Self-Completeness in Alternative Theories of Gravity

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Abstract It has recently been shown via an equivalence of gravitational radius and Compton wavelength in four dimensions that the trans-Planckian regime of gravity may be semi-classical, and that this point is defined by a minimum horizon radius commensurate with the Planck mass. We extend the formalism to modified theories of gravity to the formalisms of Randall-Sundrum and the generalized uncertainty principle.

1 Introduction

From the perspective of quantization, gravity is problematic since, among other reasons, graviton path integrals in (3 + 1)-D are readily divergent. As part of the effort to solve this, it has been shown [6, 7, 8, 12, 13] that gravity may be considered “self-complete,” in that there exists a minimum horizon scale hiding the singularity. Specifically, this distance is defined by the confluence of the classical Schwarzschild radius and the Compton wavelength,

\[ r_H = \frac{\lambda_C}{2} = \frac{2GM_{BH}}{c^2} = \frac{h}{cM_{BH}}. \]  

Letting \( M_{Pl} = \sqrt{\hbar c/G} \), this gives a minimum mass:
2 Randall-Sundrum

The Randall-Sundrum model posits our Universe is a $n$-dimensional brane in a bulk with an infinite extra dimension at a distance $\ell$, the AdS curvature radius \cite{11}. Einstein’s equations in the bulk are

$$\tilde{G}_{AB} = \tilde{\kappa}^2 \left[ -\hat{A} \delta_{AB} + \delta(\chi) (-\lambda g_{AB} + T_{AB}) \right] , \tag{3}$$

where the coupling $\tilde{\kappa} = 8\pi/\tilde{M}_p^3$ is a function of the reduced $(n+1)$-dimensional Planck mass $\tilde{M}_p$. The hierarchy problem is thus resolved by assuming that originates on the extra brane, causing our effective gravitational constant to be $G_4 = G_5/\ell$, where $G_5$ is the “true” coupling strength \cite{3}.

In the case of an electrically neutral black hole, the induced Einstein equations on the brane yield a Reissner-Nordström-like solution of the form \cite{3}

$$ds_4^2 = \left( 1 - \frac{2G_4 m}{c^2 r} + \frac{Q}{r^2} \right) c^2 dt^2 + \frac{dr^2}{1 - \frac{2G_4 m}{c^2 r} + \frac{Q}{r^2} + r^2 d\Omega^2} , \tag{4}$$

with $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and the term $Q$ is the tidal charge, resulting from leakage into the bulk. Knowing that $G_4 = \hbar c/\tilde{M}_p$, we can write \cite{4} in terms of the Planck mass to obtain the (outer) black hole horizon:

$$r_H = \frac{\hbar}{c} \frac{M_{BH}}{\tilde{M}_p} \left( 1 + \sqrt{1 - \frac{c^2 M_{BH}^4}{\hbar^2 \tilde{M}_p^4} Q} \right) . \tag{5}$$

For the external horizon to be greater than the Schwarzschild radius, we require a negative $Q$. Otherwise, both radii would be smaller than the regular Schwarzschild horizon and we would get the usual self-completeness condition from (2). Furthermore, $Q < 0$ is arguably a more “physical” choice \cite{4}. Regardless of $Q$, $r_H(M_{BH})$ becomes linear for large enough mass (Fig.1).

Equating the Compton wavelength to the black hole horizon radius, we obtain an expression for the minimum black hole mass as a function of $Q$:

$$M_{BH} \geq \frac{\pi \tilde{M}_p}{\sqrt{\pi - \frac{c^2 M_{BH}^4}{\hbar^2} Q/4\hbar^2}} \tag{6}$$

\footnote{Note that \cite{3} uses $\beta$ for the tidal charge and $Q$ for the electric charge, which we take to be null.}
Fig. 1 Black hole horizon radius as a function of mass in the Randall-Sundrum model (Planck units). The plot displays Eq. (5) for different values (negative) of the tidal charge $Q$. Note that the mass is in units of $M_{\text{Pl}}$.

However, $Q$ is not really an independent variable. For small length scales, compared to the AdS radius ($r \ll \ell$), the tidal charge becomes a linear function of the brane separation distance:

$$Q \approx \frac{r}{M_{\text{Pl}} \bar{h} c \ell}.$$  \hfill (7)

Consequently, the minimum mass is also a function of $\ell$. In fact, after some basic algebra, we find

$$M_{\text{BH}} \geq A \left( B + B^{-1} - 1 \right),$$  \hfill (8)

$$A = \frac{4\pi}{3} \frac{\bar{h}}{c \ell}, \quad B = \left[ \frac{3\pi}{2} \left( \frac{M_{\text{Pl}}}{A} \right)^2 - 1 + \frac{M_{\text{Pl}}}{A} \sqrt{\left( \frac{3\pi M_{\text{Pl}}}{2 A} \right)^2 - 3\pi} \right]^{1/3}.$$  \hfill (9)

The meaning of eq. (8) can be illuminated by means of a simple expansion:

$$M_{\text{min}} = \sqrt{\pi} M_{\text{Pl}} - \frac{c \ell}{8\bar{h}} M_{\text{Pl}}^2 + \frac{5}{128} \left( \frac{c \ell}{\bar{h}} \right)^2 \pi^{-1/2} M_{\text{Pl}}^3 + O(M_{\text{Pl}}^4).$$  \hfill (10)

Again, we recover (2) for vanishing $\ell$, as expected (Fig. 2). On the other hand, note that $M_{\text{min}} \to 0$ as $\ell \to \infty$. Furthermore, because (8) is continuous for all positive values of $\ell$ (which we require in order to have $Q < 0$), Randall-Sundrum gravity can always be considered self-complete.

