Eigenfunctions in a two-particle
Anderson tight binding model

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Abstract

We establish the phenomenon of Anderson localisation for a quantum two-particle system on a lattice \( \mathbb{Z}^d \) with short-range interaction and in presence of an IID external potential with sufficiently regular marginal distribution.
1 The two-particle tight binding model. 
Decay of Green’s functions and localisation

This paper focuses on a two-particle Anderson tight binding model on lattice $\mathbb{Z}^d$ with interaction. Our goal is three-fold. First, we establish a theorem deducing exponential localisation from a property of decay of Green’s functions in the two-particle model (Theorem 1.2 below). Second, we outline the so-called multi-scale analysis (MSA) scheme for the two-particle model. Finally, we perform the initial and the inductive steps of two-particle MSA and therefore establish the phenomenon of Anderson localisation in a two-particle model for a large disorder. See our main result, Theorem 1.1.

We consider the Hilbert space of the two-particle system $\ell_2(\mathbb{Z}^d \times \mathbb{Z}^d)$. The Hamiltonian $H^{(2)}(=H^{(2)}_{U,V,g}(\omega))$ is a lattice Schrödinger operator of the form $H^0 + U + g(V_1 + V_2)$, acting on functions $\phi \in \ell_2(\mathbb{Z}^d \times \mathbb{Z}^d)$, given by

$$H^{(2)}\phi(x) = H^0\phi(x) + (U(x) + gW(x;\omega))\phi(x) = \sum_{x \in 2^d \times 2^d, \|y-x\|=1} \phi(y) + [U(x) + gW(x;\omega)]\phi(x),$$

(1.1)

Here and below we use boldface letters such as $\mathbf{x}, \mathbf{y}, \mathbf{u}$ etc. for points in $\mathbb{Z}^d \times \mathbb{Z}^d$. Next, $x_j = (x_j^{(1)}, \ldots, x_j^{(d)})$ and $y_j = (y_j^{(1)}, \ldots, y_j^{(d)})$ stand for the coordinate vectors of particles in $\mathbb{Z}^d$, $j = 1, 2$, and $\| \cdot \|$ is the sup-norm: for $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^d \times \mathbb{R}^d$:

$$\|\mathbf{v}\| = \max_{j=1,2} \|v_j\|,$$

(1.2.1)

where

$$\|v\| = \max_{i=1,\ldots,d} |v^{(i)}|, \quad \text{for } v = (v^{(1)}, \ldots, v^{(d)}) \in \mathbb{R}^d.$$

(1.2.2)

We will consider the distance on $\mathbb{R}^d \times \mathbb{R}^d$ and $\mathbb{R}^d$ generated by the norm $\| \cdot \|$.

Throughout this paper, the random external potential $V(x;\omega)$, $x \in \mathbb{Z}^d$, is assumed to be real IID, with a common distribution function $F_V$ on $\mathbb{R}$. Of course, the random variables $W(x;\omega)$ form an array with dependencies (which is the main source of difficulties in spectral analysis of multi-particle quantum systems in random environment).

A popular assumption is that $F_V$ has a probability density function (PDF). The condition on $F_V$ guaranteeing the validity of all results presented
in this paper is as follows:

\( F_V \) has a PDF \( p_V \) which is bounded and has a compact support. \( \text{(1.3)} \)

This will allow us to use, in Section 3, some results on single-particle localisation proved in [A94] with the help of the fractional-moment method (FMM), an alternative of the MSA for single-particle models; see [AM, ASFH]. We note that a number of important facts proven or used here remain true under considerably weaker assumptions on \( F_V \). For example, Wegner-type bounds (3.5), (3.6) hold under the condition that for some \( \delta > 0 \) and all \( \epsilon > 0 \),

\[
\sup_{a \in \mathbb{R}} (F_V(a + \epsilon) - F_V(a)) \leq \epsilon^\delta;
\]

see [CS1]. Moreover, we stress that with the help of a technically more elaborate argument it is possible to obtain the main result of this paper (Theorem 1.1 below) under an assumption weaker than (1.3). We would also like to note that the IID property of \( \{V(x; \omega), x \in \mathbb{Z}^d\} \) can be relaxed. See our forthcoming manuscript [CS2].

Parameter \( g \in \mathbb{R} \) is traditionally called the coupling, or amplitude, constant.

The interaction potential \( U \) is assumed to satisfy the following properties.

(i) \( U \) is a bounded real function \( \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R} \) symmetric under the permutation of variables: \( U(x) = U(\sigma x) \), where

\[
\sigma x = (x_2, x_1) \quad \text{for} \quad x = (x_1, x_2), \quad x_1, x_2 \in \mathbb{Z}^d.
\]

(ii) \( U \) obeys

\[
U(x) = 0, \quad \text{if} \quad \|x_1 - x_2\| > r_0.
\]

Here \( r_0 \in [1, +\infty) \) is a given value (the interaction range).

Let \( \mathbb{P} \) stand for the joint probability distribution of RVs \( \{V(x; \omega), x \in \mathbb{Z}^d\} \).

The main assertion of this paper is

**Theorem 1.1** Consider the two-particle random Hamiltonian \( H^{(2)}(\omega) \) given by (1.1). Suppose that \( U \) satisfies conditions (1.4) and (1.5), and the random potential \( \{V(x; \omega), x \in \mathbb{Z}^d\} \) is IID, with a marginal distribution function \( F_V \) obeying (1.3). Then there exists \( g^* \in (0, +\infty) \) such that for any \( g \) with \( |g| \geq g^* \), with \( \mathbb{P} \)-probability one, the spectrum of operator \( H^{(2)}(\omega) \) is pure point. Furthermore, there exists a nonrandom constant \( m_+ = m_+(g) > 0 \)
(the effective mass) such that all eigenfunctions $\Psi_j(x;\omega)$ of $H^{(2)}(\omega)$ admit an exponential bound:

$$|\Psi_j(x;\omega)| \leq C_j(\omega) e^{-m_+||x||}. \quad (1.6)$$

The assertion of Theorem 1.1 can also be stated in the form where $\forall$ given $m_+ > 0$, $\exists g_* = g_*(m_+) \in (0, +\infty)$ such that $\forall g$ with $|g| \geq g_*$, the eigenfunctions $\Psi_j(x;\omega)$ of $H^{(2)}(\omega)$ admit exponential bound (1.6) with effective mass $m_+ \geq m_*$.\n
The conditions of Theorem 1.1 are assumed throughout the paper. As was said earlier, the proof of Theorem 1.1 uses mainly MSA, in its two-particle version. Most of the time we will work with finite-volume approximation operators $H^{(2)}_{\Lambda_L(u)}(= H^{(2)}_{\Lambda_L(u)}(\omega))$ given by

$$H^{(2)}_{\Lambda_L(u)} = H^{(2)}\big|_{\Lambda_L(u)} + \text{Dirichlet boundary conditions} \quad (1.7)$$

and acting on vectors $\phi \in \mathbb{C}^{\Lambda_L(u)}$ by

$$H^{(2)}_{\Lambda_L(u)} \phi(x) = \sum_{y \in \Lambda_L(u)} \phi(y) + [U(x) + gW(x;\omega)] \phi(x), \quad (1.8)$$

with $W(x)$ as in (1.1). Here and below, $\Lambda_L(u)$ stands for the ‘two-particle lattice box’ (a box, for short) of size $2L$ around $u = (u_1, u_2)$, where $u_j = (u_j^{(1)}, \ldots, u_j^{(d)}) \in \mathbb{Z}^d$:

$$\Lambda_L(u) = \left( \times_{j=1}^d \times_{i=1}^d [u_j^{(i)} - L, u_j^{(i)} + L]\right) \cap (\mathbb{Z}^d \times \mathbb{Z}^d) \quad (1.9)$$

Denoting by $|\Lambda_L(u)|$ the cardinality of $\Lambda_L(u)$, $H^{(2)}_{\Lambda_L(u)}$ is a Hermitian operator in the Hilbert space $\ell_2(\Lambda_L(u))$ of dimension $|\Lambda_L(u)|$.

In fact, the approximation (1.7) can be used for any finite subset $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}^d$ of cardinality $|\Lambda|$ producing Hermitian operator $H^{(2)}_{\Lambda}$ in $\ell_2(\Lambda)$.

Hamiltonian $H^{(2)}$ and its approximants $H^{(2)}_{\Lambda}$ admit the permutation symmetry. Namely, let $S$ be the unitary operator in $\ell_2(\mathbb{Z}^d \times \mathbb{Z}^d)$ induced by map $\sigma$:

$$S\phi(x) = \phi(\sigma x). \quad (1.10)$$
Then $S^{-1}H^{(2)}S = H^{(2)}$ and $S^{-1}H^{(2)}_{\Lambda}S = H^{(2)}_{\sigma\Lambda}$ (with natural embeddings $\mathbb{C}^\Lambda, \mathbb{C}^{\sigma\Lambda} \subset \ell_2(\mathbb{Z}^d \times \mathbb{Z}^d)$). This implies, in particular, that for any finite $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}^d$, the eigenvalues of operators $H^{(2)}_{\Lambda}$ and $H^{(2)}_{\sigma\Lambda}$ are identical. This fact is accounted for in the course of presentation.

Like its single-particle counterpart, the two-particle MSA scheme involves a number of technical parameters playing roles similar to those in the paper [DK]. In this and the following section we make use of some of these parameters (to begin with, see Theorem 1.2). More precisely, given a positive number $\alpha > 1$ and starting with $L_0 > 0$ large enough and $m_0 > 0$, define an increasing sequence $L_k$:

$$L_k = L_0^{\alpha k}, \quad k \geq 1,$$

and a decreasing positive sequence $m_k$ (depending on a positive number $\gamma$):

$$m_k = m_0 \prod_{j=1}^{k} \left(1 - \gamma L_k^{-1/2}\right), \quad k \geq 1.$$  

We will also use in Theorem 1.2 parameter $p$; our assumptions on $\alpha$, $\gamma$ and $p$ in this theorem will be that

$$p > \alpha d > 1, \quad \gamma \geq 40.$$  

Note that sequence $m_k$ is indeed positive, and the limit $\lim_{k \to \infty} m_k \geq m_0/2$ when $L_0$ is sufficiently large. We will also assume that $L_0 > r_0$.

