SOME GLUEING FORMULAS FOR SPIN POLYNOMIALS
AND A PROOF OF THE VAN DE VEN CONJECTURE.*

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Abstract. We deduce some formulas for the spin invariants of the connected sum of an arbitrary 4-manifold $X$, $b_2^+(X) > 0$ with $\overline{\mathbb{CP}^2}$ in terms of the spin invariants of $X$ and apply this to prove the Van de Ven conjecture.

§0. Introduction

In this article we present some formulas for the spin-polynomial of the connected sum of an arbitrary oriented smooth simply connected four-manifold $X$ with $b_2^+(X) > 0$ with the orientation reversed complex projective plane. These formulas are similar to ones for Donaldson’s polynomials presented by Donaldson ([D]) and Morgan-Mrowka ([MM]) in the sense that they do not involve evaluating $\nu$-classes for the polynomial of $X$. One can show that these formulas also hold in the case of connected sums with arbitrary negative definite manifolds. However the case of $\overline{\mathbb{CP}^2}$ is easier from the technical point of view.

As an application one has a proof of the Van de Ven conjecture [VdV]:

**Conjecture.** The Kodaira dimension of a simply connected algebraic surface is an invariant of its smooth structure.

This has been proved to be true for all cases except possibly nonminimal surfaces of general type (cf. [FM], [K], [OVdV], [P], [PT], [Q1], [Q2]). Therefore it suffices to prove that

No surface of general type can be diffeomorphic to a rational one.

The proof is a reduction to the case of minimal surfaces of general type, which already was treated in [PT]. This reduction uses the glueing formulas and results of [FM] describing diffeomorphisms of rational surfaces. In [FQ1] a different approach to the Van de Ven conjecture is announced. The authors use a glueing formula for Donaldson polynomials (and predict our approach) and also explicitly formulate the result on diffeomorphism of rational surfaces which we use.

§1. Connected sum formulas.

We shall start with some notations.

Let $X$ be a Riemannian, simply-connected 4-manifold with a metric $g$ and $E \to X$ some complex hermitian vector bundle of rank 2 with chern classes $c_1(E), c_2(E)$.

*This is the translation of an article submitted to Izvestija of Russian Academy of Sciences, Ser.Math. While this translation was in progress there appeared a paper [FQ2] with the proofs of the results announced in [FQ1].
This vector bundle as well as the corresponding principal $U(2)$-bundle $P$ is uniquely determined by its Chern classes. Denote also by $P_{ad}$ the $PU(2)$-principal bundle $P_{ad} = P/S^1$, and by $adE$ the corresponding vector bundle.

We shall consider the space $\mathcal{A}_E$ of $U(2)$-connections on the bundle $E$ together with the action of the gauge group $\mathcal{G}_{U(2)}$ and the space of orbits of this action $\mathcal{B}_E$. The space of $SO(3)$-connections for the bundle $adE$, the gauge group, and the orbit space, we denote by $\mathcal{A}_{adE}, \mathcal{G}_{SO(3)} \mathcal{B}_{adE}$ resp. We write $\mathcal{A}_{adE}^* \subset \mathcal{A}_{adE}$ and $\mathcal{B}_{adE}^* \subset \mathcal{B}_{adE}$ for the set of irreducible connections on $adE$ and the orbits of irreducible connections.

We shall also need the space of orbits of framed connections $\widetilde{\mathcal{B}}_{adE}$, that is the orbit space of the space of connections by the subgroup $G_0$ of the gauge group acting trivially on the fibre over some fixed point $p$.

Let
\[ \mathcal{A}_\lambda = \{ A \in \mathcal{A}_E \mid \frac{1}{2} tr F_A = \lambda \} \]
for some closed differential 2-form $\lambda$ on $X$. There is an isomorphism:
\[ \mathcal{A}_\lambda / \mathcal{G}_{U(2)} = \mathcal{A}_{adE} / \mathcal{G}_{SO(3)} = PU(2)). \]  

Let $\mathcal{B}$ be some $Spin^C$-structure on the manifold $X$. This means that one fixes an integral lift $\mathcal{B}$ of the second Stiefel-Whitney class $w_2(X)$ of $X$, and this lift defines a pair of complex hermitian rank 2 vector bundles $W^+, W^-$ subject to the following conditions:
\[ T^*_C X = Hom_C(W^+, W^-), \ c_1(\Lambda^2 W^\pm) = C. \]
A choice of a connection $\nabla$ on the line bundle $\Lambda^2 W^+$ gives rise to a Dirac operator on $X$:
\[ D^{C,\nabla} : \Gamma(W^+) \to \Gamma(W^-). \]

Coupling it with connections $a \in \mathcal{A}_E$ then gives a family of Fredholm operators over $\mathcal{A}_E$:
\[ D^{C,\nabla}_a : \Gamma(W^+ \otimes E) \to \Gamma(W^- \otimes E), \]
which is equivariant with respect to the gauge group action. Using the isomorphism (1.1) we consider it as a family over $\mathcal{B}_{adE}$.

We shall always suppose that
\[ \text{ind}(D^{C,\nabla}_a) = \dim \ker (D^{C,\nabla}_a) - \dim \coker (D^{C,\nabla}_a) \leq 0. \]

Let $S(\Gamma(W^+ \otimes E))$ denote the unit sphere in the Hilbert space $\Gamma(W^+ \otimes E)$, and $\mathbb{P}(\Gamma(W^+ \otimes E))$ the projectivisation of this space. There is a tautological bundle $\mathcal{O}_{\mathbb{P}(\Gamma(W^+ \otimes E))}(1)$ over this projective space, and the coupled Dirac operator can be interpreted as a section of the bundle
\[ \Gamma(W^- \otimes E) \otimes \mathcal{O}_{\mathbb{P}(\Gamma(W^+ \otimes E))}(1). \]

The space
\[ \mathcal{B} = \mathcal{A} \times \mathbb{S}(\Gamma(W^+ \otimes E)) \]

satisfies the condition (1.2).
is the base of the following Hilbert bundle

\[ \mathcal{H}_C = A_\lambda \times \mathcal{S}_{U(2)} \text{ Tot}, \]

where \( \text{Tot} \) stands for the total space of the bundle \( \Gamma(W^- \otimes E) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{S} \otimes E)}(1) \). One can see that the family of coupled Dirac operators gives a section \( s_D \) of the bundle \( \mathcal{H}_C \to \mathcal{P} \) (details in 1.1.26, [PT]). In the same way one gets a section \( s_{\text{asd}} \) of the Hilbert bundle \( \mathcal{H}_+ \) associated to the principal bundle \( A_{adE}^* \to B_{adE}^* \) and a natural representation of the gauge group \( \mathcal{G}_{adE} \) in the Hilbert space \( \Omega_2^+(adE) \).

**Definition.** The moduli space of pairs \( \mathcal{M} \mathcal{P}^{g,\nabla}(X, p_1(E), C + c_1(E)) \) is the space of zeroes of the section \( s_D \oplus \pi^* s_{\text{asd}} \) where \( \pi : \mathcal{P}^* \to E^* \) is the natural projection.

Using (1.1) one can give a point of \( \mathcal{M} \mathcal{P}^{g,\nabla}(X, p_1(E), C + c_1(E)) \) as a pair \((a, \langle s \rangle), a \in \mathcal{M}_{\text{asd}}^g(X, p_1(E), w_2(E)) \) and \( \langle s \rangle \) is a point of the projective space \( \mathbb{P}(\Gamma(W^+ \otimes E)) \) given by a complex line spanned by a nonzero vector spinor \( s \).

