GENERATING THE TWIST SUBGROUP BY INVOLUTIONS

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Abstract. For a nonorientable surface, the twist subgroup is an index 2 subgroup of the mapping class group. It is generated by Dehn twists about two-sided simple closed curves. In this paper, we study involution generators of the twist subgroup. We give generating sets of involutions with the smallest number of elements our methods allow.

1. Introduction

Let $N_g$ denote a closed connected nonorientable surface of genus $g$. The mapping class group of $N_g$ is defined to be the group of the isotopy classes of all diffeomorphisms of $N_g$. Throughout the paper this group will be denoted by $\text{Mod}(N_g)$. Let $\Sigma_g$ denote a closed connected orientable surface of genus $g$. The mapping class group of $\Sigma_g$ is the group of the isotopy classes of orientation preserving diffeomorphisms and is denoted by $\text{Mod}(\Sigma_g)$.

In the orientable case, it is a classical result that $\text{Mod}(\Sigma_g)$ is generated by finitely many Dehn twists about nonseparating simple closed curves [3, 5, 10]. The study of algebraic properties of mapping class group, finding small generating sets, generating sets with particular properties, is an active one leading to interesting developments. Wajnryb [19] showed that $\text{Mod}(\Sigma_g)$ can be generated by two elements given as a product of Dehn twists. As the group is not abelian, this is the smallest possible. Later, Korkmaz [7] showed that one of these generators can be taken as a Dehn twist, he also proved that $\text{Mod}(\Sigma_g)$ can be generated by two torsion elements. Recently, the third author showed that $\text{Mod}(\Sigma_g)$ is generated by two torsions of small orders [20].

Generating $\text{Mod}(\Sigma_g)$ by involutions was first considered by McCarthy and Papadopoulos [13]. They showed that the group can be generated by infinitely many conjugates of a single involution (element of order two) for $g \geq 3$. In terms of generating by finitely many involutions,
Luo [12] showed that any Dehn twist about a nonseparating simple closed curve can be written as a product six involutions, which in turn implies that \( \text{Mod}(\Sigma_g) \) can be generated by \( 12g + 6 \) involutions. Brendle and Farb [1] obtained a generating set of six involutions for \( g \geq 3 \). Following their work, Kassabov [6] showed that \( \text{Mod}(\Sigma_g) \) can be generated by four involutions if \( g \geq 7 \). Recently, Korkmaz [8] showed that \( \text{Mod}(\Sigma_g) \) is generated by three involutions if \( g \geq 8 \) and four involutions if \( g \geq 3 \). Also, the third author improved his result showing that it is generated by three involutions if \( g \geq 6 \) [21].

Compared to orientable surfaces less is known about \( \text{Mod}(N_g) \). Lickorish [9, 11] showed that it is generated by Dehn twists about two-sided simple closed curves and a so-called \( Y \)-homeomorphism (or a crosscap slide). Chillingworth [2] gave a finite generating set for \( \text{Mod}(N_g) \) that linearly depends on \( g \). Szepietowski [18] proved that \( \text{Mod}(N_g) \) is generated by three elements and by four involutions.

The twist subgroup \( T_g \) of \( \text{Mod}(N_g) \) is the group generated by Dehn twists about two-sided simple closed curves. The group \( T_g \) is a subgroup of index 2 in \( \text{Mod}(N_g) \) [11]. Chillingworth [2] showed that \( T_g \) can be generated by finitely many Dehn twists. Stukow [16] obtained a finite presentation for \( T_g \) with \((g + 2)\) Dehn twist generators. Later Omori [14] reduced the number of Dehn twist generators to \((g + 1)\) for \( g \geq 4 \). If it is not required that all generators are Dehn twists, Du [4] obtained a generating set consisting of three elements, two involutions and an element of order \( 2g \) whenever \( g \geq 5 \) and odd. Recently, Yoshihara [22] was interested in the problem of finding generating sets for \( T_g \) consisting of only involutions. He proved that \( T_g \) can be generated by six involutions for \( g \geq 14 \) and by eight involutions if \( g \geq 8 \).

Our aim in this paper is to generate \( T_g \) with fewer number of involutions. It is known that any group generated by two involutions is isomorphic to a quotient of a dihedral group. Hence, \( T_g \) cannot be generated by two involutions. We are not sure whether \( T_g \) can be generated by three involutions. Based on the approach of [8], we obtain the following result:

**Main Theorem.** The twist subgroup \( T_g \) of \( \text{Mod}(N_g) \) is generated by

1. four involutions if \( g \geq 12 \) and even,
2. four involutions if \( g = 4k + 1 \geq 5 \),
3. five involutions if \( g = 4k + 3 \geq 11 \).

We also prove that the twist subgroup \( T_g \) can be generated by

4. five involutions if \( g = 8, 10 \),
5. six involutions if \( g = 6, 7 \).
Note that if a group is generated by involutions, then its first integral homology group should consist of elements of order 2. For the twist subgroup $\mathcal{T}_g$, this is the case when $g \geq 5$ [15].

The paper is organized as follows. In Section 2, we recall some basic results on $\text{Mod}(N_g)$ and its subgroup $\mathcal{T}_g$. We work with nonorientable surfaces of even genus in Section 3 and nonorientable surfaces of odd genus in Section 4.

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2. Background and Results on Mapping Class Groups

Let $N_g$ be a closed connected nonorientable surface of genus $g$. Note that the genus for a nonorientable surface is the number of projective planes in a connected sum decomposition. The mapping class group $\text{Mod}(N_g)$ of the surface $N_g$ is defined to be the group of the isotopy classes of diffeomorphisms $N_g \to N_g$. Throughout the paper we do not distinguish a diffeomorphism from its isotopy class. For the composition of two diffeomorphisms, we use the functional notation; if $g$ and $h$ are two diffeomorphisms, the composition $gh$ means that $h$ acts on $N_g$ first.

