Scattering matrices in the $\mathfrak{sl}(3)$ twisted Yangian

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Received 31 October 2014
Accepted for publication 12 January 2015
Published 3 February 2015

Abstract. A quantum spin chain with non-conventional boundary conditions is studied. The distinct nature of these boundary conditions arises from the conversion of a soliton to an anti-soliton after being reflected by the boundary, hence the appellation soliton non-preserving boundary conditions. We focus on the simplest non-trivial case of this class of models based on the twisted Yangian quadratic algebra. Our computations are performed through the Bethe ansatz equations in the thermodynamic limit. We formulate a suitable quantization condition describing the scattering process and proceed with explicitly determining the bulk and boundary scattering amplitudes. The energy and quantum numbers of the low lying excitations are also derived.

Keywords: integrable spin chains (vertex models), quantum integrability (Bethe ansatz), solvable lattice models

ArXiv ePrint: 1410.5991
1. Introduction

The description of quantum integrable systems with boundaries dates back to the works of Cherednik [1] and Sklyanin [2]. The main object is the so-called quantum reflection algebra, defined by the quadratic exchange relations

\[ R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}, \]

(1)

where \( R_{12} \) is the bulk quantum \( R \)-matrix satisfying the Yang–Baxter equation, \( K \) encodes the boundary effects, and the dependence on the spectral parameter is suppressed throughout the section. Equation (1) is interpreted as a supplementary consistency condition between the bulk \( S \)-matrices and the reflection matrix \( K \) for factorizability of \( N \)-body amplitudes into 2-body amplitudes encompassing boundary effects [2]. In this particular case one interprets the theory as a description of soliton dynamics with a single bulk collision 2-body \( S \)-matrix \( R_{12} \) and a reflection matrix \( K \) preserving the soliton after reflection, hence the characterization ‘soliton-preserving’ boundaries.

The maximal generalization of equation (1) was proposed by Freidel and Maillet in [3], see also [4]. It is parametrized by three matrices \( A, B, D \)

\[ A_{12} K_1 B_{12} K_2 = K_2 C_{12} K_1 D_{12}, \quad A_{12} A_{21} = I = D_{12} D_{21}, \quad C_{12} = B_{21}. \]

(2)

When describing the abstract quadratic exchange algebra, \( K \) is here interpreted as a matrix (on auxiliary space 1 or 2) of generators of the quadratic exchange algebra. It can be systematically constructed from the comodule structure of (2), as ‘dressing’ of an initial \( K \)-scalar solution of (2) by successive left/right ‘coproducts’ of pairs \( A/C \) or \( B/D \).

doi:10.1088/1742-5468/2015/02/P02007
Within some general assumptions it can be shown reciprocally that all representations, at least of the reflection algebra (1), are obtained precisely by the dressing of a scalar $K$-matrix by bulk quantities obeying a Yang–Baxter type equation [5].

We shall focus here on another particular case of (2), when the initial reflection on the boundary exhibits a soliton non-preserving behavior e.g. when the reflection converts a soliton into an antisoliton (see e.g. [6,7] and references therein). In this framework one is lead to identify $A_{12} = R_{12}$, $D_{12} = \bar{R}_{21}$ and $B_{12} = \bar{R}_{12}$ where physically $R_{12}$ corresponds to the soliton–soliton collision matrix, whereas $\bar{R}_{12}$ corresponds to the soliton–anti-soliton collision matrix. We get then the following structure (see also [8])

$$R_{12} K_1 \bar{R}_{21} K_2 = K_2 \bar{R}_{12} K_1 R_{21}. \quad (3)$$

A suitable double-row monodromy matrix is then defined as alternated coproducts as commented before in the general case

$$T = \ldots R_{02} \bar{R}_{01}, \quad (4)$$

and the relevant spin chain Hamiltonians are now obtained from the quantum trace formula:

$$\tau = Tr\{K^+ T\}. \quad (5)$$

Assuming $R$ possesses the regularity property

$$R_{12}(\lambda \to 0) \propto P_{12}, \quad (6)$$

with $P$ being the permutation operator and $\lambda$ denoting the spectral parameter, the Hamiltonian

$$H_1 \propto \frac{d}{d\lambda}(\ln \tau(\lambda))\bigg|_{\lambda=0}, \quad (7)$$

yields a local spin chain interaction with boundary terms. It has the following explicit form (for more details see [6,7]):

