ARENS-MICHAEL ENVELOPES OF NILPOTENT LIE ALGEBRAS, HOLOMORPHIC FUNCTIONS OF EXPONENTIAL TYPE, AND HOMOLOGICAL EPIMORPHISMS

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Abstract. Our aim is to give an explicit description of the Arens-Michael envelope for the universal enveloping algebra of a finite-dimensional nilpotent complex Lie algebra. It turns out that the Arens-Michael envelope belongs to a class of completions introduced by R. Goodman in 70s. To find a precise form of this algebra we preliminary characterize the set of holomorphic functions of exponential type on a simply connected nilpotent complex Lie group. This approach leads to unexpected connections to Riemannian geometry and the theory of order and type for entire functions.

As a corollary, it is shown that the Arens-Michael envelope considered above is a homological epimorphism. So we get a positive answer to a question investigated earlier by Dosi and Pirkovskii.

1. Introduction

A research program on Arens-Michael envelopes and homological epimorphisms was initiated by Joseph Taylor in his seminal papers [Ta72A, Ta72B]. Taylor was inspired by his previous results on multi-operator holomorphic functional calculus and some consideration that can be incorporated in those circle of ideas that is called Noncommutative Geometry nowadays. Homological epimorphisms play important role in modern attempts to find generalizations of Taylor spectrum and functional calculus for noncommutative algebras (see [Do10A, Do10B]).

We pursue two aims: to get an explicit description of the Arens-Michael envelope for $U(g)$ and prove that it is a homological epimorphism, both for a finite-dimensional nilpotent complex Lie algebra $g$. (Here $U(g)$ denotes the universal enveloping algebra of $g$.)

Arens-Michael envelopes. In this text, all vector spaces and algebras are considered over the field $\mathbb{C}$ of complex numbers. All algebras and their homomorphisms are assumed to be unital.

Recall that a complete Hausdorff locally convex topological algebra with jointly continuous multiplication is called a $\hat{\otimes}$-algebra. (Here $\hat{\otimes}$ is the sign for the complete projective tensor product of locally convex spaces.) A $\hat{\otimes}$-algebra $A$ is called an Arens-Michael algebra (or a complete $m$-convex algebra) if its topology is determined by a family of submultiplicative prenorms $(\| \cdot \|_\alpha)$ (i.e., $\|ab\|_\alpha \leq \|a\|_\alpha \|b\|_\alpha$ for all $a, b \in A$).

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The Arens-Michael envelope of a \( \hat{\otimes} \)-algebra \( A \) [He93 Chap. 5] is a pair \((\hat{A}, \iota_A)\), where \( \hat{A} \) is an Arens-Michael algebra and \( \iota_A \) is a continuous homomorphism \( A \to \hat{A} \) s.t. for any Arens-Michael algebra \( B \) and for each continuous homomorphism \( \varphi: A \to B \) there exists a unique continuous homomorphism \( \hat{\varphi}: \hat{A} \to B \) making the following diagram commutative

\[
\begin{array}{ccc}
A & \xrightarrow{\iota_A} & \hat{A} \\
\downarrow{\varphi} & & \downarrow{\hat{\varphi}} \\
& & B
\end{array}
\] (1.1)

Note that it suffices to consider only homomorphisms with values in Banach algebras. The Arens-Michael envelope always exists and is unique up to a topological isomorphism. The algebra \( \hat{A} \) is the completion of \( A \) w.r.t. the family of all continuous submultiplicative prenorms. (An arbitrary associative \( \mathbb{C} \)-algebra can be considered as a \( \hat{\otimes} \)-algebra w.r.t. the strongest locally convex topology; in this case, all prenorms are automatically continuous.)

To formulate our first main result recall some terminology and notation from [Go78, Go79, Pi06A]. Consider a finite-dimensional nilpotent complex Lie algebra \( \mathfrak{g} \) and fix a positive decreasing filtration \( \mathcal{F} \) on \( \mathfrak{g} \), i.e., a decreasing chain of subspaces

\[
\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_k \supset \mathfrak{g}_{k+1} = 0, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.
\] (1.2)

Consider a basis \((e_1, \ldots, e_m)\) in \( \mathfrak{g} \) and set

\[
w_i := \max\{j : e_i \in \mathfrak{g}_j\} \quad \text{and} \quad w(\alpha) := \sum_i w_i \alpha_i,
\] (1.3)

where \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_+^m \). In the following we assume that \((e_i)\) is an \( \mathcal{F} \)-basis, i.e., \( w_i \leq w_{i+1} \) for all \( i \), and \( \mathfrak{g}_j = \text{span}\{e_i : w_i \geq j\} \) for all \( j \). A sequence \( \mathcal{M} = \{M_\alpha : \alpha \in \mathbb{Z}_+^m\} \) of positive numbers is called an \( \mathcal{F} \)-weight sequence if \( M_0 = 1 \) and \( M_\gamma \leq M_\alpha M_\beta \) whenever \( w(\gamma) \geq w(\alpha) + w(\beta) \). Given an \( \mathcal{F} \)-weight sequence \( \mathcal{M} \), consider the space

\[
U(\mathfrak{g}, \mathcal{M}) := \left\{ x = \sum_{\alpha \in \mathbb{Z}_+^m} c_\alpha e^\alpha \in [U] : ||x||_r := \sum_\alpha |c_\alpha| \alpha! M_\alpha^r w(\alpha) < \infty \forall r > 0 \right\},
\] (1.4)

where \((e^\alpha : \alpha \in \mathbb{Z}_+^m)\) is the PBW-basis in \( U(\mathfrak{g}) \) associated with \((e_i)\), and \([U]\) is the set of formal power series w.r.t. \((e^\alpha)\). It is proved in [Go78 Th. 6.3] that the multiplication on \( U(\mathfrak{g}) \) extends to \( U(\mathfrak{g}, \mathcal{M}) \) and \( U(\mathfrak{g}, \mathcal{M}) \) is a \( \hat{\otimes} \)-algebra.

The standard choice for \( \mathcal{F} \) is the lower central series of \( \mathfrak{g} \) that is defined inductively by \( \mathfrak{g}_1 := [\mathfrak{g}, \mathfrak{g}_{-1}] \). Consider the \( \mathcal{F} \)-weight sequence \( \mathcal{M}_1 \) defined by \( M_\alpha := w(\alpha)^{-w(\alpha)} \). (This is \( \mathcal{M}_p \) as defined in [Go79 Sec. 2, Exm. 1] with \( p = 1 \).)

**Theorem 1.1.** Let \( \mathfrak{g} \) be a finite-dimensional nilpotent complex Lie algebra, \( \mathcal{F} \) the lower central series, and \((e_i)\) an \( \mathcal{F} \)-basis. Then the topology on the \( \hat{\otimes} \)-algebra

\[
U(\mathfrak{g}, \mathcal{M}_1) = \left\{ x = \sum_{\alpha \in \mathbb{Z}_+^m} c_\alpha e^\alpha \in [U] : ||x||_r = \sum_\alpha |c_\alpha| \alpha! w(\alpha)^{-w(\alpha)} r^{w(\alpha)} < \infty \forall r > 0 \right\}
\]
can be determined by system of submultiplicative prenorms, i.e., it is an Arens-Michael algebra. Moreover, the natural homomorphism \( U(\mathfrak{g}) \to U(\mathfrak{g})_{\mathfrak{h}_1} \) is an Arens-Michael envelope.

The first step of the proof of Theorem 1.1 is reduction \( \hat{U}(\mathfrak{g}) \) to the Arens-Michael envelope of the algebra \( \mathcal{A}(G) \) of analytic functionals on the corresponding simply connected complex Lie group \( G \) (Proposition 2.1). Further, we use the identification (obtained by Akbarov in [Ak08]) between the strong dual space of \( \hat{\mathcal{A}}(G) \) and the locally convex space \( \mathcal{O}_{exp}(G) \) of holomorphic functions of exponential type on \( G \).

The key technical result is Theorem 3.1, which gives estimations for growth rate of a word length function. It is not particularly original: an explicit formulation and a part of a proof can be found in [DER03, II.4.17], where the statement is given for a right invariant Riemannian distance. The reasoning is essentially contained in the proofs of [VSC92, Pr. IV.5.6 and IV.5.7] but in a latent form. Moreover, the main goal of [ibid.] is to rate volume growth; so the reader needs some additional work to extract an argument for distances. Very close results are contained in [Ka94, Th. 4.2] and [Be96, Pr. 7.25] but in variations that are not completely satisfactory for our purposes. Besides, a careful examination shows that estimates for Riemannian distances are based on estimates for length functions; thus there is a direct way to establish Theorem 3.1 which passes Riemannian geometry. So I include a complete proof in Appendix, where a connection with Riemannian distances is also explained.

