Exactly solvable cases in QED with $t$-electric potential steps

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Abstract

In this paper, we present in detail consistent QED (and scalar QED) calculations of particle creation effects in external electromagnetic field that correspond to three most important exactly solvable cases of $t$-electric potential steps: Sauter-like electric field, $T$-constant electric field, and exponentially growing and decaying electric fields. In all these cases, we succeeded to obtain new results, such as calculations in modified configurations of the above mentioned steps and detailed considerations of new limiting cases in already studied before steps. As was recently discovered by us, the information derived from considerations of exactly solvable cases allows one to make some general conclusions about quantum effects in fields for which no closed form solutions of the Dirac (or Klein-Gordon) equation are known. In the present article we briefly represent such conclusions about an universal behavior of vacuum mean values in slowly varying strong electric fields.

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### I. INTRODUCTION

Quantum field theories (QFTs) with external backgrounds (external fields) are to a certain extent the most appropriate models for calculating quantum effects in strong fields of electromagnetic, gravitational, or other nature. These calculations must be nonperturbative with respect to the interaction with strong backgrounds. One of the most interesting effect of such kind that attracts attention already for a long time is the particle creation from the vacuum by strong external backgrounds [1–6]. In the framework of the QFT, the particle creation is closely related to a violation of the vacuum stability with time. Not all
backgrounds violate the vacuum stability. For example, electromagnetic backgrounds that violate the vacuum stability have to be electriclike fields that are able to produce nonzero work when interacting with charged particles. In such backgrounds any process is accompanied by new created particles and, thus, turns out to a many-particle process. That is why any consistent consideration of quantum processes in the vacuum violating backgrounds has to be done in the framework of a QFT; a consideration in the framework of the relativistic quantum mechanics is restricted and may lead to paradoxes and even incorrect results. It should be noted that at present, methods for the above mentioned nonperturbative calculations are consistently formulated only in QED (and scalar QED) with some specific types of external backgrounds. Namely, these types are time-dependent external electric fields that are switched on and off at the initial and the final time instants respectively, see Refs. [6–9], we refer to such kind of external fields as the $t$-electric potential steps, and some time-independent inhomogeneous external electric fields, which are called conditionally, $x$-electric potential steps, see Refs. [10]. It should be stressed that the possibility of nonperturbative calculations in the both above cases is based on the existence of exact solutions of the Dirac (or Klein-Gordon) equation in the corresponding background fields. At present, there are known only few such cases, which we call exactly solvable cases in QED with $t$-electric or $x$-electric potential steps. Note that solutions of the Dirac equation with $x$-electric potential steps are quite similar to solutions obtained for $t$-electric steps where time $t$ is replaced by the spatial coordinate $x$. Some of quantum effects in each of these cases were considered in the literature using different approaches (relativistic quantum mechanics, QED) with a certain levels of consistency and details.

The case of homogeneous and constant electric field was examined by Schwinger [1], who obtained the probability for a vacuum to remain a vacuum. Such field admit analytical solutions of the Dirac and Klein-Gordon equations and was frequently used in various QFT calculations; see Ref. [11] for a review. Different semiclassical and numerical methods were applied to study Schwinger’s effective action, see Refs. [2–5] for a review. Nikishov found the mean number of pairs created by the homogeneous and constant electric field and established relation between the Schwinger’s and Feynman’ representations of a causal propagator in this external field [12, 13]. Subsequently he expanded and deepened his approach to this problem in Refs. [14–18].

In QED (and scalar QED) with $t$-electric potential steps, there exist a few exactly solvable
cases that have real physical importance. Those are Sauter-like (or adiabatic or pulse) electric field, $T$-constant electric field (a uniform electric field which effectively acts during a sufficiently large but finite time interval $T$), an exponentially decaying electric field, and their certain combinations. The particle creation effect in the cases of the Sauter-like electric field was studied in Refs. [19–21], in the case of $T$-constant electric field in Refs. [20, 22–30], and in the case of exponential electric fields in Refs. [31, 32]. One can see that quantum effects which can be studied using the exactly solvable cases are important in astrophysics, neutrino physics, cosmology, condense matter physics, and so on. In particular, the particle creation effect due to the Sauter-like and $T$-constant electric fields is crucial for understanding the conductivity of graphene or Weyl semimetals in the nonlinear regime as was reported, e.g., in Refs. [27, 33–40]. Note that the cases of a constant and exponentially decaying electric fields have many similarities with the case of the de Sitter background, e.g., see Refs. [41–43] and references therein. One can also notice that the case of harmonically alternating electric field is also exactly solvable [44, 45]. In this case the alternating direction of the electric field can also contribute to the particle creation effect, but it is not an example of $t$-electric potential step. Using exactly solvable cases one can develop new approximation methods of calculating quantum effects in QFT with unstable vacuum, see reviews [2–6]. However, many already known results are scattered over different publications and many new result were not published at all.

In this article, we present in detail consistent QED (and scalar QED) calculations of zero order\textsuperscript{1} quantum effects in external electromagnetic field that correspond to three most important exactly solvable cases of $t$-electric potential steps: Sauter-like electric field, $T$-constant electric field, and exponentially growing and decaying electric fields. In all these cases we succeeded to obtain new results, such as calculations in modified configurations of the above mentioned $t$-electric potential steps and a detailed consideration of new limiting cases (asymptotics) in already studied before $t$-electric potential steps. Considering all the cases, we tried to cite properly all previous relevant works. In Sec. II, we briefly recall basic formulas for treatment of zero-order processes in the framework of QED (and scalar QED) with $t$-electric potential steps. In Sets. III, IV and V we study quantum effects in the three most important exactly solvable cases of $t$-electric potential steps, in their modified configurations, and calculate carefully important limiting cases. As was recently discover in

\textsuperscript{1} Processes that do not involve photons.
our work\cite{53}, an information derived from considerations of exactly solvable cases allows one to make some general conclusions about quantum effects in slowly varying strong fields for which no closed form solutions of the Dirac equation are known. In Sec. VI, we briefly represent such conclusions about an universal behavior of vacuum mean values in slowly varying strong electric fields. Some asymptotic expansions are placed in the Appendix A. In the near future, we hope to present a similar work about quantum effects in external electromagnetic field that correspond to exactly solvable cases of $x$-electric potential steps.

II. VACUUM INSTABILITY DESCRIPTION BASED ON EXACT SOLUTIONS

Potentials $A^\mu (x)$, $x = (x^\mu ) = (x^0 = t, r)$, $r = (x^i)$ of external electromagnetic fields corresponding to $t$-electric potential steps are defined as

$$A^0 = 0, \quad A (t) = (A^1 = A_x (t), \quad A^l = 0, \quad l = 2, ..., D), \quad A_x (t) \underset{t \to \pm \infty}{\longrightarrow} A_x (\pm \infty),$$

where $A_x (\pm \infty)$ are some constant quantities, and the time derivative of the potential $A_x (t)$ does not change its sign for any $t \in \mathbb{R}$. For definiteness, we suppose that

$$\dot{A}_x (t) \leq 0 \implies A_x (-\infty) > A_x (+\infty). \quad (2.2)$$

The magnetic field is always zero and electric fields are homogeneous and have the form

$$E (t) = (E_x (t), 0, ..., 0), \quad E_x (t) = -\dot{A}_x (t) = E (t) \geq 0, \quad E (t) \underset{|t| \to \infty}{\longrightarrow} 0. \quad (2.3)$$

We stress that electric fields under consideration are switched off as $|t| \to \infty$ and do not have local minima.

As an example of a $t$-electric potential step, we refer to the so-called Sauter-like (or adiabatic or pulse) electric field. This field and its vector potential have the form

$$E (t) = E \cosh^{-2} (t/T_S), \quad A_x (t) = -T_S E \tanh (t/T_S). \quad (2.4)$$

where the parameter $T_S > 0$ sets time scale.

\footnote{The Greek indexes span the Minkowsisky space-time, $\mu = 0, 1, ..., D$, and the Latin indexes span the Euclidean space, $i = 1, ..., D$. In what follows, we use the system of units where $\hbar = c = 1$.}
The Dirac equation (in the Hamiltonian form) in \((d = D + 1)\)-dimensional Minkowski space-time and with an external electromagnetic field of the form (2.1) reads

\[ i\partial_t \psi(x) = H(t) \psi(x), \quad H(t) = \gamma^0 (\gamma \mathbf{P} + m), \]
\[ P_x = -i\partial_x - U(t), \quad \mathbf{P}_\perp = -i \nabla_\perp, \quad U(t) = qA_x(t), \quad (2.5) \]

where \(H(t)\) is the one-particle Dirac Hamiltonian; \(\psi(x)\) is a \(2^{[d/2]}\)-component spinor; \([d/2]\) stands for the integer part of \(d/2\); \(m \neq 0\) is the electron mass; the index \(\perp\) stands for components of the momentum operator that are perpendicular to the electric field. Here, \(\gamma^\mu\) are the \(\gamma\)-matrices in \(d\) dimensions [46],

\[ [\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \text{diag}(1, -1, -1, \ldots). \]

The number of spin degree of freedom is \(J(d) = 2^{[d/2]} - 1\).

We choose the electron as the main particle with the charge \(q = -e\), where \(e > 0\) is the absolute value of the electron charge, and we refer to \(U(t)\) as the potential energy of an electron in the electric field.

Let us consider solutions of Dirac equation (2.5) of the following form

\[ \psi_n(x) = \exp(i\mathbf{p}r) \psi_n(t), \quad n = (\mathbf{p}, \sigma), \]
\[ \psi_n(t) = \{\gamma^0 i\partial_t - \gamma^1 [p_x - U(t)] - \gamma \mathbf{P}_\perp + m\} \phi_n(t), \quad (2.6) \]

where \(\psi_n(t)\) and \(\phi_n(t)\) are time-dependent spinors. In fact, these are states with a definite momentum \(\mathbf{p} = (p_x, \mathbf{p}_\perp)\). We can separate spin variables by the substitution

\[ \phi_n(t) = \varphi_n(t) v_{\chi, \sigma}, \quad \chi = \pm 1, \quad \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{[d/2] - 1}), \quad \sigma_s = \pm 1, \quad (2.7) \]

where \(v_{\chi, \sigma}\) is a set of constant orthonormalized spinors, satisfying the following equations:

\[ \gamma^0 \gamma^1 v_{\chi, \sigma} = \chi v_{\chi, \sigma}, \quad v_{\chi, \sigma}^\dagger v_{\chi', \sigma'} = \delta_{\chi, \chi'} \delta_{\sigma, \sigma'}. \quad (2.8) \]

In the dimensions \(d > 3\), one can subject the spinors \(v_{\chi}\) to some supplementary conditions, which, for example, may be chosen as

\[ i\gamma^{2s}\gamma^{2s+1} v_{\chi, \sigma} = \sigma_s v_{\chi, \sigma}, \quad \text{for even } d, \]
\[ i\gamma^{2s+1}\gamma^{2s+2} v_{\chi, \sigma} = \sigma_s v_{\chi, \sigma}, \quad \text{for odd } d. \quad (2.9) \]
Quantum numbers $\sigma_s$ describe the spin polarization [in the dimensions $d = 2, 3$ there are no spin degrees of freedom that are described by the quantum numbers $\sigma$], and, together with the additional index $\chi$, provide a convenient parametrization of the solutions. Then the scalar functions $\varphi_n(t)$ have to obey the second order differential equation

$$\left\{ \frac{d^2}{dt^2} + [p_x - U(t)]^2 + \pi_\perp^2 - i\chi \dot{U}(t) \right\} \varphi_n(t) = 0, \quad \pi_\perp = \sqrt{p_\perp^2 + m^2}. \quad (2.10)$$

The quantization of the Dirac field in the background under consideration is based on the existence of solutions to the Dirac equation with special asymptotics as $t \to \pm \infty$, see [6, 23] for details. For instance, we let the electric field be switched on at $t_{\text{in}}$ and switched off at $t_{\text{out}}$, so that the interaction between the Dirac field and the electric field vanishes at all time instants outside the interval $t \in (t_{\text{in}}, t_{\text{out}})$, and the Dirac equation in the Hamiltonian form is given by

$$\begin{align*}
[i \partial_t - H(t)] \zeta \psi_n(x) &= 0, \quad t \in (-\infty, t_{\text{in}}] \\
[i \partial_t - H(t)] \zeta \psi_n(x) &= 0, \quad t \in [t_{\text{out}}, +\infty) .
\end{align*} \quad (2.11)$$

where the additional quantum number $\zeta = \pm$ labels asymptotic states, respectively. These asymptotic states are solutions of eigenvalue problems,

$$\begin{align*}
H(t) \zeta \psi_n(x) &= \zeta \varepsilon_n \zeta \psi_n(x) , \quad t \in (-\infty, t_{\text{in}}] , \quad \zeta \varepsilon_n = \zeta p_0(t_{\text{in}}) , \\
H(t) \zeta \psi_n(x) &= \zeta \varepsilon_n \zeta \psi_n(x) , \quad t \in [t_{\text{out}}, +\infty) , \quad \zeta \varepsilon_n = \zeta p_0(t_{\text{out}}) , \\
p_0(t) &= \sqrt{[p_x - U(t)]^2 + \pi_\perp^2} .
\end{align*} \quad (2.12)$$

In these asymptotic states, $\zeta = +$ correspond to free electrons and $\zeta = -$ corresponds to free positrons. We call the time interval $t \in (-\infty, t_{\text{in}}]$ as the in-region, where the in-set $\{ \zeta \psi_n(x) \}$ is defined. The time interval $t \in [t_{\text{out}}, +\infty)$ is called the out-region, where the out-set $\{ \zeta \psi_n(x) \}$ is defined. In these regions:

$$\zeta \varphi_n(t) = \zeta \mathcal{N} e^{-i \zeta \varepsilon_n t}, \quad t \in (-\infty, t_{\text{in}}] , \quad \zeta \varphi_n(t) = \zeta \mathcal{N} e^{-i \zeta \varepsilon_n t}, \quad t \in [t_{\text{out}}, +\infty) , \quad (2.13)$$

where $\zeta \mathcal{N}$, $\zeta \mathcal{N}'$, are normalization constants, and there exists an energy gap between the electron and positron states.

Then we construct two complete set of solutions to the Dirac equation,

$$\begin{align*}
\zeta \psi_n(x) &= \exp(i \mathbf{p} \mathbf{r}) \left\{ \gamma^0 i \partial_t - \gamma^1 [p_x - U(t)] - \gamma \mathbf{p}_\perp + m \right\} \zeta \varphi_n(t) \psi_{\chi,\sigma}, \\
\zeta \psi_n(x) &= \exp(i \mathbf{p} \mathbf{r}) \left\{ \gamma^0 i \partial_t - \gamma^1 [p_x - U(t)] - \gamma \mathbf{p}_\perp + m \right\} \zeta \varphi_n(t) \psi_{\chi,\sigma}. \quad (2.14)
\end{align*}$$
We suppose additionally that the in- \{ \zeta \psi_n (x) \} and out-sets \{ \zeta \psi_n (x) \} are complete and orthonormal with respect to the standard definition of the inner product [47],
\[
(\psi, \psi') = \int \psi^\dagger (x) \psi' (x) \, dx , \quad dx = dx^1...dx^D . \tag{2.15}
\]

In calculating \( (2.15) \), we use the standard volume regularization, in which the scattering problem is confined by a large spatial box of volume \( V_{(d-1)} = L_1 \times \cdots \times L_D \), with the Dirac spinors subject to periodic boundary conditions at the spatial walls. This inner product is time-independent. Since \( \chi \) is not a physical quantum number if \( d > 3 \) (the spin operator \( \gamma^0 \gamma^1 \) does not commute with the Dirac Hamiltonian \( (2.25) \) in case \( m \neq 0 \)), one can select a particular \( \chi \) to calculate \( (2.15) \). For simplicity, we select the same \( \chi \) in the cases of \( \psi (x) \) and \( \psi' (x) \), so that the inner product is simplified,
\[
(\psi_n , \psi_{n'}) = V_{(d-1)} \delta_{n,n'} \mathcal{I} , \quad \mathcal{I} = \varphi_n^* (t) \left( i \vec{\partial}_t - i \vec{\partial}_t \right) \left\{ i \vec{\partial}_t - \chi [p_x - U (t)] \right\} \varphi_n (t) . \tag{2.16}
\]

Then solutions \( (2.14) \) can be subject to the orthonormality conditions
\[
( \zeta \psi_n , \zeta' \psi_{n'} ) = \delta_{n,n'} \delta_{\zeta,\zeta'} , \quad \left( \zeta \psi_n , \zeta' \psi_{n'} \right) = \delta_{n,n'} \delta_{\zeta,\zeta'} , \tag{2.17}
\]
which, in particular, leads to the following expressions for the normalization constants:
\[
\zeta \mathcal{N} = \zeta CY , \quad \zeta' \mathcal{N} = \zeta' CY , \quad Y = V_{(d-1)}^{-1/2} ,
\]
\[
\zeta C = \left[ 2p_0 (t_{in}) q_{in}^\zeta \right]^{-1/2} , \quad \zeta' C = \left[ 2p_0 (t_{out}) q_{out}^{\zeta'} \right]^{-1/2} ,
\]
\[
q_{in/out}^\zeta = p_0 (t_{in/out}) - \chi \zeta [p_x - U (t_{in/out})] . \tag{2.18}
\]

Completeness relations for the in- and out-sets are given by
\[
\sum_{\zeta,n} \zeta \psi_n (t, r) \zeta' \psi_n^\dagger (t, r') = \delta (r - r') \| = \sum_{\zeta,n} \zeta \psi_n (t, r) \zeta' \psi_n^\dagger (t, r') . \tag{2.19}
\]

Due to property \( (2.16) \) inner products \( ( \zeta' \psi_t , \zeta \psi_n ) \) are diagonal in quantum numbers \( n \) and \( l \),
\[
( \zeta' \psi_t , \zeta \psi_n ) = \delta_{l,n} g ( \zeta | \zeta ' ) , \quad g ( \zeta | \zeta ') = g ( \zeta ' | \zeta )^* . \tag{2.20}
\]

The corresponding diagonal matrix elements \( g \) relate in- and out-solutions \{ \zeta \psi_n (x) \} and \{ \zeta' \psi_n (x) \} for each \( n \),
\[
\zeta \psi_n (x) = g ( + | \zeta ) \psi_n (x) + g ( - | \zeta ) \psi_n (x) ,
\]
\[
\zeta \psi_n (x) = g ( + | \zeta ) \psi_n (x) + g ( - | \zeta ) \psi_n (x) . \tag{2.21}
\]
Substituting Eqs. (2.21), into the orthonormality conditions, we derive the unitarity relations
\[
\sum_\kappa g (\zeta|\kappa) g (\kappa|\zeta') = \sum_\kappa g (\kappa|\zeta) g (\kappa'|\zeta') = \delta_{\zeta,\zeta'} .
\]  
(2.22)

Similar consideration is possible for a scalar fields that satisfies the Klein–Gordon (KG) equation with \(t\)-electric potential step. A formal transition to the case of scalar fields can be done by setting \(\chi = 0\) in Eq. (2.10). The corresponding complete set of solutions to the KG equation reads
\[
\phi_n (x) = \exp (i p r) \varphi_n (t) , \; n = p .
\]  
(2.23)

One can also define complete in- \(\{ \zeta \phi_n \}\) and out-sets \(\{ \zeta ' \phi_n \}\) of solutions orthonormal with respect to the adequate inner product [47],
\[
(\phi, \phi')_{KG} = i \int \phi^* (x) \left( \vec{\partial}_t - \vec{F} \right) \phi' (x) d\mathbf{r} .
\]  
(2.24)

Namely,
\[
(\zeta \phi_n, \zeta' \phi_n')_{KG} = \zeta \delta_{n,n'} \delta_{\zeta,\zeta'} , \quad (\zeta \phi_n, \zeta' \phi_n')_{KG} = \zeta \delta_{n,n'} \delta_{\zeta,\zeta'} ,
\]  
(2.25)

with the normalization constants
\[
\zeta C = [2p_0 (t_{in})]^{-1/2} , \quad \zeta C = [2p_0 (t_{out})]^{-1/2} .
\]  
(2.26)

Inner products \((\zeta \phi_l, \zeta \phi_n)_{KG}\) are diagonal in quantum numbers \(n\) and \(l\),
\[
(\zeta \phi_l, \zeta \phi_n)_{KG} = \delta_{l,n} g (\zeta|\zeta) , \quad g (\zeta|\zeta) = g (\zeta|\zeta)^* .
\]  
(2.27)

The corresponding diagonal matrix elements \(g\) relate in- and out-solutions,
\[
\zeta \phi_n (x) = g (|\zeta) + \phi_n (x) - g (|\zeta) - \phi_n (x) ,
\]
\[
\zeta \phi_n (x) = g (|\zeta) + \phi_n (x) - g (|\zeta) - \phi_n (x) ,
\]  
(2.28)

and satisfy the unitarity relations
\[
\sum_\kappa g (\zeta|\kappa) \times g (\kappa|\zeta') = \sum_\kappa g (\zeta|\kappa) \times g (\kappa' |\zeta') = \zeta \delta_{\zeta,\zeta'} .
\]  
(2.29)

Decomposing the Dirac operator \(\hat{\Psi} (x)\) in the complete sets of in- and out-solutions [6, 23],
\[
\hat{\Psi} (x) = \sum_n \left[ a_n (\text{in}) + \psi_n (x) + b_n^* (\text{in}) \_ \psi_n (x) \right] = \sum_n \left[ a_n (\text{out}) + \psi_n (x) + b_n^* (\text{out}) \_ \psi_n (x) \right] ,
\]  
(2.30)
we introduce in- and out-creation and annihilation Fermi operators. Their nonzero anticommutation relations are,

\[
[a_n(\text{in}), a_m^\dagger(\text{in})]_+ = [a_n(\text{out}), a_m^\dagger(\text{out})]_+ = [b_n(\text{in}), b_m^\dagger(\text{in})]_+ = [b_n(\text{out}), b_m^\dagger(\text{out})]_+ = \delta_{nm}.
\]

(2.31)

In these terms, the Heisenberg Hamiltonian is diagonalized at \( t \leq t_{in} \) and \( t \geq t_{out} \),

\[
\hat{\mathcal{H}}(t) = \sum_n \left\{ +\varepsilon_n a_n^+(\text{in})a_n(\text{in}) + | -\varepsilon_n | b_n^+(\text{in})b_n(\text{in}) \right\}, \quad t \leq t_{in},
\]

\[
\hat{\mathcal{H}}(t) = \sum_n \left\{ +\varepsilon_n a_n^+(\text{out})a_n(\text{out}) + | -\varepsilon_n | b_n^+(\text{out})b_n(\text{out}) \right\}, \quad t \geq t_{out},
\]

(2.32)

where the diverging c-number parts have been omitted, as usual. The initial \(|0,\text{in}\rangle\) and final \(|0,\text{out}\rangle\) vacuum vectors, as well as many-particle in- and out-states, are defined by

\[
a_n(\text{in})|0,\text{in}\rangle = b_n(\text{in})|0,\text{in}\rangle = 0, \quad a_n(\text{out})|0,\text{out}\rangle = b_n(\text{out})|0,\text{out}\rangle = 0,
\]

\[
|\text{in}\rangle = b_n^+(\text{in})...a_n^+(\text{in})...|0,\text{in}\rangle, \quad |\text{out}\rangle = b_n^+(\text{out})...a_n^+(\text{out})...|0,\text{out}\rangle.
\]

(2.33)

Using the charge operator one can see that \( a_n^\dagger, a_n \) are the creation and annihilation operators of electrons, whereas \( b_n^\dagger, b_n \) are the creation and annihilation operators of positrons, respectively.

