Essentially Second Modules

Inaam Mohammed Ali Hadi*, Farhan Dakhil Shyaa, Shukur Neamah Al-aewish

1Department of Mathematics, College of Education for Pure Sciences (Ibn-Al-Haitham) University of Baghdad, Baghdad, Iraq
2Department of Mathematics, College of Education, University of Al-Qadsiyah, Al-Qadsiyah, Iraq
3Department of Urban Planning, College of Physical Planning, University Of Kufa, Iraq

Abstract
In this paper, as generalization of second modules we introduce type of modules namely (essentially second modules). A comprehensive study of this class of modules is given, also many results concerned with this type and other related modules presented.

Keywords: second modules, prime module and essentially second modules.

Mathematics Subject Classification 2010: 16D10, 61N60, 16D60, 16P60

Introduction
In this research all rings are associative with identity and all modules are unitary right modules. For a right $R$-module $M$ we write $M_R$. Agayev in [1] defined and studied $r$-semisimple modules, "where an $R$-module $M_R$ is said to be $r$-semisimple if for any right ideal $I$ of $R$, $MI$ is a Direct summand of $M$ (briefly $MI \leq B(M)$ "The class of $r$-semisimple modules contains the class of semisimple modules , also contains the class of second modules, where an $R$-module $M$ is named $second$ if $M \neq 0$ and for each $r \in R$, either $Mr = 0$ or $Mr = M[2]$. Equivalently $M$ is second if for each ideal $I$ of $R$, either $MI = 0$ or $MI = M[2]$. Annine in [3], [4] introduced the class of coprime modules. "An $R$-module $M$ is coprime if $ann_R(M) = ann_R(M/N)$ for each proper submodule $N$ of $M (N < M)$, where $ann_R(M) = \{r \in R: Mr = 0\}$. Wijayanti in [5] called an $R$-module $M$ is coprime if $ann_R(M) = ann_R(M/N)$ for each fully invariant submodule $N$ of $M$, "where a submodule $N$ of $M$ is called fully invariant if for each endomorphism $f (f \in End(M)), f(N) \subseteq N) " [6]. However, coprime module (in sense of Annine), coprime modules (in sense of Wijayanti) and second modules are coinciding.

*Email: innam1976@yahoo.com
In this paper, we give another generalization of second modules. An \( R \)-module \( M \) is an essentially second (shortly ess. second) if for each ideal \( I \) of \( R \), either \( IM = 0 \) or \( IM \leq_{ess} M \), where a submodule \( N \) of \( M \) is essential (briefly \( N \leq_{ess} M \)) if whenever \( N \cap W = (0), W \leq M \), then \( W = (0) \)[7]. Equivalently \( N \leq_{ess} M \) if and only if for each \( m \in M, \exists r \in R; \quad 0 \neq mr \in N[7] \).

It is clear that every second and uniform modules are ess. second but the converses are not true, see Remarks 2.2(2),(3).

In section two, we give the basic properties of ess. second modules such as in the class of multiplication modules, ess. second modules and uniform modules are equivalent (see, Corollary 2.4). Every pure submodule (hence every direct summand) of ess. second modules is an ess. second module (Proposition 2.12), but the direct sum of ess. second modules may be not ess. second (see Remark 2.8). Also, if \( M \) is an ess. second and \( N \) is a closed submodule, then \( \frac{M}{N} \) is an ess. second module (see Proposition 2.9).

In section three we present many relationships between ess. second modules and other related concept such as prime modules, \( r \)-semisimple modules (see Proposition 3.1, Theorem 3.2 and Proposition 3.3).

2. Essentially second modules

If \( M \) is an \( R \)-module, "a submodule \( N \) of \( M \) is second submodule if for each ideal \( I \) of \( R \), either \( NI = (0) \) or \( NI = N[2] \). A module \( M_{R} \) is second if it is a second submodule of \( M \). A ring \( R \) is a second if \( R \) is a second \( R \)-module".

We define:

**Definition 2.1:** An \( R \)-module \( M \) is called essentially second (briefly ess. second) if for each ideal \( I \) of \( R \), ether \( IM = (0) \) or \( IM \leq_{ess} M \); A ring \( R \) is ess. second if \( R \) is ess. second \( R \)-module.

