Order estimates of the uniform approximations by Zygmund sums on the classes of convolutions of periodic functions

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The Zygmund sums of a function \( f \in L_1 \) are trigonometric polynomials of the form \( Z^s_{n-1}(f; t) := \sum_{k=-n+1}^{n-1} \left( 1 - \left( \frac{k}{n} \right)^2 \right) (a_k(f) \cos kt + b_k(f) \sin kt), \) \( s > 0, \) where \( a_k(f) \) and \( b_k(f) \) are the Fourier coefficients of \( f. \) We establish the exact-order estimates of uniform approximations by the Zygmund sums \( Z^s_{n-1} \) of \( 2\pi \)-periodic continuous functions from the classes \( C^p_{\beta,p} \). These classes are defined by the convolutions of functions from the unit ball in the space \( L_{p^*}, 1 \leq p < \infty, \) with generating fixed kernels \( \Psi(t) \sim \sum_{k=1}^\infty \Psi(k) \cos \left( kt + \frac{\beta n}{2} \right), \) \( \Psi \in L_{p^*}, \beta \in \mathbb{R}, \frac{1}{p} + \frac{1}{p^*} = 1. \) We additionally assume that the product \( \psi(k)^{k+1/p} \) is generally monotonically increasing with the rate of some power function, and, besides, for \( 1 < p < \infty \) it holds that \( \sum_{k=n}^\infty \psi(k)k^{r-2} < \infty, \) and for \( p = 1 \) the following condition \( \sum_{k=n}^\infty \psi(k) < \infty \) is true.

It is shown, that under these conditions Zygmund sums \( Z^s_{n-1} \) and Fejér sums \( e_{n-1} = Z^1_{n-1} \) realize the order of the best uniform approximations by trigonometric polynomials of these classes, namely for \( 1 < p < \infty \)

\[
E_n(C^p_{\beta,p})_C \asymp \mathcal{E} \left( C^p_{\beta,p}, Z^s_{n-1} \right)_C \asymp \left( \sum_{k=n}^n \psi(k)k^{r-2} \right)^{1/p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,
\]

and for \( p = 1 \)

\[
E_n(C^1_{\beta,1})_C \asymp \mathcal{E} \left( C^1_{\beta,1}, Z^1_{n-1} \right)_C \asymp \left\{ \sum_{k=n}^n \frac{\psi(k)}{\mathcal{T}_k}, \quad \cos \frac{\beta \pi}{2} \neq 0, \quad \cos \frac{\beta \pi}{2} = 0, \right. \]

where

\[
E_n(C^p_{\beta,p})_C := \max_{f \in C^p_{\beta,p}} \inf_{t_{n-1} \in \mathcal{T}_{2n-1}} \| f(\cdot) - t_{n-1}(\cdot) \|_C,
\]

and \( \mathcal{T}_{2n-1} \) is the subspace of trigonometric polynomials \( t_{n-1} \) of order \( n - 1 \) with real coefficients,

\[
\mathcal{E} \left( C^p_{\beta,p}, Z^s_{n-1} \right)_C := \max_{f \in C^p_{\beta,p}} \| f(\cdot) - Z^s_{n-1}(f; \cdot) \|_C.
\]

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1 Notations, definitions and auxiliary statements

Denote by $L_p$, $1 \leq p \leq \infty$, the space of $2\pi$-periodic summable on $[0, 2\pi]$ functions $f$ with the norm

$$
\|f\|_p = \begin{cases} 
\left( \int_0^{2\pi} |f(t)|^p \, dt \right)^{1/p}, & 1 \leq p < \infty, \\
\text{ess sup} |f(t)|, & p = \infty,
\end{cases}
$$

and by $C$ the space of $2\pi$-periodic continuous functions with the norm defined by the equality $\|f\|_C = \max |f(t)|$.

Let $f \in L_1$ and

$$
S[f](x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx),
$$

be the Fourier series of the function $f$.

If for a sequence $\psi(k) \in \mathbb{R}$ and fixed number $\beta \in \mathbb{R}$ the series

$$
\lim_{k \to \infty} \frac{1}{\psi(k)} \left( a_k(f) \cos \left( kx + \frac{\beta \pi}{2} \right) + b_k(f) \sin \left( kx + \frac{\beta \pi}{2} \right) \right)
$$

is the Fourier series of a summable function $\varphi$, then this function is called a $(\psi, \beta)$-derivative of the function $f$ and is denoted by $f_\psi^\beta$. A set of functions, for which this condition is satisfied, is denoted by $L^\psi_\beta$, and subset all continuous functions from $L^\psi_\beta$ is denoted by $C^\psi_\beta$.

If $f \in L^\psi_\beta$ and furthermore $f_\psi^\beta \in \mathfrak{M}$, where $\mathfrak{M} \subset L_1$, then we write that $f \in L^\psi_\beta \mathfrak{M}$. Let us put $L^\psi_\beta \mathfrak{M} \cap C = C^\psi_\beta \mathfrak{M}$. The concept of $(\psi, \beta)$-derivative is a natural generalization of the concept of $(r, \beta)$-derivative in the Weyl-Nagy sense and coincides almost everywhere with the last one, when $\psi(k) = k^{-r}$, $r > 0$. Namely, in this case $L^\psi_\beta \mathfrak{M} = W^r_\beta \mathfrak{M}$, $f_\psi^\beta = f_r^\beta$, where $f_\beta^r$ is the derivative in the Weyl-Nagy sense, and $W^r_\beta \mathfrak{M}$ are the Weyl-Nagy classes [22], [20]. In the case $\beta = r$, the classes $W^r_\beta \mathfrak{M}$ are the well known Weyl classes $W^r_\mathfrak{M}$, while the derivatives $f_\beta^r$ coincide almost everywhere with the derivatives in the sense of Weyl $f_r$. If, in addition, $\beta = r, r \in \mathbb{N}$, then $f_\beta^r$ coincide almost everywhere with the usual derivatives $f^{(r)}$ of the order $r$ of the function $f$

$$
f_\beta^r = f^{(r)}(x) \quad \text{and at the same time} \quad W^r_\beta \mathfrak{M} = W_r \mathfrak{M} = W^r \mathfrak{M}.
$$

