A Tight and Unified Analysis of Extragradient for a Whole Spectrum of Differentiable Games

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Abstract

We consider differentiable games: multi-objective minimization problems, where the goal is to find a Nash equilibrium. The machine learning community has recently started using extrapolation-based variants of the gradient method. A prime example is the extragradient (Korpelevich, 1976), which yields linear convergence in cases like bilinear games, where the standard gradient method fails. The full benefits of extrapolation-based methods are not known: i) there is no unified analysis for a large class of games that includes both strongly monotone and bilinear games; ii) it is not known whether the rate achieved by extragradient can be improved, e.g. by considering multiple extrapolation steps. We answer these questions through new analysis of the extragradient’s local and global convergence properties. Our analysis covers the whole range of settings between purely bilinear and strongly monotone games. It reveals that extragradient converges via different mechanisms at these extremes; in between, it exploits the most favorable mechanism for the given problem. We then present lower bounds on the rate of convergence for a wide class of algorithms with any number of extrapolations. Our bounds prove that the extragradient achieves the optimal rate in this class, and that our upper bounds are tight. Our precise characterization of the extragradient’s convergence behavior in games shows that, unlike in convex optimization, the extragradient method may be much faster than the gradient method.

1 Introduction

Gradient-based optimization methods have underpinned many of the recent successes of machine learning. The training of many models is indeed formulated as the minimization of a loss generally involving the data. However, a growing number of new frameworks rely on optimization problems that involve multiple players with different objectives. For instance, the actor-critic model (Pfau and Vinyals, 2016), generative adversarial networks (GANs) (Goodfellow et al., 2014) and automatic curricula (Sukhbaatar et al., 2018) can be cast as two-player games.

Hence games are a generalization of the standard single-objective framework. The aim of the optimization is to find Nash equilibria, that is to say situations where none of the player can unilaterally decrease its loss. However, new issues that were not present for single-objective problems arise. The presence of rotational dynamics prevent standard algorithms such as the gradient method to converge on simple bilinear examples (Goodfellow, 2016; Balduzzi et al., 2018). Furthermore, sta-

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Some recent progress in machine learning has been made by reconsidering the extragradient method originally introduced by Korpelevich (1976) in the case of variational inequalities. This method adds an extrapolation step to the standard gradient method. Though the extragradient method and the gradient method have similar performance in convex optimization, their behaviour seems to differ when applied to games: unlike gradient, extragradient does converge on the heavily studied bilinear example (Tseng, 1995; Gidel et al., 2019a; Mokhtari et al., 2019).

However, linear convergence results for extragradient have only been proven for either strongly monotone variational inequalities problems, which include strongly convex-concave saddle point problems, or in the bilinear setting separately (Tseng, 1995; Gidel et al., 2019a; Mokhtari et al., 2019). Moreover, one can also also wonder whether increasing the number of extrapolation steps can yield significant improvements in terms of convergence. Nemirovski (2004) answered this question in the negative for sublinear rates but the question has been open for linear rates.

In this paper, we study the dynamics of the gradient, the extragradient and more generally multi-step extrapolations methods for unconstrained games. Our objective is two-fold. First, taking inspiration from the analysis of the gradient method by Gidel et al. (2019b), we aim at providing a single precise analysis of the extragradient method which covers both the bilinear and the strongly monotone settings and their intermediate cases. Second, we are interested in theoretically comparing extragradient to the gradient method and general multi-step extrapolations through upper and lower bounds on convergence rates.

Our contributions can be summarized as follows:

- We perform a spectral analysis of multi-step extrapolation methods in §4.1. We derive a local rate of convergence which covers the whole range of settings between purely bilinear and strongly monotone games and which is faster than existing rates in some regimes. Our analysis also highlights the similarity of the extragradient and the proximal point methods.

- In §4.2, we derive a global convergence rate for the extragradient method with the same unifying properties as the local analysis but less precise. It shows that, while this method converges for different reasons in the convex and bilinear settings, in between it actually takes advantage of the most favorable one. Some of our convergence results are summarized in Tab. 1 along with those from the literature.

- In §5, we use the framework from Arjevani et al. (2016) to derive lower bounds for specific classes of algorithms. (i) We show in §5.1 that the previous spectral analysis of the gradient method by Gidel et al. (2019b) is tight, confirming the difference of behaviors with the extragradient method. (ii) We prove a lower bound for 1-Stationary Canonical Linear Iterative methods with any number of extrapolation steps in §5.2. This shows that increasing this number or choosing different step sizes for each does not yield significant improvements and hence extragradient can be considered as optimal among this class.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Adv.} & \text{Gen.} & \text{Tseng} & \text{Nesterov and Scrimi} & \text{Mokhtari et al.} \\
(1995) & (2006) & \text{Global} & \text{Local} \\
\hline
\frac{(\mu_1^2 + \mu_2^2)}{L_{\max}^2} & \frac{\mu_1^2}{L_1^2} & \frac{\mu_1^2}{L_{\max}^2} & \frac{\mu_1^2}{L_{\max}^2} \left| \frac{\mu_1^2 + \mu_2^2}{L_{\max}^2} \right| + \frac{\mu_1^2}{L_{\max}^2} + \frac{\mu_1^2}{L_{\max}^2} \\
\hline
\frac{\mu^2}{r^2} & \frac{\mu}{r} & \frac{\mu}{r} & \frac{\mu}{r} + \frac{\gamma^2}{r^2} & \min \frac{8\lambda}{\max |\lambda|} + \min \frac{|\lambda|^2}{\max |\lambda|^2} \\
\hline
\end{array}
\]

Table 1: Summary of convergence results for the extragradient method on a particular highly adversarial game and in the general case. See the corresponding sections for details and notations and Tab. 2 for a more complete outlook. If a result shows that the iterates converge as $O((1 - r)^t)$, the quantity $r$ is reported. The larger the better. In general $r$ is much smaller that 1. The numerical constants are dropped for clarity. We denote $\min(a, b)$ by $a \wedge b$. 

Stationary points of the gradient dynamics are not necessarily Nash equilibria (Adolphs et al., 2019; Mazumdar et al., 2019).
2 Related Work

Extragradient, Mirror-prox. The extragradient method was first introduced by Korpelevich (1976) in the context of variational inequalities. Tseng (1995) proves results which induce linear convergence rates for this method in the bilinear and strongly monotone cases. But his bound is suboptimal for the latter, as a better rate was proven by Nesterov and Scrimalli (2006) in this case. We will recover both the latter and the rate of Tseng (1995) for the bilinear case with our analysis. The extragradient method was generalized to arbitrary geometries by Nemirovski (2004) as the mirror-prox method. A sublinear rate of $O(1/t)$ was proven for monotone variational inequalities by treating this method as an approximation of the proximal point method as we will discuss later. More recently, Mertikopoulos et al. (2019) proved that, for a broad class of saddle-point problems, its stochastic version converges almost surely to a solution. Treating the extragradient method as a perturbation of the proximal point method, Mokhtari et al. (2019) gave new but still separate derivations for the standard linear rates in the bilinear and the strongly convex-concave settings. We provide unified and more precise analysis (cf. Tab. 1). Gidel et al. (2019a) interpreted GANs as a variational inequality problem and derive a variant of extragradient which avoids "wasting" a gradient. They prove a linear convergence rate for strongly monotone variational inequality problems.

GAN training as game optimization. In the same perspective as us, Mescheder et al. (2017) study the local convergence properties of gradient descent for games through the eigenvalues of the Jacobian of the vector field. According to them, the troubles of the optimization process are due to the presence of eigenvalues with small real part or big imaginary part and argue that second-order information might be beneficial to overcome this. Through a similar spectral approach, Gidel et al. (2019b) study the dynamics of the gradient method with or without momentum and advocate the use of negative momentum and alternating updates based on the analysis of bilinear examples. Our work applies this spectral analysis to the extragradient method and multi-step extrapolation methods. On the other hand, Baldazzi et al. (2018) seek to understand game dynamics by decomposing the game into a rotational and a potential part.

Lower bounds in optimization. Lower bounds in convex optimization date back to Nemirovsky and Yudin (1983) and were popularized by Nesterov (2014). These results played a fundamental role in optimization, especially the lower bound in the strongly convex case as it is a linear rate involving the condition number. One issue with these results is that they either only valid for a finite number of iterations depending on the dimension of the problem or are proven in infinite dimensional spaces. To derive lower bounds which matches more closely the (linear) rates of convergence of common methods, Arjevani et al. (2016) introduced a new framework. It encompasses methods which, applied on quadratics, compute the next iterate as fixed linear transformation of the $p$ last iterates, for some fixed $p \geq 1$. They call them $p$-Stationary Canonical Linear Iterative algorithms ($p$-SCLI). We build on and extend this framework to derive lower bounds for games for $1$-SCLI.

Our notations are presented in §A. The proofs can be found in the subsequent appendix sections.

3 Background and motivation

3.1 $n$-player differentiable games

Following Baldazzi et al. (2018), a $n$-player differentiable game can be defined as a family of twice continuously differentiable losses $l_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R}$ for $i = 1, \ldots, n$. The parameters for player $i$ are $\omega_i \in \mathbb{R}^{d_i}$ and we note $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^d$ with $d = \sum_{i=1}^n d_i$. Ideally, we are interested in finding a Nash equilibrium: that is to say a point $\omega^* \in \mathbb{R}^d$ such that

$$\forall i \in \{1, \ldots, n\}, \quad \omega_i^* \in \arg\min_{\omega_i \in \mathbb{R}^{d_i}} l_i(\omega_1^*, \ldots, \omega_{i-1}^*, \omega_i, \omega_{i+1}^*, \ldots, \omega_n^*).$$

(1)

Moreover we say that a game is zero-sum if $\sum_{i=1}^n l_i = 0$. For instance, following Mescheder et al. (2017); Gidel et al. (2019b), the standard formulation of GANs from Goodfellow et al. (2014) can be cast as a two-player zero-sum game. The Nash equilibrium corresponds to the desired situation where the generator exactly capture the data distribution, completely confusing a perfect discriminator.
Let us now define the vector field \( v(\omega) \) associated to a \( n \)-player game and its Jacobian \( \nabla v(\omega) \):

\[
v(\omega) = \begin{pmatrix} \nabla_{\omega_1} l_1(\omega) \\ \vdots \\ \nabla_{\omega_n} l_n(\omega) \end{pmatrix} \quad \text{and} \quad \nabla v(\omega) = \begin{pmatrix} \nabla^2_{\omega_1} l_1(\omega) & \cdots & \nabla_{\omega_n} \nabla_{\omega_1} l_1(\omega) \\ \vdots & \ddots & \vdots \\ \nabla_{\omega_1} \nabla_{\omega_n} l_n(\omega) & \cdots & \nabla^2_{\omega_n} l_n(\omega) \end{pmatrix}.
\]  

(2)

We say that \( v \) is \( L \)-Lipschitz continuous for some \( L \geq 0 \) if \( \|v(\omega) - v(\omega')\| \leq L\|\omega - \omega'\| \) for all \( \omega, \omega' \in \mathbb{R}^d \), that \( v \) is \( \mu \)-strongly monotone for some \( \mu \geq 0 \) if \( \mu\|\omega - \omega'\|^2 \leq (v(\omega) - v(\omega'))^T (\omega - \omega') \) for all \( \omega, \omega' \in \mathbb{R}^d \) and that \( v \) is monotone if it is 0-strongly monotone.

A Nash equilibrium is always a stationary point of the gradient dynamics, i.e., a point \( \omega \in \mathbb{R}^d \) such that \( v(\omega) = 0 \). However, as shown by Adolphs et al. (2019); Mazumdar et al. (2019), stationary points are not necessarily Nash equilibria in general but if \( v \) is monotone, these two notions are equivalent. Hence, in this work we focus on finding stationary points.

Gidel et al. (2019b) and Balduzzi et al. (2018) mentioned two particular classes of games, which can be seen as the two opposite ends of a spectrum. As the definitions vary, we only give the intuition for these two categories. The first one is adversarial games, where the Jacobian has eigenvalues with small real parts and large imaginary parts and the cross terms \( \nabla_{\omega_i} \nabla_{\omega_j} l_j(\omega) \), for \( i \neq j \), are dominant.

The prime example of such game, which has been heavily studied is a simple bilinear example:

\[
\min_{x \in \mathbb{R}^m} \max_{y \in \mathbb{R}^m} x^T A y + b^T x + c^T y \quad A \in \mathbb{R}^{m \times m}, \ b \in \mathbb{R}^m, \ c \in \mathbb{R}^m,
\]

(3)

whose Jacobian is anti-symmetric and so has only imaginary eigenvalues. If \( A \) is non-singular, there is an unique stationary point which is also the unique Nash equilibrium. The gradient method is known not to converge in such game while the proximal point and extragradient methods converge Rockafellar (1976); Tseng (1995). This setting is of particular interest to us as Mescheder et al. (2017) has shown that at the optimum, the eigenvalues of the Jacobian of GANs are almost imaginary and so around this point GANs behave like adversarial games. The other category is cooperative games, where the Jacobian has eigenvalues with large positive real parts and small imaginary parts and the diagonal terms \( \nabla^2_{\omega_i} l_i(\omega) \) are dominant. Convex optimization problems are the archetype of such games. Our hypotheses, for both the local and the global analyses, encompass all these settings.

### 3.2 Methods and convergence analysis

**Convergence theory of fixed-point iterations.** Seeing optimization algorithms as the repeated application of some operator allows us to deduce their convergence properties from spectrum of this operator. This point of view was presented in Polyak (1987); Bertsekas (1999) and recently used in Arjevani et al. (2016); Mescheder et al. (2017); Gidel et al. (2019b) for instance. It relies on the following classical result:

**Theorem 1** (Polyak (1987)). Let \( F : \mathbb{R}^d \rightarrow \mathbb{R}^d \) be continuously differentiable and let \( \omega^* \in \mathbb{R}^d \) be a fixed point of \( F \). If \( \rho(\nabla F(\omega^*)) < 1 \), then for \( \omega_0 \) in a neighborhood of \( \omega^* \), the iterates \( (\omega_t) \), defined by \( \omega_{t+1} = F(\omega_t) \) for all \( t \geq 0 \) converge linearly to \( \omega^* \) at a rate of \( O((\rho(\nabla F(\omega^*)) + \epsilon)^t) \) for all \( \epsilon > 0 \).

Note that, if the operator \( F \) is linear, the result is stronger, as shown in Thm. 8 in §D.1. This theorem means that, to derive a local rate of convergence for a given method, one needs only to focus on bounding the eigenvalues of the Jacobian of the corresponding operator at a fixed point.

**Gradient method.** Following Gidel et al. (2019b), we define the gradient method as the application of the operator \( F_\eta(\omega) := \omega - \eta v(\omega) \), for \( \omega \in \mathbb{R}^d \). One can easily see that the fixed points of \( F_\eta \) are exactly the stationary points of the dynamics. The authors prove a bound on the spectral radius of the operator which can be simplified to give:

**Theorem 2** (Gidel et al. (2019b)). If the eigenvalues of \( \nabla v(\omega^*) \) all have positive real parts, then, for \( \eta = \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re(1/\lambda) \) one has the following upper bound on the spectral radius of \( F_\eta \):

\[
\rho(\nabla F_\eta(\omega^*)) \leq 1 - \frac\min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re(1/\lambda) \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re(\lambda).
\]

(4)
Proximal point method. For $v$ monotone (Minty, 1962; Rockafellar, 1976), the proximal point operator can be defined as $P_v(\omega) = (\Id + \eta v)^{-1}(\omega)$ for $\omega \in \mathbb{R}^d$. Therefore the proximal point method can be seen as an implicit scheme: $\omega_{t+1} = \omega_t - \eta v(\omega_{t+1})$. It was studied for variational inequalities by Rockafellar (1976), who proved a linear convergence rate for a broad class of problems, which include both bilinear and strongly monotone problems.

