BLOCKS WITH TRANSITIVE FUSION SYSTEMS

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Abstract. Suppose that all nontrivial subsections of a \( p \)-block \( B \) are conjugate (where \( p \) is a prime). By using the classification of the finite simple groups, we prove that the defect groups of \( B \) are either extraspecial of order \( p^3 \) with \( p \in \{3, 5\} \) or elementary abelian.

1. Introduction

Let \( p \) be a prime, and let \( \mathcal{F} \) be a saturated fusion system on a finite \( p \)-group \( P \) (cf. [1] and [8]). We call \( \mathcal{F} \) transitive if any two nontrivial elements in \( P \) are \( \mathcal{F} \)-conjugate. In this case, \( P \) has exponent \( \exp(P) \leq p \), and \( \text{Aut}_\mathcal{F}(P) \) acts transitively on \( Z(P) \setminus \{1\} \). This paper is motivated by the following:

**Conjecture 1.1.** (cf. [23]) Let \( \mathcal{F} \) be a transitive fusion system on a finite \( p \)-group \( P \) where \( p \) is a prime. Then \( P \) is either extraspecial of order \( p^3 \) or elementary abelian.

Moreover, if \( P \) is extraspecial of order \( p^3 \) then results by Ruiz and Viruel [26] imply that \( p \in \{3, 5, 7\} \). Note that the conjecture is trivially true for \( p = 2 \) since groups of exponent 2 are abelian. Thus Conjecture 1.1 is only of interest for \( p > 2 \). The aim of this paper is to prove the conjecture above for saturated fusion systems coming from blocks.

**Theorem 1.2.** Let \( p \) be a prime, and let \( B \) be a \( p \)-block of a finite group \( G \) with defect group \( P \). If the fusion system \( \mathcal{F} = \mathcal{F}_P(B) \) of \( B \) on \( P \) is transitive then \( P \) is either extraspecial of order \( p^3 \) or elementary abelian.

If \( P \) is extraspecial of order \( p^3 \) then the results in [26] and [20] imply that \( p \in \{3, 5\} \). We call a block \( B \) with defect group \( P \) and transitive fusion system \( \mathcal{F}_P(B) \) fusion-transitive. Whenever \( B \) has full defect then the theorem is a consequence of the results.

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in [23]. In our proof of the theorem above, we will make use of the classification of the finite simple groups.

2. Saturated fusion systems

Proposition 2.1. Let $p$ be a prime, and let $\mathcal{F}$ be a transitive fusion system on a finite $p$-group $P$ where $|P| \geq p^4$. Suppose that $P$ contains an abelian subgroup of index $p$. Then $P$ is abelian.

Proof. We assume the contrary. Then $p > 2$.

Suppose first that $P$ contains two distinct abelian subgroups $A, B$ of index $p$. Then $AB = P$, $A \cap B \subseteq Z(P)$ and $|P : A \cap B| = p^2$. Since $P$ is nonabelian we conclude that $|P : Z(P)| = p^3$. Thus $1 \neq P' \subseteq Z(P)$. Since $\text{Aut}_\mathcal{F}(P)$ acts transitively on $Z(P) \setminus \{1\}$, we conclude that $P' = Z(P)$. Hence there are $x, y \in P$ such that $P = \langle x, y \rangle$. Then $P' = \langle [x, y] \rangle$ (cf. III.1.11 in [17]); in particular, we have $|P'| = p$ and $|P| = p^3$, a contradiction.

It remains to consider the case where $P$ contains a unique abelian subgroup $A$ of index $p$. Let $Z$ be a subgroup of order $p$ in $Z(P)$, and let $B$ be an arbitrary subgroup of order $p$ in $A$. By transitivity, there is an isomorphism $\phi : B \to Z$ in $\mathcal{F}$. By definition, $Z$ is fully $\mathcal{F}$-normalised. Thus, by Proposition 4.20 in [8], $Z$ is also fully $\mathcal{F}$-automised and receptive. Hence $\phi$ extends to a morphism $\psi : N_\phi \to P$ in $\mathcal{F}$. Since $|B| = p$ we have

$$A \subseteq N_P(B) = C_P(B) \subseteq N_\phi$$

(cf. p. 99 in [8]). Since $\psi(A)$ is also an abelian subgroup of index $p$ in $A$ we conclude that $\psi(A) = A$. Thus $\psi|A \in \text{Aut}_\mathcal{F}(A)$, and $\psi|A$ maps $B$ to $Z$. This shows that $\text{Aut}_\mathcal{F}(A)$ acts transitively on the set of subgroups of order $p$ in $A$.

In the following, we view $A$ as a vector space over $\mathbb{F}_p$ and $G := \text{Aut}_\mathcal{F}(A)$ as a subgroup of $\text{GL}(A)$. If $S$ denotes the group of scalar matrices in $\text{GL}(A)$ then $H := GS$ is a transitive subgroup of $\text{GL}(A)$. The transitive linear groups were classified by Hering (cf. [16] or Remark XII.7.5 in [18]). We are going to use the list in Theorem 15.1 of [27].

Before we do this, we observe the following. By the uniqueness of $A$, $A$ is fully $\mathcal{F}$-automised, i.e. $P/A = N_P(A)/C_P(A) \in \text{Syl}_p(\text{Aut}_\mathcal{F}(A))$. Thus $G = \text{Aut}_\mathcal{F}(A)$ and $H = GS$ both have a Sylow $p$-subgroup of order $p$.

Now we write $|A| = p^n$ and go through the list in Theorem 15.1 of [27]:

(i) $H \subseteq \Gamma L_1(p^n)$; in particular, $|H|$ divides $|\Gamma L_1(p^n)| = n(p^n - 1)$.

In this case we can identify $A$ with the finite field $L := \mathbb{F}_{p^n}$. Moreover, $P$ is the semidirect product of $L$ with $B = \langle \beta \rangle$ where $\beta$ is a field automorphism of $L$. For $x \in L$, we have $x\beta \in P$ and

$$1 = (x\beta)^p = x\beta x\beta \ldots x\beta = x\beta(x)\beta^2(x) \ldots \beta^{p-1}(x) = N_K^L(x)$$
where $K$ is the fixed field of $\beta$. However, it is known that $N_K^L(L) = K$, a contradiction.

(ii) $n = km$ where $k \geq 2$ and $\text{SL}_k(p^m) \leq H$.

Since the Sylow $p$-subgroups of $H$ have order $p$, we conclude that $m = 1$ and $k = 2$. Then $n = 2$ and $|P| = p^2$, a contradiction.

(iii) $n = km$ where $k \geq 4$ is even and $\text{Sp}_k(p^m)' \leq H$.

Since $p > 2$ we have $\text{Sp}_k(p^m)' = \text{Sp}_k(p^m)$. Thus $\text{Sp}_k(p^m)$ has a Sylow $p$-subgroup of order $p^{k^2/4} \geq p^4$, a contradiction.

(iv) $n = 6m$, $p = 2$ and $G_2(2^m)' \leq H$.

This case is impossible as $p > 2$.

(v) $n = 2$ and $p \in \{5, 7, 11, 19, 23, 29, 59\}$.

Then $|P| = p^3$ which is again a contradiction.

(vi) $n = 4$, $p = 2$ and $H \cong \mathfrak{A}_7$.

This case is also impossible as $p > 2$.

(vii) $n = 4$, $p = 3$ and $H$ is one of the groups in Table 15.1 of [27].

In this case we have $|P| = 3^5 = 243$. Then Proposition 15.12 in [27] leads to a contradiction.

(viii) $n = 6$, $p = 3$ and $H \cong \text{SL}_2(13)$.

In this case we have $|P| = 3^7 = 2187$. However, one can check that $P$ has exponent 9 in this case, a contradiction.

$\square$

**Proposition 2.2.** Let $P$ be a nonabelian $p$-group with a transitive fusion system. Then $P$ is indecomposable (as a direct product).

**Proof.** Let $P = N_1 \times \cdots \times N_k$ be a decomposition into indecomposable factors $N_i \neq 1$. Assume by way of contradiction that $k \geq 2$. Since $P$ carries a transitive fusion system we have

$$Z(N_1) \times \cdots \times Z(N_k) = Z(P) \subseteq P' = N_1' \times \cdots \times N_k'.$$

Let $1 \neq x \in Z(N_i)$. By hypothesis there exists $\alpha \in \text{Aut}(P)$ such that $\alpha(x) \in Z(P) \setminus (Z(N_1) \cup \ldots \cup Z(N_k))$. By the Krull-Remak-Schmidt Theorem (see Satz I.12.5 in [17]) there is a normal automorphism $\beta$ of $P$ such that $\beta(N_i) = \alpha(N_i)$ for some $i \in \{1, \ldots, k\}$. In particular, there is $y \in Z(N_i)$ such that $\beta(y) = \alpha(x)$. By Hilfssatz I.10.3 in [17], for every $g \in P$ there is a $z_g \in Z(P)$ such that $\beta(g) = g z_g$. Obviously the map $P \to Z(P)$, $g \mapsto z_g$, is a homomorphism. Since $Z(N_i) \subseteq N_i'$, we obtain $z_y = 1$. This gives the contradiction $\alpha(x) = \beta(y) = y \in Z(N_i)$.

