Properties of twisted ferromagnetic filaments

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Abstract. The full set of equations for twisted ferromagnetic filaments is derived. The linear stability analysis of twisted ferromagnetic filament is carried out. Two different types of the buckling instability are found - monotonous and oscillatory. The first in the limit of large twist leads to the shape of filament reminding pearls on the string, the second to spontaneous rotation of the filament, which may constitute the working of chiral microengine.

1. Introduction

Chains of magnetic particles determine the properties of magnetorheological suspensions and under certain conditions may be described by the model of semiflexible filament. Many features of these filaments are determined by their twist. Twist can be easily applied to magnetic filaments due to the torques acting on the magnetic particles. Twist plays important role in the behavior of biopolymers and in living world [1]. For example in cilia twist is induced by the action of molecular motors and leads to their rotation due to the chiral symmetry breaking [2]. Chiral symmetry breaking due to the induced vortical flows is presumably responsible for left-right symmetry breaking at the embryonic stage of development of vertebrates [3]. Here we describe several phenomena caused by the action of twisting torques on the ferromagnetic filaments. We illustrate that similarly to cilia the chiral symmetry breaking takes place for flexible ferromagnetic filaments under the action of twist. Ferromagnetic filaments exist in the living world and are used by magnetotactic bacteria for the orientation in the magnetic field of the Earth [4,5]. They can be made artificially by linking commercially available ferromagnetic particles functionalized by streptavidin with biotinized DNA fragments [6].

2. Model

The theoretical model, which describes flexible ferromagnetic filaments is based on the Kirchhoff model of elastic rod extended to account for interaction energy of filament with external field (l is the contour length of the filament, subscript ... denotes derivative) [7].

\[ E = \frac{1}{2} A \int (\Omega_{\text{11}}^1 + \Omega_{\text{22}}^2) dl + \frac{1}{2} C \int \Omega_{\text{33}}^3 dl - M H \int \varepsilon_3 \cdot \vec{h} dl - \int \Lambda dl \]  

(1)

where the orts of material frame \( \varepsilon_i \) obey \( \varepsilon_{i,l} = \Omega \times \varepsilon_i \). \( A \) and \( C \) are the bend and twist elastic constants, \( M \) is the magnetization of the filament per its unit length, \( H \) is the applied magnetic field strength, and \( \vec{h} \) is the unit vector along the applied magnetic field. Inextensibility of the filament is enforced by the term \( - \int \Lambda dl \) in (1). Introducing the complex curvature
\[ \psi = (-i\Omega_1 + \Omega_2) \exp (i \int \Omega_3 dl') \quad \text{and} \quad \vec{e} = (\vec{e}_1 + i\vec{e}_2) \exp (i \int \Omega_3 dl') \]

the following expressions for the stress \( \bar{F} \) and momentum stress \( \bar{T} \) are derived

\[
\bar{F} = -A\Re(\vec{e}^* \psi) - \frac{1}{2} A(\Omega_1^2 + \Omega_2^2)\vec{e}_3 + \Re(iC \Omega_3 \vec{e}^* \psi) - MH(\vec{e}_1 \cdot \vec{h}\vec{e}_1 + \vec{e}_2 \cdot \vec{h}\vec{e}_2) - \Lambda \vec{e}_3
\]

\[
\bar{T} = A\Omega_1 \vec{e}_1 + A\Omega_2 \vec{e}_2 + C\Omega_3 \vec{e}_3
\]

We are considering the Rouse dynamics (\( \zeta \) is the drag coefficient per unit length)

\[
\zeta \ddot{\psi} = \bar{K} = \bar{F},
\]

here

\[
\bar{K} = (-\Lambda_1 + M H \Re(\vec{e}^* \cdot \vec{h}\psi))\vec{e}_3 + \Re(\vec{e}^* F_\perp)
\]

and

\[
F_\perp = -A(\psi, \psi) + |\psi|^2 \psi/2 + iC(\Omega_3 \psi, l) - \Lambda \psi + MH \vec{e}_3 \cdot \vec{h}\psi
\]

\( F_\perp \) besides the terms characteristic for nonmagnetic filaments [8,9] contains term \( MH \vec{e}_3 \cdot \vec{h}\psi \) due to the torque from the magnetic field. Equations for the complex curvature and twist are the following (\( \zeta_r \) is the rotational drag coefficient per unit length)

\[
\zeta \psi_t = F_{\perp, \mu} + |\psi|^2 F_\perp - \psi_t \Lambda_1 + \psi_\mu M H \Re(\vec{e}^* \cdot \vec{h}\psi) + i\psi \int F_\perp^{*} \psi, dl'
\]

\[
\zeta_r \Omega_{3, t} = C\Omega_{3, \mu} - \frac{\zeta_r}{\zeta} \Im(\psi F_\perp^{*})
\]

Inextensibility condition \( \vec{e}_3 \cdot \vec{K}, t = 0 \) gives

\[
-\Lambda_{\mu} = -MH\bar{h} \cdot \vec{e}_3, \mu + \Re(F_\perp \psi^*)
\]