Extragradient. The extragradient method was introduced by Korpelevich (1976) in the context of variational inequalities. Its update rule is

$$\omega_{t+1} = \omega_t - \eta v(\omega_t - \eta v(\omega_t)).$$

(5)

It can be seen as an approximation of the implicit update of the proximal point method. Indeed Nemirovski (2004) showed a rate of $O(1/t)$ for extragradient by treating it as a "good enough" approximation of the proximal point method. To see this, fix $\omega \in \mathbb{R}^d$. Then $P_v(\omega)$ is the solution of $z = \omega - \eta v(z)$. Equivalently, $P_v(\omega)$ is the fixed point of

$$\varphi_v : z \mapsto \omega - \eta v(z),$$

(6)

which is a contraction for $\eta > 0$ small enough. From Picard’s fixed point theorem one gets that the proximal point operator, $P_v(\omega)$, can be obtained as the limit of $\varphi_v^k(\omega)$ when $k$ goes to infinity. What Nemirovski (2004) showed is that $\varphi^2_{v,\omega}(\omega)$, that is to say the extragradient update, is close enough to the result of the fixed point computation to be used in place of the proximal point update without affecting the sublinear convergence speed. Our analysis of multi-step extrapolation methods will encompass all the iterates $\varphi^k_{v,\omega}$ and we will show that a similar phenomenon happens for linear convergence rates. On the other hand, Tseng (1995) proved linear convergence results for extrapolation by using the projection-type error bound Tseng (1995, Eq. 5) which, in the unconstrained case, i.e. for $v(\omega^*) = 0$, can be written as,

$$\gamma \|\omega - \omega^*\|_2 \leq \|v(\omega)\|_2 \quad \forall \omega \in \mathbb{R}^d.$$  

(7)

The author then shows that this condition holds for the bilinear game of (3) and that it induces a convergence rate of $1 - c \sigma_{\min}(A)^2/\sigma_{\max}(A)^2$ for some constant $c > 0$. He also shows that this condition is implied by strong monotonicity and this yields a convergence rate of $1 - c' \mu^2/2L^2$ for some constant $c' > 0$. However, this analysis is suboptimal as Nesterov and Scrim ali (2006) showed that extragradient converges at a rate of $1 - \mu/(2L)$ in this setting. Our analysis builds on the results from Tseng (1995) and extends them to cover the whole range of games and recover the optimal rate of Nesterov and Scrim ali (2006).

3.3 Is the gradient method locally faster than extragradient?

In this subsection, we consider an unconstrained saddle-point problem: $\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} f(x, y)$, where the parameters of the two players have the same dimension. If $f$ is two times continuously differentiable and convex-concave, one can interpret the spectral rate of the gradient method mentioned earlier in terms of the standard strong convexity and Lipschitz-smoothness constants. There are several cases but one of them is of special interest to us as it demonstrates the precision of spectral bounds. Consider such a saddle-point problem with $f$ satisfying

$$\mu_1 I \preceq \nabla^2 f \preceq L_1 I, \quad \mu_2 I \preceq -\nabla^2 f \preceq L_2 I \quad \text{and} \quad \mu_{12}^2 I \preceq (\nabla_x \nabla_y f')^T(\nabla_x \nabla_y f) \preceq L_{12}^2 I,$$

(8)

where $\mu_1, \mu_2$ and $\mu_{12}$ are non-negative constants. In particular $f$ is strongly convex-concave and there is an unique stationary point $\omega^* = (x^*, y^*)$. Assume that, at this point, $\nabla^2 f(\omega^*), \nabla^2 f(\omega^*), (\nabla_x \nabla_y f(\omega^*)), (\nabla_x \nabla_y f(\omega^*))^T$ commute. This is the case if, for instance, the objective is separable, i.e. $f(x, y) = \sum_{i=1}^m f_i(x_i, y_i)$. In particular, if the game is highly adversarial, more exactly if $\mu_{12} > 2 \max(L_1 - \mu_2, L_2 - \mu_1)$, then we can upper bound the rate of Thm. 2 as follows:

**Corollary 1.** Under the assumptions of Thm. 2 and the previous hypotheses, if in addition $\mu_{12} > 2 \max(L_1 - \mu_2, L_2 - \mu_1)$, then the spectral radius of the gradient operator is upper bounded by

$$\rho(\nabla F_\eta(\omega^*))^2 \leq 1 - \frac{1}{4} \frac{(\mu_1 + \mu_2)^2}{L_{12}^2 + L_1 L_2}.$$  

(9)
What is somewhat surprising about this result is that in some regimes, it induces faster local convergence rates than the existing global ones for extragradient. Indeed, in this case, the rate of Nesterov and Scrima (2006) gives
\[
1 - \frac{\min(\mu_1, \mu_2)}{2L_{\max}},
\]
where \(L_{\max} = \max(L_1, L_2, L_12)\). If, say, \(\mu_2\) goes to zero, that is to say the game becomes unbalanced, the rate of extragradient goes to 1 while the one of (9) stays bounded by a constant which is strictly less than 1. Indeed, the rate of Cor. 1 involves the arithmetic mean of \(\mu_1\) and \(\mu_2\), which is roughly the maximum of them, while (10) makes only the minimum of the two appear. This adaptivity to the best strong convexity constant is not present in the standard convergence rates of the extragradient method. We now remedy this situation with a new analysis of extragradient.

4 Convergence analysis of extragradient

4.1 Spectral analysis of extrapolation methods

In this subsection, we study the local dynamics of a family of algorithms which include the extragradient method. Define a \(k\)-extrapolation method by the operator
\[
F_{k,\eta} : \omega \mapsto \varphi^k_{\eta,\omega} (\omega), \quad \text{where} \quad \varphi^k_{\eta,\omega} : z \mapsto \omega - \eta v(z).
\]

We are essentially considering all the iterates of the fixed point computation discussed in §3.2. Note that \(F_{1, \eta}\) is the gradient method and \(F_{2, \eta}\) is the extragradient method. We aim at studying the local convergence properties of these methods at stationary points of the gradient dynamics, so fix \(\omega^* \in \mathbb{R}^d\) s.t. \(v(\omega^*) = 0\) and let \(\sigma^* = \text{Sp} \nabla v(\omega^*)\). We compute the spectra of these operators at this point and this immediately yields a bound on the spectral radius on the proximal point operator:

**Lemma 1.** Assuming that the eigenvalues of \(\nabla v(\omega^*)\) all have non-negative real parts, the proximal point operator \(F_{1, \eta}\) is continuously differentiable in a neighborhood of \(\omega^*\). Moreover, the spectra of the \(k\)-extrapolation operator and the proximal point operator are given by:
\[
\text{Sp} \nabla F_{\eta, k}(\omega^*) = \{ \sum_{j=0}^{k} (-\eta \lambda)^j \mid \lambda \in \sigma^* \} \quad \text{and} \quad \text{Sp} \nabla P_{\eta}(\omega^*) = \{ (1 + \eta \lambda)^{-1} \mid \lambda \in \sigma^* \},
\]

Hence, for all \(\eta > 0\), the spectral radius of the operator of the proximal point method is bounded by
\[
\rho(\nabla P_{\eta}(\omega^*))^2 \leq 1 - \frac{2\eta \Re \lambda + \eta^2 |\lambda|^2}{|1 + \eta \lambda|^2}.
\]

Again, this shows that a \(k\)-extrapolation method is essentially an approximation of the proximal point method for small step sizes as \((1 + \eta \lambda)^{-1} = \sum_{j=0}^{k} (-\eta \lambda)^j + \mathcal{O}\left(|\eta \lambda|^{k+1}\right)\). This could suggest that increasing the number of extrapolations might yield better methods but we will actually see that \(k = 2\) is enough to achieve a similar rate to proximal. We then bound the spectral radius of \(\nabla F_{\eta, k}(\omega^*)\):

**Theorem 3.** Let \(\sigma^* = \text{Sp} \nabla v(\omega^*)\). If the eigenvalues of \(\nabla v(\omega^*)\) all have non-negative real parts, the spectral radius of the \(k\)-extrapolation method for \(k \geq 2\) satisfies:
\[
\forall \eta \leq \frac{1}{4 \max_{\lambda \in \sigma^*} |\lambda|}, \quad \rho(\nabla F_{\eta, k}(\omega^*))^2 \leq 1 - \frac{2\eta \Re \lambda + \frac{7}{16} \eta^2 |\lambda|^2}{|1 + \eta \lambda|^2},
\]

For \(\eta = \left(\frac{\max_{\lambda \in \sigma^*} |\lambda|}{\eta_{\text{min}}}\right)^{-1}\), this can be simplified as:
\[
\rho(\nabla F_{\eta, k}(\omega^*))^2 \leq 1 - \frac{1}{4} \left(\frac{\min_{\lambda \in \sigma^*} \Re \lambda}{\max_{\lambda \in \sigma^*} |\lambda|} + \frac{1}{16} \min_{\lambda \in \sigma^*} |\lambda|^2 \right).
\]

First, note that the bound of (14) is surprisingly close to the one of the proximal method (13). However, one can wonder why the proximal point converges with any step size — and so arbitrarily fast — while it is not the case for the \(k\) extrapolation method, even as \(k\) goes to infinity. The reason for this difference is that for the fixed point iterates to converge to the proximal point operator, one needs \(\varphi_{\eta,\omega}\) to be a contraction and so to have \(\eta\) small enough, at least \(\eta < \left(\frac{\max_{\lambda \in \sigma^*} |\lambda|}{\eta_{\text{min}}}\right)^{-1}\) for local guarantees. This explains the bound on the step size for extrapolation methods.
The bound of (15) involves two terms: the first term can be seen as the strong monotonicity of the problem, which is predominant in convex optimization problems, while the second shows that even in the absence of it, this method still converges, such as in bilinear games. What is more, in situation in between, this bound shows that the extragradient method exploits the biggest of these quantities as they appear as a sum.

Moreover, note that if \( v \) is \( \mu \)-strongly monotone and \( L \)-Lipschitz, this bound is at least as precise as the standard one \( 1 - \mu/(2L) \) as \( \mu \) lower bounds the real part of the eigenvalues of the Jacobian, and \( L \) upper bounds their magnitude, as shown in Lem. 9 in §E.2. On the other hand, Thm. 3 also recovers the standard rates for the bilinear problem\(^4\), as shown below:

**Corollary 2** (Bilinear game). Consider the bilinear game of (3). The iterates of the \( k \)-extrapolation method with \( k \geq 2 \) converge globally to \( \omega^* \) at a linear rate of \( \mathcal{O}\left(\left(1 - \frac{1}{4} \|\sigma_{\min}(A)^2\|\right)^t\right) \).

Note that this rate is very similar – up to the constant in front of the condition number – to the one derived by Gidel et al. (2019b) for alternating gradient descent with negative momentum. This raises the question of whether general acceleration still exists for games, as we would have expected from its literature of (4L\(^{-1}\)). For this we use the usual strong monotonicity assumption as well as the condition \((\eta \mu)^2 \geq \lambda^2\) which is left as future work. So, instead of (7), we consider:

\[
\gamma \|\omega - \omega'\|_2 \leq \|v(\omega) - v(\omega')\|_2, \quad \forall \omega, \omega' \in \mathbb{R}^d. \tag{16}
\]

This can be seen as a "weak strong monotonicity" as it is implied by strong monotonicity, with \( \gamma = \mu \), but it also holds for a square non-singular bilinear example of (3) with \( \gamma = \sigma_{\min}(A) \). This condition can be related to the properties of \( \nabla v \) as follows:

**Lemma 2.** Let \( v \) be continuously differentiable and \( \gamma > 0 : (16) \) holds if and only if \( \sigma_{\min}(\nabla v) \geq \gamma \).

We can now state our global convergence result:

**Theorem 4.** Let \( v : \mathbb{R}^d \to \mathbb{R}^d \) be continuously differentiable and (i) \( \mu \)-strongly monotone for some \( \mu \geq 0 \), (ii) \( L \)-Lipschitz, (iii) such that \( \sigma_{\min}(\nabla v) \geq \gamma \) for some \( \gamma \geq 0 \). Then, for \( \eta \leq (4L)^{-1} \), the iterates of the extragradient method \( (\omega_t)_t \) converge linearly to \( \omega^* \) the unique stationary point of \( v \),

\[
\|\omega_t - \omega^*\|_2^2 \leq \left(1 - \left(\eta \mu + \frac{7}{16} \eta^2 \gamma^2\right)\right)^t \|\omega_0 - \omega^*\|_2^2 \quad \forall t \geq 0. \tag{17}
\]

For \( \eta = (4L)^{-1} \), this can be simplified as: \( \|\omega_t - \omega^*\|_2^2 \leq \left(1 - \frac{1}{4} \left(\frac{\mu}{L} + \frac{7}{16} \gamma^2\right)\right)^t \|\omega_0 - \omega^*\|_2^2 \).

As Thm. 3, this result not only recovers both the bilinear and the strongly monotone case, but shows that the extragradient method actually gets the best of both world when in between. Indeed, this bound involves the maximum of two condition numbers and therefore this method takes advantage of the best conditioning. Furthermore this rate is surprisingly similar to the result of Thm. 3 though less precise: \( \mu \) is a lower bound on the real part of the eigenvalues of \( \nabla v \), \( \gamma \) a lower bound on their magnitude and \( L \) an upper bound on the latter (see Lem. 9 in §E.2).

\(^4\)Note that by exploiting the special structure of the bilinear game and the fact that \( k = 2 \), one could derive a better constant in the rate.
Remark 1 (Interpretation of the condition numbers). As in the previous section, this rate of convergence for the extragradient method is similar to the rate of the proximal point method for a small enough step size, as shown by Prop. 1 in §E.2. Moreover, the proof of the latter gives insight into the two quantities appearing in the rate of Thm. 4. Indeed, the convergence result for the proximal point method is obtained by bounding the singular values of $\nabla P_\eta$, the Jacobian of the operator, and so by bounding the eigenvalues of $(\nabla P_\eta)^T (\nabla P_\eta)$. But the latter is equal to:

$$
(\nabla P_\eta)^T \nabla P_\eta = \left( I_d + \eta S(\nabla v) + \eta^2 \nabla v \nabla v^T \right)^{-1} \text{ where } S(\nabla v) = \frac{\nabla v + \nabla v^T}{2}.
$$

(18)

We dropped the dependence on $\omega$ for compactness. This explains why it is the quantities $L/\mu$ and $L^2/\gamma^2$ which appear in the convergence rate, as the first corresponds to the condition number of $S(\nabla v)$ and the second to the condition number of $\nabla v \nabla v^T$. Thus, the proximal point method uses information from both matrices to converge, and so does extragradient, explaining why it takes advantage of the best conditioning. Furthermore, $\nabla v \nabla v^T$ is often positive definite, even when $S(\nabla v)$ is not. That’s why these methods do converge even in the absence of strong monotonicity.

In the next section, we show that the rates we proved for the extragradient method are tight and optimal among general extrapolation methods.

5 Lower bounds for the convergence rate of extrapolation methods

In this section, we use the framework of Arjevani et al. (2016) to derive lower bounds for game optimization. Though it was originally designed for convex optimization, the tools we rely on can be extended to games, as discussed in §F. The idea of this framework is again to see algorithms as the iterated application of an operator. If the vector field is linear, this transformation is linear too and so its behavior when iterated is mainly governed by its spectral radius. As shown below, the spectral radius of the operator of a method yields a lower bound on the speed of convergence of the iterates of this method. So to show a lower bound for a class of algorithms, we consider a very general parametric operator which encompasses all the methods of this class and lower bound its spectral radius. This way, we get a lower bound on the spectral radius of all the operators of the methods of this class and therefore a lower bound on their speed of convergence.

We consider $\mathcal{V}_d$ the set of affine vector fields $v : \mathbb{R}^d \to \mathbb{R}^d$. The class of algorithms we consider is the class of 1-Stationary Canonical Linear Iterative algorithms (1-SCLI). Such an algorithm is defined by a mapping $\mathcal{N} : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$. The associated update rule can be defined through,

$$
F_{\mathcal{N}}(\omega) = w + \mathcal{N}(\nabla v)\omega \quad \forall \omega \in \mathbb{R}^d,
$$

(19)

This form of the update rule is required by the consistency condition of Arjevani et al. (2016) which is necessary for the algorithm to converge to stationary points, as discussed in §F. Also note that 1-SCLI are first-order methods which uses only the last iterate to compute the next one. Accelerated methods such as Accelerated Gradient Descent (Nesterov, 2014) or the Heavy Ball method (Polyak, 1964) belong in fact to the class of 2-SCLI, which encompass methods which uses the last two iterates.

As announced above, the spectral radius of the operator gives a lower bound on the speed of convergence of the iterates of the method on affine vector fields, which is sufficient to include bilinear games, quadratics and so strongly monotone settings too.

**Theorem 5 (Arjevani et al. (2016)).** For all $v \in \mathcal{V}_d$, for almost every\(^5\) initialization point $\omega_0 \in \mathbb{R}^d$, if $(\omega_t)_t$ are the iterates of $F_{\mathcal{N}}$ starting from $\omega_0$,

$$
\|\omega_t - \omega^*\| \geq \Omega(\rho(\nabla F_{\mathcal{N}})^d \|\omega_0 - \omega^*\|).
$$

(20)

5.1 Tightness of the analysis of the gradient method

In this subsection, we show that the rate of Thm. 2 from Gidel et al. (2019b) is tight.