A simple closed curve on a nonorientable surface $N_g$ is said to be one-sided if a regular neighbourhood of it is homeomorphic to a Möbius band. It is called two-sided if a regular neighbourhood of it is homeomorphic to an annulus. If $a$ is a two-sided simple closed curve on $N_g$, to define the Dehn twist $t_a$, we need to fix one of two possible orientations on a regular neighbourhood of $a$ (as we did for the curve $a_1$ in Figure 1). Following [8] the right-handed Dehn twist $t_a$ about $a$ will be denoted by the corresponding capital letter $A$.

Recall the following properties of Dehn twists: let $a$ and $b$ be two-sided simple closed curves on $N_g$ and let $f \in \text{Mod}(N_g)$.

- **Commutativity:** If $a$ and $b$ are disjoint, then $AB = BA$.
- **Conjugation:** If $f(a) = b$, then $fAf^{-1} = B^s$, where $s = \pm 1$ depending on whether $f$ is orientation preserving or orientation reversing on a neighbourhood of $a$ with respect to the chosen orientation.

Consider the surface $N_g$ shown in Figure 1. The Dehn twist generators of Omori can be given as follows (note that we do not have the curve $d_r$ when $g$ is odd).
Theorem 2.1. [14] The twist subgroup $T_g$ is generated by the following $(g + 1)$ Dehn twists

1. $A_1, A_2, B_1, \ldots, B_r, C_1, \ldots, C_{r-1}$ and $E$ if $g = 2r + 1$ and
2. $A_1, A_2, B_1, \ldots, B_r, C_1, \ldots, C_{r-1}, D_r$ and $E$ if $g = 2r + 2$.

Consider a basis $\{x_1, x_2, \ldots, x_{g-1}\}$ for $H_1(N_g; \mathbb{R})$ such that the curves $x_i$ are one-sided and disjoint as in Figure 2. It is known that every diffeomorphism $f : N_g \to N_g$ induces a linear map $f_* : H_1(N_g; \mathbb{R}) \to H_1(N_g; \mathbb{R})$. Therefore, one can define a homomorphism $D : \text{Mod}(N_g) \to \mathbb{Z}_2$ by $D(f) = \det(f_*)$. The following lemma from [9] tells when a mapping class falls into the twist subgroup $T_g$.

Lemma 2.2. Let $f \in \text{Mod}(N_g)$. Then $D(f) = 1$ if $f \in T_g$ and $D(f) = -1$ if $f \notin T_g$.

3. The even case

For $g = 2r + 2$, we work with the models in Figure 3. This surface is obtained from a genus $r$ orientable surface by deleting the interiors of two disjoint disks and identifying the antipodal points on the boundary.
Moreover, the genus $r$ surface minus two disks is embedded in $\mathbb{R}^3$ in such a way that each genus is in a circular position with the second genus on the $+z$-axis and the rotation $R$ by $\frac{2\pi}{r}$ about $x$-axis maps the curve $b_i$ to $b_{i+1}$ for $i = 1, \ldots, r-1$ and $b_r$ to $b_1$.

We use the explicit homeomorphism constructed in [15, Section 3] to identify the models in Figure 1 and 3. On the left hand side of Figure 3, one of the crosscaps is centered on the $+x$-axis and the other one is obtained by rotating the first one by $\pi$ about the $z$-axis. The model on the right hand side is obtained from the model on the left hand side by sliding crosscaps via a diffeomorphism, say $\phi$.

Let $\tau$ be the blackboard reflection of $N_g$ for the model in the left hand side in Figure 3. If $r$ is odd, we consider the reflection $\tau$ and if $r$ is even, we consider the reflection $\phi\tau\phi^{-1}$. The surface $N_g$ is invariant under the reflections $\tau$ and $\phi\tau\phi^{-1}$. Abusing the notation, we keep writing $\tau$ instead of $\phi\tau\phi^{-1}$. Note that $D(\tau) = -1$.

Note that the surface $N_g$ is invariant under the two rotations $\rho_1'$ and $\rho_2'$ where $\rho_1'$ is the rotation by $\pi$ about $z$-axis and $\rho_2'$ is the rotation by $\pi$ about the line $z = \tan(\frac{\pi}{r})y, x = 0$ as in Figure 3. The rotations $\rho_1'$ and $\rho_2'$ satisfy $D(\rho_1') = D(\rho_2') = -1$, which implies that the twist subgroup $\mathcal{T}_g$ does not contain $\rho_1'$ and $\rho_2'$. Let $\rho_1 = \rho_1'\tau$ and $\rho_2 = \rho_2'\tau$. Then the involutions $\rho_1$ and $\rho_2$ are contained in $\mathcal{T}_g$ by Lemma 2.2. Observe that the rotation $R = \rho_2\rho_1$. 

**Figure 3.** The models for $N_g$ if $g = 2r + 2$. 

3.1. Generating sets for the twist subgroup $\mathcal{T}_g$. Recently, Korkmaz [8] introduced new generating sets for the mapping class group of an orientable surface. We follow the outline of his proofs. Especially, since the curves $a_i$, $b_i$ and $c_i$ are exactly the same as in [8], statements about these curves follows directly from [8]. Before we state our result, let us recall the above mentioned theorem of Korkmaz. Recall that $A_i$, $B_i$, $C_i$, $E$ and $F$ represent the Dehn twists about the corresponding lower case letters in Figure 1 and 3.

**Theorem 3.1.** [8] Let $\Sigma_g$ denote a closed connected oriented surface of genus $g$. Then, if $g \geq 3$, $\text{Mod}(\Sigma_g)$ is generated by the four elements $R, A_1A_2^{-1}, B_1B_2^{-1}$ and $C_1C_2^{-1}$.

Using the above theorem, we give a generating set for $\mathcal{T}_g$ when $g$ is even.

**Theorem 3.2.** Let $r \geq 3$ and $g = 2r + 2$. Then the twist subgroup $\mathcal{T}_g$ is generated by the elements $R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}, D_r$ and $E$ if $g = 2r + 2$.

**Proof.** Let $G$ be the subgroup of $\mathcal{T}_g$ generated by the set 

$$\{R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}, D_r, E\}$$

if $g = 2r + 2$.