$\square$

**Proposition 2.3.** Let $P = \prod_{i=1}^m P_{i}$ where $P_i = C_{p_i} \times C_{p_i} \times \cdots \times C_p$ (i factors in the wreath product) and $a_i \in \mathbb{N}_0$, $i \in \mathbb{N}$ for $i \in \mathbb{N}$. Moreover, let $U$ be a normal subgroup of $P$ such that $P/U$ is cyclic, and let $Z$ be a cyclic subgroup of $Z(U)$. Suppose that $R := U/Z$ supports a transitive fusion system. Then $R$ has order $p^4$ or is elementary abelian.

**Proof.** We assume the contrary. Then $|R| \geq p^4$ and $p > 2$. 


Suppose first that $r_j > 1$ for some $j > 1$. Since $p > 2$, $P'$ contains a subgroup isomorphic to $C_{p^j} \times C_{p^j}$. Since $P' \subseteq U$ we conclude that $\exp(R) \geq p^2$, a contradiction.

Thus $r_j = 1$ for $j > 1$, and $P_j$ is the iterated wreath product of $j$ copies of $C_p$ in this case.

Suppose next that $a_j > 0$ for some $j \geq 3$. Since $p > 2$, $P'$ contains a subgroup isomorphic to $P_{j-1} \times P_{j-1}$. By Satz III.15.3 in [17], $P_{j-1}$ has exponent $p^{j-1} \geq p^2$. Since $P' \subseteq U$ we conclude that $\exp(R) \geq p^2$, a contradiction again.

Thus $P = P_{a_1}^{a_1} \times P_{a_2}^{a_2}$ where $P_1 = C_{p^j}$, and $P_2 = C_p \wr C_p$. If $a_2 \leq 1$ then $P$ and $R$ contain abelian subgroups of index $p$. In this case Proposition 2.2 gives a contradiction.

Hence we may assume that $a_2 \geq 2$. Let $\pi : P \rightarrow P_2^{a_2}$ be the relevant projection. Since $\exp(P_2) = p^2$ we cannot have $\pi(U) = P_2^{a_2}$. On the other hand, $P_2/P_2'\pi$ is elementary abelian. Since $P_2^{a_2}/\pi(U)$ is cyclic, $\pi(U)$ is a maximal subgroup of $P_2^{a_2}$.

Let $\pi_1 : P_2^{a_2} \rightarrow P_2^{a_2-1}$ be the projection onto the direct product of the first $a_2 - 1$ copies of $P_2$, and let $\pi_2 : P_2^{a_2} \rightarrow P_2^{a_2-1}$ be the projection onto the direct product of the last $a_2 - 1$ copies of $P_2$.

Now suppose that $a_2 \geq 3$. Then an argument similar to the one above shows that $\pi_1(\pi(U))$ is a maximal subgroup of $P_2^{a_2-1} = \pi_1(P_2^{a_2})$. Thus $\ker(\pi_1) \subseteq \pi(U)$ and, similarly, $\ker(\pi_2) \subseteq \pi(U)$. Thus $\pi(U)$ contains a subgroup isomorphic to $P_2^{a_2}$. Hence $\exp(R) \geq p^2$, a contradiction.

We are left with the case $a_2 = 2$, i.e. $P = A \times P_2 \times P_3$ where $A = P_1^{a_1} \cong C_{p^j}'$ is abelian. Since $\pi(U)$ is a maximal subgroup of $P_2 \times P_3$, we see that $A \times \pi(U)$ is a maximal subgroup of $P$. Let $x \in P$ such that $P = U\langle x \rangle$. Then $U\langle x^p \rangle \subseteq A \times \pi(U)$.

Since $|P : U\langle x \rangle| \leq p$ we conclude that $U\langle x^p \rangle = A \times \pi(U)$. Note that $x^p \in \bar{U}(P) \subseteq Z(P)$.

Suppose that $\exp(A) > p$, and choose an element $a \in A$ of maximal order. We write $x = x_1 x_2$ with $x_1 \in A$ and $x_2 \in P_2^2$, we write $a = u x^p$ with $u \in U$ and $i \in \mathbb{Z}$, and we write $u = u_1 u_2$ with $u_1 \in A$ and $u_2 \in P_2^2$. Then $a^p = u^p x^{xp} = u_1^p x_1^{xp_1} u_2^p x_2^{xp_2} = u_1^p x_1^{xp_1} u_2^p$. We conclude that $u_2^p = 1$ and $a^p = u_1^p x_1^{xp_1}$. Thus $p < \exp(A) = |\langle a \rangle| = |\langle u_1 \rangle| = |\langle u \rangle|$, and $1 \neq u^p \in U \cap A$.

By Aufgabe III.15.36 in [17], the elements of order $1$ or $p$ form a union of two maximal subgroups. Thus $P_2^2$ contains $p^{2p-2}(2p - 1)^2 < p^{2p+1}$ elements of order $1$ or $p$. Hence $\pi(U)$ contains elements of order $p^2$; in particular, $\bar{U}(U)$ is noncyclic. Since $\bar{U}(U) \subseteq Z$, this is a contradiction.

This contradiction shows that $\exp(A) \leq p$, i.e. $P = A \times P_2 \times P_2$ where $A$ is elementary abelian. Hence $P/P'$ is elementary abelian. Since $P/U$ is cyclic we conclude that $U$ is a maximal subgroup of $P$. Thus $U = A \times \pi(U)$ and $\bar{U}(U) \subseteq \pi(U)$.

Since $\pi(U)$ contains elements of order $p^2$, we have $1 \neq \bar{U}(U) \subseteq Z$. On the other hand, Satz III.15.4 in [17] implies that $Z(U)$ is elementary abelian. Thus $|Z| = p$ and
Z = \mathcal{U}(U) \subseteq \pi(U). Since R supports a transitive fusion system we have
\[ AZ/Z \subseteq Z(U)/Z \subseteq Z(R) \subseteq R' = U'/Z = \pi(U)'Z/Z \subseteq \pi(U)/Z. \]

Therefore \( A = 1 \), i.e. \( P = P_2 \times P_2 \). Recall that \( U \) is a maximal subgroup of \( P \) and that \( \pi_1, \pi_2 : P \to P_2 \) denote the two projections. Without loss of generality we have \( \pi_1(U) = P_2 \). Since \( \mathcal{U}(U) \) is cyclic, \( K_1 := \text{Ker}(\pi_1) \) has order \( p^p \) and exponent \( p \).

If \( \pi_2(U) \neq P_2 \) then \( U = P_2 \times \pi_2(U) \) and \( \exp(\pi_2(U)) = p \). Thus \( Z = \mathcal{U}(U) \subseteq P_2 \times 1 \) and \( R \cong P_2/Z \times \pi_2(U) \), a contradiction to Proposition 2.2.

Thus we must also have \( \pi_2(U) = P_2 \). Then also \( K_2 := U \cap \text{Ker}(\pi_2) \) has order \( p^p \) and exponent \( p \). Moreover, we have \( K_1 \times K_2 \subseteq U \).

We may choose elements \( x, y \in U \) such that \( \pi_1(x) \) and \( \pi_2(x) \) have order \( p^2 \). Since \( \langle x^p \rangle = Z = \langle y^p \rangle \) we see that \( \pi_2(x) \) and \( \pi_1(y) \) have order \( p^2 \). However, we may choose \( y \) such that \( yK_1 \) contains an element \( y' \) such that \( \pi_2(y') \) has order \( p \). Since \( \pi_1(y) = \pi_1(y') \) still has order \( p^2 \), we have a final contradiction. \( \square \)

3. Blocks

We now present the proof of Theorem 1.2.

Proof. Suppose that the result is false. Then \( P \) is nonabelian with \( |P| \geq p^4 \) and \( p > 2 \).

By [1, Proposition IV.6.3] we may assume that \( B \) is quasiprimitive. This means that, for any normal subgroup \( H \) of \( G \), \( B \) covers a unique \( p \)-block of \( H \).