$$\mathcal{H} \propto \sum_{j=1}^{L} \bar{R}_{2j-1 \,2j} \bar{R}_{2j-1 \,2j} + \sum_{j=1}^{L-1} \bar{R}_{2j+1 \,2j+2} \bar{R}_{2j+1 \,2j+2} \bar{R}_{2j+1 \,2j+2}$$

$$+ \sum_{j=1}^{L-1} \bar{R}_{2j+1 \,2j+2} \bar{R}_{2j-1 \,2j} \bar{R}_{2j-1 \,2j+2} \bar{R}_{2j-1 \,2j+2} \bar{R}_{2j-1 \,2j} \bar{R}_{2j+1 \,2j+2}$$

$$+ \sum_{j=1}^{L-1} \bar{R}_{2j+1 \,2j+2} \bar{R}_{2j-1 \,2j} \bar{R}_{2j-1 \,2j+2} \bar{R}_{2j-1 \,2j+2} \bar{R}_{2j-1 \,2j} \bar{R}_{2j+1 \,2j+2}$$

$$+ Tr_{0} \bar{R}_0 \bar{R}_{2L} \bar{R}_{2L-1} \bar{R}_0 \bar{R}_{2L-1} \bar{R}_{2L-1} \bar{R}_{2L-1} \bar{R}_{2L} + \bar{R}_{12} \bar{R}_{12} \bar{R}_{12}, \quad (8)$$

The prime denotes the derivative with respect to the spectral parameter and $\bar{R} = P \cdot R$. Note that this type of unconventional boundary conditions in the quantum spin chain framework were first studied in [6] and later generalized in [7]. These boundary conditions were originally known, albeit in a classical framework, in the context of affine Toda field theories [9]. It is worth pointing out that the implementation of these boundary conditions, based on the twisted Yangian, in the quantum spin chain frame provides a resolution of a long lasting misunderstanding regarding the various types of boundary conditions in integrable classical field theories versus integrable lattice models. More precisely, until
the full study of all possible conditions in field theories [10] and quantum spin chains [6] only boundary conditions associated to the reflection algebra were known in the spin chain context, whereas in affine Toda field theories only boundary conditions associated to the classical twisted Yangian were known.

Here we propose for the first time to study such systems in the thermodynamic limit aiming at this time at computing the bulk and boundary scattering amplitudes after implementing a novel quantization condition related to the particular models. We shall here concentrate on the special case associated to \( \mathfrak{sl}(3) \). In addition, we consider a case where the conjugate \( R \)-matrix \( \bar{R} \) is obtained from \( R \) by

\[
\bar{R}_{12} = V_1 R_{12}^t V_1 .
\]

The relevant algebraic structure (3) is now identified as a twisted Yangian (rational \( R \)-matrix) or twisted quantum Yangian (trigonometric case).

Remark that a natural construction of representations of the twisted Yangian consists in starting from the bulk monodromy matrix \( T \) obeying the fundamental quadratic relation [11]

\[
R_{12} T_1 T_2 = T_2 T_1 R_{12} ,
\]

and define the ‘folded’ or twisted generic \( K \)-matrix as:

\[
\mathbb{K}(\lambda) = T(\lambda) K(\lambda) T^t(-\lambda + \kappa) ,
\]

where \( K \) is \( c \)-number solution of the twisted Yangian equation, \( \kappa \) is a constant associated with the Lie algebra of the chosen \( R \)-matrix, and \( ^t \) denotes the transposition taken on the auxiliary space only. This natural ‘folding’ structure is also seen in the formula (4), and will have consequences on the structure of the vacuum, the eigenvectors as well as the exact symmetry of the corresponding integrable system.

This article is organized as follows. In the next section we focus on the \( \mathfrak{sl}(3) \) twisted Yangian model and study its thermodynamic limit. We compute the energy of an excitation and study the quantum numbers in order to ensure the validity of our results. Section 3 contains the main results of our work. The key result is the formulation of a quantization condition for twisted Yangian spin chains; we then prove the factorization of the bulk scattering amplitude and explicitly compute the boundary scattering amplitude. Note that the results of this section are completely new. We conclude with a short discussion.

