The Cabibbo’s angle, or equivalently the CKM matrix element $|V_{us}|$, is one of the fundamental parameters of the Standard Model. The most precise determination of $|V_{us}|$ comes from $K \rightarrow \pi \ell \nu$ decays. The PDG(2008) quotes $|V_{us}| = 0.2255(19)$ [2]. It is based on the new, very accurate experimental determination of the product $|V_{us}|f_K(0) = 0.21668(45)$ [2, 3] and on the old estimate of the vector form factor at zero-momentum transfer $f_+(0) = 0.961(8)$ given in Ref. [1].

The determination of $f_+(0)$ using lattice QCD started only recently with the quenched calculation of Ref. [3], where it was shown how $f_+(0)$ can be determined at the physical point with a ≃ 1% accuracy. The findings of Ref. [3] triggered various unquenched calculations of $f_+(0)$, namely those of Refs. [4, 5, 6] with $N_f = 2$ and pion masses above ≃ 500 MeV and the recent one of Ref. [4] with $N_f = 2 + 1$ and pion masses starting from 330 MeV.

In this Letter we present a new lattice result for $f_+(0)$ obtained from simulations with two flavors of dynamical twisted-mass quarks, using pion masses from 260 MeV up to 580 MeV. Our determination of $f_+(0)$ includes the estimates of all sources of systematic errors: discretization, finite size effects (FSE’s), $q^2$-dependence, chiral extrapolation and the effects of the quenched strange quark.

The chiral extrapolation and the related uncertainty are investigated using both SU(3) and, for the first time, SU(2) Chiral Perturbation Theory (ChPT). Within the former one can perform a systematic expansion of $f_+(0)$ of the type $f_+(0) = 1 + f_2 + f_3 + ...$, where $f_n = \mathcal{O}[M_K^n/(4\pi f_\pi)^n]$ and the first term is equal to unity due to the current conservation in the SU(3) limit. Because of the Ademollo-Gatto (AG) theorem [10], the first correction $f_2$ does not receive contributions from the local operators of the effective theory and can be computed unambiguously in terms of the kaon and pion masses ($M_K$ and $M_\pi$) and the pion decay constant $f_\pi$. It takes the value $f_2 = -0.0226$ at the physical point [1]. The task is thus reduced to the problem of finding a prediction for

\[
\Delta f ≡ f_+ + f_2 + ... = f_+(0) - (1 + f_2).
\]  

(1)

Recently SU(2) ChPT at the next-to-leading order (NLO) has been applied to study the quark-mass dependence of $f_+(0)$ [11]. In SU(2) ChPT the strange quark field does not satisfy chiral symmetry and the dependence on the strange quark mass, $m_s$, is absorbed into the low-energy constants (LEC’s) of the effective theory. The convergence of SU(2) ChPT is expected to be good when the $u/d$ quark mass is significantly smaller than $m_s$. In the case of $f_+(0)$ one gets the NLO result [11]

\[
f_+(0) = F_+ - \frac{3}{4} \frac{M_\pi^2}{(4\pi f_\pi)^2} \log\left(\frac{M_\pi^2}{\mu^2}\right) + c_+ M_\pi^2 + \mathcal{O}(\mu^4) \tag{2}
\]

where $F_+$ and $c_+$ are LEC’s functions of $m_s$ and $c_+$ depends also on the renormalization scale $\mu$ in such a way that the whole NLO result [2] is independent on $\mu$.

For the extrapolation of our lattice data to the physical point we apply both SU(2) and SU(3) ChPT obtaining consistent results, which help constraining the uncertainty of the chiral extrapolation.

We perform simulations with $N_f = 2$ flavors of dynamical twisted-mass quarks [12] generated with the tree-level Symanzik improved gauge action at a lattice spacing $a = 0.0883(6)$ fm [13, 14] ($\beta = 3.9$), for six values of the (bare) sea quark mass, namely $am_{\text{sea}} = 0.0030, 0.0040, 0.0064, 0.0085, 0.0100, 0.0150$ (see Ref. [15]). The valence light-quark mass is always kept equal to the sea quark mass (unitary pions) and the simulated pion masses goes from $\simeq 260$ to $\simeq 575$ MeV. For each pion mass we use three values of the (bare) strange quark mass, namely $am_s = 0.015, 0.022, 0.027$, to allow for a smooth, local interpolation of our results to the physical strange quark mass ($am_s^{\text{phys}} \simeq 0.021$).

