Electroweak Sphalerons with Spin and Charge

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I. INTRODUCTION

Like magnetic monopoles, sphalerons arise as classical solutions in theories which undergo spontaneous symmetry breaking. The Higgs field then defines a map from the two-sphere at infinity to the vacuum manifold, associated with the symmetry breaking. But while this mapping is non-trivial for monopoles, it is trivial for sphalerons. Consequently, lacking a conserved topological charge, sphalerons represent unstable solutions associated in a semiclassical approximation not with particles but with transition rates.

In the standard model the electroweak sector gives rise to sphalerons [1, 2, 3]. The static sphaleron solution of Weinberg-Salam theory represents the top of the energy barrier between topologically inequivalent vacua. Since the standard model does not absolutely conserve baryon number [4], at finite temperature baryon number violating processes can arise because of thermal fluctuations of the fields large enough to overcome the energy barrier between distinct vacua. The rate for baryon number violating processes is then largely determined by a Boltzmann factor, containing the height of the barrier at a given temperature [5, 6, 7, 8]. The sphaleron itself carries baryon number $Q_B = 1/2$ [2].

At finite weak mixing angle the static electroweak sphaleron possesses a large magnetic moment but does not carry electric charge. Adding electric charge to the configuration should lead to a non-vanishing Poynting vector and thus a finite angular momentum density of the system, and consequently give rise to a branch of spinning electrically charged sphalerons. Carrying non-vanishing baryon number as well, these configurations would then also entail baryon number violating processes.

Here we explicitly construct this branch of spinning electrically charged sphalerons, and study the dependence on the weak mixing angle. We show that the angular momentum and the electric charge of the solutions are proportional [9]. Electroweak sphalerons thus present the first spinning configuration, based on non-Abelian gauge fields, which corresponds to a single localized lump. Previously, only composite configurations such as monopole-antimonopole pairs were known to rotate [10, 11], whereas magnetic monopoles or dyons were shown to exclude slow rotation [12, 13].

In section 2 we present the action, the Ansatz and the boundary conditions. In section 3 we consider the relevant physical properties and, in particular, derive the linear relation between angular momentum and electric charge. We present and discuss the numerical results in section 4.

II. ACTION AND ANSATZ

We consider the bosonic sector of Weinberg-Salam theory

$$\mathcal{L} = -\frac{1}{2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4} f_{\mu\nu}f^{\mu\nu} - (D_\mu \Phi)^\dagger (D^\mu \Phi) - \lambda (\Phi^\dagger \Phi - v^2)^2$$ (1)

with su(2) field strength tensor

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + ig[V_\mu, V_\nu],$$ (2)

su(2) gauge potential $V_\mu = V^a_\mu \tau_a/2$, u(1) field strength tensor

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$ (3)

and covariant derivative of the Higgs field

$$D_\mu \Phi = \left( \partial_\mu + igV_\mu + i\frac{g'}{2} A_\mu \right) \Phi,$$ (4)
where $g$ and $g'$ denote the $SU(2)$ and $U(1)$ gauge coupling constants, respectively, $\lambda$ denotes the strength of the Higgs self-interaction and $\nu$ the vacuum expectation value of the Higgs field.

The gauge symmetry is spontaneously broken due to the non-vanishing vacuum expectation value of the Higgs field

$$\langle \Phi \rangle = \frac{\nu}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (\Phi)$$

leading to the boson masses

$$M_W = \frac{1}{2} g \nu, \quad M_Z = \frac{1}{2} \sqrt{(g^2 + g'^2)} \nu, \quad M_H = \nu \sqrt{2 \lambda}. \quad (6)$$

tan $\theta_w = g'/g$ determines the weak mixing angle $\theta_w$, defining the electric charge $e = g \sin \theta_w$.