**Remarks 2.2:**

1- Obviously each second module is ess. second, but not conversely, as one can see by: The \( Z \)-module \( Z_{4} \) is clearly an ess. second, and it is not second, for if \( I = 2Z \), then \( Z_{4}(2Z) = \{Z \} \neq \emptyset \) and \( Z_{4}(2Z) \neq Z_{4} \).

2- Every uniform module is ess. second, but not conversely: The \( Z \)-module \( M = Z \oplus Z \) is ess. second since for each ideal \( I \neq (0) \) of \( Z \), \( I = nZ, n \in Z_{4} \), so \( M_{1} = (Z \oplus Z) nZ = nZ \oplus nZ \leq_{ess} M \). If \( I = (0) \), then \( M_{1} = (0) \). Thus \( M \) is an ess. second, but it is clear that \( M \) is not uniform.

3- If \( R \) is an ess. second ring, then \( R \) is uniform.

4- Let \( M, M' \) be \( R \)-modules such that \( M \cong M' \), then \( M \) is ess. second if and only if \( M' \) is ess. second.

5- Let \( A \) be an ideal of \( R \) and \( M \) be an \( R \)-module such that \( MA = (0) \). Where \( M \) is ess. second module \( R \)-if and only if \( M \) is \( \frac{R}{A} \)-ess. second module.

**Proof:** Let \( r \neq r + A \in \frac{R}{A} \). Then \( r \in R, r \neq 0 \). Since \( M \) is an ess. second \( R \)-module, either \( Mr = 0 \) or \( Mr \leq_{ess} M \). If \( Mr = 0 \), then \( r \in A \) and \( M(r + A) = 0 \). If \( Mr \leq_{ess} M \), then \( M(r + A) = Mr \leq_{ess} M \). Thus \( M \) is an ess. second \( \frac{R}{A} \)-module. The proof of converse is similarly.

6- \( r \)-semisimple module and ess. second module are independent concepts. For examples The \( Z \)-module \( Z_{6} \) is \( r \)-semisimple but it is not ess. second. While The \( Z \)-module \( Z_{4} \) is an ess. second module, but it is not \( r \)-semisimple. Also, it is not second.

7- Let \( M \) be a torsion free \( R \)-module and \( R \) is ess. second. Then for each \( m \in M, mR \) is an ess. second.

The pursue is a characterization of ess. second modules.

**Theorem 2.3:** For an \( R \)-module \( M_{R} \), the following statements are equivalent:

1- \( M \) is ess. second;

2- If \( 0 \neq N \subseteq M, N = M[N:M] \), then \( N \leq_{ess} M \);

3- For each \( r \in R \), either \( Mr = 0 \) or \( Mr \leq_{ess} M \).

**Proof:** (2) \( \Rightarrow \) (1) Let \( I \) be an ideal of \( R \). Assume \( M_{I} \neq 0 \). Set \( N = M_{I} \). It is clear that \( M_{I} = M[M_{I}:M] \); that is \( N = M[N:M] \) and so by (2) \( N = M_{I} \leq_{ess} M \).

(1)\( \Rightarrow \) (3) It is obvious.

(3)\( \Rightarrow \) (2) Let \( 0 \neq N = M[N,M] \). Then there exists \( r \in [N:M] \) such that \( Mr \neq (0) \), so that \( Mr \leq_{ess} M \) by condition (3). But \( Mr \leq M[N:M] = N \). This implies \( N \leq_{ess} M \).
it is known that: an R-module M is called a multiplication module provided for each submodule N of M there exists an ideal I of R such that N=Ml . [8]

Corollary 2.4: For a multiplication module M over a ring R. The pursue are equivalent:
1- M is ess. second;
2- M is uniform;
3- For each r ∈ R either Mr = 0 or Mr ≤ess M.

Note that the condition M is multiplication can’t be dropped from Corollary 2.4, since the Z-module Q⊗Q is ess. second and it is not uniform.

Corollary 2.5: Let R be a commutative ring. Then R is ess. second if and only if R is uniform.

