Adiabatic quantum estimation

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ABSTRACT
In this paper, we address the adiabatic technique for quantum estimation of the azimuthal orientation of a magnetic field. Exactly solving a model consisting of a two-qubit system, where one of which is driven by a static magnetic field while the other is coupled with the magnetic field rotating adiabatically, we obtain the analytical expression of the quantum Fisher information (QFI). We investigate how the two-qubit system can be used to probe the azimuthal direction of the field and analyze the roles of the intensities of the magnetic fields, Dzyaloshinskii-Moriya interaction, spin-spin coupling coefficient, and the polar orientation of the rotating field on the precision of the estimation. Moreover, we discuss how the azimuthal direction of the rotating field can be estimated using only the qubit not affected by that field and investigate the conditions under which this strategy is more efficient than use of the qubit locally interacting with the adiabatically rotating field. Surprisingly, in the one-qubit scenario, it was found that an increase in spin-spin coupling may suppress the QFI, and hence estimating the direction of the magnetic field becomes more inaccurate.

KEYWORDS
XX Model; density matrix; adiabatic approximation; quantum metrology; quantum Fisher information.

1. Introduction

Estimating unknown parameters of a quantum system is a necessary task for almost all branches of science and technology. Evidently, any estimation would be associated with errors because of various factors such as natural stochasticity of the event in question or imperfection of measurement devices, and hence estimated values are usually inaccurate. In the process of estimating an unknown parameter of a quantum system, the quantum Fisher information (QFI) [1–4] is a pivotal quantity depending on the state of the system and its derivate with respect to the unknown parameter. In fact, the QFI, a fundamental notion in quantum metrology [5–10], plays an important role in quantum detection because it provides a bound to the accuracy of quantum estimation. The QFI, originally introduced by Fisher [11] and defined by the Cramér-Rao bound [12], is an extension of the Fisher information (FI) in the quantum regime. The FI concept, originating from the statistics, quantifies the accuracy of estimating the
parameters and characterizes the optimal rate at which the neighbouring states may be distinguished by measurement. On the other hand, the QFI is defined by maximizing the FI over all possible positive operator valued measurements (POVM).

A quantum probe is usually a microscopic physical system prepared in a quantum superposition. Therefore, the system may become very sensitive to decoherence effects and, in particular, to fluctuations affecting one or more parameters which should be estimated [13]. Hence, quantum sensing [14] is the art of exploiting the inherent fragility of quantum systems to design the best quantum protocols for metrological tasks. Up to now, it has been proved that the quantum probes are useful in various branches of quantum metrology such as gravitational wave detectors [15], interferometry [16] frequency spectroscopy [17], atomic clocks [18], thermometry [19–23], and even in sensing the magnetic field [24–32] (avian magnetoreception [33–40] essentially belongs to this field).

The exact measurement of magnetic fields has many applications, from archaeology to the monitoring of brain activity, the local mapping of fields in a nanodevice and submarine detection [41–45]. One of the most important applications of magnetic field sensors has been found in the area of biomagnetism [42, 46] defined as detecting the weak magnetic fields produced by the human organs such as brain and heart. In particular, measurements of the magnetic field originated from the brain are applied to diagnose epilepsy, and to investigate neural responses to auditory and visual stimuli. On the other hand, the highest sensitivities are obtained by superconducting quantum interference devices [47] utilizing large atomic ensembles or macroscopic pick-up coils [48, 49], at the expense of detection volumes in the $cm^3$ range. A much higher spatial resolution, but less sensitivity, is achieved by nitrogen-vacancy-centre sensors [24] or atoms [50–53], ions [54] or solid-state nanodevices [55], ultimately at the single-particle level, as shown in pioneering important experiments [56–59].

In this paper, we apply a two-qubit system to probe the azimuthal direction of a magnetic field adiabatically rotating. We exactly solve the Heisenberg model, a simple spin chain model utilized to simulate many physical quantum systems such as quantum dots [60], nuclear spins [61], superconductors [62], and optical lattices [63]. Moreover, Heisenberg model is ideal for generation of qubit states, thus attracting attention recently for realization of solid-based quantum computers. A steady state in the presence of a correlated dissipative Markovian noise to estimate an unknown magnetic field has been studied in [29]. Here we show that the adiabatic state is another rich resource for the parameter estimation.

