Quantum Solitons in Non-Linear Optics: Resonant Dielectric Media

André LeClair
Newman Laboratory
Cornell University
Ithaca, NY 14853

It is known that classical electromagnetic radiation at a frequency in resonance with energy splittings of atoms in a dielectric medium can be described using the classical sine-Gordon theory. In this paper we quantize the electromagnetic field and compute quantum corrections to the classical results by using known results from the sine-Gordon quantum field theory.
1. Introduction

The importance of integrable non-linear partial differential equations in classical non-linear optics has been recognized for some time. The most well-known example concerns weakly non-linear dielectric media, where only the first non-linear susceptibility is considered important. In this situation, the envelope of the electric field satisfies the non-linear Schrodinger equation. This was first understood theoretically by Hasegawa and Tappert\[1\]. The solitons predicted in \[1\] were observed experimentally in \[2\]. The occurrence of these classical solitons in common optical fibers promises to revolutionize high-speed telecommunications.

Our interest in this subject concerns the possibly interesting quantum effects which arise when the electromagnetic field is quantized. The latter quantization amounts to studying an interacting quantum field theory in the classical variables which satisfy the non-linear differential equation. These quantum integrable models have been studied extensively, and many of their properties have already been exactly computed. A priori, one expects quantum effects to be small for macroscopically large objects such as solitons. Nevertheless, using known exact results from the quantum non-linear Schrodinger theory, quantum effects have been predicted and measured\[3\]\[4]\[5]\[6].

It is well-known that the non-linear dielectric susceptibilities are enhanced when the radiation is in resonance with the energy splitting of quantum states of the atoms of the sample. Near resonance, one is no longer in the weakly non-linear regime (higher susceptibilities involving higher powers of the electric field become as important) and the physics is no longer well described by the non-linear Schrodinger equation. Remarkably, as was understood and demonstrated experimentally by McCall and Hahn\[7\], the system is well described classically by another famous integrable equation, the sine-Gordon (SG) equation.

In this paper, we study the quantum effects which arise when electric fields in resonance with a dielectric medium are quantized by using known exact results for the SG quantum field theory. Though these quantum effects are theoretically interesting, as we will show, they are unfortunately probably too small to be measurable at the present time.

\[1\] In fiber optic systems the soliton consists of a cluster of $10^8$ or more photons.
2. Classical Theory

In this section we review the manner in which the classical sine-Gordon equation arises in resonant dielectric media\cite{7}.

We consider electromagnetic radiation of frequency $\omega$ propagating through a collection of atoms, where the frequency $\omega$ is in resonance with an energy splitting of the atomic states. For simplicity, we suppose each atom is a two state system described by the hamiltonian $H_0$ with the following eigenstates: $H_0|\psi_1\rangle = -\frac{1}{2}\hbar\omega_0|\psi_1\rangle$, $H_0|\psi_2\rangle = \frac{1}{2}\hbar\omega_0|\psi_2\rangle$, such that $\hbar\omega_0$ is the energy difference of the two states. In the presence of radiation, the atomic hamiltonian is

$$H_{\text{atom}} = H_0 - \vec{p} \cdot \vec{E}, \quad (2.1)$$

where $\vec{p} = e \sum_i \vec{r}_i$ is the electric dipole moment operator.

We assume the radiation is propagating in the $\hat{z}$ direction, and $\vec{E} = \hat{n}E(z,t)$, where $\hat{n} \cdot \hat{z} = 0$. The only non-zero matrix elements of the operator $\vec{p} \cdot \vec{E}$ can be parameterized as follows:

$$\langle \psi_2 | \vec{p} \cdot \vec{E} | \psi_1 \rangle = pE(z,t)e^{-i\alpha}, \quad (2.2)$$

where $p$ and $\alpha$ are constants which depend on the atom in question. (We have assumed spherical symmetry.) It is convenient to introduce the Pauli matrices $\sigma_i$, and write the hamiltonian as

$$H_{\text{atom}} = -\frac{1}{2}\hbar\omega_0 \sigma_3 - E(z,t)(p_1\sigma_1 + p_2\sigma_2), \quad (2.3)$$

where $p_1 = p \cos \alpha$, $p_2 = p \sin \alpha$.

