Solution of the first basic physically nonlinear problem of elasticity theory for anisotropic bodies

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Abstract. The study looks upon the process of the physically non-linear deformation of transversely isotropic homogeneous continuous solid bodies made from composites where the reinforcing elements are far more rigid than the binder. The solution of the physically non-linear problems employs the simplified theory of plasticity as proposed by B.E. Pobedra. The study proposes an approach to the writing out of an explicit solution that builds on the small parameter method. Ilyushin plasticity functions (that fall within the generalized Hooke’s law) are assigned discrete small values, and the resulting equations are then decomposed into power series in small parameters. The values of such small parameters are the measures of the deflection of the non-linear vs. the linear medium. Such decomposition produces an analytical coordinate & small parameters function enabling an immediate solution of the non-linear problem using just one linear elasticity field, which, in turn, is also created exclusively in reliance on the method of boundary states. Below are the results of solutions of test problems featuring a transversely isotropic cylinder with homogeneous boundary conditions. High precision in this case is achieved as early on as the third iteration. In problems with non-trivial boundary conditions, maximal precision is achieved at the first iteration and heavily depends on small parameter values. Each of the problems presented provides a detailed convergence analysis and graphic illustrations of the results.

1. Introduction

Today’s mechanical engineering heavily employs materials with elastic properties that vary for different directions and have a non-linear elongation-contraction curve. Calculating the exact strength and hardness of solids made of such materials is a challenge that calls for brand-new or improved mechanics methods. Recent advances in the solution of mechanics problems give one a fairly realistic idea of the distribution of stresses within a solid body and enable the consideration of materials with more complex structure and rheological parameters, including anisotropic materials whose elastic properties are not fully symmetric. When testing bodies made of such materials for uniaxial elongation or pure shear, one also discovers that the resulting stress-strain curve is non-linear. This is why trying to solve such problems by using elasticity equations typically applied to linear elastic media would be inappropriate.

Physically nonlinear problems are addressed by quite an extensive body of research. Study [1] addresses a geometrically and physically nonlinear problem featuring a bending three-layer plate with a soft anisotropic filler. Study [2] builds resolving equations for a plane-deformation theory of plasticity, which are described through mathematical models where stress-strain relations take the
form of random cross-relationships between the invariants of stress and strain tensors. Study [3] offers a solution to a problem where plates come in contact with a nonlinear medium. Study [4] describes the process of elastoplastic deformation of transversely isotropic composites with cavities. Study [5] sets out the basic principles of the theory of plasticity of anisotropic materials and offers fairly new models of continuous media. Study [6] regards physical nonlinearity coupled with a material's inhomogeneity and solves a homogeneous and an inhomogeneous problem featuring a thick-wall cylinder in an axially symmetric setting. Study [7] looks for a fundamental solution to a nonlinear problem. Multiple studies search for specific methods for solving nonlinear problems. In study [8], the perturbation method is used for the solution of eigenvalue problems arising in nonlinear fracture mechanics. Study [9] compares different methods of integration of nonlinear differential equations and offers a precision analysis. Study [10] obtains, in a physically and geometrically nonlinear setting, integral representations of regular solutions to two-dimensional boundary value problems, which underpin the evolution of the boundary element method. The study solves a problem where a stress-strain state is determined in the vicinity of a crack apex, and another one addressing fatigue crack propagation. Study [11] deals with an averaging method used in physically nonlinear problems featuring layered plates in equilibrium and provides a solution for a problem where a layered plate is under dynamic bending load.

Boundary value problems in mechanics with mass forces for a transversely isotropic cylinder are reviewed in studies [12–13]. In study [15] the method of boundary states is used for the solution of problems in which anisotropic infinite bodies with irregular cross-section shapes are subjected to bending and twisting forces. A combination of the method of boundary states and the reverse elasticity theory method is used to solve thermoelasticity problems with surface and mass forces [16–17]. In study [18], the method of boundary states is used to explore the equilibrium of an anisotropic plane multiconnected domain and solve the first and second basic problems of the elasticity theory. In study [19], the method of boundary states combined with the small-value perturbations method is used to solve plane anisotropic problems of the theory of elasticity where the medium's physical properties are implicit in the solution, making it possible to promptly produce elastic fields when changing materials.

