Abstract. This paper introduces the notion of an excellent quotient, which is stronger than a universal geometric quotient. The main result is that for an action of a connected solvable group $G$ on an affine scheme Spec($R$) there exists a semi-invariant $f$ such that Spec($R_f$) → Spec(($R_f)^G$) is an excellent quotient. The paper contains an algorithm for computing $f$ and ($R_f)^G$. If $R$ is a polynomial ring over a field, the algorithm requires no Gröbner basis computations, and it also computes a presentation of ($R_f)^G$. In this case, ($R_f)^G$ is a complete intersection. The existence of an excellent quotient extends to actions on quasi-affine schemes.

Introduction

In the theory of connected algebraic groups, two cases stand out as being well understood: reductive groups and solvable groups. While the invariant theory of reductive groups is well-behaved and, in many aspects, well understood, this is not the case for solvable and, in particular, unipotent groups. For example, invariant rings of unipotent groups need not be finitely generated, and even even if they are, categorical quotients need not exist (see Ferrer Santos and Rittatore [9, Example 4.10]). However, a result of Rosenlicht [27] tells us that any variety $X$ with an action of an algebraic group has a dense open subset $U \subseteq X$ that admits a geometric quotient. A constructive version, involving huge Gröbner basis computations, can by found in Kemper [21]. This raises the question if more can be said for actions of special classes of groups, and if computations become easier for such groups. This brings us back to the case of unipotent groups, for which some further reaching results have indeed been obtained. In fact, quite a few authors have studied invariant theory of unipotent groups, e.g. Hochschild and Mostow [18], Grosshans [15], Fauntleroy [7, 8], and Bérczi et al. [1]; but the papers on the subject that are relevant in our context are Greuel and Pfister [13, 14] and Sancho de Salas [28]. Among other results, these papers contain the following key statement: If a connected unipotent group acts on $X$, there is a nonzero invariant $f$ such that $X_f$ admits a geometric quotient $X_f \rightarrow Y$. (More specifically, in [13] $X$ is a quasi-affine scheme over a field of characteristic 0 and the authors also show that $X_f \cong \mathbb{A}^n \times Y$ as schemes over $Y$, while in [28] $X$ is an affine scheme over a field of any characteristic; but see Remark 2.8 below about these statements.)

The above statement of Greuel and Pfister leads to the definition, made in this paper as Definition 2.1, of an “excellent quotient,” which is essentially a universal geometric quotient $X \rightarrow Y$ with the additional property that $X \cong F \times Y$ as schemes over $Y$, with $F$ another scheme. Unsurprisingly, an excellent quotient is better than a geometric quotient. For example, an excellent quotient implies the existence of a cross section $Y \rightarrow X$ (meaning that the composition $Y \rightarrow X \rightarrow Y$ is the identity), and if $X = \text{Spec}(R)$ and $Y = \text{Spec}(R^C)$, then the cross section means that $R^C$ is the image of a ring map from $R$ to $R$. So the invariant ring tends to have exceptionally few generators with an easy way of computing them.

This paper goes beyond unipotent groups by considering connected solvable groups. This is of interest not only because solvable groups naturally extend the class of unipotent groups, but also because if $G$ is a connected algebraic group acting on an affine variety $X$ and $B$ is a Borel subgroup, then $K[X]^G = K[X]^B$ (see Humphreys [19, Exercise 21.8]); so computing invariant rings of connected solvable groups means computing invariant rings of all connected groups. The main results of the paper, to be found in detail in Theorem 5.7 and Remark 5.8, have already
been stated in the abstract above. To put them in the context of the existing literature, it should be mentioned that already Rosenlicht [25, Theorem 10] showed that the quotient $X \to Y$ by a connected solvable group, viewed as a rational map, has a cross section. However, working only with rational maps, he did not consider geometric quotients in the modern setup, and his proof is far from constructive. Popov [23, Theorem 3] proved that if a connected solvable group $G$ acts on an irreducible algebraic variety $X$ over an algebraically closed field, then $X$ has a $G$-stable dense open subset that admits an excellent quotient. Again, the result is not constructive.

Thus the results of this paper extend the earlier results mentioned above in generality ($X$ need not be integral, the ground ring $K$ need not be a field), in scope (solvable groups instead of unipotent groups), and because the results are fully algorithmic. (Sancho de Salas [28] presents algorithms for the additive group and gives ideas towards algorithms for unipotent groups.) The result about complete intersections (see Theorem 5.7(b)) seems to be entirely new.

It might seem that generalizing from unipotent to solvable groups is meaningless since a solvable group consists of a unipotent group $U$ and a torus on top, for which the invariant theory is easy and harmless. However, for the group $U$, an excellent quotient (or even a categorical quotient) only exists after passing to $\text{Spec}(R_f)$ with $f$ an invariant, and for most choices of such an $f$, the torus will not act on $R_f^{G_a}$. In fact, the most technically involved part of this paper is the proof that $f$ can be chosen as a semi-invariant of the torus, and that such a choice can be made in the general situation assumed here and at a low computational cost.

The first section of this paper is devoted to actions of the additive group on an affine scheme. Such actions have been studied in various papers, e.g. Tan [29], van den Essen [6], Derksen and Kemper [5], Freudenburg [10], and Tanimoto [30]. Although the main ideas of the section are already present in these papers (particularly in [30]), none of them reaches the level of generality we require: a general ring, of any characteristic, as ground ring, and actions on possibly non-integral schemes. Section 1 introduces a variant of the notion of a “local slice” and gives a simplified algorithm for computing it. The main result, Theorem 1.6, is the algebraic way of saying that $\text{Spec}(R_f) \to \text{Spec}(R_f^{G_a})$ is an excellent quotient.

Section 2 introduces the notion of an excellent quotient and studies some basic properties: An excellent quotient is a universal geometric quotient, and excellent quotients can be put on top of each other along a chain of normal subgroups.

Sections 3 and 4 are rather technical and address the question, mentioned above, how a local slice can be found such that the denominator $f \in R$ is a semi-invariant. The setup is that the additive group appears as a normal subgroup of an ambient group, which in Section 4 is assumed to be connected and solvable. More precisely, the ambient group is assumed to be “in standard solvable form” according to Definition 4.1, a hypothesis that is automatically satisfied when the ground ring is an algebraically closed field.

Section 5 starts by dealing with actions of the multiplicative group $G_m$. The results bear an uncanny resemblance to those about $G_a$-actions. Putting all the strands together then yields the main results of the paper (Theorem 5.7 and Remark 5.8) and its algorithmic version (Algorithm 5.6). The algorithm has been implemented in the computer algebra system MAGMA [4], though not in complete generality. It turns out that the excellent quotient by a connected solvable group has fibers that are isomorphic, as a scheme without group structure or group action, to another connected solvable group.

The final section contains a sort of a converse: If a group action restricts to an open subset $X_f$ where the orbits are all of the type described above, then the action is “essentially solvable” (Theorem 6.2).

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1. Additive group actions

In this section we consider a morphic action of the additive group \( \mathbb{G}_a = \text{Spec}(K[z]) \) over a ring \( K \) on an affine scheme \( \text{Spec}(R) \), with \( R \) a \( K \)-algebra. Such an action induces a homomorphism \( \varphi : R \to R[z] \) of \( K \)-algebras. If \( s \in R \) and \( g := \varphi(s) \), then

\[
g(0) = s \quad \text{and} \quad \varphi(g(w)) = g(w + z),
\]

(1.1)

where in the second equality \( \varphi \) is applied to the polynomial ring \( R[w] \) coefficient-wise. Let us call \( \text{deg}(s) := \text{deg}_z(g) \) the degree of \( s \). The invariant ring is \( \mathcal{R}^{\mathbb{G}_a} := \ker(\varphi - \text{id}) \). If \( g = \sum_{i=0}^d c_i z^i \) with \( c_i \in R \), \( c_d \neq 0 \), it follows that

\[
\varphi(c_i) = \sum_{j=0}^{d-i} \binom{i+j}{i} c_{i+j} z^j.
\]

(1.2)

In particular, \( c := c_d \in \mathcal{R}^{\mathbb{G}_a} \) is an invariant. As we will see, throughout the paper \( c \) is the invariant that was denoted by “\( f \)” in the abstract and introduction. (Here we need the letter \( f \) for another purpose.) In fact, we can form the localization \( R_c \) of \( R \) with respect to the multiplicative set \( \{1, c, c^2, \ldots \} \) and extend \( \varphi \) to a homomorphism \( R_c \to R_c[z] \), which we will also call \( \varphi \) and which satisfies (1.1).

I learned the following argument, leading up to the proof of Proposition 1.1, from Tanimoto [30]. It is presented here for the convenience of the reader and since our situation is slightly different. Let \( a \in R \) be another ring element and set \( f := \varphi(a) \in R[z] \). Since \( c \in \mathcal{R}^{\mathbb{G}_a} \) is the highest coefficient of the above polynomial \( g \), we can perform division with remainder in \( R_c[z] \), which gives

\[
f = qg + h \tag{1.3}
\]

with \( q, h \in R_c[z], \ \text{deg}_z(h) \leq d - 1 \) and \( \text{deg}_z(q) \leq \text{deg}(a) - d \) (where we assign the degree \(-\infty\) to the zero polynomial). Using (1.1), we obtain

\[
g(w + z)(q(w + z) - \varphi(q(w))) + (h(w + z) - \varphi(h(w))) =
\]

\[
g(w + z)q(w + z) + h(w + z) - \varphi(g(w))\varphi(q(w)) - \varphi(h(w)) = f(w + z) - \varphi(f(w)) = 0
\]

Considering this as an equality of polynomials in \( w \) and using the \( w \)-degree, we conclude that \( q(w + z) = \varphi(q(w)) \) and \( h(w + z) = \varphi(h(w)) \). Substituting \( w = 0 \) yields \( \varphi(g(0)) = q \) and \( \varphi(h(0)) = h \), so \( \text{deg}(g(0)) \leq \text{deg}(a) - d \) and \( \text{deg}(h(0)) \leq d - 1 \). We can write \( q(0) = r/c^m \) and \( h(0) = b/c^m \) with \( r, b \in R \), choosing the integer \( m \) large enough such that \( r \) and \( b \) have the same degrees as \( q(0) \) and \( h(0) \), respectively. Now substituting \( z = 0 \) in (1.3) and possibly choosing \( m \) even larger yields \( c^m a = rs + b \). In summary, we obtain the following “division with remainder principle” in \( R \):

**Proposition 1.1.** For \( s, a \in R \), let \( c \in \mathcal{R}^{\mathbb{G}_a} \) be the highest coefficient of \( \varphi(s) \). Then there exist \( r, b \in R \) and \( m \in \mathbb{N}_0 \) such that

\[
c^m a = rs + b, \quad \text{deg}(r) \leq \text{deg}(a) - \text{deg}(s), \quad \text{and} \quad \text{deg}(b) \leq \text{deg}(s) - 1.
\]

If it is possible to perform addition, multiplication, and zero testing of elements of \( R \), then \( m, r \) and \( b \) can be computed.

**Definition 1.2.** A local slice of degree \( d \) with denominator \( c \) is a noninvariant \( s \in R \setminus \mathcal{R}^{\mathbb{G}_a} \) of degree \( d \) with \( c \in \mathcal{R}^{\mathbb{G}_a} \) the highest coefficient of \( \varphi(s) \), such that every \( a \in R_c \) with \( \text{deg}(a) < d \) lies in \( \mathcal{R}^{\mathbb{G}_a} \).

The significance of this notion lies in the fact that if \( s \) is a local slice, then \( b \) from Proposition 1.1 is an invariant in \( \mathcal{R}^{\mathbb{G}_a} \).

**Remark 1.3.** If \( s \in R \setminus \mathcal{R}^{\mathbb{G}_a} \) has minimal degree among all noninvariants, it is a local slice. This shows the existence of local slices if the action is nontrivial. If \( R \) is a domain, the converse holds. So our definition of a local slice is consistent with the one from Freudenburg [10] and Tanimoto [30], who only considered domains.
If the characteristic of $K$ is 0 or a prime and if $R$ is reduced, it is not hard to see from (1.2) that the degree $d$ of a local slice must be a power of the characteristic of $K$. In particular, if $\text{char}(K) = 0$ and $R$ is reduced, an element is a local slice if and only if it has degree 1. The following example shows that local slices of degree $> 1$ occur.

**Example 1.4.** (1) Let $R = K[x, y]$ be a polynomial ring over a field of characteristic $p > 0$ and define $\varphi: R \to R[z]$ by

$$\varphi(x) = x + yz + z^p, \quad \varphi(y) = y.$$  

Then $s = x$ is a local slice of degree $p$.

(2) With $K$ a ring of any characteristic, let $R = K[x, y]/(y^n)$ with $n \geq 2$ and $\varphi(x) = x + \overline{y}z$, $\varphi(\overline{y}) = \overline{y}$. Then for $0 < d < n$, we see that $s = x^d$ is a local slice of degree $d$ with denominator $c = \overline{y}^d$, since $R_c = \{0\}$. Perhaps more significantly, all local slices have nilpotent denominators, so $R_c$ is always the zero ring.

Algorithms for finding a local slice (in the case that $R$ is a finitely generated domain) were given by Sancho de Salas [28] and Tanimoto [30]. The following algorithm for the same purpose is simpler, and does not require $R$ to be a domain.

**Algorithm 1.5** (Computation of a local slice).

**Input:** A nontrivial morphic $\mathbb{G}_a$-action on $\text{Spec}(R)$ for a finitely generated $K$-algebra $R = K[a_1, \ldots, a_n]$, given by $\varphi: R \to R[z]$ as above. We assume that it is possible to perform addition, multiplication, and zero testing of elements of $R$.

**Output:** A local slice $s \in R$.

1. For $i = 1, \ldots, n$, set $b_i = a_i$. Repeat steps 2–3 until all $b_i$ have degree 0.

2. Choose $s \in R$ as a noninvariant coefficient of one of the $\varphi(b_i)$ such that $d := \text{deg}(s)$ becomes minimal. One can use (1.2) for determining $\varphi(s)$. If $\text{char}(K) = 0$, then automatically $d = 1$, so $s$ is a local slice and we are done.

3. This step updates the $b_i$. For $i = 1, \ldots, n$, find elements $r_i, b_i \in R$ and $m_i \in \mathbb{N}_0$ such that

$$c^{m_i}a_i = r_is + b_i \quad \text{and} \quad \text{deg}(b_i) \leq d - 1 \quad \text{(1.4)}$$

according to Proposition 1.1.

**Proof of correctness of Algorithm 1.5.** The choice of $s$ implies $d \leq \text{deg}(b_i)$ for all $i$, so updating the $b_i$ in step 3 strictly decreases their degree. So the algorithm will reach its termination point where all $b_i$ have degree 0.

