ON THE APPROXIMATION PROPERTIES OF CESÀRO MEANS OF NEGATIVE ORDER FOR THE DOUBLE VILENKIN–FOURIER SERIES

T. Tepnadze

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We establish approximation properties of Cesàro \((C, -\alpha, -\beta)\) means with \(\alpha, \beta \in (0, 1)\) for the Vilenkin–Fourier series. This result enables one to establish a condition sufficient for the convergence of the means \(\sigma_{n,m}^{\alpha,\beta}(x, y, f)\) to \(f(x, y)\) in the \(L^p\)-metric.

By \(N_+\) we denote the set of positive integers; \(N := N_+ \cup \{0\}\). Let \(m := (m_0, m_1, \ldots)\) denote a sequence of positive integers not smaller than 2. Moreover, by \(Z_{m_k} := \{0, 1, \ldots, m_k - 1\}\) we denote an additive group of integers modulo \(m_k\). A group \(G_m\) is defined as the complete direct product of the groups \(Z_{m_j}\) with the product of discrete topologies of \(Z_{m_j}\’s\).

The direct product of the measures

\[\mu_k(\{j\}) := \frac{1}{m_k}, \quad j \in Z_{m_k},\]

is the Haar measure on \(G_m\) with \(\mu(G_m) = 1\). If the sequence \(m\) is bounded, then \(G_m\) is called a bounded Vilenkin group. In the present paper, we consider only bounded Vilenkin groups. The elements of \(G_m\) can be represented by sequences \(x := (x_0, x_1, \ldots, x_j, \ldots), x_j \in Z_{m_j}\). The group operation \(+\) in \(G_m\) is introduced as follows:

\[x + y = ((x_0 + y_0) \text{mod} m_0, \ldots, (x_k + y_k) \text{mod} m_k, \ldots),\]

where \(x := (x_0, \ldots, x_k, \ldots)\) and \(y := (y_0, \ldots, y_k, \ldots) \in G_m\). The inverse of \(+\) is denoted by \(-\).

It is easy to give a base for the neighborhoods of \(G_m\):

\[I_0(x) := G_m,\]
\[I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}\]

for \(x \in G_m, n \in N\). We define \(I_n := I_n(0)\) for \(n \in N_+\). We set \(e_n := (0, \ldots, 0, 1, 0, \ldots) \in G_m\). The \((n+1)\)th coordinate of \(e_n\) is equal to 1 and the remaining coordinates are zeros \((n \in N)\).

If we define the so-called generalized number system based on \(m\) in the following way: \(M_0 := 1, M_{k+1} := m_k M_k, k \in N\), then every \(n \in N\) can be uniquely expressed in the form

\[n = \sum_{j=0}^{\infty} n_j M_j,\]

Javakhishvili Tbilisi State University, Tbilisi, Georgia; e-mail: tsitsinotefnadze@gmail.com.

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where \( n_j \in \mathbb{Z}_{m_j}, \ j \in \mathbb{N}_+ \), and only finite many \( n_j \)’s differ from zero. We use the following notation:

\[
|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}
\]

(i.e., \( M_{|n|} \leq n < M_{|n|+1} \)).

Further, on \( G_m \), we introduce an orthonormal system, which is called the Vilenkin system. First, we define complex-valued functions \( r_k(x) : G_m \to \mathbb{C} \) (the generalized Rademacher functions) as follows:

\[
r_k(x) := \exp \frac{2\pi i x_k}{m_k}, \quad i^2 = -1, \quad x \in G_m, \quad k \in \mathbb{N}.
\]

Further, we define the Vilenkin system \( \psi := (\psi_n : n \in \mathbb{N}) \) on \( G_m \) in the following form:

\[
\psi_n(x) := \prod_{k=0}^{\infty} r_{n_k}(x), \quad n \in \mathbb{N}.
\]

In particular, this system is called the Walsh–Paley system if \( m = 2 \).

The Dirichlet kernels are defined as

\[
D_n := \sum_{k=0}^{n-1} \psi_k, \quad n \in \mathbb{N}_+.
\]

Recall that (see [3] or [14])

\[
D_{M_n}(x) = \begin{cases} 
M_n & \text{for } x \in I_n, \\
0 & \text{for } x \notin I_n.
\end{cases}
\]

The Vilenkin system is orthonormal and complete in \( L^1(G_m) \) [1].

We also introduce some notation from the theory of two-dimensional Vilenkin systems. Let \( \tilde{m} \) be a sequence of the form \( m \). The relationship between the sequences \((\tilde{m}_n)\) and \((M_n)\) is the same as between the sequences \((m_n)\) and \((M_n)\). The group \( G_m \times G_{\tilde{m}} \) is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by \( \mu \) as in the one-dimensional case. We also suppose that \( m = \tilde{m} \) and \( G_m \times G_{\tilde{m}} = G_m^2 \).

The norm in the space \( L^p(G_m^2) \) is defined by

\[
\|f\|_p := \left( \int_{G_m^2} |f(x,y)|^p \, d\mu(x,y) \right)^{1/p}, \quad 1 \leq p < \infty.
\]

Denote by \( C(G_m^2) \) a class of continuous functions on the group \( G_m^2 \) endowed with the supremum norm. For the sake of brevity, we agree to write \( L^\infty(G_m^2) \) instead of \( C(G_m^2) \).

