STABILITY OF METRIC MEASURE SPACES WITH INTEGRAL RICCI CURVATURE BOUNDS

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ABSTRACT. In this article we study stability and compactness w.r.t. measured Gromov-Hausdorff convergence of smooth metric measure spaces with integral Ricci curvature bounds. More precisely, we prove that a sequence of $n$-dimensional Riemannian manifolds subconverges to a metric measure space that satisfies the curvature-dimension condition $CD(K, n)$ in the sense of Lott-Sturm-Villani provided the $L^p$-norm for $p > \frac{2}{n}$ of the part of the Ricci curvature that lies below $K$ converges to 0. The results also hold for sequences of general smooth metric measure spaces $(M, g_M, e^{-f} \text{vol}_M)$ where Bakry-Emery curvature replaces Ricci curvature. Corollaries are a Brunn-Minkowski-type inequality, a Bonnet-Myers estimate and a statement on finiteness of the fundamental group. Together with a uniform noncollapsing condition the limit even satisfies the Riemannian curvature-dimension condition $RCD(K, N)$. This implies volume and diameter almost rigidity theorems.

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1. Introduction

Stability and compactness properties of families of Riemannian manifolds satisfying a uniform estimate on the $L^p$-norm of the part of their Ricci curvature that lies below a given threshold $K \in \mathbb{R}$ have been a topic of increasing interest in recent years, e.g. [PW97, Aub07]. The crucial quantity for a compact Riemannian manifold $M$ is the integral curvature excess

$$k[M](\kappa, p, K) = \left( \frac{(\text{diam} M)^{2p}}{\text{vol}_M(M)} \int (\kappa - K)^p d\text{vol}_M \right)^{\frac{1}{p}}$$

where $(\kappa - K)_- = -\min\{\kappa - K, 0\}$, $\text{ric}_M \geq \kappa$ for $\kappa \in C(M)$, $p > \frac{N}{2}$ and $\dim M \leq N < \infty$.

There have been numerous publications that study Riemannian manifolds with bounded integral curvature excess, e.g. [Gal88, Yan92a, Yan92b, PW97, PW01, PLSW97, DPW00, DW04, Aub07, Ros17, DWZ18, RW20]. Remarkable properties are pre-compactness under Gromov-Hausdorff convergence, effective diameter and volume growth estimates and estimates on the spectral gap. These results resemble corresponding statements for lower Ricci curvature bounds and typically involve error terms that depend on the integral curvature excess provided it is sufficiently small. One can construct examples by gluing small cusps of arbitrary negative curvature on Riemannian manifold with a lower Ricci curvature bound. These cusps can be constructed such that the curvature excess is arbitrarily small and a subsequence will converge in Gromov-Hausdorff sense to the original manifold.

In general the precompactness property suggests that there is a theory of non-smooth limit spaces with integral curvature bounds. Moreover, the effective estimates indicate some stability property under measured Gromov-Hausdorff convergence. One conjectures that the measured Gromov-Hausdorff limit of a sequence of Riemannian manifolds with vanishing integral curvature excess $k[M](\kappa, p, K)$ is a non-smooth metric measure space with Ricci curvature bounded from below by $K$. A non-smooth theory of lower Ricci curvature bounds is in fact provided by the class of metric measure spaces that satisfy a curvature-dimension condition $CD(K, N)$ for $K \in \mathbb{R}$ and $N \geq 1$. This condition was introduced in celebrated works by Lott, Sturm and Villani [Stu06a, Stu06b, LV09].

In this article we confirm the previous conjecture. Our main result reads as follows.

Theorem 1.1. Let $D \geq 0$, $N \geq 2$ and $K \in \mathbb{R}$.

For every $\epsilon > 0$ there exists $\delta > 0$ such that the following holds. Let $(M, g_M)$ be a Riemannian manifold such that $\text{ric}_M \geq \kappa$ for $\kappa \in C(M)$, $\text{diam} M \leq D$, $\dim M \leq N$ and

$$k[M, d_M, \text{vol}_M](\kappa, p, K) < \delta \quad \text{for } p > \frac{N}{2} \quad \text{and } p \geq 1 \quad \text{if } N = 2.$$

Then there exists a metric measure space $X$ that satisfies the curvature-dimension condition $CD(K, N)$ and

$$\mathcal{D}([M, d_M, \text{vol}_M], [X]) < \epsilon.$$
Of course the theorem does not follow from the compactness property of a family of metric measure spaces that satisfy a lower Ricci curvature bound in the sense of Lott-Sturm-Villani since there is indeed no uniform lower bound for the Ricci curvature assumed.

Now let us recall that a bound of the form \( \text{ric}_M \geq \kappa \) for \( \kappa \in C(M) \) together with \( N \geq \dim M \) can be characterized in terms of a curvature-dimension condition \( \text{CD}(\kappa, N) \) that was introduced by the author in [Ket17]. For smooth metric measure spaces \((M, d_M, m_M)\), that is a Riemannian manifold equipped with a smooth measure \( e^{-f} \text{vol}_M =: m_M \) such that \( f \in C^\infty(M) \), the same characterization holds with \( \text{ric}_M \) replaced by the so-called \( N \)-Bakry-Emery curvature. In [Ket17] it was observed that the condition \( \text{CD}(\kappa, N) \) for a continuous function \( \kappa \) makes sense for any metric measure space \( X \) and generalizes the theory of Lott-Sturm-Villani to a setup of variable lower Ricci curvature bounds.

Then Theorem 1.1 is a special case of the following theorem for sequences of pointed, smooth, metric measure spaces \((M, o_i)\) where \( o_i \) is a fixed base point in \( M \).

For this setup we consider the integral curvature excess centered at \( o \) with radius \( R > 0 \):

\[
\left( \frac{R^{2p}}{m_M(B_1(o))} \int_{B_R(o)} (\kappa - K)^p \, d m_M \right)^{\frac{1}{p}} =: k_{[M,o]}(\kappa, p, K, R).
\]

We say a pointed metric measure space \((X, o)\) is normalized if \( m_X(B_1(o)) = 1 \). We prove the following result.

**Theorem 1.2.** Let \( N \in [2, \infty) \) and \( K \in \mathbb{R} \).

Let \( \{(M_i, o_i)\}_{i \in \mathbb{N}} \) be a sequence of smooth, normalized, pointed metric measure spaces that satisfy a condition \( \text{CD}(\kappa_i, N) \) for \( \kappa_i \in C(X_i) \) and

\[
k_{[M_i, o_i]}(\kappa_i, p, K, R) \to 0 \text{ when } i \to \infty \text{ for all } R > 0, p > \frac{N}{2} \text{ and } p \geq 1 \text{ if } N = 2.
\]

Then the isomorphism classes \( \{[M_i, o_i]\}_{i \in \mathbb{N}} \) subconverge in pointed measured Gromov sense to the isomorphism class of a pointed, normalized metric measure space \((X, o)\) that satisfies the condition \( \text{CD}(K, N) \).

The assumption \( p > \frac{N}{2} \) for \( N > 2 \) is sharp since Aubry showed in [Aub07] that compact \( N \)-dimensional Riemannian manifolds with \( k_{[M]}(\kappa, \frac{N}{2}, K) < \epsilon \) are dense w.r.t. Gromov-Hausdorff convergence among all compact length spaces for any \( \epsilon > 0 \).

Instead of requiring that the part of the Ricci curvature below \( K \) is in some \( L^p \)-space for \( p > \frac{N}{2} \), one can also assume that it satisfies a Kato condition. This condition is strictly weaker than the previous \( L^p \)-condition but Riemannian manifolds satisfying such a condition still have properties that resemble the ones under integral curvature bounds, e.g. [RS17, RS18]. Hence, an extension of our theorem in this direction seems to be tangible.

As part of the proof of Theorem 1.2 we derive a displacement convexity inequality (Theorem 6.1) that implies the following, new Brunn-Minkowski-type inequality under integral curvature bounds.

**Corollary 1.3** (Brunn-Minkowski inequality). Let \( M \) be a smooth, normalized mm space that satisfies \( \text{CD}(\kappa, N) \) for \( \kappa \in C(M) \) and \( N \geq 2 \). Let \( p > \frac{N}{2} \) such
that \( k_{[M]}(p,0) < \infty \) and let \( A_0, A_1 \subset M \). Then there exists a positive constant \( C = C(p,N,K) \) (see Remark 4.3) such that
\[
m_{M}(A_t)^\frac{1}{p} \geq (1 - t) m_{M}(A_0)^\frac{1}{p} + t m_{M}(A_1)^\frac{1}{p} - 2C^\frac{1}{N} k_{[M]}(\kappa,p,0)^\frac{1}{p} \forall t \in (0,1)
\]
where \( A_t = \{ \gamma(t) : \gamma \in \mathcal{G}(M), \gamma(0) \in A_0, \gamma(1) \in A_1 \} \).

Moreover we prove the following precompactness result for general, nonsmooth and non-branching metric measure spaces that was proved for Riemannian manifolds in [PW97].

**Theorem 1.4.** Consider the family \( \mathcal{X}(p,K,N) \) of isomorphism classes of essentially nonbranching pointed metric measure spaces \((X,o)\) satisfying a condition \( CD(\kappa,N) \) for some \( \kappa \in C(X) \) such that \( k_{[X,o]}(\kappa,p,K,D) \leq f(D) \) for all \([X,o] \in \mathcal{X}(p,K,N), \) for all \( D \geq 1, \ p > \frac{N}{2} \) and some function \( f : [1,\infty) \to \mathbb{R} \). Then \( \mathcal{X}(p,K,N) \) is precompact in the sense of pointed measured Gromov convergence.

The precompactness indicates that there might exist a version of our main theorem for general metric measure spaces with variable lower curvature bounds and uniformly bounded integral curvature excess. A limit space then presumably satisfies a generalized \( CD(\kappa,N) \) condition with \( \kappa \in L^p(m_X) \) rather than \( \kappa \in C(X) \).

We have the following conjecture.

**Conjecture 1.5.** There is a class of metric measure spaces \( X \) that satisfy a curvature-dimension condition \( CD(\kappa,N) \) for \( \kappa \in L^p(m_X) \) and \( p > \frac{N}{2} \). A subset of metric measure spaces that admit a uniform bound on the \( L^p \)-norm of \( \kappa \) is compact w.r.t. pointed measured Gromov convergence and the lower curvature bound \( \kappa \) is stable w.r.t. \( L^p \)-strong convergence in the sense of [GMS15].

In this respect we also mention recent work by Sturm [Stu19] where a class of metric measure spaces with lower curvature bounds in distributional sense is introduced.

In [TZ16] Tian and Zhang develop a regularity theory for limits of \( n \)-dimensional Riemannian manifolds \( M \) such that
\[
\text{ric}_M \geq \kappa, \ k_{[M]}(\kappa,p,0) \leq \Lambda < \infty \text{ with } p > \frac{N}{2}.
\]
They introduce the following uniform (and infinitesimal) noncollapsing condition: There exists \( \varkappa > 0 \) such that
\[
\text{vol}_{M_i}(B_r(x)) \geq \varkappa r^n \forall x \in M_i, \forall r \in (0,1) \text{ and } \forall i \in \mathbb{N}.
\]
This property is then used to develop a Cheeger-Colding-Naber type theory for limits of a sequence of manifolds satisfying (1) and (2). An example by Yang [Yan92a] shows that a noncollapsing condition on a definite scale is not sufficient and the splitting theorem fails. Assuming the uniform noncollapsing condition (2) Tian and Zhang develop a satisfying regularity theory for limits that arise from manifolds satisfying \( k_{[M]}(\kappa,p,0) \leq \Lambda < \infty \). In particular they prove an almost splitting theorem and an almost-volume-cone-implies-almost-metric-cone theorem. One can apply these results to our situation. The almost splitting theorem translates to a splitting theorem of the limit space in Theorem 1.2. We therefore obtain the following theorem.
Theorem 1.6. Let \( \{(M_i, o_i)\}_{i \in \mathbb{N}} \) be a sequence of \( n \)-dimensional, pointed Riemannian manifolds that satisfy the condition \( CD(\kappa_i, N) \) for \( \kappa_i \in C(X_i) \) such that
\[
k_{[M_i, o_i]}(\kappa_i, p, K, R) \to 0 \quad \text{when } i \to \infty \quad \forall R > 0
\]
with \( K \in \mathbb{R}, \ p > \frac{N}{2}, \ p \geq 1 \) if \( N = 2 \) and \( 2 \) holds. Then \( \{(M_i, o_i)\}_{i \in \mathbb{N}} \) subconverges in pointed measured Gromov sense to the isomorphism class of a pointed metric measure space \( (X, o) \) satisfying the Riemannian curvature-dimension condition \( RCD(K, N) \).

Corollary 1.7. Let \( X \) be a measured Gromov-Hausdorff limit of a sequence of Riemannian manifolds satisfying \( 1 \) and \( 2 \). Then every tangent space \( T_x X \) for \( x \in X \) satisfies the condition \( RCD(0, N) \). In particular, \( T_x X \) is an Euclidean cone over some \( RCD(N - 2, N - 1) \) space \( Y \).

We note that the example in [Yan92a] does not satisfy \( k_{[M]}(p, K) \to 0 \). Therefore we expect that our main theorem can be improved. We raise the following question.

Question 1.8. In Theorem 1.7 can we replace the condition \( CD(K, N) \) with the Riemannian condition \( RCD(K, N) \), or can we remove assumption \( 2 \) in Theorem 7.4

A theorem for \( RCD(K, 2) \) spaces by Lytchak and Stadler [LS18] also yields the following corollary.

Corollary 1.9. Let \( \{(M_i, o_i)\}_{i \in \mathbb{N}} \) be a sequence of \( 2 \)-dimensional, pointed Riemannian manifolds that satisfy the condition \( CD(\kappa_i, 2) \) for \( \kappa_i \in C(X_i) \) such that
\[
k_{[M_i, o_i]}(\kappa_i, 1, K, R) \to 0 \quad \text{when } i \to \infty \quad \forall R > 0
\]
with \( K \in \mathbb{R} \) and \( 2 \) holds. Then \( \{(M_i, o_i)\}_{i \in \mathbb{N}} \) subconverges in pointed Gromov-Hausdorff sense to a pointed \( 2 \)-dimensional Alexandrov space with curvature bounded from below by \( K \).

Let us briefly explain the main ideas in the proof of Theorem 1.2. Assume for simplicity \( K = 0 \). At the core of our proof is a new displacement convexity inequality for the \( N \)-Renyi entropy functional (Theorem 6.1). This inequality is similar to corresponding inequalities under lower Ricci curvature bounds but involves an error term that explicitly depends on the integral curvature excess. The proof of this result consists of three steps. First, we analyse carefully the \( 1 \)-dimensional model case. This allows us to prove estimates for the modified distortion coefficients \( \tau_{\kappa,N}^{(t)}(\theta) \) involving \( L^p \) integrals of \( \kappa \). Here, \( \kappa \) is a continuous function \([0, \theta]\). The coefficients \( \tau_{\kappa,N}^{(t)}(\theta) \) (before Definition 2.11) play a crucial role in the definition of the condition \( CD(\kappa, N) \) for a metric measure space \( X \) and a variable lower curvature bound \( \kappa : X \to \mathbb{R} \). Second, in Section 5 we will apply the Area and Co-area formula to a transport Kantorovich potential and derive two disintegrations of the reference measure \( e^{-V} \text{vol}_M \) (Proposition 5.6 and Lemma 5.4). Proposition 5.6 resembles a similar disintegration obtained by \( L^1 \) optimal transport (see [CM17a] and in particular [CM16]). However, since we are interested in sequences of smooth spaces, we can use classical tools of geometric analysis that are sufficient for this setup. Finally, starting from the localized version of the \( CD(\kappa, N) \) condition we put together the previous steps to obtain the desired displacement-convexity-type inequality (Section 6).
1.1. Applications. Here we present some immediate consequences that derive from the main theorems and its corollaries. First, we can prove a Bonnet-Myers diameter type bound that improves and generalizes a result by Aubry \cite{Aub07}.

**Corollary 1.10.** Let \( \{(M_i,\alpha_i)\}_{i\in\mathbb{N}} \) be a sequence of smooth, normalized pmm spaces that satisfy the condition \( CD(\kappa_i, N) \) for \( \kappa_i \in C(X_i) \) such that

\[
\kappa_{\{M_i\},\alpha_i}(\kappa_i, p, K, R) \to 0 \text{ as } i \to \infty \quad \forall R > 0
\]

with \( K > 0 \) and \( p > \frac{N}{2} \). For every \( \epsilon > 0 \) there exists \( i_\epsilon \in \mathbb{N} \) such that \( M_i \) is compact for every \( i \geq i_\epsilon \) and \( \text{diam } M_i \leq \pi_{K/(N-1)} + \epsilon \).

The proof is straightforward by arguing by contradiction.

A Bonnet-Myers type estimate for a Riemannian manifold satisfying a Kato condition was obtained in \cite{Ros19, CR18}.

As a consequence from the previous corollary we also obtain the following statement on finiteness of the fundamental group.

**Corollary 1.11.** There exists a function \( f : (0, \infty) \to (0, \infty) \) such that the following holds. If \( (M, \alpha) \) is a smooth metric measure space that satisfies the condition \( CD(\kappa, N) \) for \( \kappa \in C(M) \) and

\[
k_{\{M\},\alpha}(\kappa, p, R, K) \leq f(R) \text{ for all } R > 0
\]

with \( K > 0 \) and \( p > \frac{N}{2} \), then \( M \) has finite fundamental group.

**Proof.** Assume the statement fails. Then, there exists a sequence of smooth pointed metric measure spaces \( (M_i, \alpha_i) \) satisfying \( CD(\kappa_i, N) \) such that \( k_{\{M_i\},\alpha_i}(\kappa_i, p, R, K) \to 0 \) for \( i \to \infty \) and \( \forall R > 0 \) and such that the fundamental group is not finite.

First, by the previous Corollary we can assume that \( M_i \) is compact \( \forall i \in \mathbb{N} \) and we can normalize the metric measure space \( M_i \). Then we still have \( k_{\{M_i\},\alpha_i}(\kappa_i, p, K) \to 0 \).

Let \( \hat{M}_i \) be the Riemannian universal cover of \( M_i \) equipped with the pull back measure \( \hat{m}_{\hat{M}_i} \) under the covering map \( p_i : \hat{M}_i \to M_i \). For every \( i \in \mathbb{N} \) we choose a base point \( o_i \in \hat{M}_i \). Since the fundamental group is not finite, it follows \( \text{diam } \hat{M}_i = \infty \). Moreover it follows that the metric measure space \( \hat{M}_i \) satisfies the condition \( CD(\kappa_i \circ p_i, N) \) and

\[
k_{\{\hat{M}_i\},\alpha_i}(\kappa_i \circ p_i, p, K, R) \to 0 \text{ as } i \to \infty \quad \forall R > 0.
\]

By the previous corollary there exists \( i_\epsilon \in \mathbb{N} \) such that \( \text{diam } \hat{M}_{i_\epsilon} \leq \pi_{K/(N-1)} + \epsilon \) for \( i \geq i_\epsilon \). Hence, \( \hat{M}_{i_\epsilon} \) is compact. That is a contradiction. \( \square \)

A theorem on finiteness of the fundamental group of Riemannian manifold under an integral curvature condition appears in \cite{Aub07} and under a Kato condition in \cite{CR18,Ros19}. Let us point out that our theorem even improves Aubry’s result in the Riemannian case since we do not require a priori that \( (\kappa - K)_- \) is \( L^p(\text{vol}_{M}) \) integrable for \( p > \frac{N}{2} \). A result for weighted graphs satisfying the Kato condition appears in \cite{MR19}.

