On the distribution of $\alpha p^2$ modulo one
with prime $p$ of a special form

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Abstract: Let $\mathcal{P}_r$ denote an almost–prime with at most $r$ prime factors, counted according to multiplicity. In this paper, it is proved that for $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $\beta \in \mathbb{R}$ and $0 < \theta < 10/1561$, there exist infinitely many primes $p$, such that $\|\alpha p^2 + \beta\| < p^{-\theta}$ and $p + 2 = \mathcal{P}_4$, which constitutes an improvement upon the previous result.

Keywords: Distribution modulo one; linear sieve; exponential sum; almost–prime

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1 Introduction and main result

Let $\mathcal{P}_r$ denote an almost–prime with at most $r$ prime factors, counted according to multiplicity. The famous prime twins conjecture states that there exist infinitely many primes $p$ such that $p + 2$ is a prime, too. Up to now, this conjecture is still open, but many approximations about this conjecture were established. One of the most interesting results is due to Chen [3], who showed, in 1973, that there exist infinitely many primes $p$ such that $p + 2 = \mathcal{P}_2$.

In 1981, Heath–Brown [11] showed that there exist infinitely many arithmetic progressions of four different terms, three of which are primes, and the fourth is $\mathcal{P}_2$. In 2006, Green and Tao [5] established that there exist infinitely many arithmetic progressions consisting of three different primes $p_1 < p_2 < p_3$ such that $p_j + 2 = \mathcal{P}_2$ for each $j = 1, 2, 3$. Later, in 2008, Green and Tao [6] showed that for any $k \geq 3$ there exist infinitely many arithmetic progressions consisting of $k$ different primes $p_1 < p_2 < \cdots < p_k$.
such that \( p_j + 2 = \mathcal{P}_2 \) for each \( j = 1, 2, \ldots, k \).

Suppose that there is a problem including primes and let \( r \geq 2 \) be an integer. Having in mind Chen’s result, one may consider the problem with primes \( p \), such that \( p + 2 = \mathcal{P}_r \). Many authors investigated several kinds of problems of this type, such as Peneva and Tolev [20], Peneva [21], Tolev [25, 26, 27], etc.

Let \( \alpha \) be a irrational real number and \( \|x\| \) denote the distance from \( x \) to the nearest integer. Earlier work about the distribution of the fractional parts of the sequence \( \{\alpha p\} \) was first considered by Vinogradov [30], who showed that for any real number \( \beta \), there are infinitely many primes \( p \) such that for \( \theta = 1/5 - \varepsilon \), then

\[
\|\alpha p + \beta\| < p^{-\theta},
\]

where and below \( \varepsilon \) denotes arbitrarily small positive number. After that, the first improvement on (1.1) was due to Vaughan [28], who obtained \( \theta = 1/4 \) in (1.1), and who also required an additional factor \((\log p)^8\) on the right hand side of (1.1). Since then, many authors improved the upper bound of the exponent \( \theta \), such as Harman [8, 10], Jia [14, 15], Heath–Brown and Jia [13], etc. So far the best result is given by Matomäki [17] with \( \theta = 1/3 - \varepsilon \). Moreover, it seems very natural to consider the sequence \( \{\alpha p^k\} \) for \( k \geq 2 \), where \( p \) denotes a prime variable. Also, many authors studied the fractional parts of the sequence \( \{\alpha p^k\} \) for \( k \geq 2 \), such as Baker and Harman [1], Harman [9], Wong [31], etc.

In 2010, Todorova and Tolev [24] considered the distribution of \( \alpha p \mod 1 \) with primes of the form specified above, and showed that for \( \theta = 1/100 \), there are infinitely many solutions in primes \( p \) to (1.1) such that \( p + 2 = \mathcal{P}_4 \). Later, Matomäki [18] showed that this result actually holds with \( p + 2 = \mathcal{P}_2 \) and \( \theta = 1/1000 \). After that, Shi [22] continue to improve the result of Matomäki [18], and showed that there are infinitely many solutions in primes \( p \) to (1.1) such that \( p + 2 = \mathcal{P}_2 \) and \( \theta = 3/200 \).

Moreover, for the case \( k = 2 \), Shi and Wu [23] established the result that there exist infinitely many primes \( p \), which satisfy \( \|\alpha p^2 + \beta\| < p^{-\theta} \), such that \( p + 2 = \mathcal{P}_4 \) and \( \theta = 2/375 - \varepsilon \).

In this paper, we shall continue to improve the result of Shi and Wu [23] and establish the following theorem.

**Theorem 1.1** Suppose that \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), \( \beta \in \mathbb{R} \) and \( 0 < \theta < 10/1561 \). Then there exist infinitely many primes \( p \), which satisfy \( p + 2 = \mathcal{P}_4 \), such that

\[
\|\alpha p^2 + \beta\| < p^{-\theta}.
\]
Remark. According to the work of Shi and Wu [23], our improvement comes from using the methods developed by Tolev [27] with more delicate iterative techniques and various bounds for exponential sums, combining with a version of the Lemma 2.2 of Bombieri and Iwaniec [2], while the previous method in dealing exponential sum, e.g. [23], is based on the traditional pattern of exponential sum estimates.

2 Notation.

Let $X$ be a sufficiently large real number. Set

$$\delta = 0.307708, \quad \rho = 0.23077, \quad \eta = 0.076928, \quad \kappa = 1.4999676, \quad 0 < \theta < \frac{10}{1561}. \quad (2.1)$$

Also, we put

$$z = X^\eta, \quad y = X^\rho, \quad D = X^\delta, \quad \Delta = \Delta(X) = X^{-\theta}, \quad H = \Delta^{-1}\log^2 X. \quad (2.2)$$

Throughout this paper, we always denote primes by $p$ and $q$. $\varepsilon$ always denotes an arbitrary small positive constant, which may not be the same at different occurrences. As usual, we use $\Omega(n), \varphi(n), \mu(n), \Lambda(n)$ to denote the number of prime factors of $n$ counted according to multiplicity, Euler’s function, Möbius’ function, and Mangold’s function, respectively. We denote by $\tau_k(n)$ the number of solutions of the equation $m_1m_2\ldots m_k = n$ in natural variables $m_1, \ldots, m_k$. Especially, we write $\tau_2(n) = \tau(n)$.

Let $(m_1, m_2, \ldots, m_k)$ and $[m_1, m_2, \ldots, m_k]$ be the greatest common divisor and the least common multiple of $m_1, m_2, \ldots, m_k$, respectively. Also, we use $[x]$ and $\|x\|$, respectively, to denote the integer part of $x$ and the distance from $x$ to the nearest integer. $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$; $e(x) = e^{2\pi ix}$; $\mathcal{L} = \log X$. $\mathcal{P}_r$ always denotes an almost–prime with at most $r$ prime factors, counted according to multiplicity.

3 Preliminary Lemmas

Lemma 3.1 Let $M \leq N < N_1 \leq M_1$ and $a_n$ be any complex numbers. Then we have

$$\left| \sum_{N<n\leq N_1} a_n \right| \leq \int_{-\infty}^{+\infty} K(\theta) \left| \sum_{M<n\leq M_1} a_ne^{\theta n} \right| d\theta,$$

where

$$K(\theta) = \min \left( M_1 - M + 1, \frac{1}{\pi |\theta|}, \frac{1}{\pi^2 \theta^2} \right),$$

which satisfies

$$\int_{-\infty}^{+\infty} K(\theta) d\theta \leq 3 \log(2 + M_1 - M). \quad (3.1)$$
Proof. See Lemma 2.2 of Bombieri and Iwaniec [2].

**Lemma 3.2** Let \(3 \leq u < v < w < X\) and suppose that \(w - \frac{1}{2} \in \mathbb{N}\), and that \(w \geq 4u^2, X \geq 64u^2v, v^3 \geq 32X\). Assume further that \(f(n)\) is a complex-valued function. Then the sum

\[
\sum_{\frac{w}{2} < n \leq X} \Lambda(n)f(n)
\]

can be decomposed into \(O(\log^{10} X)\) sums, each of which either of Type I:

\[
\sum_{M < m \leq M_1} a_m \sum_{L < \ell \leq L_1} f(m\ell)
\]

with \(M < M_1 \leq 2M, L < L_1 \leq 2L, \ell \geq w, a_m \ll m^\varepsilon, ML \asymp X\), or of Type II:

\[
\sum_{M < m \leq M_1} a_m \sum_{L < \ell \leq L_1} b_{\ell}f(m\ell)
\]

with \(M < M_1 \leq 2M, L < L_1 \leq 2L, u \leq L \leq v, a_m \ll m^\varepsilon, b_{\ell} \ll \ell^\varepsilon, ML \asymp X\).

Proof. See Lemma 3 of Heath–Brown [12].

**Lemma 3.3** For \(P \geq 1\), we have

\[
\sum_{1 \leq n \leq P} e(\alpha n) \leq \min\left(P, \frac{1}{2\|\alpha\|}\right).
\]

Proof. See Lemma 4 of Chapter VI of Karatsuba [16].

**Lemma 3.4** Suppose that \(Y_1, Y_2, \alpha\) are real numbers with \(Y_1 \geq 1, Y_2 \geq 1\), and that \(|\alpha - a/q| \leq q^{-2}\) with \((a, q) = 1\). Then we have

\[
\sum_{n \leq Y_1} \min\left(\frac{Y_1Y_2}{n}, \frac{1}{\|\alpha n\|}\right) \ll Y_1Y_2\left(\frac{1}{q} + \frac{1}{Y_2} + \frac{q}{Y_1Y_2}\right) \log(2Y_1q).
\]

Proof. See Lemma 2.2 of Vaughan [29].

