On the slice genus of links

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Abstract We define Casson-Gordon $\sigma$-invariants for links and give a lower bound of the slice genus of a link in terms of these invariants. We study as an example a family of two component links of genus $h$ and show that their slice genus is $h$, whereas the Murasugi-Tristram inequality does not obstruct this link from bounding an annulus in the 4-ball.

AMS Classification 57M25; 57M27

Keywords Casson-Gordon invariants, link signatures

1 Introduction

A knot in $S^3$ is slice if it bounds a smooth 2-disk in the 4-ball $B^4$. Levine showed [Le] that a slice knot is algebraically slice, i.e. any Seifert form of a slice knot is metabolic. In this case, the Tristram-Levine signatures at the prime power order roots of unity of a slice knot must be zero. Levine showed also that the converse holds in high odd dimensions, i.e. any algebraically slice knot is slice. This is false in dimension 3: Casson and Gordon [CG1, CG2, G] showed that certain two-bridge knots in $S^3$, which are algebraically slice, are not slice knots. For this purpose, they defined several knot and 3-manifold invariants, closely related to the Tristram-Levine signatures of associated links. Further methods to calculate these invariants were developed by Gilmer [Gi3, Gi4], Litherland [Li], Gilmer-Livingston [GL], and Naik [N]. Lines [L] also computed some of these invariants for some fibered knots, which are algebraically slice but not slice. The slice genus of a link is the minimal genus for a smooth oriented connected surface properly embedded in $B^4$ with boundary the given link.

The Murasugi-Tristram inequality (see Theorem 2.1 below) gives a lower bound on the slice genus of a link in terms of the link’s Tristram-Levine signatures and related nullity invariants. The second author [Gi1] used Casson-Gordon invariants to give another lower bound on the slice genus of a knot. In particular
he gave examples of algebraically slice knots whose slice genus is arbitrarily large. We apply these methods to restrict the slice genus of a link.

We study an example a family of two component links, which have genus \( h \) Seifert surfaces. Using Theorem 4.1, we show that these links cannot bound a smoothly embedded surface in \( B^4 \) with genus lower than \( h \), while the Murasugi-Tristram inequality does not show this. In fact there are some links with the same Seifert form that bound annuli in \( B^4 \). We work in the smooth category.

The second author was partially supported by NSF-DMS-0203486.

2 Preliminaries

2.1 The Tristram-Levine signatures

Let \( L \) be an oriented link in \( S^3 \), with \( \mu \) components, and \( \theta_S \) be the Seifert pairing corresponding to a connected Seifert surface \( S \) of the link. For any complex number \( \lambda \) with \(|\lambda| = 1\), one considers the hermitian form \( \theta^\lambda_S := (1 - \lambda)\theta_S + (1 - \overline{\lambda})(\theta_S)^T \). The Tristram signature \( \sigma_L(\lambda) \) and nullity \( n_L(\lambda) \) of \( L \) are defined as the signature and nullity of \( \theta^\lambda_S \). Levine defined these same signatures for knots [Le]. The Alexander polynomial of \( L \) is \( \Delta_L(t) := \text{Det}(\theta_S - t(\theta_S)^T) \).

As is well-known, \( \sigma_L \) is a locally constant map on the complement in \( S^1 \) of the roots of \( \Delta_L \) and \( n_L \) is zero on this complement. If \( \Delta_L = 0 \), it is still true that the signature and nullity are locally constant functions on the complement of some finite collection of points.

The Murasugi-Tristram inequality allows one to estimate the slice genus of \( L \), in terms of the values of \( \sigma_L(\lambda) \) and \( n_L(\lambda) \).

\textbf{Theorem 2.1} [M, T] Suppose that \( L \) is the boundary of a properly embedded connected oriented surface \( F \) of genus \( g \) in \( B^4 \). Then, if \( \lambda \) is a prime power order root of unity, we have

\[ |\sigma_L(\lambda)| + n_L(\lambda) \leq 2g + \mu - 1. \]

2.2 The Casson-Gordon \( \sigma \)-invariant

In this section, for the reader convenience, we review the definition and some of the properties of the simplest kind of Casson-Gordon invariant. It is a reformulation of the Atiyah-Singer \( \alpha \)-invariant.
Let $M$ be an oriented compact three manifold and $\chi: H_1(M) \to \mathbb{C}^*$ be a character of finite order. For some $q \in \mathbb{N}^*$, the image of $\chi$ is contained a cyclic subgroup of order $q$ generated by $\alpha = e^{2i\pi/q}$. As $\text{Hom}(H_1(M), C_q) = [M, B(C_q)]$, it follows that $\chi$ induces $q$-fold covering of $M$, denoted $\tilde{M}$, with a canonical deck transformation. We will denote this transformation also by $\alpha$. If $\chi$ maps onto $C_q$, the canonical deck transformation sends $x$ to the other endpoint of the arc that begins at $x$ and covers a loop representing an element of $(\chi)^{-1}(\alpha)$.

As the bordism group $\Omega_3(B(C_q)) = C_q$, we may conclude that $n$ disjoint copies of $M$, for some integer $n$, bounds bound a compact 4-manifold $W$ over $B(C_q)$. Note $n$ can be taken to be $q$. Let $\tilde{W}$ be the induced covering with the deck transformation, denoted also by $\alpha$, that restricts to $\alpha$ on the boundary. This induces a $\mathbb{Z}[C_q]$-module structure on $C_*(\tilde{W})$, where the multiplication by $\alpha \in \mathbb{Z}[C_q]$ corresponds to the action of $\alpha$ on $\tilde{W}$.

The cyclotomic field $\mathbb{Q}(C_q)$ is a natural $\mathbb{Z}[C_q]$-module and the twisted homology $H^i_t(W; \mathbb{Q}(C_q))$ is defined as the homology of $C_*(\tilde{W}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q)$.

