Pseudo-Schwarzschild Spherical Accretion as a Classical Black Hole Analogue

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Abstract. We demonstrate that a spherical accretion onto astrophysical black holes, under the influence of Newtonian or various post-Newtonian pseudo-Schwarzschild gravitational potentials, may constitute a concrete example of classical analogue gravity naturally found in the Universe. We analytically calculate the corresponding analogue Hawking temperature as a function of the minimum number of physical parameters governing the accretion flow. We study both the polytropic and the isothermal accretion. We show that unlike in a general relativistic spherical accretion, analogue white hole solutions can never be obtained in such post-Newtonian systems. We also show that an isothermal spherical accretion is a remarkably simple example in which the only one information—the temperature of the fluid, is sufficient to completely describe an analogue gravity system. For both types of accretion, the analogue Hawking temperature may become higher than the usual Hawking temperature. However, the analogue Hawking temperature for accreting astrophysical black holes is considerably lower compared with the temperature of the accreting fluid.

1 Introduction

1.1 Analogue gravity and analogue Hawking temperature

Classical black-hole analogues are dynamical fluid analogues of general relativistic black holes. Such analogue gravity effects may be observed when acoustic perturbations propagate through a classical transonic fluid. At the transonic point of the fluid flow the so-called acoustic horizon is formed. The acoustic horizon resembles a black-hole event horizon in many ways. In particular, the acoustic horizon emits a quasi-thermal phonon spectrum similar to Hawking radiation.

In his pioneering work Unruh (1981) showed that the scalar field representing acoustic perturbations in a transonic barotropic irrotational fluid, satisfied a differential equation of the same form as the Klein-Gordon equation for the massless scalar field propagating in curved space-time with a
metric that closely resembles the Schwarzschild metric near the horizon. The acoustic propagation through a transonic fluid forms an analogue event horizon located at the transonic point. Acoustic waves with a quasi-thermal spectrum will be emitted from the acoustic horizon and the temperature of such acoustic emission may be calculated as (Unruh 1981)

\[ T_{ah} = \frac{\hbar}{4\pi k_B} \left[ \frac{1}{a_s} \frac{\partial^2}{\partial n^2} \right]_{ah} , \]

where \( k_B \) is Boltzmann’s constant, \( a_s \) the speed of sound, \( u_\perp \) the component of the flow velocity normal to the acoustic horizon and \( \partial/\partial n \) represents the normal derivative. The subscript ‘ah’ denotes that the quantity should be evaluated at the acoustic horizon. The temperature \( T_{ah} \) defined by (1) is an acoustic analogue of the usual Hawking temperature \( T_H \):

\[ T_H = \frac{hc^3}{8\pi k_B GM_{bh}} \]

and hence \( T_{ah} \) is referred to as the analogue Hawking temperature or simply analogue temperature. Similarly, the radiation from the acoustic (analogue) black hole is dubbed analogue Hawking radiation. Note that the sound speed in equation (1) in Unruh’s original treatment was assumed constant in space.

Unruh’s work was followed by other important papers (Jacobson 1991, Unruh 1995, Visser 1998, Jacobson 1999, Bilić 1999). As shown by Visser (1998), the equation of motion for the acoustic disturbance in a barotropic, inviscid fluid is identical to the d’Alembertian equation of motion for a minimally coupled massless scalar field propagating in (3+1) Lorentzian geometry. In particular, the acoustic metric for a point sink was shown to be conformally related to the Painlevé-Gullstrand-Lemaître form of the Schwarzschild metric. The corresponding expression for the analogue temperature was shown to be (Visser 1998)

\[ T_{ah} = \frac{\hbar}{4\pi k_B} \left[ \frac{1}{a_s} \frac{\partial}{\partial n} \left( a_s^2 - u_\perp^2 \right) \right]_{ah} . \]

Hence, to determine the analogue Hawking temperature of a classical analogue system, one needs to know the location of the acoustic horizon and the values of the speed of sound, the fluid velocity and their space gradients on the acoustic horizon.

