LÉVY LAPLACIANS IN HIDA CALCULUS AND MALLIAVIN CALCULUS

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Abstract: Some connections between different definitions of Lévy Laplacians in the stochastic analysis are considered. Two approaches are used to define these operators. The standard one is based on the application of the theory of Sobolev-Schwartz distributions over the Wiener measure (the Hida calculus). One can consider the chain of Lévy Laplacians parametrized by a real parameter with the help of this approach. One of the elements of this chain is the classical Lévy Laplacian. Another approach to define the Lévy Laplacian is based on the application of the theory of Sobolev spaces over the Wiener measure (the Malliavin calculus). It is proved that the Lévy Laplacian defined with the help of the second approach coincides with one of the elements of the chain of Lévy Laplacians, which is not the classical Lévy Laplacian, under the imbedding of the Sobolev space over the Wiener measure into the space of generalized functionals over this measure. It is shown which Lévy Laplacian in the stochastic analysis is connected to the gauge fields.

key words: Lévy Laplacian, Yang-Mills equations, Hida calculus, Malliavin calculus

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Introduction

An infinite-dimensional Laplacian, given by the formula

\[ \Delta^{\{e_n\}}_L f(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \langle f''(x)e_k, e_k \rangle, \quad (1) \]

where the function \( f \) is defined on a separable Hilbert space \( H \) and \( \{e_n\} \) is an orthonormal basis in \( H \), is called the Lévy Laplacian or the Lévy-Laplace operator. The Hida calculus or the white noise analysis is the theory of Sobolev-Schwartz distributions over the (abstract) Wiener measure. The Malliavin calculus is the theory of Sobolev spaces over measures on infinite-dimensional spaces, in particular, over the Wiener measure. The goal of the present paper is to show that there are two infinite-dimensional Laplacians in the stochastic analysis, which are natural analogues of operator \( \Delta^{\{e_n\}}_L \). The first Laplacian is defined using the Hida calculus. This operator will be called the classical Lévy Laplacian. An extensive literature is devoted to the study of this operator (see the review \[21\], and also \[30\] \[5\] and the papers cited there). The second Laplacian is defined using the Malliavin calculus. This operator is connected with the gauge fields. In the Hida calculus one can consider a chain of infinite-dimensional Laplacians parametrized by a real parameter. One of the elements of this chain is the classical Lévy Laplacian. In the paper it is proved that the Lévy Laplacian, defined with the help of the Malliavin calculus, is isomorphic to the element of this chain under the embedding of the Sobolev space over the Wiener measure into the space of generalized functionals over this measure. This element does not coincide with the classical Lévy Laplacian.

One of the main reasons for interest in the Lévy Laplacian and differential operators defined by analogy is their connection to the Yang-Mills equations. In the papers \[1\] \[2\]
by Accardi, Gibilisco and Volovich an analogue of the Lévy Laplacian on the space of functions on the space of paths in \( \mathbb{R}^d \) has been introduced. This analogue is also called the Lévy Laplacian. For this operator in \([2]\) the following has been proved. A connection in a vector bundle over \( \mathbb{R}^d \) is a solution to the Yang-Mills equations if and only if the parallel transport generated by this connection is a solution to the Laplace equation for the Lévy Laplacian. In the paper \([22]\) by Leandre and Volovich the Lévy Laplacians on the space of functions on the space of paths in a compact Riemannian manifold and on the Sobolev space over the Wiener measure on the space of paths in a compact Riemannian manifold have been introduced. It has been shown that the theorem on the connection between the Lévy Laplacian and the gauge fields is also satisfied in these cases. In \([29]\) by the author the relationship between the Lévy Laplacian and instantons has been studied.

In \([1, 2, 22]\) the Lévy Laplacian has been defined not as the Cesàro mean of the second derivatives but as an integral functional generated by a special form of the second derivative. This approach to define the Lévy Laplacian also goes back to the original P. Lévy’s works. In the paper \([1]\) the following problem has been posed: can one represent the Lévy Laplacian associated with the gauge fields in a form similar to \([1]\). In the deterministic planar case, it has been shown in work \([29]\) that this is indeed the case (see also \([7]\)). In the author’s work \([31]\), using the Malliavin calculus, the Lévy Laplacian, defined as the Cesàro mean of second partial derivatives, has been introduced on the Sobolev space over the Wiener measure and its relation to gauge fields has been studied. It should be noted that, unlike the deterministic case, the Lévy Laplacians on the Sobolev spaces over the Wiener measure, introduced in the works \([22]\) and \([31]\), operate in different ways.

One of the motivations for studying the Hida calculus was the development of the harmonic analysis for the Lévy Laplacian (see \([15]\), and also \([19, 16, 20, 17]\)). If a Sobolev space over a Wiener measure is a natural domain of the Gross-Volterra Laplacian, then the space of generalized Hida functionals is a sufficiently wide space to contain the domain of the classical Lévy Laplacian. In addition, using the Hida calculus, certain generalizations of the Lévy Laplacian has been studied: the so-called exotic and nonclassical Lévy Laplacians (see \([4, 5, 30]\)). The family of the exotic Lévy Laplacians \( \Delta^{(s)}_{\text{exotic}} \), where \( s \geq 0 \), has been introduced in \([6]\). This family has the following properties: \( \Delta^{(0)}_{\text{exotic}} \) is the Gross-Volterra Laplacian; \( \Delta^{(1)}_{\text{exotic}} \) is the classical Lévy Laplacian; if \( s_1 < s_2 \), then the domain of \( \Delta^{(s_1)}_{\text{exotic}} \) belongs to the kernel of \( \Delta^{(s_2)}_{\text{exotic}} \). We can consider the chain of Lévy Laplacians \( \Delta^{(s)}_L \) of order \( s \in \mathbb{R} \) such that \( \Delta^{(s)}_L = s \Delta^{(s)}_{\text{exotic}} \). Thus, the chain of the exotic Laplacians \( \Delta^{(s)}_{\text{exotic}} \) can be extended for negative \( s \). In this paper we consider the Lévy Laplacian \( \Delta^{(-1)}_L \) in the Hida calculus. As already mentioned above, the main result of this article is as follows. We show that the Lévy Laplacian, introduced in \([31]\) using the Malliavin calculus, coincides with \( \pi^2 \Delta^{(-1)}_L \) under the natural embedding of the Sobolev space into the space of generalized Hida functionals. Unlike the classical Lévy Laplacian \( \Delta^{(1)}_L \) in the Hida calculus, the operator \( \Delta^{(-1)}_L \) is little studied.

In a particular case some of the results of this paper have been obtained in \([32]\).