This study proposes an approach to the writing out of an approximation for a problem featuring physically nonlinear transversely isotropic finite solids of revolution where the nonlinear strain curve is little different from the linear one. The analytical form of the resulting formulae makes it possible to promptly arrive at ready solutions of non-linear problems for bodies of a given shape.

2. Problem posing
The problem explores the equilibrium of a transversally isotropic homogeneous continuous solid body with one or several coaxial revolution surfaces (figure 1). The surface is subjected to axisymmetric stresses \( p_v = \{p_r^v, p_z^v\} \) (independent of the angle \( \theta \) in a cylindrical coordinate system \( r, Q, z \)) (the first basic problem).

![Figure 1. A transversely isotropic solid of revolution.](image-url)
The material the cylinder is made of is prone to non-linear strain only in the r-direction (rather than in all directions), which is typical for fibre composites whose reinforcing fibres oriented along the z axis are far more rigid than the binder. This assumption makes it possible to address physically non-linear problems through the simplified theory of plasticity as proposed by prof. B.E. Pobedra [20]. The stress-strain state arising in the body under the surface forces is to be found.

3. Linear elasticity equations for a transversely isotropic medium
As we are dealing with an axially symmetric strain in a solid of revolution, point displacements only happen in meridian planes, and the components of the stress-strain state are independent of the angle $\theta$, which means that the stresses $\tau_{r\theta}$ & $\tau_{z\theta}$ and the strains $\gamma_{r\theta}$ & $\gamma_{z\theta}$ are zero.

Homogeneous transversely isotropic linear-elastic materials are characterized by the following equations [21].

Differential equilibrium equations:

$$\frac{\partial \tau_{r\theta}}{\partial z} + \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0;$$
$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{\tau_{r\theta}}{r} + Z = 0. \quad (3.1)$$

Cauchy formulae:

$$\varepsilon_z = \frac{\partial w}{\partial z}; \quad \varepsilon_r = \frac{\partial u}{\partial r}; \quad \varepsilon_\theta = \frac{u}{r}; \quad \gamma_{r\theta} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}; \quad (3.2)$$

Generalized Hooke's law:

$$\varepsilon_z = \frac{1}{E_z} [\sigma_z - V_z (\sigma_r + \sigma_\theta)];$$
$$\varepsilon_r = \frac{1}{E_r} (\sigma_r - V_r \sigma_\theta) - \frac{V_z}{E_z} \varepsilon_z; \quad (3.3)$$
$$\varepsilon_\theta = \frac{1}{E_r} (\sigma_\theta - V_r \sigma_r) - \frac{V_z}{E_z} \varepsilon_z; \quad \gamma_{r\theta} = \frac{1}{G_z} \tau_{r\theta}.$$

Solid boundary conditions:

$$\begin{cases} \sigma_z n_r + \tau_{r\theta} n_\theta = p_\theta^r; \\ \sigma_r n_r + \tau_{r\theta} n_\theta = p_r^r. \end{cases} \quad (3.4)$$

Here: $u$ and $w$ are the components of the displacement vector $u$ along the r-axis and z-axis, respectively; $\varepsilon_r$, $\varepsilon_\theta$, $\varepsilon_z$, and $\gamma_{r\theta}$ are the components of the strain tensor; $\sigma_r$, $\sigma_\theta$, $\sigma_z$ and $\tau_{r\theta}$ are the components of the stress tensor; $n_r$ and $n_\theta$ are the components to the normal to the boundary $S$; $E_z$ and $E_r$ are the elastic moduli for the z-axis and the isotropic plane, respectively; $V_z$ is the Poisson ratio that characterizes the compression along r under elongation along z; $V_r$ is the Poisson ratio that characterizes lateral contraction in an isotropy plane under elongation in the same plane; and $G_z$ and $G_r$ are the shear modulus in the isotropy plane and the plane perpendicular to it, respectively.

