COUNTING OF SIEBER-RICHTER PAIRS OF PERIODIC ORBITS

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Abstract. In the framework of the semiclassical approach the universal spectral correlations in the Hamiltonian systems with classical chaotic dynamics can be attributed to the systematic correlations between actions of periodic orbits which (up to the switch in the momentum direction) pass through approximately the same points of the phase space. By considering symbolic dynamics of the system one can introduce a natural ultrametric distance between periodic orbits and organize them into clusters. Each cluster consists of orbits approaching closely each other in the phase space. We study the distribution of cluster sizes for theacker’s map in the asymptotic limit of long trajectories. This problem is equivalent to the one of counting degeneracies in the length spectrum of the de Bruijn graphs. Based on this fact, we derive the probability $P_k$ that $k$ randomly chosen periodic orbits belong to the same cluster. Furthermore, we find asymptotic behaviour of the largest cluster size $|C_{\text{max}}|$ and derive the probability $P(t)$ that a random periodic orbit belongs to a cluster of the size smaller than $t|C_{\text{max}}|$, $t \in [0, 1]$.

1. Introduction

In their seminal paper [1] Bohigas Giannoni and Schmit conjectured that the local energy spectrum statistics of quantum systems with fully chaotic classical dynamics are universal and can be described by standard ensembles of Random Matrix Theory (RMT). To explore the origin of such universality semiclassical techniques based on the applications of the Gutzwiller trace formula was introduced by Berry in [2]. By this approach the correlations between energy levels of a quantum Hamiltonian system can be related to the correlations between periodic orbit actions in the corresponding classical system. Within the diagonal approximation, where only correlations between periodic orbits themselves are taken into account, Berry managed to obtain the leading order of the universal spectral form factor. The diagonal approximation, however, turned out to be insufficient to reproduce full RMT result, whose derivation had remained a distinguished challenge for yet a long time [3,4]. The breakthrough was achieved in 2001 when Sieber and Richter discovered a non-trivial mechanism of correlations between periodic orbit actions [5]. They showed that the next to the leading order term of the universal spectral form factor can be obtained by taking into account correlations between long periodic orbits with one self-crossing under a small angle (usually referred as encounter) and the associated partner orbits, see fig. 1a. Such pairs of orbits have close actions and contribute systematically into the spectral correlations. Later, this approach was extended to include correlations between periodic orbits having an arbitrary number of encounters which culminated in the derivation of the full RMT result [6].
In a nutshell Sieber-Richter pairs and its many-encounter analogs are nothing more than bunches of periodic orbits running through almost the same points of the phase space up to the switch in the momentum direction. Furthermore, in the case of broken time reversal symmetry only orbits passing close to each other with the same momentum direction are of relevance, see fig. 1b. In the present paper we restrict consideration only to the latter case. Due to the hyperbolic nature of the dynamics the action difference between such trajectories is small and determined by the lengths of the encounters. Equally important, the correlation mechanism of Sieber-Richter pairs is robust to perturbations of the dynamical system. In fact, any hyperbolic system contains a large number of long periodic orbits with close actions which are not of Sieber-Richter type. However, it might be expected that, in general, the differences between their actions fluctuate enormously under perturbations of the system. Therefore, the contribution from a generic pair of periodic orbits is washed out after averaging (e.g., over ensemble of systems) and only pairs of Sieber-Richter type contribute systematically to the spectral correlations.

![Figure 1](image.png)

**Figure 1.** a) Schematic picture of Sieber-Richter pair of periodic orbits with one encounter region being marked out by a box. b) Pair of periodic orbits with two encounters. As opposed to the case (a), here two orbits pass close to each other in the phase space of the system. Only such orbits contribute to the spectral correlations of systems with broken time-reversal symmetry.

The robustness of Sieber-Richter correlation mechanism can be easily understood by considering symbolic dynamics of the system. Assuming that a finite Markov partition exists, any periodic orbit can be encoded by a finite sequence of symbols $x = x_1x_2\ldots x_n$ from some alphabet. The fact that two periodic orbits come close to each other in the phase space has a natural interpretation on the level of the corresponding symbol sequences. Namely, let $x$ and $y$ be two sequences of length $n$ such that any subsequence of length $p$ occurs in $x$ exactly the same number of times as in $y$. This property of sequences $x$ and $y$ is referred below as $p$-closeness. It is straightforward to see that any pair of $p$-close sequences defines in fact a pair of periodic orbits which are close with respect to the Euclidean metric of the phase space. Their metric closeness is controlled by $p$: the larger parameter $p$ is, the closer two periodic orbits approach each other. Accordingly, all periodic orbits of the system can be organized into a disjoint union of clusters of $p$-close orbits. Each cluster is composed of periodic orbits with close actions running through approximately the same points of the phase space in a different time order.
Motivated by the application to quantum chaos, questions regarding the number of periodic trajectories with close actions/lengths were previously addressed both in physics and mathematics literature in the context of several different models of chaotic dynamical systems: geodesics flows on manifolds of negative curvature [8], billiards [3, 10], quantum maps [11] and quantum graphs [12–15]. In the present paper we study the phenomenon of clustering of periodic orbits on the level of symbolic dynamics. Our consideration is restricted to the simplest possible grammar assuming a two-letter alphabet, \( x_i \in \{0, 1\} \), and absence of pruning, i.e. each symbol in the sequence can be followed by any other symbol. These grammar rules is met, for instance, in the baker’s map [16]. The main question to be addressed below can be informally stated as follows: given an integer \( p \), what is the probability that a randomly picked up periodic sequence (equiv. orbit) of length \( n \) has a certain number of \( p \)-close partners when \( n \) is large enough?

2. Definitions and main results

2.1. The baker’s map. In what follows we consider clustering of periodic orbits within the paradigm model of chaotic system – the baker’s map \( \mathcal{T} \). Explicitly, the action of \( \mathcal{T} \) on the points \( v = (q, p) \) of the two-dimensional phase space \( V = [0, 1) \times [0, 1) \) is given by:

\[
\mathcal{T} \cdot v = \begin{cases} 
(2q, \frac{1}{2}p) & \text{if } q \in [0, \frac{1}{2}), \\
(2q - 2, 1 - \frac{1}{2}p) & \text{if } q \in [\frac{1}{2}, 1), 
\end{cases}
\]

where \( q \) and \( p \) play the role of the coordinate and momentum respectively, see e.g., [16] for details. The baker’s map has an advantage of having a particularly simple symbolic dynamics. This allows to avoid cumbersome notation and makes the exposition more transparent.

2.2. Symbolic dynamics and periodic orbits. Symbolic dynamics is a standard tool widely used in the theory of hyperbolic dynamical systems. By this approach each point of the system phase space is identified with a sequence of symbols from a certain alphabet. Given such a representation the time evolution of the system takes a simple form.

To introduce symbolic dynamics it is necessary first to define Markov partition of the phase space \( V \). A standard choice for the baker’s map is \( V_0 \sqcup V_1 \), where \( V_0 = [0, \frac{1}{2}) \times [0, 1) \) and \( V_1 = [\frac{1}{2}, 1) \times [0, 1) \). Then any point \( v = (q, p) \in V \) can be uniquely encoded by a two-sided sequence \( x_- x_+ \) of zeros and ones: \( x_+ = x_1 x_2 \ldots, x_- = \ldots x_{-2} x_{-1} x_0, x_i \in \{0, 1\} \) using a simple algorithm: \( x_i = 0 \) if \( \mathcal{T}^i v \in V_0 \) and \( x_i = 1 \) if \( \mathcal{T}^i v \in V_1 \) for \( i \in \mathbb{Z} \). Positive \( i \) correspond to the “future” evolution of \( v \), which is written in the \( x_+ \) subsequence. The coordinate \( q \) expresses through this subsequence by the formula \( q = 0.x_+ \). The “past” history of \( v \) and the momentum \( p \) are defined by \( x_- \) subsequence: \( p = 0.x_- \).

Within the above symbolic representation the time evolution of the system is given by the shift map \( \sigma : \ldots x_- x_0 x_1 x_2 x_3 \ldots \to \ldots x_- x_0 x_1 x_2 x_3 \ldots \).
which moves the separation point “.” between the future and the past in the sequence of symbols step by step.

All infinite periodic sequences composed of one and the same finite piece \( x \in X_n \) correspond to periodic orbits of the system. Here and below symbol \( X_n \) stands for the set of all possible sequences of zeroes and ones having the length \( n \). Let \( \gamma_x \) denotes the periodic orbit of the backer’s map associated with the sequence \( x \in X_n \). Note, that two sequences \( x, x' \) correspond to the same periodic trajectory if and only if they are related by the cyclic shift, i.e. if \( x = x_1 x_2 \ldots x_n \) and \( x' = x_{i+1} \ldots x_n x_1 \ldots x_i \) for some \( i \in \{1, \ldots n - 1\} \) then correspond to one and the same periodic trajectory \( \gamma_x = \gamma_x' \). In what follows we will also consider the quotient set \( X_n := X_n/ \sim \) with respect to the cyclic shift \( x \sim x' \). It is convenient to think of the elements belonging to the set \( X_n \) as of sequences from \( X_n \) with the “glued” ends. Importantly, the elements from \( X_n \) and all periodic orbits having the period \( n \) are in one to one correspondence according to the remark made above in this paragraph.

