Revising Ontologies via Models: The $\text{ALC}$-formula Case

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Abstract. Most approaches for repairing description logic (DL) ontologies aim at changing the axioms as little as possible while solving inconsistencies, incoherences and other types of undesired behaviours. As in Belief Change, these issues are often specified using logical formulae. Instead, in the new setting for updating DL ontologies that we propose here, the input for the change is given by a model which we want to add or remove. The main goal is to minimise the loss of information, without concerning with the syntactic structure. This new setting is motivated by scenarios where an ontology is built automatically and needs to be refined or updated. In such situations, the syntactical form is often irrelevant and the incoming information is not necessarily given as a formula. We define general operations and conditions on which they are applicable, and instantiate our approach to the case of $\text{ALC}$-formulae.

1 Introduction

Formal specifications often have to be updated either due to modelling errors or because they have become obsolete. When these specifications are description logic (DL) ontologies, it is possible to use one of the many approaches to fix missing or unwanted behaviours. Usually these methods involve the removal or replacement of formulae responsible by the undesired aspect \[2, 16, 20, 28, 29\].

The problem of changing logical representations of knowledge upon the arrival of new information is the subject matter of Belief Change \[13\]. The theory developed in this field provides constructions suitable for various formalisms and applications \[13, 23, 24, 27\]. In most approaches for Belief Change and for repairing ontologies, it is assumed that a set of formulae represents the entailments to be added or removed. However, in some situations, it might be easier to obtain this information as a model instead. This idea relates with Model Checking \[3\] whose main problem is to determine whether a model satisfies a set of constraints; and with the paradigm of Learning from Interpretations \[4\], where a
formula needs to be created or changed so as to have certain interpretations as part of its models and remove others from its set of models. Example 1 illustrates the intuition behind using models as input.

Example 1. Suppose that a system, which serves a university, uses an internal logical representation of the domain with a open world behaviour and unique names. Let $B$ be its current representation:

$$B = \{\text{Professors} : \{\text{Mary}\}, \text{Courses} : \{\text{DL, AI}\},$$

$$\{\text{teaches} : \{(\text{Mary, AI}), (\text{Mary, DL})\}\}.$$ 

Assume that a user finds mistakes in the course schedule and this is caused by the wrong information that Mary teaches the DL course. The user may lack knowledge to define the issue formally. An alternative would be to provide the user with an interface where one can specify, for instance, that the following model should be accepted $M = \{\text{Professors} = \{\text{Mary}\}, \text{Courses} = \{\text{DL, AI}\}, \text{teaches} = \{(\text{Mary, AI})\}\}$, (in this model Mary does not teach the DL course). Given this input, the system should repair itself (semi-)automatically.

We propose a new setting for Belief Change, in particular, contraction and expansion functions which take models as input. We analyse the case of $\mathcal{ALC}$-formula using quasimodels as a mean to define belief change operations. This logic satisfies properties which facilitate the design of these operations and it is close to $\mathcal{ALC}$, which is a well-studied DL. Additionally, we identify the postulates which determine these functions and prove that they characterise the mathematical constructions via representation theorems. The remaining of this work is organised as follows: in Section 2 we introduce the concepts from Belief Change which our approach builds upon and detail the paradigm we propose here. Section 3 presents $\mathcal{ALC}$-formula, the belief operations that take models as input and their respective representation theorems. In Section 4, we highlight studies which share similarities with our proposal and we conclude in Section 5.

2 Belief Change

2.1 The Classical Setting

Belief Change [1, 13] studies the problem of how an agent should modify its knowledge in light of new information. In the original paradigm of Belief Change, the AGM theory, an agent’s body of knowledge is represented as a set of formulae closed under logical consequence, called a belief set, and the new information is represented as a single formula. Belief sets, however, are not the only way for representing an agent’s body of knowledge, and another way of representing an agent’s knowledge is via belief bases: arbitrary sets of formulae, not necessarily closed under logical consequence [11]. In the AGM paradigm, an agent may modify its current belief base $B$ in response to a new piece of information $\varphi$ through three kinds of operations:
**Expansion:** \( \text{ex}(B, \varphi) \), simply add \( \varphi \) to \( B \);

**Contraction:** \( \text{con}(B, \varphi) \), reduce \( B \) so that it does not imply \( \varphi \);

**Revision:** \( \text{rev}(B, \varphi) \), incorporate \( \varphi \) and keep consistency of the resulting belief base, as long as \( \varphi \) is consistent.

When modifying its body of knowledge an agent should rationally modify its beliefs conserving most of its original beliefs. This principle of minimal change is captured in Belief Change via sets of rationality postulates. Each of the three operations (expansion, contraction and revision) presents its own set of rationality postulates which characterize precisely different classes of belief change constructions. The AGM paradigm was initially proposed for classical logics that satisfy specific requirements, dubbed AGM assumptions, among them Tarskian-icity, compactness and deduction. See [6, 24] for a complete list of the AGM assumptions and a discussion on the topic. Recently, efforts have been applied to extend Belief Change to logics that do not satisfy such assumptions. For instance, logics that are not closed under classical negation of formulae (such as is the case for most DLs) [24, 26], and temporal logics and logics without compactness [21–23].

In what follows, we define kernel contraction [13], one of the most studied constructions in Belief Change and which is closely related to the most common ways to repair ontologies. Kernel operations rely on calculating the minimal implying sets (MinImps), also known as justifications [15] or kernels [13]. A MinImp is a minimal subset that does entail a formula \( \varphi \). The set of all MinImps of a belief base \( B \) w.r.t. a formula \( \varphi \) is denoted by \( \text{MinImps}(B, \varphi) \). A kernel contraction removes from each MinImp at least one formula using an incision function.

**Definition 2.** Given a set of formulae \( B \) of language \( L \), a function \( f \) is an incision function for \( B \) iff for all \( \varphi \in L \) : (i) \( f(\text{MinImps}(B, \varphi)) \subseteq \bigcup \text{MinImps}(B, \varphi) \) and (ii) \( f(\text{MinImps}(B, \varphi)) \cap X \neq \emptyset \), for all \( X \in \text{MinImps}(B, \varphi) \).

Kernel contraction operators are built upon incision functions. The application of an incision function to a set of MinImps is a hitting set [2, 16].

**Definition 3.** Let \( L \) be a language and \( f \) an incision function. The kernel contraction on \( B \subseteq L \) determined by \( f \) is the operation \( \text{con}_f : 2^L \times L \to 2^L \) defined as:

\[
\text{con}_f(B, \varphi) = B \setminus f(\text{MinImps}(B, \varphi)).
\]

Kernel contraction operations are characterised precisely by a set of rationality postulates, as shown in the following representation theorem:

**Theorem 4 ([14]).** Let \( \text{Cn} \) be a consequence operator satisfying monotonicity and compactness defined for a language \( L \). Then \( \text{con} : 2^L \times L \to 2^L \) is an operation of kernel contraction on \( B \subseteq L \) iff for all sentences \( \varphi \in L \):

- (success) if \( \varphi \notin \text{Cn}(\emptyset) \), then \( \varphi \notin \text{Cn}(\text{con}(B, \varphi)) \),
- (inclusion) \( \text{con}(B, \varphi) \subseteq B \),
- (core-retainment) if \( \psi \in B \setminus \text{con}(B, \varphi) \), then there is some \( B' \subseteq B \) such that \( \varphi \notin \text{Cn}(B') \) and \( \varphi \in \text{Cn}(B' \cup \psi) \),
- (uniformity) if for all subsets \( B' \) of \( B, \ \varphi \in \text{Cn}(B') \) iff \( \psi \in \text{Cn}(B') \), then \( \text{con}(B, \varphi) = \text{con}(B, \psi) \).
2.2 Changing Finite Bases by Models

The Belief Change setting discussed in this section represents an epistemic state by means of a finite base. While this essentially differs from the traditional approach [1, 11], it aligns with the KM paradigm established by Katsuno and Mendelzon [17]. In Section 4 we discuss other studies in Belief Change which also take finite representability into account.

In this work, unlike the standard representation methods in Belief Change, we consider that an incoming piece of information is represented as a finite model. Belief Change operations defined in this format will be called model change operations. Recall that a model $M$ is simply a structure used to give semantics to an underlying logic language. The set of all possible models is given by $\mathfrak{M}$. Moreover, we assume a semantic system that, for each set of formulae $\mathcal{B}$ of the language $\mathcal{L}$ gives a set of models $\text{Mod}(\mathcal{B}) := \{ M \in \mathfrak{M} \mid \forall \varphi \in \mathcal{B} : M \models \varphi \}$. Let $\mathcal{P}_{\text{fin}}(\mathcal{L})$ denote the set of all finite bases in $\mathcal{L}$. We also say that a set of models $M$ is finitely representable in $\mathcal{L}$ if there is a finite base $B \in \mathcal{P}_{\text{fin}}(\mathcal{L})$ such that $\text{Mod}(B) = M$. Additionally, if for all $\varphi \in \mathcal{L}$ it holds that $M \models \varphi$ iff $M' \models \varphi$ then we write $M \equiv^\mathcal{L} M'$. We also define $[M]^\mathcal{L} := \{ M' \in \mathfrak{M} \mid M' \equiv^\mathcal{L} M \}$.

When compared to traditional methods in Belief Change and Ontology Repair [2, 13, 16], where the incoming information comes as a single formula, our approach receives instead a single model as input. Although, the initial body of knowledge is represented as a finite base, the operations we define do not aim to preserve its syntactic structure.

The first model change operation we introduce is model contraction, which eliminates one of the models of the current base (which in Section 3 is instantiated as an ontology). Model contraction is akin to a belief expansion, where a formula is added to the belief set or base, reducing the set of accepted models. The counterpart operation, model expansion, changes the base to include a new model. This relates to belief contraction, in which a formula is removed, and thus more models are seen as plausible.

We rewrite the rationality postulates that characterize kernel contraction [14], considering an incoming piece of information represented as a model instead of a single formula.

**Definition 5 (Model Contraction).** Let $\mathcal{L}$ be a language. A function $\text{con} : \mathcal{P}_{\text{fin}}(\mathcal{L}) \times \mathfrak{M} \mapsto \mathcal{P}_{\text{fin}}(\mathcal{L})$ is a finitely representable model contraction function iff for every $B \in \mathcal{P}_{\text{fin}}(\mathcal{L})$ and $M \in \mathfrak{M}$ it satisfies the following postulates:

- (success) $M \notin \text{Mod}(\text{con}(B, M)) = \emptyset$,
- (inclusion) $\text{Mod}(\text{con}(B, M)) \subseteq \text{Mod}(B)$,
- (retainment) if $M' \in \text{Mod}(B) \setminus \text{Mod}(\text{con}(B, M))$ then $M' \equiv^\mathcal{L} M$,
- (extensionality) $\text{con}(B, M) = \text{con}(B, M')$, if $M \equiv^\mathcal{L} M'$.

