The Baby Skyrme Models and Their Multi-Skyrmions

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Abstract

We analyse the structure of minimal-energy solutions of the baby Skyrme model for any topological charge \( n \); the baby multi-skyrmions. Unlike in the (3+1)D nuclear Skyrme model, a potential term must be present in the (2+1)D baby Skyrme model to ensure stability. The form of this potential term has a crucial effect on the existence and structure of baby multi-skyrmions. The simplest holomorphic baby Skyrme model has no known stable minimal-energy solution for \( n \) greater than one. The other baby Skyrme model studied in the literature possesses non-radially symmetric minimal-energy configurations that look like ‘skyrmion lattices’ formed by skyrmions with \( n = 2 \). We discuss a baby Skyrme model with a potential that has two vacua. Surprisingly, the minimal-energy solution for every \( n \) is radially-symmetric and the energy grows linearly for large \( n \). Further, these multi-skyrmions are tighter bound, have less energy and the same large \( r \) behaviour than in the model with one vacuum. We rely on numerical studies and approximations to test and verify this observation.

1 Introduction

The baby Skyrme model is a modified version of the (2+1)D \( S^2 \) sigma model. The addition of a potential and a Skyrme term to the lagrangian ensures stable solitonic solutions. The Skyrme term has its origin from the nuclear Skyrme model proposed in \cite{11} and the baby Skyrme model can therefore be viewed as its (2+1)D analogue. However, in (2+1) dimensions, a potential term is necessary in the baby Skyrme models to ensure stability of skyrmions; this term is optional in the (3+1)D nuclear Skyrme model.

The form of the potential term is largely arbitrary and gives rise to a multitude of possible baby Skyrme models. In the literature, two specific models have been studied in great detail (see \cite{7},\cite{6}). The simplest holomorphic model does not seem to admit stable n-skyrmions where \( n \) is greater than one. (We define an \( n \)-skyrmion to be the minimal-energy solution with topological charge \( n \).) And the ‘baby Skyrme model’ \cite{1} with a very simple potential possesses rather beautiful, non-radially symmetric multi-skyrmions. A new\footnote{it is called baby Skyrme model!.. we call it old baby Skyrme model to avoid confusion.} slightly modified potential gives rise to a remarkably different structure for multi-skyrmions: we call it the new baby Skyrme model. In fact, we show that the energy density of all \( n \)-skyrmions turns out to be radially symmetric configurations, namely rings...
of larger and larger radii. Clearly, the choice of the potential term has a major impact on the formation of multi-skyrmions and their shape.

We start with a short introduction to the baby Skyrme models. The earlier results for the two baby Skyrme models are re-calculated and reviewed in the light of their multi-skyrmion structure. We present a numerical and theoretical study of the new baby Skyrme model. We create \( n \)-skyrmions by putting \( n \) 1-skyrmions in an attractive channel. They form a bound state which we relax to the minimal-energy state. We find radially-symmetric new baby multi-skyrmions solutions. Therefore, we are led to look for static hedgehog solutions. Finally, we make some general comments about the existence and structure of multi-skyrmions depending on the choice of the potential term.

2 Baby Skyrme models

2.1 Non-trivial Topology & ‘Stable’ Lagrangian

Baby Skyrme models\(^4\) admit stable field configurations of finite energy and solitonic nature. These baby skyrmions are topological solitons. Their existence is a consequence of the non-trivial topology of the mapping of physical space into field space at a given time \( t \):

\[
\mathcal{M}: \mathcal{S}^2 \rightarrow \mathcal{S}^2. \tag{1}
\]

Here, physical space \( \mathcal{R}^2 \) is compactified to \( \mathcal{S}^2 \) by requiring spatial infinity to be equivalent in each direction. This one-point compactification is necessary to ensure a non-trivial mapping. The target manifold (or internal space) is described by a three-dimensional vector \( \vec{\phi} \) with \( \vec{\phi} \cdot \vec{\phi} = 1 \). The non-trivial topology allows to classify maps into equivalence classes. Each of which has a unique conserved quantity: the topological charge

\[
T = \frac{1}{4\pi} \epsilon^{abc} \int dxdy \phi_a (\partial_x \phi_b) (\partial_y \phi_c) \tag{2}
\]
given in integer units.