According to [20, Statement 3.8.3], if the series

$$
\lim_{k \to \infty} \frac{1}{\psi(k)} \cos \left( kt - \frac{\beta \pi}{2} \right), \quad \beta \in \mathbb{R},
$$

is the Fourier series of the function $\Psi_\beta \in L_1$, then the elements $f$ of the classes $L^\psi_\beta \mathfrak{M}$ for almost every $x \in \mathbb{R}$ are represented as the convolution

$$
f(x) = \frac{a_0}{2} + (\Psi_\beta \ast \varphi)(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \Psi_\beta(x - t) \varphi(t) \, dt, \quad a_0 \in \mathbb{R}, \varphi \perp 1, \varphi \in \mathfrak{M},
$$

(1)

where $\varphi$ almost everywhere coincides with $f_\psi^\beta$.

As sets $\mathfrak{M}$ we will consider the unit balls of the spaces $L_p$:

$$
U_p = \{ \varphi \in L_p : \|\varphi\|_p \leq 1 \}, \quad 1 \leq p \leq \infty.
$$
Then put: \( L^\Psi_{\beta,p} := L^\Psi_{\beta} U_p \), \( C^\Psi_{\beta,p} := C^\Psi_{\beta} U_p \), \( W^\Psi_{\beta,p} := W^\Psi_{\beta} U_p \).

According to [20, Statement 1.2], if the fixed kernel \( \Psi_\beta \) of the classes \( L^\Psi_{\beta,p} \) and \( C^\Psi_{\beta,p} \) satisfies the inclusion \( \Psi_\beta \in L^p, \frac{1}{p} + \frac{1}{p'} = 1, 1 \leq p \leq \infty \), then the convolutions of the form (1) are continuous functions, where \( \mathfrak{M} = U_p \). It is clear that in this case for \( f \in C^\Psi_{\beta,p} \) the equality (1) is fulfilled for all \( x \in \mathbb{R} \).

We assume that the sequences \( \psi(k) \) are traces on the set of natural numbers \( \mathbb{N} \) of some positive continuous convex downwards functions \( \psi(t) \) of the continuous argument \( t \geq 1 \), that tends to zero for \( t \to \infty \). The set of all such functions \( \psi(t) \) is denoted by \( \mathfrak{M} \).

To classify functions \( \psi \) from \( \mathfrak{M} \) on their speed of decreasing to zero it is convenient to use the following characteristic

\[
\alpha(t) = \alpha(\psi; t) = \frac{\psi(t)}{t|\psi'(t)|}, \quad \psi'(t) := \psi'(t + 0).
\]

With its help we consider the following subsets of the set \( \mathfrak{M} \) (see, e.g. [20])

\[
\mathfrak{M}_0 := \{ \psi \in \mathfrak{M} : \exists K > 0 \ \forall t \geq 1 \ 0 < K \leq \alpha(\psi; t) \},
\]

\[
\mathfrak{M}_C := \{ \psi \in \mathfrak{M} : \exists K_1, K_2 > 0 \ \forall t \geq 1 \ 0 < K_1 \leq \alpha(\psi; t) \leq K_2 \}.
\]

It is clear that \( \mathfrak{M}_C \subset \mathfrak{M}_0 \).

Zygmund sums of the order \( n - 1 \) of the function \( f \in L_1 \) are the trigonometric polynomials of the form

\[
Z^s_{n-1}(f; t) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left( 1 - \left( \frac{k}{n} \right)^s \right) (a_k(f) \cos kt + b_k(f) \sin kt), \quad s > 0,
\]

(2)

where \( a_k(f) \) and \( b_k(f) \) are Fourier coefficients of the function \( f \).

In the case \( s = 1 \) polynomials \( Z^1_{n-1} \) are Fejér sums

\[
Z^1_{n-1}(f; t) =: \sigma_{n-1}(f; t) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left( 1 - \frac{k}{n} \right) (a_k(f) \cos kt + b_k(f) \sin kt).
\]

In this paper we consider the following approximation characteristics

\[
\mathcal{E} \left( C^\Psi_{\beta,p}; Z^s_{n-1} \right)_C = \sup_{f \in C^\Psi_{\beta,p}} \| f(\cdot) - Z^s_{n-1}(f; \cdot) \|_C, \quad 1 \leq p \leq \infty, \quad \beta \in \mathbb{R},
\]

(3)

and solve the problem of establishing the order of decreasing to zero as \( n \to \infty \) of the mentioned quantities with respect to relations between parameters \( \psi, \beta, p \) and \( s \). It is clear that we can make conclusion about the approximation ability of a linear polynomial approximation method (including Fejér \( \sigma_{n-1} \) and Zygmund \( Z^1_{n-1} \) methods) on the class \( C^\Psi_{\beta,p} \) after comparison the rate of decreasing of the exact upper bounds of uniform deviations of trigonometric sums, which are generated by this method, on the set \( C^\Psi_{\beta,p} \) with the rate of decreasing of the best uniform approximations of the class \( C^\Psi_{\beta,p} \) by trigonometric polynomials \( t_{n-1} \) of order not higher than \( n - 1 \), namely the quantities of the form

\[
E_n(C^\Psi_{\beta,p})_C = \sup_{f \in C^\Psi_{\beta,p}} \inf_{t_{n-1}} \| f(\cdot) - t_{n-1}(\cdot) \|_C, \quad 1 \leq p \leq \infty,
\]
where \( \mathcal{T}_{2n-1} \) is the subspace of trigonometric polynomials \( t_{n-1} \) of order \( n - 1 \) with real coefficients. In this case, since always the following estimate holds

\[
E_n\left( C_{\beta,p}^\psi \right)_C \leq E \left( C_{\beta,p}^\psi ; Z_n^{s_{n-1}} \right)_C, \quad n \in \mathbb{N},
\]

(4)

it is important to know under which restrictions on the parameters \( \psi, s, \beta \) and \( p \) the following equality takes place

\[
E_n\left( C_{\beta,p}^\psi \right)_C = E \left( C_{\beta,p}^\psi ; Z_n^{s_{n-1}} \right)_C.
\]

(5)

The notation \( A(n) \asymp B(n) \) means, that \( A(n) = O(B(n)) \) and at the same time \( B(n) = O(A(n)) \), where by the notation \( A(n) = O(B(n)) \) we mean, that there exists a constant \( K > 0 \) such that the inequality \( A(n) \leq K(B(n)) \) holds.