**4 Proof of Theorem 1.1**

As is shown in [24], we take a periodic function \(\chi(t)\) with period 1 such that

\[
\begin{cases}
0 < \chi(t) < 1, & \text{if } -\Delta < t < \Delta, \\
\chi(t) = 0, & \text{if } \Delta \leq t \leq 1 - \Delta,
\end{cases}
\]  

(4.1)
which has a Fourier series

\[
\chi(t) = \Delta + \sum_{|k| > 0} g(k)e(kt) \tag{4.2}
\]

with coefficients satisfying

\[
g(0) = \Delta, \\
g(k) \ll \Delta, \quad \text{for all } k, \\
\sum_{|k| > H} |g(k)| \ll X^{-1}. \tag{4.3}
\]

The existence of such a function is a consequence of a well known lemma of Vinogradov. For instance, one can see Chapter I, §2 in [16]. Consider the sum

\[
\Gamma := \Gamma(X) = \sum_{2 < p \leq X} \chi(\alpha p^2 + \beta) W_p \log p, \tag{4.4}
\]

where

\[
P(z) = \prod_{2 < p \leq z} p, \tag{4.5}
\]

and

\[
W_p = 1 - \kappa \sum_{z < q \leq y, q|p+2} \left(1 - \frac{\log q}{\log y}\right). \tag{4.6}
\]

Let \( \Gamma_1 \) denote the sum of the terms of \( \Gamma(X) \) in which \( W_p > 0 \). Then we have

\[
\Gamma(X) \leq \Gamma_1.
\]

If we denote by \( \Gamma_2 \) the sum of the terms of \( \Gamma_1 \) in which \( \mu(p + 2) = 0 \). It is easy to see that

\[
0 \leq \Gamma_2 \ll \sum_{q \geq z} \sum_{n \leq X \mod q^2} \log n \ll (\log X) \sum_{z \leq q \leq \sqrt{X+2}} \left(\frac{X}{q^2} + 1\right)
\]

\[
\ll X^{1+\varepsilon} z^{-1} + X^{\frac{1}{2}+\varepsilon} \ll X^{1-\eta+\varepsilon}.
\]

By noting the fact that the contribution of the terms (if such terms exist) in \( \Gamma_1 \) for which \( X - 2 < p \leq X \) is \( O(\log X) \), we deduce that

\[
\Gamma \leq \Gamma_3 + O(X^{1-\eta+\varepsilon}), \tag{4.7}
\]

where

\[
\Gamma_3 = \sum_{\frac{X}{2} < p \leq X - 2} \chi(\alpha p^2 + \beta) W_p \log p.
\]
On one hand, if we assume that
\[ \Gamma(X) \gg \frac{\Delta X}{\log X}, \]
then from (4.7), we get
\[ \Gamma_3 \gg \frac{\Delta X}{\log X}, \]
and thus \( \Gamma_3 > 0 \). Hence there exists a prime \( p \), which satisfies
\[ \frac{X}{2} < p \leq X - 2, \quad \mathcal{W}_p > 0, \quad \mu^2(p + 2) = 1, \quad (p + 2, P(z)) = 1, \]
and such that
\[ \chi(\alpha p^2 + \beta) > 0. \]
Combining (4.1), (4.8) and (4.9), we can see that this prime \( p \) satisfies
\[ \|\alpha p^2 + \beta\| < p^{-\theta}. \]

On the other hand, by the properties of the weights \( \mathcal{W}_p \) (for example, one can see Chapter 9 of [7]), it is easy to see that if \( p \) satisfies (4.8), then
\[ \Omega(p + 2) = \sum_{\substack{q > z \\ q \nmid p + 2}} 1 < \frac{1}{\kappa} + \sum_{\substack{q > z \\ q \nmid p + 2}} \frac{\log q}{\log y} = \frac{1}{\kappa} + \frac{\log(p + 2)}{\log y} \leq \frac{1}{\kappa} + \frac{1}{p} < 5, \]
which implies \( p + 2 = P_4 \). Therefore, in order to prove Theorem 1.1, it is sufficient to show that there exists a sequence \( \{X_j\}_{j=1}^\infty \), which satisfies
\[ \lim_{j \to \infty} X_j = +\infty, \quad \Gamma(X_j) \gg \frac{\Delta(X_j)X_j}{\log X_j}, \quad j = 1, 2, 3, \ldots. \]

By (4.4) and (4.6), we can write \( \Gamma \) as follows
\[ \Gamma = \Psi - \kappa \Phi, \]
where
\[ \Psi = \sum_{\substack{\frac{X}{2} < p \leq X \\ (p + 2, P(z)) = 1}} \chi(\alpha p^2 + \beta) \log p, \]
and
\[ \Phi = \sum_{\substack{\frac{X}{2} < p \leq X \\ (p + 2, P(z)) = 1}} \chi(\alpha p^2 + \beta)(\log p) \sum_{\substack{z < q \leq y \\ q \nmid p + 2}} \left(1 - \frac{\log q}{\log y}\right). \]

Next, we shall give lower bound estimate of \( \Psi \) and upper bound estimate of \( \Phi \) by using lower bound linear sieve and upper bound linear sieve, respectively. First, we consider
Ψ. Let \( \lambda^-(d) \) be the lower bounds for Rosser’s weights of level \( D \). Hence for any positive integer \( d \), there holds

\[
|\lambda^-(d)| \leq 1, \quad \lambda^-(d) = 0 \quad \text{if} \quad d > D \quad \text{or} \quad \mu(d) = 0,
\]

(4.13)

\[
\sum_{d | n} \lambda^-(d) \leq \sum_{d | n} \mu(d) = \begin{cases} 1, & \text{if} \quad n = 1, \\ 0, & \text{if} \quad n \in \mathbb{N}, \ n > 1. \end{cases}
\]

(4.14)

Also, we shall use the fact if \( 2 < s < 4 \), then there holds

\[
\sum_{d | P(z)} \lambda^-(d) \frac{\varphi(d)}{\varphi(d)} \geq \Pi(z) \left( \frac{2e^7 \log(s-1)}{s} + O\left( (\log X)^{-1/3} \right) \right),
\]

(4.15)

where

\[
\Pi(z) = \prod_{2 < p \leq z} \left( 1 - \frac{1}{p-1} \right).
\]

Now, we take

\[
s = \frac{\log D}{\log z} = \frac{\delta}{\eta} = \frac{76927}{19232} \in (2, 4)
\]

(4.16)

in (4.15). By (4.11) and (4.14), we obtain

\[
\Psi = \sum_{\frac{z}{2} < p \leq X} \chi(\alpha p^2 + \beta)(\log p) \sum_{d | (p+2, P(z))} \mu(d)
\]

\[
\geq \sum_{\frac{z}{2} < p \leq X} \chi(\alpha p^2 + \beta)(\log p) \sum_{d | (p+2, P(z))} \lambda^-(d)
\]

\[
= \sum_{d | P(z)} \lambda^-(d) \sum_{\frac{z}{2} < p \leq X \atop p+2 \equiv 0(\text{mod } d)} \chi(\alpha p^2 + \beta) \log p
\]

\[
= \Psi_1, \quad \text{say.}
\]

From (4.2), we have

\[
\Psi_1 = \sum_{d | P(z)} \lambda^-(d) \sum_{\frac{z}{2} < p \leq X \atop p+2 \equiv 0(\text{mod } d)} \left( \Delta + \sum_{|k|>0} g(k)e(\alpha p^2 k + \beta k) \right) \log p
\]

\[
= \Delta \sum_{d | P(z)} \lambda^-(d) \sum_{\frac{z}{2} < p \leq X \atop p+2 \equiv 0(\text{mod } d)} \log p
\]

\[
+ \sum_{d | P(z)} \lambda^-(d) \sum_{|k|>0} g(k)e(\beta k) \sum_{\frac{z}{2} < p \leq X \atop p+2 \equiv 0(\text{mod } d)} e(\alpha p^2 k) \log p
\]

\[
= \Delta \sum_{d | P(z)} \lambda^-(d) \sum_{\frac{z}{2} < p \leq X \atop p+2 \equiv 0(\text{mod } d)} \log p
\]
+ \Delta \sum_{d \mid P(z)} \lambda^{-}(d) \sum_{0 < |k| \leq H} \left( \Delta^{-1} g(k) e(\beta k) \right) \sum_{\frac{X}{2} < p < X} e(\alpha p^2 k) \log p \\
+ \sum_{d \mid P(z)} \lambda^{-}(d) \sum_{|k| > H} g(k) e(\beta k) \sum_{\frac{X}{2} < p < X} e(\alpha p^2 k) \log p.

By (4.3), and the fact that \( \lambda^{-}(d) = 0 \) for \( d > D \), we obtain

\[
\sum_{d \mid P(z)} \lambda^{-}(d) \sum_{|k| > H} g(k) e(\beta k) \sum_{\frac{X}{2} < p < X} e(\alpha p^2 k) \log p \\
\ll \sum_{d \mid P(z)} |\lambda^{-}(d)| \sum_{|k| > H} |g(k)| \sum_{\frac{X}{2} < p < X} \log p \ll \sum_{d \leq D} \frac{1}{\varphi(d)} \ll \log D \ll \log X.
\]

Therefore, we get

\[
\Psi_1 = \Delta (\Psi_2 + \Psi_3) + O(\log X), \tag{4.18}
\]

where

\[
\Psi_2 = \sum_{d \mid P(z)} \lambda^{-}(d) \sum_{\frac{X}{2} < p < X} \log p,
\]

\[
\Psi_3 = \sum_{d \mid P(z)} \lambda^{-}(d) \sum_{0 < |k| \leq H} c(k) \sum_{\frac{X}{2} < p < X} e(\alpha p^2 k) \log p, \tag{4.19}
\]

\[
c(k) = \Delta^{-1} g(k) e(\beta k) \ll 1.
\]

For \( \Psi_2 \), by Bombieri–Vinogradov’s mean value theorem (See Chapter 28 of [4]) and (4.13), we derive that

\[
\Psi_2 = \frac{X}{2} \sum_{d \mid P(z)} \lambda^{-}(d) \frac{1}{\varphi(d)} + O \left( \frac{X}{\log^2 X} \right). \tag{4.20}
\]

It follows from Mertens’ prime number theorem (See [19]) that

\[
\Pi(z) \asymp \frac{1}{\log z}. \tag{4.21}
\]

Then from (4.15), (4.20) and (4.21), we obtain

\[
\Psi_2 \geq e^7 X \Pi(z) \frac{\log(s - 1)}{s} + O \left( \frac{X}{\log^{4/3} X} \right), \tag{4.22}
\]

where \( s \) is defined by (4.16). For \( \Psi_3 \), we shall investigate it in the next section.
Now, we study the sum \( \Phi \), which is defined by (4.12). We rewrite \( \Phi \) in the following form

\[
\Phi = \sum_{z < q < y} \left( 1 - \frac{\log q}{\log y} \right) \sum_{\frac{x}{q} < p \leq X \atop p+2 \equiv 0 \pmod{q}} \chi(\alpha p^2 + \beta) \log p. \tag{4.23}
\]

In order to give an upper bound estimate of \( \Phi \), we shall apply an upper bound linear sieve. Let \( \lambda_q^+(d) \) be the upper bounds for Rosser’s weights of level \( D/q \). Hence for any positive integer \( d \), we have

\[
|\lambda_q^+(d)| \leq 1, \quad \lambda_q^+(d) = 0 \quad \text{if} \quad d > \frac{D}{q} \quad \text{or} \quad \mu(d) = 0, \tag{4.24}
\]

\[
\sum_{d \mid n} \lambda_q^+(d) \geq \sum_{d \mid n} \mu(d) = \begin{cases} 
1, & \text{if } n = 1, \\
0, & \text{if } n \in \mathbb{N}, n > 1.
\end{cases} \tag{4.25}
\]