1.1.1. Almost rigidity results. Recall the following definitions. For \( K > 0 \) and \( N > 1 \) the 1-dimensional model space is

\[
I_{K,N} = \left(\left[0, \pi_{K/(N-1)}\right], 1, [0, \pi_{K/(N-1)}] \sin^{N-1}_{K/(N-1)} L^1\right)
\]

where \([0, \pi_{K/(N-1)}]\) is equipped with the restriction of the standard metric \(| \cdot |\) on \( \mathbb{R} \).

The metric measure space \( I_{K,N} \) satisfies \( CD(K, N) \) \cite{Stu06b} Example 1.8].
Corollary 1.13. For every \( I \in \text{Ket13} \) it was proved that if the warping function \( f \) satisfies

\[
\Phi \in M^n \Raw Space\ f : I_{K,N} \to [0, \infty) \text{ is defined as the metric completion of the weighted Riemannian manifold} (I_{K,N} \times M, h, m_C) \text{ where } h = (\cdot, \cdot)^2 + f^2 g \text{ and } m_C = f^{N-1}L^1_{|I_{K,N}} \otimes m.
\]

In \([\text{Ket13}]\) it was proved that if the warping function \( f \) satisfies

\[
f'' + \frac{K}{N-1} f \leq 0 \text{ and } (f')^2 + \frac{K}{N-1} f^2 \leq L \text{ on } I_{K,N}
\]

and \((M, d_M, m)\) satisfies \( CD(L(N-2), N-1) \) then \( I_{K,N} \times f^{N-1}M \) satisfies \( CD(K, N) \). This applies in particular when \( f = \sin_{K/(N-1)} \) and \( L = 1 \). Then the corresponding warped product is a spherical suspension. For instance, we can choose \( M = I_{N-2,N-1} \). If \( n \in \mathbb{N} \) we can choose \( M = S^n_{1-\frac{\pi}{L}} \) and we get that \( I_{K,n} \times f^{N-1}S^n_{\sin_{K/(n-1)}} \)

More generally, one can define warped products in the context of metric measure spaces. In \([\text{Ket15}]\) it was proved that \( I_{K,N-1} \times ^{N-1}Y \) satisfies the condition \( RCD(K, N) \) if and only if \( Y = (Y, dy, m_Y) \) satisfies the condition \( RCD(N-2, N-1) \).

Rigidity statements for \( RCD \) spaces together with Theorem 7.4 yield the following almost rigidity statements for smooth metric measure spaces with integral curvature bounds.

Corollary 1.12. For every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that the following holds. If \( M^n \) is a compact Riemannian manifold that satisfies \([2]\), \( k_{[M]}(p, n-1) < \delta \) for \( p > \frac{n}{2} \) and \( \text{diam} M \geq \pi - \delta \), then there exists an \( RCD(n-2, n-1) \) space \( Y \) such that

\[
\mathbb{D}([M, d_M, \text{vol}_M], [0, \pi] \times ^n Y) \leq \epsilon.
\]

Corollary 1.13. For every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that the following holds. If \((M^n, \alpha)\) is a pointed Riemannian manifold that satisfies \([2]\), \( k_{[M]}(p, 0) < \delta \) for \( p > \frac{n}{2} \) and

\[
\frac{\text{vol}_M(B_r(o))}{\text{vol}_r(B_r(0))} \geq (1 - \delta) \frac{\text{vol}_M(B_r(o))}{\text{vol}_r(B_r(0))}
\]

for some \( r > 0 \), then there exists an \( RCD(n-2, n-1) \) space \( Y \) such that

\[
\mathbb{D}([M, d_M, \text{vol}_M], [0, r] \times ^r Y) \leq \epsilon.
\]

The second corollary appears with weaker assumptions also in \([\text{TZ16}]\). But our result yields in addition that the cross section is an \( RCD \) space.

Instead of the noncollapsing condition one can also add an upper curvature bound. This will also force the limit to become \( RCD \) by \([\text{KK19b} \text{ KK19b} \text{ KKK19}]\).

Corollary 1.14. Let \( \{(M_i, \alpha_i)\}_{i \in \mathbb{N}} \) be a sequence of \( n \)-dimensional, pointed Riemannian manifolds that satisfy the condition \( CD(\kappa_i, N) \) for \( \kappa_i \in C(X_i) \) such that

\[
k_{[M_i, \alpha_i]}(p, K, R) \to 0 \text{ when } i \to \infty \text{ } \forall R > 0
\]

with \( K \in \mathbb{R}, p > \frac{n}{2}, \kappa_i \geq 1 \) if \( N = 2 \) and \( M_i \) satisfies a CAT(\( \bar{K} \)) condition with \( \bar{K} \in \mathbb{R} \). Then \( \{(M_i, \alpha_i)\}_{i \in \mathbb{N}} \) subconverges in pmG sense to the isomorphism class of a pmM space \( (X, \alpha) \) satisfying the mixed curvature condition \( RCD(K, N) + \text{CAT}(\bar{K}) \).
1.2. Plan of the paper. In section 2 we recall preliminaries on convergence of metric measure spaces and various notions of convergence together with some general results. We introduce the curvature-dimension condition $CD(\kappa, N)$ for general metric measure spaces and $\kappa \in C(X)$ and give a self-contained proof that this condition is equivalent to $\text{ric}_M \geq \kappa$ for Riemannian manifolds. We also introduce the Riemannian curvature-dimension condition $RCD(K, N)$ for $K \in \mathbb{R}$.

In section 3 we prove that uniform bounds on the integral curvature quantity $K|X|(\kappa, p, K)$ for $CD(\kappa, N)$ spaces with $p > \frac{N}{2}$ yields precompactness under measured Gromov-Hausdorff and measured Gromov convergence. A similar statement holds for the pointed case.

In section 4 we derive estimates for the case of 1-dimensional metric measure spaces.

In section 5 we present some technical observations that derive from the Area and the Co-Area formula.

In section 6 we use the 1-dimensional estimates and the technical lemma from the previous section to derive a displacement convexity inequality for smooth metric measure spaces with integral curvature bounds.

In section 7 we prove the main theorem where we consider the cases $K \leq 0$ and $K > 0$ separately. We finish with a list of straightforward applications.

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2. Preliminaries

2.1. Metric measure spaces. We follow [GMS15]. Let $(X, d_X)$ be a complete and separable metric space. We denote by $\mathcal{M}_{loc}(X)$ the collection of Borel measures on $X$ which are finite on bounded sets, by $\mathcal{M}(X)$ the subset of finite Borel measures, and by $\mathcal{P}(X)$ the collection of Borel probability measures. We say a sequence $(\mu_i)_{i \in \mathbb{N}}$ of measures in $\mathcal{M}_{loc}(X)$ converges weakly to $\mu_\infty \in \mathcal{M}_{loc}(X)$ if

$$\lim_{i \to \infty} \int f \, d\mu_i = \int f \, d\mu_\infty \quad \text{for every } f \in C_{bs}(X)$$

where $C_{bs}(X)$ is the set of bounded continuous functions with bounded support. If $(\mu_i)_{i \in \mathbb{N}} \subset \mathcal{P}(X)$, then this is equivalent to require (3) with $f \in C_b(X)$, the set of bounded continuous functions.

Let $\mu_X \in \mathcal{M}_{loc}(X)$. We call the triple $(X, d_X, \mu_X)$ a metric measure space (mm space). The case $\mu_X(X) = 0$ is excluded.

If $A \subset X$ is measurable with $\mu_X(A) < \infty$, we set $\mu_A := \mu_X|_A$ and $\bar{\mu}_A = \mu_X(A)^{-1} \mu_A$. If $\mu_X(X) = 1$, we say the mm space $X$ is normalized. If we fix a point $o \in \text{supp} \mu_X$, we call $(X, o)$ a pointed metric measure space (pmm space).

Two mm spaces $X_i$, $i = 0, 1$, are called isomorphic if there exists an isometric embedding $\iota : \text{supp} \mu_{X_0} \to X_1$ such that $\iota_* \mu_{X_0} = \mu_{X_1}$. For pmm spaces $(X_i, o_i)$, $i = 0, 1$ we further require $\iota(o_0) = o_1$. We shall denote by $[X]$ the corresponding isomorphism class of an mm space $X$, and with $[X, o]$ the isomorphism class of a pmm space $(X, o)$.

The isomorphism class $[X]$ is invariant under $\mu_X \mapsto r \cdot \mu_X$, $r \in \mathbb{R}$. Hence, as a representative of $[X]$ we will usually pick one with $\mu_X(X) = 1$. In this case we
say \( X \) is normalized. Similar for the isomorphism class of \([X, o]\) we pick \((X, o)\) such that \(m_X(B_1(o)) = 1\). In this case we call \((X, o)\) normalized.

### 2.2. Convergence of metric measure spaces

Set \( \mathbb{N} = \mathbb{N} \cup \{\infty\} \). We collect some results on convergence of mm spaces that will be needed.

**Definition 2.1.** A sequence \((X_i, d_{X_i})\)\( \in \mathbb{N} \) of compact metric spaces converges in Gromov-Hausdorff (GH) sense to a compact metric space \((X, d_X)\) if there is a compact metric space \((Z, d_Z)\) and isometric embeddings \(\iota_i : X_i \to Z, \iota : X \to Z\) such that \(\iota_i(X_i)\) converges in Hausdorff sense to \(\iota(X)\).

A sequence of compact mm spaces \((X_i)\)\( \in \mathbb{N} \) with finite \(m_{X_i}\) converges in measured Gromov-Hausdorff (mGH) sense to a compact mm space \(X\) if there exists a compact metric space \((Z, d_Z)\) and distance preserving embeddings \(\iota_i, \iota : X_i, X \to Z\) as before such that the corresponding metric spaces converge in Gromov-Hausdorff sense and \((\iota_i)_* m_{X_i} \to (\iota)_* m_X\) weakly in \(\mathcal{M}(Z)\).

**Definition 2.2.** We say pointed mm spaces \((X_i, o_i)\), \(i \in \mathbb{N}\), converge in pointed measured Gromov-Hausdorff (pmGH) sense to a pointed mm space \((X, o)\) if for every \(R > 0\) and every \(\epsilon > 0\) there exists \(i_{R, \epsilon}\), such that for \(i \geq i_{R, \epsilon}\) there are measurable maps \(f_i^{R, \epsilon} : X \to X\) such that

1. \(f_i^{R, \epsilon}(o_i) = o\),
2. \(\sup_{x, y \in B_R(o_i)} |d_X(x, y) - d_X(f_i^{R, \epsilon}(x), f_i^{R, \epsilon}(y))| < \epsilon\),
3. \(B_{R, \epsilon}(o_i) \subset B_{R, \epsilon}(f_i^{R, \epsilon}(B_R(o_i)))\),
4. \((f_i^{R, \epsilon})_* m_{B_R(o_i)}\) converges weakly to \(m_{B_R(o)}\) as \(i \to \infty\).

Let \((X, d_X)\) be a metric space and \(\epsilon > 0\). A subset \(S \subset X\) is called an \(\epsilon\)-net of \(A \subset X\) if \(A \subset \bigcup_{x \in S} B_\epsilon(x)\).

A family of metric spaces \(\mathcal{X}\) is called uniformly totally bounded if the following two statements hold. There exists \(D\) such that for all \(X \in \mathcal{X}\) \(\text{diam}_X \leq D\). For every \(\epsilon > 0\) there exists \(N(\epsilon) \in \mathbb{N}\) such that every \(X \in \mathcal{X}\) contains an \(\epsilon\)-net of not more than \(N(\epsilon)\) points.

A family of pointed metric spaces \(\mathcal{X}_0\) is called uniformly totally bounded if for every \(R > 0\) and for every \(\epsilon > 0\) there exists \(N(R, \epsilon) \in \mathbb{N}\) such that the ball \(B_R(o)\) admits an \(\epsilon\)-net of not more than \(N(R, \epsilon)\) points for all \((X, d_X, o) \in \mathcal{X}_0\).

**Theorem 2.3.** A sequence of mm spaces \((X_i)\)\( \in \mathbb{N} \) such that the corresponding family of metric spaces is uniformly totally bounded and \(\sup_{i \in \mathbb{N}} m_{X_i}(X_i) \leq C < \infty\) admits a subsequence that converges in mGH sense to a mm space \(X_\infty\).

A sequence of pmms spaces \((X_i, o_i)\)\( \in \mathbb{N} \) such that the corresponding family of pointed metric spaces is uniformly totally bounded and

\[\sup_{i \in \mathbb{N}} m_{X_i}(B_R(o_i)) \leq C(R) < \infty \quad \forall R > 1,\]

subconverges in pmGH sense to pmms space \((X_\infty, o_\infty)\).

**Definition 2.4.** A sequence of isomorphism classes \([X_i]_i\)\( \in \mathbb{N} \) of mm spaces with finite \(m_{X_i}\) converges in measured Gromov (mG) sense to the isomorphism class \([X]\) of an mm space if there exists a complete and separable metric space \((Z, d_Z)\) and isometric embeddings \(\iota_i : X_i \to Z, \iota : X \to Z\) such that \((\iota_i)_* m_{X_i} \to (\iota)_* m_X\) weakly in \(\mathcal{M}(Z)\).
In [Stu06a] Sturm introduces a distance $D$ on the space of isomorphism classes of normalized mm spaces that metrizes convergence in mG sense: Consider $[X]$ and $[Y]$ for mm spaces $X$ and $Y$, normalize the measures $\mu_X$ and $\mu_Y$ and define

$$D([X],[Y]) = \inf_{\iota_X,\iota_Y : X,Y \to Z} W_2((\iota_X)_\# \mu_X,(\iota_Y)_\# \mu_Y).$$

The infimum is w.r.t. all distance preserving embeddings $\iota_X, \iota_Y$ into a complete and separable metric space $(Z,d_Z)$, and $W_2$ is the Wasserstein distance in $(Z,d_Z)$ that is introduced in the next subsection.

**Definition 2.5.** Let $(X_i,o_i)_{i \in \mathbb{N}}$ be a sequence of pmm spaces. We say that the corresponding sequences of isomorphism classes converges in pointed measured Gromov (pmG) sense to the isomorphism class of a pmm space $(X,\iota)$ provided there exists a complete and separable metric space $(Z,d_Z)$ and isometric embeddings $\iota_i : X_i \to Z$ for $i \in \mathbb{N}$ such that $(\iota_i)_* \mu_{X_i} \to (\iota_\#)_* \mu_X$ weakly in $\mathcal{M}_{loc}(X)$, and $\iota_i(o_i) \to \iota_\#(o_\#)$ in $(Z,d_Z)$.

**Theorem 2.6** ([Stu06a], [GMS15]). If a sequence $(X_i)$ of mm spaces converges in mGH sense to a mm space $X$, then the corresponding equivalence classes converge in mG sense.

If pmm spaces $(X_i,o_i)$ converge in pmGH sense to $(X,\iota)$ then the equivalence classes converge in pmG sense.

**Theorem 2.7.** If $(X_i,o_i)_{i \in \mathbb{N}}$ converges in pmG sense to $[X,\iota]$ and $(X,o_\#)$ is uniformly totally bounded, then $(X_i,o_i)_{i \in \mathbb{N}}$ converges in pmGH sense to a pmmm space $(X,o)$ such that $[X,o] = [X,\iota]$.

In particular, if $\text{supp} \mu_{X_\#} = X_\#$, then $(X_i,o_i)$ converges in pmGH to $(X,\iota)$.

### 2.3. Wasserstein space

Let $(X,d_X)$ be a complete and separable metric space. The set of constant speed geodesics $\gamma : [0,1] \to X$ is denoted by $\mathcal{G}(X)$, and it is equipped with the topology of uniform convergence. $e_t : \gamma \mapsto \gamma(t)$ denotes the evaluation map at time $t$.

The $L^2$-Wasserstein space of Borel probability measures with finite second moment and the Wasserstein distance are denoted by $\mathcal{P}^2(X)$ and $W_X$, respectively. $\mathcal{P}^2_b(X)$ and $\mathcal{P}^2_{\text{loc}}(X)$ denote the subset of probability measures with bounded support and the family of $m_X$-absolutely continuous probability measures, respectively.

We call a set $\Gamma \subset X \times \frac{1}{2}d_X^2$-monotone or just monotone if for any finite collection $(x^1,y^1),\ldots,(x^k,y^k) \in \Gamma$, $k \in \mathbb{N}$, we have

$$\left(\sum_{i=1}^k \frac{1}{2}d_X^2(x^i,y^i) \leq \sum_{i=1}^k \frac{1}{2}d_X^2(x^i,y^{\sigma(i)})\right)$$

for any permutation $\sigma$ of $\{1,\ldots,k\}$. The support $\text{supp} \pi$ of an optimal coupling $\pi$ is a monotone set. If we replace $\frac{1}{2}d_X^2$ in (4) by a continuous function $c : X^2 \to \mathbb{R}$, we call $\Gamma$ a $c$-monotone set.

A coupling or plan between probability measures $\mu_0$ and $\mu_1$ is a probability measure $\pi \in \mathcal{P}(X^2)$ such that $(p_i)_i = \mu_1$ where $(p_i)_{i=0,1}$ are the projection maps. We denote by $\text{Cpl}(\mu_0,\mu_1)$ the set of couplings between $\mu_0, \mu_1 \in \mathcal{P}^2(X)$. A coupling $\pi \in \text{Cpl}(\mu_0,\mu_1)$ is called optimal if

$$\int_{X^2} d_X(x,y)^2 d\pi(x,y) = W_X(\mu_0,\mu_1)^2.$$
A probability measure $\Pi \in \mathcal{P}(\mathcal{G}(X))$ is called an optimal dynamical coupling if $(\epsilon_0, \epsilon_1)_t \Pi$ is an optimal coupling between its marginal distributions. Let $(\mu_t)_{t \in [0,1]}$ be a Wasserstein geodesic in $\mathcal{P}^2(X)$. We say an optimal dynamical coupling $\Pi$ is a lift of $\mu_t$ if $(\epsilon_t)_* \Pi = \mu_t$ for every $t \in [0,1]$. If $\Pi$ is the lift of a Wasserstein geodesic $\mu_t$, we call $\Pi$ itself a Wasserstein geodesic.

2.4. Curvature-dimension condition. Let $X$ be a mm space. Given $N \geq 1$ the $N$-Rényi entropy functional $S_N : \mathcal{P}(X) \to (-\infty, 0)$ with respect to $m_X$ is given by

$$
\mu = \rho \, m_X + \nu^* \mapsto S_N(\mu) := S_N(\mu|m_X) = - \int \rho^{\frac{1}{N}} \frac{d\nu}{d\mu} dm_X.
$$

In the case $N = 1$ the 1-Rényi entropy is $S_1(\mu) = -m_X(\text{supp } \rho)$. If $m_X$ is finite, then $-m_X(X) \leq S_N(\cdot) \leq 0$. Moreover, if $N > 1$ then $(\mu, \nu) \in \mathcal{P}(X) \times \mathcal{P}(X) \mapsto S_N(\mu|\nu)$ is lower semi-continuous w.r.t. weak convergence.

**Definition 2.8** (generalized sin-functions). Let $\kappa : [0, L] \to \mathbb{R}$ be a continuous function. The generalized sin function $\sin_{\kappa} : [0, L] \to \mathbb{R}$ is the solution of

$$
v'' + \kappa v = 0.
$$

such that $\sin_{\kappa}(0) = 0$ and $\sin'_{\kappa}(0) = 1$. The generalized cos-function is $\cos_{\kappa} = \sin'_{\kappa}$.

