Photon-number tomography of multimode states and positivity of the density matrix

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Abstract

For one-mode and multimode light, the photon-number tomograms of Gaussian quantum states are explicitly calculated in terms of multivariable Hermite polynomials. Positivity of the tomograms is shown to be necessary condition for positivity of the density matrix.

1 Introduction

The photon-number tomography was introduced in [1–3]. The name “photon-number tomography” was first used in [3] due to the physical meaning of measuring the quantum state by measuring number of photons (i.e., photon statistics). Photon-number tomography is the method to reconstruct the density operator of a quantum state employing measurable probability distribution function (photon statistics) called tomogram. Photon-number tomography differs of the optical tomography method [4, 5] and the symplectic tomography scheme [6–8] where continuous homodyne quadratures are measured for reconstructing the quantum state. In photon-number tomography, a discrete random variable is measured for reconstructing quantum state (photon density matrix). The other tomography, where probability distributions of discrete random variables are used, is spin tomography [9–15]. In spin tomography, the discrete random variables (spin
projections) vary in the finite domain $-j \leq m \leq j$. In photon-number tomography, the discrete random variables (number of photons) vary in infinite domain $0 \leq n \leq \infty$.

The aim of the paper is to discuss properties of photon-number tomograms for Gaussian quantum states in the one-mode and multimode cases and find a criterion of positivity of the density matrix.

The paper is organized as follows.

In Sec. 2 we review properties of the photon-number tomograms while in Sec. 3 we discuss the photon-number tomograms of multimode light. Positivity of the density matrix and its relation to positivity of the photon-state tomographic symbol is studied in Sec. 4. Conclusions and perspectives of the approach are given in Sec. 5.

## 2 One-Mode Case

The photon-number tomogram defined by the relation

$$\omega(n, \alpha) = \langle n \mid \hat{D}(\alpha) \hat{\rho} \hat{D}^{-1}(\alpha) \mid n \rangle$$

is the function of integer photon number $n$ and complex number

$$\alpha = \text{Re} \alpha + i \text{Im} \alpha,$$

where $\hat{\rho}$ is the state density operator and $\hat{D}(\alpha)$ is the Weyl displacement operator

$$\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}).$$

It is known [16] that the Wigner function, which corresponds to the density operator $\hat{\rho}$, is given by the expression

$$W_{\hat{\rho}}(q, p) = 2 \text{Tr} \left[ \hat{\rho} \hat{D}(\beta) (-1)^{\hat{a}^\dagger \hat{a}} \hat{D}(-\beta) \right],$$

$$\omega(n, \alpha) = \langle n \mid \hat{D}(\alpha) \hat{\rho} \hat{D}^{-1}(\alpha) \mid n \rangle.$$
where $\hat{D}(\beta)$ is the Weyl displacement operator with complex argument

$$\beta = \frac{1}{\sqrt{2}}(q + ip),$$

with $\hat{a}$ and $\hat{a}^\dagger$ being photon annihilation and creation operators.

Let us introduce the displaced density operator

$$\hat{\rho}_\alpha = \hat{D}^{-1}(\alpha)\hat{\rho}\hat{D}(\alpha). \quad (3)$$

The Wigner function, which corresponds to the displaced density operator, is of the form

$$W_{\hat{\rho}_\alpha}(q, p) = 2 \text{Tr} \left[ \hat{\rho}_\alpha \hat{D}(\beta) (-1)^{\hat{a}\hat{a}^\dagger} \hat{D}(-\beta) \right]. \quad (4)$$

By inserting the expression for the displaced density operator into (4), one arrives at

$$W_{\hat{\rho}_\alpha}(q, p) = 2 \text{Tr} \left[ \hat{D}^{-1}(\alpha)\hat{\rho}\hat{D}(\alpha)\hat{D}(\beta) (-1)^{\hat{a}\hat{a}^\dagger} \hat{D}(-\beta) \right]. \quad (5)$$

In view of the properties of the Weyl displacement operator

$$\hat{D}(\beta)\hat{D}(\alpha) = \hat{D}(\beta + \alpha) \exp \left[ i \text{Im} (\beta^* \alpha) \right],$$

$$\hat{D}^{-1}(\alpha) = \hat{D}(-\alpha),$$

$$\hat{D}^{-1}(\alpha)\hat{D}^{-1}(\beta) = (\hat{D}(\beta)\hat{D}(\alpha))^{-1},$$

formula (5) can be simplified

$$W_{\hat{\rho}_\alpha}(q, p) = W_{\hat{\rho}} \left( q + \sqrt{2} \text{Re} \alpha, \ p + \sqrt{2} \text{Im} \alpha \right). \quad (6)$$

