PERMUTATION TOTALLY SYMMETRIC SELF-COMPLEMENTARY
PLANE PARTITIONS

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Abstract. Alternating sign matrices and totally symmetric self-complementary plane partitions are equinumerous sets of objects for which no explicit bijection is known. In this paper, we identify the subset of totally symmetric self-complementary plane partitions corresponding to permutations by giving a statistic-preserving bijection to permutation matrices, which are a subset of alternating sign matrices. This bijection maps the inversion number of the permutation, the position of the one in the last column, and the position of the one in the last row to natural statistics on these plane partitions. We use this bijection to define a new partial order on permutations, and prove this new poset contains both the Tamari lattice and the Catalan distributive lattice as subposets. We also study a new partial order on totally symmetric self-complementary plane partitions arising from this perspective and show is a distributive lattice related to Bruhat order when restricted to permutations.

1. Introduction

Alternating sign matrices (ASM) with \( n \) rows and \( n \) columns, descending plane partitions (DPP) with largest part at most \( n \), and totally symmetric self-complementary plane partitions (TSSCPP) inside a \( 2n \times 2n \times 2n \) box are equinumerous sets that lack an explicit bijection between any two. (See [18] [19] [1] [33] [16] for these enumerations and bijective conjectures and [6] for the story behind these papers.) In [28], we gave a bijection between permutation matrices (which are a subset of ASM) and descending plane partitions with no special parts in such a way that the inversion number of the permutation matrix equals the number of parts of the DPP. In this paper, we complete the solution to this bijection problem in the special case of permutations by identifying the subset of TSSCPP corresponding to permutations and giving a bijection which yields a direct interpretation for the inversion number on these permutation TSSCPP. See Figure 1. We then use this bijection to prove several poset-theoretic results on TSSCPP and permutations, with connections to Bruhat order and partial orders on Catalan objects.

We highlight here the main new definitions and theorems of this paper. We define a new object which we show in Proposition 2.13 to be in bijection with TSSCPP.

Definition 2.12. A TSSCPP boolean triangle of order \( n \) is a triangular integer array \( \{b_{i,j}\} \) for \( 1 \leq i \leq n - 1, \ n - i \leq j \leq n - 1 \) with entries in \( \{0, 1\} \) such that the diagonal partial sums satisfy

\[
1 + \sum_{i=j+1}^{i'} b_{i,n-j-1} \geq \sum_{i=j}^{i'} b_{i,n-j}.
\]

We use this new object to characterize the permutation subset of TSSCPP.

Definition 3.1. Let permutation TSSCPP of order \( n \) be all TSSCPP inside a \( 2n \times 2n \times 2n \) box whose corresponding boolean triangles have weakly decreasing rows.

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Totally Symmetric Self-Complementary Plane Partition

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

⇔

Descending Plane Partition

\[
\begin{pmatrix}
6 & 6 & 6 & 6 \\
5 & 4 & 4 & 4 \\
3 & 3
\end{pmatrix}
\]

Figure 1. A permutation matrix (center) and the corresponding totally symmetric self-complementary plane partition (left) and descending plane partition (right). The first correspondence is the subject of this paper; the second is the subject of [28].

In Lemma 3.3 and Theorem 3.4, we translate this characterization directly to TSSCPP and magog triangles, which are triangular arrays in bijection with TSSCPP (see Definition 2.9).

Our main theorem is as follows.

**Theorem 3.5.** There is a natural, statistic-preserving bijection between $n \times n$ permutation matrices with inversion number $p$ whose one in the last row is in column $k$ and whose one in the last column is in row $\ell$ and permutation TSSCPP boolean triangles of order $n$ with $p$ zeros, exactly $n - k$ of which are contained in the last row, and for which the lowest one in diagonal $n - 1$ is in row $\ell - 1$.

We use this characterization of permutation TSSCPP to define, in Definition 4.6, $T_{n}^{\text{Perm}}$, a new partial order on permutations, as a restriction of the natural partial order on magog triangles. We show that $T_{n}^{\text{Perm}}$ contains two non-isomorphic Catalan subposets.

**Theorem 4.11.** $T_{n}^{\text{Perm}}$ contains the Tamari lattice, $\text{Tam}_{n}$, as a subposet. In particular, the subposet of $T_{n}^{\text{Perm}}$ consisting of the 132-avoiding permutations is isomorphic to $\text{Tam}_{n}$.

**Theorem 4.13.** $T_{n}^{\text{Perm}}$ contains the Catalan distributive lattice, $\text{Cat}_{n}$, as a subposet. In particular, the subposet of $T_{n}^{\text{Perm}}$ consisting of the 213-avoiding permutations is isomorphic to $\text{Cat}_{n}$.

We also define a new poset on TSSCPP, $T_{n}^{\text{Bool}}$, using TSSCPP boolean triangles. Of note is the following corollary of Theorem 3.5.

**Corollary 4.18.** $T_{n}^{\text{Bool}}$ (the induced subposet of $T_{n}^{\text{Bool}}$ consisting of all the permutation TSSCPP boolean triangles) equals $[2] \times [3] \times \cdots \times [n]$, that is, the product of chains with 1, 2, \ldots, $n-1$ elements. Thus, this is a partial order on permutations which sits between the weak and strong Bruhat orders on the symmetric group.

The outline is as follows. In Section 2, we define totally symmetric self-complementary plane partitions and alternating sign matrices and give bijections within their respective families. In Section 2.1, we recall the standard bijection from an ASM to a monotone triangle. We give, in Section 2.2, known bijections from a TSSCPP to a magog triangle and a certain nest of non-intersecting lattice paths. We transform this last object to a new object we call a boolean triangle.

In Section 3.1, we identify the permutation TSSCPP subset in terms of the boolean triangles of Section 2.2 and then translate this condition to the other objects in the TSSCPP family, including magog triangles. In Section 3.2, we use this characterization to present in Theorem 3.5...
a direct, statistic-preserving bijection between this TSSCPP subset and permutation matrices. In Section 3.3, we discuss the outlook of the general ASM–TSSCPP bijection problem and compare the bijection of this paper with the two-diagonal subset bijection of P. Biane and H. Cheballah [4].

Finally, in Section 4, we contrast various poset structures on ASM and TSSCPP as well as their permutation and Catalan subposets. In Section 4.1, we review known distributive lattices on ASM and TSSCPP. In Section 4.2, we define and study a new partial order on permutations which appears as the permutation subposet of the TSSCPP distributive lattice from Section 4.1. We prove Theorems 4.11 and 4.13 identifying two Catalan subposets of this permutation order. In Section 4.3, we study a new partial order on TSSCPP obtained via boolean triangles. We show in Corollary 4.19 that the subposet of permutation boolean triangles is an especially nice distributive lattice that sits between the weak and strong Bruhat orders. In Section 4.4, we summarize the results of Section 4.

Note this paper is the full version of the extended abstract [29].

2. The objects and their alter egos: ASM / monotone triangle, TSSCPP / non-intersecting lattice paths / boolean triangle

In this section, we first define alternating sign matrices and recall the standard bijection to monotone triangles. We then define totally symmetric self-complementary plane partitions and give bijections with non-intersecting lattice paths and new objects we call boolean triangles. Then in Section 3, we give a bijection in the permutation case via these intermediary objects.

2.1. The ASM family.

Definition 2.1 ([18]). An alternating sign matrix (ASM) is a square matrix with entries in \{0, 1, −1\} whose rows and columns each sum to one and such that the nonzero entries along each row or column alternate in sign.

See Figure 2 for the seven 3 × 3 ASM. It is clear that the alternating sign matrices with no negative entries are the permutation matrices.

Alternating sign matrices are well-known to be in bijection with monotone triangles. We then define totally symmetric self-complementary plane partitions and give bijections with non-intersecting lattice paths and new objects we call boolean triangles. Then in Section 3, we give a bijection in the permutation case via these intermediary objects.

