An approximate global stationary axisymmetric solution of Einstein equations

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We present an approximate global solution of Einstein’s equations which describes a simple star model: a self-gravitating perfect fluid with constant density and rigid rotation (Born). In order to do this we carry out a second order post–minkowskian approximation of the problem, and then we use a slow rotation subapproximation. The resulting metric depends on three arbitrary constants: density, rotational velocity and stellar radius in the non-rotating limit. Mass, angular momentum, quadrupole and other moments are expressed in terms of these three constants. As a side result we show that this type of fluid cannot be a source of the Kerr metric.

1 Introduction

In this work we present an approximate solution of the Einstein equations describing a global model for the gravitational field generated by a bounded, self–gravitating stationary and axisymmetric body rotating rigidly with constant angular velocity. The exterior metric corresponds to an asymptotically flat stationary, axisymmetric vacuum solution, while the interior is a perfect–fluid solution with the above properties. Both solutions (exterior and interior) are matched at the surface of zero pressure using the Lichnerowicz matching conditions, i. e., the global solution is $C^1$ at the boundary. To carry out the matching we use global harmonic coordinates, since these coordinates provide the necessary arbitrary constants.

1.1 Generic expresions of the metric

Let us consider a stationary metric with axial symmetry. As usual, the metric tensor will be referred to a system of spherical coordinates \( \{t, r, \theta, \varphi\} \) adapted to the symmetries and to the Papapetrou structure, so that the radial and polar coordinates are defined on the 2-D surfaces orthogonal to the surfaces of transitivity. For convenience, we shall use an orthonormal cobasis, with respect to the Minkowskian metric, associated to these coordinates:
\[ g = g_{00} \omega^0 \otimes \omega^0 + g_{03} (\omega^0 \otimes \omega^3 + \omega^3 \otimes \omega^0) + g_{33} \omega^3 \otimes \omega^3 \\
+ g_{11} \omega^1 \otimes \omega^1 + g_{12} (\omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1) + g_{22} \omega^2 \otimes \omega^2 \]  

(1)

\[ \omega^0 = dt \ , \ \omega^1 = dr \ , \ \omega^2 = r \, d\theta \ , \ \omega^3 = r \sin \theta \, d\phi \]  

(2)

However, for computational reasons, we shall write the metric tensor \( g \) with respect to another tensor basis

\[ g = g_{00} t_{00} + g_{03} t_{03} + f_d t^d + f_0 t^0 + f_1 t^1 + f_2 t^2 \]  

(3)

where \( \{ t^i \} \) are the "cartesian" coordinates associated to the spherical ones \( \{ r, \theta, \phi \} \) and the covectors \( \{ k_i, e_i, m_i \} \) are defined as

\[ d(r \sin \theta) = k_i \, dx^i \ , \ dx^3 = e_i \, dx^i \ , \ \omega^3 = r \sin \theta \, d\phi = m_i \, dx^i \]  

(5)

that is to say, they represent respectively the unitary radial, axial and azimuthal cylindrical covectors. The tensor basis \( \{ t^{00}, t^{03}, t^d, t^0, t^1, t^2 \} \) appears quite naturally in the linear and higher order approximations of Einstein's equations, which require a Poisson-like equation to be solved, which in turn requires the introduction of the tensor spherical harmonics (we shall see that they are directly related to the tensor basis we are considering). Notice that the tensor \( t^d \) is pure trace, while \( \{ t^0, t^1, t^2 \} \) are traceless (with respect to the flat 3-dimensional metric).

The relationship between the metric components on the spherical co-basis (2) and the coefficients of the metric with respect to the "pure trace–traceless" tensor basis (4) is:

\[ f_d = \frac{1}{3} (g_{11} + g_{22} + g_{33}) \]

\[ f_0 = \frac{1}{6} \left[ g_{11} (1 - 3 \cos^2 \theta) + 3g_{12} \sin 2\theta + g_{22} (1 - 3 \sin^2 \theta) + g_{33} \right] \]

\[ f_1 = \frac{1}{2} (g_{11} - g_{22}) \sin 2\theta + g_{12} \cos 2\theta \]

\[ f_2 = \frac{1}{2} (g_{11} \sin^2 \theta + g_{12} \sin 2\theta + g_{22} \cos^2 \theta - g_{33}) \]  

