Universal Low Temperature Asymptotics of the Correlation Functions of the Heisenberg Chain

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Abstract. We calculate the low temperature asymptotics of a function \( \gamma \) that generates the temperature dependence of all static correlation functions of the isotropic Heisenberg chain.

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1 Introduction

Over the past few years the mathematical structure of the static correlation functions of the \( XXZ \) chain was largely resolved. After an appropriate regularization by a disorder parameter they all factorize into polynomials in only two functions \( \rho \) and \( \omega \) \cite{[3]}. These are the one-point function and a special neighbor two-point function which, in turn, can be represented as integrals over solutions of certain linear and non-linear integral equations \cite{[2]}. This resembles much the situation with free fermions, and what is behind is indeed a remarkable fermionic structure on the space of quasi-local operators acting on the spin chain \cite{[5]}. It allows us, for instance, to calculate short-range correlators with high numerical precision directly in the thermodynamic limit \cite{[1]} \cite{[12]}.

The low temperature asymptotics of \( \rho \) and \( \omega \) universally determines the low temperature properties of all static correlation functions. In this short note we obtain the low temperature asymptotics in the special case of the isotropic Hamiltonian

\[
\mathcal{H} = J \sum_j (\sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \sigma_{j-1}^z \sigma_j^z)
\]

with no magnetic field applied and vanishing disorder parameter. Then \( \rho = 1 \) and we are left with only one function (and its derivatives) which, up to a trivial factor, is the function \( \gamma \) defined in \cite{[3]}.

2 Definition of the basic function \( \gamma \)

For our purpose here it is convenient to introduce the function \( \gamma \) within the context of a special realization of a six-vertex model (see e.g. \cite{[4]}) and its associated quantum transfer matrix \cite{[10]}.

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By definition the latter has $2(N+M)$ vertical lines alternating in direction and carrying spectral parameters

$$\underbrace{u, -u, u, -u, \ldots, -u, u'}_{2N}; \underbrace{\mu_1, -\mu_1, \mu_1, -\mu_1, \ldots, \mu_1 - u'}_{2M}.$$ 

The spectral parameter on the horizontal line will be denoted $\mu_2$. We consider this system in the limit $N,M \to +\infty$ with the fine tuning $uN = i\frac{\pi}{4}$ and $u'M = i\frac{\pi}{4}$. With an appropriate overall normalization the largest eigenvalue $\Lambda(\mu_2, \mu_1)$ is given by

$$\ln(\Lambda(\mu_2, \mu_1)) = \frac{4\pi J}{T} K(\mu_2) + \frac{4\pi \delta}{T} K(\mu_2 - \mu_1)$$

$$+ \int_{-\infty}^{\infty} dt \ln \left[ \frac{(1 + b(t, \mu_1))(1 + \overline{b}(t, \mu_1))}{2 \cosh(\pi(\mu_2 - t))} \right].$$

(2.1)

Let us note that we recover the familiar system of equations, allowing us to study the thermodynamical properties of the Hamiltonian \([1]\), by setting $\delta = 0$. The function $K(x)$ is defined as

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{e^{-ikx}}{1 + e^{\pi k}}$$

$$= \frac{1}{4\pi} \left( \psi \left( 1 - \frac{x}{2} \right) - \psi \left( 1 + \frac{ix}{2} \right) - \psi \left( 1 - \frac{ix}{2} \right) + \psi \left( 1 + \frac{x}{2} \right) \right),$$

where $\psi$ is the digamma function. The auxiliary functions $b(x, \mu)$ and $\overline{b}(x, \mu)$ are solutions of a pair of non-linear integral equations given by

$$\ln(b(x, \mu_1)) = -\frac{2\pi J}{T \cosh(\pi x)} - \frac{2\pi \delta}{T \cosh(\pi(x - \mu_1))} + \int_{-\infty}^{\infty} dt K(x - t) \ln(1 + b(t, \mu_1))$$

$$- \int_{-\infty}^{\infty} dt K(x - t + i) \ln(1 + \overline{b}(t, \mu_1))$$

(2.2)

and a similar equation obtained by exchanging $b \leftrightarrow \overline{b}$ and $i \leftrightarrow -i$ in \([22]\). The function $\gamma$ can now be introduced as

$$\gamma(\mu_1, \mu_2) = -1 + (1 + (\mu_1 - \mu_2)^2) T \frac{\partial}{\partial \delta} \ln(\Lambda(\mu_2, \mu_1)) \bigg|_{\delta = 0}.$$  

