NEW KAKEYA ESTIMATES USING THE
POLYNOMIAL WOLFF AXIOMS

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Abstract. We obtain new bounds for the Kakeya maximal conjecture in most dimensions $n < 100$, as well as improved bounds for the Kakeya set conjecture when $n = 7$ or 9. For this we consider Guth and Zahl’s strengthened formulation of the maximal conjecture, concerning families of tubes that satisfy the polynomial Wolff axioms. Our results give improved estimates for this strengthened formulation when $n = 5$ or $n \geq 7$.

1. Introduction

For $n \geq 2$ and small $\delta > 0$, a $\delta$-tube is a cylinder $T \subset \mathbb{R}^n$ of unit height and radius $\delta$, with arbitrary position and arbitrary orientation $\text{dir}(T) \in S^{n-1}$. A family $\mathcal{T}$ of $\delta$-tubes is direction-separated if $\{\text{dir}(T) : T \in \mathcal{T}\}$ forms a $\delta$-separated subset of the unit sphere.

**Conjecture 1.1** (Kakeya maximal conjecture). Let $p \geq \frac{n}{n-1}$. For all $\varepsilon > 0$, there exists a constant $C_{\varepsilon,n} > 0$ such that

$$\left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{L^p(\mathbb{R}^n)} \leq C_{\varepsilon,n} \delta^{-(n-1-n/p) - \varepsilon} \left( \sum_{T \in \mathcal{T}} |T| \right)^{1/p}$$

whenever $0 < \delta < 1$ and $\mathcal{T}$ is a direction-separated family of $\delta$-tubes.

By an application of Hölder’s inequality, one may readily verify that if (1) holds for $p = \frac{n}{n-1}$, then, for all $\varepsilon > 0$, there exists a constant $c_{\varepsilon,n} > 0$ such that

$$| \bigcup_{T \in \mathcal{T}} T | \geq c_{\varepsilon,n} \delta^n \sum_{T \in \mathcal{T}} |T|.$$

This can be interpreted as the statement that any direction-separated family of $\delta$-tubes is ‘essentially disjoint’. A more refined argument shows that if (1) holds for a given $p$, then every Kakeya set in $\mathbb{R}^n$ (that is, every compact set that contains a unit line segment in every direction) has Hausdorff dimension at least $p'$, the conjugate exponent of $p$. Thus, Conjecture 1.1 would imply the Kakeya set conjecture, that Kakeya sets in $\mathbb{R}^n$ have Hausdorff dimension $n$; see, for instance, [3, 34, 23].

For $n = 2$, the set conjecture was proven by Davies [8] and the maximal conjecture was proven by Córdoba [7] in the seventies. Both conjectures remain challenging and important open problems in higher dimensions; for partial results, see [6, 3, 33, 27, 31, 4, 21, 19, 22, 2, 9, 11, 12, 10, 24] and references therein. We highlight, in particular, the classical work of Wolff [33], which considers more general families of tubes satisfying the following hypothesis.

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Definition 1.2. We say that $\mathcal{T}$ satisfies the linear Wolff axiom if there is a constant $N \geq 1$, depending only on $n$, such that
\[
\# \{ T \in \mathcal{T} : T \subseteq E \} \leq N\delta^{-(n-1)}|E|
\]
whenever $E \subset \mathbb{R}^n$ is a rectangular box of arbitrary dimensions.

In [33], Wolff showed that (1) holds for the restricted range $p \geq \frac{n+2}{n}$ whenever $\mathcal{T}$ satisfies the linear Wolff axiom.\(^1\) Furthermore, it is not difficult to see that any direction-separated $\mathcal{T}$ satisfies the linear Wolff axiom and so his result provides similar progress for Conjecture 1.1.

Interestingly, there exist examples of tube families $\mathcal{T}$ in dimensions $n \geq 4$ that satisfy the linear Wolff axiom, but for which (1) fails to hold for the whole range $p \geq \frac{n+2}{n}$; see [30]. In particular, when $n = 4$ one may construct such $\mathcal{T}$ for which (1) is only valid in Wolff’s range $p \geq 3/2$. Examples of this kind are not direction-separated and therefore do not provide counterexamples to Conjecture 1.1.

To go beyond $p \geq \frac{3}{2}$ in four dimensions, Guth and Zahl [17] considered families of tubes which satisfy a more restrictive version of the linear Wolff axiom.

Definition 1.3. We say that $\mathcal{T}$ satisfies the $(D, N)$-polynomial Wolff axiom if
\[
\# \{ T \in \mathcal{T} : |T \cap E| \geq \lambda |T| \} \leq N\delta^{-(n-1)}\lambda^{-n}|E|
\]
whenever $\lambda \geq \delta$ and $E \subset \mathbb{R}^n$ is a semialgebraic set of complexity at most $D$.

In [20], Katz and the second author showed that for all $D \in \mathbb{N}$ and all $\varepsilon > 0$ there is a constant $C_{\varepsilon, n, D}$ such that any direction-separated family $\mathcal{T}$ satisfies the $(D, N)$-polynomial Wolff axiom with $N = C_{\varepsilon, n, D}\delta^{-\varepsilon}$ (see also [13] and [36] for similar results in three and four dimensions, respectively). Thus, the following conjecture of Guth and Zahl [17, Conjecture 1.1] is stronger than the Kakeya maximal conjecture.

Conjecture 1.4 (Guth–Zahl [17]). Let $p \geq \frac{n+2}{n}$. For all $\varepsilon > 0$, there is a complexity $D = D_{\varepsilon, n} \in \mathbb{N}$ and a constant $C_{\varepsilon, n} > 0$ such that
\[
\left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{L^p(\mathbb{R}^n)} \leq C_{\varepsilon, n} N^{1-1/p}\delta^{-(n-1-n/p)-\varepsilon} \left( \sum_{T \in \mathcal{T}} |T| \right)^{1/p}
\]
whenever $0 < \delta < 1$, $N \geq 1$ and $\mathcal{T}$ satisfies the $(D, N)$-polynomial Wolff axiom.\(^2\)

It is easy to adapt Córdoba’s $L^2$ argument [7] to prove Conjecture 1.4 for $n = 2$. Guth and Zahl [17] showed that in four dimensions, under the polynomial Wolff axiom, the $p \geq 3/2$ bound can be improved to $p \geq 85/57$.\(^2\) In all other dimensions the Wolff bound $p \geq \frac{n+2}{n}$ provides the previous best known result under the polynomial Wolff axioms alone. Our main result improves this range in high dimensions.

Theorem 1.5. Conjectures 1.1 and 1.4 are true in the range $p \geq p_n$, where
\[
p_n := 1 + \min_{2 \leq k \leq n} \max \left\{ \left( \frac{n}{n-k} \right)^{n-k} \cdot \frac{n-1}{n-k+1} \right\} \frac{1}{n-1}.
\]

\(^1\)Strictly speaking, Wolff’s theorem [33] holds under a slightly less restrictive condition referred to simply as the Wolff axiom. See [17] for a comparison of these conditions.

\(^2\)Strictly speaking, the conjecture of [17] is slightly weaker than Conjecture 1.4 in some regards and stronger in others. The positive results of [17] are also stated in a slightly different form.
The range of exponents $p \geq p_n$ is larger than Wolff’s when $n = 5$ or $n \geq 7$. To see this, note that for any $0 < r < 1$ there exists some integer $2 \leq k \leq n$ satisfying $k \in [r(n-1) + 1, r(n-1) + 2)$. Writing $p_n = 1 + \alpha_n \frac{1}{n-1}$, it follows that

$$\alpha_n < \inf_{0 < r < 1} \max \left\{ \left(1 + \frac{1}{n-1}\right)^{(n-1)(1-r)}, \frac{1}{1-r} \right\} \leq \Omega^{-1} = 1.763...$$

Here the omega constant $\Omega \in (1/2, 1)$ is the solution to $e^\Omega = \Omega^{-1}$. In particular, Theorem 1.5 implies that Conjecture 1.4 is true in the range $p \geq 1 + \Omega^{-1} \frac{1}{n-1}$, yielding an improvement over Wolff’s bound when $n \geq 9$. Calculating the precise value of $p_n$ for lower $n$, we find that Theorem 1.5 also improves the state-of-the-art for Conjecture 1.4 in dimensions $n = 5, 7, 8$; see Figure 1 for explicit values of $p_n$.

On the other hand, Katz and Tao [22] confirmed Conjecture 1.1 in the range $p \geq 1 + \frac{7}{4} \frac{1}{n-1}$. One may refine the above observations to show that $\alpha_n \to \Omega^{-1}$ as $n$ tends to infinity and so, as $7/4 = 1.75 < \Omega^{-1}$, Theorem 1.5 does not constitute an improvement for Conjecture 1.1 in high dimensions. However, by explicitly calculating values of $p_n$, we obtain improvements for Conjecture 1.1 in all dimensions $n$ belonging to the following list:

5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34, 35, 37, 39, 40, 41, 42, 44, 46, 47, 48, 49, 51, 53, 55, 56, 58, 60, 62, 65, 67, 69, 72, 74, 76, 81, 83, 90, 97.

Finally, recall that maximal estimates imply bounds for the dimension of Kakeya sets, and in particular we obtain the following corollary.

**Corollary 1.6.** Kakeya sets in $\mathbb{R}^7$ have Hausdorff dimension at least $1 + \frac{64}{7^3}$ and Kakeya sets in $\mathbb{R}^9$ have Hausdorff dimension at least $1 + \frac{8^5}{9^4}$.

Corollary 1.6 improves the previous best known lower bound of $(2-\sqrt{2})(n-4)+3$, also due to Katz and Tao [22] (note that this bound is better than the one that can be obtained from their maximal estimate).

As noted above, in high dimensions the Kakeya maximal bounds of Katz–Tao [22] are asymptotically superior to those of Theorem 1.5. Nevertheless, the bounds for Conjecture 1.1 given by Theorem 1.5 are perhaps of interest in this regime since they are obtained via a completely different approach from that used in [22]. The Katz–Tao [22] bound is proved using the sum-difference method from additive combinatorics, building upon earlier work of Bourgain [4]. The proof of Theorem 1.5, by contrast, is based on the polynomial partitioning method. This method was introduced by Guth and Katz [15] in their resolution of the Erdös distance conjecture, and was inspired by Dvir’s solution to a finite field analogue of the Kakeya

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3This list was compiled using the following Maple [25] code:

```maple
printlevel := 0: N := [insert dimension]:
p_broad := 1+(n/(n-1))^(n-k)/(n-1): p_limit := 1+1/(n-k+1):
p_Wolff := (n+2)/n: p_KT := 1+(7/4)/(n-1):
for n from 2 to N do
    p_seq := [seq(max(eval(p_broad, k = i), eval(p_limit, k = i)), i = 2 .. n)]:
    new_exponent := min(p_seq):
    if new_exponent < min(p_Wolff, p_KT) then print(n) end if:
end do:
```

4The sum-difference approach heavily exploits the direction separation hypothesis and therefore does not appear to yield estimates under the more general polynomial Wolff axiom hypothesis.
problem [9]. In recent years, polynomial partitioning techniques have been substantially developed so as to apply to a wide variety of problems in combinatorics and harmonic analysis; see, for instance, [28, 13, 14, 32, 18]. In the present article, an argument of Guth [13, 14], which was previously used to study oscillatory integral operators, is adapted so as to directly apply to the Kakeya problem. The structure of the proof and the presentation of the paper follows closely that of the companion article [18], where Guth’s arguments were extended in the oscillatory integral context so as to take into account polynomial Wolff axiom information in all dimensions.

The article is organised as follows:

- After fixing some notation in Section 2, in Section 3 the problem is reduced to estimating the so-called $k$-broad norms for the Kakeya maximal function, paralleling work on oscillatory integrals from [5, 13, 14, 18].
- In Sections 4 and 5, the basic tools for the proof of Theorem 1.5 are recalled from the literature and, in particular, the theory of $k$-broad norms and the polynomial partitioning theorem from [14] are reviewed.
- In Section 6, a recursive algorithm is described which can be interpreted as a structural statement of algebraic nature concerning extremal configurations of tubes for the Kakeya problem.
- In Section 7, the polynomial Wolff axioms are combined with the recursive algorithm to conclude the proof.

