THE HYPERPLANE IS THE ONLY STABLE, SMOOTH SOLUTION TO THE ISOPERIMETRIC PROBLEM IN GAUSSIAN SPACE

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Abstract. We study stable smooth solutions to the isoperimetric type problem for a Gaussian weight on Euclidean Space. That is, we study hypersurfaces $\Sigma^n \subset \mathbb{R}^{n+1}$ that are second order stable critical points of minimizing
$$\int_{\Sigma} e^{-\frac{|x|^2}{4}} \, dA$$
for compact variations that preserve weighted volume. Such variations are represented by $u \in C^\infty_0(\Sigma)$ such that $\int_{\Sigma} e^{-\frac{|x|^2}{4}} u \, dA = 0$. We show that such $\Sigma$ satisfy the curvature condition $H = \langle x, N \rangle/2 + C$ where $C$ is a constant. We also derive the Jacobi operator $L$ for the second variation of such $\Sigma$.

Our first main result is that for non-planar $\Sigma$, bounds on the index of $L$, acting on volume preserving variations, gives us that $\Sigma$ splits off a linear space. A corollary of this result is that hyperplanes are the only stable smooth complete solutions to this gaussian isoperimetric type problem, and that there are no hypersurfaces of index one. Finally, we show that for the case of $\Sigma^2 \subset \mathbb{R}^3$, there is a gradient decay estimate for fixed bound $|C| \leq M$, where $C$ is from the curvature condition, and $\Sigma$ obeying an appropriate volume growth bound. This shows that, in the limit as $R \to \infty$, stable $(\Sigma, \partial \Sigma) \subset (B_2R(0), \partial B_{2R}(0))$ with good volume growth bounds approach hyperplanes.

Introduction

The isoperimetric problem has been studied in a variety of different settings and is of general interest in mathematics. The problem consists of minimizing the area of hypersurfaces $\Sigma^n$ enclosing a fixed volume inside a fixed Riemannian manifold $M^{n+1}$. The problem has two parts:

1. For a given fixed enclosed volume $V$, what is the infimum of the surface area taken over all hypersurfaces $\Sigma$ with enclosed volume $V$?
2. Is this infimum actually realized by a hypersurface $\Sigma$?

A classic result of Schwarz [13] and Steiner [14] is that the round $n$-sphere solves the isoperimetric problem in $\mathbb{R}^{n+1}$. As Ros [12] comments, the work of Almgren [1], Gruter [9], and Gonzalez, Massari, and Tamanini [8] may be combined to give a fundamental result on the existence and regularity of isoperimetric surfaces. Taken together, their work implies that any compact manifold $M^{n+1}$ (possibly with boundary) has a hypersurface $\Sigma$ that is a solution to the isoperimetric problem for all fixed volumes $V \leq \text{Vol}(M)$. Furthermore, for all $n \leq 6$, the hypersurface $\Sigma$ will be smooth.

For a general introduction to the isoperimetric problem, we direct the reader to a survey by Ros [12].

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Ball [3], Borell [5], and Sudakov & Tsirel’son [15] studied isoperimetric type problems on Gaussian weighted Euclidean space. Let 
\[ \mu(S) = \int_S e^{-\frac{|x|^2}{4}} \, dx \]
be a choice of gaussian measure on \( \mathbb{R}^{n+1} \). For a fixed gaussian weighted volume \( V \), let \( P_V \) be a choice of half-space of \( \mathbb{R}^{n+1} \) such that \( \mu(P_V) = V \). For any measurable subset \( S \subset \mathbb{R}^{n+1} \) let \( S_\epsilon \) be the \( \epsilon \)-neighborhood (with respect to the Euclidean metric) of \( S \). Borell [5] and Sudakov & Tsirel’son [15] show that for any measurable \( S \) with \( \mu(S) = V \), \( \mu(S_\epsilon) \geq \mu(P_V_\epsilon) \) for all \( \epsilon \). Furthermore, we have equality only in the case of \( S \) being a half-space. Therefore, half-spaces are the minimizers for the isoperimetric type problem of minimizing \( S_\epsilon \) for all \( \epsilon \) and \( \mu(S) = V \).

Let us consider the related problem of minimizing \( \mu(\Sigma_\epsilon) \) for the epsilon neighborhood \( \Sigma_\epsilon \) of smooth hypersurfaces \( \Sigma \) with embedded tubular neighborhoods. By applying (1.6), the formula for the first variation for the gaussian volume, we see that
\[ \mu(\Sigma_\epsilon) = \left(2 \int_\Sigma e^{-\frac{|x|^2}{4}} \, dA\right) \epsilon + O_\Sigma(\epsilon^2). \]
Therefore, when comparing such smooth \( \Sigma \), we see that to minimize \( \mu(\Sigma_\epsilon) \) for small \( \epsilon \), it is necessary to minimize \( \int_\Sigma e^{-\frac{|x|^2}{4}} \, dA \).

Hence, we are lead to study the problem of minimizing \( \int_\Sigma e^{-\frac{|x|^2}{4}} \, dA \) among all \( \Sigma \) enclosing a fixed Gaussian weighted volume.

Let \( A_\mu(\Sigma) \) be the functional defined by
\[ A_\mu(\Sigma) = \int_\Sigma e^{-\frac{|x|^2}{4}} \, dA. \]
Since we will only need the derivatives of weighted volume, we don’t need to consider \( \Sigma \) enclosing a specific weighted volume, but instead just consider variations of \( \Sigma \) that preserve the weighted volume. We also drop the hypothesis that \( \Sigma \) have an embedded tubular neighborhood. From the formula for the variation of the weighted volume (1.6) and lemma [2] we are lead to study \( \Sigma \) satisfying
\[ \delta A_\mu(u) = 0 \text{ for } u \in C^\infty_0(\Sigma) \text{ satisfying } \int_\Sigma e^{-\frac{|x|^2}{4}} u \, dA = 0. \]

One should note that this variational problem is not the same as minimizing the surface area with a fixed volume for a conformal change of the Euclidean metric. For such a conformal change, the weights in the integrals for area and volume should not be the same.

In section 3 we show that critical points of (0.1) satisfy the curvature condition \( H = \langle x, N \rangle / 2 + C \) where \( x \) is the position vector, \( N \) is the normal to \( \Sigma \), and \( C \) is a constant. This should also be compared with the result of Colding-Minicozzi [7] that self-shrinkers of mean curvature flow \( (H = \langle x, N \rangle / 2) \) are critical points for minimizing \( \int_\Sigma e^{-\frac{|x|^2}{4}} \, dA \) with no volume constraint.

For any isoperimetric type problem, one is naturally lead to consider the problem of identifying local minimizers. For the above variational problem, we wish to identify those critical points that are minimizers to second order. That is, we wish to identify the critical points \( \Sigma \) that are stable with respect to the second variation for volume preserving variations. This process gives us another question for our isoperimetric type problem, and this is the question that we address in our paper:

(3) In a given manifold \( M^{n+1} \), are stable solutions to the isoperimetric problem necessarily classical solutions (minimizers)?

The answer is not positive for every manifold, as one can see in figure [1].

As mentioned earlier, in weighted Gaussian space, Borell [5] and Sudakov & Tsirel’son [15] showed that the only classical minimizing solutions to the isoperimetric problem are hyperplanes. From (3.4), we see that a surface is stable if
and only if the operator $Lu \equiv \Delta u - (1/2)\nabla_x^T u + |A|^2 u + (1/2) u$ is negative semi-definite for $u \in C^\infty_0(\Sigma)$ satisfying $\int_\Sigma e^{-\frac{|x|^2}{4}} u \, dA = 0$. In section 4, we give analytical reasons for why hyperplanes are stable. The main theorem of this paper details that these are also the only stable solutions. In fact, our theorem gives a result for how the structure of $\Sigma$ depends on the index of $L$ for functions $\int_\Sigma e^{-\frac{|x|^2}{4}} u \, dA = 0$. Here, the index of $L$ is defined as the largest dimension of subspace $V \subset \{u \in C^\infty_0(\Sigma) : \int_\Sigma e^{-\frac{|x|^2}{4}} u \, dA = 0\}$ such that $L$ is negative definite on $V$.

