Structural Controllability of Networked Relative Coupling Systems under Fixed and Switching Topologies

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Abstract—This paper studies controllability of networked systems in which subsystems are of general high-order linear dynamics and coupled through relative state variables, from a structure perspective. The purpose is to search conditions for subsystem dynamics and subsystem interaction topologies, under which there exists a set of weights for the interaction links such that the associated networked system can be controllable (i.e., structural controllability). Three types of subsystem interaction fashions are considered, which are 1) each subsystem is single-input-single-output (SISO), 2) each subsystem is multiple-input-multiple-output (MIMO), and the interaction weights for different channels between two subsystems can be different, and 3) each subsystem is MIMO but the interaction weights between two subsystems are the same. Necessary and/or sufficient conditions for structural controllability are given. These conditions indicate that, under certain conditions on the subsystem dynamics, the whole system is structurally controllable, if and only if the network topology is globally input-reachable. Finally, these results are extended to the case where subsystem dynamics are fixed but the interaction topologies are switching. A promising point of the structure analysis taken in this paper is that, it can handle certain subsystem heterogeneities, which are illustrated by some practical systems, including the liquid-level systems, the power networks and the mechanical systems.

Index Terms—Relative coupling, structural controllability, networked systems, switching topologies, fixed mode

I. INTRODUCTION

Relative sensing/coupling is a ubiquitous mechanism existing in many real-world dynamic systems, ranging from natural systems to human-made ones. For example, in a thermal system, the heat propagates from the hotter spot to the colder one in a rate proportional to the relative temperature [1]. Similar phenomena occur in the liquid flow systems [1]. In highway traffic, drivers make decisions whether to accelerate or decelerate depending on the relative distances between themselves and their proceeding vehicles [2]. In addition, many human-made complex networked systems are embedded with relative sensing/controlling/measuring to coordinate subsystems to accomplish certain tasks or function normally, such as consensus based unmanned aerial vehicle formation systems with static feedback [3], multi-agent systems (MASs) via the nearest neighboring rule [4], extremely large telescope control systems via distributed relative sensing [5], etc.

With regard to networked relative coupling systems, there are many scientific topics that have been devoted to by the control community, including consensus [3, 4], synchronization [6, 7], stability [8], etc. Among them, a fairly fundamental property, controllability/observability, has also attracted many researchers’ interest. As is known to all, controllability of a networked system means that one can actuate partial nodes/subsystems to drive the high-dimensional states of the whole system in the corresponding state-space arbitrarily. This property is not only theoretically significant, as itself is often related to both algebraical and topological properties of the networked systems [9, 10], but also relevant to other important system performances, such as stabilization, existence of an optimal controller [11], designing formation protocols [4], etc.

We mention here the relevant literature from two aspects. One is controllability of multi-agent systems, mainly focusing on controllability of a system with graph Laplacian related system matrices. The relative coupling mechanism naturally induces the graph Laplacian. Hence, many works study controllability of MASs from the perspective of graphs or spectra of the Laplacian matrices [4, 9, 12, 13]. Particularly, controllability of MASs running the nearest neighboring rule is studied in [9] using the equitable partitions from graph theory. It is shown that controllability fails if certain symmetries exist in the network topology. Some graph-theoretic characterizations for controllability of Laplacian-based leader-follower systems are reported in [12], where graphs are classified into three classes, namely, the classes of essentially controllable, completely uncontrollable, and conditionally controllable graphs. The authors in [13] study controllability of relative coupling networks using the almost equitable partitions and give some lower and upper bounds for the controllable subspaces. However, most of these works are built upon networked systems with first-order subsystems.

The other aspect is controllability of networked systems where the coupling mechanisms are general. This topic is not new [14], but seems to renew much research interest since [10], which studies controllability of complex networks using the matching theory and the cavity methods from statistical physics. Apart from studying networks of first-order systems, significant efforts have been paid to networks of general high-order linear dynamics. Relevant works include [15] on networks of networks, [16, 18] on networked identical systems, and [19–21] on networked systems with general heterogeneous subsystems. These works are built upon completely deterministic system models and seek to find relations between system controllability and network topology as well as subsystem dynamics. It is very recent that controllability of a networked system is considered in [22–23] in which the subsystem dynamics are partially or completely fixed but the subsystem interaction weights can take values independently. More specially, in [22] general heterogeneous subsystem dynamics with indeterminate parameters are considered while in [23] identical subsystems are considered. Note that the independent subsystem interaction weight assumption adopted.
therein prevents their results being directly applied to the networked relative coupling systems, as there exist zero row sum constraints in the associated Laplacian matrices.

In this paper, we study controllability of networked relative coupling systems in which subsystems are of general high-order linear dynamics, from a structure perspective. Our purpose is to search conditions for subsystem dynamics and network topologies, under which there exists a set of weights for the interaction links (interaction weights) such that the associated system is controllable (i.e., structural controllability). Three types of subsystem interaction fashions are considered, including 1) each subsystem is single-input-single-output (SISO), 2) each subsystem is multiple-input-multiple-output (MIMO), but the interaction channels between two subsystems can be differentially weighted, and 3) each subsystem is MIMO with equally weighted interaction channels. Our main contributions are as follows. For each of the three types of interaction fashions, we demonstrate that, under certain conditions on the subsystem dynamics, the whole system is structurally controllable, if and only if the network topology satisfies certain connectivity properties (i.e., input-reachability of every vertex). These results generalize [24–27] where the consensus based networks of single integrators are considered, in two aspects: each subsystem is of general high-order dynamics, and the interaction fashions are more general. Particularly, for the first interaction fashion, a weight design procedure is also given to construct a controllable networked system, and for the second one, we borrow some concepts from decentralized stabilization theory [28] to characterize subsystem dynamics. We then extend our results to the case where subsystem dynamics are fixed but the network topologies are switching. Some sufficient conditions for structural controllability are obtained based on graph union. Although our results are derived upon the condition that each subsystem has identical dynamics, the structure analysis taken here can handle certain subsystem heterogeneities (Section VII), which are illustrated by some typical practical systems, including the liquid-level systems, the power networks and the mechanical systems.

The rest of this paper is organized as follows. Section II gives the problem formulation. Sections III IV and V deal with structural controllability with three different subsystem interaction fashions, respectively. The extension of some results to the case with switching topologies is given in Section VII. Extensions with subsystem heterogeneities are provided in Section VII with three practical examples given in Section VII. Section IX ends this paper with some concluding remarks.

Notations: Given a directed graph \( G \), let \( V(G) \) denote the set of vertices of \( G \), and \( E(G) \) the set of edges of \( G \). For a set, \( \cdot \) denotes its cardinality. A matrix \( L \) is also denoted by \( L = [l_{ij}] \), which means \( l_{ij} \) is the entry in the \( i \)th row and \( j \)th column of \( L \). By \( \sigma(M) \) we denote the set of eigenvalues of the square matrix \( M \), and \( \text{diag}\{X_i|_{i=1}^n\} \) the block diagonal matrix whose \( i \)th diagonal block is \( X_i \).

II. Problem Formulation

Consider a networked system consisting of \( N \) subsystems. Let \( G_{sys} = (V_{sys}, E_{sys}) \) be the graph without self-loops describing the subsystem interaction topology (i.e., the network topology), with \( V_{sys} = \{1, ..., N\} \), and \( (i, j) \in E_{sys} \) if the \( j \)th subsystem is directly influenced by the \( i \)th one. The \( i \)th subsystem, denoted by \( S_i \), \( i \in \{1, ..., N\} \), has the following dynamics

\[
\dot{x}_i(t) = Ax_i(t) + Bv_i(t)
\]

where \( A \in \mathbb{R}^{n \times n} \), \( B = [b_1, ..., b_r] \) with \( b_j \in \mathbb{R}^{n \times 1} \) for \( j = 1, ..., r \), \( x_i(t) \in \mathbb{R}^n \) is the state vector, \( v_i(t) \in \mathbb{R}^r \) is the input injected to the \( i \)th subsystem. The input \( v_i(t) \) may contain both subsystem interactions (i.e., internal inputs) and the external control inputs. In this regard, the \( k \)th component of \( v_i(t) \), denoted by \( v_{ik}(t) \), is expressed as

\[
v_{ik}(t) = \sum_{j=1,j\neq i}^N l_{ij}^k c_k(x_j(t) - x_i(t)) + \delta_i u_{ik}(t), \forall k \in \{1, ..., r\}.
\]

Here, \( u_{ik}(t) \) is the \( k \)th component of the external input \( u_i(t) \in \mathbb{R}^r \) injected to \( S_i \), i.e., \( u_i(t) = [u_{i1}(t), ..., u_{ir}(t)]^T \), \( \delta_i \in \{0, 1\} \) in which \( \delta_i = 1 \) means that \( S_i \) is directly controlled by the external input \( u_i(t) \), and \( \delta_i = 0 \) means the contrary, \( c_k \in \mathbb{R}^{1 \times n} \) is the \( k \)th output vector which outputs the \( k \)th linear combination of the relative states \( (x_j(t) - x_i(t)) \), and \( l_{ij}^k \in \mathbb{R} \) is the weight imposed on \( c_k(x_j(t) - x_i(t)) \). Define \( C = (c_1^T, ..., c_r^T)^T \). In addition, for each \( k \in \{1, ..., r\} \), \( l_{ij}^k \neq 0 \) only if \( (j, i) \in E_{sys} (i \neq j) \).

Let \( l_{ij}^k = -\sum_{j=1,j\neq i}^N l_{ij}^k \), and \( L_k = [-l_{ij}^k] \). Then, \( L_1, ..., L_r \) are all Laplacian matrices associated with the subsystem interaction graph \( G_{sys} \). Let \( \Delta = \text{diag}\{\delta_i|_{i=1}^N\} \), \( u(t) = [u_1^T(t), ..., u_r^T(t)]^T \), \( x(t) = [x_1^T, ..., x_r^T]^T \). The lumped state-space representation of the considered networked system (1-2) is

\[
\dot{x}(t) = A_{sys}x(t) + B_{sys}u(t),
\]

with

\[
A_{sys} = I \otimes A - \sum_{k=1}^r L_k \otimes (b_k c_k), B_{sys} = \Delta \otimes B,
\]

where \( \otimes \) denotes the Kronecker product. Through this paper, without losing of generality, assume that \( b_i \neq 0 \) and \( c_i \neq 0 \) for each \( i = 1, ..., r \).

