REGULARITY FOR FULLY NONLINEAR INTEGRO-DIFFERENTIAL OPERATORS WITH KERNELS OF VARIABLE ORDERS

MINHYUN KIM AND KI-AHM LEE

Abstract. We consider fully nonlinear elliptic integro-differential operators with kernels of variable orders, which generalize the integro-differential operators of the fractional Laplacian type in \cite{2}. Since the order of differentiability of the kernel is not characterized by a single number, we use the constant
\[ C_\varphi = \left( \int_{\mathbb{R}^n} \frac{1 - \cos y_1}{|y|^{n\varphi(|y|)}} \, dy \right)^{-1} \]
instead of \(2 - \sigma\), where \(\varphi\) satisfies a weak scaling condition. We obtain the uniform Harnack inequality and Hölder estimates of viscosity solutions to the nonlinear integro-differential equations.

1. Introduction

In this paper we consider fully nonlinear elliptic integro-differential operators. By the Lévy Khintchine formula, the generator of an \(n\)-dimensional pure jump process is given by
\[ Lu(x) = \int_{\mathbb{R}^n} (u(x + y) - u(x) - \nabla u(x) \cdot y \chi_{B_1}(y)) \, d\mu(y), \tag{1.1} \]
where \(\mu\) is a measure such that \(\int_{\mathbb{R}^n} |y|^2/(1 + |y|^2) \, d\mu(y) < \infty\). Note that the value of \(Lu(x)\) is well-defined as long as \(u\) is bounded in \(\mathbb{R}^n\) and \(C^{1,1}\) in a neighborhood of \(x\). Since the operators are given in too much generality, we restrict ourselves to the operators given by symmetric kernels \(K\). In this case, the operator \eqref{1.1} can be written as
\[ Lu(x) = \int_{\mathbb{R}^n} (u(x + y) + u(x - y) - 2u(x)) K(y) \, dy, \tag{1.2} \]
and the kernel \(K\) satisfies
\[ \int_{\mathbb{R}^n} |y|^2/(1 + |y|^2) K(y) \, dy < \infty. \tag{1.3} \]
For the notational convenience we write \(\delta(u, x, y) = u(x + y) + u(x - y) - 2u(x)\) in the sequel. Nonlinear integro-differential operators such as
\[ Iu(x) = \sup_{\alpha} L_\alpha u(x) \quad \text{or} \quad Iu(x) = \inf_{\beta} \sup_{\alpha} L_{\alpha\beta} u(x) \]
arise in the stochastic control theory and the game theory. A characteristic property of these operators is that
\[ \inf_{\alpha\beta} L_{\alpha\beta} v(x) \leq I(u + v)(x) - Iu(x) \leq \sup_{\alpha\beta} L_{\alpha\beta} v(x). \]

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Caffarelli and Silverstre \cite{2} introduced the concept of ellipticity for more general nonlinear operators $I$: for a class of linear integro-differential operators $\mathcal{L}$ it holds that
\[ \mathcal{M}^+(u - v)(x) \leq Iu(x) - Iv(x) \leq \mathcal{M}^-(u - v)(x), \]
where $\mathcal{M}^+$ and $\mathcal{M}^-$ are a maximal and a minimal operator with respect to $\mathcal{L}$, defined by
\[ \mathcal{M}^+_L u(x) = \sup_{L \in \mathcal{L}} Lu(x) \quad \text{and} \quad \mathcal{M}^-_L u(x) = \inf_{L \in \mathcal{L}} Lu(x), \]
respectively. See \cite{1} for elliptic second-order differential operators. We adopt this concept and will give a precise definition in Section 3.

Caffarelli and Silverstre \cite{2} considered fully nonlinear integro-differential operators with kernels comparable to those of fractional Laplacian to obtain regularity results. That is, they considered the class of operators of the form (1.2) with
\[ (2 - \sigma) \frac{\lambda}{|y|^{n+\sigma}} \leq K(y) \leq (2 - \sigma) \frac{\Lambda}{|y|^{n+\sigma}}, \]
where $0 < \sigma < 2$. They obtained regularity estimates that remain uniform as the order of the equation $\sigma$ approaches 2 and therefore made the theory of integro-differential equations and elliptic differential equations appear somewhat unified. More generally, in \cite{4} the authors generalized these results to fully nonlinear integro-differential operators with regularly varying kernels. More precisely, they considered the class of operators of the form (1.2) with
\[ (2 - \sigma) \frac{l(|y|)}{|y|^{n+\sigma}} \leq K(y) \leq (2 - \sigma) \frac{\Lambda l(|y|)}{|y|^{n+\sigma}}, \]
where $l : (0, \infty) \to (0, \infty)$ is a locally bounded, regularly varying function at zero with index $-\sigma$. In both cases, the constant $2 - \sigma$ plays a very important role in uniform regularity estimates. They used the constant $2 - \sigma$ instead of the constant in the fractional Laplacian
\[ C(n, \sigma) = \left( \int_{\mathbb{R}^n} \frac{1 - \cos y_1}{|y|^{n+\sigma}} \, dy \right)^{-1} = \frac{2\sigma \Gamma\left(\frac{n+\sigma}{2}\right)}{\pi^{n/2} \Gamma\left(\frac{n}{2}\right)} \]
because two constants $2 - \sigma$ and $C(n, \sigma)$ have the same asymptotic behavior as $\sigma$ approaches 2 and they focused on regularity estimates which remain uniform as $\sigma$ approaches 2.

In this paper we will consider kernels of variable orders. In this case the order of the kernel cannot be characterized in a single number. This implies that we need to consider the constant which contains all information of the kernel to generalize the results of \cite{2}. We will define this constant in Section 1.1.

1.1. **Integro-differential Operators.** In order to obtain regularity results, we need to impose some assumptions on the kernel $K$. Throughout this paper, we will assume that the kernel $K$ satisfies
\[ C_{\varphi} \frac{\lambda}{|y|^{n, \varphi}(|y|)} \leq K(y) \leq C_{\varphi} \frac{\Lambda}{|y|^{n, \varphi}(|y|)} \]
for some constants $0 < \lambda \leq \Lambda < \infty$, where a function $\varphi : (0, \infty) \to (0, \infty)$ and a constant $C_{\varphi}$ will be defined below.
We first assume that the function \( \varphi \) satisfies a weak scaling condition with constants \( a \geq 1 \) and \( 0 < \underline{\sigma} \leq \bar{\sigma} < 2 \), i.e.,

\[
a^{-1} \left( \frac{R}{r} \right)^{\underline{\sigma}} \leq \frac{\varphi(R)}{\varphi(r)} \leq a \left( \frac{R}{r} \right)^{\bar{\sigma}} \quad \text{for all } 0 < r \leq R < \infty.
\]

The simplest example of this function is \( \varphi(r) = r^\sigma \) with \( \sigma \in (0, 2) \), which corresponds to the fractional Laplacian. However, more general functions such as \( \varphi(r) = r^{\underline{\sigma}} + r^{\bar{\sigma}}, \varphi(r) = r^{\underline{\sigma}}(\log(1 + r^{-2}))^{-(2-\underline{\sigma})/2} \), and \( \varphi(r) = r^{\bar{\sigma}}(\log(1 + r^{-2}))^{2/2} \) are covered.

We next observe that if we take the Fourier transform to the operator

\[
L_0 u(x) = \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^n \varphi(|y|)} \, dy,
\]

then

\[
-\hat{\mathcal{F}}(L_0 u)(\xi) = -\int_{\mathbb{R}^n} \mathcal{F}(u(\cdot + y) + u(\cdot - y) - 2u(\cdot))(|\xi|) \, dy
\]

\[
= -\int_{\mathbb{R}^n} \frac{e^{i\xi \cdot y} + e^{-i\xi \cdot y} - 2}{|y|^n \varphi(|y|)} \, dy \hat{\mathcal{F}} u(|\xi|)
\]

\[
= 2 \int_{\mathbb{R}^n} \frac{1 - \cos(\xi \cdot y)}{|y|^n \varphi(|y|)} \, dy \hat{\mathcal{F}} u(|\xi|).
\]

Since the function

\[
\xi \mapsto \int_{\mathbb{R}^n} \frac{1 - \cos(\xi \cdot y)}{|y|^n \varphi(|y|)} \, dy
\]

is rotationally symmetric, we have

\[
-\mathcal{F}(L_0 u)(\xi) = 2 \int_{\mathbb{R}^n} \frac{1 - \cos(|\xi| y_1)}{|y|^n + \sigma} \, dy \hat{\mathcal{F}} u(|\xi|).
\]

Note that when \( \varphi(r) = r^\sigma \) the integral in (1.6) can be represented as

\[
\int_{\mathbb{R}^n} \frac{1 - \cos(|\xi| y_1)}{|y|^n + \sigma} \, dy = \int_{\mathbb{R}^n} \frac{1 - \cos y_1 \, dy}{|\xi|^{n+\sigma}} = C(n, \sigma)^{-1}|\xi|^\sigma,
\]

and hence the fractional Laplacian is defined with the constant \( C(n, \sigma) \) as

\[
-(-\Delta)^{\sigma/2} u(x) = \frac{1}{2} C(n, \sigma) \int_{\mathbb{R}^n} \frac{\delta(u, x, y)}{|y|^{n+\sigma}} \, dy.
\]

Thus, in the general case, it is natural to define

\[
C_\varphi = \left( \int_{\mathbb{R}^n} \frac{1 - \cos y_1}{|y|^n \varphi(|y|)} \, dy \right)^{-1}
\]

as a normalizing constant. Then the operator \( \frac{1}{2} C_\varphi L_0 \) generalizes the fractional Laplacian \(-(-\Delta)^{\sigma/2}\). In Section 2, we will prove asymptotic properties of the constant \( C_\varphi \) and the operator \( \frac{1}{2} C_\varphi L_0 \).
1.2. Main Results. In this paper, we are concerned with the nonlinear integro-differential operator

\[(1.7) \quad Iu := \inf \sup_{\beta} L_{\alpha \beta} u, \quad L_{\alpha \beta} \in \mathcal{L}_0, \]

where \(\mathcal{L}_0\) denotes the class of linear integro-differential operators of the form (1.2) with symmetric kernels \(K\) satisfying (1.3) and (1.4).

We define functions \(\underline{C}, \overline{C} : (0, \infty) \rightarrow \mathbb{R}\) by

\[
\underline{C}(R) := \underline{C}_\sigma(R) := \int_0^R \frac{1}{r \phi(r)} \, dr \quad \text{and} \quad \overline{C}(R) := \overline{C}_\sigma(R) := \int_R^{\infty} \frac{1}{r \phi(r)} \, dr.
\]

They correspond to \(\frac{R^{2-\alpha}}{2-\sigma}\) and \(\frac{R^{2-\alpha}}{2-\sigma}\) for the case of fractional Laplacian, respectively. We will denote by \(\underline{C} = \underline{C}(1)\) and \(\overline{C} = \overline{C}(1)\).

Now we present our main results which generalize the uniform regularity results in [2]. Throughout this paper we denote \(B_R := B_R(0)\) for \(R > 0\).

**Theorem 1.1** (Harnack inequality). Let \(\sigma_0 \in (0, 2)\) and assume \(\underline{\sigma} \geq \sigma_0\). Let \(u \in C(B_{2R})\) be a nonnegative function in \(\mathbb{R}^n\) such that

\[
\mathcal{M}^-_{\underline{\sigma} \sigma} u \leq C_0 \quad \text{and} \quad \mathcal{M}^+_{\underline{\sigma} \sigma} u \geq -C_0 \quad \text{in} \ B_{2R}
\]

in the viscosity sense. Then there exists a uniform constant \(C > 0\), depending only on \(n, \lambda, \Lambda, \alpha, \) and \(\sigma_0\), such that

\[(1.8) \quad \sup_{B_R} u \leq C \left( \inf_{B_R} u + C_0 \frac{(C + \overline{C}) R^2}{\overline{C}(R)} \right). \]

**Theorem 1.2** (Hölder regularity). Let \(\sigma_0 \in (0, 2)\) and assume \(\underline{\sigma} \geq \sigma_0\). Let \(u \in C(B_{2R})\) be a function in \(\mathbb{R}^n\) such that

\[
\mathcal{M}^-_{\underline{\sigma} \sigma} u \leq C_0 \quad \text{and} \quad \mathcal{M}^+_{\underline{\sigma} \sigma} u \geq -C_0 \quad \text{in} \ B_{2R}
\]

in the viscosity sense. Then \(u \in C^\alpha(B_R)\) and

\[(1.9) \quad R^\alpha [u]_{C^\alpha(B_R)} \leq C \left( \|u\|_{L^{\infty}(B_R)} + C_0 \frac{(C + \overline{C}) R^2}{\overline{C}(R)} \right)
\]

for some uniform constants \(\alpha > 0\) and \(C > 0\) which depend only on \(n, \lambda, \Lambda, \alpha, \) and \(\sigma_0\).