Since $\mathbb{Q}(C_q)$ is flat over $\mathbb{Z}[C_q]$, we get an isomorphism $H^i_t(W; \mathbb{Q}(C_q)) \simeq H_*(\tilde{W}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q)$.

Similarly, the twisted homology $H^i_t(M; \mathbb{Q}(C_q))$ is defined as the homology of $C_*(\tilde{M}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q)$.

Let $\bar{\phi}$ be the intersection form on $H_2(\tilde{W}; \mathbb{Q})$ and define

$$\phi_\chi(W): H^2_t(W; \mathbb{Q}(C_q)) \times H^2_t(W; \mathbb{Q}(C_q)) \to \mathbb{Q}(C_q)$$

so that, for all $a, b$ in $\mathbb{Q}(C_q)$ and $x, y$ in $H_2(\tilde{W})$,

$$\phi_\chi(W)(x \otimes a, y \otimes b) = \pi b \sum_{i=1}^q \bar{\phi}(x, \alpha^i y) \pi^i,$$

where $a \to \bar{a}$ denotes the involution on $\mathbb{Q}(C_q)$ induced by complex conjugation.

**Definition 2.2** The Casson-Gordon $\sigma$-invariant of $(M, \chi)$ and the related nullity are

$$\sigma(M, \chi) := \frac{1}{n} \left( \text{Sign}(\phi_\chi(W)) - \text{Sign}(W) \right)$$

$$\eta(M, \chi) := \dim H^1_t(M; \mathbb{Q}(C_q)).$$
If $U$ is a closed 4-manifold and $\chi : H_1(U) \to C_q$ we may define $\phi_{\chi}(U)$ as above. One has that modulo torsion the bordism group $\Omega_4(B(C_q))$ is generated by the constant map from $CP(2)$ to $B(C_q)$. If $\chi$ is trivial, one has that $\text{Sign}(\phi_{\chi}(U)) = \text{Sign}(U)$. Since both signatures are invariant under cobordism, one has in general that $\text{Sign}(\phi_{\chi}(U)) = \text{Sign}(U)$. The independence of $\sigma(M, \chi)$ from the choice of $W$ and $n$ follows from this and Novikov additivity. One may see directly that these invariants do not depend on the choice of $q$. In this way Casson and Gordon argued that $\sigma(M, \chi)$ is an invariant. Alternatively one may use the Atiyah-Singer G-Signature theorem and Novikov additivity [AS].

We now describe a way to compute $\sigma(M, \chi)$ for a given surgery presentation of $(M, \chi)$.

**Definition 2.3** Let $K$ be an oriented knot in $S^3$. Let $A$ be an embedded annulus such that $\partial A = K \cup K'$ with $\text{lk}(K, K') = f$. A $p$-cable on $K$ with twist $f$ is defined to be the union of oriented parallel copies of $K$ lying in $A$ such that the number of copies with the same orientation minus the number with opposite orientation is equal to $p$.

Let us suppose that $M$ is obtained by surgery on a framed link $L = L_1 \cup \cdots \cup L_\mu$ with framings $f_1, \ldots, f_\mu$. One shows that the linking matrix $\Lambda$ of $L$ with framings in the diagonal is a presentation matrix of $H_1(M)$ and a character on $H_1(M)$ is determined by $\alpha^p_i = \chi(m_{L_i}) \in C_q$ where $m_{L_i}$ denotes the class of the meridian of $L_i$. Let $\vec{p} = (p_1, \ldots, p_\mu)$. We use the following generalization of a formula in [CG2, Lemma (3.1)], where all $p_i$ are assumed to be 1, that is given in [Gi2, Theorem(3.6)].

**Proposition 2.4** Suppose $\chi$ maps onto $C_q$. Let $L'$ with $\mu'$ components be the link obtained from $L$ by replacing each component by a non-empty algebraic $p_i$-cable with twist $f_i$ along this component. Then, if $\lambda = e^{2ir\pi/q}$, for $(r, q) = 1$, one has

$$\sigma(M, \chi') = \sigma_{L'}(\lambda) - \text{Sign}(\Lambda) + 2\frac{r(q-r)}{q^2} \vec{p}^\top \Lambda \vec{p},$$

$$\eta(M, \chi') = \eta_{L'}(\lambda) - \mu' + \mu.$$

The following proposition collects some easy additivity properties of the $\sigma$-invariant and the nullity under the connected sum.

**Proposition 2.5** Suppose that $M_1, M_2$ are connected. Then,
for all $\chi_i \in H^1(M_i; C_q)$, $i = 1, 2$, we have
\[ \sigma(M_1 \# M_2, \chi_1 \oplus \chi_2) = \sigma(M_1, \chi_1) + \sigma(M_2, \chi_2). \]
If both $\chi_i$ are non-trivial, then
\[ \eta(M_1 \# M_2, \chi_1 \oplus \chi_2) = \eta(M_1, \chi_1) + \eta(M_2, \chi_2) + 1. \]
If one $\chi_i$ is trivial, then
\[ \eta(M_1 \# M_2, \chi_1 \oplus \chi_2) = \eta(M_1, \chi_1) + \eta(M_2, \chi_2). \]

Proposition 2.6 For all $\chi \in H_1(S^1 \times S^2; C_q)$, we have
\[ \sigma(S^1 \times S^2, \chi) = 0. \]
If $\chi \neq 0$, then $\eta(S^1 \times S^2, \chi) = 0$. If $\chi = 0$, then $\eta(S^1 \times S^2, \chi) = 1$.

Proposition 2.6 for non-trivial $\chi$ can be proved for example by the use of Proposition 2.4, since $S^1 \times S^2$ is obtained by surgery on the unknot framed $0$. However it is simplest to derive this result directly from the definitions.