2.3. Clusters of periodic orbits. To define the clusters of periodic orbit, firsts, we need to introduce the notion of their closeness. Take two \( n \)-periodic orbits \( \gamma_x, \gamma_y \) composed of \( n \) points \( \{\gamma_x(i)\}_{i=0}^{n-1}, \{\gamma_y(i)\}_{i=0}^{n-1} \) in the phase space. It is natural to think about two orbits \( \gamma_x, \gamma_y \) as of close ones if in a vicinity of any point \( \gamma_x(i) \), \( i = 0, \ldots n - 1 \) one can find some point from the set \( \{\gamma_y(i)\}_{i=0}^{n-1} \) and vice versa. In other words, two trajectories pass through approximately the same parts of the phase space but perhaps in a different order. To put this picture on a more solid background, we say \( \gamma_x \) is in the \( p \)-neighborhood of \( \gamma_y \) if there exist exactly \( n \) pairs \( (\gamma_x(i_k), \gamma_y(i'_k)), k = 1, \ldots n \) such that for each \( k \) distance between points is bounded by \( ||\gamma_x(i_k) - \gamma_y(i'_k)|| \leq 2^{-p} \), where the distance \( ||v, v'|| \) in the phase space between \( v = (q, p) \) and \( v' = (q', p') \) is defined as \( ||v, v'|| := \max\{|q - q'|, |p - p'|\} \).

The above notion of metric closeness between periodic trajectories can be carried over to the topological space \( X_n \). We will say that two sequences \( x, y \in X_n \) are \( p \)-close if any sequence of \( p \leq n \) consecutive symbols \( a_1 a_2 \ldots a_p, a_i \in \{0, 1\} \) appears the same number of times (which might be also zero) both in \( x \) and \( y \). Speaking informally \( x \) and \( y \) are \( p \)-close if their local content (of the length \( p \)) is exactly the same in both sequences. This equivalence relation is denoted below as \( x \sim y \). It is straightforward to see that \( \gamma_x \) is in the \( p \)-neighborhood of \( \gamma_y \) whenever \( x \sim y \).

There are two simple but important properties of the equivalence relation \( x \sim y \) which should be emphasized:

I The relations \( x \sim y \) and \( x \sim z \) also imply that \( z \sim y \);

II The relation \( x \sim y \) implies that \( x \sim z \).

According to the first property all periodic sequences (resp. periodic orbits) can be separated into a number of clusters \( C_i^{(p)}, i = 1, \ldots N_p \) such that two sequences \( x \) and \( y \) (resp. \( \gamma_x, \gamma_y \)) belong to the same cluster if and only if \( x \sim y \). For instance, for given \( p = 2 \) three sequences \([1101000], [1100010], [1100100]\) belong to the same cluster, see fig. 2. In a completely analogous way one can consider clusters \( C_i^{(p)}, i = 1, \ldots N_p \) of sequences from the set \( X_n \). The connection between clusters is given
by: \( C^{(p)}_i = \frac{C^{(p)}_i}{\sim} \). In other words, each \( x \in C^{(p)}_i \) corresponds to a set of sequences from \( C^{(p)}_i \) which are related to each other by a cyclic shift.

The second property of \( x \sim y \) allows to organize the clusters of periodic sequences in a tree like structure. The \( p \)-th level of the tree contains clusters of \( p \)-close periodic sequences, see figs. 2, 3. One can introduce a distance in the space of sequences based on this hierarchical structure. The distance \( d(x, y) \) between two elements \( x \) and \( y \) being proportional to the maximal level of the tree where \( x \) and \( y \) belong to the same cluster: \( d(x, y) = n - \max \{ p | x \sim y \} \) (\( d(x, x) = 0 \)) satisfies the ultrametric property \( d(x, y) \leq \max \{ d(x, z), d(z, y) \} \). By the identification of periodic orbits \( \gamma_x, \gamma_y \) with the corresponding sequences \( x, y \), one can lift the distance \( d(x, y) \) to the space of periodic orbits. And we come to the conclusion that the space \( \mathcal{X}_n \) (or equivalently the space of periodic orbits) acquires a natural ultrametric structure.

2.4. Cluster distribution. The primary goal of the present paper is to understand how the distribution of the cluster sizes \( |C^{(p)}_i| \) depends on the level \( p \) in the limit \( n \to \infty \). To this end, we need to estimate the moments of the cluster sizes:

\[
Z_k = \sum_{i=1}^{N_\nu} |C^{(p)}_i|^k.
\]
Figure 3. Scheme of hierarchical structure of periodic sequences. The $p$-th level of the tree is composed of clusters (depicted by circles) of $p$-close trajectories. Each cluster at level $p$ is split into a number of clusters at the next level $p + 1$.

It turns out, however, that a more convenient object to consider is

$$Z_k = N_p \sum_{i=1}^{N_p} |C_i^{(p)}|^k,$$

where the sum runs over the clusters of sequences from $X_n$ (rather than sequences from $X_n'$). The connection between cluster sizes in $X_n$ and $X_n'$ is particularly simple if $n$ is a prime number. In this case for each $x \in X_n$ except $x^{(0)} = [00 \ldots 0]$ and $x^{(1)} = [11 \ldots 1]$, there are exactly $n$ sequences from $X_n$ which are related by the cyclic shift. Two special clusters corresponding to sequences $x^{(0)}$, $x^{(1)}$ contain just one element. For all other clusters:

$$|C_i^{(p)}| = n|C_i^{(p)}|.$$  

This yields

$$Z_k = Z_k - \frac{2}{n^k} + 2, \quad d_n = \frac{2^n - 2}{n} + 2,$$

where $d_n$ stands for the total number of elements in $X_n$. In the case when $n$ is not a prime number, the connection between cluster sizes in $X_n$ and $X_n'$ is not anymore trivial due to the presence of periodic orbits with a period less than $n$. However, if one includes only prime periodic orbits (whose period is exactly $n$) into clusters $C_i^{(p)}$, $C_i^{(p)}$ then the connection (2.4) remains valid. As a matter of fact, the exclusion of non-prime periodic orbits can be justified on the ground of a standard argument that their number is negligible in comparison with the number of prime periodic orbits in the limit $n \to \infty$. We arrive to the following relationship between (2.2)

1The number of prime and non-prime periodic orbits scales as $2^n$ and $2^{n/r}$, respectively, where $r \geq 2$ is a minimal divisor of $n$. 
and (2.3):

\[ \zeta_k = \frac{Z_k}{n^k} \left( 1 + O(n^{-1}) \right) \] \quad \[ d_n = \frac{2^n}{n} \left( 1 + O(n^{-1}) \right) . \]

Note also that for a typical cluster \(|C^{(p)}_i| (1 + O(n^{-1})) = n|C^{(p)}_i|.

It worth of mentioning that the rescaled moments have a simple interpretation as probabilities of finding a number of periodic orbits in the same cluster. Indeed, let \( \gamma_i, i = 1, \ldots k \) be a set of \( k \geq 2 \) orbits randomly chosen from the total set \( \Gamma_n = \{ \gamma_x \mid x \in \mathcal{X}_n \} \) of periodic orbits of length \( n \). Then the probability that all \( k \) orbits belong to the same cluster is given by

\[ \mathcal{P}_k = \frac{Z_k}{d_n^k} . \]

In particular, the probability \( \mathcal{P}_2 \) that two periodic orbits belong to the same cluster is given by \( Z_2/d_n^2 \).

