Abstract

Essential properties of semiclassical approximation for quantum mechanics are viewed as axioms of an abstract semiclassical mechanics. Its symmetry properties are discussed. Semiclassical systems being invariant under Lie groups are considered. An infinitesimal analog of group relation is written. Sufficient conditions for reconstructing semiclassical group transformations (integrability of representation of Lie algebra) are discussed. The obtained results may be used for mathematical proof of Poincare invariance of semiclassical Hamiltonian field theory and for investigation of quantum anomalies.
1 Introduction

Semiclassical approximation is widely used in quantum mechanics and field theory. There are only few cases when the Schrödinger equation possesses exact solutions. It is then necessary to develop different approximation techniques in order to investigate evolution equations. Semiclassical method is universal: it may be applied, provided that the coefficients of all derivative operators are small, of the order $O(\lambda)$ as $\lambda \to 0$, while an explicit form of the Hamiltonian may be arbitrary.

There are different semiclassical ansatzes that approximately satisfy quantum mechanical equations. They are reviewed in section 2. The most popular semiclassical substitution is the WKB-ansatz. However, there are other wave functions (for example, the Maslov complex-WKB ansatz [1, 2]) that conserve their forms under time evolution in the semiclassical approximation.

Semiclassical conception can be formally applied to quantum field theory (QFT) under certain conditions [3]. Examples of application of semiclassical conceptions are soliton quantization [3, 4, 20], quantum field theory in a strong external background [7, 8], one-loop [9, 10, 11, 12], time-dependent Hartree-Fock [13, 14] and Gaussian approximations [15, 16, 17, 18].

Unfortunately, ”exact” quantum field theory is constructed mathematically for a restricted class of models only (see, for example, [19, 20, 21, 22]). Therefore, formal approximate methods such as perturbation theory seem to be ways to quantize the field theory rather than to construct approximations for the exact solutions of quantum field theory equations. The conception of field quantization within the perturbation framework is popular [23, 24]. One can expect that the semiclassical approximation plays an analogous role.

An important axiom of QFT is the property of Poincare invariance [25, 26]. There are also other symmetries in QFT, as well as in different models of quantum mechanics.

The purpose of this paper is to introduce a notion of a symmetry transformation in the semiclassical mechanics, as well as to investigate infinitesimal properties of groups of semiclassical transformations, especially for the case of field systems.

2 Semiclassical mechanics

2.1 Semiclassical substitutions to quantum mechanical equations

This subsection deals with a review of semiclassical substitutions for the finite-dimensional equations of the form

$$i\lambda \frac{\partial \psi_t(x)}{\partial t} = H_t(x, -i\lambda \frac{\partial}{\partial x})\psi_t(x), \quad t \in \mathbb{R}, x \in \mathbb{R}^d, \quad (2.1)$$

where $H_t(q, p)$ is an arbitrary function.

2.1.1 WKB and Maslov complex-WKB wave functions

The most famous semiclassical approach is the WKB-method. It is as follows. The initial condition for eq. (2.1) is chosen to be

$$\psi_0(x) = \varphi_0(x)e^{iS_0(x)}, \quad (2.2)$$

where $S_0$ is a real function. The WKB-result [27] is that the solution of eq. (2.1) at time moment $t$ has the same type (2.2) up to $O(\lambda)$,

$$||\psi_t - \varphi_t e^{iS_t}|| = O(\lambda).$$

The Hamilton-Jacobi equation for $S_t$ and the transport equation for $\varphi_t$ can be written [27].

However, we are not obliged to choose the initial condition for eq. (2.1) in a form (2.2). There are other substitutions to eq. (2.1) that conserve their forms under time evolution as $\lambda \to 0$. For example,
consider the Maslov complex-WKB wave function \([\text{[1]} \text{[2]} \text{[29]}]\),

\[
\psi_0(x) = \text{const} \xi_0 e^{\frac{i}{\hbar} P_0(x-Q_0)} f_0 \left(\frac{x - Q_0}{\sqrt{\lambda}}\right) \equiv (K_{\xi_0, Q_0, P_0} f_0)(x),
\]

(2.3)

which corresponds to uncertainties of coordinates and momenta of the order \(O(\sqrt{\lambda})\), since the smooth function \(f_0(\xi)\) is chosen to damp rapidly at spatial infinity. Quantities \(Q_0, P_0 \in \mathbb{R}^d\) may be interpreted as classical values of coordinates and momenta.

It happens that the initial condition \(\psi_0\) conserves its form under time evolution up to \(O(\sqrt{\lambda})\) \([\text{[1]} \text{[2]}]\),

\[
\psi_t(x) = \text{const} \xi_0 e^{\frac{i}{\hbar} P_t(x-Q_t)} f_t \left(\frac{x - Q_t}{\sqrt{\lambda}}\right) + O(\sqrt{\lambda}).
\]

(2.4)

Moreover, \(Q_t, P_t\) satisfy the classical Hamiltonian equations, \(S_t - S_0\) is the action on the classical trajectory, while \(f_t(\xi)\) satisfies the Schrödinger equation with a quadratic Hamiltonian.

Semiclassical state \(\psi_0\) may be interpreted as a point on a bundle ("semiclassical bundle" \([29, 30]\)). The base of the bundle is a manifold \(\mathcal{X} = \{(S_0, Q_0, P_0)|S_0 \in \mathbb{R}, Q_0, P_0 \in \mathbb{R}^d\}\) which may be called as an extended phase space. If the point \(X_0 = (S_0, Q_0, P_0) \in \mathcal{X}\) is given, the "classical" state is specified. However, one should also specify the function \(f_0(\xi)\) (the "shape" of the wave packet). This corresponds to choice of an element of the fibre. The fibres of the bundle are spaces \(\mathcal{S}(\mathbb{R}^d)\), so that the bundle is trivial. If the point \((X_0 \in \mathcal{X}, f_0 \in \mathcal{S}(\mathbb{R}^d))\) is given, the semiclassical wave function \(\psi_0\) is completely specified.

The semiclassical evolution transformation may be viewed as an automorphism of the semiclassical bundle, since the evolution transformation of \((S, Q, P)\) does not depend on \(f_0\). The transformations \(u_t : \mathcal{X} \rightarrow \mathcal{X}\) and unitary operators \(U_t^0(u_t X \leftarrow X) : f_0 \mapsto f_t\) are then given. One can consider the completion \(\mathcal{F} = L^2(\mathbb{R}^d)\) of the space \(\mathcal{S}(\mathbb{R}^d)\) and extend the unitary operators \(U_t^0(u_t X \leftarrow X)\) to \(\mathcal{F}\). One obtain then operators \(U_t(u_t X \leftarrow X) : \mathcal{F} \rightarrow \mathcal{F}\).

2.1.2 Maslov theory of Lagrangian manifolds with complex germ

The wave function \(\psi(x)\) rapidly oscillates with respect to all variables. The wave function \(\psi_0\) rapidly damps at \(x - Q_t \gg O(\sqrt{\lambda})\). One should come to the conclusion that there exists a wave function asymptotically satisfying eq.\((2.1)\) which oscillates with respect to one group of variables and damps with respect to other variables. The construction of such states is given in the Maslov theory of Lagrangian manifolds with complex germ \([\text{[1]} \text{[2]}]\). Let \(\alpha \in \mathbb{R}^k, (P(\alpha), Q(\alpha)) \in \mathbb{R}^{2d}\) be a \(k\)-dimensional surface in the 2\(d\)-dimensional phase space, \(S(\alpha)\) be a real function, \(f(\alpha, \xi), \xi \in \mathbb{R}^d\) is a smooth function. Set \(\psi(x)\) to be not exponentially small if and only if the distance between point \(x\) and surface \(Q(\alpha)\) is of the order \(\leq O(\sqrt{\lambda})\). Otherwise, set \(\psi(x) \simeq 0\). If \(\min_{\alpha} |x - Q(\alpha)| = |x - Q(\overline{\alpha})| = O(\sqrt{\lambda})\), set

\[
\psi(x) = c_\lambda e^{\frac{i}{\hbar} S(\overline{\alpha})} e^{\frac{i}{\hbar} P(\overline{\alpha})(x-Q(\overline{\alpha}))} f(\overline{\alpha}, \frac{x - Q(\overline{\alpha})}{\sqrt{\lambda}}).
\]

(2.5)

One can note that wave functions \((2.2)\) and \((2.3)\) are partial cases of the wave function \((2.5)\). Namely, for \(k = 0\) the manifold \((P(\alpha), Q(\alpha))\) is a point, so that the functions \((2.3)\) coincide with \((2.3)\). Let \(k = d\). If the surface \((P(\overline{\alpha}), Q(\overline{\alpha}))\) is in the general position, for \(x\) in some domain one has \(x = Q(\overline{\alpha})\) for some \(\overline{\alpha}\). Therefore,

\[
\psi(x) = c_\lambda e^{iS(\overline{\alpha})} f(\overline{\alpha}, 0).
\]

We obtain the WKB-wave function. Thus, WKB and wave-packet asymptotic formulas \((2.2)\) and \((2.3)\) are partial cases of the wave function \((2.5)\) appeared in the theory of Lagrangian manifolds with complex germ.

The lack of formula \((2.3)\) is that the dependence of \(\overline{\alpha}\) on \(x\) is implicit and too complicated. However, under certain conditions formula \((2.3)\) is invariant if \(\overline{\alpha}\) is shifted by a quantity of the order \(O(\sqrt{\lambda})\). In
this case, the point \( \vec{\alpha} \) can be chosen in arbitrary way such that the distance of \( x \) and \( Q(\vec{\alpha}) \) is of the order \( O(\sqrt{\lambda}) \).

Namely,

\[
e^{\frac{\partial S(\vec{\alpha})}{\partial \alpha_i}} e^{\frac{\partial P(\vec{\alpha})}{\partial Q(\vec{\alpha})}} f(\vec{\alpha}, x - Q(\vec{\alpha}))
\]

(2.6)

\[
\sim e^{\frac{\partial S(\vec{\alpha})}{\partial \alpha_i} - \frac{\partial P(\vec{\alpha})}{\partial Q(\vec{\alpha})} \frac{\partial Q(\vec{\alpha})}{\partial x}} f(\vec{\alpha}, x - Q(\vec{\alpha}))
\]

if

\[
\frac{\partial S}{\partial \alpha_i} = P \frac{\partial Q}{\partial \alpha_i}
\]

(2.7)

\[
e^{\beta(\xi \frac{\partial P}{\partial \xi} - \frac{\partial Q}{\partial \xi})} f = f
\]

(2.8)

To obtain eqs. (2.7) and (2.8), one should expand left-hand side of eq. (2.6). Considering rapidly oscillating factors, we obtain eq. (2.7). To obtain eq. (2.8), it is sufficient to consider the limit \( \lambda \to 0 \).

Conditions (2.7), (2.8) simplify the check [1] that the wave function (2.5) approximately satisfies eq. (2.1) if the functions \( S, P, Q, f \) are time-dependent.

### 2.1.3 Composed semiclassical states

The form (2.3) of the semiclassical state appeared in the theory of Lagrangian manifolds with complex germ is not convenient for generalization to systems of infinite number of degrees of freedom. It is much more convenient to consider to consider wave function (2.3) as an "elementary" semiclassical state and wave function (2.3) as a "composed" semiclassical state presented as a superposition of elementary semiclassical states:

\[
\psi(x) = C_\lambda \int d\alpha e^{\frac{\partial S(\alpha)}{\partial \alpha_i} + \frac{\partial P(\alpha)}{\partial Q(\alpha)}(x - Q(\alpha))} g(\alpha, \frac{x - Q(\alpha)}{\sqrt{\lambda}}),
\]

(2.9)

where \( g(\alpha, \xi) \) is a rapidly damping function as \( \xi \to \infty \). Superpositions of such type were considered in [31, 32, 33]; the general case was investigated in [28, 34]. The composed semiclassical states for the abstract semiclassical theory were studied in [33].

To show that expression (2.9) is in agreement with formula (2.5), notice that the wave function (2.3) is exponentially small if the distance between \( x \) and the surface \( Q(\vec{\alpha}) \) is of order \( O(\sqrt{\lambda}) \). Let \( \min_\alpha |x - Q(\alpha)| = O(\sqrt{\lambda}) \) and \( |x - Q(\vec{\alpha})| = O(\sqrt{\lambda}) \). Consider the substitution \( \alpha = \vec{\alpha} + \sqrt{\lambda} \beta \). We find

\[
\psi(x) = C_\lambda \lambda^{k/2} \int d\beta e^{\frac{\partial S(\vec{\alpha} + \beta \sqrt{\lambda})}{\partial \alpha_i} + \frac{\partial P(\vec{\alpha} + \beta \sqrt{\lambda})}{\partial Q(\vec{\alpha} + \beta \sqrt{\lambda})} g(\vec{\alpha} + \beta \sqrt{\lambda}, x - Q(\vec{\alpha} + \beta \sqrt{\lambda})),
\]

If the condition (2.7) is not satisfied, this is an integral of a rapidly oscillating function. It is exponentially small. Under condition (2.7) one can consider a limit \( \lambda \to 0 \) and obtain the expression (2.3), provided that

\[
c_\lambda = C_\lambda \lambda^{k/2}
\]

and

\[
f(\vec{\alpha}, \xi) = \int d\beta e^{\frac{\partial S(\vec{\alpha})}{\partial \alpha_i} + \frac{\partial P(\vec{\alpha})}{\partial Q(\vec{\alpha})} \frac{\partial Q(\vec{\alpha})}{\partial \xi_s}} g(\vec{\alpha}, \xi)
\]

(2.10)

Integral representation (2.3) simplifies substitution of the wave function to eq. (2.1) and estimation of accuracy.

It follows from eq. (2.10) that the function \( f \) is invariant under the following change of the function \( g \) ("gauge transformation"):

\[
g(\alpha, \xi) \to g(\alpha, \xi) + \left( \frac{\partial P_m}{\partial \alpha_s} \xi_m - \frac{\partial Q_m}{\partial \alpha_s} \frac{1}{i} \frac{\partial}{\partial \xi_m} \right) \chi_s(\alpha, \xi).
\]

(2.11)

Thus, the semiclassical state is specified at fixed \( S(\alpha), P(\alpha), Q(\alpha) \) not by the function \( g \) but by the class of equivalence of functions \( g \): two functions are equivalent if they are related by the transformation (2.11).
This fact can be also illustrated if we evaluate the inner product $||\psi||^2$ as $\lambda \to 0$:

$$||\psi||^2 = C_2^2 \int d\alpha d\gamma \int dx e^{-\frac{i}{\lambda}S(\alpha)}e^{-\frac{i}{\lambda}P(\alpha)(x-Q(\alpha))}g^\ast(\alpha, \frac{x-Q(\alpha)}{\sqrt{\lambda}}) \int e^{\frac{i}{\lambda}S(\gamma)}e^{\frac{i}{\lambda}P(\gamma)(x-Q(\gamma))}g(\gamma, \frac{x-Q(\gamma)}{\sqrt{\lambda}}).$$

The integral over $x$ is not exponentially small if $\alpha - \gamma = O(\sqrt{\lambda})$. After substitution $\gamma = \alpha + \beta \sqrt{\lambda}$, $x - Q(\alpha) = \xi \sqrt{\lambda}$ and considering the limit $\lambda \to 0$, we find

$$||\psi||^2 \simeq C_2^2 \lambda \frac{\xi}{\lambda} \int d\alpha (g(\alpha, \cdot), \prod_{s=1}^{k} 2\pi \delta \left( \frac{\partial P_m}{\partial \alpha_s} \xi_m - \frac{\partial Q_m}{\partial \alpha_s} \frac{\partial}{\partial \xi_s} \right) g(\alpha, \cdot)). \quad (2.12)$$

The $k$-dimensional surface $\{(S(\alpha), P(\alpha), Q(\alpha))\}$ ("isotropic manifolds") in the extended phase space has the following physical meaning. Consider the average value of a semiclassical observable $A(x, -i \lambda \partial/\partial x)$. As $\lambda \to 0$, one has

$$(\psi, A(x, -i \lambda \partial/\partial x)\psi) \simeq C_2^2 \lambda \frac{\xi}{\lambda} \int d\alpha A(Q(\alpha), P(\alpha))(g(\alpha, \cdot)), \quad \prod_{s=1}^{k} 2\pi \delta \left( \frac{\partial P_m}{\partial \alpha_s} \xi_m - \frac{\partial Q_m}{\partial \alpha_s} \frac{\partial}{\partial \xi_s} \right) g(\alpha, \cdot).$$

We see that only values of the corresponding classical observable on the surface $\{(Q(\alpha), P(\alpha))\}$ are relevant for calculations for average values as $\lambda \to 0$. This means that the Blokhintsev-Wigner density function (Weyl symbol of the density matrix) corresponding to the composed semiclassical state is proportional to the delta function on the manifold $\{(Q(\alpha), P(\alpha))\}$.