2.3 The Casson-Gordon $\tau$-invariant

In this section, we recall the definition and some of the properties of the Casson-Gordon $\tau$-invariant. Let $C_\infty$ denote a multiplicative infinite cyclic group generated by $t$. For $\chi^+: H_1(M) \to C_q \oplus C_\infty$, we denote $\bar{\chi}: H_1(M) \to C_q$ the character obtained by composing $\chi^+$ with projection on the first factor. The character $\chi^+$ induces a $C_q \times C_\infty$-covering $M_\infty$ of $M$.

Since the bordism group $\Omega_3(B(C_q \times C_\infty)) = C_q$, bounds a compact 4-manifold $W$ over $B(C_q \times C_\infty)$ Again $n$ can be taken from to be $q$.

If we identify $\mathbb{Z}[C_q \times C_\infty]$ with the Laurent polynomial ring $\mathbb{Z}[C_q][t, t^{-1}]$, the field $\mathbb{Q}(C_q)(t)$ of rational functions over the cyclotomic field $\mathbb{Q}(C_q)$ is a flat $\mathbb{Z}[C_q \times C_\infty]$-module. We consider the chain complex $C_\ast(\tilde{W}_\infty)$ as a $\mathbb{Z}[C_q \times C_\infty]$-module given by the deck transformation of the covering. Since $W$ is compact, the vector space $H^1_2(W; \mathbb{Q}(C_q)(t)) \simeq H_2(\tilde{W}_\infty) \otimes_{\mathbb{Z}[C_q]}[t, t^{-1}] \mathbb{Q}(C_q)(t)$ is finite dimensional.

We let $J$ denote the involution on $\mathbb{Q}(C_q)(t)$ that is linear over $\mathbb{Q}$ sends $t^i$ to $t^{-i}$ and $\alpha^i$ to $\alpha^{-i}$. As in [G], one defines a hermitian form, with respect to $J$,
\[ \phi_\chi^+: H^1_2(W; \mathbb{Q}(C_q)(t)) \times H^1_2(W; \mathbb{Q}(C_q)(t)) \to \mathbb{Q}(C_q)(t), \]
such that
\[ \phi_{\chi^+}(x \otimes a, y \otimes b) = J(a) \cdot b \cdot \sum_{i \in \mathbb{Z}} \sum_{j=1}^{q} \phi^+(x, t^i a^j y) \overline{\tau^i} t^{-i}. \]

Here \( \phi^+ \) denotes the ordinary intersection form on \( \widetilde{W}_\infty \). Let \( \mathcal{W}(\mathbb{Q}(C_q)(t)) \) be the Witt group of non-singular hermitian forms on finite dimensional \( \mathbb{Q}(C_q)(t) \) vector spaces. Let us consider \( H_2(W; \mathbb{Q}(C_q)(t))/\text{Radical}(\phi_{\chi^+}) \). The induced form on it represents an element in \( \mathcal{W}(\mathbb{Q}(C_q)(t)) \), which we denote \( w(W) \).

Furthermore, the ordinary intersection form on \( H_2(W; \mathbb{Q}) \) represents an element of \( \mathcal{W}(\mathbb{Q}) \). Let \( w_0(W) \) be the image of this element in \( \mathcal{W}(\mathbb{Q}(C_q)(t)) \).

**Definition 2.7** The Casson-Gordon \( \tau \)-invariant of \( (M, \chi^+) \) is
\[ \tau(M, \chi^+) := \frac{1}{n} (w(W) - w_0(W)) \in \mathcal{W}(\mathbb{Q}(C_q)(t)) \otimes \mathbb{Q}. \]

Suppose that \( nM \) bounds another compact 4-manifold \( W' \) over \( B(C_q \times C_\infty) \). Form the closed compact manifold over \( B(C_q \times C_\infty) \), \( U := W \cup W' \) by gluing along the boundary. By Novikov additivity, we get \( w(U) = w_0(U) = (w(W) - w_0(W')) - (w(W') - w_0(W')) \). Using [CF], the bordism group \( \Omega_4(B(C_q \times C_\infty)) \), modulo torsion, is generated by \( CP(2) \), with the constant map to \( B(C_q \times C_\infty) \).

We have that \( w(CP(2)) = w_0(CP(2)) \). Since \( w(U) \), and \( w_0(U) \) only depend on the bordism class of \( U \) over \( B(C_q \times C_\infty) \), it follows that \( w(U) = w_0(U) \) and \( \tau(M, \chi^+) \) is independent of the choice of \( W \). Using the above techniques, one may check \( \tau(M, \chi^+) \) is independent of \( n \).

If \( A \in \mathcal{W}(\mathbb{Q}(C_q)(t)) \), let \( A(t) \) be a matrix representative for \( A \). The entries of \( A(t) \) are Laurent polynomials with coefficients in \( \mathbb{Q}(C_q) \). If \( \lambda \in S^1 \subset \mathbb{C} \), then \( A(\lambda) \) is hermitian and has a well defined signature \( \sigma_\lambda(A) \). One can view \( \sigma_\lambda(A) \) as a locally constant map on the complement of the set of the zeros of \( \det A(\lambda) \). As in [CG1], we re-define \( \sigma_\lambda(A) \) at each point of discontinuity as the average of the one-sided limits at the point.

We have the following estimate [Gi3, Equation (3.1)].

**Proposition 2.8** Let \( \chi^+ : H_1(M) \to C_\infty \otimes C_q \) and \( \bar{\chi} : H_1(M) \to C_q \) be \( \chi^+ \) followed by the projection to \( C_q \). We have
\[ |\sigma_1(\tau(M, \chi^+)) - \sigma(M, \bar{\chi})| \leq \eta(M, \bar{\chi}). \]
2.4 Linking forms

Let $M$ be a rational homology 3-sphere with linking form $l: H_1(M) \times H_1(M) \to \mathbb{Q}/\mathbb{Z}$.

