Effective Field Theory for Goldstone Bosons in Nonrelativistic Superfluids

Jens O. Andersen

Institute for Theoretical Physics, University of Utrecht,
Leuvenlaan 4, 3584 CE Utrecht, The Netherlands

(Dated: January 3, 2022)

We consider nonrelativistic superfluids where the global $U(1)$-symmetry is spontaneously broken. At sufficiently long wavelengths, the relevant degree of freedom is the massless Goldstone mode and we construct an effective low energy theory for the Goldstone boson. The damping rate of collective excitations at low energy is calculated. In the case of a weakly interacting Bose gas, we recover the results by Beliaev, and by Hohenberg and Martin.

PACS numbers: PACS numbers: 03.75.Fi, 67.40.-w, 32.80.Pj

I. INTRODUCTION

Spontaneous symmetry breaking is central in our description of many phenomena in condensed matter and high-energy physics (see e.g. Ref. [1] and references therein). An example of spontaneous symmetry breaking is the breaking of the global $U(1)$-symmetry in $^3$He and the appearance of superfluidity below a certain critical temperature. Associated with the breaking of the symmetry, there is a nonzero value of an order parameter and a condensate of particles in the zero-momentum state. When a global continuous symmetry is broken, the Goldstone theorem [2] states that there is a gapless excitation for each generator that does not leave the ground state invariant. In the case of nonrelativistic Bose gases, one identifies the Goldstone mode with the phonons.

At long wavelengths where the relevant degrees of freedom are the massless Goldstone bosons, one would naturally like to have an effective low energy theory that contains only these degrees of freedom. Such an effective field theory may simplify calculations of the long-wavelength properties of the system and one can obtain model independent predictions with a minimum of assumptions.

An example of a low energy theory for massless Goldstone modes is chiral perturbation theory in QCD [3]. In massless QCD, chiral symmetry is broken and one identifies the Goldstone bosons with the pions. However, QCD is a confining and strongly interacting theory at low energies. Thus the coefficients of the chiral Lagrangian cannot be calculated from QCD. Instead, the coefficients of the effective Lagrangian are fixed by experiments. In other cases, one can determine the coefficients of the low-energy theory as functions of the coupling constants in the underlying theory e.g. by a matching procedure. One calculates physical quantities at low energies perturbatively and demand they be the same in the full and in the effective theory. The coefficients of the effective theory then encode the physics at short distances. Nonrelativistic QED [4] is an example of such an effective field theory that is tailored to perform low-energy (bound state) calculations, where the coefficients encode the effects of relativistic momenta.

In this paper, we construct an effective low energy theory for the Goldstone boson in nonrelativistic superfluids with a broken global $U(1)$-symmetry. The effective Lagrangian allows one to calculate physical quantities in the long-wavelength limit with ease without assuming weak coupling.

II. EFFECTIVE LAGRANGIAN

The effective theory for the Goldstone field $\phi$ can be constructed using the methods of effective field theory [3]. Once the symmetries of the theory have been identified, one writes down the most general Lagrangian $L_{\text{eff}}$ that is consistent with these symmetries. Examples of symmetries are time-reversal invariance and Galilean invariance, and these symmetries severely restrict the possible terms in $L_{\text{eff}}$. For instance, Galilean invariance implies that the terms in the effective theory are powers of the combination $\partial_\tau \phi - \frac{1}{2} (\nabla \phi)^2$ ($h = k_B = m = 1$ henceforth). The Euclidean Lagrangian can then be written as

$$L_{\text{eff}} = -c_1 \left[ i \partial_\tau \phi - \frac{1}{2} (\nabla \phi)^2 \right] - \frac{1}{2} c_2 \left[ i \partial_\tau \phi - \frac{1}{2} (\nabla \phi)^2 \right]^2 + d_1 \left[ \partial_\tau \phi - \frac{1}{2} (\nabla \phi)^2 \right]^3 + \delta L_{\text{eff}},$$

where $c_1, c_2, ...$ are parameters that must be determined either by experiment or by matching. The first term is a total derivative and can be omitted if one is not interested in topologically nontrivial field configuration such as vortices. The term $\delta L_{\text{eff}}$ contains all terms that are higher order in the field $\phi$ and derivatives thereof. We have also omitted the unit operator $f$ [6], which is necessary to include if one is interested in the equation of state. The unit operator $f$ can be interpreted as the contribution to the free energy from large momenta. We will
briefly discuss this issue below. The terms in Eq. (1) that we have shown explicitly, gives rise to a linear dispersion relation for the Goldstone bosons, and the effective Lagrangian is therefore valid only for momenta where this is a good approximation. At larger momenta, the dispersion relation is no longer linear and terms of the form $(\nabla^2 \phi)^2$ must be included. The scale at which the effective Lagrangian no longer can be applied is given by the scale where other degrees of freedom than the Goldstone boson become important. In the case of a weakly interacting Bose gas, this scale is given by the coherence length $\xi$.

