Modular Classes of Loday Algebroids

Mathieu Stiénon ∗
E.T.H. Zürich
stienon@math.ethz.ch

Ping Xu †
Penn State University
ping@math.psu.edu

Abstract

We introduce the concept of Loday algebroids, a generalization of Courant algebroids. We define the naive cohomology and modular class of a Loday algebroid, and we show that the modular class of the double of a Lie bialgebroid vanishes. For Courant algebroids, we describe the relation between the naive and standard cohomologies and we conjecture that they are isomorphic when the Courant algebroid is transitive.

1 Naive Cohomology

Given a Courant algebroid \((E, \rho, \{\cdot, \cdot\}, \langle\cdot, \cdot\rangle)\), let \(\Gamma(\bigwedge^k \ker \rho)\) denote the space of smooth sections of the (possibly singular) vector bundle \(\bigwedge^k \ker \rho\) (i.e. smooth sections \(\alpha\) of \(\bigwedge^k E\) such that \(\alpha|_m \in \bigwedge^k \ker \rho\) for each \(m \in M\)). The extension of the pseudo-metric \(\langle\cdot, \cdot\rangle\) to \(\bigwedge^k E\) naturally induces an isomorphism \(\Xi : \bigwedge^k E \to \bigwedge^k E^*\). Since, by definition, \(\langle Df, e \rangle = \frac{1}{2} \rho(e)f\), the sections of \(\bigwedge^k \ker \rho\) are characterized as the elements \(\hat{\varepsilon} \in \Gamma(\bigwedge^k E)\) such that \(\bar{i}_{Df} \hat{\varepsilon} = 0\), \(\forall f \in C^\infty(M)\). Here \(\bar{i}_{Df} = \Xi^{-1} \cdot i_{Df} \Xi\), where \(i_{Df} : \Gamma(\bigwedge^{k+1} E^*) \to \Gamma(\bigwedge^k E^*)\) is the usual contraction of exterior forms with the section \(Df \in \Gamma(E)\). Define an operator \(\hat{d} : \Gamma(\bigwedge^k \ker \rho) \to \Gamma(\bigwedge^{k+1} E)\) by

\[
(\hat{d}\alpha)(e_0, \ldots, e_k) = \sum_{i=0}^k (-1)^i \rho(e_i)\alpha(e_0, \ldots, \hat{e}_i, \ldots, e_k) \\
+ \sum_{i<j} (-1)^{i+j} \alpha([e_i, e_j], e_0, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_k),
\]

for all \(\alpha \in \Gamma(\bigwedge^k \ker \rho)\) and \(e_0, \ldots, e_k \in \Gamma(E)\). Here the pairing between \(\Gamma(\bigwedge^k \ker \rho)\) and \(\Gamma(\bigwedge^k E)\) is via the identification \(\Xi : \bigwedge^k E \to \bigwedge^k E^*\). The following Lemma follows from the Courant algebroid properties, in particular the relations \(\rho(Df) = 0\) and \([Df, e] + D(Df, e) = 0\).

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Lemma 1.1 We have $\tilde{d}\Gamma(\wedge^k \ker \rho) \subset \Gamma(\wedge^{k+1} \ker \rho)$. Moreover, $(\Gamma(\wedge^* \ker \rho), \tilde{d})$ is a cochain complex.

The cohomology of this cochain complex is called the naive cohomology of $E$ and is denoted $H^*_{\text{naive}}(E)$.

Remark 1.2 It is easy to see that a 1-cocohain $\theta \in \Gamma(\ker \rho)$ is a 1-cocycle if, and only if, $\langle \theta, [a, b] \rangle = \rho(a)\langle \theta, b \rangle - \rho(b)\langle \theta, a \rangle$ for all $a, b \in \Gamma(E)$, and a 1-coboundary if, and only if, $\theta = \mathcal{D}f$ for some $f \in \mathcal{C}^\infty(M)$.

