Ultracold Bosons with 3-Body Attractive Interactions in an Optical Lattice

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We study the effect of an optical lattice (OL) on the ground-state properties of one-dimensional ultracold bosons with three-body attraction and two-body repulsion, which are described by a cubic-quintic Gross-Pitaevskii equation with a periodic potential. Without the OL and with a vanishing two-body interaction term, soliton solutions of theTownes type are possible only at a critical value of the three-body interaction strength, at which an infinite degeneracy of the ground-state occurs; a repulsive two-body interaction makes such localized solutions unstable. We show that the OL opens a stability window around the critical point when the strength of the periodic potential is above a critical threshold. We also consider the effect of an external parabolic trap, studying how the stability of the solitons depends on matching between minima of the periodic potential and the minimum of the parabolic trap.

I. INTRODUCTION

In the current studies of ultracold quantum gases, a great deal of interest has been drawn to the study of Bose-Einstein condensates (BECs) loaded into optical lattices (OLs), i.e., spatially periodic potentials induced by the interference between counterpropagating laser beams \[1,2,3\]. Besides playing a crucial role in effectively tuning the interaction strength in the condensate, i.e., the ratio between the kinetic and interaction energies \[4\], OLs offer an extremely useful tool for studies of the transition between the superfluid and Mott-insulator states \[5\], and for the investigation of effects in the matter-wave dynamics due to the interplay between nonlinearity and the quasi-discreteness, which is induced by a deep lattice potential \[6\].

The mean-field dynamics of the BEC loaded into the OL is described by the cubic Gross-Pitaevskii equation (GPE) with the periodic potential \[1,2,3\]. The respective Bogoliubov’s excitation spectrum features a band structure, similar to electronic Bloch bands in solid state. If the OL potential is deep enough, the lowest-band dynamics may be approximated by the discrete nonlinear Schrödinger (NLS) equation \[6\]. Using this correspondence, the BEC dynamics was studied in the framework of the nonlinear-lattice theory, see works \[6,7,8\] and short review \[9\]. The presence of the OL gives raise to energetic and dynamical instabilities, which have been predicted theoretically \[10,11,12,13,14,15,16,17\] and studied experimentally \[18,19\].

An important application of the OLs is their use for the creation and stabilization of matter-wave solitons. In particular, the periodic potential gives rise to localized gap solitons in the case of repulsive two-body interactions, as was predicted theoretically \[20\] and demonstrated experimentally \[21\] (with the attractive interactions, bright matter-wave solitons were created and observed in condensates of \(^7\)Li \[22,23\] and \(^85\)Rb \[24\] atoms). More generally, the use of time- and space-modulated fields acting on atoms is a powerful tool for the control of soliton properties \[25\]; for instance, when the GPE without external potentials admits stable soliton solutions only in the 1D geometry \[26,27\], OL potentials can stabilize solitons in any higher dimension \[28,29,30,31\]. Unlike 1D solitons, a necessary existence condition for their multidimensional counterparts, stabilized by means of OLs, is that the soliton’s norm must exceed a certain threshold value.

Another very useful tool frequently used in experiments with ultracold atomic gases is the control of the strength and sign of two-body interactions by means of an external magnetic field near the Feshbach resonance \[1,2\]. Further, recent works proposed to exploit the possibility to control the strength of three-body interactions between atoms, independently from the control of the two-body collisions \[32,33\]. One motivation for such studies is related to the possibility of creating new exotic strongly correlated phases in ultracold gases. Indeed, quantum phases, such as topological ones or spin liquids, turn out to be ground-states of the Hamiltonian including three- or multi-body-interaction terms, an example being fractional quantum-Hall states described by Pfaffian wave functions \[34\]. In a recent work \[35\], a 1D Bose gas with \(\mathcal{N}\)-body attractive interactions was studied in the mean-field approximation, with the objective to create highly degenerate ground-states of Hamiltonians including many-body terms. For the three-body interactions \(\mathcal{N} = 3\), the system is described by a quintic GPE, i.e., the respective term in the energy density is proportional to \(|\psi|^6\), where \(\psi\) is the single-atom mean-field wave function (in the general case, a similar term is proportional to \(|\psi|^{2\mathcal{N}}\)).