It is important to note that in the regularity estimates (1.8) and (1.9) the constants are independent of \(\underline{\sigma}\) and \(\overline{\sigma}\), but the term \(\frac{(C + \overline{C}) R^2}{\overline{C}(R)}\) in the right-hand side of (1.8) and (1.9) still depends on \(\underline{\sigma}\) and \(\overline{\sigma}\). For the fractional Laplacian case this term corresponds to \(\frac{2}{\underline{\sigma}} R^\sigma\) and it can be further estimated as

\[
\frac{2}{\sigma} R^\sigma \leq \frac{2}{\underline{\sigma}} R^\underline{\sigma}.
\]

In our case, we can also estimate the term \(\frac{(C + \overline{C}) R^2}{\overline{C}(R)}\) using Lemma 2.4 and Lemma 2.3 as

\[(1.10) \quad \frac{(C + \overline{C}) R^2}{\overline{C}(R)} \leq C(n, a) \frac{(C(R) + \overline{C}(R)) R^2}{\overline{C}(R)} \leq C(n, a) \left( R^2 + \frac{2a^2}{\underline{\sigma}_0} \right), \]

which is independent of \(\underline{\sigma}\) and \(\overline{\sigma}\). Notice that it has the same blow up rate with the fractional Laplacian case. Nevertheless, we leave (1.8) and (1.9) as they are because the estimate (1.10) has a different scale with respect to \(R\).
This paper is organized as follows. In Section 2 we study asymptotic properties of the constant $C_{\phi}$ and the operator $\frac{1}{2}C_{\phi}L_0$, which play crucial roles in the forthcoming regularity results. Some bounds for the constant $C_{\phi}$ are also given in this section. Section 3 is devoted to the definitions of viscosity solutions and the notion of ellipticity for nonlinear integro-differential operators. In Section 4.1 we prove the ABP estimates, which is the main ingredient in the proof of Harnack inequality. We construct a barrier function in Section 4.2 and then use this function and ABP estimates to provide the measure estimates of super-level sets of the viscosity subsolutions to elliptic integro-differential equations in Section 4.3. We establish the Harnack inequality and Hölder estimates of viscosity solutions to elliptic integro-differential equations in Section 4.4 and 4.5.

2. Asymptotics of the Constant $C_{\phi}$

It is well-known that the constant $C(n, \sigma)$ for the fractional Laplacian has the following asymptotic properties:

\begin{equation}
\lim_{\sigma \to 2^-} C(n, \sigma) = \frac{2}{\omega_n} \quad \text{and} \quad \lim_{\sigma \to 0^+} C(n, \sigma) = \frac{1}{n \omega_n},
\end{equation}

where $\omega_n$ denotes the volume of the n-dimensional unit ball, and that the fractional Laplacian $(-\Delta)^{\sigma/2}$ has the following properties:

\begin{equation}
\lim_{\sigma \to 2^-} (-\Delta)^{\sigma/2}u = -\Delta u \quad \text{and} \quad \lim_{\sigma \to 0^+} (-\Delta)^{\sigma/2}u = u.
\end{equation}

See [3] for the proofs. In this section, we prove the analogues of (2.1) and (2.2), which will imply that the constant $C_{\phi}$ generalizes the constant of the fractional Laplacian $C(n, \sigma)$.

To state the analogues of (2.1) and (2.2), we must consider a sequence of operators

\[ L_k u(x) = \frac{1}{2} C_{\phi_k} \int_{\mathbb{R}^n} \frac{\delta(u, x, y)}{|y|^n \phi_k(|y|)} dy, \]

where functions $\phi_k : (0, \infty) \to (0, \infty)$ satisfy weak scaling conditions (1.5) with constants $a_k \geq 1$ and $0 < \underline{\phi}_k \leq \overline{\phi}_k < 2$. We will assume that

\begin{equation}
\lim_{k \to \infty} a_k = 1
\end{equation}

throughout this section. The following lemma and proposition correspond to (2.1) and (2.2), respectively. Recall that $\underline{C}(R) = \frac{R^{2-\sigma}}{2-\sigma}$ and $\overline{C}(R) = \frac{R^{-\sigma}}{\sigma}$ in the fractional Laplacian case.

Lemma 2.1. Assume that (2.3) holds. If $\lim_{k \to \infty} \underline{\phi}_k = \lim_{k \to \infty} \overline{\phi}_k = 2$, then

\begin{equation}
\lim_{k \to \infty} C_{\phi_k} \underline{C}_{\phi_k}(R) = \frac{2}{\omega_n},
\end{equation}

and if $\lim_{k \to \infty} \underline{\phi}_k = \lim_{k \to \infty} \overline{\phi}_k = 0$, then

\begin{equation}
\lim_{k \to \infty} C_{\phi_k} \overline{C}_{\phi_k}(R) = \frac{1}{n \omega_n}.
\end{equation}
Proposition 2.2. Assume that (2.3) holds, and let \( u \in C_c^\infty(\mathbb{R}^n) \). If \( \lim_{k \to \infty} \sigma_k = 2 \), then
\[
\lim_{k \to \infty} -L_k u = -\Delta u,
\]
and if \( \lim_{k \to \infty} \sigma_k = \lim_{k \to \infty} \sigma_k = 0 \), then
\[
\lim_{k \to \infty} -L_k u = u.
\]

The following estimates for the functions \( C_\varphi(R) \) and \( \overline{C}_\varphi(R) \) will be used frequently in the sequel.

Lemma 2.3. It holds that
\[
\frac{1}{a(2-\sigma)} \frac{R^2}{\varphi(R)} \leq C_\varphi(R) \leq \frac{a}{2-\sigma} \frac{R^2}{\varphi(R)}, \tag{2.6}
\]
and that
\[
\frac{1}{a \varphi(R)} \leq \overline{C}_\varphi(R) \leq \frac{a}{\varphi(R)}. \tag{2.7}
\]
Moreover, for \( t \in (0, 1) \) it holds that
\[
\frac{C_\varphi(R)}{C_\varphi(tR)} \leq 1 + a^2 t^{-2+\sigma}. \tag{2.8}
\]

Proof. Using the weak scaling condition (1.5) we see that
\[
C_\varphi(R) \geq \int_0^R \frac{1}{a \varphi(R)} \left( \frac{R}{r} \right)^{\sigma} \frac{R^2}{\varphi(R)} \, dr = \frac{1}{a(2-\sigma)} \frac{R^2}{\varphi(R)},
\]
which is the first inequality in (2.6). The second inequality in (2.6) and the inequalities in (2.7) can be proved in the same manner. The last inequality follows from (1.5) and (2.6) that
\[
\frac{C_\varphi(R)}{C_\varphi(tR)} = 1 + \frac{1}{C_\varphi(tR)} \int_{tR}^R \frac{r}{\varphi(r)} \, dr \leq 1 + a(2-\sigma) \frac{\varphi(tR)}{(tR)^2} \int_{tR}^R a \frac{\varphi(tR)}{\varphi(r)} r^{1-\sigma} \, dr \leq 1 + a^2 (tR)^{-2+\sigma} (R^{2-\sigma} - (tR)^{2-\sigma}) \leq 1 + a^2 t^{-2+\sigma}.
\]

Next we will prove Lemma 2.1 using Lemma 2.3 and the fact that the constant \( C_\varphi \) can be represented by
\[
C_\varphi^{-1} = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{1 - \cos y_1}{|y_1|^n \left( 1 + \frac{|y'|^2}{|y_1|^2} \right)^{n/2}} \varphi \left( |y_1| \left( 1 + \frac{|y'|^2}{|y_1|^2} \right)^{1/2} \right) \, dy' \, dy_1
\]
\[
= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{1 - \cos y_1}{|y_1| (1 + |y'|^2)^{n/2} \varphi(|y_1|) (1 + |y'|^2)^{1/2})} \, dy' \, dy_1
\]
\[
= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{1 - \cos \zeta}{|r| \varphi(|r|)} \, dr \, dy',
\]
where \( \zeta = \zeta(y') = (1 + |y'|^2)^{1/2} \).
Proof of Lemma 2.1. We first assume that
\[
\lim_{k \to \infty} \sigma_k = \lim_{k \to \infty} \sigma_k = 2. \tag{2.9}
\]
We use the inequality (2.7) to compute
\[
0 \leq \int_{|r| \geq R} \frac{1 - \cos \frac{\zeta}{|r|}}{|r| \varphi_k(|r|)} \, dr \leq \int_{R}^{\infty} \frac{4}{r \varphi_k(r)} \, dr \leq \frac{4a_k}{\sigma_k \varphi_k(R)}.
\]
Using the assumptions (2.3), (2.9), and the inequality (2.6), we obtain
\[
0 \leq \lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \int_{|r| \geq R} \frac{1 - \cos \frac{\zeta}{|r|}}{|r| \varphi_k(|r|)} \, dr \leq \lim_{k \to \infty} 4a_k^2 \frac{2 - \sigma_k}{\sigma_k} R^2 = 0.
\]
On the other hand, using the weak scaling condition (1.5), we have
\[
\left| \int_{|r| < R} \frac{1 - \cos \frac{\zeta}{|r|}}{|r| \varphi_k(|r|)} \, dr - \int_{|r| < R} \frac{r^2}{2\zeta^2 |r| \varphi_k(|r|)} \, dr \right| \leq \frac{1}{24} \int_{|r| < R} \frac{r^4}{\zeta^4 |r| \varphi_k(|r|)} \, dr \leq \frac{a_k}{12 \zeta^4 (4 - \sigma_k) \varphi_k(R)}.
\]
Since
\[
0 \leq \lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \frac{a_k}{12 \zeta^4 (4 - \sigma_k) \varphi_k(R)} \leq \lim_{k \to \infty} \frac{a_k^2}{12 \zeta^2 (4 - \sigma_k)} R^2 = 0,
\]
we obtain
\[
\lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \int_{R}^{\infty} \frac{1}{|r| \varphi_k(|r|)} \, dr = \lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \frac{1}{\zeta^2} \int_{0}^{R} \frac{r}{\varphi_k(r)} \, dr = \frac{1}{\zeta^2}.
\]
Therefore, we conclude that
\[
\lim_{k \to \infty} C_{\varphi_k} C_{\varphi_k}(R) = \lim_{k \to \infty} \left( \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{1 - \cos \frac{\zeta}{|r|}}{|r| \varphi_k(|r|)} \, dr \, dy \right)^{-1} C_{\varphi_k}(R)
= \left( \int_{\mathbb{R}^{n-1}} \lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \int_{\mathbb{R}} \frac{1 - \cos \frac{\zeta}{|r|}}{|r| \varphi_k(|r|)} \, dr \, dy \right)^{-1}
= \left( \int_{\mathbb{R}^{n-1}} \frac{1}{\zeta^{n+2}} \, dy \right)^{-1} = \frac{2}{\omega_n}.
\]
See [3, Corollary 4.2] for the last equality.

We next assume that
\[
\lim_{k \to \infty} \sigma_k = \lim_{k \to \infty} \sigma_k = 0. \tag{2.10}
\]
We use the inequality (2.6) to compute
\[
0 \leq \int_{|r| < R} \frac{1 - \cos \frac{\zeta}{|r|}}{|r| \varphi_k(|r|)} \, dr \leq \frac{1}{\zeta^2} \int_{0}^{R} \frac{r}{\varphi_k(r)} \, dr \leq \frac{a_k}{\zeta^2 (2 - \sigma_k) \varphi_k(R)} R^2.
\]
Using the assumptions (2.3), (2.10), and the inequality (2.7), we obtain
\[
\lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \int_{|r| < R} \frac{1 - \cos \frac{\zeta}{|r|}}{|r| \varphi_k(|r|)} \, dr \leq \lim_{k \to \infty} \frac{a_k^2}{\zeta^2 (2 - \sigma_k)} R^2 = 0. \tag{2.11}
\]
On the other hand, observe that for any integer \( m \geq 1 \) we have

\[
\left| \int_{2m\zeta_π}^{2(m+1)\zeta_π} \frac{\cos \frac{r}{k}}{r\varphi_k(r)} \, dr \right| \leq \int_{2m\zeta_π}^{2(m+1)\zeta_π} \left| \frac{1}{r\varphi_k(r)} \right| \, dr
\]

(2.12)

\[
= \int_{2m}^{2m+1} \left| \frac{1}{r\varphi_k(r)} \right| \, dr.
\]

For the notational convenience, let us write \( A = \frac{1}{r\varphi_k(\zeta\pi)} - \frac{1}{r(\zeta\pi)\varphi_k(\zeta\pi(1+1))} \). If \( A \geq 0 \), then by the weak scaling condition (1.5), we have for \( r \in [2m, 2m+1] \)

\[
|A| \leq \frac{a_k}{\varphi_k((2m+1)\zeta\pi)} \left( \frac{2m+1}{r} \right)^{\frac{1}{k}} - \frac{1}{a_k\varphi_k((2m+1)\zeta\pi) r + 1} \left( \frac{2m+1}{r+1} \right)^{\frac{1}{k}}
\]

\[
= \frac{a_k^2(r+1)^{1+\frac{1}{k}} - r^{1+\frac{1}{k}}}{a_k\varphi_k((2m+1)\zeta\pi) r + 1}
\]

\[
\leq \frac{1}{2a_km\varphi((2m+1)\zeta\pi)} \left( \frac{2m+1}{2m} \right)^{\frac{1}{k}} a_k^2(r+1)^{1+\frac{1}{k}} - r^{1+\frac{1}{k}}
\]

\[
\leq \frac{9}{8a_km\varphi((2m+1)\zeta\pi)} (r+1)^{1+\frac{1}{k}} - r^{1+\frac{1}{k}}.
\]

