ANALYTIC CAPACITY AND DIMENSION
OF SETS WITH PLENTY OF BIG PROJECTIONS

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Abstract. Our main result marks progress on an old conjecture of Vitushkin. We show that a compact set in the plane with plenty of big projections (PBP) has positive analytic capacity, along with a quantitative lower bound. A higher dimensional counterpart is also proved for capacities related to the Riesz kernel, including the Lipschitz harmonic capacity. The proof uses a construction of a doubling Frostman measure on a lower content regular set, which may be of independent interest.

Our second main result is the Analyst’s Traveling Salesman Theorem for sets with plenty of big projections. As a corollary, we obtain a lower bound for the Hausdorff dimension of uniformly wiggly sets with PBP. The second corollary is an estimate for the capacities of subsets of sets with PBP, in the spirit of the quantitative solution to Denjoy’s conjecture.

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1. Introduction

The aim of this paper is to study sets with plenty of big projections.

Definition 1.1. We say that \( E \subset \mathbb{R}^n \) has plenty of \((d\text{-dimensional})\) big projections (abbreviated to PBP, or \(d\)-PBP) if there exists a constant \( 0 < \delta \leq 1 \) such that the following holds. For all \( x \in E \) and \( 0 < r < \text{diam}(E) \) there exists \( V_{x,r} \in \mathcal{G}(n,d) \) such that
\[
\mathcal{H}^d(\pi_V (E \cap B(x,r))) \geq \delta r^d \quad \text{for all } V \in B(V_{x,r}, \delta).
\]
Here $G(n, d)$ is the Grassmannian manifold of $d$-dimensional (linear) planes in $\mathbb{R}^n$, $\pi_V : \mathbb{R}^n \to V$ is the orthogonal projection, and the ball $B(V_x, \delta)$ is defined with respect to the standard metric on $G(n, d)$ (see §2.1).

This definition first appeared in [DS93], although originally David and Semmes assumed additionally that sets with PBP are Ahlfors regular.

**Definition 1.2.** A set $E \subset \mathbb{R}^n$ is called **Ahlfors $d$-regular** if there exists a constant $C \geq 1$ such that for all $x \in E$ and $0 < r < \text{diam}(E)$ we have

$$C^{-1} r^d \leq \mathcal{H}^d(E \cap B(x, r)) \leq C r^d.$$ 

The smallest constant $C$ for which the above holds is called the **Ahlfors regularity constant** of $E$.

In [DS93] David and Semmes conjectured that Ahlfors regular sets with PBP satisfy the so-called **big pieces of Lipschitz graphs condition**, a strong quantitative rectifiability property (see Definition 2.4). Their conjecture was recently solved in a breakthrough work of Orponen [Orp21] (see Theorem 2.5 for the precise statement).

In this article we use Orponen’s result to further study sets with PBP. The main novelty of our work stems from the fact that we do not assume our sets to be Ahlfors regular. In fact, some of our results remain valid for sets with non-$\sigma$-finite $\mathcal{H}^d$-measure.

### 1.1. Analytic capacity: Vitushkin’s conjecture and quantitative Denjoy’s conjecture

Recall that a compact set $E \subset \mathbb{C} \simeq \mathbb{R}^2$ is said to be **removable for bounded analytic functions** if all bounded analytic functions $f$ defined on $\mathbb{C} \setminus E$ are constant. In the 40s Ahlfors [Ahl47] managed to quantify the notion of removability by introducing **analytic capacity**. Recall that the analytic capacity of a compact set $E \subset \mathbb{C}$ is defined as

$$\gamma(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \to \mathbb{C}$ with $\|f\|_{\infty} \leq 1$, and $f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty))$. Ahlfors proved that a set is removable for bounded analytic functions if and only if $\gamma(E) = 0$.

In 1967 Vitushkin [Vit67] conjectured a geometric characterization of removability in terms of orthogonal projections. He asked the following: is it true that for compact sets $E \subset \mathbb{C}$ one has

$$\gamma(E) = 0 \iff \text{Fav}(E) = 0,$$

where $\text{Fav}(E)$ is the Favard length of $E$, defined as

$$\text{Fav}(E) = \int_0^{\pi} \mathcal{H}^1(\pi_\theta(E)) \, d\theta.$$

In the above $\pi_\theta : \mathbb{C} \to \ell_\theta$ is the orthogonal projection to $\ell_\theta = \text{span}((\sin \theta, \cos \theta))$.

It has been known for a very long time that sets with $\mathcal{H}^1(E) = 0$ are removable, while sets with $\text{dim}_H(E) > 1$ are non-removable, so Vitushkin’s conjecture holds for such sets. The case of sets $E$ satisfying $0 < \mathcal{H}^1(E) < \infty$ was much more difficult to establish, but the answer to Vitushkin’s question is also positive: the implication $(\Rightarrow)$ in (1.1) is due to Calderón [Cal77], while $(\Leftarrow)$ was shown by David [Dav98]. Finally, the case of sets with $\sigma$-finite $\mathcal{H}^1$-measure follows from the finite measure case together with subadditivity of analytic capacity, which is due to Tolsa [Tol03].
In 1986 Mattila [Mat86] showed that Vitushkin’s conjecture fails for sets with Hausdorff dimension 1 and non-σ-finite $\mathcal{H}^1$-measure. It wasn’t clear from Mattila’s proof which of the implications in (1.1) is false. Soon thereafter Jones and Murai [JM88] constructed an example of a set with $\gamma(E) > 0$ and $\text{Fav}(E) = 0$. A simpler example was found later on by Joyce and Mörters [JM00]. Hence, the implication ($\Leftarrow$) in (1.1) is false. For more information on analytic capacity and Vitushkin’s conjecture see the books [To14, Dud11, Paj02]. See also recent surveys on related topics [Ver21, Mat21].

After all these extraordinary developments one question remains: what about the implication ($\Rightarrow$) in (1.1)? Equivalently, is it true that

$$\text{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0? \quad (1.2)$$

A quantitative version of this question is the following: is it true that

$$\gamma(E) \gtrsim \text{Fav}(E)? \quad (1.3)$$

As far as we know, the only partial result related to these open problems is due to Chang and Tolsa [CT20] (more on that in Remark 1.8). In this article we make further progress on (1.2) and (1.3). If the assumption $\text{Fav}(E) > 0$ is replaced by the (significantly stronger) PBP assumption, then the analytic capacity is positive.

**Theorem 1.3.** Let $E \subset \mathbb{R}^2$ be a compact set with 1-PBP. Then,

$$\gamma(E) \gtrsim \text{diam}(E), \quad (1.4)$$

where the implicit constant depends only on the PBP constant.

**Remark 1.4.** In the theorem above we do not assume $\mathcal{H}^1(E) < \infty$. However, our result bears new information even in the $\mathcal{H}^1(E) < \infty$ case. Recall that in this regime Vitushkin’s conjecture is true, so in particular $\text{Fav}(E) > 0$ implies $\gamma(E) > 0$. However, the existing proof does not offer any quantitative lower bound on $\gamma(E)$ in terms of $\text{Fav}(E)$. Roughly speaking, the reason is the following: the Besicovitch projection theorem states that if $\text{Fav}(E) > 0$, then there exists a 1-dimensional Lipschitz graph $\Gamma$ such that $\mathcal{H}^1(E \cap \Gamma) > 0$, and then $\gamma(E) \geq \gamma(E \cap \Gamma) > 0$ follows from [Cal77]. However, the Besicovitch projection theorem does not say anything about the size of $\mathcal{H}^1(E \cap \Gamma)$, and so deriving a quantitative lower bound for $\gamma(E)$ is impossible as long as one follows this strategy. In other words, the question (1.3) remains wide open even in the $\mathcal{H}^1(E) < \infty$ case. Although Theorem 1.3 is quite far from establishing (1.3), it is (as far as we know) the first quantitative lower bound for $\gamma(E)$ which depends only on the projections of $E$.

In the finite measure case, we are able to prove more: we show a quantitative bound for the analytic capacity of subsets of sets with 1-PBP. This is a form of quantitative Denjoy’s conjecture, where rectifiable curves are substituted with sets with PBP of finite measure. Recall that Denjoy’s conjecture, which predates Vitushkin’s conjecture by about 60 years, stated that if $\Gamma \subset \mathbb{R}^2$ is a rectifiable curve, and a compact set $E \subset \Gamma$ satisfies $\mathcal{H}^1(E) > 0$, then $\gamma(E) > 0$. This statement can be seen as a special case of Vitushkin’s conjecture, if one restricts attention to the implication (1.2) and sets of finite length. Denjoy’s conjecture was confirmed by Calderón in [Cal77]. A quantitative version of Calderón’s result was established by Murai [Mur87], who showed that if $\Gamma$ is
a 1-rectifiable graph, and \( E \subset \Gamma \) is compact, then

\[
\gamma(E) \gtrsim \frac{\mathcal{H}^{1}_{\infty}(E)^{3/2}}{\mathcal{H}^{1}(\Gamma)^{1/2}}.
\]

Later on Verdera observed that Murai’s estimate can be obtained for arbitrary rectifiable curves \( \Gamma \) using Menger curvature and Jones’ Travelling Salesman Theorem, see \cite[Theorem 4.31]{Tol14}. The exponent \( 3/2 \) above is optimal, see \cite{Mur90} and \cite[§4.8]{Tol14}.

Using an Analyst’s Travelling Salesman Theorem for sets with PBP (see Theorem \ref{Theorem 1.10} below) we obtain the following.

**Theorem 1.5.** Let \( \Gamma \subset \mathbb{R}^{2} \) be a compact set with 1-PBP and \( 0 < \mathcal{H}^{1}(\Gamma) < \infty \). Then, for any compact subset \( E \subset \Gamma \)

\[
\gamma(E) \gtrsim \frac{\mathcal{H}^{1}_{\infty}(E)^{3/2}}{\mathcal{H}^{1}(\Gamma)^{1/2}},
\]

where the implicit constant depends only on the PBP constant.

Theorem 1.5 is interesting for two reasons: first, as mentioned above, it applies to subsets of sets with PBP. These subsets could very well have vanishing density in many places, and have very small projections. Note that a similar statement cannot be true for sets \( \Gamma \) with infinite length: for example, the unit square \([0,1]^{2}\) has PBP, but the 4-corner Cantor set \( K \subset [0,1]^{2} \) satisfies \( \mathcal{H}^{1}(K) \sim 1 \) and \( \gamma(K) = 0 \).

The second interesting aspect of Theorem 1.5 is that it suggests that, for many geometric problems, sets with PBP and finite measure are just as good as rectifiable curves. Let us however make the following remark: from the Analyst’s TST for sets with PBP (Theorem 1.10 below) it follows that a set with PBP and finite \( \mathcal{H}^{1} \)-measure can be covered by a rectifiable curve of comparable length. Thus, Theorem 1.5 can be derived directly from Theorem 1.10 together with the previous result of Murai and Verdera. However, in higher dimension (see the section below) the result is altogether new, since no result analogous to that of Murai was known for capacities associated to vector-valued Riesz kernel.

### 1.2. Higher dimensional variants

Higher dimensional variants of Theorem 1.3 and Theorem 1.5 are also true. Instead of analytic capacity one has to consider certain capacities associated to the vector-valued Riesz kernel. Let \( 0 < d < n \) be integers. Given a compact set \( E \subset \mathbb{R}^{n} \) the capacity \( \Gamma_{n,d}(E) \) is defined as

\[
\Gamma_{n,d}(E) = \sup |\langle T, 1 \rangle|,
\]

where the supremum is taken over all real distributions \( T \) supported in \( E \) such that

\[
\frac{x}{|x|^{d+1}} * T \in L^{\infty}(\mathbb{R}^{n}) \quad \text{and} \quad \left\| \frac{x}{|x|^{d+1}} * T \right\|_{L^{\infty}(\mathbb{R}^{n})} \leq 1.
\]

We remark that by \cite{Tol03} one has \( \Gamma_{2,1}(E) \sim \gamma(E) \). In the codimension 1 case the capacity \( \Gamma_{d+1,d}(E) \) is also called the Lipschitz harmonic capacity, and it is often denoted by \( \kappa(E) \). It was introduced by Paramonov, who observed that sets with \( \kappa(E) = 0 \) are precisely the sets removable for Lipschitz harmonic functions, see \cite[Remark 2.4]{Par90} and \cite{MP95}. The counterpart of Vitushkin’s conjecture for \( \kappa(E) \) and sets with
$\mathcal{H}^d(E) < \infty$ was established by Nazarov, Tolsa and Volberg in [NTV14b, NTV14a]. More specifically, they showed that for a compact set $E \subset \mathbb{R}^{d+1}$ with $\mathcal{H}^d(E) < \infty$ one has $\kappa(E) = 0$ if and only if $E$ is purely $d$-unrectifiable (equivalently, $\mathcal{H}^d(\pi_V(E)) = 0$ for a.e. $V \in \mathcal{G}(n,d)$). Whether an analogous statement is true for $\Gamma_{n,d}$ with $1 < n < d - 1$ is an open problem.

The higher dimensional variants of Theorem 1.3 and Theorem 1.5 are the following.

**Theorem 1.6.** Let $E \subset \mathbb{R}^n$ be a compact set with $d$-PBP. Then,

$$\Gamma_{n,d}(E) \gtrsim \text{diam}(E)^d,$$

where the implicit constant depends only on $n, d,$ and the PBP constant.

**Theorem 1.7.** Let $\Sigma \subset \mathbb{R}^n$ be a compact set with $d$-PBP and $0 < \mathcal{H}^d(\Sigma) < \infty$. Then for any compact subset $E \subset \Sigma$,

$$\Gamma_{n,d}(E) \gtrsim \frac{\mathcal{H}^d_x(\Sigma)^{\frac{d}{2} \frac{n}{d} (\Sigma)}}{\mathcal{H}^d(\Sigma)^{\frac{d}{2}}},$$

where the implicit constant depends only on $n, d,$ and the PBP constant.

**Remark 1.8.** Theorems 1.3 and 1.6 should be compared with the results of Chang and Tolsa [CT20], who proved the following. If $I \subset [0, \pi)$ is an interval and $E \subset \mathbb{C}$ is a compact set supporting a probability measure $\mu$ such that $\pi_\theta \mu \in L^2(\ell_\theta)$ for a.e. $\theta \in I$, then

$$\gamma(E) \gtrsim \frac{1}{\int_I \|\pi_\theta \mu\|_{L^2}^2 d\theta},$$

with the implicit constant depending on $\mathcal{H}^1(I)$. They also prove a higher dimensional analogue of this estimate for capacities $\Gamma_{n,d}$. The main advantage of [CT20] over our results is that their “$L^2$-projections” assumption is single-scale, while the PBP condition is multi-scale. On the other hand, our result may be easier to apply as it derives a lower bound for $\gamma(E)$ directly from the information on the projections of $E$, without the need of constructing a measure supported on $E$ and studying its projections.

Even more importantly, note that the $L^2$-projections assumption $\pi_\theta \mu \in L^2(\ell_\theta)$ implies a big projection $\mathcal{H}^1(\pi_\theta(E)) \geq ||\pi_\theta \mu||_{L^2}^2$ using the Cauchy-Schwarz inequality

$$1 = \mu(E) = ||\pi_\theta \mu||_{L^1} \leq ||\pi_\theta \mu||_{L^2} \mathcal{H}^1(\pi_\theta(E))^{1/2}.$$  

The difference between $L^2$-projections and big projections is fundamental. A characterization of Ahlfors regular sets with big pieces of Lipschitz graphs in terms of $L^2$-projections has been achieved by Martikainen and Orponen in [MO18], but it took another major breakthrough [Orp21] to find an analogous characterization in terms of PBP. Moreover, a “single-scale version” of [Orp21] is an open problem, see [Orp21, §1.3], whereas [MO18] contains the relevant single-scale result, see [MO18, Theorem 1.7].

**Remark 1.9.** The proof of Theorem 1.6 is robust, and our techniques are likely to be useful in the future work on problems (1.2) and (1.3).

In particular, suppose that the following strengthening of Orponen’s result (Theorem 2.5) was available: “if a set $E \subset \mathbb{R}^n$ is Ahlfors $d$-regular and it has uniformly large Favard length (there exists $C > 0$ such that for all $x \in E$ and all $0 < r < \text{diam}(E)$ we
have $\text{Fav}(E \cap B(x, r)) \geq C r^d$, then $E$ has big pieces of Lipschitz graphs.” With such result at hand, the proof of Theorem 1.6 would immediately yield that any set $E \subset \mathbb{R}^d$ with uniformly large Favard length satisfies

$$\Gamma_{n,d}(E) \gtrsim \text{diam}(E)^d.$$  

A related question is of course the following: suppose that $E \subset [0,1]^2$ is a 1-Ahlfors regular set with $\text{Fav}(E) \geq \delta$. Is it then true that there exists a Lipschitz graph $\Gamma$ so that $H^1(E \cap \Gamma) \gtrsim \delta$? A recent related result by A. Chang, T. Orponen and the authors [CDOV22] shows that if $E$ has almost maximal Favard length, then it is contained in a Lipschitz graph with small constant, save for a tiny subset. A natural question is whether this result can be used to obtain a new estimate for analytic capacity.

1.3. Analyst’s Traveling Salesman Theorem. Another main result of this article is the Analyst’s Travelling Salesman Theorem (TST) for sets with PBP. Recall that the original Analyst’s TST is due to Jones [Jon90], and it is a characterization of sets $E \subset \mathbb{R}^2$ such that there exists a rectifiable curve $\Gamma$ containing $E$, along with a quite precise estimate on $H^1(\Gamma)$. Much work has been put into proving analogs of this result in other spaces; for example, Okikiolu [OK92] proved an Analyst’s TST for curves in $\mathbb{R}^d$, and in [Sch07] Schul further generalized it to the Hilbert space setting. Another research direction consists in finding statements analogous to Jones’ TST but for higher dimensional sets rather than curves. While we are still lacking a theorem as complete as that of Jones, in recent years there has been much progress in this direction. Two major difficulties were: first, the coefficients used by Jones simply do not work for higher dimensional set; second, it is not obvious what to use as an analogue to curves (e.g. topological sphere do not work, see [Vil19 Introduction]). These issues have largely been overcome in the series of papers [AS18, AV21, Vil19, Hyd20, Hyd21]. We refer the reader to Sections 1 and 3 of [Vil19] for a more in-depth discussion of Analyst’s TST and its relevance. Let us mention that while these results are related to the rectifiability of sets, they do not say much about whether these sets can be parameterised. This is a major open problem in the area.

Our present result continues the line of work of Azzam, Schul, and the second author [AS18, AV21]. Let $D$ be a system of Christ-David cubes (see Lemma 2.1), and set

$$\beta(Q_0) = \beta_{E,C_0,p,d}(Q_0) := \sum_{Q \in D(Q_0)} \beta_{E,p}^{d,p}(C_0 B_Q)^2 \ell(Q)^d,$$  

(1.6)

where the coefficients $\beta_{E,p}^{d,p}$ are a variant of Jones’ $\beta$-numbers (1.9) introduced by Azzam and Schul in [AS18], see Definition 2.12.

**Theorem 1.10.** Let $E \subset \mathbb{R}^n$ be a set with $d$-PBP with constant $\delta > 0$, $D$ be a system of Christ-David cubes, $Q_0 \in D$ and $C_0 \geq 3$. Let $1 \leq p < p(d)$, where $p(d) = \frac{2d}{d-2}$ for $d > 2$ and $p(d) = \infty$ if $d \leq 2$. Then

$$\text{diam}(Q_0)^d + \beta(Q_0) \sim H^d(Q_0),$$  

(1.7)

where the implicit constant depends on $\delta$, $C_0$, $p$, $n,d$ and on the constants from the Azzam-Schul TST (see Theorem A.1 in [AV21]).
Remark 1.11. What we really prove here is one direction of the inequality (bound on the $\beta$ sum with the measure), as the other one follows from [AS18]. See Proposition 4.2.

Estimates similar to (1.7) have been proven in [AS18] and [AV21] for general lower content regular sets (see §2.3 for the definition; sets with PBP are lower content regular), with the following crucial caveat: in general, an additional error term needs to be added to the right hand side of (1.7). In [Vil19] the second author proved that the error term may be omitted if one assumes that $E$ is a topologically stable $d$-surface (a condition satisfied e.g. by Reifenberg flat sets, or Semmes surfaces), see [Vil19] Theorem 3.6. In Theorem 1.10 we prove that the error term disappears also in the case of sets with PBP. In other words, both PBP and topological stability give the set enough rigidity so that one can estimate its $\beta$-numbers using the Hausdorff measure.

To see why the estimate (1.7) may be useful, let us mention that in [AS18] Theorem II it was shown that a lower content regular set with $\beta(E) < \infty$ is rectifiable. Thus, any class of lower content regular sets for which (1.7) holds satisfies

$$\mathcal{H}^d(E) < \infty \implies \text{rectifiability}. \quad (1.8)$$

In particular, (1.8) holds for topologically stable surfaces, and for sets with PBP (of course, for sets with PBP (1.8) easily follows from the Besicovitch projection theorem, without the need to refer to $\beta$-numbers and TST). Identifying classes of sets for which (1.8) holds is an interesting problem. For example, while (1.8) is true for connected one dimensional sets, there exist 2-dimensional sets, homeomorphic to the 2-sphere, with finite $\mathcal{H}^2$-measure, and containing a purely 2-unrectifiable set of positive measure (see [Vil19, Figure 1]). See also [DK19] and [DLD20] in connection with (1.8) and quantitative topological conditions.

Theorem 1.10 is used in the proofs of Theorems 1.5 and 1.7. Another application is an estimate for the dimension of wiggly sets.

