Rational quantum integrable systems of $D_N$ type with polarized spin reversal operators

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Abstract

We study the spin Calogero model of $D_N$ type with polarized spin reversal operators, as well as its associated spin chain of Haldane–Shastry type, both in the antiferromagnetic and ferromagnetic cases. We compute the spectrum and the partition function of the former model in closed form, from which we derive an exact formula for the chain’s partition function in terms of products of partition functions of Polychronakos–Frahm spin chains of type $A$. Using a recursion relation for the latter partition functions that we derive in the paper, we are able to numerically evaluate the partition function, and thus the spectrum, of the $D_N$-type spin chain for relatively high values of the number of spins $N$. We analyze several global properties of the chain’s spectrum, such as the asymptotic level density, the distribution of consecutive spacings of the unfolded spectrum, and the average degeneracy. In particular, our results suggest that this chain is invariant under a suitable Yangian group, and that its spectrum coincides with that of a Yangian-invariant vertex model with linear energy function and dispersion relation.

Keywords: Exactly solvable quantum models, Spin chains with long-range interactions, Polarized spin reversal operator

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1. Introduction

In a recent paper, a novel type of spin Calogero models and their associated spin chains of Haldane–Shastry type was introduced [1]. The distinguishing feature of these models is that they are constructed using a new representation of the Weyl group of the $BC_N$ root system, obtained by replacing the standard spin reversal operators by an arbitrarily polarized version thereof. As shown in the latter reference, these models are exactly solvable for all such representations and, in particular, the partition function of the spin chains can be exactly computed using Polychronakos’s freezing trick [2,3]. In this

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paper, we shall extend the above results to spin Calogero models and their corresponding spin chains of Haldane–Shastry type based on the $D_N$ root system.

In order to present our work in an appropriate context, let us briefly recall the origin and significance of the latter models. The Haldane–Shastry (HS) spin chain, introduced independently by these authors in the late eighties \[4, 5\], is perhaps the best known example of an exactly solvable one-dimensional lattice model with long-range interactions. More precisely, this model describes a circular array of equispaced spins with two-body interactions inversely proportional to the square of the (chord) distance between the spins. The motivation for introducing this chain was the construction of a simple model with an exact ground state given by the $U \to \infty$ limit of Gutzwiller’s variational wavefunction for the ground state of the one-dimensional Hubbard model \[6–8\]. Over the years, the HS chain has appeared in many areas of interest both in Physics and Mathematics, such as fractional statistics and one-dimensional anyons \[9–12\], quantum entanglement \[13\], characterization of integrability vs. quantum chaos \[14–17\], quantum integrability via the asymptotic Bethe ansatz \[18–20\], Yangian quantum groups \[12, 21–23\], and conformal field theory \[18, 24–26\].

One of the key properties of the HS chain —already noted by Haldane and Shastry in their original papers— is its intimate connection with the scalar (trigonometric) Sutherland model \[27, 28\]. This connection was subsequently elucidated by Polychronakos in Ref. \[2\], who showed how to derive the HS chain from the spin Sutherland model \[29–31\] by a technique that he called the “freezing trick”. The main idea behind this technique is to note that when the coupling constant in the spin Sutherland model goes to infinity the particles tend to concentrate on the coordinates of the (essentially unique) minimum of the scalar part of the interaction potential, which are precisely the sites of the HS chain. Thus, in this limit the dynamical and the spin degrees of freedom decouple, and the latter are governed by the chain’s Hamiltonian. Using this idea it is straightforward, for instance, to obtain the first integrals of the HS chain from their well-known counterparts for the spin Sutherland model. In fact, Polychronakos showed that applying the same procedure to the (rational) Calogero model \[32\] and its spin version \[31\] one obtains an integrable chain with non-equispaced sites and long-range interactions inversely proportional to the distance between the spins \[2\]. The spectrum of this chain —known in the literature as the Polychronakos–Frahm (PF) chain— was first studied numerically by Frahm \[33\] and then exactly computed by Polychronakos \[3\], who derived an exact formula for the partition function by means of the freezing trick. On the other hand, the partition function of the HS chain was only evaluated more than a decade later by some of the authors \[14\].

Both the Sutherland and Calogero models (and their corresponding HS and PF spin chains) are associated with the $A_{N-1}$ root system, where $N$ is the number of particles. Indeed, in these models the interactions only depend on the difference between the coordinates, and the spin operators appearing in the Hamiltonian are permutation operators, and thus generate a realization of the Weyl group of $A_{N-1}$ type. In fact, there are versions of the Sutherland and Calogero models associated to any (extended) classical root system \[34\]. Among these systems, those of $BC_N$, $B_N$, $C_N$ and $D_N$ type are by far the most studied in the literature, since they make it possible to construct integrable models with an arbitrary number of particles. By applying the freezing trick to the spin version of these models one obtains the corresponding generalizations of the HS and PF chains, that we shall collectively refer to as spin chains of HS type \[35–41\]. One of the funda-
mental features of the $BC_N$ root system and its $B_N$ and $C_N$ degenerations is the fact that its Weyl algebra contains a family of reflection operators $S_i$ ($i = 1, \ldots, N$) satisfying $S_i^2 = 1$. (In the case of the $D_N$ root system, the Weyl group only contains products $S_i S_j$ with $i \neq j$.) In the spin chains studied in Refs. [35–41], the operators $S_i$ are represented by spin reversal operators $P_i$ (acting on the Hilbert space of the $i$-th particle), but this is by no means the only possible choice. As a matter of fact, in the novel version of the spin Calogero model of $BC_N$ type and its corresponding (PF) chain introduced in Ref. [1], the operators $S_i$ are represented instead by arbitrarily polarized spin reversal operators (PSRO) $P_i^{(m_1,m_2)}$, which act as the identity on the first $m_1$ elements of the spin basis and as minus the identity on the rest. These operators are equivalent under a similarity transformation to the usual spin reversal operators $P_i$ only when $m_1 = m_2$ or $m_1 = m_2 \pm 1$, i.e., when there is minimal polarization. For the remaining values of the discrete parameters $m_1$ and $m_2$, the systems constructed in the latter reference differ from the standard Calogero and PF models of $BC_N$-type. In particular, when $m_1$ or $m_2$ are zero, the corresponding spin dynamical model reduces to the $su(\infty)$-invariant extension of the Calogero model studied by Simons and Altshuler [42]; see also [43].

In this paper we introduce the spin Calogero model of $D_N$-type with PSRO and its corresponding spin chain of HS type, i.e., the PF chain of $D_N$ type with PSRO. As explained in Ref. [39], these models are singular limits of their corresponding $BC_N$ counterparts, so that their spectrum cannot be obtained by setting to zero the parameter $\beta$ in the latter models (cf. Eqs. (2.5) and (2.14)). This is also apparent at the level of the Hilbert space, which is the direct sum of the Hilbert spaces of two $BC_N$ models with opposite chiralities. Thus, the models studied in this paper are not limiting cases of their $BC_N$ versions in Ref. [1].

Our main result is the derivation of a closed-form expression for the partition function of the PF chain of $D_N$ type with PSRO in terms of products of partition functions of type-$A$ PF chains. Our approach is based on the computation of the spectrum and partition function of the corresponding spin Calogero model, from which the chain’s partition function follows by a standard freezing trick argument. The structure of this partition function turns out to be more involved than that of its $BC_N$ counterpart. In particular, it is not manifest that it is a polynomial in $q \equiv e^{-1/(k_B T)}$, as follows from the freezing trick. Using the explicit expression for the partition function, we shall study several global properties of the chain’s spectrum, such as the behavior of the level density and the average degeneracy when the number of spins tends to infinity. In particular, the fact that the number of distinct levels grows polynomially with the number of spins suggests that this model is isospectral to a Yangian-invariant vertex model of the kind studied in Ref. [44].

The paper is organized as follows. In Section 2 we recall the definition and main properties of the polarized spin reversal operators $P_i^{(m_1,m_2)}$, and construct the Hamiltonians of the $D_N$-type spin Calogero model with PSRO and its associated spin chain. Section 3 is devoted to the derivation of the closed-form expression of the chain’s partition function, as explained in the previous paragraph. Using this expression, in Section 4 we analyze several global properties of the spectrum, providing strong numerical evidence of the Gaussian character of the level density when the number of spins is large enough. In Section 5 we extend the above results to the ferromagnetic version of the models under consideration. The paper ends with a brief section summarizing our main
results and presenting our conclusions, and a short technical Appendix establishing a useful recursion relation for the partition function of the PF chain of type $A_{N-1}$.

2. Construction of the models

For the purpose of describing the $D_N$-type Calogero model with polarized spin reversal operators, it is convenient to briefly summarize the construction of its $BC_N$ counterpart \[1\]. To this end, let denote the internal spin space for $[1]$. To this end, let

$$ S = \langle |s_1, \ldots, s_N \rangle \mid s_i \in \{1, 2, \ldots, m\} \rangle. \tag{2.1} $$

denote the internal spin space for $N$ particles. As usual, the action of the spin exchange operator $P_{ij}$ on $S$ is defined as

$$ P_{ij}|s_1, \ldots, s_i, \ldots, s_j, \ldots, s_N \rangle = |s_1, \ldots, s_j, \ldots, s_i, \ldots, s_N \rangle. \tag{2.2} $$

Let us denote the PSRO associated with the $i$-th particle as $P_i^{(m_1, m_2)}$, where $m_1$ and $m_2$ are two nonnegative integers satisfying the relation $m_1 + m_2 = m$. The action of $P_i^{(m_1, m_2)}$ on $S$ is given by \[1\]

$$ P_i^{(m_1, m_2)}|s_1, \ldots, s_i, \ldots, s_N \rangle = (-1)^{f(s_i)}|s_1, \ldots, s_i, \ldots, s_N \rangle, \tag{2.3} $$

where

$$ f(s_i) = \begin{cases} 0, & 1 \leq s_i \leq m_1 \\ 1, & m_1 + 1 \leq s_i \leq m_1 + m_2. \end{cases} \tag{2.4} $$

In terms of these operators, the Hamiltonian of the $BC_N$-type Calogero model with PSRO is defined as \[1\]

$$ H_{B, \epsilon}^{(m_1, m_2)} = -\sum_i \frac{\partial^2}{\partial x_i^2} + a \sum_{i \neq j} \left( \frac{a + P_{ij}}{(x_{ij})^2} + \frac{a + \tilde{P}_{ij}^{(m_1, m_2)}}{(x_{ij})^2} \right) + \beta a \sum_i \frac{\beta a - \epsilon P_i^{(m_1, m_2)} x_i^2}{4} + \frac{a^2}{4}, \tag{2.5} $$

where the sums run from 1 to $N$, $a > \frac{1}{\beta}$, $\beta > 0$, $\epsilon = \pm 1$, $x_{ij}^\pm = x_i \pm x_j$, $r^2 = \sum_i x_i^2$, and

$$ \tilde{P}_{ij}^{(m_1, m_2)} = P_i^{(m_1, m_2)} P_j^{(m_1, m_2)} P_{ij}. $$

It can be shown that when $m$ is even (resp. odd) and $m_1 = m_2$ (resp. $m_1 = m_2 \pm 1$), the PSRO in \[2.3\] is equivalent via a similarity transformation to the usual spin reversal operator $P$, which changes $s_i$ into $m - s_i + 1$. As a result, for these special choices of $m_1$ and $m_2$, the Hamiltonian \[2.5\] reduces to that of the standard $su(m)$ spin Calogero model of $BC_N$ type studied in Ref. \[35\]. As mentioned in the Introduction, another interesting special case is $m_1 = m$, $m_2 = 0$, for which the Hamiltonian \[2.5\] reduces to the Simons–Altshuler extension of the Calogero model.