From Theorem 3.1 we obtain an explicit description of \( \mathcal{O}_{exp}(G) \) (Theorem 3.2) and show that the functions of exponential type forms exactly the dual space of \( U(\mathfrak{g})_{\mathfrak{h}_1} \), which implies our assertion. (Theorem 3.2, which plays only supporting role in our argument, is of independent interest itself. This result is an essential part of the description of the space of holomorphic functions of exponential type on an arbitrary connected complex Lie group — the subject that is discussed in [Ar19].)

**Homological epimorphisms.** Let \( A \) be a \( \hat{\otimes} \)-algebra. Recall that an \( A \hat{\otimes} \)-bimodule is a complete Hausdorff locally convex space endowed with a structure of a unital \( A \)-bimodule s.t. both left and right multiplications are jointly continuous. Below \( \hat{\otimes}_A \) denotes the projective tensor product of \( A \)-modules.

A homomorphism of \( \hat{\otimes} \)-algebras \( A \to B \) is called a homological epimorphism if the induced functor between the bounded derived categories of \( \hat{\otimes} \)-modules is fully faithful. This condition is equivalent to the following: for some (or what is the same, for each) admissible projective resolution \( 0 \leftarrow A \leftarrow L_\bullet \) in the relative category of \( A \)-\( \hat{\otimes} \)-bimodules the complex

\[
0 \leftarrow B \hat{\otimes}_A B \leftarrow B \hat{\otimes}_A L_0 \hat{\otimes}_A B \leftarrow \cdots \leftarrow B \hat{\otimes}_A L_n \hat{\otimes}_A B \leftarrow \cdots
\]

is admissible [Pi08, Rem. 6.4]. We follow the terminology from [ibid.]; the alternative terminologies: \( A \to B \) is a localization or \( B \) is stably flat over \( A \) are used in [Pi06A]. Definitions of homological notions for \( \hat{\otimes} \)-algebras can be found in [He89, He93, He00]. We do not need details here because the only necessary fact on homological epimorphisms is Theorem 1.3 below.

Taylor [Ta72B] proved that the Arens-Michael envelope is a homological epimorphism for the polynomial algebra in \( n \) generators and for the free algebra in \( n \) generators. The natural next step is to consider the universal enveloping algebra...
U(\mathfrak{g}) of a finite-dimensional complex Lie algebra \mathfrak{g}. The first and a bit disappointing result in this direction, also due to Taylor [La72B], asserts that if \mathfrak{g} is semisimple, then the Arens-Michael envelope \iota_U : U(\mathfrak{g}) \to \hat{U}(\mathfrak{g}) fails to be a homological epimorphism, in contrast to the abelian case. Many years later the results for solvable \mathfrak{g} began to appear. Dosiev [Do03, Th. 10] proved that \iota_U is a homological epimorphism provided \mathfrak{g} is metabelian (i.e., [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0). Later, Pirkovskii generalized this result to positively graded Lie algebras [Pi06A, Th. 6.19] and Dosiev generalized it to nilpotent Lie algebras satisfying a condition of ”normal growth” [Do09]. A natural conjecture became that the same is true for each nilpotent Lie algebra \mathfrak{g}. On the other hand, it was shown in [Pi06B] that \iota_U can be a homological epimorphism only when \mathfrak{g} is solvable.

Another approach was introduced Pirkovskii in [Pi08]. An Ore extension iteration gives sometimes a direct construction of \hat{U}(\mathfrak{g}) and a method to prove that \iota_U is a homological epimorphism. This method requires some technical work; nonetheless, it gives the only known up to date example of a solvable non-nilpotent \mathfrak{g} s.t. \iota_U is a homological epimorphism, namely, the two-dimensional solvable non-abelian Lie algebra.

Now we formulate our second main result.

**Theorem 1.2.** Let \mathfrak{g} be a finite-dimensional nilpotent complex Lie algebra. Then the Arens-Michael envelope \iota_U : U(\mathfrak{g}) \to \hat{U}(\mathfrak{g}) is a homological epimorphism.

The proof is based on Theorem 1.1 and the following Pirkovskii’s theorem. (The definition of an entire \mathcal{F}-weight sequence see below in (1.1).)

**Theorem 1.3.** [Pi06A, Th. 7.3] Let \mathfrak{g} be a finite dimensional nilpotent complex Lie algebra, and let \mathcal{H} be an entire \mathcal{F}-weight sequence for some positive filtration \mathcal{F}. Then the natural homomorphism U(\mathfrak{g}) \to U(\mathfrak{g})_{\mathcal{H}} is a homological epimorphism.

The present paper is organized as follows. In section 2, some preliminary results on the algebra of analytic functionals, submultiplicative weights, and holomorphic functions of exponential type are collected. In section 3, a result on growth of word length functions (Theorem 3.1) is formulated and applied to characterize holomorphic functions of exponential type on a simply connected nilpotent Lie group (Theorem 3.2). The proofs of Theorems 1.1, 1.2 and some application are contained in section 4. Appendix includes the proof of Theorem 3.1 and the explanation how this assertion is connected with growth rate of invariant Riemannian distances.

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## 2. Preliminary results

**Reduction to the algebra of analytic functionals.** Let G be a complex Lie group with Lie algebra \mathfrak{g}. Consider the Fréchet algebra \mathcal{O}(G) of holomorphic functions on G and its strong dual space \mathcal{O}(G)' endowed with the convolution multiplication. This \mathcal{S}\mathcal{O}-algebra is denoted by \mathcal{A}(G) and is called the *algebra of analytic functionals* on G (for a general complex Lie group, \mathcal{A}(G) is introduced by G. Litvinov in [Li09]).

Consider elements of U(\mathfrak{g}) as left-invariant differential operators on \mathcal{O}(G) and define the homomorphism \tau by

\[ \tau : U(\mathfrak{g}) \to \mathcal{A}(G) : \langle \tau(X), f \rangle := \langle \delta_e, Xf \rangle \quad (X \in U(\mathfrak{g}), f \in \mathcal{O}(G)), \] (2.1)
where $\delta_e$ is denoted the delta-function at $e$.

Recall that the Arens-Michael envelope is a functor from the category of $\hat{\otimes}$-algebras to the category of Arens-Michael algebras. For any $\hat{\otimes}$-algebra homomorphism $\theta$ we denote the corresponding homomorphism between the Arens-Michael envelopes by $\hat{\theta}$.

**Proposition 2.1.** Let $G$ be a simply connected complex Lie group with Lie algebra $g$. Then $\hat{\tau}: \hat{U}(g) \to \hat{\mathcal{A}}(G)$ is an Arens-Michael algebra isomorphism.

**Remark 2.2.** The similar result is valid for real Lie groups. If $G$ is a simply connected real Lie group and $g_C$ denotes the complexification of its Lie algebra, then $U(g_C)$ and the algebra $\mathcal{E}'(G)$ of compactly supported distribution have the same Arens-Michael envelope [Ta72B, p. 250].

To prove Proposition 2.1 we need the following lemma.

**Lemma 2.3.** Let $\theta: A \to B$ be an epimorphism of $\hat{\otimes}$-algebras. Suppose that there exists a continuous homomorphism $j: B \to \hat{A}$ s.t. $\iota_A = j \theta$. Then $\hat{\theta}: \hat{A} \to \hat{B}$ is an Arens-Michael algebra isomorphism.

**Proof.** The homomorphisms $\iota_A$ and $\iota_B$ have dense ranges; so they are epimorphisms. Since, by assumption, $\theta$ is an epimorphism, it follows from $\hat{\theta} \iota_A = j \theta \iota_B$ that $\hat{\theta}$ also is an epimorphism.

By the universal property of the Arens-Michael envelope, there is a continuous homomorphism $\alpha: \hat{B} \to \hat{A}$ s.t. $j = \alpha \theta$. Therefore, $\hat{\alpha} \hat{\theta} = \hat{\iota}_B \theta = \iota_B \theta = \iota_A$.