(6)
1.2 Post–minkowskian approximation

In this section we shall briefly describe the post–minkowskian approximation using the harmonic condition (we could also use the so called q–armonic condition [1]). We begin by defining the deviation of the metric in “cartesian” coordinates

\[ h_{\alpha\beta} := g_{\alpha\beta} - \eta_{\alpha\beta} \]  

(7)

Let us consider the Einstein equations and harmonicity conditions

\[
\begin{cases}
R_{\alpha\beta} = 8\pi \left( T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T \right) \equiv 8\pi T_{\alpha\beta} \\
g^{\lambda\mu}\Gamma^{\rho}_{\lambda\mu} = 0
\end{cases}
\]  

(8)

(we assume \( G = c = 1 \)). As it is known the Ricci tensor and the l.h.s. of the harmonic condition can be written in terms of the deviation \( h_{\alpha\beta} \) as follows:

\[
R_{\alpha\beta} = -\frac{1}{2} \Box h_{\alpha\beta} + \frac{1}{2} \partial_{\alpha} \left[ \partial^{\rho} \left( h_{\rho\beta} - \frac{1}{2} \eta_{\rho\beta} h \right) \right] + \frac{1}{2} \partial_{\beta} \left[ \partial^{\rho} \left( h_{\rho\alpha} - \frac{1}{2} \eta_{\rho\alpha} h \right) \right] + N_{\alpha\beta}
\]  

(9)

\[
\Gamma_{\alpha} \equiv \eta_{\alpha\rho} g^{\lambda\mu} \Gamma^{\rho}_{\lambda\mu} = \partial^{\rho} \left( h_{\rho\alpha} - \frac{1}{2} \eta_{\rho\alpha} h \right) + H_{\alpha}
\]  

(10)

where \( N_{\alpha\beta} \) and \( H_{\alpha} \) are at least quadratic in \( h_{\alpha\beta} \). We also use the notation \( h = \eta^{\alpha\beta} h_{\alpha\beta} \), \( \partial^{\rho} \equiv \eta^{\rho\lambda} \partial_{\lambda} \) and \( \Box \equiv \eta^{\lambda\mu} \partial_{\lambda} \partial_{\mu} \) (usual d’Alembertian operator). Consequently the Einstein equations using harmonic coordinates can be written as:

\[
\begin{cases}
\Box h_{\alpha\beta} = -16\pi T_{\alpha\beta} + 2 \left[ N_{\alpha\beta} - \partial_{(\alpha} H_{\beta)} \right] \\
\partial^{\rho} \left( h_{\rho\alpha} - \frac{1}{2} \eta_{\rho\alpha} h \right) = -H_{\alpha}
\end{cases}
\]  

(11)

and then the equations for the linear stationary approximation can be written as follows (where \( \triangle \) is the flat laplacian operator):

\[
\begin{cases}
\triangle h^{(1)}_{\alpha\beta} = -16\pi T^{(1)}_{\alpha\beta} \\
\partial^{k} \left[ h^{(1)}_{k\alpha} - \frac{1}{2} h^{(1)} \eta_{k\alpha} \right] = 0
\end{cases}
\]  

(12)

where, as shown below, the tensor \( T^{(1)}_{\alpha\beta} \) associated to the energy–momentum tensor requires only the flat Minkowski metric. Once a solution of these equations is fixed we can solve the postlinear approximation from the equations

\[
\begin{cases}
\triangle h^{(2)}_{\alpha\beta} = -16\pi T^{(2)}_{\alpha\beta} + 2 N^{(2)}_{\alpha\beta} - \partial_{\alpha} H^{(2)}_{\beta} - \partial_{\beta} H^{(2)}_{\alpha} \\
\partial^{k} \left[ h^{(2)}_{k\alpha} - \frac{1}{2} h^{(2)} \eta_{k\alpha} \right] = -H^{(2)}_{\alpha}
\end{cases}
\]  

