Research article

A fuzzy inference method for image fusion/refinement of CT images from incomplete data

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A B S T R A C T

The quality of computed-tomography (CT) images deteriorates when images are reconstructed from incomplete data. This work makes use of the a priori knowledge inherent in the membership functions and the logical rules of a fuzzy inference system (FIS) to compensate for the missing data. It is shown that a fuzzy inference system can be used to improve the quality of reconstructed CT images, particularly when the images are reconstructed from incomplete data. It is proposed to reconstruct a coarser image for which the data is over-complete, and use the histograms of this image and that of the original finer image to generate the membership functions required in FIS. The two images are then fused, with the aid of logical rules based on the knowledge that the two images possess the same distinct attributes (pixel values). In order to avoid the difference in spatial resolution between the original fine image and the reconstructed coarse image, a modified FIS method is introduced to refine the fine image. Results are presented, showing visually and quantitatively that this FIS refinement process improves the quality of the original fine image.

1. Introduction

Image fusion usually combines two or more images of the same object formed in different manner or modality, in order to produce a more informative image [1]. In essence, each image in the fusion process is formed from a different set of data, so that each image presents another set of attributes. In this work, we use the same set of data to improve the quality of images reconstructed in computed tomography (CT) [2]. The quality of an image is subjectively determined by its visual appearance, and is quantified by metrics that measure its numerical accuracy, composition (structure, pixel intensity and contrast) and fidelity (degree of realism).

In computed tomography, image quality is determined by the quality of measurements (affected by statistical fluctuation and noise) and the quantity of measured collected points (or projections). The later factor is particularly relevant when the image reconstruction problem is incomplete, as in when a fine (small pixel-size) image is reconstructed using a number of data points less than the number of pixels (i.e. problem unknowns). Fuzzy inference can help improve such images by incorporating logical rules that rely on the fact that similar membership functions should produce the same image attributes determined by the corresponding most probable values. The objective of this work is to take advantage of the a priori knowledge, introduced in a fuzzy inference system (FIS) through membership functions and logical rules, to improve the quality of CT images reconstructed from incomplete data.

We propose to generate the membership functions using the image histograms of the reconstructed fine image and a coarse, but more reliable image, reconstructed from the same data but with a coarser spatial resolution in an over-complete image reconstruction problem. We then fuse the fine image with the coarse one using a fuzzy inference system. This process is improved by developing a fuzzy inference method that refines the fine image while overcoming the drawback of fusing two images reconstructed with different pixel size (spatial resolution).

We first define the incomplete image reconstruction problem in CT and present some of the methods used for its solution and discuss the need for improving such solution. We then present the method used to over-complete the problem so that a more reliable solution is obtained, albeit with a coarse spatial resolution. Taking advantage of the attribute-distributions provided by image histograms, a fuzzy inference system is presented, which fuses the fine image obtained in the incomplete problem with the coarser but more accurate image resulting from the over-complete image reconstruction. The drawback of the latter approach, namely the fusing of a fine image with a coarse one, is overcome with a modified method that uses the histogram of the fine image, along with either the histograms of the fine or coarse images, to refine the fine image.

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image. This image refinement method is presented and applied on different images, showing how and why it results in a more quantitatively accurate image. The effect of data noise and degree of incompleteness are also discussed.

2. Incomplete problems in CT

The image reconstruction problem in CT in a discretized form can be represented by the matrix formulation:

\[ Y = AX \]  

(1)

where \( Y \) is an \( M \times 1 \) vector of normalized measurements, \( X \) is an \( N^2 \times 1 \) vector of the unknown image attributes of an image section with a square domain of \( N \times N \) pixels, and \( A \) is a pre-determined \( M \times N^2 \) system matrix. The linearity of the problem is enabled by using the negative of the logarithm of radiation-transmission measurements recorded in the presence of the object normalized to those recorded in its absence. An element \( A_{ij} \) in the matrix \( A \) is then the length of intersection distance of a radiation beam \( i \) with a pixel \( j \) in the imaged section in a parallel-beam configuration. Note that a fan/cone-beam configuration can be transformed to a parallel-beam one using the transformation method of Feldkamp et al. [3].

