A SUSTAINABILITY CONDITION FOR STOCHASTIC FOREST MODEL

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Abstract. A stochastic forest model of young and old age class trees is studied. First, we prove existence, uniqueness and boundedness of global nonnegative solutions. Second, we investigate asymptotic behavior of solutions by giving a sufficient condition for sustainability of the forest. Under this condition, we show existence of a Borel invariant measure. Third, we present several sufficient conditions for decline of the forest. Finally, we give some numerical examples.

1. Introduction. In 1975, Antonovsky [1] introduced a mono-species forest model with two age classes of trees:

\[
\begin{align*}
\frac{du}{dt} &= \rho v - \gamma(v)u - fu, \\
\frac{dv}{dt} &= fu - hv.
\end{align*}
\]  

Here, \(u\) and \(v\) denote the tree densities of young and old age classes, respectively. The parameters \(\rho, h\) and \(f\) are reproduction rate, mortality of old trees, and aging rate of young trees, respectively; while \(\gamma(v) = a(v - b)^2 + c\) is a mortality of young trees, which is allowed to depend on the old-tree density. In addition, \(a, b, c, \rho, f\) and \(h\) are assumed to be positive constants.

It is not difficult to see that for any pair \((u_0, v_0)\) of nonnegative initial values \(u_0\) and \(v_0\), the system (1) possesses a nonnegative and global solution. Furthermore, (1) possesses nonnegative stationary solutions given by

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1. \( O = (0,0) \)

2. \( P^- = \left( \frac{h}{f}(b - \sqrt{\frac{p_f - h(c + f)}{ah}}), b - \sqrt{\frac{p_f - h(c + f)}{ah}} \right) \) (if \( h \in [h_*, h^*] \))

3. \( P^+ = \left( \frac{h}{f}(b + \sqrt{\frac{p_f - h(c + f)}{ah}}), b + \sqrt{\frac{p_f - h(c + f)}{ah}} \right) \) (if \( h \in (0, h^*) \))

where \( h_* = \frac{\rho f}{ab^2 + c + f}, h^* = \frac{\rho f}{c + f} \). The stability and instability of these solutions depends strongly on the magnitude of the mortality \( h \) of old age class trees (see Table 1).

| \( h \) | \((0, h_*)\) | \((h_*, h^*)\) | \((h^*, \infty)\) |
|-------|-------------|-------------|-------------|
| \( O \) | unstable | stable | glob. asymp. stable |
| \( P^+ \) | stable | stable | — |
| \( P^- \) | — | unstable | — |

**Table 1.** Stability and instability of stationary solutions of (1)

On the basis of (1), Kuznetsov et al. [11] introduced a mathematical model of mono-species forest with two age classes which takes into account the seed production and dispersion. The third author studied that model with his colleagues (see, e.g., [3, 4, 5, 15] and [17, Chapter 11]). It is shown that \( h \) plays a crucial role in the asymptotic behavior of solutions.

In the real world, the parameters in the model may be random variables due to unpredictability resulting from environmental, ecological and biological disturbances. In principle, the deterministic forest model can not handle randomness. Investigating the role of fluctuation of parameters by using stochastic models should be an interesting problem in environmental and ecological sciences.

As mentioned above, the asymptotic behavior of solutions to the deterministic forest model depends strongly on the magnitude of \( h \). Therefore, in this paper we restrict ourselves to consider a stochastic forest model, where \( h \) is perturbed by (Gaussian) white noise. Since Gaussian white noises can be expressed as the generalized derivative of a Brownian motion, we make a substitution:

\[ h \sim h - \sigma dw_t \]

in (1), where \( \{w_t, t \geq 0\} \) is a one-dimensional Brownian motion defined on a filtered complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \), and \( \sigma > 0 \) is the intensity of the white noise. Our stochastic forest model is then formulated by Itô stochastic differential equations in \( \mathbb{R}^2 \):

\[
\begin{align*}
    du_t &= \left( \rho v_t - [a(v_t - b)^2 + c + f]u_t \right) dt, \\
    dv_t &= (fu_t - hv_t)dt + \sigma v_t dw_t. \\
\end{align*}
\]

(2)

In this paper, we study the stochastic forest model (2). We prove existence of unique global solutions to (2) and then study their asymptotic behavior. On one hand, we present a sufficient condition for sustainability of the forest. Under this condition, we also prove existence of a non-trivial Borel invariant measure. On the other hand, we give several sufficient conditions for decline of the forest. The results are illustrated by a few numerical examples.
To prove existence of non-trivial invariant measures to (2), a common method is to find four one-dimensional processes, namely $u^1, u^2, v^1$ and $v^2$, which satisfy two conditions:

(i) $u_t$ and $v_t$ are bounded by these processes, i.e. $u^1(t) < u_t < u^2(t)$ and $v^1(t) < v_t < v^2(t)$ for $0 < t < \infty$

(ii) These four processes do not hit the boundaries in the sense that there exist $\epsilon > 0$ and $M > 0$ such that

$$\left\{ \begin{array}{l}
\epsilon < u^1(t) < u^2(t) < M & \text{a.s., } 0 < t < \infty, \\
\epsilon < v^1(t) < v^2(t) < M & \text{a.s., } 0 < t < \infty
\end{array} \right.$$  

However, this cannot be done because

$$\lim_{t \to \infty} v_t = 0 \quad \text{a.s. (see Theorem 2.3)}.$$ 

To overcome this difficulty, we use the semigroup method presented in [6, 13]. First, we establish some estimates for the average of integrals of solutions (see (19) and (20)). Then, we construct a strongly continuous semigroup generated by solutions of (2). Using these estimates and a theorem in [13], we show that the semigroup enjoys an invariant measure.

The organization of the present paper is as follows. Section 2 proves existence and boundedness of unique global nonnegative solutions to (2). Section 3 investigates sustainability of the forest and existence of a Borel invariant measure. To the contrary, Section 4 presents some sufficient conditions for decline of the forest. Finally, Section 5 gives some numerical examples.

2. Global solutions. In this section, we prove existence of unique global nonnegative solutions to (2) and show boundedness of solutions.

Put $\mathbb{R}^2_+ = \{(u, v); u > 0, v > 0\}$. Denote by $\overline{\mathbb{R}^2_+}$ the closure of $\mathbb{R}^2_+$ in $\mathbb{R}^2$. Put

$$M_0 = \inf \{ u; \rho v - [\gamma(v) + f]u < 0 \quad \text{for every } v > 0 \}.$$ 

Then,

$$M_0 = \inf \left\{ u; \frac{\rho v}{\gamma(v) + f} - u < 0 \quad \text{for every } v > 0 \right\}$$

$$= \sup_{v > 0} \frac{\rho v}{\gamma(v) + f} \left( \frac{\rho \sqrt{ab^2 + c + f}}{\sqrt{a} \left( \sqrt{ab^2 + c + f} - \sqrt{ab} \right) + f} \right).$$ (3)

For biological reasons, throughout this paper, initial values for (2) are taken from $\overline{\mathbb{R}^2_+}$.