Let $S$ denote the set of isotopy classes of two-sided non-separating simple closed curves on $N_g$. Define a subset $\mathcal{G}$ of $S \times S$ as

$$\mathcal{G} = \{(a, b) : AB^{-1} \in G\}.$$

The set $\mathcal{G}$ defines an equivalence relation on $S$ which satisfies $G$-invariance property, that is,

if $(a, b) \in \mathcal{G}$ and $H \in G$ then $(H(a), H(b)) \in \mathcal{G}$.

Then it follows from the proof of Theorem 3.1 that the Dehn twists $A_i$ and $B_i$ for $i = 1, \ldots, r$ are contained in $G$. Also, $G$ contains $C_j$ for $j = 1, \ldots, r - 1$. Since all generators given in Theorem 2.1 are contained in the group $G$. We conclude that $G = \mathcal{T}_g$. □

3.2. Involution generators. We consider the surface $N_g$ where $g$-crosscaps are distributed on the sphere as in Figure 4. If $g = 2r + 2$ and $r \geq 3$, there is a reflection, $\sigma$, of the surface $N_g$ in the xy-plane such that

- $\sigma(f) = a_1$, $\sigma(b_r) = d_r$,
- $\sigma(x_2) = x_3$, $\sigma(x_3) = x_5$ $\sigma(x_{g-2}) = x_g$ and
- $\sigma(x_i) = x_i$ if $i = 6, \ldots, g - 3$ or $i = 1, g - 1$. 

with reverse orientation. (Recall that $x_i$’s are the generators of $H_1(N;\mathbb{R})$ as shown in Figure 2.)

The linear map $D$ associated to $\sigma$ satisfies $D(\sigma) = 1$ if $g$ is even.

Figure 4. The involution $\sigma$ if $g = 2r + 2$.

This implies that the involution $\sigma$ is contained in $\mathcal{T}_g$ if $g$ is even.

**Theorem 3.3.** The twist subgroup $\mathcal{T}_{12}$ is generated by the involutions $\rho_1, \rho_2, \rho_1 A_1 B_2 C_4 A_3$ and $\sigma$.

**Proof.** Consider the surface $N_{12}$ as in Figure 3. Since $\rho_1(a_1) = a_3, \rho_1(b_2) = b_2$ and $\rho_1(c_4) = c_4$,

and $\tau$ reverses the orientation of a neighbourhood of a two-sided simple closed curve, we get

- $\rho_1 A_1 \rho_1 = A_3^{-1}$,
- $\rho_1 B_2 \rho_1 = B_2^{-1}$ and
- $\rho_1 C_4 \rho_1 = C_4^{-1}$.

It is easy to verify that $\rho_1 A_1 B_2 C_4 A_3$ is an involution. Let $E_1 = A_1 B_2 C_4 A_3$ and let $H$ be the subgroup of $\mathcal{T}_{12}$ generated by the set

$$\{\rho_1, \rho_2, \rho_1 E_1, \sigma\}.$$ 

Note that the rotation $R$ is in the subgroup $H$. By Theorem 3.2, we need to show that the elements $A_1 A_2^{-1}, B_1 B_2^{-1}, C_1 C_2^{-1}, D_5$ and $E$ are contained in $H$.

Let $E_2 = R E_1 R^{-1} = A_2 B_3 C_5 A_4$. It can be easily shown that

$$E_2 E_1 (a_2, b_3, c_5, a_4) = (b_2, a_3, c_5, a_4),$$

so that $E_3 = B_2 A_3 C_5 A_4$ is in $H$.

Let

$$E_4 = R^2 E_1 R^{-2} = A_3 B_4 C_1 A_5.$$
It is easy to show that
\[ E_4 E_3(a_3, b_4, c_1, a_5) = (a_3, a_4, b_2, a_5) \]
so that \( E_5 = A_3 A_4 B_2 A_5 \) are contained in \( H \). Hence,
\[ E_5 E_3^{-1} = A_5 C_5^{-1} \in H. \]
One can easily see that the elements \( A_i C_i^{-1} \) are contained in \( H \) by conjugating \( A_5 C_5^{-1} \) with powers of \( R \).
Let
\[ E_6 = R E_5 R^{-1} = A_4 A_5 B_3 A_1. \]
One can easily show that
\[ E_6 E_5(a_4, a_5, b_3, a_1) = (a_4, a_5, a_3, a_1) \]
so that \( E_7 = A_4A_5A_3A_1 \) is in \( H \). Therefore,
\[ E_7 E_6^{-1} = A_3B_3^{-1} \in H. \]
By conjugating with powers of \( R \), we get \( A_iB_i^{-1} \in H \). Hence,
\[ B_5C_5^{-1} = (B_5A_5^{-1})(A_5C_5^{-1}) \in H. \]
Again by conjugating with powers of \( R \), the elements \( B_iC_i^{-1} \) are contained in \( H \).
Let
\[ E_8 = (A_2B_2^{-1})(B_3A_3^{-1})E_1 = A_1A_2C_4B_3 \]
and
\[ E_9 = R^2F_8R^{-2} = A_3A_4C_1B_5. \]
It can also be shown that
\[ E_9E_8(a_3, a_4, c_1, b_5) = (b_3, a_4, c_1, c_4) \]
so that \( E_{10} = B_3A_4C_1C_4 \). Hence,
\[ E_9E_{10}^{-1}B_3A_3^{-1} = B_3C_4^{-1} \in H. \]
The conjugation of this with powers of \( R \) implies that \( B_{i+1}C_i^{-1} \in H \).

Hence
\[ \begin{align*}
\bullet & \quad A_1A_2^{-1} = (A_1C_1^{-1})(C_1B_2^{-1})(B_2A_2^{-1}), \\
\bullet & \quad B_1B_2^{-1} = (B_1C_1^{-1})(C_1B_2^{-1}) \text{ and} \\
\bullet & \quad C_1C_2^{-1} = (C_1B_2^{-1})(B_2C_2^{-1})
\end{align*} \]
are contained in \( H \). Also it follows from the fact that \( \sigma(a_1) = f \) and \( \sigma(b_5) = d_5 \)
with a choice of orientations of regular neighbourhoods of the curves, the element \( D_5 \) and \( F \) are contained in \( H \). By the fact that \( A_1(f) = e \), \( E \) is in \( H \). We conclude that \( H = T_{12} \).