Also, we shall use the fact, for \( 1 < s_1 < 3 \), there holds

\[
\sum_{d \mid P(z)} \lambda_q^+(d) \leq \Pi(z) \left( \frac{2e^7}{s_1} + O\left( \left( \log X \right)^{-1/3} \right) \right). \tag{4.26}
\]

For prime \( q \) in the sum \( \Phi \), we take

\[
s_1 = \frac{\log(D/q)}{\log z}. \tag{4.27}
\]

Then it is easy to check that \( 1 < s_1 < 3 \), and thus (4.26) holds. By (4.23)–(4.25), we obtain

\[
\Phi = \sum_{z < q < y} \left( 1 - \frac{\log q}{\log y} \right) \sum_{\frac{x}{q} < p \leq X \atop p+2 \equiv 0 \pmod{q}} \chi(\alpha p^2 + \beta)(\log p) \sum_{d \mid (p+2, P(z))} \mu(d)
\]

\[
\leq \sum_{z < q < y} \left( 1 - \frac{\log q}{\log y} \right) \sum_{\frac{x}{q} < p \leq X \atop p+2 \equiv 0 \pmod{q}} \chi(\alpha p^2 + \beta)(\log p) \sum_{d \mid (p+2, P(z))} \lambda_q^+(d)
\]

\[
= \sum_{z < q < y} \left( 1 - \frac{\log q}{\log y} \right) \sum_{d \mid P(z)} \lambda_q^+(d) \sum_{\frac{x}{q} < p \leq X \atop p+2 \equiv 0 \pmod{q d}} \chi(\alpha p^2 + \beta)(\log p)
\]

\[
= \sum_{m \leq D} \nu(m) \sum_{\frac{x}{q} < p \leq X \atop p+2 \equiv 0 \pmod{m}} \chi(\alpha p^2 + \beta)(\log p) =: \Phi_1, \tag{4.27}
\]

where

\[
\nu(m) = \sum_{z < q < y \atop d \mid P(z) \atop m = dq} \left( 1 - \frac{\log q}{\log y} \right) \lambda_q^+(d). \tag{4.28}
\]
If \( m \leq z \), then \( \nu(m) = 0 \). If \( z < m \leq D \), by (4.5) and (4.28) we know that the representation \( m = dq \) with \( z < q < y \) and \( d | P(z) \) is unique. Thus, it is easy to see that

\[
|\nu(m)| \leq 1. \tag{4.29}
\]

From (4.2), we get

\[
\Phi_1 = \sum_{m \leq D} \nu(m) \sum_{\substack{p \leq X \atop p+2 \equiv 0 \pmod{m}}} \left( \Delta + \sum_{|k| > 0} g(k) e(\alpha p^2 k + \beta k) \right) \log p
\]

\[
= \Delta \sum_{m \leq D} \nu(m) \sum_{\substack{p \leq X \atop p+2 \equiv 0 \pmod{m}}} \log p
\]

\[
+ \Delta \sum_{m \leq D} \nu(m) \sum_{0 < |k| \leq H} \left( \Delta^{-1} g(k) e(\beta k) \right) \sum_{\substack{p \leq X \atop p+2 \equiv 0 \pmod{m}}} e(\alpha p^2 k) \log p
\]

\[
+ \sum_{m \leq D} \nu(m) \sum_{|k| > H} g(k) e(\beta k) \sum_{\substack{p \leq X \atop p+2 \equiv 0 \pmod{m}}} e(\alpha p^2 k) \log p.
\]

By (4.3) and (4.29), we get

\[
\sum_{m \leq D} \nu(m) \sum_{|k| > H} g(k) e(\beta k) \sum_{\substack{p \leq X \atop p+2 \equiv 0 \pmod{m}}} e(\alpha p^2 k) \log p
\]

\[
\ll \sum_{|k| > H} |g(k)| \sum_{m \leq D} \sum_{\substack{p \leq X \atop p+2 \equiv 0 \pmod{m}}} \log p \ll \sum_{m \leq D} \frac{1}{\varphi(m)} \ll \log X.
\]

Thus, we derive that

\[
\Phi_1 = \Delta (\Phi_2 + \Phi_3) + O(\log X), \tag{4.30}
\]

where

\[
\Phi_2 = \sum_{m \leq D} \nu(m) \sum_{\substack{p \leq X \atop p+2 \equiv 0 \pmod{m}}} \log p,
\]

\[
\Phi_3 = \sum_{m \leq D} \nu(m) \sum_{0 < |k| \leq H} c(k) \sum_{\substack{p \leq X \atop p+2 \equiv 0 \pmod{m}}} e(\alpha p^2 k) \log p. \tag{4.31}
\]
\[ c(k) = \Delta^{-1} g(k) e(\beta k) \ll 1. \]

By Bombieri–Vinogradov’s mean value theorem and (4.29), we have
\[ \Phi_2 = \frac{X}{2} \sum_{m \leq D} \frac{\nu(m)}{\varphi(m)} + O\left( \frac{X}{\log^2 X} \right). \quad (4.32) \]

Using (4.26) and (4.28), we obtain
\[
\sum_{m \leq D} \frac{\nu(m)}{\varphi(m)} = \sum_{z < q < y} \left( 1 - \frac{\log q}{\log y} \right) \frac{1}{q - 1} \sum_{d \mid P(z)} \frac{\lambda_q^+(d)}{\varphi(qd)} \\
\leq \sum_{z < q < y} \left( 1 - \frac{\log q}{\log y} \right) \frac{1}{q - 1} \Pi(z) \left( 2 e^\gamma \left( \frac{\log(D/q)}{\log z} \right) \right)^{-1} + O\left( \frac{X}{(\log X)^{1/3}} \right). \quad (4.33)
\]

Therefore, by (4.21), (4.32) and (4.33), we have
\[ \Phi_2 \leq e^\gamma \Delta \Pi(z) \sum_{z < q < y} \left( 1 - \frac{\log q}{\log y} \right) \frac{1}{q - 1} \left( \frac{\log(D/q)}{\log z} \right)^{-1} + O\left( \frac{X}{(\log X)^{\frac{4}{3} - \varepsilon}} \right). \quad (4.34) \]

Now, we find a lower bound for the sum \( \Gamma \). From (4.10), (4.17), (4.18), (4.22), (4.27), (4.30) and (4.34), we derive that
\[ \Gamma \geq e^\gamma \Delta \Pi(z) \mathcal{G} + O\left( \frac{\Delta X}{(\log X)^{\frac{4}{3} - \varepsilon}} \right) + O\left( \Delta |\Psi_3 - \kappa \Phi_3| \right), \quad (4.35) \]

where
\[ \mathcal{G} = \frac{\log(s - 1)}{s} - \kappa \sum_{z < q < y} \left( 1 - \frac{\log q}{\log y} \right) \frac{1}{q - 1} \left( \frac{\log(D/q)}{\log z} \right)^{-1}, \quad s = \frac{\log D}{\log z}. \]

Moreover, by partial summation and the prime number theorem, it is easy to show that
\[ \mathcal{G} = \mathcal{G}_0 + O\left( \frac{1}{\log X} \right), \quad (4.36) \]

where
\[ \mathcal{G}_0 = \frac{\log(s - 1)}{s} - \kappa \eta \int_{\eta}^{\rho} \left( \frac{1}{u} - \frac{1}{\rho} \right) \frac{1}{\delta - u} du. \]

According to simple numerical calculation, we know that
\[ \mathcal{G}_0 \geq 0.000032113949. \]
From (4.21), (4.35) and (4.36), we obtain
\[
\Gamma \geq e^7 \Delta X \Pi(z) \mathcal{G}_0 + O \left( \frac{\Delta X}{(\log X)^{\frac{\Delta}{4} - \epsilon}} \right) + O \left( \Delta \left| \Psi_3 - \kappa \Phi_3 \right| \right). \tag{4.37}
\]
We shall illustrate that if \( X \) runs over a suitable sequence, which tends to infinity, then the second error term in (4.37) can be absorbed. Hence we need the following lemma.

**Lemma 4.1** Suppose that \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and \( \delta, \theta, D, H \) are defined in (2.1) and (2.2). Let \( \xi(d), c(k) \) be complex numbers defined for \( d \leq D, 0 < |k| \leq H \), respectively, which satisfy
\[
\xi(d) \ll 1, \quad c(k) \ll 1.
\]
Then there exists a sequence \( \{X_j\}_{j=1}^{\infty} \) satisfying \( \lim_{j \to \infty} X_j = +\infty \), such that the sum \( S(X) \) defined by
\[
S(X) = \sum_{d \leq D} \xi(d) \sum_{1 \leq |k| \leq H} c(k) \sum_{\substack{\chi \leq p \leq X \\ p+ \lambda \equiv 0 \mod d}} \log p e^{\alpha p^2 k} \tag{4.38}
\]
satisfies
\[
S(X_j) \ll \frac{X_j}{\log^2 X_j}, \quad j = 1, 2, 3, \ldots.
\]
The proof of Lemma 4.1 will be given in the next section. From (4.19) and (4.31), we know that \( \Psi_3 - \kappa \Phi_3 \) can be represented as a sum of type (4.38) with
\[
\xi(d) = \lambda^*(d) - \kappa \nu(d),
\]
where
\[
\lambda^*(d) = \begin{cases} 
\lambda^-(d), & \text{if } d \mid P(z), \\
0, & \text{otherwise}.
\end{cases}
\]
According to Lemma 4.1 and (4.37), there exists a sequence \( \{X_j\}_{j=1}^{\infty} \), which tends to infinity, such that
\[
\Gamma(X_j) \geq e^7 \Delta X_j \Pi(z) \mathcal{G}_0 + O \left( \frac{\Delta X_j}{(\log X_j)^{\frac{\Delta}{4} - \epsilon}} \right). \tag{4.39}
\]
From (4.21) and (4.39), we know that there exists a positive constant \( c > 0 \) such that
\[
\Gamma(X_j) \geq \frac{c \Delta(X_j) X_j}{\log X_j} > 0, \quad j = 1, 2, 3, \ldots.
\]
This completes the proof of Theorem 1.1.
5 Proof of Lemma 4.1

In this section, we shall prove Lemma 4.1. Since \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), by Dirichlet’s approximation theorem, there exist infinitely many integers \( A \) and natural numbers \( Q \) with \( (A, Q) = 1 \) such that
\[
\left| \alpha - \frac{A}{Q} \right| < \frac{1}{Q^2}.
\]

For each such \( Q \), we choose \( X \) in a suitable way, i.e., as in (5.37). In this way, we construct our sequence \( \{X_j\}_{j=1}^\infty \).