Since $\iota_A$ is an epimorphism, we have $\alpha \hat{\theta} = 1$. Therefore, $\hat{\theta} \alpha \hat{\theta} \hat{\theta} = \hat{\theta}$. But $\hat{\theta}$ is an epimorphism so $\hat{\theta} \alpha = 1$. Thus $\hat{\theta}$ is invertible. □

**Proof of Proposition 2.1** Since $G$ is simply connected, it follows from [Pi06A, Pr. 9.1] that there exists a unique continuous homomorphism $j: \mathcal{A}(G) \to \hat{U}(g)$ s.t. $\varphi(g) = j \tau$. On the other hand, $\tau$ has dense range provided $G$ is connected (because the dual map is injective; see, e.g., discussion after formula (42) in [ibid.]). So we can apply Lemma 2.3. □

**Submultiplicative weights and length functions.**

**Definition 2.4.** (A) A **submultiplicative weight** on a locally compact group $G$ is a non-negative locally bounded function $\omega: G \to \mathbb{R}$ s.t.

$$\omega(gh) \leq \omega(g) \omega(h) \quad (g, h \in G).$$

(B) A **length function** on a locally compact group $G$ is a locally bounded function $\ell: G \to \mathbb{R}$ s.t.

$$\ell(gh) \leq \ell(g) + \ell(h) \quad (g, h \in G).$$

It is not hard to check that a strictly positive submultiplicative weight maps $G$ to $[1, +\infty)$, and a length function maps $G$ to $[0, +\infty)$.

**Remark 2.5.** (A) We accept the terminology from [Wi13]. As a rule, $\ell$ (or $\omega$) is assumed to be symmetric, i.e., $\ell(e) = 0$ and $\ell(g^{-1}) = \ell(g)$ (or $\omega(e) = 1$ and $\omega(g^{-1}) = \omega(g)$); see Appendix. The short term 'weight' (see, e.g., [Sc93, Da00]) is usual but it has too many different senses. Akbarov in [Ak08] employs 'semicharacter' for 'submultiplicative weight’. L. Schweitzer in [Sc03] prefers 'gauge' for 'length.
function’. The words ‘seminorm’ for ‘submultiplicative weight’ [Wa72 Sect. 4.2.2] and ‘modulus’ for ‘length function’ [DER03] can be also used in the Lie group context.

(B) We assume that \( \omega \) or \( \ell \) is locally bounded because we follow [AK08] principally. The more common admissions that \( \omega \) or \( \ell \) is measurable (w.r.t. the Haar measure) or Borel are stronger. Indeed, in these cases, submultiplicativity or subadditivity implies that \( \omega \) or \( \ell \), resp., is bounded on compact sets and, hence, is locally bounded (see [Dz86 Pr. 2.1] and [Sc93 Th. 1.2.11]).

(C) The map \( \ell \mapsto (\omega(g) := e^{\ell(g)}) \) is a bijection between the set of length functions and the set of submultiplicative weights.

We use the following notation. For a complex manifold \( M \) and a locally bounded function \( v: M \to [1, +\infty) \) denote by \( V_v \) the closed absolutely convex hull of
\[
\{v(x)^{-1}\delta_x : x \in M\}
\]
in \( \mathcal{A}(M) := O(M)' \). It is noted in [AK08 Sect. 3.4.3] that \( V_v \) is a neighbourhood of 0 in \( \mathcal{A}(M) \); so \( V_v \) is an absorbent set. Therefore its Minkowski functional is well defined on \( \mathcal{A}(M) \); we denote it by \( \| \cdot \|_{V_v} \). Let \( \mathcal{A}_v(M) \) be the completion of \( \mathcal{A}(M) \) w.r.t. \( \| \cdot \|_{V_v} \). Also, denote by \( \mathcal{A}_v(M) \) the completion of \( \mathcal{A}(M) \) w.r.t. the sequence of prenorms \( (\| \cdot \|_{V_v^n} : n \in \mathbb{N}) \), where \( v^n(x) := v(x)^n \).

For any submultiplicative weight \( \omega \) on a complex Lie group \( G \) the prenorm \( \| \cdot \|_{\omega} \) is submultiplicative and continuous on \( \mathcal{A}(G) \) [ibid., Lem. 5.1(a)]. (Note that main results in [ibid.] is formulated for Stein groups but their proofs work for all complex Lie groups.) Thus \( \mathcal{A}_v(G) \) is a unital Banach algebra, and the natural map \( \mathcal{A}(G) \to \mathcal{A}_v(G) \) is a continuous homomorphism. Obviously, the maximum of two submultiplicative weights is a submultiplicative weight, so we have a directed projective system of unital Banach algebras

\[
(\mathcal{A}_\omega(G) : \omega \text{ is a submultiplicative weight on } G)
\]
with natural connecting homomorphisms. Note that this system is not empty because the trivial character \( g \mapsto 1 \) is a submultiplicative weight. The following result is a reformulation of [ibid., Th. 5.2(a)].

**Theorem 2.6.** If \( G \) is a complex Lie group, then the continuous homomorphism
\[
\mathcal{A}(G) \to \lim_{\omega} \mathcal{A}_\omega(G)
\]
is an Arens-Michael envelope.

Let \( U \) be a generating set for \( G \), i.e., \( e \in U \) and \( \bigcup_{n=0}^{\infty} U^n = G \), where \( U^0 := \{e\} \). Recall that a locally compact group \( G \) is called *compactly generated* if there is a relatively compact generating set \( U \). For given \( U \), we define a function \( \ell_U \) on \( G \) by
\[
\ell_U(g) := \min\{n : g \in U^n\}. \tag{2.2}
\]
It is easy to see that \( \ell_U \) is a length function, it is called a *word length function* (cf., e.g., [Sc93 Exm. 1.1.7]).

For given non-negative functions \( \tau_1 \) and \( \tau_2 \) on a set \( X \) we say that \( \tau_1 \) dominated by \( \tau_2 \) (at infinity) if there are \( C, D > 0 \) s.t.
\[
\tau_1(x) \leq C\tau_2(x) + D \quad (x \in X).
\]
Non-negative functions \( \tau_1 \) and \( \tau_2 \) on a set \( X \) are said to be *equivalent* (at infinity) if \( \tau_1 \) dominated by \( \tau_2 \) and \( \tau_2 \) dominated by \( \tau_1 \).
Proposition 2.7. (cf. [Ak08 Th. 5.3]) Let $G$ be a compactly generated complex Lie group, and let $U$ be a relatively compact generating set. Put

$$\xi(g) := e^{\ell_U(g)},$$

where $\ell_U$ is the word length function defined by (2.2). Then the sequence of submultiplicative prenorms $$(\| \cdot \|_{\xi_n}; n \in \mathbb{N})$$ determines the topology on $\overline{A}(G)$, i.e.,

$$\overline{A}(G) = \overline{A}_{\xi}(G).$$

Proof. It is easy to show (see [Sc93 Th. 1.1.21] or [Ak08 Th. 5.3]) that every length function on $G$ is dominated by $\ell_U$. Therefore for each submultiplicative weight $\omega$ on $G$ there are $C > 0$ and $n \in \mathbb{N}$ s.t. $\| \cdot \|_\omega \leq C \| \cdot \|_{\xi_n}$. □

Holomorphic functions of exponential type. For a complex Lie group $G$, denote by $O_{\exp}(G)$ the linear subspace of $O(G)$ that contains every function $f$ s.t. there is a submultiplicative weight $\omega$ satisfying $|f(g)| \leq \omega(g)$ for all $g \in G$. A holomorphic function $f$ on $G$ is of exponential type, if $f \in O_{\exp}(G)$ [Ak08 Sect. 5.3.1].

To define the topology on $O_{\exp}(G)$ we use the following notation. For a complex manifold $M$ and a locally bounded function $\nu : M \to [1, +\infty)$ denote by $O_{\nu}(M)$ the linear subspace of $O(M)$ defined by

$$O_{\nu}(M) := \left\{ f \in O(M) : |f|_\nu := \sup_{x \in M} \nu(x)^{-1}|f(x)| < \infty \right\}. \quad (2.3)$$

It is easy to see that $O_{\nu}(M)$ is a Banach space w.r.t $| \cdot |_\nu$. Put $O_{\nu,\infty}(M) := \bigcup_{n \in \mathbb{N}} O_{\nu^n}(M)$. We consider $O_{\nu,\infty}(M)$ with the inductive limit topology.

For a complex Lie group $G$, we have an inductive system of Banach spaces

$$(O_{\omega}(G) : \omega \text{ is a submultiplicative weight on } G)$$

with natural connecting homomorphisms. Note that $O_{\exp}(G) = \bigcup_\omega O_\omega(G)$. So we can consider $O_{\exp}(G)$ as a locally convex space via identification

$$O_{\exp}(G) = \lim_{\omega} O_\omega(G).$$

Proposition 2.8. (cf. [Ak08 Th. 5.3]) Let $G$ be a compactly generated complex Lie group, $U$ a relatively compact generating set, and $\xi$ defined as in Proposition 2.7. Then

$$O_{\exp}(G) = O_{\xi}(G)$$

as locally convex spaces.

