Local derivations on Rings containing a von Neumann algebra and a question of Kadison.

Don Hadwin, Jiankui Li, Qihui Li, and Xiujuan Ma

Abstract. We prove that if $M$ is a von Neumann algebra whose abelian summand is discrete, then every local derivation on the algebra of all measurable operators affiliated with $M$ is a derivation. This answers a question of Richard Kadison.

At a conference held in Texas A & M University of 2012, Richard Kadison gave a talk about his joint work with Zhe Liu [10, 11] in which they proved that the only derivation that maps the algebra $S(M)$ of closed densely defined operators affiliated with a factor von Neumann algebra $M$ of type II$_1$ into that von Neumann algebra is zero. Kadison asked whether every local derivation on $S(M)$ is a derivation.

In this note we prove a general ring-theoretic result which implies that for all von Neumann algebras $M$ whose abelian summand is discrete, every local derivation on $S(M)$ of all measurable operators is a derivation.

If $R$ is a ring (resp. algebra) and $\delta : R \rightarrow R$ is an additive (resp. linear) mapping, we say that $\delta$ is a derivation if, for all $a, b \in R$, we have

$$\delta(ab) = \delta(a)b + a\delta(b).$$

We say that an additive mapping $\delta$ is a local derivation if, for every $x \in R$ there is a derivation $\rho_x$ on $R$ such that

$$\delta(x) = \rho_x(x).$$

To prove our main result, we need two lemmas. The first is a result of [3, Corollary 4.5]. Suppose $R$ is a ring with identity 1 and $n \geq 2$ is an integer. We say that a subset $\{E_{ij} : 1 \leq i, j \leq n\} \subset R$ is a system of $n \times n$ matrix units for $R$ if and only if $\sum_{i=1}^{n} E_{ii} = 1$ and, for $1 \leq i, j, s, t \leq n$, $E_{ij}E_{st} = 0$ if $j \neq s$ and $E_{ij}E_{st} = E_{it}$ if $j = s$.

In [3, p.11], Bresar shows that for any unital ring $B$, the ring $M_n(B)$ is generated by the set of all idempotents in $B$, where $2 \leq n$.

The following Lemma 1 is a special case of [3, Corollary 4.5].

Lemma 1. If $R$ is a ring with identity 1 that possesses a system of $n \times n$ matrix units for some $n \geq 2$, then every local derivation on $R$ is a derivation.

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Note that, in general, derivations need not leave ideals invariant, e.g., differentiation on the polynomials. However, if \( p \) a central idempotent, then every derivation (hence, every local derivation) \( \delta \) leaves \( pR \) and \( (1 - p)R \) invariant, since
\[
\delta(pa) = \delta((pa)p) = pa\delta(p) + \delta(pa)p.
\]
Moreover, \( pR \) is isomorphic to \( R / (1 - p)R \). This yields the following corollary to the preceding lemma. A family \( P \) of central idempotents for a ring \( R \) is separating if and only if, for each nonzero \( x \in R \), there is a \( p \in P \) such that \( px \neq 0 \).

**Lemma 2.** Suppose \( R \) is a ring with identity and \( P \) is a separating family of central idempotents such that, for each \( p \in P \), every local derivation on \( pR \) is a derivation. Then every local derivation on \( R \) is a derivation.

**Proof.** Suppose \( \delta : R \to R \) is a local derivation, \( a \in R \) and \( p \in P \). Then there is a derivation \( \rho \) on \( R \) such that
\[
\delta(pa) = \rho(pa) = \rho(p(pa)) = pp(a) + \rho(p)pa = p[\rho(pa) + \rho(p)a] = pgr(pa) = p\delta(pa).
\]
It follows that \( \delta(pR) \subset pR \) and that \( \delta|pR \) is a local derivation. Hence \( \delta|pR \) is a derivation. Thus, for every \( a, b \in R \) and every \( p \in P \), we have
\[
p\delta(ab) = \delta(pab) = \delta((pa)(pb)) = pa\delta(pb) + \delta(pa)pb = p[a\delta(b) + \delta(a)b].
\]
Hence
\[
p[\delta(ab) - [a\delta(b) + \delta(a)b]] = 0.
\]
Since \( P \) is separating, we see that \( \delta \) is a derivation on \( R \). \( \square \)

An abelian von Neumann algebra \( M \) is discrete if it is generated by its minimal nonzero projections; equivalently, if the identity operator in \( M \) is the sum of the minimal projections in \( M \). Since \( M \) is abelian, it follows that \( QM = CQ \) for every minimal projection \( Q \) in \( M \). Every von Neumann algebra on a Hilbert space \( H \) is the direct sum of algebras \( M_n \) with \( 1 \leq n < \infty \) (the finite type \( I_n \) summands) and a von Neumann algebra \( M_\infty \), which is the direct sum of algebras of type \( I_\infty \), \( II \), and \( III \) (not all summands need be present). Call the corresponding central projections \( P_n \) with \( 1 \leq n \leq \infty \).

