Simulation of induction at low magnetic Prandtl number

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We consider the induction of magnetic field in flows of electrically conducting fluid at low magnetic Prandtl number and large kinetic Reynolds number. Using the separation between the magnetic and kinetic diffusive lengthscales, we propose a new numerical approach. The coupled magnetic and fluid equations are solved using a mixed scheme, where the magnetic field fluctuations are fully resolved and the velocity fluctuations at small scale are modelled using a Large Eddy Simulation (LES) scheme. We study the response of a forced Taylor-Green flow to an externally applied field: topology of the mean induction and time fluctuations at fixed locations. The results are in remarkable agreement with existing experimental data; a global 1/f behavior at long times is also evidenced.

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One of the strongest motivations in the study of non-linear effects in magnetohydrodynamics is that electrically conductive flows are capable of dynamo action: the stretching of magnetic field lines by the flow velocity gradients can exceed the (Joule) diffusion. A bifurcation threshold occurs, above which the self-generation of a magnetic field takes place. It has been validated in constrained flows of liquid sodium which mimic analytical models: the Karlsruhe [1] and Riga experiments [2]. The self-generation of a magnetic field in non-constrained homogeneous flows is still an open problem actively studied by many groups [3]. In this research, numerical studies have long played an important role. Kinematic dynamo simulations assume a given pattern of a stationary velocity field and study the initial linear growth rate of magnetic field perturbations. They have been extensively used to test the dynamo capacity of flow geometries and proved to be successful at determining the dynamo threshold in the Karlsruhe and Riga experiments [1] [2]. They have also shown that dynamo action is a possibility in unconstrained homogeneous flows of the von Kármán type [3] [4]. Another numerical approach is to perform Direct Numerical Simulations (DNS) of the full governing equations: the induction equation coupled with the fluid dynamical one by the Lorentz force, the flow being sustained by a given force (or equivalently an average geometry). They have confirmed that dynamo action is present in flows with differential rotations and helicity [3] [4] [10]. However, DNS are at present restricted to situations where the magnetic Prandtl number, \( P_m = \nu/\lambda \) (where \( \lambda \) is the magnetic diffusivity) is of order one, i.e. to situations where the smallest scales of the magnetic and velocity fields have the same characteristic size [11]. This is not the case in liquid metals, which have very small magnetic Prandtl number values e.g. \( P_m \sim 10^{-6} \) for liquid Gallium and \( P_m \sim 10^{-5} \) for liquid Sodium. Recall that, below the dynamo threshold, a stationary forced flow with a power input \( \epsilon \) (in Watts per kg) has a viscous dissipative scale \( \eta_v \sim (\nu^3/\epsilon)^{1/4} \) and a Joule dissipative scale \( \eta_J \sim (\lambda^3/\epsilon)^{1/4} \) — hence a ratio \( \eta_v/\eta_J \sim P_m^{3/4} \). Therefore, at low \( P_m \), the magnetic diffusive length scale is very much larger than the velocity dissipative scale. If non-linear effects are to develop, the magnetic Reynolds number \( R_m \sim UL/\lambda \) (where \( U \) and \( L \) represent the characteristic velocity and scale of the flow) must be at least of order one and thus the kinetic Reynolds number of the flow, \( Re \sim UL/\nu \sim R_m/P_m \), must be very large (turbulence is fully developed). A DNS treatment of such a system is at present out of reach.

In this paper, we present a new approach for the study of the magnetic induction in large \( Re \) - low \( P_m \) flows; we restrict ourselves to regimes below the dynamo threshold. In this parameter region, the magnetic field “lives” essentially within the large and inertial hydrodynamic scales. We thus propose to treat with a sub-grid model the velocity scales which are smaller than the magnetic diffusive length. Schemes using hyperviscosity have previously been used [1] [12]. Here, we prefer a LES approach, which has proved very successful for the simulation of turbulent flows with large scale structures and for the modelling of energy transfers [13]. In this hybrid scheme, we solve the induction equation on a fully resolved grid and we use a LES method for the velocity field, with a cut-off scale at the end of the magnetic diffusive range. We consider the response of a conductive fluid to an uniform magnetic field: topology of the mean induced field and spatio-temporal features of the magnetic fluctuations are studied. The chosen flow is a forced Taylor-Green vortex (TG). It shares many similarities with the experimental von Kármán swirling flows which have already been investigated in DNS near \( P_m \sim O(1) \) [3] [10].

