Shear-free Null Quasi-Spherical Spacetimes

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(March 24, 2022)

Abstract

We study the residual gauge freedom within the null quasi-spherical (NQS) gauge for spacetimes admitting an expanding shear-free null foliation. By constructing the most general NQS coordinates subordinate to such a foliation, we obtain both a clear picture of the geometric nature of the residual coordinate freedom, and an explicit construction of nontrivial NQS metrics representing some well-known spacetimes, such as Schwarzschild, accelerated Minkowski, and Robinson-Trautman. These examples will be useful in testing numerical evolution codes. The geometric gauge freedom consists of an arbitrary boost and rotation at each coordinate sphere — and this freedom may be used to normalise the coordinate to an “inertial” frame.

04.20,04.30
I. INTRODUCTION

The recently introduced [1–4] null quasi-spherical (NQS) coordinate condition provides a new approach to the study of the Einstein equations in exterior regions admitting an expanding null foliation. The NQS gauge is described by the metric ansatz

$$ds_{\text{NQS}}^2 = -2udz(dr + vdz) + (rd\vartheta + \beta^1dr + \gamma^1dz)^2 + (r \sin \vartheta d\varphi + \beta^2dr + \gamma^2dz)^2,$$

where \((\vartheta, \varphi)\) are the usual polar coordinates on \(S^2\), \(u > 0\) and \(v\) are real-valued functions, and

$$\beta = \beta^1 \partial_\vartheta + \beta^2 \csc \vartheta \partial_\varphi, \quad \gamma = \gamma^1 \partial_\vartheta + \gamma^2 \csc \vartheta \partial_\varphi,$$

may be considered either as vectors tangent to the spheres \((z, r) = \text{const}\) or, using a complex formalism, as spin-1 fields.

The advantages of the gauge, and its generality, are discussed in [4]. The purpose of this paper is to analyse the gauge freedoms remaining within the NQS gauge condition, for the class of spacetimes admitting a shear-free \((\beta = 0)\) null foliation. We explicitly describe the construction of such foliations in Schwarzschild, Minkowski and Robinson-Trautman spacetimes.

The examples will also be useful as test data for numerical solvers, since they involve arbitrary functions but are still simple to describe explicitly. This remark applies both to characteristic and 3+1 codes — the class of boosted Schwarzschild metrics should be particularly appropriate as test data.

The NQS gauge is best understood by comparison with other popular conditions used to describe the metric on a null foliation, due to Bondi [5] and Newman and Unti [6]. The Newman-Unti radial coordinate \(r\) is determined by a choice of affine parameter along each of the null generators; the Bondi radius is defined by the condition that the spatial volume form \(\sin \vartheta d\vartheta \wedge d\varphi\) have length \((r^2 \sin \vartheta)^{-1}\). In both cases the angular coordinates \((\vartheta, \varphi)\) are transported along the null generators. Both coordinate systems are determined by labelling and normalisation conditions at just one transverse \(S^2\) in a null hypersurface and therefore have gauge freedom corresponding to functions on a single \(S^2\) (in each null hypersurface).

By contrast, the NQS radial function \(r\) has level sets isometric to standard spheres of radius \(r\). Although it is possible to then determine the angular coordinates \((\vartheta, \varphi)\) as labelling the outgoing null generators (as in [5, 6]), it seems more geometrically natural to use the \((\vartheta, \varphi)\) determined by the isometry with \(S^2\) with metric \(r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)\).

Since the metric spheres at each radius are not unique within the null hypersurface (at least, this is the case for the standard Minkowski null cone), there is an additional coordinate freedom within the NQS gauge, consisting of a choice of Lorentz transformation at each sphere. This freedom does not have an analogue in the Bondi and Newman-Unti gauges.

The vector field \(\beta\) is referred to as the shear; that this terminology does not conflict with the accepted usage of “shear” is seen by noting that the (usual) shear of the null generator \(\ell = \partial_r - r^{-1}\beta\) of the NQS metric Eqn. (1) is given in the Newman-Penrose notation by \(\sigma_{\text{NP}} = r^{-1}\partial\beta\), where \(\partial\) is the \(\ell\)th operator on the standard \(S^2\) and we identify \(\beta \sim \frac{1}{\sqrt{2}}(\beta^1 - i\beta^2)\) with a spin-1 field on \(S^2\). Consequently, vanishing shear vector \(\beta\) implies
vanishing $\sigma_{NP}$; conversely if $\sigma_{NP} = 0$ then $\beta$ consists purely of $\ell = 1$ spin-1 spherical harmonics [8]. The role played by the the $\ell = 1$ spin-1 spherical harmonics is discussed in greater detail in the following section. In the Appendix we show that any shear-free, expanding and twist-free metric admits NQS coordinates with $\beta = 0$ — this was shown in [11,12] for vacuum metrics.

The metric form (1) with $\beta = 0$, when restricted to a coordinate null hypersurface $C$, becomes

$$\text{d}s^2_C = r^2(d\vartheta^2 + \sin^2 \vartheta \text{d}\varphi^2).$$

By identifying $C$ with the future null cone at the origin in $\mathbb{R}^{3,1}$, we can see that this form is invariant under the Lorentz group $SO_0(3,1)$ — and the Lorentz transformation may also vary with $r$, since invariance only requires that each quasi-sphere $r = \text{const.}$ is mapped isometrically. Thus, our main idea is to use explicit representations of the Lorentz group acting on the standard null cone $C_0 = \{t = |x|\}$ in Minkowski space $\mathbb{R}^{3,1}$ to describe the general transformation leaving the form $\text{d}s^2_C$ invariant.

Note that the problem of finding general quasi-spherical foliations of a null hypersurface which is not shear-free and expanding is considerably more difficult, since the explicit model of the standard cone and its associated Lorentz deformations is no longer available. However, linearisation arguments suggest strongly that the gauge freedoms of the shear-free case are mirrored in the more general setting, provided the shear is not too large. Thus we expect that the description here of the shear-free NQS freedom will provide some insight into the more general case.

At least in the shear-free case, we will show that the NQS condition has gauge freedom consisting of an $SO_0(3,1)$-valued function of the radius (on each null hypersurface); this is functionally less rigid than the Bondi and NU gauges, since it has freedom in $r$ which is lacking in these gauges. This Lorentz transformation freedom may be viewed as providing a choice of "inertial frame" normalisation at each radius, and may be used to normalise certain of the remaining metric coefficients, as described below.

This interpretation is supported by a comparison [9] between the Robinson-Trautman metrics and the NU form [11] of the Minkowski metric in coordinates using null cones with base point describing a timelike curve. This comparison may also be made in the NQS coordinates, and supports both the interpretation of Robinson-Trautman spacetimes as describing an accelerated black hole rapidly settling down to a Schwarzschild black hole in uniform motion, and the interpretation of the NQS freedom as representing a choice of reference frame at each radius and time.

In section 2 we study the metric and NQS freedom of the model cone $C_0$, by constructing the most general quasi-spherical (QS) foliation of $C_0$. The resulting metric has shear vector $\beta$ consisting solely of $\ell = 1$ spherical harmonics, and we show conversely that any null surface with such shear vector is gauge-equivalent to the standard cone. In section 3 we describe the metric in general NQS coordinates on a spacetime admitting a shear-free null foliation. Section 4 describes the application of these results to the specific examples of Schwarzschild, Minkowski and Robinson-Trautman spacetimes. Basic results on shear-free expanding null hypersurfaces are collected in the Appendix.

The computations are presented in slightly more detail than is strictly necessary, in order to facilitate the use of the example NQS metrics in benchmarking numerical codes, and in
the interpretation of general NQS numerical results.

II. MODEL CONE \( C_0 \)

Let \( C_0 = \{(x,t) \in \mathbb{R}^{3,1} : t = |x|\} \) be the standard future null cone based at the origin in Minkowski space. We may use \( x \in \mathbb{R}^3 \) as a coordinate on \( C_0 \); instead a polar representation
\[
x = r\theta, \quad r = |x|, \quad \theta = x/r = (x_i/r) \in S^2
\]
will be very useful, where we identify \( S^2 = \{x \in \mathbb{R}^3 : |x| = 1\} \) and we use the direction cosines \( \theta = (\theta_1, \theta_2, \theta_3), \ |\theta| = 1 \) to parameterise \( S^2 \). The polar coordinates on \( C_0 \) will usually be denoted by \((r,\theta)\) or \((\rho,\zeta)\).