Since $H_{B, \epsilon}^{(m_1, m_2)}$ contains the discrete parameters $m_1$, $m_2$ and $\epsilon$, it is natural to inquire whether there exists any relation between models \[2.5\] with different sets of parameters. In fact, we shall now show that $H_{B, \epsilon}^{(m_1, m_2)}$ is equivalent to $H_{B, \epsilon}^{(m_2, m_1)}$ through a unitary transformation. To this end, consider the unitary operator $T$ whose action on the spin space $S$ is given by

$$ T|s_1, \ldots, s_i, \ldots, s_N \rangle = |s'_1, \ldots, s'_i, \ldots, s'_N \rangle, \tag{2.6} $$

$$ 4 $$
where
\[ s_i' = \begin{cases} s_i + m_1, & 1 \leq s_i \leq m_2 \\ s_i - m_2, & m_2 + 1 \leq s_i \leq m_1 + m_2. \end{cases} \tag{2.7} \]

Using Eqs. (2.3) and (2.6) we easily obtain
\[ T^\dagger P_i^{(m_1, m_2)} T |s_1, \ldots, s_i, \ldots, s_N\rangle = (-1)^{f(s_i')} |s_1, \ldots, s_i, \ldots, s_N\rangle. \tag{2.8} \]

From Eq. (2.7) it follows that
\[ s_i' \in \{ m_1 + 1, \ldots, m_1 + m_2 \} \quad \text{for} \quad 1 \leq s_i \leq m_2 \quad \text{and} \quad s_i' \in \{ 1, \ldots, m_1 \} \quad \text{for} \quad m_2 + 1 \leq s_i \leq m_2 + m_1, \]
so that
\[ f(s_i') = \begin{cases} 1, & 1 \leq s_i \leq m_2 \\ 0, & m_2 + 1 \leq s_i \leq m_2 + m_1. \end{cases} \tag{2.9} \]

Equations (2.8) and (2.9) clearly imply that
\[ T^\dagger P_i^{(m_1, m_2)} T = -P_i^{(m_2, m_1)}. \tag{2.10} \]

It is also obvious from Eqs. (2.2) and (2.6) that
\[ T^\dagger P_{ij} T = P_{ij}. \tag{2.11} \]

From Eqs. (2.5), (2.10) and (2.11) we readily obtain
\[ T^\dagger H_{B,\epsilon}^{(m_1, m_2)} T = H_{B,\epsilon}^{(m_2, m_1)}, \tag{2.12} \]
as claimed. In view of the above relation, it suffices to study the Hamiltonian (2.5) in the case \( \epsilon = 1 \). However, in the paper we shall intentionally keep the parameter \( \epsilon \) in \( H_{B,\epsilon}^{(m_1, m_2)} \) in order to facilitate the comparison with its \( D_N \) counterpart that we shall introduce below.

Due to the nature of the singularities of the Hamiltonian \( H_{B,\epsilon}^{(m_1, m_2)} \), its configuration space can be taken as one of the Weyl chambers of the \( BC_N \) root system, i.e., one of the maximal open subsets of \( \mathbb{R}^N \) on which the functions \( x_i \pm x_j \) and \( x_i \) have constants signs.

We shall choose this configuration space as the principal Weyl chamber
\[ C^{(B)} = \{ x \in \mathbb{R}^N : 0 < x_1 < x_2 < \cdots < x_N \} \tag{2.13} \]
where \( x \equiv (x_1, \ldots, x_N) \). The Hamiltonian \( H_{B,\epsilon}^{(m_1, m_2)} \) is thus defined on an appropriate dense subset of the Hilbert space \( L^2(C^{(B)}) \otimes S \). When \( \epsilon = 1 \), the spectrum of \( H_{B,\epsilon}^{(m_1, m_2)} \) was computed in Ref. [2] by constructing a suitable (non-orthogonal) basis of this Hilbert space in which this Hamiltonian acts triangularly.

As explained in the latter reference, from the spin dynamical model (2.5) one can construct a PF chain of \( BC_N \) type with PSRO by applying the freezing trick. The Hamiltonian of this chain is given by
\[ H_{B,\epsilon}^{(m_1, m_2)} = \sum_i \left[ \frac{1 + P_{ij}}{(\zeta_i - \zeta_j)^2} + \frac{1 + \tilde{P}_{ij}^{(m_1, m_2)}}{(\zeta_i + \zeta_j)^2} \right] + \beta \sum_i \frac{1 - \epsilon P_{ij}^{(m_1, m_2)}}{\zeta_i^2}, \tag{2.14} \]
where the lattice sites \( \zeta_i \) are related to the zeros \( y_i \) of the Laguerre polynomial \( L_N^{\beta-1} \) by \( y_i = \zeta_i^2 / 2 \). The exact partition function of the chain (2.14) has also been computed in Ref. [1] by exploiting its connection with the spin dynamical model (2.5).

The Hamiltonian \( H^{(m_1, m_2)} \) of the \( D_N \)-type spin Calogero model with PSRO is naturally defined by dropping the term related to the roots \( x_i \) in \( H_{\beta, \epsilon}^{(m_1, m_2)} \), i.e., by setting \( \beta = 0 \) in Eq. (2.5). We thus obtain

\[
H^{(m_1, m_2)} = - \sum_i \frac{\partial^2}{\partial x_i^2} + a \sum_{i \neq j} \left[ \frac{a + P_{ij}}{(x_{ij})^2} + \frac{a + \hat{P}_{ij}^{(m_1, m_2)}}{(x_{ij}^+)^2} \right] + \frac{a^2}{4} r^2. \tag{2.15}
\]

It should be noted that, unlike its \( BC_N \) counterpart, the latter Hamiltonian does not depend on \( \epsilon \). Just as before, from Eqs. (2.10) and (2.11) it follows that \( H^{(m_2, m_1)} \) is unitarily equivalent to \( H^{(m_1, m_2)} \) under \( T \):

\[
T^\dagger H^{(m_1, m_2)} T = H^{(m_2, m_1)}. \tag{2.16}
\]

Thus we can impose without loss of generality the restriction \( m_1 \geq m_2 \). Consequently, for any given \( m \) one can construct \([m/2 + 1]\) inequivalent spin Calogero models of \( D_N \) type with PSRO, where \( [\cdot] \) denotes the integer part. Among these models, only those with \( m_1 = m_2 \) (for even \( m \)) or \( m_1 = m_2 + 1 \) (for odd \( m \)) are unitarily equivalent to the \( su(m) \) spin Calogero model of \( D_N \) type with standard spin reversal operators introduced in Ref. [39].

As is the case with the latter model, the configuration space \( C \) of the Hamiltonian (2.15) can be taken as one of the maximal open subsets of \( \mathbb{R}^N \) on which the linear functionals \( x_i \pm x_j \) have constant signs. We shall again take \( C \) as the principal Weyl chamber of the \( D_N \) root system, namely

\[
C = \{ x \in \mathbb{R}^N : |x_1| < x_2 < \cdots < x_N \}. \tag{2.17}
\]

Note that this configuration space contains its \( BC_N \) counterpart (2.13) as a subset. As before, the Hamiltonian (2.15) is defined on a suitable dense subspace of the Hilbert space \( L^2(C) \otimes S \).

We shall next explain in detail how to construct the \( D_N \)-type PF chain with PSRO associated to the spin dynamical model (2.15) by means of Polychronakos’s freezing trick. To begin with, note that the Hamiltonian \( H^{(m_1, m_2)} \) can be decomposed as

\[
H^{(m_1, m_2)} = H^{\text{sc}} + a \hat{H}^{(m_1, m_2)}(x), \tag{2.18}
\]

where

\[
H^{\text{sc}} = - \sum_i \frac{\partial^2}{\partial x_i^2} + a(a - 1) \sum_{i \neq j} \left[ \frac{1}{(x_{ij})^2} + \frac{1}{(x_{ij}^+)^2} \right] + \frac{a^2}{4} r^2 \tag{2.19}
\]

is the Hamiltonian of the scalar \( D_N \) Calogero model and

\[
\hat{H}^{(m_1, m_2)}(x) = \sum_{i \neq j} \left[ \frac{1 + P_{ij}}{(x_{ij}^-)^2} + \frac{1 + \hat{P}_{ij}^{(m_1, m_2)}}{(x_{ij}^+)^2} \right]. \tag{2.20}
\]
is a spin-dependent multiplication operator. On the other hand, in the strong coupling limit \( a \to \infty \) the coefficient of the dominant term (of order \( a^2 \)) in the Hamiltonian (2.15) is given by

\[
U(x) = \sum_{i \neq j} \left[ \frac{1}{(x_{ij})^2} + \frac{1}{(x_{ij})^2} + \frac{r^2}{4} \right]. \tag{2.21}
\]

