Mok-Siu-Yeung type formulas on contact locally sub-symmetric spaces

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Abstract

We derive Mok-Siu-Yeung type formulas for horizontal maps from compact contact locally sub-symmetric spaces into strictly pseudoconvex CR manifolds and we obtain some rigidity theorems for the horizontal pseudoharmonic maps.

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1 Introduction

The main ingredient in the harmonic maps approach of superrigidity for semi-simple Lie groups is the Mok-Siu-Yeung formula \[23, 32\] for harmonic maps defined on compact locally symmetric spaces of non-compact type (cf. also \[19\]). Actually, by means of this formula, it can be shown that a harmonic map from a compact locally symmetric space of non-compact type (with some exceptions) into a Riemannian manifold with nonpositive curvature is rigid in the sense of it is a totally geodesic isometric imbedding. The contact locally sub-symmetric spaces defined by Bieliavsky, Falbel and Gorodski \[6, 17, 18\] are the contact analogues of the riemannian locally symmetric spaces. These spaces can be characterized as contact metric manifolds for which the curvature and the torsion of the Tanaka-Webster connection are parallel in the direction of the contact distribution. Moreover, these spaces are strictly pseudoconvex CR manifolds. In an other hand, in the setting of contact metric manifolds, the analogue of harmonic maps seems to be the pseudoharmonic maps defined by Barletta, Dragomir and Urakawa \[3, 5\]. Also the main purpose of this article is to derive Mok-Siu-Yeung type formulas for horizontal maps (i.e. maps preserving the contact distributions) from compact contact locally sub-symmetric spaces into strictly pseudoconvex CR manifolds in order to obtain some rigidity theorems for horizontal pseudoharmonic maps under curvature assumptions. The plan of this article is the following. The section 2 begins to recall basic facts concerning the contact metric manifolds and the strictly pseudoconvex CR manifolds, next, we focus our attention on the pseudo-hermitian curvature tensor of a strictly pseudoconvex CR manifold, which plays a central part in the following. In section 3, we investigate the contact sub-symmetric spaces, the main result of this section (Theorem 3.3) which is related to the work of Cho \[14\], is an explicit formula for the pseudo-hermitian curvature tensor of a contact locally sub-symmetric space with non zero pseudo-hermitian torsion. The section 4 is devoted to derive Mok-Siu-Yeung type formulas for horizontal maps between strictly pseudoconvex CR manifolds (Proposition 4.2). In section 5, we extend the notion of pseudoharmonic maps defined in \[3, 5\] to the setting of horizontal maps between contact metric manifolds and we define the notion of CR-pluriharmonic maps for horizontal maps between strictly pseudoconvex CR manifolds. It is interesting to note that these two notions are strongly related to the Rumin complex \[27\]. In section 6, we obtain some rigidity theorems for the horizontal pseudoharmonic maps when the source manifold is a compact contact locally sub-symmetric space. The main result of this section (Theorem 6.1) asserts that any horizontal pseudoharmonic map $\phi$ from a compact contact locally sub-symmetric space of non-compact type, holonomy irreducible and torsionless, (with some exceptions) into a Sasakian manifold with nonpositive pseudo-Hermitian complex sectional curvature satisfies $\nabla d\phi = 0$ where $\nabla d\phi$ is the covariant derivative of $d\phi$ with respect to Tanaka-Webster connections. As application (Corollary 6.1) we deduce that $\phi$ preserves some special curves called parabolic geodesics \[15\] and therefore $\phi$ is, in some sense, totally geodesic. In section 7, we restrict our attention to CR maps from com-
pact contact locally sub-symmetric spaces into strictly pseudoconvex $CR$ manifolds and we obtain the following rigidity result (Theorem 7.1): any horizontal pseudoharmonic $CR$ map from a compact contact locally sub-symmetric space of non-compact type (with some exceptions) into a pseudo-Hermitian space form with negative pseudo-Hermitian scalar curvature is constant. This article is a first step in the study of horizontal pseudoharmonic maps from compact strictly pseudoconvex $CR$ manifolds into strictly pseudoconvex $CR$ manifolds with nonpositive pseudo-Hermitian sectional curvature. In particular, some existence results are missing for the moment (excepted if the target manifold is Tanaka-Webster flat).

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2 Connection and curvature on contact metric manifolds

A contact form on a smooth manifold $M$ of dimension $m = 2d + 1$ is a 1-form $\theta$ satisfying $\theta \wedge (d\theta)^d \neq 0$ everywhere on $M$. If $\theta$ is a contact form on $M$, the hyperplan subbundle $H$ of $TM$ given by $H = \text{Ker} \theta$ is called a contact structure. The Reeb field associated to $\theta$ is the unique vector field $\xi$ on $M$ satisfying $\theta(\xi) = 1$ and $d\theta(\xi, .) = 0$. By a contact manifold $(M, \theta)$ we mean a manifold $M$ endowed with a fixed contact form $\theta$.

If $(M, \theta)$ is a contact manifold then $TM$ decomposes as $TM = H \oplus \mathbb{R} \xi$. Consequently any $p$-tensor $t$ on $M$ decomposes as $t = t_H + t_\xi$ with $t_H = t \circ \Pi_H$ and $t_\xi = t \circ \Pi_{\mathbb{R} \xi}$ ($\Pi_H$ and $\Pi_{\mathbb{R} \xi}$ are the canonical projections on $H$ and $\mathbb{R} \xi$). The tensors $t_H$ and $t_\xi$ are respectively called the horizontal part and the vertical part of $t$. Note that for an antisymmetric $p$-tensor $\gamma$ we have $\gamma_\xi = \theta \wedge i(\xi)\gamma$. We denote by $\wedge^*_H(M)$ and $\wedge^*_\xi(M)$ the bundles of horizontal and vertical antisymmetric tensors and by $\Omega^*_H(M)$ and $\Omega^*_\xi(M)$ the horizontal and vertical forms associated to.

Let $(M, \theta)$ be a contact manifold, then there exists a riemannian metric $g_\theta$ and a $(1,1)$-tensor field $J$ on $M$ such that:

$$g_\theta(\xi, X) = \theta(X), \quad J^2 = -Id + \theta \otimes \xi, \quad g_\theta(JX, Y) = d\theta(X, Y), \quad X, Y \in TM.$$  

The metric $g_\theta$ (called the Webster metric) is said to be associated to $\theta$. We call $(\theta, \xi, J, g_\theta)$ a contact metric structure and $(M, \theta, \xi, J, g_\theta)$ a contact metric manifold (cf. Blair[7]). In the following $\omega_\theta := d\theta$. 

Let \((M, \theta, \xi, J, g_\theta)\) be a contact metric manifold. We define \(L : \Omega^k(M) \to \Omega^{k+2}(M)\) by \(L = \omega_\theta \wedge\). The restriction of \(L\) to \(\Omega^*_H(M)\) will be denoted by \(L_H\) and the adjoint of \(L_H\) for the usual scalar product on \(\Omega^*_H(M)\) by \(\wedge_H\). Recall that, for any \(\gamma_H \in \Omega^*_H(M)\),
\[
(\wedge_H \gamma_H)(X_1, \ldots, X_{p-2}) = \frac{1}{2} tr_H \gamma_H(\ldots J\ldots, X_1, \ldots, X_{p-2}),
\]
where \(tr_H\) is the trace calculated with respect to a \(g_\theta\)-orthonormal frame of \(H\).

Let \(\Omega^*_{H_0}(M) := \{\gamma_H \in \Omega^*_H(M), \wedge_H \gamma_H = 0\}\) and \(\mathcal{F}^*_e(M) := \{\gamma_\xi \in \Omega^*_e(M), \: L_H \gamma_\xi = 0\}\) be the bundle of primitive horizontal forms on \(M\) and the bundle of coprimitive vertical forms on \(M\). We recall the Lefschetz decomposition
\[
\Omega^*_e(M) = \Omega^*_{H_0}(M) \oplus L_H \Omega^*_{H_0}(M) \oplus \ldots \oplus L^d_H \Omega^*_{H_0}(M).
\]

connection and curvature

For the torsion and the curvature of a connection \(\nabla\) we adopt the conventions \(T(X, Y) = [X, Y] - \nabla_X Y + \nabla_Y X\) and \(R(X, Y) = [\nabla_Y, \nabla_X] - \nabla_{[Y, X]}\).

In the following, \(N\) is the \(TM\)-valued 2-form given by:
\[
N(Y, Z) = J^2[Y, Z] + [JY, JZ] - J[Y, JZ] - J[JY, Z] + \omega_\theta(Y, Z) \xi.
\]

**Proposition 2.1** (Generalized Tanaka-Webster connection cf. \([29, 30, 31]\)). Let \((M, \theta, \xi, J, g_\theta)\) be contact metric manifold, then there exists a unique affine connection \(\nabla\) on \(TM\) with torsion \(T\) (called the generalized Tanaka-Webster connection) such that:
(a) \(\nabla \theta = 0\), \(\nabla \xi = 0\).
(b) \(\nabla g_\theta = 0\).
(c) \(T_H = -\omega_\theta \otimes \xi\) and \(i(\xi)T = -\frac{1}{2}i(\xi)N\).
(d) \((\nabla_X \omega_\theta)(Y, Z) = g_\theta((\nabla_X J)(Y), Z) = \frac{1}{2} \omega_\theta(X, N_H(Y, Z))\) for any \(X, Y, Z \in TM\).

The endomorphism \(\tau := i(\xi)T\) is called the generalized Tanaka-Webster torsion or sub-torsion. Note that \(\tau\) is \(g_\theta\)-symmetric with trace-free and satisfies \(\tau \circ J = -J \circ \tau\).

A contact metric manifold \((M, \theta, \xi, J, g_\theta)\) for which \(J\) is integrable (i.e. \(N_H = 0\) or equivalently \(\nabla J = 0\)) is called a strictly pseudoconvex \(CR\) manifold. A strictly pseudoconvex \(CR\) manifold for which the Tanaka-Webster torsion vanishes is called a Sasakian manifold.

The curvature \(R\) of the generalized Tanaka-Webster connection \(\nabla\) satisfies the following Bianchi identities (cf. \([17, 29]\)):
(First Bianchi identity)
\[
R_H(X, Y) Z + R_H(Z, X) Y + R_H(Y, Z) X = \omega_\theta(X, Y) \tau(Z) + \omega_\theta(Z, X) \tau(Y) + \omega_\theta(Y, Z) \tau(X)
\]
(1)
\[ R(X, \xi)Z + R(\xi, Z)X = (\nabla_X \tau)(Z) - (\nabla_Z \tau)(X) \]  
(2)

with \((\nabla_X \tau)(Z) = \nabla_X \tau(Z) - \tau(\nabla_X Z)\).

Remember that any horizontal 2-tensor \( t_H \) on \( M \) decomposes into \( t_H = t_H^+ + t_H^- \), where \( t_H^\pm := \frac{1}{2}(t_H \pm J^* t_H) \) are respectively the \( J \)-invariant part and the \( J \)-anti-invariant part of \( t_H \).

If \( M \) is a strictly pseudoconvex \( CR \) manifold, then we have the decomposition:

\[ R_H = R_H^+ + R_H^- \]

with

\[ R_H^-(X,Y) = -\frac{1}{2} \left( (\tau(X))^* \wedge (JY)^* - (\tau(Y))^* \wedge (JX)^* - (J\tau(X))^* \wedge Y^* + (J\tau(Y))^* \wedge X^* \right). \]  
(3)

Also we define the pseudo-Hermitian curvature tensor \( R_W^H \) of a strictly pseudoconvex \( CR \) manifold by:

\[ R_W^H(X,Y,Z,W) = g(\theta(R_H^+(X,Y))Z,W), \quad X,Y,Z,W \in H. \]

In order to give the algebraic properties of \( R_W^H \), we recall some definitions related to the curvature algebra (cf. [11]).

Let \((V, q)\) be an euclidean space and \( Q \in \bigotimes^4 V^* \). We define the Bianchi map \( b(Q) \) by:

\[ b(Q)(X,Y,Z,W) = Q(X,Y,Z,W) + Q(Z,X,Y,W) + Q(Y,Z,X,W), \quad X,Y,Z,W \in V. \]

Recall that \( b(S^2(\wedge^2 V^*)) = \wedge^4 V^* \) and that we have the decomposition:

\[ S^2(\wedge^2 V^*) = \text{Ker} b \oplus \wedge^4 V^*. \]

Let \( h, k \in \bigotimes^2 V^* \). We define the symmetric product \( h \odot k \in S^2(\bigotimes^2 V^*) \) and the Kulkarni product \( h \boxtimes k \in \bigotimes^2(\wedge^2 V^*) \) respectively by:

\[ (h \odot k)(X,Y,Z,W) = h(X,Y)k(Z,W) + h(Z,W)k(X,Y), \]

and

\[ (h \boxtimes k)(X,Y,Z,W) = (h \odot k)(X,Z,Y,W) - (h \odot k)(X,W,Y,Z). \]

Note that if \( h, k \in S^2 V^* \) then \( h \boxtimes k \in S^2(\wedge^2 V^*) \) and \( b(h \boxtimes k) = 0 \).

If \( h, k \in \wedge^2 V^* \) then \( h \boxtimes k \in S^2(\wedge^2 V^*) \) and \( b(h \boxtimes k) = -2b(h \odot k) \).

If \( h \in S^2 V^* \) and \( k \in \wedge^2 V^* \) then \( h \boxtimes k \in \wedge^2(\wedge^2 V^*) \) and \( b(h \boxtimes k) = -2b(k \otimes h) \).

We define the Ricci contraction \( c(Q) \) by:

\[ c(Q)(X,Y) = tr Q(.X,.Y), \]

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with the trace is taken with respect to a $q$-orthonormal basis of $V$.

Let $(M, \theta, \xi, J, g_\theta)$ be a contact metric manifold and let $Q_H \in \otimes^2(\Lambda^2_H(M))$. The endomorphism $\widehat{Q}_H$ of $\Lambda^2_H(M)$ associated to $Q_H$ is defined by:

$$(\widehat{Q}_H \gamma_H)(X, Y) = \frac{1}{2} \sum_{i,j} Q_H(\epsilon_i, \epsilon_j, X, Y) \gamma_H(\epsilon_i, \epsilon_j),$$

where $\gamma_H \in \Lambda^2_H(M)$, $\{\epsilon_i\}$ is a local $g_\theta$-orthonormal frame of $H$ and $X, Y \in H$.

For $X, Y, Z, W \in H$, we have $\langle \widehat{Q}_H(X \wedge Y), Z \wedge W \rangle = Q_H(X, Y, Z, W)$.

Let $\Lambda^2_{H_0}(M)$ be the bundle of primitif horizontal antisymmetric 2-tensors. For $Q_H \in S^2(\Lambda^2_H(M))$, we define $Q_{H_0} \in S^2(\Lambda^2_{H_0}(M))$ by:

$$Q_{H_0} = Q_H - \frac{1}{d} (\widehat{Q}_H \omega_\theta) \otimes \omega_\theta + \frac{1}{2d^2} (\Lambda_H(\widehat{Q}_H \omega_\theta)) \omega_\theta \otimes \omega_\theta.$$

Note that, for $Q_H$ viewed as horizontal $\Lambda^2_H(M)$-valued 2-form, we have $\widehat{Q}_H \omega_\theta = \wedge_H Q_H$.

Let $\Lambda^2_{H_0}^\pm(M)$ be the bundle of $J$-invariant ($J$-anti-invariant) horizontal antisymmetric 2-tensors. For $Q_H \in S^2(\Lambda^2_H(M))$, we define $Q_{H_0}^\pm \in S^2(\Lambda^2_{H_0}^\pm(M))$ by:

$$Q_{H_0}^\pm(X, Y, Z, W) = \frac{1}{4} (Q_H(X, Y, Z, W) \pm Q_H(JX, JY, Z, W) \pm Q_H(X, Y, JZ, JW) + Q_H(JX, JY, JZ, JW)).$$

Note that

$$(g_{\theta_H} \otimes g_{\theta_H})_0 = g_{\theta_H} \otimes g_{\theta_H} - \frac{1}{d} \omega_\theta \otimes \omega_\theta$$

and

$$(g_{\theta_H} \otimes g_{\theta_H})^\pm = \frac{1}{2} (g_{\theta_H} \otimes g_{\theta_H} \pm \omega_\theta \otimes \omega_\theta),$$

with $g_{\theta_H} = g_{\theta/H}$.

For any $P_H, Q_H \in S^2(\Lambda^2_H(M))$, the scalar product $\langle P_H, Q_H \rangle$ is defined by:

$$\langle P_H, Q_H \rangle = \frac{1}{2} tr_H \widehat{P}_H \circ \widehat{Q}_H.$$

We have

$$\langle (g_{\theta_H} \otimes g_{\theta_H})^\pm, Q_H \rangle = tr_H \widehat{Q}_H^\pm$$

and for $Q_H \in S^2(\Lambda^2_H(M)) \cap Ker b$,

$$\langle \omega_\theta \otimes \omega_\theta, Q_H \rangle = tr_H \widehat{Q}_H^+ - tr_H \widehat{Q}_H^-.$$
If \( M \) is CR then \( R^W_H \) can be written by (3) as:

\[
R^W_H = R_H + \frac{1}{2} (\omega_\theta \otimes A_\theta - g_{\theta H} \otimes B_\theta),
\]

with \( R_H(X, Y, Z, W) = g_\theta(R_H(X, Y)Z, W), \ A_\theta(X, Y) = g_\theta(\tau(X), Y) \) and \( B_\theta(X, Y) = \omega_\theta(\tau(X), Y) \).

It follows from (1) and (4) that \( R^W_H \) satisfies the following the algebraic properties:

\[
R^W_H \in S^2(\wedge^2 H(M)) \cap Ker b \text{ and } R^W_H \in S^2(\wedge^{2+} H(M)).
\]

The pseudo-Hermitian Ricci tensor \( Ric^W_H \in S^2_H(M) \), the pseudo-Hermitian Ricci form \( \rho^W_H \in \wedge^{2+} H(M) \) and the pseudo-Hermitian scalar curvature \( s^W \) are respectively defined by:

\[
Ric^W_H = c_H(R^W_H), \quad \rho^W_H = -\hat{R}^W_H \omega_\theta, \quad s^W = tr_H Ric^W_H,
\]

with \( c_H(Q_H) \) calculated with respect to a \( g_\theta \)-orthonormal frame of \( H \).