Suppose this is the case and let $a \in R_c$ be of degree $< d$. There is a nonnegative integer $k$ such that $a = c^{-k}F(a_1, \ldots, a_n)$ with $F \in K[x_1, \ldots, x_n]$ a polynomial. In the ring $R_c$, $s$ divides $F(a_1, \ldots, a_n) - F(c^{-m_1}b_1, \ldots, c^{-m_n}b_n)$ by (1.4), so it also divides $a - c^{-k}F(c^{-m_1}b_1, \ldots, c^{-m_n}b_n)$. But this difference has degree $< d$, and since the highest coefficient of $\varphi(s)$ is invertible in $R_c$, this implies that $a = c^{-k}F(c^{-m_1}b_1, \ldots, c^{-m_n}b_n)$, which is an invariant in $R_c^{\mathbb{G}_a}$. □

The following theorem shows why local slices are useful. For example, by part (b), generators of $R_c^{\mathbb{G}_a}$ can be determined immediately if a local slice is known. While parts (a) and (b) are essentially well known (at least in more restricted settings), (c) and (d) seem to be entirely new.

**Theorem 1.6.** For a nontrivial action of the additive group $\mathbb{G}_a$ over a ring $K$ on an affine $K$-scheme $\text{Spec}(R)$, given by a homomorphism $\varphi: R \to R[z]$, let $s$ be a local slice of degree $d$ with denominator $c \in R_c^{\mathbb{G}_a}$.

(a) The homomorphism $R_c^{\mathbb{G}_a}[x] \to R_c$ sending the indeterminate $x$ to $s$ is an isomorphism. We write $\psi: R_c \to R_c^{\mathbb{G}_a}[x]$ for the inverse isomorphism.

(b) The composition

$$\pi: R_c \xrightarrow{\psi} R_c^{\mathbb{G}_a}[x] \xrightarrow{x \mapsto 0} R_c^{\mathbb{G}_a}$$

is a homomorphism of $R_c^{\mathbb{G}_a}$-algebras with $\text{ker}(\pi) = (s)$. In particular, $\pi$ is surjective. For $a \in R_c$, $\pi(a)$ is given by

$$\varphi(a) = q \cdot \varphi(s) + \pi(a)$$

with $q \in R_c[z]$ (division with remainder).
(c) The composition
\[ R_c \xrightarrow{\varphi} R_c[z] \xrightarrow{\pi} R_c^G_{\ast}[z] \]
(with \( \pi \) applied coefficient-wise) is injective and makes \( R_c^G_{\ast}[z] \) into an \( R_c \)-module that is generated by \( d \) elements. In particular, if \( d = 1 \), then it is an isomorphism.

(d) Let \( B \) be a ring with a homomorphism \( R_c^G_{\ast} \to B \). Then
\[ (B \otimes_{R_c^G_{\ast}} R)^G_{\ast} = B \otimes 1. \]

Proof. (a) Let \( f \in R_c^G_{\ast}[x] \) with \( f(s) = 0 \). Since \( \varphi \) is a homomorphism of \( R_c^G_{\ast} \)-algebras, this implies \( f(\varphi(s)) = 0 \), so \( f = 0 \) since the highest coefficient of \( g := \varphi(s) \in R_c[z] \) is invertible. This shows that the map is injective. To prove surjectivity, let \( a \in R_c \). Applying Proposition 1.1 to \( R \) instead of \( B \) yields \( a = rs + b \) with \( r, b \in R_c \) such that \( \deg(b) < d \) and \( \deg(r) \leq \deg(a) - d < \deg(a) \). By Definition 1.2, \( b \in R_c^G_{\ast} \). Now, using induction on \( \deg(a) \) yields \( a = f(s) \) with \( f \in R_c^G_{\ast}[x] \).

(b) The first statement follows from (a). For the second statement, observe that the map that is claimed to be equal to \( \pi \) is a homomorphism of \( R_c^G_{\ast} \)-algebras, since the remainder from division by \( \varphi(s) \) has degree 0. Since \( R_c = R_c^G_{\ast}[s] \), it suffices to check the equality of the maps for \( a = s \), which is immediate.

(c) Take \( a \in R_c \) that is mapped to zero. By (a) we may write \( a = f(s) \) with \( f \in R_c^G_{\ast}[x] \), so
\[ 0 = \pi(\varphi(f(s))) = f(\pi(g)) = f(\pi(g - s) + \pi(s)) = f(g - s) \]
since by (1.2) all coefficients of \( g - s \) have degree \( < d \) and are therefore invariants in \( R_c^G_{\ast} \). Since \( g - s \in R_c[z] \) has \( z \)-degree \( d > 0 \) and an invertible highest coefficient, we obtain \( f = 0 \). This establishes injectivity. The above equality also shows that \( g - s = \pi(\varphi(s)) \) lies in the image. So \( g \) satisfies the polynomial \( (g(x) - s) - (g - s) \), whose coefficients (as a polynomial in \( x \)) lie in the image. This proves the second statement.

(d) By (a), the element \( 1 \otimes s \in B \otimes_{R_c^G_{\ast}} R := R' \) is algebraically independent over \( B \), and \( R' = B[1 \otimes s] \). By definition, \( (R')^G_{\ast} = \ker(\varphi' - \text{id}) \) with \( \varphi': R' \to R'[z] \) obtained by tensoring \( \varphi \). Let \( a \in R' \) and write \( a = \sum_{i=0}^{k} b_i (1 \otimes s)^i \) with \( b_i \in B \), \( b_k \neq 0 \). With the given map \( \eta: R_c^G_{\ast} \to B \) applied to \( R_c^G_{\ast}[x] \) coefficient-wise, we obtain
\[ \varphi'(a) = \sum_{i=0}^{k} b_i (1 \otimes g)^i = \sum_{i=0}^{k} b_i (\eta(g - s) \otimes 1 + 1 \otimes s)^i. \]

If \( k > 0 \), the coefficient of \( z^{kd} \) of this is \( b_k \eta(c)^k \otimes 1 \), which is nonzero since \( \eta(c) \) is invertible in \( B \). So if \( a \in (R')^G_{\ast} \), then \( k = 0 \) and therefore \( a \in B \otimes 1 \). The reverse inclusion \( B \otimes 1 \subseteq (R')^G_{\ast} \) is clear. \( \square \)

Theorem 1.6 has a geometric interpretation. In fact, the morphism \( \text{Spec}(R_c) \to \text{Spec}(R_c^G_{\ast}) \) induced from the inclusion is a excellent quotient by \( G \) with fibers \( \text{Spec}(K[x]) \) according to Definition 2.1 in the following section (see Proposition 2.5).

2. Excellent Quotients

In the following, \( S \) is a scheme and all schemes, morphisms and fiber products will be over \( S \) unless stated otherwise.

**Definition 2.1.** Let \( G \) be a group scheme acting on a scheme \( X \) by a morphism \( \text{act}: G \times X \to X \). A morphism \( \text{quo}: X \to Y \) of schemes is called an excellent quotient (of \( X \) by \( G \) with fibers \( F \)) if there is a faithfully flat scheme \( F \) with a morphism \( \text{pt}: S \to F \) (i.e., an \( S \)-valued point of \( F \)) and an isomorphism \( \text{iso}: F \times X \to X \) of schemes over \( Y \) such that the following conditions hold.

(i) With \( \text{pr}_2: G \times X \to X \) the second projection, the diagram
\[
\begin{array}{ccc}
G \times X & \xrightarrow{\text{act}} & X \\
\text{pr}_2 \downarrow & & \downarrow \text{quo} \\
X & \xrightarrow{\text{quo}} & Y
\end{array}
\]
commutes.

(ii) The composition

\[ G \times Y \xrightarrow{\text{id}_G \times \text{id}_Y} G \times F \xrightarrow{\text{id}_G \times \text{id}_Y} G \times X \xrightarrow{\text{act}} X \]

is surjective.

(iii) Let \( Y' \to Y \) be a morphism of schemes, giving rise to morphisms \( \text{quo}' : X' := X \times_Y Y' \to Y' \) and \( \text{act}' : G \times X' \to X' \) by base change. Then the map \( \Gamma(Y', \mathcal{O}_{Y'}) \to \Gamma(X', \mathcal{O}_{X'}) \) induced by \( \text{quo}' \) has the ring \( \Gamma(X', \mathcal{O}_{X'})^G \) of \( G \)-invariant functions as its image, defined (in the usual way) as follows: An element \( f \in \Gamma(X', \mathcal{O}_{X'}) \) interpreted as an element of \( \text{Hom}_Z(X', \mathbb{A}^1) \) with \( \mathbb{A}^1 := \text{Spec}(\mathbb{Z}[x]) \), lies in \( \Gamma(X', \mathcal{O}_{X'})^G \) if and only if the diagram

\[
\begin{array}{ccc}
G \times X' & \xrightarrow{\text{act}'} & X' \\
\downarrow \text{pr}_2 & & \downarrow f \\
X' & \xrightarrow{f} & \mathbb{A}^1
\end{array}
\]

commutes.

The following remark should provide a better understanding of Definition 2.1.

Remark 2.2. (a) If \( \text{quo} : X \to Y \) is an excellent quotient, the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{(\text{pt}, \text{id})} & F \\
\downarrow \text{id} & & \downarrow \text{pr}_2 \\
Y & & \xrightarrow{\text{quo}} X
\end{array}
\]

shows that the morphism \( \text{sect} := \text{iso} \circ (\text{pt}, \text{id}_Y) : Y \to X \) satisfies

\[ \text{quo} \circ \text{sect} = \text{id}_Y, \quad (2.2) \]

so it is a cross section of the quotient. This implies that \( X \to Y \) is surjective. Condition (ii) in Definition 2.1 demands the surjectivity of

\[ G \times Y \xrightarrow{(\text{id}_G, \text{sect})} G \times X \xrightarrow{\text{act}} X, \quad (2.3) \]

If \( S \) is the spectrum of an algebraically closed field, this means that every \( G \)-orbit in \( X \) meets the image of the cross section, or, equivalently, that the fibers of the quotient are precisely the orbits.

(b) In Definition 2.1, the point \( \text{pt} \) and the isomorphism \( \text{iso} \) only appear in Condition (ii). It is not hard to show that if this condition holds for some choice of a point and an isomorphism, then it holds for all choices.

(c) If \( \text{quo} : X \to Y \) is an excellent quotient with fibers \( F \), then all fibers of \( S \)-valued points of \( Y \) are isomorphic to \( F \). This follows since \( X \) and \( F \times Y \) are isomorphic as schemes over \( Y \).

(d) Since \( G \) acts on the fiber of every \( S \)-valued point of \( Y \), it follows from (c) that every such point affords a \( G \)-action on \( F \). But these actions are in general different, so there is usually no \( G \)-action on \( F \) that makes the isomorphism \( F \times Y \to X \) into a \( G \)-isomorphism.

(e) A base change of an excellent quotient is again an excellent quotient (by the same group and with the same fibers). This follows directly from the definition and the fact that surjectivity is stable under base change (see Görtz and Wedhorn [12, Proposition 4.32]).

(f) Let \( \text{quo} : X \to Y \) be an excellent quotient by a group \( G \), and let \( U \subseteq X \) be a \( G \)-stable open subscheme. With \( \text{sect} : Y \to X \) the cross section, \( V := \text{sect}^{-1}(U) \subseteq Y \) is an open subscheme, and it is not hard to see that \( U = \text{quo}^{-1}(V) \). Therefore the restriction \( \text{quo}|_U : U \to V \) arises from \( X \to Y \) by base change. By (e), it is an excellent quotient of \( U \) by \( G \).
The purpose of the following examples is to show how strong the notion of an excellent quotient is.

**Example 2.3.** In this example we assume that \( K \) is an algebraically closed field of characteristic \( \neq 2 \). Consider the action of \( G = \text{PGL}_2 \) on \( \text{SL}_2 \) by conjugation. The only invariant is given by the trace, and it is well known that restricting to the matrices with distinct eigenvalues gives a geometric quotient

\[
X := \{ A \in \text{SL}_2 \mid \text{tr}(A) \neq \pm 2 \} \to Y := K \setminus \{ \pm 2 \}.
\]

The quotient has a cross section, given by mapping \( a \in Y \) to the matrix \( \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix} \). Next we see that all fibers are isomorphic. Indeed, the fiber of \( a \neq \pm 2 \) consists of the matrices \( \begin{pmatrix} x & y \\ z & -a^{-1} \end{pmatrix} \) satisfying

\[
0 = x^2 - ax + yz + 1 = \left( \frac{2x - a}{2} \right)^2 + yz + \frac{4 - a^2}{4} = \frac{4 - a^2}{4} \left( \frac{2x - a}{\sqrt{4 - a^2}} \right)^2 + \frac{4y}{4 - a^2} \cdot z + 1.
\]

The last form of the equation shows that all fibers are isomorphic to the \( 0 \)-fiber given by \( x^2 + yz + 1 \). So the quotient has extremely good properties, but we claim that is is not excellent.

In fact, if it were excellent, then \( X \) would be isomorphic, as a scheme over \( Y \), to \( F \times Y \), with \( F \) the surface given by \( x^2 + yz + 1 \). Since \( X \) is given by the equation \( x^2 - ax + yz + 1 \) (see above), a \( Y \)-isomorphism \( F \times Y \cong X \) would imply that the surfaces over the rational function field \( K(t) \) given by \( x^2 + yz + 1 \) and by \( x^2 - tx + yz + 1 \) are isomorphic. But they are not, and the reason for this was explained to me by Igor Dolgachev. First, homogenizing the equations defines two projective quadrics in \( \mathbb{P}^3 \), which are not isomorphic since their discriminants are in different square classes. Second (and this is the hard part), it follows from a result by Gizatullin and Danilov [11, Theorem 6] that also the original affine surfaces cannot be isomorphic over \( K(t) \).

If we consider the action of \( \text{SL}_2 \) on quadratic binary forms, we also get a geometric quotient on a subset which has a cross section, and all fibers are isomorphic, but the quotient is not excellent. The proof is virtually the same as above.

**Example 2.4.** Let \( K \) be a field in which \(-1\) is not a square. Consider the natural action of \( G = \text{SO}_2(K) = \{ A \in \text{GL}_2(K) \mid A^T A = I_2, \det(A) = 1 \} \) on \( X = \mathbb{A}^2_K \). The invariant ring \( K[X]^G \) is known to be generated by \( x_1^2 + x_2^2 \), which defines a quotient \( X \to \mathbb{A}^1_K \). But this has no cross section, even after choosing a nonempty open subset \( Y \subseteq \mathbb{A}^1_K \) and restricting to its preimage \( X \). In fact, giving such a cross section is equivalent to giving polynomials \( f, g, h \in K[x] \) such that \( f^2 + g^2 = xh^2 \) with \( h^2 \neq 0 \). It is not hard to see that this is possible (if and only if \(-1\) is a square in \( K \). It follows that the quotient \( X \to Y \) is not excellent.

The following proposition deals with the affine case and shows how Condition (iii) of Definition 2.1 can be verified.

**Proposition 2.5.** Assume the situation of Definition 2.1. If \( X = \text{Spec}(R) \) and \( Y = \text{Spec}(B) \) with \( B \subseteq R \) rings, then Condition (i) of Definition 2.1 is equivalent to \( B \subseteq R^G \), and, given (i), Condition (iii) is equivalent to the following:

(iii) For every homomorphism \( B \to B' \) of rings we have \( (B' \otimes_B R)^G \subseteq B' \otimes 1 \).

