The instanton contributions to Yang-Mills theory on the torus: 
localization, Wilson loops and the perturbative expansion

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Abstract

The instanton contributions to the partition function and to homologically 
trivial Wilson loops for a $U(N)$ Yang-Mills theory on a torus $T^2$ are analyzed. 
An exact expression for the partition function is obtained as a sum of contrib-
utions localized around the classical solutions of Yang-Mills equations, that 
appear according to the general classification of Atiyah and Bott. Explicit 
expressions for the exact Wilson loop averages are obtained when $N = 2$, 
$N = 3$. For general $N$ the contribution of the zero-instanton sector has been 
carefully derived in the decompactification limit, reproducing the sum of the 
perturbative series on the plane, in which the light-cone gauge Yang-Mills 
propagator is prescribed according to Wu-Mandelstam-Leibbrandt (WML). 
Agreement with the results coming from $S^2$ is therefore obtained, confirming 
the truly perturbative nature of the WML computations.

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I. INTRODUCTION

In the last few years a lot of attention has been devoted to quantum Yang-Mills theories on compact two-dimensional surfaces: in spite of their seeming triviality, many interesting (and highly non-trivial) results were obtained exploiting their non-perturbative solvability. The exact partition function was in fact derived long ago by Migdal [1], using a lattice formulation of the theory, and subsequently the same technique was extended to the computation of Wilson loop averages [2]. Starting from these results it has been recognized [3] that, in the large-\(N\) limit, two-dimensional \(U(N)\) Yang-Mills theory is a string theory, a fact that is widely believed to have a four-dimensional counterpart.

On the other hand, from a conventional quantum field theory point of view, YM\(_2\) exhibits an interesting and peculiar behaviour, that is actually at the basis of its exact solvability: the partition function is given by a sum over contributions localized at classical solutions of the theory. For finite \(N\) the path-integral is represented as sum over unstable instantons, where each instanton contribution is modified by a finite, but non-trivial, perturbative expansion that carries the quantum corrections. Witten has shown [4] how to understand, from a general point of view, this behaviour constructing a non abelian generalization of the Duistermaat-Heckman theorem [5] and exploiting the relation with a topological field theory.

A further intriguing aspect is the appearance, on genus zero and in the limit of large \(N\), of a third order phase transition at some critical value of the sphere area [6]: a strong coupling phase, where a pure area exponentiation for Wilson loops dominates in the decompactification limit, is distinguished from a weak coupling phase with no confining behaviour [7]. Again instantons provide a clear physical picture of this phenomenon: the explicit computation of instanton contributions performed in [8] has shown that the deconfining phase comes as a result of instantons condensation. Topologically non-trivial configurations play an analogous role on the cylinder, as noticed some time before in [9], leading to a similar phase transition.
Rather surprisingly, it has been recently realized [10] that the instanton picture is essential in understanding some controversial aspects of $\text{YM}_2$ (and $\text{QCD}_2$) on the plane, some of which dating back to the mid-seventies. The facts are the following. Confinement in $\text{QCD}_2$ is usually regarded as a perturbative feature. As a matter of fact the area exponentiation is simply obtained (for any $N$) by summing on the plane the conventional perturbative series in light-cone gauge (where the situation is trivialized by the absence of self-interaction terms), with the ’t Hooft-CPV prescription for the gluon exchange potential [11]. The insertion of dynamical quarks can be handled as well in the large $N$ limit, leading to the celebrate ’t Hooft solution for the meson spectrum [11]. On the other hand, in the same years, a different prescription for the exchange potential was proposed by T.T. Wu in [12]. It was derived from analyticity requirements and, unexpectedly, it produced a very different Bethe-Salpeter equation for the meson wave function; its numerical solution [13] exhibits a spectrum that does not resemble ’t Hooft’s one, as confirmed by the analytical computations in [14]. More recently [15] it was noticed that Wu’s prescription is not a mathematical artifact, but it is the two-dimensional version of the four-dimensional Mandelstam-Leibbrandt prescription [16], that is the only one reproducing the correct perturbative physics in $4D$ [17], when light-cone gauge is adopted.

The exact resummation of the perturbative series for the Wilson loop has been done in [18] using the Wu-Mandelstam-Leibbrandt (WML) prescription, generalizing to all orders the $O(g^4)$ computation of [15,19]; it leads, as firstly noticed in [15], to a result different from a pure area-law exponentation (which would be expected from the area-preserving diffeo-invariance of the theory plus positivity arguments), and, in particular, predicting a different value for the string tension. More dramatically, the large-$N$ limit has not a confining behaviour, while one easily realizes that, on the plane, the theory should be in the strong coupling phase (the plane being thought as decompactification of a large sphere).

In [10] it was shown that the WML computations presented in [18,15,19] are indeed perturbatively correct, in the sense that what is missing represents the instanton contribution: the WML result for the Wilson loop corresponds exactly to the zero-instanton result of the
sphere (that is manifestly gauge invariant), when the decompactification limit is taken into account. In this perspective the absence of confinement at large $N$ is not a mystery, because in the weak phase only the zero-instanton sector contributes \[8\]: it seems instead surprising that 't Hooft propagator perturbatively captures the (non-perturbative) instantonic physics of the sphere. The gauge independence of the perturbative (WML) result has been checked by using the Feynman gauge, in which the theory does not look free (the self-interaction terms are present there): due to infrared singularities the computation has been performed in $D = 2 + \epsilon$ at $O(g^4)$ order, and, apart further terms linked to a possible discontinuity at $D = 2$ (see ref. \[20\] for a detailed discussion of this point), it reproduces the zero-instanton calculus.

If the full WML result has been correctly interpreted, it should be related to the local behaviour of the theory: in particular it should be derived starting from any topology, when the decompactification limit is taken into account. To further confirm the results of \[10\], we have decided to perform the computation of the zero-instanton contribution to Wilson loops on a torus ($T^2$) and to compare it, in the limit of large volume, to the WML resummation (we have therefore to consider homologically trivial loops on $T^2$ in order to make a comparison). The calculation is more difficult than in the sphere case and it presents a certain amount of interest by itself. First of all on $T^2$, at variance with the genus zero case \[8\], no one, at least at our knowledge, has derived the explicit form of the partition function and of the Wilson loops in the instanton representation. The task is harder from a technical point of view, due to the complexity of performing for a Wilson loop the Poisson resummation (that is the common trick to derive the instanton representation from the usual lattice–heat-kernel one). Conceptually we are faced with a larger class of classical solutions and non-trivial small-coupling singularities due to the non-smooth topology of the moduli space of flat connections $\mathcal{M}_F(T^2, U(N))$.

The paper is organized in the following way. In sect. 2 we rewrite the Migdal’s partition function for $U(N)$ as a sum of terms localized over the classical solutions of Yang-Mills equations. The classical contributions appear exactly according to the general classification,
presented by Atiyah and Bott in [21], of the critical points of the Yang-Mills functional on Riemann surfaces. We discuss the structure of the quantum fluctuations and of the small-couplings singularities, related to the non smooth behaviour of $\mathcal{M}_x(T^2, U(N))$. In sect. 3 we explicitly derive the Wilson loop expectation value (for an homologically trivial loop) for $U(2)$ and $U(3)$, showing how to perform in concrete the Poisson resummation (that is not trivial due to presence of “dangerous” denominators in the Migdal’s sum). The result for the zero-instanton contribution coincides, in the decompactification limit, with the WML computation. In sect. 4 we extend the computation to generic $N$, and we find full agreement with WML if

$$\frac{1}{N+1} L_N^1(x) = \frac{1}{2\pi i} \int_{C_0} \frac{dz}{z^{N+1}} \exp \left[ \sum_{k=1}^{+\infty} \frac{L_k^0\left(\frac{kx}{N+1}\right)z^k}{k} \right],$$

$L_N^0(x)$ being the generalized Laguerre polynomials (see [22] for the conventions) and $C_0$ a small contour in the complex plane surrounding the origin: we have not been able to prove this identity for generic $N$, but we have checked it by using MAPPLE till $N = 25$. In sect. 5 we draw our conclusions and discuss possible extensions of the work.

