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Study of a Coupled System with Sub-Strip and Multi-Valued Boundary Conditions via Topological Degree Theory on an Infinite Domain

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Abstract: The existence and uniqueness of solutions for a coupled system of Liouville–Caputo type fractional integro-differential equations with multi-point and sub-strip boundary conditions are investigated in this study. The fractional integro-differential equations contain a finite number of Riemann–Liouville fractional integral and non-integral type nonlinearities, as well as Caputo differential operators of various orders subject to fractional boundary conditions on an infinite interval. At the boundary conditions, we use sub-strip and multi-point contribution. There are various techniques to solve such type of differential equations and one of the most common is known as symmetry analysis. The symmetry analysis has widely been used in problems involving differential equations, although determining the symmetries can be computationally intensive compared to other methods. Therefore, we employ the degree theory due to the Mawhin involving measure of a non-compactness technique to arrive at our desired findings. An interesting pertinent problem has also been provided to demonstrate the applicability of our results.

Keywords: measure of non-compactness; fractional differential equations; sub-strip and multi-point boundary conditions; existence results

1. Introduction

For the last few decades, the discipline of fractional calculus has gotten a lot of attention from authors. Fractional derivatives have shown to be an excellent tool for modeling a variety of issues. The rationale for this accomplishment is that fractional derivatives have a higher level of stability and efficiency than classical derivatives. Modeling memory and hereditary processes of the physical world, difficulties with higher precision, and comprehensibility than classical derivatives and integrals are intriguing features of the topic concerned. As a result, there are numerous applications of this subject of mathematics in diverse fields of research such as [1–3].

Because of its numerous applications in the scientific and social sciences, fractional calculus has become a significant topic of research. Bio-engineering [4], ecology [5], financial economics [6], chaos and fractional dynamics [7], and other fields are examples. Fractional calculus methods have advanced mathematical modeling of a variety of real-world issues. It was mostly owing to the non-local nature of fractional-order differential and integral operators. Coupled systems of fractional-order differential and integro-differential equations are common in fractional-order mathematical models.

The application of fractional calculus, specifically the fractional order derivatives, also stretch forth to the fluid mechanics. Kuslish and Lage [8] studied the time dependent problem of Newtonian fluid corresponding to time-dependent simulation. They have also
shown how fractional calculus in combination with the Laplace transformed approach can be used to lower the order of the differential equation governing the phenomenon. Moreover, Noman et al. [9] have studied the Prabhakar fractional model of Casson fluid, which is based on the generalized Fourier law for oscillating surfaces, controls momentum and thermal boundary surfaces using Laplace transform algorithms (for more study of fractional order differential equations towards fluid mechanics, please refer to references therein).

Moreover, there is no denying the fact that fractional derivatives have diverse applications towards solar energy since the creation of sustainable energy is a hot topic all around the world. Because of the rapid growth of human society, environmental pollution reduction and global energy challenges are becoming increasingly important. Solar energy has demonstrated to be the most effective precursor/source of free renewable energy with the least amount of environmental damage. Similarly, modeling in solar energy with respect to fractional derivatives is a lot more efficient than classical derivatives. This is because fractional derivatives can better explain heredity and memory phenomena.

Because of their applications in a variety of domains, arbitrary order differential equations and multi-point boundary conditions have piqued the interest of nonlinear phenomenon researchers (see, for instance, [10–12]). The existence theory primarily employs two methods: (a) fixed point theory and (b) topological degree theory. The former has been well investigated, and there are several results in the literature that attest to the existence and uniqueness of a problem’s solution (see, for example, Refs. [13–17]), and references therein. Here, we state that topological degree theory has been used by Mawhin more than fifty years ago. The theory mentioned has been applied for various types of problems including functional mapping problems and operators theory [18–20]. It has also used by the mentioned author for boundary value problems in 1993 (see details in [21]). On the contrary, a review of the literature reveals that the latter is used in a small number of publications to prove the existence and uniqueness of a solution to nonlinear fractional differential equations with integral type boundary conditions. The vast range of applications of the coupled system of fractional differential equations [22,23] to real-world problems motivates researchers to investigate it. In order to determine the existence and unique solution for the aforementioned problems, the authors used topological degree theory (see, for example, [10,24,25]).

The effects of coupled integro-differential boundary conditions on a fractional-order nonlinear mixed coupled system with coupled integro-differential boundary conditions were investigated in [23]. In a recent paper [26], the authors looked at the presence of solutions for mixed-order coupled fractional differential equations of the Caputo and Riemann–Liouville types, as well as inclusions with coupled integral fractional boundary conditions. In the latest research [27], the authors studied the existence and uniqueness of solution via fixed point theory to a new class of nonlinear coupled Liouville–Caputo type fractional integro-differential equations subject to integral type boundary conditions. The current manuscript is dedicated to addressing the existence and uniqueness of solutions to a class of nonlinear coupled Liouville–Caputo type fractional integro-differential equations

\[
\begin{align*}
\mathcal{C}D^\zeta_\eta u(\eta) + \sum_{m=1}^n \mathcal{I}^\lambda m h_m(\eta, u(\eta), v(\eta)) &= x_1(\eta, u(\eta), v(\eta)), \zeta \in (1, 2), 0 \leq \eta \leq 1 \\
\mathcal{C}D^\lambda_\eta v(\eta) + \sum_{n=1}^\zeta \mathcal{I}^\alpha n g_n(\eta, u(\eta), v(\eta)) &= x_2(\eta, u(\eta), v(\eta)), \lambda \in (1, 2), 0 \leq \eta \leq 1,
\end{align*}
\]  

subject to the following boundary conditions:

\[
\begin{align*}
u(0) &= P(u), \\
\phi_1 u(1) + \varphi_1 \mathcal{D}^\zeta_\eta u(1) &= \int_\zeta^1 (1-s)^{\zeta-1} v(s) ds + \sum_{j=1}^\zeta \xi_j \mathcal{D}^\zeta_\eta v(\mu_j), \\
\phi_2 v(1) + \varphi_2 \mathcal{D}^\lambda_\eta v(1) &= \int_\lambda^1 (1-s)^{\lambda-1} u(s) ds + \sum_{j=1}^\lambda \eta_j \mathcal{D}^\lambda_\eta u(\nu_j),
\end{align*}
\]

where \(\mathcal{C}D^\zeta_\eta, \mathcal{C}D^\lambda_\eta\) show the Caputo fractional derivative of arbitrary order \(\zeta \in (1, 2)\) and \(\lambda \in (1, 2)\), respectively, \(x_1, x_2, h_m, g_n : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} [(m = 1, \ldots, k), (n = 1, \ldots, \zeta)]\) are...
continuous functions, $B, P : [0, 1] \to \mathbb{R}$ are also continuous functions, $\varphi_1, \varphi_2, \varphi_1, \varphi_2, v_1, v_2, \xi_j$, $\tau_j \in \mathbb{R}, \omega, \mu_j \in [0, 1]$, and $j = 1, \ldots, \rho$. The general analysis is backed up by a few examples for confirmation of the findings.

2. Preliminaries

The current section recalls a few fundamental definitions from the literature, which will be useful in proving our results. Thoroughly, in this study, use the notation $\mathcal{P} = C([0, 1], \mathbb{R}), \mathcal{Q} = C([0, 1], \mathbb{R})$ to express the Banach space under the topological norm: $\| a(\eta) \| = \sup \{ | a(\eta) | : 0 \leq \eta \leq 1 \}$, and $\mathcal{B}$ denotes all the bounded subsets of $\mathcal{B}(\mathcal{P} \times \mathcal{Q})$, where the product $\mathcal{P} \times \mathcal{Q}$ is a Banach space under the norm $\|(a, b)\| = \|a\| + \|b\|$.

**Definition 1** ([24]). Let $\Delta : \mathcal{L} \to \mathcal{P}$ be a mapping which is continuous and bounded, where $\mathcal{L} \subseteq \mathcal{P}$. Then, $\Delta$ is

1. $\sigma$-Lipschitz, whenever, there is a constant $K \geq 0$, $\exists \sigma(\mathcal{S}) \leq K \sigma(\mathcal{S})$, for every bounded set $\mathcal{S} \subseteq \mathcal{L}$;
2. strict $\sigma$-contraction, whenever, there is a constant $K \in [0, 1)$, along $\sigma(\Delta(\mathcal{S})) \leq K \sigma(\mathcal{S})$, for every bounded set $\mathcal{S} \subseteq \mathcal{L}$;
3. $\sigma$-condensing, whenever, $\sigma(\Delta(\mathcal{S})) < \sigma(\mathcal{S})$, for every bounded set $\mathcal{S} \subseteq \mathcal{L}$, having $\sigma(\mathcal{S}) > 0$. Specifically, $\sigma(\Delta(\mathcal{S})) \geq \sigma(\mathcal{S})$ suggests $\sigma(\mathcal{S}) = 0$.

In addition, the mapping $\Delta : \mathcal{L} \to \mathcal{P}$ is a Lipschitz, if a constant $K > 0$, $\exists \exists$ $\| \Delta(a) - \Delta(b) \| \leq K \mid a - b \mid, \forall a, b \in \mathcal{L}$.

Furthermore, $\Delta$ is a strict contraction, whenever $K < 1$.

**Proposition 1** ([28]). Whenever the mappings, $\Delta, \Pi : \mathcal{L} \to \mathcal{P}$, are $\sigma$-Lipschitz, having constants $K_1$ and $K_2$ respectively, then $\Delta + \Pi$ is a $\sigma$-Lipschitz having constant $K_1 + K_2$.

**Proposition 2** ([28]). Whenever the mapping $\Delta : \mathcal{L} \to \mathcal{P}$ is a Lipschitz having the constant $K$, then $\Delta$ is $\sigma$-Lipschitz having the similar constant $K$.

**Proposition 3** ([28]). Whenever the mapping $\Delta : \mathcal{L} \to \mathcal{P}$ is a compact having a constant $K$. Then, $\Delta$ is $\sigma$-Lipschitz having the similar constant $K$.

**Theorem 1** ([28]). Consider $\Delta : \mathcal{L} \to \mathcal{P}$ to be a $\sigma$-condensing having $\Delta = \{ a \in \mathcal{P} : \text{there exists, } \nu \in [0, 1], \text{ with } a = \Delta \nu a \}$.

If $\Lambda$ is bounded in $\mathcal{P}$, then there $\exists$ a constant $h > 0$, $\exists \Lambda \subset \mathcal{S}_h(0)$, and then a degree can be defined as:

$$D(\mathcal{I} - \nu \Delta, \mathcal{S}_h(0), 0) = 1, \text{ for all } 0 \leq \nu \leq 1.$$ 

This confirms that a fixed point $\exists$, say $x_0$ of $\Delta$, $\exists, x_0 \in \mathcal{S}_h(0)$.

**Definition 2** ([1]). Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. The Riemann–Liouville fractional integral of order $\gamma$, for $f \in \mathcal{L}_1[0, 1]$ that exists almost everywhere on $[p, 1]$ can be defined as:

$$\mathcal{I}^\gamma f(\eta) = \frac{1}{\Gamma(\gamma)} \int_0^\eta (\eta - \nu)^{-1} f(\nu) d\nu,$$

given that the right side is point-wise defined on $\mathbb{R}$.

