Vertical Density Matrix Algorithm: A Higher-Dimensional Numerical Renormalization Scheme based on the Tensor Product State Ansatz

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We present a new algorithm to calculate the thermodynamic quantities of three-dimensional (3D) classical statistical systems, based on the ideas of the tensor product state and the density matrix renormalization group. We represent the maximum-eigenvalue eigenstate of the transfer matrix as the product of local tensors which are iteratively optimized by the use of the “vertical density matrix” formed by cutting the system along the transfer direction. This algorithm, which we call vertical density matrix algorithm (VDMA), is successfully applied to the 3D Ising model. Using the Suzuki-Trotter transformation, we can also apply the VDMA to two-dimensional (2D) quantum systems, which we demonstrate for the 2D transverse field Ising model.

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I. INTRODUCTION

Since the density matrix renormalization group (DMRG) method was invented by S.R.White, \cite{1} the method has been applied to various problems in one-dimensional (1D) quantum systems and two-dimensional (2D) classical systems. \cite{2} Such a great success of the DMRG has been stimulating us to extend the algorithm to the one which can handle higher-dimensional systems, principally 2D quantum systems and 3D classical systems. \cite{3–6}

We should recall that, in the DMRG, the matrix-product structure of the wavefunction of the target states (usually the ground state or the maximum-eigenvalue eigenstate) is essential. \cite{7} From this point of view, the tensor product state (TPS) which is a natural higher-dimensional generalization of the matrix product state, should play a key role in the “higher-dimensional DMRG”. A simple but non-trivial example of the TPS is the ground state of the 2D valence-bond-solid (VBS) type quantum spin systems, where the wavefunction is expressed as a product of local finite-dimensional tensors, with all the tensor indices being contracted. \cite{8} As for 3D classical statistical systems, the maximum-eigenvalue eigenstate of the layer-to-layer transfer matrix of the 3D classical system can exactly be represented as the TPS, if we allow the tensor dimension to be infinite. We should note that we can reduce the calculation of the expectation value of the TPS to a statistical average in a lower-dimensional classical system; \((D+1)\)-dimensional classical (or \(D\)-dimensional quantum) problem reduces to a \(D\)-dimensional classical statistical problem. \cite{9} In fact, the properties of the 2D VBS model have been studied in terms of a 2D vertex model associated with the TPS wavefunction. \cite{9,10}

When developing the TPS formulation, the most important step is the determination (optimization) of the local tensor in the TPS. In Refs. \cite{11,12}, direct variational formulations are employed for the optimization of the local tensors. As compared with these “direct” method, our novel algorithm given in the present paper for 3D classical system is more like the original DMRG. The local tensor is updated by the “block-spin basis transformation” along the vertical direction. Since this transformation is constructed in terms of the density matrix made along the “vertical direction” (transfer direction of the transfer matrix), we call this algorithm the vertical density matrix algorithm (VDMA). We apply the VDMA to the 3D Ising model and discuss its efficiency. We also report the application of the VDMA to the 2D transverse field Ising model, with help of the Suzuki-Trotter transformation \cite{13}.

This paper is organized as follows: In Sec. \textsuperscript{1} we briefly explain the VDMA for 3D classical spin systems, taking the 3D Ising model as an example. In Sec. \textsuperscript{2} we show the numerical result for the 3D Ising model and the 2D transverse field Ising model. The last section is devoted to the conclusion.

II. METHOD

Let us consider the 3D Ising model on the simple cubic lattice of the size \(N \times N \times 2L\) in \(X, Y,\) and \(Z\) directions. Suppose that \(L\) and \(N\) are sufficiently large, and the neighboring Ising spins \(\sigma\) and \(\sigma'\) have ferromagnetic interaction
$-J\sigma\sigma'$. Then the Boltzmann weight for the unit cube is written as

$$W \left( \frac{\sigma_{ij}}{\sigma_{ij}} \right) \equiv \exp \left[ -\frac{K}{2} (\sigma_{ij}\sigma_{ij'} + \sigma_{ij'}\sigma_{ij} + \sigma_{ij}\sigma_{ij'} + \sigma_{ij'}\sigma_{ij} + \bar{\sigma}_{ij}\bar{\sigma}_{ij'} + \bar{\sigma}_{ij'}\bar{\sigma}_{ij} + \sigma_{ij}\sigma_{ij'} + \sigma_{ij'}\sigma_{ij} + \bar{\sigma}_{ij}\bar{\sigma}_{ij'} + \bar{\sigma}_{ij'}\bar{\sigma}_{ij} ) \right], \quad (1)$$

where $i' = i + 1$, $j' = j + 1$, and $K = J/T$. The locations of the spin variables are shown in Fig. 1.