After having completed the computations, it has appeared a paper by Billó et alt. [23], in which the relevance of topologically non-trivial excitations for $YM_2$ on the torus are discussed and the analogy with similar contributions in Matrix String Theory (MST) [24] has been stressed. Infrared and ultraviolet properties of the partition function of MST have also been investigated in [25]: we think that these results could be understood as coming from instantons as well, and are in some way related to ours.

II. PARTITION FUNCTION AND INSTANTONS

We begin by presenting the Migdal’s heat-kernel representation for the Yang-Mills partition function on $T^2$

\[1\] I thank Massimo Pietroni for the help on this point
\[ Z^{(1)} = \sum_{R} \exp \left[ -\frac{g^2 A}{2} C_2(R) \right], \]  

where \( A \) is the area of the torus, \( g^2 \) is the coupling constant appearing in the Yang-Mills action and \( C_2(R) \) is the value of the second Casimir operator in the representation \( R \). The sum runs over the irreducible representation of the gauge group: in the \( U(N) \) case the representations \( R \) can be labelled by a set of integers \( n_i = (n_1, \ldots, n_N) \), related to the Young tableaux, obeying the ordering \( +\infty > n_1 > n_2 > \ldots > n_N > -\infty \). In terms of \( n_i \) we have for the second Casimir

\[ C_2(R) = C_2(n_1, \ldots, n_N) = \frac{N}{12} (N^2 - 1) + \sum_{i=1}^{N} (n_i - \frac{N - 1}{2})^2. \]  

The dependence on the product \( g^2 A \) is peculiar of two dimensional Yang-Mills theories, that are invariant under area-preserving diffeomorphisms.

Using the permutation symmetry we get

\[ Z^{(1)} = \frac{1}{N!} \sum_{n_1 \neq n_2 \neq \ldots \neq n_N} \exp \left[ -\frac{g^2 A}{2} C_2(n_1, \ldots, n_N) \right]; \]  

we notice that, at variance with the sphere case, we cannot let two or more \( n_i \)'s be the same. The genus zero partition function is in fact

\[ Z^{(0)} = \sum_{R} d_R^2 \exp \left[ -\frac{g^2 A}{2} C_2(R) \right], \]  

where \( d_R \) is the dimension of the representation \( R \), that can be expressed in terms of the \( n_i \)'s as

\[ d_R = \Delta(n_1, \ldots, n_N), \]  

\( \Delta \) being the Vandermonde determinant, that possess the permutation symmetry and vanishes when two arguments coincides. In the following we will see that the different structure of the classical solutions on the two topologies is reflected by this different behaviour under summation.

The easiest way to obtain the instanton representation is to perform a Poisson resummation in eqs.\( (3,4) \) \[26\]: this can be done straightforwardly in eq.\( (1) \) (see \[8\] for details),
where the sum is extended without problems over all the integers, due to the vanishing of
the Vandermonde determinant, while some care is needed for eq. (3). The partition function
can be represented in the following way:

\[ Z^{(1)} = \frac{1}{N!} \sum_{\{n_i\}=-\infty}^{+\infty} \sum_P (-1)^P \int_0^{2\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \exp \left[ -\sum_{j=1}^{N} \theta_j (n_j - n_{P(j)}) \right] \exp \left[ -\frac{g^2 A}{2} C_2(n_1, ..., n_N) \right], \]

(6)

where no restriction appears on the \(n_i\)’s. \(\sum_P\) means the sum over all elements of the
symmetric group \(S_N\), \(P(i)\) denotes the index \(i\) transformed by \(P\), while \((-1)^P\) is the parity
of the permutation. One recovers the original form eq. (3) by simply integrating over the
angles \(\theta_i\) and using the formula

\[ \sum_P (-1)^P \prod_{i=1}^{N} \delta_{n_i, n_{P(i)}} = \text{det} \delta_{n_i, n_j}. \]

(7)

The basic observation is now that, due to the symmetry of \(C_2(n_1, ..., n_N)\), only the conjugacy
classes of \(S_N\) are relevant in computing the series: to see this we use the cycle decomposition
of the elements of \(S_N\).

A conjugacy class of \(S_N\) is conveniently described by the set of non-negative integers
\(\{\nu_i\} = (\nu_1, \nu_2, ..., \nu_N)\) (we follow the description of [27]) satisfying the constraint

\[ \nu_1 + 2\nu_2 + 3\nu_3 + ... + N\nu_N = N. \]

(8)

Every element belonging to \(\{\nu_i\}\) has the same parity and can be decomposed, in a standard
way, into \(\nu_1\) one-cycles, \(\nu_2\) two-cycles, ..., \(\nu_N\) \(N\)-cycles. Due to the symmetry of \(C_2(n_1, ..., n_N)\)
all the elements of a conjugacy class give the same contribution in eq. (6), as a simple
relabelling of the \(n_i\)’s and \(\theta_i\)’s is sufficient: only the parity of the class and the number of
its elements, as function of \(\{\nu_i\}\), are therefore relevant to the computation of the partition
function. It turns out that \((-1)^{\sum_{i=\text{even}} \nu_i}\) is the parity, while the number of elements in the
conjugacy class \(\{\nu_i\}\) is

\[ M_{\{\nu_i\}} = \frac{N!}{\nu_1! \nu_2! \nu_3! \cdots \nu_N!}. \]

(9)
The next step is to use the decomposition in cycles to perform explicitly the angular integrations: the effect is to express the full series as a finite sum of series over a decreasing number of integers. One easily realizes that a two-cycle results into the identification of two \( n_i \)'s in the sum, a three-cycle into the identification of three \( n_i \)'s and so on. We end up with

\[
Z^{(1)} = e^{-\frac{g^2 A N (N^2 - 1)}{N!}} \sum_{\{\nu_i\}} \sum_{n_1 \ldots n_{\nu_i} = -\infty}^{+\infty} \frac{1}{\nu_1 + \ldots + \nu_\nu} \sum_{i = even} \Phi_{\{\nu_i\}}(m_1, \ldots, m_{\nu}) (\frac{2\pi}{g^2 A})^{\frac{\nu}{2}} Z_{\{\nu_i\}} \exp \left[ -S_{\{\nu_i\}}(m_1, \ldots, m_\nu) \right],
\]

where each conjugacy class has produced a sum over \( \nu = \nu_1 + \nu_2 + \ldots + \nu_N \) integers and

\[
C_{2}\{\nu_i\}(n_1, \ldots, n_{\nu}) = \sum_{i_1 = 1}^{\nu_1} (n_{i_1} - \frac{N - 1}{2})^2 + 2 \sum_{i_2 = \nu_1 + 1}^{\nu_1 + \nu_2} (n_{i_2} - \frac{N - 1}{2})^2 + 3 \sum_{i_3 = \nu_1 + \nu_2 + 1}^{\nu_1 + \nu_2 + \nu_3} (n_{i_3} - \frac{N - 1}{2})^2 + \ldots
\]

Of course if some \( \nu_j \) is zero, the integers \( n_{\nu_1 + \ldots + \nu_{j-1} + 1}, \ldots, n_{\nu_1 + \ldots + \nu_{j-1} + \nu_j} \) do not appear. The Poisson resummation is, at this point, almost trivial, being the set \( \{n_1, \ldots, n_{\nu}\} \) unrestricted: the simple formula

\[
\sum_{n = -\infty}^{+\infty} f(n) = \sum_{m = -\infty}^{+\infty} \int_{-\infty}^{+\infty} dx e^{2\pi imx} f(x),
\]

requires in our case only gaussian integrations. The final result, expressing the original partition function as a sum over “dual” integers \( m_i \)'s, is:

\[
Z^{(1)} = e^{-\frac{g^2 A N (N^2 - 1)}{N!}} \sum_{\{\nu_i\}} \sum_{m_1, m_{\nu} = -\infty}^{+\infty} (-1)^{\Phi_{\{\nu_i\}}(m_1, \ldots, m_{\nu})} (\frac{2\pi}{g^2 A})^{\frac{\nu}{2}} Z_{\{\nu_i\}} \exp \left[ -S_{\{\nu_i\}}(m_1, \ldots, m_\nu) \right],
\]

