Weak error estimate of a fully discrete method for stochastic Cahn-Hilliard equation with additive noise

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Abstract

We prove a weak error estimate of a fully discrete scheme for stochastic Cahn–Hilliard equation with additive noise, where the spectral Galerkin method is used in space and the backward Euler method is used in time. Compared with the Allen-Cahn type stochastic partial differential equation, the error analysis here is much more sophisticated due to the presence of the unbounded operator in front of the nonlinear term. Besides, the weak analysis is also troublesome caused by the lack of the associated Kolmogorov equation. To address such issues, a novel and direct approach has been exploited which does not rely on a Kolmogorov equation but on the integration by parts formula from Malliavin calculus. To the best of our knowledge, the rates of weak convergence, shown to be higher than the strong convergence rates, are revealed in the stochastic Cahn–Hilliard equation setting for the first time. Numerical examples are finally performed to confirm the theoretical results.

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Key Words: stochastic Cahn-Hilliard equation, weak convergence rate, Malliavin calculus, spectral Galerkin method, backward Euler method.

1 Introduction

During the last decades, there has been an overwhelming activity of numerical stochastic partial differential equation (SPDE) under globally Lipschitz condition and a fast increasing number of
studies on Allen-Cahn type SPDE with non-globally Lipschitz coefficients. However, numerical analysis of stochastic Cahn-Hilliard equation, which is another prominent SPDE model with non-globally Lipschitz coefficients, is in its beginning and is far from well understood. The Cahn–Hilliard equation is of fundamental importance in various applications, such as, the complicated phase separation and coarsening phenomena in a melted alloy [4,6], spinodal decomposition for binary mixture [5], the diffusive process of populations and an oil film spreading over a solid surface [10]. Our motivating example arises from a simplified mesoscopic physical model for phase separation. The aim of this article is to investigate the weak convergence rate of a full discretization for stochastic Cahn–Hilliard equation driven by additive noise,

\[
\begin{align*}
\{ & dX(t) + A(AX(t) + F(X(t))) \, dt = dW(t), \quad t \in (0,T], \\
& X(0) = X_0.
\end{align*}
\]  

(1.1)

Let \( \mathcal{I} \) be a bounded open set of \( \mathbb{R}^d, d = 1,2,3 \) with smooth boundary and let \( H := L^2(\mathcal{I}, \mathbb{R}) \) be the Hilbert space with the usual scalar product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). The space \( \dot{H} := \{ v \in H : \int_{\mathcal{I}} v \, dx = 0 \} \) is defined as a subspace of \( H \). We make the following assumptions.

**Assumption 1.1.** \(-A : \text{dom}(A) \subset \dot{H} \to \dot{H} \) is the Neumann Laplacian defined by \(-Au = \Delta u\).

**Assumption 1.2.** \( F : L^6(\mathcal{I}, \mathbb{R}) \to H \) is a Nemytskii operator given by

\[
F(v)(x) = f(v(x)) = v^3(x) - v(x), \quad x \in \mathcal{I}, v \in L^6(\mathcal{I}, \mathbb{R}).
\]  

(1.2)

**Assumption 1.3.** \( \{W(t)\}_{t \in [0,T]} \) is an \( \dot{H} \)-valued \( Q \)-Wiener process with the covariance operator \( Q \) satisfying

\[
\operatorname{Tr}(Q) < \infty.
\]  

(1.3)

**Assumption 1.4.** The initial value \( X_0 \) is deterministic and satisfies the following regularity,

\[
|X_0|_4 < \infty,
\]  

(1.4)

where the norm \( |\cdot|_4 \) is defined in (2.3).

We remark that the assumption on the initial datum can be relaxed, but at the expense of having the constant \( C \) later depending on \( T^{-1} \), by exploiting the smoothing effect of the semigroup \( E(t), t \in [0,T] \) and standard non-smooth data error estimates.

Based on the above assumptions, following the semigroup framework in [17], the problem (1.1) admits a unique mild solution

\[
X(t) = E(t)X_0 - \int_0^t E(t-s)AF(X(s)) \, ds + \int_0^t E(t-s) \, dW(s), \quad t \in [0,T],
\]

where \( E(t) \) denotes the analytic semigroup generated by \(-A^2\). We refer the readers to [2,5,11,13,16,18,24] for the existence and uniqueness of the mild solution for such equation. Since the exact solutions are rarely known explicitly, numerical simulations are often used to investigate the behavior of the solutions. We choose the spatial semi-discretization by the spectral Galerkin method,
i.e., projecting the equation on vector space $H_N$, which is spanned by the $N$ first eigenvectors of $A$. The approximated equation of (1.1) is in the form
\[
dX^N(t) + A(X^N(t) + P_N F(X^N(t)))dt = P_N dW(t), \ t \in (0,T]; \ X^N(0) = P_N X_0,
\]
where $P_N$ is the spectral Galerkin projection operator onto the space $H_N$. In the temporal direction, we apply the backward Euler method and get the fully discrete scheme, described by
\[
X_{t_{m}}^{M,N} - X_{t_{m-1}}^{M,N} + \tau A^2 X_{t_{m}}^{M,N} + \tau P_N AF(X_{t_{m}}^{M,N}) = P_N \Delta W_m, \quad m \in \{1,2,\cdots,M\}.
\]
Here $\Delta W_m := W(t_m) - W(t_{m-1})$, $\tau = \frac{T}{M}$ is the time stepsize and $t_m = m\tau$. The main result, concerning the weak convergence rates of the full discretization, reads as,
\[
\left| \mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X_T^{M,N})] \right| \leq C \left( \lambda_N^{-\frac{3}{2}} + \tau^{\frac{3}{2}} \right), \ \forall \Phi \in C^2_b.
\] (1.5)

Throughout this article, $C$ denotes a generic positive constant that is independent of the discretization parameters $M, N$ and may change from line to line and $C^2_b$ represents the space consisting of not necessarily bounded mappings from $H$ to $\mathbb{R}$ that have continuous and bounded Fréchet derivatives up to order 2.

The idea for error analysis goes as follows. At first, the weak error is separated into two parts, both the spatial error and the temporal error,
\[
\mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X_T^{M,N})] = (\mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X^N(T))]) + (\mathbb{E}[\Phi(X^N(T))] - \mathbb{E}[\Phi(X_T^{M,N})]).
\] (1.6)
To simplify the notation, we often write $\mathcal{O}_t$ for $\int_0^t E(t-r)dW(r)$ and $\mathcal{O}_t^{N} := P_N \mathcal{O}_t$. By introducing two processes $\tilde{X}(t) := X(t) - \mathcal{O}_t$ and $\tilde{X}^N(t) := X^N(t) - \mathcal{O}_t^{N}$, we split the spatial error as
\[
\mathbb{E}[\Phi(X(T))] - \mathbb{E}[\Phi(X^N(T))] = (\mathbb{E}[\Phi(\tilde{X}(T) + \mathcal{O}_T)] - \mathbb{E}[\Phi(\tilde{X}^N(T) + \mathcal{O}_T)])
+ (\mathbb{E}[\Phi(\tilde{X}^N(T) + \mathcal{O}_T)] - \mathbb{E}[\Phi(\tilde{X}^N(T) + \mathcal{O}_T^{N})]).
\] (1.7)
To proceed further, one relies on the Taylor expansion of the test function $\Phi$. The key argument to estimate the first term on the right hand of (1.7) is the discrepancy between $\tilde{X}^N(T)$ and $\tilde{X}(T)$ in a strong sense,
\[
\left| \mathbb{E}[\Phi(\tilde{X}(T) + \mathcal{O}_T)] - \mathbb{E}[\Phi(\tilde{X}^N(T) + \mathcal{O}_T)] \right|
\leq C \left| \mathbb{E} \int_0^1 \Phi'(X(T) + \lambda(\tilde{X}^N(T) - \tilde{X}(T)))(\tilde{X}^N(T) - \tilde{X}(T))d\lambda \right|
\leq C \|\tilde{X}(T) - P_N \tilde{X}(T)\|_{L^2(\Omega,H)} + C \|P_N \tilde{X}(T) - \tilde{X}^N(T)\|_{L^2(\Omega,H)}.
\] (1.8)
The error term $\|\tilde{X}(T) - P_N \tilde{X}(T)\|_{L^2(\Omega,H)}$ can be easily controlled owing to the higher spatial regularity of the stochastic process $\tilde{X}(T)$, in the absence of the stochastic convolution. The remaining term $e(t) := P_N \tilde{X}(t) - \tilde{X}^N(t)$, satisfying the following random PDE,
\[
\frac{d}{dt} e(t) + A^2 e(t) + P_N A [F(X(t)) - F(X^N(t))] = 0, \quad e(0) = 0,
\] (1.9)
must be carefully treated due to the presence of the unbounded operator $A$ before the nonlinear term $F$. We make full use of the monotonicity of the nonlinear term $F$ and the regularities of $X(T)$, $X^N(T)$ and $O_t$ to derive $\left\| \int_0^T |e(s)|^2 ds \right\|_{L^p(\Omega, \mathbb{R})} \leq C \lambda_N^{-3}$. Then, combining it with the mild solution of (1.9) leads to the desired weak orders (c.f. (3.31)-(3.35)). Subsequently, we turn attention to the second term in (1.7). Applying the Taylor expansion gives

\[
\left| \mathbb{E}[\Phi(X^N(T) + O_T)] - \mathbb{E}[\Phi(X^N(T))] \right| \leq \left| \mathbb{E}[\Phi'(X^N(T))(O_T - O_T^N)] \right| + \left| \mathbb{E}\left[ \int_0^1 \Phi''(X^N(T) + \lambda(O_T - O_T^N))(O_T - O_T^N, O_T - O_T^N)(1 - \lambda)d\lambda \right] \right|
\]

(1.10)

The Malliavin integration by parts formula is the key ingredient to deal with the first term (c.f. (3.37)) and the second term can be easily estimated due to the boundedness of $\Phi''$. It is now easy to explain why the weak rate of convergence is expected to be higher than strong convergence rate. As a byproduct of the weak error analysis, one can easily obtain the rate of the strong error,

\[
\| X(t) - X^N(t) \|_{L^2(\Omega, \mathcal{H})} \leq \| \bar{X}(t) - X^N(t) \|_{L^2(\Omega, \mathcal{H})} + \| O_t - O_t^N \|_{L^2(\Omega, \mathcal{H})} \leq C \lambda_N^{-1},
\]

(1.11)

which is consistent with the results in [14,24] and is lower than the weak convergence rate in (1.5), due to the presence of the second error. The basic idea to estimate temporal error is the same as that of the spatial error by essentially exploiting the discrete analogue of the arguments. The main point is that error must be uniform on the spatially discrete parameter $N$.