Hence as \( a \to \infty \) the particles concentrate at the coordinates \( \xi_i \) of the unique minimum \( \xi \) of the potential \( U(x) \) in the configuration space \( C \), and the coordinate degrees of freedom of \( H^{(m_1, m_2)} \) decouple from the internal ones. By Eq. (2.18), in this limit the eigenvalues of \( H^{(m_1, m_2)} \) are approximately given by

\[
E_{ij} \simeq E_{sc}^i + aE_j, \tag{2.22}
\]

where \( E_{sc}^i \) and \( E_j \) are two arbitrary eigenvalues of \( H^{sc} \) and

\[
\hat{H}^{(m_1, m_2)}(\xi) = \sum_{i \neq j} \left[ \frac{1}{(\xi_i - \xi_j)^2} + \frac{1}{(\xi_i + \xi_j)^2} \right]. \tag{2.23}
\]

We shall take Eq. (2.23) as the precise definition of the Hamiltonian of the \( D_N \) type PF chain with PSRO. In fact, using Eqs. (2.10) and (2.11), it is easy to show that the Hamiltonians \( \hat{H}^{(m_1, m_2)}_\beta \) and \( \hat{H}^{(m_2, m_1)}_\beta \) are related by

\[
T^\dagger \hat{H}^{(m_1, m_2)}_\beta T = \hat{H}^{(m_2, m_1)}_\beta. \tag{2.24}
\]

Thus, we may assume without loss of generality that \( m_1 \geq m_2 \), so that there are again \( [m/2 + 1] \) inequivalent PF chains of \( D_N \) type with PSRO. Since the sites of these chains depend only on the scalar potential (2.21), the above models reduce to the \( su(m) \) PF chain of \( D_N \) type with standard spin reversal operators [39] when \( m_1 = m_2 \) (for even \( m \)) or \( m_1 = m_2 + 1 \) (for odd \( m \)). See, e.g., Fig. 1 for a comparison of the spectra of the \( D_N \) chain with PSRO (2.23) with \( m_1 = 3, m_2 = 1 \) and the \( su(4) \) \( D_N \) type PF chain with standard time-reversal operators (corresponding to \( m_1 = m_2 = 2 \)) for \( N = 10 \) spins.

A brief remark on the relation between the \( D_N \) and \( BC_N \) spin chains with PSRO in Eqs. (2.23) and (2.14) is now in order. As shown in [39], the lattice sites of the former chain are given by \( \xi_1 = 0 \) and \( \xi_i = \sqrt{2y_{i-1}} \) \( (2 \leq i \leq N) \), where \( y_k > 0 \) denotes the \( k \)-th root of the generalized Laguerre polynomial \( L_N^1 \). From the well-known identity \( N L_N^1(y) = -y L_{N-1}^1(y) \) and the previous characterization of the sites \( \zeta_i \) of the \( BC_N \) chain (2.14), it immediately follows that \( \xi_i = \lim_{\beta \to 0} \zeta_i. \) Although one may naively think that the Hamiltonian \( \hat{H}^{(m_1, m_2)}_\beta \) is simply the \( \beta \to 0 \) limit of its \( BC_N \) counterpart \( \hat{H}^{(m_1, m_2)}_\beta \), this is certainly not the case. The point is that, although the roots \( \zeta_i \) with \( 2 \leq i \leq N \) tend to finite nonzero limits when \( \beta \to 0 \), the first root \( \zeta_1 \) tends to 0 in this limit. As a consequence, the \( i = 1 \) term of the last sum in Eq. (2.14) need not vanish as \( \beta \to 0 \), and in fact it can be shown [39] that

\[
\lim_{\beta \to 0} \frac{\beta}{\zeta_1} = \frac{N}{2}. \tag{2.25}
\]

Letting \( \beta \to 0 \) in Eq. (2.14) and using the latter identity we immediately obtain

\[
\lim_{\beta \to 0} \hat{H}^{(m_1, m_2)}_{\beta, \epsilon} = \hat{H}^{(m_1, m_2)} + \frac{N}{2}(1 - \epsilon P^{(m_1, m_2)}). \tag{2.26}
\]
Thus, the $\beta \to 0$ limit of the Hamiltonian $H^{(m_1,m_2)}_{DN}$ differs from its $D_N$ counterpart $H^{(m_1,m_2)}_{DN}$ by the surface term or impurity interaction $N(1 - \epsilon P_{1}^{(m_1,m_2)})/2$. It is easy to see that this term vanishes only for $\epsilon = 1$, $m_1 = m$, $m_2 = 0$ (or, equivalently, $\epsilon = -1$, $m_1 = 0$, $m_2 = m$). For any other choice of $m_1$ and $m_2$, this surface term is nonzero and does not commute with the Hamiltonian $H^{(m_1,m_2)}_{DN}$. Thus, except in the previously noted special cases, the spectrum of $H^{(m_1,m_2)}_{DN}$ cannot be obtained from that of $H^{(m_1,m_2)}_{BC_N}$ by taking the $\beta \to 0$ limit. This fact is illustrated in Fig. 2, which shows that the spectra of these chains with $m_1 = 3$, $m_2 = 1$ and $N = 10$ spins are clearly different.
3. Spectrum and partition function

In this section, we shall compute in closed form the spectrum and partition function of the spin Calogero model of $D_N$ type with PSRO in Eq. (2.15). This will enable us to compute the partition function $Z^{(m_1,m_2)}_{(m_1,m_2)}$ of the $D_N$-type PF chain with PSRO (2.23) by a standard freezing trick argument. Indeed, from Eq. (2.22) it is straightforward to derive the following exact formula for $Z^{(m_1,m_2)}_{(m_1,m_2)}$ in terms of the partition functions $Z^{(m_1,m_2)}_{(m_1,m_2)}$ and $Z$ of the spin dynamical model (2.15) and of its scalar counterpart (2.19):

$$Z^{(m_1,m_2)}_{(m_1,m_2)}(T) = \lim_{a \to \infty} \frac{Z^{(m_1,m_2)}_{(m_1,m_2)}(aT)}{Z(aT)}.$$

Since $Z$ has already been computed in Ref. [39], Eq. (3.1) provides an effective way of evaluating $Z^{(m_1,m_2)}_{(m_1,m_2)}$ once $Z^{(m_1,m_2)}_{(m_1,m_2)}$ is known.

The key idea for deriving the spectrum of the spin Hamiltonian (2.15) is to observe that it can be obtained by applying a suitable projection to a simpler differential-difference operator $H'$ acting on scalar functions. The spectrum of $H'$ can be readily computed by constructing a (non-orthogonal) basis of its Hilbert space on which this operator acts triangularly. The spectrum of $H^{(m_1,m_2)}_{(m_1,m_2)}$ is then easily determined by projecting onto the Hilbert space of the latter operator.

More precisely, the auxiliary operator $H'$ is given by [39]

$$H' = -\sum_i \frac{\partial^2}{\partial x_i^2} + a \sum_{i \neq j} \left[ \frac{a}{(x_i - x_j)^2} (a - K_{ij}) + \frac{a}{(x_i + x_j)^2} (a - \tilde{K}_{ij}) \right] + \frac{a^2}{4} r^2,$$

where $K_{ij}$ and $K_i$ are coordinate exchange and sign reversing operators, defined by

$$K_{ij} f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N),$$
$$K_i f(x_1, \ldots, x_i, \ldots, x_N) = f(x_1, \ldots, -x_i, \ldots, x_N),$$

and $\tilde{K}_{ij} = K_i K_j K_{ij}$. The domain of the operator $H'$ is of course a suitable dense subset of the Hilbert space $L^2(\mathbb{R}^N)$. The operator $H'$ can be expressed in terms of the $D_N$-type rational Dunkl operators [46]

$$J_i^- = \frac{\partial}{\partial x_i} + a \sum_{j \neq i} \left[ \frac{1}{x_i - x_j} (1 - K_{ij}) + \frac{1}{x_i + x_j} (1 - \tilde{K}_{ij}) \right]$$

as [47]

$$H' = \rho(x) \left[ -\sum_i (J_i^-)^2 + a \sum_i x_i \frac{\partial}{\partial x_i} + E_0 \right] \rho(x)^{-1},$$

where

$$\rho(x) = e^{-\frac{2}{\beta} r^2} \prod_{i<j} |x_i^2 - x_j^2|^a$$

is the ground state of the scalar Calogero model of $D_N$-type and

$$E_0 = Na \left( a(N-1) + \frac{1}{2} \right)$$

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is its ground-state energy. A basis of this Hilbert space on which \( H' \) acts triangularly is provided by the functions

\[
\phi_n(x) = \rho(x) \prod_i x_i^{n_i}, \quad n \equiv (n_1, \ldots, n_N),
\]

where the \( n_i \)'s are arbitrary non-negative integers. Indeed, since \( J^- \) lowers the degree of the monomial \( \prod_i x_i^{n_i} \), from Eqs. \( (3.3) \) and \( (3.4) \) it immediately follows that

\[
H' \phi_n(x) = E'_n \phi_n(x) + \sum_{|m|<|n|} c_{mn} \phi_m(x),
\]

where the coefficients \( c_{mn} \) are real constants and

\[
E'_n = a|n| + E_0.
\]

As the diagonal elements of any upper triangular operator coincide with its eigenvalues, the spectrum of \( H' \) is given by Eq. \( (3.8) \).

