A q-Deformed Schrödinger Equation

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Abstract We found hermitian realizations of the position vector \( \vec{r} \), the angular momentum \( \vec{\Lambda} \) and the linear momentum \( \vec{p} \), all behaving like vectors under the \( su_q(2) \) algebra, generated by \( L_0 \) and \( L_\pm \). They are used to introduce a \( q \)-deformed Schrödinger equation. Its solutions for the particular cases of the Coulomb and the harmonic oscillator potentials are given and briefly discussed.
1. Introduction

The general framework of the present study is the theory of quantum $su_q(2)$ algebra which has been the subject of extensive developments. Our purpose is to derive a $q$-deformed Schrödinger equation invariant under the $su_q(2)$ algebra. Here we discuss the case of spinless particles. So far, a general procedure (see for example [1]-[3]) was to write down the Hamiltonian in spherical coordinates and replace the $su(2)$ Casimir operator $C = \vec{L}^2$ by $C_q + f(q)$ where $q$ is the deformation parameter, $C_q$ the Casimir operator of the $su_q(2)$ algebra and $f(q)$ an arbitrary function with the property $f(q) \to 0$ when $q \to 1$. Of course this method introduces arbitrariness through the function $f$ and sometimes anomalies as for example a bound spectrum [2] for the free Hamiltonian. Here we aim at removing such kind of arbitrariness and anomalies.

The novelty of our study is that we search for hermitian realizations of the position, momentum and angular momentum operators behaving as vectors with respect to $su_q(2)$ algebra, generated by the operators $L_0$ and $L_\pm$. We shall show that the angular momentum entering the expression of the Hamiltonian has components $\Lambda_0$ and $\Lambda_\pm$, different from $L_0$ and $L_\pm$. This leads to a proper behaviour of the free Hamiltonian. Here we consider two cases of central potentials: the harmonic oscillator and the Coulomb potential. Once the Hamiltonian is constructed we are able to derive both the spectrum and the eigenfunctions in a consistent way for each case. Our arguments are as follows.

The usual quantum mechanics of a point-like particle is constructed from two vectors: the position vector $\vec{r}$ and the linear momentum $\vec{p} = -i\hbar \vec{\nabla}$. These two vectors are used to build all the other quantities, as e.g. the angular momentum, the interaction potentials, etc., according to the classical rules. In general, these operators do not commute, their commutation relations following from the commutation relations of $\vec{r}$ and $\vec{p}$.

In a $q$-deformed quantum mechanics the commutation relations between the generators of the $su_q(2)$ algebra, $L_i$, and the position vector $\vec{r}$ are well defined inasmuch as $\vec{r}$ is considered a $q$-tensor of rank one (see next section). Therefore it is natural to take $r_i$ as the basic quantities from which all the others should be built. Then in deriving a $q$-deformed Schrödinger Hamiltonian, invariant under the $su_q(2)$ algebra, we searched for a realization of the linear momentum $\vec{p}$ entering the kinetic energy term. First it was necessary to find a realization for $\vec{r}$ and for $L_i$ as self adjoint operators obeying commutation relations characteristic to the deformed algebra. Then we looked for a realization of $\vec{p}$ in terms of $\vec{r}$ and of $L_i$. We found that $\vec{p}$ can be written as a sum of two terms which are parallel and perpendicular to $\vec{r}$ respectively. As discussed below, the parallel component of $\vec{p}$ is assumed to have the simplest possible form and is written as $-i\frac{\vec{r}}{r} \left( r \frac{\partial}{\partial r} + 1 \right)$ while the perpendicular one must be expressed as a vector product of $\vec{r}$ and of $\hat{\Lambda}$.

The paper is organized as follows. Section 2 contains the general commutation relations involving the $q$-angular momentum. We introduce some quantities having definite
transformation properties with respect to the $su_q(2)$ algebra, namely the invariants $C$, $C'$ and $c$ and the vector $\vec{\Lambda}$ related to $\vec{L}$.

In the third section we propose a realization of the position vector $\vec{r}$ and consistently of the $q$-angular momentum $\vec{L}$, in terms of spherical coordinates $r$, $x_0 = \cos \theta$ and $\varphi$, as for example in Refs. [4],[5].