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2. Notational conventions

We call an $n$-dimensional ball $B_r$ of radius $r$ an $r$-ball. The intersection of $S^{n-1}$ with a ball is called a cap. The $\delta$-neighbourhood of a set $E$ will be denoted by $N_\delta E$.

The arguments will involve the admissible parameters $n$, $p$ and $\varepsilon$ and the constants in the estimates will be allowed to depend on these quantities. Moreover, any constant is said to be admissible if it depends only on the admissible parameters. Given positive numbers $A, B \geq 0$ and a list of objects $L$, the notation $A \lesssim_L B$, $B \gtrsim_L A$ or $A = O_L(B)$ signifies that $A \leq C_L B$ where $C_L$ is a constant which

| $n$ | $p_n$ | $p \geq p_n$ | $n$ | $p_n$ | $p \geq p_n$ |
|-----|-------|--------------|-----|-------|--------------|
| 2   | 2     | Córdoba [7]  | 9   | $1 + 9^4/8^5$ | Theorem 1.5 |
| 3   | $5/3$ | Wolff [33]   | 10  | $1 + 10^9/9^6$ | Theorem 1.5 |
| 4   | $85/57$ | Guth–Zahl [17] | 11  | $7/6$ | Theorem 1.5 |
| 5   | $1 + 5^2/4^3$ | Theorem 1.5 | 12  | $1 + 12^6/11^7$ | Theorem 1.5 |
| 6   | $4/3$ | Wolff [33]   | 13  | $8/7$ | Theorem 1.5 |
| 7   | $1 + 7^4/6^4$ | Theorem 1.5 | 14  | $1 + 14^7/13^8$ | Theorem 1.5 |
| 8   | $1 + 8^4/7^5$ | Theorem 1.5 | 15  | $1 + 15^8/14^9$ | Theorem 1.5 |
where each $T$ family as a disjoint union of subcollections into finitely-overlapping caps $\tau \in B$ is also decomposed into tiny balls $\delta \ll 1$ of finitely-overlapping $K$-broad and $\delta$-broad norms over $A$.

The cardinality of a finite set $A$ is denoted by $\#A$. A set $A'$ is said to be a refinement of $A$ if $A' \subseteq A$ and $\#A' \gtrsim \#A$. In many cases it will be convenient to pass to a refinement of a set $A$, by which we mean that the original set $A$ is replaced with some refinement.

3. Reduction to k-broad estimates

Rather than attempt to prove (K_p) directly, it is useful to work with a class of weaker inequalities known as $k$-broad estimates. This type of inequality was introduced by Guth [13, 14] in the context of oscillatory integral operators (and, in particular, the Fourier restriction conjecture) and was inspired by the earlier multilinear Kakeya inequalities or Proposition 4.7 below for a precise statement relating the $k$-broad and $k$-linear theory.

In order to introduce the $k$-broad estimates, we decompose the unit sphere $S^{n-1}$ into finitely-overlapping caps $\tau$ of diameter $\beta$, an admissible constant satisfying $\delta \ll \beta \ll 1$. We then perform a corresponding decomposition of $T$ by writing the family as a disjoint union of subcollections

$$T = \bigcup_{\tau} T[\tau]$$

where each $T[\tau]$ satisfies $\text{dir}(T) \in \tau$ for all $T \in T[\tau]$. The ambient euclidean space is also decomposed into tiny balls $B_\delta$ of radius $\delta$. In particular, fix $B_\delta$ a collection of finitely-overlapping $\delta$-balls which cover $\mathbb{R}^n$. For $B_\delta \in B_\delta$ define

$$\mu_T(B_\delta) := \min_{V_1, \ldots, V_A \in \text{Gr}(k-1, n)} \left( \max_{1 \leq a \leq A} \left\| \sum_{T \in T[\tau]} \chi_T \right\|_{L^p(B_\delta)} \right),$$

where $A \in \mathbb{N}$ and $\text{Gr}(k-1, n)$ is the Grassmannian manifold of all $(k-1)$-dimensional subspaces in $\mathbb{R}^n$. Here $\angle(\tau, V_a)$ denotes the infimum of the (unsigned) angles $\angle(v, v')$ over all pairs of non-zero vectors $v \in \tau$ and $v' \in V_a$. For $U \subseteq \mathbb{R}^n$ the $k$-broad norm over $U$ is then defined to be

$$\left\| \sum_{T \in T} \chi_T \right\|_{\text{BL}_k(A, U)} := \left( \sum_{B_\delta \in B_\delta} \frac{|B_\delta \cap U|}{|B_\delta|} \mu_T(B_\delta) \right)^{1/p}.$$

The $k$-broad norms are not norms in any familiar sense, but they do satisfy weak analogues of various properties of $L^p$-norms. The basic properties of these objects are described in Section 4 below.

The main ingredient in the proof of Theorem 1.5 is the following estimate for $k$-broad norms.

**Theorem 3.1.** Let $p \geq 1 + \frac{1}{n-1} \left( \frac{n}{n-1} \right)^{n-k}$. For all $\varepsilon > 0$, there is an $A \sim 1$ and a complexity $D \in \mathbb{N}$ such that

$$\left\| \sum_{T \in T} \chi_T \right\|_{\text{BL}_k(A, \mathbb{R}^n)} \lesssim N^{1-1/p} \delta^{-(n-1-n/p)-\varepsilon} \left( \sum_{T \in T} |T| \right)^{1/p} \quad (\text{BL}_k^A)$$

whenever $0 < \delta < 1$, $N \geq 1$, and $T$ satisfies the $(D, N)$-polynomial Wolff axiom.
The proof of Theorem 3.1, which is based on the polynomial partitioning method and closely follows the arguments of [13, 14, 18], will be presented in Sections 4–7.

The key feature which distinguishes the $k$-broad norm from its $L^p$ counterpart is that the former vanishes whenever the tubes of $\mathbb{T}$ cluster around a $(k - 1)$-dimensional set (see Lemma 4.3 for a precise statement of this property). Owing to this special behaviour, the inequality $(BL_k^p)$ is substantially weaker than $(K_p)$. Nevertheless, a mechanism introduced by Bourgain and Guth [5] allows one to pass from $k$-broad to linear estimates, albeit under a rather stringent condition on the exponent.

**Proposition 3.2** (Bourgain–Guth [5], Guth [14]). Let $p \geq \frac{n + 1 + \varepsilon}{n - 1 + \varepsilon}$, $\varepsilon > 0$, $A \sim 1$ and $D \in \mathbb{N}$. Suppose that

$$
\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{BL_k^p(\mathbb{R}^n)} \lesssim N^{1 - 1/p} \delta^{-(n - 1 - n/p) - \varepsilon} \left( \sum_{T \in \mathbb{T}} |T| \right)^{1/p}
$$

whenever $0 < \delta < 1$, $N \geq 1$, and $\mathbb{T}$ satisfies the $(D, N)$-polynomial Wolff axiom. Then

$$
\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^p(\mathbb{R}^n)} \lesssim N^{1 - 1/p} \delta^{-(n - 1 - n/p) - \varepsilon} \left( \sum_{T \in \mathbb{T}} |T| \right)^{1/p}
$$

whenever $0 < \delta < 1$, $N \geq 1$, and $\mathbb{T}$ satisfies the $(D, N)$-polynomial Wolff axiom.

Thus, combining Theorem 3.1 and Proposition 3.2 yields Theorem 1.5. In contrast with the range of Lebesgue exponents in Theorem 3.1, the range in which Proposition 3.2 applies shrinks as $k$ increases. The optimal compromise between the constraints in Theorem 3.1 and Proposition 3.2 is given by (2).

We end this section with a proof of Proposition 3.2, which is a minor modification of the argument in [5] (see also [14]).

**Proof (of Proposition 3.2).** The proof is by an induction-on-scale argument.

For the base case, fix $\delta \sim 1$ and let $\mathbb{T}$ be a family of $\delta$-tubes satisfying the $(D, N)$-polynomial Wolff axiom. If $\mathcal{B}$ is a cover of $\mathbb{R}^n$ by finitely-overlapping balls of radius 1, then

$$
\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^p(\mathbb{R}^n)}^p \leq \sum_{B \in \mathcal{B}} \left\| \sum_{T \in \mathbb{T} : T \cap B \neq \emptyset} \chi_T \right\|_{L^p(B)}^p \lesssim \sum_{B \in \mathcal{B}} \# \{ T \in \mathbb{T} : T \subset 3B \}^p.
$$

The polynomial Wolff axiom hypothesis implies that $\# \{ T \in \mathbb{T} : T \subset 3B \} \lesssim N$ for each $B \in \mathcal{B}$ and so $(K_p)$ follows from Hölder’s inequality and the fact that any tube $T \in \mathbb{T}$ can belong to at most $O(1)$ of the balls $3B$.

Now let $C$ be a fixed constant, chosen sufficiently large so as to satisfy the requirements of the forthcoming argument, and fix some small $\delta > 0$.

**Induction hypothesis:** Suppose the inequality

$$
\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^p(\mathbb{R}^n)} \lesssim CN^{1 - 1/p} \delta^{-(n - 1 - n/p) - \varepsilon} \left( \sum_{T \in \mathbb{T}} |T| \right)^{1/p}
$$

holds whenever $\delta \in ([2^k, 1])$, $N \geq 1$, and $\mathbb{T}$ is a family of $\delta$-tubes satisfying the $(D, N)$-polynomial Wolff axiom.
Let $\mathbb{T}$ be a family of $\delta$-tubes satisfying the $(D, N)$-polynomial Wolff axiom. Fix a $\delta$-ball $B_0 \in B_3$ and subspaces $V_1, \ldots, V_A \in \text{Gr}(n, k - 1)$ which obtain the minimum in the definition of $\mu_T(B_0)$; thus

$$
\mu_T(B_0) = \max_{\tau : \angle(x, \nu_x) > \beta} \left\| \sum_{T \in \mathbb{T}[\tau]} \chi_T \right\|_{L^p(B_0)}^p.
$$

Since $A \sim 1$ and $\# \{\tau : \angle(x, \nu_x) \leq \beta\} \sim \beta^{n-2}$, by the triangle inequality followed by Hölder’s inequality,

$$
\int_{B_k} \left| \sum_{T \in \mathbb{T}} \chi_T \right|^p \lesssim \sum_{T \in \mathbb{T}[\tau]} \chi_T \left| \sum_{\tau : \angle(x, \nu_x) \leq \beta} \sum_{T \in \mathbb{T}[\tau]} \chi_T \right|^p
\lesssim \beta^{-(n-1)p} \mu_T(B_0) + \beta^{-(k-2)(p-1)} \sum_{\tau} \int_{B_k} \left| \sum_{T \in \mathbb{T}[\tau]} \chi_T \right|^p.
$$

Summing the estimate over all the balls $B_0 \in B_3$, we find that

$$
\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^p(\mathbb{R}^n)}^p \lesssim \beta^{-(n-1)p} \sum_{T \in \mathbb{T}} \left\| \chi_T \right\|_{L^p(B_k)}^p + \beta^{-(k-2)(p-1)} \sum_{\tau} \left\| \sum_{T \in \mathbb{T}[\tau]} \chi_T \right\|_{L^p(\mathbb{R}^n)}^p.
$$

The first term on the right-hand side of the above display is estimated using the hypothesised broad estimate. For the second term, we apply a linear rescaling $L : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$
\left\| \sum_{T \in \mathbb{T}[\tau]} \chi_T \right\|_{L^p(\mathbb{R}^n)}^p = \beta^{n-1} \left\| \sum_{T \in \mathbb{T}[\tau]} \chi_L(T) \right\|_{L^p(\mathbb{R}^n)}^p
$$

where $\{L(T) : T \in \mathbb{T}[\tau]\}$ is essentially a collection of $\delta$-tubes with $\delta := \beta^{-1}\delta$. To be more precise, let $\omega \in S^{n-1}$ denote the centre of the cap $\tau$ and choose $L$ so that it fixes the 1-dimensional space spanned by $\omega$ and acts as a dilation by a factor of $\beta^{-1}$ on the orthogonal complement $\omega^\perp$. Writing $x \in \mathbb{R}^n$ as $x = (x', x_n)$ with $x' \in \omega^\perp$, for any $T \in \mathbb{T}[\tau]$ with $v := \text{dir}(T)$ there exists some $u \in \mathbb{R}^n$ such that

$$
T \subseteq \{x \in \mathbb{R}^n : |x' - u' - tv| \leq \delta \text{ for some } |t| \leq 1 \text{ and } |x_n - u_n| \leq 1/2\},
$$

Applying $L$ one obtains

$$
L(T) \subseteq \{y \in \mathbb{R}^n : |y' - \beta^{-1}u' - t\beta^{-1}v| \leq \beta^{-1}\delta \text{ for some } |t| \leq 1 \text{ and } |y_n - u_n| \leq 1/2\}
$$

and the right-hand side can be covered by a bounded number of $\delta$-tubes. Furthermore, the defining inequality of the polynomial Wolff axiom is essentially invariant under this scaling and the family of $\delta$-tubes $L(T)$ continue to satisfy the $(D, N)$-polynomial Wolff axiom.