**Theorem 2.** If $\Sigma$ is a complete non-planar critical point of (0.1) with Index $= I$ for functions $\delta V_\mu(u) = 0$, then

$$\Sigma = \Sigma_0 \times \mathbb{R}^{n+1-i}$$

where $i \leq I$.

The main idea in proving this lemma is to use the fact that for any critical point $\Sigma$ satisfying (0.1) and any $v \in \mathbb{R}^{n+1}$, we have that $L(v, N) = (1/2)(v, N)$. Using the fact that the functions $\{\langle v, N \rangle\}$ form a vector space, we can find $v$ such that $\int_\Sigma e^{-\frac{|x|^2}{4}} \langle v, N \rangle \, dA = 0$. In fact, by being careful with our estimates we can consider the space $\text{Span}\{\langle v, N \rangle, 1\}$. By applying appropriate cut-off functions and plugging into a stability inequality, we get a lower bound on the dimension of $\{\langle v, N \rangle \equiv 0\}$.

An immediate consequence of this theorem is

**Corollary 2.** The only complete $L$ stable critical point of the variation problem (0.1) is the hyperplane. There are no critical points of index one.

In addition to our result for complete stable $\Sigma$, we also have statements for stable $(\Sigma, \partial\Sigma) \subset (B_{2R}(0), \partial B_{2R}(0))$. When $n = 2$, we get a decay estimate for $|A|$ depending only on $R$, and $M$ such that $|C| \leq M$. Let $|\cdot|_E$ designate the appropriate Euclidean measure. We have:

**Theorem 5.** Let $M > 0$ be given. Also, let $\Sigma \subset B_{2R}(0) \subset \mathbb{R}^3$ with $\partial\Sigma \subset \partial B_{2R}(0)$ be a stable critical point of (0.1) with $H = \langle x, N \rangle/2 + C$ and $|C| \leq M$. There exists constants $D(M)$, $E(M)$, and $F(M)$ such that if $R > D(M)$ and $|\Sigma \cap B_{2R}|_E <
\[ E(M) R e^{R^2/16}, \] then we have that
\[ \sup_{x \in B_{R/\delta}(0)} |A|^2 \leq F(M) R^2 e^{-R^2/8}. \]

So, we see that hyperplanes begin the only complete stable critical points of (0.1) is actually the limiting case for a fixed bound on \( C \), an appropriate volume growth bound on \( \Sigma \cap B_R \), and sending \( R \to \infty \).

0.1. Conventions and notation. We will denote the second fundamental form of \( \Sigma \) by \( A(X,Y) = \langle \nabla_X N, Y \rangle \). We then use \( H = \text{div} N \) for the mean curvature of \( \Sigma \). With this convention, the cylinder \( S^k \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1} \) with radius \( R \) has mean curvature \( H = k/R \).

As we have already remarked, in section 3, we show that critical points \( \Sigma \) for (0.1) satisfy
\[ H = \frac{1}{2} \langle x, N \rangle + C \]
where \( x \) is the position vector, \( N \) is the normal of \( \Sigma \), and \( C \) is a constant. \( C \) will always be used to denote this constant in the curvature condition. When we need to make use of other constants, we will use different letters.

The Euclidean volume form will be denoted \( dV \) and the Euclidean area form on \( \Sigma \) will be denoted by \( dA \). We will slightly abuse notation and use \( d\mu \) to be both \( d\mu = e^{-|x|^2/4} dV \) and \( d\mu = e^{-|x|^2/4} dA \). Which one we are using should be clear from context. On occasion we may also use \( |S|_E \) to denote the appropriate Euclidean measure of \( S \).

We let \( A_{ij,k} \) denote \( \nabla_{e_k} A_{ij} \). The Codazzi equation then says that \( (A_{ij,k}) \) is fully symmetric. The Riemann curvature tensor will be denoted by \( R_{ijkl} = \langle \nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k, \partial_l \rangle \). Since our ambient manifold is \( \mathbb{R}^{n+1} \), the Gauss equation is given by
\[ R_{ijkl} = A_{il} A_{jk} - A_{ik} A_{jl} \]

0.2. Structure of this paper. The paper will be organized as follows:

In Sections 1-3, we develop the problem we are trying to solve. In section 1 we study the first variation for weighted area \( A_f \) and weighted volume \( V_f \) for a general weight \( e^f \). We will also discuss what it means for a hypersurface to be a \textit{critical point for variations preserving} \( V_f \).

In section 2 we compute the second variation for \( A_f \) and \( V_f \) for variations preserving \( V_f \), and we derive analytical conditions for \( \Sigma \) to be \textit{stable with respect to variations preserving} \( V_f \).

In section 3 we consider the explicit weights \( d\mu \). This includes explicit formulas for variations and the Jacobi operator.

In section 4 we prove that hyper-planes are stable solutions of (0.1). In section 5 we prove our theorem on how bounds on the index of non-planar \( \Sigma \) for critical points (0.1) give us splitting of \( \Sigma \).

In section 6 we prove the integral curvature estimate that will be necessary to establish our curvature decay estimates. In section ref:ptest, we show pointwise curvature estimates from an integral curvature estimate. This is combined with the estimate from section 6 to get our decay of \( |A| \) estimate.

The appendices are devoted to reformulating standard minimal surface inequalities to fit critical points (0.1). In section A we show a Simons’-type inequality.
for critical points \(0,1\). In section B we show a mean value type inequality for hypersurfaces \(\Sigma\) with bounded \(|H|\).

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1. The first variation for area and volume

We wish to study compact variations of hypersurfaces \(\Sigma\) embedded in \(\mathbb{R}^{n+1}\) subject to a weighted volume constraint. We follow a setup similar to the euclidean case of Barbosa and do Carmo [4]. Specifically: let \(f: \mathbb{R}^{n+1} \to \mathbb{R}\) be a scalar function. We define a volume form on \(\mathbb{R}^{n+1}\) by

\[
\omega_f \equiv e^f \bigwedge_i dx^i.
\]

Now, consider a variation \(F(s,p) : \mathbb{R} \times \Sigma \to \mathbb{R}^{n+1}\), with \(\Sigma_0 = \Sigma\). Using this variation, we define the **weighted area functional** as

\[
A_f(s) \equiv \int_{\Sigma_s} e^f dA.
\]

We also define the **weighted volume of the variation** to be

\[
V_f(s) \equiv \int_{[0,s] \times \Sigma} F^*\omega_f.
\]

Note that this is not really the weighted volume “enclosed” by \(\Sigma_s\). Rather, this describes the change in the weighted volume between \(\Sigma_0\) and \(\Sigma_s\). Thus, \(V_f(s)\) describes the change in the weighted volume enclosed by \(\Sigma_s\), which is enough for our purposes. For this reason (and by a slight abuse of notation), we say that a variation preserves \(V_f\) if \(V_f(s) = 0\) for all \(s\).

Now, we remark on the invariance of \(A_f(s)\) and \(V_f(s)\) on “tangential” reparameterizations \(G(s,x) = F(s,H_s(x))\) where \(H(s,x)\) is a smooth family of diffeomorphisms of \(\Sigma\). Clearly \(A_f(s)\) is invariant under such a transformation as \(H(s,\cdot)\) represents a change of coordinates, if we view \(\Sigma\) as giving (topological) coordinates for \(\Sigma(s)\).

For \(V_f(s)\), we first need to make some calculations. Without loss of generality we may restrict to coordinates \(y^i\) on \(\Sigma\) such that \(F_i = \partial F/\partial y_i\) satisfy \(N \wedge (\bigwedge_i F_i)\) is positively oriented. We calculate

\[
(F^*\omega_f)_{s,y_1,\ldots,y_n} = \omega_f(\partial s F, F_1, \ldots, F_n)
\]

\[
= \pm e^f |\partial_s F \wedge (\bigwedge_i F_i)|
\]

\[
= e^f (\partial_s F, N) |N \wedge (\bigwedge_i F_i)|
\]

\[
= e^f (\partial_s F, N) |\bigwedge_i F_i|,
\]

(1.4)
where the ± is depending only on the orientation of $\partial_s F \wedge \bigwedge_i F_i$.

