Integral Equations in the Theory of Levy Processes

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Abstract
In this article we consider the Levy processes and the corresponding semigroup. We represent the generator of this semigroup in a convolution form. Using the obtained convolution form and the theory of integral equations we investigate the properties of a wide class of Levy processes (potential, quasi-potential, the probability of the Levy process remaining within the given domain, long time behavior, stable processes). We analyze in detail a number of concrete examples of the Levy processes (stable processes, the variance damped Levy processes, the variance gamma processes, the normal Gaussian process, the Meixner process, the compound Poisson process).

Mathematic Subject Classification (2000). Primary 60G51, Secondary 60J45, 60G17, 45A05.

Key words. Semigroup, generator, convolution form, potential, quasi-potential, sectorial operators, long time behavior.

Introduction
In the famous article by M.Kac [11] a number of examples demonstrate the interconnection between the probability theory and the theory of integral and differential equations. In particular, in article [11] the Cauchy process was considered. Later M.Kac method was used both for symmetric stable processes [30], [19] and non-symmetric stable processes [20]-[22]. In the present article with the help of M.Kac’s idea [11] and the theory of integral equations with the difference kernels [22] we investigate a wide class of Levy processes. We note that within the last ten years the Levy processes have got a number of new important applications, particularly to financial problems. We consider separately the examples of Levy processes which are used in financial mathematics.

Now we shall formulate the main results of the article.
1. The Levy process $X_t$ defines a strongly continuous semigroup $P_t$ (see [23]). The generator $L$ of the semigroup $P_t$ is a pseudo-differential operator. We
show that for a broad class of the Levy processes the generator $L$ can be represented in a convolution type form (section 2):

$$L f = \frac{d}{dx} S \frac{d}{dx} f,$$

where the operator $S$ is defined by the relation

$$S f = \frac{1}{2} A f + \int_{-\infty}^{\infty} k(y - x) f(y) dy,$$

Such a representation opens a possibility to use the theory of integral equations with difference kernels [22].

2. We introduce the notion of the truncated generator $L_\Delta$, which is important in our theory and coincides with $L$ on the bounded system of segments $\Delta$. We define the quasi-potential $B$ by the relation $-L_\Delta B f = f$ (sections 3 and 4). Under our assumptions the operator

$$B f = \int_\Delta \Phi(x,y)f(y)dy$$

is compact. It means that the operator $B$ has a discrete spectrum $\lambda_j$ ($j = 1, 2, ...$), $\lambda_j \to 0$. Hence, the corresponding truncated generator $L_\Delta$ has a discrete spectrum too. Using the results from the theory of the integral equations with the difference kernels we represent the quasi-potential $B$ in the explicit form.

3. The probability $p(t, \Delta)$ of the Levy process remaining within the given domain $\Delta$ (ruin problem) is investigated in our paper (section 5). M.Kac [11] had obtained the first results of this type for symmetric stable processes. Later we transposed these results for the non-symmetric stable processes [20], [22]. In this paper we show that integro-differential equation of M.Kac type is true for all the Levy processes, which have a continuous density. M.Kac writes [11]: ”We are led here to integro-differential equations which offer formidable analytic difficulties and which we were able to solve only in very few cases.” M. Kac was able to overcome these difficulties only for Cauchy processes. H.Widom [30] solved these equations for symmetric stable processes. Both symmetric and non-symmetric stable processes were investigated in our works [19]-[22]. Now we develop these results and transfer them on the wide class of the Levy processes.

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4. In sections 6-8 we investigate the structure and the properties of the quasi-potential $B$. In particular, we prove that the operator $B$ is compact and the following important inequality:

$$\Phi(x,y) \geq 0$$ \hspace{1cm} (0.4)

is true. From inequality (0.4) and Krein-Rutman theorem [13] we deduce that the operator $B$ has a positive eigenvalue $\lambda_1$, which is no less in modulus than every other eigenvalue of $B$.

5. Section 9 contains formulas and estimations for the probability $p(t,\Delta)$ that a sample of the process $X_\tau$ remains inside the given domain $\Delta$, when $0 \leq \tau \leq t$. Under certain conditions we have obtained the asymptotic formula

$$p(t,\Delta) = e^{-t/\lambda_1}[c_1 + o(1)], \quad t \to \infty.$$ \hspace{1cm} (0.5)

6. In sections 10-12 we separately consider a special case of the Levy processes, i.e., stable processes. We use the notation $p(t,a) = p(t,\Delta)$ when $\Delta = [-a,a]$. We consider the important case, where $a$ depends on $t$ and $a(t) \to \infty$ when $t \to \infty$. We compare the obtained results with well-known results (the iterated logarithm law, the results for the first hitting time, the results for the most visited sites problems). Further we introduce the notation $p(t,-b,a) = p(t,\Delta)$ when $\Delta = [-b,a]$. In case of the Wiener process we found the asymptotic behavior of $p(t,-b,a)$ when $b \to \infty$. It is easy to see that $p(t,-\infty,a)$ coincides with the formula for the first hitting time.

7. We analyze in detail a number of concrete examples of the Levy processes which are used in the financial mathematics (stable processes, the variance damped Levy processes, the variance gamma processes, the normal Gaussian process, the Meixner process, compound Poisson process.)

1 Main notions

Let us consider the Levy processes $X_t$ on $R$. If $P(X_0 = 0) = 1$ then Levy-Khinchine formula gives (see[4],[23])

$$\mu(z,t) = E\{\exp[izX_t]\} = \exp[-t\lambda(z)], \quad t \geq 0,$$ \hspace{1cm} (1.1)

where

$$\lambda(z) = \frac{1}{2}Az^2 - iz - \int_{-\infty}^{\infty}(e^{ixz} - 1 - ixz1_{D(x)})d\nu(x).$$ \hspace{1cm} (1.2)
Here $A \geq 0$, $\gamma = \bar{\gamma}$, and $D = \{x : |x| \leq 1\}$ is the segment $[-1,1]$. $\nu(x)$ is a monotonically increasing function satisfying the conditions
\[
\int_{-\infty}^{\infty} \frac{x^2}{1 + x^2} d\nu(x) < \infty. \quad (1.3)
\]

By $P_t(x_0, \Delta)$ we denote the probability $P(X_t \in \Delta)$ when $P(X_0 = x_0) = 1$ and $\Delta \in \mathbb{R}$. The transition operator is defined by the formula
\[
P_t f(x) = \int_{-\infty}^{\infty} P_t(x, dy) f(y). \quad (1.4)
\]

Let $C_0$ be the Banach space of continuous functions $f(x)$ ($-\infty < x < \infty$) satisfying the condition $\lim_{|x| \to \infty} f(x) = 0$, with the norm $||f|| = \sup |f(x)|$. We denote by $C_0^n$ the set of $f(x) \in C_0$ such that $f^{(k)}(x) \in C_0$, $(1 \leq k \leq n)$. It is known that [23]
\[
P_t f \in C_0, \quad (1.5)
\]

if $f(x) \in C_0$.

Now we formulate the following important result (see [4],[23]).

**Theorem 1.1.** The family of the operators $P_t$ $(t \geq 0)$ defined by the Levy process $X_t$ is a strongly continuous semigroup on $C_0$ with the norm $||P_t|| = 1$.

Let $L$ be its infinitesimal generator. Then
\[
Lf = \frac{1}{2} A \frac{d^2 f}{dx^2} + \gamma \frac{df}{dx} + \int_{-\infty}^{\infty} (f(x+y) - f(x) - y \frac{df}{dx} 1_{D(x)}) d\nu(x), \quad (1.6)
\]

where $f \in C_0^2$.

### 2 Convolution type form of infinitesimal generator

1. In this section we prove that under some conditions the infinitesimal generator $L$ can be represented in the special convolution type form
\[
Lf = \frac{d}{dx} S \frac{d}{dx} f, \quad (2.1)
\]

where the operator $S$ is defined by the relation
\[
Sf = \frac{1}{2} Af + \int_{-\infty}^{\infty} k(y-x) f(y) dy, \quad (2.2)
\]
and for arbitrary \( M(0 < M < \infty) \) we have
\[
\int_{-M}^{M} |k(t)| dt < \infty. \tag{2.3}
\]
The representation \( L \) in form (2.1) is convenient as the operator \( L \) is expressed with the help of the classic differential and convolution operators.

By \( C_c \) we denote the set of functions \( f(x) \in C_0 \) with a compact support.

**Lemma 2.1.** Let the following conditions be fulfilled.
1. The function \( \nu(x) \) is monotonically increasing, has the derivative when \( x \neq 0 \) and
\[
\int_{-\infty}^{\infty} \frac{x^2}{1 + x^2} d\nu(x) = \int_{-\infty}^{\infty} \frac{x^2}{1 + x^2}\nu'(x) dx < \infty, \tag{2.4}
\]
\[
\nu(x) \to 0, \quad x \to \infty. \tag{2.5}
\]
2. For arbitrary \( M \ (0 < M < \infty) \) we have
\[
\int_{-M}^{M} |\nu(x)| dx < \infty, \quad \int_{-M}^{M} |x|\nu'(x) dx < \infty. \tag{2.6}
\]
3. \[
x\nu(x) \to 0, \quad x \to 0. \tag{2.7}
\]

Then the expression
\[
J = \int_{-\infty}^{\infty} [f(y + x) - f(x)]\nu'(y) dy \tag{2.8}
\]
can be represented in the convolution type form
\[
J = \frac{d}{dx} \int_{-\infty}^{\infty} f'(y)k(y - x) dy \tag{2.9}
\]
where \( f(x) \in C_0^2, \ k(x) = \int_{0}^{x} \nu(y) dy. \)

**Proof.** For every \( f(x) \in C_c \) there exists such \( M \ (0 < M < \infty) \) that
\[
f(x) = 0, \quad x \notin [-M, M]. \tag{2.10}
\]
Let us introduce the following notations
\[
J_1 = \frac{d}{dx} \int_{-\infty}^{x} f'(y)k(y - x) dy, \tag{2.11}
\]
Using (2.11) we have

\[ J_1 = -\frac{d}{dx} \int_{-M}^{x} [f(y) - f(x)] k'(y - x) dy. \]  

(2.13)

From (2.11) and (2.13) we deduce the relation

\[ J_1 = f(x) k'(-M - x) + \int_{-M}^{x} [f(y) - f(x)] k''(y - x) dy. \]  

(2.14)

When \( M \to \infty \) we obtain the equality

\[ J_1 = \int_{-\infty}^{0} [f(y + x) - f(x)] k''(y) dy. \]  

(2.15)

In the same way we deduce the relation

\[ J_2 = \int_{0}^{\infty} [f(y + x) - f(x)] k''(y) dy. \]  

(2.16)

Relation (2.9) follows directly from formulas (2.15), (2.16) and the equality \( J = J_1 + J_2 \). The lemma is proved.

Lemma 2.2. Let the following conditions be fulfilled.

1. The function \( \nu(x) \) satisfies conditions (2.4) and (2.5) of Lemma 2.1.

2. For arbitrary \( M \) \( (0 < M < \infty) \) we have

\[ \int_{-M}^{M} |k(x)| dx < \infty, \quad \int_{-M}^{M} |x\nu(x)| dx < \infty, \]  

(2.17)

where

\[ k'(x) = \nu(x), \quad x \neq 0. \]  

(2.18)

3. \( xk(x) \to 0, \quad x \to 0; \quad x^2\nu(x) \to 0, \quad x \to 0. \)  

(2.19)

Then the equality

\[ J = \int_{-\infty}^{\infty} [f(y + x) - f(x) - y\frac{df(x)}{dx}1_{D(y)}] \nu'(y) dy + \Gamma f'(x), \]  

(2.20)
is true, where \( \Gamma = \overline{\Gamma} \) and \( f(x) \in C_c \).

