Linear stability of blowup solution of incompressible Keller–Segel–Navier–Stokes system

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Abstract
In this paper, we consider the linear stability of blowup solution for incompressible Keller–Segel–Navier–Stokes system in whole space $\mathbb{R}^3$. More precisely, we show that, if the initial data of the three dimensional Keller–Segel–Navier–Stokes system is close to the smooth initial function $(0,0,u_s(t,x))^T$, then there exists a blowup solution of the three dimensional linear Keller–Segel–Navier–Stokes system satisfying the decomposition
\[
(n(t,x),c(t,x),u(t,x))^T = (0,0,u_s(t,x))^T + O(\varepsilon), \quad \forall (t,x) \in (0,T^*) \times \mathbb{R}^3,
\]
in Sobolev space $H^s(\mathbb{R}^3)$ with $s = \frac{3}{2} - 5a$ and constant $0 < a \ll 1$, where $T^*$ is the maximal existence time, and $u_s(t,x)$ given in (Yan 2018) is the explicit blowup solution admitted smooth initial data for three dimensional incompressible Navier–Stokes equations.

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Keywords: Keller–Segel–Navier–Stokes system; Blowup solution; Stability

1 Introduction and main results
The Keller–Segel system coupled to the incompressible Navier–Stokes equations arises from a biological process in which cells move towards a chemically more favorable environment [20]. In this paper, we consider the blowup analysis for the three dimensional Keller–Segel–Navier–Stokes system

\[
\begin{align*}
nt + u \cdot \nabla n &= \Delta n - \chi \nabla \cdot (n \nabla c) + \sigma n - \mu n^2, \\
c_t + u \cdot \nabla c &= \Delta c - c + n, \\
u_t + u \cdot \nabla u &= \Delta u - \nabla P + n\nabla \Phi, \\
\nabla \cdot u &= 0,
\end{align*}
\]

where $\mathbb{V}(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, $\sigma \in \mathbb{R}$ and $\mu > 0$ are given parameters, the constant $\chi$ is the chemotactic sensitivity. We assume $\chi > 0$ presuming that cells move toward increasing...
concentrations of the signal substance which is produced by themselves. The unknown scalar functions $n(t,x) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$ and $c(t,x) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$ denote the population density and the signal concentration, respectively, the vector function $u(t,x) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ denotes the 3D velocity field of the fluid, $P(t,x) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$ stands for the pressure in the fluid. Moreover, the pressure $P(t,x)$ is determined by the formula

$$P(t,x) = -\Delta^{-1} \left( \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} - \nabla \cdot (n \nabla \Phi) \right).$$

(1.2)

$\Phi(x)$ is a given potential function accounting for the effects of external forces such as gravity. The divergence free condition in the last equation of (1.1) guarantees the incompressibility of the fluid.

Chemotaxis, as a mechanism of the partially-oriented movement of cells in response to a chemical signal, plays an important role in various biological processes [3, 11]. Tuval et al. [26] observed large-scale convection patterns in a water drop sitting on a glass surface containing oxygen-sensitive bacteria, oxygen diffusing into the drop through the fluid–air interface. They established a mathematical model, the so-called chemotaxis-fluid system, to describe the dynamics of swimming bacteria, Bacillus subtilis. It has the form

\[
\begin{aligned}
    n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \chi(c) \nabla c), \\
    c_t + u \cdot \nabla c &= \Delta c - nk(c), \\
    u_t + \kappa(u \cdot \nabla u) &= \Delta u - \nabla P + n \nabla \Phi, \\
    \nabla \cdot u &= 0,
\end{aligned}
\]

where $\mathcal{D}$ is a smooth bounded domain or $\mathbb{R}^3$. Lorz [19] first showed the local existence of weak solutions for the two and three dimensional system (1.1) with non-flux and inhomogeneous Dirichlet boundary conditions, respectively. Chae-Kang-Lee [6] gave the local existence of the smooth solution in the cases of two and three dimensions and global existence of classical solution for the two dimensional case for system (1.1). Winkler [31] asserted that a solution of the two dimensional chemotaxis-Navier–Stokes system in a bounded convex domain $\Omega$ stabilizes to the spatially uniform equilibrium $([1,0,0]^T) \in L^\infty(\Omega)$. Zhang–Zheng [40] used a microlocal analysis to obtain the global existence and uniqueness of weak solutions for a two dimensional chemotaxis-Navier–Stokes system in $\mathbb{R}^3$ for a large class of initial data. One can also refer to [7, 18, 28, 29] for more results on this two dimensional model. For the spatially three dimensional case, Tao–Winkler [25] constructed locally bounded global solutions in a chemotaxis-Stokes system with nonlinear diffusion. Winkler [33] established the global weak existence theory to a chemotaxis-Navier–Stokes system in a smooth bounded convex domain under a homogeneous Neumann boundary condition. With some relaxation time, Winkler [34] has shown this model admitted eventual energy solutions, meanwhile, he got such eventual energy solutions $(n,c,u) \to (\bar{n}_0,0,0)$ uniformly in a smooth bounded convex domain after the waiting time. Due to spatial limitation, we do not list all of the interesting papers on this kind of models.

Since the Keller–Segel–Navier–Stokes system plays an important role in bioconvection processes, it has attracted great interest also at the level of mathematical theory [2, 3].
Wang [27] showed the global existence of weak solutions for a three dimensional Keller–Segel–Navier–Stokes system with subcritical sensitivity. Winkler [35] proved that the three dimensional model with logistic source admits global weak solutions and asymptotic stabilization in a smooth bounded convex domain with homogeneous Dirichlet boundary conditions. At present, there are extreme difficulties to study the properties of solutions for system (1.1). One of the reasons is the question of finite time singularity/global regularity for three dimensional incompressible Navier–Stokes equations, one of the most important open problems in mathematical fluid mechanics [9]. On the one hand, even if we get rid of the effect of a fluid \( u \equiv 0 \), the question whether blowup may occur in the three dimensional Keller–Segel system

\[
\begin{align*}
    n_t &= \Delta n - \chi \nabla \cdot (n \nabla c) + \sigma n - \mu n^2, \quad (t,x) \in \mathbb{R}^+ \times \Omega, \\
    c_t &= \Delta c - c + n, \quad (t,x) \in \mathbb{R}^+ \times \Omega,
\end{align*}
\]

(1.3)

with small positive constant \( \mu \) is still open [35]. Here \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^3 \). Lankeit [15] got the global existence of weak solutions for system (1.3). Winkler [30] proved that system (1.3) can lead to a finite time explosion even for the case \( \sigma = \mu = 0 \). One can refer to [12–14,32] for more results on this kind of models.

On the other hand, if we set \( n(t,x) = c(t,x) \equiv 0 \), then the Keller–Segel–Navier–Stokes system (1.1) is reduced to the 3D incompressible Navier–Stokes equations

\[
\begin{align*}
    u_t + u \cdot \nabla u &= \nu \Delta u - \nabla P, \\
    \nabla \cdot u &= 0.
\end{align*}
\]

(1.4)

For these equations, Leray [16] showed there is global-forward-in-time weak solution of the initial value problem. After that, non-existence of finite energy self-similar blowup solutions in \( L^3(\mathbb{R}^3) \) was obtained in [22]. As is well known, Eq. (1.4) admits a simplest non-trivial stationary solution with infinite energy, i.e. the Couette flow \((y,0,0)^T\). Bedrossian–Germain–Masmoudi [1] proved the nonlinear stability of this flow. One can refer to [4, 5, 8, 10, 17, 21, 24, 39] for more related results. Yan [36, 37] found that the three dimensional Navier–Stokes equation (1.4) admits a family of stable explicit blowup solutions with infinite energy,

\[
\begin{align*}
    u_s(t,x) &= \left( \frac{ax_1}{T^*-t} + kx_2(T^*-t)^2, \frac{ax_2}{T^*-t} - kx_1(T^*-t)^2, \frac{-2ax_3}{T^*-t} \right)^T, \\
    (t,x) &\in [0, T^*) \times \mathbb{R}^3,
\end{align*}
\]

(1.5)

with the smooth initial data

\[
\begin{align*}
    u_s(0,x) &= \left( \frac{ax_1}{T^*} + kx_2(T^*)^2, \frac{ax_2}{T^*} - kx_1(T^*)^2, \frac{-2ax_3}{T^*} \right)^T,
\end{align*}
\]

where we have the constants \( a,k \in \mathbb{R}/\{0\} \), and the positive constant \( T^* \) is the maximal existence time. Moreover, the pressure \( P \) is determined by the formula

\[
P(t,x) = -\Delta^{-1} \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}.
\]
We supplement the 3D incompressible Keller–Segel–Navier–Stokes equations (1.1) with the initial data

\[
\begin{aligned}
    n(0, x) &= n_0(x), \quad x \in \mathbb{R}^3, \\
    c(0, x) &= c_0(x), \quad x \in \mathbb{R}^3, \\
    u(0, x) &= u_0(x), \quad x \in \mathbb{R}^3.
\end{aligned}
\]  

(1.6)

We now state the main result.