At the two lowest pion masses the lattice volume is $L^3 \cdot T = 32^3 \cdot 64$ in lattice units, while at the higher ones it is $24^3 \cdot 48$ in order to guarantee that $M_\pi L \gtrsim 3.7$. 

We present new lattice results of the $K \rightarrow \pi \ell \nu$ semileptonic form factors obtained from simulations with two flavors of dynamical twisted-mass fermions, using pion masses as light as 260 MeV. Our main result is $f_+(0) = 0.9560(84)$, which, combined with the latest experimental data for $K_{e3}$ decays, leads to $|V_{us}| = 0.22675(5)\exp(20) f_+(0)$. Using the PDG(2008) determinations of $|V_{ud}|$ and $|V_{ub}|$ our result implies for the unitarity relation $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.0040(15)$. For the $O(p^0)$ term of the chiral expansion of $f_+(0)$ we get $\Delta f = f_+(0) - 1 - f_2 = -0.0214(84)$.
We perform two additional simulations: the first one at $M_\pi \simeq 300$ MeV using the smaller volume and the second at $M_\pi \simeq 470$ MeV using a finer lattice spacing $a \simeq 0.07$ fm ($\beta = 4.05$) in order to check FSE’s and discretization errors, respectively.

The 2- and 3-point correlation functions relevant in this work are calculated using all-to-all quark propagators evaluated with the “one-end-trick” stochastic procedure. All the necessary formulae can be easily inferred from Ref. [12], where the degenerate case of the pion form factor is illustrated in details. At each value of the pion mass the statistical errors are evaluated with the jackknife procedure, while a bootstrap sampling is applied in order to combine the jacknives for different pion masses.

The matrix element of the weak vector current $V_\mu$ can be written as

$$\langle \pi(p')|V_\mu|K(p)\rangle = P_\mu f_+(q^2) + q_\mu f_-(q^2),$$

(3)

where $P_\mu = p_\mu + p'_\mu$ and $q_\mu = p_\mu - p'_\mu$, and the scalar form factor $f_0(q^2)$ is defined as

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{M_K^2 - M_\pi^2} f_-(q^2).$$

(4)

Following Ref. [2] the scalar form factor at $q^2 = q_{0,\text{max}}^2 = (M_K - M_\pi)^2$ can be calculated on the lattice with very high statistical precision using a suitable double ratio of 3-point correlation functions. In the present simulations we get a precision better than $\simeq 0.2\%$ (see Table 1).

At each pion and kaon masses we determine both the vector $f_+(q^2)$ and the scalar $f_0(q^2)$ form factors for several values of $q^2 < q_{0,\text{max}}^2$ in order to interpolate at $q^2 = 0$. We take advantage of the twisted boundary conditions (see Ref. [13] for details) to achieve values of $q^2$ quite close to $q^2 = 0$. The momentum dependencies of both form factors are nicely fitted either by a pole behavior

$$f_{+,0}(q^2) = f_{+,0}(0)/(1 - s_{+,0} q^2)$$

(5)

or by a quadratic dependence on $q^2$

$$f_{+,0}(q^2) = f_{+,0}(0) \cdot (1 + s_{+,0} q^2 + c_{+,0} q^4),$$

(6)

where the condition $f_0(0) = f_+(0)$ is understood. The quality of the two fits is illustrated in Fig. 1.

The values obtained for $f_{+,0}(0)$ depend on both the pion and kaon masses. The dependence of the latter is shown in Fig. 2 at $M_\pi \simeq 435$ MeV and it appears to be quite smooth, so that an interpolation at the physical strange quark mass can be easily performed using quadratic splines. This is obtained by fixing the combination $(2M_K^2 - M_\pi^2)$ at its physical value, which at each pion mass defines a reference kaon mass $M_K^{ref}$:

$$2[M_K^{ref}]^2 - M_\pi^2 = 2[M_K^{phys}]^2 - [M_\pi^{phys}]^2$$

(7)

with $M_\pi^{phys} = 135.0$ MeV and $M_K^{phys} = 494.4$ MeV.

