{iii):} Again we use (3.4.5b). $\dot{\vartheta}_s$ is future-directed lightlike implies that it lies either in or ‘above’ the lightlike hyperplane $\left(\mathbb{R} P^{0,1}_{\dot{\gamma}}(K^s)\right)^\perp$. In the former case, there exists a constant $\alpha > 0$ such that

$$\dot{\vartheta}_s' = -\alpha P^{0,1}_{\dot{\gamma}}(K^s),$$

which implies that it is tangent to the geodesic $r \to \exp(r K_s)$. Moreover, (3.4.5b) yields $\dot{\tau} = 0$. In the other case, the left hand side of (3.4.5b) is negative and thus again $\dot{\tau} > 0$.

Roughly speaking, Theorem 3.4.4 states that the time directions on $Q$ and $M^7$ are mutually consistent. The timelike case in i) and ii) effectively describes the situation that one observer $\gamma$ ‘sees’ another observer $\vartheta'$. Then the function $\tau: s \to \tau(s)$ from (3.4.4) gives the relations between the respective proper times measured and its derivative $\dot{\tau}$ may be understood as the redshift measured by $\gamma$ (cf. [35, p. 48 sqq.;38, p. 108 sq.;46; 94, §IV]). Since $\dot{\tau} > 0$, the function $\tau$ is strictly increasing and has a smooth inverse. Therefore, the theory predicts that it is physically impossible to see another observer whose clock stands still or even runs backwards. The lightlike case in i) and iii) includes the physical situation that an observer ‘sees’ the movement of ‘light’. In the spatial case of iii), the ‘light ray’ is pointed at the observer and thus no time needs to pass for the observer to see it moving.
We now conclude our discussion of the measurement of time in the splitting and continue with the question whether the kinematic observer mapping is invertible.

As stated in the previous section 3.3, the static observer mapping $\xi_{\gamma(\tau)}$ at the point $\gamma(\tau) \in \mathcal{Q}$ need neither be injective nor an immersion. So we conclude that the same holds true for the kinematic observer mapping for the observer $\gamma$. Nonetheless, we were able to prove two propositions, namely Lemma 3.3.6 on page 65 and Corollary 3.3.11 on page 67, that gave us at least a local form of invertibility. The next theorem is the kinematic analogue of Corollary 3.3.11.

**Theorem 3.4.5 (Kinematic observer mapping inversion theorem)**

Let $(\mathcal{Q},g,\mathcal{O})$ be a spacetime and $\xi_{\gamma}$ be the kinematic observer mapping for the observer $\gamma: \mathcal{I} \to \mathcal{Q}$.

Then $\xi_{\gamma}$ is almost everywhere locally invertible. The critical set is

$$\text{crit} \xi_{\gamma} = \bigcup_{\tau \in \mathcal{I}} \text{crit} \xi_{\gamma(\tau)}.$$

(3.4.6)

**Proof** For each $Z \in T\mathcal{M}^\gamma$ there exists a curve $\vartheta$, as in (3.4.4), with $\dot{\vartheta}_0 = Z$. Recalling equation (3.4.5b) from above for $s = 0$ with $\dot{\vartheta}'_0 = \xi_{\gamma}^* Z$, we see that $\dot{\tau}(0) \neq 0$ implies that $\xi_{\gamma}^* Z \neq 0$. Hence we may restrict ourselves to the case $\dot{\tau}(0) = 0$, i.e. $Z$ is spatial with respect to $\gamma$ and thus tangent to $\{\tau(0)\} \times \mathcal{C}_{(\gamma \circ \tau)(0)}$. Consequently, $\xi_{\gamma}^* Z = 0$ if and only if $(\xi_{(\gamma \circ \tau)(0)})^* K_0 = 0$. This proves (3.4.6).

Recalling Lemma 3.3.8 on page 65 and that $\dim \mathcal{M}^\gamma = \dim \mathcal{Q}$, we need to show that the set $\text{crit} \xi_{\gamma}$ has measure zero. So we introduce coordinates $x'$ on $\mathcal{M}^\gamma$ in accordance with (3.4.3), define the canonical Borel product measure $B := B_{\mathcal{I}} \times B_{\mathbb{R}^n}$ on $\mathcal{I} \times \mathbb{R}^n$, identify $\mathcal{M}^\gamma$ with its image in $\mathcal{I} \times \mathbb{R}^n$ under $x'$ and restrict $B$ to $\mathcal{M}^\gamma$. (3.4.6) now gives a splitting of $\text{crit} \xi_{\gamma}$ into sets $\text{crit} \xi_{\gamma(\tau)}$ of measure zero with respect to $B_{\mathbb{R}^n}$. In analogy to the proof in Proposition 3.3.10, we apply ‘Fubini’s theorem in measure theory’ (cf. [6, Thm. 3.4.1]) to obtain the result.

Therefore, for each time $\tau \in \mathcal{I}$ and every ‘position’ $K \in \mathcal{M}^\gamma$, which is not a conjugate value, Theorem 3.4.5 states that there exists a neighborhood of the observed event $(\tau,K)$ in $\mathcal{M}^\gamma$ that can be identified with a subset of the spacetime. Figuratively speaking, we can then find a direct correspondence between a small part of the ‘spacetime’, as the observer sees it, and a small part of the ‘actual’ spacetime.

We believe that there also exists a kinematic analogue of Lemma 3.3.6, but so far we have been unable to prove it.

**Conjecture 3.4.6**

Let $(\mathcal{Q},g,\mathcal{O})$ be a spacetime and $\xi^\gamma$ be the kinematic observer mapping for the observer $\gamma: \mathcal{I} \to \mathcal{Q}$.

Then for every $\tau \in \mathcal{I}$ there exists an open neighborhood $\mathcal{V}$ of $(\tau,0)$ in $\gamma^* T\mathcal{Q}$, such that the restriction of $\xi^\gamma$ to $\mathcal{V} \cap \mathcal{M}^\gamma$ is a diffeomorphism onto its image.

Conjecture 3.4.6 would make it possible to view (restricted) observer coordinates as coordinates on $\mathcal{Q}$, in analogy to what Klein et al. did in their article [83].
3.4 Kinematic Splitting

3.4.2 Moving Frames of Reference

In order to give a sensible definition of the word ‘observer spacetime’, we first need to make mathematical sense of the words ‘frame of reference for an observer’, which are often colloquially used in mechanics textbooks. We will also give a rigorous, invariant definition of what it means for a frame of reference to be inertial.

The underlying idea of ‘moving frames’ in general relativity is that each tangent vector \( \dot{\gamma}_\tau \) of the observer \( \gamma \) at time \( \tau \in I \) may be normalized and taken as the zeroth vector \( (X)_0 \) of a frame of reference \( (X)_\tau \) at \( \gamma (\tau) \). Again, the frame of reference represents the ‘orientation’ of the physical observer in space. Letting \( \tau \) vary, we obtain a frame of reference field over \( \gamma \) that is ‘adapted’ to the observer in this sense and thus gives the physical observer’s orientation in space at each time. The following definition formalizes the idea in terms of principal bundles.

Definition 3.4.7 (Frame of reference bundles for observers)

Let \((Q,g,\mathcal{O})\) be a spacetime of dimension \( n + 1 \), \((\mathcal{P}, \hat{\pi}, Q, \text{Lor}_{n+1})\) be the corresponding frame of reference bundle (Definition 3.1.9 on page 49) and \( \gamma : I \to Q \) be an observer.

The \textit{frame of reference bundle for} \( \gamma \), or, equivalently, \textit{reference frame bundle for} \( \gamma \), is the tuple \((\mathcal{P}^\gamma, \text{pr}_1|_{\mathcal{P}^\gamma}, I, \text{SO}_n)\), where

\[
\mathcal{P}^\gamma := \left\{ (\tau, X) \in \gamma^*\mathcal{P} \left| X_0 = \frac{1}{c} \dot{\gamma}_\tau \right. \right\}, \tag{3.4.8a}
\]

\(\text{pr}_1|_{\mathcal{P}^\gamma}\) is the restriction of the projection \(\text{pr}_1 : I \times \mathcal{P} \to I\) to \(\mathcal{P}^\gamma\) and \(\text{SO}_n\) is the special orthogonal group in \( n \) dimensions acting canonically on \(\mathcal{P}^\gamma\) from the right via the representation

\[
\rho : \text{SO}_n \to \text{GL}_{n+1} : A \mapsto \rho(A) := \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}. \tag{3.4.8b}
\]

A frame of reference for the observer \( \gamma \) (at \( \tau \in I \)) is the second component \( X \) of an element \((\tau, X)\) of the fiber \(\mathcal{P}^\gamma_\tau := (\text{pr}_1|_{\mathcal{P}^\gamma})^{-1}(\tau)\).

Of course, we still need to show the frame of reference bundle for an observer is indeed a principal \(\text{SO}_n\)-bundle.

Lemma 3.4.8

Let \((Q,g,\mathcal{O})\) be a spacetime of dimension \( n + 1 \) and let \( \gamma : I \to Q \) be an observer.

Then there exists a unique manifold structure on the frame of reference bundle \(\mathcal{P}^\gamma\) for \( \gamma \) such that it is an embedded \(\text{SO}_n\)-reduction of the frame of reference bundle \(\gamma^*\mathcal{P}\) along \( \gamma \).

\(\diamondsuit\)

PROOF Since \(\mathcal{P}\) is a principal \(\text{Lor}_{n+1}\)-bundle, so is the pullback bundle \(\gamma^*\mathcal{P}\). We wish to apply Theorem 2.2.3 from page 13.

“ iii)” : Since \( I \) is an open subset of \( \mathbb{R} \), \(\gamma^*\mathcal{P}\) admits a global section

\[
(\cdot, X) : I \to \gamma^*\mathcal{P} : \tau \mapsto (\tau, X_\tau). \nonumber
\]

As \(\frac{1}{c} \dot{\gamma}\) is smooth, future-directed timelike and has unit ‘length’, we may chose \( X \) such that \( X_0 = \frac{1}{c} \dot{\gamma} \). Hence \( X \) takes values in \(\mathcal{P}^\gamma\).

“ i)” : This is trivial.
"ii) " Let \( X, Y \) be frames of reference for \( \gamma \) at \( \tau \in \mathcal{I} \), then \( X, Y \in \mathcal{P}_\gamma(\tau) \) and so there exists a unique \( \Lambda \in \text{Lor}_{n+1} \) with \( Y = X \cdot \Lambda \). As \( X_0 = Y_0 \), the remaining vectors lie in the same hyperplane \( (\mathbb{R}X_0)^\perp \) and thus there exists an \( A \in \text{GL}_n \) with

\[
\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.
\]

Since \( \Lambda^T \cdot \eta \cdot \Lambda = \eta \) and \( \det \Lambda = 1 \), \( A \) is indeed in \( \text{SO}_n \).

