ON $q$-EULER NUMBERS RELATED TO THE MODIFIED $q$-BERNSTEIN POLYNOMIALS

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ABSTRACT. We consider $q$-Euler numbers and polynomials and $q$-Stirling numbers of first and second kinds. Finally, we investigate some interesting properties of the modified $q$-Bernstein polynomials related to $q$-Euler numbers and $q$-Stirling numbers by using fermionic $p$-adic integrals on $\mathbb{Z}_p$.

1. Introduction

Let $C[0, 1]$ be the set of continuous functions on $[0, 1]$. The classical Bernstein polynomials of degree $n$ for $f \in C[0, 1]$ are defined by

$$B_n(f) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) B_{k,n}(x), \quad 0 \leq x \leq 1$$

where $B_n(f)$ is called the Bernstein operator and

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

are called the Bernstein basis polynomials (or the Bernstein polynomials of degree $n$) (see [17]). Recently, Acikgoz and Araci have studied the generating function for Bernstein polynomials (see [1, 2]). Their generating function for $B_{k,n}(x)$ is given by

$$F^{(k)}(t, x) = \frac{t^k e^{(1-x)t} x^k}{k!} = \sum_{n=0}^{\infty} B_{k,n}(x) \frac{t^n}{n!},$$

where $k = 0, 1, \ldots$ and $x \in [0, 1]$. Note that

$$B_{k,n}(x) = \begin{cases} \binom{n}{k} x^k (1-x)^{n-k} & \text{if } n \geq k \\ 0 & \text{if } n < k \end{cases}$$

for $n = 0, 1, \ldots$ (see [1, 2]).

Let $p$ be an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-1}$.

Throughout this paper, we use the following notation

$$[x]_q = \frac{1 - q^x}{1 - q} \quad \text{and} \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}$$

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Let $N$ be the natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$.

Let $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-1/(p-1)}$ and $x \in \mathbb{Z}_p$. Then $q$-Bernstein type operator for $f \in UD(\mathbb{Z}_p)$ is defined by (see [14, 15])

\[
B_{n,q}(f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} [x]_q^k [1 - x]_q^{n-k}
\]

for $k, n \in \mathbb{Z}_+$, where

\[
B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1 - x]_q^{n-k}
\]

is called the modified $q$-Bernstein polynomials of degree $n$. When we put $q \to 1$ in (1.5), $[x]_q^k \to x^k$, $[1 - x]_q^{n-k} \to (1 - x)^{n-k}$ and we obtain the classical Bernstein polynomial, defined by (1.2). We can deduce very easily from (1.5) that

\[
B_{k,n}(x, q) = [1 - x]_q B_{k,n-1}(x, q) + [x]_q B_{k-1,n-1}(x, q)
\]

(see [14]). For $0 \leq k \leq n$, derivatives of the $n$th degree modified $q$-Bernstein polynomials are polynomials of degree $n - 1$:

\[
\frac{d}{dx}B_{k,n}(x, q) = n(q^x B_{k-1,n-1}(x, q) - q^{1-x} B_{k,n-1}(x, q)) \frac{\ln q}{q - 1}
\]

(see [14]).

The Bernstein polynomials can also be defined in many different ways. Thus, recently, many applications of these polynomials have been looked for by many authors. In recent years, the $q$-Bernstein polynomials have been investigated and studied by many authors in many different ways (see [14, 15, 17] and references therein [4, 16]). In [16], Phillips gave many results concerning the $q$-integers, and an account of the properties of $q$-Bernstein polynomials. He gave many applications of these polynomials on approximation theory. In [1, 2], Acikgoz and Araci have introduced several type Bernstein polynomials. The Acikgoz and Araci paper to announce in the conference is actually motivated to write this paper. In [17], Simsek and Acikgoz constructed a new generating function of the $q$-Bernstein type polynomials and established elementary properties of this function. In [14], Kim, Jang and Yi proposed the modified $q$-Bernstein polynomials of degree $n$, which are different $q$-Bernstein polynomials of Phillips. In [15], Kim, Choi and Kim investigated some interesting properties of the modified $q$-Bernstein polynomials of degree $n$ related to $q$-Stirling numbers and Carlitz’s $q$-Bernoulli numbers.