Now let \( H \) be a normal subgroup of \( G \), and let \( b \) be the unique \( p \)-block of \( H \) covered by \( B \). Suppose that \( P \cap H = 1 \). (This is satisfied, for example, whenever \( H \) is a \( p' \)-subgroup.) Then \( b \) has defect zero. By Clifford theory, there exist a finite group \( G^* \), a central \( p' \)-subgroup \( H^* \) of \( G^* \), and a \( p \)-block \( B^* \) of \( G^* \) with defect group \( P^* \cong P \) such that \( \mathcal{F}_{P^*}(B^*) \) is equivalent to \( \mathcal{F} \). Thus we may replace \( G \) by \( G^* \) and \( B \) by \( B^* \).

Repeating the argument above we may therefore assume that every normal subgroup \( H \) of \( G \) with \( P \cap H = 1 \) is central. In particular, we have \( O_{p'}(G) \subseteq Z(G) \).

It is well-known that \( M := O_{p'}(G) \subseteq P \). Suppose first that \( M \neq 1 \). Since \( \mathcal{F} \) is transitive this implies \( M = P \). Then \( \Phi(P) \) is a normal subgroup of \( G \) and properly contained in \( P \). Since \( \mathcal{F} \) is transitive, we must have \( \Phi(P) = 1 \). Thus \( P \) is elementary abelian in this case.

Hence, in the following, we may assume that \( O_{p'}(G) = 1 \). Then \( F(G) = O_{p'}(G) = Z(G) \). Moreover, the layer \( E(G) \) is nontrivial. Let \( b \) be the unique \( p \)-block of \( E(G) \) covered by \( B \). Then \( b \) has defect group \( P \cap E(G) \neq 1 \). Since \( B \) is transitive, this implies that \( P \subseteq E(G) \).

Let \( L_1, \ldots, L_n \) denote the components of \( G \). Then \( E(G) = L_1 \cdots L_n \) is a central product. For \( i = 1, \ldots, n \), the unique \( p \)-block \( b_i \) of \( L_i \) covered by \( b \) has defect group \( P_i := P \cap L_i \neq 1 \). Moreover, we have \( P = P_1 \times \cdots \times P_n \). Since \( \mathcal{F} \) is transitive, this
implies that $n = 1$. Thus $E(G) = L_1 =: L$ is quasisimple, and $G/Z(G)$ is isomorphic to a subgroup of $\text{Aut}(L)$.

If $|P| = p^4$ then Proposition 15.14 in [27] gives a contradiction. Thus we may assume that $|P| \geq p^6$; in particular, $|L|$ is divisible by $p^6$. If $P$ is a Sylow $p$-subgroup of $G$ then the results of [23] imply our theorem. Hence we may assume that $|G|$ is divisible by $p^6$.

We now make use of the classification of the finite simple groups and discuss the various possibilities for the simple group $F^*(G)/Z(G) \cong L/Z(L)$. Since $F$ is transitive we have $C_L(u) \cong C_L(v)$ for any $u, v \in P \setminus \{1\}$. This will be a very useful fact.

It can be checked with GAP [13] that $L/Z(L)$ cannot be a sporadic simple group. Similarly, $L/Z(L)$ cannot be a simple group with an exceptional Schur multiplier.

Suppose that $L = \mathfrak{A}_n$ is an alternating group. Then $P$ is a defect group of a $p$-block of $\mathfrak{S}_n$. Hence $P$ is also a defect group of a $p$-block of the symmetric group $\mathfrak{S}_n$. Thus $P$ is a direct product of (iterated) wreath products of groups of order $p$. Since $C_p \wr C_p$ has exponent $p^2$ we conclude that $P$ is a direct product of groups of order $p$, and the result follows in this case.

Suppose next that $L = \hat{\mathfrak{A}}_n$ is the 2-fold cover of $\mathfrak{A}_n$. We may assume that $b$ is a faithful block of $\hat{\mathfrak{A}}_n$. In this case the defect groups of $b$ have a similar structure as those in $\mathfrak{A}_n$ (cf. [24, Theorem 5.8.8]), so we are done here by the same argument.

Suppose now that $L/Z(L)$ is a group of Lie type in characteristic $p$. Then the $p$-block $b$ of $L$ has full defect, i.e. $P$ is a Sylow $p$-subgroup of $L$. Since $F$ is transitive, every nontrivial element $u \in P$ is conjugate in $G$ to an element $v \in Z(P)$. Thus $|L : C_L(u)| = |L : C_L(v)|$ is not divisible by $p$. Therefore the results in [25] imply that $P$ is abelian.

Finally suppose that $L/Z(L)$ is a group of Lie type in characteristic $r \neq p$. First we deal with the exceptional groups of Lie type. Let $S \in \text{Syl}_p(L)$. By §10.1 in [14], $S$ contains an abelian normal subgroup $N$ such that $S/N$ is isomorphic to a subgroup of the Weyl group of $L/Z(L)$. If $|S/N| \leq p$, then Proposition 2.1 gives a contradiction. This already implies the claim for $p \geq 7$. Now let $p = 5$. Then by the same argument we may assume that $L/Z(L) \cong E_8(q)$ where $q \equiv \pm 1 \pmod{5}$. This case will be handled in Section 6. Now let $p = 3$. Here we need to discuss the following groups: $F_4$, $E_6$, $E_7$ and $E_8$. For $L/Z(L) \cong F_4(q)$ we have $|P| \leq p^6$ and the result follows by Proposition 15.13 in [27]. The remaining cases will be discussed in Section 6.

We may therefore assume that $L/Z(L)$ is a classical group. In this case our theorem follows from the results of the next section. \hfill $\square$

4. Classical Groups in non-describing characteristic

We keep the notation of the previous section. We suppose in this section that $L/Z(L)$ is a simple group of Lie type in characteristic $r$, $r \neq p$. Let $q$ be a power of $r$. Suppose that $L = L^F/Z$, where $L$ is a simple simply connected algebraic group.
defined over an algebraic closure $\overline{\mathbb{F}}_q$ of a field $\mathbb{F}_q$ of $q$ elements, $F: L \rightarrow L$ a Frobenius morphism with respect to an $\mathbb{F}_q$-structure on $L$ and $Z$ is a central subgroup of $L^F$. Note that by the classification of finite simple groups, we may assume that if $q$ is a power of 2, then $L$ is not of type $C_n$. Let $b$ be the block of $L^F$ dominating $b$ and $\tilde{P}$ be a defect group of $\tilde{b}$ such that $\tilde{P}Z/Z = P$.

We define groups $H$ as follows. If $L/Z(L) = B_n(q)$, then $H = SO_{2n+1}^+(\overline{\mathbb{F}}_q)$. If $L/Z(L) = C_n(q)$, then $H = Sp_{2n}^+(\overline{\mathbb{F}}_q)$. If $L/Z(L) = D_n^+(q)$, then $H = SO_{2n}^+(\overline{\mathbb{F}}_q)$. Here, if $q$ is a power of 2 and $L$ is of type $B_n$, then by $SO_{2n}^+(\overline{\mathbb{F}}_q)$ we mean the adjoint simple group of type $B_n$. If $q$ is a power of 2 and if $L$ is of type $D_n$, then by $SO_{2n}^+(\overline{\mathbb{F}}_q)$ we mean the simple algebraic group of type $D_n$ corresponding to the root datum $(X, \Phi, Y, \Phi^Y)$ for which the fundamental roots are $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n$ and $X = \{ \sum_{i=1}^n a_ie_i : a_i \in \mathbb{Z} \}$ for an orthonormal basis, $e_1, e_2, \ldots, e_n$, of $n$-dimensional Euclidean space. We may and will assume that $H$ is an $F$-stable quotient of $L$.

**Proposition 4.1.** Suppose that $p$ is an odd prime and $L/Z(L)$ is a classical group in non-describing characteristic different from triality $D_4$. Suppose that $B$ is a fusion-transitive block with $P$ of order at least $p^5$. Then $P$ is abelian.

**Proof.** Suppose that $L/Z(L)$ is the projective special linear group $PSL_n(q)$, so $L = SL_n(\overline{\mathbb{F}}_q)$ and $L = SL_n(q)$. Let $D$ be a defect group of a block of $GL_n(q)$ covering $\tilde{b}$ such that $\tilde{P} = D \cap SL_n(q)$. By the results of Fong and Srinivasan on blocks of finite general linear groups [12, Theorem (3C)], $D$ is isomorphic to the Sylow $p$-subgroup of a direct product of general linear groups over finite extensions of $\mathbb{F}_q$. Since $Z(L)$ and $D/\tilde{P}$ are cyclic, the claim follows from Proposition 2.3. The case that $L/Z(L)$ is the projective special unitary group can be handled similarly.

Now consider the case that $L/Z(L)$ is of type $B$, $C$ or $D$. Then $\tilde{P}$ is a defect group of $L^F$. Let $1 \neq z \in Z(\tilde{P})$. Since $p$ is odd, $C_L(z)$ is a Levi subgroup of $L$. For any subset $A$ of $L$, denote by $\overline{A}$ the image of $A$ under the isogeny from $L$ onto $H$ and denote by $U$ the kernel of the isogeny. Since $U$ is a central 2-subgroup of $L$, $C_L(z) = C_H(\overline{z})$.