2.2. Monotone triangles.

Definition 2.2 ([18]). A monotone triangle of order n is a triangular array of integers \(a_{i,j}\) for \(1 \leq i \leq n, n - i \leq j \leq n - 1\) with bottom row entries \(a_{n,j} = j − 1\) and all other \(a_{i,j}\) satisfying \(a_{i+1,j-1} \leq a_{i,j} \leq a_{i+1,j}\) and \(a_{i,j} < a_{i,j+1}\).

Proposition 2.3 ([18]). Monotone triangles of order n are in bijection with \(n \times n\) ASM.

Proof. To create row \(i\) of a monotone triangle \(t\), in row \(i\) of the corresponding ASM \(A\), note which columns have a partial sum (from the top) of 1 in that row. Record the numbers of the columns in which this occurs in increasing order, thus \(a_{i,j} < a_{i,j+1}\) by construction. Since each column of \(A\) sums to one, the last row of the monotone triangle will record all column numbers 1 2 3 ··· n. Finally, by the alternating condition of the ASM, the monotone triangle entries will satisfy the diagonal inequalities \(a_{i+1,j-1} \leq a_{i,j} \leq a_{i+1,j}\). This process is clearly invertible and so is a bijection.

See Figures 3 and 4. This bijection yields the following easy corollary.
Corollary 2.4 ([18]). Monotone triangle entries $a_{i,j}$ satisfying the strict diagonal inequalities $a_{i,j-1} < a_{i-1,j} < a_{i,j}$ are in bijection with the $-1$ entries of the corresponding ASM.

We will need the notion of inversion number in an alternating sign matrix. First, recall that in a permutation $i \to \sigma(i)$, the inversion number is the number of pairs $(i, j)$ such that $i < j$ and $\sigma(j) < \sigma(i)$. We now recall the following definition of the inversion number of an alternating sign matrix; this definition extends the usual notion of inversion in a permutation.

Definition 2.5. ([18]) The inversion number of an ASM $A$ is defined as $I(A) = \sum A_{i,j}A_{k,\ell}$ where the sum is over all $i,j,k,\ell$ such that $i > k$ and $j < \ell$.

Note there is some ambiguity in the literature regarding the definition of the inversion number of an ASM. We use the above definition found in [18] and [6]; an alternative definition is this number minus the number of negative ones in the ASM, and is the definition introduced in [23] under the name positive inversions. See p. 6 of [2] for discussion. Note that in the permutation case, these two notions of inversion number are equivalent, so for this paper, the choice of convention is irrelevant.

There are many other objects in bijection with alternating sign matrices, such as height function matrices, fully-packed loop configurations, and square ice configurations; see [22].

2.2. The TSSCPP family. There are a few different ways to define plane partitions, including as two-dimensional decreasing integer arrays and as three-dimensional stacks of cubes in a corner. We will need both of these perspectives, and so define both below.

Definition 2.6. A plane partition is a two dimensional array of positive integers $\{t_{i,j}\}$ that weakly decreases across rows from left to right and down columns. We will sometimes take the array to be rectangular, in which case we complete the array to a rectangle by adding entries equal to zero. So we have the following $n \times m$ array satisfying $t_{i,j} \geq t_{i+1,j} \geq 0$ and $t_{i,j} \geq t_{i,j+1} \geq 0$.

$$
\begin{array}{cccc}
t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,m} \\
t_{2,1} & t_{2,2} & t_{2,3} & & t_{2,m} \\
\vdots & & \vdots & & \vdots \\
t_{n,1} & t_{n,2} & t_{n,3} & \cdots & t_{n,m}
\end{array}
$$

We can visualize a plane partition as a stack of unit cubes pushed into the corner of a room. If we identify the corner of the room with the origin and the room with the positive orthant and
denote each unit cube by the coordinate of its corner farthest from the origin in \( \mathbb{N}^3 \), we obtain the following equivalent definition, by which we may also define various symmetry classes.

**Definition 2.7.** A **plane partition** \( \pi \) is a finite set of positive integer lattice points \((i,j,k)\) such that if \((i,j,k) \in \pi\) and \(1 \leq i' \leq i, 1 \leq j' \leq j, \) and \(1 \leq k' \leq k\) then \((i',j',k') \in \pi\).

A plane partition \( \pi \) is **symmetric** if whenever \((i,j,k) \in \pi\) then \((j,i,k) \in \pi\) as well. \( \pi \) is **cyclically symmetric** if whenever \((i,j,k) \in \pi\) then \((j,k,i)\) and \((k,i,j)\) are in \(\pi\) as well. A plane partition is **totally symmetric** if it is both symmetric and cyclically symmetric, so that whenever \((i,j,k) \in \pi\) then all six permutations of \((i,j,k)\) are also in \(\pi\).

A plane partition is **self-complementary** inside a given bounding box \(a \times b \times c\) if it is equal to its complement in the box, that is, the collection of empty cubes in the box is of the same shape as the collection of cubes in the plane partition itself. A **totally symmetric self-complementary plane partition** (TSSCPP) inside a \(2n \times 2n \times 2n\) box is a plane partition which is both totally symmetric and self-complementary inside the box.

See Figures 5 and 6 for the seven TSSCPP inside a \(6 \times 6 \times 6\) box.

Because of the large amount of symmetry in a TSSCPP, we can record the defining information in a more compact form. That is, we take a **fundamental domain**, which amounts to the triangular region shown in Figure 7 and, more formally, the entries in Definition 2.8 below. See also Figure 5.

**Definition 2.8.** Considering a TSSCPP as the array of integers below with decreasing rows and columns

\[
\begin{array}{cccccccc}
   & t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,2n} \\
 t_{2,1} & t_{2,2} & t_{2,3} & & & t_{2,2n} \\
  \vdots & & & & \ddots & \\
 t_{2n,1} & t_{2n,2} & t_{2n,3} & \cdots & t_{2n,2n} \\
\end{array}
\]

we define the **fundamental domain** of a TSSCPP as the following entries:

\[
\begin{array}{cccccccc}
   & t_{n+1,n+1} & t_{n+1,n+2} & t_{n+1,n+3} & \cdots & t_{n+1,2n} \\
 t_{n+2,n+1} & t_{n+2,n+2} & t_{n+2,n+3} & & & t_{n+2,2n} \\
  \vdots & & & \ddots & & \\
 t_{2n,2n} \\
\end{array}
\]
We can record the information from the fundamental domain in either of two ways, namely, by counting the number of boxes in each stack or drawing lattice the paths traced out at each level. Under a slight deformation, the first way corresponds to the magog triangles of Definition 2.9 below. The second way corresponds to certain nests of non-intersecting lattice paths described in Proposition 2.11.

**Definition 2.9.** A magog triangle of order \(n\) is a triangular array of integers \(\alpha_{i,j}\) for \(1 \leq i \leq n, n - i \leq j \leq n - 1\), such that the bottom row entries are given by \(\alpha_{n,j} = j - 1\) and all other \(\alpha_{i,j}\) satisfy \(\alpha_{i+1,j-1} \leq \alpha_{i,j}\) and \(\alpha_{i,j} + 1 \geq \alpha_{i+1,j}\) and \(\alpha_{i,j} < \alpha_{i,j+1}\).

In [20], W. Mills, D. Robbins, and H. Rumsey give a bijection between the magog triangles of the above definition and TSSCPP. In D. Zeilberger’s proof of the enumeration of alternating sign matrices [33], he gave these triangles the name magog triangles (and, likewise, called monotone triangles gog triangles). See Figures 8 and 9.

**Proposition 2.10 (20).** Magog triangles of order \(n\) are in bijection with TSSCPP inside a \(2n \times 2n \times 2n\) box.

**Proof.** The magog triangle \(\alpha\) corresponding to a TSSCPP \(t\) is defined by rotating the fundamental domain and then adding \(1, 2, 3, \ldots, n\) to the diagonals. That is, \(\alpha_{i,n-j} = t_{n+j,n+i} + i - j + 1\).