**Definition 2.9** (Distortion coefficients). Consider $\kappa : [0, L] \to \mathbb{R}$ that is continuous and $\theta \in [0, L]$. Then

$$
\sigma_{\kappa}^{(\theta)}(t) = \begin{cases} 
\frac{\sin_{\kappa}(\theta t)}{\sin_{\kappa}(\theta)} & \text{if } \sin_{\kappa}(t) > 0 \text{ for all } t \in (0, \theta] \\
\infty & \text{otherwise}.
\end{cases}
$$

If $\sigma_{\kappa}^{(\theta)}(\theta) < \infty$, $t \mapsto \sigma_{\kappa}^{(\theta)}(t)$ is a solution of $u''(t) + \kappa(t) \theta^2 u(t) = 0$ satisfying $u(0) = 0$ and $u(1) = 1$. We set $\sigma_{\kappa}^{(1)}(1) = \sigma_{\kappa}^{(\theta)}$ and $\sigma_{\kappa}^{(\theta)}(\theta) = \sigma_{\kappa}^{(\theta)}$. We also define

$$
\pi_{\kappa} := \sup\{\theta \in (0, L) : \sin_{\kappa}(r) > 0 \forall r \in (0, \theta]\}.
$$

Consider a metric space $(X, d_X)$ and a continuous function $\kappa : X \to \mathbb{R}$. We set $\kappa_\gamma = \kappa \circ \tilde{\gamma}$ where $\gamma : [0, 1] \to X$ is a constant speed geodesic and $\tilde{\gamma}$ its unit speed reparametrization. We denote by $\gamma^-(t) = \gamma(1 - t)$ the reverse parametrization of $\gamma$, and we also write $\gamma = \gamma^+$ and $\kappa_\gamma^+/\gamma^+ := \kappa_{\gamma^-/\gamma^+}$.

**Proposition 2.10.** Let $\kappa : [a, b] \to \mathbb{R}$ be continuous and $u : [a, b] \to \mathbb{R}_{\geq 0}$ be an upper semi-continuous. Then the following statements are equivalent:

(i) $u'' + \kappa u \leq 0$ in the distributional sense, that is

$$
\int_a^b \varphi''(t) u(t) dt \leq - \int_a^b \varphi'(t) \kappa(t) u(t) dt
$$

for any $\varphi \in C_0^\infty((a, b))$ with $\varphi \geq 0$.

(ii) There is a constant $0 < L \leq b - a$ such that

$$
u(\gamma(t)) \geq \sigma_{\kappa_\gamma}^{(\theta)}(\theta) u(\gamma(0)) + \sigma_{\kappa_\gamma}^{(\theta)}(\theta) u(\gamma(1))
$$

for any constant speed geodesic $\gamma : [0, 1] \to [a, b]$ with $\theta = |\gamma| = L(\gamma) \leq L$.

We set $\kappa_\gamma = \kappa \circ \tilde{\gamma} : [0, \theta] \to \mathbb{R}$. $\tilde{\gamma} : [0, \theta] \to [a, b]$ denotes the unit speed reparametrization of $\gamma$. We use the convention $\infty \cdot 0 = 0$.

(iii) The statement in (iii) holds for any geodesic $\gamma : [0, 1] \to [a, b]$. 

The modified distortion coefficient along $\gamma : [0,1] \to X$ w.r.t. $\kappa \in C(X)$ and $N \in [1, \infty)$ is given by

\[
(t, \kappa, N) \mapsto \tau_{\kappa,N}^t(\gamma) = \begin{cases} t^\frac{1}{r} [\sigma_{\kappa/(N-1)}(\gamma)]^{1-\frac{1}{r}} & \text{otherwise} \\
\end{cases}
\]

where $t \cdot \infty = \infty$ for $t > 0$, $0 \cdot \infty = 0$ and $\infty \cdot \alpha = \infty$ for $\alpha > 0$. By the Sturm-Picone comparison theorem $\tau_{\kappa,N}^t(\gamma)$ is non-decreasing in $\kappa$ and non-increasing in $N$.

The following definition was introduced in \cite{Ket17}.

**Definition 2.11.** We say that an mm space $X \neq \{pt\}$ satisfies the curvature-dimension condition $CD(\kappa, N)$ for $\kappa \in C_b(X)$ and $N \geq 1$ if for each pair $\nu_0, \nu_1 \in P_b^2(m_X)$ there exists an $L^2$-Wasserstein geodesic $(\nu_t)_{t \in [0,1]} \subset P^2(m_X)$ and a dynamical optimal coupling $\Pi$ with $(\epsilon_t)_t \Pi = \nu_t$ such that

\[
\left( S_N(\nu_t) \right. - \left. \int \left[ \tau_{\kappa,N}(\gamma) \right] g_0(\epsilon_t(\gamma)) e^{-\frac{t}{2}} + \tau_{\kappa,N}(\gamma) \right] d\Pi(\gamma)
\]

for all $t \in [0,1]$ and all $N' \geq N$ where $[\nu_t]_{ac} = \rho_t$, $i = 0, 1$.

For $x \to \kappa(x) = : K \in \mathbb{R}$ the definition is exactly the curvature-dimension condition as introduced by Lott-Sturm-Villani in \cite{Stu06a, Stu06b, LV09}.

As consequence of the monotonicity of the distortion coefficients we obtain the following property.

**Proposition 2.12.** Let $(X, d_X, m_X)$ be a metric measure space which satisfies the condition $CD(\kappa, N)$ for a continuous function $\kappa : X \to \mathbb{R}$ and $N \geq 1$.

If $\kappa' : X \to \mathbb{R}$ is a continuous function such that $\kappa' \leq \kappa$, and if $N' \geq N$, then $(X, d_X, m_X)$ also satisfies the condition $CD(\kappa', N')$.

Since we assume $\kappa \in C_b(X)$, the condition $CD(\kappa, N)$ implies the Lott-Sturm-Villani curvature-dimension condition $CD(K, N)$ for $K \in \mathbb{R}$ with $\kappa \geq K$. In particular, Bishop-Gromov volume growth estimate holds \cite{Stu06b, EKS15, Ket17}, the space is locally compact and bounded sets have finite measure. Moreover, the $(\text{supp}\, m_X, d_{\text{supp}\, m_X})$ is a geodesic metric space.

**Proposition 2.13.** Let $X$ be a metric measure space which satisfies the condition $CD(\kappa, N)$ for $\kappa \in C_b(X)$ and $N \geq 1$.

(i) If there is an isomorphism $\psi : (X, d_X, m_X) \to (X', d_{X'}, m_{X'})$ onto a metric measure space $(X', d_{X'}, m_{X'})$ then $(X', d_{X'}, m_{X'})$ satisfies the condition $CD(\psi^* \kappa, N)$ with $\psi^* \kappa = \kappa \circ \psi$.

(ii) For $\alpha, \beta > 0$ the rescaled metric measure space $(X', \alpha d_{X'}, \beta m_{X'})$ satisfies $CD(\alpha^{-2} \kappa, N)$.

(iii) For each geodesically convex subset $U \subset X$ the metric measure space $(U, d_X|_{U \times U}, m_X|_{U})$ satisfies $CD(\kappa|_U, N)$.

Let $(M, g_M, e^{-V} \text{vol}_M)$ be a weighted Riemannian manifold with $V \in C^\infty(M)$. We recall that for each real number $N > n$ the Bakry-Emery $N$-Ricci tensor is defined as

\[
\text{ric}^N(v) = \text{ric}(v) - \nabla^2 V(v, v) - \frac{1}{N-n} \langle \nabla V, v \rangle^2
\]

where $v \in TM_p$. For $N = n$ we define

\[
\text{ric}^N(v) = \begin{cases} \text{ric}(v) + \nabla^2 V(v, v) & dV(v) = 0 \\
-\infty & \text{else}
\end{cases}
\]
For $1 \leq N < n$ we define $\text{ric}^{N,V}(v) := -\infty$ for all $v \neq 0$ and $0$ otherwise.

We define $\varrho(x) \in \mathbb{R} \cup \{-\infty\}$ as the smallest eigenvalue of $\text{ric}_x^{N,V}$. If $\varrho > -\infty$, then $\varrho$ is a smooth function.

The following theorem appears in [Ket17].

**Theorem 2.14.** Let $(M, g_M, e^{-V}d\text{vol}_M)$ be a weighted Riemannian manifold for $V \in C^\infty(V)$. Let $\kappa : M \to \mathbb{R}$ be continuous and $N \geq 1$.

Then, the mm space $(M, d_M, e^{-V}d\text{vol}_M)$ satisfies the condition $CD(\kappa, N)$ if and only if it has $N$-Ricci curvature bounded from below by $\kappa$.

**Proof.** For completeness we give a self-contained proof based on the Monge-Mather principle. This allows us to introduce some concepts that will be used later again.

**Theorem 2.15** (Monge-Mather principle, [Vil09] Corollary 8.2). Let $(M, g_M)$ be a Riemannian manifold, and let $E \subset M$ be a compact subset. Let $\gamma_1, \gamma_2 : [0, 1] \to E$ be two minimizing geodesics that satisfy

$$d(\gamma_1(0), \gamma_1(1))^2 + d(\gamma_2(0), \gamma_2(1))^2 \leq d(\gamma_1(0), \gamma_2(1))^2 + d(\gamma_2(0), \gamma_1(1))^2.$$ 

Then, there exists a constant $C_E > 0$ such that for any $t_0 \in (0, 1)$ we have

$$d(\gamma_1(t), \gamma_2(t)) \leq \frac{C_E}{\min(t_0, 1-t_0)}d(\gamma_1(t_0), \gamma_2(t_0)), \quad \forall t \in [0, 1].$$

1. We set $m := e^{-V}\text{vol}_M$ and let $\mu_0, \mu_1 \in \mathcal{P}_b^2(m)$. We find $R > 0$ such that $\mu_0$ and $\mu_1$ are supported in $B_{R/2}(o)$ for some $o \in M$. We set $B_R := B_R(o)$ and $\bar{B}_R := \bar{B}_R(o)$.

We first recall some general facts about $L^2$-optimal transport.

There exists a dynamical optimal plan $\Pi \in \mathcal{P}(\mathcal{G}(X))$ such that $t \mapsto (e_t)_{\#}\Pi = \mu_t$ is a $W_2$-geodesic in $\mathcal{P}^2(M)$. Let supp $\Pi = \Gamma$. The measures $\mu_t$ are supported in $(B_R)_{x_t} := (e_t)(\Gamma) \subset B_R$.

2. **Claim.** $\mu_t \in \mathcal{P}_b^2(m)$.

We fix $t_0 \in (0, 1)$. $(e_0, e_1) \circ e_t^{-1}((B_R)_{t_0})$ is a monotone subset of $B_{R/2}(o) \times B_{R/2}(o)$. Hence, for $t \in [0, 1]$ the map $e_t \circ e_t^{-1} : (B_R)_{t_0} \to M$ is a Lipschitz continuous map by the Monge-Mather principle. Let $t = 0$. By the Kirzbraun theorem $e_0 \circ e_t^{-1}$ admits a Lipschitz extension $T_{t_0}^0 : \bar{B}_R(o) \to M$.

Then, the optimal plan $\pi_{t_0}^0 = (e_{t_0}, e_0)_{\#}\Pi$ between $\mu_{t_0}$ and $\mu_0$ is induced by the Lipschitz map $T_{t_0}^0$, that is $\pi_{t_0}^0 = (\text{id}_{\bar{B}_R(o)}, T_{t_0}^0)_{\#}\mu_{t_0}$. Hence we get

$$\mu_{t_0}(A) = \pi_{t_0}^0(A, M) = \pi_{t_0}^0(A, T_{t_0}^0(A)) \leq \pi_{t_0}^0(M, T_{t_0}^0(A)) = \mu_0(T_{t_0}^0(A)).$$

If $\mathcal{R}$ is a set with $m(\mathcal{R}) = 0$, then we get $\text{vol}_M(\mathcal{R}) = 0$. By Lipschitz continuity of $T_{t_0}^0$ it follows that $\text{vol}_M(T_{t_0}^0(\mathcal{R})) = 0$, hence $m(T_{t_0}^0(\mathcal{R})) = \mu_0(T_{t_0}^0(\mathcal{R})) = 0$ and therefore $\mu_{t_0}(\mathcal{R}) = 0$. Since $t_0 \in (0, 1)$ was arbitrary so far, we see that $\mu_{t_0} \in \mathcal{P}_b^2(m)$ for all $t_0 \in (0, 1)$.

3. There exists a $\frac{1}{2}d^2$-convex function $\phi : \bar{B}_R \to \mathbb{R}$ such that the following holds. The Hamilton-Jacobi shift of $\phi$ is given by

$$\phi_t(x) = \inf_{y \in M} \frac{1}{2t}d(x, y)^2 - \phi(y), \quad x \in \bar{B}_R, \ t \in (0, 1].$$

The function $\phi_t : \bar{B}_R \to \mathbb{R}$ is $\frac{1}{2}d^2$-concave and hence Lipschitz continuous and semi-convex. Let $t_0 \in (0, 1)$ be as above. Then it is well-known [Vil09] Theorem 7.35 that the pair $(\phi_{t_0}, \phi_{t_0})$ is a solution for the dual Kantorovich problem w.r.t.
$\mu_t$ and $\mu_s$ for $s \in [0, t_0)$, and similar $(-\phi_t, \phi_s)$ is a solution of the dual Kantorovich problem w.r.t. $\mu_t$ and $\mu_s$ for $s \in (t_0, 1]$. Let $M_{t_0}$ be the set of full m-measure in $B_R$ where $\phi_{t_0}$ is differentiable. By a theorem of McCann [McC01]

$$\gamma_x(t) = \exp_x(-(t_0 - t)\nabla \phi_{t_0}) =: \tilde{T}_{t_0}(x), \quad x \in M_{t_0}$$

is the optimal map between $\mu_{t_0}$ and $\mu_s$ for all $s \in [0, 1 \setminus \{t_0\}$. By uniqueness of optimal maps it holds $T_{t_0}^t = T_{t_0}^t \mu_{t_0}$ a.e. where $T_{t_0}^t = e_{t \circ e_{t_0}}^{-1}$ from before. In particular, recall that $T_{t_0}^t : \tilde{B}(o) \to M$ is Lipschitz.

Since $\phi_t$ is semi-convex, there is a set $M_{t_0}^t \subset M_{t_0}$ of full m-measure where $\phi_{t_0}$ is twice differentiable in Alexandrov sense, and $T_{t_0}^t$ admits a weak differential $DT_{t_0}^t$ on $M_{t_0}^t$ for every $t \in [0, 1]$.  

4. Monge-Ampere inequality.

By the area formula we obtain

$$\int_U \det DT_{t_0}^t(x)\rho_t(T_{t_0}^t(x))e^{-V(T_{t_0}^t(x))}d\text{vol}_M(x) = \int_{T_{t_0}^t(U)} \mathcal{H}^n((T_{t_0}^t)^{-1}(x))\rho_t(x)d\mu_t(x)$$

for every measurable subset $U$ and $t \in [0, 1]$ and $t_0 \in (0, 1)$. On the other hand, by the measure theoretic transformation formula we deduce

$$\int_{T_{t_0}^t(U)} \rho_t(x)d\mu_t(x) = \int_{T_{t_0}^t(U)} dt(T_{t_0}^t)\#\mu_{t_0}(x) = \int_U \rho_{t_0}(x)\mathcal{H}^n(x)\mu_t(x).$$

Since $U$ was arbitrary, we obtain the Monge-Ampere inequality

$$\rho_{t_0}(x)\mathcal{H}^n(x) \leq \det DT_{t_0}^t(x)\rho_t(T_{t_0}^t(x))e^{-V(T_{t_0}^t(x))}$$

for $t_0 \in (0, 1)$ and $t \in [0, 1]$ with equality if $t \in (0, 1)$.

We set $DT_{t_0}^t(x) =: A_t(x)$.

5. Let us fix a transport geodesic $\gamma_x = T_{t_0}^t(x)$ for some $x \in M_{t_0}^t$. The vector field $J_v : t \in [0, 1] \to DT_{t_0}^t(x)v \in T_{t_0}^t(M)$ along $\gamma_x$ is smooth and satisfies the Jacobi equation for any vector $v \in T_xM$ with initial value $J_v(t_0) = v$. If $v \in (\gamma_x(t_0)^{-1})$ then $J_v \in (\gamma_x(t)^{-1})$ for every $t \in [0, 1]$. Hence, one can define a symmetric linear map $B_x(t) := DT_{t_0}^t|_{\gamma_x(t_0)^{-1}} : \gamma_x(t_0)^{-1} \to \gamma_x(t)^{-1}$. Then Jacobi field calculus yields that log det $B_x(t) =: y_x(t)$ solves

$$g_{\gamma_x}(\dot{y}_x(t), \dot{y}_x(t)) + \frac{1}{N-1} (g'_{\gamma_x}(\dot{\gamma}_x(t), \gamma_x(t)) \leq 0.$$  

Moreover, it is easy to check that $y_x(t) - v \circ \gamma_x(t) := z_x(t)$ solves

$$z''_x(t) \frac{1}{N-1} (z'_x(t))^2 + \text{ric}^N(V\gamma_x(t), z'_x(t)) \leq 0.$$  

Hence, with $\text{ric}^N \geq \kappa$ we deduce that $\det B_x(t)e^{-V\gamma_x(t)} =: \mathcal{I}_x(t)$ solves

$$\frac{d^2}{dt^2} \mathcal{I}_x(t) \frac{1}{N-1} |\dot{\gamma}_x|^2 \mathcal{I}_x(t) \frac{1}{N-1} \leq 0$$

and by Proposition [2.11] it follows

$$\mathcal{I}_x(t) \frac{1}{N-1} \geq \sigma^{(1-N)}_{\gamma_x/(N-1)}(\dot{\gamma}_x) \mathcal{I}_x(0) \frac{1}{N-1} + \sigma^{(1)}_{\gamma_x/(N-1)}(\dot{\gamma}_x) \mathcal{I}_x(1) \frac{1}{N-1}.$$  

6. On the other hand, we can consider $J_x(t) = \det A_t(x)e^{-V\gamma_x(t)}$ and $L_x(t) = J_x(t)/\mathcal{I}_x(t)$. In [Sta06, Proof of Theorem 1.7 (c)] it was shown that $L_x(t)$ is
concave. Following part (d) in the proof of Theorem 1.7 in [Stu06b] we obtain
$$
\mathcal{J}_x(t)^{\frac{1}{N}} \geq \tau_{\kappa_x^2,N}(\{\gamma_x\}) \mathcal{J}_x(0)^{\frac{1}{N}} + \tau_{\kappa_x^2,N}(\{\gamma_x\}) \mathcal{J}_x(1)^{\frac{1}{N}}.
$$

7. Thus together with the previous Monge-Ampere inequality and the measure theoretic change of variable formula we obtain
$$
-S_N(\mu_t|\mu_0) = \int_M (\rho_0(x))^{-\frac{1}{N}} \, d\mu_t(x) = \int_M (\rho_0(T_{t_0}^t(x)))^{-\frac{1}{N}} \, d\mu_{t_0}(x)
$$
$$
\geq \int_M \det A_x(t)^{\frac{1}{N}} e^{-\frac{1}{N}V(T_{t_0}^t(x))} e^{\frac{1}{N}V(x)} \rho_0(x)^{1-\frac{1}{N}} \, d\mu(x)
$$
$$
= \int_M \mathcal{J}_x(t)e^{\frac{1}{N}V(x)} \rho_0(x)^{1-\frac{1}{N}} \, d\mu(x)
$$
$$
\geq \int_M \left[ \tau_{\kappa_x^2,N}(\{\gamma_x\}) \rho_0(\gamma_x(0))^{-\frac{1}{N}} + \tau_{\kappa_x^2,N}(\{\gamma_x\}) \rho_1(\gamma_x(1))^{-\frac{1}{N}} \right] \, d\mu_t(x).
$$

Note that $\gamma_x(i) = e_i \circ e_{t_0}^{-1}(x)$, $i = 0, 1$. This proves the forward direction.

We skip the proof of the backward direction since it will not play a role in the rest of the article. □

**Example 2.16.** Let $I \subset \mathbb{R}$ be an interval, let $\kappa : I \to \mathbb{R}$ be a lower semi-continuous function and let $u : I \to [0, \infty)$ be a non-negative solution of $u'' + \frac{\kappa}{N-1} u = 0$ for $N > 1$. Then, the metric measure space $(I, | \cdot |_I, uN^{-1}d\mathcal{L}^1)$ satisfies the curvature-dimension $CD(\kappa, N)$.