We conclude that, by Theorem 2.2.3, \( \mathcal{P}_\gamma \) is an \( \text{SO}_n \)-reduction of \( \gamma^* \mathcal{P} \).

Finally, we need to show that \( \mathcal{P}_\gamma \) is embedded in \( \gamma^* \mathcal{P} \). Since \( \text{Lor}_{n+1} \) is embedded in \( \text{GL}_{n+1} \), it is sufficient to show that the restricted map \( \rho: \text{SO}_n \to \text{Lor}_{n+1} \) of (3.4.8b) is a topological embedding. Yet as the map \( \chi: \text{Lor}_{n+1} \to \mathbb{R}^{n+1}: \Lambda \to \Lambda \cdot e_0 = \Lambda^i_0 e_i \) is continuous, \( \text{SO}_n = \chi^{-1}(\{e_0\}) \) is closed in \( \text{Lor}_{n+1} \) and, by Cartan’s theorem, indeed embedded. ■

We call a map \( X: \mathcal{I} \to \mathcal{P} \) a frame of reference (field) for \( \gamma \), if \( (.,X): \mathcal{I} \to \mathcal{P}_\gamma: \tau \to (\tau,X_\tau) \) is a section of \( \mathcal{P}_\gamma \). Of course, one may also define dual frame of reference (fields) for \( \gamma \) or, equivalently, coframe of reference (fields) for \( \gamma \) by using sections of the dual bundle \( \mathcal{P}_\gamma^* \) instead.

If the spacetime is 4-dimensional, the choice of a frame of reference \( X \) for \( \gamma \) is unique up to a smooth map \( A: \mathcal{I} \to \text{SO}_3: \tau \to A(\tau) \). Physically, the map \( A \) corresponds to rotation of the frame of reference \( (X_\tau) \) at each time \( \tau \in \mathcal{I} \) to a new frame of reference \( (Y_\tau) = (X_\tau) \cdot \rho(A(\tau)). \) This naturally raises the question whether it is possible to tell when a frame of reference for an observer is rotating or not.

Indeed, this may be done via the so called Fermi-Walker derivative, named after the physicist Thomas Fermi and mathematician Arthur G. Walker for their original works [65] and [102]. We refer to [16, p. 80 sqq.;33, §2.2; 38, §1.10; 67; 88] for further reading. The following definition is borrowed from the book by Sachs and Wu [33, Prop. 2.2.1].

**Definition 3.4.9 (Fermi-Walker derivative & inertial frames)**

Let \( (\mathcal{Q},g,\mathcal{O}) \) be a spacetime of dimension \( n + 1 \), \( \gamma: \mathcal{I} \to \mathcal{Q} \): \( s \to \gamma(s) \) be a time- or spacelike curve and let \( Y \) be a vector field over \( \gamma \). Denote by \( \parallel \) and \( \perp \) the projection onto the parallel and orthogonal subspaces with respect to \( \dot{\gamma} \), respectively.

Then the **Fermi-Walker derivative of** \( Y \) (along \( \gamma \)) is given by

\[
\frac{FY}{ds} = \left( \frac{\nabla}{ds} (Y^\parallel) \right)^\parallel + \left( \frac{\nabla}{ds} (Y^\perp) \right)^\perp.
\]

(3.4.10a)

\( Y \) is called **Fermi-Walker transported along \( \gamma \)**, or **non-rotating (along \( \gamma \))**, if \( (FY/\text{ds})_s = 0 \) for all \( s \in \mathcal{I} \). For timelike \( \gamma \), \( Y \) is called **rotating (along \( \gamma \))**, if it is non-non-rotating, i.e. there exists an \( s \in \mathcal{I} \) such that \( (FY/\text{ds})_s \neq 0 \). A tangent vector \( Y_s' \in T_{\gamma(s')}^\mathcal{Q} \) over \( \gamma \) is called the **Fermi-Walker transport of** \( Y_s \in T_{\gamma(s)}^\mathcal{Q} \) to \( s' \), if there exists a Fermi-Walker transported vector field \( Y \) over \( \gamma \) taking the respective values.
3.4 Kinematic Splitting

If $X$ is a frame field over $\gamma$, then its Fermi-Walker derivative is given by differentiating the individual vector fields, i.e.

$$\frac{FX}{ds} := \frac{dX_i}{ds} \otimes e^i.$$  \hfill (3.4.10b)

The aforementioned terminology carries over to frames and frame fields over $\gamma$, that is $X_i$ satisfies the required conditions for each $i \in \{0, \ldots, n\}$.

In particular, if $X$ is a frame of reference field for an observer $\gamma$, it is called non-rotating if $\frac{FX}{d\tau} = 0$. Else it is rotating.

Additionally, if $\gamma$ is a non-accelerating observer and $X$ is a non-rotating frame of reference field for $\gamma$, then $X$ is called an inertial frame of reference (field for $\gamma$).

For timelike curves the Fermi-Walker derivative may be physically interpreted as a tool to detect the presence of (infinitesimal) rotation. For spacelike curves it detects twisting of the vector field $Y$ or change of (hyperbolic) angle relative to $\dot{\gamma}$. Both interpretations are derived from the definition, which roughly states that the Fermi-Walker derivative respects the parallel and orthogonal subspaces with respect to $\dot{\gamma}$.\footnote{One may indeed show that Fermi-Walker transport preserves angles along arbitrary time- or spacelike curves $\gamma$. The idea of proof is to first show ‘metricity’ (3.4.13) of the derivative for curves of constant length and then apply (3.4.11b) to conclude that it holds for arbitrary parametrizations. Deriving the angle formula (cf. (2.2.19) on page 22) for $\dot{\gamma}$ and non-rotating $Y$, and observing that $g(Y, Y)$ is constant, gives the result.}

**Remark 3.4.10 (Basic properties of Fermi-Walker derivative)**

Recall that for a time- or spacelike curve $\gamma: I \to Q$ and each $s \in I$ the projectors are given by

$$\left(\pi^\parallel\right)_s = \frac{\dot{\gamma}_s \otimes \dot{\gamma}_s \cdot g}{(\gamma^* g)_s (\frac{\partial}{\partial s}, \frac{\partial}{\partial s})} \quad \text{and} \quad \left(\pi^\perp\right)_s = 1_s - \left(\pi^\parallel\right)_s,$$  \hfill (3.4.11a)

where $1$ is the identity endomorphism field over $\gamma$, i.e. $1 \cdot Y = Y$ for all $Y \in TQ$ with base point in $\gamma(I) \subset Q$. Thus the Fermi-Walker derivative of a vector field $Y$ along $\gamma$ is well-defined, smooth and again yields a vector field along $\gamma$. As the projectors and the Levi-Civita connection are linear, so is the Fermi-Walker derivative. The usage of the word derivative is justified by the fact that it satisfies the *Leibniz rule*:

$$\frac{F}{ds} (fY) = \frac{df}{ds} Y + f \frac{FY}{ds},$$

for all $f \in C^\infty(I, \mathbb{R})$ and vector fields $Y$ along $\gamma$. A straightforward calculation also shows that under a change of parametrization of $\gamma$ we get

$$\frac{F}{ds} = \frac{ds'}{ds} \frac{F}{ds'},$$  \hfill (3.4.11b)

We conclude that the Fermi-Walker derivative is a covariant derivative along $\gamma$ (cf. [26, Def. 2.51]) and thus it yields a vector bundle connection on the pullback bundle $\gamma^* TQ$ with corresponding parallel transport (cf. [26, Cor. 2.59]). However, by (3.4.11a) it is impossible to extend the Fermi-Walker derivative to curves that are neither space- or timelike. Therefore the derivative does not give rise to a connection on the tangent bundle, so the terminology ‘Fermi-Walker connection’ is only admissible in the appropriate context.
In our case of interest, $\gamma$ is an observer and then the Fermi-Walker derivative takes a particularly simple form.

**Lemma 3.4.11 (Fermi-Walker derivative for observers)**

Let $(Q,g,O)$ be a spacetime, $\gamma : I \to Q : \tau \to \gamma(\tau)$ be an observer and $Y$ be a vector field over $\gamma$. Then

$$\frac{F}{d\tau} Y = \frac{\nabla}{d\tau} Y - \frac{1}{c^2} g(\dot{\gamma}, Y) \frac{\nabla\dot{\gamma}}{d\tau} + \frac{1}{c^2} g\left(\frac{\nabla\dot{\gamma}}{d\tau}, Y\right) \dot{\gamma}. \quad (3.4.12)$$

**Proof** For observers $(\gamma^* g) \left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial \tau}\right) = c^2$ and thus $\dot{\gamma} \perp (\nabla\dot{\gamma}/d\tau)$. Using the definitions (3.4.10a), (3.4.11a) and the metricity of the Levi-Civita connection, we obtain

$$\frac{FY}{d\tau} = \pi^\| \left(\frac{\nabla(\pi^\| \cdot Y)}{d\tau}\right) + \pi^\perp \left(\frac{\nabla(\pi^\perp \cdot Y)}{d\tau}\right)$$

$$= \frac{1}{c^2} \left[ \pi^\| \left(\frac{\nabla}{d\tau} (g(\dot{\gamma}, Y) \dot{\gamma})\right) + \pi^\perp \left(\frac{\nabla}{d\tau} \left(Y - \frac{1}{c^2} g(\dot{\gamma}, Y) \dot{\gamma}\right)\right)\right]$$

$$= \frac{1}{c^2} \left[ \frac{\nabla}{d\tau} (g(\dot{\gamma}, Y) \dot{\gamma}) + g(\dot{\gamma}, Y) \frac{\nabla\dot{\gamma}}{d\tau}\right]$$

$$+ \pi^\perp \left(\frac{\nabla Y}{d\tau} - \frac{1}{c^2} \frac{d}{d\tau} (g(\dot{\gamma}, Y) \dot{\gamma} - \frac{1}{c^2} g(\dot{\gamma}, Y) \frac{\nabla\dot{\gamma}}{d\tau})\right)$$

$$= \frac{1}{c^2} \left[ g\left(\frac{\nabla\dot{\gamma}}{d\tau}, Y\right) + g\left(\dot{\gamma}, \frac{\nabla Y}{d\tau}\right)\right] \dot{\gamma}$$

$$+ \left(\frac{\nabla Y}{d\tau} - \frac{1}{c^2} g(\dot{\gamma}, \frac{\nabla Y}{d\tau}) \dot{\gamma} - \frac{1}{c^2} g(\dot{\gamma}, Y) \frac{\nabla\dot{\gamma}}{d\tau}\right).$$

Equation (3.4.12) implies that for non-accelerated observers the Fermi-Walker derivative reduces to the Levi-Civita connection. Moreover, it also shows that the Fermi-Walker connection for observers is metric:

$$\frac{d}{d\tau} g(Y, Z) = g\left(\frac{FY}{d\tau}, Z\right) + g\left(Y, \frac{FZ}{d\tau}\right). \quad (3.4.13)$$

where $Y$ and $Z$ are vector fields along $\gamma$. Therefore, if $X_{\tau_0}$ is a frame of reference for $\gamma$ at $\tau_0 \in I$ and we consider the Fermi-Walker transport of each frame vector along $\gamma$, the resulting collection of vector fields will be an orthonormal frame field along $\gamma$. Since the Fermi-Walker derivative of $\dot{\gamma}$ vanishes and the Fermi-Walker transport map is continuous, this orthonormal frame field will be a frame of reference field for $\gamma$.