In particular, in the situation of Theorem 1.6, the morphism \( \text{Spec}(R_c) \to \text{Spec}(R_c^G) \) is an excellent quotient by \( \mathbb{G}_a \) with fibers \( \mathbb{A}^1_c \).

**Proof.** First assume that Condition (i) of Definition 2.1 holds, and let \( f \in B = \Gamma(Y, \mathcal{O}_Y) = \text{Hom}_Z(Y, \mathbb{A}^1) \). Viewing \( f \) as an element of \( R \) means to consider \( f \circ \text{quo} : X \to \mathbb{A}^1 \), which lies in \( \Gamma(X, \mathcal{O}_X)^G = R^G \) by (i). Conversely, assume that \( f \circ \text{quo} \) lies in \( \Gamma(X, \mathcal{O}_X)^G \) for all \( f \in \text{Hom}_Z(Y, \mathbb{A}^1) \). So

\[
f \circ \text{quo} \circ \text{act} = f \circ \text{quo} \circ \text{pr}_2.
\]

The morphisms \( \text{quo} \circ \text{act} \) and \( \text{quo} \circ \text{pr}_2 \) are both into the affine scheme \( Y \), so they are given by homomorphisms \( \varphi_1, \varphi_2 : B \to \Gamma(G \times X, \mathcal{O}_{G \times X}) \) (see Görtz and Wedhorn [12, Proposition 3.4]), and, by the above equality, for every homomorphism \( \psi : \mathbb{Z}[x] \to B \) we have \( \varphi_1 \circ \psi = \varphi_2 \circ \psi \). This implies \( \varphi_1 = \varphi_2 \) and therefore (i).
Now assume that Condition (iii) of Definition 2.1 holds, and let $B \to B'$ be a homomorphism of rings, inducing a morphism $Y' := \text{Spec}(B') \to Y$. The map $\text{quo}' : X' \to Y'$ induces the homomorphism $\varphi : B' = \Gamma(Y', \mathcal{O}_{Y'}) \to \Gamma(X', \mathcal{O}_{X'}) = B' \otimes_B R$, $b' \mapsto b' \otimes 1$, and we have 

$$(B' \otimes_B R)^G = \Gamma(X', \mathcal{O}_{X'})^G = \varphi(\Gamma(Y', \mathcal{O}_{Y'})) = B' \otimes 1.$$ 

so (iii') follows.

Conversely assume (iii') and let $Y' \to Y$ be a morphism of schemes. Since $\text{quo}' \circ \text{act}' = \text{quo}' \circ \text{pr}_2$, the image of the map $\varphi : \Gamma(Y', \mathcal{O}_{Y'}) \to \Gamma(X', \mathcal{O}_{X'})$ induced by $\text{quo}'$ is contained in $\Gamma(X', \mathcal{O}_{X'})^G$. It remains to show the reverse inclusion. Let $V \subseteq Y'$ be an open subset and let $U := (\text{quo}')^{-1}(V) \subseteq X'$ be its inverse image. Since all squares in the diagram

\begin{equation}
\begin{array}{c}
U \xrightarrow{\text{incl}} X' \xrightarrow{\text{quot}} X \xrightarrow{\text{iso}^{-1}} F \times Y \\
\text{quo}' |_U \quad \text{quot} \quad \text{quo} \quad \text{pr}_2 \quad \text{id} \quad \text{quot} \quad S
\end{array}
\end{equation}

are cartesian, so is the outer rectangle (see Görtz and Wedhorn [12, Proposition 4.16]). Therefore $\text{quo}' |_U$ is faithfully flat (see [12, Remark 14.8]), and it follows by Grothendieck [17, Corollaire 2.2.8] that the map $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ induced by it is injective. If $V$ is affine, say $V = \text{Spec}(B')$, then $U = V \times_Y X = \text{Spec}(B' \otimes_B R)$, and the map $B' = \Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U) = B' \otimes_B R$ induced by $\text{quo}' |_U$ is given by $b' \mapsto b' \otimes 1$. Let $f \in \Gamma(X', \mathcal{O}_{X'})^G$, viewed as a morphism $X' \to \mathbb{A}^1$. Then

$$f|_U \in \text{Hom}_\mathbb{Z}(U, \mathbb{A}^1)^G = (B' \otimes_B R)^G \subseteq B' \otimes 1,$$

so there exists $g_V \in B' = \text{Hom}_\mathbb{Z}(V, \mathbb{A}^1)$ with $g_V \circ \text{quo}' |_U = f|_U$. By the injectiveness of $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$, $g_V$ is uniquely determined. Now it follows from the sheaf property of $\mathcal{O}_V$ that there exists $g \in \text{Hom}_\mathbb{Z}(Y', \mathbb{A}^1)$ with $g \circ \text{quo}' = f$, i.e., $f = \varphi(g)$. This completes the proof of the equivalence.

For the last statement of the proposition, observe that $\mathbb{A}^1_K$ is faithfully flat over $\text{Spec}(K)$, and that all conditions from Definition 2.1 follow directly from Theorem 1.6 and from this proposition. 

We will now prove that an excellent quotient is a geometric quotient. Let us recall this notion. According to Mumford et al. [22, Definition 0.6], a morphism $\text{quo} : X \to Y$ of schemes (where $X$ has an action of a group scheme $G$) is a geometric quotient if:

- $(g_1)$ Condition (i) from Definition 2.1 holds.
- $(g_2)$ The morphism $(\text{act}, \text{pr}_2) : G \times X \to X \times_Y X$ is surjective. (If $S$ is the spectrum of an algebraically closed field, this means that the fibers of $\text{quo}$ are the $G$-orbits.)
- $(g_3)$ The morphism $\text{quo}$ is submersive, i.e., it is surjective and a subset $V \subseteq Y$ is open if its preimage $\text{quo}^{-1}(V) \subseteq X$ is open.
- $(g_4)$ Condition (iii) from Definition 2.1 holds for the case that $Y' = Y \subseteq Y$ is an open subset of $Y$.

By [22, Definition 0.7], the morphism is called a universal geometric quotient if for every morphism $Y' \to Y$ of schemes, the morphism $\text{quo}' : X \times_Y Y' \to Y'$ obtained by base change is a geometric quotient. Recall that a geometric quotient is always a categorical quotient. Therefore the following result implies that if an excellent quotient $X \to Y$ exists, it is unique up to isomorphism. However, the cross section $Y \to X$ (see Remark 2.2(a)) is in general not unique.

**Theorem 2.6.** Let $\text{quo} : X \to Y$ be an excellent quotient by a group scheme $G$ with fibers $F$. Then it is a faithfully flat universal geometric quotient.

**Proof.** The cartesian diagram (2.4) (without the first two columns) and the argument after it show that $X \to Y$ is faithfully flat. Since by Remark 2.2(e) excellent quotients are stable under base change, we only need to show that $X \to Y$ is a geometric quotient. The conditions $(g_1)$ and $(g_4)$ are immediate, and $(g_3)$ follows since $X \to Y$ has a cross section by Remark 2.2(a).
It remains to prove the condition \((g_2)\). We will establish the surjectivity of \((\text{act}, \text{pr}_2): G \times X \rightarrow X \times_Y X\) by proving surjectivity on the geometric points with values in fields \(L\) (see Görtz and Wedhorn \[12, Proposition 4.8\]). Fixing \(L\) with a morphism \(\text{Spec}(L) \rightarrow S\), we have a functor from the category of \(S\)-schemes to the category of sets, which assigns to an \(S\)-scheme \(A\) the set \(\hat{A} := \text{Hom}_S(\text{Spec}(L), A)\), and to a morphism \(f: A \rightarrow B\) of \(S\)-schemes the map \(\hat{f}: \hat{A} \rightarrow \hat{B}, z \mapsto f \circ z\).

It is easy to see that \((\text{pr}_1, \text{pr}_2): \hat{A} \times \hat{B} \rightarrow \hat{A} \times \hat{B}\) is a bijection between the fiber product and the cartesian product, and, more generally, for an \(S\)-scheme \(C\) and morphisms \(f: A \rightarrow C, g: B \rightarrow C\) the map
\[(\text{pr}_1, \text{pr}_2): \hat{A} \times_C B \rightarrow \{(x, y) \in \hat{A} \times \hat{B} \mid \hat{f}(x) = \hat{g}(y)\} =: \hat{A} \times_C \hat{B}\]
is a bijection (see Grothendieck \[16, (3.4.3.2)\]). In particular, \(\hat{G}\) is a group acting on \(\hat{X}\). To prove the surjectivity, take an arbitrary morphism \(\text{Spec}(K) \rightarrow X \times_Y X\) with \(K\) a field. For a field extension \(L\) this yields morphisms \(\text{Spec}(L) \rightarrow X \times_Y X\), and, by composition \(\text{Spec}(L) \rightarrow S\).

By the above, we receive a pair \((x_1, x_2) \in \hat{X} \times \hat{X}\). The claimed surjectivity will follow if we can show that \((x_1, x_2)\) corresponds to a point in the image of \((\text{act}, \text{pr}_2): \hat{G} \times \hat{X} \rightarrow \hat{X} \times_Y \hat{X}\). Using the surjectivity of \((2.3)\) and Proposition 4.8 from \[12\], we can choose \(L\) large enough such that there exist \(g_1, g_2 \in \hat{G}\) and \(y_1, y_1 \in \hat{Y}\) such that \(x_i = g_i(\text{sect}(y_i))\). We have
\[y_i = \bigl(\text{quo}(\text{sect}(y_i)) \bigr) = \bigl(\text{quo}(g_i(\text{sect}(y_i))) \bigr) = \text{quo}(x_i),\]
which with \((x_1, x_2) \in \hat{X} \times \hat{X}\) implies \(y_1 = y_2\). Therefore \(x_1 = g_1 g_2^{-1}(x_2)\), so indeed \((x_1, x_2)\) corresponds to a point in the image of \((\text{act}, \text{pr}_2): \hat{G} \times \hat{X} \rightarrow \hat{X} \times_Y \hat{X}\).

It is hardly surprising that the converse of Theorem 2.6 does not hold. The following example illustrates this.

Example 2.7. Let the cyclic group \(G\) of order 2 act on the ring \(R = K[x, x^{-1}]\) of Laurent polynomials over an integral domain in which 2 is invertible by mapping \(x\) to \(-x\). Then \(R^G = K[x^2, x^{-2}]\), and \(X := \text{Spec}(R) \rightarrow \text{Spec}(R^G) =: Y\) is a faithfully flat universal geometric quotient with all fibers of \(K\)-points isomorphic to \(F := \text{Spec}(K[x]/(x^2 - 1))\). But \(X \rightarrow Y\) is not an excellent quotient since \(X\) is irreducible, but \(F \times Y\) is not.

In fact, for finite group actions the quotient usually has no cross section \(Y \rightarrow X\).

Remark 2.8. As mentioned in the introduction, the papers by Greuel and Pfister \[13\] and Sancho de Salas \[28\] both revolve around geometric quotients. In both papers, it appears that the following argument is used (see \[13, Proof of Proposition 1.6\] and \[28, last statement of Proposition 2.3\]): If \(G = \mathbb{G}_a\) (or a connected unipotent group) acts on a ring \(R\) and \(R\) is purely transcendental over \(R^G\), then \(\text{Spec}(R) \rightarrow \text{Spec}(R^G)\) is a geometric quotient. But this is not true in general: Consider the \(\mathbb{G}_a\)-action on the polynomial ring \(R = K[x_1, x_2]\) given by mapping \(x_1\) to \(x_1 + x_2 z\) and fixing \(x_2\). Then \(R^G = K[x_2]\), but the quotient is not geometric since fiber \(x_2 = 0\) consists of 0-dimensional orbits. So the proofs in \[13\] and \[28\] seem to have a gap. But the statements are correct, the missing link being provided by Theorem 1.6(c) of this paper.

The following lemma, which we will need later, deals with invariant fields and geometric quotients, but not with excellent quotients. This may be a good place to prove it. Although the lemma is almost certainly well-known, I could not find it in the literature.

Lemma 2.9. Let \(X \rightarrow Y\) be a geometric quotient by a group scheme \(G\), with \(X\) an integral scheme. Then \(K(X)^G = K(Y)\).

Proof. We view elements of the function field \(K(X)\) as morphisms \(f: U \rightarrow \mathbb{A}^1\), where \(U \subseteq X\) is the domain of definition of the rational function \(X \rightarrow \mathbb{A}^1\) represented by \(f\) (see Görtz and Wedhorn \[12, page 235\]). The elements of \(K(X)^G\) are those where \(U\) is \(G\)-stable and the diagram \((2.1)\) (with \(X'\) replaced by \(U\) and \(\text{quo} f'\) by \(\text{quo} |_U\)) commutes.

For an element of \(K(Y)\), given by \(g: V \rightarrow \mathbb{A}^1\), the property \((g_1)\) of the geometric quotient implies that the composition \(U := \text{quo}^{-1}(V) \rightarrow \mathbb{A}^1\) defines an element of \(K(X)^G\), and so we obtain an embedding \(K(Y) \subseteq K(X)^G\). To prove equality, take an element of \(K(X)^G\), given by
Let \( V = \text{quo}(U) \) be the image of \( U \) in \( Y \). Since \( U \) is \( G \)-stable, it follows from (\( g_2 \)) that \( \text{quo}^{-1}(V) = U \) (see the argument in the proof of remark (4) in Mumford et al. [22, page 6]). By (\( g_3 \)), \( V \) is open, and now (\( g_4 \)) implies that \( f \) lies in \( K(Y) \). 

To be able to deal with a solvable group by iterating over a chain of subgroups, we need that “putting together” excellent quotients along a subgroup chain yields an excellent quotient. This is the contents of the following result.

**Theorem 2.10.** Let \( G \) be a group scheme acting on a scheme \( X \) by a morphism \( \text{act}_X : G \times X \to X \). Let \( H \subseteq G \) be a normal subgroup scheme and let \( \text{quo}_1 : X \to Y \) be an excellent quotient by \( H \) with fibers \( F_1 \). Then there exists a unique \( G \)-action on \( Y \) such that the diagram

\[
\begin{array}{ccc}
G \times X & \overset{\text{act}_X}{\longrightarrow} & X \\
\downarrow \quad & \downarrow \quad & \downarrow \quad \\
G \times Y & \overset{\text{act}_Y}{\longrightarrow} & Y \\
\end{array}
\]

(2.5)

commutes. Suppose there is an excellent quotient \( \text{quo}_2 : Y \to Z \) by \( G \) with fibers \( F_2 \). Then \( \text{quo}_2 \circ \text{quo}_1 : X \to Z \) is an excellent quotient of \( X \) by \( G \) with fibers \( F_1 \times F_2 \).

