Testing Modified Gravity in Stimulated Photon-Photon Scattering
— Dedicated to the late professor Yasunori Fujii —

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We propose a method to probe chameleon particles predicted in the $F(R)$ gravity as a model of the modified gravity based on the concept of a stimulated pulsed-radar collider. We analyze the chameleon mechanism induced by an ambient environment consisting of focused photon beams and a dilute gas surrounding a place where stimulated photon-photon scatterings occur. We then discuss how to extract the characteristic feature of the chameleon signal. We find that a chameleon with the varying mass around $(0.1 - 1)\mu$eV in a viable model of the $F(R)$ gravity is testable by searching for the steep pressure dependence of the 10th-power for the signal yield.

Introduction — A variety of independent observations have confirmed the accelerated expansion of the present Universe, which indicates an unknown energy, dark energy (DE). The modified gravity theory has been considered as one of solutions to the DE problem, instead of the ad hoc introduction of the cosmological constant to the general relativity. Among the modified gravity theories, the scalar-tensor theory introduces a new scalar field [1–3], and such a DE scalar field can couple with ordinary matter and induces the so-called fifth force. The phenomenology of the new scalar field and the induced fifth force have been developed to constrain the modified gravity: a possible deviations from the gravitational inverse-square law [4–7] and time-varying parameters of the standard model of particle physics [8–11].

It is known that a dilatonic scalar field shows up in the modified gravity for DE through the Weyl transformation of the metric [12–15]. A special class of such a dilaton is called chameleon, which is named after the remarkable feature of the environment dependence, the chameleon mechanism [16]. The coupling to matter influences the effective mass of the chameleon field, which is analogous to an in-medium mass (an effective mass) in heavy fermion materials. Consequently, the chameleon mechanism screens the fifth force in a high-density environment, which allows the modified gravity to be consistent with short-scale observations.

There have been attempts to search the chameleon field by non-gravitational experiments [17–20]. Recently, it has been suggested that the stimulated radar collider concept may be applicable to probe gravitationally coupled scalar fields [21] as illustrated in Fig.1. The method resonantly produces a low-mass scalar field $\varphi$ in quasi-parallel photon-photon scattering and simultaneously stimulates its decay by combining two different frequency photon beams along the same optical axis and focusing them into a vacuum. As a consequence of the stimulated resonant scattering, a frequency-shifted photon can be generated as a clear signal of scattering. This provides a possibility to directly explore the microscopic nature relevant to DE through the photon-photon scattering experiment, despite that the coupling strength considered to be of order of (or even less than) the gravitational constant.

In this work, we study how to examine a class of chameleon models predicted in the $F(R)$ gravity based on the aforementioned photon-photon scattering process. For testing the characteristic feature of the chameleon mechanism, we establish a theoretical basis to interface with a searching experiment which can precisely scan the pressure value in an ultra-high vacuum chamber along the optical axis of the beams where the scattering process dominantly takes place.

Chameleon in the $F(R)$ gravity — Let us consider the chameleon coupling to the matter fields in the $F(R)$ gravity. $F(R)$ stands for a function of the Ricci scalar $R$, and the Einstein-Hilbert action of the general relativity is replaced by $F(R)$. By the Weyl transformation of the metric, defined as $\hat{g}_{\mu\nu} = e^{2\sqrt{g}R/6\kappa}\hat{g}_{\mu\nu}$, $\theta_R F(R) g_{\mu\nu}$, the $F(R)$ gravity turns into the general relativity coupling to an additional scalar field $\varphi$, the scalaron:

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \hat{R} - \frac{1}{2} \hat{g}^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) - V(\varphi) \right]$$
$$+ \int d^4x \sqrt{-\hat{g}} e^{-\frac{\kappa}{4} \varphi} \mathcal{L}_M \left[ e^{-\frac{\kappa}{4} \varphi} \hat{g}_{\mu\nu}, \Phi \right],$$

where $\mathcal{L}_M(g^{\mu\nu}, \Phi)$ is the Lagrangian of an arbitrary mat-
ter field $\Phi$, $\kappa^2 = 1/M_{pl}^2$, and $M_{pl}$ is the reduced Planck mass, $M_{pl} \approx 2.44 \times 10^{19}$ [eV]. The potential $V(\varphi)$ is written as a function of the Ricci scalar $R$,

$$V(\varphi) = \frac{1}{2\kappa^2} \left( \nabla^2 \varphi \right)^2 - F(R(\varphi)) \cdot F(R(\varphi))$$ \hspace{1cm} (2)

The equation of motion of the scalaron is given by

$$\ddot{\varphi} = V_{;\varphi}(\varphi) + \frac{\kappa}{\sqrt{6}} e^{-\frac{1}{4} \kappa \varphi} T^\mu_{\mu},$$ \hspace{1cm} (3)

where $\ddot{\varphi} = \nabla^\mu \nabla_{\mu}$, and $\nabla_{\mu}$ denotes the covariant derivative composed of $g_{\mu\nu}$. We conventionally define the effective potential of the scalaron field as follows:

$$V_{\text{eff}}(\varphi) \equiv V(\varphi) + \frac{1}{4} e^{-\frac{1}{4} \kappa \varphi} T^\mu_{\mu}.$$ \hspace{1cm} (4)

Because of the dilatonic coupling between $\varphi$ and $T^\mu_{\mu}$, the effective potential and the scalaron mass acquire the environment dependence, which drives the chameleon mechanism. Hereafter, we call scalaron the chameleon.

