NOTE ON THE DISTORTION OF $(2, q)$-TORUS KNOTS

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ABSTRACT. We show that the distortion of the $(2, q)$-torus knot is not bounded linearly from below.

1. Introduction

The notion of distortion was introduced by Gromov [1]. If $\gamma$ is a rectifiable simple closed curve in $\mathbb{R}^3$, then its distortion $\delta$ is defined as

$$\delta(\gamma) = \sup_{v, w \in \gamma} \frac{d_\gamma(v, w)}{|v - w|},$$

where $d_\gamma(v, w)$ denotes the length of the shorter arc connecting $v$ and $w$ in $\gamma$ and $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^3$. For a knot $K$, its distortion $\delta(K)$ is defined as the infimum of $\delta(\gamma)$ over all rectifiable curves $\gamma$ in the isotopy class $K$. Gromov [3] asked in 1983 if every knot $K$ has distortion $\delta(K) \leq 100$. The question was open for almost three decades until Pardon gave a negative answer. His work [4] presents a lower bound for the distortion of simple closed curves on closed PL embedded surfaces with positive genus. Pardon showed that the minimal intersection number of such a curve with essential discs of the corresponding surface bounds the distortion of the curve from below. In particular for the $(p, q)$-torus knot he obtained the following bound.

**Theorem ([4]).** Let $T_{p, q}$ denote the $(p, q)$-torus knot. Then

$$\delta(T_{p, q}) \geq \frac{1}{160} \min(p, q).$$

By considering a standard embedding of $T_{p, p+1}$ on a torus of revolution one obtains $\delta(T_{p, p+1}) \leq \text{const} \cdot p$, hence for $q = p + 1$ Pardons result is sharp up to constants.

An alternative proof for the existence of families with unbounded distortion was given by Gromov and Guth [2]. In both works the answer of Gromovs question was obtained by an estimate of the conformal length, which is up to a constant a lower bound for the distortion of rectifiable closed curves. However the conformal length is in general not a good estimate for the distortion. For example one finds easily
an embedding of the $(2, q)$-torus knot with conformal length $\leq 100$
and distortion $\geq q$ by looking at standard embeddings on a torus of
revolution with suitable dimensions. In particular neither Pardon’s nor
Gromov and Guth’s arguments yield lower bounds for $\delta(T_{2,q})$. While
Pardon writes that surely $\lim_{q \to \infty} \delta(T_{2,q}) = \infty$ and that there are to his
knowledge no known embeddings of $T_{2,q}$ with sublinear distortion [4]
[p.2], Gromov and Guth [2] write that the distortion of $T_{2,q}$ appears to
be $q$ up to constants [p.33]. In this article we show that the growth
rate of $\delta(T_{2,q})$ is in fact sublinear in $q$.

**Theorem 1.** Let $q \geq 50$. Then $\delta(T_{2,q}) \leq 7q/\log q$. In particular the
distortion of the $(2, q)$-torus knot is not bounded linearly from below.

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3. PROOF

In order to prove Theorem 1 we need to give for every odd integer
$q \geq 50$ an embedding $\gamma$ of the $(2, q)$-torus knot with distortion smaller
or equal to $7q/\log q$. The idea is to use a logarithmic spiral. Let $S$
be a logarithmic spiral of unit length starting at its center $0 \in \mathbb{R}^3$
and ending at some $u \in \mathbb{R}^3$. An elementary calculation shows that its
distortion is equal to $1/|u|$. For another path $\alpha \subset \mathbb{R}^3$ of unit length
and diameter $\leq 2|u|$ with endpoints $\{v, w\} = \partial \alpha$ we get

$$\delta(\alpha) \geq \frac{d_\alpha(v, w)}{|v - w|} = \frac{1}{|v - w|} \geq \frac{1}{2|u|} = \frac{\delta(S)}{2}.$$ 

Hence up to at most a factor 2 the logarithmic spiral has the smallest
distortion among all paths for a prescribed pathlength-pathdiameter-ratio.
It seems therefore natural to pack the $q$ windings of the $(2, q)$-
torus knot into a logarithmic spiral in order to minimize distortion.