4. Mathematical model for the solution
The physically non-linear theory, like the theory of plasticity, operates such terms as stress intensity $\sigma_i$ and strain intensity $\varepsilon_i$, calculated through the components of stress and strain tensors [20, 22–23]:
Intensities of tangent stresses \( \tau_i \) and of shear strain \( \gamma_i \) (according to Huber-Mises):

\[
\tau_i = \frac{\sigma_i}{\sqrt{3}}; \quad \gamma_i = \sqrt{3} \varepsilon_i. \quad (4.3)
\]

Consider the strain process in a isotropy plane of a transversely isotropic body (where axis \( z \) is perpendicular to the isotropy plane).

The relationship between tangent stress intensity and shear strain intensity is shown in figure 2, where curve 1 represents a linear relationship and curve 2 represents a non-linear relationship.

**Figure 2.** Relationship between stress intensity and strain intensity.

In figure 2: \( G \) is the shear modulus for an isotropy plane, and \( G_c \) is the secant shear modulus for the same plane, where

\[
G_c = \frac{\tau_i}{\gamma_i}. \quad (4.5)
\]

Introduce the small parameter \( \beta \) that characterises the deviation of the secant shear modulus from the shear modulus:

\[
G_c = G(1 - \beta). \quad (4.6)
\]

Assume that the material has a nonlinear pure shear diagram described by the equation

\[
\tau_i = A \gamma_i - B \gamma_i^k, \quad (4.7)
\]

where \( A, B, \) and \( k \) are constants of the material defined by a shear test in an isotropy plane.

From (4.6) it follows that

\[
\beta = 1 - \frac{G_c}{G}. \quad (4.8)
\]

Inserting (4.5) and (4.8) in the equation (4.7), we arrive at
The relationship between tangent stress intensity and strain intensity is not stress-state dependent. From here it follows that the relationship \( \tau_i = f(\gamma_i) \) remains the same whatever stress-strain combinations we may use and can be defined by any relevant test, e.g., a pure shear test. Knowing the stress and strain values from a simple test, we can use formulae (4.1) – (4.4) to arrive at the equation \( \tau_i \sim \gamma_i \) and calculate the small parameter using formula (4.9).

Similarly, we can introduce small parameter \( \alpha \) for a plane that is perpendicular to the isotropy plane:

\[
G_z^c = G_z(1 - \alpha) ; \quad \alpha = 1 - \frac{C}{G_z} - \frac{D}{G_z} \gamma_i^{k-1} ,
\]

where \( C, B, \) and \( h \) are constants of the material defined by a shear test on the plane perpendicular to the isotropy plane, while \( G_z^c \) and \( G_z \) are the secant shear modulus and shear modulus in the same plane, respectively.

The state of the medium in the simplified theory of plasticity is governed by the generalized Hooke’s law [20]:

\[
\sigma_{rr} = (\lambda_2 + \lambda_4) \theta + \lambda_3 \epsilon_r + 2\lambda_4 (1 - \pi(p)) \frac{\epsilon_r - \epsilon_\theta}{2} ;
\]

\[
\sigma_{\theta \theta} = (\lambda_2 + \lambda_4) \theta + \lambda_3 \epsilon_r + 2\lambda_4 (1 - \pi(p)) \frac{\epsilon_\theta - \epsilon_r}{2} ;
\]

\[
\sigma_z = \lambda_3 \theta + \lambda_4 \epsilon_z ; \quad \sigma_r \theta = 2\lambda_4 (1 - \pi(p)) \epsilon_r \theta ; \quad \theta = \epsilon_r + \epsilon_\theta ;
\]

\[
\sigma_{r z} = 2\lambda_5 (1 - \chi(q)) \epsilon_{r z} ; \quad \sigma_{r \theta} = 2\lambda_5 (1 - \chi(q)) \epsilon_{r \theta} ,
\]

where \( \pi(p) \) and \( \chi(q) \) are Ilyushin plasticity functions that equal zero in an elastic zone, and \( \lambda_i \) is the parameters of the transversely isotropic medium that are linked to technical constants through the following expressions:

\[
\lambda_1 = E_z (1 - \nu) / l ; \quad \lambda_2 = E(\nu + k \nu_z^2) / [l(1 + \nu)] ;
\]

\[
\lambda_3 = E \nu_z / l ; \quad \lambda_4 = G = E / [2(1 + \nu)] ;
\]

\[
\lambda_5 = G_z ; \quad l = 1 - \nu - 2\nu_z^2 k ; \quad k = E / E_z .
\]

