STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS

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Abstract. We study the stability of coupled impedance passive regular linear systems under power-preserving interconnections. We present new conditions for strong, exponential, and non-uniform stability of the closed-loop system. We apply the stability results to the construction of passive error feedback controllers for robust output tracking and disturbance rejection for strongly stabilizable passive systems. In the case of nonsmooth reference and disturbance signals we present conditions for non-uniform rational and logarithmic rates of convergence of the output. The results are illustrated with examples on designing controllers for linear wave and heat equations, and on studying the stability of a system of coupled partial differential equations.

1. Introduction

In this paper we study the stability properties and control of regular linear systems \[ (1.1a) \quad \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in X, \]
\[ (1.1b) \quad y(t) = C_\Lambda x(t) + Du(t) \]
on a Hilbert space \( X \), where \( u(t) \) is the input of the system and \( y(t) \) is the output. Our main interest is in systems that are impedance passive \([10, 38, 40]\) (or passive for short) in the sense that their solutions satisfy
\[ \frac{d}{dt} \|x(t)\|^2 \leq 2 \Re \langle u(t), y(t) \rangle, \quad t > 0. \]

Passive systems are encountered especially in the study of mechanical or electrical systems modeled with partial differential equations. In particular, \((1.1)\) is impedance passive if \( A \) generates a contraction semigroup, \( B \) and \( C \) are bounded operators, \( C = B^* \), and \( \Re D \geq 0 \).

The paper consists of two main parts. In the first part we focus on the stability of the coupled system consisting of \((1.1)\) and another passive regular...
linear system

\[
\dot{z}(t) = A_c z(t) + B_c u_c(t), \quad z(0) = z_0 \in Z, \\
y_c(t) = C_c A z(t) + D_c u_c(t)
\]

with \(D_c^* = D_c\) under a power-preserving interconnection where \(u(t) = y_c(t), \quad u_c(t) = -y(t)\).

We study the stability of the resulting closed-loop system

\[
\dot{x}_e(t) = A_e x_e(t), \quad x_e(0) = x_{e0} \in X_e
\]
on the Hilbert space \(X_e = X \times Z\). The notation \((A_c, B_c, C_c, D_c)\) and our results on the closed-loop stability are motivated by the second part of the paper where we study robust output tracking and disturbance rejection for the system (1.1). In this situation (1.2) is an unstable dynamic feedback controller. However, our results are also applicable when the roles of the systems are reversed, i.e., when (1.2) is a system to be controlled and (1.1) is the controller, and they can also be used to study the stability of systems of partial differential equations coupled on the boundary or inside the domain. Our main interest is in the situation where \(A_c\) has a countable number of spectral points on the imaginary axis.

We study (1.3) in terms of the stability properties of the strongly continuous semigroup \(T_e(t)\) generated by \(A_e: D(A_e) \subset X_e \to X_e\). As our main results we introduce conditions under which the semigroup \(T_e(t)\) is exponentially stable, strongly stable, or non-uniformly stable [7, 36]. Among these, exponential stability is the strongest form of stability. However, in certain control applications exponential stability is unachievable, and many partial differential equations and coupled systems are known to lack exponential decay of energy. These situations arise especially in wave equations with partial damping and in coupled hyperbolic-parabolic systems [49, 6]. Recently many such coupled systems have been shown to be polynomially stable [25, 7, 8], which means that the classical solutions of the system decay at rational rates, i.e., for some constants \(M_e, \alpha, t_0 > 0\)

\[
\|T_e(t)x_{e0}\| \leq \frac{M_e}{t^{1/\alpha}} \|A_e x_{e0}\|, \quad x_{e0} \in D(A_e), \quad t \geq t_0.
\]

In this paper we introduce new results for studying polynomial and the more general non-uniform stability for coupled passive abstract linear systems (1.1) and (1.2).

Strong and exponential closed-loop stabilities of infinite-dimensional systems have been studied in the literature for passive one-dimensional boundary control systems [41, 33], coupled systems with collocated inputs and outputs [16], and passive systems coupled with finite-dimensional systems [50]. Polynomial stability of coupled systems has been studied extensively in the context of coupled linear partial differential equations [3, 1, 6, 2], and for abstract hyperbolic-parabolic systems [22].

In the second part of the paper we study the robust output regulation problem where the aim is to design a controller in such a way that the output \(y(t)\) of the system (1.1) converges to a given reference signal \(y_{\text{ref}}(t)\).
asymptotically in the sense that
\[ \| y(t) - y_{\text{ref}}(t) \| \to 0, \quad t \to \infty \]
despite possible external disturbance signals \( w_{\text{dist}}(t) \). In addition, the controller is required to be robust in the sense that it should achieve output tracking even if the parameters \( (A, B, C, D) \) experience small changes or contain uncertainties. This control problem has been studied actively in the literature for various classes of infinite-dimensional linear systems \([48, 26, 19, 35, 20, 31, 42]\) including regular linear systems \([46, 9, 32, 47, 29, 30]\) and passive systems \([35]\).

The robust output regulation problem can be solved with a dynamical error feedback controller of the form
\[
\begin{align*}
\dot{z}(t) &= A_c z(t) + B_c (y_{\text{ref}}(t) - y(t)), \quad z(0) = z_0 \in Z, \\
u(t) &= C_c \Lambda z(t) + D_c (y_{\text{ref}}(t) - y(t)).
\end{align*}
\]

One of the fundamental results of the theory, the internal model principle \([17, 15, 31, 32]\), implies that robust output tracking can be achieved by including a suitable number of copies of the frequencies \( \{\omega_k\}_{k \in I} \) of \( y_{\text{ref}}(t) \) and \( w_{\text{dist}}(t) \) into the dynamics of the controller and using the remaining parameters of (1.4) to stabilize the closed-loop system. While the inclusion of the internal model is both necessary and sufficient for robustness, the resulting closed-loop can be stabilized in various ways. Under fairly general assumptions the closed-loop stability can be achieved with observer-based design methods \([20, 29]\) leading to infinite-dimensional controllers. If the system (1.1) can be stabilized exponentially with output feedback and if \( y_{\text{ref}}(t) \) and \( w_{\text{dist}}(t) \) contain a finite number of frequencies, then \( A_c \) can be chosen to be minimal in the sense that it contains only the internal model, and the closed-loop system can be stabilized with suitable choices of \( B_c \) and \( C_c \). \([26, 19, 35]\). It was shown in \([35, \text{Thm. 1.2}]\) that if (1.1) is passive and exponentially stabilizable, then robust output regulation can be achieved in a natural way using a minimal passive controller (1.4).

In this paper we extend the passive controller design presented in \([35]\). We present a robust passive controller for systems (1.1) that are not exponentially stabilizable, but only strongly stabilizable. Such systems are encountered, for example, in control of wave equations, as illustrated in Section 6. Moreover, our design methods allow considering nonsmooth periodic reference and disturbance signals with infinite numbers of frequencies. In earlier references, the robust output regulation of nonsmooth signals has only been achieved using an observer in the controller \([20, 30]\). We solve this problem with two new robust controllers having the property that \( A_c \) contains only the internal model of the reference and disturbance signals. These controllers achieve either exponential, polynomial, or non-uniform closed-loop stability depending on the properties of the system (1.1) and the choices of the controller’s parameters. In the case of non-uniform closed-loop stability we present non-uniform rates of convergence for the output \( y(t) \) for sufficiently smooth \( y_{\text{ref}}(\cdot) \) and \( w_{\text{dist}}(\cdot) \).

One of the passive controllers presented in this paper is based on a transport equation with boundary control and observation, and under suitable assumptions on the system (1.1) (in general requiring \( D \neq 0 \)) the controller
achieves robust output regulation of all \( \tau \)-periodic reference and disturbance signals with exponential convergence rate of the output. This structure is related to the controllers used in repetitive control \([21, 46]\) and in \([23]\).

The paper is organised as follows. In Section \(2\) we state the main standing assumptions. The results on stability of the closed-loop system are presented in Section \(3\). In Section \(4\) we formulate the robust output regulation problem, and the results on construction of robust controllers are presented in Section \(5\). In Section \(6\) we illustrate the controller construction for concrete partial differential equations, including two one-dimensional wave equations and a two-dimensional heat equation. Appendix \(A\) collects helpful lemmata that are used throughout the paper.

\[ \text{2. Notation and Definitions} \]

If \( X \) and \( Y \) are Banach spaces and \( A : X \to Y \) is a linear operator, we denote by \( D(A), \mathcal{N}(A) \) and \( \mathcal{R}(A) \) the domain, kernel and range of \( A \), respectively. The space of bounded linear operators from \( X \) to \( Y \) is denoted by \( \mathcal{L}(X, Y) \). If \( A : X \to X \), then \( \sigma(A) \) and \( \rho(A) \) denote the spectrum and the resolvent set of \( A \), respectively. For \( \lambda \in \rho(A) \) the resolvent operator is \( R(\lambda, A) = (\lambda - A)^{-1} \). The inner product on a Hilbert space is denoted by \( \langle \cdot, \cdot \rangle \).

Throughout the paper we assume that \( \mathcal{L}(X, Y) \) is a linear operator, \( X, Y \) are Banach spaces, \( A : X \to Y \) is a linear operator, and \( \mathcal{L}(X, Y) \) is equipped with the graph norm of \( A \).

Throughout the paper the operators \( B \in \mathcal{L}(U, X_{-1}), B_d \in \mathcal{L}(U_d, X_{-1}) \) and \( C \in \mathcal{L}(X_1, Y) \) are admissible \([39]\) with respect to the semigroup \( T(t) \) generated by \( A : D(A) \subset X \to X \). Here \( U, U_d, \) and \( Y \) are Hilbert spaces, the space \( X_1 = D(A) \) is equipped with the graph norm of \( A \), and \( X_{-1} \) is the completion of \( X \) with respect to the norm \( \| x \|_{-1} = \| R(\lambda_0, A)x \| \) where \( \lambda_0 \in \rho(A) \) is arbitrary and fixed. We assume that the system \((A, B, B_d, C, \lambda_d, D)\) in \((2.1)\) with input \((u(t), w_{\text{dist}}(t)) \in U \times U_d \) and output \( y(t) \in Y \) is a regular linear system \([45]\) Sec. 5. We denote \( X_B = D(A) + \mathcal{R}(R(\lambda_0, A)B) \) and \( X_{B, B_d} = D(A) + \mathcal{R}(R(\lambda_0, A)[B, B_d]) \). The \( \lambda \)-extension of \( C \) is \( C_{\lambda}x = \lim_{\lambda \to \infty} \lambda CR(\lambda, A)x \), where \( D(C_{\lambda}) \) consists of those \( x \in X \) for which the limit exists. The regularity of \((2.1)\) implies that \( \mathcal{R}(R(\lambda, A)B) \subset D(C_{\lambda}) \) and \( \mathcal{R}(R(\lambda, A)B_d) \subset D(C_{\lambda}) \) for all \( \lambda \in \rho(A) \) and that the transfer functions \( P(\cdot) : \hat{u} \mapsto \hat{y} \) and \( P_d(\cdot) : \hat{w}_{\text{dist}} \mapsto \hat{y} \) have the formulas

\[
P(\lambda) = C_{\lambda} R(\lambda, A) B + D, \quad P_d(\lambda) = C_{\lambda} R(\lambda, A) B_d.
\]

Throughout the paper we assume that \( Y = U \) and that \((A, B, C, D)\) is impedance passive \([10, 38, 40]\), which is equivalent to the property that \( \text{Re}(Ax + Bu, x) \leq \text{Re}(C_{\lambda}x + Du, u) \) for all \( x \in X \) and \( u \in U \) satisfying...
Ax + Bu ∈ X [38] Thm. 4.2. Under this assumption the semigroup \( T(t) \) generated by \( A \) is contractive, \( \Re D \geq 0 \), and \( \Re P(\lambda) \geq 0 \) for all \( \lambda \in \mathbb{C}_+ \) (such transfer functions are called \textit{positive}).

We frequently use the following operator identity, see e.g. [44, Proof of Thm. 1.2]. For completeness, we give a proof of the lemma in Appendix A.

\textbf{Lemma 2.1.} Let \((A, B, C, D)\) be a regular linear system and let \( Q \in \mathcal{L}(Y, U) \) be invertible. If \( \lambda \in \rho(A) \) and if \( Q^{-1} + C_A R(\lambda, A)B \) is boundedly invertible, then \( \lambda \in \rho(A - BQC_A) \) and

\[
R(\lambda, A - BQC_A) = R(\lambda, A) - R(\lambda, A)B(Q^{-1} + C_A R(\lambda, A)B)^{-1}C_A R(\lambda, A),
\]

where \( D(A - BQC_A) = \{ x \in D(C_A) \mid (A - BQC_A)x \in X \}. \)

The system (1.2) is assumed to be another impedance passive regular linear system on a Hilbert space \( Z \) with \( D_c = D_c \). The scale spaces \( Z_1 \) and \( Z_{-1} \) are defined analogously as \( X_1 \) and \( X_{-1} \). We define \( Z_{\lambda} = D(A_{\lambda}) + \mathcal{R}(R(\lambda_0, A_{\lambda})B_{\lambda}) \) for some \( \lambda_0 \in \rho(A_{\lambda}) \) and denote the \( \Lambda \)-extension of \( C_{\lambda} \) by \( C_{\lambda} \). The passivity implies that \( \Re(A_{\lambda}z + B_{\lambda}y, z) \leq \Re(C_{\lambda}z + D_{\lambda}y, y) \) for all \( z \in Z \) and \( y \in Y \) satisfying \( A_{\lambda}z + B_{\lambda}y \in Z \), and we have \( D_c \geq 0 \). We denote the transfer function of \((A_{\lambda}, B_{\lambda}, C_{\lambda}, D_{\lambda})\) with

\[
G(\lambda) = C_{\lambda} R(\lambda, A_{\lambda})B_{\lambda} + D_{\lambda}, \quad \lambda \in \rho(A_{\lambda}).
\]

Our assumption \( D_c \geq 0 \) simplifies the analysis of the admissibility of output feedbacks of the two passive systems (1.1) and (1.2). However, many of the results also hold in the situation where \( \Re D_c \geq 0 \) as long as the appropriate feedback operators remain admissible, which is the case, e.g., if \( \|D_c - D_{\lambda}\| \) is sufficiently small.

3. Stability of Coupled Passive Systems

In this section we present our main results on the stability of the closed-loop system associated to the power-preserving interconnection of (1.1) and (1.2). Lemma 4.2 in Section 4 shows that the system operator \( A_e \) of the closed-loop system

\[
\dot{x}_e(t) = A_e x_e(t), \quad x_e(0) = x_{e0} = (x_0, z_0)^T \in X_e
\]

is given by

\[
A_e = \begin{bmatrix}
A - BD_c Q_1 C_{\lambda} & B Q_2 C_{\lambda} \\
- B_{\lambda} Q_1 C_{\lambda} & A_{\lambda} - B_{\lambda} Q_1 D C_{\lambda}
\end{bmatrix},
\]

(3.1a)

\[
D(A_e) = \begin{cases}
\{ x \mid x \in X_B \times Z_{\lambda} \mid (A - BD_c Q_1 C_{\lambda})x + B Q_2 C_{\lambda} z \in X \\
- B_{\lambda} Q_1 C_{\lambda} x + (A_{\lambda} - B_{\lambda} Q_1 D C_{\lambda}) z \in Z \}
\end{cases},
\]

(3.1b)

where \( Q_1 = (I + DD_c)^{-1} \) and \( Q_2 = (I + D_c D)^{-1} \), and that \( A_e \) generates a strongly continuous contraction semigroup \( T_e(t) \) on \( X_e \).

\textbf{Remark 3.1.} Our results assume that (1.1) is stable and its transfer function \( P(\lambda) \) satisfies certain additional conditions. However, the results are also immediately applicable when (1.1) is unstable but can be stabilized with a suitable output feedback. Indeed, if \( D_c > 0 \), we can write \( D_c = D_{c1} + D_{c2} \) with \( D_{c1} \geq 0 \) and \( D_{c2} > 0 \). Lemma A.1(d) implies that \( u(t) = -D_{c2} y(t) \)
with $D_{c2} > 0$ is an admissible feedback for $(A, B, C, D)$ and the resulting system $(A^S, B^S, C^S, D^S) = (A - BD_{c2}Q_1^SC_A, BQ_2^S, Q_1^SC_A, Q_2^SD)$ with $Q_1^S = (I + D_{c2}c)^{-1}$ and $Q_2^S = (I + D_{c2}D)^{-1}$ is regular. A direct computation shows that

$$A_c = \begin{bmatrix} A^S - B^S D_{c2}Q_3C_A^S & B^S Q_4C_A \\ -B_cQ_3C_A^S & A_c - B_cQ_3D S^S C_A \end{bmatrix}.$$ 

Since this operator has exactly the same form as the original $A_c$, each of our results it is possible to replace $(A, B, C, D)$ with the stabilized system $(A^S, B^S, C^S, D^S)$, the transfer function $P(\lambda)$ with $P_S(\lambda) = C_A^S R(\lambda, A^S)B^S + D^S$, and the feedthrough operator $D_c \geq 0$ with $D_{c1} \geq 0$. It is important to note that if $P(\lambda)$ is invertible and $\text{Re} \ P(\lambda) \geq 0$ for some $\lambda \in \rho(A)$, then for any $D_{c2} > 0$ we have $\text{Re} \ P_S(\lambda) > 0$.