The single-particle MSA scheme was used in [DK] to check, for IID potentials, decay properties of the Green’s functions (GFs). In this paper we adopt a similar strategy. For the two-particle model, the GFs in a box $\Lambda_L(u)$ are defined by:

$$G^{(2)}_{\Lambda_L(u)}(E; u, y) = \left\langle \left(H^{(2)}_{\Lambda_L(u)} - E\right)^{-1} \delta_x, \delta_y \right\rangle, \quad x, y \in \Lambda_L(u),$$

where $\delta_x(v) = 1(v = x)$ is the lattice Dirac delta-function (considered as a vector in $\mathbb{C}^{\Lambda_L(u)}$). Following [DK], we introduce

**Definition 1.1** Fix $E \in \mathbb{R}$ and $m > 0$. A two-particle box $\Lambda_L(u)$ is said to be $(E, m)$-non-singular (in short: $(E, m)$-NS) if the GFs $G^{(2)}_{\Lambda_L(u)}(E; u, u')$ defined by (1.14) for the Hamiltonian $H^{(2)}_{\Lambda_L(u)}$ from (1.8) satisfy

$$\max_{y \in \partial \Lambda_L(u)} \left|G^{(2)}_{\Lambda_L(u)}(E; u, y)\right| \leq e^{-mL}.$$  

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Otherwise, it is called \((E,m)\)-singular (or \((E,m)\)-S). Here \(\partial \Lambda_L(u)\) stands for the interior boundary (or briefly, the boundary) of box \(\Lambda_L(u)\): it is formed by points \(y \in \Lambda_L(u)\) such that there exists a site \(v \in (\mathbb{Z}^d \times \mathbb{Z}^d) \setminus \Lambda_L(u)\) with \(\|y - v\| = 1\). A similar concept can be introduced for any set \(\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}^d\), for which we use the same notation \(\partial \Lambda\).

The first step in the proof of Theorem 1.1 is Theorem 1.2 below. More precisely, Theorem 1.2 deduces exponential localisation from a postulated property of decay of two-particle GFs. The proof of Theorem 1.2 given in Section 2 follows that of its single-particle counterpart from earlier works (see Section 1 in [FMSS] and Theorem 2.3 from [DK]). Nevertheless, this theorem is an important part of our method (as in [FMSS] and [DK]); having established Theorem 1.2 one can attempt to prove two-particle localisation by analysing only GFs \(G^{(2)}_{\Lambda_L(u)}(E; u, y)\).

It is convenient to introduce the following

**Definition 1.2** A pair of two-particle boxes \(\Lambda_L(u), \Lambda_L(v)\) is called \(R\)-distant (\(R\)-D, for short) if

\[
\min \{\|u - v\|, \|\sigma u - v\|\} > 8R.
\]

(1.16)

Here, \(\sigma\) was defined in (1.4).

**Theorem 1.2** Let \(I \subseteq \mathbb{R}\) be an interval. Assume that for some \(m_0 > 0\) and \(L_0 > 1\), \(\lim_{k \to \infty} m_k \geq m_0/2\), and for any \(k \geq 0\) the following properties hold:

\[
\forall u, v \in \mathbb{Z}^d \times \mathbb{Z}^d \text{ such that } \Lambda_{L_k}(u) \text{ and } \Lambda_{L_k}(v) \text{ are } 8L_k\text{-D},
\quad \mathbb{P} \{ \forall E \in I : \Lambda_{L_k}(u) \text{ or } \Lambda_{L_k}(v) \text{ is } (m_k, E)\text{-NS} \} \geq 1 - L_k^{-2p}.
\]

(1.17)

Here \(L_k\) and \(m_k\) are defined in (1.11), (1.12), and \(\sigma\) by (1.4), with \(p, \alpha\) and \(\gamma\) satisfying (1.13). Then, with probability one, the spectrum of operator \(H^{(2)}(\omega)\) in \(I\) is pure point. Furthermore, there exists a constant \(m_+ \geq m_0/2\) such that all eigenfunctions \(\Psi_j(x; \omega)\) of \(H^{(2)}(\omega)\) with eigenvalues \(E_j(\omega) \in I\) decay exponentially fast at infinity, with the effective mass \(m_+\):

\[
|\Psi_j(x; \omega)| \leq C_j(\omega) e^{-m_+ \|x\|}.
\]

(1.18)
We stress that it is the property (DS\( k, I \)) encapsulating decay of the GFs which enables the two-particle MSA scheme to work. (Here and below, DS stands for ‘double singularity’). Clearly, Theorem 1.1 would be proved, once the validity of property (DS\( k, I \)) had been established for \( I = \mathbb{R} \) and for all \( k \geq 0 \). However, an important remark is that, to deduce Theorem 1.1, we actually need to check the conditions of Theorem 1.2 for an arbitrary interval \( I \subset \mathbb{R} \) of unit length (but of course with a fixed sequence of values \( m_k \) and \( L_k \) from (1.11), (1.12)). In fact, by covering the whole spectral line \( \mathbb{R} \) by a countable family of such intervals, we will get that the whole spectrum of \( H^{(2)} \) is pure point with \( \mathbb{P} \)-probability one, with a ‘universal’ effective mass \( m_+ > 0 \).

We will therefore focus on establishing property (DS\( k, I \)) for an arbitrary unit interval \( I \) and all \( k \geq 0 \); this is done in Sections 3–5 below. Nevertheless, many details of the presentation in Sections 3–5 do not require the assumption that the length of \( I \) is 1; we will choose appropriate conditions on an \textit{ad hoc} basis.

2 Proof of Theorem 1.2

It is well-known (see, e.g., [B, S]) that almost every energy \( E \) with respect to the spectral measure of \( H^{(2)} \) is a generalised eigenvalue of \( H^{(2)} \), i.e., solutions \( \Psi \) of the equation \( H^{(2)}\Psi = E\Psi \) are polynomially bounded. Therefore, it suffices to prove that the generalised eigenfunctions of \( H^{(2)} \) decay exponentially with \( \mathbb{P} \)-probability one.

Let \( E \in I \) be a generalised eigenvalue of Hamiltonian \( H^{(2)} \) from Eqn (1.1), and \( \Psi \) be a corresponding generalised eigenfunction. Following [DK], we will prove that

\[
\forall \tilde{\rho} \in (0, 1) : \limsup_{\|x\| \to \infty} \frac{\ln |\Psi(x; \omega)|}{\|x\|} \leq -\tilde{\rho} m, \tag{2.1}
\]

where \( m > 0 \) is the constant from the statement of Theorem 1.1.

Given \( u \in \mathbb{Z}^d \times \mathbb{Z}^d \) and an integer \( k = 0, 1, 2, \ldots, \), set

\[
R(u) = \|\sigma u - u\|, \quad b_k(u) = 1 + R(u)L_k^{-1}, \quad M_k(u) = \Lambda_{L_k}(u) \cup \sigma \Lambda_{L_k}(u); \tag{2.2}
\]

cf. (1.4). Note that \( \forall u \in \mathbb{Z}^d \times \mathbb{Z}^d, \)

\[
\forall k \geq 1: M_k(u) \subset \Lambda_{b_kL_k}(u), \quad \text{and} \quad \lim_{k \to \infty} b_k(u) = 1. \tag{2.3}
\]
Now set

\[ A_{k+1}(u) = \Lambda_{b_{k+1}L_{k+1}}(u) \setminus \Lambda_{b_kL_k}(u) \]  

(2.4)

and define the event

\[ \Omega_k(u) = \{ \exists E \in I \text{ and } x \in A_{k+1}(u) : \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(u) \text{ are } (m, E) - S \}. \]  

(2.5)

Observe that, owing to the definition of \( M_{L_{k+1}}(u) \) (see (2.2)), if \( x \in A_{k+1}(u) \), then

\[ \text{dist}(\Lambda_{L_k}(x), [\Lambda_{L_k}(u) \cup \sigma \Lambda_{L_k}(u)]) \geq 8L_k. \]

Thus, by the hypothesis of the theorem,

\[ \mathbb{P}\{ \Omega_k(u) \} \leq \frac{(2b_{k+1}L_{k+1} + 1)^{2d}}{L_k^{2p}} \leq \frac{(2b_{k+1} + 1)^{2d}}{L_k^{2p-2\alpha}}. \]  

(2.6)

Since \( p > \alpha \) and by virtue (2.3), the series

\[ \sum_{k=0}^{\infty} \mathbb{P}\{ \Omega_k(u) \} < \infty. \]  

(2.7)

Consider the event

\[ \Omega_{<\infty}(u) = \{ \forall u \in \mathbb{Z}^d \times \mathbb{Z}^d, \Omega_k(u) \text{ occurs finitely many times} \}. \]  

(2.8)

Then, owing to (2.7) and the Borel–Cantelli lemma, \( \mathbb{P}\{ \Omega_{<\infty} \} = 1 \). So, it suffices to pick a potential sample \( \{ V(y; \omega), y \in \mathbb{Z}^d \} \) with \( \omega \in \Omega_{<\infty} \) and prove exponential decay of any generalised eigenfunction \( \Psi \) of operator \( H^{(2)} \), with the respective eigenvalue \( E \in I \), for the specified \( \omega \in \Omega_{<\infty} \). From now on, the argument in the proof of Theorem becomes deterministic, and we omit symbol \( \omega \), except for the places where its presence is instructive.

Since \( \Psi \) is polynomially bounded, there exist \( C, t \in (0, +\infty) \) such that

\[ \forall x \in \mathbb{Z}^d \times \mathbb{Z}^d, |\Psi(x)| \leq C (1 + \|x\|)^t. \]  

(2.9)

Further, since \( \Psi \) is not identically zero, \( \exists u \in \mathbb{Z}^d \times \mathbb{Z}^d \) such that \( \Psi(u) \neq 0 \).

For any given \( k \), if \( E \notin \text{spec} \left( H^{(2)}_{\Lambda_{L_k}(u)} \right) \) then the values of \( \Psi \) inside \( \Lambda_{L_k}(u) \) can be recovered from its boundary values, in particular,

\[ \Psi(u) = \sum_{v \in \partial \Lambda_{L_k}(u)} \sum_{v' \in \partial^+ \Lambda_{L_k}(u)} 1_{\{\|v-v'\|=1\}} G_{\Lambda_{L_k}(u)}^{(2)}(E; u, v) \Psi(v'). \]  

(2.10)
Here and below, \( \partial^+ \Lambda_{L_k}(u) \) denotes the exterior boundary of box \( \Lambda_{L_k}(u) \) formed by the points \( y \in \partial (\mathbb{Z}^d \times \mathbb{Z}^d \setminus \Lambda_{L_k}(u)) \), i.e., the points \( y \in \mathbb{Z}^d \times \mathbb{Z}^d \setminus \Lambda_{L_k}(u) \) with dist \((y, \Lambda_{L_k}(u)) = 1\).

