Midisuperspace quantization: possibilities for fractional and emergent spacetime dimensions

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Abstract

Recently, motivated by certain loop quantum gravity (LQG) inspired corrections, it was shown that for spherically symmetric midisuperspace models infinitely many second derivative theories of gravity exist (as revealed by the presence of three arbitrary functions in the corresponding Lagrangian/Hamiltonian) and not just those allowed by spherically symmetric general relativity. This freedom can be interpreted as the freedom to accommodate certain quantum gravity corrections in these models even in the absence of higher curvature terms (at a semi-classical level, at least). For a particular choice of the arbitrary functions it is shown that the new theories map to spherically symmetric general relativity in arbitrary number of (integer) dimensions thus explicitly demonstrating that when working with midisuperspace models, one loses the information about the dimensionality of the full spacetime. In addition, it is shown that these new theories can accommodate scenarios of fractional spacetime dimensions as well as those of emergent spacetime dimensions - a possibility suggested by various approaches to quantum gravity.

1 Introduction

Symmetry reduced models of gravity, the minisuperspace [1,2] and the midisuperspace models [3,4] as they are often called, play a key role in the investigation of various approaches to quantum gravity. These models, because of the imposition of certain spacetime symmetries - homogeneity of space in the case of minisuperspace models and spherical symmetry for (one class of) midisuperspace models, result in the simplification of the Hamiltonian of general relativity. It is then hoped that the resulting simplified theory (or model) will be easier to quantize and will provide useful hints for the quantization of the full theory of general relativity.

However, the simplification of the theory for symmetry reduced models comes with a cost - one loses information about the dimensionality of the full spacetime, specifically the dimensions corresponding to symmetry directions. For instance, irrespective of the number of spacetime dimensions, the phase space of homogeneous and isotropic Friedmann-Robertson-Walker (FRW) cosmology is two dimensional, coordinatized by the scale factor and its conjugate momentum. Similarly, the phase space of a spherically symmetric spacetime is $4\infty$ dimensional (corresponding to two metric variables and their conjugate momenta) irrespective of the number of spherically symmetric directions.

This fact, though obvious when stated as above, is not always fully appreciated and this is so for a good reason. The symmetry reduced or midisuperspace models that one considers are obtained by imposing appropriate spacetime symmetries on Einstein’s theory (or its higher derivative generalizations like the Lovelock theory [5]). The corresponding Lagrangian (Hamiltonian) has a definite structure with the information about the dimensionality of the full spacetime present in the coefficients of the various terms in the Lagrangian (Hamiltonian) of the symmetry reduced theory. The information about the dimensionality of the

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spacetime can be easily retrieved from the solution of the corresponding equations of motion. For instance, the Schwarzschild solution in \( n + 2 \) dimensions \((n > 1 \text{ and } n \in \mathbb{Z})\) and where \( n \) corresponds to the number of spherically symmetric directions is of the form \( 1 - c/r^{n-1} \) and thus the form of the solution contains the information about the dimensionality of the full spacetime.

However, recently it was found that as far as spherically symmetric midisuperspace models are concerned, there is much more freedom in the structure of the theory and the symmetry reduced version of general relativity is just one among infinitely many possible second derivative theories for these spacetimes [6]. This infinite freedom is reflected in the presence of three (two) arbitrary functions of the metric coefficient \( g_{\phi \phi} \) (labeled \( E_r \) in the bulk of the paper) in the Hamiltonian of the theory in the presence (absence) of the cosmological constant. For specific functional forms of these functions, one can then obtain new spherically symmetric solutions not present in general relativity. Arguments of the previous paragraph then imply that, in general, it would not be possible to tell the dimensionality of the full spacetime to which these solutions correspond to (see section 2 and mainly section 3 below).

All this might seem inconsequential and insignificant as these new theories do not correspond to symmetry reduced classical general relativity. This would be so but for the quantum theory. As mentioned earlier, one of the main interests in symmetry reduced models is that they provide a simplified setting in which to understand various aspects of quantum gravity. And as with any quantum theory, quantum gravity will lead to quantum corrections of classical Einstein theory (including its symmetry reduced versions). These quantum corrections can appear in various forms - mass (and charge) renormalization in the case of Schwarzschild (Reissner-Nordström) black holes - but generically these will also show up in the form of field renormalization.

Now, in general, the form of quantum corrections is difficult to constrain and symmetry arguments are often required to obtain viable quantum theory. The underlying symmetry of general relativity is diffeomorphism covariance and it is expected that the smooth differential geometry structures of classical general relativity will give way to discrete structures in quantum gravity and the meaning of the usual notions of diffeomorphism invariance might not be directly applicable. In such a scenario a more algebraic notion of symmetry would be more useful. Such a notion is provided in the canonical formulation of general relativity where general covariance of the theory is encoded in the closure of the constraint algebra (of the Hamiltonian and diffeomorphism constraints) [7]. One expects that even when the classical notions of geometry are not applicable, the commutator of the operators corresponding to the classical Hamiltonian and diffeomorphism constraints should close just as in the classical theory.

