Elliptic fibrations for $SU(5) \times U(1) \times U(1)$ F-theory vacua

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Elliptic Calabi-Yau fibrations with Mordell-Weil group of rank two are constructed. Such geometries are the basis for F-theory compactifications with two abelian gauge groups in addition to non-abelian gauge symmetry. We present the elliptic fibre both as a $\mathbb{B}^2 \mathbb{P}^2[3]$-fibration and in the birationally equivalent Weierstraß form. The spectrum of charged singlets and their Yukawa interactions are worked out in generality. This framework can be combined with the toric construction of tops to implement additional non-abelian gauge groups. We utilise the classification of tops to construct $SU(5) \times U(1) \times U(1)$ gauge symmetries systematically and study the resulting geometries, presenting the defining equations, the matter curves and their charges, the Yukawa couplings and explaining the process in detail for an example. Brane recombination relates these geometries to a $\mathbb{B}^1 \mathbb{P}^2[3]$-fibration with a corresponding class of $SU(5) \times U(1)$ models. We also present the $SU(5)$ tops based on the elliptic fibre $\mathbb{B}^1 \mathbb{P}_{[1,1,2]}[4]$, corresponding to another class of $SU(5) \times U(1)$ models.

I. INTRODUCTION

Recently a lot of progress has been made in the construction of elliptically fibered Calabi-Yau 4-folds leading to four-dimensional F-theory compactifications with abelian gauge groups [1–7]. Such constructions are motivated in part by the manifold applications of abelian gauge groups in string model building. These include the prominent role of $U(1)$ selection rules in phenomenology and their relevance in particular for Grand Unified Theory (GUT) model building, where $U(1)$ selection rules can be responsible for proton stability, prevent too large $\mu$-terms or induce realistic flavour structure [12–18]. Furthermore, the construction of $U(1)$ symmetries endows one with a large class of gauge fluxes required to generate a chiral matter spectrum [2–4, 8].

F-theory compactifications with abelian symmetries are based on elliptically fibred Calabi-Yau 4-folds with Mordell-Weil group of rank one or bigger [10]. The rank of the Mordell-Weil group gives the number of independent rational sections of the fibration. While every elliptic 4-fold suitable for F-theory necessarily exhibits a universal holomorphic section that identifies the base $B_3$ of the fibration as the physical compactification space, extra rational sections are related to certain elements of $H^{1,1}(\hat{Y}_4)$, other than the class dual to $B_3$, which do not lie in $H^{1,1}(B_3)$. Such 2-forms $w_i$ give rise to a $U(1)$ gauge potential upon expanding the M-theory 3-form $C_3$ as $C_3 = A_1 \wedge w_i$ [10].

In this letter, we report on the construction of elliptically fibered Calabi-Yau 4-folds $\hat{Y}_4$ with Mordell-Weil group of rank 2. To construct fibrations with two independent extra sections, we consider an elliptic fibre described as a slightly restricted cubic in $\mathbb{P}^2$, cf. eq. [1] and FIG. 1. This fibration gives rise to $U(1) \times U(1)$ gauge symmetry with charged singlet states, whose structure and Yukawa interactions we present.

In addition to this universal $U(1) \times U(1)$ charged singlet sector, extra non-abelian gauge symmetries along divisors can be engineered. Based on the toric classification [19] of tops [20], we have explicitly worked out [9] the tops leading to an $SU(5)$ singularity over a divisor on $B_3$, corresponding in total to F-theory compactifications with $SU(5) \times U(1) \times U(1)$ gauge group. We present the fully resolved 4-folds and discuss the matter spectrum and the Yukawas.

In this letter we also provide the tops leading to $SU(5) \times U(1)$-fibrations based on the elliptic fibre $\mathbb{B}^1 \mathbb{P}_{[1,1,2]}[4]$, extending our previous studies [6].

It is interesting to note that our $SU(5) \times U(1) \times U(1)$-fibrations lend themselves also to studying models with a single $U(1)$ factor upon Higgsing a linear combination of the two $U(1)$s. This brane recombination process leads to an $\mathbb{B}^1 \mathbb{P}^2[3]$-fibration, and the charges and GUT matter curves of the associated class of $SU(5) \times U(1)$ fibrations include e.g. the model presented recently in [7].

Our analysis does not specify the base space of the fibration and is thus applicable very generally. While we reserve a detailed description and full display of our results to the companion paper [9], here we outline the main features of our approach and exemplify it with one specific fibration of the above type.