Further, stability is ensured by an appropriate choice of lagrangian terms of field derivatives and a potential. The lagrangian has the form

\[
L = \partial_{\mu} \vec{\phi} \cdot \partial^\mu \vec{\phi} - \theta_S \left[ (\partial_{\mu} \vec{\phi} \cdot \partial^\mu \vec{\phi})^2 - (\partial_{\mu} \vec{\phi} \cdot \partial_{\nu} \vec{\phi})(\partial^\mu \vec{\phi} \cdot \partial^\nu \vec{\phi}) \right] - \theta_V V(\vec{\phi}) \tag{3}
\]

and consists of three terms; from left to right: the sigma model, the Skyrme and the potential term. At a classical level, the coefficient of the sigma model term can always be set to one by re-defining \( \theta_S \) and \( \theta_V \). Thus, there are two free parameters in the model. Each term has a different scaling behaviour and, together, they ensure stability according to Derrick’s theorem[10, pages 47-48]. We require that the potential vanishes at infinity for a given vacuum field value; for example \( \vec{\phi} = (0,0,1) \). Care should be taken that the potential term is invariant under the \( SO(2) \) group transformation of \( \vec{\phi} \); this becomes vital for the use of the hedgehog ansatz. There is a further possibility that the potential is also zero for other values of the field. Actually, the fact that the potential vanishes at infinity makes the energy finite and justifies the one-point compactification of physical space discussed above.

\(^4\)see review [7]
2.2 Hedgehog Static Solutions

The static energy functional density of the baby Skyrme model is

\[ E = \left( \partial_i \vec{\phi} \cdot \partial_i \vec{\phi} \right)^2 - \left( \partial_i \vec{\phi} \cdot \partial_j \vec{\phi} \right) \left( \partial_i \vec{\phi} \cdot \partial_j \vec{\phi} \right) + \theta S V(\vec{\phi}). \] (4)

We look for solutions of the corresponding Euler-Lagrange equation. This is a very difficult task.

The hedgehog ansatz provides a starting point of our search for static solutions; in polar coordinates

\[ \vec{\phi} = \begin{pmatrix} \sin[f(r)] \sin(n\theta - \chi) \\ \sin[f(r)] \cos(n\theta - \chi) \\ \cos[f(r)] \end{pmatrix}. \] (5)

Note that \( n \) is a non-zero integer (it is the topological charge as we will discover later), \( \theta \) the polar angle, \( \chi \) a phase shift and \( f(r) \) the profile function satisfying certain boundary conditions. The hedgehog field (5) is chosen, because it is invariant under the maximal group of symmetry that leaves the energy functional invariant for non-zero topological charge (see [6, page 167]). According to the ‘Principle of Symmetric Criticality’ or ‘Coleman-Palais theorem’ ([3, pages 72-76]), we can search for static solutions invariant under any symmetry by solving the variational problem for the invariant field.

The integrated energy density takes the form

\[ E = (4\pi)^{1/2} \int_0^\infty r dr \left( f'^2 + n^2 \frac{\sin^2 f}{r^2} (1 + 2\theta S f'^2) + \theta V \bar{V}(f) \right) \] (6)

where \( f' = \frac{df}{dr} \). The energy density depends only on the profile function \( f(r) \): the invariant field. It is independent of the polar angle and has a radial symmetry. Then, the corresponding Euler-Lagrange equation with respect to the invariant field \( f(r) \) leads to a second-order ODE,

\[ \left( r + 2\theta S n^2 \frac{\sin^2 f}{r} \right) f'' + \left( 1 - 2\theta S n^2 \frac{\sin f}{r^2} f + 2\theta S n^2 \frac{\sin f f'}{r} \right) f' - \frac{n^2 \sin f \cos f}{r} - \frac{\theta V d\bar{V}(f)}{2 df} = 0, \] (7)

which we re-write in terms of the second derivative of the profile function:

\[ f'' = F(f, f', r). \] (8)

The profile function \( f(r) \) is a static solution of the baby Skyrme model. These static solutions are certainly critical points, but not necessarily global minima. However, it has been proven that the hedgehog solution of the nuclear Skyrme model is the minimal-energy solution for topological charge one (see [3, pages 80-88]). Further, an explicit hedgehog solution with the topological charge one exists for the holomorphic baby Skyrme model and has the lowest energy. Therefore, it is reasonable, but not proven here, that the hedgehog solution for topological charge one is the minimal-energy solution.

The topological charge takes the form

\[ T = -\frac{n}{2} \int_0^\infty r dr \left( \frac{f' \sin f}{r} \right) = \frac{n}{2} [\cos f(\infty) - \cos f(0)]. \] (9)
The boundary conditions for the profile function need to be fixed. Our value of the vacuum at infinity is \( \vec{\phi} = (0, 0, 1) \) and we may choose
\[
\lim_{r \to \infty} f(r) = 0.
\] (10)

Then the value of the profile function at the origin needs to be
\[
f(0) = m\pi
\] (11)
where \( m \) is an odd integer. The topological charge is \( n \) in integer units. From now on, we write all other quantities in \( 4\pi \) units. All \( m \neq 1 \) solutions probably decay into \( m = 1 \) solutions. Thus, in this paper, we concentrate our attention on solutions corresponding to \( m = 1 \).

