YANG–MILLS CONFIGURATIONS FROM 3D Riemann–Cartan Geometry

by

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Abstract

Recently, the spacelike part of the $SU(2)$ Yang–Mills equations has been identified with geometrical objects of a three-dimensional space of constant Riemann–Cartan curvature. We give a concise derivation of this Ashtekar type (“inverse Kaluza–Klein”) mapping by employing a $(3+1)$–decomposition of Clifford algebra–valued torsion and curvature two–forms. In the subcase of a mapping to purely axial 3D torsion, the corresponding Lagrangian consists of the translational and Lorentz Chern–Simons term plus cosmological term and is therefore of purely topological origin.

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1. Introduction

The formalism to be developed below describes a certain mapping between Yang–Mills configurations and a suitable Riemann–Cartan space, that is, a Riemannian space with a metric–compatible connection — and thus with a torsion. There is no unique way of constructing such a mapping, and a number of recent papers were devoted to the discussion of its definition and properties [1-10]. The case of the $SU(2)$ group was treated first [1-8], moreover, there are attempts to generalize these approaches to $SU(3)$ [9,10,7] (see also earlier models developed in [11-12]). The results obtained so far can be summarized as follows:

Yang–Mills field configurations on three– and four–dimensional manifolds generate an effective Riemann-Cartan (in certain models, Riemannian) geometry on a space (or spacetime), and vice versa, the Riemann-Cartan geometry induces Yang–Mills gauge fields. In order to understand the basic ideas involved and to formulate the main differences in the existing approaches, let us look at the Yang–Mills equations. These can formally be written without the use of any metric on an arbitrary smooth manifold $M$, see eqs.(2.2) below. The metric comes into play only when the constitutive relations are enforced between the Yang–Mills field strength $F$ and the excitation $H$ (generalizing the terminology of standard Maxwell theory), such as $*F = H$ in the case of a quadratic Yang–Mills Lagrangian. The Hodge operator $*$ is defined by some metric on $M$. In general, this metric is unrelated to the Yang–Mills field. A well known example is the Yang–Mills theory in Minkowski spacetime: The metric is fixed and does not feel dynamics of the Yang–Mills (or gauge) field. However, after the mapping the gauge field itself induces an “effective metric” on $M$ and thereby an Hodge operator on $M$, which we will denote by $\#$. In general, the $*$ and the $\#$ are different operators.

In solving the Yang–Mills equations one always needs some knowledge about the geometry of $M$. If an independent metric exists, then one writes $H = *F$ (with respect to this metric), and finally one has to solve the differential equations and to find a gauge field configuration on a prescribed background metric.

The case $* = \#$, which will be developed below, is somewhat different. One starts from a Yang–Mills field configuration $F$, formally constructs an “effective geometry” on $M$, and eventually it remains to find the geometrical variables in such a way that the vacuum Yang–Mills equation $D\#F = 0$ is satisfied automatically. This is what will be done below. One could call this approach “Yang–Mills equations without Yang–Mills equations”. More specifically, using the formalism of Clifford–valued forms, we will start from 3–dimensional (or 3D) Einstein(–Cartan) type equations with cosmological constant, (4.3). The geometry is described in terms of the frame and the torsion together with curvature two–forms. Via (4.4)–(4.5), these define the Yang–Mills gauge field. The Yang–Mills equations on this effective geometry are recovered from the Bianchi identities for curvature and torsion. In general, going in the opposite direction, from the Yang–Mills theory to the Riemann–Cartan geometry, one could have richer structures, including the case $* \neq \#$, the latter of which is discussed in [8], for instance.
2. The \((3 + 1)\)-decomposition of the Yang–Mills equations

In terms of \(SU(2)\) generators \(\tau_a := -(i/2)\sigma_a\), where \(a, b, \cdots = 1, 2, 3\) and the \(\sigma_a\) are the Pauli matrices, the \(SU(2)\) gauge connection \(A\) and the Yang–Mills field strength \(F\) read, respectively:

\[
A := A^i_a \tau_a dx^i, \quad F := dA + A \wedge A = \frac{1}{2} F^a_{ij} \tau_a dx^i \wedge dx^j. \tag{2.1}
\]

Here \(i, j, \cdots = 0, 1, 2, 3\) are (holonomic) coordinate indices of the underlying smooth manifold \(M\).

