EXTENSION OF INCOMPRESSIBLE SURFACES
ON THE BOUNDARY OF 3-MANIFOLDS

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ABSTRACT. An incompressible surface $F$ on the boundary of a compact orientable 3-manifold $M$ is arc-extendible if there is an arc $\gamma$ on $\partial M - \text{Int} F$ such that $F \cup N(\gamma)$ is incompressible, where $N(\gamma)$ is a regular neighborhood of $\gamma$ in $\partial M$. Suppose for simplicity that $M$ is irreducible, and $F$ has no disk components. If $M$ is a product $F \times I$, or if $\partial M - F$ is a set of annuli, then clearly $F$ is not arc-extendible. The main theorem of this paper shows that these are the only obstructions for $F$ to be arc-extendible.

Suppose $F$ is a compact incompressible surface on the boundary of a compact, orientable, irreducible 3-manifold $M$. Let $F'$ be a component of $\partial M - \text{Int} F$. We say that $F$ is arc-extendible (in $F'$) if there is a properly embedded arc $\gamma$ in $F'$ such that $F \cup N(\gamma)$ is incompressible. In this case $\gamma$ is called an extension arc of $F$. We study the problem of which incompressible surfaces on the boundary $M$ are arc-extendible. This is useful in, for example, finding a sequence of mutually nonparallel incompressible surfaces in a 3-manifold.

Denote by $I$ the unit interval $[0,1]$. We say that $M$ is a product $F \times I$ if there is a homeomorphism $\varphi : M \cong F \times I$ with $\varphi(F) = F \times 1$. Note that in this case $F' = \partial M - \text{Int} F$, and $F$ is not arc-extendible. A surface $F$ is diskless if it has no disk component. An incompressible surface with a disk component is always arc-extendible, unless the disk lies on a sphere component of $\partial M$. Thus to avoid trivial cases, we will only consider arc-extension of diskless surfaces.

**Theorem 1.** Let $F$ be a diskless, compact, incompressible surface on the boundary of a compact, orientable, irreducible 3-manifold $M$, and let $F'$ be a non-annular component of $\partial M - \text{Int} F$. Then either $F$ is arc-extendible in $F'$, or $M$ is a product $F \times I$.

The proof of the theorem involve some deep results about incompressible surfaces related to Dehn surgery and 2-handle additions. It breaks down into three cases. The
case that $F'$ is a thrice punctured sphere is treated in Theorem 4, which shows that if the surface obtained by gluing $F$ and $F'$ along one of the boundary curve of $F'$ is compressible for all the three boundary curves of $F'$, then $M$ must be a product. The second case is that $F'$ is parallel into $F$ (see below for definition). A similar result as above holds in this case. Theorem 9 shows that in the remaining case there is an arc $\gamma$ intersecting some circle $C$ in $F'$ at one point, so that all but at most three Dehn twists of $\gamma$ along $C$ are extension arcs of $F$. Moreover, in this case the extension arc $\gamma$ of $F$ can be chosen to have endpoints on any prescribed components of $\partial F'$. See Theorem 10 below.

Note that the irreducibility of $M$ is irrelevant to the compressibility of surfaces on $\partial M$. However, this does make the conclusion of the theorem simpler. If we drop this assumption from the theorem, the conclusion should be changed to “Either $F$ is arc-extendible in $F'$, or there is a component $F_0$ of $F$, and a homeomorphism $\varphi: M \cong F_0 \times I \# M'$ for some $M'$, such that $\varphi(F_0) = F_0 \times 1$, and $\varphi(F') = F_0 \times 0 \cup \partial F_0 \times I$.”

Given a simple closed curve $\alpha$ on a surface $S$ on the boundary of $M$, we use $M[\alpha]$ to denote the manifold obtained by adding a 2-handle to $M$ along the curve $\alpha$. More explicitly, $M[\alpha]$ is the union of $M$ and a $D^2 \times I$, with the annulus $(\partial D^2) \times I$ glued to a regular neighborhood $N(\alpha)$ of $\alpha$ on $\partial M$. Use $S[\alpha]$ to denote the surface in $M[\alpha]$ corresponding to $S$, i.e. $S[\alpha] = (S - N(\alpha)) \cup (D^2 \times \partial I)$. The following two lemmas are very useful in dealing with incompressible surfaces. Various versions of Lemma 2 have been proved by Przytycki [Pr], Johannson [Jo], Jaco [Ja], and Scharlemann [Sch]. The lemma as stated is due to Casson and Gordon [CG].

**Lemma 2.** (The Handle Addition Lemma [CG].) Let $\alpha$ be a simple closed curve on a surface $S$ on the boundary of an orientable irreducible 3-manifold $M$, such that $S$ is compressible and $S - \alpha$ is incompressible. Then $S[\alpha]$ is incompressible in $M[\alpha]$, and $M[\alpha]$ is irreducible.

**Lemma 3.** (The Generalized Handle Addition Lemma.) Let $S$ be a surface on the boundary of an orientable 3-manifold $M$, let $\gamma$ be a 1-manifold on $S$, and let $\alpha$ be a circle on $S$ disjoint from $\gamma$. Suppose $S - \gamma$ is compressible and $S - (\gamma \cup \alpha)$ is incompressible. If $D$ is a compressing disk of $S[\alpha]$ in $M[\alpha]$, then there is a compressing disk $D'$ of $S - \alpha$ in $M$ such that $\partial D' \cap \gamma \subset \partial D \cap \gamma$.