In the analogue gravity systems discussed above, the fluid flows in flat Minkowski space, whereas the sound wave propagating through the non-relativistic fluid is coupled to a curved pseudo-Riemannian metric. Phonons are null geodesics, which generate a null surface—the acoustic horizon. Introduction of viscosity may destroy Lorentz invariance and hence the acoustic analogue is best studied in a vorticity free dissipationless fluid.
1.2 Motivation: Transonic black-hole accretion as an analogue system

Astrophysical black holes are the final stage of gravitational collapse of massive celestial objects. Astrophysical black holes may be roughly classified in two categories: stellar mass black holes having a mass of a few $M_\odot$ and supermassive black holes of mass of the order of $10^6 M_\odot$ or more, where $M_\odot$ denotes the solar mass. Both the stellar mass and the supermassive astrophysical black holes accrete matter from the surroundings. Depending on the intrinsic angular momentum of the accreting material, either the spherically symmetric (with zero angular momentum) or the axisymmetric (with non-zero angular momentum) geometry is invoked to study accreting black-hole systems (for details see, e.g. Frank, King & Raine 1992).

Denoting by $u(r)$ and $a_s(r)$ the dynamical velocity and the local sound speed of the accreting fluid moving along a space curve parameterized by $r$, the local Mach number of the fluid is defined as $M(r) = u(r)/a_s(r)$. The flow is locally subsonic or supersonic if $M < 1$ or $M > 1$, respectively. The flow is said to be transonic if there exists at least one point $r_h$ such that $M(r_h) = 1$. At this point a subsonic to a supersonic transition takes place either continuously or discontinuously. The point where this crossing takes place continuously is called sonic point, and if this crossing is not continuous, the transition point is called shock or discontinuity. As a consequence of the inner boundary conditions imposed by the event horizon, the accretion onto black holes is generally transonic. The investigation of accretion processes onto celestial objects was initiated by Hoyle & Littleton (1939) who computed the rate at which pressureless matter would be captured by a moving star. Subsequently, a theory of stationary, spherically symmetric transonic hydrodynamic accretion onto a gravitating astrophysical object at rest was formulated by Bondi (1952) using a purely Newtonian approach that includes pressure effects of the accreting material.

Since the publication of the seminal paper by Bondi in 1952, the transonic behaviour of the accreting fluid onto compact astrophysical objects has been extensively studied in the astrophysics community. Similarly, the pioneering work by Unruh in 1981 initiated a substantial number of works in the theory of analogue Hawking radiation with diverse fields of application (for a review, see Novello, Visser & Volovik 2002). Despite the fact that the accreting black hole can be considered a natural example of an analogue gravity system, an attempt to bridge these two fields of research has been made only recently (Das 2004). Since both the theory of transonic astrophysical accretion and the theory of analogue Hawking radiation are related by the same physical phenomena, it is important to study the analogue Hawking radiation in a transonic accretion onto astrophysical black holes and to compute $T_{ab}$. An accreting black-hole system of classical
analogue gravity is a unique example of classical analogue gravity which exhibits both the black-hole event horizon and the analog acoustic horizon. Hence, an accreting astrophysical black hole may be considered an ideal candidate to theoretically study these two different types of horizons and to compare their properties.

1.3 Pseudo-Schwarzschild black-hole accretion

Since the relativistic effects play an important role in the regions close to the accreting black hole, a purely Newtonian gravitational potential in the form \( \Phi_N = -GM/r \) is certainly not a realistic choice to describe transonic black-hole accretion in general. However, a rigorous investigation of transonic black-hole accretion (even with spherical symmetry) may be quite complicated in a general relativistic treatment. In order to compromise between a relatively easy handling of the Newtonian description of gravity and realistic but complicated general relativistic calculations, a series of ‘modified’ Newtonian potentials have been introduced. The following potentials have been proposed:

\[
\begin{align*}
\Phi_1 &= -\frac{1}{2(r-1)}, \\
\Phi_2 &= -\frac{1}{2r} \left[ 1 - \frac{3}{2r} + 12 \left( \frac{1}{2r} \right)^2 \right], \\
\Phi_3 &= -1 + \left( 1 - \frac{1}{r} \right)^{1/2}, \\
\Phi_4 &= \frac{1}{2} \ln \left( 1 - \frac{1}{r} \right),
\end{align*}
\]

(4)

where the units \( G = c = M_{bh} = 1 \) have been used. Unless otherwise stated, hereafter we use these units. The radial distance \( r \) in (4) is measured in units of the Schwarzschild gravitational radius \( r_g = 2GM_{bh}/c^2 \), where \( M_{bh} \) is the mass of the black hole, \( G \) Newton’s gravitational constant and \( c \) is the speed of light. The potential \( \Phi_1 \) is proposed by Paczyński and Wiita (1980), \( \Phi_2 \) by Nowak and Wagoner (1991), \( \Phi_3 \) and \( \Phi_4 \) by Artemova, Björnsson & Novikov (1996).