In order to compensate for the effect of measurement noise and other uncertainties, the CT image reconstruction problem is usually over-complete, with a degree of over-determination, \( DoD = M/N^2 \) of 2 to 3 [2]. The CT problem becomes incomplete when \( DoD < 1 \), i.e. when the number of measurements is less than the number of unknowns or image pixels. Incomplete CT problems are encountered when a limited number of projections, angles or views, are used; typically to reduce radiation exposure in medical imaging or because of limited accessibility to the imaged object in two opposing directions in industrial applications.

An incomplete problem has some eigenvalues that are close to zero, i.e., its system matrix, \( A \), has a high condition number, and the inversion of Eq. (1) is an ill-posed problem. As Louis [4] stated, in a limited range of view problem, “the condition number grows exponentially with a rate depending on the size of the missing range”. That size can be measured by \( 1 – DoD \). Then \( DoD = 1 \) indicates no missing range, i.e., a complete problem in the ideal case of a noise-free problem, since noise introduces uncertainties that amount in effect to missing information. Therefore, an incomplete problem can have many solutions, i.e. a number of different values for \( X \) in Eq. (1) can produce the same value for \( Y \). This can make it difficult to determine whether the reconstructed image is a correct representation of the object. However, as Louis [4] showed, the computed tomography problem has a special eigenvalue (spectral) structure that enables the reconstruction, from incomplete data, of an image that faithfully represents the actual object, when the solution is supported by a priori knowledge of the problem. Notably, when the image is sparse, i.e. with a few dominant attributes, the compressed sensing [5] or Total variation (TV) regularization [6] methods can produce such solutions. Even then the obtained solutions are approximate because of the associated constraints and assumptions and are unlikely to produce numerically and distributively accurate images. Nevertheless, a quantitatively more accurate solution, that is unique given the linearity of the CT problem, can be obtained with the same set of data if the problem is made to be over-complete, as shown below.

3. Coarse image formation

For the same set of data, a CT image can be solved with a different spatial resolution, i.e. number of pixels, using the multi-resolution method [7, 8, 9]. Accordingly, an incomplete CT problem can be made over-complete by reducing the number of its unknowns. In other words, the problem of Eq. (1) can be reformulated as:

\[ Y = A_c X_c \]  

(2)

where \( Y \) is the same \( M \times 1 \) vector as in Eq. (1), but with \( X_c \) being an unknown \( n^2 \times 1 \) vector, such that \( n < N \) and \( DoD_c = M/n > 1 \). Then the system matrix, \( A \) becomes an \( M \times n^2 \) matrix whose elements represent intersection distances with larger pixels than in the problem represented by Eq. (1). This new image, represented by \( X_c \), is a lower spatial-resolution image in comparison to that of \( X \). To avoid pixel misalignment. \( n \) should be chosen that \( n = 2^k N \), for square pixels, with \( k \geq 1 \) being an integer, so that a coarse pixel consists of \( 2^{2k} \) corresponding contiguous pixels in the fine image. Then \( DoD_c = 2^{2k} DoD \), and for the coarse problem to be over-complete, \( DoD \) has to be \( >2^{2k} \). Given that Eq. (2) is a linear problem, an over-complete problem will have a unique solution, \( X_c \), for a given set of data, \( Y \), when a solution is obtained without any imposed constraints.

4. Image fusion with fuzzy inference

Piella [10] defined image fusion as the “process of combining multiple input images into a smaller collection of images, usually a single one, which contains the ‘relevant’ information from the inputs”. We propose here to use image fusion to produce an image that combines the fine spatial resolution of a somewhat ambiguous image obtained from incomplete data with the more reliable quantitative information of a coarser image reconstructed from the same data but in an over-complete problem. As Piella [10] indicated, image fusion can be conducted at the pixel, feature or symbol levels. The latter two methods aim at identifying and detecting certain distinguishing features enhanced by image fusion, and as such require some image processing by selection, extraction, segmentation or identification. These procedures result in feature or symbolic representations that when fused together enable either the detection of certain features or better interpretation and resolution of differences. Pixel-level fusion, on the other hand, as its name implies directly merges image attributes (values) pixel-by-pixel. In our case, pixel-level fusion is facilitated by reconstructing the coarse image with pixels that exactly overlap entire whole fine pixels. In essence, the attribute of a coarse pixel represents more accurately the average contents of the corresponding \( 2^{2k} \) fine pixels, due to the over-completeness of the coarse-image reconstruction problem.