### 2.3 The Equation of Motion

A Lagrange multiplier term \( \lambda(\vec{\phi} \cdot \vec{\phi} - 1) \) needs to be included in the lagrangian (3) to take care of the \( S^2 \) constraint (see [10, pages 48-58]). The equations of motion for each field component \( \phi_a \) and \( \lambda \) are obtained via the Euler-Lagrange equation. Solving for \( \lambda \), the equation of motion takes the form
\[
\partial_{\mu} \partial^\mu \phi_a - (\vec{\phi} \cdot \partial_{\mu} \vec{\phi}) \partial_{\mu} \phi_a - 2\theta_S[(\partial_{\nu} \vec{\phi} \cdot \partial^\nu \vec{\phi}) \partial_{\mu} \partial^\mu \phi_a + (\partial_{\mu} \vec{\phi} \cdot \partial^\nu \vec{\phi}) \partial_{\nu} \phi_a] \\
- (\partial_{\nu} \vec{\phi} \cdot \partial_{\mu} \vec{\phi}) (\vec{\phi} \cdot \partial_{\nu} \vec{\phi}) \partial_{\nu} \phi_a + (\partial_{\mu} \vec{\phi} \cdot \partial^\nu \vec{\phi}) (\partial_{\nu} \vec{\phi} \cdot \partial_{\mu} \vec{\phi}) \phi_a \\
- (\partial_{\nu} \vec{\phi} \cdot \partial_{\mu} \vec{\phi}) (\vec{\phi} \cdot \partial_{\nu} \vec{\phi}) \phi_a + 2\theta_V \frac{dV}{d\phi_3}(\delta_{31} - \phi_a \phi_3) = 0
\] (12)
which we re-write in terms of the acceleration of the field \( \phi_a \):
\[
\partial_{tt} \phi_a = K_{ab}^{-1} \mathcal{F}_b \left( \vec{\phi}, \partial_t \vec{\phi}, \partial_t \vec{\phi} \right)
\] (13)
with
\[
K_{ab} = (1 + 2\theta_S \partial_t \vec{\phi} \cdot \partial_t \vec{\phi}) \delta_{ab} - 2\theta_S \partial_a \phi_b \partial_t \phi_b.
\] (14)
We find that the inverse matrix of \( K \) exists in an explicit, but rather messy form. The equation of motion is a second order PDE.

### 3 Theoretical Prediction

Is it possible to predict the general features of the multi-skyrmions? The main obstacle is the non-linearity of the DEs even in a simpler form like in the hedgehog ansatz. Nevertheless, at special points i.e. the boundaries, approximations can be made which simplify the DEs. Derrick’s theorem allows us to give further quantitative predictions. Both serve as consistency checks for our numerical work and help our understanding of the models.

The value of the field is known at two space locations. Those special points are \( r = 0 \) and \( r = \infty \). In the hedgehog ansatz, the ODE can be approximated around these points.

- At the origin, the profile function is approximated as
\[
f \simeq \pi + C_n r^n
\] (15)
and so

$$f' \simeq nC_nr^{n-1}$$  \hspace{1cm} (16)$$
as long as $$dV(f)\bigg|_{\text{large } r} \rightarrow 0$$. Further, the energy density at the origin is

$$E(0) = \frac{1}{2} \theta_V V \mid_{n=1}$$  \hspace{1cm} (n = 1)$$
$$E(0) = \frac{1}{2} \theta_V V \mid_{n \geq 2}$$  \hspace{1cm} (n \geq 2). \hspace{1cm} (17)$$

As one would expect, the energy density of the new baby Skyrme model is zero, because there is a further vacuum at the origin; \(V[\pi] = 0\). It is non-zero only in the topological sector one. Clearly, if the hedgehog solutions are the minimal-energy solutions, the new baby \(n\)-skyrmions are ring configurations. However, in the case of the old or simplest holomorphic baby Skyrme model, the energy density is always non-zero for any static hedgehog solution. These hedgehog solutions do not seem to minimise the energy as well as in the new baby model. Our numerical results confirm this.\footnote{we just re-define \(\theta_V\) for one of the models.}

- At large \(r\), the ODE reduces to

$$f'' + \frac{1}{r} f' - \frac{n^2}{r^2} f - \frac{\theta_V}{2} \frac{dV(f)}{df} \mid_{\text{large } r} = 0.$$  \hspace{1cm} (18)$$

The last term, arising from the potential, can be neglected for some potentials in a consistent way, because it is small compared to the other terms. This is the case for the holomorphic model where this term is of order \(f^T\). However, the term has to be included for the old and new baby Skyrme model and gives \(-\frac{\theta_V}{2} f\) or \(-\theta_V f\) respectively. Looking at large \(r\), the old and the new baby skyrmions behave in the same way\footnote{also called Hobart-Derrick theorem}. Actually, the DE is that of a static Klein-Gordon field with a radially symmetric form where \(\theta_V\) plays the role of the meson mass. As discussed above, the real difference lies in the small \(r\) and medium \(r\) region. For the new baby Skyrme model, the equation \(\Box\) gives

$$f'' + \frac{1}{r} f' - f\left(\frac{n^2}{r^2} + \theta_V\right) = 0.$$  \hspace{1cm} (19)$$