The use of these potentials retains most of the features of space-time around a compact object up to a reasonably small distance from the black-hole event horizon, and helps in reproducing some crucial properties of the relativistic accretion solutions with high accuracy. Hence, these potentials might be called post-Newtonian pseudo-Schwarzschild potentials (for details, see Artemova, Björnsson & Novikov 1996 and Das 2002). These potentials were originally introduced to study the axisymmetric accretion (accretion discs). However, it was shown (Das & Sarkar 2001) that some of these potentials could efficiently describe a spherically symmetric accretion as far as comparison with the full general relativistic accretion is concerned.

In this paper we study the analogue gravity phenomena in the spherical accretion onto astrophysical black holes under the influence of vari-
ous post-Newtonian pseudo-Schwarzschild potentials described above. We use the expressions ‘post-Newtonian’ and ‘pseudo-Schwarzschild’ synonymously. Our main goal is to provide a self-consistent calculation of the analogue horizon temperature $T_{ah}$ in terms of the minimum number of physical accretion parameters, and to study the dependence of $T_{ah}$ on various flow properties. In most practical situations, a complete general relativistic treatment turns out to be almost non-tractable. In those few simple cases where the general relativistic treatment is easy, one can repeat the calculations using various pseudo-potentials, and can compare the results with those obtained in the general relativistic calculations. This comparison may help to handle more complicated accretion systems without resorting to the full general relativistic treatment.

In Section 2 we calculate the location of the acoustic horizon and evaluate the relevant accretion variables on the horizon as functions of the fundamental parameters governing the flow. Hereafter, we use the expressions ‘acoustic horizon’ and ‘analogue horizon’ synonymously. We study both the polytropic (adiabatic) and the isothermal accretion. In Section 3 we derive an analytic expression for $T_{ah}$ as a function of various accretion parameters. In Section 4 we present our results numerically and in Section 5 we give a discussion and conclusions.

2 Calculation of various sonic quantities

The non-relativistic equation of motion for spherically accreting matter in a gravitational potential denoted by $\Phi$ may be written as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\partial \Phi}{\partial r} = 0,$$

where $u$, $p$ and $\rho$ are the velocity, pressure and density of the fluid, respectively. The first term in equation (5) is the Eulerian time derivative of the dynamical velocity, the second term is the ‘advective’ term, the third term is the momentum deposition due to the pressure gradient and the last term is the gravitational force. Another equation necessary to describe the motion of the fluid is the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho ur^2) = 0.$$

To integrate the above set of equations, one also needs the equation of state that specifies the intrinsic properties of the fluid. As mentioned earlier, we will study accretion described by either a polytropic or an isothermal equation of state.
2.1 Polytropic accretion

We employ a polytropic equation of state of the form

\[ p = K \rho^\gamma, \quad (7) \]

where the polytropic index is equal to the ratio of two specific heats, \( \gamma = c_p/c_v \). The sound speed \( a_s \) is defined by

\[ a_s^2 \equiv \left. \frac{\partial p}{\partial \rho} \right|_s = \frac{\gamma p}{\rho}, \quad (8) \]

where the subscript \( s \) denotes that the derivative is taken while keeping the entropy per particle \( s = S/n \) fixed, i.e. for an isentropic process. Assuming stationarity of the flow, we find the following conservation equations:

1) Conservation of energy implies constancy of the specific energy \( E \)

\[ E = \frac{u^2}{2} + \frac{a_s^2}{\gamma - 1} + \Phi. \quad (9) \]

2) Conservation of the baryon number implies constancy of the accretion rate \( \dot{M} \)

\[ \dot{M} = 4\pi \rho u r^2. \quad (10) \]

Equation (9) is obtained from (5) with (7) and (8) and equation (10) follows directly from (6).