**Definition 3** ([1]). Consider the function $f : [0, +\infty) \to (-\infty, \infty)$. The fractional Caputo-derivative of the order $\xi$ can be demonstrated as:
\[ D^\delta f(\eta) = \frac{1}{\Gamma(n-\zeta)} \int_0^\eta (\eta - v)^{n-\zeta-1} f^{(n)}(v)dv, \eta \in [p, q]. \] (4)

**Lemma 1** ([1]). The general solution of a fractional differential equation \( ^cD^\delta f(\eta) = 0 \) for \( n-1 < \zeta < n \) is given by
\[
 f(\eta) = d_0 + d_1 \eta + d_2 \eta^2 + \ldots + d_{n-1} \eta^{n-1},
\]
where \( d_i \in \mathbb{R}, i = 0, \ldots, n-1 \).

From Lemma 1, it can be observed that
\[
 T^\delta [^cD^\delta f(\eta)] = f(\eta) + d_0 + d_1 t + \ldots + d_{n-1} t^{n-1},
\] (5)
for arbitrary \( d_i \in \mathbb{R}, i = 0, \ldots, n-1 \).

**3. Main Results**

The current section is dedicated to the existence results for the system of Equation (1) subject to the boundary conditions given as (2).

Before proceeding, consider the following lemma in which we first solve the associated linear problem in order to solve the problem (1), subject to the given boundary conditions.

**Lemma 2.** Consider for \( Y_1 \neq 0 \), and for \( x_1, x_2 \in C([0, 1], \mathbb{R}) \), the solution of the fractional differential equations
\[
\begin{case}
^cD^\delta u(\eta) &= x_1(\eta), \zeta \in (1, 2], 0 \leq \eta \leq 1, \\
^cD^\lambda v(\eta) &= x_2(\eta), \lambda \in (1, 2], 0 \leq \eta \leq 1,
\end{case}
\] (6)
subject to the boundary conditions provided in (2) given by
\[
\begin{align*}
 u(\eta) &= \int_0^\eta \frac{(\eta - s)^{\zeta-1}}{\Gamma(\zeta)} x_1(s)ds + B(u) - \frac{\eta}{Y_1} \left[ \pi_1 \int_0^1 \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} x_1(s)ds \\
 &\quad + \pi_2 x_1(s) - \pi_3 \int_0^\omega \frac{(1-s)^{\lambda-1}}{\Gamma(\lambda)} \int_0^s \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} x_2(t)dt ds \\
 &\quad - \pi_4 x_2(s) + \pi_5 \int_0^1 \frac{(1-s)^{\lambda-1}}{\Gamma(\lambda)} x_2(s)ds \\
 &\quad + \pi_6 x_2(s) + \pi_7 \int_0^\omega \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} \int_0^s \frac{(s-t)^{\zeta-1}}{\Gamma(\zeta)} x_1(t)dt ds \\
 &\quad - \pi_8 x_1(s) + Y_2 \right], \\
 v(\eta) &= \int_0^\eta \frac{(\eta - s)^{\lambda-1}}{\Gamma(\lambda)} x_2(s)ds + P(v) - \frac{\eta}{Y_1} \left[ \pi_9 \int_0^1 \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} x_1(s)ds \\
 &\quad + \pi_{10} x_1(s) - \pi_{11} \int_0^\omega \frac{(1-s)^{\lambda-1}}{\Gamma(\lambda)} \int_0^s \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} x_2(t)dt ds \\
 &\quad - \pi_{12} x_2(s) + \pi_{13} \int_0^1 \frac{(1-s)^{\lambda-1}}{\Gamma(\lambda)} x_2(s)ds \\
 &\quad + \pi_{14} x_2(s) - \pi_{15} \int_0^\omega \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} \int_0^s \frac{(s-t)^{\zeta-1}}{\Gamma(\zeta)} x_1(t)dt ds \\
 &\quad - \pi_{16} x_1(s) - Y_3 \right],
\end{align*}
\] (7) (8)
where

\[ \pi_1 = \phi_1 \left( \frac{\varphi_2}{\Gamma(1 - \lambda)} - \varphi_2 \right), \quad \pi_2 = \varphi_1 \left( \frac{\varphi_2}{\Gamma(1 - \lambda)} - \varphi_2 \right), \]

\[ \pi_3 = v_1 \left( \frac{\varphi_2}{\Gamma(1 - \lambda)} - \varphi_2 \right), \quad \pi_4 = \left( \frac{\varphi_2}{\Gamma(1 - \lambda)} - \varphi_2 \right) \sum_{j=1}^{p} \xi_j, \]

\[ \pi_5 = \varphi_2 \left( \frac{v_1}{\Gamma(\xi)} \int_{0}^{\omega} (1 - s)\xi^{-1} ds + \frac{1}{\Gamma(1 - \lambda)} \sum_{j=1}^{p} \xi_j \right), \]

\[ \pi_6 = \varphi_2 \left( \frac{v_1}{\Gamma(\xi)} \int_{0}^{\omega} (1 - s)\xi^{-1} ds + \frac{1}{\Gamma(1 - \lambda)} \sum_{j=1}^{p} \xi_j \right), \]

\[ \pi_7 = v_2 \left( \frac{v_1}{\Gamma(\xi)} \int_{0}^{\omega} (1 - s)\xi^{-1} ds + \frac{1}{\Gamma(1 - \lambda)} \sum_{j=1}^{p} \xi_j \right), \]

\[ \pi_8 = \frac{v_1}{\Gamma(\xi)} \int_{0}^{\omega} (1 - s)\xi^{-1} ds + \frac{1}{\Gamma(1 - \lambda)} \sum_{j=1}^{p} \xi_j \sum_{j=1}^{p} \tau_j, \]

\[ \pi_9 = \phi_1 \left( \frac{v_2}{\Gamma(\xi)} \int_{0}^{\omega} (1 - s)\xi^{-1} ds + \frac{1}{\Gamma(1 - \zeta)} \sum_{j=1}^{p} \tau_j \right), \]

\[ \pi_{10} = \varphi_1 \left( \frac{v_2}{\Gamma(\xi)} \int_{0}^{\omega} (1 - s)\xi^{-1} ds + \frac{1}{\Gamma(1 - \zeta)} \sum_{j=1}^{p} \tau_j \right), \]

\[ \pi_{11} = \phi_1 \left( \frac{v_2}{\Gamma(\xi)} \int_{0}^{\omega} (1 - s)\xi^{-1} ds + \frac{1}{\Gamma(1 - \zeta)} \sum_{j=1}^{p} \tau_j \right), \]

\[ \pi_{12} = \left( \frac{v_2}{\Gamma(\xi)} \int_{0}^{\omega} (1 - s)\xi^{-1} ds + \frac{1}{\Gamma(1 - \zeta)} \sum_{j=1}^{p} \tau_j \right) \sum_{j=1}^{p} \xi_j, \]

\[ \pi_{13} = \varphi_1 \left( \frac{\varphi_1}{\Gamma(1 - \xi)} \right), \quad \pi_{14} = \psi_1 \left( \frac{\varphi_1}{\Gamma(1 - \xi)} \right), \]

\[ \pi_{15} = v_2 \left( \phi_1 - \frac{\varphi_1}{\Gamma(1 - \xi)} \right), \quad \pi_{16} = \left( \phi_1 - \frac{\varphi_1}{\Gamma(1 - \xi)} \right) \sum_{j=1}^{p} \tau_j. \]

\[ Y_1 = \left( \phi_1 - \frac{\varphi_1}{\Gamma(1 - \xi)} \right) \left( \frac{\varphi_2}{\Gamma(1 - \lambda)} - \varphi_2 \right) + \left( \frac{v_1}{\Gamma(\xi)} \int_{0}^{\omega} (1 - s)\xi^{-1} ds + \frac{1}{\Gamma(1 - \lambda)} \sum_{j=1}^{p} \xi_j \right) \left( \frac{v_2}{\Gamma(\xi)} \int_{0}^{\omega} (1 - s)\xi^{-1} ds + \frac{1}{\Gamma(1 - \zeta)} \sum_{j=1}^{p} \tau_j \right), \]

\[ Y_2 = \left( \frac{\varphi_2}{\Gamma(1 - \lambda)} - \varphi_2 \right) \left[ \phi_1 B(u) + \varphi_1 D\xi B(u) - v_1 I^\lambda P(v) - D^\lambda P(v) \sum_{j=1}^{p} \xi_j \right] \]

\[ + \left( \frac{v_1}{\Gamma(\xi)} \int_{0}^{\omega} (1 - s)\xi^{-1} ds + \frac{1}{\Gamma(1 - \lambda)} \sum_{j=1}^{p} \xi_j \right) \left[ \phi_2 P(v) + \varphi_2 D^\lambda P(v) - v_2 I^\xi B(u) \right. \]

\[ - D^\xi B(u) \sum_{j=1}^{p} \tau_j \right] \]
\[
Y_3 = \left( \frac{v_2}{\Gamma(\zeta)} \int_0^\omega (1 - s)^{\zeta - 1}ds + \frac{1}{\Gamma(1 - \zeta)} \sum_{j=1}^p \frac{\tau_j}{(\mu_j)^\zeta} \right) \left[ \phi_1 B(u) + \phi_1 D^\zeta B(u) - v_1 \mathcal{I}^\zeta P(v) \right. \\
+ D^\zeta P(v) \sum_{j=1}^p \xi_j \left. \right] + \left( \phi_1 - \frac{\phi_1}{\Gamma(1 - \zeta)} \right) \left[ \phi_2 P(v) + \phi_2 D^\zeta P(v) - v_2 \mathcal{I}^\zeta B(u) \right. \\
- D^\zeta B(u) \sum_{j=1}^p \tau_j \right].
\]

**Proof.** To prove, we need to apply the fractional integrals \( \mathcal{I}^\zeta \) and \( \mathcal{I}^\lambda \) of order \( \zeta \) and \( \lambda \), respectively, on the first and second equations of (6) and then, using (2), we have

\[
u(\eta) = \int_0^\eta \frac{(\eta - s)^{\zeta - 1}}{\Gamma(\zeta)} \xi_1(s)ds - c_0 + c_1 \eta,
\]

(13)

\[
u(\eta) = \int_0^\eta \frac{(\eta - s)^{\lambda - 1}}{\Gamma(\lambda)} \xi_2(s)ds - d_0 + d_1 \eta,
\]

(14)

where \( c_0, c_1, d_0, \) and \( d_1 \) are arbitrary constants. Now using Equations (15) and (16) along with the conditions \( u(0) = B(u) \) and \( v(0) = P(v) \), consequently, from the above equations, we have

\[
u(\eta) = \int_0^\eta \frac{(\eta - s)^{\zeta - 1}}{\Gamma(\zeta)} \xi_1(s)ds + B(u) + c_1 \eta,
\]

(15)

\[
u(\eta) = \int_0^\eta \frac{(\eta - s)^{\lambda - 1}}{\Gamma(\lambda)} \xi_2(s)ds + P(v) + d_1 \eta,
\]

(16)

