ABSTRACT: There are several ways to establish and study thermal properties of black holes. I review here method of Fulling and Ruijsenaars, based on the analytic structure of Green functions on the complex plane. This method provides a clear distinction between zero and finite temperature field theories, and allows for quick evaluation of black hole temperature.

Two lectures at 3rd Danube Workshop, June 1993, Belgrade, Yugoslavia
**Introduction.**

Quantum evaporation of black holes exhibits fascinating connection between gravity, quantum mechanics and thermodynamics. In order to understand this phenomena better, and certainly in order to move on towards solution of bigger questions, like the back-reaction, the issue of the final state, or the possible evolution from pure to mixed state, (otherwise known as the “loss of quantum coherence”), it would be usefull to examine different paradigmas and different techniques that have been used in the past for study of black hole evaporation. Historically the first has been Hawking’s use of mode mixing and Bogoliubov coefficients [1]. Close to this is a technique utilizing properties of the squeezed states in black hole spacetimes, which received some attention recently [2]. Next, there is a puzzling and not yet fully understood discovery of a relationship between the flux of the outgoing radiation and the trace anomaly [3], which has been used most recently in studies of dilaton models [4]. Another approach explored in seventies has been in the physical analogy with the Klein paradox [5]. But most people consider as the most superior the method of thermal Green functions, introduced by Gibbons and Perry [6]. This is just the method that we will explore here, but our starting point will be a comprehensive study of the analytic properties of Green functions, done by Fulling and Ruijsenaars [7]. We will pragmatically follow their path, by computing first a number of relevant quantities for scalar field in flat spacetime at zero and at the finite temperature. Next we move on to curved spacetime and discuss thermal properties of Rindler and Schwarzschild spacetime. This methods allows for easy and clean evaluation of the temperature of the black hole as seen by the stationary observer outside the horizon.
Scalar field in flat spacetime

The basic relations for a massive scalar field are

\[ S[\Phi] = \int d^4x \left[ \frac{1}{2} \partial \Phi \cdot \partial \Phi + \frac{1}{2} m^2 \Phi^2 \right] ; \]

\[ \pi = \dot{\Phi} ; \quad (-\partial^2 + m^2)\Phi(x) = 0 ; \]

\[ \Phi(x) = \int d\vec{k} \left[ a_{\vec{k}} \phi_{\vec{k}}(x) + a_{\vec{k}}^\dagger \phi_{\vec{k}}^*(x) \right] ; \quad [a_{\vec{k}}, a_{\vec{k}}^\dagger] = \delta(\vec{k} - \vec{q}) . \]

\[ \phi_{\vec{k}}(x) = \frac{e^{i\vec{k} \cdot \vec{x}}}{2\omega(k)} ; \quad \omega(k) \equiv +\sqrt{\vec{k}^2 + m^2} . \]

\[ [\Phi(t, \vec{x}), \Phi(t, \vec{y})] = 0 ; \quad [\Phi(t, \vec{x}), \pi(t, \vec{y})] = i\delta(\vec{x} - \vec{y}) . \]

Suppose now that we want to compute the two point corellation function

\[ \langle \Phi(x)\Phi(y) \rangle . \quad (1) \]

The two interesting choices for the quantum state in which this expectation value is to be computed are vacuum and the thermal state. We will consider them now in turn.

**Vacuum expectation values**

If we choose to compute expectation values in the vacuum state, it follows that,

\[ \langle 0|\Phi(x), \Phi(y)|0 \rangle = \int d\vec{k} \frac{e^{i\vec{k} \cdot (\vec{x} - \vec{y}) - ik^0(x^0 - y^0)}}{2\omega(k)} . \]
Using Cauchy formula,

\[ f(z_0) = \frac{1}{2\pi i} \int_{C(z_0)} dz \frac{f(z)}{z - z_0}, \]

we can rewrite this as,

\[ \langle 0|\Phi(x)\Phi(y)|0 \rangle = (-i) \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot\Delta x}}{(k^0)^2 - \omega(k)^2}. \quad (2) \]

The correspondence between these two expressions is not unique, as the later integral is multivalued. The integrand has two poles, at \( k^0 = \pm \omega(k) \), and the final result depends on how does the contour of integration go around the poles. This choice of contour is equivalent to the choice of boundary condition for the correlation function. The residua at the two poles are \((-1)/(2k_{pole}^0)\), and the two basic contributions are

\[ \frac{-i}{2\omega(k)} e^{-i\omega(k)t}, \quad \text{and} \quad \frac{i}{2\omega(k)} e^{i\omega(k)t}. \]