**Proof.** This is essentially [Wu2, Theorem 1]. The theorem there stated that $\partial D' \cap \gamma$ has no more points than $\partial D \cap \gamma$, but the proof there gives the stronger conclusion that $\partial D' \cap \gamma \subset \partial D \cap \gamma$. □

We first study the case that the surface $F'$ in Theorem 1 is a thrice punctured sphere. Let $\alpha_1, \alpha_2, \alpha_3$ be the boundary curves of $F'$. Since $F'$ is a component of
∂M − IntF, we have \( α_i \subset \partial F \) for \( i = 1, 2, 3 \). Note that if \( \text{Int} F \cup \text{Int} F' \cup α_i \) is incompressible for some \( i \), then for any essential arc \( γ \) on \( F' \) with \( \partial γ \subset \alpha_i \), the surface \( F \cup N(γ) \) is incompressible. Hence the following theorem proves Theorem 1 in the case that \( F' \) is a twice punctured disk. However, it should be noticed that a similar statement is false if we drop the assumption that \( F' \) is a sphere with three holes.

**Theorem 4.** Let \( F \) be a diskless compact incompressible surface on the boundary of a compact, orientable, irreducible 3-manifold \( M \), and let \( F' \) be a component of \( \partial M − \text{Int} F \) which is a punctured sphere with \( \partial F' = α_1 \cup α_2 \cup α_3 \). If \( \text{Int} F \cup \text{Int} F' \cup α_i \) is compressible for \( i = 1, 2, 3 \), then \( M \) is a product \( F \times I \).

**Proof.** We fix some notation. Write \( \hat{F} = F \cup F' \). Denote by \( \hat{F}_i \) the surface obtained by gluing \( \text{Int} F \) and \( \text{Int} F' \) along \( α_i \), i.e. \( \hat{F}_i = \text{Int} F \cup \text{Int} F' \cup α_i \). Similarly, write \( \hat{F}_{ij} = \text{Int} F \cup \text{Int} F' \cup α_i \cup α_j \).

First notice that \( F' \) is incompressible. This is because each simple closed curve on \( F' \) is isotopic to one of the \( α_i \subset F \), and because \( F \) is incompressible and diskless. Since \( \text{Int} F \cap \text{Int} F' = \emptyset \), the surface \( \text{Int} F \cup \text{Int} F' \) is incompressible.

Let \( M' \) be a maximal compression body of \( \partial F \) in \( M \). Then a surface on the boundary of \( M \) is compressible in \( M \) if and only if it is compressible in \( M' \). Notice that if \( M \neq M' \), then \( M' \) is never a product \( F \times I \), so if the theorem is true for \( M' \), it is true for \( M \). Hence after replacing \( M \) by \( M' \) if necessary, we may assume without loss of generality that \( M \) is a compression body.

We claim that the curves \( α_1, α_2, α_3 \) are mutually nonparallel on \( \hat{F}_i \), that is, no component of \( F \) is an annulus with both boundary components on \( F' \). If two curves \( α_1, α_2 \), say, are parallel on \( \hat{F}_i \), then the surface \( \text{Int} F \cup \text{Int} F' = \hat{F} − α_1 \cup α_2 \cup α_3 \) is incompressible if and only if \( \hat{F}_1 = \hat{F} − α_2 \cup α_3 \) is incompressible. However, by assumption \( \hat{F}_1 \) is compressible, and we have shown that \( \text{Int} F \cup \text{Int} F' \) is incompressible. Hence the claim follows.

Since \( \hat{F}_i \) is compressible, and \( \hat{F}_i − α_i = \text{Int} F \cup \text{Int} F' \) is incompressible, we can apply the Handle Addition Lemma (Lemma 2) to \( \hat{F}_i \) and \( α_i \) to conclude that after adding a 2-handle along \( α_i \), the surface \( \hat{F}_i[α_i] \) is incompressible in \( M[α_i] \), and \( M[α_i] \) is irreducible.

Consider the surface \( \hat{F}_i[α_i] \). Notice that after adding the 2-handle, the surface \( F' \) becomes an annulus on \( \hat{F}_i[α_i] \) with boundary \( α_2 \cup α_3 \), so the two curves \( α_2, α_3 \) are parallel on \( \hat{F}_i[α_i] \). Thus, \( \hat{F}_1[α_i] = \hat{F}_i[α_i] − α_2 \cup α_3 \) being incompressible in \( M[α_i] \) implies that \( \hat{F}_i[α_i] − α_2 \) is incompressible in \( M[α_i] \). With the above notation, this says that \( \hat{F}_1[α_i] \) is incompressible in \( M[α_i] \).

By assumption \( \hat{F}_3 \) is compressible in \( M \). Let \( D \) be a compressing disk of \( \hat{F}_3 \) in \( M \). Then \( \partial D \) is disjoint from \( α_1 \cup α_2 \), because \( \partial D \subset \hat{F}_3 \). Also, \( \partial D \) is not isotopic to \( α_1 \) in
\( \hat{F}_{13} \), otherwise \( \alpha_1 \) would bound a disk in \( M \), contradicting the assumption that \( F \) is diskless and incompressible. We have shown that \( \hat{F}_{13}[\alpha_1] \) is incompressible in \( M[\alpha_1] \), so \( D \) is not a compressing disk of \( \hat{F}_{13}[\alpha_1] \) in \( M[\alpha_1] \), and hence \( \partial D \) must bound a disk in \( \hat{F}_{13}[\alpha_1] \). This is true if and only if \( \partial D \) is coplanar to \( \alpha_1 \) on \( \hat{F}_{13} \), that is, either \( \partial D \) is parallel to \( \alpha_1 \), or it bounds a once punctured torus \( T \) in \( \hat{F}_{13} \) which contains \( \alpha_1 \) as a nonseparating curve. The first possibility has been ruled out, so the second must be true. Let \( \hat{T} \) be the torus \( T \cup D \). Since we have assumed above that \( M \) is a compression body, either (i) \( \hat{T} \) is parallel to a boundary component of \( M \), or (ii) \( \hat{T} \) bounds a solid torus.