Now using Equations (15) and (16) along with the conditions

\[
\phi_1 u(1) + \phi_1 D^\zeta u(1) = \frac{v_1}{\Gamma(\zeta)} \int_0^\omega (1 - s)^{\zeta - 1}v(s)ds + \sum_{j=1}^p \xi_j D^\zeta u(\mu_j),
\]

\[
\phi_2 v(1) + \phi_2 D^\lambda v(1) = \frac{v_2}{\Gamma(\lambda)} \int_0^\omega (1 - s)^{\lambda - 1}u(s)ds + \sum_{j=1}^p \tau_j D^\lambda u(\mu_j),
\]

we obtain the system of equations in the form of unknowns as follows:

\[
\begin{align*}
K_1 c_1 - L_1 d_1 &= \Delta_1, \\
-L_1 c_1 + K_2 d_1 &= \Delta_2,
\end{align*}
\]

(17)

where

\[
K_1 = \left( \phi_1 + \frac{\phi_1}{\Gamma(1 - \zeta)} \right), \quad L_1 = \frac{v_1}{\Gamma(\zeta)} \int_0^\omega (1 - s)^{\zeta - 1}ds + \frac{1}{\Gamma(1 - \zeta)} \sum_{j=0}^p \xi_j \Gamma(\mu_j)^\zeta,
\]

\[
K_2 = \left( \frac{\phi_2}{\Gamma(1 - \lambda)} - \phi_2 \right), \quad L_2 = -\frac{v_2}{\Gamma(\lambda)} \int_0^\omega (1 - s)^{\lambda - 1}ds + \frac{1}{\Gamma(1 - \zeta)} \sum_{j=0}^p \tau_j \Gamma(\mu_j)^\lambda,
\]

\[
N_1 = \frac{\phi_1}{\Gamma(\zeta)} \int_0^1 (1 - s)^{\zeta - 1} \xi_1(s)ds + \phi_1 B(u) + \phi_1 \xi_1(s) + \phi_1 D^\zeta B(u)
\]

\[
- \frac{v_1}{\Gamma(\zeta)} \int_0^\omega (1 - s)^{\zeta - 1} \int_0^s \frac{(s - t)^{\lambda - 1}}{\Gamma(\lambda)} \xi_2(t)dt ds - v_1 \mathcal{I}^\zeta P(v) - \sum_{j=1}^p \xi_j \phi_1 
\]

\[
- D^\zeta P(v) \sum_{j=1}^p \xi_j,
\]

\[
- \phi_2 P(v) \sum_{j=1}^p \xi_j.
\]
\[ \Theta_2 = \frac{\phi_2}{\Gamma(\lambda)} \int_0^1 (1-s)^{\lambda-1} x_2(s) ds + \phi_2 P(v) + \varphi_2 \mathcal{I}_2(s) + \varphi_2 D^\lambda P(v) \]
\[ - \frac{v_2}{\Gamma(\xi)} \int_0^s (s-t)^{\xi-1} \int_0^t \frac{1}{\Gamma(\xi)} \mathcal{I}_1(t) dt ds - v_2 \mathcal{I}_2 B(u) - \mathcal{I}_1(s) \sum_{j=1}^{\nu} \tau_j \]
\[ - D^\xi B(u) \sum_{j=1}^{\nu} \tau_j. \]

By solving the system (17), we determine the following values for the constants \( c_1 \) and \( d_1 \), respectively:
\[ c_1 = \frac{K_2 \Theta_1 + L_1 \Theta_2}{K_1 K_2 - L_1 L_2}, \quad d_1 = \frac{L_2 \Theta_1 + K_1 \Theta_2}{K_1 K_2 - L_1 L_2}. \]

By putting the values of \( c_1 \) and \( d_1 \) in the Equations (13) and (14), respectively, together with the values of \( K_1, K_2, L_1, L_2, \Theta_1, \) and \( \Theta_2 \), we obtain the solution given by the Equations (7) and (8), respectively. \( \square \)

**Corollary 1.** In the sense of Lemma 2, the solution to the problem (1) subject to the boundary conditions given by (2) can be considered as a unique common solution of the following:

\[ u(\eta) = \left\{ \begin{array}{ll}
\int_0^\eta \frac{(\eta-s)^{\xi-1}}{\Gamma(\xi)} x_1(s, u(s), v(s)) ds + B(u) - \sum_{m=1}^\xi \int_0^1 \frac{(\eta-s)^{\gamma_m-1}}{\Gamma(\gamma_m)} h_m(s, u(s), v(s)) ds \\
- \frac{\eta}{\Gamma(1)} \pi_1 \int_0^1 \left( \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} x_1(s, u(s), v(s)) - \sum_{m=1}^\xi \frac{(1-s)^{\gamma_m-1}}{\Gamma(\gamma_m)} h_m(s, u(s), v(s)) \right) ds \\
+ \pi_2 \left( x_1(s, u(s), v(s)) - \sum_{m=1}^\xi \int_0^1 \frac{(1-s)^{\gamma_m-1}}{\Gamma(\gamma_m)} h_m(s, u(s), v(s)) ds \right) \\
- \pi_3 \left( \int_0^\gamma \frac{(1-s)^{\lambda-1}}{\Gamma(\lambda)} \int_0^s \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} x_2(t, u(t), v(t)) dt ds \\
+ \sum_{m=1}^\xi \int_0^\gamma \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \int_0^s \frac{(s-t)^{\xi-1}}{\Gamma(\xi)} \int_0^1 \frac{(1-f)^{\alpha_m-1}}{\Gamma(\alpha_m)} g_m(f, u(f), v(f)) df dt ds \\
- \pi_4 \left( x_2(s, u(s), v(s)) + \int_0^\gamma \sum_{m=1}^\xi \frac{(1-s)^{\xi_m-1}}{\Gamma(\xi_m)} g_m(s, u(s), v(s)) ds \right) \\
+ \pi_5 \int_0^1 \left( \frac{(1-s)^{\lambda-1}}{\Gamma(\lambda)} x_2(s, u(s), v(s)) - \sum_{m=1}^\xi \frac{(1-s)^{\lambda_m+\gamma_m-1}}{\Gamma(\lambda_m+\gamma_m)} g_m(s, u(s), v(s)) \right) ds \\
+ \pi_6 \left( x_2(s, u(s), v(s)) - \int_0^1 \sum_{m=1}^\xi \frac{(1-s)^{\lambda_m-1}}{\Gamma(\lambda_m)} g_m(s, u(s), v(s)) ds \right) \\
+ \pi_7 \left( \int_0^\gamma \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \int_0^s \frac{(s-t)^{\xi-1}}{\Gamma(\xi)} x_1(t, u(t), v(t)) dt ds + \sum_{m=1}^\xi \int_0^\gamma \frac{(1-s)^{\xi-1}}{\Gamma(\xi)} \int_0^1 \frac{(s-t)^{\xi-1}}{\Gamma(\xi)} \\
\int_0^1 \frac{(1-f)^{\gamma_m-1}}{\Gamma(\gamma_m)} h_m(f, u(f), v(f)) df dt ds \\
- \pi_8 \left( x_1(s, u(s), v(s)) + \int_0^\gamma \sum_{m=1}^\xi \frac{(1-s)^{\xi_m+\gamma_m-1}}{\Gamma(\xi_m+\gamma_m)} g_m(s, u(s), v(s)) ds \right) \\
+ Y_2 \right) \right. \]
\[ h_m(s, u(s), v(s)) ds \]
\[
\Psi(\eta) = \begin{cases} 
\int_0^\eta \frac{(\eta-s)^{\lambda-1}}{\Gamma(\lambda)} x_2(s,u(s),v(s))ds + P(v) - \sum_{n=1}^{\infty} \int_0^t \frac{(\eta-s)^{\lambda+\gamma_n-1}}{\Gamma(\lambda+\gamma_n)} g_n(s,u(s),v(s))ds \\
- \frac{\eta}{\pi} \int_0^1 \frac{1}{\Gamma(\zeta)} \frac{(\eta-s)^{\zeta-1}}{\Gamma(\zeta)} x_1(s,u(s),v(s)) - \sum_{n=1}^{\infty} \frac{(\eta-s)^{\lambda+\gamma_n-1}}{\Gamma(\lambda+\gamma_n)} g_n(s,u(s),v(s))ds \\
+ \pi_11 \left( x_1(s,u(s),v(s)) - \sum_{n=1}^{\infty} \frac{(\eta-s)^{\lambda+\gamma_n-1}}{\Gamma(\lambda+\gamma_n)} g_n(s,u(s),v(s))ds \right) \\
- \pi_{10} \left( x_1(s,u(s),v(s)) - \sum_{n=1}^{\infty} \frac{(\eta-s)^{\lambda+\gamma_n-1}}{\Gamma(\lambda+\gamma_n)} g_n(s,u(s),v(s))ds \right) \\
+ \sum_{n=1}^{\infty} \int_0^\eta \frac{(\eta-s)^{\lambda-1}}{\Gamma(\lambda)} \int_0^s \frac{1}{\Gamma(\lambda)} \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} x_2(t,u(t),v(t))dt ds \\
+ \sum_{n=1}^{\infty} \int_0^1 \frac{(\eta-s)^{\lambda-1}}{\Gamma(\lambda)} \int_0^s \frac{1}{\Gamma(\lambda)} \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} x_2(s,u(s),v(s))ds \\
+ \sum_{n=1}^{\infty} \int_0^1 \frac{(\eta-s)^{\lambda-1}}{\Gamma(\lambda)} \int_0^s \frac{1}{\Gamma(\lambda)} \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} x_2(s,u(s),v(s))ds \\
+ \sum_{n=1}^{\infty} \int_0^1 \frac{(\eta-s)^{\lambda-1}}{\Gamma(\lambda)} \int_0^s \frac{1}{\Gamma(\lambda)} \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} x_2(s,u(s),v(s))ds \\
\end{cases}
\]

Consider \( \Psi = (\Psi_1, \Psi_2) \), where \( \Psi_1 : \mathcal{P} \to \mathcal{P} \) and \( \Psi_2 : \mathcal{Q} \to \mathcal{Q} \) are operators defined by:

\[
\Psi_1(u(\eta)) = \int_0^\eta \frac{(\eta-s)^{\lambda-1}}{\Gamma(\lambda)} x_2(s,u(s),v(s))ds - \sum_{n=1}^{\infty} \int_0^\eta \frac{(\eta-s)^{\lambda+\gamma_n-1}}{\Gamma(\lambda+\gamma_n)} h_m(s,u(s),v(s))ds,
\]

\[
\Psi_2(v(\eta)) = \int_0^\eta \frac{(\eta-s)^{\lambda-1}}{\Gamma(\lambda)} x_2(s,u(s),v(s))ds - \sum_{n=1}^{\infty} \int_0^\eta \frac{(\eta-s)^{\lambda+\gamma_n-1}}{\Gamma(\lambda+\gamma_n)} g_n(s,u(s),v(s))ds.
\]