This paper is organized as follows: In Sec. II we give a brief description about the quantum adiabatic theorem and the QFI. The physical model is presented in Sec. III. We study the different scenarios for estimating the magnetic field direction in Sec. IV. Finally, in Sec. V, the main results are summarized.

2. PRELIMINARIES

2.1. The Adiabatic Approximation in Closed Quantum Systems

Let us start by reviewing the adiabatic approximation in closed quantum systems, subject to unitary evolution [64]. In this situation, the evolution is governed by the time-dependent Schrödinger equation
\[ H(t)|\psi(t)\rangle = i|\dot{\psi}(t)\rangle, \]  

in which \( H(t) \) represents the Hamiltonian and \(|\psi(t)\rangle\) denotes a quantum state in a d-dimensional Hilbert space. For simplicity, it is assumed that \( H(t) \) spectrum is entirely discrete and nondegenerate. Therefore, one can define an instantaneous basis of eigenenergies by

\[ H(t)|\xi_j(t)\rangle = \xi_j(t)|\xi_j(t)\rangle, \]  

where the set of eigenvectors \(|\xi_j(t)\rangle\) is orthonormal. In this simplest case, because there is a one-to-one correspondence between energy levels and eigenstates, the adiabaticity can be then defined as the regime associated with an independent evolution of the instantaneous eigenstates of \( H(t) \), leading to level-anti-crossing phenomenon and continuous evolution of the instantaneous eigenstates at one time to the corresponding eigenstates at later times. In fact, when the system begins its evolution in a special eigenstate \(|\xi_j(0)\rangle\), then it evolves to the instantaneous eigenstate \(|\xi_j(t)\rangle\) at a later time \( t \), without any transition to the other energy levels [65]. A general validity condition for adiabatic behaviour is given by

\[ \max_{0 \leq t \leq T} \left| \frac{\langle \xi_k|\dot{H}|\xi_j\rangle}{\xi_j - \xi_k} \right| \ll \min_{0 \leq t \leq T} |\xi_j - \xi_k|, \]  

where \( T \) is the total evolution time. In the case of the degenerate spectrum, each degenerate eigenspace of \( H(t) \), instead of individual eigenstates, has independent evolution, whose validity conditions given by Eq. (3) should be considered over eigenvectors with distinct energies.

### 2.2. Quantum Fisher Information

A standard scenario in quantum metrology may be described as follows: Firstly, the probe system, prepared in an appropriate initial state, undergoes an evolution imprinting the parameter information onto the evolved state, say \( \rho(\lambda) \), and finally a POVM measurement is performed on the probe. Repeating the overall process \( N \) times, one can infer parameter \( \lambda \) from the statistics of the measurement outcomes by choosing an unbiased estimator. The accuracy of estimating \( \lambda \) is lower bounded by the quantum Cramér-Rao inequality:

\[ \Delta \hat{\lambda} \geq \frac{1}{\sqrt{NF_Q(\lambda)}}, \]  

where the QFI associated with unknown parameter \( \lambda \) encoded in quantum state \( \rho(\lambda) \) is defined as [2, 66]

\[ F_Q(\lambda) = \text{Tr} \left[ \rho(\lambda) L^2 \right] = \text{Tr} \left[ (\partial_\lambda \rho(\lambda)) L \right], \]
in which $L$, denoting the symmetric logarithmic derivative (SLD), is given by $\partial_\lambda \rho (\lambda) = \frac{1}{2} (L \rho (\lambda) + \rho (\lambda) L)$, with $\partial_\lambda = \partial / \partial \lambda$. Utilizing the spectrum decomposition of $\rho (\lambda)$, $\rho (\lambda) = \sum p_i |\phi_i \rangle \langle \phi_i|$, where $|\phi_i \rangle$ and $p_i$ represent the eigenvectors and eigenvalues of the matrix $\rho (x)$, respectively; we can rewrite the QFI as follows [67]

$$F_Q (\lambda) = \sum_{i,j} \frac{2}{p_i + p_j} |\langle \phi_i | \partial_\lambda \rho (\lambda) | \phi_j \rangle|^2$$

$$= \sum_i \left( \frac{\partial_\lambda p_i}{p_i} \right)^2 + 2 \sum_{i \neq j} \frac{(p_i - p_j)^2}{p_i + p_j} |\langle \phi_i | \partial_\lambda \rho (\lambda) | \phi_j \rangle|^2,$$