**Lemma 3.** Let $v : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field and let $\omega \in \mathbb{R}^d$ such that the eigenvalues of $\nabla v(\omega)$ have all positive real parts. For all $\eta > 0$, the spectral radius of the gradient operator is lower bounded by:

$$
\rho(\nabla F_\eta(\omega))^2 \geq 1 - 4 \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re\left(\frac{1}{\lambda}\right) \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re(\lambda).
$$

(21)

\(^5\)For any measure absolutely continuous w.r.t. the Lebesgue measure.
This result is stronger than what we need for a standard lower bound: using Thm. 5, this yields a lower bound on the convergence of the iterates for all games with affine vector fields. In general, if \( \sigma_{\lambda} \in \text{Sp}(\nabla v(\omega^r)) \), \( \mathbb{R}(1/\lambda) \leq (\max_{\lambda \in \text{Sp}(\nabla v(\omega^r))} |\lambda|)^{-1} \) and, for adversarial games, the first term can be arbitrarily smaller than the second one. Hence, in this setting which is of special interest to us, the extrapolation method has a much faster convergence speed than the gradient method. Moreover, this lower bound shows that the difference of behaviors of the extragradient and the gradient method is not an artifact of the analysis.

5.2 Lower bounds for extrapolation methods

As shown in Lem. 11 in §G.2, for the \( k \)-extrapolation method, \( N \) is a polynomial of degree \( k - 1 \). So we consider the more general class of methods whose coefficient mapping \( N \) is any polynomial of degree at most \( k - 1 \). It includes for instance all the \( k' \)-extrapolation methods for \( k' \leq k \) with possibly different step sizes for each extrapolation step (see §G.2 for more examples). Our main result is that no method of this class cannot significantly beat the convergence speed of the extrapolation method of Thm. 3 and Thm. 4. For this, we proceed in two steps: for each of the two terms of these bounds, we provide an example matching it up to a factor. In (i) of the following theorem we give an example of convex optimization problem which matches the strong monotonicity term. Note that this example is an extension to the results of Arjevani et al. (2016) where the authors only considered only the case where \( N \) was a constant. Next, in (ii), we match the other term with a bilinear game example.

**Theorem 6.** Let \( 0 < \mu, \gamma < L \). (i) If \( d - 2 \geq k \geq 3 \), there exists \( v \in \mathbb{V}_d \) with a symmetric positive Jacobian whose spectrum is in \([\mu, L]\), such that, for any \( N \) real polynomial of degree at most \( k - 1 \),

\[
\rho(F_N) \geq 1 - \frac{4k^3}{\pi L} \cdot \tag{22}
\]

(ii) If \( d/2 - 2 \geq k/2 \geq 3 \) and \( d \) is even, there exists \( v \in \mathbb{V}_d \) \( L \)-Lipschitz with \( \min_{\lambda \in \text{Sp}(\nabla v)} |\lambda| = \sigma_{\min}(\nabla v) \geq \gamma \) corresponding to a bilinear game as described in (3) with \( m = d/2 \) such that, for any \( N \) real polynomial of degree at most \( k - 1 \),

\[
\rho(F_N) \geq 1 - \frac{k^3 \gamma^2}{2\pi L} \cdot \tag{23}
\]

First, these lower bounds show that both our convergence analyses of extragradient are tight, by looking at them for \( k = 3 \) for instance. Then, though these bounds become looser as \( k \) grows, they still show that the potential improvements are not significant in terms of conditioning, especially compared to the change of regime between the gradient and the extragradient method. Hence, they still essentially match the convergence speed of extragradient of Thm. 3 or Thm. 4. Therefore, extragradient can be considered as optimal among the general class of algorithms which uses at most a fixed number of composed gradient evaluations and only the last iterate. In particular, there is no need to consider algorithms with more extrapolation steps or with different step sizes for each of them as it only yields a constant factor improvement.

6 Conclusion

In this paper, we studied the dynamics of the extragradient method, both locally and globally. Our analyses are tight and unified as they cover the whole spectrum of games from bilinear to purely cooperative settings. They show that in between, the extragradient method enjoys the best of both worlds. We confirm that, unlike in convex optimization, the behaviors of the extragradient and the gradient method differ significantly. The other lower bounds show that extragradient can be considered as optimal among first-order methods that use only the last iterate.

Finally, as mentioned in §4.1, the rate of alternating gradient descent with negative momentum from Gidel et al. (2019b) on the bilinear example essentially matches the rate of extragradient in Cor. 2. Thus the question of acceleration for adversarial games remains open. On the other hand, for convex optimization problems, i.e. purely cooperative games, it is well known that adding positive momentum yields accelerated rates (Polyak, 1964). We leave as future work the unification of these two cases, which might be achieved through the combination of extragradient and momentum.
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A Notations

We denote by $\text{Sp}(A)$ the spectrum of a matrix $A$. Its spectral radius is defined by $\rho(A) = \max\{|\lambda| \mid \lambda \in \text{Sp}(A)\}$. We write $\sigma_{\text{min}}(A)$ for the smallest singular value of $A$, and $\sigma_{\text{max}}(A)$ for the largest. $\Re$ and $\Im$ denote respectively the real part and the imaginary part of a complex number. We write $A \preceq B$ for two symmetric real matrices if and only if $B - A$ is positive semi-definite. For a vector $X \in \mathbb{C}^d$, denote its transpose by $X^T$ and its conjugate transpose by $X^H$. $\|\|$ denotes an arbitrary norm on $\mathbb{R}^d$ unless specified. We sometimes denote $\min(a, b)$ by $a \wedge b$ and $\max(a, b)$ by $a \vee b$. For $f : \mathbb{R}^d \to \mathbb{R}^d$, we denote by $f^k$ the composition of $f$ with itself $k$ times, i.e. $f^k(\omega) = f \circ f \circ \cdots \circ f(\omega)$.

B Summary of the results

|                  | Bilinear (3) | Convex | Highly adv. §3.3 | General §4.2 |
|------------------|--------------|--------|------------------|--------------|
| Tseng (1995)     | $\sigma_{\text{min}}^2/\sigma_{\text{max}}^2$ | $\mu_1^2/\mu_2$ | $(\mu_1^2 + \mu_2^2)/\mu_1^2$ | $\mu_2^2/\mu_1$ |
| Nesterov and Scrimalli (2006) | $\times$ | $\mu_1^2/\mu_1^2$ | $\mu_1^2/\mu_1^2$ | $\mu_1^2/\mu_1^2$ |
| Mokhtari et al. (2019) | $\sigma_{\text{min}}^2/\sigma_{\text{max}}^2$ | $\mu_1^2/\mu_1^2$ | $\mu_1^2/\mu_1^2 + \mu_2^2/\mu_1^2$ | $\mu_1^2/\mu_1^2 + \mu_2^2/\mu_1^2$ |
| Thm. 4 (Global)  | $\sigma_{\text{min}}^2/\sigma_{\text{max}}^2$ | $\mu_1^2/\mu_1^2$ | $\mu_1^2/\mu_1^2 + \mu_2^2/\mu_1^2$ | $\mu_1^2/\mu_1^2 + \mu_2^2/\mu_1^2$ |
| Thm. 3 (Local)   | $\sigma_{\text{min}}^2/\sigma_{\text{max}}^2$ | $\mu_1^2/\mu_1^2$ | $\min \Re|\lambda|$ | $\mu_1^2/\mu_1^2 + \mu_2^2/\mu_1^2$ |
| Lower bounds §5.2| $\sigma_{\text{min}}^2/\sigma_{\text{max}}^2$ | $\mu_1^2/\mu_1^2$ | $\mu_1^2/\mu_1^2$ | $\mu_1^2/\mu_1^2$ |

Table 2: Summary of convergence results for the extragradient method on a particular highly adversarial game and in the general case. See the corresponding sections for details and notations. If a result shows that the iterates converge as $O((1 - r)^t)$, the quantity $r$ is reported. The larger the better. In general $r$ is much smaller than 1. The constants are dropped for clarity.

The interpretation of Thm. 4 in the context of §3.3 is done thanks to Lem. 5.

C Interpretation of spectral quantities in a two-player zero-sum game

In this appendix section, we are interested in interpreting spectral bounds in terms of the usual strong convexity and Lipschitz continuity constants in a two-player zero-sum game:

$$\min_{x \in \mathbb{R}^m} \max_{y \in \mathbb{R}^p} f(x, y)$$

with $f$ is two times continuously differentiable.

Assume,

$$\mu_1 I_m \preceq \nabla_x^2 f \preceq L_1 I_m$$

$$\mu_2 I_p \preceq -\nabla_y f \preceq L_2 I_p$$

$$\mu_{12}^2 I_p \preceq (\nabla_x \nabla_y f)^T (\nabla_x \nabla_y f) \preceq L_{12}^2 I_p$$

where $\mu_1, \mu_2$ and $\mu_{12}$ are non-negative constants. Let $\omega^* = (x^*, y^*)$ be a stationary point. To ease the presentation, let,

$$\nabla v(\omega^*) = \begin{pmatrix}
\nabla_x^2 f(\omega^*) \\
(\nabla_x \nabla_y f(\omega^*))^T
\end{pmatrix} = \begin{pmatrix}
S_1 & A \\
-A^T & S_2
\end{pmatrix}.$$  

Now, more precisely, we are interested in lower bounding $\Re(\lambda)$ and $|\lambda|$ and upper bounding $|\lambda|$ for $\lambda \in \text{Sp} \nabla v(\omega^*)$. 

12
C.1 Commutative and square case

Assume that \( p = m = \frac{d}{2} \) and that \( S_1, S_2, A \) and \( A^T \) commute. This is the case if, for instance, the objective is separable, i.e. \( f(x, y) = \sum_{i=1}^{m} f_i(x_i, y_i) \). Then, a well-known linear algebra theorem states that there exists \( U \in \mathbb{R}^{d \times d} \) unitary such that \( S_1 = U \text{diag}(\alpha_1, \ldots, \alpha_m) U^T \), \( S_2 = U \text{diag}(\beta_1, \ldots, \beta_m) V^T \) and \( A A^T = U \text{diag}(\sigma_1^2, \ldots, \sigma_p^2) U^T \) where \( \alpha_1, \ldots, \alpha_m \) are the eigenvalues of \( S_1 \), \( \beta_1, \ldots, \beta_m \) are the eigenvalues of \( S_2 \) and \( \sigma_1, \ldots, \sigma_p \) are the singular values of \( A \). See Lax (2007, p. 74) for instance.

Define,

\[
\mu = \begin{pmatrix} \mu_1 & \mu_{12} \\ -\mu_{12} & \mu_2 \end{pmatrix}
\]

\[
L = \begin{pmatrix} L_1 & L_{12} \\ -L_{12} & L_2 \end{pmatrix}.
\]

Denote by \(|\mu|\) and \(|L|\) the determinants of these matrices, and by \( \text{Tr} \mu \) and \( \text{Tr} L \) their traces.

In this case we get an exact characterization of the spectrum \( \nabla v(\omega^*) \), which we denote by \( \sigma^* = \text{Sp} \nabla v(\omega^*) \):

**Lemma 4.** \( \lambda \in \text{Sp}(\nabla v(\omega^*)) \) if and only if there exists some \( i \leq d \) such that \( \lambda \) is a root of

\[
P_i = X^2 - (\alpha_i + \beta_i)X + \alpha_i\beta_i + \sigma_i^2
\]

**Proof.** We compute the characteristic polynomial of \( \nabla v(\omega^*) \) using that \( S_2 \) and \( A^T \) commute, using the formula for the determinant of a block matrix, which can be found in Zhang (2005, Section 0.3) for instance.

\[
\begin{vmatrix} XI - S_1 & -A \\ A^T & XI - S_2 \end{vmatrix} = |(XI - S_1)(XI - S_2) + AA^T|
\]

\[
= |X^2I - X(S_1 + S_2) + S_1S_2 + AA^T|
\]

\[
= \prod_i (X^2 - (\alpha_i + \beta_i)X + \alpha_i\beta_i + \sigma_i^2)
\]

**Theorem 7.** We have the following results on the eigenvalues of \( \nabla v(\omega^*) \).

(a) For \( i \leq m \), if \( (\alpha_i - \beta_i)^2 < 4\sigma_i^2 \), the roots of \( P_i \) satisfy:

\[
\frac{\text{Tr} \mu}{2} \leq \Re(\lambda), \quad \det \mu \leq |\lambda|^2 \leq \det L, \quad \forall \lambda \in \mathbb{C} \ s.t. \ P_i(\lambda) = 0.
\]

(b) For \( i \leq m \), if \( (\alpha_i - \beta_i)^2 \geq 4\sigma_i^2 \), the roots of \( P_i \) are real non-negative and satisfy:

\[
\max \left( \mu_1 \wedge \mu_2, \frac{\det \mu}{\text{Tr} L} \right) \leq \lambda \leq L_1 \lor L_2, \quad \forall \lambda \in \mathbb{C} \ s.t. \ P_i(\lambda) = 0.
\]

(c) Hence, in general,

\[
\mu_1 \wedge \mu_2 \leq \Re \lambda, \quad |\lambda|^2 \leq 2L_{\text{max}}^2 \quad \forall \lambda \in \sigma^*,
\]

where \( L_{\text{max}} = \max(L_1, L_2, L_{12}) \).

**Proof.**

(a) Assume that \( (\alpha_i - \beta_i)^2 < 4\sigma_i^2 \), i.e. the discriminant of the polynomial \( P_i \) of Lem. 4 is negative. Consider \( \lambda \) a root of \( P_i \). Then \( \Re \lambda = \frac{\alpha_i + \beta_i}{2} \) and \( |\lambda|^2 = \alpha_i\beta_i + \sigma_i^2 \). Hence \( \Re \lambda \geq \frac{1}{2} \text{Tr} \mu \) and \( \det \mu \leq |\lambda|^2 \leq \det L \).
(b) Assume that \((\alpha_i - \beta_i)^2 \geq 4\sigma_i^2\), i.e. the discriminant of the polynomial \(P_i\) of Lem. 4 is non-negative. This implies that \(\Delta = (\text{Tr } L)^2 - 4 \det \mu \geq 0\).

Denote by \(\lambda_+\) and \(\lambda_-\) the two real roots of \(P_i\). Then

\[
\lambda_{\pm} = \frac{\alpha_i + \beta_i \pm \sqrt{(\alpha_i + \beta_i)^2 - 4(\alpha_i \beta_i + \sigma_i^2)}}{2}
\]

Hence

\[
\lambda_+ \geq \lambda_- \geq \min_{\text{max(Tr } \mu, 4 \det \mu) \leq x \leq \text{Tr } L} \frac{x - \sqrt{x^2 - 4 \det \mu}}{2}
\]

As \(x \mapsto x - \sqrt{x^2 - 4 \det \mu}\) is decreasing on its domain, the minimum is reached at \(\text{Tr } L\) and is \(\frac{\text{Tr } L - \sqrt{\Delta}}{2} \geq 0\). However this lower bound is quite loose when \(A = 0\). So note that

\[
\lambda_- = \frac{\alpha_i + \beta_i - \sqrt{(\alpha_i + \beta_i)^2 - 4(\alpha_i \beta_i + \sigma_i^2)}}{2} = \frac{\alpha_i - \beta_i}{2} = \alpha_i \lor \beta_i
\]

(33)

(34)

(35)

Similarly,

\[
\lambda_+ \leq \frac{\alpha_i + \beta_i + \sqrt{(\alpha_i - \beta_i)^2}}{2} = \alpha_i \lor \beta_i \leq L_1 \lor L_2
\]

(36)

Finally:

\[
L_1 \lor L_2 \geq \lambda_+ \geq \lambda_- \geq \max \left(\frac{\text{Tr } L - \sqrt{\Delta}}{2}, \mu_1 \land \mu_2\right)
\]

(37)

Moreover,

\[
\frac{\text{Tr } L - \sqrt{\Delta}}{2} = \frac{\left(\text{Tr } L - \sqrt{\Delta}\right)\left(\text{Tr } L + \sqrt{\Delta}\right)}{\text{Tr } L + \sqrt{\Delta}} = \frac{4 \det \mu}{\text{Tr } L + \sqrt{\Delta}} \geq \frac{2 \det \mu}{\text{Tr } L}
\]

(38)

(39)

(40)

which yields the result.

(c) These assertions are immediate corollaries of the two previous ones.

\(\square\)

We need the following lemma to be able to interpret Thm. 4 in the context of §3.3.