In essence, precisely this criteria was used to obtain the new theories referred to above (although these theories, being classical, were obtained by working with the classical phase space variables and not the corresponding operators on the Hilbert space of the symmetry reduced theory). In other words, the criteria for terming the new theories as theories of gravity is that they are diffeomorphism invariant in the \( t - r \) plane as revealed by the fact that these theories have a Hamiltonian constraint and a diffeomorphism constraint and these constraints obey the standard constraint algebra of (symmetry reduced) general relativity. The arbitrary functions appearing in the Hamiltonian of the theory can be thought of as corresponding to the possibility of accommodating some of the quantum corrections to spherically symmetric general relativity (without requiring the incorporation of higher curvature/derivative terms).

In the semi-classical regime, where one expects the notion of the metric to be meaningful,
field renormalization would presumably show up in the form of a modified metric function (see [8, 9, 10, 11, 12] for an instance, where effects of certain LQG inspired corrections were considered for Schwarzschild and Reissner-Nordström geometry). And it is precisely here that the considerations of the previous paragraphs become important as these immediately imply that if, as in the classical theory, one continues to extract the information about the dimensionality of the full spacetime from the form of the metric even in the semi-classical regime then in the presence of non-trivial quantum corrections one would no longer be able to tell the dimensionality of the original spacetime one started with.

Not only this, depending on the exact form of the quantum corrections, it might turn out that in the quantum theory the dimensionality of the full spacetime is different from that of the classical spacetime being quantized. Furthermore, generically it will turn out that the dimensionality of spacetime is not even integral but is fractional. To take a concrete example, if a quantization of the Schwarzschild spacetime (in \( n + 2 \) ‘classical’ dimensions) leads to a metric coefficient with the leading order form \( 1 + cf(r)/r^{n-1} \) with \( f(r) \) corresponding to quantum correction then, in certain cases depending on the form of \( f(r) \), the spacetime may well be regarded as having fractional dimensions (see section 4 below).

This apparent drawback of the theory can, in fact, be turned into a virtue since such a possibility gels well with the scenario of fractional and/or emergent spacetime dimensions as is suggested by various theories of quantum gravity like the causal dynamical triangulation (CDT) [13], the more recent suggestion for fractal spacetime [14] and even by considerations of gauge-gravity duality [15]. We explicitly demonstrate this by constructing an example where the exponent \( n \) in \( r^n \) is not a constant but is a function of the scale (say, \( n = n(\ell_P/r) \), \( \ell_P \) being the Planck length), so that only in the classical limit \( r \gg \ell_P \) do we recover the classically observed dimensionality of spacetime.

Below we elaborate on the ideas suggested above working in \('2 + 1' \) dimensions - the lowest spacetime dimensions in which spherical symmetry ansatz makes sense (the quotes highlighting the fact, as will be demonstrated below, that for symmetry reduced models the meaning of dimensionality is non-trivial). In the next section we present the new Hamiltonian(s) (incorporating the three arbitrary functions mentioned earlier). The arbitrariness of these functions is used to construct new solution for supposedly \( 2 + 1 \) dimensional spacetime. This solution is used in section 3 to demonstrate that the concept of the dimensionality of full spacetime is not so easy to address in midisuperspace models by showing that the solution so obtained actually corresponds to some higher dimensional solution in general relativity. To remove any ambiguity regarding the point being made we further show that with appropriate choice for the arbitrary functions, the Hamiltonian of the (supposedly \( 2 + 1 \) dimensional) non-Einsteinian midisuperspace model can be mapped to the Einsteinian (general relativistic) midisuperspace model in \( n + 2 \) dimensions. In section 4 we suggest possible implications of this observation for the quantization of midisuperspace models with regard to the appearance of fractional and/or emergent spacetime dimensionality, as suggested by various approaches to quantum gravity, illustrating the same by a simple example. We conclude in section 5.