Therefore, for a choice of variation $F_s(x)$ we have

\begin{equation}
V_f(s) = \int_0^s \left( \int_{\Sigma(t)} e^f (\partial_s F_t, N) dA(t) \right) dt.
\end{equation}

Now, given a compact variation $F(s, x)$, we may solve for a normal variation $G(s, x)$ such that $\partial_s G(s, x) = (\partial_s F(s, x), N)N$. We simply solve for $H(s, x)$ in the ODE $(DF_s)\partial_s H(s, x) + \partial_s F(s, H_s(x)) = (\partial_s F(s, x), N)N$, with initial condition $H(0, x) = x$. This can always be done for small enough $s$. Note that such a “tangential” reparameterization does not change the normal part of the variation.

Therefore, we often take our variations to be normal: $\partial_s F(s, x) = (\partial_s F(s, x))^N$.

Taking a derivative in (1.5), we have:

**Lemma 1.** For any compact variation $F(s, x)$, if $\langle \partial_s F(s), N \rangle = u$ then

\begin{equation}
V'_f(s) = \int_{\Sigma_s} e^f u \, dA.
\end{equation}

From Lemma 1 it is clear that a necessary condition for a variation to be volume preserving is that $u \equiv \langle \partial_s F, N \rangle(0)$ satisfy $\int_{\Sigma_s} e^f u \, dA = 0$. We also have that this is a sufficient condition for the existence of a volume preserving variation. Again, we follow an argument of Barbosa and do Carmo.

**Lemma 2.** Let $u$ be a $C^1_0$ function on $\Sigma$ such that $\int_{\Sigma_s} e^f u \, dA = 0$. Then, there exists a volume preserving variation $F$ of $\Sigma$ with $F'(0) = uN$.

**Proof.** Fix a function $\phi \in C^1_0(\Sigma)$ such that $\int_{\Sigma_s} e^f \phi \, dA \neq 0$. Consider the two parameter variation of the hypersurface given by, $F(s, t, p) = p + (us + \phi t)N$ along with the corresponding volume functional $V(s, t)$. We have that $\partial_t V(0, 0) = \int_{\Sigma_s} e^f \phi \, dA \neq 0$. By the Implicit Function Theorem, around $(0, 0)$ there is a function $t = \psi(s)$ such that $V(s, \psi(s)) = 0$ with $\psi(0) = 0$.

Note, that $\partial_s [V(s, \psi(s))](0) = \int_{\Sigma_s} e^f u \, dA + \psi'(0) \int_{\Sigma_s} e^f \phi \, dA$. Since $\int_{\Sigma_s} e^f \phi \, dA \neq 0$, we get that $\psi'(0) = 0$. Therefore, the volume preserving variation $G(s, p) = F(s, \psi(s), p)$ has $\partial_s G(0) = uN$, and we get the lemma.

We now compute the first variation of the weighted area functional.

**Lemma 3.** For any compact variation $F(s, x)$, if $\langle \partial_s F, N \rangle = u$, then

\begin{equation}
A'(s) = \int_{\Sigma_s} e^f u (H + \langle \nabla f, N \rangle) \, dA.
\end{equation}

**Proof.** We first consider a normal variation $F(s, x)$.

Note that $\partial_s \bigwedge_i F_i^2 = 2 \bigwedge_i F_i^2 (\partial_k F_s)^T \kappa$. Therefore,

\begin{align}
\partial_s \bigwedge_i F_i &\, = \bigwedge_i F_i |g^{kl}(\partial_k F_s, F_i) \\
&\, = \bigwedge_i F_i uH
\end{align}

\begin{equation}
(1.8)
\end{equation}
Equation (1.7) then follows from noting that \( \partial_s e^f = \langle \nabla e^f, F_s \rangle \).

Now, note that (1.7) depends only on \( \partial_s F(s) \) which is unchanged by a “tangential” reparameterization as described after (1.5). Since, \( V_f(s) \) and \( A_f(s) \) are also invariant under such a “tangential” reparameterization, we have that (1.7) holds for all variations \( F(s, x) \). □

By using pairs of approximations to the identity with opposite weights and centered at different points for \( u \), we find the curvature condition satisfied by critical points of \( A_f \) for variations preserving \( V_f \).

**Corollary 1.** If \( \Sigma \) is a hypersurface satisfying \( \delta A_f(u) = 0 \) for all \( u \in C^\infty_0(\Sigma) \) such that \( \delta V_f(u) = 0 \), then \( H + \langle \nabla f, N \rangle \) is constant on \( \Sigma \).

**2. The second variation**

Before computing the second variation, we compute the derivatives of \( N \) and \( H \) for normal variations.

**Lemma 4.** For any normal variation \( F(s, x) \), let \( u \equiv \langle \partial_s F, N \rangle \) at \( s = 0 \). Then \( \partial_s N(0) = -\nabla u \).

**Proof.** We first fix an orientation on \( \mathbb{R}^{n+1} \) such that \( N \wedge \bigwedge_i F_i \) is positive. Also, define \( * \) on \( \mathbb{R}^{n+1} \) such that \( *\omega \wedge \omega \) is positive with respect to this orientation. We then have that

\[
N = * \frac{1}{|\bigwedge_i F_i|} \bigwedge_i F_i.
\]

Now, using a computation from the proof of lemma 3, we have that

\[
\partial_s N = \frac{1}{|\bigwedge_i F_i|} \partial_s \left( * \frac{1}{|\bigwedge_i F_i|} \bigwedge_i F_i \right).
\]

This gives us

\[
\partial_s N = -uH N + \sum_L \* \left( \frac{(-1)^{L-1}}{|\bigwedge_i F_i|} \left[ \left( \partial_L F_s \right)^L F_L + \left( \partial_L F_s \right)^N N \right] \wedge \bigwedge_{i \neq L} F_i \right)
\]

\[
= -\left( uH - (\partial_s F_s)^i \right) N + \sum_L \frac{(-1)^{L-1} (\partial_L F_s)^N}{|\bigwedge_i F_i|} \* \left( N \wedge \bigwedge_{i \neq L} F_i \right).
\]

Note that \( F_L \wedge (-1)^{L-1} N \wedge \bigwedge_{i \neq L} F_i = -N \wedge \bigwedge_i F_i \).

Computing in coordinates at a point \( p \in \Sigma_0 \) with \( g_{ij} = \delta_{ij} \), we have that

\[
\partial_s N = -(uH - (\partial_s F_s)^i)N - g^{ij} \langle \partial_i F_s, N \rangle F_j.
\]

Observe that

\[
-g^{ij} \langle \partial_i F_s, N \rangle F_j = -g^{ij} \partial_i uF_j + g^{ij} \langle F_s, \partial_i N \rangle F_j
\]

\[
= -g^{ij} \partial_i uF_j,
\]

and

\[
uH - (\partial_s F_s)^i = 0.
\]
Combining what we have above, we get

\[(2.8) \quad \frac{\partial}{\partial s} N(0) = -\nabla u.\]
The last term becomes \( \int \Sigma e^f u^2 \text{Hess}_f (N, N) \). So, we have

\[
I = - \int \Sigma e^f u (\Delta u + |A|^2 u + \langle \nabla f, \nabla u \rangle - \text{Hess}_f (N, N) u - M(\nabla N f) u \) \, dA.
\]

Since \( V''(0) = 0 \), from lemma \[6\] we have that

\[
\int \Sigma e^f \langle F_s, N \rangle' (H + \langle \nabla f, N \rangle) \, dA = -M \int \Sigma e^f u^2 (\langle \nabla f, N \rangle + H) \, dA.
\]

Combining (2.16) and (2.17) gives us

\[
A''(0) = I - M \int \Sigma e^f u^2 (\langle \nabla f, N \rangle + H) \, dA + \int \Sigma e^f u^2 H \, dA
\]

\[
= - \int \Sigma e^f u (\Delta u + |A|^2 u - \text{Hess}_f (N, N) u + \langle \nabla f, \nabla u \rangle) \, dA.
\]

Note that this formula depends only on \( (\partial_s F(s))^N \). This is unchanged by “tangential” reparameterizations, as discussed after (1.5). Also, \( V_f \) and \( A_f \) are invariant under such reparameterizations. Therefore, the lemma follows.

\[\square\]

3. The Gaussian measure

Classically, \((n + 1)\)-dimensional Gaussian space is a probability space modeled as \( \mathbb{R}^{n+1} \) with measure given (up to normalization) by \( g_{ij} = e^{-x^2/(n+1)} \delta_{ij} \). For this conformal metric on \( \mathbb{R}^{n+1} \), the volume form is \( e^{-\frac{|x|^2}{4}} \, dV \). Notice that this is equivalent to setting \( f = -x^2/4 \) for the general weight of sections [1] and [2]. For this choice of \( f \), we will denote the weighted area and volume by \( A_\mu \) and \( V_\mu \) respectively. We will also denote the weighted area form by \( d\mu = e^{-\frac{|x|^2}{2}} \, dA \) and denote the weighted volume form by \( d\mu = e^{-\frac{|x|^2}{2}} \, dV \). Which we intend to use should be clear from context. Again, we remark that the weighted area form does not come from the conformal change of \( \mathbb{R}^{n+1} \).