**Remark 2.17.** The proof of the previous theorem actually shows more. If $(M, g_M, m)$ is a weighted Riemannian manifold, the condition $CD(\kappa, N)$ holds iff for each pair $\mu_0, \mu_1 \in \mathcal{P}^2(m_x)$ with bounded support there exists a Wasserstein geodesic $\Pi$ with $(\varepsilon_t)_\#\Pi = \mu_t$ such that
$$
\varrho_t(\gamma_0, \gamma_1) \geq \tau_{\kappa_x^2,N}(\{\gamma_t\}) \varrho_0(\gamma_0)^{-\frac{1}{N}} + \tau_{\kappa_x^2,N}(\{\gamma_t\}) \varrho_1(\gamma_1)^{-\frac{1}{N}}
$$
for all $t \in [0, 1]$ and $\Pi$-a.e. $\gamma \in \mathcal{G}(X)$ where $\varrho_t m_x = \mu_t$.

**2.5. The measure contraction property.**

**Definition 2.18 (Oht07, Stu06b).** We say a mm space $X$ satisfies the measure contraction property $MCP(K, N)$ for $K \in \mathbb{R}$ and $N \in (1, \infty)$ if for every $x_0 \in X$ and $A \subset X$ $(A \subset B_{\pi_{K,N-1}}(x_0))$ with $m(A) \in (0, \infty)$ there exists an optimal dynamical plan $\Pi$ such that $(\varepsilon_t)_\#\Pi = \delta_{x_0}$, $(\varepsilon_1)_\#\Pi = m_A$ and
$$
m \geq (\varepsilon_t)_\# \left( \tau_{K,N}^{(t)}(\{\gamma\}) \right)^{N-1} m(A) \Pi.
$$

Ohta shows the following in [Oht07].

**Theorem 2.19.** Let $X$ be a mm space that satisfies the $MCP(K, N)$ for $K > 0$ and $N > 1$. Then $\text{diam sup m}_X \leq \pi_{K,N}$. 

Let $X$ satisfy the $MCP(K, N)$ for $K > 0$ and $N > 1$. We say $y \in X$ is opposite to $x \in X$ if $d_X(x, y) = \pi_{K,N-1}$.

**Theorem 2.20 (Ohta, Oht07).** If $X$ satisfies the $MCP(K, N)$ for $K > 0$ and $N > 1$, then each point $x \in X$ has at most one opposite point.
2.6. Riemannian curvature-dimension condition. Let \((X,d_X)\) be a metric space. For a function \(u : X \rightarrow \mathbb{R}\) the local slope is

\[
\text{Lip}_u(x) = \lim_{y \to x} \sup_{y \neq x} \frac{|u(x)-u(y)|}{d_X(x,y)} \quad \text{if } x \in X \text{ is not isolated},
\]

\[
\text{otherwise.}
\]

For \(u \in L^2(m_X)\) the Cheeger energy is defined in [AGS13, AGS14a] via

\[
\text{Ch}^X(u) = \frac{1}{2} \inf \left\{ \liminf_{h \to \infty} \int_X (\text{Lip}_u(x))^2 \, d m_X : \|u_h - u\|_{L^2(m_X)} \to 0 \right\}.
\]

Then the \(L^2\)-Sobolev space is given by \(D(\text{Ch}^X) = \{ u \in L^2(m_X) : \text{Ch}^X(u) < \infty \}\), and we say that a mm space \(X\) is \textit{infinetimally Hilbertian} if the associated Cheeger energy is quadratic.

\textbf{Definition 2.21} ([AGS14b, Gig15, EKS15, AGMR15, AMS19, CM16]). Let \(X\) be a metric measure space, \(K \in \mathbb{R}\) and \(N \geq 1\). We say that \(X\) satisfies the \textit{Riemannian curvature-dimension condition} \(RCD(K,N)\) if \(X\) is infinetimally Hilbertian and satisfies the condition \(CD(K,N)\).

We say that an mm space \(X\) is \textit{essentially nonbranching} if for any optimal dynamical coupling \(\Pi\) there exists \(A \subset G(X)\) such that \(\Pi(A) = 1\) and for all \(\gamma, \gamma' \in A\) we have that \(\gamma(t) = \gamma'(t)\) for all \(t \in [0,c]\) and for some \(c > 0\) implies \(\gamma = \gamma'\).

In [RS14] Sturm and Rajala showed that \(RCD(K,N)\)-spaces are essentially nonbranching.

2.7. Integral Ricci curvature quantities. Let \(X\) be a mm space, let \(\kappa : X \rightarrow \mathbb{R}\) be a continuous function such that \(X\) satisfies \(CD(K,N)\). Consider

\[
\| (\kappa - K)_- \|_{L^p(m_X)} = \left( \int_A (\kappa - K)^p \, d m_X \right)^{\frac{1}{p}} \in [0,\infty],
\]

where \((\kappa - K)_- = - \min(\kappa - K,0)\). If \(m_X(A) < \infty\), we set

\[
\| (\kappa - K)_- \|_{L^p(\tilde{m}_A)} = \left( \int (\kappa - K)^p \, d \tilde{m}_A \right)^{\frac{1}{p}} \in [0,\infty].
\]

In particular, if \(m_X(X) < \infty\), for the isomorphism class \([X]\) we define

\[
k_{[X]}(\kappa,p,K) = (\text{diam } X)^2 \| (\kappa - K)_- \|_{L^p(m_X)}.
\]

We note that \(k_{[X]}(\kappa,p,K)\) behaves naturally under scaling and transformations by mm space isomorphisms. If \(\psi : X \rightarrow X'\) is an mm space isomorphism, by Proposition 2.13 we have that \(X'\) satisfies \(CD(\kappa',N)\) for \(\kappa' = \kappa \circ \psi\) and we compute

\[
k_{[X]}(\kappa,p,K)^p = \frac{1}{(\text{diam } X)^{2p}} \left( \int (\kappa - K)^p \, d \tilde{m}_X \right)^{\frac{1}{p}} = \left( \int (\kappa \circ \psi - K)^p \, d \psi^\# \tilde{m}_X \right)^{\frac{1}{p}} = \left( \int (\kappa' - K)^p \, d \tilde{m}_{X'} \right)^{\frac{1}{p}} = \frac{k_{[X]}(\kappa',p,K)^p}{(\text{diam } X')^{2p}}.
\]

Similar, if \(rX = (X, rd_X, m_X)\) is a rescaling of \(X\) by \(r \in \mathbb{R}\), one can check that

\[
k_{[X]}(\kappa,p,K) = k_{[rX]}(r^{-1} \kappa, p, r^{-1} K).
\]
Let \( o \in X \). Recall that for \([X, o]\) we pick the representative such that \( m_X(B_1(o)) = 1 \). We define
\[
k_{[X, o]}(\kappa, p, K, R) = R^2 \| (\kappa - K)_{-1} \|_{L^p(m_X)}.
\]

We can check that \( k_{[X, o]}(\kappa, p, K, R) \) again behaves naturally under scaling and under pmm space isomorphisms.

### 3. Precompactness

Let \( X \) be a metric measure space that is essentially nonbranching and satisfies \( CD(\kappa, N) \) for some admissible function \( \kappa : X \to \mathbb{R} \).

**Spherical disintegration.** We fix a point \( x_0 \in X \), and consider the disjoint decomposition \( X = \bigcup_{r > 0} X_r \) where \( X_r = \partial B_r(x_0) \). According to this decomposition the measure \( m_X \) can be disintegrated as follows
\[
m_X = \int m_r s(dr)
\]
where \( m_r \in \mathcal{P}^2(X) \) and \( s \) is a measure on \((0, \infty)\). The condition \( CD(\kappa, N) \) implies that \( r \mapsto m_X(\partial B_r(x_0)) \) is weakly differentiable. Hence, the measure \( s \) is \( L^1 \)-absolutely continuous (for instance, see the proof of Theorem 5.3 in [Ket17]). With slight abuse of notation we write \( s(dr) = s(r)\mathcal{L}^1(dr) \) and set \( m_r = s(r)m_r \). Note that
\[
m_X(\partial B_R(x_0)) = \int_0^R m_r(\partial B_R(x_0)) dr =: v(R) \quad \text{and} \quad \frac{d}{dr} m_X(\partial B_r(x_0)) = m_r(\partial B_r(x_0)) =: s(r).
\]

**Example 3.1.** Let \( I_{K/(N-1)} \) be the model space \( ([0, \pi_{K/(N-1)}], \sin_{K/(N-1)} dr) \). In this situation we choose \( x_0 = 0 \), and we have
\[
m_{K,N}(\partial B_R(x_0)) = \int_0^R \sin_{K/(N-1)}^{N-1} dr \quad \text{and} \quad \frac{d}{dr} m_X(\partial B_r(x_0)) = \sin_{K/(N-1)}^{N-1} r.
\]

We set \( v_{K,N}(r) = m_{K,N}(\partial B_r(0)) \) and \( s_{K,N}(r) = \sin_{K/(N-1)}^{N-1} r \).

**Proposition 3.2.** Let \( X \) be a mm space that is essentially nonbranching and satisfies the condition \( CD(\kappa, N) \). Let \( x_0 \in \text{supp} \ m_X \) and assume that
\[
\int_{B_D(x_0)} (\kappa - K)^p d m < \infty
\]
for some \( p > \frac{N}{2} \), \( K \leq 0 \) and \( D > 0 \).

Then, there exists a constant \( C(K, N, p, D) > 0 \) such that
\[
\frac{d}{dR} \log \left( \frac{m_X(B_R(x_0))}{\int_0^R w_{K,N-1}(r) dr} \right) \leq C(K, N, p, D) \left( \frac{R}{m_X(B_R(x_0))} \int_{B_R(x_0)} (\kappa - K)^p d m \right)^\frac{1}{p},
\]
for every \( R \in (0, D) \).

**Proof. 1.** Consider the spherical disintegration \( \{m_r\}_{r \in [0, \infty)} \) w.r.t. \( x_0 \in X \). Let \( R \in (0, \infty) \), and let \( (\mu_r)_{r \in [0, R]} \) be the constant speed \( L^2 \)-Wasserstein geodesic that connects \( \delta_{x_0} = \mu_0 \) and \( m_R = \mu_R \). Let \( \Pi \) be the induced optimal dynamical plan. Then the \( (e_t)_{\#} \Pi =: \mu_t \Pi \) is supported on \( \partial B_{\Pi R}(x_0) \).

Provided \( X \) satisfies a curvature-dimension condition for \( \kappa = \text{const} \) Cavalletti and Sturm prove that \( \mu_r \) is absolutely continuous w.r.t. \( m_r \) [CSI12 Lemma 3.2]. It is easy to check that their argument also applies in our context. We write \( d\mu_r = \hat{h}_r dm_r \). We denote by \( \Gamma \) the support of \( \Pi \). \( \Pi \)-almost every \( \gamma \in \Gamma \) has the...
length $R$ and we define $h_r : \Gamma \to \mathbb{R}$ with $h_r(\gamma) := \hat{h}_r(\gamma)$. Following again arguments of Cavalletti and Sturm [CS12, Theorem 5.2] one proves for $r_0, r_1 \in (0, R)$

$$
(10) \quad h_{(1-t)r_0 + tr_1}(\gamma) \geq \sigma^{(1-t)}_{\kappa, \gamma/N-1}(\gamma)h_{r_0}(\gamma) + \sigma^{(t)}_{\kappa, \gamma/N-1}(\gamma)h_{r_1}(\gamma)
$$

for $\Pi$-a.e. $\gamma \in G^{[0,R]}(X)$ where $\varsigma(t) = \gamma((1-t)r_0 + tr_1)$.

We set $r \in (0, R] \mapsto \omega(r) := h_r(\gamma)^{-1}$ and $g(r) = \log \omega(r)$. In particular, it follows $\omega(t)g'(t) = \omega'(t)$ and [10] is equivalent to

$$
(11) \quad g'' + \frac{1}{N-1}g^2 + \kappa \circ \gamma \leq 0 \quad \text{on} \quad (0, R] \quad \text{for} \quad \Pi\text{-a.e.} \quad \gamma.
$$

In the following we omit the dependence on $\gamma \in \Gamma$. We have equality in [10] and [11] if $h_r(\gamma)^{-1} := h_{r,K,N-1}(\gamma)^{-1} = s_{K,N-1}(r)$. In this case we write $g_{K,N-1}$ and $\omega_{K,N-1}$ for $g$ and $\omega$ respectively. Then we compute

$$
\frac{d}{dr} \omega_{K,N-1}(r) = \frac{\omega'(r)}{\omega_{K,N-1}(r)} - \frac{\omega'_{K,N-1}(r)\omega(r)}{\omega_{K,N-1}(r)^2}
$$

$$
\leq \left[ g'(r) - g'_{K,N-1}(r) \right] \frac{\omega(r)}{\omega_{K,N-1}(r)} \frac{\omega'(r)}{\omega_{K,N-1}(r)}
$$

where $\psi_{K,N-1}(r) := \max \{0, g'(r) - g'_{K,N-1}(r)\}$. Note that $\psi$ depends also on $\kappa \circ \gamma$.

2. Integration of [12] w.r.t. $r$ from $r_0 \in (0, R)$ to $R$ yields

$$
\frac{\omega(R)}{\omega_{K,N-1}(R)} \frac{\omega(r_0)}{\omega_{K,N-1}(r_0)} \leq \int_{r_0}^{R} \psi_{K,N-1}(r) \omega(r) dr.
$$

Since $K \leq 0$, we have that $r \in [0, \infty) \mapsto \omega_{K,N-1}(r)$ is monotone increasing (and positive). Hence, multiplication with $\omega_{K,N-1}(R)$ and $\omega_{K,N-1}(r_0)$ yields

$$
\omega(R)\omega_{K,N-1}(r_0) - \omega(r_0)\omega_{K,N-1}(R) \leq \omega_{K,N-1}(R) \int_{r_0}^{R} \psi_{K,N-1}(r) \omega(r) dr
$$

$$
\leq \omega_{K,N-1}(R) R \int_{0}^{1} \psi_{K,N-1}(\tau R) \omega(\tau R) d\tau.
$$

Recall that

$$
\int \omega(R) d\Pi = m_{R}(X) \quad \& \quad \int \omega(\tau R) d\Pi = \int_{\{h_{r,R} > 0\}} \hat{h}_{r,R}(x)^{-1} d\mu_{r,R}(x) \leq m_{R}(X).
$$

Therefore, integration of [13] w.r.t. $\Pi$ and application of Fubini’s theorem yield

$$
\int \omega(R) d\Pi \omega_{K,N-1}(r_0) - \int \omega(r_0) d\Pi \omega_{K,N-1}(R)
$$

$$
\leq \omega_{K,N-1}(R) R \int_{0}^{1} \psi_{K,N-1}(\tau R) \omega(\tau R) d\tau d\Pi.
$$

Oberserve

$$
R \int \int_{0}^{1} \psi_{K,N-1}(\tau R) \omega(\tau R) d\tau d\Pi
$$

$$
\leq \left( R \int \int_{0}^{1} \psi_{K,N-1}^{2p-1}(\tau R) \omega(\tau R) d\tau d\Pi \right)^{\frac{1}{2p-1}} \left( R \int_{0}^{1} \omega(\tau R) d\Pi d\tau \right)^{1-\frac{1}{2p-1}}.
$$
where \( R \int_0^1 \omega(\tau R) d\Pi d\tau \leq \int_0^R m_r \, dr = m(B_R(x_0)) \). Hence
\[
m_R(X) \omega_{K,N-1}(r_0) - m_{r_0}(X) \omega_{K,N-1}(R) \leq \omega_{K,N-1}(R) \, m(B_R(x_0)) \left( \frac{1}{m_x(B_R(x_0))} \int_0^R \psi^{2p-1}_{K,N-1}(r) \omega(r) d\Pi d\tau \right)^{\frac{1}{2p-1}}.
\]

Another integration w.r.t. \( r_0 \) from 0 to \( R \) yields
\[
m_R(X) \int_0^R \omega_{K,N-1}(r_0) dr_0 - m_x(B_R(x_0)) \omega_{K,N-1}(R) \leq R \omega_{K,N-1}(R) \, m(B_R(x_0)) \left( \frac{1}{m_x(B_R(x_0))} \int_0^R \psi^{2p-1}_{K,N-1}(r) \omega(r) d\Pi d\tau \right)^{\frac{1}{2p-1}}.
\]

The left hand side actually is
\[
\left[ \int_0^R \omega_{K,N-1}(r) dr \right]^2 \frac{d}{dR} \frac{m_x(B_R(x_0))}{\int_0^R \omega_{K,N-1}(r) dr}.
\]
Hence
\[
\left( \frac{m_x(B_R(x_0))}{\int_0^R \omega_{K,N-1}(r) dr} \right)^{-1} \frac{d}{dR} \frac{m_x(B_R(x_0))}{\int_0^R \omega_{K,N-1}(r) dr} \leq \omega_{K,N-1}(R) \left( \frac{1}{m_x(B_R(x_0))} \int_0^R \psi^{2p-1}_{K,N-1}(r) \omega(r) d\Pi d\tau \right)^{\frac{1}{2p-1}}.
\]

We estimate \( \frac{\omega_{K,N-1}(R)}{\int_0^R \omega_{K,N-1}(r) dr} \) by \( \max_{R \in [0, R]} \frac{\omega_{K,N-1}(R)}{\int_0^R \omega_{K,N-1}(r) dr} =: \Xi(K, N, D) \).

3. Recalling Proposition 4.1 we obtain
\[
\frac{d}{dR} \log \left( \frac{m_x(B_R(x_0))}{\int_0^R \omega_{K,N-1}(r) dr} \right) \leq C(K, N, p, D) \left( \frac{1}{m_x(B_R(x_0))} \int_0^R \int_0^1 R(\kappa(\tau R - K)^p \omega(\tau R) d\Pi d\tau dr \right)^{\frac{1}{2p-1}}
\]
where \( C^1(K, N, p, D) = \Xi(K, N, D) C(p, N)^{\frac{1}{2p-1}} \). Recall that \( \omega(\tau) = h_\tau(\gamma)^{-1} \) and therefore
\[
\int (\kappa(\gamma_R) - K)^p \omega(\tau R) d\Pi = \int (\kappa - K)^p h_\tau^{-1} d\Pi(x, \tau R) \leq \int (\kappa - K)^p \, d\tau R.
\]
Hence
\[
\frac{d}{dR} \log \left( \frac{m_x(B_R(x_0))}{\int_0^R \omega_{K,N-1}(r) dr} \right) \leq C(K, N, p, D) \left( \frac{R}{m_x(B_R(x_0))} \int_{B_R(x_0)} (\kappa - K)^p \, d\tau \right)^{\frac{1}{2p-1}} \quad \Box
\]

**Remark 3.3.** In the case of \( CD(K, N) \) where \( K = const \) we get that
\[
\frac{d}{dR} \frac{m_x(B_R(x_0))}{\int_0^R \sin_{K/(N-1)}^2 \tau d\tau} \leq 0.
\]
This is the classical Bishop-Gromov volume comparison.
Corollary 3.4. Let \( C \) be a mm space that is essentially nonbranching and satisfies the condition \( CD(\kappa, N) \). Let \( R > 0 \), \( r \in (0, R) \) and \( x_0 \in \text{supp}_X \) such that \( \| (\kappa - K)^{-1} B_R(x_0) \|_{L^p(\text{supp}_X)} < \infty \) for some \( p > \frac{N}{2} \), and \( K \leq 0 \). Then
\[
\frac{m_X(B_R(x_0))}{v_{K,N}(R)} \leq \frac{m_X(B_r(x_0))}{v_{K,N}(r)} C^{(K,N,p,D)}(\frac{\overline{R}^{2p}}{v_{K,N}(R)}) \int_{B_R(x_0)} (\kappa - K)^p \, dm \leq D^{2p} \int_{B_D(o)} (\kappa - K)^p \, dm = k_{[X,o]}(p,K,D)^p.
\]

Corollary 3.5. Let \((X,o)\) be a normalized pmm space that is essentially nonbranching and satisfies the condition \( CD(\kappa, N) \). Let \( D \geq 1 \) and \( p > \frac{N}{2} \). Then
\[
\frac{m_X(B_R(x))}{v_{K,N}(R)} \leq \frac{m_X(B_r(x))}{v_{K,N}(r)} C^{(K,N,p,D)}(\frac{\overline{R}^{2p}}{v_{K,N}(R)}) \int_{B_R(x_0)} (\kappa - K)^p \, dm \leq D^{2p} \int_{B_D(o)} (\kappa - K)^p \, dm = k_{[X,o]}(p,K,D)^p.
\]
for each \( x \in B_{\frac{R}{2}}(o) \cap \text{supp}_X \) and \( 0 < r < 1 \leq \frac{1}{2}R \leq 2R \leq D \).

