DYNAMICS OF WEAK SOLUTIONS FOR THE THREE DIMENSIONAL NAVIER-STOKES EQUATIONS WITH NONLINEAR DAMPING

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ABSTRACT. The main objective of this paper is to study the existence of a finite dimensional global attractor for the three dimensional Navier-Stokes equations with nonlinear damping for $r > 4$. Motivated by the idea of [1], even though we can obtain the existence of a global attractor for $r \geq 2$ by the multi-valued semiflow, it is very difficult to provide any information about its fractal dimension. Therefore, we prove the existence of a global attractor in $H$ and provide the upper bound of its fractal dimension by the methods of $\ell$-trajectories in this paper.

1. Introduction. In this paper, we mainly investigate the long-time behavior of weak solutions for the following three dimensional Navier-Stokes equations with nonlinear damping:

$$
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p + \beta |u|^{r-2} u &= f(x), \quad (x, t) \in \Omega \times \mathbb{R}^+, \\
\nabla \cdot u &= 0, \quad (x, t) \in \Omega \times \mathbb{R}^+, \\
u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times \mathbb{R}^+, \\

u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial \Omega$ and $\mathbb{R}^+ = [0, +\infty)$, $\nu > 0$ is the viscosity, $r > 4$, $\beta > 0$ is constant, $p(x, t)$ is the fluid pressure, $u(x, t)$

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is the velocity of the fluid, $f(x)$ is the given external body force, $\beta |u|^{r-2} u$ is the nonlinear damping.

The damping comes from the resistance to the motion of the flow, to which various physical phenomena such as porous media flow, drag or friction effects are related (see [2, 3, 10, 11]). In the case of $\beta = 0$, problem (1) reduces to the three dimensional incompressible Navier-Stokes equations which has been studied extensively (see [1, 6, 7, 8, 9, 15, 19, 21, 22]). However, the uniqueness of weak solutions and the global existence of strong solutions for the three dimensional incompressible Navier-Stokes equations remain open until now. Therefore, many authors turn to consider the well-posedness and the long-time behavior of solutions for problem (1) with $\beta > 0$ (see [4, 12, 13, 14, 24, 25, 26, 27, 31, 33]). In particular, in [4], the authors have proved the existence of global weak solutions for $r \geq 2$, and the existence of global strong solutions for $r \geq \frac{9}{4}$ and the uniqueness of strong solutions for $\frac{9}{4} \leq r \leq 6$, respectively, for the three dimensional Navier-Stokes equations with nonlinear damping in the whole space. The results established in [4] have been improved in [31], they proved the existence of global strong solutions for the three dimensional Navier-Stokes equations with nonlinear damping in the whole space with $u_0 \in H^1(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ under the assumptions that $r \geq 4$ and the uniqueness of strong solutions for the three dimensional Navier-Stokes equations with nonlinear damping in the whole space under the assumptions that $4 < r \leq 6$ as well as $r = 4$ and sufficiently large $\beta > 0$. In [33], the author has proved the existence of global strong solutions for the three dimensional Navier-Stokes equations with nonlinear damping in the whole space with $u_0 \in H^1(\mathbb{R}^3)$ under the assumption that $r > 4$ as well as $r = 4$ and sufficiently large $\beta > 0$, and also proved that the strong solution is unique in the class of weak solutions for any $r \geq 2$, which are significant improvements of those results established in [4]. In [24, 25, 27], the authors have established the existence of global attractors, uniform attractors and pullback attractors, respectively, in $V$ and $H^2(\Omega) \cap V$ for the three dimensional Navier-Stokes equations with nonlinear damping by combining asymptotic a priori estimates with Sobolev compactness embedding Theorem. The existence of an exponential attractor in $V$ for the three dimensional Navier-Stokes equations with nonlinear damping was proved by using the squeezing property in [26].

The authors have proved the uniqueness of weak solutions for the three dimensional Navier-Stokes equations in [28], when the weak solution additionally belongs to the class $L^8(0,T;L^4(\Omega))$, which entails that the uniqueness of weak solution for the three dimensional Navier-Stokes equations with nonlinear damping for $r \geq 8$. Therefore, we can consider the existence of a global attractor for the three dimensional Navier-Stokes equations with nonlinear damping for $r \geq 8$ by the classical theory of the infinite dimensional dynamical systems. Meanwhile, motivated by the idea of [1], we can also obtain the existence of a global attractor for the three dimensional Navier-Stokes equations with nonlinear damping for $r \geq 2$ by using the abstract theory for multi-valued semi-flow in the infinite dimensional dynamical systems, but it is very difficult to provide any information about its fractal dimension.

In this paper, we mainly consider the existence of a finite dimensional global attractor for the three dimensional Navier-Stokes equations with nonlinear damping in the case of $r > 4$. If $r \in (4,8)$, it is very tricky to prove the uniqueness of weak solutions for the three dimensional Navier-Stokes equations with nonlinear damping. Therefore, we cannot define a solution semigroup on $H$ such that
the long-time behavior of weak solutions for the three dimensional Navier-Stokes equations with nonlinear damping can be investigated by the classical theory of the infinite dimensional dynamical systems. Fortunately, we know from the work of Y. Zhou in [33] that any weak solutions of problem (1) for \( r > 4 \) will become a unique strong solution after any \( t > 0 \). Inspired by the idea of the method of \( \ell \)-trajectories for any small \( \ell > 0 \) proposed in [17], we mainly study the existence of a finite dimensional global attractor in \( H \) for the three dimensional Navier-Stokes equations with nonlinear damping by the methods of \( \ell \)-trajectories. To the best of our knowledge, the method of \( \ell \)-trajectories is based on an observation that the limit behavior of solutions to a dynamical system in an original phase space can be equivalently captured by the limit behavior of \( \ell \)-trajectories which are continuous parts of solution trajectories that are para-metrized by time from an interval of the length \( \ell \) with \( \ell > 0 \) and it can weaken the requirements on the regularity of the solution.

Throughout this paper, let \( C \) be the generic positive constants independent of initial data and let \( C(\cdot) \) be the positive constants depending on . let \( H^s(\Omega) \ (s \in \mathbb{N}) \) and \( L^q(\Omega) \ (1 \leq q \leq \infty) \), respectively, be the usual vector or scalar Sobolev spaces and Lebesgue space.

2. Preliminaries. In order to study problem (1), we introduce the space of divergence-free functions defined by

\[
\mathcal{V} = \{ u \in (C^\infty_c(\Omega))^3 : \nabla \cdot u = 0 \}.
\]

Denote by \( H \) and \( V \) the closure of \( \mathcal{V} \) with respect to the norms in \( L^2(\Omega) \) and \( H^1_0(\Omega) \), respectively, let \( X' \) be the dual space of Banach space \( X \) and let \( A = -P\Delta \) be the Stokes operator, where \( P \) is the Leray-Helmhotz projection from \( L^2(\Omega) \) onto \( H \).

In what follows, we give the definition of solutions for the three dimensional Navier-Stokes equations with nonlinear damping (1).

**Definition 2.1.** Assume that \( u_0 \in H \) and \( f \in H \). For any fixed \( T > 0 \), a function \((u(x, t), p(x, t))\) is called a weak solution of problem (1) on \((0, T)\), if

\[
\begin{align*}
    u(t) &\in L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^r(0, T; L^r(\Omega)), \\
    u_t(t) &\in L^2(0, T; V') + L^{\frac{2}{r'}}(0, T; L^{\frac{2r}{r'-2}}(\Omega))
\end{align*}
\]

satisfy

\[
\begin{align*}
    &\int_{\Omega} u(t_1) \cdot v \, dx + \nu \int_{t_0}^{t_1} \int_{\Omega} \nabla u(t) \cdot \nabla v \, dx \, dt + \int_{t_0}^{t_1} \int_{\Omega} [(u(t) \cdot \nabla)u(t)] \cdot v \, dx \, dt \\
    &\quad + \beta \int_{t_0}^{t_1} \int_{\Omega} |u(t)|^{r-2} u(t) \cdot v \, dx \, dt \\
    &= \int_{t_0}^{t_1} \int_{\Omega} f(x) \cdot v \, dx \, dt + \int_{\Omega} u(t_0) \cdot v \, dx
\end{align*}
\]

for any \( v \in V \) and any \( t_0, t_1 \in [0, T] \), and \( u(x, 0) = u_0(x) \) in the sense of trace.