Soliton solutions can be found for each \(\mathcal{N}\), but they represent the stable ground-state, with negative energy (which
The quantum many-body Hamiltonian for the 1D Bose gas with $N$-body contact attractive interactions is

$$\hat{H} = \int_{-\infty}^{+\infty} dx \left\{ \hat{\Psi}^\dagger(x) \hat{h}_0 \hat{\Psi}(x) - \frac{c}{N!} \left[ \hat{\Psi}^\dagger(x) \right]^N \left[ \hat{\Psi}(x) \right]^N \right\},$$

(1)

where $\hat{\Psi}(x)$ is the bosonic-field operator, $c > 0$ is the nonlinearity strength and

$$\hat{h}_0 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_{\text{ext}}(x)$$

(2)

is the single-particle Hamiltonian, $V_{\text{ext}}(x)$ being the external potential. The case of $N = 2$ in the homogeneous limit ($V_{\text{ext}} = 0$) corresponds to the integrable Lieb-Liniger model [40]. For attractive interactions ($c > 0$), its analytical
solution was obtained by means of the Bethe ansatz \[47\] and for a large number of particles, \(N_{\text{tot}}\), the energy of the exact ground-state solution coincides with that obtained in the mean-field approximation \[48\]. In the attractive Lieb-Liniger model, a finite ground-state energy per particle is provided by fixing product \(c N_{\text{tot}}\) to a constant value \[47, 48\], while for \(N > 2\) one has to set \(c (N_{\text{tot}})^{N-1} = \text{const}\) \[48\].

In the Heisenberg representation, the equation of motion for field \(\hat{\Psi}(x,t)\) is

\[
\frac{i \hbar}{\partial t} \frac{\partial \hat{\Psi}}{\partial t} = [\hat{\Psi}, \hat{H}] = \hbar \frac{\partial \hat{\Psi}}{\partial t} - c (\hat{\Psi})^{N-1} \left( \hat{\Psi} \right)^{N-1} \hat{\Psi}.
\]  

(3)

The mean-field approximation reduces Eq. (3) to the corresponding GPE with the power-law nonlinearity,

\[
\frac{i \hbar}{\partial t} \frac{\partial \psi(x,t)}{\partial t} = \left( \hbar \frac{\partial \psi(x,t)}{\partial t} - c |\psi(x,t)|^{\alpha}\right) \psi(x,t),
\]

(4)

where the macroscopic wave function \(\psi(x,t)\) is normalized to the total number of atoms, \(N_{\text{tot}}\), and the nonlinearity degree is related to the order of the multi-body interactions, \(N\):

\[
\alpha = 2(N-1).
\]

(5)

Thus, the usual two-body interaction \((N = 2)\) corresponds to \(\alpha = 2\), and the three-body interaction \((N = 3)\) to \(\alpha = 4\). Equation (4) conserves the energy,

\[
E = \int dx \psi^*(x) \left( \hbar \alpha + 2 |\psi(x)|^{\alpha+2} \right) \psi(x),
\]

(6)

which is the classical counterpart of quantum Hamiltonian \[1\].

In Eq. (2), \(V_{\text{ext}}(x)\) is the external trapping potential, which typically includes a superposition of an harmonic magnetic trap and periodic OL potential, \(V_{\text{ext}}(x) = V_{\text{HO}}(x) + V_{\text{OL}}(x)\), where the harmonic confining term is \(V_{\text{HO}} = m\omega^2 x^2/2\). We take the periodic potential as \(V_{\text{OL}} = \epsilon \sin^2(qx + \delta)\), where \(\epsilon\) is proportional to the power of the laser beams which build the OL, and \(q = 2\pi/\lambda\), with \(\lambda = \lambda_{\text{laser}} \sin(\theta/2)\); here, \(\lambda_{\text{laser}}\) is the wavelength of the beams, and \(\theta\) the angle between them (the period of the lattice is \(\lambda/2\)). Parameter \(\delta\) measures a mismatch between the minimum of the parabolic potential (at \(x = 0\)) and the closest local minimum of the lattice potential: when \(\delta = 0 (\delta = \pi/2)\) a minimum (maximum) of \(V_{\text{OL}}\) coincides with the minimum of \(V_{\text{HO}}\). In fact, except for Section V, we consider the situation without the parabolic trap (i.e., \(\omega = 0\)), therefore we set \(\delta = 0\) in this case.

The time-independent power-law GPE corresponding to Eq. (4) is (from now on, we use normalized units, with \(\hbar = m = 1\) and \(N_{\text{tot}} = 1\))

\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} - c |\psi(x)|^\alpha \right] \psi(x) = \mu \psi(x),
\]

(7)

where \(\mu\) is the chemical potential,

\[
V_{\text{ext}}(x) = \epsilon \sin^2(qx),
\]

(8)

and the norm of the wave function is 1. In the free-space cubic model (\(\alpha = 2\) and \(\epsilon = 0\)), Eq. (7) is the integrable NLS equation, whose multi-soliton solutions can be obtained by means of the inverse scattering method \[27\]. The commonly known single-soliton NLS solution is

\[
\psi(x) = A \text{ sech}(kx),
\]

(9)

where \(k^2 = 2|\mu|\) and \(A\) is a real amplitude, the respective value of the chemical potential being \(\mu = -c A^2/2\). For a general value of \(\alpha > 0\), the integrability is lost even in the absence of the external potential \[28\]; nevertheless, the respective single-soliton solutions can be found in an explicit form \[49, 50\].