To facilitate image fusion, the coarse image of \( k \times k \) pixels needs to be expanded into an image of \( N \times N \) pixels that matches the fine image, by assigning to each pixel in the expanded image the same image-attribute values as in the corresponding coarse pixel. This will enable direct image fusion between two images of same pixel size and structure, but at the same time results in the loss of the main advantage of the accuracy of coarse image reconstruction; because in effect it homogenizes (smooths) the expanded image by assigning the same image attribute to the same \( 2^{2k} \) contiguous pixels. This in effect challenges the effectiveness of traditional image fusion methods, such as those given in [11]. However, viewing the values generated by the coarse images as reliable definite but average values and the corresponding fine reconstruction attributes produced by the fine image as ambiguous (due to the problem’s incompleteness) and distributed values, invites the use of image fusion with a fuzzy inference system [12]. Such system is designed to accommodate ambiguities and distributions [13, 14]. Moreover, the histograms of the reconstructed images enable the formulation of the membership functions used in fuzzy inference systems [13], particularly when the images are sparse.

A fuzzy inference system, relying on the widely used Mamdani method [15], consists of three main components [16]: (1) membership functions that represent the distribution of uncertainty in the input data, needed to transform the numerical finite (crisp) data values into fuzzy variables, (2) a set of IF-THEN rules that transform certain region in the fuzzified input into regions in a fuzzy output, with the consequent of a rule determined by combining its strength with an output membership function, and (3) a defuzzification process that converts the fuzzy output into definite (crisp) output. The membership functions and the logical rules are based on a prior knowledge of the nature of the system.
around, to approximated radiation distribution membership vertical suggested were real at hand, and as such the output values are expected to better reflect the real nature of the system.

Fig. 1 shows a flowchart of two fuzzy inference system for the fusion of the coarse and fine images. The input membership functions were generated using the histograms of the fine and coarse images, as suggested by Na et al. [13]. A histogram of either these variables represents its distribution, with the horizontal axis defining values and the vertical axis giving the frequency of each reconstructed value (number of pixels having at certain value). Gaussian (normally distributed) membership function were used since uncertainties tend to follow the distribution of Y in Eqs. (1) and (2). The latter distribution is related to radiation counting, which is governed by Poisson statistics that can be approximated by a normal distribution at the high count rates encountered in CT [2]. Since Y is linearly related to X and X, in these two equations, respectively, the distributions of X and X, are also assumed to be normally distributed. The mean at each distinct (peak) value of either quantity was calculated as a frequency-weighted average of values around a peak. The standard deviation of membership values were estimated using the horizontal method reported in [17], assuming that the maximum and minimum values around a distribution span four standard deviations, i.e. covering a 95% confidence level.

The output membership functions were generated using either the histograms of the coarse image (Fig. 1(a)), or that of the fine image (Fig. 1(b)). The advantage of the first approach is that it uses, in the final defuzzification operation, the unique and more reliable values of the coarse image, but these values may be more diffused than those of the finer image due the coarser mesh size. Membership functions generated from the fine image are less definite, but are more distinct owing to the smaller pixel size.

Fuzzy rules were designed to combine similar image features, e.g. pixels that represent a particular feature of one input image are combined with the pixels with a similar feature in the other input image to form a corresponding feature in the output image. The Mean of Maximum (MOM) method was used as defuzzification operator [16], to transfer all pixels values falling in an input membership function to a central value of an output membership function. This method concurs with the fact that each pixel attribute is represented by a single value (grayscale intensity).
5. Image refinement with fuzzy inference

The main drawback of the fuzzy image fusion refinement process is its use of two images with inherently different spatial resolutions, even though the coarse image was expanded to facilitate the fusion process. Given that the coarse image was reconstructed from an over-complete set of data, its histogram is more accurate numerically but misses the fine spatial details, and the opposite is true for the histogram of the fine image. Therefore, we revised the FIS of Fig. 1 to take advantage of what each histogram offers. As Fig. 2 shows, the histogram of the fine image was employed to generate the input membership functions, to take advantage of the finer distribution it provides to produce more widely distributed fuzzy variables. As in Fig. 1 and for the same reasons discussed above, two output membership functions were attempted, those generated from the histogram of the coarse image and the one calculated from the fine image, Figs. 2(a) and 2(b), respectively. However, unlike in fuzzy image fusion, there was no need to expand the coarse image since it is not used in the same operation with the fine image, avoiding the image mismatching and smoothing problems encountered in fuzzy fusion. In this method, fuzzy rules were designed to match image features in the input image to corresponding features in the output image. Defuzzification was performed in a manner similar to the fuzzy image fusion process of Fig. 1. Again, Gaussian membership functions were used to represent the peaks of the histograms of the images. We call this an image refinement process, because it initially operates on the fine image with the goal of refining it in the final step with output membership functions.