If \( \hat{T} \) is parallel to a boundary component \( T_0 \) of \( M \), then after adding the 2-handle, the surface \( \hat{T}[\alpha_1] \) becomes a sphere which separates \( T_0 \) from \( \hat{F}[\alpha_1] \), hence is a reducing sphere of \( M[\alpha_1] \), which contradicts the irreducibility of \( M[\alpha_1] \). Similarly, if \( \hat{T} \) bounds a solid torus \( V \) but \( \alpha_1 \) is not a longitude of \( V \), then after adding the 2-handle the manifold would have a lens space or \( S^2 \times S^1 \) summand, which again contradicts the irreducibility of \( M[\alpha_1] \). (Note that \( M[\alpha_1] \) cannot be a lens space because it has some boundary components.)

We have now shown that there is a compressing disk \( D \) of \( \hat{F}_3 \) in \( M \) which cuts the manifold into two pieces, one of which is a solid torus \( V \) which contains \( \alpha_1 \) as a longitude, but is disjoint from \( \alpha_2 \). Let \( D_1 \) be a meridian disk of \( V \). Then \( \partial D_1 \cap \alpha_1 \) is a single point, and \( \partial D_1 \) is disjoint from \( \alpha_2 \) because \( \partial V \) is disjoint from \( \alpha_2 \). Notice that \( \partial D_1 \) is not coplanar to \( \alpha_2 \), for if \( \partial D_1 \) were parallel to \( \alpha_2 \) then \( \alpha_2 \) would also intersect \( \alpha_1 \), and if \( \partial D_1 \) would bound a once punctured torus containing \( \alpha_2 \) then \( \partial D_1 \) would be a separating curve on \( \partial M \), so it would intersect \( \alpha_1 \) in an even number of points, either case leading to a contradiction. Thus, after adding a 2-handle to \( M \) along \( \alpha_2 \), the disk \( D_1 \) remains a compressing disk of \( \hat{F}_2[\alpha_2] \). Since the two curves \( \alpha_1 \) and \( \alpha_3 \) are parallel in \( \hat{F}_2[\alpha_2] \), and since \( D_1 \) intersects \( \alpha_1 \) in a single point, we can isotope \( D_1 \) to another disk \( D_2 \) in \( M[\alpha_2] \) so that it intersects each of \( \alpha_1 \) and \( \alpha_3 \) in a single point. We are looking for such a disk in \( M \); however \( D_2 \) is not necessary the one because it may intersect the attached 2-handle.

Recall that the surface \( \hat{F}_2 \) is compressible, but the surface \( \hat{F}_2 - \alpha_2 = \text{Int} F \cup \text{Int} F' \) is incompressible. Hence we can apply the Generalized Handle Addition Lemma (Lemma 3, with \( S = \hat{F} \), \( \gamma = \alpha_1 \cup \alpha_3 \), and \( \alpha = \alpha_2 \)) to conclude that there is also a compressing disk \( D_3 \) of \( \hat{F} \) in \( M \), such that \( \partial D_3 \) is disjoint from \( \alpha_2 \), and \( \partial D_3 \cap (\alpha_1 \cup \alpha_3) \) is a subset of \( \partial D_3 \cap (\alpha_1 \cup \alpha_3) \).

The set \( \partial D_3 \cap (\alpha_1 \cup \alpha_3) \) is nonempty, otherwise, since \( \partial D_3 \) is also disjoint from \( \alpha_2 \), \( D_3 \) would be a compressing disk of \( \text{Int} F \cup \text{Int} F' \), contradicting the incompressibility of \( \text{Int} F \cup \text{Int} F' \). Since \( \alpha_1 \cup \alpha_2 \cup \alpha_3 \) is separating on \( \hat{F} \), the curve \( \partial D_3 \) can not intersect \( \alpha_1 \cup \alpha_2 \cup \alpha_3 \) at a single point. It follows that \( \partial D_3 \cap (\alpha_1 \cup \alpha_3) = \partial D_2 \cap (\alpha_1 \cup \alpha_3) \), that
is, $\partial D_3$ intersects each of $\alpha_1, \alpha_3$ in a single point. Such a disk is called a bigon.

Denote by $D_{13}$ the bigon $D_3$ above. Interchanging the rules of $\alpha_1$ and $\alpha_2$ in the above argument, we get another compressing disk $D_{23}$ of $\tilde{F}$ in $M$, which is disjoint from $\alpha_1$, and intersects each of $\alpha_2, \alpha_3$ in a single point. By a simple innermost circle outermost arc argument, we can isotope $D_{13}$ so that it is disjoint from $D_{23}$, and still has the same number of intersection points with each $\alpha_i$. Cutting $M$ along $D_{13} \cup D_{23}$, we get a submanifold $M'$ of $M$, in which the surface $F'$ becomes a disk $\tilde{F}' \subset F'$, and the surface $F$ becomes a surface $\tilde{F} \subset F$. It is clear that one boundary component $C$ of $\tilde{F}$ bounds a disk on $\partial M'$, namely the union of $\tilde{F}'$ and the two copies of $D_{13} \cup D_{23}$. Since $F$ is incompressible, this curve $C$ bounds a disk in $F$, so $\tilde{F}$ must be a disk. These disks together form a sphere boundary component of $M'$. Since $M$ is irreducible, $M'$ must be a 3-ball, so it is a product $\tilde{F} \times I$. Gluing back along $D_{13}$ and $D_{23}$, we see that $M$ is a product $F \times I$. This completes the proof of Theorem 4. \[\square\]

