TWISTED NOVIKOV HOMOLOGY AND JUMP LOCI IN FORMAL AND HYPERFORMAL SPACES

TOSHIKATE KOHNO AND ANDREI PAJITNOV

ABSTRACT. Let $X$ be a finite CW complex, and $\rho : \pi_1(X) \to \text{GL}(n, \mathbb{C})$ a representation. Any cohomology class $\alpha \in H^1(X, \mathbb{C})$ gives rise to a deformation $\gamma_t$ of $\rho$ defined by $\gamma_t(g) = \rho(g) \exp(t\langle \alpha, g \rangle)$. We show that the cohomology of $X$ with local coefficients $\gamma_{\text{gen}}$ corresponding to the generic point of the curve $\gamma$ is computable from a spectral sequence starting from $H^*(X, \rho)$. We compute the differentials of the spectral sequence in terms of the Massey products. We show that the spectral sequence degenerates in case when $X$ is a Kähler manifold, and $\rho$ is semi-simple.

If $\alpha \in H^1(X, \mathbb{R})$ one associates to the triple $(X, \rho, \alpha)$ the twisted Novikov homology (a module over the Novikov ring). We show that the twisted Novikov Betti numbers equal the Betti numbers of $X$ with coefficients in the local system $\gamma_{\text{gen}}$. We investigate the dependence of these numbers on $\alpha$ and prove that they are constant in the complement to a finite number of integral hyperplanes in $H^1(X, \mathbb{R})$.

1. INTRODUCTION

Let $X$ be a finite connected CW-complex; denote its fundamental group by $G$. Let $\rho : G \to \text{GL}(n, \mathbb{C})$ be a representation. Any cohomology class $\alpha \in H^1(X, \mathbb{C})$ gives rise to the following deformation of $\rho$:

$$\gamma_t : G \to \text{GL}(n, \mathbb{C}), \quad \gamma_t(g) = e^{t\langle \alpha, g \rangle} \rho(g).$$

The cohomology groups of $X$ with local coefficients $\gamma_t$ are isomorphic for all $t$ except a subset containing only isolated points. The cohomology group $H^*(X, \gamma_{\text{gen}})$ corresponding to the generic point of the curve $\gamma_t$ is the first main object of study in the present paper. We prove that there is a spectral sequence $\mathcal{F}_r^*$ starting from the homology of $X$ with coefficients in $\rho$, converging to $H^*(X, \gamma_{\text{gen}})$, and the differentials in this spectral sequence are computable in terms of some special higher Massey products with $\alpha$. The first differential in this spectral sequence is the homomorphism $L_\alpha$ of multiplication by $\alpha$ in the $\rho$-twisted cohomology of $X$. 


This type of spectral sequences appeared in the paper of S.P. Novikov [11] in the de Rham setting. It was generalized to the case of cohomology with coefficients in a field of arbitrary characteristic in the paper [13] of the second author. Our present construction is close to the original ideas of S.P. Novikov. The main technical novelty of our present approach is the systematic use of formal exponential deformations, see §§3,4; this allows to avoid the convergency issues for power series.

If $\rho$ is the trivial representation, the differentials in the spectral sequence above are the usual Massey products in the ordinary cohomology with slightly reduced indeterminacy: $d_r(x) = \langle \alpha, \ldots, \alpha, x \rangle$, see §3. Thus the spectral sequence degenerates when the space $X$ is formal, by the classical argument of P. Deligne, Ph, Griffiths, J. Morgan, D. Sullivan [3] which applies here as well (§3).

In case when $\rho$ is not trivial the situation is more complicated. The spectral sequence degenerates in particular when $X$ is a Kähler manifold and $\rho$ is a semi-simple representation (§6, Prop. 6.6). The proof uses C. Simpson’s theory of Higgs bundles [17], see also the work of H. Kasuya [8]. In this case we have

$$\text{Ker } L_\alpha / \text{Im } L_\alpha \approx H^*(X, \gamma_{\text{gen}}).$$

We introduce a class of hyperformal spaces for which all the spectral sequences $F_*$ corresponding to 1-dimensional representations degenerate in their second term.

If $\alpha$ is a real cohomology class, there is another geometric construction related to $\rho$ and $\alpha$, namely, the twisted Novikov homology introduced in the works of H. Goda and the second author see [6], [15]. This construction associates to $X$, $\rho$ and $\alpha$ a module over the Novikov ring $\hat{L}_{m,\alpha}$. The rank and torsion numbers of this module are called the twisted Novikov Betti numbers and twisted Novikov torsion numbers; they provide lower bounds for numbers of zeros of any Morse form belonging to the de Rham cohomology class $\alpha$ (see §8 for details). These invariants detect the fibered knots in $S^3$ as it follows from the recent work of S. Friedl [5]. For a given space $X$ and the representation $\rho$ the Novikov numbers depend on the cohomology class $\alpha \in H^1(X, \mathbb{R})$. The case of the torsion numbers was studied in [14] and [15]. It is proved there that the torsion numbers are constant in the open polyhedral cones formed by finite intersections of certain half-spaces in $\mathbb{R}^n$, where $n = \text{rk} H^1(X, \mathbb{Z})$. Similar analysis applies to the Novikov Betti numbers, which are of main interest to us in the present work. We prove in §8 that these numbers do not depend on $\alpha$ in the complement to a finite number of proper vector
subspaces. In general the set of all $\alpha$ for which the Novikov Betti number $\hat{b}_k^p(X, \alpha)$ is greater by $q$ than the generic value (the jump loci for the Novikov numbers) is a union of a finite number of proper vector subspaces, see §8, Prop. 8.6.

It is known that the Novikov homology and the homology with local coefficients are related to each other. This was first observed in the paper [12] of the second author, see also Novikov [11]. Similar result holds also for the twisted Novikov homology. namely we prove (see §8, Prop. 9.2) that for $\alpha \in H^1(X, \mathbb{R})$ we have

$$\hat{b}_k^p(X, \alpha) = \beta_k(X, \gamma_{gen}).$$

This implies several corollaries about both families of numerical invariants. On one hand for given $\rho$ and $\alpha$ the jumping loci for $\beta_k(X, \rho, \alpha)$ are unions of integral hyperplanes. On the other hand the twisted Novikov Betti numbers are computable from the Massey spectral sequence. In the case of degeneracy of this spectral sequence, they equal the dimension of its second term.

2. Exact couples

In this section we recall the definition of the spectral sequence of an exact couple (following [9], [7]) and give an equivalent description of the successive terms of the spectral sequence, which will be useful in the sequel.

Let $\mathcal{C} = (D, E, i, j, k)$ be an exact couple, so that we have an exact triangle

$$\begin{array}{ccc}
D & \xrightarrow{i} & D \\
\downarrow{k} & & \downarrow{j} \\
E & & 
\end{array}$$

We will usually abbreviate the notation to $\mathcal{C} = (D, E)$ and call $D$ and $E$ the first, respectively the second component of the exact couple. Following W. Massey we define the derived exact couple setting

$$E' = \text{Ker} \ (jk) / \text{Im} \ (jk), \quad D' = i(D)$$

and defining $j', k'$ suitably. Iterating the process we obtain a sequence of exact couples $\mathcal{C}_r = (D_r, E_r)$, the initial couple being numbered as $\mathcal{C}_1$; this sequence is called the spectral sequence associated to the exact couple $\mathcal{C}$.

We will need an alternative description of the groups $E_r$ and the maps $j_r, k_r$. 


Definition 2.1. 1) For \( r \geq 2 \) let \( Z_r \) be the subgroup of all elements \( x \in E \), such that \( k(x) = i^{r-1}(y) \) for some \( y \in D \). We put \( Z_1 = E \).

2) For \( r \geq 1 \) let \( B_r \) be the subgroup of all elements \( z \in E \), such that \( z = j(y) \) for some \( y \in D \) with \( i^{r-1}(y) = 0 \).

The following properties are easy to check:

\[
Z_1 = E \supset Z_2 = \text{Ker} (jk) \supset Z_3 \supset \ldots \supset Z_r \supset Z_{r+1} \supset \ldots
\]

\[
B_1 = \{0\} \subset B_2 = \text{Im} (jk) \subset B_3 \subset \ldots \subset B_r \subset B_{r+1} \subset \ldots
\]

\[
B_i \subset Z_j \quad \text{for every} \ i, j.
\]

Put \( \tilde{E}_r = Z_r/B_r \), \( D_r = \text{Im} i^r \).

