Near-thermal equilibrium with Tsallis distributions in heavy ion collisions

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Abstract. Hadron yields in high energy heavy ion collisions have been fitted with thermal models using standard (extensive) statistical distributions. These models give insight into the freeze-out conditions at varying beam energies and lead to a systematic consistent picture of freeze-out conditions at all beam energies. In this paper we investigate changes to this analysis when the statistical distributions are replaced by non-extensive Tsallis distributions for hadrons. We investigate the particle yields at SPS and RHIC energies and obtain better fits with smaller $\chi^2$ for the same hadron data, as applied earlier in the thermal fits for SPS energies but not for RHIC energies.

1. Introduction

After many years of investigating hadron-hadron and heavy ion collisions, the study of hadron production remains an active and important field of research. The lack of detailed knowledge of the microscopic mechanisms has led to the use of many different models, often from completely opposite directions. Thermal models, based on statistical weights for produced hadrons \cite{1,2,3,4}, are very successful in describing particle yields at different beam energies \cite{5,6,7,8}, especially in heavy ion collisions. These models assume the formation of a system which is in thermal and chemical equilibrium in the hadronic phase and is characterised by a set of thermodynamic variables for the hadronic phase. The deconfined period of the time evolution dominated by quarks and gluons remains hidden: full equilibration generally washes out and destroys large amounts of information about the early deconfined phase. The success of statistical models implies the loss of such information, at least for certain properties, during hadronization. It is a basic question as to which ones survive the hadronization and behave as messengers from the early (quark dominated) stages, especially if these are strongly interacting stages.

In the case of full thermal and chemical equilibrium, relativistic statistical distributions can be used, leading to exponential spectra for the transverse momentum distribution of hadrons. On the other hand, experimental data at SPS and RHIC

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energies display non-exponential behaviours at high $p_T$. One explanation of this deviation is connected to the power-like hadron spectra obtained from perturbative QCD descriptions: the hadron yield from quark and gluon fragmentation overhelms the thermal (exponential) hadron production. However, this overlap is not trivial. One can assume the appearance of near-thermal hadron distributions, which is similar to the thermal distribution at lower $p_T$, but it has a non-exponential tail at higher $p_T$. A stationary distribution of strongly interacting hadron gas in a finite volume can be characterized by such a distribution (or strongly interacting quark matter), which will hadronize into hadron matter. Tsallis distributions satisfy such criteria \[10, 11\]. In the next Section we will review the Tsallis distribution and emphasize the properties most relevant to particle yields.

2. Tsallis Distribution for Particle Multiplicities.

2.1. Relation between the Boltzmann and Tsallis distributions

Neglecting quantum statistics, the entropy of a particle of species $i$ is given by \[9\]

$$S_i = V \int \frac{d^3p}{(2\pi)^3} \left( n_i^B - n_i^B \ln n_i^B \right),$$

(1)

where the mean occupation numbers, $n_i^B$, are given by

$$n_i^B(E) \equiv g_i \exp \left( -\frac{E_i - \mu_i}{T} \right),$$

(2)

with $g_i$ being the degeneracy factor of particle $i$. The total number of particles of species $i$ is given by an integral over phase space of eq. (2):

$$N_i = V \int \frac{d^3p}{(2\pi)^3} n_i^B(E).$$

(3)

The transition to the Tsallis distribution makes use of the following substitutions \[10\]

$$\ln(x) \rightarrow \ln_q(x) \equiv \frac{x^{1-q} - 1}{1-q},$$

(4)

$$\exp(x) \rightarrow \exp_q(x) \equiv [1 + (1-q)x]^{\frac{1}{1-q}},$$

(5)

which leads to the standard result \[10, 11\]

$$n_i^T(E) = g_i \left[ 1 + (q-1)\frac{E_i - \mu_i}{T} \right]^{-\frac{1}{q-1}},$$

(6)

which is usually referred to as the Tsallis distribution \[10, 11\]. As these number densities are not normalized, we do not use the normalized $q$-probabilities which have been proposed in Ref. \[11\]. In the limit where $q \rightarrow 1$ this becomes the Boltzmann distribution:

$$\lim_{q \rightarrow 1} n_i^T(E) = n_i^B(E).$$

(7)

The particle number is now given by

$$N_i = V \int \frac{d^3p}{(2\pi)^3} (n_i^T(E))^q.$$

(8)
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Figure 1. Comparison between the Boltzmann and Tsallis distributions. The figure on the left compares the two as a function of the energy \( E \), keeping the temperature and chemical potential fixed, for various values of the Tsallis parameter \( q \). The value \( q = 1.5 \) is the maximum value that still leads to a convergent integral in Eq. (8). A derivation of the Tsallis distribution, based on the Boltzmann equation, has been given in Ref. [12]. A comparison between the two distributions is shown in Fig. 1, where it can be seen that, at fixed values of \( T \) and \( \mu \), the Tsallis distribution is always larger than the Boltzmann one if \( q > 1 \). Taking into account the large \( p_T \) results for particle production we will only consider this possibility in this paper. As a consequence, in order to keep the particle yields the same, the Tsallis distribution always leads to smaller values of \( T \) for the same set of particle yields. The dependence on the chemical potential is also illustrated on the right of Fig. 1 for a fixed temperature \( T \) and a fixed energy \( E \). As one can see, the Tsallis distribution in this case increases with increasing \( q \). The Tsallis distribution for quantum statistics has been considered in Ref. [13, 14, 15, 16].