If \( A \leq 0 \), then we have

\[
|A| \leq \frac{9}{8a_km\varphi((2m+1)\zeta\pi)} \frac{a_k^2(r+1)^{1+\frac{1}{k}} - r^{1+\frac{1}{k}}}{(r+1)^{1+\frac{1}{k}}}
\]

by similar argument. Since

\[
a_k^2(r+1)^{1+\frac{1}{k}} - r^{1+\frac{1}{k}} = \int_0^1 \frac{d}{ds} \left( (a_k^2 - 1)s(r+s)^{1+\frac{1}{k}} \right) \frac{1}{(r+1)^{1+\frac{1}{k}}} \, ds
\]

\[
= \int_0^1 (a_k^2 - 1)(r+s)^{1+\frac{1}{k}} \frac{1}{(r+1)^{1+\frac{1}{k}}} \, ds
\]

\[
+ \int_0^1 (1+\frac{1}{k}) (1+(a_k^2 - 1)s)(r+s)^{\frac{1}{k}} \frac{1}{(r+1)^{1+\frac{1}{k}}} \, ds
\]

\[
\leq a_k^2 - 1 + \frac{3a_k^2}{r+1}
\]

and

\[
a_k^2(r+1)^{1+\frac{1}{k}} - r^{1+\frac{1}{k}} = \int_0^1 \frac{d}{ds} \left( (a_k^2 + 1-s)(r+1-s)^{1+\frac{1}{k}} \right) \frac{1}{(r+1)^{1+\frac{1}{k}}} \, ds
\]

\[
= \int_0^1 (a_k^2 - 1)(r+1-s)^{1+\frac{1}{k}} \frac{1}{(r+1)^{1+\frac{1}{k}}} \, ds
\]

\[
- \int_0^1 (1+\frac{1}{k})(a_k^2 + 1-s)(r+1-s)^{\frac{1}{k}} \frac{1}{(r+1)^{1+\frac{1}{k}}} \, ds
\]

\[
\leq a_k^2 - 1,
\]

we further estimate the integrand \(|A|\) in (2.12) as

\[
(2.13) \quad |A| \leq \frac{9}{8a_km\varphi((2m+1)\zeta\pi)} \left( a_k^2 - 1 + \frac{3a_k^2}{r+1} \right)
\]
regardless of the sign of $A$.

Let $N \geq 1$ be the integer satisfying $2(N - 1)\zeta \pi < R \leq 2N\zeta \pi$. Then, from (2.12) and (2.13), we have for $m \geq N$

\[
\left| \int_{2m\zeta \pi}^{2(m+1)\zeta \pi} \frac{\cos \frac{\zeta}{r}}{r\varphi_k(r)} \, dr \right| \leq \frac{9}{8a_k} \int_{2m\zeta \pi}^{2(m+1)\zeta \pi} \left( a_k^2 - 1 + \frac{3a_k^2}{r + 1} \right) \, dr 
\leq \frac{9}{8a_k} m\varphi_k((2m + 1)\zeta \pi) \left( a_k^2 - 1 + \frac{3a_k^2}{2m + 1} \right) 
\leq \frac{9(a_k^2 - 1)}{8a_k} \frac{1}{m\varphi_k((2m + 1)\zeta \pi)} + \frac{27a_k^2}{16\varphi_k(R) m^2}.
\]

As a consequence, we have

\[
\left| \int_{R}^{\infty} \frac{\cos \frac{\zeta}{r}}{r\varphi_k(r)} \, dr \right| \leq \int_{R}^{\infty} \frac{dr}{r\varphi_k(r)} + \sum_{m=N}^{\infty} \left| \int_{2m\zeta \pi}^{2(m+1)\zeta \pi} \frac{\cos \frac{\zeta}{r}}{r\varphi_k(r)} \, dr \right| 
\leq \frac{a_k}{\varphi_k(R)} \int_{R}^{\infty} \frac{dr}{r} + \sum_{m=N}^{\infty} \left( \frac{9(a_k^2 - 1)}{8a_k} \frac{1}{m\varphi_k((2m + 1)\zeta \pi)} + \frac{27a_k^2}{16\varphi_k(R) m^2} \right) 
\leq \frac{a_k}{\varphi_k(R)} \log \frac{2N\zeta \pi}{R} + \frac{9(a_k^2 - 1)}{8a_k} \sum_{m=N}^{\infty} \frac{1}{m\varphi_k((2m + 1)\zeta \pi)} + \frac{9\pi^2 a_k^2}{32\varphi_k(R)}.
\]

Now, we claim that

\[
\lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \int_{R}^{\infty} \frac{\cos \frac{\zeta}{r}}{r\varphi_k(r)} \, dr = 0.
\]

Indeed, by using the assumptions (2.3), (2.10), and the inequality (2.7), the first and the third terms in (2.14) can be handled as

\[
0 \leq \lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \frac{a_k}{\varphi_k(R)} \log \frac{2N\zeta \pi}{R} \leq \lim_{k \to \infty} a_k \pi_k \log \frac{2N\zeta \pi}{R} = 0
\]

and

\[
0 \leq \lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \frac{9\pi^2 a_k^2}{32\varphi_k(R) \pi_k} \leq \lim_{k \to \infty} \frac{9\pi^2 a_k^3}{32 \pi_k} = 0.
\]

For the second term, we first observe that

\[
C_{\varphi_k}(R) = \int_{R}^{\infty} \frac{1}{r\varphi_k(r)} \, dr \geq \sum_{m=N}^{\infty} \int_{2m\zeta \pi}^{2(m+1)\zeta \pi} \frac{1}{r\varphi_k(r)} \, dr 
\geq \frac{1}{a_k} \sum_{m=N}^{\infty} \int_{2m\zeta \pi}^{2(m+1)\zeta \pi} \frac{1}{2(m+1)\zeta \pi \varphi_k(2(m+1)\zeta \pi)} \, dr 
= \frac{1}{a_k} \sum_{m=N}^{\infty} \frac{1}{(m+1) \varphi_k(2(m+1)\zeta \pi)}.
\]

Since $m + 1 \leq 2m$ and

\[
\frac{\varphi_k(2(m+1)\zeta \pi)}{\varphi_k((2m + 1)\zeta \pi)} \leq a_k \left( \frac{2m + 2}{2m + 1} \right) \pi_k \leq a_k \left( \frac{4}{3} \right)^2,
\]

...
we have
\[ C_{\varphi_k}(R) \geq \frac{9}{16a_k^2} \sum_{m=N}^{\infty} \frac{1}{m\varphi_k((2m+1)\zeta\pi)}, \]
which yields that
\[ \frac{1}{C_{\varphi_k}(R)} \sum_{m=N}^{\infty} \frac{1}{m\varphi_k((2m+1)\zeta\pi)} \leq \frac{16a_k^2}{9}. \]
Thus, we obtain
\[ 0 \leq \lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \sum_{m=N}^{\infty} \frac{1}{m\varphi_k((2m+1)\zeta\pi)} \leq \lim_{k \to \infty} 2a_k(a_k^2 - 1) = 0, \]
and this proves the claim. By \((2.11)\) and \((2.15)\), we have
\[ \lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \int_{|r| \geq \hat{R}} \frac{1 - \cos \frac{2}{\zeta\pi}}{2} \frac{dr}{\varphi_k(|r|)} = 2. \]
Therefore, we conclude that
\[ \lim_{k \to \infty} C_{\varphi_k} C_{\varphi_k}(R) = \lim_{k \to \infty} \left( \int_{\mathbb{R}^n-1} \int_{\mathbb{R}} \frac{1 - \cos \frac{2}{\zeta\pi}}{2} \frac{dy'}{\varphi_k(|r|)} \frac{dr}{\zeta^n} \right)^{-1} \frac{1}{C_{\varphi_k}(R)} \]
\[ = \left( \int_{\mathbb{R}^n-1} \lim_{k \to \infty} \frac{1}{C_{\varphi_k}(R)} \int_{\mathbb{R}} \frac{1 - \cos \frac{2}{\zeta\pi}}{2} \frac{dy'}{\zeta^n} \right)^{-1} \]
\[ = \left( \int_{\mathbb{R}^n-1} \frac{2}{\zeta^n} \frac{dy'}{\zeta^n} \right)^{-1} = \frac{1}{n\omega_n}, \]
which finishes the proof. See [3, Corollary 4.2] for the last equality. \(\square\)

We next prove Proposition 2.2 using Lemma 2.1.

**Proof of Proposition 2.2.** Assume first that \( \lim_{k \to \infty} \sigma_k = \lim_{k \to \infty} \overline{\sigma}_k = 2 \). In this case, we have no contribution outside the unit ball. Indeed, using inequality \((2.7)\) we have
\[ -\int_{B_1^c} \frac{\Delta(u, x, y)}{|y|^{n+2} \varphi_k(|y|)} \, dy \leq 4n\omega_n \|u\|_{L^\infty(\mathbb{R}^n)} \int_1^\infty \frac{dr}{r\varphi_k(r)} \leq \frac{4n\omega_n a_k}{\sigma_k \varphi_k(1)} \|u\|_{L^\infty(\mathbb{R}^n)}. \]
Hence, using the inequality \((2.6)\) and the limit \((2.4)\) we obtain
\[ -\frac{1}{2} \int_{B_1^c} \frac{\Delta(u, x, y)}{|y|^{n+2} \varphi_k(|y|)} \, dy = \frac{1}{2\sigma_k \varphi_k(1)} C_{\varphi_k} C_{\varphi_k}(1) \int_{B_1^c} \frac{\Delta(u, x, y)}{|y|^{n+2} \varphi_k(|y|)} \, dy \]
\[ \leq 2n\omega_n a_k^2 \frac{2 - \sigma_k}{\sigma_k} C_{\varphi_k} C_{\varphi_k}(1) \|u\|_{L^\infty(\mathbb{R}^n)} \to 0. \]
as \( k \to \infty \). On the other hand, we have
\[
\left| \int_{B_1} \frac{\delta(u, x, y) - y \cdot D^2 u(x) y}{|y|^n \varphi_k(|y|)} \, dy \right| \leq \|u\|_{C^3(\mathbb{R}^n)} \int_{B_1} \frac{|y|^3}{|y|^n \varphi_k(|y|)} \, dy \\
\quad \leq n \omega_n \|u\|_{C^3(\mathbb{R}^n)} \int_0^1 \frac{r^2}{\varphi_k(r)} \, dr \\
\quad \leq \frac{n \omega_n a_k}{\varphi_k(1)(3 - \sigma_k)} \|u\|_{C^3(\mathbb{R}^n)},
\]
and this implies that
\[
\lim_{k \to \infty} \frac{1}{2} C_{\varphi_k} \int_{B_1} \frac{\delta(u, x, y)}{|y|^n \varphi_k(|y|)} \, dy = \lim_{k \to \infty} -\frac{1}{2} C_{\varphi_k} \int_{B_1} y \cdot D^2 u(x) y \frac{1}{|y|^n \varphi_k(|y|)} \, dy.
\]
Note that if \( i \neq j \) then
\[
\int_{B_1} \frac{D_{ij} u(x) y_i y_j}{|y|^n \varphi_k(|y|)} \, dy = -\int_{B_1} \frac{D_{ij} u(x) \tilde{y}_i \tilde{y}_j}{|y|^n \varphi_k(|y|)} \, d\tilde{y},
\]
where \( \tilde{y}_j = -y_j \) and \( \tilde{y}_k = \tilde{y}_k \) for any \( k \neq j \), and hence
\[
\int_{B_1} \frac{D_{ij} u(x) y_i y_j}{|y|^n \varphi_k(|y|)} \, dy = 0.
\]
Thus, we have
\[
\int_{B_1} y \cdot D^2 u(x) y \frac{1}{|y|^n \varphi_k(|y|)} \, dy = \frac{n}{n} \int_{B_1} \frac{|y|^2}{|y|^n \varphi_k(|y|)} \, dy = \omega_n \Delta u(x) \int_0^1 \frac{r}{\varphi_k(r)} \, dr.
\]
Using (2.4) we conclude that
\[
\lim_{k \to \infty} -L_k u(x) = \left( \lim_{k \to \infty} \frac{\omega_n}{2} C_{\varphi_{\frac{1}{2}}} C_{\varphi_{\frac{1}{2}}} (1) \right) (-\Delta u)(x) = -\Delta u(x).
\]
Next, we assume that \( \lim_{k \to \infty} \mathcal{G}_k = \lim_{k \to \infty} \mathcal{G}_k = 0 \). Fix \( x \in \mathbb{R}^n \) and let \( R_0 > 0 \) be such that \( \text{supp} \, u \subset B_{R_0} \) and set \( R = R_0 + |x| + 1 \). Then using the inequality (2.6) we have
\[
\left| \int_{B_R} \frac{\delta(u, x, y)}{|y|^n \varphi_k(|y|)} \, dy \right| \leq n \omega_n \|u\|_{C^2(\mathbb{R}^n)} \int_0^R \frac{r}{\varphi_k(r)} \, dr \leq \frac{n \omega_n a_k}{2 - \sigma_k} R^2 \|u\|_{C^2(\mathbb{R}^n)}.
\]
Hence, using the inequality (2.7) and the limit (2.5) we obtain
\[
\left| -\frac{1}{2} C_{\varphi_k} \int_{B_R} \frac{\delta(u, x, y)}{|y|^n \varphi_k(|y|)} \, dy \right| \leq \frac{1}{2} C_{\varphi_k} C_{\varphi_k} (R) \left| \int_{B_R} \frac{\delta(u, x, y)}{|y|^n \varphi_k(|y|)} \, dy \right| \\
\quad \leq \frac{n \omega_n a_k^2 R^2}{2 - \sigma_k} C_{\varphi_k} C_{\varphi_k} (R) \|u\|_{C^2(\mathbb{R}^n)} \to 0
\]
as \( k \to \infty \). On the other hand, if \( |y| \geq R \), then \( |x \pm y| > R_0 \) and consequently \( u(x \pm y) = 0 \). Thus, we have
\[
-\frac{1}{2} C_{\varphi_k} \int_{B_R} \frac{\delta(u, x, y)}{|y|^n \varphi_k(|y|)} \, dy = n \omega_n C_{\varphi_{\frac{1}{2}}} C_{\varphi_{\frac{1}{2}}} (R) u(x).
\]
Therefore, using (2.5) we conclude that
\[
\lim_{k \to \infty} L_k u(x) = n \omega_n \lim_{k \to \infty} C_{\varphi_k} \phi_k(R) u(x) = u(x),
\]
which finishes the proof.