For the book keeping, let us here introduce some notations for spin variables. Write the configuration of the four spins surrounding a plaquette in the $XY$ plane as

$$\{\sigma_{ij}\} = (\sigma_{ij}\sigma_{ij'}\sigma_{ij}\sigma_{ij'}), \quad (2)$$

where the position of the plaquette can be labeled by the index $ij$. Then the Boltzmann weight is simply written as

$$W \left\{ \frac{\sigma_{ij}}{\sigma_{ij}} \right\}. \quad (3)$$

Using these notations, the transfer matrix $T$ from a layer $[\sigma]$ to the next layer $[\bar{\sigma}]$ is written as:

$$T[\bar{\sigma}|\sigma] = \prod_{i+j=even} W \left\{ \frac{\bar{\sigma}_{ij}}{\sigma_{ij}} \right\}. \quad (4)$$

In the product of Eq. (4), the spin variables are shared by the adjacent cubes in the diagonal direction and thus the Boltzmann weights make the checkerboard pattern in the $XY$-plane.(see Fig.1-(b))

Our goal is to evaluate the maximum-eigenvalue $\lambda_{\text{max}}$ of the transfer matrix $T$ and the corresponding eigenvector $|\psi_{\text{max}}\rangle$, using the TPS representation of the eigenvector. In order to analyze the TPS structure of $|\psi_{\text{max}}\rangle$, let us consider the power method briefly, which is the simplest but powerful technique to calculate $\lambda_{\text{max}}$ and $|\psi_{\text{max}}\rangle$. Define the vector $|\psi_L\rangle$ with

$$|\psi_L\rangle = T^L|\psi_0\rangle, \quad (5)$$

where $|\psi_0\rangle$ is an ‘initial’ vector that is not orthogonal to $|\psi_{\text{max}}\rangle$. Then the maximum-eigenvalue eigenvector $|\psi_{\text{max}}\rangle$ is obtained as

$$|\psi_{\text{max}}\rangle = \text{const} \times \lim_{L\to\infty} |\psi_L\rangle, \quad (6)$$

where the details of $|\psi_0\rangle$ are not important.

In Fig. 2 we show the graphical representation of $|\psi_L\rangle$, where we can see the structure of $|\psi_L\rangle$ more clearly. As is seen in this figure, the $L$ unit cubes are piling vertically up to the surface, and then the product of the vertically-arranged Boltzmann weights can be regarded as a local tensor; We define the local tensor $A_L$ at the $ij$-plaquette as,

$$A_L \left\{ \frac{\sigma^L_{ij}}{\xi^L_{ij}} \right\} = W \left\{ \frac{\sigma^1_{ij}}{\sigma_{ij}} \right\} W \left\{ \frac{\sigma^{L-1}_{ij}}{\sigma^L_{ij}} \right\} \cdots W \left\{ \frac{\sigma^1_{ij}}{\sigma^L_{ij}} \right\}. \quad (7)$$

where $\{\sigma^L_{ij}\}$ is the spin configuration at the $ij$-plaquette on the “surface layer”, and the auxiliary variable $\xi^L_{ij} \equiv (\sigma^0_{ij}\sigma^1_{ij}\sigma^2_{ij}\cdots\sigma^{L-1}_{ij})$ denotes the configuration for the spins under the surface. Using the local tensors defined above, $|\psi_L\rangle$ is represented as a TPS:

$$|\psi_L\rangle = \sum_{\xi_L} \prod_{i+j=even} A_L \left\{ \frac{\sigma^L_{ij}}{\xi^L_{ij}} \right\}. \quad (8)$$

Taking the limit $L \to \infty$, we have the maximum-eigenvalue eigenvector $|\psi_{\text{max}}\rangle$, which is now represented as the product of the local tensor $A_\infty$. 