1.4. Dimension of wiggly sets. A set $E$ is said to be uniformly wiggly of dimension $d$ and with parameter $\beta_0$ if for all balls $B$ centered on $E$ and with $0 < r(B) < \text{diam}(E)$ it holds that

$$\beta_{d,\infty}^E(B) > \beta_0. \quad (1.9)$$

Here $\beta_{d,\infty}^E$ is the $d$-dimensional version of the so-called $L^\infty$ $\beta$-number of Peter Jones [Jon90] (see Definition 2.6). Wiggly sets appear naturally in various contexts where some form of self-similarity is present. Examples include limit sets of certain Kleinian groups, some Julia sets of polynomials, and random sets (see [BP17], pp. 340-341). They were first studied via the Analyst’s TST by Bishop and Jones in [BJ97], were they proved a lower bound on the Hausdorff dimension of wiggly connected sets in the plane, assuming $d = 1$. Their result was generalised to continua in metric spaces by Azzam [Azz15]. A result in this vein was later proved by David [Dav04] (and quantified in [Vil19]), this time for uniformly non flat sets of any integer dimension, satisfying a topological condition. David’s work was motivated by a question of L. Potyagailo, concerning higher dimensional limit sets (see the introduction of [Dav04]). The result below can be seen as a version of David’s theorem with the topological condition replaced by the PBP assumption.
Theorem 1.12. Let \( n \geq 2 \) and \( 1 \leq d \leq n - 1 \). Let \( E \subset \mathbb{R}^n \) be a closed set with \( d \)-PBP (with parameter \( \delta > 0 \)) and which is uniformly wiggly of dimension \( d \) and constant \( \beta_0 \). Then

\[
\dim_H(E) \geq d + c\beta_0^{2(d+1)},
\]

where \( c \) depends on \( \delta, n, d, p \).

Remark 1.13. We defined uniformly wiggly sets in terms of \( \beta_{E,\infty} \), but, in principle, we could have used the \( L^p \) versions of the coefficients instead. Indeed, if we take a set \( E \) that satisfies the hypotheses of Theorem 1.12 with respect to the Azzam-Schul \( \beta_{E,\infty}^{A,p} \) numbers, assuming \( 1 \leq p < p(d) \) (where \( p(d) \) is defined in Theorem 1.10), then we obtain a somewhat sharper dimension estimate

\[
\dim_H(E) \geq d + c\beta_0^2,
\]

(1.11)

see Proposition 5.1. This is not unexpected, since the correct \( \beta \)-coefficients to use when working with higher dimensional sets are the averaged ones.

Another comment on definitions is in order: many interesting sets satisfy non-flatness hypotheses weaker than uniform wiggly. For example, attractors of many dynamical systems are only mean wiggly, in the sense that the \( \beta \) coefficients are large in many (but not all) scales and locations. See [GJM12, GM22] for a definition. See also [KR97] for a result similar in spirit concerning mean porous sets.

Remark 1.14. The dependence of \( \dim_H(E) \) on the PBP parameters is not a proof artifact: suppose we had a lower bound for Hausdorff dimension depending only on \( \beta_0 \) and not on \( \delta \). Consider the four corners Cantor set \( E \), constructed in the usual way, except that we dilate by constant \( \eta > 1 \) the squares in the construction. In the limit, we will obtain that a) \( \dim_H(E) = 1 + C(\eta) \) with \( C(\eta) \to 0 \) as \( \eta \to 1 \), b) \( \beta_{E,\infty}^1(B) \gtrsim 1 \geq \beta_0 > 0 \) for any ball \( B \) centered on \( E \), and c) \( E \) will have PBP with parameters depending on \( \eta \). But then, our hypothetical (and false) theorem would tell us that \( \dim_H(E) \geq 1 + c\beta_0^2 \) independently of \( \eta \). This example was pointed out by T. Orponen in a discussion on a first draft, where the exact dependence of the dimension estimate on the PBP constant had been overlooked.

In fact, after a minute’s thinking, the dependence on \( \delta \) of (1.10) appears altogether natural: the parameter \( \delta \) regulates the “rigidity” of a set with PBP, where one should think of connected set as having ‘rigidity’ 1.

1.5. Outline of the paper and proof ideas. In Section 2 we establish notation and state some quantitative rectifiability results used in the paper.

In Section 3 we prove Theorem 1.6. Since \( \Gamma_{2,1}(E) \sim \gamma(E) \) by [To03], Theorem 1.3 follows from Theorem 1.6 by taking \( n = 2, d = 1 \). To prove Theorem 1.6 we use a lower bound on \( \Gamma_{n,d} \) due to Prat [Pra12] and Girela-Sarrión [GS19], formulated in terms of measures satisfying certain flatness condition, see Theorem 2.10. We show that if \( E \) has PBP, then the Frostman measure \( \mu \) on \( E \) satisfies the flatness condition, and so Theorem 2.10 yields the desired lower bound. To show that \( \mu \) satisfies the flatness condition, we conduct a stopping time argument to construct a corona decomposition of the David-Christ lattice \( D \) into trees such that on each tree the measure \( \mu \) is approximately Ahlfors regular. For each tree, we construct an approximating measure \( \eta \) such that it is Ahlfors regular and \( \text{supp} \eta \) satisfies the PBP condition. A recent result of Orponen
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[Orp21] implies that $\eta$ satisfies the desired flatness condition. Finally, we transfer the
flatness condition back to the original Frostman measure $\mu$.

In Section 4 we prove the TST for sets with PBP, Theorem 1.10. The idea is the
following: we start off with $E$ which has PBP. Then it is easy to see that $E$ is lower
content regular. We can then use a coronisation of lower content regular sets by Ahlfors
regular sets proved in [AV21], which gives a decomposition of the cubes of $E$ into trees,
and for each tree $T$ we have an Ahlfors regular set $E_T$. The family of root cubes of
these trees satisfy a packing condition. Each $E_T$ inherits the PBP from $E$, and therefore
it is uniformly rectifiable by Orponen’s result mentioned above. By the David-Semmes
theory, each $E_T$ satisfies the strong geometric lemma. Theorem 1.10 is obtained by
carefully transferring these good estimated on $\beta$-numbers from $E_T$ to $E$.

The proofs of Theorems 1.6 and 1.10 bear many similarities. In fact, the proof of
Theorem 1.6 hinges on what can be seen as an “Analyst’s TST for measures” (Proposition
3.1 below). In turn, the proof of Proposition 3.1 is based on a coronisation of Frostman
measures of lower content regular sets by Ahlfors regular measures. It is, in spirit at
least, a version of the coronisation in [AV21], where instead of working directly with
a set $E$, we work with a Frostman measure $\mu$ on $E$. This allows us to deal with sets
with non-$\sigma$-finite Hausdorff measure, a setting where the techniques from [AV21] become
useless. Another advantage of this approach, which will be explored in future work, is
that it allows us to study rectifiability of lower content regular sets (e.g. Analyst’s TST)
without the need for Azzam-Schul $\beta$-numbers defined in terms of Hausdorff content.
These object are very useful in some respects, but difficult to work with at times.

The last two sections are dedicated to Theorem 1.12 and Theorem 1.17 (note that
Theorem 1.12 follows from Theorem 1.7 since $\gamma(E) \sim \Gamma_{2,1}(E)$). The strategy of the proof
of Theorem 1.12 is that of Bishop and Jones [BJ97], while the proof of Theorem 1.17
follows Verdera’s [To14, Theorem 4.31].

In Appendix A we construct a doubling Frostman measure supported on a lower con-
tent regular set. We use this modified Frostman measure in the proof of Theorem 1.6.
The idea of the proof is rather simple: the doubling condition is broken if two neighbour-
ing cubes have very unequal amount of mass. We fix this by recursively redistributing
“the wealth”, that is, the mass, form the cubes which are too wealthy, to the poorer
ones. We are grateful to Tuomas Orponen for helping us with the construction.

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2. Preliminaries

2.1. Notation. We gather here some notation and some results which will be used later on. We write $a \lesssim b$ if there exists an constant $C$ such that $a \leq Cb$. If the constant $C$ depends on a parameter $t$, we write $a \lesssim_t b$. By $a \sim b$ we mean $a \lesssim b \lesssim a$.

For two subsets $A, B \subset \mathbb{R}^n$, we let $\text{dist}(A, B) := \inf_{a \in A, b \in B} |a - b|$. For a point $x \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$, $\text{dist}(x, A) := \text{dist}\{x\}, A\} = \inf_{a \in A} |x - a|$. The cardinality of a set $A$ is denoted by $\# A$.

We write $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$, and, for $\lambda > 0$, $\lambda B(x, r) := B(x, \lambda r)$. At times, we may write $\mathbb{B}$ to denote $B(0, 1)$. When necessary we write $B_n(x, r)$ to distinguish a ball in $\mathbb{R}^n$ from one in $\mathbb{R}^d$, which we may denote by $B_d(x, r)$. Given a ball $B$, we denote by $r(B)$ its radius.

If $\mu$ is a Radon measure on $\mathbb{R}^n$, then the $d$-dimensional density of $\mu$ in the ball $B = B(x, r)$ is

$$\theta_\mu(B) = \theta_\mu(x, r) = \frac{\mu(B(x, r))}{r^d}.$$ 

We denote by $G(n, d)$ the Grassmanian, that is, the manifold of all $d$-dimensional linear subspaces of $\mathbb{R}^n$. A ball in $G(n, d)$ is defined with respect to the standard metric

$$d_G(V, W) = \|\pi_V - \pi_W\|_{\text{op}}.$$ 

Recall that $\pi_V : \mathbb{R}^n \to V$ is the standard orthogonal projection onto $V$. With $A(n, d)$ we denote the affine Grassmanian, the manifold of all affine $d$-planes in $\mathbb{R}^n$.

2.2. Dyadic lattice and Christ-David cubes. The family of dyadic cubes in $\mathbb{R}^n$ will be denoted by $\Delta$, and the family of dyadic cubes with sidelength $\ell(I) = 2^{-k}$ by $\Delta_k$. The $d$-dimensional skeleton of $I \in \Delta$ (i.e. the union of the $d$-dimensional faces of $I$) will be denoted by $\partial_I I$.

The idea to consider generalized dyadic cubes, that is, nested partitions of sets with nice properties, goes back to David [Dav88] and Christ [Chr90]. In many contexts it is important that these generalized cubes have thin boundaries with respect to a given measure. Since we will not need this property, we may use for example the cubes from [KRS12]. A special case of their construction gives the following.

Lemma 2.1 ([KRS12]). Let $E \subset \mathbb{R}^n$, $\rho = 1/1000$ and $c_0 = 1/500$. Then, for each $k \in \mathbb{Z}$, there is a collection $D_k$ of generalized cubes on $E$ such that the following hold.

1. For each $k \in \mathbb{Z}$, $E = \bigcup_{Q \in D_k} Q$, and the union is disjoint.
2. If $Q_1, Q_2 \in \bigcup_k D_k$ and $Q_1 \cap Q_2 \neq \emptyset$, then either $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$.
3. For $Q \in \bigcup_k D_k$, let $k(Q)$ be the unique integer so that $Q \in D_k$ and set $\ell(Q) = 5\rho^k$. Then there is $x_Q \in Q$ such that $B(x_Q, c_0 \ell(Q)) \cap E \subseteq Q \subseteq B(x_Q, \ell(Q))$. 

We introduce some notation related to cubes. Given two integers \( k < l \), we set \( \mathcal{D}_k^l = \bigcup_{i=k}^l \mathcal{D}_i \). If \( R \in \mathcal{D} \), we will denote the descendants of \( R \) by
\[
\mathcal{D}(R) = \{ Q \in \mathcal{D} : Q \subseteq R \},
\]
and \( \mathcal{D}_k(R) = \mathcal{D}(R) \cap \mathcal{D}_k \). On the other hand, \( Q^1 \) will denote the parent of \( Q \), i.e. the unique cube such that \( \ell(Q) = \rho \ell(Q^1) \) and \( Q \subseteq Q^1 \).

The child-parent relation endows \( \mathcal{D} \) with a natural tree structure. We will say that a collection \( \mathcal{T} \subset \mathcal{D} \) is a tree if
- \( \mathcal{T} \subset \mathcal{D}(R) \) for some \( R \in \mathcal{T} \). This maximal cube \( R \) will be called the root of \( \mathcal{T} \).
- for any \( Q \in \mathcal{T} \) we also have \( P \in \mathcal{T} \) for all cubes \( P \in \mathcal{D} \) with \( Q \subseteq P \subset R \), where \( R \) is the root of \( \mathcal{T} \).

The minimal cubes of \( \mathcal{T} \) will be called its stopping cubes.

For every \( Q \in \mathcal{D} \) we set \( B(Q) := B(x_Q, c_0 \ell(Q)) \) and \( B_Q = B(x_Q, \ell(Q)) \), so that
\[
B(Q) \cap E \subset Q \subset B_Q.
\]
Note that if \( P \subset Q \), then \( 2B_P \subset 2B_Q \).

Given a Radon measure \( \mu \) and a cube \( Q \in \mathcal{D} \), we define the \( d \)-dimensional density as
\[
\theta_\mu(Q) = \frac{\mu(Q)}{\ell(Q)^d}.
\]

### 2.3. Lower content regular sets.
Recall that a set \( E \subset \mathbb{R}^n \) is lower content \((d,c_1)\)-regular if, for all balls \( B \) centered on \( E \),
\[
\mathcal{H}^d_\infty(E \cap B) \geq c_1 r(B)^d.
\]
We show below that sets with \( d \)-PBP are lower content regular.

**Lemma 2.2.** Let \( E \subset \mathbb{R}^n \) be a set with \( d \)-PBP with constants \( \delta > 0 \). Then \( E \) is lower content \( d \)-regular with constant \( c \sim_d \delta \).

**Proof.** Without loss of generality, we identify \( V \) with \( \mathbb{R}^d \). For an arbitrary \( \epsilon_1 > 0 \), let \( B \) be a family of balls in \( \mathbb{R}^d \) so that
\[
\sum_{B \in B} r(B)^d \leq \mathcal{H}^d(\pi_V(E \cap B)) - \epsilon_1.
\]
Note that, since these are balls in a \( d \)-plane, \( \mathcal{H}^d(B^1 \cap \pi_V(B \cap E)) \lesssim_d r(B)^d \). Let \( \delta \) be the parameter with which \( E \) satisfied \( d \)-PBP. Fix a ball \( B \) centered on \( E \), with \( r(B) \leq \text{diam}(E) \), and a plane \( V \) in \( B(V_B, \delta) \). Then,
\[
\delta r(B)^d \leq \mathcal{H}^d(\pi_V(E \cap B)) \leq \sum_{B \in B} \mathcal{H}^d(\pi_V(E \cap B) \cap B') \lesssim_d \sum_{B' \in B} r(B')^d \leq C \mathcal{H}^d(\pi_V(E \cap B)) + C \epsilon_1
\]
\[
\leq C' \mathcal{H}^d_\infty(\pi_V(E \cap B)) + C \epsilon_1.
\]
Now, since \( \pi_V \) is 1-Lipschitz and \( \epsilon_1 \) was arbitrary, we obtain the lemma. The lower content regularity constant \( c \) depends only on \( \delta \) and \( d \), since \( C \) in the above display only depends on \( d \). \( \square \)
2.4. Frostman measure associated to \( E \). In Appendix A we construct a particularly nice Frostman measure supported on a lower content regular set \( E \). Its properties are listed in the lemma below.

**Lemma 2.3.** Let \( E \subset \mathbb{R}^n \) be a compact lower content \((d, c_1)\)-regular set. Then, there exists a measure \( \mu \) with \( \text{supp} \, \mu \subset E \) satisfying the following properties:

1. \( \mu(E) = \mathcal{H}_d^\infty(E) \gtrsim c_1 \text{diam}(E)^d \),
2. \( \mu \) has polynomial growth, that is, there exists a constant \( C_1 \geq 1 \) such that for all \( x \in E \) and \( 0 < r < \text{diam}(E) \) we have
   \[ \mu(B(x, r)) \leq C_1 r, \]
3. \( \mu \) is doubling, that is, there exists a constant \( C_{db} > 1 \) such that for all \( x \in E \) and \( 0 < r < \text{diam}(E) \) we have
   \[ \mu(B(x, 2r)) \leq C_{db} \mu(B(x, r)) \] (2.1)
4. the \( d \)-dimensional density of \( \mu \) is almost monotone, that is, there exists a constant \( A \geq 1 \) such that if \( P, Q \in D \), and \( P \subset Q \), then
   \[ \theta_\mu(P) \leq A \theta_\mu(Q). \]

In the above, \( C_1 \) may depend only on \( d, n \), while \( C_{db} \) and \( A \) may also depend on the LCR-constant \( c_1 \).

Observe that thanks to (2.1), if \( E \) is lower content regular and \( D \) is the associated David-Christ lattice, then for all \( Q \in D \)
\[ \mu(Q) \leq \mu(2BQ) \lesssim_{c_1} \mu(B(Q) \leq \mu(Q). \] (2.2)

2.5. Quantitative rectifiability and \( \beta \)-numbers. Recall that one of the quantitative notions of rectifiability introduced by David and Semmes in [DS91] is given by the big pieces of Lipschitz graphs condition.

**Definition 2.4.** An Ahlfors \( d \)-regular set \( E \subset \mathbb{R}^n \) has big pieces of Lipschitz graphs (BPLG) if there exist constants \( C_0, L > 0 \) such that for any \( x \in E \) and \( 0 < r < \text{diam}(E) \) there exists a Lipschitz graph \( \Gamma \subset \mathbb{R}^n \) with \( \text{Lip}(\Gamma) \leq L \) and
\[ \mathcal{H}^d(E \cap B(x, r) \cap \Gamma) \geq C_0 r^d. \] (2.3)

In [DS93] David and Semmes conjectured that for Ahlfors regular sets, the PBP and BPLG conditions are equivalent. This was confirmed in a recent breakthrough result of Orponen [Orp21].

**Theorem 2.5 (Orp21).** Suppose that a set \( E \subset \mathbb{R}^n \) is Ahlfors \( d \)-regular. Then, it has \( d \)-PBP if and only if it has BPLG.

We recall different variants of \( \beta \)-numbers that we will use.

**Definition 2.6 (Jones).** Let \( E \subset \mathbb{R}^n \) and \( B \) a ball. Define
\[ \beta^d_{E, \infty}(B) = \frac{1}{r(B)} \inf_{L \in A(n,d)} \sup \{ \text{dist}(y, L) : y \in E \cap B \}. \]
For \( x \in \mathbb{R}^n \) and \( r > 0 \) we set also \( \beta^d_{E, \infty}(x, r) = \beta^d_{E, \infty}(B(x, r)) \), and we will use the same notation for other types of \( \beta \)-numbers, defined below. We will also usually omit the superscript \( d \).
**Definition 2.7** (David-Semmes). Let \( \mu \) be a Radon measure on \( \mathbb{R}^n \), \( B \subset \mathbb{R}^n \) a ball, and \( 1 \leq p < \infty \). The \( L^p \) variant of Jones’ \( \beta \)-numbers is defined as

\[
\beta_{\mu,p}^d(B) = \inf_{L \in A(n,d)} \left( \frac{1}{r(B)^d} \int_B \left( \frac{\text{dist}(y,L)}{r(B)} \right)^p d\mu(y) \right)^{\frac{1}{p}}.
\]

The following is a special case of a classical result of David and Semmes.

**Theorem 2.8** ([DS91]). Let \( E \subset \mathbb{R}^n \) be a bounded Ahlfors regular set with BPLG, and let \( \mu = \mathcal{H}^d|_E \). Then, there exists a constant \( C \), depending only on \( n, d \), and the BPLG and Ahlfors regularity constants of \( E \), such that

\[
\int_E \int_0^{\text{diam}(E)} \beta_{\mu,2}(x,r)^2 \frac{dr}{r} d\mu(x) \leq C \mu(E).
\]

Together with Theorem 2.5 this gives the following.

**Corollary 2.9.** If an Ahlfors regular set \( E \) has PBP, then the surface measure \( \mu = \mathcal{H}^d|_E \) satisfies

\[
\int_E \int_0^{\text{diam}(E)} \beta_{\mu,2}(x,r)^2 \frac{dr}{r} d\mu(x) \leq C \mu(E),
\]  

where \( C > 0 \) depends only on \( n, d \), and the PBP and Ahlfors regularity constants of \( E \).

The estimate (2.4) is extremely useful for estimating \( \Gamma_{n,d}(E) \) due to the following result.

**Theorem 2.10** ([Pra12, GS19]). Let \( E \subset \mathbb{R}^n \) be compact. Then,

\[
\Gamma_{n,d}(E) \gtrsim \sup \{ \mu(E) : \mu \in \mathcal{F}(E) \},
\]

where \( \mathcal{F}(E) \) is the set of Radon measures with \( \text{supp} \mu \subset E \) satisfying the polynomial growth condition

\[
\mu(B(x,r)) \leq r^d \quad \text{for all } x \in \text{supp} \mu \text{ and } r > 0,
\]

and the flatness condition

\[
\int_0^{\infty} \beta_{\mu,2}(x,r)^2 \theta_{\mu}(x,r) \frac{dr}{r} d\mu(x) \leq \mu(E).
\]

The theorem above is a combination of two results. Prat [Pra12] related the capacity \( \Gamma_{n,d} \) with the supremum over measures whose Riesz transform is in \( L^2 \), whereas Girela-Sarrión [GS19] proved that this Riesz transform condition is true for measures \( \mu \in \mathcal{F}(E) \). Prat’s result was first proved by Tolsa [Tol03] for \( d = 1, n = 2 \), and by Volberg [Vol03] in the case \( d = n - 1 \). The result of Girela-Sarrión was first shown for \( d = 1, n = 2 \), by Azzam and Tolsa [AT15]. Finally, let us mention that while (2.5) holds for all \( 1 \leq d < n \), in the codimension-1 case \( d = n - 1 \) an estimate converse to (2.5) is also known to be true. This was shown for \( d = 1 \) by Azzam and Tolsa [Tol05, AT15] and for general \( d \in \mathbb{N} \) by Tolsa and the first author [DT21, Tol21]. Whether the same is true in codimension larger than 1 is an open problem.
In the statement of Theorem 1.10 we used the content $\beta$-numbers of Azzam and Schul, which we recall below. For $1 \leq p < \infty$ and $A \subset \mathbb{R}^n$ Borel, we define the $p$-Choquet integral as

$$\int_A f(x)^p \, d\mathcal{H}^d_\infty(x) := \int_0^\infty \mathcal{H}^d_\infty(\{x \in A : f(x) > t\}) t^{p-1} \, dt.$$  

We refer the reader to \cite{Mat95} for more details on Hausdorff measures and content and to Section 2 and the Appendix of \cite{AS18} for more details on Choquet integration.

