A GENERALIZATION OF A RESULT ON THE SUM OF ELEMENT ORDERS OF A FINITE GROUP

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ABSTRACT. Let $G$ be a finite group and let $\psi(G)$ denote the sum of element orders of $G$. It is well-known that the maximum value of $\psi$ on the set of groups of order $n$, where $n$ is a positive integer, will occur at the cyclic group $C_n$. For nilpotent groups, we prove a natural generalization of this result, obtained by replacing the element orders of $G$ with the element orders relative to a certain subgroup $H$ of $G$.

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1. Introduction

Let $G$ be a finite group. In 2009, H. Amiri, S. M. Jafarian Amiri and I. M. Isaacs introduced in their paper [1] the function

$$\psi(G) = \sum_{x \in G} o(x),$$

where $o(x)$ denotes the order of $x$ in $G$. They proved the following basic theorem:

**Theorem 1.1.** If $G$ is a group of order $n$, then $\psi(G) \leq \psi(C_n)$, and we have equality if and only if $G$ is cyclic.

Since then many authors have studied the properties of the function $\psi(G)$ and its relations with the structure of $G$ (see, e.g., [2–6] and [8]). We recall only that $\psi$ is multiplicative and

$$\psi(C_{p^n}) = \frac{p^{2n+1} - 1}{p-1}$$

when $p$ is a prime (see, e.g., of [3: Lemmas 2.2(3) and 2.9(1)]).

Given a subgroup $H$ of $G$, in what follows we will consider the function

$$\psi_H(G) = \sum_{x \in G} o_H(x),$$

where $o_H(x)$ denotes the order of $x$ relative to $H$, i.e., the smallest positive integer $m$ such that $x^m \in H$. Clearly, for $H = 1$ we have $\psi_H(G) = \psi(G)$.

By replacing $\psi(G)$ with $\psi_H(G)$, we are able to generalize the above theorem for nilpotent groups.

**Theorem 1.2.** Let $G$ be a nilpotent group of order $n$ and $H$ be a subgroup of order $m$ of $G$. Then

$$\psi_H(G) \leq \psi_{H_m}(C_n),$$

where $H_m$ is the unique subgroup of order $m$ of $C_n$. 

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Note that the inequality (1.1) can easily be proved for normal subgroups $H$. Indeed, in this case we have

$$o_H(x) = o(xH) \text{ in } G/H, \quad \text{for all } x \in G$$

and therefore

$$\psi_H(G) = |H| \psi(G/H) \leq m \psi(C_m) = \psi_{H_m}(C_n).$$

This also shows that the equality occurs in (1.1) whenever $H$ is normal and $G/H$ is cyclic.

Finally, we conjecture that Theorem 1.2 is also true for non-nilpotent groups $G$, i.e., it is true for all finite groups $G$.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [7].

2. Proof of the main result

Our first lemma collects two basic properties of the function $\psi_H(G)$.

**Lemma 2.1.**

a) If $(G_i)_{i=1}^k$ is a family of finite groups having coprime orders and $H_i \leq G_i$, $i = 1, \ldots, k$, then

$$\psi_{H_1 \times \cdots \times H_k}(G_1 \times \cdots \times G_k) = \prod_{i=1}^k \psi_{H_i}(G_i).$$

In particular, if $G$ is a finite nilpotent group, $(G_i)_{i=1}^k$ are the Sylow $p_i$-subgroups of $G$ and $H = H_1 \times \cdots \times H_k \leq G$, then

$$\psi_H(G) = \prod_{i=1}^k \psi_{H_i}(G_i).$$

b) If $G$ is a finite group and $H \trianglelefteq K \leq G$, then

$$\psi_H(G) \leq [K : H] \psi_K(G) - |K| + |H|. \quad (2.1)$$

In particular, if $G$ is a finite $p$-group and $H \leq K \leq G$ with $|H| = p^m$ and $|K| = p^{m+1}$, then

$$\psi_H(G) \leq p \psi_K(G) - p^m(p-1). \quad (2.2)$$

**Proof.**

a) Since $G_i$, $i = 1, \ldots, k$, are of coprime orders, for every $x = (x_1, \ldots, x_k) \in G_1 \times \cdots \times G_k$ we have

$$o_{H_1 \times \cdots \times H_k}(x) = \prod_{i=1}^k o_{H_i}(x_i).$$

Then

$$\psi_{H_1 \times \cdots \times H_k}(G_1 \times \cdots \times G_k) = \sum_{x=(x_1,\ldots,x_k) \in G_1 \times \cdots \times G_k} o_{H_1 \times \cdots \times H_k}(x)$$

