Spectrum of flows on discrete pair of contours

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Abstract. We consider a deterministic discrete dynamical system, which is a contour network of Buslaev type. This system contains two contours. The i-th contour contains Ni cells, i = 1, 2. There is a cluster of particles on each contour moving in accordance with given rules. The clusters contain 1 ≤ Mᵢ < Nᵢ particles respectively. There is no more than one particle in each cell at any time. There is a common point (node) of the contours. Particles cannot cross the node simultaneously. A set of repeating system states is called a spectral cycle. Average velocities of clusters correspond to spectral cycles. The set of spectral cycles and values of the average velocities have been found.

1. Introduction

A well-known transport model has been introduced in [1] (Nagel, Schreckenberg, 1992). This model and its generalizations were studied by different authors with cellular automata technique use. In the model, particles move on an infinite or closed sequence of cells in accordance with given rules. In general case, Nagel-Schreckenberg model and its versions are too complicated for analytic research and studied by simulation.

Analytic results for a simple version of Nagel-Schreckenberg model have been obtained in [2] (Belitzky, Ferrary, 2005) (a preprint of this paper has been published in 1999). These results have been obtained under assumption that, at any step, each particle moves onto a cell forward if the cell ahead is vacant. It is noted in [2] that the model is equivalent to the elementary cellular automaton 184 (CA 184) in classification of Wolfram, [3]. Results, similar to results of [2], have been obtained independently in [4] (Blank, 2000). In accordance with results of [2], [4], if the density of particles (the number of particles divided by the number of cells) is not more than 1/2, then, for any initial state, all particles move after some moment at every time. If the density is more than 1/2, then the average velocity of particles (the average number of transitions of a particle per time unit) equals (1 − ρ)/ρ, where ρ is the density. In [5] (Gray, Griffeath, 2001), analytical results have been obtained for more general traffic model. In this model, a particle moves from the cell i to a vacant cell i + 1 ahead of particle with probability depending on states of the cells i − 1, i + 2. In [5], the behavior of particles has been studied for some particular cases, and, in the general case, the formula for velocity has not obtained. In [6] (Kanai, Nishinary, Tokihiro, 2009), a formula has been obtained for a stochastic version of the traffic model. In this system, at every step, each particle moves onto a cell forward if the cell ahead is vacant. Some generalizations of results of [3-5] have been obtained in [7] (Blank, 2010).

In general case, the system state space, studied in [7], is continuous. In a particular case, the system is equivalent to the discrete system that is considered in [2, 4]. A two-dimensional traffic model with a
toroidal supporter (BML traffic model) has been introduced in [8] (Biham, Middelton, Levin, 1992). In this model, particle move in accordance with a rule, similar to the rule CA 184. Conditions of self-organization (every particle moves after some moment) and collapse (no particle moves after some moment) have obtained for BML model in [9] (D'Souza, 2005), [10] (Angel, Holroyd, Martin, Austin, Benjamini, 2006). In [12] (Bugaev, Buslaev, Kozlov, Yashina, 2011), the concept of a cluster traffic model with cluster movement has been introduced. In the discrete version of the cluster model, each contour contains a given number of cells. There are clusters of particles on each contour. In discrete case, all particles of each cluster move simultaneously in accordance with the rule CA 240. Clusters can be delayed at nodes. In the continuous version of the model a cluster is a segment moving on the contour with constant velocity in a given direction.

The concept of a contour network has been introduced in [13] (Buslaev contour networks). The supporter of a contour network is a system of contours with a network structure. Particles (clusters) move on contours in accordance with some rules. Some limitations are imposed on the system. These limitations allow us to study the system analytically. In [14] (Buslaev, Fomina, Tatashev, Yashina, 2018) the concept of spectrum of a deterministic contour networks has been introduced. In such system, a sequence of states repeated periodically from some moment. This sequence of states is called a spectral cycle. The system, considered in [14], is a closed chain of contours. Particles move on each contour in accordance with the rule of the cellular automaton 240 (CA 240). There is one cluster on each contour. The spectrum of the system is a set of spectral cycles and corresponding values of clusters velocities. The length of the contour and the length of the cluster do not depend on the index of the contour. A particular case of the system, considered in [14], was studied in [15]. It was assumed in [15] that there are two cells and a particle on each contour. A continuous analogue of the system, considered in [14], has been studied in [16]. In [17] (Buslaev, Tatashchev, 2017) and [18] (Buslaev, Tatashchev, 2018), a discrete two-contours system was considered. In this system, particles move on contours in accordance with the rule of CA 184 or CA 240. In [17], the following generalization was also considered. The supporter of the system contains N contours. There is one common point of the contours. In [17, 18] theorems have been proved for different versions of movement rules. In [17, 18], mainly, systems with contours of the same length were considered. For a system, containing contours of different lengths, in [15] conditions of self-organization (system resulting in a state of free movement from any initial state) have been obtained.