With \(\tilde{D} := d + A\) as the \(SU(2)\)-gauge covariant exterior derivative, the Yang–Mills equations are

\[
\tilde{D}F \equiv 0, \quad \tilde{D}H = J, \tag{2.2}
\]

where the excitation two–form \(H = H^a \tau_a\) and the external current three–form \(J = J^a \tau_a\) are \(su(2)\)-valued. We remind ourselves of the definition \(H = -2\partial L_{YM}/\partial F\) of the excitation in terms of the Lagrangian four–form \(L_{YM}\).

We assume that the four–dimensional manifold \(M\) admits a foliation into three–dimensional hypersurfaces \(\Sigma_t\), where \(t\) denotes a monotonically increasing parameter. By means of a vector field

\[
n := \partial_t - N^A \partial_A = N \vec{n}. \tag{2.3}
\]

(which will turn out to be time–like later on and) which is normal to a hypersurfaces \(\Sigma_t\), we can perform \((3 + 1)\)-decompositions of the Yang–Mills equations. Here \(A, B, \cdots = 1, 2, 3\) are spatial indices. Compared to other notations, it is rescaled by the lapse function \(N\), see Ref.[13] for details. With respect to this normal vector field \(n\), we define, for a \(p\)-form \(\Psi\) and arbitrary lapse \(N\) and shift \(N^A\), the normal part by

\[
\Psi := \Psi_\perp := n^i \Psi \quad \Psi_\perp := n^i \Psi, \tag{2.4}
\]

and the part tangential to the hypersurface \(\Sigma_t\) by

\[
\Psi := \Psi_\perp := (dt \wedge \Psi) = (1 - \Psi_\perp) \Psi, \quad n^i \Psi_\perp \equiv 0. \tag{2.5}
\]

These projection operators “\(\perp\)” and “\(\_\)” form a complete set, see the Appendix for further rules of calculation.

As in Maxwell’s theory, the Yang–Mills field strength \(F\) decomposes into a “magnetic” and an “electric” piece, respectively:

\[
B := B = dA + A \wedge A = B^a \tau_a, \quad E := -F_\perp = \tilde{D}A_\perp - \ell_n A = E^a \tau_a. \tag{2.6}
\]

With these abbreviations, the homogeneous Yang–Mills equation \(\tilde{D}F \equiv 0\), which is, in fact, a Bianchi type identity, decomposes as follows into constraint and propagation equation:

\[
\tilde{D}B = 0, \quad \tilde{D}E + \bar{\ell}_n B = 0. \tag{2.7}
\]
Analogously, the \((3 + 1)\)-decomposition of the inhomogeneous equation \(\tilde{D}H = J\) yields constraint and propagation equation:

\[
\tilde{D}D = J =: \rho, \quad \tilde{D}H - \tilde{L}_n D = -J_\perp =: j. \tag{2.8}
\]

Due to the specification of the charge density three–form \(\rho\) and the current two–form \(j\), we can identify

\[
D := H, \quad \text{and} \quad H := H_\perp = n \parallel H \tag{2.9}
\]

as the Yang–Mills analogues of the electric excitation two–form and the magnetic excitation one–form, respectively, cf. Hehl [14] for details in the Maxwellian \(U(1)\)–case. An overview of the eqs.(2.6)–(2.9), together with their number of independent components, is given in Table 1.