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, and the Dirichlet kernels with respect to the two-dimensional Vilenkin system are defined as follows:

\[
\widehat{f}(n_1, n_2) := \int_{G_m^2} f(x,y) \psi_{n_1}(x) \bar{\psi}_{n_2}(y) \, d\mu(x,y),
\]
\[
S_{n_1, n_2}(x, y, f) := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2) \psi_{k_1}(x) \psi_{k_2}(y),
\]

\[
D_{n_1, n_2}(x, y) := D_{n_1}(x) D_{n_2}(y).
\]

Denote
\[
S_n^{(1)}(x, y, f) := \sum_{l=0}^{n-1} \hat{f}(l, y) \psi_l(x),
\]

\[
S_m^{(2)}(x, y, f) := \sum_{r=0}^{m-1} \hat{f}(x, r) \psi_r(y),
\]

where
\[
\hat{f}(l, y) = \int_{G_m} f(x, y) \psi_l(x) \, d\mu(x)
\]
and
\[
\hat{f}(x, r) = \int_{G_m} f(x, y) \psi_r(y) \, d\mu(y).
\]

The \((c, -\alpha, -\beta)\) means of the two-dimensional Vilenkin–Fourier series are defined as
\[
\sigma_{n,m}^{-\alpha, -\beta}(x, y, f) = \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \sum_{i=0}^{n} \sum_{j=0}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \hat{f}(i, j) \psi_i(u) \psi_j(v),
\]

where
\[
A_0^\alpha = 1, \quad A_n^\alpha = \frac{(\alpha + 1) \ldots (\alpha + n)}{n!}.
\]

It is well known that [18]
\[
A_n^\alpha = \sum_{k=0}^{n} A_k^{\alpha-1}, \quad (2)
\]

\[
A_n^\alpha - A_{n-1}^\alpha = A_{n-1}^{\alpha-1}, \quad (3)
\]

\[
A_n^\alpha \sim n^\alpha. \quad (4)
\]

The dyadic partial moduli of continuity of a function \( f \in L^p(G_m^2) \) in the \( L^p \)-norm are defined by
\[
\omega_1 \left( f, \frac{1}{M_n} \right) = \sup_{u \in I_n} \| f(\cdot - u, \cdot) - f(\cdot, \cdot) \|_p,
\]

\[
\omega_2 \left( f, \frac{1}{M_n} \right) = \sup_{v \in I_n} \| f(\cdot, \cdot - v) - f(\cdot, \cdot) \|_p,
\]
while the dyadic mixed modulus of continuity is defined as follows:

\[
\omega_{1,2}\left(f, \frac{1}{M_n}, \frac{1}{M_m}\right)_p = \sup_{(u,v) \in I_n \times I_m} \| f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \|_p.
\]

It is clear that

\[
\omega_{1,2}\left(f, \frac{1}{M_n}, \frac{1}{M_m}\right)_p \leq \omega_1\left(f, \frac{1}{M_n}\right)_p + \omega_2\left(f, \frac{1}{M_m}\right)_p.
\]

The dyadic total modulus of continuity is defined by

\[
\omega\left(f, \frac{1}{M_n}\right)_p = \sup_{(u,v) \in I_n \times I_n} \| f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \|_p.
\]

The problems of summability of partial sums and Cesàro means for the Walsh–Fourier series were studied in [2, 4–13, 16]. In [17], Zhizhiashvili presented a detailed investigation of the behavior of Cesàro method of negative order for trigonometric Fourier series. Goginava [5] studied a similar problem for the Walsh system. In particular, the following theorem was proved:

**Theorem G** [5]. Let \( f \) belong to \( L^p(G_2) \) for some \( p \in [1, \infty] \) and \( \alpha \in (0, 1) \). Then, for any \( 2^k \leq n < 2^{k+1} \), \( k, n \in N \), the inequality

\[
\| \sigma_n^{-\alpha}(f) - f \|_p \leq c(p, \alpha) \left\{ 2^{k\alpha} \omega(1/2^{k-1}, f)_p + \sum_{r=0}^{k-2} 2^{r-k} \omega(1/2^r, f)_p \right\}
\]

is true.

In [15], we investigated a similar problem for the Vilenkin system.

**Theorem T.** Let \( f \) belong to \( L^p(G_m) \) for some \( p \in [1, \infty] \) and \( \alpha \in (0, 1) \). Then, for any \( M_k \leq n < M_{k+1} \), \( k, n \in N \), the inequality

\[
\| \sigma_n^{-\alpha}(f) - f \|_p \leq c(p, \alpha) \left\{ M_k^\alpha \omega(1/M_{k-1}, f)_p + \sum_{r=0}^{k-2} M_r \omega(1/M_r, f)_p \right\}
\]

is true.

Goginava [7] also studied the approximation properties of Cesàro \((c, -\alpha, -\beta)\) means with \( \alpha, \beta \in (0, 1) \) in the case of double Walsh–Fourier series. The following theorem was proved:

**Theorem G2.** Assume that \( f \) belongs to \( L^p(G_2^2) \) for some \( p \in [1, \infty] \) and \( \alpha, \beta \in (0, 1) \). Then, for any \( 2^k \leq n < 2^{k+1}, 2^l \leq m < 2^{l+1}, k, n \in N \), the inequality

\[
\left\| \sigma_{n,m}^{-\alpha,-\beta}(f) - f \right\|_p \leq c(\alpha, \beta) \left\{ 2^{k\alpha} \omega(1/2^{k-1}, f)_p + 2^{l\beta} \omega(1/2^{l-1}, f)_p \right\}
\]
\[ + 2^{k}\alpha \omega_{1,2}(f, 1/2^{k-1}, 1/2^{l-1}) \]
\[ + \sum_{r=0}^{k-2} 2^{-r-k}\omega_{1}(f, 1/2^{k}) + \sum_{s=0}^{l-2} 2^{-s-l}\omega_{2}(f, 1/2^{s}) \]

is true.