Let us first prove existence of unique global nonnegative solutions to (2). We use the following lemma.

**Lemma 2.1.** Consider the one-dimensional stochastic differential equation:

$$\begin{cases}
    dx_t = (a_2 - a_1 x_t)dt + \alpha x_t dw_t, \\
    x_t|_{t=0} = x_0 > 0,
\end{cases}$$  

where $a_1, a_2$ and $\alpha$ are positive constants. Then, there exists a unique global solution of (4) such that
Theorem 2.2. Let
\[ \log(a_3 + x_t) \]
with the convention \( \inf \) there exists a limit of this sequence, namely \( \tau \) then
\[ \bigcup \]
Since all the functions on the right-hand side of (2) are locally Lipschitz continuous, there is a unique local solution \( (u_t, v_t) \) of (2) such that \( \tau\theta \leq \alpha_{\theta,T} \), \( 0 \leq t \leq T \).

In addition, if \( 1 \leq \theta < 1 + \frac{2\alpha}{\theta} \), then \( \alpha_{\theta,T} \) is independent of \( T \).

(iii) \( \limsup_{t \to \infty} x_t = \infty \) and \( \liminf_{t \to \infty} x_t = 0 \), a.s.

Since the proof of the lemma is quite easy, we omit it.

Theorem 2.2. Let \( (u_0, v_0) \in \mathbb{R}_+^2 \). Then, there exists a unique global solution \( (u_t, v_t) \) of (2) such that \( (u_t, v_t)|_{t=0} = (u_0, v_0) \) and
\[ (u_t, v_t) \in \mathbb{R}_+^2 \] a.s., \( 0 < t < \infty \).

In addition, if \( u_0 + v_0 > 0 \), then
\[ (u_t, v_t) \in \mathbb{R}_+^2 \] a.s., \( 0 < t < \infty \).

Proof. Since all the functions on the right-hand side of (2) are locally Lipschitz continuous, there is a unique local solution \( (u_t, v_t) \) defined on an interval \( [0, \tau) \), where \( \tau \) is a stopping time having the following property (see, e.g., [2, 7]). If \( P\{\tau < \infty\} > 0 \), then \( \tau \) is an explosion time on \( \{\tau < \infty\} \), i.e.
\[ \lim_{t \to \tau} (|u_t| + |v_t|) = \infty \] a.s. on \( \{\tau < \infty\} \).

Therefore, it suffices to show that \( \tau = \infty \) a.s. and that \( u_t \geq 0, v_t \geq 0 \) a.s. for \( 0 < t < \infty \). To prove this, we use the method in [14, 16].

Consider the four cases of initial values.

Case 1: \( u_0 = v_0 = 0 \). This is a trivial case, since \( u_t = v_t = 0 \) a.s. for \( 0 \leq t < \infty \).

Case 2: \( (u_0, v_0) \in \mathbb{R}_+^2 \).

Let \( k_0 > 0 \) be a positive integer such that \( u_0 \) and \( v_0 \) lie in the interval \( [\frac{1}{k_0}, k_0] \).

Denote
\[ H_k = \left[ \frac{1}{k}, k \right] \times \left[ \frac{1}{k}, k \right], \quad k = 1, 2, \ldots, \]
then \( \cup_{k=k_0}^\infty H_k = \mathbb{R}_+^2 \). Let us define a sequence \( \{\tau_k\}_{k=k_0}^\infty \) of stopping times by
\[ \tau_k = \inf \{0 < t < \tau; (u_t, v_t) \notin H_k\} \] (5)
with the convention \( \inf \emptyset = \infty \). It is obvious that \( \{\tau_k\}_{k=k_0}^\infty \) is nondecreasing. Hence, there exists a limit of this sequence, namely \( \tau_\infty \), as \( k \to \infty \):
\[ \tau_\infty = \lim_{k \to \infty} \tau_k \leq \tau \] a.s.

Let us prove that \( \tau_\infty = \infty \) a.s. Indeed, suppose the contrary, then there would exist \( T > 0 \) and \( 0 < \epsilon < 1 \) such that
\[ P\{\tau_\infty < T\} > \epsilon. \] (6)

Consider a positive function \( V \) on \( \mathbb{R}_+^2 \), which is defined by
\[ V(u, v) = u^2 + v^2 - \log u - \log v, \quad u > 0, v > 0. \]

The Itô formula gives
\[ dV(u_t, v_t) = [LV](u_t, v_t) dt + (2\sigma v^2 - \sigma) dw_t, \]
where the infinitesimal operator $L$ is given by

$$[L](u, v) = \frac{1}{2} \sigma^2 v^2 \frac{\partial^2}{\partial v^2} + \left[ \rho \left( \gamma(v) + f \right) u \right] \frac{\partial}{\partial u} + \left( f u - h v \right) \frac{\partial}{\partial v}. \quad (7)$$

It is possibly seen that

$$[LV](u, v) = 2(\rho + f)uv - 2(\gamma(v) + f)u^2 + (\sigma^2 - 2h)v^2 + \gamma(v)$$

$$- \frac{\rho v}{u} - \frac{f u}{v} + f + \frac{\sigma^2}{2}.$$ 

In addition, there exist $M_i > 0$ ($i = 1, 2$) such that

$$[LV](u, v) < M_1V(u, v) + M_2, \quad (u, v) \in \mathbb{R}_+^2.$$ 

Therefore,

$$\int_0^{t \wedge \tau_k} dV(u_s, v_s) \leq \int_0^{t \wedge \tau_k} M_1V(u_s, v_s) + M_2 |ds + \int_0^{t \wedge \tau_k} (2\sigma v^2 - \sigma)dw_s.$$ 

Taking expectation in the two sides of the latter inequality, we obtain that

$$\mathbb{E}V(u_{t \wedge \tau_k}, v_{t \wedge \tau_k}) \leq V(u_0, v_0) + M_2(t \wedge \tau_k) + M_1 \mathbb{E} \int_0^{t \wedge \tau_k} V(u_s, v_s)ds$$

$$\leq [V(u_0, v_0) + M_2T] + M_1 \mathbb{E} \int_0^t \mathbb{E}V(u_{s \wedge \tau_k}, v_{s \wedge \tau_k})ds, \quad 0 \leq t \leq T.$$ 

