STRIATED REGULARITY OF 2-D INHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES SYSTEM WITH VARIABLE VISCOSITY

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Abstract. In this paper, we investigate the global existence and uniqueness of strong solutions to 2D incompressible inhomogeneous Navier-Stokes equations with viscous coefficient depending on the density and with initial density being discontinuous across some smooth interface. Compared with the previous results for the inhomogeneous Navier-Stokes equations with constant viscosity, the main difficulty here lies in the fact that the $L^1$ in time Lipschitz estimate of the velocity field cannot be obtained by energy method (see [11, 20, 21] for instance). Motivated by the key idea of Chemin to solve 2-D vortex patch of ideal fluid ([6, 7]), namely, striated regularity can help to get the $L^\infty$ boundedness of the double Riesz transform, we derive the a priori $L^1$ in time Lipschitz estimate of the velocity field under the assumption that the viscous coefficient is close enough to a positive constant in the bounded function space. As an application, we shall prove the propagation of $H^3$ regularity of the interface between fluids with different densities.

Keywords: Inhomogeneous Navier-Stokes equations, Littlewood-Paley theory, Striated regularity.

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1. Introduction

The purpose of this paper is first to investigate the global existence and uniqueness of strong solutions to the two-dimensional incompressible inhomogeneous Navier-Stokes equations with viscous coefficient depending on the density and with initial density being discontinuous across some smooth interface. Then we are going to study the propagation of regularity for the interface between fluids with different densities. In general, inhomogeneous incompressible Navier-Stokes system reads

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \text{div}(2\mu(\rho)\mathcal{M}(u)) + \nabla \Pi &= 0, \\
\text{div} u &= 0,
\end{align*}
\]

where $\rho, u = (u^1, u^2)$ stand for the density and velocity of the fluid respectively, $\mathcal{M}(u) = \frac{1}{2}(\nabla u + \nabla^T u)$, the stress tensor, $\Pi$ is a scalar pressure function, which guarantees the divergence free condition of the velocity field, and the viscous coefficient $\mu(\rho)$ is a smooth non-decreasing positive function on $[0, \infty)$. Such a system describes a fluid which is obtained by mixing several immiscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance.

When one assumes that the viscous coefficient is a positive constant, there are tremendous literatures on this topic. One may check [1, 10, 11, 18, 19, 20, 21, 23] and the references therein. In general, Lions [22] proved the global existence of weak solutions to (1.1) with finite
energy in any space dimension. Yet the uniqueness and regularity of such weak solutions are big open questions even in two space dimensions (see pages 31-32 of [22]).

Let $R$ be the usual Riesz transform, $Q \overset{\text{def}}{=} \nabla(-\Delta)^{-1}\text{div}$, and $\mathbb{P} \overset{\text{def}}{=} I + Q$ be the Leray projection operator to the solenoidal vector field space. Then under the additional assumptions that

\begin{equation}
\|\mu(\rho_0) - 1\|_{L^\infty(T^2)} \leq \varepsilon \quad \text{and} \quad u_0 \in H^1(T^2),
\end{equation}

Desjardins [13] introduced the so-called “pseudo-energy method” and proved the following interesting result:

**Theorem 1.1.** Let $\rho_0 \in L^\infty(T^2)$, $u_0 \in H^1(T^2)$ with $\text{div} \ u_0 = 0$. Then there exists a positive constant $\varepsilon$ such that under the assumption of (1.2), Lions weak solutions ([22]) to (1.1) satisfy the following regularity properties for all $T > 0$:

- $u \in L^\infty([0,T]; H^1(T^2))$ and $\sqrt{\rho} u_t \in L^2(0,T;\mathbb{R}^2)$;
- $\rho$ and $\mu(\rho) \in L^\infty([0,T] \times T^2) \cap C([0,T]; L^p(T^2))$ for all $p \in [1,\infty]$;
- $\nabla(\Pi - \mathcal{R}_i \mathcal{R}_j (2\mu M_{ij}))$ and $\nabla(\mathbb{P} \otimes Q(2\mu M))_{ij} \in L^2(0,T;\mathbb{R}^2)$;
- $\Pi$ may be renormalized in such a way that for some universal constant $C > 0$,

\begin{equation}
\Pi \quad \text{and} \quad \nabla u \in L^2([0,T]; L^p(T^2)) \quad \text{for all} \quad p \in [4,p^*],
\end{equation}

where

\begin{equation}
\frac{1}{p^*} = 2C\|\mu(\rho_0) - 1\|_{L^\infty}.
\end{equation}

Moreover, if $\mu(\rho_0) \geq \mu$ and $\log(\mu(\rho_0)) \in W^{1,\tilde{r}}(T^2)$ for some $\tilde{r} > 2$, there exists some positive time $\tau$ so that $u \in L^\infty([0,\tau]; H^1(T^2))$ and $\mu(\rho) \in C([0,\tau]; W^{1,\tilde{r}}(T^2))$ for any $\tilde{r} < r$.

The solution provided by Theorem 1.1 still has neither uniqueness nor regularity. However, if the initial density belongs to some Besov spaces with positive index which guarantee that the initial density is a continuous function, Abidi and the second author ([2]) and Huang and the first author ([17]) can prove not only the uniqueness but also the regularity of the solution provided by Theorem 1.1 in the whole plane.

On the other hand, Lions proposed the following open question in [22]: suppose the initial density $\rho_0 = 1_D$ for some smooth domain $D$, Theorem 2.1 of [22] provides at least one global weak solution $(\rho, u)$ of (1.1) such that for all $t \geq 0$, $\rho(t) = 1_{D(t)}$ for some set $D(t)$ with $\text{vol}(D(t)) = \text{vol}(D)$. Then whether or not the regularity of $D$ is preserved by time evolution?

When one assumes that the viscous coefficient is a positive constant, Liao and the second author [20, 21] solved the case when the system (1.1) is supplemented with the initial density, $\rho_0(x) = \eta_1 \mathbb{1}_{\Omega_0} + \eta_2 \mathbb{1}_{\Omega_0^c}$, for some pair of positive constants $(\eta_1, \eta_2)$, and for any bounded, simply connected domain $\Omega_0$ with $W^{k+2,p}(\mathbb{R}^2)$ ($p \in [2,4]$) boundary regularity. Danchin and Zhang [12] and Gancedo and Garcia-Juarez [15] proved the propagation of $C^{k+\gamma}$ regularity of the interface for $k = 1$ or $k = 2$. Lately Danchin and Mucha [11] proved the propagation of $C^{1+\gamma}$ regularity of density patch which allows vacuum.

The goal of this paper is first to study the global well-posedness of the 2-D inhomogeneous Navier-Stokes equations (1.1) with initial density having striated regularity, which in particular allows initial density to be discontinuous across some smooth interface. We point out that the main difficulty to derive the $L^1$ in time Lipschitz estimate of the velocity field lies in the fact that Riesz transform does not map continuously from $L^\infty$ to $L^\infty$ space. Motivated by the key idea of Chemin to solve the two-dimensional vortex patch of ideal flow in [6, 7], namely, striated regularity can help to get the $L^\infty$ boundedness of the double Riesz
transform, we derive the a priori $L^1$ in time Lipschitz estimate of the velocity field under the assumption that the viscous coefficient is close enough to a positive constant in the bounded function space.

In order to do so, let us first recall the following definition from [8]:

**Definition 1.1** (Definition 3.3.1 of [8]). Let $X \defeq (X_\lambda)_{\lambda \in \Lambda}$ be a family of solenoidal vector fields on $\mathbb{R}^2$. We call $X$ to be a non-degenerate family of vector fields if there holds

$$I(X) \defeq \inf_{x \in \mathbb{R}^2} \sup_{\lambda \in \Lambda} |X_\lambda(x)| > 0. \tag{1.5}$$

Our first result states as follows:

**Theorem 1.2.** Let $p \in ]2, \infty[$, $X \defeq (X_\lambda)_{\lambda \in \Lambda}$ be a non-degenerate family of vector fields in $W^{1,p}(\mathbb{R}^2)$. Let $u_0 \in H^{-2\delta}(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ for some $\delta \in ]1/p, 1/2[$, and $\rho_0 \in L^\infty(\mathbb{R}^2)$ with $\rho_0 - 1 \in L^2(\mathbb{R}^2)$ and $\sup_{\lambda \in \Lambda} \|\partial X_\lambda \mu(\rho_0)\|_{L^\infty} \leq C_{\mu, X}$. We also assume that there exist positive constants $m, M$ such that

$$m \leq \rho_0(x) \leq M \text{ for any } x \in \mathbb{R}^2. \tag{1.6}$$

Then there exists a small enough positive constant $\varepsilon_0$ such that if

$$\|\mu(\rho_0) - 1\|_{L^\infty} \leq \varepsilon_0, \tag{1.7}$$

(1.1) has a unique global solution $u \in L^\infty(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; \text{Lip}(\mathbb{R}^2))$, which satisfies

$$\|\langle t \rangle^\delta u\|_{L^\infty_t(L^2)} + \|\langle t \rangle^\delta \nabla u\|_{L^2_t(L^2)} + \|\langle t \rangle^{1/2}\partial_t u\|_{L^\infty_t(L^2)} \tag{1.8}$$

$$+ \|\langle t \rangle^{1/2} D_t u\|_{L^2_t(L^2)} + \|\sqrt{t}\langle t \rangle^{1/2} D_t u\|_{L^\infty_t(L^2)} + \|\sqrt{t}\langle t \rangle^{1/2} \nabla D_t u\|_{L^2_t(L^2)} \leq C_0.$$

Moreover, there exists a non-degenerate family of vector fields $X(t) \defeq (X_\lambda(t))_{\lambda \in \Lambda}$ for any $t > 0$ so that $X_\lambda(0)$ coincides with $X_\lambda$, $X_\lambda(t) \in L^\infty(\mathbb{R}^+; W^{1,p}(\mathbb{R}^2))$ for each $\lambda \in \Lambda$ and

$$\sup_{\lambda \in \Lambda} \|\partial X_\lambda(t) \mu(\rho)\|_{L^\infty_t(L^\infty)} \leq C_{\mu, X}.$$

Here and in all that follows, we always denote $\langle t \rangle \defeq (1 + t^2)^{1/2}$, $\partial_X f \defeq X \cdot \nabla f$, $D_t \defeq \partial_t + u \cdot \nabla$ to be the material derivative, $\delta_-$ to be any positive constant smaller than $\delta$, and $C_0$ to be a positive constant depending on $M, \|\rho_0 - 1\|_{L^2}, \|u_0\|_{H^{-2\delta}}$, and $\|u_0\|_{H^1}$.

As an application of the above result, we can prove the propagation of regularity for the interface between fluids with different densities. More precisely, let $\Omega_0$ be a simply connected $H^3(\mathbb{R}^2)$, bounded domain. Let $f_0 \in H^3(\mathbb{R}^2)$ such that $\partial \Omega_0 = f_0^{-1}(\{0\})$ and $\nabla f_0$ does not vanish on $\partial \Omega_0$. Then we can parametrize $\partial \Omega_0$ as

$$\gamma_0 : S^1 \mapsto \partial \Omega_0 \text{ via } s \mapsto \gamma_0(s) \text{ with } \partial_s \gamma_0(s) = \nabla^\perp f_0(\gamma_0(s)). \tag{1.9}$$

Let us denote

$$\Omega_\varepsilon \defeq \{x \in \mathbb{R}^2 \mid \text{dist}(x, \partial \Omega_0) < \varepsilon \}, \quad \Omega_\varepsilon^+ \defeq \{x \in \mathbb{R}^2 \setminus \Omega_0 \mid \text{dist}(x, \partial \Omega_0) > \varepsilon \},$$

and

$$\Omega_\varepsilon^- \defeq \{x \in \Omega_0 \mid \text{dist}(x, \partial \Omega_0) > \varepsilon \}.$$
We now take $\varepsilon$ sufficiently small so that $|\nabla f_0(x)| \geq c_0$ for $x \in \Omega_{2\varepsilon}$. Let $\chi_\varepsilon^+$ and $\chi_\varepsilon^-$ be smooth cut-off functions so that

$$
\chi_\varepsilon^+(x) = \begin{cases} 0, & x \in \Omega_{\varepsilon} \cup \Omega_0, \\ 1, & x \in \Omega_{2\varepsilon}^c, \end{cases} \quad \text{and} \quad \chi_\varepsilon^-(x) = \begin{cases} 0, & x \in \mathbb{R}^2 \setminus \Omega_{\varepsilon}^c, \\ 1, & x \in \Omega_{2\varepsilon}^c. \end{cases}
$$

We denote

$$
X_0 \overset{\text{def}}{=} \nabla \perp f_0, \quad X_1 \overset{\text{def}}{=} \nabla \perp (x_1 \chi_\varepsilon^+), \quad \text{and} \quad X_2 \overset{\text{def}}{=} \nabla \perp (x_1 \chi_\varepsilon^-).
$$

For any $\eta_1, \eta_2 > 0$, we take the initial density $\rho_0$ and the initial velocity $u_0$ of (1.1) as

$$
\rho_0(x) = \eta_1 \mathbf{1}_{\Omega_0}(x) + \eta_2 \mathbf{1}_{\Omega_0^c}(x), \quad u_0 \in \dot{H}^{-2\delta}(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \quad \text{with} \quad \partial_{X_\lambda} u_0 \in \dot{H}^1(\mathbb{R}^2)
$$

for some $\delta \in ]1/3, 1/2[$ and $\lambda = 0, 1, 2$.

**Theorem 1.3.** Let the initial data $(\rho_0, u_0)$ be given by (1.9) and (1.11) for some pair of positive constants $(\eta_1, \eta_2)$. Then under the smallness (1.7), (1.1) has a unique global solution $(\rho, u)$ such that $\rho(t, x) = \eta_1 \mathbf{1}_{\Omega(t)}(x) + \eta_2 \mathbf{1}_{\Omega(t)^c}(x)$, with $\Omega(t)$ being a bounded, simply connected $H^3(\mathbb{R}^2)$ domain for any $t > 0$.

**Remark 1.1.**

- **Theorem 1.3 presents an example that Theorem 1.2 works for initial density being discontinuous across a smooth interface. Thus Theorem 1.2 improves the previous results in [2, 17];**
- **In fact, if $f_0 \in W^{2, p}(\mathbb{R}^2)$ for some $p \in ]2, \infty[$ and without the additional assumption that $\partial_{X^\lambda} u_0 \in H^1(\mathbb{R}^2)$, we can prove that the time-evolved interface still belongs to the class of $W^{2, p}(\mathbb{R}^2)$ for any $t > 0$;**
- **We may also propagate higher order regularities of the interface with more striated regularities on the initial data. For simplicity, we shall not pursue on this direction here.**
- **The idea to prove the propagation of boundary regularity of the interface comes from [6, 8]. One may check [21] for more references on this topic;**

Let us end this section with the notations we shall use in the following context:

**Notations:** Let $A, B$ be two operators, we denote $[A, B] \overset{\text{def}}{=} AB - BA$, the commutator between $A$ and $B$. For $a \lesssim b$, we mean that there is a uniform constant $C$, which may be different on different lines, such that $a \leq Cb$. We denote by $(a|b)$ (or $\int_{\mathbb{R}^2} a|b|d\mathbf{x}$) the $L^2(\mathbb{R}^2)$ inner product of $a$ and $b$. We shall also use Einstein summation notation for repeated indices. Let $a > 0$, we always denote $a_-$ to be any positive constant smaller than $a$.

### 2. Outline of the Proof

In this section, we shall sketch the proof of Theorems 1.2 and 1.3.

Section 3 is devoted to the proof of the following basic energy estimates for (1.1):

**Proposition 2.1.** Let $u$ be a smooth enough solution of (1.1) on $[0, T^*]$. Then under the smallness condition (1.7), for any $t < T^*$, we have

$$
\|u\|_{L^\infty_t(L^2)}^2 + \|\nabla u\|_{L^2_t(L^2)}^2 \leq C\|u_0\|_{L^2}^2,
$$

and

$$
\|\nabla u\|_{L^\infty_t(L^2)}^2 + \int_0^t \int_{\mathbb{R}^2} \left( |D_t u|^2 + |u_t|^2 + |\mathbb{P} \cdot \mathbf{div}(\mathbf{\mu}(\rho)\mathcal{M}(u))|^2 \\
+ |\mathbb{Q} \cdot \mathbf{div}(\mathbf{\mu}(\rho)\mathcal{M}(u)) - \nabla \Pi|^2 \right) dx \, dt' \leq C\|u_0\|_{L^2}^2 \exp \left( C\|u_0\|_{L^2}^2 \right)
$$

for some $\mathbb{P}$, $\mathbb{Q}$, $\Pi$. 

With Proposition 2.1, we deduce from Proposition 2.2 of [2] that

**Proposition 2.2.** If we assume in addition that \( \rho_0 - 1 \in L^2(\mathbb{R}^2) \), \( u_0 \in \dot{H}^{-2\delta}(\mathbb{R}^2) \) for \( \delta \in ]0, 1/2[ \), then for any \( t \in [0, T^*[ \), we have

\[
\begin{align*}
\| \langle t \rangle^{\delta} u \|_{L^2_t(L^2)} + \| \langle t \rangle^{\delta} \nabla u \|_{L^2_t(L^2)} + \| \langle t \rangle^{(\frac{1}{2}+\delta)} - \nabla u \|_{L^2_t(L^2)} \\
+ \| \langle t \rangle^{(\frac{1}{2}+\delta)} - u_t \|_{L^2_t(L^2)} + \| \langle t \rangle^{(\frac{1}{2}+\delta)} - D_t u \|_{L^2_t(L^2)} \leq C_0.
\end{align*}
\]

(2.3)

In Section 4, we shall prove the following time-weighted energy estimate for \( D_t u \).

**Proposition 2.3.** Under the assumptions of Proposition 2.2 and \( \mu(\rho_0) \geq \frac{3}{4} \), for any \( t < T^* \), we have

\[
\| \sqrt{t} \langle t \rangle^{\frac{1}{2}+\delta} - D_t u \|_{L^\infty_t(L^2)} + \| \sqrt{t} \langle t \rangle^{\frac{1}{2}+\delta} - \nabla D_t u \|_{L^2_t(L^2)} \leq C_0.
\]

(2.4)

We point out that the estimate (2.4) is completely new compared with the previous references [2, 17]. Part of the estimates in Propositions 2.1 and 2.2 were obtained in [2, 17]. But the proof here will be more concise. We also emphasize that the reason why there is no Lipschitz estimate for the velocity field in Theorem 1.1 is that Riesz transform can not map continuously from \( L^\infty(\mathbb{R}^2) \) to \( L^\infty(\mathbb{R}^2) \). To overcome this difficulty, we need to appeal to the idea from [6, 7]. Toward this, for any smooth vector field \( X_0 \) with \( \text{div} X_0 = 0 \), we define the vector field \( X(t) = (X^1(t), X^2(t)) \) for \( t > 0 \) by

\[
\begin{align*}
\partial_t X + u \cdot \nabla X = X \cdot \nabla u = \partial_X u, \\
X|_{t=0} = X_0.
\end{align*}
\]

(2.5)

It is easy to check that

\[
\begin{align*}
\partial_t \text{div} X + u \cdot \nabla \text{div} X = 0 \Rightarrow \text{div} X(t) = 0, \\
[\partial_X; D_t] = 0 \quad \text{with} \quad \partial_X \overset{\text{def}}{=} X \cdot \nabla \quad \text{and} \quad D_t \overset{\text{def}}{=} \partial_t + u \cdot \nabla.
\end{align*}
\]

(2.6)

Let us denote \( \rho_X \overset{\text{def}}{=} \partial_X \rho, u_X \overset{\text{def}}{=} \partial_X u \) and \( \Pi_X \overset{\text{def}}{=} \partial_X \Pi \). Then by applying the operator \( \partial_X \) to (1.1) and using (2.6), we write

\[
\begin{align*}
\begin{cases}
\partial_t \rho_X + u \cdot \nabla \rho_X = 0, \\
\rho \partial_t u_X + \rho u \cdot \nabla u_X + \text{div}(2\mu(\rho)M(u_X)) + \nabla \Pi_X = G, \\
\text{div} u_X = \partial_i X^k \partial_k u^i = \text{div}(u \cdot \nabla X), \\
(\rho_X, u_X)|_{t=0} = (\partial_X \rho_0, \partial_X u_0).
\end{cases}
\end{align*}
\]

(2.7)

where \( G = (G_1, G_2) \) and

\[
\begin{align*}
G_j = -\rho \chi D_t u^j + \partial_j (2\partial_X \mu(\rho)M_{ij}(u)) + \partial_j X \cdot \nabla \\
- \partial_i (\mu(u)(\partial_i X \cdot \nabla u^j + \partial_j X \cdot \nabla u^i)) - \partial_i X \cdot \nabla (2\mu(\rho)M_{ij}(u)).
\end{align*}
\]

(2.8)

Observing from (2.5) that the estimate of \( \| \nabla X(t) \|_{L^p} \) depends on \( \| \nabla u_X \|_{L^p} \). In Section 5, we will find that one difficulty to derive the estimate of \( \| \nabla u_X \|_{L^1_t(L^p)} \) lies in the fact that there is no control of \( \nabla \Pi \) with \( \mu(\rho) \) belonging to the bounded function space. Indeed by applying the space divergence operator to the momentum equation of (1.1), we obtain

\[
\Pi = (-\Delta)^{-1} \text{div}(\rho D_t u) - (-\Delta)^{-1} \text{div} \text{div}(2\mu(\rho)M(u)) \overset{\text{def}}{=} \Pi_1 + \Pi_2.
\]

(2.9)

The best estimate we can expect for \( \Pi_2 \) is that \( \Pi_2 \in L^\infty(\mathbb{R}^2) \). In order to do so, we need the following proposition:
Proposition 2.4. Let $p \in ]2, \infty[$ and $X = (X_{\lambda})_{\lambda \in \Lambda}$ be a non-degenerate family of vector fields in the sense of Definition 1.1. Then for any $s \in ]2/p, 1[$, one has
\begin{equation}
\|\nabla^2(-\Delta)^{-1}g\|_{L^s} \leq C_s\left\{ \|g\|_{L^p} + \|g\|_{L^\infty} + \frac{1}{I(X)} \sup_{\lambda \in \Lambda} \left( \|X_{\lambda}\|_{L^\infty} \|g\|_{L^p} \right)^{1-\frac{2}{p}} \times \left( \|\nabla X_{\lambda}\|_{L^p} \|g\|_{L^\infty} + \|\partial X_{\lambda}g\|_{L^p} \right)^{\frac{2}{p}} + \left( \|X_{\lambda}\|_{L^\infty} \|g\|_{L^p} \right)^{1-\frac{2}{p}} \left( \|\nabla X_{\lambda}\|_{L^p} \|g\|_{L^\infty} \right)^{\frac{2}{p}} \right\},
\end{equation}
with $I(X)$ being given by \eqref{eq:1.5}.

The proof of this proposition is motivated by Theorem 3.3.1 of [8], where the vector field belongs to some Hölder space. We shall present the detailed proof of Proposition 2.4 in Section 5. As an application of Proposition 2.4, we shall prove the following estimate of $\|\nabla u_{X_{\lambda}}\|_{L^p}$.

Corollary 2.1. Let $(\rho, u)$ be a smooth enough solution of \eqref{eq:1.1} on $[0, T^*)$ and $(X_{\lambda}(t))_{\lambda \in \Lambda}$ be a non-degenerate family of vector fields where $X_{\lambda}$ satisfies \eqref{eq:2.5} with initial data $X_{\lambda}$ for each $\lambda \in \Lambda$. For a given positive constant $\mathfrak{M}$ and for some $\varepsilon_1$ sufficiently small, we define
\begin{equation}
T^* \overset{\text{def}}{=} \sup_{\lambda \in \Lambda} \{ T < T^* : \sup_{\lambda \in \Lambda} \|\nabla X_{\lambda}\|_{L^\infty(L^p)} \leq \mathfrak{M} \text{ and } \|\mu(\rho_0) - 1\|_{L^\infty} \frac{\mathfrak{M}}{I(X)} \leq \varepsilon_1 \}.
\end{equation}

Then under the assumption of \eqref{eq:1.7} and
\begin{equation}
\sup_{\lambda \in \Lambda} \|\partial X_{\lambda} \mu(\rho_0)\|_{L^\infty} \leq C_{\mu,X},
\end{equation}
for any $t \in [0, T^*]$, we have
\begin{equation}
\sup_{\lambda \in \Lambda} \|\nabla u_{X_{\lambda}}(t)\|_{L^p} \leq C \left\{ \|\nabla u\|_{L^p} \left[ 1 + \sup_{\lambda \in \Lambda} \|X_{\lambda}(t)\|_{L^\infty}^{1-\frac{2}{p}} \right] \right. \right.
\end{equation}
\begin{equation}
\left. + \left[ \|\nabla u\|_{L^p} + \|D_t u\|_{L^2} \|\nabla D_t u\|_{L^2} \right] \sup_{\lambda \in \Lambda} \|X_{\lambda}(t)\|_{L^\infty} \right.
\end{equation}
\begin{equation}
\left. + \left( \|\nabla u\|_{L^p} + \|\nabla u\|_{L^\infty} + \|D_t u\|_{L^2} \right) \sup_{\lambda \in \Lambda} \|\nabla X_{\lambda}(t)\|_{L^p} \right\},
\end{equation}
for some positive constant $C$ depending on $C_{\mu,X}$.