The realization of the linear momentum $\vec{p}$ is considered in the fourth section. We first build the part of $\vec{p}$ perpendicular to $\vec{r}$, denoted by $\vec{\partial}$. This is achieved by using the cross product $\vec{r} \times \vec{\Lambda}$. We find that the components of $\vec{\partial}$ satisfy the same type of commutation relations as the components of $\vec{r}$.

Section 5 introduces the eigenfunctions of the $q$-deformed angular momentum written as power series of $x_0 = \cos \theta$. We show that the result is a generalization of the hypergeometric functions $\, _2F_1(a, b, c; \frac{1}{2} ; x_0^2) \,$ and $\, _2F_1(a, b, c; \frac{3}{2} ; x_0^2) \,$ which can be related to the $q$-deformed spherical functions $Y_{lm}(q, x_0, \varphi)$. Some useful properties and relations satisfied by the eigenfunctions are proved. In the last section two particular cases of $q$-deformed Schrödinger equation containing a scalar potential are presented: the Coulomb and the three dimensional oscillator. Their eigensolutions are given and the removal of the accidental degeneracy is discussed.

2. The $q$-angular momentum

The $su_q(2)$ algebra is generated by three operators $L_+$, $L_0$ and $L_-$, also named the $q$-angular momentum components. They have the following commutation relations:

\[
\begin{align*}
[ L_0 , L_\pm ] &= \pm L_\pm , \\
[ L_+ , L_- ] &= [2 \, L_0] ,
\end{align*}
\]

(1)

(2)

where the quantity in square brackets is defined as

\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

(3)

In the following we shall introduce quantities having definite transformation properties with respect to the $su_q(2)$ algebra. They will further be used to build $q$-scalars and also $q$-vectors, as for instance, the $q$-linear momentum entering the expression of the Hamiltonian operator.

First of all we recall that $su_q(2)$ algebra has an invariant $C$, called the Casimir operator

\[
C = L_- \, L_+ + [L_0] \, [L_0 + 1] .
\]

(4)

Its eigenvalue associated to a $(2l + 1)$-dimensional irreducible representation is:

\[
C_l = [l] \, [l + 1].
\]

(5)
By definition a \( q \)-vector in this algebra is given by a set of three quantities \( v_k \), \( k = 0, \pm 1 \) satisfying the following relations:

\[
\left[ L_0, v_k \right] = k v_k, \tag{6}
\]

\[
\left( L_\pm v_k - q^k v_k L_\pm \right) q^{L_0} = \sqrt{2} v_{k\pm 1}, \tag{7}
\]

where \( v_{\pm 2} \) must be set equal to zero in the right-hand side of equation (7) when \( k = \pm 1 \). This definition is a particular case of an irreducible tensor of rank one (for the general case see e.g. Ref. \[3\]).

By comparing the relations (1), (2) with (6), (7) we observe that the operators \( L_k \) do not represent the components of a \( q \)-vector. Such an observation is also pointed out in Ref. \[7\] in the context of \( q \)-tensor operators for quantum groups. The situation is entirely different from the \( su(2) \) algebra where \( L_k \) form a vector in the usual sense. However, one can use the components \( L_\pm \) and \( L_0 \) to define a new vector \( \vec{\Lambda} \) in the following manner:

\[
\Lambda_{\pm1} = \mp 1 \sqrt{\frac{1}{2}} q^{-L_0} L_\pm, \tag{8}
\]

\[
\Lambda_0 = \frac{1}{2} \left( q L_+ L_- - q^{-1} L_- L_+ \right). \tag{9}
\]

It is an easy matter to show that the operators \( \Lambda_k \) satisfy the relations (6) and (7). The vector \( \vec{\Lambda} \) will be used in Section 4 to construct the transverse part of the linear momentum \( \vec{p} \).

Two \( q \)-vectors \( \vec{u} \) and \( \vec{v} \) satisfying equations (6) and (7) can be used to build a scalar \( S \), according to the following definition:

\[
S = \vec{u} \cdot \vec{v} = -\frac{1}{q} u_1 v_{-1} + u_0 v_0 - q u_{-1} v_1. \tag{10}
\]

where the coefficients appearing in the sum are the \( q \)-Clebsch-Gordan coefficients \( \langle 110 | m - m0 \rangle_q \). In this way the scalar product (10) becomes the ordinary scalar product of \( R(3) \) when \( q = 1 \). By introducing a generalization of the cross product, two \( q \)-vectors can also be used to build another \( q \)-vector required by our approach, as it will be shown in Section 4.