2 New second derivative spherically symmetric theories in \( 2+1 \) dimensions

In this section we present the new Hamiltonian(s) for spherically symmetric spacetimes as found in [6] (also see [17, 18]). We start by considering a spherically symmetric spacetime in
2 + 1 dimensions. We will be interested in the canonical formulation of the theory and for this purpose we consider the Arnowitt-Deser-Misner (ADM) metric
\[ ds^2 = -N^2 dt^2 + g_{rr}(dr + N^r dt)^2 + g_{\phi\phi} d\phi^2, \] (1)
where \( N \) and \( N^r \) are the lapse function and the shift vector respectively and \( (g_{rr}, g_{\phi\phi}) \) are the dynamical variables. However, for our purposes it would be more convenient to trade-off the metric coefficients \( g_{rr} \) and \( g_{\phi\phi} \) for two new variables \( E^\varphi = \sqrt{g_{rr}g_{\phi\phi}} \) and \( E^r = g_{\phi\phi} \) so that the metric takes the form
\[ ds^2 = -N^2 dt^2 + \frac{(E^\varphi)^2}{E^r}(dr + N^r dt)^2 + E^r d\phi^2. \] (2)
Because of the assumption of spherical symmetry, all these are functions of the coordinate time \( t \) and the radial coordinate \( r \) only (and for the same reason only the \( r \)-component of the shift vector \( N^i \) is non-zero).

The Hamiltonian and the diffeomorphism constraints corresponding to the metric (2) (obtained after performing Legendre transformation on the Einstein-Hilbert Lagrangian \( L = \int dx \sqrt{-g} R \)) are
\[ H[N] = \int dr N \left(-8G_3 K_\varphi K_r E^r - 4G_3 K^2_\varphi E^\varphi - \frac{E^\varphi}{8G_3(E^\varphi)^2} \right) \approx 0, \] \( D[N^r] = \int dr N^r (K_r E^{r'} - K^r E^\varphi) \approx 0. \) (3)

In the above expressions \( (K_\varphi, K_r) \) are the canonical momenta conjugate to \( (E^\varphi, E^r) \) respectively, and these obey the Poisson bracket relations \( \{E^\varphi(x), K_\varphi(y)\} = \delta(x,y) \) and \( \{E^r(x), K_r(x)\} = \delta(x,y) \). \( G_3 \) is Newton’s constant for \( 2 + 1 \) dimensions while \( \Lambda_3 \) is the cosmological constant (and for reasons that will become apparent in the next section we distinguish even the \( 2 + 1 \) dimensional cosmological constant by use of the subscript denoting spacetime dimensions).

A prime (′) in the above expressions denotes derivative with respect to \( r \).

These constraints satisfy the following Poisson bracket algebra
\[ \{H[N], H[M]\} = D[E^r (E^\varphi)^{-2}(NM^{-1} - N^{-1}M)], \] \( \{D[N^r], H[N]\} = H[N^r N^r], \) \( \{D[N^r], D[M^r]\} = D[N^r M^r - N^r M^r]. \) (4)
As is well known, the above algebra encodes the diffeomorphism invariance of general relativity in the canonical formulation (specifically, diffeomorphisms in the \( t - r \) plane for spherically symmetric spacetimes) [7].

Following [17] 6 [18] we can construct many more theories (actually infinite) which are quadratic in momenta and involve at most the second derivative of the metric variables and which satisfy the same constraint algebra as above. That is, these theories do not involve any higher derivative/curvature modifications of the Einstein-Hilbert Lagrangian and yet are not the symmetry reduced version of general relativity and at the same time satisfy the constraint algebra (5)-(7).

In brief, the basic idea for obtaining the new Hamiltonians is to first modify the Hamiltonian constraint (3) with \( E^r \) dependent functions \( \alpha_i(E^r) \) so that
\[ H[N] = \int dr N \left(-8G_3 \alpha_1 K_\varphi K_r E^r - 4G_3 \alpha_2 K^2_\varphi E^\varphi - \frac{\alpha_3 E^\varphi}{8G_3(E^\varphi)^2} \right) \approx 0. \] (8)
leaving the diffeomorphism constraint (4) unmodified. The next step is to evaluate the Poisson bracket

\[ \{ \hat{H}[N], \hat{H}[M] \} = \int dr \frac{E_r}{(E^r)^2} (N'M - NM') \left[ \alpha_1 (\alpha_4 K_\varphi E^\varphi - \alpha_3 K_r E^r_r) + \alpha_1 (\alpha_4 - \alpha_3) K_\varphi E^\varphi_r \right. \\
+ \left. \left( \frac{\alpha_1 \alpha_4 - \alpha_2 \alpha_3}{E^r} + \alpha_4 \frac{d\alpha_1}{dE^r} - \alpha_1 \frac{d\alpha_4}{dE^r} \right) K_\varphi E^\varphi E^r_r \right]. \tag{9} \]