**Lemma 5.** The singular values of \(\nabla v(\omega^*)\) can be lower bounded as:

\[
\mu_{12}(\mu_{12} - \max(L_1 - \mu_2, L_2 - \mu_1)) \leq \sigma_{\min}(\nabla v(\omega^*))^2.
\]

(41)

In particular, if \(\mu_{12} > 2 \max(L_1 - \mu_2, L_2 - \mu_1)\), this becomes

\[
\frac{1}{2} \mu_{12}^2 \leq \sigma_{\min}(\nabla v(\omega^*))^2.
\]

(42)

**Proof.** To prove this we compute the eigenvalues of \((\nabla v(\omega^*))^T \nabla v(\omega^*)\). We have that,

\[
(\nabla v(\omega^*))^T \nabla v(\omega^*) = \begin{pmatrix} S_2^2 + AA^T & S_1A - AS_2 \\ A^T S_1 - S_2 A^T & A^T A + S_2^2 \end{pmatrix}
\]

(43)
As in the proof of Lem. 4, using that all the blocks commute,
\[
|XI - (\nabla v(\omega^*))^T \nabla v(\omega^*)| = \left| (XI - S_1^2 - AA^T)(XI - S_2^2 - A^T A) - (S_1 - S_2)^2 AA^T \right|
\]
\[
= \prod_i \left( (XI - \alpha_i^2 - \sigma_i^2)(XI - \beta_i^2 - \sigma_i^2) - (\alpha_i - \beta_i)^2 \sigma_i^2 \right). \tag{44}
\]

Let \(Q_i(X) = (XI - \alpha_i^2 - \sigma_i^2)(XI - \beta_i^2 - \sigma_i^2) - (\alpha_i - \beta_i)^2 \sigma_i^2 \). Its discriminant is
\[
\Delta_i' = (\alpha_i^2 + \beta_i^2 + 2\sigma_i^2)^2 - 4(\alpha_i^2 + \sigma_i^2)(\beta_i^2 + \sigma_i^2) - (\alpha_i - \beta_i)^2 \sigma_i^2 \tag{46}
\]
\[
= (\alpha_i - \beta_i)^2((\alpha_i + \beta_i)^2 + 4\sigma_i^2) \geq 0. \tag{47}
\]

Hence the roots of \(Q_i\) are:
\[
\lambda_{i \pm} = \frac{1}{2} \left( \alpha_i^2 + \beta_i^2 + 2\sigma_i^2 \pm \sqrt{(\alpha_i - \beta_i)^2((\alpha_i + \beta_i)^2 + 4\sigma_i^2)} \right). \tag{48}
\]
The smallest is \(\lambda_{i -}\) which can be lower bounded by
\[
\lambda_{i -} = \frac{1}{2} \left( \alpha_i^2 + \beta_i^2 + 2\sigma_i^2 - \sqrt{(\alpha_i + \beta_i)^2((\alpha_i + \beta_i)^2 + 4\sigma_i^2)} \right) \tag{49}
\]
\[
\geq \frac{1}{2} \left( \alpha_i^2 + \beta_i^2 - |\alpha_i^2 - \beta_i^2| + 2\sigma_i(\sigma_i - |\alpha_i - \beta_i|) \right) \tag{50}
\]
\[
\geq \sigma_i(\sigma_i - |\alpha_i - \beta_i|) \tag{51}
\]
\[
\geq \mu_{12}(\mu_{12} - \max(L_1 - \mu_2, L_2 - \mu_1)). \tag{52}
\]

\[ \square \]

\section*{C.2 General case}

In this subsection, we briefly discuss the non-commutative and non-square case, and show that it is essentially the same analysis as before, though less precise. Let \(\lambda_1, \ldots, \lambda_l \in \mathbb{C}\) be the eigenvalues of \(\nabla v(\omega^*)\) and \([X_1 Y_1]^T, \ldots, [X_l Y_l]^T \in \mathbb{C}^n\) be associated eigenvectors. Let, for \(i \leq l\),
\[
\alpha_i = \begin{cases} \frac{X_i^H S_1 X_i}{||X_i||^2} & \text{if } X_i \neq 0, \\ \mu_1 & \text{otherwise} \end{cases}, \quad \beta_i = \begin{cases} \frac{Y_i^H S_2 Y_i}{||Y_i||^2} & \text{if } Y_i \neq 0, \\ \mu_2 & \text{otherwise} \end{cases}, \quad \sigma_i = \begin{cases} \frac{X_i^H A Y_i}{||X_i|| ||Y_i||} & \text{if } [X_i Y_i]^T \neq 0, \\ 0 & \text{otherwise} \end{cases}
\]

Notice that, as in the previous case, \(\mu_1 \leq \alpha_i \leq L_1, \mu_2 \leq \beta_i \leq L_2\) and \(\sigma_i^2 \leq L_{12}^2\) but \(\mu_{12}\) is no longer a lower bound for the \(\sigma_i\)s. Let \(\bar{\mu}_{12} = \min_i \sigma_i\) and
\[
\mu = \begin{pmatrix} \mu_1 & \bar{\mu}_{12} \\ -\bar{\mu}_{12} & \mu_2 \end{pmatrix}, \quad L = \begin{pmatrix} L_1 & L_2 \\ -L_2 & L_1 \end{pmatrix}
\]
The next lemma shows that the spectrum of \(\nabla v(\omega^*)\) can still be controlled with those quantities as in the previous case.

\textbf{Lemma 6.} If \(\lambda \in \text{Sp}(\nabla v(\omega^*))\), then there exists some \(i \leq l\) such that \(\lambda\) is a root of
\[
P_i = X^2 - (\alpha_i + \beta_i)X + \alpha_i \beta_i + \sigma_i^2.
\]

\textbf{Proof.} Fix \(1 \leq i \leq l\). With the notations presented above, \(\lambda_i\) is the eigenvalue associated to the eigenvector \([X_i Y_i]^T\) and so,
\[
S_1 X_i + A Y_i = \lambda_i X_i,
\]
\[
- A^T X_i + S_2 Y_i = \lambda_i Y_i.
\]

Multiplying the first equation on the left by \(X_i^H\) and the second by \(Y_i^H\) yields:
\[
\begin{pmatrix} \alpha_i - \lambda_i \\ -\lambda_i^* \end{pmatrix} \frac{X_i^H A Y_i}{||Y_i||^2} \begin{pmatrix} ||X_i||^2 \\ ||Y_i||^2 \end{pmatrix} = 0.
\]

(with the convention \(\frac{X_i^H A Y_i}{||Y_i||^2} = 0\) if \(Y_i = 0\) and \(\frac{Y_i^H A^T X_i}{||X_i||^2} = 0\) if \(X_i = 0\)). As \([X_i Y_i]^T \neq 0\), the determinant of this system is zero and this yields the desired polynomial. \(\square\)
Then the bounds of the previous subsection are still valid.

**Remark 2.** With the previous notations, \( \tilde{\mu}_{12} = \min_i \frac{\|X_i^H A Y_i\|}{\|X_i\| \|Y_i\|} \) (with the convention that \( \frac{X^H A Y_i}{\|X_i\| \|Y_i\|} = 0 \) if one of the vector is zero). Note that we cannot lower bound \( \tilde{\mu}_{12} \) with \( \mu_{12} \) in any way. Consider an example where \( S_1, S_2 \) and \( A \) are \( 2 \times 2 \) matrices and

\[
\nabla v(\omega^*) = \begin{pmatrix}
S_1 & A \\
-A^T & S_2
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 1 & 1 \\
1 & 3 & -1 & 0 \\
0 & 1 & 1 & 1 \\
-1 & 0 & 1 & 3
\end{pmatrix}.
\]

Then \( AA^T = I_2 \) so \( \mu_{12} = 1 \). But \( (1 \ 1 \ 1 \ 1)^T \) is an eigenvalue of \( \nabla v(\omega^*) \) and \( A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \). Hence \( \tilde{\mu}_{12} = 0 \).

On the other hand, \( \tilde{\mu}_{12} \) can be arbitrarily bigger than \( \mu_{12} \). Consider

\[
\nabla v(\omega^*) = \begin{pmatrix}
S_1 & A \\
-A^T & S_2
\end{pmatrix} = \begin{pmatrix}
2 - \epsilon & 1 & \epsilon & 0 \\
1 & 1 & 0 & 1 \\
-\epsilon & 0 & 2 + \epsilon & 1 \\
0 & -1 & 1 & 3
\end{pmatrix},
\]

with \( \epsilon > 0 \). Its eigenvectors are \( (1 \ 1 \ 1 \ 1)^T \) and \( (1 \ -1 \ 1 \ -1)^T \).

Hence \( \tilde{\mu}_{12} = \frac{1+\epsilon}{2} \) and \( \mu_{12} = \epsilon^2 \) so \( \tilde{\mu}_{12} \gg \mu_{12} \) when \( \epsilon \) goes to zero.

\section{Proofs of §3}

\subsection{Proofs of §3.2}

The convergence result of Thm. 1 can be strengthened if the Jacobian is constant as shown below. A proof of this classical result in linear algebra can be found in Arjevani et al. (2016) for instance.

**Theorem 8.** Let \( F : \mathbb{R}^d \rightarrow \mathbb{R}^d \) be a linear operator. If \( \rho(\nabla F) < 1 \), then for all \( \omega_0 \in \mathbb{R}^d \), the iterates \((\omega_t)\), defined as above converge linearly to \( \omega^* \) at a rate of \( O((\rho(\nabla F))^t) \).

Let us restate Thm. 2 for clarity.

**Theorem 2.** If the eigenvalues of \( \nabla v(\omega^*) \) all have positive real parts, then, for \( \eta = \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re(1/\lambda) \) one has the following upper bound on the spectral radius of \( F_\eta \):

\[
\rho(\nabla F_\eta(\omega^*))^2 \leq 1 - \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re(1/\lambda) \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re(\lambda). \tag{4}
\]

In this subsection, we quickly show how to obtain Thm. 2 from Theorem 2 of Gidel et al. (2019b), whose part which interests us now is the following:

**Theorem** (Gidel et al. (2019b, Theorem 2)). For \( \eta = \Re(1/\lambda_1) \) one has,

\[
\rho(\nabla F_\eta(\omega^*))^2 \leq 1 - \Re(1/\lambda_1) \delta \tag{53}
\]

where \( \delta = \min_{1 \leq j \leq m} |\lambda_j|^2 (2\Re(1/\lambda_j) - \Re(1/\lambda_1)) \) and \( \text{Sp} \nabla v(\omega^*) = \{\lambda_1, \ldots, \lambda_m\} \) sorted such that \( 0 < \Re(1/\lambda_1) < \Re(1/\lambda_2) < \cdots < \Re(1/\lambda_m) \).

**Proof of Thm. 2.** By definition of the order on the eigenvalues,

\[
\delta = \min_{1 \leq j \leq m} |\lambda_j|^2 (2\Re(1/\lambda_j) - \Re(1/\lambda_j) - \Re(1/\lambda_1)) \tag{54}
\]

\[
\geq \min_{1 \leq j \leq m} |\lambda_j|^2 (\Re(1/\lambda_j)) \tag{55}
\]

\[
= \min_{1 \leq j \leq m} \Re(\lambda_j) \tag{56}
\]

\end{proof}
D.2 Proofs of §3.3

Corollary 1. Under the assumptions of Thm. 2 and the previous hypotheses, if in addition \( \mu_{12} > 2 \max(L_1 - \mu_2, L_2 - \mu_1) \), then the spectral radius of the gradient operator is upper bounded by

\[
\rho(\nabla F_\eta(\omega^*))^2 \leq 1 - \frac{1}{4} \frac{(\mu_1 + \mu_2)^2}{L_1^2 + L_1 L_2}.
\]

Proof. Note that the hypotheses stated in §3.3 correspond to the assumptions of §C.1. Moreover, with the notations of this subsection, one has that \( \kappa \) will be used again later.

Hence, for all \( \eta > 0 \), the \( \eta \)-extrapolation operator and the proximal point operator are given by:

\[
\text{Sp} \nabla F_{\eta,k}(\omega^*) = \left\{ \sum_{j=0}^{k} (-\eta \lambda)^j \mid \lambda \in \sigma^* \right\} \quad \text{and} \quad \text{Sp} \nabla P_\eta(\omega^*) = \left\{ (1 + \eta \lambda)^{-1} \mid \lambda \in \sigma^* \right\}.
\]

Hence, for all \( \eta > 0 \), the spectral radius of the operator of the proximal point method is bounded by

\[
\rho(\nabla P_\eta(\omega^*))^2 \leq 1 - \min_{\lambda \in \sigma^*} \frac{2\eta |\lambda|}{1 + |\lambda|^2}.
\]

E Proofs of theorems of convergence analysis of §4

E.1 Proofs of §4.1

Lemma 1. Assuming that the eigenvalues of \( \nabla v(\omega^*) \) all have non-negative real parts, the proximal point operator \( P_\eta \) is continuously differentiable in a neighborhood of \( \omega^* \). Moreover, the spectra of the \( k \)-extrapolation operator and the proximal point operator are given by:

\[
\text{Sp} \nabla F_{\eta,k}(\omega^*) = \left\{ \sum_{j=0}^{k} (-\eta \lambda)^j \mid \lambda \in \sigma^* \right\} \quad \text{and} \quad \text{Sp} \nabla P_\eta(\omega^*) = \left\{ (1 + \eta \lambda)^{-1} \mid \lambda \in \sigma^* \right\}.
\]

Hence, for all \( \eta > 0 \), the spectral radius of the operator of the proximal point method is bounded by

\[
\rho(\nabla P_\eta(\omega^*))^2 \leq 1 - \min_{\lambda \in \sigma^*} \frac{2\eta |\lambda|^2}{1 + |\lambda|^2}.
\]

To prove the result about the \( k \)-extrapolation operator, we first show the following lemma, which will be used again later.

Recall that we defined \( \varphi_{\eta,\omega} : z \mapsto \omega - \eta v(z) \). We drop the dependence on \( \eta \) in \( \varphi_{\eta,\omega} \) for compactness.

Lemma 7. The Jacobians of \( \varphi_k(z) \) with respect to \( z \) and \( \omega \) can be written as

\[
\nabla_z \varphi_k(z) = (-\eta)^k \nabla v(\varphi_{\omega}^{k-1}(z)) \nabla v(\varphi_{\omega}^{k-2}(z)) \ldots \nabla v(\varphi_{\omega}^0(z))
\]

\[
\nabla_\omega \varphi_k(z) = \sum_{j=0}^{k-1} (-\eta)^j \nabla v(\varphi_{\omega}^{k-1}(z)) \nabla v(\varphi_{\omega}^{k-2}(z)) \ldots \nabla v(\varphi_{\omega}^{k-j}(z)).
\]

Proof. We prove the result by induction:

- For \( k = 1 \), \( \varphi_1(z) = \omega - \eta v(z) \) and the result holds.
- Assume this result holds for \( k \geq 0 \). Then,

\[
\nabla_z \varphi_{k+1}(z) = \nabla_z \varphi_{k}(\varphi_{\omega}^0(z)) \nabla_z \varphi_{k}^0(z)
\]

\[
= -\eta \nabla v(\varphi_{\omega}^0(z)) (-\eta)^k \nabla v(\varphi_{\omega}^{k-1}(z)) \ldots \nabla v(\varphi_{\omega}^0(z))
\]

\[
= (-\eta)^{k+1} \nabla v(\varphi_{\omega}^0(z)) \nabla v(\varphi_{\omega}^{k-1}(z)) \ldots \nabla v(\varphi_{\omega}^0(z)).
\]
For the derivative with respect to \( \omega \), we use the chain rule:

\[
\nabla_{\omega} \varphi_{\omega}^{-1}(z) = \nabla_{\omega} \varphi_{\omega}(\varphi_{\omega}^{k}(z)) + \nabla_{\omega} \varphi_{\omega}(\varphi_{\omega}^{k}(z)) \nabla_{\omega} \varphi_{\omega}(z) \quad (65)
\]

\[
= I_{d} - \eta v(\varphi_{\omega}^{k}(z)) \sum_{j=0}^{k-1} (-\eta)^{j} \nabla v(\varphi_{\omega}^{k-1}(z)) \cdots \nabla v(\varphi_{\omega}^{k-j}(z)) \quad (66)
\]

\[
= I_{d} + \sum_{j=0}^{k-1} (-\eta)^{j+1} \nabla v(\varphi_{\omega}^{k}(z)) \nabla v(\varphi_{\omega}^{k-1}(z)) \cdots \nabla v(\varphi_{\omega}^{k-j}(z)) \quad (67)
\]

\[
= I_{d} + \sum_{j=1}^{k} (-\eta)^{j} \nabla v(\varphi_{\omega}^{k}(z)) \nabla v(\varphi_{\omega}^{k-1}(z)) \cdots \nabla v(\varphi_{\omega}^{k-j}(z)) \quad (68)
\]

\[
= \sum_{j=0}^{k} (-\eta)^{j} \nabla v(\varphi_{\omega}^{k}(z)) \nabla v(\varphi_{\omega}^{k-1}(z)) \cdots \nabla v(\varphi_{\omega}^{k-j}(z)) \quad (69)
\]

\[
\Box
\]

In the proof of Lem. 1 and later we will use the spectral mapping theorem, which we state below for reference:

**Theorem 9 (Spectral Mapping Theorem).** Let \( A \in \mathbb{C}^{d \times d} \) be a square matrix, and \( P \) be a polynomial. Then,

\[
\text{Sp} \ P(A) = \{ P(\lambda) \mid \lambda \in \text{Sp} \ A \} .
\]

See for instance Lax (2007, Theorem 4, p. 66) for a proof.