Therefore, elementary semiclassical states describe evolution of a point particle, while composed semiclassical states (including WKB-states) describe evolution of the more complicated objects - isotropic manifolds.

### 2.2 Abstract semiclassical mechanics

Formally, the semiclassical conception can be applied to quantum field theory [28, 35]. A semiclassical complex-WKB state is specified by a set $X \in \mathcal{X}$ of classical variables (real quantity $S$, field configuration $\Phi(x)$, canonically conjugated momentum $\Pi(x)$) and a functional $f[\phi(\cdot)]$ (a "quantum state in the external field $X$"), if the functional Schrödinger representation of the canonical commutation relations is used. However, usage of this representation seems to be not rigorous. On the other hand, one can expect that it is possible to specify a semiclassical complex-WKB state by an element $f$ of a some (maybe, $X$-dependent) Hilbert space $\mathcal{F}_X$ instead of the functional $f[\phi(\cdot)]$. The structure of a semiclassical bundle remains then valid for field theory.

Let us formulate a definition of a semiclassical system. One should write down a list of essential properties ("axioms") of such systems (cf. [29, 30]). One of them is as follows.

**A1.** A locally trivial vector bundle $\pi : \mathcal{Z} \to \mathcal{X}$ called as a semiclassical bundle is specified. The base of the bundle $\mathcal{X}$ ("extended phase space") is a smooth (maybe, infinite-dimensional) manifold, while fibres $\mathcal{F}_X$, $X \in \mathcal{X}$ are Hilbert spaces.

Since the bundle is locally trivial, one can suppose without loss of generality that all spaces $\mathcal{F}_X$ coincide in a sufficiently small vicinity of each point.

Elementary ("complex-WKB") semiclassical states are viewed as points on the semiclassical bundle. Composed semiclassical states should be viewed as smooth mappings $\alpha \in \Lambda^k \mapsto (X(\alpha) \in \mathcal{X}, g(\alpha) \in \mathcal{F}_{X(\alpha)})$ for $k$-dimensional manifolds $\Lambda^k$ with given measure. However, to introduce the inner product like (2.12), it is not sufficient to use axiom A1. Therefore, additional structures on the semiclassical bundle are necessary.
2.2.1 Structures on the semiclassical bundle

An important feature of the finite-dimensional complex-WKB theory is that a \( \lambda \)-dependent quantum state \( K^\lambda_X f \) is assigned to each set \((X \in \mathcal{X}, f \in \mathcal{F}_X)\). Moreover, the mapping \( K^\lambda_X \) satisfies the following properties:

\[
(K^\lambda_X f, K^\lambda_X f) \to_{\lambda \to 0} (f, f),
\]

provided that \( \text{const} = \lambda^{-d/4} \);

\[
i\lambda \frac{\partial}{\partial X_i} K^\lambda_X = K^\lambda_X [\omega_i - \sqrt{\lambda} \Omega_i + ...] f,
\]

where

\[
\omega_i dX_i = P_j dQ_j - dS;
\]

\[
(\Omega_i dX_i f)(\xi) = (dP_j \xi_j - dQ_j \frac{\partial}{\partial \xi_j} ) f(\xi).
\]

Relation (2.14) is an important property of a semiclassical system. It allows us to introduce two additional structures on the semiclassical bundle: the differential 1-form \( \omega_i dX_i \) on the extended phase space \( \mathcal{X} \) ("action form") and the operator-valued differential 1-form \( \Omega_i dX_i \).

The commutation relations between operators \( \Omega_i \) can be obtained from eq.(2.14). Namely, apply the commutator \([i\lambda \frac{\partial}{\partial X_i}, i\lambda \frac{\partial}{\partial X_j}]\) being zero to the quantum state \( K^\lambda_X f \).

Thus, the following commutation rules are satisfied:

\[
[\Omega_i; \Omega_j] = i \left( \frac{\partial \omega_j}{\partial X_i} - \frac{\partial \omega_i}{\partial X_j} \right). \tag{2.15}
\]

It is much more convenient to present relations (2.13) in the exponential form,

\[
\exp(i\Omega_j \alpha_j) \exp(i\Omega_j \beta_j) = \exp(i\Omega_j (\alpha_j + \beta_j)) \exp\left( \frac{i}{2} \alpha_i \beta_j \left( \frac{\partial \omega_i}{\partial X_i} - \frac{\partial \omega_i}{\partial X_j} \right) \right).
\]

We come to the following requirement.

**A2.** A differential 1-form \( \omega \) and an operator-valued differential 1-form \( \Omega \) are specified on \( \mathcal{X} \): for each \( X \in \mathcal{X} \) and \( \delta X \in T_X \mathcal{X} \) the real quantity \( \omega_X[\delta X] \) and the self-adjoint operator \( \Omega_X[\delta X] \) in the space \( \mathcal{F}_X \) with a dense domain are given. The commutation relation

\[
e^{i\Omega_X[\delta X]} e^{i\Omega_X[\delta X]} = e^{i\Omega_X[\delta X_1+\delta X_2]} e^{i\frac{1}{2} \delta \omega[\delta X_1; \delta X_2]}
\]

is satisfied.

2.2.2 Composed states

1. To investigate the inner product of composed states like (2.9) in the abstract semiclassical mechanics, it is convenient to simplify the expression

\[
K^\lambda_X(\alpha + \sqrt{\lambda} \beta) g(\alpha), \quad \alpha = (\alpha_1, ..., \alpha_k)
\]

as \( \lambda \to 0 \). Let us look for a simplification in the following form:

\[
K^\lambda_X \tilde{V}_\lambda[\alpha, \beta] g(\alpha) \tag{2.17}
\]
for some operator $\tilde{V}_\lambda[\alpha, \beta]$. Applying the operators $i\frac{\partial}{\partial \beta_a}$ to expressions (2.16) and (2.17), one finds

$$K_{X(\alpha+\sqrt{\lambda})}^{\lambda} \left[ \frac{1}{\sqrt{\lambda}} \omega_i(X(\alpha + \sqrt{\lambda}) \frac{\partial X_i(\alpha + \sqrt{\lambda})}{\partial \alpha_a} - \Omega_i(X(\alpha + \sqrt{\lambda}) \frac{\partial X_i(\alpha + \sqrt{\lambda})}{\partial \alpha_a}) + \ldots \right] g(\alpha) =$$

so that

$$i \frac{\partial}{\partial \beta_a} \tilde{V}_\lambda[\alpha, \beta] = \tilde{V}_\lambda[\alpha, \beta] \left[ \frac{1}{\sqrt{\lambda}} \omega_i(X(\alpha + \sqrt{\lambda}) \frac{\partial X_i(\alpha + \sqrt{\lambda})}{\partial \alpha_a} - \Omega_i(X(\alpha + \sqrt{\lambda}) \frac{\partial X_i(\alpha + \sqrt{\lambda})}{\partial \alpha_a}) + \ldots \right]$$

After extracting a c-number factor

$$\tilde{V}_\lambda[\alpha, \beta] = \exp \left[ -i \frac{\sqrt{\lambda}}{\omega_i} \frac{\partial X_i}{\partial \alpha_a} \right] V_\lambda[\alpha, \beta],$$

one finds in the leading order in $\lambda$ the following equation on $V_\lambda[\alpha, \beta]$:

$$i \frac{\partial}{\partial \beta_a} V_\lambda[\alpha, \beta] = V_\lambda[\alpha, \beta] \left[ \frac{\partial}{\partial \alpha_b} (\omega_i \frac{\partial X_i}{\partial \alpha_a}) \beta_b - \Omega_i \frac{\partial X_i}{\partial \alpha_a} \right].$$

(2.18)

Let us look for the solution of this equation in the following form:

$$V_\lambda[\alpha, \beta] = c(\alpha, \beta) \exp[i \Omega_j \frac{\partial X_j}{\partial \alpha_a}]$$

for some c-number factor $c$. Making use of commutation relations (2.15), one takes eq.(2.18) to the form:

$$i \frac{\partial \log c}{\partial \beta_a} = \omega_j \frac{\partial^2 X_j}{\partial \alpha_a \partial \alpha_c} \beta_c + \frac{1}{2} \left( \frac{\partial \omega_i}{\partial X_j} + \frac{\partial \omega_j}{\partial X_i} \right) \frac{\partial X_i}{\partial \alpha_c} \frac{\partial X_j}{\partial \alpha_a} \beta_c,$$

so that one obtains:

$$K_{X(\alpha+\sqrt{\lambda})}^{\lambda} g(\alpha) \sim e^{-\frac{i}{\sqrt{\lambda}} \frac{\partial X_i}{\partial \alpha_a} \beta_a} e^{-\frac{i}{\sqrt{\lambda}} \frac{\partial X_i}{\partial \alpha_a} \beta_a} e^{i \Omega_j \frac{\partial X_j}{\partial \alpha_a} \beta_a} K_{X(\alpha)}^{\lambda} e^{i \Omega_j \frac{\partial X_j}{\partial \alpha_a} \beta_a} g(\alpha).$$

(2.19)

2. Consider now the composed state

$$\left( \begin{array}{c} X(\cdot) \\ g(\cdot) \end{array} \right) \equiv C_\lambda \int d\alpha K_{X(\alpha)}^{\lambda} g(\alpha), \quad \alpha = (\alpha_1, \ldots, \alpha_k).$$

Analogously to eq.(2.12), we obtain the following inner product:

$$\langle \left( \begin{array}{c} X(\cdot) \\ g(\cdot) \end{array} \right), \left( \begin{array}{c} X(\cdot) \\ g(\cdot) \end{array} \right) \rangle = |C_\lambda|^2 \lambda^{k/2} \int d\alpha d\beta (K_{X(\alpha)}^{\lambda} g(\alpha), K_{X(\alpha+\sqrt{\lambda})}^{\lambda} g(\alpha + \sqrt{\lambda})).$$

(2.20)

Consider the formal limit $\lambda \to 0$. Let us make use of eq.(2.19). One should require the isotropic condition

$$\omega_j \frac{\partial X_j}{\partial \alpha_a} = 0$$

(2.21)

to be satisfied. Otherwise, the integral (2.20) would contain a rapidly oscillating factor and be therefore exponentially small. Under condition (2.21) the inner product (2.20) takes the form

$$\langle \left( \begin{array}{c} X(\cdot) \\ g(\cdot) \end{array} \right), \left( \begin{array}{c} X(\cdot) \\ g(\cdot) \end{array} \right) \rangle = \int d\alpha (g(\alpha), \int d\beta e^{i \Omega_j \frac{\partial X_j}{\partial \alpha_a} \beta_a} g(\alpha)).$$

(2.22)

The corresponding inner product space of composed states requires additional investigations: see section 5 for details.

Note that the derivation of formulas (2.21), (2.22) is rather heuristic. More details are presented in [29, 30].
2.2.3 Symmetry transformations

An evolution transformation is viewed as an automorphism of the semiclassical bundle. If the initial elementary semiclassical state is presented as

$$K_{X_0}^\lambda f_0,$$

the semiclassical state at time moment $t$ should be presented in an analogous form

$$K_{X_t}^\lambda f_t$$

as $\lambda \to 0$. The parameters $X_t \in \mathcal{X}$ are uniquely specified by the initial values $X_0 \in \mathcal{X}$, so that the transformation $u_t : X_0 \mapsto X_t$ should be given. The vector $f_t \in \mathcal{F}_{X_t}$ should linearly depend on $f_0 \in \mathcal{F}_{X_0}$; moreover, the corresponding dependence should be unitary. Therefore, the unitary operators $U_t(X_t \leftarrow X_0) : f_0 \in \mathcal{F}_{X_0} \mapsto f_t \in \mathcal{F}_{X_t}$ should be also specified.

One can also expect that evolution $K_{X_0}^\lambda f_0 \mapsto K_{X_t}^\lambda f_t$ remains valid if $X_0$ is $\lambda$-dependent in a way $X_0 = X_0(\alpha + \sqrt{\lambda}\beta)$. It follows from eq. (2.23) that the state

$$e^{-\frac{i}{\sqrt{\lambda}} \sum \frac{\partial X_0}{\partial \alpha} \beta} e^{i \frac{\partial X_0}{\partial \alpha} \beta} K_{X_0}^\lambda f_0(\alpha).$$

is taken to

$$e^{-\frac{i}{\sqrt{\lambda}} \sum \frac{\partial X_t}{\partial \alpha} \beta} e^{i \frac{\partial X_t}{\partial \alpha} \beta} K_{X_t}^\lambda f_t(\alpha).$$

therefore, the forms $\omega$ and $\Omega$ should satisfy the relations

$$\omega_t(X_0) \frac{\partial X_0}{\partial \alpha} = \omega_t(X_t) \frac{\partial X_t}{\partial \alpha};$$

$$e^{i \Omega_t(X_t) \frac{\partial X_t}{\partial \alpha} \beta} U_t(X_t \leftarrow X_0) = U_t(X_t \leftarrow X_0) e^{i \Omega_t(X_0) \frac{\partial X_0}{\partial \alpha} \beta}.$$

We come therefore to the following definition.

**Definition 2.1.** A symmetry transformation of the semiclassical bundle is a set of a smooth mapping $u : \mathcal{X} \to \mathcal{X}$ and an unitary operator $U(uX \leftarrow X) : \mathcal{F}_X \to \mathcal{F}_{uX}$ such that

$$\omega_{uX} [u^*(uX \leftarrow X) \delta X] = \omega_X [\delta X]$$

and

$$e^{i \Omega_{uX} [u^*(uX \leftarrow X) \delta X]} U(uX \leftarrow X) = U(uX \leftarrow X) e^{i \Omega_X [\delta X]},$$

where $X \in \mathcal{X}$, $\delta X \in T_X \mathcal{X}$, $u^*(uX \leftarrow X) : T_X \mathcal{X} \to T_{uX} \mathcal{X}$ is an induced mapping of tangent spaces.

Consider the symmetry transformation of the composed state:

$$\begin{pmatrix} X(\alpha) \\ g(\alpha) \end{pmatrix} \mapsto \begin{pmatrix} uX(\alpha) \\ U(uX(\alpha) \leftarrow X(\alpha))g(\alpha) \end{pmatrix}$$

It follows from eq. (2.23) that the isotropic condition (2.21) conserves under time evolution. Eq. (2.24) implies that the inner product (2.22) also conserves under time evolution, so that a symmetry transformation of a composed state is also isometric.

Several quantum mechanical and quantum field systems are invariant under symmetry groups. Being applied to the semiclassical theory, this property means the following.

**Definition 2.2.** A semiclassical bundle is invariant under Lie group $\mathcal{G}$ if:

(i) for each $g \in \mathcal{G}$ a symmetry transformation $(u_g ; U_g(u_gX \leftarrow X))$ of the semiclassical bundle is specified;

(ii) the mapping $(g, X) \mapsto u_gX$ is smooth and the group property

$$u_{g_1 g_2} = u_{g_1} u_{g_2}$$

(2.25)
is satisfied;
(iii) the following group property takes place:

\[ U_{g_1}(u_{g_1g_2}X \leftarrow u_{g_2}X)U_{g_2}(u_{g_2}X \leftarrow X) = U_{g_1g_2}(u_{g_1g_2}X \leftarrow X). \] (2.26)

It is well-known that classical symmetries may be not hold in quantum field theory: the quantum anomalies arise in the one-loop approximation. Thus, properties (2.25), (2.26) should be carefully checked for each classical symmetry.

A useful approach for constructing group representations is an infinitesimal method: one first constructs a representation of the corresponding Lie algebra and then integrates the representation. The theory of integrability of representattions is nontrivial [36, 37, 38, 39, 40].