Define a homomorphism \( \tilde{k}_r : \tilde{E}_r \to D_r \) setting \( \tilde{k}_r(x) = k(x) \) for every \( x \in Z_r \). Define a homomorphism \( \tilde{j}_r : D_r \to \tilde{E}_r \) as follows: if \( x \in D_r \) and \( x = i^r(y) \), then put \( \tilde{j}_r(x) = [j(y)] \). It is easy to check that these homomorphisms are well-defined and give rise to an exact couple \( \mathcal{C}_r = (D_r, \tilde{E}_r) \):

\[
\begin{array}{ccc}
D_r & \xrightarrow{i} & D_r \\
\downarrow{\tilde{k}_r} & & \downarrow{\tilde{j}_r} \\
\tilde{E}_r & & \tilde{E}_r
\end{array}
\]

The proof of the next is in a usual diagram chasing:

**Proposition 2.2.** The exact couples \( \mathcal{C}_r \) and \( \tilde{\mathcal{C}}_r \) are isomorphic for any \( r \).

\[\square\]

3. **Formal deformations of differential algebras and their spectral sequences**

Let \( \mathcal{A}^* = \{\mathcal{A}^k\}_{k \in \mathbb{N}} = \{\mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \ldots\} \) be a graded-commutative differential algebra (DGA) over a field \( \mathbb{K} \) of characteristic zero. Let \( \mathcal{N}^* \) be a graded differential module (DGM) over \( \mathcal{A}^* \) (that is, \( \mathcal{N}^* \) is a graded module over \( \mathcal{A}^* \) endowed with a differential \( D \) which satisfies the Leibniz formula with respect to the pairing \( \mathcal{A}^* \times \mathcal{N}^* \to \mathcal{N}^* \)). We will use the same symbol \( d \) to denote the differentials in both \( \mathcal{A}^* \) and \( \mathcal{N}^* \), since no confusion is possible. We denote by \( \mathcal{A}^*[\![t]\!] \) the algebra of formal power series over \( \mathcal{A}^* \) endowed with the differential extended from the differential of \( \mathcal{A}^* \). Let \( \xi \in \mathcal{A}^1 \)
be a cocycle. Consider the $\mathcal{A}^*[[t]]$-module $\mathcal{N}^*[[t]]$ and endow it with the differential
\[ D_t x = dx + t\xi x. \]
Then $\mathcal{N}^*[[t]]$ is a DGM over $\mathcal{A}^*[[t]]$, and we have an exact sequence of DGMs:
\[
0 \longrightarrow \mathcal{N}^*[[t]] \xrightarrow{t} \mathcal{N}^*[[t]] \xrightarrow{\pi} \mathcal{N}^* \longrightarrow 0
\]
where $\pi$ is the natural projection $t \longrightarrow 0$. The induced long exact sequence in cohomology can be considered as an exact couple
\[
H^*(\mathcal{N}^*[[t]]) \xrightarrow{t} H^*(\mathcal{N}^*[[t]]) \xrightarrow{\pi_*} H^*(\mathcal{N}^*)
\]

**Proposition 3.1.** The spectral sequence induced by the exact couple $\mathcal{N}^*[[t]]$ depends only on the cohomology class of $\xi$.

**Proof.** Let $\xi_1, \xi_2 \in A^1$ be cohomologous cocycles, $\xi_1 = \xi_2 + df$ with $f \in A^0$. Let $D_t = d + t\xi_1$, $D'_t = d + t\xi_2$ be the corresponding differentials. Multiplication by $e^{tf} \in A^0[[t]]$ determines an isomorphism $F : \mathcal{N}^*[[t]] \rightarrow \mathcal{N}^*[[t]]$, commuting with the differentials, namely $F(D_t\omega) = D'_t(F(\omega))$. Thus the exact sequences (1) corresponding to $\xi$ and $\eta$ are isomorphic, as well as the exact couples (2) and their spectral sequences.

**Definition 3.2.** Put $\alpha = [\xi]$. The spectral sequence associated to the exact couple $\mathcal{N}^*[t]$ depends only on the cohomology class of $\xi$.

We will compute the differentials in this spectral sequence in terms of special Massey products. Denote by
\[ L_\alpha : H^*(\mathcal{N}^*) \rightarrow H^*(\mathcal{N}^*) \]
the multiplication by $\alpha$. It is clear that
\[ \mathcal{F}_2^* = \text{Ker } L_\alpha/\text{Im } L_\alpha. \]
Let $a \in H^*(\mathcal{N}^*)$. An $r$-chain starting from $a$ is a sequence of elements $\omega_1, \ldots, \omega_r \in \mathcal{N}^*$ such that
\[ d\omega_1 = 0, \quad [\omega_1] = a, \quad d\omega_2 = \xi\omega_1, \ldots, d\omega_r = \xi\omega_{r-1}. \]
For an $r$-chain $C$ put $\partial C = \xi \omega_r$; this is a cocycle in $\mathcal{N}^*$. Denote by $MZ_m^{(r)}$ the subspace of all $a \in H^*(\mathcal{N}^*)$ such that there exists an $r$-chain starting from $a$. Thus

$$MZ_m^{(1)} = H^*(\mathcal{N}^*), \quad MZ_m^{(2)} = \text{Ker} \left( L_\alpha : H^m(\mathcal{N}^*) \to H^{m+1}(\mathcal{N}^*) \right).$$

Denote by $MB_m^{(r)}$ the subspace of all $\beta \in H^*(\mathcal{N}^*)$ such that there exists an $(r - 1)$-chain $C = (\omega_1, \ldots, \omega_{r-1})$ with $\xi \omega_r$ belonging to $\beta$. By definition

$$MB_m^{(1)} = 0, \quad MB_m^{(2)} = \text{Im} \left( L_\alpha : H^{m-1}(\mathcal{N}^*) \to H^m(\mathcal{N}^*) \right).$$

It is clear that $MB_m^{(i)} \subset MZ_m^{(j)}$ for every $i, j$. Put

$$MH_m^{(r)} = MZ_m^{(r)}/MB_m^{(r)}.$$

In the next definition we omit the upper indices and write $MH_{(r)}, MZ_{(r)}$ etc. in order to simplify the notation.

**Definition 3.3.** Let $a \in H^*(\mathcal{N}^*)$, and $r \geq 1$. We say that the $(r + 1)$-tuple Massey product $\langle \xi, \ldots, \xi, a \rangle$ is defined, if $a \in MZ_{(r)}$. In this case choose any $r$-chain $(\omega_1, \ldots, \omega_r)$ starting from $a$. The cohomology class of $\partial C = \xi \omega_r$ is in $MZ_{(r)}$ (actually it is in $MZ_{(N)}$ for every $N$) and it is not difficult to show that it is well defined modulo $MB_{(r)}$.

The image of $\delta C$ in $MZ_{(r)}/MB_{(r)}$ is called the $(r + 1)$-tuple Massey product of $\xi$ and $\omega$:

$$\langle \xi, a \rangle_{(r+1)} = \left\langle \underbrace{\xi, \ldots, \xi}_{r}, a \right\rangle \in MZ_{(r)}/MB_{(r)}.$$

**Example 3.4.** The double Massey product $\langle \xi, a \rangle_{(2)}$ equals $\xi a$, the triple Massey product $\langle \xi, a \rangle_{(3)}$ equals the cohomology class of $\xi \omega_2$ where $d\omega_2 = \xi \omega_1, [\omega_1] = a$, etc.

The correspondence $a \mapsto \langle \xi, a \rangle_{(r+1)}$ gives rise to a well-defined homomorphism of degree 1

$$\Delta_r : MH_{(r)} \longrightarrow MH_{(r)}.$$

The next proposition is proved by an easy diagram chasing argument.

**Proposition 3.5.** For any $r$ we have $\Delta_r^2 = 0$, and the cohomology group $H^*(MH_{(r)}^*), \Delta_r)$ is isomorphic to $MH_{(r+1)}^*$. □

**Theorem 3.6.** For any $r \geq 2$ there is an isomorphism

$$\phi : MH_{(r)}^* \xrightarrow{\sim} \mathcal{F}_{r}^*.$$
Proof. Recall from Section 2 a spectral sequence $\tilde{\mathcal{F}}_r^*$ isomorphic to $\mathcal{F}_r^*$. It is formed by exact couples

\[
\begin{array}{ccc}
D_r & \xrightarrow{i} & D_r \\
\delta & \downarrow & \downarrow \\
\tilde{E}_r & & \tilde{E}_r \\
\end{array}
\]

where $D_1 = H^*(\mathcal{N}^*)$, $D_r = \text{Im} \left( t^{r-1} : D \to D \right)$, and $\tilde{E}_r = Z_r/B_r$ (the modules $Z_r$, $B_r$ are described in the definition 2.1).