**Theorem 3.4.** For \( g = 2r + 2 \), the twist subgroup \( T_g \) is generated by the involutions \( \rho_1, \rho_2, \rho_1A_1B_2C_{r+3}A_3 \) and \( \sigma \) if \( r \geq 7 \) and odd.

**Proof.** Consider the surface \( N_g \) as in Figure 3. We have
\[ \rho_1(a_1) = a_3, \rho_1(b_2) = b_2 \text{ and } \rho_1(c_{r+3}) = c_{r+3}. \]
Since \( \tau \) reverses the orientation of a neighbourhood of a two-sided simple closed curve, we get
\[ \bullet \quad \rho_1A_1\rho_1 = A_3^{-1} \]
\[ \rho_1 B_2 \rho_1 = B_2^{-1} \]
\[ \rho_1 C_{r+3} \rho_1 = C_{r+3}^{-1} \]

It can be shown that \( \rho_1 A_1 B_2 C_{r+3} A_3 \) is an involution. Let \( G_1 = A_1 B_2 C_{r+3} A_3 \) and let \( K \) be the subgroup of \( T_g \) generated by the set
\[ \{ \rho_1, \rho_2, \rho_1 G_1, \sigma \}. \]

Note that the rotation \( R \) is in \( K \). By Theorem 3.2, we need to show that the elements \( A_1 A_2^{-1}, B_1 B_2^{-1}, C_1 C_2^{-1}, D_r, \) and \( E \) are contained in \( K \). It follows from
\[ \begin{align*}
G_2 &= RG_1 R^{-1} = A_2 B_3 C_{r+2} A_4 \in K, \\
G_3 &= (G_2 G_1)G_2(G_2 G_1)^{-1} = B_2 A_3 C_{r+3} A_4 \in K, \\
G_4 &= RG_3 R^{-1} = B_3 A_4 C_{r+7} A_5 \in K, \\
G_5 &= (G_4 G_3)G_4(G_4 G_3)^{-1} = A_3 A_4 C_{r+7} A_5 \in K
\end{align*} \]

that
\[ G_4 G_5^{-1} = B_3 A_3^{-1} \in K. \]

Hence, the elements \( B_i A_i^{-1} \) are contained in \( K \) by conjugating \( B_3 A_3^{-1} \) with powers of \( R \). Let
\[ \begin{align*}
G_6 &= R^{r+3} G_4 R^{3-r} = B_{r+3} A_{r+2} C_2 A_{r+7} \in K, \\
G_7 &= (G_6 G_4)G_6(G_6 G_4)^{-1} = B_{r+3} A_{r+2} B_3 A_{r+7} \in K \text{ if } r > 7, \\
(G_7 = A_5 A_6 B_3 A_7 \in K \text{ if } g = 7).
\end{align*} \]

Then
\[ G_7 G_6^{-1} = B_3 C_2^{-1} \in K \text{ if } r > 7 \]
and
\[ G_7 G_6^{-1} B_5 A_5^{-1} = B_3 C_2^{-1} \in K \text{ if } r = 7. \]

Therefore, the elements \( B_{i+1} C_i^{-1} \) are contained in the group \( K \) by conjugating \( B_3 C_2^{-1} \) with powers of \( R \). Let
\[ \begin{align*}
G_8 &= R^{r+3} G_4 R^{1-r} = B_{r+5} A_{r+2} C_3 A_{r+9} \in K \text{ if } r > 7, \\
(G_8 = B_6 A_7 C_3 A_1 \in K \text{ if } r = 7), \\
G_9 &= (G_8 G_4)G_8(G_8 G_4)^{-1} = B_{r+5} A_{r+7} B_3 A_{r+9} \in K \text{ if } r > 7. \\
(G_9 = B_6 A_7 B_3 A_1 \in K \text{ if } r = 7).
\end{align*} \]

Then
\[ G_9 G_8^{-1} = B_3 C_3^{-1} \in K \text{ if } r \geq 7. \]

This implies that the subgroup \( K \) contains \( B_4 C_1^{-1} \) by conjugating \( B_3 C_3^{-1} \) with powers of \( R \). The rest of the proof is very similar to the proof of Theorem 3.3. \( \square \)

**Theorem 3.5.** For \( g = 2r + 2 \), the twist subgroup \( T_g \) is generated by the involutions \( \rho_1, \rho_2, \rho_1 A_2 C_{r+4} B_{r+4} C_{r+6} \) and \( \sigma \) if \( r \geq 6 \) and even.
By conjugating with powers of $R$ and $B$, then we get

$$\tau \text{ reverses the orientation of a neighbourhood of a two-sided simple closed curve, we have}$$

- $\rho_1 A_2 \rho_1 = A_2^{-1}$
- $\rho_1 B_{r+4} \rho_1 = B_{r+4}^{-1}$
- $\rho_1 C_{r+4} \rho_1 = C_{r+4}^{-1}$.

It can be shown that $\rho_1 A_2 C_{r+4} B_{r+4} C_{r+6}$ is an involution. Let $H_1 = A_2 C_{r+4} B_{r+4} C_{r+6}$ and let $K$ be the subgroup of $T_g$ generated by the set

$$\{\rho_1, \rho_2, \rho_1 H_1, \sigma\}.$$

Note that the rotation $R$ is in $K$. By Theorem 3.2, we need to show that the elements $A_1 A_2^{-1}, B_1 B_2^{-1}, C_1 C_2^{-1}, D_r$ and $E$ are contained in $K$. Let

- $H_2 = RH_1 R^{-1} = A_3 C_{r+2} B_{r+6} C_{r+8} B_{r+2} C_{r+8} \in K$.
- $H_3 = (H_2 H_1) H_2 (H_3 H_1)^{-1} = A_3 B_{r+4} C_{r+6} C_{r+8} B_{r+4} \in K$.
- $H_4 = RH_3 R^{-1} = A_4 B_{r+6} C_{r+8} C_{r+10} \in K$.
- $H_5 = (H_4 H_3) H_4 (H_5 H_3)^{-1} = A_4 C_{r+6} C_{r+8} C_{r+10} B_{r+4} \in K$.