Now, let \( I^C_H \in S^2(\wedge^{2+} H(M)) \cap Ker b \) given by:

\[
I^C_H = \frac{1}{8} \left( g_{\theta_H} \otimes g_{\theta_H} + \omega_\theta \otimes \omega_\theta + 2 \omega_\theta \otimes \omega_\theta \right).
\]

We have the decompositions (cf. [13]):

\[
R^W_H = \frac{s^W}{d(d+1)} I^C_H + \frac{1}{d+2} \left( \frac{1}{2} (Ric^W_{H_0} \otimes g_{\theta_H} - \rho^W_{H_0} \otimes \omega_\theta) - \rho^W_{H_0} \otimes \omega_\theta \right) + C^M_H
\]

\[
R^C_H = \frac{s^W}{d(d+1)} I^C_H + \frac{1}{2(d+2)} \left( Ric^W_{H_0} \otimes g_{\theta_H} - \rho^W_{H_0} \otimes \omega_\theta \right)_0 + C^M_H,
\]

where \( Ric^W_{H_0} \) (respectively \( \rho^W_{H_0} \)) is the traceless part of \( Ric^W_H \) (respectively the primitive part of \( \rho^W_H \)) and \( C^M_H \in S^2(\wedge^{2+} H(M)) \) \( \cap Ker b \cap Ker c_H \).

**Remark 2.1** The tensor \( C^M_H \), introduced by Chern and Moser in [13], is called the Chern-Moser tensor. Note that \( C^M_H \) is a pseudo-conformal invariant.

Now, we define the pseudo-Hermitian sectional curvature (resp. pseudo-Hermitian complex sectional curvature) of a 2-plane \( P = \mathbb{R}\{X, Y\} \subset H \) (resp. \( P = \mathbb{C}\{Z, W\} \subset H^C \)) by:

\[
K^W(P) = \frac{\langle \tilde{R}^W_H(X \wedge Y), X \wedge Y \rangle}{\langle X \wedge Y, X \wedge Y \rangle}, \quad K^W(P) = \frac{\langle \tilde{R}^W_H \omega_\theta \circ \omega_\theta \rangle}{\langle Z \wedge W, Z \wedge W \rangle},
\]
where $(\cdot,\cdot)$ and $\hat{R}^W_H$ are the natural extensions to $\wedge^2 H^C$ of $(\cdot,\cdot)$ and $R^W_H$.

The holomorphic pseudo-Hermitian sectional curvature of a holomorphic 2-plane $P = \mathbb{R}\{X,JX\} \subset H$ is defined by:

$$HK^W(P) = \frac{\langle \hat{R}^W_H(X \wedge JX), X \wedge JX \rangle}{\langle X \wedge JX, X \wedge JX \rangle}.$$

We say that a strictly pseudoconvex $CR$ manifold $(M, \theta, \xi, J, g_\theta)$ has constant holomorphic pseudo-Hermitian sectional curvature if $HK^W(P)$ is constant for any holomorphic 2-plane $P \subset H$ and for any point of $M$. In this case, we have $R^W_H = \frac{s^W}{d(d+1)} I^C_H$ with $s^W$ constant (cf. [2]). Also we call $I^C_H$ the holomorphic pseudo-Hermitian curvature tensor.

Let $(E, g^E, \nabla^E)$ be a riemannian vector bundle over a contact metric manifold $M$ and let $\Omega^p(M; E)$ (resp. $\Omega^p_H(M; E)$) be the bundle of $E$-valued forms (resp. horizontal $E$-valued forms) on $M$. We assume that $M$ is endowed with the generalized Tanaka-Webster connection $\nabla$. Remember (cf. [25]) that for any $\sigma \in \Omega^p(M; E)$,

$$(\nabla_X \sigma)(X_1, \ldots, X_p) = \nabla^E_X \sigma(X_1, \ldots, X_p) - \sum_{i=1}^p \sigma(X_1, \ldots, \nabla_X X_i, \ldots, X_p)$$

$$(d\nabla^E \sigma)(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla^E_{\dot{X}_i} \sigma(X_1, \ldots, \dot{X}_i, \ldots, X_{p+1})$$

Now, for any $\sigma_H \in \Omega^p_H(M; E)$, we define

$$(d\nabla^E_H \sigma_H)(X_1, \ldots, X_{p+1}) := (d\nabla^E_H \sigma_H)(X_1, \ldots, X_{p+1})$$

$$= \sum_{i=1}^{p+1} (-1)^{i+1} (\nabla_{\dot{X}_i} \sigma_H)(X_1, \ldots, \dot{X}_i, \ldots, X_{p+1})$$

$$(\mathcal{L}_\xi \sigma_H)(X_1, \ldots, X_p) := (i(\xi)(d\nabla^E_H \sigma_H))(X_1, \ldots, X_p)$$

$$= (\nabla_\xi \sigma_H)(X_1, \ldots, X_p) + \sum_{i=1}^p (-1)^i \sigma_H(\tau(X_i), X_1, \ldots, \dot{X}_i, \ldots, X_p)$$
\((\delta_{H} E \sigma_{H})(X_1, \ldots, X_p) = -tr_H(\nabla . \sigma_{H})(., X_1, \ldots, X_p)\)

\((R_{H}(X, Y)\sigma_{H})(X_1, \ldots, X_p) = R^E_{ii}(X, Y)\sigma_{H}(X_1, \ldots, X_p) - \sum_{i=1}^{p} \sigma_{H}(X_1, \ldots, R_{H}(X, Y)X_i, \ldots, X_p)\),

where \(R^E\) is the curvature of \(\nabla^E\).

Note that for any \(\sigma_{H} \in \Omega^p_{H}(M; E)\), we have:

\[d_{H}^{\nabla^{E}} \sigma_{H} = -L_{H}(\mathcal{L}_{\xi} \sigma_{H}) - R^{E}_{H} \wedge \sigma_{H},\]

where \(R^{E}_{H} \wedge \sigma_{H}\) is the wedge product of the horizontal \(End(E)\)-valued 2-form \(R^{E}_{H}\) with the horizontal \(E\)-valued \(p\)-form \(\sigma_{H}\). Now the horizontal \(End(H)\)-valued 2-form \(R_{H}\) satisfies the second Bianchi identity:

\[
(\nabla_{X} R_{H})(Y, Z) + (\nabla_{Z} R_{H})(X, Y) + (\nabla_{Y} R_{H})(Z, X) = -\omega_{\theta}(X, Y)(i(\xi)R)(Z) + \omega_{\theta}(Z, X)(i(\xi)R)(Y) - \omega_{\theta}(Y, Z)(i(\xi)R)(X)
\]

\[
(\nabla_{\xi} R_{H})(X, Y) - (\nabla_{X} i(\xi)R)(Y) + (\nabla_{Y} i(\xi)R)(X) = R_{H}(\tau(X, Y) + R_{H}(X, \tau(Y)).
\]  \(\text{(5)}\)

**Remark 2.2** The Bianchi identity \(\text{(5)}\) is equivalent to \(d_{H}^{\nabla^{E}} R_{H} = -L_{H}(i(\xi)R)\) and \(\mathcal{L}_{\xi} R_{H} = d_{H}^{\nabla^{E}} (i(\xi)R)\) where \(\nabla\) is the connection on \(\wedge^{2}_{H}(M)\) induced by the Tanaka-Webster connection on \(H\).

### 3 Contact sub-symmetric spaces

**Definition 3.1** A contact (locally) sub-symmetric space is a contact metric manifold \((M, \theta, \xi, J, g_{0})\) such that for every point \(x_0 \in M\) there exists an isometry (resp a local isometry) \(\psi\), called the sub-symmetry at \(x_0\), satisfying \(\psi(x_0) = x_0\) and \(d\psi(x_0)/H_{x_0} = -id\).

**Theorem 3.1** \(\text{[3]}\) Let \((M, \theta, \xi, J, g_{0})\) be a contact metric manifold endowed with its generalized Tanaka-Webster connection \(\nabla\) and let \(R\) (respectively \(T\)) be the curvature (respectively torsion) of \(\nabla\). Then:

(i) \(M\) is a contact locally sub-symmetric space if and only if \(\nabla_{H} R = \nabla_{H} T = 0\).

(ii) If \(M\) is a contact locally sub-symmetric space, \(\nabla\)-complete and simply-connected then \(M\) is a contact sub-symmetric space.

(iii) If \(M\) is a contact sub-symmetric space then \(M = G/K\) where \(G\) is the closed subgroup of \(I(M, g_{0})\) generated by all the sub-symmetries \(\psi(x_0), x_0 \in M\), and \(K\) is the isotropy subgroup at a base point (i.e. \(M\) is an homogeneous manifold).

Note that the conditions \(\nabla_{H} R = \nabla_{H} T = 0\) are equivalent to \(\nabla_{H} R_{H} = 0, i(\xi)R = 0, \nabla_{\omega_{\theta}} = 0\) and \(\nabla_{H} \tau = 0\).

**Definition 3.2** \(\text{[13]}\) A contact sub-symmetric space \(M\) is said to be irreducible if the Lie algebra \(\mathfrak{hol}(M)\) of the holonomy group \(\text{Hol}(M)\) acts irreducibly on \(H\).
The simply-connected contact sub-symmetric spaces have been classified by Bieliavsky, Falbel and Gorodski in [6].

**Theorem 3.2** Every simply-connected contact sub-symmetric space of dimension \( \geq 5 \) has the following type:

| holonomy trivial | torsionless | Compact type | Non-compact type |
|------------------|-------------|--------------|-----------------|
|                  |             | compact Hermitian (CH): | non-compact Hermitian (NCH): |
|                  |             | \( SU(p + q)/SU(p) \times SU(q) \) | \( SU(p, q)/SU(p) \times SU(q) \) |
|                  |             | \( SO(2p)/SU(p) \) | \( SO^*(2p)/SU(p) \) |
| holonomy         | torsionless | \( Sp(p)/SU(p) \) | \( Sp(p, \mathbb{R})/SU(p) \) |
| irreducible      |             | \( SO(p + 2)/SO(p) \) (\( p \geq 3 \)) | \( SO_0(p, 2)/SO(p) \) (\( p \geq 3 \)) |
|                  | with torsion | \( E_6(-78)/\text{Spin}(10) \) | \( E_6(-14)/\text{Spin}(10) \) |
|                  | \( p \geq 3 \) | \( E_7(-133)/E_6 \) | \( E_7(-25)/E_6 \) |
| holonomy         | torsionless | \( \mathcal{H}_{2p+1} \times_G CH \) \( (G \subset U(p)) \) | \( \mathcal{H}_{2p+1} \times_G NCH \) |
| not              |             | \( SO(4)/SO(2) \) | \( SO_0(2, 2)/SO(2) \) |
| irreducible      | with torsion | \( SO(3) \times \mathbb{R}^3/\text{Spin}(2) \) | \( SO_0(2, 1) \times \mathbb{R}^3/\text{SO}(2) \) |
|                  |             | \( SO(3, 1)/\text{SO}(2) \) | \( SO_0(3, 1)/\text{SO}(2) \) |

**Remark 3.1** The contact sub-symmetric spaces of compact Hermitian type (respectively non-compact Hermitian type) arise from \( S^1 \)-fibrations over irreducible Hermitian symmetric spaces of compact type (respectively non-compact type). The previous list is not complete, because \( S^1 \)-fibrations over not irreducible Hermitian symmetric spaces also produce examples of contact sub-symmetric spaces. We do not consider these examples in the following.

A contact locally sub-symmetric space \((M, \theta, \xi, J, g_\theta)\) is always a strictly pseudoconvex CR manifold (since \( \nabla \omega_\theta = 0 \)). Now we recall the notion of homogeneous strictly pseudoconvex CR manifold and symmetric strictly pseudoconvex CR manifold (cf. [20],[21]).

Let \((M, H, J, \theta, g_\theta)\) and \((N, H', J', \theta', g_{\theta'})\) be strictly pseudoconvex CR manifolds then a map \( \phi : M \to N \) such that \( d\phi(H) \subset H' \) and \( J' \circ d\phi_H = d\phi_H \circ J \) is called a CR map from \( M \) to \( N \) (the definition is also valid in the general context of CR manifolds [11]). A CR automorphism \( \phi : M \to M \) is a diffeomorphism and a CR map from \( M \) to \( M \). The group of CR automorphisms \( \text{Aut}_{CR}(M) \) is a Lie group. A CR automorphism \( \phi : M \to M \) is called a pseudo-Hermitian transformation if \( \phi^* \theta = \theta \). The group of pseudo-Hermitian transformations \( \text{PsH}(M, \theta) \) is a Lie subgroup of \( \text{Aut}_{CR}(M) \) and also a Lie subgroup of \( I(M, g_\theta) \).
Definition 3.3 A strictly pseudoconvex CR manifold \((M, H, J, \theta, g_\theta)\) is called homogeneous if there exists a closed subgroup \(G\) of \(PsH(M, \theta)\) which acts transitively on \(M\).

Definition 3.4 A (locally) symmetric strictly pseudoconvex CR manifold is a strictly pseudoconvex CR manifold \((M, H, J, \theta, g_\theta)\) such that for every point \(x_0 \in M\) there exists a pseudo-Hermitian transformation (resp a local pseudo-Hermitian transformation) \(\psi\), called the pseudo-Hermitian symmetry at \(x_0\), satisfying \(\psi(x_0) = x_0\) and \(d\psi(x_0)/H_{x_0} = -id\).

If \((M, H, J, \theta, g_\theta)\) is a symmetric strictly pseudoconvex CR manifold then \(M = G/K\) where \(G\) is the closed subgroup of \(PsH(M, \theta)\) generated by all the pseudo-Hermitian symmetries \(\psi(x_0), x_0 \in M\), and \(K\) is the isotropy subgroup at a base point. Also \(M\) is an homogeneous strictly pseudoconvex CR manifold. Note that the contact sub-symmetric spaces torsionless are symmetric Sasakian manifolds.

Let \((M, \theta, \xi, J, g_\theta)\) be a simply-connected contact sub-symmetric space and \(\Gamma\) be a cocompact discrete subgroup of \(PsH(M, \theta)\) acting freely on \(M\) then \(M/\Gamma\) is a compact contact locally sub-symmetric space.

Now we investigate the properties of the pseudo-Hermitian curvature tensor on a contact locally sub-symmetric space.

Proposition 3.1 Let \((M, \theta, \xi, J, g_\theta)\) be a contact locally sub-symmetric space endowed with its Tanaka-Webster connection \(\nabla\). Then we have \(\nabla R^W_H = 0\) and consequently \(s^W\) is constant.

Proof. Since \(\nabla H R_H = 0\) and \(\nabla J = 0\), we have \(\nabla H R^+_H = 0\). Now, we must prove that \(\nabla_\xi R^+_H = 0\). Equation (5) together with the assumption \(i(\xi)R = 0\) gives, for any \(X, Y \in H\),

\[
(\nabla_\xi R_H)(X,Y) = R_H(\tau(X), Y) + R_H(X, \tau(Y)).
\]

Since \(J \circ \tau = -\tau \circ J\), we deduce that

\[
(\nabla_\xi R^+_H)(X,Y) = R^-_H(\tau(X), Y) + R^-_H(X, \tau(Y)).
\]

If \(\tau = 0\), we have automatically \(\nabla_\xi R^+_H = 0\). Now, if \(\tau \neq 0\), the assumption \(\nabla_H \tau = 0\) implies that \(|\tau|^2\) is a strictly positive constant and that \(\tau^2 = \frac{|\tau|^2}{2d} id_H\) (cf. lemma 1 of [9]). This assumption together with (3) implies that \(R^-_H(\tau(X), Y) = -R^-_H(X, \tau(Y))\) and then (6) becomes \(\nabla_\xi R^+_H = 0\). Hence \(\nabla R^+ = 0\) and \(\nabla R^W_H = 0\). \(\square\)
In [9], Boeckx and Cho prove that a contact metric manifold \( M \) endowed with its generalized Tanaka-Webster connection \( \nabla \) satisfying the conditions \( \nabla_H J \circ \tau = 0 \) and \( \tau \neq 0 \) is a strictly pseudoconvex CR manifold (i.e. \( J \) is integrable) and a \((k,\mu)\)-space (cf. [8]). Moreover, Cho gives, in [14], a formula for the Riemannian curvature tensor of \( M \) if \( M \) has constant holomorphic pseudo-Hermitian sectional curvature. Now we obtain a formula for the pseudo-Hermitian curvature tensor of a strictly pseudoconvex CR manifold satisfying \( \nabla_H \tau = 0 \).

**Theorem 3.3** Let \( (M,\theta,\xi,J,g_\theta) \) be a strictly pseudoconvex CR manifold endowed with its Tanaka-Webster connection \( \nabla \). Assume that \( \nabla_H \tau = 0 \) and \( \tau \neq 0 \), then the pseudo-Hermitian curvature tensor and the Chern-Moser tensor of \( M \) are given by:

\[
R^W_H = \frac{s^W}{d^2} \left( I^\tau_H + \frac{2d}{|\tau|^2} T_H \right), \quad C^M_H = \frac{s^W}{d^2} \left( \frac{1}{d+1} I_{H_0}^\tau + \frac{2d}{|\tau|^2} T_{H_0} \right),
\]

with \( T_H = \frac{1}{8} \left( A_\theta \circ A_\theta + B_\theta \circ B_\theta \right) \). Moreover, if \( d \geq 2 \), then \( M \) is a contact locally sub-symmetric space.

The proof of the theorem needs the following Lemma.

**Lemma 3.1** Let \( (M,\theta,\xi,J,g_\theta) \) be a strictly pseudoconvex CR manifold such that \( \nabla_H \tau = 0 \) and \( \tau \neq 0 \). Then \( \rho^W_H = -\frac{s^W}{2d} \omega_\theta \) (i.e. \( M \) is pseudo-Einstein), and \( \nabla_\xi \tau = -\frac{s^W}{d^2} J \circ \tau \).