2.5. Main Results. The central result of the present paper is the following asymptotic formula for \( Z_k \) in the limit \( n \to \infty \):

\[ Z_k = 2^n k \left( \frac{1}{k} \right)^{2p-2} \left( \frac{2p}{\pi n} \right)^{(k-1)2p-2} \left( 1 + O(n^{-1}) \right) . \]

Using then eq. (2.6) and eq. (2.7) we obtain the probability of finding \( k \) random orbits in the same cluster

\[ \mathcal{P}_k = \left( \frac{1}{k} \right)^{2p-2} \left( \frac{2p}{\pi n} \right)^{(k-1)2p-2} \left( 1 + O(n^{-1}) \right) . \]

In addition, we show that the number of periodic orbits in the largest cluster \( C^{(p)}_{\max} \) is asymptotically given by:

\[ |C^{(p)}_{\max}| = \left( \frac{2^n}{n} \right) \left( \frac{2p}{\pi n} \right)^{2p-2} \left( 1 + O(n^{-1}) \right) . \]

Based on eqs. (2.8,2.10) we deduce probability \( P(t), t \in [0,1] \) that random periodic orbit from the set \( \Gamma_n \) belongs to a cluster with the size less then \( t|C^{(p)}_{\max}| \) and show that in the limit of \( n \to \infty \) this probability depends only on \( p \):

\[ P(t) = \int_0^t \rho(\tau) d\tau, \quad \rho(\tau) = \frac{(\log \tau)^{2p-2-1}}{(2p-2-1)!} . \]

As we show in the body of the paper the problem of counting cluster distribution of \( p \)-close periodic orbits is in fact equivalent to the one of counting degeneracies in the length spectrum of the so-called de Bruijn graphs [17]. In this context the asymptotic behavior of \( Z_2 \) and related questions have been considered previously in [12,15]. The exact connection between our results and the above mentioned works is discussed in the last section of the paper.
2.6. **Organization of the paper.** The paper is organized as follows. In Sec. 3 we show that the problem of counting cluster sizes of \(p\)-close periodic orbits in the baker’s map can be cast in the form of counting closed paths on a certain graph passing through the same edges (or vertices). Using this connection we express \(Z_k\) as a matrix integral of certain type. In Sec. 4 we evaluate these integrals in the saddle point approximation and derive eq. (2.8). In Sec. 5 we obtain a asymptotic formula for the size of the largest cluster \(C_{\text{max}}^{(p)}\). Using then the results from Sec. 4 we arrive to the probability \(P(t)\) of finding a periodic orbit in a cluster of the size smaller than \(t|C_{\text{max}}^{(p)}|\). Sec. 6 is devoted to the discussion of uniformity of periodic orbits distribution over the graph. Finally, the concluding remarks are presented in Sec. 7.

3. **Clusters of closed paths on graphs**

As we show below, the counting problem of \(p\)-close periodic orbits is equivalent to the one of counting closed paths on the de Bruijn graph \(G_p\) passing the same number of times through its edges. The graph \(G_p\) is constructed in the following way. With each sequences \(a = [a_1a_2\ldots a_p], a \in X_p\) we associate a directed edge \(e_a\) of \(G_p\) whose initial and terminal points are denoted by \(e_{(\text{in})}^a\) and \(e_{(\text{out})}^a\), respectively. The connections between \(2^p\) edges are fixed by the rule: for any pair of edges \(e_a, e_b\), defined by the sequences \(a = [a_1a_2\ldots a_p], b = [b_1b_2\ldots b_p]\) the endpoints \(e_{(\text{in})}^a, e_{(\text{out})}^b\) belong to the same vertex if and only if \(a_i = b_{i+1}\) for all \(i = 1, \ldots p-1\), see fig. 4.

![Figure 4](image)

**Figure 4.** The graph \(G_p\) for \(p = 4\) (left) and \(p = 5\) (right). Each sequence \([a_1a_2a_3a_4], a_i \in \{0,1\}\) encodes edges on \(G_4\) and vertices on \(G_5\), respectively.

It is straightforward to see that any closed path \(g\) on the graph \(G_p\) passing through \(n\) edges can be uniquely represented by a sequence \(x = [x_1x_2\ldots x_n]\) from the set \(X_n\). By such identification \(i\)th edge of \(G_p\) passed by \(g\) corresponds to the segment \([x_i,x_{i+1}\ldots x_{p-1+i}]\) of the sequence \(x\). We will use notation \(g_x\) to denote closed paths corresponding to \(x \in X_n\). For each closed path \(g_x\) let \(n(x) = \{n_a, a \in X_p\}\) be the set of integers, such that \(n_a\) is the number of times \(g_x\) passes through the edge \(a\). Then
$x \bowtie y$ if and only if $g_x, g_y$ go through every edge of $G_p$ the same number of times (but in different time order) i.e., $n(x) = n(y)$. Therefore, each cluster $C_n$ of $p$-close periodic orbits is uniquely determined by the vector of integers $n = \{n_a, a \in X_p\}$.

**Remarks 3.1.**

a) By attaching a length to each edge of the graph $G_p$ one can turn it to a metric graph. Note that each cluster $C_n$ consists then of trajectories having the same length. Accordingly, counting of cluster sizes is equivalent to counting of degeneracies in the length spectrum of the corresponding metric graph. The last problem have been studied in [12,13] for some classes of metric graphs, see also discussion in Sec. 7.

b) Counting of closed paths passing the same number of times through the edges of $G_p$, is actually equivalent to counting of closed paths passing the same number of times through the vertices of twice larger graph $G_{p+1}$. Indeed, let us enumerate the $2^p$ vertices of the graph $G_{p+1}$ in the same way as edges of $G_p$, see fig. 4. By the identification of each closed path of the length $n$ with a sequence $x \in X_n$ we obtain one-to-one correspondence between closed paths on two graphs. Correspondingly, clusters of closed paths have the same sizes.

To find the size $|C_n|$ of $n$’th cluster we need to count the number of closed paths which go through the edges $a \in X_p$ of $G_p$ exactly $n_a$ times. To this end we introduce connectivity matrix $Q$ between edges of the graph. It is convenient to use for this purpose a tensorial representation for the vector space on which $Q$ acts, see [19,20]. Let $\mathcal{H}$ be the $2^p$-dimensional linear space spanned by the vectors $|a\rangle = |a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_p\rangle$, $a_i \in \{0,1\}$. The linear operator $Q$ acts on the vectors from $\mathcal{H}$ according to the rule:

$$Q|a_1\rangle \otimes |a_2\rangle \cdots \otimes |a_p\rangle = |a_2\rangle \otimes \cdots \otimes |a_p\rangle \otimes \frac{1}{2}(|0\rangle + |1\rangle).$$

Note that this definition agrees with the connectivity rules between edges on the graph $G_p$, where each edge $e_a$, $a = [a_1a_2\ldots a_p]$ is connected with the edges $e_{a'}$, $a' = [a_2\ldots a_0]$ and $e_{a''}$, $a'' = [a_2\ldots a_p]$. In addition, with each edge $e_a$ we associate a phase $\phi_a$ and define the diagonal operator $\Lambda(\phi)$, $\phi := \{\phi_a|a \in X_p\}$:

$$\Lambda(\phi)|a_1\rangle \otimes |a_2\rangle \cdots \otimes |a_p\rangle = e^{i\phi_a}|a_1\rangle \otimes |a_2\rangle \cdots \otimes |a_p\rangle.$$

It is straightforward to see that in the matrix form $Q$, $\Lambda(\phi)$ can be written as [19]:

$$Q = \begin{pmatrix}
1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 1 \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 1
\end{pmatrix}, \quad \Lambda(\phi) = \begin{pmatrix}
e^{i\phi_1} & 0 & 0 & \ldots & 0 \\
0 & e^{i\phi_2} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & e^{i\phi_{2p-1}} & 0 \\
0 & 0 & \ldots & 0 & e^{i\phi_{2p}}
\end{pmatrix}.$$
The introduction of matrices $Q, \Lambda(\phi)$ is useful because of the following relationship between traces of their products and the sizes of the clusters $C_n$:

\[(3.3) \quad \text{Tr}(Q\Lambda(\phi))^n = \sum_n |C_n| \exp(i(n, \phi)), \quad (n, \phi) = \sum_{a \in X_p} n_a \phi_a, \quad n = \sum_{a \in X_p} n_a,\]

with the first sum running over all clusters $C_n$. Eq. (3.3) is a key component of our analysis, as it allows to express $\exp(i(n, \phi))$ through the traces of powers of matrix $Q\Lambda(\phi)$. In particular, the second moment $Z_2$ can be represented in the form of the integral over $\phi_a$:

\[(3.4) \quad Z_2 = \sum_n |C_n|^2 = \prod_{a \in X_p} \int_0^{2\pi} \frac{d\phi_a}{2\pi} |\text{Tr}(Q\Lambda(\phi))|^2.\]

Analogously, higher order moments are given by

\[(3.5) \quad Z_k = \sum_n |C_n|^k = \prod_{j=1}^{k} \prod_{a \in X_p} \int_0^{2\pi} \frac{d\phi_a}{2\pi} \text{Tr}((Q\Lambda(\phi)))^\delta \left(\sum_{l=1}^{k} |\phi_a(l)|\right).\]