One can expect that check of properties (2.25), (2.26) for the semiclassical mechanics can be also performed in analogous way. Section 3 deals with infinitesimal formulations of property (2.26) and conditions of integrability of algebra representations. Since the derivartions of [36, 37, 38, 39, 40] are not convenient for our purposes, another condition of integrability is suggested.

Usually, the infinitesimal generators of semiclassical symmetries are quadratic Hamiltonians, while operators \( \Omega_i \) are linear in coordinates and momenta. Since the most interesting examples are given by the field theory, the case of the Fock spaces \( \mathcal{F}_X \), generators being quadratic with respect to creation and annihilation operators and \( \Omega_j \) being linear combinations of them, is considered in section 4. The conditions of integrability are reformulate. Section 5 deals with group transformations of the composed states. Section 6 contains concluding remarks.

3 Infinitesimal symmetries

The purpose of this section is to investigate infinitesimal analogs of eqs. (2.25), (2.26).

3.1 From groups to algebras

1. By \( T_eG \) we denote the tangent space to the Lie group \( G \) at \( g = e \). Let \( A \in T_eG, g(\tau) \) be a smooth curve on the group \( G \) with the tangent vector \( A \) at the point \( g(0) = e \). Introduce the differential operator \( \delta[A] \) on the space of differentiable functionals \( F \) on \( \mathcal{X} \):

\[
(\delta[A]F)(X) = \frac{d}{d\tau}|_{\tau=0}F(u_{g(\tau)}X).
\] (3.1)

It is shown in a standard way that:
1. The quantity (3.1) does not depend on the choice of the curve \( g(\tau) \) with the tangent vector \( A \).
2. The following property

\[
\delta[A_1 + \alpha A_2] = \delta[A_1] + \alpha \delta[A_2], \alpha \in \mathbb{R}, A_1, A_2 \in T_eG
\]

is satisfied.

Namely, let \( g_1(\tau) \) and \( g_2(\tau) \) be smooth curves on the Lie group \( G \) such that \( g_1(0) = e, g_2(0) = e \).

One has

\[
\frac{F(u_{g_1(\tau)g_2(\tau)}X) - F(X)}{\tau} = \int_0^1 d\xi \frac{\partial}{\partial \tilde{\tau}} F(u_{g_1(\tilde{\tau})u_{g_2(\tau)}X}|_{\tilde{\tau}=\xi\tau} + \frac{F(u_{g_2(\tau)}X) - F(X)}{\tau}.
\]

Considering the limit \( \tau \to 0 \), one obtains:

\[
\frac{d}{d\tau}|_{\tau=0}F(u_{g_1(\tau)g_2(\tau)}X) = \frac{d}{d\tau}|_{\tau=0}F(u_{g_1(\tau)}X) + \frac{d}{d\tau}|_{\tau=0}F(u_{g_2(\tau)}X).
\] (3.2)
Let \( g(\tau) \) and \( \tilde{g}(\tau) \) be curves on \( \mathcal{G} \) with the tangent vector \( A \). Choose \( g_1(\tau) = g(\tau), g_2(\tau) = g^{-1}(\tau)\tilde{g}(\tau) \). Since \( g_2(\tau) - e = O(\tau^2) \), \( \frac{d}{d\tau}|_{\tau=0} F(u_{g_2(\tau)}X) = 0 \). It follows from eq(3.2) that \( \frac{d}{d\tau}|_{\tau=0} F(u_{g(\tau)}X) = \frac{d}{d\tau}|_{\tau=0} F(u_{\tilde{g}(\tau)}X) \). Thus, definition (3.1) is correct.

Furthermore, let \( g_1(\tau), g_2(\tau) \) be curves with tangent vectors \( A \) and \( B \) correspondingly. Then \( g_1(\tau)g_2(\tau) \) is a curve with the tangent vector \( A + B \). Eq.(3.2) implies that

\[
\delta[A + B] = \delta[A] + \delta[B].
\]

Finally, consider a curve \( g(\tau) \) with the tangent vector \( A \) and the curve \( \tilde{g}(\tau) = g(\alpha\tau) \) with the tangent vector \( \alpha A \). One has

\[
\frac{d}{d\tau}|_{\tau=0} F(u_{\tilde{g}(\tau)}X) = \alpha \frac{d}{d\tau}|_{\tau=0} F(u_{g(\tau)}X).
\]

Thus, \( \delta[\alpha A] = \alpha \delta[A] \).

Let \( g \in \mathcal{G}, A \in T_e\mathcal{G}, h(\tau) \) be a curve on \( \mathcal{G} \) with tangent vector \( B \) at \( h(0) = e \). Then the tangent vector for the curve \( gh(\tau)g^{-1} \) at \( h(0) = e \) does not depend on the choice of the curve \( h(\tau) \). Denote it by \( gBg^{-1} \).

Define the operator \( W_g \) on the space of functionals \( F \) as \( W_g F[X] = F[u_{g^{-1}}X] \),

We see that the following property is satisfied:

\[
W_g \delta[B]W_g^{-1}F = \delta[gBg^{-1}]F. \quad (3.3)
\]

Let \( g = g(\tau) \) be a curve with the tangent vector \( A \) at \( g(0) = e \). Differentiating expression (3.3) by \( \tau \) at \( \tau = 0 \), we obtain the following relation:

\[
([\delta[A], \delta[B]] + \delta([A, B]))F = 0. \quad (3.4)
\]

Here \([A, B]\) is the Lie-algebra commutator for the group \( \mathcal{G} \).

Namely, let \( g(\tau) \) be a smooth curve on the Lie group \( \mathcal{G} \) with tangent vector \( A \) at \( g(0) = e \). Make use of the property (3.3):

\[
W_{g(\tau)} \delta[B]W_{g^{-1}(\tau)} = \delta[g(\tau)Bg^{-1}(\tau)]F, \quad (3.5)
\]

rewrite definition (3.1) as

\[
\delta[A] = \frac{d}{d\tau}|_{\tau=0} W_{g^{-1}(\tau)},
\]

remember that the Lie commutator can be defined as

\[
[A; B] = \frac{d}{d\tau}|_{\tau=0} g(\tau)Bg^{-1}(\tau).
\]

Consider the derivatives of sides of eq(3.3) at \( \tau = 0 \). We obtain property (3.4).

2. Consider now the infinitesimal properties of the transformation \( U \). Suppose that on some dense subset \( D \) of \( \mathcal{F} \) the vector functions \( U_g[X]\Psi \equiv U_g(u_gX \leftarrow X)\Psi \) (\( \Psi \in D \)) are strongly continuously differentiable with respect to \( g \) and smooth with respect to \( X \). Define operators

\[
H(A : X)\Psi = i\frac{d}{d\tau}|_{\tau=0} U_{g(\tau)}[X]\Psi, \quad (3.6)
\]

where \( g(\tau) \) is a curve on the group \( \mathcal{G} \) with the tangent vector \( A \) at \( g(0) = e \).

We find:
1. The operator \( H(A : X) \) does not depend on the choice of the curve \( g(\tau) \) with the tangent vector \( A \).
2. The following property is satisfied:

\[
H(A_1 + \alpha A_2 : X) = H(A_1 : X) + \alpha H(A_2 : X).
\]

Let \( h(\tau) \) be a curve with the tangent vector \( B \) at \( h(0) = e \). Eq.(2.26) implies:

\[
U_g[u_{h(\tau)}X]U_{h(\tau)}[X]U_{g^{-1}}[X]\Psi = U_{gh(\tau)g^{-1}}[u_{g^{-1}}X]\Psi, \quad \Psi \in D
\]
Differentiating this identity by $\tau$ at $\tau = 0$, we obtain that for all $\Psi \in D$

$$-iU_g[X]H(B : X)U_g^{-1}[X]\Psi + (\delta[B]U_g)[X]U_g^{-1}[X]\Psi = -iH(gBg^{-1} : u_gX)\Psi.$$  \hfill (3.7)

Let $g = g(t)$ be a curve with the tangent vector $A$ at $g(0) = e$. Differentiating eq.\,(3.7) by $t$ in a weak sense, we find that on the subset $D$ the following bilinear form vanishes:

$$-[H(A : X) : H(B : X)] - i\delta[B]H(A : X) + i\delta[A]H(B : X) + iH([A; B] : X) = 0.$$  \hfill (3.8)

3. Let us investigate now infinitesimal properties of relations (2.23) and (2.24). They can be rewritten as

$$\omega_{u_gX}[u^*_g(u_gX \leftarrow X)\delta X] = \omega_X[\delta X];$$  \hfill (3.9)

$$\Omega_{u_gX}[u^*_g(u_gX \leftarrow X)\delta X]U_g(u_gX \leftarrow X) = U_g(u_gX \leftarrow X)\Omega_X[\delta X].$$  \hfill (3.10)

It is convenient to introduce the operator $\delta[A]$ for the differential 1-forms. Let $\omega$ be a differential 1-form, $g(\tau)$ be a curve on $G$ with a tangent vector $A$ at $g(0) = e$. Define:

$$(\delta[A]\omega)_X[\delta X] \equiv \frac{d}{d\tau}\big|_{\tau = 0}\omega_{u_g(\tau)X}[u^*_g(u_g(\tau)X \leftarrow X)\delta X].$$

Analogously,

$$(\delta[A]\Omega)_X[\delta X] \equiv \frac{d}{d\tau}\big|_{\tau = 0}\Omega_{u_g(\tau)X}[u^*_g(u_g(\tau)X \leftarrow X)\delta X].$$

Defining the operator $W_g$ as

$$(W^{-1}_g\omega)_X[\delta X] = \omega_{u_gX}[u^*_g(u_gX \leftarrow X)\delta X] = \omega_X[\delta X],$$

we check relations (3.3), (3.4).

Differentiating eqs. (3.9), (3.10) for $g = g(\tau)$, we find:

$$\delta[A]\omega = 0.$$  \hfill (3.11)

$$\delta[A]\Omega_X[\delta X] - i[\Omega_X[\delta X]; H(A : X)] = 0.$$  \hfill (3.12)

Here $A$ is a tangent vector to $g(\tau)$ at $\tau = 0$.

3.2 From algebras to groups

Investigate now the problem of reconstructing the group representation if the algebra representation is known. Since classical invariance is usually evident in applications, we suppose that mappings $u_g : \mathcal{X} \rightarrow \mathcal{X}$ which satisfy the condition (2.23) are already specified. Our purpose is to reconstruct the unitary operators $U_g(u_gX \leftarrow X)$ satisfying the group property (2.24), provided that operators $H(A : X)$ are known.

Without loss of generality, spaces $\mathcal{F}_X$ can be identified with the space $\mathcal{F}$ because of local triviality of the semiclassical bundle.

Let us impose the following conditions on the operators $H(A : X)$, $A \in T_e\mathcal{G}$, $X \in \mathcal{X}$.

G1. Hermitian operators $H(A : X)$ are defined on a common domain $\mathcal{D}$ which is dense in $\mathcal{F}$. $\mathcal{D}$ is a subset of domains of the operators $\Omega_X[\delta X]$.

G2. For each smooth curve $h(\alpha)$ on $\mathcal{G}$, $\delta X \in T_X\mathcal{X}$ and $\Psi \in \mathcal{D}$ the vector functions $H(A : u_{h(\alpha)}X)\Psi$ and $\Omega_{u_{h(\alpha)}X}[u^*_{h(\alpha)}(u_{h(\alpha)}X \leftarrow X)\delta X]\Psi$ are strongly continuously differentiable with respect to $\alpha$.

G3. The bilinear forms (3.8) and (3.12) vanish on $\mathcal{D}$.

Let $Z \in T_e\mathcal{G}$ be a subset of the Lie algebra of the group $\mathcal{G}$ such that $spanZ = T_e\mathcal{G}$. Let $B \in Z$, while $g_B(t)$ is a one-parametric subgroup of the Lie group $\mathcal{G}$ with the tangent vector $B$ at $t = 0$, $g_B(0) = e.$
Denote by $U^t_B(X)$ the operator taking the initial condition $\Psi_0 \in \mathcal{D}$ of the Cauchy problem for the equation

$$i\frac{\partial \Psi_t}{\partial t} = H(B : u_{gb(t)}X)\Psi_t$$

(3.13)

($\partial \Psi_t/\partial t$ is a strong derivative) to the solution $\Psi_t \in \mathcal{D}$ of the Cauchy problem, $\Psi_t = U^t_B(X)\Psi_0$. This definition is correct under the following condition.

G4. Let $B \in \mathbb{Z}$. If $\Psi_0 \in \mathcal{D}$, there exists a solution of the Cauchy problem for eq.(3.13).

Uniqueness of the solution is a corollary of the property $||\Psi_t|| = ||\Psi_0||$ which is checked directly by differentiation. The isometric operator $U^t_B(X)$ can be extended then from $\mathcal{D}$ to $\mathcal{F}$. It satisfies the property

$$U^{t_1}_{B}(u_{gb(t_2)}X)U^{t_2}_{B}(X) = U^{t_1+t_2}_{B}(X).$$

Therefore, it is invertible and unitary.

Impose also the following conditions.

G5. Let $B \in \mathbb{Z}$. For each smooth curve $h(\alpha)$ on $\mathcal{G}$ and each $\Psi_0 \in \mathcal{D}$ the quantity $||\frac{\partial}{\partial \alpha} H(A : u_{h(\alpha)}X)\Psi_t||$ is bounded uniformly with respect to $\alpha, t \in [\alpha_1, \alpha_2] \times [t_1, t_2]$ for any finite $\alpha_1, \alpha_2, t_1, t_2$.

G6. For $\Psi \in \mathcal{D}$, $B \in \mathbb{Z}$, $A \in T_0\mathcal{G}$, $\delta X \in T_X\mathcal{X}$, the following properties are satisfied:

$$||H(g_B(\tau).Ag_B^{-1}(\tau) : u_{gb(\tau)}X)[U^t_B(X)\Psi - \Psi]|| \to_{\tau \to 0} 0;$$

$$||\Omega_{u_{gb(\tau)}X}[u^{*}_{gb(\tau)}(u_{gb(\tau)}X \leftarrow X)\delta X][U^t_B(X)\Psi - \Psi]|| \to_{\tau \to 0} 0.$$  

(3.14)

(3.15)

Lemma 3.1. Let $B \in \mathbb{Z}$, $\delta X \in T_X\mathcal{X}$. Then the property

$$A^t \equiv (U^t_B(X))^{-1}\Omega_{u_{gb(t)}X}[u^{*}_{gb(t)}(u_{gb(t)}X \leftarrow X)\delta X]U^t_B(X) - \Omega_X[\delta X] = 0$$

(3.16)

is satisfied on the domain $\mathcal{D}$.

Proof. Let us consider the matrix element of the left-hand side of eq.(3.16). At $t = 0$ it vanishes. Let us calculate its time derivative,

$$\frac{1}{\tau}[(\Phi, A^{t+\tau}\Psi) - (\Phi, A^t\Psi)] =
(\Omega_{u_{gb(\tau)}X_t}[u^{*}_{gb(\tau)}(u_{gb(\tau)}X_t \leftarrow X_t)\delta X_t][\Phi_{t+\tau}, \frac{\psi_{t+\tau} - \psi_t}{\tau}]) +
(\Phi_{t+\tau}, \frac{\Omega_{u_{gb(\tau)}X_t}[u^{*}_{gb(\tau)}(u_{gb(\tau)}X_t \leftarrow X_t)\delta X_t] - \Omega_X[\delta X_t]}{\psi_{t+\tau} - \psi_t}) +
(\Phi_{t+\tau} - \Phi_t, \Omega_X[\delta X_t]\psi_t),$$

where $\Phi_t \equiv U^t_B(X)\Phi$, $\Psi_t \equiv U^t_B(X)\Psi$, $X_t = u_{gb(t)}X$, $\delta X_t = u^{*}_{gb(t)}(u_{gb(t)}X \leftarrow X)\delta X$. It follows form eq.(3.12) and property G6 that $\frac{d}{dt}(\Phi, A^t\Psi) = 0$, so that expression (3.16) vanishes as a bilinear form. Since it is defined on $\mathcal{D}$, it vanishes as an operator expression. Lemma 3.1 is proved.