Having sketched the central ideas of the weak error analysis, we review some relevant results in the literature. For the linearized stochastic Cahn-Hilliard equations, we refer to [3,7,12,15] for some strong convergence results of the finite element method. The authors in [19,21] studied the strong convergence of the fully discrete finite element approximation for Cahn-Hilliard-Cook equation under spatial regular noise, but with no rates obtained. Later, the authors in [25] derives strong convergence rates of the mixed finite element method by using a priori strong moment bounds of the numerical approximations. For unbounded noise diffusion, the existence and regularity of solution have been investigated in [2,11] and the absolute continuity has been studied in [1,13]. Recently, the strong convergence rate of the spatial spectral Galerkin method and the temporal accelerated implicit Euler method for the stochastic Cahn-Hilliard equation was obtained in [14]. For weak convergence analysis in the non-globally Lipschitz setting, we are only aware of papers [3,7,12,15] concerning the stochastic Allen-Cahn equation. To the best of our knowledge, the weak convergence rates of a fully discrete method for the stochastic Cahn-Hilliard equation are absent in the literature. It is worthwhile to point out that issues from the presence of the unbounded operator in front of the nonlinear term make the weak error analysis much more challenging. To be specific, in addition to the aforementioned difficulty in the weak analysis, the estimate of the Malliavin derivative for the spatial approximation process is also completely different, much more efforts are needed (c.f. Proposition 3.3).

The outline of the article is as follows. In the next section, we present some preliminaries, including the well-posedness and regularity of the mild solution and give a brief introduction to Malliavin calculus. Section 3 is devoted to the weak analysis of the spectral Galerkin method in space and Section 4 is concerned with the weak convergence rates of the backward Euler method in time. Finally, numerical tests are offered in Section 5 to illustrate the theoretical findings.
2 Preliminaries

In this section, the mathematical setting, well-posedness and regularity of the model and a brief introduction to Malliavin calculus are given.

2.1 Mathematical setting

Given two real separable Hilbert spaces \((H, \langle \cdot, \cdot \rangle, \| \cdot \|)\) and \((U, \langle \cdot, \cdot \rangle_U, \| \cdot \|_U)\), \(\mathcal{L}(U, H)\) stands for the space of all bounded linear operators from \(U\) to \(H\) with the operator norm \(\| \cdot \|_{\mathcal{L}(U, H)}\) and \(\mathcal{L}_2(U, H) \subset \mathcal{L}(U, H)\) denotes the space of all Hilbert-Schmidt operators from \(U\) to \(H\). For simplicity, we write \(\mathcal{L}(H)\) and \(\mathcal{L}_2(H)\) (or \(\mathcal{L}_2\) for short) instead of \(\mathcal{L}(H, H)\) and \(\mathcal{L}_2(H, H)\), respectively. A fact shows that \(\mathcal{L}_2(U, H)\) is a Hilbert space equipped with the inner product and norm,

\[
\langle T_1, T_2 \rangle_{\mathcal{L}_2(U,H)} = \sum_{i \in \mathbb{N}^+} \langle T_1 \phi_i, T_2 \phi_i \rangle,
\]

where \(\{\phi_i\}\) is an arbitrary orthonormal basis of \(U\). Let \(H = L^2(\mathcal{I}, \mathbb{R})\) and \(\dot{H} = \{v \in H : \langle v, 1 \rangle = 0\}\). \(V := C(\mathcal{I}, \mathbb{R})\) denotes the Banach space of all continuous functions with supremum norm and \(L^r(\mathcal{I}, \mathbb{R}) := \{f : \mathcal{I} \to \mathbb{R}, \int_{\mathcal{I}} |f(x)|^r \, dx < \infty\}\). We define \(P : H \to \dot{H}\) the generalized orthogonal projection by \(Pv = v - |\mathcal{I}|^{-1} \int_{\mathcal{I}} v \, dx\), then \((I - P)v = |\mathcal{I}|^{-1} \int_{\mathcal{I}} v \, dx\) is the average of \(v\).

It is easy to check that \(A\) is a positive definite, self-adjoint and unbounded linear operator on \(\dot{H}\) with compact inverse. For any \(v \in H\), we define \(Av = APv\), then there exists a family of eigenpair \(\{e_j, \lambda_j\}_{j \in \mathbb{N}}\) such that

\[
A e_j = \lambda_j e_j \quad \text{and} \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \quad \text{with} \quad \lambda_j \to \infty,
\]  

where \(e_0 = |\mathcal{I}|^{-\frac{1}{2}}\) and \(\{e_j, j = 1, \cdots\}\) forms an orthonormal basis of \(\dot{H}\). Straightforward applications of the spectral theory yields the fractional powers of \(A\) on \(\dot{H}\), e.g., \(A^\alpha v = \sum_{j=1}^\infty \lambda_j^{\alpha} \langle v, e_j \rangle e_j\), \(\alpha \in \mathbb{R}\), \(v \in \dot{H}\). The space \(\dot{H}^\alpha = \text{dom}(A^{\frac{\alpha}{2}}), \alpha \in \mathbb{R}\) is a Hilbert space with the inner product \(\langle \cdot, \cdot \rangle_\alpha\) and the associated norm \(|\cdot|_\alpha\) given by

\[
\langle v, w \rangle_\alpha = \sum_{j=1}^\infty \lambda_j^{\alpha} \langle v, e_j \rangle \langle w, e_j \rangle, \quad |v|_\alpha = \|A^{\frac{\alpha}{2}} v\| = \left(\sum_{j=1}^\infty \lambda_j^{\alpha} |\langle v, e_j \rangle|^2\right)^{\frac{1}{2}}.
\]

We also define \(\|u\|_\alpha = (|u|^2 + |\langle u, e_0 \rangle|^2)^{\frac{1}{2}}\) for \(u \in H\) and the corresponding space is \(H^\alpha := \{u \in H : |u|_\alpha < \infty\}\). A basic fact shows that for \(\alpha = 1, 2\), the norm \(|\cdot|_\alpha\) on \(\dot{H}^\alpha\) is equivalent to the standard Sobolev norm \(\| \cdot \|_{H^\alpha(\mathcal{I})}\). Since \(H^2(\mathcal{I})\) is an algebra, there is a constant \(C > 0\) such that, for any \(f, g \in \dot{H}^2\),

\[
\|fg\|_{H^2(\mathcal{I})} \leq C \|f\|_{H^2(\mathcal{I})} \|g\|_{H^2(\mathcal{I})} \leq C |f|_2 |g|_2.
\]

We recall that the operator \(-A^2\) generates an analytic semigroup \(E(t) = e^{-tA^2}\) on \(H\) due to (2.2) and we have

\[
E(t)v = e^{-tA^2}v = Pe^{-tA^2}v + (I - P)v, \quad v \in H.
\]
With aid of the eigenbasis of $A$ and Parseval’s identity, we obtain
\[\|A^\mu E(t)\|_{L^2(\mathcal{H})} \leq C t^{-\frac{\mu}{2}}, \quad t > 0, \quad \mu \geq 0,\]  
\[\|A^{-\nu}(I - E(t))\|_{L^2(\mathcal{H})} \leq C t^{\frac{\nu}{2}}, \quad t \geq 0, \quad \nu \in [0, 2],\]  
\[\int_{t_1}^{t_2} \|A^{\rho} E(s)\|^2 ds \leq C |t_2 - t_1|^{1-\rho} \|v\|^2, \quad \forall v \in \mathcal{H}, \rho \in [0, 1],\]  
\[\|A^{2\rho} \int_{t_1}^{t_2} E(t_2 - \sigma) v d\sigma\| \leq C |t_2 - t_1|^{1-\rho} \|v\|, \quad \forall v \in \mathcal{H}, \rho \in [0, 1].\]  

By Assumption 1.2 there exists a constant $C > 0$ such that
\[-(F(u) - F(v), u - v) \leq \|u - v\|^2, \quad u, v \in L^6(\mathcal{I}, \mathbb{R}),\]  
\[\|F(u) - F(v)\| \leq C (1 + \|u\|_V^p + \|v\|_V^p) \|u - v\|, \quad u, v \in V.\]

### 2.2 Well-posedness and regularity results of the model

Assumptions 1.1-1.4 are sufficient to establish well-posedness and spatio-temporal regularity of the mild solution for (1.1). Here we just state the relevant results, whose proofs can be found for example in [24, Theorem 3.5].

