Extended formulation and valid inequalities for the multi-item inventory lot-sizing problem with supplier selection

Leopoldo E. Cárdenas-Barrón * Rafael A. Melo † Marcio C. Santos ‡

February 25, 2020

Abstract

This paper considers the multi-item inventory lot-sizing problem with supplier selection. The problem consists in determining an optimal purchasing plan in order to satisfy dynamic deterministic demands for multiple items over a finite planning horizon, taking into account the fact that multiple suppliers are available to purchase from. As the complexity of the problem was an open question, we show that it is NP-hard. We propose a facility location extended formulation for the problem which can be preprocessed based on the cost structure and describe new valid inequalities in the original space of variables, which we denote \((l, S_j)\)-inequalities. Furthermore, we study the projection of the extended formulation into the original space and show the connection between the inequalities generated by this projection and the newly proposed \((l, S_j)\)-inequalities. Additionally, we present a simple and easy to implement yet very effective MIP (mixed integer programming) heuristic using the extended formulation. Computational results show that the preprocessed facility location extended formulation outperforms all other formulations for small and medium instances, as it can solve nearly all of them to optimality within the time limit. Moreover, the presented MIP heuristic is able to obtain solutions which strictly improve those achieved by a state-of-the-art method for all the large benchmark instances.

Keywords: inventory lot-sizing; supplier selection; mixed integer programming; MIP heuristics.

1 Introduction

In nowadays competitive business environment it has become more important to have excellent supplier selection and lot-sizing processes for purchasing the products required by the companies. The intention of these processes is to choose the best suppliers from which to purchase the items, the amount of the lots and the time to set the orders in a finite planning horizon. Among the problems in this context, there exists the multi-item inventory lot-sizing problem with supplier selection (MIILSPSS). These sorts of problems regarding supplier selection and lot-sizing are faced by a variety of industries since companies that manufacture or simply distribute products very often need to make these decisions in an optimized fashion.

Ustun and Demirtas (2008) propose an integration of analytic network process (ANP) and achievement scalarizing functions for a multi-objective problem of choosing suppliers and defining purchasing quantities for a single item, taking into consideration tangible-intangible criteria. Zhao and Klabjan (2012) consider a single-item lot-sizing problem with simultaneous supplier selection and provide a mixed integer programming (MIP) formulation for the problem together with a study of its underlying polytope. They provide necessary and sufficient conditions to obtain facet defining inequalities for the uncapacitated case and valid inequalities.

*Department of Industrial and Systems Engineering, School of Engineering and Sciences, Tecnológico de Monterrey, E. Garza Sada 2501 Sur, C.P. 64849, Monterrey, Nuevo León, Mexico. (lecarden@tec.mx)
†Universidade Federal da Bahia, Departamento de Ciência da Computação, Computational Intelligence and Optimization Research Lab (CInO), Salvador, Brazil. (melodcc.ufba.br)
‡Universidade Federal do Ceará, Campus Russas, Russas, Brazil. (marciocs@ufc.br)
for the capacitated one. Choudhary and Shankar (2013) propose a MIP formulation for a single-item lot-sizing and distribution problem with supplier selection. Choudhary and Shankar (2014) extend the problem studied in Choudhary and Shankar (2013) to a multi-objective setting and propose a goal programming approach. Ghaniabadi and Mazinani (2017) consider a single-item lot-sizing problem with supplier selection, backlogging and quantity discounts. The authors present a MIP formulation together with a recursive approach that can be used to solve the problem in an iterative manner. Akbalik and Rapine (2018) consider a single-item uncapacitated lot sizing problem with multi-mode replenishment and batch deliveries. The authors present an NP-hardness proof and show that the problem remains NP-hard even for very simple and strict cost structures. Additionally, they present a 2-approximation algorithm and show that the problem admits a fully polynomial time approximation scheme (FPTAS).

Kasilingam and Lee (1996) propose a MIP formulation for a multi-item supplier selection problem with lead times. Dahel (2003) considers a multi-objective supplier selection and order quantities for a multi-item problem with quantity discounts and propose a preference-based approach. Basnet and Leung (2005) studied the multi-item inventory lot-sizing problem with supplier selection (MIILSPSS), which is the problem considered in our work. In the MIILSPSS, there exists a known dynamic deterministic demand for multiple items in a finite planning horizon, which can be purchased from a set of suppliers in each of the periods. There is a fixed supplier ordering cost that is charged for each period an order is put to that supplier, as well as unitary purchasing and holding costs. There are no capacities on the amounts of purchased items. The problem consists in determining a purchasing plan minimizing the total cost. The authors proposed an exhaustive enumerative search and a heuristic based on the Wagner-Whitin algorithm which consists of a construction and an improvement phase. Wadhwa and Ravindran (2007) consider a multi-objective supplier selection problem for multiple items considering objectives as price, lead-time and quality and compare the use of three multi-objective approaches. Cárdenas-Barrón, González-Velarde, and Treviño-Garza (2015) have revisited and tackled the MIILSPSS (Basnet & Leung, 2005) with a heuristic based on the reduce and optimize approach (ROA). The authors have demonstrated through computational experiments that their heuristic obtains better solutions when compared with the methods of Basnet and Leung (2005). Ware, Singh, and Banwet (2014) formulate and solve a mixed integer nonlinear programming model to manage the dynamic supplier selection problem. A. L. Cunha, Santos, Morabito, and Barbosa-Póvoa (2018) consider the integration of a multi-item lot-sizing problem with supplier selection for raw material purchasing in a chemical industry. The authors propose a MIP formulation which is solved using a commercial solver and compare the advantages of such integrated approach over a non-integrated one in which the two problems are solved sequentially. Kirschstein and Meisel (2019) consider the application of a lot-sizing problem with supplier selection in the process industry. The authors propose a kernel search heuristic to tackle the real instances of their case study. The interested reader is referred to Aissaoui, Haouari, and Hassini (2007), Ho, Xu, and Dey (2010) and Ware, Singh, and Banwet (2012) for surveys regarding different aspects of supplier selection and order lot-sizing.

To the best of our knowledge, there are two main open research avenues for the multi-item inventory lot-sizing problem with supplier selection. The first one is to demonstrate that the problem is NP-hard, as this appears to be an open question. The second one is to develop exact approaches such as reformulations and valid inequalities with the aim of improving the previously reported solutions. Considering these aspects, the main contributions of our work can be summarized as follows. We firstly show that the multi-item inventory lot-sizing problem with supplier selection is NP-hard. Secondly, we propose a facility location extended formulation together with an effective preprocessing scheme and new valid inequalities in the original space of variables. Thirdly, we show a strong relation between these two approaches by considering the projection of the facility location extended formulation into the original space. Last but not least, considering the fact that MIP heuristics have been successfully applied for several production planning and lot-sizing problems (Akartunali & Miller, 2009; J. O. Cunha, Kramer, & Melo, 2019; Helber & Sahling, 2010; Melo & Ribeiro, 2017; Melo & Wolsey, 2012), we propose a simple and easy to implement yet very effective MIP heuristic. Computational experiments show that the preprocessed extended formulation can solve nearly all small and medium instances within around 300 seconds on average when used in a standard commercial solver. Moreover, the proposed MIP heuristic outperforms a previously state-of-the-art approach for all the large
benchmark instances.

