Bundle gerbes on supermanifolds

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Abstract

We show that every bundle gerbe on a supermanifold decomposes into a bundle gerbe over the underlying manifold and a 2-form on the supermanifold. This decomposition is not canonical, but is determined by the choice of a projection map to the underlying manifold. Along the way, we prove the familiar cohomological classification theorems for bundle gerbes and bundle gerbes with connection in the context of supermanifolds.

1 Introduction

A ‘bundle gerbe’ is a geometric object defined over a base manifold whose characteristic class sits in the third cohomology of the base [18]. It is analogous to a complex line bundle, and its characteristic class is an analogue of the Chern class in degree 2. A line bundle can be equipped with a connection, and when this is done the curvature of the connection is a closed 2-form and is a de Rham representative of the Chern class. Analogously, a bundle gerbe can also be equipped with a connection, and when this is done the curvature is a closed 3-form and is a de Rham representative of the characteristic class.

The theory of line bundles with connection has been extended to supermanifolds by Kostant [15] in his study of prequantization in supergeometry. We extend this analysis to bundle gerbes on supermanifolds, proving an analogous cohomological classification to the one on ordinary manifolds: bundle gerbes with connection are classified, up to equivalence, by the Deligne cohomology of a supermanifold.

We then analyze exactly what is gained by extending the theory of bundle gerbes from manifolds to supermanifolds: up to equivalence, every bundle gerbe on a supermanifold is determined by a bundle gerbe on the underlying manifold, called the body, and a 2-form on the supermanifold. This decomposition is not canonical, but depends on a choice of projection to the body.

To state this result explicitly, let $M$ be a supermanifold. For any supermanifold, there is a canonical inclusion of an ordinary manifold $i: M_b \to M$. Invoking the poetic terminology of DeWitt [11], we call $M_b$ the **body** of $M$, though it is more commonly called the **reduced manifold**. A bundle gerbe $G$ on $M$ can be pulled back along the inclusion $i$ to yield a bundle gerbe $i^*G$ on $M_b$. It turns out every bundle gerbe on $M_b$ can be obtained in this way, up to equivalence.

What about conversely? Can we recover $G$ from $i^*G$? To examine this, we choose a **body projection**: a map $p: M \to M_b$ satisfying $pi = 1_{M_b}$. Unlike the inclusion $i$, this map is not canonical, but it does exist in the category of smooth supermanifolds. Our main result is that every
bundle gerbe $\mathcal{G}$ on the supermanifold $M$ decomposes as follows, up to equivalence:

$$\mathcal{G} \simeq \pi^* \mathcal{G}_b \otimes I_\beta.$$  

Here, $\mathcal{G}_b$ is bundle gerbe on the body $M_b$, and $I_\beta$ is the trivial bundle gerbe with ‘curving’ given by the 2-form $\beta \in \Omega^2(M)$. Tensoring with $I_\beta$ has the effect of adding the 2-form $\beta$ to the curving of $\pi^* \mathcal{G}_b$, a part of the connection data. Moreover, $\mathcal{G}_b$ is unique—it must be equivalent to $i^* \mathcal{G}$—whereas $\beta$ is unique up to the addition of an exact 2-form, provided it is chosen so that $i^* \beta = 0$, as can always be done.

With enough experience in supergeometry, this decomposition makes intuitive sense. Folklore tells us that all the topological information in $M$ lies in the body $M_b$. Bundle gerbes without connection are topological objects, so we expect a bijection between bundle gerbes without connection on $M_b$ and those on $M$. Thus any bundle gerbe $\mathcal{G}$ on $M$ without connection must satisfy $\mathcal{G} \simeq \pi^* \mathcal{G}_b$ for some bundle gerbe $\mathcal{G}_b$ on $M_b$ without connection. Tensoring with the dual of $\mathcal{G}$, this says:

$$\mathcal{G}^* \otimes \pi^* \mathcal{G}_b \simeq \mathcal{I},$$  

where $\mathcal{I}$ is the trivial bundle gerbe without connection. Equipping $\mathcal{G}$ and $\mathcal{G}_b$ with connection, we thus have:

$$\mathcal{G}^* \otimes \pi^* \mathcal{G}_b \simeq I_\beta,$$  

because a trivial bundle gerbe with connection $I_\beta$ is a 2-form, $\beta$. This is essentially our result.

Our work is not the first to consider bundle gerbes on supermanifolds; there are two important antecedents. First, bundle gerbes are a special case of principal 2-bundles [3, 21], and Schreiber has built the machinery for principal $\infty$-bundles over supermanifolds [23]. For bundle gerbes, he constructs a higher stack denoted $\mathcal{B}^2 U(1)_{\text{conn}}$ on the site of supermanifolds, and the objects of this stack are bundle gerbes with connection. Separately, Suszek has constructed bundle gerbes over supermanifolds of relevance to string theory, focusing on the subtle interplay of bundle gerbes with supersymmetry, $\kappa$-symmetry, and İnönü–Wigner contraction [25].

This paper is organized as follows: in Section 2, we outline the basics of supergeometry. Section 3 is the core of the paper. In Section 3.1, we define bundle gerbes over supermanifolds and show they are classified, up to equivalence, by the Dixmier–Douady class in degree 3. In Section 3.2, we add connection data to our bundle gerbes, and show that bundle gerbes with connection are classified by Deligne cohomology. In Section 3.3, we define the curvature 3-form, and give necessary and sufficient conditions for a 3-form to be the curvature of a bundle gerbe. Finally, in Section 3.4, we prove our main result, decomposing a bundle gerbe on a supermanifold into a bundle gerbe on the body and a 2-form on the supermanifold.

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2 Elements of supergeometry

In the physics literature, a ‘supermanifold’, often called a ‘superspace’, is a space with both even, commuting coordinates and odd, anticommuting coordinates. For mathematicians, it is familiar
that any point of an \( n \)-dimensional manifold has a neighborhood that we can describe using \( n \) coordinate functions, \( \{x_i\} \). Since these are real-valued functions, they pairwise commute in the algebra of smooth functions on this neighborhood. The idea of the generalization from manifold to supermanifold is that any point of a \( m|n \)-dimensional supermanifold has a neighborhood we can describe using \( m \) even coordinates \( \{x_i\} \), which pairwise commute, along with \( n \) odd coordinates \( \{\theta_j\} \), which pairwise anticommute:

\[
\theta_j \theta_j' = -\theta_j' \theta_j.
\]

Since real-valued functions form a commutative algebra, the coordinates \( \{x, \theta_j\} \) are not functions. Instead, they are elements of a \( \mathbb{Z}_2 \)-graded commutative algebra, which is related to the algebra of smooth functions in a precise way: we obtain the algebra of smooth functions by quotienting out by the ideal generated by \( \{\theta_j\} \), or in other words by setting \( \theta_j = 0 \). This is the beginning of supergeometry.

In this section, we give a rapid review of those parts of supergeometry we will need. For a more leisurely introduction, we recommend the survey article of Leites [16], the books of Manin [17] or Caston, Carmeli and Fiorese [7], or the notes of Deligne and Morgan [10].

### 2.1 Superalgebra

We begin with the algebra underlying supergeometry. A super vector space \( V \) is a \( \mathbb{Z}_2 \)-graded real vector space, \( V = V_0 \oplus V_1 \). We refer to the \( \mathbb{Z}_2 \)-grading as the parity, and say elements of \( V_0 \) are even while those of \( V_1 \) are odd. As usual, elements in \( V_i \) are called homogeneous, and the parity of a homogeneous element \( v \in V_i \) is denoted by vertical braces: \( |v| = i \).

A supercommutative superalgebra \( A \) is a \( \mathbb{Z}_2 \)-graded real associative algebra with unit, \( A = A_0 \oplus A_1 \), such that \( ab = (-1)^{|a||b|}ba \) for all homogeneous elements \( a \) and \( b \) of parity \( |a| \) and \( |b| \), respectively. The key example of a supercommutative superalgebra is the Grassmann algebra \( \Lambda(\theta_1, \ldots, \theta_n) \), free on \( n \) odd generators \( \theta_i \).

A left module of a supercommutative superalgebra \( A \) is super vector space \( M \) where \( A \) acts on the left, \( A \otimes M \to M \), respecting the unit, multiplication, and parity:

- \( 1m = m \);
- \( a(bm) = (ab)m \);
- \( |am| = |a| + |m| \).

A right module is defined similarly, and the tensor product \( M \otimes_A N \) of left module \( M \) with a right module \( N \) is the usual notion of graded tensor product. Because \( A \) is supercommutative, any left module is automatically a right module, and vice versa, when we define \( am = (-1)^{|a||m|}ma \) for \( a \in A \) and \( m \in M \). Thus we will stop distinguishing left and right modules.

We say that a module \( M \) is free of finite rank if it admits a finite homogeneous basis: this is a pair of finite subsets \( \{e_1, \ldots, e_m\} \subseteq M_0 \) and \( \{f_1, \ldots, f_n\} \subseteq M_1 \) such that the union is linearly independent over \( A \) and spans \( M \). The pair of natural numbers \( m|n \) is called the rank of \( M \) and is independent of the choice of basis. In supergeometry, the rank or dimension is always an ordered pair of natural numbers, and we adopt the notation \( m|n \) for this ordered pair.

### 2.2 Supermanifolds

Having described the necessary algebra, we are ready to introduce the main objects of study in supergeometry.

**Definition 1.** A supermanifold \( M \) of dimension \( m|n \) is a pair \( (M_b, \mathcal{O}_M) \), where:

- \( M_b \) is a topological manifold of dimension \( m \), called the body of \( M \); in particular, \( M_b \) is paracompact, second countable and Hausdorff;
• $\mathcal{O}_M$ is a sheaf of supercommutative superalgebras on $M_b$, called the **structure sheaf** of $M$; the sheaf $\mathcal{O}_M$ is required to be **local**, meaning that each stalk $\mathcal{O}_{M,b}$ is a local ring, containing a unique maximal ideal;

• $M$ is equipped with an atlas extending a topological atlas of $M_b$: for each chart $(U, \varphi: U \to \mathbb{R}^n)$ in our atlas, we choose a local isomorphism of sheaves of superalgebras:

$$\phi: C^\infty_{\varphi(U)} \otimes \Lambda(\theta_1, \ldots, \theta_n) \to \mathcal{O}_M|_U.$$  

Here, $C^\infty_{\varphi(U)}$ is the sheaf of smooth functions on the open set $\varphi(U) \subseteq \mathbb{R}^n$, $\mathcal{O}_M|_U$ is the restriction of $\mathcal{O}_M$ to the open set $U \subseteq M_b$, and $\phi$ is **local** in the sense that it preserves the maximal ideals on the stalks. The triple $(U, \varphi, \phi)$ is called a **chart** on $M$, and the collection of charts is an **atlas** for $M$.

**Example 2.** Here are some first examples of supermanifolds:

• Any smooth manifold $X$ of dimension $m$ is trivially a supermanifold of dimension $m|0$, with structure sheaf the sheaf of smooth functions $C_\infty^X$.

• A **super Cartesian space** is a supermanifold of the form $\mathbb{R}^{m|n} = (\mathbb{R}^m, C^\infty_{\mathbb{R}^m} \otimes \Lambda(\theta_1, \ldots, \theta_n))$.

• A **super domain** $U^{m|n}$ of dimension $m|n$ is the restriction of the structure sheaf on $\mathbb{R}^{m|n}$ to an open set $U \subseteq \mathbb{R}^m$; in other words, $U^{m|n} = (U, C^\infty_U \otimes \Lambda(\theta_1, \ldots, \theta_n))$.

• More generally, given any supermanifold $M$ and open set $U \subseteq M_b$ of the body, $U$ becomes a supermanifold in the obvious way: $U = (U, \mathcal{O}_M|_U)$.

• Given any vector bundle $E \to X$ over a smooth manifold $X$, we obtain the supermanifold denoted $\Pi E = (X, \Lambda E^*)$, where we write $E$ for the sheaf of sections of $E$. In words, the structure sheaf of $\Pi E$ is given by sections of the exterior algebra bundle of $E^*$, the dual of $E$. If $E$ has rank $n$ and $X$ has dimension $m$, then $\Pi E$ is a supermanifold of dimension $m|n$.