In the present paper, we establish a similar assertion in the case of double Vilenkin–Fourier series.

**Theorem 1.** Assume that \( f \) belongs to \( L^p(G^2_{m}) \) for some \( p \in [1, \infty] \) and \( \alpha \in (0,1) \). Then, for any \( M_k \leq n < M_{k+1}, M_l \leq m < M_{l+1}, k, n, m, l \in N, \) the inequality

\[
\left\| \sigma_{n,m}^{\alpha, \beta}(f) - f \right\|_p \leq c(\alpha, \beta) \left( \omega_{1}(f, 1/M_{k-1})_p M_k^\alpha + \omega_{2}(f, 1/M_{l-1})_p M_l^\beta \\
+ \omega_{1,2}(f, 1/M_{k-1}, 1/M_{l-1})_p M_k^\alpha M_l^\beta \\
+ \sum_{r=0}^{k-2} \frac{M_r}{M_k}\omega_{1}(f, 1/M_{r})_p + \sum_{s=0}^{l-2} \frac{M_s}{M_l}\omega_{2}(f, 1/M_{s})_p \right)
\]

is true.

**Corollary 1.** Assume that \( f \) belongs to \( L^p \) for some \( p \in [1, \infty] \). If

\[ M_k^\alpha \omega_1 \left( f, \frac{1}{M_k} \right)_p \to 0 \quad \text{as} \quad k \to \infty, \quad 0 < \alpha < 1, \]
\[ M_l^\beta \omega_1 \left( f, \frac{1}{M_l} \right)_p \to 0 \quad \text{as} \quad l \to \infty, \quad 0 < \beta < 1, \]
\[ M_k^\alpha M_l^\beta \omega_{1,2} \left( f, \frac{1}{M_k}, \frac{1}{M_l} \right)_p \to 0 \quad \text{as} \quad k, l \to \infty, \]

then

\[ \left\| \sigma_{n,m}^{\alpha, \beta}(f) - f \right\|_p \to 0 \quad \text{as} \quad n, m \to \infty. \]

**Corollary 2.** Assume that \( f \) belongs to \( L^p \) for some \( p \in [1, \infty] \) and that \( \alpha, \beta \in (0,1), \alpha + \beta < 1 \). If

\[ \omega \left( f, \frac{1}{M_n} \right)_p = o \left( \left( \frac{1}{M_n} \right)^{\alpha + \beta} \right), \]

then

\[ \left\| \sigma_{n,m}^{\alpha, \beta}(f) - f \right\|_p \to 0 \quad \text{as} \quad n, m \to \infty. \]
The following theorem shows that Corollary 2 cannot be improved:

**Theorem 2.** For every \( \alpha, \beta \in (0, 1) \), \( \alpha + \beta < 1 \), there exists a function \( f_0 \in C(G^2_{m,n}) \) such that

\[
\omega\left(f, \frac{1}{M_n}\right)_C = O\left(\left(\frac{1}{M_n}\right)^{\alpha + \beta}\right),
\]

and

\[
\limsup_{n \to \infty} \left\| \sigma_{M_n, M_n}^{-\alpha, -\beta}(f) - f \right\|_1 > 0.
\]

In order to prove Theorem 1, we need the following lemmas:

**Lemma 1 [1].** Let \( \alpha_1, \ldots, \alpha_n \) be real numbers. Then

\[
\frac{1}{n} \int_{G_m^2} \left| \sum_{k=1}^{n} \alpha_k D_k(x) \right| d\mu(x) \leq \frac{c}{\sqrt{n}} \left( \sum_{k=1}^{n} \alpha_k^2 \right)^{1/2},
\]

where \( c \) is an absolute constant.

**Lemma 2.** Assume that \( f \) belongs to \( L^p(G^2_{m,n}) \) for some \( p \in [1, \infty] \). Then, for any \( \alpha, \beta \in (0, 1) \), the following inequality is true:

\[
I := \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} \sum_{i=0}^{M_{k-1}-1} \sum_{j=0}^{M_{l-1}-1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v)
\times \left[ f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \right] d\mu(u, v) \right\|_p
\leq c(\alpha, \beta) \left( \sum_{r=0}^{k-1} \frac{M_r}{M_k} \omega_1(f, 1/M_r)_p + \sum_{s=0}^{l-1} \frac{M_s}{M_l} \omega_2(f, 1/M_s)_p \right),
\]

where \( M_k \leq n < M_{k+1}, \ M_l \leq m < M_{l+1} \).

**Proof.** Applying Abel’s transformation, from (2), we get

\[
I \leq \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} \sum_{i=1}^{M_{k-1}-1} \sum_{j=1}^{M_{l-1}-1} A_{n-i+1}^{-\alpha-1} A_{m-j+1}^{-\beta-1} D_i(u) D_j(v)
\times \left[ f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \right] d\mu(u, v) \right\|_p
\]
+ \frac{1}{A_n^\alpha A_m^\beta} \left\| \int_{G^2_m} A_{m-M_{l+1}}^{-\beta} D_{M_{l+1}}(v) \right. \\
\times \sum_{i=1}^{M_{k-1}-1} A_{n-i+1}^{-\alpha-1} D_i(u) \left[ f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \right] d\mu(u, v) \left\|_p \\
+ \frac{1}{A_n^\alpha A_m^\beta} \left\| \int_{G^2_m} A_{n-M_{k-1}+1}^{-\alpha} D_{M_{k-1}}(u) \right. \\
\times \sum_{j=1}^{M_{l-1}-1} A_{m-j+1}^{-\beta-1} D_j(v) \left[ f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \right] d\mu(u, v) \left\|_p \\
+ \frac{1}{A_n^\alpha A_m^\beta} \left\| \int_{G^2_m} A_{n-M_{k-1}+1}^{-\alpha} A_{m-M_{l+1}}^{-\beta} \\
\times D_{M_{k-1}}(u) D_{M_{l+1}}(v) \left[ f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \right] d\mu(u, v) \left\|_p \\
= I_1 + I_2 + I_3 + I_4. \tag{5}