Transition amplitudes in the Heisenberg representation have the form \( M_{\text{in} \rightarrow \text{out}} = \langle \text{out}|\text{in}\rangle \). In particular, the vacuum-to-vacuum transition amplitude reads \( c_v = \langle 0,\text{out}|0,\text{in}\rangle \). Relative probability amplitudes of particle scattering, pair creation and annihilation are:

\[
w (+|+)_{n'n'} = c_v^{-1}\langle 0,\text{out}|a_{n'}^\dagger(\text{out})a_n^\dagger(\text{in})|0,\text{in}\rangle = \delta_{n,n'} w_n (+|+),
\]

\[
w (-|-)_{n'n'} = c_v^{-1}\langle 0,\text{out}|b_{n'}^\dagger(\text{out})b_n^\dagger(\text{in})|0,\text{in}\rangle = \delta_{n,n'} w_n (-|-),
\]

\[
w (+ - |0)_{n'n} = c_v^{-1}\langle 0,\text{out}|a_{n'}^\dagger(\text{out})b_n(\text{out})|0,\text{in}\rangle = \delta_{n,n'} w_n (+ - |0),
\]

\[
w (0| - + )_{nn'} = c_v^{-1}\langle 0,\text{out}|b_n^\dagger(\text{in})a_{n'}^\dagger(\text{in})|0,\text{in}\rangle = \delta_{n,n'} w_n (0| - +).
\]

(2.34)

The in- and out-operators are related by linear canonical transformations,

\[
a_n(\text{out}) = g(+|+)_n a_n(\text{in}) + g(+|-)_n b_n^\dagger(\text{in}), \quad b_n^\dagger(\text{out}) = g(-|+)_n a_n(\text{in}) + g(-|-)_n b_n^\dagger(\text{in}).
\]

These relations allows one to calculate the differential mean numbers of electrons \( N_n^{\text{a}}(\text{out}) \) and positrons \( N_n^{\text{b}}(\text{out}) \) created from the vacuum state as

\[
N_n^{\text{a}}(\text{out}) = \langle 0,\text{in}|a_n^\dagger(\text{out}) a_n(\text{out})|0,\text{in}\rangle = |g (-|+) |^2,
\]

\[
N_n^{\text{b}}(\text{out}) = \langle 0,\text{in}|b_n^\dagger(\text{out}) b_n(\text{out})|0,\text{in}\rangle = |g (+|-) |^2, \quad N_n^{\text{cfr}} = N_n^{\text{b}}(\text{out}) = N_n^{\text{a}}(\text{out}).
\]

(2.35)
By $N_n^{cr}$ we denote the differential numbers of created pairs. The total number of pairs created from vacuum is given by the sum

$$N = \sum_n N_n^{cr} = \sum_n |g (- |^+)|^2.$$  \hfill (2.36)

Similar consideration is possible for a quantum scalar field $\hat{\Phi}(x)$ that satisfies the KG equation with $t$-electric potential steps. Decomposing the quantum fields $\hat{\Phi}(x)$ in the complete set of exact solutions $\{\pm \phi_n(x)\}$ and $\{\pm \phi_n(x)\}$ one introduces in- and out-creation and annihilation Bose operators and obtain quite similar representations of relative probability amplitudes and the differential numbers of created pairs via the corresponding diagonal matrix elements $g$ defined by Eq. (2.27) \cite{6, 23}.

Both for fermions and bosons, relative probabilities (2.34), the vacuum-to-vacuum transition amplitude $c_v$, the probability for a vacuum to remain a vacuum $P_v$ as well as the total number $N$ of pairs created from vacuum can be expressed via the distribution $N_n^{cr}$,

$$|w_n (+ - |0)|^2 = N_n^{cr} (1 - \kappa N_n^{cr})^{-1}, \quad |w_n (-| -)|^2 = (1 - \kappa N_n^{cr})^{-1},$$

$$P_v = |c_v|^2 = \prod_n (1 - \kappa N_n^{cr})^\kappa, \quad \kappa = \begin{cases} +1 & \text{for fermions} \\ -1 & \text{for bosons} \end{cases}.$$  \hfill (2.37)

The vacuum mean electric current, energy, and momentum are defined as integrals over the spatial volume. Due to the translational invariance in the uniform external field, all these mean values are proportional to the space volume. Therefore, it is enough to calculate the vacuum mean values of the current density vector $\langle j^\mu(t) \rangle$ and of the energy-momentum tensor (EMT) $\langle T_{\mu\nu}(t) \rangle$, defined as

$$\langle j^\mu(t) \rangle = \langle 0, \text{in}|j^\mu|0, \text{in}\rangle, \quad \langle T_{\mu\nu}(t) \rangle = \langle 0, \text{in}|T_{\mu\nu}|0, \text{in}\rangle.$$  \hfill (2.38)

Here we stress the time dependence of mean values (2.38), which does exist due to a time dependence of the external field. We recall for further convenience the form of the operators of the current density and the EMT of the quantum Dirac field,

$$j^\mu = \frac{q}{2} \left[ \overline{\Psi}(x), \gamma^\mu \hat{\Psi}(x) \right], \quad T_{\mu\nu} = \frac{1}{2}(T_{\mu\nu}^{can} + T_{\nu\mu}^{can}),$$

$$T_{\mu\nu}^{can} = \frac{1}{4} \left\{ \overline{\Psi}(x), \gamma_\mu P_{\nu}\hat{\Psi}(x) + [P^{\nu}\overline{\Psi}(x), \gamma_\mu \hat{\Psi}(x)] \right\},$$

$$P_{\mu} = i\partial_\mu - qA_\mu(x), \quad \overline{\Psi}(x) = \hat{\Psi}^\dagger(x) \gamma^0.$$  \hfill (2.39)
Note that the mean values $\langle 2.38 \rangle$ depend on the definition of the initial vacuum, $|0, \text{in}\rangle$ and on the evolution of the electric field from the time $t_{\text{in}}$ of switching it on up to the current time instant $t$, but they do not depend on the further history of the system. The renormalized vacuum mean values $\langle j^\mu (t) \rangle$ and $\langle T_{\mu\nu} (t) \rangle$, $t_{\text{in}} < t < t_{\text{out}}$ are sources in equations of motion for mean electromagnetic and metric fields, respectively. In particular, complete description of the back reaction is related to the calculation of these mean values for any $t$.

Mean values and probability amplitudes are calculated by the help of different kind of propagators. The probability amplitudes are calculated using Feynman diagrams with the causal (Feynman) propagator

$$S^c (x, x') = i \langle 0, \text{out} | \hat{T} \hat{\Psi} (x) \hat{\Psi}^\dagger (x') \gamma^0 | 0, \text{in} \rangle c_v^{-1}, \quad (2.40)$$

where $\hat{T}$ denotes the chronological ordering operation. A perturbation theory (with respect to radiative processes) uses the so-called in-in propagator $S_{\text{in}}^c (x, x')$ and $S^p (x, x')$ propagator,

$$S_{\text{in}}^c (x, x') = i \langle 0, \text{in} | \hat{T} \hat{\Psi} (x) \hat{\Psi}^\dagger (x') \gamma^0 | 0, \text{in} \rangle, \quad S^p (x, x') = S_{\text{in}}^c (x, x') - S^c (x, x'). \quad (2.41)$$

All the above propagators can be expressed via the in- and out-solution as follows:

$$S^c (x, x') = i \left\{ \sum_n ^{+} \psi_n (x) \omega_n (+|+) + \bar{\psi}_n (x'), \quad t > t' \right\}, \quad (2.42)$$

$$S_{\text{in}}^c (x, x') = i \left\{ \sum_n ^{+} \psi_n (x) + \bar{\psi}_n (x'), \quad t > t' \right\} - \left\{ \sum_n ^{-} \bar{\psi}_n (x) - \psi_n (x'), \quad t < t' \right\}, \quad S^p (x, x') = -i \sum_n ^{-} \bar{\psi}_n (x) \omega_n (0|+) + \bar{\psi}_n (2.43)$$

The mean values of the operator $\langle 2.39 \rangle$ are expressed via the latter propagators as

$$\langle j^\mu (t) \rangle = \text{Re} \langle j^\mu (t) \rangle^c + \text{Re} \langle j^\mu (t) \rangle^c_P = i q \text{tr} \left[ \gamma^\mu S^{c, p} (x, x') \right]_{x=x'},$$

$$\langle T_{\mu\nu} (t) \rangle = \text{Re} \langle T_{\mu\nu} (t) \rangle^c + \text{Re} \langle T_{\mu\nu} (t) \rangle^c_P, \quad \langle T_{\mu\nu} (t) \rangle^c_P = i \text{tr} \left[ A_{\mu\nu} S^{c, p} (x, x') \right]_{x=x'} \quad (2.44)$$

Here tr stands for the trace in the $\gamma$-matrices indices and the limit $x \to x'$ is understood as follows:

$$\text{tr} [R (x, x')]_{x=x'} = \frac{1}{2} \left[ \lim_{t \to t' - 0} \text{tr} [R (x, x')] + \lim_{t \to t' + 0} \text{tr} [R (x, x')] \right]_{x=x'},$$

where $R (x, x')$ is any two point matrix function.

The function $S^p (x, y)$ vanishes in the case of a stable vacuum. In this case and only in this case $\langle j^\mu (t) \rangle = \text{Re} \langle j^\mu (t) \rangle^c, \quad \langle T_{\mu\nu} (t) \rangle = \text{Re} \langle T_{\mu\nu} (t) \rangle^c.$
III. SAUTER-LIKE ELECTRIC FIELD

Here we consider quantum effects in a $t$-electric potential step which is formed by the so-called Sauter-like electric field given by Eq. (2.4). The origin of the name of the field is the following: In his pioneer work [48] Sauter studied the Klein paradox considering the case of an inhomogeneous field given by the $x$-electric potential step $-LE \tanh (x/L)$, which is called at present the Sauter potential. The homogeneous $t$-electric step which we are going to considered here has similar form in $t$ coordinate. In a sense, solutions of the Dirac and KG equations with the Sauter-like potential are formally similar to solutions for the Sauter potential.

For the Sauter-like external field, the scalar functions $\varphi_n (t)$ (see the previous section) satisfy equation (2.10) with $U (t) = T _S e E \tanh (t/T _S)$ This field switches-on and -off adiabatically at $t _{in} \to - \infty$ and $t _{out} \to + \infty$. In in- and out-regions, Dirac spinors $\zeta \psi_n (x)$ and $\zeta \psi_n (x)$ are solutions of the eigenvalue problem (2.12) and the plane -wave frequencies are

$$\omega _\pm = p _0 (\pm \infty) = \sqrt{(p _x \pm T _S e E)^2 + \pi _\perp ^2} . \quad (3.1)$$

In the case under consideration, Eq. (2.10) is an equation for a hypergeometric function [49]. Its solutions can be written as

$$\varphi_n (t) = y ^l (1 - y)^m f (y) , \quad y = \frac{1}{2} [1 + \tanh (t/T _S)] ,$$

where $l$ and $m$ are some constants, and the function $f (y)$ is a solution of the Gauss hypergeometric differential equation [49]. We will use complete sets of solutions $\zeta \varphi_n (t)$ and $\zeta \varphi_n (t)$,

$$\zeta \varphi_n (t) = \zeta \mathcal{N} \exp (-i \zeta \omega _- t) \left[ 1 + e^{2t/T _S} \right] \frac{T _S}{2} (\zeta \omega _- - \omega _+) \zeta u (t) ,$$

$$+ u (t) = F (a , b; c; y) , \quad - u (t) = F (a + 1 - c , b + 1 - c; 2 - c; y) ;$$

$$\zeta \varphi_n (t) = \zeta \mathcal{N} \exp (-i \zeta \omega _+ t) \left[ 1 + e^{-2t/T _S} \right] \frac{T _S}{2} (\omega _- - \zeta \omega _+) \zeta u (t) ,$$

$$+ u (t) = F (a , b; a + b + 1 - c; 1 - y) , \quad - u (t) = F (c - a , c - b; c + 1 - a - b; 1 - y) ,$$

$$a = \frac{i T _S}{2} (\omega _+ - \omega _-) + \frac{1}{2} + \left( \frac{1}{4} - (e ET _S^2)^2 - i \chi e ET _S^2 \right)^{1/2} , \quad c = 1 - i T _S \omega _- ,$$

$$b = \frac{i T _S}{2} (\omega _+ - \omega _-) + \frac{1}{2} - \left( \frac{1}{4} - (e ET _S^2)^2 - i \chi e ET _S^2 \right)^{1/2} . \quad (3.2)$$
where \( F(a, b; c; y) \) is the hypergeometric series in the variable \( y \) with the normalization \( F(a, b; c; 0) = 1 \). As was already mentioned in Sec. 11, the quantity \( \chi \) can be chosen to be either \( \chi = +1 \) or \( \chi = -1 \), and \( \mathcal{N} \) and \( \mathcal{N} \) are normalization factors given by Eq. (2.18).

A formal transition to the Bose case can be done by setting \( \chi = 0 \) in Eqs. (3.2). In this case \( n = p \) and \( \mathcal{N} \) and \( \mathcal{N} \) are normalization factors given by Eq. (2.26).

In Fermi case, using Kummer’s relations and Eqs. (2.20), one can find coefficients \( g(\pm \mid -)^* \) to be

\[
g(\pm \mid -)^* = \frac{+C \Gamma(c) \Gamma(a + b - c)}{-C \Gamma(a) \Gamma(b)}, \tag{3.3}
\]

where \( +C \) and \( -C \) are constants given by Eq. (2.18) and \( \Gamma(a) \) is the Euler gamma function.

Then, using Eq. (2.35), we obtain the mean number of created pairs,

\[
N_{cr}^{\min} = \frac{\sinh \left\{ \pi T_S \left[ eET_S + \frac{1}{2} (\omega_+ - \omega_-) \right] \right\} \sinh \left\{ \pi T_S \left[ eET_S - \frac{1}{2} (\omega_+ - \omega_-) \right] \right\}}{\sinh (\pi T_S \omega_+) \sinh (\pi T_S \omega_-)}. \tag{3.4}
\]

In 3+1 QED the corresponding formula was found first in [19].

In the similar manner, in the Bose case, we obtain coefficients \( g(\pm \mid -)^* \), where \( +C \) and \( -C \) are given by Eqs. (2.20) and parameters \( a, b, \) and \( c \) are given by Eq. (3.2) at \( \chi = 0 \). Here, the mean number for created pairs is

\[
N_{cr} = \frac{\cosh^2 \left[ \pi \sqrt{(T_S^2 eE)^2 - \frac{1}{4}} \right] + \sinh^2 \left[ \frac{\pi T_S}{2} (\omega_+ - \omega_-) \right]}{\sinh (\pi T_S \omega_+) \sinh (\pi T_S \omega_-)}. \tag{3.5}
\]

It should be noted that mean numbers (3.4) and (3.5) are even functions of all the momentum \( p \). In particular it can be seen that \( p_x \to -p_x \) leads to \( \omega_+ \to -\omega_- \). The most important parameter in the case under consideration is \( T_S \). Its value determines the effective time of electric field action.

We begin our analysis by considering \( T_S \) small and constant values for the asymptotic potentials, \( U(\pm \infty) = -U(\mp \infty) = U/2 = eET_S \). In this case we deal with a very short pulse field. The corresponding potential imitates sufficiently well a \( t \)-electric rectangular potential step, and coincides with the latter as \( T_S \to 0 \). Thus, the Sauter-like potential can be considered as a regularization of the rectangular step. We assume that sufficiently small \( T_S \) for given \( \omega_\pm \) satisfies the inequalities

\[
\mathbbm{U} T_S \ll 1, \quad \text{max} \{ T_S \omega_+, T_S \omega_- \} \ll 1. \tag{3.6}
\]
In such a case mean numbers of created pairs are
\[ N^\text{cr}_n = \frac{U^2 - (\omega_+ - \omega_-)^2}{4\omega_+ \omega_-} \text{ in Fermi case,} \quad (3.7) \]
\[ N^\text{cr}_n = \frac{T_S^2 U^4 / 4 + (\omega_+ - \omega_-)^2}{4\omega_+ \omega_-} \text{ in Bose case.} \quad (3.8) \]

The number of created fermions in Eq. (3.7) does not depend on \( T_S \). However, in contrast with the Fermi case, the limit \( T_S \to 0 \) in Eq. (3.8) is possible only when the difference \((\omega_+ - \omega_-)^2\) is not very small, namely, when

\[ T_S^2 U^4 / 4 \ll (\omega_+ - \omega_-)^2 . \quad (3.9) \]

Only under the latter condition one can neglect an \( T_S \)-depending term in Eq. (3.8) to obtain

\[ N^\text{cr}_n = \frac{(\omega_+ - \omega_-)^2}{4\omega_+ \omega_-} . \quad (3.10) \]

Unlike the Fermi case, where \( N^\text{cr}_n \leq 1 \), in the Bose case, the mean number of created particles is unlimited in two ranges of the longitudinal kinetic momenta, namely when either \( \omega_+ / \omega_- \to \infty \) or \( \omega_- / \omega_+ \to \infty \),

\[ N^\text{cr}_n \approx \frac{1}{4} \max \{ \omega_+ / \omega_-, \omega_- / \omega_+ \} . \quad (3.11) \]

One can see that in the Fermi and Bose cases \( N^\text{cr}_n \to 0 \) as \( \pi_\perp \to \infty \). On a sufficiently high step and small transversal momentum, \( \pi_\perp / U \ll 1 \), one finds that the maximum mean number of bosons is only limited by the potential difference \( U \),

\[ \max N^\text{cr}_n \approx \frac{U}{4\pi_\perp} . \]

The maximum mean numbers of fermions \( N^\text{cr}_n \to 1 \) are in the range of small \( \pi_\perp \) and \(|p_x|\), when longitudinal kinetic momenta are large, \((p_x \mp U/2) \sim U/2\).

The Sauter-like potential is suitable for imitating a slowly alternating electric field. To this end the parameter \( T_S \) is taken to be sufficiently large. Let us consider just this case, supposing that

\[ T_S \gg \max \left( 1/\sqrt{eE, m/eE} \right) . \quad (3.12) \]

For both the Fermi and Bose cases, one can check that the mean numbers (3.11) and (3.5) are negligibly small,

\[ N^\text{cr}_n \ll e^{-\pi m^2/eE} , \quad (3.13) \]
for any given $p_\perp$ and for small kinetic momenta
\[
|p_x \pm eET_S| = \sqrt{eEK_S} \ll eET_S, \quad K_S \gg \max \left(1, m/\sqrt{eE}\right),
\]
where $K_S$ is any given number.