**Table 1.** \(SU(2)\) Yang–Mills field variables and equations. We display the number of independent components in \((3 + 1)\) dimensions before and after the decomposition in space and time.

| variables and field equations | components in \((3 + 1)\) dimensions | remarks |
|------------------------------|--------------------------------------|---------|
|                              | total | tangential | normal |
| potential \(A\)              | 12    | 9          | 3       | \(B \& E\) |
| field strength \(F\)         | 18    | 9          | 9       | constr. \& propag. |
| \(\tilde{D}F = 0\)           | 12    | 3          | 9       | \(D \& H\) |
| excitation \(H\)             | 18    | 9          | 9       | constr. \& propag. |
| \(\tilde{D}H = J\)           | 12    | 3          | 9       |         |

3. Clifford algebra–valued field variables in \(4D\) Riemann–Cartan geometry

The group \(SU(2)\) is isomorphic to (the covering group of) the rotation group \(SO(3)\). If one interprets the \(SO(3)\) as acting in ordinary three–dimensional Euclidean space, then the \((3 + 3)\) parameter Euclidean group \(SO(3) \subset R^3\) — with \(R^3\) as translations — represents the group of motion. The *gauging* of this external group yields a \(3D\) Riemann–Cartan geometry with curvature and torsion. Accordingly, this geometry emerges in a very natural way in the context of a gauge procedure. In order to link the \(3D\) RC–geometry to the four–dimensional manifold \(M\) discussed in Sec.1, it is plausible to start with a \(4D\) RC–geometry instead and to apply again a \((3 + 1)\) decomposition to the corresponding expressions so as to arrive at a \(3D\) RC–geometry in the end.

One can describe the \(4D\) RC–geometry by means of a very elegant formalism, namely by employing Clifford algebra–valued exterior differential forms, see Hehl et al.[15]. To this end we use the Dirac matrices \(\gamma_\alpha\) in the Bjorken and Drell convention [16] with signature \((+−−−)\). They obey the anticommutation relations

\[
\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\delta_{\alpha\beta} 1_4, \tag{3.1}
\]
where $\alpha, \beta, \cdots = \hat{0}, \hat{1}, \hat{2}, \hat{3}$ denote the (anholonomic) indices of the frame field $e_\alpha$ which is assumed to be orthonormal. The 16 matrices $\{1, \gamma_\alpha, \sigma_{\alpha\beta}, \gamma_5, \gamma_5 \gamma_\alpha\}$ form a basis of a Clifford algebra in four dimensions.

Following the notation of Hehl et al.\[15\], see Refs.\[17, 18\] for earlier approaches to “color geometrodynamics” and Ref.\[19\] for Clifford bundles, the constant $\gamma_\alpha$ matrices can be converted into Clifford algebra–valued one– or three–forms, respectively:

$$\gamma := \gamma_\alpha \vartheta^\alpha, \quad \gamma^* = \gamma^\alpha \eta_\alpha = \gamma_5 (\gamma \wedge \gamma \wedge \gamma) .$$

(3.2)

Here $\eta$ is the volume four–form and $\eta_\alpha := e_\alpha | \eta = \gamma^\alpha \eta$ is the coframe “density”. The totally antisymmetric product of Dirac matrices has now be identified with the zero–form $\gamma_5 := (i/4!) \gamma \wedge \gamma \wedge \gamma \wedge \gamma$. Another useful element of the Clifford algebra is the Lorentz generator

$$\sigma_{\alpha\beta} := \frac{i}{2} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha), \quad [\gamma_\alpha, \sigma_{\beta\gamma}] = 2i (\sigma_{\alpha\beta} \gamma_\gamma - \sigma_{\alpha\gamma} \gamma_\beta) .$$

(3.3)

and its Lie (or right) dual

$$\sigma^*_{\alpha\beta} := \sigma^{\gamma\delta} \frac{1}{2} \eta_{\alpha\beta\gamma\delta} = i \gamma_5 \sigma_{\alpha\beta} ,$$

(3.4)

where the components of the metric volume element four–form read $\eta_{\hat{0}\hat{1}\hat{2}\hat{3}} = +1$. The associated two-forms are given by

$$\sigma := \frac{1}{2} \sigma_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta = \frac{i}{2} \gamma \wedge \gamma, \quad \sigma^* = \frac{1}{2} \sigma_{\alpha\beta} \eta^{\alpha\beta} =: \sigma^* .$$