**Theorem 1.1** Let \(0 < a < \frac{1}{8}\) and \(0 < s < \frac{3}{2} - 5a\) be constants. Assume that \(\|\Phi\|_{H^{s+3}(\mathbb{R}^3)} \lesssim R \ll 1\). The three dimensional Keller–Segel–Navier–Stokes system (1.1) admits a family of linearly stable blowup solutions

\[
(n(t, x), c(t, x), u(t, x))^T = (0, 0, u_*(t, x))^T.
\]  

(1.7)

Throughout this paper, we denote the usual norm of \(L^2(\mathbb{R}^3)\) and Sobolev space \(H^s(\mathbb{R}^3)\) by \(\|\cdot\|_{L^2}\) and \(\|\cdot\|_{H^s}\) for \(s \in \mathbb{R}^+\), respectively. In fact, the Sobolev space can be equivalent to be defined via Fourier transformation, thus the fractional case is contained. The norm of the Sobolev space \(H^s(\mathbb{R}^3) := (H^s(\mathbb{R}^3))^3\) is denoted by \(\|\cdot\|_{H^s}\). The space \(L^2((0, T^*); H^s(\mathbb{R}^3))\) is equipped with the norm

\[
\|u\|_{L^2((0, T^*); H^s(\mathbb{R}^3))}^2 := \int_0^{T^*} \|u(t, \cdot)\|_{H^s}^2 \, dt.
\]

We also introduce the function spaces

\[
C^0_\mathcal{C}_s := \bigcap_{i=0}^1 C^i((0, T^*); H^{s-i}(\mathbb{R}^3)),
\]

\[
C^0_\mathcal{C}_s := \bigcap_{i=0}^1 C^i((0, T^*); H^{s-i}(\mathbb{R}^3)),
\]

with the norm

\[
\|u\|_{C^1_\mathcal{C}_s}^2 := \sup_{t \in (0, T^*)} \sum_{i=0}^1 \|\partial_i^s u\|_{H^{s-i}}^2
\]

\[
\|u\|_{C^1_\mathcal{C}_s}^2 := \sup_{t \in (0, T^*)} \sum_{i=0}^1 \|\partial_i^s u\|_{H^{s-i}}^2
\]

respectively. The symbol \(a \lesssim b\) means that there exists a positive constant \(C\) such that \(a \leq Cb\). \((a, b, c)^T\) denotes the column vector in \(\mathbb{R}^3\). The letter \(C\) with subscripts to denote dependencies stands for a positive constant that might change its value at each occurrence.
2 Proof of Theorem 1.1

The linear stability of a blowup solution is equivalent to the well-posedness of linearized equations in $\mathbb{R}^3$. For any $(t,x) \in (0,T^*) \times \mathbb{R}^3$, we recall the perturbation equations

\[
\begin{aligned}
    n_t + (v + \bar{u}) \cdot \nabla n &= \Delta n - \chi \nabla \cdot (n \nabla c) + \sigma n - \mu n^2, \\
    c_t + (v + \bar{u}) \cdot \nabla c &= \Delta c - c + n, \\
    v_t + v \cdot \nabla \bar{u} + \bar{u} \cdot \nabla v + v \cdot \nabla v &= \nabla P + v \Delta v + n \nabla \Phi, \\
    \nabla \cdot v &= 0,
\end{aligned}
\]

(2.1)

with initial data

\[
\begin{aligned}
    n(0,x) &= n_0(x), \quad x \in \mathbb{R}^3, \\
    c(0,x) &= c_0(x), \quad x \in \mathbb{R}^3, \\
    v(0,x) &= v_0(x), \quad x \in \mathbb{R}^3,
\end{aligned}
\]

(2.2)

and the boundary condition

\[
\begin{aligned}
    \lim_{|x| \to +\infty} n(t,x) &= 0, \\
    \lim_{|x| \to +\infty} c(t,x) &= 0, \\
    \lim_{|x| \to +\infty} v(t,x) &= 0,
\end{aligned}
\]

(2.3)

where $(t,x) \in (0,T^*) \times \mathbb{R}^3$, the pressure $\bar{P}$ satisfies

\[
\bar{P}(t,x) = -\Delta^{-1} \left( \frac{2}{T^* - t} \sum_{k=1}^{3} \left( \frac{\partial v_k}{\partial x_k} \right)^2 + 2k(T^* - t)^2a \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) + 2 \frac{\partial v_1}{\partial x_2} \frac{\partial v_2}{\partial x_1} \right) + 2 \frac{\partial v_1}{\partial x_2} \frac{\partial v_3}{\partial x_3} + 2 \frac{\partial v_2}{\partial x_3} \frac{\partial v_3}{\partial x_2} - \nabla \cdot (n \nabla \Phi) \right),
\]

and we have

\[
\nabla \bar{u} = \begin{pmatrix}
\frac{a}{T^* - t} & -k(T^* - t)^{2a} & 0 \\
-k(T^* - t)^{2a} & \frac{a}{T^* - t} & 0 \\
0 & 0 & -\frac{2a}{T^* - t}
\end{pmatrix}.
\]

Let $R \in (0,1)$ be a fixed constant. We define

\[
B_R := \left\{ (n(t,x),c(t,x),v(t,x))^T : \|n\|_{C^s_1,\frac{1}{2}} + \|c\|_{C^s_1,\frac{1}{2}} + \|v\|_{C^s_1,\frac{1}{2}} \leq R < 1 \right\}
\]

(2.4)

with a constant $s > 0$.

Assume that fixed functions $(n(t,x),c(t,x),v(t,x))^T \in B_R$. We linearize the nonlinear equations (2.1) around fixed functions $(n(t,x),c(t,x),v(t,x))^T$ to get the linearized equations on the unknown variables $(\Gamma(t,x),\Lambda(t,x),h(t,x))^T$ with an external force
\((f_1(t,x), f_2(t,x), g(t,x))^T\) as follows:

\[
\begin{align*}
\Gamma_t - \nabla \Gamma + (2\mu n - \sigma) \Gamma + (v + \overline{u}) \cdot \nabla \Gamma + h \cdot \nabla n \\
\frac{\partial h_i}{\partial t} - \nabla h_i + \frac{a}{T^* - t} h_1 + k(T^* - t)^{2a} h_2 + \left( \frac{ax_3}{T^* - t} + kx_2(T^* - t)^{2a} \right) \partial_x h_1 \\
\frac{\partial h_2}{\partial t} - \nabla h_2 + \frac{a}{T^* - t} h_2 + \left( \frac{ax_3}{T^* - t} + kx_2(T^* - t)^{2a} \right) \partial_x h_2 \\
\frac{\partial h_3}{\partial t} - \nabla h_3 - \frac{a}{T^* - t} h_3 + \left( \frac{ax_3}{T^* - t} + kx_2(T^* - t)^{2a} \right) \partial_x h_3
\end{align*}
\]

(2.5)

where \(\mathcal{F}_v\) denotes the Fréchet derivative on \(v\) and \((h_1(t,x), h_2(t,x), h_3(t,x))^T\).

In order to get some suitable prior estimates, we rewrite the linearized equations (2.5) as a coupled system,

\[
\Gamma_t - \nabla \Gamma + (2\mu n - \sigma) \Gamma + (v + \overline{u}) \cdot \nabla \Gamma + h \cdot \nabla n + \nabla \cdot [\Gamma \nabla c + n \nabla \Lambda] = f_1(t,x),
\]

(2.6)

\[
\Lambda_t - \nabla \Lambda + \Lambda + \nabla \cdot [\nabla (\Gamma + h) \cdot \nabla c - \Gamma] = f_2(t,x),
\]

(2.7)

with the incompressibility condition

\[
\nabla \cdot h = 0,
\]

where

\[
f(t,x) = -2\Delta^{-1} \left[ \sum_{i=1}^{3} (\partial_{x_i} w_i \partial_{x_i} h_i - \partial_{x_i} (\Gamma \partial_{x_i} \Phi)) + k(T^* - t)^{2a} (\partial_x h_2 - \nabla \cdot h_1) + \partial_{x_j} h_1 \partial_{x_j} w_2 + \partial_{x_j} h_2 \partial_{x_j} w_1 + \partial_{x_j} w_1 \partial_{x_j} h_3 + \partial_{x_j} s_3 \partial_{x_j} w_3 + \partial_{x_j} w_2 \partial_{x_j} h_3 + \partial_{x_j} s_2 \partial_{x_j} w_3 \right].
\]

(2.11)
We introduce the similarity coordinates

\[
\tau = -\ln(T^* - t) + \ln T^*, \\
y = \frac{x}{\sqrt{T^* - t}},
\]  

(2.12)

where one can see the blowup time \(T^* > 0\) has been transformed into \(+\infty\) in the similarity coordinates (2.12). So the local existence of linearized coupled system (2.6)–(2.10) with the incompressibility condition in some Sobolev space is equivalent to the global existence of linearized coupled system (2.13)–(2.17) in a related Sobolev space. This means the key point is to get the decay in time of solutions for system (2.13)–(2.17).

The linearized coupled system (2.6)–(2.10) under these coordinates is transformed into

\[
\partial_t \Gamma - \Delta_x \Gamma - \frac{y}{2} \cdot \nabla_y \Gamma + T^* e^{-\tau} (2\mu n - \sigma) \Gamma + ay_1 \partial_{x_1} \Gamma + ay_2 \partial_{x_2} \Gamma - 2ay_3 \partial_{x_3} \Gamma \\
+ k(T^*)^{2a_1} e^{-(2a_1 + 1)\tau} (y_2 \partial_{x_2} \Gamma + y_1 \partial_{x_2} \Gamma) + (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} \nabla_x \cdot \nabla_y \nu + y \sum_{i=1}^{3} \partial_{y_i} (\Gamma \partial_{y_i} \nu)
\]

\[
+ \chi \nabla_y \cdot (n \nabla_y \Lambda) = T^* e^{-\tau} f_1(T^*(1 - e^{-\tau}), (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} y),
\]

\[
\partial_t \Lambda - \Delta_x \Lambda - \frac{y}{2} \cdot \nabla_y \Lambda + T^* e^{-\tau} \Lambda + ay_1 \partial_{x_1} \Lambda + ay_2 \partial_{x_2} \Lambda - 2ay_3 \partial_{x_3} \Lambda
\]

\[
+ k(T^*)^{2a_1} e^{-(2a_1 + 1)\tau} (y_2 \partial_{x_2} \Lambda + y_1 \partial_{x_2} \Lambda) + (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} \left( \sum_{i=1}^{3} h_i \partial_{y_i} c_i - \Gamma \right)
\]

\[
= T^* e^{-\tau} f_2(T^*(1 - e^{-\tau}), (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} y),
\]

\[
\partial_x h_1 - \nu \Delta_x h_1 - \frac{y}{2} \cdot \nabla_y h_1 + ah_1 + a \partial_{y_1} h_1 + ay_2 \partial_{y_2} h_1 - 2ay_3 \partial_{y_3} h_1
\]

\[
+ k(T^*)^{2a_1} e^{-(2a_1 + 1)\tau} (h_2 + y_2 \partial_{y_2} h_1 + y_1 \partial_{y_2} h_1) - (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} \partial_{y_{1}} \Phi
\]