The group $C_H(\overline{z})$ is a direct product

$$C_H(\overline{z}) = H_0 \times \cdots \times H_r,$$

where $H_0$ is either the identity or a classical group and for $i \geq 1$, $H_i$ is a direct product of general linear groups with $F$ transitively permuting the factors. This follows easily from the standard description of the root datum of $H$. So,

$$C_H(\overline{z})^F = H_0^F \times \cdots \times H_r^F,$$

where $H_i^F$ is a finite general linear or unitary group for $i \geq 1$ and $H_0^F$ is a finite classical group (possibly the identity).
Let $L_i$ be the inverse image in $C_{L}(z)$ of $H_i$, $0 \leq i \leq r$. Then $L_i$ is a normal $F$-stable subgroup of $C_{L}(z)$, $C_{L}(z) = L_0 \cdots L_r$, and

$$[L_i, L_0 \cdots L_{i-1} L_{i+1} \cdots L_r] \leq L_i \cap (L_0 \cdots L_{i-1} L_{i+1} \cdots L_r) = U.$$

We claim that $\bar{L}^F_i$ is a normal subgroup of $H^F_i$ of 2-power index. Indeed, let $M$ be the inverse image in $L_i$ of $H^F_i$. Then $M$ is a $F$-stable subgroup since $U$ is $F$-stable. Further, $[M, F] \leq U$. Since $Z$ is central in $M$, the map $M \to U$ defined by $x \to x^{-1} F(x)$ is a group homomorphism. The kernel of this map is $L_i^F$ whence $L_i^F$ is a normal subgroup of $M$ and the index of $L_i^F$ in $M$ divides $|U|$. The claim follows since $U$ is a 2-group.

The claim implies that $L_0^F \cdots L_r^F$ is a normal subgroup of 2-power index of $C_L(z)^F$. So, $\tilde{P}$ is a defect group of $L_0^F \cdots L_r^F$. The commutator relationship given above then implies that $\tilde{P}$ is a direct product $P_0 \cdots P_r$, where $P_i$ is a defect group of $L_i^F$, $0 \leq i \leq r$. By Proposition 2.2, $\tilde{P} = P_i$ for some $i$, $1 \leq i \leq r$. Since $z$ is central in $C_L(z)$, $i \geq 1$ and $H^F_i$ is a general linear or unitary group with a central $p$-element.

Let $R = P \cap [L_i, L_i]^F$, a defect group of $[L_i, L_i]^F$. By suitably replacing $P$ by an $L_i^F$-conjugate, we may assume that the relevant block of $[L_i, L_i]^F$ is $\tilde{P}$-stable and hence that $\tilde{P}$ is a defect group of $[L_i, L_i]^F P$.

The isogeny $L_i \to H_i$ restricts to an isogeny $[L_i, L_i] \to [H_i, H_i]$ with kernel $U \cap [L_i, L_i]$. However $[H_i, H_i]$ is a simply connected semisimple group, being the direct product of special linear groups. Thus, $U \cap [L_i, L_i] = 1$ and the restriction of the isogeny to $[L_i, L_i]$ is an abstract group isomorphism from $[L_i, L_i]$ to $[H_i, H_i]$ which commutes with $F$. Consequently, $[L_i, L_i]^F \cong [H_i, H_i]^F$. Also, $U \cap [L_i, L_i]^F P = 1$ and the induced map $[L_i, L_i]^F \tilde{P} \to H_i^F$ is injective. Thus $\overline{\tilde{P}} \cong \tilde{P} \cong P$ is a defect group of $[L_i, L_i]^F \overline{\tilde{P}} \cong [H_i, H_i]^F \overline{\tilde{P}}$. Since $H_i^F$ is a finite general linear or unitary group, the result now follows from [12, Theorem (3C)] and Proposition 2.3 in the same way as for the case that $L/Z(L)$ is a projective special linear or unitary group.

5. On $A_{p-1}$-components

Lemma 5.1. Suppose that $p$ is an odd prime and let $G$ be a finite group isomorphic to one of the groups $SL_p(q)$ or $SU_p(q)$ for some prime power $q$ not divisible by $p$. Let $U$ be a non-abelian $p$-subgroup of $G$. Then $U$ contains a normal abelian subgroup $U_0$ of index $p$ such that any element of $U \setminus U_0$ has order $p$. If $|U| \geq p^{p+1}$, then $U_0$ contains an element of order $p^2$.

Proof. First, consider the case that $G$ is special linear or unitary. By replacing $q$ if necessary by some power we may assume that $U \leq SL_p(q)$ and $p$ divides $q - 1$. Let $S_0$ be the Sylow $p$-subgroup of the group of diagonal matrices of $SL_p(q)$ and let $\sigma$ be a non-diagonal, monomial matrix in $SL_p(q)$ of order $p$. Then $S = \langle S_0, \sigma \rangle$ is a Sylow $p$-subgroup of $SL_p(q)$, $S_0$ is normal in $S$, abelian, of index $p$ in $S$, rank $p - 1$ and any element of $S$ not in $S_0$ has order $p$. Let $U_0 = U \cap S_0$. Then $U_0$ has index at most $p$ in
U. On the other hand, since U is non-abelian and S_0 is abelian, U is not contained in U_0. Thus U_0 has index p in U, proving the first assertion. Now suppose that U has exponent p. Then U_0 is elementary abelian. On the other hand, U_0 \leq S_0 and the p-rank of S_0 is p - 1. Hence, |U| = p|U_0| \leq p^p. \qed

In the rest of this section, p will denote a fixed prime and G will denote a connected reductive group in characteristic r \neq p with a Frobenius morphism F with respect to some \mathbb{F}_{r'} structure for some power r' of r. In what follows, whenever we talk of a component of G, we will mean a simple component of [G, G].

We need a slight variation of the previous lemma.

**Lemma 5.2.** Suppose that p is odd. If [G, G] = \text{SL}_p, then any p-subgroup of \text{G}^F has an abelian subgroup of index p.

*Proof.* Since G = Z^\circ(G)[G, G] any element and hence any subgroup of G^F is contained in Z^\circ(G)^F[G, G]^F for some d \geq 1. This can be seen as follows. Since G = Z^\circ(G)[G, G], any element u of G can be written in the form u = xy, where x \in Z^\circ(G) and y \in [G, G]. Let \iota : G \to \text{GL}_n be an embedding. Then for some power, say F^s of F, some power, say s of r, and for all g \in G, F^s \circ \iota(g) = F_s(\iota(g)) where F_s is the standard Frobenius morphism of GL_n raising every matrix entry to the s-th power. The claim follows since for any h \in \text{GL}_n, F_s^m(h) = h for some natural number m. Since any Sylow p-subgroup of Z^\circ(G)^F[G, G]^F is of the form R_1R_2, where R_1 is a Sylow p-subgroup of Z^\circ(G)^F and R_2 is a Sylow p-subgroup of [G, G]^F, the result follows from the previous Lemma and the fact that R_1 is central in R_1R_2. \qed

**Lemma 5.3.** Suppose that p is odd. Let X = \text{SL}_p be an F-stable component of G such that X^F has a central element of order p and let Y be the product of all other components of G and Z(G). Let P be a p-subgroup of G^F such that P \cap X^F is non-abelian of order at least p^p and P is not contained in X^F Y^F. Then there exists an element of order p^2 in P. Further, if Z is a central subgroup of G^F of order p such that P/Z has exponent p, then Z \leq X^F.

*Proof.* Let \hat{P} be the inverse image of P under the surjective group homomorphism X \times Y \to G induced by multiplication. The kernel of the multiplication map is isomorphic to X \cap Y = Z(X) \cap Z(Y). Since X is a simple group of type A_{p-1}, the kernel of the multiplication map is a group of order p and in particular, \hat{P} is a finite p-group. Let P_1 \leq \hat{P} be the image of \hat{P} under the projection of X \times Y \to X. Clearly P_1 contains P \cap X^F. We claim that P \cap X^F is proper in P_1. Indeed, otherwise \hat{P} \leq (P \cap X^F) \times Y, whence P \leq (P \cap X^F)Y. This implies that P \leq (P \cap X^F)(P \cap Y^F) \leq P \cap X^F Y^F, a contradiction. Since P \cap X^F is assumed to have order at least p^p, the claim implies that |P_1| \geq p^{p+1}.

