Lang's conjectures, Conjecture H, and uniformity
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* THE PURPOSE OF THIS NOTE IS TO WISH A HAPPY BIRTHDAY TO PROFESSOR LUCIA CAPORASO.

We prove that Conjecture H of Caporaso et al. [CHarM], §6 together with Lang's conjecture implies the uniformity of rational points on varieties of general type, as predicted in [CHarM]; a few applications in arithmetic and geometry are stated. As in [8], one uses the fact that rational points on $X^n$ are $n$-tuples of points when trying to bound $n$, as well as the fact that the Noetherian induction used in [CHarM] can be wired into the definitions.

Geometers with number-theory anxiety should skip directly to the corollary on the last page.

1. Introduction

Let $X$ be a variety of general type defined over a number field $K$. It was conjectured by S. Lang that the set of rational points $X(K)$ is not Zariski dense in $X$. In the paper [CHarM] of L. Caporaso, J. Harris and B. Mazur it is shown that the above conjecture of Lang implies the existence of a uniform bound on the number of $K$-rational points of all curves of fixed genus $g$ over $K$.

The paper [CHarM] has immediately created a chasm among arithmetic geometers. This chasm, which often runs right in the middle of the personalities involved, divides between loyal believers of Lang’s conjecture, who marvel in this powerful implication, and the disbelievers, who try to use this implication to derive counterexamples to the conjecture.

In this paper we will attempt to deepen this chasm, using the techniques of [CHarM] and continuing §3, by proving more implications, some of which very strong, of various conjectures of Lang. Along the way we will often use a conjecture donned by Caporaso et al. Conjecture H (see again [CHarM], §6) about Higher dimensional varieties, which is regarded very plausible among experts of higher dimensional algebraic geometry.

Before we state any results, we need to specify various conjectures which we will apply.

1.1. A few conjectures of Lang. Let $X$ be a variety of general type over a field $K$ of characteristic 0. In view of Faltings’s proof of Mordell’s conjecture, Lang has stated the following conjectures:

Conjecture 1. 1. (Weak Lang conjecture) If $K$ is finitely generated over $\mathbb{Q}$ then the set of rational points $X(K)$ is not Zariski dense in $X$.
2. (Weak Lang conjecture for function fields) If $k \subset K$ is a finitely generated regular extension in characteristic 0, and if $X(K)$ is Zariski dense in $X$, then $X$ is birational to a variety $X_0$ defined over $k$ and the “non-constant points” $X(K) \setminus X_0(k)$ are not Zariski dense in $X$.
3. (Geometric Lang’s conjecture) Assuming only $\text{Char}(K) = 0$, there is a proper Zariski closed subset $Z(X) \subset X$, called in [CHarM] the Langian exceptional set, which is the union of all positive dimensional subvarieties which are not of general type.
4. (Strong Lang conjecture) If $K$ is finitely generated over $\mathbb{Q}$ then there is a Zariski closed subset $Z \subset X$ such that for any finitely generated field $L \supset K$ we have that $X(L) \setminus Z(L)$ is finite.

These conjectures and the relationship between them are studied in [LangAMS], [LangIII], and in the introduction of [CHarM]. For instance, it should be noted that the weak Lang conjecture together with the geometric conjecture imply the strong Lang conjecture.

It should also be remarked that the analogous situation over fields of positive characteristic is subtle and interesting. See a recent survey by Voloch [Vol].

1.2. Conjecture H. An important tool used by Caporaso et al. in [CHarM] is that of fibered powers. Let $X \to B$ be a morphism of varieties in characteristic 0, where the general fiber is a variety of general type. We denote by $X_B^n$ the $n$-th fibered power of $X$ over $B$.

Conjecture 2. (Conjecture H of [CHarM]) For sufficiently large $n$, there exists a dominant rational map $h_n : X_B^n \to W_n$ where $W_n$ is a variety of general type, and where the restriction of $h_n$ to the general fiber $(X_b)^n$ is generically finite.
This conjecture is known for curves and surfaces:

**Theorem H 1.** (Correlation theorem of [CHarM]) Conjecture H holds when $X \to B$ is a family of curves of genus $> 1$.