One can see that the Wigner function (4) corresponding to the displaced density operator is equal to the Wigner function (2) corresponding to the initial density operator but with displaced arguments.
The photon-number tomogram is the photon distribution function (the probability to have \( n \) photons) in the state described by the displaced density operator \( \hat{\rho}_\alpha \) (3), i.e.,

\[
\omega(n, \alpha) = P_n(\alpha) = \langle n | \rho_\alpha | n \rangle, \quad n = 0, 1, 2, \ldots
\]

The photon distribution function for one-mode mixed light, described by the Wigner function of generic Gaussian form

\[
W(q, p) = \frac{1}{\sqrt{\det \sigma(t)}} \exp \left( -\frac{1}{2} Q \sigma^{-1}(t) Q^T \right),
\]

where \( Q = (p - \langle p \rangle, q - \langle q \rangle) \) and the matrix \( \sigma(t) \) is real symmetric quadrature variance matrix

\[
\sigma(t) = \begin{pmatrix}
\sigma_{pp} & \sigma_{pq} \\
\sigma_{pq} & \sigma_{qq}
\end{pmatrix},
\]

was obtained explicitly in terms of the Hermite polynomials of two variables in [17]. The quadrature means and dispersions in the above formulas can depend on time.

The Hermite polynomials of two variables \( H^{(R)}_{n_1 n_2}(y_1, y_2) \), where \( n_1, n_2 \) are nonnegative integers and \( R \) is a symmetric 2x2 matrix, are determined by the generating function

\[
\exp \left[ -\frac{1}{2} (x_1 x_2) \begin{pmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + (y_1 y_2) \begin{pmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} \right] = \sum_{n_1, n_2 = 0}^{\infty} \frac{x_1^{n_1} x_2^{n_2}}{n_1! n_2!} H^{(R)}_{n_1 n_2}(y_1, y_2).
\]

The photon distribution functions of nonclassical states of light described by the Gaussian wave functions were discussed in [18–20].

Applying the scheme of calculations similar to the one used in [17] to our photon-number tomogram (7), we arrive at the photon-number tomogram as a function of the Hermite polynomial of two variables

\[
\omega(n, \alpha) = \frac{P_0(\alpha) H_{n n}^{(R)}(y_1(\alpha), y_2(\alpha))}{n!},
\]
where the matrix $R$, which determines the Hermite polynomial, reads

$$
R = \frac{1}{1 + 2T + 4d} \begin{pmatrix}
2 (\sigma_{pp} - \sigma_{qq} - 2i\sigma_{pq}) & 1 - 4d \\
1 - 4d & 2 (\sigma_{pp} - \sigma_{qq} + 2i\sigma_{pq})
\end{pmatrix}.
$$

Here $d$ is the determinant of real symmetric quadrature variance matrix $\sigma(t)$, i.e.,

$$
d = \sigma_{pp}\sigma_{qq} - \sigma_{pq}^2
$$

and $T$ is its trace

$$
T = \sigma_{pp} + \sigma_{qq}.
$$

The arguments of the Hermite polynomial are

$$
y_1(\alpha) = y_2^*(\alpha) = \frac{\sqrt{2}}{2T - 4d - 1} \left[ \left( \langle q \rangle - i \langle p \rangle + \sqrt{2} \alpha^* \right) (T - 1) + \left( \sigma_{pp} - \sigma_{qq} + 2i\sigma_{pq} \right) \left( \langle q \rangle + i \langle p \rangle + \sqrt{2} \alpha \right) \right].
$$

For the state with displaced Wigner function (4), the probability to have no photons $P_0(\alpha)$ reads

$$
P_0(\alpha) = \frac{2}{\sqrt{L}} \exp \left\{ -\frac{1}{L} \left[ (2\sigma_{qq} + 1) \left( \langle p \rangle + \sqrt{2} \text{Im} \alpha \right)^2 + (2\sigma_{pp} + 1) \left( \langle q \rangle + \sqrt{2} \text{Re} \alpha \right)^2 \right] \right\}
\times \exp \left[ \frac{4\sigma_{pq}}{L} \left( \langle p \rangle + \sqrt{2} \text{Im} \alpha \right) \left( \langle q \rangle + \sqrt{2} \text{Re} \alpha \right) \right],
$$

where $L = 1 + 2T + 4d$.