(3.5)

In terms of the Clifford algebra–valued connection and the $SO(1, 3) \cong SL(2, C)$–covariant exterior derivative

$$\Gamma := \frac{i}{4} \Gamma^{\alpha\beta} \sigma_{\alpha\beta} , \quad D = d + [\Gamma, ] ,$$

(3.6)

respectively, Clifford algebra–valued torsion and curvature two–forms can be defined as follows:

$$\Theta := D\gamma = T^\alpha \gamma_\alpha , \quad \Omega := d\Gamma + \Gamma \wedge \Gamma = \frac{i}{4} R^{\alpha\beta} \sigma_{\alpha\beta} .$$

(3.7)

The algebra–valued form commutator is given by $[\Psi, \Phi] := \Psi \wedge \Phi - (-1)^{pq} \Phi \wedge \Psi$, cf. Ref.\[18\], p.26. Then the two Bianchi identities

$$DT^\alpha \equiv R^\alpha_{\beta\gamma} \wedge \vartheta^\beta , \quad DR^{\alpha\beta} \equiv 0$$

(3.8)

of RC–spacetime take the rather simple form

$$D\Theta \equiv [\Omega, \gamma] , \quad D\Omega \equiv 0 .$$

(3.9)
These structures of a RC–space are collected in Table 2.

**Table 2.** Variables and Bianchi identities in Riemann–Cartan space. The number of independent components is displayed in (3 + 1) dimensions before and after the decomposition in space and time. The numbers in boldface count the independent components of those expressions which are involved in the mapping procedure to the tangential Yang–Mills expressions of Table 1.

| variables and Bianchi identities | components in (3 + 1) dimensions |
|----------------------------------|----------------------------------|
|                                  | total | tangential | normal |
| coframe $\gamma$                | 16    | 9+3        | 3+1    |
| connection $\Gamma$             | 24    | 9+9        | 3+3    |
| curvature $\Omega$              | 36    | 9+9        | 3+3    |
| 2nd Bianchi $D\Omega = 0$       | 24    | 3+3        | 9+9    |
| torsion $\Theta$                | 24    | 9+3        | 9+3    |
| 1st Bianchi $D\Theta = [\Omega, \gamma]$ | 16    | 3+1        | 9+3    |

For a possible mapping of the vacuum Yang–Mills equations to the Bianchi identities, see Tables 1 and 2, it is tempting to require the consistency condition

$$[\Omega, \gamma] = 0, \quad (3.10)$$

i.e. to admit only a Weyl, a symmetric tracefree Ricci, and a scalar piece in the curvature, cf.[20]. By differentiation we find

$$[\Omega, \Theta] = 0. \quad (3.11)$$

Because of the validity of the 2nd Bianchi identity $D\Omega \equiv 0$, we try the ansatz of constant curvature,

$$\Omega = \frac{\Lambda}{4} \gamma \wedge \gamma = -i\frac{\Lambda}{2} \sigma, \quad (3.12)$$

i.e., the curvature scalar is the only surviving piece of the curvature. Therefore (3.12) fulfills (3.10). Moreover, for $\Lambda \neq 0$, the 2nd Bianchi identity implies

$$[\Theta, \gamma] = 0, \quad (3.13)$$

which, in four dimensions, amounts to 24 independent conditions. Accordingly (3.13) yields vanishing torsion [21]. Thus we have to abandon, or rather to weaken, the degenerate four–dimensional ansatz (3.12).
4. Mapping Yang–Mills fields into 3D Riemann–Cartan geometry

As a weaker consistency condition, we require merely the tangential part of (3.10) to hold:

\[ [\Omega, \gamma] = 0. \] (4.1)

By differentiation we find, instead of (3.11),

\[ [\Omega, \Theta] = 0. \] (4.2)