In [12], W. Doran gave a bijection from TSSCPP inside a \(2n \times 2n \times 2n\) box to certain nests of non-intersecting lattice paths. In [11], P. Di Francesco gave a bijection to an equivalent collection of non-intersecting lattice paths. Either bijection proceeds by taking a fundamental domain of the TSSCPP, and instead of reading the number of boxes in each stack, one draws the paths going alongside those boxes. This yields a collection of non-intersecting paths with two types of steps. With a slight further deformation, namely, a quarter-turn counterclockwise, we obtain that the following objects are in bijection with TSSCPP. This is equivalent to vertically reflecting the paths in [11]. See Figures 7 and 10.
Figure 7. A TSSCPP (upper left) and its fundamental domain (upper right), along with its corresponding magog triangle (lower left), nest of non-intersecting lattice paths (middle right), and boolean triangle (lower right).

\[ \alpha_{1, n-1} \]
\[ \alpha_{2, n-2} \]
\[ \alpha_{3, n-3} \]
\[ \vdots \]
\[ \alpha_{n-1, n-1} \]
\[ 1 \]
\[ 2 \]
\[ \vdots \]
\[ n-1 \]

\[ \leq \]
\[ \geq \]

\[ \alpha_{i,j} + 1 \geq \alpha_{i+1,j} \]

Figure 8. A generic magog triangle; the “+1 \geq” symbol represents the inequality \( \alpha_{i,j} + 1 \geq \alpha_{i+1,j} \).
Proposition 2.11 ([12], [11]). Totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box are in bijection with nests of non-intersecting lattice paths starting at $(i,i)$, $i = 1, 2, \ldots, n - 1$, and ending at positive integer points on the $x$-axis of the form $(r_i, 0)$, $i = 1, 2, \ldots, n - 1$, making only vertical down steps $(0, -1)$ or southeast diagonal steps $(1, -1)$.

Definition 2.12. A TSSCPP boolean triangle of order $n$ is a triangular integer array $\{b_{i,j}\}$ for $1 \leq i \leq n - 1$, $n - i \leq j \leq n - 1$ with entries in $\{0, 1\}$ such that the diagonal partial sums satisfy

$$1 + \sum_{i=j+1}^{i'} b_{i,n-j-1} \geq \sum_{i=j}^{i'} b_{i,n-j}.$$ 

Proposition 2.13. TSSCPP boolean triangles of order $n$ are in bijection with TSSCPP inside a $2n \times 2n \times 2n$ box.

Proof. The bijection proceeds by replacing each vertical step in the nest of non-intersecting lattice paths with a one and each diagonal step with a zero. The inequality on the partial sums says that the partial sum of any northwest to southeast diagonal is not more than one larger than the partial sum of the diagonal to its left. This is equivalent to the condition that the lattice paths are non-intersecting. □

See Figure 12 for the seven TSSCPP boolean triangles of order three. Note that $\begin{array}{c} 0 \\ 1 \\ 1 \end{array}$ violates the defining inequality condition and is thus not a TSSCPP boolean triangle.
3. A bijection on permutations

In this section, we prove our main result, Theorem 3.5 giving a bijection between $n \times n$ permutation matrices and a subset of totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box which preserves the inversion number statistic and two boundary statistics. First, in Definition 3.1 and Proposition 3.2 we identify and enumerate the permutation subset of TSSCPP boolean triangles; we then translate to the other members of the TSSCPP family in Lemma 3.3 and Theorem 3.4.

3.1. Identification of permutation TSSCPP.

Definition 3.1. Let permutation TSSCPP of order $n$ be all TSSCPP inside a $2n \times 2n \times 2n$ box whose corresponding boolean triangles have weakly decreasing rows.

The terminology permutation TSSCPP is justified by Proposition 3.2 below, which counts permutation TSSCPP. In Theorem 3.5 we further justify this terminology by providing a statistic-preserving bijection between permutations and permutation TSSCPP.

Proposition 3.2. There are $n!$ permutation TSSCPP of order $n$.

Proof. The condition on a permutation TSSCPP boolean triangle that the rows be weakly decreasing means that all the ones must be left-justified, thus the defining partial sum inequality of Definition 2.12 is never violated. To construct a permutation TSSCPP, freely choose any number of left-justified ones in each row of the boolean triangle and the rest zeros; there are $i+1$ choices for row $i$, and the number of ones chosen in each row is independent of the choice for any other row. □

Though the viewpoint of non-intersecting lattice paths and boolean triangles is the most natural for characterizing permutation TSSCPP, we can use the TSSCPP family bijections discussed in the previous section to determine the permutation condition both on the magog triangle and directly on the plane partition.

Lemma 3.3. Permutation TSSCPP, as integer arrays $t_{ij}$, $1 \leq i, j \leq 2n$, are characterized as the TSSCPP such that there is no choice of integer $k \geq 0$ and fundamental domain indices $(i, j)$, $n+1 \leq i \leq j \leq 2n-1$, such that

$$t_{i,j} > t_{i,j+1} = t_{i+k,j+k+1} > t_{i+k+1,j+k+1}.$$
Alternatively, as stacks of cubes in a corner, permutation TSSCPP are the TSSCPP with no configurations of the following form inside the fundamental domain:

\[
\begin{array}{c}
t_{i,j} \\
t_{i,j+1} \\
\vdots \\
t_{i+k,j+k+1}
\end{array}
\]

Boolean triangle entries

0

1

Proof. In the bijection of Proposition 2.13 (via Proposition 2.11), the left-to-right rows of a TSSCPP boolean triangle correspond to the top-to-bottom columns of steps in the fundamental domain. The non-permutation TSSCPP boolean triangle configuration in which a zero is to the left of a one in a row corresponds directly with the configuration shown above. □

Lemma 3.3 shows that the non-permutation TSSCPP in Figures 5 and 6 is the one in the upper right.

We can also adapt this condition to characterize permutation magog triangles directly. Note that this characterization of permutations is not at all obvious from the perspective of magog triangles, but relies heavily on the connection to boolean triangles.

**Theorem 3.4.** Permutation magog triangles are characterized by the following condition: there is no entry \(\alpha_{i,j}\) and \(k \geq 0\) such that

\[
\alpha_{i,j} \geq \alpha_{i+1,j} = \alpha_{i+k+1,j-k} > \alpha_{i+k+1,j-k-1} + 1.
\]

Proof. From Proposition 2.10, we have that \(\alpha_{i,n-j} = t_{n+j,n+i} + i - j + 1\), so solving for \(t_{n+j,n+i}\) and substituting \(n + j \rightarrow i\) and \(n + i \rightarrow j\), we have \(t_{i,j} = \alpha_{j-n,2n-i} + i - j - 1\). Thus the condition of Lemma 3.3 that there is no choice of integer \(k \geq 0\) and fundamental domain indices \((i,j)\), \(n + 1 \leq i \leq j \leq 2n - 1\), such that

\[
t_{i,j} > t_{i,j+1} = t_{i+k,j+k+1} > t_{i+k+1,j+k+1}
\]

translates to

\[
\alpha_{j-n,2n-i} + i - j - 1 > \alpha_{j+1-n,2n-i} + i - j - 2 = \alpha_{j+k+1-n,2n-i-k} + i - j - 2 > \alpha_{j+k+1-n,2n-i-k-1} + i - j - 1.
\]

Simplified, this is

\[
\alpha_{j-n,2n-i} + i - j - 1 > \alpha_{j+1-n,2n-i} - 1 = \alpha_{j+k+1-n,2n-i-k} - 1 > \alpha_{j+k+1-n,2n-i-k-1}.
\]

Translating \(j - n \rightarrow i\) and \(2n - i \rightarrow j\) so that \(j \rightarrow i + n\) and \(i \rightarrow 2n - j\), we have

\[
\alpha_{i,j} > \alpha_{i+1,j} - 1 = \alpha_{i+k+1,j-k} - 1 > \alpha_{i+k+1,j-k-1},
\]

which is equivalent to the desired condition. □

We will use this characterization of permutation magog triangles in Section 4 to define a new partial order on permutations via the natural partial order on magog triangles.
3.2. The bijection. We are now ready to state and prove our main theorem.

