The Zeeman Effect for the Relativistic Bound State

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Abstract

In the context of a relativistic quantum mechanics with invariant evolution parameter, solutions for the relativistic bound state problem have been found, which yield a spectrum for the total mass coinciding with the nonrelativistic Schrödinger energy spectrum. These spectra were obtained by choosing an arbitrary spacelike unit vector $n_\mu$ and restricting the support of the eigenfunctions in spacetime to the subspace of the Minkowski measure space, for which $(x_\perp)^2 = [x - (x \cdot n)n]_\perp^2 \geq 0$. In this paper, we examine the Zeeman effect for these bound states, which requires $n_\mu$ to be a dynamical quantity. We recover the usual Zeeman splitting in a manifestly covariant form.

1 Introduction

It has been shown [1] that the replacement

$$r = \sqrt{(r_1 - r_2)^2} \rightarrow \rho = \sqrt{(r_1 - r_2)^2 - (t_1 - t_2)^2} \quad (1.1)$$

in the argument of the usual central force potentials of non-relativistic mechanics leads to a relativistic problem, yielding a mass spectrum coinciding with the nonrelativistic Schrödinger energy spectrum, in the context of a relativistic quantum mechanics with invariant parameter [2] (the correspondence is established by the fact that $t_1 \rightarrow t_2$ in the nonrelativistic limit). These spectra are obtained when one chooses a spacelike unit vector $n_\mu$, ...
(g_{\mu\nu} = \text{diag}(-1,1,1,1) \Rightarrow n^2 = +1) and restricts the support of the eigenfunctions in spacetime to the subspace of the Minkowski measure space corresponding to the condition

\[(x_{\perp})^2 = [x - (x \cdot n)n]^2 \geq 0,\] (1.2)

where we denote by \(x \equiv x^\mu\) the relative coordinates \(x_1^\mu - x_2^\mu\), for the two body system, and \(x^2 = x^\mu x_\mu\). The restricted space, called the RMS (Restricted Minkowski Space), is transitive and invariant under the O(2,1) subgroup of O(3,1) leaving \(n_\mu\) invariant and translations along \(n_\mu\).

The two-body (Poincaré invariant) Hamiltonian in this theory,

\[K = \frac{p_1^\mu p_1^\mu}{2M_1} + \frac{p_2^\mu p_2^\mu}{2M_2} + V(\rho),\] (1.3)

is quadratic in the four momenta, and one may separate variables of the center of mass motion and relative motion in the same way as in the nonrelativistic theory,

\[K = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + V(\rho),\] (1.4)

where

\[P^\mu = p_1^\mu + p_2^\mu, \quad M = M_1 + M_2\]

\[p^\mu = (M_2 p_1^\mu - M_1 p_2^\mu)/M, \quad m = M_1 M_2/M.\] (1.5)

In \cite{1, 3}, \(n_\mu\) was chosen to be the z-axis, and the relative Hamiltonian

\[K_{\text{rel}} = \frac{p^\mu p_\mu}{2m} + V(\rho)\] (1.6)

was expressed in terms of coordinates with the parameterization

\[y^0 = \rho \sinh \beta \sin \theta, \quad y^1 = \rho \cosh \beta \sin \theta \cos \phi\]

\[y^2 = \rho \cosh \beta \sin \theta \sin \phi, \quad y^3 = \rho \cos \theta\] (1.7)

for which

\[(y^1)^2 + (y^2)^2 - (y^0)^2 \geq 0.\] (1.8)

It was shown in \cite{1, 3} that the eigenfunctions of \(K_{\text{rel}}\) form irreducible representations of SU(1,1) — in the double covering of O(2,1) — parameterized by the spacelike vector \(n_\mu\).
stabilized by the particular $O(2,1)$. In [3], an induced representation of $SL(2,C)$ was con-
structed, by applying the Lorentz group to the RMS coordinates $x^\mu$ and the frame orientation $n_\mu$, and studying the action on these wavefunctions. One first observes that wavefunctions
with support on 

$$x \in \text{RMS}(n_\mu) = \left\{ x \mid [x - (x \cdot n)n]^2 \geq 0 \right\}$$

(1.9)

may be written as functions of $n_\mu$ and the coordinates of a standard frame $y \in \text{RMS}(\hat{n}_\mu)$ since, given the Lorentz transformation $L$ such that $\hat{n}_\mu = L(n_\mu)$, it follows that

$$x \in \text{RMS}(n_\mu) \quad \text{and} \quad y = L(n) x \implies y \in \text{RMS}(\hat{n}_\mu).$$

(1.10)

By choosing $\hat{n} = (0, 0, 0, 1)$ as in [1], the parameterization (1.7) may be used for $y^\mu$. Now, under Lorentz transformations labeled by $\Lambda$, the wavefunctions were shown to transform as

$$\psi_n(y) \to \psi^\Lambda_n(y) = \psi_{\Lambda^{-1}n}(D^{-1}(\Lambda, n) y)$$

(1.11)

where $\Lambda$ acts directly on $n_\mu$. The representations are moved on an orbit generated by this spacelike vector, and the Lorentz transformations act on $y^\mu$ through the $O(2,1)$ little group, represented by $D^{-1}(\Lambda, n)$, with the property

$$D^{-1}(\Lambda, n) \hat{n} = \mathcal{L}(\Lambda n) \Lambda \mathcal{L}^T(n) \hat{n} \equiv \hat{n}.$$  

(1.12)

The matrix $\mathcal{L}^T(n)$ was chosen in [3] to be a boost in the three-direction, a rotation about the two-axis, followed by a rotation about the one-axis. Thus,

$$\mathcal{L}^T(n) = e^{\gamma M^{23}} e^{\omega M^{31}} e^{\alpha M^{03}}$$

(1.13)

where

$$(\mathcal{M}^{\sigma\lambda})^{\mu\nu} = g^{\sigma\mu} g^{\lambda\nu} - g^{\sigma\nu} g^{\lambda\mu},$$

(1.14)

and so

$$\mathcal{L}^T(n) = 
\begin{pmatrix}
\cosh \alpha & 0 & 0 & \sinh \alpha \\
-\sin \omega \sinh \alpha & \cos \omega & 0 & -\sin \omega \cosh \alpha \\
\sin \gamma \cos \omega \sinh \alpha & \sin \gamma \sin \omega \cos \gamma & 0 & \sin \gamma \cos \omega \cosh \alpha \\
cos \gamma \cos \omega \sinh \alpha & \cos \gamma \sin \omega - \sin \gamma \cos \gamma \cosh \alpha
\end{pmatrix},$$

(1.15)

which provides the parameterization of $n_\mu$ as

$$n_\mu = \begin{pmatrix}
\sinh \alpha \\
-\sin \omega \cosh \alpha \\
\sin \gamma \cos \omega \cosh \alpha \\
\cos \gamma \cos \omega \cosh \alpha
\end{pmatrix}.$$  

