ON HIGH-GIRTH EXPANDER GRAPHS WITH LOCALIZED EIGENVECTORS

SHOHEI SATAKE

Abstract. The main purpose of this paper is to construct high-girth regular expander graphs with localized eigenvectors for general degrees, which is inspired by a recent work due to Alon, Ganguly and Srivastava (to appear in Israel J. Math.).

1. Introduction

For \( d \geq 2 \), let \( G \) be a \((d+1)\)-regular graph. Let \( V(G) \) and \( E(G) \) denote the vertex and edge set of \( G \), respectively. The girth of \( G \), the length of the shortest cycle of \( G \), is denoted by girth\((G)\). The adjacency matrix of \( G \), denoted by \( A(G) \), is a square 0-1 matrix with rows and columns indexed by vertices of \( G \) in which the \((u,v)\)-entry is 1 if and only if \( u \) and \( v \) are adjacent in \( G \). Throughout this paper, we consider normalized eigenvectors of \( A(G) \) with respect to \( l^2 \) norm.

Brooks and Lindenstrauss [3] proved that for any eigenvector \( v = (v_x)_{x \in V(G)} \) of \( A(G) \), real number \( \varepsilon \in (0,1) \) and subset \( S \subset V(G) \) such that \( \|v_S\|_2^2 := \sum_{x \in S} v_x^2 \geq \varepsilon \), it holds that

\[
|S| \geq \Omega_d(\varepsilon^2d^{2-7}\varepsilon^2 \text{girth}(G)) \quad (|V(G)| \to \infty).
\]

In other words, if \( A(G) \) has a \((k, \varepsilon)\)-localized eigenvector which is an eigenvector such that there exists \( S \subset V(G) \) with \( |S| = k \) and \( \|v_S\|_2^2 \geq \varepsilon \), then it must hold that \( k \geq \Omega_d(\varepsilon^2d^{2-7}\varepsilon^2 \text{girth}(G)) \). In [8], Ganguly and Srivastava improved (1.1), that is, under the same assumption, they proved the following inequality.

\[
|S| \geq \frac{\varepsilon d^{1+\varepsilon^2 \text{girth}(G)}}{2d^2}.
\]

Moreover, for arbitrary \( \varepsilon > 0 \) and \( d \geq 2 \), the authors of [8] also showed the existence of infinitely many positive integers \( m \) and \((d+1)\)-regular graphs \( G_m \) with \( m \) vertices such that girth\((G_m) \geq (1/8) \log_d(m) \) and \( A(G_m) \) has a \((k, \varepsilon)\)-localized eigenvectors with \( k = O(d^{4\varepsilon \text{girth}(G_m)}) \) as \( m \to \infty \), which shows that the bound [11] is sharp up to the constant \( \varepsilon/d^2 \).

While the above construction is based on a probabilistic method, Alon, Ganguly and Srivastava [1] gave a deterministic construction of \((d+1)\)-regular graphs with large girth and localized eigenvectors for every odd prime \( d \), improving the lower bound of girth as well as the upper bound of \( k \) in the construction in [8].
Theorem 1 ([1]). Let $d$ be an odd prime and $\alpha \in (0, 1/6)$. Then for every $\varepsilon \in (0, 1)$, there exist infinitely many positive integers $m$ and $(d+1)$-regular graphs $G_m$ with $m$ vertices satisfying the following conditions for each $m$.

1. $\text{girth}(G_m) \geq 2\alpha \log_{2d-1}(m) \geq 2\alpha \log_d(m) \cdot (1 - \frac{\log 2}{\log(2d-1)})$;

2. there exist $|\alpha \log_d(m)|$ eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{|\alpha \log_d(m)|}$ of $A(G_m)$ such that each has a corresponding $(k, \varepsilon)$-localized eigenvectors with $k = O(m^\alpha)$ as $m \to \infty$.

The main purpose of this paper is to extend Theorem 1 to more general degrees. The following is the first main theorem in this paper.

Theorem 2. Let $p_j$ denote the $j$-th prime. Let $d \in [p_j, p_{j+1}]$ be an integer and $\alpha \in (0, 1/6)$. Suppose that $\frac{\log(p_j)}{\log(p_{j+1})} > 6\alpha$. Then for every $\varepsilon \in (0, 1)$, there exist infinitely many positive integers $m$ and $(d+1)$-regular graphs $G_m$ with $m$ vertices satisfying the same conditions as Theorem 1. In particular, the same claim holds for every $d \geq p_{10} = 29$ and $\alpha \in (0, \frac{\log(p_{10})}{6\log(p_{11})})$ where $\frac{\log(p_{10})}{6\log(p_{11})} \approx 0.163$. 