The parameterisation \((\theta_i)\) of \( S^2 \) leads to a representation of tangent vector fields to \( S^2 \) as 3-vector fields on the unit sphere \( S^2 \subset \mathbb{R}^3 \) which are tangent to \( S^2 \). This will prove more convenient than using the polar coordinate basis \((\partial_\theta,\partial_\phi)\). Thus, a vector field \( Y = Y(\theta;\lambda) \), depending on \( \theta \in S^2 \) and other parameters \( \lambda \) (eg \( \lambda = (z,r) \)), may be represented as the 3-vector \( Y = (Y_i) \) satisfying \( \theta^T Y(\theta,\lambda) = \theta_i Y_i = 0 \).

Throughout we use latin indices \( i,j,\ldots \) with range 1, \ldots, 3 and the summation convention on repeated indices, not necessarily raised and lowered.

The Minkowski metric induces the rank-2 degenerate bilinear form
\[
ds_{C_0}^2 = r^2 |d\theta|^2 = r^2 \sum_{i=1}^{3} (d\theta_i)^2
\]
on \( C_0 \). Note that \( |d\theta|^2 \) is the standard metric on \( S^2 \), and \( |d\theta|^2 = \sum_{i,j=1}^{3} \Theta_{ij} dx_i dx_j \), where \( \Theta \) is the projection matrix
\[
\Theta = I - \theta\theta^T, \quad \Theta_{ij} = \delta_{ij} - \theta_i\theta_j,
\]
and \( \theta^T \) represents the transpose (row) vector.

A quasi-sphere of radius \( r \in \mathbb{R}^+ \) in \( C_0 \) is an orientation preserving embedding \( \Phi_r : S^2 \to C_0 \) such that
\[
\Phi_r^*(ds_{C_0}^2) = r^2 |d\theta|^2,
\]
and we say \( \Phi_r \) is a quasi-spherical map.

Every quasi-sphere in \( C_0 \) is determined by a unique time and space orientation preserving Lorentz transformation \( L \in SO_0(3,1) \) via the composition \( \Phi_r = L \circ i_r \) with the inclusion \( i_r : S^2 \to C_0, \theta \mapsto [r \theta^T, r]^T \),
\[
\Phi_r : \theta \in S^2 \mapsto \begin{bmatrix} r \theta \\ r \end{bmatrix} \in C_0 \mapsto L \begin{bmatrix} r \theta \\ r \end{bmatrix} \in C_0,
\]
and we now exploit this basic description.

The Lorentz transformation \( L \in SO_0(3,1) \) admits a unique boost/rotation decomposition \( L = RB \) with
\[ \mathcal{R} = \mathcal{R}(R) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}, \quad R \in SO(3), \tag{7} \]

\[ \mathcal{B} = \mathcal{B}(w) = \begin{bmatrix} W & w \\ w^T & b \end{bmatrix}, \quad w \in \mathbb{R}^3, \tag{8} \]

where \( b = \sqrt{1 + |w|^2} \geq 1 \) and \( W \) is the \( 3 \times 3 \) matrix

\[ W = I + \frac{1}{b + 1}ww^T. \tag{9} \]

It is easily checked that \( \mathcal{B} \) preserves the Minkowski metric \( \eta, \eta = \mathcal{B}^T \eta \mathcal{B} \). The quasi-spherical map \( \Phi_r \) may be described explicitly by

\[ \Phi_r(\theta) = L \begin{bmatrix} r\theta \\
    r \end{bmatrix} = \begin{bmatrix} rR\theta + r \left( 1 + \frac{\theta^T w}{b+1} \right) Rw \\
    r(b + \theta^T w) \end{bmatrix}, \tag{10} \]

where \( \theta^T w = w^T \theta \) is the usual inner product between column 3-vectors. In terms of the rectangular \( \mathbb{R}^{3,1} \) parameterisation of the target \( \mathcal{C}_0 \) we have \( \Phi_r(\theta) = [\rho(\theta, \theta)\zeta(\theta)^T, \rho(\theta, \theta)]^T \), where we define

\[ f(\theta) = f(\theta; w) := b + \theta^T w = \sqrt{1 + |w|^2} + \theta^T w, \tag{11} \]

\[ \rho(\theta, \theta) = \rho(\theta, \theta; w) := rf(\theta) \tag{12} \]

\[ \zeta(\theta) = \zeta(\theta; w, R) := f^{-1}R \left( \theta + \left( 1 + \frac{\theta^T w}{b+1} \right) w \right) = f^{-1}R(w + W\theta). \tag{13} \]

Here and elsewhere we adopt the convention that \((\rho, \zeta)\) denotes polar coordinates on the range (target) \( \mathcal{C}_0 \) of \( \Phi_r \), which leads to the description \( \Phi_r(\theta) = (\rho(\theta, \theta), \zeta(\theta)) \) of \( \Phi_r \) in polar coordinates. The metric \( ds_{\mathcal{C}_0}^2 \) on the target cone \( \mathcal{C}_0 \) in polar coordinates \((\rho, \zeta)\) is just \( \rho^2|d\zeta|^2 \) and we may verify the quasi-spherical condition \( \Phi_r^*(ds_{\mathcal{C}_0}^2) = r^2|d\theta|^2 \) by direct computation as follows.

We define the angular gradient operator \( D_{\theta} = (D_{\theta_i}) \) as the projection tangent to the unit sphere \( S^2 \) of the ordinary gradient in \( \mathbb{R}^3 \). Explicitly, let \( h_e(x) := h(x/|x|), \ x \in \mathbb{R}^3 \) be the homogeneous degree 0 extension of any \( h \in C^1(S^2) \) and define \( D_{\theta_i}h = \partial h_e/\partial x_i \). It follows that \( \theta_i D_{\theta_j}h = 0 \) by the homoogeneity condition, and \( D_{\theta_i}D_{\theta_j} = \Theta_{ij} \). Then \( \Phi_r^*(\rho^2|d\zeta|^2) = \rho(\theta, \theta)^2|\zeta_{\rho\theta}|^2 \), where \((\zeta_{\rho\theta})_{ij} = D_{\theta_i}\zeta_j \).

\[ D_{\theta_i}\zeta_j = D_{\theta_j} \left( (b + \theta^T w)^{-1}R_{ik}(w_k + W_{kl} \theta_l) \right) \]
\[ = f^{-1}R_{ik} \left( -f^{-1}(w_k + W_{kl} \theta_l) \Theta_{jm}w_m + W_{km} \Theta_{mj} \right) \]
\[ = f^{-1}R_{ik}A_{km} \Theta_{mj}, \tag{14} \]

and we have introduced the very useful matrix

\[ A := I - f^{-1} \left( \theta + (b + 1)^{-1}w \right) w^T. \tag{15} \]

By exercising a certain amount of care we may convert to a matrix notation. We adopt the convention that 3-vector quantities such as \( \theta_i, d\theta_i, w_i \), are to be treated as column vectors.
(of 1-forms, functions, etc.) and that row vectors usually will be indicated by the transpose notation, except that we regard $D_{\theta}$, $D_{\zeta}$ and their associated gradients as row vectors.

For example, with these conventions the computation (14) may be summarised as

$$\zeta,\theta = f^{-1} R A \Theta$$

and the chain rule for $h \in C^\infty(S^2, \mathbb{R})$ appears as

$$dh = h,\theta d\theta$$

$$d(h \circ \zeta) = h,\zeta \zeta,\theta d\theta$$

$$(h \circ \zeta)_,\theta = f^{-1} h,\zeta R A \Theta,$$

where both sides of the final identity are row vectors.

The matrix $A$ satisfies several useful identities,

$$A^T A = I - f^{-1}(w,\theta^T + \theta w^T),$$

$$\Theta A^T A \Theta = \Theta,$$

$$A^{-1} = I + (\theta + (b+1)^{-1}w)w^T = W + \theta w^T,$$

which may be used to show that $\Phi_r$ is quasi-spherical:

$$\Phi_r^*(\rho^2|d\zeta|^2) = r^2 f^2 |f^{-1} RA \theta|^2 = r^2 d\theta^T A^T R^T RA d\theta$$

$$= r^2 d\theta^T A^T A d\theta = r^2 |d\theta|^2,$$

since $R^T R = I$ and $\theta^T d\theta = 0$, so $d\theta = \Theta d\theta$. This confirms that $\Phi_r$ is quasi-spherical and also conformal, since $\Phi_r^*(|d\zeta|^2) = f^{-2}|d\theta|^2$. Note that from Eqs. (13),(19) we have

$$\zeta(\theta) = f^{-1} RA^{-1T} \theta,$$

where $A^{-1T}$ is the inverse transpose matrix.