**Proof.** From (2.11) we obtain the relation

\[
J_1 = f'(x)\gamma_1 - \int_{x-1}^{x} [f'(y) - f'(x)]k'(y-x)dy - \int_{-M}^{x-1} f'(y)k'(y-x)dy,
\]

(2.21)

where \( \gamma_1 = k(-1) \). We introduce the notations

\[
P_1(x,y) = f(y) - f(x) - (y-x)f'(x), \quad P_2(x,y) = f(y) - f(x).
\]

(2.22)

Integrating by parts (2.21) and passing to the limit when \( M \to \infty \) we deduce that

\[
J_1 = f'(x)\gamma_2 + \int_{x-1}^{x} P_1(x,y)k''(y-x)dy + \int_{-M}^{x-1} P_2(x,y)k''(y-x)dy,
\]

(2.23)

where \( \gamma_2 = k(-1) - k'(-1) \). It follows from (2.22) and (2.23) that

\[
J_1 = \int_{-\infty}^{x} [f(y+x) - f(x) - y \frac{df(x)}{dx} 1_{D(y)}] \nu'(y)dy + \gamma_2 f'(x).
\]

(2.24)

In the same way it can be proved that

\[
J_2 = \int_{x}^{\infty} [f(y+x) - f(x) - y \frac{df(x)}{dx} 1_{D(y)}] \nu'(y)dy + \gamma_3 f'(x),
\]

(2.25)

where \( \gamma_3 = -k(1) + k'(1) \). The relation (2.20) follows directly from (2.24) and (2.25). Here \( \Gamma = \gamma_2 + \gamma_3 \). The lemma is proved.

**Remark 2.1.** The operator \( L_0f = \frac{d}{dx}f \) can be represented in form (2.1), (2.2), where

\[
S_0f = \int_{-\infty}^{\infty} p_0(y-x)f(y)dy, \quad p_0(x) = \frac{1}{2}\text{sign}(x).
\]

(2.26)

From Lemmas 2.1, 2.2 and Remark 2.1 we deduce the following assertion.

**Theorem 2.1.** Let the conditions of Lemma 2.1 or Lemma 2.2 be fulfilled. Then the corresponding operator \( L \) has a convolution type form (2.1),(2.2).

**Proposition 2.1.** The generator \( L \) of the Levy process \( X_t \) admits the convolution type representation (2.1),(2.2) if there exist such \( C > 0 \) and \( 0 < \alpha < 2 \), \( \alpha \neq 1 \) that

\[
\nu'(y) \leq C|y|^{-\alpha-1},
\]

(2.28)
Proof. The function $\nu(y)$ has the form

$$
\nu(y) = \int_y^{-\infty} \nu'(t) dt + 1_{y<0} - \int_y^\infty \nu'(t) dt + 1_{y>0}.
$$

(2.29)

First we shall consider the case, when $1 < \alpha < 2$, and introduce the function

$$
k_0(y) = \int_y^{-\infty} (y - t) \nu'(t) dt + 1_{y<0} - \int_y^\infty (y - t) \nu'(t) dt + 1_{y>0}.
$$

(2.30)

We obtain the relation

$$
k(y) = k_0(y) + (\gamma - \Gamma)p_0(y), \quad 1 < \alpha < 2,
$$

(2.31)

where $k_0(y)$ and $p_0(y)$ are defined by (2.27) and (2.30) respectively. The constant $\Gamma$ is defined by the relation:

$$
\Gamma = k_0(-1) - k'_0(-1) - k_0(-1) + k'_0(1), \quad 1 < \alpha < 2
$$

(2.32)

It follows from (2.28)-(2.30) that the conditions of Lemma 2.2 are fulfilled. Hence the proposition is true when $1 < \alpha < 2$. Let us consider the case when $0 < \alpha < 1$. As in the previous case the function $\nu(x)$ is defined by relation (2.29). We introduce the functions

$$
k_0(y) = \int_y^{-\infty} \nu'(t) dt + 1_{y<0} + \int_y^0 \nu'(t) dt, \quad y < 0,
$$

(2.33)

$$
k_0(y) = -\int_y^{\infty} \nu'(t) dt + 1_{y>0} - \int_y^0 \nu'(t) dt, \quad y > 0,
$$

(2.34)

and

$$
k(y) = k_0(y) + \gamma p_0(y) \quad 0 < \alpha < 1.
$$

(2.35)

In view of (2.28) and (2.33),(2.34) the conditions of Lemma 2.1 are fulfilled. Hence the proposition is proved.

Corollary 2.1. If condition (2.28) is fulfilled then

$$
k_0(y) \geq 0, \quad -\infty < y < \infty, \quad 1 < \alpha < 2,
$$

(2.36)

$$
k_0(y) \leq 0, \quad -\infty < y < \infty, \quad 0 < \alpha < 1.
$$

(2.37)
Let us consider the important case when \( \alpha = 1 \).

**Proposition 2.2.** The generator \( L \) of the Levy process \( X_t \) admits the convolution type representation (2.1), (2.2) if there exist such \( C > 0 \) and \( m > 0 \) that

\[
\nu'(y) \leq C|y|^{-2}e^{-m|y|}.
\]

(2.38)

**Proof.** Using formulas (2.29)-(2.32) we see that the conditions of Lemma 2.2 are fulfilled. The proposition is proved.

**Example 2.1.** The stable processes.

For the stable processes we have \( A = 0 \), \( \gamma = \gamma \) and

\[
\nu'(y) = |y|^{-\alpha-1}(C_11_{y<0} + C_21_{y>0}),
\]

where \( C_1 > 0 \) \( C_2 > 0 \). Hence the function \( \nu(y) \) has the form

\[
\nu(y) = \frac{1}{\alpha}|y|^{-\alpha}(C_11_{y<0} - C_21_{y>0}).
\]

(2.39)

Let us introduce the functions

\[
k_0(y) = \frac{1}{\alpha(\alpha - 1)}|y|^{1-\alpha}(C_11_{y<0} + C_21_{y>0}),
\]

(2.41)

where \( 0 < \alpha < 2 \), \( \alpha \neq 1 \). When \( \alpha = 1 \) we have

\[
k_0(y) = -\log|y| (C_11_{y<0} + C_21_{y>0}).
\]

(2.42)

It means that the conditions of Theorem 2.1 are fulfilled. Hence the generator \( L \) for the stable processes admits the convolution type representation (2.1)(2.2).

**Proposition 2.3.** The kernel \( k(y) \) of the operator \( S \) in representation (2.1) for the stable processes has form (2.31), when \( 1 \leq \alpha < 2 \), and has form (2.35) when \( 0 \leq \alpha < 1 \).

**Example 2.2.** The variance damped Levy processes.

For the variance damped Levy processes we have \( A = 0 \), \( \gamma = \gamma \) and

\[
\nu'(y) = C_1e^{-\lambda_1|y|}|y|^{-\alpha-1}1_{y<0} + C_2e^{-\lambda_2|y|}|y|^{-\alpha-1}1_{y>0},
\]

(2.43)

where \( C_1 > 0 \) \( C_2 > 0 \), \( \lambda_1 > 0 \), \( \lambda_2 > 0 \) \( 0 < \alpha < 2 \). It follows from (2.43) that the conditions of Proposition 2.1 are fulfilled when \( \alpha \neq 1 \). If \( \alpha = 1 \) the conditions of Proposition 2.2 are fulfilled. Hence the generator \( L \) for the
variance damped Levy processes admits the convolution type representation (2.1),(2.2) and the kernel $k(y)$ is defined by formulas (2.30),(2.31), when $1 \leq \alpha < 2$, and by formula (2.35) when $0 < \alpha < 1$.

**Example 2.3.** The variance Gamma process.

For the variance Gamma process we have $A = 0$, $\gamma = \overline{\gamma}$ and

$$
\nu'(y) = C_1 e^{-G|y|}|y|^{-\alpha} \chi_{y<0} + C_2 e^{-M|y|}|y|^{-\alpha} \chi_{y>0},
$$

(2.44)

where $C_1 > 0$, $C_2 > 0$, $G > 0$, $M > 0$. It follows from (2.44) that the conditions of Proposition 2.2 are fulfilled and the generator $L$ of variance Gamma process admits the convolution type representation (2.1),(2.2). The kernel $k(y)$ is defined by relations (2.33) and (2.34).

**Example 2.4.** The normal inverse Gaussian process.

In the case of the normal inverse Gaussian process we have $A = 0$, $\gamma = \overline{\gamma}$ and

$$
\nu'(y) = C e^{\beta y} K_1(|y|)|y|^{-\alpha}, \quad C > 0, \quad -1 \leq \beta \leq 1,
$$

(2.45)

where $K_\lambda(x)$ denotes the modified Bessel function of the third kind with index the $\lambda$. Using equalities

$$
|K_1(|x|)| \leq M e^{-|x|/|x|}, \quad M > 0, \quad 0 < x_0 \leq |x|,
$$

(2.46)

$$
|K_1(|x|)| \leq M, \quad 0 \leq |x| \leq x_0
$$

(2.47)

we see that the conditions of Proposition 2.2 are fulfilled. Hence the corresponding generator $L$ admits the convolution type representation (2.1),(2.2) and the kernel $k(y)$ is defined by relations (2.33) and (2.34).

**Example 2.5.** The Meixner process.

For the Meixner process we have

$$
\nu'(y) = C \exp\beta x \frac{\sinh \pi x}{x},
$$

(2.48)

where $C > 0$, $-\pi < \beta < \pi$. The conditions of Proposition 2.2 are fulfilled. Hence the corresponding generator $L$ admits the convolution type representation (2.1),(2.2) and the kernel $k(y)$ is defined by relations (2.33),(2.34).

**Remark 2.1.** Examples 2.1-2.5 are used in the finance problems [24].

**Example 2.6.** Compound Poisson process.

We consider the case when $A = 0$, $\gamma = 0$ and

$$
M = \int_{-\infty}^{\infty} \nu'(y) dy < \infty.
$$

(2.49)
Using formulas (2.1) and (2.2) we deduce that the corresponding generator $L$ has the following convolution form

$$Lf = -Mf(x) + \int_{-\infty}^{\infty} \nu'(y-x)f(y)dy. \quad (2.50)$$

3 Potential

The operator

$$Qf = \int_0^\infty (P_t f) dt \quad (3.1)$$

is called potential of the semigroup $P_t$. The generator $L$ and the potential $Q$ are (in general) unbounded operators. Therefore the operators $L$ and $Q$ are defined not in the whole space $L^2(-\infty, \infty)$ but only in the subsets $D_L$ and $D_Q$ respectively. We use the following property of the potential $Q$ (see[23]).

**Proposition 3.1.** If $f = Qg$, $\ (g \in D_Q)$ then $f \in D_L$ and

$$-Lf = g. \quad (3.2)$$

**Example 3.1.** Compound Poisson process.

Let the generator $L$ has form (2.50) where

$$M = \int_{-\infty}^{\infty} \nu'(x) dx < \infty, \quad \int_{-\infty}^{\infty} [\nu'(x)]^2 dx < \infty. \quad (3.3)$$

We introduce the functions

$$K(u) = -\frac{1}{M \sqrt{2\pi}} \int_{-\infty}^{\infty} \nu'(x)e^{-iux} dx, \quad (3.4)$$

$$N(u) = \frac{K(u)}{1 - \sqrt{2\pi}K(u)}. \quad (3.5)$$

Let us note that

$$|K(u)| < \frac{1}{\sqrt{2\pi}}, \quad u \neq 0; \quad K(0) = -\frac{1}{\sqrt{2\pi}}. \quad (3.6)$$

It means that $N(u) \in L^2(-\infty, \infty)$. Hence the function

$$n(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} N(u)e^{-iux} du \quad (3.7)$$
belongs to \(L^2(-\infty, \infty)\) as well. It follows from (2.50),(3.2) and (3.7) that the corresponding potential \(Q\) has the form (see [23], Ch.11)

\[
Qf = \frac{1}{M} [f(x) + \int_{-\infty}^{\infty} f(y)n(x - y)dy].
\]

(3.8)

**Proposition 3.2.** Let conditions (3.3) be fulfilled. Then the operators \(L\) and \(Q\) are bounded in the space \(L^2(-\infty, \infty)\).

Now we shall give an example when the kernel \(n(x)\) can be written in an explicit form.