Suppose that a two-particle box \( \Lambda_{L_k}(u) \) is \((m, E)\)-NS for an infinite number of values \( k \) (i.e., for arbitrarily big values of \( k \)). Then, by Definition 1.1,
\[
\left| G_{\Lambda_{L_k}}^{(2)}(u; u, v) \right| \leq e^{-mL_k},
\]
yielding, by (2.9) and (2.10), that
\[
|\Psi(u)| \leq 4d(2L_k + 1)^{2d-1} e^{-mL_k} C(1 + \|u\| + L_k)^t \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (2.11)
\]
This would mean that, in fact, \( \Psi(u) = 0 \), which contradicts the above assumption. Therefore, \( \exists \) an integer \( k_1 = k_1(\omega, E, u) < \infty \) such that \( \forall k \geq k_1 \), box \( \Lambda_{L_k}(u) \) is \((m, E)\)-S. At the same time, owing to the choice of \( \omega \in \Omega_{<\infty} \), \( \exists k_2 = k_2(\omega, u) \) such that \( \forall k \geq k_2 \), the event \( \Omega_k(u) \) does not occur. Then
\[
\forall k \geq \max \{k_1, k_2\} : \forall \mathbf{x} \in \Lambda_{k+1}(u), \text{ box } \Lambda_{L_k}(\mathbf{x}) \text{ is } (m, E)\text{-NS}. \quad (2.12)
\]
Next, for a given \( \rho \in (0,1) \) and \( b > \frac{1+\rho}{1-\rho} \), define
\[
\tilde{\Lambda}_{k+1}(u) = \Lambda_{2b/(1+\rho)L_k}(u) \setminus \Lambda_{2/(1-\rho)L_k}(u) \subset \Lambda_{k+1}(u).
\]
Naturally, the above inclusion is valid for sufficiently large \( L_{k+1} \), or, with \( L_0 > 1 \) being fixed, for sufficiently large values of \( k \), which we will assume in this argument.

For any \( \mathbf{x} \in \tilde{\Lambda}_{k+1}(u) \), we have that
\[
\text{dist}(\mathbf{x}, \partial \Lambda_{k+1}(u)) \geq \rho \| \mathbf{x} - u \|.
\]
Furthermore, if \( \| \mathbf{x} - u \| \geq L_0/(1-\rho) \), then \( \exists k \geq 0 \) such that \( \mathbf{x} \in \tilde{\Lambda}_{k+1}(u) \).

For any \( k \geq \max \{k_1, k_2\} \), box \( \Lambda_{L_k}(u) \) must be \((m, E)\)-NS, and therefore, \( E \notin \text{spec} \left( H_{\Lambda_{L_k}(u)}^{(2)} \right) \). Hence, Eqn (2.10) holds, with \( \mathbf{x} \) instead of \( u \):
\[
\Psi(\mathbf{x}) = \sum_{v \in \partial \Lambda_{L_k}(\mathbf{x})} \sum_{v' \in \partial \Lambda_{L_k}(\mathbf{x})} 1(\|v - v'\| = 1) G_{\Lambda_{L_k}(\mathbf{x})}^{(2)}(E; u, v) \Psi(v'). \quad (2.13)
\]
Further, by virtue of non-singularity of \( \Lambda_{L_k}(u) \), \( \exists \mathbf{v}_1 \in \partial^+ \Lambda_{L_k}(u) \) such that
\[
|\Psi(\mathbf{x})| \leq 4d(2L_k + 1)^{2d-1} e^{-mL_k} |\Psi(\mathbf{v}_1)|. \quad (2.14)
\]
In fact, it suffices to apply bound (1.15) and pick, in the RHS of (2.13), a point $v'$ incident to $v$ providing the maximal absolute value of the GF $G^{(2)}_{\Lambda_{L_k}}(x; u, v)$.

Next, pick a value $\tilde{\rho} \in (0, 1)$ and write it as a product $\tilde{\rho} = \rho \rho'$, where $\rho, \rho' \in (0, 1)$. Further, pick any $b > 8 + 1 + \rho/(1 - \rho)$. We can iterate bound (2.14) at least $((L_k + 1)^{-1}\rho\|x - u\|\times, obtaining the following inequality:

$$|\Psi(x)| \leq \left(4d(2L_k + 1)^{2d-1} e^{-mL_k \rho} \right)^{(L_k + 1)^{-1}\rho\|x - u\|} C \left(1 + \|u\| + bL_{k+1}\right)^t.$$  

(2.15)

This provides an exponential decay rate of $\Psi$ arbitrarily close to $\rho$. Namely, $\exists$ an integer $k_3 \geq \max\{k_1, k_2\}$ such that $\forall k \geq k_3$, if $\|x - u\| \geq L_k/(1 - \rho)$ then

$$|\Psi(x)| \leq e^{-\rho\rho' m\|x - u\|}.$$ 

Therefore,

$$\limsup_{\|x\| \to \infty} \frac{1}{\|x\|} \ln |\Psi(x)| \leq -\rho\rho' m.$$ 

Eqn (2.1) then follows. This completes the proof of Theorem 1.2.

3 The two-particle MSA scheme.

Non-interactive pairs of singular boxes

In view of Theorem 1.2, our aim is to check property (DS. $k$, I) in Eqn (1.17). We now outline the two-particle MSA which is used for this purpose.

It bears many features borrowed from its single-particle counterpart. In both single- and two-particle versions, the MSA scheme is an elaborate induction dealing with GFs $G^{(2)}_{\Lambda_{L_k}}(u)$ and involving several mutually related parameters; some of them have been used in Sections 1 and 2. Here we give the complete list, following specifications (1.11), (1.12) of sequences $L_k$ and $m_k$ (these specifications are assumed for the rest of the paper).

- Parameter $\alpha \in (1, 2)$ determines the rate of growth of sequence $L_k$ in Eqn (1.11).
- Constant $\gamma > 0$ determines sequence $m_k$ in Eqn (1.12).
- Parameter $p > 0$ controls the power of the decay of probability of double singularity; see property (DS. $k$, I) in (1.17).
Parameters \( m_0 > 0 \) (the initial mass) and \( L_0 > 1 \) (the initial length) define the initial step of the induction. These values are related to the threshold \( g^* \in (0, \infty) \) for the coupling constant \( g \) in Theorem 1.1 roughly, by the constraint \( m_0L_0 \sim \ln g^* \); see the proof of the initial inductive step in Theorem 3.1 below. In addition, to complete the inductive step, \( L_0 \) should be large enough: \( L_0 \geq L^* \); see Theorem 3.2.

Parameters \( m_k > 0 \) and \( L_k > 1 \) (the mass and the length at step \( k \)) are chosen to follow Eqns (1.11) and (1.12) since it allows us to check Eqn (1.17) with substantial use of the single-particle MSA scheme.

Next,

- Parameter \( \beta \in (0, 1) \) controls the important property of tolerated resonances; see Eqn (3.2).

- Parameter \( q > 0 \) is responsible for decay of probability of non-tolerated resonances.

Thresholds \( L^* \) and \( g^* \) are functionally described as \( L^* = L^*(d, \beta, \alpha, m_0, p, q) \) and \( g^* = g^*(d, \beta, \alpha, m_0, p, q) \). An initial insight into the values of these thresholds is provided by writing \( L^* = \max [L^*_0, L^*_1] \) and \( g^* = \max [g^*_0, g^*_1] \). Here \( L^*_0 \) and \( g^*_0 \) are (rather explicitly) determined from Eqn (3.1) and Theorem 3.1 whereas \( L^*_1 \) and \( g^*_1 \) are encrypted into Theorem 3.2. A further insight into the values of \( L^* \) and \( g^* \) is provided in the course of the presentation below. The initial mass \( m_0 > 0 \) can be chosen at will (but of course the choice of \( m_0 \) affects that of \( L^* \) and \( g^* \)).

To start with, we assume \( L_0 \geq L^*_0 \) where \( L^*_0 \) is large enough, so that

\[
\prod_{j=1}^{\infty} \left(1 - \gamma L_j^{-1/2}\right) \geq \frac{1}{2}. \tag{3.1}
\]

Technically, it is convenient for us to run the two-particle MSA scheme under the following conditions on parameter values:

\[
p > 12d + 9, q > 4p + 12d, \beta = 1/2, \alpha = 3/2, \gamma = 40. \tag{3.2}
\]

We will assume Eqns (3.1), (3.2) for the rest of the paper, although we will use symbols \( \alpha \) and \( \beta \) to make analogies with [DK] more fulfilling.

We will also use results of the single-particle MSA formulated and proved in [DK]. The property of decay (with high probability) of GFs of the single-particle Anderson tight binding model is proved, in particular, under assumption of large disorder: \(|g| \geq \tilde{g} > 0 \), where the threshold \( \tilde{g} \) is defined in terms
of the single-particle Hamiltonian. We always assume, directly or indirectly, that the two-particle threshold $g^*$, introduced in this paper, satisfies $g^* \geq \tilde{g}$, so that for all $g$ with $|g| \geq g^*$, all results of [DK] for the single-particle model are valid. In order to avoid confusion, we will denote by $\tilde{p}$ and $\tilde{q}$ parameters analogous to $p$ and $q$ but related to the single-particle model. It is worth mentioning that, according to the results of [DK], one can choose $\tilde{p}$ and $\tilde{q}$ arbitrarily large, provided that $|g|$ is sufficiently large. Therefore, we can also assume that $\tilde{p}$ and $\tilde{q}$ are as large as required for our arguments, provided that $\tilde{g}$ is sufficiently large and

$$|g| \geq g^* \geq \tilde{g}$$

(3.3)

The initial step of the two-particle MSA scheme consists in establishing properties (S.0) and (SS.0); see Eqns (3.7) and (3.8). The inductive step of the two-particle MSA consists in deducing property (SS.$k + 1$) from property (SS.$k$); again see Eqn (3.8). These properties are equivalent to properties (DS.0, $I$), (DS.$k + 1$, $I$) and (DS.$k$, $I$), respectively, figuring in Theorem 1.2 (in the form of (SS. · ) they are slightly more convenient to deal with). Both the initial and the inductive step are done with the assistance of properties (W1) and/or (W2) (Wegner-type bounds, see Eqns (3.5) and 3.6) below) which should be established independently. In our context, i.e. for a two-particle system, properties (W1) and (W2) have been proved in [CS1].

**Definition 3.1** Given $E \in \mathbb{R}$, $v \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $L > 1$, we call the box $\Lambda_L(v)$ $E$-resonant (briefly: $E$-R) if the spectrum of the Hamiltonian $H^{(2)}_{\Lambda_L(v)}$ satisfies

$$\text{dist} \left[ E, \text{spec} \left( H^{(2)}_{\Lambda_L(v)} \right) \right] < e^{-L^2}.$$  

(3.4)

Given an $L_0 > 1$, introduce the following properties (W1) and (W2) of Hamiltonians $H^{(2)}_{\Lambda_l}$, $l \geq L_0$.

(W1) \quad $\forall \ l \geq L_0$, box $\Lambda_l(x)$ and $E \in \mathbb{R}$: $\mathbb{P} \{ \Lambda_l(x) \text{ is E-R} \} < l^{-q}$.  

(3.5)

(W2) \quad $\forall \ l \geq L_0$ and $8l$-D boxes $\Lambda_l(x)$ and $\Lambda_l(y)$, $\mathbb{P} \{ \exists E \in \mathbb{R} : \text{ both } \Lambda_l(x) \text{ and } \Lambda_l(y) \text{ are E-R} \} < l^{-q}$.  

(3.6)
Lemma 3.1 (Cf. [CS1].) Under the above assumptions on \( \{V(x; \omega)\} \) and \( U \) (see (1.3)-(1.5)), properties W1, W2 hold true.