(2.3)

It has been conjectured \([3]\) that the correlation functions of the isotropic Heisenberg chain at any finite temperature (for vanishing magnetic field) are polynomials in $\gamma$ and its derivatives evaluated at $(0,0)$. A similar statement (involving a function $\omega$ and its derivative with respect to the disorder parameter) was proved for the anisotropic XXZ chain \([5, 8, 2]\). Amazingly the isotropic limit seems non-trivial and is still a subject of ongoing work. Here we would only like to mention that the nearest- and next-to-nearest-neighbor two-point functions were expressed in terms of $\gamma$ in \([3]\) starting from the multiple integral representation for the density matrix of the Heisenberg chain obtained in \([7]\). The formulae for the longitudinal two-point functions are, for instance,

$$\langle \sigma_1^z \sigma_2^z \rangle_T = -\frac{1}{3} \gamma(0,0),$$

(2.4)

$$\langle \sigma_1^z \sigma_3^z \rangle_T = -\frac{1}{3} \gamma(0,0) - \frac{1}{6} \gamma_{xx}(0,0) + \frac{1}{3} \gamma_{xy}(0,0).$$

(2.5)

They will be used below to test our results for the low-temperature expansion. We denoted derivatives with respect to the first (resp. second) argument by the subscript $x$ (resp. $y$). Similar results for four sites can be obtained from \([1]\) in the isotropic limit. In previous work \([13]\) the high-temperature expansion (up to order 25) of the two-point functions was obtained analytically based on \([24]\) and \([25]\).
3 Low-temperature expansion

To compute the low-temperature expansion of $\gamma$, we follow the line of reasoning of the article [9], where a similar task was performed for the free energy. There are, however, two differences between the usual equations and the ones used in this note: the additional driving term in (2.2) proportional to $\delta$ and the shift $\mu_2$ in the kernel of the integration in (2.1).

The computation is based on the introduction of a shift $\mathcal{L} = \frac{1}{\pi} \ln \left( \pi J T \right)$ in the auxiliary functions:

$$ b_{\mathcal{L}}(x) = b(x + \mathcal{L}) \quad \text{and} \quad \tilde{b}_{\mathcal{L}}(x) = b(-x - \mathcal{L}). $$

In the low-temperature limit these functions satisfy

$$ \ln(b_{\mathcal{L}}(x, \mu_1)) \sim -4 e^{-\pi x} - 4 \delta e^{-\pi(x - \mu_1)} + D_{\mathcal{L}}(x) \quad + \int_{-\mathcal{L}}^{\infty} dt \left[ K(x - t) \ln(1 + b_{\mathcal{L}}(t, \mu_1)) - K(x - t + i) \ln(1 + \tilde{b}_{\mathcal{L}}(t, \mu_1)) \right], \quad (3.1) $$

where $D_{\mathcal{L}}(x)$ is the rest of the integral which does not contribute to the low-temperature limit, when the magnetic field vanishes (see [9]). A similar relation holds with $b \leftrightarrow \tilde{b}$ and $i \leftrightarrow -i$ exchanged.