For anomaly-free algebra we get the condition \( \alpha_3 = \alpha_4 \) and \( (\alpha_1 \alpha_4 - \alpha_2 \alpha_3) + E^r (\alpha_4 d\alpha_1/dE^r - \alpha_1 d\alpha_4/dE^r) = 0 \). Use of the first condition already implies that the \{ \hat{D}[N'], \hat{H}[N] \} bracket has the standard form (6). Also the first condition when used in the second condition allows, say, \( \alpha_2 \) to be expressed in terms of \( \alpha_1 \) and \( \alpha_3 \). That is, we are left with only three arbitrary functions \( (\alpha_1, \alpha_3, \alpha_5) \) instead of the five we started with. Use of these conditions further implies that the bracket

\[ \{ \hat{H}[N], \hat{H}[M] \} = D[\alpha_1 \alpha_3 E^r (E^\varphi)^{-2} (NM' - N'M)]. \tag{10} \]

The next step is to perform a canonical transformation taking \( (E^r, K_r) \rightarrow (\bar{E}^r, \bar{K}_r) \) using the generating function \( F_3 = -\alpha_1 \alpha_3 E^r \bar{K}_r \). Finally, one writes the Hamiltonian \( \hat{H}[N] \) and the diffeomorphism constraint (4) in terms of the new pair \( (\bar{K}_r, \bar{E}^r) \). It turns out that the diffeomorphism constraint retains its form even in terms of the new variables:

\[ D[N'] = \int dr N' (\bar{K}_r \bar{E}^r_r - K_\varphi^r E^\varphi_r) \approx 0. \tag{11} \]

It is also easy to check that the Hamiltonian (3) written in terms of the new variables

\[ \hat{H}[N] = \int dr N \left[ -8G_3 A_1 K_\varphi \bar{K}_r \bar{E}^r_r - 4G_3 A_2 K_\varphi^2 E^\varphi_r - \frac{E^\varphi_r \bar{E}^r_r}{8G_3 A_1 (E^r)^2} + \frac{\bar{E}^r_r}{8G_3 A_1 E^\varphi} \right. \\
\left. - \frac{A_3 (\bar{E}^r)^2}{4G_3 E^\varphi} - \frac{A_5 A_3 E^\varphi}{4G_3} \right] \approx 0, \tag{12} \]

along with the expression (11) for the diffeomorphism constraint satisfies the constraint algebra (54-57) (see [3] for a detailed discussion). In the above equation \( A_1, A_2 \) and \( A_5 \) are independent and arbitrary functions of \( \bar{E}^r \) (which have been traded for the original and equally arbitrary functions \( (\alpha_1, \alpha_3, \alpha_5) \)) and \( A_3 \) is determined in terms of \( A_1 \) and \( A_2 \) by the relation

\[ A_3 (\bar{E}^r) = \frac{1}{4A_1 E^r} + \frac{1}{2A_1^2} \frac{dA_1}{dE^r} - \frac{A_2}{4A_1^2 E^r}. \tag{13} \]

And it has to be remembered that now the ADM metric (2) is written with \( E^r \) replaced by \( \bar{E}^r \). Explicit demonstration that the resulting theory is diffeomorphism invariant can be found in [17] (although the fact that the constraint algebra closes in exactly the same way as for the symmetry reduced general relativity is a sufficient proof).

The presence of arbitrary functions \( (A_1(\bar{E}^r), A_2(\bar{E}^r), A_5(\bar{E}^r)) \) implies that the theory is non-Einsteinian (for which \( A_1 = A_2 = A_5 = 1 \) and \( A_3 = 0 \)). Actually the conclusion that the resulting models are not symmetry reduced versions of general relativity is slightly more subtle than this and following [17] one has to verify that the momenta conjugate to the metric variables in the two theories do not differ just by terms depending on the spatial geometry but also on how the spatial slice is embedded in the spacetime if the theories given by (12)
are to qualify as new theories inequivalent to symmetry reduced version of general relativity (a proof of this can be found in [6]).

Since for $A_1$, $A_2$ and $A_3$ not equal to one, the theory is not the symmetry reduced version of general relativity, one might ask as to what is the meaning of these models? We take the point of view that these new theories can be thought of as incorporating certain quantum gravity effects at the semi-classical level (in the form of arbitrary functions $A_i$ whose exact form will be given by the underlying quantum theory of gravity). Even when not incorporating all possible quantum gravity corrections to the symmetry reduced model, these functions will nevertheless be representative of certain class of quantum gravity corrections. LQG, for instance, provides a concrete example where such terms naturally arise from the so called inverse triad corrections [19, 20, 21].

As an aside we would like to note that the fact that for spherically reduced 2+1 dimensional general relativity $A_3 = 0$ means that the term involving $A_3$ (the $(\dot{E}^r)^2/E^r$ term in (12)) is completely new (compare with (3)). This is unlike what happens when one obtains new theories starting with the symmetry reduced version of 3+1 dimensional general relativity. In that case the basic terms in the new Hamiltonian continue to be the same as those in the Hamiltonian of the Einsteinian theory, with only the coefficients of these terms being arbitrary (similar to $A_i(\dot{E}^r)$ here) [6].