There are two features in the above networked system model. First, subsystems are coupled through relative state variables (i.e., \( x_j(t) - x_i(t) \)). This captures the dynamics of a large class of natural systems and human-made ones, including the interacted liquid systems [1], power networks [29], car-following behaviours in highways [2], viral infection or opinion propagation in social networks [30], consensus based unmanned aerial vehicle formation systems with static feedback [3], MASs via the nearest neighboring rule [4], extremely large telescope systems via distributed relative sensing [5], etc. Second, interactions among subsystems are in the form of multi-input multi-output (MIMO), and the relative state variables can be transmitted through multiple channels \( (v_i \in \mathbb{R}^r, r > 1) \). Each channel can have a weight (i.e., \( l_{ij}^k \))
not necessarily equal to other ones. This makes the subsystem interaction fashion more general than that of most networked MIMO system models in the existing literature [13-16-18], and enable to describe a larger class of practical systems (such as the mechanical system shown in Section VII-C which cannot be covered by the adopted model in 13-16-18). In addition, in (4) both the subsystem interactions and the external inputs involve the same input matrix \( B \). This is motivated by the observation that in many physical systems, the subsystem interactions and the external inputs are mixed up to affect one subsystem through the same channels, as shown by the three real-world examples in Section VIII. Similar settings are also adopted in [18-23].

The main problem considered in this paper is formulated as follows.

**Problem 1:** Given \( A, B, C, \Delta \) and \( G_{sys} \), verify whether there is a set of values for \( \{ l^{(k)}_{ij} \}_{(i,j) \in E_{sys}} \) with \( k = 1, \ldots, r \), such that the associated system (1)-(2) is controllable.

It can be easily verified that, if the answer to Problem 1 is Yes, then for almost all values for \( \{ l^{(k)}_{ij} \}_{(i,j) \in E_{sys}} \) with \( k = 1, \ldots, r \), the corresponding system \( (A_{sys}, B_{sys}) \) is controllable. In other words, controllability is a generic property for the pair \( (A_{sys}, B_{sys}) \) [31]. Inheriting the terminology of [32], we say \( (A_{sys}, B_{sys}) \) is structurally controllable, if the answer to Problem 1 is Yes, otherwise \( (A_{sys}, B_{sys}) \) is structurally uncontrollable. Note that in \( (A_{sys}, B_{sys}) \), there are some nonzero constants like \( A, B, C \), as well as zero sum constraints imposed on the Laplacian matrix \( L_k \) for each \( k = 1, \ldots, r \). Hence, the traditional Lin’s structural controllability theory [32] cannot be directly adopted to Problem 1.

In this paper, arising from observations on some practical systems (such as systems illustrated in Section VIII), we will consider three types of subsystem interaction fashions depending on the subsystem inputs/outputs. They are the SISO fashion, i.e., \( r = 1 \) meaning that each subsystem is SISO, the MIMO via differentially weighted channels, where, \( r > 1 \) and \( L_1, \ldots, L_r \) can take independent matrix values, and the MIMO via equally weighted channels, where \( r > 1 \) and \( L_1 = \cdots = L_r \). Namely, in the latter two cases, each subsystem can be MIMO, and the difference between them lies in whether different internal input/output variables between two subsystems are weighted separately [2]. The motivation of the second case is that, for some practical systems, different internal outputs may represent different physical variables (even with different units), and thus may be transmitted by channels with different parameters (see Section VIII-C for example). See Fig. 1 for illustrations. While the first case is a special one of the second and three ones, necessary and sufficient conditions, as well as a constructive procedure for its weight assignment, are provided. The techniques deriving conditions for these three cases also differ from each other.

To deal with the above three cases, some universal definitions are made here. Let \( I_u = \{ i : \delta_i \neq 0 \} \) be the set of indices of subsystems that are directly influenced by external inputs,  \( \delta_i \), \( i \in I_u \)

\[ A_{sys} = I \otimes A - L \otimes (bc), \quad B_{sys} = \Delta \otimes b. \]  
(5)

### III. STRUCTURAL CONTROLLABILITY WITH SISO SUBSYSTEMS

In this section, we derive conditions for the system (1)-(2) to be structurally controllable when \( r = 1 \), i.e., each subsystem is of SISO linear dynamics. As a consequence of our derivations, a design procedure is also given to construct interaction weights for the considered networked systems to be controllable. Well-encountered examples of networked systems with relative coupling SISO subsystems include the power networks and the liquid-level systems given in Section VII.

We shall assume that \( r = 1 \) through this section. In this case, for notation simplicity, let \( c = C, b = B \) and \( L = L_1 = [-I] \).

Then parameters in the system (3) can be expressed as

\[ A_{sys} = I \otimes A - L \otimes (bc), \quad B_{sys} = \Delta \otimes b. \]  
(5)

### A. Necessary and Sufficient Conditions

We first give some necessary conditions for the system (3) with parameters in (5) to be structurally controllable. These results seem to be direct derivations of several recent works,
including \([16, 21]\). To avoid the trivial case where \(|\mathcal{I}_a| = N\), assume that \(|\mathcal{I}_a| < N\). \(\square\)

**Lemma 1:** Assume that \(r = 1\) and \(|\mathcal{I}_a| < N\). Then, the system \(\mathcal{G}\) is structurally controllable, only if

1. \((A, b)\) is controllable;
2. \((A, c)\) is observable.

**Proof:** Condition 1) is a direct derivation of Theorem 1 of \([21]\). Condition 2) is a direct derivation of Theorem 4 of \([16]\).

The following theorem says that the necessary conditions in Lemma \(\mathcal{G}\) are satisfied, structural controllability of the system \(\mathcal{G}\) is solely determined by the network topology.

**Theorem 1:** Suppose that \(r = 1\) and \(|\mathcal{I}_a| < N\). Then, the system \(\mathcal{G}\) is structurally controllable, if and only if

1. \((A, b)\) is controllable and \((A, c)\) is observable;
2. \(\mathcal{G}_\text{sys}\) is globally input-reachable.

The proof can be found in the next subsection, which is constructive and self-contained. The above theorem simply indicates that, the networked system with SISO subsystems is structurally controllable, if and only if each subsystem can receive signals from at least one external input (either directly or indirectly). This result generalizes those of \([24–27]\). To avoid the trivial case where \(c_0 \neq 0\) and \(c_0 \in \mathbb{C}^{1 \times 1}\), such that \(c_0(sI - A)^{-1}b \neq 0\).

**Proof:** We resort to the theory of output controllability (Section 9.6). From \([34]\), if \((A, b)\) is controllable, then \((I_n, A, b)\) is output controllable. This requires that, the rows of transfer function matrix \((sI - A)^{-1}b\) are linearly independent in the field of complex numbers. That is, there cannot exist a \(c_0 \neq 0\) such that \(c_0(sI - A)^{-1}b = 0\).

We now give the complete proof of Theorem 1.

**Proof of Theorem 1** (Only if part) The necessity part of Condition 1 follows from Lemma \(\mathcal{G}\) for the necessity of Condition 2, suppose there is one vertex in \(\mathcal{G}_\text{sys}\) which is not input-reachable. Then, there is a permutation matrix \(P\), such that

\[
PT = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}, \quad P\Delta = \begin{bmatrix} 0 & \Delta_2 \\ 0 & 0 \end{bmatrix},
\]

where \(L_{11}\) is a \(1 \times 1\) scalar, \(L_{22}\) is a \((N - 1) \times (N - 1)\) matrix, and \(\Delta_2\) is of \((N - 1) \times N\). Let \(\hat{P} = P \otimes \Delta_2\). Then, one has \(P\Delta A_{\text{sys}} \hat{P} = I_N \otimes A - (P\Delta PT) \otimes (bc), P\Delta B_{\text{sys}} = (P\Delta \Delta) \otimes b\).

It can be verified that \((P\Delta A_{\text{sys}} \hat{P}, P\Delta B_{\text{sys}})\) has the form

\[
\begin{bmatrix} A - L_{11}bc & 0 \\ -L_{21} & I_{N-1} \otimes A - L_{22} \otimes (bc) \end{bmatrix} \begin{bmatrix} 0 \\ \Delta_2 \otimes b \end{bmatrix},
\]

which immediately means that \((A_{\text{sys}}, B_{\text{sys}})\) is not controllable for arbitrary choices of \(l_{ij}, (i, j) \in \mathcal{E}_{\text{sys}}\).

(If part: controllability of a tree) We use the mathematical induction to prove the sufficiency part. First assume that there is a spanning tree \(T\) rooted at \(u\) in \(\mathcal{G}_{\text{sys}}\). Suppose \(l_{ij} = 0\) for \((j, i) \notin E(T)\). Without losing of generality, let \(u_1\) be the root of this tree, and vertices \(u_1, 1, ..., N\) are in the order such that the parent of vertex \(k\) belongs to vertices \(k - 1, ..., 1, u_1\) in \(T\), for \(k = 1, ..., N\). Suppose that \(A_{\text{sys}}\) is permuted in accordance with the order of vertices \(1, ..., N\). Let the \(nk \times nk\) matrix \(A_\mathcal{T}\) be the submatrix of \(A_{\text{sys}}\) associated with vertices \(1, ..., k\), and \(B_\mathcal{T} = [b_1, 0_{1 \times (k-1)}]^{T}\). Consider \(A_1 = A, B_1 = b\). It is obvious that \((A_1, B_1)\) is controllable. Now suppose that \((A_1, B_1)\) is controllable for \(i = 1, ..., k\). Let \(A_{k+1}\) be partitioned as

\[
A_{k+1} = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ 0 & 0 & 0 & A_{44} \end{bmatrix},
\]

where \(A_{11} \in \mathbb{R}^{n_1 \times n_1}, A_{22} \in \mathbb{R}^{n_2 \times n_2}, A_{33} \in \mathbb{R}^{n_3 \times n_3}, n_1 + n_2 = k\), where \(n_1\) and \(n_2\) are divisible by \(n\), and \(A_{44} = A - l_{k+1}bc\), with \(l_{k+1} \in \mathbb{R}\) being the weight of the edge connecting vertex \(k + 1\) and its parent in \(T\). The first three row and column blocks of \(A_{k+1}\) form \(A_k\). We will show that, by suitably choosing \(l_{k+1}, [A_{k+1} - \lambda I, B_{k+1}]\) is of full row rank for each \(\lambda \in \mathbb{C}\), which means that \((A_{k+1}, B_{k+1})\) are structurally controllable.
is controllable by the PBH test. To this end, consider the following two cases:

Case i) $n_1 \neq 0$: in this case, $[A_{k+1} - \lambda I, B_{k+1}]$ reads as

$$
\begin{bmatrix}
A_{11} - \lambda I & 0 & 0 & b \\
A_{21} & A_{22} - \lambda I & 0 & 0 \\
A_{31} & A_{32} & A_{33} - \lambda I & 0 \\
0 & l_{k+1} bc & 0 & A_{44} - \lambda I
\end{bmatrix},
$$

where $b = [b^T, 0_{1 \times (n_1 - n_2)}]^T$. If $\lambda \notin \sigma(A_{k+1})$, as $(A_k, B_k)$ is controllable, it can be directly validated that $\text{rank}[A_{k+1} - \lambda I, B_{k+1}] = (k+1)n$. Consider $\lambda \in \sigma(A_{k+1})$. Recall $A_{44} = A - l_{k+1} bc$ and $(A, b)$ is controllable meanwhile $(A, c)$ is observable. According to Lemma 4, there exists suitable $l_{k+1}$, such that $\text{rank}(A_{k+1} - \lambda I, B_{k+1}) = 0$. Using the Schur complement (Lemma 3), $[A_{k+1} - \lambda I, B_{k+1}]$ is of full row rank, if and only if

$$
[A_{44} - \lambda I, 0] - [0, l_{k+1} bc, 0](A_k - \lambda I)^{-1} \begin{bmatrix} 0 & b \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = [A - l_{k+1} bc - \lambda I, -l_{k+1} bc(A_{22} - \lambda I)^{-1}b]
$$

is of full row rank. Note that $c(A_{22} - \lambda I)^{-1}A_{21}(A_{11} - \lambda I)^{-1}b$ is a scalar, and $A - l_{k+1} bc$ can be seen as state feedback with feedback matrix $l_{k+1} c$. As $(A, b)$ is controllable, it follows that the aforementioned condition is satisfied, if $c(A_{22} - \lambda I)^{-1}A_{21}(A_{11} - \lambda I)^{-1}b = 0$ and $l_{k+1} \neq 0$. Using the Schur complement, $c(A_{22} - \lambda I)^{-1}A_{21}(A_{11} - \lambda I)^{-1}b = 0$, and if and only if

$$
\begin{bmatrix}
A_{11} - \lambda I & 0 & b \\
A_{21} & A_{22} - \lambda I & 0 \\
0 & -c & 0
\end{bmatrix}
$$

is of full row rank, which is further equivalent to that

$$
[0, -c] \begin{bmatrix} A_{11} - \lambda I & 0 \\ A_{21} & A_{22} - \lambda I \end{bmatrix}^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix} \neq 0. \quad (6)
$$

According to Lemma 3 noting that

$$
\begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22} \\
0 & b
\end{bmatrix}
$$

is controllable (as $(A_i, B_i)$ is controllable for $i = 1, \ldots, k$), there exist only a finite number of complex values $\lambda$ such that (6) cannot be satisfied. Let $\Omega_k = \{\lambda \in \mathbb{C} : \lambda \notin \sigma(A_k), c(A_{22} - \lambda I)^{-1}A_{21}(A_{11} - \lambda I)^{-1}b = 0\}$. Therefore, from Lemma 4 by suitably choosing $l_{k+1} \neq 0$, one can always make $\sigma(A_k) \cap \sigma(A - l_{k+1} bc) = \emptyset$, and $\Omega_k \cap \sigma(A - l_{k+1} bc) = \emptyset$, noting that $\sigma(A_k)$ and $\Omega_k$ both consist of a finite number of fixed scalars. As a consequence, such $l_{k+1}$ makes $(A_{k+1}, B_{k+1})$ controllable.

Case ii) $n_1 = 0$: in this case, $[A_{k+1} - \lambda I, B_{k+1}]$ reads as

$$
\begin{bmatrix}
A_{22} - \lambda I & 0 & 0 & b \\
A_{32} & A_{33} - \lambda I & 0 & 0 \\
l_{k+1} bc & 0 & A_{44} - \lambda I & 0
\end{bmatrix}.
$$

If $\lambda \notin \sigma(A_{44})$, as $(A_k, B_k)$ is controllable, it can be directly validated that $[A_{k+1} - \lambda I, B_{k+1}]$ is of full row rank. If $\lambda \in \sigma(A_{44})$ but $\lambda \notin \sigma(A_k)$, according to Lemma 3 $[A_{k+1} - \lambda I, B_{k+1}]$ is of full row rank, if and only if

$$
[A_{44} - \lambda I, 0] - [l_{k+1} bc, 0](A_k - \lambda I)^{-1} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = [A - l_{k+1} bc - \lambda I, -l_{k+1} bc(A_{22} - \lambda I)^{-1}b]
$$

is of full row rank. Since $(A, b)$ is controllable, and $c(A_{22} - \lambda I)^{-1}b$ is a scalar, it turns out that the above condition is satisfied, if and only if $c(A_{22} - \lambda I)^{-1}b = 0$ and $l_{k+1} \neq 0$. Let $\Omega_k = \{\lambda \in \mathbb{C} : \lambda \notin \sigma(A_k), c(A_{22} - \lambda I)^{-1}b = 0\}$. Noting that $(A_{22}, b)$ is controllable, from Lemma 3 $\Omega_k$ is a finite set. Again, according to Lemma 4 by suitably choosing $l_{k+1} \neq 0$, one can always make $\sigma(A_k) \cap \sigma(A - l_{k+1} bc) = \emptyset$, and $\Omega_k \cap \sigma(A - l_{k+1} bc) = \emptyset$. Such $l_{k+1}$ makes $(A_{k+1}, B_{k+1})$ controllable.

(Controllability of the networked system) If $\tilde{G}_{\text{sys}}$ can be decomposed into more than one disjoint spanning trees rooted at $U$, let the weight of each edge connecting two different trees be zero. As these spanning trees are disjoint, each tree itself corresponds to a controllable system and the whole networked system is controllable. \hfill \Box

### C. Weight Design Procedure

In what follows, we provide a deterministic procedure, to generate the subsystem interaction weights, such that the associated networked system is controllable. For simplicity of the description, assume that $\tilde{G}_{\text{sys}}$ can be spanned by a tree $T$ rooted at $u_1$. Suppose that in the tree $T$, vertices $u_1, 1, \ldots, N$ are arranged such that vertex $k$ is reachable from one of $k-1, \ldots, 1, u_1$, for $k = 1, \ldots, N$. Let $l_k$ be the weight of the edge from the parent of vertex $k$ to it, $k = 1, \ldots, N$ ($l_1 \equiv 1$), and let the weights of edges not in $T$ be zero. Moreover, suppose that $A_{\text{sys}}$ is permuted in accordance with the order of $1, \ldots, N$, and let $A_k$ be the $kn \times kn$ submatrix of $A_{\text{sys}}$ associated with vertices $1, \ldots, k$. Partition $A_k$ as

$$
A_k = \begin{bmatrix}
A_{11} & 0 & 0 \\
A_{21} & A_{22} & 0 \\
A_{31} & A_{32} & A_{33}
\end{bmatrix},
$$

such that $A_{k+1}$ can be expressed as

$$
A_{k+1} = \begin{bmatrix}
A_{11} & 0 & 0 \\
A_{21} & A_{22} & 0 \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
$$

where $A_{11} \in \mathbb{R}^{n_1 \times n_1}, A_{22} \in \mathbb{R}^{n \times n}, A_{33} \in \mathbb{R}^{n_2 \times n_2}, n_1 + n + n_2 = kn$ with $n_1$ and $n_2$ being divisible by $n$. Then, the weights of edges in $T$ can be recursively constructed in the following way:

- $l_1 = 1$;
- for $k = 1, \ldots, N - 1$, do
  a. partition $A_k$ according to (7);
  b. if $A_{11}$ is not empty, let $\Omega_k = \{\lambda \in \mathbb{C} : \lambda \notin \sigma(A_k), c(A_{22} - \lambda I)^{-1}A_{21}(A_{11} - \lambda I)^{-1}b = 0\}$, where $b = [b^T, 0_{1 \times (n_1 - n_2)}]^T$, otherwise let $\Omega_k = \{\lambda \in \mathbb{C} : \lambda \notin \sigma(A_k), c(A_{22} - \lambda I)^{-1}b = 0\}$. Determine an $l_{k+1} \in \mathbb{R}$, such that $\sigma(A_k) \cap \sigma(A - l_{k+1} bc) = \emptyset$ and $\Omega_k \cap \sigma(A - l_{k+1} bc) = \emptyset$.\hfill \Box
The correctness of the above procedure follows the proof of Theorem 1. Concerning the implement of Step b in each iteration, the existence of $l_{k+1}$ is guaranteed by Lemma 4 and an exact numerical $l_{k+1}$ can be found using standard pole-assignment procedures [38]. A corollary can also be obtained from this procedure, which plays an important role in the proof of Theorem 3 (see Section V).

**Corollary 1:** Let $L$ be a Laplacian matrix of a graph $G$ with $N$ vertices $\{1, \ldots, N\}$. Suppose that $G$ has a spanning tree rooted at vertex 1. Let $e_i^{[N]}$ be the first column of $IN$. Then, there exists a set of weights for $G$ such that the associated $(L, e_1^{[N]})$ is controllable while $L$ has no repeated eigenvalues.

**Proof:** By setting $A = 0$, $b = c = 1$, the above procedure provides a way how such $L$ can be constructed. □

**Example 1** (Illustration of the weight assignment procedure): Consider a networked system consisting of 4 subsystems and 1 input. Subsystem parameters are $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $c = [1, 0]$. The digraph illustrations of subsystem dynamics and the subsystem interaction topology are given in Fig. 2. Let $l_1 = l_2 = 1$. Using the above weight assignment procedure, one has $\sigma(A_2) = \{0.3820, 2.6180, 1, 2\}$, and $\Omega_2 = \emptyset$. Hence, $l_3 \notin \{0, 1\}$. Choose $l_3 = 2$. One then has $\sigma(A_3) = \sigma(A_2) \cup \{0, 3\}$, and $\Omega_3 = \emptyset$. Hence, $l_4 \notin \{0, 1, 2\}$. One can choose $l_4 = 3$. It can be validated that such constructed system is controllable. On the other hand, if $l_4 = 1$, then this system is uncontrollable no matter how $l_4$ is selected.

IV. **STRUCTURAL CONTROLLABILITY WITH MIMO SYSTEMS VIA DIFFERENTIALLY WEIGHTED CHANNELS**

In this section, we generalize the results in Section III to the case of networks of MIMO subsystems via differentially weighted channels, i.e., the case where $r > 1$ and $L_1, \ldots, L_r$ can take values independently. Our approach is based on linear parameterization. It is shown that, under certain mild conditions, the whole networked system is structurally controllable, if and only if every vertex of $G_{sys}$ is input-reachable.

**A. Linear Parameterization**

In [37], controllability of a linear-parameterized pair $(A, B)$ is discussed, which is modeled as

$$A = A_0 + \sum_{i=1}^k g_i s_i h_{1i}, \quad B = B_0 + \sum_{i=1}^k g_i s_i h_{2i}. \quad (8)$$

where $A_0 \in \mathbb{R}^{n \times n}$, $B_0 \in \mathbb{R}^{n \times m}$, $g_i, h_{1i} \in \mathbb{R}^n$, $h_{2i} \in \mathbb{R}^m$, and $s_1, \ldots, s_k$ are real free parameters (indeterminates). The pair $(A, B)$ in (8) is said to be structurally controllable, if there exists one set of values for $s_1, \ldots, s_k$, such that the associated system is controllable.