3.1. **Strong Stability.** The following theorem presents sufficient conditions for the strong stability of the closed-loop system.

**Theorem 3.2.** Assume $(A, B, C, D)$ is passive and strongly stable in such a way that $i\mathbb{R} \subset \rho(A)$. Moreover, assume $(A_c, B_c, C_c, D_c)$ is passive, $D_c \geq 0$, and the following hold for some $I \subset \mathbb{Z}$.

1. $\sigma(A_c) \cap \mathbb{R} = \{i\omega_k\}_{k \in I}$ and $\text{Re} P(i\omega_k) > 0$ for all $k \in I$.
2. $I + P(i\omega)G(i\omega)$ has a bounded inverse for every $\omega \in \mathbb{R} \setminus \{i\omega_k\}_{k \in I}$ for which $\text{Re} G(i\omega)$ is not boundedly invertible.
3. $\{i\omega_k\}_{k \in I} \subset \rho(A_c - B_cD_0(I + D_cD_0)^{-1}C_c)$ whenever $\text{Re} D_0 > 0$.

Then $i\mathbb{R} \subset \rho(A_c)$ and the closed-loop system is strongly stable.

Assume in addition that $I \subset \mathbb{Z}$ is finite, $(A, B, C, D)$ is exponentially stable, and $\sup_{|\omega| \geq R} \|R(i\omega, A_c)\| < \infty$ for some $R > 0$. If we either have $\limsup_{|\omega| \to \infty} |G(i\omega)P(i\omega)| < 1$, or if $\text{Re} P(i\omega) \geq \eta(\omega) \geq 0$ and $\text{Re} G(i\omega) \geq d(\omega) \geq \eta_0 > 0$ for some constant $\eta_0 > 0$ and for all sufficiently large $|\omega|$, then the closed-loop system is exponentially stable.

**Proof.** We begin by showing that $i\mathbb{R} \subset \rho(A_c)$. Since the semigroup generated by $A_c$ is uniformly bounded by Lemma A.2, the strong stability of $T_c(t)$ then follows from the Arendt–Batty–Lyubich–Vũ Theorem [4, 27].

Lemma A.4(d) implies that $u(t) = -D_{c2}y(t)$ is an admissible output feedback for $(A, B, C, D)$, and by Lemma A.4 the resulting system $(A^d, B^d, C^d, D^d) = (A - BD_cQ_1C_A, BQ_2, Q_1C_A, Q_1D)$ is regular. The assumption $i\mathbb{R} \subset \rho(A)$ and Lemma A.3 imply $i\mathbb{R} \subset \rho(A^d)$, and by Lemma A.1(d) the transfer function $P_d(\lambda)$ is given by $P_d(i\omega) = P(i\omega)(I + D_cP(i\omega))^{-1}$ for all $\omega \in \mathbb{R}$. If $\omega \in \mathbb{R}$ and if we denote $R_{i\omega} = R(i\omega, A^d)$, then $i\omega - A_c$ has a bounded inverse given by

$$R(i\omega, A_c) = \begin{bmatrix} R_{i\omega} - R_{i\omega}B^dC_A^dS_A(i\omega)^{-1}B_cC_A^dR_{i\omega} & R_{i\omega}B^dC_A^dS_A(i\omega)^{-1} \\ -S_A(i\omega)^{-1}B_cC_A^dR_{i\omega} & S_A(i\omega)^{-1} \end{bmatrix}$$

provided that the Schur complement

$$S_A(i\omega) = i\omega - A_c + B_cD^dC_A + B_cC_A^dR(i\omega, A^d)B^dC_A$$

$$= i\omega - A_c + B_cP(i\omega)(I + D_cP(i\omega))^{-1}C_c.$$
with domain $D(S_A(iω)) = \{ z \in D(C_{CA}) \mid S_A(iω)z \in Z \}$ has a bounded inverse. If $ω = ω_n$ for some $n \in I$, then $Re P(ω_n) > 0$ and assumption (3) imply that $S_A(iω_n)$ is boundedly invertible. Thus $\{iω_k\}_{k \in I} \subset ρ(A_e)$.

Now let $ω \in \mathbb{R} \setminus \{ω_k\}_{k \in I}$. If $Re G(ω) \neq 0$, then $I + G(iω)P(iω)$ is invertible by condition (2) of the theorem. By Lemma A.1(a) the same is also true if $Re G(iω) > 0$, since $I + G(iω)P(iω) = G(iω)(G(iω)^{-1} + P(iω))$. Because

$$I + D_eP(iω) + C_{CA}R(iω, A_e)B_eP(iω) = I + G(iω)P(iω),$$

Lemma 2.1 implies that $S_A(iω)$ has a bounded inverse

\begin{equation}
S_A(iω)^{-1} = R(iω, A_e)\left[I - B_eP(iω)(I + G(iω)P(iω))^{-1}C_{CA}R(iω, A_e)\right].
\end{equation}

Thus $ω \in ρ(A_e)$ for all $ω \in \mathbb{R} \setminus \{ω_k\}_{k \in I}$. Since the semigroup $T_e(t)$ is contractive, the closed-loop system is strongly stable.

Finally, assume that $I \subset \mathbb{Z}$ is finite, $(A, B, C, D)$ is exponentially stable, and $sup_{|ω| \geq R}||R(iω, A_e)|| < \infty$ for some $R > 0$. The stability and regularity of $(A, B, C, D)$ imply that the norms $||R(\cdot, A)||$, $||R(\cdot, A)B||$, $||C_{CA}R(\cdot, A)||$, and $||P(\cdot)\rangle$ are uniformly bounded on $i\mathbb{R}$. Similarly the regularity of the controller implies that $||R(iω, A_e)||$, $||R(iω, A_e)B_e||$, $||C_{CA}R(iω, A_e)||$, and $||C_{CA}R(iω, A_e)B_e||$ are uniformly bounded with respect to $ω \in \mathbb{R}$ with $|ω| \geq R$. If $lim sup_{|ω| \to \infty}||G(iω)P(iω)|| < 1$ the norms $||P(iω)(I + G(iω)P(iω))^{-1}||$ are uniformly bounded for large $|ω|$. On the other hand, if $η(ω) = ν(ω) \geq η_0 > 0$, then Lemma A.1(b) implies $||P(iω)(I + G(iω)P(iω))^{-1}|| \lesssim η_0^{-1}$. Thus (3.2) implies that $||R(iω, A_e)||$ is uniformly bounded for large $|ω|$. Since $i\mathbb{R} \subset ρ(A_e)$ and $T_e(t)$ is contractive, the closed-loop system is exponentially stable.

**Remark 3.3.** Condition (2) is in particular satisfied if $Re G(iω) > 0$ for all $ω \in \mathbb{R} \setminus \{ω_k\}_{k \in I}$. Moreover, if $Re G(iω) \geq d_e > 0$ for some constant $d_e > 0$ and for all $ω \in \mathbb{R} \setminus \{ω_k\}_{k \in I}$, then $||P(iω)(I + G(iω)P(iω))^{-1}|| \leq d_e^{-1}$ for all $ω \in \mathbb{R} \setminus \{ω_k\}_{k \in I}$ by Lemma A.1(b).

The proof of Theorem 3.2 can be adapted to show that if $Re P(iω) > 0$ for all $ω \in \mathbb{R}$, then $T_e(t)$ is strongly stable and $i\mathbb{R} \subset ρ(A_e)$ even without assumption (2). Indeed, if $ω \in \mathbb{R} \setminus \{ω_k\}_{k \in I}$ and $Re P(iω) > 0$, then Lemma A.1(a) implies that both $P(iω)$ and $I + G(iω)P(iω) = (P(iω)^{-1} + G(iω))P(iω)$ are boundedly invertible, and $S_A(iω)$ has the bounded inverse given by the formula (3.2). Thus we again have $ω \in ρ(A_e)$. Lemma A.1(b) also shows that if $η(ω) > 0$ is such that $Re P(iω) \geq η(ω) > 0$, then $||P(iω)(I + G(iω)P(iω))^{-1}|| \leq η(ω)^{-1}||P(iω)||^2$.

The following lemma provides a sufficient condition for the assumption (3) in Theorem 3.2 for isolated spectral points under a suitable observability property.

**Lemma 3.4.** Assume $(A_e, B_e, C_e, D_e)$ is passive with $D_e > 0$. Assume further that $iω_k \in σ(A_e)$ is an isolated spectral point and $A_e$ has a spectral decomposition $A_e = A_e^0 + A_e^s$ according to $Z = N(iω_k - A_e) \oplus N(iω_k - A_e)^*$ so that $iω_k \in ρ(A_e)$, and there exists $γ > 0$ such that $||C_{CA}z|| \geq γ||z||$ for all $z \in N(iω_k - A_e)$. Then $iω_k \in ρ(A_e - B_eD_0(I + D_eD_0)^{-1}C_{CA})$ for any $D_0 \in L(U)$ with $Re D_0 > 0$. 
Proof. Let $D_0 \in \mathcal{L}(U)$ be such that $\text{Re } D_0 \geq d_0 > 0$ and denote $D_1 = D_0(I + D_0 D_0)^{-1}$. Due to the passivity of $(A_c, B_c, C_c, D_c)$ and [5], Cor. 4.3.2 we have $i \omega_k \in \sigma(A_c - B_c D_1 C_c)$ provided that $||(i \omega_k - A_c + B_c D_1 C_c)z|| \geq c||z||$ for some constant $c > 0$ and for all $z \in D(A_c - B_c D_1 C_c) \subset Z_{B_c}$. Let $z \in D(A_c - B_c D_1 C_c)$ and denote $y = (i \omega_k - A_c + B_c D_1 C_c)z$. The passivity of $(A_c, B_c, C_c, D_c)$ implies

$$
\text{Re}(y, z) = -\text{Re}(A_c z + B_c (-D_1 C_c A z), z) \geq \text{Re}(C_c z z - D_1 D_1 C_c z, D_1 C_c z)
$$

$$
= \text{Re}((I + D_0 D_0)^{-1} C_c z, D_0(I + D_0 D_0)^{-1} C_c z)
$$

$$
\geq d_0 \|I + D_0 D_0\|^{-2} \|C_c z\|^2.
$$

Thus $\|C_c z\|^2 \leq \|z\| \|y\|$. Write $z = z^k + z^c$ according to the decomposition $Z = \mathcal{N}(i \omega_k - A_c) \oplus \mathcal{N}(i \omega_k - A_c)^{\perp}$. If we apply $R_1 = R(i \omega_k + 1, A_c)$ to both sides of $y = (i \omega_k - A_c + B_c D_1 C_c)z$ and use $R_1 z^k \in \mathcal{N}(i \omega_k - A_c)$ we obtain

$$
(i \omega_k - A_c^R) R_1 z^c = R_1 y - R_1 B_c D_1 C_c z.
$$

Since $R_1 B_c \in \mathcal{L}(U, Z)$ and $i \omega_k - A_c^R$ is boundedly invertible by assumption, we have $\|R_1 z^c\|^2 \leq \|i \omega_k - A_c^R\| \|R_1 z^c\|^2 \leq \|y\|^2 + \|C_c z\|^2 \leq \|y\|^2 + \|z\| \|y\|$. Moreover, $(i \omega_k - A_c) R_1 z^c = z^c - R_1 z^c$ and $\|z^c\| \leq \|z\|$ together with (3.3) further imply

$$
\|z^c\|^2 = \|R_1 z^c + R_1 y - R_1 B_c D_1 C_c z\|^2
$$

$$
\leq \|R_1 z^c\|^2 + \|y\|^2 + \|C_c z\|^2 \leq \|y\|^2 + \|z\| \|y\|
$$

$$
\|C_c z\|^2 = \|C_c R_1 (z^c + y) - C_c R_1 B_c D_1 C_c z\|^2
$$

$$
\leq \|z^c\|^2 + \|y\|^2 + \|C_c z\|^2 \leq \|y\|^2 + \|z\| \|y\|.
$$

Finally, since $\|z^k\|^2 \leq \gamma^{-2} \|C_c z^k\|^2 \leq \gamma^{-2} (\|C_c z\|^2 + \|C_c z^c\|^2) \leq \|y\|^2 + \|z\| \|y\|$, we have $\|z\|^2 = \|z^k\|^2 + \|z^c\|^2 \leq \|y\|^2 + \|z\| \|y\|$, and thus also $\|z\| \leq \|y\|$. \qed

3.2. Exponential Stability. The following theorem presents sufficient conditions for exponential stability of the closed-loop system. The transfer function $P(i \omega)$ is allowed to be non-invertible for some values $\omega \in \mathbb{R}$ (i.e., the system $(A, B, C, D)$ may have “transmission zeros” on $i \mathbb{R}$), but such points must be uniformly disjoint from the spectrum of $A_c$. It should be noted that the result also remains valid if the conditions are satisfied for $\Omega = \mathbb{R}$. Condition (2) is in particular satisfied if $\text{Re } G(i \omega) \geq d_c > 0$ for some constant $d_c > 0$ and for all $\omega \in \mathbb{R} \setminus \Omega$. Here exponential stabilizability and exponential detectability of a regular linear system are defined as in [34] Def. 1.4–1.5 and [13], Sec. III.

Theorem 3.5. Assume $(A, B, C, D)$ is passive and exponentially stable, $\text{Re } D > 0$, and there exist $\Omega \subset \mathbb{R}$ and $\eta_0 > 0$ such that $\text{Re } P(i \omega) \geq \eta_0 > 0$ for all $\omega \in \Omega$. Moreover, assume $(A_c, B_c, C_c, D_c)$ is passive, $D_c \geq 0$, and the following hold.

1. $\sigma(A_c) \cap i \mathbb{R} \subset \partial \Omega$ and $\sup_{\omega \in \mathbb{R} \setminus \Omega} \|R(i \omega, A_c)\| < \infty$.

2. Let $\eta(\cdot), d_c(\cdot) : \mathbb{R} \setminus \Omega \rightarrow [0, 1]$ be such that $\text{Re } P(i \omega) \geq \eta(\omega) \geq 0$ and $\text{Re } G(i \omega) \geq d_c(\omega) \geq 0$ for all $\omega \in \mathbb{R} \setminus \Omega$. Assume there exist $0 < \delta < 1$ and $\eta_1 > 0$ such that for each $\omega \in \mathbb{R} \setminus \Omega$ either $\|G(i \omega) P(i \omega)\| \leq \delta < 1$ or $\eta(\omega) + d_c(\omega) \geq \eta_1 > 0$.
(3) The system \((A_c, B_c, C_c, D_c)\) is exponentially stabilizable and detectable.

Then the closed-loop system is exponentially stable.

Proof. Our aim is to show \(i\mathbb{R} \subset \rho(A_e)\) and \(\sup_{\omega \in \mathbb{R}} \|R(i\omega, A_e)\| < \infty\). First let \(\omega \in \mathbb{R} \setminus \Omega\). The proof of Theorem 3.2 shows that \(S_A(i\omega)\) has an inverse

\[
S_A(i\omega)^{-1} = R(i\omega, A_c)[I - B_cP(i\omega)(I + G(i\omega)P(i\omega))^{-1}C_AR(i\omega, A_c)].
\]

If \(\|G(i\omega)P(i\omega)\| \leq \delta < 1\), then \(\|P(i\omega)(I + G(i\omega)P(i\omega))^{-1}\| \leq \|P(i\omega)\|/(1 - \delta)\), and if \(\eta(\omega) + d_c(\omega) \geq \eta_1 > 0\), Lemma \(\boxed{\ref{lem:stab}}\) implies \(\|P(i\omega)(I + G(i\omega)P(i\omega))^{-1}\| \leq \eta_1^{-1} \max\{1, \|P(i\omega)\|\}\). Assumption (1) and the admissibility of \(B_c\) and \(C_c\) imply \(i\mathbb{R} \setminus \{i\omega \in \rho(A_e)\text{ and } \sup_{\omega \in \mathbb{R}\setminus \Omega} \|R(i\omega, A_e)\| < \infty\).

It remains to consider \(\omega \in \Omega\). We decompose \(D\) into two parts \(D = \mu D + \nu D\) with \(\mu \in (0,1)\) and \(\nu = 1 - \mu\) in such a way that the first part stabilizes \((A_e, B_e, C_e, D_e)\) exponentially and the second part can be used to show closed-loop stability. Indeed, for any \(\mu \in (0,1)\) the transfer function of the system \((A_e^\mu, B_e^\mu, C_e^\mu, D_e^\mu)\) obtained from \((A_e, B_e, C_e, D_e)\) with the admissible output feedback \(u_e(t) = -\mu D_y(t)\) is given by \(G(\lambda)(I + \mu D\mathcal{G}(\lambda))^{-1}\).

Since \(ReD > 0\), this transfer function is uniformly bounded on \(C_+\) by Lemma \(\boxed{\ref{lem:stab}}\), and since \((A_e^\mu, B_e^\mu, C_e^\mu, D_e^\mu)\) is exponentially stabilizable and detectable due to assumption (3), the semigroup generated by \(A_e^\mu\) is exponentially stable \(\boxed{\ref{cor:stab}}\).