where

\[
\Phi_{\{\nu_i\}}(m_1, \ldots, m_{\nu}) = (-1)^{\sum_{i = even} \nu_i} \exp \left[ i\pi (N - 1) \left( \sum_{j = odd} m_{ij} \right) \right],
\]

\[
Z_{\{\nu_i\}} = \frac{[\nu_{1}^{\nu_1} 2^{\nu_2} 3^{\nu_3} \ldots N^{\nu_N}]^{-\frac{1}{2}}}{\nu_1! \nu_2! \nu_3! \ldots \nu_N!},
\]

and

\[
S_{\{\nu_i\}}(m_1, \ldots, m_{\nu}) = \frac{2\pi^2}{g^2 A} \sum_{i_1 = 1}^{\nu_1} m_{i_1}^2 + \frac{\pi^2}{g^2 A} \sum_{i_2 = \nu_1 + 1}^{\nu_1 + \nu_2} m_{i_2}^2 + \frac{2\pi^2}{3g^2 A} \sum_{i_3 = \nu_1 + \nu_2 + 1}^{\nu_1 + \nu_2 + \nu_3} m_{i_3}^2 + \ldots
\]

\[
+ \frac{2\pi^2}{Ng^2 A} \sum_{i_N = \nu_1 + \ldots}^{\nu} m_{i_N}^2,
\]

where
and the explicit form of $M_{\nu_i}$ is taken into account.

These formulae have a nice interpretation in terms of instantons: we briefly recall, therefore, the general solution of Yang-Mills equations on a compact Riemann surface $\Sigma$ of genus $G$ and with gauge group $H$ \[21\]. The space of gauge equivalent connections $A$, satisfying $D_A * F(A) = 0$ ($F$ is the curvature of $A$, $*$ is the Hodge operation and $D_A$ is the usual covariant derivative respect $A$), can be conveniently described by introducing a suitable central extension $\Gamma_R$ of the fundamental group $\Pi_1(\Sigma)$ of $\Sigma$ (the central extension is universal and the center is extended to $R$). The general theorem of Atiyah and Bott states that there is a one to one correspondence between the equivalence classes of connections which are solutions of the Yang-Mills equations on $\Sigma$ and the conjugacy classes of homomorphisms $\rho: \Gamma_R \to H$.

More explicitly the connection $A^{(\rho)}$ associated to $\rho$ has curvature $F^{(\rho)} = X^{(\rho)} \otimes \omega$, where $\omega$ is the volume form of $\Sigma$; $X^{(\rho)}$ is an element of the Lie algebra of $H$ defined by the map $d \rho : R \to \text{Lie}_H$. As far as we are concerned with the evaluation of the Yang-Mills action on the classical solutions, we only have to find all the possible $X^{(\rho)}$: this can be easily done for $H = U(N)$. The key observation is that, in our case, the homomorphism $\rho$ simply provides an unitary representation of $\Gamma_R$. When this representation is irreducible, $X^{(\rho)}$ is central with respect to the action of $H$, therefore its eigenvalues are all equal to a certain real number $\lambda$: because the Chern class of a $U(N)$ bundle is an integer, $\frac{1}{2\pi} \int \text{Tr} * F = k$, we have that the direct computation

$$ \frac{V}{2\pi} \text{Tr} X_\rho = k \quad (16) $$

completely determines

$$ \lambda = \frac{2\pi}{V} \frac{k}{N} \quad (17) $$

($V$ is the volume of $\Sigma$). If $\rho$ is not irreducible we have, in general, that $X^{(\rho)}$ is central with respect to a subgroup of $U(N)$ of the type

$$ H_X = U(N_1) \otimes U(N_2) \otimes ... \otimes U(N_r), \quad (18) $$
with \( N_1 + N_2 + \ldots + N_r = N \). This effectively reflects itself into a reduction of the original bundle structure (\( P(\Sigma, H) \) is a principal \( H \)-bundle on the manifold \( \Sigma \))

\[
P(\Sigma, U(N)) \rightarrow \sum_{i=1}^{r} \oplus P(\Sigma, U(N_i)) ;
\]

(19)

the fact that the individual Chern classes must be integers implies, by repeating the argument in eq. (16), that \( X_\rho \) has eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_r \) with multiplicities \( N_1, N_2, \ldots, N_r \):

\[
\lambda_i = \frac{2\pi k_i}{V N_i}
\]

(20)

(\( \sum_i k_i = k \) is of course the total Chern class). It is possible to prove that, fixed the topology of the bundle (that for \( U(N) \) is determined by a single integer, that is the Chern class \( k \)), the absolute minimum of the Yang-Mills functional is reached when all the eigenvalues are equal (\( \lambda = \frac{2\pi k}{V N} \) for \( U(N) \)). The action at this minima is

\[
S_{\text{min}} = \frac{2\pi^2 k^2}{g^2 V N}.
\]

(21)

The general Yang-Mills connection, for a \( U(N) \) bundle, is therefore a direct sum of Yang-Mills minima for sub-bundles, according to the decomposition eq. (19). The eigenvalues of \( X_\rho \) can be rational with denominator \( N \), when \( \rho \) is irreducible, while in the opposite situation (if a \( \rho \) reducible to \( U(1)^N \) exists) they are all integers: all the intermediate possibilities according to eq. (19) could appear.

The unitary representations of \( \Gamma_R \) were classified by Narashiman and Seshadri [28]: they have shown that, provided \( G > 1 \), to any pair \((N, k)\) is associated an irreducible representation of \( \Gamma_R \), producing the Yang-Mills minimum. The genus one case is different: in fact for \( G = 1 \), the fundamental group \( \Pi_1(\Sigma) \) is abelian and so has no irreducible representation for \( N > 1 \). Thus for \( k = 0 \) and \( N > 1 \) irreducible representations of \( \Gamma_R \) do not exist, and this corresponds to the well known fact that on \( T^2 \) every flat connection is reducible [29] (we recall that flat connections are related the the homomorphisms \( \gamma : \Pi_1(\Sigma) \rightarrow H \)). As a consequence one finds that only for \((N, k)\) coprime irreducible representations of \( \Gamma_R \) do exist. This completes our excursion on the well-established (but probably not so widely known) subject of the Yang-Mills solutions on Riemann surfaces.
It is now easy to understand eq. (15) at the light of the previous discussions: $S^{\{\nu\}}(m_1, .., m_\nu)$ is the value of the Yang-Mills action on a solution determined by $\nu_1$ eigenvalues $\lambda_{i_1} = \frac{2\pi}{A} m_{i_1}$ with multiplicity 1, $\nu_2$ eigenvalues $\lambda_{i_2} = \frac{\pi}{A} m_{i_1}$ with multiplicity 2, ..., $\nu_N$ eigenvalues $\lambda_{i_N} = \frac{2\pi}{A} m_{i_N}$ with multiplicity $N$, corresponding to the reduction $U(1)^{\nu_1} \otimes U(2)^{\nu_2} \otimes \ldots \otimes U(N)^{\nu_N}$. The only subtle point in eq. (13) is that not all $S^{\{\nu\}}(m_1, .., m_\nu)$ are produced by gauge inequivalent solutions: we have to identify, as coming from the same instanton, some contributions related to different $\{\nu_i\}$.