First at all, similar to [14, (2.5) & (2.7)], we give the following lemma concerning the spatio-temporal regularity result of stochastic convolution $O_t := \int_0^t E(t - s) dW(s)$.

**Lemma 2.1.** Suppose Assumptions 1.1 and 1.3 hold. Then for all $p \geq 1$, the stochastic convolution $O_t$ satisfies
\[E \left[ \sup_{t \in [0, T]} |O_t|^p \right] + \sup_{t \in [0, T]} E \left[ |O_t|^2 \right] < \infty,\]  
and for $\alpha \in [0, 2]$,
\[\|O_t - O_s\|_{L^p(\Omega, H^\alpha)} \leq C |t - s|^{\frac{\alpha}{2}}.\]

The following theorem states the well-posedness and spatio-temporal regularity of the mild solution for stochastic Cahn-Hilliard equation (1.1).

**Theorem 2.2** (Well-posedness and regularity of the mild solution). Under Assumptions 1.1-1.4, there is a unique mild solution of (1.1) given by
\[X(t) = E(t)X_0 - \int_0^t E(t-s)AF(X(s)) ds + \int_0^t E(t-s)dW(s), \quad t \in [0, T].\]

Furthermore, for $p \geq 1$,
\[\sup_{t \in [0, T]} \|X(t)\|_{L^p(\Omega, H^2)} < \infty,\]  
and for any $\alpha \in [0, 2]$,
\[\|X(t) - X(s)\|_{L^p(\Omega, H^\alpha)} \leq C (t-s)^{\frac{\alpha}{2}}, \quad 0 \leq s < t \leq T.\]

**Corollary 2.3.** If Assumptions 1.1-1.4 are valid, then for $p \geq 1$, (2.15) and (2.16) imply
\[\sup_{t \in [0, T]} \|F(X(t))\|_{L^p(\Omega, H^2)} < \infty.\]
2.3 Introduction to Malliavin calculus

A brief introduction to Malliavin calculus is given in this subsection. For more details, one can consult the classical monograph [23]. Define a Hilbert space $U_0 = Q^2(H)$ with inner product $(u, v)_{U_0} = (Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v)$. Let $G : L^2([0, T], U_0) \to L^2(\Omega, \mathbb{R})$ be an isonormal Gaussian process. For any deterministic mapping $\phi \in L^2([0, T], U_0)$, $G(\phi)$ is centered Gaussian with the covariance structure

$$E[G(\phi_1)G(\phi_2)] = (\phi_1, \phi_2)_{L^2([0, T], U_0)}, \quad \phi_1, \phi_2 \in L^2([0, T], U_0). \quad (2.18)$$

Let $C_p^\infty(\mathbb{R}^M, \mathbb{R})$ be the space of all $C^\infty$-mappings with polynomial growth. We define the family of smooth $H$-valued cylindrical random variables as

$$S(H) = \{G = \sum_{i=1}^N g_i(G(\phi_1), \ldots, G(\phi_M))h_i : g_i \in C_p^\infty(\mathbb{R}^M, \mathbb{R}), \ h_i \in H\}. \quad (2.19)$$

The action of the Malliavin derivative on $G \in S(H)$ is given by

$$D_t G := \sum_{i=1}^N \sum_{j=1}^M \partial_j g_i(G(\phi_1), \ldots, G(\phi_M))h_i \otimes \phi_j(t), \quad (2.20)$$

where $h_i \otimes \phi_j(t)$ denotes the tensor product, that is, for $1 \leq j \leq M$ and $1 \leq i \leq N$,

$$(h_i \otimes \phi_j(t))(u) = \langle \phi_j(t), u \rangle_{U_0}h_i \in H, \quad \forall \ u \in U_0, \ h_i \in H, \ t \in [0, T]. \quad (2.21)$$

If $G$ is $\mathcal{F}_t$-measurable, then $D_s G = 0$ for $s > t$. The derivative operator $D$ is known to be closable and we define $D^{1,2}(H)$ as the closure of $S(H)$ with respect to the norm

$$\|G\|_{D^{1,2}(H)} = \left( E[\|G\|^2] + E \int_0^T \|D_t G\|^2_{L^2(U_0, H)} dt \right)^{\frac{1}{2}}. \quad (2.22)$$

We are now ready to give the Malliavin integration by parts formula. For any $G \in D^{1,2}(H)$ and $\Psi \in L^2([0, T] \times \Omega, L^2(U_0, H))$,

$$E \left[ \left( \int_0^T \Psi(t) dW(t), G \right) \right] = E \int_0^T \langle \Psi(t), D_t G \rangle_{L^2(U_0, H)} dt, \quad (2.23)$$

where the stochastic integral is Itô integral. For brevity, we write $\langle D_s G, u \rangle = D^u_s G$ to represent the derivative in the direction $u \in U_0$. The Malliavin derivative acts on the Itô integral $\int_0^t \Psi(r)dW(r)$ satisfying for all $u \in U_0$,

$$D^u_s \int_0^t \Psi(r)dW(r) = \int_0^t D^u_s \Psi(r)dW(r) + \Psi(s)u, \quad 0 \leq s \leq t \leq T. \quad (2.24)$$

Given another separable Hilbert space $\mathcal{H}$, if $\sigma \in C_b^1(H, \mathcal{H})$ and $G \in D^{1,2}(H)$, then $\sigma(G) \in D^{1,2}(\mathcal{H})$ and the chain rule holds as $D^u_t(\sigma(G)) = \sigma'(G) \cdot D^u_t G$ for $u \in U_0$. 

7
3 Weak convergence rate of the spectral Galerkin method

This section is devoted to the weak analysis of the spatial spectral Galerkin semi-discretization. In the beginning, we define a finite dimension subspace of $H$ spanned by $N$ first eigenvectors of the dominant linear operator $A$, i.e., $H^N = \text{span}\{e_1, \ldots, e_N\}$ and define the projection operator $P_N : H^\beta \to H_N$ by $P_N x = \sum_{i=1}^{N} \langle x, e_i \rangle e_i$ for $\forall x \in H^\beta, \beta \geq -2$. As a result, $A$ commutes with $P_N$ and
\[
\| (P_N - I) x \| \leq C \lambda_{N}\| x \|_\beta, \quad \forall \beta \geq 0.
\] (3.1)

Applying the spectral Galerkin approximation to (1.1) results in the finite-dimensional stochastic differential equation, given by
\[
dX^N(t) + A^2 X^N(t) + AP_N F(X^N(t))dt = P_N dW(t), \quad t \in [0, T]; \quad X^N(0) = P_N X_0,
\] (3.2)
whose unique solution, in the mild form, is written as
\[
X^N(t) = E(t)P_N X_0 - \int_0^t E(t-s)AP_N F(X^N(s))ds + \int_0^t E(t-s)P_N dW(s). \quad (3.3)
\]
Similarly, we use $O_t^N$ to represent $\int_0^t E(t-s)P_N dW(s)$, which enjoys
\[
\sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} E \left[ \| O_t^N \|_{L^p(\Omega, H^2)}^p \right] < \infty, \quad \forall p \geq 1,
\] (3.4)
and for $\alpha \in [0, 2],$
\[
\sup_{N \in \mathbb{N}} \| O_t^N - O_s^N \|_{L^p(\Omega, H^{\alpha})} \leq C |t-s|^{\frac{2-\alpha}{4}}, \quad \forall p \geq 1.
\] (3.5)

The proof of the following regularity results is referred to [24, Lemma 3.4].

**Theorem 3.1** (Spatio-temporal regularity of spatial semi-discretization). If Assumptions 1.1-1.4 are satisfied, then the mild solution of the spatial approximation process (3.3) admits for $\forall p \geq 1,$
\[
\sup_{N \in \mathbb{N}} E \left[ \sup_{t \in [0, T]} \| X^N(t) \|_{L^p(I,R)}^p \right] < \infty.
\] (3.6)
Moreover, we have
\[
\sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \| X^N(t) \|_{L^p(\Omega, H^2)} < \infty,
\] (3.7)
and for any $\alpha \in [0, 2],$
\[
\sup_{N \in \mathbb{N}} \| X^N(t) - X^N(s) \|_{L^p(\Omega, H^{\alpha})} \leq C(t-s)^{\frac{2-\alpha}{4}}, \quad 0 \leq s < t \leq T.
\] (3.8)

**Corollary 3.2.** Under Assumptions 1.1-1.4 combining (3.7) and (2.4) gives
\[
\sup_{t \in [0, T]} \| F(X^N(t)) \|_{L^p(\Omega, H^2)} < \infty, \quad \forall p \geq 1.
\] (3.9)
The next result shows that $X^N(t)$ is differentiable in Malliavin sense.