It follows from the generalized Minkowski inequality and relations (1) and (4) that

\[ I_4 \leq \frac{1}{A_n^\alpha A_m^\beta} \int_{G^2_m} \left| A_{n-M_{k-1}+1}^{-\alpha} A_{m-M_{l+1}+1}^{-\beta} D_{M_{k-1}}(u) D_{M_{l+1}}(v) \right| \\
\times \left\| f(\cdot - u, \cdot - v) - f(x, y) \right\|_p d\mu(u, v) \\
\leq c(\alpha, \beta) M_{k-1} M_{l-1} \int_{I_{k-1} \times I_{l-1}} \left\| f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \right\|_p d\mu(u, v) \\
= O(\omega_1(f, 1/M_{k-1})_p + \omega_2(f, 1/M_{l-1})_p). \tag{6}

It is evident that

\[ I_1 \leq \frac{1}{A_n^\alpha A_m^\beta} \sum_{r=0}^{k-2} \sum_{s=0}^{l-2} \left\| \int_{G^2_m} \sum_{i=M_r}^{M_{r+1}-1} \sum_{j=M_s}^{M_{s+1}-1} A_{n-i+1}^{-\alpha-1} A_{m-j+1}^{-\beta-1} D_i(u) D_j(v) \\
\times \left[ f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \right] d\mu(u, v) \right\|_p \]
Moreover, it is easy to show that

\[
I_{11} + I_{12} + I_{13}.
\]

Moreover, it is easy to show that

\[
I_{12} = 0.
\]

By using Lemma 1, for \( I_{11} \), we can write

\[
I_{11} \leq \frac{1}{A_{n}^{-\alpha} A_{m}^{-\beta}} k^{-2} l^{-2} \sum_{r=0}^{k-2} \sum_{s=0}^{l-2} \left( \int_{G_{m}^{i}} \sum_{i=M_{r}}^{M_{r+1}-1} \sum_{j=M_{s}}^{M_{s+1}-1} A_{n-i+1}^{-\alpha} A_{m-j+1}^{-\beta} D_{i}(u) D_{j}(v) \right)
\]

\[
\times \left[ f(\cdot - u, \cdot - v) - S_{M_{r}, M_{s}}(\cdot, \cdot, f) \right] \, d\mu(u, v) \right|_{p}
\]

\[
+ \frac{1}{A_{n}^{-\alpha} A_{m}^{-\beta}} k^{-2} l^{-2} \sum_{r=0}^{k-2} \sum_{s=0}^{l-2} \left( \int_{G_{m}^{i}} \sum_{i=M_{r}}^{M_{r+1}-1} \sum_{j=M_{s}}^{M_{s+1}-1} A_{n-i+1}^{-\alpha} A_{m-j+1}^{-\beta} D_{i}(u) D_{j}(v) \right)
\]

\[
\times \left[ S_{M_{r}, M_{s}}(\cdot, \cdot, f) - S_{M_{r}, M_{s}}(\cdot, \cdot, f) \right] \, d\mu(u, v) \right|_{p}
\]

\[
+ \frac{1}{A_{n}^{-\alpha} A_{m}^{-\beta}} k^{-2} l^{-2} \sum_{r=0}^{k-2} \sum_{s=0}^{l-2} \left( \int_{G_{m}^{i}} \sum_{i=M_{r}}^{M_{r+1}-1} \sum_{j=M_{s}}^{M_{s+1}-1} A_{n-i+1}^{-\alpha} A_{m-j+1}^{-\beta} D_{i}(u) D_{j}(v) \right)
\]

\[
\times \left[ S_{M_{r}, M_{s}}(\cdot, \cdot, f) - f(\cdot, \cdot) \right] \, d\mu(u, v) \right|_{p}
\]

\[
= I_{11} + I_{12} + I_{13}.
\]
\[
\times \left( \int_{G_m} \left| \sum_{j=M_s}^{M+1-\beta-1} A_{m-j+1} D_j(v) \right| d\mu(v) \right)
\]
\[
\leq c(\alpha, \beta) n^\alpha m^\beta \sum_{r=0}^{k-2} \sum_{s=0}^{l-2} (\omega_1(f, 1/M_r)_p + \omega_2(f, 1/M_s)_p)
\]
\[
\times \left( \sqrt{M_r+1} \left( \sum_{i=M_r}^{M_r+1-1} (n-i+1)^{-2\alpha-2} \right)^{1/2} \right)
\]
\[
\times \left( \sqrt{M_s+1} \left( \sum_{j=M_s}^{M_s+1-1} (m-j+1)^{-2\beta-2} \right)^{1/2} \right)
\]
\[
\leq c(\alpha, \beta) n^\alpha m^\beta \sum_{r=0}^{k-2} \sum_{s=0}^{l-2} (\omega_1(f, 1/M_r)_p + \omega_2(f, 1/M_s)_p)
\]
\[
\times \left( \sqrt{M_r+1} (n-M_{r+1})^{-\alpha-1} \sqrt{M_r+1} \right)
\]
\[
\times \left( \sqrt{M_s+1} (n-M_{s+1})^{-\beta-1} \sqrt{M_s+1} \right)
\]
\[
\leq c(\alpha, \beta) n^\alpha m^\beta \sum_{r=0}^{k-2} \sum_{s=0}^{l-2} M_{r+1}^{\alpha+1} M_{s+1}^{\beta+1} (\omega_1(f, 1/M_r)_p + \omega_2(f, 1/M_s)_p)
\]
\[
\leq c(\alpha, \beta) \left( \sum_{r=0}^{k-2} M_r \omega_1(f, 1/M_r)_p + \sum_{s=0}^{l-2} M_s \omega_2(f, 1/M_s)_p \right). \quad (9)
\]