For all sufficiently small \(\mu \in (0,1)\) the transfer function \(P_e(\lambda)\) of \((A, B, C, \nu D)\) satisfies \(\text{Re}P_e(i\omega) \geq \tilde{\eta}_0 > 0\) for some constant \(\tilde{\eta}_0 > 0\) and for all \(\omega \in \Omega\). Since \(D_e^\mu = D_e(I + \mu DD_e)^{-1}\), Lemmas \(\boxed{\ref{lem:stab}}\) and \(\boxed{\ref{lem:stab}}\) imply that we can choose \(\mu \in (0,1)\) so that \(I + \nu DD_e^\mu\) and \(I + P_e(\omega)D_e^\mu\) for all \(\omega \in \Omega\) are invertible, and \(\sup_{\omega \in \mathbb{R}} \|(I + P_e(i\omega)D_e^\mu)^{-1}\| < \infty\). Thus \(u(t) = -D_e^\mu y(t)\) is an admissible output feedback for \((A, B, C, \nu D)\). Denoting the resulting regular linear system with \((A^\mu, B^\mu, C^\mu, D^\mu) = (A - B^\mu Q_5^\mu C_\Lambda, BQ_5^\mu, Q_5^\mu C_\Lambda, \nu Q_5^\mu D)\) where \(Q_5^\mu = (I + \nu DD_e^\mu)^{-1}\) and \(Q_6^\mu = (I + \nu D_e^\mu)^{-1}\), we can write

\[
A_e = \begin{bmatrix}
A - BD_e^\mu Q_5^\mu C_\Lambda & BQ_5^\mu C^\mu C_\Lambda \\
-B_e^\mu Q_5^\mu C_\Lambda & A_e - \nu BD_e^\mu Q_5^\mu D C^\mu C_\Lambda
\end{bmatrix} = \begin{bmatrix}
A^\mu & B^\mu C^\mu \\
-B_e^\mu C_\Lambda & A_e - \nu B_e^\mu D C^\mu C_\Lambda
\end{bmatrix}.
\]

Similarly as in Lemma \(\boxed{\ref{lem:stab}}\) we can show that \(\sup_{\omega \in \Omega} \|R(i\omega, A^\mu)\| < \infty\) and the transfer function of \((A^\mu, B^\mu, C^\mu, D^\mu)\) satisfies \(P_e(\omega) = P_e(i\omega)(I + D_e^\mu P_e(i\omega))^{-1}\) for all \(\omega \in \Omega\). The transfer function of \((A^\mu, B^\mu, C^\mu, D^\mu)\) is denoted by \(G^\mu(\lambda)\).

Let \(\omega \in \Omega\). If we denote \(R^{\mu}_{i\omega} = R(i\omega, A^\mu)\), then \(i\omega - A_e\) has a bounded inverse

\[
R(i\omega, A_e) = \begin{bmatrix}
R^{\mu}_{i\omega} & R^{\mu}_{i\omega}\nu B^\mu C^\mu C_\Lambda S_A(i\omega)^{-1}B_e^\mu C_e^\mu R^{\mu}_{i\omega} R^{\mu}_{i\omega}\nu B^\mu C^\mu C_\Lambda S_A(i\omega)^{-1}

-S_A(i\omega)^{-1}B_e^\mu C_e^\mu R^{\mu}_{i\omega} R^{\mu}_{i\omega}\nu B^\mu C^\mu C_\Lambda S_A(i\omega)^{-1}
\end{bmatrix}
\]

provided that the Schur complement

\[
S_A^\mu(i\omega) = i\omega - A_e^\mu + B_e^\mu D^\mu C_e^\mu + B_e^\mu C_e^\mu R(i\omega, A^\mu)B^\mu C^\mu C_\Lambda
\]

\[
= i\omega - A_e^\mu + B_e^\mu P_e(i\omega)(I + D_e^\mu P_e(i\omega))^{-1}C_e^\mu
\]

has a bounded inverse. If \(S_A^\mu(i\omega)\) is boundedly invertible for all \(\omega \in \Omega\), then the regularity of \((A^\mu, B^\mu, C^\mu, D^\mu)\) and \(\sup_{\omega \in \mathbb{R}} \|R(i\omega, A^\mu)\| < \infty\) imply \(\sup_{\omega \in \mathbb{R}} \|R(i\omega, A_e)\| < \infty\) provided that \(\|S_A^\mu(i\omega)^{-1}\|\), \(\|S_A^\mu(i\omega)^{-1}B_e^\mu\|\),
\[ \| C_{cA}^\mu S_A^\mu(i\omega)^{-1} \|, \text{ and } \| C_{cA}^\mu S_A^\mu(i\omega)^{-1} B_c^\mu \| \text{ are uniformly bounded with respect to } \omega \in \Omega. \]

Let \( \omega \in \Omega \) be arbitrary. Since \( \text{Re} P_c(i\omega) \geq \bar{\eta}_0 > 0 \) and \( \text{Re} G^\mu(i\omega) \geq 0 \), Lemma A.1 implies that \( P_c(i\omega) \) and \( I + G^\mu(i\omega) P_c(i\omega) = (P_c(i\omega)^{-1} + G^\mu(i\omega)) P_c(i\omega) \) are boundedly invertible. Therefore the same is true for

\[ I + D_c^\mu P_c(i\omega) + C_{cA}^\mu R(i\omega, A_c^\mu) B_c^\mu P_c(i\omega) = I + G^\mu(i\omega) P_c(i\omega). \]

Lemma 2.1 implies that \( S_A^\mu(i\omega) \) has a bounded inverse

\[ S_A^\mu(i\omega)^{-1} = R(i\omega, A_c^\mu) [I - B_c^\mu P_c(i\omega) (I + G^\mu(i\omega) P_c(i\omega))^{-1} C_{cA}^\mu R(i\omega, A_c^\mu)], \]

where \( \sup_{\omega \in \Omega} \| P_c(i\omega) \| < \infty \) and \( (A_c^\mu, B_c^\mu, C_{cA}^\mu, D_c^\mu) \) is regular and exponentially stable, the norms \( \| S_A^\mu(i\omega)^{-1} \|, \| S_A^\mu(i\omega)^{-1} B_c^\mu \|, \| C_{cA}^\mu S_A^\mu(i\omega)^{-1} \|, \) and \( \| C_{cA}^\mu S_A^\mu(i\omega)^{-1} B_c^\mu \| \) are uniformly bounded with respect to \( \omega \in \Omega \). This further implies that \( \sup_{\omega \in \Omega} \| R(i\omega, A_c^\mu) \| < \infty \), and the closed-loop system is exponentially stable.

Since both \((A, B, C, D)\) and \((A_c, B_c, C_c, D_c)\) are exponentially stabilizable in Theorem 3.5, the exponential closed-loop stability could alternatively be studied using [13, Prop. 4.6].

### 3.3. Non-Uniform Closed-Loop Stability

In this section we introduce conditions for polynomial and non-uniform stability of the closed-loop system in the case where \( A_c \) is diagonal. In addition, our main result can be used as an alternative to Theorem 3.5 in showing exponential closed-loop stability. The closed-loop system is said to be non-uniformly stable when \( T_c(t) \) is uniformly bounded and \( \text{i} \mathbb{R} \subset \rho(A_c) \) but the norms \( \| R(i\omega, A_c) \| \) are not bounded with respect to \( \omega \in \mathbb{R} \). If \( M_R(\cdot) \) is a continuous non-decreasing function such that \( \| R(i\omega, A_c) \| \leq M_R(\omega) \), then there exist \( M_c, c, t_0 > 0 \) such that

\[ (3.4) \quad \| T_c(t)x_{e0} \| \leq \frac{M_c}{M_T(t)} \| A_c x_{e0} \| \quad \forall x_{e0} \in D(A_c), \quad t \geq t_0, \]

where the continuous non-decreasing function \( M_T(\cdot) : [0, \infty) \to (0, \infty) \) is determined by the results in [7, 8, 36]. In particular, if \( M_R(\omega) \lesssim 1 + \omega^\alpha \) for some \( \alpha > 0 \), we can choose \( M_T(t) = t^{1/\alpha} [8, \text{Ex. 1.6}] \).

In this section we assume \((A_c, B_c, C_c, D_c)\) is regular and passive with \( D_c \geq 0 \) on a Hilbert space \( Z = \bigotimes_{k \in \mathcal{I}} Z_k \) with norm \( \| (z_k)_{k \in \mathcal{I}} \|_Z = \sum_{k \in \mathcal{I}} \| z_k \|_{Z_k}^2 \) where \( Z_k \) are Hilbert and \( \mathcal{I} \subset Z \) is infinite. We assume \( A_c \) has the structure

\[ (3.5) \quad A_c = \text{diag}(\omega_k I Z_k)_{k \in \mathcal{I}}, \quad D(A_c) = \{ (z_k)_{k \in \mathcal{I}} \in Z \mid \sum_{k \in \mathcal{I}} |\omega_k|^2 \| z_k \|_{Z_k}^2 < \infty \}, \]

where \( \omega_k \neq \omega_l \) for \( k \neq l \) and \( \{ \omega_k \}_{k \in \mathcal{I}} \) has no finite accumulation points. Since \( A_c \) is skew-adjoint, the operators \( B_c \in \mathcal{L}(Y, Z_{-1}) \) and \( C_c \in \mathcal{L}(Z_1, Y) \) are formally adjoint, i.e., \( \langle B_c u, z \rangle_{-1,1} = \langle u, C_c z \rangle \) for all \( z \in D(A_c) \) and \( u \in Y \), and thus

\[ B_c u = (B_c^* u)_{k \in \mathcal{I}}, \quad \text{and} \quad C_c z = \sum_{k \in \mathcal{I}} B_c^* z_k, \quad z = (z_k)_{k \in \mathcal{I}} \in D(A_c). \]
for some $B_{ck} \in \mathcal{L}(Y,Z_k)$. Our main result uses wavepackets of $A_c$ \cite[Sec. 6.9]{STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS 11}.

**Definition 3.6.** Let $\omega \in \mathbb{R}$ and $\delta > 0$. An element $z = (z_k)_{k \in I} \in Z$ is a $(\omega, \delta)$-wavepacket of $A_c$ if $z_k = 0$ for those $k \in I$ for which $|\omega - \omega_k| \geq \delta$.

The following theorem is the main result of this section. The role of $\Omega_c \subset \mathbb{R}$ is to show that only the behaviour of $\text{Re} P(i\omega)$ near $\sigma(A_c) = \{i\omega_k\}_{k \in I}$ affects the asymptotic growth of $\|R(i\omega, A_c)\|$. By \cite[Cor. 2.17]{STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS 11} $\delta(\cdot)$ and $\gamma(\cdot)$ can be chosen as constant functions if and only if $(A_c, B_c)$ is exactly controllable. The assumption that $M_R(\cdot) : [0, \infty) \to (0, \infty)$ has “positive increase” means that there exists $\alpha, c, \omega_0 > 0$ such that $M_R(\lambda \omega) \geq c\lambda^\alpha M_R(\omega)$ for all $\lambda > 0$ and $\omega \geq \omega_0$ \cite[Sec. 2]{STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS 11}, and this condition is in particular satisfied if $M_R(\cdot)$ grows polynomially or exponentially. The estimation of $\|S_A(i\omega)^{-1}\|$ in the proof extends techniques developed in \cite{STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS 11}.

**Theorem 3.7.** Assume $(A,B,C,D)$ is passive and exponentially stable and the system $(A_c, B_c, C_A, D_c)$ is passive with $A_c$ of form \cite{STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS 11} and $D_c \geq 0$. Assume further that condition (2) of Theorem \cite{STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS 11} is satisfied for $\Omega = \Omega_c := \{\omega \in \mathbb{R} | \exists k \in I : |\omega - \omega_k| < \varepsilon\}$ with some $\varepsilon > 0$, and that there exist continuous non-increasing functions $\eta(\cdot), \delta(\cdot), \gamma(\cdot) : \mathbb{R}_+ \to (0,1]$ with the following properties.

- $\text{Re} P(i\omega) \geq \eta(|\omega|)$ for all $\omega \in \Omega_c$.
- $\|C_c z\| \geq \gamma(|\omega|) \|z\|$ for every $\omega \in \mathbb{R}$ and every $(\omega, \delta(|\omega|))$-wavepacket $z$ of $A_c$.

Then $T_c(t)$ is strongly stable, $i\mathbb{R} \subset \rho(A_c)$, and

$$\|R(i\omega, A_c)\| \leq M_R(\|\omega\|), \quad \text{where} \quad M_R(\cdot) = M_0 \eta(\cdot)^{-1} \gamma(\cdot)^{-2} \delta(\cdot)^{-2}$$

for some $M_0 > 0$. Moreover, the following hold.

(a) If $\sup_{\omega > 0} M_R(\omega) < \infty$, then $T_c(t)$ is exponentially stable.

(b) If $M_R(\cdot)$ is strictly increasing and has positive increase, then $\|M_T(t)\| = M_R^{-1}(ct)$ for some constants $M_R, c, t_0 > 0$.

(c) For all other $M_R(\cdot)$, \cite{STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS 11} holds with $\|M_T(t)\| = M_\log^{-1}(ct)$ for some $M_\log, c, t_0 > 0$ where $M_\log(\omega) = M_R(\omega) \left(\log(1 + M_R(\omega)) + \log(1 + \omega)\right)$ for $\omega > 0$.

**Proof.** By Theorem 3.2 and Lemma 3.4 the closed-loop system is strongly stable and $i\mathbb{R} \subset \rho(A_c)$. Once we show $\|R(i\omega, A_c)\| \leq M_R(\|\omega\|)$ the stability properties of the closed-loop system follow from the characterization of exponential stability (part (a)), from \cite[Thm. 1.1]{STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS 11} (part (b)), and from \cite[Thm. 1.5]{STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS 11} (part (c)).

Since $(A^d, B^d, C_A^d, D^d)$ is regular and exponentially stable by Lemma A.3 we have from the proof of Theorem 3.2 that for all $\omega \in \mathbb{R}$

$$\|R(i\omega, A_c)\| \lesssim \max\{\|S_A(i\omega)^{-1}\|, \|S_A(i\omega)^{-1}B_c\|, \|C_A S_A(i\omega)^{-1}\|, \|C_A S_A(i\omega)^{-1}B_c\|\},$$

where $S_A(i\omega) = i\omega - A_c + B_c P_d(i\omega) C_A$ and $P_d(i\omega) = P(i\omega)(I + D_c P(i\omega))^{-1}$. Moreover, \cite{STABILITY AND ROBUST REGULATION OF PASSIVE LINEAR SYSTEMS 11} and our assumptions imply $\sup_{\omega \in \Omega_c} \|R(i\omega, A_c)\| < \infty$ similarly as in the proof of Theorem 3.3. Thus it is sufficient to show that
for each $\omega \in \Omega_\epsilon$ the norms $\|S_A(\omega)^{-1}\|$, $\|S_A(i\omega)^{-1}B_c\|$, $\|C_{\Lambda}S_A(i\omega)^{-1}\|$, $\|C_{\Lambda}S_A(i\omega)^{-1}B_c\|$ are bounded by $M_R(\omega)$ for some constant $M_0 > 0$.

We begin by showing $\|C_{\Lambda}S_A(\omega)^{-1}B_c\| \leq M_R(\omega)$. Formula (3.2) implies that for all $\omega \in \Omega_\epsilon \setminus \{\omega_k\}_k$

$$C_{\Lambda}S_A(\omega)^{-1}B_c$$

$$= C_{\Lambda}R(i\omega, A_c)B_c[I - (I + P(i\omega)G(i\omega))^{-1}P(i\omega)C_{\Lambda}R(i\omega, A_c)B_c]$$

$$= (G(i\omega) - D_c)(I + P(i\omega)G(i\omega))^{-1}(I + P(i\omega)D_c).$$

Since $\Re P(i\omega) > 0$ and $\Re G(i\omega) \geq 0$, $I + P(i\omega)G(i\omega) = P(i\omega)(P(i\omega)^{-1} + G(i\omega))$ is boundedly invertible by Lemma [A.1](a). If we denote $Q(i\omega) = (I + P(i\omega)G(i\omega))^{-1}$, the above formula and stability of $(A, B, C, D)$ implies

$$\|C_{\Lambda}S_A(\omega)^{-1}B_c\| = \|(G(i\omega) - D_c)Q(i\omega)(I + P(i\omega)D_c)\|$$

$$\lesssim \|G(i\omega)Q(i\omega)\| + \|Q(i\omega)\|.$$ 

Here $\|G(i\omega)Q(i\omega)\| \leq \eta(\omega)^{-1}$ by Lemma [A.1](b). We claim that $\|Q(i\omega)\| \lesssim \eta(\omega)^{-1}$ for $\omega \in \Omega_\epsilon \setminus \{\omega_k\}_k$. If this is not true, then (considering $Q(i\omega)^*$) there exist sequences $(s_n)_n \subset \Omega_\epsilon \setminus \{\omega_k\}_k$ and $(u_n)_n \subset Y$ with $\|u_n\| = 1$ such that $\|G(i\omega)^{-1}||I + G(s_n)^*P(is_n)^*u_n|| \to 0$ as $n \to \infty$. Since $\sup_{\omega \in \mathbb{R}}\|P(i\omega)\| < \infty$, we have that also

$$0 \leftarrow \frac{1}{\eta(|s_n|)} \Re\langle(I + G(is_n)^*P(is_n)^*)u_n, P(is_n)^*u_n\rangle \geq \frac{\Re\langle P(is_n)^*u_n, u_n\rangle}{\eta(|s_n|)}$$

as $n \to \infty$, which is impossible since $\eta(|s_n|)^{-1}\Re\langle P(is_n)^*u_n, u_n\rangle \geq 1$ by assumption. This contradiction shows that the claim holds. Thus we have $\|C_{\Lambda}S_A(\omega)^{-1}B_c\| \lesssim \eta(\omega)^{-1} \leq M_R(\omega)$ for some $M_0 > 0$ and for all $\omega \in \Omega_\epsilon \setminus \{\omega_k\}_k$, and by continuity the same estimate holds for every $\omega \in \Omega_\epsilon$.