Lemma 2.11 (\cite{AS18}, Lemma 2.3). Let $E \subseteq \mathbb{R}^n$ be either compact or bounded and open so that $\mathcal{H}^d(E) > 0$, and let $f \geq 0$ be continuous on $E$. Then for $1 < p < \infty$,

$$\frac{1}{\mathcal{H}^d_\infty(E)} \int_E f \, d\mathcal{H}^d_\infty \lesssim_n \left( \frac{1}{\mathcal{H}^d_\infty(E)} \int_E f^p \, d\mathcal{H}^d_\infty \right)^{\frac{1}{p}}.$$  

Definition 2.12 (Azzam-Schul). Let $1 \leq p < \infty$, $E \subseteq \mathbb{R}^n$ and $B$ a ball. For a $d$-dimensional plane $L$ define

$$\beta^{d,p}_E(B, L) = \left( \frac{1}{r(B)^d} \int_{E \cap B} \left( \frac{\text{dist}(y, L)}{r(B)} \right)^p \, d\mathcal{H}_\infty^d(y) \right)^{\frac{1}{p}}.$$  

Then $\beta^{d,p}_E(B) = \inf_{L \in A(n,d)} \beta^{d,p}_E(B, L)$.

We will need the following lemma.

Lemma 2.13 (\cite{AS18}, Lemma 2.21). Let $1 \leq p < \infty$ and $E_1, E_2 \subseteq \mathbb{R}^n$. Let $x \in E_1$ and fix $r > 0$. Take some $y \in E_2$ so that $B(x, r) \subset B(y, 2r)$. Assume that $E_1, E_2$ are both lower content $d$-regular with constant $c$. Then

$$\beta^{d,p}_{E_1}(x, r) \lesssim_c \beta^{d,p}_{E_2}(y, 2r) + \left( \frac{1}{r^d} \int_{E_1 \cap B(x, 2r)} \left( \frac{\text{dist}(y, E_2)}{r} \right)^p \, d\mathcal{H}_\infty^d(y) \right)^{\frac{1}{p}}.$$  

3. \textsc{Vitushkin’s conjecture}

In this section we prove Theorem 1.10. Without loss of generality, we may assume that $\text{diam}(E) = 1$ (this follows from scaling $\Gamma_{n,d}(\lambda E) = \lambda^d \Gamma_{n,d}(E)$ for $\lambda > 0$). By Theorem 2.10 in order to show that $\Gamma_{n,d}(E) \gtrsim 1$, it suffices to construct a measure with polynomial growth supported on $E$ such that $\mu(E) \sim 1$ and

$$\iint_0^\infty \beta_{\mu,2}(x,r)^2 \, d\mu(x,r) \frac{dr}{r} \, d\mu(x) \lesssim \mu(E).$$  

Let $\mu$ be the measure from Lemma 2.3. Then it has polynomial growth, $\text{supp} \mu \subset E$, and $\mu(E) \sim 1$. We are going to show that the estimate (3.1) is satisfied by $\mu$. In fact, we will prove a somewhat stronger estimate.

Proposition 3.1. Let $E \subset \mathbb{R}^n$ be a compact set with $d$-PBP. If $\mu$ is the measure from Lemma 2.3 then

$$\iint_0^\infty \beta_{\mu,2}(x,r)^2 \, d\mu(x) \lesssim \mu(E),$$  

where the implicit constant depends only on $n, d$ and the PBP constants.
To see that (3.2) implies (3.1), recall that \( \theta_\mu(x,r) \lesssim 1 \) by the polynomial growth condition.

Observe that since we assume that \( \text{diam}(E) = 1 \), we have that \( D_0 \) consists of a single cube, namely \( D_0 = \{E\} \). By standard techniques, (3.2) is equivalent to

\[
\sum_{Q \in D} \beta_{\mu,2}(2B_Q)^2 \mu(Q) \lesssim \mu(E). 
\tag{3.3}
\]

For reader’s convenience, we sketch the proof of (3.3) \( \Rightarrow \) (3.2). Observe that if \( B_1 \subset B_2 \) are balls, and \( r(B_1) \sim r(B_2) \), then it follows from the definition of \( \beta_{\mu,2} \)-numbers that

\[
\beta_{\mu,2}(B_1) \lesssim \beta_{\mu,2}(B_2).
\]

Given \( x \in E \) and \( 0 < r < \infty \) let \( Q \in D \) be the unique cube with \( x \in Q \) and \( \rho \ell(Q) < r \leq \ell(Q) \), so that \( B(x,r) \subset 2B_Q \) and \( r(2B_Q) \sim r \). Then,

\[
\beta_{\mu,2}(x,r) \lesssim \beta_{\mu,2}(2B_Q),
\]

and it follows easily that

\[
\int_{0}^{\infty} \int_{r}^{\infty} \beta_{\mu,2}(x,r)^2 \frac{dr}{r} \, d\mu(x) \lesssim \sum_{Q \in D} \beta_{\mu,2}(2B_Q)^2 \mu(Q).
\]

Thus, our goal is to prove (3.3). Recalling that \( D_k = \{E\} \) for \( k \leq 0 \) and \( \mu(E) \sim 1 \), it follows immediately that

\[
\sum_{k \leq 0} \sum_{Q \in D_k} \beta_{\mu,2}(2B_Q)^2 \mu(Q) \lesssim \sum_{k \leq 0} \sum_{Q \in D_k} \ell(Q)^{-d} \mu(E)^3 \sim \sum_{k \leq 0} \rho^{-kd} \mu(E)^3 \lesssim \mu(E).
\]

So when proving (3.3), we may concentrate on \( Q \in D_k \) for \( k \geq 0 \). Since working with infinite sums would entail certain technicalities later on, we prefer to work with finite sums. Fix some large integer \( N > 1 \). We will show that

\[
\sum_{Q \in D_N^k} \beta_{\mu,2}(2B_Q)^2 \mu(Q) \lesssim \mu(E), \tag{3.4}
\]

with the estimate independent of \( N \). Letting \( N \to \infty \) gives the desired bound (3.3).

We are going to prove (3.4) using an appropriate corona decomposition.

3.1. Stopping time argument. We begin by conducting a stopping time argument.

Let \( R \in D_{N-1}^0 \). We will say that \( Q \in \text{LD}_0(R) \) (here LD stands for “low density”) if \( Q \in D_N^0 \), \( Q \subset R \), and

\[
\theta_\mu(Q) \leq \tau \theta_\mu(R),
\]

where \( \tau = 0.1 \). The refined family \( \text{LD}(R) \) consists of the maximal cubes from \( \text{LD}_0(R) \) (i.e., cubes \( Q \in \text{LD}_0(R) \) which are not properly contained in any other cube from \( \text{LD}_0(R) \)). We define also

\[
\text{End}(R) = \left\{ Q \in D_N : Q \subset R, \ Q \cap \bigcup_{P \in \text{LD}(R)} P = \emptyset \right\}
\]

Finally, we set

\[
\text{Stop}(R) = \text{LD}(R) \cup \text{End}(R),
\]

and

\[
\text{Tree}(R) = \left\{ Q \in D_N : Q \subset R, \text{ there exists } P \in \text{Stop}(R) \text{ such that } P \subset Q \right\}
\]
It follows immediately from the definition that $\text{Stop}(R)$ is a family of pairwise disjoint cubes covering $R$. Moreover, $R \notin \text{Stop}(R)$ (we have $R \notin \text{LD}(R)$ by the definition of $\text{LD}_0(R)$, and $R \notin \text{End}(R)$ because $R \notin \mathcal{D}_N$). Observe also that $\text{Stop}(R) \subset \text{Tree}(R)$.

In the lemma below we show that $\mu$ is $d$-ADR at the scales and locations of $\text{Tree}(R)$.

**Lemma 3.2.** Let $R \in \mathcal{D}_0^{N-1}$. Then, for all $Q \in \text{Tree}(R)$

$$\tau \theta_\mu(R) \lesssim \theta_\mu(Q) \lesssim \theta_\mu(R).$$

*Proof.* The upper estimate $\theta_\mu(Q) \lesssim \theta_\mu(R)$ follows immediately from property (4) in Lemma 2.3, and the fact that $Q \subset R$.

To see the lower bound $\tau \theta_\mu(R) \lesssim \theta_\mu(Q)$, note that this is obvious for $Q \in \text{Tree}(R) \setminus \text{LD}(R)$: for such cubes we have

$$\theta_\mu(Q) > \tau \theta_\mu(R),$$

by the stopping time condition of $\text{LD}_0(R)$.

Assume now that $Q \in \text{LD}(R)$, and denote the parent of $Q$ by $Q^1$. Then, $Q^1 \in \text{Tree}(R) \setminus \text{LD}(R)$, so that $\theta_\mu(Q^1) > \tau \theta_\mu(R)$. But now we see by the doubling property of $\mu$ (2.1) that $\theta_\mu(Q) \gtrsim \theta_\mu(Q^1)$, so that

$$\theta_\mu(Q) \gtrsim \theta_\mu(Q^1) \gtrsim \tau \theta_\mu(R).$$

□

The following key lemma is the only point in the proof where we use the PBP propety of $E$ (except for the fact that PBP implies lower content regularity of $E$, which we also use multiple times throughout the proof).

**Lemma 3.3.** For any $R \in \mathcal{D}_0^{N-1}$ we have

$$\sum_{Q \in \text{Tree}(R)} \beta_{\mu,2}(2B_Q)^2 \mu(Q) \lesssim \theta_\mu(R) \mu(R),$$

with the implicit constant depending only on $n, d$ and the PBP constants of $E$.

We defer the proof of this lemma to Subsections 3.3–3.5.

### 3.2. Corona decomposition.

We are ready to perform the corona decomposition of $\mathcal{D}$ using the stopping time argument of the previous section.

Set $\text{Top}_0 = \mathcal{D}_0 = \{ E \}$ and $\text{Top}_1 = \text{LD}(E) \setminus \mathcal{D}_N$. We proceed by induction: assume that $\text{Top}_k$ has already been defined, and that $\text{Top}_k \subset \mathcal{D}_0^{N-1}$. Then, we set

$$\text{Top}_{k+1} = \bigcup_{P \in \text{Top}_k} \text{LD}(P) \setminus \mathcal{D}_N.$$

Note that for each $k$ the family $\text{Top}_k$ consists of pairwise disjoint cubes. Observe also that there exists $1 \leq k_0 \leq N$ such that $\text{Top}_{k_0} \neq \emptyset$, and then $\text{Top}_k = \emptyset$ for all $k > k_0$.

We define

$$\text{Top} = \bigcup_{k=0}^{k_0} \text{Top}_k.$$

Remark that

$$\mathcal{D}_0^N = \bigcup_{R \in \text{Top}} \text{Tree}(R),$$
and the sum above is disjoint.

Recall that our aim is to prove (3.4), that is,
\[
\sum_{Q \in \mathcal{D}_0^N} \beta_{\mu,2}(2B_Q)^2 \mu(Q) = \sum_{R \in \text{Top}} \sum_{Q \in \text{Tree}(R)} \beta_{\mu,2}(2B_Q)^2 \mu(Q) \lesssim \mu(E).
\]

Let us show how the key estimate from Lemma 3.3 together with the corona decomposition constructed above implies (3.4).

**Proof of (3.4) using Lemma 3.3** It immediately follows from (3.5) that
\[
\sum_{Q \in \mathcal{D}_0^N} \beta_{\mu,2}(2B_Q)^2 \mu(Q) = \sum_{R \in \text{Top}} \sum_{Q \in \text{Tree}(R)} \beta_{\mu,2}(2B_Q)^2 \mu(Q) \lesssim \sum_{R \in \text{Top}} \theta_{\mu}(R) \mu(R) = \sum_{k=0}^{k_0} \sum_{R \in \text{Top}_k} \theta_{\mu}(R) \mu(R).
\]

Now observe that for $R \in \text{Top}_k$, $0 \leq k \leq k_0$, we have $\theta_{\mu}(R) \leq C \tau^k$ for some constant $C$ depending on $\mu(E)$ (and $\mu(E) \sim 1$, with the implicit constant depending on the PBP constant). Indeed, this follows from a simple induction argument: it is true for $k = 0$ because $\theta_{\mu}(E) \sim \mu(E)$. If $R \in \text{Top}_{k+1}$, then $R \in \text{LD}(R')$ for some $R' \in \text{Top}_k$. Since $\theta_{\mu}(R') \leq C \tau^k$ by the inductive assumption, we get from the definition of $\text{LD}(R)$ that
\[
\theta_{\mu}(R) \leq \tau \theta_{\mu}(R') \leq C \tau^{k+1},
\]
which closes the induction.

It follows that
\[
\sum_{k=0}^{k_0} \sum_{R \in \text{Top}_k} \theta_{\mu}(R) \mu(R) \lesssim \sum_{k=0}^{k_0} \sum_{R \in \text{Top}_k} \tau^k \mu(R) \leq \sum_{k=0}^{k_0} \tau^k \mu(E),
\]
where in the last estimate we used the fact that for each $k$ the cubes in $\text{Top}_k$ are pairwise disjoint. Recalling that $\tau = 0.1$, we get that
\[
\sum_{Q \in \mathcal{D}_0^N} \beta_{\mu,2}(2B_Q)^2 \mu(Q) \lesssim \sum_{k=0}^{k_0} \sum_{R \in \text{Top}_k} \theta_{\mu}(R) \mu(R) \lesssim \sum_{k=0}^{k_0} \tau^k \mu(E) \lesssim \sum_{k=0}^{\infty} \tau^k \mu(E) \leq 2\mu(E).
\]
This finishes the proof of (3.4). We stress that in the estimates above the implicit constants do not depend on $N$. □

In the light of the above, all that remains to do is prove Lemma 3.3. We do this in the remaining three subsections.

3.3. **Regularizing trees.** Fix $R \in \mathcal{D}_0^{N-1}$. In what follows, it will be more convenient to work with regular trees, in the sense that nearby stopping cubes have comparable sidelengths. In order to regularize $\text{Tree}(R)$ we will use a technique from [DS91] which is nowadays considered fairly standard. It will also be useful to work with certain cubes neighboring with $R$. We provide the details below.

We define the set of cubes neighboring with $R$ as
\[
\mathcal{N}(R) = \{ R' \in \mathcal{D} : \ell(R') = \ell(R), \ R' \cap 2B_R \neq \emptyset \}.
\] (3.6)
Note that \( R \in \mathcal{N}(R) \). We define also an extended version of \( \mathcal{D}(R) \):

\[
\mathcal{D}_*(R) = \bigcup_{R' \in \mathcal{N}(R)} \mathcal{D}(R').
\]

Now we perform the regularization algorithm. Given \( x \in \mathbb{R}^n \) set

\[
d_R(x) = \inf_{Q \in \text{Tree}(R)} \text{dist}(x, Q) + \ell(Q)
\]

and for \( Q \in \mathcal{D} \) set

\[
d_R(Q) = \max \left( \frac{1}{20} \inf_{x \in Q} d_R(x), 5\rho^N \right),
\]

where the parameter \( \rho = 1/1000 \) comes from the definition of \( \mathcal{D} \). Observe that the quantity \( d_R(Q) \) is "monotone" in the sense that if \( P \subset Q \), then \( d_R(P) \geq d_R(Q) \).

We define \( \text{Reg}_*(R) \) to be the family of maximal cubes \( Q \in \mathcal{D}_*(R) \) satisfying

\[
\ell(Q) \leq d_R(Q).
\]  

(3.7)

Note that the \( \ast 5\rho^N \) term in the definition of \( d_R(Q) \) ensures that the inequality above is satisfied by all \( Q \in \mathcal{D}_N \), so that \( \text{Reg}_*(R) \subset \mathcal{D}_*(R) \cap \mathcal{D}_0^N \). Observe that the cubes in \( \text{Reg}_*(R) \) are pairwise disjoint, by maximality, and also

\[
\bigcup_{Q \in \text{Reg}_*(R)} Q = \bigcup_{R' \in \mathcal{N}(R)} R' \supset 2B_R \cap E.
\]  

(3.8)

**Lemma 3.4.** If \( Q \in \text{Reg}_*(R) \) and \( x \in 5B_Q \), then \( d_R(x) \sim \ell(Q) \). Consequently, if \( Q, P \in \text{Reg}_*(R) \), and \( 5B_Q \cap 5B_P \neq \emptyset \), then \( \ell(Q) \sim \ell(P) \).

**Proof.** Let \( Q \in \text{Reg}_*(R) \) and \( x \in 5B_Q \). First we show that \( d_R(x) \geq \ell(Q) \). By the definition of \( \text{Reg}_*(R) \), we have \( \ell(Q) \leq d_R(Q) \), so it suffices to show that \( d_R(x) \geq d_R(Q) \), that is

\[
d_R(x) \geq \max \left( \frac{1}{20} \inf_{y \in Q} d_R(y), 5\rho^N \right).
\]

Suppose the maximum above is achieved by \( 5\rho^N \), i.e. \( d_R(Q) = 5\rho^N \). Then, the estimate \( d_R(x) \geq 5\rho^N \) is clear by the definition of \( d_R \), since \( \text{Tree}(R) \subset \mathcal{D}_0^N \).

Assume now that \( d_R(Q) = \frac{1}{20} \inf_{y \in Q} d_R(y) \). Since the function \( d_R \) is 1-Lipschitz, we have

\[
\frac{1}{20} d_R(x) \geq \frac{1}{20} d_R(x_Q) - \frac{1}{20} |d_R(x) - d_R(x_Q)| \geq \frac{1}{20} d_R(x_Q) - \frac{5}{20} \ell(Q) \\
\geq \frac{1}{20} \inf_{y \in Q} d_R(y) - \frac{1}{4} \ell(Q) = d_R(Q) - \frac{1}{4} \ell(Q) \geq \frac{3}{4} \ell(Q).
\]

Hence, \( d_R(x) \geq 15\ell(Q) \).

We move on to the estimate \( d_R(x) \lesssim \ell(Q) \). Recall that, by the definition of \( \text{Reg}_*(R) \), \( Q \) is a maximal cube satisfying \( \ell(Q) \leq d_R(Q) \). In particular, \( Q^1 \) satisfies

\[
\ell(Q^1) = \max \left( \frac{1}{20} \inf_{y \in Q^1} d_R(y), 5\rho^N \right) \geq \frac{1}{20} \inf_{y \in Q^1} d_R(y).
\]
Let $y \in Q^1$ be such that $\ell(Q^1) \geq 1/20 \, d_R(y)$. Since $x, y \in 2B_Q$, we may use the 1-Lipschitz property of $d_R$ to conclude that

$$
\ell(Q^1) \geq \frac{1}{20} \, d_R(y) \geq \frac{1}{20} \, d_R(x) - \frac{1}{20} \, |d_R(y) - d_R(x)| \geq \frac{1}{20} \, d_R(x) - \frac{4}{20} \, \ell(Q^1).
$$

Thus, $\ell(Q) = \rho^{-1} \, \ell(Q^1) \gtrsim d_R(x)$. □

We define the extended, regularized “tree” as

$$
\text{Tree}_*(R) = \{Q \in \mathcal{D}_*(R) : \text{there exists } P \in \text{Reg}_*(R) \text{ such that } P \subset Q\}.
$$

The family Tree$_*(R)$ might not be a “true” tree since we cannot guarantee that all $Q \in \text{Tree}_*(R)$ are contained in $R$. Nevertheless, it is a union of a bounded number of trees $\text{Tree}_*(R) \cap \mathcal{D}(R')$, $R' \in \mathcal{N}(R)$.

Remark that since $\text{Reg}_*(R) \subset \mathcal{D}_0^N$, we also have $\text{Tree}_*(R) \subset \mathcal{D}_0^N$. Below we prove that $\text{Tree}_*(R)$ is larger than the original tree $\text{Tree}(R)$.

**Lemma 3.5.** We have $\text{Tree}(R) \subset \text{Tree}_*(R)$.

**Proof.** It suffices to show that each $Q \in \text{Stop}(R)$ contains some $P \in \text{Reg}_*(R)$. To this end, observe that if $x \in Q$, then from the definition of $d_R$ we have $d_R(x) \leq \ell(Q)$. It follows that

$$
\frac{1}{20} \inf_{x \in Q} d_R(x) \leq \frac{\ell(Q)}{20}.
$$

There are two cases to consider.

**Case** $Q \in \text{Stop}(R) \cap \mathcal{D}_N$. Then $d_R(Q) = 5\rho^N = \ell(Q)$, so $Q$ satisfies (3.7), while $\ell(Q^1) > 5\rho^N = d_R(Q^1)$, so that $Q^1$ does not satisfy (3.7). Consequently, $Q \in \text{Reg}_*(R)$.

**Case** $Q \in \text{Stop}(R) \cap \mathcal{D}_0^{N-1}$. Then we have

$$
\ell(Q) > \max \left( \frac{\ell(Q)}{20}, 5\rho^N \right) \geq d_R(Q),
$$

so that $Q$ does not satisfy (3.7). Taking into account (3.8) we get that there exists $P \in \text{Reg}_*(R)$ such that $P \subset Q$. □

Recall that in Lemma 3.2 we proved that $\mu$ is AD-regular at the scales and locations of $\text{Tree}(R)$. In the lemma below we show that despite enlarging $\text{Tree}(R)$ to its regularized version $\text{Tree}_*(R)$ we did not lose this property.