\[
+ (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} \sum_{i=1}^{3} (h_i \partial_{y_i} w_i + w_i \partial_{y_i} h_1)
\]

\[
= (T^*)^{\frac{1}{2}} \partial_{y_1} f_3 + T^* e^{-\tau} g_1(T^*(1 - e^{-\tau}), (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} y),
\]

\[
\partial_x h_2 - \nu \Delta_x h_2 - \frac{y}{2} \cdot \nabla_y h_2 + ah_2 + ay_1 \partial_{y_1} h_2 + ay_2 \partial_{y_2} h_2 - 2ay_3 \partial_{y_3} h_2
\]

\[
+ k(T^*)^{2a_1} e^{-(2a_1 + 1)\tau} (\partial_{y_1} h_2 - y_2 \partial_{y_2} h_2 - y_1 \partial_{y_2} h_2) - (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} \partial_{y_2} \Phi
\]

\[
+ (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} \sum_{i=1}^{3} (h_i \partial_{y_i} w_i + w_i \partial_{y_i} h_2)
\]

\[
= (T^*)^{\frac{1}{2}} \partial_{y_2} f_3 + T^* e^{-\tau} g_2(T^*(1 - e^{-\tau}), (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} y),
\]

\[
\partial_x h_3 - \nu \Delta_x h_3 - \frac{y}{2} \cdot \nabla_y h_3 + h_3 + ay_1 \partial_{y_1} h_3 + ay_2 \partial_{y_2} h_3 - 2ay_3 \partial_{y_3} h_3
\]

\[
+ k(T^*)^{2a_1} e^{-(2a_1 + 1)\tau} (y_2 \partial_{y_2} h_3 - y_1 \partial_{y_2} h_3) - (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} \partial_{y_3} \Phi
\]
\[ + (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} \sum_{i=1}^{3} (h_i \partial_{y_i} w_3 + w_i \partial_{y_i} h_3) \]
\[ = (T^*)^{\frac{1}{2}} \partial_{y_3} \bar{y} + T^* e^{-\tau} g_3 (T^* (1 - e^{-\tau}), (T^*)^{\frac{1}{2}} e^{-\frac{1}{2} \tau} y), \tag{2.17} \]

with the incompressibility condition

\[ \nabla_y \cdot h = 0, \]

where

\[ \bar{f} = -2 \Delta_y^{-1} \left[ \sum_{i=1}^{3} (\partial_{y_i} w_i \partial_{y_i} h_1 - \partial_{y_i} (\Gamma \partial_{y_i} \Phi)) + k (T^*)^{2a} e^{-2a \tau} (\partial_{y_i} h_2 - \partial_{y_i} h_1) \right. \]
\[ \left. + \partial_{y_1} h_1 \partial_{y_2} w_2 + \partial_{y_2} h_1 \partial_{y_2} w_1 + \partial_{y_3} h_1 \partial_{y_3} w_3 + \partial_{y_2} h_2 \partial_{y_3} w_3 \right]. \tag{2.18} \]

We supplement the linearized system (2.13)–(2.17) with the initial data

\[
\begin{cases}
\Gamma(0,y) = \Gamma_0(y) \in H^s(\mathbb{R}^3), \\
\Lambda(0,y) = \Lambda_0(y) \in H^s(\mathbb{R}^3), \\
h(0,y) = h_0(y) \in H^s(\mathbb{R}^3),
\end{cases} \tag{2.19}
\]

and the boundary condition

\[
\begin{cases}
\lim_{|y| \to \infty} \Gamma(\tau,y) = 0, \\
\lim_{|y| \to \infty} \Lambda(\tau,y) = 0, \\
\lim_{|y| \to \infty} h(\tau,y) = 0.
\end{cases} \tag{2.20}
\]

We first derive prior estimates of the linearized coupled system (2.13)–(2.17) with the initial data (2.19) and condition (2.20).

**Lemma 2.1** Let \( s > 0, 0 < a \ll \frac{1}{8} \) and \( T^* \in (0,1) \) be constants. Assume that \( \| \Phi \|_{L^2_{\text{mix}}(\mathbb{R}^3)} \lesssim R \ll 1, f_i \in C^1((0,\infty), H^s(\mathbb{R}^3)) \) \((i = 1,2), g \in C^1((0,\infty), H^s(\mathbb{R}^3)) \) and \((u,c,v)^T \in B_R\). Then, for any \( \tau > 0 \), the solution \((\Gamma, \Lambda, h)^T\) of linearized coupled system (2.13)–(2.17) with the initial data (2.19) and condition (2.20) satisfies

\[
\int_{\mathbb{R}^3} (|\Gamma|^2 + |\Lambda|^2 + |h|^2) \, dy \lesssim e^{-C \tau} \left( \int_{\mathbb{R}^3} (|\Gamma_0|^2 + |\Lambda_0|^2 + |h_0|^2) \, dy \right.
\]
\[ + \int_0^\tau \int_{\mathbb{R}^3} (|f_1|^2 + |f_2|^2 + |g|^2) \, dy \, d\tau), \]

where \( C \) is a positive constant.
Proof. Multiplying both sides of (2.13)–(2.17) by $\Gamma$, $\Lambda$, $h_1$, $h_2$ and $h_3$, respectively, then integrating by parts (using the boundary condition (2.20)), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \|\Gamma\|^2_{L^2} + \|\nabla y \Gamma\|^2_{L^2} + \left( \frac{3}{4} - T^* e^{-\sigma} \right) \|\Gamma\|^2_{L^2} + 2\mu T^* e^{-\sigma} \int_{\mathbb{R}^3} n \Gamma^2 \, dy \right) \\
+ \frac{3}{4} \sum_{i=1}^3 \left( \|\partial_y \phi_i \|^2_{L^2} + n \partial_y^2 \Lambda \right) \, dy + (T^*)^\frac{1}{2} e^{-\frac{1}{2}t} \int_{\mathbb{R}^3} (\mathbf{h} \cdot \nabla_y u) \Gamma \, dy \\
= T^* e^{-\sigma} \int_{\mathbb{R}^3} \Gamma f_1 \, dy, 
\]

(2.21)

\[
\frac{1}{2} \frac{d}{dt} \left( \|\Lambda\|^2_{L^2} + \|\nabla_y \Lambda\|^2_{L^2} + \left( \frac{3}{4} + T^* e^{-\sigma} \right) \|\Lambda\|^2_{L^2} \right) \\
+ (T^*)^\frac{1}{2} e^{-\frac{1}{2}t} \int_{\mathbb{R}^3} \left( \sum_{i=1}^2 \|h_i \partial_y c_i - \Gamma\|^2 \right) \, dy = T^* e^{-\sigma} \int_{\mathbb{R}^3} \Lambda f_2 \, dy, 
\]

(2.22)

\[
\frac{1}{2} \frac{d}{dt} \|h_1\|^2_{L^2} + v \sum_{i,j=1}^3 \|\partial_y h_i\|^2_{L^2} + \left( a + \frac{3}{4} \right) \|h_1\|^2_{L^2} + k(\Gamma)^{2a+1} e^{-2(\alpha+1)\tau} \int_{\mathbb{R}^3} h_1 h_2 \, dy \\
+ (T^*)^\frac{1}{2} e^{-\frac{1}{2}t} \sum_{i=1}^3 \int_{\mathbb{R}^3} h_1 (h_i \partial_y w_1 + w_i \partial_y h_1) \, dy \\
- (T^*)^\frac{1}{2} e^{-\frac{1}{2}t} \int_{\mathbb{R}^3} \Gamma \partial_y \Phi h_1 \, dy = (T^*)^\frac{1}{2} \int_{\mathbb{R}^3} h_1 \partial_y \Phi \, dy + \int_{\mathbb{R}^3} h_1 g_1 \, dy, 
\]

(2.23)

\[
\frac{1}{2} \frac{d}{dt} \|h_2\|^2_{L^2} + v \sum_{i,j=1}^3 \|\partial_y h_i\|^2_{L^2} + \left( a + \frac{3}{4} \right) \|h_2\|^2_{L^2} - k(\Gamma)^{2a+1} e^{-2(\alpha+1)\tau} \int_{\mathbb{R}^3} h_1 h_2 \, dy \\
+ (T^*)^\frac{1}{2} e^{-\frac{1}{2}t} \sum_{i=1}^3 \int_{\mathbb{R}^3} h_2 (h_i \partial_y w_2 + w_i \partial_y h_2) \, dy \\
- (T^*)^\frac{1}{2} e^{-\frac{1}{2}t} \int_{\mathbb{R}^3} \Gamma \partial_y \Phi h_2 \, dy = (T^*)^\frac{1}{2} \int_{\mathbb{R}^3} h_2 \partial_y \Phi \, dy + \int_{\mathbb{R}^3} h_2 g_2 \, dy, 
\]

(2.24)

and

\[
\frac{1}{2} \frac{d}{dt} \|h_3\|^2_{L^2} + v \sum_{i,j=1}^3 \|\partial_y h_i\|^2_{L^2} + \left( \frac{3}{4} - 2a \right) \|h_3\|^2_{L^2} \\
+ (T^*)^\frac{1}{2} e^{-\frac{1}{2}t} \sum_{i=1}^3 \int_{\mathbb{R}^3} h_3 (h_i \partial_y w_3 + w_i \partial_y h_3) \, dy \\
- (T^*)^\frac{1}{2} e^{-\frac{1}{2}t} \int_{\mathbb{R}^3} \Gamma \partial_y \Phi h_3 \, dy = (T^*)^\frac{1}{2} \int_{\mathbb{R}^3} h_3 \partial_y \Phi \, dy + \int_{\mathbb{R}^3} h_3 g_3 \, dy. 
\]

(2.25)