• In particular, for the tangent bundle $T X \to X$, we get the **odd tangent bundle** $\Pi T X = (X, \Omega^*_X)$, whose structure sheaf is the algebra of differential forms, regarded as a supercommutative superalgebra.

We define a **smooth map** $f: M \to N$ between supermanifolds to be a pair $(f, f^*)$, where $f: M_b \to N_b$ is continuous, and $f^*: \mathcal{O}_N \to f_* \mathcal{O}_M$ is a homomorphism of sheaves of superalgebras over $N_b$. Here $f_* \mathcal{O}_M$ denotes **pushforward** of $\mathcal{O}_M$ along $f$: for any open $V \subseteq N_b$, we define $f_* \mathcal{O}_M(V) := \mathcal{O}_M(f^{-1}(V))$.

A key technical role will be played by submersions and fiber products of supermanifolds. A map of supermanifolds $\pi: Y \to M$ is a **submersion** if it is locally isomorphic to a projection map $p_1: U \times V \to U$, where $U$ and $V$ are super domains. It is immediate from this definition that any submersion admits local sections. With a bit of work, we can show that pullbacks along submersions exist in the category of supermanifolds. In particular, given a submersion $\pi: Y \to M$, the **fiber square** $Y^{[2]} = Y \times_M Y$ is defined to be the pullback of $\pi$ along itself:

$$
\begin{array}{ccc}
Y \times_M Y & \longrightarrow & Y \\
\downarrow & & \downarrow \pi \\
Y & \longrightarrow & M
\end{array}
$$

In fact, any **fiber power** $Y^{[p]}$ of $Y$ exists, and is defined to be the fiber product $Y \times_M \cdots \times_M Y$ of $Y$ with itself $p$ times. Between the fiber powers, we have the maps $\pi_i: Y^{[p+1]} \to Y^{[p]}$, the projection that omits that $i$th factor. More generally, we have the maps $\pi_{i_1, i_2, \ldots, i_q}: Y^{[p]} \to Y^{[q]}$ that project onto the $i_1$th, $i_2$th, $\ldots$, $i_q$th factors, in that order, for any $1 \leq i_1 < i_2 < \cdots < i_q \leq p$. 

4
2.3 Bundles in supergeometry

Line bundles on $M$ will play a central role in our story. As in algebraic geometry, bundles in supergeometry correspond to modules: a vector bundle on a supermanifold $M$ is a sheaf of $\mathcal{O}_M$-modules, locally free of finite rank. In particular, a complex line bundle on $M$, called simply a line bundle for short, is a sheaf $\mathcal{L}$ of modules of the complexified structure sheaf, $\mathcal{O}_M \otimes \mathbb{C}$, locally free of rank 1. This last condition means that any point $x \in M$ lies in a neighborhood $U$ on which there is an even section $s \in \mathcal{L}(U)$ such that $\mathcal{L}_U = \mathcal{O}_U \cdot s$. Here, we write $\mathcal{O}_C$ for the complexified structure sheaf $\mathcal{O}_M \otimes \mathbb{C}$, a notation we will use whenever $M$ is clear from the context. We call the section $s$ a trivializing section.

Given two vector bundles $\mathcal{E}$ and $\mathcal{F}$ on $M$, a bundle map $\varphi: \mathcal{E} \to \mathcal{F}$ is a map of sheaves preserving the parity and compatible with the $\mathcal{O}_M$-module structure. A bundle isomorphism is an invertible bundle map whose inverse is a bundle map. A bundle map is a special case of a graded bundle map: a graded bundle map of parity $p$ is a map of sheaves $\varphi: \mathcal{E} \to \mathcal{F}$ that shifts the parity of sections by $p$:

$$\varphi: \mathcal{E}_i(U) \to \mathcal{F}_{i+p}(U),$$

for any $U \subseteq M$, and that respects the $\mathcal{O}_M$-module structure up to sign:

$$\varphi(fs) = (-1)^{p(f)} \varphi(f)s, \text{ for all } f \in \mathcal{O}_M(U), \ s \in \mathcal{E}(U).$$

The sheaf $\text{Hom}(\mathcal{E}, \mathcal{F})$ of all graded bundle maps is itself a vector bundle over $M$.

We can define the usual operations on vector bundles in supergeometry [17]:

- **Duals.** Given a vector bundle $\mathcal{E}$ on $M$, its dual $\mathcal{E}^*$ is the sheaf of graded bundle maps from $\mathcal{E}$ into the structure sheaf: $\mathcal{E}^* = \text{Hom}(\mathcal{E}, \mathcal{O}_M)$.

- **Tensor product.** Given two vector bundles $\mathcal{E}$ and $\mathcal{F}$ on $M$, their tensor product $\mathcal{E} \otimes \mathcal{F}$ is a vector bundle on $M$, defined to be the sheaf with the stalks

$$\mathcal{E} \otimes \mathcal{F}_x = \mathcal{E}_x \otimes_{\mathcal{O}_{M,x}} \mathcal{F}_x,$$

for $x \in M$, any point in the body.

- **Pullback.** Given a map $f: M \to N$ of supermanifolds and a vector bundle $\mathcal{E}$ on $N$, the pullback $f^* \mathcal{E}$ is a vector bundle on $M$, defined to be the sheaf with the stalks

$$(f^* \mathcal{E})_x = \mathcal{E}_{f(x)} \otimes_{\mathcal{O}_{N,f(x)}} \mathcal{O}_{M,x},$$

for $x \in M$, any point in the body. In this tensor product, we treat the stalk $\mathcal{O}_{M,x}$ as an $\mathcal{O}_{N,f(x)}$-module using the homomorphism $f^*$.

As in classical differential geometry, given a vector bundle $\mathcal{E}$ on $M$, we can construct a supermanifold $E$, the total space of $\mathcal{E}$, and a smooth map $p: E \to M$, the bundle projection [10]. We will not work with these objects directly, but will use them as a kind of shorthand to remind the reader of the parallels with differential geometry. Thus we will sometimes speak of a vector bundle $E$ or $E \to M$, or of a line bundle $\mathcal{L}$ or $\mathcal{L} \to M$, when we mean sheaves $\mathcal{E}$ and $\mathcal{L}$ of $\mathcal{O}_M$-modules or $\mathcal{O}_C$-modules, respectively. Similarly, we will write $E^* \to M$ for the dual of $\mathcal{E}$, $E \otimes F \to M$ for the tensor product of $\mathcal{E}$ and $\mathcal{F}$, and $f^* E \to M$ for the pullback of $\mathcal{E}$ on $N$ along $f: M \to N$, but in all cases we mean the constructions with sheaves described above.

2.4 The de Rham complex

The de Rham complex on $M$ will also play a central role, so we give a lightning overview of its construction. The tangent bundle $TM$ of $M$ is the sheaf of graded derivations of $\mathcal{O}_M$, and we
refer to global sections of $\mathcal{T}M$ as vector fields on $M$. The cotangent bundle is the dual sheaf $\Omega^1(M) = \text{Hom}(\mathcal{T}M, \mathcal{O}_M)$, and we refer to global sections of $\Omega^1(M)$ as 1-forms on $M$. We write the canonical pairing between vector fields and 1-forms as:

$$\langle -, - \rangle : \mathcal{T}M \otimes \Omega^1(M) \to \mathcal{O}_M.$$ 

The de Rham complex is the $\mathbb{Z}$-graded sheaf $\Omega^\bullet(M)$ of super vector spaces whose $p$th grade is the $p$th exterior power of $\Omega^1(M)$ as an $\mathcal{O}_M$-module:

$$\Omega^p(M) := \Lambda^p_0(\Omega^1(M)).$$

Note that each grade $\Omega^p(M)$ is itself $\mathbb{Z}_2$-graded, so in fact the full complex $\Omega^\bullet(M)$ is $\mathbb{Z} \times \mathbb{Z}_2$-graded. Both degrees encode important information about a form, and we will often specify them in words. For instance, if we were to write “let $\alpha$ be an odd 2-form” or “let $\alpha$ be a 2-form of odd parity”, we would mean in both cases that $\alpha$ is an odd section of $\Omega^2(M)$. We will also write $\Omega^p$ for the sheaf of $p$-forms of parity $i$. Finally, we will also have occasion to think about the complexified de Rham complex, $\Omega^p_{\mathbb{C}} = \Omega^p \otimes \mathbb{C}$, and its even part, $\Omega^p_{\mathbb{C}, 0}$.

The full complex $\Omega^\bullet(M)$ is an algebra under the wedge product—in fact, it is a $\mathbb{Z} \times \mathbb{Z}_2$-graded commutative algebra, so that $\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$, for $\alpha$ a $p$-form of parity $p$’ and $\beta$ a $q$-form of parity $q$. As classically, there is a differential $d : \mathcal{O}_M \to \Omega^1(M)$ defined by

$$(X, df) = Xf,$$

for $X$ a local vector field and $f$ in the structure sheaf. This extends to a derivation $d$ on all of $\Omega^\bullet(M)$ such that $d^2 = 0$, the exterior derivative making $\Omega^\bullet(M)$ into a complex. The exterior derivative is defined to have bidegree $(1, 0)$. Thus for any $p$-form $\alpha$, its exterior derivative $d\alpha$ is a $(p + 1)$-form of the same parity as $\alpha$, and we have $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for any other form $\beta$.

We now prove a key lemma concerning the relationship between the de Rham complex and fiber powers of a submersion. This will play a central role when we work with connections on bundle gerbes.

The following proof briefly invokes ‘Z-points’ of the fiber power $Y^{[q]}$. For any supermanifold $Z$, a $Z$-point $(y_1, y_2, \ldots, y_q) : Z \to Y^{[q]}$ is simply a map into $Y^{[q]}$ from $Z$. By the universal property of the fiber product, any map into $Y^{[q]}$ decomposes into a $q$-tuple of maps $y_i : Z \to Y$, such that $\overset{\pi}{\pi} y_1 = \overset{\pi}{\pi} y_2 = \cdots = \overset{\pi}{\pi} y_q$. We have anticipated this in our notation by writing the $Z$-point as a $q$-tuple $(y_1, y_2, \ldots, y_q)$. The utility of $Z$-points comes from the Yoneda lemma [22]; the Yoneda Lemma tells us that in any category $C$, we can construct a morphism $f : x \to y$ between two objects $x, y \in C$ by defining a function between the sets of $z$-points $\text{Hom}(z, x) \to \text{Hom}(z, y)$ for any object $z \in C$, so long as these functions fit together into a natural transformation from the functor $\text{Hom}(-, x)$ to the functor $\text{Hom}(\cdot, y)$.

**Lemma 3.** Let $\pi : Y \to M$ be a submersion, where $Y$ and $M$ are supermanifolds. Then for any fixed natural number $p$, the following complex of $p$-forms is exact:

$$0 \to \Omega^p(M) \overset{\pi^*}{\to} \Omega^p(Y) \overset{\delta}{\to} \Omega^p(Y^{[2]}) \overset{\delta}{\to} \cdots \overset{\delta}{\to} \Omega^p(Y^{[q]}) \overset{\delta}{\to} \cdots$$

Here the differential $\delta : \Omega^p(Y^{[q]}) \to \Omega^p(Y^{[q+1]})$ is given by the sum $\delta = \sum_{i=1}^{q+1} (-1)^{i+1} \pi_i^*$, where $\pi_i : Y^{[q+1]} \to Y^{[q]}$ is the projection that omits that $i$th factor.

**Proof.** The following proof adapts an argument due to Michael Murray [19] to the context of supermanifolds.

First, it is not hard to check that $\delta \pi^* = 0$ and more generally, $\delta^2 = 0$, so $\Omega^p(Y^{[q]})$ is indeed a complex. To prove exactness, we will first assume that $\pi : Y \to M$ admits a global section,
s: \( M \rightarrow Y \). Afterwards, we will generalize to the case without a global section by using a partition of unity. In all of the following, we adopt the convention that \( Y[0] = M \) and \( \delta = \pi^* \) at this stage of the complex.