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From the practical viewpoint of the usual power method, the eigenvector is improved with the relation $|\psi_{L+1}\rangle = T|\psi_L\rangle$ iteratively. In the framework of the TPS, this power method is reformulated as the recursion relation for the local tensor:

$$A_{L+1} \left\{ \frac{\sigma_{ij}^{L+1}}{\xi_{ij}^{L+1}} \right\} = W \left\{ \frac{\sigma_{ij}^{L+1}}{\xi_{ij}^L} \right\} A_L \left\{ \frac{\sigma_{ij}^L}{\xi_{ij}^L} \right\},$$  \hspace{1cm} (9)

with which we can carry out the iterative calculation, until $A_L$ gives a good approximation of $A_\infty$. However it is generally difficult to store the tensor $A_L$ in the computer memory for a sufficiently large $L$, because if $\xi_{ij}^L$ has $M$ states, then the extended auxiliary variable $\xi_{ij}^{L+1}$ has $2M$ states.

In order to restrict the number of the auxiliary variable, we now import the idea of the DMRG into the TPS. The essence of the DMRG is that the increased number of states for $\xi_{ij}^{L+1}$ can be reduced, by using the “projection operator” generated from the “density matrix”. In the present case, the appropriate density matrix should be constructed for the spin variables $(\sigma_{ij}^{L+1}, \xi_{ij}^{L+1})$ in the vertical direction (we thus call this density matrix as “the vertical density matrix”). Introducing the “transposed” local tensor:

$$\tilde{A} \left\{ \frac{\eta_{ij}}{\sigma_{ij}} \right\} = W \left\{ \frac{\sigma_{ij}^0}{\sigma_{ij}^1} \right\} W \left\{ \sigma_{ij}^2 \right\} \cdots W \left\{ \sigma_{ij}^{L-1} \right\},$$ \hspace{1cm} (10)

the explicit form of the vertical density matrix is defined as (See Fig.

$$\rho_{kl}(\sigma_{kl}^\tau, \xi_{kl}^\tau | \sigma_{kl}^\tau, \xi_{kl}^\tau) = \sum_{[i,j],[k,l]} \prod_{i+j=even} \tilde{A} \left\{ \frac{\eta_{ij}}{\sigma_{ij}} \right\} A \left\{ \frac{\sigma_{ij}}{\xi_{ij}} \right\} A \left\{ \frac{\sigma_{kl}^{\tau''}}{\xi_{kl}^{\tau''}} \right\} \tilde{A} \left\{ \frac{\sigma_{kl}}{\xi_{kl}} \right\},$$ \hspace{1cm} (11)

where $\sum'$ denotes the configuration sum for the all spin variables except $\sigma_{kl}^{(r)}$, $\xi_{kl}^{(r)}$, and $\prod'$ means the product for the site index $(i,j)$ except at $(i,j) = (k,l)$, $(k'',l'')$ with $k'' = k - 1$ and $l'' = l - 1$. In Eq. (12), we have also used the notation for the “checked spins”:

$$\left\{ \frac{\sigma_{kl}^{k''l''}}{\xi_{kl}^{k''l''}} \right\} = \left\{ \frac{\sigma_{kl}^{k''l''}}{\xi_{kl}^{k''l''}} \right\} \left\{ \frac{\sigma_{kl}^{k''l''}}{\xi_{kl}^{k''l''}} \right\} \left\{ \frac{\sigma_{kl}^{k''l''}}{\xi_{kl}^{k''l''}} \right\}.$$

Further, we have omitted the sub(super)script $L$ if it is apparent.

For the case of the isotropic 3D Ising model, the vertical density matrix $\rho_{kl}$ becomes independent of the site index $kl$ in the thermodynamic limit. Thus we write $\rho_{kl}$ simply as $\rho$ hereafter. Moreover it should be noted that, for the isotropic case, the local tensors $\tilde{A}$ and $\tilde{A}$ satisfies the relation:

$$\tilde{A} \left\{ \frac{\eta_{ij}}{\sigma_{ij}} \right\} A \left\{ \frac{\sigma_{ij}}{\xi_{ij}} \right\} = A \left\{ \frac{\sigma_{ij}}{\xi_{ij}} \right\} \tilde{A} \left\{ \frac{\eta_{ij}}{\sigma_{ij}} \right\} .$$ \hspace{1cm} (12)