So, the general result will be,

\[ \langle 0|\Phi(x),\Phi(y)|0 \rangle_C = \int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \sum_C (-i) \frac{e^{k^0_{pole}x^0}}{2k^0_{pole}} , \]

where \( C \) stands for the contour of integration. The 2-point correlation function in Eq. (2) corresponds therefore to one particular choice for the contour of integration in (2), or to one particular choice of the boundary conditions for the solutions of Eq.’s (3-4) below. Different choice of contour leads to different vacuum expectation value in Eq. (1).

Different vacuum expectation values have been historically defined as follows:

\[ G^+(x,y) \equiv \theta(x^0 - y^0)\langle 0|\Phi(x)\Phi(y)|0 \rangle \quad \text{(positive frequency Wightman function)}; \]

\[ G^-(x,y) \equiv \theta(y^0 - x^0)\langle 0|\Phi(y)\Phi(x)|0 \rangle \quad \text{(negative frequency Wightman function)}; \]
\[ iG(x, y) \equiv \langle 0 | [\Phi(x), \Phi(y)] | 0 \rangle \] (Pauli–Jordan function);

\[ G^{(1)}(x, y) \equiv \langle 0 | \{\Phi(x), \Phi(y)\} | 0 \rangle \] (Hadamard function);

\[ G_R(x, y) \equiv (-)\theta(t_x - t_y) \ G(x, y) \] (retarded Green function);

\[ G_A(x, y) \equiv \theta(t_y - t_x) \ G(x, y) \] (advanced Green function);

\[ iG_F(x, y) \equiv \langle 0 | T(\Phi(x)\Phi(y)) | 0 \rangle \] (Feynman propagator);

All of these 2-point functions are usually called Green functions although four of them actually obey the homogeneous equation. There is a number of useful and easy to prove relations that are obeyed by these functions:

**Problem 1.** Show that,

\[ iG = G^+ - G^- ; \quad G^{(1)} = G^+ + G^- ; \quad iG_F = G^+ + G^- . \]

**Problem 2.** Show that,

\[ (-\partial_x^2 + m^2) \ G(x, y) = 0 \],

when \( G \in \{ G^+, G^-, iG, G^{(1)} \} \).

**Problem 3.** (a) Show first that, \( \partial_{tt}^2 (\theta G) = \delta \dot{G} + \theta \ddot{G} \); then, (b) show that the other three vacuum expectation values are true Green functions:

\[ (-\partial_x^2 + m^2) \ G(x, y) = \pm \delta(x - y) \]
when $G \in \{G_R, G_A, G_F\}$. Plus sign applies for the first two, minus sign for $G_F$.

Further claim is that solution to any of these equations has the following integral representation:

$$G(x, y) = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot (x-y)} G(k) \ ,$$

which obeys,

$$(-\partial_x^2 + m^2)G(x, y) = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot (x-y)} (k^2 + m^2)G(k) \ .$$

For $G \in \{G^+, G^-, iG, G^{(1)}\}$, the right hand side of this equation should vanish; when $G \in \{G_R, G_A, G_F\}$, the integrand of the Fourier transform on the right hand side should be $\pm 1$. Both can be accomplished using the multivaluedness of the complex integral (5), as the following exercises will show:

**Problem 4.** Draw all seven contours of integration for defined Green functions, and discuss boundary conditions.

**Problem 5.** Since first four functions obey the same equation, they are distinguished by different boundary conditions, and similarly for the other three:

(a) Determine boundary conditions at $\pm \infty$ for four solutions to the homogeneous equation, and justify names and notation for $G^\pm$.

(b) Determine boundary conditions for the three solutions to the inhomogeneous equation (4) by adding a source term $J(x)\Phi(x)$ to the original Lagrangean.

Let us now fix $\vec{x} - \vec{y} \to \vec{x}$ as a parameter, and consider all these Green functions as functions of time. It is immediately obvious that, by virtue of its defining expression, $G^+(t)$ may be analytically extended only to the upper half-plane, and $G^-(t)$ only to the lower. Further straightforward examination uncovers the following beautiful structure on the complex plane $z \equiv (t, \tau)$:
Problem 6. Show that there is an analytic function \( G(z) \) which coincides with the \( G^+ \) on the upper half-plane, and with \( G^- \) on the lower half-plane. Along the \( t \) axis \( \mathcal{G} \) has a branch cut for \( t > |\vec{x}| \). The jump across the cut is given by the Pauli-Jordan Green function: \( \mathcal{G}(t + i\epsilon) - \mathcal{G}(t - i\epsilon) = G(t) \), for \( \epsilon > 0 \).