Substituting \( \rho \) in terms of \( a_s \) and differentiating (10) with respect to \( r \), we obtain

\[ a_s' = \frac{a_s(1 - \gamma)}{2} \left( \frac{u'}{u} + \frac{2}{r} \right), \quad (11) \]

where \( ' \) denotes the derivative with respect to \( r \). Next we differentiate equation (9) and eliminating \( a_s' \) with the help of (11) we obtain

\[ u' = \frac{2a_s^2/r - \Phi'}{u - a_s^2/u}. \quad (12) \]

If the denominator of equation (12) vanishes at a particular radial distance \( r_h \), the numerator must also vanish at \( r_h \) to maintain the continuity of the flow. One thus finds the sonic-point condition as

\[ u_h = a_{sh} = \sqrt{\frac{r_h \Phi'_h}{2}}, \quad (13) \]

where \( r_h \) is the sonic point and the sphere of radius \( r_h \) is the acoustic horizon. Hereafter, the subscript \( h \) indicates that a particular quantity is
evaluated at \( r_h \). The location of the acoustic horizon is obtained by solving the algebraic equation

\[
E - \frac{1}{4} \left( \frac{\gamma + 1}{\gamma - 1} \right) r_h \Phi_h' - \Phi_h = 0.
\] (14)

The derivative \( u'_h \) at the corresponding sonic point is obtained by solving the quadratic equation

\[
(1 + \gamma) (u'_h)^2 + 2 (\gamma - 1) \sqrt{2 \Phi_h' r_h} u'_h + (2\gamma - 1) \frac{\Phi'_h}{r_h} + \Phi''_h = 0,
\] (15)

which follows from (12) in the limit \( r \rightarrow r_h \) evaluated with the help of l'Hospital’s rule.

Finally, the gradient of the sound speed at the acoustic horizon is obtained by substituting \( u'_h \) obtained from (15) into equation (11) at the acoustic horizon

\[
a'_s|_h = \left( \frac{1 - \gamma}{2} \right) \left( u'_h + \sqrt{2 \Phi_h'/r_h} \right).
\] (16)

Evidently, for both the Newtonian flow with \( \Phi_N = -1/r \) and the post-Newtonian flow, the location of the sonic point is identical to the location of the acoustic horizon due to the assumption of stationarity and spherical symmetry.

### 2.2 Isothermal Accretion

We employ the isothermal equation of state of the form

\[
p = \frac{RT}{\mu} \rho = c_s^2 \rho,
\] (17)

where \( T \) is the temperature, \( R \) and \( \mu \) are the universal gas constant and the mean molecular weight, respectively. The quantity \( c_s \) is the isothermal sound speed defined by

\[
c_s^2 = \left. \frac{\partial p}{\partial \rho} \right|_T = \Theta T,
\] (18)

where the derivative is taken at fixed temperature and the constant \( \Theta = \kappa_b/(\mu m_H) \) with \( m_H \approx m_p \) being the mass of the hydrogen atom. In our
model we assume that the accreting matter is predominantly hydrogen, hence \( \mu \approx 1 \). Now, the specific energy equation takes the form
\[
\mathcal{E} = \frac{u^2}{2} + \Theta T \ln \rho + \Phi ,
\]
whereas the accretion rate is given by (10) as before.

The radial change rate of the dynamical velocity is again given by (12). From equation (12) and with (18) we find the sonic point condition as
\[
u_h = \frac{\sqrt{r_h \Phi_h'}}{2} = c_s = \sqrt{\Theta T}. \tag{20}\]
The derivative of \( u \) is obtained from (12) by making use of l’Hospital’s rule as before. We find
\[
u'_h = \sqrt{-\frac{1}{2}} \left( \frac{1}{r_h \Phi_h'} + \Phi''_h \right), \tag{21}\]
where the minus sign in front of the square root indicates accretion (the plus would correspond to a wind solution). Note that the quantities in equations (20) and (21) are functions of the fluid temperature only. Hence the isothermal spherical accretion can be essentially described as a one-parameter solution of the hydrodynamical equations, parameterized by \( T \).

3 Calculation of Analogue Temperature

To calculate \( T_{ah} \) for various \( \Phi \), we first write down the general expression for \( T_{ah} \) in the relativistic form and then we reduce the expression by taking the non-relativistic post-Newtonian limit.

The relativistic acoustic metric tensor \( G_{\mu \nu} \) is defined as (Moncrief 1980, Bilić 1999)
\[
G_{\mu \nu} = \frac{n}{h a_s} \left[ g_{\mu \nu} - (1 - a_s^2) u_\mu u_\nu \right], \tag{22}\]
where \( n \) is the particle number density and \( h = (p + \rho) / n \) is the specific enthalpy. The corresponding surface gravity may be calculated as (Bilić 1999)
\[
\kappa = \sqrt{\xi^\nu \xi_\nu} \frac{\partial}{\partial n} \left( u - a_s \right) \bigg|_{ah}, \tag{23}\]
where \( \xi^\nu \) is the stationary Killing field and \( \partial / \partial n \) is the derivative normal to the acoustic horizon. This expression together with the Newtonian limit
\[
|\xi^2| = g_{00} \to \left( 1 + \frac{\Phi}{2c^2} \right) \tag{24}\]
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gives a general expression for the temperature of the analogue Hawking radiation in a spherically accreting fluid in the Newtonian as well as in any pseudo-Schwarzschild gravitational potential