Given a global section \( s: M \rightarrow Y \), we get a map \( s_{q}: Y[q] \rightarrow Y[q+1] \) defined by \( s_{q}(y_1, \ldots, y_q) = (s(\pi(y_1)), y_1, \ldots, y_q) \), for any \( Z \)-point \( (y_1, \ldots, y_q): Z \rightarrow Y[q] \). Let \( \omega \in \Omega^q(Y[q]) \) be \( \delta \)-closed:

\[
\delta \omega = 0.
\]

We claim that \( \omega = \delta s_{q-1} \omega \), proving exactness. Indeed, using \( Z \)-points it is easy to check that \( \pi_1 s_q = 1_{Y[q]} \) and \( \pi_i s_q = s_{q-1} \pi_{i-1} \) for \( i \geq 2 \). Thus, by the definition of \( \delta \) we have:

\[
s_{q} \delta = \sum_{i=1}^{q+1} (-1)^{i+1} s_{q}^* s_{i}^* = 1 - \sum_{i=2}^{q+1} (-1)^{i} \pi_{i-1}^* s_{q}^* = 1 - \delta s_{q-1}.
\]

So, because \( \delta \omega = 0 \), we must have \( s_{q}^* \delta = 0 \), which implies \( \omega - \delta s_{q-1} \omega = 0 \) by the above calculation. Thus \( \omega = \delta s_{q-1} \omega \), as claimed.

Now let us consider the case where we only have local sections. Fix an open cover \( \{U_\alpha\}_{\alpha \in I} \) of \( M \) and local sections \( s_q: U_\alpha \rightarrow Y \) \( \pi \). Given a global section \( \alpha: U_\alpha \rightarrow Y \), we get a partition of unity \( \{\varphi_\alpha \in \mathcal{O}(M)\}_{\alpha \in I} \) on \( M \) subordinate to our cover \( \{U_\alpha\}_{\alpha \in I} \), where \( \mathcal{O}(M) \) denotes the global sections of the structure sheaf \( \mathcal{O}_M \). Each fiber power \( Y[q] \) projects to \( M \), and by pulling our partition of unity back along this projection, we get a partition of unity on \( Y[q] \) subordinate to the cover \( \{Y[q]^\alpha\}_{\alpha \in I} \). Let us abuse notation slightly and denote the pullback partition of unity by \( \{\varphi_\alpha\}_{\alpha \in I} \) as well. Because they were constructed by pullback, multiplication by elements in our partition of unity commutes with \( \delta \).

Now let \( \omega \in \Omega^q(Y[q]) \) be \( \delta \)-closed:

\[
\delta \omega = 0.
\]

Restricting \( \omega \) to \( Y[q]^\alpha \), we get \( \omega_\alpha \in \Omega^q(Y[q]^\alpha) \), which is also \( \delta \)-closed, and thus \( \omega_\alpha = \delta \rho_\alpha \), by exactness. Patching the \( \rho_\alpha \) together using our partition of unity, it is immediate that \( \rho = \sum \varphi_\alpha \rho_\alpha \) satisfies \( \omega = \delta \rho \).

2.5 Connections and curvature

A connection \( \nabla \) on a line bundle \( \mathcal{L} \) on a supermanifold \( M \) is a rule that, for any vector field \( X \) on \( M \), gives a map of sheaves of super vector spaces, \( \nabla_X: \mathcal{L} \rightarrow \mathcal{L} \). This rule is \( \mathcal{O}_M \)-linear in \( X \):

\[
\nabla_X f s = f \nabla_X s,
\]

for any local section \( f \) of the structure sheaf \( \mathcal{O}_M \), and it satisfies the Leibniz rule in a graded sense:

\[
\nabla_X (fs) = (X f)s + (-1)^{|X||f|} f \nabla_X s,
\]

for any local section \( s \) of \( \mathcal{L} \) and section \( f \) of the complexified structure sheaf \( \mathcal{O}_\mathbb{C} \). The curvature \( F \) of \( \nabla \) is defined by the graded analogue of the usual formula, namely:

\[
F(X, Y)s = (\nabla_X \nabla_Y - (-1)^{|X||Y|} \nabla_Y \nabla_X - \nabla_{[X,Y]})s,
\]

for any local section \( s \) of \( \mathcal{L} \). It is a complex-valued 2-form of even parity, \( F \in \Omega^2_{\mathbb{C},0}(M) \).
2.6 The body of a supermanifold

In supermanifold theory, essentially all questions that concern the topology of $M$ reduce to questions on the body $M_b$, and we now introduce maps to pass between $M$ and its body. Despite $M_b$ being merely a topological manifold in our definition, it in fact carries a smooth structure. This arises as follows: let $\mathcal{J}_M \subseteq \mathcal{O}_M$ be the ideal generated by the odd subsheaf, $(\mathcal{O}_M)_1$. Then $(M_b, \mathcal{O}_M/\mathcal{J}_M)$ is an ordinary smooth manifold: on each chart $(U, \varphi, \phi)$ of our atlas, $\mathcal{J}_M|U$ is generated by the odd coordinates $\{\theta_j\}$, so when we quotient by this ideal we find that $(\mathcal{O}_M/\mathcal{J}_M)|U$ is isomorphic to the sheaf of smooth functions on $\varphi(U)$, giving $M_b$ a smooth structure. The inclusion map $i: M_b \hookrightarrow M$ is the map of supermanifolds that corresponds to the quotient map of structure sheaves, $\mathcal{O}_M \to \mathcal{O}_M/\mathcal{J}_M$.

One way the topology of the body appears in our work is via open covers. An open cover $\mathcal{U} = \{U_\alpha\}$ of a supermanifold $M$ is an open cover of the body $M_b$, where we equip each open $U_\alpha \subseteq M_b$ with the supermanifold structure $(U_\alpha, \mathcal{O}_M|U_\alpha)$ given by restricting the structure sheaf. We say that $\mathcal{U}$ is a good cover of $M$ if it is a good cover of $M_b$, in the sense that each nonempty finite intersection:

$$U_{\alpha_1, \alpha_2, \ldots, \alpha_k} = U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_k}$$

is diffeomorphic to $\mathbb{R}^m$, for $m$ the dimension of $M_b$.

Henceforth, we shall always regard the body $M_b$ as a smooth submanifold of $M$. Note that the inclusion of the body, $i: M_b \hookrightarrow M$, is canonical for any supermanifold, but a map from $M$ to its body is not. We say a smooth map $p: M \to M_b$ is a body projection if $p i = 1_{M_b}$. It turns out that such a map always exists [4, 12], but it is a choice of extra data.

3 Bundle gerbes

The notion of bundle gerbe is due to Michael Murray [18]. It is a geometric object defined over a base manifold whose characteristic class sits in the third cohomology of the base. It is analogous to a complex line bundle, and its characteristic class is an analogue of the Chern class in degree 2. A line bundle can also be equipped with a connection, and when this is done the curvature of the connection is a closed 2-form and a de Rham representative of the Chern class. Analogously, a bundle gerbe can be equipped with a connection, and when this is done the curvature of the connection is a closed 3-form and a de Rham representative of the characteristic class.

Bundle gerbes are just one of many constructions that geometricize 3rd degree cohomology. Shortly before Murray’s work, Brylinski developed the differential geometry of objects called ‘gerbes’, by which he meant a sheaf of groupoids [6]. In this work, Brylinski was using the original notion of gerbe due to Giraud [13], conceived as a formulation of nonabelian cohomology. Contemporaneous with Giraud, Deligne developed the tool now called Deligne cohomology [9, Sec. 2.2], the cohomology theory that we shall see classifies bundle gerbes with connection. A few years after Murray, Chatterjee and Hitchin worked with a restricted class of bundle gerbes they called ‘gerbs’ [8, 14].

In Sections 3.1 and 3.2, we invoke a dictionary between classical differential geometry and supergeometry to translate the definitions and theorems of Murray [18] and Murray–Stevenson [20] to supergeometric language. To my knowledge, this dictionary has not been made explicit before, but it is apparent in the work of Kostant [15] on line bundles in supergeometry. To wit, a line bundle with connection on an ordinary manifold $X$ has transition functions given by local sections of the sheaf $\underline{\mathcal{O}}_X$ of smooth functions valued in the nonzero complex numbers, and its connection is locally a complex-valued 1-form. On the other hand, a line bundle with connection on a supermanifold $M$ has transition elements given by local sections of $\mathcal{O}_C$, the subsheaf of even, invertible elements of the complexified structure sheaf, $\mathcal{O}_C = \mathcal{C} \otimes \mathcal{O}_M$, and the connection is locally an even, complex-valued 1-form. Bundle gerbes involve higher degree forms, but the dictionary is the same: replace the sheaf of smooth functions $\underline{\mathcal{O}}_X$ with $\mathcal{O}_C$, and the sheaf of complexified de Rham forms $\Omega^*_C$ with its even part $\Omega^*_{C,0}$.
3.1 Without connection

Recall from Section 2.2 that for any submersion $\pi: Y \to M$, the fiber powers $Y^{[p]} = Y \times_M Y \times_M \cdots \times_M Y$ ($p$ times) exist as supermanifolds. Between the fiber powers, we have the maps $\pi_i: Y^{[p]} \to Y^{[p-1]}$ which omit the $i$th factor, and more generally we have the maps $\omega_{i_1, i_2, \ldots, i_q}: Y^{[p]} \to Y^{[q]}$ that project onto the $i_1$th, $i_2$th, $\ldots$, $i_q$th factors, in that order, for any $1 \leq i_1 < i_2 < \cdots < i_q \leq p$.

**Definition 4.** A bundle gerbe on the supermanifold $M$ consists of the following data:

- a submersion, $\pi: Y \to M$.
- a complex line bundle over the fiber square of $Y$:

$$
\begin{array}{ccc}
L & \xrightarrow{\pi_1} & Y \\
\downarrow & & \downarrow \pi_2 \\
Y^{[2]} & \xrightarrow{\pi_2} & Y \\
\downarrow \pi & & \downarrow \\
M.
\end{array}
$$

- a line bundle isomorphism, the bundle gerbe multiplication, $\mu: \pi_1^* L \otimes \pi_1^* L \to \pi_2^* L$ over $Y^{[3]}$.

Note that we can equivalently write this as $\mu: \omega_{12}^* L \otimes \omega_{23}^* L \to \omega_{13}^* L$.

- The bundle gerbe multiplication satisfies an associativity condition over $Y^{[4]}$: using $\mu$, we can construct two line bundle isomorphisms over $Y^{[4]}$ from the line bundle $\omega_{12}^* L \otimes \omega_{23}^* L \otimes \omega_{34}^* L$ to the line bundle $\omega_{14}^* L$. These two maps are required to be equal. Explicitly, the following square commutes:

$$
\begin{array}{ccc}
\omega_{12}^* L \otimes \omega_{23}^* L \otimes \omega_{34}^* L & \xrightarrow{\omega_{12}^* \otimes \omega_{23}^* \mu} & \omega_{13}^* L \otimes \omega_{34}^* L \\
\downarrow \omega_{12}^* \otimes \omega_{23}^* \otimes \omega_{34}^* \mu & & \downarrow \omega_{13}^* \otimes \omega_{34}^* \mu \\
\omega_{12}^* L \otimes \omega_{24}^* L & \xrightarrow{\omega_{12}^* \otimes \omega_{24}^* \mu} & \omega_{14}^* L.
\end{array}
$$

We denote a bundle gerbe with surjective submersion $Y$, line bundle $L$ and multiplication $\mu$ by the triple $(Y, L, \mu)$, or more succinctly by a single calligraphic letter such as $\mathcal{G}$. The following is the most automatic example of a bundle gerbe, called the ‘trivial bundle gerbe’:

**Example 5** (The trivial bundle gerbe). Given a submersion $\pi: Y \to M$, choose a line bundle $L$ over $Y$. Then we can construct a line bundle $\delta L$ on $Y^{[2]}$ as follows:

$$
\delta L = \pi_1^* L \otimes \pi_2^* L^*,
$$

where we recall that $\pi_i: Y^{[2]} \to Y$ denotes the projection that omits the $i$th factor. There is a isomorphism of line bundles $\mu_{\text{can}}: \pi_3^* \delta L \otimes \pi_1^* \delta L \to \pi_2^* \delta L$ over $Y^{[3]}$, induced from the canonical pairing between $L$ and its dual $L^*$, and it is a quick calculation to check that $\mu_{\text{can}}$ is associative. Thus $(Y, \delta L, \mu_{\text{can}})$ is a bundle gerbe over $M$. We call this the **trivial bundle gerbe** over $M$, and denote it by $\mathcal{I}$. We shall also denote this gerbe by $\delta L$ when we wish to make note of the line bundle, though we shall see shortly that all trivial bundle gerbes over $M$ are equivalent to each other, independent of the choice of line bundle or submersion. Hence the notation $\mathcal{I}$ will be preferred.