In this paper, a pair of contours is studied. The lengths of the contours differ from each other. We study a discrete version of the system and a continuous version. There is a moving cluster on each contour. There exists a common point of contours (node). Delays occur at the node. We have been found spectral cycles and obtained formulas for velocities of clusters.

Assume that, the contour i contains cells, \( \iota_i = 1, 2 \). There is a moving cluster on each contour. The cluster, moving along the contour \( \iota_i \), contains particles, \( \iota_i = 1, 2 \). All particles of each contour moves onto one cell in the same direction if there is no delay occurs. There is a common point of the contours. This point is called the node. A cluster stops if it comes to the node at time such that at this time the other cluster crosses the node. If the clusters come to the node simultaneously, then one cluster moves first. This cluster is chosen in accordance with time resolution rule. A set of states such that these states are repeated periodically is called a spectral cycle. We say that the system has the property of self-organization if the system results in the state of free movement over a finite time. In this paper, we have proved that, if the condition of self-organization does not hold, then, depending on \( N_1, N_2, M_1, M_2 \), there are one or two spectral cycles. Formulas for average velocities of clusters have been obtained. A continuous version of this system was studied in our paper [19, 20].

2.  Definition of system

We consider a discrete dynamical system containing two contours \( C_1 \) and \( C_2 \). Figure 1. There are \( N_i \) cells on the contour, \( C_i = 1, 2 \). These cells are \((i, 0), (i, 1), \ldots, (i, N_i - 1), i = 1, 2 \). There is a cluster (cluster \( Cl_i \)) on the contour \( Cl_i (i = 1, 2) \). This cluster contains \( M_i < N_i \) particles, \( i = 1, 2 \). The particles of each cluster occupies neighboring cells. The contours have a common point (node). At any time \( t = \ldots \)
0, 1, 2, . . . all particles of each cluster move onto a cell in the direction of movement if there is no delay in cluster movement. Cells are numbered in reverse order to the direction of movement (by modulo \( N_i \) on the contour \( C_i \), \( i = 1, 2 \)). The node is located such that the cluster \( C_{i0} \) moving from the cell \((i, 0)\) to the cell \((i, N_i - 1)\), crosses the node, \( i = 1, 2 \). We say that the cluster \( C_i \) covers the node if there are particles in the cells \((i, 0)\) and \((i, N_i - 1)\), \( i = 1, 2 \). A system state at time \( t \) is a vector \((k_1(t), k_2(t))\) (figure 1), where \( k_i \) is the index of the cell containing the leading particle of the cluster \( C_i \), \( i = 1, 2 \). We say that the cluster \( C_i \) is at the node if there is a particle in the cell \((i, 0)\), and the cell \((i, N_i - 1)\) is vacant, \( i = 1, 2 \). Admissible states of the system are only states such that no more than one cluster covers the node. If at time \( t \) a particle of the cluster \( C_i \) is in the cell \((i, j)\), then at time \( t + 1 \) this particle is in the cell \((i, j - 1)\) (subtraction by modulo \( N_i \)), \( i = 1, 2 \), except for the following cases. At time \( t \), the cluster \( C_k \) is at 1, 2, is at the node, and the other cluster covers the node, Figure 2. If both clusters are at the node, i.e., the system is in the state \((0, 0)\), then a conflict occurs, Figure 3. In this case, only one cluster moves in accordance with the conflict resolution rule. If a conflict occurs at the initial time \( t = 0 \), or a conflict occurs at time \( t = t_0 \) and there are no delays at moments \( t = 0, 1, \ldots , t_0 - 1 \), then, at time \( t = t_0 \), the cluster \( C_i \) moves. Assume that a conflict occurs at the time \( t = t_0 \), i.e., \( k_1(t_0) = k_2(t_0) = 0 \). Suppose the latest delay in the time segment \([0, t_0 - 1]\) occurs at time \( t = t_1 \), and, at time \( t_1 \), the cluster \( C_{i0} \) does not move. Then, at time \( t_0 \), the same cluster \( C_{i0} \) does not move and the other cluster moves. The initial state of the system is given.

3. Concepts of spectral cycle, velocity and free movement, and self-organization

Since the system behavior is deterministic and the set of system states is finite, a sequence of states is repeated periodically from some moment. This sequence of states is called a spectral cycle. Suppose \( T \) is the number of system states belonging to a spectral cycle (\( T \) is the period of the cycle); \( H_i \) is the total number of transitions of cluster \( C_i \) on a spectral cycle (the cluster \( C_i \) passes the distance \( H_i \) for the period \( T_i \)); \( i = 1, 2 \). The number \( v_i = H_i / T_i \) is called the average velocity of the cluster \( C_i \) on the spectral cycle, \( i = 1, 2 \).

We say that the system is in a state of free movement at moment \( t_0 \) if both clusters move at any moment \( t \geq t_0 \). If the system results in a state of free movement, then the velocity of clusters is equal to 1. The property of the system to result in a state of free movement over a finite time from any initial state is called self-organization.