**Proof.** By Theorem 2.6 \( \text{quo}_1 \) is a universal geometric quotient, so by Mumford et al. [22, Proposition 0.1] it is a universal categorical quotient. In particular, \( G \times X \to G \times Y \) is a categorical quotient by \( H \), with \( H \) acting trivially on \( G \). We leave it to the reader to check, using the normality of \( H \), that the diagram

\[
\begin{array}{ccc}
H \times G \times X & \overset{\text{act}_X}{\longrightarrow} & G \times X \\
\downarrow \quad & \downarrow \quad & \downarrow \quad \\
G \times Y & \overset{\text{act}_Y}{\longrightarrow} & Y \\
\end{array}
\]

(with \( \text{act}_H \) standing for the \( H \)-action on \( X \)) commutes. The universal property of \( G \times X \to G \times Y \) now yields a unique morphism \( \text{act}_Y : G \times Y \to Y \) such that (2.5) commutes. By a further diagram chase, one checks that \( \text{act}_Y \) defines an action.

To show that \( \text{quo}_2 \circ \text{quo}_1 \) is an excellent quotient, we first remark that \( F_1 \times F_2 \) is faithfully flat (see Görtz and Wedhorn [12, Remark 14.8]). From the given \( S \)-valued points \( \text{pt}_i : S \to F_i \) we form the composition \( \text{pt} : S \overset{\text{pt}_1 \times \text{pt}_2}{\longrightarrow} F_1 \times F_2 = S \times F_2 \overset{\text{id} \times \text{pt}_2}{\longrightarrow} F_1 \times F_2 \). With \( \text{iso}_1 : F_1 \times Y \to X \) and \( \text{iso}_2 : F_2 \times Z \to Y \) the given isomorphisms, the diagram

\[
\begin{array}{ccc}
F_1 \times F_2 \times Z & \overset{\text{id}_{F_1}, \text{iso}_2}{\longrightarrow} & F_1 \times Y \\
\downarrow \quad & \downarrow \quad & \downarrow \quad \\
F_1 \times Z & \overset{\text{iso}_1}{\longrightarrow} & X \\
\downarrow \quad & \downarrow \quad & \downarrow \quad \\
Z & \overset{\text{quo}_2}{\longrightarrow} & Y \\
\end{array}
\]

commutes, so its upper row defines a \( Z \)-isomorphism \( \text{iso} : F_1 \times F_2 \times Z \to X \). We go on by proving (i)–(iii) from Definition 2.1.

(i) This follows since the diagram
we will use Proposition 4.8 from Götz and Wedhorn \[2.2\]
(ivii) By Remark 2.2(a) the \(\text{quo}\) have cross sections \(\text{sec}_1 = \text{iso}_1 \circ (\text{pt}_1, \text{id}_Y): Y \rightarrow X\) and \(\text{sec}_2 = \text{iso}_2 \circ (\text{pt}_2, \text{id}_Z): Z \rightarrow Y\). The commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{(\text{pt}_1, \text{id})} & F_2 \times Z \\
& \searrow & \downarrow \text{iso}_2 \\
& & Y \\
\end{array}
\]

shows that \(\text{sect} := \text{iso} \circ (\text{pt}, \text{id}_Z)\) is the cross section of \(\text{quo} := \text{quo}_2 \circ \text{quo}_1\). We know that \(H \times Y \xrightarrow{(\text{id}, \text{sect}_1)} H \times X \xrightarrow{\text{act}_X} X\) and \(G \times Z \xrightarrow{(\text{id}, \text{sect}_2)} G \times X \xrightarrow{\text{act}_X} X\) are surjective, and will deduce that \(G \times Z \xrightarrow{(\text{id}, \text{sect})} G \times X \xrightarrow{\text{act}_X} X\) is surjective. As in the proof of Theorem 2.6 we will use Proposition 4.8 from Götz and Wedhorn \[12\], and write \(\hat{X} = \text{Hom}_S(\text{Spec}(L), X)\) for \(L\) a field, and so on. Let \(\text{Spec}(K) \rightarrow X\) be a \(K\)-geometric point (with \(K\) a field), and let \(x \in \hat{X}\) be the point obtained by composing with \(\text{Spec}(L) \rightarrow \text{Spec}(K)\) for a field extension \(L\). Let \(y := \tilde{\text{quo}_1}(x) \in \tilde{Y}\). Choosing \(L\) large enough, we obtain \(g \in \tilde{G}\) and \(z \in \tilde{Z}\) with \(g(\text{sect}_2(z)) = y\), and \(h \in \tilde{H}\) such that \(\tilde{h}(\tilde{\text{sect}}(y')) = x\). It follows that

\[
y = \tilde{\text{quo}_1}(x) = \tilde{\text{quo}_1}(\tilde{\text{sect}}(y')) = y'.
\]

Set \(x' := g(\tilde{\text{sect}}(z))\). Then

\[
\tilde{\text{quo}_1}(x') = g \left( \tilde{\text{quo}_1}(\tilde{\text{sect}}(\tilde{\text{sect}}(z))) \right) = g(\tilde{\text{sect}}(z)) = y.
\]

Enlarging \(L\) again we obtain \(\tilde{h} \in \tilde{H}\) and \(\tilde{y} \in \tilde{Y}\) such that \(x' = \tilde{h}(\tilde{\text{sect}}(\tilde{y}))\). It follows that

\[
\tilde{y} = \tilde{\text{quo}_1}(\tilde{\text{sect}}(\tilde{y})) = \tilde{\text{quo}_1}(x') = y,
\]

so \(\tilde{h}^{-1}(x') = \tilde{\text{sect}}(y)\), and we obtain

\[
(h\tilde{h}^{-1}g)(\text{sect}(z)) = h(\tilde{h}^{-1}(x')) = h(\tilde{\text{sect}}(y)) = x.
\]

This shows that \(G \times Z \xrightarrow{(\text{id}, \text{sect})} G \times X \xrightarrow{\text{act}_X} X\) is surjective, as claimed.

(iii) Let \(Z' \rightarrow Z\) be a morphism of schemes, and set \(Y' = Y \times_Z Z'\) and \(X' = X \times_Y Y'\). With \(\text{quo}'\) obtained by base change, the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\text{quo}'} & Y' \\
\downarrow & & \downarrow \text{quo}' \\
X & \xrightarrow{\text{quo}_1} & Y \\
\downarrow & & \downarrow \text{quo}_2 \\
Z & & Z
\end{array}
\]
is cartesian, so $X' = X \times_Z Z'$ and $\text{quo}' = \text{quo}'_2 \circ \text{quo}'_1$. By Remark 2.2(e), the $\text{quo}'_i$ are excellent quotients, and in order to prove (iii) for $\text{quo}'$ we may replace $X'$, $Y'$ and $Z'$ by the original $X$, $Y$ and $Z$. The assertion $\Gamma(X, \mathcal{O}_X)^G = \Gamma(Z, \mathcal{O}_Z)$ would be trivial if $G$ were a group. But since it is a group scheme we need to do more.

It follows from (i) that the map $\Gamma(Z, \mathcal{O}_Z) \to \Gamma(X, \mathcal{O}_X)$ has its image inside $\Gamma(X, \mathcal{O}_X)^G$. For the converse, take $f \in \Gamma(X, \mathcal{O}_X)^G = \Gamma(Y, \mathcal{A}^1)^G$, so the diagram (2.1) (with $X'$ replaced by $X$) commutes. By restricting the action to $H$ and since $X \xrightarrow{\text{quo}} Y$ is an excellent quotient by $H$ we obtain $g \in \text{Hom}_G(Y, \mathcal{A}^1)$ with $f = g \circ \text{quo}_1$. We claim that $g \in \Gamma(Y, \mathcal{O}_Y)^G$. The diagram

commutes. We need to show that $g \circ \text{act}_Y = g \circ \text{pr}_2$. Both functions are elements of $\Gamma(G \times Y, \mathcal{O}_{G \times Y})$, and the diagram shows that mapping them into $\Gamma(G \times X, \mathcal{O}_{G \times X})$ yields the same element. But since $G \times X \to G \times Y$ is faithfully flat, the map $\Gamma(G \times Y, \mathcal{O}_{G \times Y}) \to \Gamma(G \times X, \mathcal{O}_{G \times X})$ is injective (see the argument made after (2.4)). So $g \in \Gamma(Y, \mathcal{O}_Y)^G$, and since $\text{quo}_2 : Y \to Z$ is an excellent quotient by $G$ it follows that there is $h \in \text{Hom}_G(Z, \mathcal{A}^1)$ with $g = h \circ \text{quo}_2$. So $f = h \circ \text{quo}$, and the proof is complete. \hfill \Box

3. The additive group as a normal subgroup

If the additive group $\mathbb{G}_a$ acts on an affine scheme $\text{Spec}(R)$, we know from Theorem 1.6 that by choosing a local slice with denominator $c$ one obtains an excellent quotient $\text{Spec}(R_c) \to \text{Spec}(R_{c^a})$. Now we assume that $\mathbb{G}_a$ appears as a normal subgroup in a connected solvable group $G$, and wish to build an excellent quotient by $G$ by working upwards along a chain of normal subgroups with factor groups $\mathbb{G}_a$ and $\mathbb{G}_m$ (the multiplicative group, which will be dealt with in Section 5), and using Theorem 2.10 in each step. But this only works if $c$ is chosen in such a way that $G$ acts on $R_{c^a}$. This is the case if $c$ is a semi-invariant (see after Definition 4.1). A rather straightforward strategy for producing a local slice with a semi-invariant denominator, which would work in the case that the ground ring $K$ is a field and $R$ is a domain, is the following: One shows that the denominators of local slices form a $G$-stable $K$-subspace of $R$. Choose a nonzero finite-dimensional $G$-stable subspace. Inside this, the fixed space of the unipotent radical is nonzero, and it decomposes into a direct sum of spaces of semi-invariants of the torus sitting at the top of $G$. Picking a semi-invariant in that space yields the desired denominator of a local slice. Essentially, this is the approach taken by Sancho de Salas [28] for showing that there exists a geometric quotient.

For the following reasons we choose a different, more involved approach:

(1) $K$ may not be a field and $R$ may not be a domain.

(2) We wish to obtain a fast and simple algorithm that avoids the Gröbner basis computations and even the linear algebra that would be involved in putting the above strategy into practice.

Instead of assuming $G$ to be a connected solvable group right away, it is convenient to take $G$ as an affine group scheme with an embedding of $\mathbb{G}_a$ as a normal subgroup. Under rather mild assumptions, this implies that the map $G \to G/\mathbb{G}_a =: H$ splits, i.e., there is a morphism $\text{sect} : H \to G$ of schemes (not group schemes) such that $H \xrightarrow{\text{sect}} G \to H$ is the identity, and
this yields an isomorphism $G \cong \mathbb{G}_a \times H$ of schemes (see Rosenlicht [26, Corollary 1, page 100], Kambayashi et al. [20, Splitting Lemma, page 147]). This justifies making the existence of such a splitting into an assumption. More precisely, we work in the following setup:

$G$ and $H$ are affine group schemes over a ring $K$. There is a morphism $\text{emb} : \mathbb{G}_a \to G$ of group schemes (with $\mathbb{G}_a$ the additive group over $K$) and a morphism $\text{sect} : H \times \mathbb{G}_a \to G$ of schemes such that the composition $\text{iso} : \mathbb{G}_a \times H \xrightarrow{\text{emb} \times \text{sect}} G \times \mathbb{G}_a \xrightarrow{\text{mult}} G$ is an isomorphism of schemes. It is easy to see that we may assume that $\text{sect}$ takes the identity of $H$ to the identity of $G$. We make the normal subgroup assumption precise as follows: $H$ acts on $\mathbb{G}_a$ by automorphisms, with the action given by a morphism $\text{conj} : H \times \mathbb{G}_a \to \mathbb{G}_a$, such that the diagram

$$
\begin{array}{c}
H \times \mathbb{G}_a \\
\downarrow \text{(sect, emb)} \\
\mathbb{G}_a \times H \\
\downarrow \text{(emb, sect)} \\
G \\
\downarrow \text{mult} \\
G \times G \\
\downarrow \text{mult} \\
G \\
\end{array}
$$

commutes. If we write $H = \text{Spec}(A)$ then $\text{conj}$ induces a homomorphism of $K$-algebras $K[z] \to A[z]$, and it is easy to see that $z$ must be sent to $\chi \cdot z$ with $\chi \in A$ invertible. (Viewing $\chi$ as a morphism $H \to K^*_K$, it must be a character of $H$.)

Now let $X = \text{Spec}(R)$ be an affine $K$-scheme with a morphic action $\text{act} : G \times X \to X$. Then the action $\mathbb{G}_a \times X \xrightarrow{\text{emb} \times \text{id}} G \times X \xrightarrow{\text{act}} X$ induces a homomorphism $\varphi : R \to R[z]$, and the morphism $H \times X \xrightarrow{\text{sect} \times \text{id}} G \times X \xrightarrow{\text{act}} X$ which is not an action induces $\psi : R \to A \otimes R$ (where this and all other tensor products are over $K$). With $\mathbb{G}_a$ acting trivially on $H$, it also acts on $H \times X$. The homomorphism induced by this action is $\text{id}_A \otimes \varphi : A \otimes R \to (A \otimes R)[z]$.

As we will see, the following lemma contains everything that is needed to construct a simple and fast algorithm (Algorithm 4.2) for producing a local slice with a semi-invariant denominator, as discussed above.

**Lemma 3.1.** In the above situation, let $s \in R$ be nonzero and write $\varphi(s) = \sum_{i=0}^d c_i z^i$ with $c_i \in R$, $c_d \neq 0$.

(a) We have

$$
\text{id}_A \otimes \varphi(\psi(s)) = \sum_{i=0}^d \chi^i \psi(c_i) z^i. 
$$

Moreover, $\chi^d \psi(c_d) \neq 0$. In particular, $\deg(\psi(s)) = \deg(s)$, with the degree as defined in Section 1.

(b) If $s$ is a local slice, then so is $\psi(s)$.

**Proof.**

(a) The outer edges of the commutative diagram

$$
\begin{array}{c}
H \times \mathbb{G}_a \times X \\
\downarrow \text{(id}_H, \text{emb}, \text{id}_X) \\
\text{G}_a \times H \times X \\
\downarrow \text{(id}_G, \text{sect}, \text{id}_X) \\
\text{G}_a \times G \times X \\
\downarrow \text{(id}_G, \text{act}) \\
\text{G}_a \times X \\
\downarrow \text{act} \\
X \\
\end{array}
$$

being a local slice, then so is $\psi(s)$. 

(b) If $s$ is a local slice, then so is $\psi(s)$. 

**Proof.**

(a) The outer edges of the commutative diagram

$$
\begin{array}{c}
H \times \mathbb{G}_a \times X \\
\downarrow \text{(id}_H, \text{emb}, \text{id}_X) \\
\text{G}_a \times H \times X \\
\downarrow \text{(id}_G, \text{sect}, \text{id}_X) \\
\text{G}_a \times G \times X \\
\downarrow \text{(id}_G, \text{act}) \\
\text{G}_a \times X \\
\downarrow \text{act} \\
X \\
\end{array}
$$

being a local slice, then so is $\psi(s)$. 

