Coded Caching at the Edge of Satellite Networks

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Abstract—Caching multimedia contents at the network edge is a key solution to decongest the amount of traffic in the backhaul link. In this paper, we extend and analyze the coded caching technique [1] in an unexplored scenario, i.e. at the edge of two-tier heterogeneous networks with an arbitrary number of users. We characterize the performance of such scheme by deriving a closed-form expression of the average backhaul load and reveal a significant gain compared to other benchmark caching schemes proposed in the literature.

I. INTRODUCTION

Caching has emerged as one of the key technologies for next-generation wireless systems. Bringing the desired content to the edge of the network, i.e. memorizing copies of relevant information close to users, has been shown not only to alleviate the backhaul traffic, but also to significantly reduce latency and power consumption [2].

To achieve this goal, a two-step caching strategy is typically implemented, pre-fetching the content at the edge (e.g. at base stations, LEO satellites, relays or helpers) during network off-peak periods (placement phase), so as to serve the users without consuming backhaul capacity when the network is congested (delivery phase).

Several works have recently investigated the potential benefits of caching schemes at the edge. Interesting results were obtained in [3], where authors compared two different placement approaches, i.e. encoded and uncoded content placement to reduce the download delay. Caching scheme based on maximum distance separable (MDS) codes have been studied in wireless networks to minimize the expected download time [4] or to reduce the amount of data to be sent [5].

Furthermore, Maddah-Ali et al. introduced in their pioneering work [1] the concept of coded caching, considering local caching directly on the user’s device. The idea is to deliver coded content and leverage the user’s local content to serve multiple users with a single transmission. This technique has spurred an extensive body of research providing a solid understanding of the potential and limitations of caching, e.g. [6], [7]. However, coded caching has been studied especially in setups where few cache-aided users are connected to a common server via a shared link. Instead, its potential and the trade-offs it may induce in other relevant scenarios remain unexplored to date. An example of notable practical relevance is given by two-tier networks that foresee a satellite component, which will be an integrating part of 5G and 6G systems [8]. In these settings, commonly referred to as non-terrestrial networks, terminals may not be equipped with direct satellite connectivity, and the intermediate tier is responsible for forwarding content from one end to the other.

To shed light on these relevant design aspects, the present work considers the application of coded caching in a two-tier caching satellite network for an arbitrary number of users. Unlike previous work, caching is considered at the edge of the network (e.g. relays, helpers, LEO satellites) and multiple users are connected to one or more cache-aided relays. To analyze the system performance, we derive a closed-form expression of the average backhaul transmission load when coded caching is in place. We compare our results with the benchmark given by the MDS caching scheme proposed in literature and we show a reduction in backhaul transmissions. In particular, the presented scheme triggers a gain each time the mutual difference of sets representing the file requested at each relay is not empty. To quantify such gain, the distribution of how users request for content is casted onto a combinatorial balls into bins (BiB) setting. Due to complexity of the problem, closed form expressions are derived when the distribution of files requested is uniform, while we show via Monte-Carlo that the aforementioned gain is also present when files are not equiprobable. The significant enhancement obtained encourages to investigate further relevant aspect of the edge coded caching in satellite networks.

Notation: We use capital letters, e.g. X, for discrete random variables (rvs) and their lower case counterparts, e.g. x, for their realizations. The probability mass function (pmf) of the rv X is denoted as \( p_X(x) \) and conditional pmfs as \( \Pr(X = x \mid Y = y) = p_{X|Y}(x|y) \). A set is denoted with calligraphic letters, e.g. \( S \). The cardinality of set \( S \) is indicated as \( |S| \).

II. SYSTEM MODEL

A two-tier heterogeneous network composed by a master node, two cache-enabled nodes and a number of end users are considered. While this setup applies to different network configurations, we will take as reference throughout our discussion the satellite topology illustrated in Fig. 1. Here, a satellite (S) stores a whole library \( \mathcal{F} = \{f_1, \ldots, f_N\} \) of equal-size files. On the ground, two cache-enabled relays (R_B and R_W)\(^1\) are connected via a backhaul link to S, and each one provides connectivity to some users. As typical in current satellite-aided terrestrial networks, we assume that no direct link between users and S is available\(^2\). Depending on their locations users (terminals) may be connected to one or both relays. \( U_h \) denotes the subset of users that are connected to \( h \).