Further, let \( I \subseteq \mathbb{R} \) be an interval. Given \( m_0 > 0 \) and \( L_0 > 1 \), consider property (S.0):

\[
(S.0) \quad \forall \ x \in \mathbb{Z}^d, \ \mathbb{P} \{ \exists E \in I : \ \Lambda_{L_0}(x) \text{ is } (E, m_0)\text{-}S \} < L_0^{-2p}.
\] (3.7)

Next, for interval \( I \subseteq \mathbb{R} \) and values \( L_k \) and \( m_k \), \( k \geq 0 \), as in (1.11) and (1.12), we introduce property (SS.k):

\[
(SS.k) \quad \forall \ L_k\text{-}D \text{ boxes } \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y):
\mathbb{P} \{ \exists E \in I : \text{ both } \Lambda_{L_k}(x), \Lambda_{L_k}(y) \text{ are } (E, m_k)\text{-}S \} < L_k^{-2p}.
\] (3.8)

The initial MSA step is summarised in

Theorem 3.1 \( \forall \) given \( m_0 \) and \( L_0 > 1 \) and \( \forall \) bounded interval \( I \subset \mathbb{R} \), there exists \( g_0^* = g_0^*(m_0, L_0, |I|) \in (0, +\infty) \) such that for \( |g| \geq g_0^* \), properties (S.0) and (SS.0) hold true.

Proof of Theorem 3.1 Obviously, property (S.0) implies (SS.0), so we focus on the former. Property (S.0) is established along the lines of [DK]; see [DK], Proposition A.1.2. Without loss of generality, we can assume that \( g > 0 \). Let \( E_0 \in \mathbb{R} \) be the middle point of \( I \) and \( 2\eta \) be its length: \( I = (E_0 - \eta, E_0 + \eta) \). Note that if \( \forall \ x = (x_1, x_2) \in \Lambda_{L_0}(u) \) we have

\[
|W(x) - E_0| \geq 4d + 2\eta + e^{m_0 L_0},
\]
then \( \forall \ E \in [E_0 - \eta, E_0 + \eta] \)

\[
\|G_{\Lambda_{L_0}(u)}(E)\| \leq e^{-m_0 L_0}.
\]

Next, with \( c_0 = c_0(d, \eta, m_0, L_0) := 4d + 2\eta + e^{m_0 L_0} \), observe that

\[
\mathbb{P} \{ \exists \ x \in \Lambda_{L_0}(u) : |W(x) - E_0| \leq c_0 \} = \mathbb{P} \{ \exists \ x \in \Lambda_{L_0}(u) : |g[V(x_1; \omega) + V(x_2; \omega)] - [E_0 - U(x)]| \leq c_0 \} \leq |\Lambda_{L_0}(u)| \max_{x \in \Lambda_{L_0}(u)} \mathbb{P} \{ |V(x_1; \omega) + V(x_2; \omega) - g^{-1}[E_0 - U(x)]| \leq c_0 g^{-1} \}.
\]
For \( x = (x_1, x_2) \) with \( x_1 \neq x_2 \), random variables \( V(x_1; \cdot) \) and \( V(x_2; \cdot) \) are independent and have a common bounded PDF \( p_V \) of compact support. The sum \( V(x_1; \cdot) + V(x_2; \cdot) \) has a bounded PDF \( p_V \cdot p_V \), the convolution of \( p_V \) with itself. Thus, for \( x = (x_1, x_2) \) with \( x_1 \neq x_2 \),
\[
\mathbb{P}\left\{ |V(x_1; \omega) + V(x_2; \omega) - g^{-1}[E_0 - U(x)]| \leq c_0 g^{-1}\right\} \leq c_0 \left( \max p_V \cdot p_V \right) g^{-1}.
\]
For \( x = (x_1, x_1) \), we have \( V(x_1; \omega) + V(x_2; \omega) = 2V(x_1; \omega) \), so that
\[
\mathbb{P}\left\{ |V(x_1; \omega) + V(x_2; \omega) - g^{-1}[E_0 - U(x)]| \leq c_0 g^{-1}\right\} = \mathbb{P}\left\{ |V(x_1; \omega) - (2g)^{-1}(E_0 - U(x))| \leq c_0 \cdot (2g)^{-1}\right\} \leq c_0 \left( \max p_V \right) g^{-1}.
\]
We see that in both cases \( \mathbb{P}\{ |W(x) - E_0| \leq c_0 \} \to 0 \) as \( g \to \infty \), uniformly in \( x \). Property (SS.0) then follows. □

To complete the inductive MSA step, we will prove

**Theorem 3.2** \( \forall \) given \( m_0 > 0 \), there exist \( g^*_1 \in (0, +\infty) \) and \( L^*_1 \in (0, +\infty) \) such that the following statement holds. Suppose that \( |g| \geq g^*_1 \) and \( L_0 \geq L^*_1 \). Then, \( \forall k = 0, 1, \ldots \) and \( \forall \) interval \( I \subseteq \mathbb{R} \), property (SS.k) implies (SS.k + 1).

The proof of Theorem 3.2 occupies the rest of the paper. Before we proceed with the proof, let us repeat that the property (DS.k, I) (or, equivalently, (SS.k)), for \( \forall \) \( k \geq 0 \) and \( \forall \) unit interval \( I \subseteq \mathbb{R} \), follows directly from Theorems 3.1 and 3.2.

**Proof of Theorem 3.2.** To deduce property (SS.k + 1) from (SS.k), we introduce

**Definition 3.2** Consider the following subset in \( \mathbb{Z}^d \times \mathbb{Z}^d \):
\[
\mathbb{D}_{r_0} = \{ x = (x_1, x_2) \in \mathbb{Z}^d \times \mathbb{Z}^d : \|x_1 - x_2\| \leq r_0 \}. \quad (3.9)
\]
A two-particle box \( \Lambda_x(u) \) is called interactive when \( \Lambda_x(u) \cap \mathbb{D}_{r_0} \neq \emptyset \), and non-interactive if \( \Lambda_x(u) \cap \mathbb{D}_{r_0} = \emptyset \). For a non-interactive box \( \Lambda_x(u) \), the interaction potential \( U(x) = 0 \), \( \forall x \in \Lambda_x(u) \). For brevity, we use the terms I-box and NI-box, respectively.
The procedure of deducing property $(SS. k + 1)$ from $(SS. k)$ is done here separately for the following three cases.

(I) Both $\Lambda_{k+1}(x)$ and $\Lambda_{k+1}(y)$ are NI-boxes.

(II) Both $\Lambda_{k+1}(x)$ and $\Lambda_{k+1}(y)$ are I-boxes.

(III) One of the boxes is I, while the other is NI.

In the remaining part of this section we consider case (I). Cases (II) and (III) are treated in Sections 4 and 5, respectively. We repeat that all cases require the use of property $(W1)$ and/or $(W2)$.

The plan for the rest of Section 3 is as follows. We aim to derive property $(SS. k + 1)$ for a pair of non-interactive $L_{k+1}$-D boxes $\Lambda_{L_{k+1}}(x), \Lambda_{L_{k+1}}(y)$, and we are allowed to assume property $(SS. k)$ for every pair of $L_{k}$-D boxes $\Lambda_{L_{k}}(\tilde{x}), \Lambda_{L_{k}}(\tilde{y})$, where $x, y, \tilde{x}, \tilde{y} \in \mathbb{Z}^d \times \mathbb{Z}^d$. In fact, we are able to establish property $(SS. k + 1)$ for non-interactive $L_{k+1}$-D boxes $\Lambda_{L_{k+1}}(x), \Lambda_{L_{k+1}}(y)$ directly, without referring to $(SS. k)$. (In cases (II) and (III) such a reference is needed.) An important part of our argument is a single-particle result stated as Theorem 3.3.

Let $\Lambda_{L_{k+1}}(u)$ be an NI-box, where $u = (u_1, u_2)$. We represent it as the Cartesian product

$$\Lambda_{L_{k+1}}(u) = \Lambda_{L_{k+1}}(u_1) \times \Lambda_{L_{k+1}}(u_2).$$

Here and below, for given $\ell > 1$ and $v = (v^{(1)}, \ldots, v^{(d)}) \in \mathbb{R}^d$:

$$\Lambda_v(\ell) := \left( \times_{i=1}^{d} \left[ v_j^{(i)} - \ell, v_j^{(i)} + \ell \right] \right) \cap \mathbb{Z}^d.$$  \hfill (3.11)

We call sets $\Lambda_v(\ell)$ single-particle boxes; as before, $|\Lambda_v(\ell)|$ denotes the cardinality of $\Lambda_v(\ell)$. The boundary $\partial \Lambda_v(\ell)$ is also defined in a similar fashion: it is formed by the points $y \in \Lambda_v(\ell)$ for which $\exists y' \in \mathbb{Z}^d \setminus \Lambda_v(\ell)$ with $\|y - y'\| \leq 1$.

Since the potential $U$ vanishes on $\Lambda_{L_{k+1}}(u)$, the Hamiltonian $H^{(2)}_{\Lambda_{L_{k+1}}(u)}$ takes the form

$$H^{(2)}_{\Lambda_{L_{k+1}}(u)} \phi(x) = \sum_{y \in \Lambda_{L_{k+1}}(u): \|y-x\| = 1} \phi(y) + g \sum_{j=1,2} \sum_{\omega} V(x_j; \omega) \phi(x),$$ \hfill (3.12)

or, algebraically,

$$H^{(2)}_{\Lambda_{L_{k+1}}(u)} = H^{(1)}_{\Lambda_{L_{k+1}}(u_1)} \otimes I + I \otimes H^{(1)}_{\Lambda_{L_{k+1}}(u_2)}.$$ \hfill (3.13)
Here $H^{(1)}_{j;\Lambda_{k+1}(u_j)}$ is the single-particle Hamiltonian acting on variable $x_j \in \Lambda_{L_k+1}(u_j)$, $j = 1, 2$:

\[
(H^{(1)}_{j;\Lambda_{k+1}(u_j)} \varphi)(x_j) = \sum_{y_j \in \Lambda_{k+1}(u_j): \|y_j - x_j\| = 1} \varphi(y_j) + gV(x_j; \omega)\varphi(x_j),
\]

and $I$ is the identity operator on the complementary variable.

Let $\psi_{j;s}(x)$ be the eigenvectors of operators $H^{(1)}_{j;\Lambda_{k+1}(u_j)}$ and $E_{j;s}$ be their eigenvalues, $s = 1, \ldots, |\Lambda_{L_{k+1}(u_j)}|$. Then the eigenvectors $\Psi_{s_1,s_2}$ of $H^{(2)}_{\Lambda_{k+1}(u)}$ can be represented as tensor products:

\[
\Psi_{s_1,s_2}(x) = \psi_{1;s_1}(x_1)\psi_{2;s_2}(x_2),
\]

while the eigenvalues $E_{s_1,s_2}$ of $H^{(2)}_{\Lambda_{k+1}(u)}$ are written as sums:

\[
E_{s_1,s_2} = E_{1;s_1} + E_{2;s_2},
\]

with $s_1 = 1, \ldots, |\Lambda_{L_{k+1}(u_1)}|$, $s_2 = 1, \ldots, |\Lambda_{L_{k+1}(u_2)}|$.