In non-dimensional form, the incompressible MHD equations have two classical control parameters, the magnetic and kinetic Reynolds numbers, and one has to
choose a forcing mechanism that generates the desired values of \( R_m \) and \( Re \). In order to be closer to experimental procedures, we prefer to fix the driving force and the magnetic Prandtl number. Hence, the dynamical time \( t_0 \) is set to the magnetic diffusive time scale, i.e. \( t_0 \lambda /L^2 \sim \mathcal{O}(1) \), where \( L \) is a length scale characteristic of the system size. Changes in magnetic diffusivity for real fluids would change that time scale. We write the amplitude of the system size. Changes in magnetic diffusivity for real fluids would change that time scale. We write the

\[
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nabla \cdot B = \nu \nabla^2 \mathbf{u} + \mathbf{F} + (\nabla \times \mathbf{b}) \times \mathbf{B},
\]

\[ \nabla \cdot \mathbf{u} = 0 , \ \nabla \cdot \mathbf{b} = 0 , \]

where \( \mathbf{u} \) is the velocity field, \( \mathbf{B} = \mathbf{B}_0 + \mathbf{b} \) is the net magnetic field in the flow, \( \mathbf{F} \) is the applied force and \( \mathbf{b} \) is the induced fields. Once the amplitude of the velocity fluctuations is \( u_{rms} \sim \sqrt{\mathcal{F}} \), the Reynolds number is \( Re \sim \sqrt{\mathcal{F}}/R_m \) and the magnetic Reynolds number is \( R_m \sim \sqrt{\mathcal{F}} \). When the interaction parameter is of the order of the measurement time in the inertial forces, defined as \( N \sim B_0^2/u_{rms} \sim B_0^2/\sqrt{\mathcal{F}} \), the second term on the right-hand side of the induced field on the velocity field is negligible.

The above expressions are only dimensional estimates; in practice, the characteristic flow quantities are computed as mean temporal values from the data – cf. Table 1.

We use a parallelized pseudo-spectral code in a \([0–2\pi]^3\) periodic box. Time stepping is done with an exponential forward Euler-Adams-Bashforth scheme. The LES model is of the Chollet-Lesieur type \( \text{LB} \), in which the kinematic viscosity \( \nu \) is replaced in spectral space by an eddy viscosity. In Eq. (1) the magnetic Prandtl number is then replaced by:

\[
P_m(k, t) = 0.1(1 + 5(k/K_c)^8)\sqrt{E_v(k = K_c, t)/K_c}.
\]

Here \( K_c \) is the cut-off wavenumber of the velocity field, and \( E_v(k, t) \) is the one-dimensional kinetic energy spectrum. The effective Prandtl number \( P_{m eff} \) is obtained as the temporal mean of \( P_m(0, t) \). Note that the effective fluid viscosity \( \nu_{eff} \) is of the same magnitude. A consistency condition for our approach is that the magnetic field fluctuations are fully resolved when \( 2\pi/K_c \) is smaller than the magnetic diffusive scale \( \eta_B \sim l_0/R_m^{3/4} \), \( l_0 \) being the integral scale computed from the kinetic energy spectrum. The flow is driven by the TG vortex geometry

\[
F_{TG}(k_0) = 2\mathcal{F} \left[ \begin{array}{c}
\sin(k_0 x) \cos(k_0 y) \cos(k_0 z) \\
-\cos(k_0 x) \sin(k_0 y) \cos(k_0 z) \\
0
\end{array} \right]
\]

(\( k_0, k_0, k_0 \)) is the wavevector that prescribes the velocity large scale (hereafter \( k_0 = 1 \)). The \( F_{TG} \) and \( \mathbf{B}_0 \) amplitudes are chosen such that the interaction parameter \( \mathcal{N} \) remains smaller than \( 10^{-2} \). After an initial transient \( (t < 10) \) the flow has reached a steady state: the kinetic energy fluctuates less than 3.5% around its mean value. All quantities are tracked up to \( t_{max} = 410 t_0 \) — note that \( 200 t_0 \) is of the order of the measurement time in most Sodium experiments \([14, 15, 16]\). For comparison, the eddy turnover time \( \tau_{NL} \sim t_0/u_{rms} \) is given in Table 1.