\(^1\)The subscript B and W have been chosen to facilitate the similarity between the caching scheme and BiB problem, as will become clear later.

\(^2\)Note that the setup presented is not limited to this architecture. For instance, a possible scenario may consist of a GEO satellite which acts as master node connected via backhaul links to cache-enabled LEO satellites.
relays, where $h = \{1, 2\}$, whereas $U_B$ ($U_W$) denotes the subset of users connected only to relay $R_B$ ($R_W$). Note that the set of users connected to exactly one relay is $U_1 = U_B \cup U_W$.

Each relay can store up to $M \leq N$ files locally. With $Z_B$ ($Z_W$) we indicate the files present in the cache of $R_B$ ($R_W$). During the placement phase, which is carried out offline, each file $f_j \in F$ is partitioned into $n_f$ equally long fragments, i.e. $f_j$ is fragmented as $\{f_{j}^{(1)}, \ldots, f_{j}^{(n_f)}\}$. Each cache stores $F_j$ fragments related to file $f_j$ according to one of the caching schemes that will be introduced later in this section. During the delivery phase, the network serves the users’ requests. Each user picks a file according to the file distribution considered (e.g. uniform or Zipf distribution). A user connected to $h$ relays which request for $f_j$ receives $h F_j$ fragments directly from the cached content and the $\max(0, n_f - h F_j)$ missing fragments are forwarded by the R after being retrieved by $S$ via the backhaul link. The transmission technique in the backhaul link depends on the caching scheme considered.

In such configuration, we indicate with $u$ the total number of terminals that concurrently request content from the library, each independently choosing a file to download. Specifically, we have that $u = u_1 + u_2$ where $|U_1| = u_1$, are those connected to a single relay, while $|U_2| = u_2$ are those connected to both relays. We also have that $u_1 = u_B + u_W$ where $u_B$ are the users requesting content only at $R_B$ and $u_W$ only at $R_W$. A relay directly delivers content present in its cache and retrieves content that is not available locally via the backhaul link. For simplicity we assume that all transmissions are error-free.

Following this notation, we analyze a coded caching scheme at the edge based on the strategy proposed in [1], referred to as the edge coded caching scheme (ECC). To evaluate the scheme, we derive the average backhaul transmission load, i.e. the average number of packets that $S$ should transmit in the backhaul link to satisfy user requests. We will focus on the rv $L^x$ which describes the number of transmissions required from $S$ for a given caching scheme $x$. We have that $L^x = E[L^x]$ where the operator $E$ indicates the expected value. The metric $L$ is used to compare the behavior against the benchmark given by MDS scheme. Let us discuss both schemes in the following.

**MDS Caching Scheme**

In the MDS scheme [5], the network works by caching and delivering packets that are encoded. In particular, $n_f$ fragments of file $f_j$ are used to create $n > n_f$ encoded packets using a $(n, n_f)$ MDS code. The set of encoded packets related to $f_j$ can be written as $e_j = \{e_{j}^{(1)}, \ldots, e_{j}^{(n)}\}$ where $e_{j}^{(i)}$ and $f_{j}^{(i)}$ are equally long for every $i$ and $j$. With the MDS coding technique, a user can reconstruct successfully the requested file by receiving any subset of $n_f$ encoded packets [5].

If we assume a uniform distribution of the file requested then it is also assumed that files are split into $n_f = N$ fragments. Each relay fills own cache with $F_j = M$ encoded packets per file such that $Z_B \cap Z_W = \emptyset$, i.e. relays store a different subset of encoded packets for the same file. The satellite keeps $n - 2M$ encoded packets for every file. The delivery phase is split into the following stages. First, users receive content from the relays’ cache, subsequently the missing encoded packets are sent by $S$ through the backhaul link to the $R$ which forwards them to the users. The benefit of this strategy is based on being able to serve both relays in parallel with a single transmission via the backhaul link. This occurs whenever there are requests for the same content at both relays.