We might also need to add a model to the set of models of the current base. This addition relates to classical contractions in Belief Change, which reduces the belief base.
Definition 6 (Model Expansion). Let $\mathcal{L}$ be a language. A function $\text{ex} : \mathcal{P}_{\text{fin}}(\mathcal{L}) \times \mathcal{M} \mapsto \mathcal{P}_{\text{fin}}(\mathcal{L})$ is a finitely representable model expansion iff for every $B \in \mathcal{P}_{\text{fin}}(\mathcal{L})$ and $M \in \mathcal{M}$ it satisfies the postulates:

1. **success** $M \in \text{Mod}(\text{ex}(B, M))$,
2. **persistence** $\text{Mod}(B) \subseteq \text{Mod}(\text{ex}(B, M))$,
3. **vacuity** $\text{Mod}(\text{ex}(B, M)) = \text{Mod}(B)$, if $M \in \text{Mod}(B)$,
4. **extensionality** $\text{ex}(B, M) = \text{ex}(B, M')$, if $M \equiv^E M'$.

Definition 7. Let $\mathcal{L}$ be a language and $\text{Cn}$ a Tarskian consequence operator defined over $\mathcal{L}$. Also let $\mathcal{M}$ be a fixed set of models. We say that a triple $\Lambda = (\mathcal{L}, \text{Cn}, \mathcal{M})$ is an ideal logical system if the following holds.

- For every $B \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$, $B \models \varphi$ (i.e. $\varphi \in \text{Cn}(B)$) iff $\text{Mod}(B) \subseteq \text{Mod}(\varphi)$.
- For each $M \subseteq \mathcal{M}$ there is a finite set of formulae $B$ such that $\text{Mod}(B) = M$.

If $\Lambda = (\mathcal{L}, \text{Cn}, \mathcal{M})$ is an ideal logical system, we can define a function $\text{FR}_\Lambda : 2^\mathcal{M} \mapsto \mathcal{P}_{\text{fin}}(\mathcal{L})$ and such that $\text{Mod}(\text{FR}(M)) = M$. Then, we can define model contraction as $\text{con}(B, M) = \text{FR}(\text{Mod}(B) \setminus [M]^E)$ and expansion as $\text{ex}(B, M) = \text{FR}(\text{Mod}(B) \cup [M]^E)$. The first condition in Definition 7 implies that there is a connection between the models satisfied and the logical consequences of the base obtained and the second ensures that the result always exists. An example that fits these requirements is to consider classical propositional logic with a finite signature $\Sigma$, together with its usual consequence operator and models. In this situation, we can define $\text{FR}_{\text{prop}}$ as follows:

$$\text{FR}_{\text{prop}}(M) = \bigvee_{M \in \mathcal{M}} \left( \bigwedge_{a \in \Sigma \mid M \models a} a \bigwedge_{a \in \Sigma \mid M \models \neg a} \neg a \right).$$

Next, we show that the construction proposed with FR has the properties stated in Definitions 4 and 6.

Theorem 8. Let $\Lambda = (\mathcal{L}, \text{Cn}, \mathcal{M})$ be an ideal logical system as in Definition 7. Then $\text{iCon}(B, M) := \text{FR}(\text{Mod}(B) \setminus [M]^E)$ satisfies the postulates in Definition 6.

Proof. Definition 7 ensures that the result exists and that $M \models \varphi$, for all $\varphi \in \Lambda$, giving us success. By construction we do gain models, thus we have inclusion. If $M \equiv^E M'$, then $[M]^E = [M']^E$, thus extensionality is satisfied. Also, if $M' \in \text{Mod}(\varphi) \setminus \text{Mod}(\text{iCon}(\varphi, M))$ then $M' \in [M]^E$, hence the operation satisfies retention.

Theorem 9. Let $\Lambda = (\mathcal{L}, \text{Cn}, \mathcal{M})$ be an ideal logical system as in Definition 7. Then $\text{iExp}(B, M) := \text{FR}(\text{Mod}(B) \cup [M]^E)$ satisfies the postulates in Definition 6.

Proof. Definition 7 ensures that the result exists and that $M \models \varphi$, for all $\varphi \in \Lambda$, giving us success. Due to the first condition in Definition 7 we gain vacuity: if $M \in \text{Mod}(B)$, then there will be no changes in the accepted models. By construction we do not lose models, thus we have persistance. Extensionality also holds because whenever $M \equiv^E M'$ we have then $[M]^E = [M']^E$. 
A revision operation incorporates new formulae, and removes potential conflicts in behalf of consistency. In our setting, incorporating information coincides with model contraction which could lead to an inconsistent belief state. In this case, model revision could be interpreted as a conditional model contraction: in some cases the removal might be rejected to preserve consistency. We leave the study on revision as a future work.

3 The case of \( \mathcal{ALC} \)-formula

The logic \( \mathcal{ALC} \)-formula corresponds to the DL \( \mathcal{ALC} \) enriched with boolean operators over \( \mathcal{ALC} \) axioms. As discussed in Section 2.2 in finite representable logics, such as the classical propositional logics, we can easily add and remove models while keeping the representation finite. For \( \mathcal{ALC} \)-formula, however, it is not possible to uniquely add or remove a new model \( M \) since, for instance, the language does not distinguish quantities (e.g., a model \( M \) and another model that has two duplicates of \( M \)).

Even if quantities are disregarded and our input is a class of models indistinguishable by \( \mathcal{ALC} \)-formulae, there are sets of formulae in this language that are not finitely representable. As for instance in the following infinite set:

\[ \{ C \sqsubseteq \exists r^n.T \mid n \in \mathbb{N}^>0 \} \]

where \( \exists r^n+1.T \) is a shorthand for \( \exists r.(\exists r^n.T) \) and \( \exists^1.T := \exists r.C \). As a workaround for the \( \mathcal{ALC} \)-formula case, we propose a new strategy based on the translation of \( \mathcal{ALC} \)-formulae into DNF.

3.1 \( \mathcal{ALC} \)-formulae and Quasimodels

Let \( N_C, N_R \) and \( N_I \) be countably infinite and pairwise disjoint sets of concept names, role names, and individual names, respectively. \( \mathcal{ALC} \) concepts are built according to the rule:

\[ C ::= A \mid \neg C \mid (C \sqcap C) \mid \exists r.C \],

where \( A \in N_C \) and \( r \in N_R \). \( \mathcal{ALC} \)-formulae are defined as expressions \( \phi \) of the form

\[ \phi ::= \alpha \mid \neg(\phi) \mid (\phi \land \phi) \quad \alpha ::= C(a) \mid r(a,b) \mid (C = T) \]

where \( C \) is an \( \mathcal{ALC} \) concept, \( a,b \in N_I \), and \( r \in N_R \). Denote by ind(\( \varphi \)) the set of all individual names occurring in an \( \mathcal{ALC} \)-formula \( \varphi \).

The semantics of \( \mathcal{ALC} \)-formulae and the definitions related to quasimodels are standard \[8, page 70]\. In what follows, we reproduce the essential definitions and results for this work. Let \( \varphi \) be an \( \mathcal{ALC} \)-formula. Let \( f(\varphi) \) and \( c(\varphi) \) be the set of all subformulae and subconcepts of \( \varphi \) closed under single negation, respectively.

**Definition 10.** A concept type for \( \varphi \) is a subset \( c \subseteq c(\varphi) \) such that:

1. \( D \in c \) iff \( \neg D \notin c \), for all \( D \in c(\varphi) \);
2. \( D \sqcap E \in c \) iff \( \{D, E\} \subseteq c \), for all \( D \sqcap E \in c(\varphi) \).

\[ \square \] We may omit parentheses if there is no risk of confusion. The usual concept inclusions \( C \sqsubseteq D \) can be expressed with \( T \subseteq \neg C \sqcup D \) and \( \neg C \sqcup D \sqsubseteq T \), which is \( (\neg C \sqcup D = T) \).
Definition 11. A formula type for $\varphi$ is a subset $f \subseteq f(\varphi)$ such that:

1. $\phi \in f$ iff $\neg \phi \notin f$, for all $\phi \in f(\varphi)$;
2. $\phi \land \psi \in f$ iff $\{\phi, \psi\} \subseteq f$, for all $\phi \land \psi \in f(\varphi)$.

We may omit ‘for $\varphi$’ if this is clear from the context. A model candidate for $\varphi$ is a triple $(T,o,f)$ such that $T$ is a set of concept types, $o$ is a function from $\text{ind}(\varphi)$ to $T$, $f$ a formula type, and $(T,o,f)$ satisfies the conditions: $\varphi \in f$; $C(a) \in f$ implies $C \in o(a)$; $r(a,b) \in f$ implies $\{\neg C \mid \neg \exists r. C \in o(a)\} \subseteq o(b)$.

Definition 12 (Quasimodel). A model candidate $(T,o,f)$ for $\varphi$ is a quasi-model for $\varphi$ if the following holds

- for every concept type $c \in T$ and every $\exists r. D \in c$, there is $c' \in T$ such that $\{D\} \cup \{\neg E \mid \neg \exists r. E \in c\} \subseteq c'$;
- for every concept type $c \in T$ and every concept $C$, if $\neg C \in c$ then this implies $(C = \top) \notin f$;
- for every concept $C$, if $\neg (C = \top) \in f$ then there is $c \in T$ such that $C \notin c$;
- $T$ is not empty.

Theorem 13 motivates the decision of using quasimodels to implement our operations for finite bases described in ALC-formulae.

Theorem 13 (Theorem 2.27 [8]). An ALC-formula $\varphi$ is satisfiable iff there is a quasimodel for $\varphi$.

3.2 ALC-formulae in Disjunctive Normal Form

Next, we propose a translation method which converts an ALC-formula into a disjunction of conjunctions of (possibly negated) atomic formulae. Let $S(\varphi)$ be the set of all quasimodels for $\varphi$. We define $\varphi^\dagger$ as

$$
\bigvee_{(T,o,f) \in S(\varphi)} \left( \bigwedge_{\alpha \in f} \alpha \land \bigwedge_{\neg \alpha \in f} \neg \alpha \right).
$$

where $\alpha$ is of the form $(C = \top), C(a), r(a, b)$.

Theorem 14 confirms the equivalence between a formula and its translation into DNF. As downside, the translation can be potentially exponentially larger than the original formula.

Theorem 14. For every ALC-formula $\varphi$, we have that $\varphi \equiv \varphi^\dagger$.

In the next subsections, we present finite base model change operations for ALC-formulae, i.e., functions from $L \times M \mapsto L$. We can represent the body of knowledge as a single formula because every finite belief base of ALC-formulae can be represented by the conjunction of its elements. We use our translation to add models in a “minimal” way by adding disjuncts, while removing a model amounts to removing disjuncts. We also need to obtain a model candidate relative to our translated formula, as show in Definition 15.
Definition 15 ([8]). Let $I$ be an interpretation and $\varphi$ an ALC-formula formula. The quasimodel of $I$ w.r.t. $\varphi$, symbols $qm(I) = (T, o, f)$, is

- $T := \{ c(x) \mid x \in \Delta^I \}$, where $c(x) = \{ C \in c(\varphi) \mid x \in C^I \}$,
- $o(a) = c(a^I)$, for all $a \in \text{ind}(\varphi)$,
- $f := \{ \psi \in f(\varphi) \mid I \models \psi \}$.