First, we have
\[
S(X) = W + O(HX^{\frac{1}{2}+\varepsilon}),
\]
where
\[
W = \sum_{\frac{X}{2} < n \leq X} \Lambda(n) \sum_{1 \leq |k| \leq H} c(k) e(\alpha n^2 k) \sum_{d \leq D \atop d \nmid n+2 \atop 2 \nmid d} \xi(d).
\]

According to Lemma 3.2, by taking \( u = 2^{-7}X^{\frac{5}{7}}, v = 2^7X^{\frac{3}{7}}, w = X^{\frac{1}{7}-\frac{4}{7}} \), it is easy to see that the sum \( W \) can be decomposed into \( O(\log^{10}X) \) sums, each of which either of Type I:
\[
S_I = \sum_{M < m \leq M_1} a_m \sum_{\frac{L}{2} < \ell \leq L_1} \sum_{1 \leq |k| \leq H} c(k) e(\alpha m^2 \ell^2 k) \sum_{d \leq D \atop d \nmid m\ell+2 \atop 2 \nmid d} \xi(d)
\]
with \( M_1 \leq 2M, L_1 \leq 2L, L \geq w, a_m \ll m^\varepsilon, ML \asymp X \); or of Type II:
\[
S_{II} = \sum_{M < m \leq M_1} a_m \sum_{\frac{L}{2} < \ell \leq L_1} b_\ell \sum_{1 \leq |k| \leq H} c(k) e(\alpha m^2 \ell^2 k) \sum_{d \leq D \atop d \nmid m\ell+2 \atop 2 \nmid d} \xi(d)
\]
with \( M_1 \leq 2M, L_1 \leq 2L, u \leq L \leq v, a_m \ll m^\varepsilon, b_\ell \ll \ell^\varepsilon, ML \asymp X \).

Next, we shall deal with the sums of Type I and Type II in the following subsections, respectively.

5.1 The Estimate of Type II Sums

In this subsection, we shall deal with the estimate of the sums of Type II. First, we have
\[
S_{II} = \sum_{1 \leq |k| \leq H} c(k) \sum_{M < m \leq M_1} a_m \sum_{\frac{L}{2} < \ell \leq L_1} b_\ell e(\alpha m^2 \ell^2 k) \sum_{d \leq D \atop d \nmid m\ell+2 \atop 2 \nmid d} \xi(d)
\]
\[
\ll X^\varepsilon \sum_{1 \leq |k| \leq H}\sum_{M < m \leq M_1} \left| \sum_{L < \ell \leq L_1}^{} \sum_{d \leq D}^{} b_\ell e(\alpha m^2 \ell^2 k) \sum_{d \mid m \ell + 2}^{} \xi(d) \right|.
\]

By Cauchy’s inequality, we get

\[
|S_{II}|^2 \ll X^\varepsilon HM \sum_{1 \leq |k| \leq H}\sum_{M < m \leq M_1} \left| \sum_{L < \ell \leq L_1}^{} \sum_{d \leq D}^{} b_\ell e(\alpha m^2 \ell^2 k) \sum_{d \mid m \ell + 2}^{} \xi(d) \right|^2
\]

\[
= X^\varepsilon HM \sum_{1 \leq |k| \leq H}\sum_{M < m \leq M_1} \left| \sum_{L < \ell \leq L_1}^{} \sum_{d_1, d_2 \leq D}^{} b_{\ell_1} b_{\ell_2} \xi(d_1) \xi(d_2) e(\alpha m^2 k(\ell_1^2 - \ell_2^2))\right|
\]

\[
\ll X^\varepsilon HM \sum_{1 \leq |k| \leq H}\sum_{L < \ell_1, \ell_2 \leq L_1}^{} \sum_{d_1, d_2 \leq D}^{} e(\alpha m^2 k(\ell_1^2 - \ell_2^2)) |V|
\]

where

\[
V = \sum_{M' < m \leq M_1'} e(\alpha m^2 k(\ell_1^2 - \ell_2^2)),
\]

\[
M' = \max \left( M, X \frac{X}{2\ell_1}, 2\ell_2 \right), \quad M'_1 = \min \left( M_1, X \frac{X}{\ell_1}, \ell_2 \right).
\]

If the system of the congruence

\[
\begin{align*}
m\ell_1 + 2 &\equiv 0(\text{mod} \, d_1) \\
m\ell_2 + 2 &\equiv 0(\text{mod} \, d_2)
\end{align*}
\]

has no solution, then \( V = 0 \). Assume that (5.1) has a solution. Then there exists an \( f_0 = f_0(\ell_1, \ell_2, d_1, d_2) \) such that (5.1) is equivalent to \( m \equiv f_0(\text{mod} \, [d_1, d_2]) \). In this case, we have

\[
|V| = \left| \sum_{M' < m \leq M_1'} e(\alpha m^2 k(\ell_1^2 - \ell_2^2)) \right|
\]

\[
= \left| \sum_{M' - f_0(\text{mod} \, [d_1, d_2])} e\left( \alpha f_0 + r[d_1, d_2] \right)^2 k(\ell_1^2 - \ell_2^2) \right|
\]

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Therefore, we have

\[ \sum_{M'-f_0 < r < \frac{M'-f_0}{d_1,d_2}} e\left( \alpha (r^2|d_1,d_2|^2 + 2f_0r|d_1,d_2|)k(\ell_1 - \ell_2^2) \right) \]

Moreover, by Cauchy's inequality again, we obtain

\[ \sum_{R < r < R_1} e\left( \alpha (r^2|d_1,d_2|^2 + 2f_0r|d_1,d_2|)k(\ell_1 - \ell_2^2) \right), \]

where

\[ R = \frac{M' - f_0}{d_1,d_2}, \quad R_1 = \frac{M' - f_0}{d_1,d_2}. \]

The contribution of \( V \) with \( \ell_1 = \ell_2 \) to \( |S_{II}|^2 \) is

\[ \ll X^\epsilon H M^2 \sum_{1 \leq k < H} \sum_{1 \leq L < \ell_1, \ell_2 \leq L_1} \sum_{d_1,d_2 \leq D} \frac{1}{|d_1,d_2|} \ll X^\epsilon H^2 M^2 L \sum_{d_1,d_2 \leq D} \frac{1}{|d_1,d_2|} \ll X^\epsilon H^2 M^2 L \ll X^{1+\epsilon} H^2 M. \]

Therefore, we have

\[ |S_{II}|^2 \ll X^{1+\epsilon} H^2 M + X^\epsilon H M \sum_{1 \leq k < H} \sum_{d_1,d_2 \leq D} \sum_{L < \ell_1, \ell_2 \leq L_1} \tau^2(h) \]

Moreover, by Cauchy's inequality again, we obtain

\[ |S_{II}|^4 \ll X^{2+\epsilon} H^4 M^2 + X^\epsilon H^3 M^2 \left( \sum_{1 \leq k < H} \sum_{d_1,d_2 \leq D} \sum_{L < \ell_1, \ell_2 \leq L_1} \frac{1}{|d_1,d_2|} \right) \]

\[ \ll X^{2+\epsilon} H^4 M^2 + X^\epsilon H^3 M^2 \left( \sum_{1 \leq k < H} \sum_{d_1,d_2 \leq D} \frac{1}{|d_1,d_2|} \right) \]

\[ \times \sum_{d_1,d_2 \leq D} \left( \sum_{L < \ell_1, \ell_2 \leq L_1} \frac{1}{|d_1,d_2|} \right) \]

\[ \ll X^{2+\epsilon} H^4 M^2 + X^\epsilon H^3 M^2 L^2 \left( \sum_{d_1,d_2 \leq D} \frac{1}{|d_1,d_2|} \right) \]

\[ \times \sum_{1 \leq k < H} \sum_{d_1,d_2 \leq D} \sum_{L < \ell_1, \ell_2 \leq L_1} \frac{1}{|d_1,d_2|} \]

\[ \ll X^{2+\epsilon} H^4 M^2 + X^\epsilon H^3 M^2 L^2 \left( \sum_{d_1,d_2 \leq D} \frac{1}{|d_1,d_2|} \right) \]

\[ \times \sum_{1 \leq k < H} \sum_{d_1,d_2 \leq D} \sum_{L < \ell_1, \ell_2 \leq L_1} \frac{1}{|d_1,d_2|} |V|^2 \]
\[\ll X^{2+\varepsilon} H^4 M^2 + X^{\varepsilon} H^3 M^2 L^2 \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D \atop (d_1, d_2, 2) = 1} [d_1, d_2] \sum_{L < \ell_1, \ell_2 \leq L_1 \atop \ell_1 \neq \ell_2 \atop (\ell_1, d_1) = (\ell_2, d_2) = 1} |\mathcal{V}|^2\]

\[\ll X^{2+\varepsilon} H^4 M^2 + X^{\varepsilon} H^3 M^2 L^2 \cdot \Sigma_0, \quad (5.2)\]

where

\[\Sigma_0 = \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D \atop (d_1, d_2, 2) = 1} [d_1, d_2] \sum_{L < \ell_1, \ell_2 \leq L_1 \atop \ell_1 \neq \ell_2 \atop (\ell_1, d_1) = (\ell_2, d_2) = 1} \times \sum_{R < r_1, r_2 \leq R_1} e\left(\alpha \left((r_1^2 - r_2^2)|d_1, d_2|^2 + 2f_0(r_1 - r_2)|d_1, d_2|k(\ell_1^2 - \ell_2^2)\right)\right).\]

For \(\Sigma_0\), we have

\[\Sigma_0 = \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D \atop (d_1, d_2, 2) = 1} [d_1, d_2] \sum_{L < \ell_1, \ell_2 \leq L_1 \atop \ell_1 \neq \ell_2 \atop (\ell_1, d_1) = (\ell_2, d_2) = 1} \times \sum_{s_1, s_2 \atop s_1 \equiv s_2 (\text{mod } 2) \atop 2R < s_1 + s_2 \leq 2R_1} e\left(\alpha \left(s_1 s_2|d_1, d_2|^2 + 2f_0 s_1|d_1, d_2|k(\ell_1^2 - \ell_2^2)\right)\right)\]

\[= \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D \atop (d_1, d_2, 2) = 1} [d_1, d_2] \sum_{L < \ell_1, \ell_2 \leq L_1 \atop \ell_1 \neq \ell_2 \atop (\ell_1, d_1) = (\ell_2, d_2) = 1} \times \sum_{s_1, s_2 \atop s_2 \equiv s_1 (\text{mod } 2) \atop 2R < s_2 + s_1 \leq 2R_1} e\left(2\alpha f_0 s_1|d_1, d_2|k(\ell_1^2 - \ell_2^2)\right)\]

Set

\[D_0 = X^{\frac{50}{19}}.\]

Then we divide \(\Sigma_0\) into two parts

\[\Sigma_0 = \Sigma_1 + \Sigma_2, \quad (5.3)\]

where \(\Sigma_1\) denotes the part of \(\Sigma_0\) which satisfies \([d_1, d_2] \leq D_0\), while \(\Sigma_2\) denotes the remaining part of \(\Sigma_0\) which satisfies \([d_1, d_2] > D_0\). We set \(s_2 = s_1 + 2t\) in \(\Sigma_1\) and \(\Sigma_2\),

