ESTIMATING MAXIMUM BY MOMENTS
FOR FUNCTIONS ON ORBITS

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Abstract. Let $G$ be a compact group acting in a real vector space $V$. We obtain a number of inequalities relating the $L^\infty$ norm of a matrix element of the representation of $G$ with its $L^p$ norm for $p < \infty$. We apply our results to obtain approximation algorithms to find the maximum absolute value of a given multivariate polynomial over the unit sphere (in which case $G$ is the orthogonal group) and for the multidimensional assignment problem, a hard problem of combinatorial optimization (in which case $G$ is the symmetric group).

Introduction

A general problem of optimization has to do with finding the maximum (minimum) value of a real valued function $f : X \to \mathbb{R}$. Often, the set $X$ is endowed with a probability measure $\mu$ and the function $f$ possesses a certain degree of symmetry which allows one to compute the $k$-th moment $\int_X f^k \, d\mu$ efficiently at least for small values of $k$. Thus one may ask how well the $k$-th moment approximates the maximum value. In this paper, we describe a fairly general situation where some simple and meaningful relations between the maximum and moments can be obtained. We provide two illustrations: one, continuous, has to do with optimization of multivariate polynomials on the unit sphere with possible applications to solving systems of real polynomial equations and the other, discrete, deals with optimization on the symmetric group, namely, with the multidimensional assignment problem, a hard problem of combinatorial optimization.

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(1.1) **The general setting.** Let \( G \) be a compact group with the Haar probability measure \( dg \) acting in a finite-dimensional real vector space \( V \). To avoid dealing with various technical details, we assume that the representation \( G \rightarrow GL(V) \) is continuous, where the general linear group \( GL(V) \) is considered in its standard topology.

Let us choose a vector \( v \in V \) and a linear function \( \ell : V \rightarrow \mathbb{R} \). We consider the orbit \( \{ gv : g \in G \} \) of \( v \) and the resulting function \( f : G \rightarrow \mathbb{R} \) defined by

\[
f(g) = \ell(gv).
\]

In other words, \( f \) is a matrix element in the representation of \( G \). We are interested in the relation between the following quantities:

- **The \( L^\infty \) norm of \( f \):**
  \[
  \| f \|_\infty = \max_{g \in G} |f(g)| = \max_{g \in G} |\ell(gv)|.
  \]

- **The \( L^{2k} \) norm of \( f \) for a positive integer \( k \):**
  \[
  \| f \|_{2k} = \left( \int_G f^{2k}(g) \, dg \right)^{\frac{1}{2k}} = \left( \int_G (\ell^{2k}(gv)) \, dg \right)^{\frac{1}{2k}}.
  \]

As we remarked earlier, for many examples in computational mathematics, the quantity \( \| f \|_\infty \) is of considerable interest and is hard to compute whereas \( \| f \|_{2k} \) is relatively easy to compute for moderate values of \( k \). First, we relate \( \| f \|_\infty \) and \( \| f \|_2 \).

(1.2) **Theorem.** Let \( G \) be a compact group acting in a finite-dimensional real vector space \( V \) and let \( dg \) be the Haar probability measure on \( G \). Let us fix a vector \( v \) and a linear function \( \ell : V \rightarrow \mathbb{R} \) and let us define a real-valued function \( f : G \rightarrow \mathbb{R} \) by \( f(g) = \ell(gv) \). Then

\[
\| f \|_2 \leq \| f \|_\infty \leq \sqrt{\dim V} \cdot \| f \|_2.
\]

The bounds of Theorem 1.2 are generally sharp, see Remark 2.4. To estimate how well \( \| f \|_{2k} \) approximates \( \| f \|_\infty \) for a larger \( k \), we invoke a general construction from the representation theory, see for example, Lecture 6 of [4].

(1.3) **Tensor power.** For a positive integer \( k \), let

\[
V \otimes^k = \underbrace{V \otimes \ldots \otimes V}_{k \text{ times}}
\]

be the \( k \)-th tensor power of \( V \). There is a natural action of \( G \) in \( V \otimes^k \), defined on decomposable tensors by

\[
g(v_1 \otimes \ldots \otimes v_k) = gv_1 \otimes \ldots \otimes gv_k \quad \text{for} \quad g \in G.
\]
There is a natural action of the symmetric group $S_k$ permuting the components in the tensor product. Thus, for decomposable tensors, we have

$$\sigma(v_1 \otimes \ldots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(k)} \quad \text{for} \quad \sigma \in S_k.$$  

The action of $S_k$ commutes with the action of $G$. Let $\text{Sym}_k(V)$ be the symmetric part of $V^\otimes k$ consisting of the tensors $x$ such that $\sigma x = x$ for all $\sigma \in S_k$. It is known that

$$\dim \text{Sym}_k(V) = \binom{\dim V + k - 1}{k},$$

since $\text{Sym}_k(V)$ can be thought of as the space of all real homogeneous polynomials of degree $k$ in $\dim V$ variables. Let

$$v^\otimes k = v \otimes \ldots \otimes v$$

be the $k$-th tensor power of $v$. Thus $v^\otimes k \in \text{Sym}_k(V)$ and $gv^\otimes k \in \text{Sym}_k(V)$ for all $g \in G$. It turns out that how well $\|f\|_{2k}$ approximates $\|f\|_{\infty}$ depends on the dimension $D_k$ of the subspace spanned by the orbit $\{gv^\otimes k\}$. This dimension may be different for different $v \in V$. Roughly, if $D_k$ is small then $v$ lies in a certain algebraic variety constructed from the action of $G$ in $V$ and for such $v$ the functions $f$ are “smoother” than for those $v$ for which $D_k$ is large.

Thus we obtain the following corollary of Theorem 1.2.

(1.4) **Corollary.** Let $G$ be a compact group acting in a finite-dimensional real vector space $V$ and let $dg$ be the Haar probability measure on $G$. Let us fix a vector $v$ and a linear function $\ell : V \to \mathbb{R}$ and let us define a real-valued function $f : G \to \mathbb{R}$ by $f(g) = \ell(gv)$. For a positive integer $k$, let

$$D_k = \dim \text{span}\{gv^\otimes k : g \in G\}$$

be the dimension of the span of the orbit of $v^\otimes k$ in $V^\otimes k$. Then

$$\|f\|_{2k} \leq \|f\|_{\infty} \leq (D_k)^{\frac{1}{k}} \cdot \|f\|_{2k}.$$  

Again, generally speaking, the estimates of Corollary 1.4 can not be improved, see Remark 2.5. A straightforward estimate of $D_k \leq \dim \text{Sym}_k(V)$ produces the following corollary.