2. Twisted Yangian: Bethe ansatz and thermodynamics

The twisted Yangian algebra associated to the so-called soliton non-preserving boundary conditions was first studied in the context of integrable lattice models via the Bethe ansatz formulation in [6], whereas generalizations were investigated in [7]. It was shown in [6] that the Bethe ansatz equations (BAE) of the model are given as

\[
e_1(\lambda_i)^L e_{-1}(\lambda_i) = - \prod_{j=1}^{M} e_2(\lambda_i - \lambda_j) e_2(\lambda_i + \lambda_j) e_{-1}(\lambda_i - \lambda_j) e_{-1}(\lambda_i + \lambda_j) ,
\]

where we define

\[
e_n(\lambda) = \frac{\lambda + \frac{in}{2}}{\lambda - \frac{in}{2}} .
\]
These BAE are similar to those of the \( \mathfrak{osp}(1|2) \) case \[ 12\], up to an extra boundary contribution. In fact, the case in study is the first occurrence of a more general correspondence between \( \mathfrak{sl}(2n+1) \) chains with twisted Yangian boundary conditions and \( \mathfrak{osp}(1|2n) \) open spin chains with certain boundary conditions. This correspondence was already studied in \[ 12\], and is currently under investigation \[ 13\] from the Bethe ansatz point of view.

In the usual \( \mathfrak{sl}(2n+1) \) Yangian case the ground state of the system consists of \( 2n \) filled Dirac seas. On the contrary, in the twisted Yangian case, this number is halved, due to the ‘folding’. The bulk contribution is essentially the same as that of a spin chain with \( \mathfrak{osp}(1|2n) \) symmetry, hence the intriguing correspondence mentioned above.

The \( \mathfrak{sl}(3) \) twisted Yangian quantum spin chain in particular has only one filled Dirac sea as its ground state. A hole in the Dirac sea represents an excitation in the system and incorporates both the fundamental 3 and its conjugate \( \bar{3} \) representation of \( \mathfrak{sl}(3) \), i.e. both a soliton and an anti-soliton are present in an excitation. The thermodynamic limit is performed according to the rule

\[
\frac{1}{L} \sum_{j=1}^{M} f(\lambda_j) \rightarrow \int_{0}^{\infty} d\mu \sigma(\mu) f(\mu) - \frac{1}{L} \sum_{j=1}^{\nu} f(\tilde{\lambda}_j) - \frac{1}{2L} f(0),
\]

for \( \nu \) holes in the Dirac sea with rapidities \( \tilde{\lambda}_j \), and the last term is the halved contribution at \( 0^+ \) due to the boundaries. Defining also

\[
ad_n(\lambda) = \frac{1}{2\pi} \frac{d}{d\lambda} \ln e_n(\lambda),
\]

the density of the Bethe roots as computed from the BAE (12) is given by (see also \[ 14\text{-}16])

\[
\sigma(\lambda) = a_1(\lambda) - \int_{-\infty}^{\lambda} d\mu \sigma(\mu) \left( a_2(\lambda - \mu) - a_1(\lambda - \mu) \right) + \frac{1}{2} \sum_{j=1}^{\nu} \left( a_2(\lambda - \tilde{\lambda}_j) + a_2(\lambda + \tilde{\lambda}_j) - a_1(\lambda - \tilde{\lambda}_j) - a_1(\lambda + \tilde{\lambda}_j) \right) + \frac{1}{2} \left( a_2(\lambda) - a_1(\lambda) - a_2(\lambda) \right).
\]

Taking the Fourier transform\(^1\) of the latter expression leads to

\[
\hat{K}(\omega) \hat{\sigma}(\omega) = \hat{a}_1(\omega) + \frac{1}{L} \left[ \hat{a}_2(\omega) - \hat{a}_1(\omega) - \hat{a}_2(\omega) + (\hat{a}_2(\omega) - \hat{a}_1(\omega)) \sum_{j=1}^{\nu} (e^{i\omega \tilde{\lambda}_j} + e^{-i\omega \tilde{\lambda}_j}) \right],
\]

where we have defined the kernel

\[
\hat{K}(\omega) \equiv 1 + \hat{a}_2(\omega) - \hat{a}_1(\omega) = e^{-\frac{i\omega}{2}} \frac{\cosh \frac{3\omega}{4}}{\cosh \frac{\omega}{4}}, \quad \text{and} \quad \hat{a}_n(\omega) = e^{-\frac{n\omega}{2}}.
\]