The Gronwall inequality then provides that

$$\mathbb{E}V(u_{t \wedge \tau_k}, v_{t \wedge \tau_k}) \leq [V(u_0, v_0) + M_2T]e^{M_1t}$$

$$\leq [V(u_0, v_0) + M_2T]e^{M_1T}, \quad 0 \leq t \leq T. \quad (8)$$

Hence,

$$[V(u_0, v_0) + M_2T]e^{M_1T} \geq \mathbb{E}V(u_{T \wedge \tau_k}, v_{T \wedge \tau_k})$$

$$\geq \mathbb{E}[1_{\{\tau_k < T\}}V(u_{T \wedge \tau_k}, v_{T \wedge \tau_k})]$$

$$= \mathbb{E}[1_{\{\tau_k < T\}}V(u_{\tau_k}, v_{\tau_k})]. \quad (9)$$

On the other hand, (5) gives

$$V(u_{\tau_k}, v_{\tau_k}) \geq \min \left\{ k^2 - \log k, \left( \frac{1}{k} \right)^2 - \log \left( \frac{1}{k} \right) \right\}$$

$$= \min \left\{ k^2 - \log k, \log k + \frac{1}{k^2} \right\} \quad \text{on} \{ \tau_k < \infty \}. \quad (10)$$

Thanks to (6), (9) and (10), we observe that

$$[V(u_0, v_0) + M_2T]e^{M_1T} \geq \epsilon \min \left\{ k^2 - \log k, \log k + \frac{1}{k^2} \right\}.$$ 

Letting $k \to \infty$, we arrive at a contradiction:

$$\infty > [V(u_0, v_0) + M_2T]e^{M_1T} = \infty.$$ 

Thus, $\tau_\infty = T = \infty$ a.s. Furthermore, $(u_t, v_t) \in \bigcup_{k=0}^\infty H_k = \mathbb{R}_+^2$ a.s. for $0 \leq t < \infty$.

Case 3: $u_0 > 0$, $v_0 = 0$.

Let $k_0 > 0$ be a positive integer such that $u_0$ lies in $[\frac{1}{k_0}, k_0]$. Denote

$$H^1_k = \left[ \frac{1}{k}, k \right]$$

and

$$\tau_k = \inf \left\{ 0 \leq t < \tau; u_t \notin H^1_k \right\}, \quad k = k_0, k_0 + 1, \ldots$$
Clearly, the sequence \( \{ \tau_k \}_{k=k_0}^\infty \) has a limit \( \tau_\infty \) as \( k \to \infty \):

\[
\tau_\infty = \lim_{k \to \infty} \tau_k \leq \tau \quad \text{a.s.}
\]

Let us first show that

\[
v_t > 0 \quad \text{a.s., } 0 < t \leq \tau_k.
\]

Indeed, due to the comparison theorem (see [8]), \( v_t > \bar{v}_t \) a.s. for \( 0 < t \leq \tau_k \), where \( \bar{v}_t \) is the solution of this equation:

\[
\begin{aligned}
d\bar{v}_t &= -h\bar{v}_t dt + \sigma\bar{v}_tdw_t, \\
\bar{v}_t|_{t=0} &= v_0 = 0.
\end{aligned}
\]

Obviously, \( \bar{v}_t = 0 \). Thus, (11) follows.

Let us now verify that there exists \( \alpha > 0 \) such that \( E[v_t \wedge \tau_k] \leq \alpha \) for \( 0 \leq t < \infty \).

In addition, for any \( T > 0 \), there exists \( \beta > 0 \) such that \( E[\gamma(v_t \wedge \tau_k)] \leq \beta \) for \( 0 \leq t \leq T \).

Indeed, by virtue of the first equation of (2),

\[
\frac{du}{dt} = [\gamma(v) + f] \left( \frac{\rho v}{\gamma(v) + f} - u \right) \leq [\gamma(v) + f](M_0 - u), \quad 0 \leq t < \tau,
\]

where \( M_0 \) is defined in (3). Solving this differential inequality, we obtain that

\[
u_t \leq M_0 - (M_0 - u_0)e^{-\int_0^t [\gamma(v) + f] ds}, \quad 0 \leq t < \tau.
\]

Hence,

\[
u_t \leq \max\{u_0, M_0\} \quad \text{a.s., } 0 \leq t < \tau.
\]

Using (14) and applying the comparison theorem for the second equation of (2), we observe that

\[
v_t \leq \bar{v}_t \quad \text{a.s., } 0 \leq t < \tau,
\]

where \( \bar{v}_t \) is the solution of the one-dimensional stochastic differential equation:

\[
\begin{aligned}
d\bar{v}_t &= \left[ \max\{u_0, M_0\} - hu_t \right] dt + \sigma\bar{v}_tdw_t, \\
\bar{v}_t|_{t=0} &= v_0.
\end{aligned}
\]

Thanks to Lemma 2.1–(ii) and (15), it is easily seen that (12) holds true.

Let us finally observe that

\[
\tau = \infty \quad \text{and} \quad (u_t, v_t) \in \mathbb{R}_+^2 \quad \text{a.s., } 0 < t < \infty.
\]

In view of (11), it suffices to show that \( \tau_\infty = \infty \). Indeed, suppose the contrary, then there would exist \( T > 0 \) and \( 0 < \epsilon < 1 \) such that

\[
P\{\tau_\infty < T\} > \epsilon.
\]

Consider a positive function \( H \) on \( (0, \infty) \) defined by

\[
H(u) = u^2 - \log u, \quad u > 0.
\]

The Itô formula then gives

\[
\int_0^{t \wedge \tau_k} dH(u_s) = \int_0^{t \wedge \tau_k} [2pu_s v_s - 2u_s^2 \gamma(v_s) - \frac{\rho u_s}{u_s} + \gamma(v_s)] ds
\]

\[
\leq \int_0^{t \wedge \tau_k} [2pu_s v_s \wedge \tau_k + \gamma(v_s \wedge \tau_k)] ds.
\]
Taking expectation in the two sides of this inequality and using (12) and (14), we observe that

\[ E[H(u_s \wedge \tau_k)] \leq \int_0^t (2\rho \max\{u_0, M_0\} \alpha + \beta)ds \]

\[ \leq (2\rho \max\{u_0, M_0\} \alpha + \beta)T, \quad 0 \leq t \leq T. \quad (17) \]

By using (17) instead of (8), we repeat the same argument as in Case 2 to conclude that \( \tau_\infty = \infty \) a.s.

**Case 4:** \( u_0 = 0, v_0 > 0 \). The proof for this case is similar to one for Case 3.

By the above arguments, the proof of the theorem is complete.

Let us now show boundedness for the density \( u \) of young age class trees and for moments of the density \( v \) of old age class trees.

**Theorem 2.3.** Let \((u_t, v_t)\) be a solution of (2) such that \((u_t, v_t)|_{t=0} = (u_0, v_0) \in \mathbb{R}^2_+\). Then,

(i) \( \sup_{0 \leq t < \infty} u_t \leq M^* \) a.s., where \( M^* = \max\{u_0, M_0\} \) and \( M_0 \) is defined in (3)

(ii) \( \limsup_{t \to \infty} u_t \leq M_0 \) a.s.

(iii) For any \( 1 \leq \theta < 1 + \frac{2h}{\sigma^2} \), there exists \( \alpha_\theta > 0 \) such that

\[ \limsup_{t \to \infty} \mathbb{E}[v_{\theta t}] \leq \alpha_\theta \]

(iv) \( \liminf_{t \to \infty} v_t = 0 \) a.s.