Note that at the SU(3)-symmetric point $M_K = M_\pi$ the absolute normalization $f_+(0) = 1$ is imposed automatically by the double ratio method of Ref. [3].

The results for $f_+(0)$, obtained using the pole dominance [4] or the quadratic fit [6], and interpolated at the reference kaon mass [7], are given in Table 1 for each pion mass. It can be seen that the values of $f_{+,0}(0)$ corresponding to different $q^2$-dependencies of the form factors differ by less than half of the statistical errors. In what follows we will show in the figures only the results obtained using the pole fit [5].

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$M_\pi$ (MeV) & $M_K^{ref}$ (MeV) & $f_0(q_{0,\text{max}})$ & $f_+(0)$ (pole) & $f_+(0)$ (quadratic) \\
\hline
260 & 320 & 1.00007 (224) & 0.97519 (499) & 0.97347 (505) \\
300 & 330 & 1.01923 (121) & 0.98052 (440) & 0.97950 (390) \\
375 & 555 & 1.00061 (123) & 0.98916 (264) & 0.98813 (248) \\
435 & 575 & 1.00414 (43) & 0.99343 (130) & 0.99273 (131) \\
470 & 590 & 1.00272 (34) & 0.99421 (79) & 0.99413 (85) \\
575 & 635 & 1.00016 (6) & 0.99823 (15) & 0.99827 (19) \\
\hline
\end{tabular}
\caption{Results for $f_0(q_{0,\text{max}})$ and $f_+(0)$, obtained with the pole [4] or quadratic [6] fits, interpolated at the reference kaon mass [7] for each simulated pion mass.}
\end{table}

The SU(3) chiral analysis of $f_+(0)$ starts by considering

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.pdf}
\caption{Scalar $f_0(q^2)$ and vector $f_+(q^2)$ form factors obtained at $M_\pi \simeq 300$ MeV and $M_K \simeq 580$ MeV versus $q^2$ in physical units. The solid and dashed lines are the results of the fits based on Eqs. [2] and [6], respectively.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.pdf}
\caption{Results for $f_+(0)$ versus $M_K^2$ at $M_\pi \simeq 435$ MeV. The square corresponds to the value of $f_+(0)$ obtained by local interpolation via quadratic splines (dotted line) at the reference kaon mass $M_K^{ref} \simeq 575$ MeV from Eq. [7].}
\end{figure}
the NLO term $f_2$, using the exact expression $f^{PQ}_2$ evaluated for our partially quenched (PQ) setup in Ref. [16],

$$f^{PQ}_2 = \frac{-2M_K^2 + M_π^2}{32\pi f_π^2} + \frac{3M_π^2 M_K^2 \log(M_π^2/M_K^2)}{64\pi^2 f_π^2 (M_π^2 - M_K^2)} + \frac{M_π^2 (4M_K^2 - M_π^2) \log(2 - M_π^2/M_K^2)}{64\pi^2 f_π^2 (M_π^2 - M_K^2)} \quad (8),$$

and by constructing the quantity $\Delta f$ from Eq. (1). We then carry out the extrapolation to the physical point using a simple phenomenological ansatz in terms of $M_π^2$:

$$\Delta f = \Delta_0 + \Delta_1 M_π^2 + \Delta_2 M_K^4 + \Delta_3 M_π^2 \log(M_π^2), \quad (9)$$

where $\Delta_{0,1,2,3}$ are fitting parameters.

The results obtained for $f_+(0)$ using two fits for $\Delta f$, one with $\Delta_3 = 0$ and the other with $\Delta_2 = 0$, are shown in Fig. 3(a). It can be seen that: i) the (absolute) size of $\Delta f$, whose chiral expansion starts from the NNLO term $f_2^{PQ}$ at all pion masses, and ii) the impact of the logarithmic term at NNLO is quite small.