The paper is organized as follows. The first section provides general information about the chain of the Lévy Laplacians in the deterministic case. The second section shows which elements of the chain are related to the Yang-Mills equations. In the third section the definition of the Lévy Laplacian in the Malliavin calculus and the theorem on its connection with the Yang-Mills equations are given. In the fourth section the definitions of the Lévy Laplacians in the Hida calculus are given and the theorem, that relates the Lévy Laplacian in the Malliavin calculus and the Laplacian \( \Delta^{(-1)}_L \) in the Hida calculus, is proved.
1 Lévy Laplacians

Let $V_0$ be a locally convex space (LCS) continuously embedded in $L_2([0, 1], \mathbb{R})$ in such a way that the image of $V_0$ under the embedding is dense in $L_2([0, 1], \mathbb{R})$. The symbol $\otimes_V$ denotes the projective tensor product of LCSs, and the symbol $\otimes$ denotes the Hilbert tensor product. Let $\{p_1, p_2, \ldots, p_d\}$ be an orthonormal basis in $\mathbb{R}^d$. Let $V = \mathbb{R}^d \otimes_V V_0$ and $V_\mu = \{\gamma = (\gamma^\nu)_{\nu = 1}^d \in V: \gamma^\mu \in V_0, \gamma^\nu = 0 \text{ if } \nu \neq \mu\}$. Then $V = V_1 \oplus V_2 \oplus \ldots \oplus V_d$. Let $M_\mathbb{C} (\mathbb{N})$ be the space of all complex $N \times N$-matrices. Let $\langle \cdot, \cdot \rangle_{M_\mathbb{C} (\mathbb{N})}$ be the scalar product on $M_\mathbb{C} (\mathbb{N})$, defined in the standard way: if $M_1, M_2 \in M_\mathbb{C} (\mathbb{N})$, then $(M_1, M_2)_{M_\mathbb{C} (\mathbb{N})} = \text{tr}(M_1 M_2)$. The symbol $C^2(V, M_\mathbb{C} (\mathbb{N}))$ denotes the space of two times Fréchet differentiable $M_\mathbb{C} (\mathbb{N})$-valued functions on $V$. For any $x \in V$ we have $f'(x) \in M_\mathbb{C} (\mathbb{N}) \otimes_\pi V^*$ and $f''(x) \in M_\mathbb{C} (\mathbb{N}) \otimes_\pi L(V, V^*)$. (If $X_1$ and $X_2$ are complex or real LCSs, the symbol $L(X_1, X_2)$ denotes the space of all continuous linear mappings from $X_1$ to $X_2$.) The symbol $f''_{V_\mu} V_\nu$ denotes the second partial derivative of the function $f \in C^2(V, M_\mathbb{C} (\mathbb{N}))$ along the space $V_\nu$. Let $\{e_n\}$ be an orthonormal basis in $L_2([0, 1], \mathbb{R})$. We assume that $e_n \in V_0$ for each $n \in \mathbb{N}$.

Definition 1. The Lévy Laplacian $\Delta^{\{e_n\},s}_{L}$ of order $s \in \mathbb{R}$, generalized by the basis $\{e_n\}$, is a linear mapping from $\text{Dom} \Delta^{\{e_n\},s}_{L}$ to the space of all $M_\mathbb{C} (\mathbb{N})$-valued functions on $V$ defined by

$$\Delta^{\{e_n\},s}_{L} f(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{d} \sum_{\mu=1}^{n} k^{1-s} e_n < f''_{\mu e_k} p_\mu e_k, p_\mu e_k >=$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{d} \sum_{\mu=1}^{n} k^{1-s} < f''_{V_\mu V_\nu} (x) e_k, e_k >,$$

(2)

where $\text{Dom} \Delta^{\{e_n\},s}_{L}$ is the space of all functions $f \in C^2(V, M_\mathbb{C} (\mathbb{N}))$, for which the right side of (2) exists for all $x \in V$.

This definition is motivated by the following definition of the exotic Lévy Laplacians.

Definition 2. The exotic Lévy Laplacian $\Delta^{\{e_n\},s}_{\text{exotic}}$ of order $s \geq 0$, generalized by the basis $\{e_n\}$, is a linear mapping from $\text{Dom} \Delta^{\{e_n\},s}_{\text{exotic}}$ to the space of all $M_\mathbb{C} (\mathbb{N})$-valued functions on $V$, defined by:

$$\Delta^{\{e_n\},s}_{\text{exotic}} f(x) = \lim_{n \to \infty} \frac{1}{n^s} \sum_{\mu=1}^{d} \sum_{k=1}^{n} < f''_{\mu e_k} p_\mu e_k, p_\mu e_k >,$$

(3)

where $\text{Dom} \Delta^{\{e_n\},s}_{\text{exotic}}$ is the space of all functions $f \in C^2(V, M_\mathbb{C} (\mathbb{N}))$, for which the right side of (3) exists for all $x \in V$.

Then $\Delta^{\{e_n\},0}_{\text{exotic}}$ is the Gross-Volterra Laplacian and $\Delta^{\{e_n\},1}_{\text{exotic}}$ is the classical Lévy Laplacian.

Proposition 1. If $s > 0$, then $\Delta^{\{e_n\},s}_{L} = s \Delta^{\{e_n\},s}_{\text{exotic}}$.
Proof. The proof follows directly from the following fact (see [5, 30]). Let \((a_n) \in \mathbb{R}^\infty\) and \(s > 0\). Then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} a_k k^{-s+1} = s \lim_{n \to \infty} \frac{1}{n^s} \sum_{k=1}^{n} a_k
\]
(4)
in the sense that if one side of equality (4) exists, then the other exists and equality (4) holds.

Thus, formula (2) allows us to extend the chain of the exotic Lévy Laplacians for \(s \leq 0\). The operator \(\Delta_L^{(e_n),0}\) is different from the Gross-Volterra operator.

Remark 1. For the first time formula (4) was used for study of the exotic Lévy Laplacians in [27]. The Lévy Laplacians \(\Delta_L^{(e_n),v}\) are the particular case of nonclassical Lévy Laplacians (see [8, 27]).

The following definition belongs to P. Lévy (see [23, 14, 19]).

Definition 3. An orthonormal basis \(\{e_n\}\) in \(L_2([0,1], \mathbb{R})\) is weakly uniformly dense (or equally uniformly dense), if
\[
\lim_{n \to \infty} \int_{0}^{1} h(t)\left(\frac{1}{n} \sum_{k=1}^{n} e_k^2(t) - 1\right)dt = 0
\]
for any \(h \in L_\infty([0,1], \mathbb{R})\).

Let \(h_n(t) = \sqrt{2} \sin(\pi nt)\) and \(l_n(t) = \sqrt{2} \cos(\pi nt)\) for \(n \in \mathbb{N}\) and \(l_0(t) = 1\). The orthonormal bases \(\{h_n\}_{n=1}^\infty\) and \(\{l_n\}_{n=0}^\infty\) in \(L_2([0,1], \mathbb{R})\) are weakly uniformly dense.

Let \(V_{0,0}\) be a subspace of \(V_0\) and let \(V_{0,\mu} = \{\gamma = (\gamma_\nu)_{\nu=1}^d \in V_\mu : \gamma^\mu \in V_{0,0}\}\).