The Lemma 2.1 concerns about the limiting behavior of a sequence of constants \(C_{\varphi_k}\), and does not provide an information about a fixed constant \(C_{\varphi}\). To obtain uniform regularity estimates, we need uniform bounds for the constant \(C_{\varphi}\) and these bounds will play an important role in the uniform estimates in the sequel.

**Lemma 2.4.** There exist constants \(c_1, c_2 > 0\), depending only on \(n\), such that for any \(R > 0\)
\[
(2.16) \quad \frac{c_1}{C(R) + \mathcal{C}(R)} \leq C_{\varphi} \leq \frac{ac_2}{C(1) + \mathcal{C}(1)}.
\]

**Proof.** For the lower bound, notice that from the inequality \(1 - \cos \frac{r}{\zeta} \leq \frac{r^2}{2} \zeta^2\) we have
\[
\int_{\mathbb{R}} \frac{1 - \cos \frac{r}{\zeta}}{|r| |\varphi(|r|)|} dr = 2 \int_{0}^{R} \frac{1 - \cos \frac{r}{\zeta}}{r \varphi(r)} \frac{d y'}{\zeta^2} dr + 2 \int_{R}^{\infty} \frac{1 - \cos \frac{r}{\zeta}}{r \varphi(r)} \frac{d y'}{\zeta^2} dr \leq \frac{1}{\zeta^2} C(R) + 4 \mathcal{C}(R).
\]
Thus, it follows easily that
\[
C_{\varphi} = \left( \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{1 - \cos \frac{r}{\zeta}}{|r| |\varphi(|r|)|} \frac{d y'}{\zeta^2} \right)^{-1} \geq \left( \int_{\mathbb{R}^{n-1}} \frac{1}{\zeta^2} C(R) + 4 \mathcal{C}(R) \frac{d y'}{\zeta^2} \right)^{-1} \geq \frac{c_1(n)}{C(R) + \mathcal{C}(R)}.
\]

For the upper bound, we first note that \(1 - \cos \frac{r}{\zeta} \geq \frac{r^2}{2} \zeta^2\) for \(|r| \leq 1\) since \(\zeta = (1 + |y'|)^{1/2} \geq 1\). Thus we have
\[
(2.17) \quad \int_{\mathbb{R}} \frac{1 - \cos \frac{r}{\zeta}}{|r| |\varphi(|r|)|} dr \geq \frac{1}{2 \zeta^2} C + 2 \int_{1}^{\infty} \frac{1 - \cos \frac{r}{\zeta}}{r \varphi(r)} dr.
\]

We next see that
\[
(2.18) \quad \sum_{m=0}^{\infty} \int_{2m \zeta \pi + \frac{\zeta \pi}{2}}^{2m \zeta \pi + \frac{3\zeta \pi}{2}} \frac{d r}{r \varphi(r)} \leq \int_{1}^{\infty} \frac{1 - \cos \frac{r}{\zeta}}{r \varphi(r)} dr
\]
and
\[
(2.19) \quad \sum_{m=0}^{\infty} \int_{2m \zeta \pi + \frac{\zeta \pi}{2}}^{2m \zeta \pi + \frac{3\zeta \pi}{2}} \frac{d r}{r \varphi(r)} = a \sum_{m=0}^{\infty} \int_{2m \zeta \pi + \frac{\zeta \pi}{2}}^{2m \zeta \pi + \frac{3\zeta \pi}{2}} \frac{d r}{r \varphi(r)} \leq a \int_{1}^{\infty} \frac{1 - \cos \frac{r}{\zeta}}{r \varphi(r)} dr.
\]
If \( r \geq 1 + \zeta \pi \), then \( r - \zeta \pi \geq \frac{1}{1+\zeta \pi} \cdot r \). Thus we have

\[
\int_{1}^{\frac{r}{1+\zeta \pi}} \frac{dr}{r \varphi(r)} = \int_{1+\zeta \pi}^{\frac{r}{1+\zeta \pi}} \frac{dr}{(r - \zeta \pi) \varphi(r - \zeta \pi)}
\]

\[
\leq \int_{1+\zeta \pi}^{\frac{r}{1+\zeta \pi}} \frac{a (1 + \zeta \pi)}{r \varphi(r)} \left( \frac{r}{r - \zeta \pi} \right)^{\pi} dr
\]

\[
\leq a (1 + \zeta \pi) \int_{1+\zeta \pi}^{\frac{r}{1+\zeta \pi}} \frac{dr}{r \varphi(r)} \leq a (1 + \zeta \pi)^{1+\pi} \int_{1}^{\infty} \frac{1 - \cos \frac{\pi}{r}}{r \varphi(r)} dr.
\]

Combining (2.18)-(2.20), we have

\[
\int_{1}^{\infty} \frac{dr}{r \varphi(r)} \leq \left( 1 + a + a (1 + \zeta \pi)^{3} \right) \int_{1}^{\infty} \frac{1 - \cos \frac{\pi}{r}}{r \varphi(r)} dr \leq C a \zeta \int_{1}^{\infty} \frac{1 - \cos \frac{\pi}{r}}{r \varphi(r)} dr.
\]

Putting this inequality into (2.17), we have

\[
\int_{\mathbb{R}} \frac{1 - \cos \frac{\pi}{|r|}}{|r| \varphi(|r|)} dr \geq \frac{1}{2 \zeta^2 C} + \frac{C}{a \zeta^3 C}.
\]

Therefore, we conclude that

\[
C \varphi \leq \left( \int_{\mathbb{R}^{n-1}} \left( \frac{1}{2 \zeta^2 C} + \frac{C}{a \zeta^3 C} \right) \frac{dy'}{\zeta^n} \right)^{-1} \leq \frac{ac_\zeta(n)}{C + C},
\]

which finishes the proof. \(\square\)

**3. Viscosity Solutions**

In this section, we give a definition of viscosity solutions for integro-differential equations and a notion of the ellipticity as in [2]. We refer to [1] for the local equations. We begin with the notion of \(C^{1,1}\) at the point.

**Definition 3.1.** A function \( \psi \) is said to be \( C^{1,1} \) at the point \( x \), and we denote by \( \psi \in C^{1,1}(x) \), if there is a vector \( v \in \mathbb{R}^n \) and a number \( M > 0 \) such that

\[
|\psi(x + y) - \psi(x) - v \cdot y| \leq M|y|^2 \quad \text{for } |y| \text{ small enough.}
\]

We say that a function is \( C^{1,1} \) in a set \( \Omega \) if the previous definition holds at every point \( x \in \Omega \) with a uniform constant \( M \).

We recall the definition of viscosity solutions for integro-differential equations.

**Definition 3.2.** A bounded function \( u : \mathbb{R}^n \to \mathbb{R} \) which is upper (lower) semicontinuous in \( \Omega \) is said to be a **viscosity subsolution** (**viscosity supersolution**) to \( Iu = f \), and we write \( Iu \geq f \) (\( Iu \leq f \)), when the following holds: if a \( C^2 \)-function \( \psi \) touches \( u \) from above (below) at \( x \in \Omega \) in a small neighborhood \( N \) of \( x \), i.e., \( \psi(x) = u(x) \) and \( \psi > u \) in \( N \setminus \{x\} \), then the function \( v \) defined by

\[
v := \begin{cases} 
\psi & \text{in } N, \\
u & \text{in } \mathbb{R}^n \setminus N,
\end{cases}
\]

satisfies \( Iv(x) \geq f(x) \) (\( Iv(x) \leq f(x) \)). A function \( u \) is said to be a **viscosity solution** if \( u \) is both a viscosity subsolution and a viscosity supersolution.
We can also give a definition of viscosity solutions to unbounded functions, but we will focus on bounded functions in this paper.

We next consider a collection $\mathcal{L}$ of linear integro-differential operators of the form \((1.2)\) with kernels satisfying \((1.3)\). The maximal operator and the minimal operator with respect to $\mathcal{L}$ are defined as

$$M^+_{\mathcal{L}} u = \sup_{L \in \mathcal{L}} Lu(x) \quad \text{and} \quad M^-_{\mathcal{L}} u = \inf_{L \in \mathcal{L}} Lu(x).$$

One example that we will use is the class $\mathcal{L}_0$. Recall that $\mathcal{L}_0$ is the class with kernels satisfying \((1.4)\) additionally. In this case the maximal and the minimal operators are given by

$$M^+_{\mathcal{L}_0} u(x) = C\varphi \int_{\mathbb{R}^n} \frac{\Lambda \delta(u, x, y)^+ - \lambda \delta(u, x, y)^-}{|y|^n \varphi(|y|)} \, dy \quad \text{and} \quad M^-_{\mathcal{L}_0} u(x) = C\varphi \int_{\mathbb{R}^n} \frac{\lambda \delta(u, x, y)^+ - \Lambda \delta(u, x, y)^-}{|y|^n \varphi(|y|)} \, dy.$$

Using these extremal operators, we give a general definition of ellipticity for nonlocal operators.

**Definition 3.3.** Let $\mathcal{L}$ be a class of linear integro-differential operators. An elliptic operator $I$ with respect to $\mathcal{L}$ is an operator with the following properties:

(i) If $u$ is bounded in $\mathbb{R}^n$ and is of $C^{1,1}(x)$, then $Iu(x)$ is defined classically.

(ii) If $u$ is bounded in $\mathbb{R}^n$ and is $C^2$ in some open set $\Omega$, then $Iu(x)$ is a continuous function in $\Omega$.

(iii) If $u$ and $v$ are bounded in $\mathbb{R}^n$ and are of $C^{1,1}(x)$, then

$$M^-_{\mathcal{L}_0} (u - v)(x) \leq Iu(x) - Iv(x) \leq M^+_{\mathcal{L}_0} (u - v)(x).$$

For the nonlinear integro-differential operators of the form \((1.7)\) we have the following properties: the proof can be found in \([2\), Section 3 and 4].

**Lemma 3.4.** Let $I$ be the operator of the form \((1.7)\). Then $I$ is an elliptic operator with respect to $\mathcal{L}_0$. Moreover, if $Iu \geq f$ in $\Omega$ in the viscosity sense and a function $\psi \in C^{1,1}(x)$ touches $u$ from above at $x$, then $Iu(x)$ is defined in the classical sense and $Iu(x) \geq f(x)$.

In \([2\] stability properties of viscosity solutions to the elliptic integro-differential equations with respect to the natural limits for lower-semicontinuous functions were proved. This type of limit is usually called a $\Gamma$-limit.

**Definition 3.5.** A sequence of lower-semicontinuous function $u_k$ $\Gamma$-converge to $u$ in a set $\Omega$ if the followings hold:

(i) For every sequence $x_k \to x$ in $\Omega$,

$$\liminf_{k \to \infty} u_k(x_k) \geq u(x).$$

(ii) For every $x \in \Omega$, there is a sequence $x_k \to x$ in $\Omega$ such that

$$\limsup_{k \to \infty} u_k(x_k) = u(x).$$

Note that a uniformly convergent sequence $u_k$ also converges in the $\Gamma$ sense. We refer to \([2\) for the proof of the following lemma.

**Lemma 3.6.** Let $I$ be elliptic in the sense of Definition 3.3 and $u_k$ be a sequence of functions that are uniformly bounded in $\mathbb{R}^n$ such that
(i) \( Iu_k \leq f_k \) in \( \Omega \) in the viscosity sense,
(ii) \( u_k \to u \) in the \( \Gamma \)-sense in \( \Omega \),
(iii) \( u_k \to u \) a.e. in \( \mathbb{R}^n \),
(iv) \( f_k \to f \) locally uniformly in \( \Omega \) for some continuous function \( f \).

