Improved Schemes for Asymptotically Optimal Repair of MDS Codes

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Abstract—An \((n,k,l)\) MDS code of length \(n\), dimension \(k\), and sub-packetization \(l\) over a finite field \(F\) is a set of \(n\) symbol vectors of length \(l\) over \(F\) with the property that any \(k\) vectors can recover the entire data of \(kl\) symbols. When a node fails, we can recover it by downloading symbols from the surviving nodes, and the total number of symbols downloaded in the worst case is the repair bandwidth of the code. By the cut-set bound, the repair bandwidth of an \((n,k,l)\) MDS code is at least \((n−1)/l(n−k)\).

There are several constructions of \((n,k,l)\) MDS codes whose repair bandwidths meet or asymptotically meet the cut-set bound. Letting \(r ≡ n−k\) denote the number of parities, Ye and Barg constructed \((n,k,r^n)\) Reed–Solomon codes that asymptotically meet the cut-set bound. Ye and Barg also constructed optimal bandwidth and optimal update \((n,k,r^n)\) MDS codes. A key idea in these constructions is to expand integers in base \(r\).

We show in this paper that, when \(r\) is an integral power, we can significantly reduce the sub-packetization of Ye’s and Barg’s constructions while achieving asymptotically optimal repair bandwidth. As an example, when \(r = 2^s\), we achieve the sub-packetization of \(2^m+n−1\), which improves upon the sub-packetization of \(2^m\) in Ye’s and Barg’s constructions. In general, when \(r = s^m\) for an integer \(s > 2\), our codes have sub-packetization \(l = s^m+n−1 = r^s−1\). Specifically, when \(r = s^m\), we obtain an \((n,k,s^m+n−1)\) Reed–Solomon code and an optimal update \((n,k,s^m+n−1)\) MDS code, which both have asymptotically optimal repair bandwidth. In order to obtain these results, we extend and generalize the \(r\)-ary expansion idea in Ye’s and Barg’s constructions.

Even when \(r\) is not an integral power, we can still obtain \((n,k,s^m+n−1)\) Reed–Solomon codes and optimal update \((n,k,s^m+n−1)\) MDS codes by choosing positive integers \(s\) and \(m\) such that \(s^m \leq r\). In this case, however, the resulting codes have bandwidth that is near-optimal rather than asymptotically optimal.

I. INTRODUCTION

A large file is encoded and distributed among many nodes in a distributed storage system. MDS codes are often used because of their optimal storage versus reliability trade-off. An \((n,k,l)\) MDS code of length \(n\), dimension \(k\), and sub-packetization \(l\) over a finite field \(F\) is a set of \(n\) symbol vectors of length \(l\) over \(F\) with the property that any \(k\) vectors can recover the entire data of \(kl\) symbols.

Although MDS codes can tolerate the maximum number of worst case failures for a given file size and storage space, a more common scenario is when a single node fails. In this case, a replacement node is set up to recover the content stored at the failed node by downloading information from the remaining functional nodes. We are interested in the exact repair problem of recovering the failed node exactly, and the total number of symbols downloaded in the worst case is the repair bandwidth of the code.

By the cut-set bound of [1,2], an \((n,k,l)\) MDS code has repair bandwidth at least

\[
\left(\frac{n−1}{n−k}\right)l. 
\]

(1)

Many \((n,k,l)\) MDS codes with repair bandwidth meeting or asymptotically meeting (1) have been constructed; see [3–18] and the references therein. For example, letting \(r = n−k\) denote the number of parities, Ye and Barg [12] constructed \((n,k,r^n)\) Reed–Solomon codes that asymptotically meet the cut-set bound. Ye and Barg [13] also constructed optimal bandwidth and optimal update \((n,k,r^n)\) MDS codes.

A result of [19], however, shows that in order for the cut-set bound to be achieved by an \((n,k,l)\) MDS code, the sub-packetization \(l\) must satisfy

\[
l \geq \exp\left(\sqrt{\frac{k}{2r−1}}\right). 
\]

(2)

The high-rate optimal bandwidth codes in [3–18] typically require an even larger sub-packetization than (2). For various reasons, which we do not discuss here, it is often desirable to design repair schemes that achieve low repair bandwidth without requiring a high level of sub-packetization.

Our contribution: This paper, like [10,23–27], explores a tradeoff between the sub-packetization \(l\) and the repair bandwidth of the MDS codes. When \(r\) is an integral power, we can significantly reduce the sub-packetization of Ye’s and Barg’s constructions [12,13] while achieving asymptotically optimal repair bandwidth. In the case \(r = 2^s\), for example, we achieve the sub-packetization of \(2^m+n−1\), which improves upon the sub-packetization of \(2^m\) in Ye’s and Barg’s constructions. When \(r = s^m\) for integers \(s > 2\) and \(m > 1\), our codes have sub-packetization \(l = s^m+n−1 = r^s−1\). Specifically, when \(r = s^m\), we obtain an \((n,k,s^m+n−1)\) Reed–Solomon code and an optimal update \((n,k,s^m+n−1)\) MDS code, which both have asymptotically optimal repair bandwidth.

Even when \(r\) is not an integral power, we can still obtain \((n,k,s^m+n−1)\) Reed–Solomon codes and optimal update \((n,k,s^m+n−1)\) MDS codes by choosing positive integers \(s\) and \(m\) such that \(r^s−1 \leq r\). In this case, however, the resulting codes have bandwidth that is near-optimal rather than asymptotically optimal.

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A common feature of the code constructions in [12, 13] is to expand integers in base $r$. To obtain our results, we generalize and extend the $r$-ary expansion technique in these constructions. By choosing positive integers $s$ and $m$ such that $s^m \leq r$, we improve the level of sub-packetization by expanding integers in the smaller base $s$.

In Section III, we discuss the repair of Reed–Solomon codes and, by selecting positive integers $s$ and $m$ with $s^m \leq r$, we adapt Ye’s and Barg’s code [12] to construct $(n,k,s^{m+n-1})$ Reed–Solomon codes, which have asymptotically optimal repair bandwidth when $r = s^m$. In Section III we adapt Ye’s and Barg’s code [13] to construct optimal update $(n,k,s^{m+n-1})$ MDS codes, which have asymptotically optimal repair bandwidth in the case $r = s^m$.

II. REPAIRING REED–SOLOMON CODES

In the conventional solution to the exact repair problem using Reed–Solomon (RS) codes, we split the file into $k$ blocks. Each of the $k$ blocks is represented by some element of a finite field $E$ and then viewed as the coefficient of a polynomial. In this way, the file is identified with a polynomial $f$ over $E$ with degree $k - 1$. We then distribute the file over $n$ nodes by choosing $n$ evaluation points $a_1, \ldots, a_n \in E$ and storing $f(a_i)$ at node $i$. To recover a failed node, we can download information from any $k$ remaining nodes because any $k$ evaluations $f(a_i)$ of a degree $k - 1$ polynomial exactly determine the polynomial and hence the contents of the failed node.

One can show that, in the conventional solution, downloading information from any $k$ remaining nodes is not only sufficient for repairing one node, but also necessary. At first glance, RS codes seem ill-suited for the exact repair problem because recovering the contents of a single failed node requires downloading $k$ symbols of $E$ or, equivalently, the whole file. Thus, despite the ubiquity of RS codes in storage systems, until the recent work of Guruswami and Wootters [20], these codes were regarded as poorly suited for distributed storage applications since they were thought to have a very high repair bandwidth.

To mitigate this issue, we can apply the regenerating codes framework [1] in which a replacement node may download only part of the contents of a surviving node rather than being forced to download the whole node. This is accomplished by viewing $E$ as a vector space over one of its subfields $F$ and allowing each surviving node to return one or more subsymbols of $F$. Crucially, each node may return fewer than $\log_{|F|}|E|$ symbols of $F$ when queried, and our goal is to download as few such subsymbols as possible. The (exact) repair bandwidth of the code over $E$ is the total number of subsymbols downloaded in the worst case. We assume that each node returns an $F$-linear function of its contents so that we have a linear repair scheme.