In Section 6, we shall first present the estimates of $\|X_{\lambda}(t)\|_{L^\infty}$ and $\|\nabla X_{\lambda}(t)\|_{L^p}$.

Proposition 2.5. Let $X_0 \in L^\infty \cap \dot{W}^{1,p}$ and $u \in L^1_t(Lip(\mathbb{R}^2))$ be a smooth solenoidal vector field. Let $X(t, \cdot)$ be a smooth enough solution of \eqref{eq:2.5}. Then for any $r \in [1, \infty[$, we have
\begin{equation}
\|X_0\|_{L^r} \exp \left( -\|\nabla u\|_{L^1_t(L^\infty)} \right) \leq \|X(t)\|_{L^r} \leq \|X_0\|_{L^r} \exp \left( \|\nabla u\|_{L^1_t(L^\infty)} \right).
\end{equation}
Moreover, if $(X_{\lambda}(0))_{\lambda \in \Lambda}$ be a non-degenerate family of vector fields and $X_{\lambda}(t)$ be the corresponding solution of \eqref{eq:2.5} with initial data $X_{\lambda}(0)$. Then under the assumptions of Proposition 2.2 and Corollary 2.1, for $t \leq T^*$ given by \eqref{eq:2.11}, one has
\begin{equation}
\sup_{\lambda \in \Lambda} \|\nabla X_{\lambda}(t)\|_{L^p} \leq C_0 \left[ 1 + \sup_{\lambda \in \Lambda} \|\nabla X_{\lambda}(0)\|_{L^p} + \left[ 1 + \frac{1}{I(X(0))^{p-2}} \right] \sup_{\lambda \in \Lambda} \|X_{\lambda}(0)\|_{L^\infty} \right] V(t).
\end{equation}
Here and in what follows, we always denote $V(t) = \exp \left( C \|\nabla u\|_{L^1_t(L^\infty)} \right)$.

By virtue of Propositions 2.4 and 2.5, we prove the following key ingredient used in the proof of Theorem 1.2 in Section 6:
Proposition 2.6. Let \( u \) be a smooth enough solution of (1.1) on \([0, T^*]\). Let \( (X_\lambda(0))_{\lambda \in \Lambda} \) be a non-degenerate family of vector fields. Then under the assumptions of Theorem 1.2, there exist a positive constant \( \mathcal{R} \) and some small enough constant \( \varepsilon_0 \), which depend only on \( C_0, I(X(0)), \sup_{\lambda \in \Lambda} \|X_\lambda(0)\|_{L^\infty}, \sup_{\lambda \in \Lambda} \|\nabla X_\lambda(0)\|_{L^p} \) and \( C_{\mu, X} \) so that under the smallness condition (1.7), there holds

\[
\|\nabla u\|_{L^1_t(L^\infty)} \leq \mathcal{R} \quad \text{for any} \quad t \in [0, T^*].
\]

Now we are in a position to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** We denote \( \rho_{0, \eta} \overset{\text{def}}{=} \rho_0 * j_\eta, \ u_{0, \eta} = u_0 * j_\eta, \) and \( \mu_\eta = \mu * j_\eta, \) where \( j_\eta(|x|) = \eta^{-2} j(|x|/\eta) \) is the standard Friedrich’s mollifier. Then along the same line to the proof of Propositions 2.2, 2.3, 2.5 and 2.6, we can prove that the inequalities (2.1-2.4) and (2.14-2.16) hold for the solution, \( (\rho, u, \nabla \eta) \), of (1.1) with viscous coefficient \( \mu_\eta \) and with initial data \( (\rho_{0, \eta}, u_{0, \eta}) \) on \([0, T^*_\eta]\) for some maximal time of existence \( T^*_\eta \) provided that \( \eta \) is sufficiently small. Correspondingly, for any \( \lambda \in \Lambda, (2.5) \) with initial data \( X_\lambda \) has a unique solution \( X_\lambda(t) \) for any \( t < T^*_\eta \) which satisfies the estimates (2.14) and (2.15).

By virtue of (2.16) and Theorem 4.1 of [2], we conclude that \( T^*_\eta = \infty \). Then a standard compactness argument yields the existence part of Theorem 1.2. Moreover, by virtue of (2.14) and (2.15), the limit set \( \{X_\lambda(t)\}_{\lambda \in \Lambda} \) of \( \{X_{\lambda, \eta}(t)\}_{\lambda \in \Lambda} \) as \( \eta \) goes to 0 is a non-degenerate family of vector fields and \( X_\lambda(t) \in L^\infty(\mathbb{R}^2; W^{1,p}(\mathbb{R}^2)) \) for each \( \lambda \in \Lambda \). The uniqueness part of Theorem 1.2 follows along the same line to the proof of Theorem 1.2 of [2]. We skip the details here. \( \square \)

To prove Theorem 1.3, as in [20, 21], we need to propagate striated regularity of the convection velocity field.

Proposition 2.7. Let \( \delta \in ]0, 1/2[, \ p \in ]2, 2 + 2/\delta[, \) and \( X_0 \in L^\infty \cap \dot{W}^{1,p}(\mathbb{R}^2) \). Let \( (\rho, u, X) \) be a smooth enough solution of the coupled system (1.1) with (2.5) on \([0, T^*]\). Then under the assumptions of Proposition 2.2 and \( \|\partial_{X_0} \rho_0\|_{L^\infty} \leq C_{X_0, \rho_0} \), for any \( t < T^* \), we have

\[
\|\langle t \rangle^{\delta} u_X(t)\|_{L^2}^2 + \|\langle t \rangle^{\delta} \nabla u_X\|_{L^2_t(L^2)}^2 \leq C_1 \left( 1 + \|X\|_{L^\infty_t(L^\infty)}^2 + \|\nabla X\|_{L^\infty_t(L^p)}^2 \right) V(t),
\]

Here and in all that follows, we always denote \( C_1 \) to be a positive constant depending on \( C_0, \|X_0\|_{L^\infty \cap W^{1,p}} \) and \( C_{X_0, \rho_0} \).

Proposition 2.8. Let \( p \in ]2, 2 + 2/\delta[ \). Then under the assumptions of Proposition 2.7 and (1.7), for any \( t < T^* \), one has

\[
\|\sqrt{t} \langle t \rangle^{\delta - \frac{1}{2}} \nabla u_X(t)\|_{L^2}^2 + \|\sqrt{t} \langle t \rangle^{\delta} D_t u_X(t)\|_{L^2}^2 \leq C_1 \left( 1 + \|X\|_{L^\infty_t(L^\infty)}^2 + \|\nabla X\|_{L^\infty_t(L^p)}^2 \right) V(t).
\]

If we assume moreover that \( \partial_{X_0} u_0 \in \dot{H}^1(\mathbb{R}^2) \), we have

\[
\|\langle t \rangle^{\delta + \frac{1}{2}} \nabla u_X(t)\|_{L^2}^2 + \|\langle t \rangle^{\delta + \frac{1}{2}} D_t u_X(t)\|_{L^2}^2 \leq C_1 \left( 1 + \|\nabla \partial_{X_0} u_0\|_{L^2}^2 + \|X\|_{L^\infty_t(L^\infty)}^2 + \|\nabla X\|_{L^\infty_t(L^p)}^2 \right) V(t).
\]

Propositions 2.7 and 2.8 will be proved in Sections 7 and 8 respectively. Then we shall prove in Section 9 the following propositions:
Proposition 2.9. Let $\delta \in ]1/3,1/2[$. We assume that $\partial_{X_0}\mu(\rho_0) \in L^6(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\partial^2_{X_0}\mu(\rho_0) \in L^2(\mathbb{R}^2)$. Then under the assumptions of Proposition 2.8 and (1.7), for any $t < T^*$, one has
\begin{equation}
\|\nabla \partial X(t)\|_{L^2} \leq C_2\left(1 + \|X\|_{L_v^\infty(L^\infty)} + \|\nabla X\|_{L_v^\infty(L^1)} + \|\nabla X\|_{L_v^\infty(L^8)}\right)^3
\times \nu(t) \exp(C_2\|\Pi_2\|_{L_v^1(L^\infty)}).
\end{equation}
where $\Pi_2$ is given by (2.9) and $C_2$ is a positive constant depending on $C_1$, $\|\partial_{X_0}\mu(\rho_0)\|_{L^6 \cap L^\infty}$ and $\|\partial^2_{X_0}\mu(\rho_0)\|_{L^2}$.

Now we present the proof of Theorem 1.3.

Proof of Theorem 1.3. It is easy to observe that the family of vector fields $X_\lambda$, $\lambda = 0,1,2$, given by (1.10) is a non-degenerate family of vector fields. Moreover, we deduce from (1.10)-(1.11) that
\begin{equation}
\div X_\lambda = 0, \quad X_\lambda \in (H^2)^2 \quad \text{and} \quad \min \{\eta_1, \eta_2\} \leq \rho_0 \leq \max \{\eta_1, \eta_2\}, \quad \partial^2_{X_\lambda} \rho_0 = 0
\end{equation}
for $\lambda = 0, 1, 2$ and $\ell = 1, 2$. Then the initial conditions in Theorem 1.2 are fulfilled. By virtue of Theorem 1.2, under the smallness condition (1.7), the coupled system (1.1)-(2.5) has a unique global solution $(\rho, u, \nabla \Pi, X_\lambda)$ so that the bounds (1.8) and (2.16) are valid for any $t > 0$. Moreover, Proposition 2.5 ensure that for any $p \in ]2, \infty[$ and $\Lambda = \{0, 1, 2\}$,
\begin{equation}
\left(\sup_{\lambda \in \Lambda} \|X_{\lambda}(t)\|_{L^\infty} + \sup_{\lambda \in \Lambda} \|\nabla X_{\lambda}(t)\|_{L^p}\right) \leq \mathcal{L} \quad \text{and} \quad I(X(t)) \geq I(X(0)) \exp(-\mathfrak{M}).
\end{equation}
Here and in what follows, we always denote $\mathcal{L}$ to be a positive constant depending on $I(X(0))$, $C_0$, $\mathfrak{M}$, $\|X_\lambda\|_{L^\infty}$ and $\|\nabla X_\lambda\|_{L^p}$.

In view of (2.22), we deduce from Corollary 2.1 that for any $p \in ]3, \infty[$
\begin{equation}
\int_0^\infty \sup_{\lambda \in \Lambda} \|\nabla u_{X,\lambda}(t)\|_{L^p} dt \leq \mathcal{L}.
\end{equation}

On the other hand, since $\div u = 0$, we write
\[ \Pi_2 = -(-\Delta)^{-1} \div \div (2(\mu(\rho) - 1)\mathcal{M}(u)) \quad \text{and} \quad \mathfrak{B}(\rho) \overset{\text{def}}{=} \|\mu(\rho) - 1\|_{L^\infty}, \]
so that for any $s \in ]2/p, 1[$, applying Proposition 2.4 gives rise to
\begin{equation}
\|\Pi_2\|_{L^s} \leq C_s \left\{ \mathfrak{B}(\rho)\left(\|\nabla u\|_{L^p} + \|\nabla u\|_{L^\infty}\right) + \frac{1}{I(X)} \sup_{\lambda \in \Lambda} \left(\mathfrak{B}(\rho)\|X_\lambda\|_{L^\infty}\|\nabla u\|_{L^p}\right)^{1-\frac{2}{p}} \times \left(\|\partial_{X_\lambda}\mu(\rho)\|_{L^\infty}\|\nabla u\|_{L^p} + \mathfrak{B}(\rho)\left(\|\nabla X_\lambda\|_{L^p}\|\nabla u\|_{L^\infty} + \|\nabla u_{X_\lambda}\|_{L^p}\right)\right)^{\frac{2}{p}}\right. \right. \\
+ \mathfrak{B}(\rho)^{1-\frac{2}{p}}\left.\left(\|\nabla X_\lambda\|_{L^p}\|\nabla u\|_{L^\infty}\right)^{\frac{2}{p}}\right\}. \quad \text{(2.24)}
\end{equation}

Along with (1.8), (2.22) and (2.23), for $p \in ]3, \infty[$ in (2.24), we infer
\[ \int_0^\infty \|\Pi_2(t')\|_{L^\infty} dt' \leq \mathcal{L}. \]
Then we deduce from Proposition 2.9 that
\begin{equation}
\sup_{\lambda \in \Lambda} \|\nabla \partial_{X_\lambda}(t)X_\lambda(t)\|_{L^2} \leq \mathcal{L} \quad \forall \ t \in [0, \infty[.
\end{equation}
Let us denote by $\psi(t, \cdot)$ the flow associated with the vector field $u$, that is
\[
\begin{cases}
\frac{d}{dt} \psi(t, x) = u(t, \psi(t, x)), \\
\psi(0, x) = x.
\end{cases}
\]
Then it follows from the standard theory of transport equation and (2.16) that
\[
\|\nabla \psi(t, \cdot) - \text{Id}\|_{L^\infty} \leq \mathcal{N} \exp (\mathcal{N}) \quad \forall \ t \in [0, \infty[.
\]
(2.26)

Let $\Omega(t) = \psi(t, \Omega_0)$, with the boundary parametrization $\psi(t, \gamma_0(\cdot)) : \mathbb{S}^1 \mapsto \partial \Omega(t)$. (2.26) ensures that $\partial \Omega(t)$ is of $W^{1, \infty}$ class. Furthermore, we deduce from the transport equation of (1.1) that
\[
\rho(t, x) = \eta_1 \mathbf{1}_{\Omega(t)}(x) + \eta_2 \mathbf{1}_{\Omega(t)'}(x).
\]
Next we are going to prove that $\partial \Omega(t)$ belongs to the class of $H^3$. Notice that the boundary $\partial \Omega(t)$ is the level surface of $f(t, \cdot)$ with $f$ being transported by the $\psi$-flow:
\[
\begin{cases}
\partial_t f + u \cdot \nabla f = 0, \\
f(0, x) = f_0(x).
\end{cases}
\]
So that the vector field $X_0(t, \cdot) \overset{\text{def}}{=} \nabla f(t, \cdot)$ verifies the equation (2.5) with initial data $X_0 = \nabla f_0$ and
\[
X_0(t, \psi(t, x)) = X_0(x) \cdot \nabla \psi(t, x).
\]
Then in view of (1.9), we find
\[
\partial_s X_0(t, \psi(t, \gamma_0(s))) = \partial_s \psi(t, \gamma_0(s)) \cdot \nabla X_0(t, \psi(t, \gamma_0(s)))
\]
\[
= (X_0(\gamma_0(s)) \cdot \nabla \psi(t, \gamma_0(s))) \cdot \nabla X_0(t, \psi(t, \gamma_0(s)))
\]
\[
= (X_0 \cdot \nabla X_0)(t, \psi(t, \gamma_0(s))),
\]
which together with (2.25) ensures that
\[
X_0(t, \psi(t, \gamma_0(s))) \in L^\infty(\mathbb{R}^+; \dot{H}^2(\mathbb{S}^1)).
\]
(2.28)

As a result, it comes out
\[
\partial_s (\psi(t, \gamma_0(s))) = X_0(\gamma_0(s)) \cdot \nabla \psi(t, \gamma_0(s)) = X_0(t, \psi(t, \gamma_0(s))) \in L^\infty(\mathbb{R}^+; \dot{H}^2(\mathbb{S}^1)).
\]
(2.29)
Hence $\partial \Omega(t)$ belongs to the class of $H^3$ for any $t > 0$. This completes the proof of Theorem 1.3.

Finally in the Appendix A, we shall present a commutative estimate, which will be used frequently in the whole context; while in Appendix B, we shall generalize Proposition 2.4 for elliptic equation of divergence form with bounded coefficients which may have a small gap across a surface.

3. The basic energy estimate

The goal of this section is to prove Propositions 2.1 and 2.2. Toward this and also for the $\dot{H}^1$ estimate of the tangential derivative of the velocity field, we first present the energy estimate for the linearized equation of (1.1).
Lemma 3.1. Let $u$ be a smooth solenoidal vector field and $D_t \overset{\text{def}}{=} \partial_t + u \cdot \nabla$. Let $(\rho, v, \nabla \pi)$ be a smooth enough solution to the following system on $[0, T]$:

$$
\begin{align*}
D_t \rho &= 0 \quad (t, x) \in [0, T] \times \mathbb{R}^2, \\
\rho D_t v - \text{div} \left(2\mu(\rho) \mathcal{M}(v) \right) + \nabla \pi &= F, \\
\text{div} v &= h, \\
(\rho, v)|_{t=0} &= (\rho_0, v_0).
\end{align*}
$$

(3.1)

Then under the assumption of (1.6) and (1.7), for any $t \leq T$, we have

$$
\frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho)|\mathcal{M}(v)|^2 \, dx + \int_{\mathbb{R}^2} |\sqrt{\rho} D_t v|^2 \, dx = \int_{\mathbb{R}^2} (F - \nabla \pi)|D_t v \, dx
- \int_{\mathbb{R}^2} \mu(\rho) \left((\partial_1 v^1)^2 \partial_1 u^1 + (\partial_1 v^2 + \partial_2 v^1)(\partial_1 u^1 \partial_1 v^2 + \partial_2 u^1 \partial_2 v^1) \right)
+ (\partial_2 v^2)^2 \partial_2 u^2 + (\partial_1 u^2 + \partial_2 u^1)(\partial_1 v^1 \partial_2 v^1 + \partial_1 v^2 \partial_2 v^2) \right) \, dx.
$$

(3.2)

Moreover, for any $p \in [2, \infty]$, we have

$$
\|\nabla v\|_{L^p} \leq C \left(\|h\|_{L^p} + \|\nabla v\|_{L^2}^2 \|\rho D_t v - F\|_{L^2}^{1-\frac{2}{p}} \right).
$$

(3.3)

Proof. We first deduce from (1.7) and the transport equation of (3.1) that

$$
m \leq \rho(t, \cdot) \leq M \quad \text{and} \quad \mu(\rho(t, \cdot)) = 1 - \mu(\rho_0(\cdot)) - 1.
$$

(3.4)

While by taking $L^2$ inner product of the momentum equation of (3.1) with $D_t v$ and then using integration by parts, we obtain

$$
\int_{\mathbb{R}^2} \rho |D_t v|^2 \, dx + 2 \int_{\mathbb{R}^2} \mu(\rho) \mathcal{M}(v) : \mathcal{M}(D_t v) \, dx = \int_{\mathbb{R}^2} (F - \nabla \pi)|D_t v \, dx.
$$

Note that $D_t \mu(0) = 0$, we have

$$
2 \int_{\mathbb{R}^2} \mu(\rho) \mathcal{M}(v) : \mathcal{M}(D_t v) \, dx = \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho)|\mathcal{M}(u)|^2 \, dx + 2 \int_{\mathbb{R}^2} \mu(\rho) \mathcal{M}_{ij}(v) \partial_i u \cdot \nabla v^j \, dx,
$$

which together with the fact that

$$
\int_{\mathbb{R}^2} \mu(\rho) \mathcal{M}_{ij}(v) \partial_i u \cdot \nabla v^j \, dx = \int_{\mathbb{R}^2} \mu(\rho) \left( \partial_1 u^1 \left((\partial_1 v^1)^2 + \frac{1}{2}(\partial_1 v^2 + \partial_2 v^1)\partial_1 v^2 \right)
+ \partial_2 u^2 \left((\partial_2 v^2)^2 + \frac{1}{2}(\partial_1 v^2 + \partial_2 v^1)\partial_2 v^2 \right)
+ \frac{1}{2} \left(\partial_1 v^1 \partial_2 v^1 + \partial_1 v^2 \partial_2 v^2 \right) \right) \, dx,
$$

gives rise to (3.2).

Due to $\text{div} v = h$, let $w \overset{\text{def}}{=} v + \nabla(-\Delta)^{-1} h$, we have $\text{div} w = 0$. Then it is easy to Observe that

$$
-\Delta w = \text{div} \left( (\mu(\rho) - 1) \mathcal{M}(w) \right) - \text{div}(\mu(\rho) \mathcal{M}(w)).
$$

By taking the Leray projection operator, $\mathbb{P} \overset{\text{def}}{=} I + \nabla(-\Delta)^{-1} \text{div}$, to the above equation, we obtain

$$
\nabla w = \nabla(-\Delta)^{-1} \mathbb{P} \text{div} \left( (\mu(\rho) - 1) \mathcal{M}(w) \right) - \nabla(-\Delta)^{-1} \mathbb{P} \text{div}(\mu(\rho) \mathcal{M}(w)),
$$

(3.5)
so that by virtue of (3.4), we get, by applying Gagliardo-Nirenberg inequality in 2-D, that for any $p \in [2, \infty]$, 
\[
\|\nabla w\|_{L^p} \leq C \left( (\|\mu(\rho) - 1\| \mathcal{M}(w) \|_{L^p} + \|\nabla(-\Delta)^{-1}\mathbb{P} \text{div}(\mu(\rho) \mathcal{M}(v + \nabla(-\Delta)^{-1}h)) \|_{L^p} \right) 
\]
(3.6) 
\[\leq C \left( \|\mu(\rho_0) - 1\|_{L^\infty} \|\nabla w\|_{L^p} + \|h\|_{L^p} + \|\nabla v\|_{L^2}^\frac{2}{p} \|\mathbb{P} \text{div}(\mu(\rho) \mathcal{M}(v))\|_{L^2}^{\frac{1-\frac{2}{p}}{2}} \right),
\]
which together with (1.7) ensures that 
\[
\|\nabla w\|_{L^p} \leq C \left( \|h\|_{L^p} + \|\nabla v\|_{L^2}^\frac{2}{p} \|\mathbb{P} \text{div}(\mu(\rho) \mathcal{M}(v))\|_{L^2}^{\frac{1-\frac{2}{p}}{2}} \right).
\]
We thus deduce from the momentum equation of (3.1) that 
\[
\|\nabla v\|_{L^p} \leq \|\nabla w\|_{L^p} + C \|h\|_{L^p} \leq C \left( \|\nabla v\|_{L^2}^\frac{2}{p} \|\mathbb{P} \text{div}(\mu(\rho) \mathcal{M}(v))\|_{L^2}^{\frac{1-\frac{2}{p}}{2}} \right) + C \|h\|_{L^p}.
\]
This proves (3.3) and the lemma. \qed

**Remark 3.1.** Compared with the “pseudo-energy” method introduced by Desjardins [13] (see also [2, 3]), here we take the $L^2$ inner product of the momentum equation (3.1) with $D_t v$ instead of $v_t$, which is simpler due to $D_t \mu(\rho) = 0$.

Let us now outline the proof of Propositions 2.1 and 2.2.

**Proof of Proposition 2.1.** We first get, by taking the $L^2$ inner product of the momentum equation of (1.1) with $u$, that
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\sqrt{\rho} u|^2 \, dx + 2 \int_{\mathbb{R}^2} \mu(\rho) |\mathcal{M}(u)|^2 \, dx = 0.
\]
Integrating the above equality over $[0, t]$ and using (3.4) yields (2.1).