(1.16)
By examining the generators $h_{\alpha\beta}(n)$ of (1.11), which form a representation of the O(3,1) Lie algebra (through their action on $y$ and $n$), the Casimir operators

$$\hat{c}_1 = \frac{1}{2} h_{\alpha\beta}(n) h^{\alpha\beta}(n) \quad \hat{c}_2 = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} h_{\alpha\beta}(n) h_{\gamma\delta}(n)$$

as well as the operators of the SU(2) subgroup

$$L^2(n) = \frac{1}{2} h_{ij}(n) h^{ij}(n) \quad L_1(n) = h^{23}(n) = -i \frac{\partial}{\partial y^3}$$

can be constructed as a commuting set. Moreover, the operator

$$\Lambda = \frac{1}{2} M^{\mu\nu} M_{\mu\nu} \rightarrow \ell(\ell + 1) - \frac{3}{4},$$

where $M^{\mu\nu} = y^\mu p^\nu - y^\nu p^\mu$, and the O(2,1) Casimir $N^2 = (M^{01})^2 + (M^{02})^2 + (M^{12})^2$ commute with this set and, wavefunctions were constructed which are eigenfunctions of the set

$$\{\Lambda, N^2, \hat{c}_1, \hat{c}_2, L^2(n), L_1(n)\}$$

with eigenvalues $Q = \{\ell(\ell + 1) - \frac{3}{4}, n^2 - \frac{1}{4}, c_1, c_2, L(L + 1), q\}$. The requirement that these wavefunctions lie in a unitary irreducible representation of SL(2,C) (they are in the principal series), imposes the condition $c_1 = \hat{n}^2 - 1 - c_2^2/\hat{n}^2$, where $\hat{n} = n + 1/2$.

The remaining “radial” function, after the transformation $\hat{R}(\rho) = R(\rho)/\sqrt{\rho}$ of the radial part of $\psi_n(y)$, then must satisfy an equation which is precisely of the form of the nonrelativistic Schrödinger radial equation in three dimensions (and has the same normalization). The states $\psi_n(y)$ are then eigenstates of the Lorentz invariant $K_{rel}$, whose support is on the RMS$(n)$, with the quantum numbers (1.20), and a principal quantum number $n_a$. In particular, the solutions for the problem corresponding to the Coulomb potential [1], yield bound states with a mass spectrum which coincides with the nonrelativistic Schrödinger energy spectrum. The observed energies for such systems are determined by the values of $P^\mu P_\mu$, i.e., $-E^2$ in the center of momentum frame; from (1.4) one obtains, in an expansion in orders of $1/c^2$, the nonrelativistic spectrum with relativistic corrections.

The selection rules for dipole radiation from these states have been calculated [4] and have been shown to be identical with those of the usual nonrelativistic theory, expressed in a manifestly covariant form,

$$\{\Delta \ell = \pm 1; \quad \Delta q = 0, \pm 1\}.$$
In addition to the transverse and longitudinal polarizations of the nonrelativistic theory, there is a “scalar” transition, induced by the relative time coordinate. The “scalar” polarization and the longitudinal polarization induce the same $\Delta q = 0$ transition for the relativistic case, which has a natural interpretation in terms of the Gupta-Bleuler quantization of the photon. This relationship shows that the wavefunctions act correctly as representations of the Lorentz group. Moreover, it was shown in [4] that the change in $q$, the eigenvalue of $L_1(n)$, corresponds to a change in the orientation of $n_\mu$ with respect to the polarization of the emitted or absorbed photon. That the magnetic quantum number $q$ depends on the frame orientation should not be surprising, because the operator $L_1(n)$ belongs to the SU(2) subgroup of SL(2,C), and acts on $n_\mu$, but not on the RMS coordinates (it was shown in [4] that for $\Lambda$ a rotation about the 1-axis, $D^{-1}(\Lambda, n) \equiv 1$).

In this paper, we provide a derivation of the Zeeman effect for the bound states, which requires allowing $n_\mu$ to become a dynamical quantity. We begin with a discussion of the classical $O(3,1)$ in the induced representation and obtain the group generators, which coincide with those of [3], when the momenta are understood as derivatives in the Poisson bracket sense. We construct a classical Lagrangian, in which $n_\mu$ plays an explicit dynamical role, and show that the generators are conserved. We then construct the Hamiltonian, which may be unambiguously quantized and made locally gauge invariant. Finally, it is shown that an external gauge field representing a constant magnetic field induces an energy level splitting corresponding to the usual nonrelativistic expression.

## 2 The Configuration Space

We shall be interested, in this section, in the classical relativistic mechanics of events of spacelike separation. We characterize the separation vectors by the coordinates $(n, y)$, where $n$ is the spacelike unit vector parameterized in (1.16); $y \in \text{RMS}(\hat{n})$ is parameterized in (1.7) (note that $\mathcal{L}^T(n)y \in \text{RMS}(n)$) and satisfies (1.2).

Under a Lorentz transformation $\Lambda$, we know that

$$n \rightarrow n' = \Lambda n \quad x \rightarrow x' = \Lambda x$$

(2.1)
It follows from \((1.10)\) and \((2.1)\) that
\[
x' = \Lambda x = \Lambda \mathcal{L}(n)^T y = \mathcal{L}(\Lambda n)^T \mathcal{L}(\Lambda n) \Lambda \mathcal{L}(n)^T y = \mathcal{L}(n')^T y'.
\] (2.2)
Thus \(y\) transforms as
\[
y \rightarrow y' = D^{-1}(\Lambda, n) y,
\] (2.3)
where (as in \((1.12)\)) \(D^{-1}(\Lambda, n) = \mathcal{L}(\Lambda n) \Lambda \mathcal{L}(n)^T\) belongs to the \(O(2,1)\) which leaves \(\hat{n}\) invariant, i.e.,
\[
D^{-1}(\Lambda, n) \hat{n} = \mathcal{L}(\Lambda n) \Lambda \mathcal{L}(n)^T \hat{n} = \hat{n},
\] (2.4)
and hence the relation \((1.8)\) is preserved. The coordinates thus transform as
\[
\Lambda : (n, y) \rightarrow (n, y)' = (\Lambda n, D^{-1}(\Lambda, n)y).
\] (2.5)