One should also note that the weighted area form \( d\mu = e^{-\frac{|x|^2}{2}} \, dA \) is the same functional whose critical points were shown by Colding-Minicozzi [7] to be exactly the self-shrinkers of mean curvature flow.

The first variations for \( \mu \) are

\[
V_\mu' (s) = \int \Sigma u \, d\mu,
\]

\[
A_\mu' (s) = \int \Sigma u \left( H - \frac{1}{2} \langle x, N \rangle \right) \, d\mu.
\]
So, we see that being a critical point \( [0.1] \) for volume preserving transformations is equivalent to the quantity
\[
C \equiv H - \frac{1}{2} \langle x, N \rangle
\]
being constant. We now study some readily available examples of critical points to \( [0.1] \):

- For any plane in \( \mathbb{R}^{n+1} \) (not necessarily passing through the origin), we have that \( \langle x, N \rangle \) is constant. Therefore, we immediately see that planes are critical points.

- For any sphere \( |x - x_0| = R \), we have that \( N = \frac{x - x_0}{R} \) and \( H = \frac{n}{R} \). We see that \( \langle x, N \rangle = R + \frac{1}{2R} \langle x_0, x - x_0 \rangle \). Therefore, we see that the only spheres that are critical points are the spheres centered at the origin.

- For any cylinder \( \{(x, y) \in S^k \times \mathbb{R}^{n-k} : |x - x_0| = R \} \), we have that \( N = \frac{1}{R} \langle x - x_0 \rangle \) and \( H = k/R \). Hence, \( M = k/R - R/2 - (1/2R) \langle x_0, x - x_0 \rangle \). Therefore, the only cylinders that are critical points are those that are cylinders over spheres \( S^k \) in some \((k+1)\)-plane and centered at the origin.

- If \( M = 0 \), then the hypersurfaces in question are self-shrinking under the mean curvature flow (MCF). There are many examples of self-shrinkers, including Angenent’s self-shrinking torus \([2]\) and the noncompact examples of Kapouleas, Kleene, and Møller \([10]\) and Nguyen \([11]\).

When \( f = -|x|^2/4 \) and when \( \Sigma \) is a critical point of \( A \) for all variations that preserve \( V \) (ie \( \Sigma \) satisfies \( [3.3] \)), we get the following second variation formula:
\[
A''_\mu(0) = -\int_\Sigma u \left( \Delta u + |A|^2 u + \frac{1}{2} u - \frac{1}{2} \nabla_x T u \right) d\mu
\]
where \( u \equiv \langle F_s, N \rangle \) at \( s = 0 \). Given this, we define the operators
\[
\mathcal{L} \equiv \Delta - \frac{1}{2} \nabla_x T,
\]
\[
L \equiv \Delta + |A|^2 + \frac{1}{2} - \frac{1}{2} \nabla_x T.
\]
To \( L \) we associate the corresponding quadratic functional
\[
Q(u, u) = -\int_\Sigma u L u \ d\mu.
\]
Observe that the second variation formula becomes
\[
A''_\mu(0) = -\int_\Sigma u L u \ d\mu.
\]
We then say that \( \Sigma \) is volume preserving stable (or sometimes just stable) if \( Q(f, f) \geq 0 \) for every volume preserving normal variation \( f \).
4. Stability of the Hyperplanes

Theorem 1. Hyperplanes are volume preserving stable hypersurfaces.

Proof. We begin by observing that if a hyperplane does not pass through the origin, then without any loss in generality, it may be considered to be the plane \( x_{n+1} = c \). A change of variables \( x \to x - (0, 0, \ldots, c) \) shifts this plane to pass through the origin and changes the quadratic functional \( Q \) by the constant factor \( e^{-|c|^2/4} \). Therefore, it suffices to consider the stability of a hyperplane through the origin.

Following Kapouleas, Kleene, and Møller [10], we compare the operator \( L \) to the harmonic oscillator. We temporarily remove our restriction to volume preserving variations and consider all possible variations. Since \( A \equiv 0 \), the \( L \) operator reduces to

\[
Lu = \left( \Delta + \frac{1}{2} - \frac{1}{2} \nabla_x \right) u \\
= e^{\frac{|x|^2}{8}} \left( \Delta - \frac{|x|^2}{16} + \frac{n + 2}{4} \right) e^{-\frac{|x|^2}{8}} u. \tag{4.1}
\]

Here, the new operator

\[
H_x \equiv \left( \Delta - \frac{|x|^2}{16} + \frac{n + 2}{4} \right)
\]

is being applied to \( e^{-\frac{|x|^2}{8}} u \). \( H_x \) may be viewed as a shifted version of the harmonic oscillator

\[
\tilde{H} \equiv \Delta - |x|^2 \tag{4.3}
\]

Indeed, a change of variables \( x = 2y \) gives us the operator

\[
H_y \equiv (1/4) \Delta - \frac{|x|^2}{4} + \frac{n + 2}{4} \tag{4.4}
\]

and \( \tilde{H} = 4H_y - (n + 2) \).

The eigenvalues of \( \tilde{H} \) are well-known to be \( n + 2k \) for \( k = 0, 1, 2, \ldots \), and the eigenvectors are products of \( e^{-|y|^2/2} \) and Hermite polynomials. So the eigenvalues of \( H_y \) (which are equivalent to the eigenvalues of \( L \)) take the form \((k-1)/2\). Except for the first eigenvalue, these are all positive.

Observe that \( n \) is the lowest eigenvalue of \( \tilde{H} \) and has an eigenspace spanned by \( e^{-|y|^2/2} \). Undoing the change of variables, the lowest eigenvalue of \( L \) is \(-1/2\). Furthermore, the eigenspace of \( H_x \) is spanned by \( e^{-|x|^2/8} \), so the lowest eigenspace of \( L \) is spanned by the constant functions.

On the plane, the constant functions are not volume preserving, and so we do not want to consider them. Furthermore, by the orthogonality of the eigenspaces in the weighted \( L^2 \) space, we know that all eigenfunctions corresponding to higher eigenvalues will be volume preserving. So for our purposes, it suffices to consider the stability of \( L \) ignoring the first eigenvalue.

Observe that the second eigenvalue of \( H_y \) is equal to 0. The corresponding eigenfunctions, however, are simply the coordinate functions, and the corresponding variations are simply rotations of the hyperplane in space. These are clearly volume and area-preserving. All other eigenvalues are positive, and the stability of the hyperplanes follows. \( \square \)
5. Index of nonplanar hypersurfaces

First, we consider some important identities for our calculations. Colding-Minicozzi \[7\] show that for a self-shrinker \(\Sigma\), the functions \(\langle v, N \rangle\) for any \(v \in \mathbb{R}^{n+1}\) satisfy \(L \langle v, N \rangle = \frac{1}{2} \langle v, N \rangle\). We now show that they are also eigenfunctions for critical points (0.1).