**Example 3.2.** We consider the case when

\[
\nu'(x) = e^{-|x|}, \quad -\infty < x < \infty.
\]

(3.9)

In this case \(M = 2\) and the operator \(L\) takes the form

\[
Lf = -2f(x) + \int_{-\infty}^{\infty} f(y)e^{-|x-y|}dy.
\]

(3.10)

Formulas (3.4)-(3.7) imply that

\[
Qf = \frac{1}{2}f(x) - \frac{1}{4\sqrt{2}} \int_{-\infty}^{\infty} f(y)e^{-|x-y|\sqrt{2}}dy.
\]

(3.11)

### 4 Truncated generators and quasi-potentials

Let us denote by \(\Delta\) the set of segments \([a_k, b_k]\) such that

\[a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n, \quad 1 \leq k \leq n.\]

By \(C_\Delta\) we denote the set of functions \(g(x)\) on \(L^2(\Delta)\) such that

\[g(a_k) = g(b_k) = g'(a_k) = g'(b_k) = 0, \quad 1 \leq k \leq n, \quad g''(x) \in L^p(\Delta), \quad p > 1.\]

(4.1)

We introduce the operator \(P_\Delta\) by relation \(P_\Delta f(x) = f(x)\) if \(x \in \Delta\) and \(P_\Delta f(x) = 0\) if \(x \notin \Delta\).

**Definition 4.1.** The operator

\[
L_\Delta = P_\Delta LP_\Delta
\]

(4.2)

is called a *truncated generator.*

**Definition 4.2.** The operator \(B\) with the definition domain dense in \(L^p(\Delta)\)
is called a quasi-potential if the functions \( f =Bg \) belong to definition domain of \( L_{\Delta} \) and
\[
- L_{\Delta} f = g. \tag{4.3}
\]

It follows from (4.3) that
\[
- P_{\Delta} L f = g, \quad (f = Bg). \tag{4.4}
\]

**Remark 4.1.** In a number of cases (see the next section) we need relation (4.4). In these cases we can use the quasi-potential \( B \), which is often simpler than the corresponding potential \( Q \).

**Remark 4.2.** The operators of type (4.2) are investigated in book ([22],Ch.2).

From relation (4.3) we deduce that
\[
Bg \neq 0, \quad if \quad g \neq 0, \quad g \in L^p(\Delta). \tag{4.5}
\]

**Definition 4.3.** We call the operator \( B \) a regular if the following conditions are fulfilled.

1). The operator \( B \) is compact and has the form
\[
Bf = \int_\Delta \Phi(x,y)f(y)dy, \quad f(y) \in L^p(\Delta), \quad p \geq 1, \tag{4.6}
\]

where the function \( \Phi(x,y) \) can have a discontinuity only when \( x = y \).

2). There exists a function \( \phi(x) \) such that
\[
|\Phi(x,y)| \leq \phi(x-y), \tag{4.7}
\]
\[
\int_{-R}^R \phi(x)dx < \infty \quad if \quad 0 < R < \infty. \tag{4.8}
\]

3).
\[
\Phi(x,y) \geq 0, \quad x,y \in \Delta, \tag{4.9}
\]
\[
\Phi(a_k, y) = \Phi(b_k, y) = 0, \quad 1 \leq k \leq n. \tag{4.10}
\]

4). Relation (4.5) is valid.

**Remark 4.3.** In view of condition (4.7) the regular operator \( B \) is bounded in the spaces \( L^p(\Delta) \), \( 1 \leq p \leq \infty \) (see[22],p.24).

**Remark 4.4.** If the quasi-potential \( B \) is regular , then the corresponding truncated generator \( L_{\Delta} \) has a discrete spectrum.

Further we prove that for a broad class of Levy processes the corresponding
quasi-potentials $B$ are regular.

**Example 4.1.** We consider the case when

$$\phi(x) = M|x|^{-\nu}, \quad 0 < \nu < 1. \quad (4.11)$$

**Proposition 4.1.** Let condition (4.11) be true and let the corresponding regular operator $B$ have an eigenfunction $f(x)$ with an eigenvalue $\lambda \neq 0$. Then the function $f(x)$ is continuous.

**Proof.** According to Definition 4.3 there exists an integer $N(\nu)$ such that the kernel $\Phi_N(x,t)$ of the operator

$$B^N f = \int_\Delta \Phi_M(x,y)f(y)dy, \quad f(y) \in L^p(\Delta) \quad (4.12)$$

is continuous. Hence the function $f(x)$ is continuous. The proposition is proved.

## 5 The Probability of the Levy process remaining within the given domain

In many theoretical and applied problems it is important to estimate the quantity

$$p(t,\Delta) = P\{X_\tau \in \Delta\}, \quad 0 \leq \tau \leq t, \quad (5.1)$$

i.e. the probability that a sample of the process $X_\tau$ remains inside $\Delta$ for $0 \leq \tau \leq t$ (ruin problem).

To derive the integro-differential equations corresponding to Levy processes we use the argumentation by Kac [11] and our own argumentation (see [20]-[22]). Now we get rid of the requirement for the process to be stable.

Let us consider the Levy process $X_t$ with the continuous density (see (1.1):

$$\rho(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixz} \mu(z,t) dz, \quad t > 0 \quad (5.2)$$

Now we introduce the sequence of functions

$$Q_{n+1}(x,t) = \int_0^t \int_{-\infty}^{\infty} Q_0(x-\xi,t-\tau)V(\xi)Q_n(\xi,\tau) d\xi d\tau, \quad (5.3)$$
where the function $V(x)$ is defined by relations $V(x) = 1$ when $x \notin \Delta$ and $V(x) = 0$ when $x \in \Delta$. We use the notation

$$Q_0(x, t) = \rho(x, t).$$

(5.4)

For Levy processes the following relation

$$Q_0(x, t) = \int_{-\infty}^{\infty} Q_0(x - \xi, t - \tau)Q_0(\xi, \tau)d\xi$$

(5.5)

is true. Using (5.3) and (5.5) we have

$$0 \leq Q_n(x, t) \leq t^n Q_0(x, t)/n!.$$  

(5.6)

Hence the series

$$Q(x, t, u) = \sum_{n=0}^{\infty} (-1)^n u^n Q_n(x, t)$$

(5.7)

converges. The probabilistic meaning of $Q(x, t, u)$ is defined by the relation (see [12],Ch.4)

$$E\{\exp[-u \int_0^t V(X_{\tau})d\tau], c_1 < X_t < c_2\} = \int_{c_1}^{c_2} Q(x, t, u)dx.$$  

(5.8)

The inequality $V(x) \geq 0$ and relation (5.8) imply that the function $Q(x, t, u)$ monotonically decreases with respect to the variable “$u$” and the formulas

$$0 \leq Q(x, t, u) \leq Q(x, t, 0) = Q_0(x, t) = \rho(x, t)$$

(5.9)

are true. In view of (5.2) and (5.9) the Laplace transform

$$\psi(x, s, u) = \int_0^{\infty} e^{-st}Q(x, t, u)dt, \quad s > 0.$$  

(5.10)

has the meaning. According to (5.3) the function $Q(x, t, u)$ is the solution of the equation

$$Q(x, t, u) + u \int_0^t \int_{-\infty}^{\infty} \rho(x - \xi, t - \tau)V(\xi)Q(\xi, \tau, u)d\xi d\tau = \rho(x, t)$$

(5.11)

Taking from both parts of (5.11) the Laplace transform and bearing in mind (5.10) we obtain

$$\psi(x, s, u) + u \int_{-\infty}^{\infty} V(\xi)R(x - \xi, s)\psi(\xi, s, u)d\xi = R(x, s),$$

(5.12)
where
\[ R(x, s) = \int_{0}^{\infty} e^{-st} \rho(x, t) dt. \] (5.13)

Multiplying both parts of relation (5.12) by \( \exp(ixp) \) and integrating them with respect to \( x \) \( (-\infty < x < \infty) \) we have
\[ \int_{-\infty}^{\infty} \psi(x, s, u) \exp[s + \lambda(p) + uV(x)] dx = 1. \] (5.14)

Here we use relations (1.1), (5.2) and (5.13). Now we introduce the function
\[ h(p) = \frac{1}{2\pi} \int_{\Delta} \exp f(x) dx, \] (5.15)

where the function \( f(x) \) belongs to \( C_\Delta \). Multiplying both parts of (5.14) by \( h(p) \) and integrating them with respect to \( p \) \( (-\infty < p < \infty) \) we deduce the equality
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x, s, u) \exp[s + \lambda(p)] h(p) dx dp = f(0). \] (5.16)

We have used the relations
\[ V(x)f(x) = 0, \quad -\infty < x < \infty, \] (5.17)
\[ \frac{1}{2\pi} \lim_{N \to \infty} \int_{-N}^{N} \int_{\Delta} \exp f(x) dx dp = f(0), \quad N \to \infty. \] (5.18)

Since the function \( Q(x, t, u) \) monotonically decreases with respect to "\( u \)" this is also true for the function \( \psi(x, s, u) \) according to (5.10). Hence there exists the limit
\[ \psi(x, s) = \lim \psi(x, s, u), \quad u \to \infty, \] (5.19)
where
\[ \psi(x, s) = 0, \quad x \notin \Delta. \] (5.20)

The probabilistic meaning of \( \psi(x, s) \) follows from the equality
\[ \int_{0}^{\infty} e^{-st} p(t, \Delta) dt = \int_{\Delta} \psi(x, s) dx. \] (5.21)

Using the properties of the Fourier transform and conditions (5.19) , (5.20) we deduce from (5.16) the following assertion.
Theorem 5.1. Let the considered Levy process have the continuous density. Then the relation
\[ ((sI - L_\Delta)f, \psi(x, s))_\Delta = f(0) \] is true.

Remark 5.1 For symmetric stable processes relation (5.22) was deduced by M.Kac [11].

Remark 5.2 As it is known , the stable processes, the variance damped Levy processes , the variance gamma processes, the normal inverse Gaussian process, the Meixner process have continuous densities (see([24],[31])).

Remark 5.3. So we have obtained the formula (5.21) for Laplace transform of \( p(t, \Delta) \) in terms of \( \psi(x, s) \). The double Laplace transform of \( p(t, \Delta) \) was obtained by G.Baxter and M.D.Donsker [3] for the case when \( \Delta = (-\infty, a] \).

We express the important function \( \psi(x, s) \) with the help of the quasi-potential \( B \).

Theorem 5.2. Let the considered Levy process have the continuous density and let the quasi-potential \( B \) be regular . Then in the space \( L^p(\Delta) \) \( (p > 1) \) there is one and only one function
\[ \psi(x, s) = (I + sB^*)^{-1}\Phi(0, x), \quad 0 \leq s < s_0, \] which satisfies relation (5.22).

Proof. In view of (4.4) we have
\[ -BL_\Delta f = f; \quad f \in C_\Delta. \] Relations (5.23) and (5.24) imply that
\[ ((sI - L_\Delta)f, \psi(x, s))_\Delta = -((I + sB)L_\Delta f, \psi)_\Delta = -(L_\Delta f, \Phi(0, x))_\Delta. \] Since \( \Phi(0, x) = B^*\delta(x) \), \( (\delta(x) \) is the Dirac function) then according to (5.23) and (5.25) relation (5.22) is true.

Let us suppose that in \( L(\Delta) \) there is another function \( \psi_1(x, s) \) satisfying (5.22). Then the equality
\[ ((sI - L_\Delta)f, \phi(x, s))_\Delta = 0, \quad \phi = \psi - \psi_1 \] is valid. We write relation (5.26) in the form
\[ (L_\Delta f, (I + sB^*)\phi)_\Delta = 0. \]
Due to (4.4) the range of $L_{\Delta}$ is dense in $L^p(\Delta)$. Hence in view of (5.27) we have $\phi = 0$. The theorem is proved.

The analytical apparatus for the construction and investigation of the function $\psi(x, s)$ is based on relation (5.22) and properties of the quasi-potential $B$. In the following three sections we shall investigate the properties of the operator $B$.

6 Non-negativity of the kernel $\Phi(x, y)$

In this section we deduce the following important property of the kernel $\Phi(x, y)$.