We wish now to construct a model for the Zeeman effect in this covariant framework. To do this, we recall that in the computation of the selection rules for radiative processes, as we remarked above, the restriction \(\Delta q = 0, \pm 1\) refers to a reaction of the radiation on the orientation of the coset label \(n^\mu\) of the induced representation. In the dipole approximation, the transition operator is \(x^\mu\), and in \([4]\), we demonstrated that the conservation of the eigenvalues \(L\) and \(n\) in the matrix elements of \(x^\mu\) implies the vanishing of the matrix element \(\langle \ell' n'| \sin \theta | \ell n \rangle\), leaving only the terms containing \(\langle \ell' n| \cos \theta | \ell n \rangle\) in the calculations. Since this term arises only from the \(y^3 = \rho \cos \theta\) component of \(y^\mu\), the terms of \(x^\mu\) which contribute to these matrix elements are of the form \(\mathcal{L}(n)^3 \mu y_3\). The 3-column of \(\mathcal{L}^T\) is precisely \(n^\mu\), so the calculation factors as
\[
\langle n_a\ell L' q' c'_2 | x^\mu | n_a\ell n L q c_2 \rangle = \langle n_a\ell' n' L' q' c'_2 | \rho \cos \theta \ n^\mu | n_a\ell n L q c_2 \rangle
\]
\[
= \langle n_a\ell' | \rho | n_a\ell >= \langle \ell' n| \cos \theta | \ell n \rangle < n'l' q' c'_2 | n^\mu | n L q c_2 \rangle
\]
\[
= \langle n_a\ell' | \rho | n_a\ell >= \langle \ell' n| \cos \theta | \ell n \rangle < n L q' c_2 | n^\mu | n L q c_2 \rangle
\]
\[
\times \delta_{n n'} \delta_{L L'} \delta(c_2 - c'_2).
\] (2.6)

Since \(|n_a\ell >\) refers to the radial functions and the functions \(|\ell n \rangle\) are the usual spherical harmonics, \((2.6)\) shows directly that it is the orientation of \(n_\mu\) which determines the transition in \(q\).

We deduce from this result that the vector \(n^\mu\) must be effectively coupled to the radiation field, and we shall build our model for coupling to the electromagnetic field by adding to
the Lagrangian a kinetic term for the evolution of \( n^\mu \) which, with minimal gauge invariance, provides the Zeeman coupling.

The velocity \( \dot{n} = dn/d\tau \) transforms just as \( n \) does, since \( \tau \) is invariant:

\[
n' = \Lambda n \quad \implies \quad \dot{n}' = \Lambda \dot{n} \tag{2.7}
\]

but since \( L(n) \) is now \( \tau \)-dependent, the transformation of \( \dot{y} \) is more complicated. We may write

\[
y = L(n(\tau)) x \quad \implies \quad \dot{y} = L(n)\dot{x} + \dot{L}(n)x \tag{2.8}
\]

\[
x = L(n(\tau))^T y \quad \implies \quad \dot{x} = L(n)^T\dot{y} + \dot{L}(n)^Ty \tag{2.9}
\]

and we see that since \( d/d\tau \) and the Lorentz transformation commute, (2.8) is, in fact, form invariant:

\[
\begin{align*}
(\dot{y})' &= L(n')\dot{x}' + \dot{L}(n')x' \\
&= L(\Lambda n)[\Lambda \dot{x}] + \dot{L}(\Lambda n)[\Lambda x] \\
&= L(\Lambda n)\Lambda[L(n)^T\dot{y} + \dot{L}(n)^Ty + \dot{L}(\Lambda n)[\Lambda L(n)]^Ty] \\
&= [L(\Lambda n)\Lambda L(n)^T\dot{y} + [L(\Lambda n)\Lambda \dot{L}(n)^T + \dot{L}(\Lambda n)\Lambda L(n)]^T y] \\
&= D^{-1}(\Lambda, n)\dot{y} + \dot{D}^{-1}(\Lambda, n) y \\
&= \frac{d}{d\tau}[D^{-1}(\Lambda, n) y]. 
\end{align*}
\tag{2.10}
\]

The phase space (which must include \( n, \dot{n} \)) transforms as:

\[
\Lambda : \quad \{(n, y); (\dot{n}, \dot{y})\} \longrightarrow \{(\Lambda n, D^{-1}(\Lambda, n)y); (\Lambda \dot{n}, D^{-1}(\Lambda, n)\dot{y} + \dot{D}^{-1}(\Lambda, n)y)\}. \tag{2.11}
\]

We now examine the generators of the Lorentz transformation represented in (2.3). We take

\[
\Lambda = 1 + \lambda + o(\lambda^2) \tag{2.12}
\]

and write \( \lambda \) as

\[
\lambda = \frac{1}{2} \omega_{\alpha\beta} M^{\alpha\beta} \tag{2.13}
\]

where \( \omega_{\alpha\beta}, \alpha, \beta = 0, \cdots, 3 \) is (infinitesimal) antisymmetric. The matrix generators

\[
M^{\alpha\beta} = \frac{\partial \lambda}{\partial \omega_{\alpha\beta}} \bigg|_{\omega=0} \tag{2.14}
\]
are those given in (1.14). According to (2.12) and (2.13), (2.5) becomes
\[ \Lambda : (n, y) \to (n + \lambda n, \mathcal{L}(n + \lambda n)(1 + \lambda)\mathcal{L}(n)^T y) + o(\omega^2). \] (2.15)

Defining the generators of \( \xi = (n, y) \to \xi' = (n', y') \) as
\[ X_{\alpha\beta} = \sum_{i=1}^{8} \frac{\partial \xi^{i}}{\partial \omega^{\alpha\beta}} \bigg|_{\omega=0} \frac{\partial}{\partial \xi^{i}} \] (2.16)
where for \( i = 1, \ldots, 4, \xi^{i} = n^{\mu}, \mu = 0, \ldots, 3, \) and for \( i = 5, \ldots, 8, \xi^{i} = y^{\mu}, \mu = 0, \ldots, 3. \)
Thus, for \( i = 1, \ldots, 4, \)
\[
\frac{\partial \xi^{i}}{\partial \omega^{\alpha\beta}} \bigg|_{\omega=0} = \frac{\partial}{\partial \omega^{\alpha\beta}}(n^{i} + (\lambda n)^{i}) \bigg|_{\omega=0} = \frac{\partial}{\partial \omega^{\alpha\beta}}(n^{i} + \left(\frac{1}{2} \omega^{\sigma\rho} M_{\sigma\rho} n\right)^{i}) \bigg|_{\omega=0} = \frac{1}{2}(\delta_{\alpha}^{\sigma}\delta_{\beta}^{\rho} - \delta_{\beta}^{\sigma}\delta_{\alpha}^{\rho})(M_{\sigma\rho} n)^{i} = (M_{\alpha\beta})^{\mu}_{\nu} n^{i},
\] (2.17)
so that
\[
\sum_{i=1}^{4} \frac{\partial \xi^{i}}{\partial \omega^{\alpha\beta}} \bigg|_{\omega=0} \frac{\partial}{\partial \xi^{i}} = (M_{\alpha\beta})^{\mu}_{\nu} n^{\nu} \frac{\partial}{\partial n^{\mu}}
\]
\[
= (g_{\alpha}^{\mu} g_{\beta\nu} - g_{\beta}^{\mu} g_{\alpha\nu}) n^{\nu} \frac{\partial}{\partial n^{\mu}}
\]
\[
= n_{\beta} \frac{\partial}{\partial n^{\alpha}} - n_{\alpha} \frac{\partial}{\partial n^{\beta}}
\] (2.18)
which was called \( d(\lambda_{\alpha\beta}) \) in [3].