(13)
where $T^{(2)}_{\alpha\beta}, N^{(2)}_{\alpha\beta}$ and $H^{(2)}_{\alpha}$ are constructed from the linear deviation $h^{(1)}_{\alpha\beta}$. Note that using the expressions (9,10) it obviously follows that $N^{(2)}_{\alpha\beta}$ and $H^{(2)}_{\alpha}$ may be obtained in an algorithmic way easily implemented on a computer. One only needs evaluate this expressions for the metric $g^{(1)}_{\alpha\beta} \equiv \eta_{\alpha\beta} + h^{(1)}_{\alpha\beta}$

2 Exterior solution

2.1 Linear approximation (exterior solution)

The general asymptotically flat solution of equations (12) in vacuum (exterior) with the assumed symmetries is

$$h^{(1)}_{\text{ext}} = 2 \sum_{n=0}^{\infty} \frac{M^{(1)}_{n}}{r^{n+1}} T_{n} + 2 \sum_{n=1}^{\infty} \frac{J^{(1)}_{n}}{r^{n+1}} Z_{n} + 2 \sum_{n=0}^{\infty} \frac{M^{(1)}_{n}}{r^{n+1}} D_{n} + \sum_{n=1}^{\infty} \frac{A^{(1)}_{n}}{r^{n+1}} E_{n} + \sum_{n=2}^{\infty} \frac{B^{(1)}_{n}}{r^{n+1}} F_{n}$$

where the notation:

$$
\begin{align*}
E_{n} & \equiv \frac{2}{3} n D_{n} + \frac{2}{3} n H^{0}_{n} + H^{1}_{n} \\
F_{n} & \equiv \frac{1}{2} \alpha(n-1) H^{0}_{n} + (n-1) H^{1}_{n} - \frac{1}{2} H^{2}_{n}
\end{align*}
$$

has been used, and

$$
\begin{align*}
T_{n} & \equiv P_{n}(\cos \theta) t^{00} \\
Z_{n} & \equiv P_{n}^{1}(\cos \theta) t^{03} \\
D_{n} & \equiv P_{n}(\cos \theta) t^{4} \\
H^{0}_{n} & \equiv P_{n}(\cos \theta) t^{0} \\
H^{1}_{n} & \equiv P_{n}^{1}(\cos \theta) t^{1} \\
H^{2}_{n} & \equiv P_{n}^{2}(\cos \theta) t^{2}
\end{align*}
$$

are the axially symmetric tensor spherical harmonics. The constants $M^{(1)}_{n}$ and $J^{(1)}_{n}$ represent the linear intrinsic multipole moments of Geroch and Hansen [4], while the constants $A^{(1)}_{n}$ and $B^{(1)}_{n}$ are related to the stress moments of the source in this linear approximation [6]. The constants $A$ and $B$ could be eliminated by choosing an appropriate gauge, but they are necessary for the matching problem with the Lichnerowicz conditions [8].

In the following we neglect the multipoles $M^{(1)}_{4}, J^{(1)}_{5}, A^{(1)}_{4}, B^{(1)}_{6}$ and following. We also assume equatorial symmetry which eliminates the odd ($M, A, B$) and even $J$ multipole moments. This is coherent with the approximations done in the last Section.
2.2 Post linear or second order approximation (exterior solution)

By constructing $N_{\alpha\beta}^{(2)}$ and $H_{\alpha}^{(2)}$ from (14) and solving equations (13) for vacuum assuming the stationary and axial symmetries, the following expression for the metric to second order is obtained (the calculations have been performed using computer algebra)

$$g_{\text{ext}}^{[2]} = \begin{pmatrix} -1 + \frac{2M_0^{[2]}}{r} - \frac{2M_0^{(1)^2}}{r^2} - \frac{2M_0^{(1)}A_2^{(1)}}{3r^4} - \frac{M_0^{(1)}B_2^{(1)}}{r^4} \end{pmatrix} T_0$$

$$+ \left( \frac{2M_2^{[2]}}{r^3} - \frac{4M_0^{(1)}M_2^{(1)}}{r^4} - \frac{4M_0^{(1)}A_2^{(1)}}{3r^4} \right) T_2 + \frac{2J_1^{[2]}}{r^2} Z_1 + \frac{2J_3^{[2]}}{r^4} Z_3$$