Moreover, consider \( \Phi = (\Phi_1, \Phi_2) : \mathcal{P} \times \mathcal{Q} \to \mathcal{P} \times \mathcal{Q} \) to be operators given by:
\[ \Phi_1(u, v)(\eta) = \left\{ \begin{aligned} B(u) - \frac{\eta}{Y_1} \left[ \pi_1 \int_0^1 \left( \frac{(1 - s)^{\zeta - 1}}{\Gamma(\zeta)} x_1(s, u(s), v(s)) \right) \, ds \
\sum_{m=1}^{\kappa} \frac{(1 - s)^{\zeta + \gamma_m - 1}}{\Gamma(\zeta + \gamma_m)} h_m(s, u(s), v(s)) \right] ds \
\pi_2 \left( x_1(s, u(s), v(s)) - \sum_{m=1}^{\kappa} \int_0^1 \frac{(1 - s)^{\zeta + \gamma_m - 1}}{\Gamma(\zeta + \gamma_m)} h_m(s, u(s), v(s)) \, ds \right) \
\pi_3 \left( \int_0^\omega \frac{(1 - s)^{\lambda - 1}}{\Gamma(\lambda)} \int_0^s \frac{(s - t)^{\lambda - 1}}{\Gamma(\lambda)} x_2(t, u(t), v(t)) \, dt \, ds \
\sum_{n=1}^{\varsigma} \frac{(1 - s)^{\lambda + \alpha_n - 1}}{\Gamma(\lambda + \alpha_n)} g_n(s, u(s), v(s)) \right) ds \
\pi_4 \left( x_2(s, u(s), v(s)) + \int_0^{\mu_2} \sum_{n=1}^{\varsigma} \frac{(\mu_n - s)^{\lambda + \alpha_n - 1}}{\Gamma(\lambda + \alpha_n)} g_n(s, u(s), v(s)) \, ds \right) \
\pi_5 \left( \int_0^1 \frac{(1 - s)^{\lambda - 1}}{\Gamma(\lambda)} x_2(s, u(s), v(s)) - \sum_{n=1}^{\varsigma} \frac{(1 - s)^{\lambda + \alpha_n - 1}}{\Gamma(\lambda + \alpha_n)} g_n(s, u(s), v(s)) \right) ds \
\pi_6 \left( x_2(s, u(s), v(s)) - \int_0^1 \sum_{n=1}^{\varsigma} \frac{(1 - s)^{\lambda + \alpha_n - 1}}{\Gamma(\lambda + \alpha_n)} g_n(s, u(s), v(s)) \, ds \right) \
\pi_7 \left( \int_0^\omega \frac{(1 - s)^{\zeta - 1}}{\Gamma(\zeta)} \int_0^s \frac{(s - t)^{\zeta - 1}}{\Gamma(\zeta)} x_1(t, u(t), v(t)) \, dt \, ds \
\sum_{m=1}^{\kappa} \frac{(1 - s)^{\zeta + \gamma_m - 1}}{\Gamma(\zeta + \gamma_m)} h_m(s, u(s), v(s)) \right) ds \
\pi_8 \left( x_1(s, u(s), v(s)) + \int_0^{\mu_2} \sum_{n=1}^{\varsigma} \frac{(\mu_n - s)^{\zeta + \gamma_m - 1}}{\Gamma(\zeta + \gamma_m)} h_m(s, u(s), v(s)) \, ds \right) \
+ Y_2 \right] , \end{aligned} \right. \]
\[
\Phi_2(u,v)(\eta) = \left\{ \begin{array}{lr}
P(v) - \frac{\eta}{Y_1} \left[ \tau_9 \int_0^1 \left( \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} \right) x_1(s, u(s), v(s)) ds \\
+ \sum_{m=1}^\kappa \left( \frac{(1-s)^{\zeta+\gamma_m-1}}{\Gamma(\zeta + \gamma_m)} \right) h_m(s, u(s), v(s)) \right] ds \\
+ \tau_{10} \left( x_1(s, u(s), v(s)) - \sum_{m=1}^\kappa \int_0^1 \left( \frac{(1-s)^{\zeta+\gamma_m-1}}{\Gamma(\zeta + \gamma_m)} \right) h_m(s, u(s), v(s)) ds \right) \\
- \tau_{11} \left( \int_0^\omega \left( \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} \right) x_2(t, u(t), v(t)) dt ds \\
+ \sum_{n=1}^\zeta \int_0^\omega \left( \frac{(1-s)^{\lambda-1}}{\Gamma(\lambda)} \right) x_3(s, u(s), v(s)) + \int_0^\omega \left( \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} \right) g_n(f, u(f), v(f)) df dt ds \\
- \tau_{12} \left( x_3(s, u(s), v(s)) + \sum_{n=1}^\kappa \int_0^\omega \left( \frac{(1-s)^{\lambda+\alpha_n-1}}{\Gamma(\lambda + \alpha_n)} \right) g_n(s, u(s), v(s)) ds \right) \\
+ \tau_{13} \int_0^1 \left( \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} \right) x_4(s, u(s), v(s)) - \sum_{n=1}^\zeta \left( \frac{(1-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} \right) g_n(s, u(s), v(s)) ds \\
+ \tau_{14} \left( x_4(s, u(s), v(s)) - \int_0^1 \left( \frac{(1-s)^{\zeta+\alpha_n-1}}{\Gamma(\zeta + \alpha_n)} \right) g_n(s, u(s), v(s)) ds \right) \\
+ \tau_{15} \left( \int_0^\omega \left( \frac{(s-t)^{\zeta-1}}{\Gamma(\zeta)} \right) x_5(t, u(t), v(t)) dt ds \\
+ \sum_{m=1}^\kappa \int_0^\omega \left( \frac{(1-s)^{\zeta+\gamma_m-1}}{\Gamma(\zeta + \gamma_m)} \right) h_m(f, u(f), v(f)) df dt ds \right) \\
- \tau_{16} \left( x_5(s, u(s), v(s)) + \int_0^\omega \left( \frac{(1-s)^{\zeta+\gamma_m-1}}{\Gamma(\zeta + \gamma_m)} \right) h_m(s, u(s), v(s)) ds \right) \\
+ Y_3 \right].
\]

Since \( \Omega = \Psi + \Phi \), then the system (1) can be written as:

\[
(u, v)(\eta) = \Omega(u, v)(\eta) = \Psi(u, v)(\eta) + \Phi(u, v)(\eta).
\]

Alternatively, (20) is the solution of the problem (1) subject to the boundary conditions given by (2).

Before proceeding to the existence and uniqueness results, we define the following, which will prove to be helpful in the forthcoming results.

Since \( x_1, x_2, B(u), P(v), h_m \) for \( m = 1, \ldots, \kappa \), and \( g_n \) for \( n = 1, \ldots, \zeta \) are continuous functions, then it follows that:

\[
x_1(\eta, u_\eta(\eta), v_\eta(\eta)) \to x_1(\eta, v(\eta), u(\eta)), \text{ as } r \to \infty,
\]

\[
x_2(\eta, u_\eta(\eta), v_\eta(\eta)) \to x_2(\eta, v(\eta), u(\eta)), \text{ as } r \to \infty.
\]

Then, for any \( \eta \in [0, 1] \), we find

\[
|x_1(\eta, u_\eta(\eta), v_\eta(\eta)) - x_1(\eta, v(\eta), u(\eta))| \to 0, \text{ as } r \to \infty,
\]

\[
|x_2(\eta, u_\eta(\eta), v_\eta(\eta)) - x_2(\eta, v(\eta), u(\eta))| \to 0, \text{ as } r \to \infty.
\]
Furthermore, consider
\[ h_1(\eta, u_r(\eta), v_r(\eta)) + h_2(\eta, u_r(\eta), v_r(\eta)) + \ldots + h_c(\eta, u_r(\eta), v_r(\eta)) = \Pi(\eta, u_r(\eta), v_r(\eta)), \]
\[ h_1(\eta, u(\eta), v(\eta)) + h_2(\eta, u(\eta), v(\eta)) + \ldots + h_c(\eta, u(\eta), v(\eta)) = \Pi(\eta, u(\eta), v(\eta)), \]
\[ g_1(\eta, u_r(\eta), v_r(\eta)) + g_2(\eta, u_r(\eta), v_r(\eta)) + \ldots + g_\zeta(\eta, u_r(\eta), v_r(\eta)) = \chi(\eta, u_r(\eta), v_r(\eta)), \]
\[ g_1(\eta, u(\eta), v(\eta)) + g_2(\eta, u(\eta), v(\eta)) + \ldots + g_\zeta(\eta, u(\eta), v(\eta)) = \chi(\eta, u(\eta), v(\eta)). \]

Similarly, \( h_m \) and \( g_m \) are also continuous for every \( m = 1, \ldots, \kappa \) and \( n = 1, \ldots, \zeta \), respectively. This implies that \( h_1 + \ldots + h_c \) and \( g_1 + \ldots + g_\zeta \) are continuous, respectively. Henceforth, \( \Pi \) and \( \chi \) are continuous, so by definition of continuity, we have
\[ \Pi(\eta, u_r(\eta), v_r(\eta)) \to \Pi(\eta, u(\eta), v(\eta)), \] as \( r \to \infty \)
\[ \chi(\eta, u_r(\eta), v_r(\eta)) \to \chi(\eta, u(\eta), v(\eta)), \] as \( r \to \infty. \]

Then, for any \( \eta \in [0, 1] \), we find
\[ |\Pi(\eta, u_r(\eta), v_r(\eta)) - \Pi(\eta, u(\eta), v(\eta))| \to 0, \] as \( r \to \infty \)
\[ |\chi(\eta, u_r(\eta), v_r(\eta)) - \chi(\eta, u(\eta), v(\eta))| \to 0, \] as \( r \to \infty, \]

\( Y_2 \) being the combination of continuous functions implies that \( Y_2 \) is also a continuous function. Therefore,
\[ Y_2 \to Y_2, \] implies \( |Y_{2r} - Y_2| \to 0, \) as \( r \to \infty, \]
and [24]
\[ B(u_r)(\eta) \to B(u)(\eta), \] which implies \( |B(u_r)(\eta) - B(u)(\eta)| \to 0, \) for \( \eta \in [0, 1], \) as \( r \to \infty, \)
\[ P(v_r)(\eta) \to P(v)(\eta), \] which implies \( |P(v_r)(\eta) - P(v)(\eta)| \to 0, \) for \( \eta \in [0, 1], \) as \( r \to \infty. \]

\begin{align*}
\hat{A} &= \frac{1}{|Y_1|} \left( \frac{|\pi_1|}{\Gamma(\xi + 1)} + |\pi_2| + \frac{|\pi_7|}{\Gamma(\xi + 1)\Gamma(\xi + 2)} \right), \\
\hat{B} &= \frac{1}{|Y_1|} \left( \frac{\omega(\lambda + 1)(1 - \omega)\lambda - \lambda(1 - \omega)^{\lambda + 1} + 1}{\Gamma(\lambda + 2)} \right) + |\pi_4| + \frac{|\pi_5|}{\Gamma(\lambda + 1) + |\pi_6|}, \\
\hat{C} &= \frac{1}{|Y_1|} \left( |\pi_1| \sum_{m=1}^{K} \frac{1}{\Gamma(\xi + \gamma_m + 1)} + |\pi_2| \sum_{m=1}^{K} \frac{(1 - t)^{\xi + \gamma_m - 1}}{\Gamma(\xi + \gamma_m + 1)} \right) \\
&\quad + |\pi_7| \sum_{m=1}^{K} \frac{|(1 - \omega)^{\xi + \gamma_m - 1}|}{\Gamma(\xi + \gamma_m + 2)\Gamma(\xi + 1)}, \\
\hat{D} &= \frac{1}{|Y_1|} \left( |\pi_3| \sum_{n=1}^{c} \frac{|(1 - \omega)^{\lambda + \alpha_n - 1}|}{\Gamma(\lambda + \alpha + 2)\Gamma(\lambda + 1)} \right) + |\pi_4| \sum_{n=1}^{c} \frac{|(\mu_n - \mu)|^{\lambda + \alpha_n - \mu_n}}{\Gamma(\lambda + \alpha_n + 1)} \\
&\quad + |\pi_5| \sum_{n=1}^{c} \frac{1}{\Gamma(\lambda + \alpha_n + 1)} \\
&\quad + |\pi_6| \sum_{n=1}^{c} \frac{1}{\Gamma(\alpha_n + 1)}. \end{align*}