**Proposition 3.3** (Boundedness of the Malliavin derivative). Let Assumptions 1.1-1.4 hold. Then the Malliavin derivative of $X^N(t)$ satisfies

$$E[\|D^y_sX^N(t)\|^2] \leq C\|y\|^2, \quad 0 \leq s \leq t \leq T.$$  
(3.10)

**Proof.** Taking the Malliavin derivative on the equation (3.3) in the direction $y \in U_0$ and using the chain rule yield that for $0 \leq s \leq t \leq T$,

$$D^y_sX^N(t) = E(t-s)P_Ny - \int_s^t E(t-r)APNF'(X^N(r))D^y_sX^N(r)dr.$$  
(3.11)

Thus, $D^y_sX^N(t)$ is differentiable in time and satisfies

$$\frac{dD^y_sX^N(t)}{dt} + A^2D^y_sX^N(t) + APNF'(X^N(t))D^y_sX^N(t) = 0.$$  
(3.12)

Multiplying $A^{-1}D^y_sX^N(t)$ on both sides infers

$$\left< \frac{dD^y_sX^N(t)}{dt}, A^{-1}D^y_sX^N(t) \right> + \left< A^2D^y_sX^N(t), A^{-1}D^y_sX^N(t) \right> + \left< APNF'(X^N(t))D^y_sX^N(t), A^{-1}D^y_sX^N(t) \right> = 0.$$  
(3.13)

Next, integrating (3.13) over $[s, t]$ and using Hölder’s inequality deduce

$$|D^y_sX^N(t)|^2_{-1} = |y|_{-1}^2 - 2\int_s^t |D^y_sX^N(r)|^2dr - 2\int_s^t \left< F'(X^N(r))D^y_sX^N(r), D^y_sX^N(r) \right>dr$$

$$= |y|_{-1}^2 - 2\int_s^t |D^y_sX^N(r)|_1^2dr + 2\int_s^t \left< A^2D^y_sX^N(r), A^{-1}D^y_sX^N(r) \right>dr$$

$$- 6\int_s^t \left< (X^N(r))^2D^y_sX^N(r), D^y_sX^N(r) \right>dr$$

$$\leq |y|_{-1}^2 - \int_s^t |D^y_sX^N(r)|_1^2dr + \int_s^t |D^y_sX^N(r)|^2_{-1}dr.$$  
(3.14)

Hence, by Gronwall’s inequality we have

$$|D^y_sX^N(t)|^2_{-1} \leq C|y|_{-1}^2.$$  
(3.15)

Therefore,

$$\int_s^t |D^y_sX^N(r)|_1^2dr \leq C|y|_{-1}^2.$$  
(3.16)

In the sequel, taking inner product of (3.12) by $D^y_sX^N(t)$ gives

$$\left< \frac{dD^y_sX^N(t)}{dt}, D^y_sX^N(t) \right> + \left< A^2D^y_sX^N(t), D^y_sX^N(t) \right> + \left< APNF'(X^N(t))D^y_sX^N(t), D^y_sX^N(t) \right> = 0.$$  
(3.17)
Similarly, taking integration and using Hölder’s inequality $\|fg\| \leq C\|f\|_{L^p}\|g\|_{L^q}$ and Sobolev embedding inequality $\dot{H}^{\frac{3}{2}} \subset L^6, d = 1, 2, 3$ give

$$\|D^y_sX^N(t)\|^2 = \|y\|^2 - 2\int_s^t \|AD^y_sX^N(r)\|^2dr$$
$$- 2\int_s^t \langle F'(X^N(r))D^y_sX^N(r), AD^y_sX^N(r)\rangle dr$$
$$\leq \|y\|^2 - 2\int_s^t \|AD^y_sX^N(r)\|^2dr + 2\int_s^t \|AD^y_sX^N(r)\|^2dr$$
$$+ \frac{1}{2}\int_s^t \|F'(X^N(r))D^y_sX^N(r)\|^2dr$$
$$\leq \|y\|^2 + C \left( \sup_{r \in [s,t]} \|X^N(r)\|_{L^6}^4 + 1 \right) \int_s^t \|D^y_sX^N(r)\|_{L^6}^2dr$$
$$\leq \|y\|^2 + C \|y\|^2 \left( \sup_{r \in [s,t]} \|X^N(r)\|_{L^6}^4 + 1 \right).$$

(3.18)

Taking expectation and combining it with (3.6) finish the proof.

Let us now turn to the nonlinear term $F$ and present some useful results on it.

**Lemma 3.4.** Let $F$ be the Nemytskii operator defined in Assumption 1.2 then for $d = 1, 2, 3$,

$$|F'(x)|_1 \leq C(1 + |x|^2)|y|_1, \ x \in \dot{H}^2, \ y \in \dot{H}^1,$$

(3.19)

and

$$|F'(\varsigma)\psi|_{-1} \leq C(1 + |\varsigma|^2)|\psi|_{-1}, \ \forall \varsigma \in \dot{H}^2, \ \psi \in \dot{H}. \quad (3.20)$$

**Proof.** The estimate (3.19) is an immediate consequence of [24, Lemma 3.2]. To see (3.20), we note that

$$\|A^{-\frac{3}{2}}F'(\varsigma)\| = \sup_{\|\xi\| \leq 1} \langle \psi, F'(\varsigma)A^{-\frac{3}{2}}\xi \rangle = \sup_{\|\xi\| \leq 1} \langle A^{-\frac{3}{2}}\psi, A^{-\frac{3}{2}}F'(\varsigma)A^{-\frac{3}{2}}\xi \rangle$$
$$\leq |\psi|_{-1} \sup_{\|\xi\| \leq 1} |F'(\varsigma)A^{-\frac{3}{2}}\xi|_1 \leq C(1 + |\varsigma|^2)|\psi|_{-1}. \quad (3.21)$$

This finishes the proof.

Now, we are well prepared to carry out the weak error analysis of the spatial semi-discretization.

**Theorem 3.5** (Weak convergence rate of the spatial approximation). Let $X(T)$ and $X^N(T)$ be the solution of problems (1.1) and (3.2), given by (2.14) and (3.3), respectively. Let Assumptions 1.1-1.4 hold. Then for $\Phi \in C^2_b$, there exists a constant $C > 0$ such that

$$|\mathbb{E}[\Phi(X(T))]/\mathbb{E}[\Phi(X^N(T))]| \leq C \lambda_N^{-\frac{3}{2}}. \quad (3.22)$$
Proof. By introducing two processes $\bar{X}(t) = X(t) - O_t$ and $\bar{X}^N(t) = X^N(t) - O^N_t$, we can separate the error $E[\Phi(X(T))] - E[\Phi(X^N(T))]$ into two terms as follows

$$E[\Phi(X(T))] - E[\Phi(X^N(T))]$$

$$= \left( E[\Phi(\bar{X}(T) + O_T)] - E[\Phi(\bar{X}^N(T) + O_T)] \right)$$

$$+ \left( E[\Phi(\bar{X}^N(T) + O_T)] - E[\Phi(\bar{X}^N(T) + O^N_T)] \right)$$

$$=: I_1 + I_2.$$

To estimate $I_1$, it suffices to consider the strong convergence between $\bar{X}(T)$ and $\bar{X}^N(T)$. To be specific, by the Taylor expansion and triangle inequality we have

$$|I_1| = \left| E[\Phi(\bar{X}(T) + O_T)] - E[\Phi(\bar{X}^N(T) + O_T)] \right| \leq |E[||\bar{X}(T) - \bar{X}^N(T)||]|$$

$$\leq \|\bar{X}(T) - P_N\bar{X}(T)\|_{L^2(\Omega,\mathcal{H})} + \|P_N\bar{X}(T) - \bar{X}^N(T)\|_{L^2(\Omega,\mathcal{H})}.$$ (3.24)

The regularity of $\bar{X}(T)$ can be improved due to the absence of $O_T$ in $X(T)$,

$$\|A^2\bar{X}(T)\|_{L^p(\Omega,\mathcal{H})}$$

$$\leq \|A^2E(T)X_0\|_{L^p(\Omega,\mathcal{H})} + \left\| \int_0^T A^2E(T-s)APF(X(s))\,ds \right\|_{L^p(\Omega,\mathcal{H})}$$ (3.25)

$$\leq C(|X_0|_4 + \sup_{s\in[0,T]} \|F(X(s))\|_{L^p(\Omega,\mathcal{H}^2)}) < \infty.$$

Using regularity of $\bar{X}(T)$ and (3.1), we can get

$$\|\bar{X}(T) - P_N\bar{X}(T)\|_{L^p(\Omega,\mathcal{H})} = \|(I - P_N)A^{-2}A^2\bar{X}(T)\|_{L^p(\Omega,\mathcal{H})} \leq C\lambda_N^2.$$ (3.26)

In the next step, we denote $e(t) = P_N\bar{X}(t) - \bar{X}^N(t)$, which satisfies

$$\frac{d}{dt} e(t) + A^2e(t) + AP_N\left[ F(\bar{X}(t) + O_t) - F(\bar{X}^N(t) + O^N_t) \right] = 0.$$ (3.27)

Taking inner product on both sides with $A^{-1}e(s)$ in $\hat{H}$ and using (2.10), (2.11), Hölder’s inequality, Sobolev embedding inequality and Lemma 3.4 lead to

$$\frac{1}{2} \frac{d}{ds} |e(s)|^2 + |e(s)|_1^2$$

$$= -\langle e(s), F(\bar{X}(s) + O_s) - F(P_N\bar{X}(s) + O_s) \rangle$$

$$- \langle e(s), F(P_N\bar{X}(s) + O_s) - F(\bar{X}^N(s) + O_s) \rangle$$

$$- \langle e(s), F(\bar{X}^N(s) + O_s) - F(\bar{X}(s) + O_s) \rangle$$

$$\leq \frac{3}{2} \|e(s)\|^2 + \frac{1}{2} \|F(\bar{X}(s) + O_s) - F(P_N\bar{X}(s) + O_s)\|^2$$

$$+ \frac{1}{4} |e(s)|_1^2 + |F(\bar{X}^N(s) + O_s) - F(\bar{X}(s) + O_s)|^2_1$$

$$\leq \frac{3}{2} |e(s)|^2 + \frac{9}{8} |e(s)|^2 + C\|\bar{X}(s) - P_N\bar{X}(s)\|^2(1 + |\bar{X}(s)|^2 + |O_s|^2)$$

$$+ C|O_s - O^N_s|^2 + |\bar{X}^N(s)|^2 + |O^N_s|^2. (3.28)$$
By Gronwall’s inequality, we deduce
\[
|e(T)|^2_1 + \int_0^T |e(s)|^2_1 \, ds \leq C \int_0^T \|\hat{X}(s) - P_N\hat{X}(s)\|^2_1 (1 + |\hat{X}(s)|^2_2 + |O_s|^2_2) \, ds \\
+ C \int_0^T \|O_s - O_s^N\|^2_1 (1 + |\hat{X}^N(s)|^2_2 + |O_s|^2_2) \, ds.
\]