$$+ \left( 1 + \frac{2M_0^{[2]}}{r} + \frac{4M_0^{(1)^2}}{3r^2} - \frac{2M_0^{(1)}A_2^{(1)}}{3r^4} - \frac{M_0^{(1)}B_2^{(1)}}{r^4} \right) D_0$$

$$+ \left( \frac{2M_2^{[2]}}{r^3} + \frac{64M_0^{(1)}M_2^{(1)}}{21r^4} - \frac{4M_0^{(1)}A_2^{(1)}}{3r^4} \right) D_2$$

$$+ \left( \frac{A_2^{[2]}}{r^4} + \frac{2M_0^{(1)}M_2^{(1)}}{7r^4} + \frac{2M_0^{(1)}A_2^{(1)}}{r^4} \right) E_2$$

$$+ \left( \frac{B_2^{[2]}}{r^3} - \frac{M_0^{(1)^2}}{3r^2} - \frac{4M_0^{(1)}M_2^{(1)}}{21r^4} + \frac{2M_0^{(1)}B_2^{(1)}}{r^4} \right) F_2$$

$$+ \left( \frac{B_4^{[2]}}{r^5} - \frac{M_0^{(1)}M_2^{(1)}}{7r^4} \right) F_4$$

(17)

where we used the notation $X^{[2]} = X^{(1)} + X^{(2)}$ in order to take into account the homogeneous part of the solution of (13), which is formally identical to (14). Moreover, the terms involving $1/r^6$ or higher, as well as the terms containing the products of $J$ moments by another moment, has been omitted as they are not required for the final problem.

3 Interior solution

3.1 The energy–momentum tensor

Let us describe now the stellar model. The energy-momentum tensor describing the source is a perfect fluid of constant density $\mu$, with pressure $p$ depending only on $r$ and $\theta$, i.e.
\[ T_{\alpha\beta} = \mu u_\alpha u_\beta + p (g_{\alpha\beta} + u_\alpha u_\beta) \]  
where \( u_\alpha \) is the unit 4-velocity of the fluid. Further, we shall assume that no convection takes place; so, \( u_\alpha \) is a linear combination of the two Killing vectors we are imposing

\[ u^\alpha = \psi (\xi^\alpha + \omega \zeta^\alpha) \quad , \quad \xi^\alpha = \delta_i^\alpha \quad , \quad \zeta^\alpha = \delta_\phi^\alpha \]

Finally, we assume rigid rotation (Born), that is, \( \omega \) constant. Consequently, the normalization factor \( \psi \) has the following expression.

\[ \psi = \frac{1}{\sqrt{-(g_{00} + 2\omega g_{03} r \sin \theta + \omega^2 g_{33} r^2 \sin^2 \theta)}} \]  

On the other hand, the Euler equations can be written in this case as

\[ \nabla_\alpha T^{\alpha\beta} = 0 \implies \frac{dp}{\mu + p} = d \log \psi \]  
which for constant density implies

\[ p = \mu \left( \frac{\psi}{\psi_\Sigma} - 1 \right) \]

where \( \psi_\Sigma \) is the value of \( \psi \) on the surface of the fluid, \( p = 0 \). The equation defining this surface will therefore reads [2]:

\[ \Sigma : \psi(r, \theta) = \text{const} \equiv \psi_\Sigma \]

Now, in order to handle the energy–momentum tensor in the postminkowskian approximation context, it is necessary to assign an order of magnitude to the constants appearing in this tensor: \( \mu, \omega \) and \( \psi_\Sigma \). First of all we will assume obviously that the density is a first order quantity (i.e. linear). Furthermore, for reasons which will become clear in the last section we will assume that the square of angular velocity is also of first order. So, if we perform the following substitutions

\[
\begin{align*}
g_{00} &\rightarrow -1 + h_{00}^{(1)} , \quad g_{03} \rightarrow h_{03}^{(1)} , \quad g_{33} = f_d + f_0 - f_2 \\
f_d &\rightarrow 1 + h_d^{(1)} , \quad f_0 \rightarrow h_0^{(1)} , \quad f_1 \rightarrow h_1^{(1)} , \quad f_2 \rightarrow h_2^{(1)} \\
\psi_\Sigma &\rightarrow 1 + \psi_\Sigma^{(1)}
\end{align*}
\]

we finally get the following tensor \( T \) to second order (in the cylindrical basis).
An approximate global stationary . . .