Combining (3) with the induction hypothesis we find that

$$
\left\| \sum_{T \in \mathbb{T}[\tau]} \chi_T \right\|_{L^p(\mathbb{R}^n)}^p \lesssim \beta^{n-1}C^pN^{p-1}(\beta^{-1}\delta)^{-(n-1)p+n+p}(\beta^{-1}\delta)^{n-1}\#T[\tau].
$$

Recalling that $\sum \#T[\tau] = \#\mathbb{T}$, by plugging the preceding estimate into our $L^p(\mathbb{R}^n)$-norm bound,

$$
\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^p(\mathbb{R}^n)}^p \leq C\left(\beta^{-(n-1)p} + C^p\beta^{e(p, n, k) + p}\right)N^{p-1}\delta^{-(n-1)p+n+p}(\sum_{T \in \mathbb{T}} |T|)^{p-1};
$$
here $C$ depends, amongst other things, on the implied constant in $(BL^p_k)$, and
\[ e(p, n, k) := (n - k + 1)p - (n - k + 2). \]

By assumption, $p \geq \frac{n-k+2}{n-k+1}$ and therefore $e(p, n, k) \geq 0$. Consequently, $\beta$ may be chosen sufficiently small, depending only on the admissible parameters $n, p$ and $\varepsilon$, so that
\[ C\beta^{e(p, n, k)+pe} \leq \frac{1}{2}. \]

Moreover, if $C$ is chosen sufficiently large from the outset, it follows that
\[
\left\| \sum_{T \in \mathcal{A}} \chi_T \right\|_{L^p(\mathbb{R}^n)}^p \leq C^p N^{p-1}\delta^{-(n-1)p+n-pe} \left( \sum_{T \in \mathcal{A}} |T| \right),
\]
which closes the induction and completes the proof. □

4. Basic properties of the $k$-broad norms

Vanishing property. The proof of Theorem 3.1 will involve analysing collections of tubes which enjoy certain tangency properties with respect to algebraic varieties.

Definition 4.1. Given any collection of polynomials $P_1, \ldots, P_{n-m}: \mathbb{R}^n \to \mathbb{R}$ the common zero set
\[ Z(P_1, \ldots, P_{n-m}) := \{x \in \mathbb{R}^n : P_1(x) = \cdots = P_{n-m}(x) = 0\} \]
will be referred to as a variety.\(^5\) Given a variety $Z = Z(P_1, \ldots, P_{n-m})$, define its (maximum) degree to be the number
\[ \deg Z := \max\{\deg P_1, \ldots, \deg P_{n-m}\}. \]

It will often be convenient to work with varieties which satisfy the additional property that
\[ \bigcap_{j=1}^{n-m} \nabla P_j(z) \neq 0 \quad \text{for all } z \in Z = Z(P_1, \ldots, P_{n-m}). \quad (4) \]

In this case the zero set forms a smooth $m$-dimensional submanifold of $\mathbb{R}^n$ with a (classical) tangent space $T_z Z$ at every point $z \in Z$. A variety $Z$ which satisfies (4) is said to be an $m$-dimensional transverse complete intersection.

Definition 4.2. Let $0 < \delta < r < 1$, $x_0 \in \mathbb{R}^n$ and $Z \subseteq \mathbb{R}^n$ be a transverse complete intersection. A $\delta$-tube $T \subseteq \mathbb{R}^n$ is tangent to $Z$ in $B(x_0, r)$ if
i) $T \cap B(x_0, r) \cap N_\delta Z \neq \emptyset$;
ii) If $x \in T$ and $z \in Z \cap B(x_0, 2r)$ satisfy $|z - x| \leq 4\delta$, then
\[ \angle(\text{dir}(T), T_z Z) \leq c_{\text{tang}} \frac{\delta}{r}. \]

Here $0 < c_{\text{tang}}$ is an admissible constant which is chosen small enough to ensure that, whenever i) and ii) hold,
\[ T \cap B(x_0, 2r) \subseteq N_{2\delta} Z. \quad (5) \]

The fact that such a choice is possible follows from a simple calculus exercise (see, for instance, [16, Proposition 9.2] for details of an argument of this type).

\(^5\)Note that here, in contrast with much of the algebraic geometry literature, the ideal generated by the $P_j$ is not required to be irreducible.
The raison d'être for the $k$-broad norms is the following lemma, which roughly states that the broad norms vanish if the tubes in $\mathbb{T}$ cluster around a low dimensional variety.

**Lemma 4.3** (Vanishing property). Given $\varepsilon_0 > 0$ and $0 < \beta < 1$ there exists some $0 < c < 1$ such that the following holds. Let $0 < \delta < c$, $r > \delta^{1-\varepsilon_0}$, $x_0 \in \mathbb{R}^n$ and $Z \subseteq \mathbb{R}^n$ be a transverse complete intersection of dimension at most $k - 1$. Then

$$
\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{\text{BL}_k,\mathcal{A}(B(x_0,r))} = 0
$$

whenever $T$ is a family of $\delta$-tubes which are tangent to $Z$ in $B(x_0,r)$.

**Proof.** Fix $B_\delta \in \mathcal{B}_\delta$ with $B_\delta \cap B(x_0,r) \neq \emptyset$. Recalling the definition of the $k$-broad norm, it suffices to show that there exists some $V \in \text{Gr}(k - 1, n)$ such that

$$
\max_{\tau : \angle(\tau, V) > \beta} \int_{B_\delta} \left| \sum_{T \in \mathbb{T}[\tau]} \chi_T \right|^p = 0.
$$

This would follow if $V$ has the property that

$$
\text{if } T \in \mathbb{T} \text{ satisfies } T \cap B_\delta \neq \emptyset, \text{ then } \angle(\text{dir}(T), V) \leq \beta. \tag{6}
$$

Without loss of generality, one may assume there exists some $T_0 \in \mathbb{T}$ such that $T_0 \cap B_\delta \neq \emptyset$ (otherwise (6) vacuously holds for any choice of $(k - 1)$-dimensional subspace). By the containment property resulting from the tangency hypothesis,

$$
T_0 \cap B_\delta \subseteq T_0 \cap B(x_0, 2r) \subseteq N_{2\delta}Z
$$

and therefore there exists some $z_0 \in Z$ such that $|z_0 - y_0| < 2\delta$ for some $y_0 \in T_0 \cap B_\delta$. Let $V$ be a $(k - 1)$-dimensional subspace containing $T_0, Z$. Given any $T \in \mathbb{T}$, if $x \in T \cap B_\delta$ then $|x - z_0| < 4\delta$ and property ii) of the tangency hypothesis implies

$$
\angle(\text{dir}(T), V) \leq \frac{\delta}{r}.
$$

Since $r > \delta^{1-\varepsilon_0}$, it follows that $\angle(\text{dir}(T), V) \leq \beta$ provided $\delta$ is sufficiently small depending only on $\varepsilon_0$ and $\beta$, which completes the proof. \hfill \Box

**Triangle and logarithmic convexity inequalities.** The $k$-broad norms satisfy weak variants of certain key properties of $L^p$-norms.

**Lemma 4.4** (Finite subadditivity). Let $U_1, U_2 \subseteq \mathbb{R}^n$, $1 \leq p < \infty$ and $A \in \mathbb{N}$. Then

$$
\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{\text{BL}_k,\mathcal{A}(U_1 \cup U_2)} \leq \left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{\text{BL}_k,\mathcal{A}(U_1)} + \left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{\text{BL}_k,\mathcal{A}(U_2)}
$$

whenever $T$ is a family of $\delta$-tubes.

**Lemma 4.5** (Triangle inequality). Let $U \subseteq \mathbb{R}^n$, $1 \leq p < \infty$ and $A \in \mathbb{N}$. Then

$$
\left\| \sum_{T \in \mathbb{T}_1 \cup \mathbb{T}_2} \chi_T \right\|_{\text{BL}_k,\mathcal{A}(U)} \lesssim \left\| \sum_{T \in \mathbb{T}_1} \chi_T \right\|_{\text{BL}_k,\mathcal{A}(U)} + \left\| \sum_{T \in \mathbb{T}_2} \chi_T \right\|_{\text{BL}_k,\mathcal{A}(U)}
$$

whenever $\mathbb{T}_1$ and $\mathbb{T}_2$ are families of $\delta$-tubes.

---

6Here the parameter $\beta$ appears implicitly in the definition of the $k$-broad norm.
Lemma 4.6 (Logarithmic convexity). Let $U \subseteq \mathbb{R}^n$, $1 \leq p, p_0, p_1 < \infty$ and $A \in \mathbb{N}$. Suppose that $\theta \in [0, 1]$ satisfies
\[
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.
\]
Then
\[
\left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{BL_{k,n}^p(U)} \lesssim \left( \sum_{\tau_1, \ldots, \tau_k} \left( \prod_{j=1}^k \left( \sum_{T_j \in \mathcal{T}} \chi_{N \omega_j T_j} \right) \right)^{1/k} \right)^{1/p}_{L_p(\mathbb{R}^n)}
\]
whenever $\mathcal{T}$ is a family of $\delta$-tubes.

These estimates are entirely elementary. The proofs are identical to those used to analyse broad norms in the context of the Fourier restriction problem [14]. It is remarked that the parameter $A$ appears in the definition of the $k$-broad norm to allow for these weak triangle and logarithmic convexity inequalities.

**k-broad versus k-linear estimates.** Although not required for the proof of Theorem 1.5, it is perhaps instructive to note the relationship between the $k$-broad norms and the multilinear expressions appearing in the work of Bennett–Carbery–Tao [2].

Proposition 4.7. Let $\mathcal{T}$ be a collection of $\delta$-tubes in $\mathbb{R}^n$. Then
\[
\left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{BL_{k,n}^p(U)} \lesssim \left( \sum_{\tau_1, \ldots, \tau_k} \left( \prod_{j=1}^k \left( \sum_{T_j \in \mathcal{T}} \chi_{\omega_j T_j} \right) \right)^{1/k} \right)^{1/p}_{L_p(\mathbb{R}^n)}
\]
where the sum is over all $k$-tuples $(\tau_1, \ldots, \tau_k)$ of caps of diameter $\beta$ which are $\sim \beta^{k-1}$-transversal in the sense that $|\Lambda_{j=1}^k \omega_j| \gtrsim \beta^{k-1}$ for all $\omega_j \in \tau_j$.

Thus, any $k$-linear inequality of the type featured in [2, 11, 5] is stronger than the corresponding $k$-broad estimate (given that $\beta$ is admissible).

The proof of Proposition 4.7 is a simple exercise and is omitted (see [16] for similar results in the (more complicated) context of oscillatory integral operators).

5. POLYNOMIAL PARTITIONING

In this section the algebraic and topological ingredients for the proof of Theorem 3.1 are reviewed. In particular, the key polynomial partitioning theorem is recalled, which is adapted from [13, 14] (see also [32]) and previously appeared explicitly in [18].

Given a polynomial $P : \mathbb{R}^n \rightarrow \mathbb{R}$ consider the collection $\text{cell}(P)$ of connected components of $\mathbb{R}^n \setminus Z(P)$. Each $O' \in \text{cell}(P)$ is referred to as a cell cut out by the variety $Z(P)$ and the cells are thought of as partitioning the ambient euclidean space into a finite collection of disjoint regions.