Under conditions G1-G6, we obtain:

Lemma 3.2. Let $B \in \mathbb{Z}$, $A \in T_0\mathcal{G}$, the following property is satisfied on the domain $\mathcal{D}$:

$$H(A : X) + i(U^t_B(X))^{-1}(\delta[A]U^t_B(X)) - (U^t_B(X))^{-1}H(g_B(t)Ag_B(t)^{-1} : u_{gb(t)}X)U^t_B(X) = 0.$$  

(3.17)

Proof. Let us check that under these conditions the operator $(\delta[A]U^t_B(X))$ is correctly defined, i.e. the strong derivative

$$(\delta[A]U^t_B(X))\Psi = \frac{d}{d\alpha}|_{\alpha = 0}U^t_B(u_{h(\alpha)}X)\Psi$$

exists for all $\Psi \in \mathcal{D}$, where $h(\alpha)$ is a curve on $\mathcal{G}$ with tangent vector $A$.

Denote

$$\Psi_{\alpha,t} = U^t_B(u_{h(\alpha)}X)\Psi.$$
This vector obeys the equation

\[ i \frac{\partial}{\partial t} \Psi_{\alpha,t} = H(B : u_{g_B(t)h(\alpha)}X)\Psi_{\alpha,t}, \]

so that

\[ i \frac{\partial}{\partial t}(\Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t}) = H(B : u_{g_B(t)h(\alpha+\delta\alpha)}X)(\Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t}) \]
\[ + (H(B : u_{g_B(t)h(\alpha+\delta\alpha)}X) - H(B : u_{g_B(t)}X))\Psi_{\alpha,t}. \]

Since \( \Psi_{\alpha,0} = \Psi_{\alpha+\delta\alpha,0} = \Psi \), we have

\[ \Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t} = -i \int_0^t d\tau U_{B,\gamma}^{t-\tau}(u_{g_B(\tau)h(\alpha+\delta\alpha)}X)(H(B : u_{g_B(\tau)h(\alpha+\delta\alpha)}X) - H(B : u_{g_B(\tau)}X))\Psi_{\alpha,\tau}. \]

Because of unitarity of the operators \( U_{B,t}^t \), the following estimation takes place:

\[ \| \Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t} \| \leq \int_0^t d\tau \| (H(B : u_{g_B(\tau)h(\alpha+\delta\alpha)}X) - H(B : u_{g_B(\tau)}X))\Psi_{\alpha,\tau} \|. \]

Making use of the Lesbegue theorem and condition G5, we find that \( \| \Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t} \| \to \delta\alpha = 0 \), so that the operator \( U_{B,t}^t(u_{h(\alpha)}X) \) is strongly continuous with respect to \( \alpha \).

Furthermore,

\[ \Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t} = -i \int_0^t d\tau \int_0^1 d\gamma U_{B,\gamma}^{t-\tau}(u_{g_B(\tau)h(\alpha+\delta\alpha)}X) \left( \frac{\partial}{\partial \alpha} H(B : u_{g_B(\tau)h(\alpha+\gamma\delta\alpha)}X) \right) \Psi_{\alpha,\tau}. \]

Denote

\[ \frac{\partial \Psi_{\alpha,t}}{\partial \alpha} \equiv -i \int_0^t d\tau U_{B,\gamma}^{t-\tau}(u_{g_B(\tau)h(\alpha+\delta\alpha)}X) \left( \frac{\partial}{\partial \alpha} H(B : u_{g_B(\tau)h(\alpha)X}) \right) \Psi_{\alpha,\tau}. \] (3.19)

The following estimation takes place:

\[ \| \Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t} - \frac{\partial \Psi_{\alpha,t}}{\partial \alpha} \| \leq \int_0^t d\tau \int_0^1 d\gamma \left( \left| U_{B,\gamma}^{t-\tau}(u_{g_B(\tau)h(\alpha+\delta\alpha)}X) \right| \right) \]
\[ \times \left( \left| \frac{\partial}{\partial \alpha} H(B : u_{g_B(\tau)h(\alpha)X}) \right| \right) \left| \Psi_{\alpha,\tau} \right| + \left| (U_{B,\gamma}^{t-\tau}(u_{g_B(\tau)h(\alpha+\delta\alpha)}X) - U_{B,\gamma}^{t-\tau}(u_{g_B(\tau)h(\alpha)}X)) \frac{\partial}{\partial \alpha} H(B : u_{g_B(\tau)h(\alpha)}X) \Psi_{\alpha,\tau} \right| \] (3.20)

Making use of the Lesbegue theorem, conditions G2,G5, we find that the quantity (3.20) tends to zero as \( \delta\alpha \to 0 \). Thus, the vector (3.19) is correctly defined.

It follows from the expression (3.19) that

\[ \frac{\partial}{\partial t} \frac{\partial}{\partial \alpha} U_{B,t}^t(u_{h(\alpha)}X) = -i \frac{\partial}{\partial \alpha} H(B : u_{g_B(t)h(\alpha)}X) \cdot U_{B,t}^t(u_{h(\alpha)}X) \]
\[ - i H(B : u_{g_B(t)h(\alpha)}X) \frac{\partial U_{B,t}^t(u_{h(\alpha)}X)}{\partial \alpha} \]

in a strong sense.

Let us prove now property (3.17). At \( t = 0 \) the property (3.17) is satisfied. The derivative with respect to \( t \) of any matrix element of the operator (3.17) under conditions H1-H6 vanishes. Therefore, equality (3.17) viewed in terms of bilinear forms is satisfied on \( \mathcal{D} \). Since the left-hand side of eq.(3.17) is defined on \( \mathcal{D} \), it also vanishes on \( \mathcal{D} \).

Let the property

\[ g_{B_n}(t_n(\alpha))...g_{B_1}(t_1(\alpha)) = e, \] (3.21)

be satisfy for \( \alpha \in [0,\alpha_0] \) and \( B_1,\ldots,B_n \in Z \). Here \( t_k(\alpha) \) are smooth functions. Denote \( h_k(\alpha) = g_{B_k}(t_k(\alpha)), s_k(\alpha) = h_k(\alpha)\ldots h_1(\alpha) \).

**Lemma 3.3.** Under condition (3.21) the operator

\[ U_{B_n}^{t_n(\alpha)}(u_{\delta_{n-1}(\alpha)}X)\ldots U_{B_1}^{t_1(\alpha)}(X) \] (3.22)

is \( \alpha \)-independent.

To prove lemma, denote \( U_k \equiv U_k(u_{s_{k-1}}(\alpha)X) \equiv U_{B_k}^{t(\alpha)}(u_{s_{k-1}}(\alpha)X) \). Let us use the following lemma.

**Lemma 3.4.** Let \( s(\alpha) \) be a smooth curve on the group \( \mathcal{G} \), \( t(\alpha) \) is a smooth real function, \( B \in T_s \mathcal{G} \). Then the operator function \( U_B^{t(\alpha)}(u_{s(\alpha)}X) \) is strongly differentiable with respect to \( \alpha \) on \( D \) and

\[
\frac{\partial}{\partial \alpha} U_B^{t(\alpha)}(u_{s(\alpha)}X) = -i \frac{dt}{d\alpha} H(B : u_{g_{B(t(\alpha))s(\alpha)}(\alpha)}) U_B^{t(\alpha)}(u_{s(\alpha)}X) + (\delta \frac{ds}{d\alpha} s^{-1}) U_B^{t(\alpha)}(u_{s(\alpha)}X),
\]

where \( \frac{ds}{d\alpha} s^{-1} \) is a tangent vector to the curve \( s(\alpha + \tau)s^{-1}(\alpha) \) at \( \tau = 0 \).

**Proof.** Let \( \Psi \in D \). One has

\[
\frac{1}{\delta \alpha}(U_B^{t(\alpha+\delta \alpha)}(u_{s(\alpha+\delta \alpha)}X) - U_B^{t(\alpha)}(u_{s(\alpha)}X))\Psi = \frac{1}{\delta \alpha}(U_B^{t(\alpha+\delta \alpha)}(u_{s(\alpha+\delta \alpha)}X) - U_B^{t(\alpha)}(u_{s(\alpha+\delta \alpha)}X))\Psi + \frac{1}{\delta \alpha}(U_B^{t(\alpha+\delta \alpha)}(u_{s(\alpha)}X) - U_B^{t(\alpha)}(u_{s(\alpha)}X))\Psi.
\]

The first term in the right-hand side of eq. (3.24) tends to

\[-i \frac{dt}{d\alpha} H(B : u_{g_{B(t(\alpha))s(\alpha)}(\alpha)}) \Psi\]

by definition of the operator \( U_B^{r}(X) \). Consider the second term. It can be represented as

\[
\int_0^1 d\gamma \frac{\partial}{\partial \alpha} U_B^{t(\gamma)}(u_{s(\alpha+\gamma \delta \alpha)}X)|_{\gamma = \alpha + \delta \alpha} \Psi.
\]

Making use of eq. (3.19), we take this term to the form

\[-i \int_0^1 d\gamma \int_0^{t(\alpha + \delta \alpha)} d\tau U_B^{t(\alpha + \delta \alpha) - \tau}(u_{g_{B(\tau)}s(s + \gamma \alpha)}X) \frac{\partial}{\partial \alpha} H(B : u_{g_{B(\tau)}s(s + \gamma \alpha)}X) U_B^{r}(u_{s(\alpha + \gamma \delta \alpha)}X) \Psi.
\]

Making use of the Lesbegue theorem and property \( G_6 \), we see that the vector \( (3.24) \) is strongly continuous as \( \delta \alpha \to 0 \), so that it is equal to

\[
\frac{\partial}{\partial \alpha} U_B^{t(\gamma)}(u_{s(\alpha)}X)|_{\gamma = \alpha} \Psi = ((\delta \frac{ds}{d\alpha} s^{-1}) U_B^{t(\alpha)}(u_{s(\alpha)}X))\Psi.
\]

We obtain formula (3.23).

Let us return to proof of lemma 3.3. To check formula (3.22), let us obtain that

\[
\frac{d}{d\alpha} (U_n U_{n-1}) = 0
\]

on \( D \) (the derivative is viewed in the strong sense). The property (3.23) is equivalent to the following relation:

\[
\sum_{k=1}^{n} U_n U_{k+1} \frac{\partial}{\partial \alpha} U_k U_{k-1} U_{k-1} = 0.
\]

Making use of eq. (3.23), we take eq. (3.27) to the form

\[
\sum_{k=1}^{n} U_n U_{k+1} H(-i \frac{dt}{d\alpha} B_k : u_{s_k}X) U_{k-1} U_{k-1} + \sum_{k=1}^{n} U_n U_{k+1} (\delta \frac{dt}{d\alpha} s^{-1}) U_k (u_{s_{k-1}}X) U_{k-1} U_{k-1} = 0.
\]

Applying properties (3.17) \( n - k \) times, we obtain

\[
-i H(\sum_{k=1}^{n} \frac{dt}{d\alpha} h_{n-k+1} B_k h_{k-1}^{-1} : X) + \sum_{i=1}^{n} U_n U_{i+1} (\delta \frac{dt}{d\alpha} s^{-1}) U_i (u_{s_{i-1}}X) U_{i-1} U_{i-1} - \sum_{k=1}^{n} U_n U_{i+1} \delta (h_{i-1} h_{k+1} B_k h_{k-1}^{-1} : X) U_{i-1} U_{i-1} = 0.
\]
Eq. (3.21) implies that
\[ \sum_{k=1}^{n} \frac{dt_k}{dt} h_n \ldots h_{k+1} B_k h_{k+1}^{-1} \ldots h_n^{-1} = 0; \]
\[ \frac{ds_{j+1}}{ds} s_{j+1}^{-1} = \sum_{k=1}^{l-1} h_l \ldots h_{k+1} B_k h_{k+1}^{-1} \ldots h_l^{-1} \frac{dt_k}{ds} = 0. \]

Lemma 3.3 is proved.

**Corollary.** Let \( t_k(0) = e \). Then
\[ U_{B_n}^{\alpha}(u_{x_n-1}(\alpha)) \ldots U_{B_2}^{\alpha}(X) = 1 \]
under conditions of lemma 3.3.

**Theorem 3.5.** Under conditions G1-G6 semiclassical bundle is invariant under local Lie group \( \mathcal{G} \).

**Proof.** Let \( B_1, \ldots, B_m \) be a basis on the Lie algebra. Then one can uniquely introduce the canonical coordinates of the second kind [42] on the local Lie group by the formula
\[ g = g_{B_1}(\alpha_1) \ldots g_{B_m}(\alpha_m). \]

Set
\[ U_{g}(u_g X \leftarrow X) = U_{B_1}^{\alpha_1}(u_{gB_2}(\alpha_2) \ldots g_{B_m}(\alpha_m)X) U_{B_2}^{\alpha_2}(u_{gB_3}(\alpha_3) \ldots g_{B_m}(\alpha_m)X) \ldots U_{B_m}^{\alpha_m}(X). \]

The group property is then a corollary of lemma 3.3. Namely, let
\[ g(t) = g_{B_1}(t \alpha_1) \ldots g_{B_m}(t \alpha_m); \]
\[ h(t) = g_{B_1}(t \beta_1) \ldots g_{B_m}(t \beta_m). \]

Then the relation
\[ g(t) h(t) = g_{B_1}(\gamma_1(t)) \ldots g_{B_m}(\gamma_m(t)) \]
specifies the coordinates \( \gamma_i(\alpha, \beta, t) \) in a unique fashion, since the second-kind canonical coordinates are correctly defined [42]. Moreover, the dependence \( \gamma_i(\alpha, \beta, t) \) is smooth.

It follows then from lemma 3.3 that
\[ (U_{g(t)h(t)}[u_{g(t)h(t)}X \leftarrow X])^{-1} U_{g(t)}[u_{g(t)}X \leftarrow X] U_{h(t)}[u_{h(t)}X \leftarrow X] = 1. \]

Theorem is proved.

4 Quadratic infinitesimal operators in the Fock space

Symmetries of quantum mechanical systems can be investigated “exactly” without any approximations. Realistic models of QFT are not constructed mathematically, so that investigations of semiclassical field theory may give rise to surprising results such as quantum anomalies. In this section conditions G1-G6 are reformulated, provided that \( \mathcal{F}_X \) are Fock spaces, \( H(A : X) \) are quadratic in creation and annihilation operators in the Fock space, \( \Omega_{X \delta X} \) are linear combinations of creation and annihilation operators.

Remind that the Fock space \( \mathcal{F}(L^2(\mathbb{R}^l)) \) is defined as a space of sets
\[ \Psi = (\Psi_0, \Psi_1(x_1), \ldots, \Psi_n(x_1, \ldots, x_n), \ldots) \]
of symmetric with respect to \( x_1, \ldots, x_n \) symmetric functions \( \Psi_n \) such that \( ||\Psi||^2 = \sum_{n=0}^{\infty} ||\Psi_n||^2 < \infty \). By \( A^\pm(x) \) we denote, as usual, the creation and annihilation operator distributions:
\[ (f dx A^+(x) f(x) \Psi)_n(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \Psi_{n-1}(x_1, \ldots, x_j-1, x_{j+1}, \ldots, x_n); \]
\[ (f dx A^-(x) f^*(x) \Psi)_{n-1}(x_1, \ldots, x_{n-1}) = \sqrt{n} \int dx f^*(x) \Psi_n(x, x_1, \ldots, x_{n-1}). \]
By $|0>$ we denote, as usual, the vacuum vector of the form $(1, 0, 0, \ldots)$. Arbitrary vector of the Fock space can be presented via the creation operators and vacuum vector as follows:

$$\Psi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int dx_1 \ldots dx_n \Psi_n(x_1, \ldots, x_n) A^+(x_1) \ldots A^+(x_n) |0>.$$  

Introduce the operator of number of particles $\hat{n}$ as $(\hat{n} \Psi)_n = n \Psi_n$.