(6)

In the case of a pure quantum statistical model, i.e. $\rho (\lambda) = |\psi_\lambda \rangle \langle \psi_\lambda|$, it is possible to find following simple expression [68],

$$F_Q (\lambda) = 4 |\langle \partial_\lambda \psi_\lambda | \partial_\lambda \psi_\lambda \rangle - |\langle \psi_\lambda | \partial_\lambda \psi_\lambda \rangle|^2 .$$

(7)

where $\partial_\lambda \equiv \frac{\partial}{\partial \lambda}$.

3. Physical model

The Hamiltonian describing two spin-1/2 Heisenberg XX model with antisymmetric super-exchange Dzyaloshinskii-Moriya (DM) interaction and in the presence of external magnetic fields takes the form

$$H(t) = J \left( \sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y \right) + \vec{B}_1 \cdot \vec{\sigma}_1 + \vec{B}_2(t) \cdot \vec{\sigma}_2 + \vec{D} \cdot (\vec{\sigma}_1 \times \vec{\sigma}_2),$$

(8)

in which $J$ denotes the real spin-spin coupling coefficient. This model is called ferro-magnetic for $J < 0$ and antiferromagnetic for $J > 0$. Moreover, $\vec{\sigma}_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$ where $\sigma_j^i (i = x, y, z)$ represent the Pauli operators of subsystem $j (j = 1, 2)$ and $\vec{D} \cdot (\vec{\sigma}_1 \times \vec{\sigma}_2)$ is the DM interaction. Besides, $\vec{B}_j (j = 1, 2)$ denotes the magnetic field on site $j$. We assume that $\vec{B}_1 = B_1 \hat{z}$ and $\vec{B}_2 = B_2 \hat{n}(t)$ with the unit vector $\hat{n}(t) = (\sin \theta \cos \varphi(t), \sin \theta \sin \varphi(t), \cos \theta)$ where the azimuthal angle $\varphi(t)$ changes adiabatically such that condition (3) is satisfied. Thus, spin 1 is coupled to the static magnetic field while spin 2 is driven by the adiabatically rotating magnetic field.

Choosing basis $\{|\!\!\!\uparrow\downarrow\rangle, |\!\!\!\uparrow\!\!\downarrow\rangle, |\!\!\downarrow\!\!\uparrow\rangle, |\!\!\downarrow\downarrow\rangle\}$ and assuming that the DM coupling is along $z$ axis (i.e., $\vec{D} = D \hat{z}$), we obtain the matrix representation of the Hamiltonian as follows:

$$H(t) = \begin{pmatrix}
B_1 + B_2 \cos (\theta) & B_2 \sin (\theta) e^{-i\varphi(t)} & 0 & 0 \\
B_2 \sin (\theta) e^{i\varphi(t)} & B_2 \cos (\theta) & d & 0 \\
0 & d^* & -B_1 + B_2 \cos (\theta) & B_2 \sin (\theta) e^{-i\varphi(t)} \\
0 & 0 & B_2 \sin (\theta) e^{i\varphi(t)} & B_1 - B_2 \cos (\theta)
\end{pmatrix},$$

(9)

where $d \equiv 2(J + iD)$. After lengthy calculation, we find that the eigenvalues and
eigenvectors of $H(t)$, respectively, are given by:

\[
\begin{align*}
\xi_1 &= \frac{1}{2} \sqrt{2|d|^2 + 4B_1^2 + 4B_2^2 + 2|\xi|^2}, \\
\xi_2 &= \frac{1}{2} \sqrt{2|d|^2 + 4B_1^2 + 4B_2^2 - 2|\xi|^2} = -\xi_1, \\
\xi_3 &= \frac{1}{2} \sqrt{2|d|^2 + 4B_1^2 + 4B_2^2 - 2|\xi|^2}, \\
\xi_4 &= -\frac{1}{2} \sqrt{2|d|^2 + 4B_1^2 + 4B_2^2 - 2|\xi|^2} = -\xi_3,
\end{align*}
\]