There is an exact sequence which describes the tangent bundle of moduli of pairs at a point \((a, \langle s \rangle)\):

\[ 0 \to T_{a,s} \mathcal{M} \mathcal{P}^{g,\nabla} \to (\ker D_a)/\langle s \rangle \oplus H^1_0(adE) \xrightarrow{i} \text{coker} D_a \]

\[ i(s', a) = \omega(a \ast s), \]

(here \( \ast \) is a combination of spinor multiplication and multiplication of a matrix by a vector, \( \omega \) is the orthogonal projection to the space of harmonic vector spinors \( \text{coker} D_a \)). Therefore \( T_{a,s} \mathcal{M} \mathcal{P}^{g,\nabla} \) is the kernel of the linearisation of our asd- and Dirac- equations and \( \text{coker} i \) is its cokernel. One can globalise this exact sequence to a fibre of \( \pi \) as an exact sequence of bundles over \( \mathbb{P}(\ker D_a^{C,\nabla}) \):

\[ 0 \to T \mathcal{M} \mathcal{P}^{g,\nabla}_{|\pi^{-1} a} \to T \mathbb{P}(\ker D_a^{C,\nabla}) \oplus H^1_0(adE) \xrightarrow{i} \text{coker} D_a \otimes \mathcal{O}_{\mathbb{P}(\ker D_a^{C,\nabla})}(1) \to 0. \]

An orientation of the moduli space of pairs is defined by the orientation of the moduli space of asd connection via the exact sequence (1.3).

A model for a neighborhood of the point \((a, \langle s \rangle)\) in the space \( \mathcal{M} \mathcal{P}^{g,\nabla}(X) \) is given by the zero set of some smooth section of the bundle

\[ (\text{coker} i) \oplus H^2_0(adE) \otimes \mathcal{O}_{\mathbb{P}(\ker D_a^{C,\nabla})} \]

\[ \downarrow \]

\[ \mathbb{P}(\ker D_a^{C,\nabla}) \times H^1_0(adE) \]

(cf. [PT], ch.1).

This moduli space is a source of various invariants of the smooth structure of \( X \) as described in [PT] (numerical invariants) and [T] (polynomial invariants). It is a smooth manifold provided \( b_2^+(X) > 0 \) and the metric \( g \) and the connection \( \nabla \) both are generic. For \( b_2^+(X) \) we use polynomial invariants

\[ \gamma_{p_1, C+c_1(E)}^X(\Sigma_i) = \#(\cap V_{\Sigma_i} \cap \mathcal{P} \mathcal{M}(X, p_1(E), C + c_1(E))), \]

where \( \Sigma_i \in H_2(X) \), and the \( V_{\Sigma_i} \) are codimension 2 submanifolds of \( \mathcal{P} \), defined as the zero sets of certain sections \( s_i \) of certain complex line bundles \( \mathcal{L} \) over \( \mathcal{P} \) (cf.
Such a section is defined once we have chosen a smooth Riemann surface representing the class $\Sigma$. We shall use the same notation for these representatives: $\Sigma_i$. If $b_2^+ > 1$ then these polynomials do not depend on the choice of the metric $g$ and the connection $\nabla$ on the line bundle $\Lambda^2 W^+$. If $b^+ = 1$ and $p_1 \geq -7$ or $p_1 = -8, w_2 = (c_1) \mod 2 \neq 0$ then there is a chamber structure on the space of periods $P(X)$ of metrics and the polynomials depends only on the chamber, not on the metric itself (cf. [BP]). We shall use the notation $\gamma_{p_1, C+C_1(E), C}(\Sigma_i)$ to show the dependence on the chamber $C$. In either case as a function of the cohomology classes $c_2(E), c_1(E), C$ the polynomials depend in fact only on the combinations $C + c_1(E), p_1(\text{ad}E) = c_1(E)^2 - 4c_2(E)$ of these classes.

The connected sum $X \# \overline{CP^2}$ of a manifold $X$ and a projective plane with reversed orientation $\overline{CP^2}$ can be viewed as a union

$$(X \setminus D_1) \cup (S^3 \times [-t/2, t/2]) \cup (\overline{CP^2} \setminus D_2)$$

where $D_1 \subset X, D_2 \subset \overline{CP^2}$ are small disks around some points $p \in X, q \in \overline{CP^2}$, $\partial D_i = S^3$. Define a metric $g_t$ on $X \# \overline{CP^2}$ which is generic when restricted to $X - D_1$ and $\overline{CP^2} - D_2$ and a product of $S^3$ with radius $r$ and a standard metric on the interval $[-t/2, t/2]$ for some interior of the cylinder.

When $t$ tends to infinity one gets two manifolds - $X$ and $\overline{CP^2}$ joined by an “infinite” cylinder or, in a conformally equivalent metric, two open manifolds $(X \setminus p)$ and $(\overline{CP^2} \setminus q)$.

Let $e$ be a generator of $H^2(\overline{CP^2})$. Fix some $U(2)$ vector bundle $E$ and $Spin^C$ structure on the connected sum $X \# \overline{CP^2}$, such that $(C + c_1(E))_{\overline{CP^2}} = ke$ with $-3 < k < 3$.

It is known that if one takes a sequence of metrics $g_{t_i}, t_i \to \infty$ and a sequence of connections $a_i$ on $E$ such that $a_i$ is antiselfdual (asd) with respect to $g_{t_i}$, then there exist asd-connections $a_X, a_{\overline{CP^2}}$ on $X, \overline{CP^2}$ resp., and a finite collection of points $x_1, ..., x_m$ such that after gauge transformation some subsequence of $a_i$ converges weakly over compact subsets of $X \cup \overline{CP^2} - \{p, q, x_1, ..., x_m\}$ to $a_X, a_{\overline{CP^2}}$ and the curvature densities $|F_{a_i}|$ converge to $F_{a_X} + F_{a_{\overline{CP^2}}} + \sum_i 8\pi^2 \delta_{x_i}$ (cf. ch. 7 of [DK]).

The following lemma describes the case when there is a nontrivial element of the kernel of the coupled Dirac operator $D_{a_i}$ (positive harmonic vector-spinor) for any asd-connection $a_i$.

**Lemma 1.1.** Take $E, C, a_i$ as above. If for any asd-connection $a_i$ with $i >> 0$ there exists a nonzero positive harmonic vector-spinor $s_i$ (which we shall suppose to be of unit norm), then $a_X$ also has nonzero positive harmonic vector-spinor $s_X$. Moreover a subsequence of $s_i$ converges to $s_X$ on any compact subset of $X - p$ in $C^\infty$ topology.

**Proof.** We shall use the norm $L^{8/3}$ on the space $\Gamma(W^+ \otimes E)$ of positive vector spinors which has following property: if $g' = e^{2f} g$ is a conformal change of a metric then

$$s \in \ker(D^g_a) \iff s' = e^{-\frac{3}{2}f}s \in \ker(D^{g'}_a)$$

and

$$\|s\|_{L^{8/3}(X, g)} = \|s'\|_{L^{8/3}(X, g')}. $$

This gives a possibility to derive estimates in an arbitrary conformal model of the connected sum, identifying vector spinors as above.
Let $S_t$ be a scalar curvature of $X \# \mathbb{CP}^2, g_t$. It is easy to see that it is uniformly bounded from above for all $t$. Let $\nabla_{a_i}$ be a connection on the bundle $W^+ \otimes E$ built from $a_i$ and $\nabla$. It follows from the Weitzenböck formula

$$(\mathcal{D}_{a_i})^* \mathcal{D}_{a_i} s_i = \nabla^*_{a_i} \nabla_{a_i} s_i - F_{a_i}^+ s_i + 1/4 S_t \cdot s_i + F_{a} s_i,$$

that if $\sigma$ the maximum of the norm of the sum of curvatures $1/4 S_t + F_{a}$ then there is the pointwise inequality:

$$(\nabla^*_{a_i} \nabla_{a_i} s_i, s_i) \leq (\sigma)(s_i, s_i).$$

Being combined with the standard estimate of the l.h.s. through Laplasian $\Delta$ of the function $|s_i|$

$$(\nabla^*_{a_i} \nabla_{a_i} s_i, s_i) \geq |s_i| \cdot \Delta(|s_i|),$$

which follows from the Kato inequality, it gives then

$$\Delta(|s_i|) \leq \sigma|s_i| \text{ and } \|\Delta(|s_i|)\|_{L^{s/3}} \leq \sigma \|s_i\|_{L^{s/3}}.$$

