1. INTRODUCTION

Lattice QCD allows a first principles determination of hadronic properties, including the matrix elements of operators between hadronic states. Among the more interesting matrix elements are those of the $\Delta S = 1$, four-fermion operators which are responsible for the decay of a kaon into two pions. These matrix elements are needed to relate the values of the Cabbibo-Kobayashi-Maskawa matrix in the standard model to physical measurements. Direct measurements of $K \to \pi \pi$ on the lattice are not possible [1], due to the two pions in the final state, although a recent suggestion involving tuning the lattice volume [2] looks promising. Alternatively, lowest order chiral perturbation theory can be used to relate $K \to \pi \pi$ to $K \to \pi$ and $K \to 0$ matrix elements [3], which can be measured on the lattice. This approach has been extensively studied and, while quite straightforward in principle, requires careful control over a wide range of phenomena.

Of central importance to a calculation relying on chiral perturbation theory is precise control over chiral symmetry on the lattice. A major advance in this long-standing problem is the idea of four-dimensional chiral fermions arising from mass defects in a higher-dimensional fermion formulation [4]. With improved chiral symmetry at finite lattice spacing, the lattice Ward-Takahashi identities, without complicated renormalization, are much closer to the continuum ones. Here we seek to understand how the residual chiral symmetry breaking effects for domain wall fermions enter the removal of unphysical, power divergent contributions, which is needed to get $K \to \pi \pi$ from $K \to \pi$ and $K \to 0$ matrix elements.

2. DOMAIN WALL FERMIONS

The RIKEN-BNL Research Center/BNL/Columbia (RBC) Collaboration has recently finished extensive quenched simulations of the hadron spectrum and low-energy QCD physics, using domain wall fermions [5], with a particular focus on the residual chiral symmetry breaking effects due to finite values for the fifth dimension, $L_s$ [6]. (Similar work was also reported in [7].) This work produced quantitative values for the residual quark mass $m_{\text{res}}$, which enters low energy QCD physics through the effective quark mass, $m_{\text{eff}} = m_f + m_{\text{res}}$, where $m_f$ is the input bare quark mass. The residual quark mass was shown to be small, $\sim 1/30$ the strange quark mass for $\beta = 6.0$ and $L_s = 16$.

It was also found that topological near-zero modes of the domain wall fermion operator, which are not suppressed by a fermionic determinant in quenched simulations, have pronounced effects on
quark propagators for small $m_f$. In addition, for large volumes where topological near-zero mode effects are suppressed, a non-linear dependence of $m_q^2$ on $m_f$ was found, consistent with a quenched chiral logarithm. Thus, even though the chiral properties of domain wall fermions are quite improved over other formulations, these quenched pathologies can appear in simulation results.

We have also finished measuring $K \to \pi$ and $K \to 0$ matrix elements for the 10 $\Delta S = 1$ effective four-fermion operators [8] relevant to determinations of $\epsilon'/\epsilon$ and the real parts of the $\Delta I = 1/2$ and $3/2$ amplitudes for $K \to \pi\pi$ decays. We have results for 200 quenched lattices of size $16 \times 32$ with $L_s = 16$ and $\beta = 6.0$ and a variety of light and heavy quark masses. (For more details about the simulations see [3].) The analysis of these results requires control over the residual chiral symmetry breaking effects and an understanding of the role of quenched pathologies in our data. This report is a summary of our progress to date.

A final step in determining physical matrix elements is the matching of continuum and lattice operators. This has been done using non-perturbative renormalization [10] and details can be found in [11].

3. WARD IDENTITY CHECK

Furman and Shamir [8] defined axial transformations for domain wall fermions, which lead to the Ward-Takahashi identity

$$\Delta_{\mu}(A_\mu(x)O(y)) = 2m_f(J_{sq}^0(x)O(y)) + 2(J_{sq}^a(x)O(y)) \quad (1)$$

Here $A_\mu^a(x)$ at the four-dimensional point $x$ includes spinors from all values of the the fifth coordinate. The pseudoscalar density $J_{s}^0(x)$ is constructed from four-dimensional fields in the standard way and $J_{sq}^a(x)$ is constructed from “mid-point” spinors at $s = L_s/2 - 1$ and $L_s/2$. (The notation is as in [8].)

