SIMPLE SPECTRAL BOUNDS FOR SUMS OF CERTAIN KRONECKER PRODUCTS

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Abstract. New bounds are derived for the eigenvalues of sums of Kronecker products of square matrices by relating the corresponding matrix expressions to the covariance structure of suitable bi-linear stochastic systems in discrete and continuous time.

1. Introduction

Kronecker product reduces a matrix-matrix equation to an equivalent matrix-vector form ([1] or [3, Chapter 4]). For example, consider a matrix equation $BXA^\top = C$ with known $d$-by-$d$ matrices $A, B, C$, and the unknown $d$-by-$d$ matrix $X$. To cover the most general setting, all matrices are assumed to have complex-valued entries. Introduce a column vector $\text{vec}(X) = X \in \mathbb{C}^{d^2}$ by stacking together the columns of $X$, left-to-right:

$$\text{vec}(X) = X = (X_{11}, \ldots, X_{d1}, X_{12}, \ldots, X_{d2}, \ldots, X_{1d}, \ldots, X_{dd})^\top.$$  

Then direct computations show that the matrix equation $AXB^\top = C$ can be written in the matrix-vector form for the unknown vector $X$ as

$$(A \otimes B)X = C, \quad C = \text{vec}(C),$$

where $A \otimes B$ is the Kronecker product of matrices $A$ and $B$, that is, an $d^2$-by-$d^2$ block matrix with blocks $A_{ij}B$. In other words, (1.2) means

$$\text{vec}(BXA^\top) = (A \otimes B)\text{vec}(X),$$

with $\text{vec}(\cdot)$ operation defined in (1.1).

In what follows, an $d$-dimensional column vector will usually be denoted by a lower-case bold Latin letter, e.g. $h$, whereas upper-case regular Latin letter, e.g. $A$, will usually mean an $d$-by-$d$ matrix. Then $|h|$ is the Euclidean norm of $h$ and $|A|$ is the induced matrix norm

$$|A| = \max \{|Ah| : |h| = 1\}.$$ 

For a matrix $A \in \mathbb{C}^{d \times d}$, $\overline{A}$ is the matrix with complex conjugate entries, $A^\top$ means transposition, and $A^*$ denotes the conjugate transpose: $A^* = \overline{A^\top} = \overline{A}^\top$. The same notations, $\overline{\cdot}$, $\overline{\cdot}^\top$, and $\overline{\cdot}^*$, will also be used for column vectors in $\mathbb{C}^d$. The identity matrix is $I$.

For a square matrix $A$, define the following numbers:

$$\rho(A) = \max\{|\lambda(A)| : \lambda(A) \text{ is an eigenvalue of } A\} \quad \text{(spectral radius of } A\text{)};$$

$$\alpha(A) = \max\{\Re\lambda(A) : \lambda(A) \text{ is an eigenvalue of } A\} \quad \text{(spectral abscissa of } A\text{)};$$

$$\varrho(A) = \min\{\Re\lambda(A) : \lambda(A) \text{ is an eigenvalue of } A\}.$$  

2000 Mathematics Subject Classification. Primary 15A18; Secondary 15A69.

Key words and phrases. Covariance matrix, Mean-square stability, Spectral abscissa, Spectral radius.
For a Hermitian matrix $H$,
\begin{equation}
\varrho(H)|x|^2 \leq x^* H x \leq \alpha(H)|x|^2.
\end{equation}

While eigenvalues of the matrices $A \otimes B$ and $A \otimes I + I \otimes B$ can be easily expressed in terms of the eigenvalues of the matrices $A$ and $B$ \cite[Theorems 4.2.12 and 4.4.5]{3}, there is, in general, no easy way to get the eigenvalues of the matrices
\begin{equation}
D_{A,B} = A \otimes A + \sum_{k=1}^{m} B_k \otimes B_k
\end{equation}
and
\begin{equation}
C_{A,B} = A \otimes I + I \otimes A + \sum_{k=1}^{m} B_k \otimes B_k,
\end{equation}
which appear, for example, in the study of bi-linear stochastic systems. Paper \cite{2} presents one of the first investigations of the spectral properties of \eqref{1.5} and \eqref{1.6}. The main result of the current paper provides another contribution to the subject:

**Theorem 1.1.** Given matrices $A, B_1, \ldots, B_m \in \mathbb{C}^{d \times d}$, define the matrix $D_{A,B}$ by \eqref{1.5}, the matrix $C_{A,B}$ by \eqref{1.6}, and also the matrices
\begin{equation}
N_{A,B} = A^* A + \sum_{k=1}^{m} B_k^* B_k
\end{equation}
and
\begin{equation}
M_{A,B} = A + A^* + \sum_{k=1}^{m} B_k^* B_k.
\end{equation}
Then
\begin{equation}
\varrho(N_{A,B}) \leq \rho(D_{A,B}) \leq \alpha(N_{A,B}),
\end{equation}
\begin{equation}
\varrho(M_{A,B}) \leq \alpha(C_{A,B}) \leq \alpha(M_{A,B}).
\end{equation}

In the particular case of real matrices and $m = 1$, Theorem 1.1 implies
\begin{align*}
\varrho(A^T A + B^T B) &\leq \rho(A \otimes A + B \otimes B) \leq \alpha(A^T A + B^T B), \\
\varrho(A + A^T + B^T B) &\leq \alpha(A \otimes I + I \otimes A + B \otimes B) \leq \alpha(A + A^T + B^T B).
\end{align*}

**Corollary 1.2.** If the matrix $N_{A,B}$ is scalar, that is, $N_{A,B} = \beta I$, then $\rho(D_{A,B}) = \beta$; if $M_{A,B} = \beta I$, then $\alpha(C_{A,B}) = \beta$.

The reason Theorem 1.1 is potentially useful is that the matrices $M_{A,B}$ and $N_{A,B}$ are Hermitian and have size $d$-by-$d$, whereas the matrices $C_{A,B}$ and $D_{A,B}$ are in general not Hermitian or even normal and have a much bigger size $d^2$-by-$d^2$. For example, with $m = 1$, if matrices $A$ and $B$ are orthogonal, then the matrix $D_{A,B}$ can be fairly complicated, but $N_{A,B} = 2I$, and we immediately conclude that $\rho(D_{A,B}) = 2$. Similarly, let $m = 1$, let $A = aI + S$ for a real number $a$ and a skew-symmetric matrix $S$, and let $B$ be orthogonal, then $\alpha(C_{A,B}) = 2a + 1$. Section 4 below presents more examples and further discussions.

The matrix expressions $A \otimes B$ and $A \otimes I + I \otimes B$ have designated names (Kronecker product and Kronecker sum), but there is no established terminology for \eqref{1.5} and \eqref{1.6}. 
In what follows, (1.5) will be referred to as the discrete-time stochastic Kronecker sum, and (1.6) will be referred to as the continuous-time stochastic Kronecker sum. The reason for this choice of names is motivated by the type of problems in which the corresponding matrix expressions appear.

The proof of Theorem 1.1 relies on the analysis of the covariance matrix of suitably constructed random vectors. Recall that the covariance matrix of two \( \mathbb{C}^d \)-valued random column-vectors \( \mathbf{x} = (x_1, \ldots, x_d)^\top \) and \( \mathbf{y} = (y_1, \ldots, y_d)^\top \) is \( \mathbf{U}_{x,y} = \mathbb{E} \mathbf{xy}^\top \). Also define \( r_x = \sum_{i=1}^d \mathbb{E} |x_i|^2 \), \( r_y = \sum_{i=1}^d \mathbb{E} |y_i|^2 \), and \( \mathbf{U}_{x,y} = \text{vec}(\mathbf{U}_{x,y}) \). Then \( |\mathbf{U}_{x,y}|^2 = \sum_{i,j=1}^d |\mathbb{E} x_i y_j|^2 \), and the Cauchy-Schwartz inequality \( |\mathbb{E} x_i y_j|^2 \leq \mathbb{E} |x_i|^2 \mathbb{E} |y_j|^2 \) leads to an upper bound on \( |\mathbf{U}_{x,y}| \):

\[
(1.11) \quad |\mathbf{U}_{x,y}|^2 \leq r_x r_y.
\]