Proof. First recall that the assumptions \( \nabla_H \tau = 0 \) and \( \tau \neq 0 \) imply that \( |\tau|^2 \) is a strictly positive constant and that \( \tau^2 = \frac{|\tau|^2}{2d} \text{id}_H \). Now, we have for any \( X,Y \in H \),

\[
R_H(X,Y)\tau = B_H(X,Y)\tau - \omega_\theta(X,Y)\nabla_\xi \tau,
\]

with \( B_H(X,Y) = \nabla_Y \nabla_X - \nabla_{\nabla_Y X} - (\nabla_X \nabla_Y - \nabla_{\nabla_Y X}) \). Since \( \nabla_H \tau = 0 \), we have \( R_H(X,Y)\tau = -\omega_\theta(X,Y)\nabla_\xi \tau \) and also \( R^+_{H}(X,Y)\tau = -\omega_\theta(X,Y)\nabla_\xi \tau \). We obtain

\[
g_\theta(R^W_{H}(X,Y)\tau(Z),J\tau(W)) - g_\theta(\tau(R^W_{H}(X,Y)Z),J\tau(W)) = -\omega_\theta(X,Y)g_\theta((\nabla_\xi \tau)(Z),J\tau(W)).
\]

Hence,

\[
R^W_{H}(X,Y,\tau(Z),J\tau(W)) + \frac{|\tau|^2}{2d} R^W_{H}(X,Y,Z,JW) = -\omega_\theta(X,Y)g_\theta((\nabla_\xi \tau)(Z),J\tau(W)).
\]

Let \( \{e_i\} \) be a local \( g_\theta \)-orthonormal frame of \( H \), then

\[
\sum_{1 \leq i \leq 2d} R^W_{H}(X,Y,\tau(e_i),J\tau(e_i)) + \frac{|\tau|^2}{2d} R^W_{H}(X,Y,e_i,J\tau(e_i)) = -\omega_\theta(X,Y) \sum_{1 \leq i \leq 2d} g_\theta((\nabla_\xi \tau)(e_i),J\tau(e_i)).
\]
Since $\tau^2 = \frac{|\tau|^2}{2d}id_H$, then (3) becomes

$$\sum_{1 \leq i \leq 2d} R^W_H(\epsilon_i, J\epsilon_i, X, Y) = -\omega_\theta(X, Y) \sum_{1 \leq i \leq 2d} g_\theta((\nabla_\xi \tau)(\epsilon_i), J\tau(\epsilon_i))$$

which is

$$\rho^W_H = -\wedge_H R^W_H = \omega_\theta \frac{d}{|\tau|^2}(\nabla_\xi A_\theta, B_\theta).$$

Since $\wedge_H \rho^W_H = -\frac{s^W}{2}$, we deduce that $\rho^W_H = -\frac{s^W}{2d}\omega_\theta$ and then $M$ is pseudo-Einstein. Now we have

$$(\wedge_H R^+_H)(\tau(X)) - \tau((\wedge_H R^+_H)(X)) = -d(\nabla_\xi \tau)(X).$$

The assumption $\rho^W_H = -\frac{s^W}{2d}\omega_\theta$ gives

$$(\wedge_H R^+_H)(\tau(X)) - \tau((\wedge_H R^+_H)(X)) = \frac{s^W}{d} J \circ \tau(X)$$

and then $\nabla_\xi \tau = -\frac{s^W}{d^2} J \circ \tau$. □

Proof of Theorem 3.3 Using (2), we obtain that for any $X, Y, Z \in H$

$$g_\theta(R(X, \xi)Y, Z) = g_\theta((\nabla_Z \tau)(X), Y) - g_\theta((\nabla_Y \tau)(X), Z).$$

By the assumption $\nabla_\xi \tau = 0$, we have $i(\xi)R = 0$. Now, equation (5) together with $i(\xi)R = 0, J \circ \tau = -\tau \circ J$ and $\tau^2 = \frac{|\tau|^2}{2d}id_H$ yields

$$R^+_H(\tau(X), \tau(Y)) + \frac{|\tau|^2}{2d} R^+_H(X, Y) = (\nabla_\xi R^+_H)(X, \tau(Y)).$$

(9)

By (3), we have

$$(\nabla_\xi R^+_H)(X, Y) = -\frac{1}{2}\left( ((\nabla_\xi \tau)(X))^* \wedge (JY)^* - ((\nabla_\xi \tau)(Y))^* \wedge (JX)^* 
- ((J(\nabla_\xi \tau)(X))^* \wedge Y^* + ((J(\nabla_\xi \tau)(Y))^* \wedge X^*).$$

We have $\nabla_\xi \tau = -\frac{s^W}{d^2} J \circ \tau$ (Lemma 3.1), it follows that

$$(\nabla_\xi R^+_H)(X, \tau(Y)) = \frac{s^W}{4d^3}|\tau|^2 \left( X^* \wedge Y^* + (JX)^* \wedge (JY)^* \right) + \frac{s^W}{2d^2} \left( (\tau(X))^* \wedge (\tau(Y))^* + (J \tau(X))^* \wedge (J \tau(Y))^* \right).$$
Then (9) becomes

\[ R^W_H(\tau(X), \tau(Y), Z, W) + \frac{|\tau|^2}{2d} R^W_H(X, Y, Z, W) = \frac{s^W}{8d^3} |\tau|^2 \left( g_{\theta H} \otimes g_{\theta H} + \omega_{\theta} \otimes \omega_{\theta} \right)(X, Y, Z, W) + \frac{s^W}{4d^2} \left( A_{\theta} \otimes A_{\theta} + B_{\theta} \otimes B_{\theta} \right)(X, Y, Z, W) \]  

(10)

Now, we have

\[ g_\theta(R^+_{H}(X, Y)\tau(Z), \tau(W)) - g_\theta(\tau(R^+_{H}(X, Y)Z), \tau(W)) = -\omega_\theta(X, Y)g_\theta((\nabla_\xi \tau)(Z), \tau(W)) \]

\[ = -\frac{s^W}{2d^3} |\tau|^2 \omega_\theta(X, Y)\omega_\theta(Z, W). \]

Hence

\[ R^W_H(\tau(X), \tau(Y), Z, W) - \frac{|\tau|^2}{2d} R^W_H(X, Y, Z, W) = -\frac{s^W}{4d^3} |\tau|^2 (\omega_{\theta} \otimes \omega_{\theta})(X, Y, Z, W). \]

(11)

We deduce from (10) and (11) the following expression for the pseudo-Hermitian curvature

\[ R^W_H(X, Y, Z, W) = \frac{s^W}{8d^2} \left( g_{\theta H} \otimes g_{\theta H} + \omega_{\theta} \otimes \omega_{\theta} + 2\omega_{\theta} \otimes \omega_{\theta} \right)(X, Y, Z, W) + \frac{s^W}{4d^2} |\tau|^2 \left( A_{\theta} \otimes A_{\theta} + B_{\theta} \otimes B_{\theta} \right)(X, Y, Z, W). \]

The expression for \( C^M_H \) directly follows from the decomposition of \( R^W_H \). Now we assume \( d > 1 \). Since \( \nabla_H \tau = 0 \), we have the formula \( \delta_H Ric^W_H = -\frac{1}{2} d_H s^W \). Also \( M \) pseudo-Einstein and \( d > 1 \) yields to \( s^W \) constant. Since \( \nabla_H g_\theta = \nabla_H \omega_\theta = \nabla_H A_\theta = \nabla_H B_\theta = 0 \) and \( s^W \) is constant, then we have by the previous formula for \( R^W_H \) and (11) that \( \nabla_H R_H = 0 \). Consequently \( M \) is a contact locally sub-symmetric space. \( \Box \)

**Corollary 3.1** The pseudo-Hermitian curvature tensor \( R^W_H \) of a contact locally sub-symmetric space \( M \) has the following form. If \( M \) is holonomy irreducible and torsionless, in this case \( M \) is the total space of a \( S^1 \)-fibration \( \pi \) over an irreducible Hermitian locally symmetric space \( B \) and \( R^W_H \) is given by \( R^W_H = \pi^* R^B \) where \( R^B \) is the curvature of \( B \). If \( M \) has torsion, in this case \( R^W_H \) is given by formula (7). Note that, in each case, \( M \) is pseudo-Einstein with \( s^W \) constant.

**Remark 3.2** Note that the contact sub-symmetric spaces of non-compact Hermitian type have nonpositive pseudo-Hermitian sectional curvature whereas the contact sub-symmetric spaces of compact Hermitian type have nonnegative pseudo-Hermitian sectional curvature.
4 Mok-Siu-Yeung type formulas for horizontal maps between strictly pseudoconvex CR manifolds

Let \((M, \theta, \xi, J, g, \nabla)\) be a contact metric manifold endowed with the (generalized) Tanaka-Webster connection and let \((E, g^E, \nabla^E)\) be a Riemannian vector bundle over \(M\). For any \(Q_H \in \wedge^2_H(M) \otimes \text{End}(E)\) and \(\sigma_H \in \wedge^1_H(M) \otimes E\), we define \(Q_H(\sigma_H) \in \wedge^1_H(M) \otimes E\) by:

\[
Q_H(\sigma_H)(X) = \sum_i Q_H(\epsilon_i, X)\sigma_H(\epsilon_i),
\]

where \(\{\epsilon_i\}\) is a local \(g_\theta\)-orthonormal frame of \(H\). For any \(Q_H \in S^2(\wedge^2_H(M))\) and \(s_H \in S^2_H(M) \otimes E\) (respectively \(\sigma_H \in \wedge^3_H(M) \otimes E\)), we define \(Q_H s_H \in S^2_H(M) \otimes E\) (respectively \(Q_H \sigma_H \in \wedge^3_H(M) \otimes E\)) by:

\[
(Q_H s_H)(X, Y) = \sum_{i,j} Q_H(\epsilon_i, X, Y, \epsilon_j) s_H(\epsilon_i, \epsilon_j),
\]
\[
(\widehat{Q_H} \sigma_H)(X, Y) = \frac{1}{2} \sum_{i,j} Q_H(\epsilon_i, \epsilon_j, X, Y) \sigma_H(\epsilon_i, \epsilon_j).
\]

**Proposition 4.1** Let \((M, \theta, \xi, J, g_\theta, \nabla)\) be a compact contact metric manifold and let \(Q_H \in \Gamma(S^2(\wedge^2_H(M)))\) satisfying as horizontal \(\wedge^2_H(M)\)-valued 2-form the assumptions \(\delta^H_H Q_H = 0\) and \(\wedge_H Q_H = 0\). Then, for any \(\sigma_H \in \Omega^1_H(M; E)\), we have:

\[
\int_M \left\langle Q_H \nabla^S \sigma_H, \nabla^S \sigma_H \right\rangle + \left\langle (b(Q_H) - Q_H)d_H^E \sigma_H, d_H^E \sigma_H \right\rangle v_{g_\theta} = 2 \int_M \left\langle (\widehat{Q_H} R^E_H)(\sigma_H), \sigma_H \right\rangle - \left\langle (c_H (\widehat{R_H} \circ \widehat{Q_H}))^S, \sigma_H^* g^E \right\rangle v_{g_\theta},
\]

where, for any horizontal 2-tensor \(\mu_H, \mu_H^S(X, Y) = \mu_H(X, Y) + \mu_H(Y, X)\).

Proof. Let \(Q_H \in \Gamma(S^2(\wedge^2_H(M)))\) satisfying \(\delta^H_H Q_H = 0\) and \(\wedge_H Q_H = 0\), then formula (8) of [26] gives for any \(\sigma_H \in \Omega^1_H(M; E)\)

\[
(\delta_H Q_H(\nabla))\sigma_H = \mathcal{R}^Q_H \sigma_H,
\]

where \(Q_H(\nabla)\) and \(\mathcal{R}^Q_H\) are given in a local orthonormal frame \(\{\epsilon_i\}\) of \(H\) by:

\[
Q_H(\nabla)(X) = \sum_i \widehat{Q_H}(\epsilon_i \wedge X) \nabla_{\epsilon_i} = \frac{1}{2} \sum_{i,k,l} Q_H(\epsilon_i, X, \epsilon_k, \epsilon_l) \epsilon_k \epsilon_l \nabla_{\epsilon_i},
\]

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and
\[
\mathcal{R}_H^Q = -\frac{1}{2} \sum_{i,j} \widehat{Q}_H(\epsilon_i \wedge \epsilon_j).R_H(\epsilon_i, \epsilon_j) = -\frac{1}{4} \sum_{i,j,k,l} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) \epsilon_k.\epsilon_l.R_H(\epsilon_i, \epsilon_j).
\]

By integrating, we obtain
\[
\int_M \langle Q_H(\nabla)\sigma_H, \nabla \sigma_H \rangle v_{g_0} = \int_M \langle \mathcal{R}_H^Q, \sigma_H \rangle v_{g_0}.
\]

We have
\[
\langle Q_H(\nabla)\sigma_H, \nabla \sigma_H \rangle = -\frac{1}{2} \sum_{i,j,k,l} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) \langle \nabla_{\epsilon_i} \sigma_H(\epsilon_i), \nabla_{\epsilon_j} \sigma_H(\epsilon_k) \rangle
\]

Now, for any \( X, Y \in TM \) and any \( \sigma, \gamma \in \Omega^1(M; E) \), we have (cf. [25]):
\[
\langle X.\sigma, Y.\gamma \rangle = g_\theta(X, Y) \langle \sigma, \gamma \rangle + \langle i(X)\sigma, i(Y)\gamma \rangle - \langle i(Y)\sigma, i(X)\gamma \rangle. \tag{13}
\]

We deduce from (13) that
\[
\langle Q_H(\nabla)\sigma_H, \nabla \sigma_H \rangle = -\sum_{i,j,k,l} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) \langle \nabla_{\epsilon_i} \sigma_H(\epsilon_i), \nabla_{\epsilon_j} \sigma_H(\epsilon_k) \rangle \tag{14}
\]

Now we have for any \( \sigma_H \in \Omega^1_H(M; E) \) and any \( X, Y \in H \)
\[
\langle \nabla_{X} \sigma_H \rangle(Y) = \frac{1}{2} \left( \langle \nabla^S \sigma_H(X, Y) \rangle + \langle d_{H}^{\nabla^E} \sigma_H(X, Y) \rangle \right),
\]

with \( \langle \nabla^S \sigma_H(X, Y) \rangle = \langle \nabla_X \sigma_H \rangle(Y) + \langle \nabla_Y \sigma_H \rangle(X) \) and \( \langle d_{H}^{\nabla^E} \sigma_H(X, Y) \rangle = \langle \nabla_X \sigma_H \rangle(Y) - \langle \nabla_Y \sigma_H \rangle(Y) \). Then (14) becomes
\[
\langle Q_H(\nabla)\sigma_H, \nabla \sigma_H \rangle = -\frac{1}{4} \left( \sum_{i,j,k,l} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) \langle \langle \nabla^S \sigma_H \rangle(\epsilon_i, \epsilon_l), \langle \nabla^S \sigma_H \rangle(\epsilon_j, \epsilon_k) \rangle \right)
\]
\[
+ \frac{1}{2} \left( \sum_{i,j,k,l} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) + Q_H(\epsilon_k, \epsilon_l, \epsilon_j, \epsilon_i) \langle \langle d_{H}^{\nabla^E} \sigma_H \rangle(\epsilon_i, \epsilon_l), \langle d_{H}^{\nabla^E} \sigma_H \rangle(\epsilon_j, \epsilon_k) \rangle \right)
\]
\[
+ 2 \sum_{i,j,k,l} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) \langle \langle \nabla^S \sigma_H \rangle(\epsilon_i, \epsilon_l), \langle d_{H}^{\nabla^E} \sigma_H \rangle(\epsilon_j, \epsilon_k) \rangle \right)
\]
\[
= -\frac{1}{4} \left( \sum_{i,j,k,l} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) \langle \langle \nabla^S \sigma_H \rangle(\epsilon_i, \epsilon_l), \langle \nabla^S \sigma_H \rangle(\epsilon_j, \epsilon_k) \rangle \right)
\]
\[
+ \frac{1}{2} \sum_{i,j,k,l} \left( b(Q_H)(\epsilon_i, \epsilon_l, \epsilon_j, \epsilon_k) - Q_H(\epsilon_i, \epsilon_l, \epsilon_j, \epsilon_k) \langle \langle d_{H}^{\nabla^E} \sigma_H \rangle(\epsilon_i, \epsilon_l), \langle d_{H}^{\nabla^E} \sigma_H \rangle(\epsilon_j, \epsilon_k) \rangle \right)
\]

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For the second term, we have
\[\sum_{i,j,k,l} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) \langle (\nabla^S \sigma_H)(\epsilon_i, \epsilon_l), (d_H^E \sigma_H)(\epsilon_j, \epsilon_k) \rangle\]
\[= -\frac{1}{4} \sum_{j,k} \langle ((Q_H^o \nabla^S \sigma_H)(\epsilon_j, \epsilon_k), (\nabla^S \sigma_H)(\epsilon_j, \epsilon_k) \rangle\]
\[+ \sum_{j,k} \langle (b(Q_H - Q_H^o)d_H^E \sigma_H)(\epsilon_j, \epsilon_k), (d_H^E \sigma_H)(\epsilon_j, \epsilon_k) \rangle\]
\[+2 \sum_{j,k} \langle ((Q_H \nabla^S \sigma_H)(\epsilon_j, \epsilon_k), (d_H^E \sigma_H)(\epsilon_j, \epsilon_k) \rangle\]
\[= -\frac{1}{2} \left( \langle Q_H^o \nabla^S \sigma_H, \nabla^S \sigma_H \rangle + \langle (b(Q_H - Q_H^o)d_H^E \sigma_H, d_H^E \sigma_H) \rangle \right).\]
For the second term, we have
\[\langle R^Q_H \sigma_H, \sigma_H \rangle = \frac{1}{4} \sum_{i,j,k,l} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) \langle \epsilon_l, R_H(\epsilon_i, \epsilon_j) \sigma_H, \epsilon_k, \sigma_H \rangle\]
\[= -\frac{1}{2} \sum_{i,j,k,l} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) \langle (R_H(\epsilon_i, \epsilon_j) \sigma_H)(\epsilon_k), \sigma_H(\epsilon_l) \rangle.\]
Since \(R_H(X, Y)\sigma_H(Z) = R^E_H(X, Y)\sigma_H(Z) - \sigma_H(R_H(X, Y)Z)\), we have
\[\langle R^Q_H \sigma_H, \sigma_H \rangle = -\frac{1}{2} \sum_{i,j,k,l} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) \langle R^E_H(\epsilon_i, \epsilon_j) \sigma_H(\epsilon_k), \sigma_H(\epsilon_l) \rangle\]
\[+ \frac{1}{2} \sum_{i,j,k,l,m} Q_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) R_H(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_m) \langle \sigma_H(\epsilon_l), \sigma_H(\epsilon_m) \rangle\]
\[= -\sum_{k,l} \langle (Q_H R^E_H)(\epsilon_k, \epsilon_l) \sigma_H(\epsilon_k), \sigma_H(\epsilon_l) \rangle\]
\[+ \frac{1}{2} \sum_{i,j,k,l,m} \left( c_H(\tilde{R}_H \circ \bar{Q}_H)(\epsilon_i, \epsilon_m) + c_H(\tilde{R}_H \circ \bar{Q}_H)(\epsilon_m, \epsilon_i) \right) \langle \sigma^*_H g^E(\epsilon_i, \epsilon_m) \rangle\]
\[= -(\langle Q_H R^E_H \sigma_H, \sigma_H \rangle + \langle (c_H(\tilde{R}_H \circ \bar{Q}_H))^S, \sigma^*_H g^E \rangle).\]
Hence the formula. □

Assume that \((M, \theta, \xi, J, g_\theta, \nabla)\) and \((N, \theta^', \xi^', J^', g_\theta^', \nabla')\) are contact metric manifolds endowed with their Tanaka-Webster connections and let \(\phi : M \rightarrow N\) be a differential map. Let \(\phi^*TN\) be the pull-back bundle of \(TN\) endowed with the metric and the connection induced by those of \(TN\). The covariant derivative of the \(\phi^*TN\)-valued 1-form \(d\phi_H\) is given by:
\[(\nabla_X d\phi_H)(Y) = \nabla'^*_{\phi^*TN} \phi_H(Y) - d\phi_H(\nabla_X Y),\]
where \(\nabla'^*_{\phi^*TN}\) denotes the connection induced by \(\nabla'\) on \(\phi^*TN\).
**Definition 4.1** Let $H = \text{Ker} \theta$ and $H' = \text{Ker} \theta'$. A map $\phi : M \rightarrow N$ such that $d \phi(H) \subset H'$ is called a horizontal map from $M$ to $N$. We denote by $\mathcal{H}(M, N)$ the subspace of horizontal maps from $M$ to $N$.