Presence of arbitrary functions $(A_1, A_2, A_3)$ along with the new term in the Hamiltonian suggests the possibility that using this freedom one could solve the constraints (11) and (12) along with the equations of motion $\dot{Q} = \{Q, \overline{H}[N] + D[N^r]\}$ (where $Q \equiv (\dot{E}^r, K_r, E^r, K_\varphi)$) to obtain asymptotically flat static black hole solutions even for 2+1 dimensional spacetimes - a possibility which does not exist in 2+1 dimensional general relativity.

Explicitly, the equations of motion are:

$$\dot{E}^r = -8G_3NA_1K_\varphi \dot{E}^r + N^r E^r,$$

$$\dot{E}^\varphi = -8G_3NA_1\dot{K}_r - 8G_3NA_2K_\varphi E^\varphi + N^r E^\varphi + N^r E^\varphi',$$

$$\dot{K}_\varphi = 4G_3NA_2K_\varphi^2 - \frac{N(E^r)'^2}{16G_3A_1E^r(E^\varphi)^2} + \frac{NA_2(E^r)'^2}{16G_3A_1^2E^r(E^\varphi)^2} - \frac{N^r E^r}{8G_3A_1(E^\varphi)^2} + N^r K_\varphi'$$

$$\dot{K}_r = 8G_3NA_1K_\varphi \dot{K}_r + 8G_3NK_\varphi \dot{K}_\varphi E^r \frac{dA_1}{dE^r} + 4G_3NK_\varphi E^\varphi \frac{dA_2}{dE^r} - \frac{N''}{8G_3A_1E^\varphi}$$

$$+ \frac{N'A_1\dot{E}^\varphi}{8G_3A_1(E^\varphi)^2} - \frac{N'A_2\dot{E}^\varphi}{8G_3A_1E^\varphi E^r} + \frac{N(E^r)'E^\varphi}{16G_3A_1E^r(E^\varphi)^2} - \frac{NA_2(E^r)'^2}{16G_3A_1^2E^r(E^\varphi)^2}$$

$$- \frac{NA_2E^\varphi}{8G_3A_1^2(E^\varphi)^2E^r} + \frac{N(E^r)'^2}{16G_3A_1^2E^r(E^\varphi)^2} \frac{dA_1}{dE^r} + \frac{N(E^r)'^2}{16G_3A_1^2E^rE^\varphi} \frac{dA_2}{dE^r} + \frac{NA_2(E^r)'^2}{8G_3A_1^2E^rE^\varphi} \frac{dA_1}{dE^r}$$

$$+ \frac{NA_3E^\varphi}{4G_3} \frac{dA_2}{dE^r} + N^r \ddot{K}_r + N^r \ddot{K}_\varphi'.$$

(17)
We now make the following choice for $A_1$, $A_2$ and $A_5$
\begin{align*}
A_1(E^r) &= \frac{1}{\Lambda_3 E^r}, \\
A_2(E^r) &= 0, \\
A_5(E^r) &= 1,
\end{align*}
which, from (13), implies $A_3(E^r) = -\Lambda_3/4$. With this choice the (static) solution of the equations of motion and the constraints (11), (12) (in the gauge $E^r = r^2$) is $K_\varphi = \dot{K}_r = N^r = 0$ and
\begin{align*}
N &= \left(1 - \frac{c}{r^2}\right)^{1/2}, \\
E^\varphi &= r \left(1 - \frac{c}{r^2}\right)^{-1/2}.
\end{align*}
The metric is therefore given by
\begin{equation}
\text{d}s^2 = - \left(1 - \frac{c}{r^2}\right) \text{d}t^2 + \left(1 - \frac{c}{r^2}\right)^{-1} \text{d}r^2 + r^2 \text{d}\Omega^2.
\end{equation}
Interestingly, any reference to the cosmological constant has disappeared in the metric above and we see that for $c > 0$ this metric corresponds to an asymptotically flat black hole spacetime with the horizon at $r = \sqrt{c}$. Thus it seems that, in contrast to classical general relativity in $2 + 1$ dimensions, the new theories of gravity for spherically symmetric spacetimes in $2 + 1$ dimensions have sufficient freedom to allow for asymptotically flat black hole solutions. In the next section we show that this is not the case and one has to be a bit careful in talking about the dimensionality of spacetime for symmetry reduced models.

3 Meaning of dimensionality of spacetime for symmetry reduced models?

Since we started with modifications to the classical (spherically symmetric) general relativity in $2 + 1$ dimensions, it appeared as if the modified theory (and the corresponding solution) is for $2 + 1$ dimensions. However, a look at the solution in (23) shows that the metric is that of the Schwarzschild black hole in $4 + 1$ dimensions (for a fixed value of the other two angular coordinates ($\xi, \theta$) that would be present in $4 + 1$ dimensions). This brings us to the question as to whether the theory presented above really corresponds to a $2 + 1$ dimensional theory? As should be obvious, the answer is no.