The density operator can be reconstructed from the photon-number tomogram with the help of the inverse formula [1–3]

$$
\hat{\rho} = \sum_{n=0}^{\infty} \int \frac{4 \, d^2 \alpha}{\pi (1 - s^2)} \left( \frac{s - 1}{s + 1} \right)^{(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) - n} \omega(n, \alpha),
$$

where $s$ is an arbitrary ordering parameter [16].

Within the framework of a well-known method of the star-product quantization (see, e.g., [21]) one has the following relations:
(i) The photon-number tomogram is a symbol of the density operator

$$\omega(n, \alpha) = \text{Tr} \left[ \hat{\rho} \hat{U}(x) \right], \quad (14)$$

where operator \( \hat{U}(x) \) reads

$$\hat{U}(x) = \hat{D}(\alpha) |n\rangle \langle n| \hat{D}^{-1}(\alpha), \ x = (n, \alpha);$$

(ii) The density operator can be also expressed through its symbol

$$\hat{\rho} = \sum_{n=0}^{\infty} \int d^2\alpha \omega(x) \hat{D}(x). \quad (15)$$

Comparing (13) with (15) one can see that

$$\hat{D}(x) = \frac{4}{\pi(1 - s^2)} \left( \frac{s - 1}{s + 1} \right)^{(a^+ + a^*)(\hat{a}^+ + a)} - n. \quad (16)$$

Now we consider some simple cases.

If the electromagnetic field is in the coherent state \(|\gamma\rangle\), the photon-number tomogram reads

$$\omega_{\gamma}(n, \alpha) = \frac{1}{n!} |\gamma + \alpha|^n \exp\left[-|\gamma + \alpha|^2\right]. \quad (17)$$

One has the photon-number tomogram for squeezed and correlated states of the form

$$\omega_{sq}(n, \alpha) = \frac{\tanh^n r}{n! 2^n \cosh r} \exp\left[\tanh r \sin \theta \left( \langle p \rangle + \sqrt{2} \text{Im} \alpha \right) \left( \langle q \rangle + \sqrt{2} \text{Re} \alpha \right)\right]$$

$$- \frac{1}{2} \left( \langle p \rangle + \sqrt{2} \text{Im} \alpha \right)^2 \left( 1 - \cos \theta \tanh r \right)$$

$$- \frac{1}{2} \left( \langle q \rangle + \sqrt{2} \text{Re} \alpha \right)^2 \left( 1 + \cos \theta \tanh r \right) \right]$$

$$\times \left| H_n \left\{ \frac{1}{2} e^{-i\theta/2} \sqrt{\tanh r} \left[ \langle q \rangle - i \langle p \rangle + \sqrt{2} \alpha^* + e^{i\theta} \coth r \left( \langle q \rangle + i \langle p \rangle + \sqrt{2} \alpha \right) \right] \right\} \right|^2, \quad (18)$$
where
\[
\sin \theta = \frac{2 \sigma_{pq}}{\sqrt{(\sigma_{pp} + \sigma_{qq})^2 - 1}}, \quad \cosh 2r = T.
\]

Thus for Gaussian states, we constructed tomograms, which are positive probability distributions of number of photons. The tomograms determine the density operator of the quantum state completely.

### 3 Multimode Case

Now we briefly discuss the case of a multimode mixed state of the electromagnetic field. It will be a generalization of multiparticle spin tomography with discrete random variables varying in the finite domain [22] to the case of multimode photon-number tomography with discrete random variables varying in infinite domain \(0 \leq n_1 \leq \infty, 0 \leq n_2 \leq \infty, \ldots, 0 \leq n_N \leq \infty\).