Remark 3. For all $d \geq 29$, the last claim of Theorem 2 provides infinitely many positive integers $m$ and $(d+1)$-regular graphs $G_m$ with $m$ vertices such that $\text{girth}(G_m) > 0.272 \cdot \log_d(m)$, still improving the lower bound of the girth in the construction provided in [8], while $\alpha$ must be less than 0.163. $< 0.166 \ldots = 1/6$ here.

Remarkably, the authors of [1] also proved that for each $m$ and a $(d+1)$-regular graph $G_m$ in Theorem 1, $\lambda(G_m)$, the second largest eigenvalue of $A(G_m)$ in absolute value, is less than $\frac{1}{\sqrt{2}} \sqrt{d} \approx 2.121 \sqrt{d}$. This means that Theorem 1 also provides near-Ramanujan graphs, that is, regular graphs with near-optimal spectral gap; for details of spectral gaps and Ramanujan graphs, see e.g. [6, 12] and references therein.

The following second main theorem extends Theorem 1 to almost all degrees (see Remark 10), holding the same bound of the second largest eigenvalue.

Theorem 4. Let $d \geq 2$ be an integer and $\alpha \in (0, 1/6)$. Suppose $d \in [t, p]$ for some odd prime $p$ and positive integer $t < p$ such that $p - t < (\frac{p}{2\alpha^{\alpha}} - 1)^{\sqrt{t}} \approx 0.02 \sqrt{t}$ and $\frac{\log(t)}{\log(p)} > 6\alpha$. Then, for every $\varepsilon \in (0, 1)$, there exist infinitely many positive integers $m$ and $(d+1)$-regular graphs $G_m$ with $m$ vertices satisfying the same conditions as Theorem 1. Moreover $\lambda(G_m) \leq \frac{1}{\sqrt{2}} \sqrt{d}$ holds for each $m$.

The rest of this paper is organized as follows. Section 2 gives a deterministic construction of expander regular graphs with large girth for almost all degrees, which is based on the construction due to Cioabă and Murty [4]. Section 3 proves Theorems 2 and 4. In Section 4, some concluding remarks are given.

2. A CONSTRUCTION OF EXPANDER REGULAR GRAPHS

This section introduces a construction of expander regular graphs based on the idea due to Cioabă and Murty [4]. The construction is based on the following expander regular graphs constructed by Lubotzky, Phillips and Sarnak [11].

Theorem 5 ([11]). For each odd prime $p$, there exist infinitely many positive integers $n$ and a non-bipartite $(p+1)$-regular graph $R_n$ with $n$ vertices satisfying the following conditions for each $n$. 


(1) $n$ is an even integer;
(2) $\lambda(R_n) \leq 2\sqrt{p}$;
(3) $\text{girth}(R_n) \geq \frac{3}{2} \log_p(n)$.

For our purpose, it is necessary to construct expander regular graphs with more general degrees. For each odd prime $p$, take a $(p + 1)$-regular graph $R_n$ with $n$ vertices in Theorem 5. The crucial idea is to delete a 1-factor, a 1-regular spanning subgraph, from $R_n$ to obtain a new $p$-regular graph $R'_n$ with the same vertices. Note that the existence of a 1-factor of $R_n$ is confirmed by Theorem 5 and the following theorem due to Cioab˘ a, Gregory and Haemers [5].

**Theorem 6 ([5]).** Let $H$ be a $(d + 1)$-regular graph. Suppose that $|V(H)|$ is even. Let $\lambda_3(H)$ denote the third largest eigenvalue of $A(H)$ in absolute value. Then $H$ has a 1-factor if

$$
\lambda_3(H) \leq \begin{cases} 
2.85577 & \text{if } d = 2; \\
\frac{d-1+\sqrt{(d+1)^2+12}}{2} & \text{if } d \text{ is odd;} \\
\frac{d-2+\sqrt{(d+2)^2+16}}{2} & \text{if } d \text{ is even.}
\end{cases}
$$

It follows from the construction of $R'_n$ that

$$
\lambda(R'_n) \leq 2\sqrt{p} + 1,
$$
where this follows from the Weyl’s inequality (e.g [4], [10]). Also it follows from the construction of $R'_n$ that

$$
\text{girth}(R'_n) \geq \frac{2}{3} \log_p(n) = \frac{2}{3} \frac{\log(p-1)}{\log(p)} \cdot \log_{p-1}(n).
$$

The following propositions play an important role in the next section.