The remainder of this paper is organized as follows. Section 2 formally defines the multi-item inventory lot-sizing problem with supplier selection and shows that the problem is NP-hard. Section 3 presents the facility location extended formulation together with the preprocessing scheme and describes the proposed \((l, S_j)-inequalities. Section 4 analyzes the projection of the extended formulation into the original space and shows how it relates to the \((l, S_j)-inequalities. Section 5 details the proposed MIP heuristic. Section 6 summarizes the performed computational experiments. Section 7 discusses final comments.

## 2 Problem definition and standard mixed integer programming formulation

In this section, we formally introduce the multi-item inventory lot-sizing problem with supplier selection (MIILSPSS) and describe a standard mixed integer programming formulation for the problem. After that, in Subsection 2.1, we show that the problem is NP-hard.

The MIILSPSS can be formally defined as follows. Consider \(I = \{1, \ldots, NI\}\) to be the set of items, \(J = \{1, \ldots, NJ\}\) to be the set of suppliers and \(T = \{1, \ldots, NT\}\) to be the set of periods composing the planning horizon. A determinsitic time varying demand \(d_{ij}^t \geq 0\) must be met without backlogging for each item \(i \in I\) in each period \(t \in T\). There is a unitary purchasing price \(P_{ij}\) of item \(i \in I\) from supplier \(j \in J\). A transaction cost \(O_j\) for supplier \(j \in J\) is incurred whenever any item is purchased from \(j\) in a given period. Furthermore, a per unit holding cost \(H_i\) is incurred for item \(i \in I\) in every period the item is held in stock. The problem consists in determining a purchasing plan which minimizes the total cost. Let \(d_{it}^1 = \sum_{t=1}^t d_{ij}^t\) be the cumulative demand for item \(i \in I\) in periods from \(k \in T\) up to \(t \in T\), with \(k \leq t\). It is assumed that all the costs are nonnegative and that there are no initial or final stocks.

Consider variable \(x_{ij}^t\) to be the amount of item \(i \in I\) purchased from supplier \(j \in J\) in period \(t \in T\), and variable \(y_j^t\) to be equal to one if items are purchased from supplier \(j \in J\) in period \(t \in T\) and to be equal to zero otherwise. The problem can thus be formulated as (Basnet & Leung, 2005):

\[
    z_{STD} = \min \sum_{i=1}^{NI} \sum_{j=1}^{NJ} \sum_{t=1}^{NT} P_{ij} x_{ij}^t + \sum_{j=1}^{NJ} \sum_{t=1}^{NT} O_j y_j^t + \sum_{i=1}^{NI} \sum_{t=1}^{NT} H_i \left( \sum_{j=1}^{NJ} \sum_{k=1}^{NT} x_{ij}^k - d_{it}^t \right) \tag{1}
\]

\[
    \sum_{j=1}^{NJ} x_{ij}^t \geq d_{it}^t, \quad \text{for } i \in I, \ t \in T, \tag{2}
\]

\[
    x_{ij}^t \leq M y_j^t, \quad \text{for } i \in I, \ j \in J, \ t \in T, \tag{3}
\]

\[
    y_j^t \in \{0, 1\}, \quad \text{for } j \in J, \ t \in T, \tag{4}
\]

\[
    x_{ij}^t \geq 0, \quad \text{for } i \in I, \ j \in J, \ t \in T. \tag{5}
\]

The objective function (1) minimizes the total sum of purchasing, transaction and storage costs. Constraints (2) guarantee that all the demands are satisfied. Constraints (3) ensure the setup variables are set to one whenever items are purchased from a supplier in a given period. Constraints (4) and (5) impose, respectively, the integrality and nonnegativity requirements on the variables. This formulation has \(O(NI \times NJ \times NT)\) variables and constraints.

### 2.1 NP-hardness

To the best of our knowledge, there is no available NP-hardness proof for the multi-item inventory lot-sizing problem with supplier selection. In this section, we show that the problem is NP-hard via a reduction from the uncapacitated facility location problem (Chudak & Shmoys, 2003).
The uncapacitated facility location problem (UFL) can be formally defined as follows. Consider a set \( F = \{1, \ldots, NF\} \) of potential facility locations, a set \( C = \{1, \ldots, NC\} \) of clients, a fixed cost \( q_f \) to open facility \( f \in F \), and a cost \( c_{ij} \) of serving client \( c \in C \) from facility \( f \in F \). The problem consists in obtaining a subset \( F' \subseteq F \) of the facilities to be opened and then to assign clients to these facilities while minimizing the total cost. The decision version of the problem asks whether there is a solution with cost less than or equal to a value \( K \) which is given as input.

**Theorem 1.** The multi-item inventory lot-sizing problem with supplier selection is NP-hard.

**Proof.** We show how an instance for the decision version of the MIILSPSS is obtained from an instance of the decision version of the UFL. Create a supplier for each potential facility \( f \in F \), an item for each client \( c \in C \), and set the number of periods as \( NT = 1 \). For each supplier \( f \in J \), set its transaction cost as the cost of opening the corresponding facility, i.e., \( O_f = q_f \). The cost of acquiring item \( c \in I \) from supplier \( f \in J \) is set as the cost of serving client \( c \in C \) from facility \( f \in F \), i.e., \( P_{cf} = v_{cf} \). We now show that the instance for UFL has a solution with value less than or equal to \( K \) if and only if the corresponding instance for MIILSPSS has a solution with value less than or equal to \( K \). Consider a solution \( F' \subseteq F \) for the uncapacitated facility location in which each facility \( f \in F' \) serves a set \( C_f \) of clients, with cost \( K = \sum_{f \in F'} q_f + \sum_{c \in C} v_{cf} \). Thus, there is a solution for the multi-item inventory lot-sizing problem with supplier selection with only nonzero values \( y_{1f}^j = 1 \) for \( f \in F' \) and \( x_{ij}^f = 1 \) for \( j \in F' \) and \( c \in C_f \), whose objective is \( \sum_{f \in F'} \sum_{c \in C_f} v_{cf} x_{ij}^f + \sum_{f \in F'} q_f y_{1f}^j = K \). Now consider a solution \((\hat{x}, \hat{y})\) for the multi-item lot-sizing problem with supplier selection with cost \( K = \sum_{j \in NJ} \sum_{i \in NI} \sum_{j \in NJ} \sum_{i \in NI} \sum_{t = 1}^{NT} O_{ij} y_{it}^j + \sum_{j \in NJ} \sum_{i \in NI} \sum_{t = 1}^{NT} H_{ij} \left( \sum_{u = 1}^{NT} \sum_{k = t}^{NT} X_{uk}^{ij} \right) \) and assume that \( \hat{x} \) is integral (note that such integral solution always exist when we consider solutions with the lowest possible cost). Observe that there is a corresponding solution \( F' = \{ f \in J \mid y_{1f}^j = 1 \} \) and \( C_f = \{ c \in C \mid x_{ij}^f = 1 \} \) for the uncapacitated facility location with cost \( \sum_{j \in F'} \sum_{i \in C_f} P_{ij} \).