**Lemma 3.6.** For each $Q \in \text{Tree}_*(R)$ we have

$$
\tau \, \theta_\mu(R) \lesssim \theta_\mu(Q) \lesssim \theta_\mu(R).
$$

**Proof.** If $Q \in \text{Tree}(R)$, then this was already shown in Lemma 3.2. Suppose that $Q \in \text{Tree}_*(R) \setminus \text{Tree}(R)$. Let $P \in \text{Reg}_*(R)$ be such that $P \subset Q$. By Lemma 3.4 we have $d_R(x_P) \sim \ell(P)$. By the definition of $d_R(x_P)$ there exists $P' \in \text{Tree}(R)$ such that $d_R(x_P) \sim \text{dist}(x_P, P') + \ell(P')$. Hence,

$$
\ell(P) \sim \text{dist}(x_P, P') + \ell(P').
$$

If $\ell(P') < \ell(Q)$ let $Q' \in \text{Tree}(R)$ be the ancestor of $P'$ with $\ell(Q') = \ell(Q)$, otherwise set $Q' = P'$. We claim that

$$
\ell(Q') \sim \ell(Q).
$$
This is clearly the case if $\ell(P') < \ell(Q)$. On the other hand, if $Q' = P'$, then $\ell(Q') = \ell(P') \geq \ell(Q)$, and at the same time $\ell(P') \lesssim \ell(P) \leq \ell(Q)$. This shows that $\ell(Q') \sim \ell(Q)$.

Recalling (3.10) we get that
\[
\text{dist}(Q, Q') \lesssim \ell(Q) \sim \ell(Q').
\]
Hence, $B_Q \subset C B_Q' \subset 2 C B_Q$ for some $C \sim 1$. The estimate (3.9) then follows from the fact that it is satisfied by $Q'$ (which was shown in Lemma 3.2) and from the doubling property of $\mu$.

The following auxiliary result will be useful later on.

**Lemma 3.7.** There exists $C_0 > 1$, with $C_0 \sim 1$, such that for any $Q \in \text{Tree}_*(R)$ and $P \in \text{Reg}_*(R)$ satisfying $2 B_P \cap 2 B_Q \neq \emptyset$ we have
\[
P \subset 2 B_P \subset C_0 B_Q.
\]

**Proof.** To prove the lemma it suffices to show that $\ell(P) \lesssim \ell(Q)$. If $\ell(Q) \geq \ell(P)$, we are done. Assume $\ell(Q) \leq \rho \ell(P)$, so that $Q \subset 3 B_P$.

By the definition of $\text{Tree}_*(R)$, there exists $Q' \in \text{Reg}_*(R)$ such that $Q' \subset Q$. In particular, we have $B_{Q'} \cap 3 B_P \neq \emptyset$, and since $Q', P \in \text{Reg}_*(R)$, we get from Lemma 3.4 that $\ell(P) \sim \ell(Q') \leq \ell(Q)$.

\[
\square
\]

### 3.4. Approximating measure.

In this subsection we construct a $d$-AD-regular set $\Gamma$ with $d$-PBP that approximates $E$ at the level of $\text{Tree}_*(R)$.

Recall that $\Delta$ denotes the family of usual half-open dyadic cubes in $\mathbb{R}^n$. For each $Q \in \text{Tree}_*(R)$ set
\[
\Delta_Q = \{ I \in \Delta : \ell(Q)/2 \leq \ell(I) < \ell(Q), I \cap Q \neq \emptyset \}.
\]
In particular,
\[
Q \subset \bigcup_{I \in \Delta_Q} I \subset 2 B_Q. \tag{3.11}
\]
We define also
\[
Q_\Gamma = \bigcup_{I \in \Delta_Q} \partial_d I,
\]
where $\partial_d I$ denotes the $d$-dimensional skeleton of $I \in \Delta_Q$. Observe that
\[
Q_\Gamma \subset 2 B_Q, \tag{3.12}
\]
and
\[
\mathcal{H}^d(Q_\Gamma) \sim \ell(Q)^d \tag{3.13}
\]
because $\mathcal{H}^d(\partial_d I) \sim \ell(I)^d$ for each $I \in \Delta_Q$, and $\# \Delta_Q \lesssim 1$.

Finally, we set
\[
\Gamma = \bigcup_{Q \in \text{Reg}_*(R)} Q_\Gamma.
\]
Note that
\[
\Gamma \subset \bigcup_{R' \in N(R)} 2 B_{R'} \subset 6 B_R, \tag{3.14}
\]
by (3.12) and the definition of $N(R)$ [3.16].
Lemma 3.8. We have
\[ \sum_{Q \in \text{Reg}_s(R)} 1_{Q_{\Gamma}} \sim 1_{\Gamma}. \] (3.15)

Proof. Observe that if \( Q, P \in \text{Reg}_s(R) \) are such that \( 2B_Q \cap 2B_P = \emptyset \), then \( Q_{\Gamma} \cap P_{\Gamma} = \emptyset \), by (3.12).

It may happen that \( Q_{\Gamma} \cap P_{\Gamma} \neq \emptyset \) for \( Q, P \in \text{Reg}_s(R) \) such that \( 2B_Q \cap 2B_P \neq \emptyset \). However, for any fixed \( Q \in \text{Reg}_s(R) \) there is only a bounded number of \( P \in \text{Reg}_s(R) \) where this can happen, by Lemma 3.4. \( \square \)

The set \( \Gamma \) approximates \( E \) at the scales and locations of \( \text{Tree}_s(R) \). In order to get a measure \( \nu \) approximating \( \mu \), we also define a density \( g : \Gamma \to \mathbb{R} \), so that
\[ \nu = g \cdot \mathcal{H}^d|_{\Gamma}. \]
For each \( Q \in \text{Reg}_s(R) \) we define \( g_Q : \Gamma \to \mathbb{R} \) as
\[ g_Q = \frac{\mu(Q)}{\mathcal{H}^d(Q_{\Gamma})} 1_{Q_{\Gamma}}, \]
so that
\[ \int_{\Gamma} g_Q \, d\mathcal{H}^d = \mu(Q). \] (3.16)
Note also that
\[ g_Q \sim \theta_{\mu}(Q) 1_{Q_{\Gamma}} \sim \theta_{\mu}(R) 1_{Q_{\Gamma}}, \] (3.17)
by (3.13) and Lemma 3.6.

Set
\[ g = \sum_{Q \in \text{Reg}_s(R)} g_Q \]
and
\[ \nu = g \cdot \mathcal{H}^d|_{\Gamma}. \]

Remark that for any \( x \in \Gamma \)
\[ g(x) = \sum_{Q \in \text{Reg}_s(R)} g_Q(x) \sim \theta_{\mu}(R) \sum_{Q \in \text{Reg}_s(R)} 1_{Q_{\Gamma}}(x) \sim \theta_{\mu}(R), \]
where in the last estimate we used (3.15). It follows that
\[ \nu = \theta_{\mu}(R) \cdot h \cdot \mathcal{H}^d|_{\Gamma}, \] (3.18)
where \( h(x) = g(x)/\theta_{\mu}(R) \sim 1_{\Gamma} \).

In the next two lemmas we show that \( \Gamma \) is \( d \)-AD-regular, and that it satisfies \( d \)-PBP, with constants depending only on \( n, d \) and the PBP constants of \( E \). This will allow us to use Corollary 2.9 and get estimates for the \( \beta_{n,2} \)-numbers in the following subsection.

Lemma 3.9. Measure \( \nu \) is \( d \)-AD-regular. More precisely, for \( x \in \Gamma \) and \( 0 < r < \text{diam}(\Gamma) \) we have
\[ \nu(B(x, r)) \sim \theta_{\mu}(R) r^d. \] (3.19)

In consequence, \( \Gamma \) is a \( d \)-AD-regular set, with ADR constants depending only on \( n, d \), and the PBP constants of \( E \).
Proof. Fix $x \in \Gamma$, and let $Q \in \text{Reg}_*(R)$ be a cube such that $x \in Q \cap \Gamma$; if $Q$ is non-unique, just choose one.

First we check that $\nu$ is upper ADR. Assume that $0 < r \leq \ell(Q)$. Then,

$$
\nu(B(x, r)) = \sum_{P \in \text{Reg}_*(R)} \int_{B(x, r)} g_P \, dH^d \lesssim \sum_{P \in \text{Reg}_*(R), P \cap B(x, r) \neq \emptyset} \int_{B(x, r)} \theta_\mu(R) \mathbb{1}_{Q \cap B(x, r)} \, dH^d
$$

where in the last estimate we used the AD-regularity of $Q$ similar to the one before. We have

$$
\theta_\mu(R) \sum_{P \in \text{Reg}_*(R), P \cap B(x, r) \neq \emptyset} r^d;
$$

where in the last line we used the doubling property of $\mu$.

Consequence, $B(x, r)$ is upper ADR. Assume that $0 < r < \text{diam}(\Gamma)$.

Now suppose that $\ell(Q) < r < \text{diam}(\Gamma)$.

$$
\nu(B(x, r)) = \sum_{P \in \text{Reg}_*(R)} \int_{B(x, r)} g_P \, dH^d \leq \sum_{P \in \text{Reg}_*(R), P \cap B(x, r) \neq \emptyset} \int_{g_P} H^d \mu(P).
$$

Let $Q' \in \text{Tree}_*(R)$ be the minimal cube satisfying $Q \subset Q'$ and $\ell(Q') > r$. Note that $\ell(Q') \sim r$, and $B(x, r) \subset 2BQ'$. If $P \in \text{Reg}_*(R)$ satisfies $P \cap B(x, r) \neq \emptyset$, then in particular we have $2B_P \cap 2BQ' \neq \emptyset$. Thus, by Lemma 3.7 we have $P \subset C_0BQ'$. In consequence,

$$
\nu(B(x, r)) \leq \sum_{P \in \text{Reg}_*(R), P \cap B(x, r) \neq \emptyset} \mu(P) \leq \mu(C_0BQ') \sim \mu(Q'),
$$

where in the last estimate we used the doubling property of $\mu$. We have $\mu(Q') \sim \theta_\mu(R)\ell(Q')^d$ by Lemma 3.6. Since $\ell(Q') \sim r$, we get the desired upper bound

$$
\nu(B(x, r)) \lesssim \theta_\mu(R) r^d.
$$

We turn to the lower regularity of $\nu$. The proof at scales $0 < r \leq 20 \ell(Q)$ is very similar to the one before. We have

$$
\nu(B(x, r)) = \sum_{P \in \text{Reg}_*(R)} \int_{B(x, r)} g_P \, dH^d \lesssim \int_{B(x, r)} \theta_\mu(R) \mathbb{1}_{Q \cap B(x, r)} \, dH^d
$$

where in the last line we used the AD-regularity of $Q \cap B(x, r)$. 

$$
= \theta_\mu(R) \mathcal{H}^d(Q \cap B(x, r)) \sim \theta_\mu(R) r^d,
$$

where the last line used the AD-regularity of $Q \cap B(x, r)$.
Concerning scales $20\ell(Q) \leq r < \text{diam}(\Gamma)$, we proceed as follows:

$$
\nu(B(x,r)) = \sum_{P \in \text{Reg}_c(R), P \subset B(x,r)} \int_{B(x,r)} g_P \, d\mathcal{H}^d \geq \sum_{P \in \text{Reg}_c(R), P \subset B(x,r)} \int_{B(x,r)} g_P \, d\mathcal{H}^d \tag{3.10}
$$

Let $Q' \in \text{Tree}_c(R)$ be the maximal cube satisfying $Q \subset Q'$ and $2B_{Q'} \subset B(x,r)$ (such cube exists because $2B_Q \subset B(x,r)$). Note that $\ell(Q') \sim r$.

We claim that if $P \in \text{Reg}_c(R)$ and $P \subset Q'$, then $P_b \subset B(x,r)$ (so in particular, $P$ appears in the sum on the right hand side above). Indeed, we have $P_b \subset 2B_P \subset 2B_{Q'} \subset B(x,r)$, by the definition of $Q'$. Thus,

$$
\nu(B(x,r)) \geq \sum_{P \in \text{Reg}_c(R), P \subset B(x,r)} \mu(P) \tag{3.16}
$$

where we have used Lemma 3.9 and the fact that $\ell(Q') \sim r$. This finishes the proof of $\nu$.

As a consequence of (3.18) and (3.19) we get that $\Gamma$ is a d-ADR set, with ADR constants depending only on $n, d$, and the PBP constants. \hfill \Box

In order to show that $\Gamma$ has plenty of big projections, we need the following result on $d$-dimensional skeletons of dyadic cubes.

**Lemma 3.10.** Let $I \in \Delta$, and $V \in \mathcal{G}(n,d)$. Then,

$$
\pi_V(I) \subset \pi_V(\partial_d I). \tag{3.20}
$$

**Proof.** Let $p \in \pi_V(I)$. Then $W := \pi^{-1}(\{p\})$ is an $(n-d)$-dimensional plane intersecting $I$. We are going to show that

$$
W \cap \partial_d I \neq \emptyset, \tag{3.21}
$$

so that $p \in \pi_V(\partial_d I)$, and in consequence (3.20) holds.

By rotating and translating, we may assume that

$$
W = \text{span}(e_1, \ldots, e_{n-d}) = \{(x_1, \ldots, x_{n-d}, 0, \ldots, 0) : (x_1, \ldots, x_{n-d}) \in \mathbb{R}^{n-d}\}.
$$

In this coordinates, the cube $I$ can be described by a system of inequalities

$$
I = \{x \in \mathbb{R}^n : a \leq Lx \leq b\},
$$

for some orthogonal matrix $L \in SO(n)$ and some vectors $a = (a_i)_{i=1,\ldots,n} \in \mathbb{R}^n$ and $b = (a_i + \ell(I))_{i=1,\ldots,n} \in \mathbb{R}^n$ (note that before the rotation and translation, we would have $L = \text{id}_n$, and $a$ a vector consisting of dyadic numbers).

Recall that all $d$-dimensional faces of $I$ are of the following form: for any choice of indices $A, B \subset \{1, \ldots, n\}$ with $A \cap B = \emptyset$ and $\#(A \cup B) = n - d$ we set

$$
F(A, B) = \{x \in \mathbb{R}^n : a \leq Lx \leq b, a_i = L_i \cdot x \text{ for } i \in A, L_i \cdot x = b_i \text{ for } i \in B\}.
$$
Above, $L_i$ denotes the $i$-th row of $L$. This is a one-to-one representation, in the sense that for any such $A, B$ we get a $d$-dimensional face of $I$, and each $d$-dimensional face $F$ satisfies $F = F(A, B)$ for a unique choice of $A$ and $B$.

Recalling $W = \text{span}(e_1, \ldots, e_{n-d})$, we get that

$$I \cap W = \{x = (x', 0) \in \mathbb{R}^{n-d} \times \{0\}^d : a \leq Lx \leq b\} \neq \emptyset.$$

Let $L'$ be a rectangular $(n - d) \times d$-matrix obtained by taking the first $(n - d)$ columns of $L$. Then,

$$I \cap W = \{x = (x', 0) \in \mathbb{R}^{n-d} \times \{0\}^d : a \leq L'x' \leq b\},$$

and so $I \cap W$ might be identified with the (non-empty) polytope

$$P = \{x' \in \mathbb{R}^{n-d} : a \leq L'x' \leq b\}.$$

Let $v' \in P$ be a vertex of $P$, i.e., a 0-dimensional face of $P$. We claim that the point $v = (v', 0) \in \mathbb{R}^{n-d} \times \{0\}^d$ belongs to some $d$-dimensional face of $I$. Since it clearly lies on $W$, this will give (3.21).

Since $v'$ is a vertex of $P$, we get that there exist sets of indices $A, B \subset \{1, \ldots, n\}$ with $A \cap B = \emptyset$ and $\#(A \cup B) = n - d$ such that

$$a_i = L'_{i} \cdot v' \text{ for } i \in A, \quad L'_{i} \cdot v' = b_i \text{ for } i \in B,$$

where $L'_i$ denotes the $i$-th row of $L'$. Note that the property $\#(A \cup B) = n - d$ follows from the fact that $v'$ should be a unique solution to the equations above (a 0-dimensional face is a single point), and we are in $\mathbb{R}^{n-d}$. We do not claim that the choice of $A$ and $B$ is unique, but we do not care. Finally, observe that the remaining inequalities $a_i \leq L'_{i} \cdot v' \leq b_i$, $i \in \{1, \ldots, n\} \setminus (A \cup B)$ must also be satisfied since $v \in P$.

It remains to observe that $v = (v', 0) \in F(A, B) \cap W$ because $Lv = L'v'$. In particular, $v \in \partial_{d}I \cap W$, which gives (3.21).

\[ \square \]

**Lemma 3.11.** The set $\Gamma$ has $d$-PBP, with PBP constants depending only on $n, d$, and the PBP constants of $E$.

**Proof.** Fix $x \in \Gamma$, and let $Q \in \text{Reg}_{x}(R)$ be such that $x \in Q_{\Gamma}$ (if there are many such $Q$, just choose one). We will show that for any $0 < r < \text{diam}(\Gamma)$ there exists $V_{\infty, r}^{x} \in G(n, d)$ such that for all $V \in B(V_{\infty, r}^{x}, \delta)$ we have

$$\mathcal{H}^{d}(\pi_{V}(\Gamma \cap B(x, r))) \gtrsim r^{d}. \tag{3.22}$$

Let $0 < r < 10 \ell(Q)$. Since $x \in Q_{\Gamma}$, we have $x \in \partial_{d}I$ for some $I \in \Delta_{Q}$, and so it is immediate to see that for any $V \in G(n, d)$ we have

$$\mathcal{H}^{d}(\pi_{V}(\Gamma \cap B(x, r))) \gtrsim r^{d}. \tag{3.23}$$

So the PBP property holds trivially at small scales.

Assume now that $10 \ell(Q) < r < \text{diam}(\Gamma) \sim \ell(R)$. Let $Q' \in \mathcal{D}(R)$ be the maximal cube satisfying $Q \subset Q'$ and $2B_{Q'} \subset B(x, r)$. Clearly, $\ell(Q') \sim r$. Recall that there exists a ball $B(Q')$ centered at $xQ' \in E$ with $r(B(Q')) \sim \ell(Q')$ and such that $B(Q') \cap E \subset Q'$ (see Lemma 2.1). Set

$$V_{\infty, r}^{x} := V_{B(Q')}^{E}.$$
where $V^E_{B(Q')}$ is the $d$-plane coming from the PBP property of $E$ in the ball $B(Q')$. Fix $V \in B(V^T_{x,r}, \delta)$. We will show that (3.22) holds.

Note that the PBP property of $E$ in the ball $B(Q')$ gives us
\[ H^d(\pi_V(Q')) \geq H^d(\pi_V(E \cap B(Q'))) \gtrsim \ell(Q')^d \sim r^d. \] (3.23)
We claim that
\[ \pi_V(\Gamma \cap B(x, r)) \supset \pi_V(Q'). \] (3.24)
Together with (3.23), this will give us the desired estimate (3.22).

Recall that $Q' = \bigcup_{P \in \text{Reg}_*(R), P \subset Q'} P$, and for each $P \in \text{Reg}_*(R)$ we have $P \subset \bigcup_{I \in \Delta_P} I$. In consequence,
\[ Q' \subset \bigcup_{P \in \text{Reg}_*(R), P \subset Q'} \bigcup_{I \in \Delta_P} \pi_V(I). \]
Using the above and Lemma 3.10 we get that
\[ \pi_V(Q') \subset \bigcup_{P \in \text{Reg}_*(R), P \subset Q'} \bigcup_{I \in \Delta_P} \pi_V(I) \subset \bigcup_{P \in \text{Reg}_*(R), P \subset Q'} \pi_V(P \Gamma). \]
Recalling (3.12) we see that for all $P \in \text{Reg}_*(R), P \subset Q'$, we have $P \Gamma \subset 2B_P \cap \Gamma \subset 2B_{Q'} \cap \Gamma \subset B(x, r) \cap \Gamma$;
and so
\[ \pi_V(Q') \subset \bigcup_{P \in \text{Reg}_*(R), P \subset Q'} \pi_V(P \Gamma) \subset \pi_V(\Gamma \cap B(x, r)). \]
This finishes the proof (3.24), which together with (3.23) gives (3.22). \[ \square \]

3.5. Estimating $\beta$-numbers. In this subsection we finally prove our key estimate, Lemma 3.3. We recall it for reader’s convenience.

**Lemma.** For any $R \in D_0^{N-1} \cap N$ we have
\[ \sum_{Q \in \text{Tree}(R)} \beta_{\mu,2}(2B_Q)^2 \mu(Q) \lesssim \theta_{\mu}(R) \mu(R), \] (3.25)
with the implicit constant depending only on $n, d$ and the PBP constants of $E$.

In Lemmas 3.9 and 3.11 we showed that $\Gamma$ is a $d$-AD-regular set satisfying $d$-PBP, with ADR and PBP constants depending only on $n, d$, and the PBP constants of $E$. Thus, we get from Corollary 2.9 that $\Gamma$ satisfies
\[ \int_{\Gamma} \int_0^\ell(R) \beta_{\Gamma,2}(x, r)^2 \frac{dr}{r} d\mathcal{H}^d(x) \lesssim \ell(R)^d. \]
In the above we also use that $\Gamma \subset 6B_R$, by (3.11). Recalling (3.18), we get that
\[ \int_{\Gamma} \int_0^\ell(R) \beta_{\mu,2}(x, r)^2 \frac{dr}{r} d\mathcal{H}^d(x) \lesssim \theta_{\mu}(R) \ell(R)^d. \]
Lemma 3.12. For any $Q \in \text{Tree}(R)$ we have
\[
\beta_{\nu,2}(2B_Q)^2 \lesssim \beta_{\nu,2}(C_0B_Q)^2 + \ell(Q)^{-d} \sum_{P \in \text{Reg}_*(R), 2B_P \subset C_0B_Q} \ell(P) \mu(P),
\] (3.27)
where $C_0 \sim 1$ is the constant from Lemma 3.7.

Proof. Fix $Q \in \text{Tree}(R)$, and let $L_Q$ be a $d$-plane minimizing $\beta_{\nu,2}(C_0B_Q)$.
Observe that for any $Q \in \text{Tree}(R)$, the set $2B_Q \cap E$ is fully covered by $P \in \text{Reg}_*(R)$ satisfying $P \cap 2B_Q \neq \emptyset$. Indeed, this is true for $R$ because by the definition of $\mathcal{N}(R)$ and $\text{Reg}_*(R)$ we have
\[
2B_R \cap E \subset \bigcup_{P \in \text{Reg}_*(R), P \cap 2B_Q \neq \emptyset} P.
\]
Since $2B_Q \subset 2B_R$ for any $Q \in \text{Tree}(R)$, the claim follows.