Then \( Iu \leq f \) in \( \Omega \) in the viscosity sense.

**Corollary 3.7.** Let \( I \) be elliptic in the sense of Definition 3.3 and \( u_k \) be a sequence of functions that are uniformly bounded in \( \mathbb{R}^n \) such that

(i) \( Iu_k = f_k \) in \( \Omega \) in the viscosity sense,
(ii) \( u_k \to u \) locally uniformly in \( \Omega \),
(iii) \( u_k \to u \) a.e. in \( \mathbb{R}^n \),
(iv) \( f_k \to f \) locally uniformly in \( \Omega \) for some continuous function \( f \).

Then \( Iu = f \) in \( \Omega \) in the viscosity sense.

**Lemma 3.8.** Let \( I \) be elliptic in the sense of Definition 3.3. Let \( u \) and \( v \) be bounded functions in \( \mathbb{R}^n \) such that \( Iu \geq f \) and \( Iv \leq f \) in \( \Omega \) in the viscosity sense for continuous functions \( f \) and \( g \). Then

\[ \mathcal{M}_L^+(u - v) \geq f - g \]

in \( \Omega \) in the viscosity sense.

We check the Assumption 5.1 in [2] is true when \( I \) is an elliptic operator with respect to \( \mathcal{L}_0 \) to prove the comparison principle. See also [2, Lemma 5.10].

**Theorem 3.9** (Comparison principle). Let \( I \) be an elliptic operator with respect to \( \mathcal{L}_0 \) in the sense of Definition 3.3. Let \( \Omega \in \mathbb{R}^n \) be a bounded open set. If \( u \) and \( v \) are bounded functions in \( \mathbb{R}^n \) such that \( Iu \geq f \) and \( Iv \leq f \) in \( \Omega \) in the viscosity sense for some continuous function \( f \), and \( u \leq v \) in \( \mathbb{R}^n \setminus \Omega \), then \( u \leq v \) in \( \Omega \).

**Proof.** The proof is the same as one for Theorem 5.2 in [2] if the Assumption 5.1 in [2] is provided. We claim that for every \( R \geq 4 \), there exists a constant \( \delta = \delta(R) > 0 \) such that \( Lw_R > \delta \) in \( B_R \) for any operator \( L \in \mathcal{L}_0 \), where \( w_R(x) = \min\{1, \frac{|x|^2}{4R^2}\} \).

Indeed, for \( x \in B_R \) we have

\[
\delta(w_R, x, y) = \frac{|x + y|^2}{4R^2} + \frac{|x - y|^2}{4R^2} - \frac{2|x|^2}{4R^2} = \frac{|y|^2}{2R^2} \quad \text{if} \quad x \pm y \in B_{2R}
\]

and

\[
\delta(w_R, x, y) \geq 1 - \frac{|x|^2}{2R^2} \geq 0 \quad \text{if} \quad x + y \notin B_{2R} \text{ or } x - y \notin B_{2R}.
\]

Thus for any operator \( L \in \mathcal{L}_0 \) we obtain

\[
Lw_R(x) \geq C_R \lambda \int_{B_R} \frac{|y|^2}{2R^2 |y|^\gamma \varphi(|y|)} \, dy := \delta(R) > 0,
\]

which proves the claim. \( \square \)

4. **Regularity Results**

In this section we prove Harnack inequality and Hölder regularity estimates for viscosity solutions of fully nonlinear elliptic integro-differential equations. From now on we will consider the class \( \mathcal{L}_0 \).
4.1. Aleksandrov-Bakelman-Pucci (ABP) Estimates. We start this section with an ABP estimate which generalizes [2, Theorem 8.7]. It is a fundamental tool in the proof of the Harnack inequality.

For a function $u$ that is not positive outside the ball $B_R$ we consider the concave envelope $\Gamma$ of $u^+$ in $B_{3R}$, which is defined by

$$\Gamma(x) := \begin{cases} \min \{p(x) : p \text{ is a plane such that } p \geq u^+ \text{ in } B_{3R}\} & \text{in } B_{3R}, \\ 0 & \text{in } \mathbb{R}^n \setminus B_{3R}. \end{cases}$$

We will focus on the contact set $\{u = \Gamma\} \cap B_R$ in the following lemmas.

**Lemma 4.1.** Assume that $M_{\mathcal{L}_u}^+ u \geq -f$ in $B_R$ in the viscosity sense and that $u \leq 0$ in $\mathbb{R}^n \setminus B_R$. Let $\Gamma$ be the concave envelope of $u^+$ in $B_{3R}$. Let $\rho_0 = 2^{-8n-1}$ and $r_k = \rho_02^{-1/(2-\omega)-k}R$. Then there exists a uniform constant $C > 0$, depending only on $n, \lambda$, and $\omega$, such that for each $x \in \{u = \Gamma\}$ and $M > 0$, we find $k \geq 0$ satisfying

$$|A_k| \leq C \frac{f(x)}{M} |B_{r_k}(x) \setminus B_{r_{k+1}}(x)|,$$

where

$$A_k = \left\{ y \in B_{r_k}(x) \setminus B_{r_{k+1}}(x) : u(y) < u(x) + (y - x) \cdot \nabla \Gamma(x) - M \frac{C+C}{C(r_0)} r_k^2 \right\}.$$

Here $\nabla \Gamma$ stands for an element of the superdifferential of $\Gamma$ at $x$.

**Proof.** Let $x$ be a point such that $u(x) = \Gamma(x) > 0$. By Lemma 3.4, $M_{\mathcal{L}_u}^+ u(x)$ is defined classically and $M_{\mathcal{L}_u}^+ u(x) \geq -f(x)$. Note that if $x \pm y \in B_{3R}$ then $\delta(u, x, y) \leq 0$ since $\Gamma$ is concave and lies above $u$. Moreover, if either $x + y \notin B_{3R}$ or $x - y \notin B_{3R}$ then $x \pm y \notin B_R$, which implies $u(x + y) \leq 0$ and $u(x - y) \leq 0$. In any case we have $\delta(u, x, y) \leq 0$ and hence

$$-f(x) \leq M_{\mathcal{L}_u}^+ u(x) = C_\varphi \int_{\mathbb{R}^n} -\lambda \delta^-(u, x, y) \frac{\delta^-(u, x, y)}{|y|^n \varphi(|y|)} \, dy.$$

We split the integral as

$$f(x) \geq C_\varphi \lambda \sum_{k=0}^{\infty} \int_{A_k-x} \delta^-(u, x, y) \frac{\delta^-(u, x, y)}{|y|^n \varphi(|y|)} \, dy.$$

If $y \in A_k - x$, then we have $x \pm y \in B_{3R}$ and

$$\delta(u, x, y) < \left( u(x) + y \cdot \nabla \Gamma(x) - M \frac{C+C}{C(r_0)} r_k^2 \right) + (\Gamma(x) - y \cdot \nabla \Gamma(x)) - 2u(x)$$

$$= -M \frac{C+C}{C(r_0)} r_k^2,$$

which yields

$$f(x) \geq C_\varphi \lambda \sum_{k=0}^{\infty} \int_{A_k-x} \frac{M r_k^2}{k \varphi(r_k)} |A_k|$$

with the help of the weak scaling condition (1.5).
Suppose that we cannot find $k \in \mathbb{N} \cup \{0\}$ satisfying (4.1) with some constant $C > 0$ which will be chosen later. Then we have

$$f(x) > C \varphi \lambda a^{-1} \frac{C + \overline{C}}{C(r_0)} \sum_{k=0}^{\infty} \frac{M r_k^2}{r_k^2 \varphi(r_k)} C \frac{f(x)}{M} |R_k|$$

$$= \frac{3 \omega_n \lambda}{4a} C \varphi \frac{C + \overline{C}}{C(r_0)} \sum_{k=0}^{\infty} \frac{r_k^2}{\varphi(r_k)} f(x).$$

We use the weak scaling condition to have

$$C(r_0) = \sum_{k=0}^{\infty} \int_{r_{k+1}}^{r_k} \frac{r}{\varphi(r)} \, dr \leq \sum_{k=0}^{\infty} \int_{r_{k+1}}^{r_k} \frac{a r_k^2}{\varphi(r_k)} \left( \frac{r_k}{r} \right)^n \, dr$$

$$\leq \sum_{k=0}^{\infty} \frac{a r_k (r_k - r_{k+1})}{\varphi(r_k)} \left( \frac{r_k}{r_{k+1}} \right)^2 = 4a \sum_{k=0}^{\infty} \frac{r_k^2}{\varphi(r_k)}.$$

Using the inequality (2.16) we arrive at

$$f(x) > \frac{3 \omega_n \lambda}{16a^2} c_1 C f(x),$$

which is a contradiction if we have taken $C \geq \frac{16a^2}{3 \omega_n \lambda c_1}$. \hfill \Box

We observe from (4.2) that $f(x)$ is positive for $x \in \{u = \Gamma\}$.

**Lemma 4.2.** Under the same assumptions as in Lemma 4.1, there exist uniform constants $\varepsilon_n \in (0, 1)$ and $C = C(n, \lambda, a) > 0$ such that for each $x \in \{u = \Gamma\}$, we find some $k \geq 0$ satisfying

$$\left| \left\{ y \in B_{r_k}(x) \setminus B_{r_{k+1}}(x) : u(y) < u(x) + (y - x) \cdot \nabla \Gamma(x) - C \frac{C + \overline{C}}{C(r_0)} f(x) r_k^2 \right\} \right| |B_{r_k}(x) \setminus B_{r_{k+1}}(x)| \leq \varepsilon_n$$

and

$$|\nabla \Gamma(B_{r_k/4}(x))| \leq C \left( \frac{C + \overline{C}}{C(r_0)} f(x) \right)^n |B_{r_k/4}(x)|.$$

For the proof of Lemma 4.2 we refer to [2, Lemma 8.4 and Corollary 8.5]. We next obtain a nonlocal ABP estimate.

**Theorem 4.3** (ABP estimate). Under the same assumptions as in Lemma 4.1, there is a finite, disjoint family of open cubes $Q_j$ with diameters $d_j \leq r_0$ such that the followings hold:

(i) $\{u = \Gamma\} \cap \overline{Q_j} \neq \emptyset$ for any $Q_j$,

(ii) $\{u = \Gamma\} \subset \bigcup_{j=1}^{m} \overline{Q_j}$,

(iii) $|\nabla \Gamma(\overline{Q_j})| \leq C \left( \frac{C + \overline{C}}{C(r_0)} \max_{\overline{Q_j}} f \right)^n |Q_j|$,

(iv) $\left| \left\{ y \in 32 \sqrt{n}Q_j : u(y) > \Gamma(y) - C \frac{C + \overline{C}}{C(r_0)} \left( \max_{\overline{Q_j}} f \right) d_j^2 \right\} \right| \geq \mu_n |Q_j|$ for $\mu_n = 1 - \varepsilon_n \in (0, 1)$, where $C > 0$ is a uniform constant depending only on $n, \lambda,$ and $a$. 


The proof of Theorem 4.3 can be found in [2, Theorem 3.4]. It is important to note that when \( \sigma \) is close to 2, the upper bound for the diameters \( r_0 = \rho_0^{2-1/(2-\alpha)} \) becomes very small so that Theorem 4.3 generalizes the classical ABP estimate.

### 4.2. A Barrier Function

This section is devoted to construct a barrier function at every scale to find scaling invariant uniform estimates.

#### Lemma 4.4

Let \( \kappa_1 \in (0, 1), \sigma_0 \in (0, 2) \), and assume \( \sigma \geq \sigma_0 \). There exist uniform constants \( p = p(n, \lambda, \Lambda) > n + 1 \) and \( \kappa_0 = \kappa_0(n, \lambda, \Lambda, a, \sigma_0) \in (0, \kappa_1/8) \) such that the function \( \Phi(x) = \min \{ \kappa_0 R^{-p}, |x|^{-p} \} \) satisfies

\[
\mathcal{M}_{\kappa_0} \Phi_1(x) \geq 0
\]

for \( x \in B_R \setminus B_{\kappa_1 R} \).