While due to $\text{div} u = 0$, we get, by applying Lemma 3.1, that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) |\mathcal{M}(u)|^2 \, dx + \int_{\mathbb{R}^2} |\sqrt{\rho} D_t u|^2 \, dx = - \int_{\mathbb{R}^2} \nabla \Pi \cdot D_t u \, dx \leq \int_{\mathbb{R}^2} \Pi \cdot \partial_i u \cdot \partial_j u \, dx.
\]
(3.7)

We deduce from (2.9) that 
\[
\left| \int_{\mathbb{R}^2} \Pi \partial_i u \partial_j u \, dx \right| \lesssim \|\nabla u\|_{L^2} \|\nabla u\|_{L^4}^2 + \|(-\Delta)^{-1} \text{div}(\rho D_t u)\|_{BMO} \|\nabla u_i \cdot \partial_i u\|_{H^1},
\]
where $\|f\|_{H^1}$ denotes the Hardy norm of $f$. Whereas as $\text{div} u = 0$, it follows from Theorem II.1 of [9] that
\[
\|\nabla u_i \cdot \partial_i u\|_{H^1} \lesssim \|\nabla u\|_{L^2}^2,
\]
from which, and the fact that: $\|f\|_{BMO(\mathbb{R}^2)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^2)}$, we infer
\[
\left| \int_{\mathbb{R}^2} \Pi \partial_i u \partial_j u \, dx \right| \leq C \|\nabla u\|_{L^2} \left( \|\nabla u\|_{L^4}^2 + \|\rho D_t u\|_{L^2} \|\nabla u\|_{L^2} \right).
\]
While it follow from (3.3) that 
\[
\|\nabla u\|_{L^4} \leq C \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\sqrt{\rho} D_t u\|_{L^2}^{\frac{1}{2}},
\]
(3.8) 
\[\|\nabla u\|_{L^2} \leq C \|\nabla u\|_{L^2} \|\sqrt{\rho} D_t u\|_{L^2} \|\nabla u\|_{L^2}^2.
\]
As a result, it comes out 
\[
\left| \int_{\mathbb{R}^2} \Pi \partial_i u \partial_j u \, dx \right| \leq C \|\sqrt{\rho} D_t u\|_{L^2} \|\nabla u\|_{L^2}^2.
\]
(3.9)
Inserting the above estimate into (3.7) gives rise to

$$
\frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho)|\mathcal{M}(u)|^2 \, dx + \int_{\mathbb{R}^2} |\sqrt{\rho}D_t u|^2 \, dx \leq C\|\nabla u\|_{L^2}^4.
$$

Due to (1.7) and (3.4), we have

$$
\|\nabla u\|_{L^2}^2 \leq C \int_{\mathbb{R}^2} |\mathcal{M}(u)|^2 \, dx \leq C \int_{\mathbb{R}^2} \mu(\rho)|\mathcal{M}(u)|^2 \, dx,
$$

so that applying Gronwall’s inequality to (3.10) yields

$$
\|\nabla u\|_{L^2(L^t)}^2 + \int_0^t \int_{\mathbb{R}^2} |\sqrt{\rho}D_t u|^2 \, dx \, dt' \leq C\|\nabla u_0\|_{L^2}^2 \exp \left( C\|\nabla u\|_{L^2(L^t)}^2 \right)
$$

which together with (2.1) and (3.8) implies that

$$
\|u_t\|_{L^2(L^t)} \leq \|D_t u\|_{L^2(L^t)} + \|u_t\|_{L^\infty(L^t)} \|\nabla u\|_{L^2(L^t)}
$$

$$
\leq \|D_t u\|_{L^2(L^t)} + C\|u_t\|_{L^\infty(L^t)}^\frac{1}{2} \|\nabla u\|_{L^2(L^t)}^\frac{1}{2} \|\nabla u\|_{L^2(L^t)}^\frac{1}{2} \|D_t u\|_{L^2(L^t)}^\frac{1}{2}
$$

$$
\leq C(1 + \|u_0\|_{L^2}) \|\nabla u_0\|_{L^2} \exp \left( C\|u_0\|_{L^2}^2 \right).
$$

On the other hand, by applying the Leray projection operator, $\mathbb{P}$, and the operator $\mathbb{Q}$ on the momentum equation of (3.1), we write

$$
\mathbb{P} \text{ div}(\mu(\rho)\mathcal{M}(u)) = \mathbb{P}(\rho D_t u),
$$

$$
\mathbb{Q} \text{ div}(\mu(\rho)\mathcal{M}(u)) - \nabla \Pi = \mathbb{Q}(\rho D_t u),
$$

from which and (11), we infer

$$
\int_0^t \int_{\mathbb{R}^2} \left( |\mathbb{P} \text{ div}(\mu(\rho)\mathcal{M}(u))|^2 + |\mathbb{Q} \text{ div}(\mu(\rho)\mathcal{M}(u)) - \nabla \Pi|^2 \right) \, dx \, dt' \leq C\|\nabla u_0\|_{L^2}^2 \exp \left( C\|u_0\|_{L^2}^2 \right),
$$

which together with (11) and (12) leads to (2.2). This completes the proof of Proposition 2.1. \qed

**Outline of the proof of Proposition 2.2.** With Proposition 2.1, we deduce from Proposition 2.2 of [2] that (2.3) holds except the estimate of $\|\langle t \rangle^{\frac{1}{2} + \delta} D_t u\|_{L^2(L^t)}$. Indeed, it is easy to observe from (3.8) that

$$
\|D_t u\|_{L^2} \leq \|u_t\|_{L^2} + \|u_t\|_{L^4} \|\nabla u\|_{L^4}
$$

$$
\leq \|u_t\|_{L^2} + C\|u_t\|_{L^2}^\frac{1}{2} \|\nabla u\|_{L^2}^\frac{1}{2} \|D_t u\|_{L^2}^\frac{1}{2},
$$

which implies

$$
\|D_t u\|_{L^2} \leq C(\|u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 \|\nabla u\|_{L^2}^2).
$$

Then applying Proposition 2.2 gives rise to

$$
\|\langle t \rangle^{\frac{1}{2} + \delta} D_t u\|_{L^2(L^t)}^2 \leq C \left( \|\langle t \rangle^{\frac{1}{2} + \delta} u_t\|_{L^2(L^t)}^2 + C_0 \int_0^t \langle t' \rangle^{-(1+4\delta)} \, dt' \right) \leq C_0
$$

This concludes the proof of proposition 2.2. \qed
4. THE ENERGY ESTIMATE OF $D_t u$

The goal of this section is to prove Proposition 2.3.

**Lemma 4.1.** Let $(\rho, u)$ be a smooth enough solution of (1.1) on $[0, T^*[$ with $\mu(\rho_0) \geq \frac{3}{4}$.
Then for any $t < T^*$, we have

\[
\frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{R}^2} \rho |D_t u|^2 \, dx \right) + \int_{\mathbb{R}^2} \Pi \partial_t u^2 \partial_j u^i \, dx + \|\nabla D_t u\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2 \|\sqrt{\rho} D_t u\|_{L^2}^2.
\]

**Proof.** We first get, by applying the operator $D_t$ to the momentum equation of (1.1), that

\[
\rho D_t^2 u^j - 2 \partial_i (\mu(\rho) \mathcal{M}_{ij}(D_t u)) + \partial_j D_t \Pi
\]

\[
= 2[D_t; \partial_i](\mu(\rho) \mathcal{M}_{ij}(u)) - \partial_i (\mu(\rho)(\partial_i u \cdot \nabla u^j + \partial_j u \cdot \nabla u^i)) + [\partial_j; D_t] \Pi.
\]

Taking $L^2$ inner product of the above equation with $D_t u$, we write

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \rho |D_t u|^2 \, dx + 2 \int_{\mathbb{R}^2} \mu(\rho) \mathcal{M}(D_t u)^2 \, dx
\]

\[
= - (\nabla D_t \Pi |D_t u) + 2([D_t; \partial_i](\mu(\rho) \mathcal{M}_{ij}(u)) |D_t u^j)
\]

\[
+ (\mu(\rho)(\partial_i u \cdot \nabla u^j + \partial_j u \cdot \nabla u^i) |\partial_i D_t u^j) + ([\partial_j; D_t] \Pi |D_t u^j).
\]

We now handle term by term above.

- The estimate of $(\nabla D_t \Pi |D_t u)$

  Due to $\text{div} \, u = 0$, by using integration by parts, we find

  \[
  -(\nabla D_t \Pi |D_t u) = \int_{\mathbb{R}^2} D_t \Pi \text{div} \, D_t u \, dx = \int_{\mathbb{R}^2} D_t \Pi |\partial_t u^j \partial_j u^i \, dx
  \]

  \[
  = \frac{d}{dt} \int_{\mathbb{R}^2} \Pi \partial_t u^j \partial_j u^i \, dx - 2 \int_{\mathbb{R}^2} \Pi D_t \partial_t u^j \partial_j u^i \, dx
  \]

  \[
  = \frac{d}{dt} \int_{\mathbb{R}^2} \Pi \partial_t u^j \partial_j u^i \, dx - 2 \int_{\mathbb{R}^2} \Pi D_t \partial_t u^j \partial_j u^i + \int_{\mathbb{R}^2} \Pi \partial_t u \cdot \nabla u^j \partial_j u^i \, dx.
  \]

In view of (2.9) and (3.8), we deduce from a similar derivation of (3.9) that

\[
\left| \int_{\mathbb{R}^2} \Pi \partial_t u^j \partial_j u^i \, dx \right| \leq C \left( \|\nabla u\|_{L^2}^2 \|\nabla D_t u\|_{L^2}
\]

\[
+ \|(-\Delta)^{-1} \text{div}(\rho D_t u)\|_{BMO} \|\nabla D_t u^j \cdot \partial_j u\|_{\mathcal{F}^1} \right)
\]

\[
\leq C \left( \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2} \|\sqrt{\rho} D_t u\|_{L^2} \right) \|\nabla D_t u\|_{L^2}
\]

\[
\leq C \|\nabla u\|_{L^2}^2 \|\sqrt{\rho} D_t u\|_{L^2} + \frac{1}{8} \|\nabla D_t u\|_{L^2}^2.
\]

Observing that due to $\text{div} \, u = 0$, $\partial_i u \cdot \nabla u^j \partial_j u^i = 0$. Then by inserting the above estimate into (4.4), we obtain

\[
\left| (\nabla D_t \Pi |D_t u) + \frac{d}{dt} \int_{\mathbb{R}^2} \Pi \partial_t u^j \partial_j u^i \, dx \right| \leq \frac{1}{8} \|\nabla D_t u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\sqrt{\rho} D_t u\|_{L^2}^2.
\]

- The estimate of $([D_t; \partial_i](\mu(\rho) \mathcal{M}_{ij}(u)) |D_t u^j)$

  Note that $[D_t; \partial_i] f = -\partial_i u \cdot \nabla f$, we write

  \[
  ([D_t; \partial_i](\mu(\rho) \mathcal{M}_{ij}(u)) |D_t u^j) = -(\partial_i u \cdot \nabla (\mu(\rho) \mathcal{M}_{ij}(u)) |D_t u^j)
  \]

  \[
  = (\mu(\rho) \mathcal{M}_{ij}(u) |\partial_i u \cdot \nabla D_t u^j),
  \]
which together with (3.8) implies that
\[
\left| \left( D_t; \frac{\partial_t}{\partial_t} (\mu(\rho)M_{ij}(u)) D_t u^i \right) \right| \leq C \frac{\| \nabla u \|_{L^2}^2}{\| \nabla u \|_{L^2}} \| \nabla D_t u \| _{L^2} \\
\leq C \frac{\| \nabla u \|_{L^2}^2}{\| \nabla u \|_{L^2}} \| \sqrt{\rho} D_t u \|_{L^2}^2 + \frac{1}{8} \| \nabla D_t u \|_{L^2}.
\]

- The estimate of \((\mu(\rho)(\partial_t u \cdot \nabla u^j + \partial_j u \cdot \nabla u^i) | \partial_t D_t u^i)\)

We deduce from (3.8) that
\[
\left| (\mu(\rho)(\partial_t u \cdot \nabla u^j + \partial_j u \cdot \nabla u^i) | \partial_t D_t u^i) \right| \leq C \frac{\| \nabla u \|_{L^2}^2}{\| \nabla u \|_{L^2}} \| \nabla D_t u \|_{L^2} \\
\leq C \frac{\| \nabla u \|_{L^2}^2}{\| \nabla u \|_{L^2}} \| \sqrt{\rho} D_t u \|_{L^2}^2 + \frac{1}{8} \| \nabla D_t u \|_{L^2}.
\]

- The estimate of \((\partial_j; D_t) [D_t u^j] \)

Thanks to (2.9), we write
\[
(\partial_j; D_t) [D_t u^j] = \int_{\mathbb{R}^2} \partial_j u \cdot \nabla D_t u^j \, dx \\
= \int_{\mathbb{R}^2} \partial_j u \cdot \nabla(-\Delta)^{-1} \div(\rho D_t u) D_t u^j \, dx \\
+ \int_{\mathbb{R}^2} (-\Delta)^{-1} \div(2\mu(\rho)M(u)) | \partial_j u \cdot \nabla D_t u^j \, dx,
\]
from which and (3.8), we infer
\[
\left| (\partial_j; D_t) [D_t u^j] \right| \leq C \left( \| \nabla u \|_{L^2} \| D_t u \| _{L^2}^2 + \| \nabla u \| _{L^2} \| \nabla D_t u \| _{L^2} \right) \\
\leq C \frac{\| \nabla u \|_{L^2}^2}{\| \nabla u \|_{L^2}} \| \sqrt{\rho} D_t u \|_{L^2}^2 + \frac{1}{8} \| \nabla D_t u \|_{L^2}.
\]

On the other hand, since \(\mu(\rho_0) \geq \frac{3}{4}\), we get, by using (3.4) and integration by parts, that
\[
2 \int_{\mathbb{R}^2} \mu(\rho)|M(D_t u)|^2 \, dx \geq \frac{3}{2} \int_{\mathbb{R}^2} |M(D_t u)|^2 \, dx \\
= \frac{3}{2} \int_{\mathbb{R}^2} \left( |\nabla D_t u|^2 + \partial_t D_t u^i \partial_j D_t u^j \right) \, dx \\
= \frac{3}{2} \int_{\mathbb{R}^2} \left( |\nabla D_t u|^2 + (\div(D_t u))^2 \right) \, dx \geq \frac{3}{2} \int_{\mathbb{R}^2} |\nabla D_t u|^2 \, dx.
\]

Substituting the above estimates into (4.3) leads to (4.1). \(\square\)

**Proof of Proposition 2.3.** By multiplying (4.1) by \(t - t_0\) and then integrating the resulting inequality over \([t_0, t]\), we obtain

\[
(t - t_0) \int_{\mathbb{R}^2} \rho|D_t u|^2(t) \, dx \leq 2(t - t_0) \left| \int_{\mathbb{R}^2} \Pi \partial_t u^j \partial_j u^i \, dx \right| + \int_{t_0}^t \int_{\mathbb{R}^2} \rho|D_t u|^2 \, dx \, dt' \\
+ 2 \left| \int_{t_0}^t \int_{\mathbb{R}^2} \Pi \partial_t u^j \partial_j u^i \, dx \, dt' \right| + C \int_{t_0}^t (t' - t_0) \| \nabla u \|_{L^2}^2 \| \sqrt{\rho} D_t u \|_{L^2}^2 \, dt'. \tag{4.5}
\]

In what follows, we take \(t_0 = \frac{t}{2}\) in the above inequality.
By applying (3.9) and Proposition 2.2, we get
\[
\|\sqrt{t}\langle t\rangle^{\frac{1}{2}+\delta} - \nabla D_t u\|_{L^2(\mathbb{R}^2)}^2 \leq 2(1+\delta) \int_0^t \langle t'\rangle^{(1+2\delta)_-} \int_{\mathbb{R}^2} \rho |D_t u|^2 \, dx \, dt' + 4(1+\delta) \int_0^t \langle t'\rangle^{(1+2\delta)_-} \int_{\mathbb{R}^2} \Pi \partial_i u \partial_j u^i \, dx \, dt' + C \int_0^t \langle t'\rangle^{(1+2\delta)_-} \|\sqrt{t}\|_{L^2(\mathbb{R}^2)}^2 \|\sqrt{t}\|_{L^2(\mathbb{R}^2)}^2 \, dt'.
\]

On the other hand, by multiplying (4.1) by \(t\langle t\rangle^{(1+2\delta)_-}\) and then integrating the resulting inequality over \([0, t]\), we find
\[
\int_0^t \langle t'\rangle^{(1+2\delta)_-} \int_{\mathbb{R}^2} \rho |D_t u|^2 \, dx \, dt' \leq C \int_0^t \langle t\rangle^{(1+2\delta)_-} \|D_t u\|_{L^2(\mathbb{R}^2)}^2 \, dt'.
\]

Inserting the above estimates into (4.5) gives rise to
\[
t\|D_t u(t)\|_{L^2(\mathbb{R}^2)}^2 \leq C \langle t\rangle^{(1+2\delta)_-}.
\]

Similarly, we have
\[
\int_0^t \langle t'\rangle^{(1+2\delta)_-} \|\nabla D_t u\|_{L^2(\mathbb{R}^2)}^2 \, dt' \leq C \int_0^t \langle t\rangle^{(1+2\delta)_-} \|\nabla D_t u\|_{L^2(\mathbb{R}^2)}^2 \, dt'.
\]

and
\[
\int_0^t \rho |D_t u|^2 \, dt' \leq C \int_0^t \langle t'\rangle^{(1+2\delta)_-} \|\nabla D_t u\|_{L^2(\mathbb{R}^2)}^2 \, dt'.
\]

We first deduce from Proposition 2.2 that
\[
\int_0^t \langle t'\rangle^{(1+2\delta)_-} \int_{\mathbb{R}^2} \rho |D_t u|^2 \, dx \, dt' \leq \|\langle t\rangle^{\frac{1}{2}+\delta} - D_t u\|_{L^2(\mathbb{R}^2)}^2 \leq C_0.
\]
While it is easy to observe from (3.9) and (4.6) that
\[2t(t^{1+2\delta})^{-1}\int_{\mathbb{R}^2} \Pi \partial_t u_j^i \partial_j u^i(t) \, dx \leq C t(t^{1+2\delta})^{-1} \|D_t u(t)\|_{L^2} \|\nabla u(t)\|_{L^2}^2 \leq C_0(t)^{-\delta},\]
and
\[\int_0^t \int_{\mathbb{R}^2} \Pi \partial_t u_j^i \partial_j u^i(t) \, dx \, dt' \leq \int_0^t \int_{\mathbb{R}^2} \Pi \partial_t u_j^i \partial_j u^i(t) \, dx \, dt' \leq \|\nabla u\|_{L_t^2(L^2)} \|\nabla u\|_{L_t^2(L^2)} \|D_t u\|_{L_t^2(L^2)} \leq C_0,\]
and
\[\int_0^t \|\nabla u\|_{L_t^2(L^2)}^2 \|\sqrt{\rho} D_t u\|_{L_t^2(L^2)}^2 \, dt' \leq C_0 \int_0^t \|\nabla u\|_{L_t^2(L^2)}^2 \, dt' \leq C_0.\]
Substituting the above estimates into (4.7) yields
\[\|\sqrt{7}t^{(\frac{1}{2}+\delta)} - \nabla D_t u\|_{L_t^2(L^2)} \leq C_0,\]
which together with (4.6) ensures (2.4). This completes the proof of Proposition 2.3. \(\square\)

5. \(L^\infty\) estimate of \(\nabla^2 \Delta^{-1} g\)

In this section, we shall use some basic facts on Littlewood-Paley theory. Let us recall from [4, 16] that

**Definition 5.1.** Consider a smooth radial function \(\varphi\) on \(\mathbb{R}\), supported in \([3/4, 8/3]\) such that
\[\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \tau) = 1 \quad \text{and} \quad \chi(\tau) \overset{def}{=} 1 - \sum_{j \geq 0} \varphi(2^{-j} \tau) \in C_0^\infty([0, 4/3]).\]
for any \(\tau > 0\). We denote
\[\Delta_j a = \mathcal{F}^{-1}(\varphi(2^{-j} |\xi|) \tilde{a}(\xi)) \quad \text{and} \quad S_j a = \mathcal{F}^{-1}(\chi(2^{-j} |\xi|) \tilde{a}(\xi)), \quad j \in \mathbb{Z}.\]
Let \(p \in ]1, +\infty[\) and \(s \in \mathbb{R}\). The Sobolev norms are defined as
\[\|a\|_{\dot{W}^{s,p}} \overset{def}{=} \|2^j \Delta_j a\|_{\ell^1(\mathbb{Z})} \|_{L^p}.\]
When \(p = 2\), the Sobolev spaces \(\dot{W}^{s,p}\) coincide with the classical homogeneous Sobolev spaces \(H^s\).

**Definition 5.2.** Let \(f\) be a locally integrable function. We define the maximal function \(Mf(x)\) as
\[Mf(x) = \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy\]
where \(B(x, r)\) denotes the ball with center \(x\) and radius \(r\), and \(|B(x, r)|\) the volume of ball \(B(x, r)\).

Let \(\phi_\varepsilon(x) = \frac{1}{\varepsilon^d} \phi\left(\frac{x}{\varepsilon}\right)\) for \(\varepsilon > 0\). Let \(\psi(x) = \sup_{|y| \geq |x|} |\phi(y)|\) satisfy \(\int_{\mathbb{R}^d} \psi(x) \, dx = A < \infty\). Then it follows Proposition 1.16 and Remark 1.17 of [4] that
\[(5.1) \quad \sup_{\varepsilon > 0} |(f * \phi_\varepsilon)(x)| \leq A Mf(x).\]
Lemma 5.1. Let $p \in [1, \infty]$, $X = (X^1, X^2)$ be a solenoidal vector field with $\nabla X \in L^p$ and $g \in L^\infty$ with $\partial_X g \in L^p$. Let $\sigma(\xi)$ be an infinitely differentiable function with $\text{supp} \sigma \subset \{ \xi \in \mathbb{R}^2 : |\xi| \geq \frac{1}{4} \}$, we assume that for all $\xi \in \mathbb{R}^2$ with $|\xi| \geq 1$, there holds $\sigma(\lambda \xi) = \lambda^{-1} \sigma(\xi)$. Then we have

$$
\| \partial_X \sigma(D)g \|_{L^\infty} \leq C \left( \|g\|_{L^\infty} + \frac{1}{\ln(1 + \|X\|_{L^\infty})} \right)^{1 - \frac{2}{p}} \|\nabla X\|_{L^p} \|g\|_{L^\infty} \ln \left( 1 + \frac{\|\nabla X\|_{L^p} \|g\|_{L^\infty}}{\|X\|_{L^\infty} \|g\|_{L^p}} \right).
$$

(5.2)

Proof. We first get, by using Bony's decomposition [5] and a commutator's process, that

$$
\partial_X \sigma(D)g = \sigma(D)\partial_X g + [T_{X^k}; \sigma(D)\partial_k]g - \sigma(D)T_{\partial_k g}X^k - \sigma(D)R(X^k, \partial_k g) + T_{\sigma(D)\partial_k g}X^k + R(X^k, \sigma(D)\partial_k g).
$$

(5.3)

It is easy to observe that for any $p \in [1, \infty]$,

$$
\|\sigma(D)\partial_X g\|_{W^{1,p}} \leq C \|\partial_X g\|_{L^p}.
$$

Considering the support properties to the Fourier transform of the terms in $T_{\partial_k g}X^k$, for each $x \in \mathbb{R}^2$, we deduce from (5.1) that

$$
|\Delta_j(T_{\partial_k g}X^k)(x)| \leq C \sum_{|j'| - |j| \leq 4} M \left( S_{j'-1} \partial_k g \Delta_{j'} X^k \right)(x)
$$

$$
\leq C \sum_{|j'| - |j| \leq 4} \|S_{j'-1} \partial_k g\|_{L^\infty} M \left( \Delta_{j'} X^k \right)(x)
$$

$$
\leq C \|g\|_{L^\infty} \sum_{|j'| - |j| \leq 4} M \left( 2^{j'} \Delta_{j'} X^k \right)(x),
$$

so that we have

$$
\|\sigma(D)T_{\partial_k g}X^k\|_{W^{1,p}} \leq C \|g\|_{L^\infty} \left( \sum_{j \in \mathbb{Z}} \left( \sum_{|j'| - |j| \leq 4} \left( 2^j M(\Delta_j X)(x) \right)^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
$$

$$
\leq C \|g\|_{L^\infty} \left( \sum_{j \in \mathbb{Z}} \left( 2^j M(\Delta_j X)(x) \right)^2 \right)^{\frac{1}{2}}
$$

$$
\leq C \|g\|_{L^\infty} \left( \sum_{j \in \mathbb{Z}} \left( 2^j M(\Delta_j X)(x) \right)^2 \right)^{\frac{1}{2}} \leq C \|g\|_{L^\infty} \|\nabla_X\|_{L^p}.
$$

Similarly, due to $\text{div} X = 0$ and considering the support properties to the Fourier transform of the terms in $R(X^k, \partial_k g)$, for each $x \in \mathbb{R}^2$, we have

$$
|\Delta_j R(X^k, \partial_k g)(x)| = |\partial_k \Delta_j \left( \sum_{j' \geq j - 5} \Delta_{j'} X^k \tilde{\Delta}_{j'} g \right)(x)|
$$

$$
\leq C 2^j M \left( \sum_{j' \geq j - 5} \Delta_{j'} X \tilde{\Delta}_{j'} g \right)(x)
$$

$$
\leq C \|g\|_{L^\infty} \left( \sum_{j' \geq j - 5} 2^{j-j'} M(2^{j'} \Delta_{j'} X)(x) \right),
$$
where \( \Delta_j g \overset{\text{def}}{=} \sum_{|j' - j| \leq 1} \Delta_{j'} g \). Applying Young inequality gives rise to

\[
\| \sigma(D) R(X^k, \partial_k g) \|_{\dot{W}^{1,p}} \leq \| R(X^k, \partial_k g) \|_{L^p} \leq C \| g \|_{L^\infty} \left( \sum_{\ell \geq -5} 2^{-\ell} \right) \left( \sum_{j \in \mathbb{Z}} (M(2^j \Delta_j X)(x))^2 \right)^{\frac{1}{2}} \|_{L^p} \\
\leq C \| g \|_{L^\infty} \| \nabla X \|_{L^p}.
\]