**Proposition 6.1.** Let the density $\rho(x, t)$ of Levy process $X_t$ be continuous ($t > 0$) and let the corresponding quasi-potential $B$ satisfy conditions (4.6) – (4.8) of Definition 4.3. Then the kernel $\Phi(x, y)$ is non-negative i.e.

$$
\Phi(x, y) \geq 0.
$$

\textbf{Proof.} In view of (5.9) and (5.10) we have $\psi(x, s, u) \geq 0$. Relation (5.19) implies that $\psi(x, s) \geq 0$. Now it follows from (5.23) that

$$
\Phi(0, x) = \psi(x, 0) \geq 0.
$$

Let us consider the domains $\Delta_1$ and $\Delta_2$ which are connected by relation $\Delta_2 = \Delta_1 + \delta$. We denote the corresponding truncated generators by $L_{\Delta_1}$ and $L_{\Delta_2}$, we denote the corresponding quasi-potentials by $B_1$ and $B_2$ and the corresponding kernels by $\Phi_1(x, y)$ and $\Phi_2(x, y)$. We introduce the unitary operator

$$
Uf = f(x - \delta),
$$

which maps the space $L^2(\Delta_2)$ onto $L^2(\Delta_1)$. At the beginning we suppose that the conditions of Theorem 2.1 are fulfilled. Using formulas (2.1) and (2.2) we deduce that

$$
L_{\Delta_2} = U^{-1}L_{\Delta_1}U.
$$

Hence the equality

$$
B_2 = U^{-1}B_1U
$$

is valid. The last equality can be written in the terms of the kernels

$$
\Phi_2(x, y) = \Phi_1(x + \delta, y + \delta).
$$
According to (6.2) and (6.6) we have

$$\Phi_1(\delta, y + \delta) \geq 0.$$  \hspace{1cm} (6.7)

As $\delta$ is an arbitrary real number, relation (6.1) follows directly from (6.6). We remark that an arbitrary generator $L$ can be approximated by the operators of form (2.1) (see [23], Ch.2). Hence the proposition is proved.

In view of (4.1), (4.5) and relation $Bf \in C_\Delta$ the following statement is true.

**Proposition 6.2.** Let the quasi-potential $B$ satisfy the conditions of Proposition 6.2. Then the equalities

$$\Phi(a_k, y) = \Phi(b_k, y) = 0 \quad 1 \leq k \leq n$$  \hspace{1cm} (6.8)

are valid.

### 7 Sectorial operators

1. We introduce the following notions.

**Definition 7.1.** The bounded operator $B$ in the space $L^2(\Delta)$ is called **sectorial** if

$$(Bf, f) \neq 0, \quad f \neq 0$$  \hspace{1cm} (7.1)

and

$$-\frac{\pi}{2} \beta \leq \arg(Bf, f) \leq \frac{\pi}{2} \beta, \quad 0 < \beta \leq 1.$$  \hspace{1cm} (7.2)

It is easy to see that the following assertions are true.

**Proposition 7.1.** Let the operator $B$ be sectorial. Then the operator $(I + sB)^{-1}$ is bounded when $s \geq 0$.

**Proposition 7.2.** Let the conditions of Theorem 5.2 be fulfilled. If the operator $B$ is sectorial, then formula (5.23) is valid for all $s \geq 0$.

In the present section we deduce the conditions under which the quasi-potential $B$ is sectorial. Let us consider the case when

$$\int_x^\infty yv'(y)dy < \infty, \quad (x > 0),$$  \hspace{1cm} (7.3)

$$\int_{-\infty}^x |y|v'(y)dy < \infty, \quad (x < 0).$$  \hspace{1cm} (7.4)
The corresponding kernel \( k(x) \) of the operator \( S \) (see (2.2) has the form

\[
k(x) = \int_x^{\infty} (y - x)\nu'(y)dy < \infty, \quad (x > 0),
\]

\[
k(x) = \int_{-\infty}^x (x - y)\nu'(y)dy < \infty, \quad (x < 0).
\]

Using the inequality \( \nu'(y) \geq 0 \) we obtain the following statement.

**Proposition 7.3.** Let conditions (7.3) and (7.4) be fulfilled. Then the kernel \( k(x) \) is monotone on the half-axis \((-\infty, 0)\) and on the half-axis \((0, \infty)\).

We shall use the following Pringsheim’s result.

**Theorem 7.1.** (see [25], Ch.1) Let \( f(t) \) be non-increasing function over \((0, \infty)\) and integrable on any finite interval \((0, \ell)\). If \( f(t) \to 0 \) when \( t \to \infty \), then for any positive \( x \) we have

\[
\frac{1}{2}[f(x + 0) + f(x - 0)] = \frac{2}{\pi} \int_0^\infty \cos xu \left[ \int_0^\infty f(t) \cos t u dt \right] du,
\]

\[
\frac{1}{2}[f(x + 0) + f(x - 0)] = \frac{2}{\pi} \int_0^\infty \sin xu \left[ \int_0^\infty f(t) \sin t u dt \right] du.
\]

It follows from (7.3)-(7.6) that

\[
k(x) \to 0 \quad \text{and} \quad k'(x) \to 0, \quad \text{when} \quad x \to \pm \infty.
\]

We suppose in addition that

\[
xk(x) \to 0 \quad \text{and} \quad x^2k'(x) \to 0, \quad \text{when} \quad x \to \pm 0.
\]

Using the integration by parts we deduce the assertion.

**Proposition 7.4.** Let conditions (7.3), (7.4) and (7.9), (7.10) be fulfilled. Then the relation

\[
\int_{-\infty}^{\infty} k(t) \cos xt dt = \int_{-\infty}^{\infty} \nu'(t) \frac{1 - \cos xt}{x^2} dt
\]

is true.

Relation (7.11) implies that

\[
\int_{-\infty}^{\infty} k(t) \cos xt dt > 0.
\]
The kernel \( k(x) \) of the operator \( S \) admits the representation

\[
k(x) = \int_{-\infty}^{\infty} m(t)e^{ixt} dt.
\]

(7.13)

In view of (7.12) we have

\[
\text{Re}[m(u)] > 0.
\]

(7.14)

Due to (7.13) and (7.14) the relation

\[
(Sf, f) = \int_{-\infty}^{\infty} m(u)|\int_{\Delta} f(t)e^{iut} dt|^2 du
\]

(7.15)

is valid. Hence we have

\[-\frac{\pi}{2} \leq \text{arg}(Sf, f) \leq \frac{\pi}{2}, \quad f(t) \in L^2(\Delta).
\]

(7.16)

**Proposition 7.5.** Let conditions (7.3), (7.4) and (7.9), (7.10) be fulfilled. Then the corresponding operator \( B \) is sectorial.

**Proof.** Let the function \( g(x) \) satisfies conditions (4.1). Then the relation

\[
(-Lg, g) = (Sg', g')
\]

(7.17)

holds. Equalities (4.3) and (7.17) imply that

\[
(f, Bf) = (Sg', g'), \quad g = Bf.
\]

(7.18)

Inequality (7.1) follows from relations (7.14) and (7.18). Relations (7.16) and (7.18) imply the proposition.

**Remark 7.1.** The variance damped processes (Example 2.2.) the normal inverse Gaussian process (Example 2.4.), the Meixner process (Example 2.5.) satisfy the conditions of Proposition 7.5. Hence the corresponding operators \( B \) are sectorial.

2. Now we introduce the notion of the strongly sectorial operators.

**Definition 7.2.** The sectorial operator \( B \) is called a strongly sectorial if for some \( \beta < 1 \) relation (7.2) is valid.

**Proposition 7.6.** Let the following conditions be fulfilled.

1). Relations (7.3), (7.4) and (7.9), (7.10) are valid.

2). For some \( m > 0 \) the inequality

\[
\frac{m}{|x|^2} \leq \nu'(x), \quad |x| \leq 1
\]

(7.19)
is true.

\[ \int_{-\infty}^{\infty} k(t) dt < \infty. \quad (7.20) \]

Then the corresponding operator \( B \) is strongly sectorial.

**Proof.** As it is known (see [25], Ch.1) the inequality

\[ |\int_{-\infty}^{\infty} k(t) \sin(x t) dt| \leq \frac{M}{|x|}, \quad M > 0, \quad |t| \geq 1 \quad (7.21) \]

is valid. From formulas (7.11) and (7.26) we conclude that

\[ \int_{-\infty}^{\infty} k(t) \cos(x t) dt \geq \int_{-\frac{1}{x}}^{\frac{1}{x}} v(t) \frac{1 - \cos xt}{x^2} dt \geq \frac{N}{|x|}, \quad N > 0, \quad |x| \geq 1. \quad (7.22) \]

It follows from (7.28) and (7.29) that

\[ -\frac{\pi}{2} \beta \leq \text{arg}(Sf,f) \leq \frac{\pi}{2} \beta, \quad 0 < \beta < 1. \quad (7.23) \]

Hence according to (7.18) relation (7.25) is valid. The proposition is proved.

**Remark 7.2.** The variance damped processes (Example 2.2, \( \alpha \geq 1 \)), the normal inverse Gaussian process (Example 2.4.), the Meixner process (Example 2.5.) satisfy the conditions of Proposition 7.6. Hence the corresponding operators \( B \) are strongly sectorial.

**Proposition 7.7.** Let conditions (7.3), (7.4) and (7.9), (7.10) be fulfilled. If the operator \( S \) has the form

\[ Sf = Af + \int_{\Delta} k(x-t)f(t)dt, \quad A > 0. \quad (7.24) \]

Then the corresponding operator \( B \) is strongly sectorial.

**Proof.** It is easy to see that for some \( \beta < 1 \) relation (7.23) is true. According to relation (7.18) the corresponding operator \( B \) is strongly sectorial.

**8 Quasi-potential \( B \), structure and properties**

Let us begin with the symmetric segment \( \Delta = [-c,c] \).

**Theorem 8.1.** (see [22], p.140) Let the following conditions be fulfilled
1. There exist the functions $N_k(x) \in L^p(-c,c), \quad p > 1$ which satisfy the equations
\[ SN_k = x^{k-1}, \quad k = 1, 2. \tag{8.1} \]

2. 
\[ r = \int_{-c}^{c} N_1(x) dx \neq 0 \tag{8.2} \]

Then the corresponding operator $B$ has the form
\[ Bf = \int_{-c}^{c} \Phi(x, y, c)f(y) dy \tag{8.3} \]

where
\[ \Phi(x, y, c) = \frac{1}{2} \int_{x+y}^{2c-|x-y|} q((s + x - y)/2, (s - x + y)/2] ds, \tag{8.4} \]

\[ q(x, y) = [N_1(-y)N_2(x) - N_2(-y)N_1(x)]/r. \tag{8.5} \]

It follows from (8.4) and (8.5) that
\[ \Phi(\pm c, y) = \Phi(x, \pm c) = 0. \tag{8.6} \]

Here we use the following relation
\[ q((s + x - y)/2, (s - x + y)/2] = \]
\[ [N_1((x - y - s)/2)N_2((s + x - y)/2) - N_2((x - y - s)/2)N_2((s + x - y)/2)]/r. \tag{8.7} \]

Thus
\[ q((s + x - y)/2, (s - x + y)/2] = -q((-s + x - y)/2, (-s - x + y)/2]. \tag{8.8} \]

From formulas (8.4) and (8.5) we deduce the following statement.

**Proposition 8.1.** Let the conditions of Theorem 8.1 be fulfilled. There exists a function $\phi(x)$ such that
\[ |\Phi(x, y, c)| \leq \phi(x - y), \tag{8.9} \]
\[ \int_{-R}^{R} \phi(x) dx < \infty \quad \text{if} \quad 0 < R < \infty. \tag{8.10} \]
Proof. Relation (8.4) can be written in the form
\[ \Phi(x, y, c) = \int_{c}^{c+(x-y-|x-y|)/2} q(t, t - x + y) dt. \] (8.11)

By relations
\[ N_k(x) = 0, \quad x \notin [-c, c], \quad k = 1, 2 \] (8.12)
we extend the functions \( N_k(x) \) from the segment \([-c, c]\) to the segment \([-2c, 2c]\). It follows from (8.11) and (8.12) that inequality (8.9) is valid, if
\[ \phi(x) = \int_{-c}^{c} |N_1(t)N_2(t - x)| + |N_2(t)N_1(t - x)| dt/|r|. \] (8.13)

Equality (8.13) imply that \( \phi(x) \in L^p([-2c, 2c]) \). The proposition is proved.

It follows from Proposition 8.1 that the operator \( B \) is bounded in all the spaces \( L^p(-c, c) \), \( p \geq 1 \). We shall prove that the operator \( B \) is compact.