(b) If $s$ is a local slice, then so is $\psi(s)$. 

**Proof.**

(a) The outer edges of the commutative diagram

$$
\begin{array}{c}
H \times \mathbb{G}_a \times X \\
\downarrow \text{(id}_H, \text{emb}, \text{id}_X) \\
\text{G}_a \times H \times X \\
\downarrow \text{(id}_G, \text{sect}, \text{id}_X) \\
\text{G}_a \times G \times X \\
\downarrow \text{(id}_G, \text{act}) \\
\text{G}_a \times X \\
\downarrow \text{act} \\
X \\
\end{array}
$$

being a local slice, then so is $\psi(s)$. 

(b) If $s$ is a local slice, then so is $\psi(s)$. 

**Proof.**
induce the commutative diagram

\[
\begin{array}{ccc}
R[z] & \xrightarrow{\psi \otimes \text{id}_{K[z]}} & A \otimes R \\
\downarrow & & \downarrow \text{id}_{A \otimes \varphi} \\
(A \otimes R)[z] & \xrightarrow{\chi \varphi} & (A \otimes R)[z]
\end{array}
\]

From this (3.1) follows directly.

If \( \varepsilon : A \to K \) is induced by the identity \( \text{Spec}(K) \to H \) of \( H \), then \( (\varepsilon \otimes \text{id}_R) \circ \psi = \text{id}_R \) and \( \varepsilon(\chi) = 1 \). So \( (\varepsilon \otimes \text{id}_R)(\chi^d \psi(c_d)) = c_d \neq 0 \), and \( \chi^d \psi(c_d) \neq 0 \) follows.

(b) By (a) the highest coefficient of \((\text{id}_A \otimes \varphi)(\psi(s))\) is \( \chi^d \psi(c_d) \). So we need to show that if \( a \in A \otimes R \) has deg(a) < d, then there exists \( k \) such that \( \chi^{kd} \psi(c_d)^k a \in A \otimes R^{\mathbb{G}_a} \). Consider the commutative diagram

\[
\begin{array}{cccccc}
H \times X & \xrightarrow{(\text{diag}, \text{id}_X)} & H \times X & \xrightarrow{(\text{id}_H, \text{sect}, \text{id}_X)} & H \times X & \xrightarrow{(\text{id}_H, \text{act})} & H \times X \\
pr_2 & & & & & & \\
X & \xrightarrow{\text{act}} & G \times X & \xrightarrow{(\text{mult}, \text{id}_X)} & G \times X & \xrightarrow{(\text{inv}, \text{id}_X)} & G \times X \\
\xrightarrow{G \times X} & \xrightarrow{G \times G \times X} & \xrightarrow{G \times G \times X} & \xrightarrow{(\text{emb}, \text{sect}, \text{id}_X)} & \xrightarrow{G \times \times G \times X} & \xrightarrow{H \times \times H \times X} \\
\xrightarrow{\mathbb{G}_a \times X} & \xrightarrow{\mathbb{G}_a \times G \times X} & \xrightarrow{\mathbb{G}_a \times G \times X} & \xrightarrow{G \times \times G \times X} & \xrightarrow{H \times \times H \times X}
\end{array}
\]

in which \( \text{inv} : G \to G \) is the inversion. This induces the commutative diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\varphi} & R[z] \\
\downarrow & & \downarrow \psi \otimes \text{id}_{K[z]} \\
A \otimes R & \xrightarrow{\mu \otimes \text{id}_R} & A \otimes A \otimes R \\
\downarrow \text{id}_A \otimes \varphi & & \downarrow \text{id}_A \otimes \psi \\
A \otimes R & \xrightarrow{\eta \otimes \text{id}_R} & A \otimes R
\end{array}
\]

in which \( \mu : A \otimes A \to A \) is given by multiplication and \( \eta : A[z] \to A \) is the homomorphism induced by the composition \( H \xrightarrow{\text{sect}} G \xrightarrow{\text{inv}} G \xrightarrow{\text{iso}^{-1}} \mathbb{G}_a \times X \). First let \( a \in R \) with deg(a) < d. By (1.2), all coefficients of \( \varphi(a) \) have degree < d, so by (a) the same is true for all coefficients of \( (\psi \otimes \text{id}_{K[z]})(\varphi(a)) \). It follows that

\[ b := (\eta \otimes \text{id}_R)((\psi \otimes \text{id}_{K[z]})(\varphi(a))) \in A \otimes R \]

has degree < d. Since \( s \) is a local slice, this implies that there is a nonnegative integer \( k \) such that \( (1 \otimes c_d)^k b \in A \otimes R^{\mathbb{G}_a} \). The diagram implies \( (\mu \otimes \text{id}_R)((\text{id}_A \otimes \psi)(b)) = 1 \otimes a \).

Moreover,

\[ (\mu \otimes \text{id}_R)((\text{id}_A \otimes \psi)(1 \otimes c_d)) = (\mu \otimes \text{id}_R)(1 \otimes \psi(c_d)) = \psi(c_d), \]

and we obtain

\[ \psi(c_d)^k(1 \otimes a) = (\mu \otimes \text{id}_R)((\text{id}_A \otimes \psi)((1 \otimes c_d)^k b)). \]

By (a), applying \( \psi \) to an element of \( R^{\mathbb{G}_a} \) yields an element of \( A \otimes R^{\mathbb{G}_a} \), so \( (\text{id}_A \otimes \psi)((1 \otimes c_d)^k b) \) is mapped into \( A \otimes R^{\mathbb{G}_a} \) by \( \mu \otimes \text{id}_R \). This shows that \( \psi(c_d)^k(1 \otimes a) \in A \otimes R^{\mathbb{G}_a} \), so also \( \chi^{kd} \psi(c_d)^k(1 \otimes a) \in A \otimes R^{\mathbb{G}_a} \).

Now let \( \bar{a} \in A \otimes R \) with deg(\( \bar{a} \)) < d. Write \( \bar{a} \) as a finite sum \( \bar{a} = \sum_i a_i \otimes r_i \) with \( a_i \in A \) and \( r_i \in R \) such that deg(\( r_i \)) < d. But by the above, there exists \( k \) such that \( \chi^{kd} \psi(c_d)^k(1 \otimes r_i) \in A \otimes R^{\mathbb{G}_a} \), from which the claim follows.
4. Unipotent group actions

In this section we give an algorithm, built on the previous section, that produces a local slice for an action of an additive group that appears as a normal subgroup of a connected solvable group, such that the denominator of the local slice is a semi-invariant. From this, we construct an algorithm for computing invariants of a unipotent group, which (for later purposes) is also assumed to be contained in a connected solvable group.

If $G$ is a connected solvable linear algebraic group over an algebraically closed field $K$, a lot is known about its structure (see Humphreys [19, Section 19]): The factor group $G/U$ by the unipotent radical is a torus, and $U$ has a chain of subgroups, normal in $G$, such that all factor groups are isomorphic to $G_a$. Moreover, as a variety, $U$ is isomorphic to $A_n^1$ ($n = \dim(U)$), with an isomorphism that is consistent with the subgroup chain just mentioned (see Rosenlicht [26, Corollary 2, page 101]). This justifies making this structure into an assumption for a group scheme in our more general setting, even though such examples as the group $SO_2(K)$ over a field $K$ in which $-1$ is not a square do not meet this assumption. In fact, for purposes of stating algorithms, we assume that the group scheme is given in a way that reflects the above structure. This is a mild assumption since in practice a connected solvable group will almost always be given in such a way, for example if it is defined as a closed subgroup of the group of invertible upper triangular matrices.

Definition 4.1. A group scheme $G$ over a ring $K$ is said to be in standard solvable form if $G = \text{Spec}(K[z_1, \ldots, z_l, t_1, \ldots, t_m], t_1^{-1}, \ldots, t_m^{-1})$ with $l$ and $m$ nonnegative integers such that:

1. The closed subscheme $G_i \subseteq G$ given by the ideal $(z_{i+1}, \ldots, z_l, t_1 - 1, \ldots, t_m - 1)$ $(i = 0, \ldots, l)$ is a normal subgroup. We write $U = G_i$.
2. The morphism $G_i \to G_a$ given by $K[z] \to K[G_i] = K[z_1, \ldots, z_l]$, $z \mapsto z_i$ is a morphism of group schemes.
3. With $T := \text{Spec}(K[t_1^{\pm 1}, \ldots, t_m^{\pm 1}])$ the $m$-dimensional torus, the morphism $G \to T$ given by $t_j \mapsto t_j$ is a morphism of group schemes.

If $G$ is in standard solvable form, the torus $T$ acts on each $G_i/G_{i-1} \cong G_a$ by conjugation. The actions are given by characters $\chi_i$ $(i = 1, \ldots, l)$, which are power products of the $t_j^{\pm 1}$.

It is intuitively clear that with $G_a \sim G_1 \to G$ and $H = \text{Spec}(K[z_2, \ldots, z_l, t_1^{-1}, \ldots, t_m^{\pm 1}])$ the hypotheses of Lemma 3.1 are satisfied. The formal verification of this is a bit tedious and left to the reader.

If a group scheme $G$ in standard solvable form acts on an affine scheme $\text{Spec}(R)$ with the action given by a homomorphism $\Phi: R \to R[z_1, \ldots, z_l, t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$, then an element $c \in R$ is called a semi-invariant of weight $\chi$ if $\Phi(c) = \chi \cdot c$, where $\chi = \prod_{j=1}^m t_j^{e_j}$ with $e_j$ integers. In this case the action extends to $\text{Spec}(R_c)$ by $\Phi(\frac{a}{b}) := \chi^{-k} \cdot \frac{\Phi(a)}{b}$ for $a \in R$.

We now come to the algorithm for producing a local slice whose denominator is a semi-invariant. As in Algorithm 1.5 we assume that it is possible to perform addition, multiplication, and zero testing of elements of $R$. Notice that the algorithm does not require any Gröbner basis computations and not even linear algebra (unless the underlying computations in $R$ require Gröbner bases).

Algorithm 4.2 (Computation of a local slice with semi-invariant denominator).

Input: A group scheme $G$ in standard solvable form acting on an affine scheme $\text{Spec}(R)$ with $R = K[a_1, \ldots, a_n]$ a finitely generated algebra, where the action given is by a homomorphism $\Phi: R \to R[z_1, \ldots, z_l, t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$. Assume that the characters $\chi_i \in K[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ as in Definition 4.1 are given, and that the subgroup $G_1 \cong G_a$ acts nontrivially.

Output: A local slice $s \in R$ of degree $d$ with denominator $c$ for the action of $G_1$ such that $c$ is a semi-invariant. Moreover, a homomorphism $\pi: R_c \to R_{c_1}^{G_1}$ of $R_{c_1}^{G_1}$-algebras, given by the $\pi(a_i)$, with $\ker(\pi) = (s)$. 

(1) For $i = 1, \ldots, l$, let $\varphi_i; R \to R[z_i]$ be the homomorphism obtained by composing $\Phi$ with the map fixing $z_i$, and sending the other $z_j$ to 0 and the $t_j$ to 1. Apply Algorithm 1.5 to $\varphi_1$. Let $s \in R$ be the resulting local slice of degree $d$ with denominator $c$.

(2) For $i = 2, \ldots, l$ repeat step 3.

(3) With $k$ be the degree of $\varphi_1(c)$, redefine $s$ to be the coefficient of $z^k_i$ in $\varphi_1(s)$, and $c$ to be the coefficient of $z^k_i$ in $\varphi_1(c)$. Now $s$ is a local slice of degree $d$ with denominator $c$, and $c \in R^{G_i}$.

(4) Compute $\Phi(c) \in R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ and choose a monomial $t^*$ occurring in this Laurent polynomial. If $d > 1$ and $R$ is not a domain, $t^*$ has to be chosen as the leading monomial of $\Phi(c)$ with respect to an arbitrary monomial ordering. Redefine $s$ to be the coefficient of $\chi^d_i \cdot t^*$ in $\Phi(s)$, and $c$ to be the coefficient of $t^*$ in $\Phi(c)$. Now $s$ is a local slice of degree $d$ with denominator $c$, and $c$ is a semi-invariant with weight $t^*$.

(5) With $g := \varphi_1(s)$ and $f_i := \varphi_1(a_i)$, obtain $r_i \in R_c$ by division with remainder:

\[ f_i = q_i g + r_i \]

with $q_i \in R_c[z_i]$. Then $r_i \in R^G_c$ and $\pi(a_i) = r_i$.

Remark 4.3. The requirement that $R$ be finitely generated is only used for producing a local slice by Algorithm 1.5. Since local slices always exist by Remark 1.3, the algorithm proves the existence of a local slice with semi-invariant denominator also when $R$ is not finitely generated. °

Proof of correctness of Algorithm 4.2. After step 1, $s$ is a local slice of degree $d$ with denominator $c \in R^{G_i}$. To prove the correctness of step 3, we assume, using induction on $i \geq 2$, that $s$ is a local slice of degree $d$ with denominator $c \in R^{G_i-1}$. The factor group $G_i/G_{i-1} \cong G_q$ acts on $R^{G_i-1}$ by (the restriction of) $\varphi_i$. So it follows by (1.2) that the highest coefficient $c'$ of $\varphi_i(c)$, which is the “new” $c$, lies in $R^{G_i}$. We apply Lemma 3.1 to the action of $G_i$ on $\text{Spec}(R)$. The algebra $A$ from the lemma is $A = K[z_2, \ldots, z_l]$, and we have to consider that map $\psi_1; R \to R[z_2, \ldots, z_l]$ obtained by composing with the map fixing $z_2, \ldots, z_l$ and sending $z_1, z_{i+1}, \ldots, z_l$ to 0 and all the $t_j$ to 1. The lemma tells us that $\psi_1(s)$ is a local slice of degree $d$ with denominator $\psi_1(c)$.

Since $c \in R^{G_i-1}$, we have $\psi_1(c) = \varphi_1(c)$. By Lemma 3.1, the $z^d_i$-coefficient of $\varphi_1(\psi_1(s))$ (with $\varphi_1$ applied coefficient-wise to $\psi_1(s) \in R[z_2, \ldots, z_l]$) is $\varphi_1(\psi_1(s))_d = \psi_1(c)$, so for the coefficient $c'$ of the monomial $z^k_i$ in $\psi_1(s)$ we have $\varphi_1(s)_d = c'$.

Taking the coefficient of $z^k_i$ in $\psi_1(s)$ is the same as taking the coefficient of $z^k_i$ in $\varphi_1(s)$, so $s'$ is the “new” $s$. To show that $s'$ is a local slice, let $a \in R$ with $\deg(a) < d$. Since $\varphi_1(c)$ is the denominator of the local slice $\psi_1(s)$, multiplying $a$ by a high enough power of $\varphi_1(c)$ sends it into $R^{G_i} [z]$, so multiplying it by a high enough power of $c'$ sends it into $R^{G_i}$.