After some simplifications, the density is written compactly as

\[
\hat{\sigma}(\omega) = \hat{\sigma}^{(0)}(\omega) + \frac{1}{L} \left[ \hat{r}_1(\omega) + \hat{r}_2(\omega) \sum_{j=1}^{\nu} (e^{i\omega \tilde{\lambda}_j} + e^{-i\omega \tilde{\lambda}_j}) \right],
\]

\(^1\) The following Fourier conventions are used

\[
f(\omega) = \int_{-\infty}^{\infty} dx f(x) e^{i\omega x}, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega f(\omega) e^{-i\omega x}.
\]
expressions for \( A \) with \( A \) since the first term of the density turns out to be the energy of the ground state, as will be terms \( \hat{\lambda} \) as well as their derivatives vanish for boundary scattering amplitudes. quantum numbers of the low lying excitations as well as the computation of the bulk and the latter expression will be used subsequently for the computation of the energy and ground state and the low-lying excitations. Recall that the eigenvalues of the transfer matrix are given by [6]

\[
\Lambda^{(M)}(\lambda) = (a(\lambda)\bar{b}(\lambda))^L \frac{\bar{a}(2\lambda)}{b(2\lambda)} A_1(\lambda) + (b(\lambda)\bar{b}(\lambda))^L A_2(\lambda) + (a(\lambda)\bar{b}(\lambda))^L \frac{\bar{a}(2\lambda + 2i)}{b(2\lambda)} A_3(\lambda),
\]

where

\[
a(\lambda) = \lambda + i, \quad b(\lambda) = \lambda, \quad c(\lambda) = i, \quad A_1(\lambda) = \prod_{j=1}^{M} \frac{\lambda + \mu_j - i}{\lambda + \mu_j + \frac{3}{2} \lambda - \mu_j + \frac{1}{2}},
\]

\( \{\mu_j\} \) is the set of Bethe roots and \( f(\lambda) = f(-\lambda - \frac{3}{2}) \). The terms containing \( A_2(\lambda), A_3(\lambda) \), as well as their derivatives vanish for \( \lambda = 0 \) and hence are not needed here. The exact expressions for \( A_2(\lambda), A_3(\lambda) \) can be found in [6] The first derivative of the eigenvalues with respect to the spectral parameter yields the energy of the system

\[
E(\{\mu_j\}) \propto \left. \frac{d}{d\lambda} \Lambda(\lambda, \{\mu_j\}) \right|_{\lambda=0}
\]

\[
= \left. \frac{d}{d\lambda} \left( (a(\lambda)\bar{b}(\lambda))^L \right) \right|_{\lambda=0} A_1(0) + (a(0)\bar{b}(0))^L A'_1(\{\mu_j\}),
\]

with

\[
A'_1(\{\mu_j\}) = \left. \frac{d}{d\lambda} A_1(\lambda) \right|_{\lambda=0} = \sum_{j=1}^{M} \frac{2i}{\mu_j^2 + \frac{1}{4}} = -4\pi \sum_{j=1}^{M} a_1(\mu_j).
\]

Since \( A_1(0) = 1 \), the first term contributing to the energy is independent of the Bethe roots and thus corresponds to a simple energy shift. We may then conclude that

\[
E(\{\mu_j\}) = -\sum_{j=1}^{M} a_1(\mu_j).
\]
In the thermodynamic limit and in the case of one hole present in the system, the above relation takes the form
\[ \epsilon(\tilde{\lambda}_1) = -\int_0^\infty d\mu \, a_1(\mu) \, \sigma(\mu) + \frac{1}{L} a_1(\tilde{\lambda}_1) - \frac{1}{2L} a_1(0). \] (28)

For our purpose here the boundary contribution is irrelevant, since it only contributes to the ground state. Gathering the Fourier transformed \( \frac{1}{L} \) contributions containing the rapidity of the excitation, \( \tilde{\lambda}_1 \), one concludes that
\[ \hat{\epsilon}(\omega) = \frac{\hat{a}_1(\omega)}{1 + \hat{a}_2(\omega) - \hat{a}_1(\omega)} = \hat{\sigma}^{(0)}(\omega), \] (29)
which as expected coincides with ground state density—up to boundary contributions. This is a key point for the computation of scattering amplitudes via the suitable quantization condition, which will be formulated later on in the text.