Below, $F, F'$ and $M$ will be as in Theorem 1. Using Theorem 4 we may assume that $F'$ is not a thrice punctured sphere. A curve $C'$ on $F'$ is $\partial$-nonseparating if (i) $C'$ is not parallel to a boundary curve on $F'$, and (ii) there is a proper arc $\gamma$ in $F'$ intersecting $C'$ in a single point. A sub-surface $G'$ of $F'$ is parallel into $F$ if there is a product $G' \times I \subset M$ such that $G' = G' \times 0$, and $G' \times 1 \subset F$. Similarly, a curve $C'$ on $F'$ is parallel into $F$ if there is an embedded annulus $A \subset M$ with $\partial A = C' \cup C$, where $C \subset F$.

**Lemma 5.** If $F'$ is compressible, then there is a $\partial$-nonseparating curve $C'$ on $F'$ which is not parallel into $F$.

**Proof.** Let $D$ be a compressing disk of $F'$. If $\partial D$ is non-separating on $F'$, let $C'$ be a curve in $F'$ that intersects $\partial D$ in one point. Then $C'$ is nonseparating, hence $\partial$-nonseparating on $F'$. We want to show that $C'$ is not parallel into $F$. Otherwise, let $A$ be an annulus with $\partial A = C' \cup C$, where $C \subset F$. Then $A \cap D$ is a proper 1-manifold on $D$. But $\partial (A \cap D) = (\partial A) \cap \partial D$ is a single point, which is absurd. Hence $C'$ is the curve required.

Now assume that $\partial D$ is separating on $F'$, cutting $F'$ into $F'_1$ and $F'_2$. Choose a simple loop $C_i$ on $F'_i$ as follows. If $F'_i$ is nonplanar, then there are a pair of nonseparating curves intersecting each other in one point, at least one of which is not null-homologous in $M$. Choose this one as $C_i$. If $F'_i$ is planar, choose $C_i$ to be isotopic to a boundary curve of $F'$. Note that since $F$ is incompressible and diskless, $C_i$ is not null-homotopic in $M$. Also notice that in both cases there is a properly embedded arc $\gamma$ on one of the $F'_i$ which intersects $C_1 \cup C_2$ in one point.

Now choose a band $B = I \times I$ on $F'$ such that $B \cap \partial D = I \times \frac{1}{2}$, $B \cap C_1 = I \times 0$, $B \cap C_2 = I \times 1$, and $B$ is disjoint from the arc $\gamma$ above. Such band exists because
γ is a nonseparating arc on $F'_1$. Let $C'$ be the band sum of $C_1$ and $C_2$, that is, $C' = (C_1 \cup C_2 - I \times \{0,1\}) \cup (\{0,1\} \times I)$. Then $C'$ intersects $γ$ in one point. Since $C'$ intersects $∂D$ essentially in two points, it is not parallel to any boundary component on $F'$. Therefore $C'$ is $∂$-nonseparating.

We want to show that $C'$ is not parallel into $F$. Using the property that $C_1$ are not null-homotopic in $M$, one can show by an innermost circle argument that $C'$ is not null-homotopic in $M$. Now suppose that there is an annulus $A$ in $M$ with $∂A = C' \cup C$, where $C \subset F$. Since $C'$ is not null-homotopic in $M$, $A$ is incompressible in $M$. By surgery along an innermost circle of $D \cap A$ one can eliminate all circle intersections of $A \cap D$. Since $∂(A \cap D)$ consists of two points, $A \cap D$ is a single arc, which has endpoints on the same component of $∂A$, hence it cuts off a disk $D'$ from $A$. Assume without loss of generality that $D' \cap F'$ is on $F'_1$. Let $D''$ be the disk on $D$ bounded by $(A \cap D) \cup (B \cap D)$, and let $B_1 = B \cap F'_1$. Then $D' \cup D'' \cup B_1$ is a disk with boundary $C_1$, which contradicts the fact that $C_1$ is not null-homotopic in $M$. Therefore, $C'$ is not parallel into $F$. □

**Lemma 6.** Suppose $F'$ is incompressible, and is not a thrice punctured sphere. Then either (i) there is a $∂$-nonseparating curve $C'$ on $F'$ which is not parallel into $F$, or (ii) $F'$ is parallel into $F$.

**Proof.** Since $F'$ is not a thrice punctured sphere, one can easily find a $∂$-nonseparating curve $α_0$ on $F'$. Assume that (i) is not true, so all $∂$-nonseparating curves are parallel into $F$. We want to show that $F'$ is parallel into $F$.

Since $α_0$ is parallel into $F$, the annulus $N(α_0)$ is also parallel into $F$. It is an incompressible annulus because $α_0$ is essential on $F'$ and $F'$ is incompressible. Among all connected incompressible surfaces in $\text{Int}F'$ which contain $α_0$ and are parallel into $F$, choose $G'$ such that the complexity $(χ(G'), |∂G'|)$ is minimal in the lexical-graphic order, where $χ(G')$ is the Euler characteristic of $G'$, and $|∂G'|$ is the number of boundary components of $G'$. All incompressible sub-surfaces of $F'$ have Euler characteristics bounded below by $χ(F')$, hence such $G'$ does exist.