For the attractive three-body interactions (\(\alpha = 4\)), Eq. (11) is the self-focusing quintic GPE, whose stationary version is

\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} - c |\psi(x)|^4 \right] \psi(x) = \mu \psi(x).
\]

(10)

For \(V_{\text{ext}}(x) = 0\), if one fixes coefficient \(c\) in front of the interaction term, the Townes-like solitons exist for a particular value of the norm of the wave function \[36, 51\]. On the other hand, fixing the normalization of the wave function
(recall that the norm is 1 in our units) amounts, for \( \alpha \neq 4 \), to fixing a relation among the chemical potential and the interaction strength \[\alpha \neq 4\], so that for each \( c \) it is possible to obtain a single soliton solution (although, as mentioned above, these solutions provide the ground-state in the infinite system only for \( \alpha < 4 \), i.e., for \( N < 3 \)). However, for \( \alpha = 4 \) (i.e., \( N = 3 \)) chemical potential \( \mu \) remains indefinite, assuming arbitrary negative values, while the soliton solution of the form

\[
\psi(x) = \frac{(3k^2/8c)^{1/4}}{\sqrt{\cosh(kx)}} \quad k^2 = -8\mu
\]  

(11)

satisfies the unitary normalization condition at a single (critical) value of the interaction strength \[\alpha \neq 4\],

\[
c = c^* \equiv \frac{3\pi^2}{8}.
\]  

(12)

At \( c = c^* \), all solutions (11) share a common value of the energy, which is simply \( E = 0 \) \[\alpha \neq 4\], as follows from Eqs. 6 and 12.

If the two-body interaction is added to the three-body attraction, the mean-field equation is the GPE with the CQ nonlinearity,

\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} + g|\psi(x)|^2 - c|\psi(x)|^4 + V_{\text{ext}}(x) \right] \psi(x) = \mu \psi(x).
\]  

(13)

As said above, we chiefly focus on the case of the repulsive two-body interactions, i.e., \( g \geq 0 \). A family of exact soliton solutions to Eq. (13) with \( V_{\text{ext}}(x) = 0 \) can be obtained in the exact form \[\alpha \neq 4\], which, for \( g \geq 0 \), is

\[
\psi^2(x) = \frac{A^2}{(1 + \xi A^2) \cosh(2\sqrt{2}\mu |x|) - \xi A^2},
\]  

(14)

where \( \xi \equiv g/(4|\mu|) \), and the maximum value of the density, at the soliton’s center, is

\[
A^2 = \frac{3}{c} \left( \frac{g}{4} + \sqrt{\frac{g^2}{16} + \frac{c|\mu|}{3}} \right).
\]  

(15)

A simple derivation of Eq. (14) is presented in Appendix A. Obviously, for \( g = 0 \) solution (14) reduces to Townes-like soliton (11).

Imposing the above-mentioned normalization,

\[
\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1,
\]  

(16)

on solution (14), one arrives at relation

\[
\sqrt{\frac{6}{c}} \tan^{-1} \left( \sqrt{1 + 2\xi A^2} \right) = 1,
\]  

(17)

from where it follows that, for \( g > 0 \), soliton solutions with \( \mu < 0 \) satisfying normalization condition (16) exist for \( c > c^* \). However, these solutions are unstable \[\alpha \neq 4\] (in particular, because they do not satisfy the Vakhitov-Kolokolov stability criterion \[\alpha \neq 4\]). In the following section we discuss how the OL can stabilize such localized solutions.

### III. VARIATIONAL APPROXIMATION

Both for \( \alpha = 2 \) and 4 (\( N = 2 \) and 3), and for the GPE with the mixed CQ nonlinearity, the presence of the periodic potential makes it necessary to resort to approximate methods for finding solitons. To this end, we use the VA (variational approximation) \[\alpha \neq 4\] based on the ansatz which yields exact soliton solution (11) of the quintic NLS equation in the absence of the external potential:

\[
\psi_{\text{ans}}(x) = \frac{A}{\sqrt{\cosh(x/\sigma)}}
\]  

(18)
Here, width $\sigma$ is the variational parameter to be determined by the minimization of the energy, while amplitude $A$ will be found from normalization condition \[10\]. We expect that ansatz \[13\], which does not explicitly include the modulation of the wave function induced by the OL, may give a reasonable estimate of the soliton's energy for sufficiently small values of OL strength $\epsilon$ in Eq. \[8\], cf. the known result for the 2D equation with the cubic nonlinearity ($\alpha = 2$) and OL potential \[28, 30\]. In the case of the 3D GPE which includes the cubic term and harmonic trap, this approach leads to an estimate for the critical value of the number of atoms above which the condensate collapses, that was found to be in a reasonable agreement with results produced by the numerical solution of the GPE \[56, 57\]. In 1D, the VA based on the Gaussian ansatz also provides for quite an accurate approximation to exact soliton solution \[9, 58\]. Similar analyses carried out in the 1D model including the cubic term and OL \[28, 42, 59\] have demonstrated that (unlike the 2D and 3D cases) the 1D soliton trapped in the OL potential does not have an existence threshold in terms of its norm (number of atoms).