To estimate the norms $\|S_A(\omega)^{-1}\|$, $\|S_A(i\omega)^{-1}B_c\|$, $\|C_{\Lambda}S_A(i\omega)^{-1}\|$, let $\omega \in \Omega_\epsilon$ with $|\omega| \geq 1$ and define $P_{\omega,\delta} = \text{diag}(\beta_{\delta}I_{Z_k})_{k \in \mathbb{I}} \in \mathcal{L}(\mathbb{C})$ where $\beta_{\delta} = 1$ for those $k \in \mathbb{I}$ for which $|\omega - \omega_k| < \delta(\omega)$ and $\beta_{\delta} = 0$ otherwise. The operator $P_{\omega,\delta}$ is a spectral projection of $A_c$ associated to the part $\{i\omega_k\}_k \cap (i\omega - i\delta(\omega), i\omega + i\delta(\omega))$ of its spectrum and $P_{\omega,\delta}z$ is a $(\omega, \delta(\omega))$-wavepacket of $A_c$ for every $z \in Z$. Let $u \in Y$ and $z \in \mathbb{C}$ be arbitrary and define

$$z = S_A(\omega)^{-1}(B_cu + y) \in Z_{B_c}, \text{ i.e., } (\omega - A_c + B_cP_{\omega}(i\omega)C_{\Lambda})z = B_cu + y.$$ 

Define $z_0 = P_{\omega,\delta}z$, $z_c = z - z_0$, $y_c = P_{\omega,\delta}y$, $y_c = y - y_0$. Similarly decompose $A_c = A_0^c + A_c^c$, $B_c = B_0^c + B_c^c$, and $C_{\Lambda} = C_0^c + C_c^c$. The diagonal structure of $A_c$ and the decompositions imply

$$(i\omega - A_c^c)z_c = y_c + B_c^c(u - P_{\omega}(i\omega)C_{\Lambda}z_c)$$

$$\Rightarrow z_c = R(i\omega, A_c^c)y_c + R(i\omega, A_c^c)B_c^c(u - P_{\omega}(i\omega)C_{\Lambda}z_c)$$

$$\Rightarrow C_{\Lambda}z_c = C_{\Lambda}R(i\omega, A_c^c)y_c + G_{0c}(i\omega)(u - P_{\omega}(i\omega)C_{\Lambda}z_c),$$

where we have denoted $G_{0c}(i\omega) = C_c^cR(i\omega, A_c^c)B_c^c$. The system $(A_c^c, B_c^c, C_{\Lambda}^c)$ is regular and due to the diagonal structure of $A_c$ we have $\|R(i\omega, A_c^c)\| \lesssim \delta(\omega)^{-1}$. The resolvent identity $R(i\omega, A_c^c) = R(i\omega + 1, A_c^c) + R(i\omega, A_c^c)R(i\omega +
The above expressions for $z$ and $\sup s\in\mathbb{R}|P_d(i\omega)| < \infty$ (Lemma A.2) therefore imply
\[
\|z\|^2 = \|z_c\|^2 + \|z_0\|^2 \leq \|z_c\|^2 + \gamma(|\omega|)^{-2}\|z_c z_0\|^2
\]
\[
\lesssim \|z_c\|^2 + \gamma(|\omega|)^{-2}\|C_{cA} z\|^2 + \gamma(|\omega|)^{-2}\|C_{cA} z_c\|^2
\]
\[
\lesssim (|R(i\omega, A_c)|^2 + \gamma(|\omega|)^{-2}\|C_c R(i\omega, A_c)|^2)\|y_c\|^2 + \gamma(|\omega|)^{-2}\|C_{cA} z\|^2
\]
\[
+ (|R(i\omega, A_c)^2|\|G_c(i\omega)|^2)\|y\|^2 + \|P_d(i\omega)|^2 (|\gamma| + \|P_d(i\omega)|^2 (|\gamma|)^2)
\]
\[
\lesssim \gamma(|\omega|)^{-2}\|\omega|^2\|y\|^2 + \|\gamma\|y\|^2 + \|\gamma\|y\|^2 + \|C_{cA} z\|^2).
\]

First let $u = 0$ to estimate $\|S_A(i\omega)^{-1}\|$ and $\|C_{cA} S_A(i\omega)^{-1}\|$. Then $z = S_A(i\omega)y \in D(S_A(i\omega))$. The passivity of $(A_c, B_c, C_{cA}, D_c)$ implies
\[
\text{Re}(\langle y, x \rangle) = -\text{Re}(A_c z + B_c(-P_d(i\omega)C_{cA} z), z)
\]
\[
\geq \text{Re}(C_{cA} z - D_c P_d(i\omega)C_{cA} z, P_d(i\omega)C_{cA} z)
\]
\[
= \text{Re}((I + D_c P(i\omega))^{-1} C_{cA} z, P(i\omega)(I + D_c P(i\omega))^{-1} C_{cA} z)
\]
\[
\geq \eta(|\omega|) (1 + D_c P(i\omega))^{-2} \|C_{cA} z\|^2 \geq \eta(|\omega|) \|C_{cA} z\|^2,
\]
where $M_P = 1 + \|D_c\| \sup_{\omega \in \mathbb{R}} \|P(i\omega)\| < \infty$, and thus we have $\|C_{cA} z\|^2 \lesssim \eta(|\omega|)^{-1} \|z\| \|y\|$. The above estimate for $\|z\|^2$ (again with $u = 0$) together with the scalar inequality $2ab \leq \varepsilon a^2 + b^2 / \varepsilon$ for $\varepsilon > 0$ implies
\[
\|z\|^2 \lesssim \gamma(|\omega|)^{-2}\|\gamma\|^{-2}\|y\|^2 + \|C_{cA} z\|^2
\]
\[
\lesssim \gamma(|\omega|)^{-2}\|\gamma\|^{-2}\|y\|^2 + \eta(|\omega|)^{-1}\gamma(|\omega|)^{-2}\|\gamma\|^2\|z\|\|y\|
\]
\[
\leq \gamma(|\omega|)^{-2}\|\gamma\|^{-2}\|y\|^2 + \frac{\varepsilon}{2}\|z\|^2 + \frac{1}{2\varepsilon}\eta(|\omega|)^{-2}\gamma(|\omega|)^{-4}\|\gamma\|^4\|y\|^2.
\]
Letting $\varepsilon > 0$ be small shows that $\|z\| \lesssim \eta(|\omega|)^{-1}\gamma(|\omega|)^{-2}\delta(|\omega|)^{-2}\|y\|$. Since $y \in Z$ was arbitrary, we have that $\|S_A(i\omega)^{-1}\| \leq M_0(|\omega|)$ for some $M_0 > 0$. Moreover, our earlier estimate $\|C_{cA} z\|^2 \lesssim \eta(|\omega|)^{-1} \|z\| \|y\|$ further implies
\[
\|C_{cA} S_A(i\omega)^{-1} y\|^2 = \|C_{cA} z\|^2 \lesssim \eta(|\omega|)^{-1} \|z\| \|y\|
\]
\[
\lesssim \eta(|\omega|)^{-2}\gamma(|\omega|)^{-2}\|\gamma\|^{-2}\delta(|\omega|)^{-2}\|y\|^2,
\]
and thus $\|C_{cA} S_A(i\omega)^{-1}\| \lesssim \eta(|\omega|)^{-1}\gamma(|\omega|)^{-1}\delta(|\omega|)^{-1} \leq M_R(|\omega|)$ for some $M_0 > 0$.

Finally, to estimate $\|S_A(i\omega)^{-1} B_c u\|$, let $y = 0$ and let $u \in Y$ be arbitrary. Now we have $z = S_A(i\omega)^{-1} B_c u$, and thus $\|C_{cA} z\| = \|C_{cA} S_A(i\omega) B_c u\| \lesssim \eta(|\omega|)^{-1} \|u\|$ due to our earlier estimate. Because of this, we also have
\[
\|S_A(i\omega)^{-1} B_c u\|^2 = \|z\|^2 \lesssim \gamma(|\omega|)^{-2}\|\gamma\|^{-2}\|\gamma\|^2\|z\|^2
\]
\[
\lesssim \gamma(|\omega|)^{-2}\|\gamma\|^{-2}(1 + \eta(|\omega|)^{-2})\|u\|^2.
\]
and thus \( \|S_A(i\omega)^{-1}B_c\| \lesssim \eta(|\omega|)^{-1} \gamma(|\omega|)^{-1} \delta(|\omega|)^{-1} \leq M_R(|\omega|) \) for some \( M_0 > 0 \).

In the case where \( X = \{0\} \), \( A = 0 \in \mathcal{L}(X) \), \( B = 0 \in \mathcal{L}(U, X) \), \( C = 0 \in \mathcal{L}(X, U) \), and \( D = I \in \mathcal{L}(U) \) the operator \( S_A(i\omega) \) reduces to \( i\omega - A_c + B_c(I + D_c)^{-1}C_cA \). This way Theorem 3.7 can also be used to study the non-uniform stability of semigroups generated by operators of the form \( A_c - B_cB_c^* \) and \( A_c - B_c(I + D_c)^{-1}C_cA \). This topic is considered in detail in [12].

Remark 3.8. Assume \( \{\omega_k\}_{k \in I} \) has a uniform gap, i.e., \( \inf_{k \neq l} |\omega_k - \omega_l| > 0 \), and \( \tilde{\gamma} : \mathbb{R}^+ \to (0, 1) \) is a continuous non-increasing function such that \( \inf_{\omega > 0} \tilde{\gamma}(\omega + \delta_0)/\tilde{\gamma}(\omega) > 0 \) for some \( 0 < \delta_0 < \min\{1, 1/\inf_{k \neq l} |\omega_k - \omega_l|\} \) (so that \( \tilde{\gamma}(\cdot) \) does not decrease too rapidly). If \( \|B_{ck}^*z_k\| \geq \tilde{\gamma}(|\omega_k|)||z_k|| \) for all \( k \in I \) and \( z_k \in Z_k \), then there exists a constant \( 0 < c \leq 1 \) for which the functions \( \gamma(\cdot) = c\tilde{\gamma}(\cdot) \) and \( \delta(\cdot) \equiv \delta_0 > 0 \) are such that \( \|C_cz|| \geq \gamma(|\omega|)||z|| \) for every \( \omega \in \mathbb{R} \) and every \( (\omega, \delta(|\omega|))-\text{wavepacket } z \) of \( A_c \).

4. The Robust Output Regulation Problem

We will now turn our attention to constructing passive controllers of the form (4.1) to achieve robust output tracking and disturbance rejection for a passive regular linear system (2.1). We assume the reference signal \( y_{ref}(t) \) and the disturbance signal \( w_{dist}(t) \) are of the form

\[
y_{ref}(t) = \sum_{k \in I} y_{ref}^k e^{i\omega_k t}, \quad \text{and} \quad w_{dist}(t) = \sum_{k \in I} w_{dist}^k e^{i\omega_k t},
\]

with a given set \( \{\omega_k\}_{k \in I} \subset \mathbb{R} \) of distinct frequencies with no finite accumulation points, and \( \{y_{ref}^k\}_{k \in I} \subset Y \) and \( \{w_{dist}^k\}_{k \in I} \subset U_d \). We use the notation \( w_{ext}(t) = (w_{dist}(t), y_{ref}(t))^T \) and \( w_{ext}^k = (w_{dist}^k, y_{ref}^k)^T \). We consider \( y_{ref}(t) \) and \( w_{dist}(t) \) with both finite and infinite number of frequency components, and these two classes of signals are treated separately. The latter situation is encountered in tracking and rejection of nonsmooth periodic signals [24].

If \( I \) is infinite, we assume \( \{y_{ref}^k\}_{k \in I} \in \ell^1(I; Y) \) and \( \{w_{dist}^k\}_{k \in I} \in \ell^1(I; U_d) \), which imply that \( y_{ref}(t) \) and \( w_{dist}(t) \) are uniformly continuous almost periodic functions [3, Def. 4.5.6]. In the case of real-valued \( y_{ref}(t) \) and \( w_{dist}(t) \) we have \( \omega_n \in \{\omega_k\}_{k \in I} \) for all \( n \in I \).

We make the following standing assumption on the system (2.1). Here \( P_S(\lambda) \) is the transfer function of the system \( (A^S, B^S, C^S, D^S) \) obtained from (2.1) with admissible output feedback \( u(t) = -D_{c2}y(t) \) with \( D_{c2} \geq 0 \). It should be noted that Assumption 4.1 is satisfied for some \( D_{c2} \geq 0 \) for which \( \{i\omega_k\}_{k \subset \rho(A^S) \} \) if and only if it is satisfied for all \( D_{c2} \geq 0 \) with this property. In particular, if \( i\omega_k \in \rho(A) \) for some \( k \in I \), then \( P_S(i\omega_k) \) is invertible if and only if \( P(i\omega_k) \) is invertible.

Assumption 4.1. There exists \( D_{c2} \geq 0 \) such that \( i\omega_k \in \rho(A^S) \) and \( P_S(i\omega_k) \) is boundedly invertible for all \( k \in I \).

We define the regulation error as \( e(t) = y_{ref}(t) - y(t) \). Our aim is to choose \( (A_c, B_c, C_c, D_c) \) in such a way that \( e(t) \) converges to zero in a suitable sense as \( t \to \infty \). The closed-loop system consisting of (2.1) and the controller (1.4)
with state $x_\epsilon(t) = (x(t), z(t))^T$ on $X_\epsilon = X \times Z$ is of the form

$$\dot{x}_\epsilon(t) = A_\epsilon x_\epsilon(t) + B_\epsilon w_{\text{ext}}(t), \quad x_\epsilon(0) = x_{\epsilon 0} = (x_0, z_0)^T \in X_\epsilon,$$

(4.2a)

$$e(t) = C_\epsilon x_\epsilon(t) + D_\epsilon w_{\text{ext}}(t),$$

(4.2b)

where $w_{\text{ext}}(t) = (w_{\text{dist}}(t), y_{\text{ref}}(t))^T$. If we denote $Q_1 = (I + D_\epsilon D_c)^{-1}$ and $Q_2 = (I + D_\epsilon D_c)^{-1}$, then $A_{\epsilon}$ and $D(A_{\epsilon})$ are as in (3.1) and

$$B_\epsilon = \begin{bmatrix} B_d & BD_c Q_1 \\ 0 & B_c Q_1 \end{bmatrix}, \quad C_\epsilon = \begin{bmatrix} -Q_1 C_\Lambda & -Q_1 D C_\Lambda \end{bmatrix}, \quad D_\epsilon = \begin{bmatrix} 0 & Q_1 \end{bmatrix}.$$

The following result shows that the closed-loop system is a regular linear system. The result also holds whenever $\text{Re} \ D_c \geq 0$ and $I + DD_c$ is invertible.

**Lemma 4.2.** The closed-loop system (4.2) is regular and $A_\epsilon$ in (3.1) generates a contraction semigroup.

**Proof.** Consider the regular linear system

$$\left( \begin{bmatrix} A & 0 \\ 0 & A_c \end{bmatrix}, \begin{bmatrix} B & B_d & 0 \\ 0 & 0 & B_c \end{bmatrix}, \begin{bmatrix} C_\Lambda & 0 \\ 0 & C_\Lambda \end{bmatrix}, \begin{bmatrix} D & 0 & 0 \\ 0 & 0 & D_c \end{bmatrix} \right).$$

The closed-loop system (4.2) is obtained from the above system with output feedback with $\hat{K} = \begin{bmatrix} K_d & 0 \\ 0 & K_c \end{bmatrix}$, which is an admissible feedback operator since $I + DD_c$ is boundedly invertible by Lemma A.1(d). Thus (4.2) is regular [15].

Since $A_\epsilon$ generates a semigroup $T_\epsilon(t)$ on $X_\epsilon$, the Lumer–Phillips Theorem implies that $T_\epsilon(t)$ is contractive if $A_\epsilon$ is dissipative. The estimates $\text{Re} \langle Ax + Bu, x \rangle \leq \text{Re} (C_\Lambda x + Du, u)$ and $\text{Re} \langle A_\epsilon z + B_c y, z \rangle \leq \text{Re} (C_\Lambda A_\epsilon z + D_c y, y)$ and a direct computation show that for any $x_\epsilon = (x, z)^T \in D(A_\epsilon)$ we have

$$\text{Re} \langle A_\epsilon x_\epsilon, e_\epsilon \rangle = \text{Re} \langle Ay + BQ_2 (C_\Lambda x + C_\Lambda z), x \rangle$$

$$+ \text{Re} \langle A_\epsilon z + B_c Q_1 (-C_\Lambda x - D C_\Lambda z), z \rangle$$

$$\leq \text{Re} (C_\Lambda A_\epsilon x + D Q_2 (C_\Lambda A_\epsilon x + C_\Lambda z), Q_2 (-D_c C_\Lambda x + C_\Lambda z))$$

$$+ \text{Re} (C_\Lambda A_\epsilon z + D c Q_1 (-C_\Lambda x - D C_\Lambda z), Q_1 (-C_\Lambda A_\epsilon x - D C_\Lambda z))$$

$$= 0,$$

and thus $A_\epsilon$ is dissipative. \hfill \Box

In the following we define the robust output regulation problem for the regular linear system (2.1). In the problem we consider perturbations for which the perturbed system $(\hat{A}, \hat{B}, \hat{B_d}, \hat{C}_\Lambda, \hat{D})$ and the perturbed closed-loop system remain regular. The robustness of the controller also implies that output tracking and disturbance rejection are achieved even if the operators $B_c$, $C_c$ and $D_c$ of the controller are perturbed or approximated in such a way that the closed-loop stability is preserved and the additional conditions on the perturbations stated in Section 5 are satisfied.