This time we only postulate the tangential curvature to be constant as one of the key relations of our work:

\[ \Omega = \frac{\Lambda}{4} \gamma \wedge \gamma = -\frac{i\Lambda}{2} \sigma. \] (4.3)

It fulfills (4.1). Thus

\[ D \Theta = 0. \] (4.4)

For \( \Lambda \neq 0 \), as a consequence of the tangential part of the 2nd Bianchi identity, we have

\[ [\Theta, \gamma] = 0, \] (4.5)

Now we have specified the type of \((3+1)\)-dimensional RC–geometry which we want to consider, namely the one characterized by the tangential curvature (4.3) and a tangential torsion constrained by (4.5).

It is convenient to denote

\[ \omega := \frac{i}{4} \Gamma^{AB} \sigma_{AB}, \quad \varphi := \frac{i}{4} \Gamma^{\hat{0}A} \sigma_{\hat{0}A}. \] (4.6)

Then the curvature decomposes as follows:

\[ \Omega = (d\omega + \omega \wedge \omega) + (d\varphi + \omega \wedge \varphi + \varphi \wedge \omega). \] (4.7)

Without restricting the generality of our considerations, we can fix the coframe such that \( \vartheta^0 \) is aligned with the \( t \)–axis. In this time gauge for the coframe [22],

\[ \vartheta^0 = 0, \] (4.8)

the relation (4.3) splits into two sets of equations:

\[ d\omega + \omega \wedge \omega = -\frac{i\Lambda}{2} \sigma, \] (4.9)

\[ d\varphi + \omega \wedge \varphi + \varphi \wedge \omega = 0. \] (4.10)
We take the exterior derivative of (4.10) and substitute (4.9) and (4.10):

\[ [\varphi, \sigma] = 0. \tag{4.11} \]

Eq.(4.11) has only the trivial solution

\[ \varphi = 0. \tag{4.12} \]

Hence we conclude that, in the time gauge (4.8), the postulate (4.3) yields

\[ \Gamma = \omega. \tag{4.13} \]

In particular, for the torsion we find

\[ T^\hat{0} = D \vartheta^\hat{0} = d \vartheta^\hat{0} = 0 \quad \text{and} \quad D T^\hat{0} = d d \vartheta^\hat{0} = 0. \tag{4.14} \]

Eq.(4.9) – and thus (4.3) in the time gauge (4.7) – is equivalent to the 3D Einstein equations with cosmological term. Exactly such a 3D Riemann–Cartan space follows also from the (3 + 1)–decomposition of the modified double duality ansatz \( \Omega = \ast \Omega^* - i(\Lambda/2) \sigma \) in the gauge of purely “magnetic” RC curvature, see Ref.[23], Sec.6.

Let us compare our results with the tangential column in Table 2: Because of (4.8), for the coframe 3 independent components are cancelled and 9 are left over, likewise for the connection, see (4.6) and (4.12), 9 are killed and 9 survive. According to (4.10), in the time gauge, half of the curvature components vanish, that is, 9 are only allowed for. Note, however, that in the end (4.3) admits only the 1 component of the curvature scalar as a lone survivor. The torsion constraint (4.5), in the time gauge, kicks out the 3 components of the axial piece of the torsion. Finally, the 1st Bianchi identity is made to shrink by means of (4.1) and (4.14). Thus in Table 2 only the column with the boldface numbers are left over, whereas the timelike components in the tangential column are suppressed. This is a necessary condition that the mapping, see Table 1, can be implemented.