**Proposition 7.** For each integer $d \in [p_j, p_{j+1}]$ with $p_{j+1} - p_j < (1/5) \cdot p_j$, there exist infinitely positive integers $m$ and a $(d + 1)$-regular graph $H_n$ with $n$ vertices such that

$$
\lambda(H_n) \leq \frac{2}{5} d + 2\sqrt{d},
$$

$$
\text{girth}(H_n) \geq \frac{2}{3} \frac{\log(d)}{\log(p_{j+1})} \cdot \log_d(n) > \frac{2}{3} \frac{\log(p_j)}{\log(p_{j+1})} \cdot \log_d(n).
$$

**Proof.** Let $p = p_{j+1}$ and take a $(p + 1)$-regular graph $R_n$ in Theorem 5. Let $d \in [p_j, p_{j+1}]$. It suffices to prove that the deletion of $p_{j+1} - d$ 1-factors from $R$ generates a $(d + 1)$-regular graph $H_n$ satisfying the conditions of the proposition. For $d' \in (d, p_{j+1}]$, suppose the existence of $p_{j+1} - d'$ 1-factors of $R_n$ and let $H'_n$ be the $(d' + 1)$-regular graph obtained by deleting $p_{j+1} - d'$ 1-factors from $R_n$. Then,
by (2.2),
\[
\lambda(H'_n) \leq 2\sqrt{p_j + p_j - d'} \\
= 2\sqrt{d'' + (p_{j+1} - d')} + p_{j+1} - d' \\
\leq 2\sqrt{d''} + 2(p_{j+1} - p_j) \\
\leq 2\sqrt{d''} + \frac{2}{5}p_j \\
\leq 2\sqrt{d''} + \frac{2}{5}d',
\]
where the third, fourth and fifth inequalities follow from the assumptions of \(d', \ d\) and \(p_j\). By Theorem \(6\), \(H'_n\) still has a 1-factor. Thus the induction verifies the existence of \(p_j+1-d\) 1-factors of \(R_n\). Since (2.4) follows from the above discussion, it suffices to prove (2.5), which is directly obtained by (2.3). \(\blacksquare\)

**Remark 8.** By the main theorem in \([13]\), it holds that \(p_{j+1} - p_j < (1/5) \cdot p_j\) whenever \(j \geq 10\), where \(p_{10} = 29\).

**Proposition 9.** Let \(p\) be an odd prime and \(t \geq 2\) an integer with \(0 < p - t < (\frac{5}{2\sqrt{6}} - 1)\sqrt{t}\). Then for each integer \(d \in [t, p]\), there exist infinitely many positive integers \(n\) and a \((d + 1)\)-regular graph \(H_n\) with \(n\) vertices such that

\[(2.6) \quad \lambda(H_n) \leq \frac{5}{\sqrt{6}}\sqrt{d},\]

\[(2.7) \quad \text{girth}(H_n) \geq \frac{2}{3} \cdot \frac{\log d}{\log p} \cdot \log_d(n) > \frac{2}{3} \cdot \frac{\log t}{\log p} \cdot \log_d(n).\]

**Proof.** As in Proposition \(7\) take a \((p + 1)\)-regular graph \(R_n\) in Theorem \(5\). The existence of \(p - d\) 1-factors of \(R_n\) follows from the same discussion in the proof of Proposition \(7\). Since (2.7) directly follows from (2.3), it suffices to prove (2.6), which is verified by (2.2) and since it holds that

\[
\lambda(H_n) \leq 2\sqrt{p} + p - d \\
= 2\sqrt{d} + (p - d') + p - d' \\
\leq 2\sqrt{d} + 2(p - t) \\
\leq 2\sqrt{d} + 2 \cdot \left(\frac{5}{2\sqrt{6}} - 1\right)\sqrt{d} \\
\leq 2\sqrt{d} + \left(\frac{5}{\sqrt{6}} - 2\right)\sqrt{d} = \frac{5}{\sqrt{6}}\sqrt{d},
\]

where the third, fourth and fifth inequalities follow from the assumptions of \(d, p\) and \(t\). \(\blacksquare\)

**Remark 10.** We remark that Proposition \(8\) provides \((d + 1)\)-regular graphs for almost all positive integers \(d\). Notice that if \(p_{j+1} - p_j < (\frac{5}{2\sqrt{6}} - 1)\sqrt{p_j}\), Proposition \(8\) generates \((d + 1)\)-regular graphs for all \(d \in [p_j, p_{j+1}]\). In number theory, it is known (\([9]\)) that the number of primes \(p \leq x\) such that \(p = p_j\) with \(p_{j+1} - p_j > \sqrt{p_j}\) is at most \(C_\delta x^{5/6+\delta}\) as \(x \to \infty\) for any \(\delta > 0\), where \(C_\delta > 0\) depends only \(\delta\). This implies the desired claim in this remark.
3. Proofs of Theorems \(2\) and \(3\)

The main task here is to construct \((d + 1)\)-regular graphs with the conditions of Theorems \(2\) and \(3\) by using Propositions \(7\) and \(9\) respectively.