2.2. Quantum numbers and symmetry

As was discussed in detail in [6, 7] from the study of the asymptotics of the transfer matrix one can extract the total spin:
\[ S = \sum_{j=1}^L S_j^z = L - M - \frac{1}{2}, \quad S^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \] (30)

Through the thermodynamic limit computations, for a state with \( \nu \) holes we have
\[ M = \sum_{j=1}^M 1 = L \int_0^\infty \sigma(\lambda) \, d\lambda - \nu \text{ holes} - \frac{1}{2} \text{ boundary effect} = L - \nu - \frac{1}{2}. \] (31)

For a state with one hole, the spin of this state is indeed correctly computed to be \( S = 1 \). Considering now the state with two holes, the co-product is computed
\[ S = S^z \otimes I + I \otimes S^z = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & -2 \end{pmatrix}. \] (32)

Recall that the \( \mathfrak{sl}(3) \) invariant \( R \)-matrix is given by [14]
\[ R(\lambda) = a(\lambda) \sum_{i=1}^3 e_{ii} \otimes e_{ii} + b(\lambda) \sum_{i \neq j}^3 e_{ii} \otimes e_{jj} + c(\lambda) \sum_{i \neq j}^3 e_{ij} \otimes e_{ji}, \] (33)
where \( a, b, c \) were defined in equation (24), while its conjugate is defined as
\[ \bar{R}_{12}(\lambda) = V_1 R_{12}^\dagger(-\lambda - \frac{3i}{2}) V_1, \quad V = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \] (34)
We are interested in identifying the common eigenvectors of the co-product state (32) and the product $R(\lambda)\bar{R}(\lambda)$. Let us then introduce the following basis of the real vector space $\mathbb{R}^3$

$$\left| +1 \right> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \left| 0 \right> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \left| -1 \right> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$ (35)

We identify the following common eigenvectors and associated eigenvalues:

| Spin | Eigenvector | Eigenvalue |
|------|-------------|------------|
| +2   | $| +1 \rangle \otimes | +1 \rangle$ | $a(\lambda) b(\lambda)$ |
| +1   | $| +1 \rangle \otimes | 0 \rangle + | 0 \rangle \otimes | +1 \rangle$ | $\tilde{b}(\lambda)(\bar{b}(\lambda) + c(\lambda))$ |
| 0    | $| +1 \rangle \otimes | -1 \rangle + | 0 \rangle \otimes | 0 \rangle + | -1 \rangle \otimes | +1 \rangle$ | $a(\lambda)(\bar{a}(\lambda) + 2c(\lambda))$ |
| -1   | $| -1 \rangle \otimes | 0 \rangle + | 0 \rangle \otimes | -1 \rangle$ | $\tilde{b}(\lambda)(\bar{b}(\lambda) + c(\lambda))$ |
| -2   | $| -1 \rangle \otimes | -1 \rangle$ | $a(\lambda) \tilde{b}(\lambda)$ |

(36)

3. Scattering amplitudes

The main aim in this section is the study of the bulk and boundary scattering for the $\mathfrak{sl}(3)$ twisted Yangian model. To achieve this we shall basically employ the results of the previous section together with a suitable quantization condition (see also [15–17]). Thus before we proceed with the computation of exact $S$-matrices via the twisted Yangian BAE it will be important to formulate the associated quantization condition, which describes the bulk and boundary scattering in the particular algebraic setting.

3.1. Quantization condition

Here we shall derive the suitable quantization case associated to the soliton non-preserving scattering. This is in fact one of the key points in the present article, and it is also a starting point for the investigation of the bulk and boundary scattering.

It is constructive to graphically depict the scattering matrices in order to fully comprehend the quantization condition. A soliton will be represented by a solid line and an anti-soliton by a dashed one. Let $S$ denote the soliton–soliton (or anti-soliton–anti-soliton) and $\bar{S}$ denote the soliton–anti-soliton scattering respectively. They are depicted as

$$S \quad \approx \quad \cdots \cdots \quad \text{and} \quad \bar{S} \quad \approx \quad \cdots \cdots$$

Before we discuss the quantization condition associated to the twisted Yangian let us first recall the quantization condition for the usual reflection case [17]. The double-row transfer matrix consists of two products of the bulk $S$-matrix, intertwined with the
reflection matrices $K^\pm$. A graphical illustration of such a model is given as

\[
\begin{array}{c}
K^+
\end{array}
\begin{array}{c}
S
\end{array}
\begin{array}{c}
S
\end{array}
\begin{array}{c}
K^-
\end{array}
\]

