A strong averaging principle for Lévy diffusions in foliated spaces with unbounded leaves.

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Abstract

This article extends a strong averaging principle for Lévy diffusions which live on the leaves of a foliated manifold subject to small transversal Lévy type perturbation to the case of non-compact leaves. The main result states that the existence of $p$-th moments of the foliated Lévy diffusion for $p \geq 2$ and an ergodic convergence of its coefficients in $L^p$ implies the strong $L^p$ convergence of the fast perturbed motion on the time scale $t/\varepsilon$ to the system driven by the averaged coefficients. In order to compensate the non-compactness of the leaves we use an estimate of the dynamical system for each of the increments of the canonical Marcus equation derived in [7], the boundedness of the coefficients in $L^p$ and a nonlinear Gronwall-Bihari type estimate. The price for the non-compactness are slower rates of convergence, given as $p$-dependent powers of $\varepsilon$ strictly smaller than 1/4.

Keywords: strong averaging principle; scale separation; averaging of slow-fast diffusions; Lévy jump diffusions on manifolds; foliated manifolds; Marcus canonical equation;

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1 Introduction

The literature on averaging principles for deterministic and stochastic systems reaches far back to the 18th century and is enormously rich both in theory and applications. At this point, however, we would like to refrain from a more systematic review of the long and bifurcated history of the field and restrict ourselves to the references to some classical texts. Standard texts on the deterministic field include [3], [28], [30], [31] and the references therein. For stochastic systems we refer to [9], [13], [15], [14], [27], [22], [6] and [4] and the respective bibliographies.

Loosely speaking, an averaging principle describes the observation that in a coupled slow-fast system in the limit of infinite time scale separation, the slow system is close to a system, where the fast variable is replaced by the limiting measure of its ergodic time average. In the case of stochastic differential equations rescaling time show that this problem can be restated as a problem of an ergodic system perturbed by small perturbations.

The results of this article generalize recent approaches by the authors for diffusions on finite dimensional foliated manifolds. For properties of foliated spaces consult [5], [11], [29], [32]. Motivated by [21] Gargate and Ruffino studied in [10] the case of foliated Gaussian diffusions on compact

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leaves subject to deterministic Lipschitz transversal perturbation. In Högele and Ruffino [12] the authors treat the case of foliated Lévy jump diffusions with exponential moments but still with deterministic transversal perturbation and compact leaves. This type of processes is described in terms of canonical Marcus equations.

The recent work by da Costa and Högele [7] covers the case of a general class of foliated Lévy diffusions on compact leaves perturbed by a near optimally large class of Lévy diffusions. This is carried out with the help of a nonlinear comparison principle and a fine study of the individual jump increments. However in that case the compactness still allows global estimates of the horizontal components, for instance, in the force acting on the “vertical” component of the perturbed system.

This article treats an averaging principle for the same type of foliated Lévy diffusions, however with non-compact leaves. The lack of compactness yields an almost unmitigated system of fully coupled SDEs. The strategies are once again non-linear Gronwall-Bihari type inequalities, using the $L^p$ boundedness of the drift. However, this comes at the price of slower rates of convergence. Our main result, Theorem 2.4 states that locally the transversal behavior of $X^ε_t$ can be approximated $L^p$ uniformly in time by the Lévy stochastic differential equation in the transversal space with coefficients given by the average of the deterministic transversal component of the perturbation (with respect to the invariant measure on the leaves for the original unperturbed dynamics) and the diffusion component given by the projection of the original perturbation into the transversal space. We should mention that our results cover the results by [8] as the special case of uniformly bounded jumps.

In the Section 2 we present the dynamical and stochastic framework, the main hypotheses and the main result. In Section 3 we prove the key proposition which is the basis for the proof of the main theorem, proved in Section 4. Wherever possible in the exposition without lost of coherence we refer to the article [7] in order to avoid trivial repetition.

2 Object of study and main results

2.1 The setup

The following setup is a non-compact extension of the setup on [7] and [12].

The foliated manifold: Let $M$ be a finite dimensional connected, smooth Riemannian manifold. It is known by the classical Nash theorem in [23] that any finite dimensional smooth manifold may be embedded in $\mathbb{R}^m$ with $m$ sufficiently large. We assume that $M$ is equipped with an $n$-dimensional foliation $\mathcal{F}$ in the following sense. Let $\mathcal{F} = (L_x)_{x \in M}$, with $M = \bigcup_{x \in M} L_x$ and the sets $L_x$ are equivalence classes of the elements of $M$ satisfying the following.

a. Given $x_0 \in M$ there exist a neighborhood $U \subset M$ of the corresponding leaf $L_{x_0}$ and a diffeomorphism $\varphi: U \to L_{x_0} \times V$, where $V \subset \mathbb{R}^d$ is a connected open set containing the origin $0 \in \mathbb{R}^d$.

b. For any $L_{x_0} \in \mathcal{F}$ the neighborhood $U \subset L_{x_0}$ can be taken small enough such that the coordinate map $\varphi$ is uniformly Lipschitz continuous.

Remark 2.1 The second coordinate of a point $x \in U$, called the vertical coordinate, will be denoted with the help the projection $\pi: U \to V$ by $\varphi(x) = (\bar{x}, \pi(x))$ for some $\bar{x} \in L_x$. For any fixed $v \in V$, the preimage $\pi^{-1}(v)$ is the leaf $L_x$, where $x$ is any point in $U$ such that the vertical projection satisfies $\pi(x) = v$. 
The unperturbed equation: We are interested in the ergodic behavior of the strong solution of a Lévy driven SDE with jump components which takes values in $M$ and which respects the foliation. Intuitively, a straight line increment $z$ does not cause the exit from the leaf of its current position if the entire line segment $(x_0 + \theta z)_{\theta \in [0,1]}$ is contained in it. Ordinary differential equations with a vector field $F$ on the right-hand side generalize this concept in the following sense. By definition, their solutions follow $F$ as “infinitesimally” tangents. If $F$ itself is tangential to a given manifold the integral curves remain “infinitesimally tangential” to the manifold and hence will not leave it. Therefore a straight line increment $z$ which is transformed in the stochastic integral into an integral curve following a tangential vector field $F$ of a given leaf will remain on the leaf, that is, respect the foliated structure of the space. This intuition is made rigorous in the notion of stochastic integration in the sense of a canonical Marcus equation in the sense of Kurtz, Pardoux and Protter [18]. Those equations are the equivalent for Lévy jump diffusions to the Stratonovich stochastic integration in the sense of a canonical Marcus eq uation in the sense of Kurtz, Pardoux with a vector field $Y$.

Let us consider the formal canonical Marcus stochastic differential equation

$$dX_t = F_0(X_t)dt + F(X_t) \circ dZ_t + G(X_t) \circ dB_t, \quad X_0 = x_0 \in M,$$

(1)

with the following components defined over a given filtered probability space $\Omega = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ which satisfies the usual conditions in the sense of Protter [25].

1. Let $Z = (Z_t)_{t \geq 0}$ with $Z_t = (Z^1_t, \ldots, Z^r_t)$ be a Lévy process over $\Omega$ with values in $\mathbb{R}^r$ for some $r \in \mathbb{N}$ and characteristic triplet $(0, \nu, 0)$. It is a consequence of the Lévy-Itô decomposition of $Z$ that $Z$ is a pure jumps process with respect to a Lévy measure $\nu : \mathcal{B}(\mathbb{R}^r) \to [0, \infty]$ satisfying

$$\int_{\mathbb{R}^r} (1 \wedge \|z\|^2) \nu(dx) < \infty \quad \text{and} \quad \nu(\{0\}) = 0.$$  

(2)

For details we refer to the overview article by Kunita [20] and the monographs of Sato [26] or Applebaum [2].

2. Let $F \in \mathcal{C}^2(M; L(\mathbb{R}^r; \mathfrak{M}))$ satisfying the following. The function $x \mapsto F(x)$ is $\mathcal{C}^2$ and for each $x \in M$ the linear map $F(x)$ maps a vector $z \in \mathbb{R}^r \mapsto F(x)z \in T_x L_{x}$ to the tangent space of the respective leaf. Furthermore, let $F$ and $(DF)F$ be globally Lipschitz continuous on $M$ with common Lipschitz constant $\ell > 0$.

3. Let $B = (B^1, \ldots, B^r)$ be an $\mathbb{R}^r$-valued Brownian motion on $\Omega$ and $G \in \mathcal{C}^2(M, L(\mathbb{R}^r, \mathfrak{M}))$. We assume that $G$ and $(DG)G$ are globally Lipschitz continuous on $M$ with Lipschitz constant $\ell > 0$.

Following [18] a strong solution of the formal equation (1) is defined as a random map $X : [0, \infty) \times \Omega \to M$ satisfying almost surely for all $t \geq 0$

$$X_t = x_0 + \int_0^t F_0(X_s)ds + \int_0^t G(X_s)dB_s + \frac{1}{2} \int_0^t (DG(X_s))G(X_s)d\langle B \rangle_s$$

$$+ \int_0^t F(X_{s-})dZ_s + \sum_{0 < s \leq t} (\Phi^F \Delta_s Z(X_{s-}) - X_{s-} - F(X_{s-})\Delta_s Z),$$

(3)

where $\langle B \rangle$ stands for the quadratic variation process of $B$ in $\mathbb{R}^r$ and the function $\Phi^F(x) = Y(1, x; Fz)$ and $Y(t, x; Fz)$ for the solution of the ordinary differential equation

$$\frac{d}{d\sigma} Y(\sigma) = F(Y(\sigma))z, \quad Y(0) = x \in M, \quad z \in \mathbb{R}^r.$$  

(4)
The perturbed equation: This article studies the situation where an SDE in the sense of (3), which is invariant on the leaf of the initial condition $x_0$ is perturbed by a transversal smooth vector field $\varepsilon K dt$ and stochastic differentials $\varepsilon \tilde{G} \circ dB$ and $\varepsilon \tilde{K} \circ d\tilde{Z}$, $\varepsilon > 0$, in the limit for $\varepsilon \searrow 0$. More precisely we denote by $X^\varepsilon$, $\varepsilon > 0$ the analogous solution in the sense of (3) of the perturbed formal system

$$
dX_t^\varepsilon = F_0(X_t^\varepsilon)dt + F(X_t^\varepsilon) \circ dZ_t + G(X_t^\varepsilon) \circ dB_t + \varepsilon \left( K(X_t^\varepsilon)dt + \tilde{K}(\pi(X_t^\varepsilon)) \circ d\tilde{Z}_t + \tilde{G}(\pi(X_t^\varepsilon)) \circ d\tilde{B}_t \right),
$$

where the additional coefficients are defined as follows.