In the work [27] A. Zygmund introduced trigonometric polynomials of the form (2) and found exact order estimates of the quantities \( E \left( W_{\beta,\infty}^r ; Z_n^{s_{n-1}} \right)_C \) at \( r \in \mathbb{N} \). B. Nagy investigated in [7] the quantities \( E \left( W_{\beta,\infty}^r ; Z_n^{s_{n-1}} \right)_C \) for \( r > 0, \beta \in \mathbb{Z} \), and for \( s \leq r \) he established the asymptotic equality, and for \( s > r \) he found order estimates. Later, S.A. Telyakovskii [23] obtained asymptotically exact equalities for the quantities \( E \left( W_{\beta,\infty}^r ; Z_n^{s_{n-1}} \right)_C \) for \( r > 0 \) and \( \beta \in \mathbb{R} \) for \( n \to \infty \). On the Weyl-Nagy classes, the exact order estimates of the quantities \( E \left( W_{r,\infty}^p ; Z_n^{s_{n-1}} \right)_C \) for \( 1 < p < \infty \) and \( r > 1/p \) and for \( p = 1 \) and \( r \geq 1, \beta \in \mathbb{R} \) are found in the work [6].

Concerning the Fejér sums \( \sigma_{n-1}(f;t) \) it should be noticed that the order estimates of quantities \( E \left( W_{r,\infty}^p \sigma_{n-1} \right)_C, r > 0, \beta \in \mathbb{Z} \) were found by S.M. Nikol’skii [8]; for the quantities \( E \left( W_{r,\infty}^p \sigma_{n-1} \right)_C \) for \( 1 < p \leq \infty \) and \( r > 1/p \), and also for \( p = 1 \) and \( r \geq 1 \) were found by V.M. Tikhomirov [25] and by A.I. Kamzolov [5].

Approximation properties of Zygmund sums on the classes of \( (\psi, \beta) \)-differentiable functions were studied in the works [2, 14, 15], (see also [20]). Particularly in the work [2] of D.M. Bushev the asymptotic equalities for the quantities \( E(C_{\beta,\infty}^\psi ; Z_n^{s_{n-1}})_C \) were established for some quite natural constraints on \( \psi \) and \( s \) as \( n \to \infty \). In the case, when the series \( \sum_{k=1}^{\infty} \psi^2(k) \) is convergent, the exact values of the quantities \( E \left( C_{\beta,2}^\psi ; Z_n^{s_{n-1}} \right)_C \) were established in the work [15] of A.S. Serdyuk and I.V. Sokolenko.

In the work [14], the authors found the exact order estimates of uniform approximations by Zygmund sums \( Z_n^{s_{n-1}} \) on the classes \( C_{\beta,p}^\psi \) for \( 1 < p < \infty \), when \( \psi \in \Theta_p \), and \( \Theta_p, 1 < p < \infty \), is the set of non-increasing functions \( \psi(t) \), for which there exists \( \alpha > 1/p \) such that the function \( t^\alpha \psi(t) \) almost decreases, and \( \psi(t)^{1/p - \varepsilon} \) increases on \([1, \infty)\) for some \( \varepsilon > 0 \).

Concerning the estimates of the best uniform approximations of functional compacts, it should be noticed the following. For the Weyl-Nagy classes \( W_{r,\infty}^p \) for \( r = 1/p, \beta \in \mathbb{R}, 1 \leq p \leq \infty \), the exact order estimates of the best approximations \( E_n\left( W_{r,\infty}^p \right)_C \) are known (see, e.g. [24]). Moreover, for \( p = \infty \) the exact values of the quantities \( E_n\left( W_{r,\infty}^p \right)_C \) for all \( r > 0, \beta \in \mathbb{R} \) and \( n \in \mathbb{N} \) are known (see [3]).

The order estimates of the best approximations of the classes \( C_{\beta,p}^\psi \) under certain restrictions on \( \psi, \beta \) and \( p \) were investigated in the works [4, 17, 18, 20]. In some partial cases (especially for \( p = \infty \)) the exact or asymptotically exact values of the quantities \( E_n\left( C_{\beta,p}^\psi \right)_C \) are also known (see [9–13, 16, 20]).
In this paper, we establish the exact order estimates of the quantities of the form (3) for all $1 \leq p < \infty$ and $\beta \in \mathbb{R}$, in case, when $\psi(t)^{1/p} \in M_0$, the product $\psi(k)^{k+1/p}$ generally monotonically increases, $\psi(k)^{k+1/p-\varepsilon}$ almost increases (according to Bernstein) for some $\varepsilon > 0$ and for $1 < p < \infty$

$$\sum_{k=n}^{\infty} \psi(k)k^{p'-2} < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$  

and for $p = 1$

$$\sum_{k=n}^{\infty} \psi(k) < \infty.$$  

The conditions (6) and (7) and the monotonic decreasing to zero of the sequence $\psi(k)$ ensure the inclusion $\Psi_\beta \in L_{p'}$, $1/p + 1/p' = 1, 1 \leq p < \infty$ (see, e.g. [28, Lemma 12.6.6, p. 193]).