\[
\begin{array}{|c|c|c|}
\hline
\text{RUN} & \#1 \mathbf{B}_0 = 0.1 \hat{x} & \#2 \mathbf{B}_0 = 0.1 \hat{z} \\
\hline
\text{TG} & Re = 9209 & Re = 9212 \\
\hline
k_0 = 1 & R_{m} = 6.65 & R_{m} = 6.68 \\
\hline
F = 3/2 & R_{F} = 95.94 & R_{F} = 95.96 \\
128^3 \text{ grid points} & P_{m eff} = 7.22 \times 10^{-4} & P_{m eff} = 7.26 \times 10^{-4} \\
\hline
K_c = k_{max} - 3 & N = 8.23 \times 10^{-3} & N = 8.18 \times 10^{-3} \\
k_{max} = 64 & l_0 = 2.338 & l_0 = 2.337 \\
t_{max} = 410 & l_T = 0.024 & l_T = 0.024 \\
\eta_B = 0.565 & \eta_B = 0.563 & \eta_B = 0.563 \\
\tau_{NL} = 1.217 & \tau_{NL} = 1.224 & \tau_{NL} = 1.224 \\
u_{rms} = 2.843 & u_{rms} = 2.858 & u_{rms} = 2.858 \\
b_{rms} = 0.061 & b_{rms} = 0.064 & b_{rms} = 0.064 \\
\max|\mathbf{u}| = 8.211 & \max|\mathbf{u}| = 8.249 & \max|\mathbf{u}| = 8.249 \\
\max|\mathbf{b}| = 0.160 & \max|\mathbf{b}| = 0.180 & \max|\mathbf{b}| = 0.180 \\
\hline
\end{array}
\]

Table I: Time averaged quantities: \( u_{rms} = \langle u^2 \rangle^{1/2}, b_{rms} = \langle b^2 \rangle^{1/2} \), flow integral scale \( l_0 = 2\pi \sum_k E_v(k)/\sum_k E_v(k) \), Taylor microscale \( l_T \sim k_{max} l_0 \), diffusive scale \( \eta_B \) and eddy turnover time \( \tau_{NL} \). Non-dimensional parameters: effective Prandtl number \( P_{m eff} \), kinetic Reynolds number \( Re = l_0 u_{rms}/\nu_{eff} \) (see text), and magnetic Reynolds number \( R_m = P_{m eff} R_{F} \), Taylor-based Reynolds number \( R_{F} \sim R_e^{1/2} \), interaction parameter \( N = R_m B_0^2/u_{rms}^2 \).

Figure 1 shows the power spectra of the velocity and magnetic field fluctuations with \( \mathbf{B}_0 \) applied along the \( \hat{x} \)-axis (a direction perpendicular to the rotation axis of the counter-rotating eddies of the TG cells). The kinetic energy spectrum exhibits the \( k^{-5/3} \) Kolmogorov scaling law maintained throughout the range by the LES scheme. The peak at low wavenumber is due the large scale TG forcing, also visible on the magnetic field spectrum. The magnetic inertial range is well fitted by a \( k^{-11/3} \) power law in agreement with a Kolmogorov phenomenology \([17, 18]\). The magnetic diffusive scale is reached within the computational box. The main goal of our numerical strategy is thus achieved: the magnetic fluctuations are fully resolved in a range of scales at which the velocity field follows the Kolmogorov self-similar structure of turbulence. Hence, we get the possibility to study magnetic induction in a fully developed turbulent flow at low magnetic Prandtl number.

Figure 2 displays isosurfaces of the local induced magnetic energy \( \langle b_0(x, t) \rangle_T \) averaged in the time interval \( T = [10 \sim 410] \), shown at 80% of its maximum value. For comparison, we also plot isosurfaces of the induced mag-
magnetic energy, \( \langle E_{b,\text{lin}}(x,t) \rangle_T \), obtained numerically from a linear approximation based on time averaged velocities: \( \lambda \nabla^2 b = -B_0 \nabla \langle \mathbf{v}(x,t) \rangle_T \). This is similar to numerical studies based on the averaged flow geometries [7, 19].