4. The vector field $K : M \to TM$ is smooth and globally Lipschitz continuous.

5. Let $\tilde{Z} = (\tilde{Z}^1, \ldots, \tilde{Z}^r)$ be a Lévy process on $\Omega$ with values in $\mathbb{R}^r$ with Lévy triple $(0, \nu', 0)$ for $\nu'$ being a given Lévy measure. The vector field $\tilde{K} \in C^2(V, L(\mathbb{R}^r, TM))$ satisfies that $\tilde{K}$ and $(D\tilde{K})\pi$ are globally Lipschitz continuous with Lipschitz constant $\ell > 0$.

6. Let $\tilde{B} = (\tilde{B}^1, \ldots, \tilde{B}^r)$ be a $\mathbb{R}^r$-valued Brownian motion over $\Omega$ and $\tilde{G} \in C^2(V, L(\mathbb{R}^r, TM))$ satisfy that $\tilde{G}$ and $(D\tilde{G})\pi$ are globally Lipschitz continuous with Lipschitz constant $\ell > 0$.

7. Assume that the stochastic processes $Z, B, \tilde{Z}, \tilde{B}$ are independent on $\Omega$.

Theorem 2.2 ([18], Theorem 3.2 and 5.1) 1. Under the preceding setup (items a., b., 1.-3. and 7.) there is a unique $(\mathcal{F}_t)_{t \geq 0}$ semimartingale $X$ which is a strong global solution of (7) in the sense of equation (3). It has a càdlàg version and is a (strong) Markov process.

2. Under the preceding setup (in particular items a., b. and 1.-7.) there is a unique semimartingale $X^\varepsilon$ which is a strong global solution of equation (5) in the sense of equation (3), where $F_0$ is replaced by $F_0 + \varepsilon K$ and $F$ by $(F, \varepsilon \tilde{K})$, $G$ by $(\tilde{G}, \varepsilon \tilde{G})$, $B$ by $(B, \tilde{B})$ and $Z$ by $(Z, \tilde{Z})$. The perturbed solution $X^\varepsilon$ has càdlàg paths almost surely and is a (strong) Markov process.

The support theorem: We are now in the position to apply the crucial support theorem, Proposition 4.3, in Kurtz, Pardoux and Protter [18]. Under the hypotheses of Theorem 2.2 we have for any $\varepsilon > 0$ that $x_0 \in M$ implies that $\mathbb{P}(X_t^\varepsilon(x_0) \in M \forall t \geq 0) = 1$. This result applied to the leaves of $\mathcal{M}$ yields that each solution $X$ of (11) is foliated in the sense that $X$ stays on the leaf of its initial condition, i.e. for any $x_0 \in M$ we have $\mathbb{P}(X_t(x_0) \in L_{x_0} \forall t \geq 0) = 1$.

2.2 The hypotheses and the main result

In the general setup of Subsection 2.1 we assume the following precise hypotheses.

Hypothesis 1: Integrability. There is an exponent $p \geq 2$ such that the Lévy measures $\nu$ of $Z$ and $\nu'$ of $\tilde{Z}$ satisfy

$$
\int_{\mathbb{R}^r} \|z\|^p \nu(dz) < \infty \quad \text{and} \quad \int_{\mathbb{R}^r} \|z\|^{2p} \nu'(dz) < \infty.
$$
Hypothesis 2: Foliated invariant measures.

1. Each leaf $L_{x_0} \in \mathcal{M}$ passing through $x_0 \in M$ has an associated unique invariant measure $\mu_{x_0}$ with $\text{supp}(\mu_{x_0}) = L_{x_0}$ of the unperturbed foliated system $[\mathbb{I}]$ with initial condition $x_0$.

2. For $v_0 = \pi(x_0)$ the vertical coordinate of $x_0 \in M$ we define for $h : M \to TM$

$$Q^h(v_0) := \int_{L_{x_0}} h(y)\mu_{x_0}(dy). \quad (6)$$

We assume for any globally Lipschitz continuous map $h : M \to TM$ the function

$$\mathbb{R}^d \supset V \ni v \mapsto Q^h(v) \in \mathbb{R}^d \quad (7)$$

is globally Lipschitz continuous.

Remark 2.3 Note that $L_{x_0}$ only depends on $v_0 = \pi(x_0)$. The same is true for $\mu_{x_0}$.

Hypothesis 2 guarantees that for each $x_0 \in M$, $v_0 = \pi(x_0) \in V$ the stochastic differential equation

$$dw_t = Q^{\pi K}(w_t) dt + \dot{K}(w_t) \circ d\tilde{Z}_t + \dot{G}(w_t) \circ d\tilde{B}_t, \quad w_0 = v_0 \in V \quad (8)$$

has a unique strong solution $w = (w_t(v_0))_{t \in [0,\sigma)}$ on $\Omega$, $\sigma$ being the first exit time of $w$ from $V$.

Hypothesis 3: Ergodic convergence of the vertical coefficient in $L^p$. Fix $p \geq 2$ from Hypothesis 1.

1. There are continuous functions $\eta^0 : [0,\infty) \to [0,\infty)$ and $\bar{\eta} : M \to [0,\infty)$, where $\eta^0$ is monotonically decreasing with $\eta^0(t) \to 0$ as $t \to \infty$ and $\bar{\eta}$ is globally Lipschitz continuous. For all $x_0 \in M$ and $t \geq 0$ we have

$$\left(\mathbb{E}\left[\frac{1}{t}\int_0^t \pi K(X_s(x_0)) ds - Q^{\pi K}(\pi(x_0)) \right]^p\right)^{\frac{1}{p}} \leq \bar{\eta}(x_0) \eta^0(t). \quad (9)$$

2. We assume for any $x_0 \in M$ that $\int \eta(y)\mu_{x_0}(dy) < \infty$.

It is known in the literature that there is no standard rate of convergence $[14], [17]$, which is why we assume an external rate of convergence, which decomposes by factors, see for instance $[19]$.

For $\varepsilon > 0$ and $x_0 \in M$ let $\tau^\varepsilon$ being the first exit time of the solution $X^\varepsilon(x_0)$ of equation (5) from the foliated coordinate neighborhood $U$ of item a) in Subsection 2.1

The main result of this article is the following strong averaging principle.

Theorem 2.4 Let Hypotheses 1, 2 and 3 be satisfied for some $p \geq 2$. Then for any $x_0 \in M$ and $\lambda \in (0, \frac{p-1}{p})$ there are constants $c, C > 0$ and $\varepsilon_0 \in (0, 1]$ such that $\varepsilon \in (0, \varepsilon_0]$ and $T \in [0, 1]$ imply

$$\left(\mathbb{E}\left[\sup_{t \in [0,T \wedge \varepsilon T \wedge \sigma]} |\pi(X^\varepsilon_t(x_0)) - w_t(\pi(x_0))|^p\right]\right)^{\frac{1}{p}} \leq CT \left[\varepsilon^\lambda + \eta^0(cT \ln(\varepsilon))\right]. \quad (10)$$

Remark 2.5 Our results focus on the case with only $p$-th moments, hence we set the coefficients $G$ and $\dot{G}$ to zero in the proofs.
3 The transversal perturbations

In order to prove the main theorem we need to control the error $X^\varepsilon - X$ in terms of $L^p$. This section is dedicated to the control of this error by the following result.

**Proposition 3.1** Let the assumptions of Subsection 2.1 and Hypotheses 1, 2 and 3 be satisfied for some $p \geq 2$. Then for any Lipschitz function $h : M \to \mathbb{R}$, $x_0 \in M$ and for all $T^\varepsilon : [0,1] \to [1,\infty)$ satisfying $\varepsilon T^\varepsilon \to 0$ there exist positive constants $\varepsilon_0 \in (0,1)$, $k_1, k_2, k_3 > 0$ such that $\varepsilon \in (0,\varepsilon_0]$ implies

$$
\left( \mathbb{E} \left[ \sup_{t \in [0,T]} |h(X^\varepsilon_t(x_0)) - h(X_t(x_0))|^p \right] \right)^{\frac{1}{p}} \leq k_1 \varepsilon^{\frac{p-1}{p^2}} \exp(k_2 T). \tag{11}
$$

In addition, the constant $k_1(x_0) \leq k_3(1 + \eta(x_0))$.

We apply this result for the following setting.

**Corollary 3.2** Let the assumptions of Proposition 3.1 be satisfied for some $p \geq 2$. Then for any $\lambda \in (0,\frac{1}{p^2})$ there exist positive constants $c_\lambda$, $\varepsilon_0 \in (0,1)$, $k_4, k_5 > 0$ such that for $T_\varepsilon := c_\lambda |\ln(\varepsilon)|$, $\varepsilon \in (0,\varepsilon_0]$ satisfies

$$
\left( \mathbb{E} \left[ \sup_{t \in [0,T_\varepsilon]} |h(X^\varepsilon_t(x_0)) - h(X_t(x_0))|^p \right] \right)^{\frac{1}{p}} \leq k_4 \varepsilon^\lambda. \tag{12}
$$

In addition, the constant $k_4 = k_5 k_1$.

**Proof:** Plugging $T_\varepsilon = -c \ln(\varepsilon)$ in the right-hand side of (11) we obtain $k_1 \varepsilon \exp(k_2 T_\varepsilon) = k_1 \varepsilon^{\frac{p-1}{p^2}} - ck_2$. Given $\lambda \in (0,\frac{1}{p^2})$ we fix $c_\lambda := \frac{1}{k_2} \left( \frac{p-1}{p^2} - \lambda \right)$ and infer the desired result. \(\square\)

The proof of Proposition 3.1 relies on the following lemma on positive invariant dynamical systems and the nonlinear comparison principle Corollary 5.2 given in the appendix. The main difficulty stems from the fact that the influence of the horizontal component in the vertical component cannot be estimated uniformly by the “diameter” of the leaf but has to be taken fully into account, which leads to a non-linear comparison principle.