If we substitute the secant moduli (4.6) and (4.10) for shear moduli and discrete small values \( \beta \) and \( \alpha \) for plasticity functions \( \pi(p) \) and \( \chi(q) \), respectively, Hooke’s law (4.11) will be as follows:

\[
\sigma_r = [\lambda_2 + 2\lambda_4 (1 - \beta)] \epsilon_r + \lambda_3 \epsilon_\theta + \lambda_3 \epsilon_z ;
\]

\[
\sigma_\theta = [\lambda_2 + 2\lambda_4 (1 - \beta)] \epsilon_\theta + \lambda_3 \epsilon_r + \lambda_3 \epsilon_z ;
\]

\[
\sigma_z = \lambda_3 \theta + \lambda_4 \epsilon_z ; \quad \sigma_r \theta = 2\lambda_4 (1 - \beta) \epsilon_r \theta ;
\]

\[
\sigma_{r z} = 2\lambda_5 (1 - \alpha) \epsilon_{r z} ; \quad \sigma_{r \theta} = 2\lambda_5 (1 - \alpha) \epsilon_{r \theta} .
\]

Such an assignment makes it possible to describe the real behaviour of a physically non-linear transversely isotropic medium through constants of a certain linear medium and the small parameters \( \beta \) and \( \alpha \), whose zero values correspond to a linear elastic medium.
Introduce asymptotic series using the small parameter method (Poincaré method) \[24\]:

\[
\begin{align*}
\sum_{n=0}^{\infty} \beta^n u_i^{(n)}; & \quad \varepsilon_{ij} = \sum_{n=0}^{\infty} \beta^n \varepsilon_{ij}^{(n)}; \\
\sigma_{ij} = \sum_{n=0}^{\infty} \beta^n \sigma_{ij}^{(n)}
\end{align*}
\]

(4.13)

(where the components \( \sigma_{rz} \) & \( \sigma_{z\theta} \) of the stress tensors and \( \varepsilon_{rz} \) & \( \varepsilon_{z\theta} \) of the strain tensors involve the parameter \( \alpha \) in place of \( \beta \)).

The upper indices, which are equal to the exponents of the small parameter, serve to identify the numbers of elements in the asymptotic series.

After a replacement of the summation and postulation variables with zero values for any formally non-existent element of expansion whose indices are negative (\( n < 0 \)), Hooke’s law (4.12) leads to the following consequence:

\[
\begin{align*}
\sigma_{r}^{(n)} &= \lambda_2 \theta^{(n)} + 2 \lambda_4 \varepsilon_{r}^{(n)} + \lambda_3 \varepsilon_{z}^{(n)} + \tilde{\sigma}_{r}^{(n)}; \\
\sigma_{\theta}^{(n)} &= \lambda_2 \theta^{(n)} + 2 \lambda_4 \varepsilon_{\theta}^{(n)} + \lambda_3 \varepsilon_{z}^{(n)} + \tilde{\sigma}_{\theta}^{(n)}; \\
\sigma_{z}^{(n)} &= \lambda_4 \varepsilon_{z}^{(n)} + \tilde{\sigma}_{z}^{(n)}; \\
\sigma_{rz}^{(n)} &= 2 \lambda_4 \varepsilon_{rz}^{(n)} + \tilde{\sigma}_{rz}^{(n)}; \\
\sigma_{r\theta}^{(n)} &= 2 \lambda_4 \varepsilon_{r\theta}^{(n)} + \tilde{\sigma}_{r\theta}^{(n)}; \\
\sigma_{z\theta}^{(n)} &= 2 \lambda_4 \varepsilon_{z\theta}^{(n)} + \tilde{\sigma}_{z\theta}^{(n)};
\end{align*}
\]

(4.14)

After a change of notation (tensorial and indicial notation combined)

\[
\begin{align*}
\sigma_{ij}^{(n)} &= \sigma_{ij}^{(n)} - \tilde{\sigma}_{ij}^{(n)},
\end{align*}
\]

