PROVING PARIKH’S THEOREM USING CHOMSKY-SCHÜTZENBERGER THEOREM

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Abstract. Parikh theorem was originally stated and proved in [Par]. Many different proofs of this classical theorems were produced then; our goal is to give another proof using Chomsky-Schützenberger representation theorem. We present the proof which doesn’t use any formal language theory tool at all except the representation theorem, just some linear algebra.

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1. Introduction

Recall that the context-free grammar is a 4-tuple $G = (N, \Sigma, P, S)$ where $N$ is a set of nonterminals, $\Sigma$ is an alphabet, $P$ is a collection of productions of kind $X \to \gamma$ for $X \in N$, $\gamma \in (N \cup \Sigma)^*$, and $S$ is an initial nonterminal. We say that $w \in \Sigma^*$ is derived in CFG $G$ if there exists a sequence $\{s_i\}_{i \in [1,N]}$, $s_i \in (N \cup \Sigma)^*$ such that $s_0 = S$, $s_N = w$ and every $s_i$ is obtained from $s_{i-1}$ via applying some production from $P(G)$. The set of all words $w \in \Sigma^*$ derived in $G$ is called the language generated by $G$ and is denoted by $L(G)$. The formal language $L \subset \Sigma^*$ is called context-free if $L = L(G)$ for some context-free grammar $G$.

Let $\Sigma = \{a_1, \ldots, a_k\}$. We introduce the Parikh map $\psi : \Sigma^* \to \mathbb{N}^k$:

$$\psi(w) = (\#_{a_1}(w), \ldots, \#_{a_k}(w)),$$

where $\#_x(w)$ is number of occurences of letter $x$ in word $w$. Clearly, this is a monoid homomorphism since $\psi(w_1w_2) = \psi(w_1) + \psi(w_2)$, $\psi(\epsilon) = 0$ where $0 = (0, \ldots, 0)$. For any language $L$ we define Parikh image of $L$ $\psi(L)$ by

$$\psi(L) = \{\psi(w) | w \in L\} \subset \mathbb{N}^k$$
Definition 1. A subset $S \subset \mathbb{N}^k$ is called linear if it’s a coset of a finitely generated submonoid of $\mathbb{N}^k$, i.e. there exist $v_0, v_1, \ldots v_N$ such that

$$S = \{ v_0 + \sum_{i=1}^{N} \lambda_i v_i \mid \forall i \in [1; N] \lambda_i \in \mathbb{N} \}$$

We denote this by $S = v_0 + \langle v_1, \ldots v_N \rangle$.

Definition 2. A subset $S \in \mathbb{N}^k$ is semilinear if it’s a union of finite number of linear subsets.

The following theorem is the classical one in the formal language theorem:

Theorem 1.1 (Parikh). For any context-free language $L$, $\psi(L)$ is semilinear.

The theorem was originally stated and proved by Rokhit Parikh in [Par]. Parikh’s strategy was to present all parse trees as a union of a finite number of classes each having one minimal tree and all other obtained from that minimal by insertion of a pump tree.

We’re gonna prove the theorem 1.1 using another classical theorem:

Theorem 1.2 (Chomsky-Schützenberger). Every context-free language $L \subset \Sigma^*$ admits the following presentation

$$L = h(Dyck_N \cap R),$$

where $Dyck_N$ is a language of balanced parentheses of $N$ types, $R$ is a regular language and $h : \{(1), \ldots (N)\}_N^* \rightarrow \Sigma^*$ is a language homomorphism.

Shamir tried to give a proof using that theorem, however, afterwards he regarded in [Gold] that his proof was fallacious. We, however, prefer much easier way to obtain our goal. First, we figure out that Dyck language and every regular language have semilinear Parikh images. Then we prove that if two languages have semilinear Parikh images then their intersection has semilinear image. Finally, we point out that homomorphism preserves semilinearity of Parikh images and thus the Parikh theorem is proved.

2. Regular and Dyck languages

The current subgoal is the prove the two following lemmas.

Lemma 2.1. For any $N$, $\psi(Dyck_N)$ is semilinear.

Lemma 2.2. For any regular $R$, $\psi(R)$ is semilinear.

Proof of the lemma 2.1. Consider the Dyck language alphabet $\Sigma = \{(1), \ldots (N)\}_N$. Then $\psi(Dyck_N) = \langle \{d_i \mid i \in [1; N]\} \rangle$, where $d_i = \begin{pmatrix} 0, \ldots, 0, 1, 1 \end{pmatrix}_{2(i-1)} \begin{pmatrix} 0, \ldots, 0, 1, 1, 0, \ldots, 0 \end{pmatrix}_{N-2i}$

Proof of the lemma 2.2. Recall from [Koz] that regular languages can be defined recursively:

- $\emptyset, \epsilon, \{a_i\}$ for every $a_i \in \Sigma$ are regular;
- if $A, B$ are regular languages then $A + B, AB$ and $A^*$ are regular.
Of course \( \phi(\emptyset) = \emptyset, \phi(\epsilon) = 0 \) and \( \phi(\{a_i\}) = \{(0, \ldots, 0, 1, 0, \ldots, 0)\} \), which are semilinear. So we have to prove that if \( \psi(A) \) and \( \psi(B) \) are semilinear, then \( \psi(A + B), \psi(AB) \) and \( \psi(A^*) \) are.