(1.5) **Corollary.** Let $G$ be a compact group acting in a finite-dimensional real vector space $V$ and let $dg$ be the Haar probability measure on $G$. Let us fix a vector $v$ and a linear function $\ell : V \to \mathbb{R}$ and let us define a real-valued function $f : G \to \mathbb{R}$ by $f(g) = \ell(gv)$. Let $k$ be a positive integer. Then

$$\|f\|_{2k} \leq \|f\|_{\infty} \leq \left(\frac{\dim V + k - 1}{k}\right)^{\frac{1}{k}} \cdot \|f\|_{2k}.$$  

There are examples showing that the bounds of Corollary 1.5 are “almost tight”. For instance, if $G = SO(n)$ is the orthogonal group acting in $V = \mathbb{R}^n$, computations of Section 3.3 show that the upper bound for $\|f\|_\infty$ is tight up to a factor of $\sqrt{2}$ (uniformly on $k$ and $n$).

As we remarked earlier, in many cases we are able to compute $\|f\|_{2k}$ efficiently if $k$ is not very large. Quite often (see examples of Sections 3 and 4), we can compute $\|f\|_{2k}$ in polynomial time for any fixed $k$. The following estimate shows the type of bound that we can achieve if we fix $k$ in advance.

**Corollary.** For any $\epsilon > 0$ there exists a $k_0 = k_0(\epsilon) = O(\epsilon^{-2})$ such that for any positive integer $k > k_0$, for any compact group $G$ acting in a real vector space $V$ with $\dim V \geq k$, for any linear function $\ell : V \to \mathbb{R}$, for any $v \in V$ and for the function $f(g) = \ell(gv)$, $f : G \to \mathbb{R}$, we have

$$
\|f\|_{2k} \leq \|f\|_\infty \leq \epsilon \sqrt{\dim V} \cdot \|f\|_{2k}.
$$

The paper is structured as follows. In Section 2, we prove Theorem 1.2 and Corollaries 1.4–1.6. In Section 3, we apply our results to the problem of finding the largest absolute value of a real homogeneous multivariate polynomial on the unit sphere, in which case $G = SO(n)$, the orthogonal group. In particular, we present a simple polynomial time approximation algorithm to compute the largest absolute value on the sphere of a fewnomial, that is, a polynomial having only small (fixed) number of monomials and discuss possible applications to solving systems of real fewnomial equations. In Section 4, we discuss a hard problem of combinatorial optimization, that is, the multidimensional assignment problem, in which case $G = S_n$. In particular, our results lead to an approximation algorithm for finding a bijection between vertex sets of two hypergraphs $H_1$ and $H_2$, which maximizes the number edges of $H_1$ mapped onto the edges of $H_2$.

We use the real model (see [3]) for computational complexity, counting the number of arithmetic operations performed by the algorithm. Eventually, to compute $\|f\|_{2k}$ from $\|f\|_{2k}^k$ we need to extract a root of degree $2k$, which we count as a single operation.

2. Proofs

In this section, we prove Theorem 1.2 and Corollaries 1.4–1.6. We need some standard facts from the representation theory (see, for example, [4]).

Let $G$ be a compact group acting in a finite-dimensional real vector space $V$. As is known, $V$ possesses a $G$-invariant scalar product $\langle \cdot, \cdot \rangle$:

$$
\langle u, v \rangle = \langle gu, gv \rangle \quad \text{for all} \quad u, v \in V \quad \text{and all} \quad g \in G.
$$

We introduce the corresponding Euclidean norm:

$$
\|x\| = \sqrt{\langle x, x \rangle}.
$$
The action (representation) is called irreducible if $V$ contains no proper $G$-invariant subspaces. As is known, if $G$ acts in a finite-dimensional real vector space $V$, then $V$ can be represented as a direct sum of pairwise orthogonal (with respect to a given $G$-invariant scalar product) invariant subspaces $V_i$ such that the action of $G$ in each $V_i$ is irreducible.

A somewhat “non-standard” feature of our construction is that we consider representations over the real rather than over the complex numbers. Consequently, we need a substitute for Schur’s Lemma. It comes in the form of the following observation.

Suppose that $q : V \to \mathbb{R}$ is a $G$-invariant quadratic form, that is $q(gx) = q(x)$ for all $x \in V$ and all $g \in G$. We claim that the eigenspaces of $q$ are $G$-invariant subspaces. A possible way to see that is to notice that the unit eigenvectors of $q$ are precisely the critical points of the restriction $q : S \to \mathbb{R}$ where $S = \{ x : \|x\| = 1 \}$ is the unit sphere in $V$.

Our first lemma is a real version of the orthogonality relations for matrix elements.

**Lemma.** Let $G$ be a compact group acting in a finite-dimensional real vector space $V$ endowed with a $G$-invariant scalar product $\langle \rangle$. Suppose that the representation of $G$ is irreducible and let $dg$ be the Haar probability measure on $G$. Then

$$\int_G \langle x, gv \rangle^2 \, dg = \frac{\|v\|^2 \cdot \|x\|^2}{\dim V} \quad \text{for all } x, v \in V.$$

**Proof.** Let us choose a vector $v \in V$ and let us define a quadratic form $q : V \to \mathbb{R}$ by

$$q(x) = \int_G \langle x, gv \rangle^2 \, dg.$$  

Clearly, $q(x)$ is $G$-invariant: $q(gx) = q(x)$ for all $x \in V$ and all $g \in G$. Let $\lambda$ be the largest eigenvalue of $q$ and let $W$ be the corresponding eigenspace. Then $W$ is an invariant subspace of $V$ and hence $W = V$. Thus $q(x) = \lambda\|x\|^2$ for some $\lambda \geq 0$.

To find $\lambda$, let us compute the trace of $q$. On one hand, we have $\text{tr} \ q = \lambda \dim V$.

Let $q_g(x) = \langle x, gv \rangle^2$. Then $q_g$ is a quadratic form of rank 1 with the non-zero eigenvalue $\|gv\|^2 = \|v\|^2$ which corresponds to an eigenvector $x = gv$. Hence $\text{tr} \ q_g = \|v\|^2$ and since $q(x)$ is the average of $q_g$, we have $\text{tr} \ q = \|v\|^2$. Thus $\lambda = \|v\|^2 / \dim V$.