3.3 Model Contraction for ALC-formulae

We define model contraction for ALC-formula using the notion of quasimodels discussed previously and a correspondence between models and quasimodels.

We use the following operator, denoted $\mu$, to define model contraction in Definition 16. Let $\varphi$ be an ALC-formula and let $M$ be a model. Then,

$$\mu(\varphi, M) = \text{ftypes}(\varphi) \setminus \{ f \},$$

and $\text{ftypes}(\varphi)$ is the set of all formula types in all quasimodels for $\varphi$, that is:

$$\text{ftypes}(\varphi) = \{ f \mid (T, o, f) \in S(\varphi) \}.$$

Let $\text{lit}(f) := \{ \ell \in f \mid \ell \text{ is a literal} \}$ be the set of all literals in a formula type $f$.

Definition 16. A finite base model contraction function is a function $\text{con} : \mathcal{L} \times \mathcal{M} \mapsto \mathcal{L}$ such that

$$\text{con}(\varphi, M) = \begin{cases} \bigwedge_{f \in \mu(\varphi, M)} \bigwedge_{\ell \in \text{lit}(f)} \varphi \quad & \text{if } M \models \varphi \text{ and } \mu(\varphi, M) \neq \emptyset \\ \bot \quad & \text{otherwise} \end{cases}$$

As we see later in this section, there are models $M, M'$ such that $M \not\equiv \mathcal{L} M'$ but our operations based on quasimodels cannot distinguish them. Given ALC-formulae $\varphi, \psi$, we say that $\psi$ is in the language of the literals of $\varphi$, written $\psi \in \mathcal{L}_{\text{lit}}(\varphi)$, if $\psi$ is a boolean combination of the atoms in $\varphi$. Our operations partition the models according to this restricted language. We write $M \equiv^\varphi M'$ instead of $M \equiv \mathcal{L}_{\text{lit}}(\varphi) M'$, and $[M]^\varphi$ instead of $[M]^\mathcal{L}_{\text{lit}}(\varphi)$ for conciseness.

Theorem 17. Let $M$ be a model and $\varphi$ an ALC-formula. A finite base model function $\text{con}^*(\varphi, M)$ is equivalent to $\text{con}(\varphi, M)$ iff $\text{con}^*$ satisfies:

- (success) $M \not\models \text{con}^*(\varphi, M)$,
- (inclusion) $\text{Mod}(\text{con}^*(\varphi, M)) \subseteq \text{Mod}(\varphi)$,
- (atomic retainment): For all $M' \subseteq \mathcal{M}$, if $\text{Mod}(\text{con}^*(\mathcal{B}, M)) \subset M' \subseteq \text{Mod}(\mathcal{B}) \setminus \{ M \}^\varphi$ then $M'$ is not finitely representable in ALC-formula.
- (atomic extensionality) if $M' \equiv^\varphi M$ then

$$\text{Mod}(\text{con}^*(\varphi, M)) = \text{Mod}(\text{con}^*(\varphi, M')).$$

The postulate of success guarantees that $M$ will be indeed relinquished, while inclusion imposes that no model will be gained during a contraction operation. Recall that in order to guarantee finite representability, it might be necessary to remove $M$ jointly with other models. The postulate atomic retainment captures a notion of minimal change, dictating which models are allowed to be removed together with $M$.

On the other hand, atomic extensionality imposes that if two models $M$ and $M'$ satisfy the same formulae within the literals of the current knowledge base $\varphi$, then they should present the same result.

A simpler way of implementing model contraction, also using the notion of a quasimodel,

**Definition 18.** Let $\varphi$ be an $\mathcal{ALC}$-formula and $M$ a model. Also, let $(T,o,f) = \text{qm}(\varphi,M)$. The function $\text{con}_s(\varphi,M)$ is defined follows:

$$\text{con}_s(\varphi,M) = \begin{cases} 
\varphi \land \neg(\bigwedge \text{lit}(f)) & \text{if } M \models \varphi \\
\varphi & \text{otherwise.}
\end{cases}$$

Example 19 illustrates how $\text{con}_s$ works.

**Example 19.** Consider the following $\mathcal{ALC}$-formula and interpretation $M$:

$$\varphi := P(Mary) \land C(DL) \land C(AI) \land ((\text{teaches}(Mary,DL) \land \neg\text{teaches}(Mary, AI)) \lor (\neg\text{teaches}(Mary, DL) \land \text{teaches}(Mary, AI)))$$

and $M = (\Delta^T, T)$, where $\Delta^T = \{m,d,a\}$, $C^T = \{d,a\}$, $P^T = \{m\}$, $\text{teaches}^T = \{(m,d)\}$, $Mary^T = m$, $AI^T = a$, and $DL^T = d$. Assume we want to remove $M$ from $\text{Mod}(\varphi)$. Let $\text{qm}(\varphi,M) = (T,o,f)$. Thus,

$$\text{con}_s(\varphi,M) = \varphi \land \neg \bigwedge \text{lit}(f)$$

$$\quad = \varphi \land (\neg\text{teaches}(m,a) \land \text{teaches}(m,d) \land C(d) \land C(a) \land P(m)).$$

Both model contraction operations $\text{con}$ and $\text{con}_s$ are equivalent.

**Theorem 20.** For every $\mathcal{ALC}$-formula $\varphi$ and model $M$, $\text{con}(\varphi,M) \equiv \text{con}_s(\varphi,M)$.

3.4 Model Expansion in $\mathcal{ALC}$-formulae

In this section, we investigate model expansion for $\mathcal{ALC}$-formulae. Recall that we assume that a knowledge base is represented as a single $\mathcal{ALC}$-formula $\varphi$. Expansion consists in adding an input model $M$ to the current knowledge base $\varphi$ with the requirement that the new epistemic state can be represented also as a finite formula.

**Definition 21.** Given a quasimodel $(T,o,f)$, we write $\bigwedge (T,o,f)$ as a short-cut for $\bigwedge \text{lit}(f)$. A finite base model expansion is a function $\text{ex} : \mathcal{L} \times \mathfrak{M} \rightarrow \mathcal{L}$ s.t.:

$$\text{ex}(\varphi,M) = \begin{cases} 
\varphi & \text{if } M \models \varphi \\
\varphi \lor \bigwedge \text{qm}(\neg\varphi,M) & \text{otherwise.}
\end{cases}$$
Example 22 illustrates how \( \text{ex} \) works.

Example 22. Consider the interpretation \( M \) from Example 19 and 
\[
\varphi := P(Mary) \land C(DL) \land C(AI) \land \text{teaches}(Mary, AI) \land \neg \text{teaches}(Mary, DL).
\]
Assume we want to add \( M \) to \( \text{Mod}(\varphi) \) and \( \text{qm}(\neg \varphi, M) = (T, o, f) \). Thus,
\[
\text{lit}(f) = \{ \neg \text{teaches}(m,a), \text{teaches}(m,d), C(d), C(a), P(m) \}
\]
\[
\text{ex}(\varphi, M) = \varphi \lor \bigwedge \text{lit}(f) = \varphi \lor (\neg \text{teaches}(m,a) \land \text{teaches}(m,d) \land C(d) \land C(a) \land P(m)).
\]

The operation ‘ex’ maps a current knowledge base represented as a single formula \( \varphi \) and maps it to a new knowledge base that is satisfied by the input model \( M \). The intuition is that ‘ex’ modifies the current knowledge base only if \( M \) does not satisfy \( \varphi \). This modification is carried out by making a disjunct of \( \varphi \) with a formula \( \psi \) that is satisfied by \( M \). This guarantees that \( M \) is present in the new epistemic state and that models of \( \varphi \) are not discarded. The trick is to find such an appropriate formula \( \psi \) which is obtained by taking the conjunction of all the literals within the quasimodel \( \text{qm}(\neg \varphi, M) \). Here, the quasimodel needs to be centred on \( \neg \varphi \) because \( M \not\models \varphi \), and therefore it is not possible to construct a quasimodel based on \( M \) centred on \( \varphi \). As discussed in the prelude of this section, this strategy not only adds \( M \) to the new knowledge base but also the whole equivalence class modulo the literals of \( \varphi \).

Lemma 23. For every ALC-formula \( \varphi \) and model \( M \):
\[
\text{Mod}(\text{ex}(\varphi, M)) = \text{Mod}(\varphi) \cup \{ M \}^{\varphi}.
\]
Actually, any operation that adds precisely the equivalence class of \( M \) modulo the literals is equivalent to ‘ex’. In the following, we write \( \text{ex}^*(\varphi, M) \) to refer to an arbitrary finite base expansion function of the form \( \text{ex}^*: \mathcal{L} \times \mathcal{M} \mapsto \mathcal{L} \).

Theorem 24. For every \( \text{ex}^* \), if \( \text{Mod}(\text{ex}^*(\varphi, M)) = \text{Mod}(\varphi) \cup \{ M \}^{\varphi} \) then
(i) \( \text{ex}^*(\varphi, M) \equiv \varphi \), if \( M \models \varphi \); and
(ii) \( \text{ex}^*(\varphi, M) \equiv \varphi \lor \bigwedge \text{qm}(\neg \varphi, M) \), if \( M \not\models \varphi \).

Our next step is to investigate the rationality of ‘ex*’. As expected adding the whole equivalence class of \( M \) with respect to \( \mathcal{L}_{\text{lit}}(\varphi) \) does not come freely, and some rationality postulates are captured, while others are lost:

Theorem 25. Let \( M \) be a model and \( \varphi \) an ALC-formula. A finite base model function \( \text{ex}^*(\varphi, M) \) is equivalent to \( \text{ex}(\varphi, M) \) iff \( \text{ex}^* \) satisfies:

(succes) \( M \in \text{Mod}(\text{ex}^*(\varphi, M)) \).

(persistence): \( \text{Mod}(\varphi) \subseteq \text{Mod}(\text{ex}^*(\varphi, M)) \).

(atomic temperance): For all \( \mathcal{M}' \subseteq \mathcal{M} \), if \( \text{Mod}(\varphi) \cup \{ M \}^{\varphi} \subseteq \mathcal{M}' \subseteq \text{Mod}(\text{ex}^*(\varphi, M)) \cup \{ M \} \) then \( \mathcal{M}' \) is not finitely representable in ALC-formula.
(atomic extensionality) if $M' \equiv^\varphi M$ then

$$\text{Mod}(\text{ex}^*(\varphi, M)) = \text{Mod}(\text{ex}^*(\varphi, M'))$$.

The postulates *success* and *persistence* come from requiring that $M$ will be absorbed, and that models will not be lost during an expansion. The *atomic extensionality* postulate states that if two models satisfy exactly the same literals within $\varphi$, then they should present the same results. *Atomic temperance* captures a principle of minimality and guarantees that when adding $M$, the loss of information should be minimised. Precisely, the only formulae allowed to be given up are those that are incompatible with $M$ modulo the literals of $\varphi$. Lemma 23 and Theorem 25 prove that the ‘ex’ operation is characterized by the postulates: *success, persistence, atomic temperance and atomic extensionality*.