**Theorem 3.5.** There is a natural, statistic-preserving bijection between $n \times n$ permutation matrices with inversion number $p$ whose one in the last row is in column $k$ and whose one in the last column is in row $\ell$ and permutation TSSCPP boolean triangles of order $n$ with $p$ zeros, exactly $n-k$ of which are contained in the last row, and for which the lowest one in diagonal $n-1$ is in row $\ell-1$.

**Proof.** We first describe the bijection map. An example of this bijection is shown in Figure 13.

Begin with a permutation TSSCPP of order $n$. Consider its associated boolean triangle $b = \{b_{i,j}\}$ for $1 \leq i \leq n-1$, $n-i \leq j \leq n-1$. Define $a = \{a_{i,j}\}$ for $1 \leq i \leq n$, $n-i \leq j \leq n-1$ as follows: $a_{n,j} = j+1$ and for $i < n$, $a_{i,j} = a_{i+1,j}$ if $b_{i,j} = 0$ and $a_{i,j} = a_{i+1,j}-1$ if $b_{i,j} = 1$. We claim $a$ is a monotone triangle. Clearly $a_{i,j-1} \leq a_{i-1,j} \leq a_{i,j}$. Also, $a_{i,j} < a_{i,j+1}$, since if $a_{i,j} = a_{i,j+1}$, then $a_{i,j} = a_{i+1,j}$ and $a_{i,j+1} = a_{i+1,j+1}$ so that we would need $b_{i,j} = 0$ and $b_{i,j+1} = 1$. This contradicts the fact that the rows of permutation boolean triangles must weakly decrease. Furthermore, $a$ is a monotone triangle with no negative ones in the corresponding ASM, since each entry is defined to be equal to one of its diagonal neighbors in the row below; see Corollary 2.4. This process is clearly invertible.

We now show that this map takes a permutation TSSCPP boolean triangle with $p$ zeros to a permutation matrix with $p$ inversions. Note that to convert from the monotone triangle representation of a permutation to usual one-line notation $\sigma$ such that $i \rightarrow \sigma(i)$, we set $\sigma(i)$ equal to the unique new value in row $i$ of the monotone triangle. Thus for each entry of the monotone triangle $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$, there will be an inversion in the permutation between $a_{i,j}$ and $\sigma(i+1)$. This is because $a_{i,j} = \sigma(k)$ for some $k \leq i$ and $\sigma(k) = a_{i,j} > \sigma(i)$. These entries $a_{i,j}$ such that $a_{i,j} = a_{i+1,j}$ correspond exactly to zeros in row $i$ of the boolean triangle $b$. Thus if a permutation TSSCPP has $p$ zeros in its boolean triangle, its corresponding permutation will have $p$ inversions.

Also, observe that if the number of zeros in the last row of the boolean triangle is $k$, then the one in the bottom row of the permutation matrix will be in column $n-k$. So the missing number in row $n-1$ of the monotone triangle determines where the last row of the boolean triangle transitions from ones to zeros. So by the bijection between monotone triangles and ASM, the one in the last row of the resulting permutation matrix is in column $n-k$.

Finally, if the lowest one in diagonal $n-1$ of the boolean triangle is in row $\ell-1$, this means that the entries $\{a_{i,n-1}\}$ for $\ell \leq i \leq n$ are all equal to $n$. So the one in the last column of the permutation matrix is in row $\ell$.

See Figure 13 for an example of this bijection.

3.3. Outlook on a general bijection. In [28], we discussed the obstacles to turning the bijection between permutations and descending plane partitions presented there into a bijection between all ASM and DPP. Here we discuss some of the challenges to the ASM-TSSCPP bijection in full generality.

While DPP have the property that the number of parts equals the inversion number of the ASM (this is now proved, though not bijectively [3]), TSSCPP do not have such a statistic as of yet. We showed that the number of zeros in a permutation TSSCPP boolean triangle gives the inversion number of the permutation matrix, but this is not true for general TSSCPP boolean triangles and ASM. This is because the distribution of the number of zeros over all TSSCPP boolean triangles of order $n$ does not correspond to the inversion number distribution on $n \times n$ ASM (using either the inversion number of Definition 2.5 or the alternative definition discussed in the paragraph afterward). Furthermore, while the number of special parts of a DPP corresponds to the number of negative ones in the ASM [3], there is no such TSSCPP statistic. It would seem reasonable to conjecture that the negative ones of the ASM should correspond to all instances of a zero followed by a one as you go across a row of the boolean triangle. We have calculated in Sage [24] this holds
Figure 13. An example of the bijection. Note that the matrix on the right represents the permutation 463512 which has 11 inversions. These inversions correspond to the 11 entries in the monotone triangle equal to their southeast diagonal neighbor (shown in bold), as well as the zeros in the boolean triangle and the diagonal steps of the nest of non-intersecting lattice paths.

up to $n = 4$, and it seems to hold for arbitrary $n$ in the special cases of one negative one and the maximum number of negative ones ($\lfloor \frac{n^2}{4} \rfloor$). But for the number of negative ones between one and $\lfloor \frac{n^2}{4} \rfloor$, these statistics diverge.

P. Di Francesco and P. Zinn-Justin have noted in [10] that the distribution of the number of diagonal steps in the bottom row of the TSSCPP non-intersecting lattice paths seems to correspond to the enumeration of ASM refined by the position of the unique one in the top row. So one might
hope to begin a general bijection by determining the \((n - 1)\)st row of the monotone triangle from the bottom row of the TSSCPP boolean triangle) by left-justifying all the vertical steps and then bijecting in the same way as in the permutation case. After that, though, it is unclear how to proceed. See Figure [14] for a summary of the various statistics which are preserved in the permutation case DPP-ASM-TSSCPP bijections and which should be true in full generality since the distributions correspond. (See [28] for further explanation of the DPP case.)

| DPP             | ASM        | TSSCPP boolean triangle |
|-----------------|------------|-------------------------|
| no special parts* | no \(-1\)'s | rows weakly decrease    |
| number of parts*  | number of inversions | number of zeros         |
| largest part value that does not appear  | position of 1 in last row | position of lowest 1 in last diagonal |

Figure 14. This table shows the statistics preserved by the permutation case bijections of this paper and [28]. There is a star by the DPP and TSSCPP statistics that have the same distribution as the ASM statistic in the general case. This is proved for DPP [3] and conjectured for TSSCPP [10].

We now compare this work with another bijection between subsets of ASM and TSSCPP due to P. Biane and H. Cheballah. In [4], the authors give a bijection between monotone trapezoids and magog trapezoids of two diagonals. The term trapezoid indicates the truncation of the monotone or magog triangle to a fixed number of diagonals. Their bijection is both more and less general than the one of this paper. It is more general in the sense that it includes configurations corresponding to the negative one in an ASM, whereas we consider only permutations. It is less general in that it uses only two diagonals of the triangle, where we are able to consider the full triangle corresponding to a permutation TSSCPP. Experimental evidence suggests the bijection of [4] and the bijection of this paper may coincide (up to slight deformation) in the case of permutation monotone triangles, truncated to two diagonals. Perhaps the combination of these two perspectives will provide insight on the full bijection.