With aid of the regularity of \(X(T)\) and \(X^N(T)\), (3.26), Hölder’s inequality and
\[
\|O_s - O_s^N\|_{L^p(\Omega, \dot{H}^{-1})} \leq C\|(I - P_N)A^{-\frac{1}{2}}O_s\|_{L^p(\Omega, \dot{H})} \leq C\lambda_N^{-\frac{1}{2}},
\]
one can find that
\[
\left\| \int_0^T |e(s)|^2_1 \, ds \right\|_{L^p(\Omega, R)} \leq C \int_0^T \|\hat{X}(s) - P_N\hat{X}(s)\|^2_{L^p(\Omega, \dot{H})} \, ds \\
+ C \int_0^T \|O_s - O_s^N\|^2_{L^p(\Omega, \dot{H}^{-1})} \, ds \leq C\lambda_N^{-3} + C\lambda_N^{-3} \leq C\lambda_N^{-3}.
\]

We are now ready to estimate
\[
\|e(T)\|_{L^p(\Omega, \dot{H})} = \left\| \int_0^T E(T - s)AP_N(F(X(s)) - F(X^N(s))) \, ds \right\|_{L^p(\Omega, \dot{H})} \\
\leq \left\| \int_0^T E(T - s)AP_N(F(\hat{X}(s) + O_s) - F(P_N\hat{X}(s) + O_s)) \, ds \right\|_{L^p(\Omega, \dot{H})} \\
+ \left\| \int_0^T E(T - s)AP_N(F(P_N\hat{X}(s) + O_s) - F(\hat{X}^N(s) + O_s^N)) \, ds \right\|_{L^p(\Omega, \dot{H})} \\
+ \left\| \int_0^T E(T - s)AP_N(F(\hat{X}^N(s) + O_s) - F(\hat{X}^N(s) + O_s^N)) \, ds \right\|_{L^p(\Omega, \dot{H})} \\
= e_1(T) + e_2(T) + e_3(T).
\]

Again, by (2.6), (2.11), (3.26), Sobolev embedding inequality and the regularity of \(X(t)\), we have
\[
e_1(T) = \left\| \int_0^T E(T - s)AP_N(F(\hat{X}(s) + O_s) - F(P_N\hat{X}(s) + O_s)) \, ds \right\|_{L^p(\Omega, \dot{H})} \\
\leq C \int_0^T (T - s)^{-\frac{1}{2}} \|\hat{X}(s) - P_N\hat{X}(s)\|^2_{L^p(\Omega, \dot{H}^{-2})} \, ds \\
\quad \left(1 + \sup_{s \in [0,T]} \|\hat{X}(s)\|^2_{L^p(\Omega, \dot{H}^{-2})} + \sup_{s \in [0,T]} \|O_s\|^2_{L^p(\Omega, \dot{H}^{-2})}\right) \leq C\lambda_N^{-2}.
\]
From \eqref{2.6}, \eqref{3.19} in Lemma \ref{lem:3.4}, Hölder’s inequality, \eqref{3.30} and regularity of $X(t)$ and $X^N(t)$, it follows that

$$e_2(T) \leq C \left\| \int_0^T (T-s)^{-\frac{1}{2}} \left| F(P_N \bar{X}(s) + O_s) - F(\bar{X}^N(s) + O_s) \right| ds \right\|_{L^p(\Omega, \mathbb{R})}$$

$$\leq C \left\| \int_0^T (T-s)^{-\frac{1}{2}} |e(s)|_1 (1 + |\bar{X}(s)|_2^2 + |\bar{X}^N(s)|_2^2 + |O_s|^2) ds \right\|_{L^p(\Omega, \mathbb{R})}$$

$$\leq C \left\| \int_0^T |e(s)|^2_1 ds \right\|^{\frac{1}{2}}_{L^p(\Omega, \mathbb{R})} \left( \int_0^T (T-s)^{-\frac{3}{2}} ds \right)^{\frac{1}{2}}$$

$$\leq C \lambda_N^{\frac{3}{2}}.$$

Similarly to the estimate of \eqref{3.32} with \eqref{3.20} and \eqref{3.29} instead, we obtain

$$e_3(T) = \left\| \int_0^T \left( E(T-s)A^\frac{1}{2} A^{-\frac{1}{2}} P_N (F(\bar{X}^N(s) + O_s) - F(\bar{X}^N(s) + O^N_s)) \right) ds \right\|_{L^p(\Omega, \mathcal{H})}$$

$$\leq C \int_0^T (T-s)^{-\frac{1}{2}} \|O_s - O^N_s\|_{L^{2p}(\Omega, \mathcal{H}^{-1})} ds$$

$$\leq \left( 1 + \sup_{s \in [0,T]} \|\bar{X}^N(s)\|_{L^{4p}(\Omega, \mathcal{H}^2)}^2 + \sup_{s \in [0,T]} \|O_s\|_{L^{4p}(\Omega, \mathcal{H}^2)}^2 \right)$$

$$\leq C \lambda_N^{\frac{3}{2}}.$$

Taking estimates of $e_1(T)$, $e_2(T)$ and $e_3(T)$ together yields

$$\|P_N \bar{X}(T) - \bar{X}^N(T)\|_{L^2(\Omega, \mathcal{H})} \leq C \lambda_N^{\frac{3}{2}}. \tag{3.35}$$

Combining it with \eqref{3.26} derives $|I_1| \leq C \lambda_N^{\frac{3}{2}}$. Next, we turn to the estimate of $|I_2|$,

$$|I_2| = \left| \mathbb{E} \left[ \Phi' (X^N(T))(O_T - O^N_T) \right. \right.$$

$$\left. + \int_0^1 \Phi'' (X^N(T) + \lambda(O_T - O^N_T))(O_T - O^N_T)(1 - \lambda) d\lambda \right] \right|$$

$$\leq \left| \mathbb{E} \left[ \Phi'(X^N(T))(I - P_N)O_T \right] \right| + C \mathbb{E} \left[ \|O_T - O^N_T\|^2 \right].$$

The second term can be easily bounded by utilizing \eqref{3.1}, that is

$$\mathbb{E} \left[ \|O_T - O^N_T\|^2 \right] = \mathbb{E} \left[ \|(I - P_N)O_T\|^2 \right] \leq C \lambda^{-2}_N. \tag{3.36}$$

For the first term, Proposition \ref{prop:3.3} the Malliavin integration by parts formula \eqref{2.23}, the chain
rule of the Malliavin derivative, (2.6), (3.1) and (1.3) enable us to obtain

\[
\left| \mathbb{E} \left[ \Phi'(X^N(T))(I - P_N)\Omega_T \right] \right| \\
= \left| \mathbb{E} \int_0^T \left\langle (I - P_N)E(T - s), D_s \Phi'(X^N(T)) \right\rangle \mathcal{L}_2^0 \, ds \right| \\
\leq \mathbb{E} \int_0^T \left\| (I - P_N)E(T - s) \right\|_{\mathcal{L}_2^{\frac{1}{2}}} \left\| \Phi''(X^N(T)) \right\|_{\mathcal{L}_2^{\frac{1}{2}}} \|D_s X^N(T)\|_{\mathcal{L}_2} \, ds \\
\leq \int_0^T \left\| (I - P_N)E(T - s) \right\|_{\mathcal{L}_2^{\frac{1}{2}}} \|Q_{\frac{1}{2}}\|_{\mathcal{L}_2} \, ds \\
\leq C\lambda_N^{-\frac{3}{2}} \int_0^T (T - s)^{-\frac{3}{2}} \, ds \leq C\lambda_N^{-\frac{3}{2}}.
\] (3.37)

Hence, the proof is complete. \(\square\)

4 Weak convergence rate of the backward Euler method

Based on the spatial spectral Galerkin approximation [32], this section concerns the weak error analysis of a backward Euler method in the temporal direction. We divide the interval \([0, T]\) into \(M\) equidistant subintervals with the time step-size \(\tau = \frac{T}{M}\) and denote the nodes \(t_m = m\tau\) for \(m \in \{1, \ldots, M\}, M \in \mathbb{N}^+\). Then, the fully discrete scheme is given by

\[
X^{M,N}_{t_m} - X^{M,N}_{t_{m-1}} + \tau A^2 X^{M,N}_{t_m} + \tau P_N A F(X^{M,N}_{t_m}) = P_N \Delta W_m, \quad X^{M,N}_0 = P_N X_0,
\] (4.1)

where \(\Delta W_m := W(t_m) - W(t_{m-1})\) for short. By introducing a family of operators \(\{E^{m}_{\tau,N}\}_{m=1}^M\): \(E^{m}_{\tau,N}v = (I + \tau A^2)^{-m}P_Nv = \sum_{j=1}^{N}(1 + \tau \lambda_j^2)^{-m}\langle v, e_j \rangle e_j, \forall v \in \hat{H}\), we have

\[
X^{M,N}_{t_m} = E^{m}_{\tau,N}X_0 - \tau \sum_{j=1}^{m} E^{m-j+1}_{\tau,N} AP F(X^{M,N}_{t_j}) + \sum_{j=1}^{m} E^{m-j+1}_{\tau,N} \Delta W_j.
\] (4.2)

It is easy to check that the operator \(E^{m}_{\tau,N}\) satisfies

\[
\|A^\mu E^{m}_{\tau,N}v\| \leq Ct_m^{-\frac{\mu}{2}}\|v\|, \quad \mu \in [0, 2], v \in \hat{H}, m \in \{1, 2, \ldots, M\}.
\] (4.3)

By a slight modification of the proof in [24, Theorem 4.1], the regularity of the fully discrete approximation can be derived.