4 Related Work

In the foundational paradigm of Belief Change, the AGM theory, bases have been used in the literature with two main purposes: as a finite representation of the knowledge of an agent \[5, 19\], and as a way of distinguishing agents knowledge explicitly \[11\]. Even though the AGM theory cannot be directly applied to DLs because most of these logics do not satisfy the prerequisites known as the AGM-assumptions \[7\], it has been studied and adapted to DLs \[6, 25\].

The syntactic connectivity in a knowledge base has a strong consequence of how an agent should modify its knowledge \[13\]. This sensitivity to syntax is also present in Ontology Repair and Evolution. Classical approaches preserve the syntactic form of the ontology as much as possible \[16, 28\]. However, these approaches may lead to drastic loss of information, as noticed by Hansson \[10\]. This problem has been studied in Belief Change for pseudo-contraction \[27\]. In the same direction, Troquard et al. \[29\] proposed the repair of DL ontologies by weakening axioms using refinement operators. Building on this study, Baader et al. \[2\] devised the theory of gentle repairs, which also aims at keeping most of the information within the ontology upon repair. In fact, gentle repairs are closely related to pseudo-contractions \[18\].

Other remarkable works in Belief Change in which the body of knowledge is represented in a finite way include the formalisation of revision due to Katsuno and Mendelzon \[17\] and the base-generated operations by Hansson \[12\]. In the former, Katsuno and Mendelzon \[17\] formalise traditional belief revision operations using a single formula to represent the whole belief set. This is possible because they only consider finitary propositional languages. Hansson \[12\] provides a characterisation of belief change operations over finite bases but restricted for logics which satisfy all the AGM-assumptions (such as propositional classical logic). Guerra and Wassermann \[9\] develop operations for rational change where an agent’s knowledge or behaviour is given by a Kripke model. They also provide two characterisations with AGM-style postulates.
5 Conclusion and Future Work

In this work, we have introduced a new kind of belief change operation: belief change via models. In our approach, an agent is confronted with a new piece of information in the format of a finite model, and it is compelled to modify its current epistemic state, represented as a single finite formula, either incorporating the new model, called model expansion; or removing it, called model contraction. The price for such finite representation is that the single input model cannot be removed or added alone, and some other models must be added or removed as well. As future work, we will investigate model change operations in other DLs, still taking into account finite representability. We will also explore the effects of relaxing some constraints on Belief Base operations, allowing us to rewrite axioms with different levels of preservation in the spirit of Pseudo-Contraactions, Gentle Repairs, and Axiom Weakening.

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A Proofs for Section 3

We have already given the syntax of $\mathcal{ALC}$-formulae in the main text and we provide the semantics here for the convenience of the reader. The semantics is given by interpretations. An interpretation $I$ is a pair $(\Delta_I, \cdot_I)$ where $\Delta_I$ is a countable non-empty set, called the domain of individuals, and $\cdot_I$ is a function mapping each concept name $A \in \mathbb{N}_C$ to a subset $A_I$ of $\Delta_I$ and each role name $r \in \mathbb{N}_R$ to a subset $r_I$ of $\Delta_I \times \Delta_I$. The interpretation of concepts in $I$ is

$$
\top^I = \Delta^I \quad \neg C^I = \Delta^I \setminus C^I \quad (C \cap D)^I = C^I \cap D^I \\
(\exists R.C)^I = \{ d \in \Delta^I \mid \exists d' \in C^I : (d, d') \in R^I \}.
$$

The interpretation of formulae is as expected

$I \models \neg \varphi$ iff not $I \models \varphi$ \quad $I \models (\varphi \land \psi)$ iff $I \models \varphi$ and $I \models \psi$

$I \models (C = \top)$ iff $C^I = \top^I$ \quad $I \models C(a)$ iff $a^I \in C^I$ \quad $I \models r(a, b)$ iff $(a^I, b^I) \in r^I$.

Formally, we inductively define the sets $f(\varphi)$ and $c(\varphi)$ as follows.

**Definition 26 (Subformulae).** Given an $\mathcal{ALC}$-formula $\varphi$, we have that

- if $\varphi$ is atomic then $f(\varphi) := \{ \varphi, \neg \varphi \}$;
- $f(\varphi \land \psi) := \{ \varphi \land \psi, \neg (\varphi \land \psi) \} \cup f(\varphi) \cup f(\psi)$;
- $f(\neg (\varphi \land \psi)) := f(\varphi \land \psi)$.

**Definition 27 (Subconcepts).** Given a an $\mathcal{ALC}$-formula $\varphi$, $c(\varphi)$ is the minimal set satisfying the following conditions:

- if $(C = \top) \in f(\varphi)$ then $C \in c(\varphi)$;
- if $C(a) \in f(\varphi)$ then $C \in c(\varphi)$;
- if $C \cap D \in c(\varphi)$ then $C, D \in c(\varphi)$;
- if $\exists r.C \in c(\varphi)$ then $C \in c(\varphi)$;
- $\neg C \in c(\varphi)$ iff $C \in c(\varphi)$.

**Definition 28.** Let $(T, o, f)$ be a model candidate for $\varphi$. Then, the interpretation $I_{(T, o, f)}(\varphi)$ is defined as:

- $\Delta^I := T \cup \text{ind}(\varphi)$;
Induction step:

- \( A^f := a \), for all \( a \in \text{ind}(\varphi) \);
- \( A^f := \{ c \in T \mid A \in c \} \cup \{ a \in \text{ind}(\varphi) \mid A \in o(a) \} \);
- \( (c, c') \in r^T \) iff \( \{ C \mid \neg \exists r. C \in c \} \subseteq c' \), for \( c, c' \in T \);
- \( (a, b) \in r^T \) iff \( r(a, b) \in f \), for \( a, b \in \text{ind}(\varphi) \);
- \( (a, c) \in r^T \) iff \( \{ C \mid \neg \exists r. C \in o(a) \} \subseteq c \), for \( a \in \text{ind}(\varphi) \) and \( c \in T \).

**Proof.** The proof follows by induction in the structure of \( \varphi \).

**Lemma 30.** Let \( \varphi = (C = T) \land \neg(C(a) \land \neg(r(a, b))) \). Then, we have:

\[
\begin{align*}
  f(\varphi) &= \{ \varphi, \neg \varphi, C = T, C \neq T, \neg(C(a) \land \neg r(a, b)), \neg(C(a), \neg r(a, b), r(a, b)) \\
  &= \{ \varphi, \neg \varphi, C = T, C \neq T, \neg(C(a) \land \neg r(a, b)), C(a), \neg r(a, b), r(a, b) \}
\end{align*}
\]

In any quasimodel \( (T, o, f) \) for \( \varphi \), we have that \( \varphi \in f \). However this also implies that \( (C = T), \neg(C(a) \land \neg r(a, b)) \in f \). Consequently \( (C(a) \land \neg r(a, b)) \notin f \) and thus, \( C(a) \notin f \) or \( \neg r(a, b) \notin f \). Hence, there are only three possible formula types for \( f \):

\[
f = \begin{cases} 
  \{ \varphi, \neg \varphi, C = T, \neg(C(a) \land \neg r(a, b)), C(a), r(a, b) \} & \text{or} \\
  \{ \varphi, \neg \varphi, C = T, \neg(C(a) \land \neg r(a, b)), \neg(C(a)), r(a, b) \} & \text{or} \\
  \{ \varphi, \neg \varphi, C = T, \neg(C(a) \land \neg r(a, b)), \neg(C(a)), \neg(r(a, b)) \}
\end{cases}
\]

Assuming that for each of these possible formula types there is at least one quasimodel of \( f \), we get that:

\[
\varphi \downarrow \equiv ((C = T) \land C(a) \land r(a, b)) \lor \\
((C = T) \land \neg(C(a)) \land r(a, b)) \lor \\
((C = T) \land \neg(C(a)) \land \neg(r(a, b)))
\]

It is easy to check that the formula above is equivalent to \( \varphi \).

**Example 29.** Let \( \varphi, \phi \) be \( \mathcal{ALC} \)-formulae. If \( \varphi \in f(\phi) \) then \( f(\varphi) \subseteq f(\phi) \).

**Proof.** The proof follows by induction in the structure of \( \phi \).

Then, by construction \( f(\phi) = \{ \varphi, \neg \varphi \} \). Thus, if \( \varphi \in f(\phi) \) then \( \varphi = \phi \) or \( \varphi = \neg \phi \). In either case, \( f(\varphi) = \{ \varphi, \neg \varphi \} \) which implies that \( f(\varphi) = \{ \phi, \neg \phi \} \). Thus, \( f(\varphi) \subseteq f(\phi) \).

In the following, assume that \( \phi \) is not atomic.

**Induction Hypothesis:** by construction \( \phi \) is defined as the conjunction of two formulae \( \psi \) or \( \psi' \) or the negation of such conjunction, that is, \( \phi = \psi \land \psi' \) or \( \phi = \neg(\psi \land \psi') \). Let us assume that for all \( \beta \in \{ \psi, \psi' \} \), if \( \varphi \in f(\beta) \) then \( f(\varphi) \subseteq f(\beta) \).

**Induction step:** consider the cases (i) \( \phi = \psi \land \psi' \) and (ii) \( \phi = \neg(\psi \land \psi') \).

(i) \( \phi = \psi \land \psi' \). By construction

\[
f(\phi) = f(\psi \land \psi') = \{ \psi \land \psi', \neg(\psi \land \psi') \} \cup f(\psi) \cup f(\psi').
\]

Thus, (a) \( \varphi \in \{ \psi \land \psi', \neg(\psi \land \psi') \} \) or (b) \( \varphi \in f(\psi) \) or (c) \( \varphi \in f(\psi') \).

(a) \( \varphi \in \{ \psi \land \psi', \neg(\psi \land \psi') \} \). Thus, either \( \varphi = \psi \land \psi' \) or \( \varphi = \neg(\psi \land \psi') \).

For \( \varphi = \psi \land \psi' \), we get that \( f(\varphi) = f(\psi \land \psi') = f(\phi) \) which means that \( f(\varphi) \subseteq f(\phi) \). For \( \varphi = \neg(\psi \land \psi') \), we get that \( f(\varphi) = f(\neg(\psi \land \psi')) \). By construction, \( f(\neg(\psi \land \psi')) = f(\psi \land \psi') \). Therefore, \( f(\varphi) = f(\psi \land \psi') = f(\phi) \) and so \( f(\varphi) \subseteq f(\phi) \).
Lemma 32. For every \( f \) is not a formula type for \( f \). Show that

Proof. Let then

Lemma 31. For every -formula \( \phi \) and formula type \( f \) for \( \phi \), if \( \phi, \varphi \in f \) then \( f \cap f(\varphi) \) is a formula type for \( \varphi \).

Proof. Let \( f_\varphi \) be a fixed but arbitrary formula type for \( \phi \) with \( \varphi \in f_\varphi \). We will show that \( f := f_\varphi \cap f(\varphi) \) is a formula type for \( \varphi \). Suppose for contradiction that \( f \) is not a formula type for \( \varphi \). Thus, as \( f \subseteq f(\varphi) \), either condition (1) or (2) of the formula type definition is violated:

1. There are formulae \( \psi, \neg \psi \in f(\varphi) \) such that either (a) \( \psi \notin f \) and \( \neg \psi \notin f \), or (b) \( \psi, \neg \psi \in f \).
   