Proof. Let \( x \in B_{R/2}(o) \). It holds
\[
B_1(o) \subset B_{R/2}(o) \subset B_R(x) \subset B_{2R}(o) \subset B_D(o)
\]
Therefore
\[
\frac{R^{2p}}{m_X(B_R(x_0))} \int_{B_R(x_0)} (\kappa - K)^p \, dm \leq D^{2p} \int_{B_D(o)} (\kappa - K)^p \, dm = k_{[X,o]}(p,K,D)^p.
\]
and together with the previous Corollary the claim follows. 

Corollary 3.6. Consider the family \( X(p,K,N) \) of isomorphism classes of essentially nonbranching pmm spaces \((X,o)\) satisfying a condition \( CD(\kappa, N) \) for some \( \kappa \in C(X) \) such that \( k_{[X,o]}(p,K,D) \leq f(D) \) for all \([X,o] \in X(p,K,N), \) for all \( D \geq 1, \ p > \frac{N}{2} \) and some function \( f : [1, \infty) \to \mathbb{R} \). Then \( X(p,K,N) \) is precompact in the sense of pmG convergence.

Proof. Consider \([X,o] \in X(p,K,N,k),\) and let \( D > 0. \) Let \((X,o)\) be a normalized representative of \([X,o].\) Let \( 0 < r < 1 \leq \frac{1}{2}R \leq 2R \leq D. \) By the previous corollary the number of \( r \)-balls with center in \( B_{R/2}(o) \) is bounded by constant that depends only on \( K,N,p,r \) and \( D \geq 2R. \) Hence the family of pmm spaces with isomorphism class contain in \( X(p,K,N) \) is uniformly totally bounded. Moreover
\[
\sup_{[X,o] \in X(p,K,D)} m_X(B_R(x)) \leq \frac{v_{K,N}(R)}{v_{K,N}(r)} C^{(K,N,p,D)}(\int_{B_D(o)} (\kappa - K)^p \, dm = k_{[X,o]}(p,K,D)^p).
\]
Hence, the family of mm spaces such that the corresponding isomorphism class is contained in \( X(p,K,N) \) is precompact w.r.t. pmGH convergence according to Theorem 2.3 Hence, the corresponding isomorphism classes are precompact w.r.t. pG convergence. 

4. 1-DIMENSIONAL ESTIMATES

Let \( K \in \mathbb{R} \) and \( N \in [2, \infty), \) and let \( \kappa : [0, L] \to \mathbb{R}. \) We set \( \omega(r) = \sin^{N-1} \kappa/(N-1) (r). \) The function \( g(r) = \log \omega(r) \) solves
\[
g'' + \frac{1}{N-1} (g')^2 + \kappa = 0 \quad \text{on} \quad [0, \pi_{K/(N-1)}).
\]
Let \( g_{K,N-1} : [0, L] \to \mathbb{R} \) be defined as \( g_{K,N-1} = \log \sin^{N-1} \kappa/(N-1). \) The function \( g_{K,N-1} \) is a solution of
\[
g''_{K,N-1} + \frac{1}{N-1} (g'_{K,N-1})^2 + K = 0 \quad \text{on} \quad [0, \pi_{K/(N-1)}].
\]
Let \( \psi_{K,N-1}(r) = \max \{0, g'(r) - g'_{K,N-1}(r)\} \) that solves
\[
\psi_{K,N-1}' + \frac{\psi_{K,N-1}^2}{N-1} + \frac{2\psi_{K,N-1} g_{K,N-1}}{N-1} \leq \kappa \quad \text{&} \quad \lim_{r \to 0} \psi_{K,N-1}(r) = 0.
\]

The following Theorem by Petersen/Wei/Aubry in [PW97, Lemma 2.2] and [Aub07, Lemma 3.1] is fundamental in the theory of integral Ricci curvature bounds.

**Proposition 4.1.** Let \( \omega \) and \( \psi_{K,N-1} \) be as above. Then
\[
\psi_{K,N-1}(r_0)^{2p-1} \omega(r_0) \leq C'(p, N) \int_{0}^{r_0} (\kappa(r) - K)^p \omega(r) dr.
\]
where \( r_0 \in [0, L \wedge \pi_K/(N-1)] \cap [0, \frac{1}{2} \pi_K/(N-1)] \) and
\[
\sin_{K/(N-1)}(r_0)^{4p-n-1} \psi_{K,N-1}(r_0)^{2p-1} \leq C''(p, N) \int_{0}^{r_0} (\kappa(r) - K)^p \omega(r) dr.
\]
where \( r_0 \in [0, L] \cap (\frac{1}{2} \pi_K/(N-1), \pi_K/(N-1)] \) and \( C'(p, N) = (2p - 1)^p \left( \frac{N-1}{2p-K} \right)^{p-1} \).

**Remark 4.2.** Note that \( C'(p, 2) = (2p - 1)^p \left( \frac{1}{2p-2} \right)^{p-1} \) as \( p \to 1 \).

**Remark 4.3.** For \( K > 0, \kappa > 0, L \in (0, \pi_K/(N-1) - \epsilon) \) and \( r_0 \in (0, L) \) it holds that
\[
\psi_{K,N-1}(r_0)^{2p-1} \omega(r_0) \leq C''(p, N, K, \epsilon) \int_{0}^{r_0} (\kappa(r) - K)^p \omega(r) dr.
\]
where \( C''(p, N, K, \epsilon) = \max \{C'(p, N), \sin_{K/(N-1)}(\epsilon)^{-4p+n+1}\} \). We set
\[
C := C(p, N, K, \epsilon) := \begin{cases} C'(p, N) & \text{if } K \leq 0, \\ C''(p, N, K, \epsilon) & \text{if } K > 0. \end{cases}
\]

**Corollary 4.4.** For \( K \in \mathbb{R}, N \in [2, \infty), p > \frac{N}{2}, \epsilon > 0 \) and \( \theta \in (0, L \wedge \pi_K/(N-1)) \cap (0, \pi_K/(N-1) - \epsilon) \) it holds
\[
\int_{0}^{\theta} \psi_{K,N-1}(r)^{2p-1} \omega(r) dr \leq C\theta \int_{0}^{\theta} (\kappa(r) - K)^p \omega(r) dr.
\]

**Remark 4.5.** Recall the distortion coefficient
\[
r \in [0, \theta] \mapsto \sigma^{(r/\theta)}_{\kappa/(N-1)}(\theta) = \frac{\sin_{\kappa/(N-1)}(r)}{\sin_{\kappa/(N-1)}(\theta)}
\]
for \( \theta \in [0, \pi_K/(N-1)] \) and \( \theta \in (0, L \wedge \pi_K/(N-1)) \cap (0, \frac{1}{2} \pi_K/(N-1) - \epsilon) \) we have the inequality
\[
\int_{0}^{\theta} \psi_{K,N-1}(r) \sigma^{(r/\theta)}_{\kappa/(N-1)}(\theta)^{N-1} dr \leq \left( Cg^{2p-1} \int_{0}^{\theta} (\kappa(r) - K)^p \omega_{\kappa/(N-1)}(\theta)^{N-1} dr \right)^{\frac{1}{p-1}} \left( \int_{0}^{1} \sigma^{(s)}_{\kappa/(N-1)}(\theta)^{N-1} ds \right)^{\frac{2p-2}{p-1}}.
\]
(15)
Indeed, the transformation \( t \mapsto t\theta \) and Hölder’s inequality yield
\[
\int_{t\theta}^\theta \psi_{K,N-1}(r) \sin^{N-1}_{\kappa/(N-1)}(r) dr \leq \int_t^1 \theta \cdot \psi_{K,N-1}(s\theta) \sin^{N-1}_{\kappa/(N-1)}(s\theta) ds
\]
\[
\leq \left( \int_t^1 (\theta \cdot \psi_{K,N-1}(s\theta))^{2p-1} \sin^{N-1}_{\kappa/(N-1)}(s\theta) ds \right)^{\frac{1}{2p-1}} \left( \int_t^1 \sin^{N-1}_{\kappa/(N-1)}(s\theta) ds \right)^{1-\frac{1}{2p-1}}
\]
\[
\leq \left( \theta^{2p-2} \int_0^\theta \psi_{K,N-1}(r)^{2p-1} \sin^{N-1}_{\kappa/(N-1)}(r) dr \right)^{\frac{1}{2p-1}} \left( \int_t^1 \sin^{N-1}_{\kappa/(N-1)}(s\theta) ds \right)^{1-\frac{1}{2p-1}}
\].

Then, we can apply Corollary 4.4 and divide by \( \sin^{N-1}_{\kappa/(N-1)}(\theta) \) to obtain (15).

**Lemma 4.6.** Let \( K \in \mathbb{R}, \ N \geq 2, \ \kappa : [0, L] \to \mathbb{R}, \ N \) and \( \epsilon > 0 \) be as before, and let \( \theta \in (0, L \wedge \pi_{K/(N-1)} - \epsilon) \). Then
\[
\tau_{K,N}^{(t)}(\theta) \leq \tau_{K,N}^{(t)}(\theta) + \Lambda \frac{1}{\sin^{N-1}_{\kappa/(N-1)}(\theta) N!}
\]
where the constant \( \Lambda \) is given by
\[
\Lambda := \Lambda(K,N,\epsilon) = 1 + \max_{r \in [\frac{1}{2} \pi_{K/(N-1)}, \pi_{K/(N-1)} - \epsilon]} \frac{1}{\sin^{N-1}_{\kappa/(N-1)}(r) N!}
\]
Note that \( \pi_{K/(N-1)} = \infty \) if \( K \leq 0 \). In this case we set \( \Lambda = 1 \).

**Proof.** Recall that by definition \( \tau_{K,N}^{(t)}(\theta) = \infty \) if \( \theta \geq \pi_{\kappa/(N-1)} \). Then the inequality holds. So we assume that \( L < \pi_{\kappa/(N-1)} \).

We observe that \( \log \sin^{N-1}_{\kappa/(N-1)} = g \) solves
\[
\frac{d}{dr} \sin^{N-1}_{\kappa/(N-1)} = \sin^{N-1}_{\kappa/(N-1)} \frac{d}{dr} g.
\]
Then, we compute
\[
\frac{d}{dr} \left[ \frac{\sin^{N-1}_{\kappa/(N-1)}}{\sin^{N-1}_{K/(N-1)}} \right] \leq \psi_{K,N-1} \left[ \frac{\sin^{N-1}_{\kappa/(N-1)}}{\sin^{N-1}_{K/(N-1)}} \right]^{N-1}
\]
\[
\left[ \frac{\sin^{N-1}_{\kappa/(N-1)}(\theta)}{\sin^{N-1}_{K/(N-1)}(\theta)} \right]^{N-1} - \left[ \frac{\sin^{N-1}_{\kappa/(N-1)}(t\theta)}{\sin^{N-1}_{K/(N-1)}(t\theta)} \right]^{N-1} \leq \int_{t\theta}^{\theta} \psi_{K,N}(r) \left[ \frac{\sin^{N-1}_{\kappa/(N-1)}(r)}{\sin^{N-1}_{K/(N-1)}(r)} \right]^{N-1} dr.
\]
Crossmultiplication gives
\[
\sigma_{K/(N-1)}^{(t)}(\theta)^{N-1} - \sigma_{K/(N-1)}^{(t)}(\theta)^{N-1} \leq \int_{t\theta}^{\theta} \psi_{K,N}(r) \left[ \frac{\sigma_{\kappa/(N-1)}^{(r/\theta)}(\theta)}{\sin^{N-1}_{K/(N-1)}(r)} \right]^{N-1} dr.
\]
Let us consider the left hand side of the previous inequality, and recall that \( r \in [0, L \wedge \frac{\pi_{K/(N-1)} - \epsilon}{2}] \mapsto \sin^{N-1}_{K/N}(r) \) is monotone increasing. It follows that
\[
\int_{t\theta}^{\theta} \psi_{K,N}(r) \left[ \frac{\sigma_{\kappa/(N-1)}^{(r/\theta)}(\theta)}{\sin^{N-1}_{K/(N-1)}(r)} \right]^{N-1} dr \leq \Lambda \int_{t\theta}^{\theta} \psi_{K,N}(r) \sigma_{\kappa/(N-1)}^{(r/\theta)}(\theta)^{N-1} dr.
\]
Therefore it follows
\[
\sigma_{K/(N-1)}^{(t)}(\theta)^{N-1} \leq \sigma_{K/(N-1)}^{(t)}(\theta)^{N-1} + \Lambda \theta \int_t^1 \psi_{K,N-1}(s\theta) \frac{\sigma_{\kappa/(N-1)}^{(s)}}{\sigma_{K/(N-1)}^{(s)}}(\theta)^{N-1} ds
\]
Theorem 5.1. Let $x$ be Lipschitz continuous. Since $(\alpha + \beta)^{1/N} \leq \alpha^{1/N} + \beta^{1/N}$, $\alpha, \beta > 0$, the claim follows.

We note that

$$t \int_t^1 \sigma_{\kappa/(N-1)}(\theta)^{N-1} d\theta \leq \int_t^1 \tau_{\kappa,N}(\theta)^N d\theta.$$ 

Then, by Remark 4.5 and Lemma 4.6 we obtain the following.

**Corollary 4.7.** Let $K \in \mathbb{R}$, $N \geq 2$, $p > N/2$, $\kappa : [0, L] \to \mathbb{R}$, $\epsilon > 0$ and $\theta \in (0, L \wedge \pi_{K/(N-1)} - \epsilon)$ as before. Then

$$\tau_{\kappa,N}^{(1)}(\theta) - \tau_{\kappa,N}^{(0)}(\theta) \leq \Lambda^\frac{1}{N-1} \left( C t th_p \int_0^1 (\kappa(s) - K)^p \sigma_{\kappa/(N-1)}(\theta)^{N-1} ds \right)^{\frac{1}{N-1}} \left( \int_t^1 \tau_{\kappa,N}(\theta)^N d\theta \right)^{\frac{2-N}{N-1}}.$$

5. An Application of Area and Co-area Formula

For this section we consider the following setup. Let $\Pi$ be a dynamical optimal plan with $(e_{i})_{i} \Pi = \mu_{i}$, $t \in [0, 1]$ and $\mu_{i} \in \mathcal{P}(m)$, $i = 0, 1$ such that $\mu_{i}$, $i = 0, 1$, are supported in $B_{R}(o)$ for some $o \in M$. Then $\mu_{t}$ is supported in $B_{2R}(o) = \Pi B_{R}$ for all $t \in [0, 1]$. Let us fix $t_{0} \in (0, 1)$.

Recall from paragraph 3 in the proof of Theorem 2.14 that there exists a Lipschitz map $e_{t_{0}}^{-1} : B_{R} \to \mathcal{G}(M)$ such that $e_{t_{0}}^{-1}(B_{R}) \subset \mathcal{G}(M)$ is a monotone set, $(e_{t_{0}}^{-1})_{#} \mu_{0} = \Pi$ and $e_{t_{0}}^{-1}(x) = \gamma_{x}(\cdot) = T_{t_{0}}(x)$ is given by $\gamma_{x}(s) = \exp_{x}(-(t_{0} - s)\nabla \phi_{t_{0}}(x))$ for every $x \in M_{t_{0}}$ where $\phi_{t_{0}} : B_{R} \to R$ is the Hamilton-Jacobi shift of a $\frac{1}{2}d^{2}$-convex function $\phi : B_{R} \to \mathbb{R}$. The set $M_{t}$ is the set of differentiability of $\phi_{t}$.

Since the map $(e_{0}, e_{1}) \circ e_{t_{0}}^{-1} : M_{t} \to M \times M$ is Lipschitz continuous, the function

$$x \in B_{R} \mapsto l_{t_{0}}(x) := L(e_{t_{0}}^{-1}(x)) = L(\gamma_{x}) = d_{M}(\gamma_{x}(0), \gamma_{x}(1)) = |\nabla \phi_{t_{0}}(x)|$$

is Lipschitz continuous as well. Hence $U := l_{t_{0}}^{-1}((0, \infty)) \cap B_{R}$ is an open set and $\text{vol}_{M}(U) > 0$. We also recall from paragraph 3 in the proof of Theorem 2.14 that $M_{t_{0}} \subset B_{R}$ where $\phi_{t_{0}}$ is twice differentiable, is a set of full $m$-measure in $B_{R}$. In particular, for $x \in M_{t_{0}'} \cap U$ there exists a unique gradient $\nabla \phi_{t_{0}}(x)$ and $\nabla \phi_{t_{0}}(x) \neq 0$. $M_{t_{0}'} \cap U$ is a measurable set.

**Theorem 5.1.** Let $M$ be an $n$-dimensional Riemannian manifold and let $\phi : M \to \mathbb{R}$ be Lipschitz continuous.

(i) (Coarea formula): Let $A \subset M$ be $\mathcal{H}^{n}$-measurable. Then, $A \cap \phi^{-1}(y)$ is $\mathcal{H}^{n-1}$-measurable for $L^{1}$-a.e. $y \in \mathbb{R}$, the map $y \mapsto \mathcal{H}^{n-1}(A \cap \phi^{-1}(y))$ is $L^{1}$-measurable, and it holds

$$\int_{A} |\nabla \phi|(x) d\mathcal{H}^{n}(x) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(A \cap f^{-1}(y)) dL^{1}(y).$$
(ii) (Level set integration): Let $g$ be $\mathcal{H}^n$-integrable. Assume $\text{essinf} \, |\nabla \phi| > 0$. Then, $g|_{\phi^{-1}(y)}$ is $\mathcal{H}^{n-1}$-integrable for $L^1$-a.e. $y \in \mathbb{R}$, and

$$
\int_{\{\phi \geq t\}} g(x)|\nabla \phi(x)| \mathcal{H}^n(x) = \int_t^\infty \left[ \int_{\{\phi = s\}} g(x) \mathcal{H}^{n-1}(x) \right] \, ds.
$$

We apply the coarea formula to $\phi = \phi_{i_0}$ and $A = M'_{i_0} \cap U$. It follows that for $L^1$-a.e. $a \in \mathbb{R}$ the set $\{\phi_{i_0} = a\} \cap M'_{i_0} \cap U$ is $\mathcal{H}^{n-1}$-measurable. In other words, there exists a set $Q \subset \mathbb{R}$ of full $L^1$-measure such that $\{\phi_{i_0} = a\} \cap M'_{i_0} \cap U$ is $\mathcal{H}^{n-1}$-measurable for all $a \in Q$.