Since $\Delta$ is an elliptic operator one has an a priori inequality for functions on $X \# \mathbb{CP}^2$:

$$\|s_i\|_{L^{s/3}} \leq \text{const} \|\Delta s_i\|_{L^{s/3}} + \text{const} \|s_i\|_{L^{s/3}}$$

with const independent of $t$ and $a_i$. The last two estimates give

$$\|s_i\|_{L^{s/3}} \leq \text{const}(1 + \sigma) \|s_i\|_{L^{s/3}}.$$

Applying the Sobolev embedding theorem for functions one gets

$$\|s_i\|_{L^s} \leq \text{const}' \|s_i\|_{L^{s/3}} \leq \text{const}'' \|s_i\|_{L^{s/3}}$$

with const'' independent of $t, a_i$. With this estimate in mind we now look how far the truncated spinor is from being harmonic.

There are some fixed $\theta_1, \theta_2$ with $-1/2 < \theta_1, \theta_2 < 1/2$ such that for a subsequence of $a_i$ restricted to the subcylinder $S^3 \times [\theta_1 t, \theta_2 t]$ the curvature density $|F_{a_i}|$ is small. Thus we can apply the theorem of K. Uhlenbeck on the existence of local Coulomb gauge with small connection matrix over the subcylinder.

Now let $\psi_i$ be some cut-off function with support on $X \cup S^3 \times [-t/2, \theta_2 t]$, $\text{supp}(d\psi_i) \subset S^3 \times [\theta_1 t, \theta_2 t]$ and $\|d\psi_i\|_{L^4} \leq \epsilon(t_i)$, where $\epsilon(t_i) \rightarrow 0$ when $t_i \rightarrow \infty$. We assume for simplicity of notations that there is no bubbling points on $X$ - the modification to this case consists of cutting off also in small neighborhood of the bubbling points and has been treated in [DKr, Lemma 7.1.24]. Therefore we can assume that there is an estimate for the supremum of $(a_i - a_X)$:

$$\sup(|a_i - a_X|_{\text{supp}\psi_i}) < \epsilon'(t), \quad \epsilon'(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$  

This provides the estimates

$$\|\mathcal{D}_{a_X} \psi_i s_i\|_{L^{s/3}(X)} \leq$$

$$\leq \|\mathcal{D}_{a_i} s_i\|_{L^{s/3}(X)} + \|d\psi_i s_i\|_{L^{s/3}(X)} + \sup |a_i - a_X| \cdot \|s_i\|_{L^{s/3}(X)} \leq$$

$$\leq \text{const}_1 \|\Delta s_i\|_{L^{s/3}(X)} + \text{const}_2 \|s_i\|_{L^{s/3}(X)} + \text{const}_3 \|s_i\|_{L^{s/3}(X)}$$

with constant $\text{const}_1, \text{const}_2, \text{const}_3$ dependent of $t$ and $a_i$. The last two estimates give

$$\|s_i\|_{L^{s/3}(X)} \leq \text{const}(1 + \sigma) \|s_i\|_{L^{s/3}(X)}.$$

Applying the Sobolev embedding theorem for functions one gets

$$\|s_i\|_{L^s(X)} \leq \text{const}' \|s_i\|_{L^{s/3}(X)} \leq \text{const}'' \|s_i\|_{L^{s/3}(X)}$$

with constant $\text{const}', \text{const}''$ independent of $t, a_i$. With this estimate in mind we now look how far the truncated spinor is from being harmonic.
\[ \leq \| d\psi_i \|_{L^4(X)} \| s_i \|_{L^8(\mathbb{CP}^2)} + \sup |a_i - a_X|_{\text{supp}\psi_i} \| s_i \|_{L^{8/3}(\mathbb{CP}^2)} \leq \]

\[ \leq (\text{const}'\epsilon(t) + \epsilon'(t)) \| s_i \|_{L^{8/3}(\mathbb{CP}^2)} = \epsilon_1(t_i) \| s_i \|_{L^{8/3}(\mathbb{CP}^2)} = \epsilon_1(t), \quad (1.4) \]

where \( \epsilon_1(t_i) \) tends to 0 if \( t_i \) tends to infinity.

On the other hand applying these estimates to the cut-off function \( (1 - \psi_i) \) provides a similar inequality:

\[ \| D_{a\mathbb{CP}^2}(1 - \psi_i)s_i \|_{L^{8/3}} \leq \epsilon_2. \]

But the auxiliary Lemma 1.2 shows the first eigenvalue of the operator \( D_{a\mathbb{CP}^2} \) is positive and uniformly estimated from below by some positive constant \( c \). Therefore there is an inequality

\[ \|(1 - \psi)s_i\|_{L^{8/3}} \leq \epsilon_2/c. \]

So the norm of \( (1 - \psi_i)s_i \) is arbitrarily small and hence the norm of \( \psi_is_i \) is arbitrarily close to 1. Now the first statement of the Lemma follows from inequality (1.4) and the variational characterisation of the first eigenvalue. The second statement follows from standard bootstrapping argument using the equation

\[ D_{(aX)|K}(s_i)|_K = (a_i - a_X)|_K * (s_i)|_K \]

restricted to the compact subset \( K \) of \( X - p \).

**Remark.** If one has

\[ a_i \in V_\Sigma, \Sigma \in H_2(X) \subset H_2(\mathbb{CP}^2) \]

and no bubbling occurs at \( \Sigma \), i.e. none of the \( x_i \) lies on \( \Sigma \) then for the limit connection it follows

\[ a_X \in V_\Sigma, \]

since this depends only on the restriction of the connection to a neighborhood of the Riemannian surface \( \Sigma \) where there is \( C^\infty \) convergence of connections of our sequence.

**Lemma 1.2.** Let \( F \) be a \( U(2) \) vector bundle on \( \overline{\mathbb{CP}^2} \) and \( C \) a Spin\(^c\)-structure on \( \overline{\mathbb{CP}^2} \) such that \( C + c_1(E) = ke, -3 < k < 3 \), where \( e \) is a generator of \( H^2(\overline{\mathbb{CP}^2}) \).

There exists a neighborhood \( U \) of the Fubini-Study metric on \( \overline{\mathbb{CP}^2} \) and some connection in the product of the space of metrics on \( \overline{\mathbb{CP}^2} \) and the space of connections on the line bundle \( L = \Lambda^2 W^\pm, c_1(L) = ke \), such that for any asd-connection on \( adF \) and any point \( (g, \nabla) \in U \), one has

\[ \ker D_{a\nabla}^g = 0. \]

Moreover, the first eigenvalue \( \mu_a \) of the Laplasian \( (D_{a\nabla}^g)^*D_{a\nabla}^g \) satisfies

\[ \mu_a > c \]

for some positive constant \( c \) independent of the point in \( U \) and \( a \).