**Lemma 8.** Let \(\Sigma \subset \mathbb{R}^{n+1}\) be a critical point for the variation problem (0.1) and \(v \in \mathbb{R}^{n+1}\) be a constant vector. Let \(u \equiv \langle v, N \rangle\). Then
\[
Lu = \frac{1}{2} u.
\]

**Proof.** Take a fixed point \(p\) and coordinates that are orthonormal at \(p\). Differentiating twice, applying the Codazzi equation, and then taking the trace yields
\[
\nabla e_i u = \langle v, \nabla e_i N \rangle = -A_{ij} \langle v, e_j \rangle,
\]
\[
\nabla e_k \nabla e_i u = -A_{ij,k} \langle v, e_j \rangle + A_{ij} \langle v, A_{jk} N \rangle
\]
\[
= -A_{ik,j} \langle v, e_j \rangle - A_{ij} \langle v, A_{jk} N \rangle,
\]
\[
\triangle u = \langle v, \nabla H \rangle - |A|^2 u.
\]

Differentiating (3.3) gives us
\[
\langle \nabla H, e_i \rangle = -\frac{1}{2} \langle x, A_{ij} e_j \rangle,
\]
and therefore
\[
\langle v, \nabla H \rangle = -\frac{1}{2} A_{ij} \langle x, e_j \rangle \langle v, e_i \rangle = \frac{1}{2} \langle x, \nabla u \rangle.
\]
Substituting this into (5.4) gives us
\[
Lu = \triangle u + |A|^2 u - \frac{1}{2} \langle x, \nabla u \rangle + \frac{1}{2} u = \frac{1}{2} u.
\]

Next, we consider some important identities. The first, (5.7), gives us some control on \(Q(\phi f, \phi f)\) if we have some control over \(fLf\). The second identity, (5.8), comes from the fact that, formally, if \(Lf = (1/2) f\), then \(\int |A|^2 f d\mu = 0\). The identity (5.8) measures the effect of being more rigorous by using a cut-off function. The last identity, (5.9), is a differential identity that will later be useful for our integral curvature estimate.

**Lemma 9.** For any functions \(\phi \in C_0^\infty(\Sigma)\) and \(f \in C^\infty(\Sigma)\) we have that
\[
\int_\Sigma \phi f L(\phi f) d\mu = \int_\Sigma \phi^2 f Lf d\mu - \int_\Sigma |\nabla \phi|^2 f^2 d\mu.
\]

Also, for any \(\Sigma\) satisfying \(H = \langle x, N \rangle/2 + C\) and constant vector \(v \in \mathbb{R}^{n+1}\) we have that
\[
\int_\Sigma \phi^2 |A|^2 \langle v, N \rangle d\mu = 2 \int_\Sigma \phi A(\nabla \phi, v^T) d\mu,
\]
and
\[
\mathcal{L}|x|^2 = 2n - |x|^2 - 2C \langle x, N \rangle
\]
Proof.
\[
\int_{\Sigma} \phi L(\phi f) \, d\mu = \int_{\Sigma} (f^2 \phi L \phi + \frac{1}{2} (\nabla \phi^2, \nabla f^2) + \phi^2 f L f) \, d\mu,
\]
\[
= \int_{\Sigma} (f^2 \phi L \phi - f^2 \phi L \phi - f^2 |\nabla \phi|^2 + \phi^2 f L f) \, d\mu,
\]
\[
(5.10)
\]
This shows (5.7).

To prove (5.8), consider a critical point (0.1) \( \Sigma \). Note, we have that
\[
\frac{1}{2} \int_{\Sigma} \phi (v, N) \, d\mu = \int_{\Sigma} \phi L(v, N) \, d\mu,
\]
\[
= \int_{\Sigma} (\frac{1}{2} + |A|^2) \phi (v, N) \, d\mu - \int_{\Sigma} \langle \nabla \phi, \nabla (v, N) \rangle \, d\mu.
\]
\[
(5.11)
\]
Therefore, we get that
\[
(5.12)
\]
\[
\int_{\Sigma} |A|^2 \phi (v, N) \, d\mu = \int_{\Sigma} A(\nabla \phi, \nabla^T v) \, d\mu.
\]

Substituting \( \phi \to \phi^2 \), we get (5.8).

For the final equation of the lemma, note that
\[
\frac{\partial}{\partial x^i} |x|^2 = 2n - 2\langle x, N \rangle H. \quad \text{Since } \Sigma \text{ satisfies } H = \langle x, N \rangle / 2 + C, \text{ we get } \frac{\partial}{\partial x^i} |x|^2 = 2n - 2\langle x, N \rangle C - |x^N|^2. \quad \text{This and (5.9) give us } L|x|^2 = 2n - |x|^2 - 2\langle x, N \rangle C. \quad \Box
\]

Lemma 10. For critical points \( \Sigma \) to the variational problem (0.1), there exists \( \phi \in C^\infty_0(\Sigma) \) such that \( Q \) is negative definite on \( \phi V \) and \( \text{Dim}(\phi V) = \text{Dim}V \), where
\[
V \equiv \text{Span}\{1, \langle v, N \rangle | v \in \mathbb{R}^{n+1} \}.
\]

Remark 1. The vector space \( V \) consists of all functions that are spanned by the constant function and those functions that correspond to translating the hypersurface.

Proof. Consider \( u \equiv c_0 + \langle v, N \rangle \) with \( |c_0|^2 + |v|^2 = 1 \), and consider \( Q(u \phi, u \phi) \). From (5.7) we have that
\[
(5.13)
\]
\[
Q(u \phi, u \phi) = - \int_{\Sigma} \phi^2 u L u \, d\mu + \int_{\Sigma} |\nabla \phi|^2 u^2 \, d\mu,
\]
\[
(5.14)
\]
\[
= - \int_{\Sigma} \phi^2 u c_0 (1/2 + |A|^2) \, d\mu - \int_{\Sigma} \phi^2 u (1/2) \langle v, N \rangle \, d\mu + \int_{\Sigma} |\nabla \phi|^2 u^2 \, d\mu.
\]
Using that \( u = c_0 + \langle v, N \rangle \), we get
\[
(5.15)
\]
\[
Q(u \phi, u \phi) = -(1/2) \int_{\Sigma} \phi^2 u^2 \, d\mu - \int_{\Sigma} \phi^2 |A|^2 c_0^2 \, d\mu - \int_{\Sigma} \phi^2 |A|^2 c_0 \langle v, N \rangle \, d\mu + \int_{\Sigma} |\nabla \phi|^2 u^2 \, d\mu.
\]
Now,
\[
\left| \int_{\Sigma} \varphi^2 |A|^2 c_0 \langle v, N \rangle \, d\mu \right| = 2 \left| \int_{\Sigma} \phi A(\nabla \phi, v^T) c_0 \, d\mu \right|,
\]
(5.16)
\[
\leq \int_{\Sigma} \varphi^2 |A|^2 c_0^2 \, d\mu + \int_{\Sigma} |\nabla \phi|^2 \, d\mu.
\]
Therefore,
\[
Q(\phi u, \phi u) \leq -(1/2) \int_{\Sigma} \varphi^2 u^2 \, d\mu + \int_{\Sigma} |\nabla \phi|^2 (u^2 + 1) \, d\mu.
\]
(5.17)

Now for \( R > 0 \) large, define the cut-off function \( \phi_R \) such that
\[
\phi_R(r) = \begin{cases} 
1 & r \leq R \\
1 - (1/R)(r - R) & R \leq r \leq 2R \\
0 & r \geq 2R.
\end{cases}
\]
(5.18)
and observe that \( |\nabla \phi_R| \leq 1/R \).

Now, note that the dimension of \( V \) is not necessarily \( n + 1 \). Let \( \{c_i + \langle v_i, N \rangle \} \) be a basis for \( V \) with \( |c_i|^2 + |v_i|^2 = 1 \). Define \( S \equiv \{d^i(c_i + \langle v_i, N \rangle) : \sum d_i^2 = 1\} \).