In the special case \( \mathbf{x} = \mathbf{y} \),

\[
d|\mathbf{U}_{x,y}|^2 = d \sum_{i,j=1}^d |\mathbb{E} x_i x_j^\top|^2 = d \sum_{i=1}^d \left( \mathbb{E} |x_i|^2 \right)^2 + d \sum_{i \neq j} |\mathbb{E} x_i x_j^\top|^2
\]

\[
\geq d \sum_{i=1}^d \left( \mathbb{E} |x_i|^2 \right)^2 \geq \left( \sum_{i=1}^d \mathbb{E} |x_i|^2 \right)^2,
\]

leading to a lower bound:

\[
(1.12) \quad |\mathbf{U}_{x,x}| \geq d^{-1/2} r_x.
\]

Section 2 explains how matrices of the type (1.5) appear in the analysis of discrete-time bi-linear stochastic systems and presents the proof of (1.9). Section 3 explains how matrices of the type (1.6) appear in the analysis of continuous-time bi-linear stochastic systems and presents the proof of (1.10). The connection with stochastic systems also illustrates why it is indeed natural to bound the spectral radius for matrices of the type (1.5) and the spectral abscissa for matrices of the type (1.6).

2. DISCRETE-TIME STOCHASTIC KRONESER SUM

Given matrices \( A, B_1, \ldots, B_m \in \mathbb{C}^{d \times d} \), consider two \( \mathbb{C}^d \)-valued random sequences \( \mathbf{x}(n) = (x_1(n), \ldots, x_d(n))^\top \) and \( \mathbf{y}(n) = (y_1(n), \ldots, y_d(n))^\top \), \( n = 0, 1, 2, \ldots \), defined by

\[
\mathbf{x}(n+1) = A \mathbf{x}(n) + \sum_{k=1}^m B_k \mathbf{x}(n) \xi_{n+1,k}, \quad \mathbf{x}(0) = \mathbf{u},
\]

\[
(2.1)
\]

\[
\mathbf{y}(n+1) = A \mathbf{y}(n) + \sum_{k=1}^m B_k \mathbf{y}(n) \xi_{n+1,k}, \quad \mathbf{y}(0) = \mathbf{v}.
\]

Both equations in (2.1) are driven by a white noise sequence \( \xi_{n,k} \), \( n \geq 1, k = 1, \ldots, m \) of independent, for all \( n \) and \( k \), random variables, all with zero mean and unit variance:

\[
(2.2) \quad \mathbb{E} \xi_{n,k} = 0, \quad \mathbb{E} \xi_{n,k}^2 = 1, \quad \mathbb{E} \xi_{n,k} \xi_{n,\ell} = 0 \text{ if } n \neq p \text{ or } k \neq \ell;
\]

the initial conditions \( \mathbf{u}, \mathbf{v} \in \mathbb{C}^N \) are non-random. Note that the sequences \( \mathbf{x}(n) \) and \( \mathbf{y}(n) \) satisfy the same equation and differ only in the initial conditions. In particular, \( \mathbf{u} = \mathbf{v} \) implies \( \mathbf{x}(n) = \mathbf{y}(n) \) for all \( n \geq 0 \). The term bi-linear in connection with
(2.1) reflects the fact that the noise sequence enters the system in a multiplicative, as opposed to additive, way.

**Proposition 2.1.** Define $V(n) = \mathbb{E}x(n)y^*(n)$, the covariance matrix of the random vectors $x(n)$ and $y(n)$ from (2.1), and define $r_x(n) = \mathbb{E}x^*(n)x(n) = \mathbb{E}|x(n)|^2$. Then the vector $U(n) = \text{vec}(V(n))$ satisfies

$$U(n + 1) = D^n_{A,B}U(0),$$

with the matrix

$$D_{A,B} = A \otimes A + \sum_{k=1}^{m} B_k \otimes B_k,$$

and the number $r_x(n)$ satisfies

$$|u|^2 \gamma^n \leq r_x(n) \leq |u|^2 \beta^n,$$

where $\gamma$ is the smallest eigenvalue and $\beta$ is the largest eigenvalue of the non-negative Hermitian matrix