**Proof.** The proof is based on the Weitzenböck formula:

\[ (D_{\nabla}^g)^*D_{\nabla}^g = -\Delta_{\nabla}^g + 1/4S_{\nabla}^g + F_{\nabla}^g \quad (1.5) \]
where $S$ is the scalar curvature of $CP^2$, $\nabla_a$ the connection on the bundle $W^+ \otimes F$ built from $a$ and $\nabla$. One has $F^+_a = 0$, and example 2.1.2 [H] gives that for the Fubini-Study metric, some connection $\nabla_0$ on $\Lambda^2 W^+$ and some positive number $c$ there is an estimate of $1/4S + F_{\nabla_0}$ as an endomorphism of vector-spinors:

$$1/4S + F_{\nabla_0} > 2c.$$  

Let $U$ be given by the condition $(g, \nabla) \in U$ if $1/4S + F_{\nabla} > c$. Taking the scalar product of (1.5) with $s$ and integrating over $CP^2$ gives the required estimate.

Therefore to each point of the space $MP^{g_i}(X \# CP^2, C + c_1, p_1)$ for $i \to \infty$ one can assign a point of the product

$$MP^{g_i}(X, (C + c_1)|X, k) \times MP^g(\overline{CP^2}; (w_2)_{CP^2}, m),$$

where $w_2 = c_1 \pmod{2}$, at least for one pair of negative integers $k, m; k + m \geq p_1$. Conversely, there is a procedure for glueing two asd-connections $a_X, a_{CP^2}$, defined on the summands, which gives an asd connection on the connected sum and one shows that any asd-connection on a connected sum with long enough tube is obtained in this way (cf. [DKr]). In the following we extend this construction to the case of the moduli space of pairs.

To start with we need glueing data for connections, i.e. isometries

$$\tau : T_pX \to T_qCP^2, \rho : (adE_X)_p \to (adE_{CP^2})q,$$

which for given asd-connections $a_X, a_{CP^2}$ on $X$ and $CP^2$ resp. give an $SO(3)$-asd-connection $a = I(a_X, a_{CP^2}, \rho)$ on $X \# CP^2$ with appropriate choices of some other parameters. Varying $a_X, a_{CP^2}$ in some small neighborhood, and varying the glueing parameter $\rho$ in the space of all glueing parameters $Gl$ we get a smooth map to some open set of asd connections on the connected sum.

In order to glue vector-spinors one has to glue the corresponding vector bundles of vector-spinors with a unitary map $\sigma$:

$$\sigma : (W^+_X \otimes E_X)_p \to (W_{CP^2} \otimes E_{CP^2})_q.$$

The isometries $\tau$ and $\rho$ give $\sigma$ up to a unitary scaling, which may be interpreted as a glueing parameter for glueing connections on the determinant line bundles

$$(\det(W^+_X \otimes E_X))_p \to (\det(W_{CP^2} \otimes E_{CP^2}))_q.$$

This scaling is in fact inessential since different glueing parameters for $U(1)$-connections give gauge equivalent connections provided $X$ is simply connected. Taking a cut-off function $\psi$ as in Lemma 1.1 one considers $\psi$s as a section of the glued bundle $(W^+ \otimes E)_{X \# CP^2}$ and it turns out that the point $(a = I(a_X, a_{CP^2}, \rho), \langle \psi s \rangle)$ is close to a point of $MP^{g_i}(X \# CP^2, C + c_1, p_1)$. In fact we can deform it to a point of the moduli space. The next lemma describes a neighborhood of this deformation in the space $MP^{g_i}(X \# CP^2, C + c_1, p_1)$, as the zero set of some map defined on some neighborhood in the space

$$\tilde{MP}^{g_i}(X, (C + c_1)|X, \pm 4k) \times MP^g(\overline{CP^2}; (w_2)_{CP^2}; p_1 + 4k).$$
of the preimage of the pair
\[
((a_{X}, (s)); a_{CP^2}) \in \mathcal{MP}^{g, \nabla}(X, (C + c_1)|_{X}, -4k) \times \mathcal{M}^{g}(CP^2, (w_2)_{CP^2}, p_1 + 4k)
\]
under the natural projection
\[
\mathcal{MP}^{g, \nabla}(X, (C + c_1)|_{X}, -4k) \times_{P_{U(2)}} \mathcal{M}^{g}(CP^2, (w_2)_{CP^2}, p_1 + 4k) \to \mathcal{M}^{g, \nabla}(X, (C + c_1)|_{X}, -4k) \times \mathcal{M}^{g}(CP^2, (w_2)_{CP^2}, p_1 + 4k).
\]

**Lemma 1.3.** For the point \((a_{X}, (s))\) in the space \(\mathcal{MP}^{g, \nabla}(X, (C + c_1)|_{X}, -4k)\) and \(a_{CP^2} \in \mathcal{M}^{g}(CP^2, (w_2)_{CP^2}, p_1 + 4k)\) there is a neighborhood in the space \(\mathcal{MP}^{g, \nabla}(X \# CP^2, C + c_1, p_1)\) which is diffeomorphic to the zero set of a smooth map \(\Phi\) defined on some neighborhood
\[
\mathcal{U} \subset T_{a,s}\mathcal{MP}^{g, \nabla} \times H^{1}_{a_{CP^2}}(adE) \times Gl
\]
of \(\{0\} \times \{0\} \times Gl\):
\[
\mathcal{U} \xrightarrow{\Phi} \text{coker}(i) \oplus (H^{2}_{a_{X}}(adE) \oplus H^{2}_{a_{CP^2}}(adE)) \oplus \text{coker}\mathcal{D}_{a_{CP^2}}
\]
where \(\text{coker}(i)\) is defined in (1.3).

**Proof.** We assume for simplicity that \(H^{2}_{a_{X}} = H^{2}_{CP^2} = 0\). Fix some glueing parameter \(\rho'\) and a connection \(a' = I(a_{X}, a_{CP^2}, \rho')\). Consider a neighborhood in \(\mathcal{M}^{g}(X \# CP^2, (c_1) \mod 2, p_1)\) of the subspace \(\cup_{p}I(a_{X}, a_{CP^2}, \rho)\) consisting of all connections represented in the form \(I(a_{X} + \delta a, a_{CP^2}, \rho)\) subject to an estimate of \(\delta I(a_{X} + \delta a, a_{CP^2}, \rho) - I(a_{X}, a_{CP^2}, \rho')\):
\[
\|\delta I(a_{X} + \delta a, a_{CP^2}, \rho)\|_{L^{1}} \leq \epsilon
\]
with \(\epsilon\) to be fixed in the proof. Fix lifts
\[
\xi_{X} : \text{coker}(\mathcal{D}_{X}) \to \Gamma(W^{-} \otimes E_{X}), \quad \xi_{CP^2} : \text{coker}\mathcal{D}_{a_{CP^2}} \to \Gamma(W^{-} \otimes E_{CP^2})
\]
of the obvious projections \(\pi_{X}, \pi_{CP^2}\) onto the cokernels with all section in the images \(\text{im}\xi_{X}, \text{im}\xi_{CP^2}\) supported away from the points \(p \in X, q \in CP^2\). There is a linear map
\[
\xi_{X \# CP^2} = \xi_{X} \oplus \xi_{CP^2} : \text{coker}\mathcal{D}_{a_{X}} \oplus \text{coker}\mathcal{D}_{a_{CP^2}} \to \Gamma(W^{-} \otimes E_{X \# CP^2}),
\]
which is a lift of the projection \(\pi(\cdot) = \pi_{X}(\psi\cdot) + \pi_{CP^2}(1 - \psi)\cdot\), where \(\psi\) is the cut-off function as in Lemma 1.1. Let \(s\) be a non-zero element of the kernel of \(\mathcal{D}_{a_{X}}\). Then we can make \(\|\mathcal{D}_{a'}(\psi s)\| \leq \epsilon(t)\) with \(\epsilon(t) \to 0\) as \(t \to \infty\) for suitable choices as above. Consider the map \(\phi\), which assigns to a triple consisting of an asd-connection \(a_{X} + \delta a_{X}\) on \(X\), a vector spinor \(\delta s\) on the connected sum \(X \# CP^2\) and an element \(h \in \text{coker}\mathcal{D}_{a_{X}} \oplus \text{coker}\mathcal{D}_{a_{CP^2}}\) the vector spinor on the connected sum:
\[
\phi(a_{X} + \delta a_{X}, h, \delta s) = \mathcal{D}_{U} \cdot \langle \psi s + \delta s, \xi_{X \# CP^2}(h) \rangle.
\]
The tangent map of $\phi$ at the point $(a_X,0,0)$ with respect to $\delta s, h$ is given by