(4.15)

we arrive at a generalized Hooke’s law for a transversely isotropic body expressed through constants \( \lambda_i \):

\[
\begin{align*}
s_{r}^{(n)} &= \lambda_2 \theta^{(n)} + 2 \lambda_4 \varepsilon_{r}^{(n)} + \lambda_3 \varepsilon_{z}^{(n)}; \\
s_{\theta}^{(n)} &= \lambda_2 \theta^{(n)} + 2 \lambda_4 \varepsilon_{\theta}^{(n)} + \lambda_3 \varepsilon_{z}^{(n)}; \\
s_{z}^{(n)} &= \lambda_4 \varepsilon_{z}^{(n)}; \\
s_{rz}^{(n)} &= 2 \lambda_4 \varepsilon_{rz}^{(n)}; \\
s_{r\theta}^{(n)} &= 2 \lambda_4 \varepsilon_{r\theta}^{(n)}; \\
s_{z\theta}^{(n)} &= 2 \lambda_4 \varepsilon_{z\theta}^{(n)}.
\end{align*}
\]

(4.16)

The Cauchy formula is put in the same notation:

\[
\varepsilon_{ij}^{(n)} = \frac{1}{2} \left( u_{i,j}^{(n)} + u_{j,i}^{(n)} \right).
\]

(4.17)

Using \( X_i^0 \) for volume forces and assuming as known the series

\[
X_i^0 = \sum_{n=0}^{\infty} \beta^n X_i^0(n),
\]

we rewrite the equilibrium equations in the following form:

\[
\begin{align*}
\left( s_{ij,j}^{(n)} + X_{ij}^{0(n)} \right) &= 0; \\
X_{ij}^{(n)} &= X_{ij}^{0(n)} + \tilde{\sigma}_{ij,j}^{(n)}.
\end{align*}
\]

(4.18)

Equations (4.16) – (4.18) in their form correspond to the state of strain of a linear-elastic body, which makes it possible to use linear solutions at each step.

Such a decomposition is used in study \[25\], which explores the equilibrium of a transversely isotropic cube in uniaxial elongation conditions. In study \[26\], the method of boundary states was used.
at each step to solve a specific linear elastostatics problem, which generally involved approximations. The solution also required finding the elasticity field based on fictitious volume forces, which made the job even more complicated. Given that the small parameter method per se involves approximations, the resulting accumulated error may render the final result utterly inapplicable.

It is our intention now to demonstrate an approach that uses just one elastic state and, provided it is determined with a high degree of precision, avoids error accumulation with each iteration. Assume that we have a solution \( \xi^{(0)} \) of a linear first basic problem of the theory of elasticity in the absence of volume forces \( \chi_i = 0 \). This is step zero: \( n = 0 \). Set the tensor \( \tilde{\sigma}_{ij}^{(0)} \) whose components are calculated through strain from the right-hand formulae (4.14) (where the strain's upper index \( (n-1) \) is neglected). As the given boundary conditions (BCS) are fully factored in at step zero, the next step \( n = 1 \) involves no modifications in the boundary conditions. That leads to \( \tilde{\sigma}_{ij}^{(1)} = \tilde{\sigma}_{ij}^{(0)} \). We further use Hooke's law (3) to calculate the strains and the Cesàro formulae [27] to calculate the displacements. That brings us to the condition \( \xi^{(1)} = (\sigma_{ij}^{(1)}, \epsilon_{ij}^{(1)}, u_{ij}^{(1)}) \). A similar algorithm is applied to arrive at the condition \( \xi^{(n)} \) at subsequent steps, given that \( \sigma_{ij}^{(n)} = \sigma_{ij}^{(n-1)} \). The final solution is the following series (similar to equations (4.13)):

\[
\xi = \xi^{(0)} + \xi^{(1)} \beta + \xi^{(2)} \beta^2 + \ldots + \xi^{(n)} \beta^n, \quad n = 1, 2, 3, \ldots
\]  

After a sufficient number of approximations, all that's left to do is substitute the small parameter values \( \beta \) and \( \alpha \) calculate the non-linear elastic state.