Following the spirit of the DMRG, we diagonalize $\rho$ to have the eigenvalues $w_\xi$ in the decreasing order $w_1 \geq w_2 \geq \cdots (\geq 0)$:

$$\sum_{\sigma\xi} \rho(\sigma') U(\sigma\xi | \tilde{\xi}) U(\sigma\xi | \tilde{\xi}) w_\xi = U(\sigma\xi | \tilde{\xi}) w_\xi,$$ \hspace{1cm} (13)

where $U(\sigma\xi | \tilde{\xi})$ is the eigenvector for $w_\xi$. By taking $U(\sigma\xi | \tilde{\xi})$ with $\tilde{\xi} \in 1, \cdots, M$, we construct the projection operator $U$, which is a $2M \times M$ rectangular matrix. Operating $U$ to the spin variables on each “edge” of $A$, we then make the renormalized local tensor $\tilde{A}$:

$$\tilde{A} \left\{ \frac{\tau_{ij}}{\xi_{ij}} \right\} = \sum_{\{\sigma_{ij}\}} A \left\{ \frac{\tau_{ij}}{\sigma_{ij}} \right\} A \left\{ \frac{\sigma_{ij}}{\xi_{ij}} \right\} U(\sigma_{ij}\xi_{ij} | \tilde{\xi}_{ij}) U(\sigma_{ij}\xi_{ij} | \tilde{\xi}_{ij}) U(\sigma_{ij}\xi_{ij} | \tilde{\xi}_{ij}) \times U(\sigma_{ij}\xi_{ij} | \tilde{\xi}_{ij}) .$$ \hspace{1cm} (14)

Using Eqs. (10) and (13) recursively, we can now calculate the effective local tensor $\tilde{A}_\infty$. However we encounter another problem in this process; it is also a numerically heavy problem to compute the vertical density matrix with Eq. (11).
Let us next explain how to overcome the difficulty in calculating the vertical density matrix. The key idea is that we can consider Eq. (11) as a kind of 2D classical spin system with a point defect. To see it, we here define a new tensor

\[
G \left\{ \begin{array}{c}
\eta_{ij} \\
\sigma_{ij} \\
\xi_{ij}
\end{array} \right\} \equiv A \left\{ \begin{array}{c}
\eta_{ij} \\
\sigma_{ij} \\
\xi_{ij}
\end{array} \right\} A \left\{ \begin{array}{c}
\sigma_{ij} \\
\xi_{ij}
\end{array} \right\}, \quad (15)
\]

which is graphically represented in Fig. 4.

Regarding the spin variable along the \( z \) axis as a \( 2M^2 \) state single spin:

\[
\sigma_{ij} = \begin{pmatrix}
\eta_{ij} \\
\sigma_{ij} \\
\xi_{ij}
\end{pmatrix}, \quad (16)
\]

we can see that \( G \{ \sigma_{ij} \} \) becomes the Boltzmann weight for the 2D effective classical model. In fact, the vertical density matrix \( \rho \) can be expressed as the density matrix for the point defect in the 2D classical model:

\[
\rho_{kl} (\sigma'_{kl} \xi'_{kl} | \sigma_{kl} \xi_{kl}) = \sum' \left[ \prod'_{i+j=even} G \{ \sigma_{ij} \} \right] G \{ \sigma_{k'\ell'} \} G \{ \sigma_{kl} \} \quad (17)
\]

where the meaning of the prime at the summation and the product is the same as Eq. (11), and the “checked spin” is given by

\[
\{ \sigma_{k'\ell'} \} = \begin{pmatrix}
\eta_{k'\ell'v}
\\
\sigma_{k'\ell'v}
\\
\xi_{k'\ell'v}
\end{pmatrix}.
\]

We calculate each component of the vertical density matrix (17) as a statistical average of a special local observable in 2D. To this end we apply the corner transfer matrix renormalization group (CTMRG) [15] which has been known to be quite efficient for 2D classical statistical systems.

Thus we have obtained a closed algorithm to calculate the local tensor \( A \) with Eqs. (9) and (14), assisted by the CTMRG for the 2D effective classical model. We summarize the numerical procedure as follows:

(a) For the free-boundary condition in the \( Z \) direction, define the initial local tensor \( A_1 \) as

\[
A_1 \left\{ \begin{array}{c}
\sigma_{ij} \\
\sigma'_{ij}
\end{array} \right\} = W \left\{ \begin{array}{c}
\sigma_{ij} \\
\sigma'_{ij}
\end{array} \right\}. \quad (18)
\]

For the Ferro boundary condition, define \( A_1 \) as

\[
A_1 \left\{ \begin{array}{c}
\sigma_{ij} \\
\sigma'_{ij}
\end{array} \right\} = W \left\{ \begin{array}{c}
\sigma_{ij} \\
\sigma'_{ij}
\end{array} \right\} \times W \left\{ \sigma_{ij} \ | +1 \right\}, \quad (19)
\]

where \{+1\} means that the four Ising spins are aligned upward.