Proof. Note again that each length function on $G$ is dominated by $\ell_U$. □

Remark 2.9. In [Ak08], Akbarov introduces $O_{\exp}(G)$ as an inductive limit of the system $(O_\omega(G))$, where each $O_\omega(G)$ is endowed with the topology of a Smith space, and includes his results in more general context. In particular, if $E$ is a stereotype locally convex space, he considers the dual space $E^\ast$ endowed the topology of uniform convergence on totally bounded subsets. Since each relatively compact subset of a complete metric space is totally bounded and each bounded subset of a nuclear space is relatively compact [SM99 § III.7.2, Cor. 2], we have $E^\ast = E'$ for every nuclear Fréchet space $E$. So far as $O_{\exp}(G)$ is dual to the nuclear Fréchet space $\overline{A}(G)$ (see Proposition 2.12 below), our and Akbarov’s approaches to topology are
equivalent. In addition, note that there is an alternative proof of Proposition 2.12 which based on [Ak08, Th.1.11].

**Lemma 2.10.** Let $M$ be a complex manifold. For given locally bounded function $v: M \to [1, +\infty)$, the pairing between $\mathcal{A}(M)$ and $O(M)$ induces the pairing that makes $O_v(M)$ the dual Banach space to $\mathcal{A}_v(M)$.

**Proof.** Denote by $\langle \cdot, \cdot \rangle$ the pairing between $\mathcal{A}(M)$ and $O(M)$. Evidently,

$$S_v := \{ f \in O(M) : |f(x)| \leq v(x) \forall x \in M \}$$

is the unit ball in $O_v(M)$. By [Ak08, Lem. 3.1], the set $S_v$ is the polar of $V_v$ (the closed absolutely convex hull of $\{v(x)^{-1} \delta_x : x \in M \}$) in $\mathcal{A}(M)$. Therefore,

$$|\langle \mu, f \rangle| \leq \|\mu\|_v |f|_v \quad (\mu \in \mathcal{A}_v(M), f \in O_v(M)).$$

So we have a bounded linear operator $\alpha: O_v(M) \to \mathcal{A}_v(M)'$.

On the other hand, suppose that $h \in \mathcal{A}_v(M)'$. Let $\rho_v: \mathcal{A}(M) \to \mathcal{A}_v(M)$ be the completion map. Then $h \rho_v \in \mathcal{A}(M)'$ and the corresponding function in $O(M)$ is determined by $x \mapsto h \rho_v(\delta_x)$. Denote this function by $f(x)$. Since $h$ is bounded, there is $C > 0$ s.t. $|f(x)| \leq C\|\delta_x\|_v \leq C v(x)$ for all $x \in M$; i.e., $f \in O_v(M)$ and $|f|_v \leq C$. So we have a bounded linear operator $\mathcal{A}_v(M)' \to O_v(M)$, which we denote by $\beta$.

It is obvious that $\beta \alpha = 1$. By the definition of $\|\cdot\|_v$, the linear span of $\{\delta_x : x \in M\}$ is dense in $\mathcal{A}_v(M)$, so the operator $\beta$ is injective. Therefore we have a topological isomorphism.

**Lemma 2.11.** Let $M$ be a complex manifold. For given locally bounded function $v: M \to [1, +\infty)$ and any $n \in \mathbb{N}$ consider the pairing $\langle \cdot, \cdot \rangle_{vn}$ between $\mathcal{A}(M)_{vn}$ and $O(M)_{vn}$ from Lemma 2.7. If the Fréchet space $\mathcal{A}_{vn}(M)$ is reflexive, then there is a pairing between $\mathcal{A}_{vn}(M)$ and $O_{vn}(M)$ that is compatible with all $\langle \cdot, \cdot \rangle_{vn}$ and making $\mathcal{A}(M)_{vn}$ and $O(M)_{vn}$ strong dual spaces to each other.

**Proof.** Every reflexive Fréchet space is distinguished, i.e., the strong dual is barreled [Ko83, §23.7]. For every representation $E = \lim E_n$ of a distinguished Fréchet space $E$ as a reduced (each $E \to E_n$ has dense range) projective limit of a sequence of Banach spaces, its strong dual space $E'$ equals $\lim E_n'$ (see, e.g., [BD92], Introduction). So Lemma 2.10 implies that $\mathcal{A}_{vn}(G)' \cong O_{vn}(G)$. Finally, reflexivity implies $O_{vn}(G)' \cong \mathcal{A}_{vn}(G)$. \(\square\)

**Proposition 2.12.** Let $G$ be a compactly generated complex Lie group. Then there is a pairing between $\widehat{\mathcal{A}}(G)$ and $O_{\exp}(G)$ that is compatible with all $\langle \cdot, \cdot \rangle_\omega$, where $\omega$ is a submultiplicative weight, and making $\mathcal{A}(G)$ and $O_{\exp}(G)$ strong dual spaces to each other.

**Proof.** It follows from Propositions 2.7 and 2.8 that

$$\widehat{\mathcal{A}}(G) \cong \mathcal{A}_\omega(G) \quad \text{and} \quad O_{\exp}(G) \cong O_\omega(G).$$

By [Ak08, Ths. 5.10, 6.2], $\widehat{\mathcal{A}}(G)$ is nuclear. Therefore it is reflexive and we can apply Lemma 2.11. \(\square\)

Denote by $\mathcal{P}(G)$ the algebra of polynomial functions on a simply connected complex Lie group $G$ w.r.t. the canonical coordinates of the first kind (i.e., via the identification $\exp: g \to G$ and consider the natural paring

$$\langle X, f \rangle := X f(e) \quad (X \in U(g), f \in \mathcal{P}(G)). \quad (2.4)$$
Corollary 2.13. Let \( G \) be a simply connected complex Lie group with Lie algebra \( \mathfrak{g} \). Then the locally convex spaces \( \hat{U}(\mathfrak{g}) \) and \( \mathcal{O}_{\text{exp}}(G) \) are strong dual spaces to each other w.r.t. the pairing extending the pairing between \( U(\mathfrak{g}) \) and \( \mathcal{P}(G) \).

Proof. Obviously \((X, f) = (\delta_x, X f)\), so the homomorphism \( \tau : U(\mathfrak{g}) \to \mathcal{O}(G) \) defined in \( \text{(3.1)} \) induces a pairing between \( U(\mathfrak{g}) \) and \( \mathcal{O}(G) \) that is continuously extended to \( \hat{U}(\mathfrak{g}) \). Finally, note that each simply connected complex Lie group is compactly generated \([HR79, \text{Th. 7.4}]\) and apply Propositions 2.1 and 2.12. \( \square \)

3. Growth of word length functions

Let \( \mathfrak{g} \) be a nilpotent complex or real Lie algebra and \( \mathcal{F} \) the lower central series of \( \mathfrak{g} \), which is defined by \( \mathfrak{g}_i := [\mathfrak{g}, \mathfrak{g}_{i-1}] \). Fix an \( \mathcal{F} \)-basis \( e_1, \ldots, e_m \) in \( \mathfrak{g} \). For the simply connected Lie group \( G \) associated with \( \mathfrak{g} \) consider the canonical coordinates of the first kind \((t_1, \ldots, t_m)\) and the canonical coordinates of the second kind \((\tilde{t}_1, \ldots, \tilde{t}_m)\) determined by \((e_i)\), i.e.,

\[
g = \exp\left(\sum_{i=1}^{m} t_i e_i\right) = \prod_{i=1}^{m} \exp(\tilde{t}_i e_i) \quad (g \in G).
\]

Remind the definition of \( w_i \) from \([\text{L83}]\) and set

\[
\sigma(g) := \max_i |t_i|^{1/w_i}, \quad \tilde{\sigma}(g) := \max_i |\tilde{t}_i|^{1/w_i}. \tag{3.1}
\]

In \([Go79] \) and \([Pi06A]\) \( \sigma \) is denoted by \(| \cdot | \) and is called 'homogeneous norm'.

The following theorem is the heart of our argument. See the proof in Appendix.

Theorem 3.1. Let \( G \) be a simply connected nilpotent (complex or real) Lie group, and let \( \ell \) be a word length function corresponding to a relatively compact generating set. Then \( \ell, \sigma \) and \( \tilde{\sigma} \) are equivalent (at infinity).

Now we can get the following explicit description of \( \mathcal{O}_{\text{exp}}(G) \).