$$(D_{a'} \oplus \xi)(h, \delta s) = D_{a'}(\delta s) + \xi_X \# CP^2(h),$$

and the application of the standard technique shows that it has a right inverse

$$S \oplus \pi : D_{a'} S(\eta) + \xi_X \# CP^2 \pi(\eta) = \eta$$

with the estimate of the norm of $S \oplus \pi$ independent of $t$ (compare 7.2.14, 7.2.18 of [DKr]). Since by definition $pr_2 = \xi_X \# CP^2 \pi$ is a projector, $pr_1 = D_{a'} S$ is also a projector and one can also assume that $\text{Im} S$ is transversal to $\ker D_{a'}$. This means that $S(\eta) = S(\eta_1)$, where $\eta_1 = pr_1(\eta)$. Although $pr_i$ are not orthogonal projectors one can estimate their norm by, say, $\|pr_i\| \leq 2$. Now we shall look for the solution of the equation $\phi(a_X + \delta a_X, h, \delta s) = 0$ in the form $(h, \delta s) = (\pi(\eta), S(\eta)) = (\pi(\eta_2), S(\eta_1))$:

$$D_I(a_X + \delta a_X, a_{CP^2}, \rho) \psi_s - S(\eta) + \xi(\pi(\eta)) = 0, \quad (1.6).$$

Following system of two equations is equivalent to (1.6):

$$pr_1(D_{a'}(\psi_s)) + \eta_1 + \delta I(a_X + \delta a_X, a_{CP^2}, \rho)*S(\eta) + \delta I(a_X + \delta a_X, a_{CP^2}, \rho)*S(\eta_1) = 0, \quad (1.7)$$

$$pr_2(D_{a'}(\psi_s)) + \eta_2 + \delta I(a_X + \delta a_X, a_{CP^2}, \rho)*S(\eta) + \delta I(a_X + \delta a_X, a_{CP^2}, \rho)*S(\eta_1) = 0. \quad (1.8)$$

Equation (1.7) can be rewritten in the form:

$$(1 + A)(\eta_1) = -pr_1(D_{a'}(\psi_s)) - \delta I(a_X + \delta a_X, a_{CP^2}, \rho)*S(\eta_1) = 0 \quad (1.9)$$

with the norm of $A(\eta_1) = pr_1(\delta I(a_X + \delta a_X, a_{CP^2}, \rho)*S(\eta_1))$ estimated by an inequality:

$$\|A(\eta_1)\|_{L^S/3} \leq 2\|\delta I(a_X + \delta a_X, a_{CP^2}, \rho)||L^4\|S(\eta_1)||L^S \leq 2\epsilon\|S\|\|\eta_1\|_{L^S/3} \leq \|\eta_1\|_{L^S/3}/2$$

provided $\epsilon \leq \|S\|/4$. Therefore the operator $1 + A$ is invertible and $\|\(1 + A)^{-1}\| \leq 2$. This gives the existence and uniqueness of the solution $\eta_1$ of (1.9) together with an estimate:

$$\|\eta_1\|_{L^S/3} = \|(1 + A)^{-1}(-pr_1(D_{a'}(\psi_s)) - \delta I(a_X + \delta a_X, a_{CP^2}, \rho)*S(\eta_1))\|_{L^S/3} \leq$$

$$\leq 2(\|pr_1(D_{a'}(\psi_s))\|_{L^S/3} + \|\delta I(a_X + \delta a_X, a_{CP^2}, \rho)*S(\eta_1)\|_{L^S/3}) \leq$$

$$\leq 4(\|D_{a'}(\psi_s)\|_{L^S/3} + \|\delta I(a_X + \delta a_X, a_{CP^2}, \rho)*S(\eta_1)\|_{L^S/3}) \leq$$

$$\leq 4(\epsilon(t) + \|\psi\|\|I(a_X + \delta a_X, a_{CP^2}, \rho)||L^4||S||L^S) \leq 4(\epsilon(t) + const\epsilon) \quad (1.10)$$

(we use here the estimate $\|s\|_{L^S} \leq const\|s\|_{L^S/3} \leq const\|s\|_{L^S/3}$ from Lemma 1).

Let $N$ be the image of some splitting $j$ of the exact sequence (1.3). Now consider the map $\pi : N \to \ker D_{aX} \oplus \ker D_{a_{CP^2}}$ given by

$$\delta a_X + \xi((\delta I(a_X + \delta a_X, a_{CP^2}, \rho)*S(\eta_1)) \| \xi((\delta I(a_X + \delta a_X, a_{CP^2}, \rho)*S(\eta_1)) \|$$
In order to get the proper obstruction space we shall prove that the image of the composition $pr_{im(i)}\tau$ contains an open ball centered at the origin, where $pr_{im(i)}$ is the projection to the image of the map $i$ defined in (1.3). Indeed, this follows from estimates:

$$
\|\pi_X(\delta I(a_X + \delta a_X, aCP^2, \rho) * (\psi(s))) + \pi_X(\delta I(a_X + \delta a_X, aCP^2, \rho) * S(\eta_1))\|_{L^8/3} \geq
$$

$$
\geq \|\pi_X((\delta a_X) * (\psi(s))\|_{L^8/3}/2 - \|\delta I(a_X + \delta a_X, aCP^2, \rho)\|_{L^4}\|S\| \cdot \|\eta_1\|_{L^8/3} \geq
$$

$$
\geq \|\pi_X((\delta a_X) * s)\|_{L^8/3}/4 - 2\|\delta a_X\|_{L^4}\|S\| \cdot 4(\epsilon(t) + const_1\epsilon) =
$$

$$
= \frac{1}{4}\|i_{\mathcal{N}}(\delta a_X)\|_{L^8/3} - 8\|S\| (\epsilon(t) + const_1\epsilon)\|((\delta a_X))\|_{L^4} \geq
$$

$$
\geq (\text{const}_2 - 8\|S\| (\epsilon(t) + const_1\epsilon))\|((\delta a_X))\|_{L^4},
$$

(1.11)

where the first inequality uses the estimate

$$
\|\delta I(a_X + \delta a_X, aCP^2, \rho) - \psi \cdot \delta a_X\|_{L^4} \leq \text{const} \cdot e^{-2t}
$$

(1.12)

(cf. [DKr, (7.2.37)]). The second inequality in (1.11) uses (1.12), (1.10) and the estimate of the norm of $\psi s$ from below given in Lemma 1.1. Since $i_{\mathcal{N}}$ is a linearisation of $\pi_X((\delta a_X) * s)$ (cf.1.3) and $i_{\mathcal{N}}$ is a monomorphism by the definition of $\mathcal{N}$, there is a constant $\text{const}_2$ such that for small enough $\epsilon$

$$
\|\delta a_X\| \leq \epsilon \Rightarrow \|\pi_X((\delta a_X) * s)\| \geq \frac{1}{2}\|i_{\mathcal{N}}(\delta a_X)\|_{L^8/3} \geq \text{const}_2\|((\delta a_X))\|_{L^4}
$$

(here we identify $\delta a_X$ with the corresponding element in $H^1_{a_X}$). This gives the last inequality in (1.11), and the statement about the composition $pr_{im(i)}\tau$ follows, provided $\epsilon(t)$ and $\epsilon$ are small enough. Therefore the subspace $im(\tau) + \text{coker}(i) \subset \text{coker}(\mathcal{D}_{a_X})$ contains some metric ball $B$, and there exist $\epsilon_0$ such that

$$
\epsilon(t) \leq \epsilon_0 \Rightarrow \pi(D_{a'}(\psi s)) \in B \oplus \text{coker}\mathcal{D}_{aCP^2} \subset \text{coker}\mathcal{D}_{a_X} \oplus \text{coker}\mathcal{D}_{aCP^2}.
$$

Therefore the obstruction space can be identified with $\text{coker}(i) \oplus \text{coker}\mathcal{D}_{aCP^2}$. The Lemma follows.