In the case \( \vec{u} = \vec{v} = \vec{\Lambda} \), the scalar product \( \vec{\Lambda}^2 \) defines a second invariant \[3\], \( C' \), which is not independent of \( C \). The eigenvalue of \( C' \) is:

\[
C'_l = \frac{[2l]}{2} \frac{[2l + 2]}{2}. \tag{11}
\]

One can also easily prove that there exists a third invariant \( c \), defined as

\[
c = q^{-2L_0} + \lambda \Lambda_0, \tag{12}
\]
with
\[ \lambda = q - \frac{1}{q}. \] (13)

This will be frequently used in order to write the subsequent formulae in a more compact form. Its eigenvalue is:
\[ c_l = \frac{q^{2l+1} + q^{-2l-1}}{[2]}. \] (14)

It is worth noting that in the limit \( q \to 1 \) both the \( C \) and \( C' \) turn into the Casimir invariant \( C=\hat{L}^2 \) of \( su(2) \) with the eigenvalue \( l(l+1) \), while \( c \) becomes equal to unity. The eigenvalues (11) and (14) will be used in section 4 to define the action of \( \vec{p}^2 \) on deformed spherical harmonics. The results listed in this section are valid for any realization of the \( su_q(2) \) algebra.

3. The position vector \( \vec{r} \) and a realization of \( L_0, L_{\pm} \)

In the \( R_q(3) \) space we define the position vector \( \vec{r} \) as having three noncommutative components \( r_1, r_0 \) and \( r_{-1} \), satisfying the following relations
\[ r_0 \ r_{\pm 1} = q^{r_{\pm 2}} \ r_{\pm 1} \ r_0, \] (15)
\[ r_1 \ r_{-1} = r_{-1} \ r_1 + \lambda \ r_0^2. \] (16)

These equations are similar to eqs. (3.11) of Ref. They are typical for a noncommutative algebra. The scalar quantity \( r^2 \) defined according to equation (10)
\[ r^2 = \vec{r}^2 = -\frac{1}{q} \ r_1 \ r_{-1} + r_0^2 - q \ r_{-1} \ r_1 \] (17)
commutes with all \( r_i \) and all \( L_i \) of equations (1) and (2), provided \( r_i \ (i = 0, \pm 1) \) satisfy the conditions (6) and (7) to be a vector, which is here the case. For \( q = 1 \) the scalar \( r \) is nothing else but the length of the position vector \( \vec{r} \). We shall keep this meaning for \( q \neq 1 \) too.

Searching for concrete realizations of \( r_i, L_0 \) and \( L_{\pm} \), we begin by expressing \( L_0 \) in spherical coordinates as in the \( R(3) \) case:
\[ L_0 = -i \frac{\partial}{\partial \varphi}. \] (18)

The next step is to write \( \vec{r} \) as a product of \( r \) and of a unit vector \( \vec{x} \), depending on angles so we have:
\[ r_{\pm 1} = r \ x_{\pm 1}, \] (19)
\[ r_0 = r \ x_0. \] (20)
It remains now to find a realization of $x_{\pm 1}$ in terms of the azimuthal angle $\varphi$ and of $x_0$, which is in fact equal to $\cos \theta$, just as in the classical $R(3)$ case. From the relations (16), (17), (19) and (20) one can find

\[
x_{1} x_{-1} = - \frac{1}{[2]} (1 - q^2 x_0^2),
\]
\[
x_{-1} x_{1} = - \frac{1}{[2]} (1 - q^{-2} x_0^2).
\]

This suggests that the equations (15) and (16) can be satisfied by simple forms of $x_1$ and $x_{-1}$ provided a dilatation operator $N_0$ is introduced through the commutation relations

\[
[ N_0 , x_0^n ] = n x_0^n,
\]
and having the hermiticity property

\[
N_0^+ = - N_0 - 1.
\]

Then the realization of $x_1$ and $x_{-1}$ satisfying (15) and (16) turns out to be

\[
x_1 = - e^{i\varphi} \sqrt{\frac{q}{[2]}} \sqrt{1 - q^2 x_0^2} q^{2N_0},
\]
\[
x_{-1} = e^{-i\varphi} \sqrt{\frac{1}{[2]q}} \sqrt{1 - q^{-2} x_0^2} q^{-2N_0}.
\]

Taking now into account the relations (21) and (22) and assuming

\[
x_0^+ = x_0.
\]

we get the expected hermiticity properties for $x_{\pm}$ as:

\[
x_1^+ = - \frac{1}{q} x_{-1},
\]
\[
x_{-1}^+ = - q x_1.
\]

All these arguments allow us to conclude that eqs.(19-25) define the realization of the position vector $\vec{r}$ in the $R_q(3)$ space.