The dynamics of the system is determined by Maxwell’s equations,

$$\left( \partial^2_{z} - \frac{1}{c^2} \partial^2_t \right) E(z,t) = \frac{4\pi}{c^2} \partial^2_t P(z,t), \quad (2.4)$$

where $c^2 = c^2/\epsilon_0$, $\epsilon_0$ is the ambient dielectric constant, and $\vec{P} = \hat{n}P(z,t)$ is the dipole moment per unit volume. The latter polarization can be expressed in terms of the expectations of the Pauli spin matrices $\langle \sigma_i \rangle = \langle \psi | \sigma_i | \psi \rangle$, where $| \psi \rangle$ is the atomic wavefunction. Namely,

$$\vec{P} = \hat{n}(p_1\langle \sigma_1 \rangle + p_2\langle \sigma_2 \rangle), \quad (2.5)$$
where \( n \) is the number of atoms per unit volume\(^2\). Thus, in addition to the Maxwell equation (2.4), one has dynamical equations for the polarization \( P(z, t) \) which are determined by Schrödinger’s equation for the atom:

\[
i\hbar \partial_t \langle \sigma_i \rangle = \langle \psi | [\sigma_i, H_{\text{atom}}] | \psi \rangle. \tag{2.6}\]

The latter can be expressed as

\[
\partial_t \langle \sigma_i \rangle = \sum_{j,k} \varepsilon_{ijk} V_j \langle \sigma_k \rangle, \tag{2.7}\]

where \( \varepsilon \) is the completely antisymmetric tensor with \( \varepsilon_{123} = 1 \), and

\[
V_1 = \frac{2E(z, t)}{\hbar} p_1, \quad V_2 = \frac{2E(z, t)}{\hbar} p_2, \quad V_3 = \omega_0. \tag{2.8}\]

To summarize, the dynamics is determined from the coupled equations of motion (2.4) and (2.7), wherein the atoms are treated quantum mechanically and the radiation is classical.

Let \( E(z, t) = \mathcal{E}(z, t) \cos(\omega t - kz) \) where \( \omega/k = \sigma \) and \( \mathcal{E}(z, t) \) is the envelope of the electric field. We will assume the envelope is slowly varying in comparison to the harmonic oscillations: \( \partial_t \mathcal{E} \ll \omega \mathcal{E}, \partial_z \mathcal{E} \ll k \mathcal{E} \). In this approximation, one finds

\[
\left( \partial_z^2 - \frac{1}{c^2} \partial_t^2 \right) E(z, t) \approx \frac{2\omega}{c} \left[ \left( \partial_z + \frac{1}{c} \partial_t \right) \mathcal{E}(z, t) \right] \sin(\omega t - kz). \tag{2.9}\]

Let us define \( \langle \sigma_\parallel \rangle, \langle \sigma_\perp \rangle \) as follows

\[
\langle \sigma_\parallel \rangle = \langle \sigma_1 \rangle \cos(\omega t - kz + \alpha) + \langle \sigma_2 \rangle \sin(\omega t - kz + \alpha),
\langle \sigma_\perp \rangle = -\langle \sigma_1 \rangle \sin(\omega t - kz + \alpha) + \langle \sigma_2 \rangle \cos(\omega t - kz + \alpha). \tag{2.10}\]

We make the further approximation that \( \cos(2(\omega t - kz + \alpha)) \) terms in the equations of motion for \( \partial_t \langle \sigma_i \rangle \) can be dropped in comparison to \( \cos(\omega t - kz + \alpha) \) (and similarly for the sine terms\(^3\)). (This amounts to replacing \( \sin^2(\omega t - kz + \alpha), \cos^2(\omega t - kz + \alpha) \) by \( 1/2 \)). One then finds

\[
\begin{align*}
\partial_t \langle \sigma_\parallel \rangle &= (\omega - \omega_0) \langle \sigma_\perp \rangle \tag{2.11a} \\
\partial_t \langle \sigma_\perp \rangle &= - \frac{\mathcal{E}(z, t)}{\hbar} p \langle \sigma_3 \rangle + (\omega_0 - \omega) \langle \sigma_\parallel \rangle \tag{2.11b} \\
\partial_t \langle \sigma_3 \rangle &= \frac{\mathcal{E}(z, t)}{\hbar} p \langle \sigma_\perp \rangle. \tag{2.11c}
\end{align*}
\]

\(^2\) To be more precise, \( \langle \sigma_i \rangle \) here represents average over many atoms in a small volume, and is thus a continuous field depending on \( z, t \).

\(^3\) It can be shown in perturbation theory that this is a good approximation at or near resonance.
Finally, equations (2.4) and (2.9), upon making an approximation analogous to the slowly varying envelope on the RHS of (2.4), lead to

\[
\left( \partial_z + \frac{1}{c} \partial_t \right) \mathcal{E}(z, t) \sin(\omega t - k z) = \frac{2\pi}{c^2} c n \omega p \left( \sin(\omega t - k z) \langle \sigma \rangle - \cos(\omega t - k z) \langle \sigma \rangle \right).
\]  

(2.12)