In the present paper, we consider $q$-Euler numbers, polynomials and $q$-Stirling numbers of first and second kinds. We also investigate some interesting properties of the modified $q$-Bernstein polynomials of degree $n$ related to $q$-Euler numbers and $q$-Stirling numbers by using fermionic $p$-adic integrals on $\mathbb{Z}_p$. 
2. \(q\)-Euler numbers and polynomials related to the fermionic \(p\)-adic integrals on \(\mathbb{Z}_p\)

For \(N \geq 1\), the fermionic \(q\)-extension \(\mu_q\) of the \(p\)-adic Haar distribution \(\mu_{\text{Haar}}\):

\[
\mu_{-q}(a + p^N \mathbb{Z}_p) = \frac{(-q)^a}{[p^N]_{-q}}
\]

is known as a measure on \(\mathbb{Z}_p\), where \(a + p^N \mathbb{Z}_p = \{ x \in \mathbb{Q}_p \mid |x - a|_p \leq p^{-N} \}\) (cf. [5, 8]). We shall write \(d\mu_{-q}(x)\) to remind ourselves that \(x\) is the variable of integration. Let \(UD(\mathbb{Z}_p)\) be the space of uniformly differentiable function on \(\mathbb{Z}_p\).

Then \(\mu_{-q}\) yields the fermionic \(p\)-adic \(q\)-integral of a function \(f \in UD(\mathbb{Z}_p)\):

\[
I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1 + q}{1 + q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x
\]

(cf. [8, 9]). Many interesting properties of (2.2) were studied by many authors (see [8, 9] and the references given there). Using (2.2), we have the fermionic \(p\)-adic invariant integral on \(\mathbb{Z}_p\) as follows:

\[
\lim_{q \to -1} I_q(f) = I_{-1}(f) = \int_{\mathbb{Z}_p} f(a)d\mu_{-1}(x).
\]

For \(n \in \mathbb{N}\), write \(f_n(x) = f(x + n)\). We have

\[
I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2 \sum_{l=0}^{n-1} (-1)^{n-l-1} f(l).
\]

This identity is obtained by Kim in [8] to derives interesting properties and relationships involving \(q\)-Euler numbers and polynomials. For \(n \in \mathbb{Z}_+\), we note that

\[
I_{-1}([x]^n_q) = \int_{\mathbb{Z}_p} [x]^n_q d\mu_{-1}(x) = E_{n,q},
\]

where \(E_{n,q}\) are the \(q\)-Euler numbers (see [11]). It is easy to see that \(E_{0,q} = 1\). For \(n \in \mathbb{N}\), we have

\[
\sum_{l=0}^{n} \binom{n}{l} q^l E_{l,q} = \sum_{l=0}^{n} \binom{n}{l} q^l \lim_{N \to \infty} \sum_{x=0}^{p^N-1} [x]^l_q (-1)^x
\]

\[
= \lim_{N \to \infty} \sum_{x=0}^{p^N-1} (-1)^x (q[x]_q + 1)^n
\]

\[
= \lim_{N \to \infty} \sum_{x=0}^{p^N-1} (-1)^x [x + 1]_q^n
\]

\[
= -\lim_{N \to \infty} \sum_{x=0}^{p^N-1} (-1)^x ([x]_q^n + [p^N]_q^n)
\]

\[
= -E_{n,q}.
\]

From this formula, we have the following recurrence formula

\[
E_{0,q} = 1, \quad (qE + 1)^n + E_{n,q} = 0 \quad \text{if } n \in \mathbb{N}
\]
with the usual convention of replacing \( E^l \) by \( E_{l,q} \). By the simple calculation of the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \), we see that

\[
E_{n,q} = \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + q^l},
\]

where \( \binom{n}{l} = n!/l!(n-l)! = n(n-1) \cdots (n-l+1)/l! \). Now, by introducing the following equations:

\[
[x]^n_q = q^n q^{-nx}[x]^n_q \quad \text{and} \quad q^{-nx} = \sum_{m=0}^{\infty} (1-q)^m \binom{n+m-1}{m} [x]^m_q
\]

into (2.5), we find that

\[
E_{n,1} = q^n \sum_{m=0}^{\infty} (1-q)^m \binom{n+m-1}{m} E_{n+m,q}.
\]