3. Linear stability analysis

Here we report the results of the linear stability analysis of straight twisted ferromagnetic filament with the free, unclamped ends and applied torque \( \bar{T}(\pm L, t) = C\Omega_3 \vec{e}_3 \) (2L is the filament length), which is oriented in the direction of the magnetic field (z axis). For small perturbations the relation \( \psi = \xi, \mu \) (\( \xi = x + iy \)) is valid. As a result from Eqs.(4),(5) the following equation for the small deformation of the filament is obtained (\( \vec{e}_3^0 \) is the tangent vector of straight filament)

\[
\zeta \xi_t = -A\xi, \mu \mu + iC\Omega_3 \xi, \mu \mu + MH\bar{e}_3^0 \cdot \vec{h}\xi, \mu
\]

with the boundary conditions of zero force and torque

\[
\xi, \mu |_{l=\pm L} = 0; (-A\xi, \mu \mu + iC\Omega_3 \xi, \mu \mu + MH\bar{e}_3^0 \cdot \vec{h}\xi, \mu)|_{l=\pm L} = 0
\]

Dimensionless set of equations is obtained scaling the length with \( L \), time with \( \tau = \zeta L^4/A \), introducing the magnetoelastic number \( Cm = MH\bar{e}_3^0 \cdot \vec{h}L^2/A \) and \( \chi = C\Omega_3 L/A \). Looking for the solution of Eq.(10) in form \( \xi = \exp (\lambda t) \xi(l) \) the eigenvalue problem for not self-adjoint operator is obtained:

\[
\lambda \xi = -\xi, \mu \mu + i\chi \xi, \mu \mu + Cm \xi, \mu
\]

with boundary conditions

\[
\xi, \mu(\pm 1) = 0; -\xi, \mu(\pm 1) + i\chi \xi, \mu(\pm 1) + Cm \xi, \mu(\pm 1) = 0
\]
This eigenvalue problem in the absence of the twist is investigated in [10]. It is shown that for all \( Cm < 0 \) there is positive eigenvalue, which corresponds to odd eigenmode of the filament deformation and for small \( |Cm| \) describes the overturning of the magnetic dipole at the inversion of the field. The increment of this mode diminishes with the increase of \( |Cm| \) due to the deformation of the filament. At \( |Cm| \) larger than the critical value \(-\pi^2/4\) the ferromagnetic filament becomes unstable with respect to U-like even deformation mode with increment growing with \( |Cm| \). At \( |Cm| = -(n\pi)^2/3 \), \((n = 1, 2, ...)\) the eigenvalues of the even and odd modes become equal and degenerate eigenvalue problem arises.

The eigenvalues at small \( \chi \) may be found by considering twist as perturbation near these degeneracy points. Up to the first order we have \( \lambda = \lambda_0 + \lambda_1; \xi = \xi_0 + \xi_1 \), \((\lambda_0 \) is positive eigenvalue of unperturbed problem [10]). Looking for the solution of the first order equation with \( \xi_0 = a\xi^{(e)} + b\xi^{(o)} \), where \( \xi^{(e,o)} \) are even and odd eigenfunctions at the degeneracy point, from the solvability condition we obtain

\[
\lambda_1^2 = -\chi^2 \frac{\int_{-1}^{1} \xi^{(e)} \xi_0^{(o)} dl \int_{-1}^{1} \xi^{(o)} \xi_0^{(e)} dl}{\int_{-1}^{1} \xi^{(e)}^2 dl \int_{-1}^{1} \xi^{(o)}^2 dl} \tag{14}
\]

Since the numerator in the relation Eq.14 is positive then at the degeneracy point of the spectrum pair of complex conjugated eigenvalues arise. If \( \lambda \) is a complex eigenvalue with corresponding eigenfunction \( \xi(l) \) then \( \lambda^* \) also is eigenvalue corresponding to eigenfunction \( \xi^*(-l) \). There is not more as one pair of complex eigenvalues. Variable \( \xi \) represents vector in plane perpendicular to the axis of the filament and therefore complex eigenvalues \( \lambda \) means that twisted filament rotates...
with angular velocity $\Im \lambda$. Since the corresponding eigenfunction is the mixture of odd and even eigenfunctions then for a rotating filament the symmetry is broken.

This symmetry breaking is illustrated by Fig. 1 where five consecutive shapes of the filament with broken chiral symmetry are shown at $Cm = -1.3; \chi = 3.38$ for the one period $2\pi/9.1$. As one can see the radiuses of the circles on which move both ends of the filament are different.