(b) Define the effective Boltzmann weight of the 2D classical spin system with Eq. (15).

(c) Perform the CTMRG calculation for the 2D effective classical spin system with the Boltzmann weight \( G \) and obtain the vertical density matrix \( \rho \).

(d) Diagonalize \( \rho \) and construct the projection operator \( U \).

(e) Renormalize the local tensor \( A \) with Eq. (14).

(f) Return (b), until the local tensor \( A \) is converged.

In this VDMA calculation, the accuracy is determined by the number of retained basis \( M \) for the auxiliary variables \( \xi \) and \( \eta \), and \( m \) for the CTMRG calculation in 2D classical system. We can check the convergence of the computed quantities with respect to \( M \) and \( m \).
III. RESULTS

A. The 3D Ising model

Fig. 5 shows the spontaneous magnetization $\langle \sigma \rangle$ calculated by using the VDMA. For comparison we also show the results of the 3D-version of the Kramers-Wannier (KW) approximation [11] and the Talapov and Blöte’s Monte Carlo results [16]. For each $M$, we have carried out the VDMA calculations for $m = 4, 8, 12, \text{ and } 16$, where we can find the good convergence with respect to $m$. We observe that the convergence with respect to $M$ is also sufficient in the off-critical region. Near the critical point, however, the magnetization becomes smaller as $M$ is increased, implying that large $M$ is needed for calculation in the critical region.

B. The 2D transverse field Ising model

Let us next consider the VDMA for the 2D transverse field Ising (TFI) model on the square-lattice, which is one of the fundamental quantum spin models in 2D. The Hamiltonian of the 2D TFI model is given by

$$H = -J \sum_{\langle i,j \rangle} \sigma^z_i \sigma^z_j - \Gamma \sum_i \sigma^x_i,$$

(20)

where $J(>0)$ is the ferromagnetic coupling constant, $\sigma^\alpha_i (\alpha = x, y, \text{or } z)$ are the Pauli matrices, and $\langle i, j \rangle$ denotes the nearest-neighbor pairs on the square lattice. The transverse field $\Gamma$ induces the quantum fluctuation into the system. At zero temperature, the TFI model exhibits the quantum phase transition at $\Gamma = \Gamma_c$, below which the spontaneous magnetization $\langle \sigma \rangle$ behaves as $\langle \sigma \rangle \sim (\Gamma_c - \Gamma)^\beta$. The critical field $\Gamma_c$ and the critical exponent $\beta$ have been estimated as $\Gamma_c = 3.06$ and $\beta = 0.31$ by the quantum Monte Carlo (QMC) simulation [17]. In the following we consider the VDMA for the 2D TFI model at zero temperature, and show the results of $\langle \sigma \rangle$.

As was described in the previous section, the VDMA is formulated for the 3D classical systems. In order to apply the VDMA to the 2D TFI model, we map the model to the 3D anisotropic Ising model by using the Suzuki-Trotter transformation [13]. The partition function $Z$ of the 2D TFI model is obtained as the limit of the 3D anisotropic Ising model:

$$Z = \lim_{L \to \infty} \text{Tr} \exp \left[ K_h \sum_{\tau=1}^L \sum_{\langle i,j \rangle} \sigma^z_i,_{\tau} \sigma^z_j,_{\tau} + K_v \sum_{\tau=1}^L \sum_i \sigma^x_i,_{\tau} \sigma^x_i,_{\tau+1} \right],$$  

(21)

where $\sigma^z_i,_{\tau}$ is the Ising variable at the position $i$ and imaginary time $\tau$. The effective couplings $K_h$ and $K_v$ in Eq. (21) are given by

$$K_h = \epsilon J,$$

(22)

$$K_v = -\frac{1}{2} \log \left[ \tanh(\epsilon \Gamma) \right],$$

(23)

$$\epsilon = 1/(TL),$$

(24)

where the subscripts $h$ and $v$ denote the horizontal ($X$-$Y$) direction and the vertical (Trotter) one, respectively. We can perform the VDMA calculation for this anisotropic 3D Ising model.