Let $\mathcal{H}^{\pm \pm}$ be operators in $L^2(\mathbb{R})$ with kernels $\mathcal{H}^{\pm \pm}(x, y)$. By $A^+ \mathcal{H}^{\pm \pm} A^+$ we denote the operators in the Fock space

$$A^+ \mathcal{H}^{\pm \pm} A^+ = \int dx dy A^+(x) \mathcal{H}^{\pm \pm}(x, y) A^+(y).$$

For the case of an unbounded operator $\mathcal{H}^{-+}$, the operator $A^+ \mathcal{H}^{-+} A^-$ can be defined as

$$(A^+ \mathcal{H}^{-+} A^-)_{n} = \sum_{j=1}^{n} 1^{\otimes j-1} \otimes \mathcal{H}^{-+} \otimes 1^{\otimes n-j} \Psi_n.$$

Denote also

$$A^\pm \varphi = \int dx A^\pm(x) \varphi(x).$$

Let

$$H(B : X) = \frac{1}{2} A^+ \mathcal{H}^{++}(B : X) A^+ + A^+ \mathcal{H}^{+-}(B : X) A^- + \frac{1}{2} A^- \mathcal{H}^{-+}(B : X) A^- + \mathcal{P}(B : X); \quad (4.1)$$

where $(\mathcal{H}^{++})^+ = \mathcal{H}^{--}$, $(\mathcal{H}^{+-})^+ = \mathcal{H}^{+-}$

$$\Omega_X[\delta X] = -i(A^+ \varphi_X[\delta X] - A^- \varphi_X^*[\delta X]) \quad (4.2)$$

for some $L^2(\mathbb{R})$-valued 1-form $\varphi$.

Let us formulate the conditions that are sufficient for satisfying properties G1-G6.

In QFT-applications, the operators $\mathcal{H}^{+-}(B : X)$ are unbounded. However, the singular unbounded part is $X$-independent,

$$\mathcal{H}^{-+}(B : X) = L(B) + \mathcal{H}(B : X),$$

while $\mathcal{H}(B : X)$ is a bounded operator.

Impose the following conditions on the self-adjoint operator $L$.

**F1.** There exists a positively definite self-adjoint operator $T$ such that:

(i) $\|T^{-1/2} L(B) T^{-1/2}\| < \infty$, $\|L(B) T^{-1}\| < \infty$;

(ii) for all $t_0$ there exists such a constant $C$ that $\|T^{1/2} e^{-iL(B)tT^{-1/2}}\| \leq C$, $\|Te^{-iL(B)tT^{-1}}\| \leq C$ for $t \in (-t_0, t_0)$.

**F2.** For any smooth curve $h(\alpha)$ on $G$:

(i) the function $\mathcal{P}(B : u_{h(\alpha)} X)$ is continuously differentiable;

(ii) the operator-valued function $\mathcal{H}^{++}(B : u_{h(\alpha)} X)$ is continuously differentiable in the Hilbert-Schmidt norm $\|O\| = \sqrt{Tr O^* O}$;

(iii) the operator functions $T^{-1/2} \mathcal{H}^{+-}(B : u_{h(\alpha)} X) T^{-1/2}$ and $\mathcal{H}^{-+}(B : u_{h(\alpha)} X) T^{-1}$ are continuously differentiable in the operator norm $\| \cdot \|$;

(iv) the operator function $T \mathcal{H}^{++}(B : u_{h(\alpha)} X)$ is continuous in the Hilbert-Schmidt norm;

(v) the operator functions $T \mathcal{H}(B : u_{h(\alpha)} X) T^{-1}$, $T^{1/2} \mathcal{H}(B : u_{h(\alpha)} X) T^{-1/2}$ and $\mathcal{H}(B : u_{h(\alpha)} X)$ are strongly continous;

(vi) the function $\varphi_{u_{h(\alpha)} X} [u^*_h \alpha \ [u_{h(\alpha)} X \leftarrow X ] \delta X] \in L^2(\mathbb{R})$ is strongly continuously differentiable.
**F.3.** The following commutation relations are satisfied:

\[
\begin{align*}
\mathcal{H}^{++}([A; B] : X) &= -i[\mathcal{H}^{--}(A : X)\mathcal{H}^{++}(B : X) + \mathcal{H}^{++}(B : X)(\mathcal{H}^{--}(A : X))] \\
\mathcal{H}^{+}((A; B] : X) &= -i[\mathcal{H}^{++}(B : X)(\mathcal{H}^{+}(A : X)) + \mathcal{H}^{++}(B : X) - \delta[A] \mathcal{H}^{++}(B : X) - \delta[B] \mathcal{H}^{++}(A : X)] \\
\mathcal{H}^{+}((A; B] : X) &= -i[\mathcal{H}^{++}(B : X)(\mathcal{H}^{+}(A : X)) - \mathcal{H}^{++}(A : X)(\mathcal{H}^{+}(B : X))] + \delta[A] \mathcal{H}^{++}(B : X) - \delta[B] \mathcal{H}^{++}(A : X) \\
\mathcal{H}^{+}((A; B] : X) &= -\frac{i}{2} Tr[\mathcal{H}^{++}(B : X)(\mathcal{H}^{+}(A : X)) - \mathcal{H}^{++}(A : X)(\mathcal{H}^{+}(B : X))] + \delta[A] \mathcal{H}(B : X) - \delta[B] \mathcal{H}(A : X)
\end{align*}
\]

in a sense of bilinear forms in \(D(T)\):

\[
i(\delta[A] \varphi)_X[\delta X] = \mathcal{H}^{--}(A : X)\varphi_X[\delta X] + \mathcal{H}^{++}(A : X)\varphi^*_X[\delta X].
\]

Note that condition F1 is an alternative for known conditions of integrability of Lie algebra representations [36, 37, 38, 39, 40].

Let us check properties G1-G6.

### 4.1 Some properties of the Fock space

Without loss of generality, one can suppose that \(T - 1 > 0\). Otherwise, one can change \(T \to T + 1\); properties F1-F3 will remain valid then.

Introduce the following norms in the Fock space:

\[
|||\Psi|||_m = ||(\hat{n} + 1)^n \Psi||, \quad |||\Psi|||_T = ||(A^+TA^+ + 1)^n \Psi||.
\]

**Lemma 4.1.** Let \(||\Psi||_T < \infty\). Then \(||\Psi||_m \leq ||\Psi||_T^m\).

**Proof.** It is sufficient to check that

\[
(\Psi_n, (\sum_{j=1}^n \delta_j^{-1} \otimes T \otimes 1 \otimes^{n-j} + 1)^{2m} \Psi_n) \geq (\Psi_n, (\sum_{j=1}^n 1 + 1)^{2m} \Psi_n).
\]

This relation is a corollary of the formula

\[
(\Psi_n, T^{2l_1} \otimes \ldots \otimes T^{2l_n} \Psi_n) \geq (\Psi_n, \Psi_n)
\]

for all \(l_1, \ldots, l_n \geq 0\). The latter formula is obtained from the relation \(||T^{-l_1} \otimes \ldots \otimes T^{-l_n}|| \leq 1\). Lemma B.1 is proved.

Let \(\mathcal{H}^{++}\) be a nonbounded operator in \(L^2(\mathbb{R})\) such that operators \(T^{-1/2}\mathcal{H}^{++}T^{-1/2}\) and \(\mathcal{H}^{++}T^{-1}\) are bounded.

**Lemma 4.2.** The following estimation is satisfied:

\[
|||A^+\mathcal{H}^{--}A^-\Psi||| \leq C|||\Psi|||^T_1
\]

with \(C = \max(||T^{-1/2}\mathcal{H}^{++}T^{-1/2}||, ||\mathcal{H}^{++}T^{-1}||)\).

**Proof.** One should check

\[
(\Psi_n, \mathcal{H}_i^{++}\mathcal{H}_j^{++}\Psi_n) \leq C(\Psi_n, T_i T_j \Psi_n)
\]

with \(\mathcal{H}_i^{++} = 1_i^{-1} \otimes \mathcal{H}^{++} \otimes 1^{n-i}, T_i = 1_i^{-1} \otimes T \otimes 1^{n-i}\). Denote \(T_i^{1/2} T_j^{1/2} \Psi_n = \Phi_n\). Inequality (4.3) takes the form

\[
(\Phi_n, T_i^{-1/2} T_j^{-1/2} \mathcal{H}_i^{++} \mathcal{H}_j^{++} T_i^{1/2} T_j^{1/2} \Phi_n) \leq C^2(\Phi_n, \Phi_n).
\]

For \(i \neq j\), property (4.3) is satisfied if \(C = ||T^{-1/2}\mathcal{H}^{++}T^{-1/2}||\) as a corollary of the Cauchy-Bunyakowski-Schwartz inequality. For \(i = j\), property (4.4) is satisfied if \(C = ||\mathcal{H}^{++}T^{-1}||\). Lemma 4.2 is proved.
Lemma 4.3. Consider the operator

$$\hat{\Phi} = \int dx_1...dx_m dy_1...dy_k \varphi(x_1, ..., x_n, y_1, ..., y_k) A^+(x_1)...A^+(x_m)A^-(y_1)...A^-(y_k)$$

with $\varphi \in L^2(\mathbb{R}^{l(m+k)})$. Let $\Psi \in \mathcal{F}$ and $||\Psi||_{l^{2m+k}} < \infty$. Then

$$||\hat{\Phi}\Psi||_t \leq C||\varphi||_{L^2}||\Psi||_{l^{2m+k}},$$

where $C^2 = \max\{1, (m-k)! (m-k)^2\}$.

Proof. One has

$$(\hat{\Phi}\Psi)_n(z_1, ..., z_n) = \sqrt{\frac{(n-m+k)}{(n-m)!}} \sqrt{\frac{m!}{(n-m)!}} \text{Sym} \int dy_1...dy_k \varphi(z_1, ..., z_m, y_1, ..., y_k)$$

$$\times \Psi_{n+m}(y_1, ..., y_k, z_m+1, ..., z_n)$$

where $\text{Sym}$ is a symmetrization operator. Since $||\text{Sym}\Psi_n|| \leq ||\Phi_n||$ and

$$||\int dy \varphi(z, y) \Psi(y, z')|| \leq ||\varphi|| ||\Psi||,$$

one has

$$||(\hat{\Phi}\Psi)_n|| \leq \sqrt{\frac{(n-m+k)}{(n-m)!}} \sqrt{\frac{m!}{(n-m)!}} ||\varphi|| ||\Psi|| ||\Psi_{n+m+k}||.$$  

Therefore,

$$||\hat{\Phi}\Psi||_t^2 = \sum_{n=0}^{\infty} (n+1)^2 \frac{(n-m+k)!}{(n-m)!} \frac{m!}{(n-m)!} \sum_{s=0}^{\infty} \frac{(s+m-k)!}{(s-k+1)!} \frac{m!}{(s-k+m)!} \frac{(s+1)^2}{(s+1)^{2m+k}} ||\varphi|| ||\Psi_s||^2 (s+1)^{2m+k}$$

$$\leq C^2 ||\varphi||^2 ||\Psi||^2 ||\Psi||_{l^{2m+k}},$$

where $s = m-n+k$. Lemma is proved.

Corollary.

$$||A^{\pm} \varphi \Psi|| \leq ||\varphi|| ||\Psi||_{1/2} \leq ||\varphi|| ||\Psi||_1;$$

$$||\frac{1}{2}A^{\pm} \mathcal{H}^{\pm} A^{\pm} \Psi|| \leq \frac{1}{\sqrt{2}} ||\mathcal{H}^{\pm}|| ||\Psi||_1.$$  

Therefore, we obtain the following result.

Lemma 4.4. Let properties $F1, F2$ be satisfied. Set $\mathcal{D} = \{\Psi \in \mathcal{F} ||\Psi||^T \leq \infty\}$. Let the solution for the Cauchy problem for eq. (4.3) exists for all $\Psi_0 \in \mathcal{D}$ and $\Psi_t$ be continuous in the $||\cdot||^T$-norm. Then properties $G1,G2, G5, G6$ are satisfied.

The proof is straightforward.

4.2 Evolution with quadratic Hamiltonians

Since property $G3$ is a direct corollary of $H3$, the remaining part of checking conditions $G1$-$G6$ is to prove that the Cauchy problem for the equation

$$H_t = H(B : u_{gp(t)})X = \frac{1}{2}A^{\pm} \mathcal{H}^{\pm} A^{\pm} + A^+ \mathcal{H}^+ A^- + A^- \mathcal{H}^- A^+ + \mathcal{H}_t,$$  

on the Fock vector $\Psi_t$ is correct and $\Psi_t$ is continuous in $||\cdot||^T$-norm. The strong derivative enters to eq. (1.3).

Formally, the solution for the initial condition

$$\Psi_0 = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int dx_1...dx_n A^+(x_1)...A^+(x_n) \Psi_{0,n}(x_1, ..., x_n) |0 >$$

(4.6)
is looked for in the following form

$$\Psi_t = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int dx_1 ... dx_n A^+_t(x_1) ... A^+_t(x_n) \Psi_{0,n}(x_1, ..., x_n) |0 >_t$$

(4.7)

with

$$|0 >_t = c^t \exp\left[\frac{1}{2} \int dx dy M^t(x, y) A^+(x) A^+(y)\right]|0 > .$$

(4.8)

while operators \(A^+_t(x)\) are chosen to be

$$A^+_t(x) = \int dy [A^+(y) G^+_t(y, x) - A^-(y) A^+_t(y, x)].$$

Namely, the Gaussian ansatz (4.8) formally satisfies eq.(4.9) if

$$i \frac{d\tilde{\Psi}}{dt} = \frac{1}{2} Tr H_t^+ - M^t c^t + \bar{H}_t c^t,$$

$$i \frac{dM^t}{dt} = H_t^+ + H_t^- M_t + M_t H_t^+ + M_t H_t^- M_t.$$ (4.9)

Here \(M_t\) is the operator with kernel \(M^t(x, y)\), \(H_t^+ = (H_t^-)^*\). The operators \(A^+_t(x)\) commute with \(i \frac{d}{dt} - H_t\) if

$$i \frac{dF_t}{dt} = H_t^- F_t + H_t^+ G_t,$$

$$-i \frac{dG_t}{dt} = H_t^+ G_t + H_t^- F_t.$$ (4.10)

Here \(F_t, G_t\) are operators with kernels \(F_t(x, y)\) and \(G_t(x, y)\). Note that the operator \(M_t = F_t G_t^{-1}\) formally satisfies eq.(1.9). Initial conditions (4.6) are satisfied if \(F_0 = 0, G_0 = 1\).

Let us check that eq.(4.7) is satisfied in a strong sense.

First of all, let us present some auxiliary lemmas.

**Lemma 4.5.** Let \(M\) be a Hilbert-Schmidt operator and \(\|M\| < 1\). Then

$$\exp\left[\frac{1}{2} A^+ MA^+\right]|0 >$$

(4.11)

The proof is presented in \[48\].

**Corollary.** For the state (4.11), the estimation

$$\|\Psi_n\| \leq Ae^{-\alpha n}$$

(4.12)

is satisfied under conditions of lemma 4.5 for some \(A\) and \(0 < \alpha \leq -\frac{1}{2} \log \|M\|\).

**Proof.** Since \(\|M\| < 1\), \(\|e^{2\alpha} M\| < 1\). Since expression \(\tilde{\Psi} = \exp\left[\frac{1}{2} e^{2\alpha} A^+ MA^+\right]|0 >\) specifies a Fock space vector, \(\|\Psi_{2n}\| = \|e^{2\alpha n} \Psi_{2n}\| \leq A\). Corollary is proved.