Moreover, if \( u_0 \in V \), a weak solution is called a strong solution of problem (1) on \((0, T)\), in addition, if it satisfies

\[
\begin{align*}
    u(t) &\in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)) \cap L^r(0, T; L^r(\Omega))
\end{align*}
\]
where \( \mathcal{C}([0, T]; V_w) \) denotes the space of weakly continuous functions with range in \( V \) defined on the interval \([0, T]\).

**Lemma 2.2.** ([5, 17, 18, 23]) Assume that \( p_1 \in (1, \infty) \), \( p_2 \in [1, \infty) \). Let \( X \) be a Banach space and let \( X_0, X_1 \) be separable and reflexive Banach spaces such that \( X_0 \subset \subset X \subset X_1 \). Then

\[
Y = \{ u \in L^{p_1} (0, \ell; X_0) : u' \in L^{p_2} (0, \ell; X_1) \} \subset \subset L^{p_1} (0, \ell; X),
\]

where \( \ell \) is a fixed positive constant.

**Definition 2.3.** ([16]) Let \( X \) be a Banach space and let \( \{ S(t) \}_{t \geq 0} \) be a family of operators on \( X \). We say that \( \{ S(t) \}_{t \geq 0} \) is a norm-to-weak continuous semigroup on \( X \), if \( \{ S(t) \}_{t \geq 0} \) satisfies that

1. \( S(0) = Id \) (the identity),
2. \( S(t) S(s) = S(t+s), \forall t, s \geq 0 \),
3. \( S(t_n)x_n \to S(t)x \), if \( t_n \to t \) and \( x_n \to x \) in \( X \).

**Lemma 2.4.** ([32]) Let \( X, Y \) be two Banach spaces, and let \( X^*, Y^* \) be the dual spaces of \( X, Y \), respectively. If \( X \) is dense in \( Y \), the injection \( i : X \to Y \) is continuous and its adjoint \( i^* : Y^* \to X^* \) is dense, \( \{ S(t) \}_{t \geq 0} \) be a semigroup on \( X \) and \( Y \), respectively, and assume furthermore that \( \{ S(t) \}_{t \geq 0} \) is continuous or weak continuous on \( Y \). Then \( \{ S(t) \}_{t \geq 0} \) is a norm-to-weak continuous semigroup on \( X \) if and only if \( \{ S(t) \}_{t \geq 0} \) maps compact subsets of \( X \times \mathbb{R}^+ \) into bounded sets of \( X \).

**Definition 2.5.** ([20, 29]) Let \( \{ S(t) \}_{t \geq 0} \) be a semigroup on a Banach space \( X \). A set \( A \subset X \) is said to be a global attractor if the following conditions hold:

1. \( A \) is compact in \( X \).
2. \( A \) is strictly invariant, i.e., \( S(t)A = A \) for any \( t \geq 0 \).
3. For any bounded subset \( B \subset X \) and for any neighborhood \( O = \mathcal{O}(A) \) of \( A \) in \( X \), there exists a time \( t_0 \) such that \( S(t)B \subset \mathcal{O}(A) \) for any \( t \geq t_0 \).

**Lemma 2.6.** ([18]) Let \( X \) be a (subset of) Banach space and let \( \{ S(t), X \} \) be a dynamical system. Assume that there exists a compact set \( K \subset X \) which is uniformly absorbing and positively invariant with respect to \( S(t) \). Moreover, let \( S(t) \) be continuous on \( K \). Then \( \{ S(t), X \} \) has a global attractor.

**Lemma 2.7.** ([32]) Let \( X \) be a Banach space and let \( \{ S(t) \}_{t \geq 0} \) be a norm-to-weak continuous semigroup on \( X \). Then \( \{ S(t) \}_{t \geq 0} \) has a global attractor \( A \) in \( X \) if the following conditions hold true:

1. \( \{ S(t) \}_{t \geq 0} \) has a bounded absorbing set \( B_0 \) in \( X \),
2. \( \{ S(t) \}_{t \geq 0} \) is asymptotically compact.

**Definition 2.8.** ([20, 29]) Let \( H \) be a separable real Hilbert space. For any non-empty compact subset \( K \subset H \), the fractal dimension of \( K \) is the number

\[
d_F(K) = \limsup_{\epsilon \to 0^+} \frac{\log(N_\epsilon(K))}{\log(\epsilon)},
\]

where \( N_\epsilon(K) \) denotes the minimum number of open balls in \( H \) with radii \( \epsilon > 0 \) that are necessary to cover \( K \).
Lemma 2.9. ([18]) Let $X$, $Y$ be norm spaces such that $X \subset Y$ and $A \subset Y$ be bounded. Assume that there exists a mapping $L$ such that $LA = A$ and $L : Y \to X$ is Lipschitz continuous on $A$. Then $d_{F}(A)$ is finite.

Lemma 2.10. ([18]) Let $X$ and $Y$ be two metric spaces and $f : X \to Y$ be $\alpha$-Hölder continuous on the subset $A \subset X$. Then

$$d_{F}(f(A), Y) \leq \frac{1}{\alpha}d_{F}(A, X).$$

In particular, the fractal dimension does not increase under a Lipschitz continuous mapping.

3. The existence of a global attractor.

3.1. The well-posedness of solutions. The well-posedness of solutions for the three dimensional Navier-Stokes equations with nonlinear damping was obtained in [4, 33]. Here we only state the result as follows.

Theorem 3.1. Assume that $f \in H$ and $r \geq 2$. Then for any $u_{0} \in H$, there exists at least one solution $u(t) \in C([0,T];H_{u}) \cap L_{loc}^{2}((0,T];V)$ of problem (1). Furthermore, if $r > 4$ and $u_{0} \in V$, then there exists a unique global strong solution of problem (1), which depends continuously on the initial data with respect to the topology of $H$.

Corollary 1. Assume that $f \in H$ and $u_{0m} \to u_{0}$ in $H$, let $u_{m}(t)$ be a sequence of weak solution for problem (1) such that $u_{m}(0) = u_{0m}$. For any $T > 0$, if there exists a subsequence converging ($*$-) weakly in spaces $\{v(t) \in L^{\infty}(0,T;H) \cap L^{2}(0,T;V) : v_{t} \in L^{2}(0,T;V) + L^{\infty}(0,T;L^{\infty}(\Omega))\}$ to a certain function $u(t)$. Then $u(t)$ is a weak solution on $[0,T]$ with $u(0) = u_{0}$.

3.2. The existence of a global attractor in $X_{\ell}$. In this subsection, we will consider the existence of global attractors for problem (1) by using the $\ell$-trajectory method. From Theorem 3.1, we deduce that for any $t > 0$, there exist some $t_{0} \in (0,t)$ such that $u(t_{0}) \in V$. Therefore, we infer from Theorem 3.1 that there exists a unique strong solution of problem (1) from $u(t_{0})$. However, many trajectories may start from the same initial data $u_{0} \in H$. Denote by $[\chi^{\beta}(\tau, u_{0})]_{\tau \in [0,\ell]}$, for short $\chi^{\beta}(\tau, u_{0})$ ($\beta \in \Gamma_{u_{0}}$), where $\Gamma_{u_{0}}$ is the set of indices marking trajectories starting from $u_{0}$. In the following, we first give the mathematical framework of attractor.

Definition 3.2. Let $\ell$ be a fixed positive constant. Define

$$X_{\ell} = \bigcup_{u_{0} \in H} \bigcup_{\beta \in \Gamma_{u_{0}}} \chi^{\beta}(\tau, u_{0})$$

equipped with the topology of $L^{2}(0,\ell;H)$.