### 3 Generalized Uncertainty Principle

If additional momentum dependent terms exist in the usual commutation relation, this will result in a modified uncertainty relation of the form $\Delta x \Delta p \geq \frac{\hbar}{2} (1 + \beta (\Delta p)^2)$. Such modification is known by the name of *generalized uncertainty principle* (GUP). In turn, eq. this introduces a non-zero commutator between the coordinate operators:
Fig. 2 Minimum black hole mass in Planck-mass units as a function of brane separation (solid line). The shaded region indicates the allowed values of the mass. As the correction is removed (viz. $\ell \to 0^+$), the minimum mass is again $\sqrt{\pi} M_{\text{Pl}}$ (indicated by a dot), agreeing with (2).

$$[x_i, x_j] = 2i\hbar \beta (p_i x_j - p_j x_i).$$

(11)

Because the commutator does not vanish unless $\beta = 0$, the GUP introduces a non-zero minimal uncertainty in position, which translates into the existence of a minimal length. Furthermore, this results in a a momentum integration measure

$$\int \frac{d^n p}{1 + \beta |p|} |p\rangle \langle p| = 1,$$

(12)

which presents a UV cutoff of $\sqrt{\beta}$ [10]. This has important consequences for black hole evaporation and results in remnant formation.

The GUP replaces the Dirac delta in the description of point particles of regular quantum mechanics with a wider Gaussian distribution, $e^{-|x|/\sqrt{\beta}}$. As shown in [9], we can reproduce these non-local effects by means of the GUP-inspired metric

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r} \gamma(2; r/\sqrt{\beta})\right) dt^2 - \left(1 - \frac{2GM}{c^2 r} \gamma(2; r/\sqrt{\beta})\right)^{-1} dr^2 + r^2 d\Omega^2$$

(13)

Fig. 3 Metric coefficient for GUP-inspired metric. Notice naked singularity, extremal and regular black hole cases. The Schwarzschild (SBH) case for $M = 5M_{\text{Pl}}$ is shown for comparison.
Fig. 4 GUP auxiliary function (14) for different values of minimum area $\beta$, shown in Planck units. The roots indicate the minimum black hole mass for the particular value of $\beta_0$, where $\beta_0 = \beta/\ell_P^2$. Note that there are no roots for $\beta_0 \geq 1/\pi$. This indicates that for large enough minimum areas, GUP stops being self-complete.

where $\gamma(s;x) = \int_0^x t^{s-1} e^{-t} dt$ is the lower incomplete gamma function. The metric coefficient $1/g_{rr}$ is shown in Fig. 3. Note that the extremal case happens at $M_{BH} \approx 1.66\sqrt{B}/G$ and $r_H \approx 1.73\sqrt{B}$.

It is not possible to find an explicit expression for the horizon radius corresponding to (13). However, we can naively attempt to study the self-completeness of this metric by numerically solving $1/g_{rr} = 0$ under the constraint $r = \lambda_C(M)$, viz.

$$1 - \frac{2GM}{c^2\lambda_C} = 0$$

(14)

Rather than taking the usual expression for $\lambda_C$, we follow [2] [1] by correcting the Compton wavelength to account for GUP effects:

$$\lambda_{GUP} = \frac{\hbar}{Mc}(1 + \beta M^2).$$

(15)

Note that this step is not strictly required (see [9] for a more rigorous approach). The RHS of eq. (??) is plotted in Fig. 3. The roots of this function can be interpreted as the values of $M_{BH}$ at which the horizon radius coincides with the modified Compton wavelength for a given $\beta$, i.e. a minimum black hole mass.

The allowed black hole masses are shown in Fig 4. The GUP corrections can be undone by letting $\beta \to 0$, thus recovering eq. (2). Furthermore, we find that (14) has no positive roots for $\beta/\ell_P^2 \geq 1/\pi \approx 0.318$, where $\ell_P = \sqrt{\hbar G/c^3}$ is the Planck length. Consequently, GUP is not self-complete for $\beta \geq \ell_P^2/\pi$. This can be turn into an upper bound on the minimum area: $\beta < \ell_P^2/\pi$. Note that this is a constraint several orders of magnitude stronger than those found in [5] of $\beta_0 < 10^{21}$ and the corresponding energies are too high to be tested with current experiments.

4 Conclusions

We have explored the self-completeness of gravity under two different and independent frameworks: Randall-Sundrum and GUP. In the case of Randall-Sundrum, we
Fig. 5 Black hole mass in Planck units for varying $\beta$. The allowed values correspond to the shaded region. The minimum-mass curve (solid), which was obtained numerically, presents an asymptote at $\beta = \ell_P^2 / \pi \approx 0.318$. For $\beta = 0$, we recover the GR constraint (dot).

have shown that gravity should be self-complete regardless of the AdS curvature radius and found an closed-form solution for the minimum mass, eq. (8). This is not the case for GUP: under this formalism, gravity is only self-complete as long as the minimum area satisfies $\beta_0 \leq 1 / \pi$. Such condition could be understood as a constraint on GUP. This, however, is a heuristic analysis and should be complemented by a more formal treatment (c.f. [9]).

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