4. Poset Structures

In this last section, we compare various partial orders on ASM and TSSCPP, along with their permutation and Catalan subposets. In Section 4.1 we review distributive lattices \(A_n\) of ASM and \(T_n\) of TSSCPP, defined naturally on their corresponding monotone or magog triangles [13] [32]. We note that the permutation subposet of \(A_n\) is known to be the strong Bruhat order [17]. In Section 4.2 we use the characterization of permutation TSSCPP in Theorem 3.5 to define and study the permutation subposet \(T_n^{\text{Perm}}\) of \(T_n\). While \(T_n^{\text{Perm}}\) does not coincide with any of the well-studied permutation posets, we show in Theorems 4.11 and 4.13 that \(T_n^{\text{Perm}}\) contains two different Catalan subposets: the Tamari lattice and the Catalan distributive lattice. In Section 4.3, we define a new TSSCPP partial order \(T_{\text{Bool}}\) by componentwise comparison of the TSSCPP boolean triangles. We show in Corollary 4.19 that its permutation subposet \(T_{\text{Bool}}^{\text{Perm}}\) is an especially nice product of chains distributive lattice which sits between the weak and strong Bruhat orders. Finally, we show in Corollary 4.20 that \(T_{\text{Bool}}^{\text{Perm}}\) contains the same Tamari and Catalan distributive subposets as \(T_n^{\text{Perm}}\).

4.1. ASM and TSSCPP posets via monotone and magog triangles. We start by reviewing natural ASM and TSSCPP posets defined via their monotone and magog triangles. The ASM poset was first introduced in [13] and further studied in [17] and [32]. This partial order can
be naturally and equivalently defined using any of the following objects in bijection with ASM: monotone triangles, corner sum matrices, or height functions. Here we give the definition using monotone triangles. See Figure 15 left, for an example.

**Definition 4.1.** Let $A_n$, the ASM poset of order $n$, be the partial order defined by componentwise comparison of the monotone triangles of order $n$.

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**Figure 15.** Left: The poset $A_3$ of monotone triangles of order 3, partially ordered by componentwise comparison; Center: The poset $T_3$ of magog triangles of order 3, partially ordered by componentwise comparison; Right: $T_3$ with nodes labeled by the corresponding TSSCPP boolean triangles, demonstrating the covering relations of Lemma 4.8. In all three posets, the highlighted element is the non-permutation.

$A_n$ is a lattice, which means any pair of elements has a unique meet (least upper bound) and join (greatest lower bound); $A_n$ is, furthermore, a distributive lattice, meaning the operations of meet and join distribute over each other. The fundamental theorem of finite distributive lattices says that $A_n$ is, thus, the lattice of order ideals $J(P_n)$ for some poset of join irreducibles $P_n$. (An order ideal is a subset $X$ of poset elements such that if $p \in X$ and $q \leq p$ then $q \in X$. See Chapter 3 of [20] for poset terminology.) This join-irreducible poset $P_n$ has a nice structure, which we describe in the following theorem in a different, but equivalent, way to [13] and [32]; see Figure 15 left. See Definition 8.4 of [30] for an equivalent construction of $P_n$ as a layering of successively smaller Type A positive root posets; see also the constructions in [13], [22], [32], and [31].

**Theorem 4.2** ([13], [32]). $A_n$ is the distributive lattice of order ideals of the poset $P_n$ described below. Let the elements of $P_n$ be the coordinates $(i,j,k)$ in $\mathbb{Z}^3$ such that $0 \leq i \leq n-2$, $0 \leq j \leq n-2-i$, and $0 \leq k \leq n-2-i-j$. Let the covering relations of $P_n$ be given as: $(i,j,k)$ covers $(i,j+1,k)$, $(i,j+1,k-1)$, $(i+1,j,k)$, and $(i+1,j,k-1)$, whenever these coordinates are poset elements.
The permutation subposet of $A_n$ is known to be a well-studied partial order on the symmetric group; we state here a theorem of A. Lascoux and M.-P. Schützenberger \cite{17} about the permutation subposet of $A_n$ (see Definition \ref{def:Bruhat} for the definition of Bruhat order).

**Theorem 4.3 (\cite{17}).** The induced subposet of $A_n$ on permutations of $n$, which we denote as $A_n^{\text{Perm}}$, is the strong Bruhat order. Moreover, $A_n$ is the smallest lattice to contain the strong Bruhat order as a subposet, that is, $A_n$ is the MacNeille completion of the strong Bruhat order.

We now recall a natural partial order on TSSCPP. See Figure \ref{fig:15}, center, for an example.

**Definition 4.4.** Let $T_n$, the TSSCPP poset of order $n$, be the partial order defined by componentwise comparison of the magog triangles of order $n$.

Like $A_n$, $T_n$ is also a distributive lattice. In \cite{32}, we showed that $A_n$ and $T_n$ have similar posets of join irreducibles. We also described these posets using colored vectors in $\mathbb{R}^3$ for the various different types of covering relations; we furthermore defined and studied the tetrahedral poset family of which these ASM and TSSCPP posets are members. Instead of delving into that theory here, in Theorem \ref{thm:magog} above and Theorem \ref{thm:TSSCPP} below we describe these posets in a different, but equivalent, way; see Figure \ref{fig:15}.

**Theorem 4.5 (\cite{32}).** $T_n$ is the distributive lattice of order ideals of the poset $Q_n$ described below. Let the elements of $Q_n$ be the coordinates $(i,j,k)$ in $\mathbb{Z}^3$ such that $0 \leq i \leq n-2$, $0 \leq j \leq n-2-i$, and $0 \leq k \leq n-2-i-j$. Let the covering relations of $Q_n$ be given as: $(i,j,k)$ covers $(i+1,j-1,k)$, $(i,j+1,k-1)$, and $(i+1,j,k)$, whenever these coordinates are poset elements.

In the next subsection, we study a new partial order on permutations, namely, the induced subposet of $T_n$ on permutation TSSCPP, as a consequence of the identification of permutation TSSCPP in Theorem \ref{thm:TSSCPP}.

4.2. The magog permutation poset and its Catalan subposets. Using the characterization of permutation magog triangles in Theorem \ref{thm:magog} we introduce a new partial order on permutations as the restriction of $T_n$ to permutation magog triangles. See Figure \ref{fig:15} right, with the non-permutation removed and Figure \ref{fig:17}.

**Definition 4.6.** Let the magog permutation poset $T_n^{\text{Perm}}$ on permutations of $n$ be given by componentwise comparison of the corresponding magog triangles. That is, $T_n^{\text{Perm}}$ is the induced subposet of $T_n$ consisting of the permutation magog triangles.
Though $T_n^{\text{Perm}}$ does not correspond to any of the usual permutation posets (see Figure 17), we show in Theorems 4.11 and 4.13 that $T_n^{\text{Perm}}$ contains two different Catalan posets as induced subposets: the Tamari lattice on 132-avoiding permutations and the Catalan distributive lattice on 213-avoiding permutations. Before stating and proving these theorems on subposets of $T_n^{\text{Perm}}$, we note in the following proposition a structural property of $T_n^{\text{Perm}}$.

**Proposition 4.7.** $T_n^{\text{Perm}}$ is a lattice for $n \leq 3$, but for $n \geq 4$ it is not a lattice.

*Proof.* We have computed $T_n^{\text{Perm}}$ in Sage [21] for $n \leq 5$. $T_3^{\text{Perm}}$ is the (non-distributive) lattice in Figure 17 left. $T_4^{\text{Perm}}$, shown in Figure 17 right, is not a lattice. For any $n > 1$, $T_{n-1}^{\text{Perm}}$ sits as an interval inside $T_n^{\text{Perm}}$, thus if $T_n^{\text{Perm}}$ is not a lattice, then neither is $T_n^{\text{Perm}}$. □

We will find it easier to prove Theorems 4.11 and 4.13 if we work with $T_n^{\text{Perm}}$ from the boolean triangle perspective. So we first characterize the covering relations in $T_n$ in terms of boolean triangles rather than magog triangles. See Figure 15 right.