The free propagator that corresponds to the Lagrangian \( \mathcal{L} \) is

\[
\Delta(p_0, p) = \frac{1}{c_1 p_0^2 + c_2 p^2}, \quad (2)
\]

where $p_0 = 2\pi n T$ are the Matsubara frequencies and $p = |p|$. The dispersion relation is given by the pole of the Minkowski space propagator:

\[
\epsilon(p) = \frac{c_2}{c_1} p. \quad (3)
\]

The nonrelativistic hydrodynamic speed of sound $c$ is then given by $\sqrt{c_2/c_1}$. In some cases, one does not know the underlying microscopic theory, and so one cannot determine the couplings $c_1, c_2, ...$ by matching. Instead, one performs experiments at low energies. For instance, the ratio $c_2/c_1$ is fixed by measuring the speed of sound.

In the case of a weakly interacting Bose gas, we know the underlying theory and we can calculate the coefficients $c_1, c_2, ...$ in terms of the parameters of the full theory. The Euclidean action for nonrelativistic bosons at low energy is

\[
S[\psi^\dagger, \psi] = \int_0^\beta d\tau \int \mathbf{x} \psi^\dagger \left[ \frac{\partial}{\partial \tau} - \frac{1}{2} \nabla^2 - \mu \right] \psi - \frac{1}{2} \int_0^\beta d\tau \int \mathbf{x} g(\psi^\dagger \psi)^2, \quad (4)
\]

where $\psi(\mathbf{x})$ annihilates a boson at position $\mathbf{x}$, $\mu$ is the chemical potential, and $g = 4\pi a$, where $a$ is the s-wave scattering length. If we substitute $\psi(\mathbf{x}, \tau) = \sqrt{n_0 + \sigma(x, \tau)} e^{i\phi(x, \tau)}$ into Eq. (4), we obtain the action

\[
S[\sigma, \phi] = \int_0^\beta d\tau \int \mathbf{x} \left\{ \left[ n_0 g - \mu \right] n_0 \sigma + \frac{1}{2} \left[ \sigma \frac{\partial \phi}{\partial \tau} - \phi \frac{\partial \sigma}{\partial \tau} \right] \right. \\
\left. + \frac{1}{2} \frac{1}{n_0} \nabla^2 + 2 V_0 \right\} \sigma + \frac{1}{2} \sigma \left( \nabla \phi \right)^2 + ... \}, \quad (5)
\]

where we have dropped total derivatives as well as an infinite series of higher order momentum-dependent interactions. By using the classical equation of motion for

\[
\sigma(x, \tau), \quad \text{we obtain from Eq. (5) the following action for the phase } \phi:
\]

\[
S[\phi] = \int_0^\beta d\tau \int \mathbf{x} \left[ \frac{1}{2g} (\partial_\tau \phi)^2 + \frac{1}{2} n_0 (\nabla \phi)^2 + ... \right]. \quad (6)
\]

Using Eq. (1) and $S = \int_0^\beta d\tau \int \mathbf{x} \, \mathcal{L}_{\text{eff}}$, and comparing with Eq. (6) we find the coefficients $c_1 = n_0$ and $c_2 = 1/g$.

As an application of the effective low-energy Lagrangian \( \mathcal{L} \) for the Goldstone bosons, we next calculate the damping rate of a long-wavelength excitation in the superfluid. The propagator has poles in the complex energy plane, and the real part gives the energy $\epsilon(p)$ of the excitation, while the imaginary part gives the damping rate.