In order to account for the choice of scale $\delta > 0$ appearing in the definition of the $\delta$-tubes, it will be useful to consider the family of $\delta$-shrunken cells defined by
\[
\mathcal{O} := \{O' \setminus N_\delta Z(P) : O' \in \text{cell}(P)\}. \quad (7)
\]
An important consequence of this definition is the following simple observation:

A $\delta$-tube $T$ can enter at most $\deg P + 1$ of the shrunken cells $O \in \mathcal{O}$.

Indeed, this is a simple and direct consequence of the fundamental theorem of algebra (or Bézout’s theorem) applied to the core line of $T$. 

**Theorem 5.1** (Guth [14]). Fix $0 < \delta < r$, $x_0 \in \mathbb{R}^n$ and suppose $F \in L^1(\mathbb{R}^n)$ is non-negative and supported on $B(x_0,r) \cap N_{2\delta}Z$ where $Z$ is an $m$-dimensional transverse complete intersection with $\deg Z \leq d$. At least one of the following cases holds:

**Cellular case.** There exists a polynomial $P: \mathbb{R}^n \to \mathbb{R}$ of degree $O(d)$ with the following properties:

i) $\#\text{cell}(P) \sim d^m$ and each $O \in \text{cell}(P)$ has diameter at most $r/2$.

ii) One may pass to a refinement of $\text{cell}(P)$ such that if $O$ is defined as in (7), then

$$\int_O F \sim d^{-m} \int_{\mathbb{R}^n} F \quad \text{for all } O \in \mathcal{O}.$$

**Algebraic case.** There exists an $(m-1)$-dimensional transverse complete intersection $Y$ of degree at most $O(d)$ such that

$$\int_{B(x_0,r) \cap N_{2\delta}Z} F \lesssim \log d \int_{B(x_0,r) \cap N_{\delta}Y} F.$$

This theorem is based on an earlier discrete partitioning result which played a central role in the resolution of the Erdős distance conjecture [15]. The proof is essentially topological, involving the polynomial ham sandwich theorem of Stone–Tukey [29], which is itself a consequence of the Borsuk–Ulam theorem (see, for instance, [26]), combined with a pigeonholing argument.

The theorem is applied to $k$-broad norms by taking

$$F = \sum_{B_i \in B_\delta} \mu_T(B_i) \frac{1}{\mu_{B_\delta}} \chi_{B_i}.$$

- If the cellular case holds, then it follows that

$$\left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{\text{BL}_{k,A}^p(B(x_0,r) \cap N_{2\delta}Z)} \lesssim d^{-m} \left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{\text{BL}_{k,A}^p(O)}$$

for all $O \in \mathcal{O}$ where $\mathcal{O}$ is the collection of cells produced by Theorem 5.1.

- If the algebraic case holds, then it follows that

$$\left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{\text{BL}_{k,A}^p(B(x_0,r) \cap N_{2\delta}Z)} \lesssim \log d \left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{\text{BL}_{k,A}^p(B(x_0,r) \cap N_{\delta}Y)}$$

where $Y$ is the variety produced by Theorem 5.1.

6. **Finding polynomial structure**

In this section, the recursive argument used to study the Fourier restriction problem in [18] (which, in turn, is adapted from [14]) is reformulated so as to apply to the Kakeya problem. As in [18], the argument will be presented as two separate algorithms:

- [alg 1] effects a dimensional reduction, essentially passing from an $m$-dimensional to an $(m-1)$-dimensional situation.

- [alg 2] consists of repeated application of the first algorithm to reduce to a minimal dimensional case.

The final outcome is a method of decomposing any given $k$-broad norm into pieces which are either easily controlled or enjoy special algebraic structure. This decomposition applies to arbitrary families of $\delta$-tubes. In the following section, we
will specialise to the case where the tubes satisfy the polynomial Wolff axiom and use this additional information to prove Theorem 3.1.

The first algorithm. Throughout this section let \( p \geq 1 \) and \( 0 < \varepsilon_0 \ll \varepsilon \ll 1 \) be fixed.

**Input.** [alg 1] will take as its input:

- A choice of small scale \( 0 < \delta \ll 1 \) and large scale \( r_0 \in [\delta^{1-\varepsilon_0}, \delta^{\varepsilon_0}] \).
- A transverse complete intersection \( Z \) of dimension \( m \in \{2, \ldots, n\} \).
- A family \( T \) of \( \delta \)-tubes which are tangent to \( Z \) on a ball \( B_{r_0} \) of radius \( r_0 \).
- A large integer \( A \in \mathbb{N} \).

**Output.** [alg 1] will output a finite sequence of sets \( (E_j)_{j=0}^J \), which are constructed via a recursive process. Each \( E_j \) is referred to as an ensemble and contains all the relevant information coming from the \( j \)th step of the algorithm. In particular, the ensemble \( E_j \) consists of:

- A word \( h_j \) of length \( j \) in the alphabet \( \{a, c\} \), referred to as a history. The \( a \) is an abbreviation of “algebraic” and \( c \) “cellular”. The words \( h_j \) are recursively defined by successively adjoining a single letter. Each \( h_j \) records how the cells \( O_j \subset O_j \) were constructed via repeated application of the polynomial partitioning theorem.
- A large scale \( r_j \in [\delta^{1-\varepsilon_0}, \delta^{\varepsilon_0}] \). The \( r_j \) will in fact be completely determined by the initial scales and the history \( h_j \). In particular, let \( \sigma_k : [0, 1] \to [0, 1] \) be given by
  \[
  \sigma_k(r) := \begin{cases} 
  \frac{r}{2} & \text{if the } k \text{th letter of } h_j \text{ is } c \\
  r^{1+\varepsilon_0} & \text{if the } k \text{th letter of } h_j \text{ is } a 
  \end{cases}
  \]
  for each \( 1 \leq k \leq j \). With these definitions,
  \[
  r_j := \sigma_j \circ \cdots \circ \sigma_1(r_0). 
  \]
  Note that each \( \sigma_k \) is a decreasing function and
  \[
  r_j \leq \delta^{\varepsilon_0(1+\varepsilon_0)^\#a(j)} \quad \text{and} \quad r_j \leq 2^{-\#c(j)} \delta^{\varepsilon_0} \tag{8}
  \]
  where \( \#a(j) \) and \( \#c(j) \) denote the number of occurrences of \( a \) and \( c \) in the history \( h_j \), respectively.
- A family of subsets \( O_j \subset \mathbb{R}^n \) which will be referred to as cells. Each cell \( O_j \subset O_j \) is contained in \( B_{r_0} \) and will have diameter at most \( 2r_j \).
- An assignment of a subfamily \( T[O_j] \) of \( \delta \)-tubes to each of the cells \( O_j \).
- A large integer \( d \in \mathbb{N} \) which depends only on \( \deg Z \) and the admissible parameters \( n \) and \( \varepsilon \).

Moreover, the components of the ensemble are defined so as to ensure that, for certain coefficients
\[
C_j(d) := d^{\#c(j)\varepsilon_0}d^{\#a(j)(n+\varepsilon_0)}
\]
and \( A_j := 2^{-\#a(j)}A \in \mathbb{N} \), the following properties hold:
Property I. The function \( \sum_{T \in \mathcal{T}} \chi_T \) on \( B_{r_0} \) can be compared with functions defined over the \( \mathcal{T}[O_j] \):
\[
\left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{BL^p_{k, A_j}(B_{r_0})}^P \leq C_j(d) \sum_{O_j \in \mathcal{O}_j} \left\| \sum_{T \in \mathcal{T}[O_j]} \chi_T \right\|_{BL^p_{k, A_j}(O_j)}^P.
\] (I) 

Property II. The tube families \( \mathcal{T}[O_j] \) satisfy
\[
\sum_{O_j \in \mathcal{O}_j} \# \mathcal{T}[O_j] \leq C_j(d) d \#(O_j) \# \mathcal{T}.
\] (II) 

Property III. Furthermore, each individual \( \mathcal{T}[O_j] \) satisfies
\[
\# \mathcal{T}[O_j] \leq C_j(d) d^{-\#(O_j)} \# \mathcal{T}.
\] (III) 

The initial step. The initial ensemble \( \mathcal{E}_0 \) is defined by taking:
- \( h := \emptyset \) to be the empty word;
- \( r_0 \) to be the large scale;
- \( O_0 \) the collection consisting of the single ball \( O_0 := B_{r_0} \);
- \( \mathcal{T}[O_0] := \mathcal{T} \).

All the desired properties then vacuously hold.

At this point it is also convenient to fix some large \( d \in \mathbb{N} \), to be determined later, which depends only on \( \deg Z \) and the admissible parameters \( n \) and \( \varepsilon \).

With these definitions, it is trivial to verify that Properties I, II and III hold.

The recursive step. Assume the ensembles \( \mathcal{E}_0, \ldots, \mathcal{E}_j \) have been constructed for some \( j \in \mathbb{N}_0 \) and that they all satisfy the desired properties.

Stopping conditions. The algorithm has two stopping conditions which are labelled [tiny] and [tang].

Stop: [tiny] The algorithm terminates if \( r_j \leq \delta^{1-\varepsilon} \).

Stop: [tang] Let \( C_{\text{tang}} \) and \( C_{\text{alg}} \) be fixed constants, chosen large enough to satisfy the forthcoming requirements of the proof. The algorithm terminates if the inequalities
\[
\sum_{O_j \in \mathcal{O}_j} \left\| \sum_{T \in \mathcal{T}[O_j]} \chi_T \right\|_{BL^p_{k, A_j}(O_j)}^P \leq C_{\text{tang}} \log d \sum_{S \in \mathcal{S}} \left\| \sum_{T \in \mathcal{T}[S]} \chi_T \right\|_{BL^p_{k, A_j}(B[S])}^P
\]
and
\[
\sum_{S \in \mathcal{S}} \# \mathcal{T}[S] \leq C_{\text{tang}} \delta^{-n\varepsilon} \sum_{O_j \in \mathcal{O}_j} \# \mathcal{T}[O_j];
\]
\[
\max_{S \in \mathcal{S}} \# \mathcal{T}[S] \leq C_{\text{tang}} \max_{O_j \in \mathcal{O}_j} \# \mathcal{T}[O_j]
\]
hold for some choice of:
- \( \mathcal{S} \) a collection of transverse complete intersections in \( \mathbb{R}^n \) all of equal dimension \( m - 1 \) and degree at most \( C_{\text{alg}} d \);
- An assignment of a subfamily \( \mathcal{T}[S] \) of \( \mathcal{T} \) and a \( \max\{r_j^{1+\varepsilon}, \delta^{1-\varepsilon}\} \)-ball \( B[S] \) to each \( S \in \mathcal{S} \) with the property that each \( T \in \mathcal{T}[S] \) is tangent to \( S \) in \( B[S] \) in the sense of Definition 4.2.
The stopping condition [tang] can be roughly interpreted as forcing the algorithm to terminate if one can pass to a lower dimensional situation. Indeed, by the inclusion property (5), the broad norm over $B[S]$ could instead be taken over a 26-neighbourhood of $S$.

If either of the above conditions hold, then the stopping time is defined to be $J := j$. Recalling (8), the stopping condition [tiny] implies that the algorithm must terminate after finitely many steps and, moreover,

$$\#_a(J) \lesssim \varepsilon_0^{-1} \log(\varepsilon_0^{-1}) \quad \text{and} \quad \#_c(J) \lesssim \log^{-1} \delta.$$  

Note that there can be relatively few algebraic steps $\#_a(j)$ but there can many cellular steps $\#_c(j)$. The first of the above estimates can also be used to show that $C_j(d) \lesssim d_{\varepsilon_0} - d_{\varepsilon_{\delta}}$ always holds. Furthermore, by choosing $A \geq 2^{-\varepsilon_0^2}$, say, one may ensure that the $A_j$ defined above are indeed integers.

**Recursive step.** Suppose that neither stopping condition [tiny] nor [tang] is met. One proceeds to construct the ensemble $\mathcal{O}_{j+1}$ as follows.