II. $\mathbb{B}^2 \mathbb{P}^2[3]$-FIBRATIONS

The starting point of our construction is the representation of an elliptic curve as the cubic hypersurface

$$0 = v w (c_1 w + c_2 v) + u (b_0 v^2 + b_1 v w + b_2 w^2) + u^2 (d_0 v + d_1 w + d_2 u)$$

(1)

with $[u : v : w]$ homogeneous coordinates of $\mathbb{P}^2$. If we promote $d_i$, $c_i$ and $b_i$ to sections of suitable line bundles on a 3-dimensional base $B_3$, this defines an elliptically fibered Calabi-Yau 4-fold $\hat{Y}_4$. The hypersurface [1] is a non-generic cubic within $\mathbb{P}^2$ to the extent that the coefficient of $w^3$ and $v^3$ are set to zero. As a result, the elliptic
under this map the point Sec relation explicit we first need to resolve the conifold sing-
SU find \( [9] \) that the elliptic fibre develops \( b_u^2 \) and Sec fore Sec as well as
These in turn have the following solutions:

\[
\begin{align*}
  d &= b_1^2 + 8b_0 b_2 - 4c_1 d_0 - 4c_2 d_1, \\
  c &= -8(b_0 (b_1 c_1 d_1 - b_1^2 b_2 + 2b_2 c_1 d_0 + 2b_2 c_2 d_1 - 2c_2^2 d_2) \\
  &+ (c_2 (b_1 b_2 d_0 + b_1 c_1 d_2 - 2b_2 c_2 d_2 - 2c_1 d_0 d_1) - 2b_2^2 b_2^2)), \\
  e &= 16(b_0 b_1 b_2 - b_0 c_1 d_1 - b_2 c_2 d_0 + c_1 c_2 d_2)^2.
\end{align*}
\]

Under this map the point Sec0 maps to the “zero point” \( [x : y : z] = [\lambda^2 : \lambda^3 : 0] \) of the Wei-
In fact, from the Wei-erstraß representation of (1) we find [9] that the elliptic fibre develops \( SU(2) \) singularities
of the Weierstraß model. Therefore Sec0 is related to the universal section, while Sec1 and Sec2 are related to the extra rational sections responsible for the two \( U(1) \) gauge groups. To make this relation explicit we first need to resolve the conifold
fibre developed over the codimension-two loci on \( B_3 \) given by

\[
\begin{align*}
  d_0 c_2^2 &= b_0 b_1 c_2 - b_0^2 c_1, \\
  d_1 b_0 c_2 &= b_0^2 b_2 + c_2^2 d_2
\end{align*}
\]

and

\[
\begin{align*}
  d_1 c_1^2 &= b_1 b_2 c_1 - b_2^2 c_2, \\
  d_0 b_0 b_2 c_1 &= b_0 b_2^2 + c_1^2 d_2
\end{align*}
\]

as well as

\[
\begin{align*}
  c_1^3 (d_0 c_2^2 - b_0 b_1 c_2 + b_0^2 c_1) &=
  c_1^3 (d_1 c_2^2 - b_1 b_2 c_1 + b_2^2 c_2), \\
  d_2 c_1^2 c_2 &= (c_2 (b_1 c_1 b_2 - b_2 c_2) - b_0 c_2^2) (b_0 b_2 c_2^2 + c_2 (d_1 c_1^2 - b_1 b_2 c_1 + b_2^2 c_2)).
\end{align*}
\]

These in turn have the following solutions:

1. \( C_1(1) : b_0 = c_2 = 0; \)
2. \( C_1(2) : \square \) with \( (b_0, c_2) \neq (0, 0); \)
3. \( C_1(3) : b_2 = c_1 = 0; \)
4. \( C_1(4) : \square \) with \( (b_2, c_1) \neq (0, 0); \)
5. \( C_1(5) : c_1 = c_2 = 0; \)
6. \( C_1(6) : \square \) with \( (c_1, c_2) \neq (0, 0), (b_0, c_2) \neq (0, 0) \) and \( (b_2, c_1) \neq (0, 0). \)

The singularities are resolved by two blow-ups: The fibre over the curve \( C_1(1) \) is singular in the point Sec1, which is remedied by introducing the blow-up coordinate \( s_1 \) via

\[
\begin{align*}
  u &\rightarrow s_1 u, \quad v \rightarrow s_1 v.
\end{align*}
\]

Likewise, the singularity at Sec0 in the fibre over \( C_1(3) \) is resolved via

\[
\begin{align*}
  u &\rightarrow s_0 u, \quad v \rightarrow s_0 v.
\end{align*}
\]