The observation above gives
\[
\beta_{\nu,2}(2B_Q)^2 \ell(Q)^d \lesssim \int_{2B_Q} \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 d\mu(x)
\]
\[
\leq \sum_{P \in \text{Reg}_*(R), P \cap 2B_Q \neq \emptyset} \int_P \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 d\mu(x)
\]
\[
\leq \sum_{P \in \text{Reg}_*(R), 2B_P \subset C_0B_Q} \int_P \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 d\mu(x),
\]
where in the last line we used Lemma 3.7.

Let $P \in \text{Reg}_*(R), 2B_P \subset C_0B_Q$. Then,
\[
\int_P \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 d\mu(x) = \int_\Gamma \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 g_P(x) d\mathcal{H}^d(x)
\]
\[
+ \int \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 (1_P(x) d\mu(x) - g_P(x) d\mathcal{H}^d|_\Gamma(x)) =: I_1(P) + I_2(P).
\]
It follows that
\[
\beta_{\nu,2}(2B_Q)^2 \ell(Q)^d \lesssim \sum_{P \in \text{Reg}_*(R), 2B_P \subset C_0B_Q} I_1(P) + \sum_{P \in \text{Reg}_*(R), 2B_P \subset C_0B_Q} I_2(P) =: S_1 + S_2.
\]
Estimating $S_1$ is straightforward. Recall that for $P \in \text{Reg}_s(R)$ we have supp $g_P \subset P_T \subset 2B_P$. Hence,

$$S_1 = \sum_{P \in \text{Reg}_s(R), \, 2B_P \subset C_0 B_Q} \int_{\Gamma} \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 g_P(x) \, dH^d(x)$$

$$\leq \sum_{P \in \text{Reg}_s(R)} \int_{\Gamma \cap C_0 B_Q} \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 g_P(x) \, dH^d(x)$$

$$= \int_{\Gamma \cap C_0 B_Q} \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 g(x) \, dH^d(x)$$

$$= \int_{C_0 B_Q} \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 \, d\nu(x) \sim \beta_{\nu,2}(C_0 B_Q)^2 \ell(Q)^d,$$

where in the last part we used our choice of $L_Q$. This gives us the first term from the right hand side of (3.27).

We turn to estimating $S_2$. Let $P \in \text{Reg}_s(R)$. Using the fact that $\int_{\Gamma} g_P \, dH^d = \mu(P)$ we get

$$I_2(P) = \int \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 (1_P(x) d\mu(x) - g_P(x) dH^d\mid_{\Gamma}(x))$$

$$= \int \left( \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 - \left( \frac{\text{dist}(x_P, L_Q)}{\ell(Q)} \right)^2 \right) (1_P(x) d\mu(x) - g_P(x) dH^d\mid_{\Gamma}(x)).$$

For $x \in P \cup \text{supp} g_P \subset 2B_P \subset C_0 B_Q$ we have

$$\left| \left( \frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 - \left( \frac{\text{dist}(x_P, L_Q)}{\ell(Q)} \right)^2 \right| \leq \frac{\text{dist}(x, L_Q) + \text{dist}(x_P, L_Q)}{\ell(Q)} \cdot \frac{\ell(P)}{\ell(Q)} \cdot \frac{\ell(Q)}{\ell(Q)} \leq \frac{\ell(P)}{\ell(Q)} \cdot \frac{\ell(Q)}{\ell(Q)} = \frac{\ell(P)}{\ell(Q)}.$$

Thus,

$$I_2(P) \lesssim \frac{\ell(P)}{\ell(Q)} \mu(P).$$

Summing over $P \in \text{Reg}_s(R)$ with $2B_P \subset C_0 B_Q$ yields

$$S_2 \lesssim \sum_{P \in \text{Reg}_s(R), \, 2B_P \subset C_0 B_Q} \frac{\ell(P)}{\ell(Q)} \mu(P).$$

This corresponds to the second term from the right hand side of (3.27). \qed

**Lemma 3.13.** We have

$$\sum_{Q \in \text{Tree}(R)} \beta_{\nu,2}(2B_Q)^2 \mu(Q) \lesssim \sum_{Q \in \text{Tree}(R)} \beta_{\nu,2}(C_0 B_Q)^2 \mu(Q) + \theta_{\mu}(R) \mu(R). \quad (3.28)$$
Proof. In the previous lemma we showed that for any $Q \in \text{Tree}(R)$ we have (3.27). Multiplying by $\mu(Q)$ and summing over $Q \in \text{Tree}(R)$ yields

$$\sum_{Q \in \text{Tree}(R)} \beta_{\mu,2}(2B_Q)^2 \mu(Q) \lesssim \sum_{Q \in \text{Tree}(R)} \beta_{\mu,2}(C_0B_Q)^2 \mu(Q)$$

$$+ \sum_{Q \in \text{Tree}(R)} \theta_{\mu}(Q) \sum_{P \in \text{Reg}_*(R), 2B_P \subset C_0B_Q} \frac{\ell(P)}{\ell(Q)} \mu(P).$$

Thus, in order to obtain (3.28) it suffices to prove that

$$\sum_{Q \in \text{Tree}(R)} \theta_{\mu}(Q) \sum_{P \in \text{Reg}_*(R), 2B_P \subset C_0B_Q} \frac{\ell(P)}{\ell(Q)} \mu(P) \lesssim \theta_{\mu}(R) \mu(R).$$

First, recall that by Lemma 3.2 we have $\theta_{\mu}(Q) \sim \theta_{\mu}(R)$ for all $Q \in \text{Tree}(R)$. So we only need to show

$$\sum_{Q \in \text{Tree}(R)} \theta_{\mu}(Q) \sum_{P \in \text{Reg}_*(R), 2B_P \subset C_0B_Q} \frac{\ell(P)}{\ell(Q)} \mu(P) \lesssim \mu(R). \quad (3.29)$$

Changing the order of summation transforms the left hand side to

$$\sum_{P \in \text{Reg}_*(R)} \mu(P) \sum_{Q \in \text{Tree}(R), 2B_P \subset C_0B_Q} \frac{\ell(P)}{\ell(Q)}.$$

The inner sum is essentially a geometric series: if $P \in \text{Reg}_*(R)$ is fixed, then for any generation $k \in [0, N]$, there is at most a bounded number of $Q \in \text{Tree}(R) \cap D_k$ such that $2B_P \subset C_0B_Q$ (the bound depends on $C_0 \sim 1$). Moreover, all such $Q$ necessarily satisfy $C_0 \ell(Q) \geq \ell(P)$. Hence,

$$\sum_{P \in \text{Reg}_*(R)} \mu(P) \sum_{Q \in \text{Tree}(R), 2B_P \subset C_0B_Q} \frac{\ell(P)}{\ell(Q)} \lesssim C_0 \sum_{P \in \text{Reg}_*(R)} \mu(P) \leq \mu(6B_R) \sim \mu(R),$$

where in the last two estimates we used the fact that the cubes in $\text{Reg}_*(R)$ are pairwise disjoint and contained in $6B_R$, and also the doubling property of $\mu$. This finishes the proof of (3.28). \qed

Lemma 3.13 together with (3.26) give the desired estimate

$$\sum_{Q \in \text{Tree}(R)} \beta_{\mu,2}(2B_Q)^2 \mu(Q) \lesssim \theta_{\mu}(R) \mu(R).$$

This concludes the proof of Lemma 3.3.

4. Traveling Salesman Theorem

In this section we prove the Analyst’s Traveling Salesman Theorem for sets with PBP, Theorem 1.10.
4.1. **Discrete approximation.** We recall a simplified version of the corona construction from [AV21]. Throughout this section, \(D\) will denote the Christ-David cubes from Lemma 2.1 on a fixed lower content regular set \(E\).

**Lemma 4.1 (AV21, Main Lemma).** Let \(k_0 > 0\) be an integer, and \(E \subset \mathbb{R}^n\) be a compact lower content \((d, c_1)\)-regular set. Let \(Q_0 \in D_0\) and \(D(Q_0, k_0) = D(k_0) = \bigcup_{k=0}^{k_0} \{Q \in D_k : Q \subseteq Q_0\}\). Then, there exists a family \(\text{Top}(k_0) \subseteq D(k_0)\) such that for any \(R \in \text{Top}(k_0)\) we have a tree of cubes denoted by \(\text{Tree}(R)\) with root \(R\), the trees partition \(D(k_0)\)

\[D(k_0) = \bigcup_{R \in \text{Top}(k_0)} \text{Tree}(R),\]

and this partition has the following properties:

1. There exists a constant \(\eta = \eta(d, c_1)\) so that
   \[
   \sum_{R \in \text{Top}(k_0)} \ell(R)^d \leq \eta^{-1} \mathcal{H}^d(Q_0),
   \]  
   and \(\eta \to 0\) as \(c_1 \to 0\).

2. Given \(R \in \text{Top}(k_0)\) let \(\text{Stop} = \text{Stop}(R)\) denote the minimal cubes of \(\text{Tree}(R)\) and set
   \[
   d_R(x) := \inf_{Q \in \text{Stop}} (\ell(Q) + \text{dist}(x, Q)).
   \]  
   For any \(C_0 > 4\) and \(\tau > 0\), there is a collection \(\mathcal{C}_R \subset \Delta\) of disjoint dyadic cubes covering \(C_0B_R \cap E\) so that if
   \[
   E_R := \bigcup_{I \in \mathcal{C}_R} \partial_d I,
   \]
   where \(\partial_d I\) denotes the \(d\)-dimensional skeleton of \(I\), then the following hold:
   (a) \(E_R\) is Ahlfors \(d\)-regular with constants depending on \(C_0, \tau, d,\) and \(c_1\).
   (b) We have
   \[
   C_0B_R \cap E \subseteq \bigcup_{I \in \mathcal{C}_R} I \subseteq 2C_0B_R.
   \]  
   (c) \(E\) is close to \(E_R\) in \(C_0B_R\) in the sense that
   \[
   \text{dist}(x, E_R) \lesssim \tau d_R(x) \quad \text{for all} \quad x \in E \cap C_0B_R.
   \]  
   (d) The dyadic cubes in \(\mathcal{C}_R\) satisfy
   \[
   \ell(I) \sim \tau \inf_{x \in I} d_R(x) \quad \text{for all} \quad I \in \mathcal{C}_R.
   \]

The approximating sets \(E_R\) are analogous to the surfaces \(\Gamma\) constructed in §3.4. As before, \(E_R\) inherits the PBP property from \(E\).

**Lemma 4.2.** Let \(E \subset \mathbb{R}^n\) be a set with \(d\)-PBP. Let \(E_R\) be one of the sets from Lemma 4.1. Then \(E_R\) also has \(d\)-PBP (with possibly a slightly different constant \(\delta'\) which equals to \(\delta\) up to a multiplicative constant depending only on \(d\)).

The proof is analogous to that of Lemma 3.11.

\footnote{This is not explicitly stated in [AV21], but it can be deduced from the proof, specifically see (3.4), (3.5) and (3.10) there.}
4.2. β-numbers estimates.

Remark 4.3. We will keep careful track of the dependence of the various constants on δ (the parameters from the PBP condition). On the other hand, we will usually not keep track of the dependence on n, d.

In this subsection we will prove the following.

**Proposition 4.4.** If E ⊂ \( \mathbb{R}^n \) has d-PBP with parameters \( \delta > 0 \), then for any \( Q_0 ∈ D \), we have
\[
\ell(Q_0)^d + \sum_{Q ∈ D(Q_0)} \beta_{E,2}^d (3B_Q)^2 \ell(Q)^d \lesssim_{n,d} C(\delta) \mathcal{H}^d(Q_0),
\]
where \( C(\delta) \to \infty \) as \( \delta \to 0 \).

Theorem 1.10 follows immediately from this proposition.

**Proof of Theorem 1.10.** First, by [AV21, (A.3)], we have that
\[
\ell(Q_0)^d + \sum_{Q ⊆ Q_0} \beta_{E}^{d,p} (C_0 B_Q)^2 \ell(Q)^d \sim_{p,C_0} \ell(Q_0)^d + \sum_{Q ⊆ Q_0} \beta_{E}^{d,2} (3B_Q)^2 \ell(Q)^d
\]
whenever \( C_0 > 1 \) and \( 1 ≤ p ≤ p(d) \). Together with Proposition 4.4 this establishes one of the estimates from (1.7). The converse inequality in (1.7) follows from [AS18, Theorem II], see also [AV21, Theorem A.1(1)] for a more transparent statement. □

We now focus on proving Proposition 4.4. First, recall that for \( k_0 ≥ 0 \) we defined the truncated dyadic lattice as \( \mathcal{D}(Q_0, k_0) = \bigcup_{k=0}^{k_0} \{Q ∈ \mathcal{D}_k : Q ⊆ Q_0\} \). Observe that to prove (4.6) it suffices to show
\[
\sum_{Q ∈ \mathcal{D}(Q_0, k_0)} \beta_{E}^{d,2} (3B_Q)^2 \ell(Q)^d \lesssim_{n,d} C(\delta) \mathcal{H}^d(Q_0),
\]
with bounds independent of \( k_0 ≥ 0 \). The estimate \( \ell(Q_0)^d \lesssim \delta^{-1} \mathcal{H}^d(Q_0) \) follows immediately from lower content regularity of \( E \).

Consider the coronization from Lemma 4.1 applied with \( C_0 = 6 \) and sufficiently small \( τ \), to be fixed later. Let \( R ∈ \text{Top}(k_0) \). We start off by applying Lemma 2.13 with \( E_1 = E \) and \( E_2 = E_R \). For \( Q ∈ \mathcal{D} \), recall that \( x_Q \) denotes the center of \( Q \). By the definition of \( \text{Tree}(R) \), we see that if \( Q ∈ \text{Tree}(R) \), then there exists a dyadic cube \( I ∈ \mathcal{C}_R \) with \( x_Q ∈ I \) (by Lemma 4.1(2.b)). By (1.5), \( ℓ(I) ≤ τ ℓ(Q) \). Hence, we may find a point
\[
y_Q ∈ E_R \quad \text{such that} \quad |x_Q - y_Q| ≤ ℓ(Q),
\]
and we obtain that
\[
3B_Q = B(x_Q, 3ℓ(Q)) ⊂ 6B(y_Q, 6ℓ(Q)) =: 6B_Q'.
\]
This implies that for each cube \( Q \in \text{Tree}(R) \) the hypotheses of Lemma 2.13 are satisfied (with \( E_1 = E, \ E_2 = E_R, \ 3B_Q, \) and \( 6B_Q' \)). Thus,

\[
\sum_{Q \in \text{Tree}(R)} \beta_{E}^{d,2}(3B_Q)^2 \ell(Q)^d \lesssim \sum_{Q \in \text{Tree}(R)} \beta_{E_R}^{d,2}(6B_Q')^2 \ell(Q)^d \\
+ \sum_{Q \in \text{Tree}(R)} \left( \frac{1}{\ell(Q)^d} \int_{6B_Q \cap E} \left( \frac{\text{dist}(y, E_R)}{\ell(Q)} \right)^2 d\mathcal{H}^d_\infty(y) \right) \ell(Q)^d =: I_1 + I_2. \quad (4.10)
\]

We estimate \( I_1 \). We denote by \( D_{E_R} \) a family of Christ-David cubes for \( E_R \) obtained by applying Lemma 2.1 to \( E_R \). Using (4.8) it is immediate to see that for each \( Q \in \text{Tree}(R) \) there exists \( P \in D_{E_R} \) with \( \ell(P) \sim \ell(R) \) and \( 6B'_P \subset 3B_P \), and that any \( P \in D_{E_R} \) corresponds to at most a bounded number of \( Q \in \text{Tree}(R) \). Thus, we get

\[
I_1 \lesssim \sum_{P \in D_{E_R}} \beta_{E_R}^{d,2}(3B_P)^2 \ell(P)^d. \quad (4.11)
\]

Recall that \( E_R \) is Ahlfors \( d \)-regular (Lemma 4.1) with constant \( C \) depending on \( C_0, \tau, d \) and \( c_1 \) (and thus, by Lemma 2.2 on \( \delta \)), and that it has \( d \)-PBP with parameter \( \sim \delta \) by Lemma 4.2. It follows from Corollary 2.9 that for \( \mu_R = \mathcal{H}^d \mid_{E_R} \) we have

\[
\sum_{P \in D_{E_R}} \beta_{\mu_R,2}(3B_P)^2 \ell(P)^d \lesssim_\tau C(\delta) \ell(R)^d.
\]

Since \( E_R \) is Ahlfors regular, we have \( \beta_{\mu_R,2}(x, r) \sim \beta_{E_R}^{d,2}(x, r) \) (see §1.2 in [AS18]), and so the two estimates above give

\[
I_1 \lesssim_\tau C(\delta) \ell(R)^d, \quad (4.11)
\]

where \( C(\delta) \to \infty \) as \( \delta \to 0 \).

We move on to estimating \( I_2 \). Recall from Lemma 4.1 (4.3), that

\[
6B_R \cap E = C_0 B_R \cap E \subset \bigcup_{I \in \mathcal{C}_R} I. \quad (4.12)
\]

For each \( I \in \mathcal{C}_R \), set

\[
\mathcal{F}(I) := \{ S \in D : S \cap I \neq \emptyset \text{ and } \rho(S) \cap I \leq \ell(I) \}. \quad (4.13)
\]

Then define \( \mathcal{F}(R) = \bigcup_{I \in \mathcal{C}_R} \mathcal{F}(I) \) and set \( \text{Stop}(R) \) to be the subfamily of maximal cubes in \( \mathcal{F}(R) \) (by maximality, this family is disjoint and covers \( \mathcal{F}(R) \)). Note that

\[
6B_R \cap E \subset \bigcup_{S \in \text{Stop}(R)} S. \quad (4.14)
\]

Indeed, if \( x \in 6B_R \cap E \), then by (4.12) there is an \( I \in \mathcal{C}_R \) so that \( x \in I \). But \( x \) is also contained in some Christ-David cube \( S \) such that \( \rho(S) \leq \ell(I) \), and thus (4.14) holds. Note that (4.14) trivially implies that for any \( Q \in \text{Tree}(R) \) we have \( 6B_Q \cap E \subset \bigcup_{S \in \text{Stop}(R)} S \), since \( 6B_Q \cap E \subset 6B_R \cap E \).

For any \( Q \in \text{Tree}(R) \) and \( x \in 6B_Q \cap E \), we claim that there exists a unique cube \( S \in \text{Stop}(R) \) containing \( x \), and it satisfies

\[
\text{dist}(x, E_R) \lesssim \ell(S). \quad (4.15)
\]
Indeed, by (4.14), \( x \in S \) for some \( S \in \widetilde{\text{Stop}}(R) \), and uniqueness follows from the maximality of \( \text{Stop}(R) \). By definition, there is an \( I \in \mathcal{C}_R \) so that \( S \cap I \neq \emptyset \) and \( \ell(I) \sim \ell(S) \). But then
\[
\text{dist}(x, E_R) \leq \text{diam}(S) + \text{diam}(I) \lesssim \ell(S).
\]

We can now estimate
\[
I_2 = \sum_{Q \in \text{Tree}(R)} \int_{6B_Q \cap E} \left( \frac{\text{dist}(y, E_R)}{\ell(Q)} \right)^2 d\mathcal{H}^d(y) 
\]

where we may switch the order of summation since both sums are finite.

Fix \( S \in \text{Stop}(R) \). We claim that for any \( k \geq 0 \) the number of cubes \( Q \in \text{Tree}(R) \cap \mathcal{D}_k \) with \( 6B_Q \cap S \neq \emptyset \) is bounded by some dimensional constant. Indeed, this is clear as soon as \( \ell(S) \leq \ell(Q) = \rho^k \). On the other hand, if \( \ell(S) > \rho^k \), then there is no \( Q \in \text{Tree}(R) \cap \mathcal{D}_k \) with \( 6B_Q \cap S \neq \emptyset \). To see this, observe that \( 6B_Q \cap S \neq \emptyset \) implies \( \text{dist}(S, Q) \lesssim \ell(Q) \). Also, if \( I \in \mathcal{C}_R \) is such that \( S \in \mathcal{F}(I) \), then \( \text{dist}(I, Q) \lesssim \ell(I) + \text{dist}(S, Q) \lesssim \ell(S) + \ell(Q) \). Hence
\[
\ell(S) \lesssim \ell(I) \lesssim \tau \inf_{x \in I} \text{d}_R(x) \lesssim \tau (\ell(Q) + \text{dist}(I, Q)) \lesssim \tau \ell(Q) + \tau \ell(S).
\]

Choosing \( \tau > 0 \) sufficiently small we get that \( \ell(S) \leq \ell(Q) \).