\[ T_{\text{ah}} = \frac{h}{2\pi \kappa_h} \sqrt{\frac{2c^2 + \Phi_h}{2c^2}} \left[ \frac{1}{1-a_s^2} \left| \frac{d}{dr} (a_s - u) \right| \right]_{\text{ah}}. \] (25)

The quantities required to calculate the analogue temperature (25) are obtained using the formalism presented in Section 2.1. The Newtonian and post-Newtonian polytropic spherical accretion onto astrophysical black holes is an example of an analogue-gravity system in which the value of \( T_{\text{ah}} \) can be calculated using only two measurable physical parameters, namely, \( \mathcal{E} \) and \( \gamma \).

For a particular value of \( \{\mathcal{E}, \gamma\} \), we define a dimensionless quantity \( \tau \) as the ratio of \( T_{\text{ah}} \) to \( T_H \) which turns out to be independent of the black-hole mass. In this way we can compare the properties of the acoustic versus event horizon of a spherically accreting black hole of any mass. Using equations (11)-(16) we find

\[ \tau \equiv \frac{T_{\text{ah}}}{T_H} = 4 \sqrt{\frac{2 + \Phi_h}{2}} \left( \frac{2}{2 - r_h \Phi_h} \right) \left( \frac{\gamma + 1}{2} \right) \]

\[ \sqrt{\frac{\Phi'_h}{r_h} f(\gamma) - (1 + \gamma) \Phi''_h} \] (26)

where \( f(\gamma) = (0.00075\gamma^2 - 5.0015\gamma + 3.00075) \). The quantities \( \Phi_h, \Phi'_h, \) and \( \Phi''_h \) are obtained from equation (4) for various potentials, and \( r_h \) is calculated from (14) for an astrophysically relevant choice of \( \{\mathcal{E}, \gamma\} \).

The quantity \( \mathcal{E} \) is scaled in terms of the rest mass energy. The values \( \mathcal{E} < 0 \) correspond to negative energy accretion states where a radiative extraction of the rest mass energy from the fluid is required. To make such an extraction possible, the accreting fluid has to possess viscosity or other dissipative mechanisms, which may violate Lorenz invariance. The values \( \mathcal{E} > 1 \) are mathematically allowed. However, large positive \( \mathcal{E} \) represent the flows that start from infinity with a very high thermal energy, and the \( \mathcal{E} > 1 \) accretion represents enormously hot flow configurations at very large distances from the black hole. These configurations are not conceivable in realistic astrophysical situations. Hence we set \( 0 \lesssim \mathcal{E} \lesssim 1 \).

Concerning the polytropic index, the value \( \gamma = 1 \) corresponds to the isothermal accretion in which the accreting fluid remains optically thin. This value is the physical lower limit since \( \gamma < 1 \) is not realistic in accretion astrophysics. On the other hand, \( \gamma > 2 \) is possible only for a superdense matter with a substantially large magnetic field and a direction-dependent anisotropic pressure. This requires the accreting material to be governed by
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magneto-hydrodynamic equations, dealing with which is beyond the scope of this paper. We thus set $1 \leq \gamma \leq 2$ for a polytropic flow, and separately consider the case $\gamma = 1$, corresponding to an isothermal accretion. The preferred values of $\gamma$ for a realistic polytropic black hole accretion range from $4/3$ to $5/3$ (Frank et. al. 1992).

Note that if $(a'_{\gamma} - u')_{h}$ is negative, one obtains an acoustic white-hole solution. Hence the condition for the existence of the acoustic white hole is

$$\left(\frac{\gamma + 1}{2}\right) \sqrt{\frac{\Phi'_{h}}{r_{h}}} f(\gamma) - (1 + \gamma) \Phi''_{h} < 0.$$

(27)

Since $\gamma$ and $r_{h}$ can never be negative, and since $\Phi''_{h}$ and $\Phi''_{h}$ are always real for the preferred domain of $\{\mathcal{E}, \gamma\}$, acoustic white-hole solutions are excluded in the astrophysical accretion governed by the Newtonian or post-Newtonian potentials.

