Stabilization of moduli by fluxes

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In order to lift the continuous moduli space of string vacua, non-trivial fluxes may be the essential input. In this talk I summarize aspects of two approaches to compactifications in the presence of fluxes: (i) generalized Scherk-Schwarz reductions and gauged supergravity and (ii) the description of flux-deformed geometries in terms of $G$-structures and intrinsic torsion.

1 Introduction

One of the major problem appearing in compactifications of string theory, is the emergence of a continuous moduli space of string vacua. In order to get contact, not only to the standard model of particle physics, but also to (inflationary) cosmology, these moduli have to be fixed and if supersymmetry is broken only at fairly low energies, we have to understand the mechanism while preserving at least some supersymmetries. Moduli appear in two guises: the closed string or geometrical moduli, related to deformations of the size and shape of cycles of the internal manifold and open string moduli, related to un-fixed positions of wrapped branes. Fluxes provide a mechanism to lift both moduli. On one hand, fluxes cause a gravitational force which expands/contract a cycle that is parallel/perpendicular to the flux and these competing effects yield the stabilization at values related to the strength to the different fluxes. Note, if a given cycle is parallel or perpendicular to all fluxes, it is not stabilized and the supergravity potential has a run-away mode. On the other hand, fluxes couple also to the world volume action of branes producing a potential for the open string moduli. Since the open string moduli are compact, any potential has an extremum and gives a stabilization of these moduli.

Typically, one refers to fluxes as non-zero expectation values of the RR- and NS-fields in the vacuum and there is a growing literature on compactifications in the presence of fluxes [1] – [25]. Fluxes are sourced by branes which give rise to $\delta$-function singularities in the equations of motion or Bianchi identities and in typical flux compactifications one does not take into account source terms. But note, the constraints coming

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from solving the Killing spinor equations are local equations and are not sensible to distinguish between fluxes sources by branes and background fluxes. In fact, often one can regard the background fluxes as a near horizon limit for the corresponding branes. In addition to the fluxes related to form-fields one can also consider metric fluxes (or both), which are known as twisting. All fluxes can be related to a generalized Scherk-Schwarz reduction, which one can apply to any global symmetry of the vacuum. Generically, this reduction does not commute with all supersymmetry transformations and hence supersymmetry is partially broken. The original proposal of Scherk and Schwarz was related to a global fermionic symmetry and breaks all supersymmetries [26], but when applied to (global) gauge symmetries of RR- and/or NS-forms, some supersymmetries remain unbroken. In the following we will only consider Scherk-Schwarz reductions that correspond to flux compactifications and it is straightforward to relate these reductions to gauged supergravity; see [27] – [35] for more details. This gives us a powerful tool, because in gauged supergravity we can calculate the potential explicitly and expand the fields around an extremum; although it may become technically very involved. We should also note, that Scherk-Schwarz reductions allow for a consistent truncation to the massless Kaluza-Klein spectrum.

Without relying on a specific Scherk-Schwarz reduction, the moduli problem can also be addressed in gauged supergravity directly, e.g. by asking the question: Which gauging yields a potential that fixes all moduli? In answering this question, one has to keep in mind that massive tensor or vector multiplets imply further corrections to the supergravity Lagrangian [36]. As a spin-off, one can also look for possible deSitter vacua of the explicitly known potential in gauged supergravity. Unfortunately, in this approach the lift to 10-dimensional string theory, or M-theory, remains obscure in many interesting examples. Also the (deformed) geometry of the internal space is only for very specific situations known and to address this question, one has to solve directly the 10-dimensional (Killing spinor) equations. The fluxes are then identified with specific torsion components that deform the internal geometry. Unfortunately, the reduction to 4 dimensions as well as the derivation of the Kahler and superpotential as function of the scalar fields becomes a non-trivial task in this approach.

Let us add another important remark. Having fixed the moduli may not be enough. Especially interesting are vacua that remain stable if supersymmetry is broken. Many vacua, obtained in gauged supergravity, have unfortunately some tachyonic directions and if the cosmological constant is lifted to a positive value, these models are not acceptable. Of course, this is a problem of how we break supersymmetry and what are the corrections to the potential.

We have organized this paper as follows. We start in Section 2 with a resume of the Kaluza-Klein reduction yielding a continuous (moduli) space of vacua. In Section 3 we explain the relation between Scherk-Schwarz reductions and gauged supergravity, which provides a tool to understand the lifting of the moduli space. Section 4 is devoted to the second approach, i.e. we solve the 10-dimensional Killing spinor equations and relate fluxes to torsion components and $G$-structures. Two examples are given, where the back reaction of the fluxes changes the internal geometry and which are worked out explicitly.
2 Moduli coming from Kaluza-Klein reduction

We start with a short summary of the standard Kaluza-Klein (KK) reduction of type II supergravity (without fluxes). The reader may consult the original papers [37] for further details.

In the low energy approximation, type II string theory is described by type II supergravity in 10 dimensions, which has $\mathcal{N}=2$ supersymmetry and can be chiral (type IIB) or non-chiral (type IIA) if the two gravitinos and dilatinos in the fermionic sector have equal or opposite chirality. Both supergravities have a common sector comprising the (NS-NS) fields

$$(e^m, B, \phi)$$

where $e^m$ is the vielbein 1-form, $B$ is the NS-2-form and $\phi$ denotes the dilaton. But both models differ in the RR-sector. On the IIA side, there are odd RR gauge potentials

$$\text{type IIA} : \quad (C_1, C_3) \quad (2.1)$$

which give rise to the following gauge invariant field strengths

$$F^{(2)} = mB + dC_1 \quad , \quad F^{(4)} = dC_3 + \frac{6}{m} F^{(2)} \wedge F^{(2)},$$

(2.2)

satisfying the Bianchi identities

$$dF^{(2)} = mH \quad , \quad dF^{(4)} = 12H \wedge F^{(2)}.$$  

(2.3)

We included here also the mass parameter $m$, which was introduced in supergravity by Romans [38] and is related in string theory to D8-branes (“at the end of the universe”). If $m \neq 0$, the RR-1-form $C_1$ can be gauged away giving a mass to the NS 2-form $B$. On the other hand, type IIB supergravity has even RR gauge potentials

$$\text{type IIB} : \quad (C_0, C_2, C_4^+)$$

(2.4)

and one defines the following field strengths

$$P = \frac{1}{1 - |T|^2} dT \quad , \quad G^{(3)} = \frac{1}{\sqrt{1 - |T|^2}} (F_3 - TF_3^*) \quad , \quad F^{(5)} = dC_4 - \frac{1}{4} (C_2 \wedge dB)$$

(2.5)

where the 5-form $F^{(5)}$ has to satisfy a self-duality constraint and

$$F_3 = d(B + iC_2) \quad , \quad T = \frac{1 + i\tau}{1 - i\tau} \quad , \quad \tau = C_0 + i e^{-\phi}.$$  

(2.6)

The Bianchi identities read now

$$dG^{(3)} = (i Q - P) \wedge G \quad , \quad dF^{(5)} = -\frac{1}{4} dC_2 \wedge dB.$$  

(2.7)

with the U(1) connection

$$Q = \frac{1}{1 - |T|^2} \text{Im}(dT^*)$$.
Type IIB supergravity has an SL(2,R) symmetry, which acts as: \( \tau \rightarrow \frac{a\tau + b}{c\tau + d} \) combined by a rotation of the 2-form doublet with the matrix: \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R) \). This symmetry can also be used to write the 3-form as: \( G_3 \sim i e^{\phi/2}(F_3 + \tau H_3) \), which is more standard in type IIB string theory [note, we use the Einstein frame, which explains the factor \( e^{\phi/2} \)].