4. Information from the theory of linear diophantine equations

Let us consider the equation \( ax + by + c = 0 \), where \( a \) and \( b \) are integer numbers not equal to 0, and \( c \) is an integer number. There exist integer solutions of the system if and only if the greatest divisor of numbers \( a \) and \( b \) divides the number \( c \). [21]. Let \( x = x_0; y = y_0 \) be solutions of (1). Then all solutions of the equation \( ax + by + c = 0 \) are \( x = x_0 - bt, \ y = y_0 + at, \ t = \pm 1, \pm 2 \).

Since the system behavior is deterministic and the set of system states is finite, a sequence of states is repeated periodically from some moment. This sequence of states is called a spectral cycle.

5. Condition of self-organization

Denote by \( \text{GCD}(N_1, N_2) \) the greatest common divisor of numbers \( N_1, N_2 \):

**Theorem 1.** Suppose the following condition holds
\[ M_1 + M_2 \leq GCD(N_1, N_2). \] (1)

Then the system results in the state of free movement for a finite time from any initial state (self-organization). If the condition (1) does not hold, then the system does not result in the state of free movement for a finite time from any initial state.

Theorem 1 has been proved in [17]. The proof uses the condition of that solutions of (1) exist (Section 3).

6. Optional parameters

Assume the condition of self-organization (Theorems 1) does not hold. Let us introduce optional parameters \(g_1, g_2, j_1, j_2\) and describe a way to calculate these parameters. We shall see that, if the condition of self-organization does not hold, there is one spectral cycle or there are two spectral cycles depending on values \(g_1, g_2, j_1, j_2\): The average velocities of clusters depend on these parameters. Assume that \(A\) is the set of system states such that one cluster does not move \(A_i\) is the set of system states such that the cluster \(C_i\) does not move, \(i = 1, 2\). In accordance with Lemma 1, the set \(A_1\) contains states \((0, k_2), N_2 - M_2 < k_2 < 1\). The set \(A_2\) contains states \((k_i, 0), N_1 - M_1 < k_i < N_1\). We have \(A = A_1 \cup A_2 \cup \{(0,0)\}.

Lemma 1. If (3) holds, then there exists a moment such the system is in the state \((0, N_2 - M_2)\) or in the state \((N_1 - M_1, 0)\).

Proof. Since the condition (1) does not hold, the system results in the states set \(A\) over a finite time from any initial state, and comes out of the set \(A\) through the state \((0, N_2 - M_2)\) or \((N_1 - M_1, 0)\).

Lemma 2. Let non-negative integer numbers \(x, y\) satisfy the condition \(N_2 - M_2 < N_1 x - N_2 y < N_2\). Suppose the system is in the state \((0, N_2 - M_2)\) at time \(t_0\) and no delays occur in the time interval \([t_0, t_0 + N_1 x]\), then, at the time \(t_0 + N_1 x\), a delay of the cluster \(C_i\) begins.

Proof. Under the conditions of Lemma 2, we have that, at time \(t_0 + N_1 x\), the system is in the state \((0, N_2 - j_1)\), where \(0 < j_1 = N_1 x - N_2 y - N_2 + M_2 < M_2\). From this, Lemma 2 follows.

Lemma 3. Assume that non-negative integer numbers \(x, y\) satisfy the condition \(N_2 - M_2 - M_1 < N_1 x - N_2 y < N_2 - M_2\). Suppose the system is in the state \((0, N_2 - M_2)\) at time \(t_0\) and no delays occur in the time interval \([t_0, t_0 + N_1 x; j_1]\), where \(M_1 < j_1 = N_1 x - N_2 y - N_2 + M_2 < 0\), then, at the time \(t_0 + N_1 x + |j_1|\), a delay of the cluster \(C_2\) begins.

Proof. Under the conditions of Lemma 3, we have that, at time \(t_0 + N_1 x + |j_1|\), the system is in the state \((N_1 j_1, 0)\), \(0 < j_1 < M_2\). Therefore, at time \(t_0\), the cluster \(C_2\) is at the node, and the cluster \(C_1\) covers the nodes. From this, Lemma 3 follows.

Lemma 4. Assume that non-negative integer numbers \(x, y\) satisfy the condition \(N_1 x - N_2 y = N_2 - M_2\). Suppose the system is in the state \((0, N_2 - M_2)\) at time \(t_0\) and no delays occur in the time interval \([t_0, t_0 + N_1 x]\), then, at the time \(t_0 + N_1 x\), a conflict occurs.

Proof. Under the conditions of Lemma 4, we have that, at time \(t_0 + N_1 x\), the system is in the state \((0; 0)\). Therefore, at time \(t_0\), the system is in the state \((0; 0)\). Therefore, at time \(t_0 + N_1 x\), both clusters are at the node. Thus a conflict occurs.