Theorem 4.1. Let Assumptions [1.1-1.4] be satisfied, then we have for \(\forall p \geq 1,\)

\[
\sup_{N \in \mathbb{N}} \sup_{m \in \{0, 1, \ldots, M\}} \mathbb{E} \left[ \|X^{M,N}_{t_m}\|_p \right] < \infty.
\] (4.4)
Before presenting the main theorem, we introduce the notation \( \lfloor s \rfloor := \max\{0, \tau, \cdots, m\tau, \cdots\} \cap [0, s] \), \( \lceil s \rceil := \min\{0, \tau, \cdots, m\tau, \cdots\} \cap [s, T] \) and \( [s] := \frac{|s|}{\tau} \). The fully discrete approximation operator is then defined by

\[
\Psi^{M,N}_\tau(t) := E(t)P_N - E^k_{\tau,N}, \quad t \in [t_{k-1}, t_k), \quad k \in \{1, 2, \cdots, M\}.
\] (4.5)

The following lemma of the fully discrete approximation operator plays a pivotal role in the weak convergence analysis.

**Lemma 4.2.** Under Assumption 1.1, we have the following statements.

(i) Let \( \rho \in [0, 4] \), there exists a constant \( C \) such that for \( t > 0 \),

\[
\| \Psi^{M,N}_\tau(t)u \| \leq Ct^{-\frac{\rho}{4}}|u|_{-\rho}, \quad u \in \dot{H}^{-\rho}.
\] (4.6)

(ii) Let \( \beta \in [0, 4] \), there exists a constant \( C \) such that for \( t > 0 \),

\[
\| \Psi^{M,N}_\tau(t)u \| \leq C\tau^{\frac{\beta}{4}}|u|_{\beta}, \quad u \in \dot{H}^\beta.
\] (4.7)

(iii) Let \( \alpha \in [0, 2] \), there exists a constant \( C \) such that for \( t > 0 \),

\[
\| \Psi^{M,N}_\tau(t)u \| \leq C\tau^{\frac{4-\alpha}{4}}t^{-1}|u|_{-\alpha}, \quad u \in \dot{H}^{-\alpha}.
\] (4.8)

(iv) Let \( \mu \in [2, 4] \), there exists a constant \( C \) such that for \( t > 0 \),

\[
\| \Psi^{M,N}_\tau(t)u \| \leq C\tau^{\frac{\mu-2}{4}}t^{-\frac{\mu}{4}}|u|_{\mu-4}, \quad u \in \dot{H}^{\mu-4}.
\] (4.9)

(v) Let \( \nu \in [0, 4] \), there exists a constant \( C \) such that for \( t > 0 \),

\[
\left( \int_0^t \| \Psi^{M,N}_\tau(s)u\|^2ds \right)^\frac{1}{2} \leq C\tau^{\frac{\nu}{4}}|u|_{\nu-2}, \quad u \in \dot{H}^{\nu-2}.
\] (4.10)

**Proof.** Elementary fact in [24, Lemma 5.3] yields (i), (ii), (iii) and (v). We then use the standard interpolation argument to prove (iv). For \( \mu = 2 \), it is a consequence of (i) with \( \rho = 2 \) and for \( \mu = 4 \), it is a consequence of (iii) with \( \alpha = 0 \).

For clarity of exposition, \( \mathcal{O}^{M,N}_T := \sum_{j=1}^M E^{M-j+1}_{\tau,N} \Delta W_j = \int_0^T E^{M-[s]}_{\tau,N}dW(s) \). The next lemma gives the estimate between \( \mathcal{O}^{N}_{t_m} \) and \( \mathcal{O}^{M,N}_{t_m} \).

**Lemma 4.3.** Under Assumptions 1.1 and 1.3, we have for \( p \geq 1 \),

\[
\sup_{m \in \{1, 2, \cdots, M\}} \left\| \mathcal{O}_{t_m}^{M,N} - \mathcal{O}_{t_m}^{N} \right\|_{L^p(\Omega, \dot{H}^{-\beta})} \leq C\tau^{\frac{2+\beta}{4}}, \quad \beta \in [-2, 2].
\] (4.11)
Proof. The Burkholder-Davis-Gundy inequality and (v) in Lemma 4.2 with \( \nu = 2 + \beta \) yield
\[
\| O_{t_m}^{M,N} - O_{t_m}^N \|_{L^p(\Omega, \mathcal{H}^{-\beta})} \leq C \left( \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \| \Psi^{M,N}_\tau(t_m - s) A^{-\frac{\beta}{2}} Q_{t_m}^\frac{1}{2} \|_{L^2} ds \right)^{\frac{1}{2}} \leq C \tau^{2+\frac{\beta}{4}} \| Q_{t_m}^\frac{1}{2} \|_{L^2} \leq C \tau^{2+\frac{\beta}{4}}.
\]
(4.12)
This finishes the proof. \( \square \)

The following theorem shows the weak convergence rate of the temporal semi-discretization.

**Theorem 4.4** (Weak convergence rate of the temporal approximation). \( \text{Suppose Assumptions 1.1 and 1.2 are satisfied. Let } X^N(T) \text{ and } X_T^{M,N} \text{ be given by (3.3) and (4.2), respectively. Then, we have for } \Phi \in C_b^2, \)
\[
| E[\Phi(X^N(T))] - E[\Phi(X_T^{M,N})] | \leq C \tau^{2+\frac{\beta}{4}}.
\]
(4.13)

Proof. At first, we define \( \bar{X}_T^{M,N} = X_T^{M,N} - O_T^{M,N} \) and separate the error into two terms
\[
E[\Phi(X^N(T))] - E[\Phi(X_T^{M,N})] = \left( E[\Phi(X^N(T) + O_T^N)] - E[\Phi(X^N(T) + O_T^{M,N})] \right)
+ \left( E[\Phi(X^N(T) + O_T^{M,N})] - E[\Phi(\bar{X}_T^{M,N} + O_T^{M,N})] \right)
=: K_1 + K_2.
\]
The estimate of \( K_1 \) relies on a second-order Taylor expansion that
\[
|K_1| = \left| E[\Phi(\bar{X}_T^{M,N}) - O_T^{M,N}] \right|
= \left| E[\Phi'(X^N(T))(O_T^{M,N} - O_T^N)] \right|
+ \int_0^1 \Phi''(X^N(T) + \lambda(O_T^{M,N} - O_T^N))(O_T^{M,N} - O_T^N)(O_T^{M,N} - O_T^N)(1 - \lambda)d\lambda \leq \left| E[\Phi'(X^N(T))(O_T^{M,N} - O_T^N)] \right| + C \mathbb{E}[\| O_T^{M,N} - O_T^N \|^2].
\]
(4.15)
The bound of the second term \( \mathbb{E}[\| O_T^{M,N} - O_T^N \|^2] \) can be directly obtained by Lemma 4.3 with \( \beta = 0 \). Then, we turn attention to the first term,
\[
\left| E[\Phi'(X^N(T))(O_T^{M,N} - O_T^N)] \right|
= \left| E \int_0^T \langle E(T - s)P_N - E_{r,N}^{M-[s]}, D_s \Phi'(X^N(T)) \rangle_{L^2} ds \right|
\leq E \int_0^T \| E(T - s)P_N - E_{r,N}^{M-[s]} \|_{L^2} \| \Phi''(X^N(T))D_sX^N(T) \|_{L^2} ds \leq C E \int_0^T \| E(T - s)P_N - E_{r,N}^{M-[s]} \|_{L^2}^\frac{1}{2} \| Q_{t_m}^\frac{1}{2} \|_{L^2}^2 ds
\leq C \tau^{\frac{1}{8}} \int_0^T (T - s)^{-\frac{3}{8}} ds \leq C \tau^{\frac{1}{4}},
\]
(4.16)
where \(2.23\), \(1.3\), Proposition \(3.3\) and (iv) in Lemma \(4.2\) with \(\mu = \frac{7}{2}\) were used.