The coefficient of the potential term is present here. The potential localises the skyrmion exponentially. Solving for appropriate boundary conditions, the profile function decays exponentially

$$f(r) \rightarrow \frac{1}{\sqrt{\theta_V r}} \exp(-\theta_V r).$$  \hspace{1cm} (20)$$

The Derrick theorem\footnote{seen \(\Box\), pages 52-54} provides a necessary but not sufficient condition for the existence of stable solutions. Under a simple scale transformation \(r \rightarrow \lambda r\), the total energy changes to a function of \(\lambda\) and the non-scaled energies of the three terms:

$$E[\tilde{f}(\lambda r)] = E_\sigma + \lambda^2 \theta_S E_s + \lambda^{-2} \theta_V E_V.$$

The sigma term is scale invariant. The derivative of the energy with respect to \(\lambda\) at \(\lambda = 1\) has to be zero if a stable solution exists. This implies

$$\theta_S E_s = \theta_V E_V.$$  \hspace{1cm} (22)$$
Our numerical results have to fulfill this condition. Further (21) suggests that the scaling effect can be un-done by redefining \( \theta_S \) to \( \lambda^2 \theta_S \) and \( \theta_V \) to \( \lambda^{-2} \theta_V \). In fact, writing the DE (8) in terms of \( \tilde{f}(\tilde{r}, \tilde{\theta}_S, \tilde{\theta}_V) \) and using \( \tilde{r} = \lambda r \), we find

\[
\tilde{f}(\lambda r, \lambda^{-2} \theta_S, \lambda^2 \theta_V) = f(r, \theta_S, \theta_V).
\]

(23)

Substituting \( \tilde{f} \) into the energy functional gives exactly the same energy as \( f \) does. If two models with coefficients \( \theta_S \) and \( \theta_V \) respectively, \( \tilde{\theta}_S \) and \( \tilde{\theta}_V \) satisfy

\[
\theta_S \theta_V = \tilde{\theta}_S \tilde{\theta}_V,
\]

(24)

then their stable solutions have the same energy. This is a further check on our numerical results.

4 Numerical Techniques

The baby Skyrme model is a non-integrable system and explicit solutions to its resulting differential equations are nearly impossible to find. Numerical methods are the only way forward. We need (13) for the time-evolution and relaxation of an initial configuration and use (8) to find static hedgehog solutions. These DEs are re-written as sets of two first order DEs. We discretise DEs by restricting our function to values at lattice points and by reducing the derivatives to finite differences (as explained in [8], see also [9]). We take the time step to be half the lattice spacing: \( \delta t = \frac{1}{2} \delta x \). We use fixed boundary conditions i.e. we set the derivatives to zero at the boundary. We check our numerical results via quantities conserved in the continuum limit and by changing lattice spacing and number of points. Moreover, we compare them with theoretical predictions.

Looking for static hedgehog solutions The static hedgehog solutions of (8) are found by the shooting method using the 4th order Runge-Kutta integration and the boundary conditions (10) and (11). Alternatively, one can use a relaxation technique like the Gauss-Seidel over-relaxation ([9]) applied to an initial configuration with the same boundary conditions.

Looking for multi-skyrmions We construct \( n \)-skyrmions by relaxing an initial set-up of \( n \) 1-skyrmions with relative phase shift of \( \frac{\pi}{n} \). Using the dipole picture developed by Piette et al. [1], two old (or new) baby 1-skyrmions attract each other for a non-zero value of the relative phase; phase shift of \( \frac{\pi}{2} \) for maximal attraction. A circular set-up is crucial as they maximally attract each other and 1-skyrmions do not form several states that repel each other. One possible objection to a circular set-up is its apparent discrete symmetry, the cyclic group \( Z_n \). However, the discretised PDE on a finite square lattice is not invariant under \( Z_n \) as we impose boundary conditions. Further, the linear superposition is only an approximate solution of the model and the 1-skyrmions used are produced from the hedgehog ansatz. In this sense, lattice effects and small integration errors even provide useful small perturbations. We have run simulations for non-circular set-ups, but either the 1-skyrmions take longer to fuse together or they fuse into many bound-states and repel each other. We run our simulations on grids with \( 200^2 \) or \( 300^2 \) lattice points and the lattice was \( \delta x = 0.1 \) or 0.05. However, for large topological charge, we need larger grids and the relaxation takes a long time. The corresponding hedgehog solution as an initial set-up usually works well and is faster, but biased due to its large symmetry group. The time-evolution of an initial configuration is determined by the equation of motion (13). We are using the 4th order Runge-Kutta method to evolve the initial set-up and correction techniques to keep the errors small. We relax i.e. take out kinetic energy by using a damping (or friction) term.