Mathematically, this proves that the Fermi-Walker derivative induces a principal bundle connection on the frame of reference bundle $P^\gamma$ for the observer $\gamma$. We refer to the English books by Poor [26] and Rudolph [32], as well as the German one by Baum [2] for an in-depth discussion of principal bundle connections.

We finish our discussion of frame of reference fields for observers with physically relevant examples.

**Example 3.4.12 (Frame of reference fields in Minkowski spacetime)**

We continue Example 3.1.6 from page 47.
3.4 Kinematic Splitting

i) For our prototypical, unaccelerated observer $\gamma$ we have

$$(X_0)_\tau := \frac{1}{c} \dot{\gamma}_\tau = \frac{\partial}{\partial y^0} \bigg|_{\gamma(\tau)} = (\partial_0)_{\gamma(\tau)}$$

for all $\tau \in \mathbb{R}$. Since the coordinate frame field $\partial$ on $\mathbb{R}^4$ is a frame of reference field, any frame of reference field $X$ for $\gamma$ may be written as

$$X = (\partial)_{\gamma} \cdot \rho(A),$$

where $\rho$ is the representation from (3.4.8b) for $n = 3$ and $A: \mathbb{R} \to SO_3$ is a smooth curve. By (3.4.12), the Fermi-Walker derivative reduces to the Levi-Civita derivative for unaccelerated observers. Thus, as $\partial$ is parallel, $X$ is inertial if and only if $A$ is constant. In particular, the frame of reference $X$, satisfying

$$(X_0)_\tau = (\partial)_{\gamma(\tau)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

is inertial. By a Poincaré transformation as in Example 3.3.5, any inertial frame of reference can be brought into this standard form.

ii) For our constantly accelerated observer $\gamma: \mathbb{R} \to \mathbb{R}^4$, as given by (3.1.7c) on page 48, we may assume that it ‘faces in the direction of acceleration’. In mathematical terms, we have for all $\tau \in \mathbb{R}$:

$$(X_1)_\tau := \frac{1}{a} \left( \nabla \dot{\gamma} \right)_\tau = \sinh \left( \frac{a\tau}{c} \right) (\partial_0)_{\gamma(\tau)} + \cosh \left( \frac{a\tau}{c} \right) (\partial_1)_{\gamma(\tau)},$$

by a simple calculation using (3.1.7b). One checks via (3.4.12) that this vector is non-rotating, which means that not only the magnitude, but also the direction of acceleration of $\gamma$ is constant. This is known as uniform acceleration. If we choose the smooth curve

$$A: \mathbb{R} \to SO_3: \tau \to A(\tau) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos (\omega \tau) & -\sin (\omega \tau) \\ 0 & \sin (\omega \tau) & \cos (\omega \tau) \end{pmatrix},$$

representing a rotation around the first axis in positive direction for $\omega \in \mathbb{R}_+$ and in negative direction for $\omega \in \mathbb{R}_- := (-\infty, 0)$, then $X$, given by

$$(X)_\tau := \begin{pmatrix} \cosh \left( \frac{a\tau}{c} \right) & \sinh \left( \frac{a\tau}{c} \right) \\ \sinh \left( \frac{a\tau}{c} \right) & \cosh \left( \frac{a\tau}{c} \right) \end{pmatrix} \begin{pmatrix} \cos (\omega \tau) & -\sin (\omega \tau) \\ \sin (\omega \tau) & \cos (\omega \tau) \end{pmatrix},$$

(3.4.14b)

in standard coordinates for each $\tau \in \mathbb{R}$, is a frame of reference field for $\gamma$. Unless $\omega = 0$, it is rotating. Accordingly, $X$ represents a physical observer accelerating uniformly with $a$ and rotating around its axis of acceleration with constant angular velocity $\omega$. ♦

79
3 The Splitting Construction

3.4.3 Observer Spacetime and relative Motion

After having defined the kinematic observer mapping in section 3.4.1 and frames of references for observers in section 3.4.2, we now give mathematical meaning to the word ‘observer spacetime’ and the physical concept of relative motion.

In section 3.4.1, we have already discussed the measurement of time and the assignment of time directions on the domain $M^\gamma$ of the kinematic observer mapping $\xi^\gamma$. We concluded that the 1-form $d\tau$ is sufficient to define both rigorously. Now additional ‘spatial’ geometric structures need to be carried over from the static splitting (cf. Definition 3.3.3 on page 63) to the kinematic case, i.e. we need to find natural definitions of spatial distances and right-handedness on $M^\gamma$.

To define spatial distances between events observed at different times, we require a frame of reference field $X$ for the observer $\gamma$. The underlying idea is, that taking the components of a vector $K \in M^\gamma_{\tau_0}$ with respect to $X_{\tau_0}$ for some time $\tau_0 \in I$ makes it possible to identify the event at every other time $\tau \in I$. So if $(\tau_1, K_1), (\tau_2, K_2) \in M^\gamma$ are two observed events, their spatial distance is just the distance from section 3.3 between the vectors $K_1, K_2$ ‘transported’ to some time $\tau_0$. Since this distance is invariant under the action of the rotation group $SO_n$ on $P^\gamma$, the definition is independent of the choice of $\tau_0$.

The issue of orientations is more subtle. Of course, if dim $Q = n + 1$, $\tau \in I$ and $Y_1, \ldots, Y_n$ is a right-handed basis of the tangent space $T_K M^\gamma$ with respect to the observer space orientation $O^\gamma$, then there should be at least one ‘temporal vector’ $Y_0 \in T_{(\tau,K)} M^\gamma$ such that $Y = Y_i \otimes e^i$ is a ‘right-handed’ basis in $T_{(\tau,K)} M^\gamma$. Intuitively, we take the vector field $\partial/\partial\tau$ evaluated at the point $(\tau,K)$ to obtain $Y_0$. Yet this vector field is not well-defined, as geometrically, it depends on how one ‘attaches’ the $M^\gamma_{\tau_0}$ to each other. For this reason, we require a frame of reference $X$ for $\gamma$ in this case as well.

In the following, we define both structures in terms of coordinates $x$ with respect to $X$. Invariant definitions in terms of $X$ do exist, but are complicated and not needed here.

**Definition 3.4.13 (Observer spacetime)**

Let $(Q, g, O)$ be a spacetime of dimension $n + 1$, $\gamma: I \rightarrow Q$ be an observer and $M^\gamma$ be the domain of the respective kinematic observer mapping $\xi^\gamma$. Further, let $X$ be a frame of reference field for $\gamma$.

Then the coordinates $x$ on $M^\gamma$ with respect to $X$, as given by equation (3.4.3) on page 70, are called (kinematic) observer coordinates (with respect to $X$). The tensor field

$$h = h_{ij} \; dx^i \otimes dx^j := \delta_{ab} \; dx^a \otimes dx^b = \begin{pmatrix} 0 & 1 & \ldots & 1 \end{pmatrix}$$

on $M^\gamma$ is the (kinematic) observer metric (induced by $X$) and the 1-form $d\tau$ on $M^\gamma$ is the time form (with respect to $\gamma$). The observer spacetime orientation $O$ (induced by $X$) is the $GL_n^+$-structure on $M^\gamma$ induced by the coordinate frame field $\partial$ and the representation

$$\rho: GL_n^+ \rightarrow GL_{n+1}: A \rightarrow \rho(A) := \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$  

The tuple $(M^\gamma, d\tau, h, O)$ is called the observer spacetime with respect to $X$.  

♦
As opposed to observer spaces, observer spacetimes are not Riemannian manifolds and hence are not by default equipped with an intrinsic connection. For this reason, we need to explicitly define a connection, which, as the Levi-Civita connection for Riemannian manifolds, is to be viewed as intrinsic to the observer spacetime. Obviously, the connection should coincide with the Levi-Civita connection, if restricted to the observer spaces. Physically, this assures that its auto-parallel correspond to ‘straight-line motion’, as seen by the observer.

**Definition 3.4.14 (Observer connection)**

Let \((M^\gamma, dr, h, \mathcal{O})\) be a spacetime and \(X\) be a frame of reference field for the observer \(\gamma\). If \((M^\gamma_0, d\tau, h, \mathcal{O})\) is the observer spacetime with respect to \(X\), then a covariant derivative \(N\) on \(T^1M^\gamma\) is called *(kinematic) observer connection (with respect to \(X\))*, if \(N\) is torsion-free, and compatible with \(d\tau\) and \(h\) in the following sense:

\[
N(d\tau) = 0 \quad , \quad Nh = 0 .
\]

**Lemma 3.4.15**
The observer connection exists and is unique.

**Proof** In full analogy to [99, Lem. 2.2], Lemma 3.4.15 is proven by constructing the standard Riemannian metric out of the time form and the observer metric. ■

As in the static case, kinematic observer coordinates are adapted to all geometric structures on \(M^\gamma\) for the given frame of reference field \(X\) for \(\gamma\) and thus \(N\) is simply the standard flat connection with respect to the kinematic observer coordinates.

So in practice, we choose a particular observer \(\gamma\), a frame of reference field \(X\) for \(\gamma\), which may be rotating or not, and compute the kinematic observer mapping \(\xi^\gamma\) in the kinematic observer coordinates \(x\) with respect to \(X\). Explicitly, this *kinematic observer mapping with respect to \(X\)* is the map \(\varphi^\gamma_X := \xi^\gamma \circ (x^{-1})\), i.e.

\[
\varphi^\gamma_X : \quad x (M^\gamma) \rightarrow Q \quad : \quad x = (c\tau, \vec{x}) \rightarrow \varphi^\gamma_X (c\tau, \vec{x}) := \exp ( - |\vec{x}| (X_0)_\tau + x^a (X_a)_\tau ) . \tag{3.4.17}
\]

Then, to relate events in \(Q\) to events in the observer spacetime \(M^\gamma\), this map needs to be inverted in the sense of Theorem 3.4.5 on page 74. The computation of \(\varphi \equiv \varphi^\gamma_X\) and its inverse is usually a mathematically very challenging task, but once this has been achieved, the computation of temporal and spatial distances between the events on \(Q\), as seen by the observer \(\gamma\), is very simple. We simply identify the points in \(M^\gamma\) with their coordinate values and then, in accordance with the defined geometric structures, the temporal distances between any two observed events \(x = (c\tau, \vec{x}), x' = (c\tau', \vec{x}')\) are \(|\tau - \tau'|\) and the spatial distances are \(|\vec{x} - \vec{x}'|\). The orientation is also as we would intuitively expect. Hence the geometric structures from Definition 3.4.13 and Definition 3.4.14 are only implicitly used in practice, since distances, orientations, etc. accord with ‘Newtonian intuition’.