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and derive that

\[
\Sigma_1 \leq \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D_0} \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ \ell_1 \neq \ell_2}} [d_1, d_2] \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ (\ell_1, d_1) = (\ell_2, d_2) = 1}} [d_1, d_2] \sum_{|s_1| \leq 2R_1 - 2R} \left| \sum_{R' < t \leq R_1'} e \left( 2\alpha s_1 t [d_1, d_2]^2 k (\ell_1^2 - \ell_2^2) \right) \right| \tag{5.4}
\]

and

\[
\Sigma_2 \leq \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \leq D} \sum_{\substack{\ell_1, \ell_2 \leq L_1 \\ \ell_1 \neq \ell_2}} [d_1, d_2] \sum_{\substack{L < \ell_1, \ell_2 \leq L_1 \\ \ell_1, d_1 = (\ell_2, d_2) = 1}} [d_1, d_2] \sum_{|s_1| \leq 2R_1 - 2R} \left| \sum_{R' < t \leq R_1'} e \left( 2\alpha s_1 t [d_1, d_2]^2 k (\ell_1^2 - \ell_2^2) \right) \right| , \tag{5.5}
\]

where

\[
R' = \max (R - s_1, R), \quad R'_1 = \min (R_1 - s_1, R_1).
\]

First, we consider the upper bound for \( \Sigma_1 \). Let \( \Sigma_1^{(1)} \) and \( \Sigma_1^{(2)} \) denote the contribution of the right-hand side of (5.4) for \( s_1 \neq 0 \) and \( s_1 = 0 \), respectively. Trivially, there holds

\[
\Sigma_1^{(2)} \ll HML^2 \sum_{d_1, d_2 \in D_0} 1 \ll HML^2 D_0^2 \ll D_0^2 HXL . \tag{5.6}
\]

For \( \Sigma_1^{(1)} \), by Lemma 3.3 we have

\[
\Sigma_1^{(1)} \ll \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \in D_0} [d_1, d_2] \sum_{L < \ell_1, \ell_2 \leq L_1} \sum_{\ell_1 \neq \ell_2} \min_{0 < |s| \leq \frac{2M}{|d_1, d_2|}} \left( \frac{M}{|d_1, d_2|} \frac{1}{\|2\alpha s [d_1, d_2]^2 k (\ell_1^2 - \ell_2^2)\|} \right) \ll D_0^2 \sum_{1 \leq k \leq H} \sum_{h \in D_0^2} \sum_{L < \ell_1, \ell_2 \leq L_1} \sum_{\ell_1 \neq \ell_2} \min_{0 < |s| \leq 2M} \left( \frac{M}{h} \frac{1}{\|2\alpha s h^2 k (\ell_1^2 - \ell_2^2)\|} \right) \ll D_0^2 \sum_{1 \leq k \leq H} \sum_{h \in D_0^2} \sum_{L < \ell_1, \ell_2 \leq L_1} \sum_{\ell_1 \neq \ell_2} \min_{0 < |s| \leq 2M} \left( M \frac{1}{\|2\alpha s h^2 k (\ell_1^2 - \ell_2^2)\|} \right).
\]
\[ \ll D_0^2 \sum_{1 \leq k \leq H} \sum_{h \geq D_0^2} \sum_{1 \leq |t_1| \leq L} \sum_{0 \leq |s| \leq 2M} \min \left( M, \frac{1}{\|2\alpha s h^2 k t_1 t_2\|} \right). \]

\[ \ll D_0^2 \sum_{1 \leq k \leq H} \sum_{h \geq D_0^2} \sum_{1 \leq t_1, t_2 \leq 4L} \sum_{1 \leq |s| \leq 2M} \min \left( M, \frac{1}{\|2\alpha s h^2 k t_1 t_2\|} \right). \]

\[ \ll D_0^2 \sum_{1 \leq m \leq 64D_0^3 H M L^2} \tau_7(m) \min \left( M, \frac{1}{\|\alpha m\|} \right). \] (5.7)

By Lemma 3.4, we have

\[ \sum_{1 \leq m \leq 64D_0^3 H M L^2} \tau_7(m) \min \left( M, \frac{1}{\|\alpha m\|} \right) \ll X^2 \sum_{1 \leq m \leq 64D_0^3 H M L^2} \min \left( \frac{64D_0^3 H M^2 L^2}{m}, \frac{1}{\|\alpha m\|} \right) \]

\[ \ll X^2 D_0^3 H M^2 L^2 \left( \frac{1}{Q} + \frac{1}{M} + \frac{Q}{D_0^3 H M^2 L^2} \right) \]

\[ \ll X^2 \left( \frac{H X^2 D_0^4}{Q} + H X L D_0^4 + Q \right). \] (5.8)

Combining (5.4), (5.6), (5.7) and (5.8), and by noting the fact that \( ML \ll X \), we get

\[ \Sigma_1 \ll X^2 \left( H X^2 D_0^6 Q^{-1} + H X L D_0^6 + Q D_0^2 \right). \] (5.9)

Now, we consider the estimate of \( \Sigma_2 \). According to (5.5), by a splitting argument, we have

\[ \Sigma_2 \ll \mathcal{L} \max_{D_0 \in [T, 2D^2]} (T \Sigma^{(1)}_2), \] (5.10)

where

\[ \Sigma^{(1)}_2 = \Sigma^{(1)}_2 (T) = \sum_{1 \leq k \leq H} \sum_{d_1, d_2 \in D} \sum_{L < \ell_1, \ell_2 < L_1} \sum_{T < |d_1, d_2| < 2T} \sum_{(d_1, d_2) \neq (d_2, d_2)} \sum_{(\ell_1, d_1) \neq (\ell_2, d_2)} \sum_{\ell_1 \neq \ell_2} \sum_{|s| \leq 2R_1 - 2R} \left| \sum_{R' < t < R'_1} e\left(2\alpha s t[d_1, d_2]^2 k (\ell_1^2 - \ell_2^2)\right) \right|. \]

By Lemma 3.1, we have

\[ \Sigma^{(1)}_2 = \Sigma^{(1)}_2 (T) \leq \sum_{1 \leq k \leq H} \sum_{T < |d_1, d_2| < 2T} \sum_{L < \ell_1, \ell_2 < L_1} \sum_{|s| \leq 2M} \sum_{\ell_1 \neq \ell_2} \int_{-\infty}^{+\infty} K(\theta) \left| \sum_{R' < t < R'_1} e\left(2\alpha s t[d_1, d_2]^2 k (\ell_1^2 - \ell_2^2) + \theta t\right) \right| d\theta \]
\[
\Sigma_2^{(2)}(\theta, T) = \sum_{1 \leq k \leq H} \sum_{T < |d_1, d_2| \leq 2T} \left( \sum_{d_1, d_2 \leq D} \sum_{T < |d_1, d_2| \leq 2T} \sum_{L < |\ell_1, \ell_2| \leq L_1} \sum_{|s| \leq \frac{2M}{\ell_1, \ell_2}} \sum_{\ell_1 \neq \ell_2} \right.
\times \left. \left| \sum_{\frac{\ell_1 + \ell_2}{2} < t \leq \frac{4M}{\ell_1, \ell_2}} e^{2\alpha st[d_1, d_2]^2k(\ell_1^2 - \ell_2^2) + \theta t} \right| \right|
\]

According to (3.1) and (5.11), it is easy to see that

\[
\Sigma_2^{(1)} \ll \mathcal{L} \max_{0 \leq \theta < 1} \Sigma_2^{(2)}(\theta, T). \tag{5.12}
\]

For \(\Sigma_2^{(2)}(\theta, T)\), we have

\[
\Sigma_2^{(2)}(\theta, T) = \sum_{1 \leq k \leq H} \sum_{T < |d_1, d_2| \leq 2T} \left( \sum_{d_1, d_2 \leq D} \sum_{T < |d_1, d_2| \leq 2T} \sum_{L < |\ell_1, \ell_2| \leq L_1} \sum_{|s| \leq \frac{2M}{\ell_1, \ell_2}} \sum_{\ell_1 \neq \ell_2} \right.
\times \left. \left| \sum_{\frac{\ell_1 + \ell_2}{2} < t \leq \frac{4M}{\ell_1, \ell_2}} e^{2\alpha st[d_1, d_2]^2k(\ell_1^2 - \ell_2^2) + \theta t} \right| \right|
\]

\[
\ll \sum_{1 \leq k \leq H} \sum_{T < |d_1, d_2| \leq 2T} \tau^2(h) \sum_{L < |\ell_1, \ell_2| \leq L_1} \sum_{|s| \leq \frac{2M}{\ell_1, \ell_2}} \left| \sum_{\frac{\ell_1 + \ell_2}{2} < t \leq \frac{4M}{\ell_1, \ell_2}} e^{2\alpha st[d_1, d_2]^2k(\ell_1^2 - \ell_2^2) + \theta t} \right|
\]

\[
= \sum_{1 \leq k \leq H} \sum_{T < |d_1, d_2| \leq 2T} \tau^2(h) \sum_{|s| \leq \frac{2M}{\ell_1, \ell_2}} \left| \sum_{\frac{\ell_1 + \ell_2}{2} < t \leq \frac{4M}{\ell_1, \ell_2}} e^{2\alpha stk_1 t_2 + \theta t} \right|
\]

\[
\ll \sum_{1 \leq k \leq H} \sum_{T < |d_1, d_2| \leq 2T} \tau^2(h) \sum_{1 \leq |t_1, t_2| \leq 4L} \sum_{|s| \leq \frac{2M}{t_1, t_2}} \left| \sum_{\frac{\ell_1 + \ell_2}{2} < t \leq \frac{4M}{\ell_1, \ell_2}} e^{2\alpha stk_1 t_2 + \theta t} \right|
\]

\[
\ll \mathcal{L}^2 HML_2 + \sum_{1 \leq k \leq H} \sum_{T < |d_1, d_2| \leq 2T} \tau^2(h) \sum_{1 \leq |t_1, t_2| \leq 4L} \sum_{1 \leq |s| \leq \frac{2M}{t_1, t_2}} \left| \sum_{\frac{\ell_1 + \ell_2}{2} < t \leq \frac{4M}{\ell_1, \ell_2}} e^{2\alpha stk_1 t_2 + \theta t} \right|
\]
\[
\ll \mathcal{L}^3 H M L^2 + \sum_{1 \leq k \leq H} \sum_{T - h < 2T} \tau^2(h) \sum_{1 \mid m \leq \frac{2M}{L^2}} \tau_3(\mid m \mid) \sum_{\frac{M}{4} \leq \theta \leq \frac{4M}{T}} e \left( 2\alpha h^2 km + \theta t \right) \ll \mathcal{L}^3 H M L^2 + \Sigma^{(3)}_2, \quad \text{say.}
\]