\[\text{we use the 9-point laplacian}\]
**Initial set-up:** The initial field configuration is a linear superposition of static solutions with or without initial velocity; typically we use a circular set-up of \(n\) 1-skyrmions with a \(\frac{\pi}{n}\) phase-shift between each other (for maximal attraction). The superposition is justified, because the profile function decays exponentially. The superposition is done in the complex field formalism i.e. where \(W\) is the stereographic projection of the \(\vec{\phi}\) field of \(S^2\) (see [7]). We use the profile function of a static solution (typically of topological charge one) to obtain

\[
W = \tan\left(\frac{f(r)}{2}\right) e^{-in\theta}.
\]  

This equation holds in the rest frame of a static skyrmion solution centred around its origin and \(\frac{dW}{dt} = 0\). We may introduce moving solutions by switching to a different frame of reference. This can be done by performing a Lorentz boost on the rest frame of a given \(W\), because \(W\) is a Lorentz scalar. Now it is easy to construct a linear superposition of individual, moving or not, baby skyrmion solutions \(W_\alpha\) by

\[
W(x, y) = \sum_\alpha W_\alpha(x - x_\alpha, y - y_\alpha)
\]

where \((x_\alpha, y_\alpha)\) is the location of the centre of the \(\alpha\)th skyrmion. It is important that the different skyrmions are not too close to each other. Finally, the complex field is re-written in terms of the field \(\vec{\phi}\) and its derivative.

**Correction techniques:** The integration method introduces small errors which eventually add up. In terms of the \(S^2\) constraint, this corresponds to the field leaving the two-sphere. Hence, we need to project the field back onto the sphere. The simplest and sufficient projections are

\[
\phi_a \rightarrow \frac{\phi_a}{\sqrt{\vec{\phi} \cdot \vec{\phi}}}
\]

and

\[
\partial_t \phi_a \rightarrow \partial_t \phi_a - \frac{\partial_t \vec{\phi} \cdot \vec{\phi}}{\phi \cdot \phi} \phi_a.
\]

Of course, the space derivatives may also be corrected. See [8] for further discussions.

**Relaxation technique:** A damping term in the equation of motion will gradually take the kinetic energy out of the system. The equation (13) changes to

\[
\partial_{tt} \phi_a = K_{ab}^{-1} F_b \left(\vec{\phi}, \partial_t \vec{\phi}, \partial_t \vec{\phi}\right) - \gamma \partial_t \phi_a
\]

where \(\gamma\) is the damping coefficient. We set \(\gamma\) to 0.1, but most values will do as long as they are not too large. Another approach would be to absorb the outwards travelling kinetic energy waves in the boundary region.

## 5 The Different Models

So far, the literature on baby skyrmions reports work on the holomorphic model with \(V = (1 + \phi_3)^4\) and the old baby Skyrme model with \(V = 1 - \phi_3^4\). There are no stable multi-skyrmions found in the holomorphic model. However, the old baby Skyrme model possesses non-radially symmetric minimal-energy solutions. We will show that the new baby Skyrme model with \(V = 1 - \phi_3^2\) has radially symmetric multi-skyrmions.
5.1 Holomorphic Model

The simplest holomorphic model has the potential \( V = (1 + \phi_3)^4 \) and is the first baby Skyrme model studied in the literature ([4], [12], [7]). We have re-done the calculations and agree with the literature. This agreement provides a check on our numerical methods.

The holomorphic potential is unique in the sense that its model admits an explicit solution for a skyrmion with topological charge one (we call it a 1-skyrmion). To leading order, the asymptotic behaviour does not depend on the potential. The skyrmion is polynomially localised. The force between two holomorphic skyrmions is always repulsive. This repulsion can be overcome by sending the two 1-skyrmions against each other at a sufficiently high speed. Above a critical value, they overlap and form an intermediate state. However, this state is not stable and the two 1-skyrmions scatter at 90 degrees. No multi-skyrmions are known to exist in this model.

5.2 Old Baby Skyrme Model

The old baby Skyrme model has been extensively studied in [6] and [5]. The potential \( V = 1 - \phi_3 \) gives rise to very structured multi-skyrmions. The configurations are crystal-like in the sense that its building block is a 2-skyrmion. We have re-done Piette et al.’s computations for multi-skyrmions and confirm their results. We use their coefficients i.e. add a factor \( \frac{1}{2} \) to the sigma model term, \( \theta_S = 0.25 \) and \( \theta_V = 0.1 \) [see (3)].

**Multi-skyrmions** We construct a 2-skyrmion by sending two 1-skyrmions against each other. Put close to each other, they merge into an oscillating bound-state (see next section for picture) which leads to the stable radially symmetric 2-skyrmion by numerical relaxation. We have checked its stability by evolving it in time without relaxation. We repeat the procedure and find that all higher \( n \)-skyrmions are not radially symmetric. We extend Piette et al.’s work to \( n = 7 \), \( n = 8 \) and \( n = 9 \) to make sure that the \( n \)-skyrmions are ‘skyrmion crystals’ formed by 2-skyrmions. In this paper, we only present the formation of an 8-skyrmion (see figure 1). We put eight 1-skyrmions on a circle with a phase shift of \( \pi/n \) between neighbouring 1-skyrmions. The initial configuration of eight 1-skyrmions is time-evolved and relaxed. The system starts moving to four 2-skyrmions which re-arrange themselves. Slowly, the system moves towards a stable configuration, the 8-skyrmion. The building block of this crystal-like structure is the 2-skyrmion.