Now P_1 is a finite subgroup of X, thus of some finite special linear (or unitary) group. Hence, by Lemma 5.1, there exists an element x \in P_1 of order p^2. Let y \in Y
be such that \( w = xy \in P \). Since \( P \cap X^F \) is non-abelian again by Lemma 5.1, there exists \( \sigma \in P \cap X^F \) such that \( x\sigma \) has order \( p \). Then \( w \) and \( w\sigma \in P \), \( w^p = x^py^p \) and \( (w\sigma)^p \neq y^p \). Then either \( w^p \neq 1 \) or \( (w\sigma)^p \neq 1 \), proving the first part of the result. Suppose that \( P/Z \) has exponent \( p \). Then, \( w^p, (w\sigma)^p \) are in \( Z \). Hence \( x^p \in Z \). Since \( 1 \neq x^p \) and \( Z \) has order \( p \) the second assertion follows.

**Lemma 5.4.** Let \( \mathcal{X} \) be an \( F \)-stable subset of components of \( G \). Let \( X \) be the product of all elements of \( \mathcal{X} \) and let \( Y \) be the product of \( Z^0(G) \) and all the components of \( [G, G] \) not in \( \mathcal{X} \).

(i) Let \( P \) be a defect group of a block \( b \) of \( G^F \). Then \( P \cap X^FY^F \) is a defect group of a block of \( X^FY^F \) covered by \( b \) and is of the form \( P_1P_2 \), where \( P_1 \) is a defect group of a block of \( X^F \) covered by \( b \) and \( P_2 \) is a defect group of a block of \( Y^F \) covered by \( b \). If \( Z(X)^F \cap Z(Y)^F \) has \( p \)-order, then \( P = P_1P_2 \) and the product is direct.

(ii) Let \( c \) be a \( p \)-block of \( X^FY^F \). Then the index of the stabiliser of \( c \) in \( G^F \) is prime to \( p \). Suppose further that \( Z(X)^F \cap Z(Y)^F \) is a \( p \)-group. Then \( c \) is \( G^F \)-stable, \( c \) is covered by a unique block of \( G^F \) and if \( P \) is a defect group of the block of \( G^F \) covering \( c \), then \( P \cap X^FY^F \) is a defect group of \( c \) and \( P/(P \cap X^FY^F) \cong G^F/X^FY^F \).

**Proof.** The first statement of (i) follows from the theory of covering blocks as \( X^FY^F \) is a normal subgroup of \( G^F \), \( X^F \) and \( Y^F \) centralise each other and \( X^F \cap Y^F = Z(X)^F \cap Z(Y)^F \subseteq Z(G)^F \) is central in \( X^FY^F \). The second assertion of (i) follows from the first assertion, the fact that \( |G^F| = |X^F||Y^F| \) and \( X^F \cap Y^F = Z(X)^F \cap Z(Y)^F \).

We now prove (ii). Let \( u \in G^F \) be a \( p \)-element. Then \( u = xy \), with \( x \in X \) and \( y \in Y \) such that \( x^{-1}F(x) = yF(y^{-1}) \) is an element of \( Z(X) \cap Z(Y) \). We may assume without loss of generality that \( x \) and \( y \) are \( p \)-elements. The block \( c \) of \( X^FY^F \) is a product \( c_1c_2 \) of blocks of \( c_1 \) of \( X^F \) and \( c_2 \) of \( Y^F \). Thus, it suffices to prove that \( zc_1 = c_1 \) and \( yc_2 = c_2 \).

Now consider a regular embedding \( X \leq \tilde{X} \), where \( \tilde{X} \) is a connected reductive group with connected centre containing \( X \) as a closed subgroup, such that \( [\tilde{X}, \tilde{X}] = [X, X] \) and such that \( F \) extends to a Frobenius morphism of \( \tilde{X} \). Since \( x^{-1}F(x) \in Z(\tilde{X}) \leq Z^0(\tilde{X}), x = x_1z \) for some \( x_1 \in \tilde{X}^F \), and \( z \in Z^0(\tilde{X}) \). We may assume also that \( x_1 \) is a \( p \)-element. Then \( xc_1 = c_1x_1 \). On the other hand, \( c_1 \) contains an ordinary irreducible character \( \chi \) in a Lusztig series corresponding to a semisimple element of order prime to \( p \) in the dual group of \( X \), hence the index in \( \tilde{X}^F \) of the stabiliser in \( \tilde{X}^F \) of \( \chi \) has order prime to \( p \) (see for instance [3, Corollaire 11.13]). This proves the first assertion. If \( Z(X)^F \cap Z(Y)^F \) is a \( p \)-group, then \( |G^F/X^FY^F| = |Z(X)^F \cap Z(Y)^F| \) is a power of \( p \). By the first assertion, \( c \) is \( G^F \)-stable and by standard block theory, there is a unique block of \( G^F \) covering \( c \). The second assertion of (ii) now follows from (i).
Lemma 5.5. Suppose that $p$ is odd. Let $X$ be an $F$-stable component of $G$ of type $A_{p-1}$ and let $Y$ be the product of all other components of $G$ and $Z^0(G)$. Suppose that $Z(X)^F \cap Z(Y)^F \neq 1$ and that $P$ is a defect group of $G^F$ such that $P \cap X^F$ is abelian. Then there exists an $F$-stable torus $T$ of $X$ such that $P$ is a defect group of $(YT)^F$.

Proof. In the proof, we will identify blocks with the corresponding central primitive idempotents. Let $b$ be a block of $G^F$ with $P$ as defect group and let $P_0 := P \cap X^F Y^F$. The hypothesis implies that $|Z(X)^F \cap Z(Y)^F| = p$. So, by Lemma 5.4, $b$ is a block of $X^F Y^F$, $P_0$ is a defect group of $b$ as block of $X^F Y^F$ and $P/P_0$ is isomorphic to $G^F/X^F Y^F$. Let $b = b_1b_2$, where $b_1$ is the block of $X^F$ covered by $b$ and $b_2$ is the block of $Y^F$ covered by $b$.

Let $u \in P$ generate $P$ modulo $P_0$ and write $u = xy$, $x \in X$, $y \in Y$. Since $u$ is a $p$-element, we may assume that both $x$ and $y$ are $p$-elements.

Now consider an $F$-compatible regular embedding of $X$ in $\tilde{X}$ such that $\tilde{X}^F$ is a finite general linear (or unitary) group. Since $Z(\tilde{X})$ is connected, there exists $z \in Z^0(\tilde{X})$ such that $g := xz^{-1} \in \tilde{X}^F$. Further, we may choose $z$ such that $g$ is a $p$-element. Since $u = xy$ normalises $P_1$, $x$ normalises $P_1$ and therefore $g$ normalises $P_1$. Therefore $S = \langle P_1, g \rangle \leq \tilde{X}^F$ is a $p$-group. Since $u$ normalises $b_1$ it also follows that $b_1$ is $S$-stable.

We claim that there exists a block of $\tilde{X}^F$ covering $b_1$ with a defect group $D$ containing $S$. Indeed, in order to prove the claim, it suffices to prove that $Br_S(b_1) \neq 0$. Since $b_1$ and $b_2$ are both $G^F$-stable,

$$0 \neq Br_P(b) = Br_P(b_1)Br_P(b_2)$$

and consequently $Br_P(b_1) \neq 0 \neq Br_P(b_2)$. Hence writing $b_1 = \sum_{v \in X^F} \alpha_v v$ as an element of the modular group algebra of $X^F$ there exists $v \in X^F$ with $\alpha_v$ non-zero such that $v$ centralises $P$ and in particular $v$ centralises $P_1$ and $u$. Since $z$ is central, and $y$ centralises $X$, we have that $v$ also commutes with $g$. Hence $v$ centralises $S$ and it follows that $Br_S(b_1) \neq 0$, proving the claim.

By the block theory of finite general linear (or unitary) groups (see [12]; noting that $p$ divides $q - 1$ in the linear case and that $p$ divides $q + 1$ in the unitary case) $D$ is a Sylow $p$-subgroup of the centraliser of some semisimple element of $\tilde{X}^F$. Since by hypothesis $P_1 = D \cap X^F$ is abelian, we have that $D$ is abelian. Hence $D$ is the Sylow $p$-subgroup of $T^F$ for some $F$-stable maximal torus $T$ of $X$. Set $T = X \cap T$, an $F$-stable maximal torus of $X$. Then $P_1 = D \cap X^F$ is a Sylow $p$-subgroup of $T^F$. Now $g = xz \in S \leq D \leq T$, and $z \in T$ (as $z$ is central), hence $x = gz^{-1} \in T \cap X = T$.