The application of SU(2) and SU(3) ChPT yields results for $f_+(0)$, Eqs. (10) and (11), which are consistent within the uncertainties. By averaging the two results and adding the systematic errors in quadrature we get

$$f_+^{PQ}(0)_{SU(3)} = 0.9563(53)(13). \quad (11)$$

The application of SU(2) and SU(3) ChPT yields results for $f_+(0)$, Eqs. (10) and (11), which are consistent within the uncertainties. By averaging the two results and adding the systematic errors in quadrature we get

$$f_+^{PQ}(0) = 0.9581 \pm 0.0057_{\text{stat.}} \pm 0.0035_{\text{Syst.}}. \quad (12)$$

We now present our estimates of the remaining sources of systematic effects.

Finite Size. We have performed a simulation at $M_π = 300$ MeV using the volume $24^3 \cdot 48$, which corresponds to $M_π L \approx 3.2$. We get $f_+(0) = 0.98633(362)$ using the pole-dominance fit (5) and $f_+(0) = 0.98597(337)$ using the quadratic fit (6). We combine these values with the results shown in the second row of Table 1, corresponding to the volume $32^3 \cdot 64$ with $M_π L \approx 4.2$. Assuming a volume dependence of the form $A + B e^{-M_π L}/L^{3/2}$ we obtain a residual FSE equal to 0.0018, which we add (in quadrature) to the systematic error of Eq. (12).

Discretization. We have performed a simulation at $M_π \approx 470$ MeV using a finer lattice spacing ($a \approx 0.07$ fm). We observe a systematic increase of the scalar form factor $f_0(q^2)$ at all values of $q^2$ and for all kaon masses. In particular we get $f_+(0) = 0.99555(80)$ and $0.99518(95)$ using the pole-dominance (5) and the quadratic (6) fits, respectively. We combine these values with the results shown in the fifth row of Table 1.

Assuming a linear fit in $a^2$ (which is consistent with the automatic O($a$) improvement at maximal twist (17)), we find a discretization effect equal to 0.0037, which we add both to the central value and (in quadrature) to the systematic error of Eq. (12). Clearly a more detailed study of the scaling property of $f_+(0)$ would be beneficial in order to estimate better and to reduce further the discretization error.
Quenching of the strange quark. The effect of our PQ setup can be estimated within SU(3) ChPT, because, thanks to the AG theorem, the effect of quenching the strange quark is exactly known at NLO: at the physical point $f_2 - f_2^{eq} = -0.0058$ ($\simeq 26\%$ of $f_2$). This correction is added to the central value of Eq. (12). As for the $O(p^0)$ term $\Delta f_i$, we have found evidence that the chiral logs, which are the most sensitive to quenching effects, are small compared to the contribution of the local terms (see Fig. 3(a)). We estimate that the quenching effect on $\Delta f$ is at most 50% of the same effect on $f_2$. Thus we add (in quadrature) the value 0.0028 ($\simeq 13\%$ of $\Delta f$) to the systematic error of Eq. (12). Note that this value is of the same size of the difference between our estimate of $\Delta f$ at $N_f = 2$ and the quenched one of Ref. [2].

Our final result is

$$f_+(0) = 0.9560 \pm 0.0057_{\text{stat.}} \pm 0.0062_{\text{syst.}}$$

$$= 0.9560 \pm 0.0084,$$  

(13)

which corresponds to $\Delta f = -0.0214(84)$. Our determination agrees very well with the Leutwyler-Roos result [4] and with previous lattice calculations at $N_f = 0$ [3], $N_f = 2$ [3, 8, 9] and $N_f = 2 + 1 [9]$. We have tried simple fitting functions of the form $|V_{us}|f_+(0) = 0.21668(45) \pm 0.0020 f_+(0)$, and

$$|V_{us}| = 0.2267 \pm 0.0005_{\text{stat.}} \pm 0.0020_{\text{syst.}}.$$  

(14)