**Proof.** Clearly, (i) and (ii) follow from (13) and (14). Meanwhile, applying Lemma 2.1 to the equation (16) and using the fact that \( v_t \leq \bar{v}_t \), (iii) and (iv) follow. \[ \square \]

3. **Sustainability of forest.** In this section, we present a sufficient condition for sustainability of the forest. Under this condition, we also show existence of a Borel invariant measure on \( \mathbb{R}^2_+ \) for the system (2).

3.1. **Sustainability condition.** Let us show that if the intensity of noise and mortality of old age class trees are small enough, then the forest is sustainable.

**Definition 3.1.** The system (2) is said to be sustainable if for every initial value \((u_0, v_0) \in \mathbb{R}^2_+ \setminus \{(0,0)\}\), the solution \((u_t, v_t)\) satisfies

\[ \limsup_{t \to \infty} \mathbb{E}[u_t] > 0 \quad \text{and} \quad \limsup_{t \to \infty} \mathbb{E}[v_t] > 0. \]

**Theorem 3.2.** Assume that

\[ h < \frac{\rho f}{ab^2 + c + f} \quad \text{and} \quad \sigma^2 < 2(\frac{\rho f}{ab^2 + c + f} - h) \quad (18) \]

and \((u_0, v_0) \in \mathbb{R}^2_+ \setminus \{(0,0)\}\). Let \((u_t, v_t)\) be the solution of (2) with \((u_t, v_t)|_{t=0} = (u_0, v_0)\). Then, there exists \( \epsilon > 0 \) which is independent of \((u_0, v_0)\) such that

\[ \liminf_{t \to \infty} \frac{1}{t} \int_0^t v_s ds > \epsilon, \quad \liminf_{t \to \infty} \frac{1}{t} \int_0^t v_s^2 ds > \epsilon \quad \text{a.s.} \quad (19) \]

and

\[ \liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}[u_s] ds > \epsilon, \quad \liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}[v_s] ds > \epsilon. \quad (20) \]

As a consequence, (2) is sustainable.
Indeed, (24) is equivalent to that

\[ \begin{cases} ab^2 + c + f < \kappa < \rho, \\ \sigma^2 < \frac{2(\rho - \kappa h)}{\kappa}. \end{cases} \]

Consider a function \( Q \) on \( \mathbb{R}^2_+ \backslash \{(0,0)\} \) defined by

\[ Q(u,v) = \log(u + \kappa v). \]

Theorem 2.2 and the Itô formula then provide that

\[ dQ(u_t,v_t) = [LQ](u_t,v_t)dt + \frac{\sigma \kappa v_t}{u_t + \kappa v_t} dw_t, \quad (21) \]

where the operator \( L \) is defined in (7). After some simple calculations, we obtain that

\[ [LQ](u,v) = \frac{(\kappa f - c - f)u + (\rho - \kappa h)v}{u + \kappa v} - \frac{\sigma^2 \kappa^2 v^2}{2(u + \kappa v)^2} - \frac{av(u - b)^2}{2(u + \kappa v)}. \quad (22) \]

Thereby, by using the estimate (i) of Theorem 2.3, we observe that

\begin{align*}
\log(u_t + \kappa v_t) &= \log(u_0 + \kappa v_0) - \int_0^t \frac{av^2 u_s - 2abu_s v_s}{u_s + \kappa v_s} ds + \int_0^t \frac{\sigma \kappa v_s}{u_s + \kappa v_s} dw_s \\
&\quad + \int_0^t \left[ \frac{(\kappa f - c - f - ab^2) u_s + (\rho - \kappa h)v_s}{u_s + \kappa v_s} - \frac{\sigma^2 \kappa^2 v^2_s}{2(u_s + \kappa v_s)^2} \right] ds \\
&\quad \geq \log(u_0 + \kappa v_0) - aM^* \int_0^t \frac{v^2}{M^* + \kappa v_s} ds + \int_0^t \frac{\sigma \kappa v}{u_s + \kappa v_s} dw_s \\
&\quad + \int_0^t \left[ \frac{(\kappa f - c - f - ab^2) u_s + (\rho - \kappa h)v_s}{u_s + \kappa v_s} - \frac{\sigma^2 \kappa^2 v^2_s}{2(u_s + \kappa v_s)^2} \right] ds. \quad (23)
\end{align*}

Let us show that there exists \( \varepsilon_1 > 0 \) such that for all \( (u,v) \in \mathbb{R}^2_+ \),

\[ \frac{(\kappa f - c - f - ab^2) u + (\rho - \kappa h)v}{u + \kappa v} - \frac{\sigma^2 \kappa^2 v^2}{2(u + \kappa v)^2} \geq \frac{\varepsilon_1}{2}. \quad (24) \]

Indeed, (24) is equivalent to that \( F(u,v) \geq 0 \) for all \( (u,v) \in \mathbb{R}^2_+ \), where

\[ F(u,v) = [2(\kappa f - c - f - ab^2) - \varepsilon_1] u^2 + 2[\kappa(\kappa f - c - f - ab^2) + (\rho - \kappa h) - \kappa \varepsilon_1] vu + [2\kappa(\rho - \kappa h) - \kappa^2 \sigma^2 - \kappa^2 \varepsilon_1] v^2. \]

Since \( \sigma^2 < \frac{2(\rho - \kappa h)}{\kappa} \), it is easily seen that there exists a small \( \varepsilon_1 > 0 \) satisfying the following. The quadratic equation \( F(u,v) = 0 \) in the variable \( u \) has two non-positive solutions for every \( v \geq 0 \). Thus, \( F(u,v) > 0 \) for all \( (u,v) \in \mathbb{R}^2_+ \).

**Proof for (19).** Due to (23), (24) and the fact that \( v_t \leq \bar{v}_t \), where \( \bar{v}_t \) is the solution of (16), we have

\[ \frac{\log(M^* + \kappa \bar{v}_t)}{t} + \frac{aM^*}{t} \int_0^t \frac{v^2}{M^* + \kappa v} ds \\
\quad \geq \frac{\log(u_0 + \kappa v_0)}{t} + \frac{\varepsilon_1}{2} + \frac{1}{t} \int_0^t \frac{\sigma \kappa v}{u + \kappa v} dw_s. \quad (25) \]

Put

\[ N_t = \int_0^t \frac{\sigma \kappa v_s}{u_s + \kappa v_s} dw_s. \quad (26) \]
Then, \( \{N_t\}_{0 \leq t < \infty} \) is a real-valued continuous martingale vanishing at \( t = 0 \). Furthermore, \( \{N_t\}_{0 \leq t < \infty} \) has a quadratic form given by

\[
\langle N \rangle_t = \int_0^t \frac{\sigma^2 \kappa^2 v_s^2}{(u_s + \kappa v_s)^2} ds \leq \sigma^2 t.
\]

The strong law of large numbers for martingale (see, e.g., [9, 12]) then gives

\[
\lim_{t \to \infty} \frac{N_t}{t} = 0 \quad \text{a.s.} \quad (27)
\]