Proposition 2. Let \(f \in C^2(V, M_N(\mathbb{C}))\). Let for any \(\mu \in \{1, \ldots, d\}\) and for all \(u, v \in V_{0,\mu}\) the following holds
\[
< f''_{V_{0,\mu}V_{0,\mu}^*}(x)u, v > = \int_{0}^{1} K^{V}_{\mu\mu}(x; s, t)u(t)v(s)dsdt + \int_{0}^{1} K^{L}_{\mu\mu}(x; t)u(t)v(t)dt,
\]
(5)
where \(f''_{V_{0,\mu}V_{0,\mu}^*}\) is the second partial derivative of the function \(f\) along the space \(V_{0,\mu}\), \(K^{V}_{\mu\mu}(x, \cdot, \cdot) \in L_2([0,1] \times [0,1], M_N(\mathbb{C}))\) and \(K^{L}_{\mu\mu}(x, \cdot) \in L_\infty([0,1], M_N(\mathbb{C}))\) (\(K^V\) is the Volterra part and \(K^L\) is the Lévy part). If \(\{e_n\}\) is weakly uniformly dense basis in \(L_2([0,1], \mathbb{R})\) and \(e_n \in V_{0,0}\) for all \(n \in \mathbb{N}\), then
\[
\Delta_L^{(e_n),1}f(x) = \sum_{\mu=1}^{d} \int_{0}^{1} K^{L}_{\mu\mu}(x, t)dt.
\]
(6)

Proof. Since \(\{e_n\}\) is an orthonormal basis in \(L_2([0,1], \mathbb{R})\), the sequence \(\{e_n \otimes e_n\}_{n=1}^\infty\) is an orthonormal sequence of functions in \(L_2([0,1] \times [0,1], \mathbb{R})\) and
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} \int_{0}^{1} K^{V}_{\mu\mu}(x; s, t)e_k(t)e_k(s)dsdt =
\]
\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} < K^{V}_{\mu\mu}(x), e_k \otimes e_k > = \lim_{n \to \infty} \sum_{\mu=1}^{d} < K^{V}_{\mu\mu}(x), e_n \otimes e_n > = 0.
\]
(7)
Then formula (6) follows from the fact that the basis \(\{e_n\}\) is weakly uniformly dense. \(\square\)
Remark 2. One of the approaches to define the Lévy Laplacian is as follows. The value of the Lévy Laplacian on a function is defined as an integral functional given by the special form of the second derivative of this function (see [23, 14]). If the second partial derivatives of the function $f \in C^2(V, M_N(\mathbb{C}))$ have the form (3), then the value of the classical Lévy Laplacian on $f$ can be determined by formula (6). If $V_0 = C^1([0, 1], \mathbb{R})$ and $V_{0,0} = \{ \gamma \in C^1([0, 1], \mathbb{R}) : \gamma(0) = \gamma(1) = 0 \}$, then we obtain a generalization of the definition of the Lévy Laplacian from paper [2] by Accardi, Gibilisco and Volovich. The Lévy Laplacian $\Delta^{\{e_n\},1}_L$ of the Lévy Laplacian is associated with the gauge fields.

2 Lévy Laplacians and gauge fields

Below, the Greek indices run through $\{1, \ldots, d\}$. In the paper we use the Einstein summation convention.

Let

$$W_0^{1,2}([0, 1], \mathbb{R}^d) := \{ \gamma \in AC([0, 1], \mathbb{R}^d) : \gamma(0) = 0, \dot{\gamma} \in L_2((0, 1), \mathbb{R}^d) \}.$$  

It is a Hilbert space with the scalar product

$$(\gamma_1, \gamma_2)_{W_0^{1,2}([0, 1], \mathbb{R}^d)} = \int_0^1 (\dot{\gamma}_1(t), \dot{\gamma}_2(t))_{\mathbb{R}^d} dt.$$  

Choose $V = W_0^{1,2}([0, 1], \mathbb{R}^d)$. Consider the classical Lévy Laplacian $\Delta^{\{e_n\},1}_L$ on

$$C^2(W_0^{1,2}([0, 1], \mathbb{R}^d), M_N(\mathbb{C})).$$

It is this Laplacian that is associated with the gauge fields.

Let $A(x) = A_\mu(x)dx^\mu$ be a $C^\infty$-smooth $u(N)$-valued 1-form on $\mathbb{R}^d$. It determines the connection in the trivial vector bundle over $\mathbb{R}^d$ with fibre $\mathbb{C}^N$ and structure group $U(N)$. If $\phi \in C^1(\mathbb{R}^d, u(N))$, its covariant derivative is defined by $\nabla_\mu \phi = \partial_\mu \phi + [A_\mu, \phi]$. The curvature tensor is a $u(N)$-valued 2-form $F(x) = \sum_{\mu < \nu} F_{\mu\nu}(x)dx^\mu \wedge dx^\nu$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The Yang-Mills equations on the connection $A$ are

$$\nabla_\mu F_{\mu\nu} = 0.  \quad (8)$$

The parallel transport $U_t^A(\gamma)$ along the path $\gamma \in W_0^{1,2}([0, 1], \mathbb{R}^d)$, associated with a connection $A$, is the solution to the differential equation

$$U_t^A(\gamma) = I_N - \int_0^t A_\mu(\gamma(s))U_s^A(\gamma)\dot{\gamma}^\mu(s)ds,$$

where $I_N$ is the identity $N \times N$-matrix.

Theorem 1. For the parallel transport the following equality holds

$$\Delta^{\{e_n\},1}_L U_t^A(\gamma) = -U_t^A(\gamma) \int_0^1 U_t^A(\gamma)(-1)^{\nabla_\mu F_{\mu\nu}(\gamma(t))U_t^A(\gamma)\dot{\gamma}^\nu(t)dt.  \quad (9)$$

The connection $A$ satisfies the Yang-Mills equations (8) if and only if

$$\Delta^{\{e_n\},1}_L U_t^A = 0.$$
Proof. Let 

\[ W_\mu = \{ \gamma = (\gamma^\nu)_{\nu=1}^d \in W_0^{1,2}([0,1], \mathbb{R}^d) : \gamma^\nu = 0 \text{ if } \nu \neq \mu \} \]

and \( W_{0,\mu} = \{ \gamma \in W_\mu : \gamma(1) = 0 \} \). The proof of the theorem is based on the fact that the second derivatives of the parallel transport along the spaces \( W_{0,\mu} \) have the form:

\[
< (U^A_1)^\nu W_{0,\mu}(\gamma)u, v > =
U^A_1(\gamma) \int_0^1 dt \int_0^t ds U^A_1(\gamma)^{-1} F_{\mu \nu}(\gamma(t)) \dot{\gamma}^\nu(t) U^A_1(\gamma) \times
\times U^A_s(\gamma)^{-1} F_{\mu \lambda}(\gamma(s)) \dot{\gamma}^\lambda(s) U^A_s(\gamma)(u(t)v(s) + v(t)u(s)) -
- U^A_1(\gamma) \int_0^1 U^A_1(\gamma)^{-1} \nabla_\mu F_{\nu \lambda}(\gamma(t)) \dot{\gamma}^\nu(t) U^A_1(\gamma) u(t)v(t) dt
\]

for all \( u, v \in W_{0,\mu} \). The assertion of the theorem can be proved by analogous arguments, as in Proposition 2. See [2] and [29] for the detailed proof.