\[ z_{FL} = \min_{\sum_{i=1}^{NI} \sum_{j=1}^{NJ} \sum_{t=1}^{NT} \sum_{k=1}^{NT} P_{ij} X_{tk}^{ij} + \sum_{i=1}^{NI} \sum_{t=1}^{NT} O_{ij} y_{it}^j + \sum_{i=1}^{NI} \sum_{t=1}^{NT} H_{ij} \left( \sum_{u=1}^{NT} \sum_{k=t}^{NT} X_{uk}^{ij} \right) \quad \text{(6)} \]

\[ \sum_{j=1}^{NJ} \sum_{t=1}^{NT} X_{tk}^{ij} = d_{ik}^t, \quad \text{for } i \in I, k \in T, \quad \text{(7)} \]

\[ X_{tk}^{ij} \leq d_{ik}^t y_{it}^j, \quad \text{for } i \in I, j \in J, t \in T, k \in \{t, \ldots, NT\}, \quad \text{(8)} \]

\[ y_{it}^j \in \{0, 1\}, \quad \text{for } j \in J, t \in T, \quad \text{(9)} \]

\[ X_{tk}^{ij} \geq 0, \quad \text{for } i \in I, j \in J, t \in T, k \in \{t, \ldots, NT\}. \quad \text{(10)} \]

The objective function (6) minimizes the total sum of purchase, transaction and storage costs. Constraints (7) guarantee that all the demands are satisfied. Constraints (8) enforce the setup variables to one whenever items are purchased from a supplier in a given period. Constraints (9) and (10) are integrality and nonnegativity restrictions on the variables.

3 Extended formulation and valid inequalities

In this section we present the extended formulation and valid inequalities proposed in this paper. Subsection 3.1 describes the facility location extended formulation. Subsection 3.2 introduces the new \((l, S_j)\)-inequalities.

3.1 Facility location extended formulation

Define variable \( X_{tk}^{ij} \) to be the amount of item \( i \in I \) purchased from supplier \( j \in J \) in period \( t \in T \) to satisfy demand of period \( k \in T, \) with \( t \leq k \). A facility location formulation (Krarup & Bilde, 1977) can be cast as

\[ z_{FL} = \min_{\sum_{i=1}^{NI} \sum_{j=1}^{NJ} \sum_{t=1}^{NT} \sum_{k=1}^{NT} P_{ij} X_{tk}^{ij} + \sum_{i=1}^{NI} \sum_{t=1}^{NT} O_{ij} y_{it}^j + \sum_{i=1}^{NI} \sum_{t=1}^{NT} H_{ij} \left( \sum_{u=1}^{NT} \sum_{k=t}^{NT} X_{uk}^{ij} \right) \quad \text{(6)} \]

\[ \sum_{j=1}^{NJ} \sum_{t=1}^{NT} X_{tk}^{ij} = d_{ik}^t, \quad \text{for } i \in I, k \in T, \quad \text{(7)} \]

\[ X_{tk}^{ij} \leq d_{ik}^t y_{it}^j, \quad \text{for } i \in I, j \in J, t \in T, k \in \{t, \ldots, NT\}, \quad \text{(8)} \]

\[ y_{it}^j \in \{0, 1\}, \quad \text{for } j \in J, t \in T, \quad \text{(9)} \]

\[ X_{tk}^{ij} \geq 0, \quad \text{for } i \in I, j \in J, t \in T, k \in \{t, \ldots, NT\}. \quad \text{(10)} \]
3.1.1 Preprocessing the facility location extended formulation

The facility location extended formulation has a large number of variable since, in order to proper model the problem, it considers the possibility that the demand of a period is satisfied by the production in any other period sooner in the planning horizon. Although it might be the case for certain instances that the production of the first periods are used to meet the demands on the later periods, it does not seem to be the case for real instances. In this context, we show that variables can be eliminated from the formulation based on the cost structure without losing optimality.

**Proposition 2.** Let \( X_{ij}^l \) be a variable for which \( O_j \leq (t - k) \times H_i \times d_k^l \). Thus, we can set \( X_{ij}^l = 0 \) for every \( k' \) such that \( k \leq k' \leq NT \) without losing optimality.

**Proof.** Firstly, let \( k \) be the earliest period after \( t \) for which \( O_j \leq (t - k) \times H_i \times d_k^l \) and assume there is an optimal solution in which \( X_{ij}^l = w > 0 \). Note that the right-hand side of the condition corresponds to the total storage cost for the demand of period \( k \) which was purchased in period \( t \). Thus, as the purchasing costs are time independent, we can set \( X_{ij}^l = 0 \), \( y_k^l = 1 \) and \( X_{ij}^l = w \) in order to obtain a solution which is at least as good as the previous one. Secondly, due to the property of extreme feasible solutions in fixed-charge networks, which was also observed in Basnet and Leung (2005), there exists an optimal solution in which purchasing in a given period satisfies demands of consecutive periods. Therefore, if there is an optimal solution in which \( X_{ij}^l = 0 \), there exists an optimal solution in which \( X_{ij}^l = 0 \) for every \( k < k' \leq NT \). \( \square \)

Note that Proposition 2 can be easily adapted for the case in which costs are time dependent.

3.2 The \((l, S_j)-inequalities\)

We describe the \((l, S_j)-inequalities\), which generalize the \((l, S)-inequalities\) for the uncapacitated lot-sizing (Barany, Van Roy, & Wolsey, 1984) by considering the different suppliers. Define \( L = \{1, \ldots, l\} \) with \( 1 \leq l \leq NT \), and \( S_j \subseteq L \) for each \( j \in J \).

**Theorem 3.** The \((l, S_j)-inequalities\)

\[
\sum_{j=1}^{NJ} \left( \sum_{u \in L \setminus S_j} x_{ij}^l + \sum_{u \in S_j} y_{k}^l d_{uu}^l \right) \geq d_{lt}^l, \quad \text{for } i \in I, \ l \in L, \tag{11}
\]

are valid for the multi-item inventory lot-sizing with supplier selection.