**Lemma 4.6.** Let \(M, \delta M\) be Hilbert-Schmidt operators, \(\|M\| \leq 1\), \(\|M + \delta M\| \leq 1\) and

$$\|\delta M\|_2 \leq \frac{1}{4} \log \|M\|^{-1} \|M\|^{-3/8}.$$ (4.13)

Then

$$\exp\left[\frac{1}{2} A^+ (M + \delta M) A^+\right]|0 > \geq \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{1}{2} A^+ \delta MA^+\right]^k \exp\left[\frac{1}{2} A^+ MA^+\right]|0 >$$

Proof. One should check that

$$s - \lim_{N \to \infty} \sum_{k,l,k+l \leq N} \frac{1}{2^{kl}} (A^+ \delta MA^+)^k \frac{1}{2^{kl}} (A^+ MA^+)^l |0 > =$$

$$s - \lim_{N \to \infty} \sum_{k=0}^{N} \frac{1}{2^{kl}} (A^+ \delta MA^+)^k e^{\frac{1}{2} A^+ MA^+} |0 >$$

(4.13)

Since the strong limit in the left-hand side of equality (4.13) exists, eq.(4.13) can be presented as

$$\sum_{k=0}^{N} \sum_{l=N-k+1}^{\infty} \Psi_{k,l} \to N \to \infty 0$$ (4.14)
with
\[ \Psi_{k,l} = \frac{1}{2^k k!} (A^+ \delta M A^+)^k \frac{1}{2!} (A^+ M A^+)^l |0 >. \]

Since
\[ ([A^+ \delta M A^+] \Psi)_n(x_1, ..., x_n) = \text{Sym} \sqrt{n(n-1) \delta M(x_1, x_2)} \Psi_{n-2}(x_3, ..., x_n), \]
one has
\[ ||([A^+ \delta M A^+] \Psi)_n|| \leq \sqrt{n(n-1)||\delta M||_2||\Psi||_{n-2}}. \]

By induction, one obtains:
\[ ||[A^+ \delta M A^+]^k \Psi_{n-2k}|| \leq \sqrt{\frac{n!}{(n-2k)!}} ||\delta M||^k_2 ||\Psi||_{n-2k}. \]

It follows from the estimation (I.12) that
\[ ||\Psi_{k,l}|| \leq \sqrt{(l+2k)! \frac{||\delta M||^k_2}{2^k k!} A e^{-\alpha l/2} e^{-\alpha l/2}} \]
\[ \leq \max(l+2k) e^{-\alpha (l+2k)/2} A e^{-\alpha l/2} \frac{||\delta M||^2 e^{\alpha k}}{2^k k!} = A e^{-\alpha l/2} k^k k! \alpha^{k} \left( \frac{||\delta M||^2 e^{\alpha}}{\alpha} \right)^k. \]

Since \( k! \sim (k/e)^k \sqrt{2\pi k} \) as \( k \to \infty \), one has \( e^{-k}k/k! \leq A_1 \). Therefore,
\[ ||\Psi_{k,l}|| \leq AA_1 e^{-\alpha l/2} k^k \]
with \( b = ||\delta M||^2 e^{\alpha}/\alpha \). Therefore,
\[ \sum_{k=0}^{N} \sum_{l=0}^{\infty} ||\Psi_{k,l}|| = \sum_{k=0}^{N} AA_1 b^k e^{-\frac{(N-k+1)}{2}} \frac{1}{1-e^{-\alpha/2}} \leq AA_1 \frac{e^{-\alpha (N+1)/2}}{(1-e^{-\alpha/2})(1-be^{-\alpha/2})}. \]

Therefore, for \( ||\delta M||_2 e^{3\alpha/2} \leq \alpha \) property (I.14) is satisfied. Choosing \( \alpha = -\frac{1}{3} \log ||M|| \), we obtain statement of lemma.

**Lemma 4.7.** Let \( M_t, t \in [t_1, t_2] \) be a differentiable operator function, \( ||M_t|| < \infty \).
\[ \left| \frac{M_{t+\delta t} - M_t}{\delta t} - \frac{dM_t}{dt} \right|_2 \to 0. \] (4.16)

Then
\[ \left| \frac{e^{\frac{1}{2} A^+ M_{t+\delta t} A^+}}{\delta t} |0 > - \frac{e^{\frac{1}{2} A^+ M_t A^+}}{\delta t} |0 > - \frac{1}{2} A^+ A + \frac{dM_t}{dt} A e^{\frac{1}{2} A^+ M_t A^+} |0 > \right|_m \to 0. \] (4.17)

**Proof.** Denote \( \delta M = \delta M_{t, \delta t} = M_{t+\delta t} - M_t \). It is sufficient to check the following formulas:
\[ \left| \frac{e^{\frac{1}{2} A^+ M_{t+\delta t} A^+}}{\delta t} |0 > - \frac{1}{2} A^+ M_t A^+ |0 > - \frac{1}{2} e^{\frac{1}{2} A^+ M_t A^+} |0 > \right|_m \to 0; \] (4.18)
\[ \left| \frac{A^+ \delta M}{\delta t} - \frac{dM_t}{dt} e^{\frac{1}{2} A^+ M_t A^+} \right|_m \to 0. \] (4.19)

The latter formula is a direct corollary of lemma 4.3, property \( ||e^{\frac{1}{2} A^+ M_{t+\delta t} A^+} |0 > ||_m+1 < \infty \) following from formula (I.12) and relation \( ||\frac{\delta M}{\delta t} - \frac{dM_t}{dt}||_2 \to 0 \). Formula (I.18) is a corollary of the relation
\[ \sum_{k=2}^{\infty} \sum_{l=0}^{\infty} \frac{1}{\delta t} (2k+1+l)^m ||\Psi_{k,l}|| \to 0. \] (4.20)

Making use of the estimation (4.13) and formula \( ||\delta M||^2/\delta t \to 0 \), we prove relation (4.20). Lemma 4.7 is proved.
Lemma 4.8. Let $T$ be such nonbounded self-adjoint operator in $L^2(\mathbb{R}^l)$ that $T - 1 > 0$, $D(T) \subset (H^*)^e - H^* - T^{-1}$ be uniformly bounded operator. Let the initial condition for eq. (4.3) be of the form (4.4), where $\Psi_{0,n} = 0$ as $n \geq N_0$,

$$
\Psi_{0,n}(x_1, \ldots, x_n) = \sum_{j=1}^{J_0} f_j(x_1) \cdots f_j(x_n), \quad f_j \in D(T).
$$

(4.21)

Let Hilbert-Schmidt operator $M_t$ satisfy eq. (4.8) (the derivative is defined in the Hilbert-Schmidt sense (4.10)) and initial condition $M_0 = 0$, let $c_t$ obey eq. (4.9), $F_t$ and $G_t$ be uniformly bounded operators $F_t : D(T) \to D(T)$, $G_t : D(T) \to D(T)$ satisfying eq. (4.10) in the strong sense on $D(T)$, $F_0 = 0$, $G_0 = 1$. Then the Fock vector (4.7) obeys eq. (4.5) in the strong sense.

Proof. It is sufficient to prove lemma for the initial condition

$$
\Psi_0 = \frac{1}{\sqrt{n!}} A^+ [f^1] \cdots A^+ [f^n] |0 >
$$

where $A^+ [f] = \int dx f(x) A^+(x)$. Let us show that the Fock vector

$$
\Psi_t = \frac{1}{\sqrt{n!}} A^+ [f^1] \cdots A^+ [f^n] |0 >
$$

with

$$
A^+_t [f] = \int dy [A^+(y)(G^*_t f)(y) - A^-(y)(F^*_t f)(y)]
$$

satisfies eq. (4.5). Let

$$
\dot{\Psi}_t \equiv \frac{1}{\sqrt{n!}} (\sum_{j=1}^{n} A^+_t [f^1] \cdots \dot{A}^+_t [f^j] \cdots A^+_t [f^n]) |0 >
$$

with

$$
\dot{A}^+_t [f] = \int dy [A^+(y) \frac{d}{dt} (G^*_t f)(y) - A^-(y) \frac{d}{dt} (F^*_t f)(y)],
$$

$$
\frac{d}{dt} |0 > = dt e^{\frac{1}{2} t A} \frac{d}{dt} e^{-\frac{1}{2} t A} A |0 >
$$

One has

$$
\frac{\Psi_{t+\delta t} - \Psi_t}{\delta t} - \dot{\Psi}_t = \frac{1}{\sqrt{n!}} A^+_t [f^1] \cdots A^+_t [f^n] \left( \frac{\|0^{+>t+\delta t} - 0^{+>t}\|}{\delta t} - \frac{d}{dt} |0 > \right) +
$$

$$
\frac{1}{\sqrt{n!}} \left[ A^+_t [f^1] \cdots A^+_t [f^n] - A^+_t [f^1] \cdots A^+_t [f^n] \right] |0 >
$$

$$
+ \sum_{j=1}^{n} \left[ A^+_t [f^1] \cdots A^+_t [f^{j-1}] A^+_t [f^j] - A^+_t [f^j] A^+_t [f^j] \right] |0 >
$$

$$
+ \sum_{j=1}^{n} \left[ A^+_t [f^1] \cdots A^+_t [f^{j-1}] A^+_t [f^j] - A^+_t [f^j] A^+_t [f^j] \right] |0 >
$$

It follows from lemmas 4.3, 4.7 and conditions of lemma 4.8 that

$$
|| \frac{\Psi_{t+\delta t} - \Psi_t}{\delta t} - \dot{\Psi}_t || \to \delta t \to 0.
$$

Eqs. (4.3), (4.10) imply that $\dot{\Psi}_t = - i H_t \Psi_t$. Lemma 4.8 is proved.

Denote by $D_1 \subset F$ the set of all Fock vectors $\Psi \in F$ such that $\Psi_n$ vanish at $n \geq N_0$ and have the form (4.21) as $n < N_0$. Lemma 4.8 allows us to construct the mapping $U_t : D_1 \to F$ of the form $U_t \Psi_0 = \Psi_t$. Note that the domain $D_1$ is dense in $F$.

Denote

$$
A^-_t [f] \equiv (A^+_t [f])^+ \equiv \int dy [A^- (y)(G_t f)(y) - A^- (y)(F_t f)(y)].
$$

Lemma 4.9. 1. The operators $A^+_t [f]$ obey the commutation relations

$$
[A^-_t [f], A^+_t [g]] = (f, g), \quad [A^+_t [f], A^+_t [g]] = 0.
$$

(4.22)
2. The following property is satisfied:

\[ A_t^-[f]|0 >_t= 0. \] (4.23)

3. The operator \( U_t \) is isometric.

**Proof.** The commutation relations (4.22) are rewritten as

\[
(G_t f, G_t g) - (F_t f, F_t g) = (f, g);
\]

\[
(F_t^* f, G_t g) - (G_t^* f, F_t g) = 0.
\] (4.24)

They are satisfied at \( t = 0 \). The time derivatives of the left-hand sides of eqs. (4.24) vanish because of eqs. (4.10). Statement 1 is proved.

The fact that \( U_t \) is an isometric operator is a corollary of the property \( \frac{d}{dt}(\Psi_t, \Psi_t) = 0 \).

Analogously to lemma 4.8, we find that the vector \( \Psi_t = A_t^-[f]|0 >_t \) obeys eq. (4.3) in the strong sense. Since \( \Psi_0 = 0 \) and \( ||\Psi_t|| = ||\Psi_0|| \), one has \( \Psi_t = 0 \). Property (4.23) is proved. Note that it means that

\[ M_t G_t = F_t. \] (4.25)

Lemma 4.9 is proved.

Therefore, the operator \( U_t \) can be extended to the whole space \( \mathcal{F}, U_t : \mathcal{F} \to \mathcal{F} \).

**Lemma 4.10.** Let the operator

\[
\begin{pmatrix}
G^+ & -F^+ \\
-F^T & G^T
\end{pmatrix}
\]

be invertible. Then the following relation is satisfied on \( \mathcal{D}_1 \):

\[ U_t^{-1} A^+ T A^- U_t \Psi_0 = (A^+ G_t^T + A^- F^+) T (F A^+ + G^* A^-) \Psi_0 \] (4.26)

**Proof.** It follows from lemma 4.9 that

\[
\begin{pmatrix}
G^+ & -F^+ \\
-F^T & G^T
\end{pmatrix}
\begin{pmatrix}
G & F^* \\
F & G^*
\end{pmatrix} = 1
\]

Therefore,

\[
\begin{pmatrix}
G^+ & -F^+ \\
-F^T & G^T
\end{pmatrix}^{-1} = \begin{pmatrix}
G & F^* \\
F & G^*
\end{pmatrix}
\]

and

\[ A^-(y) = \int dz (F_t(y, z) A_t^+(z) + G^*_t(y, z) A_t^-(z)), \]

\[ A^+(y) = \int dz (F_t^*(y, z) A_t^-(z) + G_t(y, z) A_t^+(z)). \]

Identity (4.26) is then a corollary of definition of the operator \( U_t \).

**Lemma 4.11.** Let \( \Psi_0 \in \mathcal{D} \). Suppose that \( T F_t \) and \( \mathcal{H}^{++} \) are continuous operator functions in the \( || \cdot ||_2 \)-norm, \( G_t, T^{1/2} G_t T^{-1/2}, TG_t T^{-1}, T^{-1/2} \mathcal{H}^{++} T^{-1/2}, \mathcal{H}^{++} T^{-1} \) are continous operator functions in the \( || \cdot ||\)-norm. Then the following statements are satisfied.

1. \( \Psi_t \equiv U_t \Psi_0 \in \mathcal{D} \).
2. \( \Psi_t \) obeys eq. (4.3) in the strong sense.
3. \( ||\Psi_t - \Psi_0||^T \to_{t \to 0} 0. \) (4.27)

**Proof.** Let \( \Psi_0 \in \mathcal{D}_1 \). For \( ||U_t \Psi_0||^T \), one has the following estimation:

\[
||U_t \Psi_0||^T = ||U_t^{-1}(T + 1) U_t \Psi_0|| \leq ||\Psi_0|| + \frac{1}{2} \{ ||A^+ G^T + A^- F^+|| T (F A^+ + G^* A^-) \Psi_0|| + (1 + ||F^+||_2 ||T F||_2) ||\Psi_0|| + (\sqrt{2} ||G^T T F||_2 + ||F^+ T F|| + ||F^+ T G||_2) ||\Psi_0|| + (||T^{-1/2} G^T T G^* T^{-1/2}|| + ||A^T T A^- T^{-1}||) ||\Psi_0|| \to_{t \to 0} 0.
\]
at \( t \in [0, t_1] \). Therefore, the operator \( U_t \) is bounded in norm \( \| \cdot \|_T^T \). The extension of the operator \( U_t \) to \( \mathcal{D} \) is then also a bounded operator in \( \| \cdot \|_T^T \) norm. One therefore has \( \Psi_t \in \mathcal{D} \).

The fact that \( \| U_t \Psi_0 - \Psi_0 \|_T^T \to_{t \to 0} 0 \) if \( \Psi_0 \in \mathcal{D}_1 \) is justified analogously to lemma 4.8. Since the operator \( U_t : \mathcal{D} \to \mathcal{D} \) is uniformly bounded at \( t \in [0, t_1] \) in \( \| \cdot \|_T^T \)-norm, the Banach-Steinhaus theorem (see, for example, [1]) implies relation (1.27).

To check the second statement, note that lemma 4.8 imply that

\[
\frac{U_{t+\delta t} - U_t}{\delta t} \to_{\delta t \to 0} dU_t \tag{4.28}
\]

in the strong sense on \( \mathcal{D}_1 \). For showing that relation (4.28) is satisfied in the strong sense on \( \mathcal{D} \), it is sufficient to show that the operator

\[
\frac{\delta U_t}{\delta t} : \mathcal{D} \to \mathcal{F}
\]

is uniformly bounded,

\[
\| \frac{\delta U_t}{\delta t} \Psi \| \leq C \| \Psi \|_T^T.
\]

One has

\[
\| \frac{\delta U_t}{\delta t} \Psi \| = \| \int_0^1 dsU_{t+s\delta t} \Psi \| = \| \int_0^1 dsH_{t+s\delta t}U_{t+s\delta t} \| \leq \max_{s \in [0,1]} [\sqrt{2}||H_{t+s\delta t}||_2 + ||T^{-1/2}H_{t+s\delta t}T^{-1/2}|| + ||H_{t+s\delta t}^{-1}||] ||U_{t+s\delta t} \Psi ||_T^T.
\]

Lemma 4.11 is proved.

Let us now check properties of operators \( F_t, F_t, M_t \).