$$N_{A,B} = A^*A + \sum_{k=1}^{m} B_k^*B_k.$$

**Proof.** By (2.1),

$$x(n + 1) = Ax(n) + \sum_{k=1}^{m} B_kx(n)\xi_{n+1,k}, \quad y^*(n + 1) = y^*(n)A^* + \sum_{k=1}^{m} y^*(n)B_k^*\xi_{n+1,k},$$

so that

$$x(n + 1)y^*(n + 1) = Ax(n)y^*(n)A^* + \sum_{k,\ell=1}^{m} B_kx(n)y^*(n)B_\ell^*\xi_{n+1,k}\xi_{n+1,\ell}$$

$$+ \sum_{k=1}^{m} Ax(n)y^*(n)B_k^*\xi_{n+1,k} + \sum_{k=1}^{m} B_kx(n)y^*(n)A^*\xi_{n+1,k}.$$ 

The vectors $x(n)$ and $y(n)$ are independent of every $\xi_{n+1,k}$. Therefore, using (2.2),

$$\mathbb{E}(Ax(n)y^*(n)B_k^*\xi_{n+1,k}) = \mathbb{E}(Ax(n)y^*(n)B_k^*)\mathbb{E}\xi_{n+1,k} = 0,$$

$$\sum_{k,\ell=1}^{m} \mathbb{E}(B_kx(n)y^*(n)B_\ell^*\xi_{n+1,k}\xi_{n+1,\ell}) = \sum_{k,\ell=1}^{m} \mathbb{E}(B_kx(n)y^*(n)B_\ell^*)\mathbb{E}(\xi_{n+1,k}\xi_{n+1,\ell})$$

$$= \sum_{k=1}^{m} B_k\mathbb{E}(x(n)y^*(n))B_k^* = \sum_{k=1}^{m} B_kV(n)B_k^*.$$

As a result,

$$V(n + 1) = AV(n)A^* + \sum_{k=1}^{m} B_kV(n)B_k^*,$$

and (2.3) follows from (1.3).
Similarly,
\[ r_x(n+1) = \mathbb{E}x^*(n) \left( A^*A + \sum_{k=1}^{m} B_k^* B_k \right) x(n). \]

Then (1.4) implies \( \gamma r_x(n) \leq r_x(n+1) \leq \beta r_x(n) \), and (2.5) follows.

Given the origin of equation (2.3), the matrix \( D_{A,B} \) from (2.4) is natural to call the **discrete-time stochastic Kronecker sum** of the matrices \( A \) and \( B_k \).

For a square matrix \( A \), denote by \( \rho \) the **spectral radius** of \( A \):
\[ \rho(A) = \max\{|\lambda(A)| : \lambda(A) \text{ is an eigenvalue of } A\}. \]

It is really very well known that
\[ \rho(A) = \lim_{n \to +\infty} |A^n|^{1/n}. \]

**Theorem 2.2.** For every matrices \( A, B_1, \ldots, B_m \in \mathbb{C}^{d \times d}, \)
\[ \varrho(N_{A,B}) \leq \rho(D_{A,B}) \leq \alpha(N_{A,B}), \]
where the matrix \( N_{A,B} = A^*A + \sum_{k=1}^{m} B_k^* B_k \), \( \varrho(N_{A,B}) \) is the smallest eigenvalue of \( N_{A,B} \), and \( \alpha(N_{A,B}) \) is the largest eigenvalue of \( N_{A,B} \).

**Proof.** Similar to Proposition 2.1 write \( \gamma = \varrho(N_{A,B}) \) and \( \beta = \alpha(N_{A,B}) \). It follows from (2.3) that
\[ |U(n)| = |D_{A,B}^n U(0)|. \]

To get the upper bound in (2.12), note that (1.11) and (2.5) imply
\[ |U(n)| \leq \sqrt{r_x(n)r_y(n)} \leq |u| |v| \beta^n. \]

Combining (2.13) and (2.14) leads to
\[ |D_{A,B}^n U(0)| \leq |u| |v| \beta^n. \]

Since \( U(0) = \text{vec}(uv^*) = v \otimes u \), and \( u \) and \( v \) are arbitrary vectors in \( \mathbb{C}^d \), it follows from (2.15) that
\[ |D_{A,B}^n| \leq a \beta^n \]
for a positive real number \( a \). Then the upper bound in (2.12) follows from (2.16) and (2.11).