Then, we get

$$H_4 H_5^{-1} = B_{r+6} C_{r+6}^{-1} \in K$$

and

$$H_2 H_3^{-1} \left( C_{r+6} B_{r+4}^{-1} \right) = C_{r+6} B_{r+4}^{-1} \in K.$$

By conjugating the elements $B_{r+6} C_{r+6}^{-1}$ and $C_{r+2} B_{r+4}^{-1}$ with powers of $R$, we conclude that $B_i C_i^{-1}$ and $C_i B_{i+1}^{-1}$ are contained in $K$. Let

- $H_6 = (B_{r+6} C_{r+6}^{-1} B_{r+6} C_{r+6}^{-1}) H_1 = B_{r+6} B_{r+6} A_2 B_{r+6} \in K$.
- $H_7 = R_{r+4}^2 H_6 R_{r+4}^{-2} = A_2 B_{r-2} B_r B_1 \in K$.
- $H_8 = (H_7 H_6) H_7 (H_7 H_6)^{-1} = B_2 B_{r-2} B_r B_1 \in K$.

Then

$$H_8 H_7^{-1} = B_2 A_2^{-1} \in K.$$

By conjugating with powers of $R$, $K$ contains $B_i A_i^{-1}$. The rest of the proof is very similar to the proof of Theorem 3.3.

In the rest of this section, we introduce involution generators for $T_g$ for $g = 6, 8$ and $10$.

We consider the models for the surface $N_{10}$, where 10-crosscaps are
Figure 6. The involution $\delta_1$ for $g = 4k + 2$.

Figure 7. The involution $\delta_2$ for $g = 4k + 2$.

Figure 8. The involution $\delta_3$ for $g = 10$.

distributed on the sphere as in Figure 6, 7 and 8. There are reflections, $\delta_1, \delta_2$ and $\delta_3$, of the surface $N_{10}$ in the $xy$-plane such that

- $\delta_1(x_i) = x_{i+1}$ if $i = 1, 5, 9$,
- $\delta_1(x_3) = x_8$, $\delta_1(x_4) = x_7$,
Recall that $x_i$ are the generators of $H_1(N_8;\mathbb{R})$ as shown in Figure 2. Note that the involutions $\delta_1, \delta_2$ and $\delta_3$ reverse the orientation of a neighbourhood of a two-sided simple closed curve. Since $D(\delta_i) = 1$, the involutions $\delta_i$ are in $T_{10}$ for $i = 1, 2, 3$.

**Theorem 3.6.** The twist subgroup $T_{10}$ is generated by five involutions $\delta_1, \delta_2, \delta_2\delta_1\delta_2A_2, \delta_1A_1, \delta_3$.

**Proof.** Let $K$ be the subgroup of $T_{10}$ generated by the set

$$\{\delta_1, \delta_2, \delta_2\delta_1\delta_2A_2, \delta_1A_1, \delta_3\}.$$  

It is clear that $\delta_2\delta_1\delta_2A_2$ and $\delta_1A_1$ are involutions. It follows from

- $A_1 = \delta_1(\delta_1A_1)$ and
- $A_2 = (\delta_2\delta_1\delta_2)(\delta_2\delta_1\delta_2A_2)$

that the elements $A_1$ and $A_2$ are in $K$. Also, It follows from

- $\delta_2(a_1) = b_1$
- $\delta_2\delta_1(b_1) = c_i$ for $i = 1, 2, 3$ and
- $\delta_2\delta_1(c_i) = b_{i+1}$ for $i = 1, 2$

that $B_i, C_i$ are contained in $K$ for $i = 1, 2, 3$. Moreover, since

- $\delta_3(c_3) = d_4$
- $\delta_1\delta_3\delta_3\delta_1\delta_3(c_1) = b_4$ and
- $A_1\delta_3(a_1) = e$

then the elements $D_4, B_4$ and $E$ are in $K$. We conclude that $K = T_{10}$ by Theorem 2.1. \hfill \Box

We consider the models for the surface $N_8$, where 8-crosscaps are distributed on the sphere as in Figure 9, 10 and 11. There are reflections, $\lambda_1, \lambda_2$ and $\lambda_3$, of the surface $N_8$ in the $xy$-plane such that

- $\lambda_1(x_i) = x_i$ if $i = 7, 8$,
- $\lambda_1(x_i) = x_{i+1}$ if $i = 1, 4$ and $\lambda_1(x_3) = x_6$,
- $\lambda_2(x_i) = x_i$ if $i = 2, 5$,
- $\lambda_2(x_1) = x_3, \lambda_2(x_4) = x_6, \lambda_2(x_7) = x_8$, and
- $\lambda_3(x_i) = x_i$ if $i = 1, 4$,
- $\lambda_3(x_2) = x_3, \lambda_3(x_5) = x_8$ and $\lambda_3(x_6) = x_7$.

Note that the involutions $\lambda_i$ reverse the orientation of a neighbourhood of a two-sided simple closed curve for $i = 1, 2, 3$. Since $D(\delta_i) = 1$, the involutions $\delta_i$ are contained in $T_8$ for $i = 1, 2, 3$. 

