BERNOULLI NUMBERS AND SUM OF POWERS OF INTEGERS OF HIGHER ORDER

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ABSTRACT. We give an expression of polynomials for higher sums of powers of integers via the higher order Bernoulli numbers.

1. INTRODUCTION

As is known the sum of powers of integers [4], [9]

\[ S_m(n) := \sum_{q=1}^{n} q^m \]

can be computed with the help of some appropriate polynomial \( \hat{S}_m(n) \) for any \( m \geq 0 \). Exponential generating function for the sums \( S_m(n) \) is given by

\[ S(n, t) = \sum_{n=1}^{\infty} e^{nt} = \frac{e^{(n+1)t} - e^t}{e^t - 1}. \] (1.1)

Expanding in series (1.1) yields an infinite set of polynomials \( \{ \hat{S}_m(n) : m \geq 0 \} \), that is,

\[ S(n, t) = \sum_{q \geq 0} \hat{S}_q(n) \frac{t^q}{q!}. \]

It is a classical result that these polynomials can be expressed as [7]

\[ \hat{S}_m(n) = \frac{1}{m+1} \sum_{q=0}^{m} (-1)^q \binom{m+1}{q} B_q n^{m+1-q}, \] (1.2)

where \( B_q \) are the Bernoulli numbers that can be derived from the exponential generating function

\[ \frac{t}{e^t - 1} = \sum_{q \geq 0} B_q \frac{t^q}{q!}. \] (1.3)

It follows from (1.3) that the Bernoulli numbers satisfy the recurrence relation

\[ \sum_{q=0}^{m} \binom{m+1}{q} B_q = \delta_{0,m}. \] (1.4)

This relation is in fact the simplest one of many known recurrence relations involving the Bernoulli numbers (see, for example, [2] and references therein). One can derive, for example, an infinite number of recurrence relations of the form

\[ \sum_{q=0}^{m} \binom{m+k}{q} S(m+k-q, k) B_q = \frac{m+k}{k} S(m+k-1, k-1), \forall k \geq 1. \] (1.5)
In this paper we investigate a class of sums that correspond to a $k$-th power of generating function (1.1) for $k \geq 1$. Our main result is a formula for polynomials allowing to calculate these sums. It turns out that these polynomials are expressed via the higher Bernoulli numbers.

2. The power sums of higher order

Let us now consider a power of the generating function (1.1):

$$(S(n, t))^k := \sum_{q \geq 0} S_q^{(k)}(n) t^q q!.$$  

We have

$$(S(n, t))^k = \left( \sum_{q=1}^n e^{qt} \right)^k = \sum_{q=k}^{kn} \binom{k}{q} n^q e^q.$$  

(2.1)

The coefficients $\binom{k}{q}$, obviously generalizing the binomial coefficients originated from Abraham De Moivre and Leonard Euler works [10], [5] and extensively studied in the literature due to their applicability. From (2.1), we see that it is a generating function for the sums of the form

$$S^{(k)}_m(n) := \sum_{q=0}^{k(n-1)} \binom{k}{q} n^{k+q}.$$  

(2.2)

It is natural to call (2.2) the sums of powers of integers of higher order. Expanding

$$\left( \frac{e^{(n+1)t} - e^t}{e^t - 1} \right)^k = \sum_{q=0}^{k(n-1)} S_q^{(k)}(n) t^q q!,$$

we get an infinite number of polynomials $\hat{S}_m^{(k)}(n)$.

Our goal in the paper is to prove that

$$\hat{S}_m^{(k)}(n) = \frac{1}{(m+k)} \sum_{q=0}^{m} (-1)^q \binom{m+k}{q} B_q^{(k)} S(m+k-q, k)n^{m+k-q},$$  

(2.3)

where $B_q^{(k)}$ are the higher order Bernoulli numbers defined as

$$\frac{t^k}{(e^t - 1)^k} = \sum_{q \geq 0} B_q^{(k)} \frac{t^q}{q!}.$$  

(2.4)