In the KK reduction one splits first the 10-dimensional space into the 4-dimensional external space \( \chi \) and the internal space \( \mathcal{Y} \) with the coordinates \( x^M = \{ x^\mu, y^m \} \) \((\mu, \nu, ..., = 0 \ldots 3, m = 4 \ldots 10)\), followed by an integration over the internal coordinates. This is straightforward if the fields do not dependent on the internal coordinates \( y^m \). In general however, this is not the case and one has to make a Fourier expansion of the fields in a complete set of harmonic eigenfunctions on the internal manifold. From the 4-dimensional point of view the higher Fourier modes correspond to massive excitations and only these excitations depend on the internal coordinates. The assumption that the fields do not depend on the internal coordinates is therefore equivalent to a truncation of the KK-spectrum on the massless sector. This is consistent only if the massless KK-fields do not act as sources for massive KK-fields, which is ensured if the Fourier expansion of the Lagrangian does not contain a coupling that is linear in the massive fields. Otherwise, the equations of motion are not solved by simply setting the massive modes to zero. This question can be answered if the internal metric is explicitly known – although the concrete calculation might be very involved. It has been done so far only for very few examples as e.g. for the sphere compactification of 11-dimensional supergravity with 4-form [39]. If the internal metric is not known, which unfortunately is often the case, the consistent truncation on the massless sector becomes a highly non-trivial problem. But one can formulate the truncation nevertheless in a weaker sense, where one admits the disturbing couplings so that massless fields appear as sources for massive fields, but only via higher derivatives. In this case, one cannot simply set to zero the massive fields, but has to integrate them out. Due to the derivative coupling, this does not change the effective low energy Lagrangian up the two derivative level but modifies higher derivatives terms (which have been neglected anyway in the 10-dimensional Lagrangian). This procedure of integrating out of massive modes is only consistent if it can be done in a finite number of steps, ie. if there are sources for only finite number of massive fields. If this is not the case, the theory remains genuinely higher dimensional and one cannot perform a KK reduction. We will assume that this is not the case for our models. For details we refer to [40] where the analysis for a Calabi-Yau space is given.

If there are no fluxes, the internal space has to be Ricci-flat and supersymmetry requires that the space has to have restricted holonomy. For a 6-dimensional internal space this means that the holonomy can be at most SU(3), ie. strictly inside SO(6) and the space is called Calabi-Yau. The amount of supersymmetry in four dimensions depends on the number of Killing spinors on the internal space and because the Killing spinors have to be singlets, this number is directly related to the holonomy. For SU(3) it is exactly one internal spinor and therefore 1/4 of supersymmetry is broken; note a 6-dimensional flat space allows four independent Weyl spinors [transforming under the 4 of \( SU(4) \cong SO(6) \)]. After the reduction, the resulting 4-dimensional theory will have eight supercharges or has \( \mathcal{N}=2 \), D=4 supersymmetry and therefore supergravity can couple to vector and hyper multiplets; in ungauged supergravity tensor multiplets are dual to hyper multiplets. Each multiplet has four bosonic degrees of freedom: the graviton and graviphoton in
the gravity multiplet; a vector field and one complex scalar in the vector multiplets and each hyper multiplet contains four real scalar fields.

The moduli are the zero modes of the scalar fields and appear in KK reduction from two sources: from the internal metric components and from the RR/NS-forms. They are in one-to-one correspondence to harmonic forms on the internal space which, for a 6-dimensional space with SU(3) holonomy, are equivalent to deformations of the complex structure and Kahler class as well as to the gauge symmetries of the RR/NS-form potentials. For example, an n-form $\omega_n \neq d\omega_{n-1}$ on the internal space with: $d\omega_n = d^*\omega_n = 0$ gives a scalar field $\phi = \phi(x)$ in the Fourier expansion $C_n = \phi(x) \omega_n + \ldots$, where $C_n$ is an n-form potential. The corresponding field strength is given by $F_n = d\phi \wedge \omega_n$ and therefore a modulus in the low energy theory is given by the constant part of $\phi$. The appearance of this modulus is of course a consequence of the gauge symmetry for the form field. The moduli coming from the internal metric are related to deformations of the Kahler 2-form $J$ and the holomorphic 3-form $\Omega$.

To be concrete the expansion goes as follows; see [37] for more details. We denote a complete basis of harmonic 3-forms by $\{\chi^k, \bar{\chi}_k\} \in H^{(3)}(Y)$ and the basis of harmonic 2-forms by $\{\omega_a\} \in H^{(2)}(Y)$, which by Hodge duality are related to 4-forms spanning $H^{(4)}(Y)$. Apart from the trivial 0-form and the volume form, there are no (regular) 1- nor 5-forms on a Calabi-Yau manifold and we have to expand all fields in these two sets of forms.

On the IIA side, we expand the RR-3-form and NS-2-form as well as the Kahler class as follows:

$$C_3 = u^k \chi_k + A^a \wedge \omega_a + cc, \quad B + iJ = z^a \omega_a \quad (2.8)$$

where we denote the Kahler class with $J$ and $A^a$ are the 4-dimensional KK gauge 1-forms. Since there are no harmonic 1-forms on a Calabi-Yau space, $C_1$ does not give rise to a KK-scalar and becomes the graviphoton upon dimensional reduction. The complex scalars $u^k$ together with another set of complex scalars $v^k$ related to complex structure deformations enter hyper multiplets and the complex scalars $z^a$ enter vector multiplets. Of the simplest cases are the rigid Calabi-Yau spaces that do not have complex structure deformations and hence none of the scalars $\{u^k, v^k\}$ are present. In this case, only one hyper multiplet is non-trivial and this is the so-called universal hyper multiplet, which consists of the dilaton $\phi$, the external $B$-field component (dualized to a scalar) and the (3,0) and (0,3) part of the RR-3-form. Although there are (rigid) Calabi Yau spaces without any harmonic (2,1)- and (1,2)-forms (e.g. $\mathbb{T}^4/\mathbb{Z}_3$ has $h^{(2,1)} = 0$), we should stress that the (3,0)- and (0,3)-forms are always non-zero. Explicit BPS solutions in 4 dimensions have a timelike or null Killing vector and the stationary case has been discussed in [41].

On the IIB side the NS-2-form and RR-forms are decomposed as follows:

$$C_4 = A^k \wedge \chi_k + cc, \quad *^a C_4 + i C_2 = v^a \omega_a, \quad B + iJ = u^a \omega_a \quad (2.9)$$

($^a C_4$ denotes the 6-dimensional Hodge dual of the 4-form). Now, the four scalars of a hyper multiplet are given by the two complex scalars: $\{u^a, v^a\}$. Since the Kahler class is always non-trivial for a Calabi-Yau space, we obtain at least two hyper multiplets. One of them is the universal hyper multiplet which

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1 As usual we distinguish between imaginary selfdual and anti-selfdual forms, which are complex conjugate to each other.
now comprises the axion-dilaton $\tau$ combined with the (dualized) external components of the NS- and RR-2-forms. On the IIB side the scalars in the vector multiplets are related to deformations of the complex structure, i.e. come from the components of the internal metric. For a rigid Calabi-Yau (i.e. $h^{(2,1)} = 0$) all vector multiplets are trivial and, apart from the gravity multiplet, we have only hyper multiplets (and the typical BPS solutions are the instanton solutions as discussed in [42, 43]).

Since the complex structure and Kahler class deformations for a given Calabi-Yau are not related to each other, the corresponding moduli spaces appear as a direct product in the low energy supergravity: $\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$, where the vector multiplet moduli space $\mathcal{M}_V$ is a special Kahler manifold and the scalars in hyper multiplets parameterize a quaternionic Kahler space $\mathcal{M}_H$. Since these Kaluza-Klein reductions do not give rise to a potential, the scalars can take any constant value in the vacuum. In order to lift this moduli space by giving a vev to the scalar fields, one has to generate a potential upon compactification. This can be done by taking into account nonzero fluxes.