The following step is to search for a realization of the $su_q(2)$ generators. The expressions we propose for $L_+$ and $L_-$ are:

\[
L_+ = \sqrt{\frac{[2]}{2}} e^{i\varphi} \tilde{x}_1^{L_0+1} \frac{1}{x_0} \frac{1 - q^{-2N_0}}{1 - q^{-2}} \tilde{x}_1^{-L_0} q^{L_0},
\]
\[
L_- = \sqrt{\frac{[2]}{2}} e^{-i\varphi} \tilde{x}_{-1}^{-L_0+1} \frac{1}{x_0} \frac{1 - q^{2N_0}}{1 - q^2} \tilde{x}_{-1}^{L_0} q^{L_0},
\]
where $\tilde{x}_\pm = e^{\mp i\varphi}x_\pm$ depend on $x_0$ only. The reason why instead of $x_\pm$ we use here $\tilde{x}_\pm$, where the phase factor has been removed, is that expressions like $x^L_\pm$ have no meaning, while $\tilde{x}^L_\pm$ are well defined as discussed below equation (30). From equations (18), (28) and (29) we can now construct the Casimir operator $C$ of equation (4). Its eigenfunctions are expected to be $q$-spherical functions as in Ref. [4]. For $q = 1$ they become ordinary spherical harmonics. Therefore they can take the form:

$$\tilde{Y}_{lm}(q, x_0, \phi) = e^{im\varphi} \tilde{x}_1^m \Theta_{lm}(x_0),$$

where $\Theta_{lm}(x_0)$ are the $q$-analogue of the associated Legendre functions. The functions (30) will be derived and normalized in Section 5.

Concerning the action of $L_+$ we note that $\tilde{x}_1^{-L_0}$ in (28) removes the factor $\tilde{x}_1^m$ in $\tilde{Y}_{lm}(q, x_0, \phi)$. In this way one prevents $q^{-2N_0}$ appearing in equations (28) and (29) from acting on $\tilde{x}_1$ and producing a troublesome result. The operator $q^{-2N_0}$ in $L_+$ acts on $\Theta_{lm}(x_0)$ only. Finally $e^{i\varphi} \tilde{x}_1^{L_0+1}$ recreates the factor $x_1^{m+1}$ required after it has been eliminated because of $\tilde{x}_1^{-L_0}$.

In the well known $R(3)$ theory of angular momentum a different mechanism prevents $\partial_\phi$ in $L_+$ from acting on $x_1^m$: the term given by $\partial_\phi x_1^m$ is exactly cancelled out by $i\text{ctg}\theta \partial_r x_1^m$, so that only the derivative $\partial_\phi \Theta_{lm}(x_0)$ remains.

It can be verified that the expressions (18), (28) and (29) satisfy the commutation relations (1) and (2) and hence one can conclude that they are the realization of the $su(q)(2)$ generators in the $R_q(3)$ space. It can also be checked that the position vector $\vec{r}$, defined by (19-22), behaves indeed as a vector in this $su(q)(2)$ algebra, since it satisfies the relations (6) and (7) with $L_\pm$ given by (28) and (29).