To clarify, consider the following numerical example.

**Example 1.** Let us assume to have two users: user 1 is connected only to $R_B$ and user 2 only to $R_W$. Consider a memory size of $M = 1$ and two equiprobable files split into $n_f = 2$ fragments

$$f_1 = \{f_1^{(1)}, f_1^{(2)}\} \text{ and } f_2 = \{f_2^{(1)}, f_2^{(2)}\}.$$ 

Let us consider a $(3, 2)$ MDS code such that we can write the encoded packets as

$$e_1 = \{e_1^{(1)}, e_1^{(2)}, e_1^{(3)}\} \text{ and } e_2 = \{e_2^{(1)}, e_2^{(2)}, e_2^{(3)}\}.$$ 

We further set $Z_B = \{e_1^{(1)}, e_1^{(2)}\}$ and $Z_W = \{e_2^{(1)}, e_2^{(2)}\}$.

To characterize the average backhaul load $L_{MDS}$, we shall consider two cases. First, we suppose that users are requesting for different content, i.e. user 1 (user 2) requests for $f_1$ ($f_2$) to relay $R_B$ ($R_W$). Since each $R$ has one encoded packet of the requested file in cache, $S$ should send one encoded packet to each $R$. Hence, the number of required backhaul transmissions, i.e. the realization of $l$ of the rv $L_{MDS}$ takes value

$$l_1 = 2.$$ 

Each user is able to reconstruct the file by receiving one encoded packet directly from the cache and the other forwarded by the relay. If, instead, both users request for the same content, $S$ can only transmit the encoded packet $e_1^{(3)}$ to both and they will successfully decode the requested content.

In this case, the number of packets to transmit is

$$l_2 = 1.$$ 

Combining the two cases, $L$ in MDS evaluates to

$$L_{MDS} = \sum_i p_L(l_i) l_i = \frac{1}{2} l_1 + \frac{1}{2} l_2 = 1 + \frac{1}{2} = \frac{3}{2}$$ 

where we sum over the $i$ possibilities on how users can request for the library content. They ask for different files with
probability \( p(l_1) = 1/2 \) while they ask for the same file with probability \( p(l_2) = 1/2 \).

**Edge Coded Caching Scheme**

In the ECC scheme, caches are filled with non-encoded fragments, while the encoding takes place in the delivery phase. \( S \) creates coded delivery opportunities so that with a unique transmission both relays are able to recover the desired information also when different content is requested.

In the placement phase, each \( R \) fills its cache with \( F_r \) exclusive fragments of file \( f_j \) so that relays have different fragments of the same file. \( S \) is aware of which content has been stored in each \( R \). In the second phase, users make their requests to the corresponding \( R \). The delivery can be split into three stages. In the first stage, a user receives fragments directly from the cached content of the associated \( R \). In the second stage, \( S \) is informed of users requests and provides missing content over the shared backhaul link by creating coded multicast opportunities transmissions when is possible. In this stage the relays decode the transmission and forward the desired packets to users. In the third and last stage, \( S \) sends the remaining content in a non-encoded transmission, and relays forward this to users.

A coded multicast opportunity allows both relays to retrieve file fragments with a single transmission. In particular, \( S \) creates a coded packet by XORing two fragments (i.e., a bitwise operation). \( S \) picks a fragment of a file requested at \( R \) and present in cache of \( R \) and, similarly, requests made from users connected only to \( R \), while balls represent requests from users connected to both relays.

With the presented examples, we observe that there exists a gain in the ECC scheme whenever there are requests for different files at relays. To understand the potential of this gain, we derive \( L \) in both schemes in a more general setting. To this aim, we start by recalling some useful results of the BiB problem, which will be later applied to our derivations.

