INSTANTON AND MERON PHYSICS IN LATTICE QCD

JOHN W. NEGELE

Center for Theoretical Physics,
Laboratory for Nuclear Science, and Department of Physics,
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139 U.S.A.
E-mail: negele@mitlns.mit.edu

Lattice field theory provides a quantitative tool to study the role of nonperturbative semiclassical configurations in QCD. This talk briefly reviews our present understanding of the role of instantons in QCD and describes in detail new developments in the study of merons on the lattice.

1 Introduction

As highlighted by this conference, despite a quarter century of effort, understanding the nonperturbative origin of confinement in QCD has remained an elusive challenge. In this talk, I will describe how combining lattice field theory with familiar semiclassical ideas is providing new insight into the role of instantons and merons in the QCD vacuum and new tools to understand the physics of confinement.

One of the great advantages of the path integral formulation of quantum mechanics and field theory is the possibility of identifying non-perturbatively the stationary configurations that dominate the action and thereby identify and understand the essential physics of complex systems with many degrees of freedom. Thus, the discovery of instantons in 1975 gave rise to great excitement and optimism that they were the key to understanding QCD. Indeed, in contrast to other many body systems in which the quanta exchanged between interacting fermions can be subsumed into a potential, it appeared that QCD was fundamentally different, with topological excitations of the gluon field dominating the physics and being responsible for a host of novel and important effects including the $\theta$ vacuum, the axial anomaly, fermion zero modes, the mass of the $\eta'$, the chiral condensate, and possibly even confinement. However, until lattice QCD became sufficiently sophisticated and sufficient resources could be devoted to the study of instanton and meron physics, it had not been possible to proceed beyond the dilute instanton gas approximation and a qualitative exploration of merons.

Using lattice field theory, our basic strategy is to reverse the usual analytical process of calculating stationary configurations and approximately summing the fluctuations around them. Rather, we use Monte Carlo sampling of the path integral for QCD on a lattice to identify typical paths contributing to
the action and then work backwards to identify the smooth classical solutions about which these paths are fluctuating.

2 The Role of Instantons

Insight into the role of instantons and their zero modes from lattice QCD has been reviewed in detail in ref. 3, and I will only summarize the highlights here. The instanton content of gluon configurations can be extracted by cooling 4, and the instanton size is consistent with the instanton liquid model and the topological susceptibility agrees with the Veneziano-Witten formula 3. We obtain striking agreement between vacuum correlation functions, ground state density-density correlation functions, and masses calculated with all gluons and with only instantons 5. Zero modes associated with instantons are clearly evident in the Dirac spectrum, and account for the $\rho$, $\pi$, and $\eta'$ contributions to vacuum correlation functions 6. Finally, we have observed directly quark localization at instantons in uncooled configurations 6. The physical picture that arises corresponds closely to the physical arguments and instanton models in which the zero modes associated with instantons produce localized quark states, and quark propagation proceeds primarily by hopping between these states 7. However, of particular relevance to this conference, it is now clear that simple configurations of instantons, such as a dilute gas 2 or a random superposition 8, do not confine color charges. It is especially important to emphasize that even an ensemble of instantons with a distribution of large instantons $\propto \rho^{-5}$ yields a static potential which approaches a constant at large distance when one is sufficiently careful in calculating the contributions of very large Wilson loops 8. Although it is possible that instanton configurations that differ essentially from a random superposition could lead to confinement, it is interesting to discuss here closely related topological gluon excitations known as merons.

3 Merons on the Lattice

In the space available in these proceedings, I would like to concentrate on new advances with James V. Steele 9 in studying merons on the lattice. Merons are solutions to the classical Yang-Mills field equations with one-half unit of localized topological charge. Since they are the (3+1)-dimensional nonabelian extension of the 't Hooft–Polyakov monopole configuration, which has been shown to lead to confinement in 2+1 dimensions 10, they have long been considered a possible mechanism for confinement 2.

A purely analytic study of merons has proven intractable for several reasons. Unlike instantons, no exact solution exists for more than two merons 11.
gauge fields for isolated merons fall off too slowly \((A \sim 1/x)\) to superpose them, and the field strength is singular. A patched Ansatz configuration that removes the singularities\(^3\) does not satisfy the classical Yang-Mills equations, preventing even calculation of gaussian fluctuations around a meron pair\(^4\). Because of these analytic impasses, we have constructed smooth, finite action configurations corresponding to a meron pair and their associated fermion zero modes on the lattice. For simplicity, we restrict our discussion to SU(2) color without loss of generality.