If all boundary components of $G'$ are parallel to some boundary components on $F'$, then either $G'$ is contained in a collar of $∂F'$, or $F' - \text{Int}G' = ∂F' \times I$. The first case does not happen because $G'$ contains the $∂$-nonseparating curve $α_0$, which by definition is not parallel to any boundary curve on $F'$. In the second case $F'$ is isotopic to $G'$, so it is parallel into $F$, and we are done. Hence we may assume that some boundary curve $β$ of $G'$ is not parallel to any boundary curve on $F'$.

We want to find a $∂$-nonseparating curve $α'$ which intersects $β$ essentially in one or two points. If $β$ is nonseparating on $F'$, choose $α'$ to be any curve on $F'$ that intersects $β$ in a single point. Then $α'$ is nonseparating, hence $∂$-nonseparating on $F'$. 
If $\beta$ separates $F'$ into $F'_1$ and $F'_2$, choose an essential arc $\alpha'_i$ on $F'_i$ with $\partial \alpha'_1 = \partial \alpha'_2 \subset \beta$. Moreover, if $F'_i$ is nonplanar, choose $\alpha'_i$ to be nonseparating on $F'_i$. Then $\alpha' = \alpha'_1 \cup \alpha'_2$ is $\partial$-nonseparating, and intersects $\beta$ essentially in two points, as required.

Isotope $\alpha'$ so that it intersects $\partial G'$ minimally. The geometric intersection number between $\alpha'$ and $\beta$ is 1 or 2, so $\alpha' \cap \partial G' \neq \emptyset$. Since $\alpha'$ is $\partial$-nonseparating, by our assumption above it is parallel into $F$, so there is an annulus $A$ with $\partial A = \alpha' \cup \alpha$, where $\alpha \subset F$. Isotope $A$ rel $\alpha'$ so that it intersects $(\partial G') \times I$ minimally. Since $G'$ is incompressible, $(\partial G') \times I$ consists of incompressible annuli in $M$, hence $A \cap ((\partial G') \times I)$ has no trivial circles. Since $F$ and $F'$ are also incompressible, one can show that $A \cap ((\partial G') \times I)$ has no trivial arcs on $A$ either. Therefore $A \cap ((\partial G') \times I)$ consists of vertical arcs $(\alpha' \cap \partial G') \times I$. These arcs cut $A$ into several squares $\alpha'_i \times I$, where each $\alpha'_i$ is the closure of a component of $\alpha' - \partial G'$. Choose $i$ so that $\alpha'_i$ lies outside of $G'$. Let $H$ be the component of $F' - \text{Int} G'$ that contains $\alpha'_i$. Then $G'' = G' \cup N(\alpha'_i)$ is a surface parallel into $F$, and $\chi(G'') = \chi(G') - 1$. The arc $\alpha'_i$ is essential on $H$, so the only case that some boundary component $\gamma$ of $G''$ bounds a disk on $F'$ is when $H$ is an annulus, and $\gamma$ is the boundary of the disk obtained by cutting $H$ along $\alpha'_i$. Since $F$ and $F'$ are incompressible and $M$ is irreducible, both ends of the annulus $\gamma \times I \subset G'' \times I \subset M$ bound disks on $F \cup F'$, which together with $\gamma \times I$ bounds a 3-ball in $M$. It follows that $G' \cup H$ is parallel into $F$. Since $G' \cup H$ has the same Euler characteristic as $G'$ but fewer number of boundary components, this contradicts the choice of $G'$. Therefore $\partial G''$ consists of essential curves on $F'$. Since $F'$ is incompressible, $G''$ is also incompressible. Since $\chi(G'') < \chi(G')$, this again contradicts the choice of $G'$. \(\square\)

Given a simple closed curve $\alpha$ and a proper arc $\gamma$ on $F'$, denote by $\tau^n_\alpha \gamma$ the curve obtained from $\gamma$ by Dehn twist $n$ times along $\alpha$, and by $N(\tau^n_\alpha \gamma)$ a regular neighborhood of $\tau^n_\alpha \gamma$ on $\partial M$. Suppose $T$ is a fixed torus boundary component of a 3-manifold $M$. Denote by $M(r)$ the manifold obtained by Dehn filling on $T$ along a slope $r$ on $T$, that is $M(r)$ is obtained by gluing a solid torus $V$ to $M$ along $T$ so that the curve $r$ on $T$ bounds a meridian disk on $V$. Denote by $\Delta(r_1, r_2)$ the minimal geometric intersection number between two slopes $r_1, r_2$. The following two theorems will be used in the proof of Theorem 9, which proves Theorem 1 in the case that $F'$ contains a $\partial$-nonseparating curve which is not parallel into $F$.

**Lemma 7.** ([Wu2], Theorem 1) Let $T$ be a torus component on the boundary of a 3-manifold $M$, and let $S$ be an incompressible surface on $\partial M - T$. Suppose there is no incompressible annulus in $M$ with one boundary component on each of $S$ and $T$. If $S$ is compressible in $M(r_1)$ and $M(r_2)$, then $\Delta(r_1, r_2) \leq 1$. In particular, $S$ is incompressible in all but at most three $M(r)$. \(\square\)
Lemma 8. ([CGLS], Theorem 2.4.3) Let $T, S, M$ be as in Lemma 7, and assume further that $M$ is irreducible. Suppose that there is an incompressible annulus $A$ in $M$ with one boundary component on $S$ and the other a curve $r_0$ on $T$. Then either $S$ is a torus and $M = S \times I$, or $S$ remains incompressible in all $M(r)$ with $\Delta(r, r_0) > 1$. □

Theorem 9. Let $\alpha$ be a $\partial$-nonseparating curve on $F'$ which is not parallel into $F$, and let $\gamma$ be a proper arc on $F'$ intersecting $\alpha$ in one point. Then $F_n = F \cup N(\tau^n_\alpha \gamma)$ is incompressible for all but at most three consecutive $n$’s.