III. BALLS INTO BINS PROBLEM APPLIED TO CACHING

To instantiate such calculations, it is convenient to map our setting onto a balls into bins setup. The general BiB problem, see e.g. [9], consists in independently throwing \( d \) balls into \( N \) bins. As illustrated in Fig. 2, this can be cast to our caching problem by having each bin associated to a file of the library, and by having balls which represent user requests. Following this parallel, the possibility for more balls to land into the same bin corresponds to have multiple users asking for a common library element. A first useful result is given by the probability of having exactly \( j \) bins out of \( N \) non empty given that \( d \) balls are thrown uniformly at random, which was derived in [10] and can be written as

\[
p(j|d) = \frac{N!}{j!(N-j)!} \left( \frac{d}{N} \right)^j \left( \frac{N-d}{N} \right)^{N-j}
\]

where \( S(d,j) = j^{-1} \sum_{i=0}^{d} (-1)^i \binom{d}{i} (j-i)^d \).

In our setup, we further need to differentiate requests made to \( R_B \) from those made to \( R_W \) and, similarly, requests made by users connected to one relay (set \( U_1 \)), from those made by users connected to two relays (set \( U_2 \)). To this aim we
distinguish requests at different relays, as illustrated in Fig.
3, by considering balls of two different colors, e.g. black and
white balls. A bin containing black and white balls indicates
that the same file is required at both relays. Having the $i$-th
bin with only black (white) balls implies that the $i$-th file was
requested only at relay $R_B$ ($R_W$).

Following this approach, a useful result is offered by the
multivariate occupancy problem assuming that there are $N$
bins and that $u_B$ black balls have been thrown and have occupied
$j$ different bins. The probability that, after throwing $u_W$ white
balls, there are exactly $k_B$ bins containing only black balls and
$k_W$ bins containing only white balls is [9]

$$
\Pr_{oc}(j, k_B, k_W, u_W) = \binom{j}{k_B} \binom{N-j}{k_W} \frac{\Delta^{bw} 0^{uw}}{N^{uw}}
$$

where $\Delta^{bw}$ is the number of bins containing the $u_W$ white balls,
i.e. $\Delta^{bw} = j - k_B + k_W$ and the quantity

$$
\Delta^{m} 0^{n} := \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m-i)^n
$$
is known as difference of zeros [9]. Note that, in our setting,
$\Pr_{oc}(j, k_B, k_W, u_W)$ provides the probability that exactly $k_B$
files are requested only to relay $R_B$ and $k_W$ files are requested
only to relay $R_W$ when in total there are $u_1 = u_B + u_W$ users
connected to exactly one relay.

### IV. AVERAGE BACKHAUL TRANSMISSION LOAD

Leaning on the parallel with the BiB problem, we now
derive the mean number of packets/fragments that $S$ needs
to send via the backhaul link to satisfy $u$ requests.

For convenience, we list in Table I the rvs needed for our
derivations together with their definition and alphabet. The
first column indicates the notation of the rv, the second its
definition and the last its alphabet. For instance, the rv $J$
denotes the number of different files requested by $u_1$ users
connected to only one relay ($U_1$) while $K_B$ denotes the number
of different files requested exclusively at $R_B$. Instead, the
notation $D_B \backslash D_W$ indicates the set difference and that is
the set of file requested at $R_B$ but not requested at $R_W$. Let us
clarify all the mentioned quantities with an example.

**Example 3.** Let us refer to Fig. 3 which illustrates a library
of $N = 10$ files (bins) and $u = 17$ users (balls). There are
$u_B = 6$ users connected only to $R_B$ (black balls), $u_W = 4$ only
to $R_W$ (white balls), while $u_2 = 7$ are connected to both relays.

We have that $u_B = 6$ users requested in total $y = 4$ different
files (represented by the four bins with black balls). Users in
$U_2$ asked in total for three files and are represented by bins
3, 4 and 5. The files requested by the $u_1$ users connected to
only one relay are in total $j = 5$, i.e. the number of bins with
black or with balls. Out of those, the files requested exclusively
at $R_B$ are $k_B = 2$, i.e. the number of bins with black balls
and without white balls, while those exclusively requested at
$R_W$ are $k_W = 1$, i.e. the number of bins with white balls
and without black balls. Since the minimum number of mono-
colour bins is one, then $z = 1$, i.e. $z = \min\{k_B, k_W\}$.