We recall that after imposing spherical symmetry on spacetime, we are effectively left with a theory only in the two dimensional $t - r$ plane (as far as spacetime dependence of the metric functions is concerned). To be more precise, the metric for spherically symmetric spacetimes has the ADM form
\begin{equation}
\text{d}s^2 = -N^2 \text{d}t^2 + \frac{(E^\varphi)^2}{E^r} (\text{d}r + N^r \text{d}t)^2 + E^r \text{d}\Omega_n^2
\end{equation}
and the only place where the information about the number of spacetime dimensions appears is in the $\text{d}\Omega_n^2$ part of the metric and this part is passive as far as the spherically reduced theory is concerned.
Although the above discussion should be enough to convince one that the information about the dimensionality of the full spacetime is lost when working with symmetry reduced models of gravity (especially in situations where the classical theory could be modified due to the presence of quantum gravity effects), this can also be demonstrated explicitly. Consider the Hamiltonian of spherically symmetric general relativity in \( n + 2 \) dimensions (where \( n \) is the number of angular dimensions and the 2 corresponds to one radial and one temporal dimension):

\[
H_{n+2} = \int \mathrm{d}r N \left[ -\frac{\tilde{G}_d K_\rho K_\rho}{n(E_r)^{(n-3)/2}} + \frac{(n-3)\tilde{G}_d K_\rho E^\rho}{4n(E_r)^{(n-1)/2}} - \frac{n(n-1)E^\phi(E_r)^{(n-3)/2}}{G_d} + \frac{nE_r^{\nu\nu}(E_r)^{(n-1)/2}}{G_d} \right].
\]

(25)

Here \( \tilde{G}_d = 4nG_d\Gamma((n+1)/2)/(n-1)\pi^{(n-1)/2} \), \( G_d \) being Newton’s constant in \( d = n + 2 \) dimensional spacetime. There is an apparent problem with this definition for \( n = 1 \) (\( d = 3 \)), and in that case we simply use the definition \( \tilde{G}_3 = 8G_3 \).

It turns out that the Hamiltonian in (12) has enough freedom in the form of functions \( A_1, A_2 \) and \( A_5 \) that it can always be mapped to (25). If we make the choice

\[
A_1 = \frac{\tilde{G}_d}{8nG_3(E_r)^{(n-1)/2}},
\]

(26)

\[
A_2 = \frac{(3 - n)\tilde{G}_d}{16nG_3(E_r)^{(n-1)/2}},
\]

(27)

\[
A_5 = \frac{4n(n-1)G_3(E_r)^{(n-3)/2}}{G_3 \Lambda_3} + \frac{8\Lambda_d G_3(E_r)^{(n-1)/2}}{G_d \Lambda_3},
\]

(28)

then equation (12) gets mapped to (25). Here we have distinguished \( \Lambda_3 \) from \( \Lambda_d \) to keep things explicit. With this choice for the arbitrary functions, if we solve the equations of motion (14)-(17) looking for static solutions, we will obtain the spacetime metric corresponding to the Schwarzschild solution in \( n + 2 \) dimensions. This explicitly demonstrates that from the perspective of the reduced theory one cannot say how many spacetime dimensions one is in.

As already mentioned, this is not very surprising since the phase space of spherically symmetric models is always \( 4\pi \) dimensional (in the absence of matter degrees of freedom) irrespective of the dimensionality of the embedding spacetime. Symmetry reduction of general relativity in different dimensions leads to Hamiltonians which only differ in the coefficients of various terms which depend on the dimensionality of the embedding spacetime (see (25)). For symmetry reduced classical general relativity this does not make much difference, since the information about the dimensionality of the full spacetime can be read from the form of the solutions of the corresponding Einstein equations.

What is new in the present case is that, as demonstrated by the existence of the Hamiltonian (12), the symmetry reduced versions of \( n + 2 \) dimensional general relativity in equation (25) are not the only allowed ones (without introducing higher derivative/curvature terms). Having amply demonstrated the fact that for symmetry reduced models the issue of the dimensionality of full spacetime is a non-trivial matter and taking the view that the new Hamiltonian (12) corresponds to a semi-classical description of certain quantum gravity effects, we next consider some of the possible implications of this observation.
4 Possible implication(s) of the new Hamiltonians

The choice (26)-(28) corresponds to one particular choice for \( (A_1, A_2, A_5) \). However, as mentioned earlier, these functions are completely arbitrary functions of \( \bar{E}^r \) and other choices for these functions will lead to corresponding solution metric such that, in general, the spacetime dimensionality inferred from them will be different from the dimensionality of the classical (and symmetry reduced) theory one started with (we repeat that, although presented as classical theories, the main point is that the actual form of functions \( A_i \) will be supplied by the quantum theory, see below).