To get the lower bound, take \( u = v \) with \( |u| = 1 \) so that \( x(n) = y(n) \) for all \( n \geq 0 \). Then (1.12) and (2.13) imply
\[ d^{-1/2} \gamma^n \leq |U(n)| \leq |D_{A,B}^n|, \]
and the lower bound in (2.12) follows from (2.11).
\[ \square \]
3. Continuous-Time Stochastic Kronecker Sum

Given matrices $A, B_1, \ldots, B_k \in \mathbb{C}^{d \times d}$, consider two $\mathbb{C}^d$-valued stochastic processes $x(t) = (x_1(t), \ldots, x_d(t))^\top$ and $y(t) = (y_1(t), \ldots, y_d(t))^\top$, $t \geq 0$, defined by the Itô integral equations

$$x(t) = u + \int_0^t A x(s) ds + \sum_{k=1}^m \int_0^t B_k x(s) dw_k(s),$$

$$y(t) = v + \int_0^t A y(s) ds + \sum_{k=1}^m \int_0^t B_k y(s) dw_k(s).$$

Both equations in (3.1) are driven by independent standard Brownian motions $w_1, \ldots, w_m$, and the initial conditions $u, v \in \mathbb{C}^d$ are non-random. Note that the processes $x(t)$ and $y(t)$ satisfy the same equation and differ only in the initial conditions. Existence and uniqueness of solution are well-known: [4, Theorem 5.2.1]. The terms $dw_k(t)$ can be considered continuous-time analogues of white noise input in (2.1). The term bilinear in connection with (3.1) reflects the fact that the noise process enters the system in a multiplicative, as opposed to additive, way.

The differential form

$$dx(t) = Ax(t) dt + \sum_{k=1}^m B_k x(t) dw_k(t), \quad dy(t) = Ay(t) dt + \sum_{k=1}^m B_k y(t) dw_k(t)$$

is a more compact, and less formal, way to write (3.1).

The peculiar behavior of white noise in continuous time, often written informally as $(dw(t))^2 = dt$, makes it necessary to modify the usual product rule for the derivatives. The result is known as the Itô formula; its one-dimensional version is presented below for the convenience of the reader.

**Proposition 3.1.** If $a, b, \sigma$, and $\mu$ are globally Lipschitz continuous functions and $f(0), g(0)$ are non-random, then

(a) there are unique continuous random processes $f$ and $g$ such that

$$f(t) = f(0) + \int_0^t a(f(s)) ds + \int_0^t \sigma(f(s)) dw(s),$$

$$g(t) = g(0) + \int_0^t b(g(s)) ds + \int_0^t \mu(g(s)) dw(s);$$

(b) the following equality holds:

$$\mathbb{E} f(t) g(t) = f(0) g(0) + \int_0^t \mathbb{E} (f(s) b(g(s)) + g(s) a(f(s)) + \sigma(f(s)) \mu(g(s))) ds. \quad (3.2)$$

**Proof.** In differential form,

$$d(fg) = fdg + gdf + \sigma \mu dt,$$

where the first two terms on the right come from the usual product rule and the third term, known as the Itô correction, is a consequence of $(dw(t))^2 = dt$. The expected
value of stochastic integrals is zero:

\[ E \int_0^t f(s)\mu(g(s))dw(s) = E \int_0^t g(s)\sigma(f(s))dw(s) = 0, \]

and then (3.2) follows. For more details, see, for example, [4, Chapter 4]. □

**Proposition 3.2.** Define \( V(t) = E\mathbf{x}(t)^*\mathbf{y}(t) \), the covariance matrix of the random vectors \( \mathbf{x}(t) \) and \( \mathbf{y}(t) \) from (3.1), and define \( r_x(t) = E\mathbf{x}(t)^*\mathbf{x}(t) \). Then the vector

\[ \mathbf{U}(t) = \text{vec}(V(t)) \]

satisfies

(3.3)

\[ \mathbf{U}(t) = e^{tC_{A,B}}\mathbf{U}(0), \]

with the matrix

(3.4)

\[ C_{A,B} = \overline{A} \otimes I + I \otimes A + \sum_{k=1}^m \overline{B}_k \otimes B_k, \]

and the number \( r_x(t) \) satisfies

(3.5)

\[ |\mathbf{u}|^2 e^{\gamma t} \leq r_x(t) \leq |\mathbf{u}|^2 e^{\beta t}, \]

where \( \gamma \) is the smallest eigenvalue and \( \beta \) is the largest eigenvalue of the Hermitian matrix