The energy to be minimized in the framework of the VA is obtained by inserting ansatz \[18\] in the GPE energy functional given by Eq. \[6\]. The kinetic and quintic-interaction energy terms in the functional both scale as $\sigma^{-2}$; then, the energy per particle computed from expression \[6\] is

$$E = \frac{\beta}{\sigma^2} + \frac{g}{\pi^2 \sigma} + \frac{\epsilon}{2} \left[1 - \text{sech}(\pi q \sigma)\right],$$  \tag{19}

where $c^*$ is defined in Eq. \[12\].

For $\epsilon = 0$ (without the OL), the scenario discussed in the previous section for the uniform CQ GPE with the attractive three-body and repulsive two-body interactions is recovered, as energy \[19\] reduces in that case to

$$E = \frac{\beta}{\sigma^2} + \frac{g}{\pi^2 \sigma}.$$  \tag{20}

For $g = 0$, the energy is positive when $c < c^*$ (i.e., $\beta > 0$, see Eq. \[20\]) and vanishes at $\sigma \to \infty$; for $c = c^*$ (i.e., $\beta = 0$) one obtains $E = 0$, in agreement with the above-mentioned exact result showing the infinite degeneracy of soliton family \[11\], while for $c > c^*$ the energy is negative and diverges (to $-\infty$) at $\sigma \to 0$, signaling, in terms of the VA, the onset of the collapse. With $g > 0$, expression \[21\] does not give rise to any minimum of the energy, which agrees with the known fact of the instability of all the solitons in this case \[38\].

A detailed study of minima of variational energy \[19\] is presented in Appendix B. In the following subsection, we consider the case of the self-focusing quintic GPE in the presence of the OL ($\epsilon > 0$, $g = 0$), while the discussion of the general case ($\epsilon > 0$, $g > 0$) is given in Section IV.

### A. Self-focusing quintic GPE with the optical-lattice potential

Here we address the stability of localized variational mode \[18\], for different values the OL parameters, strength $\epsilon$ and wavenumber $q$, keeping $g = 0$. The results of the analysis of minima of the variational energy \[19\], presented in Appendix B, can be summarized as follows (see also Fig. \[1\]): for $c \geq c^*$, the infinitely deep minimum of the energy is obtained at $\sigma \to 0$, which corresponds to the collapse, as shown in Fig. \[1\](a). For $c < c^*$, the collapse may be avoided, and three possibilities arise: there exists another special value, $c' < c^*$, such that for every $c$ between $c'$ and $c^*$ the energy has a minimum at $\sigma = \sigma_1$ and a maximum at $\sigma = \sigma_2$, while for $c < c'$ the energy does not have a minimum at any finite value of $\sigma$, see Fig. \[1\](d). Actually, two different situations should be distinguished for $c' < c < c^*$: there exists a specific value (refer to Appendix B),

$$c^{**} = c^* - \frac{3\epsilon}{2q^2} T_c,$$

where $T_c \approx 2.13$, \tag{22}

(with $c^{**} > c'$) such that, for $c^{**} < c < c^*$, the energy has a global minimum at $\sigma = \sigma_1$ (which, thus, represents the ground-state of the boson gas in this situation), while, for $c' < c < c^{**}$, the energy minimum at $\sigma = \sigma_1$ is a local one. In other words, taking into regard the fact that, as shown by Eq. \[19\], the energy-per-particle approaches value $\epsilon/2$ at large $\sigma$, we conclude that, for $c^{**} < c < c^*$ ($c' < c < c^{**}$), the energy satisfies inequality $E(\sigma_1) < \epsilon/2$ ($E(\sigma_1) > \epsilon/2$), as showed in Figs. \[1\](b,c).

From the above analysis, we infer that for $c < c^{**}$ the ground-state is a delocalized one (although the metastable state, corresponding to the above-mentioned local energy minimum, exists for $c' < c < c^{**}$), for $c^{**} < c < c^*$ the
FIG. 1: Variational energy $E$ obtained in the framework of the quintic GPE versus $\sigma$ (in units of $\epsilon/2$) for $c \geq c^*$ (a); $c^{**} < c < c^*$ (b); $c' < c < c^{**}$ (c); $c < c'$ (d). In (a) the solid (dotted) line is the energy for $c > c^*$ ($c = c^*$); in (b)-(c), points of the energy minimum and maximum, $\sigma_1$ and $\sigma_2$, are indicated.