Denote by $\epsilon_0$ the constant in the generalisation of Uhlenbeck’s local Coulomb gauge theorem to any strongly simply connected domain $\Omega$ (Prop.4.4.10 of [DKr]).

**Corollary 1.4.** There exists a real positive number $t_0$ such that for $t > t_0$ there exists a covering $\{U_i\}$ of the moduli space $\mathcal{M}^{g,\nabla}(X \# CP^2, C + c_1, p_1)$ such that for subsets

$$
\widetilde{\mathcal{M}}^\emptyset(X, (C + c_1)|X, i) \subset \widetilde{\mathcal{M}}(X, (C + c_1)|X, i)
$$

and

$$
\widetilde{\mathcal{M}}^\emptyset(CP^2, (w_2)|CP^2, p_1 + 4i) \subset \widetilde{\mathcal{M}}(CP^2, (w_2)|CP^2, p_1 + 4i)
$$

given by the condition

$$
\|F_{a_1D^4}\|_{L^2(D^4)} \leq \epsilon D_\emptyset^4
$$

where $D^4$ is a small disk around $p \in X$, $a \in CP^2$, resp., of radius $\epsilon$, one has
\[ \mathcal{U}_i \subset \widetilde{\mathcal{M}}^{rexp(-t)}(X, (C + c_1)|_X, -4i) \times p_U(2) \widetilde{\mathcal{M}}^{rexp(-t)}(\mathbb{CP}^2, (w_2)|_{\mathbb{CP}^2}, p_1 + 4i) \]

\[ \widetilde{\mathcal{M}}^r(X, (C + c_1)|_X, -4i) \times p_U(2) \uarrows{\mathcal{M}}(\mathbb{CP}^2, (w_2)|_{\mathbb{CP}^2}, p_1 + 4i) \subset \mathcal{U}_i. \]

**Proof.** Take a subdivision of the tube \( S^3 \times [-t/2, t/2] \) into \( N \) pieces

\[ \Omega_i = S^3 \times [-t/2 + lt/N, -t/2 + (l + 1)t/N], l = 0, ..., N - 1. \]

We shall use a cut off function \( \psi \) with support of \( d\psi \) in one of \( \Omega_i \) with an estimate \( \|d\psi\|_{L^4} \leq (10(t/N)^3)^{-1/4} \). Take \( N - 2 > \max\{-p_1/\epsilon_F, 10(\epsilon_0)^{-4}t^{-3}\} \), where \( \epsilon_F \leq \epsilon_{\Omega_1} \) and \( \epsilon_0 \) is the constant of Lemma 1.3. This provides the existence for any ASD-conNECTION \( a \) with first Pontrjagin class \( p_1 \) of at least one \( 0 \leq l < N - 1 \) for which

\[ \|F_a|_{\Omega_l}\|_{L^2} \leq \epsilon_F, \quad \|D_a(\psi s)|_{L^{s/3}(\Omega_l)}\| \leq \|d\psi\|_{L^4(\Omega_l)}\|s|_{\Omega_l}\| \leq (10(t/N)^3)^{-1/4}\|s|_{\Omega_l}\| \leq (10(t/N)^3)^{-1/4}N^{-1} = (10^3N)^{-1/4} \leq \epsilon_0 \]  

(1.13)

Now let \( \mathcal{U}_i \) be the open set of those connections \( a \) in \( \mathcal{M}^{g,\nabla}(X\#\mathbb{CP}^2, C + c_1, p_1) \) for which for at least one integer \( l \) (1.13) is satisfied, and the \( L^2 \)-norm of the curvature of \( a \) restricted to \( X_1 = X \cup S^3 \times [-t/2, -t/2 + lt/N] \) is close to \( 8\pi^2i \). Then by the Uhlenbeck theorem there is a trivialisation of \( E|_{\Omega_1} \) in which the connection matrix is small in \( L^2 \), and therefore one can show that norms of \( F_{\psi a}, F_{(1-\psi)a}, D_a(\psi s) \) are small. By Theorem 7.2.41 of [DKr] \( a = I(a_X, a_{\mathbb{CP}^2}, \rho), \|\psi a - a_X\|_{L^4} \leq const \epsilon_F \). If \( l < N - 1, X_1 \) is a subset of fixed compact subspace in \( X - p \) and by Lemma 1.1 one has for some \( s_X \in ker D_{a_X} \)

\[ \|(s - s_X)|_{X_1}\|_{L^8/3(X_1)} \to 0 \]

as \( t \to \infty \). Symmetrically, if \( l > 0 \) then

\[ \|s|_{(X\#\mathbb{CP}^2 - X_1)}\|_{L^8/3(X\#\mathbb{CP}^2 - X_1)} \to 0. \]

This means that

\[ \|s - \psi s_X\|_{L^8/3(X\#\mathbb{CP}^2)} \leq \|(s - s_X)|_{X_1}\|_{L^8/3(X_1)} + \|s|_{(X\#\mathbb{CP}^2 - X_1)}\|_{L^8/3(X\#\mathbb{CP}^2 - X_1)} + \|\psi\|_{L^2(\Omega_1)}\|s_X\|_{L^8/3(\Omega_1)} \to 0 \]

as \( t \to \infty \), so it follows that varying slightly \( s_X \) in \( ker D_{a_X} \) (what does not change \( S_a \)) we get \( s - \psi s_X = S_a(\eta) \) with \( \eta = D_a(s - \psi s_X) = (D_{a_X} + (\psi a - \psi a_X))(s_X) \) small enough to be in the set of solutions of (1.7 -1.8).

Now we shall consider some particular cases in which Lemma 1.3 will provide sufficient information about the moduli space of the connected sum \( X \#\mathbb{CP}^2 \) in order to compute the spin-polynomials of the connected sum in terms of those of \( X \). Consider a spin polynomial of degree \( d \) on \( X \):

\[ \omega^X_{\Sigma} = \#(\Sigma \cap \mathcal{M}(X\#\mathbb{CP}^2)). \]
which is defined only if
\[ 2d = -(3/2)k - 3(1 + b_2^+(X)) + (\omega)^2/2 - \text{sign}(X)/2 - 2 \]
for some negative integer \( k \) and some 2-dimensional cohomology class \( \omega \) on \( X \).

Let \( e \) be a generator of the second cohomology group \( H_2(\mathbb{C}P^2) \): \( e \in H_2(\mathbb{C}P^2) \subset H_2(X \# \mathbb{C}P^2) \). Take a bundle \( E \) and a \( \text{Spin}^C \)-structure \( C \) on the connected sum \( X \# \mathbb{C}P^2 \) given by one of the following three choices.

1. \( p_1(E) = k, C + c_1(E) = \omega + (\pm e) \);
2. \( p_1(E) = k + 1, C + c_1(E) = \omega + (\pm 2e) \);
3. \( p_1(E) = k + 1, C + c_1(E) = \omega \).