$$= \sum_{x_1 \in G_1} \cdots \sum_{x_k \in G_k} o_{H_1}(x_1) \cdots o_{H_k}(x_k)$$

$$= \prod_{i=1}^k \left( \sum_{x_i \in G_i} o_{H_i}(x_i) \right) = \prod_{i=1}^k \psi_{H_i}(G_i),$$

as desired.
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b) Let \( x \in G \). Then \( x^{o_K(x)} \in K \) and so \( x^{o_K(x)}H \in K/H \), implying that \( (x^{o_K(x)}H)^{[K:H]} = H \). Thus \( x^{[K:H]o_K(x)} \in H \), which leads to

\[
o_H(x) | [K : H] o_K(x)
\]

and consequently

\[
o_H(x) \leq [K : H] o_K(x).
\]

This shows that

\[
\psi_H(G) = \sum_{x \in G} o_H(x) = \sum_{x \in G \setminus H} o_H(x) + \sum_{x \in H} o_H(x)
\]

\[
\leq [K : H] \sum_{x \in G \setminus H} o_K(x) + |H|
\]

\[
= [K : H] \left( \sum_{x \in G} o_K(x) - \sum_{x \in H} o_K(x) \right) + |H|
\]

\[
= [K : H] (\psi_K(G) - |H|) + |H|
\]

\[
= [K : H] \psi_K(G) - |K| + |H|,
\]

completing the proof. \( \square \)

Remark. By taking \( H = 1 \) and \( K \leq G \) in (2.1), one obtains

\[
\psi(G) \leq |K| \psi_K(G) - |K| + 1 = |K|^2 \psi(G/K) - |K| + 1. \tag{2.3}
\]

This improves the inequality in \[4\] Proposition 2.6. Also, by taking \( K = G \) in (2.3), we get a new upper bound for \( \psi(G) \):

\[
\psi(G) \leq |G|^2 - |G| + 1. \tag{2.4}
\]

Note that we have equality in (2.4) if and only if \( G \) is cyclic of prime order.

Next we prove the inequality (1.1) for \( p \)-groups.

**Lemma 2.2.** Let \( G \) be a \( p \)-group of order \( p^n \) and \( H \) be a subgroup of order \( p^m \) of \( G \). Then

\[
\psi_H(G) \leq \psi_{H^m}(C_{p^n}).
\]

**Proof.** We will proceed by induction on \( [G : H] \). Obviously, the inequality holds for \( [G : H] = 1 \). Assume now that it holds for all subgroups of \( G \) of index less than \( [G : H] \). Since every subgroup of \( G \) is subnormal, we can choose \( K \leq G \) such that \( H \subset K \) and \( |K| = p^{n+1} \). Then, by (2.1) and the inductive hypothesis, we get

\[
\psi_H(G) \leq p \psi_K(G) - p^m(p - 1) \leq p \psi_{H_{p^{n+1}}(C_{p^n})} - p^m(p - 1)
\]

\[
= p^m \psi(C_{p^{2n-2m-1}}) - p^m(p - 1) = \left[ p^{2n-2m-1} + 1 \right] - (p - 1)
\]

\[
= p^m \frac{p^{2n-2m+1} + 1}{p + 1} = p^m \psi(C_{p^{n-m}}) = \psi_{H^m}(C_{p^n}),
\]

as desired. \( \square \)

We are now able to prove our main result.
P r o o f  o f  T h e o r e m  1.1. Let $n = p_1^{n_1} \cdots p_k^{n_k}$ be the decomposition of $n$ as a product of prime factors. Since $G$ is nilpotent, we have $G \cong G_1 \times \cdots \times G_k$, where $(G_i)_{i=1}^k$ are the Sylow $p_i$-subgroups of $G$. Moreover, any subgroup $H$ of $G$ is of type $H \cong H_1 \times \cdots \times H_k$ with $H_i \leq G_i$, $|H_i| = p_i^{m_i}$, for all $i = 1, \ldots, k$. Then, Lemmas 2.1 a), and 2.2 lead to
\[
\psi_H(G) = \prod_{i=1}^k \psi_{H_i}(G_i) \leq \prod_{i=1}^k \psi_{H_{p_i^{m_i}}}(C_{p_i^{n_i}}) = \psi_{H_m}(C_n).
\]
This completes the proof. □

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R E F E R E N C E S

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