This identity is a peculiarity of the \( p \)-adic \( q \)-Euler numbers, and the classical Euler numbers do not seem to have a similar relation. Let \( F_q(t) \) be the generating function of the \( q \)-Euler numbers. Then we obtain

\[
F_q(t) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}
\]

(2.11)

\[
= \sum_{n=0}^{\infty} \frac{2}{(1-q)^n} \sum_{l=0}^{n} (-1)^l \binom{n}{l} t^n \frac{1}{1 + q^l n!}
\]

\[
= 2e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(1-q)^k} \frac{1}{1 + q^k k!}.
\]

From (2.11) we note that

\[
F_q(t) = 2e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} (-1)^n e^{\frac{t^n}{1-q^n}} t = 2 \sum_{n=0}^{\infty} (-1)^n e^{[n]_q t}.
\]

(2.12)

It is well-known that

\[
I_{-1}([x+y]^n) = \int_{\mathbb{Z}_p} [x+y]^n d\mu_{-1}(y) = E_{n,q}(x),
\]

(2.13)

where \( E_{n,q}(x) \) are the \( q \)-Euler polynomials (see [11]). In the special case \( x = 0 \), the numbers \( E_{n,q}(0) = E_{n,q} \) are referred to as the \( q \)-Euler numbers. Thus we have

\[
\int_{\mathbb{Z}_p} [x+y]^n d\mu_{-1}(y) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \int_{\mathbb{Z}_p} [y]^k d\mu_{-1}(y)
\]

(2.14)

\[
= \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} E_{k,q}
\]

\[
= (q^x E + [x]_q)^n.
\]
It is easily verified, using (2.12) and (2.13), that the \( q \)-Euler polynomials \( \tilde{E}_{n,q}(x) \) satisfy the following formula:

\[
\sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{x^n}{n!} = \int_{\mathbb{Z}_p} e^{[x+y]t} d\mu_1(y)
\]

\[
= \sum_{n=0}^{\infty} \frac{2}{(1-q)^n} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{q^l x}{1 + q^l},
\]

\[
= 2 \sum_{n=0}^{\infty} (-1)^n [x^{n+1}]_q t^n.
\]

Using formula (2.15) when \( q \) tends to 1, we can readily derive the Euler polynomials, \( E_n(x) \), namely,

\[
\int_{\mathbb{Z}_p} e^{[x+y]t} d\mu_1(y) = 2 e^x = 2 \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}
\]

(see [8]). Note that \( E_n(0) = \tilde{E}_n(0) \) are referred to as the \( n \)-th Euler numbers. Comparing the coefficients of \( t^n/n! \) on both sides of (2.15), we have

\[
E_{n,q}(x) = 2 \sum_{m=0}^{\infty} \frac{2}{(1-q)^n} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{q^l x}{1 + q^l}.
\]

We refer to \([n]_q\) as a \( q \)-integer and note that \([n]_q\) is a continuous function of \( q \).

In an obvious way we also define a \( q \)-factorial,

\[
[n]_q! = \begin{cases} \prod_{k=0}^{n} [k]_q & n \in \mathbb{N}, \\ 1 & n = 0 \end{cases}
\]

and a \( q \)-analogue of binomial coefficient

\[
\binom{x}{n}_q = \frac{[x]_q!}{[n]_q! [x-n]_q!} = \frac{[x]_q [x-1]_q \cdots [x-n+1]_q}{[n]_q!}
\]

(cf. [10, 11]). Note that

\[
\lim_{q \to 1} \binom{x}{n}_q = \binom{x}{n} = \frac{x(x-1) \cdots (x-n+1)}{n!}
\]

It readily follows from (2.17) that

\[
\binom{x}{n}_q = \frac{(1-q)^n q^{-\binom{n}{2}}}{[n]_q!} \sum_{i=0}^{n} \binom{n}{i}_q (-1)^{n+i} q^{(n-i)x}
\]