**Lemma 3.3** For $F \in C^2(\mathbb{R}^{r+n}, L(\mathbb{R}^r, \mathbb{R}^{r+n}))$ being a globally Lipschitz continuous matrix-valued vector field and $z \in \mathbb{R}^r$ denote by $(Y(t;x,Fz))_{t \geq 0}$ the unique global strong solution of the ordinary differential equation

$$\frac{dY}{dt} = F(Y)z \quad Y(0,x,Fz) = x \in \mathbb{R}^{r+n}.$$

1) Then there exists $C > 0$ such that for any $z \in \mathbb{R}^r$ and $x, y \in M$ with $Y(t;x) = Y(t;x,Fz)$ we have

$$\sup_{t \geq 0} |(DF(Y(t;x))z)F(Y(t;x))z - (DF(Y(t;y))z)F(Y(t;y))z| \leq C |x - y| \|z\|^2.$$

2) For any $x \in M$ we have $\sup_{t \in [0,1]} \|DF(Y(t;x))F(Y(t;x))\| < \infty$.

A proof is given in [7] under Lemma 3.1.

**Proof:** (of Proposition 3.1) The first step of the proof yields the local orthogonality of the foliations and a transversal component by an appropriate change of coordinates. In a second step we estimate the transversal components with the help of the ergodic convergence of Hypothesis 3 and the nonlinear comparison principle Corollary 5.2. This is followed by the estimate of the horizontal component as the result of a classical Gronwall estimate before we conclude.
1. **Change of coordinates:** We first rewrite $X^\varepsilon$ and $X$, the solutions of equations (11) and (5), in terms of the coordinates given by the diffeomorphism $\varphi$

$$(u_t, v_t) := \varphi(X_t) \quad \text{and} \quad (u_t^\varepsilon, v_t^\varepsilon) := \varphi(X_t^\varepsilon).$$

The Lipschitz regularities of $h$ and $\varphi$ yields for $C_0 := Lip(h \circ \varphi^{-1})$ the estimate

$$|h(X_t^\varepsilon) - h(X_t)| \leq C_0(|u_t^\varepsilon - u_t| + |v_t^\varepsilon - v_t|). \quad (13)$$

The proof of the statement consists in calculating estimates for each summand on the right hand side of equation above. We define the

$$\mathcal{F}_0 := (D\varphi) \circ F_0 \circ \varphi^{-1}, \quad \mathcal{F} := (D\varphi) \circ F \circ \varphi^{-1},$$

$$\mathcal{R} := (D\varphi) \circ K \circ \varphi^{-1}, \quad \mathcal{R} := (D\varphi) \circ \tilde{K} \circ \varphi^{-1},$$

whose derivatives are uniformly bounded. Considering the components in the image of $\varphi$ we have:

$$\mathcal{R} = (\mathcal{R}_H, \mathcal{R}_V), \quad \tilde{\mathcal{R}} = (\tilde{\mathcal{R}}_H, \tilde{\mathcal{R}}_V)$$

with $\mathcal{R}_H, \tilde{\mathcal{R}}_H \in TL_{x_0}$ and $\mathcal{R}_V, \tilde{\mathcal{R}}_V \in TV \simeq \mathbb{R}^d$. The chain rule of the canonical Marcus equations mentioned in the introduction (Theorem 4.2 of [18]) yields for equation (5) the following form in $\varphi$ coordinates

$$du_t^\varepsilon = \mathcal{F}_0(u_t^\varepsilon, v_t^\varepsilon)dt + \mathcal{F}(u_t^\varepsilon, v_t^\varepsilon) \circ dZ_t + \varepsilon \mathcal{R}_H(u_t^\varepsilon, v_t^\varepsilon)dt + \varepsilon \tilde{\mathcal{R}}_H(v_t^\varepsilon) \circ d\tilde{Z}_t \quad \text{with} \quad u_t^\varepsilon \in L_{x_0}, \quad (14)$$

$$dv_t^\varepsilon = \varepsilon \mathcal{R}_V(u_t^\varepsilon, v_t^\varepsilon)dt + \varepsilon \tilde{\mathcal{R}}_V(v_t^\varepsilon) \circ d\tilde{Z}_t \quad \text{with} \quad v_t^\varepsilon \in V. \quad (15)$$

2. **Estimate of the transversal coordinate $E[\sup |v^\varepsilon - v|^p]$** Identically to [7], we start with estimates on the transversal components $|v^\varepsilon - v|$. The change of variables formula $x \mapsto g(x) := |x|^p$, $x \in \mathbb{R}^{n+d}$ using $(Dg(x), u) = p|x|^{p-2}(x, u)$ yields almost surely for $t \geq 0$

$$|v_t^\varepsilon - v_t|^p = p \int_0^t |v_s^\varepsilon - v_s|^p \varphi^{-1}(v_s^\varepsilon) \circ d\tilde{Z}_s \tag{16}$$

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2.1 Pathwise estimates: \( H_1 \): Clearly we have

\[
H_1 \leq \varepsilon p t \int_0^t |v_s^\varepsilon - v_s| ds.
\]

(17)

\( H_2 \): Young’s inequality for the conjugate indices \( p \) and \( p/(p-1) \) yields

\[
H_2 = \varepsilon p \int_0^t |v_s^\varepsilon - v_s|^{p-1} |\mathcal{R}_V(u_s, v_s)| ds
\leq \varepsilon p \sup_{[0,t]} |v^\varepsilon - v|^{p-1} \int_0^t |\mathcal{R}_V(u_s, v_s)| ds
\leq \varepsilon \sup_{[0,t]} |v^\varepsilon - v|^p + \varepsilon (p-1) t \left( \frac{1}{t} \int_0^t |\mathcal{R}_V(u_s, v_s)| ds \right)^p.
\]

(18)

\( H_3 \) and \( H_4 \): Switching to the Poisson random measure representation with respect to the compensated \( \tilde{N}' \), for instance see Kunita \[20\], we obtain

\[
H_3 \leq \varepsilon p \int_0^t \int_{\mathbb{R}^p} |v_s^\varepsilon - v_s^-|^{p-2} \langle v_s^\varepsilon - v_s^-, \tilde{\mathcal{R}}_V(v_s^-) \rangle |\tilde{N}'(dsdz) + \varepsilon C_1 \int_0^t |v_s^\varepsilon - v_s|^p ds.
\]

and

\[
H_4 \leq \varepsilon p \int_0^t \int_{\mathbb{R}^p} |v_s^\varepsilon - v_s^-|^{p-2} \langle v_s^\varepsilon - v_s^-, \tilde{\mathcal{R}}_V(v_s^-) \rangle |\tilde{N}'(dsdz) + \varepsilon C_2 \int_0^t |v_s^\varepsilon - v_s|^p ds.
\]

(19)

(20)

\( H_5 \): For the canonical Marcus terms we apply Lemma \[3.3\] statement 1) which yields a positive constant such that

\[
H_5 \leq \varepsilon^2 C_3 \int_0^t \int_{\mathbb{R}^p} |v_s^\varepsilon - v_s^-|^p |z|^2 |\tilde{N}'(dsdz) + \varepsilon^2 C_4 \int_0^t |v_s^\varepsilon - v_s|^p ds.
\]

(21)

The details can be found in \[7\].

\( H_6 \): For the last term we apply Lemma \[3.3\] statement 2), and exploit that \( \int_{\|z\|>1} \|z\|^2 \nu'(dz) < \infty \), we obtain a positive constant \( C_5 \) such that

\[
H_5 \leq \varepsilon^2 C_5 \int_0^t \int_{\mathbb{R}^p} |v_s^\varepsilon - v_s^-|^{p-1} |z|^4 |\tilde{N}'(dsdz) + \varepsilon^2 C_6 \int_0^t |v_s^\varepsilon - v_s|^{p-1} ds.
\]

(22)

Combining the estimates \([17, 22]\) we obtain

\[
|v_t^\varepsilon - v_t|^p \leq \varepsilon \sup_{[0,t]} |v^\varepsilon - v|^p + \varepsilon (p-1) t \left( \frac{1}{t} \int_0^t |\mathcal{R}_V(u_s, v_s)| ds \right)^p
+ \varepsilon (C_1 + C_2) \int_0^t |v_s^\varepsilon - v_s|^p ds
+ \varepsilon^2 C_4 \int_0^t |v_s^\varepsilon - v_s|^p ds + \varepsilon^2 C_6 \int_0^t |v_s^\varepsilon - v_s|^{p-1} ds
+ \varepsilon p \int_0^t \int_{\mathbb{R}^p} |v_s^\varepsilon - v_s^-|^{p-2} \langle v_s^\varepsilon - v_s^-, \mathcal{R}_V(v_s^-) \rangle |\tilde{N}'(dsdz)
+ \varepsilon^2 C_3 \int_0^t \int_{\mathbb{R}^p} |v_s^\varepsilon - v_s^-|^p |z|^2 |\tilde{N}'(dsdz)
+ \varepsilon^2 p C_5 \int_0^t \int_{\mathbb{R}^p} |v_s^\varepsilon - v_s^-|^{p-1} |z|^4 |\tilde{N}'(dsdz).
\]

(23)

(24)

(25)

(26)
2.2 Estimates on average: The main difference to \[7\] is found in the treatment of term \(H_2\). In the sequel we drop the superscript of \(T = T^\varepsilon\) where \(T^\varepsilon \in [1, \infty)\) satisfying \(\varepsilon T^\varepsilon \to 0\). Taking the supremum \(t \in [0, T]\) and taking the expectation yields that the term (23) can be bounded by

\[
\varepsilon \mathbb{E} \left[ \sup_{[0,T]} |v^\varepsilon - v|^p \right] + \varepsilon(p - 1)C_\infty T^p,
\]

where

\[
C_\infty = C_\infty(x_0) = \sup_{t > 0} \mathbb{E} \left[ \left( \frac{1}{T} \int_0^T |R_V(u_s(x_0), 0)| ds \right)^p \right] < \infty \tag{27}
\]
due to the convergence

\[
\mathbb{E} \left[ \left( \frac{1}{T} \int_0^T |R_V(u_s(x_0), 0)| ds - \int |R_V(y, 0)| \mu_{x_0}(dy) \right)^p \right] \to 0, \quad \text{as } t \to \infty.
\]