**Proof.** Let \((\hat{x}, \hat{y})\) be a feasible solution for (2)-(5). Firstly, consider the case in which \( \hat{y}_{k}^l = 0 \) for every \( j \in J \) and \( k \in S_j \). This implies that \( \hat{x}_{ij}^l = 0 \) for every \( j \in J \) and \( k \in S_j \), and thus constraints (2) ensure that \( \sum_{j=1}^{NJ} \sum_{u \in L \setminus S_j} \hat{x}_{ij}^l \geq d_{lt}^l \). Now, assume that \( \hat{y}_{k}^l = 1 \) for at least one \( j \in J \) and \( k \in S_j \), and let \( k' \) be the earliest period in which this happens and \( j' \) be the corresponding supplier. As constraints (2) ensure that \( \sum_{j=1}^{NJ} \sum_{u \in \{1, \ldots, k' - 1\}} \hat{x}_{ij}^l \geq d_{lt}^l \), the fact that \( \hat{y}_{k}^l = 1 \) implies that \( \sum_{j=1}^{NJ} \sum_{u \in \{1, \ldots, k' - 1\}} \hat{x}_{ij}^l + \hat{y}_{k'}^l d_{k'l}^l \geq d_{lt}^l \), and building \( S_j \) appropriately according to the choices on the inner minimum in \( O(NJ \times NT) \). This gives an \( O(NJ \times NT^2) \) algorithm to separate the inequalities for each
\[ i \in I. \] In what follows, we present an \( O(NJ \times NT \times \log NT) \) dynamic programming separation algorithm to encounter a most violated \((l, S_j)\)-inequality for each \( i \in I. \)

Given an item \( i \in I, \) define \( \alpha^i_l = \sum_{j=1}^{N_j} \sum_{k=1}^{l} \min\{\bar{x}_{i,j}^k, d_{k,l}^i\}. \) Inequality (11) is violated for \( L = \{1, \ldots, l\} \) whenever \( \alpha^i_l < d_{l,l}^i. \) Note that the nonnegativity of the demands implies \( d_{k,l}^i \leq d_{k,u}^l \) for \( k \leq l < u. \) For \( j \in J \) and \( k \in T, \) define \( l^j_k \) as the first period in which \( d_{k,j+l-1}^i \leq \bar{x}_{j}^k \leq d_{k,l}^i \) and let \( Y^j_i = \{k \in L \mid l^j_k > l\} \) and \( X^j_i = \{j \in L \mid l^j_k = l\}. \) Therefore, the value \( \alpha_l \) can be determined using the recursion

\[
\alpha^i_l = \alpha^i_{l-1} + d_{l,l}^i \left( \sum_{j \in J} \sum_{k \in X^j_i} \bar{y}_{k}^j \right) + \sum_{j \in J} \sum_{k \in X^j_i} \left( \bar{x}_{i,j}^k - d_{l,l}^i \bar{y}_{k}^j \right),
\]

with \( \alpha^i_0 = 0 \) as base case.

Considering the fact that \( Y^j_i = Y^j_{l-1} \cup \{l\} \setminus X^j_i, \) each period \( k \) enters at most once in \( Y^j, \) leaves \( Y^j \) and enters \( X^j \) at most once. Thus, all the \( \alpha_l \) values can be determined in \( O(NJ \times NT) \). Observe also that we can determine \( l^j_k \) for each \( j \in J \) and \( k \in T \) in \( O(\log NT) \) using binary search, implying a running time of \( O(NJ \times NT \times \log NT) \) for all the calculations.

### 3.2.2 The Wagner-Whitin \((l, S_j)\)-inequalities

We define the following special cases of inequalities (11) as the Wagner-Whitin \((l, S_j)\)-inequalities:

\[
\sum_{j'}=_{j}^{N_j} \left( \sum_{u=1}^{l} x_{u}^{j'} \right) + \sum_{u=1}^{l-1} x_{u}^{j'} + \sum_{u=t}^{l} y_{u}^{j} d_{u,t}^i \geq d_{l,l}^i, \quad \text{for } i \in I, \ j \in J, \ t \in L, \ l \in \{t, \ldots, NT\}. \tag{13}
\]

Note that there are \( O(NI \times NJ \times NT^2) \) Wagner-Whitin \((l, S_j)\)-inequalities.

### 4 On the projection of the facility location formulation

In this section, we study the projection of the facility location extended formulation (7)-(10) into the space of the original \((x, y)\) variables. We consider the extended formulation as a separation problem in order to describe the inequalities generated by its projection. After that, we show how they relate with the \((l, S_j)\)-inequalities, showing that the linear relaxation of the facility location extended formulation provides the same bound as that of the linear relaxation of the standard formulation together with the \((l, S_j)\)-inequalities.

Given a fractional solution \((\hat{x}, \hat{y})\) feasible for the linear relaxation of (2)-(5), we wish to find an inequality implied by the facility location extended formulation in the original space cutting off this solution. Consider the formulation

\[
z_{FLS} = \max \ 0 \tag{14}
\]

\[
\sum_{j=1}^{N_j} \sum_{k=1}^{t} X_{i,jk}^i = d_{k}^i, \quad \text{for } i \in I, \ k \in T, \tag{15}
\]

\[
X_{i,jk}^i \leq d_{k,u}^j \hat{y}_{u}^j, \quad \text{for } i \in I, \ j \in J, \ t \in T, \ k \in \{t, \ldots, NT\}, \tag{16}
\]

\[
\sum_{k=t}^{NT} X_{i,jk}^i \leq \hat{x}_{j}^i, \quad \text{for } i \in I, \ j \in J, \ t \in T, \tag{17}
\]

\[
X_{i,jk}^i \geq 0, \quad \text{for } i \in I, \ j \in J, \ t \in T, \ k \in \{t, \ldots, NT\}. \tag{18}
\]

The objective function simply maximizes an arbitrary constant. Constraints (15) ensure all the demands are satisfied. Constraints (16) limit the multicommodity purchasing variables considering the values in \( \hat{y}. \) Constraints (17) link the original facility location variables with the values assumed by the original \( \hat{x}. \) Note
that due to the nonnegativity of all the coefficients in the objective function (1), \( \sum_{j \in J} \sum_{t \in T} \tilde{x}_{ij}^t = d_{1,NT}^i \) and thus (17) will hold at equality. Constraints (18) are nonnegativity requirements on the variables.