The proper transform of (1) after these two blow-ups reads

\[
\begin{align*}
  v w (c_1 w^2 + c_2 v s_0) + u (b_0 v^2 s_0^2 + b_1 v w s_0 s_1 + b_2 w^2 s_1^2) + u^2 (d_0 v s_0^2 s_1 + d_1 w s_0 s_1^2 + d_2 u s_0^3 s_1^2) = 0.
\end{align*}
\]

and indeed describes a smooth manifold \( \hat{Y}_4 \) for generic base sections. This can be checked by exploiting the enlarged Stanley-Reisner ideal

\[
\{w s_0, w u, v s_1, s_0 s_1, v u\}.
\]

The resolved fibre ambient space is shown in FIG. 1. The

section \( S_0 = 0 \) can be viewed as the universal section of \( \hat{Y}_4 \), while \( s_1 = 0 \) and \( u = 0 \)—the section corresponding to Sec2—are the generators of the Mordell-Weil group. We will denote these sections as

\[
\begin{align*}
  S_0 : s_0 = 0, \quad S_1 : s_1 = 0, \quad S_2 : u = 0.
\end{align*}
\]

The fibre over each of the six curves \( C_1(i) \) splits into two rational curves \( P^3_A \) and \( P^3_B \) which intersect in two points, corresponding to the affine Dynkin diagram of \( SU(2) \). Consider for example the curve \( C_1(1) \). The resolved hypersurface \( [8] \) at \( b_0 = c_2 = 0 \) factorises as \( s_1(\ldots) = 0 \). The section \( S_1 \) therefore wraps, say, \( P^3_A \) of the fibre, while \( (\ldots) = 0 \) describes the second \( P^3_B \). Since \( \{s_0, s_1\} \) are in the Stanley-Reisner ideal, \( S_0 \) intersects only \( P^3_B \)—in one point. Furthermore \( S_2 \) intersects \( P^3_A \) in one point. This follows by counting common points of the various equa-
tions. In a similar way the topology over the remaining 5 curves can be understood. This behaviour is depicted for \( C_1(1) \) and \( C_1(2) \) in FIG. 2.

M2-branes wrapping either of the \( P^3 \)'s in the fibre over \( C_1(i) \) give rise to singlet states \( 1^{(0)} \) charged under the two \( U(1) \)'s of the model. To compute their charges we first

FIG. 1. The toric polygon to \( BIP^2 \) and its dual. On the
dual one we only indicated the monomials of the vertices and
omitted powers of \( s_0 \) and \( s_1 \).
The overall factor 5 appears because in section III we are considering SU(5) models.
The defining data of the remaining possibilities are presented in appendix A and described in more detail in [9]. From the associated discriminant \( \Delta = w^5(P + \mathcal{O}(w)) \) (computed most easily in the birationally equivalent Weierstraß model) with

\[
P = \frac{1}{16} b_1^2 b_0, 2c_1, 2c_1 (b_1 b_2 - d_1 c_1) \\
\times (d_2, 2b_1^2 + d_1 (b_0, 2d_1 - d_0, 2b_1))
\]  

we confirm the gauge group \( SU(5) \) along \( w = 0 \), henceforth called GUT divisor, and anticipate the existence of a 10-matter curve

\[
C_{10} = \{ b_1 = 0 \} \cap \{ w = 0 \}
\]  

as well as five 5-matter curves\(^2\)

\[
C_{5(1)} = \{ b_0, 2 = 0 \} \cap \{ w = 0 \},
\]
\[
C_{5(2)} = \{ c_2, 1 = 0 \} \cap \{ w = 0 \},
\]
\[
C_{5(3)} = \{ c_1 = 0 \} \cap \{ w = 0 \},
\]
\[
C_{5(4)} = \{ b_1 b_2 - d_1 c_1 = 0 \} \cap \{ w = 0 \},
\]
\[
C_{5(5)} = \{ d_2, 2b_1^2 + d_1 (b_0, 2d_1 - d_0, 2b_1) = 0 \} \cap \{ w = 0 \}.
\]  

From the top we read off the resolution of the \( SU(5) \) singularities. The hypersurface of the resolved 4-fold \( \hat{Y}_4 \) is described by the proper transform of (14),

\[
0 = b_0, 2e_0^2 e_1 e_2 s_1^2 s_2 w^2 + c_2, 1 e_0 e_1 e_2 s_0 w v^2 \\
+ d_0, 2e_0^2 e_1 e_3 e_4 s_1^2 s_2 u^2 + b_1 s_0 s_1 w v u +
\]
\[
+ c_1 e_1 e_2 e_3 w^2 v s_1 + d_2, 2e_0^2 e_1 e_3 e_2 s_1^2 s_2 u^3 +
\]
\[
+ d_1 e_3 e_4 s_0 s_1^2 w u^2 + b_2 e_1 e_2 e_3 e_4 s_1^2 w^2 u.
\]  