**Proof of Lem. 1.** First we compute \( \nabla F_{\eta,k}(\omega^{*}) \). As \( \omega^{*} \) is a stationary point, it is a fixed point of the extrapolation operators, i.e. \( \varphi_{\omega^{*}}^{j}(\omega^{*}) = \omega^{*} \) for all \( j \geq 0 \). Then, by the chain rule,

\[
\nabla F_{\eta,k}(\omega^{*}) = \nabla_{\omega} \varphi_{\omega^{*}}^{k}(\omega^{*}) + \nabla_{\omega} \varphi_{\omega^{*}}^{k}(\omega^{*}) (71)
\]

\[
= (-\eta \nabla v(\omega^{*}))^{k} + \sum_{j=0}^{k-1} (-\eta \nabla v(\omega^{*}))^{j} \quad (72)
\]

\[
= \sum_{j=0}^{k} (-\eta \nabla v(\omega^{*}))^{j} . \quad (73)
\]

Hence \( \nabla F_{\eta,k}(\omega^{*}) \) is a polynomial in \( \nabla v(\omega^{*}) \). Using the spectral mapping theorem (Thm. 9), one gets that

\[
\text{Sp} \ \nabla F_{\eta,k}(\omega^{*}) = \left\{ \sum_{j=0}^{k} (-\eta)^{j} \lambda^{j} \mid \lambda \in \text{Sp} \ \nabla v(\omega^{*}) \right\} . \quad (74)
\]

For the proximal point operator, first let us prove that it is differentiable in a neighborhood of \( \omega^{*} \). First notice that,

\[
\text{Sp} \ (I_{d} + \eta \nabla v(\omega^{*})) = \{ 1 + \eta \lambda \mid \lambda \in \text{Sp} \ \nabla v(\omega^{*}) \} . \quad (75)
\]

If the eigenvalues of \( \nabla v(\omega^{*}) \) all have non-negative real parts, this spectrum does not contain zero. Hence \( \omega \mapsto \omega + \eta v(\omega) \) is continuously differentiable and has a non-singular differential at \( \omega^{*} \). By the inverse function theorem (see for instance Rudin (1976)), \( \omega \mapsto \omega + \eta v(\omega) \) is invertible in a neighborhood of \( \omega^{*} \) and its inverse, which is \( P_{\eta} \), is continuously differentiable there. Moreover,

\[
\nabla P_{\eta}(\omega^{*}) = (I_{d} + \eta \nabla v(\omega^{*}))^{-1} . \quad (76)
\]

Recall that the eigenvalues of a non-singular matrix are exactly the inverses of the eigenvalues of its inverse. Hence,

\[
\text{Sp} \ \nabla P_{\eta}(\omega^{*}) = \{ \lambda^{-1} \mid \lambda \in \text{Sp} \ (I_{d} + \eta \nabla v(\omega^{*})) \} = \{ (1 + \eta \lambda)^{-1} \mid \lambda \in \text{Sp} \ \nabla v(\omega^{*}) \} , \quad (77)
\]
where the last equality follows from the spectral mapping theorem applied to $I_d + \eta \nabla v(\omega^*)$. Now, the bound on the spectral radius of the proximal point operator is immediate. Indeed, its spectral radius is:

$$
\rho(\nabla P_\eta(\omega^*))^2 = \max_{\lambda \in \sigma^*} \frac{1}{1 + \eta |\lambda|^2}
$$

(78)

$$
= 1 - \min_{\lambda \in \sigma^*} \left( \frac{2\eta |\lambda| + \eta^2 |\lambda|^2}{1 + \eta |\lambda|^2} \right)
$$

(79)

which yields the result.

**Theorem 3.** Let $\sigma^* = \text{Sp} \nabla v(\omega^*)$. If the eigenvalues of $\nabla v(\omega^*)$ all have non-negative real parts, the spectral radius of the $k$-extrapolation method for $k \geq 2$ satisfies:

$$
\forall \eta \leq \frac{1}{4} \frac{1}{\max_{\lambda \in \sigma^*} |\lambda|}, \quad \rho(\nabla F_{\eta,k}(\omega^*))^2 \leq 1 - \min_{\lambda \in \sigma^*} \frac{2\eta |\lambda| + \frac{7}{16} \eta^2 |\lambda|^2}{1 + \eta |\lambda|^2},
$$

(14)

For $\eta = (4 \max_{\lambda \in \sigma^*} |\lambda|)^{-1}$, this can be simplified as:

$$
\rho(\nabla F_{\eta,k}(\omega^*))^2 \leq 1 - \frac{1}{4} \left( \frac{\min_{\lambda \in \sigma^*} |\lambda|}{\max_{\lambda \in \sigma^*} |\lambda|} + \frac{1}{16} \min_{\lambda \in \sigma^*} |\lambda|^2 \right).
$$

(15)

**Proof.** Let $L = \max_{\lambda \in \sigma^*} |\lambda|$ and $\eta = \frac{\tau}{k}$ for some $\tau > 0$. For $\lambda \in \sigma^*$,

$$
\left| \sum_{j=0}^{k} (-\eta)^j \lambda^j \right|^2 = \frac{1 - (-\eta)^{k+1} \lambda^{k+1}}{1 + \eta |\lambda|^2}
$$

(80)

$$
= 1 + 2(1)^k \eta^{k+1} \frac{\Re(\lambda^{k+1}) + \eta^2(1+\lambda) |\lambda|^2}{1 + \eta |\lambda|^2}
$$

(81)

$$
= 1 - \frac{2\eta |\lambda| + \eta^2 |\lambda|^2 - 2(1)^k \eta^{k+1} \Re(\lambda^{k+1}) - \eta^2(1+\lambda) |\lambda|^2}{1 + \eta |\lambda|^2}
$$

(82)

$$
= 1 - \frac{2\eta |\lambda| + \eta^2 |\lambda|^2 \left( 1 - 2(1)^k \eta^{k+1} \frac{\Re(\lambda^{k+1})}{|\lambda|^2} - \eta^2(1+\lambda) |\lambda|^2 \right)}{1 + \eta |\lambda|^2}
$$

(83)

Now we focus on lower bounding the terms in between the parentheses. By definition of $\eta$, we have $\eta^{k-1} \frac{\Re(\lambda^{k+1})}{|\lambda|^2} \leq \tau^{k-1}$ and $\eta^2(1+\lambda) |\lambda|^2 \leq \tau^2(1+\lambda)$. Hence

$$
1 + 2(1)^k \eta^{k-1} \frac{\Re(\lambda^{k+1})}{|\lambda|^2} + \eta^2(1+\lambda) |\lambda|^2 \leq 1 - 2\eta^{k-1} \left( \frac{\Re(\lambda^{k+1})}{|\lambda|^2} - \eta^2(1+\lambda) |\lambda|^2 \right)
$$

(84)

$$
\geq 1 - 2\tau^{k-1} - \tau^2(1+\lambda)
$$

(85)

(86)

Notice that if $k = 1$, i.e. for the gradient method, we cannot control this quantity. However, for $k \geq 2$, if $\tau \leq (\frac{1}{4})^{\frac{k-1}{k+1}}$, one gets that

$$
1 - 2\tau^{k-1} - \tau^2(1+\lambda) \geq 1 - 1 = \frac{1}{2} - \frac{1}{16} = \frac{7}{16}
$$

(87)

which yields the first assertion of the theorem. For the second one, take $\eta = \frac{1}{16}$, i.e. the maximum step-size authorized for extragradient, and one gets that

$$
|1 + \eta |\lambda|^2 = 1 + 2\eta |\lambda| + \eta^2 |\lambda|^2 \leq 1 + 2\frac{1}{4} + \frac{1}{16} = \frac{25}{16}.
$$

(88)
Then,
\[
\frac{2\eta R \lambda + \frac{4}{\eta} R^2 |\lambda|^2}{|1 + \eta \lambda|^2} \geq \frac{1}{4} \left( \frac{2}{25} \frac{16 R \lambda}{L} + \frac{7}{100} \frac{|\lambda|^2}{L^2} \right) \quad (90)
\]
\[
\geq \frac{1}{4} \left( \frac{R \lambda}{L} + \frac{7 |\lambda|^2}{112 L^2} \right) \quad (91)
\]
\[
\geq \frac{1}{4} \left( \frac{R \lambda}{L} + \frac{1 |\lambda|^2}{16 L^2} \right), \quad (92)
\]
which yields the desired result.

\[\square\]

**Corollary 2** (Bilinear game). Consider the bilinear game of (3). The iterates of the k-extrapolation method with \( k \geq 2 \) converge globally to \( \omega^* \) at a linear rate of \( O \left( \left( 1 - \frac{1}{64} \frac{\sigma_{\min}(A)^2}{\sigma_{\max}(A)^2} \right)^k \right) \).

**Proof.** First we need to compute the eigenvalues of \( \nabla v \), which is constant here and has following the form:
\[
\nabla v = \begin{pmatrix} 0_m & A \\ -A^T & 0_m \end{pmatrix}. \quad (93)
\]
As the bilinear game we consider is a special case of square two-player zero-sum games we can use the results of §C. Moreover as the terms \( S_1 \) and \( S_2 \) are zero here, the \( S_1, S_2, A \) and \( A^T \) commute and so we can use Lem. 4. This yields:
\[
\text{Sp} \nabla v = \{ \pm i \sigma | \sigma^2 \in \text{Sp} AA^T \}. \quad (94)
\]
Hence \( \min_{\lambda \in \text{Sp} \nabla v} |\lambda|^2 = \sigma_{\min}(A)^2 \) and \( \max_{\lambda \in \text{Sp} \nabla v} |\lambda|^2 = \sigma_{\max}(A)^2 \). Using Thm. 3, we have that,
\[
\rho(\nabla F_{\eta,k}(\omega^*))^2 \leq \left( 1 - \frac{1}{64} \frac{\sigma_{\min}(A)^2}{\sigma_{\max}(A)^2} \right). \quad (95)
\]
Finally, Thm. 8 implies that the iterates of the k-extrapolation converge globally at the desired rate.

\[\square\]

**Corollary 3.** Under the assumptions of Cor. 1, the spectral radius of the n-extrapolation method operator is bounded by
\[
\rho(\nabla F_{\eta,k}(\omega^*))^2 \leq 1 - \frac{1}{4} \left( \frac{1}{2} \frac{\mu_1 + \mu_2}{\sqrt{L_1^2 + L_2^2}} + \frac{1}{16} \frac{\mu_1^2 + \mu_2^2}{L_1^2 + L_2^2} \right). \quad (96)
\]

**Proof.** This is a direct consequence of Thm. 3 and Thm. 7, as the latter gives that for any \( \lambda \in \text{Sp} \nabla v(\omega^*) \),
\[
\frac{\text{Tr} \mu}{2} \leq R \lambda, \quad |\mu| \leq |\lambda|^2 \leq |L|, \quad (97)
\]
as discussed in the proof of Cor. 1.

\[\square\]

**E.2 Proofs of §4.2**

In this subsection, \( \| \cdot \| \) denotes the Euclidean norm.

**Lemma 2.** Let \( v \) be continuously differentiable and \( \gamma > 0 \) : (16) holds if and only if \( \sigma_{\min}(\nabla v) \geq \gamma \).

Let us recall (16) here for simplicity:
\[
\| \omega - \omega' \| \leq \gamma^{-1} \| v(\omega) - v(\omega') \| \quad \forall \omega, \omega' \in \mathbb{R}^d. \quad (16)
\]
The proof of this lemma is an immediate consequence of a global inverse theorem from Hadamard (1906); Levy (1920). Let us recall its statement here:

**Theorem 10** (Hadamard (1906); Levy (1920)). Let \( f : \mathbb{R}^d \to \mathbb{R}^d \) be a continuously differentiable map. Assume that, for all \( \omega \in \mathbb{R}^d \), \( \nabla f \) is non-singular and \( \sigma_{\min}(\nabla f) \geq \gamma > 0 \). Then \( f \) is a \( C^1 \)-diffeomorphism.
A proof of this theorem can be found in Rheinboldt (1969, Theorem 3.11). We now proceed to prove the lemma.

**Proof of Lem. 2.** First we prove the direct implication. By the theorem stated above, \( v \) is a bijection from \( \mathbb{R}^d \) to \( \mathbb{R}^d \), its inverse is continuously differentiable on \( \mathbb{R}^d \) and so we have, for all \( \omega \in \mathbb{R}^d \):

\[
\nabla v^{-1}(v(\omega)) = (\nabla v(\omega))^{-1}.
\]

Hence \( \|\nabla v^{-1}(v(\omega))\| = (\sigma_{\min}((\nabla v(\omega)))^{-1} \leq \gamma^{-1}. \)

Consider \( \omega, \omega' \in \mathbb{R}^d \) and let \( u = v(\omega) \) and \( u' = v(\omega') \). Then

\[
\|\omega - \omega'\| = \|v^{-1}(u) - v^{-1}(u')\|
\]

which proves the result.

Conversely, if (16) holds, fix \( u \in \mathbb{R}^d \) with \( \|u\| = 1 \). Taking \( \omega' = \omega + tu \) in (16) with \( t \neq 0 \) and rearranging yields:

\[
\gamma \leq \left\| \frac{v(\omega + tu) - v(\omega)}{t} \right\|.
\]

Taking the limit when \( t \) goes to 0 gives that \( \gamma \leq \|\nabla v(\omega)u\| \). As it holds for all \( u \) such that \( \|u\| = 1 \) this implies that \( \gamma \leq \sigma_{\min}(\nabla v) \).

We recall the global convergence theorem for extragradient:

**Theorem 4.** Let \( v : \mathbb{R}^d \to \mathbb{R}^d \) be continuously differentiable and (i) \( \mu \)-strongly monotone for some \( \mu \geq 0 \), (ii) \( L \)-Lipschitz, (iii) such that \( \sigma_{\min}(\nabla v) \geq \gamma \) for some \( \gamma \geq 0 \). Then, for \( \eta \leq (4L)^{-1} \), the iterates of the extragradient method \( (\omega_t) \) converge linearly to \( \omega^* \) the unique stationary point of \( v \),

\[
\|\omega_t - \omega^*\|^2 \leq \left( 1 - \eta \mu + \frac{7}{16} \eta^2 \gamma^2 \right) \|\omega_0 - \omega^*\|^2 \quad \forall t \geq 0.
\]

For \( \eta = (4L)^{-1} \), this can be simplified as:

\[
\|\omega_t - \omega^*\|^2 \leq \left( 1 - \frac{1}{4} \left( \frac{L}{\mu} + \frac{1}{16} \frac{\gamma^2}{L^2} \right) \right) \|\omega_0 - \omega^*\|^2.
\]

The proof is inspired from the ones of Gidel et al. (2019a); Tseng (1995).

We will use the following well-known identity. It can be found in Gidel et al. (2019a) for instance but we state it for reference.

**Lemma 8.** Let \( \omega, \omega', u \in \mathbb{R}^d \). Then

\[
\|\omega + u - \omega'\|^2 = \|\omega - \omega'\|^2 + 2u^T(\omega + u - \omega') - \|u\|^2.
\]

**Proof.**

\[
\|\omega + u - \omega'\|^2 = \|\omega - \omega'\|^2 + 2u^T(\omega - \omega') + \|u\|^2
\]

\[
= \|\omega - \omega'\|^2 + 2u^T(\omega + u - \omega') - \|u\|^2.
\]

**Proof Thm. 4.** First note that if \( \gamma > 0 \) or \( \mu > 0 \), if \( v \) has a stationary point then it is unique.