FIG. 2: The numerically found ground-state of the quintic GPE with the periodic OL potential, for several values of nonlinearity coefficient $c$. Solid lines, starting from the narrowest configuration, refer to $c = 3.7, 3.5, 3.25, 2.5, 2, 1.9$ (recall that $c^* = 3\pi^2/8 \simeq 3.701$), and the dashed line refers to $c = 1.8$. Parameters are $\epsilon = 6$, $q = 3$ and $L = 10$. Inset: squared width $\sigma^2$ of the ground-state as a function of $c$ (the dot-dashed line is a guide to the eye). Critical value $c^{**}$ obtained from the numerical analysis is $c^{**} = 1.87(3)$, which should be compared with the corresponding value (22) predicted by the variational approximation, $c^{**} \simeq 1.57$.

ground-state is represented by a finite-size soliton configuration (in agreement with Ref. [36]) and for $c > c^*$ it is collapsing. Equation (22) shows that the width of the stability region depends on ratio $\epsilon/q^2$: keeping fixed all other parameters, the decrease of the lattice spacing (i.e., the increase of $q$) leads to a reduction of the stability region. Equation (22) also shows that for $\epsilon/q^2 = 2c^*/3T_c \approx 1.16$ the VA formally predicts $c^{**} = 0$: however, for $c = 0$, the ground-state is delocalized and the variational ansatz (18) cannot be used, as it does not take into account the modulation induced by the deep OL potential.

In Fig. 2 we plot the numerically found ground-state of the quintic GPE in a 1D box ($-L < x < L$). It is seen that, with the increase of $c^* - c$, the configuration becomes broader, until a critical value is reached, as discussed in [30]. In the inset of Fig. 2 we plot the squared width $\sigma^2 = \int_{-\infty}^{\infty} dx \psi^2(x)$ of the numerically found ground-state $\psi$ versus $c$, which makes the delocalization transition evident: for $c < c^{**}$ the width $\sigma$ is $\propto L$, while around $c \approx c^{**}$ the width suddenly decreases. Variational estimate (22) for the critical value $c^{**}$, as predicted by the VA (see Eq. (22)), is displayed in Fig. 3 together with numerical results. One observes a reasonable agreement between them, especially for small $\epsilon$, which is due both to the use of the more adequate ansatz (18), rather than a Gaussian, and also because $T_c$ is found as the value at which the global (rather than local) minimum disappears.
FIG. 3: The dotted line: the variational estimate for \( c^{**} \) as a function of \( \epsilon/q^2 \), according Eq. (22) (for the quintic GPE), the dashed line corresponding to \( c^* = 3\pi^2/8 \). Discrete symbols represent results obtained from the numerical solution of the quintic GPE. They designate the transition form the localized ground-state to the extended one (parameters are the same as in Fig. (2). According to the variational approximation, the ground-state is delocalized \((\sigma \to \infty)\) below the dotted line, and it collapses \((\sigma \to 0)\) for \( c \) above the dashed line.

IV. THE STABILITY REGION FOR THE CONDENSATE WITH COMPETING TWO- AND THREE-BODY INTERACTIONS

The most interesting situation occurs when the two-body repulsive interaction \((g > 0)\) competes with the attractive three-body collisions \((c > 0)\). As said above, all solitons in the free space \((\epsilon = 0)\) are strongly unstable in this situation [38], and the possibility of their stabilization by the OL was not studied before. The analysis of variational energy (19), presented in Appendix B, yields the following results for this case. For \( c > c^* \), the energy does not have a minimum at finite \( \sigma \), hence the OL cannot stabilize the solitons in this case. If \( c = c^* \), the energy has a global minimum at a finite value of \( \sigma \), when

\[
G \equiv \frac{2gq}{\pi \epsilon} < G_c \approx 0.663.
\]  

(23)

For \( c < c^* \), the energy features a global minimum at finite \( \sigma \) for \( c^{**}(G) < c < c^* \), where the modified critical value is

\[
c^{**}(G) \equiv c^* - \frac{3\epsilon}{2q^2} T_c(G),
\]  

(24)

cf. definition (22) for \( G = 0 \). The value \( T_c \) depends upon \( G \), vanishing for \( G \) larger than the critical value \( G_c \). This means that, to balance the destabilizing effect of the repulsive two-body interactions, the strength of the periodic potential, \( \epsilon \), must exceed its own critical value,

\[
\epsilon_{\text{crit}} = \frac{2gq}{\pi G_c}.
\]  

(25)

Otherwise, Eq. (24) yields \( c^{**} = c^* \), i.e., the OL cannot stabilize the solitons.

