Co-Tripotent Elements

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ABSTRACT:

In this paper, the concept of a co-tripotent element is introduced. The main purpose is to discuss co-tripotent elements of the ring $\mathbb{Z}_m$, where $m \in \{2^n, 2p, p^n q\}$, ($p$ and $q$ are two distinct odd primes and $n > 2$). Also co-tripotent elements have been studied of the group ring $\mathbb{Z}_2 G$, where $G$ is cyclic group of order $2^n$, $n \geq 1$.

KEY WORDS: Idempotent element, Tripotent element, Co-tripotent element.

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INTRODUCTION:

An element $x$ of a ring $R$ is called idempotent if $x^2 = x$. Idempotent elements have important role in decomposition of rings. In 2010, the concept of m-idempotent element ($m > 1$) was introduced by (Chaoling and Youghua, 2010). An element $x$ of a commutative ring $R$ is called $m$-idempotent if $x^m = x$, and $x$ is said to be a nontrivial $m$-idempotent if $m$ is the least positive integer such that $x^m = x$. Tripotent elements ($3$- idempotent) were studied by P. Hummadi and A. Mum in (Hummadi and Mum, 2010), (Hummadi and Mum, 2013).

In this paper we introduce and study the concept of co-tripotent element. A nonzero element $x$ of a ring $R$ called a co-tripotent element if $xa = x$ for some nontrivial tripotent element $a$ of $R$. The tripotent element $a$ is said to be an associated tripotent of $x$. Recall Bezout’s Lemma which state that given two integers $a$ and $b$, both are not zero, there exist integers $x$ and $y$ such that $gcd(a, b) = ax + by$ [See (David, 2006), Theorem 2.3]. We use this lemma to find the co-tripotent elements in $\mathbb{Z}_m$, where $m \in \{2^n, 2p, p^n q\}$, ($p$ and $q$ are two distinct odd primes and $n > 2$). Moreover we study co-tripotent elements in the group rings, remembering that a group ring is a free module over the ring and the same time it is a ring constructed in a natural way from any given ring and any given group (David SD, 2004).


2. Preliminaries.

For proving the main results, we need some results. We start as follows:

**Proposition 2.1.** [Hummadi and Mum, 2010, Proposition 1.3]

In $Z_n$, $n > 2, n - 1$ is a nontrivial tripotent element.

**Proposition 2.2.** [Hummadi and Mum, 2010, Proposition 1.7]

$Z_{2n}, n > 2$ has exactly three nontrivial tripotent elements, they are $(2^{n-1} - 1), (2^{n-1} + 1)$ and $(2^n - 1)$.

**Proposition 2.3.** [Hummadi and Mum, 2010, Proposition 1.8]

$Z_p^n, n \geq 1, p$ is a prime has only one nontrivial tripotent element, namely $(p^n - 1)$.

**Proposition 2.4.** [Hummadi and Mum, 2010, Proposition 1.10]

$Z_{2p}, p$ is an odd prime, has exactly two nontrivial tripotent elements, they are $(p - 1)$ and $(2p - 1)$.

**Proposition 2.5.** [Hummadi and Mum, 2010, Theorem 1.11]

$Z_{p^n q}$, where $p, q$ are two distinct odd prime numbers, has exactly five nontrivial tripotent elements, they are $tq - 1, 2tq - 1, p^tq - tq, 1 - 2tq$ and $p^tq - 1$. (where $tq - sp^n = 1$, for some integers $t$ and $s$ by Bezout’s Lemma since $gcd(q, p^n) = 1$).

2. Co-tripotent Elements

In this section, we study co-tripotent elements of the ring $Z_m$, where $m \in \{2^n, 2p, p^n q\}$, ($p$ and $q$ are two distinct odd primes and $n > 2$) and of some type of group rings.

We start by the following result.

**Lemma 3.1.** If $x$ is a co-tripotent element of a commutative ring $R$ and $y$ is a nonzero element of $R$, then $xy$ is a co-tripotent element of $R$.