Problem 7. The projection of \( G \) to the imaginary axis is called the Euclidean, or the Schwinger Green function: \( \mathcal{G}(z) \to_{t \to 0} G_E(\tau) \). Show that:

(a) \( G_E \) obeys the Euclidean version of the inhomogeneous field equation for a Green function,

\[
(\partial^2_{\tau\tau} + \nabla^2 + m^2) G_E(\tau, \vec{x}) = -\delta(\tau)\delta^{(3)}(\vec{x}) ;
\]

(b) \( G_E \) and \( G_F \) are related by the so-called Wick rotation: \( t \to i\tau \).

Results of these exercises may be stated as follows. There is Green function \( \mathcal{G}(z) \), which in upper half-plane represents the analytic continuation of \( G^+(t) \), and in lower half-plane it is the analytic continuation of \( G^-(t) \). Along the real axis \( \mathcal{G} \) has a branch cut for \( t \geq |\vec{x}| \). The jump across the branch cut is given by the Pauli-Jordan Green function, that is, by the canonical commutator for the scalar field \( \phi \). On the imaginary axis \( \mathcal{G} \) reduces to the Euclidean, or Schwinger Green function \( G_E(\tau) \). Unlike all the others mentioned so far, \( G_E \) is a true Green function: it obeys the Euclidean version of the field equation with the point source. \( G_E(\tau) \) by itself has its own analytic continuation to the real axis: its is the Feynman propagator, \( G_F(t) \), which obeys the inhomogeneous field equation with the point source. This continuation is usually called the Wick rotation.

This description, first given so eloquently by Fulling and Ruijsenaars, is the culmination of our understanding of the Green functions.

We can now compute explicitly these functions. The most efficient strategy is to compute \( G^\pm \) first, by evaluating the integral above with the appropriate boundary conditions.
Other Green functions may be then computed from $G^\pm$, by adding or subtracting the two residua according to their contours, and/or the algebraic relations between the functions. Results of the exercises below should be compared to the calculations in textbook of Bogoliubov and Shirkov [8].

**Problem 8.** Compute $G^+$ for a massless scalar field in flat spacetime. Since there is no free parameter in the Lagrangean there is no characteristic correlation scale and this Green function describes a long range correlations.

**Problem 9.** Compute $G^+$ for a massive scalar field in a flat spacetime. This Green function does have a characteristic correlation length, given by the finite mass of the particle. In the perturbation theory this would lead to the Yukawa potential.

**Problem 10.** Now compute all the other Green functions for a massless scalar field in flat spacetime.

**Problem 11.** What can we conclude from these Green functions about the physics of a massless scalar field?

**Problem 12.** Compute all the Green functions for a massive scalar field in flat spacetime.

**Problem 13.** What can we conclude from these Green functions about the physics of a massive scalar field?

---

**Thermal expectation values**

Instead of the vacuum, we now consider the thermal state,

$$
\rho = \sum_E |E\rangle e^{-\beta E} \langle E| .
$$

The 2-point correlation function is,

$$
\langle \Phi(x)\Phi(y) \rangle = \sum_E e^{-\beta E} \langle E|\Phi(x)\Phi(y)|E\rangle .
$$
It is straightforward to show that the later can be written in the path-integral form. For this we should imagine that \( \beta \) is the total interval in the imaginary time within which the “evolution” takes place: \( \beta = \tau_f - \tau_i \). Separating this interval into the infinitesimal ones, and inserting the unit operators represented through complete sums of the coordinate or momentum eigenstates will lead to the formal expression,

\[
\langle \text{out} | \Phi(x, \tau_x) \Phi(y, \tau_y) | \text{in} \rangle = \int_{\mathcal{C}(\Phi)} \left[ d\Phi(r, \tau) \right] e^{-S_E[\Phi(r, \tau)]} \Phi(x, \tau_x) \Phi(y, \tau_y) .
\]