One imposes an isomonodromy condition on the state of two holes as:

\[
\left(e^{2iPL}S(\tilde{\lambda}_1, \tilde{\lambda}_2) - 1\right)|\tilde{\lambda}_1, \tilde{\lambda}_2\rangle = 0 ,
\]

where the global scattering amplitude $S$ is given as:

\[
S(\lambda_1, \lambda_2) \equiv K^+ (\lambda_1) S(\lambda_1 - \lambda_2) K^- (\lambda_1) S(\lambda_1 + \lambda_2) .
\]

We come now to our main objective which is the derivation of a generalized quantization condition regarding the soliton non-preserving equation. Recall that the transfer matrix of the model consists of alternated coproducts, as mentioned in the introduction. A graphical illustration of a model with twisted Yangian boundary conditions will have the following form then

\[
\begin{array}{c}
K^+
\end{array}
\begin{array}{c}
S
\end{array}
\begin{array}{c}
S
\end{array}
\begin{array}{c}
K^-
\end{array}
\begin{array}{c}
S
\end{array}
\begin{array}{c}
S
\end{array}
\]

from which the momentum quantization condition follows directly again as an isomonodromy condition

\[
\left(e^{iPL}S(\tilde{\lambda}_1, \tilde{\lambda}_2) - 1\right)|\tilde{\lambda}_1, \tilde{\lambda}_2\rangle = 0 ,
\]

with the manifest factorization

\[
S(\lambda_1, \lambda_2) \equiv k^+ (\lambda_1) S(\lambda_1 - \lambda_2) S(\lambda_1 + \lambda_2) k^- (\lambda_1) S(\lambda_1 + \lambda_2) ,
\]

and $L$ being the length of the chain. Note that the phase in the exponential factor is just $L$ instead of the usual $2L$, because we deal here with ‘folding’ and not reflection, as opposed to the usual open boundary conditions. The ‘particle’—merging of $3$ and $\bar{3}$—now propagates in both directions simultaneously, hence now over a distance $L$. This factorization is expected to emerge naturally at the thermodynamic limit. Indeed, we show below that the bulk scattering amplitudes factorize appropriately, which confirms the quantization condition as formulated in (39).

From now on we consider two excitations (holes), so that $\nu = 2$. Recall that the momentum and energy are related through

\[
\epsilon(\lambda) = \frac{1}{2\pi} \frac{d\rho}{d\lambda} .
\]

Combining the momentum quantization condition (39) with the above expression, and taking into account that

\[
L \int_0^{\lambda_1} d\lambda \sigma(\lambda) \in \mathbb{Z} ,
\]

doi:10.1088/1742-5468/2015/02/P02007
we find that the scattering matrix phase, \( S = \exp(i\Phi) \), is computed through
\[
\Phi = 2\pi \int_{0}^{\lambda_1} d\lambda \left[ r_1(\lambda) + \sum_{j=1}^{2} (r_2(\lambda - \lambda_j) + r_2(\lambda + \lambda_j)) \right],
\]
(43)
or passing to momentum space to perform the computations
\[
\Phi = -\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \left( e^{-i\omega\lambda_1} \hat{r}_1(\omega) + e^{-2i\omega\lambda_1} \hat{r}_2(\omega) \right) - 2\int_{-\infty}^{\infty} \frac{d\omega}{\omega} e^{-i\omega\lambda_1} \cos(\omega\lambda_2) \hat{r}_2(\omega).
\]
(44)
The first integral provides the boundary contribution and the second one the bulk scattering. Recalling the quantization condition, one obtains
\[
k^{\pm}(\lambda) k^{-}(\lambda) = \exp \left[ -\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \left( e^{-i\omega\lambda} \hat{r}_1(\omega) + e^{-2i\omega\lambda} \hat{r}_2(\omega) \right) \right]
\]
\[
S(\lambda) = \exp \left[ -\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \hat{r}_2(\omega) e^{-i\omega\lambda} \right].
\]
(45)
where we recall that \( k^{\pm} \) correspond to the left and right boundary scattering amplitudes, and \( S \) is the bulk scattering amplitude. We have considered here for simplicity \( K^{\pm} \propto I \), so identifying the scattering amplitude \( K^{\pm}(\lambda) = k^{\pm}(\lambda)I \). As will be clear subsequently the bulk scattering factorizes into soliton–soliton amplitudes times the soliton–anti-soliton amplitude.