### 3 Fluxes, Scherk-Schwarz reduction and gauged supergravity

If we include fluxes, the dimensional reduction or compactification becomes more involved. On the other hand since flux compactifications are related to gauged supergravity not only the complete 4-dimensional Lagrangian can be constructed but also the issue of moduli stabilization can be addressed. We are now following in part the literature as given in [1, 6].

One may argue that in the vacuum all fields should be trivial and the metric is (Ricci-) flat. This strong restriction is not justified, and non-zero values of RR- and NS-fields can still be considered as a viable vacuum configuration – at least as long as they respect the 4-dimensional Poincaré symmetry. If so, we are dealing with compactifications in presence of fluxes and one can distinguish between gauge field and metric (or geometric) fluxes, that are related by supersymmetry. This means, that the form fields (2.1) or (2.4) are non-trivial in the vacuum, but nevertheless obey the equations of motion and Bianchi identities. These fluxes generate a non-zero energy-momentum tensor and hence the internal metric is in general not Ricci-flat; the resulting geometries can be quite complicated (as we will see later).

To make this more explicit let us note, that in the simplest case gauge field fluxes can be generated by a linear dependence (on the internal coordinates) of the KK scalars coming from gauge fields in (2.8) or (2.9). The gauge symmetry implies that these scalars appear only via derivatives in the Lagrangian/equations of motion and hence, a linear dependence on the internal coordinates leaves the Lagrangian still independent of the internal coordinates and one can integrate over them. This is known as the (generalized) Scherk-Schwarz reductions, which can be applied to any global symmetry and especially to those KK scalars, that parameterize an isometry of the moduli space. If the scalar field comes from an internal metric component, the underlying global symmetry is related to specific coordinate transformations and a linear dependence of these scalars is also known as metric fluxes or twisting. In general, this procedure does not commute all supersymmetry transformations and hence supersymmetry is at least partially broken. In the original Scherk-Schwarz reduction this was done with respect to a fermionic phase transformation, which did not commute with the supersymmetry transformation and hence supersymmetry was broken completely [26].
In the case here, where we apply it to isometries of the moduli spaces, some supersymmetries remain unbroken, or in other words, some supersymmetry transformations commute with the (generalized) Scherk-Schwarz reduction. Nevertheless, masses for scalars and vectors are generated and hence (part of) the moduli space is lifted. More details on these reductions are given in the literature, see [27] – [35].

From the lower dimensional point of view Scherk-Schwarz reductions correspond to a gauging of the corresponding global symmetry and following [29] let us discuss a simple example. If we just keep the axion-dilaton coupling, the type IIB supergravity action reads

\[ S \sim \int \sqrt{g} \left[ R - \frac{g^{MN} \partial_M \tau \partial_N \bar{\tau}}{|\tau - \bar{\tau}|^2} \right] \]  

which exhibits, as part of the SL(2,R) symmetry, the axionic shift symmetry

\[ \tau \rightarrow \tau + c \]

for any \( c = \text{const} \). In the Scherk-Schwarz reduction over one coordinate (say y) one assumes \( c = my \) and hence one writes

\[ \tau(x, y) = \tau(x) + my \]  

For the metric, one makes the usual KK-Ansatz

\[ ds^2 = e^{2\sigma}(dy + A_\mu dx^\mu)^2 + g_{\mu\nu}dx^\mu dx^\nu \]

where \( \partial_y \) is a Killing vector. The inverse metric becomes

\[ g^{MN} \partial_M \partial_N = e^{-2\sigma} \partial_y \partial_y + g^{\mu\nu}D_\mu D_\nu \]

with the covariant derivative

\[ D_\mu = (\partial_\mu - A_\mu \partial_y) \]  

Thus, the kinetic term yields

\[ \frac{g^{MN} \partial_M \tau \partial_N \bar{\tau}}{|\tau - \bar{\tau}|^2} = \frac{g^{\mu\nu}D_\mu \tau(x)D_\nu \bar{\tau}(x)}{|\tau(x) - \bar{\tau}(x)|^2} + \frac{m^2}{|\tau(x) - \bar{\tau}(x)|^2} e^{-2\sigma} \]

where the second term is a (run-away) potential and the covariant derivative in the kinetic term is

\[ D_\mu \tau = \partial_\mu \tau(x) - mA_\mu \]  

Therefore, the scalar field Re(\( \tau \)) is now charged with respect to the local shift transformations

\[ \tau \rightarrow \tau + c(x) \quad , \quad A \rightarrow A + \frac{1}{m}dc \]

and in the original metric the gauge transformation in A can be absorbed by a coordinate transformation \( y \rightarrow y + c(x) \). Obviously, the same result can be obtained by a gauging of the global shift symmetry in the reduced theory. The charged scalar, Re(\( \tau \)), does not enter the potential and represents a flat direction,
which is required by gauge invariance and which in turn can be used to gauge away the scalar giving a mass to the gauge boson (the kinetic term for \( \text{Re}(\tau) \) becomes a mass term for \( A_\mu \)). As we will see next there is also the dual situation where not a vector becomes massive, but an antisymmetric tensor becomes massive by “eating” a vector\(^2\). But before we come to this, let us note, that the run-away behavior of the potential, i.e. the absence of a fixed point, is related to the non-compactness of the gauged isometry and does not happen if the isometry is a U(1) action, that is not freely acting [44].

This Scherk-Schwarz reduction was related to the isometry \( \partial_y \) and generated an internal flux given by the 1-form: \( d_y \tau = mdy \) and as result the scalar field \( \text{Re}(\tau) \) became charged under the corresponding KK vector field. A general Calabi-Yau space has no isometries and the internal metric does not give rise to 4-dimensional vector fields, but nevertheless there is an analogous mechanism which relates flux compactification to gauged supergravity. To be concrete we follow now [3, 6], consider the type IIB case with fluxes for the NS-2-form \( \mathcal{B} \)

\[
B + iJ = u^a(x, y)\omega_a = [u^a(x) + c^a(y)]\omega_a.
\]

The coefficients \( c^a(y) \) are fixed by the requirement that the corresponding field strength yields a real internal 3-form (=flux), which can be expanded in the basis \( \{\chi^k, \chi_k\} \) with the constant coefficients \( m_k \), i.e. \( dc^a(y) \wedge \omega^n = m_k\chi_k = H^{\text{flux}} \) giving

\[
d(B + iJ) = du^a(x) \wedge \omega_a + (m_k\chi_k + cc).
\]

To keep the notation simple, we drop all indices \( (m^k \rightarrow m) \) and collecting the terms containing this mass deformation yields for the 5-form

\[
F_5 = dC_4 - \frac{1}{4} C_2 \wedge H = [dA - \frac{1}{4} m C^{(ext)}_2] \wedge \chi + cc + \cdots
\]

where \( C^{ext}_2 \) is the external component of the 2-form [note, a Calabi-Yau has no non-trivial 5-forms and therefore \( \omega \wedge \chi = 0 \)]. Now, the kinetic term for this 5-form yields exactly a massive 2-form coupling in 4 dimensions

\[
\left[ (dA)_{\mu\nu} - \frac{1}{8} m C^{(ext)}_{\mu\nu} \right]^2
\]

and this expression has to be dressed up with the metric of the complex structure moduli space (we suppressed the indices). This massive 2-form can be dualized to a massive vector, where the charged scalar is given by the dual of \( C^{(ext)}_2 \) and which enters the universal hyper multiplet. The potential comes now from the 3-form: \( H_3 = du \wedge \omega + m\chi + cc \), which yields after squaring a term: \( |m|^2 \equiv G^{ab}m_am_b \) (actually it might be useful to introduce here symplectic notation). It is an important property of type IIB compactifications in the presence of 3-form fluxes, that the potential is always positive definite (no-scale form) and therefore the only supersymmetric vacua are flat space vacua. This property is however lost for more general flux compactifications.