Lemma 5.2. Let $g : M \to [0, \infty)$ be measurable. Then it holds

$$
\int_{M'_{i_0} \cap U} g \, d\nu M = \int_Q \int_{\{\phi_{i_0} = a\}} 1_{M'_{i_0} \cap U} g \frac{1}{|\nabla \phi_t|} \, d\mathcal{H}^{n-1} \, da.
$$

Proof. We consider $U_\eta = \{ |\nabla \phi_{i_0}|(x) > \eta \}$. Then by level set integration we get

$$
\int_{M'_{i_0} \cap U_\eta} g \, d\nu M = \int_Q \int_{\phi_{i_0} = a} 1_{M'_{i_0} \cap U_\eta} g \frac{1}{|\nabla \phi_t|} \, d\mathcal{H}^{n-1} \, da.
$$

Since $U_\eta \uparrow U = \bigcup_{\delta > 0} U_\delta$ for $\delta \downarrow 0$, by an application of the monotone convergence theorem we obtain the desired statement.

Lemma 5.3. Let $a \in Q$. There exists a countably $\mathcal{H}^{n-1}$-rectifiable set $\Sigma_a \subset \{\phi_{i_0} = a\}$ such that $\mathcal{H}^{n-1}((\{\phi_{i_0} = a\} \cap M'_{i_0} \cap U) \setminus \Sigma_a) = 0$.

Proof. Recall the following Theorem that appears in the Appendix of [McC93].

Theorem 5.4 (Non-smooth implicit function theorem). Let $\phi : M \to \mathbb{R}$ be semiconvex. If $\nabla \phi(x_0) \neq 0$ for some $x_0 \in M$ and $\phi(x_0) = a$, then there exists $\delta > 0$ such that $\mathcal{H}^{n-1}((\{\phi = a\} \cap B_\delta(x_0)) < \infty$ and $\{\phi = a\} \cap B_\delta(x_0)$ is countably $\mathcal{H}^{n-1}$-rectifiable.

We pick $x_0 \in \{\phi_{i_0} = a\} \cap M'_{i_0} \cap U$ for $a \in Q$. Since $x_0 \in M'_{i_0}$, $\phi_{i_0}$ is twice differentiable in $x_0$. Therefore there exists a unique, non-zero gradient $\nabla \phi_{i_0}(x_0)$. By Theorem 5.4 for every $\delta > 0$ that is sufficiently small $B_\delta(x_0) \cap \{\phi_{i_0} = a\}$ is $\mathcal{H}^{n-1}$-rectifiable. The collection of all such balls $B_\delta(x_0)$ with $x_0 \in \{\phi_{i_0} = a\} \cap M'_{i_0} \cap U$ is a Vitali covering of $\{\phi_{i_0} = a\} \cap M'_{i_0} \cap U$. Therefore, the Vitali covering theorem for $\mathcal{H}^{n-1}$ implies that we can choose a countable subfamily $\{B_\delta_i(x_i)\}_{i \in \mathbb{N}}$ that still covers $\{\phi_{i_0} = a\} \cap M'_{i_0} \cap U$ up to a set of $\mathcal{H}^{n-1}$-measure 0. We define $\Sigma_a := \bigcup_{i \in \mathbb{N}} B_\delta_i(x_i)$. By construction $\Sigma_a$ is countably $\mathcal{H}^{n-1}$-rectifiable. This yields the claim.

We define the map $F : \Sigma_a \times [0, 1] \to M$ via $(x, t) \mapsto \gamma_x(t) = T_{t_0}^t(x)$. Recall that $F(x, t_0) = x$ for $x \in \Sigma_a$.

Lemma 5.5. $F$ is Lipschitz continuous.

Proof of the lemma. Not that $\{(\gamma_x(0), \gamma_x(1)) : x \in \Sigma_a\}$ is a $\frac{1}{2}d^2$-monotone set. Hence, we see by the Monge-Mather principle that

$$
d_M(F(x, t), F(y, s)) \leq \frac{C_E}{\min(1 - t_0, t_0)} d_M(x, y) + R|t - s|.
$$
This proves the claim. □

By Rademacher’s theorem there exists a measurable set $N \subset \Sigma_a \times [0, 1]$ such that $\mathcal{H}^n_{\Sigma_a \times [0, 1]}(N) = 0$ and the differential $DF(x, t)$ exists for $\forall (x, t) \in (\Sigma_a \times [0, 1]) \setminus N$. In particular, $DF(x, t)v$ exists for every $(x, t) \in (\Sigma_a \times [0, 1]) \setminus N$ and $v \in T_x \Sigma_a$.

Recall that $M'_{\ell_0}$ is the set where $\phi_{t_0}$ is twice differentiable. If $x \in M'_{\ell_0}$ then $DT^t_{\ell_0}(x) = A_x(t)$ exists for all $t \in [0, 1]$. Hence

$$DF(x, t)v = DT^t_{\ell_0}v = A_x(t)v, \quad v \in T_x \Sigma_a$$

where $\Sigma_a' = \Sigma_a \cap \{ \phi_{t_0} = a \} \cap M'_{\ell_0} \cap U$.

The vectorfield $t \in [0, 1] \mapsto A_x(t)v$ is the Jacobi field $J$ with initial conditions $J(0) = v$ and $J'(0) = \nabla^2 \phi_{t_0}v$. Moreover

$$DF(x, t)\partial_t = \dot{\gamma}_x(t) \quad \forall (x, t) \in \mathcal{S}.$$

**Proposition 5.6**. The following holds

$$\mathcal{H}^n_{\Sigma_a}(F(\mathcal{S})) = \int_0^1 \int_{\Sigma_a} |\det B_x(t)||\dot{\gamma}_x(t)|d\mathcal{H}^{n-1}(x)dt$$

If $g$ is a $\text{vol}_M$-integrable, non-negative function, it holds

$$\int_{F(\mathcal{S})} g(p)d\text{vol}_M(p) = \int_0^1 \int_{\Sigma_a} g(\gamma_x(t))|\det B_x(t)||\dot{\gamma}_x(t)|d\mathcal{H}^{n-1}(x)dt.$$

**Proof**. Recall the area formula.

**Theorem 5.7** (Area formula). Let $\mathcal{R}$ be a $\mathcal{H}^n$-rectifiable set and let $F : \mathcal{R} \to M$ be Lipschitz continuous.

(i) Let $A \subset \mathcal{R}$ be $\mathcal{H}^n$-measurable. Then

$$\int_A |JF(x)|d\mathcal{H}^n(x) = \int_M \mathcal{H}^n(A \cap F^{-1}(y))d\mathcal{H}^n(y).$$

(ii) (Change of variable formula): Let $g : \mathcal{R} \to \mathbb{R}$ be $\mathcal{H}^n$-integrable. Then

$$\int g(x)JF(x)d\mathcal{H}^n(x) = \int_M \sum_{x \in f^{-1}(y)} g(x) d\mathcal{H}^n(y).$$

$JF(x) = |\det DF(x)|$ denotes the Jacobian of the map $F$.

In our case we have $\mathcal{R} = \Sigma_a \times [0, 1]$, $F := F(x, t) = \gamma_x(t)$ and $A = S$. For $(x, t) \in S$ we observe that $DF(x, t)v = DA_x(t)v$ and $DF(x, t)\frac{d}{dt} = \dot{\gamma}_x(t)$ are perpendicular since $DA_x(t)v$ is a Jacobi field along $\gamma_x$ with initial value $v \in T_x \Sigma_a \perp \gamma_x(t_0)$. Hence

$$JF(x, t) = |\det DF(x, t)| = |\det DA_x(t)| = |\det B_x(t)||\dot{\gamma}_x(t)||$$

where $B_x(t) = A_x(t)|_{T_x \Sigma_A} : T_x \Sigma_A \to T_{\gamma_x(t)} \Sigma_a$. By the area formula it follows

$$\int_0^1 \int_{\Sigma_a} |\det B_x(t)||\dot{\gamma}_x(t)|d\mathcal{H}^{n-1}(x)dt = \int_S JF(x)d\mathcal{H}^n(x)$$

$$= \int_M \mathcal{H}^n(S \cap F^{-1}(p))d\text{vol}_M(p).$$
Of course the formula also holds when $\Sigma_a = \emptyset$. In this case all integrals become 0.

Note that $e_{h_0}^{-1}(\Sigma) \subset e_{h_0}^{-1}(M) = \Gamma_a$ is $\frac{1}{2}d^2$-monotone and hence contained in the $\frac{1}{2}d^2$-differential of the $\frac{1}{2}d^2$-convex function $t\phi_t$. One of the key observations in [Cav14] is that $\Gamma_a$ is a $d$-monotone set ([Cav14 Proposition 4.1]). And as corollary of this observation Cavalletti obtains that the family of transport segments is a partition of $\epsilon(\Omega_a \times [0,1]) \subset M$ up to a set $L$ of $\text{vol}_M$-measure 0.

Hence, we obtain

$$
\int_0^1 \int_{\Sigma_a} |\det B_x(t)||\gamma_x(t)|d\mathcal{H}^{n-1}(x)dt = \int_{\bar{F}(S)\setminus L} d\text{vol}_M.
$$

The second formula can be derived similar. 

6. Displacement convexity

**Theorem 6.1.** Let $(M,o)$ be a smooth, normalized, pointed metric measure space that satisfies $CD(\kappa,N)$ for $\kappa \in C(X)$ and $N \geq 2$. Let $K \in \mathbb{R}$, $p > \frac{N}{2}$ and $R \geq 1$ such that $k_{M,o}(p,K,2R) < \infty$. Let $\mu_0, \mu_1 \in \mathcal{P}(m_o)$ with $\mu_i(B_R) = 1$ and $|\mu_i|_{ac} = \rho_i$, $i = 0,1$. Let $\Pi$ be the dynamical optimal plan between $\mu_0$ and $\mu_1$. Let $\epsilon > 0$ and assume $\Pi((\gamma \in \mathcal{G}(M) : L(\gamma) \leq \pi_{K/(N-1)} - \epsilon)) = 1$. Then

$$
S_N((e_1)_{#}\Pi) \leq -\int \left[ \tau_{K,N}^{(1,0)}(|\gamma|)\rho_0(\gamma) - \frac{\hat{\kappa}}{2} + \tau_{K,N}^{(0)}(|\gamma|)\rho_1(\gamma) - \frac{\hat{\kappa}}{2} \right] d\Pi(\gamma)
+ 2 m_\mu(B_{2R}(o)) \hat{\kappa} + C \frac{1}{\text{vol}_M} k_{[M,o]}(p,K,2R) \frac{1}{\text{vol}_M} \forall t \in (0,1).
$$

**Proof.** By the $CD(\kappa,N)$ condition there exists a Wasserstein geodesic $\Pi$ such that

$$
\rho_t(\gamma) - \frac{\hat{\kappa}}{2} \geq \tau_{K,N}^{(1,0)}(|\gamma|)\rho_0(\gamma) - \frac{\hat{\kappa}}{2} + \tau_{K,N}^{(0)}(|\gamma|)\rho_1(\gamma) - \frac{\hat{\kappa}}{2} \text{ for } \Pi\text{-a.e. } \gamma \in \mathcal{G}(M)
$$

where $(e_1)_\Pi = \rho_t m_o$ is concentrated in $B_{2R}$. In the following we omit the dependence on $\gamma$ and write $\kappa^{+/_} = \kappa^{+/_}$. First we consider

$$
\tau_{K,N}^{(0)}(|\gamma|)\rho_1(\gamma) - \frac{\hat{\kappa}}{2}
$$

Recall that the unique Wasserstein geodesic is given by $\mu_t = (T_{t_0}^t)_{#}\mu_{t_0}$ for $t_0 \in (0,1)$ and $T_{t_0}^t(x) = \exp_x(-(t-t_0)\nabla \phi_{t_0}(x))$ for $x \in M_{t_0}$ where $\phi_{t_0}$ is the Hamilton-Jacobi shift of c-convex function $\phi : M \rightarrow \mathbb{R}$ and $M_{t_0}$ is the set of differentiable points. The dynamical plan is given by $(e_{t_0})_{#}\mu_{t_0} = \Pi$. Since $\mu_i(B_R) = 1$, $i = 0,1$, it follows that $\mu_i(B_{2R}) = 1$ for all $t \in (0,1)$.

By Corollary [4.7] we have

$$
\tau_{K,N}^{(0)}(|\gamma|)\rho_1(\gamma) - \frac{\hat{\kappa}}{2} - \tau_{K,N}^{(0)}(|\gamma|)\rho_1(\gamma) - \frac{\hat{\kappa}}{2} \leq \left( \frac{1}{\rho_1(\gamma)} \right)^\frac{\hat{\kappa}}{2} \left( \Lambda C t\hat{\kappa}^2 \int_0^1 (\kappa(\gamma(s)) - K)_{\sigma_{K,N}^{(1,0)}}(\gamma) N^{-1} ds \right)^{-\frac{1}{2N-2}} \times \left( \int_0^1 \tau_{K,N}^{(s)}(|\gamma|)N^{-1} ds \right)^{-\frac{2N-2}{2N-2}}
$$

for $\Pi\text{-a.e. } \gamma \in \mathcal{G}(M)$ where $\Lambda$ and $C$ are the constants introduced in Section 3.
Integrating \((20)\) w.r.t. \(\Pi\) and using first Jensen’s and then Hölder’s inequality yields

\[
\int \left[ \tau_{\kappa, N}^{(s)}(\|\gamma\|) \rho_t(\gamma_1)^{-\frac{1}{p}} - \tau_{\kappa, N}^{(s)}(\|\gamma\|) \rho_t(\gamma_1)^{-\frac{1}{p}} \right] d\Pi(\gamma)
\]

\[
\leq \Lambda^{2p-2} \left( C t \int |\gamma|^2 p \int_0^1 (K(\gamma(s)) - K)^{p} \sigma_{\kappa, (N-1)}^{(s)}(\|\gamma\|)^{N-1} \rho_t(\gamma_1)^{-1} ds d\Pi(\gamma) \right)^{\frac{2p-2}{N(2p-1)}}
\]

In the following we will estimate \((I)\) and \((II)\) separately. First, we consider \((II)\). It follows

\[
(II) = \int \int_1 \tau_{\kappa, N}^{(s)}(\|\gamma\|) \rho_t(\gamma_1)^{-1} ds d\Pi(\gamma) \leq \int_1 \int \rho_s(e_s(\gamma))^{-1} d\Pi(\gamma) ds
\]

\[
= \int_1 \int_0^1 \rho_s(x)^{-1} d\mu_s(x) ds = \int_1 m(\{\rho_s > 0\}) ds \leq m(B_{2R}(0)).
\]

Now, we consider \((I)\). We write \(\Pi = (\sigma_{\kappa, (N-1)}^{(s)})_{s} \mu_t\) and \(e_{\kappa, (N-1)}^{-1}(x) = \gamma_{x}. First, we decompose \(\Pi\) into \(\Pi^1 + \Pi^2 = \Pi\) where \(\Pi^1 = \Pi|_{\{\gamma \in \mathcal{G}(M) : \|\gamma\| = 0\}}\) and \(\Pi^2 = \Pi|_{\{\gamma \in \mathcal{G}(M) : \|\gamma\| > 0\}}\). \((I)\) is linear w.r.t. \(\Pi\) and vanishes on \(\Pi^2\). Therefore it is sufficient to assume \(\Pi^1 = \Pi\) and \(\mu_t = (e_t)_{t} \Pi, t \in (0, 1),\) is concentrated in the open set \(U = U_t = B_t^{-1} \{0, \infty\}\) as was considered in the previous section.

From step 4 and 6 in the proof of Theorem 2.14 we have the Monge-Ampèr equality

\[
\rho_s(T_{s}^t(x))^{-1} = J_{s}(s)\rho_t(x)^{-1}
\]

\[
= \det A_{s}(s)e^{-V_{s}T_{s}^t(x)}\rho_t(x)^{-1}
\]

\[
= \det B_{s}(s)e^{-V_{s}T_{s}^t(x)}L_{s}(s)\rho_t(x)^{-1} \quad \forall s, t \in (0, 1), \forall x \in M_{t}'.
\]

and the inequality

\[
\rho_s(T_{s}^t(x))^{-1} \leq \det B_{s}(s)e^{-V_{s}T_{s}^t(x)}L_{s}(s)\rho_t(x)^{-1} \quad \forall t \in (0, 1), \quad s = 0, 1 \forall x \in M_{t}'.
\]

By step 6 in the proof of Theorem 2.14 it also holds

\[
\det B_{s}(s) \geq \sigma_{\kappa, (N-1)}^{(s)}(\|\gamma_s\|)^{N-1} \det B_{s}(1) \quad \& \quad L_{s}(s) \geq sL_s(1)
\]

\(\forall s \in [0, 1], \forall x \in M_{t}'.\)
Therefore, after recalling that \( \mu_t = (e_t)_\# \Pi \) and \( e_t^{-1}(x) = \gamma_x(\cdot) \) with \( \gamma_x(s) = \exp(-(s-t)\nabla \phi_t(x)) \) for \( x \in M_t, \)

\[
(I) = t \int |\dot{\gamma}_x|^{2p} \int_0^1 (\kappa(\gamma_x(s)) - K)^p \sigma_{\kappa/(N-1)}(|\dot{\gamma}_x|)^{N-1} \rho_1(\gamma_x(1))^{-1} ds \mu_t(x)
\]

\[
= t \int |\dot{\gamma}_x|^{2p} \int_0^1 (\kappa(\gamma_x(s)) - K)^p \sigma_{\kappa/(N-1)}(|\dot{\gamma}_x|)^{N-1} \rho_1(T^1_t(x))^{-1} ds \mu_t(x)
\]

\[
\leq \int_{M_t \cap U} \left( \int_0^1 (\kappa(\gamma_x(s)) - K)^p |\dot{\gamma}_x|^{2p} \left\{ \sigma_{\kappa/(N-1)}(|\dot{\gamma}_x|) \right\}^{N-1} \det B_x(1)e^{-V(T^1_t(x))} \right)
\]

\[
\times tL_x(1)e^{V(x)} \rho_t(x)^{-1} ds \mu_t(x)
\]

\[
\leq \int_{M_t \cap U} \left( \int_0^1 (\kappa(\gamma_x(s)) - K)^p \det B_x(s)e^{-V(\gamma_x(s))} ds \right)
\]

\[
\times |\dot{\gamma}_x|^{2p} L_x(t)e^{V(x)} \rho_t(x)^{-1} ds \mu_t(x)
\]

\[
= \int_{M' \cap U} 1_{\{\rho_x > 0\}} \left( \int_0^1 (\kappa(\gamma_x(s)) - K)^p \det B_x(s)e^{-V(\gamma_x(s))} ds \right) |\dot{\gamma}_x|^{2p} e^{V(x)} d\mathcal{M}(x)
\]

\[
= \int_{M' \cap U} \left( \int_0^1 (\kappa(\gamma_x(s)) - K)^p \det B_x(s)e^{-V(\gamma_x(s))} ds \right) |\dot{\gamma}_x|^{2p} d\mathcal{M}(x) = (I')
\]

where we used (22) for \( s = 1 \) in the first inequality, (23) in the second inequality, \( \det B_x(t) = L_x(t) = 1 \) in the second last equality and (24) for the last equality.

We apply Lemma 5.2 to disintegrate \( \mathcal{M} \). It holds

\[
(I)' = \int_Q \int_{\Sigma'_x} \left( \int_0^1 (\kappa(\gamma_x(s)) - K)^p e^{-V(\gamma_x(s))} \det B_x(s) ds \right) |\dot{\gamma}_x|^{2p}
\]

\[
\times \frac{1}{\nabla \phi_t(x)} dH^{n-1}(x) da.
\]

where \( \Sigma'_x := M'_x \cap U \cap \{ \phi_t = a \} \cap \Sigma_a \) as was introduced in the previous section.