For the range of large longitudinal kinetic momenta,
\[
|p_x \pm eET_S| > \sqrt{eEK_S} \iff |p_x| < eET_S - \sqrt{eEK_S}, \quad \text{ (3.14)}
\]
and any given $p_\perp$, mean numbers (3.4) and (3.5) have approximately the following form
\[
N_{n}^{cr} \approx N_{n}^{as} = e^{-\pi \tau}, \quad \tau = T_S(\omega_+ + \omega_- - 2eET_S).
\]

The function $\tau$ has a minimum at $p_x = 0$,
\[
\tau_0 = \tau|_{p_x=0} = T_S \left[ 2\sqrt{\pi_\perp^2 + (eET_S)^2} - 2eET_S \right], \quad \text{(3.16)}
\]
and is growing monotonically as $|p_x|$ and $p_\perp$ grow. One can see that mean numbers $N_{n}^{as}$ are exponentially small in the range of large transversal momenta, $\pi_\perp \gg \sqrt{eEK_S}$. Therefore the following range of $\pi_\perp$ is of interest,
\[
\pi_\perp \ll \sqrt{eEK_S}. \quad \text{(3.17)}
\]

In this range, the following approximation holds true
\[
\tau \approx \frac{eET_S^2 \pi_\perp^2}{(eET_S)^2 - p_x^2}, \quad \tau_0 \approx \frac{\pi_\perp^2}{eE}. \quad \text{(3.18)}
\]

The function $\tau$ takes its maximum value
\[
\tau_{\max} = \tau|_{p_x=eET_S-\sqrt{eEK_S}} \approx \frac{\sqrt{eET_S^3\lambda}}{2K_S}
\]
as $|p_x|$ tends to its maximum. For $m \neq 0$, we see that $\tau_{\max} \to \infty$ as $\sqrt{eET_S} \to \infty$. In the wide range of transversal momenta, $\pi_\perp \ll eET_S$, the mean numbers $N_{n}^{as}$ do not depend practically on the parameter $T_S$ and coincide with differential numbers of created particles in a constant electric field [12, 13]
\[
N_{n}^{as} \approx N_{n}^{0} = e^{-\pi \lambda}. \quad \text{(3.19)}
\]

The total number of pairs created from a vacuum (defined by Eq. (2.36)) by an uniform electric field, is proportional to the space volume $V_{(d-1)}$ as $N^{cr} = V_{(d-1)}n^{cr}$ and the corresponding number density $n^{cr}$ has the form
\[
n^{cr} = \frac{1}{(2\pi)^{d-1}} \sum_{\sigma} \int d\mathbf{p} N_{n}^{cr}. \quad \text{(3.20)}
\]
In deriving Eq. (3.20) the sum over all momenta \( p \) was transformed into an integral. Then the integral in the right hand side of Eq. (3.20) can be approximated by an integral over a subrange \( \Omega \) that gives the dominant contribution with respect to the total increment to the number density of created particles,

\[
\Omega : n^\text{cr} \approx \tilde{n}^\text{cr} = \frac{1}{(2\pi)^{d-1}} \sum_\sigma \int_{p \in \Omega} d_p N^\text{cr}_n.
\] (3.21)

Let us consider the number density of pairs created from the vacuum by the Sauter-like potential with a large parameter \( T_S \). This quantity can be calculated using Eq. (3.21) with differential numbers \( N^\text{cr}_n \) approximated by Eqs. (3.15) and (3.18). In this case, the leading term, \( \tilde{n}^\text{cr} \), is formed over the range given by Eqs. (3.14) and (3.17), that is, this range is chosen as a realization of the subrange \( \Omega \) in Eq. (3.21). In this approximation, the numbers \( N^\text{as}_n \) are the same for fermions and bosons and do not depend on the spin polarization parameters \( \sigma_s \). Thus, in the Fermi case, probabilities and mean numbers summed over all \( \sigma_s \) obtain the factor \( J_d = 2^{d/2} - 1 \). We obtain that

\[
\tilde{n}^\text{cr} = \frac{J_d}{(2\pi)^{d-1}} \int_{p \in \Omega} d_p N^\text{cr}_n.
\] (3.22)

In the case of scalar bosons, \( J_d = 1 \).

Taking into account Eqs. (3.15) and (3.18), we approximate integral (3.22) as

\[
\tilde{n}^\text{cr} \approx \frac{J_d}{(2\pi)^{d-1}} \int d_{p_\perp} I_{p_\perp}, \quad I_{p_\perp} = 2 \int_0^{eET_S - \sqrt{eEK_S}} dp_x e^{-\pi \tau}.
\] (3.23)

It is convenient to introduce a variable \( t \), defined as \( \tau = \lambda t + \tau_0 \). Taken into account Eq. (3.18) we can find a relation between \( t \) and \( p_x \), and see that

\[
dp_x = \frac{1}{2} e^{ET_S} t^{-1/2}(t + 1)^{-3/2} dt.
\] (3.24)

Neglecting the contribution from \( \tau > \tau_{\text{max}} \) and using the variable \( t \), one can represent the quantity \( I_{p_\perp} \) as follows

\[
I_{p_\perp} \approx e^{ET_S} \int_0^\infty dt t^{-1/2}(t + 1)^{-3/2} e^{-\pi\lambda(t+1)}.
\] (3.25)

In particular, using Eq. (3.25), one can find the number density of created pairs with a given \( p_\perp \) for large and small \( \lambda \) in the following form

\[
I_{p_\perp} \approx \frac{e^{ET_S}}{\sqrt{\lambda}} e^{-\pi\lambda} \text{ if } \lambda \gg 1, \quad I_{p_\perp} \approx 2 e^{ET_S} \text{ if } \lambda \ll 1.
\] (3.26)
Finally, substituting Eq. (3.25) into integral (3.23) and performing the integration over $p_\perp$, we obtain

$$\tilde{n}_{cr} = \frac{J(d)T_S\delta}{(2\pi)^{d-1}}(eE)^{d/2} \exp\left(-\frac{\pi m^2}{eE}\right),$$

(3.27)

where

$$\delta = \int_0^\infty dt t^{-1/2}(t + 1)^{-(d+1)/2} \exp\left(-t\pi \frac{m^2}{eE}\right) = \sqrt{\pi}\Psi\left(\frac{1}{2}, \frac{2 - d}{2}; \pi \frac{m^2}{eE}\right).$$

(3.28)

Here $\Psi(a, b; x)$ is the confluent hypergeometric function [49]. This result was first obtained in Ref. [20]. We see that the number density $\tilde{n}_{cr}$, given by Eq. (3.27), is proportional to the total increment of the longitudinal kinetic momentum, $\Delta U_S = e|A_x(+\infty) - A_x(-\infty)| = 2eET_S$. 

From this result one can find the vacuum-to-vacuum probability $P_v$, defined by Eq. (2.37). Using the identity

$$\ln(1 \pm x) = \pm x + \ldots,$$

and performing an integration following the considerations above, one gets the following approximation

$$P_v \approx \exp\left(-\mu^S V_{(d-1)}\tilde{n}_{cr}\right), \quad \mu^S = \sum_{l=0}^\infty \frac{(-1)^{(1-\kappa)/2}\epsilon_l^S}{(l + 1)^{d/2}} \exp\left(-l\pi \frac{m^2}{eE}\right),$$

(3.29)

$$\epsilon_l^S = \delta^{-1}\sqrt{\pi}\Psi\left(\frac{1}{2}, \frac{2 - d}{2}; l\pi \frac{m^2}{eE}\right).$$

In 3+1 QED the same result was found in a different way [21].

If the Sauter-like field is weak, $m^2/eE \gg 1$, one can use asymptotic expression for the $\Psi$-function [49],

$$\Psi\left(\frac{1}{2}, \frac{2 - d}{2}; l\pi m^2/eE\right) = (eE/l\pi m^2)^{1/2} + O\left([eE/m^2]^{3/2}\right).$$

(3.30)

Then $\delta \approx \sqrt{eE/m}$, $\epsilon_l^S \approx l^{-\frac{d}{2}}$ and $\mu^S \approx 1$. In the case of a very strong field, $m^2/eE \ll 1$, one obtains from Ref. [49] that the leading term for the $\Psi$-function does not depend on the parameter $m^2/eE$,

$$\Psi\left(\frac{1}{2}, \frac{2 - d}{2}; \pi m^2/eE\right) \approx \Gamma(d/2)/\Gamma(d/2 + 1/2).$$

(3.31)

Then, for example, $\delta \approx \pi/2$ if $d = 3$ and $\delta \approx 4/3$ if $d = 4$. For the very strong field, $l\pi m^2/eE \ll 1$, the leading contribution of $\epsilon_l^S$ has a quite simple form and does not depend on the dimension, $\epsilon_l^S \approx 1$. In this case

$$\mu^S \approx \sum_{l=0}^\infty \frac{(-1)^{(1-\kappa)/2}}{(l + 1)^{d/2}}.$$
IV. $T$-CONSTANT ELECTRIC FIELD

In this section we present a detailed study of the particle creation problem from the vacuum by a $T$-constant field. This field corresponds to a regularized version of the constant field $E(t) = E$, in which the electric field remains switched on for all the time $t \in (\infty, +\infty)$. This regularization was first considered in Ref. [29] and then developed in Ref. [20]. In the present section we explore additional peculiarities concerning particle creation, supplementing the previous considerations with new details. The $T$-constant electric field is constant within the time interval $T$ and is zero outside of it,

$$E(t) = \begin{cases} 
0, & t \in I \\
E, & t \in \Pi \\
0, & t \in \Pi
\end{cases} \quad \implies A_x(t) = \begin{cases} 
-Et_{in}, & t \in I \\
-Et, & t \in \Pi \\
-Et_{out}, & t \in \Pi
\end{cases}, \quad (4.1)$$

where $I$ denotes the in-region $t \in (-\infty, t_{in}]$, II is the intermediate region where the electric field is non zero $t \in (t_{in}, t_{out})$ and III is the out-region $t \in [t_{out}, +\infty)$ and $t_{out}, t_{in}$ are constants, $t_{out} - t_{in} = T$. We choose $t_{out} = -t_{in} = T/2$. It the in-region I and in out-region III, Dirac spinors are solutions of the eigenvalue problem $(2.12)$.

For $t \in \Pi$, $U(t) = eEt$, equation $(2.10)$ can be written in the form

$$\left[ \frac{d^2}{d\xi^2} + \xi^2 - i\chi + \lambda \right] \varphi_n(t) = 0, \quad (4.2)$$

where

$$\xi = \frac{eEt - px}{\sqrt{eE}}, \quad \lambda = \frac{\pi^2}{eE}. \quad (4.3)$$

The general solution of Eq. $(4.2)$ is completely determined by an appropriate pair of the linearly independent Weber parabolic cylinder functions (WPCFs) [49]: either $D_\rho[(1 - i)\xi]$ and $D_{-1-\rho}[(1 + i)\xi]$, or $D_\rho[-(1 - i)\xi]$ and $D_{-1-\rho}[-(1 + i)\xi]$, where $\rho = i\chi/2 - (1 - \chi)/2$. Then taking into account Eq. $(2.21)$, the functions $^-\varphi_n(t)$ and $^+\varphi_n(t)$ can be presented in
the form
\[-\varphi_n (t) = \begin{cases} 
- C \exp [i p_0 (t_{in}) (t - t_{in})], & t \in I \\
- C \{a_1 D_{\rho} [(1 - i) \xi] + a_2 D_{\rho - 1} [-(1 + i) \xi]\}, & t \in II \\
g (+ | -) + C \exp [- i p_0 (t_{out}) (t - t_{out})] + \kappa g (| -) - C \exp [i p_0 (t_{out}) (t - t_{out})], & t \in III \\
\end{cases}\]
\[g (\pm | +) + C \exp [- i p_0 (t_{in}) (t - t_{in})] + \kappa g (| -) - C \exp [i p_0 (t_{in}) (t - t_{in})], & t \in I \]
\[+ \varphi_n (t) = \begin{cases} 
+ C \{a_1 D_{\rho} [(1 - i) \xi] + a_2 D_{\rho - 1} [-(1 + i) \xi]\}, & t \in II \\
+ C \exp [- i p_0 (t_{out}) (t - t_{out})], & t \in III \\
\end{cases}\]

on the whole axis $t$. Here $\kappa = 1$ and the normalization constants are given by Eqs. (2.18).

The functions $-\varphi_n (t)$ and $+\varphi_n (t)$ and their derivatives satisfy the following gluing conditions:

\[\mp \varphi_n (t_{in,\text{out}} - 0) = \pm \varphi_n (t_{in,\text{out}} + 0), \quad \partial_t \mp \varphi_n (t) |_{t=t_{in,\text{out}}-0} = \partial_t \mp \varphi_n (t) |_{t=t_{in,\text{out}}+0}. \tag{4.5}\]

Using Eq. (4.5) and the Wronskian determinant of WPCFs (4.9),

\[D_{\rho} (z) \frac{d}{dz} D_{\rho - 1} (iz) - D_{\rho - 1} (iz) \frac{d}{dz} D_{\rho} (z) = \exp \left[ - \frac{i \pi}{2} (\rho + 1) \right], \tag{4.6}\]

we find the coefficients $a_j$ and $a'_j$,

\[a_j = \frac{(-1)^j}{\sqrt{2}} \exp \left[ \frac{i \pi}{2} \left( \rho + \frac{1}{2} \right) \right] \sqrt{\xi_1^2 + \lambda f_j^{(+)} (\xi_1)}, \quad j = 1, 2, \tag{4.7}\]

\[a'_j = \frac{(-1)^j}{\sqrt{2}} \exp \left[ \frac{i \pi}{2} \left( \rho + \frac{1}{2} \right) \right] \sqrt{\xi_2^2 + \lambda f_j^{(-)} (\xi_2)}, \quad a_j = \frac{(-1)^j}{\sqrt{2}} \exp \left[ \frac{i \pi}{2} \left( \rho + \frac{1}{2} \right) \right] \sqrt{\xi_1^2 + \lambda f_j^{(+)}} (\xi_1), \quad j = 1, 2, \tag{4.8}\]

where

\[\xi_{1,2} = \xi |_{t=t_{in,\text{out}}} = \mp e E t / 2 - p_x \tag{4.8};\]

\[f_1^{(\pm)} (\xi) = \left( 1 \pm \frac{i}{\sqrt{\xi^2 + \lambda}} \frac{d}{d\xi} \right) D_{\rho - 1} [\mp (1 + i) \xi], \tag{4.8}\]

\[f_2^{(\pm)} (\xi) = \left( 1 \pm \frac{i}{\sqrt{\xi^2 + \lambda}} \frac{d}{d\xi} \right) D_{\rho} [\mp (1 - i) \xi]. \tag{4.8}\]

Note that the following relations hold: $p_0 (t_{in}) / \sqrt{e E} = \sqrt{\xi_1^2 + \lambda}$ and $p_0 (t_{out}) / \sqrt{e E} = \sqrt{\xi_2^2 + \lambda}$. Using Eqs. (4.6) one can determine the coefficients $g (\pm | +)$ and $g (\pm | -)$. It should
be noted that we need to know explicitly only the coefficients $g(\pm |\pm)$, which are

$$g(\pm |\pm) = AB \exp \left[ (\rho + 1/2) i\pi/2 \right], \quad g(- |\pm) = A'B' \exp \left[ (\rho + 1/2) i\pi/2 \right],$$

$$A = \left[ \frac{\sqrt{\xi_1^2 + \lambda \xi_2^2 + \lambda} \left( \sqrt{\xi_1^2 + \lambda + \chi \xi_2^2} \right)}{8 \left( \sqrt{\xi_1^2 + \lambda - \chi \xi_2^2} \right)} \right]^{1/2}, \quad B = f_2^{(\pm)}(\xi_1)f_1^{(\pm)}(\xi_2) - f_1^{(\pm)}(\xi_1)f_2^{(\pm)}(\xi_2),$$

$$A' = \left[ \frac{\sqrt{\xi_1^2 + \lambda \xi_2^2 + \lambda} \left( \sqrt{\xi_1^2 + \lambda + \chi \xi_2^2} \right)}{8 \left( \sqrt{\xi_1^2 + \lambda - \chi \xi_2^2} \right)} \right]^{1/2}, \quad B' = f_1^{(-)}(\xi_1)f_2^{(-)}(\xi_2) - f_2^{(-)}(\xi_1)f_1^{(-)}(\xi_2).$$

One can see that $\xi_2|_{p_x \to -p_x} = -\xi_1$ then coefficients (4.9) obey the relations

$$g(\pm |\pm)|_{p_x \to -p_x} = g(- |\pm).$$

From these relations, we see that $|g(- |\pm)|$ is an even function of momenta $p$ and does not depend on a spin polarization.

Taking into account Eq. (2.28), a formal transition to the Klein-Gordon case can be done by setting $\chi = 0$ and $\kappa = -1$ in Eqs. (4.9). In this case $n = p$ and the normalization factors are given by Eq. (2.26). In the Klein-Gordon case, the coefficients $g$ are

$$g(\pm |\pm) = \exp (-\lambda \pi/4) A_{sc} B|_{\lambda = 0}, \quad g(- |\pm) = -\exp (-\lambda \pi/4) A_{sc} B'|_{\lambda = 0},$$

$$A_{sc} = \left( \frac{1}{8} \sqrt{\xi_1^2 + \lambda \xi_2^2 + \lambda} \right)^{1/2},$$

where $B$ and $B'$ are given by Eqs. (4.9). We stress that this results are new.

The differential mean numbers of created pairs have the form $N_{cr}^{\pm} = |g(- |\pm)|^2$, see Eq. (2.35), where $g(- |\pm)$ is given by Eqs. (4.9) for Dirac particles and by Eqs. (4.11) for Klein-Gordon particles. They depend only on the values $\xi_{1,2}$ for a given $\lambda$. The $T$-constant field is a regularization for a constant uniform electric field and it is suitable for imitating a slowly varying field. That is why the $T$-constant field with a sufficiently large time interval $T$,

$$\sqrt{eET} \gg \max (1, m^2/eE),$$

is of interest. In what follows, we suppose that these conditions hold true and additionally assume that

$$\sqrt{\lambda} < K_\perp,$$

where $K_\perp$ is any given number satisfying the condition $\sqrt{eET}/2 \gg K_\perp^2 \gg \max \{1, m^2/eE\}$. 21
Let us analyze how the numbers $N_n^{cr}$ depend on the parameters $\xi_{1,2}$ and $\lambda$. Let for fermions $\chi = 1$ and $\rho = i\lambda/2 = \nu$. Since $N_n^{cr}$ are even functions of $p_z$, we can consider only the case of $p_z \leq 0$. In this case $\xi_2 = \sqrt{eET}/2$ is large, $\xi_2 \gg \max \{1, \lambda\}$, and the asymptotic expansions of WPCFs with respect to $\xi_2$ are valid. As to the parameter $\xi_1$, the whole interval $-\sqrt{eEL}/2 \leq \xi_1 \leq +\infty$ can be divided in three ranges:

$$(a) \quad -\sqrt{eET}/2 \leq \xi_1 \leq -K, \quad (b) \quad -K \leq \xi_1 < K, \quad (c) \quad \xi_1 \geq K,$$

where $K$ is any given number satisfying the condition $\sqrt{eET}/2 \gg K \gg K_1^2$. Using the asymptotic expansions of WPCFs with respect to $\xi_{1,2}$, we get the following expansions of the coefficients $f_{1,2}^{(-)}(\xi)$, given by Eqs. (4.8),

$$f_1^{(-)}(\xi) = e^{-i\xi^2/2} \left(2e^{i\pi/4}\xi\right)^{-\nu-1} \left[\frac{i}{\xi^2} + O(\xi^{-4})\right],$$

$$f_2^{(-)}(\xi) = e^{i\xi^2/2} \left(\sqrt{2}e^{-i\pi/4}\xi^\nu\right) 2 \left[1 + i\nu \frac{(1 - \nu)}{4\xi^2} + O(\xi^{-4})\right] \quad \text{if} \quad \xi \geq K;$$

$$f_1^{(-)}(\xi) = -e^{-i\xi^2/2} \left(2e^{i\pi/4}|\xi|\right)^{-\nu-1} e^{i\nu \nu} \left[2 + i \left(\frac{3}{2} \nu + \nu^2\right) \xi^{-2} + O(\xi^{-4})\right]$$

$$+ e^{i\xi^2/2} \left(\sqrt{2}e^{-i\pi/4}|\xi|\right)^\nu e^{i\nu/2} \frac{\sqrt{2\pi}}{2\Gamma(-\nu)\xi^4},$$

$$f_2^{(-)}(\xi) = ie^{-i\xi^2/2} \left(\sqrt{2}e^{i\pi/4}|\xi|\right)^{-\nu-1} e^{i\nu/2} \frac{\sqrt{2\pi}}{\Gamma(-\nu)} \left[2 + O(\xi^{-2})\right] \quad \text{if} \quad \xi < 0, |\xi| \geq K^{(4.15)}.$$

One can use Eq. (4.15) with respect to $\xi_1$ and $\xi_2$ for the cases (a) and (c). In the case (c), we find that the quantity $N_n^{cr}$ is very small,

$$N_n^{cr} \sim \max \{|\xi_1|^{-6}, |\xi_2|^{-6}\} \quad \text{if} \quad \min \{|\xi_1|, |\xi_2|\} \geq K.$$  

(4.16)

In the case (a), we obtain

$$N_n^{cr} = e^{-\pi\lambda} \left[1 + (1 - e^{-\pi\lambda})^{1/2} \frac{\sqrt{\lambda}}{2} \left(\frac{\sin \phi_1}{|\xi_1|^3} + \frac{\sin \phi_2}{|\xi_2|^3}\right) + O(|\xi_1|^{-4}) + O(|\xi_2|^{-4})\right],$$

$$\phi_{1,2} = (\xi_{1,2})^2 + \lambda \ln \left(\sqrt{2}|\xi_{1,2}|\right) - \arg \Gamma\left(i\lambda/2\right) - \pi/4.$$  

(4.17)

Consequently, the quantity (4.17) is almost constant over the wide range of longitudinal momentum $p_z$ for any given $\lambda$ satisfying Eq. (4.13). Note that the next-to-leading oscillating terms in Eq. (4.17) presented here improve an approximation obtained before in Ref. [20]. When $\sqrt{eET} \to \infty$, one obtains the result in a constant uniform electric field, given by Eq. (3.19), setting $|\xi_{1,2}| \to \infty$ in Eq. (4.17).
In the intermediate range (b), using the only asymptotics with respect to \( \xi_2 \) given by Eq. (4.15) and the exact form of \( f_1^{(-)}(\xi_1) \) given by Eq. (4.8), we find that

\[
N_{cr}^n = \frac{1}{4} e^{-\pi \lambda/4} \sqrt{\xi_1^2 + \lambda} \left( \sqrt{\xi_1^2 + \lambda - \xi_1} \right) |f_1^{(-)}(\xi_1)|^2. \tag{4.18}
\]

One can make some conclusions about the contribution of this region to the integral over the longitudinal momentum in Eq. (3.20). Taking into account that \( N_{cr}^n \) is always less than unity for fermions, one can get a rough estimation of the integral

\[
\int_{|\xi_1|<\lambda} N_{cr}^n dp_x < 2 eEK
\]

and conclude that it is not essential in comparison with the integral over the longitudinal momentum in the range (a) at \( T \to \infty \). A more accurate estimations can be made numerically. Thus, in the case of strong field, \( \lambda \lesssim 1 \), one can see that the contribution from the intermediate region (b) to the integral is much less than that given by a rough estimate. In particular, one can see that the value \( K = 3 \) is sufficiently large for the problem in question.