In this paper, it is also shown that for some conditions Zygmund sums (and at $s = 1$ also the Fejér sums) realize the orders of the best uniform approximations on the classes $C_{\beta,p}$ that is the order estimate (5) is true. Previously, this property was proved for Fourier sums [4, 18, 19, 21].

Let us formulate some necessary definitions.

A non-negative sequence $a = \{a_k\}_{k=1}^{\infty}, k \in \mathbb{N}$, is said to be generally monotonically increasing (we write $a \in GM^+$), if there exists a constant $A \geq 1$, such that for any natural $n_1$ and $n_2$ such that $n_1 \leq n_2$ the inequalities

$$a_{n_1} + \sum_{k=n_1}^{m} |a_k - a_{k+1}| \leq Aa_m, \quad m = \overline{n_1,n_2},$$

hold (see, e.g. [1, p. 811]). It is easy to see that if the positive sequence $a = \{a_k\}_{k=1}^{\infty}$ increases, starting from some number, then it generally monotonically increasing.

A non-negative sequence $a = \{a_k\}_{k=1}^{\infty}, k \in \mathbb{N}$, is said to be almost increasing (according to Bernstein, see, e.g. [26, p. 730]) if there exists a constant $K$, such that for all $n_1 \leq n_2$ we have

$$a_{n_1} \leq Ka_{n_2}.$$  

In this case, if for the sequence $a = \{a_k\}_{k=1}^{\infty}$ there exists a constant $\varepsilon > 0$, such that $\{a_k^{-\varepsilon}\}$ almost increases, then we write $a \in GA^+$. It is clear that if the sequence $a$ belongs to $GM^+$, then it is almost increasing according to Bernstein.

Let us put further $g_\delta(t) := \psi(t)^{\delta}, t \in [1, \infty)$ with $\delta > 0$.

## 2 Order estimates of the approximations by Zygmund sums on the classes of convolutions

**Theorem 1.** Let $s > 0, 1 \leq p < \infty, g_1/p \in M_0, g_{s+1/p} \in GM^+ \cap GA^+, \beta \in \mathbb{R}$ and $n \in \mathbb{N}$. In the case $1 < p < \infty$, if the condition (6) holds and the inequality

$$\inf_{t \geq 1} a(g_{1/p};t) > \frac{p'}{2}$$

holds, then the following order estimates take place

$$E_n \left(C_{\beta,p}^c \right) \prec \mathcal{E} \left(C_{\beta,p}^c \right) \prec \left(\sum_{k=n}^{\infty} \psi(k)k^{p'-2}\right)^{1/p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1;$$
in the case \( p = 1 \), if the condition (7) holds and the inequality
\[
\inf_{t \geq 1} \alpha(g_1; t) > 1
\] (12)
holds, then the following order estimates take place
\[
E_n \left( C^\psi_{\beta,1} \right) \times E \left( C^\psi_{\beta,1}; Z_{n-1}^s \right) \times \left\{ \sum_{k=n}^{\infty} \psi(k), \cos \frac{\beta \pi}{2} \neq 0, \psi(n), \cos \frac{\beta \pi}{2} = 0 \right\}
\] (13)

\textbf{Proof.} Since the operator \( Z_{n-1}^s : f(t) \rightarrow Z_{n-1}^s(f,t) \) is linear polynomial operator, which is invariant under the shift, i.e.
\[
Z_{n-1}^s (f_h, t) = Z_{n-1}^s (f, t + h), \quad f_h(t) = f(t + h), \quad h \in \mathbb{R},
\]
and norm in \( C \) and classes \( C^\psi_{\beta,p} \) also are invariant under the shift, that is
\[
\| f_h \|_C = \| f \|_C; \quad f(t) \in C^\psi_{\beta,p} \Rightarrow f_h(t) \in C^\psi_{\beta,p}
\]
then
\[
E \left( C^\psi_{\beta,p}; Z_{n-1}^s \right) = \sup_{f \in C^\psi_{\beta,p}} |f(0) - Z_{n-1}^s(f, 0)|.
\] (14)

By virtue of (1) and (2) for any function \( f \in C^\psi_{\beta,p}, 1 \leq p < \infty, \beta \in \mathbb{R}, s > 0 \), the following equality holds
\[
f(0) - Z_{n-1}^s (f, 0) = \frac{1}{\pi} \int_{-\pi}^\pi \left( \frac{1}{n^s} \sum_{k=n}^{n-1} \psi(k) k^s \cos \left( kt + \frac{\beta \pi}{2} \right) + \Psi_{-\beta,n}(t) \right) \psi(t) \, dt,
\] (15)
where \( \Psi_{-\beta,n}(t) = \sum_{k=n}^{\infty} \psi(k) \cos \left( kt + \frac{\beta \pi}{2} \right), \| \psi \|_p \leq 1, n \in \mathbb{N}. \)

Relations (14) and (15), Hölder’s inequality and triangle inequality imply that for \( 1 \leq p < \infty \)
\[
E \left( C^\psi_{\beta,p}; Z_{n-1}^s \right) \leq \frac{1}{\pi} \left\| \frac{1}{n^s} \sum_{k=1}^{n-1} \psi(k) k^s \cos \left( kt + \frac{\beta \pi}{2} \right) + \Psi_{-\beta,n}(t) \right\|_{p'},
\] (16)
\[
\leq \frac{1}{\pi n^s} \left\| \sum_{k=1}^{n-1} \psi(k) k^s \cos \left( kt + \frac{\beta \pi}{2} \right) \right\|_{p'} + \frac{1}{\pi} \| \Psi_{-\beta,n}(t) \|_{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1.
\]