The chameleon can interact with the electromagnetic field, through the quantum trace anomaly, which generates the coupling to two photons. This coupling generation can be viewed through the anomalous Weyl transformation [23–26], and the derived interaction terms can be cast into the form:

$$\mathcal{L}_{\text{M}} = \frac{1}{4} e^{-\frac{1}{4} \kappa \varphi} T^\mu_{\mu} (\text{anomaly})$$

$$- \frac{1}{4} e^{-\frac{1}{4} \kappa \varphi} \left( b_{\text{em}} \cdot \alpha_{\text{em}} F_{\mu\nu} F^{\mu\nu} \right).$$ \hspace{1cm} (5)

Here $\alpha_{\text{em}}$ is the fine-structure constant of the electromagnetic coupling, and $b_{\text{em}}$ denotes a beta function coefficient which would include all the charged particle contributions at the loop level of the standard model of particle physics: $b_{\text{em}} = 11/3$ at the one-loop level \footnote{Here the $\varphi$ dependence in $T^\mu_{\mu}$ has been disregarded, which suffices for the present study on a static environmental effect (a residual gas pressure in the experimental chamber) constructed only from matter, external to the scalaron dynamics. A similar prescription has been applied in a different context in the literature [22].}

$$\mathcal{L}_{\varphi;\gamma} = \frac{1}{4} \beta_{\gamma} \varphi F_{\mu\nu} F^{\mu\nu}, \quad \beta_{\gamma} = \frac{b_{\text{em}} \cdot \alpha_{\text{em}}}{2\sqrt{6\pi \kappa}} M_{pl}.$$ \hspace{1cm} (6)

Furthermore, various kinds of gases in an experimental chamber may also contribute to the chameleon mechanism. Thus, we have mainly two ambient sources, photon and gas densities, in scanning the effective mass of the chameleon,

$$T^\mu_{\mu} = T^\mu_{\mu}(\text{photon}) + T^\mu_{\mu}(\text{gas}).$$ \hspace{1cm} (7)

In what follows, we evaluate those two ingredients in $T^\mu_{\mu}$ in light of the proposed experimental setup.

**Sources for the chameleon mechanism** – The photon contribution in Eq. (6) goes like $T^\mu_{\mu}(\text{photon}) \propto F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$, with the electric field $E$ and magnetic field $B$. This contribution vanishes in the case with the ordinary plane electromagnetic wave where $E$ and $B$ have the same amplitude, while in the case of the focused geometry in Fig.1, amplitudes of the electric and magnetic fields are given by nontrivial spatial distribution functions (as is explained in Supplemental Material \footnote{The contents in Supplemental Material are fully based on an unpublished doctoral thesis by Yuichi Monden, which was also the ground of Refs. [27, 28].}). It turns out, however, that square of the field-strength vanishes at each point on the focal plane in the case of the circularly polarized beams [29–31]. Therefore, with the circularly polarized beams as proposed in [21], we can ignore the contribution from the background photon density \footnote{In Ref. [18] the photon contribution was simply assumed to be the same order as the gas contribution.}.

Regarding the gas contribution, we assume the conventional perfect-fluid description because of the difficulty in treating the atoms or molecules by the fundamental field-prescription. Then the trace of the energy-momentum tensor is evaluated as $T^\mu_{\mu}(\text{gas}) = -(\rho - 3P)$, where $\rho$ and $P$ represent the density and pressure, respectively. The gas in the experimental chamber exhibits a low pressure ($P \ll \rho$) and a high temperature, hence we assume the equation of state for the ideal gas. In that case the trace of the energy-momentum tensor can be approximated as follows:

$$-T^\mu_{\mu}(\text{gas}) \approx \rho \left[ \text{kg/m}^3 \right] = \frac{P \left[ \text{Pa} \right]}{RT},$$ \hspace{1cm} (8)

where $T$ [K] is the temperature, and $R$ is the specific gas constant. $R$ is defined as the molar gas constant $R'$ divided by the molar mass of the gas $M$ [kg/mol], $R = R' / M$, where $R' = 8.31446$ [m$^3$ · Pa · K$^{-1}$ · mol$^{-1}$].

In the proposed setup we assume an ultra-high vacuum chamber to reduce background frequency-shifted photons caused by the residual atoms, and such a low-density environment also weakens the chameleon mechanism. However, because the gas density is controllable by the gas pressure as in Eq. (8) over several orders of magnitude, in principle, we can measure how the signal is weakened as a function of the gas pressure which is the characteristic feature of the chameleon mechanism.

**Estimation of chameleon mass** – Next, we study the chameleon mass under the background gas pressure surrounding the scattering point, equivalently, the gas density with a fixed volume in the experimental chamber.
In this paper, we adopt the following $F(R)$ gravity model:

$$F(R) = R - \beta R_c \left[ 1 - \left( \frac{R}{R_c} \right)^{-2n} \right] + \alpha R^2,$$

where $\alpha$, $\beta$, and $n$ are positive parameters. The coefficient $\beta R_c$ in the second term can be considered as the cosmological constant $\beta R_c = 2 \Lambda$, where $\Lambda \simeq 4.2 \times 10^{-66}$ [eV$^2$] [32]. One can find the second term in the Hu-Sawicki model [33] and Starobinsky model [34] with the large-curvature limit $R/R_c \gg 1$ taken. In the laboratory environment, the typical energy scale is higher than the DE scale. The third term $\alpha R^2$ is added to cure the singularity problem [35, 36].

The second derivative of the effective potential at the potential minimum gives the chameleon mass $m_{\varphi}$. Using Eq. (2), we can write the chameleon mass formula in terms of the $F(R)$ function and its derivatives:

$$m_{\varphi}^2 = V(\varphi_{\text{min}})_{\text{eff}, \varphi} = \frac{1}{3F_{RR}(R_{\text{min}})} \left( 1 - \frac{R_{\text{min}} F_{RR}(R_{\text{min}})}{F_R(R_{\text{min}})} \right),$$

where $R_{\text{min}}$ is determined by the stationary condition $V(\varphi_{\text{min}})_{\text{eff}, \varphi} = 0$, given by $2F(R_{\text{min}}) - R_{\text{min}} F_R(R_{\text{min}}) + \kappa^2 T_{\mu \nu}^\mu = 0$. In the limit where $R/R_c \gg 1$, the stationary condition gives $R_{\text{min}} \approx -\kappa^2 T_{\mu \nu}^\mu$ in the model (9). With the simplest choice of parameters $R_c = \Lambda$ and $\beta = 2$, we obtain the chameleon mass in the analytic form [37],

$$m_{\varphi}^2 = \frac{\Lambda}{12n(2n+1)} \left( -\frac{T_{\mu \nu}^\mu}{\rho_{\Lambda}} \right)^{-2(n+1)} + 6\alpha \Lambda,$$

where we have defined the DE density as $\rho_{\Lambda} \equiv \Lambda/\kappa^2$ and taken the limit $\Lambda \ll R \ll 1/\alpha$. Depending on $T_{\mu \nu}^\mu$, Eq. (11) shows two different behaviors. In a low-density environment, where $|T_{\mu \nu}^\mu|_{\rho_{\Lambda}} \ll \left[ \frac{2n(2n+1)}{\alpha \Lambda} \right]^{2(n+1)}$, we have

$$m_{\varphi}^2 \bigg|_{\text{low\,-\,density}} \approx \frac{\Lambda}{12n(2n+1)} \left( -\frac{T_{\mu \nu}^\mu}{\rho_{\Lambda}} \right)^{2(n+1)},$$

and in a high-density environment with the condition opposite to the above,

$$m_{\varphi} \bigg|_{\text{high\,-\,density}} \approx \sqrt{\frac{1}{6\alpha}}.$$

In Fig. 2, we plot the chameleon mass given in Eq. (11) as a function of $T_{\mu \nu}^\mu$.