As we show below the number of integration and dimensions of matrices in eqs. (3.3, 3.5) can be actually reduced by the factor of two. Note that the $2^p \times 2^p$ matrix $Q$ can be represented as the product $Q = RS$ of the matrices

\[(3.6) \quad R = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}, \quad S = \begin{pmatrix}
1 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots \\
0 & 1 & 0 & \ldots & 0 & 1 & 0 & \ldots \\
0 & 0 & 1 & \ldots & 0 & 0 & 1 & \ldots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
\]

whose dimensions are $2^{p-1} \times 2^p$ and $2^p \times 2^{p-1}$, respectively. Changing the order of the matrix product in eq. (3.3) yields then

\[(3.7) \quad \text{Tr}(Q\Lambda(\phi))^n = \text{Tr}(S\Lambda(\phi)R)^n = \text{Tr}(Q'(\phi))^n,\]

where $Q'(\phi) = S\Lambda(\phi)R$ is $2^{p-1} \times 2^{p-1}$ matrix of the form

\[(3.8) \quad Q'(\phi) = \begin{pmatrix}
e^{i\phi_1} & 0 & \ldots & 0 & e^{i\phi_{2p-1}+1} & 0 & \ldots & 0 \\
e^{i\phi_2} & 0 & \ldots & 0 & e^{i\phi_{2p-1}+2} & 0 & \ldots & 0 \\
0 & e^{i\phi_3} & \ldots & 0 & 0 & e^{i\phi_{2p-1}+3} & \ldots & 0 \\
0 & e^{i\phi_4} & \ldots & 0 & 0 & e^{i\phi_{2p-1}+4} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & \ldots & e^{i\phi_{2p-1}} & 0 & 0 & \ldots & e^{i\phi_{2p}} \\
0 & 0 & \ldots & e^{i\phi_{2p-1}} & 0 & 0 & \ldots & e^{i\phi_{2p}}
\end{pmatrix}.
\]

Furthermore, using the invariance of the trace under the transformation

\[Q'(\phi) \rightarrow \Lambda(\delta)Q'(\phi)\Lambda^{-1}(\delta),\]
where $\Lambda(\delta)$ is an arbitrary diagonal matrix, we can exclude half of the integration variables in eqs. (3.4, 3.5), see Appendix A. The result is the following expression for the moments

$$Z_k = \prod_{j=1}^{k} \prod_{i=1}^{2^{p-1}} \int_{0}^{2\pi} \frac{d\phi_i^{(j)}}{2\pi} \text{Tr} \left( Q(\phi^{(j)}) \right)^n \delta \left( \sum_{l=1}^{k} \phi_i^{(l)} \right),$$

with the matrix $Q(\phi)$ given by

$$Q(\phi) = \begin{pmatrix}
\delta & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & \delta & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & e^{i\phi_{2^{p-1}-1}} & 0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & e^{i\phi_{2^{p-1}-1}} & 0 & 0 & \cdots & 1
\end{pmatrix}.$$

In the next section we will use eq. (3.9) to obtain large $n$ asymptotic of $Z_k$.

4. Clustering Probability of $k$ orbits

In order to evaluate moments $Z_k$ we will apply saddle point approximation to the integral (3.9), where $n$ will play the role of a large parameter. To make the exposition more transparent we first consider below the second moment and later extend the result to all $k > 2$.

4.1. Saddle point approximation for $Z_2$. For $k = 2$ the integral (3.9) can be written as

$$Z_2 = \left( \prod_{i=1}^{2^{p-1}} \int_{0}^{2\pi} \frac{d\phi_i}{2\pi} \right) \cdot \exp \mathcal{F}_n(\phi), \quad \mathcal{F}_n(\phi) = \log |\text{Tr}(Q(\phi))|^2.$$

It is easy to see that the global maximum of $\mathcal{F}_n(\phi)$ is attained at $\phi = 0$, where $\phi_i = 0$, $i = 1, \ldots, 2^{p-1}$ and $\mathcal{F}_n(0) = n \log 2$. We therefore need to expand $\mathcal{F}_n(\phi)$ around zero up to the second order in $\phi$ and then use saddle point approximation in (4.1). To evaluate derivatives of $\mathcal{F}_n(\phi)$ it is convenient to use the following decomposition of the matrix $Q(\phi)$:

$$Q(\phi) = \Lambda(\phi)Q_0 + Q_1.$$

Here $Q_0 = [R \ 0]$, $Q_1 = [0 \ R]$ are matrices composed of two blocks. The first (resp. second) block is given by the matrix $R$ (from eq. (3.6)) of the dimension $2^{p-1} \times 2^{p-2}$, while the second (resp. first) one is $2^{p-1} \times 2^{p-2}$ matrix of zeroes (for the element-wise definition of $Q_0$, $Q_1$, see Appendix B). Straightforward calculations give then:

$$\frac{\partial \mathcal{F}_n(\phi)}{\partial \phi_j} = 2n \text{Re} \left[ \frac{i \text{Tr}(\Lambda(\phi)P_jQ_0(Q(\phi))^{n-1})}{\text{Tr}(Q(\phi))^n} \right].$$
where \( P_j \) denotes projection matrix on \( j \) element of the basis, i.e., \((P_j)_{m,i} = \delta_{m,i} \delta_{i,j}\).

It follows immediately that \( \frac{\partial Q_j(\phi)}{\partial \phi_j} |_{\phi=0} = 0 \), as it should be for a saddle point. Taking an additional derivative in (4.2) yields

\[
\frac{\partial^2 F_n(\phi)}{\partial \phi_j \partial \phi_i} |_{\phi=0} = 2n \left[ -\sum_{k=0}^{n-2} \frac{\text{Tr} (P_i Q_0 Q^k P_j Q_0 Q^{n-k-2}) + \delta_{i,j} \text{Tr} (P_i Q_0 Q^{n-1})}{\text{Tr} Q^n} \right] \\
+ 2n^2 \left[ \frac{\text{Tr} (P_i Q_0 Q^{n-1}) \text{Tr} (P_j Q_0 Q^{n-1})}{(\text{Tr} Q^n)^2} \right],
\]

implying that

\[
\frac{\partial^2 F_n(\phi)}{\partial \phi_j \partial \phi_i} |_{\phi=0} = -n B_{i,j},
\]

where the matrix \( B \) is defined by:

\[
B = 2^{p-1} \left( \bar{Q} + \bar{Q}^T + 2I - (1 + 2p) \left( \frac{Q}{2} \right)^p \right), \quad \bar{Q} = Q_0 \sum_{k=0}^{n-1} \left( \frac{Q}{2} \right)^k.
\]

We can use now (4.1) to evaluate \( Z_2 \) in the large \( n \) limit:

\[
Z_2 = 2^{2n} \left( \prod_{j=1}^{2^{p-1}} \int_0^{2\pi} \frac{d\phi_j}{2\pi} \right) \cdot \exp \left[ -\frac{n}{2} \sum_{i,j} B_{i,j} \phi_i \phi_j \right] \left( 1 + O \left( \frac{1}{n} \right) \right)
\]

\[
= 2^{2n} (2\pi n)^{-2^{p-2}} \left( \det B \right)^{-\frac{1}{2}} \left( 1 + O \left( \frac{1}{n} \right) \right).
\]

The determinant of \( B \) can be explicitly calculated (see Appendix B) which finally leads to:

\[
Z_2(n) = 2^{2n} \left( \frac{2^{p-1}}{\pi n} \right)^{-2^{p-2}} \left( 1 + O \left( \frac{1}{n} \right) \right).
\]

4.2. Saddle point approximation for \( Z_k, \; k > 2 \). Our starting point is the representation (3.9) for \( Z_k \):

\[
Z_k = \left( \prod_{j=1}^{k-1} \int_0^{2\pi} \frac{d\phi_j^{(j)}}{2\pi} \right) \cdot \exp F_n(\{\phi^{(j)}\}),
\]

\[
F_n^{(k)}(\{\phi^{(j)}\}) = \log \left[ \text{Tr} \left( Q \left( -\bar{\phi} \right) \right)^n \prod_{i=1}^{k-1} \text{Tr} \left( Q \left( \phi^{(i)} \right) \right)^n \right],
\]

where we introduced notation:

\[
\bar{\phi} := \sum_{j=1}^{k-1} \phi^{(j)}.
\]
As in the case \( k = 2 \), the maximum of \( \mathcal{F}_n^{(k)} \) is attained when all phases vanish i.e., for \( \phi^{(j)} = 0, j = 1, \ldots k - 1 \). After taking the first derivative of \( \mathcal{F}_n^{(k)} \) we have

\[
(4.9) \quad \frac{\partial \mathcal{F}_n^{(k)}(\phi)}{\partial \phi^{(j)}_j} = n \left[ \frac{i \text{Tr} \left( A(\phi^{(l)}_l) P_j Q_0 (Q(\phi^{(l)}_l))^{n-1} \right)}{\text{Tr}(Q(\phi^{(l)}_l))} \right] - n \left[ \frac{i \text{Tr} \left( A(-\bar{\phi}_i) P_j Q_0 (Q(-\bar{\phi}))^{n-1} \right)}{\text{Tr}(Q(-\bar{\phi}))} \right]
\]

The second derivative of \( \mathcal{F}_n^{(k)} \) at \( \phi^{(j)} = 0, j = 1, \ldots k - 1 \) is then:

\[
(4.10) \quad \frac{\partial^2 \mathcal{F}_n^{(k)}(\phi)}{\partial \phi^{(j)}_j \partial \phi^{(m)}_i} \bigg|_{\phi=0} = -n B^{(k)}_{i,j,m};
\]

with the elements of the \( 2^p(k-1) \times 2^p(k-1) \) matrix \( B^{(k)} \) given by:

\[
B^{(k)}_{[i,j,m]} = \frac{1}{2} B_{i,j}(\delta_{l,m} + 1).
\]

Note that \( B^{(k)} \) can also be written as the product of two matrices composed of \( (k - 1) \times (k - 1) \) blocks:

\[
(4.11) \quad B^{(k)} = \frac{1}{2} \begin{pmatrix}
2I & 1 & \ldots & 1 \\
1 & 2I & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 2I
\end{pmatrix}
\begin{pmatrix}
B & 0 & \ldots & 0 \\
0 & B & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & B
\end{pmatrix},
\]

with \( 1 \) and \( 0 \) standing for unit and zero \( 2^p \times 2^p \) matrices, respectively.