The observation above allows us to estimate the interior sum in (4.16) as a geometric sum:
\[
I_2 \lesssim \sum_{S \in \text{Stop}(R)} \ell(S)^{d+2} \sum_{Q \in \text{Tree}(R)} \frac{1}{\ell(Q)^2} \lesssim \sum_{S \in \text{Stop}(R)} \ell(S)^d.
\]

Now, given \( S \in \widetilde{\text{Stop}}(R) \), we claim that
\[
\# \{ I \in \mathcal{C}_R : S \in \mathcal{F}(I) \} \sim 1. \quad (4.17)
\]

This follows immediately from the fact that for dyadic cubes \( I, J \) as above we have \( \ell(I) = \ell(J) \) and \( \text{dist}(I, J) \lesssim \ell(I) \). Hence, for each \( S \in \widetilde{\text{Stop}}(R) \) there is a bounded number of cubes in \( \mathcal{C}_R \) of sidelength \( \sim \ell(S) \) which intersect it. Finally, we may conclude that
\[
I_2 \lesssim \sum_{S \in \text{Stop}(R)} \ell(S)^d \sim \sum_{S \in \text{Stop}(R)} \sum_{I \in \mathcal{C}_R} \ell(I)^d
\]

and hence, using the Ahlfors \( d \)-regularity of \( E_R \),
\[
I_2 \lesssim \sum_{I \in \mathcal{E}_R} \ell(I)^d \sim \sum_{I \in \mathcal{E}_R} \mathcal{H}^d(\partial_d I) \sim \mathcal{H}^d(E_R) \lesssim \tau \delta \ell(R)^d. \quad (4.18)
\]
Putting the estimates for $I_1$ and $I_2$ together with (4.10) gives
\[
\sum_{Q \in \text{Tree}(R)} \beta_E^{d,2}(3B_Q)^2 \ell(Q)^d \lesssim C(\delta)\ell(R)^d,
\]
where $C(\delta) \to \infty$ as $\delta \to 0$. We conclude that
\[
\sum_{Q \in D(Q_0,k_0)} \beta_E^{d,2}(3B_Q)^2 \ell(Q)^d \lesssim C(\delta) \sum_{R \in \text{Top}(k_0)} \sum_{Q \in \text{Tree}(R)} \beta_E^{d,2}(3B_Q)^2 \ell(Q)^d \lesssim C(\delta) \eta^{-1} \mathcal{H}^d(Q_0) \sim C(\delta) \mathcal{H}^d(Q_0).
\]
This finishes the proof of Proposition 4.4.

5. Dimension of wiggly sets

In this section we establish our estimate for the dimension of wiggly sets with PBP, Theorem 1.12. Through this section we will work with David-Christ cubes (Lemma 2.1) associated to different sets. To avoid confusion we will write $D(E)$ to denote the system of cubes associated to a given set $E \subset \mathbb{R}^n$.

Recall that $p(d) = \frac{2d}{d-2}$ for $d > 2$ and $p(d) = \infty$ if $d \leq 2$. Theorem 1.12 will follow as an easy corollary of the following proposition.

**Proposition 5.1.** Let $E \subset \mathbb{R}^n$ be a closed set with $d$-PBP with constant $\delta > 0$ and which is uniformly wiggly of dimension $d$, constant $\beta_0$, and with respect to the Azzam-Schul $\beta_E^{d,p}$-numbers, where $1 \leq p < p(d)$. That is, $E$ satisfies
\[
\beta_E^{d,p}(x,r) \geq \beta_0 \text{ for all } x \in E, \ 0 < r < \text{diam}(E).
\]
Then
\[
\dim_H(E) \geq d + c\beta_0^2
\]
with $c$ depending on $\delta, p, n, d$.

**Proof of Theorem 1.12 using Proposition 5.1.** Let $E$ be as in the statement of the theorem. In particular, $\beta_E^{d,\infty}(B_Q) > \beta_0$. By [AS18] Lemma 2.12, Lemma 2.13, we have that
\[
\beta_E^{d,\infty}(B_Q) \lesssim d \beta_E^{d,p}(B_Q)^{\frac{1}{p-1}}.
\]
Hence $E$ is uniformly wiggly of dimension $d$ with respect to $\beta_E^{d,p}$ with constant $\beta'_0 \sim \beta_0^{d+1}$, and (1.10) follows immediately from (5.1) with $\beta_0$ replaced by $\beta'_0$. $\square$

We begin the proof of Proposition 5.1. Let $E \subset \mathbb{R}^n$ be a closed set with $d$-PBP, with constant $\delta > 0$. Without loss of generality we assume $E$ is compact, $\text{diam}(E) = 1/2$, so that $D_0 = \{E\}$, and $E \subset [0,1)^n$.

Recall that $\Delta$ is the standard dyadic lattice on $\mathbb{R}^n$, and $\Delta_k = \{I \in \Delta : \ell(I) = 2^{-k}\}$. Given a dyadic cube $I \in \Delta$ and $A \geq 1$, we will denote by $A I$ the cube with the same center as $I$ and sidelength $A \ell(I)$.
Recall that $\rho = 1/1000$ and that for $Q \in D^E_m$ we have $\ell(Q) = 5\rho^m$, see Lemma 2.11. Given $k \geq 0$ let $j(k)$ be the unique integer such that $2^{-j(k)} \leq 5\rho^k < 2^{-j(k)+1}$. We set

$$\Delta_k(E) := \left\{ I \in \Delta_j(E) : I \cap E \neq \emptyset \right\}.$$  

(5.2)

We defined $j(k)$ in such a way that if $I \in \Delta_k(E)$ and $Q \in D^E_k$ then $\ell(I) \leq \ell(Q) \leq 2\ell(I)$. In particular, $\ell(I) \sim \rho^k$.

Set

$$E_k := \bigcup_{I \in \Delta_k(E)} \partial_d I,$$  

(5.3)

where $\partial_d I$ is the $d$-dimensional dyadic skeleton of $I$. Just like in Lemma 3.11 and Lemma 4.2, one may show that since $E$ has $d$-PBP with constant $\delta$, then $E_k$ also has $d$-PBP (with constant $\delta' \sim \delta$).

**Lemma 5.2.** Let $k \geq j \geq 1$ be integers, and let $I \in \Delta_j(E)$. We have

$$\sum_{m=j}^{k} \sum_{Q \in D^E_m \atop Q \cap \partial I \neq \emptyset} \beta^d_E(B_Q)^2 \ell(Q)^d \leq C \mathcal{H}^d(E_k \cap AI),$$  

(5.4)

where $A = 24$, and $C$ depends on $\delta, n, d$ and $p$.

**Proof.** Without loss of generality we may assume that $p = 2$ (see [AV21 (A.3)]). Consider a cube $Q \in D^E_m$ with $j \leq m \leq k$ and $Q \cap 3I \neq \emptyset$. Let $P \in D^E_k$ be a cube with $\ell(P) = \ell(Q)$ and such that $B_Q \cap B_P \neq \emptyset$ (such a cube exists because $\text{dist}(x_Q, E_k) \leq 2^{-j(k)} \leq \ell(Q)$). In particular, we have $2B_Q \subset 4B_P$ and we may apply Lemma 2.13 with $E_1 = E$ and $E_2 = E_k$ to obtain

$$\beta^d_E(B_Q) \leq \beta^d_E(2B_Q) \leq \beta^d_E(4B_P) + \left( \frac{1}{\ell(Q)^d} \int_{A_{B_Q \cap E}} \left( \frac{\text{dist}(y, E_k)}{\ell(Q)} \right)^2 \, d\mathcal{H}^d \right)^{1/2}.$$  

Note that we have $B_P \cap 11I \neq \emptyset$ because $\ell(Q) \leq 2\ell(I)$, $Q \cap 3I \neq \emptyset$ and $B_P \cap B_Q \neq \emptyset$.

Multiplying by $\ell(Q)^d$ and summing over $Q \in D^E_m$ with $j \leq m \leq k$ and $Q \cap 3I \neq \emptyset$ yields

$$\sum_{m=j}^{k} \sum_{Q \in D^E_m \atop Q \cap \partial I \neq \emptyset} \beta^d_E(B_Q)^2 \ell(Q)^d \leq \sum_{m=j}^{k} \sum_{P \in D^E_k \atop B_P \cap 12I \neq \emptyset} \beta^d_E(4B_P)^2 \ell(P)^d$$

$$+ \sum_{m=j}^{k} \sum_{Q \in D^E_m \atop Q \cap \partial I \neq \emptyset} \left( \frac{1}{\ell(Q)^d} \int_{A_{B_Q \cap E}} \left( \frac{\text{dist}(y, E_k)}{\ell(Q)} \right)^2 \, d\mathcal{H}^d \right) \ell(P)^d =: S_1 + S_2.$$  

(5.5)

We begin by estimating $S_1$. Let

$$\mathcal{R} = \left\{ R \in D^E_k : B_R \cap 11I \neq \emptyset \right\}.$$  

Note that for each cube $P$ appearing in $S_1$ there exists a unique $R \in \mathcal{R}$ such that $P \subset R$. Observe also that all the cubes $R \in \mathcal{R}$ are contained in $19I$. Applying these observations
together with Theorem \([1,10]\) to the set \(E_k\) and cubes \(R \in \mathcal{R}\) yields
\[
S_1 \lesssim \sum_{R \in \mathcal{R}} \sum_{P \in \mathcal{D}(R)} \beta_{E_k}^d (4B_P)^2 \ell(P)^d \lesssim \delta \sum_{R \in \mathcal{R}} \mathcal{H}^d(R) \leq \mathcal{H}^d(E_k \cap 19I). \tag{5.6}
\]

We move on to estimating \(S_2\). This estimate is similar to that of "I_2'' in [1,10]. First, we use the definition of \(E_k\) and the subadditivity of Hausdorff content to get
\[
\int_{4B_Q \cap E} \left( \frac{\text{dist}(y,E_k)}{\ell(P)} \right)^2 \, d\mathcal{H}_\infty^d(y) \lesssim \sum_{J \in \Delta_k(E)} \ell(J)^2 \frac{\mathcal{H}_\infty^d(J \cap 4B_Q \cap E)}{\ell(Q)^2} \sum_{J \in \Delta_k(E \cap 4B_Q \neq \emptyset)} \ell(J)^{d+2}. \tag{5.7}
\]
It is easy to check that if \(Q \in \mathcal{D}_m^E\) with \(j \leq m \leq k\) and \(Q \cap 3I \neq \emptyset\), then for \(J\) as above we have \(J \subset 24I\). It follows that
\[
S_2 \lesssim \sum_{m=J}^{k} \sum_{\substack{Q \in \mathcal{D}_m^E \cap \Delta_k(E) \cap 3I \neq \emptyset \cap 4B_Q \neq \emptyset}} \ell(J)^d \frac{\mathcal{H}_\infty^d(J \cap 4B_Q \cap E)}{\ell(Q)^2} \lesssim \sum_{J \in \Delta_k(E \cap 4B_Q \neq \emptyset)} \ell(J)^{d+2} \sum_{\ell(Q) \geq \rho^2; \ell(Q) \geq \rho^2} \frac{1}{\ell(Q)^2} \lesssim \sum_{J \in \Delta_k(E \cap 4B_Q \neq \emptyset)} \ell(J)^d.
\]
We also have
\[
\sum_{J \in \Delta_k(E \cap 4B_Q \neq \emptyset)} \ell(J)^d \sim \sum_{J \in \Delta_k(E \cap 4B_Q \neq \emptyset)} \mathcal{H}^d(\partial_d I) \sim \mathcal{H}^d(E_k \cap 24I).
\]
Hence, \(S_2 \lesssim \mathcal{H}^d(E_k \cap 24I)\). Putting this together with \([5.5]\) and \([5.6]\) gives \([5.4]\). □

**Lemma 5.3.** Let \(k \geq j > 0\) be integers and \(I \in \Delta_j(E)\). We have
\[
\sum_{m=j}^{k} \sum_{\substack{Q \in \mathcal{D}_m^E \cap 3I \neq \emptyset \cap 4B_Q \neq \emptyset}} \beta_{E}^d(P_{B_Q})^2 \ell(Q)^d \geq (k - j) C \delta \beta_0^2 \rho^j d^j, \tag{5.7}
\]
where \(C\) is a constant depending only on \(n, d\), but not on \(\delta\).

**Proof.** Recall first that \(E\) is lower \((d,c_1)\)-regular (with \(c_1 \sim \delta\), see Lemma \([2.2]\)), so that in particular \(\mathcal{H}_\infty^d(E \cap 3I) \geq \delta \rho^j\). Let \(m \geq j\). Since the balls \(\{B_Q : Q \in \mathcal{D}_m^E, Q \cap 3I \neq \emptyset\}\) are a cover of \(E \cap 3I\), we get
\[
\sum_{Q \in \mathcal{D}_m^E \cap 3I \neq \emptyset} \ell(Q)^d \geq \delta \rho^j.
\]
Recalling that \(E\) is uniformly wiggly we arrive at
\[
\sum_{Q \in \mathcal{D}_m^E \cap 3I \neq \emptyset} \beta_{E}^d(P_{B_Q})^2 \ell(Q)^d \geq \beta_0^2 \sum_{Q \in \mathcal{D}_m^E \cap 3I \neq \emptyset} \ell(Q)^d \geq \delta \beta_0^2 \rho^j.
\]
Summing over \(j \leq m \leq k\) gives \([5.7]\). □
Recall that our aim was to prove that
\[ \dim_H(E) \geq d + c \beta_0^2. \]
This estimate follows immediately from Theorem 8.8 in [Mat95] and the next lemma.

**Lemma 5.4.** There exists a non-zero measure \( \mu \) with \( \text{supp} \mu \subset E \) and such that
\[ \mu(B(x,r)) \lesssim r^{d+c\beta_0^2} \quad \text{for} \quad x \in E \quad \text{and} \quad r > 0, \]
where \( c \) depends on \( \delta, p, n, d \).

**Proof.** The construction of \( \mu \) will is similar to the proof of Frostman’s lemma, as in [Mat95, Theorem 8.8]. First, we need to define a tree-like collection of dyadic cubes intersecting \( E \) which we will use in our construction.

Let \( k > j \geq 1 \) be integers, and let \( I \in \Delta_j(E) \). Using (5.4) and (5.7) we get that
\[ \mathcal{H}^d(E_k \cap AI) \gtrsim \delta^{k-j} \beta_0^2 \rho^{d-j}. \] (5.8)

Let
\[ D_k(I) := \Delta_k(E \cap AI) = \{ J \in \Delta_k : J \cap AI \cap E \neq \emptyset \}. \]
It follows easily from the definition of \( E_k \) (5.3) that
\[ E_k \cap AI \subset \bigcup_{J \in D_k(I)} \partial_d J. \]
Hence,
\[ \mathcal{H}^d(E_k \cap AI) \lesssim \rho^{dk} \# D_k(I). \]
Together with (5.8) this gives
\[ \# D_k(I) \gtrsim \delta^{k-j} \beta_0^2 \rho^{d-j-k}. \] (5.9)

Observe that \( \Delta(E) = \bigcup_{k \geq 0} \Delta_k(E) \) can be endowed with a natural tree structure, and this structure is used in the usual proof of the Frostman’s lemma. We would like to define a similar structure for the collections \( D_k(I) \) for \( I \in \Delta_j(E), k \geq j \geq 0 \), but we need to take extra care due to possible overlaps between collections \( D_k(I), D_k(J) \) when \( I, J \in \Delta_j(E) \).

To overcome this technicality, for any integers \( j, k \) with \( k \geq j \geq 1 \) and \( I \in \Delta_j(E) \) let \( D'_k(I) \subset D_k(I) \) be a maximal subfamily of \( D_k(I) \) such that for any \( J_1, J_2 \in D'_k(I) \) we have \( A'J_1 \cap A'J_2 = \emptyset \), where \( A' = A + 1 \). Note that since for each \( J \in D'_k(I) \) we have \( J \subset AI \), we get
\[ \bigcup_{J \in D'_k(I)} A'J \subset A'I. \]
Moreover, since all the cubes in \( D_k(I) \) have equal sidelength, the cubes \( \{ A'J \}_{J \in D_k(I)} \) have bounded overlap, and so it follows that
\[ \# D'_k(I) \gtrsim \# D_k(I). \]
Recalling (5.9), we get that
\[ \# D'_k(I) \gtrsim C_1(k-j) \beta_0^2 \rho^{d-j-k}. \] (5.10)
for some \( 0 < C_1 < 1 \) depending on \( \delta, p, n, d \).
Let 
\[ \kappa := [10^6 C_1^{-1} \beta_0^{-2}], \]
and observe that for \( c := C_1/10^6 \) we have 
\[ \kappa C_1 \beta_0^2 \geq 10^6 = \rho^{-2} \geq \rho^{-\kappa c \beta_0^2}, \]
where in the last inequality we use also the fact that \( C_1, \beta_0 \in (0, 1) \). Hence, if \( k \) is such that \( k = j + \kappa \), the inequality \((5.10)\) gives 
\[ \#D_{j+\kappa}(I) \geq \rho^{-\kappa(d+c\beta_0^2)}. \]  
(5.11)

We are ready to define the tree structure we will use to construct \( \mu \). Set 
\[ S_0 = \Delta_0(E) = \{[0, 1)^n\}, \]
where we used our assumption \( E \subset [0, 1)^n \). Let \( j \geq 0 \) and assume that \( S_j \) has already been defined, that \( S_j \subset \Delta_{j\kappa}(E) \), and that for every \( I, J \in S_j \), \( I \neq J \), we have \( A^I \cap A^J = \emptyset \). For each \( I \in S_j \) we set 
\[ S(I) = D_{(j+1)\kappa}(I), \]
and 
\[ S_{j+1} = \bigcup_{I \in S_j} S(I). \]

Note that for every \( J \in S_{j+1} \) there is a unique “parent” \( I \in S_j \) such that \( J \subset A I \) (and \( A^J \subset A^I \)). Moreover, for every \( I, J \in S_{j+1} \), \( I \neq J \), we have \( A^I \cap A^J = \emptyset \) (either \( I \) and \( J \) have distinct parents, or they have the same parent \( I' \) in which case we use the definition of \( D_{(j+1)\kappa}(I') \)). Finally, observe that if \( s := d + c\beta_0^2 \), then by \((5.11)\)
\[ \#S_j \geq \rho^{-j\kappa s} \quad \text{and} \quad \#S(I) \geq \rho^{-\kappa s} \quad \text{for each} \ I \in S_j. \]  
(5.12)

We are ready to construct the measure \( \mu \). It is obtained as a weak limit of measures \( \mu_j, j \geq 0 \). The definition of \( \mu_j \) follows the usual “bottom-to-top” construction of the Frostman measure, as in Theorem 8.8 of [Mat95].

First, we set 
\[ \mu_j^I = \rho^{j\kappa s} \sum_{I \in S_j} \frac{L^n|A^I}{L^n(A^I)}. \]

Assume that the measure \( \mu_j^I \) with \( 1 \leq i \leq j \) has already been defined, and that for each \( I \in S_i \) we have \( \mu_j^I(A^I) = \rho^{i\kappa s} \). We define \( \mu_j^{i-1} \) by modifying \( \mu_j^I \) at the level of \( S_{i-1} \): for each \( I \in S_{i-1} \) we set 
\[ \mu_j^{i-1}|_{A^I} := \rho^{(i-1)\kappa s} \mu_j^I(A^I)^{-1} \mu_j^I|_{A^I}. \]

Finally, we define \( \mu_j = \mu_j^0 \). Clearly, \( \mu_j([0,1)^n) = 1 \).

Observe that the quantity used in the definition of \( \mu_j^{i-1}|_{A^I} \) satisfies 
\[ \rho^{(i-1)\kappa s} \mu_j^I(A^I)^{-1} = \rho^{\kappa s} (\#S(I))^{-1} \leq 1. \]
(5.13)
Let $\mu_{j_k}$ be a subsequence of $\mu_j$ converging weakly, and define $\mu$ to be the weak limit of $\mu_{j_k}$. Since $\mu_j(\mathbb{R}^n) = 1$ for all $j$, we also have $\mu(\mathbb{R}^n) = 1$. Using (5.13), a standard argument gives $\mu(B(x, r)) \lesssim r^s$ for $x \in \mathbb{R}^n$, $r > 0$, see p.114 in [Mat95] for details. Finally, $\text{supp } \mu \subset E$ because $\text{supp } \mu_j \subset \bigcup_{I \in \Delta_j} I(\mathbb{R}^n)$.

This completes the proof of Proposition 5.1.

6. Denjoy’s conjecture

In this section we prove Theorem 1.7. Let $\Sigma \subset \mathbb{R}^{d+1}$ be a compact set with $d$-PBP with parameter $\delta > 0$, and let $E \subset \Sigma$ be compact. Let $\mathcal{D}$ denote the Christ-David cubes on $\Sigma$, as in Lemma 2.1. We are going to use Theorem 2.10 to get a lower bound on $\Gamma_{n,d}(E)$.

By Frostman’s lemma (see [Tol14, Lemma 1.23]), there exists a measure $\mu$ supported on $E$ such that $\mu(B(x, r)) \leq r^d$ for all $x \in \mathbb{R}^n$, $r > 0$, and $\mu(E) \sim \mathcal{H}_\infty^d(E)$. In fact, it follows immediately from the proof of [Tol14, Lemma 1.23] that one even has $\mu(B(x, r)) \lesssim \mathcal{H}_\infty^d(E \cap B(x, r))$.

Recall that $\Sigma$ is lower content regular with constant $\sim \delta$ (see Lemma 2.2). Using the fact that for any ball $x \in E$ and $0 < r < \infty$ we have $\mu(B(x, r)) \leq \mathcal{H}_\infty^d(E \cap B(x, r)) \leq \mathcal{H}_\infty^d(\Sigma \cap B(x, r))$ it follows from the definitions of $\beta_{\mu,2}$ and $\beta_{\Sigma,2}^{d,2}$ that

$$\beta_{\mu,2}(x, r) \lesssim \beta_{\Sigma,2}^{d,2}(x, r).$$

Indeed, given a plane $L \in A(n, d)$ infimising $\beta_{E,2}^{d,2}(x, r)$ set $E_t = E_t(x, r) := \{y \in B(x, r) \cap E : \text{dist}(y, L)^2 > tr^2\}$.

Let $\mathcal{C}$ be a covering of $E_t$ such that $\sum_{B \in \mathcal{C}} \text{diam}(B)^d \leq 2\mathcal{H}_\infty^d(E_t)$. We may assume that the covering consists of balls (see [Mat95], Chapter 5). Since $\mathcal{C}$ covers $E_t$, and $\mu$ is a Frostman measure, we have

$$\mu(E_t) \leq \sum_{B \in \mathcal{C}} \mu(B) \leq \sum_{B \in \mathcal{C}} r(B)^d \lesssim \mathcal{H}_\infty^d(E_t).$$

Hence,

$$\beta_{\mu,2}(x, r) \leq \int_0^\infty \mu(E_t) t \, dt \lesssim \int_0^\infty \mathcal{H}_\infty^d(E_t) t \, dt \sim \beta_{E,2}^{d,2}(x, r),$$

and this gives (6.1) since $E \subset \Sigma$.