**Theorem 1.** Suppose \( M \) is a von Neumann algebra on a Hilbert space \( H \). Then

1. If \( M_1 = 0 \) and \( R \) is a ring containing \( M \) with the same identity as \( M \) such that \( P = \{ P_n : 2 \leq n \leq \infty \} \) is a separating family of central idempotents for \( R \), then every local derivation on \( R \) is a derivation.
2. If \( M_1 \) is discrete, then every local derivation on the algebra of closed densely defined operators affiliated with \( M \) is a derivation.

**Proof.** (1).

By [9, Theorem 6.6.5], we know that \( P_nM \) contains an \( n \times n \) system of matrix units for \( 2 \leq n < \infty \), and it follows from [9, Lemma 6.5.6] that \( P_\infty M \) contains a \( 2 \times 2 \) system of matrix units. Since \( P_nM \subset P_nR \) is a unital embedding and \( \mathcal{P} \) is separating, it follows from the two lemmas above that every local derivation on \( R \) is a derivation.
Let $\mathcal{R}$ be the algebra of all measurable operators affiliated with $\mathcal{M}$. However, since $\mathcal{M}_1$ is discrete, $\mathcal{P}_1$ is the orthogonal sum of a family $\{Q_\lambda : \lambda \in \Lambda\}$ of minimal projections and that, for each $\lambda \in \Lambda$, $Q_\lambda \mathcal{M} = \mathbb{C} Q_\lambda$, which means that $Q_\lambda \mathcal{M}' = B(Q_\lambda H) Q_\lambda$, which, in turn, implies that $Q_\lambda \mathcal{R} = \mathbb{C} Q_\lambda$, so every local derivation on $Q_\lambda \mathcal{R}$ is a derivation. Since the elements of $\mathcal{R}$ are densely defined operators on $H$, and $\sum_{\lambda \in \Lambda} Q_\lambda + \sum_{2 \leq n \leq \infty} P_n = 1$, it follows that $\{Q_\lambda : \lambda \in \Lambda\} \cup \{P_n : 2 \leq n \leq \infty\}$ is a separating family of central idempotents for $\mathcal{R}$. Arguing as in the proof of (1) we can apply the lemmas to see that every local derivation on $\mathcal{R}$ is a derivation. □

In [1, Theorem 3.8], the authors give necessary and sufficient conditions on a commutative von Neumann algebra $\mathcal{M}$ for the existence of local derivations which are not derivations on the algebra $S(\mathcal{R})$ of measurable operators affiliated with $\mathcal{M}$.

For a von Neumann algebra $\mathcal{M}$, we can define the set $S(\mathcal{M})$ of all measurable operators affiliated with $\mathcal{M}$ and the set $LS(\mathcal{M})$ of all local measurable operators affiliated with $\mathcal{M}$.

In [12], Muratov and Chilin show that $LS(\mathcal{M})$ is a unital $*$-algebra when equipped with algebraic operations of the strong addition, multiplication, and taking the adjoint of an operator and $S(S)$ is a unital $*$-subalgebra of $LS(\mathcal{M})$.

Suppose that $\mathcal{M}$ is a von Neumann algebra with a faithful normal semi-finite trace $\tau$. Let $S(\mathcal{M}, \tau)$ denote the algebra of all $\tau$-measurable operators affiliated with $\mathcal{M}$. It is clear that $\mathcal{M} \subseteq S(\mathcal{M}) \subseteq LS(\mathcal{M})$ (see for details [1, 2, 13]).

In Theorem 1, if we choose that $\mathcal{R}$ is $\mathcal{M}$, $S(\mathcal{M})$ or $LS(\mathcal{M})$, we can obtain [1, Theorem 2.5 and Proposition 2.7].

By Theorem 1 and [1, Theorem 3.8], we can completely answer [15, Conjecture 48].

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**Department of Mathematics, University of New Hampshire, Durham, NH 03824, USA**

*E-mail address:* don@unh.edu

**URL:** http://euclid.unh.edu/~don

**Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China**

*E-mail address:* jiankuili@yahoo.com

**Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China**

*E-mail address:* lqh991978@gmail.com

**Department of Mathematics, Hebei University of Technology, Tianjing, 300130, China**

*E-mail address:* mxjsusan@hebut.edu.cn