For an isothermal flow, the quantity $c'_{s}$ is zero and using (21) we find

$$\tau = 4 \sqrt{2} \left(\frac{1}{2 - r_{h}\Phi'_{h}}\right) \sqrt{-\left(1 + \frac{\Phi_{h}}{2}\right) \left(\Phi''_{h} + \Phi''_{h} \frac{r_{h}}{r_{h}}\right)},$$

(28)

where $r_{h}$ should be evaluated using (20). Clearly, the fluid temperature $T$ completely determines the analogue Hawking temperature. Hence, a spherical isothermally accreting astrophysical black hole provides a simple system where analogue gravity can be theoretically studied using only one free parameter.

4 Results

4.1 Polytropic accretion

Given a potential $\Phi$, we calculate $r_{h}$ as a function of $\{\mathcal{E}, \gamma\}$ using equation (15). The physically motivated choice of the domain $\{\mathcal{E}, \gamma\}$ is discussed in Section 3. Note that the physically acceptable values of $r_{h}$ must be larger than 1 since the black-hole event horizon is located at $r = 1$. However, we show the results only for $r_{h} \geq 2$ since the pseudo-potentials (4) may not accurately mimic the Schwarzschild space-time too close to the black-hole event horizon. We find that $r_{h}$ anti-correlates with $\{\mathcal{E}, \gamma\}$, which is obvious from equation (14). Physically, this effect is due to a subtle competition between the magnitudes and radial gradients of the velocity field and the speed of sound. On the one hand, increase of $\mathcal{E}$ at fixed $\gamma$ means increase of the asymptotic energy configuration of the flow which is basically thermal in nature. As a result, the larger the energy $\mathcal{E}$, the closer to the accretor must the flow approach in order to gain sufficiently large dynamical velocity
Fig. 1. The ratio of the analogue to the Hawking temperature $\tau = T_{ah}/T_H$ as a function of the specific flow energy $\mathcal{E}$ and the polytropic index $\gamma$ for the Newtonian (marked with $\Phi_N$) and four pseudo-Schwarzschild potentials (marked with $\Phi_1$, $\Phi_2$, $\Phi_3$ and $\Phi_4$).

and its gradient. On the other hand, keeping $\mathcal{E}$ fixed, increase of $\gamma$ means increase of the heat capacity of the accreting fluid and in turn its speed of sound. As a result, the flow becomes supersonic at a smaller distance to the black hole.

Once we have calculated $r_h$, we calculate the ratio $\tau = T_{ah}/T_H$ as a function of two flow parameters $\mathcal{E}$ and $\gamma$ using (26) and plot the results in figure 1 for various potentials. Note that for all potentials except $\Phi_N$, the ratio $\tau$ may be larger than unity for some $\{\mathcal{E}, \gamma\}$ and increases both with increasing $\mathcal{E}$ and with increasing $\gamma$. In the region $1 < r_h < 2$, the ratio $\tau$ may become very large, and a small zone may appear where $\tau$ is again less than one. In the Newtonian case, $\tau$ becomes larger than unity only in the region $1 < r_h < 2$. However, as discussed earlier, we do not rely on the results obtained in this region.

In figure 2 we classify the $\{\mathcal{E}, \gamma\}$ parameter space to show some distinct important regions. The region marked with $\tau < 1$ is a range of values of $\{\mathcal{E}, \gamma\}$ for which the analogue temperature is always lower than the Hawking temperature. Similarly, $\tau > 1$ denotes a range of values of $\{\mathcal{E}, \gamma\}$.
Fig. 2. Classification of the \(\{E, \gamma\}\) parameter space for a polytropic spherical accretion flow in the Newtonian (marked with \(\Phi_N\)) and in four pseudo-Schwarzschild potentials (marked with \(\Phi_1, \Phi_2, \Phi_3\), and \(\Phi_4\)). The parameter space is divided in four regions in which the analogue temperature \(T_{ah}\) is always lower than the Hawking temperature \(T_H\) (marked with \(\tau < 1\)), \(T_{ah}\) is higher than \(T_H\) (marked with \(\tau > 1\)), the use of the pseudo-potentials is unreliable (marked with \(1 < r_h < 2\)), the accretion flow is always subsonic (marked with \(\text{NS}\)).

for which \(T_{ah}\) is higher than \(T_H\). In these regions we have \(r_h \geq 2\). The region marked with \(1 < r_h < 2\) represents the values of \(\{E, \gamma\}\) for which a physical solution may be obtained for \(1 < r_h < 2\), and \(\tau\) may be larger than (mostly) and less than (rarely) unity. The regions marked with \(\text{NS}\) represent the \(\{E, \gamma\}\) subset for which no physical solution \(r_h > 1\) exists. Hence, in this region of the parameter space there is no transonic flow and the acoustic horizon does not form. Note that such a situation may occur only with \(\Phi_3\).
and with purely Newtonian potentials. With all other potentials, physical values \((r_h > 1)\) exist in the entire domain of \(\{\xi, \gamma\}\).

