Directional Statistics for Polarization Observations of Individual Pulses from Radio Pulsars

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**ABSTRACT**

Radio polarimetry is a three-dimensional statistical problem. The three-dimensional aspect of the problem arises from the Stokes parameters Q, U, and V, which completely describe the polarization of electromagnetic radiation and conceptually define the orientation of a polarization vector in the Poincaré sphere. The statistical aspect of the problem arises from the random fluctuations in the source-intrinsic polarization and the instrumental noise. A simple model for the polarization of pulsar radio emission has been used to derive the three-dimensional statistics of radio polarimetry. The model is based upon the proposition that the observed polarization is due to the incoherent superposition of two, highly polarized, orthogonal modes. The directional statistics derived from the model follow the Bingham-Mardia and Fisher family of distributions. The model assumptions are supported by the qualitative agreement between the statistics derived from it and those measured with polarization observations of the individual pulses from pulsars. The orthogonal modes are thought to be the natural modes of radio wave propagation in the pulsar magnetosphere. The intensities of the modes become statistically independent when generalized Faraday rotation (GFR) in the magnetosphere causes the difference in their phases to be large. A stochastic version of GFR occurs when fluctuations in the phase difference are also large, and may be responsible for the more complicated polarization patterns observed in pulsar radio emission.

*Subject headings:* Pulsars; Polarization; Analytical methods

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1. Introduction

The four fundamental measurements made in astronomy are the intensity, flux density, or surface brightness of the electromagnetic radiation emitted by a celestial object, the wavelength, or frequency, of the radiation, its location on the sky, and the polarization of the radiation. Measurements of the latter two, location and polarization, follow the statistics of direction. The association of directional statistics with the measurement of location is obvious, but the application of directional statistics to polarization measurements is not immediately apparent until one recalls that the Stokes parameters $Q$, $U$, and $V$ describe the orientation of a polarization vector within the Poincaré sphere. The Stokes parameter $V$ defines the circular polarization of the radiation and establishes the “$z$-coordinate” of the polarization vector in the Poincaré sphere. The Stokes parameters $Q$ and $U$ describe the radiation’s linear polarization and establish the vector’s $x$- and $y$-coordinates, respectively. Here, polarization measurements are shown to follow directional statistics, and these statistics are applied to polarization observations of radio pulsars.

Pulsars are rapidly rotating, highly magnetized neutron stars. Their rotation periods range between about 1ms and 10s, and the strength of the magnetic field at their surfaces ranges from $10^8$ G for the oldest pulsars to over $10^{12}$ G for the youngest. A beam of radio emission is emitted from each of the star’s magnetic poles. A pulse of radio emission is observed as the star’s rotation causes the beam to sweep across an observer’s line of sight. Pulsar radio emission is generally thought to originate from charged particles streaming along open magnetic fields lines above the star’s magnetic pole, but unlike other astrophysical radiative processes (e.g. synchrotron radiation, maser emission, and thermal radiation), it is poorly understood. Polarization observations of the individual pulses from pulsars are made in an attempt to understand the radio emission mechanism and to study the propagation of radio waves in ultra-strong magnetic fields.

Polarization observations of individual pulses (Lyne et al. 1971; Manchester et al. 1975; Backer & Rankin 1980; Stinebring et al. 1984) show that the radiation can be highly elliptically polarized and highly variable, if not stochastic. In many cases, the mean of the polarization position angle varies in an S-shaped pattern across the pulse. But histograms of position angle created from the single pulse observations show the angles follow the pattern in two parallel paths separated by about 90 degrees (Stinebring et al. 1984). Furthermore, histograms of fractional linear polarization show that the radiation is significantly depolarized at pulse locations where these orthogonally polarized (OPMs) modes occur. The OPMs are thought to be the natural modes of wave propagation in pulsar magnetospheres (Allen & Melrose 1982; Barnard & Arons 1986). The narrow bandwidths and short sampling intervals used in single pulse observations cause the instrumental noise in these observations to be
large. The narrow bandwidths are used to overcome pulse smearing effects caused by the dispersion measure of, and multipath scattering in, the interstellar medium. The short sampling intervals, typically of order 100us, are needed to adequately resolve the short duration radio pulse. The combination of the stochastic nature of the intrinsic emission and the high instrumental noise suggests that a statistical approach is needed to analyze the single pulse data.