Corollary 2.6: For a faithful multiplication module over a ring R. The pursue are equivalent:
1- M is an ess. Second ;
2- M is uniform ;
3- R is uniform ;
4- R is ess.second .

Proposition 2.7: Let M be an ess. second module and let N ≤⊗ M. Then N is ess. second.

Proof: Let M1 ≤⊗ M. Then M = M1 ⊕ M2 for some M2 ≤ M. For any ideal I of R, either M1 I = 0 or M1 I ≠ 0. If M1 I ≠ 0 then M1 I ⊕ M2 I ≠ 0. Hence MI ≤ess M and this implies M1 I ≤ess M1, by [7, Prop. 1.1.P.16] Thus M1 is an ess. second module.

Remark 2.8: The direct sum of ess. second modules is not necessary ess. second, for example: Each of the Z-module Z3 and Z4 is an ess. second module but Z3 ⊕ Z4 ≃ Z12 is not an ess. second module since Z12(3Z) =< 3 > ≃ess Z12 and Z12(3Z) ≠< 0 >.

Proposition 2.9: For any ess. second module M, Σi∈I M_i (M_i = M, for each i ∈ I) is an ess. second.

Proof: It is easy.

A submodule N of an R-module M is closed if N has no proper essential extension, [7].

Proposition 2.10: Let N be a closed submodule of an ess. second module M. Then M/N is an ess. second module.

Proof: Let I be an ideal of R. Since M is an ess. second module, either MI = (0) or MI ≤ess M. If MI = (0), then M/N I = M + N/MI = (0). If MI ≤ess M, then MI + N ≤ess M, and since N is closed in M, then M/N I ≤ess M/N by [7, Proposition 1.4(a⇒b)]. It follows that M/N I ≤ess M/N. Thus M/N is an ess. second.

Remark 2.11: The condition (N is closed in M) is a necessary condition in Proposition 2.10, for example. The Z-module Z is an ess. second (since it is second). But Z/12Z is not ess. second and 12Z is not closed in Z.

Corollary 2.12: Let f: M → M’ be an epimorphism such that Ker(f) is closed and M is an ess. second. Then M’ is an ess. second.

By applying Proposition 2.10 we can give a different proof of Proposition 2.7 as follows

Proof: Since N ≤⊗ M, then N⊗W = M for some W ≤ M. But W ≤⊗ M, implies W is closed submodule of M [7,Exc.3.P.19] Hence M/NW is an ess. second by Proposition 2.10 and this implies N is an ess. second since N ≃ M/N.

A submodule N of an R-module M is called pure if MI ∩ N = NI for each ideal I of R, [9]

Proposition 2.13: Every pure submodule of ess. second module is an ess. second.

Proof: Let N be a pure submodule of M, let I be an ideal of R. Since M is an ess. second either MI = (0), or MI ≤ess M. If MI = (0), then NI = (0)(since N ≤ M), if MI ≤ess M, then MI ∩ N ≤ess M ∩ N = N and so NI ≤ess N. Thus N is an ess. second.

Since every direct summand of a module is pure, we can also get Proposition 2.7 directly, by Proposition 2.13.

Proposition 2.14: Let M be an R-module. M is an ess.second as a left E-module if and only if for each 0 ≠ f ∈ Hom(M,N), N ≤ M implies N ≤ess M. Where E = End(M).

Proof: Let 0 ≠ f ∈ Hom(M,N). Then iof ∈ E, where i is the inclusion mapping from N to M. Since M is an ess. second E-module, either (i o f)(M) = (0) or (i o f)(M) ≤ess M. But (i o f)(M) = (0) implies f = 0 which is a contradiction, hence (i o f)(M) ≤ess M; that is f(M) ≤ess M. But f(M) ≤ N, so that N ≤ess M.
To prove \( M \) is an ess. second \( E \)-module. That is to prove for each \( f \in E \), either \( f(M) = (0) \) or \( f(M) \leq_{ess} M \). suppose that \( f(M) \neq (0) \) that is \( f \neq 0 \). Put \( N = f(M) \), hence \( f \in \text{Hom}(M, N) \) and by hypothesis \( N \leq_{ess} M \). Thus \( f(M) \leq_{ess} M \).