**Proof.** Suppose $x$ is a co-tripotent element of $R$. Then there exists a nontrivial tripotent element $\alpha \in R$, such that $x\alpha = x$. Therefore for each $0 \neq y \in R$, $(xy)\alpha = (x\alpha)y = xy$. Hence $xy$ is a co-tripotent element.

Now we study co-tripotent elements of the ring $Z_n$.

**Proposition 3.2.**

$Z_p^n, n \geq 1, p$ is an odd prime number, has no co-tripotent element.

**Proof.** Suppose $x$ is a co-tripotent element in $Z_p^n, n \geq 1$. Then $x\alpha = x (mod p^n)$, for some nontrivial tripotent $\alpha$. By Proposition 2.3, the only nontrivial tripotent element in $Z_p^n, n \geq 1$, is $p^n - 1$. Hence $x(p^n - 1) \equiv x(mod p^n)$.

**Proposition 3.3.**

$Z_{2p}, p$ is an odd prime number, has only one co-tripotent element, namely $p^n - 1$.

**Proof.** By Proposition 2.4, $Z_{2p}, p$ is an odd prime number has exactly two nontrivial tripotent elements $p - 1$ and $2p - 1$. If $x = p$, take $\alpha = 2p - 1$.

Now,

$p(2p - 1) \equiv 2p^2 - p(mod 2p)$

$\equiv 2p - p(mod 2p)$

$\equiv p(mod 2p)$.

Therefore $p$ is a co-tripotent element.

Suppose that $x$ is any co-tripotent element different from $p$, then $x\alpha = x (mod 2p)$, for some nontrivial tripotent element $\alpha$. Then either $x(p - 1) \equiv x (mod 2p)$ or $x(2p - 1) \equiv x (mod 2p)$.

If $x(p - 1) \equiv x (mod 2p)$, then $xp - 2x \equiv 0(mod 2p)$, which means $2p|x(p - 2)$. Since $p \nmid (p - 2)$, hence $p|x$. Contradiction.

If $x(2p - 1) \equiv x (mod 2p)$, then $2xp - x \equiv x(mod 2p)$, which means that...
2x ≡ 0 (mod 2p), that is p|x. Contradiction. ■

**Proposition 3.4.**

x is a co-tripotent element in $Z_{2n}$, n > 2 if and only if $x = 2k$, for $k \in Z^+$.

**Proof.** (⇒)

By Proposition 2.2, $Z_{2n}$, n > 2 has exactly 3 nontrivial tripotent elements, they are $2^{n-1} - 1$, $2^{n-1} + 1$, and $2^n - 1$

Suppose $x = 2k$ for $k \in Z^+$. Take $\alpha = 2^{n-1} + 1$.

Now, $2k(2^{n-1} + 1) \equiv 2^n k + 2k (mod 2^n) \equiv 2k (mod 2^n)$.

Therefore $x = 2k$, is a co-tripotent element.

(⇒) Now for other direction we use contrapositive and suppose $x = 2k + 1$, for $k \in Z^+$ is a co-tripotent element.

Then $x\alpha = x (mod 2p)$, for some nontrivial tripotent element $\alpha$.

If $\alpha = 2^{n-1} - 1$, we get

$(2k + 1)(2^{n-1} - 1) \equiv 2k + 1 (mod 2^n)$

$2^n k + 2k + 2^{n-1} - 1 \equiv 2k + 1 (mod 2^n)$

$2^n - 1 - (4k + 2) \equiv 0 (mod 2^n)$

Thus $2^n|(2^{n-1} - (4k + 2))$, which is impossible

If $\alpha = 2^{n-1} + 1$, then

$(2k + 1)(2^{n-1} + 1) \equiv 2k + 1 (mod 2^n)$

$2^n k + 2k + 2^{n-1} + 1 \equiv 2k + 1 (mod 2^n)$

$2^n - 1 \equiv 0 (mod 2^n)$.