In the case of multimode light, the tomogram reads
\[
\omega(n, \vec{\alpha}) = \langle n|\hat{D}(\vec{\alpha})\hat{\rho}\hat{D}^{-1}(\vec{\alpha})|n\rangle = \langle n|\hat{\rho}_{\alpha}|n\rangle. \tag{19}
\]

Components of the vector \(n\) are the integer photon numbers in different modes
\[
n = (n_1, n_2, \ldots, n_N)
\]
and components of the vector \(\vec{\alpha}\) are complex numbers
\[
\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N),
\]
where
\[
\alpha_k = \text{Re} \alpha_k + i \text{Im} \alpha_k, \quad k = 1, \ldots, N.
\]
The displacement operator \( \hat{D}(\vec{\alpha}) \) is a product of the Weyl displacement operators of each mode

\[
\hat{D}(\vec{\alpha}) = \prod_{k=1}^{N} \hat{D}(\alpha_k) = \prod_{k=1}^{N} \exp \left( \alpha_k \hat{a}_k^\dagger - \alpha_k^* \hat{a}_k \right).
\]

The Wigner function, which corresponds to the density operator \( \hat{\rho} \), is given by the expression

\[
W_{\hat{\rho}}(q_1, \ldots, q_N, p_1, \ldots, p_N) = 2^N \text{Tr} \left[ \hat{\rho} \hat{D}(\vec{\beta}) (-1)^{\hat{a}^\dagger_1 \hat{a}_1 + \cdots + \hat{a}^\dagger_N \hat{a}_N} \hat{D}^{-1}(\vec{\beta}) \right],
\]

(20)

where \( \hat{D}(\vec{\beta}) \) is a product of the Weyl displacement operators for each mode. The vector \( \vec{\beta} \) has complex components \( \beta_k \) \((k = 1, \ldots, N)\), i.e.,

\[
\vec{\beta} = (\beta_1, \ldots, \beta_N).
\]

The Weyl displacement operator, which corresponds to the \( k \)th mode, \( \hat{D}(\beta_k) \) has complex argument

\[
\beta_k = \frac{1}{\sqrt{2}} (q_k + ip_k),
\]

and it reads

\[
\hat{D}(\beta_k) = e^{\beta_k \hat{a}_k^\dagger - \beta_k^* \hat{a}_k}.
\]

Repeating the scheme of calculations used for the one-mode case, we obtain

\[
W_{\hat{\rho}_\alpha}(q_1, \ldots, q_N, p_1, \ldots, p_N) = W_{\hat{\rho}}(q_1 + \sqrt{2} \text{Re} \alpha_1, \ldots, q_N + \sqrt{2} \text{Re} \alpha_N, p_1 + \sqrt{2} \text{Im} \alpha_1, \ldots, p_N + \sqrt{2} \text{Im} \alpha_N).
\]

(21)

One can see that the Wigner function, which corresponds to the displaced density operator in the multimode case, is equal to the Wigner function corresponding to the initial density operator but with displaced arguments.

The photon distribution function of \( N \)-mode mixed state of light described by the Wigner function of a generic Gaussian form

\[
W(Q') = \frac{1}{\sqrt{\det \sigma(t)}} \exp \left( -\frac{1}{2} Q' \sigma^{-1}(t) Q' \right),
\]

(22)
where 2N-dimensional vector $Q'$ is

$$Q' = \left( p_1 - \langle p_1 \rangle, \ldots, p_N - \langle p_N \rangle, q_1 - \langle q_1 \rangle, \ldots, q_N - \langle q_N \rangle \right)$$

and the matrix $\sigma(t)$ is $2N \times 2N$ real symmetric quadrature variance matrix, can be calculated explicitly in terms of the Hermite polynomials of 2N variables [23].

The Hermite polynomial of $N$ variables $H_{n_1 \ldots n_N}^{[R]}(y)$ is determined by the generating function

$$\exp\left(-\frac{1}{2}xRx^T + xRy^T\right) = \sum_{n_1,\ldots,n_N=0}^{\infty} \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_N^{n_N}}{n_N!} H_{n_1 \ldots n_N}^{[R]}(y),$$

where $n_k$ ($k = 1, \ldots, N$) are nonnegative integers and $R$ is a symmetric $N \times N$ matrix.