Since RS codes are MDS codes, the cut-set bound [1] and the lower bound [2] from [19] apply. Very recently, Tamo, Ye, and Barg [22] improved [2] for RS codes, and showed that any RS code meeting the cut-set bound has a sub-packetization $l$ that satisfies

$$l \geq \exp\left((1 + o(1))k \log k\right).$$

They also explicitly constructed in [22] RS codes meeting the cut-set bound whose sub-packetization is given by

$$l = \exp\left((1 + o(1))n \log n\right).$$

We note that the lower bound on the sub-packetization in [3] does not apply to RS codes whose repair bandwidth meets the cut-set bound only asymptotically as $n \to \infty$. For example, Ye and Barg [12] have previously constructed such RS codes (whose repair bandwidth asymptotically meets the cut-set bound) with sub-packetization $l = r^n$. If $r$ is fixed, while $n \to \infty$, then $r^n$ could be significantly lower than $3$.

When the sub-packetization $l$ is small, we cannot hope to meet the cut-set bound. However, Guruswami and Wootters [20] showed that an $(n,k,l)$ MDS code with a linear repair scheme must have bandwidth at least

$$(n - 1) \log_{|F|} \left(\frac{(n - 1)|E|}{(r - 1)(|E| - 1) + (n - 1)}\right).$$

Subsequently, Dau and Milenkovic [21] refined and improved [4] in some cases[4]. Moreover, full-length RS codes meeting [4] were explicitly constructed in [20] and [21]. Necessarily, these codes have small sub-packetization $l$; the optimal bandwidth RS code in [20] has sub-packetization $l = \log_{s^m/r}(n)$, for example. In Table I, we summarize the tradeoffs between sub-packetization and repair bandwidth for explicit RS code

| Code construction | Repair Bandwidth | Sub-packetization | Meets Cut-Set Bound |
|-------------------|------------------|-------------------|---------------------|
| $(n,k)$ RS code [20] | $n - 1$ | $l = \log_{s^m/r} n$ | No, but meets (4) |
| $(n,k)$ RS code [21] | $(n - 1)!|(1 - \log_{s^m} r)$ | $\log_{s^m} n$ | No, but meets (4) |
| $(n,k)$ RS code (this paper) | < $\frac{(n-1+3/r)l}{r}$ for $s^m$ | $l = s^{m+n-1}$ | Asymptotically meets (1) for $r = s^m$ |
| $(n,k)$ RS code [12] | < $(n+1)!l$ | $l \approx n^r$ | Asymptotically meets (1) |
| $(n,k)$ RS code [22] | $(n-1)!l$ | $l \approx n^r$ | Yes, meets (1) |

TABLE I
Tradeoff between repair bandwidth and sub-packetization for Reed–Solomon codes.

\[\text{The bound in [4] is Dau’s and Milenkovic’s derivation of Guruswami’s and Wootters’ bound.}\]
constructions. Further work on repairing RS codes can be found in [28–31].

Our contribution: We show in Theorem 3 that, when $r$ is an integral power, the sub-packetization of RS codes can be significantly reduced, while maintaining asymptotically optimal repair bandwidth. More precisely, when $r = s^n$ for a positive integer $s \geq 2$, our RS codes have sub-packetization $l = s^n + n - 1 = rs^{n-1}$. This improves on the sub-packetization of $l = r^n$ in Ye’s and Barg’s construction [12] of RS codes with asymptotically optimal repair bandwidth. Even when $r$ is not an integral power, we show how to obtain RS codes with near-optimal bandwidth in Corollary 7.

The rest of this section is organized as follows. In Section II-A, we discuss Guruswami and Wootters’ characterization of linear exact repair schemes for MDS codes [20], on which our result and the results in [12, 20–22, 28–31] are based. We present our RS code with improved sub-packetization in Section II-B.

A. Linear Repair Schemes for RS Codes

In a RS code, a codeword is a sequence of function values of a polynomial of degree less than $k$. Given a finite field $E$, let $E[x]$ denote the ring of polynomials over $E$.

Definition 1. A generalized Reed–Solomon code, denoted $GRS(n, k, A, v)$, of dimension $k$ over the finite field $E$ using $n$ evaluation points $A = \{\alpha_1, \ldots, \alpha_n\} \subseteq E$ is the set of vectors

$$\left\{(v_1f(\alpha_1), \ldots, v_nf(\alpha_n)) : f \in E[x], \deg(f) < k\right\} \subseteq E^n,$$

where $v = (v_1, \ldots, v_n)$ are some nonzero coefficients in $F$. When $v = (1, \ldots, 1)$, the corresponding generalized Reed–Solomon code $GRS(n, k, A, 1) = RS(n, k, A)$ is called a Reed–Solomon code.

We have that $RS(n, k, A, 1)$, the dual of a Reed-Solomon code $RS(n, k, A)$, is a generalized Reed-Solomon code $GRS(n, k, A, v)$, where

$$v_i = \prod_{j \neq i}(\alpha_i - \alpha_j)^{-1} \quad (5)$$

by [32, Theorem 4 in Chapter 10].

We formalize the definition of a linear repair scheme for the Reed-Solomon code $RS(n, k, A)$ that we discussed above. Recall that each node $n$ returns an $F$-linear function of its contents $f(\alpha)$ where $F$ is a subfield of $E$. One example of an $F$-linear function from $E$ to $F$ is the field trace $tr_{E/F}$.

Definition 2. Let $E = GF(q^d)$ be an extension of degree $l$ of the field $F = GF(q)$. The field trace $tr_{E/F}$ is defined to be

$$tr_{E/F}(\beta) = \beta + \beta^q + \beta^{q^2} + \cdots + \beta^{q^{l-1}}.$$  

Conversely, one can show that the $F$-linear functions from $E$ to $F$ are precisely the trace functionals $tr_{E/F} : E \rightarrow F$ that are given by $L_{\gamma}(\beta) = tr_{E/F}(\gamma \beta)$ for $\gamma \in E$.

In a linear repair scheme, a node that stores $f(\alpha)$ therefore returns elements of $F$ of the form $L_{\gamma}(f(\alpha))$. The field elements $\gamma \in E$ used by each node thus describe a linear repair scheme for $RS(n, k, A)$. The following definition of a linear exact repair scheme is from Guruswami and Wootters [20].

Definition 3. A linear exact repair scheme for $RS(n, k, A)$ over a subfield $F \subseteq E$ consists of

- For each $\alpha_i \in A$ and for each $\alpha_j \in A \setminus \{\alpha_i\}$, a set of queries $Q_{ij}(i) \subseteq E$.
- For each $\alpha_i \in A$, a linear reconstruction algorithm that computes

$$f(\alpha_i) = \sum_{h=1}^{l} \lambda_h \mu_h \quad (6)$$

for coefficients $\lambda_h \in F$ and a basis $\mu_1, \ldots, \mu_l$ for $E$ over $F$ so that the coefficients $\lambda_h$ are $F$-linear combinations of the queries

$$\bigcup_{j \in [n] \setminus \{i\}} \{L_{\gamma}(f(\alpha_j)) : \gamma \in Q_{ij}(i)\}. \quad (7)$$

The repair bandwidth $b$ of the linear exact repair scheme is the total number of subsymbols in $F$ returned by each node in the worst case

$$b = \max_{i \in [n]} \sum_{j \in [n] \setminus \{i\}} |Q_{ij}(i)|. \quad (8)$$

Recall that $l = \log_E|E|$ is the dimension of $E$ as a vector space over $F$. Guruswami and Wootters [20] show that specifying a linear repair scheme for $RS(n, k, A)$ over $F$ is equivalent to finding, for each $\alpha_i \in A$, a set of $l$ polynomials $P_i \subset E[x]$ of degree less than $n - k$ such that $\{p(\alpha_i) : p \in P_i\}$ is a basis for $F$ over $E$ when $i = j$ and spans a low-dimensional subspace over $F$ when $i \neq j$. Specifically, the following theorem is due to Guruswami and Wootters [20].

Theorem 1. Let $F \subseteq E$ be a subfield so that the degree of $E$ over $F$ is $l$ and let $A \subseteq E$ be any set of evaluation points. The following are equivalent.