The resolution divisors \( c_i = 0 \), \( i = 1, \ldots, 4 \) are \( P^1 \)-fibrations over the GUT divisor and combine with \( e_0 \), the proper transform of \( w = 0 \), into the affine Dynkin diagram of \( SU(5) \). As in the Weierstraß model, there are several possibilities to resolve the singularities. For definiteness we choose a resolution whose SR-ideal includes, in addition to (9), the elements

\[
\{ w e_0, w e_1, w e_3, w e_4, s_1 e_0, s_1 e_1, s_1 e_4, s_0 e_1, s_0 e_2, \\
\quad s_0 e_3, s_0 e_4, u e_2, e_0 e_2, e_2 e_4, v e_3, v e_4, u e_1, e_0 e_3 \}.
\]  

Indeed it is now possible, with the techniques presented in [3]—see [22] for different approaches—to analyse the fibre splitting over the matter curves. This confirms the appearance of the affine Dynkin diagrams of \( SO(10) \) and \( SU(6) \) respectively. An explicit construction of

the weight vectors associated with the respective representations proves the appearance of the 10- and 5-representations of \( SU(5) \) at these loci.

Further, the generators \( w_i \in H^{1,1}(\hat{Y}_4) \) must be modified such that the roots of \( SU(5) \) are uncharged under \( U(1)_i \). This amounts to the condition

\[
\int_{\hat{Y}_4} w_i \wedge E_k \wedge \omega_b = 0
\]  

in addition to transversality. Here \( E_k \) are the 2-forms dual to the resolution divisors \( \{ e_k = 0 \} \). As in [3] we ensure this by adding suitable combinations of \( E_i \). As a result of the intersection numbers

\[
\int_{\hat{Y}_4} S_i \wedge E_k \wedge \omega_b = \delta_{kA} \int_{B_3} W \wedge \omega_b
\]  

with \( A = (0, 3, 4) \) for \( i = (0, 1, 2) \) and \( W \) the class associated with the GUT divisor on \( B_3 \), the solution is

\[
w_1 = 5(S_1 - S_0 - \bar{K}) + m^1 E_i,
\]
\[
w_2 = 5(S_2 - S_0 - \bar{K} - [c_1]) + i^1 E_i,
\]  

with \( m^i = (2, 4, 6, 3)^T \) and \( i^i = (1, 2, 3, 4)^T \). The overall normalisation is chosen to ensure integer charges. The result of the computation of the \( U(1)_i \) charges of the \( SU(5) \) representations, as described in [3] [9], is [9]

\[
C_{10} : 10_{-1, 2},
\]
\[
C_{5(1)} : 5_{-3, 1}, C_{5(2)} : 5_{-2, -4}, C_{5(3)} : 5_{2, 6},
\]
\[
C_{5(4)} : 5_{-3, -4}, C_{5(5)} : 5_{2, 1}.
\]  

The complete charged massless matter spectrum consists of these states together with the \( SU(5) \) singlets [12]. It is interesting to note that these charges can be accommodated in a local 2-2-1 split spectral cover model as constructed in [13] [15]. The same applies to top 7 in appendix A.

At the intersection of the matter curves the fibre structure changes due to the split of some of the \( P^1 \)-s and Yukawa interactions between the matter states are localised. The analysis of the \( SU(5) \) charged interactions results in the following interactions:

| Point | Yukawa coupling |
|------|-----------------|
| \( \{ w = b_1 = c_2, 1 = 0 \} \) | 10_{-1, 2} 10_{-1, 2} 5_{2, -4} |
| \( \{ w = b_1 = b_0, 2 = 0 \} \) | 10_{-1, 2} 5_{-2, -4} 5_{3, -1} |
| \( \{ w = b_1 = c_1 = 0 \} \) | 10_{-1, 2} 5_{3, 4} 5_{2, 6} |
| \( \{ w = b_1 = d_1 = 0 \} \) | non-flat fibre |

From the perspective of an \( SU(5) \) GUT model, the structure of Yukawa couplings implies \( 5_{H^*} = 5_{2, -4} \), while both \( (\bar{5}_m, 5_{H^*}) = (\bar{5}_{-2, -1}, 5_{3, -1}) \) or \( (\bar{5}_m, 5_{H^*}) = (\bar{5}_{3, 4}, 5_{-2, -6}) \) (and the other way round in each case) are possible. Extra Yukawa couplings exist between the charged singlets and the 5-matter curves at the intersection points of the singlet curves with the GUT divisor \( w = 0 \). The pattern we find is in perfect agreement with

\[2\] To distinguish between the 10- and 5-curve we would also need the next order in \( w \) of the discriminant, which we do not display here for brevity. Note that in the following we analyse the enhancement loci in the resolved Calabi-Yau, from which we observe the difference as well.
the $U(1)$ charge assignments and will be presented in detail in [9].