The spectrum of the spin Hamiltonian \( H^{(m_1, m_2)} \) can be derived from that of \( H' \) by noting that these Hamiltonians are formally related by

\[
H^{(m_1, m_2)} = H'|_{K_{ij} \to -P_{ij}, K_i K_j \to P_i^{(m_1, m_2)} P_j^{(m_1, m_2)}}.
\]

In order to take advantage of this observation, we introduce the operator \( \Lambda^{(m_1, m_2)} \) projecting the Hilbert space \( L^2(\mathbb{R}^N) \otimes S \) onto states that are antisymmetric under particle permutations and symmetric under the action of \( K_i K_j P_i^{(m_1, m_2)} P_j^{(m_1, m_2)} \) for any \( i \neq j \). In other words, the projector \( \Lambda^{(m_1, m_2)} \) is determined by

\[
\pi_{ij} \Lambda^{(m_1, m_2)} = -\Lambda^{(m_1, m_2)}, \quad \pi_{i}^{(m_1, m_2)} \pi_{j}^{(m_1, m_2)} \Lambda^{(m_1, m_2)} = \Lambda^{(m_1, m_2)},
\]

where

\[
\pi_{ij} \equiv K_{ij} P_{ij}, \quad \pi_{i}^{(m_1, m_2)} \equiv K_i P_i^{(m_1, m_2)},
\]

so that

\[
K_{ij} \Lambda^{(m_1, m_2)} = -P_{ij} \Lambda^{(m_1, m_2)}, \quad K_i K_j \Lambda^{(m_1, m_2)} = P_i^{(m_1, m_2)} P_j^{(m_1, m_2)} \Lambda^{(m_1, m_2)}.
\]

We shall now outline the construction of the projector \( \Lambda^{(m_1, m_2)} \) in terms of the analogous projectors \( \Lambda_{B,\pm}^{(m_1, m_2)} \) for the \( BC_N \)-type spin Calogero model with PSRO \( (2.11) \) with chirality \( \varepsilon = \pm 1 \) (cf. \[2\]). To this end, recall that \( \Lambda_{B,\pm}^{(m_1, m_2)} \) projects from the Hilbert space \( L^2(\mathbb{R}^N) \otimes S \) onto spin wavefunctions antisymmetric under particle permutations and with parity \( \pm 1 \) under \( \pi_{i}^{(m_1, m_2)} \), i.e.,

\[
\pi_{ij} \Lambda_{B,\pm}^{(m_1, m_2)} = -\Lambda_{B,\pm}^{(m_1, m_2)}, \quad \pi_{i}^{(m_1, m_2)} \Lambda_{B,\pm}^{(m_1, m_2)} = \pm \Lambda_{B,\pm}^{(m_1, m_2)}.
\]

The projector \( \Lambda_{B,\pm}^{(m_1, m_2)} \) can then be expressed as

\[
\Lambda_{B,\pm}^{(m_1, m_2)} = \frac{1}{2^N N!} \prod_{j=1}^{N} \left( 1 \pm \pi_{j}^{(m_1, m_2)} \right) \cdot \sum_{l=1}^{N!} \varepsilon_l P_l,
\]

where
where $\mathcal{P}_l$ denotes an element of the realization of the permutation group generated by the operators $\pi_{ij}$ and $\varepsilon_l$ is the signature of $\mathcal{P}_l$. From Eqs. (3.10) and (3.13) we conclude that
\begin{equation}
\Lambda^{(m_1,m_2)} = \Lambda^{(m_1,m_2)}_{B,+} + \Lambda^{(m_1,m_2)}_{B,-}. \tag{3.15}
\end{equation}
Indeed, the right-hand side of the latter equation is clearly a projector, since
\begin{equation}
\Lambda^{(m_1,m_2)}_{B,+} \Lambda^{(m_1,m_2)}_{B,-} = \Lambda^{(m_1,m_2)}_{B,-} \Lambda^{(m_1,m_2)}_{B,+} = 0,
\end{equation}
and it satisfies (3.10) on account of (3.13). Thus the space
\[ V \equiv \Lambda^{(m_1,m_2)}(L^2(\mathbb{R}^N) \otimes \mathcal{S}) \]
decomposes as the direct sum
\begin{equation}
V = V_{B,+} \oplus V_{B,-}, \quad V_{B,\pm} \equiv \Lambda^{(m_1,m_2)}_{B,\pm}(L^2(\mathbb{R}^N) \otimes \mathcal{S}). \tag{3.16}
\end{equation}
We have already mentioned that, due to the impenetrable nature of the singularities of the Hamiltonian $H^{(m_1,m_2)}$, its Hilbert space can be taken as the space $L^2(C) \otimes \mathcal{S}$ of spin wavefunctions square integrable on the open set $C$ in Eq. (2.17). On the other hand, any point in $\mathbb{R}^N$ not lying on the singular subset $x_i \pm x_j = 0, 1 \leq i < j \leq N$, can be mapped in a unique way to a point in $C$ by a suitable element of the $D_N$ Weyl group, which is generated by coordinate permutations and sign reversals of an even number of coordinates $[48]$. Using this fact, it can be shown that $L^2(C) \otimes \mathcal{S}$ is actually isomorphic to the space $V$, and $H^{(m_1,m_2)}$ is equivalent to its natural extension to the latter space which (with a slight abuse of notation) we shall also denote by $H^{(m_1,m_2)}$. With this identification, in view of Eq. (3.12) we can write
\begin{equation}
H^{(m_1,m_2)} = H^{(m_1,m_2)} \Lambda^{(m_1,m_2)} = H' \Lambda^{(m_1,m_2)}, \tag{3.17}
\end{equation}
where $H'$ acts trivially (as the identity) on $\mathcal{S}$.

We shall now explain how the spectrum of $H^{(m_1,m_2)}$ can be derived from that of $H'$ using the previous equation. To this end, note that by Eq. (3.16) the Hilbert space $V$ is the closure of the linear subspace spanned by the spin wavefunctions
\begin{equation}
\psi^{\epsilon}_{n,s}(x) = \Lambda^{(m_1,m_2)}_{B,\epsilon}(\phi_n(x)|s), \quad \epsilon = \pm, \tag{3.18}
\end{equation}
where $|s \rangle \equiv |s_1, \ldots, s_N \rangle$ is an arbitrary element of the canonical spin basis. In fact, the wavefunctions (3.18) with fixed $\epsilon$ span a subspace whose closure is the Hilbert space $V_{B,\epsilon}$. Clearly, the functions (3.18) are not linearly independent. Indeed, using Eq. (3.13) it is easy to show that these functions satisfy the relations
\begin{equation}
\psi^{\epsilon'}_{n,s'}(x) = -\psi^{\epsilon'}_{n',s'}(x), \quad \psi^{\epsilon}_{n,s}(x) = \epsilon(-1)^{m_1+f(s)} \psi^{\epsilon}_{n,s}(x), \tag{3.19}
\end{equation}
where $n'$ and $s'$ are respectively obtained from $n$ and $s$ by permuting any two of their components (the same for both). Due to these identities, the sets $\{\psi^{\epsilon}_{n,s}(x)\}$ and $\{\psi^{\epsilon}_{n,s}(x)\}$ are both linearly independent provided that the following three conditions are imposed on the quantum numbers $n$ and $s$: 

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i) To avoid overcounting, and for later convenience, we shall order the components of \( \mathbf{n} \) as follows:

\[
\mathbf{n} = (n_c, n_s) = \left( \frac{k_2}{p_1}, \ldots, p_1, \ldots, 2p_s, \ldots, 2p_s, \frac{k_2}{p_1}, \ldots, p_1, \ldots, 2p_s, \ldots, 2p_s \right),
\]

where \( 0 \leq s, t \leq N, p_1 > p_2 > \cdots > p_s > 0 \) and \( q_1 > q_2 > \cdots > q_t > 0 \).

ii) By the second equation in (3.19), the allowed values of \( s_i \) corresponding to each \( n_i \) are given by

\[
s_i \in \begin{cases} 
\{1, 2, \ldots, m_1\}, & \text{for even } n_i, \\
\{m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\}, & \text{for odd } n_i,
\end{cases}
\]

for the set \( \{\psi^+_{n, s}(x)\} \), and by

\[
s_i \in \begin{cases} 
\{1, 2, \ldots, m_1\}, & \text{for odd } n_i, \\
\{m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\}, & \text{for even } n_i.
\end{cases}
\]

for the set \( \{\psi^-_{n, s}(x)\} \).

iii) If \( n_i = n_j \) and \( i < j \) we shall take \( s_i > s_j \), again to avoid overcounting.

If the above conditions are satisfied, each of the sets \( \{\psi^\epsilon_{n, s}(x)\} \) (\( \epsilon = \pm \)) is a non-orthogonal basis of the corresponding subspace \( V_{B, \epsilon} \), and the union of these sets provides a non-orthogonal basis of the whole Hilbert space \( V \). We shall next show that \( H^{(m_1, m_2)} \) leaves invariant each of the subspaces \( V_{B, \epsilon} \), and that it acts triangularly on the corresponding basis \( \{\psi^\epsilon_{n, s}(x)\} \) provided that we (partially) order it by the total degree \( |\mathbf{n}| \).

Indeed, using Eqs. (3.15) and (3.17), and taking into account that \( [H', \Lambda^{(m_1, m_2)}_{B, \epsilon}] = 0 \) we obtain

\[
H^{(m_1, m_2)} \psi^\epsilon_{n, s}(x) = \Lambda_{B, \epsilon}^{(m_1, m_2)} (H' \phi_{n}(x)) |s\rangle. \tag{3.20}
\]

From this equation and Eqs. (3.7) and (3.18) it readily follows that

\[
H^{(m_1, m_2)} \psi^\epsilon_{n, s}(x) = E^\epsilon_{n, s} \psi^\epsilon_{n, s}(x) + \sum_{|m| < |n|} C^{\epsilon}_{mn} \psi^\epsilon_{m, s}(x), \tag{3.21}
\]

where the \( C^{\epsilon}_{mn} \)'s are real constants and \( s' \) is a permutation of \( s \) such that \( (m, s') \) satisfies conditions i)--iii) above; see [31] for more details. By Eq. (3.21), the action of \( H^{(m_1, m_2)} \) on the whole Hilbert space \( V = V_{B, +} \oplus V_{B, -} \) is the direct sum of two upper triangular actions on each of the subspaces \( V_{B, \pm} \). Consequently, the eigenvalues of this operator are given by

\[
E^\epsilon_{n, s} = E^\epsilon_{n} = a|\mathbf{n}| + E_0, \tag{3.22}
\]

where \( \epsilon = \pm \) and \( (n, s) \) satisfies conditions i)--iii) above. Since the RHS of Eq. (3.22) does not depend on \( \epsilon \) and \( s \), the eigenvalue associated with the quantum number \( \mathbf{n} \) has
an intrinsic degeneracy $d_n^{(m_1, m_2)}$ coming from the two possible chiralities and the spin degrees of freedom. This intrinsic degeneracy is in fact the sum
\[
d_n^{(m_1, m_2)} = d_{n,+}^{(m_1, m_2)} + d_{n,-}^{(m_1, m_2)},
\]
where $d_n^{(m_1, m_2)}$ is the number of spin states satisfying conditions i)–iii) for the given $n$ and $\epsilon$. Using these conditions we readily obtain \[1\]
\[
d_{n,+}^{(m_1, m_2)} = \prod_{i=1}^{s} \left( \frac{m_1}{k_i} \right) \prod_{j=1}^{t} \left( \frac{m_2}{l_j} \right), \quad d_{n,-}^{(m_1, m_2)} = d_{n,+}^{(m_2, m_1)},
\]
and therefore
\[
d_n^{(m_1, m_2)} = \prod_{i=1}^{s} \left( \frac{m_1}{k_i} \right) \prod_{j=1}^{t} \left( \frac{m_2}{l_j} \right) + \prod_{i=1}^{s} \left( \frac{m_2}{k_i} \right) \prod_{j=1}^{t} \left( \frac{m_1}{l_j} \right).
\]
Thus the spectrum of the $D_N$-type spin Calogero model with PSRO is given by the RHS of Eq. \[2.15\], where each level possesses an intrinsic degeneracy given by Eq. \[3.25\]. Of course, the actual degeneracy of an energy $a|n| + E_0$ is the sum
\[
\sum_{|n'|=|n|} d_{n'}^{(m_1, m_2)},
\]
where the sum is over all multindices $n'$ satisfying condition i) above.