Define \( \phi \), \( \gamma \) and \( \theta \) to be the dual variables associated to constraints (15), (16) and (17), respectively. The dual of (14)-(18) can thus be written as

\[
    z_{DFLS} = \min \sum_{j \in J} \sum_{t \in T} \theta_{ij} i^t + \sum_{i = 1}^N \sum_{j = 1}^J \sum_{t = 1}^T \gamma_{ijk} d_k \hat{y}_t^j + \sum_{i = 1}^N \sum_{t = 1}^T \phi_i^i d_t
\]

(19)

\[
    \theta_{ij} + \gamma_{ij} + \phi_i^i \geq 0, \quad \text{for } i \in I, j \in J, t \in T, k \in \{t, \ldots, NT\},
\]

(20)

\[
    \theta_{ij} \geq 0, \quad \text{for } i \in I, j \in J, t \in T;
\]

(21)

\[
    \gamma_{ijk} \geq 0, \quad \text{for } i \in I, j \in J, t \in T, k \in \{t, \ldots, NT\}.
\]

(22)

Note that variables \( \phi \) are the only negative ones in a extreme ray (19) with negative cost. Thus, we normalize the extreme rays by assuming without loss of generality that \( \phi_i^i \geq -1 \) for \( i \in I \) and \( t \in T \). We formalize the inequalities obtained via (19)-(22) as

\[
    \sum_{j \in J} \sum_{t \in T} \theta_{ij} i^t + \sum_{i = 1}^N \sum_{j = 1}^J \sum_{t = 1}^T \gamma_{ijk} d_k \hat{y}_t^j + \sum_{i = 1}^N \sum_{t = 1}^T \phi_i^i d_t \geq 0.
\]

(23)

In what follows, we want to show that the matrix associated with constraints (20) is totally unimodular, and in order to do so, we use the next two results.

**Theorem 4.** A matrix \( A \) is TU iff: (a) the transpose matrix \( A^T \) is TU iff (b) the matrix \( (A, I) \) is TU, where \( I \) denotes the identity matrix. (Hoffman & Kruskal, 1957)

**Theorem 5.** A matrix \( A \) is TU if: (a) \( a_{ij} \in \{ -1, 0, +1 \} \) for all \( i, j \), and (b) for any subset \( M \) of the rows, there exists a partition \( (M_1, M_2) \) of \( M \) such that each column \( j \) satisfies \( \sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} \leq 1 \). Ghouila-Houri (1962)

**Theorem 6.** The matrix associated with constraints (20) is totally unimodular.

**Proof.** Denote \( A \) the matrix associated to constraints (20). Let \( A = (B, I) \), where \( B \) is the submatrix with the columns corresponding to variables \( \theta \) and \( \phi \) and \( I \) is the identity submatrix with those columns related to the \( \gamma \) variables. Using Theorem 4, we can concentrate on \( B \), as \( A = (B, I) \) is totally unimodular if \( B \) is totally unimodular. Furthermore, we focus on \( B^T \) and show that the properties in Theorem 5 hold. Property (a) clearly holds. Now, given \( M \) we add to \( M_1 \) the lines associated to the \( \phi \) variables and to \( M_2 \) those associated to the \( \theta \) variables. Thus the result holds.

We now analyze nondominated inequalities (23) obtained via (19)-(22).

**Lemma 7.** Nondominated inequalities are only related to a single item \( i \in I \).

**Proof.** Constraints (20)-(22) do not relate variables connected to different items. This implies that (19)-(22) can be solved separately for each item. Thus, any inequality (23) which contains more than one item can be obtained as a linear combination of the constraints related to each item separately.

**Lemma 8.** In a nondominated inequality, whenever \( \phi_{ik}^i = -1 \), for each period \( t \leq k \) either \( \theta_{ij} = 1 \) or \( \gamma_{ijk} = 1 \), but not both.

**Proof.** Note that both \( \theta_{ij} = 1 \) and \( \gamma_{ijk} = 1 \) have nonnegative coefficients in the objective function (19). With \( \phi_{ik}^i = -1 \), constraints (20) require that \( \theta_{ij} + \gamma_{ijk} \geq 1 \) for every \( t \leq k \). Whenever \( \theta_{ij} = 1 \), \( \gamma_{ijk} \) can be set to zero. Additionally, note that whenever \( \gamma_{ijk} = 1 \), \( \gamma_{ijk} = 1 \) for every \( k' \geq k \) for which \( \phi_{ik'}^i = -1 \).

**Lemma 9.** For a given item \( i \in I \), if there is a most violated inequality (23) in which \( \phi_{ik}^i = -1 \) for a given \( k > 1 \), then there is a must violated inequality in which \( \phi_{ik'}^i = -1 \) for every \( k' < k \).

7
Proof. Assume there is a most violated inequality obtained as (19) represented by a solution \((\hat{\phi}, \hat{\gamma}, \hat{\theta})\) in which \(\hat{\phi}_k^t = -1\). We want to show that we can set \(\hat{\phi}_k^t = -1\) and obtain another most violated inequality. If we set \(\hat{\phi}_k^t = -1\), observe that constraints (20) are already satisfied for every \(t \leq k'\) such that \(\hat{\theta}_i^j = 1\). Now consider the periods \(t \leq k'\) such that \(\hat{\theta}_i^j = 0\) and note that \(\hat{\gamma}_ij = 1\). Let \(T'\) be formed by all these periods. Observe that the summation \(-d^*_k + \sum_{t \in T'} (\hat{\gamma}_ij t_k^j) d_k^j\) is less or equal than zero as the inequality is a most violated one. Thus \(-d^*_k + \sum_{t \in T'} (\hat{\gamma}_ij t_k^j) d_k^j\) is also nonnegative. Thus, setting \(\hat{\phi}_k^t = -1\) and also \(\hat{\gamma}_ij = 1\) for every \(t \in T'\) leads to an inequality which is at least as violated as the original one. As this is true for any \(k' < k\), the result holds.

\begin{theorem}
Every \((l, S_j)\)-inequality can be obtained as (23).
\end{theorem}

\begin{proof}
Consider an \((l, S_j)\)-inequality obtained for a given \(i \in I\) and \(l \in T\), with sets \(S_j\) for each \(j \in J\). This inequality can be obtained as (23) by considering as only nonzero values:
\begin{itemize}
  \item \(\phi_k^t = -1\) for every \(k \in L\);
  \item \(\theta_i^j = 0\) for every \(j \in J\) and \(t \in L \setminus S_j\);
  \item \(\gamma_{ij} = 1\) for every \(j \in J\), \(t \in S_j\) and \(k \in L\) with \(t \geq k\).
\end{itemize}
\end{proof}

\begin{corollary}
Let \(\bar{z}_{STD^+}\) be the value of the linear relaxation of (1)-(5) with the addition of the inequalities (11), and \(\bar{z}_{FL}\) be the value of the linear relaxation of (6)-(10), then \(\bar{z}_{STD^+} = \bar{z}_{FL}\).
\end{corollary}

Corollary 12 follows from Theorems 10 and 11.

\section{A simple MIP heuristic}

In this section, we show how to use the facility location formulation, which often provides strong relaxations, in a heuristic way. Note that its \(O(NI \times NJ \times NT^2)\) variables and constraints turns the formulation prohibitive for being used to deal with large instances.