This implies in particular that

\[
C_\infty(x_0) \leq \int |R_V(y, 0)| \mu_{x_0}(dy) + \eta_0(0) \bar{\eta}(x_0). \tag{28}
\]

We obtain the integral inequality

\[
\mathbb{E} \left[ \sup_{[0,T]} |v^\varepsilon - v|^p \right] \leq \varepsilon C_7 \mathbb{E} \left[ \sup_{[0,T]} |v^\varepsilon - v|^p \right] + \varepsilon(p - 1)C_\infty T^p + \varepsilon C_8 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |v^\varepsilon - v|^p \right] ds
\]

\[
+ \varepsilon C_9 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |v^\varepsilon - v|^{p-1} \right] ds
\]

\[
\leq \varepsilon \mathbb{E} \left[ \sup_{[0,T]} |v^\varepsilon - v|^p \right] + \varepsilon(p - 1)C_\infty T^p + \varepsilon C_8 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |v^\varepsilon - v|^p \right] ds
\]

\[
+ \varepsilon C_9 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |v^\varepsilon - v|^{p-1} \right] \frac{p-1}{p} ds.
\]

Hence for any value \(\varepsilon \in (0, \frac{1}{2})\) we eliminate the first term

\[
\mathbb{E} \left[ \sup_{[0,T]} |v^\varepsilon - v|^p \right] \leq 2\varepsilon(p - 1)C_\infty T^p + 2\varepsilon C_8 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |v^\varepsilon - v|^p \right] ds
\]

\[
+ 2\varepsilon C_9 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |v^\varepsilon - v|^{p-1} \right] \frac{p-1}{p} ds.
\]

That is, for \(\Psi(T) = \mathbb{E} \left[ \sup_{[0,T]} |v^\varepsilon - v|^p \right]\) we have

\[
\Psi(T) \leq \varepsilon C_{10} T^p + \varepsilon C_{11} \int_0^T \Psi(s) ds + \varepsilon C_{12} \int_0^T \Psi(s) \frac{p-1}{p} ds.
\]

Using the nonlinear extension of the Gronwall-Bihari inequality in Corollary [5.2] in the appendix essentially given by Pachpatte [24], Theorem 2.4.2, which we adapt to our case we obtain a global constant \(C > 0\) such that using that \(\varepsilon_0 T\) is sufficiently small implies for all \(\varepsilon \in (0, \varepsilon_0]\)

\[
\Psi(T) \leq C \left( \varepsilon T^p + \varepsilon \frac{p-1}{p} T^{p-1} \right). \tag{29}
\]
3. Estimate of the horizontal component \( \mathbb{E}[\sup |u^\epsilon - u|^p] \): For convenience of notation we restart with the numbering of constants. Formally we obtain

\[
\begin{align*}
  u^\epsilon_t - u_t &= \int_0^t \left( \mathfrak{F}_0(u^\epsilon_s, v^\epsilon_s) - \mathfrak{F}_0(u_s, v_s) \right) ds + \int_0^t \left( \mathfrak{F}(u^\epsilon_{s-}, v^\epsilon_{s-}) - \mathfrak{F}(u_{s-}, v_{s-}) \right) \circ dZ_s \\
  &+ \varepsilon \int_0^t \left( \tilde{\mathfrak{R}}_H(u^\epsilon_s, v^\epsilon_s) - \tilde{\mathfrak{R}}_H(u_s, v_s) \right) ds + \varepsilon \int_0^t \tilde{\mathfrak{R}}_H(u^\epsilon_s, v^\epsilon_s) \circ d\tilde{Z}_s.
\end{align*}
\]

For further details consult [7] where we obtain with the help of the change of variable formula for the following equality in \( \mathbb{R}^n \) almost surely for \( t \geq 0 \)

\[
|u^\epsilon_t - u_t|^p = p \int_0^t |u^\epsilon_s - u_s|^{p-2} \langle u^\epsilon_s - u_s, \mathfrak{F}_0(u^\epsilon_s, v^\epsilon_s) - \mathfrak{F}_0(u_s, v_s) \rangle ds + p \int_0^t |u^\epsilon_{s-} - u_{s-}|^{p-2} \langle u^\epsilon_{s-} - u_{s-}, (\mathfrak{F}(u^\epsilon_{s-}, v^\epsilon_{s-}) - \mathfrak{F}(u_{s-}, v_{s-})) \circ dZ_s \rangle ds \\
+ p \sum_{0 < s \leq t} |u^\epsilon_{s-} - u_{s-}|^{p-2} \langle u^\epsilon_{s-} - u_{s-}, \Phi \Delta_s Z(u^\epsilon_{s-}, v^\epsilon_{s-}) - \Phi \Delta_s Z(u_{s-}, v_{s-}) \rangle \\
- (u^\epsilon_{s-} - u_{s-}, v^\epsilon_{s-} - v_{s-}) - \langle \mathfrak{F}(u^\epsilon_{s-}, v^\epsilon_{s-}) - \mathfrak{F}(u_{s-}, v_{s-}) \rangle \Delta_s Z \rangle
\]

\[
+ \varepsilon p \int_0^t |u^\epsilon_s - u_s|^{p-2} \langle u^\epsilon_s - u_s, \mathfrak{R}_H(u^\epsilon_s, v^\epsilon_s) - \mathfrak{R}_H(u_s, v_s) \rangle ds
\]

\[
+ \varepsilon p \int_0^t |u^\epsilon_{s-} - u_{s-}|^{p-2} \langle u^\epsilon_{s-} - u_{s-}, \tilde{\mathfrak{R}}_H(v^\epsilon_{s-}) \rangle ds
\]

\[
+ \varepsilon p \int_0^t |u^\epsilon_{s-} - u_{s-}|^{p-2} \langle u^\epsilon_{s-} - u_{s-}, \tilde{\mathfrak{R}}_H(v^\epsilon_{s-}) \rangle ds
\]

\[
+ p \sum_{0 < s \leq t} |u^\epsilon_{s-} - u_{s-}|^{p-2} \langle u^\epsilon_{s-} - u_{s-}, \Phi \Delta_s \tilde{Z}(v^\epsilon_{s-}) - \Phi \Delta_s \tilde{Z}(v_{s-}) \rangle \\
- (v^\epsilon_{s-} - v_{s-}) - \varepsilon \tilde{\mathfrak{R}}_H(v^\epsilon_{s-}) \Delta_s \tilde{Z} \rangle
\]

\[
=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\]

In fact, we shall use the following estimate

\[
|u^\epsilon_t - u_t|^{2p} \leq 8^{p-1} \sum_{i=1}^8 I_i^2.
\]

Now, we estimate each of the eight preceding summands on the right-hand side. The estimates of \( I_1 \) and \( I_4 \) are direct Lipschitz estimates. For the stochastic Itô terms we use the different kinds of maximal inequalities, see for instance [2] and [20]. The estimate of the canonical Marcus terms \( I_3, I_7 \) and \( I_8 \) is the most difficult task in which we use the result of Lemma [5.3]. The term \( I_5 \) is straightforward.

3.1 Estimate of the stochastic Itô integral terms \( I_2 \) and \( I_6 \): \( I_2 \): Due to the existence of moments of order at least 1, \( I_2 \) has the following representation with respect to the compensated Poisson random measure associated to \( Z \)

\[
\int_0^t |u^\epsilon_{s-} - u_{s-}|^{p-2} \langle u^\epsilon_{s-} - u_{s-}, (\mathfrak{F}(u^\epsilon_{s-}, v^\epsilon_{s-}) - \mathfrak{F}(u_{s-}, v_{s-})) \circ dZ_s \rangle
\]
\[\begin{align*}
&\int_0^t \int_{\mathbb{R}} |u_{s-}^\varepsilon - u_{s-}|^{p-2} \langle u_{s-}^\varepsilon, u_{s-} \rangle (\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})) z) \tilde{N}(ds dz) \\
&\quad + \int_0^t \int_{||z||>1} |u_{s-}^\varepsilon - u_{s-}|^{p-2} \langle u_{s-}^\varepsilon, u_{s-} \rangle (\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})) z) \nu'(dz) ds.
\end{align*}\] (33)

For the first term (33) we exploit the embedding \(L^2 \subset L^1\), Kunita’s maximal inequality (see [2] or [20]) for exponent equal to 2, and the Young inequality for the exponents \(p/2\) and \(p/(p - 2)\) combined with inequality (29) and obtain

\[\begin{align*}
E \left[ \sup_{t \in [0,T]} \int_0^t \int_{\mathbb{R}} |u_{s-}^\varepsilon - u_{s-}|^{p-2} \langle u_{s-}^\varepsilon, u_{s-} \rangle (\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})) z) \tilde{N}(ds dz) \right]^2 \\
&\leq E \left[ \sup_{t \in [0,T]} \int_0^t \int_{\mathbb{R}} |u_{s-}^\varepsilon - u_{s-}|^{p-2} \langle u_{s-}^\varepsilon, u_{s-} \rangle (\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})) z) \tilde{N}(ds dz) \right]^2 \\
&= E \left[ \int_0^T \int_{\mathbb{R}} |u_{s-}^\varepsilon - u_{s-}|^{2(p-2)} \langle u_{s-}^\varepsilon, u_{s-} \rangle (\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})) z) \tilde{N}(ds dz) \right]^2 \\
&\leq C_1 E \left[ \int_0^T \int_{\mathbb{R}} |u_{s-}^\varepsilon - u_{s-}|^{2(p-1)} \langle |u_{s-}^\varepsilon - u_{s-}|^2 + |v_{s-}^\varepsilon - v_{s-}|^2 \rangle ||z||^2 \tilde{N}(ds dz) \right] \\
&\leq C_1 \left( \int_{\mathbb{R}} ||z||^2 \tilde{N}(dz) \right) E \left[ \int_0^T (|u_{s-}^\varepsilon - u_{s-}|^{2p} + |u_{s-}^\varepsilon - u_{s-}|^{p-2} |v_{s-}^\varepsilon - v_{s-}|^2) ds \right] \\
&\leq C_2 \left( \int_0^T E \left[ \sup_{[0,s]} |u_{s-}^\varepsilon - u_{s-}|^{2p} \right] ds \right) + C \left( \varepsilon T^{2p} + \varepsilon^{2p-1} T \right)^{2(p+1)+1}. (35)
\end{align*}\]