Note that a horizontal map $\phi$ satisfies $\phi^* \theta' = f \theta$ with $f \in C^\infty(M, \mathbb{R})$.

**Lemma 4.1** For any horizontal map $\phi : M \rightarrow N$, we have:

\[ d\nabla' H \phi_H = -\omega' \otimes (d \phi(\xi))_{H'}, \quad (15) \]

and

\[ \mathcal{L}_\xi^\nabla' \phi_H = \nabla'_{H} (d \phi(\xi))_{H'} - f \tau' \circ d \phi_H. \quad (16) \]

Proof. For any map $\phi : M \rightarrow N$, we have $d\nabla' d \phi = -\phi^* T'$ where $T'$ is the torsion of $\nabla'$. Hence

\[ d\nabla' d \phi_H = d\nabla' (d \phi - \theta \otimes d \phi(\xi)) = -\phi^* T' - \omega' \otimes d \phi(\xi) + \theta \wedge \nabla'^{\phi^* TN} \phi(\xi) \]

\[ = -(\phi^* T')_H - \omega' \otimes d \phi(\xi) + \theta \wedge (\nabla'^{\phi^* TN} \phi(\xi) - i(\xi) \phi^* T'). \]

We deduce that

\[ d\nabla' d \phi_H = (d\nabla' d \phi_H)_H = -(\phi^* T')_H - \omega' \otimes d \phi(\xi) \]

and

\[ \mathcal{L}_\xi^\nabla' d \phi_H = i(\xi)(d\nabla' d \phi_H) = \nabla'_{H} (d \phi(\xi)) - i(\xi) \phi^* T'. \]

Now, let $\phi : M \rightarrow N$ be a horizontal map, then we have $\phi^* \theta' = f \theta$ and $\phi^* \omega'_g = f \omega_g - \theta \wedge df_H$. Since $T' = -\omega'_g \otimes \xi' + \theta \wedge \tau'$, we deduce that $\phi^* T' = -f \omega_g \otimes \xi' + \theta \wedge (df_H \otimes \xi' + f \tau \circ d \phi_H)$ and consequently

\[ d\nabla' d \phi_H = \omega' \otimes (f \xi' - d \phi(\xi)) = -\omega' \otimes (d \phi(\xi))_{H'}. \]

Now,

\[ \mathcal{L}_\xi^\nabla' d \phi_H = \nabla'_{H} (d \phi(\xi) - df_H \otimes \xi' - f \tau' \circ d \phi_H) \]

\[ = \nabla'_{H} (d \phi(\xi) - f \xi' - f \tau' \circ d \phi_H = \nabla'_{H} (d \phi(\xi))_{H'} - f \tau' \circ d \phi_H. \quad \square \]

In the following, for any horizontal symmetric 2-tensor $\mu_H$, we denote by $\mu_{H_0}$ its traceless part.
Proposition 4.2 (Mok-Siu-Yeung type formulas for horizontal maps between strictly pseudo-convex CR manifolds) Let \((M, \theta, \xi, J, g_\theta, \nabla)\) and \((N, \theta', \xi', J', g_{\theta'}, \nabla')\) be strictly pseudo-convex CR manifolds with the assumption \(M\) compact. For any \(Q^{+}_{H_0} \in \Gamma(S^2(\wedge^2_{H_0}^{+}(M)))\) (resp. \(Q^-_H \in \Gamma(S^2(\wedge^2_{H}^{-}(M)))\)) satisfying \(\delta^V_H Q^{+}_{H_0} = 0\) and \((c_H(Q^{+}_{H_0}))_0 = 0\) (resp. \(\delta^V_H Q^-_H = 0\) and \((c_H(Q^-_H))_0 = 0\)) and any horizontal map \(\phi\) from \(M\) to \(N\), we have:

\[
\int_M \langle Q^{+}_{H_0} (\nabla^S d\phi_H)_0, (\nabla^S d\phi_H)_0 \rangle - \frac{tr_H Q^{+}_{H_0}}{d^2} \left( |\delta^V_H d\phi_H|^2 + d^2 |(d\phi(\xi))_{H'}|^2 \right) v_{g_\theta} = 4 \int_M 2(\langle Q^{+}_{H_0}, (\phi^* R^W_{H'})_H \rangle) - \frac{1}{2} ((c_H(R^W_H \circ Q^{+}_{H_0}))^S, (\phi^* g_{\theta'})_H) - \langle Q^{+}_{H_0}, (\phi^* B_{\theta'} H - B_{\theta'})_0, (\phi^* g_{\theta'})_H \rangle v_{g_\theta}. \tag{17}
\]

\[
\int_M \langle Q^-_H (\nabla^S d\phi_H)_0, (\nabla^S d\phi_H)_0 \rangle - \frac{tr_H Q^-_H}{d^2} \left( |\delta^V_H d\phi_H|^2 - d^2 |(d\phi(\xi))_{H'}|^2 \right) v_{g_\theta} = 4 \int_M 2(\langle Q^-_H, (\phi^* R^W_{H'})_H \rangle) - \langle Q^-_H, ((\phi^* B_{\theta'})_H - B_{\theta}) + \frac{tr_H Q^-_H}{d} B_{\theta}, (\phi^* g_{\theta'})_H \rangle v_{g_\theta}. \tag{18}
\]

Proof. Let \(\phi : M \to N\) be a horizontal map. For any \(Q_H \in \Gamma(S^2(\wedge^2_{H}^{+}(M)))\), we obtain, using the relation \(\nabla^S d\phi_H = (\nabla^S d\phi_H)_0 - \frac{1}{d} g_{\theta_H} \otimes \delta^V_H d\phi_H\), that

\[
\langle Q_H, \nabla^S d\phi_H, \nabla^S d\phi_H \rangle = \langle Q_H (\nabla^S d\phi_H)_0, (\nabla^S d\phi_H)_0 \rangle + \frac{2}{d} (c_H(Q_H) \otimes \delta^V_H d\phi_H, (\nabla^S d\phi_H)_0)
- \frac{tr_H Q_H}{d^2} |\delta^V_H d\phi_H|^2.
\]

The assumption \(\langle c_H(Q_H)_0 \otimes \delta^V_H d\phi_H, (\nabla^S d\phi_H)_0 \rangle = 0\). Hence we have

\[
\langle Q_H, \nabla^S d\phi_H, \nabla^S d\phi_H \rangle = \langle Q_H (\nabla^S d\phi_H)_0, (\nabla^S d\phi_H)_0 \rangle - \frac{tr_H Q_H}{d^2} |\delta^V_H d\phi_H|^2. \tag{19}
\]

Since \(\phi : M \to N\) is horizontal then \([15]\) yields \(d_H^V d\phi_H = -\omega_\theta \otimes (d\phi(\xi))_{H'}\). Now we have

\[
\langle ((b(Q_H^-) - Q^-_H)d^V_H d\phi_H, d^V_H d\phi_H) \rangle = \langle (b(Q_H^-) - Q^-_H)\omega_\theta, \omega_\theta \rangle |(d\phi(\xi))_{H'}|^2
\]

\[
= \langle b(Q_H^-) - Q^-_H, \omega_\theta \otimes \omega_\theta \rangle |(d\phi(\xi))_{H'}|^2
\]

\[
= (tr_H Q_H^+ - tr_H Q_H^-) |(d\phi(\xi))_{H'}|^2.
\]

For \(Q^{+}_{H_0} \in \Gamma(S^2(\wedge^2_{H_0}^{+}(M)))\) and \(Q^-_H \in \Gamma(S^2(\wedge^2_{H}^{-}(M)))\), we have

\[
\langle ((b(Q^{+}_{H_0}) - Q^{+}_{H_0})d^V_H d\phi_H, d^V_H d\phi_H) \rangle = -tr_H Q^{+}_{H_0}. \tag{20}
\]
and
\[ \langle (b\widetilde{Q_H} - Q_H)d\phi_H, d\phi_H \rangle = tr HQ_H. \] (21)

Now for \( Q_H \in \Gamma(S^2(\Lambda^2_H(M))) \), we have
\[ \langle (\widetilde{Q_H}(Q^{\phi^*}_{H^*})T)(d\phi_H), d\phi_H \rangle = 2\langle Q_H, (\phi^* R^\phi_{H^*})_H \rangle. \]

Since \( R^\gamma_{H^*} = R^\gamma_{H^*} - \frac{1}{2}(\omega_{\gamma^*} \otimes A_{\gamma^*} - g_{\gamma^*} \otimes B_{\gamma^*}) \), we have
\[ \langle (\phi^* R^\phi_{H^*})_H \rangle_H = \langle (\phi^* R^\phi_{H^*})_H \rangle_H + \frac{1}{2}\langle \phi^* g_{\gamma^*} \rangle_H \otimes (\phi^* B_{\gamma^*})_H - \frac{1}{2}\langle \phi^* \omega_{\gamma^*} \rangle_H \otimes (\phi^* A_{\gamma^*})_H. \]

Since \( (\phi^* \omega_{\gamma^*})_H \otimes (\phi^* A_{\gamma^*})_H \in \Gamma(\Lambda^2(\Lambda^2_H(M))) \), we have
\[ \langle (\widetilde{Q_H}(Q^{\phi^*}_{H^*})T)(d\phi_H), d\phi_H \rangle = 4\langle Q_H, (\phi^* R^\phi_{H^*})_H \rangle + 2\langle Q_H, (\phi^* g_{\gamma^*})_H \otimes (\phi^* B_{\gamma^*})_H \rangle
\]
\[ = 4\langle Q_H, (\phi^* R^\phi_{H^*})_H \rangle - 2\langle Q_H, (\phi^* B_{\gamma^*})_H, (\phi^* g_{\gamma^*})_H \rangle. \] (22)

The relations \( g_{\theta_H} \otimes B_{\theta} \circ \omega_{\theta} \otimes \omega_{\theta} = 2\omega_{\theta} \otimes A_{\theta} \) and \( g_{\theta_H} \otimes B_{\theta} \circ g_{\theta_H} \otimes g_{\theta_H} = 2g_{\theta_H} \otimes B_{\theta} \) yield to
\[ \widetilde{R_H} = R^W_{H^*} + \frac{1}{2}g_{\theta_H} \otimes B_{\theta} \circ (g_{\theta_H} \otimes g_{\theta_H})^{-}. \]

Since for any \( T^\pm_H, Q^\pm_H \in S^2(\Lambda^2_H(M)) \), \( T^\pm_H \circ \widetilde{Q_H} = 0 \) and \( (g_{\theta_H} \otimes g_{\theta_H})^{-} \circ \widetilde{Q_H} = 2\widetilde{Q_H} \) then
\[ \langle (c_H(\widetilde{R_H} \circ \widetilde{Q_H})(\phi^* g_{\gamma^*})_H \rangle = \langle (c_H(\widetilde{R^W_{H^*}} \circ \widetilde{Q_H}))(\phi^* g_{\gamma^*})_H \rangle, \] (23)

and
\[ \langle (c_H(\widetilde{R_H} \circ \widetilde{Q_H}))^S, (\phi^* g_{\gamma^*})_H \rangle = \langle (c_H(g_{\theta_H} \otimes B_{\theta} \circ \widetilde{Q_H}))(\phi^* g_{\gamma^*})_H \rangle. \]

Now, we have
\[ \langle (c_H(g_{\theta_H} \otimes B_{\theta} \circ \widetilde{Q_H}))(\phi^* g_{\gamma^*})_H \rangle = \langle (c_H(\widetilde{Q_H}))^S \circ (\phi^* g_{\gamma^*})_H \rangle - 2\langle \widetilde{Q_H} B_{\theta}, (\phi^* g_{\gamma^*})_H \rangle, \]

where \( \mu_H^\circ \) is the symmetric endomorphism associated by \( g_{\theta_H} \) to the symmetric 2-tensor \( \mu_H \). Using the assumption \( (c_H(\widetilde{Q_H}))^S = 0 \), we deduce that
\[ \langle (c_H(\widetilde{R_H} \circ \widetilde{Q_H}))^S, (\phi^* g_{\gamma^*})_H \rangle = 2\langle \frac{\text{tr} \widetilde{Q_H}}{d} B_{\theta} - \mu_H^\circ, (\phi^* g_{\gamma^*})_H \rangle. \] (24)

We obtain the formulas by replacing (19), (20), (21), (22), (23) and (24) in (12). \( \square \)

As applications of formulas (17) and (18), we recover, in a different way, the Siu formula given in [25] and we derive a Tanaka-Weitzenbock formula. First we have the following lemma whose proof is left to the reader.
Lemma 4.2 We have for any $s_H \in S^2_H(M) \otimes E$ the relations:

$$g_{\theta_H} \otimes g_{\theta_H} s_H = 2(s_H - g_{\theta_H} \otimes tr_H s_H), \quad \omega_{\theta} \otimes \omega_{\theta} s_H = \omega_{\theta} \otimes \omega_{\theta} s_H = -2(s_H^+ - s_H^0),$$

$$c_H(g_{\theta_H} \otimes g_{\theta_H}) = 2(2d - 1)g_{\theta_H}, \quad c_H(\omega_{\theta} \otimes \omega_{\theta}) = c_H(\omega_{\theta} \otimes \omega_{\theta}) = 2g_{\theta_H},$$

$$tr_H g_{\theta_H} \otimes g_{\theta_H} = 2d(2d - 1), \quad tr_H \omega_{\theta} \otimes \omega_{\theta} = tr_H \omega_{\theta} \otimes \omega_{\theta} = 2d.$$

Proposition 4.3 For any horizontal map $\phi$ from a compact Sasakian manifold $M$ to a Sasakian manifold $N$, we have:

$$\int_M |(\nabla^S d\phi_H)_0|^2 + \frac{1}{2}(1 - \frac{1}{d})|\delta_H^\phi d\phi_H|^2 + d(d - 1)|\langle d\phi(\xi) \rangle_H^2|v_{g_\theta} = 4 \int_M \sum_{i,j \leq d} \langle (\phi^* R_H^\phi)^{C}_{-1} \rangle_{H} (Z_i \wedge Z_j), Z_i \wedge Z_j |v_{g_\theta} = 0. \quad (25)$$

$$\int_M |(\nabla^S d\phi_H)_0|^2 + \frac{1}{2}(1 - \frac{1}{d})|\delta_H^\phi d\phi_H|^2 + d(d - 1)|\langle d\phi(\xi) \rangle_H^2|v_{g_\theta} = 4 \int_M \sum_{i,j \leq d} \langle (\phi^* R_H^\phi)^{C}_{-1} \rangle_{H} (Z_i \wedge Z_j), Z_i \wedge Z_j |v_{g_\theta} = 0. \quad (26)$$

where, in an adapted frame $\{\epsilon_1, \ldots, \epsilon_d, J\epsilon_1, \ldots, J\epsilon_d\}$ of $H$, \n
$$tr^2_{H}((\phi^* R_H^\phi)^{C}_{-1}) = \sum_{i,j \leq d} ((\phi^* R_H^\phi)^{C}_{-1})_{H} (Z_i \wedge Z_j), Z_i \wedge Z_j |v_{g_\theta} = 0.$$

and \n
$$tr^1_{H}((\phi^* R_H^\phi)^{C}_{-1}) = \sum_{i,j \leq d} ((\phi^* R_H^\phi)^{C}_{-1})_{H} (Z_i \wedge Z_j), Z_i \wedge Z_j |v_{g_\theta} = 0.$$

with \n
$$Z_i = \frac{1}{\sqrt{d}}(\epsilon_i - \sqrt{1}J\epsilon_i).$$

Proof. Let $Q_H^+ \in \Gamma(S^2(\wedge^2 H^{-1}(M)))$ and $Q_{H_0}^+ \in \Gamma(S^2(\wedge^2 H_{0}^{-1}(M)))$ defined by:

$$Q_H^- = (g_{\theta_H} \otimes g_{\theta_H})^- = \frac{1}{2}(g_{\theta_H} \otimes g_{\theta_H} - \omega_{\theta} \otimes \omega_{\theta})$$

and \n
$$Q_{H_0}^+ = (g_{\theta_H} \otimes g_{\theta_H})^+ = \frac{1}{2}(g_{\theta_H} \otimes g_{\theta_H} + \omega_{\theta} \otimes \omega_{\theta} - \frac{2}{d} \omega_{\theta} \otimes \omega_{\theta}).$$