(cf. [11, 12]). It can be readily seen that

\[
q^{lx} = ([x]_q (q-1) + 1)\frac{l}{m} = \sum_{m=0}^{l} \binom{l}{m} (q-1)^m [x]_q^m.
\]

Thus by (2.13) and (2.19), we have

\[
\int_{\mathbb{Z}_p} \binom{x}{n}_q \, d\mu_1(x) = \frac{(q-1)^n}{[n]_q!} \sum_{i=0}^{n} \binom{n}{i}_q (-1)^{n-i} \sum_{j=0}^{n-i} \binom{n-i}{j}_q (q-1)^j E_{j,q}.
\]
From now on, we use the following notation

\[
(2.21) \quad \frac{[x]_q^l}{[x-k]_q} = q^{-\binom{l}{2}} \sum_{l=0}^{k} \binom{k}{l}_q (x)_q^{l} (1-q)^l, \quad k \in \mathbb{Z}_+, \]

\[
(2.22) \quad [x]_q^n = \sum_{k=0}^{n} q_{2,q}(n,k) \frac{[x]_q^l}{[x-k]_q}, \quad n \in \mathbb{Z}_+ \]  

(see [12]). From (2.21), (2.22) and (2.19), we calculate the following consequence

\[
[x]_q^n = \sum_{k=0}^{n} q_{2,q}(n,k) \frac{[x]_q^l}{(1-q)^k} \sum_{l=0}^{k} \binom{k}{l}_q q_{1,q}^{l} (1-k)^l (1-q)^l \]

\[
\times \sum_{m=0}^{l} \binom{l}{m}_q (q-1)^m [x]_q^m \]

\[
= \sum_{k=0}^{n} q_{2,q}(n,k) \frac{1}{(1-q)^k} \sum_{l=0}^{k} \binom{k}{l}_q q_{1,q}^{l} (1-k)^l (1-q)^l \]

\[
\times \sum_{m=0}^{l} (q-1)^m \left( \sum_{i=m}^{l} \binom{i}{m}_q q_{1,q}^{l+1}(1-k) \left( \frac{l}{m} \right) (1-q)^l \right) [x]_q^m. \]

Therefore, we obtain the following theorem.

**Theorem 2.1.** For \( n \in \mathbb{Z}_+ \),

\[
E_{n,q} = \sum_{k=0}^{n} \sum_{m=0}^{n} \sum_{l=m}^{k} q_{2,q}(n,k) (q-1)^m-k \binom{k}{l}_q q_{1,q}^{l+1}(1-k) \left( \frac{l}{m} \right) (1-q)^l \]  

\[
\times \left( \sum_{i=m}^{l} \binom{i}{m}_q q_{1,q}^{l+1}(1-k) \right) E_{m,q}. \]

By (2.23) and simple calculation, we find that

\[
\sum_{m=0}^{n} \binom{n}{m}_q (q-1)^m E_{m,q} = \int_{\mathbb{Z}_p} q^{nx} d\mu_{-1}(x) \]

\[
= \sum_{k=0}^{n} (q-1)^k \binom{n}{k}_q \int_{\mathbb{Z}_p} \prod_{i=0}^{k-1} [x-i]_q d\mu_{-1}(x) \]

\[
= \sum_{k=0}^{n} (q-1)^k \binom{n}{k}_q \sum_{q=0}^{k} s_{1,q}(k,m) \int_{\mathbb{Z}_p} [x]^m d\mu_{-1}(x) \]

\[
= \sum_{m=0}^{n} \left( \sum_{k=m}^{n} (q-1)^k \binom{n}{k}_q s_{1,q}(k,m) \right) E_{m,q}. \]

Therefore, we deduce the following theorem.