Two known solutions to the classical Yang-Mills equations in four Euclidean dimensions are instantons and merons. Both have topological charge, can be interpreted as tunneling solutions, and can be written in the general form (for the covariant derivative \(D_\mu = \partial_\mu - iA_\mu^a \sigma^a/2\))

\[
A_\mu^a(x) = \frac{2 \eta_{\alpha\mu} x^\nu}{x^2} f(x^2),
\]

with \(f(x^2) = x^2/(x^2 + \rho^2)\) for an instanton and \(f(x^2) = \frac{1}{2}\) for a meron.

Conformal symmetry of the classical Yang-Mills action, in particular under inversion \(x_\mu \rightarrow x_\mu/x^2\), shows that in addition to a meron at the origin, there is a second meron at infinity. These two merons can be mapped to arbitrary positions, which we define to be the origin and \(d_\mu\). After a gauge transformation, the gauge field for the two merons takes the simple form\(^5\)

\[
A_\mu^a(x) = \eta_{\alpha\mu} \left[ \frac{x^\nu}{x^2} + \frac{(x - d)^\nu}{(x - d)^2} \right].
\]

Similar to instantons\(^6\), a meron pair can be expressed in singular gauge by performing a large gauge transformation about the mid-point of the pair, resulting in a gauge field that falls off faster at large distances \((A \sim x^{-3})\). A careful treatment of the singularities shows that the topological charge density is\(^7\)

\[
Q(x) \equiv \frac{\text{Tr}}{16\pi^2} \left( F_{\mu\nu} F^{\mu\nu} \right) = \frac{1}{2} \delta^4(x) + \frac{1}{2} \delta^4(x - d),
\]

yielding total topological charge \(Q = 1\), just like the instanton.

The gauge field Eq. (3) has infinite action density at the singularities \(x_\mu = \{0, d_\mu\}\). Hence, a finite action Ansatz has been suggested\(^8\)

\[
A_\mu^a(x) = \eta_{\alpha\mu} x^\nu \begin{cases} 
\frac{2}{x^2 + r^2}, & \sqrt{x^2} < r, \\
\frac{1}{x^2}, & r < \sqrt{x^2} < R, \\
\frac{2}{x^2 + R^2}, & R < \sqrt{x^2}.
\end{cases}
\]
Here, the singular meron fields for $\sqrt{x^2} < r$ and $\sqrt{x^2} > R$ are replaced by instanton caps, each containing $\frac{1}{2}$ topological charge. The radii $r$ and $R$ are arbitrary and we will refer to the three separate regions defined in Eq. (4) as regions I, II, and III, respectively. The action for this configuration is
$$S = \frac{8\pi^2}{g^2} + \frac{3\pi^2}{g^2} \ln \frac{R}{r},$$
which shows the divergence in the $r \to 0$ or $R \to \infty$ limit. There is no angular dependence in this patching, and so the conformal symmetry of the meron pair is retained. Although this patching of instanton caps is continuous, the derivatives are not, and so the equations of motion are violated at the boundaries of the regions in Eq. (4).

Applying the same transformations used to attain Eq. (2) to the instanton cap solution, regions I and III become four-dimensional spheres each containing half an instanton. The geometry of the instanton caps is shown in Fig. 1 for the symmetric choice of displacement $d = \sqrt{Rr}$ along the $z$-direction. Note that the action for the complicated geometry of Fig. 1 with $d = \sqrt{Rr}$ can be rewritten as
$$S = S_0 \left( 1 + \frac{3}{4} \ln \frac{d}{r} \right),$$
with $S_0 = \frac{8\pi^2}{g^2}$. We will concentrate on this case below, and generalization to a different $d$ is straightforward.

The original positions of the two merons $x_\mu = \{0, d_\mu\}$ are not the centers of the spheres, nor are they the positions of maximum action density, which occurs within the spheres with $S_{\text{max}} = (48/g^2)(R+r)^4/d^8$ at
$$(M_I)_\mu = \frac{r^2}{r^2 + d^2} d_\mu, \quad (M_{III})_\mu = \frac{R^2}{R^2 + d^2} d_\mu.$$
However, in the limit $r \to 0$ and $R \to \infty$ holding $d$ fixed, the spheres will shrink around the original points reducing to the bare meron pair in Eq. (2). In the opposite limit $R \to r$, the radii of the spheres increase to infinity, leaving an instanton of size $\rho = d$.

An instanton can therefore be interpreted as consisting of a meron pair. This complements the fact that instantons and antiinstantons have dipole interactions between each other. If there exists a regime in QCD where meron entropy contributes more to the free energy than the logarithmic potential.
between pairs, like the Kosterlitz-Thouless phase transition, instantons could break apart into meron pairs. The intrinsic size of the instanton caps would be determined by the original instanton scale $\rho$.