Proof. Let $K$ be the knot obtained by pushing $\alpha$ slightly into $M$. There is an embedded annulus $A_0$ in $M$ with $\partial A_0 = \alpha \cup K$. Consider the manifold $M_K = M - \text{Int} N(K)$, where $N(K)$ is a regular neighborhood of $K$ in $M$. Let $T$ be the torus $\partial N(K)$, and let $(m, l)$ be the meridian-longitude pair on $T$ such that $l = A_0 \cap T$. Denote by $M_K(p/q)$ the manifold obtained by Dehn filling on $T$ along the slope $pm + ql$. The Dehn twist $\tau^n_\alpha$ on $F'$ extends to a Dehn twist of $M_K$ along the annulus $A = A_0 \cap M_K$, which sends the meridian slope $m$ of $T$ to the slope $m - nl$, so it extends to a homeomorphism $\varphi_n : M = M_K(1/0) \cong M_K(-1/n)$, which maps the curve $\tau^n_\alpha \gamma$ to the curve $\gamma$, and hence the surface $F_n$ to the surface $F_0 = F \cup N(\gamma)$. It follows that $\varphi_n$ is a homeomorphism of pairs

$$\varphi_n : (M, F_n) \rightarrow (M_K(-1/n), F_0).$$

Therefore to prove the theorem we need only show that for all but at most three consecutive integers $n$, the surface $F_0$ is incompressible in $M_K(-1/n)$.

CLAIM 1. $T = \partial N(K)$ is incompressible in $M_K$, and $M_K$ is irreducible.

If $D$ is a compressing disk of $T$ in $M_K$, then $\partial D$ must intersect the meridian $m$ of $K$ in one point, because otherwise after the trivial Dehn filling, $M = M_K(1/0)$ would contain a lens space or $S^2 \times S^1$ summand, contradicting the irreducibility of $M$. It follows that $K$, and hence $\alpha$, bounds a disk in $M$. In this case $\alpha$ is parallel to a trivial curve on $F$, which contradicts the assumption that $\alpha$ is not parallel into $F$. Similarly, if $M_K$ is reducible, then since $M$ is irreducible, $K$ is contained in a ball in $M$, so $\alpha$ would be null-homotopic. Using Dehn’s Lemma, we see that $\alpha$ bounds a disk in $M$, hence is parallel to a trivial circle in $F$, contradicting the assumption that $\alpha$ is not parallel into $F$.

CLAIM 2. $F_0$ is incompressible in $M_K$.

Recall that $A$ denotes the annulus $A_0 \cap M_K$. Since $\alpha$ intersects $\gamma$ in a single point, $A \cap F_0$ is a single arc $C$ on the boundary curve $\alpha$ of $A$. Let $D$ be a compressing disk of $F_0$, chosen so that $|D \cap A|$, the number of components in $D \cap A$, is minimal. After disk swapping along disks on $A$ bounded by innermost circles, we can assume that no component of $D \cap A$ is a trivial circle on $A$. Since $T$ is incompressible by Claim 1, the
annulus $A$ is also incompressible, so $D \cap A$ contains no essential circle component on $A$ either. Hence $D \cap A$ consists of arcs only. If some arc $e$ of $D \cap A$ is parallel to a sub-arc on $C = A \cap F_0$, then after boundary compressing $D$ along a disk $\Delta$ cut off by an outermost such arc we will get two disks $D_1, D_2$ with boundary on $F_0$, at least one of which has boundary an essential curve on $F_0$, hence is a compressing disk of $F_0$. Since $|D_1 \cap A| < |D \cap A|$, this contradicts the minimality of $|D \cap A|$. Therefore, all arcs of $D \cap A$ are essential relative to $C$, in the sense that it is not parallel to an arc on $C$. See Figure 1(a). Notice that $|D \cap A| \neq 0$, otherwise $D$ would be a compressing disk of $F$, contradicting the incompressibility of $F$.

Consider an outermost disk $\Delta$ on $D$, as shown in Figure 1(b). Then $\partial \Delta$ consists of two arcs $e_1, e_2$, where $e_1$ is an arc on $A$ which is essential relative to $C$, and $e_2$ is an arc on $F_0$ with interior disjoint from $C$. Thus $e_2 \cap N(\gamma)$ consists of two arcs $e'_2, e''_2$. Let $t_1$ be the subarc of $C$ connecting the two ends of $e'_2 \cup e''_2$ on $C$, and let $t_2$ be the subarc on $\partial N(\gamma)$ connecting the other two ends of $e'_2 \cup e''_2$. Then $e'_2 \cup t_1 \cup e''_2 \cup t_2$ bounds a disk $\Delta'$ on $N(\gamma)$. Now $A' = \Delta \cup \Delta'$ is an annulus in $M$, with one boundary component $e_1 \cup t_1$ an essential circle on $A$, which is parallel to $\alpha$, and the other component $e_2 \cup t_2$ a curve on $F$. This contradicts the assumption that $\alpha$ is not parallel into $F$.