Here \( \mathcal{C}(\Phi) \) stands for the class of the fields over which the path integral is taken. In this case they are those with the periodic boundary condition in the imaginary time:

\[
\mathcal{C}(\Phi) = \{ \Phi(r, \tau) \mid \Phi(r, \tau_i + \beta) = \Phi(r, \tau_i); \tau_i \leq \tau \leq \tau_f = \tau_i + \beta \} .
\]

This kind of reasoning is widely used, but it usually leaves some ambiguity as to what is exactly the relationship between the fields and the Physics in the real and imaginary times. Practice has shown what is the safe way to interpret this relation in many cases, but the situation often gets ambiguous when the curved spacetime is introduced. It is in this methodological respect that the construction of Fulling and Ruijsenaars appears to us so effective. We will now repeat the preceding analysis from the vacuum case to determine the analytic structure of the Green functions in the case of a free field in the flat spacetime, but at the finite temperature \( T \equiv \beta^{-1} \).

The two Whightman functions are defined as follows:

\[
G^+_{\beta}(t, x, y) \equiv Z^{-1} \sum_n Tr \left\{ e^{-\beta H} \Phi(t, x) \Phi^\dagger(0, y) \right\} ;
\]

\[
G^-_{\beta}(t, x, y) \equiv Z^{-1} \sum_n Tr \left\{ e^{-\beta H} \Phi^\dagger(0, y) \Phi(t, x) \right\} .
\]

The usual interpretation is that \( G^+_{\beta} \) is an amplitude to create particle at \((0, y)\) and annihilate it at \((t, x)\), all in the athermal bath (rather than vacuum).
Let \( N^+_n \) denote the number of particles with positive energy \( \omega_n > 0 \), and \( N^-_n \) number of antiparticles with the same energy. The partition function is given as,

\[
Z = \sum_{\{N^+_n, N^-_n\}} e^{-\beta \sum_n (N^+_n + N^-_n) \omega_n} = \prod_n (1 - e^{-\beta \omega_n})^{-2} .
\]

The power of two appears here if we are dealing with a complex, non-hermitean field. If \( \Phi = \Phi^\dagger \), then,

\[
Z = \prod_n (1 - e^{-\beta \omega_n})^{-1} .
\]

As in the vacuum case, where \( \langle 0\vert a a \vert 0 \rangle = \langle 0\vert a\dagger a\dagger \vert 0 \rangle = 0 \), we have \( \text{Tr}\{\rho_{aa}\} = \text{Tr}\{\rho_{a\dagger a\dagger}\} = 0 \). However, instead of \( \langle 0\vert a\dagger a \vert 0 \rangle = 0 \), we have the finite occupation numbers for thermal state:

\[
\langle a_n a_m\dagger \rangle = \delta_{nm} \left( 1 - \omega_n^{-1} \frac{\partial}{\partial \beta} \log Z \right) = \delta_{nm} \left( 1 - e^{-\beta \omega_n} \right)^{-1} ,
\]

so, that,

\[
\langle a_m\dagger a_n \rangle = 1 - \langle a_n a_m\dagger \rangle = (e^{\beta \omega_n} - 1)^{-1} .
\]

Using the expansion for the quantum field with time dependent piece explicitly written out (the spatial piece depends on the choice of the spatial coordinates and topology),

\[
\Phi(t, \vec{x}) = \sum_n f_n(\vec{x}) \sqrt{2 \omega_n} \left( a_n e^{-i \omega_n t} + a_n^\dagger e^{i \omega_n t} \right) ,
\]

we find the following expressions for the Whightman functions:

\[
G^+_\beta(t; \vec{x}, \vec{y}) = \sum_n f_n(\vec{x}) f^*_n(\vec{y}) \frac{\omega_n}{2} \left[ \frac{e^{-i \omega_n t}}{1 - e^{-\beta \omega_n}} + \frac{e^{i \omega_n t}}{e^{\beta \omega_n} - 1} \right] ;
\]

\[
G^-\beta(t; \vec{x}, \vec{y}) = \sum_n f_n(\vec{x}) f^*_n(\vec{y}) \frac{\omega_n}{2} \left[ \frac{e^{-i \omega_n t}}{e^{\beta \omega_n} - 1} + \frac{e^{i \omega_n t}}{1 - e^{-\beta \omega_n}} \right] .
\]

Let us now look at the analytic continuation of these functions. The first step is to establish their relationship to the Pauli-Jordan function.
Problem 14. Check that still,

\[ G^+_\beta - G^-_\beta = [\Phi(t, \vec{x}), \Phi(t, \vec{y})] . \]