Exactly along the same line, for each \( x \in \mathbb{R}^2 \), there holds

\[
| \Delta_j R(X^k, \sigma(D) \partial_k g)(x) | \leq C \| g \|_{L^\infty} \sum_{j' \geq j - 5} M(\Delta_{j'} X)(x) \\
\leq C \| g \|_{L^\infty} 2^{j - j} \sum_{j' \geq j - 5} 2^{j - j'} M(2^{j'} \Delta_{j'} X)(x),
\]

from which, we infer

\[
\| R(X^k, \sigma(D) \partial_k g) \|_{\dot{W}^{1,p}} \leq C \| g \|_{L^\infty} \| \nabla X \|_{L^p}.
\]

Next, we consider the estimate of the commutator, \( [T_{X^k}; \sigma(D) \partial_k] g \). Let us denote \( \theta(\xi) \overset{\text{def}}{=} \frac{\partial}{\partial x} \sigma(\xi \varphi(\xi)) \). It is easy to observe that

\[
(5.4) \quad [T_{X^k}; \sigma(D) \partial_k] g = \sum_{j \in \mathbb{Z}} (S_{j - 1} X^k \Delta_j \sigma(D) \partial_k g - \sigma(D) \partial_k (S_{j - 1} X^k \Delta_j g)).
\]

Whereas by applying Abel rearrangement techniques, we get

\[
S_{j - 1} X^k \Delta_j \sigma(D) \partial_k g - \sigma(D) \partial_k (S_{j - 1} X^k \Delta_j g) \\
= \sum_{|j' - j| \leq 4} (S_{j - 1} X^k \Delta_j \sigma(D) \partial_k \Delta_{j'} g - \sigma(D) \partial_k \Delta_{j'} (S_{j - 1} X^k \Delta_j g)) \\
= \sum_{|j' - j| \leq 4} \left( S_{j - 1} X \cdot (S_j \theta(2^{-\ell} D) g - S_{j - 1} \theta(2^{-\ell} D) g) \\
- \theta(2^{-\ell} D) \cdot (S_{j - 1} X (S_j g - S_{j - 1} g)) \right) \\
= - \sum_{|j' - j| \leq 4} (\Delta_j X \cdot S_j \theta(2^{-\ell} D) g - \theta(2^{-\ell} D) \cdot (\Delta_j X S_j g)),
\]

which implies for each \( x \in \mathbb{R}^2 \),

\[
(5.5) \quad (S_{j - 1} X^k \Delta_j \sigma(D) \partial_k g - \sigma(D) \partial_k (S_{j - 1} X^k \Delta_j g))(x) \\
= \sum_{|j' - j| \leq 4} 2^{2\ell} \int_{\mathbb{R}^2} \hat{\theta}(2^\ell z) (\Delta_j X(x - z) - \Delta_j X(x)) S_j g(x - z) \, dz \\
= \sum_{|j' - j| \leq 4} 2^{2\ell} \int_{\mathbb{R}^2} \hat{\theta}(2^\ell z) \int_0^1 \Delta_j \nabla X(x - \tau z) \cdot z \, d\tau S_j g(x - z) \, dz.
\]
Then for \( \Psi(z) = \tilde{\theta}(z)|z| \), we deduce from (5.1) that
\[
\left| (S_{j-1}X^k \Delta_j \sigma(D) \partial_k g - \sigma(D) \partial_k (S_{j-1}X^k \Delta_j g)) (x) \right|
\leq C \|g\|_{L^\infty} \sum_{|\ell-j| \leq 4} 2^j \int_0^1 \int_{\mathbb{R}^2} |\Psi(2^j z)| |\Delta_j \nabla X (x - \tau z)| \, dz \, d\tau
\]
\[
= C 2^{-j} \|g\|_{L^\infty} \sum_{|\ell-j| \leq 4} \int_0^1 \left( \frac{2^j}{\tau} \right)^2 \int_{\mathbb{R}^2} |\Psi(2^j z/\tau)| |\Delta_j \nabla X (x - z)| \, dz \, d\tau
\]
\[
\leq C 2^{-j} \|g\|_{L^\infty} M(\Delta_j \nabla X) (x).
\]
Considering the support properties to terms in (5.4), we conclude that
\[
\| [T_{X^k; \sigma(D) \partial_k g}]_{\dot{W}^{1, p}} \| \leq C \|g\|_{L^\infty} \left( \sum_{j \in \mathbb{Z}} \left( M(\Delta_j \nabla X) (x) \right)^2 \right)^{1/2} \|_{L^p}
\]
\[
\leq C \|g\|_{L^\infty} \|\nabla X\|_{L^p}.
\]
By summing up the estimates, we achieve
\[
\| \partial_X \sigma(D) g - T_{\sigma(D) \partial_k g} X^k \|_{\dot{W}^{1, p}} \leq C (\|\partial_X g\|_{L^p} + \|\nabla X\|_{L^p} \|g\|_{L^\infty}),
\]
from which and the 2-D interpolation inequality
\[
\|a\|_{L^\infty} \leq C \|a\|_{L^p}^{1-\frac{2}{p}} \|\nabla a\|_{L^p}^{\frac{2}{p}},
\]
we deduce that
\[
(5.6) \quad \| \partial_X \sigma(D) g - T_{\sigma(D) \partial_k g} X^k \|_{L^\infty} \leq C (\|\nabla X\|_{L^\infty} \|g\|_{L^p})^{1-\frac{2}{p}} \left( \|\nabla X\|_{L^p} \|g\|_{L^\infty} + \|\partial_X g\|_{L^p} \right)^{\frac{2}{p}}.
\]
Here we used the fact that
\[
(5.7) \quad \| \Delta_j (T_{\sigma(D) \partial_k g} X^k) \|_{L^p} \leq C \|g\|_{L^p} \|X\|_{L^\infty}.
\]
Indeed it is easy to observe that
\[
\| \Delta_j T_{\sigma(D) \partial_k g} X^k \|_{L^p} \leq \sum_{|j'-j| \leq 4} \|S_{j'-1} (\sigma(D) \partial_k g)\|_{L^p} \|\Delta_j X^k\|_{L^\infty} \quad \text{and}
\]
\[
\|S_{j'-1} (\sigma(D) \partial_k g)\|_{L^p} \leq C \|g\|_{L^p},
\]
which implies (5.7).

While note that \( \text{supp} \sigma \subset \{ \xi \in \mathbb{R}^2 : |\xi| \geq \frac{1}{2} \} \), we have
\[
\| \Delta_j \sigma(D) \partial_k g \|_{L^\infty} \leq C \|g\|_{L^\infty} \Rightarrow \|S_{j'-1} (\sigma(D) \partial_k g)\|_{L^\infty} \leq C j \|g\|_{L^\infty},
\]
so that we find
\[
(5.8) \quad \| \Delta_j (T_{\sigma(D) \partial_k g} X^k) \|_{L^p} \leq \sum_{|j'-j| \leq 4} \|S_{j'-1} (\sigma(D) \partial_k g)\|_{L^\infty} \|\Delta_j X^k\|_{L^p}
\]
\[
\leq C j 2^{-j} \|g\|_{L^\infty} \|\nabla X\|_{L^p}.
\]
Then for any integer \( N \), we deduce from Bernstein Lemma and (5.7), (5.8) that
\[
\|T_{\sigma(D) \partial_k g} X^k\|_{L^\infty} \lesssim \sum_{j \leq N} 2^{\frac{2j}{p}} \|X\|_{L^\infty} \|g\|_{L^p} + \sum_{j > N} j 2^{-j} \left( 1-\frac{2}{p} \right) \|\nabla X\|_{L^p} \|g\|_{L^\infty}
\]
\[
\lesssim 2^{\frac{2N}{p}} \|X\|_{L^\infty} \|g\|_{L^p} + N 2^{-N} \left( 1 - \frac{2}{p} \right) \|\nabla X\|_{L^p} \|g\|_{L^\infty}.
\]
Taking $N$ so that $2^N \sim \max \{ \| g \|_{L^\infty} \| \nabla X \|_{L^p}^{\frac{1}{p}} \}^2$ in the above inequality gives rise to
\begin{equation}
\| T_{(D)} \partial_{x_y} X^k \|_{L^\infty} \leq C \left( \| X \|_{L^\infty} \| g \|_{L^p} \right)^{\frac{1}{2}} \left( \| \nabla X \|_{L^p} \| g \|_{L^\infty} \right)^{\frac{1}{p}} \ln \left( e + \frac{\| \nabla X \|_{L^p} \| g \|_{L^\infty}}{\| X \|_{L^\infty} \| g \|_{L^p}} \right).
\end{equation}
Together with (5.6), we conclude the proof of (5.2), and thus Lemma 5.1.

**Remark 5.1.** Note for any $s \in \mathbb{R}^2$, $1 \leq N^2 \| (-\Delta)^{-\frac{1}{2}} \|_{L^p} \leq C_s \| (-\Delta)^{-\frac{1}{2}} \|_{L^p}$, so that we deduce from the proof of (5.9) that
\begin{equation}
\| T_{(D)} \partial_{x_y} X^k \|_{L^\infty} \leq C \left( \| X \|_{L^\infty} \| g \|_{L^p} \right)^{\frac{1}{2}} \left( \| \nabla X \|_{L^p} \| g \|_{L^\infty} \right)^{\frac{1}{p}}.
\end{equation}

**Proof of Proposition 2.4.** The proof of this proposition is motivated by that of Theorem 3.3.1 of [8]. Indeed for any vector field $X(x) = (X^1(x), X^2(x))$, there hold
\begin{align*}
|X(x)| \partial_2^2 &= \frac{X^1(x)X(x,D) \partial_1 - X^2(x)X(x,D) \partial_2 + (X^2(x))^2 \Delta}{|X(x)|}, \\
|X(x)| \partial_2^2 &= \frac{X^2(x)X(x,D) \partial_2 - X^1(x)X(x,D) \partial_1 + (X^1(x))^2 \Delta}{|X(x)|}, \\
|X(x)| \partial_1 \partial_2 &= \frac{X^1(x)X(x,D) \partial_2 + X^1(x)X(x,D) \partial_2 - X^1(x)X^2(x) \Delta}{|X(x)|}.
\end{align*}
Since $|X^i(x)| \leq |X(x)|$, we infer that for each $x \in \mathbb{R}^2$, there holds
\begin{equation}
|X(x)| \| \nabla^2 (-\Delta)^{-\frac{1}{2}} g(x) \| \leq C \left( \| X(x) \|_{L^\infty} + \| X(x,D) \nabla (-\Delta)^{-\frac{1}{2}} g \|_{L^\infty} \right).
\end{equation}
Let us denote
\begin{equation*}
\mathcal{U}_L \triangleq \{ x \in \mathbb{R}^2 : |X_\lambda(x)| \geq I(X) \}.
\end{equation*}
Then we deduce from (5.11) that
\begin{equation*}
\| \nabla^2 (-\Delta)^{-\frac{1}{2}} g \|_{L^\infty(\mathcal{U}_L)} \leq C \left( \| g \|_{L^\infty} + \frac{1}{I(X)^2} \| \partial_X \nabla (-\Delta)^{-\frac{1}{2}} g \|_{L^\infty} \right).
\end{equation*}
from which and Definition 1.1, we deduce that
\begin{equation}
\| \nabla^2 (-\Delta)^{-\frac{1}{2}} g \|_{L^\infty} \leq C \left( \| g \|_{L^\infty} + \frac{1}{I(X)^2} \sup_{\lambda \in \Lambda} \| \partial_X \nabla (-\Delta)^{-\frac{1}{2}} g \|_{L^\infty} \right).
\end{equation}
Let us now split $\nabla^2 (-\Delta)^{-\frac{1}{2}} g$ as
\begin{equation*}
\nabla^2 (-\Delta)^{-\frac{1}{2}} g = \nabla^2 (-\Delta)^{-\frac{1}{2}} S_0 g + \nabla^2 (-\Delta)^{-\frac{1}{2}} (1 - S_0) g.
\end{equation*}
It is easy to observe that
\begin{equation}
\| \nabla^2 (-\Delta)^{-\frac{1}{2}} S_0 g \|_{L^\infty} \leq C \left( \| \nabla^2 (-\Delta)^{-\frac{1}{2}} S_0 g \|_{L^p} + \| \nabla^3 (-\Delta)^{-\frac{1}{2}} S_0 g \|_{L^p} \right) \leq C \| g \|_{L^p}.
\end{equation}
Whereas it follows from (5.12) that
\begin{equation*}
\| \nabla^2 (-\Delta)^{-\frac{1}{2}} (1 - S_0) g \|_{L^\infty} \leq C \left( \| (1 - S_0) g \|_{L^\infty} + \frac{1}{I(X)^2} \sup_{\lambda \in \Lambda} \| \partial_X \nabla (-\Delta)^{-\frac{1}{2}} (1 - S_0) g \|_{L^\infty} \right)
\end{equation*}
from which, (5.2) and (5.10), for any $s \in \mathbb{R}^2$, we infer that
\begin{equation*}
\| \nabla^2 (-\Delta)^{-\frac{1}{2}} (1 - S_0) g \|_{L^\infty} \leq C \left( \| g \|_{L^\infty} + \frac{1}{I(X)^2} \sup_{\lambda \in \Lambda} \left( \| \nabla X_\lambda \|_{L^p} \| g \|_{L^\infty} + \| \partial_X \partial_X \|_{L^p} \right)^{\frac{1}{2}} \right) \left( \| X_\lambda \|_{L^\infty} \| g \|_{L^p} \right)^{\frac{1}{2}}
\end{equation*}
Along with (5.13), we complete the proof of (2.10).
As an application of Proposition 2.4, let us now present the proof of Corollary 2.1.

**Proof of Corollary 2.1.** Let $X$ be a smooth enough solution of (2.5), we first deduce from the transport equation of (2.6) that for any $\ell \in \mathbb{N},$

$$\partial_t \partial_X^\ell \mu(t) + u \cdot \nabla \partial_X^\ell \mu(t) = 0,$$

which (2.12) implies

$$\|\partial_X^{\ell} \mu(t)\|_{L^p} = \|\partial_X^{\ell} \mu(\rho_0)\|_{L^p}, \quad \forall p \in [1, \infty]\quad \text{and} \quad \sup_{\lambda \in \Lambda} \|\partial_X^{\lambda} \mu(t)\|_{L^\infty \leq C_{X,\mu}}.$$

In what follows, let us take any $X(t) \overset{\text{def}}{=} (X_\lambda(t))_{\lambda \in \Lambda}$ where $X_\lambda(t)$ solves (2.5) with initial data $X_\lambda$ for any fixed $\lambda \in \Lambda.$ We denote $h \overset{\text{def}}{=} -\nabla (-\Delta)^{-1} \text{Tr}(\nabla X \nabla u).$ Then in view of (3.5), we write

$$\nabla (u_X - h) = \nabla (-\Delta)^{-1} P \text{div}((\mu(t) - 1)M(u_X - h)) - \nabla (-\Delta)^{-1} P \text{div}(\mu(t)M(u_X - h)), $$

from which and (2.7), we infer

$$\nabla u_X = \nabla h + \nabla (-\Delta)^{-1} P \text{div}(\mu(t)M(h)) - \nabla (-\Delta)^{-1} P (\rho D_t u_X - G) + \nabla (-\Delta)^{-1} P \text{div}((\mu(t) - 1)M(u_X - h)).$$

Yet it follows from (2.8) that

$$\rho D_t u_X^j - G^j = \partial_X^j (\rho D_t u^j) - \partial_j X \cdot \nabla \Pi - \partial_i (2\partial_X \mu(t)M_{ij}(u)) + \partial_i (\mu(t) \partial_X \nabla u^j + \partial_j X \cdot \nabla u^i) + \partial_i X \cdot \nabla (2\mu(t)M_{ij}(u)),$$

we then deduce from (5.15) that

$$\|\nabla u_X\|_{L^p} \leq C\left(\|\nabla h\|_{L^p} + \|\nabla u\|_{L^p} + \|\mu(t) - 1\|_{L^\infty} \|\nabla u_X\|_{L^p}
\right. $$

$$\left. + \|\nabla u\|_{L^\infty} \|\nabla X\|_{L^p} + \|\nabla (-\Delta)^{-1} P (\partial_X (\rho D_t u) - \nabla X^k \partial_k \Pi)\|_{L^p}\right).$$

Then under the condition (1.7), we deduce from (3.4) that

$$\|\nabla u_X\|_{L^p} \leq C\left(\|\nabla h\|_{L^p} + \|\nabla u\|_{L^p} + \|\nabla u\|_{L^\infty} \|\nabla X\|_{L^p}
\right. $$

$$\left. + \|\nabla (-\Delta)^{-1} P (\partial_X (\rho D_t u) - \nabla X^k \partial_k \Pi)\|_{L^p}\right).$$

Due to $\text{div} X = 0,$ we find

$$\|\nabla (-\Delta)^{-1} P \partial_X (\rho D_t u)\|_{L^p} = \|\nabla (-\Delta)^{-1} \partial_X (X^k \rho D_t u)\|_{L^p} \leq C\|X\|_{L^\infty} \|D_t u\|_{L^p} \leq C\|X\|_{L^\infty} \|D_t u\|_{L^2}^2 \|\nabla D_t u\|_{L^2}^{1-\frac{2}{p}}.$$

Whereas according to (2.9), we infer

$$\|\nabla (-\Delta)^{-1} P (\nabla X^k \partial_k \Pi_1)\|_{L^p} \leq C\|\nabla X\|_{L^p} \|\nabla \Pi_1\|_{L^2} \leq C\|\nabla X\|_{L^p} \|D_t u\|_{L^2},$$

and

$$\|\nabla (-\Delta)^{-1} P (\nabla X^k \partial_k \Pi_2)\|_{L^p} \leq C\|\nabla X\|_{L^p} \|\Pi_2\|_{L^\infty}.$$
For any $s \in [2/p, 1]$, by substituting the above estimates into (5.17) and using (2.24), we obtain
\[
\|\nabla u_X(t)\|_{L^p} \leq C_\delta \left\{ \|\nabla u\|_{L^p} + \|D_t u\|_{L^2}^2 \|\nabla D_t u\|_{L^2}^{1-\frac{2}{p}} \|X\|_{L^\infty} + \frac{\sup_{\lambda \in \Lambda} \|X_\lambda(t)\|_{L^\infty}^{1-\frac{2}{p}}}{I(X(t))} \|\nabla u\|_{L^p} \right. \\
+ \left( \|\nabla u\|_{L^p} + \|\nabla u\|_{L^\infty} + \|D_t u\|_{L^2} \right) \|\nabla X\|_{L^p} \\
+ \|\mu(\rho_0) - 1\|_{L^\infty} \frac{\|\nabla X\|_{L^p}}{I(X(t))} \sup_{\lambda \in \Lambda} \left( \|\nabla X_\lambda\|_{L^p} \|\nabla u\|_{L^\infty} + \|\nabla u_{X_\lambda}\|_{L^p} \right)^{\frac{2}{p}} \\
\times \left( \|X_\lambda\|_{L^\infty} \|\nabla u\|_{L^p} \right)^{1-\frac{2}{p}} + \left( \|\nabla X_\lambda\|_{L^p} \|\nabla u\|_{L^\infty} \right)^{\frac{2}{p}} \left( \|X_\lambda\|_{L^\infty} \|\nabla u\|_{L^p} \right)^{1-\frac{2}{p}} \right\}. 
\]
Then under the assumption of (2.11), by taking $X = X_\lambda$ in the above inequality and take supremum of the resulting inequality for $\lambda \in \Lambda$, we achieve
\[
\sup_{\lambda \in \Lambda} \|\nabla u_{X_\lambda}(t)\|_{L^p} \leq C \left( \|\nabla u\|_{L^p} + \|D_t u\|_{L^2}^2 \|\nabla D_t u\|_{L^2}^{1-\frac{2}{p}} \sup_{\lambda \in \Lambda} \|X_\lambda(t)\|_{L^\infty} \right. \\
+ \left( \|\nabla u\|_{L^p} + \|\nabla u\|_{L^\infty} + \|D_t u\|_{L^2} \right) \sup_{\lambda \in \Lambda} \|\nabla X_\lambda(t)\|_{L^p} \\
+ \left[ 1 + \frac{\sup_{\lambda \in \Lambda} \|X_\lambda(t)\|_{L^\infty}^{1-\frac{2}{p}}}{I(X(t))} \|\nabla u\|_{L^p} \right] + \frac{1}{2} \sup_{\lambda \in \Lambda} \|\nabla u_{X_\lambda(t)}(t)\|_{L^p},
\]
which implies (2.13). This completes the proof of the Corollary 2.1. \qed

6. The $L^1_t(Lip)$ estimate of the velocity field

Let us first present the proof of Proposition 2.5.

Proof of Proposition 2.5. Due to $\text{div} \ u = 0$, for any $r \in [1, \infty[$, we get, by taking $L^2$ inner product of (2.5) with $|X|^{r-2} X$, that
\[
\left| \frac{d}{dt} \|X(t)\|_{L^r}^r \right| = r \left| (X \cdot \nabla u) |X|^{r-2} X \right| \\
\leq r \|\nabla u\|_{L^\infty} \|X\|_{L^r}^r,
\]
which implies the first inequality of (2.14) $r \in [1, \infty[$. The case for $r = \infty$ can be proved similarly.

Since $(X_\lambda(0))_{\lambda \in \Lambda}$ is a non-degenerate family of vector fields, we deduce from (2.14) that $(X_\lambda(t))_{\lambda \in \Lambda}$ is also a non-degenerate family of vector fields. Moreover, there holds
\[
(6.1) \quad I(X(0)) \exp \left( -\|\nabla u\|_{L^1_t(L^\infty)} \right) \leq I(X(t)) \leq I(X(0)) \exp \left( \|\nabla u\|_{L^1_t(L^\infty)} \right).
\]

Let us now turn to the proof of (2.15). We first take $\partial_i$ to the System (2.5) to get
\[
\partial_i \partial_j X^j + u \cdot \nabla \partial_i X^j = -\partial_i u \cdot \nabla X^j + \partial_i u_X^j.
\]
By multiplying the above equation by $(\partial_i X^j)^{p-1}$ and integrating the resulting equality over $\mathbb{R}^2$, we obtain
\[
\frac{d}{dt} \|\nabla X(t)\|_{L^p} \leq \|\nabla u\|_{L^\infty} \|\nabla X\|_{L^p} + \|\nabla u_X\|_{L^p}.
\]
Thanks to Corollary 2.1, by taking $X(t) = X_\lambda(t)$ in the above inequality and integrating the inequality over $[0, t]$, and then inserting (2.13) to the resulting inequality, we achieve

$$
\sup_{\lambda \in \Lambda} \| \nabla X_\lambda(t) \|_{L^p} \leq \sup_{\lambda \in \Lambda} \| \nabla X_\lambda(0) \|_{L^p} + C \int_0^t \| \nabla u \|_{L^p} \left( 1 + \frac{\sup_{\lambda \in \Lambda} \| X_\lambda(t') \|_{L^\infty}^{1 - \frac{2}{p}}}{I(X(t'))} \right) dt' + C \int_0^t \left( \| \nabla u \|_{L^p} + \| D_t u \|_{L^2} \sup_{\lambda \in \Lambda} \| X_\lambda(t') \|_{L^\infty} \right) dt' + C \int_0^t \left( \| \nabla u \|_{L^p} + \| \nabla u \|_{L^\infty} + \| D_t u \|_{L^2} \sup_{\lambda \in \Lambda} \| \nabla X_\lambda(t') \|_{L^p} \right) dt'.
$$

(6.2)

It is easy to observe from (3.3) and Propositions 2.2 and 2.3 that for any $r \in [2, \infty[$

$$
\| \nabla u(t) \|_{L^r} \leq C \| \nabla u(t) \|_{L^2} \| D_t u(t) \|_{L^2}^{1 - \frac{2}{r}} \leq C_1 t^{\frac{1}{r} - \frac{1}{2}} \langle t \rangle^{-(\frac{1}{2} + \delta)}.
$$

(6.3)

By virtue of (6.3), as long as $\delta > \frac{1}{p}$, we have

$$
\int_0^t \| \nabla u(t') \|_{L^p} dt' \leq C_0 \int_0^t \left( t' \right)^{\frac{1}{p} - \frac{1}{2}} \langle t' \rangle^{-(\frac{1}{2} + \delta)} dt' \leq C_0.
$$

Whereas it follows from Proposition 2.2 that

$$
\int_0^t \| D_t u(t') \|_{L^2} dt' = \int_0^t \langle t' \rangle^{-(\frac{1}{2} + \delta)} \cdot \| D_t u(t') \|_{L^2} dt' \leq C \| \langle t' \rangle^{\frac{1}{2} + \delta} - D_t u(t') \|_{L^2(L^2)} \leq C_0,
$$