Then we have $\nabla_H Q_H^+ = \nabla_H Q_H^- = 0$ (since $\nabla g_{\theta_H} = \nabla \omega_{\theta} = 0$). Moreover, using Lemma 4.2, we have $c_H(Q_H^-) = 2(2d - 1)g_{\theta_H}$ and $c_H(Q_{H_0}^+) = 2d \left(1 - \frac{1}{d^2}\right) g_{\theta_H}$. Also $c_H(Q_H^-) = (c_H(Q_{H_0}^+))^0 = 0$. Moreover, we have $(c_H(R_{H_0}^\phi \circ Q_{H_0}^+))^S = 4(1 - \frac{1}{d}) Ric_{H_0}. \quad (26)$

Let $\phi$ be a horizontal map from $M$ to $N$, by Lemma 4.2, we have \n
$$\langle Q_{H_0}^+ (\nabla^S d\phi_H)_0, (\nabla^S d\phi_H)_0 \rangle^2 = 2\langle (\nabla^S d\phi_H)_0^+ \rangle^2$$

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In an adapted frame \( \{\epsilon_1, \ldots, \epsilon_d, J\epsilon_1, \ldots, J\epsilon_d\} \) of \( H \), we have

\[
tr_H(\phi^*R^W_{H'})^\pm = \sum_{i,j \leq d} ((\phi^*R^W_{H'})^\pm_H(\epsilon_i, \epsilon_j, \epsilon_i, \epsilon_j) + (\phi^*R^W_{H'})^\pm_H(\epsilon_i, J\epsilon_j, \epsilon_i, J\epsilon_j)).
\]

Now we have, for any \( T_H \in S^2(\wedge^2_H(M)) \cap \text{Ker} \, b \), the relations

\[
\widehat{T_H}^c(Z \wedge W, Z \wedge W) = T_H^-(X, Y, X, Y) + T_H^-(X, JY, X, JY)
\]

and

\[
\widehat{T_H}^c(Z \wedge W, Z \wedge W) = T_H^+(X, Y, X, Y) + T_H^+(X, JY, X, JY),
\]

with \( Z = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX) \), \( W = \frac{1}{\sqrt{2}}(Y - \sqrt{-1}JY) \). Since \( (\phi^*R^W_{H'})_H \in S^2(\wedge^2_H(M)) \cap \text{Ker} \, b \), we deduce that

\[
\langle Q_H^-, (\phi^*R^W_{H'})_H \rangle = \sum_{i,j \leq d} ((\phi^*R^W_{H'})^c_H(Z_i \wedge Z_j, \overline{Z_i} \wedge \overline{Z_j})
\]

and

\[
\langle Q_{H_0}^+, (\phi^*R^W_{H'})_H \rangle = \sum_{i,j \leq d} \left( \frac{1}{d}((\phi^*R^W_{H'})^c_H(Z_i \wedge Z_j, \overline{Z_i} \wedge \overline{Z_j}) \right.
\]

\[
+ \left. (1 - \frac{1}{d})((\phi^*R^W_{H'})^c_H(Z_i \wedge \overline{Z_j}, \overline{Z_i} \wedge Z_j)) \right),
\]

with \( Z_i = \frac{1}{\sqrt{2}}(\epsilon_i - \sqrt{-1}J\epsilon_i) \). By replacing in (17) and (18) together with the assumptions \( M, N \) Sasakian yields the formulas. □

5 Horizontal pseudoharmonic maps, CR-pluriharmonic maps and Rumin complex

. Pseudoharmonic maps
In [3] and [5] Barletta, Dragomir and Urakawa have introduced the notion of pseudoharmonic maps from a compact contact metric manifold into a Riemannian manifold. Now we extend this notion to horizontal maps between contact metric manifolds.

Assume that \((M, \theta, \xi, J, g_\theta, \nabla)\) and \((N, \theta', \xi', J', g_{\theta'}, \nabla')\) are contact metric manifolds endowed with their Tanaka-Webster connections and that \(M\) is compact. For any differential map \(\phi : M \to N\), we define \(d\phi_{H,H'}(X) = (d\phi_H(X))_{H'}\) with \(X \in H\) and the horizontal energy \(E_{H,H'}(\phi)\) by:

\[
E_{H,H'}(\phi) = \frac{1}{2} \int_M |d\phi_{H,H'}|^2 v_{g_\theta}.
\]

**Proposition 5.1** For any variation \(\phi_t\) of \(\phi\), we have:

\[
\frac{d}{dt} E_{H,H'}(\phi_t)_{t=0} = \int_M g_{\theta'}(\delta_H d\phi_{H,H'} + i(\phi^*\theta') \tau' \circ d\phi_{H,H'} - \xi' tr_H(\phi^* A_{\theta'} H), v) v_{g_{\theta'}},
\]

with \(v = \frac{\partial \phi}{\partial t} |_{t=0}\).

Proof. Let \(\{\phi_t\}_{|t|<\epsilon}\) be a variation of \(\phi\). We consider the map \(\Phi : [-\epsilon, \epsilon] \to M \to N\) given by \(\Phi(t, x) = \phi_t(x)\) and the pull-back bundle \(\Phi^* TN \to [-\epsilon, \epsilon] \times M\) of \(TN\) by \(\Phi\). Let \(\{\epsilon_i\}\) be a local \(g_{\theta'}\)-orthonormal frame of \(H\), then we have

\[
\frac{1}{2} \frac{d}{dt} |d\phi_{H,H'}|^2 = \frac{1}{2} \frac{\partial}{\partial t} \sum_i g_{\theta'}((d\Phi(\epsilon_i))_{H'}, (d\Phi(\epsilon_i))_{H'}) = \sum_i g_{\theta'}(\nabla^{\Phi*TN}_{\frac{\partial}{\partial t}} (d\Phi(\epsilon_i))_{H'}, (d\Phi(\epsilon_i))_{H'}).\]

We have

\[
(d\nabla^\Phi d\Phi)\frac{\partial}{\partial t}, \epsilon_i = \nabla^{\Phi*TN}_{\frac{\partial}{\partial t}} d\Phi(\epsilon_i) - \nabla^{\Phi*TN}_{\epsilon_i} d\Phi(\frac{\partial}{\partial t}) - d\Phi([\frac{\partial}{\partial t}, \epsilon_i]) = -T'(d\Phi(\frac{\partial}{\partial t}), d\Phi(\epsilon_i)),
\]

where \(T'\) is the torsion of \(\nabla'\). Since \([\frac{\partial}{\partial t}, \epsilon_i] = 0\) and \(\nabla^{\Phi*TN}_{\epsilon_i}\) preserves \(H'\), we obtain

\[
\nabla^{\Phi*TN}_{\frac{\partial}{\partial t}} (d\Phi(\epsilon_i))_{H'} - \nabla^{\Phi*TN}_{\epsilon_i} (d\Phi(\frac{\partial}{\partial t}))_{H'} = -(T'(d\Phi(\frac{\partial}{\partial t}), d\Phi(\epsilon_i))_{H'}).\]

Using \(T' = -\omega_{\theta'} \otimes \xi' + \theta' \wedge \tau'\), we obtain

\[
\frac{1}{2} \frac{d}{dt} |d\phi_{H,H'}|^2 = \sum_i g_{\theta'}(\left(\nabla^{\Phi*TN}_{\epsilon_i} (d\Phi(\frac{\partial}{\partial t}))_{H'} - \theta' (d\Phi(\frac{\partial}{\partial t}))_{H'}\right) + \theta'(d\Phi(\epsilon_i))_{H'} (d\Phi(\frac{\partial}{\partial t})), (d\Phi(\epsilon_i))_{H'})
\]

\[
= \sum_i \left(\epsilon_i g_{\theta'}((d\Phi(\frac{\partial}{\partial t}))_{H'}, (d\Phi(\epsilon_i))_{H'}) - g_{\theta'}((d\Phi(\frac{\partial}{\partial t}))_{H'}, (d\Phi(\nabla_{\epsilon_i})_{H'})\right).
\]
\[-\sum_i g_{\theta'}((d\Phi_i \frac{\partial}{\partial t})_{H'})\nabla_{\epsilon_i}^{\phi_{\theta \prime} TN}(d\Phi(\epsilon_i))_{H'} - (d\Phi(\nabla_{\epsilon_i} \epsilon_i))_{H'}\]
\[-(\Phi^* \theta')(\frac{\partial}{\partial t}) \sum_i A_{\theta'}((d\Phi(\epsilon_i))_{H'},(d\Phi(\epsilon_i))_{H'})\]
\[+ \sum_i (\Phi^* \theta')(\epsilon_i) A_{\theta'}((d\Phi(\epsilon_i))_{H'},(d\Phi(\epsilon_i))_{H'})\]
\[= -\delta_{H} \alpha_{(d\Phi(\frac{\partial}{\partial t}))_{H'},(d\Phi(X))_{H'}}\]
\[\sum_i ((\nabla_{\epsilon_i}^{\phi_{\theta \prime} TN} d\Phi(\epsilon_i))_{H'},v_{H'})\]
\[-\delta_{H} \alpha_{\theta \prime}(v_{H'}) + g_{\theta'}(\sum_i (\Phi^* \theta')(\epsilon_i) A_{\theta'}((d\Phi_{H',H'}(\epsilon_i),v_{H'}))\]
\[= -\delta_{H} \alpha_{\theta \prime}(v_{H'}) + g_{\theta'}(\delta_{H} d\phi_{H',H'} + i(\phi^* \theta') \tau' o d\phi_{H',H'} - \xi tr_{H}(\phi^* A_{\theta'}))_{H'},v).\]

The result follows by integrating. □

**Definition 5.1** A map \( \phi : M \to N \) is called a pseudoharmonic map if it is a critical point of \( E_{H',H'} \).

A map \( \phi : M \to N \) is pseudoharmonic if and only if
\[\delta_{H} d\phi_{H',H'} + i(\phi^* \theta') \tau' o d\phi_{H',H'} = 0 \quad \text{and} \quad tr_{H}(\phi^* A_{\theta'})_{H} = 0.\]

A horizontal map \( \phi : M \to N \) is pseudoharmonic if and only if \( \delta_{H} d\phi_{H} = 0 \) and \( tr_{H}(\phi^* A_{\theta'})_{H} = 0.\)

- CR-pluriharmonic maps

Let \( (M, \theta, \xi, J, g_{\theta}) \) be a strictly pseudoconvex \( CR \) manifold of dimension \( 2d + 1 \). A real function \( h \) on \( M \) is called a CR-pluriharmonic function if \( h \) is the real part of a CR function on \( M \).

We have the following equivalent characterizations for the CR-pluriharmonic functions.

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Theorem 5.1 (Lee [21])
The following assertions are equivalent:
(i) $h$ is CR-pluriharmonic.
(ii) There exists a real function $\lambda$ such that $d(J^*dh_H + \lambda \theta) = 0$.
(iii) $(d_HJ^*dh_H)_0 = 0$ and $(\mathcal{L}_\xi + \frac{1}{d}d_H\delta_{H,J})J^*dh_H = \mathcal{L}_\xi J^*dh_H + \frac{1}{d}d_H\delta_{H,J}dh_H = 0$,

where $(d_HJ^*dh_H)_0$ is the primitive part of $d_HJ^*dh_H$ and $\delta_{H,J} = [\land_H, d_H]$.

Note that, if $d > 1$, then the assumption $(d_HJ^*dh_H)_0 = 0$ implies that $\mathcal{L}_\xi J^*dh_H + \frac{1}{d}d_H\delta_{H,J}dh_H = 0$ and, if $d = 1$, then the assumption $(d_HJ^*dh_H)_0 = 0$ is always satisfied for any $h$.

Definition 5.2 Let $(M, \theta, \xi, J, g_\theta, \nabla)$ and $(N, \theta', \xi', J', g_{\theta'}, \nabla')$ be strictly pseudoconvex CR manifolds endowed with their Tanaka-Webster connections together with $\dim M > 3$. A horizontal map $\phi : M \to N$ such that $(d_H^\theta J^*d\phi_H)_0 = 0$ is called a CR-pluriharmonic map from $M$ to $N$.

Proposition 5.2 (i) A map $\phi : M \to N$ is CR-pluriharmonic if and only if

$$(\nabla^S d\phi)_0^+ := (\nabla^S d\phi)^+ + \frac{1}{d}g_{\theta_H} \otimes \delta_H^\prime d\phi_H = 0.$$ 

(ii) Any CR-pluriharmonic map $\phi : M \to N$ satisfies:

$$\mathcal{L}_\xi J^*d\phi_H + \frac{1}{d}d_H^\prime \delta_H^\prime d\phi_H = \frac{2}{d-1}tr_H (R_H^{\phi_\times TN})(\mathcal{D})d\phi(.). - f\tau' \circ J^*d\phi_H.$$ 

(iii) Any CR map $\phi : M \to N$ is CR-pluriharmonic and we have

$$\delta_H^\prime J^*d\phi_H = J' \delta_H^\prime d\phi_H = d(\phi(\xi))_{H'}.$$ 

Proof. Recall that for $\gamma_H \in \Omega^2_H(M)$, its primitive part $\gamma_{H_0} \in \Omega^1_{H_0}(M)$ is given by $\gamma_{H_0} = \gamma_H - \frac{1}{d}L_H \land_H \gamma_H$. We have

$$(d_H^\theta J^*d\phi_H)_0 = d_H^\prime J^*d\phi_H - \frac{1}{d}L_H \land_H d_H^\prime J^*d\phi_H = d_H^\prime J^*d\phi_H - \frac{1}{d}L_H \delta_{H,J}^\prime J^*d\phi_H.$$ 

Now since $\delta_{H,J}^\prime J^*d\phi_H = \delta_{H,J}^\prime (d\phi_H \circ J) = \delta_H^\prime d\phi_H$, we deduce that

$$(d_H^\prime J^*d\phi_H)_0 = d_H^\prime J^*d\phi_H - \frac{1}{d}\omega_{\theta} \otimes \delta_H^\prime d\phi_H.$$ 

Now, using (15), we have

$$(d_H^\theta J^*d\phi_H)(JX,Y) = (\nabla_{JX}d\phi_H)(JY) + (\nabla_X d\phi_H)(Y) = \frac{1}{2}((\nabla^S d\phi_H)(X,Y) + (\nabla^S d\phi_H)(JX,JY))$$

$$= (\nabla^S d\phi_H)^+(X,Y).$$ 

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Hence by (28) we obtain
\[ (d_H^\nabla J^*d\phi_H)_0(JX,Y) = (\nabla^gd\phi_H)^+(X,Y) + \frac{1}{d}g_{\theta_H}(X,Y)\delta_H^\nabla d\phi_H. \]

The assumption (i) directly follows. Now we have
\[ \delta_H^\nabla J^*d\phi_H - \frac{1}{d}\wedge_H d_H^\nabla J^*d\phi_H - \frac{1}{d}\wedge_H d_H^\nabla J^*d\phi_H. \]

Since \(d_H^\nabla J^*d\phi_H = -L_H(\mathcal{L}^\nabla^E J^*d\phi_H) - R_H^{\phi TN} \wedge J^*d\phi_H \) and \([d_H^\nabla^E, L_H] = 0\), we obtain
\[ \delta_H^\nabla J^*d\phi_H = -\wedge_H (R_H^{\phi TN} \wedge J^*d\phi_H) - \wedge_H L_H(\mathcal{L}^\nabla^E J^*d\phi_H) - \frac{1}{d}\wedge_H L_H d_H^\nabla J^*d\phi_H. \]

Let \(\{\epsilon_i\}\) be a local \(g_\theta\)-orthonormal frame of \(H\), then we have
\[ \wedge_H(R_H^{\phi TN} \wedge J^*d\phi_H) = \frac{1}{2} \sum_i \left( R_H^*(d\phi_H(\epsilon_i), d\phi_H(J\epsilon_i))d\phi_H(JX) + 2R_H^*(d\phi_H(\epsilon_i), d\phi_H(X))d\phi_H(\epsilon_i) \right) \]

Using (11), we obtain that
\[ \sum_i R_H^*(d\phi_H(\epsilon_i), d\phi_H(J\epsilon_i))d\phi_H(JX) = \sum_i R_H^*(d\phi_H(JX), d\phi_H(J\epsilon_i))d\phi_H(\epsilon_i) + 2f(d-1)\tau'((d\phi_H \circ J)(X)). \]

Hence we have
\[ \wedge_H(R_H^{\phi TN} \wedge J^*d\phi_H) = -\sum_i \left( R_H^*(d\phi_H(X), d\phi_H(\epsilon_i))d\phi_H(\epsilon_i) - R_H^*(d\phi_H(JX), d\phi_H(J\epsilon_i))d\phi_H(\epsilon_i) \right) \]
\[ + f(d-1)\tau'((d\phi_H \circ J)(X)) \]
\[ = -2\sum_i (R_H^{\phi TN})^-(X,\epsilon_i)d\phi_H(\epsilon_i) + f(d-1)(\tau' \circ J^*d\phi_H)(X). \]

The assumption \(\phi\) CR-pluriharmonic together with (29) and (30) gives the formula. Hence (ii). Let \(\phi : M \rightarrow N\) be a CR map then \(J^*d\phi_H = d\phi_H \circ J\). Consequently, we have
\[ J^* \circ (d_H^\nabla J^*d\phi_H)_0 = (d_H^\nabla J^*(\phi \circ J))_0 \]
and
\[ J^* \delta_H^\nabla d\phi_H = \delta_H^\nabla (\phi \circ J). \]

By (15) we have \((d_H^\nabla J^*d\phi_H)_0 = 0\) and we obtain that \((d_H^\nabla J^*(\phi \circ J))_0 = 0\). Now we have
\[ \delta_H^\nabla (\phi \circ J) = -\delta_H^\nabla J^*(d\phi_H) = -\wedge_H (d_H^\nabla J^*d\phi_H) = d(d(\phi(\xi)))_H'. \]

The following theorem holds for pseudoharmonic maps between Sasakian manifolds.
Theorem 5.2 Let $M$ and $N$ be Sasakian manifolds. Assume that $M$ is compact and $N$ has nonpositive pseudo-Hermitian complex sectional curvature. Then:

(i) Any horizontal pseudo-harmonic map $\phi$ from $M$ to $N$ is CR-pluriharmonic.