**Lemma 4.8.** Suppose $t'$ covers $t$ in $T_n$. Let $b$ and $b'$ be the TSSCPP boolean triangles corresponding to $t$ and $t'$, respectively. Then $b$ and $b'$ differ only in one of the following:

1. $b'_{i+1,j} = 0 = b_{i+1,j}$ and $b'_{i+1,j} = 1 = b_{i+1,j}$ for some $1 \leq i \leq n - 2$, $2i - 1 \leq j \leq n - i$, or
2. $b'_{n-1,j} = 0$ and $b_{n-1,j} = 1$ for some $1 \leq j \leq n - 1$.

That is, to create $b'$ from $b$, either (1) swap a one with its southeast diagonal neighboring zero, or (2) replace a one in the bottom row with a zero.

*Proof.* The magog covering relations on TSSCPP (as stacks of cubes in a corner) consist in a box being added within the fundamental domain. This means in the nest of non-intersecting lattice paths, either (1) two adjacent steps of different types swap to add the box, or (2) a box is added along the center border of the fundamental domain, in which case a step at the end of a path changes from one type to the other. These correspond to cases (1) and (2) above. □

The Tamari lattice is a natural partial order often defined on the Catalan objects binary trees or triangulations of a convex polygon. The covering relations on binary trees are given by a local move called *tree rotation*; the triangulation covering relations are given by *flipping a diagonal*. See [21] for further references. Rather than using the definition in terms of binary trees or triangulations, we will use the original definition, often called the *bracket vector encoding*, given by S. Huang and D. Tamari in [14]. See Figure 18 left.

**Definition 4.9 ([14]).** Let the *Tamari lattice* of order $n$, which we denote as $\text{Tam}_n$, be defined as the set of sequences of $n$ positive integers $x_1, x_2, \ldots, x_n$, such that $i \leq x_i \leq n$ and if $i \leq j \leq x_i$ then $x_j \leq x_i$, ordered by reverse componentwise comparison.

Note that for convenience, we use the reverse componentwise comparison, but as the Tamari lattice is self-dual, this convention is inconsequential.

We show in Theorem 4.11 below that $T_n^{\text{Perm}}$ contains $\text{Tam}_n$ as the induced subposet of permutations that avoid the pattern 132. So we will first need to define pattern avoidance in permutations.

**Definition 4.10.** The permutation $\pi = \pi(1)\pi(2)\ldots\pi(k)$ is contained in the permutation $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$, if there is a substring $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)$, $i_1 < i_2 < \cdots < i_k$ whose values have the same relative order as $\pi(1), \pi(2), \ldots, \pi(k)$. A permutation $\sigma$ avoids the pattern $\pi$ if it does not contain $\pi$.

**Theorem 4.11.** $T_n^{\text{Perm}}$ contains $\text{Tam}_n$ as a subposet. In particular, the subposet of $T_n^{\text{Perm}}$ consisting of the 132-avoiding permutations is isomorphic to $\text{Tam}_n$. 16
Proof. Let \( b \) be a permutation TSSCPP boolean triangle. Define a sequence of positive integers \( x_1, x_2, \ldots, x_n \) where \( x_i = i + \sum_{j=n-i}^{n-1} b_{i,j} \), that is, \( x_i \) equals \( i \) plus the sum of row \( n - i \), where we consider the sum of the empty row zero to be zero. This is a bijection since the rows of a permutation TSSCPP boolean triangle must be decreasing.

We wish to show that \( x_1, x_2, \ldots, x_n \) satisfies the inequalities in Definition 4.9 and is thus a bracket vector. By construction, \( i \leq x_i \leq n \). Now we wish to show that the 132-avoiding condition on \( b \) corresponds to the property that if \( i \leq j \leq x_i \) then \( x_j \leq x_i \). In the bijection of Theorem 3.5, the permutation 132 corresponds to the TSSCPP boolean triangle \([1,0] \). More generally, if a permutation is 132-avoiding, for each boolean triangle entry equaling zero, \( b_{i,j} = 0 \), at most one of the following may be true: \( b_{i,j-1} = 1 \) or \( b_{i',j} = 1 \) for some \( i' < i \). This is because in the corresponding monotone triangle, the configuration \( b_{i,j} = 0, b_{i,j-1} = 1, \) and \( b_{i',j} = 1 \) for some \( i' < i \) means that the corresponding monotone triangle entries satisfy \( a_{i,j} = a_{i+1,j}, a_{i,j-1} = a_{i+1,j-2}, \) and \( a_{i',j} = a_{i+1,j-1} \). The first two equalities together mean that \( a_{i+1,j-1} \) must be the new value in row \( i + 1 \), since the new value will always correspond to the entry horizontally between the ones and zeros in the previous line. So in the one-line notation \( i \rightarrow \sigma(i) \) for the permutation, we have \( \sigma(i + 1) = a_{i+1,j-1} \). But since \( a_{i,j} = a_{i+1,j-1} \), we have \( \sigma(i' + 1) > a_{i+1,j-1} \geq a_{i+1,j-1} = \sigma(i+1) \). But then \( a_{i+1,j-1} = \sigma(i'') \) for some \( i'' < i' + 1 \). So \( \sigma(i''), \sigma(i'+1), \sigma(i+1) \) form a 132-pattern.

Suppose now \( i < j \). Let the sum of row \( k \) be denoted as \( \Sigma_k \). Note that if \( \Sigma_k = m \) this means \( b_{k,n-k+m-1} = 1 \) and \( b_{k,n-k+m} = 0 \), provided these entries exist. Now recall \( x_i = i + \Sigma_i \) and \( x_j = j + \Sigma_j \). We wish to show that the Tamari condition (if \( i \leq j \leq x_i \) then \( x_j \leq x_i \)) is equivalent to the 132-avoiding condition (if \( b_{i,j} = 0 \) then at most one of \( b_{i,j-1} = 1 \) or \( b_{i',j} = 1 \) for some \( i' < i \) may be true) on the corresponding boolean triangle \( b \). Suppose the 132-avoiding condition holds on \( b \) and \( i \) and \( j \) are such that \( i \leq j \leq x_i \). We wish to show \( x_j \leq x_i \). Now \( x_i \geq j \) if and only if \( i + \Sigma_{n-i} \geq j \), that is, \( \Sigma_{n-i} \geq j - i \). So there are at least as many ones in row \( n - i \) as the distance from row \( n - i \) to row \( n - j \). Now if \( b_{n-i,k} = 0 \) for some \( k \), then by the 132-avoiding condition, \( b_{n-j,k} = 0 \) as well. Thus \( \Sigma_{n-j} + j - i \leq \Sigma_{n-i} \). So \( x_j = j + \Sigma_{n-j} \leq j + (i - j + \Sigma_{n-i}) = x_i \). Thus the 132-avoiding permutation TSSCPP are in bijection with the bracket vectors of the Tamari lattice.

Finally, we must show that the reverse componentwise comparison covering relation of the bracket vectors \( x_1, x_2, \ldots, x_n \) equals the magog partial order on 132-avoiding permutation TSSCPP boolean triangles, as in Lemma 4.8. The covering relations on boolean triangles in any subposet of \( T_n \) are given by composing the covering relations in Lemma 4.8 until another element in the subposet is obtained. Now by Lemma 4.8, \( b' \) covers \( b \) in \( T_n \) if they differ in the following way: either a bottom-row zero of \( b' \) turns into a one in \( b \), or a one of \( b' \) swaps with a zero to its northwest to create \( b \). Thus, \( b' \) covers \( b \) in \( T^\text{Perm}_n \) restricted to 132-avoiding permutations if to transform \( b' \) into \( b \), one or more ones moves into and/or up a diagonal until the new boolean triangle does not violate the 132-avoiding condition. But once there is a new one in a diagonal, it cannot slide up to another open space leaving a zero behind, since that would violate the 132-avoiding condition. So instead, the first one slides up to a new, higher, row, and another one slides into its same place. This has the effect of incrementing an entry by some amount. So this is the reverse componentwise covering relation of the Tamari lattice.