Having described a single quasi-sphere, we may now consider the effects of $r$-dependence: a map $\Phi : C_0 \to C_0$ is said to be \textit{quasi-spherical} if $\Phi_r = \Phi \circ i_r$ is a quasi-spherical map for each $r > 0$, and if $r \mapsto \Phi_r$ is at least continuously differentiable in $r$ — although for simplicity we shall consider only smooth maps. Equivalently, $\Phi(r,\theta) = L(r) \circ i_r(\theta)$, where the Lorentz transformations $L(r) = R(R(r))B(w(r))$ are described by boost and rotation maps $w \in C^\infty(R^+, \mathbb{R}^3)$, $R \in C^\infty(R^+, SO(3))$. Note that we do not require that $\Phi$ be a diffeomorphism, and this does not follow from the condition $\Phi^* (ds^2_{C_0}) = r^2 |d\theta|^2$ since $ds^2_{C_0}$ is degenerate. Note also that $\Phi^{-1}$, when defined, is not usually quasi-spherical; neither is the composition of two quasi-spherical maps usually again quasi-spherical.

The general quasi-spherical map $\Phi : C_0 \to C_0$ is thus described by

$$\Phi(r,\theta) = R(R(r))B(w(r)) \begin{bmatrix} r\theta \\ r \end{bmatrix} = \begin{bmatrix} \rho(r,\theta) \zeta(r,\theta) \\ \rho(r,\theta) \end{bmatrix}$$

where the functions $\rho(r,\theta), \zeta(r,\theta)$ are defined by Eqs. (12),(13) with $w, R$ now depending on $r$. The pull-back metric is then
\[
\Phi^*(ds_{C_0}^2) = \rho(r,\theta)^2|\zeta_\theta d\theta + \zeta_r dr|^2 \\
= \rho^2|\zeta_\theta d\theta|^2 + 2\rho^2 d\theta^T \zeta_\theta^T \zeta_r dr + \rho^2|\zeta_r|^2 dr^2,
\]
where \(\zeta_r\) denotes the column vector of partial derivatives \(\frac{\partial}{\partial r}\zeta\). Note that we are taking the liberty of using \(\rho, \zeta\) to denote both the coordinates \(\rho, \zeta\) on \(C_0\) and their pullbacks \(\rho(r, \theta) = \Phi^*(\rho), \zeta(r, \theta) = \Phi^*(\zeta)\), which are functions determining the map \(\Phi\) — this ambiguity should not lead to any serious confusion.

We have already checked that \(\rho^2|\zeta_\theta d\theta|^2 = r^2|d\theta|^2\), and we compute from Eqn. (20)
\[
fA^{-1}R^{-1} \zeta_r = A^{-1}R^{-1}R_rA^{-1T} \theta + A^{-1}\frac{\partial}{\partial r}(A^{-1T})\theta - f^{-1}f_r(A^T A)^{-1} \theta.
\]  
(21)
The first term on the right hand side is simplified by introducing the antisymetric matrix
\[
S_1 := WR^{-1}R_r W,
\]  
(22)
and equals (bearing in mind the formulas \(A^{-1}W^{-1} = I + b^{-1}\theta w^T, W^{-1} = I - b^{-1}(b+1)^{-1}ww^T\))
\[
S_1 \theta + b^{-1}\Theta S_1 w.
\]

After some computation, we find that the second and third terms of Eqn. (21) may be combined into
\[
\Theta W^{-1}w_r + \frac{1}{b+1} \theta \times (w_r \times w),
\]
where \(a \times b = (\epsilon_{ijk} a_j b_k)\) is the usual cross product in \(\mathbb{R}^3\). Defining the 3-vectors \(s_1, t_1\) (depending on \(r\) but independent of \(\theta\))
\[
s_1 := *S_1 + \frac{1}{b+1} w_r \times w
\]  
(23)
\[
t_1 := b^{-1}s_1 w + W^{-1}w_r
\]  
(24)
where \(*S := (\frac{1}{2}\epsilon_{ijk} S_{jk})\), and the \((r, \theta)\)-dependent vector \(\beta\)
\[
\beta := r\Theta t_1 + r\theta \times s_1,
\]  
(25)
we have the identity
\[
\rho \zeta_r = RA\beta.
\]  
(26)
This may be used to simplify \(\Phi^*(ds_{C_0}^2)\):
\[
\rho^2 d\theta^T \zeta_\theta^T \zeta_r dr = rd\theta^T A^T RA\beta dr,
\]
\[
= rd\theta^T A^T A\beta dr = rd\theta^T \beta dr,
\]
since \(\theta^T \beta = 0\). Similarly we find
\[
\rho^2|\zeta_r|^2 = \beta^T A^T RA\beta = |\beta|^2,
\]
and it follows that \(\Phi^*(ds_{C_0}^2) = |rd\theta + \beta dr|^2\), which is in quasi-spherical form with shear vector \(\beta\).

To summarise:
Proposition 1 Suppose $\Phi : C_0 \to C_0$ is a $C^\infty$ quasi-spherical map. Then $\Phi$ satisfies Eqn. (10) for some Lorentz boost parameter $w \in C^\infty(\mathbb{R}^+, \mathbb{R}^3)$ and spatial rotation $R \in C^\infty(\mathbb{R}^+, SO(3))$. In the rectangular-polar coordinates $(r, \theta)$ on $C_0$ we have
\[ \Phi^*(ds^2_0) = |rd\theta + \beta dr|^2, \]
where the shear vector $\beta$ is defined in terms of $w$ and $R$ by Eqs. (22), (23), (24), (25).

The angular vector field $\beta$ consists solely of spin-1 $\ell = 1$ spherical harmonics. This follows from Eqn. (23), since for any constant vector $t \in \mathbb{R}^3$, the angular vector fields $\Theta t = t - (\theta^T t) \theta$ and $\theta \times t$ satisfy
\[ \Theta t = \text{grad}_{S^2}(\theta^T t), \]
\[ \theta \times t = j\text{grad}_{S^2}(\theta^T t), \]
where the complex structure $J : TS^2 \to TS^2$ is defined by anticlockwise rotation with respect to the outer normal $\theta$ to $S^2 \subset \mathbb{R}^3$. Now if $e_1, e_2$ is an oriented orthonormal frame on $S^2$ (so $e_1 \times e_2 = \theta$), we define the spin-1 projection of a vector field $X = X^1 e_1 + X^2 e_2$ by
\[ X \sim \xi = \frac{1}{\sqrt{2}}(X^1 - iX^2) \]
(note $JX = \theta \times X = -X^2 e_1 + X^1 e_2 \sim -i\xi$), and the operator $\eth$ by
\[ \eth = \frac{1}{\sqrt{2}}(\nabla_{e_1} - i\nabla_{e_2}), \]
where $\nabla$ is the standard covariant derivative on $S^2$. Note that as so defined, $\xi$ and $\eth$ are frame dependent, hence the use of $\sim$. Defining the basis vector $e = \frac{1}{\sqrt{2}}(e_1 - i e_2)$, we could instead write the equality $X = \xi e + \xi \eth$ and then consider $\xi$ as the coefficient of the representation of $X$ as a section of a spin-1 complex line bundle, with respect to the basis vector $e$.

If $\phi, \psi \in C^1(S^2, \mathbb{R})$ then
\[ \eth(\phi + i \psi) = \frac{1}{\sqrt{2}}(\phi_{,1} + \psi_{,2} + i(\psi_{,1} - \phi_{,2})), \]
where the subscripts $(\cdot)_{,a}$ for $a = 1, 2$ denote directional derivatives with respect to the basis vectors $e_1, e_2$, and the vector field correspondence is
\[ \text{grad}_{S^2} \phi - j\text{grad}_{S^2} \psi \sim \eth(\phi + i \psi). \]
In particular, for any constant $s, t \in \mathbb{R}^3$ we have the correspondence
\[ \Theta t + \theta \times s \sim \eth(\theta^T(t - i s)), \]
and the identity
\[ \Delta_{S^2}(\theta^T t) = \frac{1}{2}\theta^T t \]
completes the identification of $\Theta t + \theta \times s$ as a spin-1 $\ell = 1$ spherical harmonic.