For bosons \( \chi = 0 \) and \( \rho = i\lambda/2 - 1/2 \). We have to consider the same three ranges (4.14), wherein only the range (a) is essential. In this range, using the asymptotic expansions of WPCFs we obtain that the differential mean number of scalar particles created is

\[
N_{cr}^n \simeq e^{-\pi \lambda} \left\{ 1 + \frac{1}{2} \left( 1 + e^{\pi \lambda} \right)^{1/2} \left( \frac{\sin \vartheta_1}{|\xi_1|^2} + \frac{\sin \vartheta_2}{\xi_2^2} \right) \right\},
\]

\[
\vartheta_{1,2} = (\xi_{1,2})^2 + \lambda \log \left( \sqrt{2} |\xi_{1,2}| \right) + \arg \left[ \Gamma \left( \frac{1 + i\lambda}{2} \right) + \frac{3\pi}{8} \right]. \tag{4.19}
\]

Note that the next-to-leading term approximation (4.19) is presented here for the first time. When \( \sqrt{eET} \to \infty \), one obtains the same limit form (3.19) in a constant uniform electric field, setting \( |\xi_{1,2}| \to \infty \) in Eq. (4.19).

Let us consider the number density \( n_{cr}^n \) of pairs created by the \( T \)-constant field of large time duration \( T \). In general, the number density \( n_{cr}^n \) of pairs created by uniform field is given by integral (3.20) both for fermions and bosons. The parameter \( K \) plays the role of a sharp cutoff in integral (3.20). Therefore, the main contribution to this integral is due to an subrange \( D \) that is defined by Eq. (4.13) and the range (a) given by Eq. (4.14) for \( p_x \leq 0 \). Taking into account that \( n_{cr}^n \) is an even function of \( p_x \), we find the complete subrange \( D \) as

\[
D : \sqrt{\lambda} < K_\perp, \ |p_x|/\sqrt{eE} < \sqrt{eET}/2 - K,
\]

\[
\sqrt{eET}/2 \gg K \gg K_\perp^2 \gg \max \{1, m^2/eE \}. \tag{4.20}
\]
In this subrange \( N_{n}^{cr} \approx e^{-\pi \lambda} \) both for fermions and bosons. Then one can find the total number of created particles with given transversal momentum and spin polarization but with all possible values of longitudinal momentum:

\[
N_{p_{\perp},\sigma} = \frac{L}{2\pi} \int dp_{\perp} N_{n}^{cr} = \Delta_{long} e^{-\pi \lambda}, \quad \Delta_{long} = \frac{\sqrt{eEL}}{2\pi} \left[ \sqrt{eET} + O(K) \right], \tag{4.21}
\]

where \( L \) is the regularization length in the direction of the field. The factor \( \Delta_{long} \) can be interpreted as a number of quantum states with a longitudinal momentum, in which the particles can be created. If \( \sqrt{eET} \) is big enough, the dependence on \( K \) and \( K_{\perp} \) can be ignored, that is, the form of \( N_{n}^{cr} \) is unchanged in the inner subrange \( D \). Thus, the definition of the subrange \( D \) (4.20) can be also treated as the stabilization condition for \( N_{n}^{cr} \), that is, this range is chosen as a realization of the subrange \( \Omega \) in Eq. (3.21). In this approximation, the numbers \( N_{n}^{cr} \) for fermions and bosons are equal, such that one can represent the number density \( \tilde{n}^{cr} \) in the form Eq. (3.22).

Using Eq. (4.21) we approximate integral (3.22) as

\[
\tilde{n}^{cr} = \frac{J(d)}{(2\pi)^{d-1}} \int_{D} dp_{\perp} e^{-\pi \lambda} \approx \frac{J(d)}{(2\pi)^{d-1}} \int_{\lambda<K_{\perp}} dp_{\perp} I_{p_{\perp}}, \quad I_{p_{\perp}} = eET e^{-\pi \lambda}. \tag{4.22}
\]

Performing the integration over \( p_{\perp} \), and neglecting exponentially small contribution from the range \( \sqrt{\lambda} > K_{\perp} \), we finally obtain [20]

\[
\tilde{n}^{cr} = r^{cr} \left[ T + \frac{O(K)}{\sqrt{eE}} \right], \quad r^{cr} = \frac{J(d) (eE)^{d/2}}{(2\pi)^{d-1}} \exp \left\{ -\frac{m^2}{eE} \right\}. \tag{4.23}
\]

We see that the number density \( \tilde{n}^{cr} \), given by Eq. (4.23), is proportional to the total increment of the longitudinal kinetic momentum, \( \Delta U_{T} = e |A_{x} (+\infty) - A_{x} (-\infty)| = eET \). Note that this density is a function of the time duration of the field. The same result was obtained in 3+1 dimensions, using the functional Schrödinger picture [30]. The quantity \( r^{cr} = d\tilde{n}^{cr} / dT \) is often called the pairs production rate. It is constant if \( \sqrt{eET} \) is big enough. It is useful to compare this result with one obtained for the model of pair creation by the \( L \)-constant field [50]. In fact, the \( L \)-constant field is a constant uniform electric field confined between two capacitor plates separated by a finite distance \( L \). This field can create pairs with transversal momenta that satisfy the inequality \( 2\pi_{\perp} \leq eEL \). In this case the total number of created pairs, \( N^{cr} \), is a function of the field length \( L \). The \( T \)-constant and \( L \)-constant fields are physically distinct. However, if \( \sqrt{eEL} \) is big enough, the process of pair creation can be characterized by the same density \( r^{cr} \). Thus, only in the asymptotic case when \( T \to \infty \) and
\(L \to \infty\), one can consider these fields as regularizations of a constant uniform electric field given by two distinct gauge conditions on the electromagnetic potentials \(A^\mu (x)\).

Using Eq. (2.37) and performing the integration following the considerations above, we get the vacuum-to-vacuum probability \(P_v\) in the form

\[
P_v = \exp \left( -\mu^T V_{(d-1)} \tilde{n}^{cr} \right), \quad \mu^T = \sum_{l=0}^{\infty} \frac{(-1)^{(1-\kappa)/2}}{(l + 1)^{d/2}} \exp \left( -l \pi \frac{m^2}{eE} \right),
\]

where \(\tilde{n}^{cr}\) is given by Eq. (4.23). The formula above coincide, for \(d = 4\), with the well known Schwinger’s result [1] obtained for a constant electric field \(T \to \infty\).

V. PEAK ELECTRIC FIELD

In this section we present a complete discussion concerning particle creation from the vacuum by a third set of exactly solvable \(t\)-electric potential steps, namely, the peak electric field and the exponentially decaying electric field, both considered previously by us in Refs. [31, 32]. Here we supplement the former studies with new details and discussions particular to these fields.

A. General form of peak electric field

The peak electric field \(E(t)\) is composed of two parts, one of them is increasing exponentially on the time-interval \(I = (-\infty, 0]\), and reaches its maximal magnitude \(E > 0\) at the end of the interval \(t = 0\), the second part decreases exponentially on the time-interval \(II = (0, +\infty)\) having at \(t = 0\) the same magnitude \(E\). The vector potential \(A_x(t)\) and the field \(E_x(t)\) are

\[
A_x(t) = E \begin{cases} \frac{1}{k_1} (-e^{k_1 t} + 1), & t \in I \\ \frac{1}{k_2} (e^{-k_2 t} - 1), & t \in II \end{cases}, \quad E(t) = E \begin{cases} e^{k_1 t}, & t \in I \\ e^{-k_2 t}, & t \in II \end{cases},
\]

and \(A_y = A_z = E_y = E_z = 0\). Here \(k_1\) and \(k_2\) are positive constants. The field \(E(t)\) is continuous at \(t = 0\), but its time-derivative is not in the general case,

\[
\lim_{t \to -0} E(t) = \lim_{t \to +0} E(t) = E, \quad \forall t : E(t) \leq E, \quad \lim_{t \to -0} \dot{E}(t) = k_1 E \neq \lim_{t \to +0} \dot{E}(t) = -k_2 E. \quad (5.2)
\]

Note that the so-called exponentially decreasing electric field with the potential

\[
A_x^{ed}(t) = E \begin{cases} 0, & t \in I \\ \frac{1}{k_2} (e^{-k_2 t} - 1), & t \in II \end{cases},
\]

(5.3)
can be considered as a particular case of the peak field, when the latter switches on abruptly at \( t = 0 \), i.e., when \( k_1 \) is sufficiently large, \( k_1 \to \infty \). Similarly can be treated the exponentially increasing electric field.

Exact solutions of the Dirac equation with the exponentially decreasing and the peak electric fields have been obtained by us previously in Refs. \([31, 32]\). Following the same way, we introduce new variables \( \eta_j \),

\[
\eta_1 = ih_1 e^{k_1 t}, \quad \eta_2 = ih_2 e^{-k_2 t}, \quad h_j = 2eE k_j^{-2}, \quad j = 1, 2, \quad (5.4)
\]
in place of \( t \) and represent the scalar functions \( \varphi_n (t) \) as

\[
\varphi_n^j (t) = e^{-\eta_j/2} \eta_j^{\nu_j} \tilde{\varphi}^j (\eta_j), \quad \nu_j = \frac{i \omega_j}{k_j}, \quad \omega_j = \sqrt{\pi_j^2 + \pi_{\perp}^2}, \quad \pi_j = p_x - (-1)^j \frac{eE}{k_j}, \quad (5.5)
\]

where the subscript \( j \) distinguishes quantities associated to the time-intervals I and II. The functions \( \tilde{\varphi}^j (\eta_j) \) satisfy confluent hypergeometric equations \([49]\),

\[
\left[ \eta_j \frac{d^2}{d \eta_j^2} + (c_j - \eta_j) \frac{d}{d \eta_j} - a_j \right] \tilde{\varphi}^j (\eta_j) = 0, \quad c_j = 1 + 2 \nu_j, \quad a_j = \frac{1}{2} (1 + \chi) + (-1)^j \frac{i \pi_j}{k_j} + \nu_j. \quad (5.6)
\]

In accordance with Eq. (2.8), the quantity \( \chi \) can be chosen to be either \( \chi = +1 \) or \( \chi = -1 \).

A fundamental set of solutions for the latter equation consists of two linearly independent confluent hypergeometric functions \( \Phi (a_j, c_j; \eta_j) \) and \( \eta_j^{1-c_j} e^{\nu_j} \Phi (1 - a_j, 2 - c_j; -\eta_j) \), where

\[
\Phi (a, c; \eta) = 1 + \frac{a \eta}{c 1!} + \frac{a (a + 1) \eta^2}{c (c + 1) 2!} + \ldots. \quad (5.7)
\]

Thus, the general solution of Eq. (2.10) in the intervals I and II can be written as the following linear superposition:

\[
\varphi_n^j (t) = b_1^j y_1^j (\eta_j) + b_2^j y_2^j (\eta_j), \quad y_1^j (\eta_j) = e^{-\eta_j/2} \eta_j^{\nu_j} \Phi (a_j, c_j; \eta_j), \quad y_2^j (\eta_j) = e^{\nu_j} \eta_j^{1-c_j} \Phi (1 - a_j, 2 - c_j; -\eta_j), \quad (5.8)
\]

with arbitrary constants \( b_1^j \) and \( b_2^j \). The Wronskian of the functions \( y \) is

\[
y_1^j (\eta_j) \frac{d}{d \eta_j} y_2^j (\eta_j) - y_2^j (\eta_j) \frac{d}{d \eta_j} y_1^j (\eta_j) = \frac{1 - c_j}{\eta_j}. \quad (5.9)
\]

As can be seen from Eq. (5.11), the peak electric field is switched on at the infinitely remote past \( t \to -\infty \) and switched off at the infinitely remote future \( t \to +\infty \). At these
regions, the exact solutions represent free particles and the appropriate superpositions from Eq. (5.8) obey the asymptotic conditions (2.13), where \( p_0(\pm \infty) = \omega_1 \) denotes energy of initial particles at \( t \to -\infty \), \( p_0(\pm \infty) = \omega_2 \) denotes energy of final particles at \( t \to +\infty \) and \( \zeta\mathcal{N} \) and \( \zeta'\mathcal{N} \) are given by Eq. (2.18).

Using the initial conditions (2.13), we fix the constants \( b_1^j \) and \( b_2^j \), and then we find the in- and out-electron and positron states in the intervals I and II:

\[ +\varphi_n(t) = \mathcal{N} \exp \left( i\pi \nu_1/2 \right) y_2^1(\eta_1), \quad -\varphi_n(t) = -\mathcal{N} \exp \left( -i\pi \nu_1/2 \right) y_1^1(\eta_1), \quad t \in I; \]
\[ +\varphi_n(t) = +\mathcal{N} \exp \left( -i\pi \nu_2/2 \right) y_2^2(\eta_2), \quad -\varphi_n(t) = -\mathcal{N} \exp \left( i\pi \nu_2/2 \right) y_1^2(\eta_2), \quad t \in I'(5.10) \]

Taking into account the structure of exact solutions given by Eqs. (5.8) and (5.10), we represent the functions \( -\varphi_n(t) \) and \( +\varphi_n(t) \) in the form

\[ +\varphi_n(t) = \begin{cases} 
  g(+) +\varphi_n(t) + \kappa g(-) -\varphi_n(t), & t \in I \\
  +\mathcal{N} \exp \left( -i\pi \nu_2/2 \right) y_2^2(\eta_2), & t \in I', \end{cases} \]
\[ -\varphi_n(t) = \begin{cases} 
  -\mathcal{N} \exp \left( -i\pi \nu_1/2 \right) y_1^1(\eta_1), & t \in I \\
  g(-) +\varphi_n(t) + \kappa g(-) -\varphi_n(t), & t \in I', \end{cases} \]

already for any \( t \). Here coefficients \( g \) are defined by Eq. (2.20). The constant \( \kappa \) is defined by Eq. (2.37). Its introduction allows us to describe by one equation the case of scalar particles as well, which is discussed in detail below. The functions \( -\varphi_n(t) \) and \( +\varphi_n(t) \) and their derivatives satisfy the following continuity conditions:

\[ \left. \pm\varphi_n(t) \right|_{t=-0} = \left. \pm\varphi_n(t) \right|_{t=+0}, \quad \partial_t \left. \pm\varphi_n(t) \right|_{t=-0} = \partial_t \left. \pm\varphi_n(t) \right|_{t=+0}. \quad (5.13) \]

Using Eqs. (5.13) and (5.9), one can find coefficients \( g(\zeta|\zeta') \) and \( g(\zeta'|\zeta) \) from Eqs. (5.11) and (5.12),

\[ g(+) = C\Delta, \quad C = -\frac{1}{2} \sqrt{\frac{q_1}{\omega_1 q_2^* \omega_2}} \exp \left[ \frac{i\pi}{2} (\nu_1 - \nu_2) \right], \]
\[ \Delta = \left. \left[ k_1 h_1 y_1^2(\eta_2) \frac{d}{d\eta_1} y_2^1(\eta_1) + k_2 h_2 y_2^1(\eta_1) \frac{d}{d\eta_2} y_2^2(\eta_2) \right] \right|_{t=0}; \quad (5.14) \]
\[ g(-) = C'\Delta', \quad C' = -\frac{1}{2} \sqrt{\frac{q_2^*}{\omega_1 q_1^* \omega_2}} \exp \left[ \frac{i\pi}{2} (\nu_2 - \nu_1) \right], \]
\[ \Delta' = \left. \left\{ k_2 h_2 y_1^1(\eta_1) \frac{d}{d\eta_2} y_2^2(\eta_2) + k_1 h_1 y_2^2(\eta_2) \frac{d}{d\eta_1} y_1^1(\eta_1) \right\} \right|_{t=0}; \quad (5.15) \]
respectively \[32\]. Comparing Eqs. (5.14) and (5.15) one can verify that the symmetry under a simultaneous change \( k_1 \rightarrow k_2 \) and \( \pi_1 \rightarrow -\pi_2 \) holds true,

\[ g^{(+|-)} \Rightarrow g^{(-|+)} . \] (5.16)

A transition to solutions of the Klein-Gordon equation can be performed by setting \( \chi = 0 \) and \( \kappa = -1 \) in Eqs. (5.11) and (5.12), and by using the normalization constants \[2.26\]. Thus, in the case of scalar particles, the coefficient \( g^{(-|+)} \) reads

\[ g^{(-|+)} = C_{sc} \Delta |_{\chi=0} , \quad C_{sc} = (4\omega_1\omega_2)^{-1/2} \exp [i \pi (\nu_1 - \nu_2)/2] , \] (5.17)

where \( \Delta \) is given by Eq. (5.14). In this case, we have the antisymmetry

\[ g^{(+|-)} \Rightarrow -g^{(-|+)} . \] (5.18)

under the simultaneous change \( k_1 \rightarrow k_2 \) and \( \pi_1 \rightarrow -\pi_2 \).

Using \( g^{(-|+)} \) given by Eq. (5.14), we find that in the Fermi case, differential mean number of created particles is

\[ N_{n}^{cr} = |C\Delta|^2 . \] (5.19)

In the Bose case, using \( g^{(-|+)} \) given by Eq. (5.17), we find

\[ N_{n}^{cr} = \left|C_{sc} \Delta |_{\chi=0}\right|^2 . \] (5.20)

It is clear that mean numbers \( N_{n}^{cr} \) depend on modulus squared of the transversal momentum, \( p_\perp^2 \). It follows from Eqs. (5.16) and (5.18) that the numbers \( N_{n}^{cr} \) are invariant under the simultaneous change \( k_1 \rightarrow k_2 \) and \( \pi_1 \rightarrow -\pi_2 \) for fermions and bosons, respectively. Then if \( k_1 = k_2 \), the numbers \( N_{n}^{cr} \) appear to be even functions of the longitudinal momentum \( p_x \).

**B. Slowly varying field**

The inverse parameters \( k_1^{-1}, k_2^{-1} \) represent scales of time duration of the electric field in the increasing and decreasing time intervals I and II. In particular, slowly varying fields correspond to small values of \( k_1 \) and \( k_2 \), satisfying the conditions

\[ \min (h_1, h_2) \gg \max (1, m^2/eE) , \] (5.21)
where $h_1$ and $h_2$ are defined by Eq. (5.4). In this case, we have a two-parameter regularization for a constant electric field (additional to the above presented one-parameter regularizations by the Sauter-like electric field and the $T$-constant electric field).

Let us analyze how the differential numbers $N_{cr}^n$ depend on the quantities $p_x$ and $\pi_\perp$. A semiclassical consideration show that $N_{cr}^n$ are exponentially small for very large $\pi_\perp \gtrsim \min(eE_{k_1}^{-1}, eE_{k_2}^{-1})$. Then the range of fixed $\pi_\perp$ is of interest and in the following we assume that condition (4.13) holds true, where in the case under consideration any given number $K_\perp$ satisfies the inequality

$$\min (h_1, h_2) \gg K_\perp^2 \gg \max (1, m^2/eE).$$

(5.22)

By virtue of the symmetry properties of the numbers $N_{cr}^n$ discussed above, one can only consider either positive or negative $p_x$. Let us, for example, consider the interval $-\infty < p_x \leq 0$. In this case $\pi_2$ is negative, its modulus is large, $-\pi_2 \geq eE/k_2$, while $\pi_1$ varies from positive to negative values, $-\infty < \pi_1 \leq eE/k_1$. The case of negative $\pi_1$ with large modulus, $-2\pi_1/k_1 > K_1$, where $K_1$ is any given large number, $K_1 \gg K_\perp$, is quite simple. In this case, using the appropriate asymptotic expressions of the confluent hypergeometric function one can see that $N_{cr}^n$ are negligibly small. To see this, Eq. (A9) (see Appendix A) is useful in the range $h_1 \gtrsim -2\pi_1/k_1 > K_1$, while an expression for large $c_2$ with a fixed $a_2$ and $h_2$ and an expression for large $c_1$ with fixed $a_1 - c_1$ and $h_1$, given in [49], are useful in the range $-2\pi_1/k_1 \gg h_1$.