To obtain stationary rotating solutions of the bosonic sector of Weinberg-Salam theory, we employ the time-independent axially symmetric Ansatz

$$V_\mu dx^\mu = \left( B_1 \frac{\tau_r}{2g} + B_2 \frac{\tau_\theta}{2g} \right) dt - \sin \theta \left( H_4 \frac{\tau_r}{2g} + H_4 \frac{\tau_\theta}{2g} \right) d\varphi + \left( \frac{H_1}{r} dr + (1 - H_2) d\theta \right) \frac{\tau_\varphi}{2g}, \quad (7)$$

$$A_\mu dx^\mu = (a_1 dt + a_2 \sin^2 \theta d\varphi) / g', \quad (8)$$

and

$$\Phi = i \phi (\cos \psi \tau_r + \sin \psi \tau_\theta) \frac{\nu}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (9)$$

where $\tau_r$ denotes the cartesian product of the Pauli matrices and the spherical unit vector $e_r$, etc. The ten functions $B_1, B_2, H_1, \ldots, H_4, a_1, a_2, \phi, \psi$ depend on $r$ and $\theta$. With this Ansatz the full set of field equations reduces to a system of ten coupled partial differential equations in the independent variables $r$ and $\theta$. A residual $U(1)$ gauge degree of freedom is fixed by the condition $r \partial_r H_1 - \partial_\theta H_2 = 0 \quad (10)$.

Requiring regularity and finite energy, we impose the boundary conditions

$$r = 0: \quad B_1 \sin \theta + B_2 \cos \theta = 0, \quad \partial_r (B_1 \cos \theta - B_2 \sin \theta) = 0, \quad H_1 = H_3 = H_4 = 0, \quad H_2 = 1, \quad \partial_r a_1 = 0, \quad a_2 = 0, \quad \phi = 0, \quad \partial_r \psi = 0$$

$$r \rightarrow \infty: \quad B_1 = \gamma \cos \theta, \quad B_2 = \gamma \sin \theta, \quad H_1 = H_3 = 0, \quad H_2 = -1, \quad H_4 = 2, \quad a_1 = \gamma, \quad a_2 = 0, \quad \phi = 1, \quad \psi = 0$$

$$\theta = 0: \quad \partial_\theta B_1 = 0, \quad B_2 = 0, \quad H_1 = H_3 = 0, \quad \partial_\theta H_2 = \partial_\theta H_4 = 0, \quad \partial_\theta a_1 = \partial_\theta a_2 = 0, \quad \partial_\theta \phi = 0, \quad \psi = 0,$$

where the latter hold also at $\theta = \pi/2$, except for $B_1 = 0$ and $\partial_\theta B_2 = 0$. Here $\gamma$ is a constant.

### III. SPHALERON PROPERTIES

We now address the global charges of the sphaleron solution, its mass, angular momentum, electric charge, and baryon number. The mass $M$ and angular momentum $J$ are defined in terms of volume integrals of the respective components of the energy-momentum tensor. The mass is obtained from

$$M = \int T_t^t d^3r, \quad (11)$$

while the angular momentum

$$J = \int T_\varphi^t d^3r = \int \left[ 2 \text{Tr} \left( F^{t \mu} F_{\mu \nu} \right) + f^{t \mu} f_{\varphi \mu} + 2 \left( D^t \Phi \right) \left( D_\varphi \Phi \right) \right] d^3r \quad (12)$$

can be reexpressed with help of the equations of motion and the symmetry properties of the Ansatz $[9, 13, 14]$ as a surface integral at spatial infinity

$$J = \int_{S^2} \left\{ 2 \text{Tr} \left( \left( V_\varphi - \frac{\tau_\varphi}{2g} \right) F^{r t} \right) + \left( A_\varphi - \frac{1}{g'} \right) f^{r t} \right\} r^2 \sin \theta d\theta d\varphi. \quad (13)$$
The power law fall-off of the $U(1)$ field of a charged solution allows for a finite flux integral at infinity and thus a finite angular momentum. Insertion of the asymptotic expansion for the $U(1)$ field

$$a_1 = \gamma + \frac{Q}{r} + O \left( \frac{1}{r^2} \right),
$$

$$a_2 = -\frac{\mu}{r} + O \left( \frac{1}{r^2} \right),$$

(14)

and of the analogous expansion for the $SU(2)$ fields then yields for the angular momentum

$$\frac{J}{4\pi} = \frac{Q}{g^2} + \frac{Q}{g'^2} = \frac{Q}{g^2 \sin^2 \theta_w} = \frac{Q}{e^2},$$

