Hub-Based Community Finding

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This article presents a hub-based approach to community finding in complex networks. After identifying the network nodes with highest degree (the so-called hubs), the network is flooded with wavefronts of labels emanating from the hubs, accounting for the identification of the involved communities. The simplicity and potential of this method, which is presented for direct/undirected and weighted/unweighted networks, is illustrated with respect to the Zachary karate club data, image segmentation, and concept association. Attention is also given to the identification of the boundaries between communities.

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The problem of community finding in complex networks represents one of the most challenging and promising perspectives from which to approach, characterize and understand those general structures. Related to established areas in graph theory (e.g. [1]) and pattern recognition (e.g. [2]), the interest in community finding in complex networks was fostered by sociological studies (e.g. [3]) and further enhanced by the seminal articles by Wu and Huberman [4] and Newman and Girvan [5]. The latter defined the problem of community finding as ‘the division of network nodes into groups within which the network connections are dense, but between which are sparse’. That work also proposed a divisive methodology based on the concept of shortest path betweenness which has become the main reference for community finding investigations given its good performance, despite its relatively high computational demand. Other approaches include the method based on an analogy with electrical circuits [5], consideration of triangular loops in the network [6], application of super-paramagnetic clustering [7], analysis of the spectral properties of the networks [8], and spectral properties of the Laplacian matrix combined with clustering techniques [9]. Despite the growing attention focused on this issue — see [8, 10, 12, 13] for a good characterization of the problem and extensive related references — some important points remain not completely solved, including the definition of a community and the high computational demand implied by the most effective techniques.

The current work describes an alternative approach to community finding which is based on one of the most characteristic concepts underlying the new area of complex networks, namely that of a hub, i.e. a node in a network exhibiting high degree. As emphasized by the several investigations targeting complex networks, such nodes play determinant role in defining the connectivity patterns in several natural structures and systems [1]. Therefore, the consideration of hubs as starting points for network partition represents a particularly promising perspective from which to approach the community finding problem, a possibility which was preliminary considered in [14]. The current article reports on a simple and powerful hub-based community finding methodology which involves the flooding of the network with labels emanating with constant speed from the respective hubs. Such a procedure, which is related to the concept of distance transform in graphs [15] and label propagation in orthogonal lattices [6, 15, 17], provides a simple and natural means for partitioning networks, especially those organized around hubs, into coherent communities. Such a methodology, as well as a post-processing step allowing integration of border elements, is presented and illustrated in the following with respect to three representative weighted/unweighted and directed/undirected networks, namely the well-known Zachary karate club, image segmentation, and concept associations.

The network under analysis is assumed to have \( N \) nodes, labeled as \( i = 1, 2, \ldots, N \), and \( n \) edges represented as \( (i, j) \), which can have unit or general weight \( w_{ij} \) represented as \( w(j, i) \) in the respective weight matrix. The outdegree \( O_i \) of a specific node \( i \) is herein defined as the sum of the weights of the emerging edges, i.e. \( O_i = \sum_{k=1}^{N} w(i, k) \), while the indegree is defined as \( I_i = \sum_{k=1}^{N} w(k, i) \). Observe that undirected networks are characterized by \( O_i = I_i \) for any \( i \). The hubs are henceforth understood as the set of \( M \) nodes with the highest degrees. The \( d \)-ball with radius \( d \) centered at node \( i \) is defined as the subgraph containing all nodes which are connected to \( i \) through shortest paths no longer than \( d \). The label of a specific node \( i \) can be propagated through the network by identifying the \( d \)-balls centered at \( i \) with subsequent distance values \( d \). If such wavefronts are started at each of the \( M \) hubs, the respective labels are propagated as long as the nodes being reached by the wavefronts are empty, i.e. have not been visited by another front. In this work, such a label propagation is performed so that the labels emanating from the hubs with higher degree are propagated first, for the same value of \( d \), than those with lower degree.