We should make a comment about the boundary condition, before proceeding to details. As was seen in the previous section, the open boundary condition is assumed in the VDMA. However, the periodic boundary condition is imposed along the Trotter direction in Eq. (21). As far as the zero-temperature properties are concerned, the boundary condition is inessential due to the double limit $T \to 0$ and $L \to 0$, allowing us to apply the VDMA to Eq. (21). [18]

For a fixed value of $\epsilon$, we calculate the magnetization $\langle \sigma(\epsilon) \rangle$ with the VDMA for the infinite volume. After obtaining $\langle \sigma(\epsilon) \rangle$ for various $\epsilon$, we take the $\epsilon \to 0$ limit by extrapolation. In the actual calculation, we have observed the following $\epsilon$-dependence

$$\langle \sigma(\epsilon) \rangle = \langle \sigma(0) \rangle + \text{const} \times \epsilon^2,$$

(25)

which we adopt for the extrapolation formula.
In Fig. 6 we show the \( \langle \sigma \rangle \) vs. \( \Gamma \) plots at \( \Gamma = 3.0 \) as an example, which are obtained with the VDMA of the numbers of retained bases \((M,m) = (2,8)\) and \((3,18)\), where the convergence with respect to \( m \) is rapid. In the region \( \Gamma > 3.2 \), \( \langle \sigma \rangle \) converges to 0 smoothly.

In Fig. 7 we show the \( \langle \sigma \rangle \)-\( \Gamma \) curve and the results of the series expansion for comparison. In \( \Gamma < 2.6 \), the VDMA results are sufficiently reliable, where the good convergence about \( M \) and \( M \) can be seen. We can further see the good agreement with the series expansion in the small field region \((\Gamma < 2.0)\). In the vicinity of the critical point, however, the calculated magnetization exhibits the \( M \) dependence. The roughly estimated critical field from the VDMA calculation is about 3.2, which is 4% larger than the QMC one.

**IV. CONCLUSION**

In this paper we have constructed a novel higher-dimensional numerical renormalization algorithm which utilizes the natural tensor-product form of maximum-eigenvalue eigenstate \( |\psi_{\text{max}}\rangle \) of the transfer matrix. In our algorithm, called VDMA (vertical density matrix algorithm), the local tensor forming \( |\psi_{\text{max}}\rangle \) is iteratively updated using the vertical density matrix. We have successfully applied the VDMA to 3D Ising model. The VDMA can also be applied to 2D quantum systems using the Suzuki-Trotter transformation to 3D classical statistical systems, which we have demonstrated for the 2D transverse field Ising model. Application of the VDMA to other 2D quantum systems such as the Heisenberg model is an important subject of study, which we are now undertaking.

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[14] In Ref. [1] the superblock is constructed as \( B \bullet B \), where \( B \) and \( \bullet \) represent the block and the single site, respectively. In the VDMA algorithm, however, we construct the superblock as \( B \bullet B_k \). \( B_k \) represents the environment block and \( B \neq B_k \).

In this case, the environment block \( B_k \) contains much more sites than the system block \( B \bullet \) that corresponds to \( \langle \sigma_\mu \xi_\mu \rangle \).

For example, \( B_k \) contains \( \eta_k \), \( \eta_{k+1} \), \( \eta_{k-1} \), \( \eta_{k+1} \), \( \cdots \). (See Fig. 3.)

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FIG. 1. (a) The Boltzmann weight of a unit cell, and (b) the transfer matrix $T[\bar{\sigma} | \sigma] \ (N = 4)$.

FIG. 2. The graphical representation of the vector $|\psi_L\rangle$. Below the $ij$-plaquette, we find that the $L$ unit cubes are piling up.

FIG. 3. The vertical density matrix $\rho_{ij} = \langle \sigma_i' \xi_{kl}' | \sigma_k | \xi_{ij} \rangle$.

FIG. 4. The effective Boltzmann weight $G$.

FIG. 5. The spontaneous magnetization of the 3D Ising model.
FIG. 6. The extrapolation of the magnetization at $\Gamma = 3.0$. The solid lines are linear fits of data.

FIG. 7. $\epsilon = 0$ limit of $\langle \sigma \rangle - \Gamma$ curve at $T = 0$. The arrow shows the critical field obtained by the QMC simulation [17].