(15)

The field strength tensor $F_{\mu \nu}$ of the electromagnetic field $A_{\mu}$,

$$A_{\mu} = \sin \theta_w V_{\mu} + \cos \theta_w A_{\mu},$$

(16)

as given in a gauge where the Higgs field asymptotically tends to Eq. (5), then defines the electric charge $Q$

$$Q = \frac{1}{4\pi} \int_{S_2}^* F_{\theta \varphi} d\theta d\varphi = \frac{\sin \theta_w Q}{g} + \frac{\cos \theta_w Q}{g'} = \frac{Q}{e},$$

(17)

where the integral is evaluated at spatial infinity. Comparison of Eqs. (15) and (17) then yields a linear relation between the angular momentum $J$ and the electric charge $Q$

$$J = \frac{4\pi Q}{e}. \tag{18}$$

The magnetic moment $\mu$ is obtained from the asymptotic expansion Eq. (14), analogously to the electric charge.

Addressing finally the baryon number $Q_B$, its rate of change is given by

$$\frac{dQ_B}{dt} = \int d^3r \partial_j j_B^0 = \int d^3r \left[ \nabla \cdot \tilde{j}_B + \frac{1}{32\pi^2} \epsilon^{\mu \nu \rho \sigma} \left\{ g^2 \text{Tr} (F_{\mu \nu} F_{\rho \sigma}) + \frac{1}{2} g'^2 f_{\mu \nu} f_{\rho \sigma} \right\} \right]. \tag{19}$$

Starting at time $t = -\infty$ at the vacuum with $Q_B = 0$, one obtains the baryon number of a sphaleron solution at time $t = t_0 \tag{20}$,

$$Q_B = \int_{-\infty}^{t_0} dt \int_S \tilde{K} \cdot \tilde{S} + \int_{t=t_0} d^3r K^0,$$

(20)

where the $\nabla \cdot \tilde{j}_B$ term is neglected, and the anomaly term is reexpressed in terms of the Chern-Simons current

$$K^\mu = \frac{1}{16\pi^2} \epsilon^{\mu \nu \rho \sigma} \left\{ g^2 \text{Tr} \left( F_{\nu \rho} V_{\sigma} - \frac{2}{3} ig V_{\nu} V_{\rho} V_{\sigma} \right) + \frac{1}{2} g'^2 f_{\nu \rho} A_{\sigma} \right\}. \tag{21}$$

In a gauge, where

$$V_{\mu} \rightarrow \frac{i}{g} \partial_{\mu} \hat{U} \hat{U}^\dagger, \quad \hat{U}(\infty) = 1, \tag{22}$$

$\tilde{K}$ vanishes at infinity. Subject to the above ansatz and boundary conditions the baryon charge of the sphaleron solution $\hat{R}$ is then

$$Q_B = \int_{t=t_0} d^3r K^0 = \frac{1}{2}. \tag{23}$$
IV. RESULTS AND DISCUSSION

We solve the set of ten coupled non-linear elliptic partial differential equations numerically [16], subject to the above boundary conditions in compactified dimensionless coordinates, \( x = \hat{r}/(1 + \hat{r}) \), with \( \hat{r} = g vr \).

Employing the physical value for the mixing angle \( \theta_w \), and increasing the asymptotic value of the \( U(1) \) field \( \hat{\gamma} = \gamma/gv \), a branch of rotating charged sphalerons emerges smoothly from the static sphaleron. The branch ends when a limiting value \( \hat{\gamma}_{\text{max}} \) is reached [17]. Here some of the gauge field functions no longer decay exponentially, precluding localized solutions for larger values of \( \hat{\gamma} \). At \( \hat{\gamma}_{\text{max}} \) the solution has maximal spin, charge and mass. The dependence of the angular momentum \( J \) on \( \gamma \) is illustrated in Fig. 1. The figure also demonstrates the linear relation (18) between the charge \( Q \) and the angular momentum \( J \). In Fig. 2 we exhibit the dependence of the mass \( M \) and the magnetic moment \( \mu \) on the angular momentum \( J \).