Users connected to both relays requested in total for 4
further files, represented by bins 1, 2, 6 and 7. Then the files
requested only at one $R$ are bins 3, 4 and 6 so in total $k_1 = 3$,
while files requested only by users connected to both relays
are bins 6 and 7 such that $k_2 = 2$.

In the next derivations we assume that files are equiprobable,
each file is split into $n_F = N$ fragments and the number of files
stored at each relay is $F_j = M$ for all $j$.

#### A. MDS Average Transmission Load

Let us recall that $J$ different files are requested by $u_1$ users
in $U_1$. By the working principle of the MDS scheme, for each
file requested, $S$ has to send in the backhaul $n_F - M$ packets,
whereas the remaining $M$ are already provided to the user
via the relay’s cache. The overall number of packets that $S$
transmits to satisfy $u_1$ requests is then expressed by the rv

$$
L_1^{MDS} = (n_F - M) J.
$$

To complete the analysis, we derive the number of packets
needed to satisfy users connected to both relays (users in $U_2$).
Note that in this calculation it is needed to take into account
only the $K_2$ aggregated requests, i.e. the new files requested
by $U_2$ users but not requested by $U_1$. In fact, whenever a file
requested by users in $U_2$ coincides with a user request from
$U_1$, both requests are satisfied with the same backhaul transmission
already accounted for by $L_1^{MDS}$. Observing that each user in
$U_2$ receives in total $2M$ different fragments of the respective
file from relays, the number of packets that $S$ has to send for
each aggregated file is $(n_F - 2M)^+$ where $(x)^+ := \max(0, x)$.
Note that whenever $M \geq N/2$, no transmission in needed.

Combining these remarks, the transmissions that $S$ has to
perform to satisfy the aggregated requests can be expressed as

$$
L_2^{MDS} = (n_F - 2M)^+ K_2.
$$

So that the average backhaul load $L$ in the MDS is

$$
L^{MDS} = E[L_1^{MDS}] + E[L_2^{MDS}].
$$

Let us now calculate the two addends of equation (5).

The average transmission load for users in $U_1$ can be
computed by simply averaging over $J$ to obtain

**Table I**

| rv   | Definition                     | Alphabet            |
|------|--------------------------------|---------------------|
| $J$  | $|D_1|$                         | $\{1, ..., \beta_J = \min(u_1, N)\}$ |
| $Y$  | $|D_B|$                         | $\{1, ..., \beta_Y = \min(u_B, N)\}$ |
| $K_B$| $|D_B \backslash D_W|$         | $\{\alpha_B = \max(0, y - u_W + k_W), \ldots, \beta_B = \min(u_B, N)\}$ |
| $K_W$| $|D_W \backslash D_B|$         | $\{0, \ldots, \beta_W = \min(u_W, N - \beta_B)\}$ |
| $K_1$| $|D_1 \backslash D_2|$         | $\{\alpha_1 = \max(0, j - u_2), \ldots, \beta_1 = \min(u_1, N)\}$ |
| $K_2$| $|D_2 \backslash D_1|$         | $\{0, \ldots, \beta_2 = \min(u_2, N - \beta_1)\}$ |
| $Z$  | $\min\{K_B, K_W\}$            | $\{0, \ldots, \beta\}$ |
\[ L^{\text{MDS}} = \sum_{j=1}^{\beta_1} \binom{N}{j} S(u_1, j) j! \left[ j \left( 1 - \frac{M}{N} \right) + (1 - 2 \frac{M}{N}) + \sum_{k_1=\alpha_2}^{j} \binom{j}{k_1} \binom{N-j}{k_2} \sum_{i=0}^{N-b_2} (-1)^i \binom{N-b_2}{i} \left( \frac{N-b_2-i}{N} \right)^{u_2} \right]. \]  

(10)

\[ L_1^{\text{MDS}} = \mathbb{E} \left[ (n_F - M) J \right] \]
\[ = (n_F - M) \sum_{j=1}^{\beta_2} p_J(j|u_1) \cdot j \]
\[ = (N - M) \sum_{j=1}^{\beta_2} \binom{N}{j} S(u_1, j) j! \cdot j \]

(6)

where the quantity \( p_J(j|u_1) \) was derived with BiB occupancy problem, see (1).