6. Application and results

6.1. Testing setup

The approaches described in the above section were applied using the well-known Shepp-Logan phantom [18], created with the MATLAB function “phantom(‘Modified Shepp-Logan’,n)”, with “n” being the number of rows and columns in a square image [19]. This is a sparse phantom with a limited number of attributes (six) distributed over 10 elliptical regions, with attribute values within the range [0, 1]. In this work, a fine image pixel grid of 128×128 pixels was chosen. CT projection data was synthesized from 27 view-angles uniformly distributed over 0 to 180° degree, each having 185 detector bins, using a parallel-beam configuration, with the MATLAB code “parallelomo” [20]. This results in 4,995 CT measurement data points (detector bins multiplied by the number of view angles). The image reconstruction problem is then incomplete, since the degree of determination ($DoD = 4,995/128^2 = 0.305$). For a coarse image reconstructed with 64×64 pixels, i.e., with a coarse pixel encompassing four fine pixels, the degree of determination ($DoD_c = 1.22$) results in a slightly over-complete problem by
22%. For this coarse-pixel size, coarse image reconstruction is a complete problem when $DoD = 0.25$, i.e. $DoD_c = 1.0$. The lowest $DoD$ value used in this work was 0.305, was the closest to the lowest $DoD_c$ value required to obtain a reliable unique solution for the coarse image. Below this $DoD$ value, the coarse image reconstruction problem became singular (with a nearly infinite condition number), at least with the data acquisition scheme used in this work; due to the fact that a few radiation beams do not cross the imaged section and as such provide no useful information.

Fine image reconstruction was conducted using the TV-regularization method, which is well suited for this sparse phantom, utilizing the TVAL3 algorithm [21]. In this method the integral of the absolute gradient of the solution is used to form an approximately sparse image, so that a piece-wise constant image is arrived at [22]. Coarse-image reconstruction was initially performed with direct inversion without any constraints, so that a unique solution is arrived at for each set of data via the solution: $X = (A^t A_c)^{-1} A^t_c Y$, with the superscript $t$ indicating the transpose of the matrix.

6.2. Metrics

To test the efficacy of the image fusion and refinement processes, the final resulting image, $\hat{X}$, is compared to the actual image, $X$, which is available for testing purposes, using the following metrics that measure the numerical, distributional and compositional accuracy.

Relative Root Mean Square Error (RRMSE): This is a metric that measures how close the resulting image, $\hat{X}$, is to the actual image, it is given by [23]:

$$\text{RRMSE}(X, \hat{X}) = \frac{||X - \hat{X}||_2}{||X||_2} \times 100\%$$  \hspace{1cm} (3)

with $||.||_2$ defining a Euclidean norm.

Relative Entropy (RE): This metric measures the statistical distance between the distributions of two images, also called the Kullback–Leibler divergence [24], and is given by:

$$\text{RE}(\eta^X, \hat{\eta}) = \sum_{i} n_i \log \left( \frac{n_i^X}{\hat{n_i}} \right)$$  \hspace{1cm} (4)

where $\eta$ and $\eta^X$ are the histogram’s probability distributions of $X$ and $\hat{X}$, respectively, normalized such that $0 \leq \eta \leq 1$ and $0 \leq \eta^X \leq 1$. To avoid indefinite values, zero values of $\eta^X$ or $\hat{n}_i$ are replaced with a very small no-zero value. Then, $\text{RE} = 0$ indicates that the two compared images are identical pixel-by-pixel.