Recall that \( \dot{\gamma}_x(t) = \nabla \phi_t(\gamma_x(t)) \) for \( \Pi \text{-a.e. } \gamma \in \mathcal{G}(M) \). It follows that

\[
(I)' = \int_Q \int_{\Sigma'_x} \int_0^1 (\kappa(\gamma_x(s)) - K)^p e^{-V(\gamma_x(s))} \det B_x(s)|\dot{\gamma}_x|^{2p} dH^{n-1}(x) ds da
\]

\[
\leq (2R)^{2p-2} \int_Q \int_{\Sigma'_x} \int_0^1 (\kappa(\gamma_x(s)) - K)^p e^{-V(\gamma_x(s))} \det B_x(s)|\dot{\gamma}_x| dH^{n-1}(x) ds da
\]

\[
= (I)''.
\]

We can apply Proposition 5.3. It yields

\[
(I)'' = (2R)^{2p-2} \int_Q \int_{\mathcal{F}(S)} (\kappa(x) - K)^p e^{-V(x)} d\mathcal{M}(x) da
\]

\[
\leq (2R)^{2p-2} \int_Q \int_{B_{2R}(o)} (\kappa(x) - K)^p d\mathcal{M}(x) da
\]

\[
= (2R)^{2p-2} \mathcal{L}^1(Q) \| (\kappa - K)^p \|_{L^p(\mathcal{M} |_{B_{2R}(o)})}.
\]

It is not difficult to see that for \( \phi_t : \hat{B}_{2R} \to \mathbb{R} \) it holds \( \mathcal{L}^1(Q) \leq \mathcal{L}^1(\text{Im} \phi_t) \leq 4R^2 \).

It follows

\[
(I)'' \leq (2R)^{2p} \| (\kappa - K)^p \|_{L^p(\mathcal{M} |_{B_{2R}(o)})}.
\]
We conclude that
\[
\int \tau_{K,N}^{(t)}(\gamma_1) \rho_1(\gamma_1)^{-\frac{K}{\kappa_+}} d\Pi(\gamma) - \int \tau_{K,N}^{(t)}(\gamma_1) \rho_1(\gamma_1)^{-\frac{K}{\kappa_-}} d\Pi(\gamma)
\]
\[
\leq \left[ (\Lambda CR^{2p} \int (\kappa - K)^p \, dm) \frac{1}{\kappa_+} \right] \frac{1}{\kappa_-} (m(B_{2R}(o)))^{1 - \frac{1}{\kappa_-}}
\]
\[
\leq \left[ m(B_{2R}(o)) \frac{\Lambda}{\kappa_-} \right] \frac{1}{\kappa_-} \left( R^2 \left( \int (\kappa - K)^p \, dm \right) \right) \frac{1}{\kappa_-} (m(B_{2R}(o)))^{1 - \frac{1}{\kappa_-}}.
\]

The same inequality holds in the case when we replace \( \kappa_+ \) by \( \kappa_- \) and \( t \) by \( 1 - t \).

Therefore
\[
\int \rho_i^{-\frac{K}{\kappa_+}} \, d\mu_i \geq \int \left[ \tau_{K,N}^{(1-t)}(\gamma_0) \rho_0(\gamma_0)^{-\frac{K}{\kappa_+}} + \tau_{K,N}^{(t)}(\gamma_1) \rho_1(\gamma_1)^{-\frac{K}{\kappa_-}} \right] d\Pi(\gamma)
\]
\[
- 2 \Lambda \frac{1}{\kappa_-} m(B_{2R}(o)) \frac{\Lambda}{\kappa_-} \frac{1}{\kappa_-} \left( R^2 \left( \int (\kappa - K)^p \, dm \right) \right) \frac{1}{\kappa_-} (m(B_{2R}(o)))^{1 - \frac{1}{\kappa_-}}.
\]

That was to prove. \( \square \)

Following the prove of Theorem 6.1 we also obtain

**Theorem 6.2.** Let \( M \) be a smooth, normalized pmm space that satisfies \( CD(\kappa,N) \) for \( \kappa \in C(M) \) and \( N \geq 2 \). Let \( K \in \mathbb{R} \) and \( p > \frac{N}{2} \) such that \( k_{[M]}(p,K) < \infty \). Let \( \rho_0, \mu_1 \in \mathcal{P}(m_{\mathcal{M}}) \) with \( \mu_{i|\mathcal{M}} = \rho_i, i = 0,1 \). Let \( \Pi \) be the dynamical optimal plan between \( \mu_0 \) and \( \mu_1 \). Let \( \epsilon > 0 \) and assume \( \Pi(\{ \gamma : L(\gamma) \leq \pi_{K/(N-1)} - \epsilon \}) = 1 \). Then

\[
S_{\kappa}(\{(e_i)\# \Pi) \leq - \int \tau_{K,N}^{(1-t)}(\gamma_0) \rho_0(\gamma_0)^{-\frac{K}{\kappa_+}} + \tau_{K,N}^{(t)}(\gamma_1) \rho_1(\gamma_1)^{-\frac{K}{\kappa_-}} \right] d\Pi(\gamma)
\]
\[
+ 2 \frac{\Lambda}{\kappa_-} \frac{1}{\kappa_-} \left( R^2 \left( \int (\kappa - K)^p \, dm \right) \right) \frac{1}{\kappa_-} (m(B_{2R}(o)))^{1 - \frac{1}{\kappa_-}} \forall t \in (0,1).
\]

As an immediate corollary we also derive Corollary 6.3

**Remark 6.3.** The proof of Theorem 6.1 also yields the following. Let \( (M,o) \) be as in Theorem 6.1. Let \( x_0 \in B_R(o) \) and \( \mu \in \mathcal{P}(m_{\mathcal{M}}) \) be a measure supported in \( B_r(x_0) \) for \( r \in (0, \pi_{K/(N-1)} - \epsilon) \cap (0,R) \) and \( \epsilon > 0 \). Let \( \Pi \) be the dynamical optimal plan such that \( (e_1)\# \Pi = \mu \) and \( (e_0)\# \Pi = \delta_{x_0} \). Then it holds

\[
S_{\kappa}(\{(e_i)\# \Pi) \leq - \int \tau_{K,N}^{(1-t)}(\gamma_0) \rho_0(\gamma_0)^{-\frac{K}{\kappa_+}} \right] d\Pi(\gamma)
\]
\[
+ m_\mathcal{M}(B_{2R}(o)) \frac{\Lambda}{\kappa_-} \frac{1}{\kappa_-} (m(B_{2R}(o)))^{1 - \frac{1}{\kappa_-}} \forall t \in (0,1).
\]

7. **Stability**

**Theorem 7.1.** Let \( \{(M_i,o_i)\}_{i \in \mathbb{N}} \) be a sequence of smooth, normalized pmm spaces that satisfy the condition \( CD(\kappa_i,N) \) for \( \kappa_i \in C(X_i) \) such that

\[
k_{[M_i,o_i]}(\kappa_i,p,K,R) \to 0 \text{ when } i \to \infty \text{ } \forall R > 0
\]

with \( K \in \mathbb{R} \) and \( p > \frac{N}{2} \). Then \( \{(M_i,o_i)\}_{i \in \mathbb{N}} \) subconverges in \( pmG \) sense to the isomorphism class of a pmm space \( (X,o) \) satisfying the condition \( CD(K,N) \).
Proof. We first consider the case $K \leq 0$. We set $(M_i,o_i) = (X_i,o_i)$, $i \in \mathbb{N}$.

1. Our assumptions allow to extracting a subsequence $(X_i,o_i)$ that converges in pmGH sense to a pmmm space $(X_m,o_m)$. The corresponding isomorphism classes $[X_i,o_i]$ converge w.r.t. pmG convergence to $[X_m,o_m]$. Since $(X_i,o_i)$ are length spaces, $(X_m,o_m)$ is a length space too. In particular, $B_R(o_i)$ converges in GH sense to $B_R(o_m)$ for all $R > 0$ ([BH01], Remark 3.2.9 in [GMS15]) and $[B_R(o_i),m_{B_R(o_i)}]$ converge to $[B_R(o_m),m_{B_R(o_m)}]$ in mG sense.

For $i \in \mathbb{N}$ and $R > 0$ we set $B_R = B_R(o_i)$, $d_R = d_{B_R(o_i)}$, $m_R = m_{B_R(o_i)}$, $\bar{m}_R := m(B_R(o_i))^{-1} m_R$ and $m_R(X_i) = \alpha_R$. It holds $\sup_{i \in \mathbb{N}} \alpha_R < \infty$ and $\alpha_R \to \alpha_R^\infty$ as $i \to \infty$ for all $R > 0$.

Let $(Z,d_Z)$ be a compact metric spaces where mGH convergence of $(B_R^i)_{i \in \mathbb{N}}$ is realized. Then, also mGH convergence of $(B_R^i)_{i \in \mathbb{N}}$ is realized in $Z$.

2. Consider $R = n \in \mathbb{N}$. Note that for $\mu \in \mathcal{P}^2(m_{X_n})$ with $\mu(B_n^i) = 1$ we have $\mu \in \mathcal{P}^2(\bar{m}_n^i)$. We denote by $\rho$ the density w.r.t. $m_{X_n}$ and with $\bar{\rho}$ the density w.r.t. $\bar{m}_n^i$ that is $\bar{\rho}_n = m_{X_n}(B_n^i)\rho$.

Since $[B_n^i,d_n^i,\bar{m}_n^i] \to [B_n^\infty,d_n^\infty,\bar{m}_n^\infty]$ in mG sense, we find optimal couplings $q_n^i$ between $\bar{m}_n^i$ and $\bar{m}_n^\infty$ such that

\[ W_2(\bar{m}_n^i,\bar{m}_n^\infty)^2 = \int d_Z^2(x,y)d_q^i_n(x,y) = d_Z^2 \to 0. \]

As in the proof of Theorem 4.20 in [Stu06a] we define $Q_n^i : \mathcal{P}^2(\bar{m}_n^\infty) \to \mathcal{P}^2(\bar{m}_n^i)$ by

\[ Q_n^i : \bar{\rho} \to Q_n^i(\bar{\rho})m_n^i \]

where $Q_n^i(\bar{\rho})(y) = \int \bar{\rho}(x)Q_n^i(y,\chi)\,\text{d}\,
\rho\,\text{d}\,\chi$ (the density).

where $Q_n^i(\bar{\rho})(y,\chi)$ is a disintegration of $q_n^i$ w.r.t. $\bar{m}_n^i$. One can check that

\[ S_N(Q_n^i(\mu)|\bar{m}_n^i) \leq S_N(\mu|\bar{m}_n^\infty) \quad \text{and} \quad d_{z,w}(Q_n^i(\mu),\bar{m}_n^i) \to 0 \quad \text{as} \quad i \to \infty. \]

Similar, we define $P_n^i : \mathcal{P}^2(\bar{m}_n^i) \to \mathcal{P}^2(\bar{m}_n^\infty)$ by

\[ P_n^i : \bar{\rho}_n \to P_n^i(\bar{\rho}_n)m_n^\infty \]

where $P_n^i(\bar{\rho}_n)(y,\chi) \to 0$ as $i \to \infty$.

3. Let $\mu_j \in \mathcal{P}^2(m_{X_n})$ with density $\rho_j$ and $\mu_j(B_n(o)) = 1$ for $j = 0,1$. Assume $0 \leq \rho_j \leq r < \infty$. We will remove this assumption at the end of the proof. In particular, $\mu_0,\mu_1 \in \mathcal{P}^2(\bar{m}_n^\infty)$ and the $\bar{m}_n^\infty$-densities are $\bar{\rho}_{n,j}$, $j = 0,1$. Clearly

\[ \rho_j = \alpha_n^{\infty} \bar{\rho}_{n,j} \quad \text{and} \quad \bar{\rho}_{n,j} = \alpha_n^{\infty} \rho_{2n,j}. \]

Consider $Q_n^i(\mu_j) = \mu_{n,j}^i$, $j = 0,1$, and its densities $Q_n^i(\bar{\rho}_{n,j})$ w.r.t $\bar{m}_n^i$. Let $\rho_{n,j}^i$ be the density of $Q_n^i(\mu_j)$. Then

\[ \rho_{n,j}^i = \alpha_n^{\infty} Q_n^i(\rho_j) \quad \text{and} \quad Q_n^i(\rho_j) = \frac{\alpha_n^{\infty}}{\alpha_{2n}} Q_{2n}^i(\rho_j). \]

Let $\Pi^\infty$ be the unique optimal dynamical plan between $\mu_{n,j}^i$, $j = 0,1$ with $(c_j)_{\Pi^\infty} = \mu_{n,t}^i \in \mathcal{P}(\bar{m}_n^i)$. It holds that $\mu_{n,t}^i(B_n^i) = \mu_{n,t}^i(B_{2n}^i) = 1$. By Theorem 6.1 for
the $L^2$-Wasserstein geodesic $\Pi^i$ between $Q^i_t(\mu_j)_{j=0,1}$ with $(e_1)_\ast \Pi^i = \rho^i_t \mu_{M_t} = \mu^i_t$, $(e_1)_\ast \Pi^i(B_{2n})_t = 1$ and $(e_0, e_1)_\ast \Pi^i = \pi^i$ such that
\[
\int \rho_t^i(x) \frac{d}{dx} \mu_t^i(x) \geq \int \left[ \tau_{K,N}^{(t)}(\gamma_0) - \frac{d}{dx} \tau_{K,N}^{(t)}(\gamma_1) \right] d\Pi^i(\gamma) - 2\Lambda \frac{d}{dx} \left( \alpha_{2n}^i \right)^2 C \frac{d}{dx} \rho_{\gamma} \left[ k_{M,i_{\alpha}}(p,K,2n) \right] e^{-\frac{1}{2} x^2 - n}.
\]

(26)

4. The left hand side in the last inequality is $-S_N(\mu^i_t \mid m_{M_t})$. Changing the reference measure yields
\[
-S_N(\mu^i_t \mid m_{M_t}) = -\left( \alpha_{2n}^i \right)^2 \int \frac{d}{dx} S_N(\mu^i_t \mid \bar{m}^i_{2n}).
\]

Then, we map $\mu^i_t$ with $P^i_{2n}$ to $B^n_{2n}$ and obtain
\[
\left( \alpha_{2n}^i \right)^2 \int \frac{d}{dx} S_N(P^i_{2n} \mu^i_t \mid m_{\infty}) = \left( \alpha_{2n}^i \right)^2 S_N(P^i_{2n} \mu^i_t \mid \bar{m}^i_{2n}) \leq \left( \alpha_{2n}^i \right)^2 S_N(\mu^i_t \mid \bar{m}^i_{2n}).
\]

Similar, the first term in the right hand side of (20) equals
\[
\left( \alpha_{2n}^i \right)^2 \int \left[ \tau_{K,N}^{(t)}(d(x,y)Q^i_n(\rho_0)(x) - \frac{d}{dx} \tau_{K,N}^{(t)}(d((x,y)))Q^i_n(\rho_1)(x) - \frac{d}{dx} \right] d\pi^i(x,y)
\]

That is exactly $\left( \alpha_{2n}^i \right)^2$ times the expression $T_{K,N}^{(t)}(\pi^i_t \mid \bar{m}^i_n)$ in step (v) of the proof of Theorem 3.1 in [Stu06b]. In our context the reference measure is $\bar{m}^i_n$. Therefore, following (v) in the proof of Theorem 3.1 in [Stu06b] and using the maps $P^i_n$ and $Q^i_n$ we obtain
\[
\left( \alpha_{2n}^i \right)^2 T_{K,N}^{(t)}(\bar{m}^i \mid m_{\infty}) \leq \left( \alpha_{2n}^i \right)^2 T_{K,N}^{(t)}(\bar{m}^i \mid \bar{m}^i_{\infty}) + \left( \alpha_{2n}^i \right)^2 \frac{d}{dx} C \rho_{X,\alpha} \left[ k_{M,i_{\alpha}}(p,K,2n) \right] e^{-\frac{1}{2} x^2 - n}.
\]

(28)

if $i \geq i_\varepsilon$ for $i_\varepsilon$ sufficiently large. The constant $\hat{C}$ does not depend on $i$. The measure $\bar{m}^i_\infty \in \mathcal{P}(\mathbb{X}_\alpha^2)$ is a coupling between $\mu_0$ and $\mu_1$ (not necessarily optimal) such that $\bar{m}^i \to \pi$ weakly if $i \to \infty$ for an optimal coupling $\pi^i$ between $\mu_0$ and $\mu_1$. Hence, (20), (27) and (28) together imply
\[
\left( \alpha_{2n}^i \right)^2 S_N(P^i_{2n} \mu^i_t \mid m_{\infty}) \leq \left( \alpha_{2n}^i \right)^2 T_{K,N}^{(t)}(\bar{m}^i \mid m_{\infty}) + \left( \alpha_{2n}^i \right)^2 \frac{d}{dx} C \rho_{X,\alpha} \left[ k_{M,i_{\alpha}}(p,K,2n) \right] e^{-\frac{1}{2} x^2 - n}.
\]

5. As in step (vi) in the proof of Theorem 3.1 in [Stu06b] one checks that $P^i_{2n} \mu^i_{nt} \mid m_{\infty}$ converges weakly to a probability $\mu_t$ for every $t \in [0,1]$, and $\mu_t$ is as geodesic between $\mu_0$ and $\mu_1$ in $\mathcal{P}(\mathbb{X}^2)$, and the limit is $m_{\infty}$-absolutely continuous. Weak convergence of $\bar{m}^i_n$ and $\bar{m}^i_{2n}$ for $i \to \infty$ yields $\alpha_{2n}^i \to \alpha^\infty_{2n}$ for every $n \in \mathbb{N}$. Moreover, by Lemma 3.3 in [Stu06b] we have
\[
\limsup_{i \to \infty} T_{K,N}^{(t)}(\bar{m}^i_n \mid m^i_n) \leq T_{K,N}^{(t)}(\pi^i \mid m^i_n),
\]

and by lower semi-continuity of the $N$-Reny entropy
\[
\liminf_{i \to \infty} S_N(P^i \mu^i_t \mid m_{\infty}) \geq S_N(\mu_t \mid m_{\infty}).
\]
Hence, letting \(i \to \infty\) and using again lower (upper) semi-continuity of \(S_N(\mu_t | m_{X,\infty})\) (\(T_{K,N}^{(t)}(\pi | m_{X,\infty})\)), we obtain a geodesic \((\mu_t)_{t \in [0,1]}\) and a optimal coupling \(\pi\) such that
\[
S_N(\mu_t | m_{X,\infty}) \leq T_{K,N}^{(t)}(\pi | m_{X,\infty}).
\]

6. Finally, we want to remove the assumption that \(\rho_j \in L^\infty(m_{X,\infty})\) for \(j = 0, 1\). Therefore, consider general probability measures \(\rho, \mu \in \mathcal{P}^2(m_{X,\infty})\) with densities \(\rho, \mu\), and \(\mu_j(B^{\infty}_S(o)) = 1\) for \(j = 0, 1\). Fix an arbitrary optimal coupling \(\tilde{\pi}\) between \(\mu_0\) and \(\mu_1\), and set for \(r \in (0, \infty)\)
\[
E_r := \{(x_0, x_1) \in X^2_\infty : \rho_0(x_0) \leq r, \rho_1(x_1) \leq r\}, \quad \alpha_r = \hat{\pi}(E_r), \quad \hat{\pi}^r := \frac{1}{\alpha_r}\hat{\pi}|_{E_r}.
\]
The coupling \(\hat{\pi}^r\) is an optimal coupling between its marginals \(\rho^r_0\) and \(\mu^r_1\) such that for \(r = 0, 1\)
\[
W_{X,\infty}(\mu_j, \mu^r_j) \leq \epsilon \quad \text{if} \ r > 0 \quad \text{sufficiently large}.
\]
Depending on \(r > 0\) we can construct \(\mu^r_t\) and \(\pi^r\) as before such that
\[
S_N(\mu^r_t | m_{X,\infty}) \leq T_{K,N}^{(t)}(\pi^r | m_{X,\infty}).
\]
From \[(29)\] we obtain - after choosing subsequences - that \(\mu^r_t\) converges weakly to a probability \(\mu_t\) for \(t \in [0,1] \cap \mathbb{Q}\). Then, again as in step (vi) of the proof of Theorem 3.1 in [Stu06b], \(\mu_t\) extends to geodesic between \(\mu_0\) and \(\mu_1\) and
\[
\liminf_{i \to \infty} S_N(\mu^r_t | m_{X,\infty}) \geq S_N(\mu_t | m_{X,\infty})
\]
for \(t \in [0,1]\).