We make use of the following definition:

**Definition 3.3** Fix $\hat{m} > 0$ and a positive integer $\ell$. Given $v \in \mathbb{Z}^d$, consider the single-particle Hamiltonian $H^{(1)}_{\Lambda_{\ell}(v)}$ in $\Lambda_{\ell}(v)$ acting on vectors $\varphi \in \mathbb{C}^{\Lambda_{\ell}(v)}$:

\[
(H^{(1)}_{\Lambda_{\ell}(v)} \varphi)(x) = \sum_{y \in \Lambda_{\ell}(v): \|y - x\| = 1} \varphi(y) + gV(x; \omega)\varphi(x), \ x \in \Lambda_{\ell}(u).
\]

Let $\psi_s(x)$ be the normalised eigenvectors and $E_s$ the corresponding eigenvalues of $H^{(1)}_{\Lambda_{\ell}(v)}$. We say that a single-particle box $\Lambda_{\ell}(v)$ is $\hat{m}$-non-tunnelling ($\hat{m}$-NT, for short), if

\[
\max_{y \in \partial \Lambda_{\ell}(v)} \max_{s} \{ |\psi_s(v)\psi_s(y)| : E_s \in \text{spec} \left( H^{(1)}_{\Lambda_{\ell}(v)} \right) \} \leq e^{-\hat{m}\ell}.
\]

Otherwise we call it $\hat{m}$-tunnelling ($\hat{m}$-T ). A two-particle box $\Lambda_{\ell}(v)$ is called $\hat{m}$-non-tunnelling if both of its projections $\Pi_1\Lambda_{\ell}(v)$ and $\Pi_2\Lambda_{\ell}(v)$ are $\hat{m}$-non-tunnelling.
In future, the eigenvectors of finite-volume Hamiltonians appearing in arguments and calculations, will be assumed normalised.

**Remark.** Observe that (i) property \( \hat{m} \)-NT implies \( \hat{m}' \)-NT for any \( \hat{m}' \in [0, \hat{m}] \). Next, (ii) properties \( \hat{m} \)-T and \( \hat{m} \)-NT refer only to single-particle Hamiltonians. As we will see later, in our two-particle MSA inductive procedure, we can use the \((2m_0)\)-NT property while working with boxes \( \Lambda_{L_k}(x) \), \( \forall k \geq 0 \).

The following statement gives a formal description of a property of NI two-particle boxes which will be referred to as property (NIRoNS) (‘non-interactive boxes are resonant or non-singular’). As we said earlier, property (NIRoNS) is established for all \( k \geq 0 \), by combining known results from the single-particle localisation theory, established via MSA or the FMM. It is worth mentioning that a property close to (NIRoNS) was formulated in [FMSS], Proposition in Section 6, p.43. However, the context here is different.

**Lemma 3.2** Consider a pair of single-particle boxes \( \Lambda_{L_k}(u_j) \), \( j = 1, 2 \), where \( \| u_1 - u_2 \| > L_k + r_0 \). Given \( \hat{m} > 0 \), assume that \( \Lambda_{L_{k+1}}(u_1) \) and \( \Lambda_{L_{k+1}}(u_2) \) are \( \hat{m} \)-NT. Next, assume that the two-particle non-interactive box \( \Lambda_{L_k}(u) = \Lambda_{L_k}(u_1) \times \Lambda_{L_k}(u_2) \) is \( E \)-NR. If \( L_k^{-1} \left( L_k^2 + \ln \left( (2L_k + 1)^{2d} \right) \right) < 1 \), then \( \Lambda_{L_k}(u) \) is \( \hat{m}(1) \)-NS with

\[
\hat{m}(1) = \hat{m} \left( 1 - L_k^{-1+\beta} - L_k^{-1} \ln (2L_k + 1)^{2d} \right).
\]

(3.17)

In particular, if \( L_k^{-1} \ln (2L_k + 1)^{2d} \leq L_k^{-1+\beta} \), then \( \hat{m}(1) \geq \hat{m}(1-2L_k^{-1+\beta}) \).

**Proof of Lemma 3.2.** By definition of the GFs,

\[
G_{\Lambda_{L_k}(u)}^{(2)}(u, y; E) = \sum_{s_1=1}^{[\Lambda_{L_k}(u_1)]} \sum_{s_2=1}^{[\Lambda_{L_k}(u_2)]} \frac{\psi_{1,s_1}(u_1) \psi_{1,s_1}^*(y_1) \psi_{2,s_2}(u_2) \psi_{2,s_2}^*(y_2)}{E - (E_{1,s_1} + E_{2,s_2})}.
\]

(3.18)

Here, as before, \( E_{j,s} \) and \( \psi_{j,s} \), \( s = 1, \ldots, [\Lambda_{L_k}(u_j)] \), \( j = 1, 2 \), are the eigenvalues and the corresponding eigenvectors of \( H_{j;\Lambda_{L_k}(u_j)}^{(1)} \).

Since \( \Lambda_{L_k}(u) \) is \( E \)-NR, the absolute values \( |E - (E_{1,s_1} + E_{2,s_2})| \) of the denominators in (3.18) are bounded from below by \( e^{-L_k^2} \). The sum of numerators can be bounded as follows. First, note that if \( \| u - y \| = L_k \), then
either \( \|u_1 - y_1\| = L_k \), or \( \|u_2 - y_2\| = L_k \). Without loss of generality, suppose that \( \|u_2 - y_2\| = L_k \), then

\[
\left| \sum_{s_1, s_2} \psi_{1,s_1}(u_1) \bar{\psi}_{1,s_1}(y_1) \bar{\psi}_{2,s_2}(u_2) \psi_{2,s_2}(y_2) \right| \\
\leq \sum_{s_1} \left| \psi_{1,s_1}(u_1) \bar{\psi}_{1,s_1}(y_1) \right| \sum_{s_2} \left| \psi_{2,s_2}(u_2) \bar{\psi}_{2,s_2}(y_2) \right| \\
\leq |\Lambda_{L_k}(u_1)| \cdot 1 \cdot |\Lambda_{L_k}(u_2)| e^{-\tilde{m}L_k} = e^{-\tilde{m}L_k}(2L_k + 1)^{2d},
\]

owing to the hypothesis of non-tunnelling. Finally, we obtain

\[
G^{(2)}_{\Lambda_{L_k}(u)}(u, y; E) \leq (2L_k + 1)^{2d} e^{L_k^\beta - \tilde{m}L_k} \leq e^{-\tilde{m}(1)L_k}.
\]

This yields Lemma 3.2. ■

Now introduce the following property of single-particle Hamiltonians \( H^{(1)}_{\Lambda_k(v)} \):

\[
(\text{NT. } k, s) \quad P \{ \text{ single-particle box } \Lambda_{L_k}(v) \text{ is } (2m_0) - \text{NT} \} \geq 1 - L_k^{-s},
\]

where \( s > 0 \).

Lemma 3.2 implies the following

**Lemma 3.3** Assume property (W2). Suppose that \( \forall \ k \geq 0 \), the single-particle Hamiltonians \( H^{(1)}_{\Lambda_{L_k}(v)} \) satisfy (NT. \( k, s \)) with \( s \geq q \):

\[
P \{ \Lambda_{L_k}(v) \text{ is } (2m_0) - \text{NT} \} \geq 1 - L_k^{-q}.
\]

Suppose also that

\[
L_0^{-1} \ln (2L_0 + 1)^{2d} \leq L_0^{-1+\beta} \leq \frac{1}{4}.
\]

Then, \( \forall \ interval \ I \subseteq \mathbb{R}, \forall \ k \geq 0 \) and \( \forall \ pair \ of \ non-interactive \ L_k\text{-D two-particle boxes } \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y) \),

\[
P \{ \exists \ E \in I : \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y) \text{ are } (E, m_k) - \text{S} \} \leq 5L_k^{-q}.
\]
Proof of Lemma 3.3. By virtue of Lemma 3.1,
\[
\begin{align*}
\mathbb{P}\{ \exists E \in I : \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y) \text{ are } (E, m_k) - S \} & \leq \mathbb{P}\{ \Lambda_{L_k}(x) \text{ is } 2m_k - T \} + \mathbb{P}\{ \Lambda_{L_k}(y) \text{ is } 2m_k - T \} \\
& \quad + \mathbb{P}\{ \exists E \in I : \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y) \text{ are } E - R \} \\
& \leq 2 \cdot 2L_k^{-s} + L_k^{-q} \leq 5L_k^{-q}. \hspace{1cm} \blacksquare
\end{align*}
\]

The validity of (3.20) is guaranteed by

Theorem 3.3 Consider single-particle Hamiltonians \( H^{(1)}_{A_{L_k}(v)}, v \in \mathbb{Z}^d, k = 0, 1, \ldots \). Then \( \exists g_2^*, L_2^* \in (0, +\infty) \) such that when \( |g| \geq g_2^* \) and \( L_0 \geq L_2^* \), the following bound holds true for all \( k \geq 0 \):
\[
\mathbb{P}\{ \Lambda_{L_k}(v) \text{ is } (2m_0) - NT \} \geq 1 - L_k^{-s}, \quad s = \frac{\tilde{p} - 2(1 + \alpha)d}{\alpha}. \quad (3.22)
\]

In other words, Theorem 3.3 asserts property \((NT, k, s)\) with \( s = \frac{\tilde{p} - 2(1 + \alpha)d}{\alpha} \). So, it suffices to assume that \( \tilde{p} \geq \alpha q + 2(1 + \alpha)d \). Since, as we observed before, \( \tilde{p} = \tilde{p}(g) \to \infty \) as \( |g| \to \infty \), the latter inequality holds for \( |g| \) large enough.

As was said in Section 1, the reader can find, in the forthcoming manuscript [CS2], a proof of Theorem 3.3 based on an adaptation of MSA techniques from [DK] and valid under the IID assumption and condition (1.3). However, a stronger estimate was proved in [A94], with the help of the FMM, under condition (1.2) (in fact, the assumptions on the external potential \( V(x; \omega), x \in \mathbb{Z}^d \), adopted in [A94] are more general than IID, and, according to [A08], they can be further relaxed; see [AW]). Namely, bound (1.6) from [A94] implies that
\[
\mathbb{P}\{ \Lambda_{L_k}(v) \text{ is } (2m_0) - NT \} \geq 1 - e^{-\tilde{m}L_k}, \quad (3.23)
\]
where \( \tilde{m} = \tilde{m}(g) \to \infty \) as \( |g| \to \infty \). We also recall that, for one-dimensional single-particle models, exponential bounds of probability of exponential decay of eigen-functions in finite volumes were obtained in [GMP] (for Schrödinger operators on \( \mathbb{R} \)) and in [KS] (for lattice Schrödinger operators on \( \mathbb{Z} \)).