**Theorem 2.2.** For \( n \in \mathbb{Z}_+ \),

\[
\sum_{m=0}^{n} \binom{n}{m}_q (q-1)^m E_{m,q} = \sum_{m=0}^{n} \sum_{k=m}^{n} (q-1)^k \binom{n}{k}_q s_{1,q}(k,m) E_{m,q}. \]
Corollary 2.3. For \( m, n \in \mathbb{Z}_+ \) with \( m \leq n \),
\[
\binom{n}{m} (q-1)^m = \sum_{k=m}^{n} (q-1)^k \binom{n}{k} s_{1,q}(k,m).
\]

By (2.16) and Corollary 2.3, we obtain the following corollary.

Corollary 2.4. For \( n \in \mathbb{Z}_+ \),
\[
E_{n,q}(x) = 2 \frac{(1-q)^n}{(1-q)^n \sum_{l=0}^{n} \sum_{k=l}^{n} (-1)^l (q-1)^{k-l} \binom{n}{k} s_{1,q}(k,l) \frac{q^{lx}}{1+q^{l}}}. 
\]

It is easy to see that
\[
(2.25) \quad \binom{n}{k} = \sum_{l_0+\cdots+l_k=n-k} q^{\sum_{i=0}^{k} i},
\]
(cf. [12]). From (2.25) and Corollary 2.4, we can also derive the following interesting formula for \( q \)-Euler polynomials.

Theorem 2.5. For \( n \in \mathbb{Z}_+ \),
\[
E_{n,q}(x) = 2 \frac{\sum_{l=0}^{n} \sum_{k=l}^{n} (-1)^l (q-1)^{k-l} \binom{n}{k} s_{1,q}(k,l) \frac{q^{lx}}{1+q^{l}}}{(1-q)^n \sum_{l=0}^{n} \sum_{k=l}^{n} (-1)^l (q-1)^{k-l} \binom{n}{k} s_{1,q}(k,l) \frac{q^{lx}}{1+q^{l}}}. 
\]

These polynomials are related to the many branches of Mathematics, for example, combinatorics, number theory, discrete probability distributions for finding higher-order moments (cf. [10, 11, 13]). By substituting \( x = 0 \) into the above, we have
\[
E_{n,q} = 2 \frac{\sum_{l=0}^{n} \sum_{k=l}^{n} (-1)^l (q-1)^{k-l} \binom{n}{k} s_{1,q}(k,l) \frac{q^{l}}{1+q^{l}}}{(1-q)^n \sum_{l=0}^{n} \sum_{k=l}^{n} (-1)^l (q-1)^{k-l} \binom{n}{k} s_{1,q}(k,l) \frac{q^{l}}{1+q^{l}}},
\]
where \( E_{n,q} \) is the \( q \)-Euler numbers.

3. \( q \)-Euler numbers, \( q \)-Stirling numbers and \( q \)-Bernstein polynomials related to the fermionic \( p \)-adic integrals on \( \mathbb{Z}_p \).

First, we consider the \( q \)-extension of the generating function of Bernstein polynomials in [12].

For \( q \in \mathbb{C}_p \) with \( |1-q|_p < p^{-1/(p-1)} \), we obtain
\[
F_q^{(k)}(t, x) = \frac{t^{k} \left[ 1-x \right]_q [x]_q^k}{k!} 
\]
\[
= \left[ x \right]_q^{k} \sum_{n=0}^{\infty} \left( \frac{1}{k} \right) \left( \frac{n}{x} \right)_q^k \frac{t^{n+k}}{(n+k)!} 
\]
\[
= \sum_{n=k}^{\infty} \left( \frac{n}{k} \right) \left[ x \right]_q^k \frac{1-x^{n-k} t^{n}}{n!} 
\]
\[
\sum_{n=0}^{\infty} B_{k,n}(x, q) \frac{t^{n}}{n!}.
\]
which is the generating function of the modified $q$-Bernstein type polynomials (see [15]). Indeed, this generating function is also treated by Simsek and Acikgoz (see [17]). Note that $\lim_{q \to 1} F_q^{(k)}(t, x) = F^{(k)}(t, x)$. It is easy to show that

\begin{equation}
(3.2) \quad [1 - x]^n q^{-k} = \sum_{m=0}^{\infty} \sum_{l=0}^{n-k} \binom{l + m - 1}{m} \binom{n - k}{l} (-1)^l + m q^l [x]^m q^{-1} (q - 1)^m.
\end{equation}

From (1.4), (2.3), (2.15) and (3.2), we derive the following theorem.