Let \(K_{MH}\) be a constant integer given as input to the MIP heuristic. The MIP heuristic only considers variables \(X_{ij}^t\) defined for periods \(t \in T\) and \(k \in T\), with \(t \leq k\) and \(k \leq t + K - 1\), i.e., variables corresponding to an interval of size \(K_{MH}\). The MIP heuristic thus consists in solving the formulation

\begin{equation}
\bar{z}_{FL(K_{MH})} = \min \sum_{i=1}^{NI} \sum_{j=1}^{NJ} \sum_{t=1}^{NT} \sum_{k=t+K_{MH}-1}^{NT} P_{ij} X_{ik}^t + \sum_{j=1}^{NJ} \sum_{t=1}^{NT} O_j y_j^t + \sum_{i=1}^{NI} \sum_{t=1}^{NT} H(t) \sum_{u=t}^{t+K_{MH}-1} \sum_{k=t+1}^{NT} X_{uk}^j
\end{equation}

\begin{equation}
X_{ij}^t = d_k^j, \quad \text{for } i \in I, \ k \in T,
\end{equation}

\begin{equation}
X_{ij}^t \leq d_k^j y_j^t, \quad \text{for } i \in I, \ j \in J, \ t \in T, \ k \in \{t, \ldots, \min\{t+K_{MH}-1, NT\}\},
\end{equation}

\begin{equation}
y_j^t \in \{0, 1\}, \quad \text{for } j \in J, \ t \in T,
\end{equation}

\begin{equation}
X_{ij}^t \geq 0, \quad \text{for } i \in I, \ j \in J, \ t \in T, \ k \in \{t, \ldots, \min\{t+K_{MH}-1, NT\}\}.
\end{equation}

Note that the objective function and all constraints are similar to those of the facility location formulation (6)-(10), differing only by the fact that solely a subset of the variables are considered. This formulation has \(O(NI \times NJ \times NT \times K_{MH})\) variables and constraints.
6 Computational experiments

This section summarizes the computational experiments conducted to assess the performance of the proposed approaches. All computational experiments were carried out on a machine running under Ubuntu GNU/Linux, with an Intel(R) Core(TM) i7-8700 CPU @ 3.20GHz processor and 16Gb of RAM. The algorithms were coded in Julia v1.2.0, using JuMP v0.18.6. The formulations were solved using Gurobi 9.0.1 with the standard configurations, except the relative optimality tolerance gap which was set to \(10^{-6}\).

Subsection 6.1 describes the benchmark instances. Subsection 6.2 details the tested approaches. Subsections 6.3, 6.4 and 6.5 summarize the computational results for small, medium and large instances, respectively.

6.1 Benchmark instances

The computational experiments were performed using the benchmark set of instances proposed by Basnet and Leung (2005), where more details can be obtained. Instances are assembled into instance groups, which are identified as (NJ, NI, NT). Each instance group (NJ, NI, NT) is composed of 15 randomly created instances with NJ suppliers, NI items and NT periods. All the data were generated using uniform distributions. The transaction costs lie in [1000,2000], the unitary purchase prices lie in [20,50], the holding costs lie in [1,5], and the demands lie in [1,200]. The benchmark set contains 150 instances divided into three categories: small, medium and large, which are summarized in Table 1.

| Category | Instance groups |
|----------|-----------------|
| Small    | (3, 3, 10); (3, 3, 15); (4, 4, 10); (4, 4, 15); (5, 5, 20) |
| Medium   | (10, 10, 50); (15, 15, 100); |
| Large    | (20, 20, 100); (20, 20, 200); (50, 50, 200); |

Table 1: Instance groups.

6.2 Tested approaches and parameter settings

The following approaches were considered in the computational experiments:

(a) STD: the standard formulation (1)-(5);

(b) FL: the facility location formulation (6)-(10) preprocessed using the results of section 3.1.1;

(c) WWBC: a branch-and-cut using the \((l, S_j)\)-inequalities (11) based on STD reinforced by an a priori addition of a subset of the Wagner-Whitin inequalities (13). The cuts are added only at the root node;

(d) MH: the MIP heuristic presented in Section 5.

For the small and medium instances, we executed STD, FL and WWBC. For the large instances, we executed STD, FL, WWBC and MH.

6.2.1 Settings and parameters

All tests for STD, FL and WWBC were carried out with a time limit of one hour (3600s). When executing MH for the large instances, a time limit of 10 minutes (600s) was imposed for instance groups (20,20,100) and (20,20,200), while 30 minutes (1800s) were allowed for the largest instance group (50,50,200). For the runs of WWBC, we limited the size of intervals of the \((l, S_j)\)-inequalities to be separated to five periods (an interval \([k, l]\) is defined in a way that every period in \([1, k − 1]\) is forced to be in \(L \setminus S_j\) and a most violated inequality was added for each of these intervals. Only Wagner-Whitin \((l, S_j)\)-inequalities corresponding to intervals of at most three periods were added a priori in the formulation. The value of \(K_{MH}\) was set to ten for instance groups (20,20,100) and (20,20,200) and to two for instance group (50,50,200). These parameters were defined based on preliminary experiments which considered one instance of each instance group to check the viability of using each of the approaches.
6.3 Results for small instances

We tested three approaches for the small instances: STD, FL and WWBC. The results are summarized in Table 2. The values in each line represent average values over the corresponding instance group. The first column identifies the instance group, followed by the average optimal value. For each approach, columns time(s) and #nodes give, respectively, the average time and the average number of nodes processed by the commercial solver to prove optimality of the instances. The column red(%) for FL indicate the average reduction (in %) of the $X$ variables eliminated using the preprocessing techniques described in Section 3.1.1.

All instances could be solved to optimality within 1.21 seconds on average using any of the approaches. We observe that both FL and BC outperform STD when we consider the number of enumerated nodes. FL has shown to be the most effective when we consider the times to solve the instances to optimality. Note that very few nodes are reported for FL (most instances were solved in the root node or after the first branch). The preprocessing techniques were able to significantly reduce the number of variables.

| Instance group | opt  | STD  | FL  | WWBC |
|----------------|------|------|-----|------|
|                |      | time(s) | #nodes | time(s) | #nodes | red(%) | time(s) | #nodes |
| (3,3,10)       | 101939.8 | 0.06  | 53.3 | 0.01  | 0.2  | 29.4 | 0.19  | 0.9  |
| (3,3,15)       | 147162.9 | 0.11  | 150.5| 0.01  | 0.2  | 38.6 | 0.26  | 0.9  |
| (4,4,10)       | 124526.0 | 0.09  | 19.0 | 0.01  | 0.1  | 29.2 | 0.27  | 1.0  |
| (4,4,15)       | 185681.3 | 0.20  | 229.2| 0.01  | 0.1  | 44.5 | 0.35  | 1.0  |
| (5,5,20)       | 300866.4 | 1.21  | 1345.9| 0.06  | 0.5  | 53.5 | 0.75  | 10.1 |

Table 2: Mixed integer programming results for the small instances.