Theorem 3.7. The twist subgroup $T_8$ is generated by five involutions $\lambda_1, \lambda_2, \lambda_2\lambda_1\lambda_2 A_2, \lambda_1 A_1$ and $\lambda_3$. 
Proof. Let $K$ be the subgroup of $\mathcal{T}_8$ generated by the set
$$\{\lambda_1, \lambda_2, \lambda_2\lambda_1\lambda_2A_2, \lambda_1A_1, \lambda_3\}.$$  
It is clear that $\lambda_2\lambda_1\lambda_2A_2$ and $\lambda_1A_1$ are involutions. It follows from
- $A_1 = \lambda_1(\lambda_1A_1)$ and
- $A_2 = (\lambda_2\lambda_1\lambda_3)(\lambda_2\lambda_1\lambda_2A_2)$
that the elements $A_1$ and $A_2$ are in $K$. Also, It follows from
- $\lambda_2\lambda_1(a_1) = b_1$,
- $\lambda_2\lambda_1(b_i) = c_i$ for $i = 1, 2$,
- $\lambda_2\lambda_1(c_1) = b_2$,
- $\lambda_3(c_2) = d_3$,
- $\lambda_1\lambda_2\lambda_3\lambda_1\lambda_3(c_1) = b_3$ and
- $A_1\lambda_3(a_1) = e$
that all generators of $\mathcal{T}_8$ given in Theorem 2.1 are contained in $K$. This completes the proof.  

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12}
\caption{The involution $\xi_1$ for $g = 6$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure13}
\caption{The involution $\xi_2$ for $g = 6$.}
\end{figure}
We consider the models for the surface $N_6$, where 6-crosscaps are distributed on the sphere as in Figure 6, 7, 12 and 13. There are reflections $\delta_1, \delta_2, \xi_1$ and $\xi_2$ such that

- $\delta_1(x_i) = x_{i+1}$ if $i = 1, 3, 5$,
- $\delta_2(x_i) = x_i$ if $i \neq 1, 3$ and $\delta_2(x_1) = x_3$,
- $\xi_1(x_i) = x_i$ if $i \neq 2, 3$ and $\xi_1(x_2) = x_3$ and
- $\xi_2(x_i) = x_{i+1}$ if $i = 1, 4$ and $\xi_2(x_3) = x_6$.

Note that the involutions $\delta_i$ and $\xi_i$ reverse the orientation of a neighbourhood of a two-sided simple closed curve for $i = 1, 2$. We obtain that $D(\delta_i) = D(\xi_i) = 1$, the twist subgroup $T_8$ contains the involutions $\delta_i$ and $\xi_i$ for $i = 1, 2$.

**Theorem 3.8.** The twist subgroup $T_6$ is generated by six involutions $\delta_1, \delta_2, \delta_2\delta_1\delta_2A_2, \delta_1A_1, \xi_1$ and $\xi_2$.

**Proof.** Let $K$ be the subgroup of $T_6$ generated by the set

$$\{\delta_1, \delta_2, \delta_2\delta_1\delta_2A_2, \delta_1A_1, \xi_1, \xi_2\}.$$

It is clear that $\delta_2\delta_1\delta_2A_2$ and $\delta_1A_1$ are involutions. It follows from

- $A_1 = \delta_1(\delta_1A_1)$ and
- $A_2 = (\delta_2\delta_1\delta_2)(\delta_2\delta_1\delta_2A_2)$

that the elements $A_1$ and $A_2$ are in $K$. Also, It follows from

- $\delta_2(a_1) = b_1$,
- $\delta_2\delta_1(b_1) = c_1$,
- $\delta_1\delta_2\xi_2(b_1) = b_2$,
- $\xi_2(c_1) = d_2$ and
- $A_1\xi_1(a_1) = e$

that all generators of $T_6$ given in Theorem 2.1 are contained in $K$. This completes the proof. \(\square\)

### 4. The odd case

For $g = 4k + 1$, we work with two models for $N_g$: one is on the left hand side of Figure 14, the other one is depicted in Figure 15.

The model in Figure 14 is the nonorientable surface obtained from $S^2$ embedded in $\mathbb{R}^3$ and by deleting the interiors of $g$-disjoint disks and identifying the antipodal points on the boundary of each removed disks, say $C_i$. Moreover, each crosscap $C_i$ is in a circular position with the second crosscap $C_2$ on the $+z$-axis and the rotation $T$ by $\frac{2\pi}{g}$ about $x$-axis maps the crosscap $C_i$ to $C_{i+1}$. The model in Figure 15 is obtained from a genus $r$ orientable surface by deleting the interior of a disk and identifying the antipodal points on the boundary. Moreover, the genus $r$ surface minus a disk is embedded in $\mathbb{R}^3$ in such a way that
Figure 14. The involutions $\tau_1$ and $\tau_2$.

Figure 15. The involutions $\rho_1$ and $\rho_2$ for $g = 2r + 1$.

each genus is in a circular position with the second genus on the $+z$-axis and the rotation $R$ by $\frac{2\pi}{r}$ about $x$-axis maps the curve $b_i$ to $b_{i+1}$ for $i = 1, \ldots, r - 1$ and $b_r$ to $b_1$.

We use the explicit homeomorphism constructed in [15, Section 3] to
identify the models in Figure 1 and Figure 15. In Figure 15, one crosscap is on the +x-axis. Note that the surface $N_g$ is invariant under the two involutions $\rho_1$ and $\rho_2$ where $\rho_1$ is the reflection in the $xz$-plane and $\rho_2$ is the reflection in the plane $z = \tan(\frac{\pi}{r})y$ as in Figure 15. The rotations $\rho_1$ and $\rho_2$ satisfy $D(\rho_1) = D(\rho_2) = 1$ if $g = 4k + 1$. In this case, the twist subgroup $T_g$ contains $\rho_1$ and $\rho_2$. Observe that the rotation $R = \rho_2 \rho_1$.