Summing up (2.21)–(2.25), then

\[
\frac{1}{2} \sum_{i=1}^3 \frac{d}{dt} \left( \|h_i\|^2_{L^2} + \|\Gamma\|^2_{L^2} + \|\Lambda\|^2_{L^2} \right) + \|\nabla_y \Gamma\|^2_{L^2} + \|\nabla_y \Lambda\|^2_{L^2} + 3v \sum_{i,j=1}^3 \|\partial_y h_i\|^2_{L^2} \\
+ \left( \frac{3}{4} - T^* e^{-\sigma} \right) \|\Gamma\|^2_{L^2} + \left( \frac{3}{4} + T^* e^{-\sigma} \right) \|\Lambda\|^2_{L^2} + \left( a + \frac{3}{4} \right) \left( \|h_1\|^2_{L^2} + \|h_2\|^2_{L^2} + \|h_3\|^2_{L^2} \right) 
\]
+ \left( \frac{3}{4} - 2a \right) \| h_3 \|_{L^2}^2 + 2 \mu T^* e^{-\tau} \int_{\mathbb{R}^3} n \Gamma^2 \, dy - \chi \sum_{i=1}^{3} \int_{\mathbb{R}^3} \left( \Gamma \partial_{\gamma_j} c \partial_{\gamma_j} \Gamma + n \partial_{\gamma_j} \lambda \right) \, dy

+ \left( T^* \right)^{\frac{1}{2}} e^{\frac{3}{2} \tau} \int_{\mathbb{R}^3} (h \cdot \nabla \varphi) \, dy + \left( T^* \right)^{\frac{1}{2}} e^{\frac{3}{2} \tau} \int_{\mathbb{R}^3} \left( \sum_{i=1}^{3} h_i \partial_{\gamma_j} c - \Gamma \right) \, dy

- \left( T^* \right)^{\frac{1}{2}} e^{\frac{3}{2} \tau} \sum_{i=1}^{3} \int_{\mathbb{R}^3} \Gamma \partial_{\gamma_j} \Phi h_i \, dy + \left( T^* \right)^{\frac{1}{2}} e^{\frac{3}{2} \tau} \sum_{i=1}^{3} \int_{\mathbb{R}^3} h_1 (h_i \partial_{\gamma_j} w_1 + w_i \partial_{\gamma_j} h_1) \, dy

+ \left( T^* \right)^{\frac{1}{2}} e^{\frac{3}{2} \tau} \sum_{i=1}^{3} \int_{\mathbb{R}^3} h_2 (h_i \partial_{\gamma_j} w_2 + w_i \partial_{\gamma_j} h_2) \, dy

+ \left( T^* \right)^{\frac{1}{2}} e^{\frac{3}{2} \tau} \sum_{i=1}^{3} \int_{\mathbb{R}^3} h_3 (h_i \partial_{\gamma_j} w_3 + w_i \partial_{\gamma_j} h_3) \, dy

= \left( T^* \right)^{\frac{1}{2}} \sum_{i=1}^{3} \int_{\mathbb{R}^3} h_i \partial_{\gamma_j} \Phi \, dy + T^* e^{\frac{3}{2} \tau} \int_{\mathbb{R}^3} \left( \Gamma f_1 + \Lambda f_2 + \sum_{i=1}^{3} h_i g_i \right) \, dy, \quad \forall \tau > 0. \quad (2.26)

We now estimate each coupled nonlinear term in (2.26). Note that \((n, \varphi, \psi)^T \in B_\mathcal{R} \) and \( H^\frac{3}{2} (\mathbb{R}^3) \subset L^\infty (\mathbb{R}^3) \). We use Young’s inequality to derive

\[ 2 \mu T^* e^{\tau} \int_{\mathbb{R}^3} n \Gamma^2 \, dy \lesssim C_R \| \Gamma \|_{L^2}^2, \]

\[ \sum_{i=1}^{3} \int_{\mathbb{R}^3} \left( \Gamma \partial_{\gamma_j} c \partial_{\gamma_j} \Gamma + n \partial_{\gamma_j} \lambda \right) \, dy \lesssim C_R \left( \| \Gamma \|_{L^2}^2 + \sum_{i=1}^{3} \| \partial_{\gamma_j} \Gamma \|_{L^2}^2 + \| \partial_{\gamma_j} \lambda \|_{L^2}^2 \right), \]

\[ \left( T^* \right)^{\frac{1}{2}} e^{\frac{3}{2} \tau} \int_{\mathbb{R}^3} (h \cdot \nabla \varphi) \, dy \lesssim C_R \left( \| \Gamma \|_{L^2}^2 + \sum_{i=1}^{3} \| h_i \|_{L^2}^2 \right), \]

\[ \left( T^* \right)^{\frac{1}{2}} e^{\frac{3}{2} \tau} \int_{\mathbb{R}^3} \left( \sum_{i=1}^{3} h_i \partial_{\gamma_j} c \right) \, dy \lesssim \left( T^* \right)^{\frac{1}{2}} e^{\frac{3}{2} \tau} \left( 2 \| \lambda \|_{L^2}^2 + \| \Gamma \|_{L^2}^2 \right) + C_R \sum_{i=1}^{3} \| h_i \|_{L^2}^2, \]

\[ \left( T^* \right)^{\frac{1}{2}} e^{\frac{3}{2} \tau} \sum_{i=1}^{3} \int_{\mathbb{R}^3} \Gamma \partial_{\gamma_j} \Phi h_i \, dy \lesssim C_R \left( \| \Gamma \|_{L^2}^2 + \sum_{i=1}^{3} \| h_i \|_{L^2}^2 \right), \]

and

\[ \sum_{i=1}^{3} \int_{\mathbb{R}^3} h_1 (h_i \partial_{\gamma_j} w_1 + w_i \partial_{\gamma_j} h_1) \, dy \lesssim C_R \sum_{i=1}^{3} \left( \| h_i \|_{L^2}^2 + \| \partial_{\gamma_j} h_1 \|_{L^2} \right), \]

\[ \sum_{i=1}^{3} \int_{\mathbb{R}^3} h_2 (h_i \partial_{\gamma_j} w_2 + w_i \partial_{\gamma_j} h_2) \, dy \lesssim C_R \sum_{i=1}^{3} \left( \| h_i \|_{L^2}^2 + \| \partial_{\gamma_j} h_2 \|_{L^2} \right), \]

\[ \sum_{i=1}^{3} \int_{\mathbb{R}^3} h_3 (h_i \partial_{\gamma_j} w_3 + w_i \partial_{\gamma_j} h_3) \, dy \lesssim C_R \sum_{i=1}^{3} \left( \| h_i \|_{L^2}^2 + \| \partial_{\gamma_j} h_3 \|_{L^2} \right), \]

\[ \sum_{i=1}^{3} \int_{\mathbb{R}^3} \int_{B_\mathcal{R}} h_i (h_i \partial_{\gamma_j} w_1 + w_i \partial_{\gamma_j} h_1) \, dy \lesssim C_R \sum_{i=1}^{3} \left( \| h_i \|_{L^2}^2 + \| \partial_{\gamma_j} h_1 \|_{L^2} \right), \]
where $C_R$, $C_{a,R}$, $C_{\kappa,R}$ are three positive constants depending on $R$, $a_R$, $R$ and $\kappa$, $R$, respectively.

On the other hand, by (2.18) and the standard Calderon–Zygmund theory, i.e. for the Riesz operator $R$, we have $\|Rw\|_{L^p} \leq \|w\|_{L^p}$ with $1 < p < \infty$, we also use Young’s inequality to get

$$\left| \sum_{i=1}^{3} \int_{\Omega} h_i \partial_y f \, dy \right| \lesssim C_R \sum_{i,j=1}^{3} (\|\Gamma_i\|_{L^2}^2 + \|\partial_\gamma h_i\|_{L^2}^2)$$

(2.29)

and

$$\left| T^* e^{-\tau} \left( \Gamma f_1 + \Lambda f_2 + \sum_{i=1}^{3} h_i g_i \right) \right| \leq T^* e^{-\tau} \left[ b (\|\Gamma\|_{L^2}^2 + \|\Lambda\|_{L^2}^2) + b^{-1} (\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2) \right.

+ \sum_{i=1}^{3} \left( b \|h_i\|_{L^2}^2 + b^{-1} \|g_i\|_{L^2}^2 \right) \right],$$

(2.30)

where the positive constant $b < 1$.

Thus by (2.27)–(2.30), it follows from (2.26) that

$$\frac{1}{2} \sum_{i=1}^{3} \frac{d}{d\tau} \left( \|h_i\|_{L^2}^2 + (1 - C_R) \|\nabla_\gamma \Gamma_i\|_{L^2}^2 + (1 - C_R) \|\Lambda_i\|_{L^2}^2 + (3\nu - C_R) \sum_{i,j=1}^{3} \|\partial_\gamma h_i\|_{L^2}^2 \right.

+ \left( \frac{3}{4} - T^* e^{-\tau} (b + \alpha) - \frac{(T^*)^2}{2} e^{\frac{1}{2} \tau} - C_R \right) \|\Gamma_i\|_{L^2}^2 

+ \left( \frac{3}{4} + T^* e^{-\tau} (1 - b) - \frac{(T^*)^2}{2} e^{\frac{1}{2} \tau} - C_R \right) \|\Lambda_i\|_{L^2}^2 

+ \left( a + \frac{3}{4} - T^* e^{-\tau} b - C_R \right) (\|h_1\|_{L^2}^2 + \|h_2\|_{L^2}^2) + \left( \frac{3}{4} - 2a - T^* e^{-\tau} b - C_R \right) \|h_3\|_{L^2}^2 

\lesssim b^{-1} \left( \|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2 + \sum_{i=1}^{3} \|g_i\|_{L^2}^2 \right),$$

(2.31)

where $C_R$ is a positive constant depending on $R$, which can be very small if constants $R$ is small.