**The Robust Output Regulation Problem.** Choose $(A_c, B_c, C_c, D_c)$ in such a way that the following are satisfied:

(a) The semigroup $T_\epsilon(t)$ generated by $A_\epsilon$ is strongly stable.
(b) For the reference and disturbance signals of the form (4.1) and for all initial states \( x_{c0} \in X_e \) the regulation error satisfies

\[
\int_{t}^{t+1} \|e(s)\|ds \to 0 \quad \text{as} \quad t \to \infty.
\]

(c) If \((A, B, B_d, C_{\Lambda}, D)\) are perturbed to \((\tilde{A}, \tilde{B}, \tilde{B}_d, \tilde{C}_{\Lambda}, \tilde{D})\) in such a way that the perturbed closed-loop system is strongly stable, then for the signals (4.1) and for all initial states \( x_{c0} \in X_e \) the regulation error satisfies (4.3).

It follows from the results in [30, Sec. 3] that if the closed-loop system is exponentially stable, then convergence in (4.3) is uniformly exponentially fast, i.e., there exist \( M_e, \alpha > 0 \) such that \( \int_{t}^{t+1} \|e(s)\|ds \leq M_e e^{-\alpha t} (\|x_{c0}\| + 1) \) for all \( x_{c0} \in X_e \). If the input and output operators of the system and the controller are bounded, then the error convergences pointwise, i.e., \( \|y(t) - y_{ref}(t)\| \to 0 \) as \( t \to \infty \), and the rate is exponential if \( T_e(t) \) is exponentially stable.

5. Passive Controllers for Robust Output Regulation

The controller constructions in this section are based on the internal model principle [17, 31, 32] which implies that a controller solves the robust output regulation problem provided that its dynamics contain a suitable number of copies of the frequencies \( \{\omega_k\}_{k \in I} \) of the signals (4.1) and the closed-loop system is stable. If \( \dim Y < \infty \), then \((A_c, B_c, C_c, D_c)\) contains an internal model of the signals (4.1) if [30, Thm. 13]

\[
\dim \mathcal{N}(i\omega_k - A_c) \geq \dim Y \quad \forall k \in I.
\]

In the case of an infinite-dimensional output space, the controller contains an internal model if [30, Thm. 13]

\[
(5.1a) \quad \mathcal{R}(i\omega_k - A_c) \cap \mathcal{R}(B_c) = \{0\} \quad \forall k \in I,
\]

\[
(5.1b) \quad \mathcal{N}(B_c) = \{0\}.
\]

We consider three different situations: In Section 5.1 we construct a finite-dimensional robust controller for a strongly stabilizable system (2.1). If \((A, B, C, D)\) is exponentially stabilizable, then the convergence of the error is exponentially fast. In Section 5.2 we design a robust controller to track and reject nonsmooth \( \tau \)-periodic reference signals. The controller is based on a periodic transport equation, and achieves exponential closed-loop stability if the system (2.1) is exponentially stabilizable and satisfies \( \text{Re} P(i\omega) \geq \eta > 0 \) for some constant \( \eta > 0 \) near the points \( \omega_k = \frac{2\pi k}{\tau} \) for \( k \in \mathbb{Z} \). In Section 5.3 we design an infinite-dimensional robust controller for nonsmooth signals (4.1) with a general set of frequencies \( \{\omega_k\}_{k \in I} \). In general, the closed-loop system can not be stabilized exponentially, and we introduce conditions for non-uniform subexponential rates of convergence of the output.

In the constructions we choose the feedthrough of the controller to have the form \( D_c = D_{c1} + D_{c2} \), where \( D_{c2} \geq 0 \) is used to pre-stabilize the system \((A, B, C, D)\). We assume that the system \((A^S, B^S, C^S, D^S) = (A - BD_{c2}Q_1^SC_{\Lambda}, BQ_2^SC_{\Lambda}, Q_1^SC_{\Lambda}, Q_1^TD)\) where \( Q_1^S = (I + DD_{c2})^{-1} \) and \( Q_2^S = (I + D_{c2}D)^{-1} \) obtained from (2.1) with the output feedback \( u(t) = -D_{c2}y(t) \) is.
either strongly or exponentially stable. Its transfer function is denoted by
\( P_S(\lambda) \). The passivity of \((A, B, C, D)\) implies that also \((A^S, B^S, C^S, D^S)\) is passive.

5.1. A Robust Finite-Dimensional Controller. In this section we assume the signals \( \{1, 1\} \) contain a finite number of frequencies \( \{\omega_k\}_{k=1}^q \), i.e., \( I = \{1, \ldots, q\} \). The controller parameters are chosen in the following way.

**Definition 5.1.** Choose \( Z = Y^q \) and
\[
A_c = \text{diag} (i\omega_1 I_Y, \ldots, i\omega_q I_Y) \in \mathcal{L}(Z),
\]
where \( I_Y \) is the identity operator on \( Y \). Choose \( C_c \in \mathcal{L}(Z, Y) \) of the form
\[
C_c z = \sum_{k=1}^q C_{ck} z_k \quad \text{for} \quad z = (z_k)_{k=1}^q \in Z
\]
so that \( C_{ck} \in \mathcal{L}(Y) \) are boundedly invertible for all \( k \), choose \( B_c = C_c^* \), and choose \( D_c = D_{c1} + D_{c2} \) with \( D_{c1} > 0 \). Finally, choose \( D_{c2} \geq 0 \) in such a way that \((A^S, B^S, C^S, D^S)\) is passive and strongly stable with \( \lambda_c > 0 \) and \( \lambda_{c1} > 0 \). This controller is passive and it will achieve robust output regulation by Theorem 5.2 due to the fact that under the similarity transform
\[
V = \text{diag}(I_Y, V_1, \ldots, V_q), \quad V_k = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} I_Y & I_Y \\ iI_Y & -iI_Y \end{array} \right]
\]
the system \((V^* A_c V, V^* B_c C_c V, D_c)\) is of the form given in Definition 5.1.

**Theorem 5.2.** The controller in Definition 5.1 solves the robust output regulation problem. The closed-loop system is strongly stable and \( i\mathbb{R} \subset \rho(A_c) \).

If \((A^S, B^S, C^S, D^S)\) is exponentially stable, then also the closed-loop system is exponentially stable and for any \( y_{\text{ref}}(t) \) and \( w_{\text{dist}}(t) \) there exist \( M_\epsilon, \alpha > 0 \) such that
\[
\int_t^{t+1} ||e(s)|| ds \leq M_\epsilon e^{-\alpha t} (||x_0|| + 1) \quad \forall x_0 \in X_c.
\]

In both cases the controller is robust with respect to all perturbations that preserve the stability of the closed-loop system and for which \( i\mathbb{R} \subset \rho(\hat{A}_c) \).

**Proof.** The controller \((A_c, B_c, C_c, D_{c1})\) is passive and its transfer function \( G(\lambda) \) satisfies \( \text{Re} \, G(i\omega) = D_{c1} > 0 \) for all \( \omega \in \mathbb{R} \setminus \{\omega_k \}_{k=1}^q \). The operators \((A_c, B_c)\) satisfy (5.1). Indeed, the injectivity of \( B_c \) in (5.1a) follows directly from the fact that the components \( C_{ck} \) of \( B_c \) are boundedly invertible by assumption. Condition (5.1a) can be verified using the diagonal structure of \( A_c \) and the invertibility of \( C_{ck} \).
To prove closed-loop stability, we apply Theorem 3.2 to \((A^S, B^S, C^S, D^S)\) and \((A_c, B_c, C_c, D_c)\). Condition (2) of the theorem is satisfied since for any \(\omega \in \mathbb{R} \setminus \{\omega_k\}_{k=1}^\infty\) we have \(\text{Re}(i\omega) = \text{Re}(C_c)R(i\omega, A_c)B_c + D_{c1} = D_{c1} > 0\), and condition (3) is satisfied by Lemma 3.4 since \(C_{ck}\) are invertible. Thus the strong and exponential closed-loop stabilities follow from Theorem 3.2.

Finally, the conclusion that the controller solves the robust output regulation problem follows from [30] Thm. 13. The results in [30] are presented for controllers with \(D_c \geq 0\) but they are applicable since \(D_c \geq 0\) can be written as an output feedback for the system (2.1) without changing the properties of the closed-loop system. Moreover, the results are presented for an infinite set \(\{\omega_k\}_{k \in \mathbb{N}}\), but they also apply trivially when \(I = \mathbb{N}\).

**Proposition 5.3.** The regulation error in Theorem 5.2 converges pointwise, i.e., \(\|e(t)\| \to 0\) as \(t \to \infty\), for all initial states \(x_{e0} \in X_e\) satisfying \(A_c x_{e0} + B_c w_{ext}(0) \in X_e\). If the closed-loop system is exponentially stable, then for all \(y_{ref}(t)\) and \(w_{dist}(t)\) there exist \(M_e, \alpha > 0\) such that

\[\|e(t)\| \leq M_e e^{-\alpha t} (\|A_c x_{e0} + B_c w_{ext}(0)\| + 1)\]

for all \(x_{e0} \in X_e\) satisfying \(A_c x_{e0} + B_c w_{ext}(0) \in X_e\).

The proof of Proposition 5.3 is based on the following technical lemma, which is also used later in the following sections. The assumptions on \(H\) are automatically satisfied if \(I\) is finite, or if the closed-loop system is exponentially stable. In the latter case the property \(H v \in D(C_{cA})\) can be verified similarly as in the proof of Theorem 5.11.

**Lemma 5.4.** Assume the controller solves the robust output regulation problem and \(y_{ref}(t)\) and \(w_{dist}(t)\) are such that for some fixed \((f_k)_{k \in \mathbb{N}} \in \ell^2(\mathbb{C})\) the operator \(H : D(H) \subset \ell^2(\mathbb{C}) \to X_e\) defined by

\[Hv = \sum_{k \in I} f_k^{-1} R(i\omega_k, A_c)B_c w_{ext}^k v_k, \quad v = (v_k)_{k \in \mathbb{N}}\]

satisfies \(H \in \mathcal{L}(\ell^2(\mathbb{C}), X_e)\) and \(H v \in D(C_{cA})\) for all \(v \in \ell^2(\mathbb{C})\). If \(y_{ref}(t)\) and \(w_{dist}(t)\) are such that the series

\[q_{ext} = \sum_{k \in I} i\omega_k R(i\omega_k, A_c)B_c w_{ext}^k\]

converges in \(X_e\), then for all \(x_{e0} \in X_e\) satisfying \(A_c x_{e0} + B_c w_{ext}(0) \in X_e\) and for almost all \(t > 0\) we have

\[e(t) = C_{cA} T_c(t) A_c^{-1} (A_c x_{e0} + B_c w_{ext}(0) - q_{ext})\]

**Proof.** It follows from the properties of \(H\) and the results in [30] that for every \(x_{e0} \in X_e\) and almost all \(t > 0\) the regulation error is given by

\[e(t) = C_{cA} T_c(t) (x_{e0} - \sum_{k \in I} R(i\omega_k, A_c)B_c w_{ext}^k)\]

If \(A_c x_{e0} + B_c w_{ext}(0) \in X_e\), then a direct computation and \(q_{ext} \in X_e\) show

\[A_c \sum_{k \in I} R(i\omega_k, A_c)B_c w_{ext}^k = \sum_{k \in I} i\omega_k R(i\omega_k, A_c)B_c w_{ext}^k - B_c w_{ext}(0),\]

which implies the claim. \(\square\)
Proof of Proposition 5.3. Since $\mathcal{I}$ is finite, the conditions of Lemma 5.4 are satisfied. If $x_{e0} \in X_e$ is such that $A_e x_{e0} + B_e w_{\text{ext}}(0) \in X_e$, then the estimate $\|e(t)\| \leq \|C_e A_e^{-1} \| \|T_e(t)\| \|A_e x_{e0} + B_e w_{\text{ext}}(0) - q_{\text{ext}}\|$ implies both claims of the proposition.

The following sufficient condition for $A_e x_{e0} + B_e w_{\text{ext}}(0) \in X_e$ follows directly from the structures of $A_e$ and $B_e$. Later in Section 5.4 the same condition implies a non-uniform decay rate for the regulation error.

Lemma 5.5. If $B_e \in \mathcal{L}(U, X)$, $C_e \in \mathcal{L}(X, Y)$, and $w_{\text{dist}}(0) = 0$, then $A_e x_{e0} + B_e w_{\text{ext}}(0) \in X_e$ is satisfied for $x_{e0} = (x_0, z_0)^T \in D(A) \times D(A_e)$ if $C_e z_0 = D_e(C x_0 - y_{\text{ref}}(0))$.

5.2. A Robust Controller for $\tau$-Periodic Signals. In this section we will construct a regular linear controller that achieves exponentially fast output regulation of $\tau$-periodic reference and disturbance signals. The controller structure is based on a shift semigroup with periodic boundary conditions, and is related to controllers constructed in [21, 46, 23]. We assume that $\dim Y = p < \infty$, and that $y_{\text{ref}}(t)$ and $w_{\text{dist}}(t)$ are $\tau$-periodic functions, i.e., $\mathcal{I} = Z$ and $\{\omega_k\}_{k \in \mathbb{Z}} = \{2\pi k / \tau\}_{k \in \mathbb{Z}}$.

Definition 5.6. Choose the controller as

\begin{align}
(5.3a) & \quad z_t(\xi, t) = z_\xi(\xi, t), \quad \xi \in (0, \tau), \quad t \geq 0, \\
(5.3b) & \quad z(\cdot, 0) = z_0(\cdot) \in L^2(0, \tau; \mathbb{C}^p), \\
(5.3c) & \quad e(t) = 2^{-1/2}(z(\tau, t) - z(0, t)), \\
(5.3d) & \quad u(t) = 2^{-1/2}(z(\tau, t) + z(0, t)) + (D_{c1} + D_{c2})e(t)
\end{align}

where $z(\xi, t) = (z_1(\xi, t), \ldots, z_p(\xi, t))^T$ and $D_{c1} > 0$. Choose $D_{c2} \geq 0$ in such a way that $(A^S, B^S, C^S, D^S)$ is passive and exponentially stable.

To achieve closed-loop stability, we also assume that $\mathrm{Re} P_S(i\omega_k) \geq \eta > 0$ for some constant $\eta > 0$ and for all $k \in \mathbb{Z}$. If this condition is not satisfied, then exponential closed-loop stability is unachievable, but strong closed-loop stability can be studied using Theorem 5.11 in the next section.

Theorem 5.7. Let $y_{\text{ref}}(t)$ and $w_{\text{dist}}(t)$ be as in (3.1) with $\omega_k = 2\pi k / \tau$ for some $\tau > 0$. Assume there exist $\eta, \varepsilon > 0$ such that $\mathrm{Re} P_S(i\omega) \geq \eta > 0$ for $\omega \in \Omega_\varepsilon = \{\omega \in \mathbb{R} \mid \exists k \in \mathbb{Z} : |\omega - \omega_k| < \varepsilon\}$, and $\mathrm{Re} D > 0$. Then the controller in Definition 5.6 solves the robust output regulation problem in such a way that the closed-loop system is exponentially stable, and there exist $M_\varepsilon, \alpha \geq 0$ such that

$$
\int_t^{t+1} \|e(s)\|ds \leq M_\varepsilon e^{-\alpha t} (\|x_{e0}\| + 1) \quad \forall x_{e0} \in X_e.
$$

The controller is robust with respect to all perturbations that preserve the exponential closed-loop stability, and for which $u(t) = -D_{c2} y(t)$ remains an admissible output feedback and $\{i\omega_k\}_{k \in \mathbb{Z}} \subset \rho(\tilde{A}^S)$.

Proof. The controller in Definition 5.6 consists of $p = \dim Y$ independent one-dimensional periodic transport equations with boundary control and observation, and an additional feedthrough $(D_{c1} + D_{c2})e(t)$. The system (5.3)
defines a regular linear system with state $z(t) = z(\cdot, t)$ on $Z = L^2(0, \tau; \mathbb{C}^p)$ [51 Thm. 2.4], and a direct computation shows that its transfer function from $e(t)$ to $u(t)$ is

$$G_0(\lambda) = \frac{1 + e^{-\lambda \tau}}{1 - e^{-\lambda \tau}} I + D_{c1} + D_{c2}, \quad \lambda \notin \left\{ \frac{2\pi k}{\tau} \right\}_{k \in \mathbb{Z}}.$$  

Thus the controller can be written as a system $(A_c, B_c, C_c, D_c)$ on $Z$ where $A_c$ satisfying $A_c f = f'$ for $f \in D(A_c) = \{ f \in H^1(0, \tau; \mathbb{C}^p) \mid f(0) = f(\tau) \}$ generates a unitary group with spectrum $\sigma(A_c) = \{i\frac{2\pi k}{\tau} \}_{k \in \mathbb{Z}}$. We also have $\dim \mathcal{N}(i\omega_k - A_c) = \dim Y$ for every $k \in \mathbb{Z}$, and thus $A_c$ contains an internal model of the signals [41]. By [30 Thm. 13] the controller solves the robust output regulation problem if the closed-loop system is exponentially stable.