**Theorem 3.1.** For $k, n \in \mathbb{Z}_+$ with $n \geq k$,

\[\int_{\mathbb{Z}_0^+} B_{k,n}(x,q) \left( \frac{t}{k} \right)_n d\mu_{-1}(y) = \sum_{m=0}^{\infty} \sum_{l=0}^{n-k} \binom{l + m - 1}{m} \binom{n - k}{l} (-1)^l + m q^l (q - 1)^m E_{l+m+k,q},\]

where $E_{n,q}$ are the $q$-Euler numbers.

It is possible to write $[x]^n q^{-k}$ as a linear combination of the modified $q$-Bernstein polynomials by using the degree evaluation formulae and mathematical induction. Therefore we obtain the following theorem.

**Theorem 3.2** (14 Theorem 7)). For $k, n \in \mathbb{Z}_+, i \in \mathbb{N}$ and $x \in [0,1]$,

\[\sum_{k=1-i}^{n} \binom{k}{i} B_{k,n}(x,q) = [x]^n_q ([x]_q + [1 - x]_q)^{n-i}.\]

Let $i - 1 \leq n$. Then from (15), (3.2) and Theorem 3.2 we have

\begin{equation}
(3.3) \quad [x]_q^l = \frac{\sum_{k=1-i}^{n} \binom{k}{i} \binom{l}{k} [x]^k q [1 - x]_q^{n-k}}{[x]_q^{n-i} \left( 1 + \frac{1 - x}{[x]_q} \right)^{n-k}}
\end{equation}

\[= \sum_{m=0}^{\infty} \sum_{k=1-i}^{n} \sum_{l=0}^{m+n-k} \sum_{p=0}^{\infty} \binom{k}{l} \binom{n}{l} \binom{l + p - 1}{p} \binom{m + n - k}{l} \binom{n - i + m - 1}{m} (-1)^{l+p+m} q^l (q - 1)^p [x]_q^{i-n-m+k+p+l}.
\]

Using (2.13) and (3.3), we obtain the following theorem.

**Theorem 3.3.** For $k, n \in \mathbb{Z}_+$ and $i \in \mathbb{N}$ with $i - 1 \leq n$,

\[E_{i,q} = \sum_{m=0}^{\infty} \sum_{l=0}^{n} \sum_{p=0}^{m+n-k} \frac{n}{k} \binom{k}{l} \binom{n}{l} \binom{l + p - 1}{p} \binom{m + n - k}{l} \binom{n - i + m - 1}{m} (-1)^{l+p+m} q^l (q - 1)^p E_{i-n-m+k+p+l,q}.
\]

The $q$-String numbers of the first kind is defined by

\begin{equation}
(3.4) \quad \prod_{k=1}^{n} \left( 1 + [k]_q z \right) = \sum_{k=0}^{n} S_1(n, k; q) z^k,
\end{equation}
and the $q$-String number of the second kind is also defined by

\[
\prod_{k=1}^{n}(1 + [k]_q z)^{-1} = \sum_{k=0}^{n} S_2(n, k; q) z^k
\]

(see [15]). Therefore, we deduce the following theorem.

**Theorem 3.4** ([15, Theorem 4]). For $k, n \in \mathbb{Z}_+$ and $i \in \mathbb{N}$,

\[
\sum_{k=i-1}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q) = \sum_{k=0}^{i} \sum_{l=0}^{k} S_1(n, l; q) S_2(i, k; q) [x]^l.
\]

By Theorem [3.2], Theorem [3.4] and the definition of fermionic $p$-adic integrals on $\mathbb{Z}_p$, we obtain the following theorem.

**Theorem 3.5.** For $k, n \in \mathbb{Z}_+$ and $i \in \mathbb{N}$,

\[
E_{i,q} = \sum_{k=i-1}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} \int_{\mathbb{Z}_p} \frac{B_{k,n}(x, q)}{[x]_q + [1 - x]_q} d\mu_{-1}(x)
\]

\[
= \sum_{k=0}^{i} \sum_{l=0}^{k} S_1(n, l; q) S_2(i, k; q) E_{l,q},
\]

where $E_{i,q}$ is the $q$-Euler numbers.