Remark 3. Theorem 1 was first proved in [2] by Accardi, Gibilisco and Volovich for the Lévy Laplacian, defined as the integral functional given by the special kind of the second derivative.

Let \( D \) be an isometric isomorphism between the Hilbert spaces \( W_0^{1,2}([0,1], \mathbb{R}^d) \) and \( L^2([0,1], \mathbb{R}^d) \), defined by differentiation

\[ Dh(t) = \dot{h}(t), h \in W_0^{1,2}([0,1], \mathbb{R}^d). \]

Then the Lévy Laplacian \( \Delta^{(h_0)}_L \) on the space \( C^2(W_0^{1,2}([0,1], \mathbb{R}^d), M_N(\mathbb{C})) \) and the Lévy Laplacian \( \Delta^{(l_0)}_L \) on the space \( C^2(L_2([0,1], \mathbb{R}^d), M_N(\mathbb{C})) \) are connected as follows.

Proposition 3. If \( f \in C^2(W_0^{1,2}([0,1], \mathbb{R}^d), M_N(\mathbb{C})) \), then

\[
\pi^2 \Delta^{(l_0)}_L(D^{-1}f)(D^{-1}x) = (\Delta^{(h_0)}_L f)(x)
\]

for all \( x \in L^2([0,1], \mathbb{R}^d) \).

Proof. The proof follows directly from the chain rule.

In the next section we define the Lévy Laplacian \( \Delta_L \) in the Malliavin calculus. This Laplacian is analogue of the Lévy-Laplace operator \( \Delta^{(h_0)}_L \) on the space \( C^2(W_0^{1,2}([0,1], \mathbb{R}^d), M_N(\mathbb{C})) \). The classical Lévy Laplacian in the Hida calculus is the analogue of the Lévy Laplacian \( \Delta^{(l_0)}_L \) on the space \( C^2(L_2([0,1], \mathbb{R}^d), M_N(\mathbb{C})) \). In the last section we determine an analogue on the space of Hida generalized functionals of the Lévy Laplacian \( \Delta^{(l_0)}_L \) on the space \( C^2(L_2([0,1], \mathbb{R}^d), M_N(\mathbb{C})) \). This analogue relates to the operator \( \Delta_L \) in a way similar to (10).

3 Lévy Laplacian in Malliavin calculus

Let \( \{b_t\}_{t \in [0,1]} \) be a standard \( d \)-dimensional Brownian motion and \( (\Omega, \mathcal{F}, P) \) be the associated with this process probability space \( (\Omega = \{ \gamma \in C([0,1], \mathbb{R}^d) : \gamma(0) = 0 \}, \mathcal{F} \) is the \( \sigma \)-algebra generated by the Brownian motion and \( P \) is the Wiener measure). The symbols \( db \) and \( \partial b \) denote the Itô differential and the Stratonovich differential respectively. The space \( W_0^{1,2}([0,1], \mathbb{R}^d) \) is the Cameron-Martin space (the space of differentiability) of the Wiener measure \( P \).
Remark 4. In the paper [22] by Leandre and Volovich, the Lévy Laplacian has been introduced on the Sobolev space over the Wiener measure on the space of paths in a compact Riemannian manifold. This Laplacian has been defined as an integral functional given by the special form of the second derivative. The value of such a Lévy Laplacian on the

The Sobolev space $W^{r,p}(P,M_N(\mathbb{C}))$ is the completion of the space of all $C^\infty$-smooth cylindrical $M_N(\mathbb{C})$-valued functions with compact support on $\Omega$ with respect to the Sobolev norm

$$\|\Phi\|_{r,p} = \left( \sum_{k=0}^{r} \left( E \sum_{i_1,...,i_k=1}^{\infty} \|\partial_{g_{i_1}} \cdots \partial_{g_{i_k}} \Phi\|^2_{M_N(\mathbb{C})} \right)^{p/2} \right)^{1/p},$$

where $\{g_i\}$ is an arbitrary orthonormal basis in $W^{1,2}_0([0,1],\mathbb{R}^d)$ (for various definitions of the Sobolev spaces over the Wiener measure, see, e.g., [13]). For $p \geq 1$ and for any $h \in W^{1,2}_0([0,1],\mathbb{R}^d)$ the operator of differentiation along the direction $\partial_h$ can be extended by continuity as a continuous linear operator from $W^{1,p}(P,M_N(\mathbb{C}))$ to $L_p(\Omega, P; M_N(\mathbb{C}))$. We denote this extension again by the symbol $\partial_h$. The second derivative of an element from $W^{2,p}(P,M_N(\mathbb{C}))$ is defined by analogy.

An analogue of the classical Lévy Laplacian $\Delta_L^{[h_n]}$ for the Sobolev space $W^{2,2}(P,M_N(\mathbb{C}))$ is defined as follows.

**Definition 4.** The Lévy Laplacian $\Delta_L$ is a linear mapping from $\text{Dom}\Delta_L$ to $L_2(\Omega, P; M_N(\mathbb{C}))$ defined by the formula

$$\Delta_L f(b) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} \partial_{p_{\mu}b_k} \partial_{p_{\mu}b_k} f(b),$$

where the sequence in the right side of (11) convergences strongly in $L_2(\Omega, P; M_N(\mathbb{C}))$ and $\text{Dom}\Delta_L$ consists of all $f \in W^{2,2}(P,M_N(\mathbb{C}))$ for which the right side of (11) exists.

The stochastic parallel transport $U^A(b,t)$, associated with the connection $A$, is a solution to the stochastic equation in the sense of Stratonovich:

$$U^A(b,t) = I_N - \int_0^t A_\mu(b_s) U^A(b,s) db_\mu^p. $$

If $A$ and its partial derivatives of the first and the second order are bounded, this equation has the unique strong solution.

**Theorem 2.** Let a connection $A$ be bounded together with all its partial derivatives up to the third order inclusive. Then for the stochastic parallel transport the following holds

$$\Delta_L U^A(b,1) = U^A(b,1) \int_0^1 U^A(b,t)^{-1} F_{\mu\nu}(b_t) F_{\mu\nu}(b_t) U^A(b,t) dt -$$

$$- U^A(b,1) \int_0^1 U^A(b,t)^{-1} \nabla^\mu F_{\mu\nu}(b_t) U^A(b,t) db_\nu^p. \quad (12)$$

The connection $A$ is a solution to the Yang-Mills equations [5] if and only if for the stochastic parallel transport the following holds

$$\Delta_L U^A(b,1) = U^A(b,1) \int_0^1 U^A(b,t)^{-1} F_{\mu\nu}(b_t) F_{\mu\nu}(b_t) U^A(b,t) dt.$$

For the proof see [31].