   (a) \( \psi \notin f \) and \( \neg \psi \notin f \). By hypothesis, \( \varphi \in f_\varphi \). Thus, as \( f = f_\varphi \cap f(\varphi) \), and by construction \( \varphi \in f_\varphi \), we get that \( \varphi \in f \). Since \( f_\varphi \) is a formula type, we have that for all \( \psi' \in f(\varphi) \), \( \psi' \in f_\varphi \) iff \( \neg \psi' \notin f_\varphi \). As \( \varphi \in f_\varphi \subseteq f(\varphi) \), it follows from Lemma 30 that \( f(\varphi) \subseteq f(\varphi) \). Therefore, for all \( \psi' \in f(\varphi) \), \( \psi' \in f_\varphi \) iff \( \neg \psi' \notin f_\varphi \). By hypothesis, \( \neg \psi, \psi \in f(\varphi) \) which implies from above that either:

   \[
   \psi \in f_\varphi \text{ and } \neg \psi \notin f_\varphi, \text{ or } \psi \notin f_\varphi \text{ and } \neg \psi \in f_\varphi. \tag{2}
   \]

   By hypothesis, \( \neg \psi, \psi \in f(\varphi) \) but \( \neg \psi, \psi \notin f \). Thus, as \( f = f_\varphi \cap f(\varphi) \), we get \( \neg \psi, \psi \notin f_\varphi \), contradicting (2).

   (b) \( \psi, \neg \psi \in f \). By hypothesis, \( f_\varphi \) is a formula type which implies that for all \( \psi' \in f_\varphi \), \( \psi', \neg \psi' \notin f_\varphi \). Therefore, as \( f \subseteq f_\varphi \), we get that \( \psi, \neg \psi \notin f \), a contradiction.

2. Let \( \psi \land \psi' \in f(\varphi) \). We will show that \( \psi \land \psi' \in f \) iff \( \{ \psi, \psi' \} \subseteq f \) which contradicts the hypothesis that condition (2) from the formula type definition is violated. We split the proof in two cases: either (a) \( \psi \land \psi' \in f \) or (b) \( \psi \land \psi' \notin f \). If \( \psi \land \psi' \in f \), as \( f = f_\varphi \cap f(\varphi) \), we get that \( \psi \land \psi' \in f_\varphi \). Since \( f_\varphi \) is a formula type, we have that \( \{ \psi, \psi' \} \subseteq f_\varphi \). By definition of \( f(\varphi) \), if \( \psi \land \psi' \in f(\varphi) \) then \( \{ \psi, \psi' \} \subseteq f(\varphi) \). Hence, \( \{ \psi, \psi' \} \in f = f_\varphi \cap f(\varphi) \).

   Otherwise, \( \psi \land \psi' \notin f \). As \( f = f_\varphi \cap f(\varphi) \) and \( \psi \land \psi' \in f(\varphi) \), we get that \( \psi \land \psi' \notin f_\varphi \). Thus, as \( f_\varphi \) is a formula type, we get that \( \{ \psi, \psi' \} \notin f_\varphi \). Therefore, as \( f \subseteq f_\varphi \), we get that \( \{ \psi, \psi' \} \notin f \). From (a) and (b) we conclude that \( \psi \land \psi' \in f \) iff \( \{ \psi, \psi' \} \subseteq f \). But this contradicts the hypothesis that condition (2) from the formula type definition is violated.

Therefore, we conclude that \( f \) is a formula type.

Lemma 32. For every -formula \( \varphi \), \( f(\varphi) = f(\neg \varphi) \). \(^4\)

\(^4\) We silently remove double negation and treat \( \neg \neg \varphi \) as equal to \( \varphi \).
Proof. By construction ⪰ is a subformula of ϕ. We can see that f(ϕ) = f(ϕ) ∪ \{¬ϕ\}. Since f(ϕ) is closed under single negation and, by construction, ϕ ∈ f(ϕ), we have that ¬ϕ ∈ f(ϕ). Thus, f(ϕ) = f(ϕ).

**Definition 33.** Let ϕ be an ALC-formula. The set of of formula types for ϕ that has ϕ is given by the set

\[ τ(ϕ) = \{ f ⊆ f(ϕ) \mid f \text{ is a formula type for } ϕ \text{ and } ϕ \in f \}. \]

**Lemma 34.** For every ALC-formula ϕ and formula type f for ϕ, if ϕ ∈ f and ϕ ∈ f(ϕ) then f ∩ f(ϕ) ∈ τ(ϕ) ∪ τ(¬ϕ).

**Proof.** Let f be a fixed but arbitrary formula type for ϕ with φ ∈ f(ϕ). As f is a formula type (for ϕ) and ϕ ∈ f(ϕ), either (i) φ ∈ f(ϕ) or ¬φ ∈ f(ϕ):

(i) φ ∈ f(ϕ). Thus, by Lemma 31 we have that f ∩ f(ϕ) is a formula type for φ. Also, φ ∈ f(ϕ) ∩ f(ϕ). Therefore, f ∩ f(ϕ) ∈ τ(ϕ) which means that f ∩ f(ϕ) ∈ τ(ϕ) ∪ τ(¬ϕ).

(ii) ¬φ ∈ f(ϕ). Thus, by Lemma 31 we have that f ∩ f(¬φ) is a formula type for ¬φ. Also, ¬φ ∈ f(¬φ) ∩ f(¬φ). Therefore, f ∩ f(¬φ) ∈ τ(¬φ) which means that f ∩ f(¬φ) ∈ τ(¬φ) ∪ τ(¬φ). By Lemma 32, we have that f(¬φ) = f(¬φ) which implies that f ∩ f(¬φ) = f ∩ f(φ). Therefore, f ∩ f(φ) ∈ τ(ϕ) ∪ τ(¬ϕ).

**Lemma 35.** For every ALC-formula ϕ, f ∈ τ(ϕ) iff f is a formula type for ϕ and

1. if ϕ is atomic then f = \{φ\};
2. if ϕ = ψ ∧ ψ’ then f = \{ψ ∧ ψ’\} ∪ f_ψ ∪ f_ψ’, for some f_ψ ∈ τ(ψ) and f_ψ’ ∈ τ(ψ’);
3. if ϕ = ¬(ψ ∧ ψ’) then f = \{¬(ψ ∧ ψ’)\} ∪ f_ψ ∪ f_ψ’, for some f_ψ ∈ τ(ψ) ∪ τ(ψ’) such that either f_ψ ∈ τ(¬ψ) or f_ψ’ ∈ τ(¬ψ’).

**Proof.** The direction “⇒” is trivial, so we focus only on the “⇒” direction. Let f ∈ τ(ϕ). Thus, ϕ ∈ f and f is a formula type for ϕ. By construction, (I) either ϕ is atomic or (II) ϕ = ψ ∧ ψ’ or (III) ϕ = ¬(ψ ∧ ψ’):

(I) ϕ is atomic. Thus, by construction f = \{φ\} or f = \{¬φ\}. Thus, as ϕ ∈ f, we get f = \{φ\}.

(II) ϕ = ψ ∧ ψ’. As ϕ ∈ f, we get that ψ ∧ ψ’ ∈ f. Moreover, as f is a formula type for ϕ and ψ ∧ ψ’ ∈ f, it follows that ψ, ψ’ ∈ f.

Let

\[ f_ψ := f ∩ f(ψ) \text{ and } f_ψ := f ∩ f(ψ’). \]

As ψ, ψ’ ∈ f and f is a formula type for ϕ = ψ ∧ ψ’, by Lemma 31, f_ψ = f ∩ f(ψ) is a formula type for ψ and f_ψ’ = f ∩ f(ψ’) is a formula type for ψ’. We have that ψ ∈ f(ψ) and ψ’ ∈ f(ψ’) which means that ψ ∈ f_ψ and ψ’ ∈ f_ψ’. Thus, f_ψ ∈ τ(ψ) and f_ψ’ ∈ τ(ψ’). We still need to show that f = \{ψ ∧ ψ’\} ∪ f_ψ ∪ f_ψ’. For this, we will show that (i) f ⊆ \{ψ ∧ ψ’\} ∪ f_ψ ∪ f_ψ’.
and (ii) \( \{ \psi \land \psi' \} \cup f_\psi \cup f_{\psi'} \subseteq f \). The case (ii) is trivial, so we focus only on case (i). Let \( \phi \in f \). As \( f \) is a formula type for \( \varphi = \psi \land \psi' \), we get that
\[
\phi \in f \subseteq f(\psi \land \psi') = \{ \psi \land \psi', \neg(\psi \land \psi') \} \cup f(\psi) \cup f(\psi').
\]
Therefore, (a) \( \phi \in \{ \psi \land \psi', \neg(\psi \land \psi') \} \) or (b) \( \phi \in f(\psi) \) or (c) \( \phi \in f(\psi') \).
(a) \( \phi \in \{ \psi \land \psi', \neg(\psi \land \psi') \} \). As \( f \) is a formula type and \( \varphi = \psi \land \psi' \in f \), we get that \( \neg(\psi \land \psi') \notin f \). Therefore, as \( \phi \in f \), we have that \( \phi \neq \neg(\psi \land \psi') \).
Hence, \( \phi = \psi \land \psi' \), which implies that \( \phi \in \{ \psi \land \psi' \} \cup f_\psi \cup f_{\psi'} \).
(b) \( \phi \in f(\psi) \). Thus, as \( \phi \in f \), we get that \( \phi \in f_\psi = f \cap f(\psi) \) which implies that \( \phi \in \{ \psi \land \psi' \} \cup f_\psi \cup f_{\psi'} \).
(c) \( \phi \in f(\psi') \). Thus, as \( \phi \in f \), we get that \( \phi \in f_{\psi'} = f \cap f(\psi') \) which implies that \( \phi \in \{ \psi \land \psi' \} \cup f_\psi \cup f_{\psi'} \).
Thus, \( \phi \in \{ \psi \land \psi' \} \cup f_\psi \cup f_{\psi'} \).