So when the algorithm reaches step 4, $s$ is a local slice with denominator $c \in R^U$ with $U = G_l$. The factor group $G/U \cong T$ (the $m$-dimensional torus) acts on $R^U$ with the action given by $\Phi$. So $\Phi(c) \in R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$, and by Lemma 4.4, which we prove below, the coefficient $c'$ of any monomial $t^*$ is a semi-invariant with weight $t^*$. We apply Lemma 3.1, so in this case $\psi; R \to R[z_2, \ldots, z_l, t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ is the composition of $\Phi$ with sending $z_1$ to 0. As above, we obtain $\varphi_1(\psi(s))_d = \chi^d_i \psi(c)$.

This is an equality of (Laurent-)polynomials in $R[z_2, \ldots, z_l, t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$, on which $\varphi_1$ is applied coefficient-wise, so comparing the coefficients of $\chi^d_i \cdot t^*$ shows that for the coefficient $c'$ of $\chi^d_i \cdot t^*$ in $\psi(s)$ we have $\varphi_1(s)_d = c'$.

But $s'$ is also the coefficient of $\chi^d_i \cdot t^*$ in $\Phi(s)$, which is the “new” $s$. Since $c'$ is the “new” $c$, we are done if we can show that $s'$ is a local slice of degree $d$. By Lemma 3.1, the degree of $\psi(s)$ is $d$, so $\deg(s') \leq d$. But since $c' \neq 0$, the above equation shows that $\deg(s') = d$. If $d = 1$ or if $R$ is a domain, then all elements of $R$ of degree $d$ are local slices (see Remark 1.3) and we are done. If $d > 1$ and $R$ is not a domain (and so $t^*$ is the leading monomial of $\Phi(c)$), then the proof uses Lemma 3.1(b) and works as above.
The correctness of step 5 follows directly from Theorem 1.6(b).

The following lemma, which is surely folklore, was used in the above proof and will be used later, too.

**Lemma 4.4.** Let $T = \text{Spec}(K[t_{i}^{\pm 1}, \ldots, t_{m}^{\pm 1}])$ be an $m$-dimensional torus over a ring $K$, acting on an affine $K$-scheme $X = \text{Spec}(R)$ by a morphism $\text{act}: T \times X \to X$. With $\Phi: R \to R[t_{i}^{\pm 1}, \ldots, t_{m}^{\pm 1}]$ the induced homomorphism, let $a \in R$ and, for a power product $t$ of the $t_{i}^{\pm 1}$, let $a_{t} \in R$ be the coefficient of $t$ in $\Phi(a)$. Then $a_{t}$ is a semi-invariant of weight $t$.

**Proof.** The commutative diagram

$$
\begin{array}{ccc}
T \times T \times X & \xrightarrow{(\text{id}, \text{act})} & T \times X \\
\downarrow{\text{mult}} \downarrow{\text{id}, \text{act}} & & \downarrow{\text{act}} \\
T \times X & \xrightarrow{\text{act}} & X
\end{array}
$$

induces the commutative diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\Phi} & R[t_{i}^{\pm 1}] \\
\Phi_{*} & & \Phi_{*} \\
R[t_{i}^{\pm 1}] & \xrightarrow{\text{id}, \Phi_{*}} & R[t_{i}^{\pm 1}, t_{j}^{\pm 1}] \\
& & (t_{j} \mapsto s_{j} \cdot t_{j})
\end{array}
$$

in which $s_{1}, \ldots, s_{m}$ are new indeterminates and $\Phi_{*}$ is the composition $R \xrightarrow{\Phi} R[t_{i}^{\pm 1}] \xrightarrow{(t_{j} \mapsto s_{j})} R[t_{i}^{\pm 1}, t_{j}^{\pm 1}]$. So if $\Phi(a) = \sum e_{1}, \ldots, e_{m} \in \mathbb{Z} a_{e_{1}, \ldots, e_{m}} t_{1}^{e_{1}} \cdots t_{m}^{e_{m}}$, then

$$
\sum e_{1}, \ldots, e_{m} \in \mathbb{Z} \Phi(a_{e_{1}, \ldots, e_{m}}) t_{1}^{e_{1}} \cdots s_{m}^{e_{m}} = \sum e_{1}, \ldots, e_{m} \in \mathbb{Z} a_{e_{1}, \ldots, e_{m}} (s_{1} t_{1})^{e_{1}} \cdots (s_{m} t_{m})^{e_{m}}.
$$

For every $(e_{1}, \ldots, e_{n}) \in \mathbb{Z}^{m}$ the yields $\Phi(a_{e_{1}, \ldots, e_{m}}) = a_{e_{1}, \ldots, e_{m}} t_{1}^{e_{1}} \cdots t_{m}^{e_{m}}$, which was our claim. □

We can now apply Algorithm 4.2 iteratively along a chain of subgroups and obtain an algorithm for computing $R_{c}^{U}$ with $U$ a unipotent group. The algorithm requires that addition and multiplication of elements of $R$ are possible, and that for every $c \in R$, zero testing in $R_{c}$ is possible. Recall that for a group scheme $G$ in standard solvable form, $U = G_{l}$ stands for its unipotent radical, with the special case $G = U$ possible.

**Algorithm 4.5 (Unipotent group invariants).**

**Input:** A group scheme $G$ in standard solvable form acting on an affine scheme $\text{Spec}(R)$ with $R = K[a_{1}, \ldots, a_{n}]$ a finitely generated algebra, with the action given by a homomorphism $\Phi: R \to R[z_{1}, \ldots, z_{i}, t_{i}^{\pm 1}, \ldots, z_{m}^{\pm 1}]$. Assume that the characters $\chi_{i} \in K[t_{i}^{\pm 1}, \ldots, z_{m}^{\pm 1}]$ as in Definition 4.1 are given.

**Output:**

- A semi-invariant $c \in R$, nonzero if $R \neq \{0\}$.
- A homomorphism $\pi: R_{c} \to R_{c}^{U}$ of $R_{c}^{U}$-algebras given by the $b_{i} := \pi(a_{i})$, so $R_{c}^{U} = K[c^{-1}, b_{1}, \ldots, b_{n}]$.
- Elements $s_{1}, \ldots, s_{k} \in R_{c}$ such that the map $R_{c}^{U}[x_{1}, \ldots, x_{k}] \xrightarrow{x_{i} \mapsto s_{i}} R_{c}$, is an isomorphism and $\pi$ is equal to the composition $R_{c} \xrightarrow{\sim} R_{c}^{U} \xrightarrow{x_{i} \mapsto s_{i}} R_{c}$.
- In particular, $\text{ker}(\pi) = (s_{1}, \ldots, s_{k})$.

1. Set $c := 1$, $k := 0$, $b_{j} := a_{j}$ ($j = 1, \ldots, n$). For $i = 1, \ldots, l$ repeat steps 2–4.
2. If none of the $\Phi(b_{j})$ involves $z_{i}$, skip steps 3–4 and proceed with the next $i$.
3. Apply Algorithm 4.2 to $\tilde{G} := \text{Spec}(K[z_{i}, \ldots, z_{i}, t_{1}^{\pm 1}, \ldots, z_{m}^{\pm 1}])$ and $\tilde{R} := K[c^{-1}, b_{1}, \ldots, b_{n}] \subseteq R_{c}$, with $\Phi$ extended to $R_{c}$. Let $\tilde{s}$ be the resulting local slice of degree $d$ with denominator $\tilde{c}$ and $\tilde{\pi}: \tilde{R} \to \tilde{R}_{c}^{\tilde{G}}$ the resulting homomorphism.
(4) Choose a semi-invariant \( c' \in R \) such that \((R_c)_z = R_{c'}\). This can be done by choosing \( c' \) to be a numerator of \( c \) and then multiplying it by a high enough power of \( c \) such that it becomes a semi-invariant and an \( R \)-multiple of \( c \). Redefine \( c \) to be \( c' \), \( k \) to be \( k+1 \), \( b_j \) to be \( \tilde{\pi}(b_j) \) \((j = 1, \ldots, n)\), and set \( s_k := \tilde{s} \).

**Proof of correctness of Algorithm 4.5.** By induction on \( i \) assume that at the beginning of step 2 the elements \( c, b_j, \) and \( s_j \) are as claimed for the output of the algorithm, but with \( U \) replaced by \( G_{i-1} \). Also assume that \( \Phi(b_j) \subseteq \mathcal{O}^{G_{i-1}}_{i-1}[z_1, \ldots, z_l, t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \). We will show that, after step 4, the same holds with \( i \) replaced by \( i+1 \).

If none of the \( \Phi(b_j) \) involves \( z_i \), then \( \mathcal{O}^{G_{i}}_{i-1} = K[c^{-1}, b_1, \ldots, b_n] = \mathcal{O}^{G_i}_{i-1} \), so step 2 is correct. After step 3, the map \( \mathcal{O}^{G_i}_{i}[x_{k+1}] \to \mathcal{O}^{G_i}_{i}[x_{k+1}] \) is the composition \( \pi \circ \tilde{\pi} \) of \( \pi \) equals the composition \( \mathcal{O}^{G_i}_{i}[x_{k+1}] \circ \mathcal{O}^{G_i}_{i}[x_{k+1}] \) \( \mathcal{O}^{G_i}_{i}[x_{k+1}] \) by Theorem 1.6(b). Since \( \mathcal{O}^{G_i}_{i}[x_{k+1}] \) is an isomorphism, the composition \( \mathcal{O}^{G_i}_{i}[x_1, \ldots, x_k] \) is the same as applying \( \Phi \). So indeed \( \Phi(b_j) \) is a semi-invariant of weight \( e \).

The commutative diagram

\[ R_{c'} \xrightarrow{\sim} R_{c'}^{G_{i-1}}[x_1, \ldots, x_k] \xrightarrow{\sim} R_{c'}^{G_{i-1}}[x_1, \ldots, x_{k+1}] \]

shows that the “new” map \( \pi \) satisfies what is claimed for the output of the algorithm. It remains to show that the “new” \( b_j \) satisfy \( \Phi(b_j) \subseteq \mathcal{O}^{G_{i}}_{i-1}[z_1, \ldots, z_l, t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \). The \( b_j \) lie in \( \mathcal{O}^{G_{i}}_{i-1} \), so it suffices to prove the statement for any \( a \in \mathcal{O}^{G_{i}}_{i-1} \). Since \( \mathcal{O}^{G_{i}}_{i-1} \subseteq \mathcal{O}^{G_i}_{i-1} \) we have, by induction, \( \Phi(a) \in \mathcal{O}^{G_{i}}_{i-1}[z_1, \ldots, z_l, t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \). Apply Lemma 3.1 to the action of \( G/G_{i-1} \) on \( \mathcal{O}^{G_{i}}_{i-1} \), with normal subgroup \( G_i/G_{i-1} \cong \mathbb{A}_n \). The lemma says that applying \( \psi \) to a \( \mathbb{A}_n \)-invariant yields another \( \mathbb{A}_n \)-invariant, with \( \psi \) formed by applying \( \Phi \) followed by sending \( z_i \) to 0. So for elements of \( \mathcal{O}^{G_i}_{i} \), apply \( \psi \) is the same as applying \( \Phi \). So indeed \( \Phi(a) \in \mathcal{O}^{G_i}_{i}[z_1, \ldots, z_l, t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \). \( \square \)

5. **Multiplicative group and torus actions**

This section deals with actions of the multiplicative group. We define a notion of a local slice and show that its behavior parallels that of a local slice for an additive group action. Together with the results from the previous section, this leads to an algorithm for solvable group actions.

With \( t \) an indeterminate, let \( \mathbb{G}_m := \text{Spec}(K[t^{\pm 1}]) \) be the multiplicative group over a ring \( K \), acting morphically on an affine \( K \)-scheme \( \text{Spec}(R) \). The action is given by a homomorphism \( \varphi: R \to R[t^{\pm 1}] \) of \( K \)-algebras. If \( c \in R \) is a semi-invariant of weight \( \chi = t^k \) (with \( k \) an integer) then the map \( \varphi: R_c \to R_c[t^{\pm 1}] \), \( a/c^d \mapsto \chi^{-d} \varphi(a)/c^d \), which will also be written as \( \varphi \), defines a \( \mathbb{G}_m \)-action on \( \text{Spec}(R_c) \). For \( a \in R \) we write

\[ \deg(a) := \max\{|k| \mid t^k \text{ occurs in } \varphi(a)\}, \]

with \( \deg(0) := 0 \). So the invariant ring \( R_c^{\mathbb{G}_m} \) consists of the elements of degree 0. We now define the notion of a local slice for the multiplicative group, which plays a very similar role as a local slice for the additive group.

**Definition 5.1.** In the above situation, let \( 0 \neq c \in R \) be a semi-invariant. An element \( s \in R_c \) is called a local slice of degree \( d > 0 \) with denominator \( c \) if

(i) \( s \) is a semi-invariant of weight \( t^{-d} \),

(ii) \( s \) is invertible \((in R_c)\), and

(iii) every \( a \in R_c \) with \( \deg(a) < d \) lies in \( R_c^{\mathbb{G}_m} \).
Proposition 5.2. In the above situation, if the action is nontrivial, a local slice exists.

Proof. If there exists a nonzero nilpotent semi-invariant $c \in \mathcal{R}$, then $0 \neq 1 \in \mathcal{R}$ is a local slice of weight, say, $t$. So we may assume that no nonzero semi-invariant is nilpotent. Choose a semi-invariant $c \in \mathcal{R}$ of positive degree and a semi-invariant $s \in \mathcal{R}_c$ of \textit{minimal} positive degree $d$. We can choose a numerator $s' \in \mathcal{R}$ of $s$ that is a semi-invariant (see step 4 in Algorithm 4.5). Replacing $c$ by $cs'$ we may assume that $s$ is invertible, and replacing, if necessary, $s$ by $s^{-1}$, we may assume that $s$ has weight $t^{-d}$. Let $a \in \mathcal{R}_c$ be an element of degree $< d$. By Lemma 4.4, the coefficient of a monomial $t^k$ in $\varphi(a)$ occurring is a semi-invariant of weight $t^k$. Since $|k| \leq \deg(a) < d$ the minimality of $d$ implies $k = 0$, so $a \in \mathcal{R}_c^{G_m}$.

We will present an algorithm for producing a local slice, which actually does more: It produces a local slice that is a semi-invariant with respect to a torus containing the multiplicative group. Since we are still dealing with a single copy of $\mathbb{G}_m$, we postpone presenting the algorithm and first prove a theorem which shows the usefulness of local slices and which parallels Theorem 1.6 and so implies that the morphism $\text{Spec}(\mathcal{R}_c) \to \text{Spec}(\mathcal{R}_c^{G_m})$ is an excellent quotient with fibers $\text{Spec}(K[x^{\pm 1}])$.

Theorem 5.3. For a nontrivial action of the multiplicative group $\mathbb{G}_m$ over a ring $K$ on an affine $K$-scheme $\text{Spec}(\mathcal{R})$, given by a homomorphism $\varphi: \mathcal{R} \to \mathcal{R}[t^{\pm 1}]$, let $s$ be a local slice of degree $d$ with denominator $c$.