**Hedgehog Ansatz** Solutions of the hedgehog ansatz can be found numerically by solving equation (8) via the shooting method. As seen above, only the \( n = 1 \) and \( n = 2 \) hedgehog configurations are global minima. The 1-skyrmion has a hill shape and is exponentially localised. Unlike in the holomorphic model, the asymptotic behaviour of the profile function does depend on the potential term to leading order. The 2-skyrmion is a ring-like configuration. A \( n = 3 \) static hedgehog solution exists, but does not have the lowest energy in its topological sector. A time-evolution shows that it is unstable and relaxes to the non-radially symmetric 3-skyrmion.

**Energies** Table 1 reproduces Piette et al.’s results and we will compare them to the new baby model’s. The ionisation energy \( E_{kl} \) is defined as the energy you have to add to a \( n \)-skyrmion to break it up into a \( k \)-skyrmion and a \( l \)-skyrmion:

\[
E_{kl} = E_n - (E_k + E_l).
\]

As \( n \) increases, the ionisation energy decreases (but not monotonically) and the \( n \)-skyrmions become less bound. The 2-skyrmion emission is the energetically most favourable break-up mode. An
Figure 1: Contour plots of energy density: Eight 1-skyrmions merge together (pictures 1-4)

Figure 2: Contour plots of energy density: Bound state relaxes into an 8-skyrmion (pictures 5-8)
| Charge | Energy  | Energy per skyrmion | Break-up modes | Ionisation Energy |
|--------|---------|---------------------|----------------|-------------------|
| 1      | 1.549   | 1.549               | -              | -                |
| 2      | 2.907   | 1.454               | 1 + 1          | 0.191            |
| 3      | 4.379   | 1.460               | 2 + 1          | 0.077            |
| 4      | 5.800   | 1.450               | 2 + 2          | 0.014            |
|        |         |                     | 3 + 1          | 0.128            |
| 5      | 7.282   | 1.456               | 3 + 2          | 0.005            |
|        |         |                     | 4 + 1          | 0.068            |
| 6      | 8.693   | 1.449               | 4 + 2          | 0.015            |
|        |         |                     | 3 + 3          | 0.066            |
|        |         |                     | 5 + 1          | 0.138            |

Table 1: Multi-skyrmions of the old baby Skyrme model

emission of a 2-skyrmion takes the smallest amount of kinetic energy to break up an 8-skyrmion. Thus, it is justified to think about a \( n \)-skyrmion as a collection of 2-skyrmions bound together.

Note that the data comes from the full time-evolution. Using the hedgehog ansatz leads to slightly different, more accurate, values. There the energy of a 1-skyrmion is 1.564 and the energy of a 2-skyrmion is 2.936. This effect is due to the finite lattice and the fact that we can use more lattice points in the hedgehog ansatz.

5.3 New Baby Skyrme Model

The new baby Skyrme model\(^7\) exhibits a completely different structure for multi-skyrmions. In fact, the multi-skyrmions are ring-like configurations; their radii being proportional to their topological charge. The form of the potential is

\[
V = 1 - \phi_3^2 = (1 - \phi_3)(1 + \phi_3). \tag{31}
\]

The potential has two vacua; for \( \phi_3 = 1 \) and \( \phi_3 = -1 \). At infinity, the old and the new baby Skyrme models have the same vacuum \( \phi_3 = 1 \) and behave in the same way. They only differ for small \( r \). Another important fact is that the lagrangian of the new baby Skyrme model is invariant under \( \phi \rightarrow -\phi \).

Numerical Results

**Looking for multi-skyrmions** First, two new baby 1-skyrmions scatter in the same way as the old baby skyrmions do. Figure [3](#) shows how the two 1-skyrmions attract each other, form a bound-state, scatter away at 90 degrees, get slowed down by their mutual attraction, attract each other again and so on. This oscillating but stable bound-state is an excited state of the 2-skyrmion solution. Taking out the kinetic energy, the bound-state relaxes to the 2-skyrmion; a ring. We have looked at all higher \( n \)-skyrmions. Figure [4](#) presents the results of one of our simulations: the formation of a 5-skyrmion. The skyrmions attract each other and merge into intermediate states. Relaxation takes out the kinetic energy and the unstable intermediate states merge together. They form an irregular ring configuration that moves like a vibrating closed string. Slowly, the configuration settles down to a radially symmetric form due to the loss of kinetic energy.