**Theorem H 2.** (Correlation theorem of [Has]) Conjecture H holds when $X \to B$ is a family of surfaces of general type.

Using their Theorem H 1, and Lemma 1.1 of [CHarM], Caporaso et al. have shown that the weak Lang conjecture implies a uniform bound on the number of rational points on curves (Uniform bound theorem, [CHarM] Theorem 1.1).

It should be noted that the proofs of theorems H 1 and H 2 give a bit more: they describe a natural dominant rational map $X^*_B \to W$. For the case of curves, if $B_0$ is the image of $B$ in the moduli space, $M_g$, then for sufficiently large $n$ the inverse image $B_n \subset M_{g,n}$ in the moduli space of $n$-pointed curves is a variety of general type. Therefore the moduli map $X^*_B \to B_n \subset M_{g,n}$ satisfies the requirements. A similar construction works for surfaces of general type, and one may ask whether this should hold in general.

It is convenient to make the following definitions when discussing Lang’s conjectures:

**Definition:** 1. A variety $X/K$ is said to be a Lang variety if there is a dominant rational map $X_K \to W$, where $W$ is a positive dimensional variety of general type.

2. A positive dimensional variety $X$ is said to be geometrically mordellic (In short GeM) if $X_K$ does not contain subvarieties which are not of general type.

In [LangIII], in the course of stating even more far reaching conjectures, Lang defined by a notion of algebraically hyperbolic varieties which is very similar, and conjecturally the same as that of GeM varieties. I chose to use a different terminology here, to avoid confusion.

Note that the weak Lang conjecture directly implies that the rational points on a Lang variety over a number field are not Zariski dense, and that there are only finitely many rational points over a number field on a GeM variety.

### 1.3. Summary of results

An indicated in [CHarM] §6, Conjecture H together with Lang’s conjectures should have very strong implications for counting rational points on varieties of general type, similar to the uniform bound theorem of [CHarM]. Here we will prove the following basic result:

**Theorem 1.** Assume that the weak Lang conjecture as well as conjecture H hold. Let $X \to B$ be a family of GeM varieties over a number field $K$ (or any finitely generated field over $Q$). Then there is a uniform bound on $\sharp X_b(K)$.

One may refine this theorem for arbitrary families of varieties of general type, obtaining a bound on the number of points which do not lie in lang exceptional sets of fibers. If one assumes Lang’s geometric conjecture, one obtains a closed subset $Z(X_b)$ for every $b \in B$. A natural question which arises in such a refinement is: how do these subsets fit together? An answer was given in [CHarM], Theorem 6.1, assuming conjecture H as well: the varieties $Z(X)$ are uniformly bounded. We will show that, using results of Viehweg, one does not need to assume conjecture H:

**Theorem 2.** (Compare [CHarM], Theorem 6.1) Assume that the geometric Lang conjecture holds. Let $X \to B$ be a family of varieties of general type. Then there is a proper closed subvariety $\tilde{Z} \subset X$ such that for any $b \in B$ we have $Z(X_b) \subset \tilde{Z}$.

Using theorem 2, we can apply theorem 1 to any family $X \to B$ of varieties of general type, assuming that the geometric Lang conjecture holds: we can bound the rational points in the complement of $\tilde{Z}$.

We will apply our theorem 1 in various natural cases. An immediate but rather surprising application is the following theorem:

**Theorem 3.** Assume that the weak Lang conjecture as well as conjecture H hold. Let $X \to B$ be a family of GeM varieties over a field $K$ finitely generated over $Q$. Fix a number $d$. Then there is a uniform bound $N_d$ such that for any field extension $L$ of $K$ of degree $d$ and every $b \in B(L)$ we have $\sharp X_b(L) < N_d$.