(a) The homomorphism $(\mathcal{R}_c)^{G_m}[y^{\pm 1}] \to \mathcal{R}_c$, sending the indeterminate $y$ to $s$, is an isomorphism. We write $\psi: \mathcal{R}_c \to (\mathcal{R}_c)^{G_m}[y^{\pm 1}]$ for the inverse isomorphism.

(b) The composition

$$\pi: \mathcal{R}_c \xrightarrow{\psi} (\mathcal{R}_c)^{G_m}[y^{\pm 1}] \xrightarrow{y^{-1}} (\mathcal{R}_c)^{G_m}$$

is a homomorphism of $(\mathcal{R}_c)^{G_m}$-algebras with $\ker(\pi) = (s-1)$. In particular, $\pi$ is surjective. For $a \in \mathcal{R}_c$, $\pi(a)$ is given by substituting $t = \sqrt[d]{s}$ in $\varphi(a)$, which makes sense because $\varphi(a) \in \mathcal{R}[t^{\pm d}]$.

(c) The composition

$$\mathcal{R}_c \xrightarrow{\varphi} \mathcal{R}_c[t^{\pm 1}] \xrightarrow{\pi} (\mathcal{R}_c)^{G_m}[t^{\pm 1}]$$

(with $\pi$ applied coefficient-wise) is injective and makes $(\mathcal{R}_c)^{G_m}[t^{\pm 1}]$ into an $\mathcal{R}_c$-module that is generated by $d$ elements.

(d) Let $B$ be a ring with a homomorphism $(\mathcal{R}_c)^{G_m} \to B$. Then

$$(B \otimes (\mathcal{R}_c)^{G_m} \mathcal{R})^{G_m} = B \otimes 1.$$

Proof. (a) Let $f \in (\mathcal{R}_c)^{G_m}[y^{\pm 1}]$ be a Laurent polynomial with $f(s) = 0$. Then

$$0 = \varphi(f(s)) = f(\varphi(s)) = f(st^{-d}).$$

Since $s$ is invertible, this implies $f = 0$. This proves injectivity. For surjectivity, let $a \in \mathcal{R}_c$ by a semi-invariant of weight $t^k$ with $k \in \mathbb{Z}$. Obtain $k = qd + r$ with $q, r \in \mathbb{Z}$, $0 \leq r < d$ by division with remainder. It follows that $s^qa$ is a semi-invariant of degree $r$ and therefore an invariant, so $a = s^qa \cdot s^{-q} \in (\mathcal{R}_c)^{G_m}[s^{\pm 1}]$. We also obtain $r = 0$, so $k$ is divisible by $d$. For $a \in \mathcal{R}_c$ arbitrary write $\varphi(a) = \sum_k a_k t^k$. By Lemma 4.4, every $a_k$ is a semi-invariant and therefore lies in $(\mathcal{R}_c)^{G_m}[s^{\pm 1}]$, so the same is true for $a = \sum_k a_k$. This shows surjectivity.

(b) The first claim is clear. Regarding the second claim, we have shown above that $\varphi(a) \in \mathcal{R}[t^{\pm d}]$ for $a \in \mathcal{R}_c$. For showing that the map that is claimed to be equal to $\pi$ really is $\pi$, it suffices to check this for $s$, which is straightforward.

(c) Let $a \in \mathcal{R}_c$. By (a) we have $a = f(s)$ with $f \in (\mathcal{R}_c)^{G_m}[y^{\pm 1}]$. So

$$\pi(\varphi(a)) = \pi(f(\varphi(s))) = \pi(f(st^{-d})) = f(\pi(s)t^{-d}) = f(t^{-d}),$$

from which (c) follows.
(d) By (a), we have \( R' := B \otimes_{(R, \varphi \circ \varphi)} R = B[(1 \otimes s)^{\pm 1}] \). By definition, \((R')^G = \ker(\varphi' - \text{id}) \) with \( \varphi': R' \to R'[t^{\pm 1}] \) obtained by tensoring \( \varphi \). Let \( a \in R' \) and write \( a = \sum_{i \in \mathbb{Z}} b_i(1 \otimes s)^i \) with \( b_i \in B \). Then
\[
\varphi'(a) - a = \sum_i b_i((1 \otimes t^{-d})^i - (1 \otimes s)^i) = \sum_i b_i(1 \otimes s)^i(t^{-id} - 1),
\]
which is zero if and only if \( b_i = 0 \) for \( i \neq 0 \), i.e., \( a \in B \). \( \square \)

Now we come to the announced algorithm for finding a local slice that is a semi-invariant with respect to an ambient torus.

Algorithm 5.4 (A local slice for the multiplicative group with a semi-invariant denominator).

**Input:** A torus \( T = \text{Spec}(K[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]) \) over a ring \( K \) acting on an affine scheme \( \text{Spec}(R) \) with \( R = K[a_1, \ldots, a_n] \), with the action given by \( \Phi: R \to R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \). Assume that the subgroup \( T_1 \cong \mathbb{G}_m \), given by the ideal \((t_2 - 1, \ldots, t_m - 1)\) acts nontrivially.

**Output:** A local slice \( s \in R_c \) with denominator \( c \) for the action of \( T_1 \), such that \( s \) and \( c \) are semi-invariants of the group \( T \).

1. Set \( s := 1 \), \( c := 1 \), and \( d := 0 \).
2. While not all \( \Phi(a_i) \) (viewed as elements of \( R_c[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \)) lie in \( R_c[t_1^{\pm d}, t_2^{\pm 1}, \ldots, t_m^{\pm 1}] \), repeat steps 3–5.
3. Choose a monomial \( t = t_1^{e_1} \cdots t_m^{e_m} \) occurring in \( \Phi(a_i) \) with \( e_1 \) not a multiple of \( d \), and let \( b \in R_c \) be the coefficient of \( t \) in \( \Phi(a_i) \).
4. If \( d \neq 0 \), use division with remainder to obtain \( e_1 = qd + r \) with \( q, r \in \mathbb{Z} \), \( 0 < |r| \leq d/2 \). If \( d = 0 \), set \( r := e_1 \) and \( q := 0 \). In both cases, set \( s := s'b \).
5. If \( s \) is not invertible in \( R_c \), choose a numerator \( s' \in R \) of \( s \) that is a semi-invariant (see step 4 in Algorithm 4.5), and redefine \( c \) to be \( s'c \). Redefine \( s \) to be \( s \in R_c \) if \( r < 0 \) and \( s^{-1} \in R_c \) if \( r > 0 \). Finally, set \( d := |r| \).

**Remark 5.5.** What was said in Remark 4.3 also applies to Algorithm 5.4.

**Proof of correctness of Algorithm 5.4.** Since \( T_1 \) acts nontrivially, \( d \) becomes positive after the first passage through steps 3–5, and then strictly decreases with each subsequent passage. This guarantees termination. By Lemma 4.4, \( b \) from step 3 is a semi-invariant of weight \( t \). It follows that after step 5, \( s \) is a semi-invariant with weight \( t_1^{-d} \)-a power product of \( t_2, \ldots, t_m \). Moreover, in step 5, \( s'c \) is nonzero since \( b \in R_c \) is nonzero and therefore also \( s \), since \( s \) is invertible. Clearly the “new” \( s \) is invertible. (But although the “new” \( c \) is nonzero, it may happen that \( R_c \) becomes the zero-ring.) It remains to show that (iii) from Definition 5.1 holds when the algorithm terminates. An element \( a \in R_c \) can be written as \( a = f(a_1, \ldots, a_n)/c^k \) with \( f \) a polynomial over \( K \). Since also \( c \) can be written as a polynomial in the \( a_i \), the termination condition implies \( \Phi(a) \in R_c[t_1^{\pm d}, t_2^{\pm 1}, \ldots, t_m^{\pm 1}] \). So if \( a \) has degree \( < d \) (with respect to the \( T_1 \)-action), then \( a \in R_{T_1}^c \), and the proof is complete. \( \square \)

We can now apply Algorithm 5.4 iteratively to the multiplicative groups in a torus and combine it with Algorithm 4.5. Thus we obtain an algorithm for computing \( R_c^G \) with \( G \) a group scheme in standard solvable form. As Algorithm 4.5, the algorithm requires that addition and multiplication of elements of \( R \) are possible, and that for every \( c \in R \), zero testing in \( R_c \) is possible.

Algorithm 5.6 (Solvale group invariants).

**Input:** A group scheme \( G \) in standard solvable form (see Definition 4.1, whose notation we adopt), acting on an affine scheme \( \text{Spec}(R) \) with \( R = K[a_1, \ldots, a_n] \) a finitely generated algebra, with the action given by a homomorphism \( \Phi: R \to R[z_1, \ldots, z_t, t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \). Assume that the characters \( \chi_i \in K[t_1^{\pm 1}, \ldots, z_m^{\pm 1}] \) as in Definition 4.1 are given.

**Output:**
- A semi-invariant \( c \in R \), nonzero if \( R \neq \{0\} \).
- A homomorphism \( \pi: R_c \to (R_c)^G \) of \( (R_c)^G \)-algebras given by the \( b_i := \pi(a_i) \) and by \( b := \pi(c) \). So \((R_c)^G = K[b^{-1}, b_1, \ldots, b_n]\).
• Elements \(u_1, \ldots, u_k, s_1, \ldots, s_r \in R_c\) such that the map

\[
(R_c)^G[x_1, \ldots, x_k, y_1^\pm 1, \ldots, y_r^\pm 1] \xrightarrow{x_i \mapsto u_i} R_c
\]

(with \(x_1, y_j\) indeterminates) is an isomorphism, and \(\pi\) is equal to the composition

\[
R_c \xrightarrow{\sim} (R_c)^G[x_1, \ldots, x_k, y_1^\pm 1, \ldots, y_r^\pm 1] \xrightarrow{x_i \mapsto 0} (R_c)^G.
\]

So

\[
\ker(\pi) = (u_1, \ldots, u_k, s_1 - 1, \ldots, s_r - 1).
\]

(1) Apply Algorithm 4.5. Let \(c \in R\) be the resulting semi-invariant, \(b_i\) the image of the \(a_i\) under the map \(\pi\): \(R_c \to R_c^G\), and rename the elements \(s_1, \ldots, s_k\) from Algorithm 4.5 to \(u_1, \ldots, u_k\). Set \(b := c\) and \(r := 0\). For \(j = 1, \ldots, m\) repeat steps 2–4.

(2) If none of the \(\Phi(b_i)\) involves \(t_j\), skip steps 3–4 and proceed with the next \(j\).

(3) Apply Algorithm 5.4 to \(T := \text{Spec} (K[t_1^{\pm 1}, \ldots, z_m^{\pm 1}])\) acting on \(\bar{R} := K[b^{-1}, b_1, \ldots, b_n] \subseteq R_c\) by \(\Phi\), extended to \(R_c\). Let \(\bar{s}\) be the resulting local slice of degree \(d\) with denominator \(c\).

(4) Choose a semi-invariant \(c' \in R\) such that \((R_c)_{\bar{r}} = R_{c'}\) (see step 4 in Algorithm 4.5). For \(i = 1, \ldots, n\), obtain \(b_i'\) by substituting \(t_j = \sqrt[d]{\bar{s}}\) and \(t_j + 1 = \cdots = t_m = 1\) in \(\Phi(b_i)\), which lies in \(R_c[t_j^{\pm d}, t_{j+1}^{\pm 1}, \ldots, t_m^{\pm 1}]\). If \(c' = c\bar{c} \in R_{c'}\) (such an \(e\) exists by the choice of \(c'\)), obtain \(b'\) by doing the same with \(b'\bar{c}\) instead of \(b_i\). Set \(c := c', b_i := b_i', b := b', s_{r+1} := \bar{s}\), and \(r := r + 1\).

\textit{Proof of correctness of Algorithm 5.6.} By induction on \(j\) assume that at the beginning of step 2 the elements \(c, b_i, b, u_i\), and \(s_i\) are as claimed for the output of the algorithm, but with \(G\) replaced by the subgroup \(G_j^{j-1}\) given by the ideal \((t_j - 1, \ldots, t_m - 1)\). In particular, \(\bar{R} = (R_c)^G_{j-1}\). Also assume that \(b \in R_c\) is invertible. By step 1, this is true for \(j = 1\). We will show that after step 4, the same holds with \(j\) replaced by \(j + 1\).

If none of the \(\Phi(b_i)\) involves \(t_j\), then \((R_c)^G_{j-1} = \bar{R} = (R_c)^G_{j-1}\), so step 2 is correct. In step 3, Algorithm 5.4 can be applied since the restriction of \(\Phi\) to \(\bar{R}\) defines an action of \(T\) on \(\text{Spec}(\bar{R})\). After step 3, the map \((R_c)^G_{j}[[y_{r+1}^\pm 1]] \to \bar{R}_c, y_{r+1} \mapsto \bar{s}\) is an isomorphism by Theorem 5.3(a). We have \(\bar{c} \in \bar{R} = (R_c)^G_{j-1}\), so

\[
\bar{R}_c = ((R_c)^G_{j-1})_{\bar{c}} = ((R_c)_{\bar{r}})^{G_{j-1}} = (R_c)^{G_{j-1}}
\]

with \(c'\) from step 4. So the above isomorphism is a map \((R_c)^{G_{j-1}}[[y_{r+1}^\pm 1]] \xrightarrow{\sim} (R_c)^{G_{j-1}}\). It follows that also the composition

\[
(R_c)^{G_{j}}[x_1, \ldots, x_k, y_1^\pm 1, \ldots, y_{r+1}^\pm 1] \xrightarrow{y_{r+1} \mapsto \bar{s} = s_{r+1}} (R_c)^{G_{j-1}}[x_1, \ldots, x_k, y_1^\pm 1, \ldots, y_r^\pm 1] \xrightarrow{x_i \mapsto u_i} R_c
\]

is an isomorphism.

Moreover, by Theorem 5.3(b), the composition

\[
\bar{\pi}: (R_c)^{G_{j-1}} \xrightarrow{\sim} (R_c)^{G_{j}}[[y_{r+1}^\pm 1]] \xrightarrow{y_{r+1} \mapsto \bar{s}} (R_c)^{G_{j}}
\]

is given by applying \(\Phi\) and then substituting \(t_j = \sqrt[d]{\bar{s}}\) and \(t_j + 1 = \cdots = t_m = 1\). So in step 4, \(b_i' = \bar{\pi}(b_i) = \bar{\pi}(\pi(u_i))\) and \(b' = \bar{\pi}(b'c) = \bar{\pi}(\pi(c')\pi(c)) = \bar{\pi}(\pi(c'))\), where the second equality holds since \(\bar{c} \in (R_c)^{G_{j-1}}\). From this we also see that \(b'\) is invertible in \(R_c\) since \(b, \bar{c}\) and \(\bar{s}\) are invertible and they are semi-invariants, so applying \(\bar{\pi}\) means multiplying by a power of \(\bar{s}\). It remains to show that \(\bar{\pi} \circ \pi\) is equal to the composition \(R_c \xrightarrow{\sim} (R_c)^{G_{j}}[x_1, \ldots, x_k, y_1^\pm 1, \ldots, y_{r+1}^\pm 1] \xrightarrow{x_i \mapsto 0} (R_c)^{G_{j}}\).