(ii) If the pseudo-Hermitian Ricci tensor of $M$ is nonnegative, then any horizontal pseudoharmonic map $\phi$ from $M$ to $N$ satisfies $\nabla d\phi = 0$ and $|d\phi| = \text{const}$.

(iii) If the pseudo-Hermitian Ricci tensor of $M$ is positive, then any horizontal pseudo-harmonic map $\phi$ from $M$ to $N$ satisfies $d\phi_H = 0$ and consequently $r_{g_x}(\phi) \leq 1$, where $r_{g_x}(\phi)$ is the rank of $\phi$ at a point $x$ of $M$.

Proof. Let $\phi$ be a horizontal pseudo-harmonic map from $M$ to $N$. If $N$ has nonpositive pseudo-Hermitian complex sectional curvature then (26) yields to $(d\phi(\xi))_{H'} = 0$ and $(\nabla^S d\phi_H)_0^+ = 0$. In particular, $\phi$ is CR-pluriharmonic. Moreover, if $Ric^N_H$ is nonnegative, it follows from (26) that $(\nabla^S d\phi_H)^- = 0$. Consequently $\nabla^S d\phi_H = 0$ and $d\phi_H^+ = 0$. We deduce that $\nabla_H d\phi_H = 0$. The assumptions $M$ and $N$ torsionless together with $(d\phi(\xi))_{H'} = 0$, yield by (16) that $\nabla_{\xi'} d\phi_H = 0$. Taking into account that $d\phi(\xi) = f\xi'$, we obtain that $i(\xi)\phi^\ast \omega_{\theta'} = -d\phi_H = 0$ and so $f$ is constant. We immediately deduce that $\nabla d\phi = \nabla d\phi_H + \theta \otimes \nabla d\phi(\xi) = 0$ and so $|d\phi| = \text{const}$. If $Ric^N_H$ is positive, it directly follows from $\langle d\phi_H \circ Ric^N_H, d\phi_H \rangle = 0$ that $d\phi_H = 0$. Since $d\phi(\xi) = f\xi'$, we deduce that the rank of $\phi$ is less than equal 1 at each point of $M$. □

Horizontal maps and twisted Rumin pseudo-complex

Let $M$ be a strictly pseudoconvex $CR$ manifold of dimension $2d + 1$. We recall that the Rumin complex [27] is the complex:

$$0 \to \mathbb{R} \to C^\infty(M) \xrightarrow{d} \mathcal{R}^1(M) \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{R}^d(M) \xrightarrow{D_R} \mathcal{R}^{d+1}(M) \xrightarrow{d^*_R} \ldots \xrightarrow{d^*_R} \mathcal{R}^{2d+1}(M) \to 0,$$

where

$$\mathcal{R}^p(M) = \Omega^p_{H_0}(M) \quad \text{for } p \leq d$$

$$= \mathcal{F}_p^\ast(M) \quad \text{for } p \geq d + 1,$$

and

$$d_R \gamma_H = (d_H \gamma_H)_0 = (d_H - \frac{1}{d-p+1}L_H d_{H,J}^\ast) \gamma_H \quad \text{for } \gamma_H \in \mathcal{R}^p(M), \quad (p \leq d - 1)$$

$$D_R \gamma_H = \theta \wedge (\mathcal{L}_\xi + d_H d_{H,J}) \gamma_H \quad \text{for } \gamma_H \in \mathcal{R}^d(M)$$

$$d_R \gamma_\xi = \theta \wedge i(\xi)(d\gamma_\xi) \quad \text{for } \gamma_\xi \in \mathcal{R}^p(M), \quad (p \geq d + 1).$$

The formal adjoints of $d_R$ and $D_R$ for the usual scalar product are denoted by $\delta_R$ and $D^*_R$. The laplacians associated to this complex are defined by:

$$\Delta_R = (d - p)d_R \delta_R + (d - p + 1)\delta_R d_R \quad \text{on } \mathcal{R}^p(M) \quad (p \leq d - 1)$$

$$\Delta_R = D_R D^*_R + (\delta_R \delta_R)^2 \quad \text{on } \mathcal{R}^d(M)$$

$$\Delta_R = D_R D^*_R + (\delta_R d_R)^2 \quad \text{on } \mathcal{R}^{d+1}(M)$$

$$\Delta_R = (d - p + 1)d_R \delta_R + (d - p)\delta_R d_R \quad \text{on } \mathcal{R}^p(M) \quad (p \geq d + 2).$$
The fundamental fact is that, if $M$ is compact, then (cf. [27]):

$$H^*_dR(M, \mathbb{R}) = H^*_R(M, \mathbb{R}) = \text{Ker} \, \Delta_R,$$

where $H^*_dR(M, \mathbb{R})$ and $H^*_R(M, \mathbb{R})$ are respectively the cohomologies of the De Rham complex and the Rumin complex.

Let $(E, \nabla^E)$ be a vector bundle over $M$ then the previous definitions of $\mathcal{R}^p(M)$, $d_R$ and $D_R$ can be extended to $E$-twisted bundles. Also we define the sequence:

$$0 \rightarrow \mathbb{R} \rightarrow C^\infty(M) \xrightarrow{\delta^E_R} \mathcal{R}^1(M; E) \xrightarrow{\delta^E_R} \ldots \xrightarrow{\delta^E_R} \mathcal{R}^d(M; E) \xrightarrow{D^E_R} \mathcal{R}^{d+1}(M; E) \xrightarrow{\delta^E_R} \ldots \xrightarrow{\delta^E_R} \mathcal{R}^{2d+1}(M; E) \rightarrow 0.$$

Note that $\delta^E_R \sigma_H = - (R^E_H \wedge \sigma_H)_0$ for $\sigma_H \in \mathcal{R}^p(M; E)$ $(p \leq d - 2)$. Also the previous sequence is not a complex excepted if $E$ is flat. Also we call this sequence the twisted Rumin pseudo-complex.

Let $N$ be a strictly pseudoconvex $CR$ manifold, $\phi : M \rightarrow N$ a horizontal map and $E = \phi^*TN$ the pull-back bundle endowed with the connection $\nabla'$ induced by the Tanaka-Wester connection of $TN$. Then $d\phi_H \in \Omega^1_{\phi^*}(M; \phi^*TN)$ satisfies $\delta^E_R d\phi_H = (d\phi_H^E) = 0$. Moreover, if $\phi$ is a pseudoharmonic map then $\delta^E_R d\phi_H = \delta^E_H d\phi_H = 0$. Consequently, if $d > 1$, we have $\Delta^E_R d\phi_H = (d - 1) \delta^E_R \delta^E_R d\phi_H = 0$. The assumption $\phi$ pseudoharmonic yields to $\Delta^E_R d\phi_H = 0$. Note that $\delta^E_R 2\sigma_H = -(R^E_H \wedge \sigma_H)_0$ for $\sigma_H \in \mathcal{R}^p(M; E)$ $(p \leq d - 2)$. Also the previous sequence is not a complex excepted if $E$ is flat. Also we call this sequence the twisted Rumin pseudo-complex.

\textbf{Remark 5.1} The condition $\phi : M \rightarrow N$ CR-pluriharmonic is equivalent to $\delta^E_R J^*d\phi_H = 0$. Moreover, if $\text{dim} \, M = 3$, it seems natural in view of the Theorem 5.1 to define the CR-pluriharmonicity of a horizontal map $\phi : M \rightarrow N$ by the condition $D^E_R J^*d\phi_H = 0$.

\section{Rigidity results for horizontal pseudoharmonic maps defined on contact locally sub-symmetric spaces}

Now we derive Mok-Siu-Yeung type formulas for horizontal maps from compact contact locally sub-symmetric spaces into strictly pseudoconvex $CR$ manifolds. In this section we assume that $M$ is a contact locally sub-symmetric space of dimension $2d + 1 \geq 5$. First we consider the case $M$ torsionless.

\textbf{Lemma 6.1} Let $(M, \theta, \xi, J, g_\theta)$ be a contact locally sub-symmetric space torsionless with $s^W$ non-zero. The tensor $Q^+_{H_0} \in \Gamma(S^2(\Lambda^2_{H_0}(M)))$ given by $Q^+_{H_0} = c_0 J^\xi + C^M_H$ with

$$c_0 = -\frac{8d}{d - 1} \frac{|C^M_H|^2}{s^W}$$

is parallel and satisfies $(Q^+_{H_0}, R^W_{H_0}) = 0$ and $(c_H(Q^+_{H_0}))_0 = 0$. 

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Proof. First recall that $I^C_{H_0} = \frac{1}{4} (g_{\theta H} \otimes g_{\theta H})_0$. Now we determine $c_0$ in a such way that $Q^+_{H_0} = c_0 I^C_{H_0} + C^M_H$ satisfies $\langle Q^+_{H_0}, R^W_{H_0} \rangle = 0$. We have

$$\langle Q^+_{H_0}, R^W_{H_0} \rangle = \frac{c_0}{4} tr_H R^W_{H_0} + \langle C^M_H, R^W_{H_0} \rangle.$$ 

Now, we have

$$R^W_{H_0} = \frac{s^W}{d(d+1)} I^C_{H_0} + \frac{1}{2(d+2)} \left( Ric^W_{H_0} \otimes g_{\theta H} - \rho^W_{H_0} \otimes \omega \right)_0 + C^M_H.$$

Since $tr_H \left( Ric^W_{H_0} \otimes g_{\theta H} - \rho^W_{H_0} \otimes \omega \right)_0 = tr_H C^M_H = 0$ and $tr_H I^C_{H_0} = \frac{d^2 - 1}{2}$, we deduce that $tr_H R^W_{H_0} = \frac{d-1}{2d} s^W$. Using $\langle C^M_H, R^W_{H_0} \rangle = |C^M_H|^2$, we obtain that

$$\langle Q^+_{H_0}, R^W_{H_0} \rangle = c_0 \frac{d-1}{8d} s^W + |C^M_H|^2.$$

By taking $c_0 = -\frac{8d}{d-1} \frac{|C^M_H|^2}{s^W}$ we obtain that $\langle Q^+_{H_0}, R^W_{H_0} \rangle = 0$. Now we have $c_H(Q^+_{H_0}) = c_0 \frac{d}{2} \left( 1 - \frac{1}{d^2} \right) g_{\theta H}$ and then $(c_H(Q^+_{H_0}))_0 = 0$. Since $M$ is a contact locally sub-symmetric space, then $\nabla R^W = 0$ and $s^W$ constant yield to $\nabla C^M_H = 0$. Hence we have $|C^M_H|^2$ and $c_0$ constant. The parallelism of $Q^+_{H_0} = 0$ directly follows. $\Box$

**Proposition 6.1** For any horizontal map $\phi$ from a compact contact locally sub-symmetric space $M$, holonomy irreducible and torsionless, to a Sasakian manifold $N$, we have:

$$\int_M \frac{c_0}{2d} \left( |(\nabla^S d\phi)_0^+|^2 + \langle C^M_H (\nabla^S d\phi)_0^+, (\nabla^S d\phi_H)_0^+ \rangle + \frac{c_0}{2} \left( 1 - \frac{1}{d} \right) |(\nabla^S d\phi)_0^-|^2 \right)$$

$$+ \left( C^M_H (\nabla^S d\phi_H)_0^- , (\nabla^S d\phi_H)_0^- \right) - \frac{c_0}{2} \left( 1 - \frac{1}{d} \right) |\delta^S_{H_0} d\phi_H|^2 - \frac{c_0}{2} (d^2 - 1) |(d\phi(\xi))_{H_0^c}^2 v_{\theta H}$$

$$= 2 \int_M c_0 \left( \frac{1}{d} tr_H^2 \left( (\phi^* R^W_{H})^c_H \right)_0^+ \right) + \left( 1 - \frac{1}{d} \right) tr_H^1 \left( (\phi^* R^W_{H})^c_H \right)_0^+ + 4 \langle C^M_H, (\phi^* R^W_{H})_0^+ \rangle v_{\theta H} (31)$$

where $c_0 = -\frac{8d}{d-1} \frac{|C^M_H|^2}{s^W}$.

Proof. Let $Q^+_{H_0} = c_0 I^C_{H_0} + C^M_H$ with $c_0$ defined in Lemma 6.1. The horizontal $J$-invariant symmetric 2-tensor $(c_H(R^W_{H} \circ Q^+_{H_0}))^S$ is parallel. We deduce from the irreducibility of $M$ that $(c_H(R^W_{H} \circ Q^+_{H_0}))^S = \lambda g_{\theta H}$ with $\lambda \in C^\infty(M, \mathbb{R})$. Now,

$$\lambda = \frac{1}{2d} tr_H (c_H(R^W_{H} \circ Q^+_{H_0}))^S = \frac{2}{d} tr_H (R^W_{H} \circ Q^+_{H_0}) = \frac{4}{d} \langle R^W_{H}, Q^+_{H_0} \rangle = \frac{4}{d} (R^W_{H}, Q^+_{H_0}) = 0.$$
Hence \((c_H(\hat{R}_H^W \circ Q_{H_0}^+))^S = 0\). If \(\phi\) is a horizontal map from \(M\) to \(N\), we have by Lemma 4.2

\[
\langle Q_{H_0}^+ (\nabla^S d\phi H)_0, (\nabla^S d\phi H)_0 \rangle = \frac{c_0}{2} \left\{ \frac{1}{d^2} \left| (\nabla^S d\phi H)_0^\perp \right|^2 + \left( 1 - \frac{1}{d} \right) \left| (\nabla^S d\phi H)^\perp \right|^2 \right\} + \langle C^M_H (\nabla^S d\phi H)_0, (\nabla^S d\phi H)_0 \rangle
\]

\[
\text{tr}_H Q_{H_0}^+ = \frac{c_0}{2} \left( d^2 - 1 \right)
\]

\[
\langle Q_{H_0}^+, (\phi^* R_{H'}^{W^c})_H \rangle = \frac{c_0}{4} \left( \frac{1}{d^2} \text{tr}_H^0 \left( (\phi^* R_{H'}^W)^c \right)_H \right) + \left( 1 - \frac{1}{d} \right) \text{tr}_H^1 \left( (\phi^* R_{H'}^W)^c \right)_H \rangle + \langle C^M_H, (\phi^* R_{H'}^W)_H \rangle.
\]

By replacing in (14), we obtain the formula. \(\square\)

Now we consider the case of contact locally sub-symmetric spaces with torsion. Let \(M\) be a contact locally sub-symmetric spaces with torsion, we recall that we have \(\tau^2 = \frac{|\tau|^2}{2d} id_H\).

We always may assume that \(\frac{|\tau|^2}{2d} = 1\) also \(\tau\) becomes a paracomplex structure on \(H\). Now \((\tau, J \circ \tau, J)\) defines a so called bi-paracomplex structure (cf. [16]) on \(M\). Any horizontal 2-tensor \(t_H\) on \(M\) decomposes into \(t_H = t_{H_+} + t_{H_-}\), where \(t_{H_\pm} := \frac{1}{2}(t_H \mp \tau^* t_H)\) are respectively the \(\tau\)-invariant part and the \(\tau\)-anti-invariant part of \(t_H\). Let \(\wedge^2_{H_\pm}(M)\) be the bundle of \(\tau\)-(anti)invariant horizontal antisymmetric 2-tensors. For \(Q_H \in S^2(\wedge^2_{H}(M))\), we define \(Q_{H_\pm} \in S^2(\wedge^2_{H_\pm}(M))\) by

\[
Q_{H_\pm}(X,Y,Z,W) = \frac{1}{4} (Q_H(X,Y,Z,W) \pm Q_H(\tau(X),\tau(Y),Z,W) \pm Q_H(X,Y,\tau(Z),\tau(W)) \pm Q_H(\tau(X),\tau(Y),\tau(Z),\tau(W))).
\]

The tensors \((Q_H)_{\pm}^\pm \in S^2((\wedge^2_{H}(M))_{\pm}^\pm)\) are defined by \((Q_H)_{\pm}^\pm = (Q_{\pm}^\pm)_{\pm} = (Q_{\pm})_{\pm}^\pm\).