\[ T^{[2]} \simeq \frac{\mu}{2} \left[ 1 + \frac{1}{2} h^{(1)}_{00} - 3 \psi^{(1)}_{\Sigma} + \frac{7}{2} \omega^2 r^2 \sin^2 \theta \right] t^{00} - \mu \omega r \sin \theta t^{03} \]

\[ + \frac{\mu}{2} \left[ 1 + h^{(1)}_d - \frac{1}{2} h^{(1)}_{00} + \psi^{(1)}_{\Sigma} + \frac{1}{6} \omega^2 r^2 \sin^2 \theta \right] t^d \]

\[ + \frac{\mu}{2} \left[ h^{(1)}_0 + \frac{1}{3} \omega^2 r^2 \sin^2 \theta \right] t^0 + \frac{\mu}{2} h^{(1)}_1 t^1 + \frac{\mu}{2} \left[ h^{(1)}_2 - \omega^2 r^2 \sin^2 \theta \right] t^2 \]  

(25)

It should be remarked that terms containing \( h^{(1)}_0 \) has been neglected even though they are postlinear in character. This is in agreement with the remark after expression (17).

### 3.2 Linear approximation (interior solution)

Imposing regularity at the origin together with the usual symmetries, the general solution of equations (12) for the linear part of the energy–momentum tensor is

\[ h^{(1)}_{\text{int}} = \sum_{n=0}^{\infty} m_n^{(1)} r^n T_n + \sum_{n=1}^{\infty} j_n^{(1)} r^n Z_n + \sum_{n=0}^{\infty} m_n^{(1)} r^n D_n \]

\[ + \sum_{n=0}^{\infty} a_n^{(1)} r^n \tilde{E}_n + \sum_{n=0}^{\infty} b_n^{(1)} r^n \tilde{F}_n + h^{(1)}_p \]  

(26)

where:

\[
\begin{align*}
\tilde{E}_n &= E_n + \frac{2}{3} (2n+1) D_n - \frac{2}{3} (2n+1) H_n^0 \\
\tilde{F}_n &= -(2n+1) E_n + F_n - \frac{2}{3} (2n+1) D_n + \frac{1}{3} (2n+1) (2n+3) H_n^0
\end{align*}
\]

(27)

and where \( h^{(1)}_p \) is any particular solution. The constants \( m_n^{(1)}, j_n^{(1)}, a_n^{(1)} \) and \( b_n^{(1)} \) have no specific meaning as in the exterior case. However, since in the exterior we have negelled the multipoles \( M_4, J_5, A_4, B_6 \) and following, the same will be done for the coefficients of the analogous spherical harmonics in the interior. So, we have the following linear approximate interior solution

\[ g^{[1]}_{\text{int}} = \left[ -1 + m_{0}^{(1)} \right] T_0 + m_{2}^{(1)} r^2 T_2 + j_{1}^{(1)} r Z_1 + j_{3}^{(1)} r^3 Z_3 \]

\[ + \left[ 1 + m_{0}^{(1)} \right] D_0 + m_{2}^{(1)} r^2 D_2 + a_{0}^{(1)} \tilde{E}_0 + b_{0}^{(1)} \tilde{F}_0 + \]
\[ +p_2^{(1)} r^2 \vec{E}_2 + b_2^{(1)} r^2 \vec{F}_2 - \frac{4}{3} \pi \mu r^2 (T_0 + D_0) - \frac{8}{5} \pi \mu \omega r^3 Z_1 \]

\[ h_p^{(1)} \]

3.3 Post linear approximation (interior solution)

The postlinear approximation can be constructed in a similar way as for the exterior case. This would leads to a similar expression to (17), however it is much longer and will not be written here. Nevertheless we shall write down below the final expression after the corresponding approximation has been made and the matching with the exterior completed.