(10)

and

\[
|\xi_j(t)\rangle = N_j \begin{pmatrix}
f_j e^{-2i\phi(t)} \\
m_j e^{-i\phi(t)} \\
g_j e^{-i\phi(t)} \\
1
\end{pmatrix}; \quad (j = 1, 2, 3, 4),
\]

(11)

where the time-independent variables $f_j, m_j, g_j$ are presented in the Appendix and $N_j$ is the normalization coefficient of the $j$th eigenstate. We see that although the eigenvalues are time-independent, the eigenstates are clearly time-dependent.

4. Adiabatic quantum estimating the direction of the rotating magnetic field

In this section, we analyze the behaviour of the QFI associated with the azimuthal orientation of the rotating magnetic field. Starting from $|\xi_j(0)\rangle$ and assuming that the field is rotating adiabatically such that condition (3) is satisfied, we find that $|\psi(t)\rangle \cong |\xi_j(t)\rangle$ up to a calculable phase not affecting the QFI computation.

4.1. Two-qubit probe

Using both qubits for probing the azimuthal orientation $\phi$ of the rotating magnetic field, we find that the associated QFI corresponding to the $j$th adiabatic state can be computed using Eq. (7) as follows

\[
QFI_j = 4|N_j|^2 \left(|g_j|^2 + |m_j|^2 + 4|f_j|^2 - |N_j|^2 \left(|g_j|^2 + |m_j|^2 + 2|f_j|^2\right)^2\right).
\]

(12)

Our numerical calculation shows that the QFI of the first (third) adiabatic state is approximately equal to the QFI of the second (fourth) one, i.e., $QFI_1 \cong QFI_2$, $QFI_3 \cong QFI_4$. Moreover, all of which qualitatively exhibit the same behaviour, and hence we focus only on the analyses of the behaviour of $QFI_1$. In addition, it is found that

\[
QFI_j(J, D) = QFI_j(D, J); \quad QFI_j(-J) = QFI_j(J); \quad QFI_j(-D) = QFI_j(D),
\]

(13)
Figure 1. The two-qubit QFI, associated with estimating the azimuthal direction of the rotating magnetic field, versus $D$ for $J = 0.1, B_2 = 3, B_1 = 1.2$ and different values of $\theta$ lying in range (a) $[0, \pi/2]$ and (b) $[\pi/2, \pi]$.

Figure 2. The two-qubit QFI, associated with estimating the azimuthal direction of the rotating magnetic field, versus $D$ for $J = 1.3, \theta = \pi/4, B_1 = 1.5$ ($B_2 = 3$) and different values of $B_2$ ($B_1$).

Figure 1 illustrates the QFI corresponding to the azimuthal orientation of the rotating field versus the DM coupling for different values of its polar orientation. We observe that when $\theta$ lies in range $[0, \pi/2]$, increasing it causes achievement in the QFI optimum values with weaker DM coupling. However, if $\theta$ lies in range $[\pi/2, \pi]$, increasing it suppresses the QFI and hence the parameter estimation becomes more inaccurate.

The QFI variations versus $D$ for different intensities of the magnetic fields are plotted in Fig. 2. It is found that when $\theta$ lies in range $[0, \pi/2]$ ($[\pi/2, \pi]$), weakening the static field, along the positive (negative) direction of $z$ axis, we can achieve the optimal estimation with weaker DM coupling (see Fig. 2(a)). In particular, applying a weak static magnetic field interacting with the first qubit leads to optimal estimation of the azimuthal angle for $D \approx 0$. On the other hand, as seen in Fig. 2(b), intensifying the rotating magnetic field may enhance the estimation. In particular, it leads to achievement in optimal values of the QFI for stronger DM coupling.

denoting that the QFI remains invariant under exchanging the values of $J$ and $D$, or reversing their signs.
Figure 3. (Color online) The one-qubit QFI, associated with estimating the azimuthal angle, versus $D$ for $J = 0.2, B_1 = 3, B_2 = 2.2$ and different values of $\theta$.