(5.13)

It follows from Cauchy’s inequality that

\[
\left( \Sigma^{(3)}_2 \right)^2 \ll H \sum_{1 \leq k \leq H} \left( \sum_{T - h < 2T} \tau^2(h) \sum_{1 \mid m \leq \frac{2M}{L^2}} \tau_3(\mid m \mid) \sum_{\frac{M}{4} \leq \theta \leq \frac{4M}{T}} e \left( 2\alpha h^2 km + \theta t \right) \right)^2
\]

\[
\ll H \left( \sum_{T - h < 2T} \tau^4(h) \right) \left( \sum_{1 \mid m \leq \frac{2M}{L^2}} \tau_3^2(\mid m \mid) \right) \sum_{1 \leq k \leq H} \sum_{T - h < 2T}
\]

\[
\times \sum_{1 \mid m \leq \frac{2M}{L^2}} \sum_{\frac{M}{4} \leq \theta \leq \frac{4M}{T}} e \left( 2\alpha h^2 km + \theta t \right)^2
\]

\[
\ll \mathcal{L}^{23} H M L^2 \sum_{1 \leq k \leq H} \sum_{T - h < 2T} \sum_{1 \mid m \leq \frac{2M}{L^2}} \sum_{\frac{M}{4} \leq \theta \leq \frac{4M}{T}} e \left( 2\alpha h^2 km + \theta t \right)^2
\]

\[
= \mathcal{L}^{23} H M L^2 \cdot \Sigma^{(4)}_2, \quad \text{say.}
\]

(5.14)

For \( \Sigma^{(4)}_2 \), we have

\[
\Sigma^{(4)}_2 = \sum_{1 \leq k \leq H} \sum_{T - h < 2T} \sum_{1 \mid m \leq \frac{2M}{L^2}} \sum_{\frac{M}{4} \leq \theta_1, \theta_2 \leq \frac{4M}{T}} e \left( (2\alpha h^2 km + \theta)(t_1 - t_2) \right)
\]

\[
\ll \sum_{1 \leq k \leq H} \sum_{1 \mid m \leq \frac{2M}{L^2}} \sum_{\frac{M}{4} \leq t_1, t_2 \leq \frac{4M}{T}} \sum_{T - h < 2T} e \left( 2\alpha h^2 km(t_1 - t_2) \right)
\]

\[
\ll \frac{H M^2 L^2}{T} + \frac{M}{T} \sum_{1 \leq k \leq H} \sum_{1 \mid m \leq \frac{2M}{L^2}} \sum_{1 \mid n \leq \frac{4M}{T}} \sum_{T - h < 2T} e \left( 2\alpha h^2 kmn \right)
\]

\[
\ll \frac{H M^2 L^2}{T} + \frac{M}{T} \sum_{1 \leq k \leq H} \sum_{1 \mid m \leq \frac{2M}{L^2}} \sum_{1 \mid n \leq \frac{4M}{T}} \tau_3(\mid m \mid) \sum_{T - h < 2T} e \left( \alpha h^2 ks \right)
\]

\[
= \frac{H M^2 L^2}{T} + \frac{M}{T} \cdot \Sigma^{(5)}_2, \quad \text{say.}
\]
By Cauchy’s inequality, we deduce that

$$ (\Sigma_2^{(5)})^2 \ll H \sum_{1 \leq k \leq H} \left( \sum_{1 \leq s \leq \frac{256M^2L^2}{T^2}} \tau_3(s) \right) \sum_{T < h \leq 2T} \left| \sum_{\vartheta(h^2k\tau) \leq 256} e(\alpha h^2 k \tau) \right|^2 $$

$$ \ll H \left( \sum_{1 \leq s \leq \frac{256M^2L^2}{T^2}} \tau_3^2(s) \right) \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq \frac{256M^2L^2}{T^2}} \left| \sum_{T < h \leq 2T} e(\alpha h^2 k \tau) \right|^2 $$

$$ \ll \frac{\mathcal{Q}^8 HM^2L^2}{T^2} \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq \frac{256M^2L^2}{T^2}} \sum_{T < h_1, h_2 \leq 2T} e(\alpha ks(h_1^2 - h_2^2)) $$

$$ \ll \frac{\mathcal{Q}^8 HM^2L^2}{T^2} + \frac{\mathcal{Q}^8 HM^2L^2}{T^2} \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq \frac{256M^2L^2}{T^2}} \sum_{T < h_1, h_2 \leq 2T} e(\alpha ks(h_1^2 - h_2^2)) $$

$$ = \frac{\mathcal{Q}^8 HM^2L^2}{T^2} + \frac{\mathcal{Q}^8 HM^2L^2}{T^2} \cdot \Sigma_2^{(6)}, \quad \text{say.} \quad (5.16) $$

For $\Sigma_2^{(6)}$, from Lemma 3.3 we have

$$ \Sigma_2^{(6)} = \sum_{1 \leq n \leq 16HT^2} \sum_{1 \leq k \leq H} \sum_{1 \leq t_1, t_2 < 4T} \left( \sum_{T < h_1, h_2 < 2T} \sum_{h_1-h_2=t_1, h_1+h_2=t_2} \right) \sum_{1 \leq s \leq \frac{256M^2L^2}{T^2}} e(\alpha ks(t_1t_2)) $$

$$ \ll \sum_{1 \leq k \leq H} \sum_{1 \leq t_1, t_2 < 4T} \left( \sum_{1 \leq s \leq \frac{256M^2L^2}{T^2}} e(\alpha ks(t_1t_2)) \right) $$

$$ \ll \sum_{1 \leq k \leq H} \sum_{1 \leq t_1, t_2 < 4T} \min \left( \frac{M^2L^2}{T^2}, \frac{1}{\|\alpha t_1t_2\|} \right) $$

$$ \ll \sum_{1 \leq n \leq 16HT^2} \tau_3(n) \min \left( \frac{M^2L^2}{T^2}, \frac{1}{\|\alpha n\|} \right). \quad (5.17) $$

It follows from Lemma 3.4 that

$$ \sum_{1 \leq n \leq 16HT^2} \tau_3(n) \min \left( \frac{M^2L^2}{T^2}, \frac{1}{\|\alpha n\|} \right) $$

$$ \ll X^\varepsilon \sum_{1 \leq n \leq 16HT^2} \min \left( \frac{16HM^2L^2}{n}, \frac{1}{\|\alpha n\|} \right) $$

$$ \ll X^\varepsilon HM^2L^2 \left( \frac{1}{Q} + \frac{T^2}{M^2L^2} + \frac{Q}{HM^2L^2} \right) $$

$$ \ll X^\varepsilon (H \alpha^2 Q^{-1} + HT^2 + Q). \quad (5.18) $$

From (5.16), (5.17) and (5.18), we obtain

$$ \Sigma_2^{(5)} \ll X^\varepsilon \left( \frac{H \alpha^2}{T^{3/2}} + \frac{H \alpha^2}{TQ^{1/2}} + H + \frac{H^2 \alpha^2}{T} \right). \quad (5.19) $$
Combining (5.14) and (5.20), one has

\[ \Sigma_2^{(3)} \ll X^\epsilon \left( \frac{HX^2}{T} + \frac{HX^2M}{T^{5/4}} + \frac{HX^2}{T^{2/4}} + \frac{HM}{T} + \frac{H^{\frac{1}{2}}XQ^{\frac{1}{2}}M}{T^2} \right). \]  

(5.21)

Inserting (5.21) into (5.13), we derive that

\[ \Sigma_2^{(2)}(\theta, T) \ll X^\epsilon \left( HXL + \frac{HX^2L^\frac{1}{2}}{T^{1/2}} + \frac{HX^2}{T^{5/4}} + \frac{H^\frac{1}{2}X^2}{TQ^{1/4}} + \frac{H^{\frac{1}{2}}XQ^{\frac{1}{2}}}{T} \right), \]

which combines (5.10) and (5.12) to get

\[ \Sigma_2 \ll X^\epsilon \max_{D_0 \leq T \leq D^2} \left( HXL + \frac{HX^2L^\frac{1}{2}T^\frac{1}{2}}{T} + HX^2T^{-\frac{1}{4}} + H^\frac{1}{2}X^2Q^{-\frac{1}{4}} + H^{\frac{1}{2}}XQ^{\frac{1}{4}} \right) \]

\[ \ll X^\epsilon \left( HXL + \frac{HXL^\frac{1}{2}}{T} + HX^2D_0^{-\frac{1}{4}} + H^{\frac{1}{2}}X^2Q^{-\frac{1}{4}} + H^{\frac{1}{2}}XQ^{\frac{1}{4}} \right). \]

(5.22)

From (5.3), (5.9) and (5.22), we obtain

\[ \Sigma_0 \ll X^\epsilon \left( D_0^6HX^2Q^{-1} + D_0^6HXL + D_0^6Q + HXL + D_0^6Q + HX^2XQ^{-\frac{1}{4}} + H^{\frac{1}{2}}XQ^{\frac{1}{4}} \right), \]

which combines (5.1) yields

\[ S_{11} \ll X^\epsilon \left( HXL^{-\frac{1}{4}} + D_0^6HXQ^{-\frac{1}{4}} + D_0^6HX^2L^\frac{1}{2} + D_0^6Q^\frac{1}{2}H^\frac{1}{2}X^\frac{3}{2} + HX^\frac{1}{2}L^\frac{1}{2}X^\frac{1}{2} \right. \]

\[ + \left. HX^\frac{1}{2}L^\frac{1}{2}D^\frac{1}{2} + HXD_0^{-\frac{1}{4}} + H^{\frac{1}{2}}XQ^{-\frac{1}{4}} + H^{\frac{1}{2}}XQ^{\frac{1}{4}} \right) \]

\[ \ll X^\epsilon \left( HXL^{-\frac{1}{4}} + D_0^6HXQ^{-\frac{1}{4}} + D_0^6HX^2L^\frac{1}{2} + D_0^6Q^\frac{1}{2}H^\frac{1}{2}X^\frac{1}{2} + HX^\frac{1}{2}L^\frac{1}{2}D^\frac{1}{2} \right. \]

\[ + \left. HX^\frac{1}{2}Q^{-\frac{1}{4}} + HXD_0^{-\frac{1}{4}} + H^{\frac{1}{2}}XQ^{-\frac{1}{4}} + H^{\frac{1}{2}}XQ^{\frac{1}{4}} \right). \]

(5.23)

### 5.2 The Estimate of Type I Sums

In this subsection, we shall deal with the estimate of the sums of Type I. First, we have

\[ S_I = \sum_{1 \leq |k| \leq H} c(k) \sum_{d \leq D} \sum_{d \leq D} \sum_{d \leq D} a_m \sum_{m \leq M} \sum_{m \leq M} e(\alpha m^2 \ell^2 k), \]

where

\[ L' = \max \left(L, \frac{X}{2m}\right), \quad L'_1 = \min \left(L_1, \frac{X}{m}\right). \]