It is worth mentioning at this point that the spectrum of the $BC_N$-type spin Calogero model with PSRO and chirality $\epsilon$ in Eq. \[2.5\] is also given by the RHS of Eq. \[3.22\], with $E_0$ replaced by \[1\]
\[
E_{0,B} = E_0 + N\beta a^2.
\]
Moreover, the intrinsic degeneracy of the energy $a|n| + E_{0,B}$ is given by $d_n^{(m_1, m_2)}$. It follows from Eq. \[3.22\] that the $D_N$ spin Hamiltonian $H^{(m_1, m_2)}$ is (up to a constant) the direct sum of two $BC_N$-type spin Calogero models of opposite chiralities with PSRO. Using Eqs. \[3.22\] and \[3.25\], the canonical partition function of the $D_N$-type spin Calogero model with PSRO can be written as
\[
Z^{(m_1, m_2)}(aT) = q^{E_{0}/a} \sum_{n} d_n^{(m_1, m_2)} q^{|n|}, \quad q \equiv e^{-1/(k_B T)},
\]
where the sum ranges over all multindices $n$ satisfying condition i) above. Similarly, the partition functions of the corresponding $BC_N$-type models \[2.5\] are given by
\[
Z_{B,\pm}^{(m_1, m_2)}(aT) = q^{(E_{0,n})/a} \sum_{n} d_{n,\pm}^{(m_1, m_2)} q^{|n|} \equiv Z_{B,\pm}^{(m_2, m_1)}.
\]
From Eq. \[3.23\] it then follows that
\[
q^{-E_{0}/a} Z_n^{(m_1, m_2)}(aT) = q^{-(E_{0,n})/a} \left[ Z_{B,+}^{(m_1, m_2)}(aT) + Z_{B,-}^{(m_1, m_2)}(aT) \right] = q^{-(E_{0,n})/a} \left[ Z_{B,+}^{(m_1, m_2)}(aT) + Z_{B,+}^{(m_2, m_1)}(aT) \right].
\]
In order to apply the freezing trick formula (3.1), we need only recall the expression for the partition function \( Z \) of the scalar Calogero model of \( D_N \)-type derived in Ref. [39], namely

\[
q^{-E_0/a} Z(aT) = (1 + q^N) \prod_i (1 - q^{2i})^{-1} = q^{-(E_0/a)q} (1 + q^N) Z_B(aT), \tag{3.29}
\]

where \( Z_B \) denotes the partition of the scalar Calogero model of \( BC_N \) type. Dividing Eq. (3.28) by Eq. (3.29) and applying the analog of the freezing trick formula (3.1) for the partition function \( Z^{(m_1, m_2)} \) of the PF spin chain of \( BC_N \) type (2.14) we finally obtain

\[
Z^{(m_1, m_2)}(q) = \left(1 + q^N\right)^{-1} \left[ Z^{(m_1, m_2)}_B(q) + Z^{(m_2, m_1)}_B(q) \right], \tag{3.30}
\]

where from now on we shall use the variable \( q = e^{-1/(k_B T)} \) in place of \( T \). The partition function \( Z^{(m_1, m_2)}_B \) can in turn be expressed in terms of the partition function \( Z^{(m)}_A(q) \) of the \( su(m) \) PF chain of type \( A \) with \( k \) spins with Hamiltonian

\[
\mathcal{H}^{(m)}_A = \sum_{1 \leq i < j \leq k} \frac{1}{(\rho_i - \rho_j)^2}, \tag{3.31}
\]

where \( \rho_i \) is the \( i \)-th zero of the Hermite polynomial of degree \( k \). Indeed, it is shown in Ref. [1] that for \( m_2 > 0 \) we have

\[
Z^{(m_1, m_2)}_B(q) = \sum_{k=0}^{N} q^{N-k} \left[ \begin{array}{c} N \\ k \end{array} \right]_{q^2} Z^{(m_1)}_A(q^2) Z^{(m_2)}_A(q^2) - \frac{N}{k_B T} (m_2 > 0), \tag{3.32}
\]

where the \( q \)-binomial coefficient \( \left[ \begin{array}{c} N \\ k \end{array} \right]_{q^2} \) is defined as

\[
\left[ \begin{array}{c} N \\ k \end{array} \right]_{q^2} = \frac{(q^2)^N}{(q^2)_k(q^2)_{N-k}}, \quad (q^2)_j \equiv \prod_{i=1}^{j}(1 - q^{2i}). \tag{3.33}
\]

Combining Eqs. (3.30) and (3.32) we finally arrive at the following expression for the partition function of the \( D_N \)-type PF chain with PSRO (2.23) in terms of its type \( A \) counterpart as

\[
Z^{(m_1, m_2)}(q) = \sum_{k=0}^{N} f_{N,k}(q) Z^{(m_1)}_A(q^2) Z^{(m_2)}_A(q^2) - \frac{N}{k_B T} (m_2 > 0), \tag{3.34}
\]

where \( f_{N,k}(q) \) is given by

\[
f_{N,k}(q) = \frac{q^{N-k} + q^k}{1 + q^N} \left[ \begin{array}{c} N \\ k \end{array} \right]_{q^2}. \tag{3.35}
\]

The case \( m_2 = 0 \), for which \( P_i^{(m_1, 0)} = 1 \) and the Hamiltonian (2.23) reduces to the rational version of the (trigonometric) Simons–Altshuler chain [42], deserves special attention. Indeed, in this case by Eq. (3.19) the components of the multiindex \( n \) are all
even (resp. odd) for the eigenfunctions $\psi^{+}_{n,s}$ (resp. $\psi^{-}_{n,s}$). As shown in Ref. [1], this entails that for $m_2 = 0$ Eq. (3.32) should be replaced by

$$Z^{(m_1,0)}_{B,+}(q) = Z^{(m_1)}_{A,N}(q^2).$$  \hspace{1cm} (3.36)

On the other hand, since $P^{(0,m_2)}_i = -1$ we have

$$\mathcal{H}^{(0,m_2)}_{B,+} = \mathcal{H}^{(m_1,0)}_{B,+} + \sum_i \frac{2\beta}{\xi_i^2} = \mathcal{H}^{(m_1,0)}_{B,+} + N$$

by Eqs. (A2)-(A5) of Ref. [38]. From (3.36) it then follows that

$$Z^{(0,m_2)}_{B,+}(q) = q^N Z^{(m_2)}_{A,N}(q^2),$$  \hspace{1cm} (3.37)

and substituting into Eq. (3.30) we finally obtain

$$Z^{(m_1,0)}(q) = Z^{(m_1)}_{A,N}(q^2).$$  \hspace{1cm} (3.38)

Note that, as shown in Ref. [1], the RHS of the latter equation also coincides with the partition function of the $BC_N$-type chain (2.14) with $\epsilon = 1$ and $m_2 = 0$. This was to be expected, as the latter model reduces to its $DN$ counterpart (2.23) when $m_2 = \beta = 0$ and its spectrum does not depend on $\beta$.

As is well known, several equivalent closed-form expressions for the partition function of the $A_{k-1}$-type PF chain (3.31) exist in the literature [3, 16, 44, 49]. For instance, Polychronakos [3] showed that this function is given by

$$Z^{(m)}_{A,k}(q) = \sum_{k_1 + \cdots + k_m = k} q^{\frac{1}{2} \sum_{i=1}^{m} k_i(k_i - 1)} [k_1, \ldots, k_m]_q,$$  \hspace{1cm} (3.39)

where the $q$-multinomial coefficient $[k_1, \ldots, k_m]_q$ is defined by

$$[k_1, \ldots, k_m]_q = \frac{(q)_m}{\prod_{i=1}^{m} (q)_{k_i}}.$$

Another well-known expression for the partition function $Z^{(m)}_{A,k}$ was derived in Ref. [16], namely

$$Z^{(m)}_{A,k}(q) = \sum_{\mathbf{f} \in \mathcal{P}_k} d_m(\mathbf{f}) q^{\sum_{j=1}^{r-1} \mathcal{F}_j - \mathcal{F}_r} \prod_{j=1}^{r-1} (1 - q^{\mathcal{F}_j}).$$  \hspace{1cm} (3.40)

Here $\mathcal{P}_k$ represents the set of all ordered partitions $\mathbf{f} \equiv \{f_1, f_2, \ldots, f_r\}$ of the integer $k$, $d_m(\mathbf{f}) = \prod_{i=1}^{r} \binom{f_i}{i}$, $\mathcal{F}_j = \sum_{i=1}^{j} f_i$ are the partial sums of $\mathbf{f}$, and the complementary partial sums are defined as $\{\mathcal{F}_1', \mathcal{F}_2', \ldots, \mathcal{F}_{r-1}'\} \equiv \{1, 2, \ldots, k\} \setminus \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r\}$. A related expression for the partition function of the chain (3.31) can be obtained by exploiting its connection with a one-dimensional classical vertex model consisting of $k + 1$ vertices connected by $k$ intermediate bonds [44]. Any possible state for this vertex model can be represented by a path configuration given by

$$\mathbf{s} = \{s_1, s_2, \ldots, s_k\},$$  \hspace{1cm} (3.41)
where \( s_i \in \{1, 2, \cdots, m\} \) denotes the spin state of the \( i \)-th bond. The energy function associated with this spin path configuration \( \vec{s} \) is defined as
\[
E^{(m)}(\vec{s}) = \sum_{j=1}^{k-1} j \theta(s_j - s_{j+1}),
\]
(3.42)
where \( \theta \) is Heaviside’s step function, defined as
\[
\theta(x) = \begin{cases} 
0, & \text{if } x < 0, \\
1, & \text{if } x \geq 0.
\end{cases}
\]
(3.43)