As a corollary, we see that Lang’s conjecture together with conjecture H imply the existence of a bound on the number of points on curves of fixed genus $g$ over a number field $K$ which depends only on the degree of the number field.
These results have natural analogues for function fields. We will state a few of these, notably:

**Theorem 4.** Assume that Lang’s conjecture for function fields holds. Fix an integer \( g > 1 \). Then there is an integer \( N(g) \) such that for any generically smooth fibre of curves \( C \to D \) where the fibre has genus \( g \) and the base is hyperelliptic curve, there are at most \( N \) non-constant sections \( s : D \to C \).

We remind the reader that the gonality of a curve \( D \) is the minimal degree of a nonconstant rational function on \( D \) (so a curve of gonality 2 is hyperelliptic). One expects the above theorem to be generalized to the situation where “hyperelliptic curve” is replaced by “curve of gonality \( \leq d \)” for fixed \( d \).

2. Proof of theorem \( \Box \)

2.1. Preliminaries. Throughout this subsection we assume that conjecture H holds, and the base field is algebraically closed.

Observe that a positive dimensional subvariety of an GeM variety is GeM; and the normalization of an GeM variety is a Lang variety as well.

**Proposition 1.** Let \( X \to B \) be a family of GeM varieties. Let \( F \subset X \) be a reduced subscheme such that every component of \( F \) dominating \( B \) has positive fiber dimension. Then for \( n \) sufficiently large, every component of the fibered power \( F^n_B \) which dominates \( B \) is a Lang variety.

The proof will use the following lemmas:

**Lemma 1.** Let \( X \to B \) and \( F \) be as above, and assume that the general fiber of \( F \to B \) is irreducible. Then for \( n \) sufficiently large, the dominant component of \( F^n_B \) is a Lang variety.

**Proof.** Apply conjecture \( H \) to \( F \to B \), using the fact that the fibers of \( F \) are of general type.

**Lemma 2.** Let \( X \to B \) and \( F \) be as in the proposition, with \( F \) irreducible. Then for \( n \) sufficiently large, every component of the fibered power \( F^n_B \) which dominates \( B \) is a Lang variety.

**Proof.** Let \( \tilde{F} \) be the normalization of \( F \), and let \( \tilde{F} \to \tilde{B} \to B \) be the Stein factorization. Denote by \( c \) the degree of \( B \) over \( B \). Let \( G \subset \tilde{F}^n_B \) be a dominant component. Then \( G \) parametrizes \( n \)-tuples of points in the fibers of \( \tilde{F} \) over \( B \), and since \( G \) is irreducible, there is a decomposition \( \{1, \ldots, n\} = \cup_{i=1}^t J_i \) and \( G \) surjects onto the dominant component of \( \tilde{F}^n_{\tilde{B}} \). At least one of \( J_i \) has at least \( n/c \) elements. Using lemma 1 applied to \( \tilde{F} \to \tilde{B} \), we see that for \( n/c \) large enough \( G \) is a Lang variety.

**Proof of proposition.** Let \( F = F_1 \cup \ldots \cup F_n \) be the decomposition into irreducible components. Let \( G \) be a dominant component of \( F^n_B \). Then \( G \) dominates \( (F_1)^n_B \times_B \cdots \times_B (F_n)^n_B \). For at least one \( i \) we have \( n_i > n/m \), so applying the previous lemma we obtain that \( G \) is a Lang variety.

2.2. Prolongable points. We return to the setup in theorem \( \Box \).

**Definition.** 1. A point \( x_n = (P_1, \ldots, P_n) \in X^n_B(K) \) is said to be off diagonal if for any \( 1 \leq i < j \leq n \) we have \( P_i \neq P_j \). We extend this for \( n = 0 \) trivially by agreeing that any point of \( B(K) \) is off diagonal.

2. Let \( m > n \). An off diagonal point \( x_n \) is said to be \( m \)-prolongable if there is an off-diagonal \( x_m \in X^n_B(K) \) whose first \( n \) coordinates give \( x_n \).