To get at ease with the construction and also as a check of physical consistency, let us continue with our two examples from Example 3.1.6 on page 47 and Example 3.4.12 on page 78.
Example 3.4.16 (Observer spacetimes in Minkowski spacetime)

i) We first compute the kinematic observer mapping $\varphi$ with respect to the standard inertial frame of reference field $X$, as given by (3.1.7a) and (3.4.14a). Recalling the exponential map in Minkowski spacetime (3.3.10) on page 64, the values of $\varphi$ are

$$\varphi(c\tau, \vec{x}) = \left( c\tau - |\vec{x}| \right)$$

in standard coordinates $y = (ct, \vec{y})$ on $\mathbb{R}^4$ and for all $x = (c\tau, \vec{x}) \in \mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$. If we restrict $\varphi$ to its image, it has the smooth inverse $\bar{\varphi}$ with values

$$\bar{\varphi}(ct, \vec{y}) = \left( ct + |\vec{y}| \right)$$

and is therefore a diffeomorphism. In addition, the space-time splitting is global in the sense that $Q \setminus (\gamma(I))$ is contained in the image of the kinematic observer mapping. In fact, $\xi(\mathcal{M}^\gamma) = Q \setminus (\gamma(I))$ here.

In the physics literature the expression $t = \tau - |\vec{x}|/c$ is known as the retarded time and commonly occurs in the special-relativistic theory of electrodynamics. The coordinate $t = y_0/c$ is also a measure of time in the sense that it is the proper time of the family of observers given by $\tau \rightarrow \gamma(\tau) + (0, \vec{y}_0)$ with $\vec{y}_0 \in \mathbb{R}^3$. It may be called Einstein-synchronized time, since it is the result of a clock synchronization among these observers and is discussed already in Einstein’s original paper on special relativity [11]. It was famously argued by Reichenbach [27] that this choice is pure convention. Independent of one’s position on this issue, the proper time $\tau$ is not conventional.

As a diffeomorphism, we may view $\varphi$ as a coordinate transformation from $x$ to $y$, so we effectively ‘identify’ the observer spacetime with the spacetime itself. Thus, given any two events $(ct, \vec{y})$ and $(ct', \vec{y}')$ in the image of $\varphi$, their temporal distance is simply $|t - t'|$ and their spatial distance is $|\vec{y} - \vec{y}'|$. Hence the general theory applied to inertial frame of reference fields in Minkowski spacetime indeed reproduces the temporal and spatial distances from special relativity.

ii) For the uniformly accelerated observer $\gamma$ given by (3.1.7c) on page 48 with frame of reference field $X$ from (3.4.14b) above, we may again compute the kinematic observer mapping $\varphi$ with respect to $X$:

$$\varphi(c\tau, \vec{x}) = \gamma(\tau) + \left( - |\vec{x}| \right) X^i_0 (\tau) + x^a X^i_a (\tau) \, e_i$$

$$= \left( \begin{array}{c}
\frac{c^2}{a} \sinh \left( \frac{a\tau}{c} \right) - |\vec{x}| \cosh \left( \frac{a\tau}{c} \right) + x^1 \sinh \left( \frac{a\tau}{c} \right) \\
\frac{c^2}{a} \cosh \left( \frac{a\tau}{c} \right) - \frac{c^2}{a} - |\vec{x}| \sinh \left( \frac{a\tau}{c} \right) + x^1 \cosh \left( \frac{a\tau}{c} \right) \\
x^2 \cos (\omega\tau) - x^3 \sin (\omega\tau) \\
x^2 \sin (\omega\tau) + x^3 \cos (\omega\tau)
\end{array} \right),$$

This directly shows that the kinematic observer mapping may be very complicated, even in the absence of curvature.

As the past light cones $C_{\gamma(\tau)}^-$ should not intersect each other, we conjecture that in Minkowski spacetime the kinematic observer mapping is always a diffeomorphism onto its image. The difficulty in the proof of this statement is injectivity, as for
3.4 Kinematic Splitting

Each \( \tau \in I = \text{dom} \gamma \) the map \( \exp_{\gamma(\tau)} \) has full rank and one may then argue as in the proof of Theorem 3.4.5 to conclude that \( \xi' \) has full rank. Therefore, at least locally \( \phi \) is a diffeomorphism and one may use asymptotic expansions of \( \phi \) in each variable and formal series inversion to obtain an approximate local inverse. Without such an inverse, however, we can only fragmentarily relate the observer spacetime to the ‘actual’ spacetime.

We also wish to remark that, in the context of uniform acceleration in special relativity, the work of Rindler [29, §2.16] is frequently cited. The approach, however, significantly differs from ours (for \( \omega = 0 \)).

As for the static splitting in section 3.3, we may compute the differential of the kinematic observer mapping \( \phi \) with respect to the frame of reference field \( X \) in terms of Jacobi fields.

Proposition 3.4.17 (Differential of kinematic observer mapping)

Let \((\mathcal{Q}, g, \mathcal{O})\) be a spacetime of dimension \( n + 1 \) and \( \phi \) be the kinematic observer mapping with respect to the frame of reference field \( X \) for the observer \( \gamma : I \to \mathcal{Q} : \tau \to \gamma(\tau) \).

Then for all \( x = (c\tau, \vec{x}) \in \text{dom} \phi \subseteq (cI) \times (\mathbb{R}^n \setminus \{0\}) \):

\[
\left( \phi_\ast \frac{\partial}{\partial \tau} \right)_x = J_1(x),
\]

where \( J(x) : s \to J_s(x) \) is the unique Jacobi field along the geodesic

\[
s \to \exp_{\gamma(\tau)}(s(-|\vec{x}||X_0)_\tau + x^a(X_a)_\tau))
\]

with \( J_0(x) = \dot{\gamma}_\tau \) and

\[
\left( \frac{\nabla J(x)}{ds} \right)_0 = -|\vec{x}| \left( \frac{\nabla X_0}{d\tau} \right)_\tau + x^a \left( \frac{\nabla X_a}{d\tau} \right)_\tau.
\]

In particular, if \( X \) is Fermi-Walker transported along \( \gamma \):

\[
\left( \frac{\nabla J(x)}{ds} \right)_0 = \left( \frac{x^a}{c} \delta_{ab} \frac{\nabla \dot{\gamma}^b}{d\tau}(\tau) \right)(X_0)_\tau - \frac{|\vec{x}|}{c} \frac{\nabla \dot{\gamma}^a}{d\tau}(\tau)(X_a)_\tau,
\]

where we defined

\[
\frac{\nabla \dot{\gamma}^b}{d\tau} := X^b \cdot \frac{\nabla \dot{\gamma}}{d\tau}
\]

for all \( b \in \{1, \ldots, n\} \).

**Proof** In full analogy to the proof of Proposition 3.3.12, we need to consider the curve

\[
Y : I \to T\mathcal{Q} : \tau \to Y_\tau := -|\vec{x}|(X_0)_\tau + x^a(X_a)_\tau
\]

and apply Theorem 2.4.4 from page 39 to its tangent vector field. Now observe that its base curve is \( \gamma \) and compute the covariant derivative to obtain (3.4.19b). From the expression of the Fermi-Walker derivative for observers (3.4.12) follows (3.4.19c).

\[
83
\]
The vector field $\partial/\partial \tau$ has a particular physical significance for the theory, as already suggested by the definition of observer spacetimes. This significance is revealed within the subject of relative motion.

**Definition 3.4.18 (Relative motion)**

Let $(Q, g, O)$ be a spacetime of dimension $n + 1$, $\xi^\gamma: M^\gamma \rightarrow Q$ be the kinematic observer mapping for the observer $\gamma: I \rightarrow Q: \tau \rightarrow \gamma(\tau)$ and $X$ be a frame of reference field for $\gamma$. Further, let $x = (c\tau, \vec{x})$ be observer coordinates with respect to $X$ and $N$ be the observer connection. If

$$\vartheta: J \rightarrow M^\gamma: s \rightarrow \vartheta(s) = (\tau(s), K_s)$$

is a (smooth) curve such that $\xi^\gamma \circ \vartheta$ is an observer, denote by $\tilde{\vartheta}$ the reparametrized curve with respect to the coordinate $\tau$. Its tangent vector field in observer coordinates is

$$\dot{\tilde{\vartheta}} = c \frac{\partial}{\partial x^0} + v^a \frac{\partial}{\partial x^a}$$

The spatial part of $\dot{\tilde{\vartheta}}$ is called the *velocity (field) of $\vartheta$ relative to $X$* and, if evaluated at $\tau \in \text{dom} \tilde{\vartheta}$, it is called the *velocity of $\vartheta$ relative to $X$ at $\tau$*. Moreover,

$$\frac{N \tilde{\vartheta}}{d\tau}$$

is called the *acceleration field of $\vartheta$ relative to $X$* and, if evaluated at $\tau$, it is called the *acceleration of $\vartheta$ relative to $X$ at $\tau$*. We say $\vartheta$ is at rest with respect to $X$, if its velocity field vanishes.

The definition is sensible as, by Theorem 3.4.4ii), the reparametrization of $\vartheta$ exists. Definition 3.4.18 may appear very technical, but in fact it simply reproduces the definitions of velocity and acceleration given by ‘Newtonian intuition’. That is, if we write $(x^a \circ \vartheta)(\tau) \equiv x^a(\tau)$ in observer coordinates, then the components of the relative velocity at $\tau$ are

$$v^a(\tau) = \frac{dx^a}{d\tau}(\tau)$$

and the components of the acceleration at $\tau$ are

$$\dot{v}^a(\tau) = \frac{d^2x^a}{d\tau^2}(\tau).$$

**Remark 3.4.19 (Relative motion)**

i) Sometimes it is convenient to parametrize the functions $v^a$ with respect to the proper time $s$ of the ‘observed observer’. By a slight abuse of terminology, we also speak of relative velocity and acceleration in this context.

ii) In the context of relative motion, we stress the fact that the kinematic observer mapping is usually not injective. Physically, one observer at an instant of time may see another one at different (observed) events and thus the relative state of motion can differ vastly.
3.4 Kinematic Splitting

iii) Instead of considering just a curve in $M^{\gamma}$ that yields a future directed timelike curve under the kinematic observer mapping, one may attach to it a ‘frame of reference’ in the sense of Newtonian mechanics to model the orientation of the observed physical observer in space. Conversely, one may use local inverses of the kinematic observer mapping to relate a frame of reference field for a second observer to such a ‘Newtonian frame of reference’. A priori, it should also be possible to determine an ‘infinitesimal length contraction’ from this. However, we decided not to treat this problem here.