Detection sensitivity versus predicted chameleon mass – We implement the low pressure in an interval $10^{-8}$ [Pa] < $P$ < $10^{-6}$ [Pa]. In such a ultra-high vacuum chamber, the residual gas mainly consists of hydrogen molecule, and Eq. (8) for the hydrogen at $T = 300$ [K] gives

$$1.4 \times 10^{12} \lesssim -\frac{T_{\mu \nu}^\mu}{\rho_{\Lambda}} \lesssim 1.4 \times 10^{14}.$$

Here, the observed DE density $\rho_{\Lambda} \sim 2.5 \times 10^{-11}$ [eV$^4$]. In using the other gas, one can multiply the ratio of molecular weight to that of hydrogen molecule with the above $(-\frac{T_{\mu \nu}^\mu}{\rho_{\Lambda}})$. As a benchmark, in Eq. (11) we may choose $\alpha = 10^4$ [eV$^{-2}$] as in Fig. 2, and take $n = 1$ to have the most modest power-law scaling of $m_{\varphi}$, see Fig. 2. The chameleon mass is then predicted in an interval as

$$6.6 \times 10^{-10} \text{ [eV]} \lesssim m_{\varphi} \lesssim 6.6 \times 10^{-6} \text{ [eV]}.$$

We plot the predicted mass-coupling parameter space versus the expected detection sensitivity in Fig. 3.

Feasibility to extract the chameleon character – The proposed method [21] depicted in Fig.1 is based on the following stimulated two-body photon-photon scattering process:

$$\langle p_c(p_1) \rangle + \langle p_c(p_2) \rangle \rightarrow p_3 + \langle p_c(p_4) \rangle$$

where $\langle (\ ) \rangle$ indicates that incident two photons with four-momenta, $p_1$ and $p_2$, stochastically annihilate from a broad-band coherent creation beam with its central four-momentum $p_c$ while $p_4$ is created to a broad-band co-propagating inducing coherent beam with the central four-momentum $p_c$. This process yields stimulated emission of a $p_3$ signal photon via induced decay of a produced resonance state through energy-momentum conservation with $p_4$. In order to introduce the co-moving...
inducing beam, technically, \( \langle p_e \rangle \) and \( \langle p_i \rangle \) beams are combined along a common optical axis and simultaneously focused into vacuum [39]. Thus this process looks as if a frequency-shifted photon is generated from vacuum by mixing two-color beams. Indeed, so-called four-wave mixing (FWM) via residual atoms produces kinematically similar photon energies to those of the signal via energy-momentum conservation [40]. The photon yield from the atomic FWM process is known to be proportional to square of the third order polarization susceptibility \( \chi^{(3)} \). Because \( \chi^{(3)} \) is proportional to number density of atoms, hence, to gas pressure, the atomic FWM yield is expected to have a quadratic pressure dependence and the dependence has been indeed observed in the actual searching setup in laser frequencies [41–43].

The FWM yield is proportional to the cubic of beam peak intensity, \( J/s/m^2 \), at a focal point. Based on the expected yield in the most recent search with pulsed lasers with several 10-fs duration and photon energies \( \sim 1 \) eV [43], the intensity scaling to much lower peak power beams with a few ns duration and lower photon energies \( \sim 10^{-5} \) eV assumed in [21], predicts a negligible amount of FWM photons from the atomic process if \( \chi^{(3)} \) values in the two photon energies are similar. We further note that only the possible standard model process in vacuum, light-by-light scattering via the QED box diagram, is also negligible in the mass range \( 0.1 - 1 \) \( \mu \text{eV} \) due to the sixth-power dependence of the cross section on the center-of-mass system energy even if we take the effect of stimulation into account [44].

The key feature of the chameleon exchange appears in its density dependence of the stimulated resonant scattering rate. Figure 4 shows the expected number of signal photons as a function of gas pressure comprising hydrogen molecules at 300 K for the \( n = 1 \) case. The signal yield is calculated by assuming photon-chameleon coupling defined in Eq.(6) and the set of experimental parameters used in [21] with the same pulse energy of 100 J for the two beams. The dashed horizontal line shows unity above which experiments can test the characteristic feature of chameleon models. To evaluate the dependence for other types of molecules with Molar mass \( M \), one can scale the pressure values by multiplying \( M_{H_2}/M \) with the Molar mass of hydrogen molecule \( M_{H_2} \).

**Conclusion and discussion** — We have evaluated the feasibility to search for the chameleon field based on the stimulated radar collider concept by focusing on the gas pressure dependence of the chameleon mass. If two 100 J radar beams are available as designed in [21], the specific chameleon model provided in this paper with \( \beta = 2 \) and \( n = 1 \) is testable based on the steep pressure dependence with the 10th-power for the signal yield which is clearly separable from the quadratic pressure dependence of the background atomic FWM process. Therefore, in principle, the proposed method can provide a firm ground to test the chameleon mechanism in laboratory.
and astrophysical observations. Thus, we need further analysis on detectable ranges of parameters in the various models of $f(R)$ gravity. The theoretical interfaces established in our current analysis would be useful to reinterpret the existing constraints from the other experiments based on the chameleon mechanism and also allow to evaluate non-vanishing photon density contributions to the chameleon mechanism for different photon-beam polarization states from the circular one.