In the meantime, applying Lemma 2.1–(i) to the equation (16) and using Theorem 2.3–(ii), we observe that

\[
\begin{cases}
\lim_{t \to \infty} \frac{\log(M^* + \kappa \bar{v}_t)}{t} = 0, \\
u_t \leq M^* \quad \text{a.s., } 0 \leq t < \infty.
\end{cases}
\]

Taking the limit as \( t \to \infty \) in the two sides of (25), we hence obtain that

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \frac{aM^* v_s^2}{M^* + \kappa v_s} ds \geq \frac{\varepsilon_1}{2} \quad \text{a.s.}
\]

Since \( \frac{v^2}{M^* + \kappa v_s} < \frac{1}{M^*} v_s^2 \) and \( \frac{v^2}{M^* + \kappa v_s} < \frac{1}{M^*} v_s^2 \), we conclude that

\[
\begin{cases}
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \bar{v}_s ds > \frac{\kappa \varepsilon_1}{2aM^*} \quad \text{a.s.}, \\
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \bar{v}^2_s ds > \frac{\varepsilon_1}{2a} \quad \text{a.s.}
\end{cases}
\]

from which it follows (19).

**Proof for (20).** Taking expectation in the two sides of (25), we have

\[
\frac{\varepsilon_1}{2} \leq \liminf_{t \to \infty} \left[ \frac{\log(M^* + \kappa \bar{v}_t)}{t} + \frac{aM^*}{t} \mathbb{E} \int_0^t \frac{\bar{v}^2_s}{M^* + \kappa \bar{v}_s} ds \right]
\]

\[
\leq \liminf_{t \to \infty} \left[ \frac{M^* + \kappa \mathbb{E} \bar{v}_t}{t} + \frac{aM^*}{\kappa} \frac{1}{t} \int_0^t \mathbb{E} \bar{v}_s ds \right],
\]

here we used the estimate

\[
\log(M^* + x) < M^* + x, \quad 0 < x < \infty.
\]

On account of Lemma 2.1, the solution \( \bar{v} \) of (16) satisfies the estimate

\[
\mathbb{E} \bar{v}_t \leq \alpha_1, \quad 0 < t < \infty,
\]

where \( \alpha_1 \) is some positive constant. We thus have shown that

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E} \bar{v}_s ds \geq \frac{\varepsilon_1 \kappa}{2aM^*}.
\]

Meanwhile, taking expectation in the two sides of the second equation of (2), it follows that

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E} u_s ds = \liminf_{t \to \infty} \left[ \frac{\mathbb{E} u_t}{t} + \frac{h}{t} \int_0^t \mathbb{E} v_s ds \right]
\]

\[
= \liminf_{t \to \infty} \frac{h}{t} \int_0^t \mathbb{E} v_s ds \geq \frac{h \kappa \varepsilon_1}{2aM^*}.
\]
Therefore, (20) has been verified. As a consequence,
\[
\limsup_{t \to \infty} \mathbb{E} u_t > 0
\]
and
\[
\limsup_{t \to \infty} \mathbb{E} v_t > 0.
\]
This means that (2) is sustainable. We complete the proof. \(\square\)

### 3.2. Existence of Borel invariant measure

Let us show existence of a Borel invariant measure of the Itô process \((u_t, v_t)\), which concentrates on some domain of \(\mathbb{R}^2_+\) under the assumptions in Theorem 3.2.

Let \(P(\cdot, \cdot, \cdot, \cdot)\) be the transition probability of \((u_t, v_t)\):
\[
P(t, x, y, K) = \mathbb{P}\{(u_t, v_t) \in K; (u_t, v_t)|_{t=0} = (x, y)\},
\]
for \(0 \leq t < \infty, (x, y) \in \mathbb{R}^2_+\), and \(K \in \mathcal{B}(\mathbb{R}^2_+)\). It is well known that (see, e.g., [6, 13])

(i) \(P(t, x, y, \cdot)\) induces a strongly continuous semigroup \(\{P_t\}_{0 \leq t < \infty}\) of operators on the space \(C_B(\mathbb{R}^2_+)\) of bounded continuous functions:
\[
P_t f(x, y) = \int_{\mathbb{R}^2_+} f(\xi, \eta)P(t, x, y, d\xi d\eta), \quad f \in C_B(\mathbb{R}^2_+),
\]

(ii) \(P(t, x, y, \cdot)\) induces a positive contraction \([\cdot; P_t]\) on the space \(M(\mathbb{R}^2_+, \mathcal{B}(\mathbb{R}^2_+))\) of finite signed measures:
\[
[\mu P_t](K) = \int_{\mathbb{R}^2_+} P(t, x, y, K)\mu(dx dy), \quad \mu \in M(\mathbb{R}^2_+, \mathcal{B}(\mathbb{R}^2_+), K \in \mathcal{B}(\mathbb{R}^2_+))
\]

**Definition 3.3.** A Borel measure \(\nu\) on \(\mathbb{R}^2_+\) (i.e. a positive measure which is finite on any compact set of \(\mathbb{R}^2_+\)) is said to be invariant with respect to \(\{P_t\}_{0 \leq t < \infty}\) if for \(0 < t < \infty\) and \(K \in \mathcal{B}(\mathbb{R}^2_+)\),
\[
[\nu P_t](K) = \nu(K).
\]

The following result is well-known.

**Theorem 3.4 ([13, Theorem 5.7]).** Let \(X\) be a locally compact perfectly normal topological space. Let \(\{Q_t\}_{0 \leq t < \infty}\) be a strongly continuous semigroup on \(C_B(X)\) generated by a transition probability on \((X, \mathcal{B}(X))\). If there exists a nonnegative function \(g\) in the space \(C_0(X)\) of continuous functions with compact support such that
\[
\int_0^\infty Q_t g(x) dt = \infty, \quad x \in X,
\]
then there exists a Borel invariant measure for \(\{Q_t\}_{0 \leq t < \infty}\).

We are now ready to state our theorem.

**Theorem 3.5.** Let (18) be satisfied. Then, \(\{P_t\}_{0 \leq t < \infty}\) has a Borel invariant measure which concentrates on some domain of \(\mathbb{R}^2_+ \cap \{(u, v); u \leq M_0\}\).

**Proof.** To prove this theorem, we construct a function \(g \in C_0(\mathbb{R}^2_+)\) which satisfies the assumption in Theorem 3.4.