Lemma 5. Assume that the following holds. The system is in the state \((0, N_2 - M_2)\) at time \(t_0\). There are no delays occur in time interval \([t_0, t_1]\). Suppose, at time \(t_1\), the cluster \(C_2\) moves through the node, and a delay of the cluster \(C_1\) begins; then \(t_1 = t_0 + N_1 x + |j_1|\) and, at time \(t_1\), the system is in the state \((0, N_2 - j_1)\), where \(j_1 = \text{the minimum non-negative integer number } x \text{ such that there exists a non-negative integer number } y = y + 1 \text{ satisfying the condition } N_2 - M_2 < N_1 x + N_2 y < N_2\).

Proof. Under the conditions of the lemma, taking into account Lemma 2, we have that \(k_1(t_1) = 0\) and \(N_2 - M_2 < k_2(t_1) < N_2\). From this, taking into account that both clusters move at any moment belonging time interval \([t_0, t_1]\); we get Lemma 5.

Suppose \(g_1 = N_1 x + N_2 y\). If there exist no non-negative integer numbers \(x, y\) satisfying \(N_2 - M_2 < N_1 x + N_2 y < N_2\), then we assume that \(g_1 = \infty\).

Lemma 6. Let the following hold. The system is in the state \((0, N_2 - M_2)\) at time \(t_0\). There are no delays occur in time interval \([t_0, t_1]\). Suppose, at time \(t_1\), the cluster \(C_1\) moves through the node, and a
delay of the cluster Cl; begins: then \( t_1 = t_0 + N_1x_1^* + |j_1^*| \); and, at time \( t_i \), the system is in the state \((N_1 - |j_1^*|, 0)\); where \( x_1^* \) is the minimum non-negative integer value of \( x \) such that there exists a non-negative integer number \( y = y_1^* \) satisfying the condition \( N_1 - M_1 < N_2x_1^* - N_2y_1^* < N_1 - M_1 \) and \( j_1^* = N_1x_1^* - N_2y_1^* - N_2 + M_2 \).

Proof. Under the conditions of the lemma we have \( N_1 - M_1 < k(t_i) < N_1 \) and \( k(t_0) = N_1 \). From this, taking into account that both clusters move at any moment belonging time interval \([t_0, t_1]\), we get Lemma 6.

Suppose \( g_1^* = N_1x_1^* + |j_1^*| \). If there exist no non-negative integer numbers \( x, y \) satisfying the condition \( N_2 - M_2 - M_1 < N_1x_1^* - N_2y_1^* < N_2 - M_2 \), then we assume that \( g_1^* = \infty \).

**Lemma 7.** Assume that the following holds. The system is in the state \((0, N_2 - M_2)\) at time \( t_0 \). There are no delays occur in time interval \([t_0, t_1]\). Suppose, at time \( t_i \), the system is in the state \((0, 0)\); then \( t_1 = t_0 + N_1x_0^* \), where \( x_0^* \) is the minimum non-negative integer number \( x \) such that there exists a non-negative integer number \( y = y_1^* \) satisfying the condition \( N_1x_0^* - N_2y_0^* = N_2 - M_2 \).

Proof. Taking into account that, at time \( t_1 \), the system is in the state \((0, 0)\); and both clusters move at any moment belonging time interval \([t_0, t_1]\), we get Lemma 7.

Assume that \( g_0^* = N_2x_0^* \). If there exist no non-negative integer numbers \( x, y \) satisfying \( N_1x - N_2 y = N_2 - M_2 \), then we assume that \( g_0^* = \infty \).

**Lemma 8.** Let non-negative integer numbers \( x, y \) satisfy the condition \( N_1 - M_1 < N_2y - N_2x < N_1 \). Suppose the system is at the state \((N_1 - M_1, 0)\) at time \( t_0 \) and no delays occur in the time interval \([t_0, t_0 + N_2y)\); then, at time \( t_0 + N_2y \), a delay of the cluster Cl; occurs.

Proof. Under the conditions of the lemma, we have that, at time \( t_0 + cy \), the system is in the state \((0, N_1 - M_2)\), where \( 0 < j_2 = N_2y + N_1x - N_1 + M_1 < M_1 \). From this, Lemma 8 follows.

**Lemma 9.** Assume that non-negative integer numbers \( x, y \) satisfy the condition \( N_1 - M_1 < N_2y - N_1x < N_1 - M_1 \). Suppose the system is at the state \((N_1 - M_1, 0)\) at time \( t_0 \) and no delays occur in the time interval \([t_0, t_0 + N_2y + |j_2|)\), where \( j_2 = N_2y - N_1x - N_1 + M_1 \); then, at the time \( t_0 + N_1x + |j_2| \), a delay of the cluster Cl; begins.

Proof. Under the conditions of Lemma 9, we have that, at time \( t_0 + N_2x + |j_2| \), the system is in the state \((0, N_2 - |j_2|), -M_2 < j_2 < 0 \). Therefore, at time \( t_0 \), the cluster Cl; is at the node, and the cluster Cl; covers the node. From this, Lemma 9 follows.