For physics that involves momentum scales much less than the cutoff, the extra term in Eq. [1] involving $J_{sq}^a$ (the “mid-point” term), gives a contribution of the form $J_{sq}^a = m_{\text{res}}J_{s}^0$. This defines $m_{\text{res}}$ to $O(a^2)$, since the low-energy QCD physics of domain wall fermions should by describable by an effective Lagrangian. However, in matrix elements involving divergent quantities one only expects that $J_{sq}^a \sim O(m_{\text{res}})J_{s}^0$. Since divergent expressions enter in $K \to \pi$ amplitudes and we must manipulate the Ward-Takahashi identities to remove them, these effects need to be understood.

To test our understanding of $m_{\text{res}}$ effects and our data, we first consider $O(y) = J_5^0(y)$ in Eq. [1]. Summing over $x$ gives (no sum on $a$)

$$\sum_x ([m_f J_5^0(x) + J_{sq}^a(x)]J_5^0(y)) - \langle \bar{q}q \rangle = 0 \quad (2)$$

The term in the sum without $J_{sq}^a$ gives $m_f/m_q^2$ pion pole terms plus $O(m_f/a^2)$ contact terms along with constant terms and terms of higher order in $m_f$. Similarly the mid-point term gives $m_{\text{res}}/m_q^2$ pion pole terms, plus $O(m_{\text{res}}/a^2)$ contact terms as well as constant contributions. The pion pole terms enter as $(m_f + m_{\text{res}})/m_q^2$ in Eq. [2] while the contact contributions enter as $O(m_f) + O(m_{\text{res}})$. This form for the pion pole term is consistent with $m_{\text{res}} = 0$ at $m_f = m_{\text{res}}$.

Thus we are lead to expect a Gell-Mann–Oakes–Renner relation for domain wall fermions with the form

$$\frac{m_q^2 f^2}{2(m_f + m_{\text{res}})} = -(\langle \bar{q}q \rangle m_f + O(m_f/a^2)) + O(m_{\text{res}}/a^2) \quad (3)$$

This relation holds at non-zero values for $m_f$, with the $O(m_f/a^2)$ contact term contribution being cancelled by the similar term in $\langle \bar{q}q \rangle$. Of course, there may be other quenched pathologies entering this expression, such as effects from topological near-zero modes or quenched chiral logarithms.

We have measured $m_{\pi}, f, m_{\text{res}}$ and $-\langle \bar{q}q \rangle$ for $16^3 \times 32$ lattices at $\beta = 6.0$ [8] and plot our results in Figure [4]. One sees that for $L_s = 16$, the larger mass points are approximately linear, but do not seem to extrapolate towards the value of $-\langle \bar{q}q \rangle$ at $m_f = -m_{\text{res}}$. For $L_s = 24$, the extrapolation is much closer, as is expected since $m_{\text{res}}$ has decreased by almost a factor of 2. Also shown are solid lines representing the left-hand side of Eq. [1] when a quenched chiral logarithm fit is used for $m_{\text{res}}^2(m_f)$. One sees that the effect of chiral logarithms is to exacerbate the dis-
trix elements. Following [12], we define a sub-

\[
\text{K} \rightarrow \text{m} \Rightarrow \text{L}
\]

symbols are the left side of Eq. 3 and the filled

symbols are the right side of Eq. 3. The open symbols

are the left side of Eq. 4 and the solid line includes

the effects of quenched chiral logarithms in \( m_\pi \).

agreement in the extrapolation. Thus we have
evidence for residual mass contact terms in this
simple Ward-Takahashi identity.

In conclusion, we note that for \( L_s = 16 \), the quantity \( m_f^2 f^2 / \langle \pi \eta \rangle (m_f + m_{\text{res}}) \) is 1 in chiral perturbation theory, differs from 1 by around 20% for our data. Thus one should avoid introducing extraneous powers of this quantity while manipulating the data. We now turn to the Ward-Takahashi identities useful for subtracting the divergent contributions to \( K \rightarrow \pi \) matrix elements.