We expect a significant contribution for the numbers $N_{cr}^n$ in the range

$$h_1 \geq 2\pi_1/k_1 > -K_1,$$

(5.23)

This range can be divided in four subrange

(a) $h_1 \geq 2\pi_1/k_1 > h_1 \left[1 - \left(\sqrt{h_1 g_2}\right)^{-1}\right]$,

(b) $h_1 \left[1 - \left(\sqrt{h_1 g_2}\right)^{-1}\right] > 2\pi_1/k_1 > h_1 (1 - \varepsilon)$,

(c) $h_1 (1 - \varepsilon) > 2\pi_1/k_1 > h_1/g_1$,

(d) $h_1/g_1 > 2\pi_1/k_1 > -K_1$,

(5.24)

where $g_1$, $g_2$, and $\varepsilon$ are any given numbers satisfying the conditions

$$g_1 \gg 1, g_2 \gg 1, \left(\sqrt{h_1 g_2}\right)^{-1} \ll \varepsilon \ll 1.$$
We note that
\[ \tau_1 = -i h_1 / (2 - c_1) \approx \frac{h_1 k_1}{2 |\pi_1|} \]
in the subranges (a), (b), and (c) and
\[ \tau_2 = i h_2 / c_2 \approx \frac{h_2 k_2}{2 |\pi_2|} \]
in the whole range (5.23). In the subranges, \( \tau_2 \) satisfies the inequalities:

(a) \[ 1 \leq \tau_2^{-1} < \left[ 1 + \left( \sqrt{\frac{h_2 g_2}{g_1}} \right)^{-1} \right], \]
(b) \[ \left[ 1 + \left( \sqrt{\frac{h_2 g_2}{g_1}} \right)^{-1} \right] < \tau_2^{-1} < (1 + \varepsilon k_2 / k_1), \]
(c) \[ (1 + \varepsilon k_2 / k_1) < \tau_2^{-1} < \left[ 1 + k_2 / k_1 \right. \left( 1 - 1 / g_1 ) \right], \]
(d) \[ \left[ 1 + k_2 / k_1 \right. \left( 1 - 1 / g_1 ) \right] < \tau_2^{-1} \lesssim (1 + k_2 / k_1). \] (5.25)

We see that \( \tau_1 - 1 \to 0 \) and \( \tau_2 - 1 \to 0 \) in the range (a), while \( |\tau_1 - 1| \sim 1 \) in the range (c), and \( |\tau_2 - 1| \sim 1 \) in the ranges (c) and (d). In the range (b) these quantities vary from their values in the ranges (a) and (c).

We choose \( \chi = 1 \) for convenience in the Fermi case. In the range (a) we can use the asymptotic expression for the confluent hypergeometric function given by Eq. (A1) in Appendix A. Using Eqs. (A6) and (A7) (see Appendix A), we can find that the differential means numbers for fermions and bosons in the leading approximation have the same form

\[ N_{cr}^n = e^{-\pi \lambda} [1 + O (|Z_1|)], \quad \max |Z_1| \lesssim g_2^{-1}. \] (5.26)

In the range (c), the confluent hypergeometric function \( \Phi (a_2, c_2; ih_2) \) is approximated by Eq. (A8) and the function \( \Phi (1 - a_1, 2 - c_1; -ih_1) \) is approximated by Eq. (A9) given in the Appendix A. Then we find that

\[ N_{cr}^n = e^{-\pi \lambda} \left[ 1 + O (|Z_1|)^{-1} + O (|Z_2|)^{-1} \right], \]
\[ \max |Z_1|^{-1} \lesssim \sqrt{g_1 / h_1}, \quad \max |Z_2|^{-1} \lesssim g_2^{-1}. \] (5.27)

Using asymptotic expression (A1) and taking into account Eqs. (5.26) and (5.27), we obtain that in the range (b) the following estimate holds \( N_{cr}^n \sim e^{-\pi \lambda} \). In the range (d), the confluent hypergeometric function \( \Phi (a_2, c_2; ih_2) \) is approximated by Eq. (A8) and the function \( \Phi (1 - a_1, 2 - c_1; -ih_1) \) is approximated by Eq. (A10) given in Appendix A. Then, in
this range, we obtain the following leading-order approximation for the differential mean numbers

\[ N_n^{\text{cr}} \approx \exp\left[-\frac{\pi k_1}{k_2} (\omega_1 - \pi_1)\right] \times \begin{cases} \sinh \left[\pi (\omega_1 + \pi_1)/k_1\right] & \text{in Fermi case} \\ \cosh \left[\pi (\omega_1 + \pi_1)/k_1\right] & \text{in Bose case} \end{cases} \]  

(5.28)

It is clear that when \( \pi_1 \gg \pi_\perp \) the expressions given by Eqs. (5.28) take the form (5.27), \( N_n^{\text{cr}} \to e^{-\pi \lambda} \). Consequently, the result (5.28) is valid in the whole range (5.23). Assuming \( m/k_1 \gg 1 \), we see that \( N_n^{\text{cr}} \) given by Eqs. (5.28) are negligible in the range \( \pi_\perp < \pi_1 \leq eE/k_1 \) and are given the same formula

\[ N_n^{\text{cr}} \approx \exp\left[-\frac{2\pi}{k_1} (\omega_1 - \pi_1)\right]. \]  

(5.29)

both for bosons and fermions.

Considering positive \( p_x > 0 \), we can take into account that numbers \( N_n^{\text{cr}} \) are invariant under the simultaneous exchange \( k_1 \leftrightarrow k_2 \) and \( \pi_1 \leftrightarrow -\pi_2 \). In this case \( \pi_1 \) is positive and large, \( \pi_1 > eE/k_1 \), while \( \pi_2 \) varies from negative to positive values, \( -eE/k_2 < \pi_2 < \infty \). We find that a substantial contribution to \( N_n^{\text{cr}} \) are formed in the range

\[ -\hbar_2 < 2\pi_2/k_2 < K_2, \]  

(5.30)

where \( K_2 \) is any given large number, \( K_2 \gg K_\perp \). In this range, similarly to the case of negative \( p_x \), we obtain the following leading-order approximation for the differential mean numbers

\[ N_n^{\text{cr}} \approx \exp\left[-\frac{2\pi}{k_2} (\omega_2 + \pi_2)\right] \times \begin{cases} \sinh \left[\pi (\omega_2 - \pi_2)/k_2\right] & \text{in Fermi case} \\ \cosh \left[\pi (\omega_2 - \pi_2)/k_2\right] & \text{in Bose case} \end{cases} . \]  

(5.31)

Assuming \( m/k_2 \gg 1 \), we see that substantial value of \( N_n^{\text{cr}} \) are formed in the range \( -eE/k_2 < \pi_2 < -\pi_\perp \) and are given the same formula

\[ N_n^{\text{cr}} \approx \exp\left[-\frac{2\pi}{k_2} (\omega_2 + \pi_2)\right] \]  

(5.32)

both for bosons and fermions.

Consequently, for any given \( \lambda \) satisfying Eqs. (4.13) and (5.22), the quantities \( N_n^{\text{cr}} \) are quasiconstant over the wide range of longitudinal momentum \( p_x \). When \( h_1, h_2 \to \infty \), differential mean numbers coincide with (3.19) in a constant uniform electric field.
The above analysis shows that dominant contributions for mean numbers of created particles by a slowly varying field are formed in ranges of large kinetic momenta and have there asymptotic forms (5.29) for $p_x < 0$ and (5.32) for $p_x > 0$. In this case, the range $\Omega$ in Eq. (3.21) is realized as $\pi_\perp < \pi_1 \leq eE/k_1$ for $p_x < 0$ and as $-eE/k_2 < \pi_2 < -\pi_\perp$ for $p_x > 0$. Therefore, we can represent integral (3.22) as follows

$$\tilde{n}^{cr} = \frac{J(d)}{(2\pi)^{d-1}} \int_{\sqrt{\lambda} < K_\perp} dp_\perp I_{p_\perp}, \quad I_{p_\perp} = I^{(1)}_{p_\perp} + I^{(2)}_{p_\perp},$$

$$I^{(1)}_{p_\perp} = \int_{-\infty}^{0} dp_x N^{cr}_n \approx \int_{\pi_\perp}^{eE/k_1} d\pi_1 \exp \left[ -\frac{2\pi}{k_1} (\omega_1 - \pi_1) \right],$$

$$I^{(2)}_{p_\perp} = \int_{0}^{\infty} dp_x N^{cr}_n \approx \int_{\pi_\perp}^{eE/k_2} d\pi_2 |\pi_2| \exp \left[ -\frac{2\pi}{k_2} (\omega_2 - |\pi_2|) \right].$$

Using the variable changes

$$s = \frac{2}{k_1 \lambda} (\omega_1 - \pi_1) \quad \text{in} \quad I^{(1)}_{p_\perp}, \quad s = \frac{2}{k_2 \lambda} (\omega_2 - |\pi_2|) \quad \text{in} \quad I^{(2)}_{p_\perp},$$

and neglecting exponentially small contributions, we respectively represent the quantities $I^{(1)}_{p_\perp}$ and $I^{(2)}_{p_\perp}$ as

$$I^{(1)}_{p_\perp} \approx \int_{1}^{\infty} \frac{ds}{s} \omega_1 e^{-\pi_\lambda s}, \quad I^{(2)}_{p_\perp} \approx \int_{1}^{\infty} \frac{ds}{s} \omega_2 e^{-\pi_\lambda s}.$$  \quad (5.34)

The leading contributions for both integrals (5.34) are from a range near $s \to 1$, where $\omega_1$ and $\omega_2$ are approximately given by

$$\omega_1 \approx \frac{eE}{sk_1}, \quad \omega_2 \approx \frac{eE}{sk_2}.$$  \quad (5.35)

Consequently the leading term in $I_{p_\perp}$ takes the following form:

$$I_{p_\perp} \approx eE \left( \frac{1}{k_1} + \frac{1}{k_2} \right) \int_{1}^{\infty} \frac{ds}{s^2} e^{-\pi_\lambda s} = eE \left( \frac{1}{k_1} + \frac{1}{k_2} \right) e^{-\pi_\lambda} G (1, \pi_\lambda),$$

where

$$G (\alpha, x) = \int_{1}^{\infty} \frac{ds}{s^{\alpha+1}} e^{-x(s-1)} = e^{x} x^{\alpha} \Gamma (-\alpha, x),$$

and $\Gamma (-\alpha, x)$ is the incomplete gamma function.

Neglecting an exponentially small contribution, one can extend the integration limit over $p_\perp$ in Eq. (5.33) from $\sqrt{\lambda} < K_\perp$ to $\sqrt{\lambda} < \infty$. This allows us to calculate the integral over $p_\perp$ as Gaussian one. Thus, we find

$$\tilde{n}^{cr} = r^{cr} \left( \frac{1}{k_1} + \frac{1}{k_2} \right) G \left( \frac{d}{2}, \pi \frac{m^2}{eE} \right),$$

(5.37)
where \( r^{cr} \) is given by Eq. (4.23). We see that the number density \( \tilde{n}^{cr} \), given by Eq. (5.37), is proportional to the total increment of the longitudinal kinetic momentum, \( \Delta U_P = e |A_x (+\infty) - A_x (-\infty)| = e E_0 (k_1^{-1} + k_2^{-1}) \). Note that if the electric field \( E \) is weak, \( m^2/eE \gg 1 \), one obtains

\[
G \left( \frac{d}{2}, \pi \frac{m^2}{eE} \right) \approx \frac{eE}{\pi m^2}.
\]

(5.38)

If the electric field \( E \) is strong enough, \( m^2/eE \ll 1 \), the leading term of \( G \)-function, which is given by Eq. (5.36), is

\[
G \left( \frac{d}{2}, \pi \frac{m^2}{eE} \right) \approx \frac{2}{d}.
\]

(5.39)

Using the above considerations we perform the summation (integration) in Eq. (2.37) and obtain the vacuum-to-vacuum probability \( P_v \),

\[
P_v = \exp \left( -\mu^P V_{(d-1)} \tilde{n}^{cr} \right), \quad \mu^P = \sum_{l=0}^{\infty} \frac{(-1)^{(1-k_l)/2}}{(l + 1)^{d/2}} \exp \left( -l \pi \frac{m^2}{eE} \right),
\]

\[
\epsilon_l^P = G \left( \frac{d}{2}, l \pi \frac{m^2}{eE} \right) \left[ G \left( \frac{d}{2}, \pi \frac{m^2}{eE} \right) \right]^{-1},
\]

(5.40)

where \( \tilde{n}^{cr} \) is given by Eq. (5.37).

These results allow us to establish an immediate comparison with the one-parameter regularizations of the constant field, namely the \( T \)-constant and Sauter-like electric fields. The number densities of created particles in such fields, given by Eqs. (3.27), (4.23), and (5.37), are proportional to the corresponding increments of longitudinal kinetic momenta. This fact allows one to compare pair creation effects in such fields. Thus, for a given magnitude of the electric field \( E \) one can compare the pair creation effects in fields with equal increment of the longitudinal kinetic momentum, or one can determine such increments of the longitudinal kinetic momenta, for which particle creation effects are the same. Equating the number densities \( \tilde{n}^{cr} \) for Sauter-like field and for the peak field to the density \( \tilde{n}^{cr} \) for the \( T \)-constant field, we find an effective duration time \( T_{eff} \) in both cases,

\[
T_{eff} = T_8 \delta \text{ for Sauter-like field,}
\]

\[
T_{eff} = (k_1^{-1} + k_2^{-1}) G \left( \frac{d}{2}, \pi \frac{m^2}{eE} \right) \text{ for the peak field.}
\]

(5.41)

By the definition \( T_{eff} = T \) for the \( T \)-constant field. One can say that the Sauter-like and the peak electric fields with the same \( T_{eff} = T \) are equivalent to the \( T \)-constant field in pair production.
C. Short pulse field

Choosing parameters the peak field in a certain way, one can obtain electric fields that exist only for a short time in a vicinity of the time instant \( t = 0 \). The latter fields switch on and (or) switch off “abruptly” near the time instant \( t = 0 \). This situation can be realized for large parameters \( k_1, k_2 \to \infty \) with a fixed ratio \( k_1/k_2 \). The corresponding asymptotic potentials, \( U (+\infty) = eE\frac{k_1}{k} \) and \( U (-\infty) = -eE\frac{k_2}{k} \) define finite increments of the longitudinal kinetic momenta \( \Delta U_1 \) and \( \Delta U_2 \) for increasing and decreasing parts, respectively,

\[
\Delta U_1 = U (0) - U (-\infty) = eE\frac{k_1}{k}, \quad \Delta U_2 = U (+\infty) - U (0) = eE\frac{k_2}{k}. \tag{5.42}
\]

Effectively we have a very short pulse of the electric field. At the same time this configuration imitates well enough a \( t \)-electric rectangular potential step (it is an analog of the Klein step, which is an \( x \)-electric rectangular step; see Ref. \[10\]) and coincides with it as \( k_1, k_2 \to \infty \). Thus, these field configurations can be considered as regularizations of a rectangular step. We assume that sufficiently large \( k_1 \) and \( k_2 \) for any given \( \pi_1, \pi_2 \) and \( \pi_{1,2} = p_x - U (\mp\infty) \) satisfy the following inequalities:

\[
\Delta U_1/k_1 \ll 1, \quad \Delta U_2/k_2 \ll 1, \quad \max \left( \frac{\omega_1}{k_1}, \frac{\omega_2}{k_2} \right) \ll 1. \tag{5.43}
\]

In this case the confluent hypergeometric function can be approximated by the first two terms in Eq. \(5.7\), which are \( \Phi (a, c; \eta) \), \( c_j \approx 1 \), and \( a_j \approx (1 + \chi)/2 \). Then we obtain

\[
N^{cr}_n \approx \begin{cases} 
\left( \frac{\omega_1 + \pi_1}{4\omega_1\omega_2(\omega_2 - \pi_1)} \right) (\Delta U_2 + \Delta U_1 + \omega_2 - \omega_1)^2 & \text{in Fermi case} \\
\left( \frac{\omega_2 - \omega_1}{4\omega_1\omega_2} \right)^2 & \text{in Bose case}
\end{cases}. \tag{5.44}
\]

In contrast to the Fermi case, where \( N^{cr}_n \leq 1 \), in the Bose case, the differential numbers \( N^{cr}_n \) are unbounded in two ranges of the longitudinal kinetic momenta, in the range where \( \omega_1/\omega_2 \to \infty \) and in the range where \( \omega_2/\omega_1 \to \infty \). In these ranges they have the form

\[
N^{cr}_n \approx \frac{1}{4} \max \left\{ \omega_1/\omega_2, \omega_2/\omega_1 \right\}. \tag{5.45}
\]

We can treat this fact as an indication that the back reaction is big. If so, the concept of the external field in the scalar QED is limited by the condition \( \min (\omega_1/k_1, \omega_2/k_2) \gtrsim 1 \) for the fields under consideration. We do not see similar problem in the spinor QED.
If \( k_1 = k_2 \) (in this case \( \Delta U_2 = \Delta U_1 = \Delta U/2 \)), we can compare the above results with the regularization of rectangular steps by the Sauter-like potential, given by Eqs. (3.7) and (3.10). We see that both regularizations are in agreement, for fermions under the condition \( |\omega_2 - \omega_1| \ll \Delta U \), and for bosons under condition (3.9).

D. Exponentially decaying field

In the examples, considered above, increasing and decreasing phases of the fields are almost symmetric. Here we consider an essentially asymmetric configuration of the peak field, when, for example, the field switches abruptly on at \( t = 0 \), that is, \( k_1 \) is sufficiently large, while the parameter \( k_2 > 0 \) is arbitrary, including, for example, the case of smooth switching off process. Note, that due to the invariance of the mean numbers \( N_n^{cr} \) under the simultaneous change \( k_1 \leftrightarrow k_2 \) and \( \pi_1 \leftrightarrow -\pi_2 \), one can easily transform this situation to the case with a large \( k_2 \) and arbitrary \( k_1 > 0 \).

Let us assume that a sufficiently large \( k_1 \) satisfies the inequalities

\[
\Delta U_1/k_1 \ll 1, \quad \omega_1/k_1 \ll 1.
\]

(5.46)

Then Eqs. (5.19) and (5.20) can be reduced to the following form

\[
|\Delta|^2 \approx |\Delta_{ap}|^2 = e^{i\pi \nu_2} \left[ \chi \Delta U_1 + \omega_2 - \omega_1 + k_2 h_2 \left( -\frac{1}{2} + \frac{d}{d\eta_2} \right) \right] \Phi (a_2, c_2; \eta_2) \bigg|_{t=0}^2.
\]

(5.47)

Under the condition

\[
\Delta U_1/\omega_1 \ll 1,
\]

(5.48)

one can disregard the term \( \chi \Delta U_1 \) in Eq. (5.47) and set approximately \( \pi_1 \approx p_x \). Thus, \( \omega_1 \approx \sqrt{p_x^2 + \pi_1^2} \). In this approximation, leading terms do not contain \( \Delta U_1 \), so that we obtain

\[
N_n^{cr} \approx \begin{cases} 
|C_{\Delta_{ap}}|^2 & \text{for fermions} \\
|C_{sc \Delta_{ap}}|_{\chi=0}^2 & \text{for bosons}
\end{cases}.
\]

(5.49)

In fact, differential mean numbers obtained in these approximations are the same as in the so-called exponentially decaying electric field, given by the potential (5.3). Under condition (5.48), the results presented by Eqs. (5.49) for arbitrary \( k_2 > 0 \) are in agreement with ones obtained in Ref. [31].
Let us consider the most asymmetric case when Eqs. (5.49) hold and when the increment of the longitudinal kinetic momentum due to exponentially decaying electric field is sufficiently large ($k_2$ are sufficiently small),

$$h_2 = 2\Delta U_2/k_2 \gg \max(1, m^2/eE) .$$  \hspace{1cm} (5.50)

In this case only the range of $\pi_\perp$ (4.13) is essential, in which $K_\perp$ is any given number satisfying the condition

$$h_2 \gg K^2_\perp \gg \max(1, m^2/eE) .$$  \hspace{1cm} (5.51)

It should be noted that the distribution $N_n^{cr}$, given by Eqs. (5.49) for this most asymmetric case, coincides with the one obtained in our work [31]. However, the detailed study of this distribution was not performed there. The main contribution of this distribution to the total number $N^{cr}$ (3.20) was estimated in our recent work [32]. Here we consider a detailed dependence of distribution (5.49) on the physical parameters $p_x$ and $\pi_\perp$ and the corresponding consequences to the global quantities $N^{cr}$ and $P_v$. We choose $\chi = -1$ for convenience in the Fermi case.

In the case of large negative momenta $p_x$, $p_x < 0$ and $-2\pi_2/k_2 > g_1 h_2$ (where $g_1$ is any given number, $g_1 \gg 1$), using an expression for the confluent hypergeometric function with large $c_2$ and fixed $a_2$ and $h_2$, given in [49], one can verify that the mean numbers $N_n^{cr}$ are negligibly small both for fermions and bosons. The same holds true for very large positive $p_x$, such that $2\pi_2/k_2 > K_2$, where $K_2$ is any given large number, $K_2 \gg K_\perp$. We see that the mean numbers $N_n^{cr}$ differ from zero only in the range $-g_1 h_2 < 2\pi_2/k_2 < K_2$. This range can be divided in the following subranges:

(a) $(1 + \varepsilon) h_2 \leq -2\pi_2/k_2 < g_1 h_2$,

(b) $h_2 \left[1 + \left(\sqrt{h_2 g_2}\right)^{-1}\right] \leq -2\pi_2/k_2 < (1 + \varepsilon) h_2$,

(c) $h_2 \left[1 - \left(\sqrt{h_2 g_2}\right)^{-1}\right] \leq -2\pi_2/k_2 < h_2 \left[1 + \left(\sqrt{h_2 g_2}\right)^{-1}\right]$,

(d) $(1 - \varepsilon) h_2 \leq -2\pi_2/k_2 < h_2 \left[1 - \left(\sqrt{h_2 g_2}\right)^{-1}\right]$,

(e) $h_2/g_1 < -2\pi_2/k_2 < (1 - \varepsilon) h_2$,

(f) $-K_2 < -2\pi_2/k_2 < h_2/g_1$.

where $g_2$ and $\varepsilon$ are any given numbers satisfying the conditions $g_2 \gg 1$ and $\varepsilon \ll 1$. We
assume that $\varepsilon \sqrt{h_2} \gg 1$. Note that in the ranges (5.52) $\tau_2 = i h_2 / c_2 \approx \frac{h_2 k_2}{2 |\pi_2|}$. Then in the ranges from (a) to (e), $\tau_2$ varies from $1 / g_1$ to $g_1$.