To be more specific, for \( n \in \mathbb{Z} \) in equations (26)-(28), the form of the solution will be \((1 - c/r^{n-1})\). Its correspondence with the solution of the symmetry reduced general relativity then allows one to say what is the actual dimensionality of the spacetime corresponding to which the metric is a solution irrespective of the fact that one considered modifications to the Hamiltonian in \( 2 + 1 \) dimensions - the actual dimensionality of spacetime is \( d = n + 2 \) as illustrated by the example in section 2. However, for non-integral \( n \) the solution metric can have the form of the Schwarzschild solution \((1 - c/r^a)\) but with \( a \in \mathbb{R} \) which can therefore be interpreted as a black hole in a spacetime with fractional dimensions!

More generally, since we are taking the point of view that the \( A_i \)'s are given by the underlying quantum theory of gravity, the form of the functions \( A_i \) will not be that given in (26)-(28) (even with non-integral \( n \)) and will additionally depend on the Planck length \( \ell_P \). Furthermore, we expect the solution incorporating quantum corrections to go over to the corresponding classical solution in the limit when the Planck length \( \ell_P \to 0 \) and therefore the semi-classical solution will be expected to have the form \((1 - cf(r)/r^n)\) (where \( f(r) \) depends on the exact form of \( A_i \)'s, but will satisfy \( f(r) \to 1 \) when \( \ell_P \to 0 \)) instead of \((1 - c/r^a)\) which does not have the correct classical limit for a fixed \( a \).

On the other hand, a solution of the form \((1 - c/r^a)\) is still allowed by the new theories with the correct classical limit if the exponent of \( r \), instead of being a constant, is a function of \( r \), that is \( a = a(r) \) so that the dimensionality of spacetime becomes an emergent notion!\(^2\) (here we are working in the gauge \( \bar{E}^r = r^2 \); in general, the exponent will depend on \( \bar{E}^r \) instead of \( r \)). Since the exponent has to be dimensionless, the exact dependence on \( r \) (or \( \bar{E}^r \)) will come in the combination \( a(\ell_P/r) \) (or \( a(\ell_P^2/\bar{E}^r) \)). Below we give a simple illustration of how such a scenario can be realized for a suitable choice of the functions \( A_i \).

Consider the following choice for the arbitrary functions (we work in the gauge \( \bar{E}^r = r^2 \) and therefore write \( A_i \)'s as functions of \( r \) instead of \( \bar{E}^r \))

\[
A_1(r) = (\sqrt{\Lambda}r)^{-a(r)}, \quad a(r) = n - b\frac{\ell_P}{r}, \quad (n \in \mathbb{N}, b \in \mathbb{R}),
A_2(r) = (\sqrt{\Lambda}r)^{-a(r)} \left[ 1 - \frac{a(r)}{2} - \frac{ra'(r)}{2} \ln(\sqrt{\Lambda}r) \right],
A_5(r) = \frac{(\sqrt{\Lambda}r)^{a(r)} - 2}{2} \left[ a(r) + ra'(r) \ln(\sqrt{\Lambda}r) \right].
\]

\(^2\)Author thanks Martin Bojowald for suggesting this possibility.
For this choice, the solution of equations of motion (14)-(17) and the constraints (11), (12) is

\[ N = \left( 1 - c \frac{\Lambda b \ell_P / 2r}{a(r)} \right)^{1/2}, \]  

where \( a(r) = n - b \ell_P / r \) with \( n \in \mathbb{N} \), we see that in the classical limit where \( \ell_P / r \to 0 \), we recover the classical metric in \( n + 3 \) spacetime dimensions. However, away from the classical limit we find that the spacetime dimensionality will be different and, in general, will be fractional. Thus, we have here a very simple illustration of how the notion of emergent spacetime can arise even from a semi-classical perspective.

We would like to stress that we are not saying that in quantum gravity emergent spacetime arises precisely in the manner of the previous example. That example was only for illustration purpose and the specific choice made for \( a(r) \) was based on the general consideration that for \( r \gg \ell_P \) the spacetime dimensionality should be that of the classical theory which was being quantized. The choice \( a(r) = n - b \ell_P / r \) was one of the simplest possibilities to realize this expectation. As emphasized earlier, the exact form of \( a(r) \) will be given by the underlying quantum theory of gravity. What we are pointing out is that the concept of emergent spacetime, as is suggested by various approaches to quantum gravity like the causal dynamical triangulation [13] or fractal spacetime [14] (or even from the perspective of gauge-gravity duality [15]), can be easily accommodated even at a semi-classical level as indicated by the example above.