The result of such a flooding procedure is to partition the original network into \( M \) communities, which can also be understood as the Voronoi tessellation of the original
network \cite{1,13,16}. Observe that the above procedure implies that those nodes that are at the same distances from two hubs are dominated by the hub with the higher degree. Such a procedure implies that two (or more) hubs \(a\) and \(b\) with \(O_a > O_b\), sharing most connections, as is the case with nodes 33 and 34 in the Zachary club network (see Figure 1), may produce different communities. In case it is desired to merge such hubs, which is an application-dependent decision, the following post-processing can be performed. For each node, identify all its emanating direct connections, whose number is represented as \(E_i\), and identify the moda (i.e. the most frequent value) \(m\) among the labels of the nodes connected to \(i\). In case the ratio \(R_i\) given in Equation 1 is larger than a pre-specified threshold value \(T\), the node \(i\) receives the label \(m\). For weighted networks, it is also possible to consider the ratio between the sum of weights of the connected nodes with label equal to the moda value and the total sum of emerging edge weights (see Equation 2).

\[
R_i = \frac{M_i}{E_i} \quad (1)
\]

\[
R_w = \frac{\sum_{k \in \text{moda}} w(k, i)}{\sum_{k} w(k, i)} \quad (2)
\]

\[
Q = \sum_i (e_{ii} - a_i^2) \quad (3)
\]

A particularly interesting, and somewhat overlooked, feature of a community partition of a complex network is the boundaries between the identified communities. The boundaries can be defined with respect to nodes or edges. In the former case, the boundary of community \(i\) can be easily identified by looking for each node with label \(i\) which is linked to at least another node with different label. Such a boundary, which is respective to community \(i\), is henceforth called node-boundary of \(i\). The edge-boundary between two communities \(i\) and \(j\) corresponds to those edges connecting nodes of \(i\) to nodes of \(j\) (the edge direction can be or not observed in the case of directed networks).

Although the above described hub-based methodology can be immediately applied to unweighted (i.e. weights are 0 or 1) or weighted networks, some remarks regarding computational implementation should be considered. For undirected networks, it is more effective to follow the subsequent connections defined by the label flooding by looking for non-zero entries along the columns of the weight matrix and using lists for book-keeping. It can be verified that such a processing can be performed in \(O(N)\), as the nodes are checked only once during the labeling procedure. A possible means to processing weighted networks is to visit each node while identifying the shortest path \cite{11} to each of the \(M\) hubs, as resulting the label of the shortest hub. In case two (or more) hubs are found at the same shortest path distance, that with the highest node degree is selected. The computational cost of finding the shortest paths between each of the \(N\) nodes and the \(M\) hubs can be optimized by using algorithms such as Dijkstra’s, which implies \(O(N\log N)\) \cite{11}. Effective algorithms for distance transformation in graphs \cite{11} can also be considered for further enhancing the performance.

The potential of the above described hub-based community finding approach is illustrated in the following with respect to complex networks obtained for the Zachary karate club, image segmentation, and concept association. In order to rate the quality of the obtained communities, we consider the modularity index \(Q\). Let the number of nodes and edges completely contained inside community \(i\) be denoted by \(N_i\) and \(n_i\), respectively, and the number of edges with at least one vertex connected to \(i\) be represented as \(A_i\). The modularity index \(Q\) can now be defined by Equation 3 where \(e_{ii} = n_i/n\) and \(a_i = (2n_i + A_i)/(2n)\). Observe that \(Q \leq 1\), reaching null value for a random partition of the network \cite{2}.

We consider the Zachary karate club data first. The network obtained from this dataset is often considered as a benchmark for community finding methodologies \cite{2,13}. Observe that this network is unweighted (i.e. unit weights) and undirected. Figure 1 shows the communities obtained by the hub-based algorithm (small and large nodes) considering \(M = 2\), followed by the above described node merging post-processing considering \(T = 0.4\). The edge-boundary between the two communities is identified by thicker edges. Actually, the only node misclassified by the methodology (node 3), lies at the boundary between the two communities and present the same number of links with both of them. The quality of such a partition, which is precisely the same as that obtained in \cite{2}, is characterized by \(Q = 0.36\).