For $g = 4k + 3$, we work with the model on the right hand side of Figure 14. This surface is a genus-$g$ nonorientable surface obtained from $S^2$ embedded in $\mathbb{R}^3$ and by deleting the interiors of $g$-disjoint disks and identifying the antipodal points on the boundary of each removed disks, say $C_i$. Moreover, each crosscap $C_i$ for $i = 1, \ldots, g - 2$ is in a circular position with the second crosscap $C_2$ on the +z-axis, the rotation $T$ by $\frac{2\pi}{g-2}$ about x-axis maps the crosscap $C_i$ to $C_{i+1}$ for $i = 1, \ldots, g-3$. The crosscap $C_{g-1}$ is on the +x-axis and $C_g$ is obtained by rotating $C_{g-1}$ by $\pi$ about +z-axis. Note that the surface $N_g$ is invariant under the two reflections $\tau_1$ and $\tau_2$ where $\tau_1$ is the reflection in the z-axis and $\tau_2$ is the reflection in the plane $z = \tan(\frac{\pi}{r})y$ as in Figure 14. The reflections $\tau_1$ and $\tau_2$ satisfy $D(\tau_1) = D(\tau_2) = 1$ if $r$ is even, which implies that $\tau_1$ and $\tau_2$ are contained in the twist subgroup $T_g$.

Recall that in Theorem 3.2 we give a generating set for $T_g$ when $g$ is even. We have the following generators when $g$ is odd.

**Theorem 4.1.** Let $r \geq 3$ and $g = 2r + 1$. Then the twist subgroup $T_g$ is generated by the elements $R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}$ and $E$.

**Proof.** Let $G$ be the subgroup of $T_g$ generated by the set $\{R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}, E\}$ if $g = 2r + 1$. Let $S$ denote the set of isotopy classes of two-sided non-separating simple closed curves on $N_g$. Define a subset $G$ of $S \times S$ as $G = \{(a, b) : AB^{-1} \in G\}$. The set $G$ defines an equivalence relation on $S$ which satisfies $G$-invariance property, that is,

if $(a, b) \in G$ and $H \in G$ then $(H(a), H(b)) \in G$.

Then it follows from the proof of Theorem 3.1 that the Dehn twists $A_i$ and $B_i$ for $i = 1, \ldots, r$ are contained in $G$. Also, $G$ contains $C_j$ for $j = 1, \ldots, r - 1$. Since all generators given in Theorem 2.1 are contained in the group $G$. We conclude that $G = T_g$. \qed
Let \( g = 2r + 1 \) and consider the surface \( N_g \), where \( g \)-crosscaps are distributed on \( S^2 \) as in Figure 16. First, we introduce a reflection \( \beta \) on \( N_g \) in the \( xy \)-plane such that

- \( \beta(a_1) = f \),
- \( \beta(x_2) = x_3, \beta(x_4) = x_5 \) and
- \( \beta(x_1) = x_1, \beta(x_i) = x_i \) for \( i = 6, 7, \ldots, g \).

The involution \( \beta \) reverses the orientation of a neighbourhood of a two-sided simple closed curve. It satisfies \( D(\beta) = 1 \) and hence \( \beta \) is an element of \( T_g \).

For the remaining generators of the following theorem we refer to Figures 15 and 16.

**Theorem 4.2.** For \( g = 4k + 1 \) and \( k \geq 3 \), the twist subgroup \( T_g \) is generated by the four involutions \( \rho_1, \rho_2, \rho_1 A_2 C_{\frac{r}{2}} B_{\frac{r+4}{2}} C_{\frac{r+6}{2}} \) and \( \beta \), where \( r = 2k \).

**Proof.** Consider the surface \( N_g \) as in Figure 15. The involution \( \rho_1 \) satisfies

\[
\rho_1(a_2) = a_2, \rho_1(b_{\frac{r+4}{2}}) = b_{\frac{r+4}{2}} \quad \text{and} \quad \rho_1(c_{\frac{r}{2}}) = c_{\frac{r+6}{2}}.
\]

Since \( \rho_1 \) reverses the orientation of a neighbourhood of a two-sided simple closed curve, we have

- \( \rho_1 A_2 \rho_1 = A_2^{-1} \),
- \( \rho_1 B_{\frac{r+4}{2}} \rho_1 = B_{\frac{r+4}{2}}^{-1} \) and
- \( \rho_1 C_{\frac{r}{2}} \rho_1 = C_{\frac{r+6}{2}}^{-1} \).

It can be shown that \( \rho_1 A_2 C_{\frac{r}{2}} B_{\frac{r+4}{2}} C_{\frac{r+6}{2}} \) is an involution. Let \( H \) be the subgroup of Mod(\( N_g \)) generated by the set

\[
\{ \rho_1, \rho_2, \rho_1 A_2 C_{\frac{r}{2}} B_{\frac{r+4}{2}} C_{\frac{r+6}{2}}, \beta \}.
\]
Observe that \( R = \rho_1 \rho_2 \in K \). By the proof of Theorem 3.5, the elements \( A_1 A_2^{-1}, B_1 B_2^{-1} \) and \( C_1 C_2^{-1} \) belong to \( H \). Since \( A_1 \beta(a_1) = e \), the element \( E \) is in \( H \). We conclude that \( T_\gamma = H \) by Theorem 4.1. \( \square \)

Although we give a generating set of 4 involutions, for completeness of the applications of our method, first we give the following theorem.

**Theorem 4.3.** For \( g = 4k + 1 \) and \( k \geq 1 \), the twist subgroup \( T_\gamma \) is generated by the five involutions \( \tau_1, \tau_2, \tau_1 \tau_2 \tau_1 A_2, \tau_2 A_1 \) and \( \beta \).