The effective Lagrangian \( \mathcal{L} \) gives rise to three-point and four-point interactions and the one-loop Feynman diagrams contributing to the self-energy $\Pi(p_0, p)$ are shown in Fig. 1. The first diagram vanishes identically since the vertices are momentum dependent. The second and third diagrams are ultraviolet divergent and must be regularized. In this paper, we use dimensional regularization to regulate both infrared and ultraviolet divergences. In dimensional regularization, one calculates the loop integrals in $d = 3 - 2\epsilon$ dimensions for values of $\epsilon$ where the integrals converge. One then analytically continues back to $d = 3$ dimensions. With dimensional regularization, an arbitrary renormalization scale $M$ is introduced. An advantage of dimensional regularization is that it automatically sets power divergences to zero, while logarithmic divergences show up as poles in $\epsilon$. The second diagram is proportional to $p^2$ and has pentic, cubic, and linear divergences in the ultraviolet. It is therefore set to zero with dimensional regularization. The third graph has a logarithmic ultraviolet divergence. The one-loop self-energy is

\[
\Pi(p_0, p) = \frac{1}{4} c_2^2 \left( \frac{\epsilon M^2}{4\pi} \right)^\epsilon \sum \int \frac{d^d k}{(2\pi)^d} \left\{ \right.
\]

\[
\frac{1}{[c_1 k_0^2 + c_2 k^2][c_1 (p_0 + k_0)^2 + c_2 (p + k)^2]}
\times \left[ 4 (p \cdot k)^2 (p_0 + k_0)^2 + (p_0 k^2 + k_0 p^2)^2 \right]
\left. + 4(p \cdot k)(p_0 + k_0)(p_0 k^2 + k_0 p^2) \right\}, \quad (7)
\]

where $\gamma \approx 0.5772$ is the Euler-Mascharoni constant. In
the zero-temperature limit, the sum over Matsubara frequencies becomes an integral over the Euclidean energy. The next step in evaluating the self-energy is to introduce a Feynman parameter $y$. After integrating over energy, momentum, and finally over $y$, the self-energy can be expanded in powers of $\epsilon$. The result is

$$\Pi(p_0, p) = -\frac{p_0^2}{320 \pi^2 \sqrt{c_1c_2}} \left[ 49p^4 + 30 \frac{c_1}{c_2} p^2 + 5 \frac{c_2}{c_1} p^4 \right] \times \left[ \frac{1}{\epsilon} + \log \left( \frac{M^2}{p_0^2 + p^2} \right) + g(p_0, p) \right],$$

(8)

where $g(p_0, p)$ is an analytic function that is not important to us. After analytic continuation to real frequencies, $p_0 \to i\omega + \eta$, we find the the damping rate

$$\gamma = \text{Im} \Pi(-i\epsilon(p) + \eta, p)/2 \epsilon(p),$$

(9)

Note that the result is independent of the coupling $c_2$. By measuring the damping of long-wavelength excitation one can determine $c_1$, while measuring the speed of sound $c$ will determine ratio $c_1/c_2$. Using the value $c_1 = n_0$ for a weakly interacting Bose gas, we recover the result by Belayev [10]. (See also Refs. [11, 12]).

The quadratic part of action [6] has recently been used a starting point for calculating phase fluctuations in trapped Bose gases at low temperature [11, 12]. The effective field theory approach clearly shows why such an approach is incorrect. The phase-alone action is only valid for momenta less than the inverse coherence length, while in Refs. [10, 11], it was used for all momenta. The correct treatment of the phase fluctuations was given in Refs. [12, 13].

Consider next the pressure for a dilute Bose gas at zero temperature. If one calculates $\mathcal{P}$ using the effective Lagrangian [6], one gets zero. This disagrees with the one-loop result of Lee and Yang [7]. The reason is simply that one must include the unit operator $\eta$ in the effective Lagrangian [6] and that contribution to the pressure exactly equals the standard result.

### III. FINITE TEMPERATURE EFFECTS

We next consider finite-temperature effects using the effective low-energy Lagrangian [6].

The pressure at one-loop order is given by

$$\mathcal{P} = -\frac{1}{2} \left( \frac{\epsilon^2 M^2}{4\pi} \right)^2 \sum_{k_0} \int \frac{d^4k}{(2\pi)^d} \log \left[ c_1 k_0^2 + c_2 k^2 \right]$$

$$= \frac{\pi^2 T^4}{90} \left( \frac{c_1}{c_2} \right)^{3/2}.$$

(10)

For a weakly interacting Bose gas, this result is in agreement with the leading temperature-dependent term in a low-temperature expansion carried out by Lee and Yang [7] (See also Ref. [14]). This simply reflects the fact that the massless Goldstone modes are the important thermal excitations at low temperature.