Set $G_0 = TY$. We have $u = xy \in G_0^F$. Since $X \cap Y \leq Z(X) \leq T$, we have that $G_0^F \cap X^F Y^F = T^F Y^F$ and $G_0^F / T^F Y^F$ is isomorphic to $G^F / X^F Y^F$ and in particular has order $p$. Hence $G_0^F = \langle T^F Y^F, u \rangle$. Let $e$ be a block of $T^F$ such that $eb_2 \neq 0$. Since $T^F$ and $Y^F$ commute, $eb_2$ is a block of $T^F Y^F$. Since $T$ is central in $G_0$, $e$ is $G_0^F$-stable. Further, $b_2$ is $P$-stable hence $b_2$ is $G_0^F$-stable. So $eb_2$ is a $G_0^F$-stable block of $T^F Y^F$ and therefore a block of $G_0^F$. Since $P_1$ is the
Sylow $p$-subgroup of $T^F$ and $T^F$ is abelian, $P_1$ is the defect group of $e$ and $P_2$ is a defect group of $b_2$. Thus, $P_1P_2$ is a defect group of $eb_2$ as block of $T^F Y^F$. Since $\text{Br}_P(eb_2) = \text{Br}_P(e) \text{Br}_P(b_2)$ is non-zero, it follows by order considerations that $P$ is a defect group of $eb_2$. □

6. The case $p = 3, 5$

In this section we handle the remaining exceptional groups of Lie type for $p \leq 5$.

**Lemma 6.1.** Let $G$, $H$ be finite groups, $B$ a $p$-block of $G$ and $C$ a $p$-block of $H$ such that $B$ and $C$ are Morita equivalent. Let $P$ be a defect group of $B$, and $Q$ a defect group of $C$. Suppose that $P$ has exponent $p$. Then $P$ is abelian if and only if $Q$ is abelian. Further, $P$ has an abelian subgroup of index $p$ if and only if $Q$ has an abelian subgroup of index $p$.

**Proof.** By [21, Satz J], the exponent of defect groups is an invariant of Morita equivalence, hence $Q$ has exponent $p$. In particular any abelian subgroup of $P$ or of $Q$ is elementary abelian. The remaining statements follow by the fact that Morita equivalence preserves the rank of the corresponding defect groups (see [2, Theorem 2.6]). □

**Lemma 6.2.** Let $L$ be connected reductive, with Frobenius morphism $F$, and let $Z$ be a central $p$-subgroup of $L^F$. Let $b$ be a block of $L^F$ and $P$ a defect group of $b$. Suppose that $P/Z$ is non-abelian, supports a transitive fusion system and $|P/Z| \geq p^4$. Let $H$ be an $F$-stable Levi subgroup of $L$, let $c$ be a Bonnafé-Rouquier correspondent of $b$ in $H$ and let $Q$ be a defect group of $c$. Then $Q/Z$ has exponent $p$ and $Q/Z$ does not have an abelian subgroup of index $p$. In particular, a Sylow $p$-subgroup of $H^F$ does not have an abelian subgroup of index $p$.

**Proof.** Let $\tilde{b}$ be the block of $L^F/Z$ dominated by $b$ and let $\tilde{c}$ be the block of $H^F/Z$ dominated by $c$. By [10, Prop. 4.1], $\tilde{b}$ and $\tilde{c}$ are Morita equivalent. Further, $P/Z$ is a defect group of $\tilde{b}$ and $Q/Z$ is a defect group of $\tilde{c}$. The result now follows from Lemma 2.1 and Lemma 6.1. □

**Proposition 6.3.** Let $L$ be connected reductive, in characteristic $r \neq p = 3$ with Frobenius morphism $F$, and suppose that $[L, L]$ is simply connected of type $E_6$ in characteristic $r \neq 3$. Let $Z$ be a cyclic subgroup of $Z(L^F)$ of order 1 or 3 and let $P$ be a defect group of $L^F$. Suppose that $P/Z$ supports a transitive fusion system and $|P/Z| \geq 3^7$. Suppose further that either $Z = 1$ or that $L$ is simple. Then $P/Z$ is abelian.

**Proof.** Suppose that $P/Z$ is non-abelian. Let $H$ be an $F$-stable Levi subgroup of $L$ and $c$ a block of $H^F$ such that $c$ is quasi-isolated and $b$ and $c$ are Bonnafé-Rouquier correspondents. Let $s \in H^*$ be a semisimple label of $c$ (and $b$). Since $b$ and $c$ are Bonnafé-Rouquier correspondents, $C_{L^*}(s) = C_{H^*}(s)$. Let $Q$ be a defect group of $c$. By Lemma 6.2, we may assume that $Q/Z$ has exponent 3 and does not have an
abelian subgroup of index 3. Note that all components of \( L \) and hence of \( H \) are simply connected.

If \( H^F \) has a component of type \( D_4 \) or \( D_5 \), then the only other possible components are of type \( A_1 \). We get a contradiction by Lemma 5.4(i), Lemma 6.2 and the fact that finite groups of type \( D_4(q) \), \( D_5(q) \), \( 2D_4(q) \), \( 2D_5(q) \) and \( 3D_4(q) \) have a Sylow 3-subgroup with an abelian subgroup of index 3.

Thus, either all components of \( H \) are of type \( A \) or \( H \) has a component of type \( E_6 \). Let us first consider the case that all components of \( H \) are of type \( A \). In particular, \( C_{H^\cdot}(s) \) is a Levi subgroup of \( H^\cdot \) and since \( s \) has order prime to 3, \( C_{L^\cdot}(s) = C_{H^\cdot}(s) \) is connected. It follows that \( s \) is central in \( H^\cdot \), hence that \( Q \) is a defect group of a unipotent block of \( H^F \).

Suppose that \( H \) has a component \( X \) of type \( A_5 \). Then \( X \) is \( F \)-stable and is the only component of \( H \). If \( X^F \) does not contain a central element of order 3, then by Lemma 5.4(i), a Sylow 3-subgroup of \( H^F \) is a direct product of a Sylow 3-subgroup of \( X^F \) with the Sylow 3-subgroup of \( Z^F(H)^F \). Furthermore in this case a Sylow 3-subgroup of \( X^F \) has an abelian subgroup of index 3. If \( X^F \) contains a central element of order 3, then by [5, Prop. 3.3 and Theorem], the principal block is the only unipotent block of \( X^F \), and it follows that \( Q/Z \) has an element of order 9 since \( \text{PSL}_6(q) \) (respectively \( \text{PSU}_6(q) \)) has elements of order 9 if \( 3 \mid q - 1 \) (respectively \( 3 \mid q + 1 \)).

Suppose that \( H \) has a component of type \( A_4 \). Then the only other possible component is of type \( A_1 \) and it follows from Lemma 5.4(i) that a Sylow 3-subgroup of \( H^F \) has an abelian subgroup of index 3.

Suppose that \( H \) has a component \( X \) of type \( A_3 \). If all other components are of type \( A_1 \), then the above argument applies. If \( H \) has a component of type \( A_2 \), say \( Y \), then this is the only other component of \( H \). If the Sylow 3-subgroups of \( X^F \) are abelian, then Lemma 5.4(i) and Lemma 5.2 give the result. Thus, we may assume that the Sylow 3-subgroups of \( X^F \) are non-abelian. Thus, \( X^F \) is isomorphic to \( \text{SL}_4(q) \) (respectively \( \text{SU}_4(q) \)) with \( 3 \mid q - 1 \) (respectively \( 3 \mid q + 1 \)). Consequently, the principal block is the unique unipotent block of \( X^F \). In particular, \( Q \) contains a Sylow 3-subgroup of \( X^F \) and \( Q/Z \) has an element of order 9.

Thus, we may assume that all components of \( H \) are of type \( A_2 \) or \( A_1 \). By rank considerations, there can be at most two components of type \( A_2 \). By Lemma 5.4 (i) and Lemma 5.2 we may assume that there are two \( F \)-stable components \( X \) and \( Y \) of type \( A_2 \) such that both \( X^F \) and \( Y^F \) have central elements of order 3. Consequently, the principal block of \( X^F \) is the only unipotent block of \( X^F \) and similarly for \( Y^F \). The only other component of \( H \), if it exists, is of type \( A_1 \), which also has a unique unipotent block. Hence \( Q \) is a Sylow 3-subgroup of \( H^F \).

Since \( H \) is a Levi subgroup of \( L \), there is a surjective group homomorphism from \( Z(G)/Z^F(G) \) to \( Z(H)/Z^F(H) \) (see [3, Prop. 4.1]) and by hypothesis, \([L, L]\) is simple of type \( E_6 \). Hence \( Z(H)/Z^F(H) \) is cyclic of order 1 or 3. Since \( X \) and \( Y \) are the
only components of $H$ with central elements of order 3, it follows that either $Z(X)$ or $Z(Y)$ covers $Z(H)/Z^o(H)$. Thus, either $Z(X) \leq Z(Y)Z^o(H)$ or $Z(Y) \leq Z(X)Z^o(H)$.