Applying the scheme of calculations of [23] one can derive the photon-number tomogram in the case of multimode electromagnetic field as a function of the Hermite polynomial of 2N variables

$$\omega(n_1, \ldots, n_N, \alpha_1, \ldots, \alpha_N) = \frac{P_0(\bar{\alpha})H_{n_1 \ldots n_N}^{[R]}(y(\bar{\alpha}), y^*(\bar{\alpha}))}{n_1! \cdots n_N!}. \quad (23)$$

The matrix $R$, which determines the Hermite polynomial, reads

$$R = U^\dagger \left( E_{2N} - 2\sigma(t) \right) \left( E_{2N} + 2\sigma(t) \right)^{-1} U^*, \quad (24)$$

where $2N \times 2N$ matrix $U$ consists of $N$-dimensional unity matrices $E_N$ with different coefficients

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -iE_N & iE_N \\ E_N & E_N \end{pmatrix}$$

and $E_{2N}$ is $2N$-dimensional unity matrix. The argument of Hermite polynomial reads

$$y = 2U^T \left( E_{2N} - 2\sigma(t) \right)^{-1}(u,v)^T, \quad (25)$$

where

$$u = \left( \langle p_1 \rangle + \sqrt{2} \Im \alpha_1, \ldots, \langle p_N \rangle + \sqrt{2} \Im \alpha_N \right)$$
and
\[ v = \left( \langle q_1 \rangle + \sqrt{2} \text{Re} \alpha_1, \ldots, \langle q_N \rangle + \sqrt{2} \text{Re} \alpha_N \right). \]

For a state with the Wigner function \( W_{\hat{\rho}}(q_1, \ldots, q_N, p_1, \ldots, p_N) \), the probability to have no photons \( P_0 \) is given by the relation
\[ P_0 = \frac{1}{\sqrt{\det \left( \sigma(t) + \frac{1}{2}E_{2N} \right)}} \exp \left[ -(\mathbf{u}, \mathbf{v}) \left( 2\sigma(t) + E_{2N} \right)^{-1} (\mathbf{u}, \mathbf{v})^T \right]. \]  
(26)

It can be proved that the density operator can be reconstructed from the photon-number tomogram with the help of the inversion formula
\[ \hat{\rho} = \int \left[ \prod_{k=1}^{N} \frac{d \text{Re} \alpha_k \, d \text{Im} \alpha_k}{\pi} \right] \left\{ \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \left[ \prod_{k'=1}^{N} \frac{2}{(1 - s_{k'}) \left( s_{k'} - 1 \right)} \right]^{n_{k'}} \right\} \times \omega(n_1, \ldots, n_N, \alpha_1, \ldots, \alpha_N) \hat{T}, \]  
(27)

where operator \( \hat{T} \) reads
\[ \hat{T} = \prod_{k=1}^{N} \left[ \frac{2}{1 + s_k} \hat{D}^{-1}(\alpha_k) \left( \frac{s_k - 1}{s_k + 1} \right)^{\hat{a}_k} \right]. \]

Here \( s_k \) are arbitrary ordering parameters [16]. Employing the tomogram (19) one can reconstruct a generic (squeezed, correlated, and entangled) Gaussian density matrix of multimode light.

4 Positive of Density Matrix

In this section, we discuss a criterion of positivity of the density matrix. We relate properties of tomographic symbols with positivity of the Hermitian density matrix. First, we remind the conditions of positivity of a density matrix. Any Hermitian matrix \( R \) (both finite or infinite dimensional one) has real eigenvalues \( R_k \). It can be represented as a sum
\[ R = \sum_k R_k |k\rangle \langle k|, \]  
(28)
where $k$ is either discrete or continuous index (for infinite-dimensional matrix). The projectors (or projector densities) $|k⟩⟨k|$ satisfy the condition

$$R|k⟩⟨k| = R_k|k⟩⟨k|.$$  

(29)

The nonnegative Hermitian operators satisfy the condition

$$R_k \geq 0.$$  

(30)

We formulate a linear criterion of nonnegativity of the Hermitian operator $\hat{\rho}$. To do this, let us consider such operator as a density operator of a physical system. One can associate with this operator the tomographic symbol $\omega_{\hat{\rho}}$ of any kind (optical tomographic [4, 5, 24], symplectic tomographic [6–8], photon-number tomographic [1–3], or spin-tomographic one [9–15]). Since the tomographic symbols of the density operators of quantum states have the physical meaning of the probability distribution (or the probability density in the infinite dimensional case), one has the inequality

$$\omega_{\hat{\rho}} \geq 0.$$  

(31)

This inequality is necessary (and in some cases sufficient) condition of positivity of the density operator.