Since $X_{\ell} \subset C([0,\ell];H_{u})$, it makes sense to talk about the point values of trajectories. On the other hand, it is not clear whether $X_{\ell}$ is closed in $L^{2}(0,\ell;H)$ and hence $X_{\ell}$ in general is not a complete metric space. In what follows, we first give the definition of some operators.

For any $t \in [0,1]$, we define the mapping $e_{t} : X_{\ell} \to H$ by

$$e_{t}(\chi) = \chi(t\ell)$$

for any $\chi \in X_{\ell}$.

The operators $L_{t} : X_{\ell} \to X_{\ell}$ are given by the relation

$$L_{t}(\chi(\tau, u_{0})) = u(t + \tau, u_{0}), \quad \tau \in [0,\ell]$$
for any \( \chi(\tau, u_0) \in X_\ell \), where \( u(t) \) is the unique solution of problem (1) on \([0, \ell + t]\) such that \( u|_{[0,\ell]} = \chi(\tau, u_0) \), we can easily prove the operators \( \{L_t\}_{t \geq 0} \) is a semigroup on \( X_\ell \).

In what follows, we perform some a priori estimates of solutions for problem (1) to construct the existence of a positively invariant subset in \( H \) of the solution operator for problem (1).

**Theorem 3.3.** Assume that \( f \in H \). Then there exists a positive constant \( \rho_1 \) satisfying for any bounded subset \( B \subset H \), there exists a time \( \tau_1 = \tau_1(B) > 0 \) such that for any weak solutions of problem (1) with initial data \( u_0 \in B \), we have
\[
\|\nabla u(t)\|_{L^2(\Omega)}^2 + \|u(t)\|_{L^r(\Omega)}^r \leq \rho_1
\]
for any \( t \geq \tau_1 \).

**Proof.** Multiplying the first equation of (1) by \( u \) and integrating the resulting equality over \( \Omega \), we find
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(\Omega)}^2 + \nu \|\nabla u(t)\|_{L^2(\Omega)}^2 + \beta \|u(t)\|_{L^r(\Omega)}^r = \int_{\Omega} f(x) \cdot u(x, t) \, dx
\]
\[
\leq \|f\|_{L^2(\Omega)} \|u(t)\|_{L^2(\Omega)} + \frac{1}{2} \nu \|\nabla u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \beta \|u(t)\|_{L^r(\Omega)}^r,
\]
which implies that
\[
\frac{d}{dt} \|u(t)\|_{L^2(\Omega)}^2 + \frac{\nu \lambda_1}{2} \|u(t)\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|\nabla u(t)\|_{L^2(\Omega)}^2 + 2\beta \|u(t)\|_{L^r(\Omega)}^r \leq \frac{1}{\nu \lambda_1} \|f\|_{L^2(\Omega)}^2.
\]
(2)

It follows from the classical Gronwall inequality that
\[
\|u(t)\|_{L^2(\Omega)}^2 \leq e^{-\frac{\nu \lambda_1}{2} t} \|u_0\|_{L^2(\Omega)}^2 + \frac{2}{\nu \lambda_1} \|f\|_{L^2(\Omega)}^2,
\]
entails that for any bounded subset \( B \) of \( H \), there exists some time \( t_0 = t_0(B) \) such that for any \( u_0 \in B \), we have
\[
\|u(t)\|_{L^2(\Omega)}^2 \leq 1 + \frac{2}{\nu \lambda_1} \|f\|_{L^2(\Omega)}^2
\]
(3)

for any \( t \geq t_0 \).

Moreover, integrating (2) from \( t \) to \( t + 1 \), we obtain for any \( t \geq t_0 \),
\[
\frac{\nu}{2} \int_t^{t+1} \|\nabla u(s)\|_{L^2(\Omega)}^2 \, ds + 2\beta \int_t^{t+1} \|u(s)\|_{L^r(\Omega)}^r \, ds \leq 1 + \frac{1}{\nu \lambda_1} \|f\|_{L^2(\Omega)}^2 + \frac{2}{\nu \lambda_1} \|f\|_{L^2(\Omega)}^2.
\]
(4)

Taking the \( H \) inner product of the first equation of (1) with \( Au + |u|^{r-2}u \) and using Young inequality, we find
\[
\frac{d}{dt} \left( \frac{1}{2} \|\nabla u(t)\|_{L^2(\Omega)}^2 + \frac{\beta + \nu}{r} \|u(t)\|_{L^r(\Omega)}^r \right) + \nu \|\nabla u(t)\|_{L^2(\Omega)}^2 + (\beta + \nu) \int_\Omega |u(t)|^{r-2} |\nabla u(t)|^2 \, dx
\]
\[
+ \frac{4(r-2)\beta + \nu}{r^2} \int_\Omega |\nabla u(t)|^2 \, dx + \beta \|u(t)\|_{L^{2(r-2)}(\Omega)}^{2r-2}
\]
be a bounded subset of $H$, \( u(t) \) and \( \varrho \) is a positive constant. We infer from the uniform Gronwall inequality and (4)-(5) that there exists a positive constant \( q_1 = q_1(\nu, \lambda_1, \beta, r, \|f\|_{L^2(\Omega)}) \) such that for any \( t \geq t_0 \),

\[
\|\nabla u(t+1)\|^2_{L^2(\Omega)} + \|u(t+1)\|^2_{L^r(\Omega)} \leq q_1.
\]  

We infer from the uniform Gronwall inequality and (4)-(5) that there exists a positive constant \( q_1 = q_1(\nu, \lambda_1, \beta, r, \|f\|_{L^2(\Omega)}) \) such that for any \( t \geq t_0 \),

\[
\|\nabla u(t+1)\|^2_{L^2(\Omega)} + \|u(t+1)\|^2_{L^r(\Omega)} \leq q_1.
\]  

Let

\[
B_0 = \left\{ u \in V \cap L^r(\Omega) : \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^r(\Omega)} \leq \rho_1 \right\}
\]

be a bounded subset of \( H \), we infer from Theorem 3.3 that there exists a time \( t_1 = t_1(B_0) \geq 0 \) such that for any \( u_0 \in B_0 \) and any \( t \geq t_1 \), we have

\[
u(t) \in B_0,
\]

where \( u(t) \) is the solution of problem (1) with initial data \( u_0 \in B_0 \).

Define

\[
A(t, u_0) = \left\{ u(t) : u(t) \text{ is the solution of problem (1) with initial data } u_0 \right\},
\]

\[
B_1 = \bigcup_{t \in [0, t_1]} \{ A(t, u_0) : u_0 \in B_0 \},
\]

\[
B_2 = \overline{B_1}^t
\]

and

\[
B_0^\ell = \{ \chi \in X_\ell : c_0(\chi) \in B_2 \},
\]

from the proof of absorbing balls of Theorem 3.3, we deduce

\[
\{ A(t, u_0) : u_0 \in B_1 \} \subset B_1
\]

for any \( t \geq 0 \) and \( B_1 \) is a bounded subset of \( V \cap L^r(\Omega) \). Moreover, we have the following conclusion.
Proposition 1. Assume that $B_1$ is the bounded subset of $V \cap L^r(\Omega)$ defined above. Then $B_2 = B_1^H$ is also a bounded subset of $V \cap L^r(\Omega)$ satisfying

$$\{ A(t, u_0) : u_0 \in B_2 \} \subset B_2$$

for any $t \geq 0$.