(III) \( \varphi = \neg(\psi \land \psi') \). As \( \varphi \in f \), we get that \( \neg(\psi \land \psi') \in f \). Let
\[
f_\psi := f \cap f(\neg(\psi)) \text{ and } f_{\psi'} := f \cap f(\neg(\psi')).
\]
As \( \neg(\psi), \neg(\psi') \in f(\varphi = \neg(\psi \land \psi')) \), by Lemma 34, we have that
\[
f_\psi \in \tau(\psi) \cup \tau(\neg(\psi)) \text{ and } f_{\psi'} \in \tau(\psi') \cup \tau(\neg(\psi')).
\]
Moreover, as \( f \) is a formula type for \( \varphi \) and \( \varphi = \neg(\psi \land \psi') \in f \), it follows that \( \psi \land \psi' \notin f \). Therefore, \( \{ \psi, \psi' \} \notin f \). Thus, either \( \psi \notin f \) or \( \psi' \notin f \). Thus, as \( f \) is a formula type, either (i) \( \neg(\psi') \in f \) or (ii) \( \neg(\psi') \notin f \).
(i) \( \neg(\psi) \in f \). Thus, as \( f \) is a formula type for \( \varphi = \neg(\psi \land \psi') \), by Lemma 34.
\[f_\psi = f \cap f(\neg(\psi)) \text{ is a formula type for } \neg(\psi) \text{. We have that } \neg(\psi) \in f(\neg(\psi)). \] So \( \neg(\psi) \in f_\psi \). Thus, \( f_\psi \in \tau(\neg(\psi)) \).
(ii) \( \neg(\psi') \in f \). Analogously to item (i), we get that \( f_{\psi'} \in \tau(\neg(\psi')) \).
Thus,
\[
f_\psi \in \tau(\neg(\psi)) \text{ or } f_{\psi'} \in \tau(\neg(\psi')).
\]
We still need to show that \( f = \{ \neg(\psi \land \psi') \} \cup f_\psi \cup f_{\psi'} \). For this we need to show that (i) \( f \subseteq \{ \neg(\psi \land \psi') \} \cup f_\psi \cup f_{\psi'} \) and (ii) \( \{ \neg(\psi \land \psi') \} \cup f_\psi \cup f_{\psi'} \subseteq f \).
The case (ii) is trivial. So we focus only on case (i).
Let \( \phi \in f \). As \( f \) is a formula type for \( \varphi = \neg(\psi \land \psi') \), we get that
\[
\phi \in f \subseteq f(\neg(\psi \land \psi')) = f(\psi \land \psi')
= \{ \psi \land \psi', \neg(\psi \land \psi') \} \cup f(\psi) \cup f(\psi').
\]
Therefore, (a) \( \phi \in \{ \psi \land \psi', \neg(\psi \land \psi') \} \) or (b) \( \phi \in f(\psi) \) or (c) \( \phi \in f(\psi') \).
(a) \( \phi \in \{ \psi \land \psi', \neg(\psi \land \psi') \} \). As \( f \) is a formula type and \( \varphi = \neg(\psi \land \psi') \in f \), we get that \( \psi \land \psi' \notin f \). Therefore, \( \phi \in f \), we have that \( \phi \neq \psi \land \psi' \).
Therefore, \( \phi = \neg(\psi \land \psi') \), which implies that \( \phi \in \{ \neg(\psi \land \psi') \} \cup f_\psi \cup f_{\psi'} \).
(b) \( \phi \in f(\psi) \). By Lemma 32, we get \( f(\psi) = f(\neg(\psi)). \) Therefore, \( \phi \in f(\neg(\psi)) \).
Thus, as \( \phi \in f \), we get that \( \phi \in f_\psi = f \cap f(\neg(\psi')) \) which implies that \( \phi \in \{ \neg(\psi \land \psi') \} \cup f_\psi \cup f_{\psi'} \).
(c) \( \phi \in f(\psi') \). Analogously to item (b), we get \( \phi \in \{ \neg(\psi \land \psi') \} \cup f_\psi \cup f_{\psi'} \).
Thus, $\phi \in \{\neg(\psi \land \psi')\} \cup f_\psi \cup f_{\psi'}$.

**Definition 36 (Formula degree).** The degree of an ALC-formula $\phi$, denoted $\text{degree}(\phi)$, is

- 1 if $\phi$ is an atomic ALC-formula;
- $\text{degree}(\varphi) + 1$ if $\phi = \neg \varphi$; and
- $\text{degree}(\varphi) + \text{degree}(\psi)$ if $\phi = \varphi \land \psi$.

**Lemma 37 ([8]).** If $I \models \varphi$ then $\text{qm}(\varphi, I)$ is a quasimodel for $\varphi$.

To show Theorem 14, we use Lemma 38.

**Lemma 38.** Let $\varphi$ be an ALC-formula. If $f \in \tau(\varphi)$ then $\left( \bigwedge \text{lit}(f) \right) \models \varphi$.

**Proof.** The proof follows by induction in the degree of $\phi$.

**Base:** $\text{degree}(\phi) = 1$. Then $\phi$ is atomic. This implies from Lemma 35 that $f = \{\phi\}$. Thus, $\bigwedge \text{lit}(f) = \phi$. For ALC, this means that $\bigwedge \text{lit}(f) \models \phi$.

**Induction Hypothesis:** For every formula $\varphi$, and formula type $f_\varphi$ for $\varphi$, if $\varphi \in f_\varphi$ and $\text{degree}(\varphi) < \text{degree}(\phi)$ then $\bigwedge \text{lit}(f_\varphi) \models \varphi$.

**Induction Step:** Let $\text{degree}(\phi) > 1$. By construction, $\phi$ is of the form $\varphi \land \psi$ or $\neg \varphi$, for some ALC-formulae $\varphi$ and $\psi$:

1. $\phi = \varphi \land \psi$. Thus, from Lemma 35

   $$f = \{\varphi \land \psi\} \cup f_\varphi \cup f_{\psi},$$

   such that $f_\varphi \in \tau(\varphi), f_{\psi} \in \tau(\psi)$.

   Note that $\text{lit}(f) = \text{lit}(f_\varphi) \cup \text{lit}(f_{\psi})$. Therefore,

   $$\bigwedge \text{lit}(f) = \left( \bigwedge \text{lit}(f_\varphi) \right) \land \left( \bigwedge \text{lit}(f_{\psi}) \right) .$$

   By the definition of degree, we get that $\text{degree}(\phi) = \text{degree}(\varphi \land \psi) = \text{degree}(\varphi) + \text{degree}(\psi)$ and $1 \leq \text{degree}(\varphi)$ and $1 \leq \text{degree}(\psi)$. Therefore, $\text{degree}(\varphi) < \text{degree}(\phi)$ and $\text{degree}(\psi) < \text{degree}(\phi)$. By the inductive hypothesis,

   $$\bigwedge \text{lit}(f_\varphi) \models \varphi \text{ and } \bigwedge \text{lit}(f_{\psi}) \models \psi .$$

   Therefore,

   $$\bigwedge \text{lit}(f) = \bigwedge \text{lit}(f_\varphi) \land \bigwedge \text{lit}(f_{\psi}) \models \varphi \land \psi .$$

   Thus, as $\phi = \varphi \land \psi$, we get

   $$\bigwedge \text{lit}(f) \models \phi .$$
2. $\phi = \neg \varphi$. By construction, either: (a) $\varphi$ is atomic, or (b) $\varphi = \psi \land \psi'$.

(a) $\varphi$ is atomic. We get from Lemma 35 that $f = \{\neg \varphi\}$, which implies that $\land \text{lit}(f) = \{\neg \varphi\}$, and analogous to the base case, we get that $\land \text{lit}(f) \models \neg \varphi$ that is, $\land \text{lit}(f) \models \phi$.

(b) $\varphi = \psi \land \psi'$. By Lemma 35, we get that

$$f = \{\neg (\psi \land \psi')\} \cup f_\psi \cup f_\psi',$$

where $f_\psi \in \tau(\psi) \cup \tau(\neg \psi), f_\psi' \in \tau(\psi') \cup \tau(\neg \psi')$ such that either

i. $f_\psi \in \tau(\neg \psi)$. From definition of degree, we get that

$$\deg(\phi) = \deg(\neg (\psi \land \psi')) = \deg(\psi) + \deg(\psi') + 1,$$

and $\deg(\psi) \geq 1$ and $\deg(\psi') \geq 1$, and $\deg(\neg \psi) = \deg(\psi) + 1$. Thus, $\deg(\neg \psi) = \deg(\neg (\psi \land \psi')) = \deg(\neg \psi) + \deg(\psi')$. Thus, as $\deg(\psi') \geq 1$, we get

$$\deg(\neg \psi) < \deg(\phi).$$

Thus, by the inductive hypothesis, $\land \text{lit}(f_\psi) \models \neg \psi$. Note that for every formula $\beta$, $\neg \psi \models \neg (\psi \land \beta)$. Therefore, for $\beta = \psi'$

$$\land \text{lit}(f_\psi) \models \neg (\psi \land \psi')$$

From (3), we get that

$$\land \text{lit}(f) = \land \text{lit}(f_\psi) \land \land \text{lit}(f_\psi').$$

Thus, as $\land \text{lit}(f_\psi) \models \neg (\psi \land \psi')$, we get that $\land \text{lit}(f_\psi) \land \land \text{lit}(f_\psi') \models \neg (\psi \land \psi')$ which implies from above that $\land \text{lit}(f) \models \neg (\psi \land \psi')$ that is,

$$\land \text{lit}(f) \models \phi.$$

ii. $f_\psi \in \tau(\neg \psi')$. Analogous to item (i).

Theorem 20. For every ALC-formula $\varphi$ and model $M$, $\text{con}(\varphi, M) \equiv \text{con}_n(\varphi, M)$.

Proof. If $M \not\models \varphi$, then $\text{con}(\varphi, M) = \text{con}'(\varphi, M) = \varphi$. Now, suppose that $M \models \varphi$. In this case, we know from Lemma 37 that $(T', o', f') = \text{qm}(\varphi, M)$ is a quasimodel for $\varphi$. From Theorem 14 we know that:

$$\text{con}'(\varphi, M) \equiv \varphi \land \neg (\land \text{lit}(f))$$

$$= \left( \bigvee_{(T, o, f) \in S(\varphi)} (\land \text{lit}(f)) \right) \land \neg (\land \text{lit}(f))$$
For each \((T,o,f)\) in \(S(\varphi)\) either \(\text{lit}(f) = \text{lit}(f')\) or \(\text{lit}(f) \neq \text{lit}(f')\). If \(\text{lit}(f) = \text{lit}(f')\), then:
\[
\bigwedge \text{lit}(f) \land \neg (\bigwedge \text{lit}(f')) \equiv \bot.
\]
Otherwise, we know that
\[
\bigwedge \text{lit}(f) \land \neg (\bigwedge \text{lit}(f')) \neq \bot.
\]

Due to the definition of \(\mu\) and Corollary 38 we can conclude that for every \(f \in \text{ftypes}(\varphi)\), \(f \in \mu(\varphi,M)\) iff \(\text{lit}(f) \neq \text{lit}(f')\). As we can ignore inconsistent formulae in disjunctions, we get that \(\text{con}(\varphi,M) \equiv \text{con}'(\varphi,M)\).

**Theorem 14.** For every \(\mathcal{ALC}\)-formula \(\varphi\), we have that \(\varphi \equiv \varphi^\dagger\).

**Proof.** Let \(\varphi\) be an \(\mathcal{ALC}\)-formula and \(\mathcal{I}\) an interpretation.

\(\mathcal{I} \models \varphi \Rightarrow \mathcal{I} \models \varphi^\dagger\): First, suppose that \(\mathcal{I} \models \varphi\). From Lemma 37 we know that \(\text{qm}(\varphi,\mathcal{I}) = (T,o,f)\) is a quasimodel of \(\varphi\). Therefore, there is a disjunct \(\psi\) of \(\varphi^\dagger\) which is the conjunction of all atomic formulae in \(f\). By Definition 15 \(\mathcal{I} \models f\), thus we can conclude that \(\mathcal{I} \models \varphi^\dagger\).