Now recall that $\theta_{\mu}(x, r) = \mu(B(x, r)) r^{-d}$ is the $d$-density of $\mu$ in the ball $B(x, r)$. Since $\mu(B(x, r)) \leq r^d$, we have $\theta_{\mu}(x, r) \leq 1$. Arguing as we did below (3.3) and using
we get 
\[ \beta^2(\mu) := \int_0^\infty \beta_{\mu,2}(x,r)^2 \theta_\mu(x,r) \frac{dr}{r} d\mu(x) \]
\[ \leq \int_0^1 \beta_{\mu,2}(x,r)^2 \frac{dr}{r} d\mu(x) + \int_1^\infty \beta_{\mu,2}(x,r)^2 \frac{dr}{r} d\mu(x) \]
\[ \lesssim \sum_{Q \in D} \beta_{\mu,2}^2(3BQ)^2 \theta_\mu(Q) + \int_1^\infty \theta_\mu(x,r)^2 \frac{dr}{r} d\mu(x) \]
\[ \lesssim \sum_{Q \in D} \beta_{\mu,2}^2(3BQ)^2 \mu(Q) + \mu(E) \lesssim \sum_{Q \in D} \beta_{\nu,2}^2(3BQ)^2 \ell(Q)^d + \mu(E). \]
Applying Theorem 1.10 to estimate the last sum above yields
\[ \beta^2(\mu) \lesssim \sum_{Q \in D} \beta_{\nu,2}^2(3BQ)^2 \ell(Q)^d + \mu(E) \lesssim C(\delta) \mathcal{H}^d(\Sigma) + \mathcal{H}_{\infty}(E) \lesssim C(\delta) \mathcal{H}^d(\Sigma), \]
so that \( \beta^2(\mu) \leq C_1(\delta) \mathcal{H}^d(\Sigma) \) for some \( C_1(\delta) > 1 \).

Set
\[ C_2 := \left( \frac{\mu(E)}{C_1(\delta) \mathcal{H}^d(\Sigma)} \right)^{\frac{1}{2}}, \]
and define the measure \( \sigma := C_2 \mu \). Since \( C_2 \in (0,1) \), we have \( \sigma(B(x,r)) \leq \mu(B(x,r)) \leq r^d \). Moreover,
\[ \beta^2(\sigma) = C_2^2 \beta^2(\mu) \leq \left( \frac{\mu(E)}{C_1(\delta) \mathcal{H}^d(\Sigma)} \right)^{\frac{1}{2}} C_1(\delta) \mathcal{H}^d(\Sigma) = C_2 \mu(E) = \sigma(E). \]
Then, by Theorem 2.10
\[ \Gamma_{n,d}(E) \geq \sigma(E) = C_2 \mu(E) = \left( \frac{\mu(E)}{C_1(\delta) \mathcal{H}^d(\Sigma)} \right)^{\frac{1}{2}} \mu(E) \]
\[ = C_1(\delta)^{-\frac{1}{2}} \frac{\mu(E)^{\frac{1}{2}}}{\mathcal{H}^d(\Sigma)^{\frac{1}{2}}} \sim C_1(\delta)^{-\frac{1}{2}} \frac{\mathcal{H}^d_{\infty}(E)^{\frac{1}{2}}}{\mathcal{H}^d(\Sigma)^{\frac{1}{2}}}. \]
This finishes the proof of Theorem 1.7.

**Appendix A. Frostman’s lemma for lower content regular sets**

We prove the following version of the classical Frostman’s lemma.

**Theorem A.1.** Let \( E \subset [0,1]^n \) be a compact, lower content \((d,c_1)\)-regular set. Then, there exists a measure \( \mu \) satisfying the following properties:

1. \( \text{supp} \mu = E \),
2. \( \mu(E) = \mathcal{H}_{\infty}^d(E) \),
3. \( \mu \) has polynomial growth, that is, there exists a constant \( C_1 \geq 1 \) such that for all \( x \in E \) and \( 0 < r < \text{diam}(E) \) we have
\[ \mu(B(x,r)) \leq C_1 r^d, \]
(4) \( \mu \) is doubling, that is, there exists a constant \( C_{db} \geq 1 \) such that for all \( x \in E \) and \( 0 < r < \text{diam}(E) \) we have
\[
\mu(B(x, 2r)) \leq C_{db} \mu(B(x, r)). \tag{A.1}
\]
(5) the \( d \)-dimensional density of \( \mu \) is almost monotone, that is, there exists a constant \( A \geq 1 \) such that if \( P, Q \in \mathcal{D} \), and \( P \subset Q \), then
\[
\theta_\mu(P) \leq A \theta_\mu(Q).
\]

In the above, \( C_1 \) depends only on \( d, n \), while \( C_{db} \) and \( A \) also depend on the lower regularity constant \( c_1 \).

The usual proof of Frostman’s lemma is “bottom-to-top”: it starts from small scales and goes up (see [Mat95 Theorem 8.8]). An alternative, simpler proof is due to Tolsa [Tol14 Theorem 1.23], who came up with a “top-to-bottom” construction. To prove Theorem A.1 we will modify Tolsa’s construction so that the resulting Frostman measure is doubling. Roughly speaking, at each step we are going to modify the measure by redistributing the mass between cubes where the doubling condition fails.

A.1. Construction of a sequence of measures. In this subsection we construct a sequence of measures \( \mu_k \). The measure \( \mu \) from Theorem \( \text{[A.1]} \) is going to be the weak limit of \( \mu_k \).

Let \( \mathcal{D} \) be the David-Christ lattice on \( E \), as in Lemma 2.1. For a cube \( R \in \mathcal{D}_k \), we denote by \( \text{Ch}(R) \) the cubes \( Q \in \mathcal{D}_{k+1} \) with \( Q \subset R \). By \( \text{Nbd}(Q) \) we denote the cubes \( P \in \mathcal{D}_k \) so that \( \text{dist}(P, Q) \leq \ell(Q) \). Note that there exists a dimensional constant \( c_n > 1 \) such that
\[
\# \text{Ch}(Q) + \# \text{Nbd}(Q) \leq c_n. \tag{A.2}
\]
Recall that \( Q^1 \) denotes the parent cube of \( Q \), and also that the balls \( B(Q) = B(x_Q, c_0 \ell(Q)) \) and \( B_Q = B(x_Q, \ell(Q)) \) satisfy \( B(Q) \cap E \subset Q \subset B_Q \), where \( \ell(Q) = \rho^k \) for \( Q \in \mathcal{D}_k \).

**Proposition A.2.** Let \( E \subset [0, 1]^n \) be a compact, lower content \( (d, c_1) \)-regular set. There exists a constant \( C_0 = C_0(n, d, c_1) > 1 \) and sequence of Radon measures \( \mu_k, \, k \geq 0 \), absolutely continuous with respect to \( \mathcal{L}^n \), such that, for each \( k \), we have
\[
\mu_k(\mathbb{R}^n) = \mathcal{H}_\infty^d(E), \tag{A.3}
\]
\[
\text{supp } \mu_k = \bigcup_{Q \in \mathcal{D}_k} B(Q), \tag{A.4}
\]
and moreover for all \( Q \in \mathcal{D}_k \),
\[
\mu_k(B(Q)) \leq \mathcal{H}_\infty^d(Q) \leq 2 \ell(Q)^d, \tag{A.5}
\]
\[
\mu_k(B(Q)) \leq C_0 \mu_k(B(P)) \quad \text{for all } P \in \text{Nbd}(Q), \tag{A.6}
\]
\[
C_0^{-1} \mu_{k-1}(B(Q^1)) \leq \mu_k(B(Q)) \leq \mu_{k-1}(B(Q^1)) \quad \text{if } k \geq 1. \tag{A.7}
\]

Note that since for all \( P \in \text{Nbd}(Q) \) we have \( Q \in \text{Nbd}(P) \), if the property \( \text{[A.6]} \) holds for all \( Q \in \mathcal{D}_k \), then
\[
C_0^{-1} \mu_k(B(P)) \leq \mu_k(B(Q)) \leq C_0 \mu_k(B(P)) \quad \text{for all } P \in \text{Nbd}(Q). \tag{A.8}
\]
We construct \( \mu_k \) inductively. Let \( Q_0 \in D_0 \); that is, \( Q_0 \) is the top cube. We define

\[
\mu_0 := \mathcal{H}_\infty^d(Q_0) \cdot \frac{\mathcal{L}^n | B(Q_0) |}{\mathcal{L}^n(B(Q_0))}.
\]

Clearly, \( \mu_0 \) satisfies all the properties from Proposition \( \text{A.2} \). Now, assume that \( \mu_{k-1} \) has already been defined and satisfies the properties \( \text{A.3} \) – \( \text{A.7} \).

### A.1.1. Auxiliary measure \( \eta_k \)

For \( R \in D_{k-1} \) and for each \( Q \in \text{Ch}(R) \), set

\[
\eta_k | B(Q) := \left( \frac{\mathcal{H}_\infty^d(Q)}{\sum_{P \in \text{Ch}(R)} \mathcal{H}_\infty^d(P)} \cdot \mu_{k-1}(B(R)) \right) \frac{\mathcal{L}^n | B(Q) |}{\mathcal{L}^n(B(Q))}.
\]

Observe that

\[
\sum_{Q \in \text{Ch}(R)} \eta_k(B(Q)) = \mu_{k-1}(B(R)).
\]

**Lemma A.3.** We have \( \eta_k(\mathbb{R}^n) = \mu_{k-1}(\mathbb{R}^n) = \mathcal{H}_\infty^d(E) \), so that condition \( \text{A.3} \) holds for \( \eta_k \).

**Proof.** We have

\[
\eta_k(\mathbb{R}^n) = \sum_{Q \in D_k} \eta_k(B(Q)) = \sum_{R \in D_{k-1}} \sum_{Q \in \text{Ch}(R)} \eta_k(B(Q)) \overset{\text{A.10}}{=} \sum_{R \in D_{k-1}} \mu_{k-1}(B(R)) = \mu_{k-1}(\mathbb{R}^n) = \mathcal{H}_\infty^d(E).
\]

**Lemma A.4.** The polynomial growth condition \( \text{A.9} \) holds for \( \eta_k \).

**Proof.** Let \( R \in D_{k-1} \) and \( Q \in \text{Ch}(R) \). Since \( \text{A.3} \) holds for \( \mu_{k-1} \), we have \( \mu_{k-1}(B(R)) \leq \mathcal{H}_\infty^d(R) \leq \sum_{P \in \text{Ch}(R)} \mathcal{H}_\infty^d(P) \), by subadditivity of \( \mathcal{H}_\infty^d \). Then,

\[
\eta_k(B(Q)) = \mathcal{H}_\infty^d(Q) \cdot \frac{\mu_{k-1}(B(R))}{\sum_{P \in \text{Ch}(R)} \mathcal{H}_\infty^d(P)} \leq \mathcal{H}_\infty^d(Q) \leq 2\ell(Q)^d.
\]

**Lemma A.5.** The doubling condition \( \text{A.6} \) holds for \( \eta_k \) whenever both \( Q \) and \( P \) in \( D_k \) have a common parent \( R \in D_{k-1} \). Moreover, for all \( Q \in D_k \)

\[
\eta_k(B(Q)) \geq c_2 \mu_{k-1}(B(Q^1))
\]

for \( 0 < c_2 < 1 \) depending on \( c_1, n, d \). In particular \( \text{A.7} \) holds for \( \eta_k \) assuming \( C_0 \) is big enough.

**Proof.** Let \( R \in D_{k-1} \) and \( Q, P \in \text{Ch}(R) \). Using lower content regularity of \( E \) we have

\[
\mathcal{H}_\infty^d(Q) \geq \mathcal{H}_\infty^d(B(Q) \cap E) \geq c_1(0_0 \ell(Q)) = \frac{c_1(0_0)^d}{2^d} \cdot 2(\ell(R))^d \geq c_1(0_0)^d \mathcal{H}_\infty^d(R).
\]
Recall that \(\#\text{Ch}(R) \leq c_n\). We then compute

\[
\eta_k(B(Q)) = \frac{\mathcal{H}_\infty^d(Q)}{\sum_{P^2 \in \text{Ch}(R)} \mathcal{H}_\infty^d(P^2)} \cdot \mu_{k-1}(B(R)) \\
\geq \frac{c_1(c_0)^d}{2\rho^d} \cdot \frac{\mathcal{H}_\infty^d(R)}{c_n \mathcal{H}_\infty^d(R)} \cdot \mu_{k-1}(B(R)) = \frac{c_1(c_0)^d}{2c_n\rho^d} \mu_{k-1}(B(R)).
\]

This shows (A.12). To see (A.6) note that

\[
\eta_k(B(P)) \leq \mu_{k-1}(B(R)) \leq c_0 \eta_k(B(Q)) \leq C_0 \eta_k(B(Q)),
\]

so (A.6) holds true whenever \(P\) is a sibling of \(Q\). \(\square\)

The lemmas above say that \(\eta_k\) satisfies almost all the required properties, except that instead of the full doubling condition it only satisfies a “dyadic doubling” condition. We need to make some adjustments.

A.1.2. Definition of \(\mu_k\). We define families of cubes where the doubling condition (A.8) fails. For each \(Q \in \mathcal{D}_k\) let

\[
\text{Rich}(Q) = \{P \in \text{Nbd}(Q) : \eta_k(B(Q)) < 4C_0^{-1} \eta_k(B(P))\},
\]

and similarly

\[
\text{Poor}(Q) = \{P \in \text{Nbd}(Q) : \eta_k(B(Q)) > \frac{C_0}{4} \eta_k(B(P))\}.
\]

Note that

\[\#\text{Poor}(Q) + \#\text{Rich}(Q) \leq 2 \#\text{Nbd}(Q) \leq 2c_n. \quad (A.14)\]

Set

\[
\text{Poor}_k = \{Q \in \mathcal{D}_k : \text{Rich}(Q) \neq \emptyset\},
\]

\[
\text{Rich}_k = \{Q \in \mathcal{D}_k : \text{Poor}(Q) \neq \emptyset\}.
\]

Lemma A.6. We have \(\text{Poor}_k \cap \text{Rich}_k = \emptyset\).

Proof. Suppose that \(Q \in \text{Poor}_k \cap \text{Rich}_k\). Then, there exist \(P_p, P_r \in \text{Nbd}(Q)\) such that

\[
\frac{C_0}{4} \eta_k(B(P_p)) < \eta_k(B(Q)) < 4C_0^{-1} \eta_k(B(P_r)).
\]

In particular, \(\eta_k(B(P_p)) < 16C_0^{-2} \eta_k(B(P_r))\). Let \(R_p, R_r\) be the parents of \(P_p, P_r\), respectively. Note that \(R_p \in \text{Nbd}(R_r)\). Then, by (A.12)

\[
\mu_{k-1}(B(R_p)) \leq c_2^{-1} \eta_k(B(P_p)) < 16c_2^{-1}C_0^{-2} \eta_k(B(P_r)) \leq C_0^{-1} \mu_{k-1}(B(R_r)),
\]

assuming \(C_0\) big enough. This contradicts the doubling property (A.8) for \(\mu_{k-1}\). Hence, \(\text{Poor}_k \cap \text{Rich}_k = \emptyset\). \(\square\)

We define the measure \(\mu_k\) with \(\text{supp}\mu_k = \text{supp}\eta_k = \bigcup_{Q \in \mathcal{D}_k} B(Q)\) in the following way. For each \(Q \in \mathcal{D}_k\) we set

\[
\mu_k|_{B(Q)} = \begin{cases} 
(1 - C_0^{-1} \#\text{Poor}(Q)) \eta_k|_{B(Q)} & \text{if } Q \in \text{Rich}_k, \\
(1 + C_0^{-1} \sum_{P \in \text{Rich}(Q)} \frac{\eta_k(B(P))}{\eta_k(B(Q))}) \eta_k|_{B(Q)} & \text{if } Q \in \text{Poor}_k, \\
\eta_k|_{B(Q)} & \text{if } Q \notin \text{Rich}_k \cup \text{Poor}_k.
\end{cases}
\]
Lemma A.7. Property (A.7) holds for \( \mu_k \).

Proof. Observe that by (A.14), as soon as \( C_0 \) is large enough depending on \( n \), we have
\[
\mu_k \geq \frac{1}{2} \eta_k.
\] (A.15)

Thus, we immediately get from (A.12) that \( \mu_k \) satisfies
\[
\mu_k(B(Q)) \geq \frac{c_2}{2} \mu_{k-1}(B(Q^1)) \geq C_0^{-1} \mu_{k-1}(B(Q^1)).
\]

This shows one of the inequalities in (A.7). Now we prove that
\[
\mu_k(B(Q)) \lesssim \mu_{k-1}(B(Q^1)).
\] (A.16)

For \( Q \notin \text{Poor}_k \) this is trivial, because then by the definition of \( \mu_k \) and \( \eta_k \) we have \( \mu_k(B(Q)) \leq \eta_k(B(Q)) \leq \mu_{k-1}(B(Q^1)) \). On the other hand, for \( Q \in \text{Poor}_k \)
\[
\mu_k(B(Q)) = \eta_k(B(Q)) + C_0^{-1} \sum_{P \in \text{Rich}(Q)} \eta_k(B(P))
\]
\[
\leq \mu_{k-1}(B(Q^1)) + C_0^{-1} \sum_{P \in \text{Rich}(Q)} \mu_{k-1}(B(P^1)).
\]

For each \( P \in \text{Rich}(Q) \) we have \( P^1 \in \text{Nbd}(Q^1) \), and so by (A.6) applied to \( \mu_{k-1} \) we get
\[
\mu_{k-1}(B(Q^1)) + C_0^{-1} \sum_{P \in \text{Rich}(Q)} \mu_{k-1}(B(P^1))
\]
\[
\leq \mu_{k-1}(B(Q^1)) + \#\text{Rich}(Q) \mu_{k-1}(B(Q^1)) \lesssim \mu_{k-1}(B(Q^1)).
\]

This finishes the proof. \( \square \)

Lemma A.8. We have \( \mu_k(\mathbb{R}^n) = \eta_k(\mathbb{R}^n) = \mathcal{H}^d(E) \), so that condition (A.3) holds for \( \mu_k \).

Proof. We have
\[
\mu_k(\mathbb{R}^n) = \sum_{Q \in \mathcal{D}_k} \mu_k(B(Q)) = \sum_{Q \in \text{Rich}_k} \left( 1 - C_0^{-1} \cdot \#\text{Poor}(Q) \right) \eta_k(B(Q))
\]
\[
+ \sum_{Q \in \text{Poor}_k} \left( \eta_k(B(Q)) + C_0^{-1} \sum_{P \in \text{Rich}(Q)} \eta_k(B(P)) \right) + \sum_{Q \in \text{Rich}_k \cup \text{Poor}_k} \eta_k(B(Q))
\]
\[
= \sum_{Q \in \mathcal{D}_k} \eta_k(B(Q)) - C_0^{-1} \sum_{Q \in \text{Rich}_k} \#\text{Poor}(Q) \eta_k(B(Q)) + C_0^{-1} \sum_{Q \in \text{Poor}_k} \sum_{P \in \text{Rich}(Q)} \eta_k(B(P))
\]
\[
= \sum_{Q \in \mathcal{D}_k} \eta_k(B(Q)) = \eta_k(\mathbb{R}^n) = \mathcal{H}^d(E).
\]

Lemma A.9. The polynomial growth condition (A.5) holds for \( \mu_k \).

Proof. Observe that for \( Q \in \mathcal{D}_k \setminus \text{Poor}_k \) we have \( \mu_k(B(Q)) \leq \eta_k(B(Q)) \), so in this case (A.5) follows from Lemma A.4. Assume that \( Q \in \text{Poor}_k \). Then, there exists \( P \in \text{Nbd}(Q) \) such that
\[
\eta_k(B(Q)) \leq 2C_0^{-1} \eta_k(B(P)) \leq 4C_0^{-1} \ell(Q)^d,
\]
where in the last inequality we used again Lemma A.4. Then,

\[ \mu_k(B(Q)) = \eta_k(B(Q)) + C_0^{-1} \sum_{P \in \text{Rich}(Q)} \eta_k(B(P)) \]

\[ \leq 4C_0^{-1}\ell(Q)^d + \#\text{Rich}(Q) \cdot 2C_0^{-1}\ell(Q)^d \leq n\ C_0^{-1}\ell(Q)^d. \]

Recalling that \( \mathcal{H}^d_\infty(Q) \geq c_1\ell(Q)^d \) by (A.13), we get that \( \mu_k(B(Q)) \leq \mathcal{H}^d_\infty(Q) \leq 2\ell(Q)^d \) assuming \( C_0 \) large enough depending on \( c_1, n, d \).

**Lemma A.10.** The doubling condition (A.6) holds for \( \mu_k \).

**Proof.** We want to show that for any \( Q \in D_k \) and \( P \in \text{Nbd}(Q) \) we have \( \mu_k(B(Q)) \leq C_0 \mu_k(B(P)) \). First, suppose that \( Q \notin \text{Rich}_k \cup \text{Poor}_k \), so that \( \mu_k(B(Q)) = \eta_k(B(Q)) \). Let \( P \in \text{Nbd}(Q) \). We have \( P \notin \text{Poor}(Q) \), and so

\[ \mu_k(B(Q)) = \eta_k(B(Q)) \leq \frac{C_0}{4} \eta_k(B(P)) \leq C_0 \mu_k(B(P)). \]

Assume now that \( Q \in \text{Rich}_k \), and let \( P \in \text{Nbd}(Q) \). There are two cases: either \( P \in \text{Poor}(Q) \), or \( P \notin \text{Poor}(Q) \). In the first case we have \( Q \in \text{Rich}(P) \), and so

\[ \mu_k(B(Q)) \leq \eta_k(B(Q)) \leq C_0 \left( \eta_k(B(P)) + C_0^{-1} \sum_{Q' \in \text{Rich}(P)} \eta_k(B(Q')) \right) = C_0 \mu_k(B(P)). \]

If \( P \notin \text{Poor}(Q) \) then by the definition of \( \text{Poor}(Q) \)

\[ \mu_k(B(Q)) \leq \eta_k(B(Q)) \leq \frac{C_0}{4} \eta_k(B(P)) \leq C_0 \mu_k(B(P)). \]

This establishes (A.6) for \( Q \in \text{Rich}_k \).