Remark 1.3 Let $V$ be the $\mathcal{C}^\infty(M)$-module generated by $\mathcal{D}(\mathcal{C}^\infty(M))$. Since $\langle \mathcal{D}f, a \rangle = \frac{1}{2} \rho(a)f$, we have $V = \Gamma(\ker \rho^\perp)$ and $\Xi(V) = \Gamma(\ker \rho^0) = \rho^*(\Gamma(T^*M))$. Moreover $V \subset \Gamma(\ker \rho)$, for $\rho_{\mathcal{D}} = 0$. Therefore, when $(E, \rho, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ is a regular Courant algebroid (i.e. $\rho$ has constant rank), $E/\Xi^{-1}(\rho^*\Gamma^*M)$ is a Lie algebroid and $H^*_{\text{naive}}(E)$ is the cohomology of this Lie algebroid. However, in general, $\Gamma(E)/V$ is only a Lie-Rinehart algebra over $\mathcal{C}^\infty(M)$. One can consider $H^*_{\text{naive}}(E)$ as its cohomology [4].

Example 1.4 When $E = TM \oplus T^*M$ is an exact Courant algebroid, $H^*_{\text{naive}}(E)$ is isomorphic to the de Rham cohomology of $M$.

Example 1.5 If $E$ is a Courant algebroid over a point, i.e. a Lie algebra equipped with a non-degenerate ad-invariant bilinear form, $H^*_{\text{naive}}(E)$ is simply the Lie algebra cohomology.

2 Relation with standard cohomology

Courant algebroids can also be obtained as derived brackets [5][9] using degree two super-symplectic manifolds. More precisely, given a Courant algebroid $(E, \rho, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$, $E[1]$ is a super-Poisson manifold, where the Poisson structure is induced by the pseudo-metric. There is a minimal symplectic realization $X \xrightarrow{\pi} E[1]$ and a cubic function $\Theta$ on $X$ such that $\{\Theta, \Theta \} = 0$ and, for all $f \in \mathcal{C}^\infty(M)$ and $e_1, e_2 \in \Gamma(E)$,

$$\mathcal{D}f = \{\Theta, f\}$$

and

$$e_1 \circ e_2 = \{\{\Theta, e_1\}, e_2\},$$

where the symbol $\circ$ denotes the asymmetric Dorfman bracket defined by the relation $a \circ b = [a, b] + \mathcal{D}\langle a, b \rangle$. Here elements in $\Gamma(\wedge^k E)$ are viewed as functions of degree $k$ on $X$ by considering them as functions on $E[1]$ via the pseudo-metric $\langle \cdot, \cdot \rangle$ and identifying them with their pull back by $\pi$. Similarly functions on $M$ are also identified with their pull back in $X$. By $\mathcal{A}^k$ we denote the space of functions on $X$ of degree $k$. Then $(\mathcal{A}^*, \{\Theta, \cdot\})$ is a cochain complex. Its cohomology is called the standard cohomology by Roytenberg [9] and we shall denote it by $H^*_{\text{std}}(E)$.

Lemma 2.1

1. If $c \in \Gamma(\wedge^k \ker \rho)$, then $\{\Theta, c\} = \tilde{dc}$;

2. If $c \in \Gamma(\wedge^k E)$ satisfies $\{\Theta, c\} = 0$, then $c \in \Gamma(\wedge^k \ker \rho)$ and $\tilde{dc} = 0$.
for all $\delta$ properties: $f, g$ for any $\pi$ subspaces covariant differential operator on $E$ linear on the fibers of $E$.

The Lie derivative of Courant algebroids was introduced in \cite{10}. Let us recall its definition briefly. An infinitesimal automorphism of the vector bundle $E \rightarrow M$ is a vector field on $E$ — a derivation of the algebra $C^\infty(E)$ — which preserves the subspaces $\pi^*C^\infty(M)$ and $\Gamma(E)$ (whose elements are identified with functions linear on the fibers of $\pi$ through the pairing $\langle \cdot, \cdot \rangle$). In other words, it is a covariant differential operator on $E$, i.e., a pair of differential operators $\delta^0 : C^\infty(M) \rightarrow C^\infty(M)$ and $\delta^1 : \Gamma(E) \rightarrow \Gamma(E)$ satisfying

$$\delta^0(fg) = f\delta^0(g) + \delta^0(f)g \quad \text{and} \quad \delta^1(fe) = f\delta^1(e) + \delta^0(f)e,$$