Theorem 3.2. Let \( G \) be a simply connected nilpotent complex Lie group with Lie algebra \( \mathfrak{g} \), and let \((t_1, \ldots, t_m)\) be the canonical coordinates of the first kind associated with an \( \mathcal{F} \)-basis in \( \mathfrak{g} \), where \( \mathcal{F} \) is the lower central series. Then

\[
\mathcal{O}_{\text{exp}}(G) = \{ f \in \mathcal{O}(G) : \exists C > 0, \exists r \in \mathbb{R}_+ \text{ s.t. } |f(t_1, \ldots, t_m)| \leq Ce^{r \max_i |t_i|^{1/w_i}} \forall t_1, \ldots, t_m \}
\]

and we have

\[
\mathcal{O}_{\text{exp}}(G) \cong \varinjlim_{r \to \mathbb{R}_+} \mathcal{O}_{\eta^r}(G)
\]

as locally convex spaces, where \( \eta(t_1, \ldots, t_m) := e^{r \max_i |t_i|^{1/w_i}} \) and the Banach space \( \mathcal{O}_{\eta^r}(G) \) is defined as in \( \text{(2.3)} \).

Moreover, we can replace \((t_1, \ldots, t_m)\) by the canonical coordinates of the second kind.

We need the following lemma.

Lemma 3.3. Let \( \tau_1 \) and \( \tau_2 \) be non-negative locally bounded functions on a complex manifold \( M \). If \( \tau_1 \) and \( \tau_2 \) are equivalent and \( v_i(x) := e^{\tau_i(x)} \) \((i = 1, 2)\), then \( \mathcal{O}_{v_1}(M) = \mathcal{O}_{v_2}(M) \) as subset of \( \mathcal{O}(M) \) and as a locally convex space.
Proof. For each \( p \in \mathbb{N} \) there is \( q \in \mathbb{N} \) s.t. \( \mathcal{O}_{\psi}^q(M) \subset \mathcal{O}_{\psi}^p(M) \) and this inclusion is a continuous linear map. Therefore we have a continuous linear map \( \mathcal{O}_{\psi}(M) \rightarrow \mathcal{O}_{\psi}^\infty(M) \) of inductive limits. Similarly, we get a continuous linear map in the reverse direction. Evidently, these maps are inverse to each other and this completes the proof. \( \square \)

Proof of Theorem 3.2. Note that \( G \) is compactly generated and fix a relatively compact generating set \( U \). Proposition 2.8 implies that \( \mathcal{O}_{\exp}(G) = \mathcal{O}_{\xi}(G) \), where \( \xi(g) = e^{\ell_U(g)} \). Evidently, \( \eta^r(g) := e^{r\xi(g)} \ (r \in \mathbb{R}^+) \). It follows from Theorem 3.1 that \( \sigma \) is equivalent to \( \ell_U \); therefore \( \mathcal{O}_{\xi}(M) = \mathcal{O}_{\eta^\infty}(M) \) by Lemma 3.3. Obviously,
\[
\lim_{r \in \mathbb{R}^+} \mathcal{O}_{\eta^r}(G) = \lim_{n \in \mathbb{N}} \mathcal{O}_{\eta^n}(G),
\]
so the statement for \( \sigma \) is proved.

Exactly the same argument is applied to \( \tilde{\sigma} \). \( \square \)

Remark 3.4. In terms of the classical entire functions theory, Theorem 3.2 says, in particular, that \( \mathcal{O}_{\exp}(G) \) consists of entire functions in \( t_1, \ldots, t_m \) that are at most order \( 1/w_i \) and finite type w.r.t. \( t_i \) for all \( i = 1, \ldots, m \).

Example 3.5. [Ak08 Sect. 5.4.2] Let \( \mathfrak{g} \) be an abelian Lie algebra with basis \( e_1, \ldots, e_m \). Then \( w_1 = \cdots = w_m = 1 \) and \( \sigma(g) = \max\{|t_1|, \ldots, |t_m|\} \). So \( \mathcal{O}_{\exp}(G) \) coincides with the set of entire functions of exponential type (=of at most order 1 and finite type) on \( \mathbb{C}^m \) as it is defined in the classical theory of several complex variables [LG86 Def. 1.8]. This example justifies the terminology.

Example 3.6. Consider the 3-dimensional complex Heisenberg Lie algebra \( \mathfrak{g} \) with basis \( e_1, e_2, e_3 \) and relation \( [e_1, e_2] = e_3 \). The lower central series has the form
\[
\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 = \text{span}\{e_3\} \supset \mathfrak{g}_3 = 0.
\]
Then \( w_1 = w_2 = 1, w_3 = 2, \) and \( \sigma(g) = \max\{|t_1|, |t_2|, |t_3|^{1/2}\} \).

Example 3.7. Let \( \mathfrak{g} \) be the 7-dimensional Lie algebra with basis \( e_1, \ldots, e_7 \) and commutation relations
\[
[e_1, e_i] = e_{i+1} \ (i = 2, \ldots, 6),
\]
\[
[e_2, e_3] = -e_6, [e_3, e_4] = e_7, [e_2, e_4] = [e_2, e_5] = -e_7,
\]
the undefined brackets being zero (see [Fa72] or [GK96 Ch. 2, Exm. III.3(i)]) Then
\[
w_1 = w_2 = 1, w_3 = 2, w_4 = 3, w_5 = 4, w_6 = 5, w_7 = 6,
\]
and
\[
\sigma(g) = \max\{|t_1|, |t_2|, |t_3|^{1/2}, |t_4|^{1/3}, |t_5|^{1/4}, |t_6|^{1/5}, |t_7|^{1/6}\}.
\]
This algebra is exhibited in [Pi06A Rem. 6.6] as an example of a nilpotent Lie algebra that is not contractible and, hence, is not positively graded. Thus, it does not satisfy conditions of [ibid., Th. 6.19], which asserts that \( \iota: U(\mathfrak{g}) \rightarrow \tilde{U}(\mathfrak{g}) \) is a homological epimorphism for any positively graded Lie algebra \( \mathfrak{g} \). So our Theorem 1.2 is stronger that this Pirkovskii’s result.

Remark 3.8. Since we choose as \( \mathcal{F} \) the lower central series of \( \mathfrak{g} \), the dimensions of \( \mathfrak{g}_1, \ldots, \mathfrak{g}_k \) are invariants of a nilpotent Lie algebra \( \mathfrak{g} \). Therefore the sequence \( w_1, \ldots, w_m \) is also an invariant of \( \mathfrak{g} \). Of course, this invariant (as well as the isomorphism class of the \( \tilde{\mathfrak{g}} \)-algebra \( \mathcal{O}_{\exp}(G) \)) is far from sufficient to distinguish nilpotent
4. Proofs of main results and an application

With Theorem 3.2 under arms we can prove Theorems 1.1 and 1.2.

We need the following definition [Go77, Def. 2.1]: an $\mathcal{F}$-weight sequence $\mathcal{M}$ is entire if the following two conditions are satisfied

$$\sum_{\alpha} M_{\alpha} r^{w(\alpha)} < \infty \quad \text{for all } r > 0;$$

$$\sup_{a, \beta \neq 0} \left\{ A^{w(\alpha)/w(\beta)} M_{\beta}^{1/w(\beta)} M_{\alpha}^{-1/w(\alpha)} \right\} < \infty \quad \text{for some } A > 0.$$

Proof of Theorem 1.1 Let $G$ be a simply connected nilpotent complex Lie group with Lie algebra $\mathfrak{g}$. Given $z \in \mathbb{C}$, define a linear map $\delta_z : g \to g$ by $\delta_z(e_i) = z^{w_i} e_i$. We use the same symbol $\delta_z$ to denote the corresponding holomorphic endomorphism of $G$ satisfying $\delta_z \circ \exp = \exp \delta_z$.