**Table 3.** Mapping 3D Riemann-Cartan geometry to \( SU(2) \) Yang-Mills field configurations.

| \( SU(2) \) YM in 4D | YM | 3D RC | RC–notions | comp. |
|-----------------------|-----|--------|------------|-------|
| potential \( A \)     | \( A \) | \( \Gamma \) | connection | 9     |
| field strength \( F \)| \( F = B \) | \( \Omega \) | curvature | 9     |
| \( D F = 0 \)         | \( D B = 0 \) | \( D \Omega = 0 \) | 2nd Bianchi | 3     |
| excitation \( H \)    | \( H = D \) | \( \Theta \) | torsion | 9     |
| \( D H = 0 \)         | \( D D = 0 \) | \( D \Theta = 0 \) | 1st Bianchi with constr. | 3     |

A comparison of Table 1 and 2 makes it obvious that we should identify the gauge potential with the three–dimensional RC–connection as follows:

\[ A \otimes 1_2 = \Gamma. \tag{4.15} \]
Under the conditions (4.3), we can now identically satisfy the tangential Yang–Mills equations (2.7a), (2.8a) by means of the tangential Bianchi identities of RC–space, provided we make the following identifications:

\[
B \otimes 1_2 = F \otimes 1_2 = \Omega = -\frac{i\Lambda}{2} \tau_a \otimes 1_2 , \quad (4.16)
\]

\[
D \otimes 1_2 = H \otimes 1_2 = i\gamma_5 \gamma_0 \Theta . \quad (4.17)
\]

Here we have used the relation (5.10) of the Appendix, which holds for tetrads in the time–gauge. (In order to recover \( H = H^a \tau_a \), the factor \( \gamma_5 \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is needed in order to shift the \( SU(2) \) generators \( \tau_a \) in \( \gamma_a \) to the main diagonal.)

Since \( H = \ast F \) in the standard Yang–Mills case with quadratic Lagrangian, its \((3+1)\)-decomposition automatically provides a geometrical relation for the normal parts of the Yang–Mills fields:

\[
E \otimes 1_2 = -N \ast H \otimes 1_2 = -iN\gamma_5 \gamma_0 \ast \Theta , \quad (4.18)
\]

\[
H \otimes 1_2 = N \ast F \otimes 1_2 = -N \Lambda \gamma^a \tau_a \otimes 1_2 . \quad (4.19)
\]

The identification (4.16) implies that the 3D coframe is given by

\[
\vartheta^a = -\frac{1}{\Lambda} \ast B^a \quad (4.20)
\]

In the time gauge, the three–metric can now be expressed via (4.20) as

\[
\eta = o_{ab} \vartheta^a \vartheta^b = -\delta_{ab} \frac{1}{\Lambda} \ast B^a \ast B^b = -\frac{1}{4\Lambda^2} tr ( \ast \Omega \ast \Omega ) . \quad (4.21)
\]

Observe, that the Hodge star \( \ast \) is implicitly depending on the Yang–Mills field strength \( B \). This is reflected, e.g., in the determinant of the 3D coframe, which is, by definition, the volume three–form

\[
\eta = \frac{1}{3\Lambda^2} \ast B^a \wedge B_a . \quad (4.22)
\]

The relations (4.3) and (4.20) have been found already in a component approach [1,5].

The torsion constraint (4.5) is solved by

\[
\Theta = \varepsilon^{(ab)} \eta, \quad (4.23)
\]

where symmetric tensor \( \varepsilon^{(ab)} \) satisfies, in view of (4.4), the differential equation

\[
\varepsilon_{a} \mid D\varepsilon^{(ab)} = 0 . \quad (4.24)
\]
5. Concluding remarks

In general, the 3D constraint (4.3) of constant RC curvature can be derived from a Lagrangian only by imposing it via Lagrange multipliers. However, for the subcase of purely axial torsion, which corresponds to the choice $t^{(ab)} = -(2/\ell) e^{ab}$ with a constant $\ell$, one can do better: In the generalization of the DJT gravity model with torsion of Ref.[24], there occurs the 3D Lagrangian

$$V_\infty = \frac{\Lambda}{4\ell} Tr(\gamma \wedge \zeta_\gamma) - \frac{\Lambda}{8} Tr(\gamma \wedge \Omega) - \frac{1}{2} Tr(\Gamma \wedge \Omega) - \frac{1}{3} Tr(\Gamma \wedge \Gamma \wedge \Gamma).$$ (5.1)