Most results from single pulse polarization observations have been reported as histograms of fractional linear polarization, fractional circular polarization, and polarization position angle (Backer & Rankin 1980; Stinebring et al. 1984). While these display methods are extremely useful, they do not provide a complete picture of pulsar polarization because they force a separate interpretation of the circular and linear polarization, instead of a combined one as the observed elliptical polarization of the radiation would suggest. A complete, three-dimensional view of the polarization can be made by plotting the polarization measurements from a specific pulse location in the Poincaré sphere and projecting the result in two dimensions. The projections show how the orientation of the polarization vector fluctuates on the Poincaré sphere and reveal a wide variety of quasi-organized patterns. For example, in the cone emission at the edges of the pulse in PSR B0329+54 (Edwards & Stappers 2004), the patterns consist of two clusters of data points, each in a separate hemisphere of the Poincaré sphere. In the precursor to the pulsar’s central core component, the pattern is a single cluster of data points. Within the pulsar’s core emission at the center of the pulse, one of the two clusters seen in the cone emission stretches into an ellipse or bar, while the other spreads into an intriguing partial annulus. The signatures of these patterns are not apparent in histograms of fractional polarization or position angle, emphasizing the benefit of analyzing the Stokes parameters together. Any viable model of pulsar polarization must be able to replicate the observed patterns in addition to the histograms of fractional polarization.

2. Model of Pulsar Polarization

The details of the statistical model for pulsar polarization are summarized in a series of papers by McKinnon and Stinebring (McKinnon & Stinebring 1998, 2000; McKinnon 2003, 2004, 2006, 2009). The main hypothesis of the model is the radiation’s polarization is determined by the simultaneous interaction of two, highly polarized, orthogonal modes. By definition, the unit vectors representing the orthogonal modes are antiparallel in the Poincaré sphere and thus form a “mode diagonal” in the sphere. The model accounts for the statistical nature of the observed polarization fluctuations by assuming the mode intensities are independent random variables. The assumption of statistical independence requires the
difference in mode phases to be large (Melrose 1979) and greatly simplifies the model by allowing the mode intensities to be added (Chandrasekhar 1960). The model also accounts for the additive instrumental noise in each of the Stokes parameters. By assuming the mode intensities and instrumental noise are normal random variables, one can derive analytical expressions for the distributions of total intensity, polarization, and fractional polarization, as well as distributions for the orientation angles of the polarization vector.

The main result from the model for the purposes of this paper is the derivation of the conditional density of the polarization vector’s orientation angles. The conditional density is the joint probability density of the vector’s colatitude, $\theta$, and longitude, $\phi$, at a fixed value of polarization amplitude, $r_o$. It captures the functional form of the more general joint density in a simple analytical expression. It is known as the Bingham-Mardia (Bingham & Mardia 1978), or von Mises-Fisher, distribution.

$$f(\theta, \phi | r_o) = \frac{\sin \theta \exp[\pm \kappa^2 (\cos \theta \pm \gamma)^2]}{4\pi w(\kappa, \gamma)}$$

(1)

The conditional density is parameterized by the constants $\kappa$ and $\gamma$ and is normalized by the constant $w$. The constant $\kappa$ can be regarded as a signal-to-noise ratio in polarization. The constant $\gamma$ satisfies the relation $|\gamma| \leq 1$. By construction, the distribution is symmetric in longitude, which is uniformly distributed over $2\pi$. The vector’s longitude and colatitude are statistically independent of one another.

The plus signs in the argument of the exponential in Equation 1 occur when the polarization fluctuations are predominantly parallel to the mode diagonal. They are caused by the randomly varying intensities of the OPMs. In this case, the functional form of the colatitude conditional density is generally bimodal. The polarization pattern formed by a projection of the conditional density generally consists of a set of concentric circular contours in each hemisphere of the projection. The circular shape of the pattern arises from the symmetry in longitude.

The minus signs in the argument of the exponentail in Equation 1 occur when the polarization fluctuations are predominantly perpendicular to the mode diagonal. The origin of these perpendicular fluctuations is not known, but is discussed in the following section. In this case, the conditional density is always unimodal because it is normal in $\cos \theta$. The polarization pattern formed by the projection of this conditional density is generally a complete annulus in only one of the two projection hemispheres (McKinnon 2009).

The general applicability of Equation 1 can be illustrated with a few special cases. When $\kappa = 0$, the polarization fluctuations are dominated by instrumental noise, and the conditional density becomes isotropic, as one would expect for pure noise. When $\kappa \gg 1$,
the fluctuations are very small in comparison to the polarized signal, and the conditional density becomes a Fisher distribution (Fisher et al. 1987). When $\gamma = 0$ and the fluctuations are predominantly along the mode diagonal, as caused by OPMs, the mean intensities of the modes are equal, the modes occur with equal frequency, and the conditional density becomes the Watson bipolar distribution (McKinnon 2006; Fisher et al. 1987).

The joint probability density of the vector’s colatitude and longitude has been shown to be a reasonable representation of the distribution of angles that are actually observed (McKinnon 2006). The conditional density has been shown to produce projections of the Poincaré sphere that are qualitatively consistent with the polarization patterns observed in pulsar radio emission (McKinnon 2009).

3. Generalized Faraday Rotation

Two aspects of the model and its application to the observations require additional explanation. These are (1) an explanation for the mechanism that causes the difference in mode phases to be large, thereby providing additional justification for the assumption of independent mode intensities and (2) a physical explanation for the mechanism that creates the fluctuations perpendicular to the mode diagonal, which were incorporated in the model to account for annular polarization patterns. The explanation for both may reside with generalized Faraday rotation (GFR; Edwards & Stappers 2004; McKinnon 2009).