4. The $q$-linear momentum $\vec{p}$

In order to write down an expression for the linear momentum $\vec{p}$, we separate it into a part perpendicular and another one parallel to $\vec{x}$. The first one is defined with the aid of the cross product $\vec{x} \times \vec{\Lambda}$ and the second one is assumed to have the form $\vec{x}_1 \frac{1}{r} f \left( r \frac{\partial}{\partial r} + 1 \right)$, where $f$ is a function which will be defined in the following. Then the components of the transverse part, denoted by $\partial_k$, read:

$$\partial_1 = q^{-1} x_1 \Lambda_0 - q x_0 \Lambda_1 + x_1 c,$$

$$\partial_0 = x_1 \Lambda_{-1} - \lambda x_0 \Lambda_0 - x_{-1} \Lambda_1 + x_0 c,$$

$$\partial_{-1} = -q x_{-1} \Lambda_0 + q^{-1} x_0 \Lambda_{-1} + x_{-1} c,$$

where $c$ is the invariant defined in equation (12) and the terms $x_k c$ have been added to the cross product $\vec{x} \times \vec{\Lambda}$ in order to ensure the well defined character with respect
to the hermitian conjugation operation

\[ \partial_k^+ = -\left(\frac{1}{q}\right)^k \partial_{-k}. \]  

(34)

It can be checked that the quantities \( \partial_k \) form a vector as defined by equations (6) and (7). Moreover they satisfy the following relations:

\[ \partial_0 \partial_1 = q^{-2} \partial_1 \partial_0, \]  

(35)

\[ \partial_0 \partial_{-1} = q^2 \partial_{-1} \partial_0, \]  

(36)

\[ \partial_1 \partial_{-1} = \partial_{-1} \partial_1 + \lambda \partial_0^2. \]  

(37)

These equations are similar to (15) and (16) satisfied by the position vector. Equation (35) has been directly obtained by commuting \( \partial_0 \) with \( \partial_1 \). Equation (36) is the hermitian conjugate of the above one. Equation (37) can be obtained either from (35) or (36) by using the relation (7).

Also, by multiplying equations (31-33) with the corresponding \( x_k \) and taking into account the commutation relations (6) and (7) one gets:

\[ \vec{x} \vec{\partial} = -\vec{\partial} \vec{x} = c. \]  

(38)

By commuting the invariant \( c \) with \( \vec{x} \) one finds:

\[ \vec{\partial} = \lambda^{-2} [c, \vec{x}]. \]  

(39)

Taking now the matrix elements of the last relation one obtains:

\[ \langle l + 1 m' | \vec{\partial} | l m \rangle = \frac{[2l + 2]}{[2]} \langle l + 1 m' | \vec{x} | l m \rangle, \]  

(40)

\[ \langle l - 1 m' | \vec{\partial} | l m \rangle = -\frac{[2l]}{[2]} \langle l - 1 m' | \vec{x} | l m \rangle. \]  

(41)

[From parity arguments one can also write:]

\[ \langle l m' | \partial_k | l m \rangle = 0. \]  

(42)

The matrix elements of \( \vec{x} \) can be calculated (see next section) so that from replacing the matrix elements of \( \vec{\partial} \) by those of \( \vec{x} \) with the aid of eqs. (40) and (41) one can obtain the eigenvalues of \( \vec{\partial}^2 \). These are:

\[ \langle l m | \vec{\partial}^2 | l m \rangle = -\frac{[2l]}{[2]} \frac{[2l + 1]}{[2]} - c_l^2. \]  

(43)
At the beginning of this section we mentioned that the component of $\vec{p}$ parallel to $\vec{x}$ is assumed to have the form $\vec{x} \frac{1}{r} f \left(r \frac{\partial}{\partial r} + 1\right)$. For simplicity we take here $f(x) = x$. In this case the realization of the $q$-linear momentum $\vec{p}$ reads:

$$\vec{p} = -i \frac{1}{r} \left( \vec{x} \left(r \frac{\partial}{\partial r} + 1\right) - \vec{\partial}\right). \quad (44)$$

Then using equations (38) and (43) one can write:

$$\vec{p}^2 \tilde{Y}_{lm} = \left[-\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} + 1\right) + \frac{1}{r^2} \left(\frac{2l}{2} \frac{2l+2}{2} + c_1 \right)\right] \tilde{Y}_{lm}. \quad (45)$$

One can see that in the limit $q \to 1$ one recovers the action of the Laplace operator on a spherical harmonic which justifies our choice for $f$.

We mention that it is a simple but a tedious matter to calculate the commutation relations between $\vec{r}$ and $\vec{p}$ and to verify that one gets the right result for $q = 1$. We do not display these commutation relations here because they are rather intricate and unnecessary in the derivation of a covariant Schrödinger equation.

We also note that the operator $\vec{\Lambda}$, behaving as a vector under the $su_q(2)$ algebra, can be written as a cross product of $\vec{r}$ and $\vec{p}$, but this does not bring any simplification because of the commutation relations between $\vec{r}$ and $\vec{p}$.