The second term follows directly by Young’s inequality and the Lipschitz continuity of \(\mathfrak{F}\)

\[\begin{align*}
& E \left[ \sup_{t \in [0,T]} \int_0^t \int_{||z||>1} |u_{s-}^\varepsilon - u_{s-}|^{p-2} \langle u_{s-}^\varepsilon, u_{s-} \rangle (\mathfrak{F}(u_{s-}^\varepsilon, v_{s-}^\varepsilon) - \mathfrak{F}(u_{s-}, v_{s-})) z) \tilde{N}(ds dz) \right]^2 \\
&\leq \left( \int_{||z||>1} ||z|| \nu'(dz) E \left[ \sup_{t \in [0,T]} \int_0^t (|u_{s-}^\varepsilon - u_{s-}|^p + |u_{s-}^\varepsilon - u_{s-}|^{p-1} |v_{s-}^\varepsilon - v_{s-}|) ds \right] \right)^2 \\
&\leq \left( \int_{||z||>1} ||z|| \nu'(dz) \left( 2 \int_0^T E \left[ \sup_{[0,s]} |u_{s-}^\varepsilon - u_{s-}|^p \right] ds \right) + \int_0^T E \left[ |v_{s-}^\varepsilon - v_{s-}|^p \right] ds \right)^2 \\
&\leq C_3 T \int_0^T E \left[ \sup_{[0,s]} |u_{s-}^\varepsilon - u_{s-}|^{2p} \right] ds + C \left( \varepsilon T^{2p} + \varepsilon^{2p-1} T \right)^{2(p+1)+1}. (36)
\end{align*}\]

**I6:** We go over to the representation with the Poisson random measure \(\tilde{N}'\) associated to the Lévy process \(\tilde{Z}\) and obtain

\[\begin{align*}
&\sup_{t \in [0,T]} \varepsilon \int_0^t \int_{\mathbb{R}} |u_{s-}^\varepsilon - u_{s-}|^{p-2} \langle u_{s-}^\varepsilon, u_{s-} \rangle (\tilde{\mathcal{K}}_{H}(v_{s-}^\varepsilon)) d\tilde{Z}_s \\
&= \sup_{t \in [0,T]} \varepsilon \int_0^t \int_{\mathbb{R}} |u_{s-}^\varepsilon - u_{s-}|^{p-2} \langle u_{s-}^\varepsilon, u_{s-} \rangle (\tilde{\mathcal{K}}_{H}(v_{s-}^\varepsilon) - \tilde{\mathcal{K}}_{H}(v_{s-})) z) \tilde{N'}(ds dz) \quad (J_1) \\
&\quad + \sup_{t \in [0,T]} \varepsilon \int_0^t \int_{||z||>1} |u_{s-}^\varepsilon - u_{s-}|^{p-2} \langle u_{s-}^\varepsilon, u_{s-} \rangle (\tilde{\mathcal{K}}_{H}(v_{s-}^\varepsilon) - \tilde{\mathcal{K}}_{H}(v_{s-})) z) \nu'(dz) ds \quad (J_2)
\end{align*}\]
The terms $J_1$ and $J_2$ are estimated analogously to (35) and (36), where $\mathfrak{g}$ is replaced by $\mathfrak{R}_H$, which yield the following estimates

\begin{equation}
\left( \mathbb{E} \left[ \sup_{t \in [0,T]} \varepsilon \int_0^t \int_{\mathbb{R}^r} \left| u_s^\varepsilon - u_{s-}\right|^{p-2} \left( u_s^\varepsilon - u_{s-}, \mathfrak{R}_H(v_s^\varepsilon) \right) \bar{N}'(dsdz) \right] \right)^2 
\end{equation}

\begin{equation}
\leq C_4 \left( \int_0^T \mathbb{E} \left[ \sup_{[0,s]} \left| u^\varepsilon - u \right|^2 \right] ds \right)^{1/2} + C \left( \varepsilon T^{2p} + \varepsilon \frac{2p-1}{2p} T^{2(p+1)+1} \right)
\end{equation}

and

\begin{equation}
\left( \mathbb{E} \left[ \sup_{t \in [0,T]} \varepsilon \int_0^t \int_{\|z\|>1} \left| u_s^\varepsilon - u_{s-}\right|^{p-2} \left( u_s^\varepsilon - u_{s-}, \mathfrak{R}_H(v_s^\varepsilon) \right) \nu'(dz)ds \right] \right)^2 
\end{equation}

\begin{equation}
\leq C_5 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} \left| u^\varepsilon - u \right|^2 \right] ds + C \left( \varepsilon T^{2p} + \varepsilon \frac{2p-1}{2p} T^{2(p+1)+1} \right).
\end{equation}

For the term $J_3$ we observe that $u_s = 0$ consequently $\mathfrak{R}_V(v_s)$ is constant. Applying Kunita's maximal inequality for the exponent $2$, we obtain

\begin{equation}
\left( \mathbb{E} \left[ \sup_{t \in [0,T]} \varepsilon \int_0^t \int_{\mathbb{R}^r} \left| u_s^\varepsilon - u_{s-}\right|^{p-2} \left( u_s^\varepsilon - u_{s-}, \mathfrak{R}_H(v_s^\varepsilon) \right) \bar{N}'(dsdz) \right] \right)^2 
\end{equation}

\begin{equation}
\leq \varepsilon^2 \mathbb{E} \left[ \sup_{t \in [0,T]} \int_0^t \int_{\mathbb{R}^r} \left| u_s^\varepsilon - u_{s-}\right|^{p-2} \left( u_s^\varepsilon - u_{s-}, \mathfrak{R}_H(v_s^\varepsilon) \right) \bar{N}'(dsdz) \right]^2 
\end{equation}

\begin{equation}
\leq \varepsilon^2 C_6 \int_0^T \int_{\mathbb{R}^r} \mathbb{E} \left[ \left| u_s^\varepsilon - u \right|^{2(p-1)} \right] \|z\|^2 \nu'(dz)ds 
\end{equation}

\begin{equation}
\leq \varepsilon^2 C_6 \left( \int_{\mathbb{R}^r} \|z\|^2 \nu'(dz) \right) \left( \int_0^T \mathbb{E} \left[ \sup_{[0,s]} \left| u^\varepsilon - u \right|^{2p-2} \right] ds \right) 
\end{equation}

\begin{equation}
\leq \varepsilon^2 C_7 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} \left| u^\varepsilon - u \right|^{2p} \right] ds 
\end{equation}

\begin{equation}
\leq \varepsilon^2 C_7 \left( \int_0^T \mathbb{E} \left[ \sup_{[0,s]} \left| u^\varepsilon - u \right|^{2p} \right] ds + \frac{C_8}{p} T \right).
\end{equation}

The term $J_4$ is again easier, using $\varepsilon T < 1$ and $\varepsilon < 1$ we obtain

\begin{equation}
\left( \mathbb{E} \left[ \sup_{t \in [0,T]} \varepsilon \int_0^t \int_{\|z\|>1} \left| u_s^\varepsilon - u_{s-}\right|^{p-2} \left( u_s^\varepsilon - u_{s-}, \mathfrak{R}_H(v_s^\varepsilon) \right) \nu'(dz)ds \right] \right)^2 
\end{equation}

\begin{equation}
\leq \varepsilon \int_{\|z\|>1} \|z\| \nu'(dz) \mathbb{E} \left[ \sup_{[0,T]} \left| u^\varepsilon - u \right|^{p-1} \right] \|z\| \mathbb{E} \left[ \sup_{[0,s]} \left| u^\varepsilon - u \right|^{p-1} \right] ds \right)^2 
\end{equation}

\begin{equation}
\leq \varepsilon^2 C_9 \left( \int_0^T \mathbb{E} \left[ \sup_{[0,s]} \left| u^\varepsilon - u \right|^p \right] ds \right)^2 + C_9 \varepsilon^2 T^2 
\end{equation}

\begin{equation}
\leq \varepsilon^2 T C_9 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} \left| u^\varepsilon - u \right|^{2p} \right] ds + C_9 \varepsilon^2 T^2
\end{equation}
\[ \leq C_9 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^{2p} \right] ds + C_9 \varepsilon^{2p} T^2. \]

Summing up we obtain
\[ \mathbb{E} \left[ \sup_{[0,T]} |I_6|^2 \right] \leq C_{10} \left( \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^p \right] ds + \varepsilon^{\frac{2p-1}{2p}} T^{2(p+1)+1} + \varepsilon^2 T \right). \]

### 3.2 Estimate of the canonical Marcus terms \( I_3, I_7 \) and \( I_8 \):

The estimate is identical to estimate (54) in \[7\] and yields a constant \( C_{11} \) such that
\[ |I_3| \leq 2C_{11} \left( \sum_{0<s\leq t} |u^\varepsilon_{s-} - u_{s-}|^p \|\Delta_s Z\|^2 + \sum_{0<s\leq t} |v^\varepsilon_{s-} - v_{s-}|^p \|\Delta_s Z\|^2 \right). \]

Once again, the representation of this sum in terms of the Poisson random measure given in Kunita \[20\] tells us that
\[
\sum_{0<s\leq t} |u^\varepsilon_{s-} - u_{s-}|^p \|\Delta_s Z\|^2
= \int_0^t \int_{\mathbb{R}^r} |u^\varepsilon_{s-} - u_{s-}|^p \|z\|^2 \tilde{N}(dsdz) + \int_0^t \int_{\|z\|>1} |u^\varepsilon_{s-} - u_{s-}|^p \|z\|^2 \nu(dz) \, ds. \tag{39}
\]

The maximal inequality for integrals with respect to the compensated Poisson random measures and inequality \[29\] yield
\[
\mathbb{E} \left[ \sup_{[0,T]} |I_3|^2 \right] \leq C_{12} \int_0^T \int_{\mathbb{R}^r} \left( \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^{2p} \right] + \mathbb{E} \left[ |v^\varepsilon - v|^2 \right] \right) \|z\|^4 \nu(dz) \, ds
\leq C_{13} \left( \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^{2p} \right] ds + C \varepsilon T^{2p} + \varepsilon^{\frac{2p-1}{2p}} T^{2(p+1)+1} \right). \tag{40}
\]