On account of Theorem 3.2, we have
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{v \geq \frac{\epsilon}{2}\}} v ds \geq \frac{\epsilon}{2} > 0 \quad \text{a.s.}
\]
Using Theorem 2.3–(iii) and the Hölder inequality, for any $0 \leq \theta < \frac{2h}{\sigma^2}$, there exists $n_\theta > 0$ such that

\[
\frac{\epsilon}{2} \leq \liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}\{v_s \geq \epsilon \} ds \\
\leq \liminf_{t \to \infty} \left\{ \left[ \frac{1}{t} \int_0^t \mathbb{E}\{v_s \geq \epsilon \} ds \right]^{\frac{\theta}{\theta + 1}} \right\}^{\frac{1}{\theta + 1}} \\
\leq n_\theta \liminf_{t \to \infty} \left( \frac{1}{t} \int_0^t \mathbb{P}\{ v_s \geq \epsilon \} ds \right)^{\frac{1}{\theta + 1}.
\]

Thereby, there exists $\epsilon_0 > 0$ such that

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}\{ v_s \geq \epsilon \} ds > \epsilon_0. \tag{28}
\]

On the other hand, by Theorem 2.3–(iii), there exists $\alpha > 0$ such that $\mathbb{E}v_t \leq \alpha$ for $0 \leq t < \infty$. The Markov inequality then provides that

\[
\mathbb{P}\left\{ v_t \geq \frac{2\alpha}{\epsilon_0} \right\} \leq \frac{\epsilon_0}{2\alpha} \mathbb{E}v_t \leq \frac{\epsilon_0}{2}. \tag{29}
\]

Hence,

\[
\inf_{0 \leq t < \infty} \mathbb{P}\left\{ v_t < \frac{2\alpha}{\epsilon_0} \right\} \geq 1 - \frac{\epsilon_0}{2}. \tag{29}
\]

Put

\[
K = \{(u, v); 0 \leq u \leq M^*, \frac{\epsilon_0}{2} \leq v \leq \frac{2\alpha}{\epsilon_0}\}.
\]

Let us show that

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}\{ (u, v) \in K \} ds \geq \frac{\epsilon_0}{2} > 0. \tag{30}
\]

Indeed, suppose the contrary, then there would exist an increasing sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

\[
\frac{1}{t_n} \int_0^{t_n} \mathbb{P}\{ (u, v) \in K \} ds < \frac{\epsilon_0}{2}, \quad n = 1, 2, 3, \ldots
\]

Theorem 2.3–(i) and (29) then give

\[
\frac{1}{t_n} \int_0^{t_n} \mathbb{P}\{ v_s < \epsilon \} ds = \frac{1}{t_n} \int_0^{t_n} \mathbb{P}\{ v_s < \frac{2\alpha}{\epsilon_0} \} ds - \frac{1}{t_n} \int_0^{t_n} \mathbb{P}\{ v_s \in \left[ \epsilon, \frac{2\alpha}{\epsilon_0} \right] \} ds \\
= \frac{1}{t_n} \int_0^{t_n} \mathbb{P}\{ v_s < \frac{2\alpha}{\epsilon_0} \} ds - \frac{1}{t_n} \int_0^{t_n} \mathbb{P}\{ (u, v) \in K \} ds \\
> 1 - \frac{\epsilon_0}{2} - \frac{\epsilon_0}{2} = 1 - \epsilon_0.
\]

Combining this and (28), we arrive at a contradiction:

\[
1 = \frac{1}{t_n} \int_0^{t_n} \mathbb{P}\{ v_s < \epsilon \} ds + \frac{1}{t_n} \int_0^{t_n} \mathbb{P}\{ v_s \geq \epsilon \} ds > 1 - \epsilon_0 + \epsilon_0 = 1.
\]

Therefore, (30) holds true.

Let us fix a nonnegative function $g \in C_0(\mathbb{R}_+^2)$ such that

\[
g(x, y) = \begin{cases} 
1, & (x, y) \in K, \\
0, & (x, y) \in \mathbb{R}_+^2 \setminus K,
\end{cases}
\]
where $K_1$ is some bounded open set of $\mathbb{R}^2_+$ such that $K_1 \supset K$. In view of (30), we have
\[
\int_0^t P_s g(x, y) ds = \int_0^t \int_{\mathbb{R}^2_+} g(\xi, \eta) P(s, x, y, d\xi d\eta) ds \\
\geq \int_0^t \int_K g(\xi, \eta) P(s, x, y, d\xi d\eta) ds \\
= \int_0^t \mathbb{P}\{(u_s, v_s) \in K\} ds \to \infty \quad \text{as } t \to \infty.
\]

Thanks to Theorem 3.4, we conclude that there exists a Borel invariant measure $\nu$ on $\mathbb{R}^2_+$ for $\{P_t\}_{0 \leq t < \infty}$ such that $\nu(K) > 0$. By Theorems 2.2–2.3, $\nu$ concentrates on some domain of $\mathbb{R}^2_+ \cap \{(u, v); u \leq M_0\}$. The proof is now complete. 

4. Decline of forest. In this section, we show decline of the forest when either the mortality $h$ of old age class trees or the intensity $\sigma$ of noise is large. More precisely, if either
\[
h \geq \min \left\{ \frac{\rho f}{c + f} \left( \frac{f + 2abM^*}{ab^2 + c + f} \right) \right\}
\]
or
\[
\sigma^2 > \frac{(\rho + c - h)^2}{2c},
\]
than the forest falls into the decline. Here, $M^*$ is defined in Theorem 2.3–(i).

**Theorem 4.1.** Let $(u_t, v_t)$ be the solution of (2) with $(u_t, v_t)_{|t=0} = (u_0, v_0) \in \mathbb{R}^2_+ \setminus \{(0, 0)\}$. Assume that $h \geq \min \left\{ \frac{\rho f}{c + f} \left( \frac{f + 2abM^*}{ab^2 + c + f} \right) \right\}$. Then, as $t \to \infty$, $u_t$ and $v_t$ converge to 0 in expectation, i.e.
\[
\lim_{t \to \infty} \mathbb{E}u_t = \lim_{t \to \infty} \mathbb{E}v_t = 0. \quad (31)
\]
In particular, $u_t$ and $v_t$ converge to 0 in probability:
\[
\lim_{t \to \infty} \mathbb{P}\{u_t \geq C\} = \lim_{t \to \infty} \mathbb{P}\{v_t \geq C\} = 0, \quad C > 0. \quad (32)
\]
Furthermore,
\[
\lim_{t \to \infty} \mathbb{P}\{(u_t, v_t) \in A\} = 0 \quad \text{for any compact set } A \subset \mathbb{R}^2_+. \quad (33)
\]

**Proof.** Let us first prove that $u_t$ and $v_t$ converge to 0 in expectation. Consider the two cases of the mortality $h$.

**Case 1:** $h \geq \frac{\rho f}{c + f}$. It follows from (2) that
\[
\begin{cases}
\mathbb{E}u_t \leq u_0 + \int_0^t \left[\rho \mathbb{E}v_s - (c + f) \mathbb{E}u_s\right] ds, \\
\mathbb{E}v_t = v_0 + \int_0^t \left[f \mathbb{E}u_s - h \mathbb{E}v_s\right] ds.
\end{cases}
\]
Since the functions $\varpi_1$ and $\varpi_2$ defined by
\[
\varpi_1(X, Y) = \rho Y - (c + f)X, \quad \varpi_2(X, Y) = fX - hY
\]
are non-decreasing with respect to arguments $Y$ and $X$, respectively, the comparison theorem applied to the latter system provides that
\[
\begin{cases}
\mathbb{E}u_t \leq x_t, \quad 0 \leq t < \infty, \\
\mathbb{E}v_t \leq y_t, \quad 0 \leq t < \infty,
\end{cases} \quad (34)
\]
where \((x_t, y_t)\) is the positive solution to the linear system:

\[
\begin{align*}
\frac{dx_t}{dt} &= \rho y_t - (c + f)x_t, \\
\frac{dy_t}{dt} &= fx_t - hy_t,
\end{align*}
\]

with \((x_t, y_t)|_{t=0} = (u_0, v_0)\).