To estimate \(K_2\), it suffices to bound \(\|X^N(T) - \bar{X}_T^{M,N}\|_{L^2(\Omega, \mathcal{H})}\). To this end, we introduce an auxiliary process \(Y_\tau^{M,N}\) by

\[
Y_\tau^{M,N} = E_{\tau,N}^m X_0 - \tau \sum_{j=1}^{m} E_{\tau,N}^{m-j+1} A F(X^N(t_j)) + \sum_{j=1}^{m} E_{\tau,N}^{m-j+1} \Delta W_j
\]

(4.17)

and define \(\bar{Y}_\tau^{M,N} = Y_\tau^{M,N} - \mathcal{O}_\tau^{M,N}\). A standard argument gives

\[
\|Y_\tau^{M,N}\|_{L^p(\Omega, \mathcal{H}^2)} < \infty.
\]

(4.18)

Subsequently, by triangle inequality, we have

\[
\|X^N(T) - \bar{X}_T^{M,N}\|_{L^2(\Omega, \mathcal{H})} \leq \|X^N(T) - Y_T^{M,N}\|_{L^2(\Omega, \mathcal{H})} + \|Y_T^{M,N} - \bar{X}_T^{M,N}\|_{L^2(\Omega, \mathcal{H})}.
\]

(4.19)

The error term \(\|X^N(T) - Y_T^{M,N}\|_{L^p(\Omega, \mathcal{H})}\) can be further divided into three terms

\[
\|X^N(T) - Y_T^{M,N}\|_{L^p(\Omega, \mathcal{H})} = \|(E(T)P_N - E_{\tau,N}^M)X_0
\]

\[
- \left( \int_0^T (E(T-s)P_N AF(X^N(s))ds - \tau \sum_{j=1}^{M} E_{\tau,N}^{M-j+1} A F(X^N(t_j))) \right) \|_{L^p(\Omega, \mathcal{H})}
\]

\[
\leq \|(E(T)P_N - E_{\tau,N}^M)X_0\|_{L^p(\Omega, \mathcal{H})}
\]

\[
+ \left( \int_0^T (E(T-s)P_N - E_{\tau,N}^{M-[s]}) AF(X^N(s))ds \right) \|_{L^p(\Omega, \mathcal{H})}
\]

\[
+ \left( \int_0^T E_{\tau,N}^{M-[s]} A (F(X^N(s)) - F(X^N([s])))ds \right) \|_{L^p(\Omega, \mathcal{H})}
\]

\[
=: K_{21} + K_{22} + K_{23}.
\]

By (ii) of Lemma \(4.2\) with \(\beta = 4\) and Assumption \(1.4\), we deduce

\[
K_{21} \leq C \tau |X_0|_4 \leq C \tau.
\]

(4.21)

Concerning the term \(K_{22}\), by use of \(3.9\) and (iv) of Lemma \(4.2\) with \(\mu = \frac{7}{2}\), we observe that

\[
K_{22} \leq \int_0^T \left\| (E(T-s)P_N - E_{\tau,N}^{M-[s]}) AF(X^N(s)) \right\|_{L^p(\Omega, \mathcal{H})} ds
\]

\[
\leq C \tau \bar{\beta} \int_0^T (T-s)^{-\bar{\beta}} ds \sup_{s \in [0,T]} \|F(X^N(s))\|_{L^2p(\Omega, \mathcal{H}^2)}
\]

\[
\leq C \tau \bar{\beta}.
\]

(4.22)
To handle $K_{23}$, we decompose it into four terms with the aid of the Taylor expansion and the mild form of $X^N(t)$

$$K_{23} \leq \left\| \int_0^T E^{M-[s]}_{\tau,N} A \left( F'(X^N(s))(E([s] - s) - I)X^N(s) \right) ds \right\|_{L^p(\Omega,\hat{H})}
+ \left\| \int_0^T E^{M-[s]}_{\tau,N} A \left( F'(X^N(s)) \int_s^{[s]} E([\xi] - \tau)P_NAF(X^N(\xi))d\xi \right) ds \right\|_{L^p(\Omega,\hat{H})}
+ \left\| \int_0^T E^{M-[s]}_{\tau,N} A \left( F'(X^N(s)) \int_s^{[s]} E([\xi] - \tau)P_NdW(\xi) \right) ds \right\|_{L^p(\Omega,\hat{H})}
+ \left\| \int_0^T E^{M-[s]}_{\tau,N} A \left( \int_0^1 F''(X^N(s) + \lambda(X^N(\xi) - X^N(s))) \right) \\
(X^N([s]) - X^N(s), X^N([s]) - X^N(s))(1 - \lambda)d\lambda) ds \right\|_{L^p(\Omega,\hat{H})}
=: K_{231} + K_{232} + K_{233} + K_{234}. $$

The smoothness of $E^{m}_{\tau,N}$ in (4.3), (2.7), (3.20) and the regularity of $X^N(t)$ lead to

$$K_{231} = \left\| \int_0^T E^{M-[s]}_{\tau,N} A^{\frac{2}{p}} A^{-\frac{1}{q}} \left( F'(X^N(s))(E([s] - s) - I)X^N(s) \right) ds \right\|_{L^p(\Omega,\hat{H})}
\leq C \int_0^T (T - [s])^{-\frac{2}{p}} \left\| (1 + |X^N(s)|^2) \left( E([s] - s) - I \right)X^N(s) \right\|_{L^p(\Omega,\hat{H})} ds
\leq C \tau. \quad (4.23)$$

Following similar approach as above and utilizing (3.9) yield

$$K_{232} \leq C \int_0^T (T - [s])^{-\frac{2}{p}} \left\| (1 + |X^N(s)|^2) \right. \times \int_s^{[s]} \left\| E([\xi] - \tau)P_NAF(X^N(\xi)) \right\|_{L^p(\Omega,\hat{H})} d\xi \left\|_{L^p(\Omega,\hat{H})} ds
\leq C \tau \int_0^T (T - [s])^{-\frac{2}{p}} ds \sup_{r \in [0,T]} \left\| F(X^N(r)) \right\|_{L^{2p}(\Omega,\hat{H}^2)}
\leq C \tau. \quad (4.24)$$

From the stochastic Fubini theorem, the Burkholder–Davis–Gundy inequality and Hölder’s in-
Finally, we turn our attention to the estimate of the error term equality, it follows that

\[
K_{233} = \left\| \sum_{j=1}^{M} \int_{t_{j-1}}^{t_j} \chi_{[s,t_j]}(r) E^{M-[s]}_{\tau,N} A f(X^N(s)) E([s] - r) P_N dW(r) ds \right\|_{L^p(\Omega, \mathcal{H})}
\]

\[
= \left\| \sum_{j=1}^{M} \int_{t_{j-1}}^{t_j} \chi_{[s,t_j]}(r) E^{M-[s]}_{\tau,N} A f(X^N(s)) E([s] - r) P_N dW(r) \right\|_{L^p(\Omega, \mathcal{H})}
\]

\[
\leq \left( \sum_{j=1}^{M} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} \chi_{[s,t_j]}(r) E^{M-[s]}_{\tau,N} A f(X^N(s)) E([s] - r) ds \right)^{\frac{1}{2}} \leq C \tau \quad \quad \quad (4.26)
\]

where \( \chi_{[0,t]} \) denotes the indicator function on \([0,t]\). Owing to Hölder’s inequality, the Sobolev embedding inequality \( \dot{H}^\delta \subset V \) for \( \frac{3}{2} < \delta < 2 \) and Theorem 3.1 we obtain

\[
K_{234} = \left\| \int_0^T E^{M-[s]}_{\tau,N} A \left( \int_0^1 F''(X^N(s) + \lambda(X^N([s]) - X^N(s))) - X^N([s]) - X^N(s)) (1 - \lambda) d\lambda \right) ds \right\|_{L^p(\Omega, \mathcal{H})}
\]

\[
\leq C \int_0^T (T - [s])^{-\frac{\alpha + 2}{2}} \|X^N([s]) - X^N(s)\|^2_{L^p(\Omega, \mathcal{H})} \left( 1 + \sup_{s \in [0,T]} \|X^N(s)\|_{L^p(\Omega, V)} \right) ds
\]

\[
\leq C \tau. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (4.27)
\]

Therefore, gathering the above estimates together gives

\[
\|\tilde{X}^N(T) - \tilde{Y}_T^{M,N}\|_{L^p(\Omega, \mathcal{H})} \leq C \tau^\frac{3}{2}. \quad \quad \quad \quad \quad \quad \quad \quad \quad (4.28)
\]

Finally, we turn our attention to the estimate of the error term \( \|\tilde{Y}_T^{M,N} - X_T^{M,N}\|_{L^2(\Omega, \mathcal{H})} \). By denoting \( e_t^{M,N} = \tilde{Y}_t^{M,N} - \tilde{X}_t^{M,N} \), we have

\[
e_t^{M,N} = e_{t_{m-1}}^{M,N} + \tau A_2 e_{t_{m}}^{M,N} - \tau P_N A f(X_{t_{m}}^{M,N}) - \tau P_N A f(X_{t_{m}}^{M,N}). \quad \quad \quad (4.29)
\]

Multiplying both sides by \( A_t^{-1} e_{t_{m}}^{M,N} \) shows

\[
\left( e_{t_{m}}^{M,N} - e_{t_{m-1}}^{M,N} \right) + \tau \left( A_2 e_{t_{m}}^{M,N} - A_2 e_{t_{m-1}}^{M,N} \right) = \tau \left( F(\tilde{X}_t^{M,N} + \mathcal{O}_{t_{m}}^{M,N}) - F(\tilde{Y}_{t_{m}}^{M,N} + \mathcal{O}_{t_{m}}^{M,N}), e_t^{M,N} \right)
\]

\[
+ \tau \left( F(\tilde{Y}_{t_{m}}^{M,N} + \mathcal{O}_{t_{m}}^{M,N}) - F(\tilde{X}_t^{M,N} + \mathcal{O}_{t_{m}}^{M,N}), e_t^{M,N} \right)
\]

\[
+ \tau \left( F(\tilde{X}_t^{M,N} + \mathcal{O}_{t_{m}}^{M,N}) - F(\tilde{X}_t^{M,N} + \mathcal{O}_{t_{m}}^{M,N}), e_t^{M,N} \right). \quad \quad \quad (4.30)
\]
Thus, by $\langle e_{t_{m}}^{M,N} - e_{t_{m-1}}^{M,N}, A^{-1}e_{t_{m}}^{M,N} \rangle \geq \frac{1}{2}(|e_{t_{m}}^{M,N}|_{-1}^{2} - |e_{t_{m-1}}^{M,N}|_{-1}^{2})$ and using similar approach as in (3.23), we further obtain

\[
\frac{1}{2}(|e_{t_{m}}^{M,N}|_{-1}^{2} - |e_{t_{m-1}}^{M,N}|_{-1}^{2}) + \tau|e_{t_{m}}^{M,N}|_{1}^{2} \leq \frac{3}{4} \tau|e_{t_{m}}^{M,N}|_{1}^{2} + \frac{9}{8} \tau|e_{t_{m}}^{M,N}|_{-1}^{2} + C \tau \|F(Y_{t_{m}}^{M,N} + O_{t_{m}}^{M,N}) - F(\bar{X}^{N}(t_{m}) + O_{t_{m}}^{M,N})\|^{2} + \tau \|F(\bar{X}^{N}(t_{m}) + O_{t_{m}}^{M,N}) - F(\bar{X}^{N}(t_{m}) + O_{t_{m}}^{N})\|_{-1}^{2}.
\]