(3.6)

\[ M_{A,B} = A + A^* + \sum_{k=1}^m B_k^* B_k. \]

**Proof.** In differential form,

\[ d\mathbf{x}(t) = A\mathbf{x}(t)dt + \sum_{k=1}^m B_k\mathbf{x}(t)dw_k(t), \]

\[ d\mathbf{y}^*(t) = \mathbf{y}^*(t)A^*dt + \sum_{k=1}^m \mathbf{y}^*(t)B_k^*dw_k(t). \]

By the Itô formula,

\[ V(t) = V(0) + \int_0^t \left( AV(s) + V(s)A^* + \sum_{k=1}^m B_kV(s)B_k^* \right) ds, \]

and (3.3) follows from (1.3).

Similarly,

\[ r_x(t) = r_x(0) + \int_0^t \mathbb{E}\mathbf{x}(s)^*M_{A,B}\mathbf{x}(s)ds, \]

and then, for every real number \( a \),

\[ r_x(t) = r_x(0) + \int_0^t ar_x(s)ds + \int_0^t f_a(s)ds, \]

where

\[ f_a(s) = \int_0^s \mathbb{E}\left( \mathbf{x}^*(s)M_{A,B}\mathbf{x}(s) - a\mathbf{x}^*(s)\mathbf{x}(s) \right) ds. \]

In other words,

\[ r_x(t) = |\mathbf{u}|^2 e^{at} + \int_0^t e^{a(t-s)} f_a(s)ds. \]
If \( a = \gamma \) (the smallest eigenvalue of \( M_{A,B} \)), then \( f_a(s) \geq 0 \) and the lower bound in (3.5) follows; if \( a = \beta \) (the largest eigenvalue of \( M_{A,B} \)), then \( f_a(s) \leq 0 \), and the upper bound in (3.5) follows.

Given the origin of equation (3.3), the matrix \( C_{A,B} \) is natural to call the continuous-time stochastic Kronecker sum of the matrices \( A \) and \( B_k \).

For a square matrix \( A \), denote by \( \alpha \) the spectral abscissa of \( A \):

\[
\alpha(A) = \max \{ \Re \lambda(A) : \lambda(A) \text{ is an eigenvalue of } A \}.
\]

It is known [5, Theorem 15.3] that

\[
\alpha(A) = \lim_{t \to +\infty} \frac{1}{t} \ln |e^{tA}|.
\]

**Theorem 3.3.** For every matrices \( A, B_1, \ldots, B_m \in \mathbb{C}^{d \times d} \),

\[
\varrho(M_{A,B}) \leq \alpha(C_{A,B}) \leq \alpha(M_{A,B}).
\]

**Proof.** As in Proposition 3.2, we write \( \beta = \alpha(M_{A,B}), \ \gamma = \varrho(M_{A,B}) \). It follows from (3.3) that

\[
|U(t)| = |e^{tC_{A,B}} U(0)|.
\]

By (1.11),

\[
|U(t)| \leq \sqrt{r_x(t)r_y(t)},
\]

and then (3.5) implies

\[
|e^{tC_{A,B}} U(0)| \leq |u| |v| e^{\beta t}.
\]

Since \( U(0) = \text{vec}(uv^*) = v \otimes u \), and \( u \) and \( v \) are arbitrary vectors in \( \mathbb{C}^d \), it follows from (3.11) that

\[
|e^{tC_{A,B}}| \leq be^{\beta t}
\]

for a positive real number \( b \). Then the upper bound in (3.8) follows from (3.12) and (3.7).