The construction follows from Alon, Ganguly and Srivastava \(1\).

**Construction 1.** Fix \(\alpha \in (0, 1/6)\). Suppose we have a \((d + 1)\)-regular graph \(H = H_n\) with \(n\) vertices provided in Proposition \(7\) or \(9\). Let \(r = \lfloor \alpha \log_d(n) \rfloor\). Now construct a \((d + 1)\)-regular graph \(G\) from \(H\) by using a \(d\)-ary tree of depth \(r\), a finite tree such that all vertices except for leaves have degree \(d + 1\) and every leaf is at distance \(r\) from the root. First choose a vertex \(u \in V(H)\) and take all vertices of \(H\) which are at distance at most \(r\). By the definition of \(r\), these vertices induce a \(d\)-ary tree \(T_1\) of depth \(r\) with root \(u\). Let \(L_1\) denote the set of leaves of \(T_1\) and \(V_1 = V(T_1) \setminus L_1\). Take a set of \(|L_1|\) vertices, denoted by \(L_2\), which are at distance \(r+1\) such that there exists a perfect matching \(M\) between \(L_1\) and \(L_2\). Then remove the matching \(M\) from \(H\), and add new \(d\)-ary tree \(T_2\) of depth \(r\) to \(H\) so that the set of leaves of \(T_2\) coincides \(L_1\) and other vertices are taken as new vertices outside of \(V(H)\). Here suppose that the graph \(G_{1,2}\) on \(V(T_1) \cup V(T_2)\) satisfies that
\[
girth(G_{1,2}) \geq 2 \log_{2d-1}((d + 1)d^{r-1}),
\]
which is possible by Lemma 2.1 in \(1\). Finally add another \(d\)-ary tree \(T_3\) of depth \(r\) to \(H\) so that the set of leaves of \(T_3\) coincides \(L_2\) and other vertices are taken as new vertices outside of \(V(H) \cup V(T_2)\). The resulting \((d + 1)\)-regular graph is the desired graph \(G\).

Note that it holds by the definition of \(G\) that
\[
m := |V(G)| = n + 2 \left\{ 1 + \sum_{1 \leq l \leq r-1} (d + 1)d^{l-1} \right\} = n + O(n^\alpha).
\]

To prove Theorems \(2\) and \(3\) we have to show that the above construction provides \((d + 1)\)-regular graphs satisfying the conditions of the theorems.

First we evaluate the girth of \(G\).

**Lemma 11.** Suppose that \(d \in [p_j, p_{j+1}]\) with \(\frac{\log(p_j)}{\log(p_{j+1})} > 6\alpha\). Then, if \(n\) is sufficiently large, it holds that
\[
girth(G) \geq 2 \log_{2d-1}((d + 1)d^{r-1}) = 2 \alpha \log_{2d-1}\left(\left(1 + \frac{1}{d}\right)n\right).
\]

In particular, \(girth(G) \geq 2 \alpha \log_{2d-1}(m)\) if \(m\) is sufficiently large.

**Proof.** By the definition of \(G\), in the graph \(H \setminus V_1\), the distance between two distinct vertices of \(L_1\) is greater than \(2r\) since \(girth(H) > 4r\) which follows from the assumption of \(\alpha\) and \(r\). The same claim also holds for each pair of two distinct vertices of \(L_2\) in the graph \(H \setminus V_1\). The lemma is proved by \(2.7\), \(3.1\) and since it holds that \(\frac{\log(p_j)}{\log(p_{j+1})} \cdot \log_d(n) > 4r\), which follows from the assumption that \(\frac{\log(p_j)}{\log(p_{j+1})} > 6\alpha\). The last claim is directly obtained by \(3.2\).

Secondly, the following lemma proves the claim about localized eigenvectors.

