On confidence intervals for the power of F-tests

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Abstract

This note points out how confidence interval estimates for standard deviation transform into confidence interval estimates for the power of F-tests at fixed alternative means. An application is shown for the test of a two-sided hypothesis for the mean of a normal distribution.

Keywords: F-test; power; confidence intervals.

Let \( Y \) be a random variable distributed \( P_{\sigma} \) and \( (a(Y), b(Y)) \) be a 100(1 – \( \gamma \))% confidence interval estimator for the parameter \( \sigma \). If \( \omega = f(\sigma) \) is a strictly increasing function of \( \sigma \), then \((f(a(Y)), f(b(Y)))\) is a 100(1 – \( \gamma \))% confidence interval estimator for \( \omega \) and if \( \omega = f(\sigma) \) is a strictly decreasing function of \( \sigma \), then \((f(b(Y)), f(a(Y)))\) is a 100(1 – \( \gamma \))% confidence interval estimator for \( \omega \).

Tarasinska (2005) uses the above idea in discussing confidence interval estimation of the power of the one-sided t-test based on confidence interval estimation of standard deviation. We point out how the power of the F-test is easily handled as well, a result that subsumes the two-sided t-test.

Consider the power \( \omega \) of the typical \( \alpha \) level F-test of the mean vector based on \( X_1, X_2, \ldots, X_n \) independent, normally distributed random variables with common standard deviation \( \sigma \). Under the standard model, the F-statistic has the distribution of \( (U/u)/(V/v) \) where \( U \) is distributed
noncentral chi-square with \( u \) degrees of freedom and noncentrality parameter \( \delta \), \( V \) is distributed central chi-square with \( v \) degrees of freedom, and \( U \) and \( V \) are independent. At fixed alternative mean vectors, the noncentrality parameter \( \delta = \delta(\sigma) \) is a strictly decreasing function of the scale parameter \( \sigma \). (See Scheffé (1959, p. 39).) As an example consider the two-way, normal theory ANOVA with \( K \) observations in each of \( IJ \) cells, and the F-test that all interactions are 0. Here \( u = (I-1)(J-1) \), \( v = (K-1)IJ \) and the noncentrality parameter is \( \delta(\sigma) = \sqrt{K\sum(\alpha\beta\gamma^2/\sigma)} \).

Let \( G_{u,v,\delta} \) denote the cdf of \( (U/u)/(V/v) \), that is, the noncentral F-distribution. It is easy to see from the representation \( (U/u)/(V/v) \) that

\[
\text{Power} = \omega = 1 - G_{u,v,\delta}(c) = 1 - P(U \leq cv) = 1 - E[F_{u,\delta}(cuV/v)] \tag{1}
\]

where the expectation \( E \) is with respect to \( V \), \( F_{u,\delta} \) denotes the cdf of \( U \), and \( c \) is the \( 1-\alpha \) quantile of \( G_{u,v,0} \). Now \( F_{u,\delta}(cuV/v) \) is the probability content of an origin centered hypersphere of radius \( r = \sqrt{cuV/v} \) under translation of an origin centered spherical multivariate normal distribution by a distance \( \delta \). That this is monotone decreasing in \( \delta \) may be taken as obvious by some. It follows from more general results in Anderson (1955). For a direct proof start with the fact that

\[
F_{u,\delta}(r^2) = \delta^{1-u/2}e^{-\delta^2/2} \int_0^r x^{u/2}e^{-x^2/2}I_{u/2-1}(\delta x)dx \tag{2}
\]

where \( I_w \) denotes the modified Bessel function of the first kind and order \( w \). (For example, see Ruben (1962, (3.5))). The derivative with respect to \( \delta \) is calculated in Gilliland (1964) where we find

\[
dF_{u,\delta}(r^2)/d\delta = -r^{u/2}\delta^{1-u/2}e^{-(r^2+\delta^2)/2}I_{u/2}(r\delta). \tag{3}
\]

Since \( I_{u/2}(r\delta) > 0 \) for \( r\delta > 0 \), we see that \( F_{u,\delta}(r^2) \) is a strictly decreasing function of \( \delta \). Together with (1) and the fact that \( \delta = \delta(\sigma) \) is strictly decreasing in \( \sigma \), we see that the power \( \omega \) is a strictly decreasing function of \( \sigma \). As seen, all the necessary monotonicities are easily established if not very widely known.