Let \( E^{(m)}(m) \) be the set of \( m \)-prolongable points on \( X^n_B \), and \( F^{(m)}_n \) be the Zariski closure. Let \( F_n = \cap_{m > n} E^{(m)}_n \). By the Noetherian property of the Zariski topology we have \( F_n = F^{(m)}_n \) for some \( m \).

All we need to show is \( F_n = \emptyset \) for some \( n \).

**Lemma 3.** We have a surjection \( F_{n+1} \to F_n \).

**Proof.** The set \( E^{(m)}_{n+1} \) surjects to \( E^{(m)}_n \) for any \( m > n + 1 \).

**Lemma 4.** Every fiber of \( F_{n+1} \to F_n \) is positive dimensional.

**Proof.** Suppose that over an open set in \( F_n \) the degree of the map is \( d \). Then \( E^{(n+d+1)}_n \) cannot be dense in \( F_n \); if \( (y_1, \ldots, y_{n+d+1}) \) is an off diagonal prolongation of \( (y_1, \ldots, y_{n+d+1}) \in E^{(n+d+1)}_n \), then for \( n+1 \leq j \leq n+d+1 \) we have that the points \( (y_1, \ldots, y_n, y_j) \in E^{(n+d+1)}_{n+1} \) are distinct, therefore the degree of the map is at least \( d+1 \).
2.3. Proof of theorem. We show by induction on \(i\) that for any \(n\) and \(i\) the dimension of any fiber of \(F_{n+i} \to F_n\) is at least \(i + 1\). Lemma 4 shows this for \(i = 0\). Assume it holds true for \(i - 1\), let \(n \geq 0\) and let \(G\) be a component of \(F_{n+i}\) such that the fiber dimension of \(F_{n+i+1}\) over \(G\) is \(i\). Applying the inductive assumption to each \(F_{n+j} \to F_{n+j+1}\), we have that the dimension of every fiber of \(F_{n+k}\) over \(F_0\) is at least \(ik\). On the other hand, \(F_{n+k}\) is a subscheme of the fibered power \((F_{n+i+1})_{\mathbb{K}_i}\), so over \(G\) it has fiber dimension precisely \(ik\). Therefore there exists a component \(H_k\) of \(F_{n+k}\) dominant over \(G\) of fiber dimension \(ik\), which is therefore identified as a dominant component of the fibered power \((F_{n+i+1})_{\mathbb{K}_i}\). By proposition 1, for \(k\) sufficiently large we have that \(H_k\) is a Lang variety. Lang’s conjecture implies that \(H_k(K)\) is not dense in \(K\), contradicting the definition of \(F_{n+k}\).

3. A FEW REFINEMENTS AND APPLICATIONS IN ARITHMETIC AND GEOMETRY

3.1. Proof of Theorem 3 Assume that \(X \to B\) is a family of varieties of general type. By Hironaka’s desingularization theorem, we may assume that \(B\) is a smooth projective variety. Let \(L\) be an ample line bundle on \(B\), let \(n \gg 0\) be a sufficiently large integer and let \(H\) be a smooth divisor of \(L^\otimes n\). Let \(B_1 \to B\) be the cyclic cover ramified to order \(n\) along \(H\). Then by adjunction, \(B_1\) is a variety of general type. Let \(X_1 \to X\) be the pullback of \(X\) to \(B_1\). By the main theorem (Satz III) of [Vit], the variety \(X_1\) is of general type. Assuming the geometric Lang conjecture, Let \(Z_1(X_1)\) be the Langian exceptional set. Let \(\tilde{Z}\) be the image of \(Z_1(X_1)\) in \(X\). Then for any \(b \in B\), we have by definition that \(Z(X_b) \subseteq \tilde{Z}\).

It has been noted in [HarM] that Viehweg’s work goes a long way towards proving conjecture H. It is therefore not surprising that it may be used on occasion to replace the assumption of conjecture H.

3.2. Uniformity in terms of the degree of an extension. Let \(X \to B\) be a family of GeM varieties over \(K\). Assuming the conjectures, theorem 1 gave us a uniform bound on the number of rational points over finite extension fields in the fibers. We will now see that this in fact yields a much stronger result, namely our theorem 3 the uniform bound only depends on the degree of the field extension.