In Fig. 4 we plot the numerically found ground-state of CQ GPE (13) for several values of \( \epsilon \). It is seen that, at small \( \epsilon \), the wave function \( \psi \) remains delocalized, until a critical value is reached. In the inset of Fig. 4 the squared width of the numerically generated ground-state is plotted versus \( \epsilon \). In Fig. 5 we compare critical value \( \epsilon_{\text{crit}} \), as given by Eq. (25), with numerical results: for small \( g \), the predicted linear dependence of \( \epsilon_{\text{crit}} \) on \( g \) is well corroborated by the numerical results, the relative error in the slope being \( \sim 20\% \). In principle, the comparison between variational estimate (25) and numerical results might be further improved by choosing a variational wave function which, in the limit of \( \epsilon = 0 \) (uniform space) would reproduce exact CQ soliton (14). However, the calculations with such an ansatz are extremely cumbersome.
FIG. 4: The numerically found ground-state of the cubic-quintic GPE for several values of $\epsilon$. Solid lines, starting from the narrowest wave function, refer to $\epsilon = 4.0$, $3.0$, $2.8$, $2.6$, $2.4$, and the dashed line refers to $\epsilon = 2.3$. Parameters are $c = 3.65$, $g = 1$, $q = 3$, $L = 5$. Inset: the squared width of the ground-state versus $\epsilon$ (the dot-dashed line is a guide to the eye). Critical value $\epsilon_{\text{crit}}$ obtained from the numerical data is $\epsilon_{\text{crit}} = 2.35(5)$, which should be compared to the variational prediction given by Eq. (25), which is $\epsilon_{\text{crit}} \simeq 2.88$.

FIG. 5: Solid line: $\epsilon_{\text{crit}}$ versus $g$, as given by Eq. (25). Symbols refer to results obtained from the numerical solution for the ground-state of the cubic-quintic GPE. They represent the delocalization transition. The parameters are the same as in Fig. (4).

V. THE EFFECT OF THE HARMONIC TRAP

In this section we aim to use the variational approximation based on ansatz (18) for examining the combined effect of the parabolic trapping potential acting along with an OL, i.e., we take Eq. (10) with external potential

$$V_{\text{ext}}(x) = \frac{\omega^2 x^2}{2} + \epsilon \sin^2 (qx + \delta),$$

(26)

cf. Eq. (8), and disregard binary collisions ($g = 0$). Value $\delta = 0$ ($\delta = \pi$) corresponds to the matching (largest mismatch) between the minimum of the harmonic potential and a local minimum of the lattice potential. The respective variational energy is obtained from (6) with potential (26):

$$E = \frac{\beta}{\sigma^2} + \frac{\pi^2 \omega^2 \sigma^2}{8} + \frac{\epsilon}{2} \left[ 1 - \cos (2\delta) \text{sech}(\pi q \sigma) \right].$$

(27)

With $\cos(2\delta) \geq 0$, the soliton is stable for $c < c^*$, and it collapses otherwise. With $\cos(2\delta) < 0$, a richer behavior is predicted by the VA. The system does stabilize for $c < c^*$, while, for $c > c^*$, the presence of the mismatched harmonic
Fig. 6: The critical line separating in the \((q, c - c^*)\) plane of the model (including the parabolic trap) the metastable region from the unstable one. The parameters are \(\epsilon = 1\), \(\cos(2\delta) = -0.5\), and \(\omega = 1\).

trap gives rise to a metastability region. Since \(E \to -\infty\) as \(\sigma \to 0\) and \(E \to +\infty\) as \(\sigma \to \infty\), one can encounter two possibilities: either \(\partial E/\partial \sigma\) is positive for all \(\sigma\) (and there are no energy minima), or equation \(\partial E/\partial \sigma = 0\) has two roots, corresponding to a local minimum and a maximum. The equation for the value of \(\sigma\) at which energy \(E\) reaches the local minimum is

\[
|\beta| = \frac{\epsilon |\cos(2\delta)|}{4\pi^2q^2} f(\theta),
\]

where \(\theta \equiv \pi q \sigma\), and

\[
f(\theta) \equiv \theta^3 \left( \frac{\sinh \theta}{\cosh^2 \theta} - \eta \theta \right),
\]

\[
\eta \equiv \frac{\omega^2}{2\epsilon q^2 |\cos(2\delta)|}.
\]

One can see that, for \(c = c^*\) (i.e., \(\beta = 0\)), Eq. \(28\) does not have a nonvanishing solution if \(q\) is smaller than a critical value,

\[
q^{(\text{cr})} = \frac{\omega}{\sqrt{2\epsilon |\cos(2\delta)|}},
\]

while it has a nonvanishing solution for \(q > q^{(\text{cr})}\).

Actually, for \(c > c^*\) (i.e., \(\beta < 0\)), Eq. \(28\) with \(q > q^{(\text{cr})}\) has two nonvanishing roots, one of which is a local minimum, while such roots do not exist for \(q < q^{(\text{cr})}\). For \(q > q^{(\text{cr})}\), the right-hand side of Eq. \(28\) has a maximum value, which fixes the maximum value of \(\beta\), i.e., the maximum value of \(c\), which we refer to as \(c^{***}\). Then, for \(c > c^{***}\), the variational energy does not have a local minimum. For \(c^* < c < c^{***}\) there appears a finite metastability region, in terms of wavenumber \(q\), as illustrated by Fig. 6. In other words, for fixed \(c\), metastable states appear at large values of \(\epsilon\).