The Atiyah-Singer index theorem states that a fermion in the presence of a gauge field with topological charge $Q$, described by the equation

$$\mathcal{D} \psi = \lambda \psi,$$

with $\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$, has $n_R$ right-handed and $n_L$ left-handed zero modes (defined by $\lambda = 0$) such that $Q = n_L - n_R$. This statement can often be strengthened by a vanishing theorem. Applying $\mathcal{D}$ twice to $\psi$ decouples the right- and left-hand components, and focusing on the equation for $\psi_R$, gives

$$\left( D^2 + \frac{1}{2} \eta_{\mu\nu} \sigma^a F^{\mu\nu} \right) \psi_R = 0 ,$$

where $\eta_{\mu\nu}$ denotes the 't Hooft symbol, $\sigma^a$ acts on the spin indices of $\psi_R$, and $F^{\mu\nu}$ acts on the color indices. For a self-dual gauge field (like an instanton), the second term in Eq. (8) is zero; and since $D^2$ is a negative definite operator, there are no normalizable right-handed zero modes, implying $Q = n_L$. Although the meron pair is not self-dual, the second term can be shown to be negative definite as well, leading to the same conclusion.

In general, a gauge field in Lorentz gauge with $Q = 1$ can be written in the form

$$A^a_\mu(x) = -\eta_{\mu\nu} \partial_\nu \ln \Pi(x) .$$

This has a fermion zero mode given by

$$\psi = \begin{pmatrix} 0 \\ \phi \end{pmatrix} , \quad \phi^a = \mathcal{N} \Pi^{3/2} \varepsilon^a_\alpha ,$$

with normalization $\mathcal{N}$ and $\varepsilon = i\sigma_2$ coupling the spin index $\alpha$ to the color index $a$ (both of which can be either 1 or 2) in a singlet configuration. The gauge field for the meron pair with instanton caps corresponds to

$$\Pi(x) = \begin{cases} \frac{2\xi d^2}{x^2 + \xi^2(x-d)^2}, & \text{for regions } i = I, III, \\ \frac{d^2}{\sqrt{x^2(x-d)^2}}, & \text{region II}, \end{cases}$$

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Figure 2: The action density $S(x)$ for the regulated meron pair, normalized by the instanton action $S_0 = 8\pi^2/g^2$, sliced through the center of the configuration in the direction of separation. Shown are the analytic (solid), initial lattice (open circles) and cooled lattice (filled circles) configurations. The same is shown for the topological charge density $Q(x)$ and fermion zero mode density $\psi^\dagger \psi(x)$.

with $\xi_1 = r/d$ and $\xi_{\text{III}} = R/d$. The normalized solution to the zero mode is then Eq. (10) with

$$N^{-1} = 2\pi d^2 \left[ 2 - \sqrt{\frac{r}{R}} \right]^{1/2}.$$  \hspace{1cm} (12)

Note that the unregulated meron pair ($r \to 0, R \to \infty$) has a normalizable zero mode itself,

$$\phi_\alpha^\varphi(x) = \frac{d \varepsilon_\alpha^\varphi}{2\sqrt{2\pi} (x^2 (x - d)^2)^{3/4}}.$$  \hspace{1cm} (13)

The gauge-invariant zero mode density $\psi^\dagger \psi(x)$ has a bridge between two merons that falls off like $x^{-3}$ in contrast to the $x^{-6}$ fall-off in all other directions. This behavior can be used to identify merons when analyzing their zero modes on the lattice, similar to what was done for instantons in Ref. [1].

As mentioned above, the patching of instanton caps to obtain an explicit analytic solution has unavoidable and unphysical discontinuities in the action density. Therefore, in order to study this solution further, we put the gauge field on a space-time lattice of spacing $a$ in a box of size $L_0 \times L_1 \times L_2 \times L_3$. 


The gauge-field degrees of freedom are replaced by the usual parallel transport, $U_\mu(x) = \text{P} \exp \left[ -i \int_x^{x+a_\mu} \epsilon_\nu A_\nu(z) dz' \right]$. The exponentiated integral can be performed analytically within the instanton caps, producing arctangents. Outside of the caps, the integral is evaluated numerically by dividing each link into as many sub-links as necessary to reduce the $O(a^3)$ path-ordering errors below machine precision.