It is now straightforward to consider fluxes also for the 2-form \( \mathcal{C}_2 \), which makes the external components of the NS \( B \)-field massive. In general, one can also consider a combination of both 3-form fluxes, but due

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\(^2\) Note, in 4 dimensions a massive vector is dual to a massive tensor.
to the Chern-Simons terms, we cannot consider independent massive deformations with respect to both 2-forms. This is also reflected by the fact, that both shifts do not correspond to two commuting isometries on the quaternionic space parameterized by the universal hyper multiplet.

One should have expected that each flux compactification is related to a specific vacuum of gauged supergravity, although the concrete embedding might be involved. The opposite statement, ie. whether every vacuum obtained in gauged supergravity can be embedded into a specific flux compactification, is far from clear – most likely this is not possible. We have to keep in mind that one can also consider metric fluxes, which are related to Scherk-Schwarz reductions with respect to axionic scalars of the internal metric, which we did not discussed here, see [30, 32]. This becomes very involved, if one wants to do it explicitly and it might not be necessary because the realization of flux compactifications within gauged supergravity opens the possibility to understand the moduli stabilization within gauged supergravity and we shall summarize in the following some essentials.

The starting point is the Lagrangian of ungauged supergravity with $N=2$ supersymmetry in 4 dimensions, which is obtained from standard Kaluza-Klein reduction as discussed in the previous section giving rise to a continuous moduli space: $\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$, where $\mathcal{M}_V$ and $\mathcal{M}_H$ are parameterized by the scalars belonging to the vector multiplets and to the hyper multiplets, respectively. Potentials that are consistent with $N=2$ supersymmetry are obtained by performing a gauging of various global symmetries. There are two different types of gaugings, namely (i) one can either gauge some of isometries of the moduli space of ungauged $N=2$ supergravity or (ii) one can gauge (part of) the $SU(2)$ R-symmetry, which only acts on the fermions. We are interested in a gauging that generate a potential for both types of scalars, because we want to derive constraints for lifting the complete moduli space. In the following we will therefore discuss gaugings of isometries of $\mathcal{M}_H$ and refer to [45] for a detailed description of $N=2, D=4$ gauged supergravity.

Scalar fields in hyper multiplets parameterize a quaternionic Kahler manifold $\mathcal{M}_H$ and these spaces possess three complex structures $J^x$ as well as a triplet of Kahler two-forms $K^x$ ($x = 1, 2, 3$ denotes the $SU(2)$ index). The holonomy group of these spaces is $SU(2) \times Sp(n_H)$ and the Kahler forms are covariantly constant with respect to the $SU(2)$ connection. The isometries of $\mathcal{M}_H$ are generated by a set of Killing vectors $k_I = k^u_I \partial_u$

\[ q^u \rightarrow q^u + k^u_I \epsilon_I \]  

(3.4)

where “$I$” counts the different isometries and $q^u$ are the scalar fields of hyper multiplets. The gauging of (some of) the Abelian isometries gives gauge covariant derivatives $dq^u \rightarrow dq^u + k^u_I A^I$ so that the vector field become massive. In order to maintain supersymmetry, the gauging has to preserve the quaternionic structure, which implies that the Killing vectors have to be tri-holomorphic, which is the case whenever it is possible to express the Killing vectors in terms of a triplet of real Killing prepotentials $P_I$ as follows:

\[ K^x_{uv} h^v_I = -\nabla_u P^x_I = -\partial_u P^x_I - \epsilon^{xyz} \omega^y_u P^z_I \]  

(3.5)

where $\omega^y_u$ is the $SU(2)$ connection giving the Kahler forms by $K^x_{uv} = -\nabla_{[v} \omega^x_{u]}$. By using the Pauli matrices $\sigma^x$ one can also use a matrix notation: $P_I = \sum_{x=1}^3 P^x_I \sigma^x$. With these Killing prepotentials one
can define an SU(2)-valued superpotential by [46, 44]

\[ W^x = X^I \mathcal{P}_I^x \equiv X^I(z) \mathcal{P}_I^x(q) , \quad (3.6) \]

where \( \{z, q\} \) denote collectively the scalars from vector and hyper multiplets. A real valued superpotential can be defined as \( W^2 = e^K \det(W^x \sigma^x) \), where \( K \) is the Kahler potential of the special Kahler manifold \( \mathcal{M}_V \). Supersymmetric vacua are extrema of the real superpotential, which are equivalent to a covariantly constant superpotential \( W^x \). This gives as constraints for supersymmetric vacua

\[ \begin{align*}
(i) & \quad (\nabla_A X^I) \mathcal{P}_I^x = 0 , \\
(ii) & \quad X^I(\nabla_u \mathcal{P}_I^x) = K_{uv}^x (X^I k_u^v) = 0 \\
\end{align*} \quad (3.7) \]

where \( \{\nabla_A, \nabla_u\} \) denote the Kahler/SU(2)-covariant derivatives with respect to the scalars \( \{z^A, q^u\} \) in vector/hyper multiplets and \( X^I = X^I(z) \) is part of the symplectic section \( (X^I, F_I) \) [\( F_I \) is the derivative of the prepotential \( F(X) \) with respect to \( X^I \)].

For concrete models one has to know the moduli spaces, which are are more or less well known in the classical limit and many gaugings have been considered already. On the vector multiplet side, all corrections (perturbative and non-perturbative) are included in a prepotential, which on the 2-derivative level is holomorphic and homogeneous of degree two. Much less is known about quantum corrections for the hyper multiplet moduli space; the 1-loop correction has recently been found in [47] and instanton corrections are discussed in [43]. Classically, it is given by the coset space \( SU(2,1)/U(2) \), which is one of the few spaces that are quaternionic and Kahler at the same time and its Kahler potential can be written as

\[ K = - \log[S + \bar{S} - 2(C + \bar{C})^2] . \quad (3.8) \]

This coset space has two commuting Abelian isometries which are generated by the Killing vectors associated to shifts in the imaginary parts of \( S \) and \( C \) and their gauging has been discussed in [1, 46, 44, 48]. The gaugings of these two shift symmetries correspond exactly to the Scherk-Schwarz reductions that we discussed at the beginning of this section (the \( S \) shift corresponds to the \( \tau \) shift and the \( C \)-shift to the 3-form flux).

**Can one fix all moduli in gauged supergravity?**

In order to fix the moduli from the vector as well as hyper multiplet it was important that we gauged an isometry of the quaternionic space \( \mathcal{M}_H \) or equivalently to add fluxes (3-form flux on the IIB e.g.) which make a scalar of a hyper multiplet massive. This is only the minimal requirement, on top of this gauging one may also consider to gauge isometries of vector multiplet moduli space \( \mathcal{M}_V \). The resulting superpotential obtained from gauged quaternionic isometries was given in (3.6) with \( X^I = X^I(z) \) as the “electric” part of the symplectic section \( V = (X^I, F_I) \). It is a known problem, that gauged supergravity prefers the electric part and does not produces the magnetic part of the superpotential. But by taking into account also (massive) tensor multiplets, one can promote it to a manifestly symplectic expression [36]. An important property of this setup is however, that by a symplectic transformation one can always go into a strictly perturbative regime where all magnetic charges vanish so that the potential in (3.6) can always
be considered. This property that the electric and magnetic charges are mutually local is a consequence of
supersymmetric Ward identities [36].