There exists a sufficiently small positive constant $b \in (0, 1)$ such that

$$1 - C_R > 0, \quad 1 - C_R > 0, \quad 3\nu - C_R > 0,$$

$$\frac{3}{4} - T^* e^{-\tau} (b + \alpha) = \frac{(T^*)^2}{2} e^{\frac{1}{2} \tau} - C_R > 0,$$

$$\frac{3}{4} + T^* e^{-\tau} (1 - b) = \frac{(T^*)^2}{2} e^{\frac{1}{2} \tau} - C_R > 0,$$

$$a + \frac{3}{4} - T^* e^{-\tau} b - C_R > 0, \quad \frac{3}{4} - 2a - T^* e^{-\tau} b - C_R > 0.$$
Hence, applying Gronwall’s inequality to (2.31), there exists a positive constant C such that

$$\parallel \Gamma \parallel _{L^2}^2 + \parallel \Gamma \parallel _{L^2}^2 + \sum _{i=1}^3 \parallel h_i \parallel _{L^2}^2 \lesssim e^{-Ct} \int _{\mathbb{R}^3} \left( \Gamma _0^2 + A_0^2 + \sum _{i=1}^3 h_{0i}^2 \right) dy$$

$$+ e^{-Ct} \int _0^\infty \int _{\mathbb{R}^3} \left( f_1^2 + f_2^2 + \sum _{i=1}^3 g_i^2 d\tau \right) dy, \quad \forall \tau > 0. \quad \Box$$

In what follows, we plan to carry out higher order derivative estimates to the solutions of linearized system (2.8)–(2.10). For a fixed constant \( s > 0 \), applying \( \nabla s = \partial _y^s \) to both sides of (2.22)–(2.10), we obtain

$$\partial _t \nabla _y^s \Gamma - \nabla _y^s \partial _t \Gamma = - \frac{y}{2} \cdot \nabla _y ^{s+1} \Gamma + \left( T^s e^{-t} (2\mu n - \sigma) - \frac{s}{2} \right) \nabla _y^s \Gamma + \frac{\partial _t \nabla _y^s \Gamma}{\partial _t \nabla _y^s \Gamma} + \frac{\partial _t \nabla _y^s \Gamma}{\partial _t \nabla _y^s \Gamma}$$

$$- 2ay_3 \partial _y \nabla _y^s \Gamma + k (T^s)^{2a} e^{-2(2a+1)t} \left( 2s \nabla _y^s \Gamma + y_2 \partial _y \nabla _y^s \Gamma + y_1 \partial _y \nabla _y^s \Gamma \right)$$

$$+ (T^s)^{1/2} e^{-\frac{t}{2}} \nabla _y^s (h \cdot \nabla _y n) + \chi \sum _{i=1}^3 \partial _y \nabla _y^s (\Gamma \partial _y \psi) + \nabla _y^s \left( \nabla _y (n \nabla _y \Lambda) \right)$$

$$= T^s e^{-t} \nabla _y^s f_1, \quad (2.32)$$

$$\partial _t \nabla _y^s \Lambda - \nabla _y^s \partial _t \Lambda = - \frac{y}{2} \cdot \nabla _y ^{s+1} \Lambda + \left( T^s e^{-t} - \frac{s}{2} \right) \nabla _y^s \Lambda + \frac{\partial _t \nabla _y^s \Lambda}{\partial _t \nabla _y^s \Lambda} + \frac{\partial _t \nabla _y^s \Lambda}{\partial _t \nabla _y^s \Lambda}$$

$$- 2ay_3 \partial _y \nabla _y^s \Lambda + k (T^s)^{2a} e^{-2(2a+1)t} \left( 2s \nabla _y^s \Lambda + y_2 \partial _y \nabla _y^s \Lambda + y_1 \partial _y \nabla _y^s \Lambda \right)$$

$$+ (T^s)^{1/2} e^{-\frac{t}{2}} \nabla _y^s (h \cdot \nabla _y n) + \chi \sum _{i=1}^3 \partial _y \nabla _y^s (\Gamma \partial _y \psi) + \nabla _y^s \left( \nabla _y (n \nabla _y \Lambda) \right)$$

$$= T^s e^{-t} \nabla _y^s f_2, \quad (2.33)$$

$$\partial _t \nabla _y^s h_1 - v \nabla _y \nabla _y^s h_1 = - \frac{y}{2} \partial _y \nabla _y^s h_1 + \left( a - \frac{s}{2} \right) \nabla _y h_1 + \partial _y \nabla _y^s h_1 + \partial _y \nabla _y^s h_1$$

$$- 2ay_3 \partial _y \nabla _y^s h_1 + k (T^s)^{2a} e^{-2(2a+1)t} \left( y_2 \nabla _y^s h_1 + y_1 \partial _y \nabla _y^s h_1 \right)$$

$$- (T^s)^{1/2} e^{-\frac{t}{2}} \nabla _y^s (\Gamma \partial _y \Phi) + (T^s)^{1/2} e^{-\frac{t}{2}} \sum _{i=1}^3 \left( \nabla _y^s h_i \partial _y \psi \right) + w_i \partial _y \nabla _y^s h_1 \right) = \tilde{g}_1, \quad (2.34)$$

$$\partial _t \nabla _y^s h_2 - v \nabla _y \nabla _y^s h_2 = - \frac{y}{2} \partial _y \nabla _y^s h_2 + \left( a - \frac{s}{2} \right) \nabla _y h_2 + \partial _y \nabla _y^s h_2 + \partial _y \nabla _y^s h_2$$

$$- 2ay_3 \partial _y \nabla _y^s h_2 + k (T^s)^{2a} e^{-2(2a+1)t} \left( - y_1 \nabla _y^s h_2 - y_2 \partial _y \nabla _y^s h_2 \right)$$

$$- (T^s)^{1/2} e^{-\frac{t}{2}} \nabla _y^s (\Gamma \partial _y \Phi) + (T^s)^{1/2} e^{-\frac{t}{2}} \sum _{i=1}^3 \left( \nabla _y^s h_i \partial _y \psi \right) + w_i \partial _y \nabla _y^s h_2 \right) = \tilde{g}_2, \quad (2.35)$$

$$\partial _t \nabla _y^s h_3 - v \nabla _y \nabla _y^s h_3 = - \frac{y}{2} \partial _y \nabla _y^s h_3 - \left( 2a + \frac{s}{2} \right) \nabla _y h_3 + \partial _y \nabla _y^s h_3 + \partial _y \nabla _y^s h_3$$

$$- 2ay_3 \partial _y \nabla _y^s h_3 + k (T^s)^{2a} e^{-2(2a+1)t} \left( y_2 \partial _y \nabla _y^s h_3 - y_1 \partial _y \nabla _y^s h_3 \right)$$

$$+ (T^s)^{1/2} e^{-\frac{t}{2}} \sum _{i=1}^3 \left( \nabla _y^s h_i \partial _y \psi \right) + w_i \partial _y \nabla _y^s h_3 \right) - (T^s)^{1/2} e^{-\frac{t}{2}} \nabla _y^s (\Gamma \partial _3 \Phi) = \tilde{g}_3, \quad (2.36)$$
where

\[
\begin{align*}
\tilde{g}_1 &= (T^*)^{1 \over 2} \partial_y \nabla_y^T f + T^* e^{-T} \nabla_y^T g_1 \\
&\quad - (T^*)^{1 \over 2} e^{-1 \over 2T} \sum_{j=1}^3 \sum_{l_1+l_2 \leq 1} \left( \nabla_y^2 h_j \partial_{y_j} \nabla_y^l w_1 + \nabla_y^l w_j \partial_{y_j} \nabla_y^2 h_1 \right),
\end{align*}
\]

(2.37)

\[
\begin{align*}
\tilde{g}_2 &= (T^*)^{1 \over 2} \partial_y \nabla_y^T f + T^* e^{-T} \nabla_y^T g_2 \\
&\quad - (T^*)^{1 \over 2} e^{-1 \over 2T} \sum_{j=1}^3 \sum_{l_1+l_2 \leq 1} \left( \nabla_y^2 h_j \partial_{y_j} \nabla_y^l w_2 + \nabla_y^l w_j \partial_{y_j} \nabla_y^2 h_2 \right),
\end{align*}
\]

(2.38)

\[
\begin{align*}
\tilde{g}_3 &= (T^*)^{1 \over 2} \partial_y \nabla_y^T f + T^* e^{-T} \nabla_y^T g_3 \\
&\quad - (T^*)^{1 \over 2} e^{-1 \over 2T} \sum_{j=1}^3 \sum_{l_1+l_2 \leq 1} \left( \nabla_y^2 h_j \partial_{y_j} \nabla_y^l w_3 + \nabla_y^l w_j \partial_{y_j} \nabla_y^2 h_3 \right).
\end{align*}
\]

(2.39)

We now have the following higher order derivatives estimates.

**Lemma 2.2** Let \(0 < a \ll {1 \over 2}\) and \(0 < s < {1 \over 2} - 5a\) be constants. Assume that \(||\Phi||_{L^2(\mathbb{R}^3)} \lesssim R \ll 1, f_i \in C^1((0, +\infty), H^1(\mathbb{R}^3)) (i = 1, 2), g \in C^1((0, +\infty), H^1(\mathbb{R}^3))\) and \((n, c, v)^T \in B_R\). Then, for any \(\tau > 0\), the solution \((\Gamma, \Lambda, h)^T\) of the linearized coupled system (2.13)–(2.17) with the initial data (2.19) and condition (2.20) satisfies

\[
\int_{\mathbb{R}^3} \left( |\nabla_y^2 \Gamma|^2 + |\nabla_y^2 \Lambda|^2 + \sum_{i=1}^3 |\nabla_y^2 h_i|^2 \right) dy \lesssim e^{-C_R \tau^\epsilon T} \int_{\mathbb{R}^3} \left( |\nabla_y^2 \Gamma_0|^2 + |\nabla_y^2 \Lambda_0|^2 + \sum_{i=1}^3 |\nabla_y^2 h_0|^2 \right) dy
\]

\[
+ e^{-C_R \tau^\epsilon T} \int_0^{\infty} \left( \|\nabla_y f_1\|_{L^2} + \|\nabla_y f_2\|_{L^2} + \sum_{i=1}^3 \|\nabla_y g_i\|_{L^2} \right) d\tau, \quad \forall \tau > 0,
\]

where \(C_R T^*\) is a positive constant depending on constants \(R, T^*\).