We then calculate the action density $S(x)$ for a meron pair with instanton caps using the improved action of Ref. [14] and the topological charge density $Q(x)$ using products of clovers. These (open circles) are compared with the patched Ansatz results (solid line) in Fig. [2] for $r = 9$ and $d = 21$ (in units of the lattice spacing) on a $32^3 \times 40$ lattice. We use the Arnoldi method to solve for the zero mode of this gauge configuration and hence the density $\psi^\dagger \psi(x)$, which is also compared with the analytic result in the same figure, showing excellent agreement.
Two important features of the patched Ansatz also evident with the lattice representation are the discontinuities in the action density at the boundary of the instanton caps and vanishing of the topological charge density outside of the caps as given by Eq. (3). The discontinuity in the action density is unphysical and can be smoothed out by using a relaxation algorithm to iteratively minimize the lattice action $S$. On a sweep through the lattice, referred to as a cooling step, each link is chosen to locally minimize the action density. Since a single instanton is already a minimum of the action, this algorithm would leave the instanton unchanged (for a suitably improved lattice action). The result for the regulated meron pair after ten cooling steps is represented in Fig. 2 by the filled circles, showing the discontinuities in the action density have already been smoothed out. In Fig. 3, we also plot side-by-side the cooled $Q(x)$ and $\psi^\dagger \psi(x)$ in the $(z,t)$-plane for three different meron pair separations. As the separation vanishes, the zero mode goes over into the well-known instanton zero mode result.

Like instanton-antiinstanton pairs, however, a meron pair is not a strict minimum of the action, since it has a weak attractive interaction Eq. (5) and under repeated relaxation will coalesce to form an instanton. Nevertheless, just as it is important to sum all the quasi-stationary instanton-antiinstanton configurations to obtain essential nonperturbative physics, meron pairs may be expected to play an analogous role. A precise framework for including quasi-stationary meron pairs is to introduce a constraint $Q[A]$ on a suitable collective variable $q$ (which we chose to be the quadrupole moment $3z^2 - x^2 - y^2 - t^2$ of the topological charge) as follows

$$Z = \int DA \ exp \{-S[A]\} = \int dq \ Z_q, \quad (14)$$

$$Z_q = \int DA \ exp \left\{-S[A] - \lambda (Q[A] - q)^2\right\}. \quad (15)$$

The meron pair is then a true minimum of the effective action with constraint, allowing for a semiclassical treatment of $Z_q$. Afterwards, $q$ is integrated to obtain the full partition function $Z$.

The criterion for a good collective variable $q$ of the system is that the gradient in the direction of $q$ is small compared to the curvature associated with all the quantum fluctuations. In this case, we have an adiabatic limit in which relaxation of the unconstrained meron pair slowly evolves through a sequence of quasi-stationary solutions, each of which is close to a corresponding stationary constrained solution. Detailed comparison of the quasi-stationary and constrained solutions shows this adiabatic limit is well satisfied. Therefore, we will present here the action of a meron pair as it freely coalesces into an
Figure 4: Action of a regulated meron pair as a function of $d/r$ for initial lattice configurations of $d_{\text{init}} = 12, 14, 18, 22$ and $r_{\text{init}} = 4$ on a $32^3 \times 40$ lattice. The circles show cooling trajectories for each cooling step.

To compare with the patched Ansatz, we need the meron separation $d$ and radii $r$ and $R$ of the lattice configuration. We find these by first measuring the separation of the two maxima in the action density $\Delta \equiv |M_I - M_{III}|$ (using cubic splines) and their values (which are the same in the symmetric case, denoted by $S_{\text{max}} = 48/g^2w^4$), and then using Eq. (6) to give

$$d^2 = \Delta^2 + 4w^2,$$

$$\{r, R\} = \frac{2wd}{d \pm \Delta}. \quad (16)$$

In Fig. 4, we plot the total action of a meron pair as a function of $d/r$ (solid line), which for the analytic case is given by Eq. (5). Also shown are cooling trajectories for four lattice configurations with different initial meron pair separations. Each case starts with a patched Ansatz of a given separation and the lattice action agrees with that of the analytic Ansatz.

The first few cooling steps primarily decrease the action without changing the collective variable (which is now effectively $d/r$), as observed above in Fig. 2. Further cooling gradually decreases the collective variable, tracing out a new logarithmic curve for the total action (dashed line in Fig. 4), which is about 0.25 lower than the analytic case. This curve represents the total action of the smooth adiabatic or constrained lattice meron pair solutions. The essential property of merons that could allow them to dominate the path integral and confine color charge is the logarithmic interaction Eqs. (5) which is weak enough to be dominated by the meron entropy. Hence, the key physical
result from Fig. 4 is the fact that smooth adiabatic or constrained lattice meron pair solutions clearly exhibit this logarithmic behavior.

In summary, we have found stationary meron pair solutions, shown that they have a logarithmic interaction, and shown that they have characteristic fermion zero modes localized about the individual merons. Hence the presence and role of merons in numerical evaluations of the QCD path integral should be investigated, and the study of fermionic zero modes and cooling of gauge configurations provide possible tools to do so.

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