3. Essential Second Modules and other related concept

In this section many connections between ess. second modules and other related concepts are presented.

First we have

**Proposition 3.1:** An \( R \)-module \( M \) is an ess. second and \( r \)-semisimple iff \( M \) is second.

**Proof:** \( \Rightarrow \) Let \( I \) be an ideal of \( R \). If \( IM = (0) \), then nothing to prove. If \( IM \neq (0) \), then \( IM \leq_{ess} M \), since \( M \) is ess. second. But \( M \) is an \( r \)-semisimple, so that \( IM \leq_{ess} M \). It follows that \( IM = M \). Thus \( M \) is second.

\( \Leftarrow \) It is obvious.

An \( R \)-module \( M \) is prime if \( ann(M) = ann(N) \) for each \( (0) \neq N \leq M \) \([10] \). A proper submodule \( N \) of an \( R \)-module is prime if whenever \( x \in M, r \in R, xr \in N \) implies \( x \in N \) or \( r \in [N : M] \) \([10] \). \( M \) is a prime. if and only if \( (0) \) is a prime submodule of \( M \).

**Theorem 3.2:** Let \( M \) be a prime. over a commutative ring. \( R \) and let \( N < M \) such that \( N \) is an ess. second submodule. Then \( N \) is a prime submodule.

**Proof:** Let \( x \in M, r \in R \) with \( xr \in N \). Suppose \( x \not\in N \), so we must prove \( r \in [N : M] \). Since \( N \) is an ess. second, either \( Nr = (0) \) or \( Nr \leq_{ess} N \). If \( Nr = (0) \), then \( r \in ann(N) = ann(M) \) and this implies \( r \in [N : M] \). If \( Nr \leq_{ess} N \), then there exists \( a \in R \) such that \( 0 \neq xra \in Nr \). Thus \( xra = nr \) for some \( n \in N \). Since \( R \) is commutative, \( xra = xar \), hence \( xar = nr \) which implies \( (xa - n)r = 0 \); that is \( r \in ann(xa - n) \). But \( ann(xa - n) = ann(M) \) (since \( M \) is prime.). Therefore \( r \in ann(M) \subseteq [N : M] \). Thus \( N \) is a prime submodule.

**Proposition 3.3:** Let \( M \) be a prime \( R \)-module, \( N = xR \) for some \( x \in M \). If \( N \) is an ess. second \( R \)-module, then \( M \) is an ess. second.

**Proof:** Let \( r \in R \). Suppose \( Mr \neq (0) \) \((r \notin ann(M)) \). Hence \( r \notin ann(N) \) (since \( M \) is a prime.). So \( Nr \neq (0) \), but \( N \) is an ess. second module implies \( Nr \leq_{ess} N = xR \). Now \( x \in N \), hence there exists \( r' \in R \) such that \( 0 \neq xra \in Nr \). It follows that \( xra = xar \) for some \( a \in R \). Thus \( x(r' - ar) = 0 \); that is \( r' - ar \in ann(X) = ann(M) \). Hence for each \( m \in M \), \( mr' = mar \) and \( 0 \neq mr' \) (because if \( mr' = 0 \) then \( r \in ann(m) = ann(M) \) and \( Mr = 0 \) which is a contradiction). Therefore, \( \forall m \in M \), there exists \( r' \in R \) such that \( 0 \neq mr' = mar \in Mr \). Thus \( Mr \leq_{ess} M \) and \( M \) is an ess. second module.

**Proposition 3.4:** Let \( N \leq_{ess} M \), \( ann(M) = ann(N) \). If \( N \) is an ess. second submodule of \( M \). Then \( M \) is an ess. second module.

**Proof:** Let \( r \in R \). Since \( N \) is an ess.second submodule, then either \( Nr = (0) \) or \( Nr \leq_{ess} N \). If \( Nr = (0) \), then \( Mr = (0) \)(since \( ann(M) = ann(N) \) by hypothesis). If \( Nr \leq_{ess} N \), then \( Nr \leq_{ess} M \) since \( N \leq_{ess} M \). But \( Mr \supseteq Nr \), hence \( Mr \leq_{ess} M \). Thus \( M \) is ess. second.