We have that $l$ is non-singular, that is the adjoint of $l$ is an isomorphism $\iota: H_1(M) \to \text{Hom}(H_1(M), \mathbb{C}^*)$. Let $\nu$ denote the map $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}^*$ that sends $\frac{a}{b}$ to $e^{2\pi i a/b}$. So we have an isomorphism $\iota: H_1(M) \to H_1(M)^*$ given by $x \mapsto \nu \circ \iota(x)$. Let $\beta: H_1(M)^* \times H_1(M)^* \to \mathbb{Q}/\mathbb{Z}$ be the dual form defined by $\beta(x, y) = -l(x, y)$.

**Definition 2.9** The form $\beta$ is metabolic with metabolizer $H$ if there exists a subgroup $H$ of $H_1(M)^*$ such that $H^\perp = H$.

**Lemma 2.10** [Gi1] If $M$ bounds a spin 4-manifold $W$ then $\beta = \beta_1 \oplus \beta_2$ where $\beta_2$ is metabolic and $\beta_1$ has an even presentation with rank $\dim H_2(W; \mathbb{Q})$ and signature $\text{Sign}(W)$. Moreover, the set of characters that extend to $H_1(W)$ forms a metabolizer for $\beta_2$.

2.5 Link invariants

Let $L = L_1 \cup \cdots \cup L_\mu$ be an oriented link in $S^3$. Let $N_2$ be the two-fold covering of $S^3$ branched along $L$ and $\beta_L$ be the linking form on $H_1(N_2)^*$, see previous section.

We suppose that the Alexander polynomial of $L$ satisfies
\[
\Delta_L(-1) \neq 0.
\]

Hence, $N_2$ is a rational homology sphere. Note that if $\Delta_L(-1) \neq 1$, then $H_1(N_2; \mathbb{Z})$ is non-trivial.

**Definition 2.11** For all characters $\chi$ in $H_1(N_2)^*$, the Casson-Gordon $\sigma$-invariant of $L$ and the related nullity are (see Definition 2.2):
\[
\sigma(L, \chi) := \sigma(N_2, \chi),
\]
\[
\eta(L, \chi) := \eta(N_2, \chi).
\]

**Remark 2.12** If $L$ is a knot, then Definition 2.11 coincides with $\sigma(L, \chi)$ defined in [CG1, p.183].
3 Framed link descriptions

In this section, we study the Casson-Gordon $\tau$-invariants of the two-fold cover $M_2$ of the manifold $M_0$ described below.

Let $S^3 - T(L)$ be the complement in $S^3$ of an open tubular neighborhood of $L$ in $S^3$ and $P$ be a planar surface with $\mu$ boundary components.

Let $S$ be a Seifert surface for $L$ and $\gamma_i$ for $i = 1, \ldots, \mu$ be the curves where $S$ intersects the boundary of $S^3 - T(L)$. We define $M_0$ as the result of gluing $P \times S^1$ to $S^3 - T(L)$, where $P \times 1$ is glued along the curves $\gamma_i$. Let $*$ be a point in the boundary of $P$.

A recipe for drawing a framed link description for $M_0$ is given in the proof of Proposition 3.1.

**Proposition 3.1**

\[ H_1(M_0) \simeq \mathbb{Z} \oplus \mathbb{Z}^{\mu - 1} \simeq \langle m \rangle \oplus \mathbb{Z}^{\mu - 1}, \]

where $m$ denotes the class of $* \times S^1$ in $P \times S^1$.

**Proof**  Form a 4-manifold $X$ by gluing $P \times D^2$ to $D^4$ along $S^3$ in such a way that the total framing on $L$ agrees with the Seifert surface $S$. The boundary of this 4-manifold is $M_0$. We can get a surgery description of $M_0$ in the following way: pick $\mu - 1$ paths of $S$ joining up the components of $L$ in a chain. Deleting open neighborhoods of these paths in $S$ gives a Seifert surface for a knot $L'$ obtained by doing a fusion of $L$ along bands that are neighborhoods of the original paths. Put a circle with a dot around each of these bands (representing a 4-dimensional 1-handle in Kirby’s [K] notation), and the framing zero on $L'$. This describes a handlebody decomposition of $X$.

One can then get a standard framed link description of $M_0$ by replacing the circle with dots with unknots $T_1, \ldots, T_{\mu - 1}$ framed zero. This changes the 4-manifold but not the boundary. Note also that $lk(T_i, T_j) = 0$ and $lk(T_i, L') = 0$ for all $i = 1, \ldots, \mu - 1$. Hence $H_1(M_0) \simeq \mathbb{Z}^{\mu}$ and $m$ represents one of the generators.

We now consider an infinite cyclic covering $M_\infty$ of $M_0$, defined by a character $H_1(M_0) \to C_\infty = \langle t \rangle$ that sends $m$ to $t$ and the other generators to zero. Let us denote by $M_2$ the intermediate two-fold covering obtained by composing this character with the quotient map $C_\infty \to C_2$ sending $t$ to $-1$. Let $m_2$ denote the loop in $M_2$ given by the inverse image of $m$. A recipe for drawing a framed link description for $M_2$ is given in the proof of Remark 3.3.
Proposition 3.2 There is an isomorphism between $H_1(N_2)$ and the torsion subgroup of $H_1(M_2)$, which only depends on $L$. Moreover
\[ H_1(M_2) \simeq H_1(N_2) \oplus \mathbb{Z}^\mu \simeq H_1(N_2) \oplus \langle m_2 \rangle \oplus \mathbb{Z}^{\mu-1}. \]