**Figure 1**

**CLAIM 3.** There is no incompressible annulus $P$ in $M_K$ with one boundary component $C_1$ on $F_0$ and the other component $C_2$ a curve on $T$ which is disjoint from $l = A \cap T$.

The proof is similar to that of Claim 2. Choose $P$ so that $|P \cap A|$ is minimal. Using the fact that $P$ is incompressible, one can show as above that $P \cap A$ has no trivial circle component. Note that since $C_2$ is disjoint from $l$, $P \cap A$ has no arc component with endpoints on $l = A \cap T$. If $P \cap A$ had some essential circle component, choose such
a component \( t \) which is closest to \( l \) on \( A \). By cutting and pasting along the annulus on \( A \) bounded by \( t \cup l \), one would get another incompressible annulus \( P' \) which has fewer intersection components with \( A \). As in the proof of Claim 2 one can eliminate all arc components of \( P \cap A \) which are inessential relative to \( C = A \cap F_0 \). Hence \( P \cap A \) consists of arcs with ends on \( C \) and are essential relative to \( C \), as shown in Figure 1(a). Also, since \( P \) is disjoint from \( l \), \( P \cap A \) are inessential arcs on \( P \). Now one can use a disk \( \Delta \) cut off by an outermost arc on \( P \), proceed as in the proof of Claim 2 to get an annulus with one boundary on \( \alpha \) and the other on \( F \), and get a contradiction. Finally, if \( P \cap A = \emptyset \) then \( P \) extends to an annulus with one boundary on \( \alpha \) and the other on \( F \), contradicting the assumption that \( \alpha \) is not parallel into \( F \). This completes the proof of Claim 3.

We now continue with the proof of Theorem 9. We have shown that \( F_0 \) is incompressible in \( M_K \). If there is no essential annulus in \( M_K \) with one boundary component on each of \( F_0 \) and \( T \), then by Lemma 7 we know that \( F_0 \) is incompressible in \( M_K(r) \) for all but at most three slopes \( r \) with mutual intersection number 1. In particular, it is incompressible in \( M_K(-1/n) \) for all but at most two consecutive \( n \)'s, so the theorem follows. Now suppose there is an essential annulus \( P \) in \( M_K \) with one boundary component on \( F_0 \) and the other on \( T \). Since \( F_0 \) is not a closed surface, it is not a torus. Hence by Lemma 8, \( F_0 \) remains incompressible in \( M_K(-1/n) \) unless \( \Delta(-1/n, r_0) \leq 1 \). By Claim 3, \( r_0 \) is not the longitude slope \( 0/1 \), therefore, \( \Delta(-1/n, r_0) \leq 1 \) holds for at most three consecutive integers \( n \). This completes the proof of Theorem 9. \( \square \)

**Proof of Theorem 1.** By Theorem 4, Lemmas 5 and 6, and Theorem 9, we can now assume that \( F' \) is incompressible and is parallel into \( F \). We want to show that either \( F \) is arc-extendible in \( F' \), or \( M \) is a product \( F \times I \). As in the proof of Theorem 4, we may assume without loss of generality that \( M \) is a compression body, so all closed incompressible surfaces of \( M \) are boundary parallel. Let \( \alpha_1, \ldots, \alpha_k \) be the boundary curves of \( F' \). Let \( F' \times I \) be a product in \( M \) such that \( F' = F' \times 0 \) and \( F' \times 1 \subset F \). Write \( \alpha_i^1 = \alpha_i \times 1 \), which is a curve on \( F \) isotopic to \( \alpha_i \) in \( M \).

We have assumed above that \( F' \) is incompressible in \( M \), so \( \text{Int} F \cup \text{Int} F' \) is incompressible in \( M \). Write \( \widehat{F}_i = \text{Int} F \cup \text{Int} F' \cup \alpha_i \). If \( \widehat{F}_i \) is incompressible for some \( i \), then \( F \cup N(\gamma) \) is incompressible for any essential arc \( \gamma \) in \( F' \) with endpoints on \( \alpha_i \), and we are done. (Such an arc exists because \( F' \) is not an annulus or disk.) So assume that \( \widehat{F}_i \) is compressible for all \( i \). By the Handle Addition Lemma (Lemma 2), after adding a 2-handle to \( M \) along \( \alpha_i \), the surface \( \widehat{F}_i[\alpha_i] \) is incompressible, and \( M[\alpha_i] \) is irreducible. Notice that since \( F' \) is incompressible, the curve \( \alpha_i^1 = \alpha_i \times 1 \) in \( F \) is essential on \( F \). But after adding the 2-handle, \( \alpha_i^1 \) bounds a disk in \( M[\alpha_i] \), so it must also bound a
disk on \( \hat{F}_i[\alpha_i] \) because \( \hat{F}_i[\alpha_i] \) is incompressible. By definition \( \hat{F}_i[\alpha_i] \) is obtained from 
\((\text{Int}F \cup \text{Int}F') - \text{Int}N(\alpha_i)\) by capping off the two copies of \( \alpha_i \) with disks, hence \( \alpha_i^1 \cup \alpha_i \) 
bounds an annulus \( A_i \) on \( F_i \). Denote by \( A_i' \) the annulus \( \alpha_i \times I \subset F' \times I \subset M \). Then 
\( T_i = A_i \cup A_i' \) is a torus in \( M \). Since we have assumed above that \( M \) is a compression body, either \( T_i \) bounds a solid torus \( V_i \), or it is parallel to some torus component of \( \partial M \). However, since \( M[\alpha_i] \) is irreducible, one can show as in the proof of Theorem 4 that \( V_i \) is a solid torus, and \( \alpha_i \) is a longitude of \( V_i \). This is true for all \( i \). It is now easy to see that \( M \) is a product \( F \times I \). □

The following theorem supplements Theorem 1. It says that in most case there are 
extension arcs with endpoints on any prescribed boundary components of \( F' \).