**I_7:** For \( I_7 \) we apply Lemma \[5.3\] statement 1) and Young’s inequality and obtain the analogous result
\[
\sum_{0<s\leq t} (|u^\varepsilon_{s-} - u_{s-}|^p + |v^\varepsilon_{s-} - v_{s-}|^p) \|\Delta_s \tilde{Z}\|^2
\leq \varepsilon^2 C_{14} \left( \sum_{0<s\leq t} (|u^\varepsilon_{s-} - u_{s-}|^p + |v^\varepsilon_{s-} - v_{s-}|^p) \|\Delta_s \tilde{Z}\|^2 \right).
\]

Rewriting the last expression in terms of the (compensated) Poisson random measure \( \tilde{N}' \) we obtain
\[
\sum_{0<s\leq t} (|u^\varepsilon_{s-} - u_{s-}|^p + |v^\varepsilon_{s-} - v_{s-}|^p) \|\Delta_s \tilde{Z}\|^2
\leq \varepsilon^2 \int_0^t \int_{\mathbb{R}^r} (|u^\varepsilon_{s-} - u_{s-}|^p + |v^\varepsilon_{s-} - v_{s-}|^p) \|z\|^2 \tilde{N}'(dsdz) \tag{41}
\]
\[ + \int_0^t \int \mathbb{I}_{\|z\|>1} \left( |u_s^\varepsilon - u_s|^p + |v_s^\varepsilon - v_s|^p \right) \|z\|^{2\nu}(dz) ds. \] (42)

Kunita’s maximal inequality for the exponent 2 yields

\[
\mathbb{E} \left[ \sup_{[0,T]} \int_0^t \int_{\mathbb{R}^r} \left( |u_{s-}^\varepsilon - u_{s-}|^p + |v_{s-}^\varepsilon - v_{s-}|^p \right) \|z\|^{2\nu}(dz) ds \right]
\leq C_{15} \int_0^T \int_{\mathbb{R}^r} \mathbb{E} \left[ |u_s^\varepsilon - u_s|^{2p} + |v_s^\varepsilon - v_s|^{2p} \right] \|z\|^{2\nu}(dz) ds
\leq C_{16} \left( \int_{\mathbb{R}^r} \|z\|^{2\nu}(dz) \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^{2p} \right] ds + \int_0^T \mathbb{E} \left[ |v_s^\varepsilon - v_s|^{2p} \right] ds \right)
\leq C_{16} \int_{\mathbb{R}^r} \|z\|^{2\nu}(dz) \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^{2p} \right] ds + C_{17} (\varepsilon T^{2p} + \varepsilon \frac{2p-1}{2p} T^{2(p+1)+1}),
\]

where \(C_{17} = C\) from (29). The term (42) is treated obviously such that

\[
\mathbb{E} \left[ \sup_{[0,T]} |I_7|^2 \right] \leq \varepsilon^2 C_{18} \int_{\mathbb{R}^r} \|z\|^{4\nu}(dz) \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^{2p} \right] ds + C_{17} (\varepsilon T^{2p} + \varepsilon \frac{2p-1}{2p} T^{2(p+1)+1}). \tag{43}
\]

**I_8:** For \(I_8\) Lemma 3.3 statement 2), yields

\[
\begin{align*}
\sum_{0<s\leq t} |u_{s-}^\varepsilon - u_{s-}|^{p-2} & \langle u_{s-}^\varepsilon - u_{s-}, \Phi^{\varepsilon\tilde{H}} \Delta_s \tilde{Z}(v_{s-}) - v_{s-} - \varepsilon \tilde{H}(v_{s-}) \Delta_s \tilde{Z} \rangle \\
\leq & \sum_{0<s\leq t} |u_{s-}^\varepsilon - u_{s-}|^{p-1} |\Phi^{\varepsilon\tilde{H}} \Delta_s \tilde{Z}(v_{s-}) - v_{s-} - \varepsilon \tilde{H}(v_{s-}) \Delta_s \tilde{Z}| \\
\leq & \varepsilon^2 C_19 \sum_{0<s\leq t} |u_{s-}^\varepsilon - u_{s-}|^{p-1} \|\Delta_s \tilde{Z}\|^{2},
\end{align*}
\]

such that again Kunita’s inequality with exponent 2 and elementary Young’s estimate for parameters \(\frac{p-2}{p}\) and \(\frac{p}{2}\) yield

\[
\begin{align*}
\mathbb{E} \left[ \sup_{[0,T]} |I_8|^2 \right] & \leq \varepsilon^2 C_{20} \int_{\mathbb{R}^r} \|z\|^{4\nu}(dz) \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^{2p-2} \right] ds \\
& \leq \varepsilon C_{21} \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^{2p} \right] ds + C_{21} \varepsilon^2 T. \tag{44}
\end{align*}
\]

**3.3 Estimate of \(I_5\):**

\[
\begin{align*}
\int_0^T |u_s^\varepsilon - u_s|^p & \langle u_s^\varepsilon - u_s, \varepsilon \tilde{H}(u_s, v_s) \rangle ds \leq C_{22} \int_0^T \varepsilon |u_s^\varepsilon - u_s|^{p-1} ds \\
& \leq C_{22} \int_0^T \varepsilon |u_s^\varepsilon - u_s|^p ds + C_{22} \varepsilon^p T
\end{align*}
\]

such that

\[
\begin{align*}
\mathbb{E} \left[ \sup_{[0,T]} |I_5|^2 \right] & \leq \varepsilon C_{23} T \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^{2p} \right] ds + C_{23} \varepsilon^{2p} T^{2} \tag{45}
\end{align*}
\]
Finally a positive constant $C$ where $h$ globally Lipschitz continuous function and $Q$ function and $t$.

**Proposition 4.1** Let the assumptions of Proposition 3.1 be satisfied for fixed $p$ such that for $p > 1$ and $\varepsilon > 0$ we have

\[
E \left[ \sup_{[0,T]} |\delta_{\varepsilon}^h - u|^p \right] \leq C_{24} \left( \int_0^T E \left[ \sup_{[0,s]} |\delta_{\varepsilon}^h - u|^p \right] ds + \varepsilon T^{2p} + \varepsilon^{\frac{2p-1}{2p}} T^{2(p+1)} + (\varepsilon^n T)^2 \right).
\]

Finally

\[
E \left[ \sup_{[0,T]} |\delta_{\varepsilon}^h - u|^p \right] \leq E \left[ \sup_{[0,T]} |\delta_{\varepsilon}^h - u|^p \right] \leq C_{24} \varepsilon^{\frac{2p-1}{2p}} T^{2p+3} e^{C_{25} T}.
\]

**4. Conclusion:** The estimates of the sum of the vertical and the horizontal estimate yield

\[
E \left[ \sup_{[0,T]} |h(X_\varepsilon) - h(X)|^p \right] \leq C_{25} \left( E \left[ \sup_{[0,T]} |\delta_{\varepsilon}^h - u|^p \right] + E \left[ \sup_{[0,T]} |\delta_{\varepsilon}^h - v|^p \right] \right)
\]

\[
\leq C_{26} \varepsilon^{\frac{2p-1}{2p}} T^{2p+3} e^{C_{25} T} + C (\varepsilon^n T^p + \varepsilon^{\frac{2p-1}{2p}} T^{p+1})
\]

\[
\leq C_{27} \varepsilon^{\frac{2p-1}{2p}} e^{C_{27} T}.
\]

We finally note that the only dependence on the initial conditions stems from $C_\infty$ and hence by \[\text{the estimate } C_{27} \leq C_{28}(1 + \eta^0(0)\tilde{\eta}(x_0)). \text{ This finishes the proof.} \]

**4 The averaging error and the proof of the main result**

For convenience we fix the following notation. Given $h : M \rightarrow \mathbb{R}^n$ a globally Lipschitz continuous function and $Q^h : V \rightarrow \mathbb{R}^n$ its average on the leaves defined in definition \[\text{[3].} \text{ For } t \geq 0, x_0 \in M \text{ and } \varepsilon \in (0, 1] \text{ we write}

\[
\delta_{x_0}^h (\varepsilon, t) := \int_0^T h(X_\varepsilon^\pi (x_0)) - Q^h (\pi (X_\varepsilon^\pi (x_0))) ds.
\]

**Proposition 4.1** Let the assumptions of Proposition 3.1 be satisfied for fixed $p \geq 2$. Then for any globally Lipschitz continuous function $h : M \rightarrow \mathbb{R}^n$, $\lambda \in (0, \frac{1}{p-1})$ and $x_0 \in M$ there exist constants $b_1 > 0$ and $\varepsilon_0 \in (0, 1]$ such that for $\varepsilon \in (0, \varepsilon_0]$ and $T \in [0, 1]$ we have

\[
\left( E \left[ \sup_{s \in [0,T]} |\delta_{x_0}^h (\varepsilon, s)|^p \right] \right)^{\frac{1}{p}} \leq b_1 T \left[ \varepsilon^\lambda + \eta^0 (cT |\ln \varepsilon|) \right],
\]

where $c := \frac{1}{k_2} (\frac{p-1}{p} - \lambda) \wedge \ell \text{Lip}(\psi^{-1})$ is given in Corollary \[\text{[3,4].} \text{ and } \eta^0 \text{ is the temporal factor of the ergodic rate of convergence given in equation \[\text{[2].} \text{ by Hypothesis 3.} \]

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Proof of Proposition (4.1): Fix \( x_0 \in M \). For \( \varepsilon \in (0, 1) \) and \( T > 0 \) we define the partition
\[
t_0 = 0 < t_1^\varepsilon < \cdots < t_{N_\varepsilon}^\varepsilon \leq \frac{T}{\varepsilon} \wedge \tau^\varepsilon
\]
with the following step size
\[
\Delta_\varepsilon := -cT \ln(\varepsilon) \quad \text{for some } c > 0.
\]
The grid points of the partition are given by \( t_n^\varepsilon := n\Delta_\varepsilon \wedge \tau^\varepsilon \) for \( 0 \leq n \leq N_\varepsilon \) for \( \varepsilon \in (0, 1] \) with \( N_\varepsilon = \lfloor -\varepsilon \rfloor \). The term \( \delta_n^h(\varepsilon, t) \) can be estimated by the following three sums
\[
|\delta_n^h(\varepsilon, T)| \leq |A_1(T, \varepsilon)| + |A_2(T, \varepsilon)| + |A_3(T, \varepsilon)|,
\]
where
\[
A_1(T, \varepsilon) := \varepsilon \sum_{n=0}^{N_\varepsilon} \int_{t_n}^{t_{n+1}} [h(X_{s}^\varepsilon(x_0)) - h(X_{s-t_n}(X_{t_n}^\varepsilon(x_0)))] \, ds,
\]
\[
A_2(T, \varepsilon) := \varepsilon \sum_{n=0}^{N_\varepsilon} \int_{t_n}^{t_{n+1}} [h(X_{s-t_n}(X_{t_n}^\varepsilon(x_0))) - \Delta_\varepsilon Q(\pi(X_{t_n}^\varepsilon(x_0)))] \, ds,
\]
\[
A_3(T, \varepsilon) := \sum_{n=0}^{N_\varepsilon} \varepsilon \Delta_\varepsilon Q(\pi(X_{t_n}^\varepsilon(x_0))) - \int_0^{t_{N_\varepsilon+1}} \varepsilon Q(\pi(X_{t_n}^\varepsilon(x_0))) \, ds.
\]

The following lemmas estimate the preceding terms one-by-one. For convenience of the reader we number the constants \( C_i \).