1. There is a linear repair scheme for $RS(n, k, A)$ over $F$ with bandwidth $b$.
2. For each $\alpha_i \in A$, there is a set $P_i \subset E[x]$ of $l$ polynomials of degree less than $n - k$ such that

$$\dim_F(\{p(\alpha_i) : p \in P_i\}) = l \quad (9)$$

and the sets $\{p(\alpha_i) : p \in P_i\}$ for $j \neq i$ satisfy

$$b \geq \sum_{j \in [n] \setminus \{i\}} \dim_F(\{p(\alpha_i) : p \in P_j\}). \quad (10)$$

The RS code constructed in Section II-B as well as the RS codes constructed in [12, 20–22, 28–31], rely on the fact that the second statement in Theorem 1 implies the first, so we sketch the proof.

Suppose that the codeword symbol $f(\alpha)$ in a codeword $(f(\alpha_1), \ldots, f(\alpha_n)) \in RS(n, k, A)$ is erased.

Lemma 2. For a basis $\{\xi_1, \ldots, \xi_l\}$ for $E$ over $F$, the value of $f(\alpha)\xi$ can be uniquely recovered from the values

$$\{tr_{E/F}(\xi_h f(\alpha_i))\}_{h=1}^l.$$
Proof. If \( \{ \mu_1, \ldots, \mu_l \} \) is the dual (trace-orthogonal) basis of the basis \( \{ \xi_1, \ldots, \xi_l \} \), then
\[
 f(\alpha_i) = \sum_{h=1}^{l} \text{tr}_{E/F}(\zeta_h f(\alpha_i))\mu_h. \tag{11}
\]

Suppose we have \( l \) codewords \( \{ c_{h,1}^+, \ldots, c_{h,m}^+ \} \) in \( \text{RS}(n,k,A) \) such that \( \{ c_{h,1}^+, \ldots, c_{h,m}^+ \} \) is a basis for \( E \) over \( F \). By Lemma 2, to find the value of \( f(\alpha_i) \), it suffices to find the values of \( \{ \text{tr}_{E/F}(c_{h,j}^+ f(\alpha_i)) \} \) for all \( h \).

Lemma 3. From \( \{ (\text{tr}_{E/F}(c_{h,j}^+ f(\alpha_i))) \} \) \( h=1 \) \( j \in [n] \setminus \{ i \} \), we can recover the values of \( \{ \text{tr}_{E/F}(c_{h,j}^+ f(\alpha_i)) \} \) \( h=1 \).

Proof. By duality and because \( \text{tr}_{E/F} \) is \( F \)-linear, we have for all \( h \in [l] \) that
\[
 \text{tr}_{E/F}(c_{h,j}^+ f(\alpha_i)) = - \sum_{j \neq i} \text{tr}_{E/F}(c_{h,j}^+ f(\alpha_i)). \tag{11}
\]

Similarly, Lemma 4 holds because \( \text{tr}_{E/F} \) is \( F \)-linear.

Lemma 4. Define \( Q_j(i) \) to be a maximum linearly independent subset of the set \( \{ c_{h,j}^+ \} \) \( h=1 \). We can find the values of
\[
 \{ \text{tr}_{E/F}(c_{h,j}^+ f(\alpha_i)) \} \quad (h=1)
\]
for \( j \in [n] \setminus \{ i \} \) from the values of
\[
 \{ \text{tr}_{E/F}(\gamma f(\alpha)) : \gamma \in Q_j(i) \} = \{ L(\gamma f(\alpha)) : \gamma \in Q_j(i) \}.
\]

Now we show how to specify a linear exact repair scheme for \( \text{RS}(n,k,A) \) over \( F \) as in Definition 3.

Lemma 5. A set of \( l \) codewords \( \{ c_{h,1}^+, \ldots, c_{h,m}^+ \} \) in \( \text{RS}(n,k,A) \) such that \( \{ c_{h,1}^+, \ldots, c_{h,m}^+ \} \) is a basis for \( E \) over \( F \), suffices to specify a linear exact repair scheme for \( \text{RS}(n,k,A) \) over \( F \) as in Definition 3.

Proof. Setting \( \lambda_h = \text{tr}_{E/F}(c_{h,1}^+ f(\alpha_i)) \) and letting \( \mu_1, \ldots, \mu_l \) be a dual basis for the basis \( \{ c_{h,1}^+, \ldots, c_{h,m}^+ \} \), we have that (11) holds by (11). Moreover, defining \( Q_j(i) \) as in Lemma 3, we see that the coefficients \( \lambda_h \) are \( F \)-linear combinations of the queries in (11) by Lemma 3 and Lemma 4.

Since \( \text{RS}(n,k,A) = \text{GRS}(n,k,A,v) \), where \( v \) is given by (5), we see that the hypothesis of Lemma 5 is equivalent to (9) and that the right hand sides of (8) and (10) are equal. Hence, the second statement of Theorem 1 implies the first.

B. Repair Schemes with Improved Sub-Packetization
Recall that Ye and Barg [12] explicitly constructed \( \text{RS}(n,k,r^n) \) RS codes whose bandwidth asymptotically meets the cut-set bound. In this section, we generalize their construction and improve their sub-packetization. Theorem 5 gives a precise statement of our main result for RS codes.

Theorem 6. Let \( n \) and \( k \) be arbitrary fixed integers and suppose that \( n-k = s^m \) where \( s \geq 2 \) and \( m \geq 1 \). Let \( F \) be a finite field and let \( h(x) \) be a degree \( l \) irreducible polynomial over \( F \) where \( l = s^{m+n-1} \). Let \( \beta \) be a root of \( h(x) \) and set the symbol field \( E = F(\beta) \) to be the field generated by \( \beta \) over \( F \). Choose the set of evaluation points to be \( A = \{ \beta^0, \beta^1, \ldots, \beta^{s^{m-1}} \} \). The exact repair bandwidth of the code \( \text{RS}(n,k,A) \) over \( F \) is at most
\[
 (n - 1 + 3s^{m-1} + 2s^{m-2} + \cdots + 2s - (m-4)) l, \tag{12}
\]
and hence asymptotically meets (11) for fixed \( n - k \) as \( n \to \infty \).

Note that the construction in Theorem 6 generalizes Ye’s and Barg’s construction in [12] because setting \( s = n-k \) and \( m = 1 \) in Theorem 6 yields their result. When \( r = s^m \), instead of expanding integers in base \( r \) as Ye and Barg [12] do, we will expand integers in base \( s \).

Even if \( r \) is not an integral power, we can still use the ideas in the proof of Theorem 6. For example, we can choose any positive integers \( s \) and \( m \) such that \( s^m \leq r \). The statement and proof of Theorem 6 still hold if we replace by \( s^m \) every occurrence of \( n-k \). For fixed \( r \), as \( n \to \infty \), the ratio between the repair bandwidth of the resulting RS codes and (11) would be \( r/s^m \), which is at most 2. We summarize this result in the following corollary.

Corollary 7. Let \( n \) and \( k \) be arbitrary fixed integers and suppose that \( n - k \geq s^m \) where \( s \geq 2 \) and \( m \geq 1 \). The exact repair bandwidth over \( F \) of the code \( \text{RS}(n,k,A) \) constructed in Theorem 6 is at most
\[
 (n - 1 + 3s^{m-1} + 2s^{m-2} + \cdots + 2s - (m-4)) l. \tag{13}
\]
For fixed \( n - k \), as \( n \to \infty \), the ratio between (13) and (11) would be \( (n-k)/s^m \), which is at most 2.

In this section, we prove Theorem 8, a version of Theorem 6 with a slightly weaker bound on the repair bandwidth.

Theorem 8. The exact repair bandwidth over \( F \) of the code \( \text{RS}(n,k,A) \) constructed in Theorem 6 is at most
\[
 (n - 1 + 2(m-1)s^m - 2m + 6) l, \tag{14}
\]
and hence asymptotically meets (11) for fixed \( n - k \) as \( n \to \infty \).

More involved counting and case analysis yield the repair bandwidth bound (12) in Theorem 6 which we present in Section IV. We use the notation \( [x,y] = \{ x, x+1, \ldots, y \} \) for integers \( x < y \).