Let us show that, if \( g_{s+1/p} \in GM^+ \cap GA^+, \) where \( g_{s+1/p} = \{ \psi(k) k^{s+1/p} \}_{k=1}^{\infty}, \) then
\[
\left\| \sum_{k=1}^{n-1} \psi(k) k^s \cos \left( kt + \frac{\beta \pi}{2} \right) \right\|_{p'} = O(\psi(n)n^{s+1/p}), \quad 1 \leq p < \infty.
\] (17)

Applying Abel transformation to the function \( \sum_{k=1}^{n-1} \psi(k) k^s \cos \left( kt + \frac{\beta \pi}{2} \right) \), we have
\[
\sum_{k=1}^{n-1} \psi(k) k^s \cos \left( kt + \frac{\beta \pi}{2} \right) = \sum_{k=1}^{n-2} \left( \psi(k) k^s - \psi(k+1)(k+1)^s \right) D_{k,\beta}(t)
\]
\[
+ \psi(n-1)(n-1)^s D_{n-1,\beta}(t) - \frac{1}{2} \cos \frac{\beta \pi}{2},
\] (18)
where
\[ D_{k,\beta}(t) := \frac{1}{2} \cos \frac{\beta \pi}{2} + \sum_{v=1}^{k} \cos \left( vt - \frac{\beta \pi}{2} \right). \]

Then, in view of \( |D_{k,\beta}(\cdot)| \leq O(k^{-1/p}) = O(k^\beta), 1 \leq p < \infty, k \in \mathbb{N}, \beta \in \mathbb{R} \), (see, e.g. [4]) from (18) we get
\[
\left\| \sum_{k=1}^{n-1} \psi(k)k^s \cos \left( kt + \frac{\beta \pi}{2} \right) \right\|_{p'} = O(1) + O \left( \sum_{k=1}^{n-2} |\psi(k)k^s \psi(k+1)(k+1)^s| k_1^{1/2} \right)
+ O \left( \psi(n-1)(n-1)^{s+1/2} \right). \tag{19}
\]

Since \( g_{s+1/p} \in GM^+ \), then, by using the triangle inequality, inequality (8) and Lagrange theorem, we have
\[
\sum_{k=1}^{n-2} |\psi(k)k^s - \psi(k+1)(k+1)^s| k_1^{1/2} \leq \sum_{k=1}^{n-2} |\psi(k)k^{s+1/2} - \psi(k+1)(k+1)^{s+1/2}|
+ \sum_{k=1}^{n-2} |\psi(k+1)(k+1)^{s+1/2} - \psi(k+1)(k+1)^{s+1/2}|
\leq A\psi(n-1)(n-1)^{s+1/2} + \frac{1}{p} \sum_{k=1}^{n-2} \psi(k+1)(k+1)^{s+1/2} k_1\tag{20}
\leq A\psi(n-1)(n-1)^{s+1/2} + 2 \sum_{k=2}^{n-1} \frac{\psi(k)k^{s+1/2}}{k}.
\]

According to the condition \( g_{s+1/p} \in GA^+ \), there exits \( \varepsilon > 0 \) such that the sequence \( \{g_{s+1/p}(k)k^{-\varepsilon}\} = \{\psi(k)k^{s+1/p-\varepsilon}\} \) almost increases, and hence taking into account (9), we obtain
\[
\sum_{k=2}^{n-1} \frac{\psi(k)k^{s+1/p}}{k} = \sum_{k=2}^{n-1} \frac{\psi(k)k^{s+1/p-\varepsilon}}{k^{1-\varepsilon}} \leq K\psi(n-1)(n-1)^{s+1/p-\varepsilon} \sum_{k=2}^{n-1} \frac{1}{k^{1-\varepsilon}}
< K\psi(n-1)(n-1)^{s+1/p-\varepsilon} \int_{1}^{n-1} \frac{dt}{t^{1-\varepsilon}} \leq \frac{K}{\varepsilon} \psi(n-1)(n-1)^{s+1/p}. \tag{21}
\]

From (20) and (21) we get the following inequality
\[
|\psi(k)k^s - \psi(k+1)(k+1)^s| k_1^{1/2} \leq \left( A + \frac{2K}{\varepsilon} \right) \psi(n-1)(n-1)^{s+1/p}. \tag{22}
\]

From (19) and (22) we obtain the estimation (17).

To estimate the norm \( \|\Psi_{-\beta,n}(\cdot)\|_{p'} \) for \( 1 < p' < \infty \) we use the statement, which was established in [18], and according to which in the case when \( \{a_k\}_{k=1}^\infty \) is the monotonically non-increasing sequence of positive numbers such that \( \sum_{k=1}^\infty a_k^p k^{p'-2} < \infty \), then for arbitrary \( n \in \mathbb{N} \) and \( \gamma \in \mathbb{R} \) the following estimate holds
\[
\left\| \sum_{k=n}^\infty a_k \cos (kx + \gamma) \right\|_{p'} = O \left( \left( \sum_{k=n}^\infty a_k^p k^{p'-2} + a_n^p n^{p'-1} \right)^{1/p'} \right). \tag{23}
\]
Putting in (23) \(a_k = \psi(k), \gamma = \frac{\beta \pi}{2}\) we obtain that for \(1 < p < \infty, \beta \in \mathbb{R}\) and \(n \in \mathbb{N}\)
\[
\|\Psi_{-\beta,n}(\cdot)\|_{p'} = O\left(\sum_{k=n}^{\infty} \psi'(k) k^{p'-2} + \psi'(n) n^{p'-1}\right)^{1/p'}.
\] (24)

Then, using [18, Lemma 3], we conclude that for \(1 < p' < \infty, n \in \mathbb{N}\), under condition (6) and imbedding \(g_{1/p} \in \mathcal{M}_0\) the following estimate holds
\[
\psi'(n) n^{p'-1} = O\left(\sum_{k=n}^{\infty} \psi'(k) k^{p'-2}\right).
\] (25)