for any $f, g \in C^\infty(M)$ and $e \in \Gamma(E)$. It is known \cite{21} that the Lie algebra $\mathfrak{aut}(E)$ of infinitesimal automorphisms of the Courant algebroid $E$ consists of those covariant differential operators $\delta = (\delta^0, \delta^1)$ on $E$ which satisfy the additional properties:

$$\delta^0(e_1, e_2) = (\delta^1 e_1, e_2) + (e_1, \delta^1 e_2) \quad \text{and} \quad \delta^1[e_1, e_2] = [\delta^1 e_1, e_2] + [e_1, \delta^1 e_2],$$

for all $e_1, e_2 \in \Gamma(E)$.

\textbf{Proof.} (i) It suffices to prove the case when $k = 1$. The general situation follows from the Leibniz rule. Now since $\rho(c) = 0$, we have $\forall e_1, e_2 \in \Gamma(E),$

$$\langle c \circ e_2, e_1 \rangle - (\partial c)(e_1, e_2)$$

$$= \langle - (e_2 \circ c, e_1) + 2\langle D\langle c, e_2 \rangle, e_1 \rangle \rangle - \langle \rho(e_1)\langle c, e_2 \rangle - \rho(e_2)\langle c, e_1 \rangle \rangle - \langle e_1\langle e_2, e_2 \rangle \rangle$$

$$= \rho(e_2)\langle c, e_1 \rangle - \langle e_2 \circ c, e_1 \rangle - \langle c, e_2 \circ e_1 \rangle$$

$$= 0.$$

It thus follows that $\{\{\Theta, c\}, e_2\}, e_1 \} = 0$, which implies that $\{\Theta, c\} = \partial c$.

(ii) Since $\partial_D f = \{Df, c\} = \{\Theta, f\}, e_2\} = 0$ for all $f \in C^\infty(M)$, we have $c \in \Gamma(\wedge^k \ker \rho)$. $\Box$

As a consequence, we have a homomorphism $\phi : H^\bullet_{\text{naive}}(E) \rightarrow H^\bullet_{\text{std}}(E)$. Lemma \cite{21} also implies that $\phi$ is an isomorphism in degrees 0 and 1. It is natural to ask when $\phi$ is an isomorphism in all degrees. When $E$ is a Courant algebroid over a point, $\phi$ is clearly an isomorphism. On the other hand, when $E$ is the standard Courant algebroid $TM \oplus T^* M$, both $H^\bullet_{\text{naive}}(E)$ and $H^\bullet_{\text{std}}(E)$ are isomorphic to the de Rham cohomology of $M$. Hence $\phi$ is also an isomorphism. This leads to the following

\textbf{Conjecture} When $E$ is a transitive Courant algebroid, $\phi$ is an isomorphism.

\section{Lie derivatives and Loday algebroids}

The Lie derivative of Courant algebroids was introduced in \cite{10}. Let us recall its definition briefly. An infinitesimal automorphism of the vector bundle $E \rightarrow M$ is a vector field on $E$ — a derivation of the algebra $C^\infty(E)$ — which preserves the subspaces $\pi^*C^\infty(M)$ and $\Gamma(E)$ (whose elements are identified with functions linear on the fibers of $\pi$ through the pairing $\langle \cdot, \cdot \rangle$). In other words, it is a covariant differential operator on $E$, i.e., a pair of differential operators $\delta^0 : C^\infty(M) \rightarrow C^\infty(M)$ and $\delta^1 : \Gamma(E) \rightarrow \Gamma(E)$ satisfying

$$\delta^0(fg) = f\delta^0(g) + \delta^0(f)g \quad \text{and} \quad \delta^1(fe) = f\delta^1(e) + \delta^0(f)e,$$

for any $f, g \in C^\infty(M)$ and $e \in \Gamma(E)$. It is known \cite{21} that the Lie algebra $\mathfrak{aut}(E)$ of infinitesimal automorphisms of the Courant algebroid $E$ consists of those covariant differential operators $\delta = (\delta^0, \delta^1)$ on $E$ which satisfy the additional properties:

$$\delta^0(e_1, e_2) = (\delta^1 e_1, e_2) + (e_1, \delta^1 e_2) \quad \text{and} \quad \delta^1[e_1, e_2] = [\delta^1 e_1, e_2] + [e_1, \delta^1 e_2],$$

for all $e_1, e_2 \in \Gamma(E)$. 