Similarly, we can prove that

\[
I_{13} \leq c(\alpha, \beta) \left( \sum_{r=0}^{k-2} \frac{M_r}{M_k} \omega_1(f, 1/M_r)_p + \sum_{s=0}^{l-2} \frac{M_s}{M_l} \omega_2(f, 1/M_s)_p \right). \quad (10)
\]

Combining (7)–(10), for \( I_1 \), we get

\[
I_1 \leq c(\alpha, \beta) \left( \sum_{r=0}^{k-2} \frac{M_r}{M_k} \omega_1(f, 1/M_r)_p + \sum_{s=0}^{l-2} \frac{M_s}{M_l} \omega_2(f, 1/M_s)_p \right). \quad (11)
\]
For $I_2$, we can write

$$I_2 \leq \frac{1}{A_n^\alpha A_m^\beta} \left\| \int_{G_m} A_{m-M_{l-1}+1}^{-\beta} D_{M_{l-1}}(v) \sum_{i=1}^{M_{k-1}-1} A_{n-i+1}^{-\alpha-1} D_i(u) \right\|_{L_1}
\times \left[ f(\cdot - u, \cdot - v) - f(\cdot - u, \cdot) \right] d\mu(u, v)

+ \frac{1}{A_n^\alpha A_m^\beta} \left\| \int_{G_m} A_{m-M_{l-1}+1}^{-\beta} D_{M_{l-1}}(v) \sum_{i=1}^{M_{k-1}-1} A_{n-i+1}^{-\alpha-1} D_i(u) \right\|_{L_1}
\times \left[ f(\cdot - u, \cdot) - f(\cdot, \cdot) \right] d\mu(u, v) = I_{21} + I_{22}. \quad (12)$$

By using the generalized Minkowski inequality and relations (1) and (4), we obtain

$$I_{21} \leq c(\alpha, \beta) \frac{M_{l-1}}{A_n^{-\alpha}} \int_{l-1}^{M_{k-1}-1} \left( \int_{G_m} \left| \sum_{i=1}^{M_{k-1}-1} A_{n-i+1}^{-\alpha-1} D_i(u) \right| \right)
\times \left\| f(\cdot - u, \cdot - v) - f(\cdot - u, \cdot) \right\|_{L_1} d\mu(v)

\leq c(\alpha, \beta) n^\alpha \omega_2(f, 1/M_{l-1}) \left( \int_{G_m} \left| \sum_{i=1}^{M_{k-1}-1} A_{n-i+1}^{-\alpha-1} D_i(u) \right| d\mu(u) \right)

\leq c(\alpha, \beta) n^\alpha \omega_2(f, 1/M_{l-1}) \left( \sqrt{M_{k-1}} \left( \sum_{i=1}^{M_{k-1}-1} (n-i+1)^{-2\alpha-2} \right)^{1/2} \right)

\leq c(\alpha, \beta) n^\alpha \omega_2(f, 1/M_{l-1}) \left( \sqrt{M_{k-1}} (n-M_{k-1})^{-\alpha-1} \sqrt{M_{k-1}} \right)

\leq c(\alpha, \beta) \omega_2(f, 1/M_{l-1}). \quad (13)$$

The estimate for $I_{22}$ is similar to the estimate for $I_1$. Hence, we find

$$I_{22} \leq c(\alpha, \beta) \sum_{r=0}^{k-2} \frac{M_r}{M_k} \omega_1(f, 1/M_r). \quad (14)$$
Thus, combining (12)–(14), for $I_2$, we get

$$I_2 \leq c(\alpha, \beta) \left( \sum_{r=0}^{k-2} \frac{M_r}{M_k} \omega_1(f, 1/M_r) + \omega_2(f, 1/M_{k-1}) \right).$$

(15)

The estimate for $I_3$ is analogous to the estimate for $I_2$. Therefore, we get

$$I_3 \leq c(\alpha, \beta) \left( \sum_{s=0}^{l-2} \frac{M_s}{M_l} \omega_2(f, 1/M_s) + \omega_1(f, 1/M_{k-1}) \right).$$

(16)

Combining (5), (6), (10), (15), and (16), we complete the proof of Lemma 2.

**Lemma 3.** Assume that $f$ belongs to $L^p(G^2_m)$ for some $p \in [1, \infty]$. Then, for every $\alpha, \beta \in (0, 1)$, the following estimates hold:

$$II := \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G^2_n} \sum_{i=M_{k-1}}^{n} \sum_{j=0}^{M_{l-1}-1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right. \left. \times \left[ f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \right] d\mu(u) d\mu(v) \right\|_p$$

$$\leq c(\alpha, \beta) \omega_1(f, 1/M_{k-1}) p M_k^\alpha,$$

$$III := \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G^2_n} \sum_{i=0}^{M_{k-1}-1} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right. \left. \times \left[ f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \right] d\mu(u) d\mu(v) \right\|_p$$

$$\leq c(\alpha, \beta) \omega_2(f, 1/M_{l-1}) p M_l^\beta,$$

where $M_k \leq n < M_{k+1}$ and $M_l \leq m < M_{l+1}$.