Structural similarity index (SSIM): This metric compares the image structure, pixel intensity and contrast of the two images, and is calculated as [25]:

$$\text{SSIM}(X, \hat{X}) = \frac{(2 \mu(X) \mu(\hat{X}) + C_1) (2 \text{Cov}(X, \hat{X}) + C_2)}{\mu^2(X) + \mu^2(\hat{X}) + C_2 + \sigma^2(X) + \sigma^2(\hat{X}) + C_2}$$  \hspace{1cm} (5)

where $\mu(.)$ is the mean of an image, $\sigma(.)$ is its standard deviation, Cov is the covariance between the two images and $C_1 = (0.01L)^2$ and $C_2 = (0.03L)^2$, with $L$ being the dynamic range of the image domain. The closer the value of SSIM to unity, the more the two images resemble each other in terms of their composition.
6.3. Idealized case: DoD = 0.305

We will start with an idealized example where the synthesized data for \( Y \) is error-free, that is the equality in Eq. (1) is perfectly satisfied, to show the best possible performance of the image fusion and refinement processes. We then add random noise to \( Y \) to investigate their sensitivity to the uncertainties in data that accompany actual measurements.

The images produced for the idealized (noise-free) case at \( \text{DoD} = 0.305 \) are shown in Fig. 3. The TV-regularization method visually produced an image almost identical to the actual image, while the coarse images and its expanded form showed the smearing effect of reconstructing the image over larger pixels. This smearing effect is visually reduced to some extent in both fuzzy-fused images. On the other hand, the fine details that were blurred in the other images were restored in the refined images, resulting in images that better resemble the actual image. In either the fuzzy-fused or refined images, the use of the fine or coarse images for the output membership functions did not seem to visually make much difference.

The metrics reported in Table 1 for the reconstructed coarse image are equal or nearly equal to zero, indicating that the reconstructed image is identical to the actual image. On the other hand, the metrics for the fine image were not as close in value to zero, but are sufficiently low to indicate that the fine image was reconstructed correctly. As expected, the RRMSE error in the expanded coarse image was high, reflecting the associated smoothing effect of pixels as each four contiguous pixels in this image were assigned the same pixel value. However, the distribution values of the expanded coarse image was almost identical to the actual coarse image, as indicated by the RE metric, and its overall composition is also quite similar to its actual counterpart, as the SSIM values show. The improvement of the refinement process over fuzzy-fusion is reflected in the better metric values for the former process. However, the metric values in either process indicated improved images when the coarse image was used to generate the output membership functions due to the higher accuracy of that image over the fine image. Although, the un-fused/un-refined fine image had the lowest RRMSE value (0.76%) among all fine images, it had a higher RE value.

![Fig. 5. Actual Shepp-Logan phantom of size (128 × 128) versus fine (TV-regularization) and coarse (unconstrained) images, along with fused and refined images for DoD = 0.429 with noise-free data, omf: output membership function.](image)

![Table 1. Image metrics for the images shown in Figs. 3 (DoD = 0.305, ideal), omf: output membership function.](table)
than that of the refined image with a coarse output membership function (0.27 versus $3 \times 10^{-3}$) and the same SSIM magnitude (0.999). This is further affirmed by the histograms corresponding to those images, shown in Fig. 4. This indicates that the refinement process reflected better the pixel-distribution values than the initial reconstructed fine image, which proves the efficacy of the refinement process in restoring the fidelity of the images.

While the TV-regularization method with its is piece-wise constant constraint produced numerically and compositionally an accurate image in this error-free problem, it did not exactly preserve the distribution of the actual image, as visually evident by comparing the high-value end of the histogram of Fig. 4(b) to that of Fig. 4(a), where the latter has one peak and the former has two adjacent peaks at the far end of their histograms. On the other hand, the refinement process with the accurate coarse output membership functions restored the actual distribution, as Figs. 4(g) shows when compared to Fig. 4(a), while that with the fine membership functions, Fig. 4(h)), almost restored the actual distribution; albeit both at the expense of minor degradation in numerical and composition accuracy. The ability of the refinement process to better recover the image distribution is mostly due to the avoidance of using the expanded smooth coarser image and its associated effect on the rules. The image fusion process was unable to have the same effect, see Figs. 3(e) and 3(f), because of its use of the expanded coarse image with its smoothed pixel attributes.