\(\mathcal{I} \models \varphi^\dagger \Rightarrow \mathcal{I} \models \varphi\): Now, assume that \(\mathcal{I} \models \varphi^\dagger\). This means that there is one disjunct \(\psi\) of \(\varphi^\dagger\) such that \(\mathcal{I} \models \psi\). By construction, this disjunct is a conjunction of atomic formulae in the formula type of a quasimodel \((T,o,f)\) for \(\varphi\). Using Lemma 38 we can conclude that \(\mathcal{I} \models f\). As \(\varphi \in f\) we get that \(\mathcal{I} \models \varphi\). Hence, \(\mathcal{I} \models \varphi\) iff \(\mathcal{I} \models \varphi^\dagger\), i.e., \(\varphi \equiv \varphi^\dagger\).

Corollary 39 is a direct consequence of the definition of a formula type.

**Corollary 39.** Let \((T,o,f)\) and \((T',o',f')\) be quasimodels for an \(\mathcal{ALC}\)-formula \(\varphi\). Then, \(\text{lit}(f) = \text{lit}(f')\) iff \(f = f'\).

Given \(\mathcal{ALC}\)-formulae \(\varphi,\psi\), we say that \(\psi\) is in the language of the literals of \(\varphi\), written \(\psi \in \mathcal{L}_{\text{lit}}(\varphi)\), if \(\psi\) is a boolean combination of the atoms in \(\varphi\).

**Lemma 40.** Let \(M,M'\) be models and \(\varphi\) an \(\mathcal{ALC}\)-formula. Also let \((T,o,f) := \text{qm}(\varphi,M)\) and \((T',o',f') := \text{qm}(\varphi,M')\). Then, \([M]^{\varphi} = [M']^{\varphi}\) iff \(f = f'\).

**Proof.** First, assume that \([M]^{\varphi} = [M']^{\varphi}\). Then we know that for every \(\alpha \in \mathcal{L}_{\text{lit}}(\varphi)\), \(M \models \alpha\) iff \(M' \models \alpha\). With Corollary 39 we can conclude that \(f = f'\).

Now, assume that \(f = f'\). Corollary 39 implies that \(\text{lit}(f) = \text{lit}(f')\). In other words, for every atomic subformula \(\alpha \in \mathcal{L}_{\text{lit}}(\varphi)\) we have that \(\varphi, M \models \alpha\) iff \(M' \models \alpha\), that is, \([M]^{\varphi} = [M']^{\varphi}\).

**Lemma 41.** Let \(M\) be a model and \(\varphi\) an \(\mathcal{ALC}\)-formula. Then, the following holds: \(\text{Mod}(\varphi) \setminus [M]^{\varphi} = \text{Mod(\text{con}(\varphi,M))}\).
Proof. Let \( (T, o, f) := \text{qm}(\varphi, M) \) and \( (T', o', f') := \text{qm}(\varphi, M') \). First, suppose that \( M' \in \text{Mod}(\varphi \setminus [M]^\varphi \). We know that \( M' \models \varphi \) and by Lemma \( \ref{lem:M} \) we get that \( \text{qm}(\varphi, M') \) is a quasimodel for \( \varphi \). We also know that \( M' \not\in [M]^\varphi \). Thus, from Lemma \( \ref{lem:lem1} \) we obtain \( f \neq f' \). Therefore, \( f' \in \mu(\varphi, M) \). Hence, \( M' \in \text{Mod}(\text{con}(\varphi, M)) \) and so \( \text{Mod}(\varphi \setminus [M]^\varphi \subseteq \text{Mod}(\text{con}(\varphi, M)) \).

Now, let \( M' \in \text{Mod}(\text{con}(\varphi, M)) \). This means that there is at least one \( f'' \in \mu(\varphi, M) \) such that \( M' \models \land \text{lit}(f'') \). But as consequence of the definition of formula type, this implies that \( M' \in \text{Mod}(\varphi) \) and thus \( (T', o', f') \in \mathcal{S}(\varphi) \). We also know that \( M \not\in [M]^\varphi \), otherwise \( f' = f \) due to Lemma \( \ref{lem:lem1} \). Therefore, \( M' \in \text{Mod}(\varphi \setminus [M]^\varphi \) and we can conclude that \( \text{Mod}(\text{con}(\varphi, M)) \subseteq \text{Mod}(\varphi \setminus [M]^\varphi \).

Finally, we obtain: \( \text{Mod}(\text{con}(\varphi, M)) = \text{Mod}(\varphi \setminus [M]^\varphi \).

**Theorem 17.** Let \( M \) be a model and \( \varphi \) an \( \text{ALC} \)-formula. A finite base model function \( \text{con}^*(\varphi, M) \) is equivalent to \( \text{con}(\varphi, M) \) iff \( \text{con}^* \) satisfies:

**(success)** \( M \not\equiv \text{con}^*(\varphi, M) \),

**(inclusion)** \( \text{Mod}(\text{con}^*(\varphi, M)) \subseteq \text{Mod}(\varphi) \),

**(atomic retainment)**: For all \( M' \subseteq \mathcal{M} \), if \( \text{Mod}(\text{con}^*(B, M)) \subset \text{Mod}(B) \setminus [M]^\varphi \) then \( M' \) is not finitely representable in \( \text{ALC} \)-formula.

**(atomic extensionality)** if \( M' \equiv^\varphi M \) then \( \text{Mod}(\text{con}^*(\varphi, M)) = \text{Mod}(\text{con}^*(\varphi, M')) \).

Proof. Assume that \( \text{con}^*(\varphi, M) \equiv \text{con}(\varphi, M) \). From Lemma \( \ref{lem:M} \) we have that \( \text{Mod}(\text{con}^*(\varphi, M)) = \text{Mod}(\varphi \setminus [M]^\varphi \), hence success and inclusion are immediately satisfied. To prove atomic retainment, assume that \( M' \not\in \text{Mod}(\text{con}^*(\varphi, M)) \) and that there is a set of models \( M' \) with \( M' \in \text{M'} \), \( \text{Mod}(\text{con}^*(\varphi, M)) \subseteq M' \subseteq \text{Mod}(\varphi) \setminus [M]^\varphi \) and that is finitely representable in \( \text{ALC} \)-formula. Lemma \( \ref{lem:M} \) implies that \( \text{Mod}(\text{con}^*(\varphi, M)) = \text{Mod}(\varphi \setminus [M]^\varphi \). Hence, \( M' \in [M]^\varphi \), a contradiction as we assumed that \( M' \subseteq \text{Mod}(\varphi) \setminus [M]^\varphi \). Therefore, no such \( M' \) could exist, and thus, \( \text{con}^* \) satisfies atomic retainment.

Let \( M' \equiv^\varphi M \). Since \( \text{Mod}(\text{con}^*(\varphi, M)) = \text{Mod}(\varphi \setminus [M]^\varphi \) and \( [M']^\varphi = [M]^\varphi \), we have that: \( \text{Mod}(\varphi \setminus [M]^\varphi = \text{Mod}(\varphi \setminus [M']^\varphi = \text{Mod}(\text{con}^*(\varphi, M')) \). Hence, atomic extensionality is also satisfied.

On the other hand, suppose that \( \text{con}^*(\varphi, M) \) satisfies the postulates stated.

Let \( M' \in \text{Mod}(\varphi \setminus [M]^\varphi \) and assume that \( M' \not\in \text{Mod}(\text{con}(\varphi, M)) \). Due to atomic retainment, this means that there is no set \( M' \) finitely representable in \( \text{ALC} \)-formula such that \( \text{Mod}(\text{con}^*(\varphi, M)) \subseteq M' \subseteq \text{Mod}(\varphi \setminus [M]^\varphi \) and \( M' \in M' \). But we know from Lemma \( \ref{lem:M} \) that \( \text{Mod}(\varphi \setminus [M]^\varphi \) is finitely representable in \( \text{ALC} \)-formula and includes \( M' \) by assumption, a contradiction. Thus, no such \( M' \) could exist and \( \text{Mod}(\varphi \setminus [M]^\varphi \subseteq \text{Mod}(\text{con}^*(\varphi, M)) \).

Now, let \( M' \in \text{Mod}(\text{con}^*(\varphi, M)) \). By inclusion \( M' \in \text{Mod}(\varphi) \) and by success \( M' \not\equiv M \). We will show that \( M' \not\in [M]^\varphi \). By contradiction, suppose that \( M' \in [M]^\varphi \). Due to atomic extensionality \( \text{Mod}(\text{con}^*(\varphi, M)) = \text{Mod}(\text{con}^*(\varphi, M')) \), but success implies that \( M' \not\in \text{Mod}(\text{con}(\varphi, M')) \). This contradicts our initial assumption that \( M' \in \text{Mod}(\text{con}^*(\varphi, M)) \). Therefore \( M' \in \text{Mod}(\varphi \setminus [M]^\varphi \) and we can conclude that \( \text{Mod}(\text{con}^*(\varphi, M)) \subseteq \text{Mod}(\varphi \setminus [M]^\varphi \).

Hence, Lemma \( \ref{lem:M} \) yields \( \text{con}(\varphi, M) \equiv \text{con}^*(\varphi, M) \).
Lemma 43. Let $M$ be a model such that $M = \bigwedge \text{lit}(f)$, and $\psi \in \mathcal{L}_{\text{lit}}(f)$.

Proof. Let $f$ be a formula type, $M$ a model such that $M = \bigwedge \text{lit}(f)$, and $\psi$ and $\mathcal{ALC}$-formula such that $M \models \psi$. The proof is by induction on the degree of $\psi$.

Base: $\text{degree}(\psi) = 1$. Thus, from its definition, $\psi$ has to be an atomic formula. As $f$ is a formula type, we have that $\varphi \in f$ iff $\neg \varphi \not\in f$. Let us suppose for contradiction that $\psi \not\in f$. This implies that $\bigwedge f \models \neg \psi$. Thus, as $M \models \bigwedge \text{lit}(f)$, we have that $M \models \neg \psi$. This contradicts the hypothesis that $M \models \psi$. Thus, we conclude that $\psi \in f$. Therefore, $\bigwedge \text{lit}(f) \models \psi$.

Induction Hypothesis: For every formula $\varphi$, if $\text{degree}(\varphi) < \text{degree}(\psi)$ and $M \models \varphi$ then $\bigwedge \text{lit}(f) \models \varphi$.

Induction Step: Let $\text{degree}(\psi) > 1$. By construction, $\psi$ is of the form (1) $\varphi \land \varphi'$ or (2) $\neg \varphi$, for some $\mathcal{ALC}$-formulae $\varphi$ and $\varphi'$:

1. $\psi = \varphi \land \varphi'$. From definition, $\text{degree}(\varphi \land \varphi') = \text{degree}(\varphi) + \text{degree}(\varphi')$. Recall from definition of degree that $\text{degree}(\beta) > 1$, for every formula $\beta$. Therefore, $\text{degree}(\varphi) < \text{degree}(\varphi \land \varphi')$ and $\text{degree}(\varphi') < \text{degree}(\varphi \land \varphi')$. This means that $\text{degree}(\varphi) < \text{degree}(\psi)$ and $\text{degree}(\varphi') < \text{degree}(\psi)$. From hypothesis, $M \models \psi = \varphi \land \varphi'$. Thus, $M \models \varphi$ and $M \models \varphi'$. This implies from IH that $\bigwedge \text{lit}(f) \models \varphi$ and $\bigwedge \text{lit}(f) \models \varphi'$.