Proof of theorem 3: for \(n = 1\) or 2, Let \(Y_n = \text{Sym}^d(X'B)\), and \(Y_0 = \text{Sym}^d(B)\). Then we have natural maps \(p_n : Y_n \to Y_{n-1}\). Let \(\Gamma\) be the branch locus of the quotient map \(X' \to Y_1\), namely the set of points which are fixed by some permutation. If \(P \not\in \Gamma\) then \(p_2^{-1}(P)\) is a GeM variety, isomorphic over \(K\) to the product of \(d\) fibers of \(X\). Denote \(Y'_1 = Y_1 \setminus \Gamma_1\), and \(Y'_2 = p_2^{-1}Y'_1\). Then \(Y'_2 \to Y'_1\) is a family of GeM varieties, and by Theorem 1 we have a bound on the cardinality of \((Y'_2)_y(K)\) uniform over \(y \in Y'_1(K)\).

By induction, it suffices to bound the number of points in \(X_b(L)\) over any field \(L\) of degree \(d\) over \(K\), which are defined over \(L\) but not over any intermediate field. If \(\sigma_1, \ldots, \sigma_d\) are the distinct embeddings of \(L\) in \(K\), and \(P \in X_b(L)\) not defined over an intermediate field, then the points \(\sigma_i(P) \in X_{\sigma_i(K)}(K) \subset X_{\sigma_i(K)}(K)\) are distinct. If \((P_1, P_2) \in X_2(K)\) is a pair of such points, then the Galois orbit \[\{\sigma_i(P_1, P_2), \sigma = 1, \ldots, d\}\] is Galois stable, therefore it gives rise to a point in \(Y_2(K)\). This point has the further property that its image in \(Y_1\) does not lie in \(\Gamma_1\), so it gives rise to a point in \(Y_1(K)\). The previous paragraph shows that the number of points on a fiber is bounded.

Applying theorem 3 where \(X \to B\) is the universal family over the Hilbert scheme of 3-canonical curves of genus \(g\) (as in [HarM], §§1.2), we obtain the following:

Corollary 1. Assume that the weak Lang conjecture as well as conjecture H hold. Fix integers \(d, g > 1\) and a number field \(K\). Then there is a uniform bound \(N_d\) such that for any field extension \(L\) of \(K\) of degree \(d\) and every curve \(C\) of genus \(g\) over \(L\) we have \(\#C(L) < N_d\).

We remark that in the cases of degrees \(d \leq 3\) one does not need to assume conjecture H: this was proven in [Voj], using the fact that conjecture H holds for families of curves or surfaces. A similar result is being worked out by P. Pacelli for arbitrary \(d\).

Here is a special case: let \(f(x) \in \mathbb{Q}(x)\) be a polynomial of degree \(> 4\) with distinct complex roots. Then, assuming the weak Lang conjecture, the number of rational points over any quadratic field on the curve \(C : y^2 = f(x)\) is bounded uniformly. We remark that, if \(\deg f > 6\), this in fact may be deduced using a combination of [HarM] and a theorem of Vojta [Voj], which says that all but finitely many quadratic points on \(C\) have rational \(x\) coordinate. One then applies [HarM] which gives a uniform bound on the rational points on the twists \(ty^2 = f(x)\).

Following the suggestion of [HarM], §6 one can apply Theorem 1 to symmetric powers of curves. Since conjecture H is known for surfaces, one obtains the following (stated without proof in [HarM], Theorem 6.2):
Corollary 2. (Compare [CHarM], Theorem 6.2) Assume that the weak Lang conjecture holds. Fix a number field $K$. Then there is a uniform bound $N$ for the number of quadratic points on any nonhyperelliptic, non-bielliptic curve $C$ of genus $g$ over $K$.