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By a splitting argument, there holds
\[ S_I \ll X^\varepsilon \cdot \max_{1 \leq T \leq D} \Sigma_3, \tag{5.24} \]
where
\[ \Sigma_3 = \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{M < m \leq M_1 \atop (d,2)=1} \sum_{m \ell + 2 \equiv 0 \pmod{d}} L_j' \left( \sum_{\ell' \leq \ell' + 1} e\left(\alpha m^2 \ell^2 k\right) \right). \]

For \((m, d) = 1\), there exists \(m\), which satisfies \(0 \leq m \leq d - 1\), such that \(m\bar{m} \equiv 1 \pmod{d}\). Therefore, the equation \(m\ell + 2 \equiv 0 \pmod{d}\) is equivalent to \(\ell \equiv -2m \pmod{d}\), i.e. \(\ell = -2m + dr\) for some \(r \in \mathbb{Z}\). Then it follows from Cauchy's inequality that
\[
\left(\Sigma_3\right)^2 \ll HMT \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{M < m \leq M_1 \atop (d,2)=1} \sum_{m \ell + 2 \equiv 0 \pmod{d}} L_j' \left( \sum_{\ell' \leq \ell' + 1} e\left(\alpha m^2 \ell^2 k\right) \right)^2 
\]
\[
\ll HMT \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{M < m \leq M_1 \atop (d,2)=1} \sum_{m \ell + 2 \equiv 0 \pmod{d}} \sum_{r \leq r' < r + \frac{L_j' + 2m}{d}} e\left(\alpha m^2 (-2m + dr)^2 k\right)^2 
\]
\[
= HMT \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{M < m \leq M_1 \atop (d,2)=1} \sum_{m \ell + 2 \equiv 0 \pmod{d}} \sum_{r \leq r' < r + \frac{L_j' + 2m}{d}} e\left(\alpha m^2 (d^2 r_1^2 - r_2^2) - 4md(r_1 - r_2)\right)k. 
\]

Set
\[
R = \frac{L_j' + 2m}{d}, \quad R_1 = \frac{L_j' + 2m}{d}. 
\]

Then we have
\[
\left(\Sigma_3\right)^2 \ll X^\varepsilon H^2 M^2 L^2 + HMT \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{M < m \leq M_1 \atop (d,2)=1} \sum_{m \ell + 2 \equiv 0 \pmod{d}} \sum_{r \leq r_1, r_2 < R \atop r_1 \neq r_2} e\left(\alpha m^2 (d^2 r_1^2 - r_2^2) - 4md(r_1 - r_2)\right)k 
\]
\[
\ll X^\varepsilon H^2 M^2 L^2 + HMT \cdot \left| \Sigma_3^{(1)} \right|, \tag{5.25} 
\]
where
\[
\Sigma_3^{(1)} = \sum_{1 \leq k \leq H} \sum_{T < d \leq 2T} \sum_{M < m \leq M_1 \atop (d,2)=1} \sum_{r \leq r_1, r_2 < R \atop r_1 \neq r_2} e\left(\alpha m^2 (d^2 r_1^2 - r_2^2) - 4md(r_1 - r_2)\right)k. 
\]
For $\Sigma_k^{(1)}$, we have

$$\Sigma_k^{(1)} = \sum_{1 \leq k \leq H \cdot T < d \leq 2T} \sum_{M < m \leq M_{k+1}} \sum_{s_l \leq s_2} \left( \sum_{R < r_1, r_2 \leq R} \sum_{s_l - s_2 = s_1} 1 \right) e \left( \alpha m^2 \left( d^2 s_1 s_2 - 4mds_1 \right) k \right)$$

$$= \sum_{1 \leq k \leq H \cdot T < d \leq 2T} \sum_{M < m \leq M_{k+1}} \sum_{s_l \leq s_2} e \left( \alpha m^2 \left( d^2 s_1 s_2 - 4mds_1 \right) k \right)$$

$$\ll \sum_{1 \leq k \leq H \cdot T < d \leq 2T} \sum_{M < m \leq M_{k+1}} \sum_{s_l \leq s_2} \left( \sum_{R < r_1, r_2 \leq R} \sum_{s_l - s_2 = s_1} 1 \right) e \left( \alpha m^2 \left( 2d^2 s_1 s_2 \right) k \right)$$

$$\ll \sum_{1 \leq k \leq H \cdot T < d \leq 2T} \sum_{M < m \leq M_{k+1}} \sum_{s_l \leq s_2} \left( \sum_{R < r_1, r_2 \leq R} \sum_{s_l - s_2 = s_1} 1 \right) e \left( 2\alpha m^2 \left( d^2 s_1 k \right) \right). \quad (5.26)$$

Next, we will discuss the estimate of the right-hand side of (5.26) in two cases.

**Case 1.** Suppose that $MT \leq D_0$, and under this condition, there holds $1 \ll M, T \ll D_0$. By Lemma 3.4, we have

$$\Sigma_k^{(1)} \ll \sum_{1 \leq k \leq H \cdot T < d \leq 2T} \sum_{M < m \leq M_{k+1}} \sum_{s_l \leq s_2 \leq 8L} \min \left( L, \frac{1}{\|2\alpha m^2 d^2 sk\|} \right)$$

$$\ll \sum_{1 \leq k \leq H \cdot T < d \leq 2T} \sum_{M < m \leq M_{k+1}} \sum_{s_l \leq s_2 \leq 8L} \min \left( \frac{256HM^2 T^2 L^2}{2m^2 d^2 sk}, \frac{1}{\|2\alpha m^2 d^2 sk\|} \right)$$

$$\ll \sum_{1 \leq n \leq 256HM^2 T^2 L} \tau_7(n) \min \left( \frac{HM^2 T^2 L^2}{n}, \frac{1}{\|\alpha n\|} \right)$$

$$\ll X^\varepsilon H X^2 T^2 \left( \frac{1}{Q} + \frac{1}{L} + \frac{Q}{H X^2 T^2} \right)$$

$$\ll X^\varepsilon \left( \frac{H X^2 D_0^2}{Q} + H X (MT) T + Q \right)$$

$$\ll X^\varepsilon \left( \frac{H X^2 D_0^2}{Q} + H X D_0^2 + Q \right). \quad (5.27)$$

From (5.24), (5.25) and (5.27), we derive that, under the condition $MT \leq D_0$, there holds

$$S_1 \ll X^\varepsilon \left( H X^2 D_0^2 + H X D_0^2 Q^{-\frac{1}{2}} + H X^2 D_0^2 + H X^2 D_0^2 Q_{+}^{\frac{1}{2}} \right). \quad (5.28)$$

**Case 2.** Now, we suppose that $MT > D_0$. Set

$$R'_1 = \max \left( R, R - s_1 \right), \quad R'_1 = \min \left( R_1, R_1 - s_1 \right).$$
Applying Lemma 3.1 to (5.26), we have

\[
\Sigma_3^{(1)} \ll \sum_{1 \leq k \leq H} \sum_{T < d < 2T} \sum_{M < m \leq M_1} \sum_{1 \leq |s| \leq \frac{H}{d}} \sum_{R' < t < R'_1} \sum_{R' < t' < R'_1} e(2am^2d^2s_1t) \\
\ll \sum_{1 \leq k \leq H} \sum_{T < d < 2T} \sum_{M < m \leq M_1} \sum_{1 \leq |s| \leq \frac{H}{d}} \int_{-\infty}^{+\infty} K_1(\theta) \left| \sum_{\frac{1}{d} < t < \frac{4H}{d}} e(2am^2d^2s_1t + \theta t) \right| d\theta \\
= \int_{-\infty}^{+\infty} K_1(\theta) \cdot \Sigma_3^{(2)}(\theta, T) d\theta,
\]

(5.29)

where

\[K_1(\theta) = \min \left( \frac{15L}{4T} + 1, \frac{1}{|\theta|}, \frac{1}{\pi^2|\theta|^2} \right),\]

and

\[\Sigma_3^{(2)}(\theta, T) = \sum_{1 \leq k \leq H} \sum_{T < d < 2T} \sum_{M < m \leq M_1} \sum_{1 \leq |s| \leq \frac{H}{d}} \sum_{\frac{1}{d} < t < \frac{4H}{d}} e(2am^2d^2s_1t + \theta t).\]

According to (3.1) and (5.29), it is easy to see that

\[\Sigma_3^{(1)} \ll L \cdot \max_{0 \leq \theta \leq 1} \Sigma_3^{(2)}(\theta, T).\]

(5.30)

For \(\Sigma_3^{(2)}(\theta, T)\), we have

\[\Sigma_3^{(2)}(\theta, T) \ll \sum_{1 \leq k \leq H} \sum_{MT \leq h \leq 4MT} \tau(h) \sum_{1 \leq s \leq \frac{H}{h}} \sum_{\frac{1}{d} < t < \frac{4H}{d}} e(2\alpha h^2stk + \theta t).\]

It follows from Cauchy’s inequality that

\[
\left( \Sigma_3^{(2)}(\theta, T) \right)^2 \ll H \left( \sum_{MT \leq h \leq 4MT} \tau^2(h) \right) \left( \sum_{1 \leq s \leq \frac{H}{h}} \right) \sum_{1 \leq k \leq H} \sum_{MT \leq h \leq 4MT} \sum_{\frac{1}{d} < t < \frac{4H}{d}} e(2\alpha h^2stk + \theta t)^2 \\
\ll X^2 HML \sum_{1 \leq k \leq H} \sum_{MT \leq h \leq 4MT} \sum_{1 \leq s \leq \frac{H}{h}} \sum_{\frac{1}{d} < t_1, t_2 < \frac{4H}{d}} e((2\alpha h^2stk + \theta)(t_1 - t_2)) \\
\ll X^2 HML \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq \frac{H}{h}} \sum_{\frac{1}{d} < t_1, t_2 < \frac{4H}{d}} \sum_{MT \leq h \leq 4MT} e(2\alpha h^2stk(t_1 - t_2)) \\
\ll \frac{X^2 H^2 M^2 L^3}{T} + X^2 HML \cdot \Sigma_3^{(3)},
\]