(24)

Prior to the main result, we establish the undermentioned suppositions:
(A₁) There exist constants $0 \leq C₁ < 1$, and $0 \leq C₂ < 1$, such that, for $u₁, u₂, v₁, v₂ \in \mathbb{R}$, we have

$$|x₁(η, u₁(η)) - x₁(η, u₂(η))| \leq C₁\|u₁ - u₂\|,$$

$$|x₂(η, v₁(η)) - x₂(η, v₂(η))| \leq C₂\|v₁ - v₂\|.$$

(Another Equation)

(A₂) There exist constants $T₁, T₂ \in [0, 1)$, and $M₁, M₂ \geq 0$,  $\exists$ for any $u, v \in \mathbb{R}$, and we find that

$$|x₁(u(η))| \leq T₁\|u\| + M₁,$$

$$|x₂(v(η))| \leq T₂\|v\| + M₂.$$

(A₃) There exist constants $\tilde{b}_1, \tilde{b}_2, \tilde{N}_1, \tilde{N}_2, \tilde{N}_B, \tilde{N}_P$ all greater than zero, such that, for each $u, v \in \mathbb{R}$, we find

$$|B(u(η))| \leq \tilde{b}_1\|u\| + \tilde{N}_B,$$

$$|P(v(η))| \leq \tilde{b}_2\|v\| + \tilde{N}_P,$$

$$|x₁(η, u(η), v(η))| \leq \tilde{b}_1\|u\| + \|v\| + \tilde{N}_1,$$

$$|x₂(η, u(η), v(η))| \leq \tilde{b}_2\|u\| + \|v\| + \tilde{N}_2.$$

(A₄) There exist constants $\tilde{c}_1, \tilde{c}_2$ such that, for $u₁, u₂, v₁, v₂ \in \mathbb{R}$, we have

$$|\Pi(η, u₁(η), v(η)) - \Pi(η, u₂(η), v(η))| \leq \tilde{c}_1(\|u₁ - v₁\| + \|u₂ - v₂\|),$$

$$|\chi(η, u₁(η), v(η)) - \chi(η, u₂(η), v(η))| \leq \tilde{c}_2(\|u₁ - v₁\| + \|u₂ - v₂\|).$$

(A₅) There exist constants $\tilde{b}_1, \tilde{b}_2, \tilde{N}_1, \tilde{N}_1$ and $N₁, N₂$ that are all greater than zero, such that, for any $u, v \in \mathbb{R}$, we find

$$|\Pi(η, u(η), v(η))| \leq \tilde{b}_1(\|u\| + \|v\|) + \tilde{N}_1,$$

$$|\chi(η, u(η), v(η))| \leq \tilde{b}_2(\|u\| + \|v\|) + \tilde{N}_2.$$

(A₆) There exist $\tilde{O}_1, \tilde{O}_2$, such that, for any $u₁, u₂, v₁, v₂ \in \mathbb{R}$, we have

$$|x₁(η, u₁(η), v₁(η)) - x₁(η, u₂(η), v₂(η))| \leq \tilde{O}_1(\|u₁ - u₂\| + \|v₁ - v₂\|),$$

$$|x₂(η, u₁(η), v₁(η)) - x₂(η, u₂(η), v₂(η))| \leq \tilde{O}_2(\|u₁ - u₂\| + \|v₁ - v₂\|).$$

(A₇) There exist non-negative constants $N₁, N₂, \tilde{O}_1, \tilde{O}_2$, and $\tilde{O}_P < 1$, such that

$$|B(u₁)(η) - B(u₂)(η)| \leq \tilde{O}_1\|u₁ - u₂\|,$$

$$|P(v₁)(η) - P(v₂)(η)| \leq \tilde{O}_2\|v₁ - v₂\|,$$

$$|Y₁| - Y₂| \leq N₁\|u₁ - u₂\|,$$

$$|Y₃| - Y₄| \leq N₂\|v₁ - v₂\|.$$

**Lemma 3.** The operator $\Psi$ is a Lipschitzian, that is, it satisfies the Lipschitz condition

$$|\Psi(u₁, v₁)(η) - \Psi(u₂, v₂)(η)| \leq C(∥Y₁, Z₁∥ - ∥Y₂, Z₂∥),$$

where $C > 0$. 

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Proof. Consider for arbitrary \((u_1, v_1), (u_2, v_2) \in \mathcal{P} \times \mathcal{Q}\) that we have

\[
|\Psi_1(u_1)(\eta) - \Psi_1(u_2)(\eta)| = \sup_{\eta \in [0,1]} \left| \int_0^\eta \frac{(\eta - s)^{\zeta-1}}{\Gamma(\zeta)}(x_1(s, u_1(s)) - x_1(s, u_1(s)))ds \right.
\]

\[
- \sum_{m=1}^k \int_0^\eta \frac{(\eta - s)^{\zeta+\gamma_m-1}}{\Gamma(\zeta + \gamma_m)}(h_m(s, u_1(s)) - h_m(s, u_2(s)))ds
\]

\[
\leq \sup_{\eta \in [0,1]} \frac{\eta}{\Gamma(\zeta + 1)} \left| x_1(s, u_1(s)) - x_1(s, u_2(s)) \right|
\]

\[
\leq \frac{1}{\Gamma(\zeta + 1)} \| u_1 - u_2 \|.
\]

If \(\frac{1}{\Gamma(\zeta + 1)} \leq C_1\), then the last inequality becomes;

\[
|\Psi_1(u_1)(\eta) - \Psi_1(u_2)(\eta)| \leq C_1 \| u_1 - u_2 \|.
\] (25)

Similarly, we can have

\[
|\Psi_2(v_1)(\eta) - \Psi_2(v_2)(\eta)| \leq C_2 \| v_1 - v_2 \|,
\] (26)

where \(\frac{1}{\Gamma(\zeta + 1)} \leq C_2\).

Now, from the inequalities (25) and (26), we have

\[
|\Psi(u_1, v_1)(\eta) - \Psi(u_2, v_2)(\eta)| \leq \sup_{\eta \in [0,1]} |\Psi_1(u_1)(\eta) - \Psi_1(u_2)(\eta)| + \sup_{\eta \in [0,1]} |\Psi_2(v_1)(\eta) - \Psi_2(v_2)(\eta)|
\]

\[
\leq C_1 \sup_{\eta \in [0,1]} |\Psi_1(u_1)(\eta)\Psi_1(u_2)(\eta)| + C_2 \sup_{\eta \in [0,1]} |Ppsi_2(v_1)(\eta)\Psi_2(v_2)(\eta)|
\]

\[
= C_1 \| u_1 - u_2 \| + C_2 \| v_1 - v_2 \|.
\]

If \(C = \max\{C_1, C_2\}\), then, from the last inequality, we have

\[
|\Psi(u_1, v_1)(\eta) - \Psi(u_2, v_2)(\eta)| \leq C \| (u_1, v_1) - (u_2, v_2) \|.
\]

Henceforth, the operator \(\Psi\) is a Lipschitzian, i.e., it satisfies the Lipschitz condition. Therefore, by Proposition 2, the operator \(\Psi\) is \(\sigma\)-Lipschitz having constant \(C\). \(\Box\)

Lemma 4. The operator \(\Phi: \mathcal{P} \times \mathcal{Q} \rightarrow \mathcal{P} \times \mathcal{Q}\) is continuous.

Proof. In a bounded set, let \(\{(u_r, v_r)\}\) be a sequence:

\[
\mathcal{B}_r = \{ \| (u, v) \| \leq \ell : (u, v) \in \mathcal{P} \times \mathcal{Q} \},
\]

such that \((u_r, v_r) \rightarrow (u, v) \in \mathcal{B}_r\) as \(r \rightarrow \infty\). To prove \(\Phi\) is continuous, we need to show that

\[
\| \Phi(u_r, v_r) - \Phi(u, v) \| \rightarrow 0, \text{ whenever } r \rightarrow \infty.
\]
Now, for any arbitrary \((u_1, v_1), (u_2, v_2) \in \mathcal{P} \times \mathcal{Q}\), and for \(f, t, s, \in [0, \eta]\), we have

\[
\Phi_3(u, v)(\eta) - \Phi(u, v)(\eta) = \left| \mathcal{B}(u, v) - \frac{\eta}{\mathcal{Y}_1} \int_0^{\eta} \left( \frac{(1-\eta)^{n-1}}{\Gamma(\xi + \gamma_n)} x_1(s, u(s), v(s)) \right. \right.
\]

\[
- \sum_{n=1}^{k} \int_0^{\eta} \frac{(1-\eta)^{n-1}}{\Gamma(\xi + \gamma_n)} h_n(s, u(s), v(s)) ds + \mathcal{Y}_2 \left( x_1(s, u(s), v(s)) \right)
\]

\[
- \sum_{n=1}^{k} \int_0^{\eta} \frac{(1-\eta)^{n-1}}{\Gamma(\xi + \gamma_n)} h_n(s, u(s), v(s)) ds
\]

\[
- \mathcal{Y}_2 \left( x_1(s, u(s), v(s)) \right)
\]

\[
\left. \left. + \int_0^{\eta} \frac{(1-\eta)^{n-1}}{\Gamma(\xi + \gamma_n)} \left( \int_0^{\eta} \frac{(1-\eta)^{n-1}}{\Gamma(\xi + \gamma_n)} \mathcal{Y}_2(t, u(t), v(t)) dt \right) ds \right| \}
\]

\[
\mathcal{Y}_2 - \mathcal{Y}_2
\]
which, upon simplification, gives
\[
|\Phi_1(u_r, v_r)(\eta) - \Phi(u, v)(\eta)| \leq |B(u_r)(\eta) - B(u)(\eta)|
\]
\[
+ \frac{\eta}{|Y_1|} \left| \frac{1}{\Gamma(\zeta + 1)} \left| x_1(s, u_r(s), v_r(s)) - x_1(s, u(s), v(s)) \right| \right|
\]
\[
+ |\Pi(s, u_r(s), v_r(s)) - \Pi(s, u(s), v(s))| \sum_{m=1}^{\kappa} \frac{1}{|\Gamma(\zeta + \gamma_m + 1)|}
\]
\[
+ |\pi_2| \left( |x_1(s, u_r(s), v_r(s)) - x_1(s, u(s), v(s))| + |\Pi(s, u_r(s), v_r(s)) - \Pi(s, u(s), v(s))| \right)
\]
\[
+ |\pi_3| \left( \left| \frac{\omega(\lambda + 1)(1 - \omega) - \lambda(1 - \omega)^{\gamma + 1} + 1}{\Gamma(\lambda + 1)\Gamma(\lambda + 2)} \right| \times |x_2(t, u_r(t), v_r(t)) - x_2(\eta, u(t), v(t))| + |\chi(f, u_r(f), v_r(f)) - \chi(f, u(f), v(f))| \right)
\]
\[
+ |\pi_4| \left( |x_2(s, u_r(s), v_r(s)) - x_2(s, u(s), v(s))| + |\chi(s, u_r(s), v_r(s)) - \chi(s, u(s), v(s))| \right)
\]
\[
+ |\pi_5| \left( \left| \frac{1}{\Gamma(\lambda + 1)} \right| |x_2(s, u_r(s), v_r(s)) - x_2(s, u(s), v(s))| \right)
\]
\[
+ |\pi_6| \left( |x_2(s, u_r(s), v_r(s)) - x_2(s, u(s), v(s))| \right)
\]
\[
+ |\pi_7| \left( |x_1(t, u_r(t), v_r(t)) - x_1(t, u(t), v(t))| \right)
\]
\[
+ |\pi_8| \left( |x_1(s, u_r(s), v_r(s)) - x_1(s, u(s), v(s))| \right)
\]
\[
+ |\Pi(f, u_r(s), v_r(f)) - \Pi(s, u(s), v(s))| \sum_{m=1}^{\kappa} \frac{|(1 - \omega)^{\gamma} - 1|}{|\Gamma(\zeta + \gamma_m + 2)|}
\]
\[
+ |Y_2 - Y_2|
\]

Since \(Y_1 > 0\), thus, using the assumptions (21)–(26) on the preceding inequality and taking the sup norm, it can be deduced that, consequently,
\[ \| \Phi_1(u_r, v_r) - \Phi_1(u, v) \| \rightarrow 0, \text{ whenever, } r \rightarrow \infty. \]

In a similar fashion, it can be proved that
\[ \| \Phi_2(u_r, v_r) - \Phi_2(u, v) \| \rightarrow 0, \text{ whenever, } r \rightarrow \infty. \]

from which it can be deduced that \( \Phi_1 \) and \( \Phi_2 \) are continuous. Subsequently, \( \Phi \) is a continuous operator.