Hence $2^n|2^{n-1}$, which is not true.

Finally if $\alpha = 2^n - 1$, then

$(2k + 1)(2^n - 1) \equiv 2k + 1 (mod 2^n)$

$2^{n+1}k - 2k + 2^n - 1 \equiv 2k + 1 (mod 2^n)$

$4k + 2 \equiv 0 (mod 2^n)$.

Hence $2^{n-1}|(2k + 1)$. Contradiction for $2k + 1$ is an odd number. This completes the proof. ■

**Proposition 3.5.**

Let p and q be two distinct odd prime numbers.

Then x is a co-tripotent element of $Z_{p^n q}$ (n ≥ 1) if and only if $x = kp^n$ or $x = kq$, for $k \in Z^+$, in which $kp^n, kq \leq p^n q$.

**Proof.**

By Proposition 2.5, $Z_{p^n q}$ has exactly 5 nontrivial tripotent elements, they are $tq - 1, 2tq - 1, p^n q - tq, 1 - 2tq$ and $p^n q - 1$ (where $tq - sp^n = 1$, for some integers $t$ and $s$)

Suppose $x = kp^n$, for $k \in Z^+$. Take $\alpha = 1 - 2tq$. Then we get

$kp^n(1 - 2tq) \equiv kp^n - 2tkp^n (mod p^n q) \equiv kp^n (mod p^n q)$

Therefore $x = kp^n$ is a co-tripotent element.

Now suppose $x = kq$, for $k \in Z^+$ and take $\alpha = 2tq - 1$. Since $tq - sp^n = 1, that is 2tq - 1 = 1 + 2sp^n$.

$kq(2tq - 1) \equiv kq(1 + 2sp^n) (mod p^n q) \equiv kq (mod p^n q)$.

Therefore $x = kq$ is a co-tripotent element.

Conversely, suppose that $x$ is a co-tripotent element with $q \nmid x$ and $p^n \nmid x$. Thus $x = kp^n + l$, for some $l, 0 < l < p^n$ and $q \nmid x$.

Hence, $x\alpha \equiv x (mod p^n q)$, for some nontrivial tripotent element $\alpha$.

If $\alpha = tq - 1$, then

$(kp^n + l)(tq - 1) \equiv kp^n + l (mod p^n q)$

$(kp^n + l)(tq - 2) \equiv 0 (mod p^n q)$

Thus $p^n q|(kp^n + l)(tq - 2)$. Since $q \nmid (tq - 2)$, hence $q|(kp^n + l)$, that is $q|x$. Contradiction.

If $\alpha = 2tq - 1$, then

$(kp^n + l)(2tq - 1) \equiv kp^n + l (mod p^n q)$

$(kp^n + l)(2tq - 2) \equiv 0 (mod p^n q)$

As before $q \nmid (2tq - 2)$, hence $q|(kp^n + l)$.

Contradiction.

If $\alpha = 1 - 2tq$, then

$(kp^n + l)(1 - 2tq) \equiv kp^n + l (mod p^n q)$

$(kp^n + l)(-2tq) \equiv 0 (mod p^n q)$

This means that $p^n q|(kp^n + l)(-2tq)$ implies $p^n q|(-2t)$. Contradiction for p is an odd prime number.

If $\alpha = p^n q - tq$, then

$(kp^n + l)(p^n q - tq) \equiv kp^n + l (mod p^n q)$

$(kp^n - l)(p^n q - tq - 1) \equiv 0 (mod p^n q)$

Clearly $q|(p^n q - tq - 1)$, so $q|(kp^n - l)$.

Contradiction.

Similar arguments can be used for the case $\alpha = p^n q - 1$.

Now, suppose that $x$ is a co-tripotent element with $q \nmid x$ and $p^n \nmid x$. Thus $x = kq + l$, for some $l, 0 < l < q$ and $p^n \nmid x$. Hence, $x\alpha \equiv x (mod p^n q)$, for some nontrivial tripotent element $\alpha$.