**Lemma 10.** Assume that non-negative integer numbers \( x, y \) satisfy the condition \( N_2y - N_1x = N_2 - M_1 \). Suppose the system is at the state \((N_1 - M_1, 0)\) at time \( t_0 \) and no delays occur in the time interval \([t_0, t_0 + N_2y)\), then, at the time \( t_0 + N_2y \), a conflict occurs.

Proof. Under the condition of Lemma 10, we have that at time \( t_0 + N_2y \), the system is in the state \((0, 0)\). Therefore, at time \( t_0 + N_2y \), both clusters are at the node. Thus a conflict occurs. Lemma 10 has been proved.

**Lemma 11.** Assume that the following holds. The system is in the state \((N_1 - M_1, 0)\) at time \( t_0 \). There are no delays occur in time interval \([t_0, t_1]\). Suppose, at time \( t_1 \), the cluster Cl; moves through the node, and a delay of the cluster Cl; begins; then \( t_1 = t_0 + N_2y_2^* \), and, at time \( t_1 \), the system is in the state \((N_1 - j_2^*, 0)\), where \( y_2^* \) is the minimum non-negative integer number \( y^* \) such that there exists a non-negative integer number \( x = x_2^* \) satisfying the condition \( N_1 - M_1 < N_2y_2^* - N_1x_2^* < N_1 - M_1 \) and \( j_2^* = N_2y_2^* - N_1x_2^* - N_2 + M_1 \). From this, taking into account that both clusters move at any moment belonging time interval \([t_0, t_1]\), we get Lemma 11.

Suppose \( g_2^* = N_2y_2^* \). If there exist no non-negative integer numbers \( x, y \) satisfying the condition \( N_1 - M_1 < M_2 < N_2y - N_1x < N_1 - M_1 \), then we assume that \( g_2^* = \infty \).

**Lemma 12.** Let the following hold. The system is in the state \((N_1 - M_1, 0)\) at time \( t_0 \). There are no delays occur in time interval \([t_0, t_1]\). Suppose, at time \( t_1 \), the cluster 2 moves through then ode, and a delay of e cluster 1 begins; then \( t_1 = N_2x_2^* + |j_1^*| \), and, at time \( t_1 \), the system is in the state \((N_1 - |j_1^*|, 0)\) where \( y_2^* \) is the minimum non-negative integer value of \( y \) such that there exists a non-negative integer number \( x = x_2^* \) satisfying the condition \( N_1 - M_1 < N_2y_2^* - N_1x_2^* < N_1 - M_1 \) and
\[ j_2^- = N_2 y_2^- - N_1 x_2^- - N_1 + M_1. \]

Proof. Under the conditions of the lemma, we have \( N_1 - M_1 < \kappa_1(t_0) < N_2 \) and \( \kappa_2(t_0) = 0 \). From this, taking into account that both clusters move at any moment belonging time interval \([t_0, t_1]\), we get Lemma 12.

Assume that \( g_2^- = N_2 y_2^- + \| j_2^- \| \).

If there exist no non-negative integer numbers \( x, y \) satisfying the condition \( N_2 - M_2 < N_1 x - N_2 y < N_2 \), then we assume that \( g_2^- = \infty \).

**Lemma 13.** Assume that the following holds. The system is in the state \((N_1 - M_1, 0)\) at time \( t_0 \). There are no delays occur in time interval \([t_0, t_1]\). Suppose, at time \( t_1 \), the system is in the state \((0, 0)\); then \( t = c_1 x_0^2 \), and, at time \( t_0 \), the system is in the state \((N_1 - j_2^+, 0)\) where \( x_0^2 \) is the minimum non-negative integer number \( x \) such that there exists a non-negative integer number \( y = y_2^0 \) satisfying the condition \( N_2 y_2^0 - N_1 x_0^2 = N_1 - M_1 \).

The proof of Lemma 13 is the same as the proof of Lemma 7.

Assume that \( g_2^0 = N_2 y_2^0 \). If there exist no non-negative integer numbers \( x, y \) satisfying \( N_2 y_2^0 - N_1 x_0^2 = N_1 - M_1 \), then we assume that \( g_2^0 = \infty \).

Suppose \( g_1 = \min(g_1^+, g_1^+, g_2^0) \), \( g_2 = \min(g_2^+, g_2^+, g_2^0) \) Denote by \( j_1^+ \) value \( j_1^+, j_1^- \), or \( j_1^0 \) if \( g_1 = g_1^+, g_2 = g_2^+ \) or \( g_2 = g_2^0 \) respectively. Denote by \( j_2^+ \) the value, \( j_2^+, j_2^- \), or \( j_2^0 \) if \( g_2 = g_2^+, g_2 = g_2^+ \) or \( g_2 = g_2^0 \) respectively.

7. The behavior of system in the case of no self-organization

We shall prove theorems about spectral cycles and average velocities of particles.