**Lemma 6.2** Let \(M\) be a contact locally sub-symmetric space with torsion and \(s_H \in S^2_H(M) \otimes E\). We have the relations:

\[
\begin{align*}
A_{\theta} \circ A_{\theta} & \equiv 2(\tau^* s_H - A_{\theta} \otimes \text{tr}_H c_H(A_{\theta} \otimes s_H)), \\
B_{\theta} \circ B_{\theta} & \equiv 2((- J \circ \tau)^* s_H - B_{\theta} \otimes \text{tr}_H c_H(B_{\theta} \otimes s_H)), \\
c_H(A_{\theta} \otimes A_{\theta}) & = c_H(B_{\theta} \otimes B_{\theta}) = -\frac{|\tau|^2}{d} g_{\theta H}, \quad \text{tr}_H A_{\theta} \otimes A_{\theta} = \text{tr}_H B_{\theta} \otimes B_{\theta} = -|\tau|^2.
\end{align*}
\]
Proposition 6.2 For any horizontal map $\phi$ from a compact contact locally sub-symmetric space with torsion $M$ to a Sasakian manifold $N$, we have:

$$
\int_M \left| (\nabla^S d\phi_H)_{+}^\top \right|^2 + \left| (\nabla^S d\phi_H)_{-}^\top \right|^2 + \left| (\nabla^S d\phi_H)_{+}^\bot \right|^2 + \left| (\nabla^S d\phi_H)_{-}^\bot \right|^2 + \frac{1}{d} A_\theta \otimes \delta_H^\top (d\phi_H \circ \tau) \right|^2 \\
- \left| (\nabla^S d\phi_H)_{-}^\bot \right|^2 + \frac{1}{d} B_\theta \otimes \delta_H^\bot (d\phi_H \circ J \circ \tau) \right|^2 \\
+ \left(1 - \frac{1}{d}\right) \left( |\delta_H^\top (d\phi_H \circ J \circ \tau)|^2 - |\delta_H^\bot (d\phi_H \circ \tau)|^2 \right) + 2 \left| (d\phi(\xi))_{H'} \right|^2 \\
= 8 \int_M tr_H((\phi^* R^W_{H'})^\top_{H'})_{+}^\top \psi_{g_S},
$$

(32)

Proof. Let the tensors $Q^-_H \in \Gamma(S^2(\wedge^2_{H}^- (M)))$ and $Q^+_H \in \Gamma(S^2(\wedge^2_{H}^+ (M)))$ defined by:

$$
Q^-_H = (g_\theta \otimes g_\theta^*)^\top = \frac{1}{4} (g_\theta \otimes g_\theta^* - \omega_\theta \otimes \omega_\theta + A_\theta \otimes A_\theta - B_\theta \otimes B_\theta)
$$

and

$$
Q^+_H = \frac{1}{4} ((g_\theta \otimes g_\theta^*)_0^+ = (I^C_{H})_0^+ = \frac{1}{2} (I^C_{H} - T_{H}),
$$

with $T_{H_0} = \frac{1}{8} \left( A_\theta \otimes A_\theta + B_\theta \otimes B_\theta + 2 \omega_\theta \otimes \omega_\theta \right)$. Since $M$ is a contact locally sub-symmetric space, we have $\nabla_H A_\theta = \nabla_H B_\theta = 0$ and then $\nabla_H Q^-_H = \nabla_H Q^+_H = 0$. From Lemmas 4.2 and 6.2, we have

$$
c_H(Q^-_H) = (d - 1)g_\theta, \quad c_H(Q^+_H) = \frac{(d - 1)(d + 2)}{4d} g_\theta,
$$

and

$$
tr_H \widehat{Q^-_H} = d(d - 1), \quad tr_H \widehat{Q^+_H} = \frac{(d - 1)(d + 2)}{4}.
$$

It directly follows that $(c_H(Q^-_H))_0 = (c_H(Q^+_H))_0 = 0$, and that $\frac{tr_H \widehat{Q^-_H}}{d} B_\theta - Q^-_H B_\theta = 0$.

Now since $R^W_{H_0} = \frac{2s^W}{d^2} (I^C_{H})_+^\top$ then

$$
\widehat{R}^W_H \circ Q^+_H = \widehat{R}^W_{H_0} \circ Q^+_H = \frac{2s^W}{d^2} (I^C_{H})_+^\top \circ (I^C_{H})_0^- = 0.
$$
Also \((c_H(\widehat{R}^\phi_{H_0} \circ \widehat{Q}^\phi_{H_0}))^S = 0\). Now, let \(\phi\) be a horizontal map from \(M\) to \(N\), by Lemmas 4.2 and 6.2, we have

\[
\langle Q^\phi_{H_0} (\nabla^S d\phi_H)_0, (\nabla^S d\phi_H)_0 \rangle = \frac{1}{4} \left( (1 + \frac{1}{d}) \left( |(\nabla^S d\phi_H)_+|^2 + |(\nabla^S d\phi_H)_-|^2 \right) + \frac{1}{d} A_\theta \otimes \delta^H_H (d\phi_H \circ \tau)|^2 \right)
- \frac{1}{4} |tr_H c_H (A_\theta \otimes \nabla^S d\phi_H)|^2 + \frac{1}{d} |tr_H c_H (B_\theta \otimes \nabla^S d\phi_H)|^2
\]

\[
\langle Q^\phi_{H_0} (\nabla^S d\phi_H)_0, (\nabla^S d\phi_H)_0 \rangle = \frac{1}{4} \left( \left( 1 + \frac{1}{d} \right) \left( |(\nabla^S d\phi_H)_+|^2 + |(\nabla^S d\phi_H)_-|^2 \right) + \frac{1}{d} |\delta^H_H (d\phi_H \circ \tau)|^2 \right)
- \frac{1}{4} |tr_H c_H (A_\theta \otimes \nabla^S d\phi_H)|^2 + \frac{1}{d} |tr_H c_H (B_\theta \otimes \nabla^S d\phi_H)|^2
\]

Now we have

\[tr_H c_H (A_\theta \otimes \nabla^S d\phi_H) = -2 \delta^H_H (d\phi_H \circ \tau) \quad \text{and} \quad tr_H c_H (B_\theta \otimes \nabla^S d\phi_H) = -2 \delta^H_H (d\phi_H \circ J \circ \tau)\]

Moreover

\[|(\nabla^S d\phi_H)_+|^2 = |(\nabla^S d\phi_H)_+| + \frac{1}{d} A_\theta \otimes \delta^H_H (d\phi_H \circ \tau)|^2 + \frac{1}{d} |\delta^H_H (d\phi_H \circ \tau)|^2\]

and

\[|(\nabla^S d\phi_H)_-|^2 = |(\nabla^S d\phi_H)_-| + \frac{1}{d} B_\theta \otimes \delta^H_H (d\phi_H \circ J \circ \tau)|^2 + \frac{1}{d} |\delta^H_H (d\phi_H \circ J \circ \tau)|^2\]

Then

\[
\langle Q^\phi_{H} (\nabla^S d\phi_H)_0, (\nabla^S d\phi_H)_0 \rangle = \frac{1}{4} \left( (1 + \frac{1}{d}) \left( |(\nabla^S d\phi_H)_+|^2 + |(\nabla^S d\phi_H)_-|^2 \right) + \frac{1}{d} A_\theta \otimes \delta^H_H (d\phi_H \circ \tau)|^2 \right)
- |(\nabla^S d\phi_H)_-|^2 + \frac{1}{d} B_\theta \otimes \delta^H_H (d\phi_H \circ J \circ \tau)|^2
\]

\[
\langle Q^\phi_{H_0} (\nabla^S d\phi_H)_0, (\nabla^S d\phi_H)_0 \rangle = \frac{1}{4} \left( \left( 1 + \frac{1}{d} \right) \left( |(\nabla^S d\phi_H)_+|^2 + |(\nabla^S d\phi_H)_-|^2 \right) + \frac{1}{d} A_\theta \otimes \delta^H_H (d\phi_H \circ \tau)|^2 \right)
+ \left( 1 - \frac{1}{d} \right) \left( |\delta^H_H (d\phi_H \circ \tau)|^2 - |\delta^H_H (d\phi_H \circ \tau)|^2 \right)
\]

\[
|\delta^H_H (d\phi_H \circ \tau)|^2 = |\delta^H_H (d\phi_H \circ \tau)|^2 + \frac{1}{d} |\delta^H_H (d\phi_H \circ J \circ \tau)|^2.
\]
We have also
\[ \langle Q_{H^*}, (\phi^* R_{H'}^W)_{H'} \rangle = \langle (g_{\theta_H} \otimes g_{\theta_H})_+, (\phi^* R_{H'}^W)_{H'} \rangle = tr_H((\phi^* R_{H'}^W)_{H'})_+ \]
\[ \langle Q_{H_0^*}, (\phi^* R_{H'}^W)_{H'} \rangle = \frac{1}{4} ((g_{\theta_H} \otimes g_{\theta_H})_+ - \frac{1}{d} \omega_{g} \otimes \omega_{g}, (\phi^* R_{H'}^W)_{H'}) \]
\[ = \frac{1}{4} \left( (1 - \frac{1}{d}) tr_H((\phi^* R_{H'}^W)_{H'})_+ - \frac{1}{d} tr_H((\phi^* R_{H'}^W)_{H'})_+ + \frac{1}{d} tr_H((\phi^* R_{H'}^W)_{H'})_- \right) \]
\[ = \frac{1}{4} \left( (1 - \frac{1}{d}) tr_H((\phi^* R_{H'}^W)_{H'})_+ + \frac{1}{d} tr_H((\phi^* R_{H'}^W)_{H'})_- - tr_H((\phi^* R_{H'}^W)_{H'})_+ \right). \]

By replacing in (17) and (18), we obtain the formulas. □

Now we deduce some rigidity results for the contact sub-symmetric space of non-compact type.

In the following we denote respectively by \( g, \mathfrak{t}, l \) the Lie algebras of the Lie groups \( G, K, L \). Let \( \tilde{M} = G/K \) be a simply-connected contact sub-symmetric space of non-compact Hermitian type. Then \( \tilde{M} \) is the total space of a \( S^1 \)-fibration \( \pi \) over an irreducible Hermitian symmetric space of non-compact type \( \tilde{B} = G/L \). To the Hermitian symmetric space \( G/L \), it is naturally associated an irreducible Hermitian orthogonal involutive Lie algebra \( (g, s, \beta_{/p}) \), where \( s \) is an involutive automorphism of \( g \) such that \( l=-1 \)-eigenspace of \( s \), \( p=-1 \)-eigenspace of \( s \) and \( \beta_{/p} \) is the \( ad_{\mathfrak{t}} \)-invariant inner product on \( p \) given by the restriction of the Killing form of \( g \) to \( p \) (we refer to Falbel-Gorodski [17] for the precise definition). Also it follows a so-called irreducible sub-torsionless Hermitian sub-orthogonal involutive Lie algebra \( (g, s, \mathfrak{t}, \beta_{/p}) \) associated to the sub-symmetric space \( G/K \). Concerning the facts hold \( \mathfrak{t} = \mathfrak{l}, \mathfrak{l}, \{p, p\} = \mathfrak{l} \) and \( l = < \xi^* > \oplus \mathfrak{t} \) with \( \xi^* \) in the center of \( \mathfrak{l} \).

The ideal \( \mathfrak{t} \) of \( l \) is either a simple ideal or \( \mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2 \) with \( \mathfrak{t}_1 \) and \( \mathfrak{t}_2 \) are simple ideals of \( l \). The Killing form \( \beta \) is negative definite on \( l \) and we have the orthogonal decomposition of \( g \) relatively to \( \beta \),

\[ g = < \xi^* > \oplus \mathfrak{t} \oplus p. \]

The endomorphism \( J^* = ad_{\xi^*/p} \) of \( p \) defines a \( ad_{\mathfrak{t}} \)-invariant complex structure on \( p \).

The \( ad_{\mathfrak{t}} \)-invariant skew-symmetric bilinear form \( \beta(J^*.,.) \) on \( p \) is non-degenerate and coincides with \( \Pi^*(d\theta) \) where \( \Pi : G \to G/K \) is the natural projection.

Now the curvature \( \check{R}_H^W \) of \( \tilde{M} \) is given by \( \check{R}_H^W = \pi^* R^\check{B} \) where \( R^\check{B} \) is the curvature of \( \check{B} \). Also \( \tilde{M} \) has nonpositive pseudo-Hermitian sectional curvature and negative pseudo-Hermitian scalar curvature. The Lie algebra expression of \( \check{R}_H^W \) is given (cf. (18)), for any \( X^*_i \in p \), by:

\[ \check{R}_H^W(d\Pi(X^*_1), d\Pi(X^*_2), d\Pi(X^*_3), d\Pi(X^*_4)) = \beta([X^*_1, X^*_2], [X^*_3, X^*_4]). \]
Let \( c'_0 = -4\frac{|\tilde{R}^W_H|^2}{s^W} = -4\frac{|\tilde{R}^B|}{s^B} > 0 \) and \( \kappa(\tilde{M}) \) be the lowest eigenvalue of the quadratic form \( s_{H_0} \to \langle \tilde{R}^H_H, s_{H_0}, s_{H_0} \rangle = \langle \pi^* \tilde{R}^B, s_{H_0}, s_{H_0} \rangle \) associated to \( \tilde{R}^H_H \) for horizontal traceless symmetric 2-tensors. The following tabular, coming from those obtained in [10], [12] and [22] for the irreducible Hermitian symmetric spaces of non-compact type, gives the values of \( c'_0 \) and \( \kappa(\tilde{M}) \) for the simply-connected contact sub-symmetric spaces of non-compact Hermitian type.

| Type                                  | \( d \) | \( c'_0 \)                      | \( \kappa(\tilde{M}) \) |
|---------------------------------------|---------|---------------------------------|--------------------------|
| \( SU(p, q)/SU(p) \times SU(q) \)    | \( pq \) | \( \frac{pq+1}{(p+q)^2} \)     | \(-\frac{1}{p+q}\)       |
| \( SO^*(2p)/SU(p) \)                 | \( \frac{p(p-1)}{2} \) | \( \frac{1}{4} + \frac{3p}{4(p-1)^2} \) | \(-\frac{1}{2(p-1)}\) |
| \( Sp(p, \mathbb{R})/SU(p) \)       | \( \frac{p(p+1)}{2} \) | \( \frac{1}{4} + \frac{3p}{4(p+1)^2} \) | \(-\frac{1}{2(p+1)}\) |
| \( SO_0(p, 2)/SO(p) \)              | \( p \)  | \( \frac{1}{2} + \frac{1}{p^2} \) | \(-\frac{1}{p}\)         |
| \( E_{6(-14)}/Spin(10) \)           | 16      | \( \frac{162}{16} \)           | \(-\frac{18}{16}\)       |
| \( E_{7(-25)}/E_6 \)                | 27      | \( \frac{162}{16} \)           | \(-\frac{18}{16}\)       |

**Theorem 6.1** Let \( \tilde{M} \) be a simply-connected contact sub-symmetric space of non-compact Hermitian type other than \( SU(d, 1)/SU(d) \) and let \( \Gamma \) be a cocompact discrete subgroup of \( PsH(\tilde{M}) \). Any horizontal pseudoharmonic map \( \phi \) from \( M = \tilde{M}/\Gamma \) to a Sasakian manifold \( N \) with nonpositive pseudo-Hermitian complex sectional curvature satisfies \( \nabla d\phi = 0 \).

Proof. Let \( \phi \) be a horizontal pseudoharmonic map from \( M \) to \( N \). Since \( M \) and \( N \) are torsionless, then [25] together with the assumption on the curvature of \( N \) yields to

\[
(d\phi(\xi))_{H'} = 0, \quad (\nabla^S d\phi_H)_{H'} = 0 \quad \text{and} \quad \text{tr}_H^{2,0}(\langle \phi^* R^W_H \rangle_H^C) = 0.
\]

Now, using the irreducibility of \( M \), we have by equation [31]:

\[
\int_M \frac{c_0}{2} \left(1 - \frac{1}{d}\right) \langle (\nabla^S d\phi_H)^{-1} - (C^M_H (\nabla^S d\phi_H)^{-1}, (\nabla^S d\phi_H)^{-1})v_{g_0} \rangle + 4\langle C^M_H, (\phi^* R^W_H)^C \rangle_H v_{g_0} = 0.
\]

(33)

Now, since \( M \) is pseudo-Einstein, then \( C^M_H = R^W_H - s^W \frac{I^C_H}{d(d+1)} \) and \( |C^M_H|^2 = |R^W_H|^2 - \frac{s^W}{4d(d+1)} \).

We deduce by Lemma 4.2 that

\[
\frac{c_0}{2} \left(1 - \frac{1}{d}\right) \langle (\nabla^S d\phi_H)^{-1} - (C^M_H (\nabla^S d\phi_H)^{-1}, (\nabla^S d\phi_H)^{-1}) \rangle = c'_0 \langle (\nabla^S d\phi_H)^{-1} - (R^W_H, (\nabla^S d\phi_H)^{-1}) \rangle.
\]

(34)
and
\[ c_0 \left( 1 - \frac{1}{d} \right) tr_H^1 \left( (\phi^* R^W_H)^C \right) = 4(C^M, (\phi^* R^W_H)) = 2c_0 tr_H^1 \left( (\phi^* R^W_H)^C \right) + 4(R^W_H, (\phi^* R^W_H)). \]
(35)

By the comparison between \( c'_0 \) and \( \kappa(\tilde{M}) \), we deduce that (34) is positive excepted if \((\nabla^S d\phi_H)^- = 0\). Moreover, using the Lie algebra expression of \( \tilde{R}^W_H \), we can prove as Jost-Yau do in [19] p 257-273, that (35) is always nonpositive. Also it follows from (33) that \((\nabla^S d\phi_H)^- = 0\). Consequently \( \nabla^S d\phi_H = 0 \) and \( d_H^\Gamma d\phi_H = 0 \). The end of the proof follows from the proof of Theorem 5.2. □

Let \( I \) be an open interval of \( \mathbb{R} \) containing 0, recall that a regular curve \( c : I \rightarrow M \) on a strictly pseudoconvex \( CR \) manifold \( M \) is called a parabolic geodesic (cf. [15]) if \( \dot{c}(0) \in H_{c(0)} \) and if there exists \( \alpha \in \mathbb{R} \) such that \( \nabla_{\dot{c}(t)} \dot{c}(t) = \alpha \xi_{c(t)} \) for \( t \in I \). As a consequence of the previous theorem, we have:

**Corollary 6.1** Let \( M = \tilde{M}/\Gamma \) as above. Any horizontal pseudoharmonic map \( \phi \) from \( M \) to a Sasakian manifold \( N \) with nonpositive pseudo-Hermitian complex sectional curvature maps parabolic geodesics of \( M \) to parabolic geodesics of \( N \).