**Lemma 3.7.** 1) $MZ_{(r)} = Z_r$, 2) $MB_{(r)} = B_r$.

Proof. We will prove 1), the proof of 2) is similar. Let $\zeta \in H^*(\mathcal{N}^*)$ and $z$ be a cocycle belonging to $\zeta$. Then $\delta(\zeta)$ equals the cohomology class of $\xi_z \in \mathcal{N}^*[[t]]$; and $\zeta \in Z_r$ if and only if there is $\mu \in \mathcal{N}^*[[t]]$ such that

\[
\xi_z - D_t \mu \in t^{r-1} \mathcal{N}^*[[t]].
\]

This condition is clearly equivalent to the existence of a sequence of elements $\mu_0, \mu_1, \ldots \in \mathcal{N}^*[[t]]$ such that

\[
\xi_z = d\mu_0, \quad d\mu_i + \xi \mu_{i-1} = 0 \quad \text{for every} \quad 0 \leq i \leq r-2.
\]

The condition (4) is in turn equivalent to the existence of an $r$-chain starting from $\zeta$. □

The Lemma implies that $\tilde{E}_r^* \approx MH_{(r)}^*$ and it is not difficult to prove that this isomorphism is compatible with the boundary operators. □

In view of the Proposition 3.1 we obtain the next Corollary.

**Corollary 3.8.** Let $\xi \in \mathcal{A}^1$ be a cocycle. The graded groups $MH_{(r)}^*$ defined above depend only on the cohomology class of $\xi$, which will be denoted by $\alpha$. □

Therefore the differentials in the spectral sequence $\{ \mathcal{F}_r^* \}$ are equal to the higher Massey products with the cohomology class of $\xi$. Observe that these Massey products, defined above, have smaller indeterminacy than the usual Massey products. The second term of the spectral sequence is described therefore in terms of multiplication by the cocycle $[\xi]$. It is convenient to give a general definition.
Definition 3.9. Let $\mathcal{K}^*$ be a differential graded algebra, and $\theta$ be an element of odd degree $s$. Denote by $L_\theta : \mathcal{K}^* \to \mathcal{K}^{*+s}$ the homomorphism of multiplication by $\theta$. The quotient

$$\text{Ker } L_\theta / \text{Im } L_\theta$$

is a graded module which will be denoted by $\mathcal{H}^*(\mathcal{K}^*, \theta)$ and called $\theta$-cohomology of $\mathcal{K}^*$. The dimension of $\mathcal{H}^k(\mathcal{K}^*, \theta)$ will be denoted by $B_k(\mathcal{K}^*, \theta)$.

We have therefore

$$F_2^*(\mathcal{N}^*, \alpha) \approx \mathcal{H}^*(H^*(\mathcal{N}^*), \alpha).$$

Let us consider some examples, which will be important for the sequel.

Example 3.10. Let $\mathcal{K}^* = \mathcal{A}^*$. The homomorphism $L_\alpha$ is the multiplication by $\alpha$ in the cohomology $H^*(\mathcal{A}^*)$, and the differentials in the spectral sequence $F_r$ are the higher Massey products induced by the ring structure in $H^*(\mathcal{A}^*)$.

Example 3.11. Let $\mathcal{A}^*$ be a DGA and $\eta \in \mathcal{A}^1$ be a cocycle. Endow the algebra $\mathcal{A}^*$ with the differential $d_\eta$ defined by the following formula:

$$d_\eta(x) = dx + \eta x;$$

we obtain a DGM over $\mathcal{A}$, which we will denote by $\tilde{\mathcal{A}}^*$. For an element $\alpha \in H^1(\mathcal{A}^*)$ we obtain a spectral sequence $F_r^*$ with

$$F_1^* = H^*(\tilde{\mathcal{A}}^*); \quad F_2^* = \mathcal{H}^*(H^*(\tilde{\mathcal{A}}^*), \alpha).$$

Example 3.12. Let $M$ be a connected $C^\infty$ manifold, and $E$ be an $n$-dimensional complex flat bundle over $M$. Denote by $\rho : \pi_1(M) \to \text{GL}(n, \mathbb{C})$ the monodromy of $E$. Let $A^*(M)$ be the algebra of complex differential forms on $M$. The space $A^*(M, E)$ of the differential forms with coefficients in $E$ is a DGM over $A^*(M)$; its cohomology is isomorphic to the cohomology $H^*(M, \rho)$ with local coefficients with respect to the representation $\rho$. We obtain a spectral sequence $F_r$ with

$$F_1^* = H^*(M, \rho); \quad F_2^* = \mathcal{H}^*(H^*(M, \rho), \alpha).$$

Now let us consider some cases when the spectral sequences constructed above, degenerate in its second term. Recall that a differential graded algebra $\mathcal{A}^*$ is called \textit{formal} if it has the same minimal model as its cohomology algebra. Here is a useful characterization of minimal formal algebras.
Theorem 3.13. (P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan, [3], Th. 4.1) Let \( A^\ast \) be a minimal algebra over a field of characteristic zero, generated (as a free graded-commutative algebra) in degree \( k \) by a vector space \( V_k \); denote by \( C_k \subset V_k \) the subspace of the closed generators. The algebra \( A^\ast \) is formal if and only if in each \( V_k \) there is a direct complement \( N_k \) to \( C_k \) in \( V_k \), such that any cocycle in the ideal generated by \( \bigoplus_k (N_k) \) is cohomologous to zero.

This theorem leads to the proof of the well-known property that in formal algebras all Massey products vanish (see [3]). We will show that a similar result holds for the special Massey products introduced above, for the setting of the Example 3.10.

Theorem 3.14. Let \( A^\ast \) be a formal differential algebra, \( \alpha \in H^1(A^\ast) \). Then the spectral sequence \( F^\ast_2(A^\ast, \alpha) \) degenerates at its second term \( F^\ast_2 = H^\ast(H^\ast(A^\ast), \alpha) \).

Proof. It suffices to establish the property for the case of formal minimal algebras. Let \( A^\ast \) be a formal minimal algebra with the space of generators \( V_k \) in dimension \( k \) decomposed as \( V_k = C_k \oplus N_k \), see 3.13. We will prove that \( MZ(2) = MZ(3) = \ldots = MZ(r) \) and \( MB(2) = MB(3) = \ldots = MB(r) \) for every \( r \geq 2 \).

Choose \( \xi \in A^1 \) representing \( \alpha \in H^1(A^\ast) \). Let \( a \in Z_r \), and \( C = (\omega_1, \ldots, \omega_r) \) be an \( r \)-chain starting from \( a \) so that
\[
d \omega_1 = 0, \quad [\omega_1] = a, \quad d \omega_2 = \xi \omega_1, \ldots, d \omega_r = \xi \omega_{r-1}.
\]

Denote by \( \Lambda \) the algebra generated by the space of closed generators \( C_i \), so that \( M = \Lambda \oplus I(N_s) \). Write \( \omega_r = \omega_r^0 + \omega_r^1 \) with \( \omega_r^0 \in \Lambda, \omega_r^1 \in I(N_s) \). Then \( \xi \omega_{r-1} = d \omega_r^1 \), and
\[
d(\xi \omega_r^1) = d(\xi \omega_r) = \xi^2 \omega_{r-1} = 0,
\]
so that \( \xi \omega_r^1 \) is a cocycle belonging to \( I(N_s) \). Therefore \( \xi \omega_r^1 = d \omega_{r+1} \) for some \( \omega_{r+1} \in I(N_s) \) and we obtain an \( (r + 1) \)-chain starting from \( a \), so that \( a \in MZ(r+1) \).

A similar argument shows that \( B_2 = B_r \) for every \( r \geq 2 \). Therefore the spectral sequence degenerates at the term \( E_2 \).

The next proposition gives a sufficient condition for the degeneracy of the spectral sequence associated with a DGM over a DG-algebra \( A^\ast \). It will be used in Section 3 while studying the case of Kähler manifolds.

Definition 3.15. A DG-module \( N^\ast \) over a DG algebra \( A^\ast \) is called formal if it is a direct summand of a formal DGA \( B^\ast \) over \( A^\ast \), that is,
\[
B^\ast = N^\ast \oplus K^\ast,
\]
where both $\mathcal{N}^*$ and $\mathcal{K}^*$ are DG-submodules of $\mathcal{B}^*$.

**Proposition 3.16.** Let $\mathcal{N}^*$ be a formal DG-module over $\mathcal{A}^*$, and $\alpha \in H^1(\mathcal{A}^*)$. Then the spectral sequence $\mathcal{F}_r^*(\mathcal{N}^*, \alpha)$ degenerates at its second terms.

**Proof.** The direct sum decomposition (5) implies that $\mathcal{F}_r^*(\mathcal{B}^*, \alpha) = \mathcal{F}_r^*(\mathcal{N}^*, \alpha) \oplus \mathcal{F}_r^*(\mathcal{K}^*, \alpha)$; the spectral sequence $\mathcal{F}_r^*(\mathcal{B}^*, \alpha)$ degenerates by the previous proposition and the result follows. \hfill \Box

4. A SPECTRAL SEQUENCE CONVERGING TO THE TWISTED HOMOLOGY

Let $X$ be a connected topological space, $G = \pi_1(X)$. Let $B$ be an integral domain and $\rho : G \to \text{GL}(n, B)$ a representation. The cohomology of the cochain complex $C^\ast(X, \rho) = \text{Hom}_G(C^\ast(X), B^n)$ is a $B$-module called the twisted cohomology of $X$ with respect to $\rho$ and denoted by $H^\ast(X, \rho)$. Denote by $\{B\}$ the fraction field of $B$. The dimension over $\{B\}$ of the localization $H^k(X, \rho) \otimes \{B\}$ will be called the $k$ the Betti number of $X$ with respect to $\rho$ and denoted by $\beta_k(X, \rho)$.