Proof Let $R$ be the result of gluing $P \times D^2$ to $S^3 \times I$ along $L \times 1 \subset S^3 \times 1$ using the framing given by the Seifert surface. Thus $R$ is the result of adding $\mu - 1$ 1-handles to $S^3 \times I$ and then one 2-handle along $L'$, as in the proof above. Then $X$ in the proof above can be obtained by gluing $D^4$ to $R$ along $S^3 \times 0$. Since $D^2$ is the double branched cover of itself along the origin, $P \times D^2$ is the double branched cover of itself along $P \times 0$. Let $R_2$ denote the double branched cover of $R$ that is obtained by gluing $P \times D^2$ to $N_2 \times I$ along a neighborhood of the lift of $L \times 1 \subset S^3 \times 1$. We have that $\partial R_2 = -N_2 \sqcup M_2$, where $R_2$ is the result of adding $\mu - 1$ 1-handles to $N_2 \times I$ and then one 2-handle along the lift $L'$. Moreover this lift of $L'$ is null-homologous in $N_2$. It follows that $H_1(R_2)$ is isomorphic to $H_1(N_2) \oplus \mathbb{Z}^{\mu-1}$, with the inclusion of $N_2$ into $R_2$ inducing an isomorphism $i_N$ of $H_1(N_2)$ to the torsion subgroup of $H_1(R_2)$. Turning this handle decomposition upside down we have that $R_2$ is the result of adding to $M_2 \times I$ one 2-handle along a neighborhood of $m_2$ and then $\mu - 1$ 3-handles. It follows that $H_1(R_2) \oplus \mathbb{Z} = H_1(R_2) \oplus \langle m_2 \rangle$ is isomorphic to $H_1(M_2)$ with the inclusion of $M_2$ in $R_2$ inducing an isomorphism $i_M$ of the torsion subgroup $H_1(M_2)$ to the torsion subgroup of $H_1(R_2)$. Thus $(i_M)^{-1} \circ i_N$ is an isomorphism from $H_1(N_2)$ to the torsion subgroup of $H_1(M_2)$ and this isomorphism is constructed without any arbitrary choices. \( \square \)

Remark 3.3 We could have proved Proposition 3.1 in a similar way to the proof of Proposition 3.2. We could have also proved Proposition 3.2 (except for the isomorphism only depending on $L$) in a similar way to the proof of Proposition 3.1 as follows. We can find a surgery description of $M_2$ from a surgery description of $N_2$. The procedure of how to visualize a lift of $L$ and the surface $S$ in $N_2$ is given in [AK]. One considers the lifts of the paths chosen in the proof of Proposition 3.1, on the lift of $S$. One then fuses the components of the lift of $L$ along these paths, obtaining a lift of $L'$. The surgery description of $M_2$ is obtained by adding to the surgery description of $N_2$ the lift of $L'$ with zero framing together with $\mu - 1$ more unknotted zero-framed components encircling each fusion. The linking matrix of this link is a direct sum of that of $N_2$ and a $\mu \times \mu$ zero matrix.
Let $i_T$ denote the inclusion of the torsion subgroup of $H_1(M_2)$ into $H_1(M_2)$, and let $\psi: H_1(N_2) \to H_1(M_2)$ denote the monomorphism given by $i_T \circ (i_M)^{-1} \circ i_N$.

**Theorem 3.4** Let $\chi^+: H_1(M_2) \to C_q \oplus C_\infty$. Let $\chi: H_1(N_2) \to C_q$ be $\chi^+ \circ \psi$ composed with the projection to $C_q$. We have that:

$$|\sigma_1(\tau(M_2, \chi^+)) - \sigma(L, \chi)| \leq \eta(L, \chi) + \mu.$$

**Remark 3.5** If $L$ is a knot, then $\tau(M_2, \chi^+)$ coincides with $\tau(L, \chi)$ defined in [CG1, p.189].

**Proof of Theorem 3.4** We use the surgery description of $M_2$ given in Remark 3.3. Let $P$ be given by the surgery description of $M_2$ but with the component corresponding to $L'$ deleted. Hence,

$$P = N_2 \# (\mu - 1) S^1 \times S^2.$$

$\chi^+$ induces some character $\chi'$ on $H_1(P)$. According to Section 2.3, we let $\chi \in H^1(M_2; C_q)$ and $\chi' \in H^1(P; C_q)$ denote the characters $\chi^+$ and $\chi'$ followed by the projection $C_q \oplus C_\infty \to C_q$. Using Propositions 2.5 and 2.6, one has that

$$\sigma(P, \chi') = \sigma(L, \chi) \text{ and } \eta(P, \chi') = \eta(L, \chi) + \mu - 1.$$

Moreover, since $M_2$ is obtained by surgery on $L'$ in $P$, it follows from [Gi3, Proposition (3.3)] that

$$|\sigma(P, \chi') - \sigma(M_2, \chi)| + |\eta(M_2, \chi') - \eta(P, \chi')| \leq 1 \text{ or}$$

$$|\sigma(L, \chi) - \sigma(M_2, \chi)| + |\eta(M_2, \chi) - \eta(L, \chi) - \mu + 1| \leq 1.$$

Thus

$$|\sigma(L, \chi) - \sigma(M_2, \chi)| \leq \eta(L, \chi) + \mu - \eta(M_2, \chi).$$

Finally, one gets, by Theorem 2.8,

$$|\sigma_1(\tau(M_2, \chi^+)) - \sigma(L, \chi)| \leq |\sigma_1(\tau(M_2, \chi^+)) - \sigma(M_2, \chi)| + |\sigma(M_2, \chi) - \sigma(L, \chi)|$$

$$\leq \eta(M_2, \chi) + \eta(L, \chi) + \mu - \eta(M_2, \chi) = \eta(L, \chi) + \mu.$$


4 The slice genus of links

See Section 2.5 for notations.