**Theorem 10.** Let \( F, F', M \) be as in Theorem 1. Suppose \( M \) is not a product \( F \times I \), 
and suppose \( F' \) is not parallel into \( F \) and is not a thrice punctured sphere. Then it 
contains an extension arc \( \gamma \) of \( F \) with endpoints on any prescribed components of \( \partial F' \).

**Proof.** If \( F' \) is nonplanar, then by the proof of Lemmas 5 and 6, there is a \( \partial \)-
nonseparating circle \( \alpha \) (denoted by \( C' \) there) on \( F' \) which is not parallel into \( F \), 
and is actually nonseparating on \( F' \). Hence given any boundary components \( \partial_1, \partial_2 \) of \( F' \), (possibly \( \partial_1 = \partial_2 \)), there is an arc \( \gamma \) with endpoints on \( \partial_1 \) and \( \partial_2 \), intersecting \( \alpha \) in one point. By Theorem 9, for all but at most three integers \( n \), the arc \( \gamma_n = \tau_n^\alpha \gamma \) is an 
extension arc of \( F \).

Now suppose \( F' \) is planar with \( |\partial F'| \geq 4 \). First assume that \( \partial_1, \partial_2 \) are distinct 
boundary components of \( F' \). By the proof of Lemmas 5 and 6, the curve \( \alpha \) is a band 
sum of two boundary components of \( F' \). From the proofs one can see that we can 
always choose \( \alpha \) to be the band sum of \( \partial_1 \) and \( \partial_3 \), with \( \partial_3 \neq \partial_1, \partial_2 \). Hence there is an 
arc \( \gamma \) from \( \partial_1 \) to \( \partial_2 \) intersecting \( \alpha \) in one point. We can then apply Theorem 9 to get 
an extension arc \( \gamma_n \) with one endpoint on each of \( \partial_1 \) and \( \partial_2 \).

We now proceed to find an extension arc in \( F' \) with boundary on the same 
component \( \partial_1 \) of \( \partial F' \). By the proof of Lemmas 5 and 6, we can choose the curve \( \alpha \) above 
to be the band sum of of \( \partial_2 \) and \( \partial_3 \), with \( \partial_1 \neq \partial_2, \partial_3 \). Recall that \( \alpha \) is not parallel 
into \( F \). Choose an arc \( \gamma \) as follows. Let \( \partial_2' \) be a curve on \( F' \) parallel to \( \partial_2 \), let \( \gamma' \) be 
an arc connecting \( \partial_2' \) to \( \partial_1 \) intersecting \( \alpha \) in one point, and let \( Q \) be the sub-surface 
\( N(\gamma' \cup \partial_2') \) of \( F' \). Then \( \gamma \) is the closure of the arc component of \( \partial Q \cap \text{Int}F' \), that is, 
\( \gamma \) is the arc component of the frontier of \( Q \) in \( F' \), see Figure 2 below. Consider the 
surface \( F_0 = F \cup N(\gamma) \), and observe that \( F_0 \) is isotopic to the surface \( F \cup Q \). After 
Dehn twist along \( \alpha \), it is isotopic to the surface \( F \cup N(\tau_n^\alpha \gamma) \); hence to show that all 
but at most three \( \tau_n^\alpha \gamma \) are extension arcs of \( F \) in \( F' \), we need only show that \( F \cup Q \) is 
incompressible after all but at most three Dehn twist along \( \alpha \). Since \( F \cup Q \) intersects \( \alpha \)
in a single arc, the argument in the proof of Theorem 9 is still valid, with the following easy modifications. We use the notations in that proof.

The proof of Claim 2 needs the following modifications. (i) The arc $e_2$ on the boundary of the outermost disk $\Delta$ may be on $Q$. In this case, notice that the other arc $e_1$ on $\partial D$ is isotopic to an arc $\alpha_1$ on $\alpha$, and $e_2 \cup \alpha_1$ is isotopic in $F'$ to the curve $\partial_3$, so the fact that $e_1 \cup e_2$ bounds a disk $\Delta$ would imply that $\partial_3$ bounds a disk. Since $\partial_3$ is also on $\partial F$, this contradicts the fact that $F$ is incompressible and diskless. (ii) The compressing disk $D$ of $F \cup Q$ could be disjoint from the annulus $A$. But since $F$ is incompressible, this would imply that $\partial D$ lies on $Q$, hence is isotopic to $\partial_2$, which would imply that $\partial_2$ bounds a disk, again contradicting the assumption that $F$ is incompressible and diskless.

The proof of Claim 3 applies to show that the annulus $P$ there can be modified to be disjoint from the annulus $A$. Then notice that the component of $\partial P$ on $F \cup Q$ is either in $F$, or in $Q$ and hence parallel to $\partial_2$. Since $\partial_2 \subset F$, in either case $P$ can be extended to an annulus with one boundary component on $\alpha$ and the other on $F$, which contradicts the assumption that $\alpha$ is not parallel into $F$.

The rest part of the proof of Theorem 9 follows verbatim to show that $F \cup Q$ is incompressible after all but at most three Dehn twist along $\alpha$. □

Remark. Theorem 10 is not true if $F'$ is a thrice punctured sphere.

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