Figure 4. (Color online) The one-qubit QFI, associated with estimating the azimuthal angle, versus $D$ for $\theta = 0.5, B_1 = 3, B_2 = 2.5$ and different values of $J$.

Figure 5. (Color online) The one-qubit QFI, associated with estimating the azimuthal direction of the rotating magnetic field, versus $D$ for $J = 0.5, \theta = 1.1, B_1 = 3$ ($B_2 = 2.2$) and different values of $B_2 (B_1)$.

4.2. one-qubit probe

It is interesting to estimate the azimuthal direction of the rotating magnetic field adiabatically interacting with the second qubit, using the first spin (qubit A) as the quantum probe. Applying the adiabatic approximation, we find that the evolution of qubit A is described by the following reduced density matrix obtained by tracing the
adiabatic states on qubit B:

\[ \rho^A_j(t) = \text{Tr}_B(\xi_j \xi_j^\dagger) = |N_j|^2 \left( \sum_j \left( |f_j|^2 + |m_j|^2 \right) \left( f_j g_j^* + m_j \right) e^{-i\phi(t)} \right) \left( |g_j|^2 + 1 \right). \]  \hspace{1cm} (14)

Using Eqs. (6) and (14), one can find the following analytical expression for the QFI with respect to the azimuthal direction of the rotating field

\[ q\Omega^A_j = 4 |N_j|^4 |m_j + f_j g_j^*|^2, \]  \hspace{1cm} (15)

Our numerical computation shows that the QFIs associated with different adiabatic states are approximately equal, i.e.,

\[ q\Omega^A_1 \approx q\Omega^A_2 \approx q\Omega^A_3 \approx q\Omega^A_4. \]  \hspace{1cm} (16)

Moreover, relations similar to (13), extracted for two-qubit QFI, can be presented for one-qubit scenario.

As plotted in Fig. 3, it is seen that a decrease in \( \theta \) does not shift the optimal point of the QFI versus D. Moreover, decreasing polar angle \( \theta \) raises the optimal value of the QFI, and hence enhances the optimal precision of estimating the azimuthal direction of the rotating field.

Figure 4 illustrates the effects of the spin-spin coupling on the parameter estimation. We see that although strengthening the coupling leads to achievement in optimal values of the estimation with weaker DM interaction, an increase in spin-spin coupling may suppress the QFI in strong coupling regime, and hence we can extract less information about the azimuthal orientation of the rotating field using the other qubit as the probe.

Although in two-qubit scenario an increase in the intensity of the static field affecting the first qubit does not considerably decrease the optimal value of the QFI, in one-qubit scenario intensification of \( B_1 \) leads to the suppression of the QFI, thus reducing the optimal precision of estimation (see Fig. 5(a)). Nevertheless, as demonstrated in Fig. 5(b), intensifying the rotating field may enhance notably the precision of the estimation.

Now an important question arises: under what conditions is use of the qubit not affected by the rotating field more efficient for probing the azimuthal orientation of \( \vec{B}_2 \) than use of the qubit driven by that field? First, it should be noted that the two-qubit scenario for the estimation is always more efficient than the one-qubit one, i.e., \( \text{QFI} \geq q\Omega \). On the other hand, the following results have been obtained using numerical calculation:

a) when \( B_1 = 0 \) or \( J \gg 1 \) or \( D \gg 1 \), we find that the information extracted from the first qubit can lead to better estimation of the azimuthal angle, than the information achieved from the second one. In fact, under those conditions, we obtain \( q\Omega^A_j \geq q\Omega^B_j \); Moreover, if \( J \gg 1 \), the optimal estimation occurs for \( D = 0 \).

b) If \( J \) or \( D \) equals zero and \( B_1 = B_2 \), we find that \( q\Omega^B_j \geq q\Omega^A_j \) for \( j = 1, 2 \) while \( q\Omega^A_j \geq q\Omega^B_j \) for \( j = 3, 4 \).
5. Summary and conclusions