Corfmat and Morse [37] first gave some necessary and sufficient conditions for the linear-parameterized pair $(A, B)$ to be structurally controllable. To present their results, one needs to first construct an auxiliary digraph, which is defined as follows. Define two transfer function matrices $G_{zv}(\lambda) = [h_{11}(\lambda), \ldots, h_{1k}(\lambda)](\lambda I - A_0)^{-1} [g_1, \ldots, g_k]$, $G_{zu}(\lambda) = [h_{11}(\lambda), \ldots, h_{1k}(\lambda)](\lambda I - A_0)^{-1} B_0 + [h_{21}(\lambda), \ldots, h_{2k}(\lambda)]$. The auxiliary digraph $G_{aux}$ associated with $[G_{zv}(\lambda), G_{zu}(\lambda)]$ is defined as $G_{aux} = (\mathcal{V}_{aux}, \mathcal{E}_{aux})$, where the vertex set $\mathcal{V}_{aux} = \{z_1, \ldots, z_k\} \cup \{u_1, \ldots, u_m\}$, and the edge set $\mathcal{E}_{aux} = \{\{z_i, z_j\} : [G_{zv}(\lambda)]_{ji} \neq 0\} \cup \{\{u_i, z_j\} : [G_{zu}(\lambda)]_{ji} \neq 0\}$. We say a vertex $z_i \in \{z_1, \ldots, z_k\}$ is input-reachable, if there exists a path starting from one vertex of $\{u_1, \ldots, u_m\}$ ending at $z_i$ in $G_{aux}$. A cycle of $G_{aux}$ is input-reachable, if every vertex of this cycle is input-reachable (note that a cycle of $G_{aux}$ can only consist of vertices of $z_1, \ldots, z_k$). Based on these arguments, the following theorem gives necessary and sufficient conditions for $(A, B)$ in (8) to be structurally controllable.

**Lemma 6** ([37], [22]): The pair $(A, B)$ in (8) is structurally controllable, if and only if

1. Every cycle is input-reachable in $G_{aux}$;
2. For each $\lambda_0 \in \sigma(A_0)$, $\text{grank}[\lambda_0 I - A_0 - \sum_{i=1}^k g_i s_i h_{1i}^T, B_0 + \sum_{i=1}^k g_i s_i h_{2i}] = n$.

In [22], it points out that, Condition 1) of Lemma 6 means that there is no parameter-dependent uncontrollable mode for $(A, B)$, i.e., the uncontrollable mode that depends on the values of parameters $s_1, \ldots, s_k$ and Condition 2) of the same lemma means that there is no fixed uncontrollable mode for $(A, B)$, i.e., the uncontrollable mode that is independent of the values of parameters $s_1, \ldots, s_k$. The linear-parameterization will play a key role in our following derivations.

To make notations simple, given a matrix $[H, P]$ where $H$ and $P$ are with the dimensions of $n \times n$ and $n \times m$ respectively, we will use $G_{aux}(H, P)$ to denote the auxiliary graph associated with $[H, P]$, which is defined as follows: $G_{aux}(H, P) = (\mathcal{V}_H \cup \mathcal{V}_P, \mathcal{E}_{HH} \cup \mathcal{E}_{PH})$, where $\mathcal{V}_H = \{v_1, \ldots, v_m\}$, $\mathcal{V}_P = \{p_1, \ldots, p_m\}$, $\mathcal{E}_{HH} = \{(v_i, v_j) : H_{ji} \neq 0\}$ and $\mathcal{E}_{PH} = \{(p_i, v_j) : P_{ji} \neq 0\}$. With a little abuse of terminology, if for each $v_i \in \mathcal{V}_H$, there is a path staring from one vertex of $\mathcal{V}_P$ ending at $v_i$, we say that every vertex in $G_{aux}(H, P)$ is input-reachable.

3Although the second condition in Theorem 1 of [37] is not of the form presented here, from the spirits of its proof, it is equivalent to Condition 2) presented here.
B. Structural Controllability with MIMO Subsystems

We are now deriving conditions for structural controllability using the linear parameterization. Notice that the sum of each row of the Laplacian matrix $L_i$ is zero, which introduces some dependencies for the nonzero entries of $L_i$, $i = 1, ..., r$. Hence, we need to diagonalize $L_i$. To this end, define the incidence matrix $K_f$ associated with $G_{sys}$ as follows: $K_f$ is a $|\mathcal{E}_{sys}| \times |\mathcal{V}_{sys}|$ matrix, and $[K_f]_{ij} = 1$ if $j$ is the starting vertex (ending vertex) of the $i$th edge of $\mathcal{E}_{sys}$, and the remaining entries are zero. Afterwards, define a $|\mathcal{V}_{sys}| \times |\mathcal{E}_{sys}|$ matrix $K$ as follows:

$$K_{ij} = \begin{cases} 1, & [K_f]_{ij} = -1 \\ 0 & \text{otherwise} \end{cases}$$

Then, it can be validated that $L_i = -KA_cK_f$, where $A_c$ is a diagonal matrix whose $j$th diagonal equals the weight of the $j$th edge of $\mathcal{E}_{sys}$ associated with $L_i$, $j = 1, ..., |\mathcal{E}_{sys}|$.

Using the diagonalization on $L_i$, one has

$$A = I \otimes A + \sum_{k=1}^{r} (\Delta_k A_k) \otimes (b_k c_k)$$

$$= I \otimes A + \sum_{k=1}^{r} (K \otimes b_k) \Lambda_k (K \otimes c_k).$$

Hence, $(A_{sys}, B_{sys})$ can be recast as

$$[A_{sys}, B_{sys}] = [I \otimes A, \Delta \otimes B] + [K \otimes b_1, ..., K \otimes b_r] \quad \text{diag}\{A_1, ..., A_r\} \begin{bmatrix} K_f \otimes c_1, ..., K_f \otimes c_r \end{bmatrix}^T, 0].$$ (9)

To extend the SISO case to the MIMO one, we shall draw the notion of fixed mode from decentralized stabilization [28] to replace the conditions in Lemma 1 for subsystems.

**Definition 1 (Fixed mode):** Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, and $C \in \mathbb{R}^{r \times n}$, let $K \subseteq \mathbb{R}^{r \times r}$ be the set of all $r \times r$ diagonal matrices. Then $(A, B, C)$ is said to have no fixed mode with respect to $K$ if $\bigcap_{K \subseteq K} \sigma(A + BK) = \emptyset$.

Fixed mode has the following properties.

**Lemma 7:** Given $A \in \mathbb{R}^{n \times n}$, $B = \begin{bmatrix} b_1, ..., b_r \end{bmatrix}$, and $C = [c_1, ..., c_r]^T$, where $b_i, c_i \in \mathbb{R}^{n \times 1}$ for $i = 1, ..., r$, if $(A, B, C)$ has no fixed mode, then 1) for each $i \in \{1, ..., r\}$, there exists at least one $j \in \{1, ..., r\}$, such that $c_i(\lambda I - A)^{-1}b_j \neq 0$; 2) for each $k \in \{1, ..., r\}$, there exists at least one $l \in \{1, ..., r\}$, such that $c_k(\lambda I - A)^{-1}b_l \neq 0$.

**Proof:** See the appendix. \hfill \Box

To proceed with our derivations, we need the following immediate result, whose proof is postponed to the appendix.

**Lemma 8:** Given four matrices $H, P, G$ and $A$, suppose the following conditions hold: 1) $H, P \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{n \times n}$, where $H, P$ are the dimensions of $k \times n$, $n \times m$ and $m \times n$ respectively; 2) $[GH]_{ij} \neq 0$ if and only if there exists one $l \in \{1, ..., k\}$ such that $G_{lj} \neq 0$ and $H_{il} \neq 0$; 3) $A$ is a diagonal matrix whose diagonal entries are free parameters. Then, every cycle is input-reachable in $G_{aux}(GH, GP)$, if and only if such property holds in $G_{aux}(H\Lambda G, P)$.

Based on the above arguments, we have the following theorem, which gives a necessary and sufficient condition on the network topology for structural controllability of networked relative coupling MIMO systems via differentially weighted channels, under the absence of fixed mode for subsystem dynamics.

**Theorem 2:** For the networked system with relative coupling MIMO subsystems described by (4), suppose that $(A, [b_1, ..., b_r], [c_1, ..., c_r]^T)$ has no fixed mode. Then, the networked system is structurally controllable, if and only if $G_{sys}$ is globally input-reachable.

**Proof:** The necessity of input-reachability of each vertex of $G_{sys}$ follows similar arguments to the proof of Theorem 1 which is thus omitted here for space consideration.

We are now proving the sufficiency part. We shall use the linear parameterization presented in Lemma 6 based on (9). Recall that $L_i = -KA_cK_f$.

For Lemma 6 to be used, direct algebraic manipulations show that the associated transfer function matrices are

$$G_{zv}(\lambda) = [K_f \otimes c_1 \quad \lambda I \otimes A^{-1} \quad \lambda I \otimes b_1]$$

$$= \begin{bmatrix} K_f \otimes c_1 \\ K_f \otimes c_2 \\ \vdots \\ K_f \otimes c_r \end{bmatrix} \begin{bmatrix} \lambda I - A^{-1}b_1 \\ \lambda I - A^{-1}b_2 \\ \vdots \\ \lambda I - A^{-1}b_r \end{bmatrix}$$

$$G_{zu}(\lambda) = [K_f \otimes c_1 \quad \lambda I \otimes (\lambda I - A)^{-1}B]$$

$$= \begin{bmatrix} K_f \otimes c_1 \\ K_f \otimes c_2 \\ \vdots \\ K_f \otimes c_r \end{bmatrix} \begin{bmatrix} \lambda I - A^{-1}B \end{bmatrix}$$

Partition $G_{zv}(\lambda)$ into $r \times r$ blocks, where the $(i, j)$th block is $(K_f)_{ij} \otimes (c_i(\lambda I - A)^{-1}b_j)$. From Lemma 7 there is at least one nonzero block in each row block and one nonzero block in each column block in $G_{zv}(\lambda)$. Suppose that the $(i, \sigma_i)$th block is nonzero, $i = 1, ..., r$, where $\sigma_1, ..., \sigma_r$ is a permutation of $1, ..., r$. Let $\bar{G}_{zv}(\lambda)$ be the matrix obtained from $G_{zv}(\lambda)$ by preserving the $(i, \sigma_i)_{i=1}^{r} \lambda I$th blocks and making the rest zero. Similarly, partition $G_{zu}(\lambda)$ into $r \times 1$ blocks, where the $i$th row block is $\lambda I \otimes (\lambda I - A)^{-1}B_i$. Again from Lemma 7 each row block is nonzero. Let $\bar{G}_{zu}(\lambda) = [\lambda I \otimes (\lambda I - A)^{-1}B]$.