We conclude that any integral curve $\tilde{\vartheta}$ of the vector field $\partial/\partial \tau$ corresponds to a physical observer at rest relative to the reference frame field $X$, provided $\xi^\gamma \circ \tilde{\vartheta}$ is timelike (see Theorem 3.4.4i)). As the next example shows, the latter condition is not guaranteed in general.

Example 3.4.20 (Non-existence of physical observers at rest)
We consider Example 3.4.16ii) for $a \to 0$, i.e. where we have an unaccelerated observer with rotating frame of reference field. The map $\varphi$ is then given by

$$\varphi (c \tau, \vec{x}) = \begin{pmatrix} c \tau - |\vec{x}| \\ x^1 \\ x^2 \cos(\omega \tau) - x^3 \sin(\omega \tau) \\ x^2 \sin(\omega \tau) + x^3 \cos(\omega \tau) \end{pmatrix} = \begin{pmatrix} y^0 \\ \vec{y} \end{pmatrix}. $$

Since $|\vec{y}| = |\vec{x}|$, we may easily compute a global inverse, but this is not needed here. The integral curves of $\partial/\partial \tau$ under $\varphi$ are timelike if and only if $\varphi^* (\partial/\partial \tau)$ is timelike. It is thus sufficient to compute

$$g_{\varphi(x)} \left( \varphi^* \frac{\partial}{\partial \tau}, \varphi^* \frac{\partial}{\partial \tau} \right) = \eta_{ij} \frac{\partial \varphi^i}{\partial \tau} (x) \frac{\partial \varphi^j}{\partial \tau} (x) = c^2 - \left( \frac{\partial \vec{y}}{\partial \tau} (x) \right)^2 = c^2 - \omega^2 \left( (x^2)^2 + (x^3)^2 \right)$$

for $x \in \text{dom} \varphi$. This is negative for large enough values of $x^2$ and $x^3$, and indeed to be expected: To keep up with the rotation of the ‘observing’ physical observer for increasing spatial distances (perpendicular to the axis of rotation), a far away ‘observed’ physical observer would eventually need to move faster than light. An impossibility.

The subject of non-accelerating, constantly rotating ‘observers’ in Minkowski spacetime has been widely discussed in the relativity literature, see e.g. [48, 50, 75, 76]. To our knowledge, however, the approach followed here has not been pursued elsewhere.

We will continue our discussion of the subject of relative motion in the following chapter.
4 The Newtonian Limit

Any physical theory, which is not able to reproduce empirically supported results in their domain of validity, must necessarily be false. This statement, which might appear as a platitude at first sight, reveals itself as a powerful tool of falsifying physical theories already on the theoretical level. In our case, this implies that a mathematical theory of separating space and time in general relativity has to be able to reproduce Newtonian mechanics within its domain of validity in some mathematically admissible approximation. Such an approximation procedure for obtaining Newtonian mechanics out of general relativity via a theory of space-time splitting is what we philosophically define as the Newtonian limit.

Einstein himself was well aware of the fact that the existence of the Newtonian limit would be a crucial requirement for the physical feasibility of his general theory of relativity. In the special theory of relativity, the proof of the existence of the Newtonian limit, though naive, was straightforward (see e.g. [99, §2] for a detailed discussion), but for the general one more sophisticated reasoning had to be applied. Einstein decided that he had to generalize the Gauß’ law for gravity [11, p. 87] to arrive at a law for the spacetime curvature and then find a procedure to rederive the Gauß’ law as an approximation. His success in this endeavor in 1915 [55] marks a historic event: The sought-after equation is nowadays known as the Einstein (field) equation and the procedure is called the weak-field approximation. Accounts of his reasoning can be found in many introductory books on general relativity, see e.g. the books by Carroll [10, §4.1 & 4.2] and Wald [39, §4.4(a)].

Unfortunately, the standard approaches to the Newtonian limit both in the special and the general theory of relativity are worthy of criticism, the main defect being a reliance on coordinates rather than on geometric structures. In particular, from a mathematician’s perspective, the weak field approximation is a heuristic rather than a rigorous method. Though these coordinate methods might be shown to be justifiable under a more careful mathematical analysis, by themselves they constitute inadmissible evidence for the existence of the Newtonian limit. Hence for the general theory other approaches have been found trying to address these issues, notably due to Élie Cartan and Jürgen Ehlers. A discussion thereof, including references to the original works, can be found in Maren Reimold’s work [28]. Our discussion here, however, takes a different route and relates the Newtonian limit to the construction in the previous chapter. On one hand this provides a rigorous approach to the Newtonian limit without relying on somewhat arbitrary coordinates or the ad-hoc introduction of geometric structures, on the other hand the existence of the Newtonian limit is required to give physical credibility to the splitting formalism itself. Scientific care dictates that the Newtonian limit is to be derived by mathematically and philosophically sound methods. Its existence may not be taken for granted a priori, as doing so would deprive the theorist of one of the primary means of falsifying the theory.

Having this discussion in mind, in section 4.1 we start with a general a priori approach to the Newtonian limit employing the splitting construction. Important problems required
for a proof of the existence of the Newtonian limit (in our sense) for the general theory of relativity are brought forward. In section 4.2 we voice criticism towards the standard approach to the Newtonian limit in the special theory of relativity and show that for inertial frame of reference fields in Minkowski spacetime the Newtonian limit indeed exists. In addition, we give lower order correction terms.

We wish to remark that our discussion only applies to the Newtonian limit of point masses for a given spacetime, observer and frame of reference field. The issue of the Newtonian limit of general-relativistic field theories (e.g. magneto-hydrodynamics, quantum theories) depends highly on the theory under consideration and a discussion thereof would go beyond the scope of this thesis. It should, however, be said that for field theories the relevant geometric structures have to be identified and a coherent, critical understanding of what precisely constitutes a Newtonian limit in the respective case needs to be attained.

4.1 General Newtonian Limit

The two main ingredients of Newtonian mechanics are a model of space and time with associated geometric structures, called Newtonian spacetime, as well as a law determining the dynamics of mass points therein, that is Newton’s second law of motion. A careful mathematical axiomatization of the physical concept of Newtonian spacetime has been given in [99, §2] and shall not be repeated here. For our purposes, it is sufficient to recall that the observer spacetime models ‘the world as the physical observer sees it’ and thus the Newtonian spacetime needs to be somehow related to 4-dimensional observer spacetimes. Indeed, it directly follows from the respective definitions that, mathematically speaking, every observer spacetime $(\mathcal{M}^\gamma, d\tau, h, O)$ in a spacetime $(\mathcal{Q}, g, O)$ of dimension 4 is a Newtonian spacetime under the identification of $\mathcal{M}^\gamma$ with the domain of the respective kinematic observer coordinates $x$, i.e. an open subset of $\mathbb{R}^4$. In particular, this statement is independent of the spacetime, observer $\gamma$ or the frame of reference field $X$.

If we transcribe the implicit law of Newtonian mechanics that all clocks run at the same rate to this setting, then for a curve $\vartheta: J \to \mathcal{M}^\gamma: s \to \vartheta(s) = (\tau(s), K_s)$, representing physical motion, we must demand $\dot{\tau} \equiv d\tau(\dot{\vartheta}) = 1$. If this holds, the curve parameters $s$ and $\tau$ are equal (up to a shift) and then the observer connection $N$ may be used to reformulate Newton’s second law

$$\vec{F} = m \frac{N\dot{\vartheta}}{d\tau}, \tag{4.1.1}$$

where $m \in \mathbb{R}_+$ is the mass of the ‘observed object’ traveling along $\vartheta$ and $\vec{F}$ is a (spatial) vector field along $\vartheta$. That is, given $\vartheta$ such that $\gamma := \xi^\gamma \circ \vartheta$ is an observer, equation (4.1.1) yields the relative force $\vec{F}$. Expressions for $\vec{F}$ in observer coordinates ought to be sensible within the Newtonian theory.

We have thus obtained two necessary conditions for the existence of the Newtonian limit. Since both conditions in conjunction with respective initial conditions specify the motion entirely, the two conditions are in fact sufficient. However, as the spacetime $\mathcal{Q}$ represents ‘the world as it is’, neither the condition $\dot{\tau} = 1$ nor equation (4.1.1) ought to be viewed as equations of motion in relativity theory, but rather serve as indicators of
4 The Newtonian Limit

how well the relativistic model may be cast into the framework of Newtonian mechanics. Their approximate validity proves the existence of the Newtonian limit mathematically - under the assumptions necessary to make these approximations.

Let us further specify the model case. In practical situations, we first determine the spacetime \((\mathcal{Q}, g, \mathcal{O})\), second we require an observer \(\gamma\) and an appropriate frame of reference field \(X\) for \(\gamma\). In the third step, we compute the kinematic observer mapping \(\varphi\) with respect to \(X\). In the fourth step, a second observer \(\gamma' : \mathcal{J} \to \mathcal{Q}\) needs to be found, which satisfies the dynamical law

\[
F' = m \frac{\nabla \gamma'}{ds},
\]  

(4.1.2)

where \(m \in \mathbb{R}_+\) is the mass of \(\gamma'\) and \(F'\) denotes the ‘actual’ force acting on \(\gamma'\). If \(\gamma'\) does not lie in the image of \(\varphi\), then \(\gamma\) does not ‘see’ \(\gamma'\) and thus the question of the existence of the Newtonian limit is meaningless. Therefore, we are only interested in the case, where \(\gamma'\) lies fully in the image of \(\varphi\). Indeed, there is a simple sufficient condition for the local existence of a smooth curve \(s \to x(s) = (c\tau(s), \bar{x}(s))\) in \(\text{dom} \varphi \subset \mathbb{R}^4\) with \((\varphi \circ x)(s) = \gamma'(s)\) for all admissible \(s \in \mathbb{R}\).

Lemma 4.1.1

Let \((\mathcal{Q}, g, \mathcal{O})\) be a spacetime, and let \(\xi^\gamma : \mathcal{M}^\gamma \to \mathcal{Q}\) be the kinematic observer mapping for the observer \(\gamma\). Further, let \(\gamma' : \mathcal{J} \to \mathcal{Q}\) be another observer and let there exist an \(s_0 \in \mathcal{J}\) such that \(\gamma'(s_0)\) lies in the image of \(\xi^\gamma\) and is a regular value of \(\xi^\gamma\). Then there exists an open neighborhood \(\mathcal{J}'\) of \(s_0\) in \(\mathcal{J} \subseteq \mathbb{R}\) and a smooth curve

\[
\theta : \mathcal{J}' \to \mathcal{M}^\gamma : s \to \theta(s) = (\tau(s), K_s),
\]

such that \(\gamma'|_{\mathcal{J}'} = \xi^\gamma \circ \theta\) and \(d\tau/ ds > 0\).

\[\Box\]

**Proof** This is a direct corollary of the kinematic observer mapping inversion theorem (Theorem 3.4.5): Since \(\gamma'(s_0)\) is a regular value lying in the image, there exists an \((\tau', K') \in \mathcal{M}^\gamma\) with open neighborhood \(V\) such that \(\xi^\gamma|_V\) is a diffeomorphism onto its image. Since \(\xi^\gamma|_V\) is open and \(\gamma\) is continuous, \(\mathcal{J}'\) exists and \(\theta := (\xi^\gamma|_V)^{-1} \circ (\gamma'|_{\mathcal{J}'})\) does the job. \(d\tau/ ds > 0\) follows from the consistency of time directions (Theorem 3.4.4ii)).