**Proof** Taking the inner product of both sides of (2.32)–(2.36) by \(\nabla_y^2 \Gamma, \nabla_y^2 \Lambda, \nabla_y^2 h_1, \nabla_y^2 h_2\) and \(\nabla_y^2 h_3\), respectively, then integrating by parts, we have

\[
{1 \over 2} {d \over d\tau} \left( \|\nabla_y^2 \Gamma\|_{L^2}^2 + \|\nabla_y^{2+1} \Gamma\|_{L^2}^2 + \left( {3 \over 4} - \frac{s}{2} - T^* e^{-T} \sigma + + 2\kappa (T^*)^{2a+1} e^{-(2a+1)T} \right) \|\nabla_y^2 \Gamma\|_{L^2}^2 \right)
\]

\[
+ 2\mu T^* e^{-T} \int_{\mathbb{R}^3} n |\nabla_y \Gamma|^2 dy + (T^*)^{1 \over 2} e^{-1 \over 2T} \int_{\mathbb{R}^3} \nabla_y^2 (h \cdot \nabla_y n) \cdot \nabla_y^2 \Gamma dy
\]

\[
+ \chi \sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_{y_i} \nabla_y^2 (\Gamma \partial_{y_i} \psi) \cdot \nabla_y^2 \Gamma dy + \chi \int_{\mathbb{R}^3} \nabla_y^2 (\nabla_y (n \nabla_y \Lambda)) \cdot \nabla_y^2 \Gamma dy
\]

\[
= T^* e^{-T} \int_{\mathbb{R}^3} \nabla_y^2 f_1 \cdot \nabla_y^2 \Gamma dy,
\]

(2.40)
Summing up (2.42)–(2.44), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla_y \Lambda \|_{L^2}^2 + \| \nabla_y^{s+1} \Lambda \|_{L^2}^2 + \left( \frac{3}{4} - \frac{s}{2} + T^s e^{-\frac{t}{2}} + 2\kappa \left( T^s \right)^{2s+1} e^{-\frac{(2s+1)\tau}{2}} \right) \| \nabla_y \Gamma \|_{L^2}^2 \right)
\]

\[
+ (T^s)^{\frac{1}{2}} e^{-\frac{t}{2}} \int_{\mathbb{R}^3} \left( \sum_{i=1}^{3} \nabla_y (h_i \partial_y d_i) - \nabla_y \Gamma \right) \cdot \nabla_y \lambda dy
\]

\[
= T^s e^{-\frac{t}{2}} \int_{\mathbb{R}^3} \nabla_y f_2 \cdot \nabla_y \lambda dy,
\]

(2.41)

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla_y h_1 \|_{L^2}^2 + \left( a + \frac{3}{4} - \frac{s}{2} \right) \| \nabla_y h_1 \|_{L^2}^2 \right)
\]

\[
+ k (T^s)^{2s+1} e^{-\frac{(2s+1)\tau}{2}} \int_{\mathbb{R}^3} \nabla_y h_1 \cdot \nabla_y h_2 dy
\]

\[
+ (T^s)^{\frac{1}{2}} e^{-\frac{t}{2}} \sum_{i=1}^{3} \int_{\mathbb{R}^3} \nabla_y h_1 \cdot \left( \nabla_y h_i \partial_y w_i + w_i \partial_y \nabla_y h_i \right) dy
\]

\[
- (T^s)^{\frac{1}{2}} e^{-\frac{t}{2}} \int_{\mathbb{R}^3} \nabla_y h_1 \cdot \nabla_y (\Gamma \partial_y \Phi) dy
\]

\[
= \int_{\mathbb{R}^3} \nabla_y h_1 \cdot \tilde{g}_1 dy,
\]

(2.42)

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla_y h_2 \|_{L^2}^2 + \left( a + \frac{3}{4} - \frac{s}{2} \right) \| \nabla_y h_2 \|_{L^2}^2 \right)
\]

\[
- k (T^s)^{2s+1} e^{-\frac{(2s+1)\tau}{2}} \int_{\mathbb{R}^3} \nabla_y h_1 \cdot \nabla_y h_2 dy
\]

\[
+ (T^s)^{\frac{1}{2}} e^{-\frac{t}{2}} \sum_{i=1}^{3} \int_{\mathbb{R}^3} \nabla_y h_2 \cdot \left( \nabla_y h_i \partial_y w_i + w_i \partial_y \nabla_y h_i \right) dy
\]

\[
- (T^s)^{\frac{1}{2}} e^{-\frac{t}{2}} \int_{\mathbb{R}^3} \nabla_y h_2 \cdot \nabla_y (\Gamma \partial_y \Phi) dy
\]

\[
= \int_{\mathbb{R}^3} \nabla_y h_2 \cdot \tilde{g}_2 dy,
\]

(2.43)

and

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla_y h_3 \|_{L^2}^2 + \left( a + \frac{3}{4} - \frac{s}{2} \right) \| \nabla_y h_3 \|_{L^2}^2 \right)
\]

\[
+ (T^s)^{\frac{1}{2}} e^{-\frac{t}{2}} \sum_{i=1}^{3} \int_{\Omega} \nabla_y h_3 \cdot \left( \nabla_y h_i \partial_y w_i + w_i \partial_y \nabla_y h_i \right) dy
\]

\[
- (T^s)^{\frac{1}{2}} e^{-\frac{t}{2}} \int_{\Omega} \nabla_y h_3 \cdot \nabla_y (\Gamma \partial_y \Phi) dy = \int_{\Omega} \nabla_y h_3 \cdot \tilde{g}_3 dy.
\]

(2.44)

Summing up (2.42)–(2.44), we have
We now estimate each nonlinear term in (2.45). On the one hand, note that \((\mu, c, \nu)^T \in B_R\). We employ Young’s inequality, \(H^\frac{2}{3}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)\) and integrating by parts to derive

\[
\begin{align*}
2\mu T^* e^{-\tau} \int_{\mathbb{R}^3} n|\nabla_j \Gamma|^2 \, dy & \leq C_R \|\nabla_j \Gamma\|_{L^2}^2, \\
(T^*)_T e^{-\tau} \int_{\mathbb{R}^3} \sum_{j=1}^3 \nabla_j (\h \cdot \nabla_j h) \cdot \nabla_j \Gamma \, dy & \leq C_R \left( \|\nabla_j \Gamma\|_{L^2}^2 + \sum_{i=1}^3 \left( \|h_i\|_{L^2}^2 + \|\nabla_j h_i\|_{L^2}^2 \right) \right), \\
\chi \int_{\mathbb{R}^3} \nabla_j (\nabla_j (\nabla_j \cdot (n \nabla_j \Lambda))) \cdot \nabla_j \Gamma \, dy & \leq C_R \left( \|\nabla_j \Gamma\|_{L^2}^2 + \|\nabla_j \nabla_j \Gamma\|_{L^2}^2 \right), \\
(T^*)_T e^{-\tau} \int_{\mathbb{R}^3} \sum_{j=1}^3 \nabla_j (h_i \partial_j d_i) - \nabla_j \Gamma \, dy & \leq \frac{(T^*)_T}{2} e^{-\tau} \left( \|\nabla_j \Gamma\|_{L^2}^2 + \|\nabla_j \nabla_j \Gamma\|_{L^2}^2 \right) + C_R \sum_{i=1}^3 \left( \|h_i\|_{L^2}^2 + \|\nabla_j h_i\|_{L^2}^2 \right).
\end{align*}
\]
and

\[
\sum_{i=1}^{3} \int_{\mathbb{R}^{3}} \nabla_{j} h_{1} \cdot \left( \nabla_{j} h_{1} \partial_{y_{i}} w_{1} + w_{i} \partial_{y_{j}} \nabla_{j} h_{1} \right) dy \leq \left( \sum_{k=1}^{3} \left( \| \partial_{y_{k}} w_{1} \|_{L^{\infty}} + \| w_{i} \|_{L^{\infty}} \right) \right) \sum_{i=1}^{3} \int_{\mathbb{R}^{3}} \left( |\nabla_{j} h_{1}|^2 + |\partial_{y_{j}} \nabla_{j} h_{1}|^2 \right) dy
\]

\[
\leq C_{R} \sum_{i=1}^{3} \int_{\mathbb{R}^{3}} \left( |\nabla_{j} h_{1}|^2 + |\partial_{y_{j}} \nabla_{j} h_{1}|^2 \right) dy,
\]

(2.47)

\[
\sum_{i=1}^{3} \int_{\mathbb{R}^{3}} \nabla_{j} h_{2} \cdot \left( \nabla_{j} h_{1} \partial_{y_{i}} w_{2} + w_{i} \partial_{y_{j}} \nabla_{j} h_{2} \right) dy \leq \left( \sum_{k=1}^{3} \left( \| \partial_{y_{k}} w_{2} \|_{L^{\infty}} + \| w_{i} \|_{L^{\infty}} \right) \right) \sum_{i=1}^{3} \int_{\mathbb{R}^{3}} \left( |\nabla_{j} h_{1}|^2 + |\partial_{y_{j}} \nabla_{j} h_{2}|^2 \right) dy
\]

\[
\leq C_{R} \sum_{i=1}^{3} \int_{\mathbb{R}^{3}} \left( |\nabla_{j} h_{1}|^2 + |\partial_{y_{j}} \nabla_{j} h_{2}|^2 \right) dy,
\]

(2.48)

\[
\sum_{i=1}^{3} \int_{\mathbb{R}^{3}} \nabla_{j} h_{3} \cdot \left( \nabla_{j} h_{1} \partial_{y_{i}} w_{3} + w_{i} \partial_{y_{j}} \nabla_{j} h_{3} \right) dy \leq \left( \sum_{k=1}^{3} \left( \| \partial_{y_{k}} w_{3} \|_{L^{\infty}} + \| w_{i} \|_{L^{\infty}} \right) \right) \sum_{i=1}^{3} \int_{\mathbb{R}^{3}} \left( |\nabla_{j} h_{1}|^2 + |\partial_{y_{j}} \nabla_{j} h_{3}|^2 \right) dy
\]

\[
\leq C_{R} \sum_{i=1}^{3} \int_{\mathbb{R}^{3}} \left( |\nabla_{j} h_{1}|^2 + |\partial_{y_{j}} \nabla_{j} h_{3}|^2 \right) dy,
\]

(2.49)

\[
\left( T^{*} \right)^{1/2} e^{-\frac{R^{2}}{4}} \sum_{i=1}^{3} \int_{\mathbb{R}^{3}} \nabla_{j} h_{1} \cdot \nabla_{j} (\Gamma \delta_{y} \Phi) dy \leq C_{R} \left( \sum_{i=1}^{3} \| \nabla_{j} h_{1} \|_{L^{2}}^{2} + \| \nabla_{j} \Gamma \|_{L^{2}}^{2} \right)
\]

(2.50)

where the \( C_{R} \) are positive constants depending on \( R \), which are small constants as \( R \) is small.

