Fermi breakup and the Statistical Multifragmentation Model

B.V. Carlson, R. Donangelo, S.R. Souza, W.G. Lynch, A.W. Steiner and M.B. Tsang

1 Instituto Tecnológico de Aeronáutica, Pça. Mal. Eduardo Gomes, 50 12228-900 São José dos Campos, Brazil
2 Instituto de Física, Universidade Federal do Rio de Janeiro, Cidade Universitária, CP 68528, 21941-972, Rio de Janeiro, Brazil
3 Instituto de Física, Facultad de Ingeniería Universidad de la República, Julio Herrera y Reissig 565, 11300 Montevideo, Uruguay
4 Instituto de Física, Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves 9500, CP 15051, 91501-970, Porto Alegre, Brazil
5 Joint Institute for Nuclear Astrophysics, National Superconducting Cyclotron Laboratory, and the Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824, USA
E-mail: brett@ita.br

Abstract. We discuss the equivalence of a generalized Fermi breakup model, in which densities of excited states are taken into account, and the microcanonical Statistical Multifragmentation Model used to describe the desintegration of highly excited fragments of nuclear reactions.

1. Introduction
Both the Fermi breakup[1] (FBM) and the Statistical Multifragmentation[2, 3] (SMM) models provide prescriptions for calculating mass and charge distributions and multiplicities of the fragments emitted in the breakup of an excited nuclear system. Although they are usually formulated in different terms and applied in very different regions of mass and excitation energy, they are essentially one and the same model. We make this close relationship explicit and comment on how the FBM/SMM model might be extended to provide a comprehensive description of statistical nuclear decay.

2. Generalized Fermi breakup
In applications of the FBM to nuclear decay[3], the phase-space integral that determines the density of final states of a configuration of \( n \) fragments is usually written as

\[
\omega_n = \prod_{l=1}^{k} \frac{1}{N_l!} \left( \frac{V_n}{(2\pi \hbar)^3} \right)^{n-1} \prod_{j=1}^{n} g_j \int d^{3} p_j \delta \left( \sum_{j=1}^{n} \vec{p}_j \right) \times \delta \left( \varepsilon_0 - B_0 - E_{c0} - \sum_{j=1}^{n} \frac{p_j^2}{2m_j} - B_j - E_{cj} \right) ,
\]

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where the sums and products $j = 1, \ldots, n$ run over all fragments of the breakup mode, while the sum $l = 1, \cdots, k$ runs over the distinct fragments and takes into account their multiplicities $N_l$. Here, $\varepsilon_0$ is the excitation energy of the decaying nucleus, $B_0$ its binding energy and $E_{c0}$ is a term associated with the Wigner-Seitz correction to the Coulomb energy of the system. $V_n$ is the volume in which the momentum states are normalized and is usually defined as

$$V_n = (1 + \chi) V_0,$$

where $V_0$ is the ground state volume of the decaying nucleus and the expansion factor $\chi$ is usually taken to be $\chi = 1$. For the fragments, $B_j$ is the binding energy of fragment $j$ and $g_j$ is its spin multiplicity, while the $E_{cj}$ represent the remaining Wigner-Seitz corrections to the Coulomb energy, taken to be

$$E_{cj} = \frac{C_{Coul} Z_j^2}{A_j^{1/3}}.$$

Conservation of nucleon number and charge requires that

$$A_0 = \sum_{j=1}^{n} A_j = \sum_{l=1}^{k} N_l A_l,$$

and

$$Z_0 = \sum_{j=1}^{n} Z_j = \sum_{l=1}^{k} N_l Z_l,$$

where $Z_j$ and $A_j$ are the charge and mass number, respectively, of fragment $j$. The FBM assumes that the fragments are emitted in their ground states or in (almost) particle-stable excited states.

As the total excitation energy is increased, other particle-unstable excited states that are long-lived in comparison to the initial decaying nucleus could also be included and can make significant contributions to the phase space integral. These can be incorporated compactly using the densities of excited states of the fragments. Such an extension of the Fermi breakup integral takes the form

$$\omega_n = \prod_{l=1}^{k} \frac{1}{N_l!} \left( \frac{V_n}{(2\pi \hbar)^3} \right)^{n-1} \int \prod_{j=1}^{n} d^3 p_j \, \delta \left( \sum_{j=1}^{n} \vec{p}_j \right) \times \int \prod_{j=1}^{n} (\omega_j (\varepsilon_j) d\varepsilon_j) \, \delta \left( \varepsilon_0 - B_0 - E_{c0} - \sum_{j=1}^{n} \left( \frac{p_j^2}{2m_j} + \varepsilon_j - B_j - E_{cj} \right) \right)$$

where $\varepsilon_j$ is the excitation energy of fragment $j$, $\omega_j (\varepsilon_j)$ its density of states and $B_j$ is now its ground-state binding energy. Note that this expression does not contain the fragment spin multiplicities, $g_j$, which are now assumed to be incorporated in the density of states. For a particle with no excited states, we have $\omega_j (\varepsilon_j) = g_j \delta (\varepsilon_j)$.