5. Solving cylinder problems
The proposed methodology is tested on a relatively simple first basic problem for a circular cylinder made of the unidirectional composite boron-aluminium where boron is the reinforcing fibre and duraluminium is the binding agent [28]. After a non-dimensionization similar to the procedure presented in [29], the body occupies the region \( V = \{ (r, z) \mid 0 \leq r \leq 1, -2 \leq z \leq 2 \} \). The material's technical constants are: \( E_r = 1.3992; \quad E_z = 2.6682; \quad \nu_r = 0.0682; \quad \nu_z = 0.248; \quad G_r = 0.6549; \quad G_z = 0.5396; \quad A = 0.5; \quad B = 0.2; \quad C = 0.4; \quad D = 0.1; \quad k = 2; \quad \varepsilon_i = 0.1 \). The small parameters are: \( \beta = 0.206026; \quad \alpha = 0.240178 \).

The body is subjected to the following forces:

\[
\begin{cases}
(1,0); & r = 1, -2 \leq z \leq 2; \\
(0,-1); & z = -2, 0 \leq r \leq 1; \\
(0,1); & z = 2, 0 \leq r \leq 1.
\end{cases}
\]

There are no volume forces at work: \( X^0 = 0 \).

The solution of the linear elasticity problem is plain: \( u = 0.573 r; \quad w = 0.18889 z \); \( \sigma_r = \sigma_\theta = \sigma_z = 1; \quad \sigma_r = \sigma_\theta = \sigma_r \theta = 0 \) and matches equations (3.1) – (3.4).

The desired approximate solution (4.19) reads as follows (to get around the awkwardness of the formulae involved, we will provide the components of the displacement vector in a truncated form, specifying the formulae at \( \beta \) up to \( n = 4 \)):

\[
\begin{align*}
    u &= 0.573 r + 0.4998 r \beta + 0.436 r \beta^2 + 0.3803 r \beta^3 + 0.331 r \beta^4 + \ldots; \\
    w &= 0.1889 z - 0.1395 z \beta - 0.1217 z \beta^2 - 0.1061 z \beta^3 - 0.092 z \beta^4 - \ldots;
\end{align*}
\]  

(4.1)
\[
\sigma_r = \sigma_\theta = \sigma_z = 1; \quad \sigma_{rc} = \sigma_{r\theta} = \sigma_{z\theta} = 0.
\]

Once small parameters are substituted and the components are kept at \( n = 3 \), the strains calculated using Cauchy formulae (3.2) are as follows:

\[
\varepsilon_r = \varepsilon_\theta = 0.697819; \quad \varepsilon_z = 0.0697819; \quad \varepsilon_{rz} = \varepsilon_{r\theta} = \varepsilon_{z\theta} = 0.
\]

To estimate the error, compare the strains of the resulting state to those of the elastic state with technical constants corresponding to secant shear modules (4.6), (4.10):

\[
E_r = \frac{4\lambda_4(\beta - 1)(\lambda_3^2 - \lambda_4(\lambda_2 + \lambda_4 - \beta \lambda_4))}{\lambda_3^2 - \lambda_4 \lambda_2 + 2(\beta - 1)\lambda_4^2} = 1.12778;
\]

\[
E_z = \lambda_1 - \frac{\lambda_2^2}{\lambda_2 + \lambda_4} = 2.62834;
\]

\[
\nu_r = \frac{\lambda_3^2 - \lambda_4 \lambda_2}{\lambda_3^2 - \lambda_4(\lambda_2 - 2\lambda_4(\beta - 1))} = 0.084403;
\]

\[
\nu_z = \frac{\lambda_3}{2(\lambda_2 + \lambda_4(1 - \beta))} = 0.297818;
\]

\[
G_z = \lambda_5(1 - \alpha) = 0.42; \quad G_r = \frac{E_r}{2(1 + \nu_r)} = 0.52.
\]

In the last scenario,

\[
\varepsilon_r = \varepsilon_\theta = 0.698547; \quad \varepsilon_z = 0.153848; \quad \varepsilon_{rz} = \varepsilon_{r\theta} = \varepsilon_{z\theta} = 0.
\]

The error values for the strains are: \( \varepsilon_r \) and \( \varepsilon_\theta = 0.1\% ; \varepsilon_z = 0.13\% . \) i.e., three iterations are sufficient for attaining high precision.

For now we assume that the material’s reinforcement is small, and diagram 2 in figure 2 is materially non-linear. Assume that: \( A = 0.2 ; \quad B = 0.5 ; \quad C = 0.2 ; \quad D = 0.1 ; \quad k = 2 ; \) and \( \varepsilon_i = 0.1 . \) In this case the small parameters are: \( \beta = 0.618282 \); \( \alpha = 0.610823 \).