In order to solve these equations, note that \( \partial_t \left( \sum_i \langle \sigma_i \rangle \langle \sigma_i \rangle \right) = 0 \). Thus, if the atoms start out in their ground state with \( \langle \sigma_3 \rangle = \frac{1}{2} \), one has \( \sum_i \langle \sigma_i \rangle^2 = \frac{1}{4} \) for all time. On resonance, when \( \omega = \omega_0 \), eq. (2.11a) implies \( \langle \sigma \parallel \rangle = 0 \) for all time. The constraint \( \sum_i \langle \sigma_i \rangle^2 = \frac{1}{4} \) can be imposed with the following parameterization:

\[
\langle \sigma \parallel \rangle = \frac{1}{2} \sin(\beta_{cl} \phi(z, t)), \quad \langle \sigma_3 \rangle = \frac{1}{2} \cos(\beta_{cl} \phi(z, t)).
\]  

(2.13)

The parameter \( \beta_{cl} \) is arbitrary at this stage, but will be fixed in the next section. Equations (2.11b,c) now imply

\[
\partial_t \phi = - \frac{p}{\beta_{cl} \hbar} \mathcal{E}(z, t).
\]  

(2.14)

Inserting (2.14) into (2.12) and defining \( x = 2z - ct \), one obtains the SG equation:

\[
(\partial_t^2 - \frac{c^2}{\bar{c}} \partial_x^2) \phi = -\mu^2 \beta_{cl} \cos(\beta_{cl} \phi),
\]  

(2.15)

where

\[
\mu^2 = \frac{2\pi n p^2 \omega}{\hbar \epsilon_0}.
\]  

(2.16)

3. Quantum Effects

In this section we proceed to quantize the electromagnetic field. In order to do this, the SG field \( \phi \) must be properly normalized such that the energy of soliton solutions corresponds to the physical energy; this amounts to properly fixing the constant \( \beta_{cl} \).

The action which gives the classical SG equation of motion is

\[
S_{SG} = \frac{1}{c} \int dx dt \left( \frac{1}{2} (\partial_t \phi)^2 - \frac{c^2}{2} (\partial_x \phi)^2 + \frac{\mu^2}{\beta_{cl}^2} \cos(\beta_{cl} \phi) \right).
\]  

(3.1)

On the other hand, the properly normalized Maxwell action is

\[
S_{Maxwell} = \frac{1}{c^2} \int d^3 x dt \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \ldots \right) = \frac{1}{c^2} \int d^3 x dt \left( \frac{1}{2} (\partial_t A)^2 + \ldots \right)
\]  

(3.2)
where the vector potential is $\vec{A} = \hat{n}A$ and as usual $\vec{E} = -\frac{1}{c} \partial_t \vec{A}$. (One can show that $A_0$ can be set to zero.) To normalize the field $\phi$, one needs only compare the kinetic terms in (3.1) and (3.2). The dimensional reduction is made by assuming simply that $A$ is independent of $y, z$ and $\int dydz = A$, where $A$ is an effective cross-sectional area perpendicular to the direction of propagation (as in the cross-sectional area of a fiber). From (2.14) one has

$$S_{\text{Maxwell}} = \frac{A}{2} \left( \frac{\beta_{cl} \hbar}{p} \right)^2 \int dxdt \frac{1}{2} (\partial_t \phi)^2 + ....$$

(3.3)

Comparing with (3.1), one fixes $\beta_{cl}^2 = \frac{2p^2}{(A \hbar^2 \sigma)}$.

Finally, as is conventionally done, we rescale $\phi \rightarrow \sqrt{\hbar} \phi$, so that $S_{\text{SG}}/\hbar$ takes the form (3.1) with $\beta_{cl}$ replaced by $\beta$, where

$$\beta^2 = \hbar \beta_{cl}^2 = \frac{2p^2 \sqrt{\varepsilon_0}}{A \hbar c}.$$  

(3.4)

The constant $\beta$ is the conventional dimensionless coupling constant in the quantum SG theory, and is allowed to be in the range $0 \leq \beta^2 \leq 8\pi$, where $\beta^2 = 8\pi$ corresponds to a phase transition[8]. The limit where the radiation is classical but the atom is still quantum mechanical corresponds to the limit $\beta \rightarrow 0$.

The magnitude of the quantum corrections to classical results is determined by the parameter $\beta^2/8\pi$. One can obtain an order of magnitude estimate by taking $p \approx eR_{\text{bohr}}$, where $R_{\text{bohr}}$ is the Bohr radius. One finds

$$\frac{\beta^2}{8\pi} \approx 10^{-21} \frac{\sqrt{\varepsilon_0}}{A},$$

(3.5)

where $A$ is in cm$^2$. Disappointingly, for realistic $A$, the constant $\beta^2/8\pi$ is exceedingly small. For the remainder of this paper we describe quantum effects that are at least in principle measureable, though in practice probably too small to observe.