In the range (a), the confluent hypergeometric function $\Phi (a_2, c_2; i h_2)$ is approximated by Eq. (A8) given in Appendix A. In this range the differential mean numbers in the leading-order approximation are

$$N_n^{cr} \approx \frac{\omega_1 - |p_x|}{2 \omega_1} \left[ 1 + O \left( |Z_2|^{-1} \right) \right] \times \begin{cases} 1 \text{ for fermions} \\ \frac{\omega_1 - |p_x|}{|p_x|} \text{ for bosons} \end{cases}, \quad (5.53)$$

where $\max |Z_2|^{-1} \sim (\varepsilon \sqrt{h_2})^{-1} \lesssim \sqrt{g_1 / h_2}$. We see from Eq. (5.53) that $N_n^{cr} \to 0$ if $|p_x| \gg \pi_\perp$. Note that $\varepsilon e E / k_2 < |p_x| < (g_1 - 1) e E / k_2$. Taking into account the inequality (4.13), we see that the numbers (5.53) are negligibly small if $\varepsilon \sqrt{h_2} \gg K_\perp$.

In the range (c), $\tau_2 - 1 \to 0$ and, using Eqs. (A2), (A3), and (A4) given in Appendix A we find that

$$N_n^{cr} = \frac{\omega_1 + p_x}{2 \omega_1} e^{-\pi \lambda / 2} \left[ 1 + O \left( |Z_2|^{-1} \right) \right] \times \begin{cases} \cosh \left( \frac{\pi \lambda}{4} \right) \text{ for fermions} \\ \frac{\pi (\omega_1 + p_x)}{\sqrt{8 e E |\Gamma (3 + i \lambda / 4)|}} \text{ for bosons} \end{cases}, \quad (5.54)$$

where $\max |Z_2|^{-1} \lesssim g_2^{-1}$. Note that $N_n^{cr}$ given by Eq. (5.54) are finite and restricted, $N_n^{cr} \leq 1$ for fermions and $N_n^{cr} \lesssim 1 / g_2$ for bosons. In the range (b) the distributions $N_n^{cr}$ vary between their values in the ranges (a) and (c).

In the range (e), parameters $\eta_2$ and $c_2$ are large with $a_2$ fixed and $\tau_2 > 1$. In this case, using the asymptotic expression of the confluent hypergeometric function given by Eq. (A9) in Appendix A we find that

$$N_n^{cr} = \exp \left[ -\frac{2 \pi}{k_2} (\omega_2 + \pi_2) \right] \left[ 1 + O \left( |Z_2|^{-1} \right) \right], \quad (5.55)$$

where $Z_2$ is given by Eq. (A2) in the Appendix A both for fermions and bosons. We note that modulus $|Z_2|^{-1}$ varies from $|Z_2|^{-1} \sim (\varepsilon \sqrt{h_2})^{-1}$ to $|Z_2|^{-1} \sim \left( (g_1 - 1) \sqrt{h_2} \right)^{-1}$. Approximately, expression (5.55) can be written as

$$N_n^{cr} \approx \exp \left( -\frac{\pi \pi_2^2}{k_2 |\pi_2|} \right). \quad (5.56)$$

Note that $e E / g_1 < k_2 |\pi_2| < (1 - \varepsilon) e E$ in the range (e) and the distribution $N_n^{cr}$ given by Eq. (5.56) has the following limiting form:

$$N_n^{cr} \to e^{-\pi \lambda} \quad \text{as} \quad k_2 |\pi_2| \to (1 - \varepsilon) e E. \quad (5.57)$$
Thus, the distribution (3.19) is reproduced in the case of an exponentially decaying electric field in the wide range of a large increment of the longitudinal kinetic momentum, $-\pi_2 \sim eE/k_2$. Taking into account the condition $\varepsilon \sqrt{\hbar_2} \gg 1$, we see that $p_x/\sqrt{eE} \gg 1$ in this range. Thus, condition (5.48) holds if

$$\Delta U_1/\sqrt{eE} \ll 1.$$  \hspace{1cm} (5.58)

Under this condition, the form of the distribution $N_n^{cr}$ does not depend on the details of the switching on before the time instant $t = 0$. In the range (d), the distributions $N_n^{cr}$ vary from their values in the ranges (c) and (e) for fermions and bosons.

In the range (f), we can use an asymptotic expression of the confluent hypergeometric function for large $\hbar_2$ at fixed $a_2$ and $c_2$ given by Eq. (A10) in Appendix A to get the following result:

$$N_n^{cr} \approx \exp\left[-\frac{\pi}{k_2} (\omega_2 + \pi_2)\right] \times \begin{cases} \sinh [\pi (\omega_2 - \pi_2)/k_2] & \text{for fermions} \\ \cosh (\pi (\omega_2 - \pi_2)/k_2) & \text{for bosons} \end{cases}$$ \hspace{1cm} (5.59)

in the leading-order approximation. The same distribution takes place in a slowly varying field for $p_x > 0$, see Eq. (5.31). In the range (f) the form of $N_n^{cr}$ given by Eq. (5.59) does not depend on details of switching on at $t = 0$ if condition (5.58) holds true. For $m/k_2 \gg 1$, distribution (5.59) is approximated by Eq. (5.55).

Note that WKB approximation holds true under the condition $(\omega_2 + \pi_2)/k_2 \gg 1$ for $N_n^{cr}$ given by Eq. (5.55). In the range (e) $N_n^{cr}$ given by (5.55) coincides exactly with an estimation, obtained previously in [51, 52] in the framework of the semiclassical consideration. We stress that in our consideration (which does not use the WKB) the approximation (5.55) holds for any value of $(\omega_2 + \pi_2)/k_2$ and exact results given by Eqs. (5.53), (5.54), and (5.59) are quite different from the corresponding semiclassical ones.

Now, we can estimate the number density $n^{cr}$ of pairs created by an exponentially decaying electric field, defined by Eq. (3.20). To this end, we represent the leading terms of integral (3.20) as a sum of two contributions, one due to the ranges (e) and (f) and another one due to the ranges (b), (c), and (d):

$$n^{cr} \approx \frac{J(d)}{(2\pi)^{d-1}} \int_{\sqrt{\lambda_<K_\bot}} d\pi_\bot I_{\pi_\bot}, \quad I_{\pi_\bot} = I_{\pi_\bot}^{(1)} + I_{\pi_\bot}^{(2)},$$

$$I_{\pi_\bot}^{(1)} = \int_{\pi_2 \in \text{(b)\cup(c)\cup(d)}} d\pi_2 N_n^{cr}, \quad I_{\pi_\bot}^{(2)} = \int_{\pi_2 \in \text{(e)\cup(f)}} d\pi_2 N_n^{cr}.$$  \hspace{1cm} (5.60)

Note that the mean numbers $N_n^{cr}$ given by Eq. (5.53) in the total range (a) and the mean numbers given by Eqs. (5.59) in the range $-\pi_2 \lesssim \pi_\bot$ are negligibly small. The main contri-
tribution to the number density (5.60) is due to the wide range \((e) \cup (f)\) of a large increment of the longitudinal kinetic momentum \(\pi_2\) with a relatively small transversal momentum \(|p_\perp|\). The contribution to this quantity from the relatively narrow momentum ranges \((b), (c),\) and \((d)\) is finite and the corresponding integral \(I_{p_\perp}^{(1)}\) is of the order \(\sqrt{eE/g_2}\). The integral \(I_{p_\perp}^{(2)}\) can be taken from Eq. (5.33). Using the results of Sec. V B, we can find the leading term in \(I_{p_\perp}^{(2)}\),

\[
I_{p_\perp}^{(2)} \approx \frac{eE}{k_2} \int_1^\infty ds s^2 e^{-\pi\lambda s} \frac{eE}{k_2} e^{-\pi\lambda} G(1, \pi\lambda),
\]

(5.61)

where \(G(\alpha, x)\) is given by Eq. (5.36). The integral \(I_{p_\perp}^{(1)}\) in Eq. (5.60) is much less than the integral \(I_{p_\perp}^{(2)}\) (5.61), which represents the dominant contribution, \(I_{p_\perp} \approx I_{p_\perp}^{(2)}\). In this case, the range \(\Omega\) in Eq. (3.22) is realized by the condition \(\sqrt{\lambda < K_\perp}\) and \(\pi_2 \in (e) \cup (f)\). Then, calculating the Gaussian integral, we find

\[
\tilde{n}_{cr} = \frac{r_{cr} G\left(\frac{d}{2} \pi \frac{m^2}{eE}\right)}{k_2},
\]

(5.62)

where \(r_{cr}\) is given by Eq. (4.23). We see that \(\tilde{n}_{cr}\) given by Eq. (5.62) is the \(k_2\)-dependent part of the number density of pairs created in the slowly varying peak field (5.37).

Calculating the vacuum-to-vacuum probability, we obtain

\[
P_v = \exp\left(-\mu^P V_{(d-1)} \tilde{n}_{cr}\right),
\]

(5.63)

where \(\tilde{n}_{cr}\) is given by Eq. (5.62) and \(\mu^P\) is given by Eq. (5.40).

As it was mentioned above, in the ranges of dominant contribution \((e)\) and \((f)\) under condition (5.58), the form of \(N_{n}^{cr}\) does not depend on the details of the switching on at \(t = 0\). Therefore, calculations in an exponentially decaying field are quite representative for a large class of decaying electric fields switching on abruptly.

VI. UNIVERSAL BEHAVIOR OF THE VACUUM MEAN VALUES IN SLOWLY VARYING ELECTRIC FIELDS

A. Total density of created pairs

As was recently discovered in our work [53], an information derived from considerations of exactly solvable cases allows one to make some general conclusions about quantum effects in slowly varying strong fields for which no closed form solutions of the Dirac equation are
known. Below, we briefly represent such conclusions about an universal behavior of vacuum mean values in slowly varying strong electric fields.

We note that in all these cases the quantity \( N_{n}^{cr} \) is quasiconstant over the wide range of the longitudinal momentum \( p_{x} \) for any given \( \lambda \), namely \( N_{n}^{cr} \sim e^{-\pi \lambda} \). Pair creation effects in such fields are proportional to large increments of the longitudinal kinetic momentum, \( \Delta U = e |A_{x}(+\infty) - A_{x}(-\infty)| \). Defining the slowly varying regime in general terms, one can observe an universal character of vacuum effects caused by strong electric field.

We call \( E(t) \) a slowly varying electric field on a time interval \( \Delta t \) if the following condition holds true:

\[
\left| \frac{\bar{E}(t) \Delta t}{\bar{E}(t)} \right| \ll 1, \quad \Delta t / \Delta t_{st}^{m} \gg 1, \quad (6.1)
\]

where \( \bar{E}(t) \) and \( \dot{\bar{E}}(t) \) are mean values of \( E(t) \) and \( \dot{E}(t) \) on the time interval \( \Delta t \), respectively, and \( \Delta t \) is significantly larger than the time scale \( \Delta t_{st}^{m} \) which is

\[
\Delta t_{st}^{m} = \Delta t_{st} \max \left\{ 1, m^{2} / e \bar{E}(t) \right\}, \quad \Delta t_{st} = e^{\bar{E}(t)} - 1 / 2. \quad (6.2)
\]

We are primarily interested in strong electric fields, \( m^{2} / e \bar{E}(t) \lesssim 1 \). In this case, inequality (6.2) is simplified to the form \( \Delta t / \Delta t_{st} \gg 1 \), in which the mass \( m \) is absent. In such cases, the potential of the corresponding electric steps hardly differs from the potential of a constant electric field,

\[
U(t) = -e A_{x}(t) \approx U_{c}(t) = e \bar{E}(t) t + U_{0}, \quad (6.3)
\]

on the time interval \( \Delta t \), where \( U_{0} \) is a given constant. This behavior is inherent to the fields of exact solvable cases presented above.

If the electric field is not very strong, mean numbers \( N_{n}^{cr} \) of created pairs (or distributions) at the final time instant are exponentially small, \( N_{n}^{cr} \ll 1 \). In this case the probability of the vacuum to remain a vacuum and probabilities of particle scattering and pair creation have simple representations in terms of these numbers,

\[
|w_{n}(+) - |0)\right|^2 \approx N_{n}^{cr}, \quad |w_{n}(|-)\right|^2 \approx (1 + N_{n}^{cr}), \quad P_{v} \approx 1 - \sum_{n} N_{n}^{cr}. \quad (6.4)
\]

The latter relations are often used in semiclassical calculations to find \( N_{n}^{cr} \) and the total number of created pair, \( N^{cr} = \sum_{n} N_{n}^{cr} \), from the representation of \( P_{v} \) given by Schwinger’s effective action.
However, when the electric field cannot be considered as a weak one (e.g., in some situations in astrophysics and condensed matter), the mean numbers $N_{n}^{cr}$ can achieve their limited values $N_{n}^{cr} \to 1$ already at finite time instants $t$ and the sum $N^{cr}$ cannot be considered as a small quantity. Moreover, for slowly varying strong electric fields this sum is proportional to the large parameter $T_{eff}/\Delta t_{st}$. In such a case relations (6.4) are not correct anymore. However, as shown next, for arbitrary slowly varying strong electric field one can derive in the leading-term approximation an universal form for the total density of created pairs.

Let us define the range $D(t)$ as follows:

$$ D(t) : \langle P_{x}(t) \rangle < 0, \quad |\langle P_{x}(t) \rangle| \gg \pi_{\perp}. \quad (6.5) $$

In this range the longitudinal kinetic momentum $\langle P_{x}(t) \rangle = p_{x} - U(t)$ is negative and big enough. If $p_{x}$ components of the particle momentum belongs to the range $D(t)$, then the particle energy is in main determined by an increment of the longitudinal kinetic momentum, $U(t) - U(t_{in})$, during the time interval $t - t_{in}$ and $\langle P_{x}(t) \rangle = \langle P_{x}(t_{in}) \rangle - [U(t) - U(t_{in})]$. Note that $D(t) \subset D(t')$ if $t < t'$. The leading term of the total number density of created pairs, $\tilde{n}^{cr}$, is formed over the range $D(t_{out})$, that is, the range $D(t_{out})$ is chosen as a realization of the subrange $\Omega$ in Eq. (3.22).

In the case when the electric field does not switch abruptly on and off, that is, the field slowly weakens at $t \to \pm \infty$ and one of the time instants $t_{in}$ and $t_{out}$, or both are infinite $t_{in} \to -\infty$ and $t_{out} \to \infty$, one can ignore exponentially small contributions to $\tilde{n}^{cr}$ from the time intervals $\left[ t_{in}, t_{in}^{eff} \right]$ and $\left( t_{out}^{eff}, t_{out} \right)$, where electric fields are much less than the maximum field $E$, $E\left( t_{in}^{eff} \right), E\left( t_{out}^{eff} \right) \ll E$. Thus, in the general case it is enough to consider a finite interval $\left[ t_{in}^{eff}, t_{out}^{eff} \right]$. Denoting $t_{1} = t_{in}^{eff}$ and $t_{M+1} = t_{out}^{eff}$, we divide this interval into $M$ intervals $\Delta t_{i} = t_{i+1} - t_{i} > 0$, $i = 1, ..., M$, $\sum_{i=1}^{M} \Delta t_{i} = t_{out}^{eff} - t_{in}^{eff}$. We suppose that Eqs. (6.1) and (6.2) hold true for all the intervals, respectively. That allows us to treat the electric field as approximately constant within each interval, $\overline{E(t)} \approx \overline{E(t_{i})}$, for $t \in (t_{i}, t_{i+1})$. Note that inside of each interval $\Delta t_{i}$ abrupt changes of the electric field $E(t)$ whose duration is much less than $\Delta t_{i}$, cannot change significantly the total value of $\tilde{n}^{cr}$, since $N_{n}^{cr} \leq 1$ for fermions. Using Eq. (4.23) for the case of $T$-constant field, we can
represent \( \tilde{n}^{\text{cr}} \) as the following sum

\[
\tilde{n}^{\text{cr}} = \sum_{i=1}^{M} \Delta \tilde{n}_{i}^{\text{cr}}, \quad \Delta \tilde{n}_{i}^{\text{cr}} \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{eE(t_i)}^{eE(t_i + \Delta t_i)} dp_x \int _{\sqrt{\kappa E(t_i)} < K_\perp} dp_{\perp} N_{n}^{(i)},
\]

where \( K_\perp \) is any given number satisfying the condition \( \sqrt{eE(t_i)} \Delta t_i \gg K_\perp^2 \gg \max \{ 1, m^2/eE(t_i) \} \).

Taking into account Eq. (6.5), we represent the variable \( p_x \) as follows

\[
p_x = U(t), \quad U(t) = \int_{t_{\text{in}}}^{t} dt' eE(t') + U(t_{\text{in}}).
\]

Then neglecting small contributions to the integral (6.6), we find the following universal form for the total density of created pairs in the leading-term approximation for a slowly varying, but otherwise arbitrary strong electric field

\[
\tilde{n}^{\text{cr}} \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{t_{\text{in}}}^{t_{\text{out}}} dt eE(t) \int dp_{\perp} N_{n}^{\text{uni}}, \quad N_{n}^{\text{uni}} = \exp \left[ -\pi \frac{\pi^2}{eE(t)} \right].
\]

Note that \( N_{n}^{\text{uni}} \) is written in an universal form which can be used to calculate any total characteristics of the pair creation effect. After the integration over \( p_{\perp} \), we finally obtain

\[
\tilde{n}^{\text{cr}} = \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{t_{\text{in}}}^{t_{\text{out}}} dt \left[ eE(t) \right]^{d/2} \exp \left\{ -\pi \frac{m^2}{eE(t)} \right\}.
\]

These universal forms can be derived for bosons as well, if to restrict forms of external electric fields. Namely, by fields that have no abrupt variations of \( E(t) \) that can produce significant grow of \( n_{n}^{\text{cr}} \) on a finite time interval. In fact, in this case we have to include in the range \( D(t) \) the only subranges where \( n_{n}^{\text{cr}} \leq 1 \). In this case the universal forms for bosons are the same (6.8) and (6.9) assuming that \( J_{(d)} \) is the number of the boson spin degrees of freedom, in particular, \( J_{(d)} = 1 \) for scalar particles and \( J_{(4)} = 3 \) for vector particles.

Using the identity \( -\kappa \ln (1 - \kappa N_{n}^{\text{uni}}) = N_{n}^{\text{uni}} + (1 - \kappa)^{(1-\kappa)/2} (N_{n}^{\text{uni}})^2 \ldots \), in the same manner one can derive an universal form of the vacuum-to-vacuum transition probability \( P_{v} \) defined for fermions (\( \kappa = +1 \)) and bosons (\( \kappa = -1 \)) by Eq. (2.37). Performing the integration over \( p_{\perp} \), we obtain that

\[
P_{v} \approx \exp \left\{ -\frac{V_{(d-1)}J_{(d)}}{(2\pi)^{d-1}} \sum_{l=0}^{\infty} \int_{t_{\text{in}}}^{t_{\text{out}}} dt \left( -1 \right)^{(1-\kappa)/2} \left[ eE(t) \right]^{d/2} \left( l + 1 \right)^{d/2} \exp \left[ -\pi \frac{(l + 1) m^2}{eE(t)} \right] \right\}.
\]

Using Eqs. (6.9) and (6.10), one obtains the same expressions (3.27), (4.23), and (5.37) for the total densities and expressions (3.29), (4.24), and (5.40) for the vacuum-to-vacuum
transition probabilities that were found in the corresponding exactly solvable cases. Thus, we have an independent confirmation of the universal forms obtained above. These representations do not require knowledge of corresponding solutions of the relativistic wave equations.

The representation (6.10) coincides with the leading term approximation of derivative expansion results from field-theoretic calculations obtained in Refs. [21, 54, 55] for \( d = 3 \) and \( d = 4 \), respectively. In this approximation the probability \( P_v \) was derived from a formal expansion in increasing numbers of derivatives of the background field strength for Schwinger’s effective action:

\[
S = S^{(0)}[F_{\mu\nu}] + S^{(2)}[F_{\mu\nu}, \partial_\mu F_{\nu\rho}] + \ldots
\]

where \( S^{(0)} \) involves no derivatives of the background field strength \( F_{\mu\nu} \) (that is, \( S^{(0)} \) is a locally constant field approximation for \( S \)), while the first correction \( S^{(2)} \) involves two derivatives of the field strength, and so on, see Ref. [11] for a review. It was found that

\[
P_v = \exp \left(-2\text{Im}S^{(0)}\right).
\]

In the derivative expansion the fields are assumed to vary very slowly and satisfy the condition (6.1). A very convenient formalism for doing such an expansion is the worldline formalism, see [56] for the review, in which the effective action is written as a quantum mechanical path integral.

However, for a general background field, it is extremely difficult to estimate and compare the magnitude of various terms in the derivative expansion. Only under the assumption \( m^2/eE > 1 \), one can demonstrate that the derivative expansion is completely consistent with the semiclassical WKB analysis of the imaginary part of the effective action [57]. It is shown only for a constant electric field that Eq. (6.12) is given exactly by the semiclassical WKB limit when the leading order of fluctuations is taken into account [58].