Proof. From the definition of $B_2$, we infer that for any $x \in B_2$, there exists a sequence $\{ x_n \}_{n=1}^\infty \subset B_1$ such that

$$x_n \to x \text{ in } H, \text{ as } n \to \infty.$$

Since $x_n$ is uniformly bounded in $V \cap L^r(\Omega)$ and $V \cap L^r(\Omega)$ is a reflexive Banach space, we deduce that there exist some $y \in V \cap L^r(\Omega)$ and a subsequence $\{ x_{n_j} \}_{j=1}^\infty$ of $\{ x_n \}_{n=1}^\infty$ such that

$$x_{n_j} \to y \text{ in } V \cap L^r(\Omega), \text{ as } j \to \infty.$$

From the compactness of $V \cap L^r(\Omega) \subset H$ and the lower semi-continuity of $\| \cdot \|$, we obtain

$$x = y$$

and

$$\| x \|_{V \cap L^r(\Omega)} \leq \liminf_{n \to +\infty} \| x_n \|_{V \cap L^r(\Omega)}.$$

Therefore, $B_2$ is a bounded subset of $V \cap L^r(\Omega)$.

For any $x \in B_2$ and any fixed $t > 0$, there exists a sequence $\{ x_n \}_{n=1}^\infty \subset B_1$ such that $x_n \to x$ in $H$ as $n \to \infty$, we infer from Theorem 3.1 that $A(t, x_n) \to A(t, x)$ in $H$ as $n \to \infty$. Notice that $A(t, x_n) \in B_1$ for any $n \in \mathbb{Z}^+$, we obtain $A(t, x) \in B_2$.

Therefore, we obtain

$$\{ A(t, u_0) : u_0 \in B_2 \} \subset B_2$$

for any $t \geq 0$. \qed

From Theorem 3.3, we immediately obtain the following result.

Corollary 2. Assume that $f \in H$. Then for any bounded subset $B^\ell \subset X_\ell$, there exists a time $\tau_2 = \tau_2(B^\ell) > 0$ such that for any weak solutions of problem (1) with short trajectory $\chi \in B^\ell$, we have

$$\| \nabla u(t) \|_{L^2(\Omega)}^2 + \| u(t) \|_{L^r(\Omega)}^r \leq \rho_1$$

for any $t \geq \tau_2$.

Next, we establish the existence of a positively invariant, compact absorbing set of the semigroup $\{ L_\ell \}_{\ell \geq 0}$ in $X_\ell$.

Theorem 3.4. Assume that $f \in H$. Then there exists a positive constant $\rho_2$ satisfying for the $B^\ell_0$, there exists some time $\tau_3 = \tau_3(B^\ell_0) > 0$ such that for any weak solutions of problem (1) with short trajectory $\chi(\tau, u_0) \in B^\ell_0$, we have

$$\int_0^\ell \| \nabla u(t+\tau) \|_{L^2(\Omega)}^2 d\tau + \int_0^\ell \| u(t+\tau) \|_{L^r(\Omega)}^r d\tau + \int_0^\ell \| Au(t+\tau) \|_{L^2(\Omega)}^2 d\tau$$

$$+ \int_0^\ell \int \| \nabla u(t+\tau) \|_{L^2(\Omega)}^2 d\tau + \int_0^\ell \| u(t+\tau) \|_{L^{2r-2}(\Omega)}^{2r-2} d\tau + \int_0^\ell \| u(t+\tau) \|_{L^2(\Omega)}^2 d\tau \leq \rho_2$$

for any $t \geq \tau_3$. 


Proof. From (2), we know
\[
\frac{d}{ds} \left( e^{\frac{\nu}{2\lambda_1}} \|u(s)\|_{L^2(\Omega)}^2 \right) \leq \frac{1}{\nu\lambda_1} \|f\|_{L^2(\Omega)}^2 e^{\frac{\nu}{2}\tau}.
\]
(7)

For any \( \tau \in (0, \ell) \), integrating (7) with respect to \( s \) from \( \tau \) to \( t + \tau \) and integrating the resulting inequality over \( (0, \ell) \) with respect to \( \tau \), we obtain
\[
\int_0^\ell \|u(t + \tau)\|_{L^2(\Omega)}^2 d\tau \\
\leq e^{-\frac{\nu}{2} \ell} \int_0^\ell \|u(\tau)\|_{L^2(\Omega)}^2 d\tau + \frac{2\ell}{\nu^2\lambda_1^2} \|f\|_{L^2(\Omega)}^2
\]
\[
\leq e^{-\frac{\nu}{2} \ell} \left( \frac{2}{\nu\lambda_1} \|u_0\|_{L^2(\Omega)}^2 + \frac{2}{\nu^2\lambda_1^2} \|f\|_{L^2(\Omega)}^2 \right) + \frac{2\ell}{\nu^2\lambda_1^2} \|f\|_{L^2(\Omega)}^2.
\]
(8)

From Proposition 1 and inequality (8), we deduce that there exists some time \( t_2 = t_2(B_0^\ell) \) such that
\[
\int_0^\ell \|u(t + \tau)\|_{L^2(\Omega)}^2 d\tau \leq \frac{3\ell}{\nu^2\lambda_1^2} \|f\|_{L^2(\Omega)}^2
\]
for any \( t \geq t_2 \).

Integrating (2) between \( t - s \) and \( t + \ell \) with \( t \geq s \) and \( s \in (0, \frac{\ell}{2}) \), we obtain
\[
\frac{\nu}{2} \int_0^\ell \|\nabla u(t + \tau)\|_{L^2(\Omega)}^2 d\tau + 2\beta \int_0^\ell \|u(t + \tau)\|_{L^r(\Omega)}^r d\tau
\]
\[
\leq \frac{1}{\nu\lambda_1} \|f\|_{L^2(\Omega)}^2 (s + \ell) + \|u(t - s)\|_{L^2(\Omega)}^2.
\]
(10)

Integrating (10) with respect to \( s \) over \( (0, \frac{\ell}{2}) \) and using (9), we find for any \( t \geq t_2 + \frac{\ell}{2} \),
\[
\frac{\nu\ell}{4} \int_0^\ell \|\nabla u(t + \tau)\|_{L^2(\Omega)}^2 d\tau + 4\beta \int_0^\ell \|u(t + \tau)\|_{L^r(\Omega)}^r d\tau
\]
\[
\leq \frac{5\ell^2}{8\nu\lambda_1} \|f\|_{L^2(\Omega)}^2 + \int_0^{\frac{\ell}{2}} \|u(t - s)\|_{L^2(\Omega)}^2 ds
\]
\[
\leq \frac{5\ell^2}{8\nu\lambda_1} \|f\|_{L^2(\Omega)}^2 + \int_0^{\frac{\ell}{2}} \|u(t - \frac{\ell}{2} + s)\|_{L^2(\Omega)}^2 ds
\]
\[
\leq \left( \frac{5\ell^2}{8\nu\lambda_1} + \frac{3\ell}{\nu^2\lambda_1^2} \right) \|f\|_{L^2(\Omega)}^2
\]
which implies that
\[
\frac{\nu}{2} \int_0^\ell \|\nabla u(t + \tau)\|_{L^2(\Omega)}^2 d\tau + 2\beta \int_0^\ell \|u(t + \tau)\|_{L^r(\Omega)}^r d\tau
\]
\[
\leq \left( \frac{5\ell}{4\nu\lambda_1} + \frac{6}{\nu^2\lambda_1^2} \right) \|f\|_{L^2(\Omega)}^2
\]
(11)
for any \( t \geq t_2 + \frac{\ell}{2} \).