To get the lower bound, take \( u = v \) with \( |u| = 1 \), so that \( x(t) = y(t) \) for all \( t \geq 0 \). Then (1.12) and (2.14) imply

\[
d^{-1/2}e^{\gamma t} \leq |U(n)| \leq |e^{tC_{A,B}}|,
\]

and the lower bound in (3.8) follows from (3.7).

\[\square\]

4. **Examples and Further Discussions**

Without additional information about the matrices \( A \) and \( B \), it is not possible to know how tight the bounds in (1.9) and (1.10) will be. As an example, consider two real matrices

\[
A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}.
\]
The corresponding stochastic systems are

\[ x_1(n+1) = ax_1(n), \quad x_2(n+1) = bx_2(n) + \sigma x_1(n)\xi_{n+1} \]

in discrete time, and \( dx_1(t) = ax_1(t)dt, \quad dx_2(t) = bx_2(t)dt + \sigma x_1(t)dw(t) \) in continuous time. Then

\[
D_{A,B} = A \otimes A + B \otimes B = \begin{pmatrix}
a^2 & 0 & 0 & 0 \\
0 & ab & 0 & 0 \\
0 & 0 & ab & 0 \\
\sigma^2 & 0 & 0 & b^2
\end{pmatrix},
\]

\[
N_{A,B} = A^\top A + B^\top B = \begin{pmatrix}
a^2 + \sigma^2 & 0 \\
0 & b^2
\end{pmatrix};
\]

\[
C_{A,B} = A \otimes I + I \otimes A + B \otimes B = \begin{pmatrix}
2a & 0 & 0 & 0 \\
0 & a + b & 0 & 0 \\
0 & 0 & a + b & 0 \\
\sigma^2 & 0 & 0 & 2b
\end{pmatrix},
\]

\[
M_{A,B} = A^\top + A + B^\top B = \begin{pmatrix}
2a + \sigma^2 & 0 \\
0 & 2b
\end{pmatrix}.
\]

In particular, both \( \rho(D_{A,B}) \) and \( \alpha(C_{A,B}) \) do not depend on \( \sigma \):

\[
\rho(D_{A,B}) = \max(a^2, b^2), \quad \alpha(C_{A,B}) = \max(2a, 2b),
\]

whereas

\[
\alpha(N_{A,B}) = \max(a^2 + \sigma^2, b^2) \quad \text{and} \quad \alpha(M_{A,B}) = \max(2a + \sigma^2, 2b)
\]

can be arbitrarily large.

An important question in the study of stochastic systems is whether the matrices \( D_{A,B} \) and \( C_{A,B} \) are stable, that is, \( \rho(D_{A,B}) < 1 \) and \( \alpha(C_{A,B}) < 0 \). One consequence of Propositions 2.1 and 3.2 is that stability of the stochastic Kronecker sum matrix is equivalent to the mean-square asymptotic stability of the corresponding stochastic system:

\[
\rho(D_{A,B}) < 1 \iff \lim_{n \to \infty} \mathbb{E}|x(n)|^2 = 0,
\]

\[
\alpha(C_{A,B}) < 1 \iff \lim_{t \to +\infty} \mathbb{E}|x(t)|^2 = 0.
\]

The example shows that it is possible to have this stability even when the matrices \( N_{A,B} \) and \( M_{A,B} \) are not stable: \( D_{A,B} \) is stable if (and only if) \( \max(|a|, |b|) < 1 \), and \( C_{A,B} \) is stable if (and only if) \( \max(a, b) < 0 \); this is also clear by looking directly at the corresponding stochastic system.

One can always use the lower bounds in (1.9) and (1.10) to check if the the matrices \( D_{A,B} \) and \( C_{A,B} \) (and hence the corresponding systems) are not stable. In the above example, if

\[
\rho(N_{A,B}) = \min(a^2 + \sigma^2, b^2) > 1,
\]

then \( |b| > 1 \) and \( D_{A,B} \) is certainly not stable; similarly, if

\[
\rho(M_{A,B}) = \min(2a + \sigma^2, 2b) > 0,
\]

then \( b > 0 \) and \( C_{A,B} \) is certainly not stable.
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