Assume that $Z(X) \leq Z(Y)Z^o(H)$. Let $U$ be the product of all components of $H$ other than $X$ and $Z^o(H)$. Then, $Z(X)^F \leq (Z(Y)Z^o(H))^F \leq U^F$ and hence $3 \mid |X^F \cap U^F|$. Since $Q$ is a Sylow 3-subgroup of $H^F$ and $|H^F| = |X^F||U^F|$, $Q$ is not contained in $X^F U^F$. Further, $Q \cap X^F$ is a Sylow 3-subgroup of $X^F$ and in particular is non-abelian of order at least $3^3$. By Lemma 6.2, $Q/Z$ has exponent 3. So, by Lemma 5.3, $1 \neq Z \leq Z(X)$ whence $Z = Z(X)$. Since $Z \neq 1$, $L$ is simple by hypothesis. In particular, $Z = Z(X)$ covers $Z(G)/Z^o(G)$. It follows that $Z(Y) \leq Z(X)Z^o(H)$. By the same argument as above with $Y$ replacing $X$, we get that $Z = Z(Y)$. In particular $Z(X) = Z(Y)$, a contradiction since $X \cap Y = 1$.

Finally, consider the case that $H$ has a component of type $E_6$. Then $H = L$ and $b = c$. Let $b_0$ be a block of $[L, L]^F$ covered by $b$ and let $P_0 = P \cap [L, L]^F$ be a defect group of $b_0$. Let $R$ be the Sylow 3-subgroup of $Z^o(L)^F$. By Lemma 5.4(i) applied with $X = [L, L]$ and $Y = Z^o(L)$, $P \cap [L, L]^F Z^o(L)^F = P_0 R$. So, $P/R_0 R$ is a subgroup of $L^F/(L, L]^F Z^o(L)^F)$. Since $L^F/(L, L]^F Z^o(L)^F)$ is either trivial or has order 3, we have that $P_0 R$ has index at most 3 in $P$. If $P_0$ is abelian, then $P$ and hence $P/Z$ has an abelian subgroup of index 3. Thus, $P_0$ is non-abelian. We claim that $R \leq P_0$. Indeed, by hypothesis, either $Z = 1$ or $[L, L] = L$. If $L = [L, L]$, then $R = 1$ and the claim holds trivially. If $Z = 1$, then $P$ supports a transitive fusion system. Hence $R \leq Z(P) \leq [P, P] \leq [L, L]^F$ and the claim is proved. Thus, $P_0 = PR$ has index at most 3 in $P$.

Assume first that $b_0$ is unipotent. The unipotent 3-blocks of exceptional groups have been described in [11]. If $b_0$ is the principal block, then $P/Z$ has exponent greater than 3. So, $b_0$ is non-principal and $P_0$ is non-abelian. By [11] (last part of the proofs for Tableau 1), $P_0$ is the extension of a homocyclic group, say $T$, of rank 2 by a group of order 3. If $T$ is not elementary abelian, then $TZ/Z$ has exponent at least 9 and hence so does $P/Z$. Thus, we may assume that $T$ is elementary abelian. So, $|P_0| = 3^3$ and $|P| \leq 3^4$, a contradiction.

So, we may assume that $b_0$ is quasi-isolated but not unipotent. Here the blocks are described in [19, Section 4.3]. In particular, $b_0$ corresponds to one of lines 13, 14, or 15 of Table 4 of [19] (and the corresponding Ennola duals; see the last remark of Section 4 of [19]). If $b_0$ corresponds to line 15, then $P_0$ is abelian. If $b_0$ corresponds to line 14, then $P_0$ is the extension of a homocyclic group, say $T$, of rank 4 by a group of order 3. If $T$ is not elementary abelian, then $TZ/Z$ has exponent at least 9 and if $T$ is elementary abelian, then $|P_0| \leq 3^5$, whence $|P| \leq 3^6$, a contradiction. If $b_0$ corresponds to line 13, then $P_0$ contains a subgroup isomorphic to a Sylow 3-subgroup of $SL_6(q)$ with $3 \mid q - 1$. In particular, $\mathfrak{U}^1(P)$ is not cyclic. On the other hand, since $P/Z$ has exponent 3, $\mathfrak{U}^1(P) \leq Z$. This is a contradiction as $Z$ is cyclic. □
Proposition 6.4. Suppose that either \( p = 3 \) and \( L \) is simple and simply connected of type \( E_7 \) or \( E_8 \) in characteristic \( r \neq 3 \) or that \( p = 5 \) and \( L \) is simple of type \( E_8 \) in characteristic \( r \neq 5 \). Let \( F \) be a Frobenius morphism on \( L \) and let \( P \) be a defect group of a \( p \)-block of \( L^F \). Suppose that \( P \) supports a transitive fusion system and \( |P| \geq 3^7 \) if \( p = 3 \). Then \( P \) is abelian.

Proof. Suppose if possible that \( P \) is not abelian. As before \( P \) has exponent \( p \), and is indecomposable and \( P \) does not have an abelian subgroup of index \( p \). Let \( z \in Z(P) \). Since \( L \) is simply connected, \( H := C_L(z) \) is a connected reductive subgroup of \( L \) of maximal rank and of semisimple rank at most 8 and by [24, Chapter 5, Theorem 9.6], \( P \) is a defect group of \( H^F \). The possible components of \( H \) are of type \( A, D, E_6 \) or \( E_7 \).

Let \( \mathcal{X} \) be an \( F \)-stable subset of elements of \( H \) and let \( X \) be the product of the \( \mathcal{X} \)'s. Suppose that \( X^F \) does not have a central element of order \( p \). By Lemma 5.4(i), \( P = (P \cap X^F) \times (P \cap Y^F) \) where \( Y \) is the product of \( Z^\circ(H) \) and all components of \( H \) other than those in \( \mathcal{X} \). The indecomposability of \( P \) implies that either \( P \leq X^F \) or \( P \leq Y^F \). Since \( z \) is a central \( p \)-element of \( H^F \), and \( X^F \) does not have a central element of order \( p \), it follows that \( P \leq Y^F \). By replacing \( H \) by \( Y \), we may assume that the fixed points of every \( F \)-orbit of components of \( H \) have central elements of order \( p \) (\( Y \) may have rank less than \( H \)). Thus, if \( p = 5 \) the only possible components are of type \( A_4 \) and if \( p = 3 \), then the only possible components are of type \( A_2, A_5, A_8 \) or \( E_7 \).

Suppose that \( H \) has an \( F \)-stable component \( X \) of type \( A_{p-1} \). Let \( Y \) be the product of all components of \( H \) other than those in \( X \) with \( Z^\circ(H) \). By Lemma 5.4(i) and the indecomposability of \( P \), we may assume that \( Z(X)^F \cap Z(Y)^F \) and hence \( H^F / X^F Y^F \) has order \( p \). So, by Lemma 5.4(ii), \( P \) is not contained in \( X^F Y^F \). By Lemma 5.5, we may assume that \( P \cap X^F \) is not abelian since otherwise we can replace \( X \) by a torus. Since \( X^F \) has a central element of order \( p \), \( X^F \) is a special linear (respectively unitary) group. The only non-abelian defect groups of a finite special linear (or unitary) group of degree \( p \) in non-describing characteristic are Sylow \( p \)-subgroups and \( P \cap X^F \) is a non-abelian defect group of \( X^F \). Thus, \( P \cap X^F \) is a Sylow \( p \)-subgroup of \( X^F \) and consequently has order at least \( p^2 \). Since we have shown above that \( P \) is not contained in \( X^F Y^F \), by Lemma 5.3, \( P \) has an element of order \( p^2 \), a contradiction. Thus, we may assume that any component of \( H \) of type \( A_{p-1} \) lies in an \( F \)-orbit of size at least 2.

If \( p = 5 \), the only case left to consider is that \( H \) has two components of type \( A_4 \) (and these are the only ones) transitively permuted by \( F \). In this case, by rank considerations, \( Z^\circ(H) \) is trivial, and hence \( H^F \) is isomorphic to a special linear or unitary group. In particular the Sylow 5-subgroups of \( H^F \) have an abelian subgroup of index 5, a contradiction. This completes the proof for the case that \( p = 5 \).
Now assume that $p = 3$. Let us first consider the case that there is a component $X$ of $H$ of type $A_8$. Then $H = X = \text{SL}_8$ and we may argue as in the first part of the proof of Proposition 4.1.