Now for \( i = 5, \ldots, 8, \)
\[
\frac{\partial \xi^{i}}{\partial \omega^{\alpha\beta}} \bigg|_{\omega=0} = \frac{\partial}{\partial \omega^{\alpha\beta}} \left[ \mathcal{L}(n + \lambda n)(1 + \lambda)\mathcal{L}(n)^T y \right]^{i} \bigg|_{\omega=0}
\]
\[
= \left[ \frac{\partial}{\partial \omega^{\alpha\beta}} \mathcal{L}(n + \lambda n) \bigg|_{\omega=0} \right] \mathcal{L}(n)^T y + \left[ \mathcal{L}(n) \frac{\partial}{\partial \omega^{\alpha\beta}} \lambda \bigg|_{\omega=0} \right] \mathcal{L}(n)^T y \bigg|^{i}
\]
\[
= \left[ \frac{\partial}{\partial n^{\mu}} \mathcal{L}(n) \frac{\partial}{\partial \omega^{\alpha\beta}} (\lambda n)^{\mu} \bigg|_{\omega=0} \right] \mathcal{L}(n)^T y + \left[ \mathcal{L}(n) M_{\alpha\beta} \mathcal{L}(n)^T y \right]^{i}
\]
\[
= \left[ -(M_{\alpha\beta})^{\nu}_{\mu} \mathcal{L}(n) \frac{\partial}{\partial n^{\mu}} \mathcal{L}(n)^T + \mathcal{L}(n) M_{\alpha\beta} \mathcal{L}(n)^T \right]^{ij} y_{j}
\] (2.19)
where we have used the fact that

\[ L(n)L(n)^T = 1 \implies \left( \frac{\partial}{\partial n^\mu} L(n) \right) L(n)^T + L(n) \frac{\partial}{\partial n^\mu} L(n)^T = 0. \] (2.20)

Thus, we find that

\[ \sum_{i=5}^8 \left. \frac{\partial \xi_i}{\partial \omega^{\alpha\beta}} \right|_{\omega=0} = \left[ L(n)M_{\alpha\beta}L(n)^T - (M_{\alpha\beta})^{\mu\nu} n^\nu L(n) \frac{\partial}{\partial n^\mu} L(n)^T \right]^{\rho\sigma} y^\rho \frac{\partial}{\partial y^\sigma} \] (2.21)

Using (1.14) for \( M_{\alpha\beta} \), we obtain

\[ \sum_{i=5}^8 \left. \frac{\partial \xi_i}{\partial \omega^{\alpha\beta}} \right|_{\omega=0} = L_{\sigma\beta} L^\rho_\alpha (y^\rho \frac{\partial}{\partial y^\sigma} - y^\sigma \frac{\partial}{\partial y^\rho}) - n_\beta L^\rho_\sigma \frac{\partial}{\partial n^\alpha} L^\zeta_\rho (y^\sigma \frac{\partial}{\partial y^\rho} - y^\rho \frac{\partial}{\partial y^\sigma}) \] (2.22)

which was called \( g(\lambda_{\alpha\beta}) \) in [3]. So finally, we obtain

\[ X_{\alpha\beta} = L_{\sigma\beta} L^\rho_\alpha (y^\rho \frac{\partial}{\partial y^\sigma} - y^\sigma \frac{\partial}{\partial y^\rho}) - n_\beta L^\rho_\sigma \frac{\partial}{\partial n^\alpha} L^\zeta_\rho (y^\sigma \frac{\partial}{\partial y^\rho} - y^\rho \frac{\partial}{\partial y^\sigma}) + n_\beta \frac{\partial}{\partial n^\alpha} - n_\alpha \frac{\partial}{\partial n^\beta} \] (2.23)

which was called \( i\hbar n(\lambda_{\alpha\beta}) \) in [3]. It was shown that these generators satisfy the Lie algebra of SL(2,C). We will maintain the matrix notation for \( M_{\alpha\beta} \) so that (2.23) may be written as

\[ X_{\alpha\beta} = \left[ L(n)M_{\alpha\beta}L(n)^T \right]^{\mu\nu} y^\nu \frac{\partial}{\partial y^\mu} - \left[ L(n)M_{\alpha\beta}L(n)^T \right]^{\mu\nu} y^\nu \frac{\partial}{\partial n^\mu} - (M_{\alpha\beta})^{\mu\nu} n^\nu \frac{\partial}{\partial n^\mu} \]

\[ = -y^T [L(n)M_{\alpha\beta}L(n)^T] \nabla_y - y^T L(n)[n^T M_{\alpha\beta} \nabla_n] L^T \nabla_y - n^T M_{\alpha\beta} \nabla_n \] (2.24)

where \( (\nabla_y)_\mu = \frac{\partial}{\partial y^\mu} \). By defining the four matrices

\[ S_\mu = L \frac{\partial}{\partial n^\mu} L^T \] (2.25)

(which by (2.20) are antisymmetric) equation (2.24) becomes

\[ X_{\alpha\beta} = - \left\{ y^T [L(n)M_{\alpha\beta}L(n)^T] \nabla_y + n_\mu (M_{\alpha\beta})^{\mu\nu} [y^T S_\nu \nabla_y + (\nabla_n)_\nu] \right\} \] (2.26)

3 Classical and Quantum Mechanics of the Generalized Phase Space

For classical dynamical systems whose potential depends only on \( \rho \) (given by (1.1)), we would like to write a Lagrangian for the reduced “one-body problem” which includes an explicit kinetic term for \( n \). A possible choice is

\[ L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \lambda \dot{n}^2 - V(x^2) \] (3.1)
where $\lambda$ is a length scale required because $n$ is a unit vector. Notice that when $\dot{n} = 0$, the dynamics depend only on $\dot{x}$ for fixed $n_\mu$ and so the relative coordinate remains within $\text{RMS}(n)$. Rewriting (2.9) as,

$$\dot{x} = \mathcal{L}^T \dot{y} + \dot{\mathcal{L}}^T y = \mathcal{L}^T[y + \mathcal{L}^T y]$$  (3.2)

we may write (3.1) in the form

$$L = \frac{1}{2} m [\dot{y} + \mathcal{L} \dot{y}]^2 + \frac{1}{2} \lambda \dot{n}^2 - V(x^2).$$  (3.3)