\(^7\) see also [1](#)
Figure 3: Contour plot of energy density: Formation of 2-skyrmion

Figure 5 shows the final configuration of multi-skyrmions from charge two to five. Actually, the larger rings are slightly deformed. This effect is due to boundary effects and reduced by using larger grids. Thus, we have convinced ourselves of the radial symmetry of \( n \)-skyrmions.

In the hedgehog ansatz Clearly, the \( n \)-skyrmions are radially symmetric and we can study them in the hedgehog ansatz. Numerically speaking, the problem is reduced to one dimension and, effectively, we can take as many lattice points as we want. We want to compare the multi-skyrmions to those of the old baby Skyrme model. We add a factor of \( \frac{1}{2} \) to the sigma term. The coefficient \( \theta_V \) is set to half of the value of \( \theta_V \) in the old baby Skyrme model: \( \theta_V = 0.05 \). Now, the old baby Skyrme model has exactly the same large \( r \) behaviour i.e same pion mass. Then, we set \( \theta_S \) so that the energy of the new baby 1-skyrmion is now approximately the same as the old baby 1-skyrmion in the hedgehog ansatz: \( \theta_S = 0.44365 \). This convention puts both models on an equal footing. Note that there is a certain ambiguity about the choice of the coefficients. The energy is a function of \( \theta_S \theta_V \) and a compensating change of both coefficients gives the same energy.

Again, we find the hedgehog solutions by solving (8) using the shooting method. Figure 6 shows all solutions up to topological charge 10. The higher the charge the more difficult it becomes to find solutions numerically. The shooting method becomes more and more sensitive to the numerically determined value of its derivative at the origin i.e. \( C_{10} \approx 10^{-10} \) (see [5]). The numerical results are very interesting. The peaks of the energy density of rings converge to an asymptotic height and their position shifts by an asymptotically constant amount. This observation deserves some further understanding.

Relation between energy peak and its location Our numerical results show that the profile function \( f \) at the position of the energy density peak, \( r = d(n) \), approaches the value \( \frac{\pi}{2} \) for
Figure 4: Contour Plot of Energy Density: Formation of 5-skyrmion

Figure 5: Contour Plot of Energy Density: Rings of multi-skyrmions from $n = 2$ to $n = 5$
Figure 6: The energy density of new baby skyrmions up to charge 10

large \( n \). The energy density at this point reduces to

\[
\mathcal{E}[d(n), n] = f'_d + \frac{n^2}{d^2}(1 + 2\theta_S f'_d) + \theta_V. \tag{32}
\]

Its value depends on the derivative of the profile function at \( d(n) \) and the ratio between \( n \) and \( d(n) \). Now, figure 6 shows that the height of the energy density of the peak is approximately a constant for large \( n \) i.e.

\[
\lim_{n \to \infty} \mathcal{E}[d(n), n] = \text{constant}. \tag{33}
\]

The larger the topological charge the more the multi-skyrmion approaches the peak and shape of the ‘asymptotic multi-skyrmion’. Using this empirical knowledge leads us to conclude that, in the large \( n \) limit,

\[
\lim_{n \to \infty} f'_d = \alpha^2 \tag{34}
\]

and

\[
\lim_{n \to \infty} \frac{n^2}{d(n)^2} = \beta^2, \tag{35}
\]

where \( \alpha \) and \( \beta \) are constants. And, the peak of the ring shifts by a fixed amount from a skyrmion of charge \( n \) to one of charge \( n + 1 \) i.e.

\[
d(n) = \frac{1}{\beta} n. \tag{36}
\]

This relation agrees with our numerical work and provides a good consistency check. The energy density becomes

\[
\mathcal{E}[d(\infty)] = \alpha^2 + \beta^2(1 + 2\theta_S \alpha^2) + \theta_V. \tag{37}
\]

The shape of the \( n \)-skyrmions approaches that of the ‘asymptotic multi-skyrmion’. We can approximate the configurations by a finite box of height \( \mathcal{E}_d \) around the point \( r = d(n) \). This gives us the dependence of the total energy on the topological charge i.e.

\[
E = \int_{d(n) - \alpha}^{d(n) + \alpha} drr\mathcal{E}_d = 2\mathcal{E}_d da(n) = (2\mathcal{E}_d \alpha \beta^2)n \tag{38}
\]
| Charge | Energy | Energy per skyrmion | Break-up modes | Ionisation Energy |
|--------|--------|---------------------|----------------|------------------|
| 1      | 1.564  | 1.564               | -              | -                |
| 2      | 2.809  | 1.405               | 1 + 1          | 0.319            |
| 3      | 4.112  | 1.371               | 2 + 1          | 0.262            |
|        |        |                     | 1 + 1 + 1      | 0.580            |
| 4      | 5.433  | 1.358               | 2 + 2          | 0.186            |
|        |        |                     | 3 + 1          | 0.243            |
| 5      | 6.761  | 1.352               | 3 + 2          | 0.160            |
|        |        |                     | 4 + 1          | 0.235            |
| 6      | 8.094  | 1.349               | 3 + 3          | 0.130            |
|        |        |                     | 4 + 2          | 0.148            |

Table 2: Multi-skyrmions of the new baby Skyrme model

using (36). Asymptotically, the total energy grows linearly with $n$. Linear dependence suggests that radially symmetric solutions are global minima. The system does not switch to less symmetric configurations, because they do not have linear dependence on $n$.