If $\alpha = tq - 1$, then

$(kq + l)(tq - 1) \equiv kq + l (mod p^n q)$

$(kq + l)(tq - 2) \equiv 0 (mod p^n q)$

Thus $p^n q|(kq + l)(tq - 2)$. Since $p^n \nmid (kq + l)$, hence $p^n q|(tq - 2)$. But $tq = 1 + sp^n$, that is $tq - 2 = sp^n - 1$, so $p^n q|(sp^n - 1)$.

Contradiction.

If $\alpha = 1 - 2tq$, then

$(kq + l)(1 - 2tq) \equiv kq + l (mod p^n q)$

$(kq + l)(-2tq) \equiv 0 (mod p^n q)$

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Thus \( p^n \mid (kq + l)(2t) \). Since \( p^n \not\mid (kq + l) \), hence \( p^n \nmid (2t) \). Contradiction for an odd prime \( p \).

If \( \alpha = p^n q - t q \), then
\[
(kq + l)(p^n q - t q) \equiv kq + l \pmod{p^n q} \\
(kq + l)(p^n q - t q - 1) \equiv 0 \pmod{p^n q}
\]
Clearly \( p^n \nmid (p^n q - t q - 1) \), so \( p^n \nmid (kq + l) \). Contradiction.

Similar arguments can be used for the case \( \alpha = p^n q - 1 \).

Thus \( x \) is not a co-tripotent element. ■

In order to prove a characterization co-tripotent elements of group rings, we need the following lemma.

**Lemma 3.6.** Let \( Z_2 G \) be the group ring of \( G \) over \( Z_2 \), where \( G \) is a cyclic group of order \( 2^n, n \geq 2 \), generated by \( g \). Then for any integers \( 0 < t_1 < t_2 < ... < t_k < 2^{n-1}, k \geq 1 \), the following two elements are nontrivial tripotent
\[
g^{t_1} + g^{t_2} + ... + g^{t_k} + g^{2(n-1)} + g^{t_1+2(n-1)} + \\
g^{t_2+2(n-1)} + ... + g^{t_k+2(n-1)}, \text{ and}
\]
\[
1 + g^{t_1} + g^{t_2} + ... + g^{t_k} + g^{t_1+2(n-1)} + \\
g^{t_2+2(n-1)} + ... + g^{t_k+2(n-1)}
\]
Moreover the number of nontrivial tripotent elements is \( 2^{2(n-1)} - 1 \).

**Proof.** Let \( t_1, t_2, ..., t_k \), be any \( k \) distinct integers with \( 0 < t_1 < t_2 < ... < t_k < 2^{n-1} \).

If
\[
F_k = g^{t_1} + g^{t_2} + ... + g^{t_k} + g^{2(n-1)} + g^{t_1+2(n-1)} + \\
g^{t_2+2(n-1)} + ... + g^{t_k+2(n-1)},
\]
then
\[
F_k^2 = g^{2t_1} + g^{2t_2} + ... + g^{2t_k} + g^{2(n-1)} + g^{2t_1+2n} + \\
g^{2t_2+2n} + ... + g^{2t_k+2n} = 1 \neq F_k.
\]
Hence \( F_k^2 \neq F_k \), and therefore \( F_k \) is a nontrivial tripotent element.

Now, if
\[
H_k = 1 + g^{t_1} + g^{t_2} + ... + g^{t_k} + g^{t_1+2(n-1)} + \\
g^{t_2+2(n-1)} + ... + g^{t_k+2(n-1)}
\]
then
\[
H_k^2 = 1 + g^{2t_1} + g^{2t_2} + ... + g^{2t_k} + g^{2t_1+2n} + \\
g^{2t_2+2n} + ... + g^{2t_k+2n}.
\]
Hence \( H_k \neq H_k \), and therefore \( H_k \) is a nontrivial tripotent element.