Combining this value with $|V_{td}| = 0.97418(27)$ and $|V_{tb}| = 0.00393(36)$ from PDG2008 [2], we get from (13)

$$|V_{us}|^2 + |V_{ud}|^2 + |V_{ub}|^2 = 1.0004 \pm 0.0015.$$  

(15)

In conclusion we present our results for the slopes of the scalar ($s_0$) and vector ($s_+$) form factors. Their light-quark mass dependence is illustrated in Fig. 4 and it appears to be quite mild. We have tried simple fitting functions of the form

$$s_j = a_j + b_j M^2 + c_j M^4 + d_j M^6 \log(M^2),$$

(16)

where $a_j, b_j, c_j$ and $d_j$ are fitting parameters and $j = +, 0$. The results of two fits, one with $d_{+,0} = 0$ and the other with $c_{+,0} = 0$, are shown in Fig. 4.

In terms of the dimensionless quantities $\lambda_{+,0} = M^2 s_{+,0}$ the extrapolation to the physical point and the evaluation of the systematic uncertainties yield

$$\lambda_0 = (12.8 \pm 2.2_{\text{stat.}} \pm 4.5_{\text{syst.}}) \cdot 10^{-3},$$

$$\lambda_+ = (23.7 \pm 2.3_{\text{stat.}} \pm 2.1_{\text{syst.}}) \cdot 10^{-3}.$$  

(17)

where the large systematic error on $\lambda_0$ is dominated by discretization effects. Our results for both $\lambda_0$ and $\lambda_+$ agree very well with the latest experimental averages

$$\lambda_0^{\text{exp.}} = (13.4 \pm 1.2) \cdot 10^{-3} \text{ and } \lambda_+^{\text{exp.}} = (24.9 \pm 1.1) \cdot 10^{-3},$$

obtained in Ref. [3] using data from KLOE, KTeV, IS-TRA+ and NA48 experiments.

We thank all the ETMC members for fruitful discussions and the apeNEXT computer centres in Rome and Zeuthen for their invaluable technical help. One of us (F.M.) also acknowledges the Consolider-Ingenio 2010 Program CPAN (CSD2007-00042).

[1] N. Cabibbo, Phys. Rev. Lett. 10 (1963) 531; M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49 (1973) 652.
[2] C. Amsler et al. [Particle Data Group], Phys. Lett. B 667 (2008) 1.
[3] M. Antonelli et al. [FlaviaNet Working Group on Kaon Decays], arXiv:0801.1817 [hep-ph].
[4] H. Leutwyler and M. Roos, Z. Phys. C 25 (1984) 91.
[5] D. Becirevic et al., Nucl. Phys. B 705 (2005) 339.
[6] N. Tsutsui et al. [JLQCD collaboration], PoS LAT2005 (2006) 357.
[7] C. Dawson, T. Iizubuchi, T. Kaneko, S. Sasaki and A. Soni, Phys. Rev. D 74 (2006) 114502.
[8] D. Brommel et al. [The QCDSF collaboration], PoS LAT2007 (2007) 364.
[9] P. A. Boyle et al., Phys. Rev. Lett. 100 (2008) 141601.
[10] M. Ademollo and R. Gatto, Phys. Rev. Lett. 13 (1964) 264.
[11] J. M. Flynn and C. T. Sachrajda [RBC/ UKQCD collaboration], Nucl. Phys. B 812 (2009) 64.
[12] R. Frezzotti, P. A. Grassi, S. Sint and P. Weisz [Alpha collaboration], JHEP 0108 (2001) 058.
[13] R. Frezzotti, V. Lubicz and S. Simula, Phys. Rev. D 79 (2009) 074506.
[14] P. Dimopoulos et al., [ETM collaboration], arXiv:0810.2873 [hep-lat].
[15] Ph. Boucaud et al. [ETM collaboration], Phys. Lett. B 650 (2007) 304; Comput. Phys. Commun. 179 (2008) 695.
[16] D. Becirevic, G. Martinelli and G. Villadoro, Phys. Lett. B 633 (2006) 84.
[17] R. Frezzotti and G. C. Rossi, JHEP 0408 (2004) 007.