Comparison is now to be made with the state of a cylinder made of a material where: \( E_r = 0.58044; \quad E_z = 2.46816; \quad \nu_r = 0.16089; \quad \nu_z = 0.49798; \quad G_r = 0.25; \) and \( G_z = 0.21 . \) In this scenario the following is true:

\[
u = 1.24386r \quad w = 0.00163z.
\]

After the substitution of the small parameters in series (5.1), the shears are:

- at \( n = 3 \): \( u = 1.13861r \); \( w = 0.031z \);
- at \( n = 10 \): \( u = 1.24246r \); \( w = 0.00202z \);
- at \( n = 16 \): \( u = 1.24382r \); \( w = 0.00164z \).

i.e., greater precision is achievable by further iterations.

Consider a problem where the material of the same cylinder with non-trivial boundary conditions (figure 3) is subjected to a deformation according to law (4.7) with the constants \( A = 0.56 ; \quad B = 0.2 ; \quad C = 0.44 ; \quad D = 0.4 ; \quad k = 2 ; \) and \( \varepsilon_i = 0.1 . \) The small parameters are: \( \beta = 0.114414 \); \( \alpha = 0.110452 \).
The process for mapping elasticity fields for a linear problem using the method of boundary states is described at length in study [17].

The solution is obtained at a first approximation. To get around the awkwardness of the formulas involved, we will provide the components of the displacement vector in a truncated form, focusing on their structure:

\[
\begin{align*}
    u & = -0.02456 r - 0.01363 r^3 + \ldots + 0.02108 rz^2 - 0.015 r^3 z^2 + \ldots + \\
    w & = 0.12812 z + 0.07272 r^2 z + \ldots - 0.01373 z^3 + 0.032 r^2 z^3 + \ldots + \\
    v & = (-0.02142 r - 0.01247 r^3 + \ldots + 0.08716 rz^2 - 0.1077 r^3 z^2 + \ldots) \beta ; \\
    \end{align*}
\]

The precision of the solution is estimated based on the state of a cylinder made of a material with the following elastic constants: 

\[
\begin{align*}
    E_r &= 1.2485 ; \quad E_z = 2.6478 ; \quad \nu_r = 0.0763 ; \quad \nu_z = 0.2734 ; \quad G_r = 0.58 ; \quad \text{and} \quad G_z = 0.48 .
\end{align*}
\]

Verify through the shears on the body's boundaries and present the result in a graphical form (figure 4). The dotted line represents the precision-state shear, and the solid line reflects the approximate-state shear. The shears on the graphs are scaled, i.e., the true value \( u \) in figure 4 is equal to the graph value times \( \kappa \).
As seen from the charts, the approximate-state shear curves match those for precision-state shears, with the sole exception of the $u$ component at the boundary $S_2$.

Compare the findings for this boundary problem at $\beta = 0.206026$; $\alpha = 0.240178$ (figure 5, shown for $u$ and $w$ at the boundary $S_2$).

Figure 5 shows that the error grows as small parameters are amplified.

It should be noted that the error already occurs at the first iteration, and so a greater $n$ does not lead to better precision, as even second-power components of small parameters have no material influence on the output.

6. Output analysis

The proposed approach significantly saves computational resources as it only uses one elastic state and involves no need to find the elasticity field (either based on boundary conditions or based on fictitious
volume forces) at every step, as was the case in study [25]. It also avoids error accumulation that approximate solutions of the above problems entail.

The method's weakness is in its limited precision, as, in the case of problems with heterogeneous boundary states (computation problems), the error occurs as early on as the first iteration, and multiple iterations do nothing to reduce it in any meaningful way. Importantly, small parameters for computation problems should not exceed the order of smallness of 0.1.

In the case of problems with relatively simple boundary conditions causing homogeneous stressed states, small parameters can reach comparison values, as precision is manageable and is achieved through a direct increase in the number of approximations.

The study's findings are well applicable to engineering calculations for solids made of transversely anisotropic physically-nonlinear materials where precision requirements are looser. The equations (power series) have an analytical form and allow for a variety of analytical applications.

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