To understand this point let us consider the simplest cases, $U(2)$ and $U(3)$. The partition function for $U(2)$ is written explicitly

$$Z^{(1)}(2) = e^{-\frac{\beta^2}{2}} \left( \sum_{m_1, m_2 = -\infty}^{+\infty} \exp \left[ -\frac{2\pi^2}{g^2 A} (m_1^2 + m_2^2) \right] (-1)^{m_1+m_2} \left( 2\pi \frac{\beta}{g^2 A} \right) \right) + \sum_{m = -\infty}^{+\infty} \exp \left[ -\frac{\pi^2}{g^2 A} m^2 \right] \left( -1 \right)^m \left( 2\pi \frac{\beta}{g^2 A} \right)^{\frac{1}{2}};$$

(22)

the first term corresponds to solutions coming from the reduction $U(2) \rightarrow U(1) \otimes U(1)$, while the second one receives contributions only from minima (with all eigenvalues equal, proportional to $\frac{m}{2}$). Actually we have connections in both the sum producing the same value of the action (by taking $m_1 = m_2 = \hat{m}$ in the first sum and $m = 2\hat{m}$ in the second one, both representing the minimum value of the action for the Chern class $k = 2\hat{m}$): are these connections gauge inequivalent or not? The answer is negative because, as we have seen before, for $G = 1$ only when $N$ and $k$ are coprime the solutions are irreducible: the minima, when the Chern class is even, originate, therefore, from reducible connections (of type $U(1) \otimes U(1)$). The partition function is better rewritten as

$$Z^{(1)}(2) = e^{-\frac{\beta^2}{2}} \left( \sum_{m_1, m_2 = -\infty}^{+\infty} \exp \left[ -\frac{2\pi^2}{g^2 A} (m_1^2 + m_2^2) \right] \left( -1 \right)^{m_1+m_2} \left( 2\pi \frac{\beta}{g^2 A} \right) + \delta_{m_1, m_2} \left( 2\pi \frac{\beta}{g^2 A} \right)^{\frac{1}{2}} \right)$$

$$- \sum_{m = -\infty}^{+\infty} \exp \left[ -\frac{\pi^2}{g^2 A} (2m + 1)^2 \right] \frac{1}{\sqrt{2}} \left( 2\pi \frac{\beta}{g^2 A} \right)^{\frac{1}{2}},$$

(23)

where every classical contribution appears together the polynomial part coming from the quantum fluctuations: incidentally reducible and irreducible solutions are disentangled in this way. Repeating the exercise for $U(3)$ we get
\[ Z^{(1)}(3) = \frac{e^{-g^2A}}{6} \left( \sum_{m_i=-\infty}^{+\infty} \exp \left[ -\frac{2\pi^2}{g^2A}(m_1^2 + m_2^2 + m_3^2) \right] \left[ \frac{2\pi}{g^2A} \right]^\frac{3}{2} \right) - \delta_{m_2,m_3} \frac{3}{\sqrt{2}} \left( \frac{2\pi}{g^2A} \right)^{\frac{3}{2}} \\
+ \frac{2}{\sqrt{3}} \left( \frac{2\pi}{g^2A} \right)^\frac{1}{2} \delta_{m_1,m_2} \delta_{m_2,m_3} \left( \sum_{m_1,m_2=-\infty}^{+\infty} \exp \left[ -\frac{2\pi^2}{g^2A}m_1^2 - \frac{\pi^2}{g^2A}(2m_2 + 1)^2 \right] \right) \left( \frac{2\pi}{g^2A} \right)^{\frac{3}{2}} \\
+ \sum_{m=-\infty}^{+\infty} \left( \exp \left[ -\frac{\pi^2}{3g^2A}(3m + 1)^2 \right] + \exp \left[ -\frac{\pi^2}{3g^2A}(3m + 2)^2 \right] \right) \left( \frac{2\pi}{g^2A} \right)^{\frac{3}{2}}. \tag{24} \]

It is now clear that eq. (13) can be fully rewritten in this way, grouping together the classical action for gauge equivalent solutions: we prefer to maintain the form of eq.(13), that is better for our purpose.

Some remarks are now in order; first of all we notice the difference with the sphere case. There, only instantons coming from the \( U(1)^N \)-reduction were present, and consequently only integer numbers labelled the classical solutions; here we have, in general, rational numbers associated to instantons. This is the geometrical counterpart of the different algebraic form of the sums in eqs. (3,4), the exclusion of some integers in the first one resulting in a richer instantons structure. Next we remark that the genus one properties provide a simple identification for gauge equivalent contributions to the classical action (no irreducible representation is available for \( (N,k) \) not coprime). Third we notice that for \( g^2A \to 0 \), only the zero-instanton sector survives, as it should: let us discuss this limit, that is actually the important one for our final computations. Taking all the \( m_i \)'s to zero in eq.(13) we have (we disregard the exponential term that simply represents a contribution to the cosmological constant and can be eliminated by choosing a suitable renormalization condition)

\[ Z^{(1)}(0) = \sum \{ \nu_i \} \left[ \frac{2\pi}{g^2A} \right]^{\frac{3}{2}} \left[ \frac{1}{\nu_1! \nu_2! \nu_3! \cdots \nu_N!} \right] \left( \frac{2\pi}{g^2A} \right)^{\frac{3}{2}}. \tag{25} \]

To compute the sum we observe that the constraint \( \nu_1 + 2\nu_2 + \cdots + N\nu_N = N \) can be implemented by introducing an angular variable to obtain

\[ Z^{(1)}(0) = \int_0^{2\pi} \frac{d\theta}{2\pi} \exp \left[ i(\nu_1 + 2\nu_2 + \cdots + N\nu_N - N)\theta \right] \left( \frac{\pi}{\alpha} \right)^{\nu_1+\nu_2+\cdots+\nu_N} \frac{1}{\nu_1! \nu_2! \cdots \nu_N!} \left( \frac{2\pi}{g^2A} \right)^{-\frac{3}{2}}, \tag{26} \]

where we have defined \( \frac{g^2A}{2} = \alpha \). The sum over \( \nu_i \)'s is simple, giving
\[ Z_{(0)}^{(1)} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-iN\theta} \exp \left[ -\sqrt{\frac{\pi}{\alpha}} \sum_{k=1}^{N} \frac{e^{-ik\theta}(-1)^k}{k^2} \right], \tag{27} \]

that can be expressed as a contour integral in the complex plane

\[ Z_{(0)}^{(1)} = \frac{1}{2\pi i} \int_{C_0} \frac{dz}{z^{N+1}} \exp \left[ z\Phi(-z; 3/2; 1) e^{-iN\theta} \right], \tag{28} \]

where \( C_0 \) rounds the origin anticlockwise, sufficiently close so that the function \( \Phi(z; s; \mu) \)

\[ \Phi(z; s; \mu) = \sum_{k=0}^{+\infty} \frac{z^k}{(k+\mu)^s}, \tag{29} \]

is analytic. Expanding the exponential we finally have

\[ Z_{(0)}^{(1)} = \sum_{k=1}^{N} \frac{a_k(N)}{k!} \left( \frac{\pi}{\alpha} \right)^{\frac{k}{2}} \]

\[ a_k(N) = \frac{1}{2\pi i} \int_{C_0} \frac{dz}{z^{N-k}} \Phi(-z; 3/2; 1)^k. \tag{30} \]

According to general analysis \([4]\) the terms that do not vanish exponentially must be interpreted as the contribution of the flat connections to the localization formula. This can be naively understood observing that as \( g^2 A \to 0 \) the Yang-Mills partition function should be equal to the \( BF \) one

\[ Z_{Y,M} = \int \mathcal{D}A \exp \left[ -\frac{1}{2g^2} \int \text{Tr} \left[ F^* F \right] \right] = \int \mathcal{D}A \mathcal{D}B \exp \left[ \int \left( \text{Tr} \left[ BF \right] + \omega g^2 \text{Tr} \left[ B^2 \right] \right) \right] \]

\[ Z_{BF} = \int \mathcal{D}A \mathcal{D}B \exp \left[ \int \text{Tr} \left[ BF \right] \right] = \int \mathcal{D}A \delta(F). \tag{31} \]

If the gauge group \( H \) and the Riemann surface \( \Sigma \) are such that the space of flat connections is smooth and the gauge group acts freely on it, the zero-instanton sector is a polynomial in \( g^2 A \). The zero-order term is the symplectic volume of \( \mathcal{M}_F(\Sigma, H) \) while the coefficients of the other terms have an interpretation as intersection numbers on the moduli space. When \( H \) and \( \Sigma \) are such that the space of flat connections is not smooth or the gauge group does not act freely on it (as in the case of \( U(N) \) on \( T^2 \)) non-analyticity appears in the limit \( g^2 A \to 0 \) (as we have observed in our specific case). It could be possible, as suggested in \([4]\), to interpretate the singularities in eq.\(30\) as coming from the singular structure of \( \mathcal{M}_F(\Sigma, H) \), but this is beyond the purposes of the present paper. We simply note, that, because we are

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interested to take eventually the decompactification limit on the zero-instanton sector, only
the term proportional to \( \left( \frac{\pi}{\alpha} \right)^{\frac{1}{2}} \) will be relevant
\[
Z^{(1)}_{(0)} \rightarrow \frac{(-1)^{N+1}}{N^{\frac{3}{2}}} \left( \frac{2\pi}{g^2 A} \right)^{\frac{1}{2}}
\] (32)
when \( A \rightarrow +\infty \).