**Proof.** Let \( K \) be the subgroup of \( T_\gamma \) generated by the set

\[ \{ \tau_1, \tau_2, \tau_1 \tau_2 \tau_1 A_2, \tau_2 A_1, \beta \} \]

Note that the rotation \( T = \tau_1 \tau_2 \) is contained in \( K \). It follows from

- \( A_1 = \tau_2 \tau_2 A_1 \) and
- \( A_2 = (\tau_1 \tau_2) \tau_1 (\tau_1 \tau_2 \tau_1) A_2 \)

that the elements \( A_1 \) and \( A_2 \) are in \( K \). By conjugating \( A_1 \) with powers of \( T \), \( T_\gamma \) contains the elements \( B_i \) and \( C_i \). Moreover, it follows from \( \beta(a_1) = f \) that the element \( F \) is in \( K \). Since \( A_1(f) = e \), we get \( E \in K \). This finishes the proof by Theorem 2.1. \( \square \)

In the next theorem, we present four involutions to generate particularly \( T_5 \) and \( T_6 \). This completes the case \( g = 4k + 1 \) and \( k \geq 1 \). First, recall that \( A_1, A_2, B_1, B_2, C_1 \) and \( E \) generate \( T_5 \) and \( A_1, A_2, B_1, B_2, B_3, B_4, C_1, C_2, C_3 \) and \( E \) generate \( T_6 \). We use the following three involutions \( \gamma, S\gamma \) and \( S^{2k-2}(S\gamma)S^{2-2k} A_2 \) of the generating set given in [18, Theorem 5]. The involution \( \gamma \) is defined as the reflection in the \( xz \)-plane where the crosscaps are distributed along the equator on \( S^2 \). The map \( S \) is defined as the composition \( B_{2k} C_{2k-1} B_{2k-1} \cdots C_1 B_1 A_1 \). Note that \( D(\gamma) = D(S\gamma) = D(S^{2k-2}(S\gamma)S^{2-2k} A_2) = 1 \).

**Theorem 4.4.** The twist subgroups \( T_5 \) and \( T_6 \) can be generated by the involutions \( \gamma, S\gamma, S^{2k-2}(S\gamma)S^{2-2k} A_2 \) and \( \beta \) for \( k = 1, 2 \).

**Proof.** The generator \( A_1 \) can be obtained by \( S \) and \( A_2 \) [7, Theorem 5]. By conjugating with powers of \( S \), it is easy to see that the elements \( B_i \) and \( C_i \) belong to \( T_\gamma \). Also, the generator \( E \) is contained in \( T_\gamma \) since \( A_1 \beta(a_1) = e \). \( \square \)

Now, let \( g = 4k + 3 \) and consider \( N_g \), where \( g \)-crosscaps are distributed over \( S^2 \) as in Figure 17. The surface \( N_g \) is symmetrical in the \( xy \)-plane. Let \( \mu \) be the reflection in the \( xy \)-plane. Note that the linear map associated to the involution \( \mu \) satisfies \( D(\mu) = 1 \) if \( k \geq 2 \). Therefore, the involution \( \mu \) is in \( T_\gamma \) for \( k \geq 2 \).
Theorem 4.5. For \( g = 4k + 3 \) and \( k \geq 2 \), the twist subgroup \( T_g \) is generated by the five involutions \( \tau_1, \tau_2, \tau_1\tau_2A_2, \tau_2A_1 \) and \( \mu \).

Proof. Let \( K \) be the subgroup of \( T_g \) generated by the set
\[
\{\tau_1, \tau_2, \tau_1\tau_2A_2, \tau_2A_1, \mu\}.
\]
Note that the rotation \( T = \tau_1\tau_2 \) is contained in \( K \). It follows from
\begin{itemize}
  \item \( A_1 = \tau_2(\tau_2A_1) \)
  \item \( A_2 = (\tau_1\tau_2\tau_1)(\tau_1\tau_2\tau_1A_2) \)
\end{itemize}
that the elements \( A_1 \) and \( A_2 \) are in \( K \). By conjugating \( A_1 \) with powers of \( T \), \( T_g \) contains the elements \( B_i \) for \( i = 1, \ldots, 2k \) and \( C_j \) for \( j = 1, \ldots, 2k - 1 \).
Let \( T(b_{2k}) = x \) and \( \mu(x) = y \). Then the elements \( X \) and \( Y \) are contained in \( K \) by the fact that \( B_{2k} \) is in \( K \).
It follows from
\begin{itemize}
  \item \( T^{-1}(y) = c_{2k} \)
  \item \( \mu(b_{2k}) = b_{2k+1} \)
\end{itemize}
that \( C_{2k} \) and \( B_{2k+1} \) are contained in \( K \). This completes the proof by Theorem 2.1.

For the surface \( N_7 \), we introduce two involutions, \( \sigma_1 \) and \( \sigma_2 \), shown in Figure 18 and 19. In these figures, the surface is symmetric with respect to the \( xy \)-plane. Both \( \sigma_1 \) and \( \sigma_2 \) are reflections in the \( xy \)-plane and \( D(\sigma_1) = D(\sigma_2) = 1 \). Hence, both \( \sigma_1 \) and \( \sigma_2 \) belong to \( T_7 \). For the remaining generators in the following theorem we refer to the model on the right hand side of Figure 14.

Theorem 4.6. The twist subgroup \( T_7 \) is generated by the six involutions \( \tau_1, \tau_2, \tau_1\tau_2\tau_1A_2, \tau_2A_1, \sigma_1 \) and \( \sigma_2 \).
Figure 18. The involution $\sigma_1$ for $g = 7$.

Figure 19. The involution $\sigma_2$ for $g = 7$.

Proof. Let $K$ be the subgroup of $\mathcal{T}_g$ generated by the set
\[ \{ \tau_1, \tau_2, \tau_2 \tau_1 A_2, \tau_2 A_1, \sigma_1, \sigma_2 \}. \]
Note that the rotation $T = \tau_1 \tau_2$ is contained in $K$. It follows from
- $A_1 = \tau_2 (\tau_2 A_1)$ and
- $A_2 = (\tau_1 \tau_2 \tau_1)(\tau_1 \tau_2 \tau_1 A_2)$
that the elements $A_1$ and $A_2$ are in $K$. By conjugating $A_1$ with powers of $T$, $\mathcal{T}_g$ contains the elements $B_1, C_1$ and $B_2$.

Let $T(b_2) = x$ and $\sigma_2(x) = y$. Then the elements $X$ and $Y$ are contained in $K$. It follows from
- $T^{-1}(y) = c_2$,
- $\sigma_2(b_2) = b_3$
that $C_2$ and $B_3$ are contained in $K$. Moreover, since $A_1 \sigma_1(a_1) = e$, $E \in K$, which completes the proof by Theorem 2.1. \qed
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