(4.31)

By iteration in $m$ and Gronwall’s inequality, we obtain

\[
|e_{T}^{M,N}|_{-1}^{2} + \tau \sum_{j=1}^{M} |e_{t_{j}}^{M,N}|_{1}^{2} \leq C \tau \sum_{j=1}^{M} \left( \|F(Y_{t_{j}}^{M,N} + O_{t_{j}}^{M,N}) - F(\bar{X}^{N}(t_{j}) + O_{t_{j}}^{M,N})\|^{2} + \|F(\bar{X}^{N}(t_{j}) + O_{t_{j}}^{M,N}) - F(\bar{X}^{N}(t_{j}) + O_{t_{j}}^{N})\|_{-1}^{2} \right).
\]

(4.32)

Taking expectation and employing (2.11), (3.20), (4.28), Lemma 4.3 and regularity of $Y_{t_{m}}^{M,N}$ and $X^{N}(t)$ result in

\[
\left\|\tau \sum_{j=1}^{M} |e_{t_{j}}^{M,N}|_{1}^{2}\right\|_{L^{p}(\Omega, \mathbb{R})} \leq C \tau \sum_{j=1}^{M} \left( \|Y_{t_{j}}^{M,N} - \bar{X}^{N}(t_{j})\|^{2} + \|\bar{X}^{N}(t_{j})\|_{V}^{4} + \|O_{t_{j}}^{M,N}\|_{V}^{4} \right) \left\|1 + \|\bar{X}^{N}(t_{j})\|_{V}^{4} + \|O_{t_{j}}^{M,N}\|_{V}^{4} \right\|_{L^{p}(\Omega, \mathbb{R})} + C \tau \sum_{j=1}^{M} \left( \|O_{t_{j}}^{M,N} - O_{t_{j}}^{N}\|_{-1}^{2} + \|O_{t_{j}}^{M,N}\|_{2}^{4} + \|O_{t_{j}}^{N}\|_{2}^{4} \right) \left\|1 + \|\bar{X}^{N}(t_{j})\|_{V}^{4} + \|O_{t_{j}}^{M,N}\|_{V}^{4} \right\|_{L^{p}(\Omega, \mathbb{R})} \leq C \tau^{\frac{1}{2}}.
\]

(4.33)

Furthermore, we split $\|e_{T}^{M,N}\|_{L^{p}(\Omega, \mathbb{H})}$ into three parts

\[
\|e_{T}^{M,N}\|_{L^{p}(\Omega, \mathbb{H})} \leq \tau \left\|\sum_{j=1}^{M} E^{M-j+1}_{t_{j},N} A(F(\bar{X}^{N}(t_{j}))) - F(\bar{X}^{M,N}_{t_{j}})\right\|_{L^{p}(\Omega, \mathbb{H})} + \tau \left\|\sum_{j=1}^{M} E^{M-j+1}_{t_{j},N} A(F(\bar{Y}^{N}_{t_{j}} + O_{t_{j}}^{N}) - F(\bar{Y}^{M,N}_{t_{j}} + O_{t_{j}}^{N})\right\|_{L^{p}(\Omega, \mathbb{H})} + \tau \left\|\sum_{j=1}^{M} E^{M-j+1}_{t_{j},N} A(F(\bar{Y}^{M,N}_{t_{j}} + O_{t_{j}}^{M,N}) - F(\bar{X}^{M,N}_{t_{j}} + O_{t_{j}}^{N})\right\|_{L^{p}(\Omega, \mathbb{H})}
\]

=: Err_{1} + Err_{2} + Err_{3}.
Analogously to the above estimate but with (3.20) instead, we derive

\[
E_{rr1} \leq C \tau \sum_{j=1}^{M} t_{M-j+1}^{\frac{1}{2}} \| \tilde{X}^N(t_j) - \tilde{Y}^M(t_j) \|_{L^2(p, \hat{H})} \leq C \tau^{\frac{3}{2}}.
\]

Analogously to the above estimate but with (3.20) instead, we derive

\[
E_{rr2} \leq C \tau \sum_{j=1}^{M} t_{M-j+1}^{\frac{3}{2}} \| O^N(t_j) - O^M(t_j) \|_{L^2(p, \hat{H}^{-1})} \leq C \tau^{\frac{3}{2}}.
\]

At last, combining (3.19), Hölder’s inequality, (4.33) and regularity of \(Y^M_{t_m}\) and \(X^M_{t_m}\) leads to

\[
E_{rr3} \leq C \left\| \tau \sum_{j=1}^{M} t_{M-j+1}^{\frac{1}{4}} |F(\tilde{Y}^M(t_j) + O^M(t_j)) - F(\tilde{X}^M(t_j) + O^M(t_j))| \right\|_{L^p(\Omega, \mathbb{R})} \\
\leq C \left\| \tau \sum_{j=1}^{M} t_{M-j+1}^{\frac{1}{4}} e_{t_j}^{M,N} |1 + |\tilde{Y}^M(t_j)|^2 + |\tilde{X}^M(t_j)|^2 + |O^M(t_j)|^2| \right\|_{L^p(\Omega, \mathbb{R})} \\
\leq C \left\| \tau \sum_{j=1}^{M} |e_{t_j}^{M,N}|^2 \right\|_{L^p(\Omega, \mathbb{R})}^{\frac{1}{2}} \tau^{\frac{3}{2}}.
\]

This completes the proof of this theorem.

**Remark 4.5.** As a by-product of the weak error analysis, one can easily obtain the rates of the strong error, for \(m \in \{1, 2, \cdots, M\},\)

\[
\| X(t_m) - X^M_{t_m} \|_{L^p(\Omega, \hat{H})} \leq \| \tilde{X}(t_m) - \tilde{X}^M_{t_m} \|_{L^p(\Omega, \hat{H})} + \| O_{t_m} - O^M_{t_m} \|_{L^p(\Omega, \hat{H})} \leq C(\lambda_{N}^{\frac{1}{2}} + \tau^{\frac{1}{2}}),
\]

which is lower than the weak error, due to the presence of the second error.
5 Numerical experiments

This section is devoted to numerically illustrate the main assertions obtained above. Let us focus on the following spatial one-dimensional stochastic Cahn-Hilliard equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 w}{\partial x^2} + \dot{W}, & (t, x) \in (0, T) \times (0, 1), \\
w &= -\frac{\partial^2 u}{\partial x^2} + u - u^3, & x \in (0, 1), \\
u(0, x) &= \sqrt{2} \cos(\pi x), & x \in (0, 1), \\
\frac{\partial u}{\partial x} \bigg|_{x=0} &= \frac{\partial u}{\partial x} \bigg|_{x=1} = 0, & t \in (0, T], \\
\frac{\partial w}{\partial x} \bigg|_{x=0} &= \frac{\partial w}{\partial x} \bigg|_{x=1} = 0, & t \in (0, T], \\
\end{align*}
\]

(5.1)

where \( \{W(t)\}_{t\in[0,T]} \) is a cylindrical \( Q \)-wiener process. We apply the fully discrete method (4.1) to approximate the numerical example (5.1) and take the test function \( \Phi(X) = \sin(\|X\|) \) to measure the weak error at the endpoint \( T = 1 \). The expectations are approximated by computing averages over 1000 samples.

In what follows, the covariance operator \( Q \) is chosen to satisfy

\[
Qe_1 = 0, \quad Qe_i = \frac{1}{i \log(i)^2} e_i \quad \forall i \geq 2.
\]

(5.2)

Obviously, (5.2) guarantees the condition (1.3) in Assumption 1.3. The spatial weak convergence order is shown in the left hand of Fig. 1, where we plot the weak errors against \( 1/Ns = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8} \) on a log-log scale. Since the exact solutions are not available, we use a small stepsize solution \( \tau_{\text{exact}} = 2^{-17} \) and \( N = 2^9 \) to simulate the exact solution. It can be clearly observed that the resulting weak errors decrease at a slope close to 3, which is consistent with the predicted convergence order in Theorem 3.5.

![Figure 1: Weak convergence rates for the full discretization.](image-url)

In the next step, we test the weak convergence order in the temporal direction by fixing \( N = 100 \) and using different time stepsizes \( \tau = 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9} \). The ‘exact’ solution are identified with the numerical ones using a small stepsize \( \tau_{\text{exact}} = 2^{-14} \). From the right hand of Fig. 1 the expected weak convergence order \( 3/4 \) in Theorem 4.4 is numerically confirmed.
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