**Proof.** It follows from the generalized Minkowski inequality that

$$II = \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G^2_n} \sum_{i=M_{k-1}}^{n} \sum_{j=0}^{M_{l-1}-1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) f(\cdot - u, \cdot - v) d\mu(u, v) \right\|_p$$
\[
\frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left| \int_{G_m^2} \sum_{i=M_{k-1}}^{n} \sum_{j=0}^{M_{l-1}} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right. \\
\left. \times \left[ f(\cdot - u, \cdot - v) - S_{M_{k-1}}^{(1)}(\cdot - u, \cdot - v, f) \right] d\mu(u,v) \right|_p
\]

\[
\leq \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \int_{G_m^2} \left| \sum_{i=M_{k-1}}^{n} \sum_{j=0}^{M_{l-1}} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right|
\left. \times \left| f(\cdot - u, \cdot - v) - S_{M_{k-1}}^{(1)}(\cdot - u, \cdot - v, f) \right| d\mu(u,v) \right|
\]

\[
+ \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \int_{G_m^2} \left| \sum_{i=M_{k}}^{n} \sum_{j=0}^{M_{l-1}} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right|
\left. \times \left| f(\cdot - u, \cdot - v) - S_{M_{k-1}}^{(1)}(\cdot - u, \cdot - v, f) \right| d\mu(u,v) \right|
\]

\[
= II_1 + II_2.
\] (17)

In [15], it was shown that the inequality

\[
\int_{G_{m}} \left| \sum_{v=M_{k-1}}^{M_{l-1}} A_{n-v}^{-\alpha} \psi_v(u) \right| d\mu(u) \leq c(\alpha), \quad k = 1, 2, \ldots
\] (18)

is true.

By using Lemma 1 and relations (4) and (18), for \(II_1\), we can write

\[
II_1 \leq c(\alpha, \beta) n^\alpha m^\beta \omega_1(f, 1/M_{k-1})_p \left( \int_{G_{m}} \left| \sum_{i=M_{k-1}}^{M_{l-1}} A_{n-i}^{-\alpha} \psi_i(u) \right| d\mu(u) \right)
\]

\[
\times \left( \int_{G_{m}} \left| \sum_{j=1}^{M_{l-1}} A_{m-j+1}^{-\beta} \psi_{j-1}(v) \right| d\mu(v) \right)
\]

\[
\leq c(\alpha, \beta) n^\alpha m^\beta \omega_1(f, 1/M_{k-1})_p \left( \sqrt{M_{l-1}} \left( \sum_{i=1}^{M_{l-1}} (m - j + 1)^{-2\beta-2} \right)^{1/2} \right)
\]
\[ \leq c(\alpha, \beta)n^\alpha m^\beta \omega_2(f, 1/M_{k-1})_p \left( \sqrt{M_{l-1}} (n - M_{l-1})^{-\beta - 1} \sqrt{M_{l-1}} \right) \]

\[ \leq c(\alpha, \beta) \omega_1(f, 1/M_{k-1})_p M_k^\alpha. \]  \hspace{1cm} (19)

The estimate for \( II_2 \) is analogous to the estimate for \( II_1 \) and, hence, we get

\[ II_2 \leq c(\alpha, \beta) \omega_1(f, 1/M_{k-1})_p M_k^\alpha. \]  \hspace{1cm} (20)

Combining (17)–(20), we obtain

\[ II \leq c(\alpha, \beta) \omega_1(f, 1/M_{l-1})_p M_l^\beta. \]  \hspace{1cm} (21)

Similarly, we can prove that

\[ III \leq c(\alpha, \beta) \omega_2(f, 1/M_{l-1})_p M_l^\beta. \]  \hspace{1cm} (22)

Combining (21) and (22), we complete the proof of Lemma 3.

**Lemma 4.** Assume that \( f \) belongs to \( L^p(G_n^2) \) for some \( p \in [1, \infty] \). Then, for every \( \alpha, \beta \in (0, 1) \), the following inequality is true:

\[ IV := \frac{1}{A_n^\alpha A_m^\beta} \left\| \int_{G_n^2} \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^\alpha A_{m-j}^\beta \psi_i(u) \psi_j(v) \right. \]

\[ \times \left[ f(\cdot - u, \cdot - v) - f(\cdot, \cdot) \right] d\mu(u) d\mu(v) \right\|_p \]

\[ \leq c(\alpha, \beta) \omega_{1,2}(f, 1/M_k, 1/M_l)_p M_k^\alpha M_l^\beta, \]

where \( M_k \leq n < M_{k+1} \) and \( M_l \leq m < M_{l+1} \).

**Proof.** It follows from the generalized Minkowski inequality and relations (1) and (4) that

\[ IV = \frac{1}{A_n^\alpha A_m^\beta} \left\| \int_{G_n^2} \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^\alpha A_{m-j}^\beta \psi_i(u) \psi_j(v) f(\cdot - u, \cdot - v) d\mu(u, v) \right\|_p \]

\[ \leq c(\alpha, \beta) \omega_{1,2}(f, 1/M_k, 1/M_l)_p M_k^\alpha M_l^\beta. \]
\[
\times \left[ S_{M_{k-1}, M_{l-1}}(-u, -v, f) - S_{M_{k-1}}^{(1)}(-u, -v, f) \\
- S_{M_{l-1}}^{(2)}(-u, -v, f) + f(-u, -v) \right] d\mu(u, v) \] 
\]

\[
\leq \frac{1}{A_{n}^{-\alpha} A_{m}^{-\beta}} \int_{G_{n}^{2}} \left| \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_{i}(u) \psi_{j}(v) \right| 
\times \left| S_{M_{k-1}, M_{l-1}}(-u, -v, f) - S_{M_{k-1}}^{(1)}(-u, -v, f) \\
- S_{M_{l-1}}^{(2)}(-u, -v, f) + f(-u, -v) \right| d\mu(u, v) 
\]

\[
\leq c(\alpha, \beta) n^{\alpha} m^{\beta} \omega_{1,2}(f, 1/M_{k-1}, 1/M_{l-1})_{p} 
\]

\[
\times \left[ f(-u, -v) - f(\cdot, \cdot) \right] d\mu(u, v) 
\]

\[
+ \frac{1}{A_{n}^{-\alpha} A_{m}^{-\beta}} \int_{G_{n}^{2}} \left| \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_{i}(u) \psi_{j}(v) \right| 
\times \left[ f(-u, -v) - f(\cdot, \cdot) \right] d\mu(u, v) 
\]