Proof of Theorem 8. From Theorem 6 it suffices to find, for each \( i \in [0,n-1] \), a set of \( l \) polynomials \( \{ f_{ij} \} \) satisfying \( \text{deg}(f_{ij}) < n-k \) so that \( f_{ij}(\beta^{s^j}) \) form a basis for \( E \) over \( F \) and so that
\[
 \sum_{0 \leq i \leq n-1} \dim_F \left( \{ f_{ij}(\beta^{s^j}) \} \right) \leq l. \tag{15}
\]

is bounded above by \( (14) \).

Given \( a \in [0,l-1] \), we can write its \( s \)-ary expansion as \( (a_{m+n-2}, \ldots, a_0) \); that is
\[
 a = \sum_{j=0}^{m+n-2} a_j s^j, \tag{16}
\]

where $a_j \in [0, s-1]$ for $j \in [0, m+n-2]$. Define the set
$S_i$ for $i \in [0, n-1]$ by
$$S_i = \{a \in [0, l-1] : a_i = \cdots = a_{i+m-1} = 0\},$$
define the set of $l$ polynomials
$$\{ f_{i,j} \}_{j=1}^l = \{ \beta^a z^j : a \in S_i, z \in [0, n-k-1] \}.$$ Notice that, for each $i$, we have defined $l$ polynomials because $|S_i| = s^{n-1}$ and there are $n-k = s^m$ choices for $z$.

**Claim 9.** For $i \in [0, n-1]$, the set $\{ f_{i,1}(\beta^s), \ldots, f_{i,l}(\beta^s) \}$ is a basis for $E$ over $F$.

**Proof.** Since $n-k = s^m$ and $z \in [0, n-k-1]$, we can write the $s$-ary expansion of $z$ as
$$z = \sum_{j=0}^{m-1} z_j \beta^j,$$
where $z_j \in [0, s-1]$ for $j \in [0, m-1]$. Consequently,
$$\beta^a (\beta^s)^z = \beta^a (\beta^s)^{\sum_{j=0}^{m-1} z_j \beta^j} = \beta^a + \sum_{j=0}^{m-1} z_j \beta^{s+j}.$$ By considering $s$-ary expansions, we see that as $a$ ranges over $S_i$ and as $z_j$ ranges over $[0, s-1]$ for $j \in [0, m-1]$, we have $a + \sum_{j=0}^{m-1} z_j \beta^{s+j}$ ranges over $[0, l-1]$. We thus have
$$\{ f_{i,1}(\beta^s), \ldots, f_{i,l}(\beta^s) \} = \{1, \beta, \ldots, \beta^l\},$$
where the latter set is clearly a basis for $E$ over $F$.

Our remaining task is to bound (15) from above by (14). We demonstrate upper bounds for $\dim_F(\{ f_{i,j}(\beta^s) \}_{j=1}^l)$ in two cases according to whether $t < i$ or $t > i$.

**Claim 10.** If $t \in [0, i-1]$ and $i-t \geq m$, then
$$\dim_F(\{ f_{i,j}(\beta^s) \}_{j=1}^l) \leq \frac{l}{n-k} + \frac{l}{s^{i-t}}.$$ 

**Proof.** Define the set
$$S_{i,t} = \{u \in [0, l-1] : u_{i+m-1} = \cdots = u_{i+1} = 0, u_i = 1, u_{i-1} = \cdots = u_{t+m} = 0\}.$$ We claim that if $t \in [0, i-1]$ and $i-t \geq m$, then
$$\{ f_{i,j}(\beta^s) \}_{j=1}^l = \{ \beta^a + z \beta^s : a \in S_i, z \in [0, n-k-1] \} \subseteq \{ \beta^a : u \in S_i \cup S_{i,t} \}.$$ Please refer to Figure 1. By considering the $s$-ary expansions of $a$ and $zs^t$, we see that if $u = a + zs^t \notin S_i$, then there are carries from coordinate $t+m-1$ to coordinate $m+n-1$, which explains (16). Claim 10 follows because
$$|S_{i,t}| = \frac{l}{n-k}.$$

**Claim 11.** If $t \in [i+1, n-1]$ and $t-i \geq m$, then
$$\dim_F(\{ f_{i,j}(\beta^s) \}_{j=1}^l) \leq \frac{l}{n-k} + \frac{l}{s^{m+n-t-1}}.$$ 

**Proof.** Define the set
$$S_{i,t} = \{u \in [0, s^{m+n-1}] : u_{m+n-1} = 1, u_{m+n-2} = \cdots = u_{m+t} = 0, u_{i+m-1} = \cdots = u_i = 0\}.$$ We claim that if $t \in [i+1, n-1]$ and $t-i \geq m$, then
$$\{ f_{i,j}(\beta^s) \}_{j=1}^l = \{ \beta^a + z \beta^s : a \in S_i, z \in [0, n-k-1] \} \subseteq \{ \beta^a : u \in S_i \cup S_{i,t} \}.$$ Please refer to Figure 2. By considering the $s$-ary expansions of $a$ and $zs^t$, we see that if $u = a + zs^t \notin S_i$, then there are carries from coordinate $m+t-1$ to coordinate $m+n-1$, which explains (17) and (18). Claim 11 follows because
$$|S_{i,t}| = \frac{l}{s^{m+n-t-1}}.$$
Finally, we bound (15) from above by (14). An upper bound on (15) is
\[
\frac{(n - 2(m - 1) + 2(m - 1)(n - k))}{n - k} + \sum_{t=1}^{i-m} \frac{l}{s^t} + \sum_{t=i+m}^{n-1} \frac{l}{s^m+n-t}.
\tag{19}
\]
If \( i < m \) then the first sum in (19) is omitted. Similarly, if \( i > n - 1 - m \), then the last sum in (19) is omitted.

The first term in (19) comes from summing the \( l / (n - k) \) in Claim 10 and Claim 11 and using the trivial bound
\[
\dim_F\left(\{f_{ij}(\beta^s)\}_{j=1}^l\right) \leq l
\tag{20}
\]
for all \( t \in [i - (m - 1), i + (m - 1)] \). The second and third terms in (19) come from Claim 10 and Claim 11 respectively. Using well-known formulas for geometric series, we can bound the second term in (19) from above by
\[
\sum_{t=1}^{i-m} \frac{l}{s^t} < \frac{2l}{s^m}.
\tag{21}
\]
Finally, using well-known formulas for geometric series, we can bound the last term in (19) by
\[
\sum_{t=i+m}^{n-1} \frac{l}{s^m+n-t} < \frac{2l}{s^m}.
\tag{22}
\]
Summing the first term of (19) and the right-hand-sides of (21) and (22) yields that (15) is bounded above by (14). This completes the proof of Theorem 8.

### III. Optimal Bandwidth and Optimal Update Codes

In an MDS code, each parity node is a function of the entire information stored in the system. Consequently, when an information element changes its value, each parity node needs to update at least one of its elements. An **optimal update code** is one in which each parity node needs to update exactly one of its elements when an information element changes value. Optimal update codes are desirable since updating is a frequent operation.

One way to construct optimal update MDS codes is to encode the parity nodes with diagonal encoding matrices. In other words, each parity node \( C_{k+i} \in F^l \) for \( i \in [r] \) is defined by
\[
C_{k+i} = \sum_{j=1}^{k} D_{ij} C_j,
\tag{23}
\]
where \( C_1, \ldots, C_k \in F^l \) are the systematic nodes and \( D_{ij} \) is an \( l \times l \) diagonal matrix. In [13], Ye and Barg construct optimal bandwidth and optimal update \((n, k, r^n)\) MDS codes with diagonal encoding matrices.

**Our contribution:** In this section, we show that when \( r = s^m \) is an integral power, we can adapt Ye’s and Barg’s construction [13] to obtain optimal update \((n, k, s^{m+n-1})\) MDS codes with diagonal encoding matrices and asymptotically optimal repair bandwidth. We construct our code in Construction 12 and show that it has asymptotically optimal repair bandwidth in Theorem 13. Even when \( r \) is not an integral power, we show how to obtain optimal update MDS codes with near-optimal bandwidth in Corollary 14.