By construction, (3.3) is Lorentz invariant, and so is invariant under the transformations induced by (2.24). Therefore, applying Noether’s theorem

$$0 = \delta L = \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i + \frac{\partial L}{\partial \dot{\xi}_i} \delta \dot{\xi}_i$$

$$= \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i + \frac{\partial L}{\partial \dot{\xi}_i} \frac{d}{d\tau} \delta \dot{\xi}_i$$

$$= \left[ \frac{\partial L}{\partial \dot{\xi}_i} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{\xi}_i} \right] \delta \dot{\xi}_i + \frac{d}{d\tau} \left[ \frac{\partial L}{\partial \delta \dot{\xi}_i} \right],$$  (3.4)

where the first term vanishes for solutions to the Euler-Lagrange equation, and taking the variation to be $\delta \xi_i = \frac{1}{2} \omega^{\alpha\beta} X_{\alpha\beta} \xi_i$, one obtains the conservation law

$$\frac{d}{d\tau} \left[ p_\mu X_{\alpha\beta} y_\mu + \pi_\mu X_{\alpha\beta} n_\mu \right] = 0$$  (3.5)

where

$$p_\mu = \frac{\partial L}{\partial \dot{y}_\mu} \quad \text{and} \quad \pi_\mu = \frac{\partial L}{\partial \dot{n}_\mu},$$  (3.6)

using the notation $p_\mu$ for the variable conjugate to $y_\mu$ (for each $n_\mu$). Since the variables $y_\mu$ are bounded by the RMS parameterization (1.7), the $p_\mu$ are symmetric but not self-adjoint. These operators, however, occur in combinations which have self-adjoint extensions. We discuss these questions elsewhere. Using (2.24) for $X_{\alpha\beta}$, (3.5) becomes,

$$\frac{d}{d\tau} \left\{ y^T \mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}^T y + n_\mu (\mathcal{M}_{\alpha\beta})^{\mu\nu} [y^T S_\nu p + \pi_\nu] \right\} = 0.$$  (3.7)

If we understand $\pi_\nu$, in the Poisson bracket sense, as a derivative with respect to $n_\mu$, then the quantum operators $h_n(\lambda_{\alpha\beta})$ of [3] now appear as classical constants of the motion for the Lagrangian (3.1).
To obtain the Hamiltonian, we first observe that \( L \) depends on \( \tau \) only through \( n \), so
\[
L \dot{\mathcal{L}}^T = \mathcal{L} \left( \dot{n}^\nu \frac{\partial}{\partial n^\nu} \mathcal{L}^T \right) = \dot{n}^\nu S_{\nu} \quad (3.8)
\]
Applying (3.6) to (3.3),
\[
p_\mu = \frac{\partial L}{\partial \dot{y}_\mu} = m[\dot{y}_\mu + (L \dot{\mathcal{L}}^T y)_\mu] \quad \Rightarrow \quad p = m[\dot{y} + \dot{n}^\nu S_{\nu} y] \quad (3.9)
\]
and
\[
\pi_\mu = \frac{\partial L}{\partial \dot{n}_\mu} = \lambda \dot{n}_\mu + m[\dot{y} + \dot{n}^\nu S_{\nu} y]^T \frac{\partial}{\partial \dot{n}_\mu} [\dot{y} + \dot{n}^\nu S_{\nu} y] = \lambda \dot{n}_\mu - y^T S_{\mu} p \quad (3.10)
\]
where we used (3.9) and the antisymmetry of \( S_{\mu} \) to obtain (3.10). Equations (3.9) and (3.10) may be inverted to eliminate \( (\dot{n}, \dot{y}) \):
\[
\dot{n}_\mu = \frac{1}{\lambda} [\pi_\mu + y^T S_{\mu} p] \quad (3.11)
\]
and
\[
\dot{y} = \frac{1}{m} p - \dot{n}^\mu S_{\mu} y = \frac{1}{m} p - \frac{1}{\lambda} [\pi_\mu + y^T S_{\mu} p] S_{\mu} y \quad (3.12)
\]
which may be used to write the Hamiltonian as
\[
K = \dot{y} \cdot p + \dot{n} \cdot \pi - L = p^T \left( \frac{1}{m} p - \frac{1}{\lambda} [\pi^\mu + y^T S^\mu p] S_{\mu} y \right) + \left( \frac{1}{\lambda} [\pi_\mu + y^T S_{\mu} p] \right)^\mu - \frac{1}{2} m \left( \frac{1}{m} p^2 \right)
- \frac{1}{2} \lambda \left( \frac{1}{\lambda} \right)^2 (\pi^\mu + y^T S^\mu p)(\pi_\mu + y^T S_{\mu} p) + V
= \frac{p^2}{2m} + \frac{1}{2\lambda} (\pi^\mu + y^T S^\mu p)(\pi_\mu + y^T S_{\mu} p) + V \quad (3.13)
\]
Since \( S^\mu \) is antisymmetric, we may regard (3.13) as a quantum Hamiltonian without ordering ambiguity in the operator \( y^T S^\mu p \). The Schrödinger equation is then
\[
i \partial_\tau \psi = K \psi = \left[ \frac{p^2}{2m} + \frac{1}{2\lambda} (\pi^\mu + y^T S^\mu p)(\pi_\mu + y^T S_{\mu} p) + V \right] \psi, \quad (3.14)
\]
where we take as quantum operators
\[
p_\mu = i \frac{\partial}{\partial y^\mu} \quad \pi_\mu = i \frac{\partial}{\partial n^\mu} \quad (3.15)
\]
We require that (3.14) be locally gauge invariant in the coordinate space \( (n, y) \), that is, under transformations of the form
\[
\psi \rightarrow e^{-i e \Theta(n, y)} \psi; \quad (3.16)
\]
this can be accomplished through the minimal coupling prescription

\[ p_\mu \rightarrow p_\mu - eA_\mu^{(n)} \quad \pi_\mu \rightarrow \pi_\mu - e\chi_\mu \]  

(3.17)

together with the requirement that under gauge transformation

\[ A_\mu^{(n)} \rightarrow A_\mu^{(n)} + \frac{\partial}{\partial y^\mu} \Theta \quad \chi_\mu \rightarrow \chi_\mu + \left( \frac{\partial}{\partial n^\mu} + y^T S_\mu \nabla_y \right) \Theta. \]  

(3.18)

Note that \( A_\mu^{(n)} \) transforms under \( O(3,1) \) as an induced (over \( O(2,1) \)) representation; it transforms as \( p_\mu \) under Lorentz transformations (i.e., under the \( O(2,1) \) little group) and so, since the Maxwell equations are Lorentz invariant, it satisfies the Maxwell equation in the \( y^\mu \) variables. Under gauge transformation,

\[ (p - eA^{(n)})e^{-ie\Theta} \psi = e^{-ie\Theta}(p + e\nabla_y \Theta - eA^{(n)})\psi = e^{-ie\Theta}(p - eA^{(n)})\psi \]  

(3.19)

and

\[ (\pi + y^T S_\mu p - e\chi')e^{-ie\Theta} \psi = e^{-ie\Theta}(\pi + y^T S_\mu p + e\frac{\partial}{\partial m^\mu} \Theta + ey^T S_\mu \nabla_n \Theta - e\chi')\psi \]  

(3.20)

so that the gauge invariant form of (3.14) is

\[ i\partial_\tau \psi = K \psi = \left[ \frac{1}{2m} (p - eA^{(n)})^2 + \frac{1}{2\lambda} (\pi + y^T S_\mu p - e\chi')(\pi + y^T S_\mu p - e\chi') + V \right] \psi \]  