**Lemma 5.** The operators \( \Psi \) and \( \Phi \) satisfy the growth conditions, i.e., for any \((u, v) \in P \times Q\), we have
\[
\| \Psi(u, v) \| \leq T\| (u, v) \| + M, \quad (27) \]
\[
\| \Phi(u, v) \| \leq \tilde{B}\| (u, v) \| + \tilde{H}, \quad (28) \]

respectively, where \( T = \max\{T_1, T_2\}, M = \max\{M_1, M_2\}, \tilde{B} = 9\tilde{K}, \text{ and } \tilde{H} = 2\tilde{L} + N_1 + N_2. \)

**Proof.** To prove that the operator \( \Psi \) satisfies the growth condition, we begin with
\[
\| \Psi(u, v) \| = \| \Psi_1(u), \Psi_2(v) \| \\
\leq \left[ \int_0^{\eta} \frac{(\eta - s)^{\zeta - 1}}{\Gamma(\zeta)} x_1(s, u(s), v(s))ds \right] \sum_{m=1}^{\kappa} \int_0^{\eta} \frac{(\eta - s)^{\gamma_m - 1}}{\Gamma(\zeta + \gamma_m)} h_m(s, u(s), v(s))ds \\
+ \left[ \int_0^{\eta} \frac{(\eta - s)^{\lambda - 1}}{\Gamma(\lambda)} x_2(s, u(s), v(s))ds \right] \sum_{n=1}^{\xi} \int_0^{\eta} \frac{(\eta - s)^{\zeta_n - 1}}{\Gamma(\lambda + \zeta_n)} g_n(s, u(s), v(s))ds, \\
\leq \frac{1}{\Gamma(\zeta + 1)} \| x_1(u) \| + \frac{1}{\Gamma(\lambda + 1)} \| x_2(v) \|. \]

Now, using the assumption \( A_2 \) on the last inequality, we have
\[
\| \Psi(u, v) \| \leq \frac{T_1}{\Gamma(\zeta + 1)} \| u \| + M_1 + \frac{T_2}{\Gamma(\lambda + 1)} \| v \| + M_2. \quad (29) \]

Let \( T = \max\{T_1, T_2\} \) and \( M = \max\{M_1, M_2\} \). Then, \( 29 \) becomes
\[
\| \Psi(u, v) \| \leq T \| u \| + T \| v \| + M, \\
= T(\| u \| + \| v \|) + M. \]

from which it can be deduced that
\[
\| \Psi(u, v) \| \leq T \| u, v \| + M. \quad (30) \]

Henceforth, the operator \( \Psi \) satisfies the growth condition. Proceeding, we show that the operator \( \Phi \) also satisfies the growth condition:
\[ \| \Phi_1(u, v) \| = \left\| B(u) - \frac{\eta}{Y_1} \left[ \pi_1 \int_0^1 \left( \frac{(1-s)^{\gamma-1}}{\Gamma(\xi)} x_1(s, u(s), v(s)) - \sum_{m=1}^{\kappa} \frac{(1-s)^{\gamma_m}}{\Gamma(\xi + \gamma_m)} h_m(s, u(s), v(s)) \right) ds \right. \right. \]
\[ + \left. \pi_2 \left( x_1(s, u(s), v(s)) - \sum_{m=1}^{\kappa} \int_0^1 \frac{(1-s)^{\gamma_m}}{\Gamma(\xi + \gamma_m)} h_m(s, u(s), v(s)) ds \right) \right. \]
\[ - \left. \pi_3 \left( \int_0^\omega \frac{(1-s)^{\lambda-1}}{\Gamma(\lambda)} \int_0^t \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} \nu_{f_0}(f, u(f), v(f)) df dt ds \right) \right. \]
\[ + \left. \sum_{n=1}^{\xi} \int_0^\omega \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} x_1(t, u(t), v(t)) dt ds \right) \left. \right. \]
\[ + \left. \sum_{m=1}^{\kappa} \int_0^\omega \frac{(1-s)^{\gamma_m}}{\Gamma(\xi + \gamma_m)} h_m(s, u(s), v(s)) ds \right) \left. \right. \]
\[ \left. + \left( x_1(s, u(s), v(s)) - \int_0^1 \frac{(1-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} g_n(s, u(s), v(s)) ds \right) \right] \]
\[ \leq |B(u)| + \frac{1}{Y_1} \left[ |\pi_1| \left( \frac{1}{\Gamma(\xi + 1)} \right) \| x_1(u, v) \| - \sum_{m=1}^{\kappa} \frac{1}{\Gamma(\xi + \gamma_m + 1)} \right] \]
\[ + |\pi_2| \left( \| x_1(u, v) \| + \sum_{m=1}^{\kappa} \frac{(1-t)^{\gamma_m}}{\Gamma(\xi + \gamma_m + 1)} \right) + |\pi_3| \left( \| x_1(u, v) \| \right) \]
\[ \left( \frac{\omega(\lambda+1)(-1-\omega\\lambda-\lambda((-1-\omega)^{\alpha_n+1} - 1))}{\Gamma(\lambda+1)\Gamma(\lambda+2)} \right) \]
\[ + |\pi_4| \left( \| x_1(u, v) \| + \| x_1(u, v) \| \sum_{n=1}^{\xi} \frac{\nu_n}{\Gamma(\lambda + \alpha_n + 1)} \right) \]
\[ + |\pi_5| \left( \frac{1}{\Gamma(\lambda+1)} \right) \| x_1(u, v) \| \]
\[ + |\pi_6| \left( \frac{1}{\Gamma(\lambda+1)} \right) \| x_1(u, v) \| \]
\[ + |\pi_7| \left( \frac{\omega(\xi+1)(-1-\omega)^{\xi-\xi((-1-\omega)^{\xi+1} + 1)} \Gamma(\xi+1)\Gamma(\xi+2)}{\Gamma(\xi+1)\Gamma(\xi+2)} \right) \]
\[ + |\pi_8| \left( \| x_1(u, v) \| + \sum_{m=1}^{\kappa} \frac{(1-t)^{\gamma_m}}{\Gamma(\xi + \gamma_m + 1)} \right) \]
\[ \leq |B(u)| + |A| \| x_1(u, v) \| + |B| \| x_2(u, v) \| + |C| \| \Pi(u, v) \| \]
\[ + D \| x_1(u, v) \| + \frac{1}{Y_1} \| x_1(u, v) \|. \]
Now, using the assumptions $A_3$ and $A_5$ [24], we get the last inequality we obtain
\[
\|\Phi_1(u,v)\| \leq \tilde{a}_1 \|u\| + \check{A}\tilde{b}_1(\|u\| + \|v\|) + \tilde{N}_1 + \tilde{Q}_1(\|u\| + \|v\|) + \tilde{N}_1 + \tilde{Q}_1(\|u\| + \|v\|) + \tilde{N}_1 + \tilde{Q}_1(\|u\| + \|v\|) + \tilde{N}_1 + \tilde{Q}_1(\|u\| + \|v\|) + \tilde{N}_1 + \tilde{Q}_1(\|u\| + \|v\|),
\] \hspace{1cm} (31)

If we take $\tilde{K} = \max\{\tilde{a}_1, \check{A}\tilde{b}_1, \tilde{Q}_1(\|u\| + \|v\|)\}$, $L = \max\{\tilde{N}_1, \tilde{Q}_1(\|u\| + \|v\|)\}$, and $\left|\frac{\tilde{N}_2}{\tilde{Q}_1}\right| < \tilde{N}_1$. Then, (31) can be expressed as
\[
\|\Psi_1(u,v)\| \leq 5\tilde{K}\|u\| + 4\tilde{K}\|v\| + \tilde{L} + \tilde{N}_1.
\]

In a similar manner, we have
\[
\|\Phi_2(u,v)\| \leq 4\tilde{K}\|u\| + 5\tilde{K}\|v\| + \tilde{L} + \tilde{N}_2,
\]
where $h' = \max\{h_1, h_2\}$. Furthermore,
\[
\|\Phi(u,v)\| \leq \|\Phi_1(u,v)\| + \|\Phi_2(u,v)\|,
\]
\[
\leq 5\tilde{K}\|u\| + 4\tilde{K}\|v\| + \tilde{L} + \tilde{N}_1 + 4\tilde{K}\|u\| + 5\tilde{K}\|v\| + \tilde{L} + \tilde{N}_2,
\]
\[
= 9(\|u\|\tilde{K} + \|v\|\tilde{K}) + 2\tilde{L} + \tilde{N}_1 + \tilde{N}_2.
\]

The last inequality implies
\[
\|\Phi(u,v)\| \leq \tilde{B}\|u,v\| + \tilde{H}.
\] \hspace{1cm} (32)

From (32), it can be deduced that $\Phi$ satisfies the growth condition. □

**Lemma 6.** The operator $\Phi : P \times Q \to P \times Q$ is compact.

**Proof.** Consider $D$ to be a bounded subset of $B_1$. Consider a sequence $\{u_r, v_r\}_{r \in \mathbb{N}} \in D$; then, from Equation (28), we have
\[
\|\Phi(u_r, v_r)\| \leq \tilde{B}\ell + \tilde{H},
\]
which shows that $\Phi(D)$ is bounded. Now, we only want to prove that $\Phi$ is equi-continuous. Consider, for any $\{(u_r, v_r)\}_{r \in \mathbb{N}} \in D$, and $\epsilon > 0$, that we have
\[ |\Phi_1(u, v, (\theta, \phi)) - \Phi_2(u, v, (\theta, \phi))| = |R(u, v) - \frac{\partial}{\partial v} \left[ \int_0^1 \left( \frac{1 - \epsilon(t)^{-1}}{t + \gamma_0} \right)^J (u, v, (\theta, \phi)) \right] \]
It is obvious that, by taking the limit $\bar{\eta} \to \eta$, we have
\[ |\Phi(u_r, v_r)(\eta) - \Phi(u_r, v_r)(\bar{\eta})| \to 0. \]
Consequently, there exists, $\epsilon > 0$, such that
\[ |\Phi(u_r, v_r)(\eta) - \Phi(u_r, v_r)(\bar{\eta})| \leq \frac{\epsilon}{2}. \tag{33} \]
Likewise, it can be proved that
\[ |\Phi_2(u_r, v_r)(\eta) - \Phi_2(u_r, v_r)(\bar{\eta})| \leq \frac{\epsilon}{2}. \tag{34} \]
Consequently, from inequalities (33) and (34), and taking the sup norm, we have
\[ \|\Phi(u_r, v_r) - \Phi(u_r, v_r)\| \leq \epsilon. \]
from which it can be concluded that $\Phi(D)$ is equicontinuous. Henceforth, using the Arzela–Ascoli theorem, the operator $\Phi(D)$ is compact, and therefore by Proposition 2, $\Phi$ is $\sigma$-Lipschitz having constant zero. \qed

**Theorem 2.** While using the assumptions $A_1$, $A_3$, and $A_5$, the problem (1) possesses a solution in $P \times Q$, given that $\rho(T + \hat{B}) < 1$. Additionally, the set of solutions of the system (1) is bounded in $P \times Q$.