Using the Yangian quantum group symmetry of the model (3.31), it can be shown that its partition function coincides with that of the one-dimensional vertex model with energy function (3.42) (cf. [44]). Thus \( Z^{(m)}_{A,k}(q) \) can be expressed as
\[
Z^{(m)}_{A,k}(q) = \sum_{\vec{s}} q^{E^{(m)}(\vec{s})},
\]
(3.44)
where the sum has runs over all possible \( m^k \) spin path configurations. In particular, from Eq. (3.44) it follows that
\[
Z^{(1)}_{A,k}(q) = q^{\frac{1}{2} k(k-1)}.
\]
Thus the partition function (3.34) with \( m_2 = 1 \) reduces to
\[
Z^{(m_1,1)}(q) = \sum_{k=0}^{N} q^{(N-k)(N-k-1)} f_{N,k}(q) Z^{(m_1)}_{A,k}(q^2).
\]
(3.45)

It is obvious from any of the expressions (3.39), (3.40) or (3.44) that the partition function \( Z^{(m)}_{A,k}(q) \) is a polynomial in \( q \). In particular, from Eq. (3.38) it follows that \( Z^{(m_1,0)} \) is an even polynomial in \( q \), and its energies are therefore even nonnegative integers. By Eq. (3.44), to show that the partition function of the \( D_N \)-type PF chain with PSRO is a polynomial in \( q \) when \( m_2 > 0 \) it suffices to prove that the coefficients \( f_{N,k}(q) \) in Eq. (3.35) depend polynomially on \( q \). Although it is well known that the \( q \)-binomial coefficient \( \left[ \begin{array}{c} N \\ k \end{array} \right]_{q^2} \) in (3.33) is indeed an even polynomial in \( q \) of degree \( 2k(N - k) \), it is not clear whether \( f_{N,k}(q) \) is also a polynomial. In fact, we have verified that this is the case for a wide range of values of \( N \) and all \( k \leq N \). We conjecture that this is true in general, so that when \( m_2 > 0 \) the energies of the spin chain (2.23) are also nonnegative integers. Note that the latter fact also follows from the freezing trick formula (2.22), Eq. (3.22) for the spectrum of the spin dynamical model (2.15) and the analogous formula for the scalar \( D_N \)-type Calogero model in Ref. [39].

4. Statistical properties of the spectrum

A characteristic property of all spin chains of Haldane–Shastry type is the fact that their level density approaches a Gaussian distribution as the number of spins tends to infinity. This property has been rigorously proved for the chains of \( A_{N-1} \) type and their
related one-dimensional vertex models \[51, 52\], and has been numerically checked for
the \( B_N \), \( BC_N \) and \( D_N \) type chains with standard spin reversal operators \[37–39, 41\].
More recently, it has been established that the level density of the \( BC_N \)-type PF chain
with PSRO shows a similar behavior \[1\]. It is therefore of interest to ascertain whether
the level density of the \( D_N \)-type spin chain with PSRO in Eq. (2.23) becomes normally
distributed as the number of spins tends to infinity. In fact, Figs. 1 and 2 clearly suggest
that this is actually the case. We shall restrict ourselves in the rest of this section to the
case \( m_2 > 0 \), since for \( m_2 = 0 \) the spectrum of the chain (2.23) is twice that of an su(\( m_1 \))
PF chain of \( A_{N-1} \) type (with the same degeneracies) on account of Eq. (3.38).

The spectrum of the spin chain (2.23) can be determined for any fixed \( N \) by evaluating
its partition function (3.34) with the help of, e.g., Mathematica. It turns out that the
most efficient way to compute the partition function

\[
Z^{(m)}_{A,k}(q) = \sum_{l=1}^{\min(m,k)} \binom{m}{l} q^{k-l} \prod_{i=1}^{l-1} (1 - q^{k-i}) \cdot Z^{(m)}_{A,k-l}(q)
\]

with the initial condition \( Z^{(m)}_{A,0}(q) = 1 \) (see Appendix A). In this way it is possible to
evaluate the partition function \( Z^{(m_1,m_2)}(q) \) on a standard desktop computer for relatively
high values of \( N \) (of the order of 50) and, say, \( m_1 + m_2 \leq 4 \). Our computations show that
the energy levels of the \( D_N \)-type spin chain with PSRO are always a set of consecutive
integers. This result is consistent with the fact that the spectrum of all previously studied
rational spin chains of HS type is a set of consecutive integers \[3, 38, 39\], including the
rational spin chain of \( BC_N \) type with PSRO introduced in Ref. [1]. For this reason, in
order to test the Gaussian character of the level density of the chain (2.23) as
\( N \to \infty \) one can compare directly its normalized level density

\[
f(E) = m^{-N} \sum_{i=1}^{L} d_i \delta(E - E_i), \quad m \equiv m_1 + m_2,
\]

where \( E_1 < \cdots < E_L \) are the distinct energy levels and \( d_i \) is the degeneracy of \( E_i \), with
the Gaussian distribution

\[
g(E) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(E-\mu)^2}{2\sigma^2}}
\]

with parameters \( \mu \) and \( \sigma \) given by the mean and standard deviation of the spectrum, re-
spectively. More precisely, the level density of the chain (2.23) is asymptotically Gaussian
provided that

\[
\frac{d_i}{mN} \simeq g(E_i), \quad N \gg 1.
\]

In order to check the validity of the latter equation for any given \( m_1, m_2 \) and \( N \) we
need to compute the corresponding values of \( \mu \) and \( \sigma \). We shall next show that, as is
the case with other spin chains of HS type, these parameters can be easily evaluated in
closed form from their definition

\[
\mu = m^{-N} \text{ tr } H^{(m_1,m_2)}, \quad \sigma^2 = m^{-N} \text{ tr } \left[ (H^{(m_1,m_2)})^2 \right] - \mu^2.
\]
The traces appearing in (4.4) can be computed in essentially the same way as for the BC\textsubscript{N}-type PF chain with PSRO (2.14), using the traces of the spin operators $P_{ij}$, $P_{ij}^{(m_1,m_2)}$ and $\bar{P}_{ij}^{(m_1,m_2)}$ given in Ref. [1]. Proceeding in this way we obtain

$$
\mu = \left(1 + \frac{1}{m}\right) \sum_{i \neq j} (h_{ij} + \bar{h}_{ij}),
$$
(4.5)

$$
\sigma^2 = 2 \left(1 - \frac{1}{m^2}\right) \sum_{i \neq j} (h_{ij}^2 + \bar{h}_{ij}^2) + \frac{4}{m^2} (t^2 - 1) \sum_{i \neq j} h_{ij} \bar{h}_{ij},
$$
(4.6)

where $t \equiv m_1 - m_2$ and

$$h_{ij} = (\xi_i - \xi_j)^{-2}, \quad \bar{h}_{ij} = (\xi_i + \xi_j)^{-2}.$$

The sums in Eqs. (4.5)-(4.6) can be evaluated by taking the $\beta \to 0$ limit of the corresponding formulas in Appendix A of Ref. [38]. We thus obtain

$$
\mu = \frac{1}{2} \left(1 + \frac{1}{m}\right) N(N - 1),
$$
(4.7)

$$
\sigma^2 = \frac{1}{36} \left(1 - \frac{1}{m^2}\right) N(N - 1)(4N + 1) + \frac{1}{4m^2} N(N - 1)(t^2 - 1).
$$
(4.8)

Figure 3: Left: level density of the chain (2.23) with $m_1 = 3$, $m_2 = 1$ and $N = 10$ (blue dots) compared to the Gaussian distribution (4.3) (continuous red line). Right: analogous plot for $N = 20$ spins.

We have checked that the normalized level density of the spin chain (2.23) is indeed in excellent agreement with the Gaussian distribution (4.3) for different values of $m_1$, $m_2$, and even moderately large values of $N \gtrsim 15$. As an example, in Fig. 3 we compare the normalized level density of the chain (2.23) with $m_1 = 3$, $m_2 = 1$, for $N = 10$ and $N = 20$ spins, respectively, with the corresponding Gaussian distribution (4.3).

It is apparent from these plots that the fit, already quite good for $N = 10$, improves significantly for $N = 20$. This is confirmed by computing the RMSE errors for both fits, which are respectively equal to $3.66 \times 10^{-2}$ and $2.18 \times 10^{-2}$. For comparison purposes, we note that this error decreases to $1.11 \times 10^{-2}$ for $N = 50$ spins.

Another interesting property of the spectrum of the chain (2.23) is connected to the distribution of the spacings between consecutive levels of the unfolded spectrum [54].
which in this case is given by
\[ s_i = (\eta_{i+1} - \eta_i) / \Delta, \quad i = 1, \ldots, L - 1, \]
where \( \Delta = (\eta_L - \eta_1) / (L - 1), \eta_i = \eta(\mathcal{E}_i), \) and
\[ \eta(\mathcal{E}) = \int_{-\infty}^{\mathcal{E}} g(\mathcal{E}') d\mathcal{E}' = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\mathcal{E} - \mu}{\sqrt{2} \sigma} \right) \right]. \]

According to a celebrated conjecture due to Berry and Tabor [54], the distribution of these spacings for a “generic” quantum integrable system should be Poissonian, i.e., \( p(s) = e^{-s} \).