(3.21)

Consider the derivative operator which acts on \( \Theta(n,y) \) in the transformation of the gauge field \( \chi_\mu \) in (3.18). We denote this operator by

\[ D_\mu = (\nabla_n)_\mu + y^T S_\mu \nabla_y \]  

(3.22)

and we notice that \( D_\mu \) also appears in the Lorentz generators \( X_{\alpha\beta} \) (2.26). From (3.11) we see that \( D_\mu \) may be regarded as the quantum operator corresponding to \( \lambda \dot{n} \). Using (3.22) in (2.26), the generators assume the simpler form

\[ X_{\alpha\beta} = -\{ y^T [L(n)M_{\alpha\beta}L^T] \nabla_y + n_\mu (M_{\alpha\beta})^{\mu\nu} D_\nu \} \]  

(3.23)

which, in light of (3.11) and the definitions of \( p_\mu \) and \( \pi_\mu \), suggests the analog

\[ X_{\alpha\beta} \sim i [x^T M_{\alpha\beta}(m\dot{x}) + n^T M_{\alpha\beta}(\lambda \dot{n})]. \]  

(3.24)
In fact, using (3.9) and (3.11) in (3.7), we find for the classical conservation law, that
\[
\frac{d}{d\tau} \left\{ y^T \mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}^T p + n_\mu (\mathcal{M}_{\alpha\beta})^{\mu\nu} [y^T S_{\nu p} + \pi_\nu] \right\} = 0
\]
\[
= \frac{d}{d\tau} \left\{ m y^T \mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}^T [\dot{y} + \dot{\eta}^\nu S_{\nu y}] + n^T (\mathcal{M}_{\alpha\beta}) [\dot{\lambda} n] \right\}
\]
\[
= \frac{d}{d\tau} \left\{ m y^T \mathcal{L}(n) \mathcal{M}_{\alpha\beta} \mathcal{L}^T [\dot{y} + \mathcal{L}^T \dot{y}] + n^T (\mathcal{M}_{\alpha\beta}) [\dot{\lambda} n] \right\}
\]
\[
= \frac{d}{d\tau} \left\{ m x^T \mathcal{M}_{\alpha\beta} [\mathcal{L}^T \dot{y} + \mathcal{L}^T \dot{y}] + n^T (\mathcal{M}_{\alpha\beta}) [\dot{\lambda} n] \right\}
\]
\[
= \frac{d}{d\tau} \left\{ x^T \mathcal{M}_{\alpha\beta} [\dot{m} \dot{x}] + n^T (\mathcal{M}_{\alpha\beta}) [\dot{\lambda} n] \right\}
\]
providing the generators with the form of a generalized angular momentum in terms of the relative Minkowski variables and the frame orientation variables.

The Hamiltonian (3.13) also assumes a simple form when expressed in terms of (3.22):
\[
K = -\frac{1}{2m} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y_\mu} - \frac{1}{2\lambda} D_\mu D^\mu + V. \tag{3.26}
\]
Suppose that a function \( f(n, y) \) is defined in such a way that its dependence on \( n \) is only through \( \mathcal{L}(n)^T y \) (which is to say that \( f \) is a function of \( x \) alone, even as \( n \) varies in \( \tau \)). Then we find that
\[
\frac{\partial}{\partial y^\mu} f = \left. \frac{df}{d\xi^\alpha} \right|_{\xi = \mathcal{L}(n)^T y} \frac{\partial}{\partial y^\mu} (\mathcal{L}_\beta^\alpha y^\beta) = \mathcal{L}_\mu^\alpha \frac{df}{d\xi^\alpha} \Big|_{\xi = \mathcal{L}(n)^T y} \tag{3.27}
\]
and
\[
\frac{\partial}{\partial n^\mu} f = \left. \frac{df}{d\xi^\alpha} \right|_{\xi = \mathcal{L}(n)^T y} \frac{\partial}{\partial n^\mu} (\mathcal{L}_\beta^\alpha y^\beta) \tag{3.28}
\]
so that
\[
D_\mu f = \left( \frac{\partial}{\partial n^\mu} + y^T S_{\mu} \nabla_y \right) f
\]
\[
= \left[ \frac{\partial}{\partial n^\mu} + y^\beta \mathcal{L}^\beta_\gamma \left( \frac{\partial}{\partial n^\mu} \mathcal{L}^\alpha_\gamma \right) \frac{\partial}{\partial y^\alpha} \right] f
\]
\[
= \left. \frac{df}{d\xi^\sigma} \right|_{\xi = \mathcal{L}(n)^T y} y^\beta \left[ \frac{\partial}{\partial n^\mu} \mathcal{L}^\beta_\sigma + \mathcal{L}^\gamma_\beta (\frac{\partial}{\partial n^\mu} \mathcal{L}^\alpha_\gamma) \mathcal{L}^\sigma_\alpha \right]
\]
\[
= \left. \frac{df}{d\xi^\sigma} \right|_{\xi = \mathcal{L}(n)^T y} y^\beta \left[ \frac{\partial}{\partial n^\mu} \mathcal{L}^\beta_\sigma + \mathcal{L}^\gamma_\beta (\mathcal{L}^T)^\sigma_\alpha \frac{\partial}{\partial n^\mu} \mathcal{L}^\alpha_\gamma \right]
\]
\[
= \left. \frac{df}{d\xi^\sigma} \right|_{\xi = \mathcal{L}(n)^T y} y^\beta \left[ \frac{\partial}{\partial n^\mu} \mathcal{L}^\beta_\sigma - \mathcal{L}^\gamma_\beta \mathcal{L}^\alpha_\gamma \frac{\partial}{\partial n^\mu} \mathcal{L}^\sigma_\alpha \right]
\]
\[
\equiv 0 \tag{3.29}
\]
where we have used (2.20). Thus, \( D_\mu \) acts as a kind of covariant derivative which vanishes on functions of \( x \) alone. In particular, \( D_\mu \) vanishes on the eigenstates discussed in [1] and [3], in which case the Hamiltonian (3.13, 3.26) reduces to the RMS Hamiltonian discussed in [1]. The dynamical effects that we shall discuss in the next section are associated with the evolution of the wave function of the system to a form which does not depend only on \( x^\mu \).

Notice also that

\[
\dot{n}^\mu D_\mu = \left( \dot{n}^\mu \frac{\partial}{\partial n^\mu} + y^T \dot{n}^\mu S_\mu \nabla y \right) = (\dot{n} \cdot \nabla n - y^T \dot{\mathcal{L}}^T \nabla y) = (\dot{n} \cdot \nabla n + \dot{y} \cdot \nabla y - \dot{x} \cdot \nabla x) \tag{3.30}
\]

We may rewrite this expression as

\[
dx \cdot \nabla x + dn^\mu D_\mu = dy \cdot \nabla y + dn \cdot \nabla n \tag{3.31}
\]

which shows in yet another way that \( \nabla x \) and \( D_\mu \) generate the changes induced by \( dx \) and \( dn \) (with \( x^\mu \) held constant), just as \( \nabla y \) and \( \nabla n \) generate the changes induced by \( dy \) and \( dn \) (with \( y^\mu \) held constant).