(6.4)

and

$$
\int_0^t \| D_t u(t') \|_{L^2}^{\frac{2}{p}} \| \nabla D_t u(t') \|_{L^2}^{1 - \frac{2}{p}} dt' = \int_0^t \left( t' \right)^{-\frac{1}{2} (1 - \frac{2}{p})} \langle t' \rangle^{-(\frac{1}{2} + \delta)} - \| \langle t' \rangle^{\frac{1}{2} + \delta} - D_t u(t') \|_{L^2}^{\frac{2}{p}} \| \nabla D_t u(t') \|_{L^2}^{1 - \frac{2}{p}} dt' \leq C \| \langle t' \rangle^{\frac{1}{2} + \delta} - D_t u \|_{L^2(L^2)}^{\frac{2}{p}} \| \nabla D_t u \|_{L^2(L^2)}^{1 - \frac{2}{p}} \leq C_0.
$$

Hence by applying Gronwall’s inequality to (6.2), we arrive at

$$
\sup_{\lambda \in \Lambda} \| \nabla X_\lambda(t) \|_{L^p} \leq C_0 \left( 1 + \sup_{\lambda \in \Lambda} \| \nabla X_\lambda(0) \|_{L^p} + \sup_{\lambda \in \Lambda} \| X_\lambda \|_{L^\infty(L^\infty)} \right. + \left. \sup_{\tau \in [0, t]} \sup_{\lambda \in \Lambda} \| X_\lambda(\tau) \|_{L^\infty}^{1 - \frac{2}{p}} \frac{1}{I(X(\tau))} \right) V(t).
$$

Inserting the Inequality (6.1) to the above inequality leads to (2.15). This completes the proof of the proposition. \( \square \)

Proof of Proposition 2.6. Let $S_0$ be the partial sum operator introduced in Definition 5.1. We denote

$$
\mathfrak{A} \overset{\text{def}}{=} (-\Delta)^{-1} (I - S_0) \text{div} \left( (\mu(\rho) - 1) \mathcal{M}(u) \right) \quad \text{and} \quad \mathfrak{B}(\rho) \overset{\text{def}}{=} \| \mu(\rho) - 1 \|_{L^\infty}.
$$

(6.5)

Due to $\text{div} u = 0$, according to the momentum equation of (1.1), we write

$$
-\Delta u = -\mathbb{P}(\rho D_t u) + \mathbb{P} \text{div} \left( (\mu(\rho) - 1) \mathcal{M}(u) \right),
$$
which implies

\[(6.6) \quad (I - S_0)\nabla u = -\nabla (-\Delta)^{-1} P (I - S_0) (\rho D_t u) + \nabla P A.\]

By virtue of the the 2-D interpolation inequality that

\[
\|f\|_{L^\infty} \leq C \|f\|_{L^4}^{\frac{1}{2}} \|\nabla f\|_{L^4}^{\frac{1}{2}} \leq C \|f\|_{H^1}^{\frac{1}{2}} \|\nabla f\|_{L^1}^{\frac{1}{2}},
\]

and that

\[
\|\nabla (-\Delta)^{-1} P (I - S_0) (\rho D_t u)\|_{H^1} \leq C \|\rho D_t u\|_{L^2},
\]

we infer

\[
\|\nabla (-\Delta)^{-1} P (I - S_0) (\rho D_t u)\|_{L^\infty} \leq C \|\rho D_t u\|_{L^\infty}^{\frac{1}{2}} \|\rho D_t u\|_{L^4}^{\frac{1}{2}}
\]

\[
\leq C \|D_t u\|_{L^2} \|\nabla D_t u\|_{L^2}^{\frac{1}{2}}.
\]

(6.7)

On the other hand, let \(X_\lambda(t)\) be the corresponding solution of (2.5) with initial data \(X_\lambda(0)\). Thanks to (6.1), \((X_\lambda(t))_{\lambda \in A}\) is a non-degenerate family of vector fields. Recall that \(P = Id + \nabla (-\Delta)^{-1} \text{div}, \) for any \(s \in [2/p, 1[\), we get, by applying Proposition 2.4, that

\[
\|\nabla P A\|_{L^\infty} \lesssim \mathfrak{B}(\rho) \|\nabla u\|_{L^p} + \|\nabla A\|_{L^\infty} + \frac{1}{I(X)} \sup_{\lambda \in A} \left\{ \mathfrak{B}(\rho) \|X_\lambda\|_{L^\infty} \|\nabla u\|_{L^p} )^{1 - \frac{2}{p}} \right\}^{\frac{1}{2}}
\]

\[
\times \left( (\|\nabla X_\lambda\|_{L^p} \|\nabla A\|_{L^\infty} + \|\partial_{X_\lambda} \nabla A\|_{L^p} )^{\frac{2}{p}} + \mathfrak{B}(\rho) \|X_\lambda\|_{L^\infty} \|\nabla u\|_{L^p} \right)^{\frac{1}{2}} \left( \|\nabla X_\lambda\|_{L^p} \|\nabla A\|_{L^\infty} \right)^{\frac{2}{p}}.
\]

(6.8)

It follows from Proposition A.1 that

\[
\|\partial_{X_\lambda} \nabla A\|_{L^p} \lesssim \mathfrak{B}(\rho) \left( (\|\nabla X_\lambda\|_{L^p} \|\nabla u\|_{L^\infty} + \|\nabla u_{X_\lambda}\|_{L^p} + \|\partial_{X_\lambda} \mu(\rho)\|_{L^\infty} \|\nabla u\|_{L^p} ) + \|\nabla u\|_{L^p} \right).
\]

\[
\text{so that in view of (5.14), we get, by applying Young's inequality, that}
\]

\[
\frac{1}{I(X)} \left( \mathfrak{B}(\rho) \|X_\lambda\|_{L^\infty} \|\nabla u\|_{L^p} \right)^{1 - \frac{2}{p}} \|\partial_{X_\lambda} \nabla A\|_{L^p}^{\frac{1}{2}}
\]

\[
\lesssim \mathfrak{B}(\rho) \left( I(X)^{-\frac{p}{p-2}} \|\nabla u\|_{L^p} \|X_\lambda\|_{L^\infty} + \|\nabla u\|_{L^\infty} \|\nabla X_\lambda\|_{L^p} + \|\nabla u_{X_\lambda}\|_{L^p} \right) + \|\nabla u\|_{L^p}.
\]

While we deduce from Proposition 2.4 that for any \(s \in [2/p, 1[\),

\[
\|\nabla A\|_{L^\infty} \lesssim \mathfrak{B}(\rho) \left( (\|\nabla u\|_{L^p} + \|\nabla u\|_{L^\infty} ) + \frac{1}{I(X)} \sup_{\lambda \in A} \left( \mathfrak{B}(\rho) \|X_\lambda\|_{L^\infty} \|\nabla u\|_{L^p} )^{1 - \frac{2}{p}} \right) \right.
\]

\[
\times \left( \mathfrak{B}(\rho) \left( (\|\nabla u\|_{L^\infty} \|\nabla X_\lambda\|_{L^p} + \|\nabla u_{X_\lambda}\|_{L^p} + \|\partial_{X_\lambda} \mu(\rho)\|_{L^\infty} \|\nabla u\|_{L^p} )^{\frac{2}{p}} + \mathfrak{B}(\rho) \|X_\lambda\|_{L^\infty} \|\nabla u\|_{L^p} \right)^{\frac{1}{2}} \left( \|\nabla u\|_{L^\infty} \|\nabla X_\lambda\|_{L^p} \right)^{\frac{2}{p}} \right),
\]

Applying Young's inequality yields

\[
\|\nabla A\|_{L^\infty} \lesssim (\|\nabla u\|_{L^p} + \mathfrak{B}(\rho) \left( (\|\nabla u\|_{L^\infty} \right)
\]

\[
+ \sup_{\lambda \in A} \left( C(s, p, X) \|\nabla u\|_{L^p} \|X_\lambda\|_{L^\infty} + \|\nabla u\|_{L^\infty} \|\nabla X_\lambda\|_{L^p} + \|\nabla u_{X_\lambda}\|_{L^p} \right) \right).
\]

Here and in what follows, we always denote

\[
(6.9) \quad C(s, p, X) \overset{\text{def}}{=} I(X)^{-\frac{p}{p-2}} + I(X)^{-\frac{2}{p-2}}.
\]
The same estimate holds for
\[ \frac{1}{T(X)} \sup_{\lambda \in \Lambda} (B(\rho)) \| X^\lambda \|_{L^\infty} \| \nabla u \|_{L^p} \right)^{1 - \frac{2}{p}} \left( \| \nabla X^\lambda \|_{L^p} \| \nabla \mathfrak{A} \|_{L^\infty} \right)^{\frac{2}{p}}. \]

Inserting the above inequalities into (6.8) gives rise to
\[ \| \nabla \mathfrak{A} \|_{L^\infty} \leq \text{C} \left\{ \| \nabla u \|_{L^p} + B(\rho) \left( \| \nabla u \|_{L^\infty} + \sup_{\lambda \in \Lambda} (C(s, p, X) \| \nabla u \|_{L^p}) \| X^\lambda \|_{L^\infty} \right) + \sup_{\lambda \in \Lambda} (C(s, p, X) \| \nabla u \|_{L^p} \| X^\lambda \|_{L^\infty} \right) \right\}. \]

By virtue of (6.6), (6.7) and (6.10), we achieve
\[ \| \nabla u \|_{L^\infty} \leq \left[ \| S_0 \nabla u \|_{L^\infty} + \| (I - S_0) \nabla u \|_{L^\infty} \right] \]
\[ \leq \text{C} \left\{ \| \nabla u \|_{L^p} + \| D_t u \|_{L^2} \| \nabla D_t u \|_{L^2} + B(\rho) \left( \| \nabla u \|_{L^\infty} \right) \right. \]
\[ \left. + \sup_{\lambda \in \Lambda} (C(s, p, X) \| \nabla u \|_{L^p} \| X^\lambda \|_{L^\infty} \right) \left. \right\}. \]

Whereas for some \( \mathfrak{M} \), which we shall fix later on, we define \( T^* \) by (2.11). Then under the smallness condition (1.7), for any \( t \in [0, T^*] \), we get, by inserting the estimate (2.13) to the above inequality, that
\[ \| \nabla u(t) \|_{L^\infty} \leq \text{C} \left\{ \| \nabla u \|_{L^p} + \| D_t u \|_{L^2} \| \nabla D_t u \|_{L^2} \right\} \]
\[ \leq \text{C} \int_0^{T^*} \left( \| \nabla u \|_{L^p} + \| D_t u \|_{L^2} \| \nabla D_t u \|_{L^2} \right) \right\} dt' \leq \text{\frac{\text{\mathfrak{M}}}{2}}. \]

Then we deduce from (2.14), (2.15), (3.4) and (6.11) that
\[ \| \nabla u \|_{L^\infty(L^\infty)} \leq \frac{\mathfrak{M}^2}{2} \left( 1 + \mathfrak{M}^2 \right) \exp \left( C \| \nabla u \|_{L^\infty(L^\infty)} \right) \]
\[ \mathfrak{M} \overset{\text{def}}{=} \left( 1 + (1 + C(s, p, X_0)) \sup_{\lambda \in \Lambda} \| X^\lambda(0) \|_{L^\infty} \right) \sup_{\lambda \in \Lambda} \| \nabla X^\lambda(0) \|_{L^p}. \]

In particular, if \( \varepsilon_0 \) in (1.7) is so small that
\[ \varepsilon_0 \leq \frac{1}{2} \mathfrak{M}^{-1} \exp \left( -C \mathfrak{M} \right), \]
we conclude that
\[ \| \nabla u(t) \|_{L^\infty(L^\infty)} \leq \frac{3 \mathfrak{M}}{4} \quad \text{for any } t \in [0, T^*]. \]

We now take
\[ \mathfrak{M} \overset{\text{def}}{=} 2C_0 \left( 1 + (1 + C(s, p, X_0)) \sup_{\lambda \in \Lambda} \| X^\lambda(0) \|_{L^\infty} \right) \exp \left( C \mathfrak{M} \right). \]

We claim that \( T^* = T^* \). Otherwise, if \( T^* < T^* \), we deduce from (2.15) that
\[ \| \nabla X^\lambda \|_{L^\infty(L^p)} \leq \mathfrak{M}/2 \quad \text{for any } t \in [0, T^*], \]
under the assumption that

\[
\varepsilon_0 \leq \min\left\{ \frac{\varepsilon_1 I(X_0)}{2N}, \frac{1}{2} R^{-1} \exp(-C\varepsilon) \right\}.
\]

This shows that under the assumption (6.17), (6.14) and (6.16) hold on \([0, T^*[\), which contradicts with the definition of \(T^*\) given by (2.11). This in turn shows that \(T^* = T^*\). We complete the proof of the Proposition 2.6. \(\square\)

### 7. The energy estimate of \(u_X\)

**Lemma 7.1.** Let

\[
\mathcal{Y}(t) = \frac{1}{2} \int_{\mathbb{R}^2} \rho|u_X|^2 \, dx + \int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(\rho u_X) |u \cdot \nabla X| \, dx.
\]

Then under the assumptions of Proposition 2.7, for any \(p \in [2, \infty]\) and any \(t < T^*\), there holds

\[
\frac{d}{dt} \mathcal{Y}(t) + \|\nabla u_X\|^2_{L^2} \leq C\left( \|\nabla u\|_{L^\infty} + \|D_t u\|_{L^2} + (t)^{-1}\right) \mathcal{Y}(t)
\]

\[
+ C\left( (t)^{1+\frac{2}{p}} \|D_t u\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \left( \|D_t u\|_{L^2}^{1+\frac{2}{p}} + \|\nabla u\|^2_{L^2}^{\frac{2}{p}} \right) \right)
\]

\[
\|\nabla X\|^2_{L^p} + (\|\nabla u\|_{L^\infty} + \|D_t u\|_{L^2} + (t)^{-1} + \|u\|^2_{L^\infty} \|\nabla X\|^2_{L^p})
\]

**Proof.** We first get, by taking \(L^2\) inner product of \(u_X\) with the momentum equation of (2.7), that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \rho|u_X(t)|^2 \, dx + 2 \int_{\mathbb{R}^2} \mu(\rho)|\mathcal{M}(u_X)|^2 \, dx = \int_{\mathbb{R}^2} (G - \nabla \Pi_X) |u_X| \, dx.
\]

To handle the term \(\Pi_X\), we take the divergence operator to the momentum equation of (2.7) to get

\[
\Pi_X = (-\Delta)^{-1} \text{div}(\rho D_t u_X - \text{div}(2\mu(\rho)\mathcal{M}(u_X))) - \text{div}(-\Delta)^{-1} G.
\]

Then we have

\[
\int_{\mathbb{R}^2} (G - \nabla \Pi_X) |u_X| \, dx = \int_{\mathbb{R}^2} \mathcal{P} G |u_X| \, dx
\]

\[
- \int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(\rho D_t u_X - \text{div}(2\mu(\rho)\mathcal{M}(u_X))) |u_X| \, dx.
\]

Next let us handle term by term above. By using integration by parts and \(D_t \rho = 0\), we find

\[
\int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(\rho D_t u_X) |u_X| \, dx = \frac{d}{dt} \int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(\rho u_X) |u_X| \, dx
\]

\[
+ \int_{\mathbb{R}^2} \left[ \nabla(-\Delta)^{-1} \text{div}(u \cdot \nabla) u_X \right] |u_X| \, dx - \int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(\rho u_X) |D_t u_X| \, dx.
\]

Due to \(\text{div} u_X = \text{div}(u \cdot \nabla X)\), one has

\[
\int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(\rho u_X) |u_X| \, dx = - \int_{\mathbb{R}^2} (-\Delta)^{-1} \text{div}(\rho u_X) |u_X| \, dx
\]

\[
= \int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(\rho u_X) |u \cdot \nabla X| \, dx.
\]

While it follows from commutator’s estimate that

\[
\left| \int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(u \cdot \nabla) u_X \right| \leq C \|\nabla u\|_{L^\infty} \|u_X\|^2_{L^2}.
\]
Whereas note that $\text{div } X = 0$ and $\text{div } u_X = \partial_t X^k \partial_k u^i$, we deduce from (2.5) that

$$\text{div } D_t u_X = \partial_t \text{div } u_X + \text{div}(u \cdot \nabla u_X) = \partial_t (\partial_t X^k \partial_k u^i) + \partial_t u \cdot \nabla u_X \tag{7.7}$$

$$= [D_t; \partial_t] X^k \partial_k u^i + \partial_t D_t X^k \partial_k u^i + \partial_t X^k D_t \partial_k u^i + \partial_t u \cdot \nabla u_X^i,$$

so that, by using integration by parts, we achieve

$$- \int_{\mathbb{R}^2} \nabla (-\Delta)^{-1} \text{div}(\rho u_X)|D_t u_X| \, dx = \int_{\mathbb{R}^2} (-\Delta)^{-1} \text{div}(\rho u_X)|\text{div } D_t u_X| \, dx$$

$$= - \int_{\mathbb{R}^2} \nabla (-\Delta)^{-1} \text{div}(\rho u_X)|(2u_X \cdot \nabla u + D_t u \cdot \nabla X) \, dx,$$

from which and the following interpolation inequality

$$\|a\|_{L^q} \leq C\|a\|_{L^2}^{\frac{2}{q}}\|\nabla a\|_{L^2}^{1-\frac{2}{q}} \quad \forall \ q \in ]2, \infty[,$$

we infer for any $\eta > 0$,

$$\left| \int_{\mathbb{R}^2} \nabla (-\Delta)^{-1} \text{div}(\rho u_X)|D_t u_X| \, dx \right|$$

$$\leq C\left( \|\nabla u\|_{L^\infty} \|u_X\|^2_{L^2} + \|u_X\|^{\frac{2p}{L^2p}}_{L^{\frac{p}{2}}} |D_t u|_{L^2} \|\nabla X\|_{L^p} \right)$$

$$\leq C\left( \|\nabla u\|_{L^\infty} \|u_X\|^2_{L^2} + \|u_X\|_{L^2}^{-\frac{p}{2}} \|\nabla u_X\|_{L^2}^{\frac{2}{p}} |D_t u|_{L^2} \|\nabla X\|_{L^p} \right)$$

$$\leq \eta \|\nabla u_X\|^2_{L^2} + C_\eta \left( \|\nabla u\|_{L^\infty} + |D_t u|_{L^2} \right) \|u_X\|^2_{L^2} + |D_t u|_{L^2}^{1+\frac{2}{p}} \|\nabla X\|^2_{L^p}.$$

Substituting the above estimates into (7.6) yields

$$\left| \int_{\mathbb{R}^2} \nabla (-\Delta)^{-1} \text{div}(\rho D_t u_X)|u_X| \, dx - \frac{d}{dt} \int_{\mathbb{R}^2} \nabla (-\Delta)^{-1} \text{div}(\rho u_X)|u \cdot \nabla X| \, dx \right|$$

$$\leq \eta \|\nabla u_X\|^2_{L^2} + C_\eta \left( \|\nabla u\|_{L^\infty} + |D_t u|_{L^2} \right) \|u_X\|^2_{L^2} + |D_t u|_{L^2}^{1+\frac{2}{p}} \|\nabla X\|^2_{L^p}.$$

To handle the last term in (7.5), by using integration by parts and $\text{div } u_X = \text{tr}(\nabla X \nabla u)$, we obtain

$$\left| \int_{\mathbb{R}^2} \nabla (-\Delta)^{-1} \text{div}(2\mu(\rho) \mathcal{M}(u_X))|u_X| \, dx \right|$$

$$= \left| \int_{\mathbb{R}^2} (-\Delta)^{-1} \text{div}(2\mu(\rho) \mathcal{M}(u_X))|\text{tr}(\nabla X \nabla u)| \, dx \right|$$

$$\leq C\|\mu(\rho)\|_{L^\infty} \|\nabla u_X\|_{L^2} \|\nabla X\|_{L^p} \|\nabla u\|_{L^\frac{2p}{p-2}}$$

$$\leq \eta \|\nabla u_X\|^2_{L^2} + C_\eta \|\nabla u\|^2_{L^\frac{2p}{p-2}} \|\nabla X\|^2_{L^p}.$$
It remains to deal with the term $\int_{\mathbb{R}^2} \mathbb{P} G |u_X| \, dx$. Indeed it follows from the transport equation of (2.7) that
\[
\left| \int_{\mathbb{R}^2} \mathbb{P}(\rho_X D_t u) |u_X| \, dx \right| \leq C \|\rho_X\|_{L^\infty} \|D_t u\|_{L^2} \|u_X\|_{L^2} \\
\leq C(t)^{-1+} \|u_X\|^2_{L^2} + C(t)^{1+} \|D_t u\|^2_{L^2},
\]
\[
\left| \int_{\mathbb{R}^2} \mathbb{P} \text{div}(2\partial_X \mu(\rho) \mathcal{M}(u)) |u_X| \, dx \right| \leq C \|\partial_X \mu(\rho)\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla u_X\|_{L^2} \\
\leq \eta \|\nabla u_X\|^2_{L^2} + C_\eta \|\nabla u\|^2_{L^2}.
\]

While in view of (2.9) and $\text{div} X = 0$, we write
\[
\int_{\mathbb{R}^2} \mathbb{P}(\nabla X^k \partial_k \Pi) |u_X| \, dx = \int_{\mathbb{R}^2} \mathbb{P}(\nabla X^k \partial_k (-\Delta)^{-1} \text{div}(\rho D_t u)) |u_X| \, dx \\
+ \int_{\mathbb{R}^2} \mathbb{P}(\nabla X^k (-\Delta)^{-1} \text{div}(2\mu(\rho) \mathcal{M}(u))) |\partial_k u_X| \, dx,
\]
so that we obtain
\[
\left| \int_{\mathbb{R}^2} \mathbb{P}(\nabla X^k \partial_k \Pi) |u_X^i| \, dx \right| \leq C \|\nabla X^i\|_{L^p} \left( \|D_t u\|_{L^2} \|u_X\|_{L^p \frac{2p}{p-2}} \|\nabla u_X\|_{L^2} \right) \\
\leq \eta \|\nabla u_X\|^2_{L^2} + C_\eta \left( \|D_t u\|_{L^2} \|u_X\|_{L^p \frac{2p}{p-2}} \right) \|\nabla X^i\|^2_{L^p},
\]

Finally due to the $\text{div} X = 0$, we get, by using integration by parts, that
\[
\left| \int_{\mathbb{R}^2} \mathbb{P}(\partial_i X \cdot \nabla (\mu(\rho) \mathcal{M}_{ij}(u))) |u_X| \, dx \right| \leq C \|\nabla X\|_{L^p} \|\nabla u\|_{L^p \frac{2p}{p-2}} \|\nabla u_X\|_{L^2} \\
\leq \eta \|\nabla u_X\|^2_{L^2} + C_\eta \left( \|D_t u\|^2_{L^2} + \|\nabla u\|^2_{L^p \frac{2p}{p-2}} \right) \|\nabla X\|^2_{L^p},
\]
\[
\left| \int_{\mathbb{R}^2} \mathbb{P}(\partial_i (\mu(\rho) (\partial_i X \cdot \nabla u + \nabla X \cdot \nabla u^i))) |u_X| \, dx \right| \leq C \|\nabla X\|_{L^p} \|\nabla u\|_{L^p \frac{2p}{p-2}} \|\nabla u_X\|_{L^2} \\
\leq \eta \|\nabla u_X\|^2_{L^2} + C_\eta \left( \|D_t u\|^2_{L^2} + \|\nabla u\|^2_{L^p \frac{2p}{p-2}} \right) \|\nabla X\|^2_{L^p}.
\]

As a result, it comes out
\[
\left| \int_{\mathbb{R}^2} \mathbb{P} G |u_X| \, dx \right| \leq 4\eta \|\nabla u_X\|^2_{L^2} + C_\eta \left( \|D_t u\|_{L^2} + (t)^{-1+} \right) \|u_X\|^2_{L^2} \\
+ C_\eta \left( t \right)^{1+} \|D_t u\|^2_{L^2} + \|\nabla u\|^2_{L^2} + C_\eta \left( t \right)^{1+ \frac{2p}{p-2}} \|\nabla u\|^2_{L^p \frac{2p}{p-2}} \|\nabla X\|^2_{L^p}.
\] (7.10)

Notice that
\[
\left| \int_{\mathbb{R}^2} \nabla (-\Delta)^{-1} \text{div}(\rho u_X) |u \cdot \nabla X| \, dx \right| \leq C \|u_X\|_{L^2} \|u\|_{L^p \frac{2p}{p-2}} \|\nabla X\|_{L^p},
\]
we have
\[
\mathcal{Y}(t) \geq c \|u_X\|^2_{L^2} - C \|u\|^2_{L^p \frac{2p}{p-2}} \|\nabla X\|^2_{L^p}.
\]