It follows that any 100\((1-\gamma)\)% confidence interval estimator for \( \sigma \) based on the residual sum of squares \( V \) transforms to a 100\((1-\gamma)\)% confidence interval estimator for the power \( \omega \). The usual 100\((1-\gamma)\)% CI interval estimate for \( \sigma \) is \( a < \sigma < b \) where \( a = \sqrt{V/B} \) and \( b = \sqrt{V/A} \) with \( A \) and \( B \) such that \( F_{v,0}(B) - F_{v,0}(A) = 1-\gamma \). The typical choices are \( A = F_{v,0}^{-1}(\gamma/2) \) and
\[ B = F_{v,0}^{-1}(1 - \gamma/2). \] The corresponding 100(1 - \gamma)\% confidence interval estimator for \( \omega \) is

\[ 1 - G_{u,v,\delta(b)}(c) < \omega < 1 - G_{u,v,\delta(a)}(c). \] (4)

Tarasinska (2005, pp. 126-127, Table 1) proposes using positions \( A \) and \( B \) to minimize the length of the confidence interval for the power of the one-sided t-test. The idea applied to the power of the F-test would be to choose \( A \) and \( B \) to minimize

\[ L = G_{u,v,\delta(\sqrt{V/A})}(c) - G_{u,v,\delta(\sqrt{V/B})}(c) \] (5)

subject to the constraint \( F_{v,0}(B) - F_{v,0}(A) = 1 - \gamma \). In this case, the minimizing \( A \) and \( B \) depend upon \( V \) and the resulting intervals for power \( \omega \) and their corresponding intervals for \( \sigma \) are not shown to have 100(1 - \gamma)\% coverage probability (See Gilliland and Li (2007)).

Let \( Y_1, \ldots, Y_n \) be a random sample from a normal population with mean \( \mu \) and variance \( \sigma^2 \). Consider the test for the null hypothesis \( H_0 : \mu = \mu_o \) against the two-sided alternative hypothesis \( H_1 : \mu \neq \mu_o \) with the level of significance \( \alpha \). The null hypothesis \( H_0 \) is rejected if

\[ |\bar{Y} - \mu_o| > \frac{S}{\sqrt{n - 1}} \sqrt{c} \]

in which \( \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \), \( S^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \) and \( c \) are the sample mean, the sample variance and the \( 1 - \alpha \) quantile of the central F-distribution \( G_{1,n-1,0} \), respectively.

The power function of the test (Lehmann, 1991) is defined as:

\[ P(|\bar{Y} - \mu_o| > \frac{S}{\sqrt{n - 1}} \sqrt{c}) = 1 - G_{1,n-1,\delta(\sigma)}(c). \] (6)

where \( \delta(\sigma) = \sqrt{n} \left| \frac{\mu - \mu_o}{\sigma} \right| \) is non-centrality parameter. By using the invariant properties of Maximum Likelihood Estimators (MLE), the MLE of power function in (6) is

\[ 1 - G_{1,n-1,\delta(S)}(c). \] (7)

Let \( (a,b) \) be any 100(1 - \gamma)\% confidence interval for \( \sigma \), then by (4) the confidence interval for power function is

\[ \{1 - G_{1,n-1,\delta(b)}(c), 1 - G_{1,n-1,\delta(a)}(c)\}. \] (8)

The curve of the confidence intervals in (8) and the MLE estimates in (7) for the power function, as functions of \( \frac{\mu - \mu_o}{S} \) for \( n = 10 \) and \( \alpha = 0.05 \), are given in Figure 1.
Figure 1. CI curves for the test power and the estimate of the power as the functions of $\frac{\mu - \mu_0}{\sigma}$.

Acknowledgement: We are grateful to the editor and referee for their helpful comments and suggestions which improved the presentation of the result.

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