All the usual operations on line bundles generalize to bundle gerbes:

**Duals.** Given a bundle gerbe $\mathcal{G} = (Y, L, \mu)$ over $M$, its dual $\mathcal{G}^*$ is the bundle gerbe $(Y, L^*, (\mu^{-1})^*)$ over $M$. 

---

9
Tensor product. Given two bundle gerbes \( G = (Y, L, \mu) \) and \( H = (Z, Q, \nu) \), the tensor product \( G \otimes H \) is the bundle gerbe \( (Y \times_M Z, L \otimes Q, \mu \otimes \nu) \). Here \( L \otimes Q \) denotes the result of tensoring the pullbacks of \( L \) and \( Q \) to the fiber square \( (Y \times_M Z)[2] \), but we treat the pullbacks as implicit.

Pullback. Given a bundle gerbe \( G = (Y, L, \mu) \) over the supermanifold \( N \), and a map of supermanifolds \( f: M \rightarrow N \), the pullback \( f^*G \) is the bundle gerbe \( (f^*Y, f^*L, f^*\mu) \), where we have abused notation to write \( f^*L \) for the pullback of the line bundle \( L \) to \( (f^*Y)[2] \), and similarly for the multiplication.

The collection of all bundle gerbes over \( M \) forms a bicategory rather than a mere category. In a full treatment, we would now describe the 1-morphisms and 2-morphisms in this bicategory. This has been done by Stevenson [24] and Waldorf [26] for bundle gerbes over smooth manifolds, however, and we do not anticipate the 1- and 2-morphisms over supermanifolds to differ significantly. Nonetheless, we will need one piece of information about the bicategory of bundle gerbes: we need to know when two bundle gerbes over \( M \) are equivalent. We turn to this now.

A trivialization of a bundle gerbe \( G = (Y, L, \mu) \) is a choice of line bundle \( T \rightarrow Y \) and an isomorphism \( \tau: L \rightarrow \delta T \) of line bundles over \( Y[2] \) compatible with the bundle gerbe multiplication on \( G \) and \( \delta T \). We say that a bundle gerbe is trivializable if a trivialization exists.

Definition 6. Two bundle gerbes \( G \) and \( H \) over \( M \) are equivalent if \( G \otimes H^* \) is trivializable.

We write \( G \simeq H \) when \( G \) and \( H \) are equivalent. Equivalence of bundle gerbes is also called stable isomorphism in the literature. With this definition in hand, it is easy to see that the trivial bundle gerbe is unique up to equivalence.

Proposition 7. Let \( Y \rightarrow M \) and \( Z \rightarrow M \) be submersions, and let \( L \) and \( Q \) be line bundles over \( Y \) and \( Z \), respectively. Then the trivial bundle gerbes \( (Y, \delta L, \mu_{\text{can}}) \) and \( (Z, \delta Q, \mu_{\text{can}}) \) are equivalent.

Proof. Write \( \delta L \) and \( \delta Q \), respectively, for the above bundle gerbes. We merely need to show that \( \delta L \otimes (\delta Q)^* \) is trivializable. The submersion for this tensor product of gerbes is \( X = Y \times_M Z \), and in fact we have an isomorphism of line bundles

\[
\delta L \otimes (\delta Q)^* \cong \delta(L \otimes Q^*)
\]

over the fiber square, \( X[2] \). This is the desired trivialization.

\[\square\]

Corollary 8. The trivial bundle gerbe \( \mathcal{I} \) is equivalent to the following bundle gerbe:

\[(M, \mathcal{O}_C, \mu_{\text{can}})\]

where the submersion \( M \rightarrow M \) is the identity, the line bundle over \( M[2] = M \) is the trivial line bundle \( \mathcal{O}_C \), and the multiplication \( \mu_{\text{can}} \) is the usual multiplication on the complexified structure sheaf, \( \mathcal{O}_C \).

Let \( \mathcal{G}(M) \) denote the collection of all equivalence classes of bundle gerbes on \( M \). The tensor product operation equips \( \mathcal{G}(M) \) with an abelian group structure, with duals as inverses and the trivial bundle gerbe \( \mathcal{I} \) as the identity. In fact, we shall see that \( \mathcal{G}(M) \) depends only on the topology of the body:

\[\mathcal{G}(M) \cong H^3(M_b, \mathbb{Z}).\]

For a given bundle gerbe \( G \), the corresponding class in \( H^3(M_b, \mathbb{Z}) \) is called the ‘Dixmier–Douady class of \( G \), and denoted DD(\( G \)). It is the analogue for bundle gerbes of the Chern class of a line bundle. As in the line bundle case, the Dixmier–Douady class is constructed using Čech cohomology.
We perform the construction of DD(\mathcal{G}) for \mathcal{G} = (Y, L, \mu) as follows. Choose a good cover \{U_\alpha\} of M such that the submersion \pi: Y \to M admits local sections \{s_\alpha: U_\alpha \to Y\}. On each double intersection \alpha\beta = U_\alpha \cap U_\beta, the universal property of the fiber product yields a map:

\[(s_\alpha, s_\beta): U_{\alpha\beta} \to Y^{[2]}\]

We use the map \((s_\alpha, s_\beta)\) to pull back our line bundle L to \(U_{\alpha\beta}\), yielding a line bundle \(L_{\alpha\beta} = (s_\alpha, s_\beta)^* L\). Because \(U_{\alpha\beta}\) is contractible, \(L_{\alpha\beta}\) is trivializable [15], so there is a section \(\sigma_{\alpha\beta} \in \mathcal{L}_{\alpha\beta}(U_{\alpha\beta})\) such that \(\mathcal{L}_{\alpha\beta} = \mathcal{O}_C \cdot \sigma_{\alpha\beta}\). On the triple intersection \(U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma\), the bundle gerbe multiplication induces an isomorphism \(\mu: L_{\alpha\beta} \otimes L_{\beta\gamma} \to L_{\alpha\gamma}\). We thus have two sections of \(L_{\alpha\gamma}\) over a triple intersection, namely \(\sigma_{\alpha\gamma}\) and \(\mu(\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma})\), both of which are trivializing. Thus we must have:

\[\mu(\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma}) = g_{\alpha\beta\gamma} \sigma_{\alpha\gamma},\]

for some even, invertible element of the complexified structure sheaf, \(g_{\alpha\beta\gamma} \in \mathcal{O}_C(U_{\alpha\beta\gamma})\). The associativity of \(\mu\) implies that \(g_{\alpha\beta\gamma}\) is a Čech 2-cocycle. If we write \(\mathcal{O}_C^*\) for the subsheaf of even, invertible elements in the complexified structure sheaf \(\mathcal{O}_C\), then we have constructed a class in the 2nd Čech cohomology of this sheaf:

\[\left[g_{\alpha\beta\gamma}\right] \in \check{H}^2(M_b, \mathcal{O}_C^*).\]

By standard arguments in Čech cohomology, we can check that the class \([g_{\alpha\beta\gamma}]\) is independent of the choices made.

Finally, we can construct an isomorphism \(\check{H}^2(M_b, \mathcal{O}_C^*) \cong H^3(M_b, \mathbb{Z})\). This is the Dixmier–Douady class of \(\mathcal{G}\), denoted DD(\mathcal{G}). Thanks to the isomorphism with \(\check{H}^2(M_b, \mathcal{O}_C^*)\), we will also think of DD(\mathcal{G}) as living in this group.

We construct the isomorphism \(\check{H}^2(M_b, \mathcal{O}_C^*) \cong H^3(M_b, \mathbb{Z})\) in the same way as one constructs the classical isomorphism \(\check{H}^p(X, \mathcal{C}_X) \cong H^{p+1}(X, \mathbb{Z})\) for \(X\) a smooth manifold and \(\mathcal{C}_X\) the sheaf of smooth functions valued in the invertible complex numbers, \(\mathbb{C}^*\). Specifically, we use a short exact sequence of sheaves on \(M_b\):

\[0 \to \mathbb{Z}_{M_b} \to (\mathcal{O}_C)_0 \xrightarrow{\text{exp}} \mathcal{O}_C^* \to 0.\]

Indeed, Kostant proved that if we define \(\text{exp}(f) = \sum_{n=0}^{\infty} \frac{f^n}{n!}\), then this series converges in a suitable topology and defines a homomorphism \(\text{exp}: (\mathcal{O}_C)_0(U) \to \mathcal{O}_C^*(U)\) for any open set \(U\), which is surjective when \(U\) is simply connected. Moreover, he showed this descends to a sheaf homomorphism \(\text{exp}: (\mathcal{O}_C)_0 \to \mathcal{O}_C^*\), with kernel given by the constant sheaf on the integers [15].

With this short exact sequence of sheaves in hand, we get a long exact sequence upon passing to sheaf cohomology. Because \((\mathcal{O}_C)_0\) is a soft sheaf, the connecting homomorphism provides the desired isomorphism \(\check{H}^2(M_b, \mathcal{O}_C^*) \cong H^3(M_b, \mathbb{Z})\).

To close this circle of ideas, we now prove that the Dixmier–Douady class just constructed is a complete invariant of a bundle gerbe up to equivalence. This is the content of the following theorem. Its proof follows the proof of the analogous result over smooth manifolds by Murray [18] and Murray–Stevenson [20], adapted to the language of structure sheaves.

**Theorem 9.** Let \(M\) be a supermanifold, \(\mathcal{G}(M)\) the abelian group of equivalence classes of bundle gerbes over \(M\), and let DD: \(\mathcal{G}(M) \to \check{H}^2(M, \mathcal{O}_C^*)\) be the map sending the equivalence class of a bundle gerbe on \(M\) to its Dixmier–Douady class, regarded as a Čech cohomology class. Then:

1. DD is onto;
2. DD(\mathcal{G}) = 0 if and only if \mathcal{G} is trivial;
3. DD(\mathcal{G} \otimes \mathcal{H}) = DD(\mathcal{G}) + DD(\mathcal{H});
\[DD(\mathcal{G}^*) = -DD(\mathcal{G});\]

where \(\mathcal{G}\) and \(\mathcal{H}\) are bundle gerbes on \(M\).

II. If \(f: M \to N\) is a map of supermanifolds, then \(DD\) is natural with respect to pullback along \(f\):

\[DD(f^*\mathcal{G}) = f^*DD(\mathcal{G}),\]

for any bundle gerbe \(\mathcal{G}\) on \(N\).

Proof. Let \(\mathcal{G} = (Y, L, \mu)\) be a bundle gerbe on \(M\), and let \(DD(\mathcal{G}) = [g_{\alpha\beta\gamma}]\) be the Čech cohomology class constructed as above using a good cover \(\{U_\alpha\}\), sections \(s_\alpha: U_\alpha \to Y\), and trivializing sections \(\sigma_{\alpha\beta} \in \mathcal{L}_{\alpha\beta}(U_{\alpha\beta})\). As we noted above, the class \(DD(\mathcal{G})\) is independent of the choices made by standard arguments in Čech cohomology.

The fact that \(DD\) respects pullbacks is also standard: construct \([g_{\alpha\beta\gamma}\gamma]\) using a good cover of \(\{U_\alpha\}\) of \(N\), and choose a good cover \(\{V_\alpha\}\) on \(M\) that is a refinement of the cover \(\{f^{-1}(U_\alpha)\}\). The pullback of \([g_{\alpha\beta\gamma}]\) to this refinement represents the class of \(DD(f^*\mathcal{G})\), as desired.