3
For any \( e \in \Gamma(E) \), the pair \( \delta_e = (\delta^0_e, \delta^1_e) \) defined by the relations \( \delta^0_e(f) = \rho(e)f \) and \( \delta^1_e(x) = e_0 x \) is an infinitesimal automorphism of the Courant algebroid \( E \), i.e. \( \delta_e \in \mathfrak{aut}(E) \). Let us denote the (local) flow generated by the vector field on \( E \) corresponding to \( \delta_e \) by \( \phi_t \). By abuse of notations, we use the same symbol \( \phi_t \) (resp. \( \phi^i_t \)) to denote its induced flow on the tensor bundles \( E^i_j = (\otimes^i E) \otimes (\otimes^j E^*) \) (resp. the induced action on the spaces of sections of the \( E^i_j \)'s). For any section \( \sigma \in \Gamma(E^i_j) \), define the Lie derivative \( L_\sigma \in \Gamma(E^i_j) \) by \( L_\sigma = \frac{\partial}{\partial t} \phi^i_t \sigma |_{t=0} \). Thus we have the usual identity: \( \frac{\partial}{\partial t} \phi^i_t \sigma |_{t=0} = \phi^i_t(L_\sigma \sigma) \). In the following proposition, we give a list of important properties of this Lie derivative.

**Proposition 3.1** For all \( f, g \in C^\infty(M) \) and \( x, y, z \in \Gamma(E) \), we have:

\[
\begin{align*}
L_z f &= \rho(z)f, \\
L_z x &= z \circ x, \\
L_{Df} x &= 0, \\
L_x \circ Df &= D L_x f, \\
[\delta, L_z] &= L_{\delta z}, \\
L_z (\sigma \circ \tau) &= L_z \sigma \circ \tau + \sigma \circ L_z \tau, \\
L_z [x, y] &= [L_z x, y] + [x, L_z y], \\
L_{f_{\tau}} x &= f L_{\tau x} - (\rho(y)f) x + 2\langle x, y \rangle Df, \\
L_z \langle x, y \rangle &= \langle L_z x, y \rangle + \langle x, L_z y \rangle.
\end{align*}
\]

**Definition 3.2** A Loday algebroid consists of a vector bundle \( \pi : E \to M \), a pseudo-metric \( \langle \cdot, \cdot \rangle \) on the fibers of \( \pi \), a bundle map \( \rho : E \to TM \) and an \( \mathbb{R} \)-bilinear operation \( \circ \) on \( \Gamma(E) \) satisfying

\[
\begin{align*}
e_1 \circ (e_2 \circ e_3) &= (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3), \\
\rho(e_1 \circ e_2) &= [\rho(e_1), \rho(e_2)], \\
e_1 \circ (f e_2) &= (\rho(e_1)f)e_2 + f(e_1 \circ e_2), \\
e_1 \circ e_2 + e_2 \circ e_1 &= 2D(e_1, e_2), \\
Df \circ e &= 0,
\end{align*}
\]

where \( D : C^\infty(M) \to \Gamma(E) \) is the \( \mathbb{R} \)-linear map defined by \( \langle Df, e \rangle = \frac{1}{2} \rho(e) f \).

**Remark 3.3**

1. According to [2], for Courant algebroids, axioms [3] and [4] are redundant. It would be interesting to investigate if it is also the case for Loday algebroids.

2. The Leibniz algebroids studied by several authors [6, 7, 12] are a more general notion.

3. A Courant algebroid is a Loday algebroid satisfying the additional axiom \( \rho(e) e_1, e_2 = \langle e \circ e_1, e_2 \rangle + \langle e_1, e \circ e_2 \rangle \).

**Lemma 3.4** If \( (E, \rho, \circ, \langle \cdot, \cdot \rangle) \) is a Loday algebroid, then \( \rho(Df) = 0 \) and \( [Df, e] + D\langle Df, e \rangle = 0 \), for all \( f \in C^\infty(M) \) and \( e \in \Gamma(E) \). Here \( [x, y] = \frac{1}{2}(x \circ y - y \circ x) \) as in a Courant algebroid.
depends on the module $S$. Thus the class $\theta$ for the line bundle $S$.