We thus come to the following conclusion.
Theorem 3.4 \(\forall\) given interval \(I \subseteq \mathbb{R}\) and \(k = 0, 1, \ldots\), property \((SS.k)\) holds for all pairs of \(L_k\)-D non-interactive boxes \(\Lambda_{L_k}(x), \Lambda_{L_k}(y)\).

Summarising the above argument: the validity of property \((SS.k + 1)\) for a pair of two-particle NI-boxes did not require us to assume \((SS.k)\). However, in the course of deriving \((SS.k + 1)\) for NI-boxes we used property (3.20) for single-particle boxes, as well as the Wegner-type property \((W2)\).

This completes the analysis of the case (I) where both boxes \(\Lambda_{L_{k+1}}(x)\) and \(\Lambda_{L_{k+1}}(y)\) are NI.

For future use, we also give

Lemma 3.4 Consider a two-particle box \(\Lambda_{L_{k+1}}(u)\). Let \(M(\Lambda_{L_{k+1}}(u); E)\) be the maximal number of \((E, m_k)\)-S, pair-wise \(L_k\)-D NI-boxes \(\Lambda_{L_k}(u^{(j)}) \subset \Lambda_{L_{k+1}}(u)\). The following property holds

\[
\mathbb{P} \{ \exists E \in I : M(\Lambda_{L_{k+1}}(u); E) \geq 2 \} \leq L_k^{2d(1+\alpha)} \cdot 5L_{k-2}\delta < L_k^{4d\alpha} \cdot L_k^{2p}. \quad (3.24)
\]

Proof of Lemma 3.4 The number of possible pairs of centres \(u^{(1)}, u^{(2)}\) is bounded by

\[
(2L_{k+1} + 1)^{2d} \leq (2L_k^a + 1)^{2d} \leq (L_k^{a+1})^{2d} = L_k^{2d(a+1)},
\]

while for a given pair of centres one can apply Theorem 3.3. ■

4 Interactive pairs of singular boxes

Speaking informally, case (II) corresponds to a two-particle system with ‘confinement’: in both boxes \(\Lambda_{L_{k+1}}(x)\) and \(\Lambda_{L_{k+1}}(y)\), particles are at a distance \(\leq 2L_k + r_0\) from each other, and form a ‘compound quantum object’ which can be considered as a ‘single particle’ subject to a random external potential. It is not entirely surprising, then, that such a compound object should feature localisation properties resembling those from the single-particle theory. The reader may see that the analysis needed to cover case (II) is rather similar to that in [DK]. It relies essentially upon properties \((W1)\) and \((W2)\) (Wegner-type estimates). However, it is worth mentioning that the derivation of estimates \((W1)\) and \((W2)\) required new ideas due to strong dependencies in the random potential \(gV(x_1; \omega) + gV(x_2; \omega)\). As was said before, these
dependencies do not decay as $\|x_1 - x_2\| \to \infty$. Our proofs given in [CS1] are based on Stollmann’s lemma (cf. [St1], [St2]) rather than on the original ideas of Wegner.

The main outcome in case (II) is Theorem 4.1 placed at the end of this section. Before we proceed further, let us state a geometric assertion (see Lemma 4.1 below) which we prove in Section 6. Given a two-particle box $\Lambda_L(u)$, with $u = (u_1, u_2)$, and $u_j = (u_j^{(1)}, \ldots, u_j^{(d)}) \in \mathbb{Z}^d$, set

$$
\Pi \Lambda_L(u) = \Pi_1 \Lambda_L(u) \cup \Pi_2 \Lambda_L(u) \subset \mathbb{Z}^d.
$$

Here $\Pi_1 \Lambda_L(u)$ and $\Pi_2 \Lambda_L(u)$ denote the projections of $\Lambda_L(u)$ to the first and the second factor in $\mathbb{Z}^d \times \mathbb{Z}^d$:

$$
\Pi_j \Lambda_L(u) = \left( \times \frac{d}{i=1} [u^{(i)}_j - L, u^{(i)}_j + L] \right) \cap \mathbb{Z}^d, \ j = 1, 2,
$$

cf. (1.9). In other words, $\Pi_j \Lambda_L(u)$ describes a ‘supporting domain’ of the single-particle external potential $\{V(x), \ x \in \mathbb{Z}^d\}$ contributing into the potential field $W(x)$, $x \in \Lambda_L(u)$.

**Lemma 4.1** Let be $L > r_0$ and consider two interactive $8L$-D boxes $\Lambda_L(u')$ and $\Lambda_L(u'')$, with $\text{dist} [\Lambda_L(u'), \Lambda_L(u'')] > 8L$. Then

$$
\Pi \Lambda_L(u') \cap \Pi \Lambda_L(u'') = \emptyset.
$$

Lemma 4.1 is used in the proof of Lemma 4.2 which, in turn, is important in establishing Theorem 4.1. Actually, it is a natural complement to Lemma 2.2 in [CS1]. Let $I \subseteq \mathbb{R}$ be an interval. Consider the following assertion

$$(\text{IS.} k) : \forall \text{ pair of interactive } L_k\text{-D boxes } \Lambda_{L_k}(x) \text{ and } \Lambda_{L_k}(y): \forall E \subseteq I : \text{both } \Lambda_{L_k}(x), \Lambda_{L_k}(y) \text{ are } (E, m_k\text{-S}) \leq L_k^{-2p}.$$  

**Lemma 4.2** Given $k \geq 0$, assume that property $(\text{IS.} k)$ holds true. Consider a box $\Lambda_{L_{k+1}}(u)$ and let $N(\Lambda_{L_{k+1}}(u); E)$ be the maximal number of $(E, m_k\text{-S})$, pair-wise $L_k\text{-D}$-boxes $\Lambda_{L_k}(u^{(j)}) \subset \Lambda_{L_{k+1}}(u)$. Then $\forall \ n \geq 1,$

$$
\mathbb{P} \left\{ \exists E \subseteq I : N(\Lambda_{L_{k+1}}(u); E) \geq 2n \right\} \leq L_k^{-2np}.
$$

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Proof of Lemma 4.2. Suppose $\exists$ I-boxes $\Lambda_{L_k}(u^{(1)}), \ldots, \Lambda_{L_k}(u^{(2n)})$ $\subset \Lambda_{L_k+1}(u)$ such that any two of them are $L_k$-D, i.e., are at the distance $> 8L_k$.

By virtue of Lemma 4.1, it is readily seen that
(a) $\forall$ pair $\Lambda_{L_k}(u^{(2i-1)}), \Lambda_{L_k}(u^{(2i)})$, the respective (random) operators $H^{(2)}_{\Lambda_{L_k}(u^{(2i-1)})}(\omega)$ and $H^{(2)}_{\Lambda_{L_k}(u^{(2i)})}(\omega)$ are independent, and so are their spectra and GFs.

(b) Moreover, the pairs of operators,
$$
\left( H^{(2)}_{\Lambda_{L_k}(u^{(2i-1)})}(\omega), H^{(2)}_{\Lambda_{L_k}(u^{(2i)})}(\omega) \right), \quad i = 1, \ldots, n,
$$
form an independent family.

Indeed, operator $H^{(2)}_{\Lambda_{L_k}(u^{(i)})}$, with $i \in \{1, \ldots, 2n\}$, is measurable relative to the sigma-algebra $\mathcal{B}_i$ generated by random variables $\{V(x), x \in \Pi \Lambda_{L_k}(u^{(i)})\}$, with
$$
\Pi \Lambda_{L_k}(u^{(i)}) = \Pi_1 \Lambda_{L_k}(u^{(i)}) \cup \Pi_2 \Lambda_{L_k}(u^{(i)}) \subset \mathbb{Z}^d.
$$

Now, by Lemma 4.2, the sets $\Pi \Lambda_{L_k}(u^{(i)})$, $i \in \{1, \ldots, 2n\}$, are pairwise disjoint, so that all sigma-algebras $\mathcal{B}_i, i \in \{1, \ldots, 2n\}$, are independent.

Remark. This property formalises the observation made in the beginning of this section: a pair of particles corresponding to an interactive box of size $2L_k$ forms a "compound quantum object" of size $< 8L_k$, and their analysis is quite similar to that from the single-particle MSA.

Thus, any collection of events $A_1, \ldots, A_{n-1}$ related to the corresponding pairs
$$
\left( H^{(2)}_{\Lambda_{L_k}(u^{(2i-1)})}, H^{(2)}_{\Lambda_{L_k}(u^{(2i)})} \right), \quad i = 1, \ldots, n,
$$
also form an independent family.

Now, for $i = 1, \ldots, n - 1$, set
$$
A_i = \{ \exists E \in I: \text{both} \, \Lambda_{L_k}(u^{(2i-1)}), \, \Lambda_{L_k}(u^{(2j+2)}) \text{ are } (E, m_k)-S \}.
$$

Then, by virtue of (IS.$k$),
$$
P \left\{ A_j \right\} \leq L_k^{-2p}, \quad 0 \leq j \leq n - 1,
$$
and by virtue of independence of events $A_0, \ldots, A_{n-1}$, we obtain
$$
P \left\{ \bigcap_{j=0}^{n-1} A_j \right\} = \prod_{j=0}^{n-1} P \left\{ A_j \right\} \leq (L_k^{-2p})^n.
$$
To complete the proof, note that the total number of different families of $2n$ boxes $\Lambda_{L_k} \subset \Lambda_{L_{k+1}}(u)$ with required properties is bounded from above by

$$\frac{1}{(2n)!} \left(2(L_k + r_0 + 1)L_{d+1}^d\right)^{2n} \leq \frac{1}{(2n)!} \left(4L_kL_{d+1}^d\right)^{2n} \leq L_k^{2n(1+d\alpha)},$$

since their centres must belong to the subset $\mathcal{D}_{L_k+r_0} \cap \Lambda_{L_{k+1}}(u)$. Here

$$\mathcal{D}_{L_k+r_0} = \left\{ (x_1, x_2) \in \mathbb{Z}^d \times \mathbb{Z}^d : \|x_1 - x_2\| \leq L_k + r_0 \right\}$$

is a ‘layer’ of width $2(L_k + r_0)$ adjoint to the set $\mathcal{D} := \{x = (x, x), x \in \mathbb{Z}^d\}$, the diagonal in $\mathbb{Z}^d \times \mathbb{Z}^d$.

Recall also that $r_0 < L_0 \leq L_k$, $k \geq 0$, by our assumption. This yields Lemma 4.2.