Next, we wish to show that \(DD\) is an isomorphism of abelian groups. First of all, the fact that \(DD(I) = 0\) for the trivial bundle gerbe \(I\) and that \(DD(\mathcal{G}^*) = -DD(\mathcal{G})\) is natural with respect to pullback along \(f\). So, it remains to check that \(DD\) is a bijection. To show it is onto, fix a class \(g \in \tilde{H}^2(M, \mathcal{O}_C)\), and choose a good cover \(\{U_\alpha\}\) for which \(g\) is represented by sections \(g_{\alpha\beta\gamma} \in \mathcal{O}_C(U_{\alpha\beta\gamma})\) satisfying the Čech 2-cocycle condition. Let our submersion be \(Y = \coprod(U_\alpha, \mathcal{O}_{U_\alpha})\) with the obvious projection to \(M\). Then the fiber square is \(Y^{[2]} = \coprod(U_{\alpha\beta\gamma}, \mathcal{O}_{U_{\alpha\beta\gamma}})\). For our bundle gerbe \(\mathcal{G}\), we take the trivial line bundle on \(Y^{[2]}\) and use \(g\) to define the bundle gerbe multiplication over \(Y^{[3]}\), which is associative because \(g_{\alpha\beta\gamma}\) is a 2-cocycle. We have that \(DD(\mathcal{G}) = g\) by construction.

Finally, it remains to check that \(DD\) is one-to-one, we check that it has trivial kernel. Indeed, \(DD(\mathcal{G}) = 0\) means our Čech 2-cocycle \([g_{\alpha\beta\gamma}]\) is a coboundary:

\[g_{\alpha\beta\gamma} = f_{\alpha\beta}f_{\beta\gamma}f_{\gamma\alpha},\]

for some even, invertible sections \(f_{\alpha\beta} \in \mathcal{O}_C(U_{\alpha\beta})\). In fact, without loss of generality, we can assume \(g_{\alpha\beta\gamma} = 1\). Simply replace the trivializing sections \(\sigma_{\alpha\beta} \in \mathcal{L}_{\alpha\beta}(U_{\alpha\beta})\) with \(\frac{1}{f_{\alpha\beta}}\sigma_{\alpha\beta}\). For these trivializing sections, we indeed have \(g_{\alpha\beta\gamma} = 1\).

We can now use these data to construct a line bundle \(T \to Y\) by patching together line bundles \(T_\alpha\) on \(Y_\alpha = \pi^{-1}(U_\alpha)\). Each \(T_\alpha\) is defined as follows: note that there are two maps from \(Y_\alpha\) to \(Y\): the inclusion \(i: Y_\alpha \to Y\) and the composite of the projection followed by the section, \(s_\alpha\pi: Y_\alpha \to Y\). By the universal property of the fiber product, this gives us a map \((s_\alpha\pi, i): Y_\alpha \to Y^{[2]}\). So, we can pull \(L\) back along this map, and define \(T_\alpha = (s_\alpha\pi, i)^*L\). These glue together to give a line bundle \(T\) on all of \(Y\). To see this, note that using the sections \(\sigma_{\alpha\beta}\) on each intersection \(Y_{\alpha\beta}\), we can define a line bundle isomorphism:

\[\tilde{\sigma}_{\alpha\beta}: T_\alpha \to T_\beta,\]

How? Because \(\sigma_{\alpha\beta} \in \mathcal{L}_{\alpha\beta}\), and using the bundle gerbe multiplication, this is isomorphic to \((s_\alpha\pi, i)^*L \otimes (s_\beta\pi, i)^*L^* = T_\alpha \otimes T_\beta^\ast\). Finally, because we have normalized the sections \(\sigma_{\alpha\beta}\) so that \(g_{\alpha\beta\gamma} = 1\), these isomorphisms satisfy:

\[\tilde{\sigma}_{\alpha\beta}\tilde{\sigma}_{\beta\gamma}\tilde{\sigma}_{\gamma\alpha} = 1\]

and we conclude that the \(T_\alpha\) do indeed glue to give a line bundle \(T \to Y\). By construction, \(L \cong \delta T\) and this is compatible with the bundle gerbe multiplication, so \(T\) defines a trivialization of \(\mathcal{G}\). \(\square\)
3.2 With connection

Definition 10. A connection on a bundle gerbe $\mathcal{G} = (Y, L, \mu)$ consists of the following data:

- a connection $\nabla$ on the line bundle $L$, called the line bundle connection;
- an even, complex-valued 2-form $B \in \Omega^2_{\mathbb{C}, 0}(Y)$, called the curving; here $\Omega^2_{\mathbb{C}}$ denotes the complexification $\Omega^p \otimes \mathbb{C}$, and $\Omega^2_{\mathbb{C}, 0}$ denotes its even part;

such that:

- the gerbe multiplication $\mu: \pi_1^* L \otimes \pi_1^* L \to \pi_2^* L$ is an isomorphism of line bundles with connection over $Y$[3], for the connection induced by $\nabla$;
- the line bundle connection and curving satisfy the descent equation on $Y$[2]:

$$F_\nabla = \pi_1^* B - \pi_2^* B,$$

where $F_\nabla$ is the curvature 2-form of $\nabla$.

We write a connection as the pair $(\nabla, B)$ consisting of the line bundle connection and the curving. We denote a bundle gerbe $\mathcal{G}$ with connection $(\nabla, B)$ by the triple $(\mathcal{G}, \nabla, B)$, or simply by $\mathcal{G}$. While we call the pair $(\nabla, B)$ a connection, some authors use the word “connection” to refer to the line bundle connection $\nabla$ alone [18, 20]. In the theory of 2-bundles [3], which generalizes the theory of bundle gerbes [21], the pair $(\nabla, B)$ is called a 2-connection [1, 2].

The first example of a connection is on the trivial bundle gerbe, and we can produce such a connection starting from any even 2-form on $M$. This is analogous to the fact that an even 1-form defines a connection on the trivial line bundle.

Example 11 (Connections on the trivial bundle gerbe). By Corollary 8, the trivial bundle gerbe $\mathcal{I}$ has the following simple form, up to equivalence:

$$\mathcal{I} = (M, \mathcal{O}_\mathbb{C}, \mu_{\text{can}})$$

where the submersion $M \to M$ is the identity, the line bundle over $M[2] = M$ is the trivial line bundle $\mathcal{O}_\mathbb{C}$, and the multiplication $\mu_{\text{can}}$ is the usual multiplication on the complexified structure sheaf, $\mathcal{O}_\mathbb{C}$.

Now fix an even, complex-valued 2-form $b \in \Omega^2_{\mathbb{C}, 0}(M)$. Using $b$, we equip the trivial bundle gerbe with a connection as follows: take the line bundle connection to be the canonical flat connection on $\nabla^0$ on $\mathcal{O}_\mathbb{C}$, and take the curving to be the 2-form $b$. We denote the trivial bundle gerbe with this connection by $\mathcal{I}_b$, and call it the trivial bundle gerbe with connection 2-form $b$. For $b = 0$, we continue to write $\mathcal{I}$ rather than $\mathcal{I}_0$.

A trivial line bundle with connection, considered up to isomorphism, is the same as a 1-form. For bundle gerbes, however, the analogous fact is not true: $\mathcal{I}_b \simeq \mathcal{I}_{b'}$ does not imply that $b = b'$, but rather that $b - b'$ is a closed, integral 2-form, as we explain later.

All the usual operations on bundle gerbes can be easily modified to incorporate connections:

**Duals.** Given a bundle gerbe $\mathcal{G}$ over $M$ with connection $(\nabla, B)$, its dual $\mathcal{G}^*$ has the connection $(\nabla^*, -B)$.

**Tensor product.** Given two bundle gerbes $\mathcal{G} = (Y, L, \mu)$ and $\mathcal{H} = (Z, Q, \nu)$ with connections $(\nabla, B)$ and $(\nabla', B')$, respectively, recall their tensor product is

$$\mathcal{G} \otimes \mathcal{H} = (Y \times_M Z, L \otimes Q, \mu \otimes \nu).$$

This has the connection $(\nabla + \nabla', B + B')$. Here $B + B'$ denotes the result of pulling $B$ and $B'$ back to $Y \times_M Z$ and adding them, but we treat these pullbacks as implicit.
**Pullback.** Given a bundle gerbe $\mathcal{G}$ with connection $(\nabla, B)$ over the supermanifold $N$, and a map of supermanifolds $f: M \to N$, the pullback $f^*\mathcal{G}$ has the connection $(f^*\nabla, f^*B)$.

Much like bundle gerbes without connection, the collection of bundle gerbes with connection on $M$ form a bicategory $[24, 26]$. For our purposes, it is enough to know when two bundle gerbes with connection are equivalent in this bicategory. For bundle gerbes without connection, $\mathcal{G}$ and $\mathcal{H}$ are equivalent when $\mathcal{G} \otimes \mathcal{H}^*$ is trivializable, and the definition with connection is the same. We need only specify how connections behave under trivialization.

A **trivialization** of a bundle gerbe $\mathcal{G} = (Y, L, \mu)$ with connection $(\nabla, B)$ is a choice of the following data:

- a line bundle $T \to Y$ with connection;
- an isomorphism $\tau: L \to \delta T$ of line bundles with connection over $Y$.

These data must satisfy:

- The isomorphism $\tau$ respects the bundle gerbe multiplication on $\mathcal{G}$ and $\delta T$;
- The curvature $F_T$ of the connection on $T$ equals the curving: $F_T = B$.

We say that a bundle gerbe with connection is **trivializable** if a trivialization exists.

**Definition 12.** Two bundle gerbes $\mathcal{G}$ and $\mathcal{H}$ with connection over $M$ are **equivalent** if $\mathcal{G} \otimes \mathcal{H}^*$ is trivializable.

As before, we write $\mathcal{G} \simeq \mathcal{H}$ when $\mathcal{G}$ and $\mathcal{H}$ are equivalent.

Let $\mathbb{G}^\nabla(M)$ denote the collection of all equivalence classes of bundle gerbes on $M$. The tensor product operation equips $\mathbb{G}^\nabla(M)$ with an abelian group structure, with duals as inverses and $\mathbb{I}$ as the identity. In contrast to the case without connection, $\mathbb{G}^\nabla(M)$ depends on the supermanifold structure of $M$, not just on the topology of the body. Like their line bundle cousins, bundle gerbes with connection are classified by ‘Deligne cohomology’:

$$\mathbb{G}^\nabla(M) \cong H^2(M, \mathcal{O}_C \xrightarrow{d \log} \Omega^1_{C,0} \xrightarrow{d} \Omega^2_{C,0}).$$

Given a bundle gerbe $\mathcal{G}$, we denote the corresponding cohomology class by $\text{Del}(\mathcal{G})$, and call it the ‘Deligne class of $\mathcal{G}$.’ To describe its construction, we must first define Deligne cohomology for supermanifolds.

Recall that for $X$ a smooth manifold, the **kth smooth Deligne complex** is the following complex of sheaves on $X$ (with $\mathcal{O}_X$ in degree 0):

$$\mathcal{O}_X \xrightarrow{d \log} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k_X.$$ 

We then define **smooth Deligne cohomology** on $X$ as the hypercohomology of this complex.

Concretely, we can compute Deligne cohomology using Čech hypercohomology: fix a good cover $\{U_\alpha\}$ of $X$, and take the Čech complex for each sheaf in the Deligne complex. In this way, we obtain a double complex of abelian groups, called the **Čech–Deligne complex**. Totalizing and taking cohomology yields Deligne cohomology.

It is not difficult to generalize this definition to supermanifolds, with one minor complication: we will need only the even part of the de Rham complex. So, on a supermanifold $M$, define the **kth smooth Deligne complex** $\mathcal{D}(k)$ to be the following complex of sheaves (with $\mathcal{O}_C$ in degree 0):

$$\mathcal{O}_C \xrightarrow{d \log} \Omega^1_{C,0} \xrightarrow{d} \Omega^2_{C,0} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k_{C,0}.$$
As usual, $\mathcal{O}^*_\mathbb{C}$ is the subsheaf of even, invertible elements of the complexified structure sheaf $\mathcal{O}_\mathbb{C}$, and $\Omega^p_{\mathbb{C},0}$ is the sheaf of even, complexified $p$-forms. We define the operation $d \log$ to mean

$$d \log(f) = \frac{df}{f},$$

for any $f \in \mathcal{O}^*_\mathbb{C}(U)$. Because sections of $\mathcal{O}^*_\mathbb{C}$ are even and invertible, this definition is unambiguous.