Given $O_j \in \mathcal{O}_j$, apply the polynomial partitioning theorem with degree $d$ to

$$\left\| \sum_{T \in \mathbb{V}[O_j]} c_T \right\|_{BL_{p, A_j}^p(O_j)}^p = \left\| \sum_{T \in \mathbb{V}[O_j]} c_T \right\|_{BL_{p, A_j}^p(O_j)}^p.$$

For each $O_j \in \mathcal{O}_j$, either the cellular or the algebraic case holds, as defined in Theorem 5.1. Let $\mathcal{O}_{j, \text{cell}}$ denote the subcollection of $\mathcal{O}_j$ consisting of all cells for which the cellular case holds and $\mathcal{O}_{j, \text{alg}} := \mathcal{O}_j \setminus \mathcal{O}_{j, \text{cell}}$. Thus, by (I)$_j$, one may bound $\left\| \sum_{T \in \mathbb{T}} c_T \right\|_{BL_{p, A_j}^p(B_{r_0})}$ by

$$C_j(d) \left[ \sum_{O_j \in \mathcal{O}_{j, \text{cell}}} \left\| \sum_{T \in \mathbb{V}[O_j]} c_T \right\|_{BL_{p, A_j}^p(O_j)}^p + \sum_{O_j \in \mathcal{O}_{j, \text{alg}}} \left\| \sum_{T \in \mathbb{V}[O_j]} c_T \right\|_{BL_{p, A_j}^p(O_j)}^p \right];$$

the analysis is splits into two cases depending on which term in the above sum dominates.

**Cellular-dominant case.** Suppose that the inequality

$$\sum_{O_j \in \mathcal{O}_{j, \text{alg}}} \left\| \sum_{T \in \mathbb{V}[O_j]} c_T \right\|_{BL_{p, A_j}^p(O_j)}^p \leq \sum_{O_j \in \mathcal{O}_{j, \text{cell}}} \left\| \sum_{T \in \mathbb{V}[O_j]} c_T \right\|_{BL_{p, A_j}^p(O_j)}^p$$

holds so that

$$\left\| \sum_{T \in \mathbb{T}} c_T \right\|_{BL_{p, A_j}^p(B_{r_0})} \lesssim 2C_j(d) \sum_{O_j \in \mathcal{O}_{j, \text{cell}}} \left\| \sum_{T \in \mathbb{V}[O_j]} c_T \right\|_{BL_{p, A_j}^p(O_j)}^p. \quad (9)$$

Definition of $\mathcal{O}_{j+1}$. Define $\mathcal{O}_{j+1}$ by adjoining the letter $c$ to the word $\mathcal{O}_j$. Thus, it follows from the definitions that

$$r_{j+1} = \frac{1}{2} r_j, \quad \#_c(j+1) = \#_c(j) + 1 \quad \text{and} \quad \#_a(j+1) = \#_a(j). \quad (10)$$

The next generation of cells $\mathcal{O}_{j+1}$ arise from the cellular decomposition guaranteed by Theorem 5.1. Fix $O_j \in \mathcal{O}_{j, \text{cell}}$ so that there exists some polynomial $P: \mathbb{R}^n \to \mathbb{R}$ of degree $O(d)$ with the following properties:

i) $\#_{\text{cell}}(P) \sim d^n$ and each $O \in \text{cell}(P)$ has diameter at most $2r_{j+1}$. 

ii) One may pass to a refinement of cell($P$) such that if
\[ O_{j+1}(O_j) := \{ O \setminus N_\delta(Z) : O \in \text{cell}(P) \} \]
denotes the corresponding collection of $\delta$-shrunken cells, then
\[ \left\| \sum_{T \in T[O_j]} \chi_T \right\|_{\text{BL}^p_{k,A_j}(O_j)}^p \lesssim d^m \left\| \sum_{T \in T[O_{j+1}]} \chi_T \right\|_{\text{BL}^p_{k,A_j}(O_{j+1})}^p \]
for all $O_{j+1} \in O_{j+1}(O_j)$. Given $O_{j+1} \in O_{j+1}(O_j)$, define
\[ T[O_{j+1}] := \{ T \in T[O_j] : T \cap O_{j+1} \neq \emptyset \}. \]
Recall that, by the fundamental theorem of algebra (or Bézout’s theorem), any $\delta$-tube $T$ can enter at most $O(d)$ cells $O_{j+1} \in O_{j+1}(O_j)$ and, consequently,
\[ \sum_{O_{j+1} \in O_{j+1}(O_j)} \# T[O_{j+1}] \lesssim d \cdot \# T[O_j]. \tag{11} \]
By the pigeonhole principle, one may pass to a refinement of $O_{j+1}(O_j)$ such that
\[ \# T[O_{j+1}] \lesssim d^{-(m-1)} \# T[O_j] \quad \text{for all } O_{j+1} \in O_{j+1}(O_j). \tag{12} \]
Finally, define
\[ O_{j+1} := \bigcup_{O_j \in O_{j,\text{cell}}} O_{j+1}(O_j). \]
This completes the construction of $\mathcal{O}_{j+1}$ and it remains to check that the new ensemble satisfies the desired properties. In view of this, it is useful to note that
\[ C_j(d) = d^{-\varepsilon} C_{j+1}(d) \quad \text{and} \quad A_j = A_{j+1}, \tag{13} \]
which follows immediately from (10) and the definition of the $C_j(d)$ and $A_j$.

**Property I.** Fix $O_j \in O_{j,\text{cell}}$ and observe that
\[ \# O_{j+1}(O_j) \sim d^m \]
and
\[ \left\| \sum_{T \in T[O_j]} \chi_T \right\|_{\text{BL}^p_{k,A_j}(O_j)}^p \lesssim d^m \left\| \sum_{T \in T[O_{j+1}]} \chi_T \right\|_{\text{BL}^p_{k,A_j}(O_{j+1})}^p \]
for all $O_{j+1} \in O_{j+1}(O_j)$. Averaging,
\[ \left\| \sum_{T \in T[O_j]} \chi_T \right\|_{\text{BL}^p_{k,A_j}(O_j)}^p \lesssim \sum_{O_{j+1} \in O_{j+1}(O_j)} \left\| \sum_{T \in T[O_{j+1}]} \chi_T \right\|_{\text{BL}^p_{k,A_j}(O_{j+1})}^p \]
and, recalling (9) and (13), one deduces that
\[ \left\| \sum_{T \in T} \chi_T \right\|_{\text{BL}^p_{k,A}(B_{\delta_0})}^p \lesssim C d^{-\varepsilon} C_{j+1}(d) \sum_{O_{j+1} \in O_{j+1}(O_j)} \left\| \sum_{T \in T[O_{j+1}]} \chi_T \right\|_{\text{BL}^p_{k,A_j}(O_{j+1})}^p. \]
Provided $d$ is chosen large enough so as to ensure that the additional $d^{-\varepsilon}$ factor absorbs the unwanted constant $C$, one deduces (I)$_{j+1}$. This should be compared with the approach of Solymosi and Tao to polynomial partitioning [28].
Property II. By the construction,
\[
\sum_{O_{j+1} \in O_{j+1}} \# T[O_{j+1}] = \sum_{O_j \in O_j \cap O_{j+1}(O_j)} \sum_{O_{j+1} \in O_{j+1}(O_j)} \# T[O_{j+1}] \leq d \sum_{O_j \in O_j} \# T[O_j],
\]
where the inequality follows from a term-wise application of (11). Thus, \((II)_j\), (10) and (13) imply that
\[
\sum_{O_{j+1} \in O_{j+1}} \# T[O_{j+1}] \lesssim d^{-\varepsilon} C_{j+1}(d) d^{\# \varepsilon(j+1)} \# T.
\]
Provided \(d\) is chosen sufficiently large, one deduces \((II)_{j+1}\).

Property III. Fix \(O_{j+1} \in O_{j+1}(O_j)\) and recall from (12) that
\[
\# T[O_{j+1}] \lesssim d^{-(m-1)} \# T[O_j].
\]
Thus, \((III)_j\), (10) and (13) imply that
\[
\# T[O_{j+1}] \lesssim d^{-\varepsilon} C_{j+1}(d) d^{-\# \varepsilon(j+1)(m-1)} \# T[O_j].
\]
Provided \(d\) is chosen sufficiently large as before, one deduces \((III)_{j+1}\).

\(\blacktriangleright\) **Algebraic-dominant case.** Suppose the hypothesis of the cellular-dominant case fails so that
\[
\left\| \sum_{T \in T} \chi_T \right\|_{BL_k,A_j(B_{\alpha})}^p \lesssim 2 C_j(d) \sum_{O_j \in O_{j,alg}} \left\| \sum_{T \in T[O_j]} \chi_T \right\|_{BL_k,A_j(O_j)}^p.
\]
Each cell in \(O_{j,alg}\) satisfies the condition of the algebraic case of Theorem 5.1; this information is used to construct the \((j+1)\)-generation ensemble.

Definition of \(d_{j+1}\). Define \(d_{j+1}\) by adjoining the letter \(a\) to the word \(d_j\). Thus, it follows from the definitions that
\[
d_{j+1} = d_j^{1+\varepsilon}, \quad \# c(j+1) = \# c(j) \quad \text{and} \quad \# a(j+1) = \# a(j) + 1. \tag{15}
\]
The next generation of cells is constructed from the varieties which arise from the algebraic case in Theorem 5.1. Fix \(O_j \in O_{j,alg}\) so that there exists a transverse complete intersection \(Y_j\) of dimension \(m-1\) and \(\deg Y_j \leq C_{alg} d\) such that
\[
\left\| \sum_{T \in T[O_j]} \chi_T \right\|_{BL_k,A_j(O_j)}^p \lesssim \log d \left\| \sum_{T \in T[O_j]} \chi_T \right\|_{BL_k,A_j(O_j \cap N_s Y_j)}^p.
\]
Let \(B(O_j)\) be a cover of \(O_j \cap N_s Y_j\) consisting of finitely-overlapping balls of radius \(\max\{r_{j+1}, \delta^{1-\varepsilon}\}\). For each \(B \in B(O_j)\) let \(T_B\) denote the family of \(T \in T[O_j]\) for which \(T \cap B \cap N_s Y_j \neq \emptyset\). This set is partitioned into the subsets
\[
T_{B,tang} := \{ T \in T_B : T \text{ is tangent to } Y_j \text{ on } B \}, \quad T_{B,trans} := T_B \setminus T_{B,tang};
\]
here the notion of tangency is that given in Definition 4.2.

By hypothesis, \([tang]\) fails and, consequently, one may deduce that
\[
\sum_{O_j \in O_{j,alg}} \left\| \sum_{T \in T[O_j]} \chi_T \right\|_{BL_k,A_j(O_j)}^p \lesssim \log d \sum_{O_j \in O_{j,alg}} \left\| \sum_{T \in T_{B,trans}} \chi_T \right\|_{BL_k,A_{j+1}(B_j)}^p \tag{16}
\]
where, for notational convenience, \( B_j := B \cap N_\delta Y_j \). Indeed, provided \( C_{\text{tang}} > 0 \) is sufficiently large,

\[
\sum_{O_j \in \mathcal{O}_{j,\text{alg}}} \sum_{B \in \mathcal{B}(O_j)} \# T_{B,\text{tang}} \leq C_{\text{tang}} \delta^{-n \varepsilon_3} \sum_{O_j \in \mathcal{O}_j} \# T[O_j];
\]

\[
\max_{O_j \in \mathcal{O}_{j,\text{alg}}} \max_{B \in \mathcal{B}(O_j)} \# T_{B,\text{tang}} \leq \max_{O_j \in \mathcal{O}_j} \# T[O_j].
\]

Consequently, the failure of the stopping condition \([\text{tang}]\) forces

\[
\log d \sum_{O_j \in \mathcal{O}_{j,\text{alg}}} \sum_{B \in \mathcal{B}(O_j)} \left\| \sum_{T \in \mathcal{T}_{B,\text{tang}}} \chi_T \mathbb{P} \right\|_{\mathcal{L}^p_{\text{BL}, A_j+1}(B)} \leq \frac{1}{C_{\text{tang}}} \sum_{O_j \in \mathcal{O}_j} \left\| \sum_{T \in \mathcal{T}[O_j]} \chi_T \mathbb{P} \right\|_{\mathcal{L}^p_{\text{BL}, A_j}(O_j)}
\]

(since the estimates in (17) show all other conditions for \([\text{tang}]\) are met for \( \mathcal{S}, T[S] \) and \( B[S] \) appropriately defined). On the other hand, by the triangle inequality for broad norms (Lemma 4.5), using the fact that \( A_{j+1} = A_j/2 \), the left-hand side of (16) is dominated by

\[
\log d \sum_{O_j \in \mathcal{O}_{j,\text{alg}}} \sum_{B \in \mathcal{B}(O_j)} \left\| \sum_{T \in \mathcal{T}_{B,\text{tang}}} \chi_T \mathbb{P} \right\|_{\mathcal{L}^p_{\text{BL}, A_{j+1}}(B_j)} + \left\| \sum_{T \in \mathcal{T}_{B,\text{trans}}} \chi_T \mathbb{P} \right\|_{\mathcal{L}^p_{\text{BL}, A_{j+1}}(B_j)}.
\]

For a suitable choice of constant \( C_{\text{tang}} \), combining the information in the two previous displays yields (16).