**Theorem 4.1** Suppose \( L \) is the boundary of a connected oriented properly embedded surface \( F \) of genus \( g \) in \( B^4 \), and that \( \Delta_L(-1) \neq 0 \). Then, \( \beta_L \) can be written as a direct sum \( \beta_1 \oplus \beta_2 \) such that the following two conditions hold:

1) \( \beta_1 \) has an even presentation of rank \( 2g + \mu - 1 \) and signature \( \sigma_L(-1) \), and \( \beta_2 \) is metabolic.

2) There is a metabolizer for \( \beta_2 \) such that for all characters \( \chi \) of prime power order in this metabolizer,

\[
|\sigma(L, \chi) + \sigma_L(-1)| \leq \eta(L, \chi) + 4g + 3\mu - 2.
\]

**Proof** We let \( b_i(X) \) denote the \( i \)th Betti number of a space \( X \). We have \( b_1(F) = 2g + \mu - 1 \).

Let \( W'_0 \), with boundary \( M'_0 \), be the complement of an open tubular neighborhood of \( F \) in \( B^4 \). By the Thom isomorphism, excision, and the long exact sequence of the pair \((B^4, W'_0)\), \( W'_0 \) has the homology of \( S^1 \) wedge \( b_1(F) \) 2-spheres. Let \( W'_2 \) with boundary \( M'_2 \) be the two-fold covering of \( W'_0 \). Note that if \( F \) is planar, \( M'_0 = M_0 \), and \( M'_2 = M_2 \) (see Section 3).

Let \( V_2 \) be the two-fold covering of \( B^4 \) with branched set \( F \). Note that \( V_2 \) is spin as \( w_2(V_2) \) is the pull-up of a class in \( H^2(B^4, \mathbb{Z}_2) \), by [Gi5, Theorem 7], for instance. The boundary of \( V_2 \) is \( N_2 \). As in [Gi1], one calculates that \( b_2(V_2) = 2g + \mu - 1 \). One has \( \text{Sign}(V_2) = \sigma_L(-1) \) by [V].

By Lemma 2.10, \( \beta_L \) can be written as a direct sum \( \beta_1 \oplus \beta_2 \) as in condition 1) above, such that the characters on \( H_1(N_2) \) that extend to \( H_1(V_2) \) form a metabolizer \( H \) for \( \beta_2 \). We now suppose \( \chi \in H \) and show that Condition 2) holds for \( \chi \).

We also let \( \chi \) denote an extension of \( \chi \) to \( H_1(V_2) \) with image some cyclic group \( C_q \) where \( q \) is a power of a prime integer (possibly larger than those corresponding to the character on \( H_1(N_2) \)). Of course \( \chi \in H^1(V_2, C_q) \) restricted to \( W'_2 \) extends \( \chi \) restricted to \( M'_2 \). We simply denote all these restrictions by \( \chi \).

Let \( W'_\infty \) denote the infinite cyclic cover of \( W'_0 \). Note that \( W'_2 \) is a quotient of this covering space. \( \chi \) induces a \( C_q \)-covering of \( V_2 \) and thus of \( W'_2 \). If we pull the \( C_q \)-covering of \( W'_2 \) up to \( W'_\infty \), we obtain \( \tilde{W'}_\infty \), a \( C_q \times C_\infty \)-covering of \( W'_2 \). If we identify properly \( F \times S^1 \) in \( M'_2 \), this covering restricted to \( F \times S^1 \) is given by
a character $H_1(F \times S^1) \simeq H_1(F) \oplus H_1(S^1) \to \mathbb{C}_q \times \mathbb{C}_\infty$ that maps $H_1(F)$ to zero in $\mathbb{C}_\infty$, $H_1(S^1)$ to zero in $\mathbb{C}_q$ and isomorphically onto $\mathbb{C}_\infty$. For this note: since $\text{Hom}(H_1(F), \mathbb{Z}) = H^1(F) = [F, S^1]$, we may define diffeomorphisms of $F \times S^1$ that induce the identity on the second factor of $H_1(F \times S^1) \approx H_1(F) \oplus \mathbb{Z}$, and send $(x, 0) \in H_1(F) \oplus \mathbb{Z}$, to $(x, f(x)) \in H_1(F) \oplus \mathbb{Z}$, for any $f \in \text{Hom}(H_1(F), \mathbb{Z})$.

As in [Gi1], choose inductively a collection of $g$ disjoint curves in the kernel of $\chi$ that form a metabolizer for the intersection form on $H_1(F)/H_1(\partial F)$. By taking a tubular neighborhood of these curves in $F$, we obtain a collection of $S^1 \times I$ embedded in $F$. Using these embeddings we can attach round 2-handles $(B^2 \times I) \times S^1$ along $(S^1 \times I) \times S^1$ to the trivial cobordism $M_2' \times I$ and obtain a cobordism $\Omega$ between $M_2$ and $M_2'$.

Let $U = W'_2 \cup_{M'_2} \Omega$ with boundary $M_2$. The $\mathbb{C}_q \times \mathbb{C}_\infty$-covering of $W'_2$ extends uniquely to $U$. Note that $\Omega$ may also be viewed as the result of attaching round 1-handles to $M_2$.