Integrating (5) between \( t - s \) and \( t + \ell \) with \( t \geq s \) and \( s \in (0, \frac{\ell}{2}) \), we obtain
\[
\nu \int_0^\ell \|Au(t + \tau)\|_{L^2(\Omega)}^2 d\tau + (\beta + \nu) \int_0^\ell \int_\Omega |u(t + \tau)|^{-2} |\nabla u|^2 dx d\tau
\]
\[
+ \frac{8(r-2)(\beta + \nu)}{r^2} \int_0^t \int_\Omega |\nabla u(t+\tau)|^2 d\tau + \beta \int_0^t \|u(t+\tau)\|_{L^{2r-2}(\Omega)}^{2r-2} d\tau \\
\leq \left(\frac{2}{\nu} + \frac{1}{\beta}\right)\|f\|_{L^2(\Omega)}^2 (s+\ell) + C(\beta, \nu, r) \int_0^{\frac{\ell}{2}+t} \|\nabla u(t+\tau)\|_{L^2(\Omega)}^2 d\tau \\
+ \|\nabla u(t-s)\|_{L^2(\Omega)}^2 + \frac{2}{r} \|u(t-s)\|_{L^r(\Omega)}^r.
\]

Furthermore, integrating (12) with respect to \(s\) over \((0, \frac{\ell}{2})\) and using (11), we find for any \(t \geq t_2 + \ell\),
\[
\frac{\nu \ell}{2} \int_0^t \|Au(t+\tau)\|_{L^2(\Omega)}^2 d\tau + \left(\frac{\beta + \nu}{\nu}\right) \frac{\ell}{\nu} \int_0^t \int_\Omega |u(t+\tau)|^{r-2} |\nabla u(t+\tau)|^2 d\tau d\tau \\
+ \frac{4(r-2)(\beta + \nu) \ell}{r^2} \frac{\ell}{\nu} \int_0^t \int_\Omega |\nabla u(t+\tau)|^2 d\tau d\tau + \frac{\beta \ell}{2} \int_0^t \|u(t+\tau)\|_{L^{2r-2}(\Omega)}^{2r-2} d\tau \\
\leq \left(\frac{2}{\nu} + \frac{1}{\beta}\right) \frac{5\ell^2}{8} \|f\|_{L^2(\Omega)}^2 + C(\beta, \nu, r) \int_0^{\frac{\ell}{2}+t} \|\nabla u(t+\tau)\|_{L^2(\Omega)}^2 d\tau ds \\
+ \frac{2}{r} \|\nabla u(t-s)\|_{L^2(\Omega)}^2 ds + \frac{2}{r} \int_0^\frac{\ell}{2} \|u(t-s)\|_{L^r(\Omega)}^r ds \\
\leq \left(\frac{2}{\nu} + \frac{1}{\beta}\right) \frac{5\ell^2}{8} \|f\|_{L^2(\Omega)}^2 + \left(\frac{2C(\beta, \nu, r) \ell}{\nu} + \frac{\nu}{\nu} + \frac{1}{\beta r}\right) \left(\frac{5\ell^2}{4\nu \lambda_1} + \frac{6 \nu^2 \lambda_1^2}\right) \|f\|_{L^2(\Omega)}^2,
\]

which implies that
\[
\nu \int_0^t \|Au(t+\tau)\|_{L^2(\Omega)}^2 d\tau + \left(\frac{\beta + \nu}{\nu}\right) \int_0^t \int_\Omega |u(t+\tau)|^{r-2} |\nabla u(t+\tau)|^2 d\tau d\tau \\
+ \frac{8(r-2)(\beta + \nu)}{r^2} \int_0^t \int_\Omega |\nabla u(t+\tau)|^2 d\tau d\tau + \beta \int_0^t \|u(t+\tau)\|_{L^{2r-2}(\Omega)}^{2r-2} d\tau \\
\leq \left(\frac{2}{\nu} + \frac{1}{\beta}\right) \frac{5\ell^2}{4} \|f\|_{L^2(\Omega)}^2 + \left(\frac{4C(\beta, \nu, r)}{\nu} + \frac{4}{\nu \ell} + \frac{2}{\beta r \ell}\right) \left(\frac{5\ell^2}{4\nu \lambda_1} + \frac{6 \nu^2 \lambda_1^2}\right) \|f\|_{L^2(\Omega)}^2
\]

for any \(t \geq t_2 + \ell\).

For any \(v \in H\), we deduce from Hölder inequality that
\[
\langle u(t), v \rangle \leq \nu \|Au(t)\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \beta \|u(t)\|_{L^{2r-2}(\Omega)}^{r-1} \|v\|_{L^2(\Omega)} \\
+ \left(\int_\Omega |u(t)|^2 |\nabla u(t)|^2 dx\right)^\frac{1}{2} \|v\|_{L^2(\Omega)},
\]

entails that
\[
\|u(t)\|_{L^2(\Omega)} \leq \nu \|Au(t)\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \beta \|u(t)\|_{L^{2r-2}(\Omega)}^{r-1} \\
+ \left(\int_\Omega |u(t)|^2 |\nabla u(t)|^2 dx\right)^\frac{1}{2}.
\]

Integrating (14) over \((t, t+\ell)\) and combining (11) with (13), we deduce that there exists a positive constant \(\varphi_2\) such that
\[
\int_0^\ell \|u(t+\tau)\|_{L^2(\Omega)}^2 d\tau \leq \varphi_2
\]

for any \(t \geq t_0 + \ell\). \(\square\)
Let
\[ Y = \{ \chi \in X_\ell : \chi \in L^2(0, \ell; V), \chi_\ell \in L^2(0, \ell; H) \}, \]
equipped with the following norm
\[ \|\chi\|_Y = \left\{ \int_0^\ell (\|\nabla\chi_\tau\|_{L^2(\Omega)}^2 + \|\chi_\ell(\tau)\|_{L^2(\Omega)}^2) \, d\tau \right\}^{\frac{1}{2}}. \]
Define
\[ B^\ell_t = \{ \chi \in X_\ell : \|\chi\|_Y^2 \leq \rho_2 \}. \]
From Proposition 1 and Theorem 3.4, we know that \( L_tB^\ell_0 \subset B^\ell_0 \) for any \( t \geq 0 \) as well as \( L_tB^\ell_0 \subset B^\ell_t \) for any \( t \geq \tau_3 \).

**Lemma 3.5.** Assume that \( f \in H \). Then \( \overline{L_tB^\ell_0L^2(0, \ell; H)} \subset B^\ell_0 \) for any \( t \geq 0 \).

**Proof.** Thanks to \( L_tB^\ell_0 \subset B^\ell_0 \) for any \( t \geq 0 \), it is enough to prove that
\[ \overline{L_tB^\ell_0L^2(0, \ell; H)} \subset B^\ell_0. \]
For any \( \chi_0 \in \overline{B^\ell_0L^2(0, \ell; H)} \), there exists a sequence of trajectories \( \chi_n \in B^\ell_0 \) such that \( \chi_n \to \chi_0 \) in \( L^2(0, \ell; H) \). Since \( e_0(\chi_n) \) is bounded in \( V \) by Proposition 1 for any \( n \geq 1 \), there exists a subsequence \( \{e_0(\chi_{n_j})\}_{j=1}^\infty \) of \( \{e_0(\chi_n)\}_{n=1}^\infty \) and \( u_0 \in V \cap L^r(\Omega) \) such that \( e_0(\chi_{n_j}) \to u_0 \) in \( V \cap L^r(\Omega) \) and \( e_0(\chi_{n_j}) \to u_0 \) in \( H \). From the proof of the existence of weak solutions for problem (1), we deduce that for any \( T > 0 \), there exists a subsequence converging \((\ast\ast)\) weakly in spaces \( \{u(t) \in L^\infty(0, T; H) \cap L^2(0, T; V) : u(t) \in L^2(0, T; V') + L^{\frac{3r}{r-2}}(0, T; L^{\frac{3r}{r-2}}(\Omega)) \} \) to a certain function \( u(t) \) with \( u(0) = u_0 \). Therefore, we obtain \( \chi_0 \in X_\ell \) from Corollary 1. Since \( B_2 \) is strongly closed in \( H \), we deduce that \( e_0(\chi) \in B_2 \). Therefore, we obtain \( \chi_0 \in B^\ell_0 \). \( \square \)

**Lemma 3.6.** Assume that \( f \in H \). Then the mapping \( L_t : X_\ell \to X_\ell \) is locally Lipschitz continuous on \( B^\ell_1 \) for all \( t \geq 0 \).