We also derive from Eqn. (29) that
\[ \Delta_{S^2}(\theta^T t)^2 - \frac{1}{4}|t|^2 = -6((\theta^T t)^2 - \frac{1}{4}|t|^2) \]
and thus $\theta^T t \theta^T s - \frac{1}{4}|t|^2 s$ is an $\ell = 2$ spherical harmonic for any $s, t \in \mathbb{R}^3$.

The relations (23), (24) between $s_1, t_1$ and $S_1, w, \chi$ may be inverted, since by direct computation we have the following lemma.
Lemma 2 Suppose \( s, t, w, \tilde{w}, \sigma \in \mathbb{R}^3 \) and let \( b := \sqrt{1 + |w|^2} \) and \( W^{-1} = I - \frac{1}{b(b+1)} w w^T \). The equations
\[
\begin{align*}
  t &= b^{-1} w \times \sigma + W^{-1} \tilde{w} \\
  s &= \sigma + \frac{1}{b+1} \tilde{w} \times w
\end{align*}
\] (35)
are equivalent to
\[
\begin{align*}
  \tilde{w} &= b t + s \times w \\
  \sigma &= b W^{-1} s + \frac{b}{b+1} w \times t.
\end{align*}
\] (36)

Applying this lemma with \( s = s_1, t = t_1, \tilde{w} = w_r, \sigma = *S_1 \) and assuming (23),(24) gives the ordinary differential equations
\[
\begin{align*}
  w_r &= b t_1 + s_1 \times w \\
  R_r &= RW^{-1} (b * (W^{-1} s_1) + \frac{b}{b+1} * (w \times t_1)) W^{-1},
\end{align*}
\] (37) (38)
where for any vector \( t \) we define the matrix *\( t = (\epsilon_{ijk} t_k) \). (So the two star operations interchange vectors with antisymmetric matrices.) Consequently we have the following reconstruction results.

Proposition 3 Suppose \( s_1, t_1 \in C^\infty(\mathbb{R}^+, \mathbb{R}^3) \) are given functions. Given initial conditions
\[
\begin{align*}
  w(r_0) &= w_0 \in \mathbb{R}^3 \\
  R(r_0) &= R_0 \in SO(3),
\end{align*}
\] (39)
there is a unique quasi-spherical map \( \Phi : C_0 \to C_0 \) with parameters \( w \in C^\infty(\mathbb{R}^+, \mathbb{R}^3), R \in C^\infty(\mathbb{R}^+, SO(3)) \) satisfying \( w(r_0) = w_0, R(r_0) = R_0 \) and such that the shear vector \( \beta \) satisfies
\[
\beta(r, \theta) = \Theta t_1(r) + \theta \times s_1(r).
\]

Proof: With \( s_1, t_1 \) given functions of \( r \) and \( b = \sqrt{1 + |w|^2} \), Eqn. (37) gives an ordinary differential equation for \( w \), with initial condition \( w(r_0) = w_0 \). Since for \( r \in [r_0, r_1] \)
\[
|b(r)t_1(r) + s_1(r) \times w(r)| \leq 2 (|w(r)| + 1) \sup_{[r_0, r_1]} (|t_1| + |s_1|),
\]
by Gronwall’s inequality the solution of Eqn. (37) is locally bounded and may be continued to all \( r \in \mathbb{R}^+ \). Substituting \( w(r) \) into Eqn. (38) gives an ode for \( R(r) \in SO(3) \) with initial condition \( R(r_0) = R_0 \), which similarly has a global solution \( R \in C^\infty(\mathbb{R}^+, SO(3)) \). The quasi-spherical map defined by the solutions \( w(r), R(r) \) via Eqn. (10) has the required shear vector \( \beta \) by previous computations. \( \Phi \) is the unique map satisfying the initial conditions at \( r_0 \) since any quasi-spherical map may be put into the form (10) and the parameters are then uniquely determined by the initial value problem (37),(38),(39). QED.
**Corollary 4** Suppose $|rd\theta + \beta dr|^2$ is a quasi-spherical form on $\mathcal{C}_0$, with shear vector $\beta$ consisting solely of $\ell = 1$ spherical harmonics (i.e., $\beta$ may be expressed in the form Eqn. (23)). Then there is a quasi-spherical map $\Phi : \mathcal{C}_0 \to \mathcal{C}_0$, $\Phi(r, \theta) = (\rho, \zeta)$, such that
\[ \Phi^*(\rho^2|d\zeta|^2) = |rd\theta + \beta dr|^2. \] (40)

Furthermore, $\Phi$ is unique up to a rigid Lorentz transformation of $\mathcal{C}_0$: if $\tilde{\Phi} : \mathcal{C}_0 \to \mathcal{C}_0$ is any map satisfying Eqn. (40), then there is $L_0 \in SO_0(3,1)$ such that $\tilde{\Phi} = L_0 \circ \Phi$.

**Proof:** Since $\beta$ is pure $\ell = 1$, the $\ell = 1$ spherical harmonic coefficient functions $s_1(r), t_1(r)$ are uniquely determined by Eqn. (23), and an appropriate quasi-spherical map $\tilde{\Phi}$ may be constructed using Proposition 3 and initial conditions $w(r_0) = 0$, $R(r_0) = I$ at some radius $r_0$. If $\tilde{\Phi} : \mathcal{C}_0 \to \mathcal{C}_0$ also satisfies Eqn. (40) then $\tilde{\Phi}$ is quasi-spherical and hence may be parameterised by Lorentz transformations $L(r)$, with parameter functions $\tilde{w}(r)$ and $\tilde{R}(r)$. Let $w_0 = \tilde{w}(r_0)$ and $R_0 = \tilde{R}(r_0)$ and let $L_0$ be the corresponding Lorentz transformation. Because $L_0^*(r^2|d\theta|^2) = r^2|d\theta|^2$, the map $L_0 \circ \Phi$ is also quasi-spherical satisfying Eqn. (40), and has parameters $\tilde{w}(r)$, $\tilde{R}(r)$ satisfying the initial conditions $\tilde{w}(r_0) = w_0$, $\tilde{R}(r_0) = R_0$. Since the parameters $s_1, t_1$ are determined uniquely from $\beta$ in Eqn. (40), uniqueness of the solution of the initial value problem Eqs. (37), (38), (39) implies $\tilde{\Phi} = L_0 \circ \Phi$ as required. QED.

Note that $\tilde{w}(r), \tilde{R}(r)$ can be computed in terms of $w(r), R(r)$ and $w_0, R_0$ from the identity
\[ \tilde{L} = R(\tilde{R})B(\tilde{w}) = R(R_0)B(w_0)R(R)B(w). \]

However it is not true in general that the composition of quasi-spherical maps is again quasi-spherical — $\tilde{L}$ in this identity defines a quasi-spherical map only when $w_0, R_0$ are constant.

**III. DEFORMATION OF SPACETIME METRICS**

We consider now those spacetimes whose metric can be placed in the form
\[ ds^2_{SF} = -2U\, dz (dr + V\, dz) + |r\, d\theta + \Gamma\, dz|^2, \] (41)

where $\Gamma = \Gamma(z, r, \theta)$ is an angular vector field, so $\Gamma$ satisfies $\theta^T \Gamma = 0$. We may verify that the null congruence defined by the coordinate tangent vector $\partial_r$ is expanding, shear-free and twist-free. This class includes the Schwarzschild, Robinson-Trautman and accelerated Minkowski spacetimes, and will be further discussed in the following section. For the present, we use the techniques of the previous section to study the effect of quasi-spherical Lorentz deformations of metrics of the form (41).

We regard $ds^2_{SF}$ as defined on (a subset of) $\mathbb{R} \times \mathcal{C}_0$, and then the metric induces the standard form $r^2|d\theta|^2$ on $\mathcal{C}_0$. The form (41) is in fact the most general metric form compatible with this property and such that the coordinate $z$ is null (characteristic). Extending previous definitions, for any domain $\Omega \subset \mathbb{R} \times \mathbb{R}^+$ with coordinates $(z, r)$, we say that $\Phi : \Omega \times S^2 \to \mathbb{R} \times \mathcal{C}_0$. 