Hence

$$q(x) = \frac{\|v\|^2 \cdot \|x\|^2}{\dim V}$$

and the proof follows. \qed

Now we use that every representation is a sum of irreducible representations.

**Lemma.** Let $G$ be a compact group acting in a finite-dimensional real vector space $V$ endowed with a $G$-invariant scalar product $\langle \rangle$. Let $dg$ be the Haar probability
measure on $G$. Let us fix a vector $v \in V$. Then there exists a decomposition $V = V_1 \oplus \ldots \oplus V_k$ of $V$ into the direct sum of non-zero pairwise orthogonal invariant subspaces such that for every $x \in V$ we have

$$\int_G \langle x, gv \rangle^2 \, dg = \sum_{i=1}^k \frac{\|x_i\|^2 \cdot \|v_i\|^2}{\dim V_i},$$

where $x_i$ and $v_i$ are the orthogonal projections onto $V_i$ of $x$ and $v$ respectively.

**Proof.** Let us define a quadratic form $q : V \to \mathbb{R}$ by

$$q(x) = \int_G \langle x, gv \rangle^2 \, dg.$$

Then $q$ is $G$-invariant, $q(gx) = q(x)$ for all $g \in G$ and all $x \in V$. Thus the eigenspaces of $q$ are $G$-invariant subspaces of $V$. Let us write every eigenspace as a direct sum of pairwise orthogonal invariant subspaces $V_i$ such that the action of $G$ in each $V_i$ is irreducible. Thus we obtain the decomposition $V = V_1 \oplus \ldots \oplus V_k$ and we have

$$q(x) = \sum_{i=1}^k \lambda_i \|x_i\|^2,$$

where $x_i$ is the orthogonal projection of $x$ onto $V_i$ and $\lambda_i$ are non-negative numbers. To find $\lambda_i$, let us choose a non-zero $x \in V_i$. Then $\langle x, gv \rangle = \langle x, gv_i \rangle$ and by Lemma 2.1, we get

$$q(x) = \lambda_i \|x\|^2 = \frac{\|v_i\|^2 \cdot \|x\|^2}{\dim V_i},$$

from which

$$\lambda_i = \frac{\|v_i\|^2}{\dim V_i}.$$

The proof now follows. \qed

(2.3) **Remark.** A decomposition $V = V_1 \oplus \ldots \oplus V_k$ of a representation into the direct sum of pairwise orthogonal irreducible components is *not* unique as long as some irreducible representation appear with a multiplicity greater than 1 (which means that the representations of $G$ in some subspaces $V_i$ are isomorphic). One can construct some simple examples showing that the decomposition of Lemma 2.2 indeed depends on $v$.

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** The inequality

$$\|f\|_2 \leq \|f\|_\infty$$
is quite standard. Let us prove that
\[ \|f\|_{\infty} \leq \sqrt{\dim V} \cdot \|f\|_2. \]
Let \( e \) be the identity in \( G \). We note that it suffices to prove that
\[ |f(e)| = |\ell(v)| \leq \sqrt{\dim V} \cdot \|f\|_2, \]
because the inequality for \( f(g) = \ell(gv) \) would follow then by choosing a new vector \( v \):
\[ \text{new } v := g(\text{old } v). \]

Let us introduce a \( G \)-invariant scalar product \( \langle \rangle \) in \( V \) so that \( \ell(x) = \langle c, x \rangle \) for some \( c \in V \) and all \( x \in V \). Applying Lemma 2.2, we obtain a decomposition \( V = V_1 \oplus \ldots \oplus V_k \) into the direct sum of pairwise orthogonal invariant subspaces such that
\[ \|f\|_2^2 = \int_G \langle c, gv \rangle^2 \, dg = \sum_{i=1}^k \frac{\|c_i\|^2 \cdot \|v_i\|^2}{\dim V_i}, \]
where \( c_i \) and \( v_i \) are the orthogonal projections onto \( V_i \) of \( c \) and \( v \) respectively. We have
\[ f(e) = \langle c, v \rangle = \sum_{i=1}^k \langle c_i, v_i \rangle \]
and hence
\[ |f(e)| \leq \sum_{i=1}^k |\langle c_i, v_i \rangle| \leq \sum_{i=1}^k \|c_i\| \cdot \|v_i\|. \]
Let
\[ \alpha_i = \frac{\|c_i\| \cdot \|v_i\|}{\sqrt{\dim V_i}}. \]
Then
\[ |f(e)|^2 \leq \left( \sum_{i=1}^k \alpha_i \sqrt{\dim V_i} \right)^2 \leq \left( \sum_{i=1}^k \alpha_i^2 \right) \left( \sum_{i=1}^k \dim V_i \right) = \left( \dim V \right) \cdot \|f\|_2^2. \]
and the proof follows. \( \square \)

\( (2.4) \) Remark. Analyzing the proof of Theorem 1.2, it is not hard to find out when the bound \( \|f\|_{\infty} \leq \sqrt{\dim V} \cdot \|f\|_2 \) is sharp. In particular, the bound is sharp for the class of linear functions on the orbit of \( v \) as long as the orbit of \( v \) spans \( V \). Here are some natural cases when the bound is attained.

Suppose, for example, that we have an absolutely irreducible representation \( \rho \) of \( G \) in a real vector space \( W \) (that is, the representation remains irreducible after complexification). Thus, for every \( g \in G \), \( \rho(g) \) is an operator in \( W \). We interpret
\( \rho(g) \) as a point in the space \( V = \text{End}(W) \) of all linear transformations \( W \to W \). Let \( \chi(g) = \text{tr}(g) \) be the character of the representation. We think of \( \chi(g) \) as of a linear function on the orbit of the identity operator \( I \in \text{End}(W) \) under the action \( g(x) = \rho(g)x \) for all \( x \in \text{End}(W) \). Then \( \|\chi\|_\infty = \dim W = \sqrt{\dim V} \). The orthogonality relations for the characters (see, for example, Chapter 2 of [4]) state that \( \|\chi\|_2 = 1 \) and hence the bound of Theorem 1.2 holds with equality.