**Remark 3.5:** The condition \( ann(M) = ann(N) \) is necessary condition, for example. Let \( M \) be the \( Z \)-module \( M = Z_2 \oplus Z_4 \). Let \( N = Z_2 \oplus <2> \leq_{ess} M, ann(M) = 4Z \neq ann(N) = 2Z \). But \( N \approx Z_2 \oplus Z_2 \) so that \( N \) is an ess. second. But \( M \) is not an ess. second module since \( M(2Z) = (0) \oplus <2> \neq 0 \) and \( M(2Z) \leq_{ess} M \).

An \( R \)-module \( M \) is called coquasi-Dedekind if \( Hom(M, N) = (0) \) for each \( N \leq M \)[11]. Equivalently \( M \) is coquasi-Dedekind if for each \( 0 \neq f \in End(M) \), \( f \) is an epimorphism".

We present the following

**Definition 3.6:** An \( R \)-module \( M \) is to be essentially coquasi-Dedekind if for each \( f \in End(M) \), \( Im f \leq_{ess} M \).

Note that Sahra in [11] gave the following: an \( R \)-module \( M \) is called essentially coquasi-Dedekind if for each \( 0 \neq f \in End(M) \), \( Ker(f) \leq_{ess} M \). However our definition is different of that was given in [11].
Examples 3.7:
1- Every simple module (and the $Z$-modules $Z,Q$) are ess. coquasi-Dedekind in sense of Definition 3.6, but it is not ess. coquasi-Dedekind in sense of [11].
2- Consider $Z_{12}$ as $Z$-modules, is an ess. couasi-Dedekind in sense of [11]. But it is not ess. coquasi-Dedekind in sense of Definition 3.6, since there exists $f: Z_{12} \rightarrow Z_{12}$ define by $f(x) = 6x$ for each $x \in Z_{12}$ and $Imf = 6 \not\leq \mathcal{E}_{ess} Z_{12}$.

**Remark 3.8:** Every ess. coquasi-Dedekind module is ess. second.

**Proof:** Let $r \in R$. If $Mr \not= (0)$. Define $f: M \rightarrow M$ by $f(m) = mr$ for each $m \in M, 0 \not= f$. Then $Imf = Mr$. But $Im(f) \leq_{ess} M$ since $M$ is ess. coquasi-Dedekind. Thus $Mr \leq_{ess} M$.

Note that the reverse is not achievable in public as: let $M = Q \oplus Q$ as $Z$-module. $M$ is ess. second module , but it is not ess. coquasi- Dedekind since $\exists f \in End(M)$ such that $f(x,y) = (x,0)$, for each $(x,y) \in M$ and so $Im(f) = Q \oplus (0) \not\leq_{ess} M$.

An $R$-module $M$ is scalar module if for each $f \in End(M)$, $\exists 0 \not= r \in R, f(m) = mr, \forall m \in M$ [12].

**Proposition 3.9:** Let $M$ be a scalar module. Then $M$ is an ess. coquasi-Dedekind iff $M$ is an ess. second module.

**Proof:** It is easy, so is omitted.

The following result follows directly.

**Proposition 3.10:** Let $M$ be an $R$-module. Then $M$ is an ess. coquasi-Dedekind iff $M$ is an ess.second left $E$-module, where $E = End(M)$.

By combining Proposition 3.10 and Proposition 2.13, we have the following:

**Corollary 3.11:** For an $R$-module $M$. The pursue are synonymous:
1- $M$ is an ess. coquasi-Dedekind $R$-module ;
2- $Hom(M,N) \not= 0$ (where $N \leq M$) implies $N \leq_{ess} M$;
3- $M$ is an ess. second left $E$-module.

As we mention in the introduction the second module is called coprime by some authors, see[2,13], Sahera in [11] introduced the concept ess. coprime as a generalization of coprime (second module) where an $R$-module is referred by an ess. coprime if for each $r \in R$, either $Mr = M$ or $ann_{M}(r) \leq_{ess} M$, where $ann_{M}(r) = \{m \in M: mr = 0\}$.