Let us start with a given representation $\rho : G \to \text{GL}(n, \mathbb{C})$, pick a cohomology class $\alpha \in H^1(X, \mathbb{C})$ and consider the exponential deformation of $\rho$:

$$\gamma_t : G \to \text{GL}(n, \mathbb{C}), \quad \gamma_t(g) = \rho(g)e^{t\langle \alpha, g \rangle} \quad (t \in \mathbb{C}).$$

It is not difficult to prove that for a given $k$ the Betti number $\beta_k(X, \gamma_t)$ does not depend on $t$ in the complement to some subset $S_k \subset \mathbb{C}$ consisting of isolated points. Let $\Lambda = \mathbb{C}[\lbrack t \rbrack]$ and consider the representation $\tilde{\gamma}$ of $G$ defined as follows:

$$\tilde{\gamma} : G \to \text{GL}(n, \Lambda), \quad \tilde{\gamma}(g) = \rho(g)e^{t\langle \alpha, g \rangle} \in \text{GL}(n, \Lambda)$$

(where $e^{t\langle \alpha, g \rangle}$ is understood as a formal power series). This representation will be called formal exponential deformation of $\rho$. It is clear that

$$\beta_k(X, \tilde{\gamma}) = \beta_k(X, \gamma_t) \quad \text{for} \ t \notin S_k.$$ 

Thus the Betti number corresponding to the generic point of the exponential deformation curve equals the Betti number of $X$ with respect to the formal exponential deformation. The exact sequence

$$0 \longrightarrow \Lambda \overset{t}{\longrightarrow} \Lambda \longrightarrow \mathbb{C} \longrightarrow 0$$




gives a short exact sequence of complexes

\[ 0 \longrightarrow C^*(X, \hat{\gamma}) \overset{t}{\longrightarrow} C^*(X, \gamma) \longrightarrow C^*(X, \rho) \longrightarrow 0 \]

which in turn induces a long exact sequence in cohomology, that can be interpreted as an exact couple

\[ (6) \]

\[ H^*(X, \hat{\gamma}) \overset{t}{\longrightarrow} H^*(X, \gamma) \]

\[ H^*(X, \rho) \]

Denote by \( \mathcal{L}_r^*(X, \rho, \alpha) = (U_r^*, W_r^*) \) the induced spectral sequence.

**Proposition 4.1.** Let \( X \) be a finite connected CW-complex. Then

\[ \dim C W^k_{\infty} = \beta_k(X, \hat{\gamma}) = \beta_k(X, \gamma_t) \] for generic \( t \).

**Proof.** The ring \( \Lambda \) is principal, and the \( \Lambda \)-module \( H^*(X, \hat{\gamma}) \) is a finite direct sum of cyclic modules. Only the free summands survive to \( E_\infty \) and the contribution of each such summand to \( \dim \mathbb{C} W^k_{\infty} \) equals 1.

5. COMPARING THE TWO SPECTRAL SEQUENCES

We are interested in the formal deformations of differential algebras related to topological spaces, mainly manifolds. Let \( M \) be a connected \( C^\infty \) manifold, and \( E \) a flat \( n \)-dimensional vector bundle over \( M \). Denote \( \pi_1(M) \) by \( G \), and let \( \rho : G \to \text{GL}(n, \mathbb{C}) \) be the monodromy representation of \( E \).

In this section we will work with the module \( \mathcal{N}^* = A^*(M, E) \) of \( E \)-valued \( C^\infty \) differential forms on \( M \). This vector space is a DGM over the algebra \( A^*(M) \) of the \( \mathbb{C} \)-valued \( C^\infty \) differential forms on \( M \). Let \( \alpha \in H^1(M) \). The corresponding exact couple

\[ (7) \]

\[ H^*(\mathcal{N}^*[t]) \overset{t}{\longrightarrow} H^*(\mathcal{N}^*[t]) \]

\[ H^*(\mathcal{N}^*) \]

\[ \pi_* \]

gives rise to the deformation spectral sequence \( \mathcal{F}_r^*(\mathcal{N}^*, \alpha) \) (see \( \S \)). Let \( \hat{\gamma} \) be the formal exponential deformation of \( \rho \) corresponding to the class \( \alpha \):

\[ \hat{\gamma}(g) = \rho(g)e^{t(\alpha, g)} \in \text{GL}(n, \Lambda), \quad \Lambda = \mathbb{C}[[t]]. \]
We associate to this deformation the spectral sequence $\mathcal{L}^\ast(M, \rho, \alpha)$ (see [4]).

**Theorem 5.1.** The spectral sequences $\mathcal{F}_r^\ast(N^\ast, \alpha)$ and $\mathcal{L}_r^\ast(M, \rho, \alpha)$ are isomorphic.

**Proof.** The $A^\ast(M)$-module $A^\ast(M, E)[[t]]$ can be considered as the vector space of exterior differential forms on $M$ with values in $E[[t]]$. Let $\pi : \tilde{M} \to M$ be the universal covering of $M$. The vector bundle $E$ is isomorphic to $\tilde{M} \times \mathbb{C}^n$, and $\pi^*E \cong M \times \mathbb{C}^n$. Denote by $T^\ast(M)$ the $A^\ast(M)$-submodule of $A^\ast(\tilde{M}, \mathbb{C}^n[[t]])$, consisting of the differential forms on $\tilde{M}$ which are equivariant with respect to the representation $\hat{\gamma}$. Choose a closed 1-form $\xi$ within the class $\alpha$. Let $F : \tilde{M} \to \mathbb{C}$ be a $C^\infty$ function such that $\pi^*\xi = dF$. The next lemma is obvious.

**Lemma 5.2.** The homomorphism
$$\Phi : A^\ast(M, E)[[t]] \longrightarrow T^\ast(M); \quad \Phi(\omega) = e^{tF} \pi^*\omega$$
is an isomorphism of DG-modules over $A^\ast(M)$.

For a manifold $N$ we denote by $S^\infty_\ast(N)$ the graded group of all singular $C^\infty$-chains on $N$. The integration map determines a homomorphism
$$I : T^\ast(M) \longrightarrow \text{Hom}_G(S^\infty_\ast(\tilde{M}), \mathbb{C}^n[[t]]) .$$
Here $\mathbb{C}^n[[t]]$ is endowed with the structure of a $G$-module via the representation $\hat{\gamma} : G \to \text{GL}(n, \Lambda)$.

**Proposition 5.3.** The induced map in the cohomology groups
$$I^\ast : H^\ast(T^\ast(M)) \longrightarrow H^\ast(M, \hat{\gamma})$$
is an isomorphism.

**Proof.** The argument follows the usual sheaf-theoretic proof of the de Rham theorem. Consider the sheaf $T^k$ on $M$ whose sections over an open subset $U \subset M$ are $\hat{\gamma}$-equivariant $k$-forms on $\pi^{-1}(U)$ with values in $\mathbb{C}^n[[t]]$. Denote by $\mathcal{A}$ the sheaf on $M$, whose sections over $U$ are $\hat{\gamma}$-equivariant locally constant functions $\pi^{-1}(U) \to \mathbb{C}^n[[t]]$. We have an exact sequence of sheaves
\begin{equation}
\mathcal{R}_1 = \{0 \longrightarrow \mathcal{A} \longrightarrow T^0 \overset{d}{\longrightarrow} T^1 \longrightarrow \ldots \}.
\end{equation}
Consider the sheaf $\mathcal{Z}^\ast$ of $\hat{\gamma}$-equivariant singular cochains:
$$\mathcal{Z}^\ast(U) = \text{Hom}_G(S^\infty_\ast(\pi^{-1}(U)), \mathbb{C}^n[[t]]) .$$
The sequence

\[ \mathcal{R}_2 = \{ 0 \rightarrow \mathcal{A} \rightarrow \mathcal{Z}^0 \xrightarrow{\delta} \mathcal{Z}^1 \xrightarrow{\delta} \ldots \} \]

(where \(\delta\) is the coboundary operator on singular cochains) is also exact, and we have two soft acyclic resolutions of the sheaf \(\mathcal{A}\). The integration map \(I\) induces a homomorphism \(\mathcal{R}_1 \rightarrow \mathcal{R}_2\) which equals identity on \(\mathcal{A}\). The standard sheaf-theoretic result implies that the induced homomorphism in the cohomology groups of the complexes of global sections is an isomorphism (see for example [19], Corollary 3.14). The proof of the Proposition is complete. □

The isomorphism

\[ I_* \circ \Phi_* : H^* \left( A^* (M, E)[[t]] \right) \rightarrow H^* (M, \hat{\gamma}) \]

induces an isomorphism of the exact couple

\[ H^* \left( A^* (M, E)[[t]] \right) \xrightarrow{t} H^* \left( A^* (M, E)[[t]] \right) \]

\[ H^* (M, E) \xrightarrow{\pi_*} \]

\[ H^* (M, \hat{\gamma}) \]

\[ H^* (M, E) \xrightarrow{\pi_*} \]

(10) to the exact couple

\[ H^* (M, \hat{\gamma}) \xrightarrow{t} H^* (M, \hat{\gamma}) \]

\[ H^* (M, E) \xrightarrow{\pi_*} \]

and therefore the isomorphism of the spectral sequences

\[ \mathcal{F}_r^* (\mathcal{N}^*, \alpha) \cong \mathcal{L}_r^* (M, \rho, \alpha). \]

Now let us proceed to general topological spaces. The rational homotopy theory of D. Sullivan (see [18], [3], [2]) associates to a connected topological space \(X\) a minimal algebra \(\mathcal{M}^* (X)\) over \(\mathbb{C}\), well defined up to isomorphism.