Table 3 analyzes the linear relaxation bounds provided by FL for the small instances. The first and second columns are the same as in Table 3. Columns $\bar{z}_{FL}$ and gap$\bar{z}_{FL}$(%) report, respectively, the average linear relaxation bound and the average optimality gap provided by $\bar{z}_{FL}$, the latter determined for each instance as $100 \times \frac{opt - \bar{z}_{FL}}{opt}$. One can observe that FL provides very strong bounds, which are quite close to the optimal, as they are within 0.02% on average for all instance groups.

| Instance group | opt  | $\bar{z}_{FL}$ | gap$\bar{z}_{FL}$(%) |
|----------------|------|----------------|--------------------|
| (3,3,10)       | 101939.8 | 101920.1 | 0.02               |
| (3,3,15)       | 147162.9 | 147150.9 | 0.01               |
| (4,4,10)       | 124526.0 | 124517.9 | 0.01               |
| (4,4,15)       | 185681.3 | 185674.2 | <0.01             |
| (5,5,20)       | 300866.4 | 300792.8 | <0.01             |

Table 3: Results regarding the linear relaxation of the facility location formulation for the small instances.

6.4 Results for medium instances

As for the small instances, we tested STD, FL and WWBC for the medium instances. The results comparing the approaches are summarized in Table 4. The values in each line represent average values over the corresponding instance group. The first column identifies the instance group. For each of the approaches, column ub gives the average solution value, column time gives the average time to solve those instances which were solved to optimality, column #opt shows the number of instances solved to optimality, and column gap(%) gives the average open gap for those instances not solved to optimality, which is determined for each instance as $100 \times \frac{ub - lb}{ub}$, where $lb$ is the lower bound achieved at the end of the execution. Additionally, column red(%) represents the same as in Table 2. The results show that FL and WWBC could solve to optimality all the instances of group (10,10,50). It can be seen that the valid inequalities in WWBC were effective in improving STD for the instances in this group. FL could solve all the instances in this group within an average time of 3.2 seconds. For group (15,15,100), FL was again the best performing approach.
as it could reach optimality for 12/15 instances and for the unsolved instances the average gap was only 0.06%. WWBC did not perform very well for these instances, and the reason appears to be that the number of inequalities to be added did not allow the solver to achieve good upper bounds using heuristic solutions.

Table 5 reports the results regarding the linear relaxation bounds produced by FL for the medium instances. The first column identifies the instance group. Column best gives the average of the best values encountered using any of the approaches (STD, FL, WWBC), columns $\bar{z}_{FL}$ and $\text{gap}_{FL}(\%)$ report, correspondingly, the average linear relaxation bound and the average optimality gap provided by $\bar{z}_{FL}$, the latter determined for each instance as $100 \times \frac{\text{best} - \bar{z}_{FL}}{\text{best}}$. Again, it can be seen that FL provides very strong bounds, as they are within 0.06% on average for the two instance groups.

### 6.5 Results for large instances

We tested STD, FL, WWBC and MH for the large instances. The results are also compared with those obtained by the heuristics of Cárdenas-Barrón et al. (2015), which are denoted CGT15.

Table 6 summarizes the results. For STD, FL and WWBC, the columns represent the same information as the corresponding ones in Table 4. For MH, $\text{ub}$ gives the average best solution encountered, $\text{time}(s)$ informs the average running time in seconds, and $\text{gap}_{MH}(\%)$ gives the average optimality gap, obtained for each instance as $100 \times \frac{\text{ub} - \text{best}lb}{\text{ub}}$, where $\text{best}lb$ represents the best lower bound amongst those obtained with STD, FL and WWBC. We remark that $\text{best}lb$ was obtained using FL for instance groups (20,20,100) and (20,20,200) and with STD for instance group (50,50,200). For CGT15, $\text{ub}$ gives the average of the best heuristic result obtained in (Cárdenas-Barrón et al., 2015) (note that this takes into consideration for each instance the best amongst the two variants of the heuristic described in their work) and $\text{time}$ informs the average time in seconds reported by the authors. Cells with '-' indicate that there were instances for which the solver halted due to memory limitations. Note that we do not report times for STD, FL and WWBC. All executions of STD reached the time limit. The executions of WWBC either reached the time limit or halted due to memory limitations.

The results show that the average gap for FL when applied to instance group (20,20,100) is very small. In fact, seven of the instances in this group were solved to optimality with an average time of 1497.3 seconds. However, FL encountered issues for the other two larger groups due to its enormous size, even after the preprocessing. Again, WWBC faced trouble to obtain good quality feasible solutions. Most importantly, the results also show the considerable improvement of the average upper bound achieved by MH when compared to CGT15. As a matter of fact, the MIP heuristic obtained a remarkable result as it strictly improved all the solutions encountered by CGT15.
| Instance group | STD        | FL          | WWBC       |
|----------------|------------|-------------|------------|
|                | ub         | time | #opt | gap(%) | ub  | time | #opt | gap(%) | red(%) | ub   | time | #opt | gap(%) |
| (10,10,50)     | 1336981.5  | 540.0 | 14   | 0.02   | 1336981.5 | 3.2  | 15   | 0.00   | 77.5   | 1336981.5 | 143.3 | 15   | 0.00   |
| (15,15,100)    | 3802247.6  | 1740.6 | 4    | 0.15   | 3800974.8 | 300.8 | 12   | 0.06   | 88.9   | 3964885.8 | 2261.5 | 1    | 4.00   |

Table 4: Mixed integer programming results for the medium instances.

| Instance group | best | FL |
|----------------|------|----|
|                | z    | z  |
| (10,10,50)     | 1336981.5 | 1336538.6 | 0.03 |
| (15,15,100)    | 3800974.8 | 3798505.4 | 0.06 |

Table 5: Results regarding the linear relaxation of the facility location formulation for the medium instances.

| Instance group | STD   | FL    | WWBC   | MH   | CGT15 |
|----------------|-------|-------|--------|------|-------|
|                | ub    | gap(%)| ub     | gap(%)| ub    | time | gapMH(%)| ub  | time |
| (20,20,100)    | 5030120.6 | 4.53  | 4949373.9 | 0.07 | 88.3  | 5892766.6 | 17.08 | 4949399.1 | 542.5 | 0.04 | 4975148.5 | 1800.0 |
| (20,20,200)    | 10578643.5 | 12.60 | -      | -    | 94.2  | 24335149.9 | 55.3  | 9818168.7 | 600.0 | 0.75 | 9914547.6 | 1800.0 |
| (50,50,200)    | 25140386.3 | 16.17 | -      | -    | -     | 23056349.6 | 1800.0 | 8.61  | 23457449.4 | 1800.0 |

Table 6: Results for the large instances.
7 Final conclusions

In this paper, we considered the multi-item inventory lot-sizing problem with supplier selection. The complexity of the problem was an open question and thus we have shown that it is NP-hard. Moreover, we have proposed a facility location extended formulation together with a preprocessing scheme, valid inequalities in the original space of variables and an easy to implement mixed integer programming (MIP) heuristic.