**Proof.** Without loss of generality we may assume that \( x = R_0 e_1 \) for \( \kappa_1 R \leq R_0 < R \). We need to compute

\[
\mathcal{M}_{\kappa_0} \Phi_1(x) = C \int_{\mathbb{R}^n} \frac{\lambda \delta^+ (\Phi_1, x, y) - \Lambda \delta^- (\Phi_1, x, y)}{|y|^n \varphi(|y|)} \, dy
\]

\[
= C \int_{\mathbb{R}^n} \frac{\lambda \delta^+}{|y|^n \varphi(|y|)} \, dy + C \int_{B_{R_0/2}} \frac{\lambda \delta^+ - \Lambda \delta^-}{|y|^n \varphi(|y|)} \, dy
\]

\[
+ C \int_{\mathbb{R}^n \setminus B_{R_0/2}} \frac{\lambda \delta^+ - \Lambda \delta^-}{|y|^n \varphi(|y|)} \, dy =: C \varphi (I_1 + I_2 + I_3).
\]

For \( |y| \leq R_0/2 \), we have

\[
\delta(\Phi_1, x, y) = R_0^{-p} \left( \frac{x}{R_0} + \frac{y}{R_0} \right)^{-p} + \left( \frac{x}{R_0} - \frac{y}{R_0} \right)^{-p} - 2
\]

\[
\geq p R_0^{-p} \left( - \left( \frac{y}{R_0} \right)^2 + (p + 2) \frac{y_1^2}{R_0^2} - \frac{1}{2} (p + 2)(p + 4) \frac{y_1^2 |y|^2}{R_0^4} \right).
\]

We choose \( p = p(n, \lambda, \Lambda) > n + 1 \) large enough so that

\[
(p + 2) \frac{\lambda}{2} \int_{\partial B_1} y_1^2 \, d\sigma(y) - \Lambda |\partial B_1| \geq 0.
\]

Then we obtain

\[
I_2 \geq p R_0^{-p} \int_{B_{R_0/2}} \left( \frac{\lambda}{2} (p + 2) \frac{y_1^2}{R_0^2} - \Lambda \left( \left( \frac{y_1^2}{R_0^2} + \frac{(p + 2)(p + 4)}{2 R_0^4} \frac{y_1^2 |y|^2}{R_0^4} \right) \right) \right) \frac{1}{|y|^n \varphi(|y|)} \, dy
\]

\[
\geq -p R_0^{-p} \Lambda (p + 2)(p + 4) \frac{c_n}{2 R_0^4} \int_{0}^{R_0/2} \frac{r^3 \varphi(r)}{\varphi(r)} \, dr,
\]

where \( c_n = \int_{\partial B_1} y_1^2 \, d\sigma(y) > 0 \) is a constant depending only on \( n \). Using the weak scaling condition (1.5), we have

\[
\int_{0}^{R_0/2} \frac{r^3 \varphi(r)}{\varphi(r)} \, dr \leq \frac{a}{4 - 16 \varphi(R_0/2)} \leq \frac{a R_0^4}{32 \varphi(R_0/2)}
\]

and hence

\[
I_2 \geq - \frac{\Lambda p (p + 2)(p + 4)c_n a}{64 R_0^p \varphi(R_0/2)}.
\]
On the other hand, using the inequality (2.7) we estimate

\[(4.4) \quad I_3 \geq - \int_{B_{R_0/2}} 2\lambda R_0^{-p} |y|^n \varphi(|y|) \, dy = -2n\omega_n \lambda R_0^{-p} \mathcal{C}(R_0/2) \geq -\frac{2n\omega_n \lambda}{a\sigma R_0^p \varphi(R_0/2)}.\]

Now we will make \( I_1 \) sufficiently large by selecting \( \kappa_0 > 0 \) small. We have

\[
I_1 \geq \frac{\lambda}{4} \int_{B_{R_0/4}(x)} \frac{|x-y|^{-p} - 2R_0^{-p}}{|y|^n \varphi(|y|)} \, dy \geq \frac{\lambda}{4} \int_{B_{R_0/4}(x) \setminus B_{\kappa_0 R}(x)} \frac{|x-y|^{-p}}{|y|^n \varphi(|y|)} \, dy
\]

\[
= \frac{\lambda}{4} \int_{B_{R_0/4}(0) \setminus B_{\kappa_0 R}(0)} \frac{|x+z|^n \varphi(|x+z|)}{|z|^p} \, dz
\]

\[
\geq \frac{\lambda n \omega_n}{2^{n+2} R_0^n} \left( \min_{r \in \left[\frac{\kappa_0}{3n}, \frac{1}{3n}\right]} \frac{1}{\varphi(r)} \right) \int_{\kappa_0 R} r^{-p+n-1} \, dr.
\]

If we have taken \( \kappa_0 \in (0, \kappa_1/8) \), then we have

\[
\int_{\kappa_0 R} r^{-p+n-1} \, dr = \frac{(\kappa_0 R)^{-p+n} - (R_0/4)^{-p+n}}{p-n} \geq \frac{(\kappa_0 / \kappa_1)^{p-n} - 4^{p-n}}{p-n} R_0^{-p} \geq \frac{1}{2(p-n) \kappa_0} R_0^{-p}.
\]

We use the weak scaling condition (1.5) to obtain

\[
\min_{r \in \left[\frac{\kappa_0}{3n}, \frac{1}{3n}\right]} \frac{1}{\varphi(r)} \geq \frac{1}{a \varphi(3R_0/2)} \geq \frac{1}{a^2 3^n \varphi(2R_0/2)} \geq \frac{1}{9a^2 \varphi(R_0/2)},
\]

and hence

\[(4.5) \quad I_1 \geq \frac{\lambda n \omega_n}{9 \cdot 2^{n+3} a^2 (p-n)} \kappa_1 \frac{1}{\kappa_0 \varphi(R_0/2)}.\]

Combining (4.3)-(4.5), we have

\[
I_1 + I_2 + I_3 \geq \left( \frac{\lambda n \omega_n}{9 \cdot 2^{n+3} a^2 (p-n)} \kappa_1 \frac{\lambda p(p+2)(p+4)c_n a}{64} + \frac{2\Lambda n \omega_n}{a\sigma_0} \right) \frac{R_0^{-p}}{\varphi(R_0/2)}.
\]

By taking \( \kappa_0 = \kappa_0(n, \lambda, \Lambda, a, \sigma_0) \in (0, \kappa_1/8) \) small enough, we have \( \mathcal{M}_{L_0} \Phi_1(x) \geq 0 \) for \( x \in B_R \setminus B_{\kappa_1 R} \).

4.3. **Power Decay Estimates.** In this section we establish the measure estimates of super-level sets of the viscosity supersolutions to fully nonlinear elliptic integro-differential equations with respect to \( L_0 \) using the ABP estimates and the barrier function constructed in Lemma 4.4. Let \( Q_R = Q_R(0) \) denote a dyadic cube of side \( R \) centered at 0 in the sequel.

**Lemma 4.5.** Assume \( \underline{\varphi} \geq \sigma_0 > 0 \). There exist uniform constants \( \varepsilon_0, \mu_0 \in (0, 1) \) and \( M_0 > 1 \), depending on \( n, \lambda, \Lambda, a \) and \( \sigma_0 \), such that if a nonnegative function \( u \) satisfies \( \inf_{Q_R} u \leq 1 \) and \( M \mathcal{L}_0 u \leq \frac{C(R)}{R^{n+2} \mathcal{L}_0(1)} \varepsilon_0 \) in \( Q_{2R} \) in the viscosity sense, then

\[
\left| \left\{ u \leq M_0 \right\} \cap Q_{\frac{R}{2}} \right| > \mu_0 \left| Q_{\frac{R}{2}} \right|.
\]


Proof. Let $\Phi_1$ be the function in Lemma 4.4 with $\kappa_1 = \rho_0$. Define

$$\Phi(x) := c_0 \begin{cases} P(x) \\ (\kappa_0 R)^p (\Phi_1(x) - \Phi_1(R)) = (\kappa_0 R)^p (|x|^{-p} - R^{-p}) \end{cases}$$

for $x \in B_{\kappa_0 R}$,

$$\Phi(x) = 0$$

for $x \notin B_{\kappa_0 R}$, for $x \in B_R \setminus B_{\kappa_0 R}$,

where $c_0 = \frac{2^p}{\kappa_0^p (|4/3|^p - 1)}$ and $P(x) := -a|x|^2 + b$ with $a = \frac{1}{2}p(\kappa_0 R)^{-2}$ and $b = 1 - \kappa_0^p + \frac{1}{2}p$. Then $\Phi$ is a $C^{1,1}$ function on $B_R$ and $\Phi \ge 2$ in $B_{3R/4}$. If $x \in B_R \setminus B_{\kappa_0 R}$, then

$$\delta(\Phi, x, y) = \Phi(x + y) + \Phi(x - y) - 2\Phi(x) \ge c_0(\kappa_0 R)^p (\Phi_1(x + y) - R^{-p} + \Phi_1(x - y) - R^{-p} - 2\Phi_1(x) + 2R^{-p}) = c_0(\kappa_0 R)^p \delta(\Phi_1, x, y).$$

Thus, we have

$$\mathcal{M}_{\mathcal{L}_0} \Phi(x) = C_\varphi \int_{\mathbb{R}^n} \frac{\lambda \delta^+(\Phi, x, y) - \Lambda \delta^-(\Phi, x, y)}{|y|^n \varphi(|y|)} dy \ge C_\varphi c_0(\kappa_0 R)^p \int_{\mathbb{R}^n} \frac{\lambda \delta^+(\Phi_1, x, y) - \Lambda \delta^-(\Phi_1, x, y)}{|y|^n \varphi(|y|)} dy \ge 0.$$

If $x \in B_{\kappa_0 R}$, then we have $\delta(\Phi, x, y) \ge 0$ and hence $\mathcal{M}_{\mathcal{L}_0} \Phi(x) \ge 0$. Finally, if $x \in B_{\kappa_0 R}$, then we have

$$\mathcal{M}_{\mathcal{L}_0} \Phi(x) \ge -C_\varphi \Lambda \int_{\mathbb{R}^n} \frac{\delta^-(\Phi, x, y)}{|y|^n \varphi(|y|)} dy \ge -C_\varphi \Lambda \int_{B_R} \frac{cR^{-2}|y|^2}{|y|^n \varphi(|y|)} dy - 2bc_0C_\varphi \Lambda \int_{B_{\kappa_0 R}} \frac{1}{|y|^n \varphi(|y|)} dy = -C_\varphi c\Lambda \omega_R R^{-2}C(R) - 2bc_0C_\varphi \Lambda \omega_R C(R)$$

since $D^2\Phi \ge -cR^{-2}I$ a.e. in $B_R$ for some constant $c > 0$. This implies that

$$\mathcal{M}_{\mathcal{L}_0} \Phi \ge -\psi$$

for some function with supp $\psi \in B_{\rho_0 R}$ and a uniform bound

$$\psi \le C_\varphi c\Lambda \omega_R R^{-2}C(R) + 2bc_0C_\varphi \Lambda \omega_R C(R).$$

(4.6)

We now consider the function $v := \Phi - u$. It satisfies that $v \le 0$ outside $B_R$, max$_{B_R} v \ge 1$, and

$$\mathcal{M}_{\mathcal{L}_0}^+ v \ge \mathcal{M}_{\mathcal{L}_0} \Phi - \mathcal{M}_{\mathcal{L}_0}^- u \ge -\psi - \frac{C(R)}{R^2(C + \bar{C})} \epsilon_0$$
in $B_R$. For the concave envelope $\Gamma$ of $u^+$ in $B_{3R}$, by Theorem 4.3, we have
\[
\frac{1}{R} \leq \frac{1}{R} \max_{B_R} v \leq C|\nabla \Gamma(B_R)|^{1/n} \leq C \left( \sum_j |\nabla \Gamma(Q_j)| \right)^{1/n} \\
\leq C \left( \sum_j C \left( \frac{C + C}{C(r_0)} \left( \psi + \frac{C(R)}{R^2(C + C)} \varepsilon_0 \right) \right)^n |Q_j| \right)^{1/n} \\
\leq \frac{C}{R^2} \left( \sum_j \left( \frac{C + C}{C(r_0)} R^2 \psi + \frac{C(R)}{C(r_0)} \varepsilon_0 \right)^n |Q_j| \right)^{1/n} .
\]
Since $\text{supp} \psi \subset B_{\rho_0 R}$ and $\sum_j |Q_j| \leq C|B_R|$, it follows that
\[
\frac{1}{R} \leq \frac{C}{R^2} \left( \sum_{Q_j \cap B_{\rho_0 R} \neq \emptyset} \left( \frac{C(R)}{C(r_0)} \frac{R^2 \psi + C(R)}{C(r_0)} \varepsilon_0 \right)^n |Q_j| \right)^{1/n} + C \frac{C(R)}{R C(r_0)} \varepsilon_0 .
\]
Using (4.6) and (2.16) we have
\[
\frac{1}{R} \leq \frac{C}{R^2} \left( \sum_{Q_j \cap B_{\rho_0 R} \neq \emptyset} \left( \frac{C(R)}{C(r_0)} + \frac{R^2 C(R)}{C(r_0)} \right)^n |Q_j| \right)^{1/n} + C \frac{C(R)}{R C(r_0)} \varepsilon_0 .
\]
We use the inequality (2.8) to obtain
\[
\frac{C(R)}{C(r_0)} \leq 1 + a^2 \left( \frac{R}{r_0} \right)^{2-\sigma} = 1 + \frac{2a^2}{\rho_0^{2-\sigma}} \leq 1 + \frac{2a^2}{\rho_0^\sigma} ,
\]
and use the inequalities (2.6), (2.7) to obtain
\[
\frac{R^2 C(R)}{C(r_0)} \leq R^2 \left( \frac{a}{\varphi(R)\sigma} \right) \left( \frac{r_0^2}{a \varphi(r_0)(2 - \sigma)} \right)^{-1} \\
= a^2 \frac{2 - \sigma}{\sigma} \frac{R^2 \varphi(r_0)}{r_0^2 \varphi(R)} \leq 2a^3 \sigma_0 \left( \frac{R}{r_0} \right)^{2-\sigma} = \frac{4a^3}{\rho_0^{2-\sigma}} .
\]
Therefore, it follows that
\[
1 \leq \frac{C}{R} \left( \sum_{Q_j \cap B_{\rho_0 R} \neq \emptyset} |Q_j| \right)^{1/n} + C \varepsilon_0 .
\]
By taking $\varepsilon_0 > 0$ small, we have
\[
\left| Q_{\frac{R}{2n^2}} \right| \leq C \sum_{Q_j \cap B_{\rho_0 R} \neq \emptyset} |Q_j| ,
\]
for some constant $C > 0$ depending on $n, \lambda, \Lambda, a$, and $\sigma_0$. We now use Theorem 4.3 to obtain
\[
\mu|Q_j| \leq \left| \{ y \in 32\sqrt{n} Q_j : v(y) > \Gamma(y) - C \frac{C + C}{C(r_0)} C \varphi \left( R^{-2} C(R) + C(R) \right) d_y \} \right| \\
\leq \left| \{ y \in 32\sqrt{n} Q_j : u(y) \leq \Phi(y) \} \right| \leq \left| \{ y \in 32\sqrt{n} Q_j : u(y) \leq M_0 \} \right|.
\]
Theorem 4.7 follows.