4 The matching

4.1 Expansion of the constants and matching surface

This subsection is the key to understand the analysis carried out in the present paper. First we take an adimensional parameter \( \lambda \) which essentially reproduces the post–minkowskian approximation carried out above. Then we use another adimensional parameter \( \Omega \) describing a slow rotation approximation \([5] [10] [12]\) whose first term has spherical symmetry (“post–spherical approximation”). Specifically the parameters we have used are:

A) \( \lambda \) is the ratio between \( m \), a constant with units of mass, later identified as the “newtonian mass” of the source, and a length \( r_0 \) measuring the radius of the star in absence of rotation (i.e. spherical symmetry).

B) \( \Omega \) is the ratio between the centrifugal and gravitational forces at the surface of the star. It is therefore proportional to the rotational velocity of the source and measures its deviation from spherical symmetry. This parameter is used to perform, at each order of \( \lambda \), a second approximation (slow rotation).

\[ \begin{align*}
\lambda &= \frac{m}{r_0} : \text{Post–minkowskian approximation} \\
\omega r_0 &= \lambda^{1/2} \Omega , \quad \Omega : \text{Slow rotation approximation}
\end{align*} \]

Notice that the exponents of \( \lambda \) for the \( Z_n \) coefficients will be half-integers. Nevertheless, with respect to the post–minkowskian approximation the order of magnitude of the corresponding terms will equal the integer part of those exponents. The final approximation will consist on imposing \( \lambda^{5/2} \to 0 \) and \( \Omega^4 \to 0 \).

The fundamental hypothesis of this work is demonstrated in the following expansions for the “exact” constants \( M_n, J_n, A_n \) and \( B_n \) (the formal meaning of the
term “exact” in our case is $X_n = \lim_{r \to \infty} X_n^r$). In general, these are expansions in powers of $\lambda$, each coefficient of which is, in turn, expanded in powers of $\Omega$. This hypothesis is based on the Newtonian analysis of a selfgravitating rotating object, which has, as is well-known, the Maclaurin spheroids as possible solutions [9].

$$\frac{M_0}{r_0} = \lambda \left[ 1 + M_0^{(1,2)} \Omega^2 + O(\Omega^4) \right] + \lambda^2 \left[ M_0^{(2,0)} + M_0^{(2,2)} \Omega^2 + O(\Omega^4) \right] + O(\lambda^3)$$

$$\frac{M_2}{r_0} = \lambda \Omega^2 \left[ M_2^{(1,2)} + O(\Omega^2) \right] + \lambda^2 \Omega^2 \left[ M_2^{(2,2)} + O(\Omega^2) \right] + O(\lambda^3)$$

$$\frac{J_1}{r_0} = \lambda^{3/2} \Omega \left[ J_1^{(1,1)} + J_1^{(1,3)} \Omega^2 + O(\Omega^4) \right] + O(\lambda^{5/2})$$

$$\frac{J_3}{r_0} = \lambda^{3/2} \Omega^3 \left[ J_3^{(1,3)} + O(\Omega^2) \right] + O(\lambda^{5/2})$$

(30)

$$\frac{A_2}{r_0} = \lambda \Omega^2 \left[ A_2^{(1,2)} + O(\Omega^2) \right] + \lambda^2 \Omega^2 \left[ A_2^{(2,2)} + O(\Omega^2) \right] + O(\lambda^3)$$

$$\frac{B_2}{r_0} = \lambda \left[ B_2^{(1,0)} + B_2^{(1,2)} \Omega^2 + O(\Omega^4) \right] + \lambda^2 \left[ B_2^{(2,0)} + B_2^{(2,2)} \Omega^2 + O(\Omega^4) \right] + O(\lambda^3)$$

$$\frac{B_4}{r_0} = \lambda \Omega^2 \left[ B_4^{(1,2)} + O(\Omega^2) \right] + \lambda^2 \Omega^2 \left[ B_4^{(2,2)} + O(\Omega^2) \right] + O(\lambda^3)$$

(31)

For $M_0$ and $B_2$, the first terms are independent of $\Omega$ because they already appear in spherical symmetry [6] [11]. For $J_1$ (angular momentum) we can assume that the lowest term is linear in $\Omega$, which corresponds to a sphere put into rotation. However, $M_2$ and $J_3$ represent true deformations of sphericity due to rotation; therefore, we assume that the expansions start as indicated. Finally, for $A_2$ and $B_4$, we assume a similar structure to $M_2$; this is based on the fact that these constants are associated to the spherical harmonics of the same order as the quadrupole moment.