$\mathbb{R} \times C_0$ is quasi-spherical if the restrictions $\Phi_{(z,r)}$ map $S^2 \to C_0$ and are quasi-spherical, for each $(z, r) \in \Omega$. As previously, we shall assume $\Phi$ is $C^\infty$.

In order to compute the pullback $\Phi^*(ds^2_{SF})$ using the above techniques, we first rename the polar coordinates in Eqn. (41) from $(r, \theta)$ to $(\rho, \zeta)$. Thus we now regard the metric parameters $U, V, \Gamma$ as functions of the coordinates $(z, \rho, \zeta)$ on the range $\mathbb{R} \times C_0$ of $\Phi$, and we reserve $(z, r, \theta)$ for coordinates on the domain $\Omega \times S^2$.

The map $\Phi$ may be described using the Lorentz boost and rotation functions $w \in C^\infty(\Omega, \mathbb{R}^3), R \in C^\infty(\Omega, SO(3))$ via

$$\Phi(z, r, \theta) = (z, \rho, \zeta) = (z, \rho(z, r, \theta), \zeta(z, r, \theta)), \tag{42}$$

where as before,

$$\rho(z, r, \theta) = r f(\theta; w(z, r)) = r (\sqrt{1 + |w|^2 + \theta^T w}), \tag{43}$$
$$\zeta(z, r, \theta) = \zeta(\theta; w(z, r), R(z, r)) = f^{-1} R(w + W \theta). \tag{44}$$

To compute the pullback $\Phi^*(ds^2_{SF})$ we use Eqn. (37) and the definitions Eqs. (22), (23), (24), (25):

$$\rho_r = \frac{\partial \rho}{\partial r} = f + r(\theta + b^{-1}w)^T w_r$$
$$= f + r(\theta + b^{-1}w)^T (bt_1 + s_1 \times w)$$
$$= f + rw^T (\Theta t_1 + \theta \times s_1) + r(b + w^T \theta) \theta^T t_1$$
$$= f(1 + r\theta^T t_1) + w^T \beta. \tag{45}$$

Defining the 3-vector quantities $s_0, t_0, \gamma$ by

$$S_0 := WR^{-1} R_z W \tag{46}$$
$$s_0 := *S_0 + \frac{1}{b+1} w \times w \tag{47}$$
$$t_0 := b^{-1} S_0 w + W^{-1} w_z = b^{-1}(w_z - s_0 \times w) \tag{48}$$
$$\gamma := r(\Theta t_0 + \theta \times s_0), \tag{49}$$

we similarly find

$$\rho_z = rf_z r + r f \theta^T t_0 + w^T \gamma \tag{50}$$

and thus

$$\Phi^*(d\rho) = (f(1 + r\theta^T t_1) + w^T \beta) dr + (r f \theta^T t_0 + w^T \gamma) dz + r w^T d\theta. \tag{51}$$

Using the identities Eqs. (16), (26) and the analogous

$$\rho \zeta, z = RA \gamma, \tag{52}$$

we also have

$$\Phi^*(d\zeta) = \zeta, z dr + \zeta, z dz + \zeta, \theta d\theta = \rho^{-1} RA(\beta dr + \gamma dz + r d\theta). \tag{53}$$
We denote the pullbacks of the metric functions $U,V,\Gamma$ by a tilde, so for example $\tilde{U} = \Phi^*(U)$ and $\tilde{U}(z,r,\theta) = U(z,\rho(z,r,\theta),\zeta(z,r,\theta))$. Substituting Eqs. (51,53) into Eqn. (11) with coordinates $(\rho,\zeta)$ replacing $(r,\theta)$ as already mentioned, gives

$$\Phi^*(ds^2_{\text{SF}}) = -2\tilde{U}(f(1 + r\theta^T t_1) + w^T\beta) dz dr
-2\tilde{U}(\tilde{V} + r f\theta^T t_0 + w^T\bar{\gamma}) dz^2 - 2\tilde{U}^T d\theta dz
+ |RA(r d\theta + \beta dr + (\bar{\gamma} + A^{-1}R^T \bar{\Gamma}) dz)|^2. \quad (54)$$

Moreover, $\Phi$ is spherical with respect to the expanding shear-free NQS metric

Now recall that $\Gamma = \Gamma(z,\rho,\zeta)$ is angular with respect to the $(\rho,\zeta)$ coordinates, and note that the pullback of $\zeta^T\Gamma = 0$ simplifies using Eqn. (20) to

$$0 = f^{-1}\theta^T A^{-1}R^T \bar{\Gamma},$$

where $\bar{\Gamma} = \Phi^*(\Gamma) = \Gamma(z,\rho(z,r,\theta),\zeta(z,r,\theta))$. This shows that the vector $A^{-1}R^T \bar{\Gamma}$ is purely angular in the $(r,\theta)$ coordinates, and the final term of Eqn. (54) becomes

$$|r d\theta + \beta dr + (\bar{\gamma} + A^{-1}R^T \bar{\Gamma}) dz|^2.$$ 

Introducing the angular vector

$$\gamma := \bar{\gamma} + A^{-1}R^T \bar{\Gamma} = \tilde{U}\bar{\Theta}w = \bar{\gamma} + \Theta(A^T R^T \bar{\Gamma} - \tilde{U})w \quad (56)$$

and noting that $w^T A^{-1} = fw^T$, the pullback metric becomes

$$\Phi^*(ds^2_{\text{SF}}) = |r d\theta + \beta dr + \gamma dz|^2 - 2\tilde{U}(1 + r\theta^T t_1) dr dz
-2\tilde{U}(\tilde{V} + r f\theta^T t_0 + \frac{1}{2}\tilde{U}|\Theta w|^2 - fw^T R^T \bar{\Gamma}) dz^2. \quad (57)$$

Comparing this metric with the general NQS metric Eqn. (11), which may be written in 3-vector notation as

$$ds^2_{\text{NQS}} = -2udz (dr + v dz) + |rd\theta + \beta dr + \gamma dz|^2,$$ 

we obtain the main transformation result for shear-free metrics.

**Proposition 5** Suppose $\Phi \in C^\infty(\Omega \times S^2, \mathbb{R} \times C_0)$ for some domain $\Omega \subset \mathbb{R}^2$ and $\Phi$ is quasi-spherical with respect to the expanding shear-free NQS metric $ds^2_{\text{SF}}$ given by Eqn. (11), with null coordinate $z$. Then $\Phi$ is described by Eqn. (12) with Lorentz boost and rotation functions $w \in C^\infty(\Omega, \mathbb{R}^3)$, $R \in C^\infty(\Omega, SO(3))$, and the pullback $\Phi^*(ds^2_{\text{SF}})$ is given by Eqn. (57).

Defining the pullbacks $\tilde{U} = \Phi^*(U)$, $\tilde{V} = \Phi^*(V)$, $\tilde{\Gamma} = \Phi^*(\Gamma)$ and the derived vectors $s_0, s_1, t_0, t_1$ in terms of $w, R$ via Eqs. (23), (24), (24), and (40), (47), (48), the NQS parameters $u, v, \beta, \gamma$ of the metric $\Phi^*(ds^2_{\text{SF}})$ are given explicitly by

$$u = \tilde{U}(f(1 + r\theta^T t_1)) \quad (59)$$

$$uv = \tilde{U}(\tilde{V} + r f\theta^T t_0 + \frac{1}{2}\tilde{U}|\Theta w|^2 - fw^T R^T \bar{\Gamma}) \quad (60)$$

$$\beta = r\Theta t_1 + r\theta \times s_1, \quad (61)$$

$$\gamma = r\Theta t_0 + r\theta \times s_0 + A^{-1}R^T \bar{\Gamma} - \tilde{U}\Theta w. \quad (62)$$

Moreover, $\Phi$ is a diffeomorphism if the vector $t_1(z,r)$ defined by Eqn. (24) satisfies

$$r |t_1| < 1, \quad \forall \ (z,r) \in \Omega. \quad (63)$$
Proof: Because \( ds^2_{SF} \) is non-degenerate (by assumption), \( \Phi \) will be a diffeomorphism iff the pullback is also non-degenerate, and this holds exactly when \( u \), the coefficient of \( dz \, dr \) in \( \Phi^* (ds^2_{SF}) \), is non-zero. But \( \tilde{U} \neq 0 \) by assumption, and \( f = \sqrt{1 + |w|e^2 + \theta^T w} > 0 \) for all \( w \in \mathbb{R}^3 \), \( \theta \in S^2 \), so the condition reduces to \( 1 + r \theta^T t_1 > 0 \). Clearly this holds for all \( \theta \in S^2 \) if and only if Eqn. (63) is satisfied. All other statements of the proposition follow from previous computations. QED.