**Energies** Table 2 shows our numerical results for the energies of the multi-skyrmions up to $n = 6$. First, the 2-skyrmion has a lower energy than an old baby 2-skyrmion, but looks exactly the same at large $r$. The multi-skyrmions do not break up via a 2-skyrmion emission (see last section) but into two similar configurations i.e. $5 \rightarrow 3 + 2$ or $6 \rightarrow 3 + 3$. This can be seen from the ionisation energy.

Further the ionisation energy and the energy per skyrmion decreases to an asymptotic value for large $n$ (unlike the old baby multi-skyrmions). The monotonic decrease of the ionisation energy shows that the large $n$-skyrmions become less stable: a smaller addition of kinetic energy can break them apart. Nevertheless, they are much tighter bound and more stable than their old baby analogues.

### 6 Summary and Open Questions

Clearly, the choice of the potential term has a crucial effect on the structure of multi-skyrmions. The comparison between the new and the old baby Skyrme model has proved to be very interesting. Both models have the same asymptotic behaviour, but possess completely different multi-skyrmion structures. The new baby Skyrme model has radially symmetric minimal-energy solutions for all topological charges whereas the old baby multi-skyrmions are 'skyrmion lattices' formed by 2-skyrmions. New baby multi-skyrmions are tighter bound and have less energy than their old baby analogues.

We have backed up our numerical results by monitoring conserved quantities (like energy, topological charge, $S^2$ constraints), comparing with approximations for small and large $r$, checking the relation between energy peaks and their position and verifying conditions imposed by Derrick's theorem.

Obviously, a general framework that predicts the structure of multi-skyrmions for a given potential is desirable. However, we have not been able to achieve this goal. Rather, we state some empirical laws derived from our numerical experiments.
**Existence of multi-skyrmions.** Baby Skyrme models seem to admit stable multi-skyrmions only if the force between two 1-skyrmions can be attractive. Two 1-skyrmions overlap, form an intermediate state and scatter at 90 degrees. Only an attractive force between them can overcome the energy due to the scattering and lead to a bound state. A related observation leads us to the conjecture that if the asymptotic behaviour does not depend on the potential coefficient, the force between two 1-skyrmions is repulsive; see the holomorphic model.

**Structure of multi-skyrmions.** Clearly, the potential shapes the structure of multi-skyrmions. Unfortunately, we can only show this fact by our numerical results. The existence of more than one vacuum seems to be crucial to the radial symmetric shape of the new baby $n$-skyrmions. It might well be possible to prove that potentials with more than one vacuum lead to radially symmetric multi-skyrmions. However, we were not able to back such claims by a theoretical study.

To conclude, some interesting questions arise from the study of multi-skyrmions in the baby Skyrme models and are worth investigating further.

- The choice for potential terms is largely arbitrary. The study of other potentials and in particular those with multiple vacua could help to clarify the issues surrounding the existence and structure of multi-skyrmions. Further, on one hand, potentials with multiple vacua can still have the same large distance behaviour like old baby skyrmions. On the other hand, their multi-skyrmion solutions should have rather exotic shapes, because the dynamics tries to have multiple vacua. Unlike the conventional smooth lump 1-skyrmion, their 1-skyrmion lumps may have riddles in them. Or, one might like to rephrase our question and ask whether multi-vacua models have circular domain walls, too.

- Of course, the application of the baby Skyrme model in the quantum regime requires an appropriate quantization scheme. The radially symmetric new baby multi-skyrmion solutions simplify this task. Unlike the old baby Skyrme model, the mass correction of all multi-skyrmions can be calculated numerically without considerable computer power. The next step should be the quantization of the baby Skyrme model.

- Are there any applications? If there are 90 degrees scattering phenomena observed in experiments, then the analysis of bound-states can lead to the determination of the potential responsible for these phenomena.

- The choice of the potential crucially shapes the structure of multi-skyrmions. This is probably not true in the nuclear Skyrme model, because the sigma term is not scale invariant. Often, an ‘old baby Skyrme’-type potential is included to have a pion mass and exponential decay. There is no argument why the potential term used cannot have two vacua. What happens in a nuclear Skyrme model with a ‘new baby Skyrme potential’? Is the energy of the $n$-skyrmions lower?

It is possible to view the ring-like multi-skyrmions of the new baby Skyrme model as circular domain walls separating the two vacua. This is an interesting and conceptually clearer viewpoint. Indeed, at the location of the peak of the energy density, the field switches from one vacuum value to the other. These domain walls also have a topological charge.

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