**Proposition 8.2.** Let the conditions of Theorem 8.1 be fulfilled. Then the operator \( B \) is compact in all the spaces \( L^p(-c, c), \quad p \geq 1 \).

**Proof.** Let us consider the operator \( B^* \) in the space \( L^q(-c, c) \), \( 1/p + 1/q = 1 \).
Using relation (8.3) we have
\[ B^* f_n = \int_{-c}^{c} \Phi(y, x, c)f_n(y)dy \] (8.14)
where the functions \( f_n(x) \rightarrow 0 \) in the weak sense. Relation (8.14) can be represented in the following form
\[ B^* f_n = \int_{-c}^{c} f_n(y) \int_{y}^{c+(y-x-|x-y|)/2} q(t, t - y + x) dt \, dy. \] (8.15)

By interchanging the order of the integration in (8.15) we see that \( ||B^* f_n|| \rightarrow 0 \), i.e. the operator \( B^* \) is compact. Hence the operator \( B \) is compact too. The proposition is proved.

Using formulas (8.5) and (8.11) we obtain the assertion.

**Proposition 8.3** Let the conditions of Theorem 8.1 be fulfilled. If the functions \( N_1(x) \) and \( N_2(x) \) can have a discontinuity only when \( x = \pm c \) then the function \( \Phi(x, y, c) \) can have a discontinuity only when \( x = y \).

**Corollary 8.1.** Let the conditions of Proposition 8.3 be fulfilled. Then the eigenvectors of the corresponding operator \( B \) are continuous.
We use the following assertion (see[22],p.73).
Proposition 8.4. Let the following conditions be fulfilled.

1). The kernel \( k(x) \) of the operator \( S \) has the form
\[
k(x) = \log \frac{A}{2|x|} + h(x), \quad -2c \leq x \leq 2c, \tag{8.16}
\]
where \( A > 0, \ A \neq c \).

2).
\[
\int_{-2c}^{2c} |h'(u)|^q (2c - |u|) du < \infty, \quad q > 2. \tag{8.17}
\]

3. The equation
\[
Sf = \int_{-c}^{c} \left[ \log \frac{A}{2|\xi - t|} + h(\xi - t) \right] f(t) dt = 0 \tag{8.18}
\]
has only the trivial solution in \( L^p(-c,c) \), where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Then the equation
\[
Sf = g, \quad g'(x) \in L^p(-c,c) \tag{8.19}
\]
has one and only one solution in \( L^p(-c,c) \).

Corollary 8.2. Let the conditions of Proposition 8.4 be valid. Then there exist the functions \( N_1(x) \) and \( N_2(x) \) which satisfy the equations
\[
SN_k = x^{k-1}, \quad N_k(x) \in L^p(-c,c), \quad k = 1, 2. \tag{8.20}
\]
The functions \( N_1(x) \) and \( N_2(x) \) can have a discontinuity only when \( x = \pm c \).

Remark 8.1. If conditions (7.13) and (7.14) are fulfilled then according to (7.15) we have \( (f, Sf) \neq 0 \), when \( ||f|| \neq 0 \). In particular the relation \( (N_1, SN_1) = r \neq 0 \) is true.

Remark 8.2. In view of (6.4) and (6.5) Proposition 8.1 is valid not only in the case of the symmetric segment \([-c,c]\) but in the general case \([-a,b]\) too.

9 Long time behavior

1. In order to investigate the asymptotic behavior of \( p(t, \Delta) \) when \( t \to \infty \), we use the non-negativity of the kernel \( \Phi(x,y) \). We apply the following Krein-Rutman theorem (see [13], section 6).

Theorem 9.1. If a linear compact operator \( B \) leaving invariant a cone
$K$, has a point of the spectrum different from zero, then it has a positive eigenvalue $\lambda_1$ not less in modulus than any other eigenvalues $\lambda_k$, $(k > 1)$. To this eigenvalue $\lambda_1$ corresponds at least one eigenvector $g_1 \in K, (Bg_1 = \lambda_1 g_1)$ of the operator $B$ and at least one eigenvector $h_1 \in K^*, (B^* h_1 = \lambda_1 h_1)$ of the operator $B^*$. We remark that in our case the cone $K$ consists of non-negative functions $f(x) \in L^p(\Delta)$. Hence we have

$$g_1(x) \geq 0, \quad h_1(x) \geq 0. \quad (9.1)$$

We introduce the following normalizing condition

$$(g_1, h_1) = \int_{\Delta} g_1(x) h_1(x) dx = 1,$$  \quad (9.2)

Let the interval $\Delta_1$ and the point $x_0$ be such that

$$x_0 \in \Delta_1 \in \Delta. \quad (9.3)$$

Together with quantity $p(t, \Delta)$ we consider the expression

$$p(x_0, \Delta_1, t, \Delta) = P((X_t \in \Delta_1) \cap (X_t \in \Delta), \quad (9.4)$$

where $x_0 = X_0$. If the relations $x_0 = 0, \Delta_1 = \Delta$ are true, then $p(x_0, \Delta_1, t, \Delta) = p(t, \Delta)$. In this section we investigate the asymptotic behavior of $p(x_0, \Delta_1, t, \Delta)$ and $p(t, \Delta)$ when $t \to \infty$.

**Theorem 9.2.** Let the considered Lévy process have the continuous density and let the corresponding quasi-potential $B$ be regular and strongly sectorial. And let the operator $B$ have a point of the spectrum different from zero. Then the asymptotic equality

$$p(t, \Delta) = e^{-t/\lambda_1} [q(t) + o(1)], \quad t \to +\infty \quad (9.5)$$

is true. The function $q(t)$ has the form

$$q(t) = c_1 + \sum_{k=2}^{m} c_k e^{i\nu_k} \geq 0, \quad (9.6)$$

where $\nu_k$ are real.

**Proof.** The spectrum $(\lambda_k, \quad k > 1)$ of the operator $B$ is situated in the sector

$$-\frac{\pi}{2} \beta \leq \arg z \leq \frac{\pi}{2} \beta, \quad 0 \leq \beta < 1, \quad |z| \leq \lambda_1. \quad (9.7)$$
We introduce the domain $D_\epsilon$:

$$\frac{-\pi}{2}(\beta + \epsilon) \leq \arg z \leq \frac{\pi}{2}(\beta + \epsilon), \quad |z - (1/2)\lambda_1| < (1/2)(\lambda_1 - r),$$

where $0 < \epsilon < 1 - \beta$, $\ r < \lambda_1$. If $z$ belongs to the domain $D_\epsilon$ then the relation

$$\text{Re}(1/z) > 1/\lambda_1$$

holds. As the operator $B$ is compact only a finite number of eigenvalues $\lambda_k, \ 1 < k \leq m$ of this operator does not belong to the domain $D_\epsilon$. We denote the boundary of domain $D_\epsilon$ by $\Gamma_\epsilon$. Without loss of generality we may assume that the points of spectrum $\lambda_k \neq 0$ do not belong to $\Gamma_\epsilon$. Taking into account the equality

$$(\Phi(0, x), g_1(x)) = \lambda_1 g_1(0),$$

we deduce from formulas (5.21) and (5.23) the relation

$$p(t, \Delta) = \sum_{k=1}^{m} \sum_{j=0}^{n_k} e^{-t/\lambda_k} t^j c_{k,j} + J,$$

where $n_k$ is the index of the eigenvalue $\lambda_k$,

$$J = -\frac{1}{2i\pi} \int \frac{1}{z} e^{-t/z} ((B^* - zI)^{-1} \Phi(0, x), 1) dz.$$

We note that

$$n_1 = 1.$$  

Indeed, if $n_1 > 1$ then there exists such a function $f_1$ that

$$B f_1 = \lambda_1 f_1 + g_1.$$  

In this case the relations

$$(B f_1, h_1) = \lambda_1 (f_1, h_1) + (g_1, h_1) = \lambda_1 (f_1, h_1)$$

are true. Hence $(g_1, h_1) = 0$. The last relation contradicts condition (9.2). It proves equality (9.13).

Relation (8.9) implies that

$$\Phi(0, x) \in L^p(\Delta).$$
We denote by $W(B)$ the numerical range of $B$. The closure of the convex hull of $W(B)$ is situated in the sector (9.7). Hence the estimation
\[
||(B^* - zI)^{-1}||_p \leq M/|z|, \quad z \in \Gamma_\varepsilon
\] (9.17)
is true (see [26] for the Hilbert case $p = 2$ and [16],[28] for the Banach space $p \geq 1$). By $||B||_p$ we denote the norm of the operator $B$ in the space $L^p(\Delta)$.

It follows from estimation (9.17) that the integral $J$ exists.

Among the numbers $\lambda_k$ we choose for which $\text{Re}(1/\lambda_k), \quad (1 \leq k \leq m)$ has the smallest value $\delta$. Among the obtained numbers we choose $\mu_k, \quad (1 \leq k \leq \ell)$ the indexes $n_k$ of which have the largest value $n$. We deduce from (9.10)-(9.12) that
\[
p(t, \Delta) = e^{-t\delta} t^n \left[ \sum_{k=1}^{\ell} e^{-t/\mu_k} c_k + o(1) \right], \quad t \to \infty.
\] (9.18)

We note that the function
\[
Q(t) = \sum_{k=1}^{\ell} e^{it\text{Im}(\mu_k^{-1})} c_k
\] (9.19)
is almost periodic (see [14]). Hence in view of (9.18) and the inequality $p(t, \Delta) > 0, \quad t \geq 0$ the following relation
\[
Q(t) \geq 0, \quad -\infty < t < \infty
\] (9.20)
is valid.

First we assume that at least one of the inequalities
\[
\delta < \lambda_1^{-1}, \quad n > 1
\] (9.21)
is true. Using (9.21) and the inequality
\[
\lambda_1 \geq \lambda_k, \quad k = 2, 3, ...
\] (9.22)
we have
\[
\text{Im}\mu_j^{-1} \neq 0, \quad 1 \leq j \leq \ell.
\] (9.23)

It follows from (9.19) that
\[
c_j = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Q(t)e^{-it(\text{Im}\mu_j^{-1})} dt, \quad T \to \infty.
\] (9.24)
In view of (9.20) we obtain the relations

$$|c_j| \leq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Q(t) dt = 0, \quad T \to \infty,$$

(9.25)
i.e. $c_j = 0$, $1 \leq j \leq \ell$. This means that relations (9.21) are not true. Hence the equalities

$$\delta = \lambda_1^{-1}, \quad n = 1$$

(9.26)
are true. From (9.18) and (9.19) we get the asymptotic equality

$$p(t, \Delta) = e^{-t/\lambda_1} [q(t) + o(1)] \quad t \to \infty,$$

(9.27)
where the function $q(t)$ is defined by relation (9.6) and

$$c_k = g_k(0) \int_{\Delta} h_k(x) dx, \quad \nu_k = \text{Im}(\mu^{-1}).$$

(9.28)
Here $g_k(x)$ are the eigenfunctions of the operator $B$ corresponding to the eigenvalues $\lambda_k$, and $h_k(x)$ are the eigenfunctions of the operator $B^\ast$ corresponding to the eigenvalues $\overline{\lambda_k}$. The following conditions are fulfilled

$$(g_k, h_k) = \int_{\Delta} \overline{g_k(x)} h_k(x) dx = 1,$$

(9.29)
$$(g_k, h_\ell) = \int_{\Delta} \overline{g_k(x)} h_\ell(x) dx = 0, \quad k \neq \ell.$$  

(9.30)
Using the almost periodicity of the function $q(t)$ we deduce from (9.27) the inequality

$$q(t) \geq 0.$$  

(9.31)
The theorem is proved.

**Corollary 9.1.** Let the conditions of Theorem 9.2 be fulfilled. Then all the eigenvalues $\lambda_j$ of $B$ belong to the disk

$$|z - (1/2)\lambda_1| \leq (1/2).$$

(9.32)
All the eigenvalues $\lambda_j$ of $B$ which belong to the boundary of disc (9.32) have the indexes $n_j = 1$.

**Remark 9.1.** The exponential decay of the transition probability $P_t(x, B)$ was proved by P. Tuominen and R.L.Tweedie [29]. Theorem 9.2. gives the
exponential decay of $p(t, \Delta)$. These two results are independent. Using formula (9.11) we obtain the following assertion.