The **smooth Deligne cohomology** of $M$ is the hypercohomology of the $k$th Deligne complex for a given $k$. We will focus on the second cohomology group of the second smooth Deligne complex, $H^2(M, \mathcal{D}(2))$, because this is the group that classifies bundle gerbes with connection. For brevity, we call this group the **Deligne cohomology** of $M$.

We now construct a class in the Deligne cohomology of $M$ from a bundle gerbe $\mathcal{G} = (Y, L, \mu)$ with connection $(\nabla, B)$. We use Čech hypercohomology to do so. The choices involved are the same as in the previous section, but we recall them here: choose a good open cover $U = \{U_\alpha\}$ of $M$ such that the submersion $\pi: Y \to M$ admits sections $s_\alpha: U_\alpha \to Y$. On each double intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$, the universal property of the fiber product yields a map:

$$(s_\alpha, s_\beta): U_{\alpha\beta} \to Y^{[2]}.$$  

As in the last section, in the map $s_\alpha: U_\alpha \to Y$ and the induced map $(s_\alpha, s_\beta): U_{\alpha\beta} \to Y^{[2]}$, we are treating open subsets of the body $M_0$ as supermanifolds. We do this by restricting the structure sheaf $\mathcal{O}_M$ to the given open set.

Now use the map $(s_\alpha, s_\beta)$ to pull back our line bundle $L$ to $U_{\alpha\beta}$, yielding a line bundle $L_{\alpha\beta} = (s_\alpha, s_\beta)^* L$. Because $U_{\alpha\beta}$ is contractible, $L_{\alpha\beta}$ is trivializable [15], so there is a section $\sigma_{\alpha\beta} \in L_{\alpha\beta}(U_{\alpha\beta})$ such that $L_{\alpha\beta} = \mathcal{O}_\mathbb{C} \cdot \sigma_{\alpha\beta}|_{U_{\alpha\beta}}$.

Having made these choices, we obtain our degree 2 class in Deligne cohomology. In detail, we take the Čech–Deligne double complex

$$C^\bullet(\mathcal{U}, \mathcal{D}(2))$$

associated to the cover $\mathcal{U}$, and pass to the total complex. Degree 2 of the total complex is the following direct sum:

$$\prod_{\alpha, \beta, \gamma} \mathcal{O}^*_\mathbb{C}(U_{\alpha\beta\gamma}) \oplus \prod_{\alpha, \beta} \Omega^2_{\mathbb{C},0}(U_{\alpha\beta}) \oplus \prod_{\alpha} \Omega^2_{\mathbb{C},0}(U_\alpha).$$

Inside this direct sum we find our representative $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha)$. The first element $g_{\alpha\beta\gamma}$ is the same as in the last section: on the triple intersection $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, the bundle gerbe multiplication induces an isomorphism $\mu: L_{\alpha\beta} \otimes L_{\beta\gamma} \to L_{\alpha\gamma}$. We thus have two sections of $L_{\alpha\gamma}$ over a triple intersection, namely $\sigma_{\alpha\gamma}$ and $\mu(\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma})$, both of which are trivializing. Thus we must have:

$$\mu(\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma}) = g_{\alpha\beta\gamma}\sigma_{\alpha\gamma},$$

for some even, invertible element of the complexified structure sheaf, $g_{\alpha\beta\gamma} \in \mathcal{O}^*_\mathbb{C}(U_{\alpha\beta\gamma})$.

The even 1-form $A_{\alpha\beta} \in \Omega^1_{\mathbb{C},0}(U_{\alpha\beta})$ on each double intersection $U_{\alpha\beta}$ comes from our connection. Specifically, we pull $\nabla$ back to $L_{\alpha\beta}$, and define $A_{\alpha\beta}$ to be the unique even 1-form such that the following equation holds:

$$\nabla(\sigma_{\alpha\beta}) = A_{\alpha\beta} \otimes \sigma_{\alpha\beta}.$$  

Finally, the even 2-form $B_\alpha \in \Omega^2_{\mathbb{C},0}(U_\alpha)$ on each open $U_\alpha$ is simply the pullback of the curving $B \in \Omega^2_{\mathbb{C},0}(Y)$ along $s_\alpha$:

$$B_\alpha = s_\alpha^* B.$$  

The triple $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha)$ is a degree 2 element in the Čech–Deligne total complex. With some work, we can use the definition of bundle gerbe with connection to show that it is a 2-cocycle. With some more work, we can show that the cohomology class of this 2-cocycle is independent of the
choices we have made. Thus we obtain a class $\text{Del}(\mathcal{G}) \in H^2(M, \mathcal{D}(2))$ in Deligne cohomology, the **Deligne class of the bundle gerbe with connection** $\mathcal{G}$.

The Deligne class just constructed is a complete invariant of a bundle gerbe with connection, considered up to equivalence. We prove this in Theorem 13. Its proof follows the proof of the analogous result over smooth manifolds by Murray [18] and Murray–Stevenson [20], adapted to the language of structure sheaves.

**Theorem 13.** Let $M$ be a supermanifold, $\mathbb{G}^\nabla(M)$ the abelian group of equivalence classes of bundle gerbes with connection over $M$, and let $\text{Del}: \mathbb{G}^\nabla(M) \to H^2(M, \mathcal{D}(2))$ be the map sending the equivalence class of a bundle gerbe with connection on $M$ to the corresponding Deligne cohomology class. Then:

I. $\text{Del}: \mathbb{G}^\nabla(M) \to H^2(M, \mathcal{D}(2))$ is an isomorphism of abelian groups, i.e.,

- $\text{Del}$ is onto;
- $\text{Del}(\mathcal{G}) = 0$ if and only if $\mathcal{G}$ is trivial;
- $\text{Del}(\mathcal{G} \otimes \mathcal{H}) = \text{Del}(\mathcal{G}) + \text{Del}(\mathcal{H})$;
- $\text{Del}(\mathcal{G}^*) = -\text{Del}(\mathcal{G})$;

where $\mathcal{G}$ and $\mathcal{H}$ are bundle gerbes with connection on $M$.

II. If $f: M \to N$ is a map of supermanifolds, $\text{Del}$ is natural with respect to pullback by $f$:

$$\text{Del}(f^*\mathcal{G}) = f^*\text{Del}(\mathcal{G}),$$

for any bundle gerbe $\mathcal{G}$ with connection on $N$.

**Proof.** The proof of this theorem is similar to the proof of Theorem 9, but differs when it comes to showing that $\text{Del}$ is injective. We thus focus on this part. Because $\text{Del}$ is a group homomorphism, it suffices to check that it has trivial kernel. In other words, that $\text{Del}(\mathcal{G}) = 0$ implies $\mathcal{G}$ has a trivialization.

So let $\mathcal{G} = (Y, L, \mu)$ be a bundle gerbe on $M$ with connection $(\nabla, B)$. Let $\text{Del}(\mathcal{G}) = [g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_{\alpha}]$ be the Deligne class constructed as above using a good cover $\{U_\alpha\}$, sections $s_\alpha: U_\alpha \to Y$, and trivializing sections $\sigma_{\alpha\beta} \in L_{\alpha\beta}(U_{\alpha\beta})$. The equation $\text{Del}(\mathcal{G}) = 0$ means our representative 2-cocycle $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_{\alpha})$ is a coboundary, which in turn means that

$$
g_{\alpha\beta\gamma} = f_{\alpha\beta}f_{\beta\gamma}f_{\gamma\alpha},$$
$$A_{\alpha\beta} = z_\alpha - z_\beta + f_{\alpha\beta}^{-1}df_{\alpha\beta},$$
$$B_{\alpha} = dz_\alpha,$$

for some collection of even, invertible elements of the complexified structure sheaf, $f_{\alpha\beta} \in \mathcal{O}^*_C(U_{\alpha\beta})$, and some collection of even 1-forms, $z_\alpha \in \Omega^1_{C,0}(U_\alpha)$.

Forgetting the connection $\nabla$ and the associated cocycle data $(A_{\alpha\beta}, B_{\alpha})$, we are in precisely the situation of Theorem 9. As in the proof of that theorem, we can construct a line bundle $T \to Y$ and a line bundle isomorphism $\tau: L \to \delta T$ compatible with multiplication. To finish constructing a trivialization of $(\mathcal{G}, \nabla, B)$, we need to:

- Choose a connection $\nabla_T$ on $T$ such that, with the induced connection on $\delta T$, the map $\tau$ becomes an isomorphism of line bundles with connection;
- Check that $F_T = B$, for $F_T$ the curvature of $\nabla_T$, and $B$ the curving.
Finding the connection is straightforward: to begin, choose any connection $\nabla_T$ on $T$. Let $\delta \nabla_T$ denote the induced connection on $\delta T$. Using the isomorphism to identify $L$ and $\delta T$, the two line bundle connections differ by an even 1-form:

$$\nabla = \delta \nabla_T + a,$$

for some $a \in \Omega^1_{\mathbb{C},0}(Y^{[2]})$. The fact that $\nabla$ is compatible with multiplication implies $\delta a = 0$, and an application of Lemma 3 tells us that $a = \delta a'$ for some even 1-form $a' \in \Omega^1_{\mathbb{C},0}(Y)$. Redefining $\nabla_T$ to be $\nabla_T - a'$, we have $\nabla = \delta \nabla_T$, as desired.

Alas, with this choice $(T, \nabla_T)$ of line bundle and connection, we cannot guarantee that the curvature equals the curving, $F_T = B$. Instead, we can check that $F_T - B = \pi^*b$, for some 2-form $b$ on $M$, as follows: because $(\nabla, B)$ is a connection, it satisfies the descent equation, $F_T = \delta B$. Moreover, because $\nabla = \delta \nabla_T$, we also have $F_T = \delta F_T$. Combining these two facts, we see that $F_T - B$ is closed under $\delta$, so another application of Lemma 3 gives us the 2-form $b$.

To finish constructing our trivialization, we note there is a line bundle with connection $(Q, \nabla_Q)$ on $M$ whose curvature is the 2-form $b$: $F_Q = b$. To see this, pull our line bundle $T$ back to the open set $U_\alpha$, $T_\alpha = s_\alpha^* T$, and choose trivializing sections $\sigma_\alpha \in T_\alpha(U_\alpha)$. The connection $\nabla_T$ induces a connection $\nabla_{T_\alpha}$, and we let $a_\alpha \in \Omega^1_{\mathbb{C}}(U_\alpha)$ denote the corresponding connection 1-form:

$$\nabla_{T_\alpha} \sigma_\alpha = a_\alpha \otimes \sigma_\alpha.$$

Finally, redefine the trivializing sections $\sigma_{\alpha\beta}$ of $L_{\alpha\beta}$ to be given by the difference $\sigma_{\alpha\beta} = \frac{2\pi}{\sigma_\beta}$. With these choices, we have:

$$g_{\alpha\beta\gamma} = 1, \quad A_{\alpha\beta} = a_\alpha - a_\beta.$$

Combining this with the fact that this 2-cocycle is a coboundary, we see:

$$1 = f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha},$$

$$a_\alpha - a_\beta = z_\alpha - z_\beta + f_{\alpha\beta}^{-1} df_{\alpha\beta},$$

$$B_{\alpha\beta} = dz_\alpha.$$

Thus, there is a line bundle $(Q, \nabla_Q)$ on $M$ with descent data $(f_{\alpha\beta}, a_\alpha - z_\alpha)$. It is a quick exercise to check that $F_Q = b$ using the descent data.

Now, with the line bundle $(Q, \nabla_Q)$ in hand, note that $T \otimes \pi^* Q^*$ with its induced connection $\nabla_T \otimes \pi^* Q^*$ has curvature equal to the curving $B$. Moreover, because $\delta \pi^* = 0$, note that $\delta (T \otimes \pi^* Q^*) \cong \delta T$ and this isomorphism takes $\delta \nabla_T \otimes \pi^* Q^*$ to $\delta \nabla_T$. Thus we can redefine $T$ to be $T \otimes \pi^* Q^*$, and this yields our trivialization of $(G, \nabla, B)$.