5. The eigenfunctions of the $q$-angular momentum

By definition, the basis vectors $\Phi_{lm}(q, x_0, \varphi)$ forming an invariant subspace for a $(2l + 1)$-dimensional irreducible representation of $su_q(2)$ are eigenfunctions of $L_0$ and of the Casimir operator $C$ of equation (4). We begin by writing them as polynomials in $x_0$ multiplied by $x_1^m$:

$$\Phi_{lm}(q, x_0, \varphi) = x_1^m \sum_{k \geq 0} a_k x_0^k. \quad (46)$$

where the sum runs either over $k$ even when $l - m$ is even or over $k$ odd when $l - m$ is odd. In both cases it runs up to $l - m$ but it starts at zero for $l - m$ even and at 1 for $l - m$ odd.

As for the $R(3)$ case, the basic equation which determines the matrix elements of $L_+$ and $L_-$ reads:

$$L_+ L_- \Phi_{lm}(q, x_0, \varphi) = \left[l + m\right] \left[l - m + 1\right] \Phi_{lm}(q, x_0, \varphi). \quad (47)$$

This equation leads to the recursion relation:

$$a_{k+2} = -q^{-2m} \frac{\left[l - m - k\right] \left[l + m + k + 1\right]}{\left[k + 1\right] \left[k + 2\right]} a_k. \quad (48)$$
Then taking \( a_0 = 1 \) we obtain for \( l - m \) even:

\[
\Phi_{lm}(q, x_0, \varphi) = x_1^m \left\{ 1 - \frac{[l-m][l+m+1]}{[2]!} \left( q^{-m}x_0 \right)^2 \right. \\
+ \frac{[l-m][l-m-2][l+m+1][l+m+3]}{[4]!} \left( q^{-m}x_0 \right)^4 - \ldots \left\},
\]

(49)

while for \( l - m \) odd we get:

\[
\Phi_{lm}(q, x_0, \varphi) = x_1^m \left\{ \frac{1}{[1]!} \left( q^{-m}x_0 \right) - \frac{[l-m-1][l+m+2]}{[3]!} \left( q^{-m}x_0 \right)^3 \\
+ \frac{[l-m-1][l-m-3][l+m+2][l+m+4]}{[5]!} \left( q^{-m}x_0 \right)^5 - \ldots \right\}.
\]

(50)

In order to express these results in terms of a \( q \)-hypergeometric series it is necessary to write all the \( q \)-numbers \([n] \) in the form

\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = [2] \frac{(q^2)^{\frac{n}{2}} - (q^2)^{-\frac{n}{2}}}{q^2 - q^{-2}} = [2] \left[ \frac{n}{2} \right] q^2.
\]

(51)

For \( l - m \) even we have then:

\[
\Phi_{lm}(q, x_0, \varphi) = x_1^m \ {}_2F_1 \left( q^2; \frac{l+m+1}{2}, \frac{-l+m}{2}; \frac{1}{2}; q^{-m}x_0^2 \right),
\]

(52)

while for \( l - m \) odd we get:

\[
\Phi_{lm}(q, x_0, \varphi) = x_1^m q^{-m}x_0 \ {}_2F_1 \left( q^2; \frac{l+m+2}{2}, \frac{-l+m+1}{2}; \frac{3}{2}; q^{-m}x_0^2 \right).
\]

(53)

The argument \( q^2 \) in \( {}_2F_1 \) specifies that all the \( q \)-numbers in the series expansion of \( {}_2F_1 \) must be calculated with \( q^2 \) instead of \( q \).

Moreover we found that the functions \( \Phi_{lm}(q, x_0, \varphi) \) satisfy the following simple relations:

\[
x_1 \frac{1}{x_0} \frac{1 - q^{-2N_0}}{1 - q^{-2}} \Phi_{lm}(q, x_0, \varphi) = - [l-m][l+m+1] \Phi_{l \ m+1}(q, x_0, \varphi),
\]

(54)

for \( l - m \) even, and

\[
x_1 \frac{1}{x_0} \frac{1 - q^{-2N_0}}{1 - q^{-2}} \Phi_{lm}(q, x_0, \varphi) = \Phi_{l \ m+1}(q, x_0, \varphi),
\]