It should be stressed that unlike to the authors of Refs. [21, 54, 55], Eq. (6.10) is derived in the framework of the general exact formulation of strong-field QED [6, 23], where \( P_v \) are defined by Eq. (2.37). Therefore, the obtained result holds true for any strong field under consideration. In particular, it is proven that Eq. (6.12) is given exactly by the semiclassical WKB limit for arbitrary slowly varying electric field.
B. Time evolution of vacuum instability

In this section details of the time evolution of vacuum instability effects are of interest. In particular, the study of the time evolution of the mean electric current, energy, and momentum provides us with new characteristics of the effect, related, in particular, to the back reaction. Due to the translational invariance of the spatially uniform external field, all the corresponding mean values are proportional to the space volume. Therefore, it is enough to calculate the vacuum mean values of the current density vector $\langle j^\mu(t) \rangle$ and of the EMT $\langle T_{\mu\nu}(t) \rangle$, defined by Eq. (2.38). Note that these densities depend on the initial vacuum, on the evolution of the electric field from the initial time instant up to the current time instant $t$, but they do not depend on the further history of the system and definition of particle-antiparticle at the time $t$.

Let us consider the time dependence of the current density vector $\langle j^\mu(t) \rangle$ and of the EMT $\langle T_{\mu\nu}(t) \rangle$, given by Eqs. (2.44). Due to the uniform character of the distributions $N^{cr}_n$, the only diagonal matrix elements of EMT differ from zero and the only longitudinal current components are not zero if $d \neq 3$. In $d = 3$ dimensions, a non-zero current component $\langle j^2(t) \rangle$ can exist, this fact is related to the so-called Chern-Simons term in the effective action, see details in Ref. [27]. However, if there are both fermion species in a model, as it takes place, for example, in the Dirac model of the graphene, then $\langle j^2(t) \rangle = 0$.

It follows from Eqs. (2.43) and (2.44) that the nonzero terms $\text{Re}\langle j^\mu(t)\rangle^p$ and $\text{Re}\langle T_{\mu\nu}(t)\rangle^p$ appear due to the vacuum instability. These terms are growing with time due to an increase of the number of states that are occupied by created pairs. In any system of Fermi particles the mean value $\langle j^2(t) \rangle$ is finite.

As a consequence of Eq. (6.5), we have

$$i\partial_t \pm \varphi_n(t) \approx \pm |\langle P_x(t) \rangle| \pm \varphi_n(t),$$

(6.13)

which means that at the time $t$ we deal with an ultrarelativistic particle and its kinetic momentum $\langle P_x(t) \rangle$ can be considered as a large parameter. Considering time dependence of means $\text{Re}\langle j^1(t)\rangle^p$ and $\text{Re}\langle T_{\mu\nu}(t)\rangle^p$, we suppose that the time difference $t - t_{in}$ is big enough to satisfy Eq. (6.13). Using exact relation Eq. (2.21) to express solutions $\pm \psi_n$ via $\pm \psi_n$, and neglecting strongly oscillating terms, we find that leading contribution to the function $S^p(x,x')$ (defined by Eq. (2.43)) at $t \sim t'$ can be represented by the following
expression

\[ S^p(x, x') \approx -i \sum_n N_n^{cr} \left[ \psi_n(x) \bar{\psi}_n(x') - \psi_n(x) \bar{\psi}_n(x') \right]. \quad (6.14) \]

It is clear that for any large enough difference \( t - t_{in} \) the integral over momentum \( p \) in the right hand side of Eq. (6.14) can be approximated by an integral over the range \( D(t_{out}) \) that gives the dominant contribution to the mean number of created particles with respect to the total increment of the longitudinal kinetic momentum. Moreover, taking into account Eqs. (6.5) and (6.7), we see that \( D(t) \subset D(t') \subset D(t_{out}) \) if \( t < t' < t_{out} \) and for a given difference \( t - t_{in} \) the dominant contribution to the right hand side of Eq. (6.14) is from a subrange \( D(t) \subset D(t_{out}) \).

We recall that, according to Eq. (2.14), one can choose the corresponding in- and out-Dirac solutions either with \( \chi = +1 \) or with \( \chi = -1 \). Using this possibility, we choose \( \chi = +1 \) for \( +\psi_n(x) \) and \( \chi = -1 \) for \( -\psi_n(x) \). With such a choice, taking into account that \( p \in D(t) \), we simplify essentially the matrix structure of the representation (6.14). Thus, after a summation over spin polarizations \( \sigma \), we obtain the following result:

\[ S^p(x, x') \approx (\gamma P + m)\Delta^p(x, x'), \quad (6.15) \]

where the function \( \Delta^p(x, x') \) reads

\[
\Delta^p(x, x') = -i \sum_{p \in D(t)} N_n^{cr} |\langle P_x(t) \rangle| \exp [i p (x - x')]
\times \left\{ (1 + \gamma^0 \gamma^1) \left[ +\varphi_n(t) + \varphi_n^*(t') \right] \big|_{\chi = +1} + (1 - \gamma^0 \gamma^1) \left[ -\varphi_n(t) - \varphi_n^*(t') \right] \big|_{\chi = -1} \right\}.
\]

Using Eq. (6.15) in Eq. (2.44), we find the following representations for the vacuum means of current density and EMT components:

\[
\langle j^1(t) \rangle^p \approx 2e^{V_{(d-1)}J_{(d)}} \int_{p \in D(t)} dp N_n^{cr} \rho(t) |\langle P_x(t) \rangle|;
\]

\[
\langle T_{00}(t) \rangle^p \approx \langle T_{11}(t) \rangle^p \approx \frac{V_{(d-1)}J_{(d)}}{(2\pi)^{d-1}} \int_{p \in D(t)} dp N_n^{cr} \rho(t) \langle P_x(t) \rangle^2,
\]

\[
\langle T_{il}(t) \rangle^p \approx \frac{V_{(d-1)}J_{(d)}}{(2\pi)^{d-1}} \int_{p \in D(t)} dp N_n^{cr} \rho(t) p_l^2, \quad l = 2, ..., D,
\]

\[
\rho(t) = 2|\langle P_x(t) \rangle| \left\{ |\varphi_n(t)|^2 \big|_{\chi = +1} + |\varphi_n(t)|^2 \big|_{\chi = -1} \right\}. \quad (6.16)
\]

In what follows we show that some universal behavior of the densities \( \langle j^1(t) \rangle^p \) and \( \langle T_{\mu\mu}(t) \rangle^p \) can be derived from general forms (6.16) for any large difference. We begin
the demonstration of this fact with the case of a finite interval of time when the electric field potential can be approximated by a potential of a constant electric field \((6.3)\). At the same time, we assume that \(\langle P_x (t) \rangle\) satisfies condition \((6.5)\) at the time \(t\). It is convenient to compare the cases of \(T\)-constant and exponentially decaying fields, which both are abruptly switching on but their ways of switching off may be different.

In the case of exponentially decaying field, the functions \(\pm \varphi_n (t)\) in Eq. \((6.16)\) are given by the second line in Eq. \((5.10)\) and approximation \((6.3)\) holds if \(k_2 t \ll 1\). Then \(|\langle P_x (t) \rangle| \ll |\pi_2|\).

To obtain functions \(\pm \varphi_n (t)\) in such an approximation we use the asymptotic representation \((A8)\). Thus, we obtain

\[
\rho (t) = \left[ V_{(d-1)} |\langle P_x (t) \rangle| \right]^{-1}.
\]

In the range \(D(t)\), the distribution \(N_n^{\text{cr}}\) is approximately given by Eq. \((3.19)\). Finally we obtain

\[
\langle j^1(t) \rangle^p \approx 2er^{\text{cr}} \Delta t,
\]

\[
\langle T_{00}(t) \rangle^p \approx \langle T_{11}(t) \rangle^p \approx eEr^{\text{cr}} \Delta t^2,
\]

\[
\langle T_{ll}(t) \rangle^p \approx \pi^{-1}r^{\text{cr}} \ln \left( \sqrt{eE} \Delta t \right) \text{ if } l = 2, ..., D,
\]

where \(\Delta t = t - t_{\text{in}}\) is the duration time of a constant field. In this case \(t_{\text{in}} = 0\).

The field potential of the \(T\)-constant field \((4.1)\) has the form \((6.3)\) in the intermediate region II. For sufficiently large times \(t < t_{\text{out}}\), when the longitudinal kinetic momentum belongs to the range \(D(t)\), the distribution \(N_n^{\text{cr}}\) is approximately given by Eq. \((3.19)\). In this case, exact expressions for the functions \(\varphi_n (t)\), given by Eq. \((4.3)\), and similar expressions for the functions \(- \varphi_n (t)\) can be approximated as the following WPCFs:

\[
\pm \varphi_n (t) \approx V_{(d-1)}^{-1/2} CD_{-1}^{-1/p_x} [(1 + i) \xi], \quad - \varphi_n (t) \approx V_{(d-1)}^{-1/2} CD_{-1}^{-1/p_x} [(1 - i) \xi],
\]

\[
\xi = (eEt - p_x) (eE)^{-1/2}, \quad C = (2eE)^{-1/2} \exp \left( - \pi \lambda / 8 \right).
\]

Then we find from Eq. \((6.16)\) that the densities \(\langle j^1(t) \rangle^p\) and \(\langle T_{\mu\mu}(t) \rangle^p\) have the same form \((6.18)\) with \(t_{\text{in}} = -T/2\).

Note that the above results are obtained by using functions \(\pm \varphi_n (t)\), which have in and out-asymptotics at \(t_{\text{out}}\). Nevertheless, these results show also that densities \((6.18)\) are not affected by evolution of the functions \(\pm \varphi_n (t)\) from \(t\) to \(t_{\text{out}}\) in the range \(p \in D(t)\), assuming that the corresponding electric field exists during a macroscopically large time period \(\Delta t\), satisfying Eq. \((6.1)\). This fact is closely related with a characteristic property of the kernel of
integrals (6.16), which will be derived from an universal form of the total density of created pairs given by Eq. (6.8). Let \( t'_{\text{out}} < t_{\text{out}} \) is another possible final time instant, then

\[
\tilde{n}^{cr} (t'_{\text{out}}) \approx \frac{J(d)}{(2\pi)^{d-1}} \int_{t_{\text{in}}}^{t'_{\text{out}}} dt [eE (t)]^{d/2} \exp \left\{ -\pi \frac{m^2}{eE (t)} \right\} \tag{6.20}
\]

Eq. (6.20) corresponds to the assumption that in the range \( p \in D (t'_{\text{out}}) \subset D (t_{\text{out}}) \) the electric field is switched on at \( t_{\text{in}} \) and switched off at \( t'_{\text{out}} \). Then instead of functions \( \zeta \psi_n (x) \) satisfying the eigenvalue problem (2.12), we have to use solutions of the following eigenvalue problem

\[
H (t) \zeta \psi_n (t'_{\text{out}}) (x) = \zeta \varepsilon_n \zeta \psi_n (t'_{\text{out}}) (x) , \ t \in [t'_{\text{out}}, +\infty ) , \ \zeta \varepsilon_n = \zeta p_0 (t'_{\text{out}}).
\]

Using the representation

\[
\zeta \psi_n (t'_{\text{out}}) (x) = [i\partial_t + H (t)] \gamma^0 \exp (ipr) \zeta \varphi_n (t'_{\text{out}}) (t) v_{\chi, \sigma}
\]

we obtain

\[
\zeta \varphi_n (t'_{\text{out}}) (t) = \zeta N(t'_{\text{out}}) \exp \left\{ -i \zeta p_0 (t'_{\text{out}}) (t - t'_{\text{out}}) \right\} , \ t \in [t'_{\text{out}}, +\infty ), \\
\zeta N(t'_{\text{out}}) = (2p_0 (t'_{\text{out}}) \{p_0 (t'_{\text{out}}) - \chi \zeta [p_x - U (t'_{\text{out}})] \} V_{(d-1)})^{-1/2} . \tag{6.21}
\]

Thus, leading contribution to the function \( S^p (x, x') \) (defined by Eq. (2.43)) at \( t' \sim t < t'_{\text{out}} \) can be expressed via \( \zeta \psi_n (t'_{\text{out}}) (x) \) as follows

\[
S^p (x, x') \approx -i \sum_{\chi, \sigma, p \in D(t)} N_n^{cr} \left[ + \tilde{J}_n (t'_{\text{out}}) (x) + \tilde{J}_n (t'_{\text{out}}) (x') - \tilde{J}_n (t'_{\text{out}}) (x) - \tilde{J}_n (t'_{\text{out}}) (x') \right] . \tag{6.22}
\]

Then \( \rho (t) \) in Eq. (6.16) can be represented as

\[
\rho (t) = 2 \left\langle \left| \langle P_x (t) \rangle \right| \left| + \varphi_n (t'_{\text{out}}) (t) \right|_{\chi=+1}^2 + \left| - \varphi_n (t'_{\text{out}}) (t) \right|_{\chi=-1}^2 \right\rangle .
\]

Taken into account Eq. (6.21), we can see that Eq. (6.17) holds for any large time difference \( t - t_{\text{in}} \). Using the universal form of the differential numbers of created pairs, \( N_n^{cr} \approx N_n^{\text{uni}} \), given by Eq. (6.8), changing the variable according to Eq. (6.7), and performing the integration over \( p_\perp \), we find from Eq. (6.16) that the vacuum mean values of current and EMT components have the following universal behavior for any large difference \( t - t_{\text{in}} \):

\[
\langle j^1 (t) \rangle^p \approx 2e\tilde{n}^{cr} (t) , \\
\langle T_{00} (t) \rangle^p \approx \langle T_{11} (t) \rangle^p \approx \frac{J(d)}{(2\pi)^{d-1}} \int_{t_{\text{in}}}^{t} dt' [U (t) - U (t')] [eE (t')]^{d/2} \exp \left\{ -\pi m^2 \frac{eE (t')}{} \right\} , \\
\langle T_{ll} (t) \rangle^p \approx \frac{J(d)}{(2\pi)^d} \int_{t_{\text{in}}}^{t} \frac{dt' \ [eE (t')]^{d/2+1}}{[U (t) - U (t')] \exp \left\{ -\pi m^2 \frac{eE (t')}{} \right\}} , \ l = 2, ..., D . \tag{6.23}
\]
Here \( \bar{n}_{cr} (t) \) is given by Eq. (6.20).

For \( t > t_{out} \), the pair production stops, vacuum polarization effects disappear, and quantities (6.23) for \( t > t_{out} \) maintain their values at \( t = t_{out} \),

\[
\langle j^1 (t) \rangle |_{t > t_{out}} \approx \langle j^1 (t_{out}) \rangle^p, \quad \langle T_{\mu\mu} (t) \rangle |_{t > t_{out}} \approx \langle T_{\mu\mu} (t_{out}) \rangle^p.
\] (6.24)

Note that \( \bar{n}_{cr} (t_{out}) = \bar{n}_{cr} \) is the number density of created real pairs, given by Eq. (6.9), that is, it is the number density of electrons and positrons detectable at any time instant after switching of an electric field off. The quantities \( \langle j^1 (t_{out}) \rangle^p \) and \( \langle T_{\mu\mu} (t_{out}) \rangle^p \) are the mean current density and the EMT components of real pairs created from the vacuum. The energy density \( \langle T_{00} (t_{out}) \rangle \) is equal to the pressure \( \langle T_{11} (t_{out}) \rangle \) along the direction of the electric field at the time instant \( t_{out} \). This equality is a natural equation of state for noninteracting particles accelerated by an electric field to relativistic velocities.

In particular, for fields admitting exactly solvable cases, we find from Eqs. (6.23) and (6.24):

(a) For \( T \)-constant field

\[
\langle T_{00} (t_{out}) \rangle^p \approx \langle T_{11} (t_{out}) \rangle^p \approx e E \bar{n}_{cr} (t_{out} - t_{in})^2, \\
\langle T_{ll} (t_{out}) \rangle^p \approx \pi^{-1} r_{cr} \ln \left[ \sqrt{e E (t_{out} - t_{in})} \right], \quad l = 2, ..., D.
\] (6.25)

(b) For the peak field:

\[
\langle T_{00} (t_{out}) \rangle^p \approx \langle T_{11} (t_{out}) \rangle^p \approx e E r_{cr} \left[ k_{-1}^{-1} \right. \\
\times \left. \left( k_{-2}^{-1} - k_{1}^{-1} \right) G \left( \frac{d}{2} + 1, \frac{\pi m^2}{eE} \right) + k_{1}^{-1} G \left( \frac{d}{2}, \frac{\pi m^2}{eE} \right) \right], \\
\langle T_{ll} (t_{out}) \rangle^p \approx \frac{r_{cr}}{2\pi} \left[ G \left( \frac{d}{2} - 1, \frac{\pi m^2}{eE} \right) + \frac{k_{2}}{k_{1}} G \left( \frac{d}{2}, \frac{\pi m^2}{eE} \right) \right], \quad l = 2, ..., D.
\] (6.26)

Densities (6.26) correspond to the case of an exponentially decaying field as \( k_{1}^{-1} \to 0 \).

(c) For Sauter-like field:

\[
\langle T_{00} (t_{out}) \rangle^p \approx \langle T_{11} (t_{out}) \rangle^p \approx e E r_{cr} S_{\delta} \left[ G \left( \frac{d}{2}, \frac{\pi m^2}{eE} \right) \right], \\
\langle T_{ll} (t_{out}) \rangle^p \approx \frac{r_{cr}}{2\pi} \left[ \sqrt{\pi E} \left( \frac{1}{2}, 2 - \frac{d}{2}, \frac{\pi m^2}{eE} \right) + G \left( \frac{d}{2} - 1, \frac{\pi m^2}{eE} \right) \right].
\] (6.27)

Note that using the differential mean numbers of created pairs given by Eqs. (3.15), (3.19), (5.29), and (5.32) for the exactly solvable cases [without the use of the universal form given by Eq. (6.8)], we obtain from Eq. (6.16) literally expressions (6.25) (earlier
obtained in Refs. [25–27], (6.26), and (6.27). It is an independent confirmation of universal form (6.23).

The obtained results show that the scale \( \Delta t_{\text{st}} \) plays the role of the stabilization time for the densities \( \langle j^1(t) \rangle^p \) and \( \langle T_{\mu\nu}(t) \rangle^p \). The characteristic parameter \( m^2/eE \) can be represented as the ratio of two characteristic lengths:

\[
c^3m^2/\hbar eE = (c\Delta t_{\text{st}}/\Lambda_C)^2,
\]

where \( \Lambda_C = \hbar/mc \) is the Compton wavelength. In strong electric fields, \( (c\Delta t_{\text{st}}/\Lambda_C)^2 \lesssim 1 \), inequality (6.2) is simplified to the form \( \Delta t/\Delta t_{\text{st}} \gg 1 \), in which the Compton wavelength is absent. We see that the scale \( \Delta t_{\text{st}} \) plays the role of the stabilization time for a strong electric field. This means that \( \Delta t_{\text{st}} \) is a characteristic time scale which allows us to distinguish fields that have microscopic or macroscopic time change, it plays similar role as the Compton wavelength plays in the case of a weak field. Therefore, calculations in a \( T \)-constant field are quite representative for a large class of slowly varying electric fields.

Under natural assumptions, the parameter \( eE(t)\Delta t^2 \) is limited. Considering problems of high-energy physics in \( d = 4 \), it is usually assumed that just from the beginning there exists an uniform classical electric field with a given energy density. This field can be modelled by the \( T \)-constant field. The system of particles interacting with this field is closed, that is, the total energy of the system is conserved. Under such an assumption, the pair creation is a transient process and, for example, the applicability of the constant field approximation is limited by the smallness of the back reaction, which implies the following restriction from above:

\[
(\Delta t/\Delta t_{\text{st}})^2 \ll \frac{\pi^2}{J\alpha} \exp \left( \frac{\pi^3m^2}{\hbar eE} \right), \tag{6.28}
\]
on time \( \Delta t \) for a given electric field strength \( E \). Here \( \alpha \) is the fine structure constant and \( J \) is the number of the spin degrees of freedom, see [22]. Thus, there is a range of the parameters \( E \) and \( \Delta t \) where the approximation of the constant external field is consistent.

It is well known that at certain conditions (the so-called charge neutrality point) electronic excitations in graphene monolayer behave as relativistic Dirac massless fermions in \( 2 + 1 \) dimensions, with the Fermi velocity \( v_F \simeq 10^6 \) m/s playing the role of the speed of light, see details in recent reviews [59, 60]. Then in the range of the applicability of the Dirac model to the graphene physics, any electric field is strong. There appears a timescale specific to graphene (and to similar nanostructures with the Dirac fermions), \( \Delta t_{\text{st}}^g = (eEv_F/\hbar)^{-1/2} \), which plays the role of the stabilization time in the case under consideration. The \( T \)-constant field is suitable for imitating a slowly varying field under condition \( \Delta t/\Delta t_{\text{st}}^g \gg 1 \). The
transport in graphene can be considered as ballistic then the ballistic flight time $T_{bal}$ to be the effective time duration, $\Delta t = T_{bal}$. The external constant electric field can be considered as a good approximation of the effective mean field as long as the field produced by the induced current of created particles, $\langle j^1(t) \rangle^p$ given by Eq. (6.13), is negligible compared to the applied field. This implies the consistency restriction $\Delta t \ll \Delta t_{br} = \Delta t_{st}^K/4\alpha [27]$. Thus, in this case there is a window in voltages, $7 \times 10^{-4} \text{V} \ll V \ll 8 \text{V}$, where the model with constant external field is consistent. These voltages are in the range typically used in experiments with the graphene.