![Figure 1](image1.png)

Figure 1: (a) The asymptotic value of the \( U(1) \) field \( \hat{\gamma} = \gamma/gv \) and (b) the \( U(1) \) charge \( Q \) versus the angular momentum \( J \) \((J_0 = 4\pi/g^2)\) for several values of the mixing angle \( \theta_w \). The asterisk marks the extrapolated maximal value for \( J \) for \( \theta_w = 0 \).

![Figure 2](image2.png)

Figure 2: Same as Fig. 1 for (a) the mass \( M \) \((M_0 = v/g)\) and (b) the magnetic moment \( \mu \) \((\mu_0 = 2\sin^2 \theta_w/gv)\).

To gain further understanding of these rotating solutions, we now address the mixing angle dependence, varying \( \theta_w \) in the range \( 0 \leq \theta_w \leq \pi/2 \). We recall that, as \( \theta_w \) is increased, the mass of the static sphaleron decreases, and its energy density, being spherical at \( \theta_w = 0 \), becomes increasingly prolate [18].

Choosing a fixed value of \( \theta_w \) beyond the physical value, the respective branch of rotating charged solutions exhibits a slower increase of angular momentum \( J \) with \( U(1) \) parameter \( \gamma \), and thus a smaller maximal value of \( J \). In the limit \( \theta_w \to \pi/2 \), the relation between \( J \) and \( \gamma \) becomes almost linear, as seen in Fig. 1 for \( \theta_w = 0.49\pi \). Note, that the limit cannot be obtained numerically.
Let us now consider smaller values of $\theta_w$, and in particular the limit $\theta_w \to 0$. In this limit the sphaleron was expected not to rotate \cite{13}. The numerical data, as shown in Fig. 1, however indicate the presence of a rotating branch of solutions in this limit. In fact, for a given value of $\gamma$, the angular momentum increases with decreasing $\theta_w$, assuming its largest value in the limit. The charge on the other hand, decreases to zero in this limit, thus provoking the question as to what then allows for the rotation?

Analysis then shows that the $U(1)$ field becomes trivial in the limit $\theta_w \to 0$, except that $a_1$ assumes a finite constant value, $a_1 = \gamma$. Thus $\gamma$ enters the covariant derivative of the Higgs field, and provides non-trivial boundary conditions for the time-component of the $SU(2)$ gauge field, thus giving rise to a non-vanishing $SU(2)$ Poynting vector and, consequently, angular momentum.

These limiting solutions can also be considered from an alternative point of view. Giving the Higgs field a time-dependent phase, as discussed in \cite{19}, $\gamma$ enters as a frequency parameter, analogous as in non-topological solitons. This permits an otherwise identical set of solutions with the same $\gamma$-dependence of the angular momentum and the mass.

Let us finally consider the effect of spin and charge on the energy density and thus the shape of the configuration. The effect of charge is to spread the energy density further out, while reducing its central magnitude, as also seen in dyons, for instance. This effect is quite pronounced for large charge. On the other hand, the expected effect of angular momentum, i.e., a relative centrifugal flattening of the shape of the energy density, is barely noticable even at maximal spin. In particular, for larger $\theta_w$, the prolate deformation of the solutions is retained, and only marginally reduced in the presence of rotation.

Concluding, we have shown that the static electroweak sphaleron gives rise to a branch of rotating electrically charged solutions, whose angular momentum and charge are proportional. Carrying baryon number $Q_B = 1/2$, they can be associated with baryon number violating processes. Their presence may thus affect the calculations of the generation of the baryon number asymmetry of the universe within the standard model \cite{5, 6, 7}. The inclusion of rotation and charge in more general solutions of Weinberg-Salam theory, such as multisphalerons or sphaleron-antisphaleron systems \cite{20}, is currently under study. Further insight is expected from the study of the fermion modes in the background of these solutions \cite{21}.

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