As in [Gi1], $\text{Sign}(W'_2) = \text{Sign}(W_2)$. Since the intersection form on $\Omega$ is zero, we get $\text{Sign}(U) = \text{Sign}(W'_2) = \text{Sign}(W_2) = \sigma_L(-1)$. The $\mathbb{C}_q \times \mathbb{C}_\infty$-covering of $\Omega$, restricted to each round 1-handle is $q$ copies of $B^2 \times I \times \mathbb{R}$ attached to the trivial cobordism $\tilde{M}'_\infty \times I$ along $q$ copies of $S^1 \times I \times \mathbb{R}$. Using a Mayer-Vietoris sequence, one sees that the inclusion induces an isomorphism (which preserves the Hermitian form)

$$H_2(U; \mathbb{Q}(\mathbb{C}_q)(t)) \simeq H_2(W'_2; \mathbb{Q}(\mathbb{C}_q)(t)).$$

Thus, if $w(W'_2)$ denotes the image of the intersection form on $H_2(W'_2; \mathbb{Q}(\mathbb{C}_q)(t))$ in $\mathcal{W}(\mathbb{Q}(\mathbb{C}_q)(t))$, we get $\sigma_1(\tau(M_2, \chi^+)) = \sigma_1(w(W'_2)) - \sigma_L(-1)$.

If $q$ is a prime power, we may apply Lemma 2 of [Gi1] and conclude that $H_i(\tilde{W}'_\infty; \mathbb{Q})$ is finite dimensional for all $i \neq 2$. Thus, $H_2(W'_2; \mathbb{Q}(\mathbb{C}_q)(t))$ is zero for all $i \neq 2$. Since the Euler characteristic of $W'_2$ with coefficients in $\mathbb{Q}(\mathbb{C}_q)(t)$ coincides with those with coefficients in $\mathbb{Q}$, we get $\dim H_2(W'_2; \mathbb{Q}(\mathbb{C}_q)(t)) = \chi(W'_2) = 2\chi(W'_0) = 2(1 - \chi(F)) = 2b_1(F)$. Thus $|\sigma_1(\tau(M_2, \chi^+) + \sigma_L(-1)| \leq 2b_1(F)$. Hence,

$$|\sigma(L, \chi) + \sigma_L(-1)| \leq |\sigma(L, \chi) - \sigma_1(\tau(M_2, \chi^+))| + |\sigma_1(\tau(M_2, \chi^+) + \sigma_L(-1)| \leq \eta(L, \chi) + \mu + 2(2g + \mu - 1) = \eta(L, \chi) + 4g + 3\mu - 2$$

by Theorem 3.4.

5 Examples

Let $L = L_1 \cup L_2$ be the link with two components of Figure 1 and $S$ be the Seifert surface of $L$ given by the picture. The squares with $K$ denote two
parallel copies with linking number 0 of an arc tied in the knot $K$. Note that $L$ is actually a family of examples. Specific links are determined by the choice of two parameters: a knot $K$ and a positive integer $h$. Since $S$ has genus $h$, the slice genus of $L$ is at most $h$.

![Figure 1: The link $L$](image)

One calculates that $\sigma_L(\lambda) = 1$, and $n_L(\lambda) = 0$ for all $\lambda$. Thus, the Murasugi-Tristram inequality says nothing about the slice genus of $L$. In fact, if $K$ is a slice knot, then one can surger this surface to obtain a smooth cylinder in the 4-ball with boundary $L$. Thus there can be no arguments based solely on a Seifert pairing for $L$ that would imply that the slice genus is non-zero.

**Theorem 5.1** If $\sigma_K(e^{2i\pi/3}) \geq 2h$ or $\sigma_K(e^{2i\pi/3}) \leq -2h - 2$, then $L$ has slice genus $h$.

**Proof** Using [AK], a surgery presentation of $N_2$ as surgery on a framed link of $2h + 1$ components can be obtained from the surface $S$ (see Figure 2).

Let $Q$ be the 3-manifold obtained from the link pictured in Figure 2. Here $K'$ denotes $K$ with the string orientation reversed. Since $RP(3)$ is obtained by surgery on the unknot framed 2, we get:

$$N_2 = RP(3)\#_h Q.$$

The linking matrix of the framed link of the surgery presentation of $N_2$ is

$$\Lambda = [2] \bigoplus^h \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}.$$  \Lambda is a presentation matrix of $(H_1(N_2^*), \beta_L)$; we obtain

$$H_1(N_2^*) \cong \mathbb{Z}_2 \bigoplus^{2h} \mathbb{Z}_3$$
and $\beta_L$ is given by the following matrix, with entries in $\mathbb{Q}/\mathbb{Z}$:

$$[1/2] \bigoplus \bigoplus^h \begin{bmatrix} 0 & 1/3 \\ 1/3 & 0 \end{bmatrix}.$$ 

By Theorem 4.1, if $L$ bounds a surface of genus $h - 1$ in $B^4$, then $\beta_L$ must be decomposed as $\beta_1 \oplus \beta_2$ where:

1) $\beta_1$ has an even presentation matrix of rank $2h - 1$, and signature 1 (all we really need here is that it has a rank $2h - 1$ presentation.)

2) $\beta_2$ is metabolic and for all characters $\chi$ of prime power order in some metabolizer of $\beta_2$, the following inequality holds:

$$(*) \quad |\sigma(L, \chi) + 1| - \eta(L, \chi) \leq 4h.$$ 

As $\mathbb{Z}_2 \bigoplus \bigoplus^{2h}\mathbb{Z}_3$ does not have a rank $2h - 1$ presentation, $\beta_2$ is non-trivial. As metabolic forms are defined on groups whose cardinality is a square, $\beta_2$ is defined on a group with no 2-torsion. Thus the metabolizer contains a non-trivial character of order three satisfying $\beta_L(\chi, \chi) = 0$.

The first homology of $Q$ is $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, generated by, say, $m_1$ and $m_2$, positive meridians of these components. Each of these components is oriented counterclockwise. We first work out $\sigma(Q, \chi)$ and $\eta(Q, \chi)$ for characters of order three.