For \( O_j \in \mathcal{O}_{j,\text{alg}} \) define

\[
\mathcal{O}_{j+1}(O_j) := \{ B \cap N_\delta Y_j : B \in \mathcal{B}(O_j) \}
\]

and let \( T[O_{j+1}] := \mathcal{T}_{B,\text{trans}} \) for \( O_{j+1} = B \cap N_\delta Y_j \in \mathcal{O}_{j+1}(O_j) \). The collection of cells \( \mathcal{O}_{j+1} \) is then given by

\[
\mathcal{O}_{j+1} := \bigcup_{O_j \in \mathcal{O}_{j,\text{alg}}} \mathcal{O}_{j+1}(O_j).
\]

It remains to verify that the ensemble \( \mathcal{E}_{j+1} \) satisfies the desired properties. In view of this, it is useful to note that

\[
C_j(d) = d^{-(n+\varepsilon_3)} C_{j+1}(d),
\]

which follows directly from the definition of \( C_j(d) \) and (15).

**Property I.** By combining (16) together with the various definitions one obtains

\[
\sum_{O_j \in \mathcal{O}_{j,\text{alg}}} \left\| \sum_{T \in \mathcal{T}[O_j]} \chi_T \mathbb{P} \right\|_{\mathcal{L}^p_{\text{BL}, A_j}(O_j)} \leq \log d \sum_{O_{j+1} \in \mathcal{O}_{j+1}} \left\| \sum_{T \in \mathcal{T}[O_{j+1}]} \chi_T \mathbb{P} \right\|_{\mathcal{L}^p_{\text{BL}, A_{j+1}}(O_{j+1})}.
\]

Recalling (14) and (18), if \( c(d) := Cd^{-(n+\varepsilon_3)} \log d \) for an appropriate choice of admissible constant \( C \), then

\[
\left\| \sum_{T \in \mathcal{T}} \chi_T \mathbb{P} \right\|_{\mathcal{L}^p_{\text{BL}, A_j}(B_{\nu})} \leq c(d) C_{j+1}(d) \sum_{O_{j+1} \in \mathcal{O}_{j+1}} \left\| \sum_{T \in \mathcal{T}[O_{j+1}]} \chi_T \mathbb{P} \right\|_{\mathcal{L}^p_{\text{BL}, A_{j+1}}(O_{j+1})}.
\]

Provided \( d \) is sufficiently large, \( c(d) \leq 1 \) and one thereby deduces (I)\(_{j+1} \).
Property II. Fix $O_j \in \mathcal{O}_{j, \text{alg}}$ and note that

$$
\sum_{O_{j+1} \in \mathcal{O}_{j+1}(O_j)} \# \mathcal{T}[O_{j+1}] = \sum_{B \in \mathcal{B}(O_j)} \# \mathcal{T}_{B, \text{trans}}
$$

(19)

by the definition of $\mathcal{T}[O_{j+1}]$. To estimate the latter sum one may invoke the following algebraic-geometric result of Guth, which appears in Lemma 5.7 of [14].

**Lemma 6.1 ([14]).** Suppose $T$ is an infinite cylinder in $\mathbb{R}^n$ of radius $\delta$ and central axis $\ell$ and $Y$ is a transverse complete intersection. For $\alpha > 0$ let

$$
Y_{> \alpha} := \{ y \in Y : \angle(T_y Y, \ell) > \alpha \}.
$$

The set $Y_{> \alpha} \cap T$ is contained in a union of $O((\deg Y)^n)$ balls of radius $\delta \alpha^{-1}$.

Since $T \cap B \cap N_{\delta} Y \neq \emptyset$ by the definition of $\mathcal{T}_B$, a tube $T \in \mathcal{T}_B$ belongs to $\mathcal{T}_{B, \text{trans}}$ if and only if the angle condition ii) from Definition 4.2 fails to be satisfied. Thus, given any $T \in \bigcup_{B \in \mathcal{B}} \mathcal{T}_{B, \text{trans}}$, it follows from the definitions that

$$
\angle(\text{dir}(T), T_y Y) \gtrsim \frac{\delta}{r_{j+1}}
$$

for some $y \in Y \cap 2B$ with $|y - x| \lesssim \delta$ for some $x \in T$. This implies that

$$
N_{C_0 \delta T \cap 2B \cap Y_{> \alpha_{j+1}}} \neq \emptyset
$$

where $\alpha_{j+1} \sim \delta/r_{j+1}$. Consequently, by Lemma 6.1, any $T \in \bigcup_{B \in \mathcal{B}(O_j)} \mathcal{T}_{B, \text{trans}}$ lies in at most $O(d^n)$ of the sets $\mathcal{T}_B$ and so

$$
\sum_{B \in \mathcal{B}(O_j)} \# \mathcal{T}_{B, \text{trans}} \lesssim d^n \# \mathcal{T}[O_j].
$$

Combining this inequality with (19) and summing over all $O_j \in \mathcal{O}_{j, \text{alg}}$,

$$
\sum_{O_{j+1} \in \mathcal{O}_{j+1}} \# \mathcal{T}[O_{j+1}] \lesssim d^n \sum_{O_j \in \mathcal{O}_j} \# \mathcal{T}[O_j].
$$

Applying (II)$_j$, (15) and (18), one concludes that

$$
\sum_{O_{j+1} \in \mathcal{O}_{j+1}} \# \mathcal{T}[O_{j+1}] \lesssim d^{-\varepsilon} C_{j+1}(d) d^{\#(j+1)} \# \mathcal{T}.
$$

Provided $d$ is chosen to be sufficiently large to absorb the implicit constant, one deduces (II)$_{j+1}$.

Property III. Fix $O_j \in \mathcal{O}_{j, \text{alg}}$ and $O_{j+1} \in \mathcal{O}_{j+1}(O_j)$. By definition, $\mathcal{T}[O_{j+1}] \subseteq \mathcal{T}[O_j]$ and so

$$
\# \mathcal{T}[O_{j+1}] \leq \# \mathcal{T}[O_j] \leq C_{j+1}(d) d^{-\varepsilon(j+1)(m-1)} \# \mathcal{T},
$$

by (III)$_j$ and (15).
The second algorithm. The algorithm [alg 1] is now applied repeatedly in order to arrive at a final decomposition of the $k$-broad norm. This process forms part of a second algorithm, referred to as [alg 2].

Throughout this section let $p_k$, with $k \leq \ell \leq n$, denote some choice of Lebesgue exponents satisfying $p_k \geq p_{k+1} \geq \cdots \geq p_n =: p \geq 1$. The numbers $0 \leq \Theta, 1$ are then defined in terms of the $p_k$ by

$$\Theta := \left(1 - \frac{1}{p} \right)^{-1} \left(1 - \frac{1}{p} \right)$$

so that $\Theta = 1$. Also fix $0 < \varepsilon, \varepsilon \ll 1$ as in the previous section.

There are two stages to [alg 2], which can roughly be described as follows:

- **The recursive stage:** $\sum_{T \in T} \chi_T$ is repeatedly decomposed into pieces with favourable tangency properties with respect to varieties of progressively lower dimension.
- **The final stage:** $\sum_{T \in T} \chi_T$ is further decomposed into very small scale pieces.

To begin, the recursive stage of [alg 2] is described.

**Input.** [alg 2] will take as its input:

- A choice of small scale $0 < \delta \ll 1$.
- A large integer $A \in \mathbb{N}$.
- A family of $\delta$-tubes $T$ which are non-degenerate in the sense that

$$\left\| \sum_{T \in T} \chi_T \right\|_{\text{BL}^p_{k,A}(\mathbb{R}^n)} \neq 0. \quad (20)$$

Note that the process applies to essentially arbitrary families of $\delta$-tubes (in particular, the polynomial Wolff axiom hypothesis does not appear at this stage).

**Output.** The $(n + 1 - \ell)$th step of the recursion will produce:

- An $(n + 1 - \ell)$-tuple of:
  - scales $\vec{\delta} = (\delta_n, \ldots, \delta_\ell)$ satisfying $\delta_n > \cdots > \delta_\ell > \delta_1 - \varepsilon;$
  - large and (in general) non-admissible parameters $\vec{D} = (D_n, \ldots, D_\ell)$;
  - integers $\vec{A} = (A_n, \ldots, A_\ell)$ satisfying $A = A_n > A_{n-1} > \cdots > A_\ell$.

Each of these $(n + 1 - \ell)$-tuples is formed by adjoining a component to the corresponding $(n - \ell)$-tuple from the previous stage.

- A family $\vec{S}_\ell$ of $(n + 1 - \ell)$-tuples of transverse complete intersections $\vec{S}_\ell = (S_n, \ldots, S_\ell)$ satisfying $\dim S_i = i$ and $\deg S_i = O(1)$ for $\ell \leq i \leq n$.
- An assignment of a $\delta$-ball $B[\vec{S}_\ell]$ and a subfamily $T[\vec{S}_\ell]$ of $\delta$-tubes to each $S_\ell \in \vec{S}_\ell$ with the property that the tubes $T \in T[\vec{S}_\ell]$ are tangent to $S_\ell$ in $B[\vec{S}_\ell]$ (here $S_\ell$ is the final component of $\vec{S}_\ell$).

This data is chosen so that the following properties hold:

**Notation.** Throughout this section a large number of harmless $\delta^{-\varepsilon}$-factors appear in the inequalities. For notational convenience, given $A, B \geq 0$ let $A \lesssim B$ or $B \gtrsim A$ denote $A < \delta^{-\varepsilon} B$ for some $c > 0$ depending only on $n$ and $p$. 
Property 1. The inequality
\[
\left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{BL_{k,A}(\mathbb{R}^n)} \lesssim C(\bar{D}_\ell; \bar{\delta}_\ell) [\delta^n \# \mathcal{T}]^{1-\Theta_\ell} \left( \sum_{\bar{S}_\ell \in \bar{\mathcal{S}}_\ell} \left\| \sum_{T \in \mathcal{T}[\bar{S}_\ell]} \chi_T \right\|_{BL_{k,A}(B[\bar{S}_\ell])} \right)^{\Theta_\ell}
\]
holds for
\[
C(\bar{D}_\ell; \bar{\delta}_\ell) := \prod_{i=\ell}^{n-1} \left( \frac{\delta_i}{\delta} \right)^{\Theta_{i+1} - \Theta_i} D_i^{1+\varepsilon_i} (\Theta_{i+1} - \Theta_i).
\]

Property 2. For \( \ell \leq n-1 \), the inequality
\[
\sum_{\bar{S}_\ell \in \bar{\mathcal{S}}_\ell} \# \mathcal{T}[\bar{S}_\ell] \lesssim D^{1+\varepsilon_\ell} \sum_{S_{\ell+1} \in S_{\ell+1}} \# \mathcal{T}[\bar{S}_{\ell+1}]
\]
holds.

Property 3. For \( \ell \leq n-1 \), the inequality
\[
\max_{\bar{S}_\ell \in \bar{\mathcal{S}}_\ell} \# \mathcal{T}[\bar{S}_\ell] \lesssim D^{1-\ell+\varepsilon_\ell} \max_{\bar{S}_{\ell+1} \in S_{\ell+1}} \# \mathcal{T}[\bar{S}_{\ell+1}]
\]
holds.

By the inclusion property (5), the broad norms over \( B[\bar{S}_\ell] \) on the right-hand side of (21) could be replaced by broad norms over 2\( \delta \)-neighbourhoods of \( S_\ell \).