But this follows from the commutative diagram
This completes the proof. □

Algorithm 5.6 has been implemented in the computer algebra system MAGMA [4]. The implementation is limited to the case that $R$ is a polynomial ring over a field of characteristic 0.

We finish this section by giving a summary of the results obtained about quotients by solvable groups.

Theorem 5.7. Let $G$ be a group scheme over a ring $K$ in standard solvable form acting on a nonempty affine $K$-scheme $\text{Spec}(R)$. (Recall that every connected solvable linear algebraic group over an algebraically closed field can be brought into standard solvable form.)

(a) There exists a nonzero semi-invariant $c \in R$ such that the morphism $X := \text{Spec}(R_c) \to Y := \text{Spec}((R_c)^G)$ is an excellent quotient by $G$ with fibers $F = \text{Spec}(K[x_1, \ldots, x_k, y_1^{\pm 1}, \ldots, y_r^{\pm 1}])$. If $G$ is unipotent, then $r = 0$. In particular, $X \to Y$ is a universal geometric quotient, and it is a faithfully flat morphism.

(b) Assume $R_c \neq \{0\}$. The $(R_c)^G$-homomorphism $\pi: R_c \to (R_c)^G$ induced by the cross section $Y \to X$ (see Remark 2.2(a)) has a kernel that is generated by $k + r$ elements. Moreover, $\dim((R_c)^G) = \dim(R_c) - (k + r)$. So if $K$ is a field and $R$ or $R_c$ is a complete intersection, then also $(R_c)^G$ is a complete intersection.

(c) If $R$ is finitely generated over $K$ and if it is possible to carry out the multiplication and addition of elements of $R$ and zero testing of elements of $R_c'$ for $c' \in R$, then Algorithm 5.6 computes $(R_c)^G$ and a map $(R_c)^G[x_1, \ldots, x_k, y_1^{\pm 1}, \ldots, y_r^{\pm 1}] \to R_c$ defining the isomorphism $X \xrightarrow{\sim} F \times Y$ that comes with the excellent quotient. The algorithm also computes the above homomorphism $\pi: R_c \to (R_c)^G$ of $(R_c)^G$-algebras and its kernel. So $(R_c)^G$ requires at most as many generators as $R_c$. Algorithm 5.6 does not require any Gröbner basis computations, unless they are necessary for the operations in $R$.

(d) If $R$ is an integral domain, then $\text{Quot}(R)^G = \text{Quot}((R_c)^G)$. So Algorithm 5.6 also computes the invariant field $K(X)^G = \text{Quot}(R)^G$.

Proof. (a) The existence of $c$ follows by induction on $l + m$ (the number of factors of type $G_a$ or $G_m$ in $G$), using Remarks 4.3 and 5.5 for the existence of local slices with semi-invariant denominators, Theorems 1.6 and 5.3 to show that the quotient by the first normal subgroup of type $G_a$ or $G_m$ is excellent, and Theorem 2.10 to set the induction in motion. The other properties of the quotient follow from Theorem 2.6.

(b) By definition, $\pi$ is the composition of the isomorphism

$$R_c \xrightarrow{\sim} K[x_1, \ldots, x_k, t_1^{\pm 1}, \ldots, t_r^{\pm 1}] \otimes (R_c)^G = (R_c)^G[x_1, \ldots, x_k, t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$$

with the homomorphism $(R_c)^G[x_1, \ldots, x_k, t_1^{\pm 1}, \ldots, t_r^{\pm 1}] \to (R_c)^G$ sending $x_i$ to 0 and $t_j$ to 1. The latter map has kernel $(x_1, \ldots, x_k, t_1 - 1, \ldots, t_r - 1)$. It follows that also the kernel of $\pi$ is generated by $k + r$ elements. The statement on dimension follows from the above isomorphism. If $R$ is a complete intersection, then so is $R_c$ since $R_c \cong R[x]/(cx - 1)$ and $\dim(R_c) = \dim(R)$, the equality following from the fact that complete intersections are equidimensional. Now the statement on complete intersections is a consequence of the other statements from (b).

(c) See the proof of correctness of Algorithm 5.6.

(d) This follows from (a) and Lemma 2.9. □
Example 2.4 shows that the hypothesis that \( G \) be in standard solvable form cannot be dropped from Theorem 5.7: If \( K \) is not an algebraically closed field, it does not suffice that \( G \) is connected and solvable.

**Remark 5.8.** The existence of an excellent quotient extends to quasi-affine schemes. In fact, let \( X \) be a Noetherian scheme and \( U \subseteq X \) an open subscheme. Then \( U \) is isomorphic to a (schematically) dense open subscheme of \( \tilde{X} := \text{Spec}(\Gamma(U, O_U)) \) (see Görtz and Wedhorn [12, Proposition 13.80]) and we may assume \( U \subseteq \tilde{X} \). Moreover, let \( G \) be a group scheme acting on \( U \). By the definition of \( \tilde{X} \), the action extends to it. Observe that \( \tilde{X} \) need not be of finite type even if \( X \) is, but that is not an obstacle to the validity of Theorem 5.7(a). So if \( G \) is in standard solvable form, there exists a nonzero semi-invariant \( c \in \Gamma(U, O_U) =: R \) and an excellent quotient \( \text{Spec}(R_c) \to Y \). If \( U \) is reduced then \( \text{Spec}(R_c) \) is nonempty, and the same is true for \( U_c := U \cap \text{Spec}(R_c) \). In any case, \( U_c \) is a \( G \)-stable open subscheme of \( U \) and of \( \tilde{X} \). By Remark 2.2(f), \( U_c \) admits an excellent quotient by \( G \) with fibers \( F = \text{Spec}(K[x_1, \ldots, x_k, y_1^{\pm 1}, \ldots, y_r^{\pm 1}]) \).

Moreover, when we are working over an algebraically closed field, the following is true: If \( X \) is an irreducible algebraic variety with an action of a connected solvable group \( G \), then by Popov [23, Theorem 2] there exists an affine \( G \)-variety \( Y \) that is isomorphic (as a \( G \)-variety) to a dense open subset \( U \subseteq X \). So as above \( X \) has a \( G \)-stable dense open subset that admits an excellent quotient. Thus we recover a result of Popov [23, Theorem 3].

6. A converse

In this section we ask whether the assertions of Theorem 5.7 are limited to actions of connected solvable groups. It is fairly clear (and stated in Theorem 6.2(a)) that most parts of Theorem 5.7 extend to the case in which the \( G \)-orbits are in fact orbits of a connected solvable subgroup. Extended in this way, Theorem 5.7 actually has a converse, which is stated in Theorem 6.2(b).

Since the proof of the converse requires a result of Borel [2], which is only proved over algebraically closed fields, we assume for the rest of this paper that \( K \) is an algebraically closed field. We also assume that varieties and algebraic groups are reduced. We need the following terminology to state our result.

**Definition 6.1.** A morphic action \( G \times X \to X \) of a linear algebraic group on an affine variety is said to be essentially solvable if there exists a nonzero \( d \in K[X] \) such that the open subset \( X_d \) where \( d \) does not vanish is \( G \)-stable, and for every \( x \in X_d \) the \( G \)-orbit of \( x \) coincides with the \( R(G) \)-orbit of \( x \), where \( R(G) \) is the radical. If we can replace \( R(G) \) by the unipotent radical \( R_u(G) \), then the action is said to be essentially unipotent.

**Theorem 6.2.** Let \( G \times X \to X \) be a morphic action of a linear algebraic group on an affine variety.

(a) If the action is essentially solvable, then the assertions of Theorem 5.7(a), (b) and (d) hold, except that the element \( c \in K[X] \) need not be a semi-invariant, but has the property that \( X_c \) is \( G \)-stable. Moreover, we have \( (K[X]_c)^G = (K[X]_c)^{R(G)} \).

(b) If there is a nonzero \( c \in K[X] \) with \( X_c \) \( G \)-stable such that all \( G \)-orbits in \( X_c \) are isomorphic (as varieties) to a product \( \Delta^{r_c} \times T \) with \( T \) a torus, then the action is essentially solvable. If \( T \) is trivial for all orbits, the action is essentially unipotent.

**Remark.** (a) Notice that the assertion of Theorem 5.7(a) imply the hypothesis of part (b), so (b) contains the converse of (a).

(b) If \( G \) is connected and \( X \) is irreducible, then every \( d \in K[X] \) such that \( X_d \) is \( G \)-stable has to be a semi-invariant (see Popov and Vinberg [24, Theorem 3.1]). So in this case the elements \( c \) and \( d \) from Definition 6.1 and Theorem 6.2 are semi-invariants.

**Proof of Theorem 6.2(a).** Applying Theorem 5.7 to the action of \( R(G) \) on \( X_d \) yields a nonzero \( R(G) \)-semi-invariant \( c \in K[X_d] \). For every \( x \in X_d \) and \( \sigma \in G \) there exists \( \tau \in R(G) \) with \( \sigma(x) = \tau(x) \), so if \( c(x) \neq 0 \), then

\[
c(\sigma(x)) = c(\tau(x)) = (\tau^{-1}(c))(x) = \chi(\tau)^{-1}c(x) \neq 0,
\]
with $\chi$ the character belonging to $c$. This shows that $(X_d)_c$ is $G$-stable. Choose a numerator of $c \in K[X]_d$, multiply it by $d$ and replace $c$ by the product. Then $(X_d)_c$ is replaced by $X_c$ and $K[X_d]_c$ is replaced by $K[X]_c$. So Theorem 5.7(a) tells us that $X_c \to Y := \Spec((K[X]_c)^R(G))$ is an excellent quotient by $R(G)$ with fibers $\Spec(K[x_1 + \cdots + x_k, y_1^{\pm 1}, \ldots, y_r^{\pm 1}]$. Since the orbits of $G$ and $R(G)$ coincide on $X_c$, we have $(K[X]_c)^G = (K[X]_c)^{R(G)}$, and it is easy to check that the quotient is also an excellent quotient by $G$. Now also Theorem 5.7(d) with $R(G)$ replaced by $G$ follows, and so does Theorem 5.7(b), which only makes statements about the ring and the invariant ring.

Part (b) of Theorem 6.2 follows directly from the following lemma. The main ideas of the proof were shown to me by Hanspeter Kraft. The last statement is due to Borel [2].

**Lemma 6.3.** Let $G$ be a linear algebraic group over an algebraically closed field $K$, acting transitively on the affine variety $F = \mathbb{A}^n_K \times T$, where $T$ is a torus (regarded as a variety). Then also the radical $R(G)$ acts transitively on $F$. If $T$ is trivial, then the unipotent radical $R_u(G)$ acts transitively on $F$.

**Proof.** A short argument shows that since $F$ is irreducible, the identity component $G^0$ acts transitively on $F$, so we may assume $G$ to be connected.

Choose $v \in \mathbb{A}^n_K$ and define $\sigma \colon G \times T \to T$ as the composition

$$G \times T \xrightarrow{(g,v) \mapsto (g,v,t)} G \times \mathbb{A}^n_K \times T \xrightarrow{\text{act}} \mathbb{A}^n_K \times T \xrightarrow{pr_2} T.$$ 

This does not depend on the choice of $v$, since for fixed $g \in G$ and $t \in T$, mapping $v \in \mathbb{A}^n_K$ to $pr_2(g(v,t))$ yields a morphism $\mathbb{A}^n_K \to T$. But this must be constant, since the induced homomorphism $K[T] = K[x_1, x_1^{-1}, \ldots, x_l, x_l^{-1}] \to K[\mathbb{A}^n_K] = K[y_1, \ldots, y_n]$ maps invertible elements to invertible elements, so it maps the $x_i$ to constants. The map $\sigma$ defines a $G$-action on $T$ since for $g, h \in G$ and $t \in T$ we have

$$\sigma(g h, t) = pr_2(g(h(v,t))) = pr_2(g(v', t'))$$

with $(v', t') := h(v, t)$, and

$$\sigma(g, \sigma(h, t)) = \sigma(g, t') = pr_2(g(v', t')),$$

so the independence of the choice of $v$ implies $\sigma(g h, t) = \sigma(g, \sigma(h, t))$. The transitivity of the action on $F$ implies the transitivity of the action on $T$.

Let $\varphi \colon T \to T$ be an automorphism. Then for $t = (t_1, \ldots, t_l) \in T$ with $0 \neq t_i \in K$ we have

$$\varphi(t) = \left( a_1 \prod_{j=1}^l t_1^{e_{i,j}}, \ldots, a_l \prod_{j=1}^l t_1^{e_{i,j}} \right)$$

with $0 \neq a_i \in K$ and $e_{i,j} \in \mathbb{Z}$. Fix a $t$ whose components $t_i$ are $m$th roots of unity. Then the same follows for the components of $\varphi(t)\varphi(1)^{-1}$ (with component-wise multiplication and $1 := (1, \ldots, 1)$), and therefore also for $\varphi(t)\varphi(1)^{-1}t^{-1}$. This leaves only finitely many possibilities for $\varphi(t)\varphi(1)^{-1}t^{-1}$. So since $G$ is connected, the morphism $G \to T$, $g \mapsto g(t)g(1)^{-1}t^{-1}$ is constant, so $g(t)g(1)^{-1}t^{-1} = 1$ for all $g \in G$. This holds for all $t \in T$ whose components are $m$th roots of unity for some $m$. But since these $t$ form a dense subset of $T$, it extends to all $t \in T$. So $g(t) = g(1)t$ for $g \in G$, $t \in T$. This implies that the morphism $\pi \colon G \to T$, $g \mapsto g(1)$ is a homomorphism of algebraic groups. Since $G$ acts transitively on $T$, it is surjective. So by Borel [3, Corollary 14.11], also the restriction $\pi|_{R(G)}$ is surjective.

It is straightforward to check that for $H := \ker(\pi) \subseteq G$ the composition

$$H \times \mathbb{A}^n_K \xrightarrow{(h,v) \mapsto (h,v,1)} H \times \mathbb{A}^n_K \times T \xrightarrow{\text{act}} \mathbb{A}^n_K \times T \xrightarrow{pr_2} \mathbb{A}^n_K$$

defines an action. This action is transitive since for $v, v' \in \mathbb{A}^n_K$ there is $g \in G$ with $g(v, 1) = (v', 1)$, so $g \in H$. Therefore also $H^0$ acts transitively on $\mathbb{A}^n_K$, and from this it follows by Borel [2] that the same is true for $R_u(H)$. Now let $(v, t) \in \mathbb{A}^n_K \times T$. By the above there exists $g \in R(G)$ such that $\pi(g) = t$, which implies $g^{-1}(t) = 1$ and $g^{-1}(v, t) = (v', 1)$ with $v' \in \mathbb{A}^n_K$. If $T$ is trivial
we may choose \( g \) to be the identity. There also exists \( h \in R_u(H) \) such that \( h(v', 1) = (0, 1) \), so 
\((hg^{-1})(v, t) = (0, 1)\). Since \( R_u(H) \subseteq R_u(G) \subseteq R(G) \), the claim follows. \( \Box \)

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