**Lemma 4.2** For any \( \lambda \in (0, \frac{p-1}{p}) \) there exist positive constants \( b_2 > 0 \) and \( \varepsilon_0 \in (0, 1) \) such that for any \( \varepsilon \in (0, \varepsilon_0] \) and \( T \geq 0 \)
\[
\left( \mathbb{E} \left[ \sup_{s \in [0, T]} |A_1(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq b_2 T \varepsilon^\lambda.
\]

**Proof:** Using the Markov property analogously to [7] and Corollary 4.2 we obtain
\[
\mathbb{E} \left[ \sup_{[0, T]} A_1(s, \varepsilon) \right]^{\frac{1}{p}} \leq \varepsilon \sum_{n=0}^{N_\varepsilon-1} \mathbb{E} \left[ \left| \int_{t_n}^{t_{n+1}} [h(X_{s}^\varepsilon(x_0)) - h(X_{s-t_n}(X_{t_n}^\varepsilon(x_0)))ds] \right|^p \bigg| \mathcal{F}_{t_n} \right]^{\frac{1}{p}}
\]
\[
= \varepsilon \sum_{n=0}^{N_\varepsilon-1} \mathbb{E} \left[ \left| \int_{t_n}^{t_{n+1}} [h(X_{s-t_n}(y)) - h(X_{s-t_n}(y))ds] \right|^p \bigg| y = X_{t_n}^\varepsilon(x_0) \right]^{\frac{1}{p}}
\]
\[
\leq \varepsilon (N_\varepsilon + 1) \Delta_\varepsilon \max_{n=0, \ldots, N_\varepsilon} \mathbb{E} \left[ \left| \sup_{s \in [0, t_n]} [h(X_{s}^\varepsilon(y)) - h(X_{s-t_n}(y))] \right|^p \bigg| y = X_{t_n}^\varepsilon(x_0) \right]^{\frac{1}{p}}
\]
\[
\leq T \varepsilon^\lambda \max_{n=0, \ldots, N_\varepsilon} \mathbb{E} [k_4(X_{t_n}^\varepsilon(x_0))].
\]
Note that by Corollary \[32\] we have
\[
\max_{n=0,\ldots,N^\varepsilon} \mathbb{E}[k_4(X_{t_n}^\varepsilon(x_0))] \leq k_3 k_5 (1 + \max_{n=0,\ldots,N^\varepsilon} \mathbb{E}[\bar{\eta}(X_{t_n}^\varepsilon(x_0))]).
\]
It remains to bound the last summand. We estimate as follows for any \(n \in \mathbb{N}\)
\[
\mathbb{E}[\bar{\eta}(X_{t_n}^\varepsilon(x_0))]
\leq \mathbb{E}[\bar{\eta}(X_{t_n}^\varepsilon(x_0)) - \bar{\eta}(X_{t_n}(x_0))] + \mathbb{E}[\bar{\eta}(X_{t_n}(x_0))]
\leq \bar{\ell} \mathbb{E}[[X_{t_n}^\varepsilon(x_0) - X_{t_n}(x_0)] + \int \bar{\eta}(y)\mu_{x_0}(dy) + \sup_{t \geq 0} \mathbb{E}[[\int \bar{\eta}(y)\mu_{x_0}(dy) - \bar{\eta}(X_{t}(x_0))]]
\leq \bar{\ell} \mathbb{E}[[X_{t_n}^\varepsilon(x_0) - X_{t_n}(x_0)]^p]^{\frac{1}{p}} + C_1.
\]
For the first term in the preceding expression we derive a recursion formula. Using Theorem 3.2 in Kunita \[20\] it yields for the horizontal component
\[
\mathbb{E}\left[\sup_{s \in [0,T]} |X_s(x_1) - X_s(x_2)|^p \right] \leq e^{\ell \text{Lip}(\varphi^{-1}) T} |x_1 - x_2|^p,
\]
which implies the inequality
\[
\mathbb{E}[|X_{t_1}(x_1) - X_{t_1}(x_2)|^p] \leq C_2 \varepsilon^\lambda |x_1 - x_2|^p.
\]
We estimate
\[
\mathbb{E}[|X_{t_n}^\varepsilon(x_0) - X_{t_n}(x_0)|^p]^{\frac{1}{p}}
\leq \mathbb{E}[|X_{t_n}^\varepsilon(x_0) - X_{t_n-t_{n-1}}(X_{t_{n-1}}^\varepsilon(x_0))|^p]^{\frac{1}{p}} + \mathbb{E}[|X_{t_n}(X_{t_{n-1}}^\varepsilon(x_0)) - X_{t_n}(x_0)|^p]^{\frac{1}{p}}
\leq C_3 \varepsilon^\lambda \mathbb{E}[k_1(X_{t_{n-1}}^\varepsilon(x_0))] + C_2 \varepsilon^\lambda \mathbb{E}[|X_{t_{n-1}}^\varepsilon(x_0) - X_{t_{n-1}}(x_0)|^p]^{\frac{1}{p}}
\leq C_4 \varepsilon^\lambda \left(1 + \mathbb{E}[\bar{\eta}(X_{t_{n-1}}^\varepsilon(x_0))] \right) + C_2 \varepsilon^\lambda \mathbb{E}[|X_{t_{n-1}}^\varepsilon(x_0) - X_{t_{n-1}}(x_0)|^p]^{\frac{1}{p}}
\leq C_4 \varepsilon^\lambda \left(C_1 + \bar{\ell} \mathbb{E}[|X_{t_{n-1}}^\varepsilon(x_0) - X_{t_{n-1}}(x_0)|^p]^{\frac{1}{p}} \right) + C_2 \varepsilon^\lambda \mathbb{E}[|X_{t_{n-1}}^\varepsilon(x_0) - X_{t_{n-1}}(x_0)|^p]^{\frac{1}{p}}
= C_5 \varepsilon^\lambda \mathbb{E}[|X_{t_{n-1}}^\varepsilon(x_0) - X_{t_{n-1}}(x_0)|^p]^{\frac{1}{p}} + C_6 \varepsilon^\lambda.
\]
That is, for \(\psi_n := \mathbb{E}[|X_{t_n}^\varepsilon(x_0) - X_{t_n}(x_0)|^p]^{\frac{1}{p}}\) we have then
\[
\psi_n \leq C_7 \varepsilon^\lambda \psi_{n-1} + C_7 \varepsilon^\lambda,
\]
which gives the following estimate for any \(n \in \mathbb{N}\) and \(k \in \{1, \ldots, n\}\)
\[
\psi_n \leq (C_7 \varepsilon^\lambda)^{n-k} + \sum_{i=1}^{n-k} (C_7 \varepsilon^\lambda)^i.
\]
For \(C_7 \varepsilon^\lambda < \frac{1}{2}\) we obtain for any \(n \in \mathbb{N}\) the estimate
\[
\psi_n \leq C_7 \varepsilon^\lambda + \sum_{i=1}^{\infty} (C_7 \varepsilon^\lambda)^i \leq 3C_7 \varepsilon^\lambda < \infty.
\]
Under these assumptions, we obtain for all \( \varepsilon \in (0, \varepsilon_0) \)

\[
\max_{n=0, \ldots, N_\varepsilon} \mathbb{E} \left[ \bar{\eta}(\bar{X}_t^\varepsilon(x_0)) \right] \leq \max_{n=0, \ldots, N_\varepsilon} \left( \ell \mathbb{E} \left[ |X_{t_n}^\varepsilon(x_0) - X_{t_n}^\varepsilon(x_0)|^p \right] \right)^{\frac{1}{p}} + C_1 \leq C_7 \varepsilon^\lambda + C_1 < \infty. \quad (47)
\]

Going back to our main estimate, we obtain \( C_8 > 0 \) such that

\[
\mathbb{E} \left[ \sup_{s \in [0, T]} A_1(s, \varepsilon)^p \right]^{\frac{1}{p}} \leq T \varepsilon^\lambda k_3 k_5 \left( 1 + \max_{n=0, \ldots, N_\varepsilon} \mathbb{E} \left[ \bar{\eta}(X_{t_n}^\varepsilon(x_0)) \right] \right) \leq C_8 T \varepsilon^\lambda.
\]

\[ \blacksquare \]

**Lemma 4.3** For any \( \lambda \in (0, \frac{p-1}{p}) \) there exist positive constants \( b_3 > 0 \) and \( \varepsilon_0 \in (0, 1) \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and \( T \geq 0 \)

\[
\left( \mathbb{E} \left[ \sup_{s \in [0, T]} |A_2(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq b_3 T \eta^0(cT |\ln(\varepsilon)|).
\]