The classical soliton solutions to the SG equation are characterized by a topological charge $T = \pm 1$, where

$$T = \frac{\beta}{2\pi} (\phi(x = \infty) - \phi(x = -\infty)).$$

Solitons of either charge correspond to solutions where at fixed $z$ the atoms in the far past are in their ground state, and are all in their excited state at some intermediate time: $\langle \sigma_3 \rangle_{t=\pm\infty} = 1/2$. These classical solitons have been observed experimentally in[7]. What
distinguishes solitons \((T = 1)\) from antisolitons \((T = -1)\) is the sign of the envelope of the electric field. From the known classical soliton solutions and (2.14) one finds for \(T = \pm 1\),
\[
\mathcal{E}(x,t) = \pm \frac{2\hbar \mu}{p} \sqrt{\frac{\mathcal{E} + v}{\mathcal{E} - v}} \left( \cosh \left( \frac{\mu(x - vt)}{\sqrt{\mathcal{E}^2 - v^2}} \right) \right)^{-1}.
\]
Thus the electric fields for the soliton versus the antisoliton are out of phase by \(\pi\).

The classical soliton mass is given by \(M_s = 8m/\beta^2\), where \(m = \mu \hbar/\tau^2\). From (3.4) one can express \(\beta^2/8\pi\) in terms of the classical soliton mass:
\[
\frac{\beta^2}{8\pi} = 8\sqrt{\varepsilon_0} \left( \frac{\hbar \omega}{M_s c^2} \right) \left( \frac{\hbar n A c}{M_s c^2} \right).
\]
Thus,
\[
\frac{\beta^2}{8\pi} \sim \frac{1}{N_\gamma} \left( \frac{\lambda_c}{L_{\text{atom}}} \right),
\]
where \(N_\gamma = \hbar \omega/M_s c^2\) roughly corresponds classically to the number of photons that comprise the soliton, \(\lambda_c\) is the Compton wavelength of the soliton, and \(L_{\text{atom}} = 1/nA\) is the inter-atomic spacing. The above equation summarizes where one expects quantum effects to be important: when the soliton is comprised of small numbers of photons, or when the Compton wavelength is large compared to the space between the atoms.

We now derive quantum corrections to the frequency and density dependence of the soliton mass. Classically, \(M_s \propto \sqrt{\omega n}\). In the quantum theory, short distance singularities are removed by suitably normal ordering the \(\cos(\beta \phi)\) potential\[8\]:
\[
\frac{\mu^2}{\beta^2} \cos(\beta \phi) \rightarrow \lambda : \cos(\beta \phi) :
\]
The anomalous scaling dimension of the operator \(:\cos(\beta \phi) :\) is \(\beta^2/4\pi\), so that \(\lambda\) has mass dimension \(2(1 - \frac{\beta^2}{8\pi})\). Therefore,
\[
M_s = c(\beta) (\sqrt{\lambda})^{1/(1 - \frac{\beta^2}{8\pi})}
\]
for some constant \(c(\beta)\). The latter constant was computed exactly in \[9\]. Since \(\lambda \propto \omega n\), one finds
\[
M_s \propto \sqrt{\omega n} \left( 1 + \frac{\beta^2}{16\pi} \log(\omega n) + O(\beta^4) \right).
\]
The classical scattering matrix for the solitons has been computed by comparing N-soliton solutions in the far past and far future\[10\]. The exact quantum S-matrix is also
known. Since $\beta^2/8\pi$ is small here, the quantum corrections to classical scattering are most easily determined by incorporating the one-loop corrections to the classical scattering, which amounts to the replacement $\beta^2 \rightarrow \gamma = \beta^2/(1 - \beta^2/8\pi)$. The necessary formulas can be found in the above papers.

The most interesting aspect of the quantum scattering of solitons is that the so-called reflection amplitude is a purely quantum effect, analogous to barrier penetration. Namely, one considers an in-state consisting of a soliton of momentum $p_1$ and an antisoliton of momentum $p_2$ which scatters into an out state where the momenta $p_1$ and $p_2$ are interchanged. The $T = +1$ soliton is thus reflected back with momentum equal to that of the incoming $T = -1$ antisoliton. In the semi-classical approximation, this reflection amplitude is

$$S_R(\theta) = \frac{1}{2} \left( e^{16\pi^2 i/\gamma} - 1 \right) e^{-8\pi |\theta|} S(\theta)$$

(3.11)

where

$$S(\theta) = \exp \left( \frac{8}{\gamma} \int_0^\pi d\eta \log \left[ \frac{e^{\theta - i\eta} + 1}{e^{\theta + i\eta}} \right] \right)$$

(3.12)

and $\theta = \theta_1 - \theta_2$, where $p_{1,2} = M_s \sinh \theta_{1,2}$. Since $1/\gamma \approx 1/\beta^2$ is very large, the oscillatory factor $\exp(16\pi^2 i/\gamma) - 1$ in (3.11) will make the detection of reflection processes difficult, since even in a small range of $\beta^2$, this factor averages to zero.

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