It is now easy to see that, if every vertex in $G_{aux}(\bar{G}_{zv}(\lambda), \bar{G}_{zu}(\lambda))$ is input reachable, then such property holds in $G_{aux}(\bar{G}_{zv}(\lambda), \bar{G}_{zu}(\lambda))$.

Define $\Delta_U = [\Delta_1, ..., \Delta_r]^T$. Decompose

$$\text{diag}(K_f^T) \begin{bmatrix} \Delta_1 & \cdots & \Delta_r \end{bmatrix}^T.$$
follows, we will prove that the former condition holds.

Assume that there is a spanning tree rooted at \( U \) in \( \mathcal{G}_{\text{sys}} \), denoted by \( T \). Suppose that \( 0, 1, \ldots, N \) is the topological ordering of vertices in \( T \) in the sense that the parent of vertex \( k \), denoted by \( \text{Par}(k) \), is among vertices \( 0, 1, \ldots, k - 1 \), i.e., \( \text{Par}(k) \in \{0, 1, \ldots, k - 1\} \), for \( k = 1, \ldots, N \), where 0 represents the root in \( U \). Without losing any generality, permute \( L \) in accordance with the ordering of \( 1, \ldots, N \). Recall that \( L_U \) and \( \Delta_U \) have the dimensions of \( rN \times rN \) and \( rN \times 1 \) respectively. Denote the vertex associated with the \( j \)th column in the \( i \)th block of \( L_U \) by the pair \( (i, j) \). Construct a digraph \( \mathcal{G}_U = (V_U, E_U) \) associated with \( (i, \sigma_i)_{i=1}^r \), with \( V_U = \{1, \ldots, r\} \) and \( E_U = \{(\sigma_i, i), i \in V_U\} \). Let \( \text{CPa}(i) \) be the in-neighbor of each \( i \in V_U \). As \( \sigma_1, \ldots, \sigma_r \) is a permutation of \( 1, \ldots, r \), \( \mathcal{G}_U \) consists of a collection of disjoint cycles \([35]\).

Hence, for each \( i \in V_U \), \( \text{CPa}(i) \) is not empty. Based on these definitions, it can be validated that \( \text{CPa}(i) \) is always an in-neighbor of each \( i \) in \( V_U \) as the parent of \( \sigma_i \). Similarly, \( \sigma_j \) is the in-neighbor of \( \sigma_i \), \( j = 1, \ldots, r \), \( i \in V_U \). Moreover, vertex \( \{\text{CPa}(i), 1\} \) has an incoming edge from vertex 0, which is thus input-reachable, \( i \in V_U \). From these observations, for each vertex \( (i, j) \), \( i \in V_U \), \( j = 1, \ldots, r \), there is a path starting from \( \text{CPa}(i), 1 \) to \( \text{CPa}(i), 1 \) ending it. Noting that \( \prod_{j=1}^{r-1} \text{CPa}(\ldots(\text{CPa}(i), \ldots), j) = 1 \), it concludes that every vertex of \( \mathcal{G}_{\text{aux}}(L_U, \Delta_U) \) is input-reachable. Consequently, there is no input-unreachable cycle in \( \mathcal{G}_{\text{aux}}(L_U, \Delta_U) \). The case that \( \mathcal{G}_{\text{sys}} \) can be decomposed into more than one spanning trees follows a similar way to the above arguments.

As for Condition 2) of Lemma 6, again assume that \( \mathcal{G}_{\text{sys}} \) has a spanning tree rooted at \( v_1 \in U \), and denote this tree by \( T \). Let the weights of edges not in \( E(T) \) be zero, i.e., \( l_{ij}^T = 0 \) for \( (j, i) \notin E(T), k = 1, \ldots, r \). Then, the \( j \)th diagonal block of \( A_{\text{sys}} \) can be expressed as \( A_{\text{sys}} = B \text{diag}(\{l_{ij}^k\}_{k=1}^N) \), \( j = 2, \ldots, N \). From the definition of fixed mode, for each \( \lambda_i \in \sigma(A) \), it holds that \( \text{rank}(\lambda_i I - A - B \text{diag}(\{l_{ij}^k\}_{k=1}^N) \), \( j = 2, \ldots, N \) \). Hence, after some row and column permutations, \( \lambda_i I - A_{\text{sys}}, B_{\text{sys}} \) have a block lower triangular form, whose 1st diagonal block, being \( \lambda_i I - A_{\text{sys}}, B_{\text{sys}} \), and its 2nd \( N \)th diagonal blocks, are all of full row generic rank. Hence, \( \text{rank}(\lambda_i I - A_{\text{sys}}, B_{\text{sys}}) = nN \). The case that \( \mathcal{G}_{\text{aux}} \) can be decomposed into more than one disjoint trees can be proved similarly. Therefore, Condition 2) of Lemma 6 is satisfied.

By Lemma 6, this finishes the proof.

**Example 2:** To illustrate the first part of the proof of Theorem 2 consider \( L = \begin{bmatrix} 0 & 0 & 0 \\ l_{22} & 0 & 0 \\ l_{33} & 0 & l_{33} \end{bmatrix} \), \( r = 2 \) with \( \sigma_1 = 2 \), \( \sigma_2 = 1 \), and \( L_{U} = \{1\} \). The graph \( \mathcal{G}_{U} \) and the subgraph of \( \mathcal{G}_{\text{aux}}(L_U, \Delta_U) \) (this subgraph is associated with the matrix obtained from \( L_U, \Delta_U \) by zeroing each diagonal entry in each nonzero blocks of \( L_U \)) are given together in Fig. 3.

It is easy to see that Theorem 1 is a special case of Theorem 2. While the condition that \((A, [b_1, \ldots, b_r], [c_1^T, \ldots, c_r^T]^T)\) has no fixed mode is necessary for the networked system 4 with SISO subsystems to be structurally controllable (provided that \(|\mathcal{S}| < N)\), it is not necessary for the case with MIMO subsystems. In fact, in the latter case, the controllability of \((A, [b_1, \ldots, b_r])\) necessary while the observability of \((A, [c_1^T, \ldots, c_r^T]^T)\) is not. This means that, allowing the MIMO interaction fashions may make conditions for subsystems less restrictive to achieve controllability.

**Remark 2:** If \((A, B)\) is controllable but \((A, B, C)\) has some fixed modes, from the proof of Theorem 2 the existence of a spanning in \( \mathcal{G}_{\text{sys}} \) is sufficient to eliminate the parameter-dependent uncontrollable modes. However, it seems that the tree topology is usually not sufficient to eliminate the fixed uncontrollable modes. For this case, some further efforts need to be made.

V. STRUCTURAL CONTROLLABILITY WITH MIMO SUBSYSTEMS VIA EQUALLY WEIGHTED CHANNELS

In the above section, we have discussed structural controllability of networked relative coupling MIMO systems, in which the weights of interaction links between two subsystems can be heterogeneous. In this section, we study the same problem in which the interaction weights between two subsystems are identical as shown in Fig. 1(c).

For notation simplicity, let \( L_1 = \cdots = L_r = L = [-l_{ij}] \), and rewrite (4) as

\[
A_{\text{sys}} = I \otimes A - L \otimes (BC), \quad B_{\text{sys}} = \Delta \otimes B. \tag{10}
\]

Note that unlike (4), each indeterminate in \( A_{\text{sys}} \) may have a coefficient matrix whose rank is larger than one if \( \text{rank}(BC) > 1 \). Hence, the methodology based on the linear parameterization (Lemma 6) may not be suitable for this case. In fact, there is in general no readily available efficient method for structural controllability analysis of LTI systems when the coefficient matrices of some undeterminates have ranks larger than one \([22]\). However, by exploring the structure peculiarity of (10), some concise results could be obtained.

Motivated by the fixed mode in Theorem 2 we make a condition on the subsystem dynamics \((A, B, C)\), i.e., \( \int_{\mathbb{R}} \sigma(A+IBC) = 0 \), which means that \( A+IBC \) has no fixed eigenvalues across \( l \in \mathbb{R} \). Under this condition, we are able to give a necessary and sufficient condition for structural controllability of the system (3)-(10).

\(^4\)One can verify that, if \((A, B)\) is controllable and \((A, C)\) is observable, then \((A, B, C)\) has no fixed mode.
Theorem 3: Given the system (3)-(10), suppose that \( \bigcap_{l \in \mathbb{R}} \sigma(A + lBC) = \emptyset \). Then, the system (10) is structurally controllable, if and only if every vertex is input-reachable in \( \mathcal{G}_{\text{sys}} \).

Proof: (Only if part) The only if part follows similar arguments to those of the proof of Theorem 1. Details are omitted here.

(If part) First assume that \( \mathcal{G}_{\text{sys}} \) can be spanned by a tree rooted at \( u_1 \in U \). Let \( u_1, 1, \ldots, N \) be the topological ordering of vertices in such tree, i.e., the parent of vertex \( k \) is among vertices \( u_1, 1, \ldots, k-1, k = 1, \ldots, N \). Moreover, let the weights of edges not in that tree be zero. Without losing any generality, assume that \( L \) has been permuted in accordance with the ordering of \( 1, \ldots, N \). Then, \( A_{\text{sys}} = I \otimes A - L \otimes (BC) \) has the following block triangular form:

\[
\begin{bmatrix}
A & 0 & \cdots & 0 \\
-l_{22}BC & A + l_{22}BC & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & \cdots & \cdots & A + lNNBC
\end{bmatrix}
\]

where * denotes a block which is not of interest. It is easy to see that, the set of eigenvalues of \( A_{\text{sys}} \) is \( \bigcup_{i=1}^{\ldots,N} \sigma(A + l_{ii}BC) \), where \( l_{11} = 0 \). For a fixed \( l_{ii} \in \mathbb{R} \), let \( \lambda^{[i]}_j \) be the \( j \)th distinct eigenvalue of \( A + l_{ii}BC \), \( i \in \{1, \ldots, N\}, j \in \{1, \ldots, n_i\} \), where \( n_i \) is the number of distinct eigenvalues of \( A + l_{ii}BC \). Let \( \mu^{[i]}_{r_{ij}} \) be a collection of vectors spanning the left null space of \( \lambda^{[i]}_j I - (A + l_{ii}BC) \), where \( r_{ij} \) is the geometric multiplicity of \( \lambda^{[i]}_j \). Note that, for any \( w \in \mathbb{C}^N \), \( l \in \{1, \ldots, r_{ij}\} \),

\[
(w^T \otimes \mu^{[i]}_{r_{ij}})^T A_{\text{sys}} = (w^T \otimes \mu^{[i]}_{r_{ij}})(I \otimes A - L \otimes (BC))
\]

\[
= w^T \otimes (\mu^{[i]}_{r_{ij}})^T A - (w^T L) \otimes (\mu^{[i]}_{r_{ij}})^T BC
\]

\[
= \lambda w^T \otimes \mu^{[i]}_{r_{ij}} + (l_{ii}w^T - w^T L) \otimes (\mu^{[i]}_{r_{ij}})^T BC
\]

(11)

If \( w^T L = -l_{ii}w^T \), i.e., \( w \) is a left eigenvector of \( L \) associated with the eigenvalue \( -l_{ii} \), then \( w^T \mu^{[i]}_{r_{ij}} \) is a left eigenvector of \( A_{\text{sys}} \) associated with the eigenvalue \( \lambda^{[i]}_j \).