We can now discuss the conditions of getting a complete lifting of the moduli space. A necessary
condition for this is that the variations of the hyperino and gaugino vanish for constant scalars which
yielded eqs. (3.7). The condition \((ii)\) is equivalent to the existence of a fixed point for the Killing vector
\[ k = X^I k_I \]
and the complete hyper multiplet moduli space is lifted if \(k\) has a NUT fixed point, i.e. if it represents a
point on \(\mathcal{M}_H\). This excludes by the way, axionic shift symmetries and requires a compact isometry [44].
The fixed point set of a Killing vector field is always of even co-dimension, which is related to the rank
of the 2-form \(dk\) calculated on the fixed point set. For a related discussion see [49]. In fact, if the rank
is maximal, i.e. \(\det(dk) \neq 0\), the fixed point set is in fact a point on the manifold and \(dk\) parameterizes a
rotation around the fixed point. Otherwise, any zero mode of \(dk\) would parameterize a shift symmetry of
the fixed point set and hence if \(\det(dk) = 0\), the potential will have some flat directions. Therefore, we get
the following two conditions for lifting the hyper multiplet moduli space
\[ |k| = 0 , \quad \text{with:} \quad \det(dk) \neq 0 . \] (3.9)

If we can find a Killing vector that satisfies both conditions, the hyper multiplet moduli space will be
lifted in the vacuum. We should place a warning here. Although, the isometries on the classical level
are well understood it is unclear whether the full quantum corrected moduli space has isometries at all,
which makes the moduli fixing issue obscure – at least from the supergravity point of view. But we do
not want to speculate here about the quantum moduli space for hyper multiplets and shall instead continue
with the discussion of the second condition in (3.7). If the hyper scalars are fixed, the Killing prepotentials
are some fixed functions of the scalars of the vector multiplets, i.e. \(P_I = P_I(q(z))\) and hence they vary
over \(\mathcal{M}_V\). If \(P_I\) would be constant, only one vacuum can occur, namely at the point where this constant
symplectic vector is a normal vector on \(\mathcal{M}_V\) [50]. But since \(P_I\) varies now, it might become normal at
different points, related to the appearance of multiple critical points as eg. the ones discussed in [44]. If we
calculate the second covariant derivatives on \(\mathcal{M}_V\) at this fixed point, i.e. \(\nabla_A \nabla_B X^I P^*_I\) and use relations
from special geometry\(^3\) we find that all these critical points are isolated – at least as long as the metric does
not degenerate. Therefore, there are no further constraints from the vector multiplet moduli space and the
crucial relations that have to be realized are the ones in (3.9).

How about supersymmetric flat space vacua with all moduli fixed?\(^4\) If the matrix \(W^x \sigma^x\) does not
degenerate, this is only possible if we impose the additional requirement that \(W^x\) vanish in the vacuum
and this implies that the Killing prepotentials have to vanish, i.e. \(0 = P^x \equiv K^x u_k \partial^u k^v\). We are interested
in the case where all moduli are fixed and hence \(dk|_{|k|=0}\) has to be an orthogonal matrix. Recall, the
holonomy of a quaternionic space was \(SU(2) \times Sp(n_H)\) and since the Kahler forms \(K^x\) are \(Sp(n_H)\)
singlets, the Killing prepotentials vanish if the rotation parameterized by \(dk|_{|k|=0}\) is in a subgroup of

\(^3\) Because: \(\nabla_A \nabla_B X^I \sim g_{AB} X^J\).

\(^4\) I am grateful to Gianguido Dall’Agata for a discussion on this point.
$Sp(n_H)$ and leaves the $SU(2)$ part invariant. We leave it open, whether there exist an appropriate Killing vector $k$ obeying all constraints.

In the discussion so far we did not mention the fact that the potential obtained in gauged supergravity is always independent of the charged scalar field(s) and we have to ask whether this indicates some flat directions of the potential. There are two reasons why this does not spoil our discussion so far. On one hand, if the fixed point set is zero-dimensional (i.e. a point) this flat direction is only an artificial angular coordinate on the moduli space. On the other hand, as we mentioned already before, the charged scalars “can be eaten” by the vector fields giving them a mass that corresponds to the eigenvalues of $(h_{uv}k^u_j k^v_j)|_{|k|=0}$. So, there are no moduli related to these scalars anymore.

In gauged supergravity the problem of fixing the moduli is rather well formulated. The question is however, to embed these models into string or M-theory, i.e. to obtain the potential by a suitable dimensional reduction. In addition, although these reductions may lift the moduli space, it is not granted that we will obtain a unique vacuum and we can end up with a landscape of string vacua [51]. Although this conclusion was reached only for specific fluxes and it is not inevitable for (most) general fluxes, it may happen that we have to rely, at least to a certain extend, on an anthropic selection for choosing the vacuum in which we live [52]. Adopting this philosophy, the problem with the cosmological constant disappears. In gauged supergravity this ambiguity is related to the different ways of gauging a given isometry with different U(1) gauge fields. On the other hand, if one takes into account general fluxes on the type IIA side, the vacuum is rather unique [18] and one may wonder whether the landscape is an “artifact” of Calabi-Yau compactifications of string theory. This brings us to another approach to address the consequences of fluxes and which allows directly to determine the resulting deformed geometry.

4 Deformed geometry and $G$-structures

The approach to flux compactifications via gauged supergravity had the advantage to yield the explicit potential and one can therefore not only investigate supersymmetric vacua, but explore also possible de-Sitter vacua. The drawback is however that this approach does not yield the 10-dimensional geometry and to obtain this flux-deformed geometry one has to solve directly the 10-dimensional supersymmetry constraints.

Supersymmetry exchanges fermionic with bosonic degrees of freedom and in a supersymmetric vacua with trivial fermions, the fermionic variations have to vanish [the variations of the bosonic field vanish identical for trivial fermionic fields]. For type II supergravity, these are the gravitino $\delta \Psi_\mu$ (spin 3/2) and dilatino $\delta \lambda$ (spin 1/2) variation. On the IIA side we can combine both Majorana-Weyl spinors of opposite chirality to a general Majorana spinor and the variations read in the string frame [53]

\begin{align}
\text{IIA:} \quad \delta \psi_M &= \left\{ D_M + \frac{1}{8} H M_1 \Gamma_{11} + \frac{1}{8} e^\phi \left[ m \Gamma_M + F^{(2)} \Gamma_M \Gamma_{11} + F^{(4)} \Gamma_M \right] \right\} \epsilon, \\
\delta \lambda &= \left\{ \partial \phi + \frac{1}{12} H \Gamma_{11} + \frac{1}{4} e^\phi \left[ 5 m + 3 F^{(2)} \Gamma_{11} + F^{(4)} \right] \right\} \epsilon
\end{align}

(4.1)

where $\epsilon$ is the Killing spinor which is also Majorana. On the IIB side, both Majorana-Weyl spinors have the same chirality and can be combined into a single (complex) Weyl spinor so that the variations can be
written in the Einstein frame$^5$ as $[54, 55]$

\[
\text{IIB : } \delta \Psi_M = \left[ D_M - \frac{i}{2} Q_M + \frac{i}{480} F^{(5)} \Gamma_M \right] \epsilon - \frac{1}{96} \left[ G^{(3)} \Gamma_M + 6 G^{(3)} \Gamma_M \right] \epsilon^*, \quad \delta \lambda = i P e^* - \frac{i}{24} G^{(3)} \epsilon
\]

(4.2)

where all fields were introduced in Section 2. In these variations all indices are contracted with $\Gamma$-matrices, i.e. we used the abbreviations

\[
\partial \equiv \Gamma^M \partial_M, \quad F^{(2)} = F^{(2)}_{MN} \Gamma^{MN}, \quad H = H_{PQR} \Gamma^{PQR}, \quad H_M = H_{MPQ} \Gamma^{PQ}, \quad \text{etc. (4.3)}
\]

[see also (2.2) and (2.5) for the definition of the fields].