The average transmission load given by the aggregated files requested by \( \mathcal{U}_2 \) can be computed by conditioning to \( J \), i.e.

\[ L_2^{\text{MDS}} = \mathbb{E} J \left[ \mathbb{E} \left[ (n_F - M)^+ K_2 | J \right] \right]. \]

(7)

Let us first focus on the inner expectation, and derive the conditional pmf \( p_{K_2}(k_2|j) \), i.e. the probability that users connected to both relays request for exactly \( k_2 \) new files given that \( j \) different files have been requested by users connected to one relay. To help the reader, we refer to Fig. 3 the sought probability can be computed in the BiB setup as the probability of having \( j + k_2 \) non empty bins after throwing \( u_2 \) (grey) balls, conditioned on having already \( j \) non empty bins occupied by \( u_1 \) balls. As discussed, this results is offered by the multivariate occupancy problem, and we have

\[ p_{K_2}(k_2|j) = \sum_{k_1=\alpha_1}^{j} \mathbb{P}_{\text{oc}}(j, k_1, k_2, u_2), \]

(8)

where the correspondent number of file requested at both relays is \( b_j = j - k_1 + k_2 \). In (8) we are summing up all the possible values that \( k_1 \) can assume (i.e. files exclusively requested at \( \mathcal{R}_B \) represented by bins with only black balls). Accordingly,

\[ L_2^{\text{MDS}} = \mathbb{E} J \left[ (n_F - M)^+ \sum_{k_2=0}^{\beta_2} k_2 p_{K_2}(k_2|j) \right] \]
\[ = (N - M)^+ \sum_{j=0}^{\beta_1} p_J(j|u_1) \sum_{k_2=0}^{\beta_2} k_2 \sum_{k_1=\alpha_1}^{j} \mathbb{P}_{\text{oc}}(j, k_1, k_2, u_2). \]

(9)

By inserting (6) and (9) onto (5) we obtain \( L^{\text{MDS}} \) and normalizing by the number of fragments \( N \), we have that the normalized average transmission load, \( \bar{L}^{\text{MDS}} = L^{\text{MDS}} / N \), in the MDS scheme is given in (10) at the top of the page.

B. ECC Average Transmission Load

Let us start by considering users in \( \mathcal{U}_1 \). Since \( M \) fragments of the requested files are obtained from the relay’s cache then each user needs \( n_F - M \) additional fragments. Let us calculate the number of packets that \( S \) should transmit to satisfy these requests by considering the coded caching opportunities. Denoting by \( Y \) the rv counting the number of different files requested by the \( u_B \) users connected only to \( \mathcal{R}_B \) and by \( K_W \) the rv counting the number of files exclusively requested by the \( u_W \) connected only to \( \mathcal{R}_W \) and not requested to \( \mathcal{R}_B \), in total users have to receive \( (n_F - M)(Y + K_W) \) fragments in order that all their requests are satisfied. However, note that transmissions given by the coded opportunities should not be counted. As for Example 2, a coded transmission opportunity takes places each time that a file is requested at one relay and not in the other and vice-versa. S by XORing the corresponding content present at each cache can make a transmission useful to both relays. Each coded transmission opportunity allows the \( \omega_1 \) XORed packets involving the two files. In particular,

\[ \omega_1 = \min(M, N - M), \]

where \( \omega_1 \) is the number of fragments per file combined in a coding opportunity and it depends on the cache size. For each transmission opportunity, when \( M \leq N/2 \); then in total \( M \) fragments per file are XORed, whereas if \( M > N/2 \) then requests are satisfied by combining \( N - M \) fragments per file.