We now show \( T^\text{Perm}_n \) contains another Catalan poset. The Catalan distributive lattice of order \( n \) is the natural containment partial order on several Catalan objects, such as Dyck paths, partitions inside a staircase, and order ideals in the Type A positive root poset. We give the following equivalent definition, written in a similar form to Definition 4.9 above. See Figure 18.

**Definition 4.12.** Let the Catalan distributive lattice of order \( n \), which we denote as \( \text{Cat}_n \), be defined as the set of sequences of \( n \) positive integers \( x_1, x_2, \ldots, x_n \), such that \( i \leq x_i \leq n \) and if \( i \leq j \) then \( x_i \leq x_j \), ordered by reverse componentwise comparison.
Figure 17. Left: $T_{3}^{\text{Perm}}$, the poset of permutation magog triangles of order 3; Right: $T_{4}^{\text{Perm}}$. In both pictures, the 132-avoiding permutations are circled; Theorem 4.11 shows the subposet of $T_{n}^{\text{Perm}}$ consisting of the 132-avoiding permutations is the Tamari lattice of order $n$.

Figure 18. Left: The poset $\text{Tam}_3$ of Definition 4.9 Right: The poset $\text{Cat}_3$ of Definition 4.12
Note that by subtracting $i$ from $x_i$, the area sequence of the corresponding Dyck path is obtained.

**Theorem 4.13.** $T_n^\text{Perm}$ contains $\text{Cat}_n$ as a subposet. In particular, the subposet of $T_n^\text{Perm}$ consisting of the $213$-avoiding permutations is isomorphic to $\text{Cat}_n$.

**Proof.** Let $b$ be a permutation TSSCPP boolean triangle. As in the proof of Theorem 4.11, define a sequence of positive integers $x_1, x_2, \ldots, x_n$ where $x_i = i + \sum_{j=n-i}^{n-1} b_{n-i,j}$, that is, $i$ plus the sum of row $n-i$, where we consider the sum of the empty row zero to be zero. This is a bijection since the rows of a permutation TSSCPP boolean triangle must be decreasing.

We wish to show that $x_1, x_2, \ldots, x_n$ satisfies the inequalities in Definition 4.12. By construction, $i \leq x_i \leq n$. In the bijection of Theorem 3.5, the permutation $213$ corresponds to the TSSCPP boolean triangle $\begin{array}{c} 1 & 0 & 1 \\ \end{array}$. More generally, we wish to show that the $213$-avoiding condition on $b$ corresponds to the property that if $i \leq j$ then $x_i \leq x_j$. Let $b$ be the boolean triangle of a $213$-avoiding permutation. We claim that if $b_{i,j} = 0$, then $b_{i',j} = 0$ for all $i' > i$. For suppose $b_{i,j} = 0$ and $b_{i+1,j} = 1$. Then in the corresponding monotone triangle, $a_{i,j} = a_{i+1,j} = a_{i+2,j-1}$. Also, $\sigma(i+2) > a_{i+2,j-1}$ since $b_{i+1,j} = 1$ and the ones are left-justified, while $\sigma(i+1) < a_{i+1,j}$ since $b_{i,j} = 0$ and the new value in a row of the monotone triangle $a$ sits horizontally between the rightmost one and leftmost zero in the previous row of the corresponding boolean triangle. Finally, since the value $a_{i,j}$ is in row $i$ of the monotone triangle, $a_{i,j} = \sigma(k)$ for some $k \leq i$. That is, $a_{i,j}$ must be the new value in $a$ either in row $i$ or in some previous row. Putting these facts together, we have $\sigma(i+1) < \sigma(k) < \sigma(i+2)$ and $k < i+1 < i+2$, so $\sigma(k), \sigma(i+1), \sigma(i+2)$ is a $213$-pattern in the permutation $\sigma$.

Now suppose $i < j$ and $x_i > x_j$. Since $i < j$, this means $n-i > n-j$. Now $x_i > x_j$ implies that in the boolean triangle $b$, $i$ plus the sum of row $n-i$ is greater than $j$ plus the sum of row $n-j$. That is, the sum of row $n-i$ is greater than $j-i$ plus the sum of row $n-j$. Thus, there must be a $k$ such that $b_{n-j,k} = 0$ and $b_{n-i,k} = 1$, since the entries equal to one in a permutation boolean triangle must be left-justified. Thus by the previous paragraph, there must be a $213$-pattern in the corresponding permutation. Therefore, the $213$-avoiding condition on the permutation corresponding to $b$ is equivalent to the property that if $i \leq j$ then $x_i \leq x_j$.

Finally, we must show that the reverse componentwise comparison covering relation of the Catalan distributive sequences $x_1, \ldots, x_n$ equals the magog partial order on $213$-avoiding permutation TSSCPP boolean triangles, whose covering relations are given in Lemma 4.8. The reverse componentwise comparison covering relation on the sequences $x_1, \ldots, x_n$ corresponds to the rightmost one in some row of $b$ turning into a zero. This one must also have a one to its northwest and a zero to its southeast (if these entries exist in $b$) so that the $213$-avoiding condition is satisfied. Thus, removing this one is equivalent to sliding it to the southeast until it leaves the matrix, which is a composition of the magog covering relations of Lemma 4.8. Also, each slide of this one a single entry to the southeast results in another boolean triangle, since this sliding will not violate the defining inequality of Definition 2.12. Thus, in the case of $213$-avoiding boolean triangles, the magog covering relations correspond to the Catalan distributive order. \hfill \square

**Remark 4.14.** A natural question is whether $T_n^\text{Perm}$ restricted to the permutations which avoid the pattern $123, 231, 312$, or $321$ yields a known Catalan poset. We have calculated these posets in Sage [24] and have noted that for $n = 4$, none of these posets are ranked and none are lattices. So these posets are not $Tam_n$ or $\text{Cat}_n$, since both of these are lattices; it is unknown whether they have some interpretation in terms of other Catalan objects.

### 4.3. The TSSCPP boolean poset and Bruhat order

We now define a new TSSCPP poset using boolean triangles. See Figure 20 for an example in the case $n = 3$. 

19
Figure 19. Left: $T^\text{Perm}_3$, the poset of permutation magog triangles of order 3; Right: $T^\text{Perm}_4$. In both pictures, the 213-avoiding permutations are circled; Theorem 4.13 shows the subposet of $T^\text{Perm}_n$ consisting of the 213-avoiding permutations is the Catalan distributive lattice of order $n$.

**Definition 4.15.** Define the boolean partial order $T^\text{Bool}_n$ (or the TSSCPP boolean poset of order $n$) by reverse componentwise comparison of the TSSCPP boolean triangles of order $n$.

Note that $T^\text{Bool}_n$ is the induced subposet of the Boolean lattice on $\binom{n}{2}$ elements given by only taking the elements corresponding to TSSCPP boolean triangles.

**Proposition 4.16.** $T^\text{Bool}_n$ is a lattice for $n \leq 3$, but for $n \geq 4$ it is not a lattice.

**Proof.** We have computed $T^\text{Bool}_n$ in Sage [24] for $n \leq 5$. $T^\text{Bool}_3$ is the (non-distributive) lattice in Figure 20. $T^\text{Bool}_4$ is not a lattice. For any $n > 1$, $T^\text{Bool}_{n-1}$ sits as an interval inside $T^\text{Bool}_n$, thus if $T^\text{Bool}_{n-1}$ is not a lattice, then neither is $T^\text{Bool}_n$. \hfill $\square$

If we further restrict this order to permutation TSSCPP, we show in Corollary 4.19 the poset formed is an exceptionally nice distributive lattice which sits between the weak and strong Bruhat orders; see Figure 21.

**Definition 4.17.** Let the TSSCPP boolean permutation poset $T^\text{Bool}^\text{Perm}_n$ on permutations of $n$ be given by the componentwise comparison of the corresponding TSSCPP boolean triangles. That is, $T^\text{Bool}^\text{Perm}_n$ is the induced subposet of $T^\text{Bool}_n$ consisting of the permutation TSSCPP boolean triangles.
We will also need the following definition; see [5].