On the other hand, a fundamental conjecture in quantum chaos due to Bohigas, Giannoni and Schmit [55] posits that the spacings distribution for a fully chaotic quantum system invariant under time reversal should follow Wigner’s law
\[ p(s) = \left( \frac{\pi s}{2} \right) \exp \left( -\frac{\pi s^2}{4} \right), \]
characteristic of the Gaussian orthogonal ensemble in random matrix theory [56]. In fact, it has been shown that the spacings distribution of a large class of integrable spin chains of Haldane–Shastry type follows neither Poisson’s nor Wigner’s law [1, 14, 16, 38, 57]. More precisely, it is shown in Refs. [16, 38, 58] that the cumulative spacings density \( P(s) \equiv \int_0^s p(s') ds' \) of a quantum system with equispaced energy levels and asymptotically Gaussian level density follows the “square root of a logarithm law”
\[ P(s) \simeq 1 - \frac{2}{\sqrt{\pi} s_{\text{max}}} \sqrt{\log \left( \frac{s_{\text{max}}}{s} \right)}, \quad s_{\text{max}} = \frac{\mathcal{E}_L - \mathcal{E}_1}{\sqrt{2} \pi \sigma}, \] (4.9)
provided that a few mild technical conditions are satisfied. We have just shown that the energy levels of the rational \( D_N \) chain with PSRO (2.23) are equispaced and its level density is asymptotically Gaussian, and it can be easily checked using the formulas for \( \mathcal{E}_1 \) and \( \mathcal{E}_L \) below that the technical assumptions in Ref. [58] are satisfied. Thus the spacings distribution of this chain is again approximately given by Eq. (4.9). It should be noted that for a more precise test of the validity of the Berry–Tabor conjecture one should restrict oneself to eigenspaces with well-defined quantum numbers corresponding to the main symmetries of the model. On the other hand, the fact that the spacings distribution of the whole spectrum is not Poissonian suggests that the Berry–Tabor conjecture does not hold in these eigenspaces, since the superposition of even a small number of Poissonian distributions is also Poissonian [59].

One of the characteristic properties of both the original Haldane–Shastry and the Polychronakos–Frahm spin chains of \( A_{N-1} \) type is their invariance under the quantum group \( Y(\mathfrak{sl}(m)) \). From the existence of such a large symmetry group one should expect that the spectrum of these chain exhibits a high degree of degeneracy. In fact, it is shown in Ref. [60] that the spectrum of these models is far more degenerate than that of a generic Yangian-invariant system, due to their equivalence to a vertex model of the form (3.42) with a very simple dispersion relation. Indeed, as shown in the latter reference, the number \( \nu^{(m)} \) of distinct levels of a generic \( Y(\mathfrak{sl}(m)) \)-invariant spin system with a large number of sites \( N \) behaves as \( \lambda^N \), where \( 1 < \lambda_m < 2 \) is the highest real root of the polynomial \( \lambda^m - \lambda^{m-1} - \cdots - 1 \). In contrast, \( \nu^{(m)} \) grows as a polynomial in \( N \) for
all spin chains of HS type associated with the $A_{N-1}$ root system. For instance, in the
case of the type $A_{N-1}$ PF chain this polynomial is simply given by $\mathcal{E}_l - \hat{\mathcal{E}}_1 + 1$, since its
spectrum is a set of consecutive integers. From the explicit expressions for the maximum
and minimum energies of this model in Ref. [16] we easily obtain

$$\nu^{(m)} = \frac{1}{2} \left( 1 - \frac{1}{m} \right) N^2 + \frac{l(m - l)}{2m} + 1 \quad \text{(PF chain)},$$

where $l = N \mod m$. The situation is far less clear for spin chains of HS type associated
to other root systems, with either standard or polarized spin reversal operators. On the
one hand, the presence of these spin reversal operators breaks $su(m)$ invariance, so that
it is not obvious whether these models are invariant under a suitable quantum group, let
alone $Y(sl(m))$. On the other hand, it has been observed that the spectrum of some of
these chains is also highly degenerate, which seems to indicate the presence of a large
symmetry group.

In the particular case of the $D_N$-type chain with PSRO in Eq. (2.23), the number of
distinct energy levels can again be exactly computed under the assumption (which we
have numerically checked) that the spectrum consists of consecutive integers. Indeed, it
suffices to evaluate the maximum and minimum energies $\mathcal{E}_{\max}^{(m_1, m_2)}$ and $\mathcal{E}_{\min}^{(m_1, m_2)}$, in terms
of which the number $\nu^{(m_1, m_2)}$ of distinct energy levels is given by

$$\nu^{(m_1, m_2)} = \mathcal{E}_{\max}^{(m_1, m_2)} - \mathcal{E}_{\min}^{(m_1, m_2)} + 1.$$ 

In the first place, the maximum energy can be easily computed by taking into account
that $P_{ij}$ and $\tilde{P}_{ij}^{(m_1, m_2)}$ are self-adjoint operators whose square is the identity, so that
their eigenvalues are $\pm 1$. Moreover, it is clear that a state of the form $|s, s, \ldots, s\rangle$ is a
simultaneous eigenvector of all the operators $P_{ij}$ and $\tilde{P}_{ij}^{(m_1, m_2)}$ with eigenvalue 1. Hence
the maximum energy of the chain (2.23) is given by

$$\mathcal{E}_{\max}^{(m_1, m_2)} = 2 \sum_{i \neq j} [(\xi_i - \xi_j)^{-2} + (\xi_i + \xi_j)^{-2}] = N(N - 1),$$

where the sum was evaluated in [39]. On the other hand, by Eq. (3.36) the minimum
energy is given by

$$\mathcal{E}_{\min}^{(m_1, m_2)} = \min \left( \mathcal{E}_{B,+}^{(m_1, m_2)}, \mathcal{E}_{B,-}^{(m_2, m_1)} \right),$$

where $\mathcal{E}_{B,+}^{(m_1, m_2)}$ is the minimum energy of the $BCN$-type chain (2.14) with $\epsilon = +1$. The
latter energy was computed in Ref. [11], with the result

$$\mathcal{E}_{B,+}^{(m_1, m_2)} = (N - l)(N + l - m_1)/m + (l - m_1)\theta(l - m_1)$$

where $l \equiv N \mod m$ and $\theta$ is Heaviside’s function (cf. Eq. (3.43)). Using the above
relation it is straightforward to check that if $m_1 \geq m_2$ we have $\mathcal{E}_{B,+}^{(m_1, m_2)} \leq \mathcal{E}_{B,+}^{(m_2, m_1)}$, and therefore

$$\mathcal{E}_{\min}^{(m_1, m_2)} = \mathcal{E}_{B,+}^{(m_1, m_2)}, \quad m_1 \geq m_2.$$ 

From Eqs. (4.11) and (4.14), and the assumption that the energy levels are equispaced,
we finally obtain the following closed formula for the number of distinct energy levels of
the $D_N$ chain (2.23):
\[\nu^{(m_1,m_2)} = \left(1 - \frac{1}{m}\right)N^2 - \frac{m_2}{m} N + \frac{l(l-m_1)}{m} - (l-m_1)\theta(l-m_1) + 1.\] (4.15)

Thus, it is apparent that $\nu^{(m_1,m_2)}$ is a quadratic polynomial in $N$, as is the case with the PF chain of $A_{N-1}$ type (cf. Eq. (4.10)). In particular, the spectrum of the chain (2.23) exhibits a very high degeneracy, much larger than that of a generic Yangian-invariant $\text{su}(m)$ spin model; see, e.g., Fig. 4.

The low number $\nu^{(m_1,m_2)}$ of distinct energy levels of the model (2.23) entails an extremely high average degeneracy $d^{(m_1,m_2)} \equiv mN / \nu^{(m_1,m_2)}$, which in turn suggests the existence of a large symmetry group. More precisely, it was shown in Ref [60] that the polynomial growth of the number of distinct energy levels of the spin chains of HS type associated to the $A_{N-1}$ root system is ultimately due to the equivalence of these chains to a Yangian-invariant vertex model of the form (3.42) with a suitable dispersion relation. This observation makes it reasonable to conjecture that the $D_N$-type spin chain with PSRO (2.23) is also invariant under a suitable Yangian group, and that its spectrum coincides with that of a vertex model analogous to (3.42) with an appropriate energy function.

5. The ferromagnetic models

We shall consider in this section the ferromagnetic counterparts of the $D_N$-type spin Calogero model with PSRO (2.15) and its corresponding spin chain (2.23), with Hamiltonians respectively given by
\[H^F_{(m_1,m_2)} = -\sum_i \frac{\partial^2}{\partial x_i^2} + a \sum_{i \neq j} \left[ \frac{a - P_{ij}}{(x_{ij}^+)^2} + \frac{a - \tilde{P}^{(m_1,m_2)}_{ij}}{(x_{ij}^-)^2} \right] + \frac{a^2}{4} r^2 \] (5.1)
and
\[H^F_{(m_1,m_2)} = \sum_{i \neq j} \left[ \frac{1 - P_{ij}}{(\xi_i - \xi_j)^2} + \frac{1 - \tilde{P}^{(m_1,m_2)}_{ij}}{(\xi_i + \xi_j)^2} \right]. \] (5.2)
The spectrum of the ferromagnetic spin Calogero model (5.1) can be studied in a similar way as its antiferromagnetic counterpart, following the procedure described in Section 3. To begin with, we note that the Hamiltonian (5.1) and the auxiliary operator (3.2) are related by

\[ H^{(m_1, m_2)}_F = H|_{K_{ij} \rightarrow P_{ij}, K_i \rightarrow P^{(m_1, m_2)}_i P^{(m_1, m_2)}_j}. \]  

Hence, the operator \( \Lambda^{(m_1, m_2)} \) in Section 3 should be replaced by the projector \( \Lambda^{(m_1, m_2)}_i \) onto states symmetric under simultaneous exchange of the particles’ spatial and spin coordinates, and with parity +1 under the product of an even number of operators \( \pi_i^{(m_1, m_2)} \) (cf. (3.11)). The new projection operator is the sum

\[ \Lambda^{(m_1, m_2)}_i = \Lambda^{(m_1, m_2)}_{B, s,+} + \Lambda^{(m_1, m_2)}_{B, s,-} \]

of the symmetric analogs of the \( BC_N \)-type projectors in Section 3, determined by

\[ \pi_{ij} \Lambda^{(m_1, m_2)}_{B, s, \pm} = \Lambda^{(m_1, m_2)}_{B, s, \pm}, \quad \pi_{i}^{(m_1, m_2)} \Lambda^{(m_1, m_2)}_{B, s, \pm} = \pm \Lambda^{(m_1, m_2)}_{B, s, \pm}. \]  

As explained in Section 3 for the antiferromagnetic case, the operator \( H^{(m_1, m_2)}_F \) is equivalent to its natural extension to the Hilbert space

\[ V = V_{B, s,+} \oplus V_{B, s,-}, \quad V_{B, s, \pm} \equiv \Lambda^{(m_1, m_2)}_{B, s, \pm}(L^2(\mathbb{R}^N) \otimes \mathcal{S}). \]  

A set of (non-orthogonal) vectors whose linear span is dense in each of the Hilbert spaces \( V_{B, s, \pm} \) can be constructed in much the same way as in the antiferromagnetic case, replacing \( \Lambda^{(m_1, m_2)}_{B, s, \pm} \) by \( \Lambda^{(m_1, m_2)}_{B, s, \pm} \) in (3.18). Due to the symmetry of \( \Lambda^{(m_1, m_2)}_{B, s, \pm} \) under permutations, in order to obtain a basis of these Hilbert spaces we must replace condition iii) in Section 3 by iii’) \( s_i \geq s_j \) if \( n_i = n_j \) and \( i < j \).