### 3.2. Bulk scattering amplitude: factorization

Let us first focus on the bulk scattering and verify that the scattering factorizes into the two amplitudes mentioned above. After some algebra, it can be shown that the integrand in the bulk scattering amplitude appearing in equation (45) is given by
\[
\hat{r}_{s}(\omega) = \hat{r}_2(\omega) = \frac{\left( e^{\frac{i\omega}{2}} - e^{-\frac{i\omega}{2}} \right) \left( e^{-\frac{i\omega}{2}} - e^{\frac{i\omega}{2}} \right)}{2\sinh \frac{3\omega}{2}}.
\]
(46)
This expression should be compared with the expressions computed in the Yangian \( sl(3) \) model. More specifically, the soliton–soliton and soliton–anti-soliton amplitudes in that model are given by the following expressions
\[
\hat{r}_{s}(\omega) = \hat{a}_2(\omega) \hat{R}_{11}(\omega) - \hat{a}_1(\omega) \hat{R}_{12}(\omega) = \frac{e^{\frac{i\omega}{2}} - e^{-\frac{i\omega}{2}}}{2\sinh \frac{3\omega}{2}},
\]
\[
\hat{r}_{s}(\omega) = \hat{a}_2(\omega) \hat{R}_{12}(\omega) - \hat{a}_1(\omega) \hat{R}_{11}(\omega) = \frac{1 - e^{-i\omega}}{2\sinh \frac{3\omega}{2}},
\]
(47)
where \( \hat{R}_{ij}(\omega) \) denotes the inverse of the kernel for the bulk \( sl(3) \) scattering [18]
\[
\hat{R}_{ij}(\omega) = e^{\frac{i\omega}{2}} \frac{\sinh \left( \min(i,j) \frac{|\omega|}{2} \right) \sinh \left( (3 - \max(i,j)) \frac{|\omega|}{2} \right)}{\sinh \frac{|\omega|}{2} \sinh \frac{3|\omega|}{2}}.
\]
(48)
A quick inspection of relations (46) and (47) reveals that
\[
\hat{r}_{s}(\omega) = \hat{r}_{s}(\omega) + \hat{r}_{s}(\omega) \Rightarrow S(\lambda) = S'(\lambda) \overline{S}(\lambda),
\]
(49)
where we define:
\[
X(\lambda) = \exp \left[ -\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \hat{r}_{X}(\omega) e^{-i\omega\lambda} \right], \quad X \in \{ S, S, \overline{S} \}.
\]
(50)
Relation (49) expresses the expected factorization of the bulk amplitude into two separate ones, the soliton–soliton and soliton–anti-soliton amplitude, (see [19] and references therein), which correspond to $S(\lambda)$ and $\bar{S}(\lambda)$. This fact confirms the validity of the form of the quantization condition as formulated in (40).

### 3.3. Boundary scattering amplitude

Let us now come to the study of the boundary scattering. Recalling the first relation of (45), we denote the boundary amplitude as

$$k^+(\lambda) k^-(\lambda) = \exp \left[ - \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \left( e^{-i\omega \lambda} \hat{r}_1(\omega) + e^{-2i\omega \lambda} \hat{r}_2(\omega) \right) \right] = \exp \left[ A_1 + A_2 \right].$$

(51)

It is convenient here as in the bulk case to express the boundary scattering amplitudes in terms of $\Gamma$-functions. For that purpose we use the identity

$$\frac{1}{2} \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{e^{-\frac{\omega}{2}}}{\cosh \frac{\omega}{2}} = \ln \frac{\Gamma\left(\frac{\omega+1}{4}\right)}{\Gamma\left(\frac{\omega+3}{4}\right)},$$

(52)

and we therefore express the amplitude in the form

$$A_1 = -\frac{1}{2} \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{e^{-i\omega \lambda}}{\cosh \frac{3\omega}{4}} + \int_{0}^{\infty} \left( \lambda \rightarrow -\lambda \right).$$

(53)