- \( \psi(A + B) = \psi(A) \cup \psi(B) \), so \( \psi(A + B) \) is a finite union of linear sets, hence it’s semilinear;
- \( \psi(AB) = \psi(A) + \psi(B) \), a Minkowski sum of Parikh images of \( A \) and \( B \); because of formula [\( \square \)], we see that \( \psi(AB) \subseteq \psi(A) + \psi(B) \), and every element \( v \in \psi(A) + \psi(B) \) can be presented as Parikh image of concatenation of two words from \( A \) and \( B \) — we just need to express

\[
\begin{align*}
v &= v_1 + v_2, \ \exists w_1 \in A, w_2 \in B \ \\
\psi(w_1) &= v_1, \psi(w_2) &= v_2 \ \\
\psi(A + B) \text{ is semilinear because of}
\end{align*}
\]

\[
\begin{align*}
\left( v_0 + \sum_{i \in [1:N]} v_i \right) + \left( v_0' + \sum_{i \in [1:M]} v_i' \right) &= \left( v_0 + v_0' + \sum_{i \in [1:N]} v_i \right) \\
\text{and} \quad (\bigcup_{i} A_i) + (\bigcup_{j} B_j) &= \bigcup_{i,j} (A_i + B_j)
\end{align*}
\]

- If \( \psi(A) = \bigcap_{i=0}^{N} S_i \) where \( S_i = v_0^{(i)} + \sum_{j=1}^{i-1} v_0^{(j)} \) then \( \psi(A^*) = \sum_{i=1}^{N} v_0^{(i)} + S_1 + \ldots + S_N \). Indeed, every \( v \in \sum_{i=1}^{N} v_0^{(i)} + S_1 + \ldots + S_N \) can be presented by a word \( w_1 \ldots w_N u_1 \ldots u_N \) where \( \psi(w_i) \in S_i \) and \( \psi(u_i) = C_i v_0^{(i)} \) and clearly \( \psi(A^*) \subseteq \sum_{i=1}^{N} v_0^{(i)} + S_1 + \ldots + S_N \). Since \( \psi(A^*) \) is a Minkowski sum of semilinear sets it’s linear.

\[\square\]

### 3. Semilinearity of Intersections

**Lemma 3.1.** If \( S_1, S_2 \subseteq \mathbb{N}^k \) are semilinear, then \( S_1 \cap S_2 \) is semilinear.

**Proof.** The result is a classical one, its fully linear algebraic proof can be found in [LW]. However, we present yet another proof. We prove that the intersection of two linear sets is semilinear, then the result follows immediately. Let \( S_1 = v_0 + \sum_{i=1}^{N} v_i \) and \( S_2 = v_0' + \sum_{i=1}^{M} v_i' \); consider two \( \mathbb{Q} \)-cones \( \langle v_1, \ldots, v_N \rangle \) and \( \langle v_1', \ldots, v_M' \rangle \); their intersection is a finitely generated \( \mathbb{Q} \)-cone spanned by \( \tilde{s}_1, \ldots, \tilde{s}_m \). Take

\[
s_i = C_i \tilde{s}_i, \ C_i = \min \{ \lambda \in \mathbb{N} | \exists \lambda_j, \mu_l | \lambda \tilde{s}_i = \sum \lambda_j v_j = \sum \mu_l v_l' \}
\]

Let’s define \( s \in \mathbb{N}^k \) as *substractible* if \( s = \sum \lambda_i s_i \) for natural \( s_i \). For \( x, y \in S_1 \cap S_2 \) we call \( x \leq y \) iff \( y = x + s \) for some *substractible* \( s \). For every \( s_i \) and \( S \subseteq \mathbb{N}^k \) we may define \( W_i(S) = \{ x \in S | x - s \in \mathbb{N} \} \) and \( X_i(S) \) as the maximal sum of coordinates of points in \( W_i(S) \). Every \( X_i(S_1) \) and \( X_i(S_2) \) actually exists because \( X_i(S_1), X_i(S_2) \leq m | \max_{j \in [1:k]} (s_{ij}) | \);
thus $X_i(S_1 \cap S_2)$ exists. So there exist only finitely many $\leq$-minimal points since their sum of coordinates can be bounded by $m \sum_{i=1}^{m} X_i(S_1 \cap S_2)$. For every point in intersection of linear sets we can find the minimal one; and so
\[(4) \quad S_1 \cap S_2 = \bigcup_{x \text{ is } \leq\text{-minimal}} (x + (s_1, \ldots, s_m))\]
and hence it’s semilinear.

4. PROOF OF THE MAIN THEOREM

**Lemma 4.1.** For every homomorphism $\phi : \Sigma^* \to \Gamma^*$ semilinearity $\psi(L)$ implies semilinearity of $\psi(\phi(L))$.

**Proof.** Homomorphism $\phi : \Sigma^* \to \Gamma^*$ induces the linear map
$$\Phi : \mathbb{N}^{|\Sigma|} \to \mathbb{N}^{|\Gamma|}, \quad (0, \ldots, 0, 1, 0, \ldots, 0) \mapsto (#_1(\Phi(a_i)), \ldots, #_{|\Gamma|}(\Phi(a_i)))$$
so we just linearly change generators of linear subsets and preserve the semilinearity. □

**Proof of the main theorem.** If $L$ is context-free then it can be presented as $h(Dyck_N \cap R)$. According to lemmas §2.1 and §2.2 $Dyck_N$ and $R$ have semilinear Parikh images so their intersection does and so is it homomorphic image which is $L$. □

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I came up with this when I was preparing a home task for my MIPT second-year undergraduate students; I wanted to give a problem which could use Parikh or Chomsky-Schützenberger theorem. Then I realized that this very proof was never met in literature before. I gratefully thank those brilliant sharp-minded kids, especially from group 673, who I teach with such great delight and wish they will discover something quite powerful and elegant in the future.

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