**Proof of Theorem 1.** Let $q$ be an odd integer greater or equal to 50,
and $k = \log(q)/2\pi q$. We define the embedding $\gamma$ as the union of a
segment of the logarithmic spiral with slope $k$, denoted by $S$, and a
piecewise linear part, denoted by $L$, see Figure 1. The segment of the
logarithmic spiral $S$ is contained in the yellow painted $(x, z)$ plane and
parametrized by
\[ \varphi : [0, \pi q] \to \mathbb{R}^2, \quad \varphi(s) = e^{ks} \cdot \left( \begin{array}{c} \cos(s) \\ \sin(s) \end{array} \right), \]
see Figures 1 and 2. The segment of the piecewise linear part \( L \) is in the green painted \((x, y)\) plane, see Figures 1 and 3. Note that
\[ |\varphi(\pi q)| = e^{k\pi q} = \sqrt{q} \quad \text{and} \quad |\varphi(0)| = 1, \]
hence the lengths defining \( L \) in Figure 3 are chosen such that the union \( \gamma \) of \( S \) and \( L \) is the simple closed curve illustrated in Figure 1. The linear segments \( L_1 \) and \( L_2 \) indicated in Figure 3 are named because of their special role in the following computations.

**Figure 1.** The embedding \( \gamma \) for \( q = 7 \).

**Figure 2.** The logarithmic spiral \( S \) in the \((x, z)\) plane.
To see that the obtained curve is an embedded $(2, q)$-torus knot, we
perturb $\gamma$, see Figure 4. This simple closed curve is ambient isotopic in
$\mathbb{R}^3$ to $\gamma$ and if we project it onto the $(x, y)$ plane, we see a well known
diagram of the $(2, q)$-torus knot, see Figure 5.

We now estimate the distortion of $\gamma$. One has to show that
\[ \frac{d_\gamma(v, w)}{|v - w|} \leq \frac{7q}{\log q} \]
for all pairs of points $v, w \in \gamma$. A calculation shows that
\[ \frac{1}{k} \cdot \sqrt{2k^2 + 1} = \frac{2\pi q}{\log q} \cdot \sqrt{2(\log q/2\pi q)^2 + 1} \leq \frac{7q}{\log q} \]
for all positive integers. Therefore, it suffices to show that
\[ \frac{d_\gamma(v, w)}{|v - w|} \leq \frac{\sqrt{2k^2 + 1}}{k}. \]
In order to do this, we distinguish four cases.
Figure 5. Projection onto the \((x, y)\) plane.

Case 1: \(v, w \in S\). Let \(0 \leq s \leq t \leq \pi q\), \(v = \varphi(s), w = \varphi(t)\). From

\[
|\varphi'(r)| = \left| \begin{pmatrix} \cos(r) & -\sin(r) \\ \sin(r) & \cos(r) \end{pmatrix} \begin{pmatrix} ke^{kr} \\ e^{kr} \end{pmatrix} \right| = \left| \begin{pmatrix} ke^{kr} \\ e^{kr} \end{pmatrix} \right| = \sqrt{k^2 + 1} \cdot e^{kr},
\]

we get

\[
d_\gamma(v, w) \leq d_S(v, w)
= \int_s^t |\varphi'(r)| dr
= \int_s^t \sqrt{k^2 + 1} e^{kr} dr
= \frac{\sqrt{k^2 + 1}}{k} \cdot (e^{kt} - e^{ks})
= \frac{\sqrt{k^2 + 1}}{k} \cdot (|\varphi(t)| - |\varphi(s)|)
= \frac{\sqrt{k^2 + 1}}{k} \cdot (|w| - |v|).
\]

Since \(|w - v| \geq |w| - |v|\), we conclude that

\[
\frac{d_\gamma(v, w)}{|v - w|} \leq \frac{\sqrt{k^2 + 1}}{k} \cdot \frac{(|w| - |v|)}{|w| - |v|} = \frac{\sqrt{k^2 + 1}}{k}.
\]

Case 2: \(v \in L_1 \cup L_2, \ w \in S\). We consider the case where \(v \in L_1\).