As the proof suggests, finding the curve \(x\) in practice requires inverting the kinematic observer mapping \(\varphi\) and thus the curve need neither be unique nor can we guarantee that \(\mathcal{J} = \mathcal{J}'\). Moreover, \(\gamma'(s_0)\) need not be a regular value of \(\varphi\). In fact, it is even possible that \(\gamma'(s)\) is a critical value for every \(s \in \mathcal{J}\). An example of such a curve can be constructed in the plane wave spacetimes, again we refer to the articles by Perlick [96, §5.11] and the original one by Penrose [98].

Without the existence of a smooth curve \(x\) satisfying \(\varphi \circ x = \gamma'|_{\mathcal{J}'}\), the question of the existence of the Newtonian limit is again superfluous. Indeed, Newton’s second law and the condition \(\dot{\tau} \approx 1\) are only sensible in this setting, if such a curve exists.

We may therefore continue with the assumption that a curve \(x : \mathcal{J} \to \text{dom} \varphi\) is given and that \(\gamma' := \varphi \circ x\) is an observer. Then, defining \(\alpha_s := (\varphi^* g)|_s\) for any \(s \in \mathcal{J}'\) and taking the components of the relative velocity field to be \(v^i := dx^i/ d\tau\), we may find a general expression for \(\dot{\tau} > 0\) by computing

\[
c^2 = g \left( (\varphi \circ x), (\varphi \circ x) \right) = \alpha \left( \dot{x}, \dot{x} \right) = \alpha_{ij} \dot{x}^i \dot{x}^j = \alpha_{ij} \dot{\tau}^2 v^i v^j,
\]
which yields
\[ \dot{\tau} = \frac{1}{\sqrt{\alpha_{00} + 2\alpha_{0a} \frac{v^a}{c} + \alpha_{ab} \frac{v^a v^b}{c^2}}} \] (4.1.3)
with \(a, b \in \{1, 2, 3\}\). It is obvious that this expression is not identically 1 in most cases, yet we only require an approximate validity.

To determine which approximation to use, we recall that, empirically, Newtonian mechanics is known to provide an adequate description of phenomena at length and time scales familiar to everyday human experience. With respect to these scales the speed of light \(c\) is in general very large and we may thus consider \(\varepsilon := 1/c\) as a perturbation parameter. Yet mindlessly expanding all equations in \(\varepsilon\) will yield wrong results as the occurrence or non-occurrence of \(c\) in a physical equation depends on the particular conventions used. Philosophically, the problem of what to expand boils down to the question which convention is ‘most natural’ in the sense that it does not ‘artificially’ introduce factors of \(c\). We claim that a natural convention is the one where all expressions are written in terms of the coordinates \((\tau, \vec{x})\). In order to avoid philosophically deep discussions on naturalness, we justify this choice by observing that it yields reasonable results in the example treated in the concluding section. Nonetheless, the approximation can only be made if the dependence of the \(\alpha_{ij}\) on \(c\) is known, so additional assumptions are needed to make statements on the existence of the Newtonian limit in the general case.

**Example 4.1.2**

In the particular case where \(\alpha_{00} > 0\) and all \(\alpha_{ij}\) are independent of \(\varepsilon\), a second-order Taylor expansion of the above expression (4.1.3) in \(\varepsilon = 1/c\) around 0 yields:
\[ \dot{\tau} = \frac{1}{(\alpha_{00})^{1/2}} - \frac{\alpha_{0a} \frac{v^a}{c}}{(\alpha_{00})^{3/2}} + \frac{\left(3\alpha_{0a}\alpha_{0b} - \alpha_{00}\alpha_{ab}\right) \frac{v^a v^b}{c^2}}{2(\alpha_{00})^{5/2}} + O\left(\frac{1}{c^3}\right) \] (4.1.4)

Here we have used the common ‘big O notation’ to indicate the order of the approximation (cf. [30, §1.B]). Therefore in this case the Newtonian limit can only exist, if \(\alpha_{00} = 1\). Note that additional approximations may be needed for this to hold. Unless all \(\alpha_{0a}\) vanish, the Newtonian limit needs to be attained in the zeroth order approximation in \(1/c\).  

Expansion terms that give corrections to the expression \(\dot{\tau}\) as well as \(F^c\) in the Newtonian limit are correspondingly called *relativistic correction terms*.

For the second condition (4.1.1), we need to relate the ‘actual dynamics’ (4.1.2) of \(\gamma' = \varphi \circ x\) to the ‘observed dynamics’ of \(x\). In general this can only be done in a 1-1 manner, if \(\varphi\) restricted to an open neighborhood \(\mathcal{V}\) of the curve \(x\) in \(\text{dom} \varphi \subset \mathbb{R}^4\) is a diffeomorphism onto its image. For convenience, we assume that \(\varphi(\mathcal{V}) \subseteq \mathcal{Q}\) is contained in the domain of the coordinate map \(\kappa\). Then we may understand \(\varphi\) as a coordinate transformation from observer coordinates \(x\) with respect to \(X\) to the coordinates \(\kappa\). We denote the inverse of \(\varphi|\mathcal{V}\) by \(\varphi^{-1}\) and assume that \(\partial/\partial \kappa^0\) is timelike.

In the most common situations, we are given the functions \(F'^a := \partial \kappa^a \cdot F'\) with \(a \in \{1, 2, 3\}\) and we still need to calculate \(F^a := \partial \kappa^0 \cdot F'\). This is obtained from
\[
0 = g\left(\dot{\gamma}', F'\right) = \alpha \left(\dot{x}, \varphi_* F'\right) = \alpha_{ij} \dot{x}^i \varphi^j \varphi_{ik} F'^k
= \dot{\tau}\left(F'^0 \left(\alpha_{0i} c \varphi^i,0 + \alpha_{ai} v^a \varphi^i,0\right) + \left(\alpha_{0i} c \varphi^i,a F'^a + \alpha_{ai} v^a \varphi^i,a F'^a\right)\right),
\]
where we used the notation $\bar{\varphi}_{i,j} := \partial \kappa^i / \partial x^j$. Since $\partial / \partial \kappa^0$ is timelike:

$$g \left( \gamma', \partial / \partial \kappa^0 \right) = \alpha_{ij} \dot{x}^i \dot{x}^j = \dot{\tau} \left( \alpha_{0i} c \bar{\varphi}^i,0 + \alpha_{ai} v^a \bar{\varphi}^i,0 \right) \neq 0,$$

and therefore

$$F^{00} = - \frac{\left( \alpha_{0i} + \alpha_{bi} v^b \right) \bar{\varphi}^i,0 F^{0b}}{\left( \alpha_{0j} + \alpha_{cj} v^c \right) \bar{\varphi}^j,0}.$$

(4.1.5)

In accordance with (4.1.1), the ‘observed dynamics’ in observer coordinates is given by

$$m \frac{d^2 x^c}{d\tau^2} = m \frac{dv^c}{d\tau} = F^c$$

and we need to determine the components $F^c$ of the relative force $\vec{F}$ from the ‘actual dynamics’ (4.1.2) in observer coordinates:

$$\frac{d^2 x^c}{ds^2} + \Upsilon_{cij} \frac{dx^i}{ds} \frac{dx^j}{ds} = \frac{1}{m} \bar{\varphi}^c,i F^{i},$$

(4.1.6)

where the $\Upsilon_{cij}$ are the relevant Christoffel symbols. These are obtained from the Christoffel symbols $\Gamma^k_{ij}$ in coordinates $\kappa$ by the usual transformation formula:

$$\Upsilon^c_{ij} = \frac{\partial x^c}{\partial \kappa^k} \Gamma^k_{lij} \frac{\partial \kappa^l}{\partial x^i} \frac{\partial \kappa^j}{\partial x^j} + \frac{\partial x^c}{\partial \kappa^l} \frac{\partial^2 \kappa^l}{\partial x^i \partial x^j}.$$ (4.1.7)

Of course, they can also be calculated from the pullback $\varphi^* g$ restricted $\mathcal{V}$. If we define $\bar{\alpha} := (\alpha)^{-1}$ and denote partial derivatives by a comma, this is done via

$$\Upsilon^c_{ij} = \bar{\alpha}_{cl} \left( \alpha_{li,j} + \alpha_{lj,i} - \alpha_{ij,l} \right),$$ (4.1.8)

but here one requires the algebraic inverse of $\varphi^* g$. Once all $\Upsilon_{cij}$ have been found, we compute

$$\frac{d^2 x^c}{ds^2} = \frac{d}{ds} \left( \dot{\tau} \frac{dx^c}{d\tau} \right) = \dot{\tau} \frac{dx^c}{d\tau} + \dot{\tau}^2 \frac{d^2 x^c}{d\tau^2},$$

and after splitting the left hand side of (4.1.6) into ‘temporal’ and ‘spatial’ parts along with some rearrangement of terms, we ultimately get an expression for $F^c$:

$$m \frac{dv^c}{d\tau} = \sum_{\text{actual forces}} \text{pseudo-forces}$$

(4.1.9)

Thus the relative force is always the sum of the ‘actual’ forces acting on the observer $\gamma'$ and the pseudo-forces, which, by definition, are of purely geometric origin and so always contain the mass of $\gamma'$ as a simple factor. Physically, (actual) forces have the property that they cause an absolute acceleration, while pseudo-forces only lead to relative accelerations. In physics textbooks one sometimes reads the claim that ‘pseudo-forces only occur in non-inertial frames of reference’, but, at least in the presence of curvature, equation (4.1.9) shows that this is incorrect. In particular, (4.1.9) states that gravity is a pseudo-force.