By inserting the Estimates (7.8), (7.9) and (7.10) into (7.5) and then substituting the resulting inequality into (7.3) and taking $\eta$ sufficiently small, we achieve (7.2). This completes the proof of the lemma. \qed
Proof of Proposition 2.7. By multiplying (7.2) with \( \langle t \rangle^{2\delta-} \), we find
\[
\frac{d}{dt} \langle t \rangle^{2\delta-} \mathcal{Y}(t) + \| \langle t \rangle^{\delta-} \nabla u_X \|^2_{L^2} \leq C \left( \| \nabla u \|_{L^\infty} + \| D_t u \|_{L^2} + \langle t \rangle^{-1+} \right) \langle t \rangle^{2\delta-} \mathcal{Y}(t)
\]
(7.11)
\[
+ 2\delta \langle t \rangle^{2\delta-1} \mathcal{Y}(t) + C \langle t \rangle^{2\delta-} \left( \langle t \rangle^{1+} \| D_t u \|_{L^2}^{2} + \| \nabla u \|_{L^2}^{2} \
+ \left( \| \nabla u \|_{L^\infty} + \| D_t u \|_{L^2} + \langle t \rangle^{-1+} \right) \| u \|^2_{L^2} \right)^{\frac{1+2\delta}{2}} + \| D_t u \|_{L^2}^{1+} \| \nabla u \|_{L^2}^{2} \| \nabla X \|_{L^p}^2.
\]

It is easy to observe from Proposition 2.2 that
\[
\int_0^t \langle t' \rangle^{2\delta-1} \mathcal{Y}(t') \, dt' \leq \int_0^t \langle t' \rangle^{2\delta-1} \left( \| \nabla u \|_{L^2}^{2} + \| X \|_{L^\infty} \| \nabla \|_{L^p} \| u \|^2_{L^2} \right) \, dt' 
\leq C_0 \left( \| X \|_{L^\infty}^{2} \| (L^\infty) \| + \| \nabla X \|_{L^\infty(L^p)}^2 \right),
\]
and due to \( p < 2 \left( 1 + 1/\delta \right) \), one has
\[
\int_0^t \langle t' \rangle^{2\delta-} \| D_t u(t') \|^2_{L^2} \, dt' \leq \left( \int_0^t \langle t' \rangle^{2\delta-} \| D_t u(t') \|^2_{L^2} \, dt' \right)^{\frac{2}{p+2}} \langle t \rangle \langle t \rangle^{\delta-} \| D_t u \|_{L^2(L^2)}^{\frac{p+2}{2}} \leq C_0.
\]
Similarly it follows from (3.3) and Proposition 2.1 that
\[
\int_0^t \langle t' \rangle^{2\delta-} \| \nabla u(t') \|^2 \| L^p \, dt' \leq C \langle t \rangle^{\delta} \| \nabla u \|_{L^2(L^2)}^{2\left( 1 - \frac{2}{2} \right)} \| \nabla \|_{L^2(L^2)}^{\frac{2}{2} - 2} \| D_t u \|_{L^2(L^2)}^{\frac{4}{2}} \leq C_0.
\]
Hence thanks to Proposition 2.2 and the fact that \( \mathcal{Y}(0) \leq \left( \| X(0) \|_{L^\infty} + \| \nabla X(0) \|^2_{L^p} \right) \| u \|^2_{H^1} \), by applying Gronwall’s inequality to (7.11), we achieve (2.17).

\[\square\]

8. ENERGY ESTIMATE OF \( \nabla u_X \)

Lemma 8.1. Under the assumptions of Lemma 7.1, for any \( t < T^* \), one has
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) |\mathcal{M}(u_X)|^2 \, dx + \int_{\mathbb{R}^2} \rho |D_t u_X|^2 \, dx 
= \frac{d}{dt} \int_{\mathbb{R}^2} \nabla X^k (-\Delta)^{-1} \text{div} \text{div} (2\mu(\rho) \mathcal{M}(u)) \| \partial_k p u_X \| \, dx 
- \frac{d}{dt} \left( \int_{\mathbb{R}^2} \partial_X \mu(\rho) \mathcal{M}(u) : \mathbb{P} \nabla u_X \, dx + \int_{\mathbb{R}^2} \mu(\rho) \partial_t X^k \mathcal{M}_{ij}(u) [\mathbb{P}^j \partial_k u_i^j] \, dx \right) 
- \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) \left( \partial_t X \cdot \nabla u + \nabla X \cdot \nabla u^i \right) |\partial_k p u_X \| \, dx + r,
\]
where \( \mathbb{P} = (\mathbb{P}^1, \mathbb{P}^2) = Id + \nabla (-\Delta)^{-1} \text{div} \text{div} \) denotes the leray projection operator to the solenoidal vector field spaces, and \( r \) satisfies
\[
| r | \leq C \left( \| \nabla u \|_{L^\infty} + \| \nabla u \|^2_{L^2_L} \right) \| \nabla u_X \|^2_{L^2} + \| D_t u \|^2_{L^2_{\frac{2p}{p-2}}} \| \nabla X \|^2_{L^p} + \| D_t u \|^2_{L^2} 
+ C \left( \| \nabla D_t u \|_{L^2} + \| \nabla u \|^2_{L^2} \right) \| \nabla X \|_{L^p} + \| \nabla u \|_{L^p} \| \nabla u_X \|_{L^2} \) \| \nabla u_X \|^2_{L^2_{\frac{2p}{p-2}}} 
+ C \left( \| \nabla u \|^2_{L^2_{\frac{2p}{p-2}}} + \| \nabla u \|_{L^\infty} \| \nabla u \|^2_{L^2_{\frac{2p}{p-2}}} \right) \| \nabla X \|_{L^p} 
+ \| \nabla u \|^2_{L^2} + \| \nabla u \|_{L^2} \| \nabla u \|_{L^\infty} + \| \nabla D_t u \|_{L^2} \) \| \nabla u_X \|_{L^2}.
Proof. In view of (2.7), we get, by applying Lemma 3.1, that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho)|\mathcal{M}(u_X)|^2 \, dx + \int_{\mathbb{R}^2} \rho |D_t u_X|^2 \, dx \\
\leq C \|\nabla u\|_{L^\infty} \|\nabla u_X\|_{L^2}^2 + \int_{\mathbb{R}^2} (G - \nabla \Pi_X)|D_t u_X| \, dx,
\]
from which and (7.4), we infer
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho)|\mathcal{M}(u_X)|^2 \, dx + \int_{\mathbb{R}^2} \rho |D_t u_X|^2 \, dx \leq C \|\nabla u\|_{L^\infty} \|\nabla u_X\|_{L^2}^2 \\
+ \left| \int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div} \text{div}(2\mu(\rho)\mathcal{M}(u_X))|D_t u_X| \, dx \right| \\
+ \left| \int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(\rho D_t u_X)|D_t u_X| \, dx \right| + \left| \int_{\mathbb{R}^2} \mathbb{P}G|D_t u_X| \, dx \right|.
\]
(8.2)

Thanks to (7.7), we get, by using integration by parts, that
\[
\left| \int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(\rho D_t u_X)|D_t u_X| \, dx \right| = \left| \int_{\mathbb{R}^2} (-\Delta)^{-1} \text{div}(\rho D_t u_X)|\text{div} D_t u_X| \, dx \right| \\
= \left| \int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(\rho D_t u_X)|(2u \cdot \nabla u_X + D_t u \cdot \nabla X)| \, dx \right| \\
\leq C \|\rho D_t u_X\|_{L^2} \left( \|u\|_{L^\infty} \|\nabla u_X\|_{L^2} + \|D_t u\|_{L^{\frac{2p}{p-2}}} \|\nabla X\|_{L^p} \right),
\]
Similarly, one has
\[
\left| \int_{\mathbb{R}^2} \nabla(-\Delta)^{-1} \text{div}(2\mu(\rho)\mathcal{M}(u_X))|D_t u_X| \, dx \right| = \left| \int_{\mathbb{R}^2} (-\Delta)^{-1} \text{div}(2\mu(\rho)\mathcal{M}(u_X)) \left( 2\text{tr}(\nabla u_X \nabla u) + \text{tr}(\nabla X \nabla D_t u) \right) \, dx \right| \\
\leq C \left( \|\nabla u\|_{L^\infty} \|\nabla u_X\|_{L^2}^2 + \|\nabla u_X\|_{L^{\frac{2p}{p-2}}} \|\nabla X\|_{L^p} \|\nabla D_t u\|_{L^2} \right).
\]

It remains to deal with the last term in (8.2), which we shall handle term by term below.

- The estimate of \( \mathbb{P}(\nabla X^k \partial_k \Pi)|D_t u_X| \)

It follows from (2.9) that
\[
\int_{\mathbb{R}^2} \mathbb{P}(\nabla X^k \partial_k \Pi)|D_t u_X| \, dx = \int_{\mathbb{R}^2} \nabla X^k \partial_k (-\Delta)^{-1} \text{div}(\rho D_t u) \|\mathbb{P}D_t u_X\| \, dx \\
- \int_{\mathbb{R}^2} \nabla X^k \partial_k (-\Delta)^{-1} \text{div} (2\mu(\rho)\mathcal{M}(u)) \|\mathbb{P}D_t u_X\| \, dx.
\]

It is easy to observe that
\[
\left| \int_{\mathbb{R}^2} \nabla X^k \partial_k (-\Delta)^{-1} \text{div}(\rho D_t u)|\mathbb{P}D_t u_X| \, dx \right| \leq C \|\nabla X\|_{L^p} \|D_t u\|_{L^{\frac{2p}{p-2}}} \|\nabla u_X\|_{L^2}.
\]

By using integration by parts, one has
\[
- \int_{\mathbb{R}^2} \nabla X^k \partial_k (-\Delta)^{-1} \text{div} (2\mu(\rho)\mathcal{M}(u)) \|\mathbb{P}D_t u_X\| \, dx \\
= \int_{\mathbb{R}^2} \nabla X^k (-\Delta)^{-1} \text{div} (2\mu(\rho)\mathcal{M}(u)) \big( \langle \mathbb{P}(\partial_k u \cdot \nabla u_X) + \mathbb{P}; u \cdot \nabla \rangle \partial_k u_X + D_t \mathbb{P}\partial_k u_X \big) \, dx.
\]
We first deduce that
\[ | \int_{\mathbb{R}^2} \nabla X^k \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) [\mathbb{P}(\partial_t u \cdot \nabla u)] \, dx | \leq C \| \nabla u \|_{L^{4p \sigma}}^2 \| \nabla X \|_{L^p} \| \nabla u_X \|_{L^2}. \]

Applying classical commutator’s estimate yields
\[ | \int_{\mathbb{R}^2} \nabla X^k \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) [\mathbb{P} ; u \cdot \nabla] | \partial_k u_X | \, dx | \leq C \| \nabla X \|_{L^p} \| \nabla u \|_{L^{4p \sigma}} \| \nabla u_X \|_{L^2}. \]

While we get, by using integration by parts, that
\[ \int_{\mathbb{R}^2} \nabla X^k \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) \, dx = \frac{d}{dt} \int_{\mathbb{R}^2} \nabla X^k \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) \, dx \]
\[ - \int_{\mathbb{R}^2} (D_t \nabla X^k \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) + \nabla X^k \nabla u_X \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u))) | \partial_k \mathbb{P} u_X | \, dx. \]

By virtue of (2.5), we find
\[ \int_{\mathbb{R}^2} D_t \nabla X^k \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) \, dx \]
\[ = \int_{\mathbb{R}^2} (-\nabla u \cdot \nabla X^k \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) + \nabla u_X^k \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u))) | \partial_k \mathbb{P} u_X | \, dx, \]

which gives rise to
\[ | \int_{\mathbb{R}^2} D_t \nabla X^k \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) \, dx | \leq C \left( \| \nabla u \|_{L^{4p \sigma}}^2 \| \nabla X \|_{L^p} + \| \nabla u \|_{L^p} \| \nabla u_X \|_{L^{4p \sigma}} \right) \| \nabla u_X \|_{L^2}. \]

Whereas by using a commutator’s argument, we write
\[ \int_{\mathbb{R}^2} \nabla X^k D_t \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) | \partial_k \mathbb{P} u_X | \, dx \]
\[ = \int_{\mathbb{R}^2} (\nabla X^k | u \cdot \nabla \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) + \nabla X^k \Delta^{-1} \text{div} \text{div} (2\mu(\rho)D_t \mathcal{M}(u))) | \partial_k \mathbb{P} u_X | \, dx, \]

which together with the classical commutator estimate implies
\[ | \int_{\mathbb{R}^2} \nabla X^k D_t \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) | \partial_k \mathbb{P} u_X | \, dx | \leq C \| \nabla X \|_{L^p} \| \nabla u \|_{L^{4p \sigma}} \| \nabla u_X \|_{L^2} + (\| \nabla D_t u \|_{L^2} + \| \nabla u \|_{L^4}) \| \nabla u_X \|_{L^{4p \sigma}}. \]

By summarizing the above estimates, we obtain
\[ (8.3) \int_{\mathbb{R}^2} \mathbb{P}(\nabla X^k \partial_k \Pi) | D_t u_X | \, dx = \frac{d}{dt} \int_{\mathbb{R}^2} \nabla X^k \Delta^{-1} \text{div} \text{div} (2\mu(\rho)\mathcal{M}(u)) | \partial_k \mathbb{P} u_X | \, dx + r_1 \]
with \( r_1 \) satisfying
\[
|r_1| \leq C\|\nabla X\|_{L^p} \left( \|D_t u\|_{L^\infty}^2 \|D_t u_X\|_{L^2} + \|\nabla u\|_{L^\infty}^2 \right) + C\|\nabla u\|_{L^p} \|\nabla u_X\|_{L^2}.
\]

\[
+ \left( \|\nabla D_t u\|_{L^2} + \|\nabla u\|_{L^2}^2 \right) \|\nabla u_X\|_{L^\infty} + C\|\nabla u\|_{L^p} \|\nabla u_X\|_{L^2}.
\]

**By using integration by parts, we write**
\[
\int_{\mathbb{R}^2} \mathbb{P} \text{div} (\partial_X \mu(\rho) M(u)) |D_t u_X| \, dx = -\int_{\mathbb{R}^2} \partial_X \mu(\rho) M(u) : \nabla \mathbb{P} D_t u_X \, dx
\]

\[
= -\int_{\mathbb{R}^2} \partial_X \mu(\rho) M(u) : \left( \mathbb{P} \left( \nabla u \cdot \nabla u_X \right) + \left[ \mathbb{P}; u \cdot \nabla \right] \nabla u_X + D_t \mathbb{P} \nabla u_X \right) \, dx,
\]

and
\[
\int_{\mathbb{R}^2} \partial_X \mu(\rho) M(u) : D_t \mathbb{P} \nabla u_X \, dx = \frac{d}{dt} \int_{\mathbb{R}^2} \partial_X \mu(\rho) M(u) : \mathbb{P} \nabla u_X \, dx
\]

\[
- \int_{\mathbb{R}^2} \partial_X \mu(\rho) D_t M(u) : \nabla u_X \, dx,
\]

This leads to
\[
\int_{\mathbb{R}^2} \mathbb{P} \text{div} (\partial_X \mu(\rho) M(u)) |D_t u_X| \, dx = -\frac{d}{dt} \int_{\mathbb{R}^2} \partial_X \mu(\rho) M(u) : \mathbb{P} \nabla u_X \, dx + r_2,
\]

with \( r_2 \) satisfying
\[
|r_2| \leq C \left( \|\nabla u\|_{L^1}^2 + \|\nabla u\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla D_t u\|_{L^2} \right) \|\nabla u_X\|_{L^2}.
\]

**Again by using integration by parts, we write**
\[
\int_{\mathbb{R}^2} \mathbb{P} \partial_t \left( \mu(\rho) \left( \partial_t X \cdot \nabla u + \nabla X \cdot \nabla u^i \right) \right) |D_t u_X| \, dx
\]

\[
= -\int_{\mathbb{R}^2} \mu(\rho) \left( \partial_t X \cdot \nabla u + \nabla X \cdot \nabla u^i \right) \mathbb{P} \partial_t D_t u_X \, dx
\]

\[
= -\int_{\mathbb{R}^2} \mu(\rho) \left( \partial_t X \cdot \nabla u + \nabla X \cdot \nabla u^i \right) \left( \mathbb{P} \left( \partial_t u \cdot \nabla u_X \right) + \left[ \mathbb{P}; u \cdot \nabla \right] \partial_t u_X + D_t \partial_t \mathbb{P} u_X \right) \, dx,
\]

and
\[
\int_{\mathbb{R}^2} \mu(\rho) \left( \partial_t X \cdot \nabla u + \nabla X \cdot \nabla u^i \right) |D_t \partial_t \mathbb{P} u_X| \, dx
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) \left( \partial_t X \cdot \nabla u + \nabla X \cdot \nabla u^i \right) \partial_t \mathbb{P} u_X \, dx
\]

\[
- \int_{\mathbb{R}^2} \mu(\rho) \left( D_t \partial_t X \cdot \nabla u + D_t \nabla X \cdot \nabla u^i \right) \partial_t \mathbb{P} u_X \, dx
\]

\[
- \int_{\mathbb{R}^2} \mu(\rho) \left( \partial_t X \cdot D_t \nabla u + \nabla X \cdot D_t \nabla u^i \right) \partial_t \mathbb{P} u_X \, dx.
\]
Note from (2.5) that $D_t \partial_t X = -\partial_t u \cdot \nabla X + \partial_t u_X$ and $D_t \nabla u = -\nabla u \cdot \nabla + D_t u$, we deduce that
\[
\int_{\mathbb{R}^2} \mathbb{P} \partial_t (\mu(\rho)(\partial_t X \cdot \nabla u + \nabla X \cdot \nabla u')) |D_t u_X| \, dx
\]
\[(8.5)\]
\[
= -\frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) (\partial_t X \cdot \nabla u + \nabla X \cdot \nabla u') |\partial_t \mathbb{P} u_X| \, dx + r_3,
\]
with $r_3$ satisfying
\[
|r_3| \leq C \left( \left( \|\nabla X\|_{L^p} \left( \|\nabla u\|_{L^\frac{2p}{p-2}}^2 + \|\nabla u\|_{L^\frac{2p}{p-2}} \|\nabla u\|_{L^\infty} \right) + \|\nabla u\|_{L^\infty} \|\nabla u_X\|_{L^2} \right) \|\nabla u_X\|_{L^2}^2 + \|\nabla X\|_{L^p} \|\nabla D_t u\|_{L^2} \|\nabla u_X\|_{L^\frac{2p}{p-2}} \right).
\]

- The estimate of \([\mathbb{P}^j (\partial_t X \cdot \nabla (\mu(\rho) M_{ij}(u))) |D_t u_X^j]\)

Along the same line to the proof of (8.5), we write
\[
\int_{\mathbb{R}^2} \mathbb{P}^j (\partial_t X \cdot \nabla (\mu(\rho) M_{ij}(u))) |D_t u_X^j| \, dx
\]
\[
= -\int_{\mathbb{R}^2} \mu(\rho) \partial_t X^k M_{ij}(u) \left( [\mathbb{P}^j (\partial_t u \cdot \nabla u_X^j)] + [\mathbb{P}^j u \cdot \nabla] \partial_t u_X^j + D_t \mathbb{P}^j \partial_t u_X^j \right) \, dx,
\]
and
\[
\int_{\mathbb{R}^2} \mu(\rho) \partial_t X^k M_{ij}(u) |D_t \mathbb{P}^j \partial_t u_X^j| \, dx = \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) \partial_t X^k M_{ij}(u) |\mathbb{P}^j \partial_t u_X^j| \, dx
\]
\[
- \int_{\mathbb{R}^2} \mu(\rho) D_t \partial_t X^k M_{ij}(u) |\mathbb{P}^j \partial_t u_X^j| \, dx - \int_{\mathbb{R}^2} \mu(\rho) \partial_t X^k D_t M_{ij}(u) |\mathbb{P}^j \partial_t u_X^j| \, dx.
\]
Hence we obtain
\[
\int_{\mathbb{R}^2} \mathbb{P}^j (\partial_t X \cdot \nabla (\mu(\rho) M_{ij}(u))) |D_t u_X^j| \, dx
\]
\[(8.6)\]
\[
= -\frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) \partial_t X^k M_{ij}(u) |\mathbb{P}^j \partial_t u_X^j| \, dx + r_4,
\]
with $r_4$ shares the same estimate as $r_3$.

Finally it is trivial to note that
\[
\left| \int_{\mathbb{R}^2} \mathbb{P} (\rho_X D_t u) |D_t u_X| \, dx \right| \leq \|\rho_X\|_{L^\infty} \|D_t u\|_{L^2} \|D_t u_X\|_{L^2}.
\]
Inserting the above estimate and (8.3-8.6) into (8.2) leads to (8.1). This completes the proof of the lemma.

**Lemma 8.2.** Let $p \in \left] 4, 2 (1 + 1/\delta) \right]$. Then under the assumptions of Lemma 7.1 and (1.7), for any $t < T^*$, we have
\[
\|\nabla u_X\|_{L^\frac{2p}{p-2}} \leq C \left( \|\nabla X\|_{L^p} \left( \|\nabla u\|_{L^\frac{2p}{p-2}}^2 + \|D_t u\|_{L^\frac{2p}{p-2}} \right)
\]
\[
+ \|\nabla u\|_{L^\frac{2p}{p-2}} \left( \|D_t u\|_{L^2} + \|D_t u_X\|_{L^2} \right)^\frac{3}{2}
\]
\[
\times \left( \|\nabla u\|_{L^2} + \|\nabla u_X\|_{L^2} + \|\nabla X\|_{L^p} \|\nabla u\|_{L^\frac{2p}{p-2}} + \|X\|_{L^\infty} \|D_t u\|_{L^2} \right)^{1-\frac{2}{p}} \right).
\]
Proof. We first get, by a similar derivation of (5.17), that
\[
\|\nabla u\|_{L^{2p}} \leq C\left(\|\nabla h\|_{L^{2p}} + \|\nabla u\|_{L^{2p}} + \|\nabla u\|_{L^{2p}} \|\nabla X\|_{L^p} + \|\nabla (-\Delta)^{-\frac{1}{2}}P(\partial_X(pD_t u) - \nabla X^k \partial_k \Pi)\|_{L^{2p}}\right).
\]
(8.8)

Recall that \( h := -\nabla (-\Delta)^{-\frac{1}{2}} \text{Tr}(\nabla X \nabla u) \), one has
\[
\|\nabla h\|_{L^{2p}} \leq C\|\nabla X\|_{L^p} \|\nabla u\|_{L^{2p}}.
\]
Whereas note from the momentum equation of (1.1) that
\[
P(\partial_X(pD_t u) - \nabla X^k \partial_k \Pi) = P\partial_X \text{div}(\mu(\rho)M(u)) = P\partial_k [X^k; \partial_i]\mu(M(i)(u)) + P\text{div}(\partial_X \mu(\rho)M(u) + \mu(\rho)\partial_X M(u)),
\]
so that
\[
\|\nabla (-\Delta)^{-\frac{1}{2}}P(\partial_X(pD_t u) - \nabla X^k \partial_k \Pi)\|_{L^2} \leq C\left(\|\nabla u\|_{L^{2p}} \|\nabla X\|_{L^p} + \|\nabla u\|_{L^2} + \|\nabla u_X\|_{L^2}\right).
\]
While thanks to (2.9), we have
\[
\|\nabla (-\Delta)^{-\frac{1}{2}}P(\nabla X^k \partial_k \Pi)\|_{L^2} \leq \|\nabla (-\Delta)^{-\frac{1}{2}}P\partial_k (X^k \nabla (-\Delta)^{-\frac{1}{2}} \text{div}(\rho D_t u))\|_{L^2} + \|\nabla (-\Delta)^{-\frac{1}{2}}P\partial_k (\nabla X^k (-\Delta)^{-\frac{1}{2}} \text{div}(\mu(\rho)M(u)))\|_{L^2} \leq C\left(\|X\|_{L^\infty} \|D_t u\|_{L^2} + \|\nabla X\|_{L^p} \|\nabla u\|_{L^{2p}}\right).
\]
As a result, it comes out
\[
\|\nabla (-\Delta)^{-\frac{1}{2}}P\partial_X(pD_t u)\|_{L^2} \leq C\left(\|\nabla u\|_{L^2} + \|\nabla u_X\|_{L^2}\right) + \|\nabla X\|_{L^p} \|\nabla u\|_{L^{2p}} + \|X\|_{L^\infty} \|D_t u\|_{L^2}.
\]
(8.9)