**Proof:** We have

\[
\left( \mathbb{E} \left[ \sup_{s \in [0, T]} |A_2(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq \varepsilon \left[ \mathbb{E} \left[ \sum_{n=0}^{N_\varepsilon} \left| \int_{t_n}^{t_{n+1}} h(X_{s-t_n}(X_{t_n}^\varepsilon(x_0))) ds - \Delta_\varepsilon Q^{\beta}(\pi(X_{t_n}^\varepsilon(x_0))) \right|^p \right]^{\frac{1}{p}} \right]
\]

\[
\leq \varepsilon \Delta_\varepsilon \sum_{n=0}^{N_\varepsilon} \left[ \mathbb{E} \left[ \frac{1}{\Delta_\varepsilon} \int_{t_n}^{t_{n+1}} h(X_{s-t_n}(X_{t_n}^\varepsilon(x_0))) ds - Q(\pi(X_{t_n}^\varepsilon(x_0))) \right]^p \right]^{\frac{1}{p}}.
\]

We apply the Markov property for all \( n = 0, \ldots, N_\varepsilon \). By Hypothesis 3 the two terms inside the modulus converge to each other when \( \Delta_\varepsilon \) goes to infinity with rate of convergence bounded by \( \bar{\eta}(X_{t_n}^\varepsilon(x_0)) \eta^0(\Delta_\varepsilon) \). Hence, for small \( \varepsilon \) we have

\[
\left( \mathbb{E} \left[ \sup_{s \in [0, T]} |A_2(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq \varepsilon N_\varepsilon \Delta_\varepsilon \eta^0(\Delta_\varepsilon) \max_{n=0, \ldots, N_\varepsilon} \mathbb{E}[\bar{\eta}(X_{t_n}^\varepsilon(x_0))]
\]

\[
\leq T \eta^0(cT |\ln(\varepsilon)|) \max_{n=0, \ldots, N_\varepsilon} \mathbb{E}[\bar{\eta}(X_{t_n}^\varepsilon(x_0))].
\]

Therefore, using (47), we obtain for \( \varepsilon \in (0, \varepsilon_0) \) the estimate

\[
\left( \mathbb{E} \left[ \sup_{s \in [0, T]} |A_2(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq C_8 T \eta^0(cT |\ln(\varepsilon)|). \]

\[ \blacksquare \]

**Lemma 4.4** For any \( \lambda \in (0, \frac{p-1}{p}) \) there exist positive constants \( b_4 > 0 \) and \( \varepsilon_0 \in (0, 1) \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and \( T \geq 0 \)

\[
\left( \mathbb{E} \left[ \sup_{s \in [0, T]} |A_3(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq b_4 T \varepsilon^\lambda.
\]

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Proof: We calculate
\[
|A_3(T, \varepsilon)| = \left| \sum_{n=0}^{N_\varepsilon} \varepsilon \Delta \xi Q^{\pi K}(\pi(X^n_{\xi})) - \int_0^{\sum_{n=0}^{N_\varepsilon} \Delta \xi} Q^{\pi K}(\pi(X^n_{\xi})) \, ds \right|
\]
\[
\leq \varepsilon \sum_{n=0}^{N_\varepsilon} \Delta \xi \sup_{t_n \leq s < t_{n+1}} |Q^{\pi K}(\pi(X^n_{\xi})) - Q^{\pi K}(\pi(X^n_{\xi}))| 
\]
\[
\leq \varepsilon \Delta \xi C_1 \sum_{n=0}^{N_\varepsilon} \sup_{t_n \leq s < t_{n+1}} |v^n_s - v^n_{t_n}|. \tag{48}
\]

By Minkowski's inequality, the Markov property, Proposition \ref{prop5.1} and \ref{prop5.2} (with the appropriate constant $C_8$) we have
\[
E[ \sup_{s \in [0, T]} |A_3(s, \varepsilon)|^p]^{1/p} \leq TC_1 \max_{n \in \{0, \ldots, N_\varepsilon\}} E[\sup_{t_n \leq s < t_{n+1}} |v^n_s - v^n_{t_n}(y) - v^n_0(y)|^p \mid y = F_{t_n}]]^{1/p}
\]
\[
\leq TC_1 \max_{n \in \{0, \ldots, N_\varepsilon\}} E[\sup_{t_0 \leq s < t_1} |v^n_s(y) - v^n_0(y)|^p \mid y = X^n_{t_n}(x_0)]^{1/p}
\]
\[
\leq TC_2 \varepsilon \lambda E[k_4(X^n_{t_n}(x_0))]
\]
\[
\leq TC_2 C_8 \varepsilon^\lambda.
\]

This ends the proof of Proposition \ref{prop5.1}.

Proof of the main Theorem \ref{thm2.4} With the help of Proposition \ref{prop5.1} the proof of Theorem \ref{thm2.4} is identical the one given in Section 5 of \cite{7}.

5 Appendix: Nonlinear comparison principle

Proposition 5.1 (Pachpatte \cite{24}) Let $u, f, g$ and $h$ be nonnegative continuous functions defined on $\mathbb{R}^+$. Let $v$ be a continuous non-decreasing subadditive and submultiplicative function defined on $\mathbb{R}^+$ and $v(u) > 0$ on $(0, \infty)$. Let $e, \varphi$ be continuous and nondecreasing functions defined on $\mathbb{R}^+$ with $p$ being strictly positive and $\varphi(0) = 0$. If
\[
u(t) \leq e(t) + g(t) \int_0^t f(s)v(u(s))ds + \varphi\left( \int_0^t h(s)v(u(s))ds \right)
\]
for all $t \geq 0$, then for any $0 \leq t \leq t_2$
\[
u(t) \leq a(t) \left[ e(t) + \varphi\left( F^{-1}(F(A(t)) + \int_0^t h(s)v(a(s))ds) \right) \right],
\]
where
\[
a(t) := 1 + g(t) \int_0^t f(s) \exp \left( \int_s^t g(\sigma)f(\sigma)d\sigma \right)ds,
\]
\[
A(t) := \int_0^t h(s)v(a(s)\varphi(s))ds,
\]

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\[ F(t) := \int_0^t \frac{ds}{v(\varphi(s))}, \]

\( F^{-1} \) is the inverse of \( F \) and \( t_2 \in \mathbb{R}^+ \) such that

\[ F(A(t)) + \int_0^t h(s)v(a(t))ds \in \text{dom}(F^{-1}) \quad \text{for all } 0 \leq t \leq t_2. \]

In the following special case of coefficients it is possible to drop the continuity assumption on \( u \).

**Corollary 5.2** Let \( \Psi \) a non-negative, measurable, increasing function and \( h \) be nonnegative, continuous, increasing function on the interval \([0,T]\) satisfying for \( p \geq 2, \varepsilon > 0, c > 0 \) and any \( t \in [0,T] \) the inequality

\[ \Psi(t) \leq \varepsilon ct^p + \varepsilon c \left( \int_0^t \Psi(s) + \Psi(s)^{\frac{p-1}{p}} ds \right), \quad t \in [0,T]. \]  

(49)

Then there is a constant \( k > 0 \) such that for any \( \varepsilon_0 \in (0,1] \) such that \( \varepsilon_0 T < k \) we have for all \( t \in [0,T] \) and \( \varepsilon \in (0,\varepsilon_0] \)

\[ \Psi(t) \leq C \left( \varepsilon t^p + t^p(\varepsilon t)^{\frac{p-1}{p}} \right). \]

**Proof:** For \( e(t) = c \varepsilon t^p, \ g \equiv 1, \ f, h \equiv \varepsilon c, \ \varphi(t) = t, \ w(t) = t^{\frac{p-1}{p}} \) we calculate the coefficients of Proposition 5.1

\[ a(t) := 1 + \varepsilon c \int_0^t \exp(\varepsilon c(t - s)) ds = \exp(\varepsilon ct) \]

and in the limit of \( \varepsilon t \) being small (\( \varepsilon t \ll 1 \)) we have

\[ \varepsilon \int_0^t a(s)^{\frac{p-1}{p}} ds = \varepsilon t \left( \frac{\exp(\varepsilon c^{\frac{p-1}{p}} \varepsilon t) - 1}{c^{\frac{p-1}{p}} \varepsilon t} \right) \leq \varepsilon t \ll 1 2\varepsilon t. \]

Applying the change of parameter \( r = \varepsilon s \) it follows that

\[ A(t) := \int_0^t \exp(\varepsilon c^{\frac{p-1}{p}} s(e(s))^{\frac{p-1}{p}} ds = \int_0^t \exp(c^{\frac{p-1}{p}} \varepsilon s) (c^{\frac{p-1}{p}} \varepsilon s)^{\frac{p-1}{p}} ds \]

\[ = \varepsilon^{\frac{p-1}{p}} \int_0^{\varepsilon t} \exp(c^{\frac{p-1}{p}} r) c^{\frac{p-1}{p}} \frac{r}{\varepsilon}^{p-1} dr \leq t \frac{1}{\varepsilon^p t^2} \int_0^{\varepsilon t} \exp(c^{\frac{p-1}{p}} r) (cr)^{\frac{p-1}{p}} dr \]

\[ \leq \varepsilon t \ll 1 2t \exp(c^{\frac{p-1}{p}} \varepsilon t)(c^{\varepsilon t})^{\frac{p-1}{p}} \leq C_1 t \exp(c^{\frac{p-1}{p}} \varepsilon t)(\varepsilon t)^{\frac{p-1}{p}}. \]

Finally, we obtain

\[ F(t) := \int_0^t s^{\frac{p-1}{p}} ds = t^{\frac{1}{p}}, \quad \text{and} \quad F^{-1}(t) := t^p. \]

In the sequel we follow the proof of Theorem 2.4.2 in Pachpatté [24] and define the continuous, positive, non-decreasing function

\[ n(t) := e(t) + \varphi \left( \int_0^t h(s)w(u(s))ds \right) = e(t) + \varepsilon c \int_0^t h(s)u(s)^{\frac{p-1}{p}} ds, \quad t \geq 0, \]
such that inequality (49) can be restated as

\[ u(t) \leq n(t) + g(t) \int_0^t f(s)u(s)ds = e(t) + \varepsilon c \int_0^t u(s)ds. \]

It is well-known, see for instance [1], that this integral estimate implies the following Gronwall-Bellmann inequality also in the case of \( u \) being merely positive measurable. The main reason is that the integral is absolutely continuous with a bounded density. This result yields

\[ u(t) \leq a(t)n(t), \quad t \geq 0. \]

The remainder of the proof of Theorem 2.4.2 in [24] does use the continuity of \( u \) and remains intact.

\[ \blacksquare \]

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