We turn to the issue of when $\mathcal{I}_b \simeq \mathcal{I}_{b'}$. Let us write $\mathbb{I}^\nabla(M)$ for equivalence classes of trivial bundle gerbes with connection; more precisely, this is the set of all equivalence classes of bundle gerbes with connection that have vanishing Dixmier–Douady class:

$$\mathbb{I}^\nabla(M) = \{ [G] \in \mathbb{G}^\nabla(M) : DD(G) = 0 \}.$$

This is a group under tensor product, and we will now compute it. To facilitate this, we need a definition: we say that a closed, complex-valued $p$-form $\alpha \in \Omega^p_\mathbb{C}(M)$ is integral if its de Rham class $[\alpha]$ lies in the image of the map $H^p(M_b, 2\pi i \mathbb{Z}) \to H^p(M_b, \mathbb{C})$ from integral cohomology.

**Proposition 14.** The abelian group $\mathbb{I}^\nabla(M)$ of trivial bundle gerbes with connection is isomorphic to the group of even 2-forms modulo the group of even, closed, integral 2-forms, which we denote by $\Omega^2_\mathbb{C}(M)$:

$$\mathbb{I}^\nabla(M) \cong \Omega^2_\mathbb{C}(M)/\Omega^2_{\mathbb{C},0}(M).$$
Proof. Consider the map
\[ \Omega^2_{\mathbb{C},0}(M) \rightarrow \mathcal{P}(M) \]
\[ b \mapsto \mathcal{I}_b, \]
where \( \mathcal{I}_b = (M, \mathcal{O}_C, \mu_{\text{can}}) \) is the trivial bundle gerbe with curving \( b \in \Omega^2_{\mathbb{C},0}(M) \), as constructed in Example 11. Because the curvings add under tensor product, this map is a homomorphism. We wish to show that it is onto with kernel \( \Omega^2_{\mathbb{Z}}(M) \).

To show that it is onto, let \( G = (Y, L, \mu) \) be a bundle gerbe with connection \( (\nabla, B) \) and vanishing Dixmier–Douady class. By Theorem 9, there is a line bundle \( T \rightarrow Y \) and an isomorphism of line bundles \( L \cong \delta T \) respecting bundle gerbe multiplication. We henceforth identify the line bundles \( L \) and \( \delta T \) using this isomorphism. By the same argument as in the proof of Theorem 13, we can equip \( T \) with a connection \( \nabla_T \) such that \( \nabla = \delta \nabla_T \). Also by the same argument, there is a 2-form \( b \) on the base such that:
\[ B = F_T + \pi^* b. \]
Thus \( G \cong \mathcal{I}_b \).

Finally, to show the kernel is \( \Omega^2_{\mathbb{Z}}(M) \), we must check that \( \mathcal{I}_b \) is trivializable if and only if \( b \) is closed and integral. On the one hand, if \( b \) is closed and integral, then there is a line bundle \( T \rightarrow M \) with connection and curvature \( b \) [15]. This provides a trivialization. On the other hand, if there is a trivialization, \( T \rightarrow M \), then \( b = F_T \), so \( b \) is closed and integral.

3.3 Curvature

To any connection on a bundle gerbe, we can associate a 3-form, called the ‘curvature’. The curvature 3-form is closed, and we shall see it is a de Rham representative of the Dixmier–Douady class.

Definition 15. Let \( G = (Y, L, \mu) \) be a bundle gerbe with connection \( (\nabla, B) \) over the supermanifold \( M \). We define the curvature \( \text{curv}(G) \) as the unique complex-valued 3-form \( \text{curv}(G) \in \Omega^3_{\mathbb{C}}(M) \) whose pullback to \( Y \) is the exterior derivative of the curving:
\[ \pi^* \text{curv}(G) = dB. \]

The existence and uniqueness of this 3-form is a consequence of the descent equation,
\[ F_{\nabla} = \pi_1^* B - \pi_2^* B. \]
Indeed, recall that in the notation of Lemma 3, we write \( \delta : \Omega^2_{\mathbb{C}}(Y) \rightarrow \Omega^2_{\mathbb{C}}(Y[2]) \) for the map taking \( B \) to \( \pi_1^* B - \pi_2^* B \). With this notation, the descent equation reads \( F_{\nabla} = \delta B \). Because the exterior derivative commutes with pullback, we quickly compute:
\[ \delta dB = d\delta B = dF_{\nabla} = 0. \]
So, the 3-form \( dB \) is \( \delta \)-closed, and Lemma 3 guarantees a unique 3-form \( H \) on \( M \) such that \( dB = \pi^* H \).
This 3-form \( H \) is the curvature.
It is immediate from the definition that the curvature is closed, so it represents a class in de Rham cohomology. The central fact about the curvature is that this class is the same as the Dixmier–Douady class, up to a pesky sign:
\[ \text{DD}(G) = -[\text{curv}(G)]. \]
In order to make sense of this equation, we need to say how the Dixmier–Douady class sits in de Rham cohomology. Recall the Dixmier–Douady class was constructed as a degree-2 class in Čech cohomology, \( \text{DD}(G) \in H^2(M_b, \mathcal{O}_C^*) \). We then used a short exact sequence of sheaves:
\[ 0 \rightarrow \mathbb{Z}_{M_b} \rightarrow (\mathcal{O}_C)_0 \rightarrow \mathcal{O}_C^* \rightarrow 0, \]
to construct an isomorphism $\bar{H}^3(M_b, O^*_C) \cong H^3(M_b, 2\pi i \mathbb{Z})$. Here, the $2\pi i$ arises naturally from its appearance in the short exact sequence of sheaves. Of course, as abelian groups, $H^3(M_b, 2\pi i \mathbb{Z}) \cong H^3(M_b, \mathbb{Z})$, but it is useful to keep the factor of $2\pi i$ when comparing $DD(\mathcal{G})$ with the curvature.

Finally, because of the inclusion $2\pi i \mathbb{Z} \hookrightarrow \mathbb{C}$, we get a map $H^3(M_b, 2\pi i \mathbb{Z}) \to H^3(M_b, \mathbb{C})$ from integral to complex coefficients. Abusing notation, we continue to write $DD(\mathcal{G}) \in H^3(M_b, \mathbb{C})$ for the image of this map.

**Proposition 16.** As cohomology classes in $H^3(M_b, \mathbb{C})$, we have $DD(\mathcal{G}) = -[\text{curv}(\mathcal{G})]$.

**Proof.** The core of the proof is a standard double complex argument such as one might find in Bott and Tu [5], but we give it for completeness, following the proof in Waldorf’s thesis [27, Theorem 1.5.3].

We want to compare the Dixmier–Douady class $DD(\mathcal{G})$, constructed using Čech cohomology, with the de Rham class of the curvature $\text{curv}(\mathcal{G})$, and the natural way to do this is to work in the Čech–de Rham double complex:

$$C^\bullet(\mathcal{U}, \Omega^\bullet_C),$$

for some good cover $\mathcal{U} = \{U_\alpha\}$ of our supermanifold $M$. In supergeometry, we have both partitions of unity and the Poincaré lemma, so the usual arguments show that this complex has exact rows and columns. We take the total differential to be $D = \delta + (-1)^p d$, where $\delta$ is the Čech differential, $d$ is the exterior derivative, and $p$ is the Čech degree.

The result follows if we show that representatives of $DD(\mathcal{G})$ and $-\text{curv}(\mathcal{G})$ are cohomologous in the Čech–de Rham complex, and we can do this starting with the Deligne class:

$$\text{Del}(\mathcal{G}) = [g_{a\beta\gamma}, A_{a\beta}, B_\alpha].$$

Because the cover is good, the exponential map $\exp: (O_C)_0(U_{a\beta\gamma}) \to O^*_C(U_{a\beta\gamma})$ is surjective [15], so we can choose a logarithm $h_{a\beta\gamma} \in (O_C)_0(U_{a\beta\gamma})$ for each $g_{a\beta\gamma} \in O^*_C(U_{a\beta\gamma})$.

Having made this choice, the usual construction of the connecting homomorphism in Čech cohomology tells us that $k_{a\beta\gamma\delta} = (\delta h)_{a\beta\gamma\delta}$ is a Čech representative of the Dixmier–Douady class $DD(\mathcal{G})$, or more precisely of its image in $H^3(M_b, \mathbb{C})$. Similarly, restricting the curvature $H = \text{curv}(\mathcal{G})$ to each open gives a family of 3-forms $H_\alpha = H|_{U_\alpha}$ that represents the de Rham class $[H]$ in the Čech–de Rham complex. Now, a calculation shows:

$$D(h_{a\beta\gamma}, A_{a\beta}, B_\alpha) = (k_{a\beta\gamma\delta}, 0, 0, 0, H_\alpha).$$

The right-hand side represents $DD(\mathcal{G}) + [H]$, and the equation says this is cohomologous to zero. This completes the proof.

Because it lies in the image of $H^3(M_b, 2\pi i \mathbb{Z}) \to H^3(M_b, \mathbb{C})$, the curvature 3-form is quite special: its de Rham class lies in a distinguished lattice inside the larger vector space. In general, we say that a closed, complex-valued $p$-form $\alpha \in \Omega^p_C(M)$ is integral if its de Rham class $[\alpha]$ lies in the image of the map $H^p(M_b, 2\pi i \mathbb{Z}) \to H^p(M_b, \mathbb{C})$ from integral cohomology.

Clearly, the curvature 3-form is integral. In fact, every even, closed, integral 3-form is the curvature 3-form of some bundle gerbe:

**Theorem 17.** If $H \in \Omega^3_C(M)$ is an even, closed, integral, complex-valued 3-form on the supermanifold $M$, then there is a bundle gerbe $\mathcal{G}$ with connection on $M$ such that the curvature of $\mathcal{G}$ is $H$.

**Proof.** This proof is similar to that of Proposition 16, but running in reverse: in the Čech–de Rham complex, we know that $H_\alpha = H|_{U_\alpha}$ is cohomologous to a Čech representative $k_{a\beta\gamma\delta} \in O_C(U_{a\beta\gamma\delta})$ that, by hypothesis, is valued in $2\pi i \mathbb{Z}$. Thus:

$$D(h_{a\beta\gamma}, A_{a\beta}, B_\alpha) = (-k_{a\beta\gamma\delta}, 0, 0, 0, H_\alpha),$$

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for some Čech–de Rham 2-cochain \((h_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha)\). Setting \(g_{\alpha\beta\gamma} = \exp(h_{\alpha\beta\gamma})\), a calculation shows that \((g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha)\) is a 2-cocycle in the Čech–Deligne complex. Letting \(\mathcal{G}\) be the bundle gerbe with this Deligne class, we have \(\text{curv}(\mathcal{G}) = H\).

\[\square\]

### 3.4 Body and soul

We have noted that every supermanifold contains an ordinary manifold: the body. While the inclusion of the body into the full supermanifold is canonical, there is no canonical projection from the supermanifold down to the body. In fact, for holomorphic supermanifolds, such a projection need not even exist.

We are working with smooth supermanifolds, so a projection does exist, but it is a choice of extra data. In this section we fix such a choice. For a supermanifold \(M\), fix a **body projection**: a map \(p: M \to M_b\) such that \(p i = 1_{M_b}\), where \(i: M_b \hookrightarrow M\) is the canonical inclusion of the body \(M_b\).

In his treatment of supergeometry, it was DeWitt who christened the submanifold \(M_b\) the ‘body’ of \(M\). He then called the directions in \(M\) transverse to the body \(M_b\) the ‘soul’ [11]. Although our framework for supergeometry is based on locally ringed spaces and is distinct from DeWitt’s formalism, we adopt his poetic terminology.