**Remark 4.** In the paper [22] by Leandre and Volovich, the Lévy Laplacian has been introduced on the Sobolev space over the Wiener measure on the space of paths in a compact Riemannian manifold. This Laplacian has been defined as an integral functional given by the special form of the second derivative. The value of such a Lévy Laplacian on the
stochastic parallel transport does not have the first term in the right-hand side of (12). Thus, for this Lévy Laplacian the theorem on the equivalence of the Yang-Mills equations and the Laplace equation for the Lévy Laplacian is satisfied. It would be interesting to study the relationship between the Laplacian given by Definition 4 and the Laplacian introduced by Leandre and Volovich.

Remark 5. In the paper [21], a divergence corresponding to the Laplacian \( \Delta_L \) has been introduced. It has been shown that a stochastic parallel transport is a solution to an equation containing such a divergence if and only if the associated connection is a solution to the Yang-Mills equations. The resulting equation for the stochastic parallel transport is an analogue of the equation of motion of chiral fields (cf. [10]).

Remark 6. It would be interesting to investigate whether it is possible to use the Levy-Laplacian approach in some areas connected to the theory of gauge fields (see, e.g., [11, 27, 24, 35, 13]).

At the end of this section, we give some general information about the Fock spaces and the Wiener-Ito-Segal isomorphism, which are needed for the proof of the theorem in the next section.

Let \( \mathcal{F} \) be a separable Hilbert space. The symbol \( \otimes \) denotes the symmetric tensor product. If \( F \in \mathcal{F}^\otimes n \) and \( f \in \mathcal{F}^\otimes k \), than the contraction \( F \otimes_k f \) is an element from \( \mathcal{F}^\otimes (n-k) \) such that for any \( h \in \mathcal{F}^\otimes k \) holds \( < F, h \otimes_k f > = < F \otimes_k f, h > \). The following estimates for the norms hold (see, e.g., [26])

\[
\| F \otimes f \|_{\mathcal{F}^\otimes (n+k)} \leq \| F \|_{\mathcal{F}^\otimes n} \| f \|_{\mathcal{F}^\otimes k}, \quad (13)
\]

\[
\| F \otimes_k f \|_{\mathcal{F}^\otimes (n-k)} \leq \| F \|_{\mathcal{F}^\otimes n} \| f \|_{\mathcal{F}^\otimes k}, \quad \text{if} \ n \geq k. \quad (14)
\]

The (boson) Fock space \( \Gamma(\mathcal{F}) \) over \( \mathcal{F} \) is a Hilbert space with the Hilbert norm \( \| \cdot \|_{\Gamma(\mathcal{F})} \), defined by

\[
\Gamma(\mathcal{F}) = \{ f = (f_n)_{n=0}^{\infty}; \ f_n \in \mathcal{F}^\otimes n, \| f \|_{\Gamma(\mathcal{F})} = \sum_{n=0}^{\infty} n! |f_n|_{\mathcal{F}^\otimes n}^2 < \infty \}.
\]

The tensor product \( M_N(\mathbb{C}) \otimes \Gamma(\mathcal{F}) \) and the associated Hilbert norm are of the form

\[
M_N(\mathbb{C}) \otimes \Gamma(\mathcal{F}) = \{ F = (F^n)_{n=0}^{\infty}; \ F^n \in M_N(\mathbb{C}) \otimes \mathcal{F}^\otimes n, \| F \|_{M_N(\mathbb{C}) \otimes \Gamma(\mathcal{F})} = \sum_{n=0}^{\infty} n! |F^n|_{M_N(\mathbb{C}) \otimes \mathcal{F}^\otimes n}^2 < \infty \}.
\]

Let \( T_n = T_n(\mathbb{R}^d, M_N(\mathbb{C})) \) be the space of all \( M_N(\mathbb{C}) \)-valued tensors of type \((0, n)\) on \( \mathbb{R}^d \), endowed with the standard structure of a Hilbert space. Let the space \( L_2^{sym}(\{0, 1\}^n, T_n) \) consists of all functions \( F^n \in L_2([0, 1]^n, T_n) \), for which the following holds

\[
F^n_{\sigma(1)\sigma(2)\cdots\sigma(n)}(t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(n)}) = F^n_{i_1i_2\cdots i_k}(t_1, t_2, \ldots, t_n),
\]

where \( \sigma \) is a permutation of the first \( n \) natural numbers. If \( \mathcal{F} = L_2([0, 1], \mathbb{R}^d) \), then \( M_N(\mathbb{C}) \otimes \mathcal{F}^\otimes n \) coincides with the space \( L_2^{sym}(\{0, 1\}^n, T_n) \).

The unitary Wiener-Ito-Segal isomorphism \( J_1 \) between \( M_N(\mathbb{C}) \otimes \Gamma(H) \) and \( L_2(\Omega, P; M_N(\mathbb{C})) \) acts as follows. If \( F = (F^n)_{n=0}^{\infty} \in M_N(\mathbb{C}) \otimes \Gamma(H) \), then

\[
J_1(F) = \sum_{n=0}^{\infty} I_n(F^n),
\]

\[4\text{If } F^n \in T_n, \text{ then } \| F^n \|_{T_n}^2 = \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} tr(F^n_{i_1i_2\cdots i_n}(F^n_{i_1i_2\cdots i_n})^*) \]
where $I_0(F^0) = F^0$,

$$I_n(F^n) = \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} n! \int_0^1 \cdots \int_0^1 F^n_{i_1 \cdots i_n}(t_1, \ldots, t_n) db_{i_1}^1 \cdots db_{i_n}^n$$

and the series (15) converges strongly in $L_2(\Omega, P; M_N(\mathbb{C}))$. It is known (see, e.g., [25, 13]) that if $\mathcal{J}_1(F) \in W^{1,2}(P, M_N(\mathbb{C}))$ and $h \in W^{1,2}_0([0, 1], \mathbb{R}^d)$, then

$$\partial_h(\mathcal{J}_1(F)) = \sum_{n=1}^{\infty} I_{n-1}(F^n \otimes_1 h).$$  

(16)

### 4 Lévy Laplacians in Hida calculus

We need to generalize Definition 1 from Section 1 to determine the Lévy Laplacian on the space of generalized Hida functionals.