### 6.4. Idealized case: $DoD = 0.429$

When the degree of incompleteness was decreased, by increasing the $DoD$ value to 0.429, the image quality slightly improved (crisper) as visually shown in Fig. 5 for noise-free data. The metrics of Table 2 reflected this improvement, when compared to those of Table 1, except for the expanded coarse image and the fuzzy-fused image with output membership functions generated using the coarse image. The metrics for the expanded coarse image was identical to that for $DoD = 0.305$, since the coarse images in both cases were almost identical to the actual coarse image, since their solution was obtained for an error-free overdetermined problem. The slightly higher RMSE value in the fused image with the membership functions of the coarse image, compared to that using the membership functions of the fine image, is due to the slightly bigger mismatch between the RMSE values, for the expanded coarse image and the fine image, than in the same metric for $DoD = 0.305$ shown in Table 1. Once more, the refinement process with the coarse output membership function produced a histogram identical to that of the actual image, while that with the fine output member-

### Table 2. Image metrics for the images shown in Figs. 5 ($DoD = 0.429$, ideal), omf: output membership function.

| Image                  | RRMSE (%)  | RE   | SSIM  |
|------------------------|------------|------|-------|
| Fine                   | 0.7218     | 0.1580 | 0.9908 |
| Coarse                 | 9.2872 $\times 10^{-11}$ | 0.0 | 1 |
| Expanded Coarse        | 49.478     | 1.2813 $\times 10^{-4}$ | 0.8999 |
| Fuzzy Fusion           |            |      |       |
| With Coarse omf        | 25.70      | 1.0518 | 0.9414 |
| With Fine omf          | 23.79      | 1.1857 | 0.9483 |
| Fuzzy Refinement       |            |      |       |
| With Coarse omf        | 0.7130     | 3.2853 $\times 10^{-6}$ | 0.9991 |
| With Fine omf          | 0.7190     | 0.03277 | 0.9991 |
ship functions was nearly identical, as shown in Fig. 6, which is also reflected by the corresponding RE values.

6.5. Effect of noise

Noise was introduced to the projections, so that Eq. (1) takes the form:

\[ Y(I + \epsilon) = AX \]  

(6)

where \( I \) is the identity vector and \( \epsilon \) is a vector of relative noise with zero mean and a standard deviation that reflects the level of statistical variation in radiation counting. Radiation counting as indicated above is governed by Poisson statistics, which can be approximated by a normal distribution at the high count rates of CT systems [2]. The MATLAB function “randn” [26] was used to generate a random vector from a standard normal distribution (of zero mean and a unity standard deviation), which was then multiplied by the relative noise level to produce the vector \( \epsilon \) in Eq. (6). Typically the noise level in CT measurements has a relative standard deviation of about 5% [27].

When a 5% noise was added to data for the \( D_oD = 0.429 \) problem, the fine image reconstructed by TV-regularization reflected visually well the actual image (Fig. 7), and that was quantitatively confirmed by the RRMSE and SSIM metrics in Table 3. This is because of the method’s low susceptibility to noise [6]. The histogram for the fine image also resembled well that of the actual image (Fig. 8), but not exactly as also reflected by the value of the RE metric in Table 3. The unconstrained reconstruction of the coarse image was much more sensitive to noise as shown in the above mentioned figures and table. This made it difficult to produce sharp (narrow) membership functions using the histograms of the coarse image or its expanded version. The result is less accurate visual images, histograms and metric values when using the smaller number of output membership functions generated from the coarse image; see Figs. 7 and 8 and Table 3. As the same Figures and

![Image](https://via.placeholder.com/150)

Fig. 7. Actual Shepp-Logan phantom of size (128 x 128) versus fine (TV-regularization) and coarse (unconstrained) images, along with fused and refined images for \( D_oD = 0.429 \) with 5% noise in data, omf: output membership function.

| Image          | RRMSE (%) | RE   | SSIM  |
|----------------|-----------|------|-------|
| Fine           | 0.7553    | 0.1865 | 0.9991 |
| Coarse         | 13.181    | 33.793 | 0.9999 |
| Expanded Coarse| 51.34     | 34.952 | 0.4756 |
| Fuzzy Fusion   |           |       |       |
| With Coarse omf| 21.137    | 42.705 | 0.7251 |
| With Fine omf  | 9.7688    | 9.4366 | 0.9766 |
| Fuzzy Refinement|         |       |       |
| With Coarse omf| 5.4781    | 24.668 | 0.9085 |
| With Fine omf  | 0.9052    | 0.0660 | 0.9988 |

Table 3. Image metrics for the images shown in Figs. 7, omf: output membership function (\( D_oD = 0.429 \), 5% noise).
Table demonstrate, when the fine image was used to produce the output membership functions, the refinement process improved the fine image in terms of fidelity (RE), with a minor degradation in accuracy (RRMSE) and composition (SSME), as was the case in the error-free cases discussed above.