\textbf{Killing spinors and $G$-structures}

The number of unbroken supersymmetries is given by the zero modes of these equations, i.e. the number of Killing spinors for which these variations vanish. The 10-dimensional spinor can be expanded in all independent internal and external spinors so that we can always write

\[
\epsilon = \theta_l \otimes \eta_l
\]

(4.4)

with $\theta_l$ and $\eta_l$ denoting the four- and six-dimensional spinors, respectively. On the IIA side, $\epsilon$ has to be Majorana whereas on the IIB side it is Weyl. In order to have a well-defined KK-reduction, the internal spinors have to be singlets under the structure group $G \subset SO(6)$. The internal space can admit up to four Weyl spinors which transform under the 4 of $SU(4) \simeq SO(6)$ and therefore are singlets only under the trivial identity (i.e. structure group has to be trivial). On the other hand, no singlet spinors are possible if the structure group is the whole $SU(4)$; for $SU(3)$ we find one singlet spinor, which appears as the singlet in the $SU(3)$ decomposition: $4 \rightarrow 1 + 3$. In the same way, if we have two independent internal spinors, one can find an $SU(2) \subset SU(3)$ under which both are singlets. If we do not take into account brane sources, the Killing spinors have to be globally well-defined and the existence of these spinors is in one-to-one correspondence to the existence of globally well-defined differential forms. These differential forms which are also singlet under the structure group $G$, define $G$-structures and can be written as fermionic bi-linears

\[
\Lambda^{(n)}_{kl} = \eta^*_k \gamma^{(n)} \eta_l, \quad \Sigma^{(n)}_{kl} = \eta^*_k \gamma^{(n)} \eta_l
\]

(4.5)

where $\gamma^{(n)} = \gamma_{m_1 m_2 \cdots m_n}$. By using the Killing spinor equations one derives differential equations for the internal spinors which in turn give differential equations for these forms, see $[7]$. If we are interested in a 4-dimensional flat vacuum, the external spinor is covariantly constant. The internal spinor on the other hand, cannot be covariantly constant (due to the fluxes) and solving the corresponding differential equation yields not only the spinor but also the geometry of the internal space. Note, by imposing the constraint that the spinor is a singlet under the group $G$, the Killing spinor equations give a set of first order differential equations for the vielbeine.

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$^5$ Which is more appropriate on the IIB side, because it makes the $SL(2,R)$ symmetry manifest.
Having only one internal spinor, SU(3)-structures are given by the 2-form and a 3-form
\[ \eta^I \gamma_{mn} \eta = i J_{mn}, \quad \eta^T \gamma_{mnp} \eta = i \Omega_{mnp} \] (4.6)
with \( 1 = \eta^\dagger \eta \) where \( J \) is a symplectic form with \( J^2 = -I \) and can be used to define (anti) holomorphic coordinates and \( \Omega \) is then the holomorphic 3-form. All other fermionic bi-linear vanish as result of identities for 6-d \( \gamma \)-matrices and hence no further (regular) singlet forms can be build. Being an SU(3) singlet spinor, \( \eta \) satisfies the projectors
\[
\begin{align*}
(\gamma_m + i J_{mn} \gamma^n) \eta &= 0, \\
(\gamma_{mn} - i J_{mn}) \eta &= i \frac{1}{2} \Omega_{mnp} \gamma^p \eta^*, \\
(\gamma_{mnp} - 3i J_{[mn} \gamma_{p]}) \eta &= i \Omega_{mnp} \eta^* .
\end{align*}
\]
(4.7)
If the spinor is covariantly constant, these forms are closed and the structure group is identical to the holonomy – if not, the holonomy is not inside SU(3) and the space cannot be Calabi-Yau (not even complex in general). The failure of the structure group to be the holonomy is measured by torsion classes. Following the literature [56, 7, 9, 10], one introduces five classes \( \mathcal{W}_i \) by
\[
\begin{align*}
dJ &= \frac{3}{4} (\mathcal{W}_1 \bar{\Omega} - \bar{\mathcal{W}}_1 \Omega) + \mathcal{W}_3 + J \wedge \mathcal{W}_4 , \\
d\Omega &= \mathcal{W}_1 J \wedge J + J \wedge \mathcal{W}_2 + \Omega \wedge \mathcal{W}_5 \quad (4.8)
\end{align*}
\]
with the constraints: \( J \wedge J \wedge \mathcal{W}_2 = J \wedge \mathcal{W}_3 = \Omega \wedge \mathcal{W}_3 = 0 \). Depending on which torsion components \( \mathcal{W}_i \) are non-zero, one can classify the geometry of the internal space. E.g., if only \( \mathcal{W}_1 \neq 0 \) the space is called nearly Kahler, for \( \mathcal{W}_2 \neq 0 \) almost Kahler, the space is complex if \( \mathcal{W}_1 = \mathcal{W}_2 = 0 \) and it is Kahler if only \( \mathcal{W}_5 \neq 0 \).

Before we will give examples, let us also comment on the SU(2) case, we refer to [19, 18] for more details. In this case, we have two Weyl spinors \( \eta^1 \) and \( \eta^2 \) and we can define three 2-forms, that are supported on a 4-dimensions subspace, and one holomorphic vector, which defines a fibration over this 4-dimensional base space. The three 2-forms come on an equal footing and one can pick one of them to use it as symplectic form and the remaining two can be combined into one holomorphic 2-form so that the 6-dimensional geometry is fixed by the triplet \( (v, J^{(0)}, \hat{\Omega}^{(2,0)} \) ). If there are no fluxes, both spinors are covariantly constant the internal space has SU(2) holonomy and is therefore equivalent to \( \mathbb{T}^2 \times K3 \), where the three (anti-selfdual) 2-forms are supported on K3 and the \( \mathbb{T}^2 \) is identified by the vector field. Naively, one would argue that SU(2) structures can be relevant only for the very specific examples where the Euler number of the 6-dimensional space vanishes; because only in this case a globally well-defined vector field exist. But one has to take the statement of “globally well defined” with a grain of salt, because any wrapped brane can violate this requirement so that the singular behavior is related to the location of the brane source. In fact, instanton corrections coming from string world sheets and/or instantonic 3-branes (which are wrapped on a 4-cycle) are typical examples that imply SU(2) instead of SU(3) structure. Note, the structures are related to certain fibrations of the manifold and have no direct consequences of the amount of supersymmetry.

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For the 4-dimensional external space, supersymmetry requires that it has to be, up to warping, flat or anti-deSitter and hence we make the Ansatz for the metric

\[ ds^2 = e^{2A(y)} \left[ g_{\mu\nu} dx^\mu dx^\nu + h_{mn}(y) dy^m dy^n \right] \]  

(4.9)

where \( g_{\mu\nu} \) is either flat or \( AdS_4 \) and \( h_{mn} \) is the metric on \( \mathcal{Y} \) and the warp factor depend only on the coordinates of the internal space. In the vacuum all off-diagonal terms should vanish and the fluxes should have only internal components or are proportional to the 4-dimensional volume form. These constraints are required by 4-dimensional Poincaré invariance.

Let us now discuss two flux vacua with SU(3) structure in more detail.

**Type IIA flux vacuum with SU(3) structure**

On the type IIA side the list of literature about flux vacua is not as long as on the IIB side, which is in part due to the stronger back reaction of fluxes on the geometry and the difficulties in formulating quantized string theory on these geometries. The internal geometry on the IIB side on the other hand, remains complex which on the IIA side this is not the case. We want to summarize here a configuration that has been presented in [18, 23, 15] (for related discussions see also [8, 13, 16, 22, 23]).