Applying now saddle point approximation to (4.8) we obtain

\[
(4.12) \quad Z_k = 2^{kn} \left( \prod_{l=1}^{k-1} \prod_{j=1}^{2^{p-1}} \int_0^{2\pi} d\phi^{(l)}_j \frac{1}{2\pi} \right) \cdot \exp \left[ -\frac{n}{2} \sum_{i,j} B^{(k)}_{i,j,m} \phi^{(l)}_i \phi^{(m)}_j \right] \left( 1 + O \left( \frac{1}{n} \right) \right)
\]

\[
= 2^{kn} (2\pi n)^{-k(k-1)/2^{p-2}} (\det B^{(k)})^{\frac{k}{2}} \left( 1 + O \left( \frac{1}{n} \right) \right).
\]

By eq. (4.11) the determinant of \( B^{(k)} \) can be explicitly evaluated:

\[
\det B^{(k)} = \left( \frac{k}{2^{k-1}} \right)^{2^p} (\det B)^{k-1}.
\]

Substituting this expression into eq. (4.12) gives then

\[
(4.13) \quad Z_k = \frac{2^{2k}}{k^{2^{p-2}}} \left( \frac{2^p}{\pi n} \right)^{(k-1)/2^{p-2}} \left( 1 + O \left( \frac{1}{n} \right) \right).
\]

To verify the above asymptotic formula we evaluated the ratio between the leading order term of (4.13) and the exact value of \( Z_k \) obtained by numerical calculations of \(|C_n|\). The results for \( p = 3 \) are presented on fig. (5). As can be observed, this ratio is close to one for large \( n \).
5. Distribution of cluster sizes

We will use now the results of the previous section to evaluate the probability $P(t)$ that a randomly chosen periodic orbit belongs to a cluster of the size $t|C_{\text{max}}|$, $t \in [0, 1]$, where $|C_{\text{max}}| = \max_n |C_n|$ is the size of the largest cluster. To this end we first need to establish the asymptotic behavior of $|C_{\text{max}}|$. By eq. (3.3) the maximal size is given by:

$$|C_{\text{max}}| = \max_n \left( \prod_{j=1}^{2^{p-1}} \int_0^{2\pi} \frac{d\phi_j}{2\pi} \right) \cdot \exp F(\phi),$$

where

$$F(\phi) = -i(n, \phi) + \log \text{Tr} (Q \Lambda(\phi)).$$

To evaluate the above expression we can once again use saddle point approximation. Taking the first derivative of $F(\phi)$ leads to the following equation for the saddle point

$$(5.2) \quad n_i = n \frac{\text{Tr} (\Lambda(\phi)P_i Q_0(Q(\phi))^{n-1})}{\text{Tr}(Q(\phi))^n},$$

with $n_i$ being $i$'s component of the vector $n$. It can be expected that the cluster of the maximum size is provided by the most homogeneous vector $\vec{n}$ among all admissible vectors $n$. For the sake of simplicity take $n$, such that $n \mod 2^p = 0$. In this case $\vec{n} = 2^{-p}(1, 1, \ldots 1)$ and the saddle point equation (5.2) is satisfied when $\phi = 0$. At this point $F(0) = n \log 2$. Since $|F(\phi)| \leq n \log 2$ for any $\phi$, it is clear that $C_{\vec{n}}$ is, indeed, has (at least asymptotically) the largest size among all clusters. Repeating then the same calculations for the second derivative of $F(\phi)$, as in the
previously, we obtain after saddle point approximation:
(5.3) $|C_{\text{max}}| = Z_2(n/2) (1 + O(1/n)),$
where $Z_2(n)$ is given by eq. (4.7).

To calculate the probability $P(t)$, it is useful to notice that the moments $Z_k$ can be represented as integrals:

(5.4) $Z_k = \int_0^1 dt \rho(t) t^{k-1},$

where $\rho(t) \Delta t$ is the probability to find a random periodic orbit in a cluster $C_n$ whose size belongs to the interval $|C_{\text{max}}| \cdot t \leq |C_n| \leq |C_{\text{max}}| \cdot (t + \Delta t)$. The probability to find a random periodic orbit in the cluster of the size smaller than $|C_{\text{max}}| t$ is therefore:

$$P(t) = \int_0^t dt \rho(\tau).$$

It remains to find $\rho(\tau)$. After substituting into eq. (5.4) the asymptotic expressions for $Z_k$ and $|C_{\text{max}}|$ we obtain in the limit $n \to \infty$:

(5.5) $\int_0^1 dt \rho(t) t^{k-1} = k^{-2p^{-2}}.$

By taking the Laplace transform on both sides of eq. (5.5) one has

(5.6) $\rho(t) = \frac{(\log t)^{2p^{-2}-1}}{(2p^{-2} - 1)!}.$

In particular, for $p = 2, 3$ this gives:

(5.7) $P(t) = t$ (for $p = 2$) ; $P(t) = t(\log t - 1)$ (for $p = 3$).

The comparison of the above result with the direct numerical simulation is shown on fig. 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{The exact distribution of cluster sizes for $n = 70$ (upper green) and $n = 47$ (lower red) is shown in comparison with the asymptotic expression $P(t) = t(\log t - 1)$ (dashed blue line) in the case $p = 3$.}
\end{figure}
6. Anisotropy properties of clusters

The method used in the previous sections to calculate moments $Z_k$ can be also applied to obtain a more refined information on the distribution of periodic orbits in clusters with regard to each edge individually. One can ask for instance how many times a sequence $a \in X_p$ appears in a random sequence $x$ of length $n$. As one can expect, if all orbits are weighted in the same way the result does not depend on $a$:

$$< n_a > := 2^{-n} \sum_n n_a |C_n| = -2^{-n} i \partial_{\phi_a} \text{Tr} (Q\Lambda(\phi)) |_{\phi=0} = n/2^p. \quad (6.1)$$

In other words, the periodic orbits are ergodically distributed over the graph $G_p$. However, if only periodic orbits from large clusters (or small clusters) are considered the question regarding homogeneity of their distribution over the edges of the graph, does not seem to have a trivial answer. For instance it is clear that smallest clusters are dominated by periodic orbits with either large number of zeroes or ones. Therefore, among the periodic orbits belonging to small clusters there should be enhanced probability to meet a subsequence consisting of all zeroes (or ones).

The purpose of the present section is to investigate the anisotropy properties of the graph with regard to distribution of periodic orbits belonging to clusters of different size. To this end let us consider the sum

$$< n_a >_2 := \frac{\sum_n n_a |C_n|^2}{\sum_n |C_n|^2}. \quad (6.2)$$

As opposed to $< n_a >$, the above quantity gives a larger weight to the periodic orbits from larger clusters. Thus, in principle, $< n_a >_2$ might depend on $a$. To verify this, we observe that (6.2) can be written as

$$F_n(a)(\phi) = \log \left[ \text{Re}(-i \partial_{\phi_a} \text{Tr}(Q(\phi)))^n \text{Tr}(Q(-\phi))^n \right] = \log \left[ n \text{Re}(\langle a | \Lambda(\phi)Q_0Q(\phi)^{n-1} |a \rangle \text{Tr}(Q(-\phi))^n) \right],$$

where $|a\rangle$ is the state corresponding to the sequence (edge) $a$. We can now apply saddle point approximation to eq. (6.3). Expanding $F_n(a)(\phi)$ up to the second order in $\phi_j$’s gives

$$F_n(a)(\phi) = \log(n 2^{2n-p}) - \frac{n}{2} \left( \sum_{i,j} B_{i,j} \phi_i \phi_j + O(\phi_i^4) \right), \quad (6.4)$$

with the matrix $B$ given by eq. (4.5). Substituting then (6.4) into (6.3) leads to

$$< n_a >_2 = n/2^p + O(n^0), \quad (6.5)$$

where only the subleading term (possibly) depends on $a$. This form of $< n_a >_2$ indicates that to the leading order in $n$ the periodic orbits are equidistributed uniformly over the graph. The asymmetry shows up only in the second and higher order terms of the asymptotic expansion. In fig. 7 we plotted $< n_a >_2$ as a function of $n$ for
different edges of the graph. It is clearly visible that existing asymmetry between different edges, essentially, does not grow with $n$.