III. INSTANTON CONTRIBUTIONS TO WILSON LOOPS: THE \( U(2) \) AND \( U(3) \)
CASE

The next step is to derive the instanton representation for an homologically trivial, non
self-intersecting Wilson loop on \( T^2 \): we want to consider, in particular, the zero-instanton
result and eventually take the limit of infinite area, in order to make a comparison with \[10\].
In genus one the general formula \[2\] gives
\[
\mathcal{W}^{(1)} = \frac{1}{Z^{(1)} N} \sum_{R,S} \frac{d_R}{d_S} \exp \left[ -\frac{g^2 A_1}{2} C_2(R) - \frac{g^2 A_2}{2} C_2(S) \right] \int dU \text{Tr}[U] \chi_R(U) \chi_S^\dagger(U),
\] (33)
where \( A_1 + A_2 = A \) are the areas singled out by the loop, \( A_1 \) being the simply connected one;
the integral in eq. (33) is over the \( U(N) \) group manifold while \( \chi_{R,S}(U) \) is the character of
the group element \( U \) in the \( R(S) \) representation. We would like to express eq. (33) through
the set of integers defining \( R \) and \( S \), to compute the integral and, at the end, to perform a
Poisson resummation. The first two steps are easily done, using the explicit formula for the
characters of \( U(N) \) in terms of the \( n_i \)’s and the symmetry properties of the Vandermonde
determinants:
\[
\mathcal{W}^{(1)} = \frac{1}{Z^{(1)} N} \sum_{n_1 \neq n_2 \neq \ldots \neq n_N} \frac{N!}{(n_1 - n_j - 1)} \exp \left[ -\frac{g^2 A}{2} \sum_{i=1}^{N} \left( n_i - \frac{N - 1}{2} \right)^2 - g^2 A_1 (n_1 - \frac{N}{2}) \right]
\] (34)
we have neglected the cosmological constant contribution, that does not play any role and
actually cancels out in the final result with the same term in the partition function. Un-
fortunately the Poisson resummation cannot be performed simply following the lines of the
previous section because of the presence of the denominators in eq. (34): when \( n_1 \) is equal to some \( n_i \)'s the denominator develops a potential singularity, and therefore we need a more careful analysis (we cannot extend the sum everywhere and then subtract the coincident subsums). Let us consider, for the moment, the simplest case, \( U(2) \)

\[
\mathcal{W}^{(1)} = \frac{1}{2} Z^{(1)} \sum_{n_1 \neq n_2} \frac{(n_1 - n_2 - 1)}{(n_1 - n_2)} \exp \left[ -\frac{g^2 A}{2} (n_1^2 + n_2^2) + \frac{g^2 A}{2} (n_1 + n_2 - \frac{1}{2}) - g^2 A_1 n_1 \right].
\]  

(35)

The idea is to use a contour representation for this sum: the basic identity we need is

\[
\sum_{n \neq m} \frac{f(m)}{m - n} = \sum_{n = -\infty}^{+\infty} \frac{1}{2i} \int_C dz \cot(\pi z) f(z) - \sum_{n = -\infty}^{+\infty} f'(n),
\]

(36)

where \( C \) is a contour on the complex plane enclosing the real axis anti-clockwise (we assume, of course, that \( f(z) \) has a good behaviour at \( \pm \infty \) near the real axis and has no singularities inside the contour). We have therefore two different contributions in eq. (35):

\[
\mathcal{W}^{(1)} = \frac{1}{2} Z^{(1)} \left[ B_1 + B_2 \right],
\]

(37)

with

\[
B_1 = \sum_{n = -\infty}^{+\infty} \exp \left[ -\alpha (n^2 - n + \frac{1}{2}) \right] \left( \frac{1}{2i} \int_C dz \cot(\pi z) \frac{z - n + \frac{1}{2}}{z - n} \exp \left[ -\alpha z^2 + z(\alpha - 2\alpha_1) \right] \right),
\]

(38)

\[
B_2 = \sum_{n = -\infty}^{+\infty} \exp \left[ -\alpha (n^2 - n + \frac{1}{2}) \right] \frac{d}{dz} \left( (z - n + 1) \exp \left[ -\alpha z^2 + z(\alpha - 2\alpha_1) \right] \right)_{z = n},
\]

(39)

being \( \alpha_1 = \frac{g^2 A}{2} \). The Poisson resummation can be done without problems on \( B_2 \), obtaining

\[
B_2 = \left( \frac{\pi}{2\alpha} \right)^{\frac{1}{2}} \exp \left[ -\alpha_1 + \frac{\alpha_1^2}{2\alpha} \right] \sum_{m = -\infty}^{+\infty} \exp \left[ -\frac{\pi^2 m^2}{2\alpha} + i\pi m \frac{\alpha_1}{\alpha} \right] (-1)^m (1 - \alpha_1 + \pi im).
\]

(40)

\( B_1 \) deserves a deeper study: using the Laurent expansion for \( \cot(\pi z) \) and explicitly parametrizing \( C \) with a real parameter \( t \), one easily gets the Poisson sum:

\[
B_1 = \sum_{m_1 = -\infty}^{+\infty} \sum_{m_2 = -\infty}^{+\infty} \int_{-\infty}^{+\infty} dx dy \exp \left[ 2\pi i (m_1 x + m_2 y) \right] \exp \left[ -\alpha x^2 + \alpha x - \frac{\alpha}{2} \right] \frac{1}{\pi} \int_{-\infty}^{+\infty} dt(t - x + 1) \text{Im} \left[ \frac{1}{t - y + ie} \frac{1}{t - x + ie} \right] \exp \left[ -\alpha t^2 + (\alpha - 2\alpha_1)t \right].
\]

(41)
One can check, evaluating the imaginary part, that the only non vanishing contribution is

\[
B_1 = \sum_{m_1, m_2 = -\infty}^{+\infty} \int_{-\infty}^{+\infty} dx \, dy \exp \left[ 2\pi i (m_1 x + m_2 y) - \alpha (x^2 - x + \frac{1}{2}) - \alpha (y^2 - y) - 2\alpha_1 y \right] \frac{1}{\pi} (y - x + 1) \text{CPV} \left( \frac{1}{y - x} \right).
\]

We can arrive to a compact formula by using the Schwinger representation of the denominator in eq. (42) and integrating on \( x \) and \( y \):

\[
B_1 = \frac{\pi}{2\alpha} \exp \left[ -\alpha_1 + \frac{\alpha_1^2}{2\alpha} \right] \sum_{m_1, m_2 = -\infty}^{+\infty} \exp \left[ -\frac{\pi^2 (m_1^2 + m_2^2)}{\alpha} + 2i\pi m_1 \frac{\alpha_1}{\alpha} \right] (-1)^{m_1 + m_2} \left[ 1 - \int_0^{+\infty} dt \exp \left[ -\frac{t^2}{2\alpha} \right] \sin \left[ t \left( \frac{\alpha_1}{\alpha} + i\pi (m_2 - m_1) \right) \right] \right].
\]

The final result can be presented as

\[
\mathcal{W}^{(1)} = \frac{\exp \left[ -\alpha_1 \right]}{2 \mathcal{Z}^{(1)}} \left[ \exp \left[ \frac{\alpha_1^2}{2\alpha} \right] \sum_{m_1, m_2 = -\infty}^{+\infty} \exp \left[ -\frac{\pi^2 (m_1^2 + m_2^2)}{\alpha} + 2i\pi m_1 \frac{\alpha_1}{\alpha} \right] (\frac{\pi}{\alpha}) W(m_1, m_2) + \exp \left[ \frac{\alpha_1^2}{2\alpha} \right] \sum_{m = -\infty}^{+\infty} \exp \left[ -\frac{\pi^2 m^2}{2\alpha} + i\pi m \frac{\alpha_1}{\alpha} \right] (\frac{\pi}{2\alpha})^{\frac{1}{2}} W(m) \right].
\]

with

\[
W(m_1, m_2) = (-1)^{m_1 + m_2} \left[ 1 - (\alpha_1 + i\pi (m_2 - m_1)) \right] _1 F_1 \left( 1; \frac{3}{2}; -\frac{(\alpha_1 + i\pi (m_2 - m_1))^2}{2\alpha} \right),
\]

\[
W(m) = (-1)^{m+1} (1 - \alpha_1 - i\pi m),
\]