Let $T$ be an interval in $\mathbb{R}$ and $[0, 1] \subset T$. Let $\mathcal{V}_0$ be a complex LCS continuously embedded in $L_2(T, \mathbb{C})$ in such a way that the image of $\mathcal{V}_0$ under the embedding is dense in $L_2(T, \mathbb{C})$. Let $\mathcal{V} = \mathbb{C}^d \otimes \mathcal{V}_0$ and $H_C = L_2(T, \mathbb{C}^d)$. Let $C_2(T, M_N(\mathbb{C}))$ be the space of all two times Fréchet complex differentiable $M_N(\mathbb{C})$-valued functions on $\mathcal{V}$ (for the definition of the Fréchet complex differentiability see, e.g., [28]), the second derivative of which has the form

$$< f''(x)u, v > = \int_T \int_T K^V_{\mu \nu}(x; s, t)u^\mu(t)v^\nu(s)dt ds + \int_T K^L_{\mu \nu}(x; t)u^\mu(t)v^\nu(t)dt$$

for all $u, v \in \mathcal{V}$, where $K^V(x, \cdot; \cdot) \in L_2^{sym}(T \times T, \mathbb{C})$, $K^L(x; \cdot) \in L_\infty(T, M_N(\mathbb{C}))$ and $K^L_{\mu \nu} = K^L_{\nu \mu}$ ($K^V$ is the Volterra part and $K^L$ is the Lévy part as in Section 1). Then for each $x \in \mathcal{V}$ it is possible to extend by continuity $f''(x)$ to the element from $M_N(\mathbb{C}) \otimes L(H_C, H_C)$, which we will denote by the same symbol $f''(x)$. Let $\{e_n\}$ be an orthonormal basis in $L_2([0, 1], \mathbb{R})$, whose elements may not belong to the space $\mathcal{V}_0$. (We identify $e_n$ with the function from $L_2([0, 1], \mathbb{R})$ that is equal to $e_n(t)$ for $t \in [0, 1]$ and is equal to zero for $t \in T \setminus [0, 1]$.) For $s = \{-1, 1\}$ the definition of the Lévy Laplacian $\Delta_L^{\{e_n\}, s}$ can be generalized as follows.

**Definition 5.** The Lévy Laplacian $\Delta_L^{\{e_n\}, s}$ of order $s \in \{-1, 1\}$ is a linear mapping from $\text{Dom} \Delta_L^{\{e_n\}, s}$ to the space of all $M_N(\mathbb{C})$-valued functions on $\mathcal{V}$ defined by:

$$\Delta_L^{\{e_n\}, s}f(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} k^{1-s} < f''(x)p_{\mu}e_k, p_{\mu}e_k >,$$

where $\text{Dom} \Delta_L^{\{e_n\}, s}$ is the space of all functions $f \in C_2^2(\mathcal{V}, M_N(\mathbb{C}))$, for which the right side of (17) exists for any $x \in \mathcal{V}$.

Let $\{e_n\}$ be an orthonormal basis in $L_2(T, \mathbb{R})$. Let $A$ be a self-adjoint operator on $H_C$ acting by the formula:

$$A(e_n \otimes p_{\mu}) = \lambda_n(e_n \otimes p_{\mu}),$$

where $\{\lambda_n\}$ is an increasing sequence of real numbers such that

$$1 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \text{ and } \sum_{k=1}^{\infty} \lambda_k^{-2} < \infty.$$
Then the operator $A^{-1}$ is a Hilbert-Schmidt operator. For any $p \geq 0$ the Hilbert space $E_p$ with the Hilbert norm $|\cdot|_p$ is defined by

$$E_p = \{ \xi \in H_C : |\xi|^2_p = \sum_{k=1}^{\infty} \sum_{\mu=1}^{d} \lambda_k^{2p}|(\xi, e_k \otimes p_\mu)_{H_C}|^2 < \infty \}.$$ 

For any $p < 0$ the norm $|\cdot|_p$ is defined on all $H_C$ by

$$|\xi|^2_p = \sum_{k=1}^{\infty} \sum_{\mu=1}^{d} \lambda_k^{2p}|(\xi, e_k \otimes p_\mu)_{H_C}|^2.$$ 

For $p < 0$ the Hilbert space $E_p$ is the completion of $H_C$ with respect to this norm. For all $p \in \mathbb{R}$ we denote the Hilbert norm on $E_p^{\otimes n}$ also by the symbol $|\cdot|_p$. The space $E_0$ coincides with $H_C$. Below we denote the Hilbert norm on $M_N(\mathbb{C}) \otimes H_C^{\otimes n}$ also by the symbol $|\cdot|_0$. This will not lead to any confusion.

Denote the projective limit $\text{proj lim}_{p \to +\infty} E_p$ by $E_C$. The space $E_C$ is a nuclear Fréchet space and, hence, is a reflexive space. Its conjugate space $E_C^*$ is the inductive limit $\text{ind lim}_{p \to +\infty} E_{-p}$. We obtain the complex rigged Hilbert space:

$$E_C \subset H_C \subset E_C^*.$$ 

Using the restriction of the operator $A$ to $H_\mathbb{R} = L_2(\mathbb{T}, \mathbb{R}^d)$ in a similar way we obtain the real rigged Hilbert space:

$$E_\mathbb{R} \subset H_\mathbb{R} \subset E_\mathbb{R}^*.$$ 

The norms on the spaces $\Gamma(E_p)$ and $M_N(\mathbb{C}) \otimes \Gamma(E_p)$ are denoted by the symbol $\|\cdot\|_p$. Denote the projective limit $\text{proj lim}_{p \to +\infty} \Gamma(E_p)$ by the symbol $\mathcal{E}$. Then $\mathcal{E}^* = \text{ind lim}_{p \to +\infty} \Gamma(E_{-p})$. The space $M_N(\mathbb{C}) \otimes_{\pi} \mathcal{E} = \text{proj lim}_{p \to +\infty} M_N(\mathbb{C}) \otimes \Gamma(E_p)$ is the space of $M_N(\mathbb{C})$-valued Hida test functionals (white noise test functionals) and the space $(M_N(\mathbb{C}) \otimes_{\pi} \mathcal{E})^*$ is the space of $M_N(\mathbb{C})$-valued Hida generalized functionals (white noise generalized functionals). Denote the canonical bilinear form on $(M_N(\mathbb{C}) \otimes_{\pi} \mathcal{E})^* \times (M_N(\mathbb{C}) \otimes_{\pi} \mathcal{E})$ by the symbol $\langle \langle \cdot, \cdot \rangle \rangle$.

**Remark 7.** If $T = \mathbb{R}$ and

$$A = 1 + i^2 - \frac{d^2}{dt^2},$$

then $E_C = S(\mathbb{R}, \mathbb{C}^d)$ is the Schwartz space of rapidly decreasing functions and $E_C^* = S^*(\mathbb{R}, \mathbb{C}^d)$ is the space of generalized functions of slow growth. The case, where $T = [0, 1]$ and $\{e_n\}$ is a basis consisting of trigonometric functions, is also often considered (see [3, 4, 7, 8]).

By the Minlos-Sazonov theorem there is a Gaussian probability measure $\mu_T$ on $\sigma$-algebra generated by $E_\mathbb{R}$-cylindrical sets on $E_\mathbb{R}^*$ such that its Fourier transform has the form $\widetilde{\mu}_T(\xi) = e^{-\frac{\langle \xi, \xi \rangle_T}{2}}$. An element from $\mathcal{E}$

$$\psi_\xi = (1, \xi, \frac{\xi \otimes 2}{2}, \ldots, \frac{\xi \otimes n}{n!}, \ldots),$$

where $\xi \in E_C$, is called a coherent state.