Note that the fibre over the points $\{w = b_1 = d_1 = 0\}$ becomes 2-dimensional. If these points are present, the fibration is non-flat. To be on the safe side we can always restrict ourselves to fibrations over base spaces $B_3$ where the set $\{w = b_1 = d_1 = 0\}$ is empty as a consequence of the intersection of the associated divisor classes. It is understood that the base $B_3$ has this property. This restriction is to be interpreted as a constraint on the base space to give rise to a well-defined F-theory compactification with the desired $SU(5) \times U(1) \times U(1)$ structure. The appearance of such a constraint is of course by no means unexpected and simply reflects the well-known phenomenon that given a specific brane configuration not every compactification space is automatically compatible with it.

$SU(5) \times U(1)$ via recombination

The $U(1) \times U(1)$-fibrations presented in this article lend themselves to studying brane recombination processes that Higgs the $U(1) \times U(1)$ to some linear combination. This way the $SU(5) \times U(1) \times U(1)$ models presented in this article give rise to a large class of $SU(5) \times U(1)$ models.

Independently of the non-abelian gauge symmetry, we can consider e.g. the breaking of $U(1) \times U(1)$ to the sum of both abelian factors by giving a VEV to the singlet 1.5.5 and its conjugate localised along the curve $C_{1(5)}$ in a D-flat manner. In brane language this corresponds to a recombination process that renders the fibration more generic. The analogous process interpolating from the models of [1] with one $U(1)$ group to fibrations without abelian gauge symmetry is well understood [12]. As it turns out, in the present situation a non-zero VEV for 1.5.5 plus conjugate corresponds to “switching on” the monomial $v^3$ in the cubic [1]. This leads to a $Bl^1\mathbb{P}^2[3]$-fibration.

Apart from breaking the abelian gauge group corresponding to the difference of the two $U(1)$ factors this leads to a recombination of the matter curves intersecting $C_{1(5)}$. This follows already from field theory in view of the presence of a corresponding Yukawa coupling involving 1.5.5. Therefore the two singlet curves $C_{1(4)}$ and $C_{1(5)}$ combine into a single one along which one type of singlets of combined diagonal $U(1)$ charge 10 lives.

In the presence of $SU(5)$ singularities the recombination also affects the $SU(5)$ charge matter sector. The curve $C_{1(4)}$ intersects the 5-matter curves $C_{5(1)}$ and $C_{5(2)}$ in the point $w = b_{0,2} = c_{2,1}$, where the Yukawa coupling 1.5.5 5 1.5 2.4 is localised. A VEV for $1^{(1)}$ thus leads to the recombination of $C_{5(1)}$ and $C_{5(2)}$ into a single curve along which a 5-state of diagonal charge −2 localises. All the remaining $SU(5)$ curves are unaffected, except that the states are charged only under the remaining diagonal $U(1)$. This structure of $SU(5)$ matter curves and their charges agrees with the corresponding data of the model in [7]. This is not unexpected since the top appearing in [7] agrees with the top in the main part of this article, apart from the additional monomial $\alpha v^3$ which is responsible for the brane recombination. There are seven tops for such a $Bl^1\mathbb{P}^2[3]$-fibration, see [9]. Furthermore, note that the additional section of the $Bl^1\mathbb{P}^2[3]$-fibration is not realised torically unlike the $Bl^1\mathbb{P}[1,1,2][4]$ case [21]. More details will be presented in [9].

IV. OUTLOOK

Based on the polygon in FIG. 1 we have analysed the 8 additional tops compatible with $SU(5)$ symmetry. The first six of these lead to a structure of one 10- and five 5-matter curves and corresponding Yukawa points similar to the pattern presented in the previous section, albeit each with a different $U(1) \times U(1)$ charge assignment. We collect the main results of these models in appendix A.

In [6] we have described, inspired by [5], $SU(5) \times U(1)$-fibrations based on the elliptic fibre $\mathbb{P}[1,1,2][4]$. This geometry likewise falls under the class of fibrations considered in [9]. In appendix B we present the analogous data for $SU(5)$ tops associated with this geometry.