We have

\[ \left| Q_{\frac{R}{2\sqrt{n}}} \right| \leq C \sum_{j, \sigma \in B_{\rho \cdot a} \neq 0} |Q_j| \leq C \left| \{ u \leq M_0 \} \cap B_{\frac{R}{2\sqrt{n}}} \right| \leq C \left| \{ u \leq M_0 \} \cap Q_{\frac{R}{2\sqrt{n}}} \right| . \]

Thus, we have \( |\{ u \leq M_0 \} \cap Q_{\frac{R}{2\sqrt{n}}} | > \mu_0 |Q_{\frac{R}{2\sqrt{n}}} | \). □

Corollary 4.6. Under the same assumptions as in Lemma 4.5, we have

\[ \left| \{ u > M_0^k \} \cap Q_{\frac{R}{2\sqrt{n}}} \right| \leq (1 - \mu_0)^k |Q_{\frac{R}{2\sqrt{n}}} | \]

for all \( k \in \mathbb{N} \), and hence

\[ \left| \{ u > t \} \cap Q_{\frac{R}{2\sqrt{n}}} \right| \leq CR^a t^{-\varepsilon} \]

for all \( t > 0 \), where \( C \) and \( \varepsilon \) are uniform constants.

By the standard covering argument, we deduce the weak Harnack inequality as follows.

Theorem 4.7 (Weak Harnack inequality). Assume \( \sigma \geq \sigma_0 > 0 \). Let \( u \) be a nonnegative function in \( \mathbb{R}^n \) such that

\[ \mathcal{M}_{\mathcal{L}_0} u \leq C_0 \text{ in } B_{2R} \]

in the viscosity sense. Then we have

\[ \left| \{ u > t \} \cap B_R \right| \leq CR^a \left( u(0) + C_0 R^2 \frac{C + \overline{C}}{C(R)} \right)^{\varepsilon} t^{-\varepsilon} \text{ for all } t > 0, \]

and hence

\[ \left( \int_{B_R} |u|^p \right)^{1/p} \leq C \left( u(0) + C_0 R^2 \frac{C + \overline{C}}{C(R)} \right) , \]

where \( C > 0, \varepsilon > 0, \) and \( p > 0 \) are uniform constants depending only on \( n, \lambda, \Lambda, a, \) and \( \sigma_0 \).

4.4. Harnack Inequality. This section is devoted to the proof of Harnack inequality for fully nonlinear elliptic integro-differential operators with respect to \( \mathcal{L}_0 \), where the constant depends only on \( n, \lambda, \Lambda, a, \) and \( \sigma_0 \).

Theorem 4.8 (Harnack inequality). Assume \( \sigma \geq \sigma_0 > 0 \). Let \( u \in C(B_{2R}) \) be a nonnegative function in \( \mathbb{R}^n \) such that

\[ \mathcal{M}_{\mathcal{L}_0}^- u \leq C_0 \frac{C(R)}{(C + \overline{C}) R^2} \text{ and } \mathcal{M}_{\mathcal{L}_0}^+ u \geq -C_0 \frac{C(R)}{(C + \overline{C}) R^2} \text{ in } B_{2R} \]

in the viscosity sense. Then there exists a uniform constant \( C > 0 \), depending only on \( n, \lambda, \Lambda, a, \) and \( \sigma_0 \), such that

\[ \sup_{B_{R/2}} u \leq C (u(0) + C_0) . \]
Proof. We may assume that $u > 0$, $u(0) \leq 1$, and $C_0 = 1$. Let $\varepsilon > 0$ be the constant as in Theorem 4.7 and let $\gamma = (n + 2)/\varepsilon$. Consider the minimal value of $\alpha > 0$ such that
\[ u(x) \leq h_\alpha(x) := \alpha \left( 1 - \frac{|x|}{R} \right)^{-\gamma} \text{ for all } x \in B_R, \]
so that there exists $x_0 \in B_R$ satisfying $u(x_0) = h_\alpha(x_0)$. It is enough to show that $\alpha$ is uniformly bounded.

Let $d = R - |x_0|$, $r = d/2$, and let $A = \{ u > u(x_0)/2 \}$. Then $u(x_0) = h_\alpha(x_0) = \alpha R^\gamma d^{-\gamma}$. By the weak Harnack inequality, we have

\[ |A \cap B_r| \leq C R^n \left( \frac{4}{u(x_0)} \right)^\varepsilon \leq C \alpha^{-\varepsilon} R^n \left( \frac{d}{R} \right)^\gamma \leq C \alpha^{-\varepsilon} d^n. \]

Since $B_r(x_0) \subset B_R$ and $r = d/2$, we obtain

\[ |\{ u > u(x_0)/2 \} \cap B_r(x_0)| \leq C \alpha^{-\varepsilon} |B_r(x_0)|. \]

We will show that there exists a uniform constant $\theta > 0$ such that $|\{ u < u(x_0)/2 \} \cap B_{\theta r/4}(x_0)| \leq \frac{1}{4} |B_{\theta r/4}|$ for a large constant $\alpha > 1$, which yields that $\alpha > 0$ is uniformly bounded.

We first estimate $|\{ u < u(x_0)/2 \} \cap B_{\theta r}(x_0)|$ for small $\theta > 0$, which will be chosen uniformly later. For every $x \in B_{\theta r}(x_0)$,

\[ u(x) \leq h_\alpha(x) \leq \alpha \left( \frac{d - \theta r}{R} \right)^{-\gamma} = \alpha \left( \frac{d}{R} \right)^{-\gamma} \left( 1 - \theta \right)^{-\gamma} u(x_0). \]

Consider the function
\[ v(x) := \left( 1 - \frac{\theta}{2} \right)^{-\gamma} u(x_0) - u(x). \]

Note that $v$ is nonnegative in $B_{\theta r}(x_0)$. To apply the weak Harnack inequality to $w := u^+$, we compute $\mathcal{M}_{\mathcal{L}_0}^+ w$ in $B_{\theta r}(x_0)$. For $x \in B_{\theta r}(x_0)$,

\begin{align*}
\mathcal{M}_{\mathcal{L}_0}^- w(x) &\leq \mathcal{M}_{\mathcal{L}_0}^- v(x) + \mathcal{M}_{\mathcal{L}_0}^+ v^-(x) \\
&\leq -\mathcal{M}_{\mathcal{L}_0}^- u(x) + C_\varphi \int_{\mathbb{R}^n} \frac{\Lambda v^- (x + y) + \Lambda v^- (x - y)}{|y|^n \varphi(|y|)} \, dy \\
&= \frac{C(R)}{R^2(C + C)} + 2\Lambda C_\varphi \int_{\{v(x+y) < 0\}} \frac{v^- (x + y)}{|y|^n \varphi(|y|)} \, dy \\
&\leq \frac{C(R)}{R^2(C + C)} + 2\Lambda C_\varphi \int_{B^+_{\theta r}(x_0 - x)} \frac{(u(x + y) - (1 - \frac{\theta}{2})^{-\gamma} u(x_0))^+}{|y|^n \varphi(|y|)} \, dy \tag{4.7}
\end{align*}

in the viscosity sense.

Consider the largest number $\beta > 0$ such that
\[ u(x) \geq g_\beta(x) := \beta \left( 1 - \frac{|4x|^2}{R^2} \right)^+, \]
and let $x_1 \in B_R$ be a point such that $u(x_1) = g_\beta(x_1)$. This is possible because we have assumed that $u > 0$ in $B_{2R}$. We observe that $\beta \leq 1$ since $u(0) \leq 1$. We
estimate
\[ C_\varphi \int_{\mathbb{R}^n} \frac{\delta^-(u, x_1, y)}{|y|^n \varphi(|y|)} \, dy \leq C_\varphi \int_{\mathbb{R}^n} \frac{\delta^-(g_\beta, x_1, y)}{|y|^n \varphi(|y|)} \, dy \]
\[ \leq C C_\varphi \left( \int_{B_R} \frac{|y|^2}{R^2} \frac{1}{|y|^n \varphi(|y|)} \, dy + \int_{\mathbb{R}^n \setminus B_R} \frac{dy}{|y|^n \varphi(|y|)} \right) \]
\[ \leq \frac{C}{C + \overline{C}} \left( \frac{C(R)}{R^2} + \overline{C}(R) \right). \]

Since
\[ \frac{C(R)}{R^2(C + \overline{C})} \geq \mathcal{M}_{x_0} u = C_\varphi \int_{\mathbb{R}^n} \frac{\lambda \delta^+(u, x_1, y) - \Lambda \delta^-(u, x_1, y)}{|y|^n \varphi(|y|)} \, dy, \]
we obtain
\[ C_\varphi \int_{\mathbb{R}^n} \frac{\delta^+(u, x_1, y)}{|y|^n \varphi(|y|)} \, dy \leq \frac{C}{C + \overline{C}} \left( \frac{C(R)}{R^2} + \overline{C}(R) \right). \]

Since \( u(x_1) \leq \beta \leq 1 \) and \( u(x - y) > 0 \), we have
\[ C_\varphi \int_{\mathbb{R}^n} \frac{(u(x_1 + y) - 2)^+}{|y|^n \varphi(|y|)} \, dy \leq \frac{C}{C + \overline{C}} \left( \frac{C(R)}{R^2} + \overline{C}(R) \right). \]

If \( u(x_0) \leq 2 \), then \( \alpha = u(x_0)^d \leq 2 \), which gives a uniform bound for \( \alpha \). Assume that \( u(x_0) > 2 \), then we can estimate the second term of (4.7) for \( x \in B_{\theta r/2}(x_0) \) as follows:
\[
\int_{B_{\theta r}(x_0 - x)} \frac{(u(x + y) - (1 - \frac{2}{\theta r})^{-\gamma} u(x_0))^+}{|y|^n \varphi(|y|)} \, dy \\
\leq \int_{B_{\theta r}(x_0 - x)} \frac{(u(x + y) - 2)^+}{|y|^n \varphi(|y|)} \, dy \\
= \int_{B_{\theta r}(x_0 - x)} \frac{(u(x_1 + x + y - x_1) - 2)^+ |x + y - x_1|^n \varphi(|x + y - x_1|)}{|y|^n \varphi(|y|)} \, dy.
\]

We see that for \( x \in B_{\theta r/2}(x_0) \) and \( y \in \mathbb{R} \setminus B_{\theta r}(x_0 - x) \),
\[ \frac{|x + y - x_1|^n}{|y|^n} \leq \left( \frac{|x - x_0| + |x_0| + |x_1|}{|y|} + 1 \right)^n \leq \left( \frac{CR}{\theta r} + 1 \right)^n \leq C \left( \frac{R}{\theta r} \right)^n \]
and
\[ \frac{\varphi(|x + y - x_1|)}{\varphi(|y|)} \leq \frac{\varphi(CR + |y|)}{\varphi(|y|)} \leq a \left( \frac{R}{\theta r} \right)^{\overline{C}} \leq a \left( \frac{R}{\theta r} \right)^2. \]

Therefore, we obtain
\[ \mathcal{M}_{x_0} w(x) \leq \frac{C(R)}{R^2(C + \overline{C})} + \frac{C}{C + \overline{C}} \left( \frac{R}{\theta r} \right)^{n+2} \left( \frac{CR}{R^2} + \overline{C}(R) \right) \]
\[ \leq \frac{C}{C + \overline{C}} \left( \frac{R}{\theta r} \right)^{n+2} \left( \frac{C(R)}{R^2} + \overline{C}(R) \right). \]
Now we apply the weak Harnack inequality to \( w \) in \( B_{\theta r/2}(x_0) \) to obtain that
\[
\left| \left\{ u < \frac{u(x_0)}{2} \right\} \cap B_{\theta r}(x_0) \right| = \left| \left\{ w > \left( \frac{1 - \theta}{2} \right)^{-\gamma} - \frac{1}{2} u(x_0) \right\} \cap B_{\theta r}(x_0) \right|
\leq C(\theta r)^n \left( w(x_0) + C \left( \frac{R}{\theta r} \right)^{(n+2)\varepsilon} \left( 1 + \frac{R^2 C(R)}{C(R)} \right)^\varepsilon \left( \frac{1 - \theta}{2} \right)^{-\gamma} - \frac{1}{2} \right)^{\varepsilon} \frac{1}{u(x_0)^\varepsilon}.
\]