Let us next consider the case that there is a component $X$ of $H$ of type $A_5$. If $X$ also has a component of type $A_2$, then by rank consideration this is the unique component of type $A_2$ and we have ruled out this situation above. Thus $X$ is the unique component of $H$. Let $P_0$ be a defect group of a covered block of $X^F$. The Sylow $3$-subgroup of $Z^o(H)^F$ is contained in $Z(P)$ and $Z(P) \leq [P, P] \leq [X, X] \cap H^F \leq X^F$, hence we have that the Sylow $3$-subgroup of $Z^o(H)^F$ is contained in $X^F$ and in particular has order at most $3$. Thus, $P_0$ has index at most $3$ in $P$. In particular $P_0$ is non-abelian. Now $X = M/Z$, where $M$ is a special linear group of degree $6$ (with a compatible $F$-action) and $Z$ is a central subgroup. Since $Z(M)$ is cyclic of order $6$ (or $3$ if $r = 2$) and since $X$ has a central element of order $3$, $Z$ is either trivial or of order $2$, $Z$ is $F$-stable and $Z^F = Z$. Further, $M^F/Z$ is a normal subgroup of $X^F = (M/Z)^F$ of index $|Z|$. Thus $P_0$ is a defect group of $M^F/Z$ and up to isomorphism a defect group of $M^F$ and $M^F = \text{SL}_6(q)$ (respectively $\text{SU}_6(q)$). Since $M^F/Z$ has index prime to $3$, $M^F/Z$ contains the $3$-part of the centre of $X^F$, hence $M^F$ has a central element of order $3$. Thus, $P_0$ is the intersection with $X^F$ of a Sylow $3$-subgroup of the centraliser of a semisimple $3'$-element of $\text{GL}_6(q)$ (or $\text{GU}_6(q)$). Since $P_0$ has exponent $3$ and is non-abelian, the possible structures of semisimple centralisers in $\text{GL}_6(q)$ (or $\text{GU}_6(q)$) force that the centraliser in $\text{GL}_6(q)$ (respectively $\text{GU}_6(q)$) has the form $\text{GL}_3(q^2)$. Hence $|P_0| \leq p^3$ and $|P| \leq p^4$, a contradiction.

Suppose $H$ has a component of type $E_6$. Arguing as in the previous case $H$ has no components of type $A_2$ and hence the $E_6$-component is the unique component of $H$. This component is of simply connected type since as explained in the beginning of the proof we may assume that the $F$-fixed point subgroup of every $F$-orbit of components of $H$ has central elements of order $3$ and we are done by Proposition 6.3 (note that we apply Proposition 6.3 here in the case that $Z = 1$).

The only case left to consider is that all components of $H$ are of type $A_2$ and no component is $F$-stable. By rank considerations and the fact that groups of type $E_8$ do not have semisimple centralisers with component type $A_2^3$ (see the tables in [9]), we are left with two possibilities: either $H$ has exactly three components, all of type $A_2$ and in a single $F$-orbit or $H$ has exactly two components both of type $A_2$ and in a single $F$-orbit. In any case, $[H, H]^F$ has a quotient or subgroup $H_0$ isomorphic to $\text{PSL}_3(q)$ (respectively $\text{PSU}_3(q)$) for some $q$ such that $|[H, H]^F|/|H_0|$ equals $1$ or $3$. Let $P_0 = P \cap [H, H]$ and let $P'_0$ be either the intersection of $P_0$ with $H_0$ or the image of $P_0$ in $H_0$. Then $P'_0$ has exponent $3$. Since any $3$-subgroup of a finite projective special linear or unitary group of degree $3$ has an abelian subgroup of index $3$ and since the $3$-rank of these groups is $2$, it follows that $|P_0'| \leq 3^3$. Hence $|P_0| \leq 3^4$.

We claim that the index of $P_0$ in $P$ is at most $3$. Indeed, let $R$ be the Sylow $3$-subgroup of $Z^o(H)^F$. Then $R \leq Z(P) \leq [P, P] \leq [H, H]$, that is $R \leq P_0$. On the
other hand, $|P/P_0R|$ divides $|Z([H,H]^F)|_3$ and we have seen from the structure of $[H,H]^F$ that $Z([H,H]^F)$ has order at most 3. This proves the claim. Hence $|P| \leq 3^5$, a contradiction. □

7. Consequences

We note some consequences of Theorem 1.2.

**Theorem 7.1.** Let $B$ be a block of a finite group such that $k(B) - l(B) = 1$ (e. g. a block with multiplicity 1). Then $B$ has elementary abelian defect groups.

**Proof.** See proof of Theorem 3.6 in [23]. □

**Corollary 7.2.** Let $B$ be a block of a finite group such that $k(B) = 3$. Then $B$ has elementary abelian defect groups.

**Proof.** We have $l(B) \in \{1, 2\}$. In case $l(B) = 1$ it was shown by Külshammer [22] that the defect groups of $B$ have order 3. The remaining case $l(B) = 2$ follows from Theorem 7.1. □

**References**

[1] M. Aschbacher, R. Kessar and B. Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, Vol. 391, Cambridge University Press, Cambridge, 2011.

[2] C. Bessenrodt and W. Willems, *Relations between complexity and modular invariants and consequences for p-soluble groups*, J. Algebra 86 (1984), 445–456.

[3] C. Bonnafé, *Sur les caractères des groupes réductifs finis a centre non connexe: applications aux groupes spéciaux linéaires et unitaires*, Astérisque 306 (2006).

[4] C. Bonnafé, R. Rouquier, *Catégories dérivées et variétés de Deligne-Lusztig*, Publ. Math. Inst. Hautes Études Sci. 97 (2003), 1–59.

[5] M. Cabanes and M. Enguehard, *On unipotent blocks and their ordinary characters*, Invent. Math. 117 (1994), 149–164.

[6] M. Cabanes and M. Enguehard, *On blocks of finite reductive groups and twisted induction*, Advances Math. 145 (1999), 189–229.

[7] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.

[8] D. A. Craven, *The Theory of Fusion Systems*, Cambridge Studies in Advanced Mathematics, Vol. 131, Cambridge University Press, Cambridge, 2011.

[9] D. I. Deriziotis, *Centraлизаторы простых элементов из групп Чевалье группы $E_7$ и $E_8$*, Tokyo. J. Math 6 (1983), 191–216.

[10] C. Eaton, R. Kessar, B. Külshammer, B. Sambale, *2-blocks with abelian defect groups* Adv. Math. 254 (2014), 706–735.

[11] M. Enguehard, *Sur les l-blocs unipotents des groupes réductifs finis quand l est mauvais*, J. Algebra 230 (2000), 334–377.

[12] P. Fong and B. Srinivasan, *The blocks of finite general linear and unitary groups*, Inv. Math. 69 (1982), 101–153.

[13] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.6.4*, 2013, http://www.gap.system.org.
[14] D. Gorenstein and R. Lyons, *The local structure of finite groups of characteristic 2 type*, Mem. Amer. Math. Soc. 42 (1983).

[15] D. Gorenstein, R. Lyons and R. Solomon, *The classification of finite simple groups 3*, Mathematical Surveys and Monographs 40, American Mathematical Society (1998).

[16] C. Hering, *Transitive linear groups and linear groups which contain irreducible subgroups of prime order*, Geom. Dedic. 2 (1974), 425–460.

[17] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin 1967.

[18] B. Huppert and N. Blackburn, *Finite groups III*, Springer-Verlag, Berlin 1982.

[19] R. Kessar and G. Malle, *Quasi-isolated blocks and Brauer’s height zero conjecture*, Ann. of Math. (2) 178 (2013) 321–384.

[20] R. Kessar and R. Stancu, *A reduction theorem for fusion systems of blocks*, J. Algebra 319 (2008), 806–823.

[21] B. Külshammer, *Bemerkungen über die Gruppenalgebra als symmetrische Algebra II*, J. Algebra 75 (1982), 59-69.

[22] B. Külshammer, *Symmetric local algebras and small blocks of finite groups*, J. Algebra 88 (1984), 190–195.

[23] B. Külshammer, G. Navarro, B. Sambale and P. H. Tiep, *Finite groups with two conjugacy classes of p-elements and related questions for p-blocks*, Bull. London Math. Soc. 46 (2014), 305–314.

[24] H. Nagao and Y. Tsushima, *Representations of finite groups*, Academic Press Inc., Boston, MA, 1989.

[25] G. Navarro and P. H. Tiep, *Abelian Sylow subgroups in a finite group*, J. Algebra 398 (2014), 519-526.

[26] A. Ruiz and A. Viruel, *The classification of p-local finite groups over the extraspecial group of order $p^3$ and exponent p*, Math. Z. 248 (2004), 45–65.

[27] B. Sambale, *Blocks of finite groups and their invariants*, Lecture Notes in Math. Vol. 2127, Springer-Verlag, Berlin, 2015.

[28] R. Steinberg, *Lectures on Chevalley Groups*, Notes by J. Faulkner and R. Wilson, Mimeographed notes, Yale University Mathematics Department (1968).