First of all, consider the Cauchy problem

\[
\begin{align*}
 i\dot{f}_t &= Y_t f_t + Z_t g_t, \\
 -i\dot{g}_t &= Z_t^* f_t + Y_t^* g_t, \\
 f_0 &= 0, g_0 &= 1,
\end{align*}
\tag{4.29}
\]

where \( g_t \) is a bounded operator functions, \( f_t \) is a Hilbert-Schmidt operator function. The derivatives in (4.29) are understood as

\[
\| (\frac{g_{t+\delta t} - g_t}{\delta t} - \dot{g}_t) \varphi \| \to_{\delta t \to 0} 0, \quad \| (\frac{f_{t+\delta t} - f_t}{\delta t} - \dot{f}_t) \|_2 \to_{\delta t \to 0} 0. \tag{4.30}
\]

**Lemma 4.12.** Let \( Y_t \) be a strongly continuous operator function, while \( \| Z_{t+\tau} - Z_t \|_2 \to_{\tau \to 0} 0, \| TZ_t \|_2 \leq a_1^2, \| TY_t T^{-1} \| \leq a_1^2, \| T^{1/2} Y_t T^{-1} \| \leq a_2^3 \) for smooth functions \( a_k \). Then there exist a solution to the Cauchy problem (4.29) such that

\[
\| T f_t \|_2 \leq a_4^2, \quad \| T^{1/2} g_t T^{-1/2} \| \leq a_5^2, \quad \| T g_t T^{-1} \| \leq a_6^2, \quad \| g_t \| \leq a_7^2 \tag{4.31}
\]

for smooth functions \( a_k \).

**Proof** (cf. [8]). Let us look for the solution to the Cauchy problem in the following form:

\[
f_t = \sum_{n=0}^{\infty} f_t^n, \quad g_t = \sum_{n=0}^{\infty} g_t^n. \tag{4.32}
\]

where \( f_0^0 = 0, g_0^0 = 1, \)

\[
f_t^{n+1} = -i \int_0^t d\tau (Y_\tau f_t^n + Z_\tau g_t^n),
\]

\[
g_t^{n+1} = -i \int_0^t d\tau (Y_\tau^* g_t^n + Z_\tau^* f_t^n). \tag{4.33}
\]

By induction we find that \( \| f_t^n \|_2 \leq C_1 t^n / n!, \| g_t^n \|_2 \leq C_1 t^n / n! \) for \( t \in [0, t_1] \). Here \( C_1 \) is a constant.

Therefore, the series (4.32) converge. \( f_t \) is a Hilbert-Schmidt operator, while \( g_t \) is a bounded operator. Analogously, we show

\[
\| T f_t^n \|_2 \leq \frac{C_2 t^n}{n!}, \quad \| T g_t^n T^{-1} \| \leq \frac{C_2 t^n}{n!}, \quad \| T^{1/2} g_t^n T^{-1/2} \| \leq \frac{C_2 t^n}{n!},
\]

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where \( t \in [0, t_1] \). Therefore, properties (4.31) are satisfied.

To check relations (4.30), note that
\[
\begin{align*}
  f_t &= -i \int_0^t d\tau (Y_\tau f_\tau + Z_\tau g_\tau), \\
  g_t &= i \int_0^t d\tau (Z_\tau^* f_\tau + Y_\tau^* g_\tau).
\end{align*}
\] (4.34)

Eqs. (4.34) imply that the operator functions \( f_t, g_t \) obey properties
\[
||T(f_{t+\delta t} - f_t)||_2 \to \delta t \to 0 0, \quad ||(g_{t+\delta t} - g_t)||_2 \to \delta t \to 0
\]

Therefore,
\[
\begin{align*}
  ||(i \frac{f_{t+\delta t} - f_t}{\delta t} - Y_t f_t - Z_t g_t)||_2 &\leq \int_0^1 ds ||Y_{t+s\delta t} f_{t+s\delta t} + Z_{t+s\delta t} g_{t+s\delta t} - Y_t f_t - Z_t g_t||_2, \\
  ||(-i \frac{g_{t+\delta t} - g_t}{\delta t} - Z_t^* f_t - Y_t^* g_t)\varphi_t|| &\leq \int_0^1 ds ||(Z_{t+s\delta t}^* f_{t+s\delta t} + Y_{t+s\delta t}^* g_{t+s\delta t} - Z_t^* f_t - Y_t^* g_t)\varphi_t||.
\end{align*}
\]

Since the integrands are uniformly bounded functions, the Lesbesgue theorem (see, for example, [11]) tells us that it is sufficient to check that
\[
\begin{align*}
  ||Y_{t+\tau} f_{t+\tau} - Y_t f_t||_2 &\to \delta t \to 0, \quad ||Z_{t+\tau} g_{t+\tau} - Z_t g_t||_2 \to \delta t \to 0, \\
  s - \lim_{\tau \to 0} Z_{t+\tau}^* f_{t+\tau} = Z_t^* f_t, \\
  s - \lim_{\tau \to 0} Y_{t+\tau}^* g_{t+\tau} = Y_t^* g_t.
\end{align*}
\]

These relations are corollaries of conditions of lemma 4.12 and formulas (4.33).

**Lemma 4.13.** Let \( \mathcal{H}_t^{+} = L + \mathcal{H}_t, \mathcal{H}_t, T^{1/2} \mathcal{H}T^{-1/2}, T \mathcal{H}^{-1} \) be strongly continous operator functions, \( ||\mathcal{H}_t^{+} - \mathcal{H}_t^{+}||_2 \to \delta t \to 0 0, L \) be a t-independent (maybe nonbouned) self-adjoint operator, such that \( ||LT^{-1}|| < \infty, \) while \( ||T^{1/2}e^{-iLt}T^{-1/2}|| < \infty, ||Te^{-iLt}T^{-1}|| < \infty. \) Then there exists a solution to the Cauchy problem for system (4.10) for the initial condition \( F_0 = 0, G_0 = 1: \)
\[
\begin{align*}
  ||(i \frac{F_{t+\delta t} - F_t}{\delta t} - \mathcal{H}_t^{+} + F_t - \mathcal{H}_t^{+} G_t)||_2 &\to \delta t \to 0, \\
  ||(-i \frac{G_{t+\delta t} - G_t}{\delta t} - \mathcal{H}_t^{+} - G_t - \mathcal{H}_t^{+} F_t)\varphi|| &\to \delta t \to 0, \quad \varphi \in D(T).
\end{align*}
\] (4.35)

Moreover,
\[
\begin{align*}
  ||TF_t||_2 &\leq b(t), \quad ||TG_tT^{-1}|| \leq b(t), \quad ||T^{1/2}G_tT^{-1/2}|| \leq b(t), \quad ||G_t|| \leq b(t)
\end{align*}
\] (4.36)

for some smooth function \( b(t) \) on \( t \in [0, t_1] \). The properties (4.25) are also satisfied.

**Proof.** Consider the operator functions
\[
\begin{align*}
  F_t &= e^{-iLt} f_t, \quad G_t = e^{iLt} g_t,
\end{align*}
\]

where \((f_t, g_t)\) is a solution to the Cauchy problem (4.29) with \( Y_t = e^{iLt} H_t e^{-iLt}, Z_t = e^{iLt} H_t^{+} e^{iLt}, \) \( f_0 = 0, g_0 = 1. \) Check of properties (4.36) is straightforward. Let us prove relations (4.35). One has
\[
\begin{align*}
  i \frac{F_{t+\delta t} - F_t}{\delta t} - (L + \mathcal{H}_t) F_t - \mathcal{H}_t^{+} + G_t = (i e^{-i\delta t T^{-1}} - L T^{-1}) T F_t + i e^{-iLt} (\frac{f_{t+\delta t} - f_t}{\delta t} - \dot{f}_t), \\
  -i \frac{G_{t+\delta t} - G_t}{\delta t} - (L^* + \mathcal{H}_t^*) G_t - \mathcal{H}_t^{+} - F_t = (-i e^{-i\delta t T^{-1}} - L^* T^{-1}) T F_t + i e^{-iLt} (\frac{g_{t+\delta t} - g_t}{\delta t} - \dot{g}_t).
\end{align*}
\]

Since
\[
\begin{align*}
  ||(i e^{-iLtT^{-1}} - L T^{-1})\varphi|| \leq \int_0^1 ds ||(e^{-iLt\tau} - 1) L T^{-1}\varphi|| \to \tau \to 0 0,
\end{align*}
\]
we obtain relations (4.33).

Property (4.25) is proved analogously to [13]: one should consider the convergent in \( ||\cdot||\)-norm series
\[
\begin{align*}
  \begin{pmatrix} G & F^* \\
  F & G^*
\end{pmatrix}^{-1} = \sum_{n=0}^{\infty} \begin{pmatrix} G^{(-n)} & F^{(-n)*} \\
  F^{(-n)} & G^{(-n)*}
\end{pmatrix} \begin{pmatrix} e^{iL^*t} & 0 \\
  0 & e^{-iLt}
\end{pmatrix}
\end{align*}
\] 24
with
\[
\begin{pmatrix}
G_t^{(-n)} \\
F_t^{(-n)} \\
G_t^{(-n)*}
\end{pmatrix} = i \int_0^t dt \begin{pmatrix}
G_t^{(-n+1)} \\
F_t^{(-n+1)} \\
G_t^{(-n+1)*}
\end{pmatrix} + \begin{pmatrix}
Y_\tau \\
-Z_\tau^* \\
-Y_\tau^*
\end{pmatrix}
\]

Lemma 4.13 is proved.

**Lemma 4.14.** Under conditions of lemma 4.13 there exists a solution to the Cauchy problem for eq. (4.3) with the initial condition \( M_0 = 0 \).

**Proof.** It follows from \( \|G^{-1}\| < 1 \) that the matrix \( G \) is invertible and \( \|G^{-1}\| < 1 \). Consider the operator\( M_t = F_t G_t^{-1} \). Note that \( \|TM_t\|_2 < \infty \), \( \|LM_t\| < \infty \). One has
\[
M_{t+\delta t} - M_t = M_{t+\delta t}(G_t - G_{t+\delta t})G_t^{-1} + (F_{t+\delta t} - F_t)G_t^{-1},
\]
so that \( \|T(M_{t+\delta t} - M_t)\|_2 \to \delta t \to 0 0 \). Therefore,
\[
M_{t+\delta t} TT^{-1}(\frac{G_t - G_{t+\delta t}}{\delta t} - \dot{G}_t)G_t^{-1} + (M_{t+\delta t} - M_t) TT^{-1} \dot{G}_t G_t^{-1} + \frac{(\frac{F_{t+\delta t} - F_t}{\delta t} - \dot{F}_t)G_t^{-1}}{\delta t}.
\]

Analogously to lemmas 4.12, 4.13, one finds
\[
\|\frac{G_t^+ - G_t^+}{\delta t} - \dot{G}_t^+ T^{-1} \varphi\| \to \delta t \to 0.
\]
Therefore,
\[
\|\frac{M_{t+\delta t} - M_t}{\delta t} - \hat{M}_t\|_2 \to \delta t \to 0.
\]

Lemma 4.14 is proved.

Therefore, we have proved the following theorem.

**Lemma 4.15.** Let \( T, L \) be self-adjoint operators in \( L^2(\mathbb{R}^l) \) such that
\[
\|T^{-1/2}LT^{-1/2}\| < \infty, \quad \|LT^{-1}\| < \infty, \quad \|T^{1/2}e^{-iLt}T^{-1/2}\| \leq C, \quad \|Te^{-iLt}T^{-1}\| \leq C, \quad t \in [0, t_1].
\]

Let \( T - \omega \) be positively definite for some positive constant \( \omega \), \( \mathcal{H}^+ = L + \mathcal{H}_t \), \( \mathcal{H}^{++} \) be operator-valued functions such that \( \|T(\mathcal{H}^+_t - \mathcal{H}^{++}_t)\|_2 \to \delta t \to 0 \), \( \mathcal{H}_t, TH_T T^{-1}, T^{1/2}H_T T^{-1/2} \) are strongly continuos operator functions, \( \mathcal{H}_t \) be a continous function. Then there exists a unique solution \( \Psi_t \) to the Cauchy problem \( \mathbf{(4.3)} \), provided that \( \Psi_0 \in \mathcal{D} \equiv \{\Psi \in \mathcal{F}|||\Psi||_T^2 < \infty\} \) it satisfies the properties \( \Psi_t \in \mathcal{D} \) and \( ||\Psi_t - \Psi_0||_T^2 \to \delta t \to 0 \).

Thus, properties G1-G6 are obtained as corollaries of F1-F3.

### 5 Composed semiclassical states

We have already mentioned that composed semiclassical states are specified by a set \( \left( X(\alpha)\right) g(\alpha) \in \mathcal{F}_{X(\alpha)} \); \( \alpha \in \Lambda^k \). The inner product is given by eq. (2.22). For the case \( \mathcal{F}_{X(\alpha)} = \mathcal{F}(L^2(\mathbb{R}^l)) \) and \( \omega \) of the form (4.2), expression (2.22) takes the form
\[
\int d\alpha(g(\alpha), \int d\beta e^{i\beta s(A^+B_s - A^-B_s^* g(\alpha))}
\]
with
\[
B_s(\alpha, \cdot) = \varphi^*_X \left( \frac{\partial X}{\partial \alpha_s} \right).
\]

Let us investigate the inner product space of composed states in more details.
5.1 Constrained Fock space

The purpose of this subsection is to investigate the properties of the inner product

\[ <Y_1, Y_2> = \int d\beta(Y_1, \exp[\sum_{s=1}^{k} \beta_s \int dx(B_s(x)A^+(x) - B^*_s(x)A^-(x))]Y_2) \]  

(5.3)

for the Fock vectors \( Y_1, Y_2 \). Suppose the functions \( B_1, ..., B_k \) to be linearly independent. Since the inner product (5.3) resembles the inner products for constrained systems [47], we will call the space under construction as a constrained Fock space.

First of all, investigate the problem of convergence of the integral (5.3). Note that the operator

\[ U[B] = \exp[\int dx(B(x)A^+(x) - B^*(x)A^-(x))] \]

is a well-defined unitary operator [43], provided that \( B \in L^2(\mathbb{R}^l) \), and obey the relations

\[ A^-(x)U[B] = U[B](A^-(x) + B(x)); \]
\[ A^+(x)U[B] = U[B](A^+(x) + B^*(x)). \]

Lemma 5.1. (cf. [28]). The following estimation is satisfied:

\[ ||B||^m(Y_1, U[B]Y_2) \leq \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} ||Y_1||^{k/2}||Y_2||^{(m-k)/2}. \]  

(5.4)

Proof. One has

\[ [\int dx B^*(x)A^-(x); U[B]] = ||B||^2U[B], \]

so that

\[ ||B||^2(Y_1, U[B]Y_2) = (\int dx B(x)A^+(x)Y_1, U[B]Y_2) - (Y_1, U[B]\int dx B^*(x)A^-(x)Y_2). \]

Applying this identity \( m \) times, we obtain:

\[ ||B||^m(Y_1, U[B]Y_2) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!(m-k)!} \left( \int dx B(x)A^+(x) \right)^k Y_1, \]
\[ U[B]\left( \int dx B^*(x)A^-(x) \right)^{m-k} Y_2. \]

Making use of the result of lemma 4.3,

\[ ||\int dx B(x)A^\pm(x)Y|| \leq ||B||||Y||^{l+1/2}, \]

we find:

\[ ||B||^m(Y_1, U[B]Y_2) \leq \sum_{s=0}^{m} \frac{m!}{s!(m-s)!} ||B||^m||Y_1||^{s/2}||Y_2||^{(m-s)/2}. \]

Lemma 5.1 is proved.

Corollary 1. Let \( B_1, ..., B_k \) be linearly independent functions. Then for some constant \( C_1 > 0 \) the following estimation is satisfied:

\[ ||\beta||^m(Y_1, U[\sum_s \beta_s B_s]Y_2) \leq C_1^m||Y_1||^{m/2}||Y_2||^{m/2}. \]

Proof. It is sufficient to notice that for linearly independent \( B_1, ..., B_k \) the matrix \( (B_m, B_s) \) is not degenerate, so that \( \frac{1}{2} \sum_m \beta_m B_m ||^2 \geq C_1^{-1}||\beta||^2 \) for some \( C_1 \). Applying the property \( ||Y||^{s/2} \leq ||Y||^{m/2} \) for \( s \leq m \), making use of eq. (5.4), we prove corollary 1.
and integral (5.3) converges.

**Corollary 2.** Let \( \|Y_1\|_{m/2} < \infty, \|Y_2\|_{m/2} < \infty \) for some \( m > k \), Then the integrand entering to eq. (5.3) obeys the relation
\[
\|(Y_1, U[\sum_s \beta_s B_s]Y_2)\| \leq \frac{\text{const}}{(|\beta| + 1)^m}
\]
and integral (5.3) converges.