Recall that our choice of an $\mathcal{F}$-weight sequence is $\mathcal{M}_1$ defined by $M_{\alpha} := w(\alpha)^{-w(\alpha)}$. Consider the growth function $\Phi$ associated with $\mathcal{M}_1$ given by

$$\Phi(g) := \sum_{\alpha} M_{\alpha} \sigma(g)^{w(\alpha)} = \sum_{\alpha} \left( \frac{\sigma(g)}{\sigma(\alpha)} \right)^{w(\alpha)},$$

where the function $\sigma$ is defined in (3.1). (Since $\mathcal{M}_1$ is entire [Go79, Sect. 2, Exm. 1], the function $\Phi$ is well defined.) Denote by $\mathcal{O}_{\mathcal{M}_1}(G)$ the linear subspace of $\mathcal{O}(G)$ that contains every function $f$ s.t. there are $C > 0$ and $r > 0$ and $|f(g)| \leq C \Phi(\delta_z g)$ is satisfied for all $g \in G$. To make $\mathcal{O}_{\mathcal{M}_1}(G)$ a locally convex space consider, for $r > 0$, the space

$$\mathcal{O}_{\mathcal{M}_1,r}(G) := \left\{ f \in \mathcal{O}(G) : N_r(f) := \sup_{g \in G} |\Phi(\delta_z g) - f(g)| < \infty \right\}.$$
On the other hand, by Corollary 2.13 \( \hat{U}(g) \) is the strong dual space of \( \mathcal{O}_{exp}(G) \) via the same paring. Thus \( U(g)_{\mathcal{M}_i} \) and \( \hat{U}(g) \) are isomorphic as \( \mathbb{R} \)-algebras, and \( U(g) \to U(g)_{\mathcal{M}_i} \) is an Arens-Michael envelope. 

**Proof of Theorem 1.3.** Since \( \mathcal{M}_i \) is entire \[ Go79 \] Sect. 2, Exm. 1, we can refer to Pirkovskii’s Theorem 1.3 which asserts that, in this case, the natural map \( U(g) \to U(g)_{\mathcal{M}_i} \) is a homological epimorphism. Thus, the assertion follows Theorem 1.1. □

As an application of Theorem 1.1 we obtain an estimation of a submultiplicative norm for powers of elements in a nilpotent complex Lie algebra \( g \). Let \( \mathcal{F} \) denote, as usual, the lower central series. Given \( X \in g \) s.t. \( X \neq 0 \), set \( w(X) := \max \{ j : X \in g_j \} \).

If \( \| \cdot \| \) is a submultiplicative prenorm on \( U(g) \), then denote by \( A \) the completion of \( U(g) \) w.r.t. \( \| \cdot \| \) and by \( \lambda : g \to A \) the corresponding Lie algebra homomorphism.

It is easy to see from the spectral properties of Banach algebras that, for any \( X \in [g, g] \), the element \( \lambda(X) \) is topologically nilpotent, i.e., \( \| \lambda(X)^n \|^{1/n} = o(1) \). (In fact, \( \lambda([g, g]) \) is contained in the Jacobson radical of \( A \) \[ Tu84 \].) Theorem 1.1 gives us the following decay estimation.

**Proposition 4.1.** Let \( A \) be a unital Banach algebra with a submultiplicative norm \( \| \cdot \| \), let \( g \) be a nilpotent complex Lie algebra with lower central series \( \mathcal{F} \) as a positive filtration, and let \( \lambda : g \to A \) be a Lie algebra homomorphism. Then for each \( X \in g \setminus \{ 0 \} \),

\[
\| \lambda(X)^n \|^{1/n} = O\left( \frac{1}{n^{w(X)-1}} \right) \quad (n \in \mathbb{N}).
\]

Note that in the case when \( w(X) = 1 \) we have the trivial assertion that \( \| \lambda(X)^n \|^{1/n} \) is bounded but for \( w(X) > 1 \) the statement is more interesting.

**Proof.** Consider the system of prenorms \( (\| \cdot \|_r : r > 0) \) on \( U(g) \) from Theorem 1.1. By the universal property, one can extend \( \lambda \) to a homomorphism \( U(g) \to A \). Then \( \| \lambda(\cdot) \| \) is a submultiplicative prenorm on \( U(g) \). It follows from Theorem 1.1 that \( \| \lambda(\cdot) \| \) is continuous w.r.t. \( (\| \cdot \|_r : r > 0) \). Thus it is sufficient to show that \( \| X^n \|^{1/n} = O(n^{1-w(X)}) \) for every \( X \neq 0 \) and every \( r > 0 \).

Fix \( X \in g \setminus \{ 0 \} \). We claim that there is an \( \mathcal{F} \)-basis \( (e_i) \) in \( g \) s.t. \( X = e_i \) for some \( i \). Indeed, let \( j := w(X) \). Then \( X \in g_j \) and \( X \notin g_{j+1} \). Consider an arbitrary \( \mathcal{F} \)-basis \( (e_i) \) in \( g \). Since \( g_j = \text{span}\{ e_i : w_i \geq j \} \), we have \( X = \sum_{w_i \geq j} c_i e_i \) for some \( c_i \in \mathbb{C} \) s.t. there is \( i \) with \( w_i = j \) and \( c_i \neq 0 \). So we can replace \( e_i \) by \( g_j \) and get an \( \mathcal{F} \)-basis again.

If \( \alpha := (0, \ldots, n, \ldots) \), where \( n \) is on the \( i \)th place, then \( w(\alpha) = nw_i \) and \( \alpha! = n! \). Therefore,

\[
\| e_i^n \|_r = n! \left( \frac{r}{nw_i} \right)^{nw_i}.
\]

Hence, \( \| e_i^n \|^{1/n} = O(n^{1-w_i}) \). Finally, remind that \( e_i = X \) with \( w_i = w(X) \). □

5. **Appendix. Proof of Theorem 3.1 and relations with Riemannian distances**

The case of Theorem 3.1 when \( g \) is complex is easily reduced for the real case, so below we suppose that \( g \) is a nilpotent real Lie algebra.
The argument consists of three parts and we need three lemmas. Both statements of
the first lemma are partial cases of \[VSC92\] Lem. IV.5.1.

**Lemma 5.1.** (A) For each \(n \in \mathbb{N}\) there are \(N \in \mathbb{N}, s_1, \ldots, s_N \in \{1, \ldots, n\}\), and \(\alpha_1, \ldots, \alpha_N \in \mathbb{R}\) s.t. for arbitrary \(Y_1, \ldots, Y_n \in \mathfrak{g}\)
\[
\exp \left( \sum_{s=1}^{n} Y_s \right) = \prod_{i=1}^{N} \exp (\alpha_i Y_s). \tag{5.1}
\]

(B) For each \(n \in \mathbb{N}\) and each word \(U = (u_1, \ldots, u_j)\) in \(\{1, \ldots, n\}\) there are \(N' \in \mathbb{N}, s'_1, \ldots, s'_{N'} \in \{1, \ldots, n\}\), and \(\alpha'_1, \ldots, \alpha'_{N'} \in \mathbb{R}\) s.t. for arbitrary \(Y_1, \ldots, Y_n \in \mathfrak{g}\)
\[
\exp(Y_U) = \prod_{p=1}^{N'} \exp(\alpha'_p Y_{s'_p}), \tag{5.2}
\]
where \(Y_U := [Y_{u_1}, [Y_{u_2}, \ldots, Y_{u_j}] \ldots]\).

**Lemma 5.2.** There are \(C, D \geq 0\) s.t.
\[
\sigma(g_1 g_2) \leq C \max \{\sigma(g_1), \sigma(g_2)\} + D \quad (g_1, g_2 \in G) \tag{5.3}
\]
i.e., \(e^\sigma\) is 'sub-polynomial' in the terminology of \[SC93\] (1.3.1)]

**Proof.** Let \(k\) be the positive integer s.t. \(g_k \neq 0\) and \(g_{k+1} = 0\). For \(X, Y \in \mathfrak{g}\) denote by \(X \ast Y\) the Hausdorff product, i.e., \(\exp(X \ast Y) = \exp X \exp Y\). It follows from the Baker-Campbell-Hausdorff formula that, for each word \(U\) in \(\{1, \ldots, 2m\}\) of length at most \(k\), there is \(\beta_U \in \mathbb{R}\) s.t.
\[
\left( \sum_{i=1}^{m} Y_i \right) \ast \left( \sum_{i=m+1}^{2m} Y_i \right) = \sum_{U} \beta_U Y_U \tag{5.4}
\]
for every \(Y_1, \ldots, Y_{2m} \in \mathfrak{g}\).

Write \(g_1 = \exp(\sum_{i=1}^{m} t_i e_i)\) and \(g_2 = \exp(\sum_{i=1}^{m} t_{m+i} e_i)\). Substituting in (5.4), we obtain
\[
\left( \sum_{i=1}^{m} t_i e_i \right) \ast \left( \sum_{i=m+1}^{2m} t_{m+i} e_i \right) = \sum_{U} \beta_U t_{u_1} \cdots t_{u_s} E_U,
\]
where \(E_U = [e_{u_1}, [e_{u_2}, \ldots, e_{u_s}] \ldots]\). Denote \(u_i\) modulo \(\bar{u}_i\). Then \(E_U \in \mathfrak{g}_{j(U)}\), where \(j(U) := \sum_{i=1}^{s} w_{\bar{u}_i}\). So we have
\[
E_U = \sum_{w_p \geq j(U)} \alpha_{U, p} e_p
\]
for some \(\alpha_{U, p}\).