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SUPPLEMENT 1: ELECTROMAGNETIC FIELD IN A FOCUSED PHOTON BEAM

Setup for Focusing Optics

We consider the electric field \( \mathbf{E} \) and magnetic field \( \mathbf{B} \) in the focusing optics [29–31]. The wave equation in vacuum goes like

\[
\Box \mathbf{F}(r, t) = 0. \tag{17}
\]

Its solution \( \mathbf{F}(r, t) \) is expressed in terms of the complex amplitude as

\[
\mathbf{F}(r, t) = \frac{1}{2} \hat{\mathbf{F}}(r)e^{-i\omega t} + \text{c.c.}. \tag{18}
\]

Then the amplitude \( \hat{\mathbf{F}}(r) \) satisfies the Helmholtz equation,

\[
(\nabla^2 + k^2) \hat{\mathbf{F}}(r) = 0, \tag{19}
\]

where \( k = \omega/c \) is the wave number \( k = |k| \) with \( k = (k_x, k_y, k_z) \). Because \( k_z = \sqrt{k^2 - k_x^2 - k_y^2} \) is determined for a given wave number, the Fourier expansion of \( \hat{\mathbf{F}}(r) \) goes like

\[
\hat{\mathbf{F}}(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y f(k_x, k_y, z) e^{ik_x x + ik_y y} \tag{20}
\]

By substituting Eq. (20) into Eq. (19), the Fourier kernel function \( f(k_x, k_y, z) \) is written as

\[
f(k_x, k_y, z) = \mathbf{U}(k_x, k_y) e^{ik_z z} + \mathbf{V}(k_x, k_y) e^{-ik_z z}, \tag{21}
\]

so that

\[
\hat{\mathbf{F}}(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \left[ \mathbf{U}(k_x, k_y) e^{ik_z z} + \mathbf{V}(k_x, k_y) e^{-ik_z z} \right] e^{ik_x x + ik_y y}. \tag{22}
\]

Next, we consider the case of focusing the photon beam. See Fig. 5. Writing the electromagnetic field of the focused photon beam with the frequency \( \omega \) as

\[
\mathbf{E}(r, t) = \frac{1}{2} \hat{\mathbf{E}}(r)e^{-i\omega t} + \text{c.c.},
\]

\[
\mathbf{B}(r, t) = \frac{1}{2} \hat{\mathbf{B}}(r)e^{-i\omega t} + \text{c.c.}, \tag{23}
\]

we pay attention to the complex amplitudes \( \hat{\mathbf{E}}(r) \) and \( \hat{\mathbf{B}}(r) \). Equation (22) allows us to express the amplitude of the electric field \( \mathbf{E}(r) \) as

\[
\hat{\mathbf{E}}(r) = \int_{-\infty}^{\infty} ds_x ds_y \left[ \mathbf{U}(s_x, s_y) e^{iks_x z} + \mathbf{V}(s_x, s_y) e^{-iks_x z} \right] e^{iks_x x +iks_y y}, \tag{24}
\]

where \( s_z = \sqrt{1 - s_x^2 - s_y^2} \). Considering the case where \( s_x^2 + s_y^2 \leq 1 \), we find

\[
\hat{\mathbf{E}}(r) = \int_{s_x^2 + s_y^2 \leq 1} ds_x ds_y \left[ \mathbf{U}(s_x, s_y) e^{iks_x z} + \mathbf{V}(s_x, s_y) e^{-iks_x z} \right] e^{iks_x x +iks_y y}. \tag{25}
\]

By introducing the variables,

\[
u_x = \frac{x}{f}, \quad u_y = \frac{y}{f}, u_z = \frac{z}{f} = -\sqrt{1 - u_x^2 - u_y^2}, \tag{26}
\]

\( \hat{\mathbf{E}}(r) \) at a point \( r_1 = (x_1, y_1, z_1) \) on the sphere \( S_1 \) is evaluated as

\[
\hat{\mathbf{E}}(r_1) = \int_{s_x^2 + s_y^2 \leq 1} ds_x ds_y \left[ \mathbf{U}(s_x, s_y) e^{ikf \Phi_+(s_x, s_y)} + \mathbf{V}(s_x, s_y) e^{-ikf \Phi_-(s_x, s_y)} \right], \tag{27}
\]

where \( f \) stands for the focal distance, and

\[
\Phi_\pm(s_x, s_y) = s_x x + s_y y \pm \sqrt{1 - s_x^2 - s_y^2} u_z. \tag{28}
\]

Assuming the focal distance is larger than the wave length \( f k \gg 1 \), we apply the stationary phase method in evaluating the integration in Eq. (27). We first consider
the first term in Eq. (27), to find that a stationary point $(s_{x_0}, s_{y_0})$ satisfies
\[ \frac{\partial \Phi_+ (s_x, s_y)}{\partial s_x} = u_x - \frac{s_x}{\sqrt{1 - s_x^2 - s_y^2}} u_z = 0, \]
\[ \frac{\partial \Phi_+ (s_x, s_y)}{\partial s_y} = u_y - \frac{s_y}{\sqrt{1 - s_x^2 - s_y^2}} u_z = 0. \]  
(29)

Because $s_{x_0} < 0$, one finds
\[ s_{x_0} = -u_x, \quad s_{y_0} = -u_y. \]  
(30)

The Taylor expansion of $\Phi_+ (s_x, s_y)$ around the stationary point $(s_{x_0}, s_{y_0})$ gives
\[
\Phi_+ (s_x, s_y) = \Phi_+ (s_{x_0}, s_{y_0}) + \frac{w_1}{2} (s_x - s_{x_0})^2 \\
+ \frac{w_2}{2} (s_y - s_{y_0})^2 + w_3 (s_x - s_{x_0}) (s_y - s_{y_0}) + \cdots, \]
(31)

where we have defined
\[
w_1 = -\frac{1 - u_x^2}{(1 - s_x^2 - s_y^2)^{3/2}} u_z, \\
w_2 = -\frac{1 - u_y^2}{(1 - s_x^2 - s_y^2)^{3/2}} u_z, \\
w_3 = -\frac{u_x u_y}{(1 - s_x^2 - s_y^2)^{3/2}} u_z. \]  
(32)

with $u_z < 0$, $w_1 > 0$, and $w_2 > 0$.