\(_1 F_1 (\gamma; \beta; z)\) being a confluent hypergeometric function [22]. In this form, the result for the Wilson loop closely resembles the structure of eq. (22); we have the classical action and the holonomy phase evaluated on the instanton solutions, and the quantum contributions around them, represented by \( W(m_1, m_2), W(m) \). Notice that they naturally appear ordered by their singularity degree in \( \alpha \), a fact that is welcome because we would like to discuss the asymptotic behaviour in \( \alpha \). The zero-instanton sector gives, of course, the non-exponentially suppressed contribution as \( \alpha \to 0 \)

\[
\mathcal{W}^{(1)}_0 = \frac{\exp \left[ -\frac{g^2 A_1}{2} \right]}{1 - \sqrt{2} (\frac{g^2 A_1}{2\pi})^{\frac{1}{2}}} \left[ \exp \left[ \frac{g^2 A_1^2}{2A} \right] (1 - \frac{g^2 A_1^2}{2} _1 F_1 (1; \frac{3}{2}; -\frac{g^2 A_1^2}{4A})) - \exp \left[ \frac{g^2 A_1^2}{4A} \right] \sqrt{2} (\frac{g^2 A_1}{2\pi})^{\frac{1}{2}} (1 - \frac{g^2 A_1}{2}) \right].
\]

(46)
Before discussing eq. (46), it is better to recall the results of [10], where the analogous quantity was derived on the genus zero (for arbitrary $N$). There we obtained

$$\mathcal{W}_0^{(0)} = \frac{1}{N} \exp \left[ -\frac{g^2 A_1 A_2}{2A} \right] L_{N-1}^1 \left( \frac{g^2 A_1 A_2}{A} \right). \tag{47}$$

In the decompactification limit $A \to \infty$, $A_1$ fixed ($A_1$ and $A_2$ are the areas singled out by the loop on the sphere, $A = A_1 + A_2$), the quantity in the equation above exactly coincides, for any value of $N$, with eq.(11) of ref. [18], which was derived following completely different considerations. We recall indeed that their result was obtained by a full resummation at all orders of the perturbative expansion of the Wilson loop in terms of Yang-Mills propagators in light-cone gauge, endowed with the WML prescription. We notice that $\mathcal{W}_0^{(0)}$ does not exhibit the usual area-law exponentiation; actually, in the large-$N$ limit, exponentiation (and thereby confinement) is completely lost, as first noticed in [18]. As a matter of fact, from eq.(47), taking the limit $N \to \infty$, we easily get

$$\lim_{N \to \infty} \mathcal{W}_0 = \sqrt{\frac{A_1 + A_2}{g^2 A_1 A_2}} J_1 \left( \sqrt{\frac{4g^2 A_1 A_2}{A_1 + A_2}} \right) \tag{48}$$

with $g^2 = g^2 N$. At this stage, however, this is no longer surprising since $\mathcal{W}_0^{(0)}$ does not contain any genuine non perturbative contribution. If on the sphere $S^2$ we consider the weak coupling phase $g^2 A < \pi^2$, instanton contributions are suppressed. As a matter of fact, eq.(18) provides the complete Wilson loop expression in the weak coupling phase [3,7]. In turn confinement occurs in the strong coupling phase [8]. For any value of $N$ the pure area law exponentiation follows, after decompactification, from resummation of all instanton sectors, changing completely the zero sector behaviour and, in particular, the value of the string tension.

Taking into account these considerations, we can now proceed to discuss eq. (46), that is the exact analogous on $T^2$ of eq. (17) (in the $N = 2$ case). Here the zero-instanton contribution is much more complicated, as we could expect due to the presence of a non trivial moduli space of flat connections and to an intrinsic asymmetry between $A_1$ and $A_2$. Moreover no phase transition has been found, at large $N$, on the torus, so we do not expect that the
zero-instanton contribution is related to some regime of the complete theory. Nevertheless when \( A \to +\infty \), with \( A_1 \) fixed, we exactly recover the decompactification limit of eq. (47)

\[
\mathcal{W}^{(1)}_0 \to \frac{1}{2} \exp \left[ -\frac{g^2 A_1}{2} \right] \left( 1 - \frac{g^2 A_1}{2} \right),
\]

(49)

confirming the independence of the zero-instanton contribution, in the decompactification limit, from the topology chosen and coinciding with the WML computation on the plane. We notice that, looking at eq. (45), the quantum fluctuation around the irreducible instantons is

\[
\mathcal{W}^{(1)}_{\text{irr.}} = \exp \left[ -\frac{\alpha_1}{2} \right] \exp \left[ -\frac{\alpha_1^2}{2\alpha} \right] \sum_{m=-\infty}^{+\infty} \exp \left[ -\frac{\pi^2(2m+1)^2}{2\alpha} + 2i\pi(m+\frac{1}{2})\frac{\alpha_1}{\alpha} \right] W(m),
\]

(50)

that is as well expressed through the same Laguerre polynomial. In other words, when the Chern class is odd, the contribution to the Wilson loop coming from fluctuations around the minimum of the Yang-Mills action seems to mimic the WML result.

Coming back to eq. (46), it is possible to recover WML even not performing explicitly the decompactification limit: taking \( g^2 A \) and \( g^2 A_1 \) fixed and performing an expansion in \( \frac{g^2 A_1^2}{A} \), the first term is again eq. (49). We don’t know the meaning, if any, of this kind of expansion.

The computation for \( U(2) \) can be extended to \( U(3) \), with an increasing complexity in the non-irreducible sector. The expression to be Poisson resummed is this time:

\[
\mathcal{W}^{(1)} = \frac{1}{6Z^{(1)}} \sum_{n_1 \neq n_2 \neq n_3} \left[ 1 + \frac{2}{(n_1 - n_2)} + \frac{1}{(n_1 - n_2)(n_1 - n_3)} \right] \exp \left[ -\alpha \sum_{n_i=1}^{3} n_i^2 + \alpha_1 n_1 \right],
\]

(51)

and everything goes as in the \( U(2) \) case, except that the third term needs a bit more work: we have to extract a second derivative and the square of a first derivative, but nothing conceptually novel happens. We simply obtain a somewhat more complicated quantum contribution, involving integrals similar to the previous ones. The final expression is nevertheless cumbersome, reproducing the expected expansion in the same instantons of eq. (24): we present only the result for the zero-instanton sector.
\[ \mathcal{W}_0^{(1)} = \frac{\exp[-\alpha_1]}{1 - \frac{3}{\sqrt{2}} \pi \frac{\alpha}{\sqrt{3}} + \frac{2 \alpha}{\sqrt{3}} \pi} \left[ \exp\left[ -\frac{\alpha_1^2}{3 \alpha} \right] \frac{2 \alpha}{\sqrt{3}} \pi \left( 1 - 2 \alpha_1 + \frac{(2 \alpha_1)^2}{6} \right) \right. \\
- \exp\left[ \frac{\alpha_1^2}{2 \alpha} \right] 3 \left( \frac{\alpha}{2 \pi} \right)^{\frac{3}{2}} \left[ 1 - \frac{2}{3} \alpha_1 - \frac{4}{3} \alpha_1 F_1(1; \frac{3}{2}; -\frac{\alpha_1^2}{2 \alpha}) + \frac{4}{3} \alpha_1 F_1(1; \frac{3}{2}; -\frac{\alpha_1^2}{6 \alpha}) \right] \\
+ \exp\left[ \frac{\alpha_1^2}{\alpha} \right] \left( 1 - 2 \alpha_1 F_1(1; \frac{3}{2}; -\frac{\alpha_1^2}{2 \alpha}) + \alpha_1^2 I_1 \right) \right] \]

where

\[ I_1 = \int_0^{+\infty} dt_1 dt_2 \exp\left[ -\frac{t_1^2}{2 \alpha} - \frac{t_2^2}{2 \alpha} - t_1 t_2 \right] \sin(t_1 \frac{\alpha_1}{\alpha}) \sin(t_2 \frac{\alpha_1}{\alpha}). \]

In spite of the complicated expression, the limit of infinite area is simple

\[ \mathcal{W}_0^{(1)} \to \frac{1}{3} \exp\left[ -\frac{g^2 A_1}{2} \right] \left( 1 - g^2 A_1 + \frac{(g^2 A_1)^2}{6} \right) = \exp\left[ -\frac{g^2 A_1}{2} \right] \frac{1}{3} L_2^1(g^2 A_1); \]

the WML result is again recovered. Explicitly working out eq. (52), one can check that the expansion in \( \frac{g^2 A_1^2}{A} \) still reproduces the same Laguerre polynomial.