C. Vacuum polarization

In which follows we use the example of the $T$-constant field to consider the contributions $\text{Re} \langle j^\mu(t) \rangle^c$ and $\text{Re} \langle T_{\mu\nu}(t) \rangle^c$ to the mean values of the current density $\langle j^\mu(t) \rangle$ and the EMT $\langle T_{\mu\nu}(t) \rangle$, given by Eqs. (2.44). Note that the mean current density $\langle j^\mu(t) \rangle$ and the physical part of the mean value $\langle T_{\mu\nu}(t) \rangle$ are zero for any $t < t_{in}$. For $t > t_{in}$, we are interested in these mean values only for a large time periods $\Delta t = t - t_{in}$ satisfying Eq. (6.1). In this case, the longitudinal kinetic momentum belongs to the range (6.5) and distributions $N_n^{cr}$ are approximated by Eq. (3.19). Using approximation (6.19), the functions $-\varphi_n(t)$, given by Eq. (4.4), and similar functions $+\varphi_n(t)$ can be taken in the following form

$$-\varphi_n(t) = V^{-1/2}_{(d-1)CD_{-1-\rho}}[-(1+i)\xi], \quad +\varphi_n(t) = V^{-1/2}_{(d-1)CD_{\rho}}[-(1-i)\xi]. \quad (6.29)$$

In the same approximation, the causal propagator $S^c(x, x')$ (2.42) can be calculated using solutions $\pm \psi_n(x)$ and $\pm \psi_n(x)$ with scalar functions given by Eqs. (6.19) and (6.29) in the range (6.5). It can be shown that the main contributions to $\text{Re} \langle j^\mu(t) \rangle^c$, $\langle j^2(t) \rangle$ and $\text{Re} \langle T_{\mu\nu}(t) \rangle^c$ are formed in the range (6.5) for a large time period $\Delta t$. It is important that these contributions are independent of the interval $\Delta t$, that is, the densities $\text{Re} \langle j^\mu(t) \rangle^c$, $\langle j^2(t) \rangle$, and $\text{Re} \langle T_{\mu\nu}(t) \rangle^c$ are local quantities describing only vacuum polarization effects. Then we integrate in Eq. (2.42) over all the momenta. Thus, we see that in the case under consideration, the propagator $S^c(x, x')$ can be approximated by the propagator in a constant uniform electric field.

The propagator $S^c(x, x')$ in a constant uniform electric field can be represented as the
Fock–Schwinger proper time integral:

\[ S^c(x, x') = (\gamma P + m)\Delta^c(x, x') \]

\[ \Delta^c(x, x') = \int_0^\infty f(x, x', s) ds, \]  

(6.30)

see [12] and [61, 62], where the Fock–Schwinger kernel \( f(x, x', s) \) reads

\[ f(x, x', s) = \exp \left( i\frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu} s \right) f^{(0)}(x, x', s), \]

(6.31)

\[ f^{(0)}(x, x', s) = -\frac{eEs^{-d/2+1}}{(4\pi i)^{d/2} \sinh(eEs)} \times \exp \left[ -i(\epsilon + m^2 s) + \frac{1}{4\epsilon}(x - x')eF \coth(eFs)(x - x') \right]. \]

Here \( \coth(eFs) \) is the matrix with the components \( \coth(eFs)_{\mu\nu} \), \( F_{\mu\nu} = E (\delta^i_\mu \delta^j_\nu - \delta^i_\nu \delta^j_\mu) \), and \( \Lambda = (t + t')(x_1 - x'_1)/2 \), see [1, 63].

It is easy to see that \( \langle j^1(t) \rangle^c = 0 \), as should be expected due to the translational symmetry. If \( d = 3 \) there is a transverse vacuum-polarization current,

\[ \langle j^2(t) \rangle = \pm \frac{e^2}{4\pi^{3/2}} \gamma \left( \frac{1}{2}, \frac{\pi m^2}{eE} \right) E, \]  

(6.31)

for each \( \pm \) fermion species, [27], where \( \gamma(1/2, x) \) is the incomplete gamma function. Note that the transverse current of created particles is absent, \( \langle j^2(t) \rangle = 0 \) if \( t > t_{\text{out}} \). The factor in the front of \( E \) in Eq. (6.31) can be considered as a nonequilibrium Hall conductivity for large duration of the electric field. In the presence of both \( \pm \) species in a model, \( \langle j^2(t) \rangle = 0 \) for any \( t \).

Using Eq. (6.30), we obtain components of the EMT for the \( T \)-constant field in the following form

\[ \text{Re}\langle T_{00}(t) \rangle^c = -\text{Re}\langle T_{11}(t) \rangle^c = E_0 \frac{\partial \text{Re} \mathcal{L}[E]}{\partial E} - \text{Re} \mathcal{L}[E], \]

\[ \text{Re}\langle T_{ll}(t) \rangle^c = \text{Re} \mathcal{L}[E], \]  

\[ l = 2, ..., D, \]  

(6.32)

where

\[ \mathcal{L}[E] = \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{tr} f(x, x, s), \text{tr} f(x, x, s) = 2^{[d/2]} \cosh(eEs)f^{(0)}(x, x, s). \]  

(6.33)

The quantity \( \mathcal{L}[E] \) can be identified with a non-renormalized one-loop effective Euler-Heisenberg Lagrangian of the Dirac field in an uniform constant electric field \( E \). Note that components \( \text{Re}\langle T_{\mu\nu}(t) \rangle^c \) do not depend of the time duration \( \Delta t \) of the \( T \)-constant field if \( \Delta t \) is sufficiently large.
This result can be generalized to the case of arbitrary slowly varying electric field. To this end we divide as before the finite interval \( (t_{\text{eff.in}}, t_{\text{eff.out}}] \) into \( M \) intervals \( \Delta t_i = t_{i+1} - t_i > 0 \), such that Eq. (6.1) holds true for each of them. That allows us to treat the electric field as approximately constant within each interval, \( \bar{E}(t) \approx \bar{E}(t_i) \) for \( t \in (t_i, t_{i+1}] \). In each such an interval, we obtain expressions similar to the ones \( (6.32) \) and \( (6.33) \), where the constant electric field \( E \) has to be substituted by \( \bar{E}(t_i) \). Then components of the EMT for arbitrary slowly varying strong electric field \( E(t) \) in the leading-term approximation can be represented as

\[
\text{Re}\langle T_{00}(t)\rangle^c = -\text{Re}\langle T_{11}(t)\rangle^c = E(t) \frac{\partial \text{Re}\mathcal{L}[E(t)]}{\partial E(t)} - \text{Re}\mathcal{L}[E(t)],
\]

\[
\text{Re}\langle T_{ll}(t)\rangle^c = \text{Re}\mathcal{L}[E(t)], \quad l = 2, \ldots, D,
\]

where

\[
\mathcal{L}[E(t)] = \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{tr} \tilde{f}(x, x, s), \quad \text{tr} \tilde{f}(x, x, s) = 2^{d/2} \cosh [eE(t)s] \tilde{f}^{(0)}(x, x, s),
\]

\[
\tilde{f}^{(0)}(x, x, s) = -\frac{eE(t)s^{-d/2} \exp (-im^2s)}{(4\pi i)^{d/2} \sinh [eE(t)s]}.
\]

Note that \( \mathcal{L}[E(t)] \) evolves in time due to the time dependence of the field \( E(t) \).

The quantity \( \mathcal{L}[E(t)] \) describes the vacuum polarization. The quantities \( (6.34) \) are divergent due to the real part of the effective Lagrangian \( (6.35) \), which is ill defined. This real part must be regularized and renormalized. In low dimensions, \( d \leq 4 \), \( \text{Re}\mathcal{L}[E(t)] \) can be regularized in the proper-time representation and renormalized by the Schwinger renormalizations of the charge and the electromagnetic field \([1]\). In particular, for \( d = 4 \), the renormalized effective Lagrangian \( \mathcal{L}_{\text{ren}}[E(t)] \) is

\[
\mathcal{L}_{\text{ren}}[E(t)] = \int_0^\infty ds \frac{\exp (-im^2s)}{8\pi^2s} \left\{ eE(t) \coth [eE(t)s] - \frac{1}{s^2} - \frac{[eE(t)s]^2}{3} \right\}.
\]

In higher dimensions, \( d > 4 \), a different approach is required. One can give a precise meaning and calculate the one-loop effective action using zeta-function regularization, see details in Ref. \([27]\). Making the same renormalization for \( \langle T_{\mu\nu}(t)\rangle^c \), we can see that for the renormalized EMT the following relations hold true

\[
\text{Re}\langle T_{00}(t)\rangle^c_{\text{ren}} = -\text{Re}\langle T_{11}(t)\rangle^c_{\text{ren}} = E(t) \frac{\partial \text{Re}\mathcal{L}_{\text{ren}}[E(t)]}{\partial E(t)} - \text{Re}\mathcal{L}_{\text{ren}}[E(t)],
\]

\[
\text{Re}\langle T_{ll}(t)\rangle^c_{\text{ren}} = \text{Re}\mathcal{L}_{\text{ren}}[E(t)], \quad l = 2, 3, \ldots, D.
\]
In the strong-field case \((m^2/eE(t) \ll 1)\), the leading contributions to the renormalized EMT are
\[
\text{Re} \langle T_{\mu\mu}(t) \rangle_{\text{ren}}^c \sim \begin{cases} 
[eE(t)]^{d/2}, & d \neq 4k \\
[eE(t)]^{d/2} \ln [eE(t)/M^2], & d = 4k
\end{cases}
\quad \text{(6.38)}
\]

The final form of the vacuum mean components of the EMT are
\[
\langle T_{\mu\mu}(t) \rangle_{\text{ren}} = \text{Re} \langle T_{\mu\mu}(t) \rangle_{\text{ren}}^c + \text{Re} \langle T_{\mu\mu}(t) \rangle^p,
\quad \text{(6.39)}
\]
where the components \(\text{Re} \langle T_{\mu\mu}(t) \rangle_{\text{ren}}^c\) and \(\text{Re} \langle T_{\mu\mu}(t) \rangle^p\) are given by Eqs. (6.37) and (6.23), respectively. For \(t < t_{\text{in}}\) and \(t > t_{\text{out}}\) the electric field is absent such that \(\text{Re} \langle T_{\mu\mu}(t) \rangle_{\text{ren}}^c = 0\).

On the right hand side of Eq. (6.39), the term \(\text{Re} \langle T_{\mu\mu}(t) \rangle^p\) represents contributions due to the vacuum instability, whereas the term \(\text{Re} \langle T_{\mu\mu}(t) \rangle_{\text{ren}}^c\) represents vacuum polarization effects. For weak electric fields, \(m^2/eE \gg 1\), contributions due to the vacuum instability are exponentially small, so that the vacuum polarization effects play the principal role. For strong electric fields, \(m^2/eE \ll 1\), the energy density of the vacuum polarization \(\text{Re} \langle T_{00}(t) \rangle_{\text{ren}}^c\) is negligible compared to the energy density due to the vacuum instability \(\langle T_{00}(t) \rangle^p\);
\[
\langle T_{\mu\mu}(t) \rangle_{\text{ren}} \approx \text{Re} \langle T_{\mu\mu}(t) \rangle^p.
\quad \text{(6.40)}
\]
The latter density is formed on the whole time interval \(t - t_{\text{in}}\), however, dominant contributions are due to time intervals \(\Delta t_i\) with \(m^2/eE(t_i) < 1\) and the large dimensionless parameters \(\sqrt{eE(t_i)} \Delta t_i\).

We note that effective Lagrangian (6.35) and its renormalized form \(L_{\text{ren}}[E(t)]\) coincide with leading term approximation of derivative expansion results from field-theoretic calculations obtained in Refs. [21, 54, 55] for \(d = 3\) and \(d = 4\). In this approximation the \(S^{(0)}\) term of the Schwinger’s effective action, given by the expansion (6.11), has the form
\[
S^{(0)}[F_{\mu\nu}] = \int dx L_{\text{ren}}[E(t)].
\quad \text{(6.41)}
\]
It should be stressed that unlike to the authors of Refs. [21, 54, 55], expression (6.35) and its renormalized form were derived in the framework of the general exact formulation of strong-field QED [6, 23], using QED definition of the mean value of the EMT, given by Eq. (2.44). Therefore \(L_{\text{ren}}[E(t)]\) is obtained independently from the derivative expansion approach and the obtained result holds true for any strong field under consideration. Moreover, it is proven that in this case not only the imaginary part of \(S^{(0)}\) but its real part as
well is given exactly by the semiclassical WKB limit. It is clearly demonstrated that the imaginary part of the effective action, $\text{Im}S^{(0)}$, is related to the vacuum-to-vacuum transition probability $P_v$ and can be represented as an integral of $\mathcal{L}_{\text{ren}} [E(t)]$ over the total field history, whereas the kernel of the real part of this effective action, $\text{Re} \mathcal{L}_{\text{ren}} [E(t)]$, is related to the local EMT which defines the vacuum polarization. Obtained results justify the derivative expansion as an asymptotic expansion that can be useful to find the corrections for mean values of the EMT components. We also note that some authors have argued that the locally constant field approximation, which amounts to limiting oneself to the leading contribution of the derivative expansion of the effective action, allows for reliable results for electromagnetic fields of arbitrary strength; cf., e.g., [64, 65].

VII. CONCLUDING REMARKS

We have presented in detail consistent QED calculations of zero order quantum effects in external electromagnetic field that correspond to the most important three exactly solvable cases of $t$-electric potential steps, namely, the Sauter-like electric field, the $T$-constant electric field, and the exponentially growing and decaying electric fields. In all the cases, we present some new important details, unpublished so far. Nontrivial details underlying the derivation of number density of pairs created from vacuum due to the strong Sauter-like case are presented. Next-to-leading term approximation for the differential mean number of pair created, $N_{\text{cr}}^n$, due to $T$-constant electric field of long duration is obtained. A detailed study of differential mean numbers of pair created in the most asymmetric case of the exponentially growing and decaying electric fields is presented. On the base of exactly solvable cases, we consider in detail distributions $N_{\text{cr}}^n$ as functions on the particle momenta and establish the ranges of dominant contributions for mean numbers of created particles due to a slowly varying field. This allows us to gain new insights on the universal behavior of the vacuum mean values in slowly varying intense electric fields. Comparing results for three exactly solvable cases, one can see the appearance of a large parameter, which is an increment of the longitudinal kinetic momentum, and which corresponds to a large number of quantum states, in which particles can be created. One can define the slowly varying regime in general terms. Using the cases of the $T$-constant and exponentially growing and decaying electric fields, we find universal forms of the vacuum mean values of current, EMT components and
the total density of created pairs in the leading-term approximation for any large duration of an electric field. One can find a close relation of obtained universal forms with a leading term approximation of derivative expansion results in field-theoretic calculations. In fact, it is the possibility to adopt a locally constant field approximation which makes an effect universal. These results allow one to formulate semiclassical approximations, that are not restricted by smallness of differential mean numbers of created pairs, and could be helpful for the development of numerical methods in strong-field QED.

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Appendix A: Some asymptotic expansions

The asymptotic expression of the confluent hypergeometric function for large \( \eta \) and \( c \) with fixed \( a \) and \( \tau = \eta/c \sim 1 \) is given by Eq. (13.8.4) in [66] as

\[
\Phi(a,c;\eta) \simeq e^{a/2} e^{\frac{\eta^2}{4}} F(a,c;\tau), \quad Z = -(\tau - 1) W(\tau) \sqrt{c},
\]

\[
F(a,c;\tau) = \tau W^{1-a} D_{-a}(Z) + R D_{1-a}(Z),
\]

\[
R = (W^a - \tau W^{1-a}) / Z, \quad W(\tau) = \left[ 2(\tau - 1 - \ln \tau) / (\tau - 1)^2 \right]^{1/2}
\]

where \( D_{-a}(Z) \) is the Weber parabolic cylinder function (WPCF) [49]. Using Eq. (A1) we present the functions \( y_1^2(\eta_2), y_2^2(\eta_1) \) and their derivatives at \( t = 0 \) as

\[
y_2^1(\eta_1)|_{t=0} \simeq e^{i\eta_1/2} (i\eta_1)^{-\nu_1} (2-c_1)^{(1-a_1)/2} e^{\frac{\eta_1^2}{4}} F(1-a_1,2-c_1;\tau_1),
\]

\[
Z_1 = -(\tau_1 - 1) W(\tau_1) \sqrt{2-c_1}, \quad \tau_1 = -ih_1 / (2-c_1),
\]

\[
\frac{\partial y_2^1(\eta_1)}{\partial \eta_1} \bigg|_{t=0} \simeq e^{i\eta_1/2} (i\eta_1)^{-\nu_1} (2-c_1)^{(1-a_1)/2} e^{\frac{\eta_1^2}{4}} \left[ -\frac{1}{2 i h_1} - \frac{1}{2-c_1} \frac{\partial}{\partial \tau_1} \right] F(1-a_1,2-c_1;\tau_1);
\]

\[
y_2^2(\eta_2)|_{t=0} \simeq e^{-i\eta_2/2} (i\eta_2)^{\nu_2} c_2^{a_2/2} e^{\frac{\eta_2^2}{4}} F(a_2,c_2;\tau_2),
\]

\[
\frac{\partial y_2^2(\eta_2)}{\partial \eta_2} \bigg|_{t=0} \simeq e^{-i\eta_2/2} (i\eta_2)^{\nu_2} c_2^{a_2/2} e^{\frac{\eta_2^2}{4}} \left[ -\frac{1}{2 i h_2} + \frac{1}{c_2} \frac{\partial}{\partial \tau_2} \right] F(a_2,c_2;\tau_2).
\]

Assuming \( \tau - 1 \to 0 \), one has

\[
W^{1-a} \approx 1 + \frac{a - 1}{3} (\tau - 1), \quad R \approx \frac{2(a+1)}{3\sqrt{c}}, \quad Z \approx -(\tau - 1) \sqrt{c},
\]

\[
\frac{\partial F(a,c;\tau)}{\partial \tau} \approx \frac{2 + a}{3} D_{-a}(Z) + \frac{\partial D_{-a}(Z)}{\partial \tau} + R \frac{\partial D_{1-a}(Z)}{\partial \tau}.
\]
Expanding WPCFs near $Z = 0$, in the leading approximation at $Z \to 0$ one obtains that
\[
\frac{\partial F(a, c; \tau)}{\partial \tau} \approx -\sqrt{\eta} D'_{-a}(0) + O(\eta),
\]
\[
F(a, c; \tau) \approx D_{-a}(0) + O(e^{-1/2}),
\]
(A3)
and
\[
D_{-a}(0) = \frac{2^{-a/2} \sqrt{\pi}}{\Gamma\left(\frac{a+1}{2}\right)}, \quad D'_{-a}(0) = \frac{2^{(1-a)/2} \sqrt{\pi}}{\Gamma\left(\frac{a}{2}\right)},
\]
(A4)
where $\Gamma(z)$ is the Euler gamma function. We find under condition (5.21) that
\[
\omega_{1,2} \approx |\pi_{1,2}|(1 + \lambda/h_{1,2}), \quad a_{1,2} \approx (1 + \chi)/2 + i\lambda/2,
\]
\[
2 - c_1 \approx 1 - i \left(\lambda + \frac{2\pi_1}{k_1}\right), \quad c_2 \approx 1 + i \left(\lambda - \frac{2\pi_2}{k_2}\right),
\]
\[
\tau_1 - 1 \approx -\frac{1}{h_1} \left(i + \lambda + \frac{2p_x}{k_1}\right), \quad \tau_2 - 1 \approx \frac{1}{h_2} \left(i - \lambda + \frac{2p_x}{k_2}\right).
\]
(A5)
Using Eqs. (A2), (A3), and (A5) we represent Eq. (5.19) in the form
\[
N^\text{cr}_n = e^{-\pi\lambda/2} \left[|\delta_0|^2 + O\left(h_{1}^{-1/2}\right) + O\left(h_{2}^{-1/2}\right)\right],
\]
\[
\delta_0 = e^{i\pi/4} D_{-a_2}(0) D'_{a_2-1}(0) - e^{-i\pi/4} D'_{-a_2}(0) D_{a_2-1}(0).
\]
(A6)
Assuming $\chi = 1$ for fermions and $\chi = 0$ for bosons, and using the relations of the Euler gamma function we find that
\[
\delta_0 = \exp\left(i\pi/2 + i\pi\chi/4\right) e^{-\pi\lambda/4}.
\]
(A7)
Assuming $|\tau - 1| \sim 1$, one can use the asymptotic expansions of WPCFs in Eq. (A1), e.g., see [49, 66]. Note that arg $(Z) \approx \frac{1}{2} \operatorname{arg} (c)$ if $1 - \tau > 0$. Then one finds that
\[
\Phi(a, c; \eta) = (1 - \tau)^{-a} \left[1 + O\left(|Z|^{-1}\right)\right] \quad \text{if} \quad 1 - \tau > 0.
\]
(A8)
In the case of $1 - \tau < 0$, one has
\[
\arg (Z) \approx \begin{cases} 
\frac{1}{2} \operatorname{arg} (c) + \pi & \text{if} \quad \operatorname{arg} (c) < 0 \\
\frac{1}{2} \operatorname{arg} (c) - \pi & \text{if} \quad \operatorname{arg} (c) > 0.
\end{cases}
\]
Then one obtains finally that
\[
\Phi(a, c; \eta) = \begin{cases} 
(\tau - 1)^{-a} e^{-i\pi a} \left[1 + O\left(|Z|^{-1}\right)\right] & \text{if} \quad \operatorname{arg} (c) < 0 \\
(\tau - 1)^{-a} e^{i\pi a} \left[1 + O\left(|Z|^{-1}\right)\right] & \text{if} \quad \operatorname{arg} (c) > 0.
\end{cases}
\]
(A9)
The asymptotic expression of the confluent hypergeometric function $\Phi(a, c; \pm ih)$ for large real $h$ with fixed $a$ and $c$ is given by Eq. (6.13.1(2)) in [49] as

$$\Phi(a, c; \pm ih) = \frac{\Gamma(c)}{\Gamma(c-a)} e^{\frac{\pm i\pi a}{2} - a} + O(|h|^{a-1}) + \frac{\Gamma(c)}{\Gamma(a)} e^{\pm i\pi/2h} e^{\pm i\pi/2h} a-c + O(|h|^{a-c-1}).$$

(A10)

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