![Figure 7](image-url)

**Figure 7.** Distribution of periodic orbits over edges of the graph for $p = 3$. The upper figure shows the graph of $< n_a >_2$ as a function of $n$ for three different edges $a_1 = [011]$, $a_2 = [101]$ and $a_3 = [000]$. The lower figures show the graphs (for the same edges) of $\bar{n}_a(t)$ for $n = 82$ (left) and $n = 70$ (right), respectively. Here the dotted lines correspond to the constant value $n/8$. Note, that up to a shift in the vertical direction, these two graphs are close to each other.

In an analogous way one can estimate averages:

$$< n_a >_k := \sum_n n_a |C_n|^k \sum_n |C_n|^k$$

for an arbitrary $k$. As in the case of $k = 2$, the leading order term of the asymptotic expansion of $< n_a >_k$ is equal to $n/2^p$. This can be interpreted to the extent that periodic orbits uniformly (to the leading order of $n$) pass through the edges of the graph $G_p$, independently of the size of the clusters they belong. The non-uniformity, however, does appear in the next order $n^0$ of the asymptotic expansion. To demonstrate this, we calculate numerically the averages:

$$\bar{n}_a(t) := \sum_{|C_n| < t|C_{max}|} n_a |C_n| \sum_{|C_n| < t|C_{max}|} |C_n|,$$
where the sum runs over clusters with sizes less than $t|C_{\text{max}}|$, $t \in [0, 1]$. It follows straightforwardly from the definition of $\tilde{n}_a(t)$ and eq. (6.1) that

$$\sum_{a \in X_p} \tilde{n}_a(t) = n \text{ for } t \in [0, 1], \quad \tilde{n}_a(1) = n/2^p.$$ 

In the case $p = 3$, the resulting plot of $\tilde{n}_a(t)$ is shown on fig. (7) for $n = 82$ and $n = 60$, respectively. As one can see, there is enhanced probability for periodic orbits from small clusters $t \ll 1$ to pass through the edge $a = [000]$, and suppressed probability to pass through the edge $[011]$. On the other hand this picture depends very little on $n$.

7. Discussion

To summarize, the problem of counting $p$-close periodic orbits of the backer’s map can be cast in an equivalent form of finding degeneracies in the length spectrum of the de Bruijn graph $G_p$. The latter problem has previously attracted attention in [13], where an asymptotic expression for $Z_2$ has been obtained for a generic graph. Related counting problems were also considered in [12] for fully connected graphs and in [14] for binary directed graphs. In comparison to [13] we go somewhat further, as we derive asymptotics for all moments $Z_k$ and explicitly obtain the leading term prefactors depending on $p$. Our considerations have been restricted to a simplest possible symbolic dynamics which occurs in the backer’s map (leading to a specific binary graph). As a matter of fact, the results can be straightforwardly generalized to the alphabets with a larger number of symbols. Moreover, the present approach can be extended to the symbolic dynamics with non-trivial grammar rules, when certain symbolic subsequences are forbidden. In that case the resulting graph $G_p$ would typically have a non-homogeneous structure, where the number of outgoing and incoming edges depend on a specific vertex.

The rescaled moments $Z_k$ can be interpreted as the probabilities $P_k$ that $k$ randomly chosen periodic sequences are $p$-close. Making use of information on $P_k$, we obtain the asymptotics of the probability $P(t)$ to find a random periodic sequence in a cluster of the size smaller than $t|C_{\text{max}}|$, $t \leq 1$, with $C_{\text{max}}$ being the largest cluster. Most significantly, in the large $n$ limit, $P(t)$ does not depend on the length of sequence, but only on $p$. It is worth noting that $P(t)$ is basically determined by the distribution of “large” clusters, whose size is of the order $|C_{\text{max}}|$. “Small” clusters $C_i$, whose size is $|C_i| \ll |C_{\text{max}}|$ do not affect (asymptotically) the moments $Z_k$ and therefore, do not contribute to $P(t)$. A natural question arises: what is the number of “large” clusters in comparison to the number of “small” clusters? As we show in Appendix C the total number of clusters $N_p$ is proportional to $n^{2p-1}$. On the other hand, based on eqs. (5.3, 4.13) a rough estimation of “large” clusters yields

$$N_p^{(l)} \sim Z_k/|C_{\text{max}}|^k \sim n^{2p-2}.$$ 

This shows that the number of “large” clusters scales as $N_p^{(l)} \sim \sqrt{N_p}$. In other words most of the clusters are “small” in the limit $n \to \infty$, although their influence on $Z_k$ is negligible.
We have also studied the distribution of the periodic orbits over the graph $G_p$. A priori, it might be expected that subsequences consisting of only zeroes (or ones) appear more often than others in periodic sequences belonging to “small” clusters. As we show, this is indeed so, but such inhomogeneities vanish asymptotically. To the leading order of $n$ periodic orbits cover the graph $G_p$ uniformly. In other words, in long sequences belonging to clusters of size comparable with $|C_{\text{max}}|$ each subsequence of $p$ symbols occurs on average the same number of times.

One of the main motivations for the present study comes from the theory of quantum chaos, where clusters of $p$-close orbits play an important role. It is worth mentioning, however, that the limit considered in the present paper, $n \to \infty$ with a fixed $p$, is somewhat different from the usual semiclassical limit. In the last case spectral correlations are determined by pairs of long periodic orbits whose encounter lengths are proportional to the logarithm of their total lengths [6]. This corresponds to the limit of both $n \to \infty$, $p \to \infty$ such that the ratio $n/2^p$ is fixed. In this limit one might expect completely different behavior for $P_k$. In particular the number of $p$-close pairs should grow faster than $2^n$. The exact behavior of $P_k$ in the semiclassical limit is of great interest and we leave it for future investigations.

Acknowledgments

We thank P. Braun and M. Kieburg for valuable discussions. This work was supported by the Sonderforschungsbereich Transregio 12

Appendix A. Change of integration variables in (3.4) and (3.5).

Here we demonstrate how the total number of integration variables in the integral (3.4) can be effectively reduced by half. To perform the change of variables we use the invariance of the trace under the multiplication of each matrix $Q \Lambda(\phi)$ by a diagonal matrix $\Lambda(\xi) = \text{diag}\{e^{i\xi_1}, \ldots, e^{i\xi_{2^p}}\}$ and its conjugate from the left and from the right side correspondingly:

$$Q' \Lambda(\phi) \to e^{-i\xi} \Lambda(\xi)Q' \Lambda(\phi)\Lambda(\xi)^\dagger.$$  

Any such transformation with an arbitrary $\xi = (\xi_1, \ldots, \xi_{2^{p-1}})$ and $\xi$ leaves the integrand in (3.4) intact. Note also that this transformation does not affect the structure of the matrix (3.8), but only its phases. Using this one can eliminate the phases $\phi'_{k} \equiv \phi_{2^{p-1}+1}, \ldots, \phi'_{2^p-1} \equiv \phi_{2^p}$ on the “right side” of the matrix $Q'$ by imposing the system of linear equations on $\xi_k, k = 1, \ldots, 2^{p-1}$

\begin{align*}
\tilde{\phi}_{2k-1} - \xi_{2k-1} + \xi_{2^{p-1}+k} - \xi &= 0; \\
\tilde{\phi}_{2k} - \xi_{2k} + \xi_{2^{p-1}+k} - \xi &= 0.
\end{align*}

On the “left side” of the matrix $Q'$ the above transformation induces new phases

\begin{align*}
\varphi_{2k-1} &= \phi_{2k-1} - \xi_{2k-1} + \xi_k - \xi; \\
\varphi_{2k} &= \phi_{2k} - \xi_{2k} + \xi_k - \xi;
\end{align*}

where $\xi_i$’s are determined by eq. (A.1). After fixing the common phase $\xi$ to be a symmetric combination of other variables $\xi_k$, i.e. $\xi = 2^{-p} \sum_k \xi_k$ the system of
equations (A.1-A.2) can be cast into the matrix form:

\[ \phi' = F \xi; \]  
\[ \varphi = \phi - G \xi, \]

where the matrices \( F \) and \( G \) are expressed in terms of \( Q_0, Q_1, \) and \( Q \):

\[ G = 1 - Q_0 + (Q/2)^p; \quad F = 1 - Q_1 + (Q/2)^p. \]

Combining now eq. (A.3) with eq. (A.4) we define the following linear transformation

\[ \begin{pmatrix} \phi \\ \phi' \end{pmatrix} = \begin{pmatrix} 1 & GF^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}, \]

which exists since the determinants of both \( F \) and \( G \) are different from zero (see proposition B.1). By construction, the resulting matrix \( e^{-itA(\xi)Q'\Lambda(\phi)\Lambda(\xi)} \) has the required form (3.10) with the phases given by \( \varphi \). Furthermore, since the Jacobian of the linear transformation (A.6) equals one, the variables \( \varphi' \) do not enter the integrand at all and can be integrated out. In the case of higher moments \( Z_k, k > 2 \) one has to perform the similar transform for each set of variables \( \phi^{(j)} \). The resulting expression for the integral is given then by the equations (3.9) and (3.10).