The Bernoulli numbers of higher order appeared in [11] in connection with a theory of finite differences and then was investigated by many authors from different points of view (see, for example, [3]). These numbers are known to satisfy [11]

$$B_n^{(k+1)} = \frac{k-n}{n} B_n^{(k)} - nB_n^{(k)}.$$  

The number $B_n^{(k)}$ with fixed $n \geq 0$ turns out to be some polynomial in $k$. These kind of polynomials are known as Nörlund polynomials. One can find a number of these polynomials in [11]. For convenience, we have written out several Nörlund polynomials in the
Appendix. The numbers $S(n, k)$ in (2.3) are the Stirling numbers of the second kind that satisfy recurrence relation
\[ S(n, k) = S(n-1, k-1) + kS(n-1, k) \] (2.5)
with appropriate boundary conditions [15], [4].

It is easy to prove that the higher order Bernoulli numbers satisfy the recurrence relation
\[ \sum_{q=0}^{m} \binom{m+k}{q} S(m+k-q, k) B^{(q)}_q = \delta_{0,m}. \] (2.6)

The most general relation involving (1.4), (1.5) and (2.6) as particular cases is
\[ \sum_{q=0}^{m} \binom{m+k}{q} S(m+k-q, k-r) B^{(q)}_r = \delta_{0,m+k} \] (2.7)

As is known $S(m+k, k)$, for any fixed $m \geq 0$, is expressed as a polynomial $f_m(k)$ of degree $2m$, which satisfy the identity
\[ f_m(k) - f_m(k-1) = kf_m(k-1) \]
following from the identity (2.5). Therefore we can replace $S(m+k-q, k)$ by $f_{m-q}(k)$ in (2.3). In the literature the polynomials $f_m(k)$ are known as the Stirling polynomials [6], [8]. These are known to be expressed via the Nörlund polynomials as (see, for example, [1])
\[ f_m(k) = \binom{m+k}{m} B^{(-k)}_m. \]

The following proposition also gives the relationship of the higher Bernoulli numbers with the Stirling numbers.

**Proposition 2.1.** One has
\[ B^{(k)}_m = \sum_{q=1}^{m} \frac{s(q+k, k)}{(q+k)} S(m, q). \] (2.8)

In (2.8), $s(n, k)$ stands for the Stirling numbers of the first kind [14].

It is evident that in the case $k = 1$, (2.8) becomes
\[ B_m = \sum_{q=1}^{m} (-1)^q \frac{q!}{q+1} S(m, q), \]
while in the case $k = 2$, it takes the following form:
\[ B^{(2)}_m = 2 \sum_{q=1}^{m} (-1)^q \frac{(q+1)!H_{q+1}}{(q+1)(q+2)} S(m, q), \]
where $H_m$ are harmonic number defined by $H_m := \sum_{q=1}^{m} 1/q$.

To prove (2.3), we need the following lemma:
Lemma 2.2. By virtue of (2.7) we have
\begin{equation}
R^{(k,r)}_m(n) := \sum_{q=0}^{m}(1)q \binom{m+k}{q} S(m+k-q,k)\tilde{S}^{(r)}_q(n) \tag{2.9}
\end{equation}

\begin{equation}
= \frac{1}{\binom{k}{r}} \sum_{j=0}^{m}(1)^j \binom{m+k}{m+k-r-j} S(m+k-r-j,k-r) \times S(r+j,r)n^{r+j}, \quad \forall m \geq 0, \quad k \geq r. \tag{2.10}
\end{equation}

It should be remarked that in the case \( k = r \), (2.10) becomes
\begin{equation}
R^{(k,k)}_m(n) = (1)m S(m+k,k)n^{m+k}. \tag{2.11}
\end{equation}

Proof of lemma 2.2. We can rewrite (2.9) as
\begin{equation}
R^{(k,r)}_m(n) = \sum_{0 \leq j \leq q \leq m} a_q b_{q,j} n^{r+q-j},
\end{equation}

where
\begin{equation}
a_q := \binom{m+k}{q} r^{r+q} S(m+k-q,k)
\end{equation}

and
\begin{equation}
b_{q,j} := (1)^{q-j} \binom{r+q}{r} B^{(r)}_{q-j} S(r+q-j,r).
\end{equation}