In terms of the shifted functions the largest eigenvalue becomes

$$ \ln(\Lambda(\mu_2, \mu_1)) \sim \frac{4\pi J}{T} K(\mu_2) + \frac{4\pi \delta}{T} K(\mu_2 - \mu_1) \quad + \frac{T}{J\pi} \int_{-\mathcal{L}}^{\infty} dt e^{-\pi(\mu_2 - t)} \ln \left[ (1 + b_{\mathcal{L}}(t, \mu_1))(1 + \tilde{b}_{\mathcal{L}}(t, \mu_1)) \right] \quad + \frac{T}{J\pi} \int_{-\mathcal{L}}^{\infty} dt e^{-\pi(\mu_2 + t)} \ln \left[ (1 + b_{\mathcal{L}}(t, \mu_1))(1 + \tilde{b}_{\mathcal{L}}(t, \mu_1)) \right]. $$

To evaluate these integrals we compute

$$ I = \int_{-\mathcal{L}}^{\infty} dt \left[ \ln(1 + b_{\mathcal{L}}(t, \mu_1)) \ln(b_{\mathcal{L}}(t, \mu_1))' + \ln(1 + \tilde{b}_{\mathcal{L}}(t, \mu_1)) \ln(\tilde{b}_{\mathcal{L}}(t, \mu_1))' \right] $$

using two different methods. Here the prime stands for the derivative with respect to $t$. First, we compute it explicitly using the change of variables $z = \ln(b_{\mathcal{L}})$ or $z = \ln(\tilde{b}_{\mathcal{L}})$, respectively, which results in

$$ I = 2 \int_{-\infty}^{0} \ln(1 + e^z) dz = \frac{\pi^2}{6}. $$

Second, we replace $\ln(b_{\mathcal{L}}(t, \mu_1))$ and $\ln(\tilde{b}_{\mathcal{L}}(t, \mu_1))$ by their scaling limits (3.1) and simplify the resulting expression by taking into account that the derivative of $K(x)$ is odd and contributions by double integrals cancel pairwise. This way we obtain

$$ I = 4\pi \left( 1 + \frac{\delta}{J} e^{\pi \mu_1} \right) \int_{-\mathcal{L}}^{\infty} dt e^{-\pi t} \ln \left[ (1 + b(t, \mu_1))(1 + \tilde{b}(t, \mu_1)) \right]. $$

The same type of manipulation can be performed for the functions $\tilde{b}$, and a similar result is obtained with $\mu_1$ replaced by $-\mu_1$.

Gathering these findings we obtain the asymptotic form of the largest eigenvalue,

$$ \ln(\Lambda(\mu_2, \mu_1)) \sim \frac{4\pi J}{T} K(\mu_2) + \frac{4\pi \delta}{T} K(\mu_2 - \mu_1) \quad + \frac{T}{24 J} \left( e^{\pi \mu_2} + \frac{e^{-\pi \mu_2}}{1 + \frac{\delta}{J} e^{\pi \mu_1}} + e^{-\pi \mu_2} - \frac{\delta}{J} e^{-\pi \mu_1} \right). $$
Thus, using (2.3), the function $\gamma$ behaves asymptotically for small temperatures as

$$\gamma(\mu_1, \mu_2) \sim -1 + (1 + (\mu_1 - \mu_2)^2) \left(4\pi K(\mu_2 - \mu_1) - \frac{T^2}{12J^2} \cosh(\pi(\mu_1 + \mu_2))\right).$$

This is our main result.

Using (2.4) and (2.5), we obtain the low-temperature expansion of the longitudinal correlation functions

$$\langle \sigma_i^z \sigma_j^z \rangle_T \sim \frac{1}{3} - \frac{4}{3} \ln(2) + \frac{T^2}{J^2} \frac{1}{36},$$
$$\langle \sigma_i^z \sigma_j^z \rangle_T \sim \frac{1}{3} - \frac{16}{3} \ln(2) + 3\zeta(3) - \frac{T^2}{J^2} \frac{1}{36} \left(\frac{\pi^2}{2} - 4\right).$$

The constant terms (independent of the temperature) in these expansions are in agreement with those originally found in [11, 6]. In the figure we compare the combined low- and high-temperature results for the next-to-nearest neighbor $zz$-correlation functions with the full numerical curve obtained by implementing the linear and non-linear integral equations that determine $\gamma$ and its derivatives [3] on a computer. The high-temperature data and some additional Monte-Carlo data are taken from [14]. We find that the numerical curves (NLIE, QMC) are amazingly well approximated by its low- and high-temperature approximations.

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