As a result, the spectrum of the ferromagnetic model (5.1) is still given by Eq. (3.22), but the corresponding degeneracy factor \( d^{(m_1, m_2)}_{F, n} \) in (3.25) should be replaced by

\[ d^{(m_1, m_2)}_{F, n} = \prod_{i=1}^{s} \left( m_1 + k_i - 1 \right) \prod_{j=1}^{t} \left( m_2 + l_j - 1 \right) \]

\[ + \prod_{i=1}^{s} \left( m_2 + k_i - 1 \right) \prod_{j=1}^{t} \left( m_1 + l_j - 1 \right). \]  

Using this formula for the degeneracy factor and proceeding as in Section 3 we find that the partition function of the \( D_N \)-type ferromagnetic spin chain (5.2) is given by the following analog of Eq. (3.30):

\[ Z^{(m_1, m_2)}_F(q) = (1 + q^N)^{-1} \left[ Z^{(m_1, m_2)}_{\text{B,F,}+}(q) + Z^{(m_2, m_1)}_{\text{B,F,}+}(q) \right], \]  

where \( Z^{(m_1, m_2)}_{\text{B,F,}+} \) denotes the partition function of the ferromagnetic counterpart of the rational \( BC_N \)-type chain (2.14). Proceeding as in Ref. [1] one can readily prove the ferromagnetic version of Eq. (3.32), namely

\[ Z^{(m_1, m_2)}_{\text{B,F,}+}(q) = \sum_{k=0}^{N} q^{N-k} \left[ \begin{array}{c} N \\ k \end{array} \right] q^2 Z^{(m_1)}_{A,F,k}(q^2) Z^{(m_2)}_{A,F,N-k}(q^2) \quad (m_2 > 0). \]
Here \( Z_{A,F,k}^{(m)} \) denotes the partition function of the ferromagnetic version of the su(\( m \)) PF chain of type \( A_{N-1} \), with \( k \) spins, obtained replacing \( P_{ij} \) by \( -P_{ij} \) in the latter equation. Finally, from Eqs. (5.7) and (5.8) we immediately obtain the following explicit formula for the partition function of the ferromagnetic chain (5.2):

\[
Z_{F}^{(m_1,m_2)}(q) = \sum_{k=0}^{N} f_{N,k}(q) Z_{A,F,k}^{(m_1)}(q^2) Z_{A,F,N-k}^{(m_2)}(q^2) \quad (m_2 > 0), \quad (5.9)
\]

where \( f_{N,k}(q) \) is again given by (3.35). For \( m_2 = 0 \), proceeding exactly as in Section 3 we obtain

\[
Z_{F}^{(m_1,0)}(q) = Z_{A,F,N}^{(m_1)}(q^2). \quad (5.10)
\]

Several explicit expressions for the partition function \( Z_{A,F,k}^{(m)} \) of the ferromagnetic PF chain of \( A_{k-1} \) type appearing in the previous formulas are known in the literature. The first of these expressions is the analog of Eq. (3.39), namely

\[
Z_{A,F,k}^{(m)}(q) = \sum_{k_1 + \cdots + k_m = k} [k_1, \ldots, k_m]_q. \quad (5.11)
\]

Alternatively, \( Z_{A,F,k}^{(m)} \) may be obtained from Eq. (3.40) replacing \( d_m(f) \) by its ferromagnetic version \( d_{F,m}(f) = \prod_{i=1}^{m} (m+f_i-1) \). Finally, \( Z_{A,F,k}^{(m)} \) is also given by the RHS of Eq. (3.44) with \( \theta(x) \) replaced by \( 1 - \theta(x) \) in the definition (3.42) of \( E^{(m)}(x) \). From any of these explicit formulas for \( Z_{A,F,k}^{(m)}(q) \), it follows that this function is a polynomial in \( q \).

As is well known, the partition functions of the \( A_{N-1} \)-type ferromagnetic and antiferromagnetic PF spin chains satisfy a certain duality relation [9] [61] [62]. In fact, a similar relation also holds for PF chains associated with other root systems [11] [38] [39]. In order to establish a duality relation between the partition functions of the ferromagnetic and antiferromagnetic spin chains of \( D_N \) type with PSRO, it suffices to observe that their Hamiltonians (5.2) and (2.23) are related by

\[
\mathcal{H}_{F}^{(m_1,m_2)} + \mathcal{H}^{(m_1,m_2)} = 2 \sum_{i \neq j} [\xi_i - \xi_j]^{-2} + (\xi_i + \xi_j)^{-2} = N(N-1) \quad (5.11)
\]

(cf. Eq. (4.11)). This obviously implies that the eigenvalues of \( \mathcal{H}_{F}^{(m_1,m_2)} \) and \( \mathcal{H}^{(m_1,m_2)} \) are also related by (5.11), so that their partition functions satisfy the duality relation

\[
Z_{F}^{(m_1,m_2)}(q) = q^{N(N-1)} Z^{(m_1,m_2)}(q^{-1}). \quad (5.12)
\]

6. Conclusions and outlook

We introduce the \( D_N \) spin Calogero model with PSRO and its associated spin chain of HS type, namely the \( D_N \) PF chain with PSRO. We solve the former model by finding a suitable (non-orthonormal) basis of its Hilbert space on which its Hamiltonian acts triangually. From the spectrum of this model we are able to compute its partition
function in closed form, which yields the partition function of the spin chain via Polychronakos's freezing trick. More precisely, we show that the latter partition function can be expressed in terms of the partition function of the type-A PF chain. Since the type-A partition function can be efficiently evaluated using a simple recursion formula that we also derive in this paper, we are able to exactly compute the spectrum of the $D_N$-type chain for relatively high values of $N$. In this way, we are able to study several global properties of the spectrum of the latter chain. In particular, we provide strong numerical evidence showing that its energy levels are a sequence of consecutive integers, and that its level density becomes normally distributed when the number of spins tends to infinity. From these facts we conclude that the spacings between consecutive levels of the unfolded spectrum follows a “square-root-of-a-logarithm” distribution, characteristic of most spin chains of HS type. We also determine the number of distinct energy levels of the spin chain, showing that it is a second-degree polynomial in $N$, as is the case with the PF chain of $A_{N-1}$ type. For spin chains of HS type related to the $A_{N-1}$ root system, it is known [60] that the polynomial growth of the number of distinct levels is a consequence of the fact that these models are equivalent to a Yangian-invariant vertex model with linear energy function and polynomial dispersion relation. Our results strongly suggest that this is also the case for the present model, a conjecture which certainly deserves further study. In particular, the validity of this conjecture would also point out at the existence of a suitable Yangian symmetry for both the $D_N$-type spin chain and the spin Calogero model with PSRO, as is the case with the rational and trigonometric Calogero–Sutherland models of $A_{N-1}$-type and their associated spin chains.

The present work suggests some possible future developments. Among them, the most natural one would be to address the extension of our results to the Sutherland (both trigonometric and hyperbolic) models of $BC_N$, $B_N$ and $D_N$ type and their related spin chains. From a more mathematical standpoint, the fact that the chain’s spectrum consists of integers leads us to conjecture that the function $f_{N,k}(q)$ in Eq. (3.35) is a polynomial in $q$. Although this conjecture can be easily checked numerically, we have not been able to find an analytic proof thereof using the properties of $q$-binomial coefficients.

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Appendix A. Recursion relation for the partition function of the PF chain of $A_{k-1}$ type

We shall provide in this Appendix a short derivation of the recursion relation (4.1) satisfied by the partition function $Z^{(m)}_{A,k}$ of the su$(m)$ PF chain of $A_{k-1}$ type. The main idea behind the proof is to decompose the multiindex $f \in \mathcal{P}_k$ in Eq. (3.40) as

$$f = (f_1, \ldots, f_{r-1}, l) \equiv (\bar{f}, l),$$

with $1 \leq l \leq \min(m, k)$ and $\bar{f} \in \mathcal{P}_{k-1}$. Setting $s = r - 1$ we have

$$F_1 + \cdots + F_{r-1} = \tilde{F}_1 + \cdots + \tilde{F}_{s-1} + F_{r-1} = \tilde{F}_1 + \cdots + \tilde{F}_{s-1} + k - l,$$

(A.1)
and therefore

\[ \{ F'_1, \ldots, F'_{k-r} \} = \{ \tilde{F}'_1, \ldots, \tilde{F}'_{k-l-s} \} \cup \{ k-l+1, \ldots, k-1 \}. \quad (A.2) \]

Substituting (A.1) and (A.2) into Eq. (3.40) we obtain

\[ Z^{(m)}_{A,k}(q) = \min(m,k) \sum_{l=1}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) q^{k-l} \prod_{i=1}^{l-1} (1 - q^{k-i}) \cdot \sum_{f \in P_{k-l}} \prod_{i=1}^{s} \left( \begin{array}{c} m \\ f_i \end{array} \right) \tilde{F}'_{i+1} \cdots \tilde{F}'_{k-l-s} \prod_{i=1}^{k-l-s} (1 - q^{\tilde{F}'_i}) \]

as claimed. As to the initial condition, from Eq. (3.40) with \( k = 1 \) it easily follows that

\[ Z^{(m)}_{A,1}(q) = m. \]

From the recursion relation (4.1) with \( k = 1 \) we easily obtain

\[ Z^{(m)}_{A,0}(q) = 1. \]

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