Now write \(g_1 g_2 = \exp(\sum_{i=1}^{m} t'_i e_i)\). Then the coefficient \(t'_{U}\) is bounded by
\[
\sum_{U} |\beta_U| |\alpha_{U, p}| |t_{u_1} \cdots t_{u_s}|.
\]
Note that \(w_p \geq j(U)\) implies
\[
|t_{u_1} \cdots t_{u_s}|^{1/w_p} \leq |t_{u_1} \cdots t_{u_s}|^{1/j(U)} + 1 \leq \sum_{i=1}^{s} |t_{u_i}|^{1/w_{\bar{u}_i}} + 1.
\]
Hence there exist \(C'\) and \(D'\) s.t.
\[
\sum_{p=1}^{m} |t'_{p}|^{1/w_p} \leq C' \left( \sum_{i=1}^{m} |t_i|^{1/w_i} + \sum_{i=1}^{m} |t_{m+i}|^{1/w_i} \right) + D',
\]
which implies (5.3).

Fix subspaces \(v_1, \ldots, v_k\) s.t. \(v_i \oplus g_{i+1} = g_i\). Evidently, \(g = \oplus_{i=1}^{k} v_i\). It can easily be checked that this decomposition can assumed compatible with an \(\mathcal{F}\)-basis. Set \(v^{(1)} := v_1\) and \(v^{(j)} := [v_1, v^{(j-1)}]\) for \(j > 1\). It is not hard to see that

\[
\mathcal{g}_j = v^{(j)} + \cdots + v^{(k)}.
\]

(5.5)

Lemma 5.3. For any \(C > 0\) there is a norm \(\| \cdot \|\) on \(g\) (as a real linear space) s.t. the following conditions are satisfied.

1. If \(V = \sum_s V_s\), where \(V_s \in v_s\), then \(\|V_s\| \leq \|V\|\) for all \(s\).
2. For any \(p \in \mathbb{N}\), \(u_1, \ldots, u_p \in \{1, \ldots, k\}\), and \(V_s \in v_s\) (\(s = 1, \ldots, p\)), one has

\[
\| [V_{u_1}, [V_{u_2}, \cdots, V_{u_p}] \cdots ] \| \leq C\|v_{u_1}\|\|V_{u_2}\| \cdots \|V_{u_p}\|.
\]

Proof. Fix a norm \(\| \cdot \|\) on each \(v_s\) and consider norms on \(g\) of the form \(\| \sum_s V_s \| = \sum \lambda_s\|V_s\|\) (\(\lambda_s > 0\)). Obviously, any such norm satisfies to (1). Proceeding by induction on \(p\) it is not hard to show that there is a norm of this form that satisfies to (2).

Proof of Theorem 3.1 (the real case). Let \(\ell\) and \(\ell'\) be the word length functions corresponding to the generating sets

\[
\bigcup_{i=1}^{m} \{\exp(t_i e_i) : |t_i| \leq 1\} \quad \text{and} \quad \{\exp(X) : X \in v_1, \|X\| \leq 1\},
\]

resp. The first set is generating, since for each \(i\) the linear span of \(\{e_i, \ldots, e_m\}\) is a subalgebra of \(g\). To see that the second set is generating one can apply (5.3) with \(j = 1\), the surjectivity of the exponential map, and both parts of Lemma 5.1. Since \(G\) is compactly generated, all word length functions are equivalent [Sc93, Th. 1.1.21] (cf. also Corollary 5.5 below). Thus, it suffices to show that \(\ell \leq \tilde{\sigma} \leq \sigma \leq \ell'\), where \(\leq\) means ”is dominated by”.

1. First, we prove that \(\ell \leq \tilde{\sigma}\). Denote by \(S\) the subset of indices \(p\) s.t. \(w_p = 1\) (eq., \(e_p \in v_1\)). For any word \(U = (u_1, \ldots, u_j)\) in \(S\) set \(\lambda(U) = j\) and consider the \(j\)th commutator \(E_\ell(U) := [t_{e_{u_1}}, t_{e_{u_2}}, \ldots, t_{e_{u_j}}] \cdots\) for \(t \in \mathbb{R}\).

Fix \(j \in \{1, \ldots, k\}\) and \(e_i\) with \(w_i = j\). Since \(e_i \in g_j\), it follows from (5.5) that \(e_i\) is a linear combination \(\sum_U \mu_U E_\ell(1)\), where \(U\) runs all words in \(S\) with \(j \leq \lambda(U) \leq k\). Therefore, for any \(t \in \mathbb{R}\),

\[
t e_i = \text{sgn}(t) \sum_{j \leq \lambda(U) \leq k} \mu_U E_\ell([t]^{1/\lambda(U)}).
\]

(5.6)

For simplicity, we assume that \(t > 0\); for negative \(t\) the following estimates are the same.

Enumerating all words in the sum above as \(U_1, \ldots, U_n\) and applying Part (A) of Lemma 5.1 for \(n\), we can substitute \(\sum \mu_U E_\ell([t]^{1/\lambda(U)})\) in (5.1) and get from (5.6) the equality

\[
\exp(t e_i) = \prod_{r=1}^{N} \exp(\alpha_r \mu_{U_r} E_{U_r}([t]^{1/\lambda(U_r)})).
\]

Further, applying Part (B) of Lemma 5.1 to each factor in the product with \(Y_1 = \alpha_r \mu_{U_r} [t]^{1/\lambda(U_r)} e_1\) and \(Y_s = [t]^{1/\lambda(U_r)} e_s\) for \(s \geq 2\), we have that there are \(N''\) and
\(\beta_1, \ldots, \beta_N\) independent in \(t\) with some positive integers \(\lambda_1, \ldots, \lambda_N\) non less that \(j\) s.t.

\[
\exp(te_i) = \prod_{p=1}^{N'} \exp\left(\|t\|^{1/\lambda_p} \beta_p e_{i_p}\right).
\]

Hence, for each \(t \geq 0\),

\[
\ell(\exp te_i) \leq \sum_p \ell(\exp\left(\|t\|^{1/\lambda_p} \beta_p e_{i_p}\right)) \leq \sum_p \left(\|t\|^{1/\lambda_p} \beta_p + 1\right).
\]

Since \(\|t\|^{1/\lambda_p} \leq \|t\|^{1/j}\) for \(\|t\| \geq 1\), we obtain \(\ell(\exp(te_i)) \leq C_j \|t\|^{1/j} + D_j\) for all \(t\), where \(C_j\) and \(D_j\) depend only on \(j\).

Finally, write any \(g \in G\) as

\[
g = \prod_{i=1}^{m} \exp(\bar{t}_i e_i),
\]

where \(\bar{t}_1, \ldots, \bar{t}_m \in \mathbb{R}\). Therefore

\[
\ell(g) \leq \sum_i \ell(\exp(\bar{t}_i e_i)) \leq \sum_i C_{w_i} \|\bar{t}_i\|^{1/w_i} + D_{w_i}.
\]

Thus, \(\ell\) is dominated by \(\bar{\sigma}\).

(2) Secondly, we prove that \(\bar{\sigma} \leq \sigma\) (cf. the proof of [DER03, II.4.17]). We show by induction on \(i\) in the reverse order that for each \(i = 1, \ldots, m\) there are constants \(A_i\) and \(B_i\) s.t. for every \(g = \exp(\sum_{s=1}^{m} t_s e_s)\) the estimate \(\bar{\sigma}(g) \leq A_i \sigma(g) + B_i\) holds.

Obviously, if \(i = m\), then \(\bar{\sigma}(g) = \sigma(g)\). Suppose that the induction assumption is satisfied for \(i\). The Baker-Campbell-Hausdorff formula implies that

\[
(-t_{i-1} e_{i-1}) \ast \left(\sum_{s=i-1}^{m} t_s e_s\right) = \sum_{s=i}^{m} t'_s e_s
\]

for some \(t'_s\). Write \(\exp(\sum_{s=i}^{m} t'_s e_s) = \prod_{s=i}^{m} \exp(t'_s e_s)\). Then

\[
\exp\left(\sum_{s=i-1}^{m} t_s e_s\right) = \prod_{s=i-1}^{m} \exp(t_s e_s),
\]

where \(\bar{t}_{i-1} := t_{i-1}\). Applying Lemma 5.7 to 5.7, we get

\[
\max_{1 \leq s \leq m} \|t'_s\|^{1/w_s} \leq C \max_{1 \leq s \leq m} \|t_s\|^{1/w_s} + D.
\]

By the inductive assumption, we have

\[
\max_{1 \leq s \leq m} \|\bar{t}_s\|^{1/w_s} \leq A_i \max_{1 \leq s \leq m} \|t'_s\|^{1/w_s} + B_i.
\]

Combining the two estimates we obtain

\[
\max_{1 \leq s \leq m} \|\bar{t}_s\|^{1/w_s} \leq A_{i-1} \max_{1 \leq s \leq m} \|t'_s\|^{1/w_s} + B_{i-1}.
\]

for some \(A_{i-1}\) and \(B_{i-1}\) depending only on \(i\). The induction is complete.