To show closed-loop stability, we will verify the conditions of Theorem 3.5 for the systems $(A^S, B^S, C^S, D^S)$ and $(A_c, B_c, C_c, D_{c1})$ with $\Omega = \Omega_c$. For this we will consider the controller with inputs and outputs

$$u_c(t) = 2^{-1/2}(z(\tau, t) - z(0, t)),$$

$$y_c(t) = 2^{-1/2}(z(\tau, t) + z(0, t)) + (D_{c1} + D_{c2})u_c(t).$$

The feedthrough operator of the controller is given by $D_c = \lim_{\lambda \to \infty} G_0(\lambda) = I + D_{c1} + D_{c2}$. Without the component $(D_{c1} + D_{c2})u_c(t)$ of the feedthrough the solutions of (5.3) satisfy $\frac{d}{dt} \| z(t) \|_{L^2}^2 = 2 \text{Re} \langle u_c(t), y_c(t) \rangle$, and thus the controller is passive by [38 Thm. 4.2]. Let $d_c > 0$ be such that $D_{c1} \geq d_c > 0$. The transfer function $G(\lambda)$ of $(A_c, B_c, C_{cA}, I + D_{c1})$ satisfies $\text{Re} \, G(i\omega) = D_{c1} \geq d_c > 0$ for all $\omega \in \mathbb{R} \setminus \{ \omega_k \}_{k \in \mathbb{Z}}$, and thus condition (2) of Theorem 3.5 is satisfied. To show that condition (3) of Theorem 3.5 is satisfied, it is sufficient to show that for any $D_0 \in \mathcal{L}(U)$ with $\text{Re} \, D_0 > 0$ the system $(A_c, B_c, C_{cA}, I + D_{c1})$ is stabilized exponentially with feedback $u_c(t) = -D_0 y_c(t)$. The feedback leads to a partial differential equation

$$z(t, \xi, t) = z(\xi, t), \quad \xi \in (0, \tau), \quad t \geq 0,$$

$$(I + D_{tot}) z(\tau, t) = (I - D_{tot}) z(0, t).$$

where $D_{tot} = D_0(I + D_{c1} + D_0)^{-1}$. The exponential stability of this system follows from a straightforward application of [41 Thm. III.2], since $\text{Re} \, D_{tot} > 0$ by Lemma 3.5(c). Thus Theorem 3.5 shows that the closed-loop system is exponentially stable.

Remark 5.8. The results in [30] also show that if $(y_{ref})_k = (a_k y_k)_k$ and $(w_{dist})_k = (a_k w_k)_k$ where $(y_k)_k \in \ell^2(Y)$, $(w_k)_k \in \ell^2(U_d)$ are fixed, and $(a_k)_k \in \ell^2(\mathbb{C})$, then there exist $M_c, \alpha > 0$ such that

$$\int_t^{t+1} \| e(s) \| ds \leq M_c e^{-\alpha t} (\| x_{c0} \| + \| (a_k)_k \|_{\ell^2})$$

for all $x_{c0} \in X_c$ and $(a_k)_k \in \ell^2(\mathbb{C})$.

Lemma 5.4 implies the following result on the pointwise convergence of $\| e(t) \|$. The conditions require that $y_{ref}(t)$ and $w_{dist}(t)$ have a sufficient levels of smoothness.
Corollary 5.9. If the signals \( \sigma \) are such that \((k_y^{\text{ref}})_k \in l^1(Y) \) and \((k w^{\text{dist}})_k \in l^1(U_d) \), then in Theorem 5.7 there exist \( M, \alpha > 0 \) such that for all \( x_{e0} \in X_e \) satisfying \( A_x x_{e0} + B_e w_{\text{ext}}(0) \in X_e \) we have
\[
\|e(t)\| \leq M e^{-\alpha t}\left(\|A_x x_{e0} + B_e w_{\text{ext}}(0)\| + 1\right).
\]

If \( P(i \mu_j) \) is not invertible for some \( \{i \mu_j\}^N_{j=1} \subset \{i \frac{2\pi k}{\tau}\}_{k \in \mathbb{Z}} \), for example for \( \mu_j = 0 \), then the robust output regulation problem is not solvable for signals \( y_{\text{ref}}(t) \) and \( w_{\text{dist}}(t) \) containing these frequencies. In this situation we can modify the controller in Definition 5.6 by replacing (5.3a) with
\[
z_t(\xi, t) = z_\xi(\xi, t) - \frac{1}{\tau} \sum_{j=1}^N \sum_{k=1}^p e_k \cdot e^{i \mu_j \xi} \int_0^\tau z_k(s, t) e^{-i \mu_j s} ds, \quad \xi \in (0, \tau),
\]
where \( \{e_k\}_{k=1}^p \) are the Euclidean basis vectors of \( \mathbb{C}^p \). This corresponds to stabilizing the eigenvalues \( \{i \mu_j\}^N_{j=1} \) of the transport system (5.3), and the resulting controller has the property \( \sigma(A_e) \cap i \mathbb{R} = \{i \frac{2\pi k}{\tau}\}_{k \in \mathbb{Z}} \setminus \{i \mu_j\}^N_{j=1} \).

With this modification the system operator of the controller is of the form
\[
A_c = A_0^0 - B_0 B_0^* \quad \text{with} \quad B_0 \in \mathcal{L}(\mathbb{C}^N, Z).
\]
The controller is again passive and is stabilized exponentially with feedback \( u_c(t) = -D_0 y_c(t) \) with \( \text{Re} \ D_0 > 0 \), and the exponential closed-loop stability follows from Theorem 3.5.

5.3. A Robust Controller for Nonsmooth Signals. In this section we construct an infinite-dimensional diagonal controller for signals (4.1) with a general set \( \{\omega_k\}_{k \in \mathbb{Z}} \) of distinct frequencies with no finite accumulation points. The controller can also be used for systems with an infinite-dimensional output space \( Y \). If \( y_{\text{ref}}(t) \) and \( w_{\text{dist}}(t) \) are \( \tau \)-periodic and \( \text{dim} \ Y < \infty \), then the controller is of similar form as in Definition 5.6.

Definition 5.10. Choose \( Z = l^2(\mathbb{I}; Y) \) and
\[
A_c = \text{diag}(i \omega_k I Y)_{k \in \mathbb{I}}, \quad D(A_c) = \{(z_k)_{k \in \mathbb{I}} \mid (|\omega_k| \|z_k\|)_k \in l^2(\mathbb{C})\},
\]
where \( I_y \) is the identity operator on \( Y \). Let \( D_{c1} = D_{c2} + D_{c1} \geq 0 \) and \( D_{c2} \geq 0 \). Choose admissible \( B_c \in \mathcal{L}(Y, Z_{-1}) \) and \( C_c \in \mathcal{L}(Z_1, Y) \) as
\[
B_c y = (B_c y_k)_{k \in \mathbb{I}} \quad \forall y \in Y, \quad C_c z = \sum_{k \in \mathbb{I}} B_{ck}^* z_k \quad \forall z \in D(A_c),
\]
with boundedly invertible \( B_{ck} \in \mathcal{L}(Y) \) so that \((A_c, B_c, C_c, D_{c1})\) is a regular linear system whose transfer function \( G(\lambda) \) satisfies \( \text{Re} \ G(i \omega) \geq d_c > 0 \) for some constant \( d_c > 0 \) and for all \( \omega \in \mathbb{R} \setminus \{\omega_k\}_{k \in \mathbb{I}} \). Finally, choose \( D_{c2} \geq 0 \) in such a way that \((A^S, B^S, C^S, D^S)\) is passive and strongly stable with \( i \mathbb{R} \subset \rho(A^S) \).

If \( \text{dim} \ Y < \infty \) and \( \{\omega_k\}_{k \in \mathbb{I}} \) has a uniform gap, i.e., \( \inf_{k \neq l}|\omega_k - \omega_l| > 0 \), then \([39]\) Cor. 5.2.5, Prop. 5.3.5 imply that \( B_c \) and \( C_c \) are admissible with respect to \( A_c \) if \((\|B_{ck}\|_{k \in \mathbb{I}}) \in l^\infty(\mathbb{C}) \) and \((\|C_{ck}\|_{k \in \mathbb{I}}) \in l^\infty(\mathbb{C}) \). For more general conditions for admissibility, see \([39]\) Sec. 5.3. The system \((A_c, B_c, C_c, D_{c1})\) is regular whenever \( B_c \) and \( C_c \) are admissible and there exists \( \varepsilon > 0 \) such that \((1 + |\omega_k|)^{-1/2 + \varepsilon}\|B_{ck}\|_k \in l^2(\mathbb{C}) \) \([14]\) Prop. 4.1. However, there are also regular linear systems, such as the controller in Definition 5.6 for which neither of these conditions is satisfied. If \( \{\omega_k\}_{k \in \mathbb{I}} \)
has a uniform gap, \(|\omega_k|^\varepsilon \|B_{ck}\|)_k \in L^\infty(\mathbb{C})\) for some \(\varepsilon > 0\) and \(D_{c1} > 0\), then \((A_c, B_c, C_c, D_{c1})\) satisfies the conditions of Definition 5.10.

Due to the lack of exponential closed-loop stability, the solvability of the robust output regulation problem requires additional conditions on the reference and disturbance signals. These conditions relate the behaviour of the coefficients \(y_{ref}^k\) and \(w_{dist}^k\) to the behaviour of the transfer functions \(P(\lambda)\) and \(P_d(\lambda)\) on the frequencies \(\{\omega_k\}_{k \in I}\). We pose conditions on the sequences \(\Pi_{ext} = (\Pi_{ext}(k))_{k \in I} \subset X_{B_c} \times Y\) consisting of the elements \(\Pi_{ext}(k) = (\Pi_{ext}^1(k), \Pi_{ext}^2(k))\) with

\[
\Pi_{ext}^1(k) = R(i\omega_k, A^S)B_{ck}^S u_k + R(i\omega_k, A^S)B_{wk}^k,
\]

\[
\Pi_{ext}^2(k) = (B_{ck}^S)^{-1}(u_k - D_{c2}y_{ref}^k),
\]

where \(u_k = P_S(i\omega_k)^{-1}y_{ref}^k - P_S(i\omega_k)^{-1}C_{A}^S R(i\omega_k, A^S)B_{wk}^k\). In the case of a perturbed system, we define \(\tilde{\Pi}_{ext} = (\tilde{\Pi}_{ext}(k))_{k \in I}\) analogously. Alternate ways of expressing \(\Pi_{ext}(k)\) are presented in Lemma 5.12. Note in particular that if \((A^S, B^S, C^S, D^S)\) is exponentially stable, then \((5.4)\) are satisfied provided that \((\|u_k\|_{k \in I}) \in \ell^1(\mathbb{C})\) and \((\|B_{ck}^S\|\|u_k - D_{c2}y_{ref}^k\|_{k \in I}) \in \ell^2(\mathbb{C})\).

**Theorem 5.11.** Assume \(\Re P_S(i\omega_k) > 0\) for all \(k \in I\). The controller in Definition 5.10 solves the robust output regulation problem for all \(y_{ref}(t)\) and \(w_{dist}(t)\) whose coefficients satisfy

\[(5.4)\quad (\Pi_{ext}^1(k))_{k \in I} \in \ell^1(X), \quad (\Pi_{ext}^2(k))_{k \in I} \in \ell^2(Y), \quad (u_k)_{k \in I} \in \ell^1(U)\]

The closed-loop system is strongly stable and \(i\mathbb{R} \subset \rho(A_c)\).

The controller is robust with respect to all perturbations \((\hat{A}, \hat{B}, \hat{B}_d, \hat{C}, \hat{D})\) for which \(u(t) = -D_{c2}y(t)\) remains an admissible output feedback, the strong closed-loop stability is preserved, \(\{\omega_k\}_{k \in I} \subset \rho(\hat{A}_c) \cap \rho(\hat{A}^S)\), \(P_S(i\omega_k)\) are invertible for \(k \in I\), and \((\tilde{\Pi}_{ext}(k))_{k \in I}\) satisfies \((5.4)\).

If the closed-loop system is exponentially stable, then \((5.4)\) are satisfied automatically, and there exist \(M_e, \alpha > 0\) such that \(\int_0^{t+1} \|e(s)\| ds \leq M_e e^{-\alpha t}(\|x_0\| + 1)\) for all \(x_0 \in X_e\).

**Proof.** The proof is based on the application of [30] Thm. 13]. The diagonal structure of the controller and the invertibility of \(B_{ck}\) imply that \(A_c\) and \(B_c\) satisfy the conditions \((5.1)\). To show that the closed-loop system is strongly stable, we apply Theorem 3.2 for the systems \((A^S, B^S, C^S, D^S)\) and \((A_c, B_c, C_c, D_{c1})\). Conditions (1) and (2) are satisfied due to the construction in Definition 5.10 and condition (3) is satisfied by Lemma 3.4 since \(C_{ck} = B_{ck}^s\) are invertible. Thus by Theorem 3.2 the closed-loop system is strongly stable and \(i\mathbb{R} \subset \rho(A_c)\).

To apply [30] Thm. 13] directly, we would need \(R(i\omega_k, A_c)B_c w_{ext}^k \in \ell^1(X_c)\). However, in [30] this property is used as a sufficient condition for the existence of \((f_k)_{k \in I} \in \ell^2(\mathbb{C})\) such that the operator \(H : D(H) \subset \ell^2(\mathbb{C}) \rightarrow X_e\) in Lemma 5.4 satisfies \(H \in \mathcal{L}(\ell^2(\mathbb{C}), X_e)\) and \(\mathcal{R}(H) \subset D(C_{A_c})\). Here we will verify that the sequence \((f_k)_{k \in I} \in \ell^2(\mathbb{C})\) with

\[
f_k = \begin{cases} 
\frac{1}{2-|k|} (\|\Pi_{ext}^2(k)\| + \|u_{ext}^k\| + \|\Pi_{ext}^1(k)\| + \|u_k\|)^{1/2} & \text{if } w_{ext}^k \neq 0 \\
0 & \text{if } w_{ext}^k = 0 
\end{cases}
\]
has this property. If \( k \in \mathcal{I} \) and \( x_e^k = (\Pi^1_{\text{ext}}(k), z_k) \in X_{B,B_d} \times Z_{B_e} \) where
\[
z_k = (z_k^j)_{j \in \mathcal{I}}, \quad z_k^k = \Pi^2_{\text{ext}}(k), \quad z_k^j = 0, \ j \neq k,
\]
then it is straightforward to verify that \( (\omega_k - A_c)x_e^k = B_c w^k_{\text{ext}} \) and thus we have \( R(\omega_k, A_c)B_c w^k_{\text{ext}} = (\Pi^1_{\text{ext}}(k), z_k) \). Now \( (f^{-1}_k(\|w^k_{\text{ext}}\| + \|\Pi_{\text{ext}}^1(k)\|) + \|u_k\|)k) \in \ell^2(\mathcal{C}) \) and \( (f^{-1}_k\Pi^2_{\text{ext}}(k))k) \in \ell^\infty(\mathcal{Y}) \). These properties and the structure of \( R(\omega_k, A_c)B_c w^k_{\text{ext}} \) imply that \( Hv \) is well-defined for every \( v \in \ell^2(\mathcal{C}) \), and
\[
\|Hv\|^2 = \left\| \sum_{k \in \mathcal{I}} f^{-1}_k\Pi^1_{\text{ext}}(k)v_k \right\|_X^2 + \left\| (f^{-1}_k\Pi^2_{\text{ext}}(k)v_k)_k \right\|_{\ell^2(\mathcal{Y})}^2 \\
\leq \|v\|^2 \left( \left\| (f^{-1}_k\Pi^1_{\text{ext}}(k))k \right\|_X^2 + \|v\|^2 \left( f^{-1}_k\Pi^2_{\text{ext}}(k) \right)_k \right)_{\ell^\infty(\mathcal{Y})}^2
\]
implies \( H \in \mathcal{L}(\ell^2(\mathcal{C}), X_e) \). It remains to show \( \mathcal{R}(\Sigma) \subseteq D(C_{c\lambda}) \). If we denote \( P_0(\lambda) = C_cR(\lambda, A_c)B_c \), then \( P_0(\omega_k)w^k_{\text{ext}} = -Q_1(C_\lambda\Pi^1_{\text{ext}}(k) + D(\omega_k - D_{c\mathcal{Y}}y^k_{\text{ref}})) \) for every \( k \in \mathcal{I} \). The regularity of \( (A^5, B^5, C^5, D^5) \) and (5.3) imply \( (f^{-1}_kP_0(\omega_k)w^k_{\text{ext}})_k \in \ell^1(\mathcal{Y}) \). If \( v \in \ell^2(\mathcal{C}) \) and \( \lambda > 0 \), the resolvent identity implies
\[
\lambda C_{c\lambda}R(\lambda, A_c)Hv = \sum_{k \in \mathcal{I}} \frac{\lambda f^{-1}_k v_k}{\lambda - i\omega_k} P_0(\omega_k)w^k_{\text{ext}} - P_0(\lambda) \sum_{k \in \mathcal{I}} \frac{\lambda f^{-1}_k v_k}{\lambda - i\omega_k} w^k_{\text{ext}} \\
\rightarrow - \sum_{k \in \mathcal{I}} f^{-1}_k P_0(\omega_k)w^k_{\text{ext}} v_k
\]
as \( \lambda \to \infty \) since \( (A_c, B_c, C_c) \) is regular and since \( (f^{-1}_kP_0(\omega_k)w^k_{\text{ext}})_k \in \ell^1(\mathcal{Y}) \) and \( (f^{-1}_k w^k_{\text{ext}}v_k)_k \in \ell^1(U_d \times X) \). Thus \( Hv \in D(C_{c\lambda}) \) by definition. An analogous argument shows that for perturbed systems \( (\tilde{A}, \tilde{B}, \tilde{B}_d, \tilde{C}, \tilde{D}) \) the sequence \( (f_k)_k \) can again be chosen so that \( H \) has the required properties. Thus the claims of the theorem follow from [30, Thm. 13]. If the closed-loop system is exponentially stable, then \( (\Pi^1_{\text{ext}}(k), z_k) = R(\omega_k, A_c)B_e w^k_{\text{ext}} \) implies \( (\Pi^1_{\text{ext}}(k))_k \in \ell^1(X \times Y) \), which also shows \( (\|u_k\|)_k \in \ell^1(\mathcal{C}) \). \( \square \)