Since exponents with both signs are present in the expressions for \( G^\pm_\beta \), if we now substitute \( z = t + i\tau \) for \( t \), we can see that neither of the function is analytic on the whole half-plane, either upper or lower. However, observe that if we fix \( \tau > 0 \), for \( \omega_n \to \infty \),

\[ G^+_\beta \sim e^{\omega_n \tau} , \quad G^-_\beta \sim e^{\omega_n (\tau - \beta)} . \]

Similarly, for \( \tau < 0 \), the leading pieces are,

\[ G^+_\beta \sim e^{-\omega_n (\tau + \beta)} , \quad G^-_\beta \sim e^{-\omega_n \tau} . \]

Therefore, \( G^+_\beta \) may be extended from the real axis to the strip \( \tau \in [-\beta, 0] \), and \( G^-_\beta \) to the strip \( \tau \in [0, \beta] \).

Furthermore:

Problem 15. Observe that,

\[ G^+_\beta (\tau - \beta) = G^-_\beta (\tau) ; \quad G^-_\beta (\tau + \beta) = G^+_\beta (\tau) . \]

Using this result it is easy to see that one can do analytic continuation to the whole plane, using strips of the width \( \beta \) in imaginary direction as Weierstrass circles. One therefore arrives on the following analytic structure on the complex plane: there is a complex function \( G_\beta(z, \vec{x}) \), periodic in imaginary direction with the period \( \beta \). Within each strip \([\tau, \tau + \beta]\), \( G \) has two representations, one through \( G^+_\beta \), another through \( G^-_\beta \), but with identical values. This realization within one strip is repeated within all other strips. Along the real axis, for \( t > |\vec{x}| \), \( G_\beta \) has a branch cut. The jump across the cut is given by the canonical commutator, that is, by the Pauli-Jordan Green function at
the zero temperature. Due to the periodicity in $\tau$ this branch cut is therefore present at $\tau = 0, \pm \beta, \pm 2\beta, \pm 3\beta, \ldots$.

Let us now consider the restriction to the imaginary axis, the so-called thermal Euclidean function, or thermal Schwinger function,

$$G_\beta(\tau, \vec{x}) = \sum_n g(\vec{x}) \left[ \frac{e^{\omega_n \tau}}{e^{\beta \omega_n} - 1} + \frac{e^{-\omega_n \beta}}{1 - e^{-\omega_n \beta}} \right].$$

In zero temperature case $G_E$ was an even function, $G_E(-\tau) = G_E(\tau)$. Here we find the same result:

**Problem 16.** Show that $G_E^\beta(-\tau) = G_E^\beta(\tau + \beta) = G_E^\beta(\tau)$.

**Problem 17.** Show that those two properties, periodicity in $\tau$ and even nature of $G_E^\beta$, combined together make this function periodic with a period $\beta/2$!

As people say, $G_E^\beta$ is reflected around $\tau = n\beta/2$. This is then repeated through all the other $\beta$-wide stripes.

**Problem 18.** Draw the analytic structure of thermal Green function $G_\beta(z)$ in the complex plane $z = t + i\tau$.

**Rindler spacetime**

Let us now consider the case of a flat spacetime as seen by an uniformly accelerated observer. The trajectory of such an observer moving in $X$ direction is given by,

$$t = A^{-1} \sinh AT, \quad X = A^{-1} \cosh AT. \quad \text{(6)}$$

Here $A \in [0, \infty]$ is constant proper acceleration, and $T \in (-\infty, \infty)$ is his proper time. For future convenience let us introduce the dimensionless affine parameter $\lambda \equiv AT$, and the “radius” $a \equiv A^{-1}$. 

12
If we consider the set of all the observers with all possible values of constant proper acceleration $A$ we see that pair $(\lambda, a)$ may be used as a new set of coordinates in place of $(t, X)$. The line element may be written as,

$$ds^2 = -a^2 d\lambda^2 + da^2 + dl_2^2,$$

where the last term stays for the length in two unchanged coordinates. As it is well known, due to limitations on the speed to which observer may accelerate to, these coordinates cover just the two wedges in the $(t, X)$ plane, within the straight lines $X = \pm t$. But within these wedges the spacetime looks flat in new coordinates $(\lambda, a)$.