Similarly, it was shown in [AH], lemma 1 that if the gonality of a curve $C$ is $> 2d$ then $\text{Sym}^d(C)$ is GeM. Recall that a closed point $P$ on $C$ is said to be of degree $d$ over $K$ if $[K(P) : K] = d$. We deduce the following:

Corollary 3. Assume that the weak Lang conjecture holds. Fix a number field $K$ and an integer $d$. Then there is a uniform bound $N$ for the number of points of degree $d$ over $K$ on any curve $C$ of genus $g$ and gonality $> 2d$ over $K$.

3.3. The geometric case. One can use the same methods using Lang’s conjecture for function fields of characteristic 0, say over $C$. Given a fibration $X \to B$ where the generic fiber is a variety of general type, a rational point $s \in X(K_B)$ over the function field of $B$ is called constant if $X$ is birational to a product $X_0 \times B$ and $s$ corresponds to a point on $X_0$. Lang’s conjecture for function fields says that the non-constant points are not Zariski dense.

In this section we will restrict attention to the case where the base is the projective line $\mathbb{P}^1$. We will only assume the following statement: if $X$ is a variety of general type, then the rational curves in $X$ are not Zariski dense. It is easy to see that this statement in fact follows from the geometric Lang conjecture, as well as from Lang’s conjecture for function fields.

We would like to apply this conjecture to obtain geometric uniformity results. One has to be careful here, since the conjecture does not apply to Lang varieties, and one has to use a variety of general type directly.

As stated in the introduction, if $X \to B$ is a family of curves of genus $> 1$ the appropriate variety $W$ of general type dominated by $X_B^g$ is identified in [CHarM] as the image $B_n \subset M_{g,n}$ of $X_B^g$ by the moduli map. We use this in the proof of the following proposition:

Proposition 2. Assume that Lang’s conjecture for function fields holds. Fix an integer $g > 1$. Then there is a bound $N$ such that for any generically smooth family of curves $C \to \mathbb{P}^1$ of genus $g$ there are at most $N$ non-constant sections $s : \mathbb{P}^1 \to C$.

Proof. First note that if $s : \mathbb{P}^1 \to C$ is a nonconstant section whose image in $M_{g,1}$ is a point, then $s$ becomes a constant section after a finite base change $D \to \mathbb{P}^1$. This implies that $s$ is fixed by a nontrivial automorphism of $C$, and the number of such points is bounded in terms of $g$. Therefore it suffices to bound the number of sections whose image in $M_{g,1}$ is non-constant. We will call such sections strictly non-constant.

Let $B_0 \subset M_g$ be a closed subvariety, and choose such that $B_n \subset M_{g,n}$ is of general type. If a family $C \to \mathbb{P}^1$ has moduli in $B_0$, then for any $n$-tuple of strictly non-constant sections $s_i : \mathbb{P}^1 \to C$, we obtain a non-constant rational map $\mathbb{P}^1 \to B_n$. Let $F \subset B_n$ be the Zariski closure of the images of the collection of non-constant rational maps obtained this way.

Since $B_n$ is of general type, Lang’s conjecture implies that $F \neq B_n$. Applying lemma 1.1 of [CHarM] we obtain that there is an closed subset set $F_0 \subset B_0$ and an integer $N$ such that, given a family of curve $C \to \mathbb{P}^1$ such that the rational image of $\mathbb{P}^1$ in $M_g$ lies in $B_0$ but not in $F_0$, there are at most $N$ non-constant sections of $C$. Noetherian induction gives the theorem.

Choosing a coordinate $t$ on $\mathbb{P}^1$ we can pull back the curve $C$ along the map $\mathbb{P}^1 \to \mathbb{P}^1$ obtained by taking $n$-th roots of $t$. Let $\mathbb{C}(t^{1/\infty}) = \mathbb{C}\{t^{1/n}, n \geq 1\}$, the field obtained by adjoining all roots of $t$. If one restricts attention to non-isotrivial curves, one obtains the following amusing result (suggested to the author by Felipe Voloch):

Corollary 4. Assume that the Lang conjecture for function fields holds. Fix an integer $g > 1$. Then there is a bound $N$ such that for any smooth nonsotrivial curve $C$ over $\mathbb{C}(t)$ of genus $g$ there are at most $N$ points in $C(\mathbb{C}(t^{1/\infty}))$.