**Proof.** For any fixed \( t > 0 \) and any \( \chi^1, \chi^2 \in B^\ell_1 \), let \( u_1(t + \tau) = L_t\chi^1, u_2(t + \tau) = L_t\chi^2 \) and let \( u = u_1 - u_2 \), then \( u \) satisfies the following equation
\[ \frac{d}{dt}u(t) + \nu\nabla u(t) + P[(u_1 \cdot \nabla)u] + P[(u_1 \cdot \nabla)u] + \beta P[(|u_1|^{-2}u_1 - |u_2|^{-2}u_2)] = 0. \] (16)
Multiplying the first equation of (16) by \( u \) and integrating by parts, we find
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(\Omega)}^2 + \nu \|\nabla u(t)\|_{L^2(\Omega)}^2 + \beta \int_{\Omega} (|u_1(t)|^{-2}u_1(t) - |u_2(t)|^{-2}u_2(t)) \cdot u(t) \, dx \]
\[ = \int_{\Omega} (u(t) \cdot \nabla)u(t) \cdot u_2(t) \, dx \]
\[ \leq \|u_2(t)\|_{L^{3r}(\Omega)} \|\nabla u(t)\|_{L^{3r}(\Omega)} \|u(t)\|_{L^{\frac{3r}{r-2}}(\Omega)} \|u_2(t)\|_{L^{\frac{3r}{r-2}}(\Omega)} . \] (17)
Thanks to Lemma 2.2 in [30], we obtain
\[ \int_{\Omega} (|u_1|^{-2}u_1 - |u_2|^{-2}u_2) \cdot u \, dx \geq C(r) \|u\|_{L^{3r}(\Omega)} \]
and from Gagliardo-Nirenberg inequality, we infer
\[ \|u\|_{L^{\frac{3r}{r-2}}(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}^{1-\frac{2}{r}} \|\nabla u\|_{L^2(\Omega)}^{\frac{1}{r}}), \] (19)
we infer from (17)-(19) and Young inequality that
\[
\frac{d}{dt} \|u(t)\|_{L^2(\Omega)} + \nu \|\nabla u(t)\|_{L^2(\Omega)}^2 + C(r)\|u(t)\|_{L^2(\Omega)}^r \\
\leq C \left( \|u_2(t)\|_{L^2(\Omega)}^2 + \|u_2(t)\|_{L^2(\Omega)}^{2T} \right) \|u(t)\|_{L^2(\Omega)}^2 \\
\leq C \left( 1 + \|u_2(t)\|_{L^2(\Omega)}^2 \right) \|u(t)\|_{L^2(\Omega)}^2
\]
for any \( r \geq 3 \).

From the classical Gronwall inequality, we deduce for any \( s \in (0, \ell) \) and any \( t \geq 0 \),
\[
\|u(t + s)\|_{L^2(\Omega)}^2 \leq \|u(s)\|_{L^2(\Omega)}^2 \exp \left( C \int_s^{t+s} \left( 1 + \|u_2(\tau)\|_{L^2(\Omega)}^{2T} \right) d\tau \right) \\
\leq \|u(s)\|_{L^2(\Omega)}^2 \exp \left( C \int_0^{t+\ell} \left( 1 + \|u_2(\tau)\|_{L^2(\Omega)}^{2T} \right) d\tau \right). \tag{21}
\]
Integrating (21) with respect to \( s \) for 0 to \( \ell \), we obtain
\[
\int_0^\ell \|u(t + s)\|_{L^2(\Omega)}^2 ds \\
\leq \exp \left( C \int_0^{t+\ell} \left( 1 + \|u_2(\tau)\|_{L^2(\Omega)}^{2T} \right) d\tau \right) \int_0^\ell \|u(s)\|_{L^2(\Omega)}^2 ds. \tag{22}
\]
Taking the \( H \) inner product of the first equation of (1) with \( Au \) and using Young inequality, we find
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2(\Omega)}^2 + \nu \|Au(t)\|_{L^2(\Omega)}^2 + \beta \int_\Omega |u(t)|^{r-2} |\nabla u(t)|^2 dx \\
+ \frac{4(r - 2)\beta}{r^2} \int_\Omega \|\nabla u(t)\|^2 dx \\
= \int_\Omega f(x) \cdot Au(t) dx - \int_\Omega [(u \cdot \nabla)u(t)] \cdot Au(t) dx \\
\leq \|f\|_{L^2(\Omega)} \|Au(t)\|_{L^2(\Omega)} + \|Au(t)\|_{L^2(\Omega)} \left( \int_\Omega |u(t)|^2 |\nabla u(t)|^2 dx \right)^{\frac{1}{2}} \\
\leq \frac{1}{\nu} \|f\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|Au(t)\|_{L^2(\Omega)}^2 + \frac{1}{\nu} \int_\Omega |u(t)|^2 |\nabla u(t)|^2 dx \\
\leq \frac{1}{\nu} \|f\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|Au(t)\|_{L^2(\Omega)}^2 + \frac{\beta}{2} \int_\Omega |u(t)|^{r-2} |\nabla u(t)|^2 dx + C(\beta, \nu, r) \|\nabla u(t)\|_{L^2(\Omega)}^2,
\]
which implies that
\[
\frac{d}{dt} \|\nabla u(t)\|_{L^2(\Omega)}^2 + \nu \|Au(t)\|_{L^2(\Omega)}^2 + \beta \int_\Omega |u(t)|^{r-2} |\nabla u(t)|^2 dx \\
+ \frac{8(r - 2)\beta}{r^2} \int_\Omega \|\nabla u(t)\|^2 dx \\
\leq \frac{2}{\nu} \|f\|_{L^2(\Omega)}^2 + C(\beta, \nu, r) \|\nabla u(t)\|_{L^2(\Omega)}^2. \tag{23}
\]
We infer from the Classical Gronwall inequality that
\[
\|\nabla u(t)\|_{L^2(\Omega)}^2 \leq \left( \frac{2T}{\nu} \|f\|_{L^2(\Omega)}^2 + \|\nabla u_0\|_{L^2(\Omega)}^2 \right) e^{C(\beta, \nu, r)T} \tag{24}
\]
for any $t \in [0, T]$.

Moreover, we obtain
\[
\nu \int_0^t \| Au(s) \|_{L^2(\Omega)}^2 \, ds + \beta \int_0^t \int_\Omega |u(s)|^{r-2} \nabla u(s)^2 \, dx \, ds \\
+ \frac{8(r-2)\beta}{r^2} \int_0^t \int_\Omega |\nabla u(s)|^r \, dx \, ds \\
\leq \left( \frac{2T}{\nu} \| f \|_{L^2(\Omega)}^2 + \| \nabla u_0 \|_{L^2(\Omega)}^2 \right) \left( 1 + C(\beta, \nu, r) T e^{C(\beta, \nu, r) T} \right)
\]
for any $t \in [0, T]$.

We deduce from the Sobolev embedding inequality $\|u\|_{L^q(\Omega)} \leq \lambda_0 \| \nabla u \|_{L^2(\Omega)}$ for any $u \in V$ and some $\lambda_0 > 0$ that
\[
\int_0^t \| u(s) \|_{L^q(\Omega)}^r \, ds \\
\leq \frac{\lambda_0^r \lambda_0^2}{8(r-2)\beta} \left( \frac{2T}{\nu} \| f \|_{L^2(\Omega)}^2 + \| \nabla u_0 \|_{L^2(\Omega)}^2 \right) \left( 1 + C(\beta, \nu, r) T e^{C(\beta, \nu, r) T} \right)
\]
for any $t \in [0, T]$.

Since $e_0(\chi^2)$ is uniformly bounded in $V \cap L^r(\Omega)$ for any $\chi^2 \in B^1_2$, we conclude from (26) that
\[
M(t) := \exp \left( C \int_0^{t+\ell} \left( 1 + \| u_2(\tau) \|_{L^{3r}(\Omega)} \right) \, d\tau \right)
\]
is a finite number depending on $e_0(\chi^2) \in B_2$ and $t > 0$. Therefore, the mapping $L_t : X_t \to X_t$ is locally Lipschitz continuous on $B^1_2$ for all $t \geq 0$.

We can immediately obtain the existence of a global attractor in $X_t$ from Lemma 2.6 stated as follows.