Since \( \text{Dim}V < \infty \), we have that \( S \) is compact. This implies there exists \( R_0 \) such that for all \( u \in S \) we have that \( B_{R_0} \cap \{u \neq 0\} \neq \emptyset \); if not, there would exist a sequence of \( u_j = d^i_j(c_i + \langle v_i, N \rangle) \in S \) such that \( u_j \equiv 0 \) on \( B_j \). By passing to a subsequence and taking a limit \( d^i_j \to d^i \), this implies that \( d^i_\infty(c_i + \langle v_i, N \rangle) \equiv 0 \in S \) which is a contradiction. Therefore, we have that for \( R \geq R_0 \) and all \( u \in S \) that
\[
\int_{\Sigma} \varphi_R^2 u^2 \, d\mu \geq M_R > 0.
\]
(5.19)

Note that \( M_R \) is increasing in \( R \), and that \( \text{Dim}(\phi_R V) = \text{Dim}V \).

Now, since \( V \) is a finite dimensional subspace of \( L^2_\mu \), we have that the integral \( \int_{\Sigma} |\nabla \phi_R|^2 (u^2 + 1) \, d\mu \to 0 \) uniformly in \( u \in V \) as \( R \to 0 \). By (5.17), we may choose an \( R > R_0 \) such that
\[
Q(\phi_R u, \phi_R u) \leq -M_{R_0}/4 < 0
\]
for \( u \in S \). Hence, \( Q \) is negative definite on \( \phi_R V \).

\textbf{Theorem 2.} \textit{If} \( \Sigma \) \textit{is a complete non-planar critical point of} \( 0 \) \textit{with Index} \( I \) \textit{for functions} \( \delta V_\mu(u) = 0 \), \textit{then}
\[
\Sigma = \Sigma_0 \times \mathbb{R}^{n+1-i}
\]
(5.21)
\textit{where} \( i \leq I \).

\textbf{Proof.} Let \( V \equiv \text{Span}\{1, \langle v, N \rangle\}_{v \in \mathbb{R}^{n+1}} \). First, we comment on \( \text{Dim}V \). Consider the case that the constant function \( c_0 \in \text{Span}\{\langle v, N \rangle\}_{v \in \mathbb{R}^{n+1}} \). We have that \( Lc_0 = (1/2)c_0 \), but \( c_0 \) is also constant, so \( Lc_0 = (1/2 + |A|^2)c_0 \). Therefore, \( |A|^2c_0 \equiv 0 \), and since \( \Sigma \) is non-planar, we have that \( c_0 = 0 \). Hence,
\[
\text{Dim}V = 1 + \text{DimSpan}\{\langle v, N \rangle\}_{v \in \mathbb{R}^{n+1}}.
\]
(5.22)
By Lemma 10, we have for some $\phi \in C_0^\infty(\Sigma)$ that $\dim \phi V = \dim V$ and that $Q$ is negative definite on $\phi V$. Now, note that by counting dimensions, we have $\dim(\phi V \cap 1^\perp) \geq \dim \text{Span}\{\langle v, N \rangle \}_{v \in \mathbb{R}^{n+1}}$. Hence, $\dim \text{Span}\{\langle v, N \rangle \}_{v \in \mathbb{R}^{n+1}} \leq I$. Therefore,

$$n + 1 - i \equiv \dim \{v : \langle v, N \rangle \equiv 0\} \geq n + 1 - I.$$ 

Finally, note that

$$\Sigma = \Sigma_0 \times \{v : \langle v, N \rangle \equiv 0\}. \quad \square$$

**Corollary 2.** The only complete $L$ stable critical point of the variation problem (0.1) is the hyperplane. There are no critical points of index one.

6. AN INTEGRAL CURVATURE ESTIMATE

Since, the functions $\langle v, N \rangle$ for $v \in \mathbb{R}^{n+1}$ play a key role in the proof of theorem 2, it is not surprising that they play a key role in creating an integral estimate for the non-complete case. We will use these functions with appropriate cut-off functions to prove our estimate.

First, we need some notation. For two sided $\Sigma$ and any $\phi \in C_0^\infty(\Sigma)$ such that $\phi \geq 0$ and $|\{\phi \neq 0\}| > 0$, let

$$N_\phi \equiv \frac{\int \phi N \, d\mu}{\int \phi \, d\mu}.$$ 

We find a version of the stability inequality that is valid for any $\phi \in C_0^\infty(\Sigma)$ such that $|\{\phi \neq 0\}| > 0$. We don’t require $\int_\Sigma \phi \, d\mu = 0$.

**Lemma 11.** Let $\Sigma \subset \mathbb{R}^{n+1}$ be a stable critical point of (0.1), such that $H = \langle x, N \rangle/2 + C$. Let $\phi \in C_0^\infty(\Sigma)$ such that $|\{\phi \neq 0\}| > 0$. Then, we have

$$\int_\Sigma \phi^2|N - N_\phi|^2 \, d\mu + |N_\phi|^2 \int_\Sigma \phi^2|A|^2 \, d\mu \leq B(n) \int_\Sigma |\nabla \phi|^2 \, d\mu.$$ 

**Proof.** Let $v \in \mathbb{R}^{n+1}$ such that $|v| = 1$, and note that $\int_\Sigma \phi \langle v, N - N_\phi \rangle \, d\mu = 0$. Therefore, we may plug $\phi \langle v, N - N_\phi \rangle$ into the stability inequality. Using (5.7), we have

$$\int_\Sigma \phi^2 \langle v, N - N_\phi \rangle L(v, N - N_\phi) \, d\mu \leq \int_\Sigma |\nabla \phi|^2 \langle v, N - N_\phi \rangle^2 \, d\mu,$$

$$\leq 4 \int_\Sigma |\nabla \phi|^2 \, d\mu.$$ 

Note that $L(\langle v, N \rangle) = \frac{1}{2} \langle v, N \rangle$, and $L(\langle v, N_\phi \rangle) = (\frac{1}{2} + |A|^2)\langle v, N_\phi \rangle$. Therefore, we get

$$\int_\Sigma \phi^2 \langle v, N - N_\phi \rangle^2 \, d\mu - \int_\Sigma \phi^2 |A|^2 \langle v, N_\phi \rangle \langle v, N - N_\phi \rangle \, d\mu \leq \int_\Sigma 4|\nabla \phi|^2 \, d\mu.$$
Applying a Cauchy inequality to (5.8), we get

\[ \langle v, N_\phi \rangle \int_{\Sigma} |A|^2 |\phi^2(v, N)| \, d\mu \leq \frac{1}{2} \int_{\Sigma} \phi^2(v, N_\phi)^2 |A|^2 \, d\mu + 2 \int_{\Sigma} |\nabla \phi|^2 |v|^2 \, d\mu. \]

Combining (6.4) and (6.5) gives us

\[ \int_{\Sigma} \phi^2(v, N - N_\phi)^2 \, d\mu + \langle v, N_\phi \rangle^2 \int_{\Sigma} \phi^2 |A|^2 \, d\mu \leq 12 \int_{\Sigma} |\nabla \phi|^2 \, d\mu. \]

We sum over a constant orthonormal frame for \( \mathbb{R}^{n+1} \) to prove the lemma.

Now, we use this modified stability inequality and a volume area bound [3] to obtain an integral estimate for \( |A|^2 \). In the following lemma, \( |\cdot|_E \) represents measure taken with respect to the usual Euclidean measure.