As another example, let us consider a finite group \( G \) of cardinality \( |G| \) and an arbitrary function \( f : G \to \mathbb{R} \). Of course, in this case, the inequality \( \|f\|_\infty \leq \sqrt{|G|} \cdot \|f\|_2 \) is the best we can hope for (take \( f \) to be the delta-function of an element of \( G \)). The function \( f \) can be thought of as a linear function on the orbit of a point in the regular representation of \( G \). The space \( V \) in this case is the vector space of all linear functions \( f : G \to \mathbb{R} \) where \( G \) acts by shifts: \( gf(x) = f(g^{-1}x) \).

Let \( v \in V \) be the delta-function at the identity: \( v(e) = 1 \) where \( e \) is the identity in \( G \) and \( v(g) = 0 \) for all \( g \neq e \). Then \( f \) is a linear function on the orbit of \( v \) and \( \dim V = |G| \).

To prove Corollary 1.4, we use the construction of the tensor power (see Section 1.3).

**Proof of Corollary 1.4.** Let us define a function \( h : G \to \mathbb{R} \) by

\[
h(g) = f^k(g) = \ell \otimes^k (v \otimes^k).
\]

Thus \( h \) is a linear function on the orbit of \( v \otimes^k \). Let

\[
W = \text{span}\{gv \otimes^k : g \in G\}
\]

be the span of the orbit of \( v \otimes^k \). Hence \( \dim W = D_k \). Applying Theorem 1.2 to the linear function \( h \) on the orbit of \( v \otimes^k \) in \( W \), we get

\[
\|h\|_2 \leq \|h\|_\infty \leq \sqrt{D_k} \cdot \|h\|_2.
\]

Now we note that \( \|h\|_\infty = \|f\|_\infty^k \) and that \( \|h\|_2 = \|f\|_2^k \).

\((2.5)\) **Remark.** The bound \( \|f\|_\infty \leq (D_k)^{\frac{1}{2k}} \|f\|_2^k \) is rarely sharp. One example when it is sharp is provided by a generic orbit in the regular representation of a finite group \( G \), see Remark 2.4. In Section 3.3, we present a series of examples of matrix functions for \( G = SO(n) \) for which the estimate is sharp up to a constant factor uniformly on \( k \) (and uniformly on \( n \)).

Corollary 1.5 follows by a general estimate of \( D_k \).

**Proof of Corollary 1.5.** We apply Corollary 1.4. The orbit \( \{gv \otimes^k\} \) lies in the symmetric part \( \text{Sym}_k(V) \) of the tensor product \( V \otimes^k \) and hence

\[
D_k = \dim \text{span}\{gv \otimes^k\} \leq \dim \text{Sym}_k(V) = \binom{\dim V + k - 1}{k}.
\]
(2.6) Remark. As follows from Section 3.3, the upper bound for \( \|f\|_\infty \) is sharp up to a constant factor uniformly on \( k \) if \( G = SO(n), V = \mathbb{R}^n \) and \( G \) acts in \( V \) by its defining representation.

We describe below classes of functions \( f : G \rightarrow \mathbb{R} \) for which some sharp estimates can be obtained. Let us fix a representation \( \rho \) of \( G \) in a real vector space \( V \) and let \( F_\rho \) be the vector space spanned by the matrix elements of \( \rho \). Then for some constant \( C(\rho, k) \) and for all \( f \in F_\rho \) we have \( \|f\|_\infty \leq C(\rho, k)\|f\|_{2k} \). Let us assume that \( \rho \) is absolutely irreducible (cf. Remark 2.4). In principle, the best possible value of \( C(\rho, k) \) can be computed from the representation theory of \( G \) as follows. Let us choose an \( f \in F_\rho \). Shifting \( f \), if necessary, we may assume that the maximum absolute value of \( f \) is attained at the identity \( e \) of \( G \). Let us define \( h : G \rightarrow \mathbb{R} \) by

\[
h(x) = \int_G f(g^{-1}xg) \, dg \quad \text{for all} \quad x \in G.
\]

Then \( \|h\|_\infty = \|f\|_\infty \) and \( \|h\|_{2k} \leq \|f\|_{2k} \) for all positive integers \( k \). Thus the largest ratio \( \|f\|_\infty / \|f\|_{2k} \) for \( f \in F_\rho \) is attained when \( f \) satisfies \( f(g^{-1}xg) = f(x) \) for all \( g \in G \) and all \( x \in G \) from which it follows that \( f \) is a multiple of the character \( \chi(g) = \text{tr} \, \rho(g) \), see Remark 2.4. We observe that \( \|\chi\|_\infty = \dim V \). Moreover, the orthogonality relations (see Lecture 6 of [4]) imply that \( \|\chi\|_{2k}^2 \) is the sum of squares of multiplicities of the irreducible components of the tensor power \( \rho^\otimes k \). Summarizing, we conclude that in order to be able to compute the best possible constant \( C(\rho, k) \) such that \( \|f\|_\infty \leq C(\rho, k)\|f\|_{2k} \) for any linear combination \( f \) of matrix elements of \( \rho \), it suffices to know how the tensor power \( \rho^\otimes k \) decomposes into the sum of absolutely irreducible representations.

Finally, Corollary 1.6 follows by an estimate of the binomial coefficient.

Proof of Corollary 1.6. Let us choose a \( k_0 \) such that \( (k!)^{1/k} > 2\epsilon^{-2} \) for all \( k > k_0 \). By Stirling’s formula, we can choose \( k_0 = O(\epsilon^{-2}) \). Then

\[
\left( \frac{\dim V + k - 1}{k} \right)^{\frac{1}{2k}} \leq \left( \frac{2k \dim^k V \cdot k!}{\dim V} \right)^{\frac{1}{2k}} \leq \epsilon \sqrt{\dim V}.
\]

The proof follows by Corollary 1.5.

3. Applications to Polynomials

In this section, we apply our results to approximate the maximum absolute value of a homogeneous multivariate polynomial on the unit sphere.
Let $p$ be a homogeneous polynomial of degree $d$ in $n$ real variables $\xi_1, \ldots, \xi_n$. Thus we can write

$$p(x) = \sum_{1 \leq i_1, \ldots, i_d \leq n} \gamma_{i_1 \ldots i_d} \xi_{i_1} \cdots \xi_{i_d} \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n)$$

where $\gamma_{i_1 \ldots i_d}$ are some real numbers.