**Theorem 3.7.** Assume that $f \in H$. Then the semigroup $\{L_t\}_{t \geq 0}$ generated by problem (1) possesses a global attractor $A_t$ in $X_t$ and $e_t(A_t)$ is uniformly bounded in $V \cap L^r(\Omega)$ with respect to $t \in [0, 1]$, where
\[
e_t(A_t) = \{ e_t(\chi) : \chi \in A_t \}
\]
for any $t \in [0, 1]$.

From Corollary 2, Theorem 3.4, Lemma 2.2, Lemma 3.6 and Lemma 2.4, thanks to the compactness of $H^2(\Omega) \cap V \subset L^p(\Omega)$ for any $p > 2$, we conclude that the semigroup $\{L_t\}_{t \geq 0}$ is norm-to-weak continuity on $L^2(0, \ell; L^p(\Omega))$ and
\[
B^1_2 = \left\{ \chi \in X_t : \int_0^\ell \| A\chi(\tau) \|_{L^2(\Omega)}^2 \, d\tau + \int_0^\ell \| \chi(\tau) \|_{L^2(\Omega)}^2 \, d\tau \leq \rho_2 \right\}
\]
is a compact absorbing set in $L^2(0, \ell; L^p(\Omega))$, we immediately deduce from Lemma 2.7 the following conclusion.

**Corollary 3.** Assume that $f \in H$. Then the semigroup $\{L_t\}_{t \geq 0}$ generated by problem (1) possesses a global attractor $A^p_\nu$ in $L^2(0, \ell; L^p(\Omega))$ for any $p > 2$.

In what follows, we prove the smooth property of the semigroup $\{L_t\}_{t \geq 0}$ to estimate the fractal dimension of the global attractor $A_t$. 
Theorem 3.8. Assume that \( f \in H \), let \( \lambda^1 \) and \( \lambda^2 \) be two short trajectories belonging to \( \mathcal{A}_\ell \). Then there exists a positive constant \( \kappa \) independent of \( t \) such that for arbitrary \( t \geq \ell \), we have

\[
\|L_t \chi^1 - L_t \chi^2\|_{Y_1}^2 \leq \kappa \mathcal{M}_\ell(t) \int_0^\ell \|\chi^1(t) - \chi^2(t)\|_{L^2(\Omega)}^2 \, dt,
\]
where \( \mathcal{M}_\ell(t) \) is given in (27) and

\[
Y_1 = \{\chi \in X_\ell : \chi \in L^2(0, \ell; V), \chi_t \in L^2(0, \ell; (H^2(\Omega) \cap V'))\}
\]
equipped with the following norm

\[
\|\chi\|_{Y_1} = \left\{ \int_0^\ell \|\nabla \chi(\tau)\|_{L^2(\Omega)}^2 + \|\chi_t(\tau)\|_{(H^2(\Omega) \cap V')}^2 \, d\tau \right\}^{\frac{1}{2}}.
\]

Proof. For any \( \lambda^1, \lambda^2 \in \mathcal{A}_\ell \), let \( u_1(t + \tau) = L_t \chi^1 \), \( u_2(t + \tau) = L_t \chi^2 \) and let \( u = u_1 - u_2 \). Since \( e_t(\lambda^1) \) and \( e_t(\lambda^2) \) is uniformly bounded in \( V \cap L'(\Omega) \) with respect to \( t \in [0, 1] \) for any \( \lambda^1, \lambda^2 \in \mathcal{A}_\ell \), from the proof of Lemma 3.6, we conclude

\[
\frac{d}{dt}\|u(t)\|_{L^2(\Omega)}^2 + \nu \|\nabla u(t)\|_{L^2(\Omega)}^2 + C(r)\|u(t)\|_{L^r(\Omega)}^r \leq C \left( 1 + \|u_2(t)\|_{L^2(\Omega)} \right) \|u(t)\|_{L^2(\Omega)}^2.
\]
(28)

For any \( t \geq \ell \), integrating (28) from \( t-s \) to \( t+\ell \) with \( s \in [0, \frac{\ell}{2}] \), we conclude

\[
\|u(t+\ell)\|_{L^2(\Omega)}^2 + \nu \int_{t-s}^{t+\ell} \|\nabla u(\tau)\|_{L^2(\Omega)}^2 \, d\tau + C(r) \int_{t-s}^{t+\ell} \|u(\tau)\|_{L^r(\Omega)}^r \, d\tau \leq C \int_{t-s}^{t+\ell} \left( 1 + \|u_2(\tau)\|_{L^2(\Omega)} \right) \|u(\tau)\|_{L^2(\Omega)}^2 \, d\tau + \|u(t-s)\|_{L^2(\Omega)}^2.
\]

It follows from the classical Gronwall inequality that

\[
\|u(t+\ell)\|_{L^2(\Omega)}^2 + \nu \int_{t-s}^{t+\ell} \|\nabla u(\tau)\|_{L^2(\Omega)}^2 \, d\tau + C(r) \int_{t-s}^{t+\ell} \|u(\tau)\|_{L^r(\Omega)}^r \, d\tau \leq \exp \left( C \int_{t-s}^{t+\ell} \left( 1 + \|u_2(\tau)\|_{L^2(\Omega)} \right) \, d\tau \right) \|u(t-s)\|_{L^2(\Omega)}^2.
\]
(29)

We deduce from the classical Gronwall inequality and (28) that for any \( t \geq \ell \) and any \( s \in [0, \frac{\ell}{2}] \),

\[
\|u(t-s)\|_{L^2(\Omega)}^2 \leq \|u(s)\|_{L^2(\Omega)}^2 \exp \left( C \int_{t-s}^{t+\ell} \left( 1 + \|u_2(\tau)\|_{L^2(\Omega)} \right) \, d\tau \right) \leq \|u(s)\|_{L^2(\Omega)}^2 \exp \left( C \int_{0}^{t+\ell} (1 + \|u_2(\tau)\|_{L^2(\Omega)}^2) \, d\tau \right).
\]
(30)

Combining (29) with (30), we obtain

\[
\nu \int_0^\ell \|\nabla u(t+\tau)\|_{L^2(\Omega)}^2 \, d\tau + C(r) \int_0^\ell \|u(t+\tau)\|_{L^r(\Omega)}^r \, d\tau \leq \exp \left( C \int_0^{t+\ell} (1 + \|u_2(\tau)\|_{L^2(\Omega)}^2) \, d\tau \right) \|u(s)\|_{L^2(\Omega)}^2.
\]