And from (14), it follows that $\theta$ satisfies $\langle D(fg) = gDf + fDg \rangle$ implies that $\rho(\langle \rangle) = 0$. The other relation follows immediately from (12) and (11). □

As a consequence, the definition of the naive cohomology extends from Courant algebroids to Loday algebroids.

Let $(E, \rho, \circ, \langle \cdot, \cdot \rangle)$ be a Loday algebroid. Given a section $z \in \Gamma(E)$, set $L_z f = \rho(z)f$ for $f \in C^\infty(M)$ and $L_z x = z \circ x$ for $x \in \Gamma(E)$ and extend $L_z$ to $\Gamma(\Lambda^k E)$ by the Leibniz rule.

**Proposition 3.5** Identities (4), (5) and (6) still hold for any Loday algebroid.

**Remark 3.6** It is unknown if the standard cohomology can be defined for Loday algebroids. Indeed, it would be interesting to see if there exists a derived bracket in the sense of Kosmann-Schwarzbach (7) for a Loday algebroid.

## 4 Modular classes

A Loday algebroid module is a vector bundle $S \rightarrow M$ endowed with an $\mathbb{R}$-linear map $\Gamma(E) \otimes \Gamma(S) \rightarrow \Gamma(S) : e \otimes s \mapsto \nabla_e s$ satisfying

\[
\begin{align*}
\nabla_D f s &= 0 \\
\nabla f e s &= f \nabla e s \\
\nabla (e_1 \nabla e_2 s) - \nabla e_1 (\nabla e_2 s) &= \nabla [e_1, e_2] s
\end{align*}
\]

(13) for any $f \in C^\infty(M), e, e_1, e_2 \in \Gamma(E)$ and $s \in \Gamma(S)$.

Now let $S$ be a real line bundle which is a module of the Loday algebroid $E$. Assume that there exists a nowhere zero section $s \in \Gamma(S)$. The relation $D_s s = \langle \theta_s, e \rangle s$ defines a section $\theta_s \in \Gamma(E)$. From $\nabla_D f s = 0$, it follows that $\rho(\theta_s) = 0$. And from (11), it follows that $\theta_s$ is a naive 1-cocycle. Finally, (13) implies that, for any nowhere vanishing function $f \in C^\infty(M)$, $\theta f_s = f \theta_s + 2D(\ln |f|)$.

Thus the class $[\theta_s] \in H^1_{\text{naive}}(E)$ is independent of the chosen section $s$ and only depends on the module $S$. We will denote this class by $\theta_S$. As in (11), when the line bundle $S$ is not trivial, we set $\theta_S = \frac{1}{2} \theta_S \otimes s$, where $S \otimes s$ is necessarily a trivial real line bundle. We call $\theta_S$ the modular class of the module $S$.

**Theorem 4.1** Given a Loday algebroid $(E, \rho, \circ, \langle \cdot, \cdot \rangle)$, $\Lambda^{top} E$ is an $E$-module with $\nabla = L$.

**Proof.** It remains to prove that $L_{f e} s = f L_e s$ for any $f \in C^\infty(M)$ and $s \in \Gamma(\Lambda^{top} E)$. According to Proposition 3.1, for any $f \in C^\infty(M)$ and $e, a \in \Gamma(E)$, we have

\[
L_{f e} a = f L_e a - (\rho(a)f)e + 2(e, a)Df = f L_e a - 2\langle Df, a \rangle e + 2(e, a)Df = (f L_e - 2(e \wedge) i_Df + 2(Df \wedge) i_e)(a)
\]
Note that, as differential operators on $\Gamma(\wedge^* E)$, $\mathcal{L}_{f e}$, $f \mathcal{L}_e$, $2(e \wedge) \bar{s}_D f$ and $2(Df \wedge) \bar{s}_e$ all are derivations of degree 0 with respect to the wedge product on $\Gamma(\wedge^* E)$. Since $\mathcal{L}_{f e}$ and $f \mathcal{L}_e - 2(e \wedge) \bar{s}_D f + 2(Df \wedge) \bar{s}_e$ are equal when acting both on sections of $E$ and on functions on $M$, they are also equal when extended to $\Gamma(\wedge^* E)$. In particular, if $s \in \Gamma(\wedge^{top} E)$,

$$\mathcal{L}_{f e}s = f \mathcal{L}_e s - 2(e \wedge) \bar{s}_D f s + 2(Df \wedge) \bar{s}_e s = f \mathcal{L}_e s - 2(e, Df)s + 2(Df, e)s = f \mathcal{L}_e s.$$ 