Computational experiments have shown that the facility location formulation was very effective when solving small and medium instances to optimality as nearly all of them could be solved to optimality within a few minutes. Also, the preprocessing scheme was able to reach considerable reduction in the amount of variables considered for optimization. The valid inequalities implemented in a branch-and-cut approach could successfully improve the capacity of the solver to deal with small and medium instances of group (10,10,50). However, this performance was substantially reduced for the larger instances. Finally, the proposed MIP heuristic was able to encounter high quality results, outperforming those obtained by a state-of-the-art approach for all the tested large instances.

The multi-item inventory lot-sizing problem with supplier selection treated in this paper is for a two echelon supply chain composed of one buyer and multiple suppliers. Thus, it would be interesting to explore this problem in a multi-echelon supply chain. Additionally, one could consider to include constraints on production, storage, budget, among others. Another challenging extension would be to allow shortages with full backordering. One main characteristic in the demands of the items is that they are known in advance, what is not always true in practice. For this reason, it would be also interesting to model the uncertainty in the demand. These are some research avenues that could investigated in the future.

Acknowledgments: Work of Rafael A. Melo was supported by Universidade Federal da Bahia, the Brazilian Ministry of Science, Technology, Innovation and Communication (MCTIC); the State of Bahia Research Foundation (FAPESB); and the Brazilian National Council for Scientific and Technological Development (CNPq).

References

Aissaoui, N., Haouari, M., & Hassini, E. (2007). Supplier selection and order lot sizing modeling: A review. Computers & Operations Research, 34(12), 3516 - 3540.
Akartunalı, K., & Miller, A. J. (2009). A heuristic approach for big bucket multi-level production planning problems. European Journal of Operational Research, 193(2), 396–411.
Akbalik, A., & Rapine, C. (2018). Lot sizing problem with multi-mode replenishment and batch delivery. Omega, 81, 123–133.
Barany, I., Van Roy, T., & Wolsey, L. A. (1984). Uncapacitated lot-sizing: The convex hull of solutions. In Mathematical programming at Oberwolfach II (pp. 32–43). Springer.
Basnet, C., & Leung, J. M. (2005). Inventory lot-sizing with supplier selection. Computers & Operations Research, 32(1), 1 – 14.
Choudhary, D., & Shankar, R. (2013). Joint decision of procurement lot-size, supplier selection, and carrier selection. Journal of Purchasing and Supply Management, 19(1), 16 - 26.
Choudhary, D., & Shankar, R. (2014). A goal programming model for joint decision making of inventory lot-size, supplier selection and carrier selection. Computers & Industrial Engineering, 71, 1–9.
Chudak, F. A., & Shmoys, D. B. (2003). Improved approximation algorithms for the uncapacitated facility location problem. SIAM Journal on Computing, 33(1), 1–25.
Cunha, A. L., Santos, M. O., Morabito, R., & Barbosa-Póvoa, A. (2018). An integrated approach for production lot sizing and raw material purchasing. European Journal of Operational Research, 269(3), 923 - 938.
Cunha, J. O., Kramer, H. H., & Melo, R. A. (2019). Effective matheuristics for the multi-item capacitated lot-sizing problem with remanufacturing. Computers & Operations Research, 104, 149 - 158.
Cárdenas-Barrón, L. E., González-Velarde, J. L., & Treviño-Garza, G. (2015). A new approach to solve the multi-product multi-period inventory lot sizing with supplier selection problem. *Computers & Operations Research, 64*, 225 – 232.

Dahel, N.-E. (2003). Vendor selection and order quantity allocation in volume discount environments. *Supply Chain Management: An International Journal, 8*(4), 335–342.

Ghaniabadi, M., & Mazinani, A. (2017). Dynamic lot sizing with multiple suppliers, backlogging and quantity discounts. *Computers & Industrial Engineering, 110*, 67–74.

Ghouila-Houri, A. (1962). Caractérisation des matrices totalement unimodulaires. *Comptes Redus Hebdomadaires des Séances de l’Académie des Sciences (Paris), 254*, 1192–1194.

Helber, S., & Sahling, F. (2010). A fix-and-optimize approach for the multi-level capacitated lot sizing problem. *International Journal of Production Economics, 123*(2), 247–256.

Ho, W., Xu, X., & Dey, P. K. (2010). Multi-criteria decision making approaches for supplier evaluation and selection: A literature review. *European Journal of Operational Research, 202*(1), 16–24.

Hoffman, A., & Kruskal, J. (1957). Integral boundary points of convex polyhedra. In *Linear inequalities and related systems* (Vol. 38, pp. 223–246). Princeton University Press.

Kasilingam, R. G., & Lee, C. P. (1996). Selection of vendors - A mixed-integer programming approach. *Computers & Industrial Engineering, 31*(1-2), 347–350.

Kirschstein, T., & Meisel, F. (2019). A multi-period multi-commodity lot-sizing problem with supplier selection, storage selection and discounts for the process industry. *European Journal of Operational Research, 279*(2), 393 - 406.

Krarup, J., & Bilde, O. (1977). Plant location, set covering and economic lot size: An $O(mn)$-algorithm for structured problems. In *Numerische Methoden bei Optimierungsaufgaben Band 3* (pp. 155–180). Springer.

Melo, R. A., & Ribeiro, C. C. (2017). Formulations and heuristics for the multi-item uncapacitated lot-sizing problem with inventory bounds. *International Journal of Production Research, 55*(2), 576-592.

Melo, R. A., & Wolsey, L. A. (2012). MIP formulations and heuristics for two-level production-transportation problems. *Computers & Operations Research, 39*(11), 2776 - 2786.

Ustun, O., & Demirtas, E. A. (2008). Multi-period lot-sizing with supplier selection using achievement scalarizing functions. *Computers & Industrial Engineering, 54*(4), 918–931.

Wadhwa, V., & Ravindran, A. R. (2007). Vendor selection in outsourcing. *Computers & Operations Research, 34*(12), 3725–3737.

Ware, N. R., Singh, S., & Banwet, D. (2012). Supplier selection problem: A state-of-the-art review. *Management Science Letters, 2*(5), 1465–1490.

Ware, N. R., Singh, S., & Banwet, D. (2014). A mixed-integer non-linear program to model dynamic supplier selection problem. *Expert Systems with Applications, 41*(2), 671–678.

Zhao, Y., & Klabjan, D. (2012). A polyhedral study of lot-sizing with supplier selection. *Discrete Optimization, 9*(2), 65–76.