Consider now the thermal expectation value as measured by such an observer: $G^R_\beta(\lambda, a)$. Without any calculation, we know from the preceding discussion the analytic structure of this function, in particular we know that it must be periodic in imaginary proper time $S : T \rightarrow S$ with a period $\beta$.

Let us compare this with the vacuum expectation value measured by the inertial observer using Minkowski coordinates $(t, X)$. This would allow us to compare the two physical pictures as seen by the two observers. The punch line is, of course, that those two pictures correspond to the same physical situation, as we shall see shortly. To facilitate such comparison we only need to do a very innocent step, just to perform the change of coordinates in one function to those naturally used for another. We must also take care of the proper dimensions, as it will become apparent.

We will start with the vacuum expectation value as calculated by the Minkowski observer. Given two points, $(t_1, X_1, Y_1, Z_1)$ and $(t_2, X_2, Y_2, Z_2)$, the vacuum expectation values must depend only on $\Delta s^2 = -\Delta t^2 + \Delta X^2 + \Delta l_2^2$, due to the Lorentz invariance.

After substitution of the Rindler coordinates, viz. Eq. (6), we obtain function that depends only on a finite line element expressed as, $a_1^2 + a_2^2 - 2a_1 a_2 \cosh(\lambda_1 - \lambda_2) + \Delta l_2^2$. 

13
One can now make a series of simple observations.

First, note that upon change to the purely imaginary Rindler time, \( T \to iS \), equivalently, \( \lambda \to i\sigma \), the line element becomes, \( ds^2 = da^2 + a^2 d\sigma^2 \). This looks like the line element in polar coordinates, and there will be no singularity as \( a \to 0 \) if \( \sigma \) is indeed a polar angle. But in that case it must be periodic, with a period \( 2\pi \). The corresponding imaginary proper time \( S \) must then also be periodic, with a period \( 2\pi/A \). That is, we use \( \lambda \equiv AT \) generalizes to \( \lambda + i\sigma \equiv A(T + iS) \), so that \( \sigma \equiv AS \). In that case every function of \( S \) must also have the same period, or fraction of it. In particular, we can see that for the vacuum expectation value evaluated above the Grand vacuum Green function function \( G \) has the form,

\[
G(a_1^2 + a_2^2 - 2a_1a_2 \cos(\sigma_1 - \sigma_2) + dl_2^2),
\]

along the imaginary axis. This is the same periodicity as the thermal Green function in flat spacetime at finite temperature \( A/(2\pi)! \)

This type of the reasoning is used very often: periodicity of the metric in imaginary time is interpreted as the presence of thermal features. However, to make reliable statement one has to investigate the analytic structure of Green functions.

The elementary observation is that \( G(t, X) \), being the vacuum expectation value, is holomorphic within the strip \( \Delta t < \Delta X \), i.e., for \( \Delta s^2 > 0 \). But this line element is invariant under the change of coordinates, hence, this expectation value, as seen by the observer using Rindler coordinates \( (\lambda, a) \) has the same holomorphic strip and the same periodicity in imaginary direction as what he would call the thermal expectation value.

As a final confidence test, one can that the following statement holds:

*Problem 19.* Show by a direct coordinate substitution from (6) that the equation for the Euclidean Green function in Minkowski space becomes the equation for Euclidean Green function in Rindler coordinates.
To summarize: After transformation to Rindler coordinates, $\mathcal{G}(t+i\tau, \Delta x)$ has the same periodicity, and the same analyticity strip as $\mathcal{G}_\beta(T+iS, \Delta x)$ for $\beta = 2\pi/A$; Euclidean Green functions in Minkowski and Rindler coordinates are solutions of the two equations related by the simple change of coordinates; the same must be said about the boundary conditions they obey; thus, it is the same Green function; analytic continuation of Euclidean Green function from one streep to another along the imaginary axis uniquely defines $\mathcal{G}$ over the whole complex plane; therefore,

$$\mathcal{G}(t+i\tau, \Delta x) = \mathcal{G}_{2\pi/A}(T+iS, \Delta x),$$

in the entire complex plane.

On basis of that statement we conclude that while observer using Minkowski coordinates thinks all the time that he is computing averages with respect to the vacuum state, the Minkowski vacuum, observer using Rindler coordinates, the accelerated observer, notices that expectation values for a system (the scalar field) in the same state look like thermal expectation values, computed for a finite temperature field theory at temperature $A/(2\pi)$. In other words, Rindler observer sees Minkowski vacuum as thermal density matrix, temperature being proportional to its proper acceleration.