In general terms, Faraday rotation is the physical process that alters the difference between the phases of the modes as they propagate through a plasma (Melrose 1979). The modes are incoherent when the difference in their phases at a given wavelength is large ($\Delta \chi \gg 1$) and are coherent (coupled) as long as the phase difference is small ($\Delta \chi < 1$). The modes retain their individual polarization identity in an observation when they are incoherent, but effectively lose their individual identity when they are coherent. Faraday rotation can become stochastic when the fluctuations in phase difference are large ($\sigma_\chi \gg 1$; Melrose & Macquart 1998).

GFR alters the component of the radiation’s polarization vector that is perpendicular to the polarization vectors of the plasma’s wave propagation modes. For any plasma, the unit vectors representing the polarization states of the two modes are anti-parallel on a diagonal through the Poincaré sphere. For the cold, weakly-magnetized plasma that is the interstellar medium (ISM), the propagation modes are circularly polarized, and the mode diagonal defined by their polarization vectors connects the poles of the Poincaré sphere. Faraday rotation in the ISM causes the orientation of the radiation’s polarization to vary
in a plane perpendicular to the mode diagonal, either on the Poincaré sphere’s equator or on a small circle parallel to it, depending upon the polarization state of the plasma-incident radiation. For the relativistic plasma in the strong magnetic field of a pulsar’s magnetosphere, the modes are thought to be linearly polarized (Allen & Melrose 1982; Barnard & Arons 1986; Melrose 1979) so that the mode diagonal lies in the equatorial plane of the Poincaré sphere. Similar to Faraday rotation in the ISM, GFR in a pulsar’s magnetosphere causes the polarization vector to rotate on a small circle in the Poincaré sphere that is perpendicular to and centered on the mode diagonal (e.g. see Fig. 3 of Kennett & Melrose 1998). Random fluctuations in $\Delta \chi$ (i.e. stochastic GFR) would appear as a partial annulus around the mode diagonal, as is observed in the core component of PSR B0329+54 (Edwards & Stappers 2004).

Figure 1 is a plot of $\sigma_\chi$ versus $\Delta \chi$ and summarizes the discussion above. The plot is divided into four regions, I through IV, that define the conditions under which OPM and stochastic GFR can occur. OPMs can occur only in regions III and IV, to the right of $\Delta \chi = 1$, where the modes are incoherent. The modes are coherent when $\Delta \chi < 1$; therefore, OPMs will not be observed when conditions in the pulsar magnetosphere (or the ISM) are consistent with those in regions I and II. The Faraday rotation that is typically observed in the ISM or in the lobes of extragalactic radio jets occurs in region II where the modes are coherent, but the fluctuations in $\Delta \chi$ are small. Stochastic GFR can occur only under the conditions specific to region I, where the modes are coherent but the fluctuations in $\Delta \chi$ are large. Returning now to the observations, the bimodal polarization pattern observed in the cone emission of PSR B0329+54 arises from OPMs. The mean and standard deviation of $\Delta \chi$ at this location of the pulse would reside in region III or IV of Figure 1. The properties of $\Delta \chi$ in the pulsar’s core precursor also likely reside in region III or IV of the figure, even though the polarization pattern at this pulse location consists of a single cluster of data points. OPMs clearly occur everywhere else within the pulse. OPMs may also occur in the precursor, but one of the modes may be so strong that the other mode is never detected. However, one cannot rule out the possibility that the properties of $\Delta \chi$ in the precursor reside within region II of the figure. The polarization pattern in the core component of PSR B0329+54 is much more complicated because both modes are present, but one of them reveals itself as a partial annulus. The statistical model described here cannot completely explain this behavior. The pattern may arise from a condition that falls on the regional boundaries of Figure 1 where the modes are occasionally coherent with large fluctuations in $\Delta \chi$ (i.e. in region I of the figure), thus explaining the partial annulus, but are otherwise incoherent (i.e. in region III or IV) to account for the bimodal aspect of the polarization pattern.
Fig. 1.— Requirements on mode phase difference, $\Delta \chi$, and its fluctuations, $\sigma_\chi$, for OPM and stochastic GFR to occur.
4. Conclusions

A statistical model has been developed for the polarization of pulsar radio emission. The model can explain a wide variety of polarization patterns observed in the radio emission. The observations are thus consistent with the model’s hypothesis that the polarization of the radiation is determined by the simultaneous interaction of two, highly polarized, orthogonal modes. The analysis of the polarization data shows that polarization signatures of physical processes can become apparent when the Stokes parameters are analyzed together, instead of separately. An interpretation of the model’s assumptions and its application to the observations suggest that generalized Faraday rotation may be operative in pulsar magnetospheres. The model shows, in a rigorous way, that polarization measurements follow the statistics of direction.

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