IV. EXAMPLES

A. Spherically symmetric spacetimes

The metric form
\[
\text{ds}^2_{SS} = -2U \, dz (dr + V \, dr) + r^2 |d\theta|^2,
\]  
with \( U, V \) functions of \((z, r)\) only, includes the Schwarzschild metric as the special case \( U = 1, V = \frac{1}{2}(1 - 2M/r), M \in \mathbb{R} \). The geometric mass function \( m = \frac{1}{2}(1 - \eta_{ab} r_a r_b) \) for the general metric (64) is given by
\[
2m(z, r) = r (1 - 2V/U).
\]  
Again switching from \((r, \theta)\) to \((\rho, \zeta)\) coordinates in Eqn. (64), Proposition 4 describes \( \text{ds}^2_{SS} \) in general Lorentz transformed NQS coordinates, with in particular \( \tilde{\Gamma} = 0 \) and
\[
\tilde{U}(z, r, \theta) = U(z, \rho(z, r, \theta)), \quad \tilde{V}(z, r, \theta) = V(z, \rho(z, r, \theta)),
\]  
where \( \rho(z, r, \theta) = rf(\theta; w(z, r)) = r(b + \theta^T w), b = \sqrt{1 + |w|^2} \). In general the angular dependence of \( u \) and \( uv \) will be rather complicated, due to the effects of \( \rho \)-dependence of \( \tilde{U}, \tilde{V} \) in Eqn. (66).

In the special case of the Schwarzschild metric \( \tilde{U} = 1, 2\tilde{V} = 1 - 2M/\rho \), and the fields \( \beta, \gamma \) given by Eqs. (61), (62) with \( \tilde{\Gamma} = 0 \) are both pure \( \ell = 1 \) spin-1 spherical harmonics, and \( u, v \) satisfy
\[
u = (b + \theta^T w)(1 + r \theta^T t_1) = f(1 + r \theta^T t_1)
\]
\[
2uv = -\frac{2M}{r(b + \theta^T w)} + (b + \theta^T w)(b - \theta^T w + 2r \theta^T t_0).
\]  
If the boost \( w \in \mathbb{R}^3 \) is constant and \( R = I \), then we obtain the rather simple NQS metric parameters
\[
\beta = 0, \quad \gamma = -\Theta w,
\]
\[
u = b + \theta^T w, \quad 2v = b - \theta^T w - \frac{2M}{r(b + \theta^T w)},
\]
which describe the Schwarzschild spacetime in rigidly boosted coordinates.

Since in general the functions \( w(z, r) \in \mathbb{R}^3, R(z, r) \in SO(3) \) are arbitrary, subject only to smoothness and the size condition Eqn. (63), in order to construct challenging exact solutions for numerical relativity benchmarking, we might choose \( w, R \) in any reasonable
manner, keeping \( w = 0 \) and \( R = I \) in regions where we wish the solution to remain explicitly equal to the standard Schwarzschild metric. Note that asymptotic decay conditions may also be readily determined: for example the natural conditions
\[
w_r = O(r^{-2}) \quad w_z = O(r^{-1})
\]
and similarly for \( R \), give bounded \( u, v \) and \( \gamma \) with \( \beta \to 0 \) as \( r \to \infty \).

**B. Accelerated Minkowski metric**

By moving the base point of the standard future light cone along a timelike curve in Minkowski space, we may construct another class of shear-free NQS metrics. The Minkowski metric associated with an accelerated null cone foliation was discussed in [10], using a special choice of affine parameter on the null rays to determine the radius function. Let \( z \mapsto (p(z), \tau(z)) \in \mathbb{R}^{3,1} \) be a future-timelike curve in Minkowski space and denote the tangent vector by \((\dot{p}, \dot{\tau})\), with \((\dot{\cdot})\) indicating \( d/dz \). Two possible normalisations for \( z \) are \( \dot{\tau} = 1 \) and \( \dot{\tau} = \sqrt{1 + |\dot{p}|^2} \). Define \( \Psi : \mathbb{R}^{3,1} \to \mathbb{R}^{3,1} \) by
\[
\Psi(z, r, \theta) = \begin{bmatrix} X \\ T \end{bmatrix} = \begin{bmatrix} p(z) + r\theta \\ \tau(z) + r \end{bmatrix}
\]
(70)
where \((z = t - r, r, \theta)\) are null-polar coordinates and \((X, T)\) are rectangular coordinates on \( \mathbb{R}^{3,1} \). Note that \( \Psi \) maps the future null cone \( z = \text{const} \) to the future null cone based at \((x, t) = (p(z), \tau(z))\). The accelerated Minkowski metric \( ds_{AM}^2 := \Psi^*(-dT^2 + |dX|^2) \) may be written
\[
\begin{align*}
ds_{AM}^2 &= -(\dot{\tau} dz + dr)^2 + |\dot{p} dz + \theta dr + r d\theta|^2 \\
&= -2(\dot{\tau} - \theta^T \dot{p}) dz (dr + \frac{1}{2}(\dot{\tau} + \theta^T \dot{p}) dz) + |r d\theta + \Theta \dot{p} dz|^2,
\end{align*}
\]
(71)
which is a metric in the shear-free NQS form (69), with coefficient functions
\[
\begin{align*}
U &= U(z, \theta) = \dot{\tau}(z) - \theta^T \dot{p}(z) \\
V &= V(z, \theta) = \frac{1}{2}(\dot{\tau}(z) + \theta^T \dot{p}(z)) \\
\Gamma &= \Gamma(z, \theta) = \Theta \dot{p}(z).
\end{align*}
\]
(72-74)
Note that the timelike condition \( \dot{\tau} > |\dot{p}| \) ensures that \( U, V \) are both strictly positive.

Proposition 3 gives the NQS coefficients for the Lorentz-transformed metric \( \Phi^*(ds_{AM}^2) \) (with \( ds_{AM}^2 \) written in \((z, \rho, \zeta)\) as before), and we may simplify as follows. The shear \( \beta \) is given simply by Eqn. (61) and because \( \Theta A^T R^T \dot{\Gamma} = \Theta A^T R^T \dot{p} \), we find that
\[
\gamma = r(\Theta t_0 + \theta \times s_0) + \Theta(-\dot{\tau} w + W R^T \dot{p}).
\]
(75)
Since \( f\zeta = R(w + W \theta) = RA^{-1T} \theta \), we have
\[
\dot{\tilde{U}} = \dot{\tau} - \zeta^T \dot{p} = \dot{\tau} - f^{-1}(w + W \theta)^T R^T \dot{p}
\]
and
\[ \tilde{V} = \frac{1}{2}(\dot{\tau} + \zeta^T \dot{p}) = \frac{1}{2}(\dot{\tau} + f^{-1}(w + W\theta)^T R^T \dot{p}) \]

and \[ \tilde{\Gamma} = (I - \zeta \zeta^T) \dot{p}. \] This gives immediately that

\[ u = (1 + r\theta^T t_1) (f \dot{\tau} - (w + W\theta)^T R^T \dot{p}), \]

and after some computations,

\[ 2v = \frac{1}{1 + r\theta^T t_1} \left( [(b - \theta^T w) \dot{\tau} - (w - W\theta)^T R^T \dot{p} + 2r\theta^T t_0] \right). \]

Notice that the spherical harmonic decompositions of \( u, w \) contain terms with \( \ell = 0, 1, 2 \) whereas \( \beta, \gamma \) are both pure \( \ell = 1 \).