This is the main result that we wish to demonstrate here. Let us just mention that an apparent puzzle of seeing pure state as thermal may be explained by invoking horizons for the Rindler observer. Similarly, there is no problem of a finite state. With an unlimited supply of energy, Rindler observer may indefinitely maintain its constant acceleration. Once it stops accelerating, horizons disappear, and the two observers agree that they are seeing the vacuum state, as both are inertial observers.
**Schwarzschild black hole**

This case offers considerably more puzzle, but thanks to its similarity with the Rindler spacetime it is easy at least to take a first step and establish the existence of thermal features.

The standard form of the line element is,

$$ds^2 = -\left(1 - \frac{r_S}{r}\right)dt^2 + \frac{dr^2}{1 - r_S/r} + r^2d\Omega^2_2.$$  \hspace{1cm} (8)

The Schwarzschild radius is $r_S = 2GM$, $M$ being the mass of the black hole.

Consider now an observer keeping himself at a fixed radius $r$ above the horizon of the black hole. In order to do so he must accelerate away from the hole, with an acceleration just equal to the gravitational acceleration provided by the hole. Thus, such an observer is Rindler observer.

The line element (8) may easily be transformed into the Rindler form in case of an observer not too far from the horizon. Let us have $r = r_S + \delta r \equiv r_S(1 + x)$, with $x > 0$. The line element becomes

$$ds^2 = -xdt^2 + r_S^2\frac{dx^2}{x}.$$  

This can be transformed to the Rindler form $-a^2d\lambda^2 + da^2$ if we define $a \equiv 2r_Sx^{1/2}$, and consequently,

$$\lambda \equiv \frac{t}{2r_S}.$$  

Upon extension to the complex plane we should find, as before, the periodicity of the Green functions in imaginary dimensionless time $\sigma$, with period of $2\pi$. The corresponding period in the imaginary component of the Schwarzschild time is then $\beta = 4\pi r_S$. Thus, the observer at a constant distance above the horizon of the Schwarzschild black hole sees the
thermal bath at temperature,

\[ T_{BH} = \frac{1}{4\pi r_s}. \]

What is this temperature of? If the observer that maintains constant distance from black hole is Rindler observer, then inertial observer is obviously the one in a free fall towards the hole. Indeed, by equivalence principle, its vacuum state must be identical to the vacuum of the observer at infinity, which does not detect gravitationally the existence of the black hole. Thus, if initial state of the quantum field is vacuum with respect to the empty space, once black hole moves in, (or we move towards the hole), we perceive quantum fluctuations in that state as decohered thermal fluctuations.

And why do we call this temperature the black hole temperature? One can show, just as in the case of a Rindler observer, that horizon may be understood as the source of the black-body radiation associated with this temperature. By overall energy conservation this radiation must lead to the decrease in the mass of the hole, hence to its evaporation.

Conclusion

As said before, there are several ways to exhibit the thermal features of black holes. There is probably no the best way, and there is no need for one. The full understanding of the black hole evaporation most likely will require shifting between different paradigmas and methods of calculation. But the method reviewed here is the one that is most firmly rooted in the quantum field theory. It clarifies what does it mean to have finite temperature, and allows for a simple and reliable calculations. Thus, the use of thermal Green functions ought to be usefull in future studies of the final state of the evaporation, and the questions about the “loss of the quantum coherence.” The main obstacle is how to proceed with what is essentially a non-equilibrium configuration, and how to take into account the back-reaction.
Acknowledgement

The organizers of this Workshop must be praised for their effort and such a splendid job under such extraordinary circumstances.

References:

1. S.W. Hawking, *Nature*, 248, 30, (1974).
2. L. Grishchuk, WUSL preprint, (1993).
3. P. Davies, S. Fulling, and W. Unruh, *Phys. Rev.*, D13, 2720, (1976).
4. C. Callan et al., *Phys. Rev.*, D45, R1005, (1992).
5. T. Damour and R. Ruffini, *Phys. Rev. Lett.*, 35, 463, (1975).
6. G. Gibbons and M. Perry, *Proc. Roy. Soc.*, A 358, 467, (1978).
7. S. Fulling and S. Ruijseenaars, *Phys. Rep.*, 152, 136, (1987).
8. N. Bolgoliubov and D. Shirkov, *Vvedenije v teoriju kvantovyh poljej*, Nauka, (1973).