4. TOWARDS \( \langle \pi^+ \pi^- | O_n | K^0 \rangle \)

Here we discuss our tests of a particular method
for removing the divergent effects in \( K \rightarrow \pi \) matrix elements. Following [12], we define a sub-

tracted operator

\[
\tilde{O}_n = O_n + \eta_n [(m_d + m_s) \bar{s}d + (m_d - m_s) \bar{s}\gamma_5 d] \quad (4)
\]

where \( O_n \) is a \( \Delta S = 1 \) four-fermion operator and \( \eta_n \) is chosen such that

\[
\langle 0 | \tilde{O}_n | K^0 \rangle = 0 \quad (5)
\]

Then following [3], but noting that contact terms in the Ward-Takahashi identities coming from \( m_{\text{res}} \) effects will enter, we find

\[
\langle \pi^+ \pi^- | O_n | K^0 \rangle = \frac{i(m_{K_0}^2 - m_{\pi^+}^2) \left( d \langle \pi^+ | \tilde{O}_n | K^+ \rangle \right)}{dm_{\pi}^2} \quad (6)
\]

where the derivative removes any contact terms proportional to \( m_{\text{res}} \).

The successful removal of the effects of mixing with the dimension three operators implemented in Eq. 4 requires chiral symmetry. We first check this with a simpler Ward-Takahashi identity, obtained by inserting an operator \( O = \pi(y)(1 - \gamma_5) d(y) \bar{u}(z) \gamma_5 s(z) \) in Eq. 1. The pion pole terms in the Ward-Takahashi identity then lead to

\[
\langle \pi | \tilde{s}d | K \rangle \frac{2(m_f + m_{\text{res}})}{m_{\pi}^2} = 1 + O(m_f) + O(m_{\text{res}}) + \cdots \quad (7)
\]

There are no power divergent operators in this Ward-Takahashi identity, so the coefficient of the \( m_f \) and \( m_{\text{res}} \) contact terms are finite, but there may still be quenched chiral logarithm effects present. In Figure 2 we plot the left-hand side of Eq. 6 versus \( m_f \). We see that for larger \( m_f \), the points appear to extrapolate reasonably well to 1, but there is noticeable curvature for small quark masses. The figure includes results for axial vector sources, which agree with those for pseudoscalar sources, indicating that topological near-zero modes are not the origin of this non-linearity. We are investigating whether this can be explained as a quenched chiral logarithm.

We determine \( \eta_n \) from measurements of \( \langle 0 | O_n | K^0 \rangle \) for non-degenerate masses for the strange and down quarks. The upper panel of Figure 3 shows a plot of \( \langle 0 | O_2 | K^0 \rangle / \langle 0 | \tilde{s} \gamma_5 d | K^0 \rangle \) versus \( m_s - m_d \). We see good linearity which allows for a reliable determination of \( \eta_2 \), with statistically errors at the few percent level for this particular operator.
With the value of \( \eta_2 \) determined, \( \tilde{O}_2 \) is known and Figure 3 shows

\[
\frac{\langle \pi^+ | O_2^\frac{1}{2} + \eta_2 (m_u + m_d) \bar{s}d | K^+ \rangle}{\langle \pi | P | 0 \rangle \langle 0 | P | K \rangle}.
\]  

(8)

where \( O_2^\frac{1}{2} \) is the \( \Delta I = 1/2 \) part of \( O_2 \). In lowest order chiral perturbation theory, the denominator is a constant, so the desired \( K \to \pi \pi \) matrix element is given by the slope and the intercept is related to \( m_{\text{res}} \). We are looking carefully for non-linearities in our data, paying particular attention to the quenched chiral logarithm effects recently calculated in [13].

5. CONCLUSIONS

We have seen that the Ward-Takahashi identities for domain wall fermions include \( O(m_{\text{res}}/a^2) \) contact terms which can be removed by taking derivatives with respect to \( m_f \). Our data also show evidence for non-linear dependence on \( m_f \), some of which may be due to quenched chiral logarithms. We have shown that the subtracted operator \( \tilde{O}_2 \) can be determined and are now doing the subtractions for all the operators.

The calculations reported here were run on the QCDSP computers at Columbia University and the RIKEN-BNL Research Center.

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