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and let $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ be a point. Then $V = (\mathbb{R}^n)^\otimes d$ can be identified with the space $\mathbb{R}^{n^d}$. The coordinates of a typical point (tensor) $X \in V$ are

$$\left( X_{i_1 \ldots i_d} : 1 \leq i_1, \ldots, i_d \leq n \right)$$

and the scalar product in $V$ is defined by

$$\langle X, Y \rangle = \sum_{1 \leq i_1, \ldots, i_d \leq n} X_{i_1 \ldots i_d} Y_{i_1 \ldots i_d}.$$

For $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, the coordinates of $x^\otimes d$ are

$$\left( \xi_{i_1} \cdots \xi_{i_d} \quad \text{for} \quad 1 \leq i_1, \ldots, i_d \leq n \right).$$

Therefore, we can write

$$p(x) = \langle c, x^\otimes d \rangle \quad \text{where} \quad c = \left( \gamma_{i_1 \ldots i_d} \right).$$

Let $G = SO(n)$ be the group of orientation preserving orthogonal transformations of $\mathbb{R}^n$. Then $G$ acts in $V$ by the $d$-th tensor power of its defining representation in $\mathbb{R}^n$. Let us choose $w = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Then, for any $g \in G$, we have

$$\langle c, gw^\otimes d \rangle = p(gw)$$

and the orbit $\{ gw : g \in G \}$ is the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Thus the values of $p(x)$, as $x$ ranges over the unit sphere in $\mathbb{R}^n$, are the values of the linear function

$$\ell(gw^\otimes d) = \langle c, gw^\otimes d \rangle = \langle c, x^\otimes d \rangle$$

as $g$ ranges over the orthogonal group $SO(n)$.

Moreover, the push-forward of the Haar probability measure $dg$ on $G$ is the probability measure $dx$ on $S^{n-1}$. Thus we connect the values of a polynomial on the unit sphere with the values of a linear function on the orbit of the group $G = SO(n)$. 
(3.1) Corollary. Let \( p \) be a homogeneous polynomial of degree \( d \) in \( n \) real variables, let \( S^{n-1} \) be the unit sphere in \( \mathbb{R}^n \) and let \( dx \) be the rotation invariant probability measure on \( S^{n-1} \). For a positive integer \( k \), let us define the \( L^{2k} \) norm of \( p \) by

\[
\|p\|_{2k} = \left( \int_{S^{n-1}} p^{2k}(x) \, dx \right)^{\frac{1}{2k}}
\]

and the \( L^\infty \) norm by

\[
\|p\|_\infty = \max_{x \in S^{n-1}} |p(x)|.
\]

Then

\[
\|p\|_{2k} \leq \|p\|_\infty \leq \left( \frac{kd + n - 1}{kd} \right)^{\frac{1}{2k}} \|p\|_{2k}.
\]

Proof. We apply Corollary 1.5. Let \( w = (1, 0, \ldots, 0) \) be as above. Then, for \( v = w \otimes d \), we can write \( p(gw) = \ell(gv) \) for some linear functional \( \ell : V \to \mathbb{R} \) and all \( g \in G \). The dimension \( D_k \) of the span of the orbit \( \{gv^{\otimes k} = gw^{\otimes kd} : g \in G \} \) is that of the space of homogeneous polynomials of degree \( kd \) in \( n \) variables. Hence

\[
D_k = \binom{kd + n - 1}{kd}.
\]

We have

\[
\int_G \ell^{2k}(gv) \, dg = \int_{S^{n-1}} p^{2k}(x) \, dx.
\]

The proof now follows. \( \square \)

One way to integrate polynomials over the unite sphere is to take the sum of the integrals of the monomials. The following result is certainly known, but for the sake of completeness, we sketch its proof here.

(3.2) Lemma. Let \( p(x) = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \) be a monomial. If at least one of \( \alpha_i \)'s is odd then

\[
\int_{S^{n-1}} p(x) \, dx = 0.
\]

If \( \alpha_i = 2\beta_i \), where \( \beta_i \) are non-negative integers for \( i = 1, \ldots, n \), then

\[
\int_{S^{n-1}} p(x) \, dx = \frac{\Gamma(n/2) \prod_{i=1}^{n} \Gamma(\beta_i + 1/2)}{\pi^{n/2} \Gamma(\beta_1 + \ldots + \beta_n + n/2)},
\]

where \( dx \) is the Haar probability measure on \( S^{n-1} \).

Sketch of Proof. If \( \alpha_i \) is odd then

\[
p(\xi_1, \ldots, \xi_{i-1}, -\xi_i, \xi_{i+1}, \ldots, \xi_n) = -p(\xi_1, \ldots, \xi_{i-1}, \xi_i, \xi_{i+1}, \ldots, \xi_n)
\]

and

\[
p(\xi_1, \ldots, \xi_{i-1}, 0, \xi_{i+1}, \ldots, \xi_n) = \frac{1}{\sqrt{2}} \left( p(\xi_1, \ldots, \xi_{i-1}, \xi_i, \xi_{i+1}, \ldots, \xi_n) + p(\xi_1, \ldots, \xi_{i-1}, -\xi_i, \xi_{i+1}, \ldots, \xi_n) \right)
\]

with \( \xi_i \) fixed. The integrals of the monomials with \( \alpha_i \) odd are then zero.
and hence the average value of $p$ over the unit sphere is 0.

Assuming that $\alpha_i = 2\beta_i$ for $i = 1, \ldots, n$, we get

$$\int_{\mathbb{R}^n} p(x)e^{-\|x\|^2} \, d\mu = \prod_{i=1}^{n} \int_{\mathbb{R}} \xi^{2\beta_i}e^{-\xi^2} \, d\xi = \prod_{i=1}^{n} \Gamma(\beta_1 + 1/2),$$

where $\mu$ is the standard Lebesgue measure in $\mathbb{R}^n$.

On the other hand, passing to the polar coordinates and using that $p$ is homogeneous of degree $d = 2(\beta_1 + \ldots + \beta_n)$, we get

$$\int_{\mathbb{R}^n} p(x)e^{-\|x\|^2} \, d\mu = |S^{n-1}| \cdot \left( \int_{S^{n-1}} p(x) \, dx \right) \cdot \left( \int_0^{\infty} r^{d+n-1}e^{-r^2} \, dr \right),$$

where $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ is the Euclidean volume of the unit sphere in $\mathbb{R}^n$. The proof now follows. $\square$

The estimates of Corollary 3.1 are probably not optimal (apart from the case of $k = 1$), but the following simple example shows that in some sense, they are close to being optimal.