One can also try to prove uniformity results analogous to theorem 3. Using the results in [AH] we can refine proposition 2 and obtain theorem 4.

Proof of theorem 4. The proof is a slight modification of the theorem of [AH], keeping track of the dominant map to a variety of general type.

As in the proof of theorem 3, it suffices to look at sections $s : D \to C$ which are not pullbacks of sections of a family over $\mathbb{P}^1$. 
In an analogous way to the proof of theorem [1], we say that an $n$-tuple of distinct, strictly non-constant sections is $m$-prolongable if it may be prolonged to an $m$-tuple of distinct, strictly non-constant sections, none of which being the pullback from a family over $\mathbb{P}^d$. Any $n$-tuple of distinct sections $s_i : D \to C$ over a hyperelliptic curve $D$ gives rise to a rational map $\mathbb{P}^d \to \text{Sym}^2(\mathbf{M}_{g,n})$. We define $F_n^{(m)}$ to be the closure in $\text{Sym}^2(\mathbf{M}_{g,n})$ of the images of $m$-prolongable sections, and $F_n = \cap_{m > n} F_n^{(m)}$.

As in Lemma 1, we have that the relative dimension of any fiber of $F_{n+1} \to F_n$ is positive. We have two cases to consider: either for high $n$ there is a component of $F_{n+1}$ having fiber dimension 1 over $F_n$, or for all $n$ the fiber dimension is everywhere 2.

In case the fiber dimension is 1, we will see that there is a component of $F_{n+k}$ which is a variety of general type. Assuming Lang’s conjecture for function fields this contradicts the fact that the images of non-constant sections are dense. Fix a general fiber $f$ of $F_{n+1}$ over $F_n$. The curve $f$ lies inside a surface isomorphic to the product of two curves $C_{b_1} \times C_{b_2}$. By the definition of $m$-prolongable sections, and analogously to lemma 1, we obtain that there is a component $f'$ of $f$ which maps surjectively to both $C_{b_1}$ and $C_{b_2}$. Therefore as either $b_1$ or $b_2$ moves in $B_0$, the moduli of $f'$ move as well.

Let $F'$ be a component of $F_{n+1}$ whose fibers have this property. If we follow the proof of proposition 1 and use the moduli description of the dominant map to a variety of general type $m : (F')^k_{F_n} \to W$, we see that if $E$ is a general curve in $(F')^k_{F_n}$ lying in a fiber of $m$, then $E$ projects to a point in $B_0$; moreover, by the definition of prolongable points, $E$ projects to an off diagonal point in some $(F''')^k_{F_n}$. But the fibers over off-diagonal points are GeM varieties, therefore the general fiber of the map $m$ is of general type. By the main theorem of [Vie], $(F')^k_{F_n}$ is itself a variety of general type, and therefore $F_{n+k}$ has a component of general type, contradicting Lang’s conjecture.

In case of fiber dimension 2, we use proposition 1 of [3]: let $B \subset \text{Sym}^2(\mathbf{M}_g)$. Then for high $n$, the inverse image $B_n \subset \text{Sym}^2(\mathbf{M}_{g,n})$ of $B$ is a variety of general type. Since the images of non-constant sections are dense in $F_n$, this again contradicts Lang’s conjecture.

If one restricts attentions to trivial fibrations, one obtains as an immediate corollary:

**Corollary 5.** Assume that the Lang conjecture for function fields holds. Fix an integer $g > 1$. Then there is an integer $N$ such that for any curve $C$ of genus $g$ and any hyperelliptic curve $D$ there are at most $N$ non-trivial morphisms $f : D \to C$.

It should be noted that the theory of Hilbert schemes gives the existence of a bound depending on the genus of $D$, which is clearly not as strong. As in the arithmetic case, I expect that work in progress of Pacelli should generalize these results to the case where $D$ is d-gonal, for fixed $d$.

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