**Corollary 9.2.** Let the considered Levy process have the continuous density and let the corresponding quasi-potential $B$ be regular and strongly sectorial. And let the operator $B$ have no points of the spectrum different from zero. Then the equality

$$\lim_{t \to +\infty} [p(t, \Delta)e^{t/\lambda}] = 0,$$

(9.33)

is true for any $\lambda > 0$.

2. Now we find the conditions under which the operator $B$ has a point of the spectrum different from zero. We represent the corresponding operator $B$ in the form $B = B_1 + iB_2$ where the operators $B_1$ and $B_2$ are self-adjoint. We assume that $B_1 \in \Sigma_p$, i.e.

$$\sum_{n=1}^{\infty} |s_n|^{-p} < \infty,$$

(9.34)

where $s_n$ are eigenvalues of the operator $B_1$ and $p > 1$. As operator $B$ is sectorial, then

$$B_1 \geq 0.$$  

(9.35)

**Theorem 9.3.** Let the considered Levy process have the continuous density and let the corresponding quasi-potential $B$ be regular and strongly sectorial. If $B_1 \in \Sigma_p, \ p > 1$ and

$$1/p > \beta,$$

(9.36)

then the operator $B$ has a point of the spectrum different from zero.

**Proof.** It follows from estimation (9.17) that

$$||(I - zB)^{-1}||_{p} \leq M, \quad |\arg z| \geq \beta + \epsilon.$$  

(9.37)

Let us suppose that the formulated assertion is not true, i.e. the operator $B$ has no points of the spectrum different from zero. We set

$$A(r, B) = \sup_{0 \leq \theta \leq 2\pi} ||(I - re^{i\theta}B)^{-1}||.$$

(9.38)

It follows (see [9]) from condition $B_1 \in \Sigma_p$ that $B_2 \in \Sigma_p$ and

$$\log A(r, B) = O(r^p).$$

(9.39)

According Phragmen-Lindelof theorem and relations (9.36)-(9.39) we have

$$||(I - zB)^{-1}|| \leq M.$$  

(9.40)
The last relation is possible only when \( B = 0 \). But in our case \( B \neq 0 \). The obtained contradiction proves the theorem.

**Proposition 9.1.** Let the kernel of \( \Phi(x, y) \) of the corresponding operator \( B \) be bounded. If this operator \( B \) is strongly sectorial, then it has a point of the spectrum different from zero.

**Proof.** As in Theorem 9.3 we suppose that the operator \( B \) has no points of the spectrum different from zero. Using the boundedness of the kernel \( \Phi(x, y) \) we obtain the inequality

\[
TrB_1 < \infty.
\] (9.41)

It follows from relations (9.35) and (9.41) that (see the triangular model of M. Livshits [15]) \( \rho = 1 \). Since \( \beta < 1 \) all the conditions of Theorem 9.3 are fulfilled. Hence the proposition is proved.

3. Now we shall consider the important case when

\[
\text{rank} \lambda_1 = 1. \quad (9.42)
\]

**Theorem 9.4.** Let the conditions of theorem 9.2 be fulfilled. In the case (9.42) the following relation

\[
p(t, \Delta) = e^{-t/\lambda_1}[c_1 + o(1)], \quad t \to + \infty
\] (9.43)

is true.

**Proof.** In view of (9.31) we have

\[
\lim \frac{1}{T} \int_0^T q(t)dt \geq \lim \frac{1}{T} \int_0^T q(t)e^{-it(\text{Im} \beta_j)}dt, \quad T \to \infty,
\] (9.44)

i.e.

\[
g_1(0) \int_\Delta h_1(x)dx \geq |g_j(0)| \int_\Delta h_j(x)dx. \quad (9.45)
\]

In the same way we can prove that

\[
g_1(x_0) \int_{\Delta_1} h_1(x)dx \geq |g_j(x_0)| \int_{\Delta_1} h_j(x)dx,
\] (9.46)

where

\[
x_0 \in \Delta_1 \in \Delta. \quad (9.47)
\]
It follows from (9.46) that
\[ g_1(x_0)h_1(x) \geq |g_j(x_0)h_j(x)|. \]  
(9.48)

We introduce the normalization condition
\[ g_1(x_0) = g_j(x_0). \]  
(9.49)

Due to (9.46) and (9.48) the inequalities
\[ \int_{\Delta_1} h_1(x)dx \geq |\int_{\Delta_1} h_j(x)dx|. \]  
(9.50)

\[ h_1(x) \geq |h_j(x)| \]  
(9.51)

are true. The equality sign in (9.50) and (9.51) can be only if
\[ h_j(x) = |h_j(x)|e^{ia}. \]  
(9.52)

It is possible only in the case when \( j = 1 \). Hence there exists such a point \( x_1 \) that
\[ h_1(x_1) > |h_j(x_1)| \]  
(9.53)

Thus we have
\[ 1 = \int_{\Delta_1} g_1(x)h_1(x)dx > \int_{\Delta_1} g_j(x)h_j(x)dx = 1, \]  
(9.54)

where \( x_1 \in \Delta_1 \). The received contradiction (9.54) means that \( j = 1 \). Now the assertion of the theorem follows directly from (9.5).

**Corollary 9.3.** Let conditions of Theorem 9.2 be fulfilled. If \( \text{rank} \lambda_1 = 1 \) and \( x_0 \in \Delta_1 \in \Delta \) then the asymptotic equality
\[ p(x_0, \Delta_1, t, \Delta) = e^{-t/\lambda_1}g_1(x_0) \int_{\Delta_1} h_1(x)dx[1 + o(1)], \quad t \to +\infty \]  
(9.55)

is true.

The following Krein-Rutman theorem [13] gives the sufficient conditions when relation (9.42) is valid.

**Theorem 9.5.** Suppose that the non-negative kernel \( \Phi(x,y) \) satisfies the condition
\[ \int_{\Delta} \int_{\Delta} |\Phi(x,y)|^2dxdy < \infty \]  
(9.56)
and has the following property: for each $\epsilon > 0$ there exists an integer $N = N(\epsilon)$ such that the kernel $\Phi_N(x, y)$ of the operator $B^N$ takes the value zero on a set of points of measure not greater than $\epsilon$. Then

$$\text{rank} \lambda_1 = 1; \quad \lambda_1 > \lambda_k, \quad k = 2, 3, \ldots$$

(9.57)

It is easy to see that the following assertion is valid.

**Proposition 9.2.** Let the inequality

$$\Phi(x, y) > 0,$$  

(9.58)

be true, when $x \neq a_k, \ x \neq b_k, \ y \neq a_k, \ y \neq b_k$.

Then

$$g_1(x) > 0$$  

(9.59)

, when $x \neq a_k, \ x \neq b_k$.

4. Let us consider separately the case when the operator $B$ is regular and

$$k(x) = k(-x).$$  

(9.60)

The corresponding operator $S$ is self-adjoint. Hence the operator $B$ is self-adjoint and strongly sectorial. In this case equality (9.11) can be written in the form

$$p(t, \Delta) = \sum_{k=1}^{\infty} e^{-t/\lambda_k} g_k(0) \int_{\Delta} g_k(x) dx.$$  

(9.61)

### 10 Stable Processes, Main Notions

1. Let $X_1, X_2, \ldots$ be mutually independent random variables with the same law of distribution $F(x)$. The distribution $F(x)$ is called strictly stable if the random variable

$$X = (X_1 + X_2 + \ldots + X_n)/n^{1/\alpha}$$  

(10.1)

is also distributed according to the law $F(x)$. The number $\alpha$ ($0 < \alpha \leq 2$) is called a characteristic exponent of the distribution. The homogeneous process $X(\tau)$ ($X(0) = 0$) with independent increments is called a stable process if

$$E[\exp (i\xi X(\tau))] = \exp \{-\tau|\xi|^\alpha[1 - i\beta(\text{sign}\xi)(\tan \frac{\pi\alpha}{2})]\},$$  

(10.2)
where $0 < \alpha < 2, \alpha \neq 1, -1 \leq \beta \leq 1, \quad \tau > 0$. When $\alpha = 1$ we have

$$E[\exp (i\xi X(\tau))] = \exp \{-\tau|\xi|[1 + \frac{2i\beta}{\pi}(\text{sign}\xi)(\log |\xi|)]\}, \quad (10.3)$$

where $-1 \leq \beta \leq 1, \quad \tau > 0$. The stable processes are a natural generalization of the Wiener processes. In many theoretical and applied problems it is important to estimate the value

$$p_{\alpha}(t, a) = P(\sup |X(\tau)| < a), \quad 0 \leq \tau \leq t. \quad (10.4)$$

For the stable processes Theorem 9.1. was proved before (see [19]-[22]). The value of $p_{\alpha}(t, a)$ decreases very quickly by the exponential law when $t \to \infty$. This fact prompted the idea to consider the case when the value of $a$ depends on $t$ and $a(t) \to \infty, \quad t \to \infty$. In this paper we deduce the conditions under which one of the following three cases is realized:

1) $\lim p_{\alpha}(t, a(t)) = 1, \quad t \to \infty.$
2) $\lim p_{\alpha}(t, a(t)) = 0, \quad t \to \infty.$
3) $\lim p_{\alpha}(t, a(t)) = p_{\infty}, \quad 0 < p_{\infty} \leq 1, \quad t \to \infty.$

We investigate the situation when $t \to 0$ too.

**Remark 10.1.** In the famous work by M.Kac [11] the connection of the theory of stable processes and the theory of integral equations was shown. M.Kac considered in detail only the case $\alpha = 1, \quad \beta = 0$. The case $0 < \alpha < 2, \quad \beta = 0$ was later studied by H.Widom [30]. As to the general case $0 < \alpha < 2, \quad -1 \leq \beta \leq 1$ it was investigated in our works [19]-[22]. In all the mentioned works the parameter $a$ was fixed. Further we consider the important case when $a$ depends on $t$ and $a(t) \to \infty, \quad t \to \infty.$

### 11 Stable Processes, Quasi-potential.

1. In this section we formulate some results from our paper [20] (see also [22], Ch.7). Here $\psi_{\alpha}(x, s, a)$ is defined by the relation

$$\psi_{\alpha}(x, s, a) = (I + sB_{\alpha}^{*})^{-1}\Phi_{\alpha}(0, x, a), \quad (11.1)$$

The quasi-potential $B_{\alpha}$ and its kernel $\Phi_{\alpha}(x, y, a)$ will be written later in the explicit form.

Further we consider the three cases.

**Case 1.** $0 < \alpha < 2, \quad \alpha \neq 1, \quad -1 < \beta < 1.$
Case 2. \(1 < \alpha < 2, \quad \beta = \pm 1\).

Case 3. \(\alpha = 1, \quad \beta = 0\).

Now we introduce the operators

\[
B_\alpha f = \int_{-a}^{a} \Phi_\alpha(x, y, a) f(y) \, dy
\]

acting in the space \(L^2(-a, a)\).

In case 1 the kernel \(\Phi_\alpha(x, y, a)\) has the following form (see \[20\], \[22\])

\[
\Phi_\alpha(x, y, a) = C_\alpha (2a)^{\mu - 1} \int_{|x-y|}^{|a^2-xy|} [z^2 - a^2(x - y)^2]^{-\rho} [z - a(x - y)]^{2\rho - \mu} \, dz,
\]

where the constants \(\mu, \rho,\) and \(C_\alpha\) are defined by the relations \(\mu = 2 - \alpha, \quad \sin \pi \rho = \frac{1 - \beta}{1 + \beta} \sin (\mu - \rho), \quad 0 < \mu - \rho < 1,\)

\[
C_\alpha = \frac{\sin \pi \rho}{(\sin \pi \alpha/2)(1 - \beta) \Gamma(1 - \rho) \Gamma(1 + \rho - \mu)}.
\]

Here \(\Gamma(z)\) is Euler’s gamma function. We remark that the constants \(\mu, \rho,\) and \(C_\alpha\) do not depend on parameter \(a\).