(55)

for \( l - m \) odd.
The normalized eigenfunctions of C and $L_0$ take now the form:

$$Y_{lm}(q, x_0, \varphi) = (-1)^{l-m} \frac{[2l + 1]}{4\pi} \left( \frac{[l - m - 1]!!}{[l - m]!!} \cdot \frac{[l + m - 1]!!}{[l + m]!!} \right)^{1/2} \frac{2^{l+1}}{2^{l+3}} \Phi_{l,m}(q, x_0, \varphi),$$

(56)

for $l - m$ even, and

$$Y_{lm}(q, x_0, \varphi) = (-1)^{\frac{l-m-1}{2}} \frac{[2l + 1]}{4\pi} \left( \frac{[l - m]!!}{[l - m - 1]!!} \cdot \frac{[l + m]!!}{[l + m - 1]!!} \right)^{1/2} \frac{2^{l}}{2^{l-1}} \Phi_{l,m}(q, x_0, \varphi),$$

(57)

for $l - m$ odd. Their orthogonality relation becomes:

$$\int Y_{l^+m'}(q, x_0, \varphi) Y_{lm}(q, x_0, \varphi) d\varphi d[x_0] = \delta_{ll'} \delta_{mm'},$$

(58)

where the integral over $\varphi$ is the same as for spherical harmonics, while the integral over $d[x_0]$ defined on the interval (-1,1) is the sum of

$$\int_0^1 x_0^n d[x_0] = \frac{1}{n+1},$$

(59)

and of

$$\int_{-1}^0 x_0^n d[x_0] = (-1)^n \frac{1}{n+1}.$$

(60)

The phase appearing in the right-hand side of the integral (60) is due to parity arguments. The relation (59) is in fact the result of a discrete integration of $f(x_0) = x_0^n$, performed by dividing the integration interval (0,1) in an infinite set of segments located between two successive points $x_k = q^k$ and $x_k = q^{k+1}$ where $q < 1$

$$\int_0^1 f(x_0) d[x_0] = \sum_{k=0}^{\infty} f(x_{2k+1}) (x_{2k+1} - x_{2k+2}).$$

(61)

Looking now for the properties of $Y_{lm}$, just as in the $R(3)$ case, we found that the product $x_k Y_{lm}$ can be expressed in terms of $Y_{l+1, m+k}$ or $Y_{l-1, m+k}$ as follows:

$$x_1 Y_{lm}(q, x_0, \varphi) = q^{l-m} \sqrt{\frac{[l + m + 1][l + m + 2]}{[2][2l + 1][2l + 3]}} Y_{l+1 m+1}(q, x_0, \varphi)$$

$$- q^{-l-m-1} \sqrt{\frac{[l - m][l - m - 1]}{[2][2l + 1][2l - 1]}} Y_{l-1 m+1}(q, x_0, \varphi),$$

(62)

$$x_0 Y_{lm} = q^{-m} \sqrt{\frac{[l - m + 1][l + m + 1]}{[2l + 1][2l + 3]}} Y_{l+1 m}(q, x_0, \varphi)$$
\[ -q^{-m} \sqrt{\frac{(l-m)(l+m)}{2l+1}[2l-1]} Y_{l-1}^{m}(q,x_0,\varphi), \] (63)

\[ x_{-1} Y_{lm}(q,x_0,\varphi) = q^{l-m} \sqrt{\frac{(l-m+1)(l+m+2)}{2(2l+1)[2l+3]}} Y_{l+1}^{m-1}(q,x_0,\varphi) \]

\[ - q^{l-m+1} \frac{(l+m)(l+m-1)}{2(2l+1)[2l+1]} Y_{l-1}^{m-1}(q,x_0,\varphi). \] (64)

In addition, we have found three relations which express the non-commutativity of \(x_k\) with \(Y_{lm}\) and represent a generalization of the equations (15) and (16):

\[ x_0 Y_{lm}(q,x_0,\varphi) = q^{-2m} Y_{lm}(q,x_0,\varphi) x_0, \] (65)

\[ x_1 Y_{lm}(q,x_0,\varphi) = Y_{lm}(q,x_0,\varphi) x_1 \]

\[ + \frac{\lambda}{\sqrt{2}} q^{-m-1} \sqrt{(l-m)(l+m+1)} Y_{l+1}^{m+1}(q,x_0,\varphi) x_0, \] (66)

\[ x_{-1} Y_{lm}(q,x_0,\varphi) = Y_{lm}(q,x_0,\varphi) x_{-1} \]

\[ - \frac{\lambda}{\sqrt{2}} q^{l-m+1} \sqrt{(l+m)(l-m+1)} Y_{l-1}^{m-1}(q,x_0,\varphi) x_0. \] (67)

The last two equations have been obtained from (65) by acting with \(L_+\) or \(L_-\) which leads to a rising or lowering of \(m\) in \(Y_{lm}\).