It will be useful to examine the classical Lagrangian in the presence of the fields \( A^{(n)}_\mu \) and \( \chi_\mu \), which we may find by treating the Hamiltonian in (3.21) as a classical functional and evaluating

\[
\dot{n}^\mu = \frac{\partial}{\partial \pi^\mu} K = \frac{1}{\lambda} (\pi^\mu + y^T S_\mu p - e\chi^\mu) \tag{3.32}
\]

and

\[
\dot{y}_\mu = \frac{\partial}{\partial p^\mu} K = \frac{1}{m} (p_\mu - eA^{(n)}_\mu) + \frac{1}{\lambda} (\pi_\nu + y^T S_\nu p - e\chi_\nu) \frac{\partial}{\partial \pi^\mu} (y^T S_\nu p) = \frac{1}{m} (p_\mu - eA^{(n)}_\mu) - \dot{n}_\nu (S^\nu)_{\mu\sigma} y^\sigma. \tag{3.33}
\]

Recalling (3.8), we find that

\[
L = p \cdot \dot{y} + \pi \cdot \dot{n} - K = \frac{1}{2} m [\dot{y} + \mathcal{L}^T y]^2 + \frac{1}{2} \lambda \dot{n}^2 + e[(\dot{y} + \mathcal{L}^T y) \cdot A^{(n)} + \dot{n} \cdot \chi] - V(x^2). \tag{3.34}
\]

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From (3.2), we have

\[ \dot{y} + \mathcal{L} \dot{\mathcal{L}}^T y = \mathcal{L} \dot{x}, \]  

so that we may write (3.34) in the form

\[ L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \lambda \dot{n}^2 + e [\dot{x} \cdot (\mathcal{L}^T \Lambda (n)) + \dot{n} \cdot \chi] - V(x^2). \]  

In order for \( L \) to be a Lorentz scalar, \( \mathcal{L}^T \Lambda (n) \) must transform under the full Lorentz group \( O(3,1) \). Since \( \Lambda (n) \) was introduced as a field which transforms under the \( O(2,1) \) little group, we may write

\[ \Lambda (n)' = D^{-1}(\Lambda, n) \Lambda (n) \Lambda \mathcal{L}^T \Lambda (n) A(n) = \mathcal{L}^T (\Lambda n) A(n) \rightarrow \mathcal{L}^T (\Lambda n) A(n)' = \Lambda \mathcal{L}^T (n) A(n) \]  

verifying that the combination \( \mathcal{L}^T \Lambda (n) \) transforms as a four vector under \( \Lambda \).

4 The Zeeman Effect

In [3], the spacelike vector \( n \) played no particular role in the dynamics and could be chosen arbitrarily, because the systems under discussion were \( O(3,1) \)-symmetric and no direction in spacetime was intrinsic to the problem (other than the axis of the bound state). That situation generalizes the nonrelativistic spherically symmetric central force problem, in which the absence of a preferred direction in space leads to the degeneracy of the energy spectrum with respect to the magnetic quantum number (which characterizes the orientation of the angular momentum). In [4], it was shown that the vector \( n \) plays a role in dipole radiation from the bound state, because conservation of angular momentum and the spin-1 nature of the electromagnetic field impose an orientation dependence on the interaction. Thus, the photon carries off spin provided by the bound state transition, and that transition depends on the orientation of the angular momentum of the state (determined by \( n \)) and the photon polarization.

In the Zeeman effect, one lifts the degeneracy of the bound state spectrum by placing the state in a constant external magnetic field, which interacts with the magnetic moment (angular momentum) of the system and thereby provides a preferred direction in space. In the semiclassical picture, the atom will tend to rotate. The interaction angular momentum is intimately connected with the rotation generators, and for the bound states discussed
here, these generators are elements of the rotation subgroup of the induced representation of O(3,1). Since the rotation group O(3) ⊂ O(3,1) acts on the vector \( n \) as well as the RMS variables \( y \), the relativistic Zeeman effect can clearly only be described in the context of a theory which explicitly permits the generators to act directly on all the variables in the theory. In this section, we provide such a description in the context of the Hamiltonian theory given in the Section 4.

In the nonrelativistic case, the Zeeman effect is obtained as a first order perturbation of the hydrogen atom bound state, by a vector potential

\[
A(r) = -\frac{1}{2} B \times r \tag{4.1}
\]

which leads to the constant magnetic field

\[
(\nabla \times A)^i = \epsilon^{ijk} \frac{\partial}{\partial r^j} \left( -\frac{1}{2} \epsilon_{kln} B_l r_m \right) = B^i. \tag{4.2}
\]

The Hamiltonian becomes

\[
H = \frac{1}{2m}(p - eA)^2 + V \\
= \frac{p^2}{2m} + V + \frac{e}{2m}(p \cdot A + A \cdot p) + o(e^2) \\
= H_0 + \frac{e}{m} A \cdot p + o(e^2) \\
= H_0 - \frac{e}{2m}(B \times r) \cdot p + o(e^2) \\
= H_0 - \frac{e}{2m} B \cdot \mathbf{L} + o(e^2) \tag{4.3}
\]

where \( \mathbf{L} = r \times p \) is the angular momentum operator. Thus taking \( \mathbf{B} \) in the direction of the diagonal angular momentum operator (usually the \( z \)-axis), the observed Zeeman splitting is obtained from (4.3) as

\[
E_{ln} \rightarrow E_{lnq} = E_{ln} - \frac{eB}{2m} q. \tag{4.4}
\]

where \( q \) is the eigenvalue of the operator \( L_z \).