Proof. For any curve \( c : I \subset \mathbb{R} \rightarrow M \) and any map \( \phi \) from \( M \) to \( N \), we have
\[
\nabla'_{(\phi \circ c)(t)} (\phi \circ c)(t) = (\nabla_{\dot{c}(t)} d\phi)(\dot{c}(t)) + d\phi(\nabla_{\dot{c}(t)} \dot{c}(t)).
\]
If \( \phi \) is a horizontal pseudoharmonic map from \( M \) to \( N \), we have \( \nabla d\phi = 0 \). Consequently, for a parabolic geodesic \( c : I \subset \mathbb{R} \rightarrow M \), we obtain that \( \nabla'_{(\phi \circ c)(t)} (\phi \circ c)(t) = \alpha d\phi(\xi_{c(t)}) = \alpha f\xi_{(\phi \circ c)(t)} \). Hence \( \phi \circ c \) is a parabolic geodesic of \( N \). □

**Corollary 6.2** Let \( M = \tilde{M}/\Gamma \) as above. Any horizontal pseudoharmonic map \( \phi \) from \( M \) to a Tanaka-Webster flat Sasakian manifold \( N \) satisfies \( d\phi_H = 0 \).

Proof. Let \( \phi \) be a horizontal pseudoharmonic map from \( M \) to \( N \). Since \( N \) is Tanaka-Webster flat then Theorem 6.1 together with equation (26), yields to \( \langle d\phi_H \circ \tilde{R}^W_H, d\phi_H \rangle = 0 \). Now, since \( M \) is pseudo-Einstein with \( s^W < 0 \), then \( d\phi_H = 0 \). □

If \( M \) is a compact strictly pseudoconvex \( CR \) manifold, then \( b_1(M) = \dim Ker \triangle_R \). Also, we have using formulas similar to (25), (26) and (31):

**Corollary 6.3** Let \( \tilde{M} \) be a simply-connected contact sub-symmetric space of non-compact Hermitian type other than \( SU(d,1)/SU(d) \) then \( b_1(M/\Gamma) = 0 \).

**Remark 6.1** We can observe that the previous corollary directly follows from the two following facts. First \( M \) is the total space of a \( S^1 \)-fibration over a compact irreducible Hermitian locally symmetric space of non-compact type \( B \) and then \( b_1(M) = b_1(B) \). Second by the Matsushima Theorem [22] we have \( b_1(B) = 0 \).
7 Rigidity results for CR maps defined on contact locally sub-symmetric spaces

In this section we suppose that $N$ is a strictly pseudoconvex CR manifold and $\phi$ is a CR map from $M$ to $N$.

**Proposition 7.1** (i) For any CR map $\phi$ from a compact contact locally sub-symmetric space torsionless $M$ to $N$, we have:

$$
\int_M \frac{c_0}{2} (1 - \frac{1}{d}) |(\nabla^S d\phi_H)^{-1}|^2 + |(C_H^M (\nabla^S d\phi_H)^{-1}, (\nabla^S d\phi_H)^{-1}) - c_0(d^2 - 1)|(d\phi(\xi))_{H'}|\|^2 v_{g_{\phi}} = 2 \int_M \frac{c_0}{2} (1 - \frac{1}{d}) (HBK_{\phi}^{W})_{H} + 4(C_H^M, (\phi^* R^W_{H'}))_{H'} v_{g_{\phi}}. \tag{36}
$$

(ii) For any CR map $\phi$ from a compact contact locally sub-symmetric space with torsion $M$ to $N$, we have:

$$
\int_M (1 - \frac{2}{d}) \left( |(\nabla^S d\phi_H)^{-1} + \frac{1}{d} A_{\phi} \otimes \delta_H^{\prime} (d\phi_H \circ \tau)|^2 + |(\nabla^S d\phi_H)^{-1} + \frac{1}{d} B_{\phi} \otimes J' \delta_H^{\prime} (d\phi_H \circ \tau)|^2 \right) + 2 \left(1 - \frac{1}{d}\right) \left( |\delta_H^{\prime} (d\phi_H \circ \tau)|^2 - d^2 |(d\phi(\xi))_{H'}|^2 \right) v_{g_{\phi}} = 8 \int_M \left(1 - \frac{1}{d}\right) (HBK_{\phi}^{W})_{H} - (K_{\phi}^{W})_{H} v_{g_{\phi}}, \tag{37}
$$

with

$$(K_{\phi}^{W})_{H} = \sum_{i,j \leq d} R^W_{H'}(d\phi_H(\xi_i), d\phi_H(\xi_j), d\phi_H(\xi_i), d\phi_H(\xi_j)),
$$

and

$$(HBK_{\phi}^{W})_{H} = \sum_{i,j \leq d} R^W_{H'}(d\phi_H(\xi_i), J' d\phi_H(\xi_j), d\phi_H(\xi_j), J' d\phi_H(\xi_j)).$$

Proof. Let $\phi$ be a CR map from $M$ to $N$ then $J' \circ d\phi_H = d\phi_H \circ J$. Also we have $\delta_H^{\prime} d\phi_H = -dJ' (d\phi(\xi))_{H'}$ by (27) and $\delta_H^{\prime} (d\phi_H \circ J \circ \tau) = J' \delta_H^{\prime} (d\phi_H \circ \tau)$. Moreover,

$$(\phi^* g_{\theta'})_{H}(X, Y) = (\phi^* \omega_{\theta'})_{H}(X, J' Y) = f\omega_{\theta}(X, J' Y) = f g_{\theta h}(X, Y)
$$

and

$$(\phi^* B_{\theta'})_{H}(JX, JY) = B_{\theta'}(J' d\phi_H(X), J' d\phi_H(Y)) = -B_{\theta'}(d\phi_H(X), d\phi_H(Y)) = -(\phi^* B_{\theta'})_{H}(X, Y).$$

Hence we have $(\phi^* g_{\theta'})_{H} = f g_{\theta h}$ and $(\phi^* B_{\theta'})_{H} = 0$. It follows that for each $Q^+_H$ defined in Propositions 6.1 and 6.2, we have

$$
\langle (c_H(R^W_{H} \circ Q^+_H))^{S}, (\phi^* g_{\theta'})_{H} \rangle = \frac{1}{2} ftr_H(c_H(R^W_{H} \circ Q^+_H))^{S} = 4(R^W_{H}, Q^+_H) = 0,
$$

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Now, we have \((\phi^* g_\theta)_H\) and \(({\tilde{W}}^* g_\theta)_H\). We can choose an adapted frame \((\phi^* B_{\theta'})_H\) and \((\phi^* B_{\theta'})_{\hat{H}}\). Consequently \(tr_H((\phi^* B_{\theta'})_H) = 0\) and \(tr_H((\phi^* B_{\theta'})_{\hat{H}}') = 0\).

Now the formulas directly follow from equations (17) and (18) combined to (31) and (32).

We can choose an adapted frame \(\{e_1, \ldots, e_d, J_1, \ldots, J_d\}\) of \(H\) such that \(\tau(e_i) = e_i\) and \(\tau(J_\epsilon) = -J_\epsilon\), also we obtain

\[
tr_H((\phi^* R^W_{H'})_H) = \sum_{i,j \leq d} R^W_{H'}(d\phi_H(e_i), J'd\phi_H(e_i), d\phi_H(e_j), J'd\phi_H(e_j)).
\]

Now the formulas directly follow from equations (17) and (18) combined to (31) and (32). \(\square\)

A strictly pseudoconvex CR manifold with constant holomorphic pseudo-Hermitian sectional curvature is called a pseudo-Hermitian space form. The sphere \(S^{2d+1} = SU(d+1)/SU(d)\) viewed as the total space of the Hopf fibration over \(\mathbb{C}P^d\) and its non-compact dual \(SU(d,1)/SU(d)\), are examples of Sasakian pseudo-Hermitian space forms with respectively \(s^W > 0\) and \(s^W < 0\). The Heisenberg group \(H_{2d+1}\) with its standard pseudo-Hermitian structure is an example of flat Sasakian pseudo-Hermitian space form whereas the unit tangent bundle of the hyperbolic space \(H^{d+1}\), \(T_1H^{d+1}\) with its standard pseudo-Hermitian structure is an example of flat non-Sasakian pseudo-Hermitian space form (cf. [14]). Note that all these examples are examples of contact sub-symmetric spaces.

**Theorem 7.1** (i) Let \(\tilde{M}\) be a simply-connected contact sub-symmetric space of non-compact Hermitian type other than \(SU(d,1)/SU(d)\) and let \(\Gamma\) be a cocompact discrete subgroup of \(PSH(\tilde{M})\). Then any horizontal pseudoharmonic CR map \(\phi\) from \(M = \tilde{M}/\Gamma\) to a strictly pseudoconvex CR manifold \(N\) with nonpositive pseudo-Hermitian complex sectional curvature satisfies \(\nabla_Hd\phi = 0\).

(ii) Let \(\tilde{M}\) be a simply-connected contact sub-symmetric space of non-compact type other than \(H_{2p+1} \times C NCH\), then any horizontal pseudoharmonic CR map \(\phi\) from \(M = \tilde{M}/\Gamma\) to a pseudo-Hermitian space form \(N\) with \(s^W < 0\) is constant.

Proof. A horizontal pseudoharmonic CR map \(\phi\) from \(M\) to \(N\) satisfies, by Proposition 5.2, \(\delta^W_H d\phi = (d\phi(\xi))_{H'} = 0\) and \((\nabla^S d\phi_H)^+ = 0\). Since \(M\) is torsionless, then it directly follows from (30) that \((\nabla^S d\phi_H)^- = 0\). We deduce from the previous assumptions that
\( \nabla_H d\phi = 0 \). Hence (i) is proved. Now we assume that \( N \) is a pseudo-Hermitian space form, then we have \( R^\mathcal{W}_H = \frac{s^{\mathcal{W}'} c}{d'(d + 1)} I^\mathcal{C}_H \). Since \( \phi \) is a CR map, we obtain

\[
(\phi^* R^\mathcal{W}_H^+) = (\phi^* R^\mathcal{W}_H^+) = \frac{s^{\mathcal{W}'} c}{d'(d + 1)} (\phi^* I^\mathcal{C}_H) = \frac{f^2}{d'(d + 1)} s^{\mathcal{W}'} I^\mathcal{C}_H.
\]

Hence we have \( \langle C^\mathcal{M}_H, (\phi^* R^\mathcal{W}_H^+) \rangle = 0 \) and

\[
(HBK^\mathcal{W}_\phi)_H = tr_H(\phi^* R^\mathcal{W}_H^+) = \frac{f^2}{2 d'(d + 1)} s^{\mathcal{W}'} I^\mathcal{C}_H,
\]

\[
(K^\mathcal{W}_\phi)_H = tr_H((\phi^* R^\mathcal{W}_H^+) = \frac{f^2}{2 d'(d + 1)} s^{\mathcal{W}'} I^\mathcal{C}_H,
\]

and

\[
\left(1 - \frac{1}{d}\right)(HBK^\mathcal{W}_\phi)_H - (K^\mathcal{W}_\phi)_H = \frac{f^2}{d'(d + 1)} s^{\mathcal{W}'}.
\]

Since \( s^{\mathcal{W}'} < 0 \), the right hand sides of (36) and (37) are nonpositive, whereas the left hand sides are nonnegative. We deduce from (36) that \( (HBK^\mathcal{W}_\phi)_H = 0 \) and from (37) that \( \left(1 - \frac{1}{d}\right)(HBK^\mathcal{W}_\phi)_H - (K^\mathcal{W}_\phi)_H = 0 \). In each case, we obtain that \( f = 0 \). Since \( \langle \phi^* g_{\theta}' \rangle = f g_{\theta} \) and \( d\phi(\xi) = f\xi' \), then \( \phi \) is constant. \( \Box \)

A regular curve \( c : I \to M \) on a strictly pseudoconvex CR manifold \( M \) is called a Carnot-Caratheodory geodesic (cf. [4], [27], [28]) if \( \dot{c}(t) \in H_{\dot{c}(t)} \) and if there exists a function \( \alpha : I \to \mathbb{R} \) with \( \dot{\alpha}(t) = A_0(\dot{c}(t), \dot{c}(t)) \) such that \( \nabla_{\dot{c}(t)}(\dot{c}(t)) = -\alpha(t) J(\dot{c}(t)) \) for \( t \in I \).

Also, we have

**Corollary 7.1** Let \( \tilde{M} \) be a simply-connected contact sub-symmetric space of non-compact Hermitian type other than \( SU(d,1)/SU(d) \) and \( M = \tilde{M}/\Gamma \). Then any horizontal pseudo-harmonic CR map \( \phi \) from \( M \) to a Sasakian manifold \( N \) with nonpositive pseudo-Hermitian complex sectional curvature maps Carnot-Caratheodory geodesics of \( M \) to Carnot-Caratheodory geodesics of \( N \).

**Proof.** Let \( c : I \subset \mathbb{R} \to M \) be a Carnot-Caratheodory geodesic, then \( \nabla_{(\dot{c}(t))}(\dot{c}(t)) = -\alpha J(\dot{c}(t)) \) with \( \alpha \) constant (since \( M \) is torsionless). Let \( \phi \) be a horizontal pseudoharmonic CR map from \( M \) to \( N \), we obtain using \( \nabla_H d\phi = 0 \) that \( \nabla_{((\phi \circ c)(t))}(\dot{c}(t)) = -\alpha d\phi_H(\dot{c}(t)) = -\alpha J(\dot{c}(t)) = -\alpha J((\phi \circ c)(t)) \). Also \( \phi \circ c \) is a Carnot-Caratheodory geodesic of \( N \). \( \Box \)
References

[1] D. V. ALEKSEEVSKY and A. F. SPIRO, Compact homogeneous CR manifolds, *J. Geom. Anal.* 12 (2002), 183-201.

[2] E. BARLETTA, On the pseudohermitian sectional curvature of a strictly pseudoconvex CR manifold, preprint, arXiv:math.cv/0605394.

[3] E. BARLETTA and S. DRAGOMIR, Differential equations on contact Riemannian manifolds, *Ann. Scuola. Norm. Sup. Pisa.* 30 (2001), 63-95.

[4] E. BARLETTA and S. DRAGOMIR, Jacobi fields of the Tanaka-Webster connection on Sasakian manifolds, *Kodai. Math. J.* 29 (2006), 406-454.

[5] E. BARLETTA, S. DRAGOMIR and H. URAKAWA, Pseudoharmonic maps from a nondegenerate CR manifolds into a Riemannian manifold, *Indiana. Univ. Math. J.* 50 (2001), 719-746.

[6] P. BIELIAVSKY, E. FALBEL and C. GORODSKI, The classification of simply-connected contact sub-Riemannian symmetric spaces, *Pac. Math. J.* 188 (1999), 65-82.

[7] D. E. BLAIR, Contact manifolds in Riemannian Geometry, Lecture Notes in Math., Vol. 509, Springer-Verlag, New York, 1976.

[8] D. E. BLAIR, T. KOUFOGIORGOS and B. J. PAPANTONIOU, Contact metric manifolds satisfying a nullity condition, *Israel. J. Math.* 91 (1995), 189-214.

[9] E. BOECKX and J. T. CHO, \( \eta \)-parallel contact metric spaces, *Diff. Geom. Appl.* 22 (2005), 275-285.

[10] A. BOREL, On the curvature tensor of the Hermitian symmetric manifolds, *Ann. of. Math.* 71 (1960), 508-521.

[11] J. P. BOURGUIGNON, Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d’Einstein, *Invent. Math.* 63 (1981), 263-286.

[12] E. CALABI and E. VESENTINI, On compact locally symmetric Kahler manifolds, *Ann. of. Math.* 71 (1960), 472-507.

[13] S. S. CHERN and J. MOSER, Real hypersurfaces in complex manifolds, *Acta. Math.* 133 (1974), 48-69.

[14] J. T. CHO, Geometry of contact strongly pseudo-convex CR manifolds, *J. Korean. Math. Soc.* 43 (2006), 1019-1045.
[15] S. DRAGOMIR, On pseudo-Hermitian immersions between strictly pseudoconvex CR manifolds, *Amer. J. of Math.* **117** (1995), 169-202.

[16] F. ETAYO and R. SANTAMARIA, Connections functorially attached to almost complex product structures, arXiv:math.cv/0502319.

[17] E. FALBEL and C. GORODSKI, On contact sub-Riemannian symmetric spaces, *Ann. scient. Ec. Norm. Sup.* **28** (1995), 571-589.

[18] E. FALBEL, C. GORODSKI and M. RUMIN, Holonomy of sub-Riemannian manifolds, *Intern. J. Math.* **8** (3) (1997), 317-344.

[19] J. JOST and S. T. YAU, Harmonic maps and superrigidity, *Proc. Symp. Pure Math.* **54** (1993), 245-280.

[20] W. KAUP and D. ZAITSEV, On Symmetric Cauchy-Riemann Manifolds, *Adv. Math.* **49** (2000), 145-181.

[21] J.M. LEE, Pseudo-Einstein structures on *CR* manifolds, *Amer. J. of Math.* **110** (1988), 157-178.

[22] Y. MATSUSHIMA, On the first Betti number of compact quotient spaces of higher-dimensional symmetric spaces, *Ann. of Math.* **75** (1962), 312-330.

[23] N. MOK, Y. T. SIU and S. K. YEUNG, Geometric superrigidity, *Invent. Math.* **113** (1993), 57-83.

[24] E. MUSSO, Homogeneous pseudo-Hermitian Riemannian manifolds of Einstein type, *Amer. J. of Math.* **113** (1991), 219-241.

[25] R. PETIT, Harmonic maps and strictly pseudoconvex *CR* manifolds, *Comm. Anal. Geom.* **10** (2002), 575-610.

[26] R. PETIT, Spinc-structures and Dirac operators on contact manifolds, *Diff. Geom. Appl.* **22** (2005), 229-252.

[27] M. RUMIN, Formes différentielles sur les variétés de contact, *J. Diff. Geom.* **39** (1994), 281-330.

[28] R. S. STRICHERTZ, Sub-Riemannian geometry, *J. Diff. Geom.* **24** (1986), 221-263.

[29] N. TANAKA, A Differential Geometric Study on strongly pseudoconvex *CR* manifolds, Lecture Notes in Math., Vol. 9, Kyoto University, 1975.

[30] S. TANNO, Variational problems on contact riemannian manifolds, *Trans. Amer. Math. Soc.* **314** (1989), 349-379.
[31] S. WEBSTER, Pseudo-Hermitian structures on a real hypersurface, *J. Diff. Geom.* 13 (1978), 25-41.

[32] S. K. YEUNG, On vanishing theorems and rigidity of locally symmetric manifolds, *Geom. Anal. Func. Appl* 11 (2001), 175-198.