3. Relation between the Tsallis parameter \( q \) and temperature fluctuations

The parameter \( q \) plays a central role in the Tsallis distribution and a physical interpretation is needed to appreciate its significance. To this end we follow the
analysis of Ref. \cite{17} and write the Tsallis distribution as a superposition of Boltzmann distributions

\[ n^T(E) = \int_0^\infty d\left(\frac{1}{T_B}\right) e^{-(E-\mu)/T_B} f\left(\frac{1}{T_B}\right), \tag{9} \]

where the detailed form of the function \( f \) is given in \cite{17}. The parameter \( T_B \) is the standard temperature as it appears in the Boltzmann distribution. It is straightforward to show \cite{17} that the average value of \( 1/T_B \) is given by the Tsallis temperature:

\[ \left\langle \frac{1}{T_B} \right\rangle = \int_0^\infty d\left(\frac{1}{T_B}\right) \left(\frac{1}{T_B}\right) f\left(\frac{1}{T_B}\right) = \frac{1}{T}, \tag{10} \]

while the fluctuation in the temperature is given by the deviation of the Tsallis parameter \( q \) from unity:

\[ \frac{\left\langle \left(\frac{1}{T_B}\right)^2 \right\rangle - \left\langle \frac{1}{T_B} \right\rangle^2}{\left\langle \frac{1}{T_B} \right\rangle^2} = q - 1 \tag{11} \]

which becomes zero in the Boltzmann limit. The above leads to the interpretation of the Tsallis distribution as a superposition of Boltzmann distributions with different temperatures. The average value of these (Boltzmann) temperatures is the temperature \( T \) appearing in the Tsallis distribution. This is the interpretation of the Tsallis temperature that we will follow. The other parameter in the Tsallis distribution, \( q \), describes the spread around the average value of the (Boltzmann) temperature \( T \). For \( q = 1 \) we have an exact Boltzmann distribution, for values of \( q \) which deviate from 1, we have a corresponding deviation. From this point of view the Tsallis distribution describes a distribution of (Boltzmann) temperatures. A deviation from \( q = 1 \) means that a spread of temperatures is needed instead of a single value.

4. Thermal Fit Details

In order to identify the energy dependence of the deviation from ideal gas behaviour, thermal fits were performed on yields measured at the CERN SPS in central Pb-Pb collisions at 40 AGeV, 80 AGeV and 158 AGeV (using the same data as analyzed in \cite{8}) and yields measured at RHIC in central Au-Au collisions at \( \sqrt{s} = 130 \) AGeV (using the same data as analyzed in \cite{19}) and at \( \sqrt{s} = 200 \) AGeV.

In the CERN SPS fits, the thermal parameters \( T, \mu_B, \gamma_S \) and \( R \) were fit to the data, while \( \mu_Q \) and \( \mu_S \) were fixed by the initial baryon-to-charge ratio and strangeness content in the colliding system, respectively.

In the case of the RHIC analysis we again fit \( T, \mu_B, \mu_S \) and \( \gamma_S \) to the data. The use of mid-rapidity data here led to the relaxing of the constraints on \( \mu_S \) and \( \mu_Q \) typical in analyzes of \( 4\pi \) data. Instead, \( \mu_Q \) was set to zero as justified by the observed \( \pi^+/\pi^- \) ratio. The following expression was used to calculate primordial particle yields,

\[ n_i^T(E, \gamma_s) = \gamma_{s_i}^{\left|\gamma_S\right|} \left[ 1 + (q-1) \left(\frac{E-\mu}{T}\right)\right]^{-1/(q-1)}, \tag{12} \]
where $|S_i|$ is the number of valence strange quarks and anti-quarks in species $i$. The value $\gamma_s = 1$ obviously corresponds to complete strangeness equilibration. All calculations were done using the THERMUS package [18].