Let $i - 1 \leq n$. It is easy to show that

\[
[x]_q^i ([x]_q + [1 - x]_q)^{n-i}
\]

\[
= \sum_{l=0}^{n-i} \binom{n-i}{l} [x]^{l+i} [1 - x]_q^{n-i-l}
\]

\[
= \sum_{l=0}^{n-i} \sum_{m=0}^{n-i-l} \binom{n-i}{l} \binom{n-i-l}{m} (-1)^m q^m [x]_q^{m+l} (1 - q)^{-s} [x]_q^{m+i+l,s}.
\]

From (3.6) and Theorem [3.2] we have the following theorem.

**Theorem 3.6.** For $k, n \in \mathbb{Z}_+$ and $i \in \mathbb{N}$,

\[
\sum_{k=i-1}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} \int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_{-1}(x) = \sum_{l=0}^{n-i} \sum_{m=0}^{n-i-l} \sum_{s=0}^{\infty} \binom{n-i}{l} \binom{n-i-l}{m} (m + s - 1) \times (-1)^m q^m (1 - q)^s E_{m+i+l+s,q},
\]

where $E_{i,q}$ are the $q$-Euler numbers.

In the same manner, we can obtain the following theorem.
Theorem 3.7. For \( k, n \in \mathbb{Z}_+ \) and \( i \in \mathbb{N} \),
\[
\int_{\mathbb{Z}_p} B_{k,n}(x,q) d\mu_{-1}(x) = \sum_{j=k}^{n} \sum_{m=0}^{\infty} \binom{n}{j} \binom{j-1}{m} (-1)^{j} a^i \cdot \frac{q^{jx}}{(-q^{h-r+1}; q^{-1})_r},
\]
where \( E_{i,q} \) are the \( q \)-Euler numbers.

4. FURTHER REMARKS AND OBSERVATIONS

The \( q \)-binomial formulas are known,
\[
(a; q)_n = (1 - a) (1 - a q) \cdots (1 - a q^{n-1}) = \sum_{i=0}^{n} \binom{n}{i}_q q^i (-1)^i a^i,
\]
(4.1)
\[
\frac{1}{(a; q)_n} = \frac{1}{(1 - a) (1 - a q) \cdots (1 - a q^{n-1})} = \sum_{i=0}^{\infty} \binom{n+i-1}{i}_q a^i.
\]

For \( h \in \mathbb{Z}, n \in \mathbb{Z}_+ \) and \( r \in \mathbb{N} \), we introduce the extended higher-order \( q \)-Euler polynomials as follows 11:
\[
E_{n,q}^{(h,r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{i=1}^{r} q^{\sum_{j=1}^{r}(h-j)x} [x + x_1 + \cdots + x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).
\]

Then
\[
E_{n,q}^{(h,r)}(x) = \frac{2^r}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{l}_q (-1)^l \frac{q^{lx}}{(-q^{h-r+1}; q^{-1})_r},
\]
(4.3)
\[
= \frac{2^r}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{l}_q (-1)^l \frac{q^{lx}}{(-q^{h-r+1}; q^{-1})_r}.
\]

Let us now define the extended higher-order Nörlund type \( q \)-Euler polynomials as follows 11:
\[
E_{n,q}^{(h,-r)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{l}_q (-1)^l
\]
(4.4)
\[
\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{r}(h-j)x} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).
\]

In the special case \( x = 0 \), \( E_{n,q}^{(h,-r)} = E_{n,q}^{(h,-r)}(0) \) are called the extended higher-order Nörlund type \( q \)-Euler numbers. From 11, we note that
\[
E_{n,q}^{(h,-r)}(x) = \frac{1}{2^r (1-q)^n} \sum_{i=0}^{n} \binom{n}{l}_q (-1)^l q^{lx} (-q^{h-r+1}; q)_r
\]
(4.5)
\[
= \frac{1}{2^r} \sum_{m=0}^{r} q^{(m)}_q (h-r)_m \binom{r}{m}_q [m + x]^n.
\]