Fix any \( \omega_0 \in \mathbb{R}^d \), and denote \( \omega_1 = \omega_0 - \eta v(\omega_0) \) and \( \omega_2 = \omega_0 - \eta v(\omega_1) \). Applying Lem. 8 for \((\omega, \omega', u) = (\omega_0, \omega^*, -\eta v(\omega_1))\) and \((\omega, \omega', u) = (\omega_0, \omega_2, -\eta v(\omega_0))\) yields:

\[
\|\omega_2 - \omega^*\|^2 = \|\omega_0 - \omega^*\|^2 - 2\eta v(\omega_1)^T(\omega_2 - \omega^*) - \|\omega_2 - \omega_0\|^2.
\]

\[
\|\omega_1 - \omega_2\|^2 = \|\omega_0 - \omega_2\|^2 - 2\eta v(\omega_0)^T(\omega_1 - \omega_2) - \|\omega_1 - \omega_0\|^2.
\]
Summing these two equations gives:
\[
\|\omega_2 - \omega^*\|^2 = (108)
\]
\[
\|\omega_0 - \omega^*\|^2 - 2\eta \nu (\omega_0)^T (\omega_2 - \omega^*) - 2\eta \nu (\omega_0)^T (\omega_1 - \omega_2) - \|\omega_1 - \omega_0\|^2 - \|\omega_1 - \omega_2\|^2 (109)
\]
Then, rearranging and using that \(\nu(\omega^*) = 0\) yields that,
\[
2\eta \nu (\omega_1)^T (\omega_2 - \omega^*) + 2\eta \nu (\omega_0)^T (\omega_1 - \omega_2) (110)
\]
\[
= 2\eta (\nu(\omega_1))^T (\omega_1 - \omega^*) + 2\eta (\nu(\omega_0) - \nu(\omega_1))^T (\omega_1 - \omega_2) (111)
\]
\[
= 2\eta \nu (\omega_1 - \nu(\omega^*))^T (\omega_1 - \omega^*) + 2\eta (\nu(\omega_0) - \nu(\omega_1))^T (\omega_1 - \omega_2) (112)
\]
\[
\geq 2\eta \mu \|\omega_1 - \omega^*\|^2 - 2\eta \|\nu(\omega_1) - \nu(\omega_0)\| \|\omega_1 - \omega_2\| (113)
\]
where the first term is lower bounded using strong monotonicity and the second one using Cauchy-Schwarz’s inequality. Using in addition the fact that \(\nu\) is Lipschitz continuous we obtain:
\[
2\eta \nu (\omega_1)^T (\omega_2 - \omega^*) + 2\eta \nu (\omega_0)^T (\omega_1 - \omega_2) (114)
\]
\[
\geq 2\eta \mu \|\omega_1 - \omega^*\|^2 - 2\eta L \|\omega_0 - \omega_1\| \|\omega_1 - \omega_2\| (115)
\]
\[
\geq 2\eta \mu \|\omega_1 - \omega^*\|^2 - (\eta^2 L^2 \|\omega_0 - \omega_1\|^2 + \|\omega_1 - \omega_2\|^2) (116)
\]
where the last inequality comes from Young’s inequality. Using this inequality in (108) yields:
\[
\|\omega_2 - \omega^*\|^2 \leq \|\omega_0 - \omega^*\|^2 - 2\eta \mu \|\omega_1 - \omega^*\|^2 + (\eta^2 L^2 - 1) \|\omega_0 - \omega_1\|^2. (117)
\]
Now we lower bound \(\|\omega_1 - \omega^*\|\) using \(\|\omega_0 - \omega^*\|\). Indeed, from Young’s inequality we obtain
\[
2\|\omega_1 - \omega^*\|^2 \geq \|\omega_0 - \omega^*\|^2 - 2\|\omega_0 - \omega_1\|^2. (118)
\]
Hence, we have that,
\[
\|\omega_2 - \omega^*\|^2 \leq (1 - \eta \mu) \|\omega_0 - \omega^*\|^2 + (\eta^2 L^2 + 2\eta \mu - 1) \|\omega_0 - \omega_1\|^2. (119)
\]
Note that if \(\eta \leq \frac{1}{4\mu}, \) as \(\mu \leq L, \frac{\eta^2 L^2}{2} + 2\eta \mu - 1 \leq -\frac{\eta}{16}. \) Therefore, with \(c = \frac{\eta}{16}, \)
\[
\|\omega_2 - \omega^*\|^2 \leq (1 - \eta \mu) \|\omega_0 - \omega^*\|^2 - c \|\omega_0 - \omega_1\|^2 (120)
\]
\[
= (1 - \eta \mu) \|\omega_0 - \omega^*\|^2 - c\eta^2 \|\nu(\omega_0)\|^2. (121)
\]
Finally, using (iii) and Lem. 2, we obtain:
\[
\|\omega_2 - \omega^*\|^2 \leq (1 - \eta \mu - c\eta^2 \gamma^2) \|\omega_0 - \omega^*\|^2 (122)
\]
which yields the result.

With the next lemma, we relate the quantities appearing in Thm. 4 to the spectrum of \(\nabla \nu. \) Though the first part of the proof is standard — it can be found in Facchinei and Pang (2003, Prop. 2.3.2) for instance — and we include it only for completeness, we have not seen the bounds on the eigenvalues in the literature.

**Lemma 9.** Let \(\nu : \mathbb{R}^d \to \mathbb{R}^d\) be continuously differentiable and (i) \(\mu\)-strongly monotone for some \(\mu \geq 0, \) (ii) \(L\)-Lipschitz, (iii) such that \(\sigma_{\text{min}}(\nabla v) \geq \gamma\) for some \(\gamma \geq 0.\) Then, for all \(\omega \in \mathbb{R}^d, \)
\[
\mu \|u\|^2 \leq (\nabla v(\omega) u)^T u, \quad \|\nu(u)\| \leq \|\nabla v(\omega) u\| \leq L \|u\|, \quad \forall u \in \mathbb{R}^d, (123)
\]
and
\[
\mu \leq \Re(\lambda), \quad \gamma \leq |\lambda| \leq L, \quad \forall \lambda \in \text{Sp} \nabla v(\omega). (124)
\]

**Proof.** By definition of \(\mu\)-strong monotonicity, and \(L\)-Lipschitz one has that, for any \(\omega, \omega' \in \mathbb{R}^d, \)
\[
\mu \|\omega - \omega'\|^2 \leq (\nu(\omega) - \nu(\omega'))^T (\omega - \omega') (125)
\]
\[
\|\nu(\omega) - \nu(\omega')\| \leq L \|\omega - \omega'\|. (126)
\]
Fix \(\omega \in \mathbb{R}^d, u \in \mathbb{R}^d\) such that \(\|u\| = 1.\) Taking \(\omega' = \omega + tu\) for \(t > 0\) in the previous inequalities and dividing by \(t\) yields
\[
\mu \leq \frac{1}{t} (\nu(\omega) - \nu(\omega + tu))^T u (127)
\]
\[
\frac{1}{t} \|\nu(\omega) - \nu(\omega + tu)\| \leq L. (128)
\]
Letting $t$ goes to 0 gives
\[
\mu \leq (\nabla v(\omega)u)^T u \tag{129}
\]
\[
\|\nabla v(\omega)u\| \leq L. \tag{130}
\]
Furthermore, by the properties of the singular values,
\[
\|\nabla v(\omega)u\| \geq \gamma. \tag{131}
\]
Hence, by homogeneity, we have that, for all $u \in \mathbb{R}^d$,
\[
\mu\|u\| \leq (\nabla v(\omega)u)^T u, \quad \gamma\|u\| \leq \|\nabla v(\omega)u\| \leq L\|u\|. \tag{132}
\]
Now, take $\lambda \in \text{Sp} \nabla v(\omega)$ an eigenvalue of $\nabla v(\omega)$ and let $Z \in \mathbb{C}^d \setminus \{0\}$ be one of its associated eigenvectors. Note that $Z$ can be written as $Z = X + iY$ with $X, Y \in \mathbb{R}^d$. By definition of $Z$, we have
\[
\nabla v(\omega)Z = \lambda Z. \tag{133}
\]
Now, taking the real and imaginary part yields:
\[
\begin{cases}
\nabla v(\omega)X &= \Re(\lambda)X - \Im(\lambda)Y \\
\nabla v(\omega)Y &= \Im(\lambda)X + \Re(\lambda)Y
\end{cases} \tag{134}
\]
Taking the squared norm and developing the right-hand sides yields
\[
\begin{align*}
\|\nabla v(\omega)X\|^2 &= \Re(\lambda)^2\|X\|^2 + \Im(\lambda)^2\|Y\|^2 - 2\Re(\lambda)\Im(\lambda)X^T Y \\
\|\nabla v(\omega)Y\|^2 &= \Im(\lambda)^2\|X\|^2 + \Re(\lambda)^2\|Y\|^2 + 2\Re(\lambda)\Im(\lambda)X^T Y.
\end{align*} \tag{135}
\]
Now summing these two equations gives
\[
\|\nabla v(\omega)X\|^2 + \|\nabla v(\omega)Y\|^2 = |\lambda|^2(\|X\|^2 + \|Y\|^2). \tag{136}
\]
Finally, apply (132) for $u = X$ and $u = Y$:
\[
\gamma^2(||X||^2 + ||Y||^2) \leq |\lambda|^2(||X||^2 + ||Y||^2) \leq L^2(||X||^2 + ||Y||^2). \tag{137}
\]
As $Z \neq 0$, $||X||^2 + ||Y||^2 > 0$ and this yields $\gamma \leq |\lambda| \leq L$. To get the inequality concerning $\gamma$, multiply on the left the first line of (134) by $X^T$ and the second one by $Y^T$:
\[
\begin{cases}
X^T(\nabla v(\omega)X) &= \Re(\lambda)||X||^2 - \Im(\lambda)X^T Y \\
Y^T(\nabla v(\omega)Y) &= \Im(\lambda)Y^T X + \Re(\lambda)||Y||^2.
\end{cases} \tag{138}
\]
Again, summing these two lines and using (132) yields:
\[
\mu(||X||^2 + ||Y||^2) \leq \Re(\lambda)(||X||^2 + ||Y||^2). \tag{139}
\]
As $Z \neq 0$, $||X||^2 + ||Y||^2 > 0$ and so $\mu \leq \Re(\lambda)$.

**Proposition 1.** *Under the assumptions of Thm. 4, the iterates of the proximal point method method $(\omega_t)$, with $\eta > 0$ converge linearly to $\omega^*$ the unique stationary point of $v$,
\[
\|\omega_t - \omega^*\|^2 \leq \left(1 - \frac{2\eta\mu + \eta^2\gamma^2}{1 + 2\eta\mu + \eta^2\gamma^2}\right)^t \|\omega_0 - \omega^*\|^2 \quad \forall t \geq 0. \tag{140}
\]

**Proof.** To proof this convergence result, we upper bound the singular values of the proximal point operator $P_{\eta}$. As $v$ is monotone, by Lem. 9, the eigenvalues of $\nabla v$ have all non-negative real parts everywhere. As in the proof of Lem. 1, $\omega \mapsto \omega + \eta v(\omega)$ is continuously differentiable and has a non-singular differential at every $\omega_0 \in \mathbb{R}^d$. By the inverse function theorem, $\omega \mapsto \omega + \eta v(\omega)$ has a continuously differentiable inverse in a neighborhood of $\omega_0$. Its inverse is exactly $P_{\eta}$ and it also satisfies
\[
\nabla P_{\eta}(\omega_0) = (I_d + \eta \nabla v(\omega_0))^{-1}. \tag{141}
\]
The singular values $\nabla P_{\eta}(\omega_0)$ are the eigenvalues of $(\nabla P_{\eta}(\omega_0))^T (\nabla P_{\eta}(\omega_0))$. The latter is equal to:
\[
(\nabla P_{\eta}(\omega_0))^T (\nabla P_{\eta}(\omega_0)) = (I_d + \eta \nabla v(\omega_0) + \eta (\nabla v(\omega_0))^T + \eta^2(\nabla v(\omega_0))^T (\nabla v(\omega_0)))^{-1}. \tag{142}
\]
Now, let $\lambda \in \mathbb{R}$ be an eigenvalue of $(\nabla P_{\eta}(\omega_0))^T (\nabla P_{\eta}(\omega_0))$ and let $X \neq 0$ be one of its associated eigenvectors. As $\nabla P_{\eta}(\omega_0)$ is non-singular, $\lambda \neq 0$ and applying the previous equation yields:

$$\lambda^{-1} X = (I_d + \eta \nabla v(\omega_0) + \eta (\nabla v(\omega_0))^T + \eta^2 (\nabla v(\omega_0))^T (\nabla v(\omega_0))) X .$$  \hfill (143)

Finally, multiply this equation on the left by $X^T$:

$$\lambda^{-1} \|X\|^2 = \|X\|^2 + \eta X^T (\nabla v(\omega_0) + (\nabla v(\omega_0))^T) X + \eta^2 \|\nabla v(\omega_0)\| X^2 .$$  \hfill (144)

Applying the first part of Lem. 9 yields

$$\lambda^{-1} \|X\|^2 \geq (1 + 2\eta\mu + \eta^2 \gamma^2) \|X\|^2 .$$  \hfill (145)

Hence, as $X \neq 0$, we have proven that,

$$\sigma_{\text{max}}(\nabla v(\omega_0)) \leq (1 + 2\eta\mu + \eta^2 \gamma^2)^{-1} .$$  \hfill (146)

This implies that, for all $\omega, \omega' \in \mathbb{R}^d$,

$$\|P_{\eta}(\omega) - P_{\eta}(\omega')\|^2 = \left| \int_0^1 \nabla v(\omega' + t(\omega - \omega')) (\omega - \omega') \right|^2 \leq (1 + 2\eta\mu + \eta^2 \gamma^2)^{-1} \|\omega - \omega'\|^2 .$$  \hfill (147)

Hence, as $P_{\eta}(\omega^*) = \omega^*$, taking $\omega' = \omega^*$ gives the desired global convergence rate.

F  The $p$-SCLI framework for game optimization

The approach we use to prove our lower bounds comes from Arjevani et al. (2016). Though, their whole framework was developed for convex optimization, a careful reading of their proof shows that most of their results carry on to games, at least those in their first three sections. However, we work only in the restricted setting of 1-SCLI and so we actually rely on a very small subset of their results, more exactly two of them.

The first one is Thm. 5 and is crucially used in the derivation of our lower bounds. We state it again for clarity.

Theorem 5 (Arjevani et al. (2016)). For all $v \in V_d$, for almost every\(^6\) initialization point $\omega_0 \in \mathbb{R}^d$, if $(\omega_t)_t$ are the iterates of $F_{N'}$ starting from $\omega_0$,

$$\|\omega_t - \omega^*\| \geq \Omega(\rho(\nabla F_N)^t \|\omega_0 - \omega^*\|) .$$  \hfill (20)

Actually, as $F_{N'} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an affine operator and $\omega^*$ is one of its fixed point, this theorem is only a reformulation of Arjevani et al. (2016, Lemma 10), which is a standard result in linear algebra. So we actually do not rely on their most advanced results which were proven only for convex optimization problems. For completeness, we state this lemma below and show how to derive Thm. 5 from it.

Lemma 10 (Arjevani et al. (2016, Lemma 10)). Let $A \in \mathbb{R}^{d \times d}$. There exists $c > 0$, $d \geq m \geq 1$ integer and $r \in \mathbb{R}^d$, $r \neq 0$ such that for any $u \in \mathbb{R}^d$ such that $u^T r \neq 0$, for sufficiently large $t \geq 1$ one has:

$$\|A^t u\| \geq ct^{m-1} \rho(A)^t \|u\| .$$  \hfill (149)

Proof of thm. 5. $F_{N'}$ is affine so it can be written as $F_{N'}(\omega) = \nabla F_{N'}\omega + F_{N'}(0)$. Moreover, as $v(\omega^*) = 0$, $F_{N'}(\omega^*) = \omega^* + N(\nabla v)(\omega^*) = \omega^*$. Hence, for all $\omega \in \mathbb{R}^d$,

$$F_{N'}(\omega) - \omega^* = F_{N'}(\omega) - F_{N'}(\omega^*) = \nabla F_{N'}(\omega - \omega^*) .$$  \hfill (150)

Therefore, for $t \geq 0$,

$$\|\omega_t - \omega^*\| = \left\| (\nabla F_{N'})^t (\omega - \omega^*) \right\| .$$  \hfill (151)

Finally, apply the lemma above to $A = \nabla F_{N'}$. The condition $(\omega_0 - \omega^*)^T r \neq 0$ is not satisfied only on an affine subset of dimension 1, which is of measure zero for any measure absolutely continuous

---

\(^6\)For any measure absolutely continuous w.r.t. the Lebesgue measure.
w.r.t. the Lebesgue measure. Hence for almost every $\omega_0 \in \mathbb{R}^d$ w.r.t. to such measure, $(\omega_0 - \omega^*)^T r \neq 0$ and so one has, for $t \geq 1$ large enough,

$$
\|\omega_t - \omega^*\| \geq c t^{m-1} \rho(\nabla F_{N})^t \|\omega_t - \omega^*\|
$$

(152)

$$
\geq c \rho(\nabla F_{N})^t \|\omega_t - \omega^*\|
$$

(153)

which is the desired result. \qed

The other result we use is more anecdotal: it is their consistency condition, which is a necessary condition for an $p$-SCLI method to converge to a stationary point of the gradient dynamics. Indeed, general 1-SCLI as defined in Arjevani et al. (2016) are given not by one but by two mappings $C, N : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ and the update rule is

$$
F_{N}(\omega) = C(\nabla v)w + N(\nabla v)v(0) \quad \forall \omega \in \mathbb{R}^d.
$$

(154)

However, they show in Arjevani et al. (2016, Thm. 5) that, for a method to converge to a stationary point of $v$, at least for convex problems, that is to say symmetric positive semi-definite $\nabla v$, $C$ and $N$ need to satisfy:

$$
I_d - C(\nabla v) = -N(\nabla v)\nabla v.
$$

(155)

If $C$ and $N$ are polynomials, this equality for all symmetric positive semi-definite $\nabla v$ implies the equality on all matrices. Injecting this result in (154) yields the definition of 1-SCLI we used.