The following alternate expressions for \( \Pi^1_{\text{ext}}(k) \) can be verified using standard operator identities and Lemma 2.1

\textbf{Lemma 5.12.} If \( i\omega_k \in \rho(A) \) for some \( k \in \mathcal{I} \), then
\[
\Pi^1_{\text{ext}}(k) = R(i\omega_k, A)B_d w^k_{\text{dist}} + R(i\omega_k, A)B\tilde{u}_k \\
\Pi^2_{\text{ext}}(k) = (B^{-1}_{ck})^{-1}\tilde{u}_k, \quad u_k = \tilde{u}_k + D_{c\mathcal{Y}}y^k_{\text{ref}}
\]
where \( \tilde{u}_k = P(i\omega_k)^{-1}y^k_{\text{ref}} - P(i\omega_k)^{-1}P_d(i\omega_k)w^k_{\text{ext}} \). If \( D \) is boundedly invertible, then \( \Pi^1_{\text{ext}}(k) = R_k^B B_d w^k_{\text{dist}} + R(i\omega_k, A^5)B^5 P_5(i\omega_k)^{-1}y^k_{\text{ref}} \) for all \( k \in \mathcal{I} \), where \( R_k^B = R(i\omega_k, A^5 - B^5(D^{-1}C^5)) \).

The following result shows that pointwise convergence is achieved for sufficiently smooth signals \( y_{\text{ref}}(t) \) and \( w_{\text{dist}}(t) \) and for suitable initial states.

\textbf{Proposition 5.13.} Assume \( y_{\text{ref}}(t) \) and \( w_{\text{dist}}(t) \) are such that \( (\omega_k\Pi^1_{\text{ext}}(k))_k \in \ell^1(X) \) and \( (\omega_k\Pi^2_{\text{ext}}(k))_k \in \ell^2(\mathcal{Y}) \). If \( x_0 \in X_e \) and \( A_c x_0 + B_c w_{\text{ext}}(0) \in X_e \), then the regulation error in Theorem 5.11 satisfies \( \|e(t)\| \to 0 \) as \( t \to \infty \).
If the closed-loop system is exponentially stable, then there exist $M_e, \alpha > 0$ such that
\[ \|e(t)\| \leq M_e e^{-\alpha t} (\|A_e x_{e0} + B_e w_{ext}(0)\| + 1) \]
for all $x_{e0} \in X_e$ satisfying $A_e x_{e0} + B_e w_{ext}(0) \in X_e$.

Proof. As in the proof of Theorem 5.11 $R(i\omega, A_e)B_e w_{ext}^k = (\Pi_{ext}^1(k), z_k)$ where $z_k = (z^i_k)_j$ is such that $z^i_k = \Pi_{ext}^2(k)$ and $z^j_k = 0$ for $j \neq k$. This structure, $(\omega_k \Pi_{ext}^1(k))_k \in \ell^1(X)$, and $(\omega_k \Pi_{ext}^2(k))_k \in \ell^2(Y)$ imply that $q_{ext}$ in (5.2) satisfies $q_{ext} \in X_e$. Since the required properties of $H$ were verified in the proof of Theorem 5.11, the claims follow from Lemma 5.4. \hfill $\Box$

5.4. Non-Uniform Convergence Rates of the Regulation Error. We will now use Theorem 3.7 to derive convergence rates for the regulation error in Theorem 5.11. The estimates are valid for reference and disturbance signals with sufficient levels of smoothness. In particular, we assume $\{\omega_k\}_{k \in \mathcal{I}}$ has a uniform gap and the coefficients of $y_{ref}(t)$ and $w_{dist}(t)$ satisfy
\[ (5.5) \quad (\omega_k \Pi_{ext}^1(k))_{k \in \mathcal{I}} \in \ell^1(X), \quad (\omega_k \Pi_{ext}^2(k))_{k \in \mathcal{I}} \in \ell^2(Y), \]
which is a strictly stronger condition than the first two parts of (5.4).

**Theorem 5.14.** Assume $(A^S, B^S, C^S, D^S)$ is passive and exponentially stable, the controller is as in Definition 5.10, and the conditions of Theorem 5.11 are satisfied.

Assume there exists $0 < \varepsilon < \frac{1}{2} \inf_{k \neq l} |\omega_k - \omega_l|$ such that $\text{Re} P_S(i\omega) > 0$ for all $\omega \in \Omega_e = \{\omega \in \mathbb{R} \mid \exists k \in \mathcal{I} : |\omega - \omega_k| < \varepsilon\}$. Let $\eta(\cdot), \gamma(\cdot) : \mathbb{R}_+ \to (0, 1]$ be continuous non-increasing functions with the property $\inf_{\omega > 0} \gamma(\omega + \delta_0)/\gamma(\omega) > 0$ for some $0 < \delta_0 < \min \{1, \varepsilon\}$ such that the following hold.

- $\text{Re} P_S(i\omega) \geq \eta(|\omega|)$ for all $\omega \in \Omega_e$.
- $\|B_k^u y\| \geq \gamma(|\omega_k|) \|y\|$ for all $k \in \mathcal{I}$ and $y \in Y$.

Then the controller solves the robust output regulation problem and there exists $M_0 > 0$ such that $\|R(i\omega, A_e)\| \leq M_R(\|\omega\|)$ with $M_R(\cdot) = M_0 \eta(\cdot)^{-1} \gamma(\cdot)^{-2}$. If $\sup_{\omega > 0} M_R(\omega) < \infty$, then the closed-loop system is exponentially stable. More generally, there exist $M_e^* > 0 \geq 1$ such that if (5.5) hold, then for all $x_{e0} \in X_e$ satisfying $A_e x_{e0} + B_e w_{ext}(0) \in X_e$ we have
\[ (5.6) \quad \int_{t_1}^{t+1} \|e(s)\| ds \leq \frac{M_e^*}{M_T(t)} (\|A_e x_{e0} + B_e w_{ext}(0)\| + M_{ext}), \quad t \geq t_0, \]
where $M_T(t)$ is determined by parts (b)–(c) of Theorem 3.7 and $M_{ext}^2 = \|((\omega_k \Pi_{ext}^1(k))^2 + (\omega_k \Pi_{ext}^2(k))^2)_{k \in \mathcal{I}}. In particular, if $\eta(\omega)^{-1} \gamma(\omega)^{-2} = O(\omega^\alpha)$ for some $\alpha > 0$, then (5.6) holds with $M_T(t) = t^{1/\alpha}$.

Proof. Theorem 5.11 shows that the controller solves the robust output regulation problem, and $\|R(i\omega, A_e)\| \leq M_R(\|\omega\|)$ follows from Theorem 3.7 and Remark 3.8. Thus (5.4) holds $M_T(\cdot)$ and for some $M_e, t_0 > 0$. As shown in the proofs of Theorem 5.11 and Lemma 5.13, the conditions of Lemma 5.4 are satisfied whenever $y_{ref}(t)$ and $w_{dist}(t)$ are such that (5.4) and (5.5) hold. If $x_{e0} \in X_e$ is such that $A_e x_{e0} + B_e w_{ext}(0) \in X_e$, then
The admissibility of $C_{eA}$ and \([3.4]\) imply
\[
\int_t^{t+1} \|e(s)\|ds \lesssim \|T_c(t)A_e^{-1}(A_e x_0 + B_e w_ext(0) - q_{ext})\|
\]
\[
\leq \frac{M_e}{M_T(t)} (\|A_e x_0 + B_e w_ext(0)\| + \|q_{ext}\|),
\]
which implies the claim since $\|q_{ext}\|^2 \leq M^2_{ext}$.
\[\square\]

If $C \in \mathcal{L}(X,Y)$ and $C_e \in \mathcal{L}(Z,U)$ in Theorem \(5.14\) then \((5.6)\) can be replaced with a pointwise rate $\|e(t)\| \leq \frac{M_e}{M_T(t)} (\|A_e x_0 + B_e w_ext(0)\| + M_{ext})$ for $t \geq t_0$. If $w_{ext}(0) = 0$ and $B_e \in \mathcal{L}(Z,U)$, then Lemma \(5.5\) gives a sufficient condition for initial states $z_0 \in Z$ that achieve the convergence rate \((5.6)\).

The following result presents necessary conditions for exponential closed-loop stability with controllers satisfying the conditions \((5.1)\), which in turn are necessary for robustness by \(30\), Thm. 13.

**Proposition 5.15.** Assume $(A^S, B^S, C^S, D^S)$ is strongly stable, $\{i\omega_k\}_{k \in \mathcal{I}} \subset \rho(A^S)$, and $(A_e, B_e, C_e, D_e)$ satisfies \((5.1)\). If the closed-loop system is exponentially stable, then $\sup_{k \in \mathcal{I}} \|P_S(i\omega_k)^{-1}\| < \infty$.

**Proof.** It follows from the proof of Lemma \(4.2\) that $B^0_e = \begin{bmatrix} 0 & B_e \end{bmatrix}$ and $C^0_e = [0, C_{eA}]$ are admissible with respect to $A_e$. The proof of Theorem \(3.2\) implies $C^0_e R(i\omega_k, A_e)B^0_e = C_{eA} S_A(i\omega_k)^{-1} B_e$ where $S_A(i\omega_k) = i\omega_k - A_e + B_e P_d(i\omega_k) C_{eA}$ and $P_d(i\omega_k) = P_S(i\omega_k)(I + D_{eA} P_S(i\omega_k))^{-1}$. Since the closed-loop system is exponentially stable, we must have
\[
\sup_{k \in \mathcal{I}} \|C_{eA} S_A(i\omega_k)^{-1} B_e\| < \infty.
\]
Let $y \in Y$ and denote $z = S_A(i\omega_k)^{-1} B_e y \in Z_{B_e}$, which implies $(i\omega_k - A_e)z = B_e(y - P_d(i\omega_k) C_{eA} z)$. The conditions \((5.1)\) show that we must have $y = P_d(i\omega_k) C_{eA} z$. Thus $C_{eA} S_A(i\omega_k)^{-1} B_e y = P_d(i\omega_k)^{-1} y = (P_S(i\omega_k)^{-1} + D_{eA}) y$ for all $y \in Y$, and the claim follows from \((5.7)\). \[\square\]

6. Examples

6.1. A Wave Equation with Boundary Control. We consider a one-dimensional undamped wave equation with boundary control and observation,
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2}(\xi, t) &= w_{\xi\xi}(\xi, t), \quad \xi \in (0, 1) \\
\frac{\partial u}{\partial t}(\xi, 0) &= w_0(\xi), \quad w_t(\xi, 0) = w_1(\xi), \\
u(t) &= -w_2(0, t), \quad w_\xi(1, t) = 0, \\
y(t) &= w_t(0, t).
\end{align*}
\]
The results in \(5.1\) show that \((6.1)\) defines a regular linear system with state $x(t) = (w_\xi(\cdot, t), w_t(\cdot, t))^T$ on $X = L^2(0, 1) \times L^2(0, 1)$. Its transfer function is given by
\[
P(\lambda) = \frac{1 + e^{-2\lambda}}{1 - e^{-2\lambda}}, \quad \lambda \neq i\pi k, \quad k \in \mathbb{Z}
\]
and $D = 1$. In particular, we have $\text{Re} P(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$. We will construct a controller that achieves exponential closed-loop stability and robust output regulation for 1-periodic signals of the form $y_{\text{ref}}(t) = \sum_{k \in \mathbb{Z}} y_{\text{ref}}^k e^{i 2 \pi k t}$ with $(y_{\text{ref}}^k)_k \in \ell^1(\mathbb{C})$. For this we will use a controller based on the transport equation presented in Section 5.2 with $\tau = 1$.

The system (6.1) can be stabilized exponentially with negative output feedback $u(t) = -D_{c2} y(t)$ with $D_{c2} > 0$. For $\lambda \in \mathbb{C}_+$ the transfer function $P_S(\lambda)$ of the stabilized system $(A^S, B^S, C^S, D^S)$ is given by

$$P_S(\lambda) = P(\lambda)(I + D_{c2} P(\lambda))^{-1} = \frac{1 + e^{-2\lambda}}{1 + D_{c2} + (D_{c2} - 1)e^{-2\lambda}}$$

and $\text{Re} P_S(i\omega) = \frac{D_{c2} \cos(\omega)^2}{1 + (D_{c2} - 1)\cos(\omega)^2}$. Now $\text{Re} P_S(i\omega) = 0$ if and only if $\omega = (k + 1/2)\pi$ for some $k \in \mathbb{Z}$. Therefore for any fixed $0 < \varepsilon < \pi/2$ there exists $\eta > 0$ such that $\text{Re} P_S(i\omega) \geq \eta > 0$ for all $\omega \in \Omega_\varepsilon = \{ \omega \in \mathbb{R} \mid \exists k \in \mathbb{Z} : |\omega - 2\pi k| < \varepsilon \}$.

The conditions of Theorem 5.7 are satisfied, and thus the controller in Definition 5.6 solves the robust output regulation problem for all 1-periodic signals of the form $y_{\text{ref}}(t) = \sum_{k \in \mathbb{Z}} y_{\text{ref}}^k e^{i 2 \pi k t}$ with $(y_{\text{ref}}^k)_k \in \ell^1(\mathbb{C})$. For this we will use a controller based on the transport equation presented in Section 5.2 with $\tau = 1$.

Now $\text{Re} P_S(i\omega) = 0$ if and only if $\omega = (k + 1/2)\pi$ for some $k \in \mathbb{Z}$. For this we will use a controller based on the transport equation presented in Section 5.2 with $\tau = 1$.

In this example we consider one-dimensional wave equation, now with distributed control and observation,

(6.2a) $w_{tt}(\xi, t) = w_{\xi\xi}(\xi, t) + b(\xi) u(t), \quad \xi \in (0, 1)$

(6.2b) $w(0, t) = 0, \quad w(1, t) = 0,$

(6.2c) $w_\xi(0, t) = w_\xi(1, t) = w(\xi, t) = w_\xi(\xi, t) = 0,$

(6.2d) $y(t) = \int_0^1 b(\xi) w_\xi(\xi, t) d\xi,$

where $b(\xi) = 2(1 - \xi)$. Equation (6.2) determines a passive linear system with state $x(t) = (w(\cdot, t), w_\xi(\cdot, t))^T$ on $X = H^1_0((0, 1) \cap L^2(0, 1)$ with bounded input and output operators satisfying $C = B^*$. The transfer function $P(\lambda)$ can be computed as in [13, Sec. II]. Negative output feedback $u(t) = -D_{c2} y(t)$ stabilizes the system strongly for any $D_{c2} > 0$, but the system is not exponentially stabilizable. However, the semigroup generated by $A^S$ is polynomially
stable since \( \int_0^1 b(\xi) \sin(k\pi \xi) d\xi = \frac{2}{k\pi} \) implies \( \|R(i\omega, A - BDcC)\| = O(\omega^2) \) for \( Dc > 0 \) by [THM Thm. 1].

Our aim is to design a controller to achieve robust output tracking of \( y_{\text{ref}}(t) = \sin(\pi t) + \frac{1}{4} \cos(2\pi t) \). The frequencies of the signal \( y_{\text{ref}}(t) \) are \( \{\pm \pi, \pm 2\pi\} \). Due to robustness, the controller will be able to track any reference signal with these frequencies. Since \( \text{dim} \ Y = p = 1 \), we can construct a passive feedback controller in Definition 5.1 on \( Z = \mathbb{R}^4 \) by choosing

\[
A_c = \text{blockdiag}(J_1, J_2), \quad J_1 = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 2\pi \\ -2\pi & 0 \end{bmatrix},
\]

\( C_c = [k_1, 0, k_2, 0], \ B_c = C_c^*, \) and \( D_c > 0 \). The values of \( k_1, k_2 \in \mathbb{R} \) and \( D_c \) affect the stability properties of the closed-loop system. In this example we choose \( k_1 = k_2 = 3 \) and \( D_c = 35 \). By construction the controller is robust with respect to perturbations in the system provided that the strong stability of the closed-loop is preserved. Since \( B \) and \( C \) are bounded operators, Proposition 5.3 shows that \( \|e(t)\| \to 0 \) as \( t \to \infty \) for all initial states \( x_0 \in D(A) \) and \( z_0 \in Z \).