A simple manipulation shows that
\[
q^{(m)}_q \binom{r}{m}_q = \frac{q^{(n)}_q [r]_q \cdots [r-m+1]_q}{[m]_q!} = \frac{1}{[m]_q!} \prod_{k=0}^{m-1} ([r]_q - [k]_q)
\]
(4.6)
Lemma 4.1. For \( h \in \mathbb{Z}, n \in \mathbb{Z}_+ \) and \( r \in \mathbb{N} \),

\[
E_{n,q}^{(h,-r)}(x) = \frac{1}{2^r[m]_q!} \sum_{m=0}^{n} q^{(h-r)m} S_1(m - 1, k; q)(-1)^k [r]_q^{m-k} [x + m]_q^n.
\]

From (2.19), we can easily see that

\[
[x + m]_q^n = \frac{1}{(1 - q)^n} \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} \frac{1}{(1 - q)^j} (-1)^{j+l} (1 - q)^l q^{mj} [x]_q^l.
\]

Using (2.13) and (4.8), we obtain the following lemma.

Lemma 4.2. For \( m, n \in \mathbb{Z}_+ \),

\[
E_{n,q}(m) = \frac{1}{(1 - q)^n} \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} \frac{1}{(1 - q)^j} (-1)^{j+l} (1 - q)^l q^{mj} E_{l,q}.
\]

By Lemma 4.1, Lemma 4.2, and the definition of fermionic \( p \)-adic integrals on \( \mathbb{Z}_p \), we obtain the following theorem.

Theorem 4.3. For \( h \in \mathbb{Z}, n \in \mathbb{Z}_+ \) and \( r \in \mathbb{N} \),

\[
\int_{\mathbb{Z}_p} E_{n,q}^{(h,-r)}(x) d\mu_1(x) = \frac{1}{2^r[m]_q!} \sum_{m=0}^{n} q^{(h-r)m} S_1(m - 1, k; q)(-1)^k [r]_q^{m-k} E_{n,q}(m)
\]

\[
= \frac{1}{2^r[m]_q!} \sum_{m=0}^{n} q^{(h-r)m} S_1(m - 1, k; q)(-1)^k [r]_q^{m-k}
\]

\[
\times \frac{1}{(1 - q)^n} \sum_{j=0}^{n} \sum_{l=0}^{j} \binom{n}{j} \frac{1}{(1 - q)^j} (-1)^{j+l} (1 - q)^l q^{mj} E_{l,q}.
\]

Put \( h = 0 \) in (4.4). We consider the following polynomials \( E_{n,q}^{(0,-r)}(x) \):

\[
E_{n,q}^{(0,-r)}(x) = \sum_{l=0}^{n} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q(x_1 + \cdots + x_r) q^{-\sum_{j=1}^{r} j x_j} d\mu_{r-1}(x_1) \cdots d\mu_1(x_r).
\]

Then

\[
E_{n,q}^{(0,-r)}(x) = \frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} q^{-r m} [m + x]_q^n.
\]

A simple calculation of the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) shows that

\[
\int_{\mathbb{Z}_p} E_{n,q}^{(0,-r)}(x) d\mu_1(x) = \frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} q^{-r m} E_{n,q}(m).
\]

Using Theorem 4.3, we can also prove that

\[
\int_{\mathbb{Z}_p} E_{n,q}^{(0,-r)}(x) d\mu_1(x) = \frac{1}{2^r[m]_q!} \sum_{m=0}^{n} q^{-r m} S_1(m - 1, k; q)(-1)^k [r]_q^{m-k} E_{n,q}(m).
\]
Therefore, we obtain the following theorem.

**Theorem 4.4.** For $m \in \mathbb{Z}_+$, $r \in \mathbb{N}$ with $m \leq r$,

$$
\binom{r}{m}q^{\binom{m}{2}} - rm = 1 + \sum_{k=0}^{m} q^{-rm} S_1(m-1,k;q)(-1)^k[r]_q^{m-k}.
$$

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