Note that, with our level of sub-packetization, our MDS codes cannot meet the cut-set bound (1) because a result of [33] shows that an optimal bandwidth \((n, k, l)\) MDS code with diagonal encoding matrices satisfies \( l \geq r^k \). For fixed \( r = s^m \), as \( n \to \infty \), we have \( s^{m+n-1} < r^k \), so the most we can hope for is asymptotically optimal repair bandwidth.

Let \( C \) be an \((n, k, l)\) MDS code with nodes \( C_i \in F^l \) represented as column vectors for \( i \in [n] \). We consider codes defined in the following parity-check form
\[
C = \left\{(C_1, \ldots, C_n) : \sum_{i=1}^{n} A_{ij} C_i = 0, t \in [r]\right\},
\tag{24}
\]
where \( A_{ij} \) is an \( l \times l \) matrix over \( F \) for \( t \in [r] \) and \( i \in [n] \). Given positive integers \( r \) and \( n \), define an \((n, k, l)\) MDS code \( C \) by setting in (24)
\[
A_{ij} = A_{i-1,j}, t \in [r], i \in [n],
\tag{25}
\]
where \( A_{1,j} \) are \( l \times l \) matrices that will be specified in Construction 12. We use the convention \( A^0 = I \).

**Definition 4.** Let \( s \geq 2 \) and \( m \geq 1 \) be positive integers. Let \( l = s^{m+n-1} \). Given \( a \in [0, l-1] \), we can write its \( s \)-ary expansion as \((a_{m+n-1}, \ldots, a_0)\); that is
\[
a = \sum_{j=0}^{m+n-1} a_j s^j - 1,
\]
where \( a_j \in [0, s-1] \) for \( j \in [m + n - 1] \). For \( i \in [n] \), let \( a_{(i+m-1, \ldots, i)} \) be the unique \( x \in [0, s^{m-1}] \) such that the \( s \)-ary representation of \( x \) is \((a_{i+m-1, \ldots, i})\).

To illustrate Definition 4, consider the following example.

**Example 1:** Let \( s = 2, m = 2, n = 10 \), and \( l = 2^{13} \). Let \( a = 6 \) whose binary expansion is \((0, 0, 0, 1, 0, 1)\). We have \( 6_{(2,1)} = 2, 6_{(3,2)} = 3, 6_{(4,3)} = 1, 6_{(i+1, j)} = 0 \) if \( i \in [4, 10] \).

We now show how to adapt Ye’s and Barg’s Construction 1 in [13].

**Construction 12.** Let \( n \) and \( k \) be fixed integers and suppose that \( n - k \geq s^m \), where \( s \geq 2 \) and \( m \geq 1 \). Let \( F \) be a finite field of size \(|F| \geq s^m n \) and let \( l = s^{m+n-1} \). Let \( \lambda_{(i,j)} \in [n,j] \in [0,s^{m-1}] \) be \( s^m n \) distinct elements in \( F \). Consider the code family given by (24) and (25) where we take
\[
A_i = \sum_{a=0}^{l-1} \lambda_{i,a(i+m-1, \ldots, i)} e_a e_i^*, \quad i \in [n].
\]
Here, \( \{e_a : a \in [0, l-1]\} \) is the standard basis of \( F^l \) over \( F \).

Since the \( A_i \) for \( i \in [n] \) are diagonal matrices, we can write out the parity-check equations coordinatewise. Letting \( c_{i,a} \) denote the \( a^{th} \) coordinate of the column vector \( C_i \), we have for all integers \( a \in [0, l-1] \) and \( t \in [0, r-1] \) that
\[
\sum_{i=1}^{n} A_{i,a(i+m-1, \ldots, i)} c_{i,a} = 0.
\tag{26}
\]
To illustrate (26) consider the following example.

**Example 2:** Suppose \( s = 2, m = 2, r = 4, n = 10, \) and \( l = 2^{11} \). For \( n \in \{0, 2, 4, 6\} \) and \( t \in [0, 3] \), the equations in (26) are
\[
\begin{align*}
\lambda_{1,0}^t c_{1,0} & + \lambda_{2,0}^t c_{2,0} + \lambda_{3,0}^t c_{3,0} + \lambda_{4,0}^t c_{4,0} + \sum_{j=5}^{10} \lambda_{j,0}^t c_{j,0} = 0. \\
\lambda_{1,2}^t c_{1,2} + \lambda_{2,2}^t c_{2,2} + \lambda_{3,2}^t c_{3,2} + \lambda_{4,2}^t c_{4,2} + \sum_{j=5}^{10} \lambda_{j,2}^t c_{j,2} = 0. \\
\lambda_{1,4}^t c_{1,4} + \lambda_{2,4}^t c_{2,4} + \lambda_{3,4}^t c_{3,4} + \lambda_{4,4}^t c_{4,4} + \sum_{j=5}^{10} \lambda_{j,4}^t c_{j,4} = 0. \\
\lambda_{1,6}^t c_{1,6} + \lambda_{2,6}^t c_{2,6} + \lambda_{3,6}^t c_{3,6} + \lambda_{4,6}^t c_{4,6} + \sum_{j=5}^{10} \lambda_{j,6}^t c_{j,6} = 0.
\end{align*}
\]

Theorem 13 addresses the repair bandwidth of the code in Construction 12 when \( r = s^m \). The full proof of Theorem 13 is presented in Section V.

**Theorem 13.** If \( n - k = s^m \), the exact repair bandwidth of the code in Construction 12 is at most
\[
\left( \frac{n - 1 + 2 \sum_{i=1}^{s^m-1} (s^i - 1)}{n - k} \right) l,
\]
and hence asymptotically meets (1) for fixed \( n - k \) as \( n \to \infty \).

As in Corollary 2, even if \( r \) is not an integral power, we can still use the ideas in Theorem 13. We can choose any positive integers \( s, m, r, n \) such that \( s^m \leq r \). The statement and proof of Theorem 13 still hold if we replace \( s^m \) every occurrence of \( n - k \). For fixed \( r \), as \( n \to \infty \), the ratio between the repair bandwidth of the resulting codes and (1) would be \( r/s^m \), which is at most 2. We summarize this result in the following corollary.

**Corollary 14.** If \( n - k \geq s^m \), the exact repair bandwidth of the code in Construction 12 is at most
\[
\left( \frac{n - 1 + 2 \sum_{i=1}^{s^m-1} (s^i - 1)}{n - k} \right) l,
\]
For fixed \( n - k \), as \( n \to \infty \), the ratio between (28) and (1) would be \( (n-k)/s^m \), which is at most 2.

In this section, we prove Theorem 15 with a slightly weaker bound on the repair bandwidth. In conjunction with the proof of Theorem 15 the reader may find it useful to consult Example 3, which illustrates the notation in Theorem 15.

**Theorem 15.** If \( n - k = s^m \), the exact repair bandwidth of the code in Construction 12 is at most
\[
\left( \frac{n - 1 + 2(m - 1)s^m - 2m + 2}{n - k} \right) l,
\]
and hence asymptotically meets (1) for fixed \( n - k \) as \( n \to \infty \).