Now we draw attention to the simple image in Figure 2(a), which contains a floppy-disk, a coin and a pen.
The objective here is to segment the image into reasonable regions of interest, namely the three objects \( \mathcal{R} \). As in [14], each image pixel is understood as a node, and the absolute difference between the gray-levels at any two pixels \( i \) and \( j \) is taken as the respective weight \( w(i, j) \). Therefore, two pixels with similar gray-level are connected by an edge with small weight, which can be understood as the similarity between those pixels \( \mathcal{S} \). Unlike in [14], such a fully connected graph is not thresholded, therefore avoiding one adjustable parameter, and the identification of the hubs is not performed sequentially along the processing, but as its first step. As such, the obtained network is weighted and undirected (the difference between pixels is symmetric). It should be observed that the consideration of image segmentation as a community finding benchmark is particularly interesting, not only because of the easy visualization of the obtained results therefore afforded, but also for the possibility to immediately check the coherence and quality of the obtained communities, which should correspond to the main regions in the original image. In order to quantify the quality of the obtained partitions, the template image in Figure 2 is considered as the reference for the correct classes. Such a template was obtained by a human operator by considering the original, higher resolution, image from which the image in (a) was derived by subsampling \( \mathcal{R} \). The results obtained by the hub-based approach considering \( M = 2 \), shown in Figure 2(c), can be found to be in good agreement with the template in (b). It should be observed that, as several hubs are obtained for the same region as a consequence of the weight-assignment procedure (which produces a fully-connected graph as a result), the two hubs were sampled manually from each of the two regions. The obtained value of \( Q \) for such partitioning was found to be equal to 0.007, which is so low because of the several original connections between the two classes implied by the procedure adopted in order to obtain the weight matrix, which is fully connected.

Finally, we consider the concept association experiment reported in [20] (see also [12]), which involved word associations by a human subject. A weighted, directed network is obtained by considering each distinct word as a node, while the weight of the edge between node \( i \) and a node \( j \) corresponds to the number of times the word associated to \( i \) was followed by that associated to \( j \). The hub-based community finding algorithm was applied with \( M = 10 \) and \( T = 0.4 \). Table II shows five of the principal hub-words and some of the words falling on the respectively defined communities, which include directly (shown in italics) and indirectly associated words. The word ‘fast’, for instance, was included into the community dominated by the hub \textit{animal} through the following stream of associations \textit{animal} \( \rightarrow \) butter\textit{fly} \( \rightarrow \) wing \( \rightarrow \) airplane \( \rightarrow \) fast. The values of \( Q \) for \( M = 2 \) to 50 with and without the node-merging scheme is shown in Figure 3. It is clear from this curve that such post-processing is highly effective in increasing the quality of the network partitioning.

All in all, the prospects of using the network hubs as references for finding communities along the network, which can be obtained through label propagation, has been found to provide a natural and powerful means for partitioning complex networks, especially those organized around hubs (e.g. scale-free networks) into coherent subgraphs. The potential of the reported approach has been fully substantiated with respect to three case-examples of weighted/unweighted and directed/undirected networks. Given its low computational demand (order \( N \)), this methodology presents
FIG. 3: The values of the modularity $Q$ for $M = 2$ to 50 without (solid line) and with node-merging post-processing with $T = 0.4$ (dotted line).

| sun (18) | drink (15) | cold (15) | way (15) | animal (15) |
|----------|------------|-----------|----------|-------------|
| pyramid  | soft       | water     | easy     | cat         |
| round    | eat        | sky       | rough    | horse       |
| yellow   | well       | wool      | good     | brown       |
| circle   | much       | air       | one      | butterfly   |
| hot      | few        | pullover  | brief    | wing        |
| triangle | sheep      | thin      | single   | fast        |
| drawing  |            |           |          |             |

TABLE I: Five of the hubs with highest degree (indicated within parenthesis) and some of the related concepts included in the respective communities. Directly associated concepts are shown in italics.

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