According to the conditions of Theorem 1 we have that \(g_{1/p} \in \mathcal{M}_0\), so taking into account (25), from (24), we obtain
\[
\|\Psi_{-\beta,n}(\cdot)\|_{p'} = O\left(\sum_{k=n}^{\infty} \psi'(k) k^{p'-2}\right)^{1/p'}, \quad 1 < p' < \infty, \quad \beta \in \mathbb{R}, \quad n \in \mathbb{N}.
\] (26)

Combining (16), (17) and (26) in the case when \(g_{1/p} \in \mathcal{M}_0\), and \(g_{s+1/p} \in GM^+ \cap GA^+\), we arrive at the estimate
\[
\mathcal{E}\left(C_{\beta,p'}^{\psi}; Z_n\right) = O\left(\sum_{k=n}^{\infty} \psi'(k) k^{p'-2}\right)^{1/p'}, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.
\] (27)

As follows from [18, Corollary 1 and 2], for \(1 < p < \infty, 1/p + 1/p' = 1, n \in \mathbb{N}\) and \(\beta \in \mathbb{R}\), under conditions (6) and (10) and imbedding \(g_{1/p} \in \mathcal{M}_0\) for \(E_n\left(C_{\beta,p}^{\psi}\right)_C\) we arrive at the following order estimates
\[
E_n\left(C_{\beta,p}^{\psi}\right)_C \lesssim \left(\sum_{k=n}^{\infty} \psi'(k) k^{p'-2}\right)^{1/p'}.
\] (28)

Therefore, by virtue of inequality (4) and relations (27) and (28) we obtain order equality (11).

Further, let us consider the case \(p = 1\). Let us establish the estimate of the norm \(\|\Psi_{-\beta,n}(\cdot)\|_{p'} = \|\Psi_{-\beta,n}(\cdot)\|_{\infty}\). It is obvious that for any \(\beta \in \mathbb{R}\) the following inequality holds
\[
\|\Psi_{-\beta,n}(\cdot)\|_{\infty} = \left\|\sum_{k=n}^{\infty} \psi(k) \cos\left(kt + \frac{\beta \pi}{2}\right)\right\|_{\infty} \leq \sum_{k=n}^{\infty} \psi(k).
\] (29)

If \(\beta = 2k + 1, k \in \mathbb{Z}\), then following estimate takes place
\[
\|\Psi_{-\beta,n}(\cdot)\|_{\infty} = \left\|\sum_{k=n}^{\infty} \psi(k) \sin kt\right\|_{\infty} \leq (\pi + 2)\psi(n)n
\] (30)
(see, e.g. [21, relation (82)]).

According to [21, Lemma 3], if \(g_1 \in \mathcal{M}_0\), where \(g_1 = \{\psi(k)k\}_{k=1}^{\infty}\) and the condition (7) holds, then the following estimates are true
\[
\psi(n)n = O\left(\sum_{k=n}^{\infty} \psi(k)\right).
\] (31)
If \( g_1 \in \mathcal{M}_0 \) and the conditions (7) hold, then combining (16), (17), (29) – (31), we obtain the following estimates

\[
E_\left(C^\psi_{\beta,1}; Z_{n-1}^s\right)_C = \left\{ \begin{array}{ll}
O\left(\frac{\psi(k)}{\sqrt[k]{k}}\right), & \cos \frac{\beta \pi}{2} \neq 0, \\
O\left(\psi(n) n\right), & \cos \frac{\beta \pi}{2} = 0.
\end{array} \right.
\] (32)

To estimate the quantity \( E_\left(C^\psi_{\beta,1}; Z_{n-1}^s\right)_C \) from below, we use [21, Theorems 3 and 4], according to which, if \( g_1 \in \mathcal{M}_0 \) and the conditions (7) and (12) are true, then for \( n \in \mathbb{N} \) and \( \beta \in \mathbb{R} \) the following order equalities take place

\[
E_n\left(C^\psi_{\beta,1}\right)_C \asymp \left\{ \begin{array}{ll}
\sum_{k=n}^{\infty} \psi(k), & \cos \frac{\beta \pi}{2} \neq 0, \\
\psi(n) n, & \cos \frac{\beta \pi}{2} = 0.
\end{array} \right.
\] (33)

The estimate (13) follows from the inequality (4), estimates (32) and (33).

Assume that the conditions of Theorem 1 take place, moreover, more stronger imbedding \( g_1/p \in \mathcal{M}_C \) holds. As it follows from [18, Lemma 3], if \( g_1/p \in \mathcal{M}_C \) and the condition (6) holds, then for \( 1 < p < \infty \) the following estimates take place

\[
\sum_{k=n}^{\infty} \psi(p')(k)(k')^{p'-2} \asymp \psi'(n)n^{p'-1}.
\] (34)

In addition, as it was shown in [21, Lemma 3], if \( g_1 \in \mathcal{M}_C \) and the condition (7) holds, then the following order estimates are true

\[
\sum_{k=n}^{\infty} \psi(k) \asymp \psi(n)n.
\] (35)

Formulas (34) and (35), and Theorem 1 allow us to write the following statement.

**Theorem 2.** Let \( s > 0, 1 \leq p < \infty, g_{1/p} \in \mathcal{M}_C, g_{s+1/p} \in GM^+ \cap GA^+, \beta \in \mathbb{R} \) and \( n \in \mathbb{N} \).