Since it consists out of a cosmological term plus the tangential part of the translational and Lorentz Chern–Simons term, it is of purely topological origin. Variation with respect to $\gamma$ and $\Gamma$ yields the field equations

$$\Theta = \frac{2}{\ell} \zeta_\gamma, \quad \Omega = -\frac{i}{2} \Lambda \sigma,$$ (5.2)

which is just (4.3) for purely axial torsion. Moreover, in that case we find from (4.16), (4.18) that not only the “magnetic” but also the “electric” component of the Yang–Mills field strength is proportional to the triad, i.e. $E = -(4N/\ell) \tilde{\gamma}^a \epsilon_a$. Such a mapping is considered, e.g., by Lunev [1].

We have found [24] a peculiar symmetry in the equations of these 3D models, which has the suggestive Clifford algebra correspondence

$$\gamma_5 \gamma \sim \ell \Gamma.$$ (5.3)

Then, the 3D metric can also be expressed via $g = (1/4) Tr(\gamma \gamma)$ in terms of the Yang–Mills connection $A = \Gamma$. This is actually related to the identifications proposed in Refs. [9,10,11].

Using a quaternionic representation, Dolan [25] has already earlier derived a similar mapping in the special case of self– or anti–selfdual Yang–Mills configurations. In particular, he could show that then the underlying geometry is even a 4D Einstein space (with Euclidean signature).

In our approach via Clifford algebra–valued forms, the mapping from Yang–Mills to 3D Einstein–Cartan space is rather transparent and simple geometrically. It is akin to Ashtekar’s Hamiltonian approach [26] to the Einstein’s general relativity, except for that here the traceless 3D torsion plays an essential role. In Refs. [27, 28], however, new canonical variables for the teleparallelism equivalent of general relativity are generated such that the complexified torsion, as the field strength of translations, carries the gravitational degrees of freedom in a Yang–Mills type fashion.
6. Appendix: The \((3 + 1)\)-decomposition of coframe, torsion, curvature, and Bianchi identities

In order to apply these \((3 + 1)\)-decompositions to field theory or geometry, the following rules with respect exterior multiplication and the forming of the Hodge dual are instrumental:

\[
\begin{align*}
\perp (\Psi \wedge \Phi) &= \perp \Psi \wedge \Phi + \Psi \wedge \perp \Phi \\
&= dt \wedge (\Psi_\perp \wedge \Phi_\perp + (-1)^p \Psi_\perp \wedge \Phi_\perp), \\
\Psi_\perp \wedge \Phi &= \Psi \wedge \Phi,
\end{align*}
\]

\[
\perp (\star \Psi) = (-1)^p N dt \wedge \perp \star \Psi, \\
\star \Psi = -\frac{1}{N} \perp \star \Psi.
\]

Here \(\perp\) denotes the Hodge dual in three dimensions which, in the Bjorken and Drell conventions, is an anti–involutive operator:

\[
\perp \star \Psi = -\Psi.
\]

The exterior derivative of a \(p\)-form decomposes as follows:

\[
\perp (d\Psi) = dt \wedge (\ell_n \Psi - d\Psi_\perp), \\
d\Psi = d\Psi.
\]

In the formulae above, the Lie derivative of \(p\)-forms along a vector field \(\xi\) is defined by

\[
\ell_\xi \Psi := \xi \mid d\Psi + d(\xi \mid \Psi).
\]

For Lie algebra–valued \(p\)-forms, the gauge–covariant Lie derivative

\[
L_\xi \Psi^{\alpha\cdots} := \xi \mid D\Psi^{\alpha\cdots} + D(\xi \mid \Psi^{\alpha\cdots}) = \ell_\xi \Psi^{\alpha\cdots} + (\xi \mid \Gamma^\beta{}_{\alpha}) \Psi^{\beta\cdots} + \cdots
\]

is more convenient.