**Proof.** By Theorem 1, the operator $\Psi$ is Lipschitz have constant $C \in [0,1]$, and, by Lemma 6, the operator $\Phi$ is compact and $\sigma$-Lipschitz having a constant zero. Therefore, Proposition 2, it follows that $\Omega$ is a contraction along with the constant $C$. Consider the following set:
\[ \hat{O} = \{(u, v) \in P \times Q : \text{there exists } \hat{\rho} \in [0,1] \text{ with } (u, v) = \hat{\rho}\Omega(u, v)\}. \]
Now, we show that $\hat{O}$ is bounded. To prove this, consider $(u, v) \in \hat{O}$ and using the inequalities (27) and (28), we obtain
\[
\|(u, v)\| = \|\hat{\rho}\Omega(u, v)\|,
\leq |\hat{\rho}|\left(\|\Psi(u, v)\| + \|\Phi(u, v)\|\right),
\leq \hat{\rho}(T\|(u, v)\| + \hat{B}\|(u, v)\| + \hat{\mathcal{H}}),
\leq \hat{\rho}(T + \hat{B})\|(u, v)\| + \hat{\rho}\hat{\mathcal{M}} + \hat{\mathcal{H}}.
\]
From the above analysis, it can be inferred that $\hat{O}$ is bounded in $P \times Q$. Consequently, by using Theorem 1, $\Psi$ has a fixed point. Furthermore, the set of fixed points is bounded in $P \times Q$. \qed

**Theorem 3.** While using the assumptions $A_1$, $A_4$, $A_6$, and $A_7$, the system (1) possesses a unique solution, given that $C + \hat{G} + \hat{I} < 1$.

**Proof.** To prove that the problem (1) possesses a unique solution, we show that $\Omega$ defined in (20) satisfies the Banach contraction theorem. To proceed, consider that $(u_1, v_1), (u_2, v_2) \in P \times Q$; then, from Lemma 3, we have
\[
|\Psi(u_1, v_1) - \Psi(u_2, v_2)| \leq C\|(u_1, v_1) - (u_2, v_2)\|. \tag{35}
\]
The assumptions $A_1, A_4, A_6,$ and $A_7$ will be utilized for $t, f, s \in [0, \eta]$ in the process. For this, consider
\[ [\Phi_1(u_1, v_1) - \Phi_2(u_2, v_2)] = \left| B(u_1) - \frac{\eta}{T_1} \int \left( \frac{1 - s\zeta}{1 - s\zeta} \right) x_1(s, u_1(s), v_1(s)) \right| ds \]

\[-\sum_{n=1}^{\infty} \frac{(1-s)^{\zeta+n-1}}{\Gamma(\zeta + n)} g_n(s, u_1(s), v_1(s)) \] 

\[+ \frac{\pi_2}{T_1} \int (1-s)^{\zeta+1} \int_0^{\infty} \frac{(s-t)^{\zeta+1}}{\Gamma(\zeta + 1)} b_n(s, u_1(t), v_1(t)) dt ds \]

\[+ \pi_2 \int_0^{\infty} \frac{(s-t)^{\zeta+1}}{\Gamma(\zeta + 1)} b_n(s, u_1(t), v_1(t)) dt ds \]

\[+ \frac{\pi_2}{T_1} \int (1-s)^{\zeta+1} \int_0^{\infty} \frac{(s-t)^{\zeta+1}}{\Gamma(\zeta + 1)} b_n(s, u_1(t), v_1(t)) dt ds \]

\[+ \frac{\pi_2}{T_1} \int (1-s)^{\zeta+1} \int_0^{\infty} \frac{(s-t)^{\zeta+1}}{\Gamma(\zeta + 1)} b_n(s, u_1(t), v_1(t)) dt ds \]

\[+ \pi_2 \int_0^{\infty} \frac{(s-t)^{\zeta+1}}{\Gamma(\zeta + 1)} b_n(s, u_1(t), v_1(t)) dt ds \]

\[+ \frac{\pi_2}{T_1} \int (1-s)^{\zeta+1} \int_0^{\infty} \frac{(s-t)^{\zeta+1}}{\Gamma(\zeta + 1)} b_n(s, u_1(t), v_1(t)) dt ds \]

\[+ \pi_2 \int_0^{\infty} \frac{(s-t)^{\zeta+1}}{\Gamma(\zeta + 1)} b_n(s, u_1(t), v_1(t)) dt ds \]

\[\leq \left| B(u_2) - \frac{\eta}{T_1} \int \left( \frac{1 - s\zeta}{1 - s\zeta} \right) x_1(s, u_2(s), v_2(s)) \right| ds \]

\[-\sum_{n=1}^{\infty} \frac{(1-s)^{\zeta+n-1}}{\Gamma(\zeta + n)} g_n(s, u_2(s), v_2(s)) \]
\[ + |\tau_3| \left( \frac{[\omega(\lambda + 1)(1 - \omega)^{l} - \lambda((1 - \omega)^{l+1} + 1)]}{\Gamma(\lambda + 1)\Gamma(\lambda + 2)} \right) |x_2(t, u_1(t), v_1(t)) - x_2(\eta, u_2(t), v_2(t))| \\
+ |\tau_4| \left( |\chi(f, u_1(f), v_1(f)) - \chi(f, u_2(f), v_2(f))| + \sum_{n=1}^{\xi} \frac{[(1 - \omega)^{l} - 1]^{l+\alpha_n+1}}{\Gamma(\lambda + \alpha_n + 2)\Gamma(\lambda + 1)} \right) \\
+ |\tau_5| \left( |\chi(s, u_1(s), v_1(s)) - \chi(s, u_2(s), v_2(s))| + \sum_{n=1}^{\xi} \frac{1}{\Gamma(\lambda + \alpha_n + 1)} \right) \\
+ |\tau_6| \left( |\chi(s, u_1(s), v_1(s)) - \chi(s, u_2(s), v_2(s))| \right) \\
+ |\tau_7| \left( |\chi(s, u_1(s), v_1(s)) - \chi(s, u_2(s), v_2(s))| \right) \\
+ |\tau_8| \left( |\chi(s, u_1(s), v_1(s)) - \chi(s, u_2(s), v_2(s))| \right) + |\Pi(s, u_1(s), v_1(s)) - \Pi(s, u_2(s), v_2(s))| \\
+ \sum_{m=1}^{\mu} \left( \frac{(\mu_m - \mu_j)^{l+\tau_m} - \mu_m}{\Gamma(\xi + \tau_m + 1)} \right) \right] + |Y_{2|} - Y_{2'}| \right) \\
\leq \\tilde{O}_B\|u_1 - u_2\| + \tilde{A}\tilde{O}_{x_2}((|u_1 - u_2| + \|v_1 - v_2\|) + \tilde{B}\tilde{O}_{x_2}((|u_1 - u_2| + \|v_1 - v_2\|), \\
+ \tilde{C}_1((|u_1 - u_2| + \|v_1 - v_2\|) + \tilde{D}_1\|u_1 - u_2\| + \tilde{N}_x((|u_1 - u_2| - \|v_1 - v_2\|), \\
\leq (\tilde{O}_B + \tilde{A}\tilde{O}_{x_2} + \tilde{B}\tilde{O}_{x_2} + \tilde{C}_1 + \tilde{D}_1 + \tilde{N}_x)\|u_1 - u_2\| + (\tilde{A}\tilde{O}_{x_2} + \tilde{B}\tilde{O}_{x_2} + \tilde{C}_1 + \tilde{D}_1 + \tilde{N}_x)\|v_1 - v_2\|. \\
\right)
\]

Taking \( \tilde{G} = \max\{\tilde{O}_B + \tilde{A}\tilde{O}_{x_2} + \tilde{B}\tilde{O}_{x_2} + \tilde{C}_1 + \tilde{D}_1 + \tilde{N}_x, \tilde{A}\tilde{O}_{x_2} + \tilde{B}\tilde{O}_{x_2} + \tilde{C}_1 + \tilde{D}_1 + \tilde{N}_x\} \),
so from the above inequality, we have

\[ \|\Phi_1(u_1, v_1) - \Phi_1(u_2, v_2)\| \leq \tilde{G} \|u_1 - u_2\| - \|v_1 - v_2\|, \]

\[ \leq \tilde{G} \|u_1 - v_1, u_2 - v_2\|, \]

from which it can be deduced that

\[ \|\Phi_1(u_1, v_1) - \Phi_1(u_2, v_2)\| \leq \tilde{G} \|u_1 - v_1, u_2 - v_2\|. \] (36)

In a similar fashion, one can prove that

\[ \|\Phi_2(u_1, v_1) - \Phi_2(u_2, v_2)\| \leq \tilde{f} \|u_1 - v_1, u_2 - v_2\| \] (37)

where \( \tilde{f} = \max\{\tilde{A}\tilde{O}_{x_1} + \tilde{B}\tilde{O}_{x_2} + \tilde{C}_1 + \tilde{D}_1, \tilde{O}_B + \tilde{A}\tilde{O}_{x_1} + \tilde{B}\tilde{O}_{x_2} + \tilde{C}_1 + \tilde{D}_1 + \tilde{N}_x\} \). Collectively, from inequalities (36) and (37), it can be deduced that

\[ \|\Phi_1(u_1, v_1) - \Phi_1(u_2, v_2)\| = \|\Phi_1(u_1, v_1) - \Phi_1(u_2, v_2)\| + \|\Phi_2(u_1, v_1) - \Phi_2(u_2, v_2)\|, \]

\[ \leq \tilde{G} \|u_1 - v_1, u_2 - v_2\| + \tilde{f} \|u_1 - v_1, u_2 - v_2\|, \]

\[ = (\tilde{G} + \tilde{f}) \|u_1 - v_1, u_2 - v_2\|. \] (38)
Now, from inequalities (35) and (38), we have
\[
|Ω(u_1, v_1) - Ω(u_2, v_2)| \leq \|Ψ(u_1, v_1) - Ψ(u_2, v_2)\| + \|Φ(u_1, v_1) - Φ(u_2, v_2)\|,
\]
subject to the given boundary conditions:
\[
C\|u_1 - u_2\| + (\tilde{G} + \tilde{f})\|v_1 - v_2\| = (C + \tilde{G} + \tilde{f})\|(u_1, v_1) - (u_2, v_2)\|. \tag{39}
\]
From the inequality (39), it can be concluded that Ω is a Banach contraction. Subsequently, the problem (1) possesses a unique solution. \(\square\)

For the validation of the above analysis, consider the examples below.