**VI. A \(q\)-deformed Schrödinger equation**

Taking into account all the above results, we assume that the Hamiltonian entering the \(q\)-deformed Schrödinger equation is:

\[ \mathcal{H} = \frac{1}{2} \vec{\mathcal{P}}^2 + V(r) \] (68)

where operator \(\vec{\mathcal{P}}\) has been defined in the fourth section. The eigenfunctions of this Hamiltonian are:

\[ \Psi(r,x_0,\varphi) = r^L u_L(r) Y_{lm}(q,x_0,\varphi) \] (69)
where $L$ is the solution of the following equation:

$$L(L + 1) = \frac{[2l][2l + 2]}{[2][2]} + c_l^2 - c_l$$

obtained from the requirement that $u_L(r)$ remains finite in the limit $r \to 0$.

This Schrödinger equation has simple solutions for the Coulomb potential $V(r) = -r^{-1}$ and for the oscillator potential $V(r) = \frac{1}{2} r^2$. The eigenvalues of the two Hamiltonians are:

$$(E_{nl})_{\text{Coulomb}} = -\frac{1}{2(n + L + 1)^2}$$

for the Coulomb potential and

$$(E_{nl})_{\text{oscillator}} = (2n + L + \frac{3}{2}).$$

for the oscillator potential, $n$ being the radial quantum number and $L$ the solution of the equation (70), usually not an integer. We notice that the spectrum is degenerate with respect to the magnetic quantum number $m$, i.e. the essential degeneracy subsists. But the eigenvalues (71) and (72) depend on two quantum numbers so that the accidental degeneracy of the $q = 1$ case is removed. The dependence of eigenvalues on $q$ can be obtained through solving equation (70) for $L$.

The solution of the wave equation which does not depend on $\theta$ and $\varphi$ gives for the expectation value of $x_0^2$ the value $R^2/3$ instead of $R^2/3$ obtained in the case of spherical symmetry. The quantity $R^2$ denotes the expectation value of the operator $r^2$ in each case. It then results that the quadrupole moment as well as all the $2^{2n}$-poles are different from zero, although the wave function does not depend on $\theta$ and $\varphi$. This clearly shows that the Hamiltonian (68-70) has lost the spherical symmetry. One can mention however that it gained another one, namely the symmetry under the $su_q(2)$ algebra which may have new physical implications.

We remark that there are three sources producing differences in the eigenvalue problem between the case of $q$-deformed Schrödinger equation and the case of spherical symmetry. The first one is that the $q$-functions $Y_{lm}(q, x_0, \varphi)$ differ from the spherical harmonics $Y_{lm}(\theta, \varphi)$ as shown in Section 5. The second reason is that the coefficient of the centrifugal potential in the radial Schrödinger equation is proportional to $L(L + 1)$, with $L$ given by eq. (70), and not to $l(l + 1)$, as in the sperial case. The third source is that in the $q$-deformed case the integral over $x_0$ is performed according to the relations (58)-(60).

As a final comment let us recall that for $l = 0$ one has $c_l = 1$, hence $L = 0$. As a consequence the $l = 0$ levels are independent of the deformation parameter both for the harmonic oscillator and the Coulomb potential. An important physical aspect is that the centrifugal barrier disappears for $l = 0$ in contrast to the Hamiltonian $H_q$ of Ref. [2]. Moreover the whole Coulomb spectrum of Ref. [2] is different from ours. It is not surprising because this work is based on the results of Ref. [3] where the realization of
the $su_q(2)$ generators and the basis vectors of $su_q(2)$ irreps are entirely different from ours.

Physical applications with numerical examples of the $q$-deformed Coulomb and harmonic oscillator spectra will be considered elsewhere.

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