VI. CONCLUSIONS

In this work we have studied the effect of the OL (optical lattice) on the 1D Bose gas with attractive three-body and repulsive two-body interactions, described by the GPE (Gross-Pitaevskii equation) with the CQ (cubic-quintic) nonlinearity. Actually, the effective quintic attractive term in the GPE may be induced by the residual deviation of the condensate, tightly trapped in a cigar-shaped confining potential, from the one-dimensionality (when the three body losses are negligible) \(37, 38\) or by three-body interaction terms between atoms according to recent proposals \(32, 33\).
In the absence of an external potential, soliton solutions to this equation with the CQ nonlinearity are known in the exact form, but they all are strongly unstable. We have demonstrated that the OL opens a stability window for the solitons, provided that the OL strength, $\epsilon$, exceeds a finite minimum value. The size of the stability window depends on $\epsilon/q^2$, where $q$ is the OL’s wavenumber. We have also considered effects of the additional harmonic trap, finding that, if the quintic nonlinearity is strong enough ($c \geq c^*$), a metastability region may arise, depending on the mismatch between minima of the periodic potential and harmonic trap.

**APPENDIX A: LOCALIZED SOLUTIONS OF THE CUBIC-QUINTIC GROSS-PITAEVSKII EQUATION**

Assuming that $\psi(x)$ is real, we look for localized solutions to the CQ NLS equation,

$$-\frac{1}{2} \frac{d^2 \psi}{dx^2} + g\psi^3 - c\psi^5 = \mu \psi$$  \hspace{1cm} (A1)

with $c > 0$ and $g \geq 0$. Interpreting $x$ as a formal time variable and $\psi(x)$ as the coordinate of a particle, Eq. (A1) formally corresponds to the Newton’s equation of motion of this particle,

$$M \frac{d^2 \psi}{dx^2} = -\frac{\partial V}{\partial \psi},$$  \hspace{1cm} (A2)

where the effective mass is $M = 1/2$, and the potential is

$$V(\psi) = \frac{\mu}{2} \psi^2 - \frac{g}{4} \psi^4 + \frac{c}{6} \psi^6,$$  \hspace{1cm} (A3)

with an arbitrary additive constant chosen so as to have $V(0) = 0$. Potential (A3) for $\mu < 0$, which corresponds to normalizable solutions, is plotted in Fig. 7. Condition $V(\pm A) = 0$ yields expression (15) for the soliton’s amplitude.

Further, we make use of the conservation of the corresponding Hamiltonian,

$$H = \frac{M}{2} \left( \frac{d\psi}{dx} \right)^2 + V(\psi).$$  \hspace{1cm} (A4)

The boundary conditions for localized solutions, $\psi(x \to \infty) \to 0$, $d\psi/dx(x \to \infty) \to 0$, select $H = 0$ in Eq. (A4). Taking into regard the fact that $V(\psi) \leq 0$ for $0 \leq \psi \leq A$, and looking for solutions with $d\psi/dx < 0$ at $x > 0$, one obtains from here the soliton solution in an implicit form,

$$x = \int_{\psi(x)}^{A} \frac{d\psi}{2\sqrt{-V(\psi)}}.$$  \hspace{1cm} (A5)
It further follows from Eq. \((A5)\) that
\[
\mathcal{E} = \frac{\psi^2(x)}{A^2} - \frac{2a^2 + b^2A^2}{2a^2 + b^2\psi^2(x) + 2a\sqrt{a^2 + b^2\psi^2(x)} - \psi^4(x)},
\]
with \(\mathcal{E} \equiv e^{-2\sqrt{2}\mu|x|}x\). In Eq. \((A6)\), we use notation \(a^2 = 3|\mu|/c\) and \(b^2 = 3g/2c\). Thus, from Eq. \((A6)\) one obtains
\[
\psi^2(x) = \frac{4a^2A^2(2a^2 + b^2A^2)\mathcal{E}}{[2a^2 + b^2A^2(1 - \mathcal{E})]^2 + 4a^2A^4\mathcal{E}^2}. \tag{A7}
\]
One can easily check that this expression yields \(\psi^2(x) = A^2 / \cosh(2\sqrt{2}\mu|x|)\) for \(g = 0\), and that \(\psi(0) = A\), as it must be. Finally, using relation \(a^2 + b^2A^2 = A^4\), one obtains Eq. \((14)\) from Eq. \((A7)\), after a straightforward algebra.