When $kf \gg 1$, the integrand in the vicinity of the stationary point $(s_{x_0}, s_{y_0})$ mainly contributes to the integration, and thus, we find
\[
\mathbf{W}_1 (r_1) \approx \int_{s^2 + s'^2 \leq 1} ds_x ds_y \mathbf{U} (s_x, s_y) e^{ikf \Phi_+ (s_x, s_y)} \\
\approx \mathbf{U} (s_{x_0}, s_{y_0}) e^{ikf \Phi_+ (s_{x_0}, s_{y_0})} \int_{s^2 + s'^2 \leq 1} ds_x ds_y \\
e^{ikf \left[ \frac{w_1^2}{2} (s_x - s_{x_0})^2 + \frac{w_2^2}{2} (s_y - s_{y_0})^2 + w_3 (s_x - s_{x_0}) (s_y - s_{y_0}) \right]}, \]
(33)
The infinitesimal region $(s_x - s_{x_0})^2 + (s_y - s_{y_0})^2 \leq \epsilon \ll 1$ dominates over the target integration domain $s_x^2 + s_y^2 \leq 1$. Defining $\epsilon_x \equiv s_x - s_{x_0}$ and $\epsilon_y \equiv s_y - s_{y_0}$, we find
\[
\mathbf{W}_1 (r_1) \approx \mathbf{U} (s_{x_0}, s_{y_0}) e^{ikf \Phi_+ (s_{x_0}, s_{y_0})} \\
\times \int_{\epsilon_x^2 + \epsilon_y^2 \leq \epsilon} d\epsilon_x d\epsilon_y e^{ikf \left[ \frac{w_1^2}{2} \epsilon_x^2 + \frac{w_2^2}{2} \epsilon_y^2 + w_3 \epsilon_x \epsilon_y \right]} \\
= \mathbf{U} (s_{x_0}, s_{y_0}) e^{ikf \Phi_+ (s_{x_0}, s_{y_0})} \\
\approx \int_{-\epsilon}^{\epsilon} d\epsilon_y \int_{-\sqrt{\epsilon^2 - \epsilon_y^2}}^{\sqrt{\epsilon^2 - \epsilon_y^2}} d\epsilon_x e^{ikf \left[ \frac{w_1^2}{2} \epsilon_x^2 + \frac{w_2^2}{2} \epsilon_y^2 + w_3 \epsilon_x \epsilon_y \right]}.
\]  
(34)

Performing change of variables, $X \equiv \sqrt{kf} \epsilon_x$ and $Y \equiv \sqrt{kf} \epsilon_y$, the integration can be approximated as
\[
\mathbf{W}_1 (r_1) \approx \mathbf{U} (s_{x_0}, s_{y_0}) e^{ikf \Phi_+ (s_{x_0}, s_{y_0})} \\
\times \frac{1}{kf} \int_{-\infty}^\infty dY \int_{-\infty}^\infty dX e^{ikf \left[ \frac{w_1^2}{2} X^2 + \frac{w_2^2}{2} Y^2 + w_3 XY \right]}.
\]  
(35)

Noting
\[
\int_{-\infty}^\infty dY \int_{-\infty}^\infty dX e^{ikf \left[ \frac{w_1^2}{2} X^2 + \frac{w_2^2}{2} Y^2 + w_3 XY \right]} = -2\pi i u_x, \]
(36)
and using $\Phi_+ (s_{x_0}, s_{y_0}) = -1$, we find
\[
\mathbf{W}_1 (r_1) \approx -\frac{2\pi i u_x}{kf} U (-u_x, -u_y) e^{-ikf}. \]
(37)

In the same way, we can compute the second term in Eq. (27). Thus we find
\[
\hat{\mathbf{E}} (r_1) \approx \frac{2\pi i u_x}{kf} \left[ -U (-u_x, -u_y) e^{-ikf} + \mathbf{V} (u_x, u_y) e^{ikf} \right]. \]
(38)

If there is no aberration in the focusing optics so that
\[
s_x = -\frac{x_1}{f} = -u_x, \]
\[
s_y = -\frac{y_1}{f} = -u_y, \]
\[
s_x = -\frac{z_1}{f} = -u_z, \]
(39)
then we have
\[
\hat{\mathbf{E}} (r_1) \approx \frac{2\pi i s_x}{kf} \left[ -U (s_x, s_y) e^{-ikf} + \mathbf{V} (-s_x, -s_y) e^{ikf} \right]. \]
(40)

Next, we compute the coefficients $U (s_x, s_y)$ and $\mathbf{V} (-s_x, -s_y)$ in Eq. (40). Because the amplitude of the spherical wave is proportional to the inverse of distance, we can express $\hat{\mathbf{E}} (r_1)$ as
\[
\hat{\mathbf{E}} (r_1) = \frac{a_1 (s_x, s_y)}{f} e^{ik \phi (r_1)}. \]
(41)

Here, $a_1 (s_x, s_y)$ is a vector perpendicular to the light ray from the sphere $S_1$ to the focus $O$, and $\phi (r_1)$ is the eikonal function to describe the optical path length from a point $O'$ to $r_1$ in the object space. Assuming the optical path length of the spherical wave with its source at point $O'$ which converges at the focus $O$ through the point $r_1$ is expressed by the constant $C$ to be
\[
\phi (r_1) = C - |r_1| = C - f. \]
(42)
Thus, we find the relation between $a_1(s_x, s_y)$ and the coefficients $U(s_x, s_y)$, $V(-s_x, -s_y)$ as follows

$$U(s_x, s_y) = -\frac{ik}{2\pi} \frac{a_1(s_x, s_y)}{s_z} e^{ikC} \tag{43}$$

$$V(s_x, s_y) = 0.$$ 

Ignoring the constant factor $e^{ikC}$ and substituting the above into Eq. (25), we obtain the amplitude of the electric field,

$$\hat{E}(r) = -\frac{ik}{2\pi} \int \int_{s_x^2 + s_y^2 < 1} ds_x ds_y \frac{a_1(s_x, s_y)}{s_z} e^{ik\mathbf{s} \cdot \mathbf{r}}, \tag{44}$$

and, in a similar way, that of the magnetic field goes like

$$\hat{B}(r) = -\frac{ik}{2\pi} \int \int_{s_x^2 + s_y^2 < 1} ds_x ds_y \frac{a_2(s_x, s_y)}{s_z} e^{ik\mathbf{s} \cdot \mathbf{r}}. \tag{45}$$