From this system and a fact that \(h \geq \rho f\), we observe that

\[
\begin{align*}
\frac{d(hx_t + \rho y_t)}{dt} &= [\rho f - h(c + f)]x_t \leq 0, \\
\frac{d[fx_t + (c + f)y_t]}{dt} &= [\rho f - h(c + f)]y_t \leq 0.
\end{align*}
\]

Hence, \(hx_t + \rho y_t\) and \(fx_t + (c + f)y_t\) are non-increasing as \(t\) increases. As a consequence, there exist two nonnegative constants \(\beta_1\) and \(\beta_2\) such that

\[
\begin{align*}
\lim_{t \to \infty} (hx_t + \rho y_t) &= \beta_1, \\
\lim_{t \to \infty} [fx_t + (c + f)y_t] &= \beta_2.
\end{align*}
\]

It is then seen that

\[
\begin{align*}
\lim_{t \to \infty} x_t &= \frac{\beta_1(c + f) - \beta_2\rho}{h(c + f) - \rho f}, \\
\lim_{t \to \infty} y_t &= \frac{\beta_2 h - \beta_1 f}{h(c + f) - \rho f}.
\end{align*}
\]

This means that \((\beta_1(c + f) - \beta_2\rho, \beta_2 h - \beta_1 f)\) is a stationary solution of (35). Substituting this solution for \((x_t, y_t)\) in (35), we obtain that

\[
\begin{align*}
\rho(\beta_2 h - \beta_1 f) - (c + f)[\beta_1(c + f) - \beta_2\rho] &= 0, \\
f[\beta_1(c + f) - \beta_2\rho] - h(\beta_2 h - \beta_1 f) &= 0.
\end{align*}
\]

Solving this system of algebraic equations, we arrive at

\[
\beta_1 = \beta_2 = 0.
\]

Hence,

\[
\lim_{t \to \infty} x_t = \lim_{t \to \infty} y_t = 0. \quad (36)
\]

Combining (34) and (36), we conclude that

\[
\lim_{t \to \infty} \mathbb{E}u_t = \lim_{t \to \infty} \mathbb{E}v_t = 0.
\]

**Case 2:** \(h \geq \frac{f(\rho + 2abM^*)}{ab^2 + c + f}\). From (2) and Theorem 2.3–(i), we have

\[
\begin{align*}
\mathbb{E}u_t &\leq u_0 + \int_0^t [\rho \mathbb{E}v_s + 2ab \mathbb{E}(u_s v_s) - (ab^2 + c + f)\mathbb{E}u_s]ds \\
&\leq u_0 + \int_0^t [(\rho + 2abM^*)\mathbb{E}v_s - (ab^2 + c + f)\mathbb{E}u_s]ds, \\
\mathbb{E}v_t &= v_0 + \int_0^t [f\mathbb{E}u_s - h\mathbb{E}v_s]ds.
\end{align*}
\]

Using the same argument as in Case 1, we conclude that

\[
\lim_{t \to \infty} \mathbb{E}u_t = \lim_{t \to \infty} \mathbb{E}v_t = 0.
\]

Let us now verify (32) and (33). For any \(0 < c_1 < c_2, 0 < d_1 < d_2\),

\[
\mathbb{P}\{(u_t, v_t) \in [c_1, c_2] \times [d_1, d_2]\} \leq \mathbb{P}\{u_t \geq c_1\},
\]

where \((x_t, y_t)\) is the positive solution to the linear system:
and
\[ P\{u_t \geq c_1\} \leq \frac{1}{c_1} E u_t, \quad P\{v_t \geq c_1\} \leq \frac{1}{c_1} E v_t. \]
This together with (31) derives (32) and (33). It completes the proof of the theorem.

Under somewhat stronger assumptions than those of Theorem 4.1, we can show almost sure convergence of \( u_t \) and \( v_t \) to 0. Consider two functions \( F_1 \) and \( F_2 \) defined by
\[
F_1(x) = f^2 x^4 + 2f(\sigma^2 + h - c - f)x^3 + [(c + f - h)^2 - 2\rho f - 2(c + f)\sigma^2]x^2 + 2\rho(c + f - h)x + \rho^2, \\
F_2(x) = fx^2 - (c + f + h)x + \rho.
\]
Assume that either
\[
\inf_{x \in (0, \frac{c + f}{\kappa})} F_1(x) < 0, \quad (37)
\]
or
\[
\frac{2\rho}{\sigma^2 + 2h} < \frac{c + f}{\kappa} \quad \text{and there exists } \lambda \text{ such that} \\
\frac{2\rho}{\sigma^2 + 2h} < \lambda < \frac{c + f}{\kappa}, F_1(\lambda) > 0 \text{ and } F_2(\lambda) < 0 \quad (38)
\]
holds true. Then, the following theorem shows such convergence.

**Theorem 4.2.** Let \((u_t, v_t)\) be the solution of (2) with \((u_t, v_t)_{t=0} = (u_0, v_0) \in \mathbb{R}_+^2 \setminus \{(0,0)\}\). Under (37) or (38), \( \lim_{t \to \infty} u_t = \lim_{t \to \infty} v_t = 0 \) a.s.

**Proof.** We again use the function \( Q \) defined by \( Q(u, v) = \log(u + \kappa v) \) as in the proof of Theorem 3.2, where \( \kappa \) is a positive constant that will be fixed below.

Let us first show that under (37) or (38), there exists a small \( \epsilon > 0 \) such that
\[
[LQ](u, v) \leq -\frac{\epsilon}{2} \quad \text{for all } (u, v) \in \mathbb{R}_+^2,
\]
where \([LQ]\) is defined in (22). Indeed, it is easily seen that a sufficient condition for this (in fact, it is also a necessary condition) is that there exists \( \epsilon > 0 \) such that
\[
F(u, v) = 2(c + f - \kappa f - e)u^2 - 2[\kappa(\kappa f - c - f) + \rho - \kappa h + \kappa]vu \\
+ [\sigma^2 \kappa^2 + 2\kappa(\kappa h - \rho) - \kappa^2 e]v^2 \geq 0 \quad \text{for all } (u, v) \in \mathbb{R}_+^2. \quad (39)
\]
If (37) takes place, choose \( \kappa \) such that
\[
0 < \kappa < \frac{c + f}{\kappa} \quad \text{and} \quad F_1(\kappa) < 0.
\]
It is then easily seen that there exists a small \( \epsilon > 0 \) such that the quadratic equation \( F(u, v) = 0 \) in \( u \) has a non-positive discriminant for all \( v \geq 0 \). This implies (39). In the meantime, if (38) takes place, choose \( \kappa = \lambda \) in (38). Similarly, it is seen that there exists \( \epsilon > 0 \) such that the equation \( F(u, v) = 0 \) has non-positive two solutions for all \( v \geq 0 \). This also derives (39).