Finally, suppose that \( Q \in \text{Poor}_k \) and \( P \in \text{Nbd}(Q) \). Since for any \( R \in \text{Nbd}(Q) \) we have \( R^1 \in \text{Nbd}(P^1) \), we get

\[ \mu_k(B(Q)) = \eta_k(B(Q)) + C_0^{-1} \sum_{R \in \text{Rich}(Q)} \eta_k(B(R)) \]

\[ \leq \eta_k(B(Q)) + C_0^{-1} \sum_{R \in \text{Nbd}(Q)} \eta_k(B(R)) \leq \eta_k(B(Q)) + C_0^{-1} \sum_{R' \in \text{Nbd}(P^1)} \sum_{R \in \text{Ch}(R')} \eta_k(B(R)) \]

\[ \leq C_0 \eta_k(B(Q)) + C_0^{-1} \sum_{R' \in \text{Nbd}(P^1)} \mu_{k-1}(B(R')). \]

Concerning the first term on the right hand side, note that since \( Q \in \text{Poor}_k \), by Lemma A.6 we have \( Q \notin \text{Rich}_k \) and consequently \( P \notin \text{Poor}(Q) \), so that \( \eta_k(B(Q)) \leq \frac{C_0}{4} \eta_k(B(P)) \). To deal with the second term we use the doubling property (A.6) for \( \mu_{k-1} \).
and the fact that \( \#\text{Nbd}(P^1) \leq c_n \):

\[
\mu_k(B(Q)) \leq \eta_k(B(Q)) + C_0^{-1} \sum_{R' \in \text{Nbd}(P^1)} \mu_{k-1}(B(R'))
\]

\[
\leq C_0 \frac{4}{\eta_k(B(P))} + \#\text{Nbd}(P^1) \mu_{k-1}(B(P^1)) \leq C_0 \frac{4}{\eta_k(B(P))} + c_n \mu_{k-1}(B(P^1)) \tag{A.12}
\]

\[
\leq \frac{C_0}{4} \eta_k(B(P)) + \frac{c_n}{C_0} \eta_k(B(P)) \leq \frac{C_0}{2} \eta_k(B(P)) \leq C_0 \mu_k(B(P)), \tag{A.13}
\]

assuming \( C_0 \) large enough. This gives the desired inequality (A.6) for \( Q \in \text{Poor}_k \).

□

We have checked that \( \mu_k \) satisfies properties (A.3)–(A.7), and so the proof of Proposition A.2 is complete.

We prove two more properties of \( \mu_k \) that will be useful later on. Recall that the sequence of measures \( \nu_k \) constructed in the classical Frostman lemma has the following pleasant property: if \( Q \in D_k \) and \( j \geq k \), then \( \nu_j(Q) = \nu_k(Q) \) – the mass never escapes \( Q \) after step \( k \) of the construction. Our modified sequence of measures \( \mu_k \) doesn’t satisfy this property, but it’s not too far off.

Recall that \( B(Q) \cap E \subset Q \subset B_Q = B(x_Q, \ell(Q)) \).

Lemma A.11. If \( j \geq k \geq 0 \) and \( Q \in D_k \), then

\[
\mu_j(2B_Q) \geq \mu_k(B(Q)) \tag{A.17}
\]

and

\[
\mu_j(B_Q) \leq \mu_k(10B_Q). \tag{A.18}
\]

Proof. In the definition of \( \mu_{k+1} \) we transfer the mass of \( \eta_{k+1} \) only between neighbors. It follows easily that for any family \( \mathcal{F} \subset D_k \)

\[
\sum_{Q \in \mathcal{F}} \mu_k(B(Q)) = \sum_{Q \in \mathcal{F}} \sum_{P \in \text{Ch}(Q)} \eta_{k+1}(B(P)) \leq \sum_{R \in \mathcal{A}(\mathcal{F})} \mu_{k+1}(B(R)), \tag{A.19}
\]

where

\[
\mathcal{A}(\mathcal{F}) := \bigcup_{P \in \mathcal{F}} \bigcup_{R \in \text{Ch}(P)} \text{Nbd}(R) \subset D_{k+1}.
\]

Let \( Q \in D_k \). Set \( \mathcal{A}_0(Q) := \{ Q \} \), and then inductively

\[
\mathcal{A}_i(Q) := \mathcal{A}(\mathcal{A}_{i-1}(Q)) = \bigcup_{P \in \mathcal{A}_{i-1}(Q)} \bigcup_{R \in \text{Ch}(P)} \text{Nbd}(R) \subset D_{k+i}.
\]

Applying (A.19) \( i \)-times yields

\[
\mu_k(B(Q)) \leq \sum_{P \in \mathcal{A}_i(Q)} \mu_{k+i}(B(P)). \tag{A.20}
\]

Observe that if \( R \in D \), then \( \bigcup_{R' \in \text{Nbd}(R)} 2B_{R'} \subset 5B_R \). It follows that for all \( i \geq 0 \)

\[
\bigcup_{P \in \mathcal{A}_i(Q)} 2B_P \subset 2B_Q.
\]
Indeed, this is clear for \( i = 0 \), and then by induction

\[
\bigcup_{P \in A_i(Q)} 2B_P = \bigcup_{P \in A_{i-1}(Q)} \bigcup_{R \in \text{Ch}(P)} 2B_R \subset \bigcup_{P \in A_{i-1}(Q)} \bigcup_{R \in \text{Ch}(P)} 5B_R \subset \bigcup_{P \in A_{i-1}(Q)} 2B_P \subset 2B_Q.
\]

Together with (A.20) this gives (A.17).

Similarly, if for \( \mathcal{F} \subset \mathcal{D}_i \) we define

\[
B(\mathcal{F}) := \bigcup_{P \in \mathcal{F}} \text{Nbd}(P^1) \subset \mathcal{D}_{i-1},
\]

then by the definition of \( \mu_i \)

\[
\sum_{Q \in \mathcal{F}} \mu_j(B(Q)) \leq \sum_{P \in \bigcup_{Q \in \mathcal{F}} \text{Nbd}(Q)} \eta_j(B(P)) \leq \sum_{R \in B(\mathcal{F})} \sum_{P \in \text{Ch}(R)} \eta_j(B(P)) \underset{(A.10)}{=} \sum_{R \in B(\mathcal{F})} \mu_{j-1}(B(R)). \quad (A.21)
\]

Let \( Q \in \mathcal{D}_k \) and fix \( j \geq k \). Set \( \mathcal{B}_0(Q) = \{ P \in \mathcal{D}_j : B(P) \cap B_Q \neq \emptyset \} \), and for \( 0 < i \leq j - k \) we define inductively

\[
\mathcal{B}_i(Q) := B(\mathcal{B}_{i-1}(Q)) = \bigcup_{P \in \mathcal{B}_{i-1}(Q)} \text{Nbd}(P^1) \subset \mathcal{D}_{j-i}.
\]

By (A.21)

\[
\mu_j(B_Q) \leq \sum_{P \in \mathcal{B}_i(Q)} \mu_j(B(P)) \leq \sum_{P \in \mathcal{B}_{j-k}(Q)} \mu_k(B(P)). \quad (A.22)
\]

Now, we claim that

\[
\bigcup_{P \in \mathcal{B}_{j-k}(Q)} B(P) \subset 10B_Q.
\]

To see this, note that each \( P \in \mathcal{B}_i(Q) \) has a neighbor \( P' \) such that \( \text{Ch}(P') \cap \mathcal{B}_{i-1}(Q) \neq \emptyset \).

Hence,

\[
\text{dist}(P, \bigcup_{R \in \mathcal{B}_{i-1}(Q)} R) \leq \text{dist}(P, P') + \text{diam}(P') \leq 3\ell(P) = 15\rho^{j-i}.
\]

Fix \( P \in \mathcal{B}_{j-k}(Q) \), and let \( P = P_{j-k}, P_{j-k-1}, \ldots, P_0 \) be such that for each \( i \) we have \( P_i \in \mathcal{B}_i(Q) \) and \( \text{dist}(P_i, \bigcup_{R \in \mathcal{B}_{i-1}(Q)} R) = \text{dist}(P_i, P_{i-1}) \). Then,

\[
\text{dist}(P, \bigcup_{R \in \mathcal{B}_i(Q)} R) \leq \sum_{i=j-k}^{1} \text{dist}(P_i, \bigcup_{R \in \mathcal{B}_{i-1}(Q)} R) + \ell(P_{i-1}) \leq \sum_{i=j-k}^{1} 4\ell(P_i) = \sum_{i=j-k}^{1} 20\rho^{j-i} \leq 25\rho^k = 5\ell(Q).
\]

It is easy to see that \( \bigcup_{R \in \mathcal{B}_0(Q)} R \subset 3B_Q \), and so it follows that

\[
B(P) \subset 10B_Q.
\]

Together with (A.22), this gives (A.18). \( \square \)
Another nice property of \( \mu_k \) we are going to need is related to the approximate monotonicity of densities.

**Lemma A.12.** Assume that \( Q \in \mathcal{D}_k \). Then, there exists \( R \in \text{Nbd}(Q^1) \subset \mathcal{D}_{k-1} \) with

\[
\frac{\mu_k(B(Q))}{\mathcal{H}_\infty^d(Q)} \leq \frac{\mu_{k-1}(B(R))}{\mathcal{H}_\infty^d(R)}.
\]

**Proof.** There are two cases to consider. First, if \( \mu_k(B(Q)) \leq \eta_k(B(Q)) \), then by the definition

\[
\mu_k(B(Q)) \leq \eta_k(B(Q)) = \frac{\mathcal{H}_\infty^d(Q)}{\sum_{P \in \text{Ch}(Q^1)} \mathcal{H}_\infty^d(P)} \mu_{k-1}(B(Q^1)) \leq \frac{\mathcal{H}_\infty^d(Q)}{\mathcal{H}_\infty^d(Q^1)} \mu_{k-1}(B(Q^1)).
\]

Hence, taking \( R = Q^1 \) gives \( (A.23) \).

Now suppose that \( \mu_k(B(Q)) > \eta_k(B(Q)) \), i.e. \( Q \in \text{Poor}_k \). Let \( P \in \text{Rich}(Q) \) be the cube maximizing \( \eta_k(B(P)) \). Then,

\[
\mu_k(B(Q)) = \eta_k(B(Q)) + C_0^{-1} \sum_{S \in \text{Rich}(Q)} \eta_k(B(S)) \leq 4C_0^{-1} \eta_k(B(P)) + \#\text{Rich}(Q)C_0^{-1} \eta_k(B(P)) \leq C_0^{-1}(4 + c_n) \frac{\mathcal{H}_\infty^d(P)}{\mathcal{H}_\infty^d(P^1)} \mu_{k-1}(B(P^1)).
\]

Using lower content regularity of \( E \) we have \( \mathcal{H}_\infty^d(Q) \geq \mathcal{H}_\infty^d(E \cap B(Q)) \geq c_1(c_0 \ell(Q))^d \), so that

\[
\frac{\mu_k(B(Q))}{\mathcal{H}_\infty^d(Q)} \leq C_0^{-1}(4 + c_n) \frac{\mathcal{H}_\infty^d(P)}{\mathcal{H}_\infty^d(Q)} \frac{\mu_{k-1}(B(P^1))}{\mathcal{H}_\infty^d(P^1)} \leq C_0^{-1}(4 + c_n) 2^d c_1^{-1} c_0^{-d} \mu_{k-1}(B(P^1)) \leq \mu_{k-1}(B(P^1)),
\]

assuming \( C_0 \) large enough depending on \( c_1, n, d \). Since \( P^1 \in \text{Nbd}(Q^1) \), choosing \( R = P^1 \) gives \( (A.23) \).

**A.2. The limit measure.** For each \( k \) we have \( \mu_k(\mathbb{R}^n) = \mu_k([0,1]^n) = \mathcal{H}_\infty^d(E) < \infty \), and so there exists a subsequence \( \mu_{k_j} \) converging in the weak sense to a Radon measure \( \mu \) which also satisfies \( \mu(\mathbb{R}^n) = \mathcal{H}_\infty^d(E) \). This shows property (2) from Theorem A.1. In the next few lemmas we prove that \( \mu \) satisfies all the other required properties.

The following two basic facts about weak limits of measures will often be used without explicit mention: if \( \nu_j \to \nu \) weakly, then for any open \( U \subset \mathbb{R}^n \) and any compact \( K \subset \mathbb{R}^n \)

\[
\nu(U) \leq \liminf_{j \to \infty} \nu_j(U) \quad \text{and} \quad \limsup_{j \to \infty} \nu_j(K) \leq \nu(K).
\]

See [Mat95, Theorem 1.24].

**Lemma A.13.** We have \( \text{supp} \mu = E \), so that property (1) in Theorem A.1 is satisfied.
Proof. Let $x \in \mathbb{R}^n \setminus E$. Since $E$ is compact, for sufficiently small $r > 0$ and sufficiently large $k \in \mathbb{N}$ we have $B(x, r) \cap \mathcal{N}_k(E) = \emptyset$, where $\mathcal{N}_k(E)$ is the open $5\rho_k$ neighborhood of $E$. Note that for any $Q \in \mathcal{D}_k$ the ball $B(Q)$ is compactly contained in $\mathcal{N}_k(E)$. It follows from (A.4) that $\mu_k(B(x, r)) = 0$ for sufficiently large $k$. Thus,

$$\mu(B(x, r)) \leq \liminf_{j \to \infty} \mu_{k_j}(B(x, r)) = 0.$$ 

Thus, $x \notin \text{supp} \mu$. This shows $\text{supp} \mu \subset E$.

Now, let $x \in E$ and $r > 0$. For $k \in \mathbb{N}$ large enough we have that if $Q \in \mathcal{D}_k$ contains $x$, then $2B_Q \subset B(x, r)$. Fix such $k$ and $Q \in \mathcal{D}_k$. By (A.17) we get that for all $j \geq k$

$$\mu_{j}(B(x, r)) \geq \mu_{j}(2B_Q) \geq \mu_{k}(B(Q)) > 0.$$

In consequence,

$$\mu(B(x, r)) \geq \limsup_{j \to \infty} \mu_{k_j}(B(x, r)) \geq \mu_{k}(B(Q)) > 0.$$ 

So $x \in \text{supp} \mu$. \hfill \qed

**Lemma A.14.** The measure $\mu$ satisfies $\mu(B(x, r)) \lesssim r^d$. In particular, property (3) from Theorem A.1 is satisfied.

Proof. Let $x \in E$ and $r > 0$. If $r > 5\rho^2$, then we just use the fact that $\mu(\mathbb{R}^d) = \mathcal{H}^d_{\infty}(E) \lesssim 1$. Suppose that $r \leq 5\rho^2$, so that there exists $k \in \mathbb{N}$ such that $5\rho^{k+2} \leq r \leq 5\rho^{k+1}$. Let $Q \in \mathcal{D}_k$ be such that $x \in Q$. It follows that $B(x, r) \subset B_Q$, and then

$$\mu(B(x, r)) \leq \left(\text{lim inf}_{j \to \infty} \mu_{k_j}(B(x, r)) \right) \leq \text{lim inf}_{j \to \infty} \mu_{k_j}(B_Q) \leq \mu_{k}(10B_Q) \leq \sum_{P \in \mathcal{D}_k} \mu_{k}(B(P)) \leq \sum_{P \in \mathcal{D}_k} 2\ell(P)^d \lesssim \ell(Q)^d \sim r^d.$$ 

\hfill \qed

**Lemma A.15.** The measure $\mu$ satisfies $\mu(B(x, 2r)) \lesssim_{C_0} \mu(B(x, r))$ for all $x \in E$ and $r > 0$. In particular, property (4) in Theorem A.1 is satisfied.

Proof. Let $x \in E$ and $r > 0$, and let $Q \in \bigcup_{k=0}^{\infty} \mathcal{D}_k$ be the largest cube with $2B_Q \subset B(x, r)$. Let $k \in \mathbb{N}$ be such that $Q \in \mathcal{D}_k$. Observe that

$$\mu(B(x, r)) \geq \mu(2B_Q) \geq \limsup_{j \to \infty} \mu_{k_j}(2B_Q) \geq \mu_k(B(Q)). \quad (A.24)$$

If $r \geq 1$, then $\ell(Q) \sim 1$ and $k \lesssim 1$, so that using (A.7) $k$-times yields

$$\mu(B(x, r)) \geq \mu_k(B(Q)) \gtrsim_{C_0} \mu_0(E) = \mathcal{H}^d_{\infty}(E) \geq \mu(B(x, 2r)).$$

Now suppose that $0 < r < 1$, so that $\ell(Q) \sim r$. Let

$$\mathcal{F} = \{P \in \mathcal{D}_k : B_P \cap B(x, 2r) \neq \emptyset\}.$$
Note that \( \# \mathcal{F} \lesssim 1 \). Using the fact that for each \( P \in \mathcal{F} \) we have \( 10B_P \subset B(x,12r) \), and that \( \{10B_P\}_{P \in \mathcal{F}} \) have bounded overlap, we estimate

\[
\mu(B(x,2r)) \leq \sum_{P \in \mathcal{F}} \mu(B_P) \leq \liminf_{j \to \infty} \sum_{P \in \mathcal{F}} \mu_{k_j}(B_P) \lesssim \sum_{P \in \mathcal{F}} \mu_k(10B_P) \lesssim \sum_{R \in \mathcal{G}} \mu_k(B(R)),
\]

where 
\[
\mathcal{G} = \{ R \in \mathcal{D}_k : B(R) \cap B(x,12r) \neq \emptyset \}.
\]

Note that for \( R \in \mathcal{G} \) we have \( \ell(R) = \ell(Q) \) and \( \text{dist}(Q,R) \lesssim 1 \). It follows that 
\( Q^1 \in \text{Nbd}(R^1) \), so that using (A.17) and (A.8) yields

\[
\mu_k(B(Q)) \gtrsim C_0^{-1}\mu_{k-1}(B(Q^1)) \gtrsim C_0^{-2}\mu_{k-1}(B(R^1)) \gtrsim C_0^{-2}\mu_k(B(R)).
\]

Together with (A.21), (A.25), and the fact that \( \# \mathcal{G} \lesssim 1 \), this gives

\[
\mu(B(x,r)) \gtrsim C_0 \sum_{R \in \mathcal{G}} \mu_k(B(R)) \gtrsim \mu(B(x,2r)).
\]

\[\square\]

**Lemma A.16.** If \( P,Q \in \mathcal{D} \) and \( P \subset Q \), then \( \theta_\mu(P) \lesssim_{c_1,c_{db}} \theta_\mu(Q) \). In particular, \( \mu \) satisfies property (5) from Theorem A.1.

**Proof.** Let \( P,Q \in \mathcal{D} \) with \( P \subset Q \). Then \( P \in \mathcal{D}_j \) and \( Q \in \mathcal{D}_k \) for some \( j \geq k \). Let

\[
\mathcal{F} = \{ R \in \mathcal{D}_j : B(R) \cap 10B_P \neq \emptyset \}.
\]

Note that \( \mathcal{F} \lesssim 1 \). For every \( R \in \mathcal{F} \) we apply Lemma A.12 \((j-k)\)-many times to obtain a sequence of cubes \( R = R_0, R_1, R_2, \ldots, R_{j-k} \) such that \( R_i \in \text{Nbd}(R_{i-1}^1) \), and

\[
\frac{\mu_j(B(R_i))}{\mathcal{H}^d_\infty(R_i)} \leq \frac{\mu_j-B(R_{i+1})}{\mathcal{H}^d_\infty(R_{i+1})}.
\]

Using this inequality \((j-k)\)-many times, together with lower content regularity of \( E \), and the fact that \( \ell(R) = \ell(P), \ell(R_{j-k}) = \ell(Q) \), yields

\[
\frac{\mu_j(B(R))}{\ell(P)^d} \lesssim \mu_j(B(R)) \mathcal{H}^d_\infty(R) \leq \frac{\mu_k(B(R_{j-k}))}{\ell(Q)^d} \lesssim_{c_1} \frac{\mu_k(B(R_{j-k}))}{\ell(Q)^d}.
\]

By Lemma A.11 for any \( i \geq j \geq k \) we get

\[
\frac{\mu_i(B_P)}{\ell(P)^d} \lesssim \frac{\mu_i(10B_P)}{\ell(P)^d} \lesssim \sum_{R \in \mathcal{F}} \mu_j(B(R)) \mathcal{H}^d_\infty(R) \lesssim_{c_1} \sum_{R \in \mathcal{F}} \frac{\mu_k(B(R_{j-k}))}{\ell(Q)^d} \lesssim \sum_{R \in \mathcal{F}} \frac{\mu_i(B_{R_{j-k}})}{\ell(Q)^d}.
\]

Note that \( \text{dist}(R_{j-k},Q) \lesssim \ell(Q) = \ell(R_{j-k}) \) for every \( R \in \mathcal{F} \), so that \( 2B_{R_{j-k}} \subset CB_Q \) for some absolute \( C \). Hence, for all \( i \geq j \)

\[
\frac{\mu_i(B_P)}{\ell(P)^d} \lesssim_{c_1} \frac{\mu_i(CB_Q)}{\ell(Q)^d}.
\]

Passing to the limit we get

\[
\frac{\mu(B_P)}{\ell(P)^d} \leq \liminf_{j \to \infty} \frac{\mu_{k_j}(B_P)}{\ell(P)^d} \lesssim_{c_1} \limsup_{j \to \infty} \frac{\mu_{k_j}(CB_Q)}{\ell(Q)^d} \lesssim \frac{\mu(CB_Q)}{\ell(Q)^d}.
\]

Using the doubling property of \( \mu \) finishes the proof. \[\square\]
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