(3.3) Powers of linear functions. Let $p$ be the power of a linear function, for example, $p(x) = \xi^d_1$. Then $\|p\|_\infty = 1$ and, by Lemma 3.2,

$$\|p\|_2^k = \left( \frac{\Gamma(n/2)\Gamma(kd + 1/2)}{\sqrt{\pi}\Gamma(kd + n/2)} \right)^{1/k}.$$

Then Corollary 3.1 gives us the estimate

$$\|p\|_\infty \leq \left( \frac{\Gamma(n/2)\Gamma(kd + 1/2)\Gamma(kd + n)}{\sqrt{\pi}\Gamma(kd + n/2)\Gamma(n)\Gamma(kd + 1)} \right)^{1/k} \leq \left( \frac{\Gamma(n/2)\Gamma(kd + n)}{\Gamma(kd + n/2)\Gamma(n)} \right)^{1/k} \leq 2^d.$$ 

Hence, among all homogeneous polynomials of a given degree $d$, powers of linear functions give the largest ratio $\|f\|_\infty/\|f\|_2^k$ up to a constant factor depending on the degree of $f$ and independent of the number of variables $n$ and the value of $k$. This may serve as an indication that the bound of Corollary 1.5 are not too bad, cf. Remarks 2.5 and 2.6. G. Blekherman [2] pointed out to the author that the powers, in general, do not provide exactly the largest ratio $\|f\|_\infty/\|f\|_2^k$ among all polynomials of a given degree $d$. Such “extremal” polynomials $f$ were computed by G. Blekherman when some of the parameters $n, d$ and $k$ are small.

Suppose we want to approximate $\|p\|_\infty$ by $\|p\|_2^k$ for a sufficiently large $k$. Let us see what trade-off between between the computational complexity and accuracy can we achieve.
**Low degree polynomials.** Let us fix the degree $d$ and allow the number of variables to vary. Suppose that we are given a homogeneous polynomial $p$ of degree $d$ and that we want to estimate $\|p\|_\infty$. This problem is provably computationally hard already for $d = 4$ (one can infer it from results of Part 1 of [3]) and is suspected to be hard for $d = 3$.

Let $m$ be the number of monomials in $p$, so $m = O(n^d)$. We observe that for any fixed $k$, the direct computation of $p^{2k}(x)$ and computing $\|p\|_{2k}$ via Lemma 3.2 has $O(m^{2k})$ complexity. One the other hand, using Corollary 3.1, we get that

$$\|p\|_\infty \leq C(k)n^{d/2}\|p\|_{2k}$$

where $C(k) = O(k^{-1/2})$.

In other words, for any fixed $\epsilon > 0$ there is a polynomial time algorithm estimating $\|p\|_\infty$ within a factor of $\epsilon n^{d/2}$. If we want a better estimate, we have to take a larger $k$. Thus, for any constant $C > 1$, from Corollary 3.1 (cf. also Corollary 1.6), we get that

$$\|p\|_\infty \leq C\|p\|_{2k}$$

for some $k = O(n)$.

Since $p^{2k}(x)$ contains at most $(\sum_{2kd}^{2kd+n-1})$ monomials, we can compute $\|p\|_{2k}$ by Lemma 3.2 in $O(n^{k+2})$ time. Summarizing, for any $C > 1$ there exists a $\gamma > 0$ such that we can approximate $\|p\|_\infty$ within a factor $C$ in $2^{\gamma n}$ time.

**Fewnomials.** Suppose that we do not fix the degree $d$ of $p$ but fix instead the number $m$ of monomials in $p$. Thus we can write

$$p(x) = \sum_{i=1}^{m} p_i(x),$$

where

$$p_i(x) = \gamma_i \xi_1^{\alpha_{i1}} \cdots \xi_n^{\alpha_{im}}$$

are monomials. For a positive integer $k$, by the multinomial expansion, we get

$$p^{2k} = \sum_{\substack{r_1, \ldots, r_m \geq 0 \\ r_1 + \cdots + r_m = 2k}} \frac{(2k)!}{r_1! \cdots r_m!} p_1^{r_1} \cdots p_m^{r_m}.$$

Thus $p^{2k}$ contains at most $(\sum_{m+2k-1}^{m+2k-1})$ monomials, which is a polynomial in $k$ when $m$ is fixed. Using Lemma 3.2, we compute $\|p\|_{2k}$ in $O(dn(2k)^m)$ time. Given an $\epsilon > 0$, let us choose an integer $k = O(\epsilon^{-1}n^2 \ln d)$ such that

$$\frac{n - 1}{2k} \ln(kd + 1) < \ln(1 + \epsilon).$$

Using Corollary 3.1 and a simple estimate

$$\binom{kd + n - 1}{n - 1} = \frac{(kd + 1)(kd + 2) \cdots (kd + n - 1)}{1 \cdot 2 \cdots (n - 1)} \leq (kd + 1)^{n-1}$$
we conclude that
\[ \|p\|_{2k} \leq \|p\|_{\infty} \leq (1 + \epsilon) \cdot \|p\|_{2k}. \]
Hence as long as the number of monomials is fixed, we get a polynomial time approximation algorithm, which, for any given \( \epsilon > 0 \) computes the maximum absolute value of a given polynomial (“fewnomial”) over the unit sphere within a relative error of \( \epsilon \), in time polynomial in \( \epsilon^{-1} \), the number of variables \( n \) and the degree \( d \) of the polynomial. In fact, the only place where we have to use polynomially many in \( d \) arithmetic operations is when we compute gamma-functions (factorials) in Lemma 3.2. Apart from that, the running time of the algorithm is polynomial in \( \ln d \).