This may be confirmed exactly by considering the energy balance equation for the linear problem. Multiplying the real and imaginary parts of Eq.(10) by $x_{t}$ and $y_{t}$ correspondingly after adding the results and integration along the filament we obtain in dimensionless units ($\xi(l,t) \rightarrow \exp (i\lambda t)\xi(l)$)

$$|\lambda|^{2} \int_{-1}^{+1} |\xi|^{2}dl = -2\Re(\lambda)E - \frac{\chi \Im(\lambda)}{2} |\xi|^{2}_{-1}^{+1}$$

(15)

where $E$ is the sum of bending, magnetic and twist energies:

$$E = \frac{1}{2} \int_{-1}^{+1} |\xi_{,\mu}|^{2}dl + \frac{1}{2} Cm \int_{-1}^{1} |\xi_{,\mu}|^{2}dl - \frac{\chi}{2} \int_{-1}^{1} \Im(\xi_{,\mu}^{*}\xi_{,\mu})dl$$

From relation (15) follows that in the case of neutral complex eigenvalues ($\Re(\lambda)$=0) the work of twisting torque is dissipated into heat due to the rotation of the filament.

The neutral curves of monotonous perturbations are given by equations ($k$ is natural number)

$$Cm = \frac{\chi^{2} - (k\pi)^{2}}{4}$$

(16)

Shapes of filaments which correspond to neutral modes have $k - 1$ nodes, where the curvature of the filament is equal to zero and which coincide with the nodes of buckled filament at $Cm < 0$ in the absence of twist. Its torsion is constant and equals to $\chi/2$ even in the limit of straight filament. An illustration of emerging shape, which reminds the pearls on the string, is shown in Fig. 2 for $k = 2$.

Eigenvalues for arbitrary values of parameters are calculated numerically. Numerical algorithm is based on splines and differentiation matrices with nodes in zeros of classical orthogonal polynoms [11]. Obtained results show that at $\pi < \chi < \chi_{c}$ application of field parallel to magnetization diminishes the critical twist for buckling instability and antiparallel stabilizes the buckling instability due to the twist ($\chi_{c}$ is critical value of $\chi$ when twisted filament buckles in zero field and is given by equation $\tan \chi_{c} = \chi_{c}$). For $0 < \chi < \pi$ the situation is opposite to just described.

4. Singular perturbations

Described exchange of stability may be understood by perturbation theory analysis of the eigenvalues of not self-adjoint operator. Eigenvalue $\lambda_{0} = 0$ of the problem Eqs.(12) and (13) at $Cm = 0$ has twofold degeneracy. Eigenmodes $\xi_{0}^{(1)} = 1$ and $\xi_{0}^{(2)} = l$ correspond to the solid displacement of the filament and its rotation. In the magnetic field degeneracy is broken. The eigenvalue problem in the first order with the respect to $Cm$ reads ($\xi = \xi_{0} + \xi; \lambda = \lambda_{0} + \lambda_{1}$)

$$\hat{L}\xi = -\xi_{,\mu\mu} + i\chi \xi_{,\mu} = \lambda_{1}\xi_{0}^{(2)} - Cm\xi_{0}^{(2)}$$

(17)

at boundary conditions

$$\xi_{,\mu}(\pm 1) = 0; \xi_{,\mu\mu}(\pm 1) = Cm\xi_{0}^{(2)}(\pm 1)$$

(18)

Adjoint operator $\hat{L}^{+}$ is determined as

$$\hat{L}^{+}u = -u_{,\mu\mu\mu} + i\chi u_{,\mu\mu}$$

(19)
with the different from $\hat{L}$ boundary conditions

$$u_{,lll}(\pm 1) - i\chi u_{,ll}(\pm 1) = 0; \quad u_{,ll}(\pm 1) - i\chi u_{,l}(\pm 1) = 0$$

(20)

Operator $\hat{L}^+$ has double eigenvalue $\lambda = 0$. The corresponding eigenfunctions are $u_1 = 1$ and $u_2 = \exp(i\chi l)$. Solvability condition of problem Eqs.(17),(18) reads

$$\lambda_1 \int_{-1}^{1} u_{2,2}^{(0)}(l) dl = - Cmu_{2,0,1}^{(2)} \bigg|_{-1}^{+1}$$

which gives

$$\lambda_1 = \frac{Cm\chi^2 \sin \chi}{\chi \cos \chi - \sin \chi}$$

(21)

Relation (21) shows that at $\pi < \chi < \chi_c$ $\lambda_1 > 0$ for $Cm > 0$, which corresponds to the decrease of the critical twist for the buckling of the filament in the field parallel to magnetization of the filament. At $Cm < 0$ and $\pi < \chi < \chi_c$ in agreement with numerical data $\lambda_1 < 0$. At $2\pi > \chi > \chi_c$ application of the field along the magnetization stabilizes the buckling of the filament and applied opposite to the magnetization destabilizes it. From the relation (21) we see that in the point $(Cm = 0; \chi = \chi_c)$ the perturbation theory approach breaks down. This corresponds to the fact that in singular point $(Cm = 0; \chi = \chi_c)$ the algebraic multiplicity of the solution of dispersion equation $\lambda = 0$ is 4 instead of 3 for $0 < \chi < \chi_c$. Analysis of this singular perturbation problem is pending for the future publications.

In conclusion we have demonstrated that twisted ferromagnetic filaments can work as chiral microengines. Competing magnetic torque and twist induced buckling instabilities lead to the rich family of equilibrium shapes, which in the case of stabilizing field direction reminds pearls on the string. There are special points in the parameter plane, where the perturbations are singular.

5. References

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