The idea is to find the maximum of

\[
\frac{d_\gamma(v, w)}{|v - w|}
\]
for fixed $w$ and varying $v$. Let $t = |v - \varphi(0)|$, $a = |\varphi(0) - w|$, and $b = d_S(\varphi(0), w)$, see Figure 6.

Note that

$$|v - w| = \sqrt{t^2 + a^2}$$

and

$$d_\gamma(v, \varphi(0)) = |v - \varphi(0)| = t.$$

We get

$$d_\gamma(v, w) \leq \frac{d_\gamma(v, \varphi(0)) + d_S(\varphi(0), w)}{|v - w|} = \frac{t + b}{\sqrt{t^2 + a^2}} =: f(t).$$

Deriving $f$ with respect to $t$ yields a unique critical point at $t = a^2/b$:

$$0 = f'(t) = \frac{a^2 - bt}{(a^2 + t^2)^{3/2}} \iff t = a^2/b.$$

Since $a^2/b$ is the only critical point, $f(\infty) = 1 \leq b/a = f(0)$ and

$$f(0) = \frac{b}{a} \leq \frac{\sqrt{a^2 + b^2}}{a} = \frac{\frac{a^2}{b} + b}{\sqrt{(\frac{a^2}{b})^2 + a^2}} = f(a^2/b),$$

$a^2/b$ must be a global maximum. Consequently we get
\[
\frac{d_\gamma(v, w)}{|v - w|} \leq \frac{\sqrt{a^2 + b^2}}{a} = \sqrt{1 + \left(\frac{b}{a}\right)^2} = \sqrt{1 + \left(\frac{d_S(\varphi(0), w)}{|\varphi(0) - w|}\right)^2} \leq \sqrt{1 + \left(\frac{\sqrt{k^2 + 1}}{k}\right)^2} = \frac{\sqrt{2k^2 + 1}}{k}.
\]

In the case where \( v \in L_2 \), we make the estimate with the path that connects \( v \) with \( w \) through \( \varphi(\pi q) \). It works exactly the same and yields the same estimate.

**Case 3:** \( v, w \in L \). Consider Figure 3 and note that all pairs of points \( v, w \in L \) that could cause big distortion are of euclidean distance at least 1. Therefore we get
\[
\frac{d_\gamma(v, w)}{|v - w|} \leq l(L) = 11\sqrt{q} + 1.
\]
A calculation shows that
\[
11\sqrt{q} + 1 \leq \frac{2\pi q}{\log q} = \frac{1}{k}
\]
for \( q \) greater or equal to 50.

**Case 4:** \( v \in L \setminus (L_1 \cup L_2) \), \( w \in S \). Note that for these pairs of points we have
\[
|v - w| \geq |w|.
\]
We estimate \( d_\gamma(v, w) \) using results of Case 1 and 3:
\[
d_\gamma(v, w) \leq d_L(v, \varphi(0)) + d_S(\varphi(0), w) \leq \frac{1}{k} + \frac{\sqrt{k^2 + 1}}{k} \cdot (|w| - 1) \leq \frac{\sqrt{k^2 + 1}}{k} \cdot |w|.
\]
We conclude that
\[
\frac{d_\gamma(v, w)}{|v - w|} \leq \frac{\frac{\sqrt{k^2 + 1}}{k} \cdot |w|}{|w|} = \frac{\sqrt{k^2 + 1}}{k}.
\]
which finishes the proof.

With the same technique and somewhat more effort one can give an embedding $\gamma_q$ of $T_{2,q}$ with $\delta(\gamma_q) \sim \frac{\pi - q}{2 \log q}$. In addition a more technical proof yields that this asymptotical upper bound for $\delta(T_{2,q})$ is sharp for those embeddings of $T_{2,q}$ that project to a standard knot diagram via a linear projection. This let the author to the following.

**Question.** Is $\delta(T_{2,q})$ up to a constant asymptotically equal to $q/\log q$? And if yes, is the constant equal to $\pi/2$?

**References**

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