On the other hand, by (2.18), we know the highest order derivatives on \( h_{1} \) of \( \partial_{y_{1}} \nabla_{j} \bar{f} \) is \( s \).
So we can use the standard Calderon–Zygmund theory, Young’s inequality and integrating
by parts to derive

\[
\sum_{i=1}^{3} \int_{\mathbb{R}^{3}} \nabla_{j} h_{i} \cdot \partial_{y_{i}} \nabla_{j} \bar{f} dy \leq C_{R} \left( \sum_{i=1}^{3} \| \nabla_{j} h_{i} \|_{L^{2}}^{2} + \| \nabla_{j} \Gamma \|_{L^{2}}^{2} \right),
\]

(2.51)
furthermore, by (2.37)–(2.39), we have

\[ |T^* e^{-T} \int_{\mathbb{R}^3} (\nabla_j f_1 \cdot \nabla_j \Gamma + \nabla_j f_2 \cdot \nabla_j \Lambda) | \]

\[ \lesssim \frac{1}{2} \left( \sum_{i=1}^{3} \left\| \nabla_j f_i \right\|^2_{L^2} + \left\| \nabla_j \Gamma \right\|^2_{L^2} + \left\| \nabla_j \Lambda \right\|^2_{L^2} \right), \]

\[ \sum_{i=1}^{3} \int_{\mathbb{R}^3} (\nabla_j h_i \cdot \tilde{g}_i) dy \]

\[ \lesssim \left( C_{R, T^*} + \frac{1}{2} \right) \sum_{i=1}^{3} \left( \left\| \nabla_j h_i \right\|^2_{L^2} + \left\| \nabla_j \Gamma \right\|^2_{L^2} + \left\| \nabla_j \Lambda \right\|^2_{L^2} \right) + 4 \sum_{i=1}^{3} \left\| \nabla_j g_i \right\|^2_{L^2}, \]

where \( C_{R, T^*} \) is a positive constant depending on \( R, T^* \), which is a small constant as \( R \) small.

Hence we can apply estimates (2.46)–(2.52) to (2.45), then

\[ \frac{1}{2} \sum_{i=1}^{3} \frac{d}{d\tau} \left( \left\| \nabla_j \Gamma \right\|^2_{L^2} + \left\| \nabla_j \Lambda \right\|^2_{L^2} + \left\| \nabla_j h_i \right\|^2_{L^2} \right) \]

\[ + (1 - C_R) \left\| \nabla_j^{s+1} \nabla_j \Gamma \right\|^2_{L^2} + (1 - C_R) \left\| \nabla_j^{s+1} \nabla_j \Lambda \right\|^2_{L^2} + (3\nu - C_R) \sum_{i,j=1}^{3} \left\| \partial_{ij} \nabla_j h_j \right\|^2_{L^2} \]

\[ + \left( \frac{3}{4} - \frac{s}{2} - T^* e^{-T} (b + \sigma) - \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2}T} - C_R \right) \left\| \Gamma \right\|^2_{L^2} \]

\[ + \left( \frac{3}{4} - \frac{s}{2} + T^* e^{-T} (1 - b) - \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2}T} - C_R \right) \left\| \Lambda \right\|^2_{L^2} \]

\[ + \left( a + \frac{3}{4} - \frac{s}{2} - T^* e^{-T} b - C_R \right) \left( \left\| \nabla_j h_1 \right\|^2_{L^2} + \left\| \nabla_j h_2 \right\|^2_{L^2} \right) \]

\[ + \left( \frac{3}{4} - \frac{s}{2} - 2a - T^* e^{-T} b - C_R \right) \left\| \nabla_j h_3 \right\|^2_{L^2} \]

\[ \lesssim \frac{1}{2} (\| f_1 \|^2_{L^2} + \| f_2 \|^2_{L^2}) + 4 \sum_{i=1}^{3} \left\| \nabla_j g_i \right\|^2_{L^2}. \]

Since \( 0 < a < \frac{1}{2} \) and \( 0 < s < \frac{3}{2} - 5a \) are constants, there exists a sufficient small positive constant \( R \) such that

\[ 1 - C_R > 0, \quad 1 - C_R > 0, \quad 3\nu - C_R > 0, \]

\[ \frac{3}{4} - \frac{s}{2} - T^* e^{-T} (b + \sigma) - \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2}T} - C_R > 0, \]

\[ \frac{3}{4} - \frac{s}{2} + T^* e^{-T} (1 - b) - \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2}T} - C_R > 0, \]

\[ a + \frac{3}{4} - \frac{s}{2} - T^* e^{-T} b - C_R > 0, \]

\[ \frac{3}{4} - \frac{s}{2} - 2a - T^* e^{-T} b - C_R > 0. \]
Hence, applying Gronwall’s inequality to (2.53), there exists a positive constant $C_{R,T^*}$ depending on $R$ and $T^*$ such that

$$
\sum_{i=1}^{3} \left( \| \nabla_j \Gamma \|_{L^2}^2 + \| \nabla_j \Lambda \|_{L^2}^2 + \| \nabla_j h_i \|_{L^2}^2 \right) \\
\lesssim e^{-C_{R,T^*} \tau} \sum_{i=1}^{3} \left( \| \nabla_j \Gamma_0 \|_{L^2}^2 + \| \nabla_j \Lambda_0 \|_{L^2}^2 + \| \nabla_j h_0 \|_{L^2}^2 \right) \\
+ e^{-C_{R,T^*} \tau} \int_0^{+\infty} \left( \| f_1 \|_{L^2}^2 + \| f_2 \|_{L^2}^2 + \sum_{i=1}^{3} \| \nabla_j g_i \|_{L^2}^2 \right) d\tau, \quad \forall \tau > 0.
$$

Furthermore, we have the following result.

**Lemma 2.3** Let $0 < a \ll \frac{1}{5}$ and $0 < s < \frac{3}{2} - 5a$ be constants. Assume that $\| \Phi \|_{L^{2s+5}(\mathbb{R}^3)} \lesssim R \ll 1, f_i \in C^1((0, +\infty), H^s(\mathbb{R}^3))$ $(i = 1, 2), g \in C^1((0, +\infty), H^s(\mathbb{R}^3))$ and $(n, c, v)^T \in B_R$. Then, for any $\tau > 0$, the solution $(\Gamma, \Lambda, h)^T$ of the linearized coupled system (2.13)–(2.17) with the initial data (2.19) and condition (2.20) satisfies

$$
\int_{\mathbb{R}^3} \left( |\nabla_j \partial_t \Gamma|^2 + |\nabla_j \partial_t \Lambda|^2 + \sum_{i=1}^{3} |\nabla_j \partial_t h_i|^2 \right) dy \\
\lesssim e^{-C_{a,R,\kappa,v,\mu,\delta} \tau} \int_{\mathbb{R}^3} \left( |\nabla_j \partial_t \Gamma_0|^2 + |\nabla_j \partial_t \Lambda_0|^2 + \sum_{i=1}^{3} |\nabla_j \partial_t h_0|^2 \right) dy \\
+ e^{-C_{a,R,\kappa,v,\mu,\delta} \tau} \int_0^{+\infty} \left( \| f_1 \|_{L^2}^2 + \| f_2 \|_{L^2}^2 + \sum_{i=1}^{3} \| \nabla_j g_i \|_{L^2}^2 \right) d\tau, \quad \forall \tau > 0,
$$

where $C_{a,R,\kappa,v,\mu,\delta}$ is a positive constant depending on the constants $a, R, \kappa, v, \mu, \delta$.

**Proof** Similar to getting the estimate in Lemma 2.2, we apply the operator $\partial_t \nabla_j$ to both sides of (2.8)–(2.10), then using a similar process to the proof of Lemma 2.2, we can obtain this result. \hfill \Box

**Proposition 2.1** Let $0 < a \ll \frac{1}{5}$ and $0 < s < \frac{3}{2} - 5a$ be constants. Assume that $\| \Phi \|_{L^{2s+5}(\mathbb{R}^3)} \lesssim R \ll 1, f_i \in C^1((0, +\infty), H^s(\mathbb{R}^3))$ $(i = 1, 2), g \in C^1((0, +\infty), H^s(\mathbb{R}^3))$ and $(n, c, v)^T \in B_R$. Then, for any $\tau > 0$, the linearized coupled system (2.13)–(2.17) with the initial data (2.19) and condition (2.20) admits a solution

$$
\Gamma \in C^0_0 := \bigcap_{i=0}^{1} C^i((0, +\infty); H^{s-i}), \\
\Lambda \in C^0_0 := \bigcap_{i=0}^{1} C^i((0, +\infty); H^{s-i}), \\
h \in C^0_0 := \bigcap_{i=0}^{1} C^i((0, +\infty); H^{s-i}).
$$
Moreover,

\[
\|\Gamma\|_{C_0}^2 + \|\Lambda\|_{C_0}^2 + \|h\|_{C_0}^2 \lesssim \|\Gamma_0\|_{C_0}^2 + \|\Lambda_0\|_{C_0}^2 + \|h_0\|_{C_0}^2 + \|f_1\|_{C_0}^2 + \|f_2\|_{C_0}^2 + \|g\|_{C_0}^2,
\]

\[\forall \tau > 0. \tag{2.54}\]