**Proof.** For \( (w_m, \ldots, w_1) \in [0, s-1]^m \) and \( a \in [0, l-1] \), define \( a(i; w_m, \ldots, w_1) \) to be the integer whose

s-ary representation is obtained from the s-ary representation of \( a \) by replacing the \( m \) coordinates \( a_{i+m-1}, \ldots, a_i \) with \( w_m, \ldots, w_1 \) respectively; that is
\[
a(i; w_m, \ldots, w_1) = (a_{n+m-1}, \ldots, a_{i+m}, w_m, \ldots, w_1, a_{i-1}, \ldots, a_1).
\]

For \( i \in [n] \) and \( a \in [0, l-1] \), let \( S_{ai} \) be the set of integers in \([0, l-1]\) whose s-ary representation is obtained from the s-ary representation of \( a \) by replacing the \( m \) coordinates \( a_{i+m-1}, \ldots, a_i \) with \( w_m, \ldots, w_1 \) respectively for all \((w_m, \ldots, w_1) \in [0, s-1]^m\); that is
\[
S_{ai} = \{ a(i; w_m, \ldots, w_1) : (w_m, \ldots, w_1) \in [0, s-1]^m \}.
\]

For \( a \in [0, l-1] \) and \( j \in [n] \) \( \setminus \{i\} \), we define elements in \( F \) by
\[
u_{i,j}^{(a)} = \sum_{a' \in S_{ai}} c_{i,a'}. \tag{32}
\]

We will show that for any \( i \in [n] \) and \( a \in [0, l-1] \), the \( r \) coordinates
\[
\{ c_{i,a'} : a' \in S_{ai} \} \tag{33}
\]
in \( C_i \) are functions of the following set \( D_i^{(a)} \subset F \),
\[
D_i^{(a)} = \{ u_{i,j}^{(a)} : |j - i| \geq m \} \cup \{ c_{i,a'} : |j - i| < m, a' \in S_{ai} \}.
\]

To repair \( r \) coordinates in the failed node, a surviving node \( j \) needs to transmit one scalar in \( F \) if \( |j - i| \geq m \) and \( r \) scalars otherwise. Consequently,
\[
|D_i^{(a)}| \leq (n - 1) - 2(m - 1) + 2(m - 1)r,
\]
and so the repair bandwidth of the code in Construction 12 is at most (29).

We write (26) for \( t \in [0, r-1] \) and sum over \( a' \in S_{ai} \). When \( t = 0 \), we obtain
\[
\sum_{a' \in S_{ai}} c_{i,a'} = -\sum_{j \neq i} u_{i,j}^{(a)} \tag{34}
\]
Notice that if \( |j - i| \geq m \), then for all \( a' \in S_{ai}, \) the value of \( a'(j + m - 1, \ldots, j) \) is the same. For \( j \) such that \( |j - i| \geq m \) define \( l_{ai} \in (0, r-1] \) to be the value of \( a'(j + m - 1, \ldots, j) \) for any \( a' \in S_{ai} \). When \( 1 \leq t \leq r - 1 \), we obtain
\[
\sum_{a' \in S_{ai}} \lambda_{i,a'}^{t} c_{i,a'} = -\sum_{j \leq |j - i| \geq m} \lambda_{i,j}^{t} u_{i,j}^{(a)} - \sum_{|j - i| < m} \sum_{a' \in S_{ai}} \lambda_{i,a'}^{t} c_{i,a'} \tag{35}
\]
For \( t = 0 \), let \( B_0 \) denote the right-hand-side of (34). Similarly, for \( 1 \leq t \leq r - 1 \), let \( B_t \) denote the right-hand-side of (35). Writing the system of equations given by (34) and (35) in matrix form yields
\[
\begin{bmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
\lambda_{i,0}^{t-1} & \cdots & \lambda_{i,r-1}^{t-1}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
c_{i,a(0,0,0)} \\
\vdots \\
c_{i,a(s-1,s-1)}
\end{array}
\end{bmatrix}
= \begin{bmatrix}
B_0 \\
\vdots \\
B_{r-1}
\end{bmatrix}. \tag{36}
\]
Since $B_0, \ldots, B_{r-1}$ are functions of the elements in $D_1^{(a)}$ and since $\lambda_{i,0}, \ldots, \lambda_{i,r-1}$ are distinct, we can solve the system in (33) for the $r$ coordinates in (33) given $D_1^{(a)}$.

To illustrate Theorem 15, consider the following example.

**Example 3:** Suppose $s = 2$, $m = 2$, $r = 4$, $n = 10$, and $l = 2^{11}$. Suppose node 2 has failed so $i = 2$. Letting $a = 0$, we have in (30) that

$$0(2; 0, 0) = 0, 0(2; 0, 1) = 2, 0(2; 1, 0) = 4, 0(2; 1, 1) = 6.$$ 

Hence, $S_{0,2} = \{0, 2, 4, 6\}$ and

$$u_{j,2} = c_{j,0} + c_{j,2} + c_{j,4} + c_{j,6}. \quad (37)$$

The four coordinates in (33) are $\{c_{2,0,}, c_{2,2,}, c_{2,4,}, c_{2,6}\}$ and we claim these coordinates are functions of the set $D_2^{(a)} = \{u_{j,2} : 4 \leq j \leq 10\} \cup \{c_{j,a} : j = \{1, 3\}, a' \in \{0, 2, 4, 6\}\}.$

To see why the coordinates $c_{2,0}, c_{2,2}, c_{2,4}, c_{2,6}$ are functions of the set $D_2^{(a)}$, sum the equations in Example 2. When $t = 0$, we obtain

$$c_{2,0} + c_{2,2} + c_{2,4} + c_{2,6} = -\sum_{j \neq 2} u_{j,2}^{(a)}. \quad (38)$$

We have $l_{0,j} = 0$ for $4 \leq j \leq 10$, so when $1 \leq t \leq 3$, we obtain

$$\lambda_{2,0}^{l_{0}(2,2)} + \lambda_{2,2}^{l_{0}(2,2)} + \lambda_{2,4}^{l_{0}(2,2)} + \lambda_{2,6}^{l_{0}(2,2)} \quad (39)$$

We now show that the code in Construction 12 is MDS.

**Theorem 16.** The code $C$ given by Construction 12 is MDS.

**Proof.** Writing the parity check equations coordinate-wise, we have for all $a \in [0, l - 1]$ that

$$\begin{bmatrix} 1 & \cdots & 1 \\ \lambda_{1,a(m,\ldots,1)} & \cdots & \lambda_{n,a(n+m-1,\ldots,n)} \\ \cdots & \cdots & \cdots \\ \lambda_{r-1,a(m,\ldots,1)} & \cdots & \lambda_{r-1,a(n+m-1,\ldots,n)} \end{bmatrix} \begin{bmatrix} c_{1,a} \\ \vdots \\ c_{n,a} \end{bmatrix} = 0. \quad (39)$$

Since every $r$ columns of the parity-check matrix in (39) have rank $r$, any $k$ out of $n$ elements in the set $\{c_{1,a}, \ldots, c_{n,a}\}$ can recover the whole set. As this holds for all $a \in [0, l - 1]$, any $k$ nodes of a codeword in $C$ can recover the whole codeword.

Finally, we show that the code constructed in Construction 12 has diagonal encoding matrices and is thus optimal update.

**Theorem 17.** The code $C$ given by Construction 12 has diagonal encoding matrices and is thus optimal update.

**Proof.** We assume that the first $k$ nodes $C_{1,\ldots,k}$ are the systematic nodes, and we seek diagonal encoding matrices $D_{i,j}$ satisfying (23). Let

$$V_1^{(a)} = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_{k+1,a(m,\ldots,1)} & \cdots & \lambda_{n,a(n+m-1,\ldots,n)} \\ \cdots & \cdots & \cdots \\ \lambda_{k+1,a(n+m-1,\ldots,1)} & \cdots & \lambda_{n,a(n+m-1,\ldots,n)} \end{bmatrix}$$

$$V_2^{(a)} = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_{1,a(m,\ldots,1)} & \cdots & \lambda_{k,a(k+m-1,\ldots,1)} \\ \cdots & \cdots & \cdots \\ \lambda_{1,a(n,\ldots,1)} & \cdots & \lambda_{k,a(n+m-1,\ldots,1)} \end{bmatrix}.$$ 

By (39), we have

$$V_1^{(a)} = -V_2^{(a)}, \quad (40)$$

or equivalently,

$$\begin{bmatrix} c_{k+1,a} \\ \vdots \\ c_{n,a} \end{bmatrix} = -V_2^{(a)} - 1 \begin{bmatrix} c_{1,a} \\ \vdots \\ c_{k,a} \end{bmatrix}.$$ 

Letting $M^{(a)} = -V_2^{(a)} - 1$, we have for each $i \in [r]$ and $a \in [0, l - 1]$ that

$$c_{k+i,a} = M^{(a)}(i,1)c_{1,a} + \cdots + M^{(a)}(i,k)c_{k,a},$$

where $M^{(a)}(i,j)$ is the $(i,j)$ entry of the matrix $M^{(a)}$. Defining an $l \times l$ diagonal matrix $D_{i,j}$ by

$$D_{i,j}(a, a) = M^{(a)}(i,j),$$

we have that (23) is satisfied.