In case 2 when \(\beta = 1\) the relation \([20],[22]\)

\[
\Phi_\alpha(x, y, a) = \frac{(\cos \pi \alpha / 2)}{(2a)^{\alpha - 1} \Gamma(\alpha)} \{ [a(|x - y| + y - x)]^{\alpha - 1} - (a - x)^{\alpha - 1} (a + y)^{\alpha - 1} \}
\]

holds. In case 2 when \(\beta = -1\) we have \([20],[22]\)

\[
\Phi_\alpha(x, y, a) = \frac{(\cos \pi \alpha / 2)}{(2a)^{\alpha - 1} \Gamma(\alpha)} \{ [a(|x - y| + x - y)]^{\alpha - 1} - (a + x)^{\alpha - 1} (a - y)^{\alpha - 1} \}
\]

Finally, in case 3 according to M.Kac \[11\] the equality

\[
\Phi_1(x, y, a) = \frac{1}{4} \log \frac{a^2 - xy + \sqrt{(a^2 - x^2)(a^2 - y^2)}}{a^2 - xy - \sqrt{(a^2 - x^2)(a^2 - y^2)}}
\]

is valid.

The important assertion (see \[22\], Ch.7) follows from formulas (11.2)-(11.8):
Proposition 11.1 Let one of the following conditions be fulfilled:
I. \(0 < \alpha < 2, \quad \alpha \neq 1, \quad -1 < \beta < 1.\)
II. \(1 < \alpha < 2, \quad \beta = \pm 1.\)
III. \(\alpha = 1, \quad \beta = 0.\)
Then the corresponding operator \(B_\alpha\) is regular and strongly sectorial.

2. Let us introduce the denotation
\[
p_\alpha(t,-b,a) = P(-b < X(\tau) < a), \tag{11.9}
\]
where \(a > 0, \quad b > 0, \quad 0 \leq \tau \leq t.\) We consider in short the case when the parameter \(b\) is not necessary equal to \(a.\) As in case \((-a,a)\) we have the relation
\[
\int_0^\infty e^{-su}p_\alpha(u,-b,a)du = \int_{-b}^a \psi_\alpha(x,s,-b,a)dx. \tag{11.10}
\]
Here \(\psi_\alpha(x,s,-b,a)\) is defined by relation
\[
\psi_\alpha(x,s,-b,a) = (I + sB_\alpha^*)^{-1}\Phi_\alpha(0,x,-b,a), \tag{11.11}
\]
Now the operator \(B_\alpha\) has the form
\[
B_\alpha f = \int_{-b}^a \Phi_\alpha(x,y,-b,a)f(y)dy \tag{11.12}
\]
and acts in the space \(L^2(-b,a).\) The kernel \(\Phi_\alpha(x,y,-b,a)\) is connected with \(\Phi_\alpha(x,y,a)\) (see (11.7)) by the formula
\[
\Phi_\alpha(x,y,-b,a) = \Phi_\alpha(x + \frac{b-a}{2}, y + \frac{b-a}{2}, \frac{a+b}{2}). \tag{11.13}
\]
In this way we have reduced the non-symmetric case \((-b,a)\) to the symmetric one \((-\frac{a+b}{2}, \frac{a+b}{2}).\) Let us consider separately the case \(0 < \alpha < 2, \quad \beta = 0.\) In this case the operator \(B_\alpha\) is self-adjoint. We denote by \(\lambda_j, \quad (j = 1, 2, \ldots)\) the eigenvalues of \(B_\alpha\) and by \(g_j(x)\) the corresponding real normalized eigenfunctions. Then we can write the new formula (see [20]) for \(p_\alpha(t,-b,a)\) which is different from (9.10):
\[
p_\alpha(t,-b,a) = \sum_{j=1}^\infty g_j(0) \int_{-b}^a g_j(x)dx e^{-\mu_j t}, \tag{11.14}
\]
where \(\mu_j = 1/\lambda_j.\)
12 On sample functions behavior of stable processes

From the scaling property of the stable processes we deduce the relations

\[ p_\alpha(t, a) = p_\alpha\left(\frac{t}{a\alpha}, 1\right), \quad (12.1) \]

\[ \lambda_k(a, \alpha) = a\alpha \lambda_k(1, \alpha). \quad (12.2) \]

We introduce the notations

\[ \lambda_\alpha(1) = \lambda_\alpha, \quad p_\alpha(t, 1) = p_\alpha(t), \quad g_\alpha(x, 1) = g_\alpha(x), \quad h_\alpha(x, 1) = h_\alpha(x). \quad (12.3) \]

Using relations (12.1), (12.2) and notations (12.3) we can rewrite Theorem 9.1 in the following way.

**Theorem 12.1.** Let one of the following conditions be fulfilled:

I. \(0 < \alpha < 2\), \(\alpha \neq 1\), \(-1 < \beta < 1\).
II. \(1 < \alpha < 2\), \(\beta = \pm 1\).
III. \(\alpha = 1\), \(\beta = 0\).

Then the asymptotic equality holds

\[ p_\alpha(t, a) = e^{-t/[a\alpha\lambda_\alpha]}g_\alpha(0) \int_{-1}^{1} h_\alpha(x)dx[1 + o(1)], \quad t \to \infty. \quad (12.4) \]

**Proof.** The corresponding operator \(B_\alpha\) is regular and strongly sectorial (see Proposition 11.1). The stable processes have the continuous density (see [31]). So all conditions of Theorem 9.1 are fulfilled. It proves the theorem.

**Remark 12.1.** The operator \(B_\alpha\) is self-adjoint when \(\beta = 0\). In this case \(h_\alpha = g_\alpha\).

**Remark 12.2.** The value \(\lambda_\alpha\) characterizes how fast \(p_\alpha(t, a)\) converges to zero when \(t \to \infty\). The two-sided estimation for \(\lambda_\alpha\) when \(\beta = 0\) is given in [17] (see also [22], p.150).

3. Now we consider the case when the parameter \(a\) depends on \(t\). From Theorem 12.1 we deduce the assertions.

**Corollary 12.1.** Let one of conditions I-III of Theorem 12.1 be fulfilled and

\[ \frac{t}{a^n(t)} \to \infty, \quad t \to \infty. \quad (12.5) \]
Then the following equalities are true:

1) \( p_\alpha(t, a(t)) = e^{-t/|a(t)|\lambda_\alpha} g_\alpha(0) \int_{-1}^{1} h_\alpha(x) dx [1 + o(1)], \quad t \to \infty. \)  \( (12.6) \)

2) \( \lim_{t \to \infty} p_\alpha(t, a) = 0, \quad t \to \infty. \)  \( (12.7) \)

3) \( \lim_{t \to \infty} P[\sup |X(\tau)| > a(t)] = 1, \quad 0 \leq \tau \leq t, \quad t \to \infty. \)  \( (12.8) \)

**Corollary 12.2.** Let one of conditions I-III of Theorem 12.1 be fulfilled and 
\[
\frac{t}{|a(t)|^\alpha} \to 0, \quad t \to 0.
\]  \( (12.9) \)

Then the following equalities are true:

1) \( \lim_{t \to 0} p_\alpha(t, a(t)) = 1, \quad t \to 0. \)  \( (12.10) \)

2) \( \lim_{t \to 0} P[\sup |X(\tau)| > a(t)] = 0 \quad 0 \leq \tau \leq t, \quad t \to 0. \)  \( (12.11) \)

Corollary 12.2 follows from (12.1) and the relation
\[
\lim_{t \to 0} p_\alpha(t) = 1, \quad t \to 0.
\]  \( (12.12) \)

**Corollary 12.3.** Let one of conditions I-III of Theorem 12.1 be fulfilled and 
\[
\frac{t}{|a(t)|^\alpha} \to T, \quad 0 < T < \infty, \quad t \to \infty.
\]  \( (12.13) \)

Then the following equality is true:
\[
\lim_{t \to \infty} p_\alpha(t, a(t)) = p_\alpha(T), \quad t \to \infty.
\]  \( (12.14) \)

Corollary 12.3 follows from (12.1).

**13 Wiener Process**

1. We consider separately the important special case when \( \alpha = 2 \) (Wiener process). In this case the kernel \( \Phi_2(x, t, -b, a) \) of the operator \( B_2 \) coincides with the Green’s function (see [4], [11]) of the equation
\[
- \frac{1}{2} \frac{d^2 y}{dx^2} = f(x), \quad -b \leq x \leq a
\]  \( (13.1) \)
with the boundary conditions
\[ y(-b) = y(a) = 0, \quad b > 0, \quad a > 0. \] (13.2)

It is easy to see that
\[ \Phi_2(x, t, -b, a) = \frac{2}{a + b} \begin{cases} (t + b)(a - x) & -b \leq t \leq a \\ (a - t)(b + x) & -b \leq x \leq t \leq a \end{cases} \] (13.3)

Equality (12.1) is also true when \( \alpha = 2 \) and when \( b = a \), i.e.
\[ p_2(t, a) = p_2(t/a^2, 1). \] (13.4)

The eigenvalues of problem (13.1), (13.2) have the form
\[ \mu_n = \left( \frac{n\pi}{a + b} \right)^2 / 2, \quad n = 1, 2, 3... \] (13.5)

The corresponding normalized eigenfunctions are defined by the equality
\[ g_n(x) = \sqrt{\frac{2}{a + b}} \sin \left[ \left( \frac{n\pi}{a + b} \right)(x + b) \right]. \] (13.6)

Using formulas (13.5) and (13.6) we have
\[ p_2(t, -b, a) = \sum_{m=0}^{\infty} \frac{4}{(2m + 1)\pi} \sin \left( \frac{(2m + 1)b\pi}{a + b} \right) e^{-t(\frac{(2m+1)\pi}{a+b})^2/2} \] (13.7)

**Remark 13.1.** If \( b = a = 1 \) then relation (13.7) takes the form
\[ p_2(t) = \sum_{m=0}^{\infty} (-1)^m \frac{2}{\pi(m + 1/2)} e^{-t(\pi(m+1/2))^2/2}. \] (13.8)

Series (13.8) satisfies the conditions of Leibniz theorem. It means that \( p_2(t, a) \) can be calculated with a given precision when the parameters \( t \) and \( a \) are fixed.

From (13.4) and (13.8) we deduce that
\[ p_2(t, a) = \frac{4}{\pi} e^{-t\pi^2/8[a(t)]^2} (1 + o(1)), \] (13.9)
where $t/|a(t)|^2 \to \infty$.

**Proposition 13.1.** Theorem 12.1 and Corollaries 12.1 – 12.3 are true in the case when $\alpha = 2$ too.

**Remark 13.2.** From the probabilistic point of view it is easy to see that the function $p_2(t), (t > 0)$ is monotonic decreasing and

$$0 < p_2(t) \leq 1; \quad \lim_{t \to 0} p_2(t) = 1, \quad t \to 0. \quad (13.10)$$

2. Now we shall describe the behavior of $p(t, -b, a)$ when $b \to \infty$. To do it we consider

$$\frac{d}{dt}p_2(t, -b, a) = -\frac{2\pi}{(a + b)^2} \sum_{m=0}^{\infty} (2m + 1) \sin(2m+1) \frac{b\pi}{a + b} e^{-t(2m+1)^2/2}. \quad (13.11)$$

We use the following Poisson result (see[7]).

**Theorem 13.1.** If the function $F(x)$ satisfies the inequalities

$$\int_0^\infty |F(x)|dx < \infty, \quad \int_0^\infty |F'(x)|dx < \infty \quad (13.12)$$

then the equality

$$\sum_{m=0}^\infty F(m) = \frac{1}{2} F(0) + \int_0^\infty F(x)dx + 2 \sum_{m=1}^\infty \int_0^\infty F(x) \cos 2\pi mx dx \quad (13.13)$$

is true.