$\square$

The modular class $[\theta_{\wedge^{top} E}] \in H^1_{\text{naive}}(E)$ of the $E$-module $\wedge^{top} E$ is called the modular class of the Loday algebroid $E$.

## 5 Examples

Let $E = A \oplus A^*$ be the double of a Lie bialgebroid $(A, A^*)$ \[5\]. In this case, $D = \frac{1}{2}(d + d_e)$ and, for all $X, Y \in \Gamma(A)$ and $\xi, \eta \in \Gamma(A^*)$, the bracket on $\Gamma(E)$ is defined by

$$[X, \xi] = (-L_\xi X + \frac{1}{2}d_e(\xi, X)) + (L_X \xi - \frac{1}{2}d(\xi, X)), \quad \{X, Y\} = [X, Y], \quad \{\xi, \eta\} = [\xi, \eta].$$

Now $\wedge^{top} E \cong (\wedge^{top} A) \otimes (\wedge^{top} A^*)$ is a trivial line bundle. For the sake of simplicity, we assume that there exists a nowhere vanishing section $V \in \Gamma(\wedge^{top} A)$. Let $\Omega \in \Gamma(\wedge^{top} A^*)$ be its dual section. For any $X \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$, one has $\mathcal{L}_X \xi = [X, \xi] + D\langle X, \xi \rangle = -L_\xi X + d_e(\xi, X) + L_X \xi = -i_\xi d_e X + L_X \xi$. It follows from the Leibniz rule (see Proposition 5.1) that, for any $\sigma \in \Gamma(\wedge^k A^*)$, $\mathcal{L}_X \sigma = L_X \sigma + \lambda$ where $\lambda \in \Gamma(A \otimes (\wedge^{k-1} A^*))$. On the other hand, since $A$ is isotropic with respect to $\langle \cdot, \cdot \rangle$, we have that $\mathcal{L}_X \tau = L_X \tau$ if $\tau \in \Gamma(\wedge^k A)$. It thus follows that

$$\mathcal{L}_X (V \wedge \Omega) = (\mathcal{L}_X V) \wedge \Omega + V \wedge (\mathcal{L}_X \Omega) = (L_X V) \wedge \Omega + V \wedge (L_X \Omega) = 0.$$ 

Similarly we have $\mathcal{L}_\xi (V \wedge \Omega) = 0$, for all $\xi \in \Gamma(A^*)$. Thus we have proved

**Theorem 5.1** If a Courant algebroid is the double of a Lie bialgebroid, then its modular class vanishes.

**Example 5.2** If $E = TM \oplus T^* M$ is an exact Courant algebroid, the Courant bracket is given by

$$[X + \xi, Y + \eta] = [X, Y] + i_X \wedge Y \phi + L_X \eta - L_Y \xi + \frac{1}{2}d((\xi, Y) - (\eta, X)),$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$. Here $\phi$ is a closed 3-form.

Take a nowhere zero $V \in \Gamma(\wedge^{top} TM)$ and its dual $\Omega \in \Omega^{top}(M)$. One easily sees that $\mathcal{L}_X V = L_X V + V'$, where $V' \in \Gamma(\wedge^{top-1} TM)$ and $\mathcal{L}_X \Omega = L_X \Omega$. Thus it follows that $\mathcal{L}_X (V \wedge \Omega) = L_X (V \wedge \Omega) = 0, \forall X \in \Gamma(TM)$. One also sees that $\mathcal{L}_\xi (V \wedge \Omega) = 0, \forall \xi \in \Gamma(T^* M)$. Therefore the modular class vanishes.
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