Finally, we rewrite $a_1(s_x, s_y)$ and $a_2(s_x, s_y)$ in terms of angles in the polar coordinate. Considering $e_0(\theta, \varphi)$ as the electric field (with the phase removed) of the incident light ray on the plane $S_0$ and $e_1(\theta, \varphi)$ as that on the sphere $S_1$, we write $\hat{e}_0(\varphi)$ and $\hat{e}_1(\theta, \varphi)$ as unit vectors parallel to $e_0(\theta, \varphi)$ and $e_1(\theta, \varphi)$, respectively. One then finds

$$e_1(\theta, \varphi) = \frac{a_1(\theta, \varphi)}{f}. \tag{46}$$

Let $\delta S_0$ be the infinitesimal circular on $S_0$, and $\delta S_1$ be the area of the projection of $\delta S_0$ onto $S_1$. Then the following relation holds between $\delta S_0$ and $\delta S_1$:

$$\delta S_0 = \delta S_1 \cos \theta. \tag{47}$$

Using the energy conservation law

$$|e_0(\theta, \varphi)|^2 \delta S_0 = |e_1(\theta, \varphi)|^2 \delta S_1,$$

we obtain

$$e_1(\theta, \varphi) = |e_0(\theta, \varphi)| \cos \frac{1}{2} \theta \hat{e}_1(\theta, \varphi). \tag{49}$$

Let the two vectors $g_0(\varphi)$ and $g_1(\theta, \varphi)$ be unit vectors, that are perpendicular to the incident light ray and the focused light ray, respectively. They lie on a plane including both the light ray and the optical axis (see Fig. 6). $g_0(\varphi)$ and $g_1(\theta, \varphi)$ are given as

$$g_0(\varphi) = \cos \varphi \hat{x} + \sin \varphi \hat{y}$$

$$g_1(\theta, \varphi) = \cos \theta \cos \varphi \hat{x} + \cos \theta \sin \varphi \hat{y} + \sin \theta \hat{z} \tag{50}$$

$$s(\theta, \varphi) = -\sin \theta \cos \varphi \hat{x} - \sin \theta \sin \varphi \hat{y} + \cos \theta \hat{z},$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are $x$, $y$, $z$-direction unit vectors. By definition, we have

$$g_0(\varphi) \times \hat{z} = g_1(\theta, \varphi) \times s(\theta, \varphi)$$

$$= \sin \varphi \hat{x} - \cos \varphi \hat{y}. \tag{51}$$

Because $\hat{e}_0(\varphi)$ is perpendicular to the $z$-axis, we can decompose it as

$$\hat{e}_0(\varphi) = [\hat{e}_0(\varphi) \cdot g_0(\varphi)] g_0(\varphi)$$

$$+ [\hat{e}_0(\varphi) \cdot (g_0(\varphi) \times \hat{z})] (g_0(\varphi) \times \hat{z}). \tag{52}$$

When $e_0(\theta, \varphi)$ is converted into $e_1(\theta, \varphi)$, the $g_0(\varphi)$-direction component is transformed into the $g_1(\varphi)$-direction component, and the $g_0(\varphi) \times \hat{z}$-direction component is done into the $g_1(\theta, \varphi) \times s(\theta, \varphi)$-direction component. Thus,

$$e_1(\theta, \varphi)$$

$$= |e_0(\theta, \varphi)| \cos \frac{1}{2} \theta \left\{ [\hat{e}_0(\varphi) \cdot g_0(\varphi)] g_1(\varphi) \right.$$

$$+ \left[ \hat{e}_0(\varphi) \cdot (g_0(\varphi) \times \hat{z}) \right] (g_1(\theta, \varphi) \times s(\theta, \varphi)) \right\}. \tag{53}$$

In the focusing optics without aberration, we can express $e_0(\theta, \varphi)$ as

$$e_0(\theta, \varphi) = e_0 l_0(\theta) \hat{e}_0(\varphi), \tag{54}$$

where $e_0$ is the maximum value of $|e_0(\theta, \varphi)|$, and $l_0(\theta)$ is the relative amplitude. Therefore, by using Eq. (46), we
obtain
\[ a_1(\theta, \varphi) = \frac{f e_0 l_0(\theta) \cos \frac{\theta}{2}}{2} \left\{ [\hat{e}_0(\varphi) \cdot \mathbf{g}_0(\varphi)] \mathbf{g}_1(\theta, \varphi) + [\hat{e}_0(\varphi) \cdot (\mathbf{g}_0(\varphi) \times \hat{z})] (\mathbf{g}_1(\theta, \varphi) \times \mathbf{s}(\theta, \varphi)) \right\} . \] (55)

The relation between the electric and magnetic fields leads to
\[ a_2(\theta, \varphi) = \mathbf{s}(\theta, \varphi) \times a_1(\theta, \varphi) = \frac{f e_0 l_0(\theta) \cos \frac{\theta}{2}}{2} \left\{ [\hat{e}_0(\varphi) \cdot (\mathbf{g}_0(\varphi) \times \hat{z})] \mathbf{g}_1(\theta, \varphi) - [\hat{e}_0(\varphi) \cdot \mathbf{g}_0(\varphi)] (\mathbf{g}_1(\theta, \varphi) \times \mathbf{s}(\theta, \varphi)) \right\} . \] (56)

In the cylindrical coordinate \((\rho, \varphi, z)\), \(\mathbf{r} = \rho \cos \varphi \hat{x} + \rho \sin \varphi \hat{y} + z \hat{z}\), and Eq. (50) tells us
\[ \mathbf{s}(\theta', \varphi') \cdot \mathbf{r} = z \cos \theta' - \rho \sin \theta' \cos (\varphi' - \varphi) . \] (57)

The Jacobian goes like
\[ \frac{\partial (s_x, s_y)}{\partial (\theta', \varphi')} = \sin \theta \cos \theta' , \quad ds_x ds_y = s_z \sin \theta' d\theta' d\varphi' . \] (58)