**Proof** Let \( P \) be the Leray projector onto the space of divergence free functions. We apply the Leray projector to system (2.5), we have

\[
\begin{cases}
\Gamma_t - \Delta \Gamma + (2\mu n - \sigma)\Gamma + (v + \bar{u}) \cdot \nabla \Gamma + h \cdot \nabla n + \chi \nabla \cdot [\Gamma \nabla c + n \nabla \Lambda] = f_1(t,x), \\
\Lambda_t - \Delta \Lambda + \Lambda + (v + \bar{u}) \cdot \nabla \Lambda + h \cdot \nabla c - \Gamma = f_2(t,x), \\
h_t - vP\Delta h + \bar{P}(h \cdot \nabla (\bar{u} + v) + (\bar{u} + v) \cdot \nabla h - \Gamma \nabla \Phi) = P g(t,x).
\end{cases} \tag{2.55}
\]

In the similarity coordinates (2.12), we can rewrite the linear system (2.55) as

\[
\partial_t U + (A + \mathcal{N}) U = T^* e^{-\tau} F,
\]

where \( U := (\Gamma, \Lambda, h_1, h_2, h_3)^T \), \( \mathcal{N}(U) := (M_1, M_2, N_1, N_2, N_3)^T \), \( F := (f_1, f_2, P g_1, P g_2, P g_3)^T \)

and the matrix operator is given by

\[
A := \begin{pmatrix}
-\mu \Delta y & 0 & 0 & 0 \\
0 & -\delta \Delta y & 0 & 0 \\
0 & 0 & -vP\Delta y & 0 \\
0 & 0 & 0 & -vP\Delta y
\end{pmatrix} \quad \text{5x5},
\]

and

\[
M_1(\Gamma, \Lambda, h) := -\Delta \Gamma - \frac{y}{2} \cdot \nabla \Gamma + T^* e^{-\tau} (2\mu n - \sigma)\Gamma + ay_1 \partial_{y_1} \Gamma + ay_2 \partial_{y_2} \Gamma
\]

\[
-2ay_3 \partial_{y_3} \Gamma + k(T^*) e^{-2(2\mu + 1)\tau} (y_2 \partial_{y_2} \Gamma + y_1 \partial_{y_1} \Gamma)
\]

\[
+ (T^*) \frac{1}{2} e^{-\frac{3}{2} \tau} h \cdot \nabla c + \chi \sum_{i=1}^{3} \partial_{y_i} \nabla y \cdot (n \nabla \Lambda),
\]

\[
M_2(\Gamma, \Lambda, h) := -\frac{y}{2} \cdot \nabla \Lambda + T^* e^{-\tau} \Lambda + ay_1 \partial_{y_1} \Lambda + ay_2 \partial_{y_2} \Lambda - 2ay_3 \partial_{y_3} \Lambda
\]

\[
+ k(T^*) e^{-2(2\mu + 1)\tau} (y_2 \partial_{y_2} \Lambda + y_1 \partial_{y_1} \Lambda) + (T^*) \frac{1}{2} e^{-\frac{3}{2} \tau} \left( \sum_{i=1}^{2} h_i \partial_{y_i} \chi_i - \Gamma \right),
\]

\[
N_1(\Gamma, \Lambda, h) := -\frac{y}{2} \cdot \nabla h_1 + ah_1 + ay_1 \partial_{y_1} h_1 + ay_2 \partial_{y_2} h_1 + 2ay_3 \partial_{y_3} h_1
\]

\[
+ k(T^*) e^{-2(2\mu + 1)\tau} (h_2 + y_2 \partial_{y_2} h_1 + y_1 \partial_{y_1} h_1) + (T^*) \frac{1}{2} e^{-\frac{3}{2} \tau} \Gamma \partial_{y_1} \Phi
\]

\[
+ (T^*) \frac{1}{2} e^{-\frac{3}{2} \tau} \sum_{i=1}^{3} (h_i \partial_{y_i} w_1 + w_i \partial_{y_i} h_1),
\]
\[
\mathbb{N}_2(\Gamma, A, \mathbf{h}) := -\frac{y}{2} \cdot \nabla_j h_2 + a y_1 \partial_j h_2 + a y_2 \partial_j h_2 - 2 a y_3 \partial_j h_2 \\
+ k(T)^{2a+1} e^{-(2a+1)t} (-h_1 + y_2 \partial_j h_2 - y_1 \partial_j h_2) - \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2} t} \Gamma \partial_j \Phi \\
+ \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2} t} \sum_{i=1}^{3} (h_i \partial_j w_2 + w_i \partial_j h_2), \\
\mathbb{N}_3(\Gamma, A, \mathbf{h}) := -\frac{y}{2} \cdot \nabla_j h_3 - a h_3 + a y_1 \partial_j h_3 + a y_2 \partial_j h_3 - 2 a y_3 \partial_j h_3 \\
+ k(T)^{2a+1} e^{-(2a+1)t} (y_2 \partial_j h_3 - y_1 \partial_j h_3) - \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2} t} \Gamma \partial_j \Phi \\
+ \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2} t} \sum_{i=1}^{3} (h_i \partial_j w_3 + w_i \partial_j h_3).
\]

Obviously, there is no singular coefficient in the linear operator \( A + \mathcal{N} \) by noticing (2.55). We follow the idea of [38] to show the linear operator \( A + \mathcal{N} \) generate a strongly continuous semigroup \( e^{(A+\mathcal{N})t} \) in Sobolev space \( \mathbb{H}^2(\mathbb{R}^3) \times \mathbb{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \). To see this, by the same process as getting (2.53), for the constants \( 0 < a \ll \frac{1}{8} \) and \( 0 < s < \frac{3}{2} - 5a \), we can obtain

\[
\int_\Omega \nabla_j U \cdot \nabla_j ((A + \mathcal{N})U) \, dy \\
\leq -\left( \frac{3}{4} - \frac{s}{2} - T^* e^{-t} (b + \sigma) - \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2} t} - C_\mathcal{R} \right) \| \nabla \|^2_{L^2} \\
- \left( \frac{3}{4} - \frac{s}{2} + T^* e^{-t} (1 - b) - \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2} t} - C_\mathcal{R} \right) \| \Lambda \|^2_{L^2} \\
- \left( a + \frac{3}{4} - \frac{s}{2} - T^* e^{-t} b - C_\mathcal{R} \right) (\| \nabla_j h_1 \|^2_{L^2} + \| \nabla_j h_2 \|^2_{L^2}) \\
- \left( \frac{3}{4} - \frac{s}{2} - 2a - T^* e^{-t} b - C_\mathcal{R} \right) \| \nabla_j h_3 \|^2_{L^2} \\
\tag{2.56}
\]

and

\[
\frac{3}{4} - \frac{s}{2} - T^* e^{-t} (b + \sigma) - \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2} t} - C_\mathcal{R} > 0, \\
\frac{3}{4} - \frac{s}{2} + T^* e^{-t} (1 - b) - \left( T^* \right)^{\frac{1}{2}} e^{-\frac{1}{2} t} - C_\mathcal{R} > 0, \\
a + \frac{3}{4} - \frac{s}{2} - T^* e^{-t} b - C_\mathcal{R} > 0, \\
\frac{3}{4} - \frac{s}{2} - 2a - T^* e^{-t} b - C_\mathcal{R} > 0.
\]

Hence by (2.56), we get

\[
\int_\Omega \nabla_j U \cdot \nabla_j ((A + \mathcal{N})U) \, dy \leq 0.
\]

Hence the linear operator \( A + \mathcal{N} \) is a linear dissipative operator in \( \mathbb{H}^2(\mathbb{R}^3) \times \mathbb{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \). Moreover, if we set

\[(A + \mathcal{N})U = 0,
\]
then, by (2.56), we know the linear operator $A + \mathcal{N}$ is injective. Furthermore, we can verify that this linear operator is surjective by using the standard theory of elliptic-type equations of the general order. Thus the linear operator $A + \mathcal{N}$ can generate a strongly continuous semigroup $e^{(A + \mathcal{N})t}$ in Sobolev space $\mathbb{H}^s(\mathbb{R}^3) \times \mathbb{H}^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ by the Lumer–Phillips theorem [23]. Therefore, the linear system (2.55) admits a solution

$$\Gamma \in C^i_t := \bigcap_{i=0}^1 C^i((0, +\infty); \mathbb{H}^{s-i}),$$

$$\Lambda \in C^i_t := \bigcap_{i=0}^1 C^i((0, +\infty); \mathbb{H}^{s-i}),$$

$$h \in C^i_t := \bigcap_{i=0}^1 C^i((0, +\infty); H^{s-i}).$$

Furthermore, it follows from Lemmas 2.2–2.3 that (2.54) holds. □

Recalling the self-similarity coordinates (2.12), the original coordinate can be expressed by the self-similarity coordinates as follows:

$$t = T(1 - e^{-\tau}), \quad x = y\sqrt{T^* - t},$$

so we can directly use Proposition 2.1 to get the following result.

**Proposition 2.2.** Let $0 < \alpha \ll \frac{1}{8}$ and $0 < s < \frac{3}{2} - 5a$ be constants. Assume that $\|\Phi\|_{L^2(\mathbb{R}^3)} \lesssim R \ll 1, f_i \in C^1((0, T^*), \mathbb{H}^s(\mathbb{R}^3)) \quad (i = 1, 2), \ g \in C^1((0, T^*), H^s(\mathbb{R}^3))$ and $(n, c, \nu)^T \in B_8$. Then the linearized coupled system (2.5) with the initial data (2.2) and condition (2.3) admits a local solution

$$\Gamma \in C^i_t := \bigcap_{i=0}^1 C^i((0, T^*); \mathbb{H}^{s-i}(\mathbb{R}^3)), $$

$$\Lambda \in C^i_t := \bigcap_{i=0}^1 C^i((0, T^*); \mathbb{H}^{s-i}(\mathbb{R}^3)), $$

$$h \in C^i_t := \bigcap_{i=0}^1 C^i((0, T^*); H^{s-i}(\mathbb{R}^3)). $$

Moreover,

$$\|\Gamma\|_{C^0_t}^2 + \|\Lambda\|_{C^0_t}^2 + \|h\|_{C^0_t}^2 \lesssim \|\Gamma_0\|_{C^0_t}^2 + \|\Lambda_0\|_{C^0_t}^2 + \|h_0\|_{C^0_t}^2 + \|f_1\|_{C^0_t}^2 + \|f_2\|_{C^0_t}^2 + \|g\|_{C^0_t}^2, \quad \forall t \in (0, T^*).$$

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