In order to be consistent with the metric Ansatz and to preserve the Poincaré symmetry in four dimensions, we allow for (general) flux components in the internal space \( \mathcal{Y} \) combined with a 4-form flux proportional to external volume form yielding a Freud-Rubin parameter \( \lambda \):

\[ F^{(2)} = \frac{1}{2} F^{(2)}_{mn} dy^m \wedge dy^n, \quad H = \frac{1}{3} H_{mnp} dy^m \wedge dy^n \wedge dy^p, \]
\[ F^{(4)} = \lambda dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + \frac{1}{4} F^{(4)}_{mnpq} dy^m \wedge dy^n \wedge dy^p \wedge dy^q. \]  

(4.10)

Since the type IIA supergravity is non-chiral we can combine both Majorana-Weyl spinors into one Majorana spinor and can take as spinor Ansatz for an \( N=1 \) vacuum (with SU(3) structures)

\[ \epsilon = (a\theta + b\theta^\ast) \otimes \eta + cc = \theta \otimes (a\eta + b^\ast \eta^\ast) + cc. \]  

(4.11)

There are two special cases: if \( ab = 0 \) the 4-dimensional spinor is Weyl and if \( b = a^\ast \) we have a Majorana spinor. The Killing spinor equations (4.1) can be solved by employing the relations (4.7) and one finds three solutions.

(i) If \( \eta \) is Majorana-Weyl, ie. \( ab = 0 \)

For this spinor Ansatz, the mass and all RR-fields have to vanish:

\[ F_{mn} = G_{mnpq} = W = dA = m = 0 \]  

(4.12)

and only the fields from the NS-sector are non-trivial

\[ J \wedge H = * d\phi, \quad H \wedge \Omega = 0. \]  

(4.13)

One might have expected this result, because the NS-sector is common to all string models, and a common solution can only be described by one 10-dimensional Majorana-Weyl (Killing) spinor. The non-zero
components of the 3-form $H$ can be decomposed with respect to SU(3) representations, i.e. we have a $(3, 0)$, $(2, 1)$, $(2, 1)_0$ part, where the subscript indicates the non-primitive part (i.e. the components encoded in $H \wedge J \neq 0$). The real forms are of course the real and imaginary part of this projected forms. As solution one finds, the holomorphic part $H^{(3,0)}$ has to vanish, $H^{(2,1)}_0$ fixes the dilaton and if the primitive part $H^{(2,1)}$ is non-zero the internal space is non-Kahlerian. A prototype example that solves these equations is the NS5-brane supergravity solution, but there are also other examples [57, 10, 12, 21].

(ii) If $\eta$ is Majorana, i.e. $a = b^*$ and $m = 0$

This massless case can be lifted to 11-dimensional dimensions and the only solution that one finds lifts the internal space to a $G_2$-holonomy space, i.e. only the 2-form is non-zero and the 4-form is trivial. This is the solution discussed in [8] which relates the warp factor and the 2-form by a monopole equation

$$e^A dA = -\frac{1}{8} e^{2\phi} * (F \wedge \bar{\Omega}) , \quad \phi = -3U ,$$

$$0 = F^{(4)} = H \wedge \Omega = F^{(2)} \wedge J . \quad (4.14)$$

Obviously, intersecting 6-branes is the prototype solution in this class and the relation to $G_2$-holomony spaces in M-theory without any 4-form flux, identifies the moduli space of this configuration with the moduli space of the corresponding 7-manifold and as before, this flux background does not fix all moduli.

(iii) If $\eta$ is Majorana, i.e. $a = b^*$ but $m \neq 0$

This is the generic situation; also if $a \neq b^*$, one gets an equivalent solution. In this case all fluxes are non-zero and the dilaton and warp factor are constant [18]

$$0 = d\phi = dA , \quad H = H_0 \text{Im} \Omega , \quad (4.15)$$

$$F^{(2)} = F_0^{(2)} J , \quad F^{(4)} = F_0^{(4)} J \wedge J , \quad (4.16)$$

with the coefficients given by

$$F_0^{(2)} = -\frac{2}{9} \lambda , \quad F_0^{(4)} = \frac{m}{20} , \quad H_0 = -\frac{2m}{5} e^{\phi} , \quad \sqrt{\frac{85}{2}} \lambda = 9m . \quad (4.17)$$

Hence, the dilaton is fixed by the ratio of the (quantized) fluxes

$$e^{\phi} = -\frac{H_0}{8 G_0} . \quad (4.18)$$

In this case the external space cannot be flat, but must be anti-deSitter with a (negative) cosmological constant

$$\Lambda = -e^{4\phi} m^2 = -\left( \frac{H_0}{8 G_0} \right)^4 m^2 .$$

The geometry of the internal space is nearly Kahler which is equivalent to weak SU(3) holonomy and these spaces can be defined by the differential equations

$$dJ \sim \text{Im} \Omega , \quad d\Omega \sim J \wedge J .$$

This means that for nearly Kahler spaces only the first torsion class $W_1$ is non-trivial and moreover, the cone over any nearly Kahler space gives a $G_2$ holonomy space [58]. This can be used to construct explicit examples.
The simplest spaces of weak SU(3) holonomy is $S^6$ or $S^3 \times S^3$; other regular examples are the twistor spaces over $S^4$ or $\mathbb{CP}^2$. These spaces have no geometrical (closed string) moduli and therefore can serve as a starting point for exploring cosmological implications as the KKLT scenario on the IIB side [59]. In doing this one has to wrap (anti) branes around supersymmetric (calibrated) cycles and the most interesting candidates on the IIA side are wrapped (anti) D6-branes, which can wrap cycles of the space $S^3 \times S^3$. The anti-D6-branes are of course a source for (negative) RR-fields and to ensure (meta) stability the vacuum should not contain the corresponding flux (ie. $dC_1 = 0$) so that these anti-D6-branes cannot directly decay with the background flux. Since the vacuum has only a massive B-field flux, the anti-6-branes cannot decay directly, but only due to the non-perturbative process where the anti-D6-branes breaking-up leaving anti-NS5-branes at their endpoints. These anti-5-branes can then decay with the NS-B-field flux. Therefore, we should expect, that meta-stable deSitter vacua should exist in the same way as on the IIB side. Moreover, since the space $S^3 \times S^3$ has three supersymmetric 3-cycles, which intersect at an SU(3) angle, the intersection of three anti-D6-branes supports chiral matter and if one wraps an equal number of branes no orientifold projections are necessary [60].

We should add, that the solution presented here may have a generalization, were further components of the 2-form are non-zero. As shown in [23] additional non-primitive (1,1) components of $F^{(2)}$ do not break supersymmetry and as consequence the internal space is not nearly Kahler anymore. But unfortunately an explicit example is not known yet.

**Type IIB fluxes with SU(3) structure**

The type IIB side has been discussed in the literature already extensively, see [2, 4, 5, 11, 14, 19, 20] and we want to summarize here only some aspects. Again we admit only fluxes that are consistent with 4-dimensional Poincaré symmetry, ie. they have components along the internal space with the only exception of the 5-form, that has to have components along the external space; required by the self-duality.

An important property on the IIB side is, that as long as one keeps SU(3) structures, the vacuum has to be flat, ie. a cosmological constant can be generated by fluxes [24]. This may indicate, that SU(3) structures always yield potentials of the no-scale form which are positive definite, but this needs to be verified for the most general fluxes consistent with SU(3) structures. Recall, the no-scale structure is only an approximation and corrections (quantum corrections, D3-instanton corrections etc.) do not respect this property and yield anti deSitter vacua and therefore these corrections have to break the SU(3) down to SU(2) structures [cp. also the discussion after eq. (4.8)]. Let us stress that we are using here only supersymmetry and therefore our approach is valid for classical and quantum geometry as long as at least four supercharges remain unbroken!