By a dimension count it follows that in the first and second case one has
\[ \dim \left( \cap V_{\Sigma_i} \cap \mathcal{PM}(X \# \mathbb{C}P^2, p_1(E), C + c_1(E)) \right) = 0, \]
and in the third -
\[ \dim \left( \cap V_{\Sigma_i} \cap \mathcal{PM}(X \# \mathbb{C}P^2, p_1(E), C + c_1(E)) \right) = 2. \]

Therefore in the first and second case the polynomial
\[ \gamma_{p_1, C + c_1(E)}^{X \# \mathbb{C}P^2}(\Sigma_i) = \# \{ \cap V_{\Sigma_i} \cap \mathcal{PM}(X \# \mathbb{C}P^2, p_1(E), C + c_1(E)) \}, \]
is defined and in the third the polynomial
\[ \gamma_{p_1, C + c_1(E)}^{X \# \mathbb{C}P^2}(\Sigma_i, e) = \# \{ \cap V_{\Sigma_i} \cap V_e \cap \mathcal{PM}(X \# \mathbb{C}P^2, p_1(E), C + c_1(E)) \} \]
is defined.

Let \( a_j \) be a sequence of connections in \( \cap V_{\Sigma_i} \) which are asd in the metric \( g_j \) resp. In each of the three cases by the remark to Lemma 1.1 the limit connection \( a_X \) on \( X \) belongs to the moduli space
\[ \cap V_{\Sigma_i} \cap \mathcal{PM}(X, k + 4m, \omega), \]
for some \( m \geq 0 \), of the virtual dimension \(-8m \). Since we took the metric on \( X \) and the connection on the bundle \( \Lambda^2 W^\pm \) to be generic, and since \( b_2^+(X) > 0 \) there are no reducible asd-connections and the moduli spaces of negative virtual dimension are empty (cf. [PT], ch.1, sec.3). Therefore the only remaining case is \( m = 0 \).

From the inequality
\[ p_1(a_X) + p_1(a_{\mathbb{C}P^2}) \geq p_1(E) \]
it follows that \( p_1(a_{\mathbb{C}P^2}) = 0 \) for the first case and \( p_1(a_{\mathbb{C}P^2}) = -1 \) for the second and the third. Moduli space of asd-connections on \( \mathbb{C}P^2 \) for a generic metric with these Pontrjagin numbers has negative virtual dimension and therefore contain only reducible connections. It follows that \( a_{\mathbb{C}P^2} \) is the trivial connection in the first case, and the only reducible one with \( p_1 = -1 \) in the second.

In this situation Lemma 1.3 provides a satisfactory description of the (compact) manifold
\[ Z = \cap V_e \cap \mathcal{PM}(X, k + 1) \times_{PM(\mathbb{C}P^2, p_1(E), p_1(\mathbb{C}P^2))} \mathcal{PM}(\mathbb{C}P^2, p_1(a_{\mathbb{C}P^2}), p_1(a_{\mathbb{C}P^2})). \]
and the bundle $\Xi$ on $Z$ which is a tensor product of the lift of the $PU(2)$-equivariant bundle

$$U(2) \times_{S^1 \times S^1} \text{coker}D_{a_{CP^2}} \rightarrow \tilde{M}(\overline{CP^2}, w_2(a_{CP^2}), p_1(a_{CP^2})) = U(2) / (S^1 \times S^1)$$

via obvious projection

$$\cap \tilde{V}_{\Sigma_i} \cap \tilde{PM}(X, k, \omega) \times_{\rho U(2)} \tilde{M}(\overline{CP^2}, w_2(a_{CP^2}), p_1(a_{CP^2}))$$

and tautological line bundle on $\cap \tilde{V}_{\Sigma_i} \cap \tilde{PM}(X, k, \omega)$ such that our, cut down moduli space

$$\cap V_{\Sigma_i} \cap \mathcal{P}M(X#\overline{CP^2}, p_1(E), C + c_1(E))$$

is embedded in $Z$ as the zero set of some generic section of $\Xi$. This gives a possibility of computing the polynomial for $X#\overline{CP^2}$ in topological terms.

In the first case our manifold $Z$ is parametrised as a family of glued connections of the type $a_X \# \theta$ with $(a_X, <s_X>) \in \cap V_{\Sigma_i} \cap \mathcal{P}M(X, p_1(E), C + c_1(E))$. Since a connected sum with the trivial connection $\theta$ does not need any glueing parameters, one has $Z = \cap V_{\Sigma_i} \cap \mathcal{P}M(X, k, \omega)$. Index of the coupled Dirac operator for $\theta$ is zero and hence $\Xi = 0$.

In the second case the reducible connection on $\overline{CP^2}$ has the stabilizer group $S^1$ and therefore

$$Z = \cap \tilde{V}_{\Sigma_i} \cap \tilde{PM}(X, k, \omega) / S^1 = \cup \mathbb{P}^1.$$ 

For each component $\mathbb{P}^1$ the restriction to it of the bundle $\Xi$ is

$$\left( (U(2) \times U(2)) \times_{S^1 \times (S^1 \times S^1)} (\mathbb{C} \oplus \text{coker}D_{a_{CP^2}}) \right) / U(2)$$

where $S^1 \times (S^1 \times S^1)$ is the subgroup given as the product of the center subgroup $S^1 \subset U(2)$ and the maximal torus $(S^1 \times S^1) \subset U(2)$. The representation of this subgroup is the product of the generating linear one of $S^1$ and the natural representation of the stabiliser subgroup $(S^1 \times S^1)$ of the reducible connection $a_{CP^2} = \lambda_1 \oplus \lambda_2$ which is the sum of representations $\oplus_i \text{coker}D_{\lambda_i}$, each $S^1$ acting on its summand with degree $\pm 1$. For the specified $Spin^C$-structure on $\overline{CP^2}$ this cokernel is one dimensional and therefore

$$\Xi|_{\mathbb{P}^1} = U(2) \times_{S^1 \times S^1} \text{coker}D_{a_{CP^2}} = \mathcal{O}_{\mathbb{P}^1}(1).$$

In the third case the space $Z$ is the same, but the index of the reducible connection on $\overline{CP^2}$ is zero, the Dirac operator coupled to this connection is an isomorphism and the bundle arises as a line bundle $\mathcal{L}_e, c_1(\mathcal{L}_e) = \mu(e)$:

$$\Xi = \mathcal{L}_e = \mathcal{O}_{\mathbb{P}^1}(-1).$$

Therefore the following theorem is proved:
Theorem. Let $k \in \mathbb{Z}, k \leq 0; \omega \in H^2(X, \mathbb{Z}), \Sigma_1, \ldots, \Sigma_d \in H_2(X, \mathbb{Z})$ such that
\[2d = -3/2k - 3(1 + b_2^+(X)) + (\omega)^2/2 - \text{sign}(X) - 2 \geq 0.\]

Then
\[
\gamma_{k,\omega}^{\#CP^2} (\Sigma_1, \ldots, \Sigma_d) = \pm \gamma_{k,\omega}^{\#CP^2} (\Sigma_1, \ldots, \Sigma_d),
\]
\[
\gamma_{k+1,\omega}^{\#CP^2} (\Sigma_1, \ldots, \Sigma_d) = \pm \gamma_{k,\omega}^{\#CP^2} (\Sigma_1, \ldots, \Sigma_d),
\]
\[
\gamma_{k+1,\omega}^{\#CP^2} (\Sigma_i, \ldots, \Sigma_d, e) = \pm \gamma_{k,\omega}^{\#CP^2} (\Sigma_1, \ldots, \Sigma_d).
\]

Remarks.
1. Signs in the r.h.s. of the formulas are determined by the choice of orientation.
2. Considering the case $b_2^+(X) = 1$ one has to take into account the chamber structure i.e. the fact that stretching the neck of the connected sum $X \# CP^2$ provides periods of our metric tending to the hyperplane of orthogonals to $e$ in the limit. However in the case of Spin-polynomials this hyperplane is not a wall by the Proposition 1.4.2 of [PT] and the negative definiteness of the intersection form on $H^2(CP^2)$. Therefore any chamber in the period space $P(X)$ for metrics on $X$ is embedded in a unique chamber in the period space $P(X\#CP^2)$.