Let \(\alpha \in H^1 (X)\); we obtain a spectral sequence \(\mathcal{F}_r^* (\mathcal{M}^* (X), \alpha)\). It is not difficult to see that for the case when \(X\) is a \(C^\infty\) manifold, this spectral sequence is isomorphic to the one considered in the previous section. Indeed, we have a homotopy equivalence \(\phi : \mathcal{M}^* (X) \rightarrow A^* (X)\); the induced homomorphism of the spectral sequences

\[ \mathcal{F}_r^* (\mathcal{M}^* (X), \alpha) \rightarrow \mathcal{F}_r^* (A^* (X), \phi_*(\alpha)) \]
is an isomorphism in the second term $\mathcal{F}_2$. Thus the two spectral sequences are isomorphic.

**Theorem 5.4.** Let $X$ be a finite CW complex. Then
$$\mathcal{F}_r^*(M^*(X), \alpha) \approx \mathcal{L}_r^*(X, \alpha).$$

**Proof.** Let $f : X \to M$ be a homotopy equivalence of $X$ to a $C^\infty$ manifold $M$ (possibly non-compact). Denote by $\alpha'$ the $(f^{-1})^*$-image of $\alpha$. The map $f$ induces a homotopy equivalence $F : M^*(X) \to A^*(M)$ of DGAs. We obtain isomorphisms of the spectral sequences:

\begin{align}
(12) & \quad \mathcal{F}_r^*(M^*(X), \alpha) \approx \mathcal{F}_r^*\left(A^*(M), (f^{-1})^*(\alpha)\right) \\
(13) & \quad \mathcal{L}^*(X, \alpha) \approx \mathcal{L}^*(M, \alpha).
\end{align}

Now apply Theorem 5.1 and the result follows. $\square$

**Remark 5.5.** Theorem 5.4 can be generalized to the case of local coefficients so as to obtain a result similar to Theorem 5.1 in the setting of general topological spaces.

6. **Formal and Hyperformal Spaces**

The next theorem follows immediately from Theorems 5.4 and 3.14.

**Theorem 6.1.** Let $X$ be a finite connected CW-complex. Assume that $X$ is formal. Then the spectral sequences
$$\mathcal{F}_r^*(M^*(X), \alpha) \approx \mathcal{L}_r^*(X, \alpha)$$
degenerate at their second term.

Thus the dimension of the homology of $X$ with coefficients in a generic point of the exponential deformation of the trivial representation can be computed from the multiplicative structure of the ordinary homology. Namely, let $\alpha \in H^1(X, \mathbb{C})$ and let $\gamma_t$ be the exponential deformation of the trivial representation:
$$\gamma_t(g) = e^{t(\alpha, g)}.$$  

Denote by $b_k(X, \gamma_{gen})$ the Betti number of $X$ with coefficients in a generic point of the curve $\gamma_t$, and let $L_\alpha$ be the operation of multiplication by $\alpha$ in $H^k(X, \mathbb{C})$. The theorem above implies that
$$b_k(X, \gamma_{gen}) = \mathcal{H}^*(H^k(X, \mathbb{C}), \alpha).$$

The case of the spectral sequence $\mathcal{L}^*(X, \rho, \alpha)$ where $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ is a non-trivial representation, is more complicated and to
guarantee the degeneracy of the spectral sequence a stronger condition is necessary. Let us first consider the case of 1-dimensional representations \( \rho \). Let \( M \) be a connected \( C^\infty \) manifold. Denote by \( G \) the fundamental group of \( M \), let \( Ch(G) \) be the group of homomorphisms \( G \to \mathbb{C}^* = GL(1, \mathbb{C}) \). For a character \( \rho \in Ch(G) \) denote by \( E_\rho \) the corresponding flat vector bundle over \( M \). Put

\[
\tilde{A}^*(M) = \bigoplus_{\rho \in Ch(G)} A^*(M, E_\rho).
\]

The pairing \( E_\rho \otimes E_\eta \approx E_{\rho \eta} \) induces a natural structure of a differential graded algebra on the vector space \( \tilde{A}^*(M) \).

**Definition 6.2.** A \( C^\infty \) manifold \( M \) is hyperformal if the algebra \( \tilde{A}^*(M) \) is formal.

**Theorem 6.3.** Let \( M \) be a hyperformal manifold, \( \rho \in Ch(G) \) and \( \alpha \in H^1(M, \mathbb{C}) \). Then the spectral sequences

\[
\mathcal{F}^r(M^*(M), \rho, \alpha) \approx \mathcal{L}^*(M, \rho, \alpha)
\]

degenerate in their second term.

**Proof.** The DG-module \( A^*(M, E_\rho) \) is formal; apply Proposition 3.16 and the proof is over. \( \square \)

Denote by \( b_k(M, \rho, \gamma_{gen}) \) the \( k \)-th Betti number of \( M \) with coefficients in a generic point of the curve

\[
\gamma_t(g) = \rho(g)e^{t(\alpha(g))}.
\]

**Corollary 6.4.** Let \( M \) be a hyperformal manifold, \( \rho \in Ch(G) \), and \( \alpha \in H^1(M, \mathbb{C}) \). Then

\[
b_k(M, \rho, \gamma_{gen}) = B_k(H^*(M, \rho), \alpha).
\]

An big class of examples of hyperformal spaces is formed by Kähler manifolds, as it follows from C. Simpson theory of Higgs bundles [17].

**Theorem 6.5.** Any compact Kähler manifold is hyperformal.

**Proof.** Let \( \rho \in Ch(\pi_1(M)) \). The flat bundle \( E_\rho \) has a unique structure of a harmonic Higgs bundle (see [1], [17]); the exterior differential \( D_\rho \) in the DG-module \( A^*(M, E_\rho) \) writes therefore as \( D_\rho = \ldots \)
$D'_\rho + D''_\rho$, and the natural homomorphisms of DG-modules

$$\begin{align*}
&\left(\text{Ker } (D'_\rho), D''_\rho\right) \\
&\left(A^*(M, E_\rho), D\right) \\
&\left(H^*_{\text{DeRham}}(M, E_\rho), 0\right)
\end{align*}$$

induce isomorphisms in cohomology (see [17], Lemma 2.2 (Formality)). Denote the DG-module $(\text{Ker } (D'_\rho), D''_\rho)$ by $K_\rho(M)$, and put

$$K^*(M) = \bigoplus_{\rho \in \text{Ch}(G)} K_\rho(M).$$

The multiplicativity properties of Higgs bundles imply that $K^*$ is a DG-algebra and we have the maps of DGAs:

$$\begin{align*}
&\left.K^*(M)\right. \\
&\left.A^*(M)\right. \\
&\left.\bigoplus_{\rho} H^*(M, E_\rho)\right.
\end{align*}$$

both inducing isomorphisms in cohomology. The theorem follows. □

Consider now the case of representations of higher rank.

**Proposition 6.6.** Let $M$ be a compact Kähler manifold and $\rho : \pi_1(M) \to \text{GL}(n, \mathbb{C})$ be a semi-simple representation. Then the DG-module $\hat{A}^*(M, E_\rho)$ is formal.

**Proof.** By [17], Theorem 1 there is a harmonic metric on the bundle $E_\rho$. The tensor powers of this metric provide harmonic metrics on the bundles $E_\rho^\otimes n$ for any $n \geq 1$. Put

$$L^*_\rho(M) = \bigoplus_{n=0}^\infty A^*(M, E_\rho^\otimes n)$$

(where $A^*(M, E_\rho^\otimes 0) = A^*(M)$ by convention). Then $L^*_\rho(M)$ is a DG-algebra. The same argument as in the proof of Theorem 6.5 implies that this algebra is formal, and it remains to observe that $\hat{A}^*(M, E_\rho)$ is a direct summand of $L^*_\rho(M)$. □
7. ON THE BETTI NUMBERS OF CHAIN COMPLEXES OVER LAURENT POLYNOMIAL RINGS

This section contains some preliminary algebraic results. We discuss the Betti numbers of complexes over Laurent polynomial rings in view of further applications to the Novikov Betti numbers and the homology with local coefficients.