G \quad Proofs of §5

G.1 \quad Proofs of §5.1

Lemma 3. Let $v : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field and let $\omega \in \mathbb{R}^d$ such that the eigenvalues of $\nabla v(\omega)$ have all positive real parts. For all $\eta > 0$, the spectral radius of the gradient operator is lower bounded by:

$$
\rho(\nabla F_\eta(\omega))^2 \geq 1 - 4 \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} (1/\lambda) \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} (\Re(\lambda)).
$$

(21)

To prove this lemma we once again rely on Gidel et al. (2019b, Theorem 2) which we recall below:

**Theorem** (Gidel et al. (2019b, Theorem 2)). The best step-size $\eta^*$, that is to say the solution of the optimization problem

$$
\min_{\eta} \rho(\nabla F_\eta(\omega))^2,
$$

(156)

satisfy:

$$
\min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re(1/\lambda) \leq \eta^* \leq 2 \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re(1/\lambda).
$$

(157)

Lem. 3 is now immediate.

**Proof of Lem. 3.** By definition of the spectral radius,

$$
\rho(\nabla F_\eta^*(\omega^*))^2 = \max_{\lambda \in \text{Sp}(\nabla v(\omega^*))} |1 - \eta^*\lambda|^2
$$

(158)

$$
= 1 - \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} 2\eta^*\Re(\lambda) - |\eta^*\lambda|^2
$$

(159)

$$
\geq 1 - \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} 2\eta^*\Re(\lambda)
$$

(160)

$$
\geq 1 - 4 \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re(\lambda) \min_{\lambda \in \text{Sp}(\nabla v(\omega^*))} \Re(1/\lambda)
$$

(161)

\qed

25
G.2 Proofs of §5.2

The class of methods we consider, that is to say the methods whose coefficient mappings $\mathcal{N}$ are any polynomial of degree at most $k - 1$, is very general. It includes:

- the $k'$-extrapolation methods $F_{k', \eta}$ for $k' \leq k$ as defined by (11).
- extrapolation methods with different step sizes for each extrapolation:
  \[ \omega \mapsto \varphi_{\eta_1, \omega} \circ \varphi_{\eta_2, \omega} \circ \cdots \circ \varphi_{\eta_k, \omega}(\omega), \]
  (162)
- cyclic Richardson iterations (Opfer and Schober, 1984): methods whose update is composed of successive gradient steps with possibly different step sizes for each
  \[ \omega \mapsto F_{\eta_1} \circ F_{\eta_2} \circ \cdots \circ F_{\eta_k}(\omega), \]
  (163)
and any combination of these with at most $k$ composed gradient evaluations.

The lemma below shows how $k$-extrapolation algorithms fit into the definition of 1-SCLI:

Lemma 11. For a $k$-extrapolation method, $\mathcal{N}(\nabla v) = -\eta \sum_{j=0}^{k-1} (-\eta \nabla v)^j$.

Proof. This result is a direct consequence of Lem. 7. For $\omega \in \mathbb{R}^d$, one gets, by the chain rule,
\[ \nabla F_{\eta, k}(\omega) = \nabla_{\omega} \varphi_{\eta, \omega}^k(\omega) + \nabla_{\omega} \varphi_{\eta, \omega}^k(\omega) \]
\[ = (-\eta \nabla v)^k + \sum_{j=0}^{k-1} (-\eta \nabla v)^j \]
\[ = \sum_{j=0}^{k} (-\eta \nabla v)^j. \]
(166)
as $\nabla v$ is constant. Hence, as expected, $F_{\eta, k}$ is linear so write that, for all $\omega \in \mathbb{R}^d$,
\[ F_{\eta, k}(\omega) = \nabla F_{\eta, k} \omega + b. \]
(167)
If $v$ has a stationary point $\omega^*$, evaluating at $\omega^*$ yields
\[ \omega^* = \sum_{j=0}^{k} (-\eta \nabla v)^j \omega^* + b. \]
(168)
Using that $v(\omega^*) = 0$ and so $(\nabla v)\omega^* = -v(0)$, one gets that
\[ b = -\eta \sum_{j=1}^{k} (-\eta \nabla v)^{j-1} v(0), \]
(169)
and so
\[ F_{\eta, k}(\omega) = \omega - \eta \sum_{j=1}^{k} (-\eta \nabla v)^{j-1} v(\omega), \]
(170)
which yields the result for affine vector fields with a stationary point. In particular it holds for vector fields such that $\nabla v$ is non-singular. As the previous equality is continuous in $\nabla v$, by density of non-singular matrices, the result holds for all affine vector fields.

Theorem 6. Let $0 < \mu, \gamma < L$. (i) If $d - 2 \geq k \geq 3$, there exists $v \in \mathcal{V}_d$ with a symmetric positive Jacobian whose spectrum is in $[\mu, L]$, such that, for any $\mathcal{N}$ real polynomial of degree at most $k - 1$,
\[ \rho(F_{\mathcal{N}}) \geq 1 - \frac{4k^3 \mu}{\pi L}. \]
(22)
(ii) If $d/2 - 2 \geq k/2 \geq 3$ and $d$ is even, there exists $v \in \mathcal{V}_d$ $L$-Lipschitz with $\min_{\lambda \in \mathfrak{sp} \nabla v} |\lambda| = \sigma_{\min}(\nabla v) \geq \gamma$ corresponding to a bilinear game as described in (3) with $m = d/2$ such that, for any $\mathcal{N}$ real polynomial of degree at most $k - 1$,
\[ \rho(F_{\mathcal{N}}) \geq 1 - \frac{k^3 \gamma^2}{2\pi L^2}. \]
(23)
To ease the presentation of the proof of the theorem, we rely on several lemmas. We first prove (i) and (ii) will follow as a consequence.

In the following, we denote by $\mathbb{R}_{k-1}[X]$ the set of real polynomials of degree at most $k - 1$.

**Lemma 12.** For $v \in \mathcal{V}_d$, 
\[
\min_{N \in \mathbb{R}_{k-1}[X]} \frac{1}{2} \rho(F_N)^2 = \min_{a_0, \ldots, a_{k-1} \in \mathbb{R}} \max_{\lambda \in \text{Sp} \nabla v} \frac{1}{2} \left| 1 + \sum_{l=0}^{k-1} a_l \lambda^{l+1} \right|^2. \tag{171}
\]

**Proof.** Recall the definition of $F_N$, which is affine by assumption, 
\[
\forall \omega \in \mathbb{R}^d, F_N(\omega) = w + N(\nabla v)(\omega). \tag{172}
\]
Then $\nabla F_N = I_d + N(\nabla v)\nabla v$. As $N$ is a polynomial, by the spectral mapping theorem (Thm. 9),
\[
\text{Sp} \nabla F_N = \{1 + N(\lambda) \mid \lambda \in \text{Sp} \nabla v\}, \tag{173}
\]
which yields the result. \qed

**Lemma 13.** Assume that $\text{Sp} \nabla v = \{\lambda_1, \ldots, \lambda_m\} \subset \mathbb{R}$. Then (171) can be lower bounded by the value of the following problem:
\[
\max \sum_{j=1}^{m} \nu_j \left( \xi_j - \frac{1}{2} \xi_j^2 \right)
\]
\[
s.t. \ \nu_j \geq 0, \ \xi_j \in \mathbb{R}, \ \forall 1 \leq j \leq m
\]
\[
\sum_{j=1}^{m} \nu_j \xi_j \lambda_j^l = 0, \ \forall 1 \leq l \leq k
\]
\[
\sum_{j=1}^{m} \nu_j = 1
\]

**Proof.** The right-hand side of (171) can be written as a constrained optimization problem as follows:
\[
\min_{t, a_0, \ldots, a_{k-1}, z_1, \ldots, z_m \in \mathbb{R}} t
\]
\[
s.t. \ t \geq \frac{1}{2} z_j^2, \ \forall 1 \leq j \leq m
\]
\[
z_j = 1 + \sum_{l=0}^{k-1} a_l \lambda_j^{l+1}, \ \forall 1 \leq j \leq m.
\]
By weak duality, see Boyd and Vandenberghe (2004) for instance, we can lower bound the value of this problem by the value of its dual. So let us write the Lagrangian of this problem:
\[
\mathcal{L}(t, a_0, \ldots, a_{k-1}, z_1, \ldots, z_m, \nu_1, \ldots, \nu_m, \chi_1, \ldots, \chi_m)
\]
\[
= t + \sum_{j=1}^{m} \nu_j \left( \frac{1}{2} z_j^2 - t \right) + \chi_j \left( 1 + \sum_{l=0}^{k-1} a_l \lambda_j^{l+1} - z_j \right). \tag{176}
\]
The Lagrangian is convex and quadratic so its minimum with respect to $t, a_0, \ldots, a_{k-1}, z_1, \ldots, z_m$ is characterized by the first order condition. Moreover, if there is no solution to the first order condition, its minimum is $-\infty$ (see for instance Boyd and Vandenberghe (2004, Example 4.5)).

One has that, for any $1 \leq j \leq m$ and $0 \leq l \leq k - 1$,
\[
\partial_t \mathcal{L} = 1 - \sum_{j=0}^{m} \nu_j \tag{177}
\]
\[
\partial_{a_l} \mathcal{L} = \sum_{j=0}^{m} \chi_j \lambda_j^{l+1} \tag{178}
\]
\[
\partial_{z_j} \mathcal{L} = \nu_j z_j - \chi_j. \tag{179}
\]
Setting these quantities to zero yields the following dual problem:

\[
\begin{align*}
\max \sum_{j=1, \nu_j \neq 0} m & \chi_j - \frac{1}{2\nu_j} \chi_j^2 \\
\text{s.t.} & \quad \nu_j \geq 0, \quad \chi_j \in \mathbb{R}, \ \forall 1 \leq j \leq m \\
& \sum_{j=1}^m \chi_j \lambda_j^l = 0, \ \forall 1 \leq l \leq k \\
& \nu_j = 0 \implies \chi_j = 0 \\
& \sum_{j=1}^m \nu_j = 1
\end{align*}
\]

Taking \(\nu_j \xi_j = \chi_j\) yields the result:

\[
\begin{align*}
\max \sum_{j=1}^m \nu_j (\xi_j - \frac{1}{2} \xi_j^2) \\
\text{s.t.} & \quad \nu_j \geq 0, \quad \xi_j \in \mathbb{R}, \ \forall 1 \leq j \leq m \\
& \sum_{j=1}^m \nu_j \xi_j \lambda_j^l = 0, \ \forall 1 \leq l \leq k \\
& \sum_{j=1}^m \nu_j = 1
\end{align*}
\]

The next lemma concerns Vandermonde matrices and Lagrange polynomials.

**Lemma 14.** Let \(\lambda_1, \ldots, \lambda_d\) be distinct reals. Denote the Vandermonde matrix by

\[
V(\lambda_1, \ldots, \lambda_d) = \begin{pmatrix}
1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{d-1} \\
1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{d-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{d-1}
\end{pmatrix}
\]

Then

\[
V(\lambda_1, \ldots, \lambda_d)^{-1} = \begin{pmatrix}
L_1^{(0)} & L_2^{(0)} & \ldots & L_d^{(0)} \\
L_1^{(1)} & L_2^{(1)} & \ldots & L_d^{(1)} \\
\vdots & \vdots & \ddots & \vdots \\
L_1^{(d-1)} & L_2^{(d-1)} & \ldots & L_d^{(d-1)}
\end{pmatrix}
\]

where \(L_1, L_2, \ldots, L_d\) are the Lagrange interpolation polynomials associated to \(\lambda_1, \ldots, \lambda_d\) and \(L_j = \sum_{t=0}^{d-1} L_j^{(t)} X^t\) for \(1 \leq j \leq d\).

A proof of this result can be found at Atkinson (1989, Theorem 3.1).

The next lemma is the last one before we finally prove the theorem. Recall that in Thm. 6 we assume that \(k + 1 \leq d\).

**Lemma 15.** Assume that \(\text{Sp} \nabla v = \{\lambda_1, \ldots, \lambda_{k+1}\}\) where \(\lambda_1, \ldots, \lambda_{k+1}\) are distinct non-zero reals. Then the problem of (171) is lower bounded by

\[
\frac{1}{2} \left(1 - \frac{\sum_{j=1}^k \lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1}) \right)^2,
\]

where \(L_1, \ldots, L_k\) are the Lagrange interpolation polynomials associated to \(\lambda_1, \ldots, \lambda_k\).
Proof. To prove this lemma, we start from the result of Lem. 13 and we provide feasible \((\nu_j)_j\) and \((\xi_j)_j\). First, any feasible \((\nu_j)_j\) and \((\xi_j)_j\) must satisfy the \(k\) constraints involving the powers of the eigenvalues, which can be rewritten as:

\[
V(\lambda_1, \ldots, \lambda_k)^T \begin{pmatrix} \nu_1 \xi_1 \lambda_1 \\ \nu_2 \xi_2 \lambda_2 \\ \vdots \\ \nu_k \xi_k \lambda_k \end{pmatrix} = -\nu_{k+1} \xi_{k+1} \begin{pmatrix} \lambda_{k+1} \\ \lambda_{k+1}^2 \\ \vdots \\ \lambda_{k+1}^k \end{pmatrix}.
\]  

(185)

Using the previous lemma yields, for \(1 \leq j \leq k\),

\[
\nu_j \xi_j = -\nu_{k+1} \xi_{k+1} \frac{1}{\lambda_j} \begin{pmatrix} L_j^{(0)} \\ L_j^{(1)} \\ \vdots \\ L_j^{(k-1)} \end{pmatrix} \begin{pmatrix} \lambda_{k+1} \\ \lambda_{k+1}^2 \\ \vdots \\ \lambda_{k+1}^k \end{pmatrix} = -\nu_{k+1} \xi_{k+1} \frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1}).
\]  

(186)

(187)

Hence the problem can be rewritten only in terms of the \((\nu_j)_j\) and \(\xi_{k+1}\). Let \(c_j = \frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1})\).
The objective becomes:

\[
\sum_{j=1}^m \nu_j \left( \xi_j - \frac{1}{2} \xi_j^2 \right) = \nu_{k+1} \xi_{k+1} \left( 1 - \sum_{j=1}^k c_j \right) - \frac{1}{2} \nu_{k+1} \xi_{k+1}^2 \left( 1 + \sum_{j=0}^k \frac{\nu_{k+1}}{\nu_j} c_j^2 \right).
\]  

(188)

Choosing \(\xi_{k+1} = \frac{1 - \sum_{j=1}^k c_j}{1 + \sum_{j=0}^k \frac{\nu_{k+1}}{\nu_j} c_j^2}\) to maximize this quadratic yields:

\[
\sum_{j=1}^m \nu_j \left( \xi_j - \frac{1}{2} \xi_j^2 \right) = \frac{1}{2} \nu_{k+1} \left( 1 - \sum_{j=1}^k c_j \right)^2 \left( 1 + \sum_{j=0}^k \frac{\nu_{k+1}}{\nu_j} c_j^2 \right).
\]  

(189)

Finally take \(\nu_j = \frac{1}{1 + \sum_{j=1}^k |c_j|} \) for \(j \leq k\) and \(\nu_{k+1} = \frac{1}{1 + \sum_{j=1}^k |c_j|} \) which satisfy the hypotheses of the problem of Lem. 13. With the feasible \((\nu_j)_j\) and \((\xi_j)_j\) defined this way, the value of the objective is

\[
\frac{1}{2} \left( 1 - \sum_{j=1}^k |c_j| \right)^2 \left( 1 + \sum_{j=1}^k |c_j| \right),
\]  

(190)

which is the desired result.

We finally prove \((i)\) of Thm. 6.

Proof of \((i)\) of Thm. 6. To prove the theorem, we build on the result of Lem. 15. We have to choose \(\lambda_1, \ldots, \lambda_{k+1} \in [\mu, L]\) positive distinct such that (184) is big. One could try to distribute the eigenvalues uniformly across the interval but this leads to a lower bound which decreases exponentially in \(k\). To make things a bit better, we use Chebyshev points of the second kind studied by Salzer (1971). However we will actually refer to the more recent presentation of Berrut and Trefethen (2004).