Let us consider this criterion for the photon-number tomograms. Since the photon-number tomogram completely determines the state, positivity of the density operator is given by the explicit relation

$$\omega_{\hat{\rho}}(n, \vec{\alpha}) \geq 0,$$  

(32)

where the tomogram $\omega_{\hat{\rho}}(n, \vec{\alpha})$ is determined in terms of the operator $\hat{\rho}$ as follows

$$\omega_{\hat{\rho}}(n, \vec{\alpha}) = \text{Tr} \left[ \hat{\rho} \hat{D}(\vec{\alpha}) |n⟩⟨n| \hat{D}(-\vec{\alpha}) \right].$$  

(33)
If one takes an Hermitian operator $\hat{R}^\dagger = \hat{R}$, which has some negative eigenvalues, one can get

$$\omega_R(n, \vec{\alpha}) < 0 \quad (34)$$

for some values of $n$ and $\vec{\alpha}$. This criterion can be expressed as a criterion for admissible Wigner functions $W(q, p)$. The Wigner function is admissible (i.e., it correspond to a state of a quantum system), if one has

$$\omega_\rho(n, \vec{\alpha}) = \int W(q + \text{Re} \vec{\alpha}, p + \text{Im} \vec{\alpha}) \prod_{k=1}^N \left( 2e^{-p_k^2 - q_k^2} L_n \left[ 2(p_k^2 + q_k^2) \right] \right) \frac{dq_k}{2\pi} \frac{dp_k}{2\pi} \geq 0 \quad (35)$$

for any vector $n = (n_1, n_2, \ldots, n_N)$ with $n_k = 0, 1, 2, \ldots$ and arbitrary complex vector $\vec{\alpha}$. In the case of the Wigner function, which is the Weyl symbol of an Hermitian (nonpositive) operator $\hat{R}$, the analogous integral can take negative values, i.e.,

$$\omega_R(n, \vec{\alpha}) < 0 \quad (36)$$

for some $n$, $\vec{\alpha}$. For Gaussian Hermitian operators, the criterion takes the form of inequality

$$\frac{P_0(\vec{\alpha}) H_{\vec{R}}(\vec{R})}{n_1! \cdots n_N!} \geq 0 \quad (37)$$

for all values of $n$ and $\vec{\alpha}$. In (37) the expressions for $P_0(\vec{\alpha})$, $\vec{R}$, and $y(\vec{\alpha})$ are given by formulas (24)–(26) for photon-number tomograms.

For Gaussian states, this criterion can be reformulated as the property of quadrature dispersion matrix $\sigma$. For example, it has to satisfy the condition

$$\det \begin{pmatrix} \sigma_{p_k p_k} & \sigma_{p_k q_k} \\ \sigma_{p_k q_k} & \sigma_{q_k q_k} \end{pmatrix} \geq \frac{1}{4} \quad (38)$$

for all $k = 1, 2, \ldots, N$. The Schrödinger–Robertson uncertainty relation

$$\det \sigma \geq \left( \frac{1}{4} \right)^N \quad (39)$$
(here $\hbar = 1$) is also necessary condition of positivity of the Gaussian state. But it is not sufficient condition. The necessary and sufficient condition of positivity of the Gaussian state is the set of inequalities (37), (38), and partial Schrödinger uncertainty relations for all modes. The set of inequalities (38) for non-Gaussian states is necessary condition of positivity (nonnegativity) of an Hermitian operator but they are insufficient for nonnegativity. For one-mode Gaussian state, the condition of nonnegativity of the density operator reads

$$\sigma_{pp}\sigma_{qq} - \sigma_{pq}^2 \geq \frac{1}{4},$$

which is nothing else as the standard Schrödinger–Robertson uncertainty relation. If this inequality is not fulfilled, one has nonpositive density operators. But such operator (or such corresponding Wigner function) can describe the classical state with Gaussian distribution function on the phase space. Thus negative probabilities related to “density operators” are appropriate to describe the classical states in the standard classical statistical mechanics. The connection of uncertainty relation for quadratures (position and momentum) with photon statistics was found in [23].

5 Conclusions

We have shown that the photon-number tomogram of a generic Gaussian (one-mode and multimode) state is expressed in terms of multivariable Hermite polynomials. This tomogram coincides with the photon-number distribution function related to the Wigner function with displaced arguments. The tomogram obtained can be used for measuring quantum states. We found the criterion of positivity of the density matrix and consider this criterion for the photon-number tomograms. We discovered necessary and sufficient condition of positivity of the density matrix in the case of
a Gaussian state. We constructed the set of inequalities which is necessary condition of positivity of the density operator in the case of non-Gaussian states.

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