Let $T$ be an integral domain and $\{T\}$ be its fraction field. Let $C_\ast$ be a finite free chain complex over $T$:

$$C_\ast = \{ 0 \leftarrow C_0 \leftarrow \ldots \leftarrow \partial_k C_k \leftarrow \partial_{k+1} C_{k+1} \leftarrow \ldots \}$$

The tensor product of this chain complex with $\{T\}$ will be denoted by $C_\ast$ and the boundary operator in this complex will be denoted by $\partial_\ast$. The Betti number $b_k(C_\ast)$ is equal to

$$\dim kC_k - rk\partial_k - rk\partial_{k+1} = rkC_k - rk\partial_k - rk\partial_{k+1}$$

(where $rk\partial_k$ stands for the maximal rank of a non-zero minor of the matrix of $\partial_k$).

Let $\phi : T \to U$ be a homomorphism to another integral domain. Denote by $b_k(C_\ast, \phi)$ the $k$-th Betti number of the tensor product $C_\ast \otimes_\phi \{U\}$. Then $b_k(C_\ast, \phi) \geq b_k(C_\ast)$. The inequality is strict if the $\phi$-images of all the $rk\partial_k$-minors of $\partial_k$ vanish, or if the $\phi$-images of all the $rk\partial_{k+1}$-minors of $\partial_{k+1}$ vanish. In general, for $q > 0$ the condition $b_k(C_\ast, \phi) \geq b_k(C_\ast) + q$ is equivalent to the existence of a number $i$, with $0 \leq i \leq q$ such that all the $(rk\partial_k - i)$-minors of $\phi(\partial_k)$ vanish and all the $(rk\partial_{k+1} - (q - i))$-minors of $\phi(\partial_{k+1})$ vanish.

We are interested in the case of the Laurent polynomial rings. Let $R$ be an integral domain and $C_\ast$ be a finite free chain complex over $L_n = R[t_1^\pm, \ldots, t_n^\pm] = R[Z^n]$. Let $p : Z^n \to Z^m$ be a group homomorphism, extend it to a ring homomorphism $L_n \to L_m$ which will be denoted by the same letter $p$.

Denote by $Q_m$ is the field of fractions of $L_m$, that is, the field of the rational functions in $m$ variables with coefficients in the fraction field of $R$.

Form the chain complex $C_\ast \otimes_p Q_m$, and denote by $b_k(C_\ast, p)$ the dimension of the vector space $H_k(C_\ast \otimes_p Q_m)$ over $Q_m$.

Observe that if $p$ is injective, then $b_k(C_\ast) = b_k(C_\ast, p)$. We will now study the dependence of $b_k(C_\ast, p)$ on $p$.

**Definition 7.1.** A subgroup $G \subset Z^n$ is called **full** if it is a direct summand of $Z^n$. We say that a homomorphism $p : Z^n \to Z^m$ is **subordinate** to a full subgroup $G \subset \text{Hom}(Z^n, Z^m)$ and we write $p \sqsubset G$, if all the coordinates of $p$ are in $G$. 
Remark 7.2. Denote by $K$ the subgroup of $\mathbb{Z}^n$ dual to $G$. Then $p \subseteq G$ if and only if $p \mid K = 0$.

Theorem 7.3. Let $C_*$ be a finite free complex over $L_n$. Let $k \geq 0, q > 0$. Then there is a finite family of proper full subgroups $G_i \subset \text{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that for $p \in \text{Hom}(\mathbb{Z}^n, \mathbb{Z}^m)$ the condition

$$b_k(C_*, p) \geq b_k(C_*) + q$$

is equivalent to the following condition: $p \subseteq G_i$ for some $i$.

Proof. Let us do the case $q = 1$, the general case is similar. Let $F$ denote the set of all the $(rk\partial_k)$-minors of the matrix $\partial_k : C_k \to C_{k-1}$, and all the $(rk\partial_{k+1})$-minors of the matrix $\partial_{k+1} : C_{k+1} \to C_k$. Let $\Delta \in F$, write $\Delta = \sum g \in \mathbb{Z}^n r_g \cdot g$.

According to our previous observation it suffices to study the set $\Sigma$ of all homomorphisms $p : \mathbb{Z}^n \to \mathbb{Z}^m$ such that $p(\Delta) = 0$. Let $\Gamma = \text{supp} \Delta$, this is a finite subset of $\mathbb{Z}^n$. Any homomorphism $p : \mathbb{Z}^n \to \mathbb{Z}^m$ with $p(\Delta) = 0$ must be non-injective on $\Gamma$. To describe the set of all such homomorphisms let us say that a subdivision $\Gamma = \Gamma_1 \sqcup \ldots \sqcup \Gamma_N$ is $\Delta$-fitted, if for any $j$ we have

$$\sum_{g_k \in \Gamma_j} r_g = 0.$$

For any $\Delta$-fitted subdivision $S$ consider the subgroup $L(S) \subset \text{Hom}(\mathbb{Z}^n, \mathbb{Z})$ consisting of all homomorphisms $h : \mathbb{Z}^n \to \mathbb{Z}$ such that $h \mid \Gamma_i$ is constant for every $i$. Then $L(S)$ is a full subgroup of $\text{Hom}(\mathbb{Z}^n, \mathbb{Z})$. Observe that $L(S)$ is a proper subgroup since every $\Gamma_i$ contains at least two elements. It is clear that a homomorphism $p : \mathbb{Z}^n \to \mathbb{Z}^m$ belongs to $\Sigma$ if and only if $p$ is constant on each component $\Gamma_i$ of a $\Delta$-fitted subdivision of $\Delta$, that is, $p \subset L(S)$. □

8. The twisted Novikov Betti numbers

Let $X$ be a finite connected CW complex and $\rho : \pi_1(X) \to \text{GL}(l, R)$ be a representation (where $R$ is an integral domain). We will discuss several numerical invariants deduced from $\rho$. Denote $\pi_1(X)$ by $G$, and $\text{rk}H_1(X, \mathbb{Z})$ by $n$. The chain complex of the universal cover $\tilde{X}$ has a natural structure of a left $G$-module; it will be convenient to consider on it a right $G$-action $x \cdot g = g^{-1}(x)$. Our first numerical invariant is the twisted Betti number of $X$ with respect to $\rho$:

$$b_k(X, \rho) = \text{dim}_{\{R\}} H_k \left( C_*(\tilde{X}) \otimes_{\rho} \{R\}^I \right).$$
Denote by $\phi$ the projection of $G$ onto $G/Tors \cong \mathbb{Z}^n$. We obtain two representations

$$
\begin{array}{c}
G \\
\phi \\
\rho \\
\mathbb{Z}^n \\
GL(l, R)
\end{array}
$$

Let $\rho^\circ : G \to GL(l, L_n)$ denote the tensor product of these two representations (where $L_n = R[\mathbb{Z}^n]$ is the ring of Laurent polynomials in $n$ variables) and put

$$b^\circ_k(X, \rho) = \dim_{\{L_n\}} H_k\left(C_*((\tilde{X}) \otimes \{L_n\}^l)\right)$$

Denote $\overline{X} \xrightarrow{\mathbb{Z}^n} X$ the maximal free abelian covering of $X$; we will consider $b^\circ_k(X, \rho)$ as the Betti number of $\overline{X}$ with $\rho$-twisted coefficients.

Let $p : \mathbb{Z}^n \to \mathbb{Z}^m$ be a homomorphism. Consider the tensor product of representations $\rho$ and $p \circ \phi$:

$$\rho_p = \rho \otimes (p \circ \phi) : G \to GL(l, L_m)$$

and let

$$b_k(X, \rho; p) = \dim_{\{L_m\}} H_k\left(C_*((\tilde{X}) \otimes \{L_m\}^l)\right)$$

It is not difficult to see that

$$b_k(X, \rho; p) \geq b^\circ_k(X, \rho)$$

for every $p$;

$$b_k(X, \rho) \geq b^\circ_k(X, \rho).$$

Theorem 7.3 of the previous section implies the following.

**Proposition 8.1.** Let $k \geq 0$, $q > 0$. Then there is a finite family of proper full subgroups $G_i \subset \text{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that for $p \in \text{Hom}(\mathbb{Z}^n, \mathbb{Z}^m)$ the condition

$$b_k(X, \rho; p) \geq b^\circ_k(X, \rho) + q$$

is equivalent to the following condition: $p \sqsubseteq G_i$ for some $i$.