Regarding the constants $m$, $j$, $a$ and $b$ in the interior solution, we take similar expansions as above. For shortness they will not be included here.

In agreement with the “post–spherical approximation” we make the supplementary hypothesis that the equation determining the surface $\Sigma$ of the star has the structure:

$$r = r_0 + r_0 \Omega^2 P_2(\cos \theta) + O(\Omega^4)$$

(32)

so that, for vanishing rotation, it becomes the equation of a sphere of radius $r_0$. In the following subsections we analyze the matching problem. Let us advance here the solution we find for this surface, which reads:
\[ \Sigma : \eta = 1 + \left( -\frac{5}{6} + \frac{5\lambda}{r} \right) \Omega^2 P_2(\cos \theta) + O(\Omega^4) \quad , \quad \eta \equiv \frac{r}{r_0} \quad (33) \]

### 4.2 Results of the matching (exterior solution)

Imposing the condition that the metric (interior and exterior) is continuous with continuous first derivatives on the surface of the fluid, the following expansions for the constants in the exterior solution are found:

\[
\begin{cases}
\frac{M_0}{r_0} = \lambda + \lambda^2 \left[ 3 + \frac{2}{5} \Omega^2 \right] + O(\lambda, \Omega^4) \\
\frac{M_2}{r_0} = -\frac{1}{2} \lambda \Omega^2 - \frac{69}{70} \lambda^2 \Omega^2 + O(\lambda, \Omega^4) \\
\frac{J_1}{r_0} = \frac{\lambda}{2} \Omega \left( \frac{2}{5} + \frac{1}{3} \Omega^2 \right) + O(\lambda^{5/2}, \Omega^5) \\
\frac{J_3}{r_0} = -\frac{1}{7} \lambda^{3/2} \Omega^3 + O(\lambda^{5/2}, \Omega^5) \\
\end{cases}
\]

In the \( M_0 \) expansion, the different contributions (newtonian mass, binding energy, rotational energy) to the total mass of the body can be observed. Plugging this expressions into (17) the following metric tensor of the exterior region is obtained

\[
g_{\text{ext}}^{[2]} = \begin{bmatrix}
-1 + \frac{2\lambda}{\eta} + 2\lambda^2 \left( 3 + \frac{2\Omega^2}{5} \right) & \frac{1}{\eta^2} \mathbf{T}_0 + \left( -\frac{\lambda}{\eta^3} - \frac{69\lambda^2}{35\eta^3} + \frac{2\lambda^2}{\eta^4} \right) \Omega^2 \mathbf{T}_2 \\
+ 2\lambda^{3/2} \Omega & \left( \frac{2}{5} + \frac{\Omega^2}{3} \right) \frac{1}{\eta^2} \mathbf{Z}_1 - \frac{2\lambda^{3/2} \Omega^3}{7\eta^4} \mathbf{Z}_3 \\
+ \left[ 1 + \frac{2\lambda}{\eta} + 2\lambda^2 \left( 3 + \frac{2\Omega^2}{5} \right) \frac{1}{\eta} + \frac{4\lambda^2}{3\eta^2} \right] \mathbf{D}_0 + \left( -\frac{\lambda}{\eta^3} - \frac{69\lambda^2}{35\eta^3} - \frac{32\lambda^2}{21\eta^4} \right) \Omega^2 \mathbf{D}_2 \\
- \frac{\lambda^2 \Omega^2}{7\eta^4} \mathbf{E}_2 + \lambda^2 \left[ \frac{-1}{3\eta^2} + \frac{4}{35} \left( 2 + \frac{2\Omega^2}{3} \right) \frac{1}{\eta^3} + \frac{2\Omega^2}{21\eta^4} \right] \mathbf{F}_2 \\
+ \lambda^2 \Omega^2 \left( \frac{1}{14\eta^4} - \frac{4}{63\eta^5} \right) \mathbf{F}_4
\end{bmatrix}
\]
We should remark that, as mentioned in the abstract, the Kerr metric does not belong to this approximate solution. Kerr’s metric can be obtained from (17) by making the substitutions $M_0 = m$, $M_2 = -ma^2$, $J_1 = ma$ and $J_3 = -\frac{1}{9}ma^3$ (here $m$ and $a$ stand for the standard parameters of the Kerr metric). It can be easily checked that these values can not be obtained by any choice of the free parameters $\lambda$, $\Omega$ and $r_0$. One might think that this need not to be true for another fluid, for instance a polytrope or differential rotation.