On the other hand, under the condition that \( \mathcal{G}_{\text{sys}} \) has a spanning tree rooted at vertex \( u_1 \), from Corollary 1 there exists \( \{l_{11}, \ldots, l_{NN}\} \), such that \( (L, e_1^{[N]}) \) is controllable while \( l_{ii} \neq l_{ij} \) for any two distinct \( i, j \in \{1, \ldots, N\} \), whose first column of \( LN \). Let \( w_1 \) be the left eigenvector of such \( L \) associated with the eigenvalue \( -l_{ii} \). Then, following similar arguments to \( (11) \), it can be validated that, any left eigenvector \( \xi \) of \( A_{\text{sys}} \) associated with \( \lambda^{[i]}_j \), \( i \in \{1, \ldots, N\}, j \in \{1, \ldots, n_i\} \), can be expressed as

\[
\xi = (w_1 \otimes \mu^{[i]}_{r_{ij}}) \alpha
\]

where \( \mu^{[i]}_{r_{ij}} = [\mu^{[i]}_{r_{ij}}, \ldots, \mu^{[i]}_{r_{ij}}] \), \( \alpha \in \mathbb{C}^{r_1} \). Suppose that the networked system associated with \( L \) is uncontrollable. According to the PBH test, there must exist some \( \lambda^{[i]}_j \), \( i \in \{1, \ldots, N\}, j \in \{1, \ldots, n_i\} \), and \( \alpha \in \mathbb{C}^{r_1} \), such that the corresponding eigenvector \( \xi \) (expressed by (12)) satisfies

\[
\xi^T (e_1^{[N]} \otimes B) = (w_1^T e_1^{[N]}) \otimes (\alpha^T \mu^{[i]}_{r_{ij}})^T B = 0
\]

(13)

However, by controllability of \( (L, e_1^{[N]}) \), \( w_1^T e_1^{[N]} \neq 0 \). To make (13) hold, it must have that \( \alpha^T \mu^{[i]}_{r_{ij}} \) \( B = 0 \). Notice that by definition, \( \mu^{[i]}_{r_{ij}} (A - \lambda^{[i]}_j I + l_{ii}BC) = 0 \). Combining the above two equalities, one has that for any \( \delta \in \mathbb{R} \), \( \alpha^T \mu^{[i]}_{r_{ij}} (A - \lambda^{[i]}_j I + (l_{ii} + \delta) BC) = 0 \), which means that \( \lambda^{[i]}_j \in \bigcap_{l \in \mathbb{R}} \sigma(A + lBC) \), causing a contradiction. Hence, the networked system associated with \( L \) must be controllable.

If \( \mathcal{G}_{\text{sys}} \) can be decomposed into more than one disjoint spanning trees, then each spanning tree corresponds to a controllable system, and the whose networked system is controllable. This finishes the proof of the if part.

The above result shows that, if the subsystem dynamics satisfies the condition in Theorem 3 and the network topology satisfies certain connectivity property (i.e., input-reachability of every vertex), then for all most all weights of the interaction links, the associated systems are controllable.

Remark 3: It can be seen that if \( \bigcap_{l \in \mathbb{R}} \sigma(A + lBC) = \emptyset \), then \((A, B, C)\) has no fixed mode. Hence, the condition of Theorem 2 is less restrictive than that of Theorem 3. This is reasonable, as allowing heterogeneous interaction weights between two subsystems permits more freedom for weight assignment for the controllability of the whole system.

Remark 4: It should be noted that the condition \( \bigcap_{l \in \mathbb{R}} \sigma(A + lBC) = \emptyset \) has also been proposed in [13]. However, different from [13] where the interaction weights are fixed and form a diagonalizable matrix, in this paper we study structural controllability where the weights are indeterminates, and we do not need the diagonalization assumption. It can be seen that the construction of a controllable \((L, e_1^{[N]})\) with no repeated eigenvalues plays an important part in the proof of Theorem 3.

VI. STRUCTURAL CONTROLLABILITY UNDER SWITCHING TOPOLOGIES

In this section, we extend our results to the networked systems with switching topologies. For space consideration, we only consider the networked SISO subsystem case. Similar considerations can be made for the MIMO cases.

Consider a networked system with relative coupling SISO subsystems under switching topologies. The dynamics of this networked system is

\[
\dot{x}(t) = (I \otimes A - L_{\sigma(t)} \otimes (bc))x(t) + (\Delta \otimes b)u(t)
\]

(14)

where \( \sigma(t) : [0, \infty) \rightarrow \{1, \ldots, p\} \) is the switching signal, and \( \sigma(t) = i \) means that the topology \( \mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i) \), with \( \mathcal{V}_i = \{1, \ldots, N\} \), whose corresponding Laplacian matrix is \( L_{si} \), is active at time instance \( t \). All the remaining symbols have the same definitions as those in Section II. Define \( A_{\text{sys}} = I \otimes A - L_{si} \otimes (bc) \), and \( B_{\text{sys}} = \Delta \otimes b \), for \( i = 1, \ldots, p \).

In the above model, the time-varying nature of the network topology is modeled by switches among \( p \) possible topologies \( \mathcal{G}_1, \ldots, \mathcal{G}_p \) from one time to another. It is assumed that weights of links of two different topologies are mutually independent, while weights of links of one topology, as well as subsystem parameters \( A, b \) and \( c \), are time invariant.

Definition 2 ([18]): Consider a switched linear system \( \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \), where \( \sigma(t) : [0, \infty) \rightarrow \{1, \ldots, p\} \) is the switching signal.
\{1, ..., p\} is the switching signal, \(A_i \in \mathbb{R}^{n \times n}\) and \(B_i \in \mathbb{R}^{n \times m}\) are called switching modes, for \(i = 1, ..., p\). This system is said to be controllable, if for any initial state \(x_0\) and any final state \(x_1\), there exists a finite time instance \(t_f\), a switching signal \(\sigma : [0, t_f] \rightarrow \{1, ..., p\}\) and a control input \(u : [0, t_f] \rightarrow \mathbb{R}\), such that \(x(0) = x_0\) and \(x(t_f) = x_1\).

Lemma 9 ([39]): Consider a switched linear system \(\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)\), where \(\sigma(t) : [0, \infty) \rightarrow \{1, ..., p\}\) is the switching signal, \(A_i \in \mathbb{R}^{n \times n}\), \(B_i \in \mathbb{R}^{n \times m}\), for \(i = 1, ..., p\). This system is controllable, if and only if the following matrix has full row rank

\[
\begin{bmatrix}
B_1, & B_2, & \cdots, & B_p, & A_1B_1, & \cdots, & A_1B_p, & A_1^2B_1, & \cdots, & A_1^2B_p, & \cdots,
A_pB_1, & \cdots, & A_pB_p, & A_p^2B_1, & \cdots, & A_p^2B_p, & \cdots, & A_p^{n-1}B_1, & \cdots, & A_p^{n-1}B_p
\end{bmatrix}
\]  

(15)

Definition 3: The networked system \([14]\) with switching topologies is said to be structurally controllable, if there exists at least one set of values for the Laplacian matrices \(L_{s1}, ..., L_{sp}\), such that the corresponding system \([14]\) is controllable in the numerical sense.

The purpose of this section is to explore necessary or sufficient conditions for structural controllability of the networked system \([14]\) with switching topologies. Note that controllability is still a generic property for such systems. The study differs from [39] in the following aspects. The work [39] focuses on the general state-space plants in which the system matrices for different switching modes are independent. In \([14]\), only the weights of different switching topologies are independent while subsystem dynamics are fixed.

Now we introduce the concept of joint input-reachability. Recall that \(\mathcal{G}_i = (\mathcal{V}_{\text{sys}}, \mathcal{E}_i), i = 1, ..., p\). Similar to Section II, let \(\mathcal{I}_u = \{i : \delta_i \neq 0\}\) be the set of indices of subsystems that are directly influenced by external inputs, and \(\mathcal{U} = \{u_i : i \in \mathcal{I}_u\}\). Then, a vertex \(i \in \mathcal{V}_{\text{sys}}\) is said to be jointly input-reachable, if there is a path starting from one vertex of \(\mathcal{U}\) and ending at \(i\) in the graph union \(\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_p = (\mathcal{V}_{\text{sys}} \cup \mathcal{U}, \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_p \cup \mathcal{E}_{\text{ux}})\). If every vertex of \(\mathcal{V}_{\text{sys}}\) is jointly input-reachable, we say that the switching system \([14]\) is globally jointly input-reachable.

Lemma 10: The system \([14]\) with switching topologies \(\mathcal{G}_1, ..., \mathcal{G}_p\) is structurally controllable, if and only if \((A, b)\) is controllable and the system is globally jointly input-reachable.

Proof: Let \(\Psi\) be the matrix obtained by replacing \(A_i\) with \(A_{\text{sys}}\) and \(B_i\) with \(B_{\text{sys}}\) in \((15)\). If \((A, b)\) is uncontrollable, there exists \(x \in \mathbb{C}^n\) satisfying \(x^T A x = \lambda x^T\) and \(x^T b = 0\). Hence, \(1_{1 \times N} \otimes x^T A_{\text{sys}} = \lambda 1_{1 \times N} \otimes x^T\), \(1_{1 \times N} \otimes x^T (\Delta \otimes b) = 0\), for \(i = 1, ..., p\). Here \(1_{1 \times N}\) denotes the \(1 \times N\) matrix with all entries being 1. Due to these equalities, one has \(1_{1 \times N} \otimes x^T \Psi = 0\), which means that \(\Psi\) is not of full row rank, leading to the uncontrollability of the networked system \([14]\).