Notice that the concept ess. second is independent with ess.coprime[11]. Like:
1- Let $M = Z_{2} \oplus Z$ as $Z$-module. It is easy to see that $M$ is an ess. coprime and it is not ess. second.
2- For the $Z$-module $M = Z \oplus Z$. $M$ is ess. second. But for any $0 \not= r \in Z$, $ann(r) = \{(a,b) \in M: (a,b)r = (0,0)\} = (0) \not\leq_{ess} M$. Also, $Mr \not= M$ for each $r \in Z, r \not= \pm 1$. Thus $M$ is not ess. coprime.

It is known that for every second $R$-module $ann_{Z}(M)$ a prime ideal. of $R$. However this is not true for ess. second module as we have: the $Z$-module $Z_{8}$ is a ess. second (since it is uniform ) and $ann_{Z}(Z_{8}) = 8Z$ which is not a prime ideal. of $Z$.

In [13] we define the concept essential prime (briefly ess. prime ) as follows : an $R$-module $M$ is said to be an ess. prime whenever $ann_{R}(M) = ann_{R}(N)$ for all $N \leq_{ess} M$.

We state and prove the pursue:

**Proposition 3.12:** Let $M$ be an ess. second $R$-module and ess. prime.. Then $ann_{R}(M)$ is a prime ideal. of $R$.

**Proof:** Let $a,b \in R$ and $a,b \in ann_{R}(M)$ ($Mab = 0$). Assume $a \not\in ann(M)$, that is $Ma \not= (0)$. Since $M$ is ess. second, then $Ma \leq_{ess} M$. on the other hand $M$ is ess. prime, so $ann_{R}(M) = ann_{R}(Ma)$. But $b \in ann_{R}(M)$ (since $Mab = (0)$) hence $b \in ann_{R}(M)$. Thus $ann_{R}(M)$ is a prime ideal.

Note that ess. the second module does not imply ess. prime., as the $Z$-module $M = Z_{4}$ is ess. second, however it is not ess. prime since $ann_{Z}(M) = 4Z \not= ann_{Z}(2) = 2Z$, and $2 \not\leq_{ess} Z_{4}$. Also, ess. prime. does not imply ess. second, as: The $Z$-module $M = Z_{2} \oplus Z$ is an ess. prime and it is not an ess. second.

**Corollary 3.13:** Let $M$ be an $R$-module and every prime ideal. of $R$ is maximal. Then the pursue are synonymous:
1- $M$ is second;
2- $M$ is prime.;
3- $M$ is an ess. prime. and ess. second;
4- $ann_{R}(M)$ is a prime ideal . of $R$.

1378
Proof: (1) $\iff$ (2). [14, Lemma 1.1]

(2) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (2) $ann_R(M) \subseteq ann_R(N)$ for each $0 \neq N \leq M$. But $ann_R(M)$ is a prime ideal. by condition (4), so $ann_R(M)$ is maximal and so $ann_R(M) = ann_R(N)$. Thus $M$ is a prime module.

(3) $\Rightarrow$ (2) By Proposition 3.12, $ann_R(M)$ is a prime ideal, hence $ann_R(M)$ is maximal by hypothesis. But $ann_R(M) \subseteq ann_R(N)$ for each $0 \neq N \leq M$ so that $ann_R(M) = ann_R(N)$. Thus $M$ is prime.

(2) $\Rightarrow$ (3) Since $M$ is prime, then $M$ is an essential prime. But $M$ is prime implies $M$ is second by (part (2) $\iff$ (1)), hence $M$ is ess. second.

It is known that if $R$ is an Artinian ring or a Boolean ring, then every prime ideal is maximal. Hence we get.

Corollary 3.14: Let $M$ be an $R$-module where $R$ is an Artinian ring or Boolean ring. Then the pursue is synonymous.

1- $M$ is second ;
2- $M$ is prime ;
3- $M$ is ess. prime and ess. second; 
4- $ann_R(M)$ is a prime ideal. Of $R$.

Proposition 3.15: Let $M$ be an $R$-module such that $ann_R(M)$ is semisimple and $ann_R(N) = ann_R(M)$, for each $N \leq M$. Then $M$ is prime and second module.