The domination of the analogue temperature over the Hawking temperature is most prominent with the Paczyński and Wiita (1980) potential, and least prominent with the pure Newtonian potential. If we denote by \(\tau^\Phi\) the analogue to the Hawking temperature ratio for a particular \(\Phi\) at a particular value of \(\{\xi, \gamma\}\) for which a physical solution \(r_h\) exists for all five potentials, then

\[
\tau^{\Phi_1} > \tau^{\Phi_2} > \tau^{\Phi_3} > \tau^{\Phi_4} > \tau^{\Phi_N}.
\]

(29)

Fig. 3. The ratio of the analogue to the Hawking temperature \(\tau = T_{ah}/T_H\) as a function of the acoustic velocity \(c_s\) (scaled in the unit of the velocity of light) at \(r_h\), for spherical isothermal accretion in the Newtonian (marked with \(\Phi_N\)) and four pseudo-Schwarzschild potentials (marked with \(\Phi_1, \Phi_2, \Phi_3\) and \(\Phi_4\)). Results are shown for \(r_h \geq 2\) only.

4.2 Isothermal accretion

As demonstrated in Section 2.2, the flow temperature \(T\) fully determines the location of the acoustic horizon \(r_h\) and other relevant quantities evaluated at \(r_h\). We take a representative fluid temperature (in degrees of K) in the
range $10^8 \leq T \leq 4 \times 10^{12}$, where the upper limit ensures that $r_h$ should never be less than 2. We find that the horizon radius $r_h$ calculated using equation (20) increases with $T$. This is obvious because the higher the temperature, the larger the speed of sound, and the flow needs to get closer to the accretor to gain the dynamical velocity $u$ and become supersonic. For a given temperature $T$, we find

$$r_h^{\Phi_N} > r_h^{\Phi_2} > r_h^{\Phi_4} > r_h^{\Phi_3} > r_h^{\Phi_1}. \tag{30}$$

In figure 3 we plot the ratio $\tau$ as a function of the isothermal acoustic velocity $c_s$ (scaled in the unit of the velocity of light) at the acoustic horizon $r_h$, for five potentials as in figure 1. The value of $c_s$ is obtained from eq. (20). The lines in the plot extend as far as a physical value $r_h \geq 2$ exists. We find that $\tau$ always increases with $T$ (hence with $c_s$) and at a particular $T$ (as well as at that corresponding $c_s$) we also find

$$\tau^{\Phi_2} > \tau^{\Phi_N} > \tau^{\Phi_4} > \tau^{\Phi_3} > \tau^{\Phi_1}. \tag{31}$$

5 Discussion

In this work we have demonstrated that a spherical accretion onto astrophysical black holes, under the influence of Newtonian and various post-Newtonian pseudo-Schwarzschild potentials, can be considered as an analogue gravity system naturally found in the Universe. Spherically accreting black holes are unique analogue gravity systems in which two kinds of horizons, black-hole event and acoustic analogue horizon, may be simultaneously present. We have calculated the analogue horizon temperature $T_{ah}$ for spherically accreting systems as a function of the minimum number of physical parameters governing the accretion flow. We have modelled the flow in a self-consistent way and studied the dependence of $T_{ah}$ on various flow properties. In our model the calculations have been performed with a position-dependent speed of sound using the relativistic formalism in curved space (Bilić 1999).

Note, however, that the analogy has been applied to describe the classical perturbation of the fluid in terms of a field satisfying the wave equation in an effective geometry. It is not our aim to provide a formulation by which the phonon field generated in this system could be quantized. To accomplish this task, one would need to show that the effective action for the acoustic perturbation is equivalent to a field theoretical action in curved space, and the corresponding commutation and dispersion relations should directly follow (see, e.g. Unruh & Schützhold 2003). Such considerations would be beyond the scope of this paper. Also note that for all types of accretions discussed here, the analogue temperature $T_{ah}$ is many orders of
magnitude lower compared with the fluid temperature of accreting matter. Hence, the black-hole accretion system may not be a good candidate for detection of the analogue Hawking radiation.