**Theorem 3.** Let \( M > 0 \) and \( n \) be given. Also, let \( \Sigma \subset B_{2R}(0) \subset \mathbb{R}^{n+1} \) with \( \partial \Sigma \subset \partial B_{2R}(0) \) be a stable critical point of (6.1) with \( H = \langle x, N \rangle / 2 + C \) and \( |C| \leq M \). There exists constants \( D(M, n), E(M, n), \) and \( F(M, n) \) such that if \( R > D(M, n) \) and \( |\Sigma \cap B_{2R}|_E < E(M, n)R^2e^{R^2/4} \), then we have that

\[ \int_{B_R} |A|^2 \, d\mu \leq F(M, n)|\Sigma \cap B_{2R}|_E^{-2}e^{-R^2/4}. \]

**Proof.** We first construct a cut off function depending only on the Euclidean \(|x|\) such that

\[ \phi(r) = \begin{cases} 
1 & r \leq R \\
\text{linear} & R \leq r \leq 2R \\
0 & 2R \leq r 
\end{cases} \]

Our modified stability inequality lemma gives us

\[ \int_{\Sigma} \phi^2 \, d\mu - 2 \int_{\Sigma} \phi^2 |N_\phi| \, d\mu + \int_{\Sigma} \phi^2(|A|^2 + 1) \, d\mu |N_\phi|^2 \leq B_n \frac{|\Sigma \cap B_{2R}|_E}{R^2} e^{-R^2/4}. \]

Note that the left hand side of this inequality is quadratic in \( |N_\phi| \), and since any quadratic with \( a > 0 \) satisfies \( au^2 + bu + c \geq c - \frac{b^2}{4a} \), we get that

\[ \int_{\Sigma} \phi^2 \, d\mu - \left( \frac{\int_{\Sigma} \phi^2 \, d\mu}{\int_{\Sigma} \phi^2(|A|^2 + 1) \, d\mu} \right)^2 \leq B_n |\Sigma \cap B_{2R}|_E R^{-2}e^{-R^2/4}. \]

So, we have

\[ \frac{\int_{\Sigma} \phi^2 \, d\mu}{} \int_{\Sigma} \phi^2 |A|^2 \, d\mu \leq B_n |\Sigma \cap B_{2R}|_E R^{-2}e^{-R^2/4}, \]

and so

\[ \left( \int_{\Sigma} \phi^2 \, d\mu - B_n |\Sigma \cap B_{2R}|_E R^{-2}e^{-R^2/4} \right) \int_{\Sigma} \phi^2 |A|^2 \, d\mu \leq B_n |\Sigma \cap B_{2R}|_E^2 R^{-2}e^{-R^2/4}. \]
Now, what we need is a lower bound on $\int_{\Sigma} \phi^2 d\mu$. In order to accomplish this, we look at getting some control over $\min |x|$ and the Euclidean mean curvature $H$ around some point realizing $\min |x|$. Let $\Sigma$ achieve $\min |x|$ at the point $p \in \Sigma$. So at $p$, we have from (5.9) that

$$2n - |x|^2 + 2M|x| \geq 2n - |x|^2 - 2C(x, N) = L|x|^2 \geq 0 \quad (6.13)$$

So, we have that there exists a constant $D(M, n)$ such that $|x| \leq D(M, n)$. Since $H = \langle x, N \rangle/2 + C$, we have that $|H| \leq D(M, n)$ on $B_{2D(M, n)}$. Therefore, by the lower bound on area, we get that $\int \phi^2 d\mu \geq D(M, n)$.

So for some choice of $E(M, n)$, if $|\Sigma \cap B_{2R}| < E(M, n)R^2e^{2R^2/4}$, then we get

$$F(M, n) \int_{\Sigma} \phi^2 |A|^2 d\mu \leq \left( \int_{\Sigma} \phi^2 d\mu - B_n |\Sigma \cap B_{2R}| e^{-2R^2/4} \right) \int_{\Sigma} \phi^2 |A|^2 d\mu. \quad (6.14)$$

So, for some constant $F(M, n)$, we have that $|\Sigma \cap B_{2R}| < E(M, n)R^2e^{2R^2/4}$,

$$\int_{\Sigma} \phi^2 |A|^2 d\mu \leq F(M, n) |\Sigma \cap B_{2R}| e^{-2R^2/4}. \quad (6.15)$$

Hence, we get the lemma. \hfill \Box

7. A POINTWISE CURVATURE ESTIMATE

To achieve a pointwise curvature estimate from an integral estimate, we will need to make use of two inequalities: a Simons-type inequality and a mean value inequality. These inequalities have well-known analogues in the theory of minimal surfaces, and we adjust these proofs to fit our needs. The statements of these inequalities are presented here, but their proofs are left for the appendices.

**Lemma 12. Simons Inequality:** For a critical point $(0, 1) \Sigma^n \subset \mathbb{R}^{n+1}$, we have

$$\nabla |A|^2 \geq -((|x|^2/4)|A|^2 - (2 + C^2)|A|^4). \quad (7.1)$$

**Lemma 13. Mean Value Inequality:** Suppose that, on a hypersurface with $|H| \leq M$, a function $f$ satisfies $f \geq 0$ and $\Delta f \geq -\lambda t^{-2}f$ for some $\lambda$ on $B_t(x) \subset B_R(0)$. Then, for $s \leq t$ and $2k = \lambda t/2M$,

$$e^{kx} - s^n \int_{B_s(x)\cap \Sigma} f \geq f(x). \quad (7.2)$$

Following an argument of Choi-Schoen [6], we get the following pointwise curvature estimate:

**Theorem 4.** There exists $\varepsilon_M > 0$ such that the following holds: Suppose $\Sigma \subset \mathbb{R}^3$ is any hypersurface satisfying $H = \langle x, N \rangle/2 + C$ with $|C| \leq M$, and suppose $x_0 \in \Sigma$. Also, suppose that for some $R \geq 1$ and some $C \leq M$ and suppose $x_0 \in \Sigma$. Also, suppose that for some $R \geq 1$ and some $r_0 < 1/R$, we have $B_{r_0}(x_0) \subset B_R(0)$ and $\partial \Sigma \subset \partial B_R(0)$. Finally, suppose that

$$\int_{B_{r_0}(x_0)} |A|^2 < \delta. \quad (7.3)$$

Then for all $0 < \sigma \leq r_0$ and $y \in B_{r_0-\sigma}(x_0)$, $|A|^2(y) \leq \delta/\sigma^2$. 

Proof. On $B_{r_0}(x_0)$, define the function
\[ F(y) = (r_0 - d(y, x_0))^2 |A|^2 \] (7.4)
where $d(y, x_0)$ is the Euclidean distance between the two points. Observe that $F \geq 0$ in $B_{r_0}$, and $F = 0$ on $\partial B_{r_0}$. Set $x_1$ to be the point where $F$ achieves its maximum. Observe that if $F(x_1) \leq \delta$ we will be done, since for $y \in B_{r_0 - \sigma}$, $\sigma^2 |A|^2 \leq F(x_1) \leq \delta$. We will now show that $F(x_1) > \delta$ gives a contradiction for some $\epsilon_M$ small enough and independent of $\Sigma, \delta$, and $R \geq 1$.

Suppose that $F(x_1) > \delta$, ie
\[ (r_0 - d(x_1, x_0))^2 |A|^2 > \delta, \] (7.5)
and fix $\sigma$ so that $\sigma^2 |A|^2 = \delta/4$. Observe that the following equations hold:
\[ \sigma \leq \frac{1}{2} (r_0 - d(x_1, x_0)) \leq \frac{1}{2R} < 1, \] (7.6)
\[ \frac{1}{2} \leq \frac{r_0 - d(y, x_0)}{r_0 - d(x_1, x_0)} \leq 2, \forall y \in B_\sigma(x_1). \] (7.7)

Using these, we compute
\[ (r_0 - d(x_1, x_0))^2 \sup_{B_\sigma(x_1)} |A|^2 \leq 4 \sup_{B_\sigma(x_1)} (r_0 - d(\cdot, x_0))^2 |A|^2 \]
\[ = 4 \sup_{B_\sigma(x_1)} F(\cdot) \leq 4F(x_1) \]
\[ = 4(r_0 - d(x_1, x_0))^2 |A|^2(x_1). \]

Therefore,
\[ \sup_{B_\sigma(x_1)} |A|^2 \leq 4|A|^2(x_1) = \frac{\delta}{\sigma^2} < \frac{1}{\sigma^2}. \] (7.8)

Plugging (7.8) into Simons Inequality (7.1) gives us
\[ \Delta|A|^2 \geq -(R^2/8)|A|^2 - (2 + C^2)/\sigma^2 |A|^2, \] (7.9)
and by using (7.6) (specifically, that $R \leq 1/\sigma$) yields
\[ \Delta|A|^2 \geq -\delta^{-2}(3 + M^2)|A|^2. \] (7.10)