The coordinate system constructed in [10] corresponds to an NQS metric constructed from the choices \( R = I, w = \dot{p}(z) \), with proper time normalisation of \( (p(z), \tau(z)) \). The metric Eqn. (12) in [10] may be transformed to NQS form with NQS parameters \( \dot{\tau} = b, \ u = 1, \ \beta = 0, \ 2v = 1 + 2r\theta^T t_0, \ \gamma = r\Theta t_0 + r\theta \times s_0 \), where \( t_0 = W^{-1} \dot{p}, \ s_0 = (b + 1)^{-1} \dot{p} \times \dot{p} \).

Thus if the acceleration \( \ddot{p} \) is non-zero then both \( v, \gamma \) will be unbounded as \( r \to \infty \). The transformation between the NU and NQS coordinates amounts to redefining the angular variables (the null cones and quasi-spheres are unchanged), and has the effect of moving the conformal isometry \( P \) (cf Eqn. (13) of [10]) to the NQS field \( \gamma \).

Alternatively, the choice \( w = R^T \dot{p}, \ \dot{\tau} = b \), with \( \dot{R} = \frac{1}{b+1} (\dot{p} \dot{p}^T - \ddot{p}^T) R \) gives a Fermi-Walker transported spatial frame. In this case we have \( *S_0 = \frac{b}{b+1} R^T \dot{p} \times \dot{p}, \ s_0 = 0, \ t_0 = W^{-1} R^T \dot{p}, \) and \( u = 1, \ 2v = 1 + 2r\theta^T t_0, \ \beta = 0, \ \gamma = r\Theta t_0. \) I am indebted to Andrew Norton for this computation. Note that in this case, the parameters \( (\tau, p) \) and \( (w, R) \) may be recovered from the metric data \( t_0(z) \) by solving \( \dot{w} = bt_0, \ \dot{R} = (b + 1)^{-1} R (wt_0^T - t_0 w^T) \) and \( \dot{p} = Rw \), with initial conditions \( w(z_0) = 0, \ R(z_0) = I \) and \( p(z_0) = 0 \) corresponding to an initial frame at rest.

### C. Robinson-Trautman metrics

It was shown by Robinson and Trautman [11] that vacuum spacetimes which contain a null geodesic congruence which is hypersurface-orthogonal, expanding and shear-free have particularly simple structure. A coordinate transformation [12] brings such metrics to the NQS form

\[ ds^2_{KT} = -2U \ dz \ (dr + \frac{1}{2} (\Delta_0 U + U - 2MU^{-2}/r) \ dz) + |rd\theta - U^T_\theta dz|^2 \]

where \( M \in \mathbb{R} \) is constant, \( \Delta_0 = \Delta_{S^2} \) is the standard metric Laplacian on \( S^2 \) and \( U = U(z, \theta) \) is independent of \( r \). The vacuum Einstein equations are satisfied by \( ds^2_{KT} \) if \( U \) satisfies the nonlinear parabolic equation

\[ 12M \frac{\partial U}{\partial z} + U^3 \Delta_0 K = 0 \]

where \( M \neq 0 \) and \( K = U^2 (\Delta_0 \log U + 1) \) is the Gauss curvature of the metric \( U^{-2} ds^2_0 \) conformal to the standard metric \( ds^2_0 \) on \( S^2 \). If \( M = 0 \) then the metric reduces to the accelerated
Minkowski metrics considered above \[11\]. A global existence theorem for Eqn. (79) has been given by Chruściel \[13\].

The RT metric has NQS parameters \(U, V = \frac{1}{2}(\Delta_0 U + U + 2MU^{-2}/r), \beta = 0\) and \(\Gamma = -U^T_{\rho},\) and after changing Eqn. (79) over to \((z, \rho, \zeta)\) coordinates, Proposition 5 describes the effect of a general Lorentz deformation of the coordinates. In particular, since \(\tilde{\Gamma}(z, r, \theta) = -U^T_{\zeta}(z, \zeta(z, r, \theta))\), using the identity

\[
\tilde{\Gamma} = D_{\theta} U(z, \zeta) = f^{-1}U, \zeta RA \Theta
\]

and Eqn. (55), we may simplify terms involving \(\tilde{\Gamma}\):

\[
\Theta A^T R^T \tilde{\Gamma} = -f \tilde{U} T_{\theta},
\]

\[
f w^T R^T \tilde{\Gamma} = w^T \Lambda^{-1} R^T \tilde{\Gamma} = -f w^T \tilde{U} T_{\theta}.
\]

Thus in the RT case Eqs. (80),(81) become

\[
\begin{align*}
uv &= \tilde{U}(V + rf \theta^T \tau_0 + \frac{1}{2}\tilde{U}|\Theta w|^2 - f w^T \tilde{U} T_{\theta}), \\
\gamma &= r \Theta \tau_0 + r \theta \times s_0 - (f \tilde{U})^T T_{\theta},
\end{align*}
\]

and \(u = f \tilde{U}(1 + r \theta^T \tau_1), \beta = r \Theta \tau_1 + r \theta \times s_1\).

Note that if the Lorentz deformation preserves \(\beta = 0\) then \(w = w(z), R = R(z)\) and \(u = f \tilde{U}, \) and \(\gamma\) will remain independent of \(r\) only if \(s_0 = t_0 = 0.\) This requires that \(w, R\) are constant, and the transformed metric will again be in the explicit RT form of Eqn. (78). This global Lorentz transformation may be used to normalise to zero the \(\ell = 1\) spherical harmonic components of \(\lim_{z \to \infty} u(z, \theta)\) (or equivalently, of \(\lim_{z \to \infty} \gamma(z, \theta)\)) — this transformation may be interpreted as defining an asymptotic rest frame for the RT spacetime \[14\], \[13\].

V. DISCUSSION

In the case of vanishing shear \(\sigma_{NP} = 0\) (and assuming non-zero expansion \(\rho_{NP} \neq 0\) and spherical sections), we have seen that the null hypersurfaces are isometric to the standard cone \(C_0\) (Proposition 8), and the residual freedom in the NQS gauge consists precisely of a Lorentz transformation at each quasi-sphere. The transformed metric has NQS shear \(\beta\) consisting purely of \(\ell = 1\) spherical harmonics, and conversely, if \(\beta\) is pure \(\ell = 1\) then there is an inverse quasi-spherical map which transforms the metric into NQS form with \(\beta = 0.\) Thus the \(\ell = 1\) spherical harmonic components of the NQS shear \(\beta\) are pure gauge.

Generalised Lorentz transformations preserving the condition \(\beta = 0\) have parameters \((w, R)\) depending only on \(z,\) since Eqn. (61) combined with Eqs. (57),(58) show that \(w, R^r\) must vanish. This remaining gauge freedom may be used to set the \(\ell = 1\) components of \(\gamma\) to zero at one fixed radius \(r_0\) as follows. The six \(\ell = 1\) coefficients of the terms \(\Theta(A^T R^T \tilde{\Gamma} - \tilde{U} w)\) of \(\gamma\) (cf. Eqn. (62)) form a nonlinear functional of \(w, R,\) so Lemma 2 may be used to solve for \(w, z, R^r\) from \(s_0, t_0,\) giving a system of ordinary differential equations

\[
\frac{d}{dz}(w(z, r_0), R(z, r_0)) = F(z, r_0; w, R),
\]

16
where $F(z, r_0; w, R)$ is linearly bounded in $w$. Consequently there exists a solution which is global in $z$, which in turn ensures (after applying the resulting Lorentz transformation) that $\gamma(z, r_0)$ has vanishing $\ell = 1$ components at each $z$.

Alternatively, it might be possible to use the gauge freedom $w(z), R(z)$ to normalise the $\ell = 1$ components of $u(z, r_0)$ using Eqn. (59), since $f = b + \theta^T w$ is pure $\ell = 0, 1$ and $t_1 = 0$ by the condition $\beta = 0$. Note that this remaining freedom is similar to that available in the Bondi and Newman-Unti gauges. In any case, it is a plausible conjecture that the gauge freedom remaining in the general NQS metric (1) may be used to eliminate the $\ell = 1$ components of $\beta$, and that the freedom remains to make a rigid Lorentz transformation on each null hypersurface.