Lemma 4 is proved.

**Proof of Theorem 1.** It is evident that

\[
\sigma_{n,m}^{-\alpha, -\beta}(f, x, y) - f(x, y) 
\]

\[
= \frac{1}{A_{n}^{-\alpha} A_{m}^{-\beta}} \int_{G_{n}^{2}} \sum_{i=0}^{M_{k-1} - 1} \sum_{j=0}^{M_{l-1} - 1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_{i}(u) \psi_{j}(v) 
\times \left[ f(-u, -v) - f(\cdot, \cdot) \right] d\mu(u, v) 
\]

\[
+ \frac{1}{A_{n}^{-\alpha} A_{m}^{-\beta}} \int_{G_{n}^{2}} \sum_{i=M_{k-1}}^{n} \sum_{j=0}^{M_{l-1} - 1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_{i}(u) \psi_{j}(v) 
\times \left[ f(-u, -v) - f(\cdot, \cdot) \right] d\mu(u, v) 
\]
\[ + \int_{G_{2n}^m} \sum_{i=0}^{M_{k-1} - 1} \sum_{j=M_{l-1}}^{m} A_{m-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \]

\[ \times [f(\cdot - u, \cdot - v) - f(\cdot, \cdot)] d\mu(u, v) \]

\[ + \int_{G_{2n}^m} \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{m-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \]

\[ \times [f(\cdot - u, \cdot - v) - f(\cdot, \cdot)] d\mu(u) d\mu(v) \]

\[ = I + II + III + IV. \]

Since

\[ \left\| \sigma_{n,m}^{-\alpha, -\beta} (f, x) - f(x) \right\|_p \leq \|I\|_p + \|II\|_p + \|III\|_p + \|IV\|_p, \]

in view of Lemmas 2–4, the proof of the theorem is completed.

**Proof of Corollary 2.** Since

\[ \omega_i \left( f, \frac{1}{M_n} \right) \leq \omega \left( f, \frac{1}{M_n} \right), \quad i = 1, 2, \]

\[ \omega_{1,2} \left( f, \frac{1}{M_n}, \frac{1}{M_m} \right) \leq 2 \omega_1 \left( f, \frac{1}{M_n} \right) \]

and

\[ \omega_{1,2} \left( f, \frac{1}{M_n}, \frac{1}{M_m} \right) \leq 2 \omega_2 \left( f, \frac{1}{M_m} \right), \]

we obtain

\[ \omega_{1,2} \left( f, \frac{1}{M_n}, \frac{1}{M_m} \right) = \left( \omega_{1,2} \left( f, \frac{1}{M_n}, \frac{1}{M_m} \right) \right)^{\frac{\alpha}{\alpha + \beta}} \left( \omega_{1,2} \left( f, \frac{1}{M_n}, \frac{1}{M_m} \right) \right)^{\frac{\beta}{\alpha + \beta}} \]

\[ \leq 2 \left( \omega \left( f, \frac{1}{M_n} \right) \right)^{\frac{\alpha}{\alpha + \beta}} \left( \omega_2 \left( f, \frac{1}{M_m} \right) \right)^{\frac{\beta}{\alpha + \beta}} \]

\[ \leq 2 \left( \omega \left( f, \frac{1}{M_n} \right) \right)^{\frac{\alpha}{\alpha + \beta}} \left( \omega \left( f, \frac{1}{M_m} \right) \right)^{\frac{\beta}{\alpha + \beta}}. \]

The validity of Corollary 2 immediately follows from Corollary 1.
**Proof of Theorem 2.** First, we set

\[ f_j(x) = \rho_j(x) = \exp \frac{2\pi i x_j}{m_j}. \]

Then we define a function

\[ f(x, y) = \sum_{j=1}^{\infty} \frac{1}{M_j^{(\alpha+\beta)}} f_j(x) f_j(y). \]

We now prove that

\[
\omega\left(f, \frac{1}{M_n}\right) = O\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right). \tag{24}
\]

Since

\[ |f_j(x-t) - f_j(x)| = 0, \quad j = 0, 1, \ldots, n-1, \quad t \in I_n, \]

we find

\[
|f(x-t, y) - f(x, y)| \leq \sum_{j=1}^{n-1} \frac{1}{M_j^{(\alpha+\beta)}} |f_j(x-t) - f_j(x)| + \sum_{j=n}^{\infty} \frac{2}{M_j^{(\alpha+\beta)}}
\]

\[ \leq \frac{c}{M_n^{(\alpha+\beta)}}. \]

Hence,

\[
\omega_1\left(f, \frac{1}{M_n}\right) = O\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right). \tag{25}
\]

Similarly, we get

\[
\omega_2\left(f, \frac{1}{M_m}\right) = O\left(\left(\frac{1}{M_m}\right)^{\alpha+\beta}\right). \tag{26}
\]

Further, by (25) and (26), we obtain (24).