(5.31)

where

\[\Sigma_3^{(3)} = \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq \frac{H}{h}} \sum_{\frac{1}{d} < t_1, t_2 < \frac{4H}{d}} \sum_{t_1 \neq t_2 \atop MT \leq h \leq 4MT} e(2\alpha h^2st(t_1 - t_2)).\]
For $\Sigma^{(3)}_3$, we have

$$\Sigma^{(3)}_3 = \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq \frac{4L}{T}} \sum_{1 \leq |r_1| \leq \frac{4L}{T}} \left( \sum_{\frac{L}{T} \leq t_1, t_2 \leq \frac{4L}{T}} 1 \right) \sum_{MT < h \leq 4MT} e(2\alpha h^2 \varepsilon r_1)$$

$$\ll \frac{L}{T} \sum_{1 \leq k \leq H} \sum_{1 \leq s \leq \frac{4L}{T}} \sum_{1 \leq |r_1| \leq \frac{4L}{T}} \left( \sum_{MT < h \leq 4MT} e(2\alpha h^2 \varepsilon r_1) \right)$$

$$\ll \frac{L}{T} \sum_{1 \leq n \leq \frac{32HL^2}{T^2}} \tau_4(n) \sum_{MT < h \leq 4MT} e(\alpha h^2 n).$$

Therefore, by Cauchy’s inequality, one has

$$\left( \Sigma^{(3)}_3 \right)^2 \ll \frac{L^2}{T^2} \left( \sum_{1 \leq n \leq \frac{32HL^2}{T^2}} \tau_4^2(n) \right) \left( \sum_{1 \leq n \leq \frac{32HL^2}{T^2}} \sum_{MT < h \leq 4MT} e(\alpha h^2 n) \right)^2$$

$$\ll \frac{X^4 HL^4}{T^4} \sum_{1 \leq n \leq \frac{32HL^2}{T^2}} \sum_{MT < h_1, h_2 \leq 4MT} e(\alpha (h_1^2 - h_2^2)n)$$

$$\ll \frac{X^2 H^2 ML^6}{T^4} + \frac{X^4 HL^4}{T^4} \cdot \Sigma^{(4)}_3,$$  \hspace{1cm} (5.32)

where

$$\Sigma^{(4)}_3 = \sum_{MT < h_1, h_2 \leq 4MT} \sum_{h_1 \neq h_2} e(\alpha (h_1^2 - h_2^2)n).$$

For $\Sigma^{(4)}_3$, by Lemma 3.3 we have

$$\Sigma^{(4)}_3 = \sum_{1 \leq |t_1|, |t_2| \leq 8MT} \left( \sum_{MT < h_1, h_2 \leq 4MT} 1 \right) \sum_{1 \leq n \leq \frac{32HL^2}{T^2}} e(\alpha t_1 t_2 n)$$

$$\ll \sum_{1 \leq t_1, t_2 \leq 8MT} \sum_{1 \leq n \leq \frac{32HL^2}{T^2}} e(\alpha t_1 t_2 n)$$

$$\ll \min \left( \frac{HL^2}{T^2}, \frac{1}{\|\alpha t_1 t_2\|} \right)$$

$$\ll \tau(t) \min \left( \frac{HL^2}{T^2}, \frac{1}{\|\alpha t\|} \right).$$ \hspace{1cm} (5.33)

It follows from Lemma 3.4 that

$$\sum_{1 \leq |t_1|, |t_2| \leq 8MT} \tau(t) \min \left( \frac{HL^2}{T^2}, \frac{1}{\|\alpha t\|} \right)$$
\[
\ll X^\varepsilon \sum_{1 \leq t \leq 64M^2T^2} \min \left( \frac{64HM^2L^2}{t}, \frac{1}{\|\alpha t\|} \right) \\
\ll X^\varepsilon HM^2L^2 \left( \frac{1}{Q} + \frac{T^2}{HL^2} + \frac{Q}{HM^2L^2} \right) \\
\ll X^\varepsilon \left( \frac{HX^2}{Q} + M^2T^2 + Q \right). 
\] (5.34)

From (5.32), (5.33) and (5.34), we derive that

\[ \Sigma_3^{(3)} \ll X^\varepsilon \left( \frac{HXL^2}{T^{5/2}} + \frac{HXL^2}{T^{2Q^{1/2}}} + \frac{HXL}{T} + \frac{HXL^2Q^{1/2}}{T^2} \right), \]

which combines (5.31) yields

\[ \Sigma_3^{(2)}(\theta, T) \ll X^\varepsilon \left( \frac{HXL^{1/2}}{T^{1/2}} + \frac{HXL^{1/2}}{T^{2Q^{1/2}}} + \frac{HXL}{TQ^{1/4}} + \frac{HXL^2Q^{1/2}}{T} \right). \] (5.35)

From (5.25), (5.30) and (5.35), we obtain

\[ \Sigma_3 \ll X^\varepsilon \left( HXL\frac{1}{2}T^{\frac{1}{2}} + HXL\frac{1}{2}T^{\frac{1}{2}} + HXL\frac{1}{2}T^{\frac{1}{2}}\right), \]

from which and (5.24), we derive that, under the condition \( MT > D_0 \), there holds

\[ S_I \ll X^\varepsilon \max_{\frac{1}{2} \leq t \leq D} \left( \frac{HXT^{1/2}}{L^{1/2}} + \frac{HXT^{1/2}}{L^{1/4}} + \frac{HXL^{1/2}}{T^{1/8}} + \frac{HX}{Q^{1/8}} + HXL \right) \]

\[ \ll X^\varepsilon \left( \frac{HXL^{1/2}}{w^{1/2}} + \frac{HXL^{1/2}}{w^{1/4}} + \frac{HX}{(MT)^{1/8}} + \frac{HX}{Q^{1/8}} + HXL \right) \]

\[ \ll X^\varepsilon \left( HX \frac{1}{w^{1/2}} + HX \frac{1}{w^{1/4}} + HX \frac{1}{w^{1/4}} + HX \frac{1}{w^{1/4}} + \frac{HX}{Q^{1/8}} + HXL \right) \] (5.36)

5.3 Proof of Lemma 4.1

From (5.23), (5.28) and (5.36), by taking

\[ Q = X^{\frac{2}{15}} \]

then we deduce that, under the condition (2.1) and (2.2), there holds

\[ S_I \ll X^{1-\varpi} \quad \text{and} \quad S_{II} \ll X^{1-\varpi} \]

for some \( \varpi > 0 \). This completes the proof of Lemma 4.1.
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References

[1] R. C. Baker, G. Harman, *On the distribution of \( \alpha p^k \) modulo one*, Mathematika, 38 (1991), no. 1, 170–184.

[2] E. Bombieri, H. Iwaniec, *On the order of \( \zeta\left(\frac{1}{2} + it\right) \)*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 13 (1986), no. 3, 449–472.

[3] J. R. Chen, *On the representation of a larger even integer as the sum of a prime and the product of at most two primes*, Sci. Sinica, 16 (1973), 157–176.

[4] H. Davenport, *Multiplicative Number Theory*, 2nd Edition, Springer–Verlag, New York, 1980.

[5] B. Green, T. Tao, *Restriction theory of the Selberg sieve with applications*, J. Théor. Nombres Bordeaux, 18 (2006), no. 1, 147–182.

[6] B. Green, T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. of Math. (2), 167 (2008), no. 2, 481–547.

[7] H. Halberstam, H. E. Richert, *Sieve Methods*, Academic Press, London, 1974.

[8] G. Harman, *On the distribution of \( \alpha p \) modulo one*, J. London Math. Soc. (2), 27 (1983), no. 1, 9–18.

[9] G. Harman, *Trigonometric sums over primes. II*, Glasgow Math. J., 24 (1983), no. 1, 23–37.

[10] G. Harman, *On the distribution of \( \alpha p \) modulo one. II*, Proc. London Math. Soc. (3), 72 (1996), no. 2, 241–260.
[11] D. R. Heath–Brown, *Three primes and an almost–prime in arithmetic progression*, J. London Math. Soc. (2), **23** (1981), no. 3, 396–414.

[12] D. R. Heath–Brown, *The Pjateckiĭ–Šapiro prime number theorem*, J. Number Theory, **16** (1983), no. 2, 242–266.

[13] D. R. Heath–Brown, C. Jia, *The distribution of $\alpha_p$ modulo one*, Proc. London Math. Soc. (3), **84** (2002), no. 1, 79–104.

[14] C. Jia, *On the distribution of $\alpha_p$ modulo one*, J. Number Theory, **45** (1993), no. 3, 241–253.

[15] C. Jia, *On the distribution of $\alpha_p$ modulo one. II*, Sci. China Ser. A, **43** (2000), no. 7, 703–721.

[16] A. A. Karatsuba, *Basic analytic number theory*, Springer–Verlag, Berlin, 1993.

[17] K. Matomäki, *The distribution of $\alpha_p$ modulo one*, Math. Proc. Cambridge Philos. Soc., **147** (2009), no. 2, 267–283.

[18] K. Matomäki, *A Bombieri–Vinogradov type exponential sum result with applications*, J. Number Theory, **129** (2009), no. 9, 2214–2225.

[19] F. Mertens, *Ein Beitrag zur analytischen Zahlentheorie*, J. Reine Angew. Math., **78** (1874), 46–62.

[20] T. P. Peneva, T. D. Tolev, *An additive problem with primes and almost–primes*, Acta Arith., **83** (1998), no. 2, 155–169.

[21] T. P. Peneva, *On the ternary Goldbach problem with primes $p_i$ such that $p_i + 2$ are almost–primes*, Acta Math. Hungar., **86** (2000), no. 4, 305–318.

[22] S. Y. Shi, *On the distribution of $\alpha_p$ modulo one for primes $p$ of a special form*, Osaka J. Math., **49** (2012), no. 4, 993–1004.

[23] S. Y. Shi, Z. X. Wu, *Distribution of $p^2$ modulo one for primes $p$ of a special type*, Chinese Ann. Math. Ser. A, **34** (2013), no. 4, 479–486.

[24] T. L. Todorova, D. I. Tolev, *On the distribution of $\alpha_p$ modulo one for primes $p$ of a special form*, Math. Slovaca, **60** (2010), no. 6, 771–786.

[25] D. I. Tolev, *Arithmetic progressions of prime–almost–prime twins*, Acta Arith., **88** (1999), no. 1, 67–98.
[26] D. I. Tolev, *Representations of large integers as sums of two primes of special type*, in: Algebraic number theory and Diophantine analysis (Graz, 1998), 485–495, de Gruyter, Berlin, 2000.

[27] D. I. Tolev, *Additive problems with prime numbers of special type*, Acta Arith., 96 (2000), no. 1, 53–88.

[28] R. C. Vaughan, *On the distribution of \( \alpha p \) modulo 1*, Mathematika, 24 (1977), no. 2, 135–141.

[29] R. C. Vaughan, *The Hardy–Littlewood Method, 2nd edn.*, Cambridge University Press, Cambridge, 1997.

[30] I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*, Translated from the Russian, revised and annotated by K. F. Roth and Anne Davenport. Reprint of the 1954 translation. Dover Publications, Inc., Mineola, NY, 2004.

[31] K. C. Wong, *On the distribution of \( \alpha p^k \) modulo 1*, Glasgow Math. J., 39 (1997), no. 2, 121–130.