We would also like to add that since the main aim of the previous example was to illustrate the possibility of emergent spacetime dimensionality, we considered the simplest model of static spacetime. More realistic scenarios of emergent spacetime would most likely also have temporal evolution towards classicality and such a situation can naturally occur in the cosmological context. By incorporating suitable matter degrees of freedom in the new theories (which will lead to another \( E^r \)-dependent arbitrary function in the matter Hamiltonian [6]) one can obtain dynamical models which mimic cosmological evolution but with fractional (and emergent) spacetime dimensionality at early times and go over to the classical limit only at late times (for instance, the spherically symmetric classical Lemaitre–Tolman–Bondi models, where the matter is in the form of pressureless dust, are often used to model inhomogeneous cosmology).

It might seem that classical spacetime dimensionality (obtained from the asymptotic behavior of \( g_{tt} \), say) is recovered only in the \( r \to \infty \) limit, which appears unrealistic since we expect such strong quantum modifications as fractional or emergent spacetime dimensions to arise in deep quantum regime. However, as already mentioned, the specific example considered was chosen for its simplicity to bring out the key point that midisuperspace models can accommodate the scenario of emergent spacetime. In specific quantum theories, for large but finite \( r \), there may still be corrections but, in general, these would keep the spacetime
dimensionality close to the classical dimensions (the difference being of the order of the Planck length). Alternatively, one can view the model considered not so much as a model for black holes but rather as a model of possible space-time structures in which case the asymptotic form of the metric can be viewed as a simpler version of spectral or other dimensions used in the examples of CDT and other theories.

If one considers quantization of midisuperspace models then, demanding diffeomorphism invariance to be a good symmetry of the quantum theory one would want the quantized theory to satisfy the classical constraint algebra (5)-(7). In such a situation quantum corrections would (presumably) lead to a modification of the spacetime metric (in the semi-classical regime one expects the notion of metric to be well defined). As seen above, for a modified (semi-classical) theory, the most natural interpretation of spacetime dimensions will probably be an emergent one, including the possibility for fractional dimensions (unless the quantum corrections only renormalize the classical parameters like the mass and charge of the black hole, including Newton’s constant, but leave the overall structure of the metric unmodified - an unlikely scenario as suggested by several studies of the effects of LQG corrections on black hole metric [8, 9, 10, 11, 12]).

5 Conclusions

In this paper we highlighted that when working with symmetry reduced models of spacetime, the so called minisuperspace and midisuperspace models, one has to be careful when considering the dimensionality of spacetime in which these models are supposedly embedded. This has to be especially so when considering the quantization of these models. Our intuition regarding the dimensionality of embedding spacetime is based on the solutions in classical general relativity. As explicitly shown in this paper, this implies that we should allow the possibility of emergent spacetime in the quantum theory.

Our discussion relied on the solutions obtained in recently constructed (new) second derivative theories of gravity for spherically symmetric spacetime. If we make the reasonable assumption that the classical version of the constraint algebra continues to hold even in quantum gravity (this can be thought of as an algebraic notion of general covariance) then, at least at the semi-classical level, quantum gravity corrections will lead to modification of the Hamiltonian of the classical theory. The new symmetry reduced models mentioned above can be seen in this light. That is, they allow possible incorporation of quantum gravity effects (in the form of functions $A_i$) without requiring the addition of higher curvature/derivative terms.

Using these models we explicitly showed that since in the presence of spherical symmetry the phase space of general relativity is of dimensions $4\infty$ irrespective of the dimensionality of the embedding spacetime, we cannot make naive conclusions about spacetime dimensions when there are quantum corrections present. To emphasize this point we also demonstrated that with a suitable choice for the functions $A_i$, a supposedly $2+1$ dimensional model can be mapped to a classical (general relativistic) model in any number of spacetime dimensions.

As a more interesting consequence of these new models we further showed that they allow for the possibility of emergent and fractional spacetime. And although this was shown for spherically symmetric models only, we expect the conclusion to hold more generally. The possibility of emergent spacetime has been suggested by various approaches to quantum gravity like the CDT [13], the approach involving fractal spacetime of [14] and the causal set.
approach [16] as well as in gauge-gravity duality [15]. For illustration purpose we considered only the case of static spacetime and the emergent nature of spacetime was apparent only with respect to the spatial scale. However, in a more realistic situation, emergent behavior in time is also expected and this can be achieved by incorporating suitable matter degrees of freedom to mimic certain cosmological scenarios.

Acknowledgements

The author would like to thank Aninda Sinha for useful discussions and especially to Martin Bojowald for useful discussions and for his careful reading of the manuscript and his comments on the same.

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