Using some probability theory, we get that the number of such nontrivial tripotent elements is \( 2^{2(n-1)} - 1 \).

Example 3.7. Consider the group ring \( Z_2 G \), where \( G \) is cyclic group of order \( 2^3 = 8 \) generated by \( g \). Then by Proposition 3.6, \( Z_2 G \) has 15 nontrivial tripotent elements, they are
\[
F_1 = g^4 \\
F_2 = g + g^4 + g^5 \\
F_3 = g^2 + g^4 + g^6 \\
F_4 = g^3 + g^7 \\
F_5 = g + g^2 + g^4 + g^5 + g^6 \\
F_6 = g + g^3 + g^4 + g^5 + g^7 \\
F_7 = g^2 + g^3 + g^4 + g^5 + g^7 \\
F_8 = g + g^2 + g^3 + g^4 + g^5 + g^6 + g^7 \\
F_9 = 1 + g + g^5 \\
F_{10} = 1 + g^2 + g^6 \\
F_{11} = 1 + g^3 + g^7 \\
F_{12} = 1 + g + g^2 + g^5 + g^6 \\
F_{13} = 1 + g + g^3 + g^5 + g^7 \\
F_{14} = 1 + g^2 + g^3 + g^6 + g^7 \\
F_{15} = g + g^2 + g^3 + g^5 + g^6 + g^7
\]

Now, we have the following characterization:

**Proposition 3.8.** In the group ring \( Z_2 G \), where \( G \) is cyclic group generated by \( g \) of an odd order \( m \), no co-tripotent elements.

**Proof.** By Proposition 2.6, since \( Z_2 G \) has no nontrivial tripotent elements, hence we have no co-tripotent elements.

**Theorem 3.9.** Let \( Z_2 G \) be the group ring of \( G \) over \( Z_2 \), where \( G \) is cyclic group of order \( 2^n, n \geq 2 \), generated by \( g \). If \( x \) has an even number of summands, then \( x \) is a co-tripotent element.

Moreover the number of co-tripotent elements is \( 2^m - 1 \), where \( m = (2^n - 1) \).

**Proof.** Suppose that \( x \) has an even number of summands. Hence
\[
x = g^{t_1} + g^{t_2} + ... + g^{t_k}, \text{ where } 0 \leq t_1 < t_2 < ... < t_k < m, \text{ and } k \text{ is even.}
\]
Take \( \alpha = g + g^2 + g^3 + ... + g^m \) which a nontrivial tripotent element is given by Lemma 3.6. Then
\[
x \alpha = (g^{t_1} + g^{t_2} + ... + g^{t_k})(g + g^2 + g^3 + ... + g^m)
\]
We describe the multiplication \( x \alpha \) by the following array say \( A \).
Let \( A = [a_{ij}]_{k \times m} \), where \( a_{ij} \) is the summand of \( x \alpha \) which is the product of the \( i \)th summand of \( x \) with \( j \)th summand of \( \alpha \). This means \( x \alpha = \sum_{i=1}^{k} \sum_{j=1}^{m} a_{ij} \).

Considering the first and the second rows of this array, we see that \( g^{t_2} \) occurs in \( r \)th column \((r = t_2 - t_1)\) and \( g^{t_1} \) occurs in the \( s \)th column \((s = (2^n - r))\). By adding the terms of these two rows, we get \( g^{t_1} + g^{t_2} \) (observing that the coefficient of each \( g^j \) is in \( Z_2 \)). By adding the terms of the third and the fourth rows, we get
\[
g^{t_3} + g^{t_4}.
\]
Continuing in this manner, by adding the terms of \( (k - 1)t \)th and \( (k)t \)th rows, we have
\[
g^{t_{k-1}} + g^{t_k}.
\]
By adding all terms of this array, we get \( g^{t_1} + g^{t_2} + \cdots + g^{t_k} \). Thus we obtain \( x \alpha = x \).