In [6] we were most interested, for the phenomenological reasons described therein, in fibrations with several 10-curves, for which we have provided also an alternative description based on a factorised Tate model. As it turned out, the appearance of more than one 10-curve requires a deviation from purely toric methods as these assume the base sections to be generic apart from factoring out overall powers of the GUT divisor. We found [6] that an $SU(4)$ model with subsequent deformations yields models of the desired type. A similar analysis for the $Bl^2\mathbb{P}[3]$-fibrations of this paper will be presented in [9].

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Appendix A: $SU(5) \times U(1) \times U(1)$ fibrations

In this appendix we briefly summarise the defining data and the resulting spectra for the remaining $SU(5)$ tops for the $Bl^2\mathbb{P}[3]$-fibration. We present here the proper transformed hypersurfaces of the resolved 4-fold $Y_4$ describing the $SU(5) \times U(1) \times U(1)$ F-theory model as well as the $SU(5)$ matter spectrum and the Yukawa
points. We also display where the fibration becomes non-flat in codimension-three. It is assumed that the base space $B_3$ chosen for the fibration does not allow these intersection loci such that the fibration remains flat everywhere. More details and derivations will appear in [9]. Note that in addition to the six classes of models presented in the sequel there are two more possible tops, which however exhibit unconventional enhancement loci so that we do not display them here.

**Top 2**

Proper transform:

\[ 0 = b_{0,3}c_1^2c_2e_1e_4u^2v^2s_2^2 + d_{0,2}c_1e_1c_2e_4u^2v^2s_3^2 + d_{2,1}c_1e_0c_2u^2v^2s_3^2 + c_{2,1}c_1e_0e_3v^2w_3 + b_{1,1}u^v w s_0 s_1 + d_1 e_1e_2c_3^2w_3s_0^2 + c_1 e_2c_3^2v^2w_3s_1 + b_{2,1}e_2c_3^2e_4^2u^2w^2s_1^2. \]

Matter curves and Yukawa points:

| Curve on $\{w = 0\}$ | Matter representation |
|------------------------|-----------------------|
| $\{b_1 = b_0, 2 = 0\}$ | $10_{-1, -2} + 10_{1, 2}$ |
| $\{c_2, 1 = 0\}$      | $5_{-3, 4} + 5_{3, -4}$ |
| $\{c_1 = 0\}$         | $5_{-3, -6} + 5_{3, 6}$ |
| $\{d_{2, 1} = 0\}$    | $5_{-3, -1} + 5_{3, 1}$ |
| $\{b_1b_2 - c_1d_1 = 0\}$ | $5_{2, 4} + 5_{2, -4}$ |
| $\{b_0, 3b_2^2 + c_{2, 1}\}$ | $5_{2, -1} + 5_{2, -1}$ |
| $\{b_0, 3b_1 - c_2d_{0, 1} = 0\}$ | non-flat fibre |

| Point on $\{w = 0\}$ | Yukawa coupling |
|------------------------|-----------------|
| $\{b_1 = c_1 = 0\}$  | $10_{1, 2} 10_{2, 4} 5_{-3, -6}$ |
| $\{b_1 = d_{1, 0} = 0\}$ | $10_{-1, -2} 10_{1, 0} 10_{2, 4}$ |
| $\{b_1 = d_{2, 1} = 0\}$ | non-flat fibre |
| $\{b_1 = c_{2, 1} = 0\}$ | non-flat fibre |

**Top 3**

Proper transform:

\[ 0 = b_{0,3}c_1c_2c_3^2u^2v^2s_0^2 + d_{0,1}c_1e_0e_3c_2u^2v^2s_0^2 + c_1e_3c_4v^2w^2s_1 + c_2, 2c_1e_1c_3c_2^2e_3^2v^2w^2s_0^2 + b_{1,1}u^v w s_0 s_1 + d_{1,1}c_1c_2c_3^2e_3^2w_0s_3^2 + c_{2,1}c_1e_0e_2v^2w_3s_0^2 + d_{2,1}c_0c_2c_3^2u^2w_3s_0^2 + b_{2,1}c_1c_2c_3^2e_4u^2w^2s_1^2. \]

Matter curves and Yukawa points:

| Curve on $\{w = 0\}$ | Matter representation |
|------------------------|-----------------------|
| $\{b_1 = b_0, 2 = 0\}$ | $10_{1, 1} + 10_{1, 0}$ |
| $\{b_1 = d_{0, 1} = 0\}$ | $5_{-3, -1} + 5_{3, 1}$ |
| $\{b_1 = d_{1, 1} = 0\}$ | $5_{2, -1} + 5_{2, -1}$ |
| $\{b_1, b_2c_1 - c_1b_2 = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |
| $\{c_1d_1 - b_1b_2 = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |
| $\{b_0, b_2c_1 - c_1b_2 = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |
| $\{b_1, c_1c_2d_{1, 0} = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |
| $\{b_1, c_1c_2d_{1, 1} = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |
| $\{b_1, c_1 = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |

**Top 4**

Proper transform:

\[ 0 = b_{0,3}c_1c_2c_3^2e_4u^2v^2s_0^2 + d_{0,1}c_1e_0e_3c_2u^2v^2s_0^2 + b_{1,1}u^v w s_0 s_1 + c_2, 2c_1c_3e_4v^2w_3s_0^2 + d_{2,1}c_0c_1c_3c_2^2e_3^2u^2w_3s_0^2 + d_{1,1}c_1c_3e_4v^2w_3s_0^2 + c_{2,1}c_1e_3c_4v^2w^2s_1 + b_{2,1}c_1c_2c_3^2e_4u^2w^2s_1^2. \]

Matter curves and Yukawa points:

| Curve on $\{w = 0\}$ | Matter representation |
|------------------------|-----------------------|
| $\{b_1 = b_0, 2 = 0\}$ | $10_{1, 1} + 10_{1, 0}$ |
| $\{b_1 = d_{0, 1} = 0\}$ | $5_{-3, -1} + 5_{3, 1}$ |
| $\{b_1 = d_{1, 1} = 0\}$ | $5_{2, -1} + 5_{2, -1}$ |
| $\{b_1, b_2c_1 - c_1b_2 = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |
| $\{b_1, c_1c_2d_{1, 0} = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |
| $\{b_1, c_1c_2d_{1, 1} = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |
| $\{b_1, c_1 = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |

**Top 5**

Proper transform:

\[ 0 = b_{0,3}c_1c_2c_3e_4u^2v^2s_0^2 + d_{0,1}c_1e_0e_3c_2u^2v^2s_0^2 + b_{1,1}u^v w s_0 s_1 + c_2, 2c_1c_3e_4v^2w_3s_0^2 + d_{2,1}c_0c_1c_3c_2^2e_3^2u^2w_3s_0^2 + b_{2,1}c_1c_2c_3^2e_4u^2w^2s_1^2. \]

Matter curves and Yukawa points:

| Curve on $\{w = 0\}$ | Matter representation |
|------------------------|-----------------------|
| $\{b_1 = b_0, 2 = 0\}$ | $10_{1, 0} + 10_{1, 0}$ |
| $\{b_1 = d_{0, 1} = 0\}$ | $5_{-3, -1} + 5_{3, 1}$ |
| $\{b_1 = d_{1, 1} = 0\}$ | $5_{2, -1} + 5_{2, -1}$ |
| $\{b_1, b_2c_1 - c_1b_2 = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |
| $\{b_1, c_1c_2d_{1, 0} = 0\}$ | $5_{-3, -1} + 5_{3, 1}$ |
| $\{b_1, c_1c_2d_{1, 1} = 0\}$ | $5_{2, -1} + 5_{2, -1}$ |
| $\{b_1, c_1 = 0\}$ | $5_{2, -2} + 5_{2, -2}$ |

Point on $\{w = 0\}$ Yukawa coupling

| $\{b_1 = b_0, 2 = 0\}$ | $10_{1, 0} 10_{1, 0}$ |
| $\{b_1 = d_{0, 1} = 0\}$ | $10_{1, 0} 10_{1, 0}$ |
| $\{b_1 = d_{1, 1} = 0\}$ | $10_{1, 0} 10_{1, 0}$ |
| $\{b_1, c_1 = 0\}$ | $10_{1, 0} 10_{1, 0}$ |

non-flat fibre
Matter curves and Yukawa points:

\[
\begin{align*}
\text{Curve on } \{w = 0\} & \quad \text{Matter representation} \\
\{b_1 = 0\} & \quad 10_{2,2} + 10_{2,-2} \\
\{b_{0,1} = 0\} & \quad 5_{-4,1} + 5_{4,-1} \\
\{b_2 = 0\} & \quad 5_{-4,-4} + 5_{4,4} \\
\{c_{1,1} = 0\} & \quad 5_{1,6} + 5_{-1,-6} \\
\{b_{0,1}c_{1,1} - b_1c_{2,2} = 0\} & \quad 5_{1,-4} + 5_{-1,4} \\
\{b_{0,1}d_1^2 - b_1d_0d_1 + b_1^2d_2, 1 = 0\} & \quad 5_{1,1} + 5_{-1,-1} \\
\end{align*}
\]