In Section 3, we introduced two gauge compensation fields, \( A^{(n)}_\mu \) and \( \chi_\mu \), required to make the Hamiltonian (3.13) locally gauge invariant. However, we now argue that just as \( n \) and \( y \) transform under inequivalent representations of the Lorentz group (\( y \) transforms under the
O(2,1) little group induced by the action of the full O(3,1), so $A^{(n)}_\mu$ and $\chi_\mu$ must be seen as inequivalent representations of the usual U(1) gauge group of electromagnetism. In the full spacelike region, a constant electromagnetic field, $F^{\mu\nu}$, can be represented through the vector potential

$$A^\mu(x) = -\frac{1}{2} F^{\mu\nu} x_\nu.$$  \hspace{1cm} (4.5)

We now restrict the support of $A^\mu$ to $x \in \text{RMS}(n)$ and express the vector potential as a vector oriented with $\text{RMS}(\hat{n})$ by writing

$$A^{(n)}_\mu(y) = \mathcal{L}_{\mu\nu} A^\nu (\mathcal{L}^T y) = -\frac{1}{2} \mathcal{L}_{\mu\nu} F^\nu_{\sigma} \mathcal{L}_\chi^\sigma y^\lambda = -\frac{1}{2} (\mathcal{L} F \mathcal{L}^T y)_\mu.$$  \hspace{1cm} (4.6)

For the field $\chi_\mu$, we choose (note that $\hat{n}$ undergoes Lorentz transform in the same way as $x$),

$$\chi_\mu(n) = b^2 A^\mu(n) = -\frac{b^2}{2} F^\nu_{\sigma} n^\sigma$$  \hspace{1cm} (4.7)

(here $b$ is another length scale, required since $A^\mu(x)$ has units of length$^{-1}$, so $F^\nu_{\sigma}$ must have units of length$^{-2}$, but $\chi_\mu$ must be without units) and we use (4.6) and (4.7) in the Schrödinger equation (3.21).

$$i \partial_\tau \psi = \left[ \frac{1}{2m} (p - eA^{(n)})^2 + \frac{1}{2\lambda} (\pi^\mu + y^T S^\mu p - e\chi^\mu) (\pi_\mu + y^T S_\mu p - e\chi_\mu) + V \right] \psi$$

$$= \left[ \frac{1}{2m} p^2 - \frac{e}{2m} (p \cdot A^{(n)} + A^{(n)} \cdot p) + \frac{1}{2\lambda} (\pi^\mu + y^T S^\mu p)^2 - \frac{e}{2\lambda} [(\pi^\mu + y^T S^\mu p)\chi_\mu + \chi^\mu (\pi_\mu + y^T S_\mu p)] + V + o(e^2) \right] \psi$$

$$= \left[ \frac{1}{2m} p^2 + \frac{1}{2\lambda} (\pi^\mu + y^T S^\mu p)^2 + V - e \left[ \frac{1}{m} A^{(n)} \cdot p + \frac{1}{\lambda} \chi^\mu (\pi_\mu + y^T S_\mu p) \right] + o(e^2) \right] \psi$$  \hspace{1cm} (4.8)

where the first three terms of (4.8) are the unperturbed Hamiltonian $K_0$.

The perturbation term to order $o(e)$, is

$$- e \left[ \frac{1}{m} A^{(n)} \cdot p + \frac{1}{\lambda} \chi^\mu (\pi_\mu + y^T S_\mu p) \right]$$

$$= -e \left[ \frac{1}{m} A^{(n)T} p + \frac{1}{\lambda} \chi^T \pi + y^T (S \cdot \chi) p \right]$$

$$= -e \left[ \frac{1}{2m} (\mathcal{L} F \mathcal{L}^T y)^T p + \frac{b^2}{2\lambda} F^\mu_{\nu} n^\nu (\pi_\mu + y^T S_\mu p) \right]$$

$$= \frac{e}{2m} [y^T \mathcal{L} F \mathcal{L}^T p + \frac{mb^2}{\lambda} n_\nu F^\nu_{\mu} (\pi_\mu + y^T S_\mu p)].$$  \hspace{1cm} (4.9)
We now expand the electromagnetic field tensor on the basis of four by four antisymmetric tensors given by the Lorentz generators $\mathcal{M}^{\mu\nu}$. Thus,

$$F = \frac{1}{2} F_{\mu\nu} \mathcal{M}^{\mu\nu}$$

may be verified through

$$(F)^{\alpha\beta} = \frac{1}{2} F_{\mu\nu} (\mathcal{M}^{\mu\nu})^{\alpha\beta} = \frac{1}{2} F_{\mu\nu} (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) = F^{\alpha\beta}.$$  (4.11)

Using (4.10) in (4.9) we find that the perturbation term to order $o(e)$ becomes

$$\frac{e}{4m} F_{\alpha\beta} \left[ y^T \mathcal{L} \mathcal{M}^{\alpha\beta} \mathcal{L}^T p + \frac{mb^2}{\lambda} n_\mu (\mathcal{M}^{\alpha\beta})^{\mu\nu} (\pi_\nu + y^T S_\nu p) \right]$$

(4.12)

We note that if $\lambda/b^2 = m$, then we may write the first order perturbation (using (3.23)) as

$$\frac{e}{4m} F_{\alpha\beta} \left[ y^T \mathcal{L} \mathcal{M}^{\alpha\beta} \mathcal{L}^T p + n_\mu (\mathcal{M}^{\alpha\beta})^{\mu\nu} (\pi_\nu + y^T S_\nu p) \right] = \frac{e}{4m} F_{\alpha\beta} X^{\alpha\beta}.$$  (4.13)

For $F^{\mu\nu} F_{\mu\nu} = 2(B^2 - E^2) > 0$, there exists a frame for which the interaction is purely magnetic. In such a frame, the perturbation becomes

$$\frac{e}{4m} F_{\alpha\beta} X^{\alpha\beta} = \frac{e}{4m} F_{ij} X^{ij} = \frac{e}{4m} \epsilon_{ijk} B^k X^{ij} = \frac{e}{2m} B^k \left[ \frac{1}{2} \epsilon_{ijk} X^{ij} \right] = \frac{e}{2m} B^k h(\lambda_k)$$

(4.14)

where $h(\lambda_k)$ are the three conserved generators of the SU(2) rotation subgroup of SL(2,C) for the phase space $\{(n, y); (\pi, p)\}$, that is, the angular momentum operator for the eigenstates of the induced representation. Notice that in the matrix element for unperturbed eigenstates, the second terms of (4.13) vanishes, so the relativistic Zeeman effect does not depend upon the values of $\lambda$ or $b$.

In [3], the diagonal angular momentum operator is $L_1(n) = h(\lambda_1) = -i \partial / \partial \gamma$, and so if we take $B = B(1, 0, 0)$ then we find that

$$K_0 \rightarrow K = K_0 - \frac{eB}{2m} h(\lambda_1)$$  (4.15)

splits the mass levels of the bound states according to

$$E_{\ell n} \rightarrow E_{\ell n} - \frac{eB}{2m} q$$  (4.16)

In going from (4.15) to (4.16), we have used the fact that the unperturbed Hamiltonian of (4.8) reduces to the unperturbed Hamiltonian of [3]. Equation (4.16) further justifies the
conclusion reached in [4] that $q$ is the magnetic quantum number. Moreover, the manifest covariance of the formalism guarantees that the splitting of the spectrum will be independent of the observer. We observe that if $F_{\mu\nu}F_{\mu\nu} < 0$, we may find a frame in which the interaction is purely electric, leading to a covariant formulation of the Stark effect. Since the electric field couples to the boost generators (which reduce to the position operator in the nonrelativistic limit) and these generators are not diagonal in this representation, the Stark effect remains formally (one really has only a resonance spectrum; the bound states are destroyed by the non-compact generator) a second order perturbation, and we will discuss it elsewhere.

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