**APPENDIX B: THE VARIATIONAL ENERGY**

In this Appendix we aim to study minima of variational energy \((19)\). When \(g = 0\), one sees that, for \(c > c^*\), the energy per particle tends to \(-\infty\) at \(\sigma \to 0\), and to \(\epsilon/2\) at \(\sigma \to \infty\). Then, with regard to \(\partial E/\partial \sigma > 0\), no local (metastable) minima exist, and variational wave function \((18)\) is not the ground-state for any finite width. For \(c = c^*\), one obtains the global minimum at \(\sigma = 0\), which implies the collapse. For \(c < c^*\), the situation is different: \(E \to \infty\) as \(\sigma \to 0\) (because \(\beta > 0\)), and \(E - \epsilon/2 \to +0\) for \(\sigma \to \infty\). Then, it is necessary to find the value of \(\beta\) at which derivative \(\partial E/\partial \sigma\) has two real zeros. Introducing the parameter
\[
T \equiv \frac{4\beta\pi^2a^2}{\epsilon}, \tag{B1}
\]
with \(\beta\) defined as per Eq. \((20)\), one can write condition \(\partial E/\partial \sigma = 0\) as
\[
T = \theta^3 \frac{\sinh \theta}{\cosh^2 \theta}, \tag{B2}
\]
where \(\theta = \pi q\sigma\), as defined above. Equation \((B2)\) can be satisfied if \(T\) is smaller than a maximum value, \(T' \approx 2.67\), and it then has two roots, \(\theta_1\) and \(\theta_2\), which correspond, respectively to the minimum at \(\sigma = \sigma_1\), and maximum at \(\sigma = \sigma_2\) (see Fig. \(1\)). For \(T > T'\), Eq. \((B1)\) has no roots, hence the variational energy has no minima at finite values of the soliton’s width, \(\sigma\). A plot of \(\theta_1\) as a function of \(T\) is presented in Fig. \(8\) where the maximum value of \(\theta_1\) is \(\theta_1^{\text{max}} \approx 3.0415\). The energy minimum at \(\theta_1\) is a global one if \(E(\theta_1) < \epsilon/2\); using Eq. \((19)\), this condition reads
\[
T - \frac{2\theta_1^2(T)}{\cosh \theta_1(T)} < 0, \tag{B3}
\]
As one can see from Fig. \(8\), condition \((B3)\) is satisfied for \(T < T_c\), where \(T_c \approx 2.1289\); then, a global minimum exists only for \(0 < T < T_c\), while for \(T_c < T < T'\) the minimum is local, corresponding to a metastable state. Using the value of \(T_c\) and definition \((B1)\), one arrives at Eq. \((22)\).

For \(g > 0\) (recall it corresponds to the two-body repulsion), variational energy \((19)\) for \(c > c^*\) does not have a minimum at finite values of \(\sigma\). However, for \(c = c^*\) a finite minimum is possible. Indeed, with definition of \(G\) as per Eq. \((23)\), condition \(\partial E/\partial \sigma = 0\) can be written as
\[
G = \frac{\theta^2 \sinh \theta}{\cosh \theta}. \tag{B4}
\]
For \(G < G' \approx 1.0341\), Eq. \((B4)\) has two roots. By imposing the condition that the value of the energy at \(\sigma = \sigma_1\) be smaller than \(\epsilon/2\), one gets \(G < G_c \simeq 0.6627\). Then, similar to the situation considered above, a global minimum exists only for \(0 < G < G_c\), while for \(G_c < G < G'\) the minimum is local.

For \(c < c^*\), condition \(\partial E/\partial \sigma = 0\) reads
\[
T = \theta^3 \frac{\sinh \theta}{\cosh \theta} - G\theta. \tag{B5}
\]
One can see that condition \((B5)\) is satisfied for \(T < T'(G)\), with \(T'(G') = 0\). Then, for \(G > G'\), i.e., for \(\epsilon\) small enough, the variational energy does not have a minimum. Imposing the condition that the minimum is global leads
FIG. 8: The solid line represents $\theta_1$ as a function of parameter $T$ (defined in Eq. (152)) for $g = 0$. The dashed line is the plot of function $T - 2\theta_1^2(T)/\cosh \theta_1(T)$ versus $T$. The maximum value of $\theta_1$ at $T = T'$ is indicated.

FIG. 9: The solid (dashed) line is the plot of $T_c(T')$ as a function of parameter $G$. 

to $T < T_c$, with $T_c(G_c) = 0$. Then, for $G > G_c$, i.e. for $\epsilon$ smaller than a critical value, the variational energy cannot have a global minimum at a finite value of $\sigma$, i.e., localized states cannot realize a global minimum. Functions $T'(G)$ and $T_c(G)$ are plotted in Fig. 9. In Fig. 10 we plot maximum value $\theta_1^{\text{max}}$ of $\theta_1$ for $T = T'(G)$, as a function of $G$.

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FIG. 10: Plot of maximum value $\theta_{1}^{\text{max}}$ of the smaller root of Eq. (12), $\theta_1$ (at $T = T'(G)$), as a function of $G$. 

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