**Theorem 2.** Suppose the condition of self-organization (2) does not hold and inequalities \( j_i \geq 0, j_2 < 0 \) hold; then there exist a unique spectral cycle, and this cycle contains the state \((0, N_2 - M_2)\). The period of the cycle is equal to \( g_1 + M_2 - j_1 \). Average velocities of clusters are equal to \( v_1 = g_1 / (g_1 + M_2 - j_1), v_2 = 1 \).

Proof. Since the condition of self-organization does not hold, the system does not result in the state of free movement. Hence, in accordance with Lemma 1, the system results in a state, belonging the set \( A_1 \), over a finite time, and, after this the systems results in the state \((0, N_2 - M_2)\), or the system results in a state, belonging the set \( A_2 \), over a finite time, and, after this, the system results in the state \((N_1 - M_1, 0)\). In accordance with Lemmas 5-7, from the state \((0, N_2 - M_2)\), the system results again in a state, belonging to the set \( A_1 \cup (0, 0) \). In accordance Lemmas 11-13, from the state \((N_1 - M_1, 0)\), the system results in a state, belonging to the set \( A_1 \). Therefore there is a unique spectral cycle, and this spectral cycle contains the state \((0, N_2 - M_2)\). On the spectral cycle, the system is in the states, not belonging to the set \( A \) (both clusters move in these states), during \( g_1 \) time units and the system is in states, belonging to the set (only the cluster \( C_1 \) moves in these states), during \( M_2 - j_1 \) time units. From this, Theorem 2 follows.

**Example 1.** Assume that \( N_1 = 3, M_1 = 2, N_2 = 5, M_2 = 3 \). The greatest common divisor of \( N_1 \) and \( N_2 \) is equal to \( d = 1 \), and therefore the inequality (1) does not hold. Hence the condition of self-organization does not hold. Let us find the values \( g_1^+, j_1^+ \). Assume that \( x = 0 \). Then there exists no non-negative integer value of \( y \) satisfying \( N_2 - M_2 < N_1 x - N_2 y < N_2 \). Assume that \( x = 1 \). Then the number \( y = y_1^+ = 0 \) satisfies (4). Therefore, \( x_1^+ = 1, y_1^+ = 0 \), and, \( g_1^+ = 3, j_1^+ = 1 \). Let us find the value \( g_1^0, j_1^0 \). If \( x = 0 \) or \( x = 1 \), then there exists no non-negative integer value of \( y \) satisfying \( N_2 - M_2 < N_1 x - N_2 y < N_2 \). Assume that \( x = 2 \). Then the number \( y = 1 \) satisfies the condition. We have \( x_1^- = 2, y_1^- = 1 \), and, \( x_1^- = 2, y_1^- = 1, j_1^+ = -1, g_1^+ = 7 \). Let us find the value \( g_1^0 \). If \( x = 0, x = 1, x = 2, \) or \( x = 3 \), then there exists no non-negative integer number \( y \) satisfying the equation \( N_1 x - N_2 y = N_2 - M_2 \). If \( x = 4 \), then the number \( y = y_2^+ = 2 \) satisfies the condition, and, \( g_1^0 = x_4^0 = 12 \). Thus, \( g_1 = g_1^+ = 3, j_1 = j_1^+ = 1 \). Let us find \( g_2^+ \) and \( j_2^- \). Assume that \( y = 0 \). Then there does not exist non-negative integer value of \( y \) satisfying \( N_2 - M_2 < N_2 y - N_1 x < N_1 \). Assume that \( y = 1 \). Then the number \( x = x_2^- = 1 \) satisfies the condition. Therefore, \( x_2^- = 1, y_2^- = 1 \), and, \( j_2^- = 1, g_2^+ = 5 \). Let us find the value \( g_2^- \). Suppose \( y = 0 \); then the number \( x = x_2^- = 0 \) satisfies \( N_1 - M_1 < N_2 y - N_1 x < N_1 - M_1 \). We have \( x_2^- = 0, y_2^- = 0 \), and, \( j_2^- = -1, g_2^- = 1 \). Let us find the value \( g_2^0, j_2^0 \).
If \( y = 0 \) or \( y = 1 \), then there exists no non-negative integer number \( y \) satisfying the equation \( N_2 y^2 - N_1 x^2 = N_1 - M_1 \). If \( y = y_2^2 = 2 \), then the number \( x = x_2^2 = 3 \) satisfies the condition, and, \( g_2^2 = 10 \). Thus, we have \( g_2 = 1, j_2 = -1 \). The conditions of Theorem 2 hold. Therefore, \( v_1 = 3/5, v_2 = 1 \).

**Theorem 3.** Suppose the condition of self-organization (2) does not hold and inequalities \( j_1 < 0, j_2 \geq 0 \); then there exists a unique spectral cycle, and this cycle contains the state \((N_1 - M_1, 0)\). The period of this spectral cycle equals \( g_2 + M_1 - j_2 \).Velocities of clusters are equal to \( v_1 = 1, v_2 = g_2 / (g_2 + M_1 - j_2) \).