Point on \(\{w = 0\}\) | Yukawa coupling
\[
\begin{align*}
\{b_1 = b_2 = 0\} & \quad 10_{2,2} + 10_{2,-2} \\
\{b_1 = c_{1,1} = 0\} & \quad 10_{2,5} - 10_{5,1} + 5_{-1,-6} \\
\{b_1 = b_{0,1} = 0\} & \quad \text{non-flat fibre} \\
\end{align*}
\]

Appendix B: SU(5) \(\times\) U(1) from BI\[3\]P[1,1,2][4]

We now display the main results for the construction of SU(5) \(\times\) U(1) F-theory compactifications based on a BI[3]P[1,1,2][4]-fibration. This is a direct continuation of our approach presented in [6], to which we refer for details of the general setup. While in [6] we were interested in SU(4) singularities degenerating further to SU(5), since these can accommodate multiple 10-curves, here we summarise the result of a direct implementation of SU(5) singularities via the tops construction. For four tops we give the hypersurface of the resolved Calabi-Yau 4-fold \(\hat{\mathcal{Y}}_4\) as well as the SU(5)-matter spectrum and Yukawa points. The homogeneous coordinates of \(\mathbb{P}[1,1,2]\) are \(\{u : v : w\}\) and \(s_1\) denotes the rational section, cf. FIG. 4. In addition, we list again the loci where the fibration becomes non-flat in codimension-three and point out that to obtain a suitable F-theory compactification one chooses the base \(B_3\) such that these loci are absent. Note that there is a fifth possible top, which we have not included due to non-standard behaviour at the enhancement loci. We refer to [9] for more details.
Top 2

Proper transform:

\[
0 = w^2 s_1 e_3 e_4 + b_{1,2} w u^2 s_1^2 c_1 e_3 e_4 + b_1 u v w s_1 \\
+ b_2 v^2 w e_1 e_2 e_3 - c_{0,4} u^4 s_1^4 c_1^2 e_3 e_4 \\
- c_{1,2} u^3 v s_1^2 e_1 e_4 - c_{2,1} u^2 v^2 s_1 e_0 e_1 e_2 \\
- c_{3,1} u v^3 e_0 e_1^2 e_2 e_3
\]

Matter curves and Yukawa points:

| Curve on \{w = 0\} | Matter representation |
|---------------------|-----------------------|
| \{b_1 = 0\}        | \(10_{-3} + 10_{-2}\) |
| \{b_2 = 0\}        | \(5_{6} + 5_{-6}\)    |
| \{b_1 c_{3,1} + b_2 c_{2,1} = 0\} | \(5_{-4} + 5_{4}\)    |
| \{b_1^2 c_{0,4} - b_{1,2} b_1 c_{1,2} - c_{1,2}^2 = 0\} | \(5_{1} + 5_{-1}\)    |

Point on \{w = 0\} | Yukawa coupling |
|------------------|----------------|
| \{b_1 = b_2 = 0\} | \(10_{-3} 5_{6}\) |
| \{b_1 = c_{3,1} = 0\} | \(10_{-3} 5_{-5} 5_{4}\) |
| \{b_1 = b_{0,1} = 0\} | non-flat fibre|

Top 3

Proper transform:

\[
0 = w^2 s_1 e_2^2 e_3 e_4 + b_{0,1} w u^2 s_1^2 c_0 e_3 e_4 + b_1 u v w s_1 \\
+ b_2 v^2 w e_1 e_2 e_3 - c_{0,3} u^4 s_1^3 c_0^2 e_3 e_4 \\
- c_{1,2} u^3 v s_1^2 e_0^2 e_4 - c_{2,1} u^2 v^2 s_1 e_0 e_1^2 e_2 \\
- c_{3,2} u v^3 e_0 e_1^2 e_2 e_3
\]

Matter curves and Yukawa points:

| Curve on \{w = 0\} | Matter representation |
|---------------------|-----------------------|
| \{b_1 = 0\}        | \(10_{-1} + 10_{1}\) |
| \{b_2 = 0\}        | \(5_{7} + 5_{-7}\)    |
| \{b_1 c_{0,3} - b_{0,1} c_{1,2} = 0\} | \(5_{2} + 5_{-2}\)    |
| \{b_2^2 c_{1,2} - b_1 b_{2,1} + b_2^2 c_{3,2} = 0\} | \(5_{-3} + 5_{3}\)    |

Point on \{w = 0\} | Yukawa coupling |
|------------------|----------------|
| \{b_1 = b_{0,1} = 0\} | \(10_{-3} 5_{-2}\) |
| \{b_1 = c_{1,2} = 0\} | \(10_{-3} 5_{-3} 5_{2}\) |
| \{b_1 = b_2 = 0\} | non-flat fibre|

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