**Lemma 4.3** Let $K(\Lambda_{L_{k+1}}(u); E)$ be the maximal number of $(E, m_k)$-S, pairwise $L_k$-D boxes $\Lambda_{L_k}(u^{(j)}) \subset \Lambda_{L_{k+1}}(u)$ (interactive or non-interactive). Then \forall \ n \geq 1,

$$\mathbb{P}\left\{ \exists E \in I : K(\Lambda_{L_{k+1}}(u); E) \geq 2n + 2 \right\} \leq L_k^{4d\alpha} \cdot L_k^{-2\tilde{p}} + L_k^{2n(1+d\alpha)} \cdot L_k^{-2\alpha}.$$

(4.9)

**Proof of Lemma 4.3** Assume that $K(\Lambda_{L_{k+1}}(u); E) \geq 2n + 2$. Let $M(\Lambda_{L_{k+1}}(u); E)$ be as in Lemma 3.4 and $N(\Lambda_{L_{k+1}}(u); E)$ as in Lemma 4.2. Obviously,

$$K(\Lambda_{L_{k+1}}(u); E) \leq M(\Lambda_{L_{k+1}}(u); E) + N(\Lambda_{L_{k+1}}(u); E).$$

Then either $M(\Lambda_{L_{k+1}}(u); E) \geq 2$ or $N(\Lambda_{L_{k+1}}(u); E) \geq 2n$. Therefore,

$$\mathbb{P}\left\{ \exists E \in I : K(\Lambda_{L_{k+1}}(u); E) \geq 2n + 2 \right\} \leq \mathbb{P}\left\{ \exists E \in I : M(\Lambda_{L_{k+1}}(u); E) \geq 2 \right\} + \mathbb{P}\left\{ \exists E \in I : N(\Lambda_{L_{k+1}}(u); E) \geq 2n \right\} \leq L_k^{4d\alpha} \cdot L_k^{-2\tilde{p}} + L_k^{2n(1+d\alpha)} \cdot L_k^{-2\alpha},$$

by virtue of (3.24) and (4.4)

An elementary calculation now gives rise to the following.
Corollary 4.1 Under assumptions of Lemma 4.3 with $n \geq 4$, $p \geq 12d + 9$, $\tilde{p} \geq 3p + 3d$, $\alpha = 3/2$, for $L_0 \geq 2$ large enough, we have

$$\mathbb{P} \{ \exists E \in I : K(\Lambda_{L_{k+1}(u)}; E) \geq 2n + 2 \} \leq L_{k+1}^{-2p-1}. \quad (4.10)$$

Remark. Our lower bounds on values of $n$, $p$ and $\tilde{p}$ are not sharp.

Definition 4.1. A box $\Lambda_{L_{k+1}}(v)$ is called $(E,J)$-completely non-resonant ($(E,J)$-CNR in brief), if the following properties are fulfilled:

(i) $\Lambda_{L_{k+1}}(v)$ is $E$-NR;
(ii) all boxes of the form $\Lambda_j(L_{k+1})(y) \subset \Lambda_{L_{k+1}}(v)$, $y \in \Lambda_{L_{k+1}}(v)$, $j = 1, \ldots, J$, are $E$-NR.

As follows from Definition 4.1 and property (W.2), we have

Lemma 4.4 Let $\Lambda_{L_{k+1}}(u') = \Lambda_{L_{k+1}}(u'')$ be two $L_{k+1}$-D boxes. Then, for $L_0 > (J + 1)^2$,

$$\mathbb{P} \{ \forall E \in I : either \Lambda_{L_{k+1}}(u') or \Lambda_{L_{k+1}}(u'') is (E,J)-CNR \} \geq 1 - (J + 1)^2 L_{k+1}^{-(q-2\alpha-1)} > 1 - L_{k+1}^{(-q'-4)} , \quad q' := q/\alpha. \quad (4.11)$$

The statement of Lemma 4.5 below is a simple reformulation of Lemma 4.2 from [DK], adapted to our notations. Indeed, the reader familiar with the proof given in [DK] can see that the structure of the external potential is irrelevant to this completely deterministic statement. So it applies directly to our model with potential $U(x) + gW(x)$. For that reason, the proof of Lemma 4.5 is omitted.

Lemma 4.5 Fix an odd positive integer $J$ and suppose that the following properties are fulfilled:

(i) $\Lambda_{L_{k+1}}(v)$ is $(E,J)$-CNR, and (ii) $K(\Lambda_{L_{k+1}(u)}; E) \leq J$.

Then for sufficiently large $L_0$, box $\Lambda_{L_{k+1}}(v)$ is $(E,m_{k+1})$-NS with

$$m_{k+1} \geq m_k \left( 1 - \frac{5J + 6}{(2L_k)^{1/2}} \right) > m_0/2 > 0. \quad (4.12)$$
Remark. In [DK], it is also assumed that $\alpha < (J + 1)(d + 1/2)$. In our case, this is automatically satisfied with $\alpha = 3/2$ and $J \geq 1$. In particular, with $J = 9$, we obtain
\[
m_{k+1} \geq m_k \left(1 - \frac{51}{(2L_k)^{1/2}}\right) > m_k \left(1 - \frac{40}{L_k^{1/2}}\right),
\]
which explains our assumption (3.1) and the recursive definition (1.13) with $\gamma = 40$.

Now comes a statement which extends Lemma 4.1 from [DK] to pairs of two-particle $L_k$-D I-boxes.

**Theorem 4.1** \(\forall\) given interval \(I \subseteq \mathbb{R}\), there exists \(L_0^* \in (0, +\infty)\) such that if \(L_0 \geq L_0^*\), then, \(\forall k \geq 0\), property (IS.\(k\)) in (4.1) implies (IS.\(k+1\)).

**Proof of Theorem 4.1.** Let \(x, y \in \mathbb{Z}^d \times \mathbb{Z}^d\) and assume that \(\Lambda_{L_k+1}(x)\) and \(\Lambda_{L_k+1}(y)\) are \(L_k\)-D I-boxes. Consider the following two events:
\[
B = \{\exists E \in I: \text{both } \Lambda_{L_k+1}(x) \text{ and } \Lambda_{L_k+1}(y) \text{ are } (E, m_{k+1})-S\},
\]
and, for a given odd integer \(J\),
\[
\Sigma = \{\exists E \in I: \text{neither } \Lambda_{L_k+1}(x) \text{ nor } \Lambda_{L_k+1}(y) \text{ is } (E, J)-\text{CNR}\}.
\]
By virtue of Lemma 4.4, we have, with \(L_0\) large enough \((L_0 \geq J + 1)^2\) and \(\alpha = 3/2\):
\[
\mathbb{P}\{\Sigma\} \leq L_{k+1}^{-(q'-4)}, \quad q' := q/\alpha.
\]  
(4.14)

Further,
\[
\mathbb{P}\{B\} = \mathbb{P}\{B \cap \Sigma\} + \mathbb{P}\{B \cap \Sigma^c\} \leq \mathbb{P}\{\Sigma\} + \mathbb{P}\{B \cap \Sigma^c\},
\]
and we know that \(\mathbb{P}\{\Sigma\} \leq L_{k+1}^{-q'+4}\). So, it suffices now to estimate \(\mathbb{P}\{B \cap \Sigma^c\}\). Within the event \(B \cap \Sigma^c\), for any \(E \in I\), one of the boxes \(\Lambda_{L_k+1}(x), \Lambda_{L_k+1}(y)\) must be \((E, J)\) -CNR. Without loss of generality, assume that for some \(E \in I\), \(\Lambda_{L_k+1}(x)\) is \((E, J)\)-CNR and \((E, m_{k+1})\)-S. By Lemma 4.5, for such value of \(E\), \(K(\Lambda_{L_k+1}(x); E) \geq J + 1\). We see that
\[
B \cap \Sigma^c \subset \{\exists E \in I: \quad K(\Lambda_{L_k+1}(x); E) \geq J + 1\}\]
and, therefore, by Lemma 4.3 with the same values of parameters as in Corollary 4.1, as before:

\[ \mathbb{P}\{B \cap \Sigma^c\} \leq \mathbb{P}\{\exists E \in I: K(\Lambda_{L_{k+1}}(x); E) \geq J + 1\} \leq L_k^{-2p}. \quad (4.15) \]

In what follows we consider \( J = 9 \) although it will be convenient to use symbol \( J \), in particular, to stress analogies with [DK].

5 Mixed pairs of singular two-particle boxes

It remains to derive the property (SS.\( k + 1 \)) in case (III), i.e., for mixed pairs of two-particle boxes (where one is I and the other NI). Here we use several properties which have been established earlier in this paper for all scale lengths, namely, (W1), (W2), (NT.\( k, s \)) with \( s \geq q \), (NIRoNS), and the inductive assumption (IS.\( k + 1 \)) which we have already derived from (IS.\( k \)) in Section 4.

A natural counterpart of Theorem 4.1 for mixed pairs of boxes is the following

**Theorem 5.1** \( \forall \) given interval \( I \subseteq \mathbb{R} \), there exists a constant \( L^*_4 \in (0, +\infty) \) with the following property. Assume that \( L_0 \geq L^*_4 \) and, for a given \( k \geq 0 \), the property (SS.\( k \)) holds: (i) \( \forall \) pair of \( L_k \)-D NI-boxes \( \Lambda_{L_k}(\bar{x}), \Lambda_{L_k}(\bar{y}) \), and (ii) \( \forall \) pair of \( L_k \)-D I-boxes \( \Lambda_{L_k}(\bar{x}), \Lambda_{L_k}(\bar{y}) \).

Let \( \Lambda_{L_{k+1}}(x), \Lambda_{L_{k+1}}(y) \) be a pair of \( L_{k+1} \)-D boxes, where \( \Lambda_{L_{k+1}}(x) \) is I and \( \Lambda_{L_{k+1}}(y) \) NI. Then

\[ \mathbb{P}\{\exists E \in I: \text{both } \Lambda_{L_{k+1}}(x), \Lambda_{L_{k+1}}(y) \text{ are } (E, m_{k+1})-S\} \leq L_{k+1}^{-2p}. \quad (5.1) \]

Before starting a formal proof we give an informal description of our strategy.

1. We are going to list several situations which may give rise to singularity of a mixed pair \( \Lambda_{L_{k+1}}(x) \) (an I-box), \( \Lambda_{L_{k+1}}(y) \) (an NI-box). Next, we show that each situation is covered by an event of (negligibly) small probability. Finally, we show that if neither of these events occurs, the pair of boxes in question cannot be \( (E, m_{k+1})-S \).
2. Given a pair of an I-box and an NI-box, which are \((E, m)\)-S for some (and the same) \(E\), we note first that, owing to \((\text{NIReNS})\), with high probability, the NI-box has to be \(E\)-R. If it is not, we count such an event as an unlikely situation which may give rise to simultaneous singularity of the pair in question.

3. Assuming that the NI-box \(\Lambda_{L_k+1}(y)\) is \(E\)-R, we apply the Wegner-type estimate \((W2)\) and conclude that, with high probability, neither the I-box \(\Lambda_{L_k+1}(x)\), nor any of its sub-boxes of size \(2L_k\) is \(E\)-R. Again, the presence of 'unwanted' \(E\)-R boxes is considered as an unlikely situation. Otherwise, we conclude that \(\Lambda_{L_k+1}(x)\) is \((E, J)\)-CNR.