Suppose there is at least one vertex in \(\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_p\), denoted by \(i^*\), which is not input-reachable. Decomposing \(\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_p\) into strongly connected components, suppose the strongly connected component which contains \(i^*\) has \(\bar{p}\) vertices. This means that, all these \(\bar{p}\) vertices are not reachable. A strongly connected component is a subset of vertices of a graph, such that every two vertices of it are mutually reachable.

\[
P L_{si} P^T = \begin{bmatrix} L_{si11} & 0 & \cdots & 0 \\ L_{si21} & L_{si22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, P \Delta = \begin{bmatrix} 0 & \cdots & 0 \\ \Delta_2 & \cdots & \Delta_2 \\ \vdots & \ddots & \vdots \\ \Delta_2 & \cdots & \Delta_2 \end{bmatrix},
\]

where \(L_{si11} \in \mathbb{R}^{\bar{p} \times \bar{p}}, L_{si21} \in \mathbb{R}^{(N-\bar{p}) \times \bar{p}}, L_{si22} \in \mathbb{R}^{(N-\bar{p}) \times (N-\bar{p})}\), for \(i = 1, ..., p\). It can be validated that \((P \otimes I_n) \Psi\) has the form of

\[
\begin{bmatrix}
0 & \cdots & 0 \\
* & \cdots & * \\
\vdots & \ddots & \vdots \\
* & \cdots & * 
\end{bmatrix},
\]

i.e., the first \(n \bar{p}\) rows of \((P \otimes I_n) \Psi\) are zeros, where * denotes the entry which is not of interest. It means that \(\Psi\) is not of full row rank. Hence, the networked system \([14]\) is not controllable.

Now suppose that \((A, c)\) is observable, we have the following necessary and sufficient condition for structural controllability.

Theorem 4: Assume that \((A, c)\) is controllable and \((A, c)\) is observable. The system \([14]\) with switching topologies \(\mathcal{G}_1, ..., \mathcal{G}_p\) is structurally controllable, if and only if it is globally jointly input-reachable.

Proof: The necessity comes from Lemma 10. For sufficiency, if the whole system is globally jointly input-reachable, let \(L_g\) be the Laplacian matrix associated with the digraph \(g = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_p\), such that \((I \otimes A - L_g \otimes (bc), \Delta \otimes b)\) is controllable. According to Theorem 1 as every vertex \(G_g\) is input-reachable, such \(L_g\) always exists. By the PBH test, \((p(I \otimes A - L_g \otimes (bc)), p\Delta \otimes b)\) is also controllable. Moreover, it is easy to see that, there exist \(L_{s1}, ..., L_{sp}\), where \(L_{si}\) is the Laplacian matrix associated with \(G_i\), \(i = 1, ..., p\), such that \(L_g = (L_{s1} + \cdots + L_{sp})\). Recalling that \(A_{\text{sys}} = I \otimes A - L_{si} \otimes (bc)\), and \(B_{\text{sys}} = \Delta \otimes b\), one has \(p(I \otimes A - L_g \otimes (bc)) = A_{\text{sys}1} + \cdots + A_{\text{sys}p}\). That is to say, \((A_{\text{sys}1} + \cdots + A_{\text{sys}p}, B_{\text{sys}1} + \cdots + B_{\text{sys}p})\) is controllable. On the other hand, from Lemma 9 and from the proof of Theorem 3.7 in [40], it can be obtained that, if \((A_1 + \cdots + A_p, B_1 + B_2 + \cdots + B_p)\) is controllable, then the system with switching modes \(\{A_1, B_1\}, \ldots, \{A_p, B_p\}\) is controllable. Hence, the system with switching modes \(\{(A_{\text{sys}1}, \Delta \otimes b), \ldots, (A_{\text{sys}p}, \Delta \otimes b)\}\) is controllable. This means that, the system \([14]\) with switching topologies \(\mathcal{G}_1, ..., \mathcal{G}_p\) is structurally controllable.

The above theorem indicates that, even if the networked system with each individual fixed topology \(G_i\) is not structurally controllable, the system with switching topologies could be. This indicates the possible advantages of switching mechanism. It should be noted that, even if \(|\mathcal{I}_u| < N\), the observability of \((A, c)\) is not necessary for controllability of the networked system \([14]\) with switching topologies, which is shown in the following Example 3. This is in contrast to the case with fixed topologies.

Example 3 (Controllable networked system with unobservable \((A, c)\) and uncontrollable individual topologies): Consider a networked system with two switching topologies, whose associated Laplacian matrices are respectively
$L_{s1} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ and $L_{s2} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$. The remaining parameters are $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $c = \begin{bmatrix} 0 & 1 \end{bmatrix}$, and $\mathcal{I}_u = \{1\}$. It can be validated that, $(A, c)$ is unobservable. Hence, the networked system with each fixed individual topology is structurally uncontrollable. However, setting all weights of edges to be 1, it can be found that the resulting networked system is controllable.

**VII. Subject to Certain Subsystem Heterogeneities**

One promising point of the structure analysis is that, our results in Sections III-V could be extended to take certain subsystem heterogeneities into consideration. This is in contrast to most existing work \[17, 18\], where the identical subsystem dynamics assumption is important as otherwise it is in general not possible to implement the eigenvalues/eigenvectors decomposition involved therein.

When modeling real-world networked systems, it is often the case that subsystems obey the same physical laws thus parameterized similarly, but possibly with different values of their elementary parameters. Here, the elementary parameters, similar to the first principle parameters mentioned in \[22\], refer to parameters that directly describe the movements of subsystems. For example, in the vehicle-spring-damper chain system illustrated in Fig. 5 of Section VIII, the mass of the vehicle, the constants of the spring and damper, could be seen as elementary parameters. For the first look, the inevitable subsystem heterogeneities caused by the variants of subsystem elementary parameters, might prevent our analysis in this paper from being applicable. However, our analysis and most results in Sections III-V are indeed applicable under certain of these heterogeneities.

There are two cases. The first case is that, one could decouple the ‘heterogeneous part’ from subsystem dynamics and put it into the subsystem interaction weights. If after such operation, the structure of the associated Laplacian matrices and the corresponding parameter independencies are preserved, then most results in Sections III-V could still be valid. See the examples in Section VIII.

The second case is that, the subsystem heterogeneities arising from the differences in values of elementary parameters could be expressed by $A + \delta A_i$, where $\delta A_i$ is a structured matrix, $i \in \{1, \ldots, N\}$, and $\delta A_1, \ldots, \delta A_N$ have the same structure, denoted by $\delta A$, whereas their nonzero entries could take values independently (both within each $\delta A_i$ and between two different $\delta A_i$ and $\delta A_j$). Moreover, nonzero entries of $\delta A_1, \ldots, \delta A_N$ are independent of those of Laplacian matrix $L$ (for brevity, we only focus on the networked SISO subsystem case). In this regard, the $i$th subsystem dynamics \[1\] could be rewritten as

$$x_i(t) = (A + \delta A_i)x_i(t) + bv_i(t).$$

**Corollary 2:** Consider the networked system described by \[16\] and \[2\]. This system is structurally controllable, if $1) (A + \delta A, b)$ is structurally controllable and $2) (A + \delta A, c)$ is structurally observable. $2) \mathcal{G}_{sys}$ is globally input-reachable.

**Proof:** If $1)$ and $2)$ are satisfied, first choose one numerical realization of $\delta A$, denoted by $\delta A$, such that $(A + \delta A, b)$ is controllable and $(A + \delta A, c)$ is controllable. Let $\delta A_i$ for each subsystem take the same value as $\delta A$, $i \in \{1, \ldots, N\}$. From Theorem \[1\] and because of $2)$, the resulting networked system is structurally controllable. 

**Remark 5:** Note that $2)$ and the first part of $1)$ is also necessary for Corollary \[2\] to hold. However, different from the homogeneous subsystem case, the second part of $1)$ is no longer necessary. This indicates that subsystem heterogeneities may often be helpful in controllability of networked systems. Some related discussions could be found in \[21\].

It should be noted that, not all subsystem heterogeneities could be handled in the way mentioned above. For more general networked systems with heterogeneous subsystems, readers could be referred to \[22\].

**VIII. Applications to Some Real-world Examples**

In this section, we show that some typical real-world systems could be modeled as networked relative coupling systems and how our theoretical results are applied to them. These systems include some fluid systems, power networks and mechanical systems. Typically, all these examples involve subsystems with heterogeneous parameters. However, through some simple manipulations mentioned in Section VIII they can be covered by the main results of this paper.

**A. Fluid Systems**

Consider the fluid-level system with $N$ interacted tanks shown in Fig. 4. Assuming small variations of the variables from the steady-state values, the dynamics of the $i$th tank could be described by \[1\]

$$h_i - h_{i+1} = q_i R_i C_i \dot{h}_i = q_{i-1} - q_i,$$

$i \in \{1, \ldots, N\}$, where $h_i$ is the head of the fluid level, $q_i$ is the outflow rate, $C_i$ and $R_i$ are the capacitance of the tank and the resistance of liquid flow in the pipe, respectively. Here, $q_0$ should be regarded as the input rate, and $h_{N+1} = 0$. See \[1\] Chap 4 for details.

Equation \[17\] could be rewritten as

$$\dot{h}_i = \frac{1}{C_i R_{i-1}} (h_{i-1} - h_i) + \frac{1}{C_i R_i} (h_{i+1} - h_i).$$

Regarding $\{C_i, R_i\}_{i=1}^N$ as independent indeterminates, there is no algebraic dependence among the nonzero off-diagonal entries of the associated matrix $L = [-l_{ij}]$. By Theorem \[1\] since the considered fluid-level system has a chain structure, we conclude that it is generically controllable.

This means that there exists one numerical realization of $\delta A$, denoted by $\delta A$, such that $(A + \delta A, b)$ is controllable and $(A + \delta A, c)$ is observable.
B. Power Networks

Consider a power network consisting of \( N \) generators. The dynamics of each generator around its equilibrium state could be described by the following linearized Swing equation \([29]\):

\[
m_i \ddot{\theta}_i + d_i \dot{\theta}_i = - \sum_{j=1}^{N} k_{ij}(\theta_i - \theta_j) + P_i, \quad (18)
\]

\( i \in \{1, ..., N\} \), where \( \theta_i \) is the phase angle, \( m_i \) and \( d_i \) are respectively the inertia and damping coefficients, \( P_i \) is the input power, all for the \( i \)th generator. In addition, \( k_{ij} \) is the susceptance of the power line from the \( j \)th generator to the \( i \)th one, whose value is often hard to accurately obtain.