Computing or approximating the maximum absolute value of a polynomial on the unit sphere can be used for testing whether a given system of real polynomial equations has a real solution, a difficult and important problem, see for example, [3] and [7]. Suppose that \( p_i: i = 1, \ldots, s \) are given homogeneous polynomials of degree \( d \) in \( n \) variables \( x = (\xi_1, \ldots, \xi_n) \) and that we would like to test whether the system
\[ p_i(x) = 0 \quad \text{for} \quad i = 1, \ldots, s \]
has a real solution \( x \neq 0 \). Let
\[ q = \sum_{i=1}^{s} p_i^2(x). \]
Thus we want to test whether
\[ \min_{x \in S^{n-1}} q(x) = 0. \]
Let us choose a
\[ \gamma > \max_{x \in S^{n-1}} q(x) \]
and let
\[ p = \gamma \|x\|^{2d} - q. \]
Thus the problem reduces to checking whether
\[ \max_{x \in S^{n-1}} |p(x)| = \gamma. \]
If the polynomials \( p_i \) of the original system did not have too many monomials, we can try to approximate \( \|p\|_{\infty} \) by \( \|p\|_{2k} \) for a reasonably large \( k \), cf. Section 3.5. Similarly, to choose an appropriate \( \gamma \), we can compute \( \|q\|_{2k} \) for a sufficiently large \( k \). The number of monomials in the system is relevant to the “topological complexity” of the set of real solutions [5], so it should not be surprising that it is also relevant to the computational complexity of the decision problem. In particular, this approach may be useful for detecting “badly unsolvable” systems (systems for which the value of \( \|p\|_{\infty} \) is substantially smaller than \( \gamma \)) of fewnomial equations.
4. Applications to Combinatorial Optimization

Let us fix a number \( d \) and let \( V = (\mathbb{R}^n)^\otimes d = \mathbb{R}^{n^d} \) be the vector space of \( d \)-dimensional arrays (tensors)

\[
X = (x_{i_1...i_d} : 1 \leq i_1, \ldots, i_d \leq n).
\]

To simplify the notation somewhat, we denote the coordinate \( s \) of \( X \) by \( x_I \), where \( I = (i_1, \ldots, i_d) \).

We introduce the scalar product by

\[
\langle X, Y \rangle = \sum_I x_I y_I \quad \text{for} \quad I = (1 \leq i_1, \ldots, i_d \leq n).
\]

Let \( G = S_n \) be the symmetric group of all permutations \( g \) of the set \( \{1, \ldots, n\} \). We introduce the action of \( S_n \) on \( V \) by the \( d \)-th tensor power of the natural action of \( S_n \) in \( \mathbb{R}^n \):

\[
Y = gX \quad \text{provided} \quad x_I = y_{gI} \quad \text{where} \quad g(i_1, \ldots, i_d) = (g(i_1), \ldots, g(i_d)).
\]

Let us choose two tensors \( A, B \in V \) and let

\[
f(g) = \langle B, gA \rangle = \sum_{1 \leq i_1, \ldots, i_d \leq n} a_{i_1...i_d} b_{g(i_1)...g(i_d)},
\]

be the corresponding matrix element.

The problem of maximizing (minimizing) \( f \) is one of the most general problems of combinatorial optimization, known as the \( d \)-dimensional assignment problem (see, for example, [6]). It is straightforward for \( d = 1 \) but already quite difficult for \( d = 2 \) (see [1]).

**(4.1) Example: hypergraphs.** Recall that a \( d \)-hypergraph \( H \) on the set \( \{1, \ldots, n\} \) is a set of subsets \( E \subset \{1, \ldots, n\} \), called edges of \( H \), such that \( |E| \leq d \) for the cardinality \( |E| \) of every edge \( E \) of \( H \). A hypergraph is called uniform provided \( |E| = d \) for every edge \( E \) of \( H \). Let \( H_1 \) and \( H_2 \) be uniform \( d \)-hypergraphs with the set of vertices \( \{1, \ldots, n\} \). Let us define the adjacency tensor \( A = (a_{i_1...i_d}) \) of \( H_1 \) by

\[
a_{i_1...i_d} = \begin{cases} 
1 & \text{if } \{i_1, \ldots, i_d\} \text{ is an edge of } H_1 \\
0 & \text{otherwise.}
\end{cases}
\]

Let us define \( B = (b_{i_1...i_d}) \) by:

\[
b_{i_1...i_d} = \begin{cases} 
\frac{1}{d!} & \text{if } \{i_1, \ldots, i_d\} \text{ is an edge of } H_2 \\
0 & \text{otherwise.}
\end{cases}
\]
A permutation $g$ of the set $\{1, \ldots, n\}$ is interpreted as a bijection between the vertices of $H_2$ and the vertices of $H_1$ and the value of

$$f(g) = \langle B, gA \rangle$$

is the number of edges of $H_2$ mapped onto the edges of $H_1$. The value of $\|f\|_\infty$ is the maximum number of edges of $H_1$ and $H_2$ that can be matched by a bijection of the vertices of $H_1$ and $H_2$. If $H_1$ and $H_2$ are not uniform, we can modify $B$ by letting

$$b_{i_1, \ldots, i_d} = \frac{k_1! \cdots k_r!}{d!}$$

provided $\{i_1, \ldots, i_d\}$ is an edge of $H_2$ and the multiplicities of the elements in the multiset $\{\{i_1, \ldots, i_d\}\}$ are $k_1, \ldots, k_r$, so that $k_1 + \ldots + k_r = d$. Then again the value of $\|f\|_\infty$ is equal to the maximum number of edges of $H_1$ and $H_2$ that can be matched by a bijection of the vertex sets.

One can extend this construction to oriented hypergraphs whose edges are ordered subsets of $\{1, \ldots, n\}$. By introducing weights on the edges of $H_1$ and $H_2$ we can introduce “prices” for matching (or mismatching) particular edges.

Applying Corollary 1.5, we get the inequality

$$\|f\|_{2k} \leq \|f\|_\infty \leq \left(\frac{n^d + k - 1}{k}\right)^{\frac{k}{2}} \|f\|_{2k}$$

for the function $f$ of a general $d$-dimensional assignment problem.

In various special cases, the bound can be somewhat improved by using Corollary 1.4. For example, if the coordinates of $A$ (or $B$) are 0’s and 1’s, one can prove that

$$\|f\|_{2k} \leq \|f\|_\infty \leq D^{\frac{k}{2k}}(n, d, k) \cdot \|f\|_{2k}$$

where $D(n, d, k) = \sum_{j=1}^{k} \left(\begin{array}{c} n \\ j \end{array} \right)$.

We claim that for small (fixed) values of $k$ the value of $\|f\|_{2k}$ can be computed relatively easily (in polynomial time). First, we observe that computation of $\|f\|_{2k}$ reduces to computation of the average of a matrix element for larger tensors.