4. Focusing on the I-box \(\Lambda_{L_k+1}(x)\), we use properties \((W2)\) and \((\text{IS}.k)\) to prove that, with high probability, it contains a limited number of distant sub-boxes of size \(2L_k\) which are \((E, m_k)\)-S. Specifically, it is unlikely that \(\Lambda_{L_k+1}(x)\) contains at least two \(L_k\)-D NI-sub-boxes of size \(2L_k\) (by \((\text{NIReNS})\) and \((W2)\)); it is also unlikely that it contains at least \((J - 1) L_k\)-D I-sub-boxes of size \(2L_k\), by virtue of \((\text{IS}.k)\).

5. Finally, if a two-particle box of width \(2L_{k+1}\) is both \((E, J)\)-CNR and contains at most \((J - 2) + (2 - 1) = J - 1\) distant sub-boxes, it must be \((E, m_{k+1})\)-NS, which is a possibility outside the event in Eqn (5.1). So, the sum of probabilities of the above-mentioned events gives an upper bound for the probability of simultaneous singularity of the given mixed pair of boxes.

**Proof of Theorem 5.1.** Recall that the Hamiltonian \(H^{(2)}_{\Lambda_{L_k+1}(y)}\) is decomposed as in Eqns (3.12), (3.13). Consider the following three events:

\[
B = \left\{ \exists E \in I : \text{both } \Lambda_{L_k+1}(x), \Lambda_{L_k+1}(y) \text{ are } (E, m_{k+1})\text{-S} \right\},
\]

\[
T = \left\{ \text{either } \Lambda_{L_k+1}(y_1) \text{ or } \Lambda_{L_k+1}(y_2) \text{ is } (2m_0)\text{-T} \right\},
\]

and

\[
\Sigma = \left\{ \exists E \in I : \text{neither } \Lambda_{L_k+1}(x) \text{ nor } \Lambda_{L_k+1}(y) \text{ is } (E, J)\text{-CNR} \right\}.
\]

Event \(B\) is the one figuring in the bound (5.1), and we are interested in estimating its probability.
Recall that by virtue of (3.22), we have
\[ P\{T\} \leq L_{k+1}^{-s}, \quad \text{where} \quad s = \frac{\tilde{p} - 2(1 + \alpha)d}{\alpha} = \frac{\tilde{p} - 5d}{\alpha}, \quad (5.2) \]
while for event \( \Sigma \) we have again, by virtue of Lemma 4.4 and inequality (4.13), with our choice of parameters \( J \) and \( L_0 \) (\( J = 9 \) and \( L_0 \) large enough),
\[ P\{\Sigma\} \leq L_{k+1}^{-q+2}. \quad (5.3) \]
Further,
\[ P\{B\} = P\{B \cap T\} + P\{B \cap T^c\} \leq P\{T\} + P\{B \cap T^c\} \leq L_{k+1}^{-s} + P\{B \cap T^c\}. \]
Now, we estimate \( P\{B \cap T^c\} \):
\[ P\{B \cap T^c\} = P\{B \cap T^c \cap \Sigma\} + P\{B \cap T^c \cap \Sigma^c\} \leq P\{\Sigma\} + P\{B \cap T^c \cap \Sigma^c\} \leq L_{k+1}^{-q+2} + P\{B \cap T^c \cap \Sigma^c\}. \]
So, it suffices to estimate \( P\{B \cap T^c \cap \Sigma^c\} \). Within the event \( B \cap T^c \cap \Sigma^c \), one of the boxes \( \Lambda_{L_{k+1}}(x), \Lambda_{L_{k+1}}(y) \) is \( E\)-NR. It cannot be the NI-box \( \Lambda_{L_{k+1}}(y) \). Indeed, by Corollary 4.1, had box \( \Lambda_{L_{k+1}}(y) \) been both \( E\)-NR and \( (2m_0)\)-NT, it would have been \( (E, m_{k+1})\)-NS, which is not allowed within the event \( B \). Thus, the I-box \( \Lambda_{L_{k+1}}(x) \) must be \( E\)-NR, but \( (E, m_{k+1})\)-S:
\[ B \cap T^c \cap \Sigma^c \subset \{ \exists E \in I : \Lambda_{L_{k+1}}(x) \text{ is } (E, m_{k+1})\)-S and \( E\)-NR}. \]
However, applying Lemma 4.5 we see that
\[ \{ \exists E \in I : \Lambda_{L_{k+1}}(x) \text{ is } (E, m_{k+1})\)-S and \( E\)-NR} \subset \{ \exists E \in I : K(\Lambda_{L_{k+1}}(x); E) \geq J + 1 \}. \]
Therefore, with the same values of parameters as in Corollary 4.1
\[ P\{B \cap T^c \cap \Sigma^c\} \leq P\{\exists E \in I : K(\Lambda_{L_{k+1}}(x); E) \geq 2n + 2\} \leq 2L_{k+1}^{-1} L_{k+1}^{-2p}. \quad (5.4) \]
Finally, we get, with \( q' := q/\alpha \),
\[ P\{B\} \leq P\{B \cap T\} + P\{\Sigma\} + P\{B \cap T^c \cap \Sigma^c\} \leq L_{k+1}^{-s} + L_{k+1}^{-q+4} + 2L_{k+1}^{-1} L_{k+1}^{-2p} \leq L_{k+1}^{-2p}. \quad (5.5) \]
if we can guarantee that

$$\max \left\{ L_{k+1}^{-s+2p}, L_{k+1}^{-q'+4+2p}, 2L_{k+1}^{-1} \right\} \leq \frac{1}{3}. \quad (5.6)$$

The bound (5.6) follows from our assumptions, provided that $L_0$ is large enough and

$$s - 2p = \bar{p} - 5d - 2p > 1, \quad q' - 2p - 4 > 1. \quad (5.7)$$

This completes the proof of Theorem 5.1.

Therefore, Theorem 3.2 is proven. In turn, this completes the proof of Theorem 1.1.

6 Proof of Lemma 4.1

Recall that we deal with two-particle boxes $\Lambda' := \Lambda_L(u')$ and $\Lambda'' := \Lambda_L(u'')$ such that

(i) $\text{dist}(\Lambda', \Lambda'') > 8L$ and (ii) $\Lambda' \cap \mathbb{D}_{r_0} \neq \emptyset \neq \Lambda'' \cap \mathbb{D}_{r_0}.$

Recall that we denote by $\mathbb{D}$ the diagonal in $\mathbb{Z}^d \times \mathbb{Z}^d$: $\mathbb{D} = \{ x = (x, x), \ x \in \mathbb{Z}^d \}$. Then property (ii) implies that

$$\Lambda_{L+r_0}(u') \cap \mathbb{D} \neq \emptyset, \ \Lambda_{L+r_0}(u'') \cap \mathbb{D} \neq \emptyset, \quad (6.1)$$

so that $\exists \overline{x}' = (\overline{x}', \overline{x}') \in \Lambda_{L+r_0}(u')$ and $\exists \overline{x}'' = (\overline{x}'', \overline{x}'') \in \Lambda_{L+r_0}(u'')$. Next, observe that

$$\text{dist}(\Lambda_{L+r_0}(u'), \Lambda_{L+r_0}(u'')) \geq \text{dist}(\Lambda_L(u'), \Lambda_L(u'')) - 2r_0 > 8L - 2r_0 > 0, \quad (6.2)$$

owing to the assumption $L > r_0$, and therefore,

$$||\overline{x}' - \overline{x}''|| = ||\overline{x}' - \overline{x}''|| = ||\overline{x}'_2 - \overline{x}''_2|| > 8L - 2r_0, \quad (6.3)$$

since $\overline{x}', \overline{x}'' \in \mathbb{D}$.

Further, for arbitrary points $x' \in \Lambda'$, $x'' \in \Lambda''$, and any $j \in \{1, 2\}$, we can write the triangle inequality as follows:

$$\text{dist}(\overline{x}'_j, \overline{x}''_j) \leq \text{dist}(\overline{x}'_j, x'_j) + \text{dist}(x'_j, x''_j) + \text{dist}(x''_j, \overline{x}''_j)$$
or, equivalently,
\[
\text{dist}(x'_j, x''_j) \geq \text{dist}(\tilde{x}'_j, x''_j) - \text{dist}(x'_j, \tilde{x}'_j) - \text{dist}(x''_j, \tilde{x}'_j)
\]
\[
> 8L - 2r_0 - (2L + 2r_0) - (2L + 2r_0) = 4L - 6r_0 \geq 2L > 0,
\]
(6.4)
since
\[
\text{dist}(\tilde{x}'_j, x'_j) \leq \text{diam}(\Lambda_{L+r_0}) = 2L + 2r_0
\]
and the same upper bound holds for \( \text{dist}(x''_j, \tilde{x}'_j) \).

We see that, for \( j = 1, 2 \),
\[
\text{dist}(\Pi_j \Lambda', \Pi_j \Lambda'') > 2L > 0,
\]
(6.5) so that \( \Pi_1 \Lambda' \cap \Pi_1 \Lambda'' = \emptyset \), \( \Pi_2 \Lambda' \cap \Pi_2 \Lambda'' = \emptyset \).

Finally, to reach the same conclusion for \( \Pi_1 \Lambda' \cap \Pi_2 \Lambda'' \) and \( \Pi_2 \Lambda' \cap \Pi_1 \Lambda'' \),
it suffices to replace \( \Lambda' \) by \( \sigma \Lambda' \) and to use the definition of \( L \)-D boxes:
\[
\text{dist}(\Pi_j (\sigma \Lambda'), \Pi_j \Lambda'') > 8L.
\]
Indeed, we have
\[
\Pi_1 (\sigma \Lambda') = \Pi_2 \Lambda', \quad \Pi_2 (\sigma \Lambda') = \Pi_1 \Lambda'
\]
so that an analogue of inequality (6.5) for boxes \( \sigma \Lambda' \) and \( \Lambda'' \) reads
\[
\text{dist}(\Pi_j (\sigma \Lambda'), \Pi_j \Lambda'') > 2L > 0,
\]
(6.6) yielding
\[
\text{dist}(\Pi_2 \Lambda', \Pi_1 \Lambda'') > 2L > 0, \quad \text{dist}(\Pi_1 \Lambda', \Pi_2 \Lambda'') > 2L > 0,
\]
(6.7) so that \( \Pi_2 \Lambda' \cap \Pi_1 \Lambda'' = \emptyset \), \( \Pi_1 \Lambda' \cap \Pi_2 \Lambda'' = \emptyset \). Now we see that
\[
(\Pi_1 \Lambda' \cup \Pi_2 \Lambda') \cap (\Pi_1 \Lambda'' \cup \Pi_2 \Lambda'') = \emptyset.
\]
(6.8)
This completes the proof of Lemma \[\blacksquare\].

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