In analogy to the expansion of $\dot{\tau}$ in $\epsilon = 1/c$, we may expand the right hand side of (4.1.9) to check for the existence of the Newtonian limit. As before, we need additional assumptions on the $c$-dependence of the $\alpha_{ij}$ to make statements on the general case.
Example 4.1.3
We continue Example 4.1.2 from above, i.e. we assume $\alpha_{00} = 1$ and all $\alpha_{ij}$, if written in terms of $\tau$ and $\vec{x}$, are independent of $c$. According to equation (4.1.8) and since $\alpha_{00,l} \equiv 0$, the relevant Christoffel symbols turn out to be

$$
\Gamma^c_{00} = \frac{1}{c} \alpha^{ca} \frac{\partial \alpha_{a0}}{\partial \tau}
$$

$$
\Gamma^c_{0a} = \frac{1}{2} \alpha^{cb} \left( \alpha_{b0,a} - \alpha_{0a,b} \right) + \frac{1}{2c} \alpha^{cb} \frac{\partial \alpha_{ab}}{\partial \tau}
$$

$$
\Gamma^c_{ab} = \frac{1}{2} \alpha^{cl} \left( \alpha_{la,b} + \alpha_{lb,a} - \delta^d_l \alpha_{ab,d} \right) - \frac{1}{2c} \alpha^{ca} \frac{\partial \alpha_{ab}}{\partial \tau}.
$$

For the existence of the Newtonian limit in the force-free case $F'_t = 0$, we require that the pseudo-forces in (4.1.9) do not diverge for $1/c \to 0$. Therefore, if we plug in the Christoffel symbols and group the terms by their $c$-dependence, we find that

$$
0 \approx \alpha^{ca} \frac{\partial \alpha_{a0}}{\partial \tau}
$$

$$
0 \approx \alpha^{cb} \left( \alpha_{b0,a} - \alpha_{0a,b} \right)
$$

needs to hold for the Newtonian limit to exist for various values of the $v^a < c$. Again, unless all $\alpha_{0a}$ vanish, the Newtonian limit needs to be attained in the zeroth order approximation in $1/c$, so taking the limit $1/c \to 0$, our force equation reads:

$$
m \frac{dv^c}{d\tau} \approx -m \alpha^{cb} \frac{\partial \alpha_{ab}}{\partial \tau} v^a - \frac{m}{2} \alpha^{cl} \left( \alpha_{la,b} + \alpha_{lb,a} - \delta^d_l \alpha_{ab,d} \right) v^a v^b.
$$

Summing up, the existence of the Newtonian limit depends on the following choices:

1) the spacetime $(\mathcal{Q}, g, \mathcal{O})$,

2) the observer $\gamma$,

3) the frame of reference field $X$ for $\gamma$,

4) the chosen (maximal) set $\mathcal{V}$ in the domain of the kinematic observer mapping $\varphi$ with respect to $X$, where $\varphi|\mathcal{V}$ is a diffeomorphism onto its image, and

5) the allowed forces $F'$, which may also depend on $c$.

Fortunately, the consistency of the general theory of relativity with Newtonian gravitational theory only requires a proof of the existence of the Newtonian limit for a few particular cases, namely those where Newtonian mechanics makes statements on gravity. Here care must be taken regarding the fundamentally different conceptions of gravity in the two theories, which are to a certain extent incommensurable.\(^9\) To our assessment, the cases relevant for Newtonian gravitational theory are:

i) inertial frame of reference field for a (non-accelerated) observer in Minkowski spacetime,

---

\(^9\) We refer to the work by Kuhn [20] and the synopsis by Pajares [97] for a discussion of the concept of mutual incommensurability of scientific theories.
4 The Newtonian Limit

ii) arbitrarily rotating frame of reference field for a constantly accelerated observer in Minkowski spacetime,

iii) non-rotating frame of reference field for certain observers in Schwarzschild spacetime sufficiently ‘far away’ from the gravitating mass.

Case i) is required to show that special relativity in the Newtonian limit agrees with Newtonian mechanics in the absence of gravity. In particular, we require 
\[ F^c \approx 0 \] if and only if \[ F' = 0. \] The case is considered in section 4.2.

Case ii) should reproduce the constant gravitational force, as well as the coriolis, centrifugal and Euler pseudo-forces in the Newtonian theory (see e.g. [7, \S 10.4] for formulas). We emphasize that the constant ‘downward’ gravitational force acting on the ‘observed’ observer \( \gamma' \) in the Newtonian ontology needs to be described by a constant 'upward' acceleration on the observer \( \gamma \) in the relativistic ontology.

Number iii) is the mathematically most challenging case, but also the most interesting one from a physical perspective. The Newtonian limit should partially reproduce the pseudo-force (field)

\[ \vec{F} = -m \frac{GM}{r^2} \frac{\partial}{\partial r}, \] (4.1.10)

where \( G \) is the gravitational constant, \( M \) the ‘active’ mass of the ‘gravitational source’ and \( r \) is an adapted coordinate representing the distance of the observer \( \gamma' \) from the ‘source’ - as viewed by the observer \( \gamma \). Note that \( r \) need not be the respective Schwarzschild coordinate. We do not know whether \( \gamma \) should be taken to be static, i.e. its tangent vector is parallel to \( (\partial/\partial t)_{\gamma} \), or unaccelerated and ‘moving around the source’ for a derivation of the Newtonian limit. We say the Newtonian limit should “partially reproduce” the above Newtonian force, because formally the (exterior) Schwarzschild spacetime from Example 3.1.4, as a solution of the vacuum equation \( R = 0 \), depends on the parameter \( R > 0 \), not on the factor \( GM \). If one assumes that \( R \) ought to depend on \( M \), then an argument of physical dimensions implies that

\[ R \propto \frac{GM}{c^2}, \]

i.e. they need to be proportional, if only the physical constants \( G \) and \( c \) are allowed. Indeed, the weak field approximation claims that \( R = 2GM/c^2 \) (see e.g. [39, p. 124]).

Remark 4.1.4 (On the Einstein equation and Newton’s law of gravity)
As stated before, the Einstein equation is a generalization of the Gauß’ law for gravity, where the relation is given by the weak field approximation. However, the Gauß’ law itself is an abstraction from Newton’s law (4.1.10) of gravity to the continuous case. A derivation can be found, for instance, in the book by Bradbury [7, \S 5.4]. So if the Newtonian limit in case ii) exists and it is also possible in case iii) to derive Newton’s law (in terms of \( R \)) in an approximation where \( \gamma \approx 1 \), the vacuum Einstein equation \( R = 0 \) alone would reproduce the bulk of Newtonian gravitational theory.

This raises the important question whether the equation \( R = 0 \) is enough to explain the empirical data. In fact, Einstein himself, together with Infeld and Hoffmann, raised this question in a 1937 article [56]:

[...] energy-momentum tensors, however, must be regarded as purely temporary and more or less phenomenological devices for representing the structure
of matter, and their entry into the equations makes it impossible to determine how far the results obtained are independent of the particular assumption made concerning the constitution of matter.

Actually, the only equations of gravitation which follow without ambiguity from the fundamental assumptions of the general theory of relativity are the equations for empty space, and it is important to know whether they alone are capable of determining the motion of bodies.

We leave it to the reader to judge whether their argument in favor of the hypothesis is convincing and proceed with our own discussion.

We first observe that modeling purely gravitational interactions between two objects indeed only requires solutions to the vacuum equation, not solutions to the full Einstein equation: If the influence of one of the (inertial) masses on the overall spacetime geometry is negligible, the model employing non-accelerating observers in the Schwarzschild spacetime is sufficient. In order to describe the simplest situation, where two objects ‘interact gravitationally’, we require a ‘non-rotating two black hole solution’ of the equation $R = 0$. To our knowledge, no such explicit solution has been found so far, but, formally, this is where the full law (4.1.10) needs to be derivable in some ‘Newtonian limit’. So the appropriate identification of the integration constants with physical parameters (e.g. expressing $R$ in terms of $M$ in the Schwarzschild model) by ‘gluing’ the ($R \neq 0$)-solutions to the respective ($R = 0$)-solutions is the only point, where an argument employing the full Einstein equation would be needed to fully reproduce Newtonian gravitational theory. For an example of such a ‘gluing’, we refer to the book by Wald [39, §6.2].

Of course, this discussion underlies the implicit assumption that (4.1.10) actually describes the empirical data, where the masses $M$, $m$ and distance $r$ are obtained by an independent procedure and not simply matched to fit the law. Due to a lack of knowledge on the subject, we cannot make definitive statements on this. According to a review article by Gillies [68] on the measurement of the gravitational constant $G$, Newton’s law does seem to be a good approximation in a variety of instances. Yet, due to mutually contradicting values of $G$ appearing in the literature, the experimental issue is not entirely settled.

In the next section we show that the Newtonian limit for i) indeed exists. For the cases ii) and iii) we have not obtained a proof so far and expect additional approximations to be necessary. These additional approximations would give further qualitative and quantitative constraints on the validity of the Newtonian theory.

We conclude this section with the remark that, if one would like to go beyond Newtonian gravitational theory (e.g. introduce electromagnetic fields), one first needs to postulate the corresponding (invariant) force $F'$ acting on the second observer $\gamma'$ and then show that it reduces to the correct Newtonian force $\vec{F}$ under the Newtonian limit. We recommend to do this first for inertial frames of reference in Minkowski spacetime and then consider more complicated situations, if necessary.

**4.2 Newtonian Limit in Special Relativity**

Before we proceed with our proof of the existence of the Newtonian limit in special relativity, we state two reasons why the common derivation of the Newtonian limit in
special relativity is naive:

i) It philosophically assumes the standard observer $\gamma$, as defined by equation (3.1.7a) on page 47, together with the standard inertial frame of reference field for $\gamma$, as given by (3.4.14a) on page 79, but these mathematical objects do not appear explicitly in the definition of the Newtonian limit.

ii) For the definition of the velocity of the ‘observed observer’ $\gamma'$, it employs the Einstein-synchronized time $t$ (see Example 3.4.16i)) with respect to $\gamma$, but this is not the time referred to in Newtonian physics. The Einstein-synchronized time may be understood to take account of the ‘finiteness of the speed of light’, whereas the time in Newtonian physics coincides with the time $\tau$ as measured by $\gamma$ at each point in space. Thus the Newtonian limit needs to employ $\tau$ rather than $t$.

Both points ought to be remedied by the approach to the Newtonian limit discussed in the preceding section. Since we are interested in the Newtonian limit for the special theory of relativity, the spacetime we need to consider is Minkowski spacetime from Example 3.1.2.

As the ontology of special relativity requires ‘inertial observers’, we conclude that we have to mathematically consider an arbitrary inertial frame of reference field $X$ for an arbitrary (non-accelerating) observer $\gamma$. Historically, the importance of inertial frames of reference for the laws of Newtonian mechanics has first been deduced by Ludwig Lange in 1885. By successfully ridding the theory of the notion of ‘absolute space’ (cf. [17, p. 140 sq.]), Lange paved the way for the development of the special theory of relativity.