Rewrite \((18)\) as

\[
\begin{bmatrix}
\dot{\theta}_1 \\
\vdots \\
\dot{\theta}_N
\end{bmatrix}
= \begin{bmatrix}
0 & \frac{1}{m_1} \\
0 & \frac{1}{m_2} \\
\vdots & \vdots \\
0 & \frac{1}{m_N}
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\vdots \\
\theta_N
\end{bmatrix}
+ \sum_{j=1}^{N} \frac{k_{ij}}{m_j} \begin{bmatrix}
0 \\
1 \\
0 \\
\ddots \\
0
\end{bmatrix}
\begin{bmatrix}
\theta_j - \theta_1 \\
\theta_j - \theta_2 \\
\vdots \\
\theta_j - \theta_N
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\ddots \\
0
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2 \\
\vdots \\
P_N
\end{bmatrix},
\]

\( \tag{19} \)

Suppose that \( \{m_i, d_i\}_{i=1}^{N} \) are mutually independent. Then, \( \frac{1}{m_i} \) and \( \frac{k_{ij}}{m_j} \) can be seen as indeterminate parameters representing subsystem heterogeneities and weights of the associated Laplacian matrix, respectively. In this regard, the considered power network model can be described by \((16)\) and \((2)\), which is a networked system with relative coupling SISO subsystems. It can be validated that, \((A + \delta A, b)\) is structurally controllable and \((A + \delta A, c)\) is structurally observable. By Corollary \([2]\) provided that there exists a path (consisting of power lines) from one input to each generator in the power system, this system is structurally controllable.

C. Mechanical Systems

Consider the vehicle-spring-damper chain system shown in Fig. 4 which is also mentioned in \([22]\). This system consists of \( N \) sub-systems. For the \( i \)th subsystem, let \( x_i \) be the displacement of the vehicle, and \( m_i, k_i \) and \( \mu_i \) be the mass of the vehicle, the constants of the spring and the damper, respectively, and \( u_i \) be the force imposed on the vehicle. The dynamics for the \( i \)th vehicle, \( i = 1, ..., N \), is

\[
x_i = \frac{m_i}{2} \frac{\ddot{x}_{i-1} - \ddot{x}_i}{k_i} + \frac{m_i-1}{2} k_i (x_{i+1} - x_i) - \frac{m_i-1}{2} k_i (x_i - x_{i-1}) + \frac{m_i}{2} u_i,
\]

with boundary conditions \( x_0 = 0 \), \( \mu_{N+1} = 0 \), \( k_{N+1} = 0 \) and \( x_{N+1} < \infty \).

Let \( x_1 = x_1, x_2 = \dot{x}_1 \). It is easy to see that \((20)\) can be rewritten as the following model of MIMO subsystems:

\[
\begin{bmatrix}
x_{1,1} \\
x_{1,2}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_{1,2} \\
x_{1,1}
\end{bmatrix}
+ \frac{k_i}{m_i} \begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
-x_{i-1,1} + x_{i+1,1} - x_{i-1,2} + x_{i+1,2}
\end{bmatrix}
+ \frac{k_i}{m_i} \begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
-x_{i,1} + x_{i+1,1} - x_{i,2} + x_{i+1,2}
\end{bmatrix}
+ \frac{\mu_i}{m_i} \begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
x_{1,2} \\
x_{1,1}
\end{bmatrix}.
\]

In the above model, every two connected subsystems are interacted through two channels with different output vectors \([c_1, c_2]\), inducing two Laplacian matrices \( L^{[1]} = [-I_{ij}^{[1]}] \) and \( L^{[2]} = [-I_{ij}^{[2]}] \). Regarding \( \{m_i, k_i, \mu_i\}_{i=1}^{N} \) as independent indeterminates, there is no algebraic dependence among the nonzero off-diagonal entries in \( L^{[1]} \) and \( L^{[2]} \). Moreover, it can be validated that the associated \((A, B, C)\) has no fixed mode. Note that the associated \( G_{\text{sys}} \) is undirected. By Theorem \([2]\) the considered mechanical system is structurally controllable by driving arbitrary one vehicle in this system.

IX. Conclusions

This paper studies structural controllability of networked relative coupling systems in which each subsystem is of fixed general high-order linear dynamics. Three types of subsystem interaction fashions are considered, including SISO, MIMO via differentially weighted channels, and MIMO via equally weighted channels. It is shown that, under certain conditions on subsystem dynamics, the whole system is structurally controllable, if and only if the network topology is globally input-reachable. Extensions to the case with switching topologies are also considered. It is also demonstrated that some results can be extended to handle certain subsystem heterogeneities. Further research includes, considering similar problems in the cases with undirected (symmetric) network topologies, or with more complicated subsystem interaction fashions such as the double Laplacian interconnections in the traffic systems \([2, 41]\).

APPENDIX

**Proof of Lemma \([7]\)** From Definition \([1]\) it is obvious that if \((A, B, C)\) has no fixed mode, then \((A, B)\) is controllable and \((A, C)\) is observable. Hence, \((I, A, B)\) is output controllable. Following Lemma \([5]\) one has that for any \( c_0 \neq 0 \), \( c_0 (\lambda I - A)^{-1}B \neq 0 \). Statement 2) follows similar arguments and the duality between controllability and observability.

**Proof of Lemma \([8]\)** Denote the set of vertices of \( G_{\text{aux}}(GH, GP) \) by \( \mathcal{V} \), \( \mathcal{U} \), where \( \mathcal{V} = \{v_1, ..., v_n\} \) is associated with the columns of \( GH \) and \( \mathcal{U} = \{u_1, ..., u_m\} \) is associated with the columns of \( GP \). Moreover, denote the set of vertices
of $G_{aux}(HAG, P)$ by $\mathcal{W} \cup \mathcal{U}$, where $\mathcal{W} = \{w_1, \ldots, w_k\}$ is associated with the columns of $HG$ and $\mathcal{U} = \{u_1, \ldots, u_m\}$ is associated with the columns of $P$.

Suppose that there is a cycle in $G_{aux}(HAG, P)$, denoted by $C_1 = \{w_{i_1} \rightarrow w_{i_2} \rightarrow \cdots \rightarrow w_{i_q} \rightarrow w_{i_1}\}$. Moreover, suppose that there is a path from $w_{i_0} \in \mathcal{U}$ to $w_q \in \{w_{i_2}, \ldots, w_{i_q}\}$, and denote such path by $\{w_{i_0} \rightarrow w_{i_1} \rightarrow \cdots \rightarrow w_q\}$. This means that the $(i_1, i_0)$-th entry of $P$, and the $(i_2, i_1)$-th,...,$(i_{q-1}, i_{q-1})$-th,...,$(i_q,i_1)$-th and $(i_1, i_0)$-th entries of $HAG$ are nonzeros. Note that $[HAG]_{i_0j} \neq 0$, if and only if there exists an $l \in \{1, \ldots, n\}$ such that $H_{il} \neq 0$ and $G_{ij} \neq 0$. Hence, there exists a sequence of integers $k_1, \ldots, k_{q-1}, k_1, k_0 \in \{1, \ldots, n\}$ (possibly with repeated values), such that $H_{i_{k_q}i_{k_{q-1}}} \neq 0$ and $G_{i_{k_{q-1}}i_{k_q}} \neq 0$ for $j = 1, \ldots, q-1$, $H_{i_{k_{q-1}}i_{k_q}} \neq 0$ and $G_{i_{k_q}i_{k_{q-1}}} \neq 0$ for $j = 1, \ldots, s$, where $i_{s+1}$ is defined to be $i_1$. Because of Condition 2) in this lemma, the above indicates that the $(k_2, k_1)$-th,...,$(k_{q-1}, k_{q-1})$-th,$(k_1, k_q)$-th,$(k_2, k_1)$-th,$(k_s, k_{s-1})$-th entries of $GH$ are nonzero, where $s \in \{1, \ldots, s\}$, and $[GP]_{k_{s-1}i_0} \neq 0$. Then, there exists a sequence of edges $(w_{i_{k_{s-1}}}v_{k_{s-1}}), \ldots,(w_{i_{k_{s-1}}}v_{k_{s-1}}),\ldots,(w_{i_{k_{s-1}}}v_{k_{s-1}}),\ldots,(w_{i_{k_{s-1}}}v_{k_{s-1}}),\ldots,(w_{i_{k_{s-1}}}v_{k_{s-1}})$ in $G_{aux}(GH, GP)$. A cycle can always be found from these edges, and this cycle is input-reachable. Since every step of the above analysis is invertible, such property still holds in the direction from $G_{aux}(GH, GP)$ to $G_{aux}(HAG, P)$.

That is to say, each cycle in $G_{aux}(HAG, P)$ corresponds to (at least) one cycle in $G_{aux}(GH, GP)$, and the input-reachability of the former cycle implies the input-reachability of the latter one, and vice versa. This further leads to proof of this lemma.

REFERENCES

[1] K. Ogata, Y. Yang, Modern Control Engineering, Vol. 4, Prentice-Hall, 2002.
[2] Y. Zhang, J. Yao, G. Chen, Towards mesoscopic analysis of inter-vehicle communications, Journal of the Franklin Institute 355 (3) (2018) 1470–1492.
[3] R. Olfati-Saber, R. M. Murray, Consensus problems in networks of agents with switching topology and time-delays, IEEE Transactions on Automatic Control 49 (9) (2004) 1520–1533.
[4] H. G. Tanner, On the controllability of nearest neighbor interconnections, in: 43rd IEEE Conference on Decision and Control, IEEE, 2004, pp. 2467–2472.
[5] A. Sarlette, R. J. Sepulchre, Control limitations from distributed sensing: Theory and extremely large telescope application, Automatica 50 (2) (2014) 421–430.
[6] L. Scardovi, R. Sepulchre, Synchronization in networks of identical linear systems, Automatica 45 (11) 2557–2562.
[7] X. F. Wang, G. Chen, Synchronization in scale-free dynamical networks: robustness and fragility, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications 49 (1) (2002) 54–62.
[8] V. Hamdipoor, Y. Kim, Partitioning of relative sensing networks: A stability margin perspective, Automatica 106 (2019) 294–300.
[9] R. Alessandri, M. Ji, M. Moshiri, et al., Controllability of multi-agent systems from a graph-theoretic perspective, SIAM Journal on Control and Optimization 48 (1) (2009) 162–186.
[10] Y. Y. Liu, J. J. Slotine, A. L. Barabasi, Controllability of complex networks, Nature 48 (7346) (2011) 167–173.
[11] F. Borrelli, T. Keviczky, Distributed lqr design for identical dynamically decoupled systems, IEEE Transactions on Automatic Control 53 (8) (2008) 1901–1912.
1980.

[37] J. Corfmat, A. S. Morse, Structurally controllable and structurally canonical systems, IEEE Transactions on Automatic Control 21 (1) (1976) 129–131.

[38] Z. Sun, S. S. Ge, T. H. Lee, Controllability and reachability criteria for switched linear systems, Automatica 48 (5) (2002) 775–786.

[39] X. Liu, H. Lin, B. M. Chen, Structural controllability of switched linear systems, Automatica 49 (12) 3531–3537.

[40] X. Liu, H. Lin, B. M. Chen, Graph-theoretic characterisations of structural controllability for multi-agent system with switching topology, International Journal of Control 86 (2) (2013) 222–231.

[41] M. Bürger, C. De Persis, Dynamic coupling design for nonlinear output agreement and time-varying flow control, Automatica 51 (2015) 210–222.