The coframe decomposes as follows:

\[
\begin{align*}
\perp \vartheta^\alpha &= n^i e_i^\alpha dt \iff \vartheta_\perp^\alpha = n^\alpha, \\
\vartheta^\alpha &= e_i^\alpha dx^i - n^\alpha dt,
\end{align*}
\]

such that

\[
\vartheta^\alpha = \perp \vartheta^\alpha + \vartheta_\perp^\alpha.
\]

For the “tetrads in the time–gauge” \([22]\), the space–like one–form \(\vartheta_0\) is vanishing. Although, we do not assume this gauge in our subsequent decompositions, it is convenient in the \((3 + 1)\) decomposition of the Lorentz generator two–form \(\sigma\). We find

\[
\sigma_\perp = \frac{i}{2} [\gamma_\perp, \gamma],
\]

where the normal part \(\gamma_\perp = n^\alpha \gamma_\alpha\) generalizes \(\gamma_0\) for arbitrary lapse and shift, and

\[
\sigma = \eta_\perp (\sigma^e \otimes 1_2).
\]
Here we have used the relation $\sigma_{ab} = \eta_{0abc}(\sigma^c \otimes 1_2) = \eta_{abc}(\sigma^c \otimes 1_2)$ for the Pauli matrices $\sigma^a$.

For the connection we obtain

$$\Gamma_{\bot}^{\alpha\beta} = \Gamma_{\bot}^{\alpha\beta} dt = n^i \Gamma_{i}^{\alpha\beta} dt, \quad \Gamma_{\parallel}^{\alpha\beta} = \Gamma_{i}^{\alpha\beta} dx^i - \Gamma_{\parallel}^{\alpha\beta} dt. \quad (6.11)$$

By evaluating Cartan’s structure equation $T^\alpha := D\vartheta^\alpha$, we find the associated decompositions of the torsion:

$$T^\alpha_{\bot} = L_n \vartheta^\alpha - Dn^\alpha, \quad T^\alpha = D \vartheta^\alpha. \quad (6.12)$$

For the Riemann–Cartan curvature $R^{\alpha\beta} := d\Gamma^{\alpha\beta} - \Gamma^{\alpha\gamma} \wedge \Gamma_{\gamma}^{\beta}$ we obtain

$$R^\alpha_{\bot} = \ell_n \Gamma_{\bot}^{\alpha\beta} - D\Gamma_{\parallel}^{\alpha\beta}, \quad R^\alpha = d\Gamma^{\alpha\beta} - \Gamma^{\alpha\gamma} \wedge \Gamma_{\gamma}^{\beta}. \quad (6.13)$$

We also exhibit the $(3+1)$–decompositions of the Bianchi identities: For the first identity (3.8a.) we find

$$L_n T^\alpha - DT^\alpha_{\bot} \equiv R_{\beta}^{\alpha} \wedge \vartheta_{\bot}^{\beta} + R_{\beta}^{\alpha} n_{\beta} \quad (6.14)$$

and

$$DT^\alpha \equiv R_{\beta}^{\alpha} \wedge \vartheta_{\beta}. \quad (6.15)$$

The second Bianchi identity (3.8b) decomposes into

$$D_D(\Gamma_{\bot}^{\alpha\beta}) \equiv R_{\gamma}^{\alpha} \Gamma_{\bot}^{\gamma\beta} - R_{\gamma}^{\beta} \Gamma_{\bot}^{\gamma\alpha} \quad \iff \quad \ell_n R^{\alpha\beta} \equiv D(\ell_n \Gamma^{\alpha\beta}), \quad (6.16)$$

and

$$D R^{\alpha\beta} \equiv 0. \quad (6.17)$$

Using the operator

$$\gamma_{\alpha}^{\beta} := \delta_{\beta}^{\alpha} - \bar{n}_{\alpha} \bar{n}_{\beta} \quad (6.18)$$

the spatial components $A, B, C, \cdots = 1, 2, 3$ of the above expressions can be readily projected out.

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