Appendix B. Calculation of the determinant of the matrix (4.5)

In this appendix we calculate the determinant of the matrix \( B \). We recall that \( B \) is defined using \( 2^p \times 2^p \) matrices \( Q_0, Q_1 \), their sum \( Q = Q_0 + Q_1 \) and \( Q_p = \left( \frac{Q}{2} \right)^p \), see (4.5). For the sake of completeness, we give their element-wise definition:

- The entries of the matrices \( Q \) and \( Q_0, Q_1 \) are given by \( (i, j = 1, \ldots 2^p): \)

\[ [Q]_{i,j} = \sum_{k=1}^{2^p-1} (\delta_{i,2^k-1} + \delta_{i,2^k})(\delta_{j,k} + \delta_{j,2^{p-1}+k}) \]

\[ [Q_0]_{i,j} = \sum_{k=1}^{2^p-1} (\delta_{i,2^k-1} + \delta_{i,2^k})\delta_{j,k}; \quad [Q_1]_{i,j} = \sum_{k=1}^{2^p-1} (\delta_{i,2^k-1} + \delta_{i,2^k})\delta_{j,2^{p-1}+k}. \]

- All elements of matrix \( Q_p \) are equal to \( 2^{-p} \).

Bellow we collect a number of matrix relations which are of use in the further analysis. They follow directly from the matrix definitions. For any natural \( k \):

\[ Q^k Q_p = Q_p Q^k = 2^k Q_p; \quad Q_0^k Q_p = Q_1^k Q_p = Q_p; \quad Q_p^k = Q_p; \]

\[ Q_0^T Q_0 + Q_1^T Q_1 = 2 \cdot 1. \]

For the traces of various matrix products we have:

\[ \text{Tr} Q_p Q_{0,1}^k = \text{Tr} Q_{0,1}^k = 1, \quad \text{Tr} Q_p^k = 2^k, \]

where (and further) \( Q_{0,1} \) stands either for \( Q_0 \) or \( Q_1 \).
Two more, useful equalities can be derived by using (B.3) 
\begin{align}
(Q_{0,1} - Q_p)^k &= (Q_{0,1} - Q_p)Q_{0,1}^{k-1}; \\
(Q - Q_p)^k &= Q^k - (2^k - 1)Q_p.
\end{align}
Both can be proved by induction. To derive (B.6) it is enough to notice that 
\(Q_p(Q_{0,1} - Q_p) = 0\). To obtain the coefficient at \(Q_p\) in eq. (B.7) we assume that 
\((Q - Q_p)^k = Q^k - a_k Q_p\) with some yet unknown \(a_k\). Then the next iteration gives 
the following recurrence relation for \(a_k\): \(a_{k+1} = 2^k + a_k\). Its solution is \(a_k = 2^k - 1\) 
as it is indicated in (B.7).

Having described the properties of the matrices \(Q_{0,1}, Q\) and \(Q_p\) we come to 
calculation of their determinants. The first auxiliary statement is:

**Proposition B.1.** For any real \(\alpha\)
\begin{align}
det [I - \alpha(Q_{0,1} - Q_p)] &= 1, \\
det [I - \alpha(Q - Q_p)] &= 1 - \alpha.
\end{align}

**Proof.** Consider the logarithm of the determinant:
\[
\log det [I - \alpha(Q_{0,1} - Q_p)] = -\sum_{k=1}^{\infty} \alpha^k \frac{\text{Tr}(Q_{0,1} - Q_p)^k}{k}.
\]
Applying now the formula (B.6) we come to the matrix \((Q_{0,1} - Q_p)Q_{0,1}^k\) which trace is 
zero for all \(k > 0\) due to the relations (B.5). This proves the first statement. In the 
same way, using the equation (B.7) and the information about traces we derive
\[
\log det [I - \alpha(Q - Q_p)] = -\sum_{k=1}^{\infty} \frac{\alpha^k}{k} \text{Tr} (Q - Q_p)^k = \log(1 - \alpha).
\]

We can use the above proposition for \(\alpha = 1\) to obtain the determinants of the 
matrices \(F\) and \(G\) defined in (A.5):

**Corollary B.2.** \[\text{det } F = \text{det } G = 1\]

In the remaining part of the appendix the determinant of \(2^p \times 2^p\) matrix (A.5)
\[
B = 2^{-p-1} \left( 2I - (1 + 2p)Q_p + Q_0 \sum_{r=0}^{p-1} \left( \frac{Q}{2} \right)^r + \sum_{r=0}^{p-1} \left( \frac{Q^T}{2} \right)^r Q_0^r \right),
\]
is evaluated. The following remark helps to perform the calculations.

**Remark B.3.** In the saddle-point calculations of the integral (3.9) made after the 
change of variables one comes to the Gaussian integral (4.6) involving matrix \(B\).
However, change of the order leads to another matrix, $2^{-p}M_{p+1}$. It has twice larger dimensionality and connected to the matrix $B$ by the relation

\[(B.9) \quad 2^{-p}M_{p+1} = \begin{pmatrix} 1 & 0 & 0 \\ GF^{-1} & B & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -GF^{-1} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} B & -BGF^{-1} \\ -(GF^{-1})^TB & (GF^{-1})^TBGF^{-1} \end{pmatrix}.\]

Explicit form of matrix $M_{p+1}$ is

\[(B.10) \quad M_{p+1} = 1 + \sum_{r=1}^{p} \left[ \left( \frac{Q}{2} \right)^r + \left( \frac{Q^T}{2} \right)^r \right] - (2p + 1)Q_{p+1},\]

where all matrices are of the size $2^{p+1} \times 2^{p+1}$.

Eq. (B.9) allows to cast the problem of calculation of $\det B$ into a “symmetric” form. Namely, by the following proposition the determinant of $B$ can be expressed through the eigenvalues of two matrices which depend only on $Q$ and $Q^T$.

**Proposition B.4.** The determinant of the matrix $B$ can be represented as

\[(B.11) \quad \det B = 2^{-2p} \frac{4 \prod_{\lambda_j \neq 0} \lambda_j}{\det M_p},\]

where $\lambda_j$ are non-zero eigenvalues of $2^{p+1} \times 2^{p+1}$ matrix $M_{p+1}$ and $\tilde{M}_p$ is $2^p \times 2^p$ matrix $\tilde{M}_p := 1 + M_p + 3Q_p$.

**Proof.** Consider the spectral problem for the matrix $M_{p+1}$:

\[M_{p+1} \begin{pmatrix} X \\ Y \end{pmatrix} = \lambda \begin{pmatrix} X \\ Y \end{pmatrix} \implies BX - BCY = \lambda X \\ -C^T BX + C^T BCY = \lambda Y,\]

where $C = GF^{-1}$. The latter system reduces to the equation

\[\lambda(C^T X + Y) = 0,\]

meaning that either $\lambda = 0$ or $-C^T X = Y$. Backward substitution results in the following eigenvalue problem

\[B(1 + CC^T)X = \lambda X\]

where $\lambda$’s are non-zero eigenvalues of the matrix $M$, so that we can write down

\[\det B = \frac{\prod_{\lambda_j \neq 0} \lambda_j}{\det(1 + CC^T)} = \frac{\prod_{\lambda_j \neq 0} \lambda_j}{\det(F^TF + G^TG)}.\]

In the last expression the result of the proposition (B.3) has been used.

One can proceed further by evolving the denominator. First we observe that it depends only on the matrix $Q$. Indeed, due to the properties (B.3), (B.4) one has

\[F^TF + G^TG = 4 \cdot 1 - Q - Q^T + 2Q_p\]

\[= 2 \left( 1 - \frac{Q}{2} + \frac{Q_p}{2} \right) \left( \frac{1}{1 - \frac{Q}{2} + \frac{Q_p}{2}} + \frac{1}{1 - \frac{Q^T}{2} + \frac{Q_p}{2}} \right) \left( 1 - \frac{Q^T}{2} + \frac{Q_p}{2} \right)\]
Expansion of each of the fractions with the help of equation (B.7),
\[
\frac{1}{1 - \frac{Q}{2} + \frac{Q_p}{2}} = \sum_{k=0}^{p-1} \left( \frac{Q - Q_p}{2} \right)^k = \sum_{k=0}^{p-1} \left( \frac{Q}{2} \right)^k - (p-2)Q_p,
\]
yields
\[
(B.12) \quad F^TF + G^TG = 2 \left( 1 - \frac{Q}{2} + \frac{Q_p}{2} \right) \left( 1 + M_p + 3Q_p \right) \left( 1 - \frac{Q^T}{2} + \frac{Q_p}{2} \right),
\]
where all matrices are of the size $2^p \times 2^p$. By taking into account (B.8) we arrive to the final expression for the determinant of $B$. \hfill \Box

It remains now to calculate the spectrum of matrices $M_{p+1}$ and $\tilde{M}_p$. Note that both matrices have a similar structure and can be treated in the same way. Their spectra is provided by the following lemma.