Then we get, by using 2-D interpolation inequality, that
\[
\|\nabla (-\Delta)^{-\frac{1}{2}}P\partial_X(pD_t u)\|_{L^{2p}} \leq C\|\nabla (-\Delta)^{-\frac{1}{2}}P\partial_X(pD_t u)\|_{L^2} \|\partial_X(pD_t u)\|_{L^{2p}}^{\frac{1}{p}} \|\partial_X(pD_t u)\|_{L^2} \leq C\left(\|D_t u\|_{L^2} + \|D_t u_X\|_{L^2}\right) \|\nabla u\|_{L^2} + \|\nabla u_X\|_{L^2} + \|\nabla X\|_{L^p} \|\nabla u\|_{L^{2p}} + \|X\|_{L^\infty} \|D_t u\|_{L^2}^{1 - \frac{2}{p}}.
\]
It remains to handle the estimate of \( \|\nabla (-\Delta)^{-\frac{1}{2}}P(\nabla X^k \partial_k \Pi)\|_{L^{2p}} \). Indeed it is easy to observe that
\[
\|\nabla (-\Delta)^{-\frac{1}{2}}P\nabla X^k \partial_k (-\Delta)^{-\frac{1}{2}} \text{div}(\rho D_t u)\|_{L^{2p}} \leq C\|\nabla X\|_{L^p} \|D_t u\|_{L^{2p}}.
\]
Along the same line, we have
\[
\|\nabla (-\Delta)^{-\frac{1}{2}}P\partial_k (\nabla X^k (-\Delta)^{-\frac{1}{2}} \text{div}(\mu(\rho)M(u)))\|_{L^{2p}} \leq C\|\nabla X\|_{L^p} \|\nabla u\|_{L^{2p}},
\]
so thanks to (2.9), we obtain
\[
\|\nabla (-\Delta)^{-\frac{1}{2}}P(\nabla X^k \partial_k \Pi)\|_{L^{2p}} \leq C\|\nabla X\|_{L^p} \left(\|D_t u\|_{L^{2p}} + \|\nabla u\|_{L^{2p}}\right).
\]
By substituting the above inequalities into (8.8) leads to the Estimate (8.7). \( \square \)

We are now in a position to present the proof of Proposition 2.8.
Proof of Proposition 2.8. We get, by first multiplying (8.1) by \((t - t_0)\) and then integrating the resulting inequality over \([0, t]\), that

\[
(t - t_0)\|\nabla u_X(t)\|_{L^2}^2 + \int_{t_0}^t (t - t_0) \int_{\mathbb{R}^2} \rho |D_t u_X|^2 \, dx \, dt' \leq C \left( \int_{t_0}^t \|\nabla u_X(t')\|_{L^2}^2 \, dt' + (t - t_0) \left( \|\nabla u\|_{L^{\frac{2p}{p-2}}} \|\nabla X\|_{L^p} + \|\nabla u\|_{L^2} \right) \|\nabla u_X\|_{L^2} + \int_{t_0}^t (t - t_0) r(t') \, dt' \right).
\]

(8.10)

However, by virtue of Lemmas 8.1 and 8.2, we write

\[
\left( t - t_0 \right) \|\nabla u_X(t)\|_{L^2}^2 + C \left( \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^{\frac{2p}{p-2}}}^2 \|\nabla u\|_{L^\infty} \right) \right)
\]

\[
\left( t - t_0 \right) \|\nabla u\|_{L^2}^2 + C \left( \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^{\frac{2p}{p-2}}}^2 \|\nabla u\|_{L^\infty} \right)
\]

(8.11)

where

\[
f_1 \overset{\text{def}}{=} \|\nabla u\|_{L^\frac{p}{p-1}}^\frac{p}{p-1} + \|\nabla u\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty},
\]

\[
f_2 \overset{\text{def}}{=} \|\nabla u\|_{L^\frac{p}{p-1}}^\frac{p}{p-1} \left( \|\nabla u\|_{L^2} + \|\nabla u\|_{L^{\frac{2p}{p-2}}} \|\nabla X\|_{L^p} + \|D_t u\|_{L^2} \|X\|_{L^\infty} \right)^\frac{p-2}{2},
\]

\[
f_3 \overset{\text{def}}{=} \|\nabla D_t u\|_{L^2} + \|\nabla u\|_{L^{\frac{2p}{p-2}}}^2 + \|D_t u\|_{L^\frac{2p}{p-2}} \|\nabla u\|_{L^\infty} \left( \|\nabla u\|_{L^{\frac{2p}{p-2}}} + \|D_t u\|_{L^\frac{2p}{p-2}} \|\nabla u\|_{L^\infty} \right) \right)^\frac{p-2}{2},
\]

\[
f_4 \overset{\text{def}}{=} \|\nabla u\|_{L^\frac{p}{p-1}}^\frac{p}{p-1} \|\nabla X\|_{L^\frac{p}{p-1}},
\]

\[
f_5 \overset{\text{def}}{=} \|\nabla D_t u\|_{L^2} + \|\nabla u\|_{L^2}^\frac{2}{5} \|\nabla X\|_{L^p},
\]

and

\[
g_1 \overset{\text{def}}{=} \left( \|\nabla D_t u\|_{L^2} + \|\nabla u\|_{L^2}^\frac{2}{5} \right) \left( \|\nabla u\|_{L^{\frac{2p}{p-2}}} + \|D_t u\|_{L^\frac{2p}{p-2}} \right)
\]

\[
+ \left( \|\nabla D_t u\|_{L^2} + \|\nabla u\|_{L^2}^\frac{2}{5} \right) \left( \|\nabla u\|_{L^{\frac{2p}{p-2}}} + \|D_t u\|_{L^2} \right)^\frac{p-2}{2} + \|D_t u\|_{L^\frac{2p}{p-2}}^2,
\]

\[
g_2 \overset{\text{def}}{=} \left( \|\nabla D_t u\|_{L^2} + \|\nabla u\|_{L^2}^\frac{2}{5} \right) \left( \|\nabla D_t u\|_{L^2} + \|\nabla u\|_{L^{\frac{2p}{p-2}}} \right)^{1-\frac{2}{5}} \left( \|\nabla u\|_{L^{\frac{2p}{p-2}}} + \|D_t u\|_{L^2} \right)^{1-\frac{2}{5}},
\]

\[
g_3 \overset{\text{def}}{=} \left( \|\nabla D_t u\|_{L^2} + \|\nabla u\|_{L^2}^\frac{2}{5} \right) \left( \|\nabla u\|_{L^{\frac{2p}{p-2}}} \right)^\frac{p-1}{p-2} \|\nabla u\|_{L^2}^\frac{p-2}{p-1},
\]

\[
g_4 \overset{\text{def}}{=} \left( \|\nabla D_t u\|_{L^2} + \|\nabla u\|_{L^2}^\frac{2}{5} \right) \left( \|\nabla u\|_{L^{\frac{2p}{p-2}}} + \|\nabla u\|_{L^2}^{1-\frac{2}{p}} \right) \left( \|\nabla D_t u\|_{L^2} \right).
\]

In what follows, we shall take \(t_0 = \frac{t}{2}\) in (8.10). Next let us handle the right-hand side of (8.10).
By virtue of (6.3), we find
\[
C t \left( \| \nabla u \|_{L^p} \| \nabla X \|_{L^p} + \| \nabla u \|_{L^2} \right) \| \nabla u_X \|_{L^2} \\
\leq \frac{t}{4} \| \nabla u_X \|_{L^2}^2 + C t \left( \| \nabla u \|_{L^p} + \| \nabla u \|_{L^2} \right) \\
\leq \frac{t}{4} \| \nabla u_X \|_{L^2}^2 + C_1 (t)^{-2\delta -} (1 + \| \nabla X \|_{L^p}^2) \).
\]

Whereas by integrating (7.2) over \([t/2, t]\) and using Propositions 2.2 and 2.7, we arrive at

\[
(8.12) \quad C \int_{t/2}^{t} \| \nabla u_X \|_{L^2}^2 \, dt' \leq \| \mathcal{Y}(t/2) \|_{L^2}^2 + C_1 (t)^{-2\delta -} (1 + \| \nabla X \|_{L^p}^2) \, V(t) \\
\leq C_1 (t)^{-2\delta -} (1 + \| \nabla X \|_{L^p}^2) \, V(t).
\]

It remains to handle the last term in (8.10), which we shall deal with term by term below. By applying Proposition 2.2, (6.3) and (8.12), we obtain

\[
\bullet \int_{t/2}^{t} (t' - t/2, \| \nabla u \|_{L^p} \| D_t u \|_{L^2}^2 \| \nabla u_X \|_{L^2}^2 (1 - \frac{1}{p}) \, dt' \\
\leq C t \left( \int_{t/2}^{t} \| \nabla u_X (t') \|_{L^2}^2 \, dt' \right)^{1-p/2} \left( \int_{t/2}^{t} \| \nabla u (t') \|_{L^p}^p \| D_t u (t') \|_{L^2}^2 \, dt' \right)^{p/2} \\
\leq C_1 (t)^{-2\delta -} (1 + \| \nabla X \|_{L^p}^2) \, V(t).
\]

Similarly, we find

\[
\bullet \int_{t/2}^{t} (t' - t/2, f_2(t') \| \nabla u_X (t') \|_{L^2}^{p-2} \, dt' \\
\leq C t \left( \int_{t/2}^{t} \| \nabla u_X (t') \|_{L^2}^2 \, dt' \right)^{2(p-2)/p} \left( \int_{t/2}^{t} f_2(t') \| D_t u (t') \|_{L^2}^2 \, dt' \right)^{2/p} \\
\leq C_1 (t)^{-2\delta -} (1 + \| \nabla X \|_{L^p}^2) \, V(t),
\]

and

\[
\bullet \int_{t/2}^{t} (t' - t/2, f_3(t') \| \nabla u_X (t') \|_{L^2} \, dt' \leq C t \left( \int_{t/2}^{t} \| \nabla u_X (t') \|_{L^2}^2 \, dt' \right)^{1/2} \left( \int_{t/2}^{t} f_3(t')^2 \, dt' \right)^{1/2} \\
\leq C_1 (t)^{-2\delta -} (1 + \| \nabla X \|_{L^p}^2) \, V(t).
\]

Note that

\[
\int_{t/2}^{t} f_4(t') \| D_t u \|_{L^2}^2 \, dt' \leq C \| \nabla X \|_{L^p}^2 \left( \int_{t/2}^{t} (\| D_t u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2) \, dt' \right) \\
\leq C_1 t^{-1} (t)^{-2(\frac{1}{2} + \delta -)} \left( 1 + \sqrt{t} (t)^{\left(\frac{1}{2} + \delta -\right)} \right) \| D_t u \|_{L^2}^2 \),
\]
we deduce from Proposition 2.3 and (8.12) that

\[
\int_{\frac{t}{2}}^{t} (t' - t/2) f_4(t') \| \nabla u_X(t') \|_{L^2}^{\frac{p-2}{p-1}} dt' \\
\leq Ct \left( \int_{\frac{t}{2}}^{t} f_4(t') \frac{2(p-1)}{p} dt' \right)^{\frac{2(p-1)}{2(p-1)}} \left( \int_{\frac{t}{2}}^{t} \| \nabla u_X(t') \|_{L^2}^2 dt' \right)^{\frac{p-2}{2(p-1)}} \\
\leq C_1(t)^{-2\delta} - (1 + \|X\|_{L^p(L^p)}^2 + \|\nabla X\|_{L^p(L^p)}^2) V(t),
\]

and

\[
\int_{\frac{t}{2}}^{t} (t' - t/2) f_5(t') \| \nabla u_X(t') \|_{L^2}^{\frac{p-2}{p}} dt' \\
\leq \| \nabla X \|_{L^p(L^p)} \left( \int_{\frac{t}{2}}^{t} (t' - t/2)^2 \left( \| \nabla D_t u \|_{L^2}^2 + \| \nabla u \|_{L^4}^4 \right) dt' \right)^{\frac{2(p-2)}{p}} \\
\times \left( \int_{\frac{t}{2}}^{t} \| D_t u \|_{L^2}^2 dt' \right)^{\frac{2(p-2)}{2p}} \left( \int_{\frac{t}{2}}^{t} \| \nabla u_X(t') \|_{L^2}^2 dt' \right)^{\frac{p-2}{2p}} \\
\leq C_1(t)^{-2\delta} - (1 + \|X\|_{L^p(L^p)}^2 + \|\nabla X\|_{L^p(L^p)}^2) V(t).
\]

Observing that

\[
\int_{\frac{t}{2}}^{t} (t' - t/2) \| D_t u \|_{L^p}^{\frac{2p}{p+2}} dt' \leq Ct \left( \int_{\frac{t}{2}}^{t} \| D_t u \|_{L^2}^2 dt' \right)^{\frac{1-p}{p}} \left( \int_{\frac{t}{2}}^{t} \| \nabla D_t u \|_{L^2}^2 dt' \right)^{\frac{2}{p}} \\
\leq C(t)^{-2(\frac{2}{p} + \epsilon)} - \left( \frac{2}{p} + \epsilon \right) - D_t u \|_{L^p(L^p)}^{\frac{1-p}{p}} \left( \sqrt{t}(t) \right)^{\frac{1}{p}} \| \nabla D_t u \|_{L^p(L^p)}^{\frac{2}{p}}.
\]

The same estimate holds for the other terms in \( g_1(t) \). As a result, it comes out

\[
\int_{\frac{t}{2}}^{t} (t' - t/2) g_1(t') \left( \| \nabla X(t') \|_{L^p}^2 + \| X(t') \|_{L^\infty}^2 \right) dt' \\
\leq C_1(t)^{-2\delta} - (1 + \|X\|_{L^p(L^p)}^2 + \|\nabla X\|_{L^p(L^p)}^2).
\]

Along the same line, we deduce from Propositions 2.2 and 2.3 that

\[
\int_{\frac{t}{2}}^{t} (t' - t/2) g_2(t') \left( \| \nabla X(t') \|_{L^p}^{2 \left( \frac{1}{p} - \frac{1}{p} \right)} + \| X(t') \|_{L^\infty}^{2 \left( \frac{1}{p} - \frac{1}{p} \right)} \right) dt' \\
\leq Ct \left( \| \nabla X \|_{L^p(L^p)} + \| X \|_{L^p(L^\infty)} \right)^{2 \left( \frac{1}{p} - \frac{1}{p} \right)} \left( \int_{\frac{t}{2}}^{t} \| \nabla D_t u \|_{L^2}^2 + \| \nabla u \|_{L^4}^4 \right) dt' \frac{1}{2} \\
\times \left( \int_{\frac{t}{2}}^{t} \| D_t u \|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left( \int_{\frac{t}{2}}^{t} \| \nabla u \|_{L^2}^{2} + \| \nabla u \|_{L^4}^{2} \right) dt' \frac{n-2}{2p} \\
\leq C_1(t)^{-2\delta} - \left( \| \nabla X \|_{L^p(L^p)} + \| X \|_{L^p(L^\infty)} \right)^{2 \left( \frac{1}{p} - \frac{1}{p} \right)}.
\]
and
\[
\{ t' - t/2 \} g_3(t') \| \nabla X(t') \|_{L_{p^{-1}}(L^p)} \| \nabla X(t') \|_{L_{p^{-1}}(L^p)} \| \nabla u(t') \|_{L^2} \| D_t u(t') \|_{L^{2(p-1)}} dt' \\
\leq C t \| \nabla X \|_{L_{p^{-1}}(L^p)} \left( \int_{\frac{t}{2}}^{t} \| \nabla D_t u \|_{L^2}^2 + \| \nabla u \|_{L^4}^4 \right) dt' \left( \int_{\frac{t}{2}}^{t} \| \nabla u(t') \|_{L^2}^2 dt' \right)^{\frac{p-2}{2(p-1)}} \\
\leq C_1 \left( t \right)^{-\frac{1}{p-1}} - 2 \delta - \| \nabla X \|_{L_{p^{-1}}(L^p)}^{p-1},
\]
and
\[
\{ t' - t/2 \} g_4(t') \| \nabla X(t') \|_{L^p} dt' \leq C t \| \nabla X \|_{L_{p^{-1}}(L^p)} \left( \int_{\frac{t}{2}}^{t} \| \nabla D_t u \|_{L^2}^2 + \| \nabla u \|_{L^4}^4 \right) dt' \frac{1}{2} \\
\times \left( \left( \int_{\frac{t}{2}}^{t} \| \nabla u \|_{L^2}^2 dt' \right)^{\frac{p}{p-1}} + \left( \int_{\frac{t}{2}}^{t} \| \nabla u(t') \|_{L^2}^2 dt' \right)^{\frac{p-2}{2(p-1)}} \left( \int_{\frac{t}{2}}^{t} \| D_t u(t') \|_{L^2}^2 dt' \right)^{\frac{2}{p}} \right) \\
\leq C_1 \left( t \right)^{-\frac{1}{p-1}} - 2 \delta - \| \nabla X \|_{L_{p^{-1}}(L^p)},
\]
and finally
\[
\{ t' - t/2 \} \left( \| D_t u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^{2(p-1)} \right) \| \nabla u \|_{L^\infty} dt' \\
\leq \langle t \rangle^{-2 \delta - \frac{p}{2}} \left( \| D_t u \|_{L^2}^2 + C_1 \| \nabla u \|_{L^2}^4 \right) dt' \\
\leq C_1 \langle t \rangle^{-2 \delta - \frac{p}{2}} \left( 1 + \| \nabla u \|_{L^2}^4 \right).
\]
Consequently, we get, by applying Gronwall’s inequality to (8.10) and then inserting the above estimates to the resulting inequality, that
\[
\| \nabla u X(t) \|_{L^2} \leq C_1 \langle t \rangle^{-2 \delta - (1 + \| \nabla u \|_{L^2}^2 + \| \nabla X \|_{L^\infty}^2) V(t) \rangle.
\]
With the above estimate, by repeating the argument following (4.6), we can prove the estimate of \( \| \nabla u X(t) \|_{L^2} \) in (2.18).

Under the additional assumption that \( \partial_{X_0} u_0 \in H^1(\mathbb{R}^2) \), we get by a similar derivation of (8.13) that
\[
\| \nabla u X \|_{L^2}^2 + \| D_t u \|_{L^2}^2 \leq C_1 \left( 1 + \| \nabla u X_0 \|_{L^2}^2 + \| \nabla X \|_{L^\infty}^2 \right) V(t),
\]
which together with (2.18) ensures (2.19). This completes the proof of Proposition 2.8. \( \square \)

9. The estimate of \( \| \nabla \partial_X X \|_{L^2} \)

Lemma 9.1. Let \((\rho, u, X)\) be a smooth enough solution of the coupled system of (1.1) with (2.5). We assume that \( \partial_{X_0} u_0 \in L^6(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \) and \( \partial_{X_0}^2 u_0(\rho_0) \in L^2(\mathbb{R}^2) \). Then under the smallness assumption (1.7), we have
\[
\| \nabla \partial_X^2 u \|_{L^2} \leq C \left( \left( \| \Pi_2 \|_{L^\infty} + \| \nabla u \|_{L^\infty} \right) \| \nabla \partial_X X \|_{L^2} + \| \nabla u \|_{L^4} \right) \| \nabla X \|_{L^4} \\
+ \| \nabla X \|_{L^8}^2 + \| \nabla u \|_{L^3} \| \nabla X \|_{L^6} + \| \partial_X u \|_{L^6} \| \nabla u \|_{L^2} \\
+ \| \nabla u \|_{L^\infty} \| \partial_X^2 \|_{L^2} + \| \nabla X \|_{L^4}^2 \\
+ \| X \|_{L^\infty} \left( \| D_t u \|_{L^2} + \| D_t u X \|_{L^2} + \| \nabla X \|_{L^4} D_t u \|_{L^4} \right),
\]
where \( \Pi_2 \) is given by (2.9).
Proof. According to (5.15), we write
\[
\nabla \partial_X^2 u = \nabla \left[ \partial_X \partial_X u + \partial_X \nabla h + \partial_X \nabla (-\Delta)^{-1} \text{div} (\mu \mathcal{M}(h)) \right] \\
- \partial_X \nabla (-\Delta)^{-1} \mathbb{P} (\rho D_t u - G) + \partial_X \nabla (-\Delta)^{-1} \mathbb{P} (\mu - 1) \mathcal{M}(u_X - h).
\]

In what follows, we shall handle the estimates of the above terms.

• The estimate of $\partial_X \nabla h$

Recall that $h \overset{\text{def}}{=} - \nabla (-\Delta)^{-1} \text{Tr}(\nabla \nabla u)$, we write
\[
\partial_X \nabla h = -[\partial_X ; \nabla^2 (-\Delta)^{-1}] \text{Tr}(\nabla \nabla u) - \nabla^2 (-\Delta)^{-1} \text{Tr}(\partial_X \nabla X \partial_X \nabla u) + \nabla \partial_X \nabla u,
\]
from which and the classical commutator’s estimate, we infer
\[
\| \partial_X \nabla h \|_{L^2} \lesssim (\| \nabla X \|_{L^4}^2 + \| \nabla \partial_X X \|_{L^2}) \| \nabla u \|_{L^\infty} + \| \nabla X \|_{L^6} \| \nabla u_X \|_{L^3}.
\]

• The estimate of $\partial_X \nabla (-\Delta)^{-1} \mathbb{P} \text{div} (\mu \mathcal{M}(h))$

We first write
\[
\partial_X \nabla (-\Delta)^{-1} \mathbb{P} \text{div} (\mu \mathcal{M}(h)) = [\partial_X ; \nabla (-\Delta)^{-1} \mathbb{P} \text{div} (\mu \mathcal{M}(h))]
\]
\[
+ \nabla (-\Delta)^{-1} \mathbb{P} \text{div} (\partial_X \mu \mathcal{M}(h)) + \mu \partial_X \mathcal{M}(h),
\]
from which and (9.3), we deduce from the classical commutator’s estimate that
\[
\| \partial_X \nabla (-\Delta)^{-1} \mathbb{P} \text{div} (\mu \mathcal{M}(h)) \|_{L^2} \lesssim \| \nabla X \|_{L^4} \| \nabla h \|_{L^4} + \| \nabla h \|_{L^2} + \| \partial_X \nabla h \|_{L^2}
\]
\[
\lesssim (\| \nabla X \|_{L^2}^2 + \| \nabla X \|_{L^2}^2 + \| \nabla \partial_X X \|_{L^2}) \| \nabla u \|_{L^\infty} + \| \nabla X \|_{L^6} \| \nabla u_X \|_{L^3}.
\]

• The estimate of $\partial_X \nabla (-\Delta)^{-1} \mathbb{P} (\mu (u_X - h))$

We first write
\[
\partial_X \nabla (-\Delta)^{-1} \mathbb{P} (\mu (u_X - h)) = [\partial_X ; \nabla (-\Delta)^{-1} \mathbb{P} \text{div} (\mu (u_X - h))]
\]
\[
+ \nabla (-\Delta)^{-1} \mathbb{P} \text{div} (\mu (u_X - h)) \partial_X \mathcal{M}(h) + \mu \partial_X \mathcal{M}(h),
\]
from which, and the commutator’s estimate, we infer
\[
\| \partial_X \nabla (-\Delta)^{-1} \mathbb{P} (\mu (u_X - h)) \|_{L^2}
\]
\[
\lesssim \| \partial_X \mu \|_{L^6} \| \nabla u_X \|_{L^4} + \| \partial_X \mu \|_{L^\infty} \| \nabla h \|_{L^2} + \| \mu (u_X - h) \|_{L^2}^2
\]
\[
\times (\| \nabla h \|_{L^4} \| \nabla X \|_{L^4} + \| \nabla u_X \|_{L^6} \| \nabla X \|_{L^6} + \| \partial_X \nabla h \|_{L^2}^2).
\]

Then by virtue of (5.14) and (9.3), we achieve
\[
\| \partial_X \nabla (-\Delta)^{-1} \mathbb{P} (\mu (u_X - h)) \|_{L^2}
\]
\[
\lesssim \| \partial_X \mu \|_{L^6} \| \nabla u_X \|_{L^4} \| \nabla u \|_{L^\infty} \| \nabla X \|_{L^2} + \| \mu (u_X - h) \|_{L^2}^2
\]
\[
\times (\| \nabla \partial_X u \|_{L^2}^2 + (\| \nabla X \|_{L^2}^2 + \| \nabla \partial_X X \|_{L^2}) \| \nabla u \|_{L^\infty} + \| \nabla X \|_{L^6} \| \nabla u_X \|_{L^3}^2).
\]

• The estimate of $\partial_X \nabla (-\Delta)^{-1} \mathbb{P} (\rho D_t u - G)$

It is easy to observe that
\[
\| \partial_X \nabla (-\Delta)^{-1} \mathbb{P} (\rho D_t u) \|_{L^2} \lesssim \| X \|_{L^\infty} \| D_t u \|_{L^2} + \| D_t u_X \|_{L^2}.
\]

While in view of (2.9), we have
\[
\| \partial_X \nabla (-\Delta)^{-1} \mathbb{P} (\partial_J X \cdot \nabla \Pi_1) \|_{L^2} \lesssim \| X \|_{L^\infty} \| \nabla X \|_{L^4} \| \nabla \Pi_1 \|_{L^4}
\]
\[
\lesssim \| X \|_{L^\infty} \| \nabla X \|_{L^4} \| D_t u \|_{L^4}.
\]
To estimate $\partial_X \nabla (-\Delta)^{-1}P(\partial_j X \cdot \nabla \Pi_2)$, we write

$$\partial_X \nabla (-\Delta)^{-1}P(\partial_j X \cdot \nabla \Pi_2) = [\partial_X ; \nabla (-\Delta)^{-1}\partial_j X \nabla \Pi_2] + \nabla (-\Delta)^{-1}P \partial_X (\partial_j X \Pi_2),$$

which implies

$$\| \partial_X \nabla (-\Delta)^{-1}P(\partial_j X \cdot \nabla \Pi_2) \|_{L^2} \lesssim \| \nabla X \|_{L^6}^2 \| \Pi_2 \|_{L^4} + \| \nabla \partial_X X \|_{L^2} \| \Pi_2 \|_{L^\infty} + \| \partial_j X \partial_X \Pi_2 \|_{L^2}. \tag{9.4}$$