That is \( x \) is a co-tripotent element. Using some probability theory, we get that the number of such co-tripotent elements is \( 2^n - 1 \).

We complete this theorem with an example to illustrate.

**Example 3.10.** Consider the group ring \( Z_2G \), where \( G \) is cyclic group of order \( 2^2 \), generated by \( g \). By Theorem 3.9, the co-tripotent elements are
\[
1 + g, \quad 1 + g^2, \quad 1 + g^3, \quad g + g^2, \quad g + g^3, \\
g^2 + g^3, \quad 1 + g + g^2 + g^3.
\]

We expect that the converse of Theorem 3.9 is true, that is every co-tripotent element has an even number of summands, equivalently if \( \alpha \) is a nontrivial tripotent element, then \( \alpha \) cannot be an associated element of an element with an odd number of summands.

To prove the above for special cases, we need to state the following lemma.

**Lemma 3.11.** In the group ring \( Z_2G \), where \( G \) is a cyclic group of order \( 2^n, n > 1 \), if \( x = \sum_{i=0}^{2^n-1} a_i g^i \) and \( \sum_{i=0}^{2^n-1} a_i = 0 \), then \( x \) has an even number of summands.

**Proof.** Clearly, since \( a_i \in Z_2 \).

**Proposition 3.12.** In the group ring \( Z_2G \), where \( G \) is a cyclic group of order \( 2^n, n > 1 \), the nontrivial tripotent element \( g^{2^{n-1}} \) is not an associated element of any element with an odd number of summands.

**Proof.** Suppose \( \alpha = g^{2^{n-1}} \) is an associated element of \( x = \sum_{i=0}^{2^n-1} a_i g^i \). Then \( ax = x \), that is
\[
g^{2^n-1}(a_0 + a_1 g + \cdots + a_{2^n-1} g^{2^{n-1}}) = a_0 + a_1 g + \cdots + a_{2^n-1} g^{2^n-1}
\]
Equating the coefficients of \( g^j \) on both sides, we get
\[
\begin{align*}
a_0 &= a_{2^n-1} \quad \Rightarrow \quad a_0 + a_{2^n-1} = 0 \quad \cdots \quad (1) \\
a_1 &= a_{2^n-1+1} \quad \Rightarrow \quad a_1 + a_{2^n-1+1} = 0 \quad \cdots \quad (2) \\
a_2 &= a_{2^n-1+2} \quad \Rightarrow \quad a_2 + a_{2^n-1+2} = 0 \quad \cdots \quad (3) \\
a_{2^n-1} &= a_{2^n-1} \quad \Rightarrow \quad a_{2^n-1} + a_{2^n-1} = 0 \quad \cdots \quad (2^{n-1})
\end{align*}
\]
Which implies \( \sum_{i=0}^{2^n-1} (a_i + a_{2^n-1+i}) = 0 \), that is \( \sum_{i=0}^{2^n-1} a_i = 0 \). By Lemma 3.11, \( x \) has an even number of summands.

**Proposition 3.13.** In the group ring \( Z_2G \), where \( G \) is cyclic group of order \( 2^n, n > 1 \), for each \( 0 < k < n - 1 \), the nontrivial tripotent element \( g^k g^{2^{n-1}} + g^{2^n+1+2k} \), is not an associated element of any element \( x = \sum_{i=0}^{2^n-1} a_i g^i \).