**Lemma B.5.** The spectrum of the matrix $M_{p+1}$ contains $2^p$ (half of the total number) zero eigenvalues, the non-zero part of the spectrum has the following structure:

\[
\lambda_k = k + 1 \quad \text{with multiplicity} \quad 2^{p-k-1}, \quad 0 \leq k \leq p - 1,
\]
and $\lambda_p = p + 1$ has multiplicity one. The non-trivial eigenvalues of $\tilde{M}_p$ are given by:

\[
\tilde{\lambda}_k = k + 2 \quad \text{with multiplicity} \quad 2^{p-k-2}, \quad 0 \leq k \leq p - 2,
\]
and $\tilde{\lambda}_p = p - 1, \quad \tilde{\lambda}_p = 4$ having multiplicity one. The remaining $2^p - 1$ eigenvalues are all equal to one.

**Proof.** To prove this lemma it is convenient to work within the tensorial representation of matrices. According to (3.1) the matrix $Q^r$ is given by
\[
(B.13) \quad Q^r = \left( s \otimes \cdots \otimes s \otimes I \otimes \cdots \otimes I \right) \cdot T^r,
\]
where $T$ is the following shift operator:

\[
T|a_1 \rangle \otimes |a_2 \rangle \cdots \otimes |a_p \rangle = |a_2 \rangle \otimes \cdots \otimes |a_p \rangle \otimes |a_1 \rangle,
\]
and the projection $s$ acts on $|0 \rangle, |1 \rangle$ as

\[
s|0 \rangle = 0; \quad s|1 \rangle = |1 \rangle.
\]

Note that the result of the action of the operator $Q^r$ on the basis vectors $|j_1 \rangle \otimes |j_2 \rangle \otimes \cdots \otimes |j_p \rangle, \ j_k \in \{0, 1\}$ essentially depends on the number of consecutive “ones” at the end of the sequence $j_1, j_2 \ldots j_p$. Whenever this number is larger than $r$, the result of the action of $Q^r$ is zero. This property allows to find all eigenvalues of the operator $M_{p+1}$.

Let $\chi^{(k)}$ be a vector having exactly $k$ “ones” at the right end of the encoding sequence, separated by two “zeroes” (if $k < p$) from the rest of the sequence, i.e.,

\[
\chi^{(k)} = |0 \rangle \otimes \omega_{p-k-1} \otimes |0 \rangle \otimes \underbrace{|1 \rangle \otimes \cdots \otimes |1 \rangle}_{k \text{ times}},
\]
where $\omega_{p-k-1}$ is defined as

\[
\omega_{p-k-1} = \left\{ \begin{array}{ll}
1 & \text{if } \omega_{p-k-1} = 1, \\
0 & \text{if } \omega_{p-k-1} = 0,
\end{array} \right.
\]
and the remaining $2^p - 1 - k$ eigenvalues are all equal to one.
where \( \omega_{p-k-1} = |j_1\rangle \otimes |j_2\rangle \otimes \cdots \otimes |j_{p-k-1}\rangle \) is an arbitrary product vector of the length \( p-k-1 \). By the definition \( \chi^{(p+1)} \) is the vector consisting of only “ones”. It is easy to see that \( \chi^{(p+1)} \) is an eigenvector of \( M_{p+1} \) with zero eigenvalue. All other eigenvectors of \( M_{p+1} \) can be constructed as linear combinations of the rotations of \( \chi^{(k)} \)'s:

\[
\sum_{j=0}^{k} \alpha_j T^j \chi^{(k)}.
\]

The action of \( M_{p+1} \) on each of these combinations \( (k \leq p) \) results in

\[
M_{p+1} \sum_{j=0}^{k} \alpha_j T^j \chi^{(k)} = \left( \sum_{j=0}^{k} \alpha_j \right) \left( \sum_{j'=0}^{k} T^{j'} \chi^{(k)} \right).
\]

Equating the right hand side of this expression with \( \lambda \sum_{j=0}^{k} \alpha_j T^j \chi^{(k)} \) we rewrite the original problem as the eigenvalue problem for the \( k \times k \) matrix consisting of all ones:

\[
\sum_{j=0}^{k} \alpha_j = \lambda \alpha_k, \quad k = 0, 1, \ldots, p.
\]

The solution is well known – all eigenvalues are zeros except one which is equal to \( k + 1 \). The degeneracy of the eigenvalues is defined by the free part \( \omega_{p-k-1} \) of the vector \( \chi^{(k)} \) and, therefore, equals to \( 2^{p-k-1} \) for \( 0 \leq k \leq p-1 \) and to one for \( k = p \). The total number of non-zero eigenvalues is

\[
1 + \sum_{k=0}^{p-1} 2^{p-k-1} = 2^p.
\]

The eigenvalues of \( \tilde{M}_p \) can be obtained by a general shift of all eigenvalues of \( M_p \) by 1 with the only exception of zero eigenvalue corresponding to \( \chi^{(p)} \) which should be shifted by 4. \( \square \)

By Lemma [B.5] one has the following chain of identities

\[
\prod_{\lambda_j \neq 0} \lambda_j = (p+1) \prod_{k=0}^{p-1} (k+1)^{2^{p-k-1}} = (p+1) \prod_{k=0}^{p-2} (k+2)^{2^{p-k-2}} = \frac{1}{4} \det \tilde{M}_p.
\]

Substituting this into (B.11) gives the final expression for the determinant of \( B \):

**Proposition B.6.**

\[ \det B = 2^{-2^p}. \]
Appendix C. Calculation of the total number of clusters

Here we estimate the total number \( N_p(n) \) of equivalence classes \( C_n \) of all sequences of the total length \( n \) generated by the equivalence relation \( x \sim y \) (\( x, y \in X^n \)). Recall that every equivalence class is uniquely parametrized by a vector of integers \( n = \{n_a, a \in X_p\} \). The elements of this vector determine the number of times a periodic orbit from \( C_n \) passes trough the corresponding edge of the graph \( G_p \). Since vector \( n \) corresponds to a real periodic orbit of the length \( n \), its components must satisfy the following constraints:

i. The total length of the trajectory is fixed:

\[
\sum_{a \in X_p} n_a = n. \tag{C.1}
\]

ii. The number of times a periodic orbit enters a vertex of the graph \( G_p \) must be equal to the number of exits from the same vertex. This balancing condition is represented by the equation

\[
Sn = n^T R. \tag{C.2}
\]

It is easy to see that for any vector \( n \) satisfying the above conditions, with \( n_a \neq 0 \) for all \( a \), it is possible to find a closed path on the graph which passes through an edge \( a \) exactly \( n_a \) times. Note also that the number of solutions with \( n_a = 0 \) for some \( a \) is smaller by a factor \( 1/n \) than the total number of solutions of (C.1,C.2), see [15]. Therefore, to find the leading asymptotics of \( N_p(n) \) it is sufficient to count vectors of positive integers satisfying the equations (C.1,C.2).

Since the system (C.2) is composed of \( 2^p - 1 \) linearly independent conditions, we can chose \( 2^p - 1 \) first elements of \( n \) freely, while the rest is then uniquely fixed by eqs. (C.1,C.2). In addition, the constraints \( n_i \geq 0, i = 1, \ldots, 2^p \) must be satisfied. These constraints define a \( 2^p - 1 \)-polytope \( V_p \) in the \( 2^p - 1 \)-dimensional space of \( n_1, \ldots, n_{2^p-1} \). Geometrically the number of clusters \( N_p(n) \) can be interpreted as the total number of points with integer coordinates encompassed by \( V_p \). Accordingly, the leading term of \( N_p(n) \) in the large-\( n \) limit is given by the volume of \( V_p \).

\[
N_p(n) = w_p n^{2^p-1} (1 + O(1/n)), \tag{C.3}
\]

where the coefficient \( w_p \) can be calculated explicitly for low values of \( p \). We illustrate this by the following example.

**Example:** For \( p = 2 \) the conditions (C.1,C.2) take the form:

\[
n_2 = n_3, \quad n_1 + n_2 + n_3 + n_4 = n.
\]

We chose two independent integer be \( n_1 = k, n_2 = m \). Since \( n_i \geq 0 \) for all \( i = 1, 2, 3, 4 \), the problem is reduced to calculation of the area of the triangle

\[
V_2 = \{x \geq 0, y \geq 0, n - 2x - y \geq 0\},
\]

in the \((x,y)\)-plane. As a result, for \( p = 2 \) we obtain \( N_2(n) = n^2/4 + O(n) \).

It worth mentioning that similar problem for non-directed graphs was considered in [15]. It was shown that the number of equivalence classes (equiv. the degeneracy...
classes in the length spectrum of the graph) in the leading order of \( n \) is proportional to \( n|E|^{-1} \), where \( |E| \) is the total number of edges in the graph. For comparison with (C.3), note that the number of edges in \( G_p \) is \( 2^p \). This reflects the fact that the number of equivalence classes in directed graphs is essentially smaller than in non-directed graphs.

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