First step. Vacuously, the tubes belonging to \( \mathcal{T} \) are tangent to the \( n \)-dimensional variety \( \mathbb{R}^n \). Let \( \mathcal{B}_\circ \) denote a collection of finitely-overlapping balls of radius \( \varepsilon_\circ \) which cover \( \bigcup_{T \in \mathcal{T}} T \) and define

- \( \delta_n := \varepsilon_\circ; \ D_n := 1 \) and \( A_n := A \); 
- \( \mathcal{S}_n \) is the collection consisting of repeated copies of the 1-tuple \( (\mathbb{R}^n) \), with one copy for each ball in \( \mathcal{B}_\circ \); 
- For each \( \bar{S}_n \in \mathcal{S}_n \) assign a ball \( B[\bar{S}_n] \in \mathcal{B}_\circ \) and let 
  \( \mathcal{T}[\bar{S}_n] := \{ T \in \mathcal{T} : T \cap B[\bar{S}_n] \neq \emptyset \} \).

By a straightforward orthogonality argument (identical to that used to establish the base case in the proof of Proposition 3.2), Property 1 can be shown to hold with \( C(\bar{D}_n; \bar{\delta}_n) = 1 \) and \( \Theta_n = 1 \).

\((n+2-\ell)\)th step. Let \( \ell \geq 1 \) and suppose that the recursive algorithm has run through \( n+1-\ell \) steps. Since each family \( \mathcal{T}[\bar{S}_\ell] \) consists of \( \delta \)-tubes which are tangent to \( S_\ell \) on \( B[\bar{S}_\ell] \), one may apply [alg 1] to bound the \( k \)-broad norm
\[
\left\| \sum_{T \in \mathcal{T}[\bar{S}_\ell]} \chi_T \right\|_{BL_{k,A}(B[\bar{S}_\ell])}.
\]
One of two things can happen: either [alg 1] terminates due to the stopping condition [tiny] or it terminates due to the stopping condition [tang]. The current recursive process terminates if the contributions from terms of the former type dominate:
Stopping condition. The recursive stage of [alg 2] has a single stopping condition, which is denoted by [tiny-dom].

Stop:[tiny-dom] Suppose that the inequality
\[
\sum_{S_t \in \tilde{S}_t} \sum_{T \in \tilde{T}[\tilde{S}_t]} \chi_T|_{BL_{k,A_t}(B[S_t])}^p \leq \frac{1}{2} \sum_{S_t \in \tilde{S}_t, \text{tiny}} \sum_{T \in \tilde{T}[\tilde{S}_t]} \chi_T|_{BL_{k,A_t}(B[S_t])}^p \tag{22}
\]
holds, where the right-hand summation is restricted to those $S_t \in \tilde{S}_t$ for which [alg 1] terminates owing to the stopping condition [tiny]. Then [alg 2] terminates.

Assume that the condition [tiny-dom] is not met. Necessarily,
\[
\sum_{S_t \in \tilde{S}_t} \sum_{T \in \tilde{T}[\tilde{S}_t]} \chi_T|_{BL_{k,A_t}(B[S_t])}^p \leq \frac{1}{2} \sum_{S_t \in \tilde{S}_t, \text{tang}} \sum_{T \in \tilde{T}[\tilde{S}_t]} \chi_T|_{BL_{k,A_t}(B[S_t])}^p ,
\tag{23}
\]
where the right-hand summation is restricted to those $S_t \in \tilde{S}_t$ for which [alg 1] does not terminate owing to [tiny] and therefore terminates owing to [tang]. Consequently, for each $\tilde{S}_t \in \tilde{S}_t, \text{tang}$ the inequalities
\[
\left\| \sum_{T \in \tilde{T}[\tilde{S}_t]} \chi_T|_{BL_{k,A_t}(B[\tilde{S}_t])}^p \right\| \lesssim \sum_{S_{t-1} \in \tilde{S}_{t-1}[\tilde{S}_t]} \| \sum_{T \in \tilde{T}[\tilde{S}_{t-1}]} \chi_T|_{BL_{k,2A_{t-1}}(B[\tilde{S}_{t-1}])}^p \right\| ,
\tag{24}
\]
and
\[
\max_{S_{t-1} \in \tilde{S}_{t-1}[\tilde{S}_t]} \#\tilde{T}[\tilde{S}_{t-1}] \lesssim D_{t-1}^{(t-1)+\varepsilon} \#\tilde{T}[\tilde{S}_t] ;
\tag{25}
\]
\[
\max_{S_{t-1} \in \tilde{S}_{t-1}[\tilde{S}_t]} \#\tilde{T}[\tilde{S}_{t-1}] \lesssim D_{t-1}^{(t-1)+\varepsilon} \#\tilde{T}[\tilde{S}_t] \tag{26}
\]
hold for some choice of:
- Scale $\delta_{t-1}$ satisfying $\delta_t > \delta_{t-1} \geq \delta^{1-\varepsilon}$; non-admissible number $D_{t-1}$ and large integer $A_{t-1}$ satisfying $A_{t-1} \sim A_t$;
- Family $\tilde{S}_{t-1}[\tilde{S}_t]$ of $(t-1)$-dimensional transverse complete intersections of degree $O(1)$;
- Assignment of a subfamily $\tilde{T}[\tilde{S}_{t-1}]=\tilde{T}[\tilde{S}_t]|_{\tilde{S}_{t-1}}$ of $\delta$-tubes for every $S_{t-1} \in S_{t-1}[\tilde{S}_t]$ such that each $T \in \tilde{T}[\tilde{S}_{t-1}]$ is tangent to $S_{t-1}$ on $B[\tilde{S}_{t-1}]$.

Each inequality (24), (25) and (26) is obtained by combining the definition of the stopping condition [tang] with Properties I, II and III from [alg 1], respectively. Indeed, we take
\[
\begin{align*}
r_0 &:= \delta_t, \quad \delta_{t-1} := \max\{r_1^{1+\varepsilon}, \delta^{1-\varepsilon}\}, \quad \text{and} \quad D_{t-1} := d\#\tilde{e}(J),
\end{align*}
\]
using the notation from [alg 1].

The $\delta_{t-1}$, $D_{t-1}$ and $A_{t-1}$ can depend on the choice of $\tilde{S}_t$, but this dependence can be essentially removed by pigeonholing. In particular, $\#\tilde{e}(J)$ depends on $\tilde{S}_t$, but satisfies $\#\tilde{e}(J) = O(\log \delta^{-1})$. Thus, since there are only logarithmically many possible different values, one may find a subset of the $\tilde{S}_{t, \text{tang}}$ over which the $D_{t-1}$ all have a common value and, moreover, the inequality (22) still holds except that the constant $1/2$ is now replaced with, say, $\delta^{-\varepsilon}$. A brief inspection of [alg 1] shows that both $\delta_{t-1}$ and $A_{t-1}$ are determined by $D_{t-1}$ and so the desired uniformity is immediately inherited by these parameters.
Letting $\mathcal{S}_{\ell-1}$ denote the structured set 
\[ \mathcal{S}_{\ell-1} := \{(\bar{S}_\ell, S_{\ell-1}) : \bar{S}_\ell \in \mathcal{S}_{\ell, \text{tang}} \text{ and } S_{\ell-1} \in S_{\ell-1}[\bar{S}_\ell]\}, \]
where $\mathcal{S}_{\ell, \text{tang}}$ is understood to be the refined collection described in the previous paragraph, it remains to verify that the desired properties hold for the newly constructed data. Property 2 follows immediately from (25) and Property 3 from (26), so it remains only to verify Property 1.

By combining the inequality (21) from the previous stage of the algorithm with (23) and (24), one deduces that 
\[ \left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{\ell^q \mathcal{B}^A_{\ell,M}(\mathbb{R}^n)} \lesssim C(\tilde{D}_\ell; \tilde{\delta}_i)[\delta^n \# T]^{1-\Theta_i} \left\| \sum_{T \in \mathcal{T}[\bar{S}_{\ell-1}]} \chi_T \right\|_{\ell^q \mathcal{B}^A_{\ell+1,M}(\bar{S}_{\ell-1})}^{1/q}, \]
where, for any $1 \leq q < \infty$ and $M \in \mathbb{N}$, we write
\[ \left\| \sum_{T \in \mathcal{T}[\bar{S}_{\ell-1}]} \chi_T \right\|_{\ell^q \mathcal{B}^A_{\ell,M}(\bar{S}_{\ell-1})} := \left( \sum_{S_{\ell-1} \in \mathcal{S}_{\ell-1}} \left\| \sum_{T \in \mathcal{T}[\bar{S}_{\ell-1}]} \chi_T \right\|_{\ell^q \mathcal{B}^A_{\ell,M}(B[\bar{S}_{\ell-1}])} \right)^{1/q}. \]

Taking $q = p_\ell$ and $M = 2A_{\ell+1}$, the logarithmic convexity inequality (Lemma 4.6) dominates the preceding expression by 
\[ \left\| \sum_{T \in \mathcal{T}[\bar{S}_{\ell-1}]} \chi_T \right\|_{\ell^{p_\ell} \mathcal{B}^A_{\ell+1,M}(\bar{S}_{\ell-1})} \lesssim \left( \delta_i^{-1} \delta_i^{\ell-1} \delta_i^n \sum_{S_{\ell-1} \in \mathcal{S}_{\ell-1}} \# T[\bar{S}_{\ell-1}] \right). \]

Observe that, trivially, one has 
\[ \left\| \sum_{T \in \mathcal{T}[\bar{S}_{\ell-1}]} \chi_T \right\|_{\ell^{p_\ell} \mathcal{B}^A_{\ell+1,M}(\bar{S}_{\ell-1})} \lesssim \left( \delta_i^{-1} \delta_i^{\ell-1} \delta_i^n \sum_{S_{\ell-1} \in \mathcal{S}_{\ell-1}} \# T[\bar{S}_{\ell-1}] \right). \]

and, by Property 2 for the tube families $\{T[\bar{S}_i] : \bar{S}_i \in \mathcal{S}_i\}$ for $\ell - 1 \leq i \leq n - 1$, it follows that 
\[ \left\| \sum_{T \in \mathcal{T}[\bar{S}_{\ell-1}]} \chi_T \right\|_{\ell^{p_\ell} \mathcal{B}^A_{\ell+1,M}(\bar{S}_{\ell-1})} \lesssim \left( \delta_i^{-1} \delta_i^{\ell-1} \delta_i^n \sum_{S_{\ell-1} \in \mathcal{S}_{\ell-1}} \# T[\bar{S}_{\ell-1}] \right). \]

One may readily verify that 
\[ C(\tilde{D}_\ell; \tilde{\delta}_i) \left( \delta_i^{-1} \prod_{i=\ell-1}^{n-1} D_i^{1+\varepsilon_0} \right)^{\Theta_i-\Theta_{\ell-1}} = C(\tilde{D}_{\ell-1}; \tilde{\delta}_{\ell-1}) \]
and so, combining the above estimates, 
\[ \left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{\ell^{p_\ell} \mathcal{B}^A_{\ell,M}(\mathbb{R}^n)} \lesssim C(\tilde{D}_{\ell-1}; \tilde{\delta}_{\ell-1})[\delta^n \# T]^{1-\Theta_{\ell-1}} \left\| \sum_{T \in \mathcal{T}[\bar{S}_{\ell-1}]} \chi_T \right\|_{\ell^{p_\ell} \mathcal{B}^A_{\ell+1,M}(\bar{S}_{\ell-1})}, \]
which is Property 1.
The final stage. If the algorithm has not stopped by the $k$th step, then it necessarily terminates at the $k$th step. Indeed, otherwise (21) would hold for $\ell = k - 1$ and families $T[S_{k-1}]$ of $\delta_{k-1}$-tubes which are tangent to some transverse complete intersection of dimension $k - 1$. By the vanishing property of the $k$-broad norms as described in Lemma 4.3, one would then have

$$\left\| \sum_{T \in T[S_{k-1}]} \chi_T \right\|_{BL_k^{k-1}(B[S_{k-1}]^\ell)} = 0,$$

which, by (21), would contradict the non-degeneracy hypothesis (20).