Thus in case (13.11) we have

$$F(x) = G(x) - G(2x), \quad (13.14)$$

where

$$G(x) = -\frac{2\pi}{(a + b)^2} x \sin \frac{xb\pi}{a + b} e^{-t(\frac{b}{a+b})^2/2}. \quad (13.15)$$

It is easy to see that conditions (13.12) are fulfilled and

$$F(0) = 0, \quad \int_0^\infty F(x)dx = \frac{1}{2} \int_0^\infty G(x)dx \quad (13.16)$$

Using (13.15) and (13.16) we deduce the equality

$$\int_0^\infty F(x)dx = -\frac{1}{\pi t} \int_0^\infty u e^{-u^2/2} \sin \frac{ua}{\sqrt{t}} du, \quad (13.17)$$

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where \( u = \frac{x}{a+b}\sqrt{t} \). Now we use the following relation from the sine transformation theory (see [25]).

\[
\int_0^\infty u e^{-u^2/2} \sin(xu) \, du = \sqrt{\frac{\pi}{2}} x e^{-x^2/2}.
\] (13.18)

In view of (13.17) and (13.18) the equality

\[
\int_0^\infty F(x) \, dx = -\frac{a}{\sqrt{2\pi}} t^{-3/2} e^{-a^2/2t}
\] (13.19)

is true. Now we calculate the integrals

\[
J_m = 2 \int_0^\infty G(2x) \cos 2\pi mx \, dx, \quad I_m = 2 \int_0^\infty G(x) \cos 2\pi mx \, dx
\] (13.20)

Using again formula (13.18) we have

\[
J_m = -\sqrt{2/\pi} t^{-3/2} [A_m e^{-A_m^2/2t} - B_m e^{-B_m^2/2t}],
\] (13.21)

where \( A_m = 2m(a+b) + a \), \( B_m = 2m(a+b) - a \). In the same way we found

\[
I_m = -\sqrt{1/2\pi} t^{-3/2} [C_m e^{-C_m^2/2t} - D_m e^{-D_m^2/2t}],
\] (13.22)

where \( C_m = m(a+b) + a \), \( D_m = m(a+b) - a \). From relation (13.7) and equality

\[
\left[ \int_{e/\sqrt{t}}^{d/\sqrt{t}} e^{-u^2/2} \, du \right]' = -\frac{1}{2} t^{-3/2} (de^{-d^2/2t} - ce^{-c^2/2t})
\] (13.23)

we obtain the following representation of \( p_2(t, -b, a) \):

\[
p_2(t, -b, a) = 1 - \sqrt{2/\pi} \int_0^\infty e^{-u^2/2t} \, du + q_a(t, -b, a),
\] (13.24)

where

\[
q_2(t, -b, a) = \sqrt{2/\pi} \sum_{m=1}^\infty [2 \int_{B_m/\sqrt{t}}^{A_m/\sqrt{t}} e^{-u^2/2t} \, du - \int_{D_m/\sqrt{t}}^{C_m/\sqrt{t}} e^{-u^2/2t} \, du]
\] (13.25)

So we have deduced two formulas (13.7) and (13.24) for \( p_2(t, -b, a) \). Formula (13.7) is useful when \( t \) is great and the parameters \( a \) and \( b \) are fixed.
Proposition 13.2. In the case of the Wiener process ($\alpha = 2$) the asymptotic equality

$$p_2(t, -b, a) = \frac{4}{\pi} \sin \frac{a\pi}{a + b} e^{-t(\pi)^2/[2(a+b)^2]}[1 + o(1)], \quad t \to \infty$$  \hspace{1cm} (13.26)

holds.

Formula (13.24) is useful when $b$ is great and parameters $a$ and $t$ are fixed.

Proposition 13.3. In the case of the Wiener process ($\alpha = 2$) the asymptotic equality $p_2(t, -b, a)$

$$= 1 - \sqrt{2/\pi} \int_{a/\sqrt{t}}^{\infty} e^{-u^2/2t} du - \sqrt{2/\pi} \int_{b/\sqrt{t}}^{(b+2a)/\sqrt{t}} e^{-u^2/2t} du[1 + o(1)], \quad (13.27)$$

where $b \to \infty$, is valid.

The well-known formula (see [8]) for the first hitting time

$$p_2(t, -\infty, a) = 1 - \sqrt{2/\pi} \int_{a/\sqrt{t}}^{\infty} e^{-u^2/2t} du$$  \hspace{1cm} (13.28)

follows directly from (13.27).

14 Iterated logarithm law, most visited sites and first hitting time

It is interesting to compare our results (Theorem 9.1, Corollaries 12.1-12.3 and Proposition 13.1 -13.3) with the well-known results mentioned in the title of the section.

1. We begin with the famous Khinchine theorem (see [4]) about the iterated logarithm law.

**Theorem 14.1.** Let $X(t)$ be stable process ($0 < \alpha < 2$). Then almost surely (a.s.) that

$$\limsup \frac{\sup |X(t)|}{(t \log t)^{1/\alpha} \log |\log t|^{(1/\alpha) + \epsilon}} = \begin{cases} 0 & \epsilon > 0 \text{ a.s.} \\ \infty & \epsilon = 0 \text{ a.s.} \end{cases} \quad (14.1)$$

We introduce the random process

$$U(t) = \sup |X(\tau)|, \quad 0 \leq \tau \leq t$$  \hspace{1cm} (14.2)
From Corollaries 12.1-12.3 and Proposition 13.1 we deduce the assertion.

**Theorem 14.2.** Let one of conditions I – III of Theorem 12.1 be fulfilled or let $\alpha = 2$ and

$$b(t) \to \infty, \ t \to \infty. \quad (14.3)$$

Then

$$b(t)U(t)/t^{1/\alpha} \to \infty \ (P), \ \ U(t)/[b(t)t^{1/\alpha}] \to 0 \ (P) \quad (14.4)$$

(It is denoted by symbol (P), that the convergence is in probability.)

In particular we have:

$$[(\log t)U(t)]/t^{1/\alpha} \to \infty \ (P), \ \ U(t)/[(\log t)t^{1/\alpha}] \to 0 \ (P), \quad (14.5)$$

when $\epsilon > 0$. We see that our approach and the classical one have some similar points (estimation of $|X(\tau)|$), but these approaches are essentially different. We consider the behavior of $|X(\tau)|$ on the interval $(0, t)$, and in the classical case $|X(\tau)|$ is considered on the interval $(t, \infty)$.

2. We denote by $V(t)$ the most visited site of stable process $X$ up to time $t$ (see [1]). We formulate the following result (see [1] and references therein).

Let $1 < \alpha < 2, \ \beta = 0, \ \gamma > 9/(\alpha - 1)$. Then the relation

$$\lim \frac{(\log t)^\gamma}{t^{1/\alpha}} |V(t)| = \infty, \ t \to \infty \ (a.s.) \quad (14.6)$$

is true. To this important result we add the following estimation.

**Theorem 14.3.** Let one of the conditions I – III of Theorem 12.1 be fulfilled or let $\alpha = 2$ and

$$b(t) \to \infty, \ t \to \infty. \quad (14.7)$$

Then

$$|V(t)(t)/[b(t)t^{1/\alpha}] \to 0 \ (P) \quad (14.8)$$

In particular we have when $\epsilon > 0$:

$$|V(t)|/[(\log t)t^{1/\alpha}] \to 0 \ (P) \quad (14.9)$$

The formulated theorem follows directly from the inequality $U(t) \geq |V(t)|$.

3. The first hitting time $T_\alpha$ is defined by the formula

$$T_\alpha = \inf (t \geq 0, X(t) \geq a). \quad (14.10)$$
It is obvious that
\[ P(T_a > t) = P[\sup X(\tau) < a], \quad 0 \leq \tau \leq t. \tag{14.11} \]
We have
\[ P(T_a > t) \geq P[-b < \sup X(\tau) < a] = p_a(t, -b, a), \quad 0 \leq \tau \leq t. \tag{14.12} \]
So our formulas for \( p(t, -b, a) \) estimate \( P(T_a > t) \) from below. It is easy to see that
\[ p(t, -b, a) \to P(T_a > t), \quad b \to +\infty. \tag{14.13} \]

**Remark 14.1.** Our results can be interpreted in terms of the first hitting time \( T_{[-b,a]} \) one of the barriers either \(-b\) or \(a\) (ruin problem). Namely, we have
\[ P(T_{[-b,a]} > t) = p(t, -b, a). \tag{14.14} \]
The distribution of the first hitting time for the Levy processes is an open problem.

**Remark 14.2.** Rogozin B.A. in his interesting work [18] established the law of the overshoot distribution for the stable processes when the existing interval is fixed.

**References**

1. **Bass R.F., Eisenbaum N. and Shi Z.**, The Most Visited Sites of Symmetric Stable Processes, Probability Theory and Related Fields, 116, (2000), 391-404.
2. **Bonsall F.F., Duncan J.**, Numerical Ranges,1-49, MAA Studies in Mathematics, v.21 (ed. Bartle R.G.) 1980.
3. **Baxter G., Donsker M.D.**, On the Distribution of the Supremum Functional for Processes with Stationary Independent Increments, Trans. Amer. Math. Soc. 8, 73-87,1957.
4. **Bertoin J.**, Levy Processes, University Press, Cambridge, 1996.
5. **Chuangyi Z.**, Almost Periodic Type Functions and Ergodicity, Beijing, New York, Kluwer, 2003.
6. **Chung K.L.**, Green, Brown and Probability , World Scientifc, 2002.
7. **Evgrafov M.A.** Asymptotic Estimates and Entire Functions , Gordon
8. **Feller W.**, An Introduction to Probability Theory and its Applications, J.Wiley and Sons, 1971.

9. **Gohberg I., Krein M.G.**, Introduction to the Theory of Non-selfadjoint Operators, Amer. Math. Soc. Providence, 1970.

10. **Ito K.**, On Stochastic Differential Equations, Memoirs Amer. Math. Soc. No.4, 1951.

11. **Kac M.**, On some Connections Between Probability Theory and Differential and Integral Equations, Proc.Sec.Berkeley Symp.Math.Stat. and Prob., Berkeley, 189-215, 1951.

12. **Kac M.**, Probability and Related Topics in Physical Sciences, Colorado, 1957.

13. **Krein M.G., Rutman M.A.**, Linear Operators Leaving Invariant a Cone in a Banach Space, Amer. Math. Soc. Translation, no.26, 1950.

14. **Levitan B.M.** Some Questions of the Theory of Almost Periodic Functions, Amer. Math. Soc.,Translation, 28, 1950.

15. **Livshits M.S.**, Operators, Oscillations, Waves, Open Systems, American Math. Society, Providence, 1973.

16. **Pietsch A.**, Eigenvalues and s-Numbers, Cambridge University Press, 1987.

17. **Pozin S.M., Sakhnovich L.A.**, Two-sided Estimation of the Smallest Eigenvalue of an Operator Characterizing Stable Processes, Theory Prob. Appl., 36, No.2,385-388, 1991.

18. **Rogozin B.A.**, The distribution of the first hit for stable and asymptotically stable walks on an interval, Theory Probab. Appl. 17, 332-338, 1972.

19. **Sakhnovich L.A.**, Abel Integral Equations in the theory of Stable Processes, Ukr.Math. Journ., 36:2, 193-197, 1984.

20. **Sakhnovich L.A.**, Integral Equations in the theory of Stable Processes, St.Peterburg Math.J., 4, No.4 1993,819-829.

21. **Sakhnovich L.A.**, The Principle of Imperceptibility of the Boundary in the Theory of Stable Processes, St.Peterburg Math.J., 6, No.6, 1995, 1219-1228.

22. **Sakhnovich L.A.**, Integral Equations with Difference Kernels, Operator Theory, v.84, 1996, Birkhauser.

23. **Sato K.**, Levy Processes and Infinitely Divisible Distributions, University Press, Cambridge, 1999.

24. **Schoutens W.**, Levy Processes in Finance, Wiley series in Probability and Statistics,2003
25. Titchmarsh E.C., Introduction to the Theory of Fourier Integrals, Oxford, 1937.
26. Stone M., Linear Transformation in Hilbert space, New York, 1932.
27. Thomas M., Barndorff O., (ed.), Levy Processes; Theory and Applications, Birkhauser, 2001.
28. Trefethen L.N., Embree M. Spectra and Pseudospectra, Princeton University Press, 2005.
29. Tuominen P., Tweedie R.L., Exponential Decay and Ergodicity of General Markov Processes and their Discrete Skeletons. Adv. in Appl. Probab. 11, 784-803, 1979.
30. Widom H., Stable Processes and Integral Equations, Trans. Amer. Math. Soc. 98, 430-449, 1961.
31. Zolotarev V.M., One-dimensional stable distribution, Providence, Amer. Math. Soc. 1986.