Therefore, $\bigwedge \text{lit}(f) \models \varphi \land \varphi' = \psi$.

2. $\psi = \neg \varphi$. We have two cases, either (i) $\varphi$ is an atomic formula or (ii) $\varphi = (\beta \land \beta')$. For the first case, analogous to the base case, we get that $\bigwedge \text{lit}(f) \models \psi$. So we focus only on the second case. From the definition of degree, we get that $\text{degree}(\neg \beta) < \text{degree}(\psi)$ and $\text{degree}(\neg \beta') < \text{degree}(\psi)$. As $M \models \psi = \neg (\beta \land \beta')$, we get that either (a) $M \models \neg \beta$ or (b) $M \models \neg \beta'$.

(a) $M \models \neg \beta$. From above, $\text{degree}(\neg \beta) < \text{degree}(\psi)$. Thus, from IH, we get that $\bigwedge \text{lit}(f) \models \neg \beta$. Thus, $\bigwedge \text{lit}(f) \models \neg (\beta \land \beta') = \psi$

(b): $M \models \neg \beta'$. Analogous to case (a).

Lemma 43. If $M \models \varphi$, $f \in \text{ftypes}(\varphi)$ and $M \models \bigwedge \text{lit}(f)$ then $\text{Mod}(\bigwedge \text{lit}(f)) = [M]^\varphi$.

Proof. We need to show that $M' \in \text{Mod}(\bigwedge \text{lit}(f))$ iff $M' \in [M]^\varphi$.

$\Rightarrow$. $M' \in \text{Mod}(\bigwedge \text{lit}(f))$. To show that $M' \in [M]^\varphi$, it suffices to show that $M' \equiv^\varphi M$. Let $\psi \in \mathcal{L}_{\text{lit}}(\varphi)$, we need to show that $M \models \psi$ if $M \models \psi$.

(a) “$\Rightarrow$” $M' \models \psi$. From Proposition 42 we have that $\bigwedge \text{lit}(f) \models \psi$. This jointly with $M' \models \bigwedge \text{lit}(f)$ implies that $M' \models \psi$.

(b) “$\Leftarrow$” $M' \models \psi$. Analogous to item (a).

$\Leftarrow$” $M' \in [M]^\varphi$. Note that $\bigwedge \text{lit}(f) \in \mathcal{L}_{\text{lit}}(\varphi)$. Thus as $M' \equiv^\varphi M$, and $M \models \bigwedge \text{lit}(f)$, we get that $M' \models \bigwedge \text{lit}(f)$.
Lemma 23. For every ALC-formula $\varphi$ and model $M$:
\[ \text{Mod}(\text{ex}(\varphi, M)) = \text{Mod}(\varphi) \cup [M]^{\varphi}. \]

Proof. We have two cases: either (i) $M \models \varphi$ or (ii) $M \not\models \varphi$.

(i) $M \models \varphi$. Then, from the definition of ex, we have that $\text{ex}(\varphi, M) = \varphi$ which implies that $\text{Mod}(\text{ex}(\varphi, M)) = \text{Mod}(\varphi)$. As $M \models \varphi$, we get that $[M]^{\varphi} \subseteq \text{Mod}(\varphi)$. Therefore, $\text{Mod}(\varphi) \cup [M]^{\varphi} = \text{Mod}(\varphi)$. This implies that $\text{Mod}(\text{ex}(\varphi, M)) = \text{Mod}(\varphi) \cup [M]^{\varphi}$.

(ii) $M \not\models \varphi$. Thus, from the definition of ex, we get that $\text{ex}(\varphi, M) = \varphi \lor \bigwedge \text{lit}(f)$, where $qm(\neg \varphi, M) = (T, o, f)$. This implies that $\text{Mod}(\text{ex}(\varphi, M)) = \text{Mod}(\varphi \lor \bigwedge \text{lit}(f))$. Note that $\text{Mod}(\varphi \lor \bigwedge \text{lit}(f)) = \text{Mod}(\varphi) \cup \text{Mod}(\bigwedge \text{lit}(f))$.

As $qm(\neg \varphi, M) = (T, o, f)$, it follows from the definition of $qm$ that $f \in \text{ftypes}(\neg \varphi)$ and $M \models \bigwedge \text{lit}(f)$. In summary, $M \models \neg \varphi$, $f \in \text{ftypes}(\neg \varphi)$ and $M \models \bigwedge \text{lit}(f)$. Thus, from Lemma 43 we have that $\text{Mod}(\bigwedge \text{lit}(f)) = [M]^{\varphi}$.

Therefore, $\text{Mod}(\text{ex}(\varphi, M)) = \text{Mod}(\varphi) \cup [M]^{\varphi}$.

Theorem 24. For every $\text{ex}^*$, if $\text{Mod}(\text{ex}^*(\varphi, M)) = \text{Mod}(\varphi) \cup [M]^{\varphi}$ then

(i) $\text{ex}^*(\varphi, M) \equiv \varphi$, if $M \models \varphi$; and
(ii) $\text{ex}^*(\varphi, M) \equiv \varphi \lor \bigwedge qm(\neg \varphi, M)$, if $M \not\models \varphi$.

Proof. We consider each case separately.

(i) $M \models \varphi$. Thus, $[M]^{\varphi} \subseteq \text{Mod}(\varphi)$ which implies that $\text{Mod}(\varphi) \cup [M]^{\varphi} = \text{Mod}(\varphi)$.

Therefore, $\text{ex}^*(\varphi, M) \equiv \varphi$.

(ii) $M \not\models \varphi$. Let $qm(\neg \varphi, M) = (T, o, f)$. Note that $f \in \text{ftypes}(\varphi)$, and from definition of $qm$ that $M \models \bigwedge \text{lit}(f)$. Thus, it follows from Lemma 43 that $\text{Mod}(\bigwedge \text{lit}(f)) = [M]^{\varphi}$. Thus, $\text{Mod}(\varphi \lor \bigwedge qm(\neg \varphi, M)) = \text{Mod}(\varphi) \cup [M]^{\varphi}$. This means that $\text{ex}^*(\varphi, M) \equiv \varphi \lor \bigwedge qm(\neg \varphi, M)$.

Theorem 25. Let $M$ be a model and $\varphi$ an ALC-formula. A finite base model function $\text{ex}^*(\varphi, M)$ is equivalent to $\text{ex}(\varphi, M)$ iff $\text{ex}^*$ satisfies:

(success) $M \in \text{Mod}(\text{ex}^*(\varphi, M))$. 

(persistence): \( \text{Mod}(\varphi) \subseteq \text{Mod}(\text{ex}^*(\varphi, M)) \).

(atomic temperance): For all \( M' \subseteq M \), if \( \text{Mod}(\varphi) \cup [M]^\varphi \subseteq M' \subseteq \text{Mod}(\text{ex}^*(\varphi, M)) \cup \{ M \} \) then \( M' \) is not finitely representable in \( \mathcal{ALC} \) formula.

(atomic extensionality) if \( M' \equiv^\varphi M \) then

\[
\text{Mod}(\text{ex}^*(\varphi, M)) = \text{Mod}(\text{ex}^*(\varphi, M')).
\]

Proof. First, assume that \( \text{ex}^*(\varphi, M) \equiv \text{ex}(\varphi, M) \). From Lemma \ref{lem:success} we have that \( \text{Mod}(\text{ex}^*(\varphi, M)) = \text{Mod}(\varphi) \cup [M]^\varphi \), hence success and persistence are immediately satisfied. To prove atomic temperance, assume that \( M' \in \text{Mod}(\text{ex}^*(\varphi, M)) \) and that there is a set of models \( M' \) with \( M' \equiv^\varphi M \) that is finitely representable in \( \mathcal{ALC} \) formula and such that \( \text{Mod}(\varphi) \cup [M]^\varphi \subseteq M' \subseteq \text{Mod}(\text{ex}^*(\varphi, M)) \). Lemma \ref{lem:success} implies that \( \text{Mod}(\text{ex}^*(\varphi, M)) = \text{Mod}(\varphi) \cup [M]^\varphi \). Hence, \( M' \not\subseteq [M]^\varphi \), a contradiction as we assumed that \( M' \subseteq \text{Mod}(\varphi) \cup [M]^\varphi \). Therefore, no such \( M' \) could exist, and thus, \( \text{ex}^* \) satisfies atomic temperance.

Let \( M' \equiv^\varphi M \). Since \( \text{Mod}(\text{ex}^*(\varphi, M)) = \text{Mod}(\varphi) \cup [M]^\varphi \) and \( [M']^\varphi = [M]^\varphi \), we have that: \( \text{Mod}(\varphi) \cup [M]^\varphi = \text{Mod}(\varphi) \cup [M']^\varphi = \text{Mod}(\text{ex}^*(\varphi, M')) \). Hence, atomic extensionality is also satisfied.

On the other hand, suppose that \( \text{ex}^*(\varphi, M) \) satisfies the postulates stated. Let \( M' \in \text{Mod}(\varphi) \cup [M]^\varphi \). If \( M' \in \text{Mod}(\varphi) \) then success ensures that \( M' \in \text{Mod}(\text{ex}^*(\varphi, M)) \). Otherwise, we have \( M' \equiv^\varphi M \), and as consequence of success and atomic extensionality we also obtain \( M' \in \text{Mod}(\text{ex}^*(\varphi, M)) \). Therefore, \( \text{Mod}(\varphi) \cup [M]^\varphi \subseteq \text{Mod}(\text{ex}^*(\varphi, M)) \).

Now, let \( M' \in \text{Mod}(\text{ex}^*(\varphi, M)) \) and assume that \( M' \not\in \text{Mod}(\varphi) \cup [M]^\varphi \). Success, persistence and atomic extensionality imply that \( \text{Mod}(\text{ex}^*(\varphi, M)) \). Atomic temperance states that there is no set of models \( M' \) that is finitely representable in \( \mathcal{ALC} \) formula with \( \text{Mod}(\varphi) \cup [M]^\varphi \subseteq M' \subseteq \text{Mod}(\text{ex}^*(\varphi, M)) \cup \{ M \} \). But we know from Lemma \ref{lem:success} that \( \text{Mod}(\varphi) \cup [M]^\varphi \) is finitely representable in \( \mathcal{ALC} \) formula and does not include \( M' \) by assumption, a contradiction. Thus, no such \( M' \) could exist and \( \text{Mod}(\text{ex}^*(\varphi, M)) \subseteq \text{Mod}(\varphi) \cup [M]^\varphi \).

Hence, Lemma \ref{lem:success} yields \( \text{ex}^*(\varphi, M) \equiv \text{ex}(\varphi, M) \).