For simulations, the system (6.2) was approximated with the Finite Element Method with \( N = 24 \) points on \([0, 1] \times [0, 1] \). Figure 1 depicts the behaviour of the error \( e(t) \) and the integrals \( \int_t^{t+1} \|e(s)\| ds \) for \( 0 \leq t \leq 24 \) for initial states \( x_0(\xi) = \xi(1 - \xi)(2 - 5\xi) \) and \( z_0 = 0 \). Figure 1 also plots the solution \( w(\xi, t) \) of the controlled wave equation for \( 0 \leq t \leq 6 \).

**Figure 1.** The solution \( w(\xi, t) \) of controlled wave equation (left) and \( e(t) \) (top right) and \( \int_t^{t+1} \|e(s)\| ds \) (bottom right).

### 6.3. Periodic Output Tracking for a Heat Equation.

In the final example we consider a two-dimensional boundary controlled heat equation on \( \Omega = [0, 1] \times [0, 1] \)

\[
\begin{align*}
(6.3a) & \quad x_t(x, t) + \Delta x(x, t), \quad x(x, 0) = x_0(x) \\
(6.3b) & \quad \frac{\partial x}{\partial n}(x, t)|_{\Gamma_1} = u(t), \quad \frac{\partial x}{\partial n}(x, t)|_{\Gamma_2} = \text{dist}(t), \quad \frac{\partial x}{\partial n}(x, t)|_{\Gamma_0} = 0 \\
(6.3c) & \quad y(t) = \int_{\Gamma_1} x(x, t) d\xi,
\end{align*}
\]
where the parts $\Gamma_0$, $\Gamma_1$, and $\Gamma_2$ of the boundary $\partial \Omega$ are defined so that $\Gamma_1 = \{ \xi = (0, \xi_2) \mid 0 \leq \xi_2 \leq 1 \}$, $\Gamma_2 = \{ \xi = (\xi_1, 1) \mid 1/2 \leq \xi_1 \leq 1 \}$, $\Gamma_0 = \partial \Omega \setminus (\Gamma_1 \cup \Gamma_2)$. By [11, Cor. 2] the heat equation defines a regular linear system with state $x(t) = x(\cdot, t)$ on $X = L^2(\Omega)$ with feedthrough $D = 0$. The system is passive,

$$P(\lambda) = \frac{\coth(\sqrt{\lambda})}{\sqrt{\lambda}}, \quad \lambda \in \mathbb{C}_+ \setminus \{0\},$$

and $|P(i\omega)^{-1}| = O(|\sqrt{\omega}|)$ for $\omega \in \mathbb{R}$ with large $|\omega|$. The system (6.3) is exponentially stabilizable with feedback $u(t) = -Dz(t)$ for any $Dz > 0$.

We will design an infinite-dimensional dynamic feedback controller that achieves robust output tracking of the 2-periodic nonsmooth reference signal $y_{\text{ref}}(t)$ in Figure 2 and rejects a suitable class of 2-periodic disturbance signals $w_{\text{dist}}(t)$. The frequencies of the signals are $\{\omega_k\}_{k \in \mathbb{Z}}$ with $\omega_k = \pi k$ for $k \in \mathbb{Z}$, and the Fourier coefficients of $y_{\text{ref}}(t)$ are such that $|y_{\text{ref}}|^2 = O(|k|^{-3})$.

We can construct the controller as in Definition 5.10 by choosing $Z = \ell^2(\mathbb{C})$, $A_c = \text{diag}(i\omega_k)_{k \in \mathbb{Z}}$, $B_c = c((1 + |k|)^{-1/2-\epsilon})_{k \in \mathbb{Z}}$ for some small $\epsilon > 0$, $C_c = B_c^*$, and $D_c = 0$. The parameters $\epsilon > 0$, $D_c = Dz > 0$ and $c > 0$ affect the stability properties of the closed-loop system. Proposition 5.15 shows that since $P(\omega_k) \to 0$ as $|k| \to \infty$, the closed-loop system can not be stabilized exponentially. However, by Theorem 3.1 the closed-loop system consisting of (2.1) and the controller with the above choices of parameters is polynomially stable. Indeed, since $\text{Re} \, P_S(i\omega) = O(|\omega|^{-1/2})$ and $|B_{ck}^{-1}| = (1 + |k|)^{1/2+\epsilon} = O(|\omega_k|^{1/2+\epsilon})$, we have from Theorem 5.14 that $\|R(i\omega, A_c)\| = O(|\omega|^{3/2+2\epsilon})$ and there exist $M_r, t_0 > 0$ such that

$$\|T_e(t)x_0\| \leq \frac{M_e}{\lambda^{1/\alpha}} \|Ae x_0\|, \quad x_0 \in D(A_c), \quad t \geq t_0,$$

where $\alpha = 3/2 + 2\epsilon$.

To verify that the controller is capable of regulating the given signals $y_{\text{ref}}(t)$ and $w_{\text{dist}}(t)$, we need to show that the conditions (5.1) are satisfied. The norms $\|R(i\omega, A)B\|$ and $\|R(i\omega, A)B_d\|$ are uniformly bounded for large $|\omega|$. Lemma 5.12 and $(B_{ck}^*)k \in \ell^2(\mathbb{C})$ imply that it is sufficient to show

$$\langle |B_{ck}|^{-1}P_S(i\omega_k)^{-1}(|y_{\text{ref}}^k| + |P_d(i\omega_k)||w_{\text{dist}}^k)| \rangle_{k \in \mathbb{Z}} \in \ell^2(\mathbb{C}).$$

The eigenfunction expansion of $A$ can be used to show $|P_d(i\omega)| = O(|\omega|^{-1})$, and since $|P(i\omega)|^{-1} = O(|\omega|^{-1/2})$, the above condition is satisfied for all $y_{\text{ref}}(t)$ and $w_{\text{dist}}(t)$ with

$$|k|^{1+\epsilon} |y_{\text{ref}}^k| \in \ell^2(\mathbb{C}) \quad \text{and} \quad |k|^\epsilon |w_{\text{dist}}^k| \in \ell^2(\mathbb{C}).$$

The condition on $(y_{\text{ref}}^k)_k$ in particular holds for $y_{\text{ref}}(t)$ in Figure 2.

Finally, we can study the rational rates of decay of $\|e(t)\|$ using Theorem 5.14. The conditions in (5.5) are both satisfied if

$$|k|^{2+\epsilon} |y_{\text{ref}}^k| \in \ell^2(\mathbb{C}) \quad \text{and} \quad |k|^{1+\epsilon} |w_{\text{dist}}^k| \in \ell^2(\mathbb{C}).$$
The first condition is satisfied for $y_{\text{ref}}(t)$ in Figure 2 whenever $0 < \varepsilon < 1/2$. Then for all $x_{t_0} \in X_e$ such that $A_c x_{t_0} + B_c v_{t_0} \in X_e$ we have

$$
(6.4) \quad \int_t^{t+1} \|e(s)\| ds \leq \frac{M \varepsilon}{\|u\|_\alpha} (\|A_c x_{t_0} + B_c w_{\text{ext}}(0)\| + M_{\text{ext}}), \quad t \geq t_0
$$

where $\alpha = 3/2 + 2\varepsilon$, and a direct estimates shows that for any fixed $\varepsilon > 0$

$$
M_{\text{ext}} \lesssim \|\|k\|^{2+\varepsilon}|y_{\text{ref}}| + |k|^{1+\varepsilon}|w_{\text{dist}}(0)|\|_{\ell^2}.
$$

For disturbance signals satisfying $w_{\text{dist}}(0) = 0$, Lemma 5.5 shows that (6.4) holds whenever $x_0 \in D(A)$ and $z_0 \in D(A_c)$ are such that $C_c z_0 = D_c (C_A x_0 - y_{\text{ref}}(0))$. Moreover, by Proposition 5.13 the regulation error satisfies $\|e(t)\| \rightarrow 0$ as $t \rightarrow \infty$ for all such initial states.

For simulations the solution of the controlled heat equation (6.3) was approximated with Finite Differences using a $N \times N$ grid with $N = 20$. The free parameters of the controller were chosen as $\varepsilon = 1/10$, $c = 8$, and $D_c = 15$. The state of the controller was approximated by truncating the infinite matrix $A_c$ to a $31 \times 31$ diagonal matrix with eigenvalues $\{i \pi k\}_{|k| \leq N_S}$ for $N_S = 15$. Figure 2 depicts the output of the controlled heat equation for $2 \leq t \leq 8$ and the behaviour of the error integrals for $0 \leq t \leq 10$ for the initial state $x_0(\xi_1, \xi_2) = -(1 + \xi_1^2/4 - \xi_1^2/6)(\cos(\pi \xi_2)/10 + 2)$ such that $x_0 \in D(A)$ and an initial state $z_0 \in D(A_c)$ satisfying $C_c z_0 = D_c (C_A x_0 - y_{\text{ref}}(0))$.

![Figure 2](image_url)

Figure 2. The reference $y_{\text{ref}}(t)$ (left, gray), the output $y(t)$ (left, blue), and $\int_t^{t+1} \|e(s)\| ds$ (right) for the heat equation.

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**Appendix A.**

**Lemma A.1.** Let $X$ be a Hilbert space and let $T, S \in \mathcal{L}(X)$ be such that Re $T \geq c \geq 0$ and Re $S \geq d \geq 0$.

(a) If $T$ is boundedly invertible, then Re $T^{-1} \geq c\|T\|^{-2}$. If $c > 0$, then $T^{-1}$ exists and $\|T^{-1}\| \leq \frac{1}{c}$.

(b) If $c > 0$ or $d > 0$, then $\|T(I + ST)^{-1}\| \leq \frac{\|T\|^2}{c + d\|T\|^2}$. If $c > 0$ and $d \geq 0$, then

$$
\text{Re} T(I + ST)^{-1} \geq \frac{c^3 + c^2 d\|T\|^2}{\|T\|^2(1 + c\|S\|)^2}.
$$
(c) If $T$ is invertible, $c \geq 0$, and $d > 0$, then $\Re T(I + ST)^{-1} \geq d(\|T^{-1}\| + \|S\|)^{-2}$.

(d) If $c \geq 0$ and $S \geq 0$, then $I + ST$ and $I + TS$ are boundedly invertible, and $\Re T(I + ST)^{-1} \geq 0$.

Proof. (a): The proof of the first part is elementary and latter claims follow from the estimate $\|Tx\|/\|x\| \geq \|Tx, x\| \geq \Re(Tx, x) \geq \|x\|^2$ for $x \in X$.

(b): If $c > 0$, we can use part (a) and $T(I + ST)^{-1} = (I^{-1} + S)^{-1}$. If $d > 0$, then an argument similar to the one used in [14, Lem. 2.3] shows that $\|T(I + ST)^{-1}\| \leq \frac{1}{d}$.

(c): The claim follows from $T(I + ST)^{-1} = (I^{-1} + S)^{-1}$ and part (a).

(d): Assume $\Re T \geq 0$ and $S \geq 0$. The invertibility of $I + ST$ implies that also $I + TS$ is invertible. It is straightforward to show that the range of $I + ST$ is dense in $X$. Thus it suffices to show that $I + ST$ is lower bounded. If $\{x_n\}$ is an admissible output feedback operator by Lemma A.1(d). Since $\sup_{\lambda \in \mathbb{C}_+} \|P(\lambda)\| < \infty$, then $\sup_{\lambda \in \mathbb{C}_+} \|T(I + ST)^{-1}\| < \infty$.

Proof. The property that $-1 \in \rho(D_cP(\lambda))$ for all $\lambda \in \mathbb{C}_+$ follows from Lemma A.2(d). Assume $\sup_{\lambda \in \mathbb{C}_+} \|P(\lambda)\| < \infty$. In order to show that $(I + D_cP(\lambda))^{-1}$ are uniformly bounded for $\lambda \in \mathbb{C}_+$, it is sufficient to show that there exists a constant $r > 0$ such that $\|\lambda(I + D_cP(\lambda))u\| \leq r\|u\|$ for all $\lambda \in \mathbb{C}_+$ and $u \in U$. If no such $r > 0$ exists, we can choose sequences $(\lambda_n)_n \subset \mathbb{C}_+$ and $(u_n)_n \subset U$ with $\|u_n\| = 1$ for all $n \in \mathbb{N}$ such that $\|\lambda_n(I + D_cP(\lambda))u_n\| \to 0$ as $n \to \infty$. Then

$$0 \leftarrow \Re\{(I + D_cP(\lambda_n))u_n, P(\lambda_n)u_n\} \geq \|D_c^{1/2}P(\lambda_n)u_n\|^2,$$

which implies $\|D_cP(\lambda_n)u_n\| \to 0$ as $n \to \infty$. However, since $\|u_n\| = 1$, we would then have $\|(I + D_cP(\lambda_n))u_n\| \not\to 0$ as $n \to \infty$, which is a contradiction.

The last lemma concerns output feedback for passive systems. Several additional results on this topic can be found in [18].

Lemma A.3. Assume $(A, B, C, D)$ is a passive regular linear system and $\sigma(A) \subset \mathbb{C}_-$ if $D_c \geq 0$, the system $(A-BD_cQ_1C_A, BQ_2, Q_1C_A, Q_1D)$ with $Q_1 = (I + DD_c)^{-1}$ and $Q_2 = (I + D_cD_c)^{-1}$ is regular, passive, and strongly stable in such a way that $\sigma(A-BD_cQ_1C_A) \subset \mathbb{C}_-$. Then $A$ generates an exponentially stable semigroup, then the same is true for $A-BD_cQ_1C_A$.

Proof. The system $(A-BD_cQ_1C_A, BQ_2, Q_1C_A, Q_1D)$ is obtained from (1.1) with output feedback $u(t) = -D_cy(t)$. The regularity follows from [45], since $-D_c$ is an admissible output feedback operator by Lemma A.2(d). Since $D_c \geq 0$, it is straightforward to verify that $(A-BD_cQ_1C_A, BQ_2, Q_1C_A, Q_1D)$ is passive. In particular $A-BD_cQ_1C_A$ generates a contraction semigroup,
and the strong stability of the semigroup follows from the Arendt–Batty–Lyubich–Vũ Theorem \[4, 27\] once we have shown

and the strong stability of the semigroup follows from the Arendt–Batty–Lyubich–Vũ Theorem \[4, 27\] once we have shown

Since \(X\) is boundedly invertible by Lemma A.1(d). Using Lemma 2.1 we therefore see that \(\lambda \in \rho(A - BD_dQ_1C_A)\) and

\[
R(\lambda, A - BD_dQ_1C_A) = R(\lambda, A) - R(\lambda, A)B(I + D_cP(\lambda))^{-1}D_cC_A R(\lambda, A).
\]

Since \(\lambda \in \mathbb{C}_+\) was arbitrary, we have \(\sigma(A - BD_dQ_1C_A) \subset \mathbb{C}_-\). If \(A\) generates an exponentially stable semigroup, then \(\sup_{\lambda \in \mathbb{C}_-}\|\lambda + D_cP(\lambda)\| < \infty\) by Lemma A.2 and the regularity and exponential stability of \((A, B, C, D)\) imply \(\sup_{\lambda \in \mathbb{C}_-}\|R(\lambda, A - BD_dQ_1C_A)\| < \infty\). Thus the semigroup generated by \(A - BD_dQ_1C_A\) is exponentially stable.

\[\square\]

**Proof of Lemma 2.1.** Let \(\lambda \in \rho(A)\) be such that \(Q^{-1} + C_A R(\lambda, A)B\) has a bounded inverse. Denote \(R_\lambda = R(\lambda, A)\) and \(R(\lambda) = R_\lambda - R_\lambda B(Q^{-1} + C_AR_\lambda B)^{-1}C_AR_\lambda\). If \(x \in X\), then \(R(\lambda)x \in X_B\) and a computation on \(X_{-1}\) shows

\[
(\lambda - A + BQC_A)R(\lambda)x = x + B\left[Q - (I + QC_AR_\lambda B)(Q^{-1} + C_AR_\lambda B)^{-1}\right]C_AR_\lambda x = x \in X.
\]

Thus \(R(\lambda)x \in D(A - BQC_A)\) and \((\lambda - A + BQC_A) R(\lambda) = I\). On the other hand, if \(x \in D(A - BQC_A)\), then \(x \in X_B\) and we can again compute on \(X_{-1}\) (considering \(R(\lambda)\) as an operator \(R(\lambda) : X \to \mathcal{R}(B) \to X\))

\[
R(\lambda)(\lambda - A + BQC_A)x = x + R_\lambda B\left[Q - (Q^{-1} + C_AR_\lambda B)^{-1}(I + C_AR_\lambda BQ)\right]C_AR_\lambda x = x.
\]

Since \(x \in D(A - BQC_A)\) was arbitrary, this completes the proof.

\[\square\]

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