As argued in Example 3.1.6i), it is sufficient to consider the standard frame of reference field $X := (\partial)_\gamma$ for the standard observer $\gamma$. This is the mathematical justification of the physical statement that ‘all inertial observers are mechanically equivalent’. From Example 3.4.16i) on page 82, we recall that the kinematic observer mapping $\varphi$ with respect to $X$ is given by

$$\varphi(c\tau, \vec{x}) = \left(\frac{c\tau - |\vec{x}|}{\vec{x}}\right)$$

for $x = (c\tau, \vec{x}) \in \mathbb{R}^4 \setminus \{(0) \times \mathbb{R}^3\}$ and is a diffeomorphism onto its image. Thus for the Newtonian limit we do not need to restrict ourselves to a particular subset of the observer spacetime, contrary to what one might need to do in different instances. We may therefore consider $\varphi$ (restricted to its image) as a coordinate transformation from observer coordinates $x$ to standard coordinates $y$ on $\mathbb{R}^4$. Its Jacobian is given by the expressions

$$\frac{\partial y^0}{\partial x^0}(x) = 1, \quad \frac{\partial y^a}{\partial x^0}(x) = 0, \quad \frac{\partial y^0}{\partial x^a}(x) = -\frac{\delta_{ab}}{|\vec{x}|}, \quad \frac{\partial y^a}{\partial x^b}(x) = \delta^a_b,$$

where $a, b \in \{1, 2, 3\}$. Writing now $s \to x(s)$ for the second observer $\gamma'$ viewed as a curve in the domain $\mathbb{R}^4 \setminus \{(0) \times \mathbb{R}^3\}$ of $\varphi$, we may compute the components of $\alpha := (\varphi^* g)_x$:

$$\alpha_{00} = \frac{\partial y^i}{\partial x^0} \eta_{ij} \frac{\partial y^j}{\partial x^0} = 1$$

$$\alpha_{0a} = \frac{\partial y^i}{\partial x^0} \eta_{ij} \frac{\partial y^j}{\partial x^a} = -\delta_{ab} \frac{x^b}{|\vec{x}|}.$$
4.2 Newtonian Limit in Special Relativity

\[ \alpha_{ab} = \frac{\partial y^i}{\partial x^a} \eta_{ij} \frac{\partial y^j}{\partial x^b} = \delta_{ac} \hat{x}^c \delta_{bd} \hat{x}^d - \delta_{ab}. \]

Here we defined \( \hat{x} := \vec{x}/|\vec{x}| \) with respective components \( \hat{x}^a \) for convenience. As in Example 4.1.2, all \( \alpha_{ij} \) are independent of \( c \), \( \alpha_{00} = 1 \) and the \( \alpha_{0a} \) do not vanish. Hence the Newtonian limit needs to be attained in the limit \( c \to \infty \).

From our general expression (4.1.3) of \( \dot{\tau} \) we find

\[ \dot{\tau} = \frac{1}{\sqrt{1 - \frac{2\delta_{ab} \hat{x}^a \hat{x}^b}{c^2} + \left( \frac{\delta_{ab} \hat{x}^a \hat{x}^b}{c^2} \right)^2 - \delta_{ab} v^a v^b}}. \]

To simplify this we may write \( \vec{u} \cdot \vec{w} := \delta_{ab} u^a w^b \) for any \( \vec{u}, \vec{w} \in \mathbb{R}^3 \), \( \vec{v} = v^a e_a \), \( v := |\vec{v}| \) and \( \hat{v} := \vec{v}/v \):

\[ \dot{\tau} = \frac{1}{\sqrt{1 - 2(\hat{x} \cdot \hat{v}) \frac{v}{c} + \left( \frac{(\hat{x} \cdot \hat{v})^2}{c^2} - 1 \right) \left( \frac{v}{c} \right)^2}}. \] (4.2.1)

It is worthwhile to look at the admissible values of \( v \). If the velocity \( \vec{v} \) of the second observer is orthogonal to its location vector \( \vec{x} \), we must have \( v < c \). Yet if \( \vec{v} \) is pointing away from the observer, i.e. \( \hat{x} \cdot \hat{v} = 1 \), \( v \) is less than \( c/2 \) and, if \( \vec{v} \) points towards the observer, \( v \) may assume any positive value. Though this seems problematic at first sight, it is in fact a reasonable prediction of special relativity, if one measures the velocity \( \vec{v} \) with respect to \( \gamma \)’s time \( \tau \) instead of the synchronized time \( t \).

To obtain the Newtonian limit together with the next two relativistic correction terms, we expand expression (4.2.1) in terms of \( 1/c \) up to second order:

\[ \dot{\tau} = 1 + (\hat{x} \cdot \hat{v}) \frac{v}{c} + \left( \frac{(\hat{x} \cdot \hat{v})^2}{c^2} + \frac{1}{2} \right) \left( \frac{v}{c} \right)^2 + O \left( \frac{1}{c^3} \right). \] (4.2.2)

As a ‘real-world comparison’, assume a jet fighter reaches a speed \( v \) of 7000 kilometers per hour relative to the observer \( \gamma \) and recall that the speed of light \( c \) is approximately 300,000 kilometers per second. Then, if we neglect the influence of the upward acceleration on the clocks ('constant gravity'), equation (4.2.2) states that the first order relativistic correction of their respective clock rates \( \dot{\tau} \) is at most \( 6.5 \cdot 10^{-6} \) in absolute value.

Let us continue with the assumption that the second observer \( \gamma’ \) moves under the influence of a force \( F’ \), which is independent of the speed of light \( c \) and small in the sense that the Newtonian limit, if it exists, remains approximately valid. In coordinates \( y \) all Christoffel symbols \( \Gamma^k_{ij} \) vanish and thus, according to (4.1.7) on page 90,

\[ \Upsilon^c_{ij} = \frac{\partial x^c}{\partial y^k} \frac{\partial^2 y^k}{\partial x^i \partial x^j} = \frac{\partial^2 y^c}{\partial x^i \partial x^j} \equiv 0. \]

By our relative force equation (4.1.9), we therefore have

\[ m \frac{d\vec{v}}{d\tau} = \frac{1}{\tau^2} \vec{F}' - m \frac{\ddot{\tau}}{\tau^2} \vec{v}, \] (4.2.3)

where \( \vec{F}' \) is the spatial part of \( F' \). Hence there does appear a pseudo-force here, despite the fact that the frame of reference is inertial. If \( \dot{\tau} \approx 1 \), then \( \ddot{\tau} \approx 0 \) and thus we obtain Newton’s second law:

\[ m \frac{d\vec{v}}{d\tau} \approx \vec{F}' \].
We conclude that the Newtonian limit of the special theory of relativity indeed exists and that $\vec{F} = \vec{F}^\prime$ in this limit.

In the remainder of this section, we calculate the first two relativistic correction terms of the force equation (4.2.3) by using the approximation (4.2.2) from above. Consequently, we require expansions of $1/\tau^2$ and of

$$\frac{\dot{\tau}}{\tau^2} = \frac{\ddot{\tau}}{\dot{\tau}^2} = \frac{d^2\tau}{d\tau^2}$$

(4.2.4)

up to second order in $1/c$. For the derivative $d\tau / d\tau$ we find

$$\left(\frac{d\dot{x}}{d\tau} \cdot \vec{v} + \dot{x} \cdot \frac{d\vec{v}}{d\tau}\right) \frac{1}{c^2} + \left(\vec{v} \cdot \frac{d\vec{v}}{d\tau} + \ddot{x} \cdot \frac{d\ddot{v}}{d\tau}\right) \frac{1}{c^4} + O\left(1/c^3\right),$$

and so we compute

$$\frac{d\dot{x}}{d\tau} = \frac{1}{\vec{x}} \left(\vec{v} - (\dot{x} \cdot \vec{v}) \dot{x}\right).$$

(4.2.5)

Combining them yields

$$\frac{d\dot{\tau}}{d\tau} = \left(\dot{x} \cdot \frac{d\vec{v}}{d\tau} + \frac{\vec{v}^2}{\vec{x}} \left(1 - (\dot{x} \cdot \vec{v})^2\right)\right) \frac{1}{c}$$

$$+ \left(\vec{v} \cdot \frac{d\vec{v}}{d\tau} + \ddot{x} + \dot{x} \cdot \frac{d\ddot{v}}{d\tau}\right) \frac{1}{c^2} + O\left(1/c^3\right).$$

(4.2.6)

Now, if $f$ and $g$ are (real) polynomial expansions in a real perturbation parameter $\varepsilon$ around 0, we may write

$$f(\varepsilon) = f_0 + f_1 \varepsilon + f_2 \varepsilon^2 + O\left(\varepsilon^3\right), \quad g(\varepsilon) = g_0 + g_1 \varepsilon + g_2 \varepsilon^2 + O\left(\varepsilon^3\right).$$

Their product $fg$ is given by

$$f(\varepsilon) g(\varepsilon) = f_0 g_0 + (f_0 g_1 + f_1 g_0) \varepsilon + (f_0 g_2 + f_1 g_1 + f_2 g_0) \varepsilon^2 + O\left(\varepsilon^3\right),$$

(4.2.7)

and so the algebraic inverse $g = 1/f$ is obtained by demanding $gf = 1$, i.e.

$$\frac{1}{f(\varepsilon)} = \frac{1}{f_0} - \frac{f_1}{f_0^2} \varepsilon + \left(- \frac{f_2}{f_0^3} + \frac{(f_0)^2}{f_0^3}\right) \varepsilon^2 + O\left(\varepsilon^3\right),$$

(4.2.8)

provided it exists. See e.g. [30, Thm. 1.6] for general formulas. From (4.2.4) we find, that we first need to invert $\dot{\tau}$ via (4.2.8) (to first order) and then multiply by the expansion (4.2.6) of $d\tau / d\tau$ via the rule (4.2.7). After some labor we obtain

$$\frac{\dot{\tau}}{\tau^2} = \left(\dot{x} \cdot \frac{d\vec{v}}{d\tau} + \frac{\vec{v}^2}{\vec{x}} \left(1 - (\dot{x} \cdot \vec{v})^2\right)\right) \frac{1}{c}$$

$$+ \left(\vec{v} \cdot \frac{d\vec{v}}{d\tau} + \ddot{x} + \dot{x} \cdot \frac{d\ddot{v}}{d\tau}\right) \frac{1}{c^2} + O\left(1/c^3\right).$$

(4.2.9)

Similarly, to obtain an expansion of $1/\tau^2$, we first compute $\dot{\tau}^2$ by plugging the expansion (4.2.2) for $\dot{\tau}$ into the multiplication rule (4.2.7), and then invert it via (4.2.8):

$$\frac{1}{\tau^2} = 1 - 2 (\dot{x} \cdot \vec{v}) \frac{\varepsilon}{c} + \left((\dot{x} \cdot \vec{v})^2 - 1\right) \left(\frac{\varepsilon}{c}\right)^2 + O\left(\frac{1}{c^3}\right).$$
Finally both expressions need to be put into the relative force law (4.2.3) from above. It should be noted, that this force law is more adequate for comparing the predictions of Newtonian mechanics with those of special relativity than for calculating trajectories. In particular, (4.2.9) shows, that for very small $|\vec{x}|$ relative to the chosen length scale, the approximation in $1/c$ may break down.
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