Proof: To prove $M$ is prime. Let $r \in ann_R(N)$. Then $Nr = 0$ and so $\frac{M}{N} r = 0$, by hypothesis; that is $Mr \subseteq N$. Thus $M^2 \subseteq Nr = 0$. Thus $M^{r^2} = 0$ which implies $Mr = 0(r \in ann(M))$ since $ann(M)$ is semi prime. Hence, $ann_R(M) = ann_R(N)$. Therefore $M$ is prime. But $ann(N) = ann_R\left(\frac{M}{N}\right)$ so that $ann_R(M) = ann_R\left(\frac{M}{N}\right)$ for each $N < M$. Hence $M$ is second.

An $R$-module $M$ is homogenous semisimple if $M$ is a direct sum of pairwise isomorphic simple submodules, [14]. In the last part of Lemma 1.1 in [14]. If $M$ is a module over a commutative $R$ such that every prime ideal is maximal, then $M$ is second if $M$ is a homogenous semisimple.

Corollary 3.16: If $M$ is an $R$-module, where $R$ is a commutative ring. such that every prime ideal is maximal (hence if $R$ is Artinian ring or Boolean or Von Neumann regular). Then the pursue are synonymous:

1- $M$ is second ;
2- $M$ is prime ;
3- $M$ is an ess. prime and ess. second module;
4- $ann(M)$ is a maximal ideal;
5- $M$ is a homogenous semisimple.

Proposition 3.17: Let $M$ be multiplication module over a ring $R$. Then $M$ is a second if and only if $M$ is a homogenous semisimple.

Proof: $\Rightarrow$ Since $M$ is a multiplication module then for each proper submodule $N$ of $M$, $N=M$ $[N:M]=M$. Because $M$ is second, $ann_{\frac{M}{N}}=ann M$, hence $N=M$ $ann M=0$ Then $M$ is simple. Thus $M$ is homogenous semisimple.

$\Leftarrow$ It is given in [14].

Corollary 3.18: Let $R$ be a commutative ring. Then $R$ is second if and only if $R$ is homogenous semisimple

References

1. NAZIM AGAYEV, CES’IM ÇEL’IK and TAH’IRE ÖZENProc. 2018. "On a generalization of semisimple modules" Indian Acad. Sci. (Math. Sci.). 128(20): 1-10.
2. Yassemi "The dual notion of prime submodules" Arch. Math. (Bron), 37 (2001): 273-288.
3. Annin, S. 2008. Attached primes over noncommutative rings. J. Pure Appl. Algebra, 21: 510–521.
4. Annine S. 2002. "Associated and attached prime over non commutative ring" Ph.D. University of Berkeley.
5. Wijayanti I.E. 2006. "Coprime Modules and comodules", ph.D. Thesis, Heinrich-Heine University Dusseldorf.
6. Wisbauer, R. 1991. "Foundations of Modules and Rings theory", reading: Gordon and Breach.
7. Goodearl K.R. 1976. "Ring Theory, Non Singular Rings and Modules", Marcel Dekker", Inc. New York and Basel.
8. Zeinab, A.EL-Bast and Patrck F.Smith, 1988. " Multiplication Modules " Communication in Algebra, 10(4): 755-779.
9. Anderson, F. W., Fuller K. R. 1992. "Rings and Categories of Modules", Second Edition, Graduate Texts in Math., Vol.13, Springer-Verlag, Berlin-Heidelberg-New York.
10. Beachy, J. 1975. Some aspects of noncommutative localization, in Noncommutative Ring Theory, Kent State University, Lecture Notes in Mathematics, Vol. 545, Springer-Verlag, Berlin-New York.
11. ahera, M. Y. 2003. "Coquasi-Dedekind Modules" Ph.D. thesis Univ. of Baghdad.
12. Shihab B.N. 2004. "Scalar Reflexive Modules". Ph.D. Thesis, College of Education Ibn AL-Haitham, University of Baghdad.
13. Enaam Mohammed Ali and Thaar Younis. 2011. Essentially prime Modules and Related. J. of Basrah researchers (sciences), 37(4), (2011).
14. Çeken S., Alkan M. and Smith P. F. 2013. Second Modules over Noncommutative Rings, Communications in Algebra, 41(1): 83-98.