3. The Statistical Multifragmentation Model

After rewriting the densities of fragment states in terms of the internal Helmholtz free energies, defined for fragment $j$ by

$$e^{-\beta_j f_j^*(\beta_j)} = \int_0^\infty d\varepsilon_j e^{-\beta_j \varepsilon_j} \omega_j (\varepsilon_j),$$
all but one of the integrals can be performed analytically. We can then write the density of final states $\omega_n$ as

$$\omega_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta \exp \left[ -\beta \left( F_n(\beta) - E_0 \right) \right],$$

where

$$E_0 = \varepsilon_0 - B_0,$$

and the total Helmholtz free energy $F_n(\beta)$ has been defined as

$$F_n(\beta) = \sum_{l=1}^{k} N_l \left( f^*_l(\beta) + f^{\text{trans}}_l(\beta) - B_l - E_{cl} \right) - \left( f^{\text{trans}}_0(\beta) - E_{c0} \right),$$

with the sum over fragments replaced by a sum over distinct fragments times their multiplicities. The translational Helmholtz free energies are given by

$$f^{\text{trans}}_l(\beta) = -\frac{1}{\beta} \left[ \ln \left( V_n \left( \frac{m_N A_l}{2\pi \hbar^2} \right)^{3/2} \right) - \ln \left( N_l! \right) \right],$$

where we write the mass of fragment $l$ as $m_l = m_N A_l$ and the mass of the decaying nucleus as $m_0 = m_N A_0$, with $m_N$ the nucleon mass.

To approximate the final integral, we use the method of steepest descent. Using the relations of the Helmholtz free energy to the entropy and energy,

$$s = -\frac{df}{dT} = \beta \frac{df}{d\beta} \quad \text{and} \quad e = f + Ts = f + \beta \frac{df}{d\beta},$$

we find that the saddle point condition is equivalent to the requirement that energy is conserved,

$$\sum_{l=1}^{k} N_l \left( \varepsilon^*_l(\beta_0) + \varepsilon^{\text{trans}}_l(\beta_0) - B_l - E_{cl} \right) - \left( \varepsilon^{\text{trans}}_0(\beta_0) - E_{c0} \right) = \varepsilon_0 - B_0.$$

At the saddle point $\beta_0$, the argument of the exponential is then the total entropy, $S_n(\beta_0)$. To complete the evaluation, we calculate the second derivative

$$\frac{d^2}{d\beta^2} \left( \beta F_n(\beta) \right) = T^2 \frac{d^2 F_n}{dT^2} = -C_{V,n} T^2,$$

where $C_{V,n}$ is the specific heat of the configuration at constant volume, and conclude that the direction of steepest descent is purely imaginary. The integral thus yields

$$\omega_n = \frac{\exp \left( S_n(T_0) \right)}{\sqrt{2\pi C_{V,n} T_0^2}}.$$

As $T_0 \to 0$, we expect the phase space integral to reduce to the original form of the FBM, for which a well-known closed-form expression exists,

$$\omega_n \to \prod_{l=1}^{k} \frac{1}{N_l!} \frac{1}{m_0^{3/2}} \prod_{j=1}^{n} g_j m_j^{3/2} \left( \frac{V_n}{(2\pi)^{3/2} \hbar^3} \right)^{n-1} \frac{E_{kin}^{3(n-1)/2-1}}{\Gamma(3(n-1)/2)}$$

where

$$E_{kin} = \varepsilon_0 - B_0 - E_{c0} + \sum_{j=1}^{n} (B_j + E_{cj})$$
is the total kinetic energy of the fragments. The steepest-descent approximation does indeed reduce to this closed-form expression, with all fragments in their ground states, as $T_0 \to 0$, except for a multiplicative factor that substitutes a Stirling approximation for the gamma function in the denominator,

$$R(k) = \frac{\Gamma(k)}{\exp[(k - 1/2) \ln(k) - k + \ln(2\pi)/2]} ,$$

where $k = 3(n - 1)/2$. This factor is approximately 1.06 for $n = 2$ ($k = 3/2$), 1.03 for $n = 3$ and decreases to one as $n$ increases.

4. Model comparison

We compare the results for the Fermi breakup of $^{16}$O at an excitation energy of 50 MeV including 1) only the ground states of the fragments, 2) the ground states and particle-bound states of the fragments and 3) the ground states and all excited states of the fragments found in the RIPL-2 nuclear level library (http://www-nds.iaea.org/RIPL-2/).

The calculations were performed using the steepest-descent approximation and furnish 16 two-fragment, 38 three-fragment, 33 small four-fragment and several negligible five and six-fragment decay channels. The mean primary fragment multiplicity decreases from 2.8 for the ground-state-only calculation to 2.4 for the particle-stable state one and to 2.3 when all excited states are included. The primary fragment production cross sections of the isotopes of nitrogen, carbon, boron and beryllium are shown in Fig. 1 as functions of the neutron excess, $N - Z$.

We draw attention to the substantial differences between the calculation including ground and particle-stable states (red squares), which corresponds to the FBM model in common use[3], and that including all states (blue circles). These differences make it clear that the restriction to particle-stable states seriously violates the assumption of statistical equilibrium, even at this fairly low value of the excitation energy, and call into question the physical relevance of such calculations.
5. Conclusion
The FBM and the SMM are essentially one and the same model. Strictly speaking, however, this is not a model of the decay of an excited nuclear system. A decay model, such as the usual compound nucleus (CN) evaporation one, furnishes partial decay widths for the different emission channels rather than probabilities. A connection between the two-fragment FBM/SMM configuration and the usual CN decay was established long ago[4]. Work on extending this to arbitrary configurations is in progress.

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