For now, assume that \(k\) is even and so \(k \geq 4\). Note that we will only use that \(d - 1 \geq k\) (and not that \(d - 1 \geq k\)). Define, for \(1 \leq j \leq k\), \(\lambda_j = \frac{d + 1}{2} \frac{k + 1}{k} \cos \frac{j - 1}{k - 1} \pi\). Using the barycentric formula of Berrut and Trefethen (2004, Eq. 4.2), the polynomial which interpolates \(f_1, \ldots, f_k\) at the points \(\lambda_1, \ldots, \lambda_k\) can be written as:

\[
P(X) = \frac{\sum_{j=1}^k \frac{w_j}{X - \lambda_j} f_j}{\sum_{j=1}^k \frac{w_j}{X - \lambda_j}},
\]  

(191)

where

\[
w_j = \begin{cases} 
(-1)^{j-1} & \text{if } 2 \leq j \leq k - 1 \\
\frac{1}{2} (-1)^{j-1} & \text{if } j \in \{1, k\}.
\end{cases}
\]  

(192)
Define \( Z(X) = \sum_{j=1}^{k} \frac{w_j}{X - \lambda_j} \).

Now, \( \sum_{j=1}^{k} \frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1}) \) can be seen as the polynomial interpolating \( \frac{\lambda_{k+1}}{\lambda_1}, \ldots, \frac{\lambda_{k+1}}{\lambda_j} \) at the points \( \lambda_1, \ldots, \lambda_j \) evaluated at \( \lambda_{k+1} \). Hence, using the barycentric formula,

\[
\sum_{j=1}^{k} \frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1}) = \frac{1}{Z(\lambda_{k+1})} \sum_{j=1}^{k} \frac{w_j}{\lambda_{k+1} - \lambda_j} \lambda_{k+1} / \lambda_j.
\] (193)

Similarly, \( \sum_{j=1}^{k} \left| \frac{\lambda_{k+1}}{\lambda_j} \right| L_j(\lambda_{k+1}) \) can be seen as the polynomial interpolating \( \left| \frac{\lambda_{k+1}}{\lambda_1} \right| \text{sign}(L_1(\lambda_{k+1})), \ldots, \left| \frac{\lambda_{k+1}}{\lambda_j} \right| \text{sign}(L_j(\lambda_{k+1})) \) at the points \( \lambda_1, \ldots, \lambda_j \) evaluated at \( \lambda_{k+1} \).

However, from Berrut and Trefethen (2004, Section 3),

\[
L_j(\lambda_{k+1}) = \left( \prod_{j=1}^{k} (\lambda_{k+1} - \lambda_j) \right) \frac{w_j}{\lambda_{k+1} - \lambda_j},
\] (194)

and by Berrut and Trefethen (2004, Eq. 4.1),

\[
1 = \left( \prod_{j=1}^{k} (\lambda_{k+1} - \lambda_j) \right) Z(\lambda_{k+1}).
\] (195)

Hence

\[
\text{sign}(L_j(\lambda_{k+1})) = \text{sign} Z(\lambda_{k+1}) \text{sign} \left( \frac{w_j}{\lambda_{k+1} - \lambda_j} \right).
\] (196)

Therefore, using the barycentric formula again,

\[
\sum_{j=1}^{k} \frac{\lambda_{k+1}}{\lambda_j} |L_j(\lambda_{k+1})| = \frac{1}{|Z(\lambda_{k+1})|} \sum_{j=1}^{k} \left| \frac{w_j}{\lambda_{k+1} - \lambda_j} \right| \lambda_{k+1} / \lambda_j.
\] (197)

Hence, (184) becomes:

\[
\frac{1}{2} \left( 1 - \sum_{j=1}^{k} \frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1}) \right)^2
\]

\[
= \frac{1}{2} \left( 1 - \frac{1}{Z(\lambda_{k+1})} \sum_{j=1}^{k} \frac{w_j}{\lambda_{k+1} - \lambda_j} \lambda_{k+1} / \lambda_j \right)^2
\]

\[
= \frac{1}{2} \left( 1 - \frac{1}{|Z(\lambda_{k+1})|} \sum_{j=1}^{k} \frac{w_j}{\lambda_{k+1} - \lambda_j} \lambda_{k+1} / \lambda_j \right)^2
\]

\[
= \frac{1}{2} \left( 1 - \frac{1}{|Z(\lambda_{k+1})|} \sum_{j=1}^{k} \frac{w_j}{\lambda_{k+1} - \lambda_j} \lambda_{k+1} / \lambda_j \right)^2
\] (198)

\[
= \frac{1}{2} \left( 1 - \frac{1}{|Z(\lambda_{k+1})|} \sum_{j=1}^{k} \frac{w_j}{\lambda_{k+1} - \lambda_j} \lambda_{k+1} / \lambda_j \right)^2
\]

\[
= \frac{1}{2} \left( 1 + \frac{1}{|Z(\lambda_{k+1})|} \sum_{j=1}^{k} \frac{w_j}{\lambda_{k+1} - \lambda_j} \lambda_{k+1} / \lambda_j \right)^2.
\] (199)

Now take any \( \lambda_{k+1} \) such that \( \lambda_1 < \lambda_{k+1} < \lambda_2 \). Then, from (195), \( Z(\lambda_{k+1}) = (-1)^{k+1} = -1 \) as we assume that \( k \) is even. By definition of the coefficients \( w_j \), \( \lambda_{k+1} / \lambda_j = +1 \). Hence \( 1 + \text{sign} Z(\lambda_{k+1}) \text{sign} \frac{w_1}{\lambda_{k+1} - \lambda_1} = 0 \). Similarly, \( \lambda_{k+1} / \lambda_2 = +1 \) and so \( 1 + \text{sign} Z(\lambda_{k+1}) \text{sign} \frac{w_2}{\lambda_{k+1} - \lambda_2} = 0 \) too.\footnote{We could do without this, but it is free and gives slightly better constants.}
As the quantity inside the parentheses of (200) is non-negative, we can focus on lower bounding it. Using the considerations on signs we get:

\[
\frac{1}{|Z(\lambda_{k+1})|} \sum_{j=1}^{k} \left| \frac{w_j}{\lambda_{k+1} - \lambda_j} \right| \frac{\lambda_{k+1}}{\lambda_j} \left( 1 + \text{sign} \left( Z(\lambda_{k+1}) \right) \text{sign} \left( \frac{w_j}{\lambda_{k+1} - \lambda_j} \right) \right)
\]

\[
= \frac{1}{|Z(\lambda_{k+1})|} \sum_{j=3}^{k} \left| \frac{w_j}{\lambda_{k+1} - \lambda_j} \right| \frac{\lambda_{k+1}}{\lambda_j} \left( 1 + \text{sign} \left( Z(\lambda_{k+1}) \right) \text{sign} \left( \frac{w_j}{\lambda_{k+1} - \lambda_j} \right) \right)
\]

\[
\leq 2 \frac{\sum_{j=3}^{k} \left| \frac{1}{\lambda_{k+1} - \lambda_j} \right| \frac{\lambda_{k+1}}{\lambda_j}}{|Z(\lambda_{k+1})|} + \sum_{j=1}^{k} \left| \frac{w_j}{\lambda_{k+1} - \lambda_j} \right| \frac{\lambda_{k+1}}{\lambda_j}
\]

\[
\leq 2 \frac{\sum_{j=3}^{k} \left| \frac{1}{\lambda_{k+1} - \lambda_j} \right| \frac{\lambda_{k+1}}{\lambda_j}}{\frac{1}{\lambda_{k+1} - \lambda_1} + \frac{\lambda_{k+1}}{\lambda_1}}
\]

\[
\leq 2 \frac{(k-2) \left| \frac{\lambda_{k+1}}{\lambda_3} \right|}{\frac{1}{\lambda_{k+1} - \lambda_1} + \frac{\lambda_{k+1}}{\lambda_1}}
\]

where we used that, for \( j \geq 3 \), \( \left| \frac{1}{\lambda_{k+1} - \lambda_j} \right| \frac{\lambda_{k+1}}{\lambda_j} \leq \left| \frac{1}{\lambda_{k+1} - \lambda_3} \right| \frac{\lambda_{k+1}}{\lambda_3} \) as \( \lambda_1 < \lambda_{k+1} < \lambda_2 < \lambda_3 < \cdots < \lambda_k \). Now, recalling that \( \lambda_1 = \mu \), and using that \( \lambda_1 < \lambda_{k+1} < \lambda_2 < \lambda_3 \) for the inequality,

\[
2 \frac{(k-2) \left| \frac{\lambda_{k+1}}{\lambda_3} \right|}{\frac{1}{\lambda_{k+1} - \lambda_1} + \frac{\lambda_{k+1}}{\lambda_1}} = 4(k-2) \frac{\mu}{\lambda_3} \frac{\lambda_{k+1} - \lambda_1}{\lambda_{k+1} - \lambda_3}
\]

\[
\leq 4(k-2) \frac{\mu}{\lambda_3} \left| \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_3} \right|
\]

\[
= 4(k-2) \frac{\mu}{\frac{1}{L(1 - \cos \frac{2\pi}{k-1})} + \frac{1}{2} \mu \left( 1 + \cos \frac{2\pi}{k-1} \right) \left| \cos \frac{\pi}{k-1} - \cos \frac{2\pi}{k-1} \right|}
\]

\[
\leq 8(k-2) \frac{\mu}{L(1 - \cos \frac{2\pi}{k-1}) \left| \cos \frac{\pi}{k-1} - \cos \frac{2\pi}{k-1} \right|}
\]

\[
= 8(k-2) \frac{\mu}{L \left| \cos \frac{\pi}{k-1} - \cos \frac{2\pi}{k-1} \right|}
\]

by definition of the interpolation points. Now, for \( k \geq 4 \), the sinus is non-negative on \( \left[ \frac{\pi}{k-1}, \frac{2\pi}{k-1} \right] \) and reaches its minimum at \( \frac{\pi}{k-1} \). Hence,

\[
\left| \cos \frac{\pi}{k-1} - \cos \frac{2\pi}{k-1} \right| = \int_{\pi/(k-1)}^{2\pi/(k-1)} \sin t \, dt
\]

\[
= \int_{\pi/(k-1)}^{2\pi/(k-1)} \sin t \, dt
\]

\[
\geq \frac{\pi}{k-1} \sin \frac{\pi}{k-1}
\]

\[
\geq \frac{2}{(k-1)^2}
\]
as \(0 \geq \frac{\pi}{\lambda_1} \geq \frac{\pi}{2}\). Putting everything together yields,

\[
\frac{1}{2} \left( \frac{1 - \sum_{j=1}^{k} \frac{\lambda_i}{\lambda_j} L_j (\lambda_{k+1})}{1 + \sum_{j=1}^{k} \left| \frac{\lambda_i}{\lambda_j} L_j (\lambda_{k+1}) \right|} \right)^2 \geq \frac{1}{2} \left( 1 - \frac{4(k-1)^2(k-2) \mu}{\pi} \right)^2
\]

(216)

\[
\geq \frac{1}{2} \left( 1 - \frac{4(k-1)^3 \mu}{\pi} \right)^2,
\]

(217)

which yields the desired result by the definition of the problem of (171).

The lower bound holds for any \(v\) such that \(\text{Sp} \, \nabla v = \{\lambda_1, \ldots, \lambda_{k+1}\}\). As \(\{\lambda_1, \ldots, \lambda_{k+1}\} \subset [\mu, L]\), one can choose \(v\) of the form \(v = \nabla f\) where \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) is a \(\mu\)-strongly convex and \(L\)-smooth quadratic function with \(\text{Sp} \, \nabla^2 f = \{\lambda_1, \ldots, \lambda_{k+1}\}\).

Now, we tackle the case \(k\) odd, with \(k \geq 3\) and \(d - 1 \geq k + 1\). Note that if \(\mathcal{N}\) is a real polynomial of degree at most \(k - 1\), it is also a polynomial of degree at most \((k + 1) - 1\). Applying the result above yields that there exists \(v \in \mathcal{V}_d\) with the desired properties such that,

\[
\rho(F_{\mathcal{N}}) \geq 1 - \frac{k^3 \gamma^2}{2\pi L^2}.
\]

(218)

Hence, (i) holds for any \(d - 2 \geq k \geq 3\).

Then, (ii) is essentially a corollary of (i).

**Proof of (ii) of Thm. 6.** For a square zero-sum two player game, the Jacobian of the vector field can be written as,

\[
\nabla v = \begin{pmatrix} 0_m & A \\ -A^T & 0_m \end{pmatrix}
\]

(219)

where \(A \in \mathbb{R}^{m \times m}\). By Lem. 4,

\[
\text{Sp} \, \nabla v = \{i\sqrt{\lambda} \mid \lambda \in \text{Sp} \, AA^T\} \cup \{-i\sqrt{\lambda} \mid \lambda \in \text{Sp} \, AA^T\}.
\]

(220)

Using Lem. 12, one gets that:

\[
\min_{N \in \mathbb{R}_{k-1}[X]} \frac{1}{2} \rho(F_{\mathcal{N}})^2 = \min_{a_0, \ldots, a_{k-1} \in \mathbb{R} \, \lambda \in \text{Sp} \, AA^T} \max \left\{ \frac{1}{2} \left( 1 + \sum_{l=0}^{k-1} a_l (i \sqrt{\lambda})^{l+1} \right)^2, \frac{1}{2} \left( 1 + \sum_{l=0}^{k-1} a_l (-i \sqrt{\lambda})^{l+1} \right)^2 \right\}
\]

(221)

\[
\geq \min_{a_0, \ldots, a_{k-1} \in \mathbb{R} \, \lambda \in \text{Sp} \, AA^T} \max \left\{ \frac{1}{2} \left( 1 + \sum_{l=0}^{k-1} a_l (i \sqrt{\lambda})^{l+1} \right)^2 \right\}
\]

(222)

\[
\geq \min_{a_0, \ldots, a_{k-1} \in \mathbb{R} \, \lambda \in \text{Sp} \, AA^T} \max \left\{ \frac{1}{2} \left( 1 + \sum_{l=0}^{\lfloor k/2 \rfloor} a_{2l} (-1)^l \lambda^l \right)^2 \right\}
\]

(223)

\[
= \min_{a_0, \ldots, a_{\lfloor k/2 \rfloor - 1} \in \mathbb{R} \, \lambda \in \text{Sp} \, AA^T} \max \left\{ \frac{1}{2} \left( 1 + \sum_{l=1}^{\lfloor k/2 \rfloor} a_{l} \lambda^l \right)^2 \right\}.
\]

(224)

Using Lem. 12 again,

\[
\min_{a_0, \ldots, a_{\lfloor k/2 \rfloor - 1} \in \mathbb{R} \, \lambda \in \text{Sp} \, AA^T} \max \left\{ \frac{1}{2} \left( 1 + \sum_{l=1}^{\lfloor k/2 \rfloor} a_{l-1} \lambda^l \right)^2 \right\} = \min_{N \in \mathbb{R}_{\lfloor k/2 \rfloor - 1}[X]} \frac{1}{2} \rho(F_{\mathcal{N}})^2.
\]

(225)

(226)
where \( \tilde{F}_N \) is the 1-SCLI operator of \( N \), as defined by (19) applied to the vector field \( \omega \mapsto AA^T \omega \).

Let \( S \in \mathbb{R}^{m \times m} \) be a symmetric positive definite matrix given by (i) of this theorem applied with \((\mu, L) = (\gamma^2, L^2)\) instead of \( k \) and so such that \( \text{Sp} \, S \subset [\gamma^2, L^2] \). Now choose \( A \in \mathbb{R}^{m \times m} \) such that \( A^T A = S \), for instance by taking a square root of \( S \) (see Lax (2007, Chapter 10)). Then,

\[
\min_{N \in \mathbb{R}_{\lfloor k/2 \rfloor - 1}[X]} \frac{1}{2} \rho(\tilde{F}_N)^2 \geq \frac{1}{2} \left( 1 - \frac{k^3 \mu}{2\pi L} \right). \tag{227}
\]

Moreover, by computing \( \nabla v^T \nabla v \) and using that \( \text{Sp} \, AA^T = \text{Sp} \, A^T A \), one gets that

\[
\min_{\lambda \in \text{Sp} \nabla v} |\lambda| = \sigma_{\min}(\nabla v) = \sigma_{\min}(A) \geq \gamma \quad \text{and} \quad \sigma_{\max}(\nabla v) = \sigma_{\max}(A) \leq L.
\]

Remark 3. Interestingly, the examples we end up using have a spectrum similar to the one of the matrix Nesterov uses in the proofs of his lower bounds in Nesterov (2014). The choice of the spectrum of the Jacobian of the vector field was indeed the choice of interpolation points. Following Salzer (1971); Berrut and Trefethen (2004) we used points distributed across the interval as a cosine as it minimizes oscillations near the edge of the interval. Therefore, this links the hardness Nesterov’s examples to the well-conditioning of families of interpolation points.