Thus Eqs. (44) and (45) are written in the polar-coordinate as follows:
\[ \hat{\mathbf{E}}(\mathbf{r}) = -\frac{ik}{2\pi} \int_0^{\theta_f} d\theta' \int_0^{2\pi} d\varphi' a_1(\theta', \varphi') \sin \theta' \left( e^{ik[z \cos \theta' - \rho \sin \theta' \cos (\varphi' - \varphi)]} \right) , \] (59)
\[ \hat{\mathbf{B}}(\mathbf{r}) = -\frac{ik}{2\pi} \int_0^{\theta_f} d\theta' \int_0^{2\pi} d\varphi' a_2(\theta', \varphi') \sin \theta' \left( e^{ik[z \cos \theta' - \rho \sin \theta' \cos (\varphi' - \varphi)]} \right) . \] (60)

**Linear Polarization**

When the incident photon beam is linearly polarized, Eq. (54) goes like
\[ \hat{\mathbf{e}}_0(\theta, \varphi) = e_0 l_q(\theta) \hat{x} , \quad \hat{\mathbf{e}}_0(\varphi) = \hat{x} , \] (61)
where \(l_q(\theta)\) expresses the Gaussian distribution,
\[ l_q(\theta) = \exp \left[ -\left( \frac{\sin \theta}{\sin \frac{\theta_f}{2}} \right)^2 \right] . \] (62)

Using
\[ \hat{\mathbf{e}}_0(\varphi) \cdot \mathbf{g}_0(\varphi) = \cos \varphi , \]
\[ \hat{\mathbf{e}}_0(\varphi) \cdot (\mathbf{g}_0(\varphi) \times \hat{z}) = \sin \varphi , \] (63)
we can derive each component of \(a_1(\theta, \varphi)\) and \(a_2(\theta, \varphi)\) in the Cartesian coordinate as follows:
\[ a_1(\theta, \varphi) = \frac{1}{2} f e_0 l_q(\theta) \cos \frac{\theta}{2} \left\{ (1 + \cos \theta) - (1 - \cos \theta) \cos \varphi \right\} \hat{x} + (\cos \theta - 1) \sin 2\varphi \hat{y} + 2 \sin \theta \cos \varphi \hat{z} , \] (64)
\[ a_2(\theta, \varphi) = \frac{1}{2} f e_0 l_q(\theta) \cos \frac{\theta}{2} \left\{ (\cos \theta - 1) \sin 2\varphi \hat{x} + [(1 + \cos \theta) + (1 - \cos \theta) \cos 2\varphi] \hat{y} + 2 \sin \theta \cos \varphi \hat{z} \right\} . \] (65)

Substituting the above into Eqs. (59) and (60), we finally obtain the amplitudes of the electric field and magnetic field as follows:
\[ \hat{E}_x(\rho, \varphi, z) = -i A [I_0(\rho, z) + \cos 2\varphi I_2(\rho, z)] , \]
\[ \hat{E}_y(\rho, \varphi, z) = -i A \sin 2\varphi I_2(\rho, z) , \]
\[ \hat{E}_z(\rho, \varphi, z) = -2A \cos \varphi I_1(\rho, z) , \]
\[ \hat{B}_x(\rho, \varphi, z) = -i A \sin 2\varphi I_2(\rho, z) , \]
\[ \hat{B}_y(\rho, \varphi, z) = -i A [I_0(\rho, z) - \cos 2\varphi I_2(\rho, z)] , \]
\[ \hat{B}_z(\rho, \varphi, z) = -2A \cos \varphi I_1(\rho, z) , \]
where \(A = k f e_0/2\), and
\[ I_0(\rho, z) = \int_0^{\theta_f} d\theta' l_q(\theta') \cos \frac{\theta'}{2} \sin \theta' \left\{ (1 + \cos \theta') J_0(k \rho \sin \theta') e^{ikz \cos \theta'} \right\} , \]
\[ I_1(\rho, z) = \int_0^{\theta_f} d\theta' l_q(\theta') \cos \frac{\theta'}{2} \sin^2 \theta' \left\{ J_1(k \rho \sin \theta') e^{ikz \cos \theta'} \right\} , \]
\[ I_2(\rho, z) = \int_0^{\theta_f} d\theta' l_q(\theta') \cos \frac{\theta'}{2} \sin \theta' \left\{ (1 - \cos \theta') J_2(k \rho \sin \theta') e^{ikz \cos \theta'} \right\} , \] (66)
where \(J_n(x) (n = 0, 1, 2)\) is Bessel function of the first kind.

**Circular Polarization**

When the incident photon beam is circularly polarized, Eq. (54) goes like
\[ \hat{\mathbf{e}}_0(\theta, \varphi) = e_0 l_q(\theta) (\cos \varphi \hat{\rho} - \sin \varphi \hat{\varphi}) , \]
\[ \hat{\mathbf{e}}_0(\varphi) = (\cos \varphi \hat{\rho} - \sin \varphi \hat{\varphi}) , \quad \varphi = \frac{\pi}{4} , \] (68)
where \( l_{bg}(\theta) \) expresses the Bessel-Gaussian distribution,

\[
l_{bg}(\theta) = J_1 \left( \frac{\sin \theta}{\sin \frac{\theta}{2}} \right) \exp \left[ - \left( \frac{\sin \theta}{\sin \frac{\theta}{2}} \right)^2 \right]. \tag{69}
\]

Here, one finds

\[
\hat{\rho} = \cos \varphi \hat{x} + \sin \varphi \hat{y}, \quad \hat{\phi} = -\sin \varphi \hat{x} + \cos \varphi \hat{y}. \tag{70}
\]