**IV. APPENDIX A: PROOF OF THEOREM 6**

In this section, we prove Theorem 6. First we improve the bound in Claim 10. If $S$ is a set and $d \in \mathbb{Z}_+$ is a positive integer, recall that $S^d$ is the Cartesian product of $S$ taken $d$ times. Define

$$X_d = [0, s - 1]^d \setminus \{(s - 1, \ldots, s - 1)\}, \quad (40)$$

$$Y_d = [0, s - 1]^d \setminus \{(0, \ldots, 0)\}. \quad (41)$$

**Claim 18.** If $t \in [0, i - 1]$ and $i - t \geq m$, then

$$\dim_F \left(\left\{f_{i,j}(S^t)^{T}\right\}_{j=1}^l\right) \leq \frac{l}{m - k} \cdot \frac{(sm - 1)}{s^{i-t+m}}.$$ 

**Proof.** Define the set

$$S_{i,t} = \{u \in [0, l - 1] : u_{i+m-1} = \cdots = u_{i+1} = 0, \quad u_i = 1, u_{i-1} = \cdots = u_{i+m} = 0 \}, \quad (42)$$

$$u_{i+m-1}, \ldots, u_i \in X_m}. \quad (43)$$

We claim that if $t \in [0, i - 1]$ and $i - t \geq m$, then

$$\left\{f_{i,j}(S^t)^{T}\right\}_{j=1}^l = \{B^{a + z_2} : a \in S_{i,t}, \ z \in [0, n - k - 1]\}$$

$$\subseteq \{B^a : u \in S_{i} \cup S_{i,t}\}. \quad (44)$$

**Claim 19.** If $t \in [0, i - 1]$ and $i - t \geq m$, then

$$\dim_F \left(\left\{f_{i,j}(S^t)^{T}\right\}_{j=1}^l\right) \leq \frac{l}{m - k} \cdot \frac{(sm - 1)}{s^{i-t+m}}.$$ 

**Proof.** Define the set
In Theorem 8, we proved a weaker version of Theorem 6.

Let $\beta$, $\eta$, and $\gamma$ lie in $[0, s-1]$, $t \in [i - m - 1, i + m - 1]$ denote the first coordinate from which there is a carry in $u$. Since $a_y$ and $c_y$ lie in $[0, s-1]$, this implies that $u_y \in [0, s-2]$ and that (42) holds. Consequently, if $u \notin S_f$, then $u \in S_{i,t}$. Claim 19 follows because

$$|S_i| = \frac{1}{n-k} \quad \text{and} \quad |S_{i,t}| = \left(\frac{s^n - 1}{s^{i-t+m}}\right) \cdot k.$$ 

In Theorem 8, we proved a weaker version of Theorem 6 by using, for all $t \in [i - (m - 1), i + (m - 1)]$, the trivial bound (20). We now improve (20) in this case.

Claim 19. If $t \in [0, i-1]$ and $i - t < m$, write $t = i - w$, where $w \in [m-1]$. We have

$$\dim_f(\{f_{ij}(\beta^{s^j})\}_{j=1}^l) \leq \frac{1}{n-k} + \left(\frac{s^n - 1}{s^{m+w}}\right).$$

Proof. Define the sets

$$S_{i,t,1} = \{u \in [0, l-1] : u_{i+m-1} = \cdots = u_{i-m+w+1} = 0, u_{i+m-w+1} = 1, u_{i+m-w+1} = \cdots = u_i = 0, (u_{i+1}, \ldots, u_{i-w}) \in X_w\}$$

and carries from coordinate $i - 1$ to coordinate $i + m - w$.

As in Claim 18, we must have that (43) holds because the position of first carry occurs in $[i - w, i - 1]$. Hence, if $u_{i+m-w} = 1$ then $u \in S_{i,t,1}$.

If $u_{i+m-w} = 0$ and $u \notin S_i$ then $u \in S_{i,t,0}$. Claim 19 holds because

$$|S_{i,t,1}| = \frac{(s^w - 1)l}{s^{m+w}}, \quad \text{and} \quad |S_{i,t,0}| = \frac{(s^m - s^w)l}{s^{m+w}}.$$ 

Claim 20. If $t \in [i + 1, n-1]$ and $t - i < m$, write $t = i + w$ where $w \in [m-1]$. We have

$$\dim_f(\{f_{ij}(\beta^{s^j})\}_{j=1}^l) \leq \frac{1}{n-k} + \left(\frac{s^w - 1}{s^{i-w}}\right).$$

Proof. Define the sets

$$S_{i,t,1} = \{u \in [0, s^{m+n-1}] : u_{m+n-1} = 1, u_{m+n-1} = \cdots = u_{m+w+1} = 0, (u_{i+1}, \ldots, u_{i+w}) \in X_w\}$$

We claim that if $t \in [i + 1, n-1]$ and $t - i < m$, then

$$\{f_{ij}(\beta^{s^j})\}_{j=1}^l = \{\beta^{a+z} : a \in S_{i,t}, z \in [0, n-k-1]\} \subseteq \{\beta^u : u \in S_i \cup S_{i,t,1} \cup S_{i,t,0}\}.$$

Please refer to Figure 3 and let $u = a + zs^t$. By considering the $s$-ary expansions of $a$ and $zs^t$, we have $u_{i+m-w} = 0$. In order for $u_{i+m-w} = 1$, we need

$$u_i = \cdots = u_{i+m-w-1} = 0$$

and carries from the coordinate $i + w + m - 1$ to the coordinate $m + n - 1$. As in Claim 18 we have that (45)
holds because the position of first carry lies in the interval \([i + m, i + w + m - 1]\). Hence, if \(u_{n+m-1} \neq 0\), we must have \(u \in S_{i,t,1}\).

If \(u_{n+m-1} = 0\) and \(u \notin S_i\) then \(u \in S_{i,t,0}\). Claim 20 holds because

\[
|S_{i,t,1}| = \frac{(s^w - 1)l}{s^{n+w-1} - 1}, \quad \text{and} \quad |S_{i,t,0}| = \frac{(s^m - 1)l}{s^m}.
\]

Finally, we bound (15) from above by (12). Let

\[ M = \min\{m - 1, n - 1 - i\} \]

denote the minimum of \(m - 1\) and \(n - 1 - i\). Summing the bounds in Claim 18, Claim 19, Claim 20, and Claim 11 respectively, an upper bound on (18) is

\[
\frac{(n-1)l}{n-k} + \sum_{t=1}^{i} \frac{(s^m-1)l}{s^{n+m+i-1} - 1} + \sum_{w=1}^{M} \frac{(s^w-1)l}{s^{n+w-1}} + \sum_{t=1}^{n-1} \frac{l}{(n+m-l-1)}.
\]

(46)

If \(i > n - 1 - m\), then the last sum in (46) is omitted.

The first sum in (46) comes from combining the bounds in Claim 18 and Claim 19 and holds regardless of whether \(i < m\) or \(i \geq m\). Using well-known formulas for geometric series, we can bound the first sum in (46) from above by

\[
\sum_{t=1}^{i} \frac{(s^m-1)l}{s^{n+m+i-1}} \leq \frac{(s^m-1)l}{s^m} \cdot (1+i).
\]

(47)

Note that the second sum in (46) only appears if \(i < n - 1\). Hence, we can bound the second sum in (46) from above by

\[
\sum_{w=1}^{M} \frac{(s^w-1)l}{s^{n+w-1}} \leq \frac{Ml}{s^{n+1}} < \frac{l}{s}.
\]

(48)

Similarly, we can bound the third sum in (46) by

\[
\sum_{w=1}^{M} \frac{(s^w-1)l}{s^{n+w-1}} \leq \frac{(s^m-1)l}{s^m} \cdot (s^m-1).
\]

(49)

Finally, using well-known formulas for geometric series, we can bound the last sum in (46) by (22). Summing the right-hand-sides of (47), (48), (49), and (22) yields that (15) is bounded above by (12). This completes the proof of Theorem 6.