Let $\chi(a_1, a_2)$ denote the character on $H_1(Q)$ sending $m_j$ to $e^{2\pi i a_j}$, where the $a_j$ take the values zero and $\pm 1$.

We use Proposition 2.4 to compute $\sigma(Q, \chi(1,0))$ and $\eta(Q, \chi(1,0))$ assuming that $K$ is trivial. For this, one may adapt the trick illustrated on a link with 2 twists between the components [Gi2, Fig (3.3), Remark (3.65b)]. In the case $K$ is the unknot, we obtain

$$\sigma(Q, \chi(1,0)) = 1 \quad \text{and} \quad \eta(Q, \chi(1,0)) = 0.$$
It is not difficult to see that inserting the knots of the type \( K \) changes the result as follows (note that \( K \) and \( K' \) have the same Tristram-Levine signatures):

\[
\sigma(Q, \chi_{(1,0)}) = 1 + 2\sigma_K(e^{2\pi i/3}) \quad \text{and} \quad \eta(Q, \chi_{(1,0)}) = 0.
\]

These same values hold for the characters \( \chi_{(-1,0)} \) and \( \chi_{(0,\pm 1)} \) by symmetry.

Using Proposition 2.4

\[
\sigma(Q, \pm \chi_{(1,1)}) = -1 - 24/9 + 4\sigma_K(e^{2\pi i/3}), \quad \eta(Q, \pm \chi_{(1,1)}) = 0
\]

\[
\sigma(Q, \pm \chi_{(1,-1)}) = 4 + 24/9 + 4\sigma_K(e^{2\pi i/3}) \quad \text{and} \quad \eta(Q, \pm \chi_{(1,-1)}) = 1.
\]

One also has

\[
\sigma(Q, \chi_{(0,0)}) = 0 \quad \text{and} \quad \eta(Q, \chi_{(0,0)}) = 0.
\]

Any order three character on \( N_2 \) that is self annihilating under the linking form is given as the sum of the trivial character on \( RP(3) \) and characters of type \( \chi_{(0,0)} \), \( \chi_{(\pm 1,0)} \) and \( \chi_{(0,\pm 1)} \) on \( Q \) and characters of type \( \pm \chi_{(1,1)} \) and \( \pm \chi_{(1,-1)} \) on \( Q \# Q \). Using Proposition 2.5, one can calculate \( \sigma(L, \chi) \) and \( \eta(L, \chi) \) for all these characters \( \chi \). It is now a trivial matter to check that for every non-trivial character with \( \beta(\chi, \chi) = 0 \), the inequality (*) is not satisfied.

\[\Box\]

References

[AK] Akbulut, S., Kirby, R., Branched covers of surfaces in 4-manifolds, Math. Ann. 252, 111-131 (1980).

[AS] Atiyah, M. F., Singer, I. M., The index of elliptic operators. III, Ann. of Math. (2) 87, 546-604 (1968).

[CF] Conner, P. E., Floyd, E.E., Differential Periodic Maps, Ergebnisse der Mathematik und ihrer Grenzgebiete, 33, Springer-Verlag, (1964).

[CG1] Casson, A. J., Gordon, C. Mc A., Cobordism of classical knots, Progr. Math., 62, A La Recherche de la Topologie Perdue, Birkhauser, Boston, MA, 181–199 (1986).

[CG2] Casson, A. J., Gordon, C. Mc A., On slice knots in dimension three, Proc. Symp. in Pure Math. XXX, 2, 39-53 (1978).

[Gi1] Gilmer, P. M., On the slice genus of knots, Invent. Math. 66, 191-197 (1982).

[Gi2] Gilmer, P. M., Configurations of surfaces in 4-manifolds, Trans. Amer. Math. Soc. 264, 353-380 (1981).

[Gi3] Gilmer, P. M., Slice knots in \( S^3 \), Quart. J. Math. Oxford 34, 305-322 (1983).

[Gi4] Gilmer, P. M., Classical knot and link concordance, Comment. Math. Helv. 68, 1-19 (1993).
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[Gi5] Gilmer, P. M., Signatures of singular branched covers, Math. Ann. 295 (4), 643–659 (1993).

[GL] Gilmer, P. M., Livingston, C., The Casson-Gordon invariant and link concordance, Topology 31, (3), 475–492 (1992).

[G] Gordon, C. McA., Some aspects of classical knot theory, Knot theory (Proc. Sem., Plans-sur-Bex, 1977), Lecture Notes in Math., 685, Springer Verlag, Berlin, 1-60 (1978).

[K] Kirby, R. C., The Topology of 4-manifolds, Lecture Notes in Math 1374 Springer Verlag, Berlin (1989).

[Le] Levine, J., Knot cobordism groups in codimension two, Comment. Math. Helv. 44 229–244 (1969).

[L] Lines D., Cobordisme de noeuds fibrés et de leur monodromie, Knots, braids and singularities (Plans-sur-Bex, 1982), 147–173, Monogr. Enseign. Math., 31, Enseignement Math., Geneva, (1983).

[Li] Litherland, R. A., Cobordism of satellite knots, Four-Manifold Theory (Durham, N.H., 1982), Contemp. Math., 35, Amer. Math. Soc., Providence, RI, 327–362 (1984).

[M] Murasugi, K., On a certain numerical invariant of link types, Trans. Amer. Math. Soc. 117 , 387–422 (1965).

[N] Naik, S. Casson-Gordon invariants of genus one knots and concordance to reverses, J. Knot Theory Ramifications 5, 661–677 (1996).

[T] Tristram, A. G., Some cobordism invariants for links, Proc, Camb. Philos. Soc., 66, 251-264 (1969).

[V] Viro, O. Ja. Branched coverings of manifolds with boundary, and invariants of links. I, Math. USSR-Izv. 7 1239–1256 (1973).

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