Finally, we prove that \(\sigma_{M_n,M_n}^{-\alpha,-\beta}(f)\) diverge in the metric of \(L^1\). It is clear that

\[
\left\| \sigma_{M_n,M_n}^{-\alpha,-\beta}(f) - f \right\|_1 \geq \int_{G_n^2} \left[ \sigma_{M_n,M_n}^{-\alpha,-\beta}(f; x, y) - f(x, y) \right] \psi_{M_k}(x) \psi_{M_k}(y) \, d\mu(x, y)
\]
\[
\left| \int_{G_m^2}^{\alpha,-\beta} (f; x, y) \psi_M(x) \psi_M(y) dx dy \right| - \left| \hat{f}(M, M) \right|
\]

\[
= \left| \frac{1}{A^{-\alpha} M^{-\beta}} \sum_{i=0}^{M_k} \sum_{j=0}^{M_k} A^{-\alpha} M^{-\beta} f(i, \bar{j}) \right|
\]

\[
\times \int_{G_m^2} \psi_i(x) \psi_j(y) \psi_M(x) \psi_M(y) \mu(x, y) \left| \hat{f}(M, M) \right|
\]

\[
= \frac{1}{A^{-\alpha} M^{-\beta}} \left| \hat{f}(M, M) \right| - \left| \hat{f}(M, M) \right|
\]

We have

\[
\hat{f}(M, M) = \int_{G_m^2} f(x, y) \psi_M(x) \psi_M(y) \mu(x, y)
\]

\[
= \sum_{j=1}^{\infty} \frac{1}{M^{(\alpha+\beta)}_j} \int_{G_m^2} \rho_j(x) \rho_j(y) \psi_M(x) \psi_M(y) \mu(x, y)
\]

\[
= \sum_{j=1}^{\infty} \frac{1}{M^{(\alpha+\beta)}_j} \int_{G_m} \rho_j(x) \psi_M(x) \mu(x) \int_{G_m} \rho_j(y) \psi_M(y) \mu(y)
\]

\[
= \frac{1}{M^{(\alpha+\beta)}_j}.
\]

Thus, we can write

\[
\left\| \sigma_{M_n,M_n}^{-\alpha,-\beta} (f) - f \right\|_1 \geq c(\alpha, \beta).
\]

Theorem 2 is proved.

REFERENCES

1. G. N. Agaev, N. Ya. Vilenkin, G. M. Dzhafarli, and A. I. Rubinshtein, *Multiplicative Systems of Functions and Harmonic Analysis on Zero-Dimensional Groups* [in Russian], Ehlm, Baku (1981).
2. N. J. Fine, “Cesàro summability of Walsh–Fourier series,” *Proc. Nat. Acad. Sci. USA*, 41, 558–591 (1995).
3. B. I. Golubov, A. V. Efimov, and V. A. Skvortsov, *Walsh Series and Transformations. Theory and Applications* [in Russian], Nauka, Moscow (1987).
4. U. Goginava, “On the uniform convergence of Walsh–Fourier series,” *Acta Math. Hungar.*, 93, No. 1-2, 59–70 (2001).
5. U. Goginava, “On the approximation properties of Cesàro means of negative order of Walsh–Fourier series,” *J. Approx. Theory*, 115, No. 1, 9–20 (2002).
6. U. Goginava, “Uniform convergence of Cesàro means of negative order of double Walsh–Fourier series,” *J. Approx. Theory*, **124**, No. 1, 96–108 (2003).

7. U. Goginava, “Cesàro means of double Walsh–Fourier series,” *Anal. Math.*, **30**, 289–304 (2004).

8. U. Goginava and K. Nagy, “On the maximal operator of Walsh–Kaczmarz–Fejér means,” *Czechoslovak Math. J.*, **61**(136), No. 3, 673–686 (2011).

9. G. Gát and U. Goginava, “A weak type inequality for the maximal operator of \((C, \alpha)\)-means of Fourier series with respect to the Walsh–Kaczmarz system,” *Acta Math. Hungar.*, **125**, No. 1-2, 65–83 (2009).

10. G. Gát and K. Nagy, “Cesàro summability of the character system of the p-series field in the Kaczmarz rearrangement,” *Anal. Math.*, **28**, No. 1, 1–23 (2002).

11. K. Nagy, “Approximation by Cesàro means of negative order of Walsh–Kaczmarz–Fourier series,” *East J. Approx.*, **16**, No. 3, 297–311 (2010).

12. P. Simon and F. Weisz, “Weak inequalities for Cesàro and Riesz summability of Walsh–Fourier series,” *J. Approx. Theory*, **151**, No. 1, 1–19 (2008).

13. F. Schipp, “Über gewisse Maximaloperatoren,” *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **18**, 189–195 (1975).

14. F. Schipp, W. R. Wade, P. Simon, and J. Pál, *Walsh Series, Introduction to Dyadic Harmonic Analysis*, Hilger, Bristol (1990).

15. T. Tepnadze, “On the approximation properties of Cesàro means of negative order of Vilenkin–Fourier series,” *Studia Sci. Math. Hungar.*, **53**, No. 4, 532–544 (2016).

16. V. I. Tevzadze, “Uniform \((C, – \alpha)\) summability of Fourier series with respect to the Walsh–Paley system,” *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, **22**, No. 1, 41–61 (2006).

17. L. V. Zhizhiashvili, *Trigonometric Fourier Series and Their Conjugates* [in Russian], Tbilisi (1993).

18. A. Zygmund, *Trigonometric Series*, Vol. 1, Cambridge Univ. Press, Cambridge, UK (1959).