Eq. (52) strongly suggests that the computation of the zero-instanton sector, with \( N \) generic, is really involved for the Wilson loops: nevertheless, in the decompactification limit, only the term proportional to \( \langle \frac{\pi}{\alpha} \rangle^{\frac{1}{2}} \) is relevant. We do not try therefore to compute the full contribution, but, in the next section, we shall focus our attention only on the leading term.

**IV. INSTANTON CONTRIBUTIONS TO WILSON LOOPS: GENERAL RELATION WITH THE PERTURBATIVE WML RESULT**

We are interested in computing the zero-instanton contribution to eq. (34) for general \( N \), in the limit in which \( g^2 A \to +\infty \). To this aim let us introduce a different set of integers and consider the group \( U(N + 1) \),

\[ n_1 \to n, \]
\[ n_1 - n_{j+1} \to -n_j, \quad \text{for} \quad j = 1, ..., N. \]

Eq. (34) takes then the form
\[ W^{(1)} = \frac{1}{\mathcal{Z}^{(1)}(N+1)!} \sum_{n_i \neq n_j \neq 0} \prod_{k=1}^{N} \left( \frac{n_k + 1}{n_k} \right) \exp \left[ -\alpha \sum_{i=1}^{N} n_i^2 \right] \]
\[
\sum_{n=-\infty}^{+\infty} \left[ -\alpha (N+1) \left( n - \frac{N}{2} \right)^2 + 2(n - \frac{N}{2}) \left( \alpha_1 - \alpha \sum_{i=1}^{N} n_i \right) \right].
\] (56)

We can Poisson resum over \( n \)
\[ W^{(1)} = \exp \left[ -\alpha_1 + \frac{\alpha_1^2}{(N+1)\alpha} \right] \left( \frac{\pi}{(N+1)\alpha} \right)^{\frac{1}{2}} \sum_{-\infty}^{+\infty} \exp \left[ -\frac{\pi^2 m^2}{(N+1)\alpha} + \frac{2\pi i m}{(N+1) \alpha} \right] (-1)^{mN} \]
\[
\sum_{n_i \neq n_j \neq 0} \prod_{k=1}^{N} \left( \frac{n_k}{n_k + 1} \right) \exp \left[ -\alpha \left( \sum_{i=1}^{N} n_i^2 + \frac{1}{N+1} \left( \sum_{i=1}^{N} n_i \right)^2 \right) - \frac{2}{N+1} \left( \alpha_1 - i m \right) \sum_{i=1}^{N} n_i \right].
\] (57)

In eq. (57) the first line is exactly what we expect for the contribution of irreducible instantons, that, when the quantum number is taken to zero, is the same, in the decompactification limit, of the zero-instanton sector (as the factor \( \frac{\pi}{(N+1)\alpha} \)^{\frac{1}{2}} clearly shows). On the other hand we can argue, from the results of the previous section, that when \( m \) is prime with respect to \( N+1 \), the full contribution to the Wilson loop is likely to be a Laguerre polynomial. The crucial point is therefore to extract, from the very complicated function obtained by Poisson resumming over the \( n_i \)'s, the part not involving other instanton numbers neither modifying the singularity behaviour in \( \alpha \). The basic object we have to study is the sum \((\xi = 2\alpha_1 - i\pi m) / (N+1)\):
\[
I(\alpha, \xi) = \sum_{n_i \neq n_j \neq 0} \prod_{k=1}^{N} \left( \frac{n_k}{n_k + 1} \right) \exp \left[ -\alpha \left( \sum_{i=1}^{N} n_i^2 + \frac{1}{N+1} \left( \sum_{i=1}^{N} n_i \right)^2 \right) - \xi \sum_{i=1}^{N} n_i \right]; \quad (58)
\]
what we need is to pass to the “dual” integers \( m_i \) by using the technique developed in the previous section and to evaluate the contribution at \( m_i = 0 \). The important observation is that, as far as we are concerned with this type of computation, the presence of \( \alpha \) in the exponential can be disregarded: the difference is relevant only in the non-leading behaviour as \( \alpha \to +\infty \): the exact \( \alpha \) dependence, for the leading term in the zero-instanton sector, is factorized in the term \( \exp \left[ -\frac{\alpha_1^2}{(N+1)\alpha} \right] \left( \frac{\pi}{(n+1)\alpha} \right)^{\frac{1}{2}} \) appearing in eq. (57). With this remark in mind, we can extract the relevant part from the series, that we call \( I(\xi) \), without worrying about the convergence (for which the presence of \( \alpha \) is instead essential). Eq. (58) can be rewritten using the conjugacy class formalism as
I(\xi) \simeq \sum_{\{\nu_i\}} \sum_{n_i \neq 0} \exp \left[ i\pi \sum_{i=\text{even}} \nu_i \right] M(\nu_i) \prod_{i_1=1}^{\nu_1} \frac{n_{i_1} + 1}{n_{i_1}} \prod_{i_2=\nu_1+1}^{\nu_1+\nu_2} \left( \frac{n_{i_2} + 1}{n_{i_2}} \right) \ldots \exp \left[ -\xi \left( \sum_{i_1=1}^{\nu_1} n_{i_1} + 2 \sum_{i_2=\nu_1+1}^{\nu_1+\nu_2} n_{i_2} + \ldots + N \sum_{i_N=\nu_1+\ldots}^{\nu} n_{i_N} \right) \right]. \tag{59}

Due to the absence of \alpha we have the miraculous factorization

I(\xi) \simeq \sum_{\{\nu_i\}} \exp \left[ i\pi \sum_{i=\text{even}} \nu_i \right] M(\nu_i) \prod_{k=1}^{N} \left[ \sum_{n \neq 0} \left( \frac{n + 1}{n} \right)^k \exp \left[ \xi kn \right] \right]^{\nu_k}. \tag{60}

Let us examine the sum \sum_{n \neq 0} \left( \frac{n + 1}{n} \right)^k \exp \left[ \xi kn \right] : it is not convergent, of course (as we said we missed the convergence factor, quadratic in the \emph{n}_i's, in the exponential and proportional to \alpha), but it captures the relevant behaviour. In fact the zero-term in the dual integers coming from the Poisson resummation originates from

\sum_{n \neq 0} \left( \frac{n + 1}{n} \right)^k \exp \left[ \xi kn \right] \simeq \frac{1}{2i} \int_{\mathcal{C}} \frac{dz}{z} \cot(\pi z) (\frac{z + 1}{z})^k \exp \left[ -\xi z \right] \nonumber \\
- \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{dz}{z} \cot(\pi z) (\frac{z + 1}{z})^k \exp \left[ -\xi z \right], \tag{61}

where \mathcal{C} is the contour considered in eq. (60) while \mathcal{C}_0 is a small circle around the origin in the complex plane (we remark again that we do not mind about the convergence of the first integral; it would be cured inserting the \alpha dependence and it would be relevant only for the non-leading behaviour in \alpha). The \emph{m}_i = 0 term comes therefore from the second integral and can be nicely evaluated to be:

\frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{dz}{z} \cot(\pi z) (\frac{z + 1}{z})^k \exp \left[ -\xi z \right] = L_k^0(\xi k), \tag{62}

\text{L}_k^0(x) \text{ being the usual Laguerre polynomial of degree } k. \text{ In eq. (60) we have to consider only terms of the this type, obtaining}