We get Theorem 3 from Theorem 2 if we renumber the contours.

**Theorem 4.** Suppose inequalities \( j_1 < 0, j_2 < 0 \) hold. Then there is a unique spectral cycle. The spectral cycle contains the states \((0, N_1 - M_1)\) and \((N_2 - M_2, 0)\). The period of the spectral cycle equals \( g_1 + g_2 + M_1 + M_2 - |j_1| - |j_2| \). Velocities of clusters are equal to \( v_1 = 1 - (M_2 - |j_2|) / (g_1 + g_2 + M_1 + M_2 - |j_1| - |j_2|) \), \( v_2 = 1 - (M_2 - |j_1|) / (g_1 + g_2 + M_1 + M_2 - |j_1| - |j_2|) \).

**Proof.** Since the condition of self-organization does not hold, the system does not result in the state of free movement. Hence, in accordance with Lemma 1, the system results in a state, belonging the set \( A_1 \), over a finite time, and, after this, the system results in the state \((0, N_2 - M_2)\), or the system results in a state, belonging the set \( A_2 \), over a finite time, and, after this, in the state \((N_1 - M_1, 0)\). In accordance Lemma 5-7, from the state \((0, N_1 - M_2)\), over a finite time, the system results in a state belonging to \( A_1 \). In accordance Lemmas 11-13, from the state \((N_1 - M_1, 0)\), over a finite time, the system results in states belonging to \( A_2 \). Therefore there is a unique spectral cycle, and this spectral cycle contains the states \((0, N_2 - M_2)\) and \((N_1 - M_1, 0)\). On the spectral cycle, the system is in the states, not belonging to the set \( A \) (both clusters move in these states), during \( g_1 + g_2 \) time units, the system is in states, belonging to the set \( A \), and in these states, during \( M_2 - |j_1| \) time units, and the system is in states, belonging to the set \( A_2 \) (only the cluster \( C_1 \) moves in these states), during \( M_2 - |j_2| \) time units. From this, Theorem 4 follows.

**Example 2.** Suppose that \( N_1 = 10, M_1 = 4, N_2 = 5, M_2 = 2 \). In this case we have \( j_1 = -3, g_1 = 3, j_2 = -1, g_2 = 6 \), \( v_1 = 10/11, v_2 = 10/11 \). There exists a unique spectral cycle. Its period is equal to \( T = 11 \).

**Theorem 5.** Suppose inequalities \( j_1 \geq 0, j_2 \geq 0 \) hold. Then there are two spectral cycles. One of these cycles contains the state \((0, N_1 - M_1)\). The period of this cycle equals \( g_1 + N_2 - j_1 \). On this cycle, velocities of clusters are equal to \( v_1 = g_1 / (g_1 + M_2 - j_1), v_2 = 1 \). The other cycle contains the state \((N_1 - M_1, 0)\). The period of this cycle equals \( g_2 + M_1 - j_2 \). On this cycle, velocities of clusters are equal to \( v_1 = g_1 / (g_1 + M_2 - j_1), v_2 = 1 \).

**Proof.** In accordance with Lemma 1, depending on the initial state, the system results in a state of the set \( A_1 \), and after this, the system results in the state \((0, N_2 - M_2)\), or a state, belonging to the set \( A_2 \), over a finite time, and, after this, in the state \((N_1 - M_1, 0)\). In accordance with Lemmas 5-7, if the system is in the state of the set, then, returning to the set \( A \), the system results in states belonging \( A_2 \cup (0; 0) \). In this case, over a finite time, the system will not be in states belonging, and the system will be only in the state belonging to the spectral cycle that contains the state \((0, N_2 - M_2)\). In this case, on the spectral cycle, the system is in the states, not belonging to the set \( A \), during time units and the system is in states, belonging to the set, during \( M_2 - j_1 \) time units, and therefore, \( v_1 = g_1 / (g_1 + M_2 - j_1), v_2 = 1 \). In accordance with Lemmas 11-13, if the system is in the state of the set \( A_1 \), then, returning to the set \( A \), the system will only in states belonging \( A_1 \cup (0; 0) \). Over a finite time, the system will not be in states belonging, and the system will be only in the states belonging to the spectral cycle, contains the state \((N_1 - M_1, 0)\). On the spectral cycle, the system is in the states, not belonging to the set \( A \), during \( g_2 \) time units and the system is in states, belonging to the set, during \( M_2 - j_2 \) time units, and therefore, \( v_1 = 1, v_2 = g_2 / (g_2 + M_2 - j_2) \).