**Corollary 3.** Let \( Y_{1,n}, Y_{2,n} \) be such sequences of Fock vectors that \( \|Y_{1,n}\|_{m/2} \to_{n \to \infty} 0 \), \( \|Y_{2,n}\|_{m/2} \leq C \) for some \( m > k \). Then \( \langle Y_1, Y_2 \rangle \to_{n \to \infty} 0 \).

Let us investigate the property of nonnegative definiteness of the inner product (5.3).

**Lemma 5.2.** Let \( \|Y\|_{m} < \infty \) for some \( m > k \) and \( \text{Im}(B_s, B_l) = 0 \). Then \( \langle Y, Y \rangle > \geq 0 \).

**Proof.** Introduce the following "regularized" inner product
\[
\langle Y, Y \rangle_{\varepsilon} = \int d\beta e^{-|\beta|^2} \langle Y, U[\sum_s \beta_s B_s]Y \rangle.
\]
It follows form estimation (5.5) and the Lesbegue theorem [41] that
\[
\langle Y, Y \rangle_{\varepsilon} \to_{\varepsilon \to 0} 0.
\]
It is sufficient then to prove that \( \langle Y, Y \rangle_{\varepsilon} > \geq 0 \). One has:
\[
e^{-|\beta|^2} = (4\pi \varepsilon)^{k/2} \int d\beta' e^{-2\varepsilon|\beta' - \beta''|^2 - 2\varepsilon|\beta''|^2}
\]
Therefore,
\[
\langle Y, Y \rangle_{\varepsilon} = \int d\beta' d\beta'' (4\pi \varepsilon)^{k/2} e^{-2\varepsilon(|\beta'|^2 + |\beta''|^2)} \langle U[\sum_s \beta''_s B_s]Y, U[\sum_s \beta'_s B_s]Y \rangle,
\]
here the shift of variable \( \beta = \beta' - \beta'' \) is made. We have also taken into account that
\[
U[\sum_s \beta'_s B_s]U[- \sum_s \beta''_s B_s] = U[\sum_s (\beta'_s - \beta''_s) B_s],
\]
provided that the operators
\[
\int dx (B_s(x)A^+(x) - B^*_s(x)A^-(x))
\]
commute (i.e. \( \text{Im}(B_s, B_l) = 0 \)). Formula (5.6) is taken to the form
\[
\langle Y, Y \rangle_{\varepsilon} = || \int d\beta (4\pi \varepsilon)^{k/4} e^{-2\varepsilon|\beta|^2} U[\sum_s \beta_s B_s]Y ||^2 \geq 0.
\]
Lemma 5.2 is proved.

The expression (5.3) depends on \( k \) functions \( B_1, ..., B_k \). However, one may perform linear substitutions of variables \( \beta \), so that only the subspace \( \text{span}\{B_1, ..., B_k\} \) is essential.

**Definition 5.1.** A \( k \)-dimensional subspace \( L_k \subset L^2(\mathbb{R}^l) \) is called as a \( k \)-dimensional isotropic plane if \( \text{Im}(B', B'') = 0 \) for all \( B', B'' \in L_k \).

Let \( L_k \) be a \( k \)-dimensional isotropic plane with an invariant under shifts measure \( d\sigma \). Let \( B_1, ..., B_k \) be a basis on \( L_k \). One can assign then coordinates \( \beta_1, ..., \beta_n \) to any element \( B \in L_k \) according to the formula \( B = \sum_s \beta_s B_s \). The measure \( d\sigma \) is presented as \( d\sigma = a d\beta_1 ... d\beta_k \) for some constant \( a \). Consider the inner product
\[
\langle Y_1, Y_2 \rangle_{L_k} = a \int d\beta \langle Y_1, U[\sum_s \beta_s B_s]Y_2 \rangle = \int d\sigma \langle Y_1, U[B]Y_2 \rangle,
\]
\( \|Y_{1,2}\|_{k/2+1} \leq \infty \). This definition is invariant under change of basis.
By $F_{k/2+1}$ we denote space of such Fock vectors $Y$ that $||Y||_{k/2+1} < \infty$. We say that $Y \not\in L_k$ if $<Y,Y>_{L_k} = 0$. Thus, the space $F_{k/2+1}$ is divided into equivalence classes. Introduce the following inner product on the factor-space $F_{k/2+1}$/$\sim$:

$$< [Y_1], [Y_2] >_{L_k} = < Y_1, Y_2 >_{L_k} \quad (5.8)$$

for all $Y_1 \in [Y_1], Y_2 \in [Y_2]$. This definition is correct because of the following statement.

**Lemma 5.3.** Let $< Y, Y >_{L_k} = 0$. Then $< Y, Y' >_{L_k} = 0$ for all $Y'$. The proof is standard (cf., for example, [11]). One has

$$0 \leq < Y' + \sigma Y, Y' + \sigma Y >_{L_k} = < Y', Y'_k >_{L_k} + \sigma^* < Y, Y' >_{L_k} + \sigma < Y', Y >_{L_k}$$

for all $\sigma \in C$, so that $< Y, Y' >_{L_k} = 0$.

**Definition 5.2.** A constrained Fock space $F(L_k, d\sigma)$ is the completeness of the factor-space $F_{k/2+1}$/$\sim$ with respect to the inner product (5.8),

$$F(L_k) = \overline{F_{k/2+1}}/\sim.$$  

### 5.2 Transformations of constrained Fock vectors

Let us investigate evolution of constrained Fock vectors. Consider the Cauchy problem for eq.(4.3). Denote $F_m = \{ \Psi \in F ||\Psi||_m < \infty \}$.

**Lemma 5.4.** Let $\Psi_0 \in F_m$. Then $\Psi_t \in F_m$.

**Proof.** Analogously to proof of lemma 4.11, one has

$$||U_t \Psi_0|| = ||U_t^{-1}(A^+ A^- + 1)^m U_t \Psi_0|| = ||(1 + (A^+ G_t^T A^- F_t^+ + G_t^* A^-))^m \Psi_0||$$

It follows from Lemmas 4.2, 4.3 that

$$||((1 + (A^+ G_t^T A^- F_t^+ + G_t^* A^-)) \Psi ||_{l+1} \leq ||\Psi ||_{l+1} + ||F_t||^2 ||\Psi||_{l+1} + ||F_t^T F_t^* + G_t^T G_t^*|| ||\Psi||_{l+1} + ||F_t^+ G_t^*||_2 ||\Psi||_{l+1} \leq C ||\Psi||_{l+1}$$

with

$$C = 1 + ||F_t||^2 + \sqrt{2} ||G_t^T F_t||_2 + ||F_t^T F_t^* + G_t^T G_t^*|| + ||F_t^+ G_t^*||_2.$$  

Applying this estimation, we obtain by induction:

$$||U_t \Psi_0||_m \leq C^m ||\Psi_0||_m.$$  

Lemma is proved.

Let $L_k$ be a $k$-dimensional isotropic plane with invariant measure $d\sigma$. Define its evolution transformation $L_k^t$ as follows. Let $(B_1, ..., B_k)$ be a basis on $L_k$. Let $B^t_s$ be solutions to the Cauchy problems

$$iB^t_s = H_t^+ B^t_s + H_t^- (B^*_s)^*;$$
$$-iB^*_s = H_t^- B^*_s + H_t^+ B^t_s;$$
$$B^0_s = B_s; B^0_s = B^*_s.$$  

they can be expressed as

$$B^t_s = F_t B^*_s + G_t^* B_s;$$
$$B^*_s = F^*_t B_s + G_t B^*_s.$$  

**Lemma 5.5.** Let $\text{Im}(B_i, B_j) = 0$. Then $\text{Im}(B^t_i, B^*_j) = 0$.  

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Proof. One has:

\[ 2i \text{Im}(B_i^*, B_j^t) = (B_i^*, B_j^t) - (B_j^*, B_i^t) = \]
\[ (F_i B_i^* + G_i B_i, F_i B_j^* + G_i B_j) - (F_j B_j^* + G_j B_j, F_i B_i^* + G_i B_i) = \]
\[ (B_i, B_j) - (B_j, B_i) = 0. \]

because of relations (2.24) of Appendix B.

Therefore, \( L_k^t \) is also an isotropic plane. Define the measure \( d\sigma \) on \( L_k^t \) as follows. For the choice of coordinates \( \beta_1, ..., \beta_k \) on \( L_k^t \) according to the formula \( B = \sum_s \beta_s B_s^t \), set \( d\sigma = d\beta_1...d\beta_k \), where \( a \) does not depend on \( t \).

**Lemma 5.6.** The inner product \( \langle \cdot, \cdot \rangle_{L_k^t} \) is invariant under time evolution:

\[ \langle \Psi_t, \Psi_t \rangle_{L_k^t} = \langle \Psi_0, \Psi_0 \rangle_{L_k^t}. \]

**Proof.** By definition, one has

\[ \langle \Psi_t, \Psi_t \rangle_{L_k^t} = a \int d\beta(\Psi_t, U(\sum_s \beta_s B_s^t) \Psi_t) = a \int d\beta(\Psi_0, U^t(\sum_s \beta_s B_s^t) U_t \Psi_0). \]

Eq.(5.11) implies that

\[ U(\sum_s \beta_s B_s^t) = \exp(\sum_s \int \beta_s dx (A_t^+(x) B_s(x) - A_t^-(x) B_s^*(x))). \]

Making use of the relation

\[ U_t^+ A_t^+(x) U_t = A_t^+(x), \]

we obtain statement of lemma 5.6.

It follows from lemma 5.6 that operator \( U_t \) takes equivalent states to equivalent. Therefore, it can be reduced to the factorspace \( F_{[k/2+1]}/\sim \). Since it is unitary, it can be extended to \( F(L_k) \).

### 5.3 Definition of a composed semiclassical state and its symmetry transformation

Let us formulate a definition of a composed semiclassical state.

Let \( \{X(\alpha), \alpha \in \Lambda^k \} \) be a smooth \( k \)-dimensional manifold in the extended phase space \( X \) with measure \( d\Sigma \) such that an isotropic condition (2.21)

\[ \omega_{X(\alpha)}\left[ \frac{\partial X(\alpha)}{\partial \alpha_a} \right] = 0. \]

is satisfied. It follows from commutation relations (2.11) that

\[ [\Omega_X \left[ \frac{\partial X}{\partial \alpha_a} \right], \Omega_X \left[ \frac{\partial X}{\partial \alpha_b} \right]] = i \left( \frac{\partial \omega_X(X(\alpha))}{\partial \alpha_a} \frac{\partial X}{\partial \alpha_b} - \frac{\partial \omega_X(X(\alpha))}{\partial \alpha_b} \frac{\partial X}{\partial \alpha_a} \right) = i \left( \frac{\partial}{\partial \alpha_a} (\omega_X \left[ \frac{\partial X}{\partial \alpha_b} \right]) - \frac{\partial}{\partial \alpha_b} (\omega_X \left[ \frac{\partial X}{\partial \alpha_a} \right]) \right) = 0, \]

so that

\[ [A^+ B_a - A^- B_a^*, A^+ B_b - A^- B_b^*] = 0, \]

where \( B_a \) have the form (5.2). Therefore,

\[ \text{Im}(B_a, B_b) = 0. \]

Define an isotropic plane \( L_k(\alpha) \equiv L_k(\alpha : \Lambda^k) \) as \( \text{span}\{B_1, ..., B_k\} \). It does not depend on the particular choice of coordinates \( \alpha_1, ..., \alpha_k \). Introduce the following measure \( d\sigma(\alpha) \) on \( L_k(\alpha) \):

\[ d\sigma(\alpha) = \frac{D\Sigma(\alpha)}{D\alpha} (\alpha)d\beta_1...d\beta_k, \quad (5.11) \]
where $\beta_1, \ldots, \beta_k$ are coordinates on $L_k(\alpha)$ which are determined as $B = \sum_s \beta_s B_s$.

Definition (5.11) is invariant under change of coordinates. Namely, let $(\alpha'_1, \ldots, \alpha'_k)$ be another set of local coordinates chosen instead of $(\alpha_1, \ldots, \alpha_k)$. Then

$$B'_i = \sum_{s=1}^k \frac{\partial \alpha_s}{\partial \alpha'_i} B_s,$$

so that property $\sum_i \beta'_i B'_i = \sum_s \beta_s B_s$ implies that coordinate sets $\beta$ and $\beta'$ should be related as follows:

$$\beta_s = \sum_i \frac{\partial \alpha_s}{\partial \alpha'_i} \beta'_i.$$

Therefore, for the choice of coordinates $\alpha'$ one has

$$d\sigma' = \frac{D \Sigma}{D \alpha} d\beta'_1 \ldots d\beta'_k = \frac{D \Sigma}{D \alpha'} d\beta_1 \ldots d\beta_k = d\sigma.$$

The invariance property is checked.

Introduce the vector (Hilbert) bundle $\pi_{\Lambda^k}$ as follows. The base of the bundle is the isotropic manifold $\Lambda^k$. The fibre that corresponds to the point $\alpha \in \Lambda^k$ is $H_\alpha = \mathcal{F}(L_k(\alpha))$. Composed semiclassical states are introduced as sections of bundle $\pi_{\Lambda^k}$.

Definition 5.2. A composed semiclassical state is a set of isotropic manifold $\Lambda^k$ and section $Z$ of the bundle $\pi_{\Lambda^k}$, such that the inner product

$$< (\Lambda^k, Z), (\Lambda^k, Z) > = \int_{\Lambda^k} d\Sigma(Z(\alpha), Z(\alpha)) \mathcal{F}(L_k(\alpha))$$

converges.

Group transformation of isotropic manifold $\Lambda^k = \{ X(\alpha) \}$ is determined as

$$u_g \{ X(\alpha) \} = \{ u_g X(\alpha) \}.$$

Section $\{ Z(\alpha) \}$ is transformed as follows. Let $Z(\alpha) = [Y(\alpha)]$. Define $U_g Z(\alpha) = [U_g Y(\alpha)]$. This definition is correct because of the results of previous subsubsection, provided that

$$L_k(\alpha : u_g \Lambda^k) = U_g L_k(\alpha : \Lambda^k). \quad (5.12)$$

It is sufficient to prove property (5.12) for the case $g = G_B(t)$. One should check that eq. (5.9)

$$i \dot{B}_s^t = \mathcal{H}^+(B : u_{gB(t)} X) B_s^t + \mathcal{H}^+(B : u_{gB(t)} X) (B_s^t)^*$$

is satisfied for

$$B_s^t = \varphi u_{gB(t)} X \left[ \frac{\partial (u_{gB(t)} X)}{\partial \alpha_s} \right].$$

However, system (5.13) is a direct corollary of property F3.

Thus, the composed semiclassical states and their group transformations are introduced.

6 Conclusions

Essential properties of the semiclassical Maslov complex-WKB approximation for quantum mechanics - a bundle structure of set of semiclassical wave packets and their behavior under small variations of classical variables - are considered as a framework of an abstract semiclassical mechanics. QFT models in the weak-coupling approximation may be viewed as examples of abstract semiclassical systems.

Symmetry properties of semiclassical systems are written in infinitesimal form. Algebraic conditions (3.4), (3.8) and (3.11), (3.12) are obtained as infinitesimal analogs of semiclassical group properties
Condition (3.8) is very important since there are quantum anomalies in
QFT-models: symmetry properties may be violated in 1-loop approximation. Therefore, satisfaction
of relation (3.8) means absence of anomalies.

It is remarkable that the fact of commutativity of the left-hand side of eq.(3.8) with all operators
$\Omega_X[\delta X]$ is a corollary of eq.(3.12) and classical symmetry properties. Thus, one can expect that identity
(3.8) is violated in quantum anomaly case in such a way that its right-hand side becomes a nontrivial
$c$-number (maybe, $X$-dependent) quantity.

Sufficient conditions for constructing operators $U_g$ are presented. For the case of $X$-independent
generators $H(A : X)$, they may be viewed as an alternative for known conditions of integrability of
Lie-algebra representations.

The obtained properties F1-F3 can be explicitly checked in proof of Poincare invariance of hamilton-
nian semiclassical field theory [44].

The semiclassical Maslov theory of Lagrangian manifolds with complex germ (including WKB-
method) may be also generalized to the case of the abstract semiclassical mechanics. The composed
semiclassical states are viewed as surfaces on the semiclassical bundle. Symmetry properties remain
valid for the composed states as well.

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