We have studied both the polytropic and the isothermal flow. For a polytropic flow, two free parameters $E$ and $\gamma$ are required to calculate $T_{\text{ah}}$. The polytropic index $\gamma$ is assumed to be constant throughout the fluid. A realistic model of the flow with no assumptions would perhaps require a variable polytropic index having a functional dependence on the radial distance, i.e. $\gamma = \gamma(r)$ instead of a constant $\gamma$. However, we have performed the calculations for a sufficiently large range of $\gamma$ and we believe that all possible astrophysically relevant polytropic indices are covered.

Das & Sarkar (2001) have shown that among all pseudo-potentials, $\Phi_1$ and $\Phi_4$ are the best because they provide an accurate approximation to the full general relativistic solution up to a reasonably close distance to the event horizon. While $\Phi_1$ is the best approximation for an ultra-relativistic flow with $\gamma = 4/3$, the potential $\Phi_4$ turns out to be the best approximation when the flow becomes non-relativistic, i.e. when $\gamma \simeq 5/3$. Hence, the temperature $T_{\text{ah}}$ calculated with $\Phi_1$ and $\Phi_4$ is comparable with the value of $T_{\text{ah}}$ calculated for a general relativistic flow. In the isothermal case, there is presently no such comparative study and hence no preference among various $\Phi$ may be set for an isothermal accretion.

From our calculations it follows that the main difference between the post-Newtonian polytropic flow (irrespective of the potential used) and the full relativistic flow is that an analogue white hole can never form in the Newtonian or post-Newtonian spherical accretion, whereas in a general relativistic accretion, analogue white-hole solutions exist for an extended parametric region of $\{E, \gamma\}$ (Das 2004). In an isothermal accretion with a constant speed of sound, the question of the analogue white-hole formation does not arise at all.

We have found that the isothermal spherical accretion onto an astrophysical black hole represents a simple analogue gravity model in which one physical parameter, the fluid temperature $T$, completely specifies the system and provides sufficient information for the calculation of the analogue temperature. We would, however, like to raise one point of caution. In a realistic astrophysical situation with perturbations of various types, isothermality of the flow is difficult to maintain if the acoustic horizon is formed at a very large distance from the event horizon. Owing to the fact that, close to a spherically accreting Schwarzschild black hole, the electron number density $n_e$ falls off as $r^{-3/2}$ while the photon number density $n_\gamma$ falls off as $r^{-2}$ (Frank et al. 1992), the ratio of $n_e$ to $n_\gamma$ is proportional to $\sqrt{r}$. Hence, the number of electrons per photon decreases with decreasing $r$, so that close to the accretor, the momentum transfer by photons on the accreting fluid may efficiently keep the fluid temperature roughly constant.
Hence, isothermality may be a justified assumption at small values of \(r_h\). However, the momentum transfer efficiency falls off with increasing \(r\) and the isothermality assumption may break down far away from the black hole.

We have found that in both the adiabatic and the isothermal flow, the analogue temperature may take over the Hawking temperature, i.e. the ratio \(\tau\) may be larger than unity. This effect has also been established in a fully relativistic flow (Das 2004). One should note that since \(\tau\) decreases with increasing \(r_h\), high values of \(\tau\) are obtained if the acoustic horizon is formed close to the event horizon, i.e. for \(1 < r_h < 2\). However, our post-Newtonian model may not be reliable in this region since the difference between a post-Newtonian metric and the general relativistic one is likely to be quite substantial in the vicinity of the event horizon. The potentials discussed here could only be used to obtain a more accurate description compared with the purely Newtonian approach. Hence very large values of \(\tau\) are consistent only in a complete general relativistic treatment of accretion.

One interesting important problem is the study of the analogue gravity and the calculation of the analogue temperature for a non-spherical axisymmetric accretion (accretion discs). In this case, the situation is more complicated because, first, a realistic disc accretion requires a more involved parametric space and, second, more than one sonic points exist for specific boundary conditions (Das 2002, Das, Pendharkar & Mitra 2003, Das 2004a, Barai, Das & Wiita 2004). A detailed study of these issues is beyond the scope of this article and is presented elsewhere (Abraham, Bilić & Das).

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