Let \( \tilde{j} = q - j \) and
\begin{equation}
b_{q,j} = (1)^{q-j} \binom{r+q}{r-j} B^{(r)}_{q-j} S(r+j, r).
\end{equation}

In what follows, for simplicity, let us write \( \tilde{j} \) without the tilde. Making use the identity
\begin{equation}
\binom{r+q}{r-j} = \binom{r+q}{r} \frac{q}{r-j} \binom{r+q}{r},
\end{equation}

we get
\begin{equation}
a_q b_{q,j} = (1)^{q-j} \frac{S(r+j, r)}{r+q} \binom{m+k}{q} \binom{q}{j} S(m+k-q,k)B^{(r)}_{q-j}
\end{equation}

and therefore
\begin{equation}
R^{(k,r)}_m(n) = \sum_{0 \leq j \leq q \leq m} a_q b_{q,j} n^{r+q-j}
\end{equation}

\begin{equation}
= \sum_{0 \leq j \leq m} (-1)^j \frac{S(r+j, r)}{(r+j)} n^{r+j} \sum_{j \leq q \leq m} \binom{m+k}{q} \binom{q}{j} S(m+k-q,k)B^{(r)}_{q-j}.
\end{equation}

In turn, making use the identity
\begin{equation}
\binom{m+k}{q} \binom{q}{j} = \binom{m+k}{j} \binom{m+k-j}{q-j},
\end{equation}

we get
\begin{equation}
R^{(k,r)}_m(n) = \sum_{0 \leq j \leq m} (-1)^j \frac{S(r+j, r)}{(r+j)} \binom{m+k}{j} n^{r+j}
\end{equation}

\begin{equation}
\times \sum_{j \leq q \leq m} \binom{m+k-j}{q-j} S(m+k-q,k)B^{(r)}_{q-j}.
\end{equation}
Finally, by virtue of (2.7), we get
\[
\sum_{j \leq q \leq m} \binom{m+k-j}{q-j} S(m+k-q,k) B_{q-j}^{(r)} = \sum_{0 \leq q \leq m-j} \binom{m+k-j}{q-j} S(m+k-j-q,k) B_q^{(r)} = \frac{(m+k-j)}{(m+k-j-r)} S(m+k-j-r,k-r)
\]
and hence
\[
R_{m}^{(k,r)}(n) = \sum_{0 \leq j \leq m} (-1)^j \binom{m+k-j}{j} \binom{m+k-j-r}{r} S(m+k-j-r,k-r) S(r+j,r)n^{r+j}.
\]

Therefore the lemma is proved. □

The recurrence relation, for example (2.11), uniquely determines an infinite set of polynomials \( \{ \hat{S}_m^{(k)}(n) : m \geq 0 \} \). We have written out some of them in the Appendix. For example, \( \hat{S}_0^{(k)}(n) = n^k \). On the other hand
\[
S_0^{(k)}(n) := \sum_{q=0}^{k(n-1)} \left( \begin{array}{c} k \\ q \end{array} \right) n^q = \left( \sum_{q=0}^{n-1} t^q \right)^k |_{t=1} = n^k.
\]

Lemma 2.3. The higher sums \( S_m^{(k)}(n) \) satisfy the same recurrence relations as in lemma 2.2, that is,
\[
\sum_{q=0}^{m} (-1)^q \binom{m+k}{q} S(m+k-q,k) S_q^{(r)}(n) = \frac{1}{(k)} \sum_{j=0}^{m-1} (-1)^j \binom{m+k}{m+k-j-r} S(m+k-j-r,k-r) S(r+j,r)n^{r+j}, \tag{2.12}
\]

In the case \( k = r = 1 \), (2.12) becomes the well-known identity for the sums of powers \[12].

Proof of lemma 2.3. This lemma is proved by using standard arguments. Let us replace an argument of generating function \( t \to -t \) to get
\[
\sum_{q=0}^{m} (-1)^q S_q^{(r)}(n) \frac{t^q}{q!} = (-1)^k \left( \frac{e^{-nt} - 1}{e^t - 1} \right)^r. \tag{2.13}
\]

Multiplying both sides of (2.13) by \( (e^t - 1)^k \) and taking into account that
\[
(e^t - 1)^k = k! \left( \sum_{q=0}^{\infty} S(q,k) \frac{t^q}{q!} \right),
\]
we get (2.12). □

Now, we are in a position to prove our theorem.