**Proof.** Suppose \( \alpha = g^k g^{2^{n-1}} + g^{2^n+1+2k} \) is an associated element of \( x = \sum_{i=0}^{2^n-1} a_i g^i \), that is
\[
(g^k g^{2^{n-1}} + g^{2^n+1+2k}) \left( \sum_{i=0}^{2^n-1} a_i g^i \right) = \sum_{i=0}^{2^n-1} a_i g^i
\]
Equating the coefficients of \( g^j \) in both sides, we get the following equations
\[
\begin{align*}
a_0 &= a_{2^n-k} + a_{2^n-1} + a_{2^n-1-k} \quad \Rightarrow \quad a_0 + a_{2^n-k} + a_{2^n-1} + a_{2^n-1-k} = 0 \quad \cdots \quad (0) \\
a_1 &= a_{2^n-k+1} + a_{2^n-1+1} + a_{2^n-1-k+1} \quad \Rightarrow \quad a_1 + a_{2^n-k+1} + a_{2^n-1+1} + a_{2^n-1-k+1} = 0 \quad \cdots \quad (1) \\
&\quad \vdots \\
a_{2^{k-1}} &= a_{2^n-1} + a_{2^n-1+2^k} \quad \Rightarrow \quad a_{2^{k-1}} + a_{2^n-1} + a_{2^n-1+2^k} + a_{2^n-1-2^k} = 0 \quad \cdots \quad (2^{n-1})
\end{align*}
\]
Equating the coefficients of \( g^j \) in both sides, we get the following equations
\[
\begin{align*}
a_0 &= a_{2^n-2k} + a_{2^n-1} + a_{2^n-1-2k} \quad \Rightarrow \quad a_0 + a_{2^n-2k} + a_{2^n-1} + a_{2^n-1-2k} = 0 \quad \cdots \quad (0) \\
a_1 &= a_{2^n-2k+1} + a_{2^n-1+1} + a_{2^n-1-2k+1} \quad \Rightarrow \quad a_1 + a_{2^n-2k+1} + a_{2^n-1+1} + a_{2^n-1-2k+1} = 0 \quad \cdots \quad (1) \\
&\quad \vdots \\
a_{2^{k-1}} &= a_{2^n-1} + a_{2^n-1+2^{k-1}} \quad \Rightarrow \quad a_{2^{k-1}} + a_{2^n-1} + a_{2^n-1+2^{k-1}} + a_{2^n-1-2^{k-1}} = 0 \quad \cdots \quad (2^{n-1})
\end{align*}
\]
Adding both sides of the above equations, we get
\[
\sum_{i=0}^{n-1} a_i = 0.
\]
By Lemma 3.11, we get that \( x \) has an even number of summands. ■

We illustrate the above Proposition by the following example.

**Example 3.14.**
Consider the group ring \( \mathbb{Z}_2G \), where \( G \) is cyclic group of order \( 2^5 \) generated by \( g \). Let
\[
\alpha = g^4 + g^{16} + g^{20},
\]
which is a nontrivial tripotent element of \( \mathbb{Z}_2G \). By Proposition 3.13 suppose \( \alpha \) is an associated element of any element \( x = \sum_{i=0}^{n-1} a_i g^i \), that is \( \alpha \sum_{i=0}^{n-1} a_i g^i = \sum_{i=0}^{n-1} a_i g^i \).

Equating the coefficient of \( g^i \), for each \( i \) in both sides, we obtain the following

\[
\begin{align*}
\alpha_0 + a_{28} + a_{16} + a_{12} &= 0 \quad \cdots (0) \\
\alpha_1 + a_{29} + a_{17} + a_{13} &= 0 \quad \cdots (1) \\
\alpha_2 + a_{30} + a_{18} + a_{14} &= 0 \quad \cdots (2) \\
\alpha_3 + a_{31} + a_{19} + a_{15} &= 0 \quad \cdots (3) \\
\alpha_4 + a_{24} + a_{20} &= 0 \quad \cdots (4) \\
\alpha_5 + a_{25} + a_{21} &= 0 \quad \cdots (5) \\
\alpha_6 + a_{26} + a_{22} &= 0 \quad \cdots (6) \\
\alpha_7 + a_{27} + a_{23} &= 0 \quad \cdots (7)
\end{align*}
\]

Adding both sides of the above equations, we get
\[
\sum_{i=0}^{n-1} a_i = 0.
\]
By Lemma 3.11, we get that \( x \) has an even number of summands.

**References**

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