Suppose the recursive process terminates at step $m$, so that $m \geq k$. For each $S_m \in \mathcal{S}_{m, \tiny}$ let $\mathcal{O}[S_m]$ denote the final collection of cells output by [alg 1] (that is, the collection denoted by $\mathcal{O}_J$ in the notation of the previous subsection) when applied to estimate the broad norm $\left\| \sum_{T \in T[S_m]} \chi_T \right\|_{BL_k^{m}(B[S_m])}$. By Properties I, II and III of [alg 1] one has

$$\left\| \sum_{T \in T[S_m]} \chi_T \right\|_{BL_k^{m}(B[S_m])} \lesssim \sum_{O \in \mathcal{O}[S_m]} \left\| \sum_{T \in T[O]} \chi_T \right\|_{BL_k^{m-1}(O)},$$

for some $A_m \sim A_m$ where the families $T[O]$ satisfy

$$\sum_{O \in \mathcal{O}[S_m]} \# T[O] \lesssim D_{m-1}^{1+\varepsilon} \# T[S_m]$$

(27)

and

$$\max_{O \in \mathcal{O}[S_m]} \# T[O] \lesssim D_{m-1}^{-(m-1)+\varepsilon} \# T[S_m]$$

(28)

for $D_{m-1}$ a large and (in general) non-admissible parameter. Once again, by pigeonholing, one may pass to a subcollection of $\mathcal{S}_{m, \tiny}$ and thereby assume that the $D_{m-1}$ (and also the $A_{m-1}$) all share a common value.

If $O$ denotes the union of the $\mathcal{O}[S_m]$ over all $S_m$ belonging to subcollection of $\mathcal{S}_{m, \tiny}$ described above, then [alg 2] outputs the following inequality.

First key estimate.

$$\left\| \sum_{T \in T} \chi_T \right\|_{BL_k^k(\mathbb{R}^n)} \lesssim C(D_m; \bar{\delta}_m)\delta^n \# T[\delta^n \# T]^{1-\Theta_m} \left( \sum_{O \in \mathcal{O}} \sum_{T \in T[O]} \chi_T \right)_{BL_k^{m-1}(O)} \frac{\Theta_m}{p_m}.$$

7. Proof of Theorem 3.1

Henceforth, fix $T$ to be a family of $\delta$-tubes in $\mathbb{R}^n$ which satisfy the $(D, N)$-polynomial Wolff axiom for some $D$, chosen sufficiently large (depending only on the admissible parameters $n$ and $\varepsilon$) so as to satisfy the forthcoming requirements of the proof. Without loss of generality, we may assume that $T$ satisfies the non-degeneracy hypothesis (20). The algorithms described in the previous section can be applied to this tube family, leading to the final decomposition of the broad norm described in the first key estimate. One therefore wishes to show, using the polynomial Wolff axiom hypothesis, that the quantity on the right-hand side of the first key estimate can be effectively bounded, provided that the exponents $p_1, \ldots, p_n$ are suitably chosen.
Since each $O \in \mathcal{O}$ is contained in a ball of radius at most $\delta^{1-\varepsilon}$, trivially one may bound
\[
\left\| \sum_{T \in \mathcal{T}[O]} \chi_T \right\|_{BL_{k,A_{m-1}}(O)}^{p_m} \lesssim \delta^n (\#\mathcal{T}[O])^{p_m}.
\]
Recalling that $\Theta_m (1 - \frac{1}{p_m}) = 1 - \frac{1}{p}$, this yields
\[
\left( \sum_{O \in \mathcal{O}} \left\| \sum_{T \in \mathcal{T}[O]} \chi_T \right\|_{BL_{k,A_{m-1}}(O)}^{p_m} \right)^{\frac{1}{p_m}} \lesssim \left( \max_{O \in \mathcal{O}} \#\mathcal{T}[O] \right)^{\frac{1}{p}} \left( \delta^n \sum_{O \in \mathcal{O}} \#\mathcal{T}[O] \right)^{\frac{1}{p}}.
\]
Now (27) and repeated application of Property 2 from [alg 2] imply
\[
\sum_{O \in \mathcal{O}} \#\mathcal{T}[O] \lesssim \left( \prod_{i=m-1}^{n-1} D_i^{1+\varepsilon} \right) \#\mathcal{T}.
\]
Combining this with the first key estimate and the definition of $C(\tilde{D}_m; \tilde{\delta}_m)$, one concludes that
\[
\left\| \sum_{T \in \mathcal{T}} \chi_T \right\|_{BL_{k,A}(\mathbb{R}^n)} \lesssim C(\tilde{D}; \tilde{\delta}) \left( \max_{O \in \mathcal{O}} \#\mathcal{T}[O] \right)^{1+\frac{1}{p}} \left( \delta^n \sum_{T \in \mathcal{T}} |T| \right)^{\frac{1}{p}} \tag{29}
\]
where, taking $\delta_{m-1} := \delta$, the constant takes the form
\[
C(\tilde{D}; \tilde{\delta}) := \prod_{i=m-1}^{n-1} \left( \delta_i^{\Theta_{i+1} - \Theta_i} D_i^{\Theta_{i+1} - (1+\frac{1}{p}) + O(\varepsilon_i)} \right).
\]
In order to bound the maximum appearing on the right-hand side of (29), by (28) and repeated application of Property 3 of [alg 2], it follows that
\[
\max_{O \in \mathcal{O}} \#\mathcal{T}[O] \lesssim \left( \prod_{i=m-1}^{\ell-1} D_i^{1+\varepsilon} \right) \max_{\tilde{S}_t \in \mathcal{T}_t} \#\mathcal{T}[\tilde{S}_t]
\]
whenever $m \leq \ell \leq n$. Recall, for each tube family $\mathcal{T}[\tilde{S}_t]$ produced by [alg 2] there exists a $\delta_t$-ball $B_{\delta_t} := B[\tilde{S}_t]$ such that every $\delta$-tube $T \in \mathcal{T}[\tilde{S}_t]$ is tangent to $S_t$ in $B_{\delta_t}$; in particular,
\[
T \cap B_{\delta_t} \cap N_{\delta S_t} \neq \emptyset \quad \text{and} \quad T \cap 2B_{\delta_t} \subseteq N_{2\delta S_t}.
\]
Here $S_t$ is a transverse complete intersection of dimension $\ell$ and $\deg S_t$ depends only on the admissible parameters $n$ and $\varepsilon$. Thus, if $D$ is chosen to be sufficiently large, then the $(D, N)$-polynomial Wolff axiom implies that
\[
\#\mathcal{T}[\tilde{S}_t] \leq \# \{ T \in \mathcal{T} : |T \cap 2B_{\delta_t} \cap N_{2\delta S_t}| \geq \delta_t |T| \} \lesssim N \delta^{-(n-1)} \delta^{-n} |2B_{\delta_t} \cap N_{2\delta S_t}|.
\]
Moreover, by Wongkew’s lemma [35],
\[
|2B_{\delta_t} \cap N_{2\delta S_t}| \lesssim \delta^{n-\ell} \delta^t,
\]
Combining these observations,
\[
\max_{\tilde{S}_t \in \mathcal{T}_t} \#\mathcal{T}[\tilde{S}_t] \lesssim N \delta^{-(n-1)} \left( \frac{\delta_t}{\delta} \right)^{-(n-\ell)}
\]
so that
\[
\max_{O \in \mathcal{O}} \#\mathcal{T}[O] \lesssim N \left( \prod_{i=m-1}^{\ell-1} D_i^{1+\varepsilon} \right) \left( \frac{\delta_t}{\delta} \right)^{-(n-\ell)} \delta^{-(n-1)}
\]
for all $m \leq \ell \leq n$. Finally, these $n - m + 1$ different estimates can be combined into a single inequality by taking a weighted geometric mean, yielding:

**Second key estimate.** Let $0 \leq \gamma_m, \ldots, \gamma_n \leq 1$ satisfy $\sum_{j=m}^n \gamma_j = 1$. Then

$$\max_{\mathcal{O} \in \mathcal{D}} \# \mathcal{O} | \mathcal{O} | \lesssim N \left( \prod_{i=m-1}^{n-1} \left( \frac{\delta_i}{\delta} \right)^{-(n-i)\gamma_i} D_i^{-i(1-\sum_{j=m}^i \gamma_j) + O(\varepsilon)} \right) \delta^{(n-1)}.$$ 

Substituting the second key estimate into (29), one obtains

$$\| \sum_{T \in \mathcal{T}} X_T \|_{BL^p_{k,A}([R^n])} \lesssim N^{1-\frac{1}{p}} \left( \prod_{i=m-1}^{n-1} \left( \frac{\delta_i}{\delta} \right)^{X I} D_i^{O(\varepsilon)} \right) \delta^{-\left(n-1-\frac{1}{p}\right)} \left( \sum_{T \in \mathcal{T}} |T| \right)^{\frac{1}{p}}$$

where

$$X_i := \Theta_{i+1} - \Theta_i - (n-i)\gamma_i \left( 1 - \frac{1}{p} \right);$$

$$Y_i := \Theta_{i+1} - \left( 1 + i \left( 1 - \sum_{j=m}^i \gamma_j \right) \right) \left( 1 - \frac{1}{p} \right).$$

One now chooses the various exponents so that $X_i, Y_i = 0$ for all $m \leq i \leq n-1$ and $Y_{n-1} = 0$. This ensures that the $(\delta_i/\delta)^{X_i}$ and $D_i^{Y_i}$ factors in the above expression are admissible but does not allow one to control the $D_i^{O(\varepsilon)}$ factors, which may still be non-admissible. To deal with the $D_i^{O(\varepsilon)}$ one may perturb the $p$ exponent which results under the conditions $X_i, Y_i = 0$, so that $Y_i$ becomes negative, and then choose $\varepsilon_0$ sufficiently small depending on the choice of perturbation. This yields an open range of $k$-broad estimates, which can then be trivially extended to a closed range via interpolation through logarithmic convexity (the interpolation argument relies on the fact that one is permitted an $\delta^{-\varepsilon}$-loss in the constants in the $k$-broad inequalities).

The condition $X_i = 0$ is equivalent to

$$\left( 1 - \frac{1}{p_{i+1}} \right)^{-1} - \left( 1 - \frac{1}{p_i} \right)^{-1} = (n-i)\gamma_i$$

whilst the condition $Y_{i-1} = 0$ is equivalent to

$$\left( 1 - \frac{1}{p_i} \right)^{-1} = i - (i-1) \sum_{j=m}^{i-1} \gamma_j.$$  

Choose $p_m := \frac{m}{m-1}$ so that (31) holds in the $i = m$ case. The remaining $p_i$ are then defined in terms of the $\gamma_j$ by the equation

$$\left( 1 - \frac{1}{p_i} \right)^{-1} = m + \sum_{j=m}^{i-1} (n-j)\gamma_j$$

so that each of the $n - m$ constraints in (30) is met.

It remains to solve for the $n - m + 1$ variables $\gamma_m, \ldots, \gamma_n$. By comparing the right-hand sides of (31) and (32), it follows that

$$\sum_{j=m}^{i-1} (n+i-j-1)\gamma_j = i - m$$

for $m+1 \leq i \leq n$.  

(33)
To solve this linear system, let $\beta_i$ denote the left-hand side of (33) and observe that
\[
\beta_{i+1} + \beta_{i-1} - 2\beta_i = n\gamma_i - (n - 1)\gamma_{i-1}
\]
for $m + 1 \leq i \leq n - 1$, where $\beta_m := 0$. On the other hand, by considering the right-hand side of (33), it is clear that
\[
\beta_{i+1} + \beta_{i-1} - 2\beta_i = 0.
\]
Combining these observations gives a recursive relation for $\gamma_j$ and from this one deduces that
\[
\gamma_j = \frac{1}{n} \prod_{i=m}^{j-1} \frac{n - 1}{n} = \frac{1}{n} \left(1 - \frac{1}{n}\right)^{j-m}
\]
for $m \leq j \leq n - 1$.

It remains to check that these parameter values give the correct value of $p_n$, corresponding to the exponent featured in Theorem 3.1. It follows from (31) that
\[
\left(1 - \frac{1}{p_n}\right)^{-1} = n - (n - 1) \sum_{j=m}^{n-1} \frac{1}{n} \left(1 - \frac{1}{n}\right)^{j-m}
\]
\[
= 1 + (n - 1) \left(1 - \frac{1}{n}\right)^{n-m}.
\]
This is largest when $m = k$, which directly yields the desired range of $p$, as stated in Theorem 3.1, completing the proof.

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