Depending on the concrete form of the spinor one finds again different solutions and the most general spinor, consistent with SU(3) structures can be written as

$$
\epsilon = a \left[ \theta \times \eta \right] + b^* \left[ \theta^* \times \eta^* \right]
$$

(4.19)

where both spinors are chiral and $a$ and $b$ are complex coefficients; we refer to [14, 19] for a classification of the different spinor Ansätze. In comparison to the IIA spinor in (4.11), this spinor is Weyl, but in general
not Majorana – only for \( a = b \), the 10-dimensional spinor \( \epsilon \) is Majorana-Weyl which gives again the NS-sector solution that we encountered in (i) on the IIA side. There is another special sub-class of solution, namely if \( ab = 0 \) which was explored to a large extend in [2, 5]. The special interest in this case comes due to the fact, that it still allows the internal space to be Calabi-Yau and in the following we will summarize this case as well as present the solution for the general case, which has recently been found in [24].

(i) Calabi-Yau flux compactifications

Let us summarize the solution in [2] and consider the spinor

\[
\epsilon = \theta \times \eta.
\]

Using the chirality of \( \eta \), both terms in the dilatino variation \( \delta \lambda \) in (4.2) have to vanish separately, yielding two equations

\[
G \eta = 0 \quad , \quad P \eta^* = 0
\]

from which one infers, due to the relations (4.7), that in holomorphic coordinates: \( G_{abc} = G_{ab}^b = 0 \) and \( P \) is a holomorphic vector. Similarly, in the gravitino variation, both terms have to vanish separately yielding the constraint for the 3-form flux

\[
G \eta^* = G_m \eta^* = 0
\]

giving: \( G_{\bar{a}\bar{b}\bar{c}} = G_a^{\bar{b}\bar{c}} \Omega_{\bar{a}\bar{b}\bar{c}d} = 0 \) and therefore the 3-form flux has to be primitive (ie. \( G \wedge J = 0 \)) and of (2,1) type. The Ansatz for the (external) 5-form components are

\[
F_5^{(ext)} = 5 e^{-4A} \text{vol}_4 \wedge dZ
\]

and selfduality property fixes the remaining components. Now, the gravitino variation vanishes for a holomorphic \( P \) only if

\[
Z \sim e^{4A}.
\]

Finally, the geometry is fixed by the differential equation obeyed by Weyl spinor which becomes in this case

\[
\nabla_m \eta = \frac{i}{2} Q_m \eta
\]

where \( Q_m \) was defined in (2.6). By inspecting the torsion components in (4.8), this spinor equation implies that only \( V_5 \neq 0 \) and all others vanish, which means that the space is Kahler and \( Q \) is the Kahler connection. Since \( Q \) is a specific function on one complex coordinate \( \tau \), only specific Kahler geometries are possible, namely Kahler spaces related to wrapped 7-branes (which can be identified with singularities of the holomorphic axion/dilaton). If the vector \( Q \) vanishes and the axion-dilaton is trivial, the spinor is covariantly constant and hence the space can have at most SU(3) holonomy and is Calabi-Yau. Since the corresponding potential has the no-scale structure, there is at least one un-fixed modulus. This can only be
fixed if one considers a flux vacuum yielding a 4-dimensional anti-deSitter vacuum, which however, was not compatible with SU(3) structures and one has instead to consider SU(2) structures. We have also to keep in mind, that fluxes that generate a 4-dimensional cosmological constant will always, if one includes the back reaction, render the internal space to a non-Kahler geometry as it should be for any compactification that fixes all moduli.

(ii) General flux compactifications

Interestingly it is possible to solve the Killing spinor equations without making any assumptions [24]. The type IIB supergravity in our notation has a local U(1) symmetry which becomes manifest if we define the phase \( e^{2i\theta} = \frac{1+i\bar{\tau}}{\sqrt{2}} \) and write the fields in (2.5) as [55]

\[
Q_M = \partial_M \theta - \frac{\partial_M \tau_{-1}}{2\tau_2} , \quad P_M = ie^{2i\theta} \frac{\partial_M \tau_{-1}}{2\tau_2} , \quad G_{(3)} = i\frac{e^{i\theta}}{\sqrt{\tau_2}}(dA_2 - \tau dB_2) .
\]  

(4.24)

The phase \( \theta \) drops out from the equations of motion as well as from the Bianchi identities and the underlying symmetry is the local U(1) gauge transformation

\[
\epsilon \to e^{i\varphi} \epsilon , \quad \theta \to \theta + g
\]  

(4.25)

for some function \( g \). This local symmetry is due to the coset \( SL(2,R)/U(1) \) which is parameterized by the scalar fields of type IIB supergravity and implies that the phase \( \theta \) can be chosen freely, one can take \( \theta = 0 \) (string theory convention) or \( e^{2i\theta} = \frac{1+i\bar{\tau}}{\sqrt{2}} \) (supergravity convention) or any other value. Recall, we are working in the Einstein frame which explains the pre-factor \( \tau_{-1/2} = e^{\phi/2} \) in the 3-form \( G_3 \).

We can write the spinor (4.19) as

\[
\epsilon = e^{A - i\omega} \left( \sin \alpha [\zeta \otimes \chi] + \cos \alpha [\zeta^* \otimes \chi^*] \right) \]  

(4.26)

where the appearance of the warp factor is a consequence of the gravitino variation [14, 24]. We absorbed the common phase of \( a \) and \( b \) into the spinor \( (\chi = e^{i\beta} \chi_0) \) and this phase drops out in most of the calculations.

The 5-form flux is again parameterized by the function \( Z \) as in (4.21), and for the 3-form flux one find the form

\[
G = \frac{1}{4} e^{-2A - i\omega} J \wedge \left( \cot \alpha P_i dz^i + \tan \alpha P_i d\bar{z}^i \right) + G_{(prim)}^{(prim)} \]  

(4.27)

with the primitive part obeying: \( J \wedge G^{(prim)} = \Omega \wedge G^{(prim)} = \bar{\Omega} \wedge G^{(prim)} = 0 \); \( P_i \) is the holomorphic part of the vector introduced in (4.24) and \( z^i \) denote the three coordinates parameterizing the internal space. Now, the solution of the Killing spinor equation is given in terms of one holomorphic function

\[
f = f(z^i) \]  

(4.28)

and can be written as [24]

\[
\begin{align*}
\tau &= c_0 + i e^{-\phi_0} \frac{|f|^2 \cos 2\alpha}{f \sin^2 \alpha + f^* \cos^2 \alpha} , \\
e^{-4A} &= \frac{\Re e f}{4|f|^2} \frac{\sin^2 2\alpha}{\cos 2\alpha} , \\
Z &= \frac{|f|^2 \cos 2\alpha}{\Re e f \sin 2\alpha} , \\
\tan(\theta + \omega) &= -\frac{\Im \Re e f}{\Re e f} \cos 2\alpha .
\end{align*}
\]  

(4.29)
Using the local symmetry (4.25) we can set $\omega$ or $\theta$ to any fixed value, but the combination $\theta + \omega$ is gauge invariant. Note, supersymmetry leaves one function (in addition to the holomorphic function $f$) free which has to be fixed by the Bianchi identities or equations of motion; this is the master function in \cite{17}. We chose here $\alpha$, which is the mixing angle between the two spinors, but one may also take $Z$ which can be fixed by the Bianchi identity $dF_5 \sim G \wedge \bar{G}$. The Calabi-Yau case is of course a special (where $\alpha \simeq 0$), where the axion-dilaton and the vector $P$ are holomorphic. For the general case, the internal geometry is a complex manifold and becomes (conformal) Kahler if:
\begin{itemize}
  \item[(i)] $G \wedge \bar{G} = 0$, which is ensured if the primitive part of $G$ is of (2,1) type,
  \item[(ii)] $dZ \wedge dA = 0$, which can be seen as a constraint on the function $f$.
\end{itemize}

Another special case are the solutions describing supergravity flows, that correspond to the case where the holomorphic function is constant $f = \text{constant}$. For more discussion we refer to \cite{24}.

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