**Definition 4.18.** The *weak order* on the symmetric group $S_n$ is the partial order on permutations of $n$ whose covering relations are given as: $\pi$ covers $\sigma$ if they differ by an adjacent transposition $(i, i+1)$ and there is an inversion between $i$ and $i+1$ in $\pi$. The *strong Bruhat order* on $S_n$ is the partial order whose covering relations are given by: $\pi$ covers $\sigma$ if they differ by a transposition $(i, j)$ and $\pi$ has one more inversion than $\sigma$.

As a corollary of Theorem 3.5, we have the following; see Figure 21.

**Corollary 4.19.** $T_{\text{Bool}}^n$ equals $[2] \times [3] \times \cdots \times [n]$, that is, the product of chains with $1, 2, \ldots, n-1$ elements. Thus, this is a partial order on permutations which sits between the weak and strong Bruhat orders on the symmetric group. That is, it contains all of the ordering relations of the weak order plus some of the additional relations of the strong order.

**Proof.** In the proof of Theorem 3.5 we noted that a zero in row $i$ of a permutation TSSCPP boolean triangle represents an inversion between $\sigma(i+1)$ and some $\sigma(k)$ with $k \leq i$ in the one-line notation of the corresponding permutation. So the number of zeros in row $i$ of the permutation TSSCPP boolean triangle equals the number of $k \leq i$ such that $\sigma(k) > \sigma(i+1)$. As we noted in Proposition 3.2, the number of zeros in a row of a permutation TSSCPP boolean triangle can be chosen independently, thus $T_{\text{Bool}}^n$ is the product of chains $[2] \times [3] \times \cdots \times [n]$; the order ideal composed of $\ell$ elements in the chain $[i]$ corresponds to row $i-1$ of the boolean triangle containing $\ell$ zeros.

It is known that the product of chains poset $[2] \times [3] \times \cdots \times [n]$ sits between the weak and strong Bruhat orders (see, for example, [27, p. 182], attributed to Gansner). We prove this directly in our context by showing the covering relations in $T_{\text{Bool}}^n$ (1) contain the covering relations of the weak order and (2) are contained in the covering relations of the strong order. To show (1), we
note that swapping \(\sigma(i) = k\) and \(\sigma(j) = k + 1, i < j\), creates a single new inversion at \(\sigma(j)\); this adds a zero to row \(j - 1\) of the boolean triangle and leaves all other entries the same. Thus, the covering relations of the weak order are also covering relations in \(T_{Bool}^n\).

To show (2), we wish to show that changing a one into a zero in row \(i\) of a permutation TSSCPP boolean triangle is a transposition on the corresponding permutation. Recall that the ones in a permutation TSSCPP boolean triangle are left-justified. Thus, increasing the number of zeros in row \(i\) can only happen by changing the last one in the row to a zero. In particular, suppose two permutation boolean triangles \(b\) and \(b'\) are equal in all entries, except \(b_{i,j} = 1\) and \(b'_{i,j} = 0\). Thus \(b'\) covers \(b\) in \(T_{Bool}^n\). Let \(a\) and \(a'\) be the monotone triangles in bijection via Theorem 3.5 with \(b\) and \(b'\), respectively. Now \(a\) and \(a'\) are equal for rows \(i + 1\) through \(n\). But \(a'_{i,j} = a_{i+1,j}\) whereas \(a_{i,j} = a_{i+1,j-1}\). Also, there is a (possibly empty) path above \(a'_{i,j}\) which now equals the value \(a_{i+1,j}\) rather than \(a_{i+1,j-1}\). Now in \(a\), the value \(a_{i+1,j-1}\) first appears in some row \(k\) for \(k \leq i\). So in \(a'\), the value \(a_{i+1,j-1}\) first appears now in row \(k\). This means in the corresponding permutations \(\sigma\) and \(\sigma'\), \(\sigma(k) = a_{i+1,j-1}\) and \(\sigma'(k) = a_{i+1,j}\) whereas \(\sigma'(i) = a_{i+1,j-1}\) and \(\sigma(i) = a_{i+1,j}\), and \(\sigma\) and \(\sigma'\) agree in all other positions. This is a transposition of \(a_{i+1,j-1}\) and \(a_{i+1,j}\), thus all covering relations in \(T_{Bool}^n\) are covering relations in the strong Bruhat order on permutations of \(n\).

\[\begin{array}{cccc}
321 & 321 & 321 & 321 \\
312 & 321 & 312 & 312 \\
132 & 132 & 132 & 132 \\
123 & 123 & 123 & 123 \\
\end{array}\]

**Figure 21.** From left to right: the weak order on permutations of 3, the boolean partial order \(T_{Bool}^3\) on permutation TSSCPP of order 3, and the strong Bruhat order on permutations of 3

Finally, we note that though \(T_n^\text{Perm}\) and \(T_{Bool}^n\) differ, they contain the same Tamari and Catalan distributive posets. So we obtain the following, as a corollary of Theorems 4.11 and 4.13.

**Corollary 4.20.** \(T_{Bool}^n\) contains both \(Tam_n\) and \(Cat_n\) as subposets. In particular, the subposet of \(T_{Bool}^n\) consisting of the 132-avoiding permutations is isomorphic to \(Tam_n\) and the subposet consisting of the 213-avoiding permutations is isomorphic to \(Cat_n\).

**Proof.** In Theorems 4.11 and 4.13 it was shown that the \(T_n\) covering relations of Lemma 4.8 on either the 132- or 213-avoiding subposets reduce to reverse componentwise comparison covering relations on the Catalan sequence \(x_1, x_2, \ldots, x_n\), where \(x_i\) was defined to equal \(i\) plus the sum of row \(n - i\) in the boolean triangle. This is equivalent to reverse componentwise comparison of the 132- or 213-avoiding permutation TSSCPP boolean triangles. Thus the 132-avoiding subposet of \(T_n^\text{Perm}\) is equivalent to the 132-avoiding subposet of \(T_{Bool}^n\), and likewise for the 213-avoiding subposets. \(\square\)
4.4. Summary. In this section, we have discussed three posets: \( A_n \) on ASM, and \( T_n \) and \( T_{\text{Bool}} n \) on TSSCPP. \( A_n \) and \( T_n \) are distributive lattices and members of the tetrahedral poset family [32], while \( T_{\text{Bool}} n \) is not a lattice in general (Proposition 4.16). \( A_{n}^{\text{Perm}} \) and \( T_{\text{Bool}} n^{\text{Perm}} \) are known permutation posets; \( A_{n}^{\text{Perm}} \) is the strong Bruhat order [17] which is not a lattice, and we have shown \( T_{\text{Bool}} n^{\text{Perm}} \) is \([2] \times [3] \times \cdots \times [n] \), which is a distributive lattice that sits between the weak and strong Bruhat orders (Corollary 4.19). While \( T_{\text{Perm}} n^{\text{Perm}} \) is neither a well-studied permutation partial order nor a lattice (Proposition 4.7), it has Catalan subposets isomorphic to the Tamari lattice (Theorem 4.11) and the Catalan distributive lattice (Theorem 4.13). Thus, \( T_{\text{Perm}} n^{\text{Perm}} \) may be an interesting new perspective from which to study permutations as well. We have also shown in Corollary 4.20 that the same Tamari and Catalan distributive subposets exist in \( T_{\text{Perm}} n^{\text{Perm}} \) as in \( T_{\text{Perm}} n \). An additional point of contrast is that \( A_{n}^{\text{Perm}} \), as the strong Bruhat order, contains the Catalan distributive lattice as an induced subposet, but not the Tamari lattice.

We hope that the study of these various ASM and TSSCPP partial orders will continue to provide insight on the combinatorics of these objects and the associated outstanding bijection problems.

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