Using the identity (52) as well as

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

we compute the boundary contribution $A_1$

$$S^{(1)} = \exp(A_1) = \tan \frac{\pi}{2} (\lambda-1) \frac{\Gamma\left(\frac{\lambda}{4} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda}{4} + \frac{3}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{5}{4}\right)}{\tan \frac{\pi}{2} (\lambda+1) \Gamma\left(\frac{\lambda}{4} + \frac{1}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{3}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{5}{4}\right)}$$

$$\times \Gamma\left(\frac{\lambda}{4} + \frac{7}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{9}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{11}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{13}{4}\right).$$

(55)

Let us also compute the other term associated to the boundary scattering amplitude:

$$A_2 = -\frac{1}{2} \int_{0}^{\infty} \frac{d\omega}{\omega} \frac{e^{-2i\omega \lambda}}{\cosh \frac{3\omega}{4}} + \int_{0}^{\infty} \left( \lambda \rightarrow -\lambda \right),$$

(56)

Using the identity (52) together with the duplication formula for the $\Gamma$-function

$$\Gamma(x) \Gamma\left(x + \frac{1}{2}\right) = 2^{-2x+1} \sqrt{\pi} \Gamma(2x),$$

(57)

we obtain

$$S^{(2)} = \exp(A_2) = \frac{\Gamma\left(\frac{\lambda}{4} + \frac{1}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{3}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{5}{4}\right) \Gamma\left(\frac{\lambda}{4} + 1\right)}{\Gamma\left(\frac{\lambda}{4} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda}{4} + \frac{3}{2}\right) \Gamma\left(\frac{\lambda}{4} + \frac{5}{2}\right) \Gamma\left(\frac{\lambda}{4} + \frac{3}{4}\right)}$$

$$\times \Gamma\left(\frac{\lambda}{4} + \frac{7}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{9}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{11}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{13}{4}\right).$$

(58)

Finally, the total boundary amplitude associated to the left and right boundary scattering is given as:

$$k^+(\lambda) k^-(\lambda) = S^{(1)} S^{(2)} = \tan \frac{\pi}{2} (\lambda-1) \Gamma\left(\frac{\lambda}{4} + \frac{1}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{3}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{5}{4}\right) \Gamma\left(\frac{\lambda}{4} + 1\right) \times \Gamma\left(\frac{\lambda}{4} + \frac{7}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{9}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{11}{4}\right) \Gamma\left(\frac{\lambda}{4} + \frac{13}{4}\right).$$

(59)

This concludes our derivation of the boundary scattering amplitude. Notice that since we have chosen the simplest reflection matrices $K^\pm \propto I$, one only needs to compute the overall physical factor (amplitude) for the left and right boundary scattering.



doi:10.1088/1742-5468/2015/02/P02007
4. Discussion

The bulk and boundary scattering in the context of the $\mathfrak{sl}(3)$ twisted Yangian is studied. The analysis in based on the Bethe ansatz methodology. In particular, the thermodynamic limit of the associated Bethe ansatz equations is studied and the ground state and excitations are determined. The scattering among the particle-like excitations gives rise to a factorized form expressed explicitly as a product of the soliton–soliton times the soliton–anti-soliton scattering amplitude of the bulk $\mathfrak{sl}(3)$ case. Moreover, the interaction of the excitation with the boundary is studied and the corresponding boundary scattering amplitude is derived. Note that we have considered here the simplest boundary matrices i.e. $K^\pm \propto \mathbb{I}$ ($K^\pm(\lambda) = k^\pm(\lambda) \mathbb{I}$). One of the key points in this investigation together with the study of the boundary scattering is the formulation of the suitable quantization condition compatible with the underlying algebraic setting as well as the corresponding physical interpretation. This is also confirmed by the fact that the bulk scattering factorizes into the product of the soliton–soliton and soliton–antisoliton scattering amplitudes.

It is worth pointing out that in the particular case under study as well as for the generic $\mathfrak{sl}(2n+1)$ case the Bethe ansatz equations are similar to the $\mathfrak{osp}(1|2n)$ case, whereas in the $\mathfrak{sl}(2n)$ case they are a bit modified. In any case, the next natural step is to generalize these computations for the $\mathfrak{sl}(n)$ case. Furthermore, the study of defects within the context of the twisted Yangian is a very interesting direction to pursue. Hopefully, the aforementioned issues will be addressed in a forthcoming publication.

Acknowledgments

AD wishes to thank University of Cergy-Pontoise, where part of this work was completed, for kind hospitality. We thank the referees for their helpful comments and suggestions on the presentation of our results.

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doi:10.1088/1742-5468/2015/02/P02007