In the case \( 1 < p < \infty \), if the conditions (6) and (10) hold, then the following order estimates take place

\[
E_n\left(C^\psi_{\beta,p}\right)_C \asymp E_\left(C^\psi_{\beta,p'; Z_{n-1}^s}\right)_C \asymp \psi(n)n^{1/p},
\] (36)

and in the case \( p = 1 \) if the conditions (7) and (12) hold, then the following order estimates take place

\[
E_n\left(C^\psi_{\beta,1}\right)_C \asymp E_\left(C^\psi_{\beta,1'; Z_{n-1}^s}\right)_C \asymp \psi(n)n.
\] (37)

**Proof.** Order estimates (36) were established in [14]. Note, that when \( 1 < p < \infty \), \( g_{1/p} \in \mathcal{M}_0 \) and

\[
\lim_{t \to \infty} \alpha(g_{1/p}; t) = \infty,
\] (38)

then the order estimates (36) do not take place, since in this case we have the following (see [18])

\[
\psi(n)n^{1/p} = o\left((\sum_{k=n}^{\infty} \psi'(k)(k')^{p'-2})^{1/p'}\right), \quad n \to \infty.
\]
Similarly, when $p = 1$, $g_{1/p} = g_1 \in \mathcal{M}_0$ and
\[
\lim_{t \to \infty} a(g_1; t) = \infty,
\] (39)
then as follows from [21, Lemma 3]
\[
\psi(n)n = o\left(\sum_{k=n}^{\infty} \psi(k)\right),
\]
in this case, for $\beta$ such that $\cos \frac{\beta \pi}{2} \neq 0$ order estimates (37) do not take place.

As example of the function $\psi(t)$, for which the conditions of Theorem 1 and the equalities (38) and (39) take place, we can use the function
\[
\psi(t) = t^{-1/p} \ln^{-\gamma}(t + K), \quad \gamma > \begin{cases} \frac{1}{p}, & 1 < p < \infty, \\ 1, & p = 1, \end{cases} \quad K > e^{\gamma p'/2}, 1/p + 1/p' = 1,
\] (40)
(see [18,21]). Let us write the order estimates for the quantities $E_n\left(C^\psi_{\beta,p}\right)_C$ and $E\left(C^\psi_{\beta,p'}; Z^s_{n-1}\right)_C$
in the case, when $\psi(t)$ has the form (40).

**Theorem 3.** Let $\psi(t) = t^{-1/p} \ln^{-\gamma}(t + K)$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$. If $1 < p < \infty$, $\gamma > 1/p'$, $K > e^{\gamma p'/2}$, $1/p + 1/p' = 1$, then
\[
E_n\left(C^\psi_{\beta,p}\right)_C \asymp E\left(C^\psi_{\beta,p'}; Z^s_{n-1}\right)_C \asymp \psi(n)n^{1/p} \ln^{1/p'} n, \quad n \geq 2;
\] (41)
if $p = 1$, $\gamma > 1$, $K > e^\gamma$, then
\[
E_n\left(C^\psi_{\beta,1}\right)_C \asymp E\left(C^\psi_{\beta,1}; Z^s_{n-1}\right)_C \asymp \begin{cases} \psi(n)n \ln n, & \cos \frac{\beta \pi}{2} \neq 0, \\ \psi(n)n, & \cos \frac{\beta \pi}{2} = 0, \end{cases} \quad n \geq 2.
\] (42)

**Proof.** We show that for the indicated function $\psi$ of the form (40) all conditions of the Theorem 1 are true. Indeed, for $1 < p < \infty$, $\gamma > 1/p'$, $K > e^{\gamma p'/2}$ we have
\[
\sum_{k=n}^{\infty} \psi'(k)k^{p' - 2} = \sum_{k=n}^{\infty} \frac{1}{k \ln^{\gamma p'}(k + K)} < \infty, \quad a(\psi_{1/p'; t}) = \frac{(t + K) \ln(t + K)}{\gamma t} > \frac{\ln(t + e^{\gamma p'/2})}{\gamma},
\]
and hence $\lim_{t \to \infty} a(g_{1/p'; t}) = \infty$ and $a(g_{1/p'; t}) > \frac{p'}{2}.

For $p = 1$, $\gamma > 1$, $K \geq e^\gamma$, we have
\[
\sum_{k=n}^{\infty} \psi(k) \leq \sum_{k=n}^{\infty} \frac{1}{k \ln(k + e^\gamma)} < \infty, \quad a(g_{1}; t) > \frac{\ln(t + e^\gamma)}{\gamma},
\]
and hence $\lim_{t \to \infty} a(g_1; t) = \infty$ and $a(g_1; t) > 1$.

It is obvious that for any $s > 0$ and $1 \leq p < \infty$ the functions $g_{s+1/p}(t) = t^s \ln^{-\gamma}(t + K)$ increase monotonically, starting from some point $t_0$. Therefore, it is not difficult to be convinced that the sequence $g_{s+1/p}(k)$ belongs to the set $GM^+ \cap GA^+$.

Therefore, the function $\psi$ of the form (40) satisfies the conditions of Theorem 1.
Further, using [18, formula (79)], we obtain

\[
\left( \sum_{k=n}^{\infty} \psi^p(k)k^{p'-2} \right)^{1/p'} \propto \left( \int_{n}^{\infty} \psi^p(t)t^{p'-2} \, dt \right)^{1/p'} \propto \left( \int_{n}^{\infty} \frac{dt}{t \ln^\gamma (t+1)} \right)^{1/p'} \propto \ln^{1/p'-\gamma} n
\]

\[
= \psi(n)n^{1/p} \ln^{1/p'} n \left( \frac{\ln^{-\gamma} n}{\ln^{-\gamma} (n+K)} \right) \propto \psi(n)n^{1/p} \ln^{1/p'} n, \quad n \geq 2.
\]

Then formula (41) follows from the estimate (11) and the above relations.

Similarly, by virtue of [21, inequality (87)] we get

\[
\sum_{k=n}^{\infty} \psi(k) \propto \int_{n}^{\infty} \psi(t) \, dt = \int_{n}^{\infty} \frac{dt}{t \ln^{\gamma} (t+1)} \propto \ln^{-1/\gamma} n \propto \psi(n)n \ln n, \quad n > 2. \tag{43}
\]

Formula (42) follows from the estimates (13) and relations (43), in the case where \( \beta \) is such that \( \cos \frac{\beta \pi}{2} \neq 0 \).