(4.2) Lemma. Let us fix two tensors $A = (a_I)$ and $B = (b_I)$ for $I = (1 \leq i_1, \ldots, i_d \leq n)$. For a positive integer $m$ (in particular, for $m = 2k$), let us define tensors $X = A^{\otimes m}$ and $Y = B^{\otimes m}$ as follows:

$$X = (x_J) \quad \text{and} \quad Y = (y_J) \quad \text{where} \quad J = (1 \leq j_1, \ldots, j_{dm} \leq n)$$

and where

$$x_J = a_{I_1} \cdots a_{I_m} \quad \text{and} \quad y_J = b_{I_1} \cdots b_{I_m} \quad \text{provided} \quad J = (I_1, \ldots, I_m).$$
Then
\[ \frac{1}{n!} \sum_{g \in S_n} \langle B, gA \rangle^m = \frac{1}{n!} \sum_{g \in S_n} \langle Y, gX \rangle. \]

**Proof.** The proof follows by observation that
\[ \langle B, gA \rangle^m = \langle B^\otimes m, gA^\otimes m \rangle = \langle Y, gX \rangle. \]
\[ \square \]

Next, we show how to compute the average.

**Lemma.** Let us fix a positive integer \( l \) (in particular, \( l = md = 2kd \)). For a partition \( \Sigma = \{ \Sigma_1, \ldots, \Sigma_r \} \) of the set \( \{1, \ldots, l\} \) into non-empty disjoint subsets, we say that a sequence \( I = (i_1, \ldots, i_l) \) has type \( \Sigma \) if for each \( \Sigma_p \) the indices \( i_j : j \in \Sigma_p \) are all equal and if for each pair of subsets \( \Sigma_p \) and \( \Sigma_q \) the indices \( i_j : j \in \Sigma_p \) and \( i_j : j \in \Sigma_q \) are different.

Let \( X = (x_I) \) and \( Y = (y_I) \), \( I = (1 \leq i_1, \ldots, i_l \leq n) \) be tensors (in particular, we can have \( X = A^\otimes m = A^\otimes 2k \) and \( Y = B^\otimes m = B^\otimes 2k \)).

**Lemma.** Let \( X = (\overline{x}_I) \) and \( Y = (\overline{y}_I) \), \( I = (1 \leq i_1, \ldots, i_l \leq n) \) be tensors defined by
\[ \overline{x}_I = \frac{(n-r)!}{n!} \sum_{J: \text{type } J = \text{type } I} x_J \quad \text{provided type } I = (\Sigma_1, \ldots, \Sigma_r) \]
and
\[ \overline{y}_I = \frac{(n-r)!}{n!} \sum_{J: \text{type } J = \text{type } I} y_J \quad \text{provided type } I = (\Sigma_1, \ldots, \Sigma_r). \]

Then
\[ \frac{1}{n!} \sum_{g \in S_n} \langle Y, gX \rangle = \langle \overline{Y}, \overline{X} \rangle. \]

**Proof.** The two index sets \( I = (i_1, \ldots, i_l) \) and \( J = (j_1, \ldots, j_l) \) belong to the same orbit of the action \( I \mapsto gI \) of \( S_n \) if and only if they have the same type \( \{ \Sigma_1, \ldots, \Sigma_r \} \). Moreover, the stabilizer of \( I \) consists of \((n-r)!\) permutations. Hence
\[ \overline{X} = \frac{1}{n!} \sum_{g \in S_n} gX \quad \text{and} \quad \overline{Y} = \frac{1}{n!} \sum_{g \in S_n} gY. \]

We have
\[ \frac{1}{n!} \sum_{g \in S_n} \langle Y, gX \rangle = \langle \frac{1}{n!} \sum_{g \in S_n} gY, \frac{1}{n!} \sum_{g \in S_n} gX \rangle \]
and the proof follows. \( \square \)
Combining Lemmas 4.2 and 4.3, we observe that as long as $d$ and $k$ are fixed, we can compute $\|f\|_{2k}$ in $O(n^{2kd})$ time, that is, in polynomial in $n$ time.

In particular, from Corollary 1.6, we conclude that for any fixed $d$ and for any fixed $\epsilon > 0$ there exists a polynomial in $n$ algorithm for estimating $\|f\|_\infty$ within a factor of $en^{d/2}$. This result seems to be new already for $d = 2$, cf. [1].

(4.4) Remark. So far we have shown how to approximate $\|f\|_\infty$ by $\|f\|_{2k}$ but we did not discuss how to find a particular permutation $g$ which gives the value of $|f(g)|$ close to $\|f\|_\infty$. In fact, it is not hard to construct a permutation $g \in S_n$ for which $|f(g)| \geq \|f\|_{2k}$ and hence $|f(g)|$ approximates $\|f\|_\infty$ within a factor of $en^{d/2}$ at the cost of some extra work, which still results in a polynomial time algorithm when $k$ is fixed. The idea is to use the “divide-and-conquer” approach. We split the symmetric group $S_n$ into the union of cosets $S_j = \{g : g(1) = j\}$ and then compute the average value of $f^{2k}$ over each coset separately (this would require some straightforward modification of Lemma 4.3). Then a coset should be chosen which gives the largest average. Thus we have determined $g(1) = j$ and we proceed to determine $g(2), \ldots, g(n)$ successively.

REFERENCES

1. E.M. Arkin, R. Hassin and M. Sviridenko, Approximating the maximum quadratic assignment problem, Inform. Process. Lett. 77 (2001), 13–16.
2. G. Blekherman, personal communication (2001).
3. L. Blum, F. Cucker, M. Shub and S. Smale, Complexity and Real Computation. With a foreword by Richard M. Karp, Springer-Verlag, New York, 1998.
4. W. Fulton and J. Harris, Representation Theory. A First Course, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.
5. A.G. Khovanskii, Fewnomials, Translations of Mathematical Monographs, vol. 88, American Mathematical Society, Providence, RI, 1991; Translated from the Russian by Smilka Zdravkovska.
6. P.M. Pardalos and L.S. Pitsoulis, Quadratic and multidimensional assignment problems, Nonlinear optimization and related topics (Erice, 1998), Appl. Optim., 36, Kluwer Acad. Publ., Dordrecht, 2000, pp. 235–256.
7. J. Renegar, Computational complexity of solving real algebraic formulae, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 1595–1606.

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