**Example 3.** Suppose \( N_1 = 4, M_1 = 2, N_2 = 6, M_2 = 2 \). In this case, we have \( j_1 = 0, g_1 = 4, j_2 = 0, g_2 = 6 \). There exist two spectral cycles. On one of these cycles, the clusters move with velocities \( v_1 = 2/3, v_2 = 1/3 \). The period of these cycles equals \( T = 6 \). On the other spectral cycle, the clusters move with velocities \( v_1 = 1, v_2 = 3/7 \). The period of these cycles equals \( T = 8 \).
8. Conclusions
We study the spectrum of two-contours system with contours of different lengths. The number of cells on the contour is equal to $N_i$, $i = 1, 2$. There is a moving cluster on the contour $C_i$, and this cluster contains $M_i$ particles, $i = 1, 2$. There is a common point of the contours. This point is called the node. Delays are due to some limitations. In accordance with limitations more than one cluster cannot cross the node simultaneously. A cluster stops if it comes to the node at time such that at this time the other cluster crosses the node. If clusters come to the node simultaneously, then the conflict occurs. In the case of a conflict, only one cluster moves in accordance with a given conflict resolution rule. A set of states such that these states are repeated periodically is called a spectral cycle.

In this paper, it has been proved that, if the condition of self-organization does not hold, then, depending on $N_1, N_2, M_1, M_2$, there are one or two spectral cycles. Formulas for average velocities of clusters have been obtained. We use approaches related to facts of linear Diophantine equations.

Acknowledgments
This work has been supported by the Russian Foundation for Basic Research, Grant No. 17-01-00821-a and Grant No. 17-07-01358-a.

References
[1] Nagel K and Schreckenberg M A 1992 Cellular automation models for freeway traffic J. Phys. I. 2(12) pp 2221-2229
[2] Belitzky V and Ferrary P A 2005 Invariant measures and convergence properties for cellular automation 184 and related processes J. Stat. Phys. 118(3) pp 589-623
[3] Wolfram S 1983 Statistical mechanics of cellular automata Rev. Mod. Phys. 55 pp 601-644
[4] Blank M L 2000 Exact analysis of dynamical systems arising in models of flow traffic Russian Math Surveys 55(5) pp 562-563
[5] Gray L and Grefeath D 2001 The ergodic theory of traffic jams. J. Stat. Phys. 105(3/4) pp 413-452
[6] Kanai M Nishinary K and Tokihiro T 2009 Exact solution and asymptotic behavior of the asymmetric simple exclusion process on a ring arXive.0905.2795v1
[7] Blank M 2010 Metric properties of discrete time exclusion type processes in continuum. J. Stat. Phys. 140(1) pp 170-197
[8] Biham O, Middleton A A and Levine D 1992 Self-organization and a dynamical transition in traffic-flow models Phys. Rev. A American Physical Society 46(10) R6124-R6127
[9] D'Souza R M (2005) Coexisting phases and lattice dependence of a cellular automaton model for traffic flow Phys. Rev. E The American Physical Society 71(6) 066112
[10] Angel O, Horloyd A E and Martin J B 2005 The jammed phase of the Biham-Middleton-Levine traffic model Electronic Communication in Probability 10 pp 167-178
[11] Austin T and Benjamini I 2006 For what number of cars must self organization occur in the Biham-Middleton-Levine traffic model from any possible starting configuration? arXiv:math/0607759
[12] Bugaev A S, Buslaev A P Kozlov V V and Yashina M V 2011 Distributed problems of monitoring and modern approaches to traffic modeling 14th International IEEE Conference on Intelligent Transactions Systems (ITSC 2011), Washington, USA, 5-7.10.2011 pp 477-481
[13] Kozlov V V, Buslaev A P and Tatashev A G 2013 On synergy of totally connected flow on chainmails CMMSE-2013 Cadis Spain 3 861-873
[14] Buslaev A P, Fomina M Yu, Tatashev A G and Yashina M V 2018 On discrete flow networks model spectra: statement, simulation, hypotheses. J. Phys: Conf. Ser. 1053(012034) pp 1-7
[15] Kozlov V V, Buslaev A P and Tatashev A G 2015 Monotonic walks on a necklace and coloured dynamic vector Int. J. Comput. Math. 92(9) pp 1910-1920
[16] Buslaev A P, Tatashev A G and Yashina M V 2018 Flows spectrum on closed trio of contours *Eur. J. Pure Appl. Math.* **11**(1) pp 260-283

[17] Buslaev A P and Tatashev A G 2017 Flows on discrete traffic flower *Journal of Mathematics Research* **9**(1) pp 98-108

[18] Buslaev A P and Tatashev A G 2018 Exact results for discrete dynamical systems on a pair of contours *Math Meth Appl Sci* **41**(17) pp 7283-7294

[19] Tatashev A G and Yashina M V 2019 Spectrum of continuous too-contours system *ITM Web of conferences* **24** 01014

[20] Tatashev A G and Yashina M V 2019 Behavior of continuous too-contours system *WSEAS Transactions on mathematics* **18** pp 28-36

[21] Buchshtab A A 1966 Number theory (Moscow: Prosvescheniye) In Russian