Finally, note that we have shown that \(\bar{\sigma}(g) \leq A_1 \sigma(g) + B_1\) for all \(g \in G\).

(3) Thirdly, we prove that \(\sigma \leq \ell'\). Let \(\|\cdot\|\) be the norm on \(g\) existing by Lemma 5.3 (the value of constant \(C\) is specified below). Note that \(\sigma(g)\) is equivalent to the function \(g \mapsto \max_j \|V_j\|^{1/j}\), where \(g = \exp(\sum_j V_j)\) with \(V_j \in v_j\). So it suffices to show that \(\ell'(g) = n\) implies \(\|V_j\| \leq n^j\) for all \(j\).
We proceed by induction. For \( n = 0 \) and \( n = 1 \) the claim is obvious. Suppose that it holds for \( n - 1 \geq 1 \). If \( \ell'(g) = n \), then \( g = g_1g_2 \), where \( \ell'(g_1) = 1 \) and \( \ell'(g_2) = n - 1 \), i.e., \( g_1 = \exp V_0 \), where \( V_0 \in \mathfrak{v}_1 \) with \( \|V_0\| \leq 1 \) and \( g_2 = \exp(\sum_j V_j) \), where \( V_j \in \mathfrak{v}_j \) with \( \|V_j\| \leq (n - 1)^j \). Write \( g_1g_2 = \exp(\sum_{j=1}^k W_j) \), where \( W_j \in \mathfrak{v}_j \). We need to show that \( \|W_j\| \leq n^j \) for all \( j \).

Note that, by the Baker-Campbell-Hausdorff formula, there are \( \gamma_U \) s.t.
\[
V_0 * (V_1 + \cdots + V_K) = \exp \left( V_0 + V_1 + \cdots + V_K + \sum_U \gamma_U V_U \right),
\]
for any \( V_0 \in \mathfrak{v}_1 \) and \( V_j \in \mathfrak{v}_j \) \((j = 1, \ldots, k)\), where \( V_U := \{V_{u_1}, V_{u_2}, \ldots, U_{u_p}\} \) and \( U = (u_1, \ldots, u_p) \) runs all words in \( \{0, 1, \ldots, k\} \) of length at least 2 and at most \( k \), and containing at least 1 of occurrence of 0.

Further, for each \( U \), there is a unique decomposition
\[
V_U = \sum_{j=|U|}^k Y_{U,j}; \quad (Y_{U,j} \in \mathfrak{v}_j),
\]
where \( |U| \) denotes the sum of \( u_1 + \cdots + u_p \) and the number of occurrence of 0 in \( U \). We get from \( (5.8) \) that \( W_1 = V_0 + V_1 \) and for \( j \geq 2 \)
\[
W_j = V_j + \sum_{U \in S_j} \gamma_U Y_{U,j}, \tag{5.9}
\]
where \( S_j \) denotes the set of words \( U \) s.t. the length of \( U \) is at least 2, \( |U| \leq j \), and \( U \) contains at least 1 of occurrence of 0.

By setting \( C := (\sum_U |\gamma_U|)^{-1} \) in Lemma \( 5.3 \) we get
\[
\left\| \sum_{U \in S_j} \gamma_U Y_{U,j} \right\| \leq \sum_{U \in S_j} |\gamma_U| \|Y_{U,j}\| \leq (by \text{ Part (A)}) \sum_{U \in S_j} |\gamma_U| \|V_U\| \leq (by \text{ Part (B)}) \sum_{U \in S_j} |\gamma_U| C \|V_{u_1}\| \|V_{u_2}\| \cdots \|V_{u_p}\| \leq \max_{U \in S_j} \{\|V_{u_1}\| \|V_{u_2}\| \cdots \|V_{u_p}\|\}. \tag{5.10}
\]
According to the inductive assumption, \( \|V_{u_s}\| \leq (n - 1)^{u_s} \) for all \( u_s \). Since \( U \in S_j \), we have \( u_1 + \cdots + u_p < |U| \leq j \). Therefore,
\[
\|V_{u_1}\| \|V_{u_2}\| \cdots \|V_{u_p}\| \leq (n - 1)^{j-1}.
\]
Finally, \( (5.9) \) and \( (5.10) \) imply \( \|W_j\| \leq 2(n - 1)^{j-1} \leq n^j \) for \( j \geq 2 \) and obviously \( \|W_1\| \leq \|V_0\| + \|V_1\| \leq n \). \[\square\]

**On left invariant Riemannian distances.** The following remarks explain why Theorem \( 3.1 \) can be reformulated in terms of left invariant (sub)-Riemannian distances; thus, we have a direct connection with results in [Be96, DER03, Ka94, VSC92].

We say that a length function \( \ell \) is *symmetric* if \( \ell(e) = 0 \) and \( \ell(g^{-1}) = \ell(g) \) for all \( g \in G \). Obviously, for given length function \( \ell \), the function \( \ell' \) defined by
\[
\ell'(g) := \max\{\ell(g), \ell(g^{-1})\}, \quad (g \neq e) \quad \text{and} \quad \ell'(e) := 1
\]
is a symmetric length function.

For a symmetric length function $\ell$ consider the following condition:

(a) there exists $C$ such that, for each $g \in G$ satisfying $\ell(g) \geq 1$, there are

$g_0, \ldots, g_n \in G$ s.t. $g_0 = e$, $g_n = g$, and $\ell(g_i^{-1}g_{i+1}) \leq 1$ ($i = 0, \ldots, n - 1$) with $n \leq C\ell(g)$.

It is trivial that the formulas $\ell d(g,h) := \ell(g^{-1}h)$ and $d\ell(g) := d(e,g)$ determine a bijection between the set of symmetric length functions on $G$ and the set of locally bounded left invariant distances on $G$. Under this correspondence our condition (a) becomes condition $(C_3)$ from p. 40 of [VSC92]. In the terminology of [CH16] Def. 3.B.1, property $(C_3)$ means that $d$ is 1-large-scale geodesic.

**Proposition 5.4.** Let $G$ be a locally compact group and let $\ell_1$ and $\ell_2$ be symmetric length functions on $G$ satisfying to condition (a). Suppose that $\ell_1$ is bounded on $\{g \in G : \ell_2(g) \leq 1\}$ and $\ell_2$ is bounded on $\{g \in G : \ell_1(g) \leq 1\}$. Then $\ell_1$ and $\ell_2$ are equivalent.

**Proof.** It is noted in [VSC92] Rem. III.4.3 that the argument from [ibid., Prop. III.4.2] can be applied in our situation.\qed

Recall that every connected locally compact group is compactly generated [HR79 Th. 7.4].

**Proposition 5.5.** Let $G$ be a connected real Lie group. Suppose that $U$ is a symmetric relatively compact generating set and $d$ is a distance determined by a left invariant Riemannian metric. Then $\ell_U$ is equivalent (at infinity) to $d\ell$.

**Proof.** It is easy to see that $\ell_U$ is bounded on $\{g \in G : d\ell(g) \leq 1\}$ and $d\ell$ is bounded on $\{g \in G : \ell_U(g) \leq 1\}$. By Proposition 5.4 it suffices to show that $\ell_U$ and $d\ell$ satisfy to (a).

Let $\ell_U(g) = n > 0$ and fix $g = h_1 \cdots h_n$ where $h_i \in U$. Set $g_0 = e$, and $g_i = h_1 \cdots h_i$ for $i = 1, \ldots, n$. Then $\ell_U(g_i^{-1}g_{i+1}) = \ell(h_{i+1}) \leq 1$ for $i < n$. Thus (a) is satisfied for $\ell_U$ with $C = 1$. Since $d$ is defined as the infimum of lengths of piecewise smooth paths, condition (a) is satisfied for $d\ell$. \qed

For a nilpotent Lie group $G$, the distance $d$ in Proposition 5.5 can be replaced by a Carnot-Carathéodory distance in the sense of [VSC92 Sect. III.4].

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