The unitary Wiener-Ito-Segal isomorphism $j_2$ between $\Gamma(H_C)$ and $L_2(E_\mathbb{R}^*, \mu_T; \mathbb{C})$ is uniquely determined by the values on coherent states

$$j_2(\psi_\xi)(x) = e^{\langle x, \xi \rangle - \langle \xi, \xi \rangle / 2}.$$
The complex rigged Hilbert space

\[ \mathcal{E} \subset \Gamma(H) \cong L_2(E^*_R, \mu_I, \mathbb{C}) \subset \mathcal{E}^* \]

is called Hida-Kubo-Takenaka space. The symbol \( \mathcal{J}_2 \) denotes the unitary isomorphism between \( M_N(\mathbb{C}) \otimes \Gamma(H) \) and \( L_2(E^*_R, \mu_I; M_N(\mathbb{C})) \), generated by \( j_2 \). We will not distinguish \( F \in M_N(\mathbb{C}) \otimes \Gamma(H) \) and \( \mathcal{J}_2(F) \in L_2(E^*_R, \mu_I; M_N(\mathbb{C})) \) and respectively the spaces \( M_N(\mathbb{C}) \otimes \Gamma(H) \) and \( L_2(E^*_R, \mu_I; M_N(\mathbb{C})) \).

**Remark 8.** Let \( \nu_I \) be the Gaussian measure with zero mean and identity correlation operator on \( L_2(T, \mathbb{R}^d) \). This measure is not \( \sigma \)-additive. The measure \( \mu_I \) is the image of a measure \( \nu_I \) under the embedding of \( L_2(T, \mathbb{R}^d) \) into \( E^*_R \). Let \( T = [0, 1] \). Let the measure \( \nu_I \circ D \) be the image of the measure \( \nu_I \) under \( D^{-1} \). Then the Wiener measure \( P \) is the image of the measure \( \nu_I \circ D \) under the embedding of \( W_0^{1,2}([0,1], \mathbb{R}^d) \) into \( \Omega \).

The Hida generalized functional \( \Phi \in \mathcal{E}^* \) can be formally written in the form (see [26, 20])

\[ \Phi = \sum_{n=0}^{\infty} \langle x^{\otimes n} : F^n \rangle, \quad (18) \]

where \( :x^{\otimes n} : \) is the Wick tensor of order \( n \), and \( F^n \in M_N \otimes_\pi (E^*_C) \) (the space \( (E^*_C) \) coincides with \( \text{ind lim}_{p \to +\infty} E_{-p}^{\otimes n} \)).

The \( S \)-transform of a generalized functional \( \Phi \in M_N(\mathbb{C}) \otimes_\pi \mathcal{E}^* \) is the function \( S\Phi: E_C \to M_N(\mathbb{C}) \) defined by

\[ (S\Phi(\xi), M)_{M_N(\mathbb{C})} = \langle \langle \Phi, M \otimes \psi_\xi \rangle \rangle, \]

for all \( \xi \in E_C \) and \( M \in M_N(\mathbb{C}) \). The generalized Hida functional is uniquely determined by its \( S \)-transform. If \( \Phi \in M_N(\mathbb{C}) \otimes_\pi \mathcal{E}^* \) has the form (18), the following holds

\[ S\Phi(\xi) = \sum_{n=0}^{\infty} \langle F^n, \xi^{\otimes n} \rangle. \quad (19) \]

A \( M_N(\mathbb{C}) \)-valued function \( G \) on \( E_C \) is the \( S \)-transform of some \( \Phi \in \mathcal{E}^* \) if and only if (see, e.g., [20, 26])

1. for any \( \zeta, \eta \in E_C \) the function \( G_{\zeta, \eta}(z) = G(z\eta + \zeta) \) is entire;

2. there exist \( C, K > 0 \) and \( p \in \mathbb{R} \) such that for all \( \xi \in E_C \) the following estimate holds

\[ \|G(\xi)\|_{M_N(\mathbb{C})} \leq Ce^{K|\xi|^2_p}. \]

A \( M_N(\mathbb{C}) \)-valued function on \( E_C \), that satisfies conditions 1 and 2, is called \( U \)-functional. The symbol \( \mathcal{F}_U \) denotes the space of \( U \)-functionals.

**Remark 9.** For the detailed theory of the vector-valued Hida distributions see [26].

**Definition 6.** The domain of the Lévy Laplacian \( \tilde{\Delta}_{L}^{(e_n)} \) of order \( s \in \{-1, 1\} \), generalized by the orthonormal basis \( \{e_n\} \) in \( L_2([0,1], \mathbb{R}) \), is the space \( \text{Dom} \tilde{\Delta}_{L}^{(e_n)} = \{ \Phi \in (M_N(\mathbb{C}) \otimes_\pi \mathcal{E}^*) : S\Phi \in \text{Dom} \tilde{\Delta}_{L}^{(e_n)}, \text{Dom} \tilde{\Delta}_{L}^{(e_n)}, S\Phi \in \mathcal{F}_U \} \). The Lévy Laplacian \( \tilde{\Delta}_{L}^{(e_n)} \) is a linear mapping from \( \text{Dom} \tilde{\Delta}_{L}^{(e_n)} \) to \( M_N(\mathbb{C}) \otimes_\pi \mathcal{E}^* \) defined by

\[ \tilde{\Delta}_{L}^{(e_n)} \Phi = S^{-1} \Delta_{L}^{(e_n)}(S\Phi). \quad (20) \]
Remark 10. In the present paper we consider only Lévy Laplacians of orders \((-1)\) and 1. The chain of the exotic Lévy Laplacians generated by a basis of trigonometric functions in the Hida calculus has been considered in papers \([5, 4, 31]\). Its extension for negative orders has been considered in the paper \([30]\). In fact, the Lévy Laplacian of order \((-1)\) was first considered in \([31]\).

The following fact is known. The proof, which we give, is taken from Theorem 6.42 from \([16]\) with minor modifications.

**Proposition 4.** If \(\Phi \in L_2\left(E_\mathbb{R}^\nu, \mu_\mathcal{F}; M_N(\mathbb{C})\right),\) then \(S\Phi \in C^2_L\left(E_\mathbb{C}, M_N(\mathbb{C})\right)\) and the Lévy part of \(S\Phi''\) vanishes, i.e. \(K''_{\mathcal{L}} = 0\). Hence, if \(\{e_n\}\) is weakly uniformly dense basis, then \(\hat{\Delta}^{\{e_n\},1}_L \Phi = 0\).