The interpretation of $\ell = 1$ spherical harmonic components as gauge terms has also been noted in the construction of the Regge-Wheeler-Zerilli equations for linearised perturbations of Schwarzschild [16], [17], [18]. The gauge-invariant quantities satisfying the RWZ equations are constructed from $\ell \geq 2$ components of the metric perturbations. Furthermore, one quantity constructed from the $\ell = 1$ components represents (non-dynamic) angular momentum [18], [2] arising from the Kerr perturbation of the Schwarzschild metric — this quantity corresponds to the linearised limit of the odd (rotational) $\ell = 1$ component of $\partial_z (\beta/r) - \partial_r (\gamma/r)$, and vanishes for the pure gauge variations constructed above.

ACKNOWLEDGMENTS

The support of the Institute for Mathematical Sciences at the Chinese University of Hong Kong during the preparation of this paper is gratefully acknowledged. I also wish to thank Andrew Norton for his careful review of this manuscript, which has helped to clarify many of the computations.

APPENDIX A: SHEAR-FREE SPACETIMES

Let $\mathcal{N}$ be a null hypersurface in some spacetime, with induced (degenerate) metric $ds^2_N = g_N$. An adapted null frame on $\mathcal{N}$ is a pair of vector fields $(l, m)$ where $l$ is a degeneracy vector for $ds^2_N$, $m \in T\mathcal{N} \otimes \mathbb{C}$, and $(l, m + \bar{m}, i (m - \bar{m}))$ form a real basis for $T\mathcal{N}$, such that $g_N(m, m) = g_N(l, l) = g_N(l, m) = 0, \quad g_N(m, \bar{m}) = 1$.

Using $\nabla$ to denote the ambient spacetime covariant derivative, we define the shear and expansion of $ds^2_N$ (with respect to the null adapted frame $(l, m)$) by

\begin{align}
\text{shear} &= \sigma_{NP} = -g(\nabla_m m, l), \\
\text{expansion} &= \rho_{NP} = -g(\nabla_m \bar{m}, l).
\end{align}

(A1) (A2)

Although we use the Newman-Penrose notation, the importance of the shear and expansion of a null geodesic congruence was known prior to [7] — see [13] for example.

**Lemma 6** Let $(l, m)$ be an adapted null frame for $\mathcal{N}$, then the shear and expansion depend only on $(l, m)$ and $ds^2_N$. In particular we have
\[ \sigma_{NP} = -g_N(m, [l, m]) \]  
\[ \rho_{NP} = -\frac{1}{2}(g_N(m, [l, \bar{m}]) + g_N(\bar{m}, [l, m])), \]  
where \([l, m]\) is the Lie bracket.

**Proof:** The identities (A3), (A4) are easily verified since \([l, m]\), being the Lie bracket of vector fields tangent to the hypersurface \(N\), is again tangent to \(N\). QED.

**Lemma 7** Suppose \((l', m')\) is a null adapted frame which presents the same orientation of \(N\) and the null generators as \((l, m)\). There are real functions \(\alpha, \lambda, \mu \in C^\infty(N)\) such that

\[ m' = e^{i\lambda}m + \alpha l, \quad l' = e^\mu l, \]  
and the shear and expansion satisfy

\[ \sigma'_{NP} = e^{\mu + 2i\lambda}\sigma_{NP} \]  
\[ \rho'_{NP} = e^{\mu}\rho_{NP}. \]  
Consequently the conditions “\(\rho_{NP} \neq 0, \sigma_{NP} = 0\) everywhere on \(N\)” are independent of the choice of adapted null frame, and we may consider \(\sigma_{NP}/\rho_{NP}\) as a section of a spin-2 complex line bundle over \(N\).

**Proof:** The representation (A3) for the frame change follows directly from the orientation and orthogonality conditions. The formula

\[ [m', l'] = e^{\mu + i\lambda}([m, l] + D_m \mu l - i D_l \lambda m) + e^\mu(\alpha D_l \mu - D_l \alpha) l \]

where \(D_l, D_m\) are the directional derivative operators, leads directly to Eqs. (A6), (A7). QED.

Using the foliation of \(N\) by null generating curves, we may introduce adapted coordinates \((\rho, x^3, x^4)\) by requiring \((x^3, x^4)\) to be constant along the null generators, and then allowing \(\rho\) to be any parameterisation of the null generators. In such coordinates the metric becomes

\[ ds^2_N = h_{ab}dx^a dx^b; \]

where the indices \(a, b\) have range 3, 4 and \(h_{ab} = h_{ab}(\rho, x^3, x^4)\). A natural choice of null frame is \(l = \partial_\rho\) and \(m = m^a \partial_a\), where \(\partial_\rho, \partial_3, \partial_4\) are the coordinate tangent vectors. Introducing the cotangent vector \(m_a dx^a\), where the \(m_a, a = 3, 4\) are defined by the requirements \(m^a m_a = 0, \bar{m}^a m_a = 1\), we have

\[ h_{ab} = m_a \bar{m}_b + \bar{m}_a m_b. \]

Direct computation using \([l, m] = \partial_\rho (m^a) \partial_a\) gives the following expressions for the shear and expansion with respect to the coordinate-based null framing \((l, m)\):

\[ \sigma_{NP} = m^a \partial_\rho(m_a) \]
\[ = \frac{1}{2} m^a m^b \partial_{\rho} h_{ab} \]
\[ \rho_{NP} = \frac{1}{2} (\bar{m}^a \partial_\rho(m_a) + m^a \partial_\rho(\bar{m}_a)) \]
\[ = \frac{1}{4} h^{ab} \partial_{\rho} h_{ab}. \]
where \([h^{ab}] = [h_{ab}]^{-1} = m^a\tilde{m}^b + \tilde{m}^a m^b\).

If \(\mathcal{N}\) is shear-free and expanding then the metric on \(\mathcal{N}\) may be brought into explicitly NQS form. It should be possible to extend this result to allow some non-zero shear, but the proof will be considerably more difficult.

**Proposition 8** Suppose \(\mathcal{N}\) is a null 3-manifold with everywhere vanishing shear and non-zero expansion, and having spatial cross-sections which are topological spheres. Then there exist polar coordinates \((r, \vartheta, \varphi)\) on \(\mathcal{N} \simeq \mathbb{R} \times S^2\) such that \(ds^2_{\mathcal{N}} = r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)\).

Note that the following argument may be easily adapted in case the spatial sections are not spheres.

**Proof:** Let \(l = \partial_{\rho}\), \(m = m^a\partial_a\) be a coordinate-based null frame for \(\mathcal{N}\). By Eqn. (A8) and the shear-free condition we have

\[
m^a m^b \partial_{\rho} h_{ab} = 0.
\]

Now \(\partial_{\rho} h_{ab}\) may be decomposed

\[
\partial_{\rho} h_{ab} = A(m_a \tilde{m}_b + \tilde{m}_a m_b) + Bm_a m_b + B\tilde{m}_a \tilde{m}_b,
\]

where \(A\) is real-valued and \(B\) is complex-valued. The shear-free condition shows that \(B = 0\) and the resulting equation \(\partial_{\rho} h_{ab} = Ah_{ab}\) may be integrated along each null generator to give

\[
h_{ab}(\rho, x) = \exp \left( \int_{\rho_0}^{\rho} A(s, x) ds \right) h_{ab}^0(x),
\]

where \(h_{ab}^0(x) = h_{ab}(\rho_0(x), x)\) is a fixed metric on \(S^2\). Now the Riemann Uniformisation Theorem [19] shows there is a diffeomorphism \(\Phi : S^2 \to S^2, x \mapsto (\vartheta(x), \varphi(x))\), and a function \(\phi \in C^\infty(S^2, \mathbb{R}^+)\) such that \(h_{ab}^0 dx^a dx^b = \phi^2(x)\Phi^*(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)\). Using the coordinates \((\vartheta, \varphi)\) to label the null generators gives the representation

\[
ds^2_{\mathcal{N}} = \exp \left( \int_{\rho_0}^{\rho} A \right) \phi^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2).
\]

Now define the positive function \(r = r(\rho, \vartheta, \varphi)\) by \(r = \exp \left( \frac{1}{2} \int_{\rho_0}^{\rho} A \right) \phi\). Since \(4\rho_{NP} = h_{ab} \partial_{\rho} h_{ab} = 2A\), it follows that

\[
\frac{\partial r}{\partial \rho} = r\rho_{NP} > 0,
\]

hence \(r\) is a valid coordinate, and \(ds^2_{\mathcal{N}}\) takes the required form in the coordinates \((r, \vartheta, \varphi)\).

QED.
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