In a way similar to the case of the linear polarization we thus get

\[
\hat{E}_x(\rho, \varphi, z) = -2A \left[ \cos \varphi_0 \cos \varphi U_0(\rho, z),
+ \sin \varphi_0 \sin \varphi U_2(\rho, z) \right],
\]

\[
\hat{E}_y(\rho, \varphi, z) = -2A \left[ \cos \varphi_0 \sin \varphi U_0(\rho, z),
- \sin \varphi_0 \cos \varphi U_2(\rho, z) \right],
\]

\[
\hat{E}_z(\rho, \varphi, z) = -2iA \cos \varphi_0 U_1(\rho, z),
\]

\[
\hat{B}_x(\rho, \varphi, z) = -2A \left[ \sin \varphi_0 \cos \varphi U_0(\rho, z),
- \cos \varphi_0 \sin \varphi U_2(\rho, z) \right],
\]

\[
\hat{B}_y(\rho, \varphi, z) = -2A \left[ \sin \varphi_0 \sin \varphi U_0(\rho, z),
+ \cos \varphi_0 \cos \varphi U_2(\rho, z) \right],
\]

\[
\hat{B}_z(\rho, \varphi, z) = -2iA \sin \varphi_0 U_1(\rho, z),
\]

where

\[
U_0(\rho, z) = \int_0^{\frac{\theta_f}{2}} d\theta' l_{bg}(\theta') \cos \frac{\theta'}{2} \theta' \sin \theta',
\]

\[
J_1(k\rho \sin \theta') e^{ikz \cos \theta'},
\]

\[
U_1(\rho, z) = \int_0^{\frac{\theta_f}{2}} d\theta' l_{bg}(\theta') \cos \frac{\theta'}{2} \theta' \sin^2 \theta',
\]

\[
J_0(k\rho \sin \theta') e^{ikz \cos \theta'},
\]

\[
U_2(\rho, z) = \int_0^{\frac{\theta_f}{2}} d\theta' l_{bg}(\theta') \cos \frac{\theta'}{2} \theta' \sin \theta',
\]

\[
J_1(k\rho \sin \theta') e^{ikz \cos \theta'}. \tag{72}
\]

Amplitude distribution

We compute the amplitudes of electric field \( E \) and magnetic field \( B \) and evaluate the field-strength squared \( F_{\mu\nu} F^{\mu\nu} \). In the following, we focus only on the root mean square of each quantity. That is,

\[
\sqrt{\langle |E(\rho, \varphi, z)|^2 \rangle} = \frac{1}{\sqrt{2}} \sqrt{\langle \mathbf{E}(\mathbf{r}) \rangle^2}, \tag{73}
\]

\[
\sqrt{\langle |B(\rho, \varphi, z)|^2 \rangle} = \frac{1}{\sqrt{2}} \sqrt{\langle \mathbf{B}(\mathbf{r}) \rangle^2}, \tag{74}
\]

\[
\sqrt{\langle |F_{\mu\nu} F^{\mu\nu}(\rho, \varphi, z)|^2 \rangle} = \frac{2}{\sqrt{2}} \sqrt{\langle \mathbf{B}^2(\mathbf{r}) - \mathbf{E}^2(\mathbf{r}) \rangle^2}. \tag{75}
\]

Based on the experimental setup, we utilize the set of parameters in Table I.

From the geometrical parameters, we compute the aperture angle \( \theta_f \) (see also Fig. 5),

\[
\theta_f = 2 \arctan \left( \frac{\text{beam radius [m]}}{\text{focal length [m]}} \right). \tag{76}
\]

Hereafter, we use the parameters of creation beam in evaluating the amplitudes of the electric and magnetic fields. Furthermore, we normalize the amplitude of the electric field for the incident beam \( c_0 = 1 \), thus \( A = k f / 2 \), to see how the electric and magnetic fields are amplified in the focused beam.

First, we consider the case of focusing the linearly polarized beam. We plot the amplitude distribution of the electric field, magnetic field, and field-strength squared on the focal plane \( (z = 0) \) in Figs 7, 8, 9. Note that \( e^{ikz \cos \theta'} \) in Eq. (67) vanishes on the focal plane. In each plot, the \( (x, y) \) coordinate is in the unit of the wavelength \( \lambda \) because the \( \rho(x, y) \)-dependence originates from \( J_n(k\rho \sin \theta') \) and \( k\rho = 2\pi\rho/\lambda \). One finds that the amplitude distributions of the electric and magnetic fields are elliptical for the linearly polarized beam. The plot of
FIG. 8. The same as Fig. 7, but for the magnetic field in linearly polarized beam.

FIG. 9. The same as Figs. 7 and 8, but for the field-strength squared in linearly polarized beam.

Next, we consider the case of focusing the circularly polarized beam. We plot the amplitude distribution of the electric field and magnetic field in Figs 10, 11. Equation (72) shows that \( U_0 \) and \( U_2 \) vanishes when \( \rho \to 0 \), and only \( \hat{E}_z \) and \( \hat{B}_z \) on \( z \)-axis \( (\rho = 0) \) contributes to the total amplitudes. From Eq. (71), one finds that the electric and magnetic fields have the same amplitude at each point on the focal plane for the circularly polarized beam \( (\varphi_0 = \pi/4) \). Therefore, the field-strength square vanishes on the focal plane for the circularly polarized beam.

FIG. 10. The root mean square of the electric field in circularly polarized beam.

FIG. 11. The root mean square of the magnetic field in circularly polarized beam.

SUPPLEMENT 2: EVALUATION OF GAS IN CHAMBER

The specific gas constants for typical residual gases are shown in Table II. As an illustration, we consider the dry air. For the room temperature \( T = 300 \text{ [K]} \) and the

| Gas                  | Molar mass [kg/mol] | \( R \) |
|----------------------|---------------------|-------|
| Hydrogen (H\(_2\))  | \( 2.0159 \times 10^{-3} \) | 4124  |
| Water vapor (H\(_2\)O) | \( 18.015 \times 10^{-3} \) | 461.5 |
| Nitrogen (N\(_2\))  | \( 28.013 \times 10^{-3} \) | 296.8 |
| Carbon Monoxide (CO) | \( 28.010 \times 10^{-3} \) | 296.8 |
| Dry air (mixture)    | \( 28.965 \times 10^{-3} \) | 287.1 |
| Carbon Dioxide (CO\(_2\)) | \( 44.010 \times 10^{-3} \) | 188.9 |

TABLE II. Specific gas constant for several gas species.
pressure $P = 10^{-7}$ [Pa], we find $\rho = 1.16 \times 10^{-12}$ [kg/m$^3$]. Note that 1 [Pa] = $4.80 \times 10^{-2}$ [eV$^4$] and 1 [kg/m$^3$] = $4.31 \times 10^{15}$ [eV$^4$], and thus, $P = 4.80 \times 10^{-9}$ [eV$^4$] and $\rho = 5.00 \times 10^3$ [eV$^4$]. Then, we can ignore the pressure in evaluating the trace of the energy-momentum tensor, thus, $T_{\mu(\text{gas})} = -\rho$. 