\[
= \mathcal{M}_\ell(t) \|u(s)\|_{L^2(\Omega)}^2,
\]
Integrating the above inequality over \((0, t]\) with respect to \(s\), we obtain
\[
\nu \int_0^t \|\nabla u(t + \tau)\|_{L^2(\Omega)}^2 d\tau + C(r) \int_0^t \|u(t + \tau)\|_{L^r(\Omega)}^r d\tau \\
\leq \frac{2M_\ell(t)}{\ell} \int_0^t \|u(\tau)\|_{L^2(\Omega)}^2 d\tau.
\]
Since \(M_\ell(t)\) is bounded for any fixed \(t\), we obtain
\[
\nu \int_0^t \|\nabla u(t + \tau)\|_{L^2(\Omega)}^2 d\tau + C(r) \int_0^t \|u(t + \tau)\|_{L^r(\Omega)}^r d\tau \\
\leq \frac{2M_\ell(t)}{\ell} \int_0^t \|u(\tau)\|_{L^2(\Omega)}^2 d\tau.
\]
For any \(v \in V \cap L^r(\Omega)\), we infer from Hölder inequality and Lemma 2.2 in [30] that
\[
\langle u(t), v \rangle \\
\leq \nu \|\nabla u(t)\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|u_1\|_{L^6(\Omega)} \|u\|_{L^3(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
+ C_2 \beta \int_\Omega (|u_1| + |u_2|)^{-2} |u||v| \, dx + \|u_2\|_{L^6(\Omega)} \|u\|_{L^3(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
\leq C \|\nabla u_1\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + C \|\nabla u_2\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
+ \nu \|\nabla u(t)\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + C_\beta (\|u_1\|_{L^r(\Omega)} + \|u_2\|_{L^r(\Omega)})^{-2} \|u\|_{L^r(\Omega)} \|v\|_{L^r(\Omega)}.
\]
Since \(e_1(\chi^1)\) and \(e_1(\chi^2)\) is uniformly bounded in \(V \cap L^r(\Omega)\) with respect to \(t \in [0, 1]\) for any \(\chi^1, \chi^2 \in A_\ell\) and \(H^2(\Omega) \cap V \subset V \cap L^r(\Omega)\), we obtain
\[
\|u(t)\|_{(H^2(\Omega) \cap V)} \leq C (1 + \|\nabla u_1\|_{L^2(\Omega)} + \|\nabla u_2\|_{L^2(\Omega)}) \|\nabla u\|_{L^2(\Omega)} \\
+ C_\beta (\|u_1\|_{L^r(\Omega)} + \|u_2\|_{L^r(\Omega)})^{-2} \|u\|_{L^r(\Omega)} \\
\leq C (\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^r(\Omega)}).
\]
Therefore, we conclude from (31)-(32) that there exists a positive constant \(\kappa_1\) such that
\[
\int_0^t \|u(t + \tau)\|_{(H^2(\Omega) \cap V)}^2 d\tau \leq \kappa_1 M_\ell(t) \int_0^t \|u(\tau)\|_{L^2(\Omega)}^2 d\tau.
\]
The proof of Theorem 3.8 is completed.

From Lemma 2.2, Lemma 2.9, Theorem 3.7 and Theorem 3.8, we immediately obtain the following result.

**Theorem 3.9.** Assume that \(f \in H\). Then the fractal dimension of the global attractor \(A_\ell\) in \(X_\ell\) of the semigroup \(\{L_\ell\}_{\ell \geq 0}\) generated by problem (1) established in Theorem 3.7 is finite.

### 3.3. The existence of a global attractor in \(H\)

In this subsection, we prove the existence of a finite dimensional global attractor in \(H\) of problem (1). From Theorem 3.1, we deduce that for any given initial condition \(u_0 \in B_2 \subset V \cap L^r(\Omega)\), there exists a unique solution of problem (1), hence solution operators \(S(t)\) restricted to \(B_2\) is a semigroup. Moreover, \(B_2\) is positively invariant with respect to \(S(t)\).
Theorem 3.10. Assume that \( f \in H \). Then the mapping \( e_1 : A_\ell \to A = e_1(A_\ell) \) is Lipschitz continuous. That is, for any two short trajectories \( \chi^1, \chi^2 \in A_\ell \), there exists a positive constant \( \theta \) dependent on \( \ell \) such that

\[
\|e_1(\chi^1) - e_1(\chi^2)\|_2^2 \leq \theta \int_0^\ell \|\chi^1(r) - \chi^2(r)\|_2^2 \, dr.
\]

Proof. For any \( \chi^1, \chi^2 \in A_\ell \), let \( u_1(t + \tau) = L_\ell \chi^1, u_2(t + \tau) = L_\ell \chi^2 \) and let \( u = u_1 - u_2 \). Since \( e_0(\chi^1) \) and \( e_0(\chi^2) \) is uniformly bounded in \( V \) for any \( \chi^1, \chi^2 \in A_\ell \), from the proof of Lemma 3.6, we conclude

\[
\frac{d}{dt}\|u(t)\|_{L^2(\Omega)}^2 + \nu \|\nabla u(t)\|_{L^2(\Omega)}^2 + C(\tau)\|u(t)\|_{L^3(\Omega)}^3 \leq C(1 + \|u_2(t)\|_{L^3(\Omega)})\|u(t)\|_{L^2(\Omega)}^2.
\]

Integrating (34) over \((0, \ell)\), we obtain

\[
\|u(\ell)\|_{L^2(\Omega)}^2 \leq \|u(s)\|_{L^2(\Omega)}^2 \exp\left(C \int_s^\ell (1 + \|u_2(\tau)\|_{L^3(\Omega)}) \, d\tau\right) \leq \|u(s)\|_{L^2(\Omega)}^2 \exp\left(C \int_0^\ell (1 + \|u_2(t)\|_{L^3(\Omega)}) \, dt\right).
\]

Thanks to (27), we know that

\[
\mathcal{M}_\ell(0) = \exp\left(C \int_0^\ell (1 + \|u_2(\tau)\|_{L^3(\Omega)}) \, d\tau\right) < +\infty,
\]

which implies that the mapping \( e_1 : A_\ell \to A \) is Lipschitz continuous. \( \square \)

Theorem 3.11. Assume that \( f \in H \). Then dynamical system \((S(t), B_2)\) possesses a global attractor \( A = e_1(A_\ell) \) in \( H \), which is compact, invariant in \( H \) and attracts every bounded subset in \( H \) with respect to the topology of \( H \). Furthermore, \( A \) is bounded in \( V \) and its fractal dimension is finite.

Proof. From Lemma 2.10, Theorem 3.7, Theorem 3.9 and Theorem 3.10, we know that \( A \) is compact and the fractal dimension of \( A \) is finite. As a result of \( L_\ell A_\ell = A_\ell \), we have

\[
S(t)A = S(t)e_1(A_\ell) = e_1(L_\ell A_\ell) = e_1(A_\ell) = A
\]

for any \( t \geq 0 \). From the definition of \( B_2 \), we deduce that for any bounded subset of \( H \), there exists some time \( \bar{t} = \bar{t}(B) \) such that for any \( t \geq \bar{t} \), we have

\[
\{A(t, u_0) : u_0 \in B\} \subset B_2.
\]

Therefore, we only need to prove that

\[
\lim_{t \to +\infty} \text{dist}_H(S(t)B_2, A) = 0.
\]

Otherwise, there exist some positive constant \( \epsilon_0 \), some sequence \( \{u_n\}_{n=1}^\infty \subset B_2 \) and some \( \{t_n\}_{n=1}^\infty \) with \( t_n \to +\infty \) as \( n \to +\infty \) such that

\[
\text{dist}_H(S(t_n)u_n, A) \geq \epsilon_0.
\]
From the definition of $B_2$, we deduce that there exists $\chi_n \in B_0^\ell$ such that
\[ u_n = e_0(\chi_n). \]
Since $\{\chi_n\}_{n=1}^\infty$ is bounded in $X_\ell$ and $\mathcal{A}_\ell$ is a global attractor in $X_\ell$ of the semigroup $\{L_t\}_{t \geq 0}$ generated by problem (1), there exist a subsequence $\{\chi_{n_j}\}_{n=1}^\infty$ of $\{\chi_n\}_{n=1}^\infty$ and a subsequence $\{t_{n_j}\}_{n=1}^\infty$ of $\{t_n\}_{n=1}^\infty$ such that
\[ L(t_{n_j}) - \epsilon \chi_{n_j} \to \chi \in \mathcal{A}_\ell \text{ in } X_\ell \text{ as } j \to +\infty. \]
Thanks to the continuity of $e_1$, we have
\[ S(t_{n_j})u_{n_j} = e_1(L(t_{n_j}) - \epsilon \chi_{n_j}) \to e_1(\chi) \in \mathcal{A} \text{ in } H \text{ as } j \to +\infty, \]
which contradicts with (35).

Since $\mathcal{A}$ is bounded in $V$, we immediately obtain the following result.

**Theorem 3.12.** Assume that $f \in H$ and $\mathcal{A}_1$ is the global attractor established in [24]. Then
\[ \mathcal{A}_1 = \mathcal{A}. \]

**Remark 1.** The result of fractal dimension of the global attractor in Theorem 3.11 is consistent with the one established in [26].

**Remark 2.** In the case that $r = 4$ and $\beta > \frac{1}{4r}$, after a minor modification, we can still obtain the conclusions in this paper.

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