V. APPENDIX B: PROOF OF THEOREM 13

In this section, we prove Theorem 13. We proved a weaker version of Theorem 13 in Theorem 15 by allowing, in the repair of node \(i\), for node \(j\) to transmit all of its elements if \(|j - i| < m\). We now show that node \(i\) can still be repaired if node \(j\) transmits only \(s^{l-i}\) scalars in \(F\) when \(|j - i| \leq m\). In conjunction with the proof of Theorem 13 the reader may find it useful to consult Example 4, which illustrates the notation in Theorem 13.

Proof of Theorem 13 For integers \(a \in [0, l - 1]\) and \(i \in [n]\), we will show that the \(r\) coordinates in (33) are functions of a set of at most \(n - 1 + 2 \sum_{w=1}^{m-w}(s^{w-1})\) elements of \(F\).

For \(a \in [0, l - 1]\), \(j \in [n]\) such that \(|j - i| = w < m\), and \((b_{m-w}, \ldots, b_1) \in [0, s - 1]^{m-w}\), we must first define a set \(T_{a,j}(b_{m-w}, \ldots, b_1)\). If \(j > i\), then \(T_{a,j}(b_{m-w}, \ldots, b_1)\) is the set of all integers in \([0, l - 1]\) whose \(s\)-ary representation is obtained from \(a\) by replacing the \(m\) coordinates \(a_{i+m-1}, \ldots, a_{i}\) with \(b_{m-w}, \ldots, b_1\) respectively for all tuples \((a_0, \ldots, a_1) \in [0, s - 1]^{m-w}\). If \(j < i\), then \(T_{a,j}(b_{m-w}, \ldots, b_1)\) is the set of all integers in \([0, l - 1]\) whose \(s\)-ary representation is obtained from \(a\) by replacing the \(m\) coordinates \(a_{i+m-1}, \ldots, a_{i}\) with \(d_{m-w}, \ldots, d_{m-w+1}, b_{m-w}, \ldots, b_1\) respectively for all \((d_{m-w}, \ldots, d_{m-w+1}, b_{m-w}, \ldots, b_1) \in [0, s - 1]^{m-w}\). For \((b_{m-w}, \ldots, b_1) \in [0, s - 1]^{m-w}, a \in [0, l - 1]\), and \(j \in [n]\) such that \(|j - i| = w < m\), and \((b_{m-w}, \ldots, b_1) \in [0, s - 1]^{m-w}\), we must define elements in \(F\) by

\[
T_{a,j}(b_{m-w}, \ldots, b_1) = \bigcup_{a' \in T_{a,j}(b_{m-w}, \ldots, b_1)} u_{i,j}^{(a)}(b_{m-w}, \ldots, b_1) = a' \in T_{a,j}(b_{m-w}, \ldots, b_1)
\]

(51)

Recall the definition of \(u_{i,j}^{(a)}\) from (32). We claim that the \(r\) coordinates in (33) are functions of the set \(E_{i}^{(a)} \subset F\),

\[
E_{i}^{(a)} = \{u_{i,j}^{(a)} : |j - i| \geq m\} \cup \{u_{i,j}^{(a)}(b_{m-w}, \ldots, b_1) : |j - i| = w < m, (b_{m-w}, \ldots, b_1) \in [0, s - 1]^{m-w}\}.
\]

(50)

It may be useful to consult Example 4, which illustrates this claim for \(a = 0\) and \(i = 2\) in our running example of \(s = 2, m = 2, r = 4, n = 10\), and \(l = 2\). To repair \(r\) coordinates in the failed node \(i\), a surviving node \(j\) needs to transmit one scalar in \(F\) if \(|j - i| \geq m\) and \(s^{l-i}\) scalars in \(F\) otherwise. Consequently, the size of \(E_{i}^{(a)}\) is at most \(n - 1 + 2 \sum_{w=1}^{m-w}(s^{w-1})\), and so the repair bandwidth of the code in Construction 12 is at most (27).

We write (26) for \(t \in \{0, r - 1\} \) and \(m \) sum over \(a' \in S_{a,i}\). When \(t = 0\), we still obtain (34). If \(|j - i| = w < m\), then the value \(a'(j + m - 1, \ldots, j)\) is the same over all \(a'\) in \(T_{a,j}(b_{m-w}, \ldots, b_1)\) for a fixed \((b_{m-w}, \ldots, b_1) \in [0, s - 1]^{m-w}\). For \(j\) satisfying \(|j - i| = w < m\) and a fixed \((b_{m-w}, \ldots, b_1) \in [0, s - 1]^{m-w}\), define \(a_{j}(b_{m-w}, \ldots, b_1) \in [0, r - 1]^{m-w}\) to be the value of \(a'(j + m - 1, \ldots, j)\) for any \(a' \in T_{a,j}(b_{m-w}, \ldots, b_1)\). Recalling the definition of \(a_{j}(b_{m-w}, \ldots, b_1)\) from the proof of Theorem 13, we obtain when \(1 \leq t \leq r - 1\) that

\[
\sum_{a' \in S_{a,j}} \lambda_{i,j}^{(a')}(b_{m-w}, \ldots, b_1) c_{i,a'} = - \sum_{j:|j-i|\geq m} \lambda_{j}^{(a)}(b_{m-w}, \ldots, b_1) \sum_{j:|j-i|=w<m} \lambda_{j}^{(a)}(b_{m-w}, \ldots, b_1). \quad (52)
\]
For $t = 0$, let $B_0$ denote the right-hand-side of (34). Similarly, for $1 \leq t \leq r - 1$, let $B_t$ denote the right-hand-side of (52). Since $B_0, \ldots, B_{r-1}$ are functions of the elements in $E_t(\delta)$ and since $\lambda_{i,0}, \ldots, \lambda_{i,r-1}$ are distinct, we can solve the system in (36) for the $r$ coordinates in (33) given $E_t(\delta)$.

To illustrate Theorem 1 consider the following example.

Example 4: Suppose $s = 2, m = 2, r = 4, n = 10$, and $l = 2^{11}$. Suppose node 2 has failed so $i = 2$. Letting $a = 0$, we have from Example 3 that $S_{0,2} = \{0, 2, 4, 6\}$ and the four coordinates in (33) are $\{c_{2,0}, c_{2,2}, c_{2,4}, c_{2,6}\}$.

In this example, equations (50) and (51) are

\[
T_{0,1,2}(0) = \{0, 4\}, \quad T_{0,1,2}(1) = \{2, 6\}, \\
T_{0,3,2}(0) = \{0, 2\}, \quad T_{0,3,2}(1) = \{4, 6\}, \\
u_{1,2}(0) = c_{1,0} + c_{1,4}, \quad u_{1,2}(1) = c_{1,2} + c_{1,6}, \\
u_{3,2}(0) = c_{3,0} + c_{3,2}, \quad u_{3,2}(1) = c_{3,4} + c_{3,6}.
\]

Recalling the definition of $u_{j,2}$ from (51), we claim these coordinates are functions of the set

\[
E_2(0) = \{u_{j,2} : 4 \leq j \leq 10\} \cup \{u_{1,2}(0), u_{1,2}(1), u_{3,2}(0), u_{3,2}(1)\}.
\]

To see why the coordinates $c_{2,0}, c_{2,2}, c_{2,4}, c_{2,6}$ are functions of the set $E_2(0)$ in (53), sum the equations in Example 2. When $t = 0$, we still obtain $\lambda_{2,0}c_{2,0} + \lambda_{2,1}c_{2,1} + \lambda_{2,2}c_{2,2} + \lambda_{2,3}c_{2,3} = -\lambda_{1,0}(c_{1,0} + c_{1,4}) - \lambda_{1,2}(c_{1,2} + c_{1,6}) - \lambda_{3,0}(c_{3,0} + c_{3,2}) - \lambda_{3,1}(c_{3,4} + c_{3,6}) - \sum_{j=4}^{10} \lambda_{j,0}(c_{j,0} + c_{j,2} + c_{j,4} + c_{j,6}) = -\delta_1(0)u_{1,2}(0) - \delta_1(1)u_{1,2}(1) - \delta_3(0)u_{3,2}(0) - \delta_3(1)u_{3,2}(1) - \sum_{j=4}^{10} \delta_j(0)u_{j,2}. \]

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