Finally we consider damping at finite $T$. The imaginary part of the self-energy can be calculated in a similar manner as before, except that the integral over Euclidean energy is replaced by a summation over Matsubara frequencies. The result is

$$\gamma = \frac{3\pi^3 T^4}{40 c_1}.$$

In the case of a weakly interacting Bose gas, this results was first obtained by Hohenberg and Martin [17]. Using the Lagrangian [6], Liu [8] obtained a more general result to lowest order in $p$. This result is valid for $p \ll n_0 g$ and reduces to $3\pi^2 T^4/(40 n_0)$ in the limit $T/n_0 g \to 0$.

### IV. DISCUSSION

In this paper, we have constructed an effective field theory for Goldstone modes at low energies for a nonrelativistic superfluid with a broken $U(1)$-symmetry. The results for the damping of collective excitations is in agreement with those of Belayev, and by Hohenberg and Martin for the weakly interacting theory. However, the results go beyond because nowhere have we assumed weak coupling. The only assumptions are Galilean invariance and low energy. Hence, the results [6, 11, 12] are valid for any nonrelativistic superfluid with a broken $U(1)$-symmetry in the long-wavelength limit.

There is another way of arriving at a low-energy theory for the Goldstone modes, namely by calculating the quantum effective action [6]. In the effective action approach, integrating out "short-distance physics" amounts to minimizing with respect to the amplitude of the field $\psi(x, \tau)$. Such an approach was used in Ref. [18] to calculate a low-energy effective Lagrangian for relativistic superfluids where the $U(1)$ baryon symmetry is spontaneously broken. In Ref. [18], it was shown that the scattering amplitudes among the Goldstone bosons can be found once the equation of state is known. Applying the formalism to the dilute Bose calculating the quantum effective action and using $\mathcal{P}(\mu) = \mu^2/2g$, one would immediately obtain Eq. (10) with the coefficients $c_1 = \mu/g = n_0$ and $c_2 = 1/g$.

### Acknowledgments

This work was supported by the Stichting voor Fundamenteel Onderzoek der Materie (FOM), which is supported by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO).
[1] S. Weinberg, *The Quantum Theory of Fields II, Modern Applications* (Cambridge University Press, Cambridge England 1996).
[2] J. Goldstone, Nuovo Cim. 19, 154 (1961).
[3] J. Gasser and H. Leutwyler, Annals Phys. 158, 142 (1984); Nucl. Phys. B 250, 465 (1985).
[4] W.E. Caswell and G.P. Lepage, Phys. Lett. B 167, 437 (1986).
[5] H. Georgi, Ann. Rev. Nucl. Part. Sci. 43, 209 (1993).
[6] E. Braaten, A. Nieto, Phys. Rev. D 51, 6990 (1995).
[7] S.T. Beliaev, Sov. J. Phys. 7, 289 (1958); 34, 299 (1958).
[8] W.V. Liu, Phys. Rev. Lett. 79, 4056 (1997).
[9] W.V. Liu, Int. J. Mod. Phys. B 12, 2103 (1998).
[10] D. S. Petrov, M. Holzmann, and G.V. Shlyapnikov, Phys. Rev. Lett. 84, 2551 (2000).
[11] D. S. Petrov, G.V. Shlyapnikov, and J.T.M. Walraven, Phys. Rev. Lett. 85, 3745 (2000).
[12] J. O. Andersen, U. Al Khawaja, and H. T. C. Stoof, Phys. Rev. Lett 88, 070407 (2002).
[13] U. Al Khawaja, J.O. Andersen, N.P. Proukakis, and H.T.C. Stoof, Phys. Rev. A 66, 013615 (2002).
[14] T.D. Lee and C.N. Yang, Phys. Rev. 105, 1119 (1957).
[15] T.D. Lee and C.N. Yang, Phys. Rev. 112, 1419 (1958).
[16] T. Haugset, H. Haugerud, and F. Ravndal, Ann. Phys. (N.Y) 226, 27 (1998).
[17] P.C. Hohenberg and P.C. Martin, Ann Phys. (N.Y.) 34, 291 (1965).
[18] D.T. Son, hep-ph/0204193.