5. Results and Conclusions

The most surprising result of our analysis is shown in Fig. (2): the quality of the fits, as measured by the $\chi^2$/d.o.f., improves at first as the Tsallis parameter $q$ increases. It reaches a minimum value around $q \approx 1.07$ for SPS beam energy of 158 AGeV. This behaviour is repeated at other SPS energies with the minima at slightly different values of $q$, i.e. 1.08 for 80 AGeV and about 1.05 at 40 AGeV beam energy. This behaviour is not seen at RHIC energies. Clearly, changes in the Tsallis parameter $q$ have only a small negligible effect on the $\chi^2$ values at RHIC energies, of course, this still leaves open the possibility for $q$ values larger than 1 [20]. However on the SPS data the effect is substantial and changes the interpretation substantially. One possible interpretation is that at SPS energies fluctuations in the freeze-out temperature are substantial.

Recently [21] a coalescence model with a Tsallis distribution for quarks was used to fit the transverse momenta spectra measured at RHIC. This fit does not include decays from resonances and therefore cannot be compared directly to ours since decays can substantially modify the transverse momenta, also the emerging hadrons are not in a Tsallis-type equilibrium gas, which is an assumption of the present analysis. The
authors obtained values for the Tsallis parameter $q$ which are remarkably similar for all particle species considered, i.e. $q \approx 1.2$ which cannot be excluded by our analysis.

The freeze-out temperature $T$ decreases, as expected, with increasing values of $q$. This can be understood from the fact that the Tsallis distribution is always larger than the Boltzmann one (as long as $q > 1$). Hence, in order to match the same particle yields one has to adjust $T$ to lower values. This is seen at all energies in Fig. 3. However, the drop in $T$, turns out to be quite drastic numerically. In fact, the decrease in particle numbers has to be compensated by increases in all other thermodynamic variables. The (modest) increase in the baryon chemical potential is shown in Fig. 3 on the right hand side.

The strangeness non-equilibrium factor $\gamma_s$ as shown in Fig. 4. It is interesting to note that the Tsallis distribution leads to a much better chemical equilibrium than the corresponding Boltzmann distribution with $q = 1$. In all cases considered the $\gamma_s$ is very close to the chemical equilibrium value of 1. Clearly, the use of the Tsallis distribution in relativistic heavy ion collisions calls for a reevaluation of the understanding gained from previous analyses [5, 7, 8].
Figure 4. The strangeness non-equilibrium factor $\gamma_s$ as a function of the Tsallis parameter $q$.

Acknowledgments

The authors acknowledge support from the Hungary - South Africa scientific cooperation programme and Hungarian OTKA grant NK62044. We acknowledge useful discussions with H.G. Miller, T.S. Biró, P. Ván and G.G. Barnaföldi.

References

[1] E. Fermi, Progress Theor. Phys., 5 (1950) 570.
[2] I.Ya. Pomeranchuk, Dokl. Akad. Nauk Ser. Fiz. 78 (1951) 889.
[3] W. Heisenberg, Naturwissenschaften, 39 (1952) 69.
[4] R. Hagedorn, Nuovo Cimento, 35 (1965) 395.
[5] J. Cleymans, H. Oeschler, K. Redlich, S. Wheaton, Phys. Rev. C 73 (2006) 034905.
[6] J. Cleymans, P. Braun-Munzinger, H. Oeschler and K. Redlich, Nucl. Phys. A 697 (2002) 902.
[7] A. Andronic, P. Braun-Munzinger, J. Stachel, Nucl. Phys. A 772 (2006) 167.
[8] F. Becattini, J. Manninen, M. Gaździcki Phys. Rev. C 73 (2006) 044905.
[9] See e.g. S.R. de Groot, W.A. van Leeuwen, Ch.G. van Weert, *Relativistic Kinetic Theory*, North Holland 1980.
[10] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[11] C. Tsallis, R.S. Mendes, A.R. Plastino, Physics A 261 (1998) 534.
[12] T.S. Biró, G. Purcsel, Phys. Rev. Lett. 95 (2005) 162302.
[13] A.M. Teweldeberhan, A.R. Plastino, H.G. Miller, Phys. Lett. A 343 (2005) 71.
[14] A.R. Plastino, A. Plastino, H.G. Miller, Phys. Lett. A 343 (2005) 71.
[15] F. Buyukkilic, D. Demirhan, Phys. Lett. A 181 (1993) 24.
[16] F. Buyukkilic, D. Demirhan, A. Gulec, Phys. Lett. A 197 (1995) 209.
[17] G. Wilk and Z. Wlodarczyk, Phys. Rev. Lett. 84 (2000) 2770.
[18] S. Wheaton, J. Cleymans, M. Hauer, Computer Physics Communications 180 (2009) 84.
[19] J. Cleymans, B. Kämpfer, M. Kaneta, S. Wheaton and N. Xu, Phys. Rev. C 71 (2005) 054901.
[20] T.S. Biró, K. Úrmössy K and G.G. Barnaföldi, J. Phys. G 35 044012
[21] T.S. Biró, K. Úrmössy, arXiv:0812.2985 [hep-ph].