In summary, each coded transmission opportunity allows \( S \) to combine \( \omega_1 \) packets where a packet is formed by two encoded fragments. Accordingly, the overall number of transmission needed in the backhaul to serve users in \( \mathcal{U}_1 \) is

\[ L_1^{\text{ECC}} = (n_F - M) (Y + K_W) - \omega_1 Z \]

where \( Z \) is the rv denoting the number of coded opportunities.

Let us now consider the users connected to both relays, i.e. the set \( \mathcal{U}_2 \). In this case, we simply observe that no gain opportunity emerges from the aggregated requests by such users. In fact, users already receive content from both caches. Therefore, the value of the backhaul transmissions is the same as computed for the MDS scheme and we get

\[ L_2^{\text{ECC}} = L_2^{\text{MDS}}. \]

The average backhaul transmission load of the ECC is

\[ L^{\text{ECC}} = \mathbb{E} \left[ L_1^{\text{ECC}} \right] + \mathbb{E} \left[ L_2^{\text{ECC}} \right]. \]

(11)

where we need to derive only \( \mathbb{E} [L_1^{\text{ECC}}] \). Conditioning on \( Y \), we have

\[ L_1^{\text{ECC}} = \mathbb{E}_Y \left[ \mathbb{E} \left[ (n_F - M)(Y + K_W) - \omega_1 Z | Y \right] \right] \]
\[ = \mathbb{E}_Y \left[ \mathbb{E} \left[ (n_F - M)(Y + K_W) | Y \right] \right] - \mathbb{E}_Y \left[ \mathbb{E} \left[ \omega_1 Z | Y \right] \right]. \]

(12)

Let us first focus on the conditional distribution of \( K_W \). Given \( Y = y \) different files requested from users in \( \mathcal{U}_B \), the probability \( p_{K_W}(k_W|y) \) of having exactly \( k_W \) files requested only at \( \mathcal{R}_W \) can be derived from the BiB multivariate occupancy problem by considering all values that \( k_B \) can assume as

\[ p_{K_W}(k_W|y) = \sum_{k_B = \alpha_B}^{y} \mathbb{P}_{\text{oc}}(y, k_B, k_W, u_W) \]

(13)

where \( b_W = y - k_B + k_W \).
Similarly, the probability \( p_{\beta}(z|y) \) of having \( Z = z \) coded transmission opportunities conditioned on \( Y = y \) files, can be computed considering two disjoint events. The first is that \( u_B \) users ask exclusively for exactly \( z \) files at \( R_B \) and \( u_W \) users have ask at least \( z \) exclusively files at \( R_W \). The second is the probability that \( u_B \) users ask exclusively for more than \( z \) files at \( R_B \) and \( u_W \) users ask exactly \( z \) exclusively files at \( R_W \). Thus, we can write

\[
p_{\beta}(z|y) = \sum_{k_W = z}^{\beta_W} \frac{1}{\binom{\beta_B}{y} \binom{\beta_W}{z}} \left( \sum_{k_B = z}^{\beta_B} \frac{\binom{\beta_B}{y} \binom{\beta_W}{z} \binom{y + k_W}{1} \binom{N - y - k_W}{1}}{\binom{N - y}{1} \binom{N - y - k_W}{1}} \right)
\]

\[
+ \sum_{k_B = z+1}^{\beta_B} \frac{1}{\binom{\beta_B}{y} \binom{\beta_W}{z}} \left( \sum_{k_W = z}^{\beta_W} \frac{\binom{\beta_B}{y} \binom{\beta_W}{z} \binom{y + k_W}{1} \binom{N - y - k_W}{1}}{\binom{N - y}{1} \binom{N - y - k_W}{1}} \right).
\]

where \( b_W = y - z + k_W \) and \( b_B = y - k_B + z \). If we now plug (13) and (14) into (11) and we remove the condition on \( Y \), we obtain

\[
L_{1}^{ECC} = \sum_{y=1}^{\beta_B} p_{\beta}(y|u_B) \left[ (N - M) \left( y + \sum_{k_W = 0}^{\beta_W} \frac{\binom{\beta_B}{y} \binom{\beta_W}{z} \binom{y + k_W}{1} \binom{N - y - k_W}{1}}{\binom{N - y}{1} \binom{N - y - k_W}{1}} \right) \right]
\]

\[
- \min(M, N - M) \sum_{z=0}^{y} \sum_{k_W = 0}^{\beta_W} \frac{\binom{\beta_B}{y} \binom{\beta_W}{z} \binom{y + k_W}{1} \binom{N - y - k_W}{1}}{\binom{N - y}{1} \binom{N - y - k_W}{1}}.
\]