Therefore,
\[ \Delta|A|^2 \geq -\sigma^{-2}\lambda|A|^2 \] (7.11)
on $B_\sigma(x_1)$, where $\lambda = \lambda(M, n)$. Then, by the Mean Value Inequality (7.2),
\[ |A|^2(x_1) \leq e^{R\sigma + M\sigma + (3 + C^2)/2\sigma^{-2}} \int_{B_\sigma(x_1) \cap \Sigma} |A|^2 \]
\[ \leq e^{M^2 + M^2/2\sigma^{-2}} \int_{B_\sigma(x_1) \cap \Sigma} |A|^2. \] (7.12)

Substituting back in our definition of $\sigma$, we get
\begin{align}
\delta/4 &= \sigma^2 |A|^2(x_1) \leq e^{M+2+M^2/2} \int_{B_\delta(x_1) \cap \Sigma} |A|^2 \\
&\leq e^{M+2+M^2/2} \int_{B_{\delta \epsilon}(x_1) \cap \Sigma} |A|^2 \\
&\leq e^{M+2+M^2/2} \delta \epsilon. \quad (7.16)
\end{align}

We see that we may choose \( \epsilon \) depending only on \( M \) such that there is a contradiction. \( \square \)

Using theorem 7.3 combined with theorem 6.7 we get a pointwise estimate for stable critical points of (0.1).

**Theorem 5.** Let \( M > 0 \) be given. Also, let \( \Sigma \subset B_{2R}(0) \subset \mathbb{R}^3 \) with \( \partial \Sigma \subset \partial B_{2R}(0) \) be a stable critical point of (0.1) with \( H = \langle x, N \rangle^2 + C \) and \( |C| \leq M \). There exists constants \( D(M), E(M), \) and \( F(M) \) such that if \( R > D(M) \) and \( |\Sigma \cap B_{2R}| \leq E(M)R e^{R^2/16} \), then we have that

\begin{equation}
\sup_{x \in B_{R/2}(0)} |A|^2 \leq F(M)R^2 e^{-R^2/8}. \quad (7.17)
\end{equation}

**Appendix A. Simons-type inequality**

Here we prove Simons’ inequality, which is lemma \( \text{(7.1)} \) or \( \text{(7.11)} \).

**Proof.** First, note that from Codazzi’s equation we have that \( \nabla A \) is symmetric. We fix a point \( p \in \Sigma \) and look at geodesic normal coordinates centered at \( p \). Therefore, at \( p \) we have that

\begin{equation}
\triangle A_{jk} = \nabla^2_{ik} A_{jk} = \nabla^2_{ij} A_{ik} = \nabla^2_{ji} A_{ik} + R_{ij} A_{ik} \\
= \nabla^2_{jk} A_{ik} - R_{jji} A_{ik} - R_{ijk} A_{il} \\
= \nabla^2_{jk} H - R_{jji} A_{ik} - R_{ijk} A_{il} \\
= \nabla^2_{jk} H + (-A_{ij} A^l_i + A_{ik} A^l_j) A_{ik} + (-A_{jk} A^l_i + A_{ik} A^l_j) A_{il} \\
= \nabla^2_{jk} H + H A^2_{jk} - |A|^2 A_{jk}. \quad (A.1)
\end{equation}

Now, from \( H = \langle x, N \rangle + C \) we have that

\begin{equation}
\nabla_k H = (1/2) A(k, x^T), \quad (A.2)
\end{equation}

and

\begin{equation}
\nabla^2_{jk} H = (1/2) \nabla_j A(k, x^T) + (1/2) A_{jk} - (1/2) \langle x, N \rangle A^2_{jk}. \quad (A.3)
\end{equation}

From this we get that

\begin{equation}
\mathcal{L} |A|^2 = |A|^2 + 2 \text{Tr} A^3 - 2 |A|^4 + 2 |\nabla A|^2. \quad (A.4)
\end{equation}
Therefore,
\[ \Delta |A|^2 = |A|^2 + 2C \text{Tr} A^3 - 2|A|^4 + 2|\nabla A|^2 + \langle A, \nabla x^T A \rangle \]
\[ \geq |A|^2 - 2C|A|^3 - 2|A|^4 + 2|\nabla A|^2 - |x||A||\nabla A| \]
\[ \geq (1 - |x|^2/8)|A|^2 - 2C|A|^3 - 2|A|^4 \]
(A.5)
\[ \geq -(|x|^2/8)|A|^2 - (2 + C^2)|A|^4. \]

\[ \square \]

**Appendix B. Mean value inequality**

To prove lemma 13 and (7.2), assume $|H| \leq M$. Recall that $\Delta |x|^2 = 2n - 2\langle x, N \rangle H$. This is true for general hypersurfaces in Euclidean space, and thus we can translate our surface in space so that we are looking inside $B_s(0)$. Then

(B.1)
\[ 2n \int_{B_s \cap \Sigma} f \, dA = \int_{B_s \cap \Sigma} f \Delta |x|^2 \, dA + 2 \int_{B_s \cap \Sigma} f \langle x, N \rangle H \, dA \]
\[ = \int_{B_s \cap \Sigma} |x|^2 \Delta f \, dA + 2 \int_{\partial B_s \cap \Sigma} f |x^T| \, dA - s^2 \int_{B_s \cap \Sigma} \Delta f \, dA + 2 \int_{\partial B_s \cap \Sigma} \langle x, N \rangle H f \, dA. \]

Let $g(s) = s^{-n} \int_{B_s \cap \Sigma} f \, dA$. From the coarea formula and (B.1), we get

(B.2)
\[ g'(s) = -ns^{-n-1} \int_{B_s \cap \Sigma} f \, dA + s^{-n} \int_{\partial B_s \cap \Sigma} f |x| |x^T| \, dA \]
\[ = -s^{-n-1} \left( \frac{1}{2} \int_{B_s \cap \Sigma} (|x|^2 - s^2) \Delta f \, dA + \int_{\partial B_s \cap \Sigma} f |x^T| \, dA + \int_{B_s \cap \Sigma} \langle x, N \rangle f H \, dA \right) \]
\[ + s^{-n} \int_{\partial B_s \cap \Sigma} f |x| |x^T| \, dA \]
\[ \geq \frac{1}{2} s^{-n+1} \int_{B_s \cap \Sigma} \Delta f \, dA - s^{-n-1} \int_{B_s \cap \Sigma} \langle x, N \rangle f H \, dA. \]

Here the final inequality assumes the positivity of $f$. Additionally, if we assume $\Delta f \geq -\lambda t^{-2} f$ on $B_t$, our bound on $|H|$ gives us

(B.3)
\[ g'(s) \geq -\frac{\lambda}{2} s^{1-n} \int_{B_s \cap \Sigma} f t^{-2} \, dA - Ms^{-1-n} \int_{B_s \cap \Sigma} sf \, dA \]
\[ \geq \left( -\frac{\lambda}{2t} + M \right) g(s) \]
for all $s \leq t$. Therefore,

(B.4)
\[ \frac{d}{ds} \left( g(s)e^{(\frac{1}{2} + M)s} \right) \geq 0. \]
Integrating (B.4) from $s_0$ to $s_1$ (both assumed to be less than $t$) and letting $s_0 \searrow 0$, we get

\[ e^{ks_1} s_1^{-n} \int_{B_{s_1} \cap \Sigma} f \geq f(p), \]

where $k = \lambda/2t + M$. This completes the proof. □

Note, that we get the following corollary (monotonicity):

**Corollary 3.** Let $p \in \Sigma$, and let $|H| \leq M$ in $B_t(p) \cap \Sigma$. Then for $s \leq t$, we have $|B_s \cap \Sigma| \geq \omega_n e^{-Ms} s^n$, where $\omega_n$ is the volume of the standard unit ball in $\mathbb{R}^n$.

**Proof:** Using the above work with $f \equiv 1$, (B.4) gives us

\[ e^{ks_1} s_1^{-n} \int_{B_{s_1} \cap \Sigma} dA \geq e^{ks_0} s_0^{-n} \int_{B_{s_0} \cap \Sigma} dA, \]

and by letting $s_0 \searrow 0$ (since $p$ is a point on $\Sigma$), the corollary follows. □

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