§2. Application

Let $S_r$ and $S_g$ be a rational surface and a surface of general type with $p_g(S_g) = 0$ resp. We assume the rational surface to be a good generic surface (cf. [FM, 1.2.1]). Denote by $p : S_g \to S_{mg}$ the map to the minimal model of $S_g$ and by $\{L_i\}$ a collection of exceptional fibres of this map. For $e \in H^2(S_r, \mathbb{Z}), (e)^2 = -1$ the wall $W^e$ will be a hyperplane of orthogonals to $e$ in $H^2(S_r, \mathbb{Z})$. One says that the wall is ordinary if the restriction of the intersection form to the wall is odd, and extraordinary otherwise. For $e, (e)^2 = -1$, the reflection $R_e$ in the wall $W^e$ is defined by $R_e(x) = x + 2(x.e)e$. It follows from [FM, Prop.3.2.4] that if $e$ is represented by an embedded (-1)-sphere, i.e. there is an embedded sphere $S^2$ with the fundamental class Poincare dual to $e, (e)^2 = -1$, there is an orientation preserving self-diffeomorphism of $S_r$ inducing $R_e$. This is the case in particular for any $e \in \mathcal{E}(S_r)$ where $\mathcal{E}(S_r)$ is a set of all cohomology classes represented by exceptional (-1)-curves on $S_r$. We shall use the notions of a P-cell and a super P-cell as defined in [FM, ch2].

Lemma 2.1. Suppose $f$ is a hypothetical diffeomorphism
\[S_r \to S_g.\]

There exists a self-diffeomorphism $\psi$ of $S_r$ and a collection of exceptional curves $E_i$ on $S_r$ such that for all (-1)-curves $L_i$ on $S_g$ we have
\[\psi^*f^*l_i = e_i.\]

\[\text{Proved also in [FQ2] as Theorem 1.7. The arguments of our proof are a minor modification of the arguments in [FM] and are of course similar to those in [FQ2].}\]
with \( l_i = [L_i], e_i = [E_i] \) denoting the 2-cohomology classes in \( H_2(S_g), H_2(S_r) \) represented by \( L_i, E_i \) resp.

**Proof.** Let \( K \) be the Kähler cone of \( S_r \). Take any polarisation \( H \in K \). Consider the projection \( \prod_i \frac{(1 + R_{f^{-1}l_i})}{2} \) onto the subspace \( \cap_i W^{f_{*}l_i} \) and consider the point

\[
p = \prod_i \left( \frac{1 + R_{f_{*}l_i}}{2} \right) H
\]

in this subspace. Since the cohomology class \( f_{*}l_i \) is represented by a \((-1)\)-sphere \( f^{-1}(L_i) \) the reflection \( R_{f_{*}l_i} \) is induced by a diffeomorphism. Thus does any reflection of the type \( R_l = \prod_{i \in J} R_{f_{*}l_i} \). Therefore by Thm.10 of [FM] it follows that \( R_l \) leave the super P-cell \( S(K) \) invariant and \( R_l(H) \in S(K) \). By the convexity of the supercell (cf. Prop. 2.5.6 of [FM]) one has therefore \( p \in S(K) \).

Take any P-cell \( P \) in \( S(K) \) which contains the point \( p \in P \) under consideration. By the definition of the super P-cell there exists an isometry \( \psi \) in the subgroup of isometries generated by reflections in the set \( \mathcal{F}_K \) of ordinary walls of \( K \) such that \( \psi(P) = K \). But these latter reflections are induced by diffeomorphisms since all such walls are given by \((-1)\)-curves on \( S_r \) (cf. Prop.3.6 and Thm10B of [FM]). Therefore, \( \psi \) is induced by a diffeomorphism.

Now, any wall \( W^e \) for \( e = \psi_{*}f_{*}l_i \) contains the point \( \psi_{*}(p) \) of the P-cell \( K \). In fact \( W^e \) is a wall of the P-cell \( K \). To establish this fact one needs to show that the intersection \( W^e \cap K \) contains an open subset of \( W^e \). By Proposition 3.4 of [FM]

\[
K = \{ y \in \mathbb{H}_{+} | y \cdot (-K_{S_r}) \geq 0, y \cdot h \geq 0 \text{ for all classes } h \in \mathcal{E}(S_r) \}.
\]

Therefore

\[
W^e \cap K = \{ y \in \mathbb{H}_{+} | y \cdot pr(-K_{S_r}) \geq 0, y \cdot pr(h) \geq 0 \text{ for all classes } h \in \mathcal{E}(S_r) \},
\]

where \( pr \) is the projection to \( W^e \). Let \( E_1, ..., E_k \) be the set of all exceptional curves satisfying \( q \in W[E_i] \) and assume that \( \pm e \not\equiv E_i \) (otherwise we are done). It follows from Lemma 1.11 of ch.2 of [FM] that \( [E_i] = pr[E_i] \in W^e \) is a set of pairwise orthogonal elements. For some neighborhood \( U_q \) of \( q \in W^e \) one has

\[
U_q \cap W^e \cap K = \{ y \in W^e \cap U_q | y \cdot pr(-K_{S_r}) \geq 0, y \cdot pr([E_i]) \geq 0 \},
\]

and this intersection is an open set in \( W^e \) if the elements \( pr(-K_{S_r}), [E_i] \in W^e \) are linearly independent. But if this is not the case one has \( K_{S_r} = \pm ae \). This is impossible since \( W^K_{S_r} \) is even and \( e \) defines an ordinary wall unless \( S_g \) is homeomorphic to \((CP^1 \times CP^1) \#CP^2\) and in this case \( K^2_{S_g} = 7 \) is not a complete square.

So \( W^\psi_{*}f_{*}l_i \) is a collection of ordinary walls of the P-cell \( K \) with a nonempty intersection \( \cap W^\psi_{*}f_{*}l_i = \psi_{*}f_{*}H^2(S_{gm}) \), and for any such collection one has \( \psi_{*}f_{*}l_i = [E_i] \), as follows from Prop3.6, ch.2 of [FM].

**Corollary 2.2.** No surface of general type can be diffeomorphic to a rational one.

**Proof.** Assume that there exists a diffeomorphism \( f : S_r \rightarrow S_g \). By Lemma 2.1 one can also assume that there exists a blow down \( S_r \rightarrow \bar{S} \) such that \( f^*(H^2(S_{gm}) = H^2(\bar{S}) \).
$H^2(\tilde{\mathcal{S}})$. It follows from [PT] that there are classes $p_1 \in H^4(S_{mg}), \beta \in H^2(S_{mg}), p_1 > -8$ and a chamber $C \subset \mathcal{P}(S_{mg})$ such that

$$\gamma_{p_1,\beta,C}^{S_{mg}} \neq 0, \gamma_{p_1,\beta}^{\tilde{S},f*,f*} = 0.$$

Now by Theorem we have a contradiction

$$0 \neq \gamma_{p_1,\beta,C}^{S_{mg}} = \gamma_{p_1,\beta}^{S_g} + \sum l_i, \gamma_{p_1,\beta}^{S_r} + \sum e_i, \gamma_{p_1,\beta}^{\tilde{S},f*,f*} = 0.$$

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