In this paper, we have investigated the adiabatic estimation of the direction of a magnetic field from the perspective of the QFI. In fact, we applied two qubits as potential probes, described by the Heisenberg XX model, such that one of which was driven by the rotating magnetic field, as adiabatic condition was satisfied, and the other one experienced a static magnetic field. Adiabatic evolution guarantees continuous evolution of the instantaneous eigenstates of the Hamiltonian at one time to the corresponding eigenstates at later times. After analytical computation of the two-qubit QFI associated with the azimuthal orientation of the rotating field, we exactly analyzed the effects of intensities of the magnetic fields, the DM interaction, and the coupling coefficient $J$ on the optimal estimation. In particular, we found that the polar orientation of the rotating magnetic field plays a key role in the process of estimating its azimuthal direction such that when $\theta$ lies in range $[0, \pi/2]$, increasing the polar angle causes achievement in the QFI optimum values with weaker DM coupling. However, if $\theta$ lies in range $[\pi/2, \pi]$, increasing it suppresses the QFI and hence the parameter estimation becomes more inaccurate.

We also discussed how we can estimate the azimuthal direction of the rotating field using the qubit not affected by that field. The one-qubit QFI was computed for this purpose and its behaviour was investigated in detail. We illustrated that an increase in the coupling between the qubits may suppress the QFI in strong coupling regime, and hence less information about the azimuthal orientation of the rotating field can be extracted using the qubit not driven by the rotating field. Moreover, we investigated under what conditions the use of the qubit not affected by the rotating field is more efficient for the estimation than the use of the qubit driven by that field.

Our approach and model can be extended to discuss about avian magnetoreception in the context of quantum mechanics [39]. This new branch of quantum biology proposes that migratory birds depend on the photoinduced radical pairs created in ocular cryptochrome proteins for navigation, which has been verified by intensive evidences and behavioral experiments using birds [69–73]. Because of the quantum mechanical nature of the radical pair model, a growing interest has arisen to use the field of quantum information for understanding the mechanism of avian magnetoreception [74–76].

Data accessibility. This paper does not have any experimental data.

Competing interests. We have no competing interests.

Authors’ contributions. All the authors conceived the work and agreed on the approach to pursue. H.R. planned and supervised the project, carried out the theoretical calculations and wrote the manuscript. L.F.-Sh., H.R., and M.G. extracted and discussed the results. All authors gave final approval for publication.

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Appendix A.

The time-independent coefficients appeared in (11) are given by:

\[ f_j = -\frac{2 B_2^2 d \sin^2 \theta}{P_j}, \]
\[ m_j = \frac{2 B_2 \sin \theta \ d (B_2 \cos \theta + B_1 - \xi_j)}{P_j}, \]
\[ g_{1,2} = -\frac{B_2 \sin \theta \ (4B_1^2 - 4B_1 \xi_{1,2} + |d|^2 + |\xi|^2)}{P_{1,2}}, \tag{A1} \]
\[ g_{3,4} = -\frac{B_2 \sin \theta \ (4B_1^2 - 4B_1 \xi_{3,4} + |d|^2 - |\xi|^2)}{P_{3,4}}, \]

where

\[ P_{1,2} = 4B_2 \left( \frac{|\xi|^2}{4} - \frac{|d|^2}{4} + B_1 (B_1 - \xi_{1,2}) \right) \cos \theta + (B_1 - \xi_{1,2})(|\xi|^2 - |d|^2) + 4B_1B_2^2, \tag{A2} \]
\[ P_{3,4} = 4B_2 \left( -\frac{|\xi|^2}{4} - \frac{|d|^2}{4} + B_1 (B_1 - \xi_{3,4}) \right) \cos \theta + (-B_1 + \xi_{3,4})(|\xi|^2 + |d|^2) + 4B_1B_2^2, \tag{A3} \]

and where

\[ |\xi|^2 = \sqrt{-4 \cos^2 \theta |d|^2 B_2^2 - 8 \cos \theta \ |d|^2 B_1 B_2 + |d|^4 + 4 \ |d|^2 B_2^2 + 16 B_1^2 B_2^2}. \tag{A4} \]

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