\left[ \sum_{n \neq 0} \left( \frac{n + 1}{n} \right)^k \exp \left[ \xi kn \right] \right]^{\nu_k} \simeq (-1)^{\nu_k} \left[ L_k^0(\xi k) \right]^{\nu_k}. \tag{63}

The exact leading contribution turns out to be

I(\xi) = \sum_{\{\nu_i\}} \exp \left[ i\pi \sum_{i=\text{even}} \nu_i \right] M(\nu_i) \prod_{k=1}^{N} \left[ L_k^0(\xi k) \right]^{\nu_k} : \tag{64}
exploiting the same trick in eq. (20) we get

\[ I(\xi) = (-1)^N \sum_{\nu_k = 0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-iN\theta} \prod_{k=1}^N \left[ \frac{L_0^0(\xi_k)}{k} \right]^{\nu_k} \exp \left[ i\theta \nu_k \right] \nu_k ! \]

\[ = (-1)^N N! \int_{C_0} \frac{dz}{z^{N+1}} \exp \left[ \sum_{k=1}^N \frac{L_0^0(\xi_k) z^k}{k} \right]. \]  

(65)

We have the elements to extract the zero-instanton contribution in the limit \( \alpha \to +\infty \): for the partition function

\[ Z_{(0)}^{(1)} \to (-1)^N \frac{1}{N + 1} \left( \frac{\pi}{(N + 1)\alpha} \right)^{\frac{1}{2}}, \]  

(66)

then it follows (for \( m = 0 \) we have \( \xi = \frac{2\alpha_1}{N + 1} \))

\[ \mathcal{W}_0^{(1)} \approx \exp \left[ -\alpha_1 + \frac{\alpha_1^2}{(N + 1)\alpha} \right] \frac{1}{2\pi i} \int_{C_0} \frac{dz}{z^{N+1}} \exp \left[ \sum_{k=1}^{\infty} \frac{L_0^0(\frac{2\alpha_1}{N + 1}) z^k}{k} \right] ; \]  

(67)

We have extended the sum in the exponential of eq. (65) to \( +\infty \), taking the contour \( C_0 \) sufficiently close to the origin. In the decompactification limit we end up with

\[ \mathcal{W}_0^{(1)} = \exp \left[ -\frac{g^2 A_1}{2} \right] \frac{1}{2\pi i} \int_{C_0} \frac{dz}{z^{N+1}} \exp \left[ \sum_{k=1}^{\infty} \frac{L_0^0(\frac{g^2 A_1}{N + 1}) z^k}{k} \right] ; \]  

(68)

that is the desired result: the calculation of an homologically trivial Wilson loop on the torus, taking into account only the zero-instanton sector and then decompactifying the torus itself.

We see that if the formula

\[ \frac{1}{N + 1} L_N^1(x) = \frac{1}{2\pi i} \int_{C_0} \frac{dz}{z^{N+1}} \exp \left[ \sum_{k=1}^{\infty} \frac{L_0^0(\frac{kx}{N + 1}) z^k}{k} \right] ; \]  

(69)

is true, we have full agreement with the WML computation on the plane and with the same calculation done from the sphere. We have not been able to prove eq. (69) for general \( N \), but we have used MAPPLE to perform the analytical check (the integral is simply the \( N \)-derivative of the exponential respect to \( z \), evaluated in \( z = 0 \)) till \( N = 25 \). The general proof should not be trivial, because a non-linearity, quickly increasing with \( N \), involves different types of Laguerre polynomials. If our conjecture is true (and we think that the MAPPLE result and the perfect agreement with WML are not accidental), we have
proven that on genus one, when only the zero-instanton sector is taken into account, the Wilson loop averages coincide with the perturbative WML resummation on the plane (in the decompactification limit). This suggests that the WML prescription is truly related to the local behaviour of YM$_2$, considering only the trivial solution in a path-integral expansion. We notice that while on the sphere flat connections do not exist (in the sense that they are all gauge equivalent to the trivial one) on the torus, although reducible, they are definitively there, complicating the expansion around $F = 0$; nevertheless the leading term as $A \to +\infty$ still reproduces WML. This fact has probably some meanings in relation to the structure of $\mathcal{M}_\mathcal{F}(T^2, U(N))$, but we do not enter into the question. Moreover we see that all the irreducible connections (minima of the Yang-Mills action when the Chern class $k$ is prime with respect to $N$) contribute to the Wilson loop with $L^1_N(g^2 A_1 - i\pi m)$, as it is evident from eq. (65). The fluctuation around these particular instantons (that are the absolute minima inside the topological class $k$ coprime to $N$) exactly produces the Laguerre polynomial, a fact that probably deserves a deeper interpretation. Actually we notice that on the sphere too there were minima contributing in a similar way: it is not difficult to show that when the Chern class is $k = mN$ the minimum of the action is obtained for $m_i = m$ and in the expression for $W(m_1, .., m_N)$ the usual Laguerre polynomial appears.

V. CONCLUSIONS

The instanton contributions to the $U(N)$ Yang-Mills partition function and to homologically trivial Wilson loops on $T^2$ have been carefully examined. We have derived the instanton representation for the partition function and identified the contributions coming from the classical solutions of Yang-Mills equations on genus one: they appear according to the general classification of Atiyah and Bott [21], allowing for rational values of the instanton numbers, at variance with the sphere case, where only integer Dirac monopoles were present. We discussed the singularities of the zero-instanton sector and their relevance in the decompactification limit. Then we developed a technique to perform the Poisson resum-
mation for homologically trivial Wilson loops: at variance with the sphere case we are faced with potential singularities in passing from the Young tableaux indices $n_i$’s (coming from the Migdal’s representation of the Wilson loop averages) to the “dual” integers $m_i$’s (labelling the instanton solutions). We presented the full result in the simplest cases, namely $U(2)$ and $U(3)$, that are nevertheless quite complicated. The Wilson loop appears as a sum over classical contributions (coming from the Yang-Mills solutions) modified by finite quantum corrections. The zero-instanton part is particularly interesting and can be expressed through confluent hypergeometric functions: in the large area limit, it reproduces the WML perturbative resummation. We have extended this computation to general $N$, finding agreement with the sphere and the plane results if a particular mathematical identity, eq. (69), holds: from a mathematical point of view, it should be non-trivial to prove it in general (we checked it analytically till $N = 25$ by using MAPPLE).

Our computation on $T^2$ has therefore confirmed the conclusion in [10], namely that there is no contradiction between the use of the WML prescription in the light-cone propagator and the pure area law exponentiation; this prescription is correct, but the ensuing perturbative calculation can only provide us with the expression for $W_0$, the result coming from the decompactification of the sphere and of the torus, when instantons are disregarded. The paradox of ref. [18] is solved by recognizing that they did not take into account the genuine $\mathcal{O}(\exp(-\frac{1}{g^2}))$ non perturbative quantities. As a matter of fact, the Migdal’s formula for the Wilson loop eq. (33) can be understood as a full summation of perturbative and non-perturbative contributions: in the limit $A \to +\infty$, $A_1$ finite, it simply reproduces the pure area law. We notice that, from the torus point of view, even the full contribution of the flat connections must be considered to recover the pure area result.

What might instead be surprising in this context is the fact that, using the instantaneous 't Hooft-CPV potential and just resumming at all orders the related perturbative series, one still ends up with the correct pure area exponentiation. This feature is likely to be linked to some peculiar properties of the light-front vacuum (we remind the reader that the light-cone CPV prescription follows from canonical light-front quantization [17]; still we know it is
perturbatively unacceptable in higher dimensions [30] and cannot be smoothly continued to any Euclidean formulation).

From another point of view, we find interesting the possibility to relate the zero-instanton sector to the structure of $\mathcal{M}_F(\Sigma, U(N))$, and, more generally, to explore homologically non-trivial loops in this context. On the other hand we think that the relation between our results (that are obtained in the spirit of the semiclassical expansion, according to the localization principle) and the recent computations of [23,25], where topological sectors appeared to play a crucial role, deserves further studies. A first step in this direction should be, for example, to understand how the instanton solutions are connected to the topological obstructions to smoothly diagonalize two-dimensional Lie-valued maps [31]. Investigations in these directions are in progress.

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