Let us now verify that
\[
\lim_{t \to \infty} Q(u_t, v_t) = -\infty \quad \text{a.s.}
\]
It follows from (21) that
\[
\frac{1}{t} Q(u_t, v_t) = \frac{1}{t} Q(u_0, v_0) + \frac{1}{t} \int_0^t [LQ](u_s, v_s) ds + \frac{1}{t} \int_0^t \frac{\sigma \nu v_s}{u_s + \kappa v_s} dw_s \\
\leq \frac{1}{t} Q(u_0, v_0) - \epsilon + \frac{1}{t} \int_0^t \frac{\sigma \nu v_s}{u_s + \kappa v_s} dw_s, \quad 0 < t < \infty. \tag{40}
\]

On account of (26) and (27),
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{\sigma \nu v_s}{u_s + \kappa v_s} dw_s = 0 \text{ a.s.}
\]
Hence, taking the limit as \( t \to \infty \) in both the hand sides of (40), we observe that
\[
\limsup_{t \to \infty} \frac{1}{t} Q(u_t, v_t) \leq -\epsilon 2 \text{ a.s.}
\]
This implies that \( \lim_{t \to \infty} Q(u_t, v_t) = -\infty \) a.s. Thus,
\[
\lim_{t \to \infty} u_t = \lim_{t \to \infty} v_t = 0 \text{ a.s.}
\]
The proof is complete. \( \Box \)

**Remark 1.** It is possibly seen that if \( F_1(1) < 0 \), then (37) takes place. After some simple calculations on the inequality \( F_1(1) < 0 \), we arrive at this condition:
\[
\sigma^2 > \frac{(\rho + c - h)^2}{2c}.
\]
According to Theorem 4.2, we therefore conclude that a noise with large intensity causes decline of the forest.

5. **Numerical examples.** Let us exhibit some numerical examples for sustainability of the forest and possibility of decline. For the computations, we used a scheme of order 1.5 (see, e.g., [10]).

5.1. **Sustainability of forest.** In the system (2), set \( a = 2, b = 1, c = 2.5, f = 4, h = 1, \rho = 5, \sigma = 0.5, \) and take initial value \( (u_0, v_0) = (2, 1) \).

Figure 1 gives sample trajectories of \( u \) and \( v \) in the phase space and in time. Figure 2 plots points \( (u_T, v_T) \) of \( 10^4 \) sample trajectories of \( (u, v) \) at time \( T = 1000 \).

By computing \( 10^3 \) sample trajectories of \( (u, v) \), Figure 3 shows a graph of the expectation of tree densities of young and old age classes.

Figure 4 gives a sample trajectory of two processes \( I \) and \( J \) defined by
\[
I(t) = \frac{1}{t} \int_0^t u_s ds, \quad J(t) = \frac{1}{t} \int_0^t v_s ds.
\]

Figure 5 demonstrates a trajectory of two probability functions \( R \) and \( S \) defined by
\[
\begin{cases}
R(t) = \mathbb{P}\{(u_t, v_t) \in A; (u_0, v_0) = (2, 1)\}, \\
S(t) = \mathbb{P}\{(u_t, v_t) \in A; (u_0, v_0) = (3, 4)\},
\end{cases}
\]
along \( t \in [50, 100] \), where \( A = [0.5, 30] \times [0.1, 20] \). These functions are calculated on the basis of 2000 sample trajectories of \( (u_t, v_t) \) corresponding to each of the two initial values.
Figure 1. Sample trajectories of $u_t$ and $v_t$ of (2) with parameters: $a = 2, b = 1, c = 2.5, f = 4, h = 1, \rho = 5, \sigma = 0.5$ and initial value $(u_0, v_0) = (2, 1)$. The left figure illustrates a sample trajectory of $(u_t, v_t)$ in the phase space; the right figure illustrates sample trajectories of $u_t$ and $v_t$ along $t \in [0, 100]$.

Figure 2. Distribution of $(u_t, v_t)$ of (2) at $t = 10^3$. The parameters and initial value are taken as in the legend of Fig. 1.

5.2. Decline of forest. First, set $a = 3, b = 4, c = 5, f = 6, h = 2, \rho = 7, \sigma = 4$ and take $(u_0, v_0) = (4, 3)$. Figure 6 gives sample trajectories of $u$ and $v$ in the phase space and in time.
Second, set $a = 3, b = 4, c = 5, f = 6, h = 3.82, \rho = 7, \sigma = 0.25$ and take $(u_0, v_0) = (4, 3)$. By computing $5 \times 10^2$ sample trajectories of $(u, v)$, Figure 7 shows a graph of expectation of tree densities of young and old age classes.

**Figure 3.** Graphs of $E_u$ and $E_v$ along $t \in [0, 20]$. The parameters and initial value are taken as in the legend of Fig. 1.

**Figure 4.** Sample trajectory of two processes $I$ and $J$ defined by $I(t) = \frac{1}{t} \int_0^t u_s ds$ and $J(t) = \frac{1}{t} \int_0^t v_s ds$ along $t \in [0, 100]$. The parameters and initial value are taken as in the legend of Fig. 1.
Figure 5. Graph of probability functions $R$ and $S$ defined by $R(t) = \mathbb{P}\{(u_t, v_t) \in A; (u_0, v_0) = (2, 1)\}$ and $S(t) = \mathbb{P}\{(u_t, v_t) \in A; (u_0, v_0) = (3, 4)\}$ along $t \in [50, 100]$, where $A = [0.5, 30] \times [0.1, 20]$ and the parameters of (2) are taken as in the legend of Fig. 1. These functions are calculated on the basis of 2000 sample trajectories of $(u_t, v_t)$ corresponding to each initial value.

Figure 6. Decline of forest under the effect of noise with large intensity $\sigma$. Here, $a = 3, b = 4, c = 5, f = 6, h = 2, \rho = 7, \sigma = 4$ and initial value $(u_0, v_0) = (4, 3)$. The left figure is a sample trajectory of $(u_t, v_t)$ in the phase space; the right figure is a sample trajectory of $u$ and $v$ along $t \in [0, 1]$. 
Figure 7. Decline of forest when the mortality $h$ of old trees is large. Here, $a = 3, b = 4, c = 5, f = 6, h = 3.82, \rho = 7, \sigma = 0.25$ and initial value $(u_0, v_0) = (4, 3)$. The figure gives a graph of $E_u$ and $E_v$ along $t \in [0, 10]$.

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