As it was already mentioned, for \( s = 1 \) the Zygmund sums \( Z_{n-1} \) coincide with the known Fejér sums \( \sigma_{n-1} \). Therefore, Theorem 1 and 2 imply the following statements.

**Proposition 1.** Let \( 1 \leq p < \infty, g_{1/p} \in \mathcal{M}, g_{1+1/p} \in GM^+ \cap GA^+, \beta \in \mathbb{R} \) and \( n \in \mathbb{N} \).

In the case \( 1 < p < \infty \), if the conditions (6) and (10) hold, then the following order estimates take place

\[
E_n(C_{\beta,p}^\psi) \propto \mathcal{E} \left( C_{\beta,p}^\psi; \sigma_{n-1} \right) \propto \left( \sum_{k=n}^{\infty} \psi^p(k)k^{p'-2} \right)^{1/p'};
\]

in the case \( p = 1 \), if the conditions (7) and (12) hold, then the following order equalities take place

\[
E_n(C_{\beta,1}^\psi) \propto \mathcal{E} \left( C_{\beta,1}^\psi; \sigma_{n-1} \right) \propto \left\{ \begin{array}{ll}
\sum_{k=n}^{\infty} \psi(k), & \cos \frac{\beta \pi}{2} \neq 0, \\
\psi(n)n, & \cos \frac{\beta \pi}{2} = 0.
\end{array} \right.
\]

**Proposition 2.** Let \( 1 \leq p < \infty, g_{1/p} \in \mathcal{M}, g_{1+1/p} \in GM^+ \cap GA^+, \beta \in \mathbb{R} \) and \( n \in \mathbb{N} \).

In the case \( 1 < p < \infty \), if the conditions (6) and (10) hold, then the following order estimates take place

\[
E_n(C_{\beta,p}^\psi) \propto \mathcal{E} \left( C_{\beta,p}^\psi; \sigma_{n-1} \right) \propto \psi(n)n^{1/p};
\]

in the case \( p = 1 \), if the conditions (7) and (12) hold, then the following order estimates take place

\[
E_n(C_{\beta,1}^\psi) \propto \mathcal{E} \left( C_{\beta,1}^\psi; \sigma_{n-1} \right) \propto \psi(n)n.
\]

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Сердюк А.С., Грабова Ю.З. Порядкові оцінки рівномірних наближень сумами Зигмунда на класах згорток періодичних функцій / Карпатські матем. публ. — 2021. — Т.13, №1. — С. 68–80.

Суми Зигмунда \( Z_{n-1}^s(f; t) \) функції \( f \in L_1 \) — це тригонометричні поліноми вигляду
\[
Z_{n-1}^s(f; t) := \frac{2}{\pi} + \sum_{k=1}^{\infty} \left( 1 - \left| \frac{k}{n} \right|^s \right)^p (a_k(f) \cos kt + b_k(f) \sin kt),
\]
s > 0, де \( a_k(f) \) і \( b_k(f) \) — коефіцієнти Фур'є функції \( f \). Отримано точні порядкові оцінки рівномірних наближень сумами Зигмунда \( Z_{n-1}^s \) на класах \( C_{p, q} \). Ці класи складаються з \( 2\pi \)-періодичних неперервних функцій \( f \), які збираються у вигляді згортки функцій, що належать однією класам просторів \( L_p \),
\[
1 \leq p < \infty, \text{ з фіксованими твірними ядрами } \Psi(t) \sim \sum_{k=1}^{\infty} \psi(k) \cos \left( kt + \frac{2\pi}{p} \right), \Psi \in L_p, \beta \in \mathbb{R},
\]
\[
\frac{1}{p} + \frac{1}{p'} = 1,
\]
в випадку, коли добуток \( \psi(k)k^{s+1/p} \) узагальнено монотонно зростає з деякою степенною швидкістю, і, крім того, при \( 1 < p < \infty \) виконується нерівність \( \sum_{k=1}^{\infty} \psi^p(k)k^{s-2} < \infty \), а при \( p = 1 \) — нерівність \( \sum_{k=1}^{\infty} \psi(k) < \infty \). Показано, що при виконанні зазначених умов суми Зигмунда \( Z_{n-1}^s \), а також суми Фейєра \( c_{n-1} = Z_{n-1}^1 \) реалізують порядки найкращих рівномірних наближень тригонометричними поліномами на вказаних функціональних класах, а саме при \( 1 < p < \infty \)
\[
E_n(C_{p, q}^f) \subset E \left( C_{p, q}^f; Z_{n-1}^s \right) \subset \left( \sum_{k=n}^{\infty} \psi^p(k)k^{s-2} \right)^{1/p'}, 1 - \frac{1}{p} + \frac{1}{p'} = 1,
\]
а при \( p = 1 \)
\[
E_n(C_{p, q}^f) \subset E \left( C_{p, q}^f; Z_{n-1}^s \right) \subset \sum_{k=n}^{\infty} \psi(k), \cos \frac{\beta \pi}{2} \neq 0,
\]
\[
E_n(C_{p, q}^f) \subset E \left( C_{p, q}^f; Z_{n-1}^s \right) \subset \sum_{k=n}^{\infty} \psi(k)n, \cos \frac{\beta \pi}{2} \neq 0,
\]
de
\[
E_n(C_{p, q}^f) := \sup_{f \in C_{p, q}^f} \inf_{t_{n-1} \in T_{2n-1}} \| f(\cdot) - t_{n-1}(\cdot) \|_C,
\]
\( T_{2n-1} \) — підпростір тригонометричних поліномів \( t_{n-1} \) порядку \( n - 1 \) з дійсними коефіцієнта-ми,
\[
E \left( C_{p, q}^f; Z_{n-1}^s \right) := \sup_{f \in C_{p, q}^f} \| f(\cdot) - Z_{n-1}^s(f; \cdot) \|_C.
\]

Ключові слова і фрази: найкраще наближення, сума Зигмунда, сума Фейєра, підпростір тригонометричних поліномів, порядкова оцінка.