For any supermanifold \(M\), pullback along the inclusion \(i: M_b \hookrightarrow M\) gives us a short exact sequence:

\[
0 \longrightarrow \Omega^\bullet(M_b) \longrightarrow \Omega^\bullet(M) \stackrel{i^*}{\longrightarrow} \Omega^\bullet(M_b) \longrightarrow 0.
\]

Here, \(\Omega^\bullet(M_b)\) is the ordinary de Rham complex on the ordinary manifold \(M_b\); we call it the **body of the de Rham complex** \(\Omega^\bullet(M)\). The kernel of the pullback, \(\Omega^\bullet(M) = \ker(i^*)\), is the subcomplex of forms that go to zero upon restriction to the body. We call this subcomplex \(\Omega^\bullet(M_b)\) the **soul of the de Rham complex** \(\Omega^\bullet(M)\).

Having fixed a body projection \(p: M \to M_b\), we get a splitting of this short exact sequence.

**Proposition 18.** Let \(M\) be a supermanifold equipped with a body projection \(p: M \to M_b\). Then the pullback along the body projection, \(p^*\), splits the short exact sequence of forms:

\[
0 \longrightarrow \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M) \stackrel{i^*}{\longrightarrow} \Omega^\bullet(M_b) \longrightarrow 0
\]

In particular, the de Rham complex \(\Omega^\bullet(M)\) decomposes into a direct sum of body and soul:

\[
\Omega^\bullet(M) \cong \Omega^\bullet(M_b) \oplus \Omega^\bullet_s(M),
\]

where the projection map \(\Omega^\bullet(M) \to \Omega^\bullet(M_b)\) is \(i^*\) and the inclusion map \(\Omega^\bullet(M_b) \hookrightarrow \Omega^\bullet(M)\) is \(p^*\).

**Proof.** To show that \(i^*\) is onto, first note that this is automatic in any coordinate patch with coordinates \((x_i, \theta_j)\): in this patch, a form on the body is just a form on the supermanifold independent of the \(\theta_j\) coordinates. Surjectivity now follows from a partition of unity argument.

To show that the sequence splits, note that \(pi = 1\) by the definition of the body projection. Thus \(i^*p^* = 1\), and we are done. \(\square\)

Using this proposition, we see that any \(k\)-form \(\omega \in \Omega^k(M)\) uniquely decomposes into its **body** \(\omega_b = p^*i^*\omega\) and its **soul** \(\omega_s = \omega - p^*i^*\omega\). The importance of the body and soul decomposition lies in the fact that the soul of the de Rham complex \(\Omega^\bullet_s(M)\) is acyclic: its cohomology is trivial in all degrees.

**Proposition 19.** \(H^\bullet(\Omega^\bullet_s(M)) = 0\). In other words, if \(\omega = \omega_s\) is a closed \(k\)-form that is pure soul, then it is exact: \(\omega = d\sigma\), for some \(\sigma \in \Omega^{k-1}_s(M)\).
Proof. In his notes on supergeometry [15], Kostant observes that restriction to the body, $i^*: \Omega^*(M) \to \Omega^*(M_b)$, is a quasi-isomorphism: it induces isomorphism on cohomology. So, the map $i^*$ is a surjective map of complexes that is also a quasi-isomorphism, and it follows that its kernel, $\Omega^*_b(M)$, is an acyclic complex.

Now let us apply the body and soul decomposition to the curvature of a bundle gerbe. If $H = \text{curv}(G)$ is the curvature 3-form of a bundle gerbe with connection, then applying the body and soul decomposition to the complexified de Rham complex, we have $H = H_b + H_s$. Both $H_b$ and $H_s$ are closed, so by the last proposition there is a pure soul 2-form $\beta \in \Omega^2_s(M)$ such that $H_s = d\beta$, and we find:

$$H = H_b + d\beta.$$ 

We noted in the last section that $H$ is integral: its cohomology class $[H] \in H^3(M_b, \mathbb{C})$ lies in the image of integral cohomology, $H^3(M_b, 2\pi i\mathbb{Z}) \to H^3(M_b, \mathbb{C})$. Because $[H] = [H_b]$, the body $H_b$ must be integral as well.

Thus, from the curvature 3-form $H$ of a bundle gerbe $G$ on a supermanifold $M$, we have constructed an ordinary, closed, integral 3-form $H_b$ on the body $M_b$. From the theory of bundle gerbes, it now follows that there is an ordinary bundle gerbe $G_b$ on the body $M_b$ with curvature $H_b = \text{curv}(G_b)$.

By “ordinary”, we mean that this bundle gerbe is constructed entirely in the category of smooth manifolds: $G_b = (Y_b, L_b, \mu_b)$ for $Y_b$, a smooth manifold and $Y_b \to M_b$ a surjective submersion, $L_b$ a line bundle on $Y_b$, and so forth.

How does the bundle gerbe $G_b$ in the category of smooth manifolds compare to the bundle gerbe $G$ that we started with in the category of supermanifolds? Using the projection $p: M \to M_b$, we can try to compare them: the pullback $p^*G_b$ is a bundle gerbe on $M$. This new bundle gerbe on $M$ has different curvature from $G$, but only just: $\text{curv}(p^*G_b) = H_b$, while $\text{curv}(G) = H_b + d\beta$. This suggests that if we add $\beta$ to the curving of $p^*G_b$, we might recover $G$. Indeed:

$$\text{curv}(p^*G_b \otimes I_\beta) = H.$$ 

Even so, $G$ need not be equivalent to $p^*G \otimes I_\beta$. As a pair of bundle gerbes on $M$ with the same curvature, they can differ by a flat bundle gerbe.

To understand this, fix a 3-form $H$ and consider the set of bundle gerbes with curvature $H$:

$$G^V_H(M) = \{ [G] \in G^V(M) : \text{curv}(G) = H \}.$$ 

Of course, this set is empty unless $H$ is even, closed and integral, so we assume this from now on. This subset of the group $G^V(M)$ is not itself a group unless $H = 0$, because curvatures add under tensor product: $\text{curv}(G \otimes H) = \text{curv}(G) + \text{curv}(H)$.

We call a bundle gerbe $G$ flat if $\text{curv}(G) = 0$. The group $G^V_0(M)$ of equivalence classes of flat bundle gerbes is an essential tool for understanding how two bundle gerbes with the same curvature can differ. This is because, for any two bundle gerbes $G$ and $H$ with the same curvature $H$, there is a unique flat bundle gerbe $G_0$ such that:

$$G \otimes G_0 \simeq H.$$ 

Indeed, tensoring both sides with $G^*$, we must have $G_0 \simeq G^* \otimes H$. We have proved:

**Proposition 20.** For any even, closed, integral 3-form $H \in \Omega_3^C(M)$, the set $G^V_H(M)$ of equivalence classes of bundle gerbes with curvature $H$ is a torsor for the group of flat bundle gerbes, $G^V_0(M)$.

Returning to the body and soul decomposition, we would like to compare the torsor of bundle gerbes of specified curvature on the body with that on the full supermanifold. To begin, let us compare the group of flat bundle gerbes $G^V_0(M_b)$ on the body with the that on the full supermanifold, $G^V_0(M)$. Using our fixed projection map $p: M \to M_b$, we can pullback flat bundle gerbes on the body. The resulting bundle gerbe on $M$ is still flat, and this is an isomorphism:
Proposition 21. Let $M$ be a supermanifold equipped with a body projection $p: M \to M_b$. Pullback of flat bundle gerbes along the body projection induces an isomorphism:

$$p^* : \mathcal{G}_0^\Sigma(M_b) \to \mathcal{G}_0^\Sigma(M)$$

between the group of flat bundle gerbes on the body $M_b$ and on the the full supermanifold $M$.

Proof. To show $p^*$ has trivial kernel, suppose $p^* \mathcal{G} \simeq \mathcal{I}$. Because $pi = 1$ for the canonical inclusion $i$, applying $i^*$ yields $\mathcal{G} \simeq \mathcal{I}$. On the other hand, let $\mathcal{G}$ be a flat bundle gerbe on the supermanifold $M$ and consider the bundle gerbe $\mathcal{G}_b = i^* \mathcal{G}$ on the body. We claim $p^* \mathcal{G}_b \simeq \mathcal{G}$. Indeed, since $p$ and $i$ act trivially on the Dixmier–Douady class, we have that $DD(p^* \mathcal{G}_b \otimes \mathcal{G}^*) = 0$. Thus from Proposition 14, we have:

$$p^* \mathcal{G}_b \otimes \mathcal{G}^* \simeq \mathcal{I}_\beta$$

for some 2-form $\beta$. Moreover, we can choose $\beta$ to be pure soul, because applying $i^*$ kills it. Now, since the left-hand side is flat, we must have $d\beta = 0$. But a closed pure soul 2-form is integral, so $\mathcal{I}_\beta \simeq \mathcal{I}$. We conclude $p^* \mathcal{G}_b \simeq \mathcal{G}$, as desired.

The proof of the previous proposition actually shows that restriction to the body $i^*$ is the inverse of $p^*$, so it turns out the isomorphism between the groups of flat bundle gerbes is independent of the choice of body projection. Nevertheless, the choice of body projection is essential for comparing the torsors $\mathcal{G}_{H^\infty}^\Sigma(M_b)$ and $\mathcal{G}_0^\Sigma(M)$, so we have chosen to emphasize it.

Theorem 22. Let $M$ be a supermanifold equipped with a body projection $p: M \to M_b$, and let $H \in \Omega^2_{\Sigma}(M)$ be an even, closed, integral 3-form. Choosing an even 2-form $\beta \in \Omega^2_{\Sigma}(M)$ such that $H_\beta = d\beta$ is the soul of $H$, we have the following isomorphism of $\mathcal{G}_{H_b}^\Sigma(M_b)$-torsors:

$$\theta : \mathcal{G}_{H_b}^\Sigma(M_b) \to \mathcal{G}_b^\Sigma(M), \quad \mathcal{G}_b \mapsto p^* \mathcal{G}_b \otimes \mathcal{I}_\beta.$$

Proof. Because any map between torsors is an isomorphism, we need only show that $\theta$ intertwines the action. For any bundle gerbe $\mathcal{G}_b$ on the body with curvature $H_b$, and any flat bundle gerbe $\mathcal{G}_0$, we have:

$$\theta(\mathcal{G}_b \otimes \mathcal{G}_0) = p^*(\mathcal{G}_b \otimes \mathcal{G}_0) \otimes \mathcal{I}_\beta \simeq (p^* \mathcal{G}_b \otimes \mathcal{I}_\beta) \otimes p^* \mathcal{G}_0,$$

using the fact that pullback respects the tensor product. Since we are using $p^*$ to identify the groups of flat bundle gerbes, this is the desired result.

As a corollary, we get the body and soul decomposition of any bundle gerbe.

Corollary 23. Let $\mathcal{G}$ be a bundle gerbe on a supermanifold $M$, and let $p: M \to M_b$ be a body projection. Then there is a bundle gerbe $\mathcal{G}_b$ on the body $M_b$ and a pure soul 2-form $\beta$ such that:

$$\mathcal{G} \simeq p^* \mathcal{G}_b \otimes \mathcal{I}_\beta.$$

Moreover, $\mathcal{G}_b \simeq i^* \mathcal{G}$, and this is the unique choice up to equivalence, while $\beta$ is unique up to the addition of an exact pure soul 2-form.

Proof. The existence of this decomposition follows from Theorem 22 with $H = \text{curv}(\mathcal{G})$. Applying $i^*$ to both sides of $\mathcal{G} \simeq p^* \mathcal{G}_b \otimes \mathcal{I}_\beta$, see that $\mathcal{G}_b \simeq i^* \mathcal{G}$, where we have used $i^* \mathcal{I}_\beta \simeq \mathcal{I}$, because $\beta$ is pure soul. Finally, if

$$p^* \mathcal{G}_b \otimes \mathcal{I}_\beta \simeq p^* \mathcal{G}_b \otimes \mathcal{I}_{\beta'}$$

for two pure soul 2-forms $\beta$ and $\beta'$, then we can tensor on both sides with $p^* \mathcal{G}_b^\Sigma$ to conclude $\mathcal{I}_\beta \simeq \mathcal{I}_{\beta'}$. From Proposition 14, we conclude that $\beta - \beta'$ is closed, and Proposition 19 tells us it is exact.
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