Set \(\tau_{K,N}^{(\epsilon)}(\pi)\rho_0(\gamma_0)^{-\frac{\alpha}{\delta}} + \tau_{K,N}^{(\epsilon)}(\gamma)\rho_1(\gamma_1)^{-\frac{\alpha}{\delta}} = \psi(\gamma)\). \(\psi\) is integrable w.r.t. \(\tilde{\pi}\), since the distortion coefficients are bounded \(\rho_0\) and \(\rho_1\) are probability densities for \(\mu_0\) and \(\mu_1\) respectively, and \(\tilde{\pi}\) is an coupling between \(\mu_0\) and \(\mu_1\). Therefore, if we set \(\pi^r = \alpha_r\pi + \tilde{\pi}|_{X^2 \setminus E_r}\), it follows that
\[
\lim_{\epsilon \to 0} \left| T_{K,N}^{(t)}(\pi^r | m_{X,\infty}) - T_{K,N}^{(t)}(\pi^r | m_{X,\infty})\right| = 0.
\]

Now, by compactness we can choose subsequence \(\epsilon_i\) such that \(\pi^{\epsilon_i}\) converges weakly to an optimal coupling \(\pi\) between \(\mu_0\) and \(\mu_1\). Since \(\pi^r\) is a coupling between \(\mu_0\) and \(\mu_1\) for every \(\epsilon > 0\), we can apply again Lemma 3.3 from [Stu06b] for upper semi-continuity of \(T_{K,N}^{(t)}(\pi | m_{X,\infty})\) in \(\pi\). Hence
\[
S_N(\mu_t | m_{X,\infty}) \leq \liminf_{i \to \infty} S_N(\mu^{\epsilon_i}_t | m_{X,\infty}) \leq \limsup_{i \to \infty} T_{K,N}^{(t)}(\pi^{\epsilon_i} | m_{X,\infty}) \leq T_{K,N}^{(t)}(\pi | m_{X,\infty}).
\]
This finishes the proof for \(K \leq 0\).

For \(K > 0\) we first prove the following preliminary result.

**Proposition 7.2.** Let \(\{(M_i, o_i)\}_{i \in \mathbb{N}}\) be a sequence of smooth, normalized pmm spaces that satisfy the condition \(CD(\kappa_i, N)\) for \(\kappa_i \in C(X_i)\) such that
\[
k_{[M_i, o_i]}(p, K, R) \to 0 \quad \text{as} \quad i \to \infty \quad \forall R > 0
\]
with \(K > 0\) and \(p > \frac{N}{2}\). Then \(\{(M_i, o_i)\}_{i \in \mathbb{N}}\) subconverges in pmG sense to the isomorphism class of a pmm space \((X, o)\) satisfying the MCP\((K, N)\).
Proof of the Proposition. We already know that a limit pmn space \((X_\infty, o_\infty)\) exists. We fix \(R \geq \pi_{K/(N-1)}\) and \(x_\infty \in B_R \subset X_\infty\), and let \(\mu \in \mathcal{P}(m_{X_\infty})\) such that for some \(\epsilon > 0\) it holds \(\mu(B_{\pi_{K/(N-1)}-\epsilon}(x_\infty)) = 1\).

Let \((Z,d_Z)\) be a metric space where the mGH convergence of \(B_{2R}^i\) to \(B_{2\infty}^i\) is realized. Then, we can find a sequence \(x_i \in B_{2R}^i\) such that \(x_i \to x_\infty\) in \(Z\) and \(\bar{B}_{\pi_{K/(N-1)}-\epsilon}(x_i)\) converges in Hausdorff sense to \(\bar{B}_{\pi_{K/(N-1)}-\epsilon}(x_0)\) in \(Z\). In particular \([B_{\pi_{K/(N-1)}-\epsilon}(x_i)]\) converges in mG sense. Moreover, for \(i \in \mathbb{N}\) sufficiently large it holds \(\bar{B}_{\pi_{K/(N-1)}-\epsilon}(x_0) \subset B_{2R}^i\). We set \(\bar{B}_{\pi_{K/(N-1)}-\epsilon}(x_i) = B^i, \bar{m}^i = m_{X_i, |B^i|}, \bar{m}^i = m^i(B^i)^{-1}m^i\) for \(i \in \mathbb{N}\).

As in step 2 of the proof for \(K \leq 0\) we construct \(Q^i : \mathcal{P}^2(\bar{m}^\infty) \to \mathcal{P}^2(\bar{m}^i)\) by
\[
Q^i : \bar{m}^\infty \mapsto Q^i(\bar{m}^i) m^i \quad \text{where} \quad Q^i(\bar{m}^i)(y) = \int \bar{m}^i(y, dx)
\]
The measure \(Q^i(\mu) = \mu^i\) that is concentrated in \(\bar{B}_{\pi_{K/(N-1)}-\epsilon}(x_i)\). By Remark 6.3 it holds
\[
S_N((e_t)_\# \Pi^i) \leq -\int \tau_{K,N}^{(1)}(\gamma) \rho_1(\gamma_1)^{-K} d\Pi^i(\gamma)
\]
(30)
\[
+ m_{\Pi^i}(B_{2R}(o)) \frac{1}{\Lambda} C_{\pi_{K/(N-1)}} k_{[M,o]}(p, K, 2R) \frac{1}{\sqrt{d-1}} \forall t \in (0,1).
\]
where \(\Pi^i\) is the unique optimal dynamical plan between \(\delta_{x_i}\) and \(\mu^i\).

At this point we can essentially repeat the steps from before and it holds
\[
S_N((e_t)_\# \Pi^\infty) \leq -\int \tau_{K,N}^{(1)}(\gamma) \rho_1(\gamma_1)^{-K} d\Pi^\infty(\gamma) \forall t \in (0,1).
\]
(31)
for an optimal dynamical plan \(\Pi^\infty\) between \(\delta_{x_\infty}\) and \(\mu\) that is the limit of \((\Pi^i)_{i \in \mathbb{N}}\).

By another limiting process we also obtain this inequality for measures \(\mu\) with \(\mu(\bar{B}_{\pi_{K/(N-1)}-\epsilon}(x_\infty)) = 1\).

Inequality (31) is a version of the measure contraction property that was used in [CM17a]. It implies the measure contraction property in the sense of Ohta by work of Rajala [Raj12].

Corollary 7.3. Let \((M_i, o_i)\) be as in the previous proposition. For every \(\epsilon > 0\) there exists \(i_\epsilon \in \mathbb{N}\) such that \(\text{diam supp } m_{M_i} < \pi_{K/(N-1)} + \epsilon\) for all \(i \geq i_\epsilon\).

Proof. We can argue by contradiction. Assume the statement is false. Then there exists a subsequence that is also denoted by \((M_i, o_i)\) and points \(x_i, y_i \in \text{supp } m_{M_i}\) such that \(d_{M_i}(x_i, y_i) > \pi_{K/(N-1)} + \epsilon\). By the previous proposition we can extract a subsequence that converges in pmG convergence to some MCP\((K, N)\) space \(X\). In particular, \(d_{M_i}(x_i, y_i) \to d_X(x, y)\) for points \(x, y \in X\). That is not possible because of the Bonnet-Myers theorem for MCP\((K, N)\) spaces.

Now we can proceed with the proof of Theorem 6.1}

7. Let \(\epsilon > 0\). By relabeling the sequence we can assume \(\text{diam } m_{M_i} \leq \pi_{K/(N-1)} + \frac{1}{2}\).

Let \(i \geq i_\epsilon\) such that \(\frac{1}{4} \leq \epsilon < 1\). We go back to step 2 from before and pick \(N \geq R \geq \pi_{K/(N-1)} + 1\). Then \(B_R^i = M_i\) and \(\bar{m}_M^i = m_M^i\). We replace \(m_{M_i}\) with its normalisation \(\bar{m}_M^i\) and we can assume that \([M_i]\) converges in Gromov sense to \([X]\).

We define the map \(Q^i = Q^i_n : \mathcal{P}(m_X^\infty) \to \mathcal{P}(m_{M_i})\) as in step 2 before. We set \(\Xi(\epsilon) = \{\gamma \in \mathcal{G}(M) : L(\gamma) \leq \pi_{K/(N-1)} - \epsilon\}$. 

Then we can follow the proof of the case $K \leq 0$. As in 3, let $\mu_j \in \mathcal{P}(m_{\infty})$ with density $\rho_j$ for $j = 0, 1$. Assume $0 \leq \rho_j \leq r < \infty$ on $\text{supp} \mu_j$. We consider $Q^i(\mu_j) = \mu^i_j$ with density $\rho^i_j$ w.r.t. $m_{M_i}$. Then

$$
\int (\rho^i_j)^{-\frac{\tau}{\gamma}} d\mu_t \geq \int_{\Xi(\epsilon)} \left[ \tau^{(1-t)}_{K,N}(\gamma) \rho_0^i(\gamma) - \frac{\tau}{\gamma} + \tau^{(1-t)}_{K,N}(\gamma) \rho_1^i(\gamma) - \frac{\tau}{\gamma} \right] d\Pi^i(\gamma)
- 2\Lambda(\epsilon) \frac{\tau}{\gamma} C K_{M_i}(\rho, K)^{\frac{\tau}{\gamma}}
$$

where $\Pi^i$ is the dynamical optimal plan between $\mu_0$ and $\mu_1$, and $(\epsilon_0)^{\#} \Pi^i = \rho^i_0 m_{M_i}$.

We estimate the first term in the right hand side. Let $\epsilon_0 \geq \epsilon$. Since $K > 0$, the map $\theta \mapsto \tau^{(1-t)}_{K,N}(\theta)$ is monotone increasing and it holds

$$
\int_{\Xi(\epsilon)} \tau^{(1-t)}_{K,N}(\gamma) \rho_0^i(\gamma) - \frac{\tau}{\gamma} d\Pi^i(\gamma)
\geq \int_{\Xi(\epsilon)} \tau^{(1-t)}_{K,N}(\gamma) \rho_0^i(\gamma) - \frac{\tau}{\gamma} d\Pi^i(\gamma)
= \int_{\Xi(\epsilon)} \tau^{(1-t)}_{K,N}(\gamma) \rho_0^i(\gamma) - \frac{\tau}{\gamma}
- \tau^{(1-t)}_{K,N}(\pi_{K/(N-1)} - \epsilon_0) \int (\rho_0^i(\gamma) - \frac{\tau}{\gamma}) d\Pi^i(\gamma)
$$

We define $\hat{\Pi}^i := \hat{\Pi}^i_0 = \Pi(\mathcal{G}(M) \setminus \Xi(\epsilon))^{-1} \Pi_{\mathcal{G}(M) \setminus \Xi(\epsilon)}$. It holds

$$
\rho_0^i = (\epsilon_0)^{\#} \Pi = (\epsilon_0)^{\#} (\Pi_{\mathcal{G}(M) \setminus \Xi(\epsilon)} + (\epsilon_0)^{\#} \Pi_{\mathcal{G}(M) \setminus \Xi(\epsilon)} \geq (\epsilon_0)^{\#} \Pi_{\mathcal{G}(M) \setminus \Xi(\epsilon)} = \Pi(\mathcal{G}(M) \setminus \Xi(\epsilon)) \rho^i_0.
$$

Hence

$$
\Pi^i(\mathcal{G}(M) \setminus \Xi(\epsilon))^{1-1} \rho_0^i \geq \hat{\rho}_0^i \ m_{M_i} \text{ a.e.}.
$$

Therefore, it follows

$$
\int_{\mathcal{G}(M) \setminus \Xi(\epsilon)} \rho_0^i(\gamma)^{-\frac{\tau}{\gamma}} d\Pi^i(\gamma) = \Pi^i(\mathcal{G}(M) \setminus \Xi(\epsilon)) \int (\rho_0^i)^{-\frac{\tau}{\gamma}}(\gamma) d\hat{\Pi}^i
\leq \Pi^i(\mathcal{G}(M) \setminus \Xi(\epsilon))^{1-\frac{\tau}{\gamma}} \int (\hat{\rho}_0^i)^{-\frac{\tau}{\gamma}}(\gamma) d\hat{\Pi}^i(\gamma)
\leq -S_N((\epsilon_0)^{\#} \hat{\Pi}^i).
$$

We obtain

$$
\int (\rho^i_j)^{-\frac{\tau}{\gamma}} d\mu_t \geq \left[ \tau^{(1-t)}_{K,N}(\hat{\theta}^i(\gamma)) \rho^i_0(\gamma) - \frac{\tau}{\gamma} + \tau^{(1-t)}_{K,N}(\hat{\theta}^i(\gamma)) \rho^i_1(\gamma) - \frac{\tau}{\gamma} \right] d\Pi^i(\gamma)
- 2\Lambda(\epsilon) \frac{\tau}{\gamma} C K_{M_i}(p, K)^{\frac{\tau}{\gamma}} + C(\epsilon_0) S_N((\epsilon_0)^{\#} \hat{\Pi}^i)
$$

where $\hat{\theta}^i(\gamma) = |\gamma| \wedge (\pi_{K/(N-1)} - \epsilon)$ and $C(\epsilon_0) = \tau^{(1)}_{K,N}(\pi_{K/(N-1)} - \epsilon_0)$ for any $\epsilon_0 \geq \epsilon$.

8. Let $\epsilon_1 = \frac{1}{2}$. We study the sequence of dynamical optimal plans $\hat{\Pi}^i =: \Pi^i_{\epsilon_1}$. After going to a subsequence $\hat{\Pi}^i_{\epsilon_1}$ converges weakly in $\mathcal{P}(\mathcal{G}(Z))$ to a dynamical optimal plan $\Pi \in \mathcal{P}(X_{\infty})$. By weak convergence $\hat{\Pi}$ is supported on $\Xi_0 = \{ \gamma \in \mathcal{G}(X_{\infty}) : |\gamma| \geq \pi_{K/(N-1)} \}$. Now $\frac{1}{2} d_{X_{\infty}}^{-2}$-monotonicity yields

$$
2\pi^2_{K/(N-1)} = d(\gamma_0, \gamma_1)^2 + d(\tilde{\gamma}_0, \tilde{\gamma}_1)^2 \leq d(\gamma_0, \tilde{\gamma}_1)^2 + d(\tilde{\gamma}_0, \gamma_1)^2 \leq 2\pi^2_{K/(N-1)}.
$$

Therefore, any coupling between $\hat{\rho}_0 = (\epsilon_0)^{\#} \hat{\Pi}$ and $\hat{\rho}_1 = (\epsilon_1)^{\#} \hat{\Pi}$ is optimal and supported on $\Xi_0$. By uniqueness of opposite points we conclude that $\hat{\rho}_j = \delta_{x_j}$.
for opposite points $x_0$ and $x_1$. Hence $S_N(\tilde{\mu}_j | m_{X_\infty}) = 0$, $j = 0, 1$, and by lower semi-continuity of $S_N$ it follows
\[
\liminf_{i \to \infty} S_N((e_0)_\# \Pi_{e_i} | m_{M_i}) \geq 0
\]

9. At this point we can follow verbatim the proof for $K \leq 0$ to obtain the following estimate for $\mu_0, \mu_1$ and dynamical optimal plan $\Pi$ in $X_\infty$:
\[
S_N((e_\ell)_\# \Pi | m_{X_\infty}) \leq - \int \left[ \tau_{K,N}^{(1-t)}(\hat{\theta}(\gamma)) \rho_0(\gamma_0)^{-\frac{t}{N}} + \tau_{K,N}^{(t)}(\hat{\theta}(\gamma)) \rho_1(\gamma_1)^{-\frac{t}{N}} \right] d\Pi(\gamma)
\]
where $\hat{\theta}(\gamma) = |\gamma| \wedge (\pi_{K/(N-1)} - \epsilon_0)$. To finish the proof we prove monotonicity of $\theta \mapsto \tau_{K,N}^{(t)}(\theta)$ for $K > 0$ and the monotone convergence theorem to let $\epsilon_0 \to 0$.

**Theorem 7.4.** Let $\{(M_i, o_i)\}_{i \in \mathbb{N}}$ be a sequence of $n$-dimensional, pointed Riemannian manifolds that satisfy the condition $C(\kappa_i, N)$ for $\kappa_i \in C(X)$ such that
\[
k_i(M_i, o_i)(p, K, R) \to 0 \text{ when } i \to \infty \quad \forall R > 0
\]
with $K \in \mathbb{R}, \ p > \frac{N}{2}$, $p \geq 1$ if $N = 2$ and (2) holds. Then $\{(M_i, o_i)\}_{i \in \mathbb{N}}$ subconverges in pmG sense to the isomorphism class of a pmnm space $(X, o)$ satisfying the Riemannian curvature-dimension condition $RCD(K, N)$.

**Proof.** We need to check that $X$ is infinitesimal Hilbertian. By [KK19a Proposition 6.7] it is enough to prove that for $m_X$-a.e. point $x \in X$ a blow up tangent space exists that is isometric to $\mathbb{R}^d$ for some $d \in \mathbb{N}$.

To see this we can follow the same strategy as in the proof of [KK19a Theorem 6.2] based on a theorem of Preiss that says for $m_X$-a.e. point $x \in X$ an iterated tangent space at $x$ shows up as tangent space a $x$, and an isometric splitting theorem for tangent spaces of $X$. The splitting theorem follows by the almost splitting theorem by Tian and Zhang in [TZ16]. For $x \in X$ and a tangent space $T_x X$ at $X$ there exists a blow up sequence $(\epsilon_i, M_i, x_i)$ that converges in measured Gromov-Hausdorff sense to $T_x X$. If $T_x X$ contains a geodesic, in a small ball around $x_i$ we can find points $p_i^+, p_i^-$ such that
\[
d_{M_i}(p_i^+, x_i) + d_{M_i}(p_i^-, x_i) - d_{M_i}(p_i^+, p_i^-) \leq \delta_i d_{M_i}(p_i^+, p_i^-).
\]
for $\delta_i \to 0$. By the almost splitting theorem [TZ16 Theorem 2.31] given $\epsilon > 0$ we can choose $\delta_i > 0$ sufficiently small such that $d_{GH}(B_\delta(x_i), B_\delta(0)) \leq \epsilon$ where $B_\delta(0) \subset \mathbb{R} \times Y$ for some metric space $Y$ with diam $Y \leq \pi$. Hence, after rescaling appropriately and taking the limit for $\epsilon \to 0$, we see that $T_x X$ splits off $\mathbb{R}$ isometrically.

By Preiss’s theorem for $m_X$-a.e. we can iterate this process and obtain some tangent cone $Z_x$ at $x$ that is isometric to $\mathbb{R}^k$ for some $k \leq n$. This finishes the proof. \[\square\]

**Corollary 7.5.** Let $X$ be a measure Gromov-Hausdorff limit of sequence of Riemannian manifolds satisfying (1) and (2). Then every tangent space $T_x X$ for $x \in X$ satisfies the condition $RCD(0, N)$. In particular, $T_x X$ is an Euclidean cone over some $RCD(N-2, N-1)$ space $Y$. 
Proof. By [TZ16, Theorem 2.33] (blow up) tangent cones are always Euclidean cones. For a blow up sequence \((\epsilon_i M_i, o_i)\) we can compute that
\[
k_{[\epsilon_i M_i, o_i]}(\epsilon_i \kappa_i, p, \epsilon_i K, R) = k_{[M_i, x_i]}(\kappa_i, p, K, \epsilon_i R)
= (\epsilon_i R)^{2p} \int (\kappa_i - K)^p \, d\mathcal{M}_i \to 0.
\]
Hence, by the previous theorem \(T_\alpha X\) satisfies \(RCD(0, N)\). The last part follows from [Ket15].

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