**Proof.** Let \(\Phi \in L_2\left(E_\mathbb{R}^\nu, \mu_\mathcal{F}; M_N(\mathbb{C})\right)\) and \(\Phi = \sum_{n=0}^{\infty} < x^{\otimes n} :, F^n > \). By the nuclear theorem (see, e.g., \([25, 12]\)) \(L(E_\mathbb{C}, E_\mathbb{C}^\ast) \cong (E_\mathbb{C} \otimes_\pi E_\mathbb{C})^\ast\). Thus one can identify \(S\Phi''(\xi)\) with the element from \(M_N(\mathbb{C}) \otimes_\pi (E_\mathbb{C} \otimes_\pi E_\mathbb{C})^\ast\). By \([19]\) we have

\[
< S\Phi''(\xi), \zeta \otimes \eta > = \sum_{n=2}^{\infty} n(n-1) < F^n, \xi^{(n-2)} \circ \zeta \otimes \eta > = \sum_{n=2}^{\infty} n(n-1) < F^n, (n-2) \circ \zeta \otimes \eta >.
\]

We consider the entire function

\[
q(z) = \sum_{n=2}^{\infty} \frac{n^2(n-1)^2}{n!} z^{n-2}.
\]

By the Schwarz inequality and estimates \([13]\) and \([13]\) we have that

\[
\sum_{n=2}^{\infty} \frac{n^2(n-1)^2}{n!} |\xi|_{0}^{2(n-2)} \leq \left( \sum_{n=2}^{\infty} \frac{n^2(n-1)^2}{n!} |\xi|_{0}^{2(n-2)} \right)^{\frac{1}{2}} \leq \|F\| \sqrt{q(|\xi|_{0})}.
\]

We obtain that the series

\[
\sum_{n=2}^{\infty} n(n-1) F^n \otimes_{(n-2)} \xi^{(n-2)}
\]

converges in \(M_N(\mathbb{C}) \otimes H_{\mathbb{C}}^\otimes\) for all \(\xi \in E_\mathbb{C}\). Then \(S\Phi''(\xi) \in M_N(\mathbb{C}) \otimes H_{\mathbb{C}}^\otimes\) and there exists \(K''(\xi;,;) \in L^2_{\text{sym}}(T \times T, T_2)\) such that

\[
< S\Phi''(\xi), \eta > = < S\Phi''(\xi), \zeta \otimes \eta > = \int_T \int_T K''_{\mathcal{L}}(\xi; s, t) \zeta^{(s)}(t) \eta^{(s)}(s) dtds.
\]

This means that \(S\Phi \in C^2_L\left(E_\mathbb{C}, M_N(\mathbb{C})\right)\) and the Lévy part of the second derivative of \(S\Phi\) vanishes.

Denote the embedding of \(M_N(\mathbb{C}) \otimes \Gamma(L_2([0, 1], \mathbb{R}^d))\) into \(M_N(\mathbb{C}) \otimes \Gamma(L_2(\mathbb{R}, \mathbb{R}^d))\) by the symbol \(\mathcal{J}_3\) and the orthogonal projection \(M_N(\mathbb{C}) \otimes \Gamma(L_2(\mathbb{R}, \mathbb{R}^d))\) on \(M_N(\mathbb{C}) \otimes \Gamma(L_2([0, 1], \mathbb{R}^d))\) by the symbol \(\mathcal{P}\). Let the linear mapping \(\mathcal{J}: L_2(\Omega, P; M_N(\mathbb{C})) \to L_2(\mathbb{R}, \mu_\mathcal{F}; M_N(\mathbb{C}))\) be defined by

\[
\mathcal{J} = \mathcal{J}_3 \mathcal{J}_1^{-1}.
\]
Theorem 3. If $\Psi \in \text{Dom}\Delta_L$, then
\[ \mathcal{J}\Delta_L\Psi = \pi^2\Delta_L^{\{1\},-1}\mathcal{J}\Psi. \] (21)

Proof. Let $\Psi = \sum_{n=0}^{\infty} I_n(F^n) \in \text{Dom}\Delta_L$. For all $\xi \in E_C$ and $M \in M_N(\mathbb{C})$ we have
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} \langle S(\mathcal{J}\partial_{p\mu h_k}\partial_{p\mu h_k}\Psi)(\xi), M \rangle_{M_N(\mathbb{C})} = \]
\[ = \lim_{n \to \infty} \langle \mathcal{J}\left(\frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} \partial_{p\mu h_k}\partial_{p\mu h_k}\Psi, M \otimes \psi \right), M \otimes \psi \rangle = \]
\[ = \lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} \partial_{p\mu h_k}\partial_{p\mu h_k}\Psi, \mathcal{P}(M \otimes \psi) \right)_{L_2(\Omega, \mu; M_N(\mathbb{C}))} = \]
\[ = (\Delta_L\Psi, \mathcal{P}(M \otimes \psi))_{L_2(\Omega, \mu; M_N(\mathbb{C}))} = \langle \mathcal{J}(\Delta_L\Psi), M \otimes \psi \rangle = \]
\[ = (S(\mathcal{J}\Delta_L\Psi)(\xi), M)_{M_N(\mathbb{C})}. \] (22)

Since $h_k(t) = \pi k l_k(t)$, equality (16) implies
\[ \partial_{p\mu h_k}\partial_{p\mu h_k}\Psi = \pi^2 k^2 \sum_{n=2}^{\infty} n(n-1) I_{n-2}(F^n \otimes_2 (p\mu l_k \otimes p\mu l_k)). \]

Then
\[ \mathcal{J}\partial_{p\mu h_k}\partial_{p\mu h_k}\Psi = \pi^2 k^2 \sum_{n=2}^{\infty} n(n-1) <; x_{\otimes n-2};, F^n \otimes_2 (p\mu l_k \otimes p\mu l_k) >. \]

Due to (19) we obtain
\[ S(\mathcal{J}\partial_{p\mu h_k}\partial_{p\mu h_k}\Psi)(\xi) = \pi^2 k^2 \sum_{n=2}^{\infty} n(n-1) <; F^n \otimes_2 (p\mu l_k \otimes p\mu l_k), \xi_{\otimes n-2} >. \]

Since $\mathcal{J}\Psi \in L_2(E_2^\infty, \mu_1; M_N(\mathbb{C}))$, Proposition 4 implies
\[ S(\mathcal{J}\Psi)^{''}(\xi) = \sum_{n=2}^{\infty} n(n-1) F^n \otimes_{(n-2)} E_{\mathbb{C}}^\otimes \in M_N(\mathbb{C}) \otimes H_{\mathbb{C}}^\otimes. \]

Then
\[ \pi^2 k^2 < S(\mathcal{J}\Psi)^{''}(\xi), p\mu l_k \otimes p\mu l_k > = \sum_{n=2}^{\infty} \pi^2 k^2 n(n-1) <; F^n \otimes_{(n-2)} \xi_{\otimes n-2}, p\mu l_k \otimes p\mu l_k > = \]
\[ = \sum_{n=2}^{\infty} \pi^2 k^2 n(n-1) <; F^n \otimes_{2} (p\mu l_k \otimes p\mu l_k), \xi_{\otimes n-2} > = S(\mathcal{J}\partial_{p\mu h_k}\partial_{p\mu h_k}\Psi)(\xi). \]

Thus
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} S(\mathcal{J}\partial_{p\mu h_k}\partial_{p\mu h_k}\Psi)(\xi) = \]
\[ = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} \pi^2 k^2 < S(\mathcal{J}\Psi)^{''}(\xi)p\mu l_k, p\mu l_k > = \pi^2 \Delta_L^{\{1\},-1} S(\mathcal{J}\Psi)(\xi). \] (23)

From (23) and (22) we obtain that
\[ S(\mathcal{J}\Delta_L\Psi)(\xi) = \pi^2 \Delta_L^{\{1\},-1} S(\mathcal{J}\Psi)(\xi). \]

This means that equality (21) is true.
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