Unconstrained image reconstruction of the coarse image is susceptible to noise as the results of Figs. 7 and 8 and Table 3 show. However, the effect of noise can be constrained by regularization of image reconstruction. Using the TV-regularization method discussed above, the coarse image with 5% noise in data was reconstructed and the results are shown in Figs. 9 and 10 and Table 4. As expected the coarse images (Figs. 9(c) and 9(d)) and histograms (Figs. 10(c) and 10(d)) improved with regularized reconstruction, in comparison to the unconstrained counterparts (Figs. 7(c) and 7(d)) and histograms (Figs. 8(c) and 8(d), respectively). This improvement is also reflected in the metrics for coarse images in Table 4, when compared to those in Table 3. Notably, though the RRMSE value for the expanded image was not reduced much by regularization. On the other hand, the fidelity and composition metrics (RE and SSIM, respectively) improved considerably, indicating that regularization was able to preserve the image’s distribution and its composition even in the presence of image smoothing caused by expanding it. The improvement in the coarse image enabled the generation of distinct histogram indicators for the coarse image (Figs. 10(c) and 10(d)), which enabled the generation of sharper membership functions. This and the better quality of the coarse image, resulted in better fused images (Figs. 9(e), 9(f), 10(e) and 10(f)), as supported by the RE and SSIM metrics of Table 4, albeit at the expense of high RRMSE values. Once more, the refinement process produced better images (Figs. 9(g), 9(h), 10(g) and 10(h)) than those with fuzzy fusion as was in other results, and better than those with unconstrained reconstruction of the coarse image (Figs. 7(g), 7(h), 8(g) and 8(h)).

### 7. Conclusions

It is shown that a computed-tomography (CT) image reconstructed from incomplete data can be improved by fusing it with a coarser image reconstructed in an over-complete problem. The histograms of the two images were used to formulate the membership functions required in image fusion with a fuzzy inference system (FIS), taking advantage of rules based on the fact that the two image should posses the same distinct image attributes. However, the difference in the spatial resolution

### Table 4. Image metrics for the images shown in Figs. 9, omf: output membership function (DoD = 0.429, 5% noise).

| Image               | RRMSE (%) | RE       | SSIM   |
|---------------------|-----------|----------|--------|
| Fine                | 0.7552    | 0.1865   | 0.9991 |
| Coarse              | 0.1216    | 0.0123   | 0.9999 |
| Expanded Coarse     | 49.47     | 0.0129   | 0.8999 |
| Fuzzy Fusion        | 42.545    | 1.0354   | 0.9122 |
| With Coarse omf     | 25.852    | 1.627    | 0.9440 |
| With Fine omf       | 0.5616    | 2.098 × 10⁻² | 0.9994 |
| Fuzzy Refinement    | 0.5726    | 0.2846   | 0.9994 |

Fig. 8. Corresponding histograms of the images shown in Fig. 7 (DoD = 0.429, 5% noise).
of the fused images prevented effective application of the fuzzy fusion process. Therefore, a fuzzy refinement process was introduced that utilized input/output membership functions generated from either image, but not both at the same stage. This refinement process was shown to result in a higher quality image than the original image or that produced by traditional image fusion.

Unlike other methods for compensating for missing data in incomplete CT problems, which rely on external constraints or knowledge, see for example Chapter 14 in [2], the method introduced in this work relies on information present in the available data. This is done via the reconstruction of a coarse image for a complete problem, and using its features to refine the image from incomplete data, with the aid of a fuzzy inference system. It should be noted that the membership functions and rules of a FIS are best formulated for images with distinct features. Future work should examine whether this approach can be extended to images with continuous features.

**Declarations**

**Author contribution statement**

Varinder Malik: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Wrote the paper.

Esam M.A. Hussein: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper.

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**Data availability statement**

Data will be made available on request.

**Declaration of interests statement**

The authors declare no conflict of interest.

**Additional information**

No additional information is available for this paper.

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Fig. 10. Corresponding histograms of the images shown in Fig. 9.

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