The final result in (16) is obtained by adding the expression of \( L_{1}^{ECC} \) to (15) which is the average transmission load of the ECC scheme normalized to the number of fragment \( N_F = N \), \( L_{ECC} = L_{ECC}/N \).

V. RESULTS

We evaluate the average backhaul load for the benchmark MDS scheme and compare it with the one of the proposed

\[
L_{ECC} = \frac{1}{N} \sum_{y=1}^{\beta_B} \sum_{k_B = 0}^{\beta_W} \binom{\beta_B}{y} \binom{\beta_W}{z} \binom{y + k_W}{1} \binom{N - y - k_W}{1} \binom{y + k_W}{1} \binom{N - y - k_W}{1}
\]

\[
+ \sum_{z=1}^{y} \left[ \frac{\beta_B}{N} \sum_{k_B = z}^{\beta_B} \binom{\beta_B}{y} \binom{\beta_W}{z} \binom{y + k_W}{1} \binom{N - y - k_W}{1} \binom{y + k_W}{1} \binom{N - y - k_W}{1} \right].
\]

In both cases quantities are normalized to the library size \( N \), i.e. \( L_{MDS} = L_{MDS}/N \) and \( L_{ECC} = L_{ECC}/N \). We assume the library size \( N = 100 \) and 20\% of the users to be connected to both relays while 80\% to a single relay. For simplicity, we consider that of half users in \( U \) are connected only to \( R_B \) and half only to \( R_W \).

In Fig. 4, the normalized average backhaul load as a function of the cache size \( M \) for different number of users \( u \) is plotted. As shown, the ECC scheme outperforms the benchmark MDS caching scheme for every number of users \( u \) considered. As expected, by fixing \( u \) requests, \( L \) decreases by increasing \( M \), since more content directly from the cache can be provided. Given \( M \), the gain between ECC and the MDS scheme is higher when the number of total users \( u \) is greater because more transmission opportunities take place. The maximum gain is obtained when \( M = N/2 \), in fact, this cache operating point encodes half of file content (the maximum portion of a file that can be combined) in a transmission opportunity.

Motivated by the good performance obtained, we also show Monte-Carlo results when file request distribution is not equiprobable. The normalized average backhaul load in this case is reported in Fig. 5. It is assumed that users request content according the Zipf distribution with \( \alpha = 0.80 \) and each relay optimizes own cache content according the algorithm given in [5]. In this set up, we can appreciate the efficiency of the caching placement due to the not uniform demands. In fact, given the number of users \( u \), a cache size \( M \) and a scheme then \( L \) is lower than in our previous scenario. A gain on the ECC with respect to MDS is still present. Due to the lower number of coding opportunities and due to the placement considered such gain is smaller with respect to our previous results.

VI. CONCLUSIONS

We applied the Maddah-Ali caching scheme at the edge of a two-tier heterogeneous satellite network with multiple users. A closed-form expression of the average backhaul transmission load for the ECC scheme was derived. The performance of the scheme was compared with those of the benchmark given by the MDS one. We quantified the nature of the transmission gain obtained by casting out problem with known results and each relay optimizes own cache content according the algorithm given in [5]. In this set up, we can appreciate the efficiency of the caching placement due to the not uniform demands. In fact, given the number of users \( u \), a cache size \( M \) and a scheme then \( L \) is lower than in our previous scenario. A gain on the ECC with respect to MDS is still present. Due to the lower number of coding opportunities and due to the placement considered such gain is smaller with respect to our previous results.
The relevant reduction of load backhaul transmission obtained validates the coded caching scheme in satellite networks and suggest its investigation in more sophisticated scenarios.

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