4. Examples

The current section provides a look into an example which authenticate the entire analysis of the above study.

Example 1. Consider the system given below:
\[
\begin{align*}
D^{4/3} u(\eta) + \sum_{m=0}^{3} T^{5/3}_m h_m(\eta, u(\eta), v(\eta)) &= x_1(\eta, u(\eta), v(\eta)), \\
D^{5/3} v(\eta) + \sum_{m=0}^{3} T^{5/3}_m g_m(\eta, u(\eta), v(\eta)) &= x_2(\eta, u(\eta), v(\eta)),
\end{align*}
\tag{40}
\]
subject to the given boundary conditions:
\[
\left\{
\begin{array}{l}
u(0) = \frac{3}{\Pi} \cos(u) + \frac{9}{\Pi^2}, \quad v(0) = \frac{9}{\Pi^2} \sin(v) + \frac{9}{\Pi}, \\
\phi_1 u(1) + \phi_2 D^{4/3} u(1) = \frac{v_1}{\Pi(4/3)} \int_0^{1/3} (1 - s)^{4/3 - 1} ds + \sum_{j=0}^{3} \tau_j D^{4/3} v(\mu_j), \\
\phi_2 v(1) + \phi_2 D^{4/3} v(1) = \frac{v_2}{\Pi(5/3)} \int_0^{1/3} (1 - s)^{5/3 - 1} ds + \sum_{j=0}^{3} \tau_j D^{5/3} u(\mu_j).
\end{array}
\right.
\]
Here, we have \(\xi = 4/3, \lambda = 5/3, \nu = 1, \mu_1 = \mu_2 = 0, \nu_1 = \nu_2 = 1, \xi_1 = 1/4, \xi_2 = 1/2, \tau_1 = 1/3, \tau_3 = 2/3, \phi_1 = 1/7, \phi_2 = 7/9, \phi_1 = 5/6, \text{and } \phi_2 = 3/4. \text{ Furthermore, we have}
\[
\begin{align*}
x_1(\eta, u(\eta), v(\eta)) &= \frac{1}{|\sin(\eta)| + 1} \left(\frac{|\cos(u)|}{|\cos(\eta)| + 2} - \frac{|\sin(v)| + 1}{\eta^3 + 1}\right), \\
x_2(\eta, u(\eta), v(\eta)) &= \frac{1}{|\cos(\eta)| + 1} \left(\frac{\sin(v) + 1/2}{|\sin(\eta)| + 2} - \frac{|u(\eta)|}{\eta^3 + 2}\right), \\
h_1(\eta, u(\eta), v(\eta)) &= \frac{1}{e^{1/2} + 1} \left(\frac{|u(\eta)|}{\eta + 1} - \frac{2|v(\eta)|}{2}\right), \\
h_2(\eta, u(\eta), v(\eta)) &= \frac{e^{1/2} + 1}{\sqrt{143 + \eta^3}} \left(\frac{2u(\eta)}{3\eta^3 + 1} - \sin(v)\right), \\
h_3(\eta, u(\eta), v(\eta)) &= \frac{1}{\sqrt{196 + \eta^2}} \left(\frac{3v(\eta) + u(\eta)}{2}\right), \\
g_1(\eta, u(\eta), v(\eta)) &= \frac{1}{\sqrt{2 + \eta^3}} \left(\frac{v(\eta)}{2} - \frac{u(\eta)}{5}\right), \\
g_2(\eta, u(\eta), v(\eta)) &= \frac{1}{\sqrt{\eta^2 + 1}} \left(\frac{|u(\eta)|}{|\sin(\eta)| + 1} - \frac{|v(\eta)|}{|\cos(\eta)| + 1}\right), \\
g_3(\eta, u(\eta), v(\eta)) &= \frac{1}{9\eta^2 + \sqrt{5}} \left(\frac{|\sin(u)|}{3\eta + 1} - \frac{2|v(\eta)|}{5}\right).
\end{align*}
\]
It can be easily be verified that the functions defined above are all bounded and continuous. Furthermore, the operators Ψ and Φ are bounded and continuous, which implies that Ω is also continuous and bounded.

In addition,
Example 2. Consider the system given below:

\[ \Psi(u_1, v_1)(\eta) - \Psi(u_2, v_2)(\eta) \leq \left| \frac{3(1 - \eta)}{7} \sin(u_1) + \frac{9(1 - \eta)}{14} \cos(v_1) - \frac{3(1 - \eta)}{7} \sin(u_2) \right| - \frac{9(1 - \eta)}{14} \cos(v_2) \right|. \]

Applying the supremum norm on both sides of the above equation, we obtain

\[ \| \Psi(u_1, v_1) - \Psi(u_2, v_2) \| \leq \frac{9}{14} \| (u_1, u_2), (v_1, v_2) \| \]

from which it is concluded that \( \Psi \) is a \( \sigma \)-Lipschitz having a constant \( \frac{9}{14} < 1 \) and \( \Phi \) is a \( \sigma \)-Lipschitz possessing a constant zero. From the above analysis, it can be deduced that \( \Omega \) is strict \( \sigma \)-Lipschitz having constant \( \frac{9}{14} \). Then,

\[ \hat{O} = \left\{ (u, v) \in C[0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \exists \beta \in [0, 1] \text{ with } (u, v) = \frac{9}{14} \Omega(u, v) \right\}. \]

Then, by simple calculations, we have

\[ \| (u, v) \| \leq 1. \]

The above calculations show that \( \hat{O} \) is bounded. Therefore, using Theorem 2, the system (40) has a solution \( (u, v) \in C[0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \). Furthermore, since we have \( C + \mathcal{G} + \mathcal{J} = 0.634891 < 1 \), by applying Theorem 3, the system (40) possesses a unique solution.

Here, we construct another example as

**Example 2.** Consider the system given below:

\[
\begin{align*}
D^{3/2}u(\eta) + \sum_{m=0}^{2} T_m h_m(\eta, u(\eta), v(\eta)) &= x_1(\eta, u(\eta), v(\eta)), \\
D^{3/2}u(\eta) + \sum_{n=0}^{2} T_n g_n(\eta, u(\eta), v(\eta)) &= x_2(\eta, u(\eta), v(\eta)),
\end{align*}
\]

subject to the given boundary conditions:

\[
\begin{align*}
u(0) &= \frac{4}{3} \sqrt{|u|} + \frac{5}{12}, & v(0) &= \frac{5}{12} \sqrt{|v|} + \frac{5}{12}, \\
\phi_1 u(1) + \varphi_1 D^{3/2} u(1) &= \frac{3}{17} (1 - s)^{3/2 - 1} ds + \sum_{j=0}^{2} \varphi_j D^{3/2} v(\mu_j), \\
\phi_2 v(1) + \varphi_2 D^{3/2} v(1) &= \frac{3}{17} (1 - s)^{3/2 - 1} ds + \sum_{j=0}^{2} \varphi_j D^{3/2} u(\mu_j).
\end{align*}
\]

Here, we have \( \zeta = 3/2, \lambda = 3/2, w = 1, \mu_1 = \mu_2 = 0, v_1 = v_2 = 1, \xi_1 = 1/5, \xi_2 = 1/2, \tau_1 = 1/3, \tau_2 = 2/3, \phi_1 = 1/7, \phi_2 = 7/9, \varphi_1 = 5/6, \) and \( \varphi_2 = 3/4. \) Furthermore, we have

\[
\begin{align*}
x_1(\eta, u(\eta), v(\eta)) &= \frac{1}{|u| + 1} \left( \frac{|u| + 2}{|u(\eta)| + 1} - \frac{|v|}{|v(\eta)| + 1} \right), \\
x_2(\eta, u(\eta), v(\eta)) &= \frac{1}{|v| + 1} \left( \frac{|u(\eta)|}{|u(\eta)| + 1} - \frac{2}{|v(\eta)| + 1} \right), \\
h_1(\eta, u(\eta), v(\eta)) &= \frac{1}{|v(\eta)| + 1} \left( \frac{|u(\eta)|}{|u(\eta)| + 1} - \frac{1}{|v(\eta)| + 1} \right), \\
h_2(\eta, u(\eta), v(\eta)) &= \frac{1}{|u(\eta)| + 1} \left( \frac{|v(\eta)|}{|v(\eta)| + 1} - \frac{1}{|v(\eta)| + 1} \right), \\
g_1(\eta, u(\eta), v(\eta)) &= \frac{1}{|v| + 1} \left( \frac{1}{|u(\eta)|} - \frac{2}{|v(\eta)| + 1} \right), \\
g_2(\eta, u(\eta), v(\eta)) &= \frac{1}{|v| + 1} \left( \frac{1}{|u(\eta)|} - \frac{2}{|v(\eta)| + 1} \right).
\end{align*}
\]

It can be easily verified that the functions defined above are all bounded and continuous.
Like the previous example, we can show that $\Psi$ and $\Phi$ are bounded and continuous, which led us to conclude that $\Omega$ is also continuous and bounded. Furthermore, we can calculate

$$\|\Psi(u_1, v_1) - \Psi(u_2, v_2)\| \leq \frac{5}{12} \|(u_1, u_2), (v_1, v_2)\|,$$

from which it is concluded that $\Psi$ is a $\sigma$-Lipschitz having a constant $\frac{5}{12} < 1$ and $\Phi$ is a $\sigma$-Lipschitz possessing a constant zero. From the above analysis, it can be deduced that $\Omega$ is strict $\sigma$-Lipschitz having constant $\frac{5}{12}$. Thus,

$$\hat{O} = \left\{(u, v) \in C[0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \exists \hat{\rho} \in [0, 1] \text{ with } (u, v) = \frac{5}{12} \Omega(u, v) \right\}.$$

Then, by simple calculations, we have

$$\|(u, v)\| \leq 1.$$

The above calculations show that $\hat{O}$ is bounded. Therefore, using Theorem 2, the system (40) has a solution $(u, v) \in C[0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Furthermore, since we have $C + \hat{G} + \hat{J} = 0.55678 < 1$, then, by applying Theorem 3, the system (40) possesses a unique solution.

5. Conclusions

In this manuscript, we have derived existence and uniqueness results for a nonlinear coupled system of Caputo–Riemann–Liouville type fractional integro-differential equations with coupled sub-strip boundary conditions using a topological degree theory method. The concerned topological theory of Mawhin is a powerful tool that relaxes the strong compact conditions of fixed point theory to some weaker one. Before this theory mentioned, such problems like this had not yet been studied. Here, we mention that problems under coupled sub-strip boundary conditions is a major area of research in fluid mechanics and hydrodynamics where people very rarely investigate existence theory. In fact, before proceeding, establishing a numerical scheme existence theory of the problem is important. Therefore, we have utilized degree theory and established sufficient adequate results for existence of solutions. In addition, by two pertinent examples, we have demonstrated the authenticity and credibility of the derived results.

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