By inequalities (2.6) and (2.7), we have
\[
\frac{R^2 C(R)}{C(R)} \leq R^2 \left( \frac{a}{\phi(R)} \right) \left( \frac{R}{a(2 - a)\phi(R)} \right)^{-1} = a^2 \frac{2 - a}{a} \leq \frac{2a^2}{\sigma_0}.
\]
Thus, we have
\[
\left| \left\{ u < \frac{u(x_0)}{2} \right\} \cap B_{\theta r}(x_0) \right|
\leq C(\theta r)^n \left( \left( \frac{1 - \theta}{2} \right)^{-\gamma} - 1 \right)^\varepsilon \left( \frac{1}{u(x_0)} \right) + C \left( \frac{R}{\theta r} \right)^{(n+2)\varepsilon} \frac{1}{u(x_0)^\varepsilon}.
\]
Since \( u(x_0) = \alpha(R/2r)^\gamma \) and \( \gamma = (n+2)/\varepsilon \), we have
\[
\left| \left\{ u < \frac{u(x_0)}{2} \right\} \cap B_{\theta r}(x_0) \right|
\leq C(\theta r)^n \left( \left( \frac{1 - \theta}{2} \right)^{-\gamma} - 1 \right)^\varepsilon C \left( \frac{R}{\theta r} \right)^{(n+2-\gamma)\varepsilon} \alpha^{-\varepsilon} \theta^{-\gamma\varepsilon}
\leq C(\theta r)^n \left( \left( \frac{1 - \theta}{2} \right)^{-\gamma} - 1 \right)^\varepsilon C\alpha^{-\varepsilon} \theta^{-\gamma\varepsilon}.
\]
We choose a uniform constant \( \theta > 0 \) sufficiently small so that
\[
C(\theta r)^n \left( \frac{1 - \theta}{2} \right)^{-\gamma} - 1 \right)^\varepsilon \leq \frac{1}{4} \left| B_{\theta r}(x_0) \right|.
\]
If \( \alpha > 0 \) is sufficiently large, then we have
\[
C(\theta r)^n \theta^{-\gamma\varepsilon} \alpha^{-\varepsilon} \leq \frac{1}{4} \left| B_{\theta r}(x_0) \right|,
\]
which implies that
\[
\left| \left\{ u < \frac{u(x_0)}{2} \right\} \cap B_{\theta r}(x_0) \right| \leq \frac{1}{4} \left| B_{\theta r}(x_0) \right|.
\]
Therefore, \( \alpha \) is uniformly bounded and the result follows. \( \square \)
4.5. Hölder Continuity. Theorem 1.2 follows from the following lemma by a simple scaling.

**Lemma 4.9.** Assume that $\sigma \geq \sigma_0 > 0$. There exists a uniform constant $\varepsilon_0 > 0$, depending only on $n, \lambda, \Lambda, a$, and $\sigma_0$, such that if $-\frac{1}{2} \leq u \leq \frac{1}{2}$ in $\mathbb{R}^n$ and

$$M^+_L u \geq -\varepsilon_0 \frac{C(R)}{R^2(C + C)} \quad \text{and} \quad M^-_L u \leq \varepsilon_0 \frac{C(R)}{R^2(C + C)} \quad \text{in } B_R,$$

then $u \in C^\alpha(B_{R/2})$ and

$$|u(x) - u(0)| \leq CR^{-\alpha}|x|^\alpha$$

for some uniform constants $\alpha > 0$ and $C > 0$.

**Proof.** We will show that there exist an increasing sequence $\{m_k\}_{k \geq 0}$ and a decreasing sequence $\{M_k\}_{k \geq 0}$ satisfying $m_k \leq u \leq M_k$ in $B_{4^{-k}R}$ and $M_k - m_k = 4^{-\alpha k}$, so that the theorem holds.

For $k = 0$ we choose $m_0 = -\frac{1}{2}$ and $M_0 = \frac{1}{2}$. Now assume that we have sequences up to $m_k$ and $M_k$. We want to show that we can continue the sequences by finding $m_{k+1}$ and $M_{k+1}$.

In the ball $B_{4^{-(k+1)}R}$, either $u \geq (M_k + m_k)/2$ in at least half of the points in measure, or $u \leq (M_k + m_k)/2$ in at least half of the points. Let us say that

$$\left| \left\{ u \geq \frac{M_k + m_k}{2} \right\} \cap B_{4^{-(k+1)}R} \right| \geq \frac{|B_{4^{-(k+1)}R}|}{2}.$$

Define the function

$$v(x) := \frac{u(x) - m_k}{M_k - m_k}/2.$$

Then $v \geq 0$ in $B_{4^{-k}R}$ by the induction hypothesis, and $|\{v \geq 1\} \cap B_{4^{-(k+1)}R}| \geq |B_{4^{-(k+1)}R}|/2$. To apply Theorem 4.7, we define $w = v^+$. Note that we still have $|\{w \geq 1\} \cap B_{4^{-(k+1)}R}| \geq |B_{4^{-(k+1)}R}|/2$. Since $M^-_L v \leq \varepsilon_0 \frac{C(R)}{R^2(C + C)}$ in $B_R$,

$$M^-_L w \leq M^-_L v + M^+_L v^- \leq \varepsilon_0 \frac{C(R)}{(M_k - m_k)/2 R^2(C + C)} + M^+_L v^- \quad \text{in } B_R.$$

To estimate $M^+_L v^-$, we claim that $v(x) \geq -2 \left( \frac{|4x|^\alpha}{(4^{-k}R)^\alpha} - 1 \right)$ in $B_{4^{-k}R}$. Indeed, for $x \in B_{4^{-(k+j)}R} \setminus B_{4^{-(k+1)}R}, 1 \leq j \leq k$,

$$v(x) = \frac{u(x) - m_k}{(M_k - m_k)/2} \geq \frac{m_k - j - M_k - j + M_k - m_k}{(M_k - m_k)/2} = -2(4^{\alpha j} - 1) \geq -2 \left( \frac{|4x|^\alpha}{(4^{-k}R)^\alpha} - 1 \right),$$

and for $x \in B_{4^{-k}R}^c$,

$$v(x) \geq \frac{-\frac{1}{2} - M_k + M_k - m_k}{(M_k - m_k)/2} = -(1 + 2M_k)4^{\alpha k} + 2 \geq -2(4^{\alpha k} - 1) \geq -2 \left( \frac{|4x|^\alpha}{(4^{-k}R)^\alpha} - 1 \right).$$
Thus, we have for $x \in B_{3,4^{-(k+1)}R}$

$$\mathcal{M}_{L_0}^+ v^-(x) \leq 2\Lambda C_\varphi \int_{\mathbb{R}^n} \frac{v^-(x+y)}{|y|^n \varphi(|y|)} \, dy = 2\Lambda C_\varphi \int_{v(x+y)<0} \frac{v^-(x+y)}{|y|^n \varphi(|y|)} \, dy$$

$$\leq 4\Lambda C_\varphi \int_{x+y \notin B_{4^{-k}R}} \left( \frac{|4(x+y)|}{(4^{-k}R)^\alpha} - 1 \right) \frac{dy}{|y|^n \varphi(|y|)}.$$ 

If $x \in B_{3,4^{-(k+1)}R}$ and $x+y \notin B_{4^{-k}R}$, then $|y| \geq |x+y| - |x| > 4^{-k}R - 3 \cdot 4^{-(k+1)}R = 4^{-(k+1)}R$ and $|x+y| \leq |x| + |y| \leq 3 \cdot 4^{-(k+1)}R + |y| \leq 4|y|$. Thus we obtain

$$\mathcal{M}_{L_0}^+ v^-(x) \leq 4\Lambda C_\varphi \int_{|y|>4^{-(k+1)}R} \left( \frac{16|y|}{4^{-k}R} \right)^\alpha \left( \frac{16r}{4^{-k}R} \right)^\alpha - 1 \frac{dr}{r \varphi(r)}$$

$$\leq CC_\varphi \alpha \left( \frac{4^{-(k+1)}R}{\varphi(4^{-(k+1)}R)} \right)^\alpha \int_{4^{-(k+1)}R}^{\infty} \left( \frac{16r}{4^{-k}R} \right)^\alpha - 1 \, r^{-1-\alpha} \, dr.$$ 

If we have taken $\alpha < \sigma_0$, then

$$\int_{4^{-(k+1)}R}^{\infty} \left( \frac{16r}{4^{-k}R} \right)^\alpha - 1 \, r^{-1-\alpha} \, dr = \left( \frac{4}{\sigma - \alpha} - 1 \right) \left( \frac{4}{\sigma} - 1 \right) \left( \frac{16}{\sigma} \right)^\alpha \left( \frac{16}{\sigma} \right)^{\frac{\sigma}{\sigma}} \left( \frac{16}{\sigma} \right)^{\frac{\sigma}{\sigma}}$$

Since

$$\frac{4}{\sigma - \alpha} - 1 = \frac{\sigma(\sigma - 1) + \alpha}{\sigma(\sigma - 1)} \leq \frac{2(\sigma - 1) + \alpha}{\sigma(\sigma - 1)} = f(\alpha, \sigma_0),$$

we have

$$\mathcal{M}_{L_0}^+ v^-(x) \leq \frac{CC_\varphi a}{\varphi(4^{-(k+1)}R)} f(\alpha, \sigma_0).$$

Hence,

$$\mathcal{M}_{L_0}^- w \leq 2\varepsilon_0 4^{\alpha k} \frac{C(R)}{R^2(\sigma + C)} + \frac{CC_\varphi a}{\varphi(4^{-(k+1)}R)} f$$

in $B_{3,4^{-(k+1)}R}$. Note that the same holds in $B_{4^{-(k+1)}R}(x)$ for $x \in B_{4^{-k}R}$. We applying Theorem 4.7 to $w$ in $B_{4^{-(k+1)}R}(x)$ to obtain

$$\frac{|B_{4^{-(k+1)}R}|}{2} \leq |\{w \geq 1\} \cap B_{4^{-(k+1)}R}|$$

$$\leq C \left( \frac{R}{4^{k+1}} \right)^n \left( w(x) + \frac{2\varepsilon_0 4^{\alpha k} C(R)}{R^2(\sigma + C)} + \frac{CC_\varphi a}{\varphi(4^{-(k+1)}R)} f \right)^\varepsilon \left( \frac{R}{4^{k+1}} \right)^n$$

$$\leq C \left( \frac{R}{4^{k+1}} \right)^n \left( w(x) + \varepsilon_0 \frac{8(4^{(k+1)}R)}{C(4^{-(k+1)}R)} + C_{\varphi(4^{-(k+1)}R)} \left( \frac{4^{-(k+1)}R}{C(4^{-(k+1)}R)} \right)^\varepsilon \right).$$

We have

$$\frac{C(R)}{C(4^{-(k+1)}R)} \leq 1 + a^2 4^{(2-\sigma)(k+1)} \leq 1 + 16a^2 4^{(2-\sigma)k}$$

by the inequality (2.8), and

$$\frac{(4^{-(k+1)}R)^2 f}{\varphi(4^{-(k+1)}R)C(4^{-(k+1)}R)} \leq a(2 - \sigma) \leq 2a$$
by the inequality (2.6). Therefore, using $\alpha < \sigma_0$, we have
\[
\theta \leq w(x) + \frac{\varepsilon_0}{8} 4^{(\alpha-2)k} (1 + 16\alpha^2 4^{(\alpha-2)}) + C a f(\alpha, \sigma_0)
\]
\[
= w(x) + \frac{\varepsilon_0}{8} 4^{(\alpha-2)k} + 2\varepsilon_0 a^2 4^{(\alpha-2)} + C a f(\alpha, \sigma_0)
\]
\[
\leq w(x) + \frac{\varepsilon_0}{8} + 2\varepsilon_0 a^2 + C a f(\alpha, \sigma_0),
\]
where $\theta > 0$ is a uniform constant. Notice that we have $\lim_{\alpha \to 0} f(\alpha, \sigma_0) = 0$. If we have chosen $\alpha$ and $\varepsilon_0$ satisfying $g(\alpha) < \theta / 4$, then
\[
M_{k+1} \geq u \geq \frac{M_k - m_k}{4} \theta = M_k - \left(1 - \frac{\theta}{4}\right)4^{-\alpha} \geq M_k - 4^{-\alpha(k+1)} = m_{k+1}
\]
in $B_{4^{-\alpha(k+1)}R}$. 

On the other hand, if $\{|u \geq (M_k + m_k)/2\} \cap B_{4^{-(k+1)}R} \geq |B_{4^{-(k+1)}R}|/2$, we define
\[
v(x) = \frac{M_k - u(x)}{(M_k - m_k)/2}
\]
and continue in the same way using that $\mathcal{M}_{L_0}^{+} u \geq -\varepsilon_0 C/R_{\text{Hitch}}^R C$.

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