**Theorem 2.4.** One has

\[ S_m^{(k)}(n) = \hat{S}_m^{(k)}(n), \]

where \( \hat{S}_m^{(k)}(n) \) be the polynomials (2.2).

**Proof.** This theorem is a simple consequence of lemma 2.2 and lemma 2.3 since the sums \( S_m^{(k)}(n) \) satisfy the same recurrence relations as the polynomials \( \hat{S}_m^{(k)}(n) \). □

3. The Relationship of the Sums \( S_m^{(k)}(n) \) to Other Sums

In [13] we considered sums of the form

\[ S_m^{(k)}(n) := \sum_{\{\lambda\} \in B_{j,k,n}} (\lambda_1^m + (\lambda_2 - n)^m + \cdots + (\lambda_k - kn + n)^m), \quad (3.1) \]

where it is supposed that \( m \) is odd. Here \( B_{k,k,n} := \{\lambda_q : 1 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq kn\} \). Let us remark that there are some terms of the form \( r^m \) with negative \( r \) in (3.1). It is evident that in this case \( r^m = -|r|^m \). By this rule, the sum (3.1) can be rewritten as

\[ S_m^{(k)}(n) = \sum_{q=0}^{kn} c_q(k,n)q^m \quad (3.2) \]

with some integer coefficients \( c_r(k,n) \).

It was conjectured in [13] that in the case of odd \( m \) the sums \( S_m^{(k)}(n) \) and \( \hat{S}_m^{(k)}(n) \) are related with each other by

\[ S_m^{(k)}(n) = \sum_{q=0}^{k-1} \binom{k(n+1)}{q} S_m^{(k-q)}(n). \quad (3.3) \]

Let us define the sums \( S_m^{(k)}(n) \) with even \( m \) by (3.2). It is evident that conjectural relation (3.3) is valid for both odd and even \( m \). More exactly, actual calculations show that

\[ S_m^{(k)}(n) - \sum_{q=0}^{k-1} \binom{k(n+1)}{q} S_m^{(k-q)}(n) = c_0(k,n)\delta_{m,0}. \]

**Appendix**

**Nörlund polynomials.** The first six of the Nörlund polynomials are given by

\[ B_0^{(k)} = 1, \quad B_1^{(k)} = -\frac{1}{2}k, \quad B_2^{(k)} = \frac{1}{12}k(3k - 1), \quad B_3^{(k)} = -\frac{1}{8}k^2(k - 1), \]

\[ B_4^{(k)} = \frac{1}{240}(15k^3 - 30k^2 + 5k + 2), \quad B_5^{(k)} = -\frac{1}{96}k^2(k - 1)(3k^2 - 7k - 2). \]
The polynomials $\hat{S}_m^{(k)}(n)$. The first six of these polynomials are given by

\begin{align*}
\hat{S}_0^{(k)}(n) &= n^k, \quad \hat{S}_1^{(k)}(n) = \frac{k}{2} n^k (n + 1), \\
\hat{S}_2^{(k)}(n) &= \frac{k}{12} n^k (n + 1) ((3k + 1)n + 3k - 1), \\
\hat{S}_3^{(k)}(n) &= \frac{k^2}{8} n^k (n + 1)^2 ((k + 1)n + k - 1), \\
\hat{S}_4^{(k)}(n) &= \frac{k}{240} n^k (n + 1) \left( (15k^3 + 30k^2 + 5k - 2)n^3 + (45k^3 + 30k^2 - 5k + 2)n^2 \\
&\quad + (45k^3 - 30k^2 - 5k - 2)n + 15k^3 - 30k^2 + 5k + 2 \right), \\
\hat{S}_5^{(k)}(n) &= \frac{k^2}{96} n^k (n + 1)^2 \left( (3k^3 + 10k^2 + 5k - 2)n^3 + (9k^3 + 10k^2 - 5k + 2)n^2 \\
&\quad + (9k^3 - 10k^2 - 5k - 2)n + 3k^3 - 10k^2 + 5k + 2 \right).
\end{align*}

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