Proceeding to the Novikov homology, let us first recall the definition of the Novikov ring. Let $\eta : \mathbb{Z}^m \to \mathbb{R}$ be a group homomorphism. The Novikov completion $\hat{L}_{m, \eta}$ of the ring $L_m = R[\mathbb{Z}^m]$ with respect to $\eta$ is defined as the set of all series of the form $\lambda = \sum_g n_g g$ (where $g \in G$ and $n_g \in R$) satisfying the following condition of finiteness in the direction corresponding to $\eta$:

$$\hat{L}_{m, \eta} = \left\{ \lambda \mid \forall C \in \mathbb{R}, \text{ the set } \text{supp } \lambda \cap \eta^{-1}([C, \infty]) \text{ is finite } \right\}.$$
In general the ring \( \hat{L}_{m,\eta} \) is rather complicated, however if \( \eta \) is a monomorphism, this ring is Euclidean by a theorem of J.-Cl. Sikorav (see [14], Th. 1.4).

Let \( \xi : G \to \mathbb{R} \) be a homomorphism. We can factor it as follows:

\[
\begin{array}{ccc}
G & \longrightarrow & H_1(X, \mathbb{Z})/\text{Tors} \\
& & \cong \mathbb{Z}^n \\
p & & \xi \\
\mathbb{Z}^m & \longrightarrow & \mathbb{R}
\end{array}
\]

where \( p \) is an epimorphism and \( \tilde{\xi} \) is a monomorphism. The Novikov completion of the ring \( R[\mathbb{Z}^m] \) with respect to \( \tilde{\xi} \) will be denoted by \( \hat{L}_{m,\xi} \); this ring is principal. Denote by \( \hat{\rho}_\xi \) the composition

\[
G \xrightarrow{\rho_p} \text{GL}(r, L_m) \xleftarrow{\hat{\rho}} \text{GL}(l, \hat{L}_{m,\xi}).
\]

**Definition 8.2.** ([6], [15]) The \( \hat{L}_{m,\xi} \)-module

\[
H_*(C_*(\tilde{X}) \otimes \hat{L}_{m,\xi}^l)
\]

is called the \( \rho \)-twisted Novikov homology of \( X \) with respect to \( \xi \). The rank and the torsion number of its \( k \)-th component are denoted respectively by \( \hat{b}^\rho_k(X, \xi) \) and \( \hat{q}^\rho_k(X, \xi) \).

The geometric reasons to consider these completions of the module \( C_*(\tilde{X}) \) are as follows. If \( X \) is a compact manifold, and \( \omega \) is a closed 1-form on \( M \) with non-degenerate zeroes, denote by \( \xi \) the period homomorphism \( [\omega] = \xi : H_1(M, \mathbb{Z}) \to \mathbb{R} \). The Morse-Novikov theory implies the following lower bounds on the number of zeroes of \( \omega \):

\[
m_k(\omega) \geq \frac{1}{r} \left( \hat{b}^\rho_k(X, \xi) + \hat{q}^\rho_k(X, \xi) + \hat{q}^\rho_{k-1}(X, \xi) \right).
\]

Compared to the other versions of the Novikov homology, the twisted Novikov homology has the advantage of being computable, and at the same time to keep the information about the non-abelian structure of \( G \) and related invariants. In a recent work [5] S. Friedl and S. Vidussi proved that the twisted Novikov homology detects fibredness of knots in \( S^3 \). At present we will need only the simplest part of the twisted Novikov homology invariants, namely the twisted Novikov Betti numbers.

**Proposition 8.3.** Let \( \xi : G \to \mathbb{R} \) be a homomorphism and \( \rho : G \to \text{GL}(l, R) \) a representation. Then for every \( k \) we have

\[
b_k(X, \rho, p) = \hat{b}^\rho_k(X, \xi)
\]
Proof. The twisted Betti Novikov number in question equals the dimension of
\[ H_k(C_*(\tilde{X}) \otimes L^1_m) \otimes \{\hat{L}_{m,\xi}\} \]
over the field of fractions \(\{\hat{L}_{m,\xi}\}\) of the Novikov ring.

The Betti number \(b_k(X, \rho; p)\) is the dimension of the vector space
\[ H_k(C_*(\tilde{X}) \otimes L^1_m) \otimes \{L_m\} \]
over the field of fractions \(\{L_m\}\). The inclusion \(L_m \subset \hat{L}_{m,\xi}\) extends to an inclusion of fields \(\{L_m\} \subset \{\hat{L}_{m,\xi}\}\) and the result follows. \(\square\)

**Definition 8.4.** For a homomorphism \(\xi : G \to \mathbb{R}\) the subgroup \(\xi(G)\) is a free finitely generated abelian group; its rank is called *irrationality degree of* \(\xi\) and denoted \(\text{Irr}\xi\). In particular \(\text{Irr}\xi = 1\) if and only if \(\xi\) is a multiple of a homomorphism \(G \to \mathbb{Z}\). We say that \(\xi\) is *maximally irrational* if \(\text{Irr}\xi = n\).

Observe that if \(\xi\) is maximally irrational, then the homomorphism \(p\) in the diagram ([16]) is an isomorphism and \(\hat{b}^\rho_k(X, \xi)\) does not depend on \(\xi\).

**Definition 8.5.** The number \(\hat{b}^\rho_k(X, \xi)\) where \(\xi\) is maximally irrational, will be denoted by \(\hat{b}^\rho_k(X, \rho)\).

It is not difficult to prove that

\begin{align*}
(17) & \quad b_k(X, \rho) \geq \hat{b}^\rho_k(X, \rho) = b^\rho_k(X, \rho), \\
(18) & \quad \hat{b}^\rho_k(X, \xi) \geq \hat{b}^\rho_k(X, \rho) \quad \text{for every } \xi : G \to \mathbb{R}.
\end{align*}

Theorem [7.3] of the previous section implies the following.

**Proposition 8.6.** Let \(k \geq 0, q > 0\). Then there is a finite family of proper full subgroups \(G_i \subset \text{Hom}(\mathbb{Z}^n, \mathbb{Z})\) such that for the condition
\[ \hat{b}^\rho_k(X, \xi) \geq b^\rho_k(X, \rho) + q \]

is equivalent to the following condition: \(\xi \in \bigcup_i G_i \otimes \mathbb{R}\).
9. Homology with local coefficients

Let us proceed now to the homology with local coefficients. Let \( \rho : G \to \text{GL}(n, \mathbb{C}) \) be a representation of \( G \) and \( \alpha \in H^1(X, \mathbb{C}) \). The cohomology class \( \alpha \) can be considered as a homomorphism \( \alpha : G \to \mathbb{C} \), which can be factored as follows

\[
\begin{align*}
G &\to H_1(X, \mathbb{Z})/\text{Tors} \xrightarrow{\sim} \mathbb{Z}^n \\
\downarrow p &\downarrow \alpha \\
\mathbb{Z}^m &\to \mathbb{R}
\end{align*}
\]

where \( p \) is an epimorphism and \( \alpha \) a monomorphism. Here \( m = \text{Irr} \alpha \); denote the coordinates of \( p \) by \( p_i : \mathbb{Z}^m \to \mathbb{Z} \), \( 1 \leq i \leq m \). We have \( \alpha = \sum_{i=1}^m \alpha_i p_i \), and the numbers \( \alpha_i \in \mathbb{C} \) are linearly independent over \( \mathbb{Q} \).

Recall the exponential and the formal exponential deformations of \( \rho \):

\[
\gamma_t : G \to \text{GL}(n, \mathbb{C}), \quad \gamma_t(g) = \rho(g) e^{t \langle \alpha, g \rangle} \quad (t \in \mathbb{C}).
\]

and the corresponding Betti numbers \( \beta_k(X, \gamma_t), \beta_k(X, \hat{\gamma}) \).

We will need a simple lemma.

Lemma 9.1. Let \( \alpha_1, \ldots, \alpha_m \) be complex numbers, linearly independent over \( \mathbb{Q} \). Then the power series \( e^{\alpha_1 t}, \ldots, e^{\alpha_m t} \in \mathbb{C}[[t]] \) are algebraically independent.