4.3 Results of the matching (interior solution)

Similarly, the matching conditions produce the following expressions for the constants of the interior solution

\[
\begin{align*}
\begin{cases}
m_0 &= 3\lambda + \frac{1}{4} \lambda^2 (21 + 2\Omega^2) + O(\lambda^3, \Omega^4) \\
m_2 r_0^2 &= -\lambda \Omega^2 - \frac{73}{70} \lambda^2 \Omega^2 + O(\lambda^3, \Omega^4)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\lambda_0 &= \lambda \Omega^2 \left(2 + \frac{2}{3} \Omega^2\right) + O(\lambda^5/2, \Omega^5) \\
\lambda_3^3 &= -\frac{2}{7} \lambda^3/2 \Omega^3 + O(\lambda^5/2, \Omega^5)
\end{cases}
\end{align*}
\]

(37)

\[
\begin{align*}
\begin{cases}
a_0 &= \frac{1}{3} \lambda^2 (21 + 2\Omega^2) + O(\lambda^3, \Omega^4) \\
a_2 r_0^2 &= -\frac{43}{15} \lambda^2 \Omega^2 + O(\lambda^3, \Omega^4)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
b_0 &= O(\lambda^3, \Omega^4) \\
b_2 r_0^2 &= -\frac{86}{105} \lambda^2 \Omega^2 + O(\lambda^3, \Omega^4)
\end{cases}
\end{align*}
\]

(38)

so that the metric in the interior region once matched to the exterior reads:

\[
g^{[2]}_{\text{int}} = \begin{bmatrix}
-1 + 3\lambda - \lambda \eta^2 + \frac{1}{2} \lambda^2 \left(\frac{21}{2} + \Omega^2\right) + \lambda^2 \left(-\frac{3}{2} + \Omega^2\right) \eta^2 + \frac{1}{2} \lambda^2 \left(\frac{1}{2} - \frac{7}{5} \Omega^2\right) \eta^4 \\
\left(-\lambda - \frac{73}{70} \lambda^2 + \frac{15}{14} \lambda^2 \eta^2\right) \Omega^3 \eta^2 T_2 + \lambda^{3/2} \left[2 + \frac{2}{3} \Omega^2 - \frac{6}{5} \eta^2\right] \Omega \eta Z_1 - \frac{2}{7} \lambda^{3/2} \lambda^3 \eta^3 Z_3 \\
\left[1 + 3\lambda - \lambda \eta^2 + \frac{1}{2} \lambda^2 \left(\frac{49}{2} + \frac{7}{3} \Omega^2\right) - \lambda^2 \left(\frac{11}{2} + \frac{1}{3} \Omega^2\right) \eta^2 + \frac{1}{6} \lambda^2 \left(\frac{7}{2} - \frac{1}{5} \Omega^2\right) \eta^4\right] D_0 \\
\left(-\lambda \eta^2 - \frac{1079}{210} \lambda^2 \eta^2 + \frac{23}{14} \lambda^2 \eta^4\right) \Omega^2 D_2 + \frac{1}{7} \lambda^2 \Omega^2 (-4\eta^2 + 3\eta^4) E_2 \\
\lambda^2 \left[\frac{1}{5} \left(-1 + \frac{18}{7} \Omega^2\right) \eta^2 + \frac{2}{21} (1 - 4\Omega^2) \eta^4\right] F_2 + \frac{1}{126} \lambda^2 \Omega^2 \eta^4 F_4
\end{bmatrix}
\]

(39)
On the other hand the expression for the pressure is the following:

\[ p = \frac{\chi^2}{8\pi r_0^2} \left[ (3 - 2\Omega^2)(1 - \eta^2) - 5\Omega^2\eta^2P_2(\cos \theta) \right] + O(\lambda^3, \Omega^4) \quad (40) \]

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