When \( B_0 \) is applied along \( \hat{z} \), in a direction parallel to the rotation axis of the TG eddies, the most intense magnetic energy structures are concentrated round the \( z = \pi/2, 3\pi/2 \) planes, in agreement with the differential rotation of the TG vortex. Moreover, the most intense structures of \( \langle E_b(x,t) \rangle_T \) and \( \langle E_{b,\text{lin}}(x,t) \rangle_T \) fields coincide. For \( B_0 \) along the \( \hat{x} \)-axis, one observes the main induction concentration around the \( z = 0, \pi \) planes, as expected from a direct inspection from the flow forcing. However, the most intense structures of the \( \langle E_b(x,t) \rangle_T \) and \( \langle E_{b,\text{lin}}(x,t) \rangle_T \) fields do not coincide everywhere in that case (see location \( (\pi/2, \pi/2, 0) \) in Fig. 2(bottom), for example). Note also that the linear calculation overestimates the time averaged magnetic fluctuations, whatever the orientation of the applied field. Altogether it shows than one should be cautious when using average velocity fields in the calculation of magnetic induction, particularly if restricted to linear effects. The difference between the fields is probably linked to the large scale electromotive force due to turbulent motions. The influence of this force, as well as the large scale induction topology and its connection with the small scale fluctuations, will be reported in a forthcoming paper [20].

Figure 3 shows the temporal fluctuations of the induced field amplitude, \( |b(x,t)| \), probed inside the flow at two locations chosen from the previous topological observations, for \( B_0 \) along the \( \hat{x} \)-axis. This is equivalent to using local probes as in laboratory experiments. The intensity of the induced magnetic field has strong local fluctuations. The point at \( (0, \pi, 0) \) is in a region of strong mean induction, whereas the point at \( (0.6\pi, 0.6\pi, 0.6\pi) \) is at location of low mean induction (cf. Fig. 2(bottom)). We observe that, occasionally, the induced field gets larger than the applied field. In fact, if small amplitude fluctuations (about 10%) are induced over time intervals of the order of the diffusive time \( t_0 \), much larger variations (~300%) can be observed over long time periods, of the order of \( 10t_0 \). These observations are in excellent qual-

**FIG. 1:** Magnetic (solid line) and kinetic (dash line) energy spectra computed at \( t = 210 \) for RUN 1 with \( B_0 = 0.1 \hat{x} \).

**FIG. 2:** Topology of the local induced magnetic energy, averaged in time, when \( B_0 \) is applied along the \( \hat{z} \)-axis (top) and along the \( \hat{x} \)-axis (bottom) - in red: \( \langle E_b(x,t) \rangle_T \); in blue: \( \langle E_{b,\text{lin}}(x,t) \rangle_T \) - (see text). The isosurfaces are plotted at 80% of the maximum values of the fields: \( \max \langle E_b \rangle_T = 0.0056 \) and \( \max \langle E_{b,\text{lin}} \rangle_T = 0.0063 \) for \( B_0 = 0.1 \hat{z} \), and \( \max \langle E_b \rangle_T = 0.0041 \) and \( \max \langle E_{b,\text{lin}} \rangle_T = 0.0063 \) for \( B_0 = 0.1 \hat{x} \).

**FIG. 3:** Time traces of \( |b(x,t)| \), for \( B_0 = 0.1 \hat{x} \), at two fixed points. In blue: \( (0, \pi, 0) \), mean value \( \langle |b(x,t)| \rangle_T / B_0 = 0.92 \), fluctuation level \( |b(x,t)|_{\text{rms}} / B_0 = 0.28 \). In red: \( (0.6\pi, 0.6\pi, 0.6\pi) \) mean value \( \langle |b(x,t)| \rangle_T / B_0 = 0.44 \), fluctuation level \( |b(x,t)|_{\text{rms}} / B_0 = 0.19 \).
We have considered here the response to an externally applied field. The time behavior of magnetic field fluctuations is found in excellent agreement with experimental measurements. It has also revealed that the $1/f$ regime detected locally traces back to the global dynamics of the flow. Future work will analyze the contribution of turbulent fluctuations to the large scale magnetic field dynamics, and the influence of the magnetic Prandtl number on the threshold of the dynamo instability.

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