Proof. If \( P \in \mathbb{C}[z_1, \ldots, z_m] \) is a polynomial such that \( P(e^{\alpha_1 t}, \ldots, e^{\alpha_m t}) = 0 \), write \( P = \sum a_I t^I \) where the sum ranges over multiindices \( I = (k_1, \ldots, k_m) \in \mathbb{N}^m \). Denote the string \( (\alpha_1, \ldots, \alpha_m) \) by \( \alpha \). The series \( \zeta = P(e^{\alpha_1 t}, \ldots, e^{\alpha_m t}) \) is then a finite sum of exponential functions of the form \( a_I e^{t \langle I, \alpha \rangle} \). Observe that \( \langle I, \alpha \rangle \neq \langle J, \alpha \rangle \) if \( I \neq J \), since \( \alpha_i \) are linearly independent over \( \mathbb{Q} \). Therefore \( \zeta \) is a finite linear combination of exponential functions \( e^{t \beta_I} \) with pairwise different \( \beta_I \). Thus \( \zeta = 0 \) implies \( a_I = 0 \) for all \( I \). \( \square \)

Proposition 9.2. \( ^\dagger \) We have

1) \( \beta_k(X, \gamma_t) = \beta_k(X, \hat{\gamma}) = b_k(X, \rho, p) \)

2) If \( \alpha \in H^1(X, \mathbb{R}) \) then \( \beta_k(X, \gamma_t) = \beta_k(X, \hat{\gamma}) = \hat{b}_k(X, \rho) \)

Proof. The representation \( \hat{\gamma} \) factors through \( \mathbb{Z}^m \) as follows

\[
G \xrightarrow{p \circ \phi} \mathbb{Z}^m \xrightarrow{\Gamma} (\mathbb{C}[\![t]\!] )^* \]

Let \( (e_1, \ldots, e_m) \) denote the canonical basis in \( \mathbb{Z}^m \); then we have \( \Gamma(e_i) = e^{t \alpha_i} \). The extension of \( \Gamma \) to a ring homomorphism \( \mathbb{Z}[\mathbb{Z}^m] \to \)

\( \dagger \) A particular case of this proposition corresponding to the vanishing Novikov Betti numbers was proved by S. Papadima and A. Suciu in [16].
$\mathbb{C}[[t]]$ is injective by the previous lemma, and therefore can be further extended to a homomorphism

$$\Gamma : \{L_m\} \to \mathbb{C}((t))$$

of the fraction fields. Therefore

$$\dim_{\{L_m\}} H_k\left(C_*(\widehat{X}) \otimes \{L_m\}^t\right) = \dim_{\mathbb{C}((t))} H_k\left(C_*(\widehat{X}) \otimes \{L_m\}^t \otimes \mathbb{C}((t))\right)$$

and the point 1) is proved. The point 2) follows from 1) and Proposition 8.3.

The next proposition follows immediately.

**Proposition 9.3.** Let $k \geq 0, q > 0$. Then there is a finite family of proper full subgroups $G_i \subset \text{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that for the condition

$$\beta_k(X, \widehat{\gamma}) \geq b_k^o(X, \rho) + q$$

is equivalent to the following condition: $\xi \in \bigcup_i G_i \otimes \mathbb{C}$.

10. **FORMAL AND HYPERFORMAL SPACES II: THE JUMP LOCI**

Let $X$ be a manifold, $\rho : \pi_1(X) \to \text{GL}(n, \mathbb{C})$ a representation, and $\alpha \in H^1(X, \mathbb{C})$. In this section we assume that the spectral sequences $\mathcal{F}_r^*$ and $\mathcal{L}_r^*$ associated to the triple $(X, \rho, \alpha)$ degenerate at their second term, so that

$$\mathcal{F}_r^*(X, \rho, \alpha) \approx \mathcal{L}_r^*(X, \rho, \alpha) \approx \mathcal{H}^*(H^*(X, \rho), \alpha)$$

According to Section 6 this holds in particular when $X$ is a Kähler manifold and $\rho$ a semi-simple representation. The following theorem follows from Theorem 5.1 together with Propositions 8.3, 9.2.

**Theorem 10.1.**

1) $\beta_k(X, \widehat{\gamma}) = \beta_k(X, \gamma_{\text{gen}}) = B_k^k(H^*(X, \rho), \alpha)$.

2) If $\alpha \in H^1(X, \mathbb{R})$ then

$$\widehat{b}_k^o(X, \alpha) = \beta_k(X, \widehat{\gamma}) = B_k^k(H^*(X, \rho), \alpha).$$

We deduce the following result concerning the jump loci of the Betti numbers:

**Proposition 10.2.** Let $k \geq 0, q > 0$. Then there is a finite family of proper full subgroups $G_i \subset \text{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that each of the following conditions (i), (ii)

\begin{align*}
(20) & \quad \beta_k(X, \widehat{\gamma}) \geq b_k^o(X, \rho) + q; \\
(21) & \quad B_k^k(H^*(X, \rho), \alpha) \geq b_k^o(X, \rho) + q
\end{align*}

is equivalent to the condition $\alpha \in \bigcup_i G_i \otimes \mathbb{C}$. 
For $\alpha \in \mathcal{H}^1(X, \mathbb{R})$ each of the following conditions (i), (ii)

\begin{align}
\hat{b}_k^{\rho}(X, \xi) & \geq b_k^{\rho}(X, \rho) + q; \\
\mathcal{B}^k(H^*(X, \rho), \alpha) & \geq b_k^{\rho}(X, \rho) + q
\end{align}

is equivalent to the condition $\alpha \in \bigcup_i G_i \otimes \mathbb{R}$. \hfill $\square$

11. Acknowledgements

The main part of the work was accomplished during the second author’s stay at the Graduate School of Mathematics, Tokyo university, in the fall of 2011. The second author is grateful to the Graduate School of Mathematics for warm hospitality.

The authors thank Hisashi Kasuya for valuable discussions concerning his work.

During the preparation of the paper, the authors became aware of the paper [4] by A. Dimca and S. Papadima, on related subjects. Theorem 6.1 of the present paper is also proved by a completely different method in [4], see Theorem E.

References

[1] D. Arapura, Higgs line bundles, Green-Lazarsfeld sets, and maps of Kahler manifolds to curves, Bull. Amer. Math. Soc. (N.S.), 26(2):310-314, 1992.
[2] A. K. Bousfield, V. K. A. M. Gugenheim, On PL de Rham theory and rational homotopy type, Mem. AMS 179, (1976).
[3] P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan, Real homotopy theory of Kahler manifolds, Invent. Math. 29 (1975), 245 - 274.
[4] A. Dimca, S. Papadima, Nonabelian cohomology jump loci from an analytic viewpoint, arXiv:1206.3773.
[5] S. Friedl, S. Vidussi, A vanishing theorem for twisted Alexander polynomials with applications to symplectic 4-manifolds, arXiv:1205.2434
[6] H. Goda, A. Pajitnov, Twisted Novikov homology and circle-valued Morse theory for knots and links, e-print: math.GT/0312374, Journal Publication: Osaka Journal of Mathematics, 42 No. 3, 2005, p. 557 – 572.
[7] S.-T. Hu, Homotopy Theory, Academic Press, 1959.
[8] H. Kasuya, Minimal models, formality and hard Lefschetz properties of solv-manifolds with local systems, arXiv:1009.1940 (2010).
[9] W. Massey, Exact Couples in Algebraic Topology (Parts I and II), Ann. Math. 56 (1952), pp. 363-396.
[10] S.P. Novikov, Many-valued functions and functionals. An analogue of Morse theory, Sov. Math. Dokl. 24 (1981), 222 – 226.
[11] S.P. Novikov, Bloch homology, critical points of functions and closed 1-forms, Soviet Math. Dokl. 287 (1986), N° 2.
[12] A. Pajitnov, An analytic proof of the real part of the Novikov inequalities DAN SSSR, 293, 1987 no. 6 (in Russian); Engl transl: Soviet Math. Dokl.35 (1987), no. 2.
[13] A. Pajitnov, Proof of the Novikov’s conjecture on homology with local coefficients over a field of finite characteristic, Soviet Math. Dokl. 37 (1988), no. 3.

[14] A. Pajitnov, On the sharpness of Novikov-type inequalities for manifolds with free abelian fundamental group, Matem. Sbornik, 180 no. 11, (1989) English translation: Math. USSR Sbornik, 68 (1991), 351 - 389.

[15] A. Pajitnov, Novikov Homology, twisted Alexander polynomials and Thurston cones, Algebra i Analiz, v. 18 N° 5 (2006), 173 – 209.

[16] S. Papadima, A. Suciu, Bieri-Neumann-Strebel-Renz invariants and homology jumping loci, Proceedings of the London Mathematical Society 100 (2010), no. 3, 795-834.

[17] C. Simpson, Higgs bundles and local systems, Publications mathematiques de l'I.H.E.S., tome 75 (1992), p. 5-95.

[18] D. Sullivan, Infinitesimal computations in topology, Publ. I.H.E.S. 47 (1977), 269 - 331.

[19] R.O. Wells, Differential Analysis on complex manifolds, Prentice-Hall, 1973

KAVLI IPMU, GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO-KU, TOKYO 153-8914, JAPAN
E-mail address: kohno@ms.u-tokyo.ac.jp

LABORATOIRE MATHEMATIQUES JEAN LERAY UMR 6629, UNIVERSITE DE NANTES, FACULTE DES SCIENCES, 2, RUE DE LA HOUSSINIÈRE, 44072, NANTES, CEDEX
E-mail address: andrei.pajitnov@univ-nantes.fr