Yet note from (2.9) that

$$\partial_X \Pi_2 = [\partial_X ; (-\Delta)^{-1} \partial_j X \nabla \Pi_2] + (-\Delta)^{-1} \partial_j X \nabla (2\mu(\rho)\mathcal{M}(u)) + (-\Delta)^{-1} \partial_j X \nabla (2\mu(\rho)\partial_X \mathcal{M}(u)), \tag{9.4}$$

from which, we infer

$$\| \partial_j X \partial_X \Pi_2 \|_{L^2} \lesssim (\| \nabla X \|_{L^8}^2 + \| \nabla X \|_{L^4} \| \Pi_2 \|_{L^4} + \| \nabla \partial_X X \|_{L^2} \| \Pi_2 \|_{L^\infty} + \| \partial_X \partial_X \Pi_2 \|_{L^2} \| \Pi_2 \|_{L^\infty}.$$ 

Inserting the above estimates into (9.4) gives rise to

$$\| \partial_X \nabla (-\Delta)^{-1}P(\partial_j X \cdot \nabla \Pi_2) \|_{L^2} \lesssim (\| \nabla X \|_{L^8}^2 + \| \nabla X \|_{L^4} \| \Pi_2 \|_{L^4} + \| \nabla \partial_X X \|_{L^2} \| \Pi_2 \|_{L^\infty} + \| \partial_X \partial_X \Pi_2 \|_{L^2} \| \Pi_2 \|_{L^\infty}. \tag{9.4}$$

Once again through a commutative argument, we write

$$\partial_X \nabla (-\Delta)^{-1}P\partial_i (2\partial_X \mu(\rho)\mathcal{M}_{ij}(u)) = [\partial_X ; \nabla (-\Delta)^{-1}P\partial_i (2\partial_X \mu(\rho)\mathcal{M}_{ij}(u)) + \nabla (-\Delta)^{-1}P\partial_i (2\partial_X \mu(\rho)\mathcal{M}_{ij}(u)) + 2\partial_X \mu(\rho)\partial_X \mathcal{M}_{ij}(u)], \tag{9.4}$$

from which, we infer

$$\| \partial_X \nabla (-\Delta)^{-1}P\partial_i (2\partial_X \mu(\rho)\mathcal{M}_{ij}(u)) \|_{L^2} \lesssim (\| \nabla X \|_{L^4} \| \Pi_2 \|_{L^4} + \| \nabla \partial_X \Pi_2 \|_{L^2} \| \Pi_2 \|_{L^\infty} + \| \partial_X \Pi_2 \|_{L^2} \| \Pi_2 \|_{L^\infty}. \tag{9.4}$$

To deal with the remaining terms in (5.16), we write

$$\partial_X \nabla (-\Delta)^{-1}P\partial_i \mu(\rho)\partial_j X \cdot \nabla u^j + \partial_j X \cdot \nabla u^j)$$

$$= [\partial_X ; \nabla (-\Delta)^{-1}P\partial_i \mu(\rho)\partial_j X \cdot \nabla u^j + \partial_j X \cdot \nabla u^j) + \nabla (-\Delta)^{-1}P\partial_i \mu(\rho)\partial_j X \cdot \nabla u^j + \partial_j X \cdot \nabla u^j) + \nabla (-\Delta)^{-1}P\partial_i \mu(\rho)\partial_j X \cdot \nabla u^j + \partial_j X \cdot \nabla u^j), \tag{9.4}$$

and

$$\partial_X \nabla (-\Delta)^{-1}P \partial_i \mu(\rho)\partial_j X \cdot \nabla (2\mu(\rho)\mathcal{M}_{ij}(u)) = [\partial_X ; \nabla (-\Delta)^{-1}P \partial_i \mu(\rho)\partial_j X \cdot \nabla (2\mu(\rho)\mathcal{M}_{ij}(u)) + \nabla (-\Delta)^{-1}P \partial_i \partial_X (2\mu(\rho)\mathcal{M}_{ij}(u)) + \nabla (-\Delta)^{-1}P \partial_i \partial_X (2\mu(\rho)\mathcal{M}_{ij}(u)) \tag{9.4},$$

from which, we deduce that

$$\| \partial_X \nabla (-\Delta)^{-1}P\partial_i \mu(\rho)\partial_j X \cdot \nabla u^j + \partial_j X \cdot \nabla u^j) \|_{L^2} \lesssim (\| \nabla X \|_{L^8}^2 + \| \nabla X \|_{L^4} \| \Pi_2 \|_{L^4} + \| \nabla \partial_X \Pi_2 \|_{L^2} \| \Pi_2 \|_{L^\infty} \| \nabla \partial_X X \|_{L^2}, \tag{9.4}$$

and

$$\| \partial_X \nabla (-\Delta)^{-1}P \partial_i \mu(\rho)\partial_j X \cdot \nabla (2\mu(\rho)\mathcal{M}_{ij}(u)) \|_{L^2} \lesssim (\| \nabla X \|_{L^8}^2 \| \Pi_2 \|_{L^4} + \| \nabla X \|_{L^4} \| \Pi_2 \|_{L^4} + \| \nabla X \|_{L^4} \| \Pi_2 \|_{L^\infty} \| \nabla \Pi_2 \|_{L^2}. \tag{9.4}$$
Hence in view of (5.16), by summarizing the above estimates, we achieve
\[
\| \partial_X \nabla (-\Delta)^{-1} \mathbb{P}(\rho \partial_t u_X - G) \|_{L^2} \lesssim (\| \Pi_2 \|_{L^\infty} + \| \nabla u \|_{L^\infty}) \| \nabla \partial_X X \|_{L^2}
\]
\[
+ \| \nabla u \|_{L^4} (\| \nabla X \|_{L^4} + \| \nabla X \|_{L^2}) + \| \nabla u \|_{L^\infty} \| \partial_X^2 \mu(\rho) \|_{L^2}
\]
\[
+ \| \nabla u_X \|_{L^4} (\| \nabla X \|_{L^8} + \| \partial_X \mu(\rho) \|_{L^8})
\]
\[
+ \| X \|_{L^\infty} (\| D_t u \|_{L^2} + \| D_t u_X \|_{L^2} + \| \nabla X \|_{L^4} D_t u \|_{L^4}).
\]

Thanks to (9.2) and (5.14), by summarizing the above estimates and using the smallness condition (1.7), we obtain (9.1). This completes the proof of Lemma 9.1. \[\Box\]

**Proof of Proposition 2.9.** Taking \( \partial_X \) to (2.5) yields
\[
D_t \partial_X X = \partial_X^2 u.
\]
Taking \( \nabla \) to the above equation and then taking \( L^2 \) inner product of the resulting equation with \( \nabla \partial_X X \), we find
\[
\frac{d}{dt} \| \nabla \partial_X X(t) \|_{L^2} \leq \| \nabla u \|_{L^\infty} \| \nabla \partial_X X(t) \|_{L^2} + \| \nabla \partial_X^2 u \|_{L^2}.
\]

On the other hand, by taking \( p = 6 \) in (8.7) and inserting the resulting inequality to (9.1), and then substitute it to (9.5), we achieve
\[
\frac{d}{dt} \| \nabla \partial_X X(t) \|_{L^2} \leq C_2 \left( \| \nabla u \|_{L^\infty} + \| \Pi_2 \|_{L^\infty} \right) \| \nabla \partial_X X(t) \|_{L^2} + \| D_t u_X \|_{L^2} \| X \|_{L^\infty}
\]
\[
+ \left[ \| D_t u \|_{L^2} + \| D_t u_X \|_{L^2} \right] \frac{1}{3} \left( \| \nabla u \|_{L^2} + \| \nabla u \|_{L^3} + \| D_t u \|_{L^2} + \| \nabla X \|_{L^2} \right)^{\frac{1}{3}}
\]
\[
+ \| \nabla u \|_{L^3} + \| \nabla u \|_{L^\infty} + \| D_t u \|_{L^2} + \| D_t u \|_{L^6} \left( 1 + \| X \|_{L^\infty}^2 + \| \nabla X \|_{L^4}^2 + \| \nabla X \|_{L^6}^2 \right).
\]

Note that for \( \delta \in [1/3, 1/2]\), we deduce from Propositions 2.7 and 2.8 that
\[
\int_0^t \| D_t u_X \|_{L^2} \| \nabla u_X \|_{L^2} dt' \leq \| (t)^{\delta + \frac{1}{2}} D_t u_X \|_{L^2}^{\frac{1}{2}} \| (t)^{\delta - \nabla u_X \|_{L^2}^{\frac{3}{2}}}
\]
\[
\times \left( \int_0^t (t')^{-(2\delta + 1)} dt' \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \| (t)^{\delta + \frac{1}{2}} D_t u_X \|_{L^2} + \| (t)^{\delta - \nabla u_X \|_{L^2}} \right).
\]

Therefore, (2.20) follows from Gronwall's inequality, Propositions 2.2 and 2.3, and Propositions 2.7 and 2.8. This completes the proof of Proposition 2.9. \[\Box\]

**Appendix A. The commutative estimate**

Let us first recall the following lemma from [14].

**Lemma A.1.** Let \( p, q \in ]1, \infty[ \) or \( p = q = \infty \). Let \( \{ f_j \}_{j \in \mathbb{Z}} \) be a sequence of functions in \( L^p(\mathbb{R}^d) \) so that \( (f_j(x))_{x \in \mathbb{Z}} \in L^p(\mathbb{R}^d) \). Then there holds
\[
\| (M(f_j)(x))_{x} \|_{L^p} \leq C \| (f_j(x))_{x} \|_{L^p}.
\]

**Proposition A.1.** Let \( p, r \in ]1, \infty[ \) and \( q \in ]1, \infty[ \) satisfying \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). Let \( X = (X^1, \cdots, X^d) \in W^{1, p}(\mathbb{R}^d) \) with \( \text{div} X = 0 \), \( g \in L^q(\mathbb{R}^d) \), \( R_i = \partial_i (-\Delta)^{-\frac{1}{2}} \) be the Riesz transform. Then one has
\[
\| [\partial_X; R_i R_j] g \|_{L^r} \leq C \| \nabla X \|_{L^p} \| g \|_{L^q}.
\]
Proof. We first get by, applying Bony’s decomposition that
\begin{equation}
[\partial_X; R_i R_j]f = [T_X^k; R_i R_j]\partial_k g + T_{\partial_k R_i R_j g} X^k - R_i R_j (T_{\partial_k g} X^k) + R(X^k, \partial_k R_i R_j g) - R_i R_j (R(X^k, \partial_k g)).
\end{equation}
(A.2)

In view of (2.27) of [4], one has
\begin{equation}
|\Delta_j (\partial_k R_i R_j g)(x)| \leq C 2^j M g(x) \implies |S_j (\partial_k R_i R_j g)(x)| \leq C 2^j M g(x),
\end{equation}
(A.3)
from which, we infer
\[ |\Delta_j (T_{\partial_k R_i R_j g} X^k)(x)| \leq C \sum_{|j'-j| \leq 4} M (S_{j'-1} (\partial_k R_i R_j g) \Delta_{j'} X^k)(x) \]
\[ \leq C \sum_{|j'-j| \leq 4} M (M g 2^j' \Delta_{j'} X^k)(x). \]

As a result, we deduce from Lemma A.1 that
\[ \left\| T_{\partial_k R_i R_j g} X^k \right\|_{L^r} \leq C \left\| \left\{ \sum_{j \in \mathbb{Z}} [M (M g 2^j' \Delta_{j'} X^k)]^2 \right\}^{\frac{1}{2}} \right\|_{L^r} \]
\[ \leq C \left\| M g \left\{ \sum_{j \in \mathbb{Z}} [2^j \Delta_j X^k]^2 \right\}^{\frac{1}{2}} \right\|_{L^r} \]
\[ \leq C \left\| M g \right\|_{L^q} \left\| \left\{ \sum_{j \in \mathbb{Z}} [2^j \Delta_j X^k]^2 \right\}^{\frac{1}{2}} \right\|_{L^p} \leq C \left\| g \right\|_{L^q} \left\| \nabla X \right\|_{L^p}. \]

The same estimate holds for $T_{\partial_k g} X^k$, so that we obtain
\[ \left\| R_i R_j (T_{\partial_k g} X^k) \right\|_{L^r} \leq C \left\| T_{\partial_k g} X^k \right\|_{L^r} \leq C \left\| g \right\|_{L^q} \left\| \nabla X \right\|_{L^p}. \]

While due to div $X = 0$ and (A.3), we write
\[ |\Delta_j R(X^k, \partial_k R_i R_j g)(x)| = |\partial_k \Delta_j R(X^k, R_i R_j g)(x)| \]
\[ \leq C 2^j M \left( \sum_{j' \geq j-5} \Delta_{j'} X^k \Delta_{j'} R_i R_j g)(x) \right) \]
\[ \leq C M \left( \sum_{j' \geq j-5} 2^{j-j'} |2^{j'} \Delta_{j'} X^k| M g)(x), \right) \]
from which, we infer
\[ \left\| R(X^k, \partial_k R_i R_j g) \right\|_{L^r} \leq \left\| \left\{ \sum_{j} \left[ \sum_{j' \geq j-5} 2^{j-j'} |2^{j'} \Delta_{j'} X^k| M g \right]^2 \right\}^{\frac{1}{2}} \right\|_{L^r} \]
\[ \leq C \left\| M g \left\{ \sum_{j \in \mathbb{Z}} [2^j \Delta_j X^k]^2 \right\}^{\frac{1}{2}} \right\|_{L^r} \leq C \left\| g \right\|_{L^q} \left\| \nabla X \right\|_{L^p}. \]

The same estimate holds for $R_i R_j (R(X^k, \partial_k g))$.

Finally let us turn to the first term on the right hand side of (A.2). We first get, by a similar derivation of (5.5), that
\[ [T_X^k; R_i R_j] \partial_k g = \sum_{|j-j'| \leq 4} 2^{2j} \int_{\mathbb{R}^2} \tilde{\theta}(2^j z) \int_0^1 \Delta_j \nabla X(x - \tau z) \cdot z \, d\tau S_j \partial_k g(x - z) \, dz, \]
where $\theta(\xi) \overset{\text{def}}{=} \xi_i \xi_j |\xi|^{-2} \phi(\xi)$. Then we deduce from (A.3) that
\[
|[T_{X^k}; R_i R_j] \partial_k g| \leq \sum_{|j-l| \leq 4} 2^l \int_{\mathbb{R}^2} \Psi(2^l z) \int_0^1 |\Delta_j \nabla X(x - \tau z)| d\tau 2^j Mg(x - z) \, dz,
\]
for $\Psi(z) \overset{\text{def}}{=} |z||\theta(z)|$. Now since $r \in ]1, \infty[$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$, we can choose $\alpha, \beta \in ]0,1[$ satisfying
\[
\alpha + \beta = 1, \quad \alpha > \frac{1}{2}, \quad p\alpha > 1 \quad \text{and} \quad q\beta > 1.
\]
We get, by applying Hölder’s inequality, that
\[
|[T_{X^k}; R_i R_j] \partial_k g(x)| \leq \sum_{|j-l| \leq 4} \left( \int_0^1 2^l \int_{\mathbb{R}^2} \Psi(2^l z) |\Delta_j \nabla X(x - \tau z)|^\frac{1}{r} \, dz \, d\tau \right)^\alpha \times \left( 2^l \int_{\mathbb{R}^2} \Psi(2^l z) [Mg(x - z)]^\frac{1}{r} \, dz \right)^\beta
\leq \sum_{|j-l| \leq 4} [M(|\Delta_j \nabla X|^{\frac{1}{r}})]^\alpha [M(Mg)^{\frac{1}{r}}]^\beta,
\]
from which, (A.4) and Lemma A.1, we deduce that
\[
\| [T_{X^k}; R_i R_j] \partial_k g \|_{L^r} \leq \left\| \left\{ \sum_{j \in \mathbb{Z}} [M(|\Delta_j \nabla X|^{\frac{1}{r}})]^{2\alpha} \right\}^{1/2} \right\|_{L^{p\alpha}}^{\alpha} \left\| M(Mg)^{\frac{1}{r}} \right\|_{L^{q\beta}}^{\beta} \leq C \left\| \left\{ \sum_{j \in \mathbb{Z}} [2^{j'} \Delta_{j'} X^{k}]^{2} \right\}^{\frac{1}{2}} \right\|_{L^p} \left\| Mg^\frac{1}{r} \right\|_{L^{q\beta}}^{\beta} \leq C \| \nabla X \|_{L^p} \| g \|_{L^q}.
\]
By summing up the above estimate, we conclude the proof of (A.1). \qed

**APPENDIX B. LIPSCHITZ ESTIMATE OF ELLIPTIC EQUATION OF DIVERGENCE FORM**

The goal of this appendix is to generalize Proposition 2.4 to elliptic equation of divergence form with bounded coefficients which may have a small gap across a surface. The main result reads

**Proposition B.1.** Let $p \in ]2, \infty[$, $X = (X_\lambda)_{\lambda \in \Lambda}$ be a non-degenerate family of vector fields in the sense of Definition 1.1 with $X_\lambda \in C^1_b(\mathbb{R}^2)$ and $\nabla X_\lambda \in L^p(\mathbb{R}^2)$ for each $\lambda \in \Lambda$. Let $a_{ij} \in L^\infty(\mathbb{R}^2)$ with $\sup_{\lambda \in \Lambda} \| \partial_{X_\lambda} a_{ij} \|_{L^\infty} \leq C$ and $\| (a_{ij})_{ij} - Id \|_{L^\infty} \leq \varepsilon_0$ for some $\varepsilon_0$ sufficiently small. We assume moreover that $f \in L^p \cap L^{2p/(p+2)}(\mathbb{R}^2)$ and $\partial_{X_\lambda} f \in L^{2p/(p+2)}(\mathbb{R}^2)$ for $\lambda \in \Lambda$. Then the following equation
\[
(B.1) \quad \sum_{i,j=1}^2 \partial_i (a_{ij} \partial_j u) = f \quad \text{for} \quad x \in \mathbb{R}^2,
\]
has a unique solution \( u \in \hat{W}^{1,p}(\mathbb{R}^2) \cap \hat{W}^{1,\infty}(\mathbb{R}^2) \) so that for any \( s \in ]2/p, 1[ \),
\[
\| \nabla u \|_{L^\infty} \leq C_s \left( \varepsilon_0 \left[ 1 + C(s,p,X) \sup_{\lambda \in \Lambda} \| X \lambda \|_{L^\infty} \left( \| \nabla X \lambda \|_{L^p}^{2/p} + \| \nabla X \lambda \|_{L^\infty}^{2/p} \right) \right] \| f \|_{L^{2/p}}^{2/p} 
+ \| f \|_{L^{2/p}} \left[ \frac{1 - \frac{2}{p}}{L^{2/p}} + \frac{C}{I(X)} \left( \varepsilon_0 \| X \lambda \|_{L^\infty} \| f \|_{L^{2/p}} \right)^{1 - \frac{2}{p}} \right] \right) 
\times \left[ 1 + \varepsilon_0 \| \nabla X \lambda \|_{L^\infty} \right] \| f \|_{L^{2/p}} \left[ \frac{2}{2 + p} + \varepsilon_0 \| \partial X \lambda f \|_{L^{2/p}} \right]^{2/p},
\]
for \( C(s,p,X) \) given by (6.9).

**Proof.** The proof of this proposition consists in the estimate of the striated regularity of the solution (B.1) and then applying Proposition 2.4. Let us denote
\[
\Theta \overset{\text{def}}{=} ((a_{ij})_{2 \times 2} - \text{Id}) \nabla u,
\]
For simplicity, we just present the *a priori* estimate for smooth enough solutions of (B.1). We first write
\[
\nabla u = \sum_{i,j=1}^{2} \nabla \left( \Delta \right)^{-1} \partial_i((a_{ij} - \delta_{ij}) \partial_j u) + \nabla \Delta^{-1} f,
\]
from which, we infer
\[
\| \nabla u \|_{L^p} \leq C \left( \| \Theta \|_{L^p} + \| f \|_{L^{2/p}} \right) \leq C \left( \varepsilon_0 \| \nabla u \|_{L^p} + \| f \|_{L^{2/p}} \right).
\]
So that by taking \( \varepsilon_0 \) sufficiently small, we obtain
\[
\| \nabla u \|_{L^p} \leq C \| f \|_{L^{2/p}}.
\]
Whereas for any \( C^1 \) vector field \( X \), we get, by applying \( \partial_X \) to (B.1), that
\[
\sum_{i,j=1}^{2} \partial_i(a_{ij} \partial_j X u) = \partial_X f - \sum_{i,j=1}^{2} \partial_i(\partial_X a_{ij} \partial_j u) - \sum_{i,j=1}^{2} \partial_i(a_{ij} \partial_j X \partial_j u).
\]
Then along the same line to proof of (B.4), we deduce
\[
\| \nabla \partial_X u \|_{L^p} \leq C \left( \| \partial_X f \|_{L^{2/p}} + \left( \| \partial_X a_{ij} \|_{L^\infty} + \| \nabla X \|_{L^\infty} \right) \| \nabla u \|_{L^p} \right)
\leq C \left( \| \partial_X f \|_{L^{2/p}} + \left( \| \partial_X a_{ij} \|_{L^\infty} + \| \nabla X \|_{L^\infty} \right) \| f \|_{L^{2/p}} \right).
\]
On the other hand, for any \( s \in ]2/p, 1[ \), we deduce from Proposition 2.4 and (B.3) that
\[
\| \nabla u \|_{L^\infty} \leq C \left( \| \Theta \|_{L^p} + \| \Theta \|_{L^\infty} \right) + \| \nabla \Delta^{-1} f \|_{L^\infty} + \frac{C_s}{I(X)} \sup_{\lambda \in \Lambda} \left( \| \nabla X \lambda \|_{L^\infty} \| \Theta \|_{L^p} \right)^{1 - \frac{2}{p}}
\times \left( \| \nabla X \lambda \|_{L^p} \| \Theta \|_{L^\infty} + \| \partial_X \Theta \|_{L^p} \right)^{2/p} + \left( \| \nabla X \lambda \|_{L^\infty} \| \Theta \|_{L^p} \right)^{1 - \frac{2}{p} \frac{2}{ps}} \left( \| \nabla X \lambda \|_{L^p} \| \Theta \|_{L^\infty} \right)^{2/p}.
\]
Due to \( p \in ]2, \infty[ \), we have
\[
\| \nabla \Delta^{-1} f \|_{L^\infty} \leq C \| f \|_{L^p} \| f \|_{L^{2/p}}^{1 - \frac{2}{p}}.
\]
Applying Young’s inequality yields
\[
\frac{C}{I(X)}(\varepsilon_0 \|X\|_{L^\infty} \|\nabla u\|_{L^p})^{1-\frac{2}{p}} (\varepsilon_0 \|\nabla X\|_{L^p} \|\nabla u\|_{L^\infty})^{\frac{2}{p}} \\
\leq C \varepsilon_0 \left( \|\nabla u\|_{L^\infty} + I(X)^{-\frac{p}{p-2}} \|X\|_{L^\infty} \|\nabla X\|_{L^p}^{\frac{2}{p-2}} \|\nabla u\|_{L^p} \right).
\]
As a result, it comes out
\[
\|\nabla u\|_{L^\infty} \leq C(\varepsilon_0 \|X\|_{L^\infty} \|\nabla u\|_{L^p})^{1-\frac{2}{p}} + C \varepsilon_0 \left( \|\nabla u\|_{L^p} + \|\nabla u\|_{L^\infty} \right) \\
+ C((s, p, X)) \|X\|_{L^\infty} \left( \|\nabla X\|_{L^p}^{\frac{2}{p-2}} + \|\nabla X\|_{L^p}^{\frac{2}{p-2}} \|\nabla u\|_{L^p} \right) \\
+ \frac{C}{I(X)} \sup_{\lambda \in \Lambda} (\varepsilon_0 \|X\|_{L^\infty} \|\nabla u\|_{L^p})^{1-\frac{2}{p}} (\|\nabla u\|_{L^p} + \varepsilon_0 \|\nabla \partial X u\|_{L^p})^{\frac{2}{p}}.
\]
By taking $\varepsilon_0$ sufficiently small and inserting the Estimates (B.4) and (B.5) to the above inequality, we achieve (B.2).

\[\square\]

**Remark B.1.** By repeating the proof of Proposition B.1, we can prove the same Lipschitz estimate (B.2) for the solutions of the following Stokes type system:
\[
\sum_{i,j=1}^{2} \partial_i (a_{ij} \partial_j u) = f - \nabla p \quad \text{and} \quad \text{div} \, u = 0 \quad \text{for} \quad x \in \mathbb{R}^2.
\]
We can even work for the above problems in the multi-dimensional case.

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