**Lemma 12.** For each \(\varepsilon \in (0, 1)\), Construction \(7\) provides \((d + 1)\)-regular graphs \(G\) with \(m\) vertices such that there exist \(\lfloor \alpha \log_d(m) \rfloor\) eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_{\lfloor \alpha \log_d(m) \rfloor}\) of \(A(G)\) such that each has a \((k, \varepsilon)\)-localized eigenvector with \(k = O(m^\alpha)\) as \(m \to \infty\).
Proof. The lemma follows from Lemmas 3.1, 3.2 and 3.3 in [8] and Lemma 3.5 in [1]. □

Now we are ready to prove Theorem 2.

Proof of Theorem 2. The proof is completed by applying Construction 1 to a \((d+1)\)-regular graph \(H_n\) obtained from Proposition 7, and using Lemmas 11 and 12. The last claim follows from Remark 8. □

The remained task is to prove Theorem 4. Now choose parameters \(p, t\) and \(d\) satisfying the assumption of Proposition 9. Let \(H_n\) be a \((d+1)\)-regular graph with \(n\) vertices in Proposition 9 and \(G\) a \((d+1)\)-regular graph with \(m\) vertices obtained from \(H_n\) by Construction 1.

To prove Theorem 4, we have to evaluate \(\lambda(G)\).

Lemma 13. For every fixed \(\beta > 0\), if \(m\) is sufficiently large, then it holds that

\[
\lambda(G) \leq \left( \frac{3d-1}{\sqrt{d^2 - 1}} + \beta \right) \sqrt{d}.
\]

In particular, \(\lambda(G) \leq \frac{3}{\sqrt{2}} \sqrt{d}\) if \(m\) is sufficiently large.

We apply the following two lemmas proved in [1] to prove Lemma 13.

Lemma 14 ([1]). Let \(\mu\) be a non-trivial eigenvalue, an eigenvalue distinct to the top eigenvalue \(d+1\), of \(A(G)\). Let \(g = (g_v)_{v \in V(G)}\) be a normalized eigenvector corresponding to \(\mu\). Then it holds that

\[
|\mu| = |g^T A(G) g| \leq \lambda(H) + (\sqrt{d} + 1) \sum_{u \in L_1 \cup L_2} g_u^2 + \frac{2(d+1)|L_1|}{n}.
\]

Lemma 15 ([1]). For any \(\beta > 0\), suppose that \(|\mu| \geq (b + \beta)\) where \(b = b(d) = \frac{3d-1}{\sqrt{d^2 - 1}}\). Then

\[
\sum_{u \in L_1 \cup L_2} g_u^2 \to 0 \quad (m \to \infty).
\]

Proof of Lemma 13. Let \(\mu\) be a non-trivial eigenvalue of \(A(G)\). Suppose that \(|\mu| \geq (b + \beta)\). By (2.6) and Lemma 14 it holds that

\[
|\mu| \leq \frac{5}{\sqrt{6}} \sqrt{d} + (\sqrt{d} + 1) \sum_{u \in L_1 \cup L_2} g_u^2 + \frac{2(d+1)|L_1|}{n}.
\]

Notice that \(|L_1| = (d+1)d^{\alpha-1} = O(n^\alpha)\), and, by (3.2), \(m \to \infty\) if and only if \(n \to \infty\). Thus, by (3.7) and Lemma 15 we have \(|\mu| \leq \frac{5}{\sqrt{6}} \sqrt{d} + o(1)\) when \(m \to \infty\), which contradicts the assumption that \(|\mu| \geq (b + \beta)\) for sufficiently large \(m\), since \(\frac{5}{\sqrt{6}} = b(2) < b + \beta\) for every \(d \geq 2\). □

Proof of Theorem 4. Note that there exists \(p(\alpha)\) such that for any \(p > p(\alpha)\) and \(t < p\) with \(p - t < \left( \frac{5}{2\sqrt{6}} - 1 \right) \sqrt{t}\), the condition \(\frac{\log(G)}{\log(p)} > 6\alpha\) holds. Thus, by Proposition 9 and Construction 1, the theorem follows from Lemmas 11, 12 and 13. □
4. Concluding remarks

- In analytic number theory, it is conjectured that the order of the magnitude of the gap $p_{j+1} - p_j$ can be bounded by a polylogarithmic function of $p_j$ for any sufficiently large $j$. For example, a famous conjecture due to Cramér predicts that $p_{j+1} - p_j = O((\log p_j)^2)$; for details, see e.g. [7]. If this is true, by using Construction 1, one can prove that there exists $d_0$ such that for every $d > d_0$ there exist infinitely many positive integers $m$ and $(d+1)$-regular graphs with $m$ vertices satisfying the all conditions of Theorem 4.

- From the viewpoint of the study of expander graphs, it would be interesting to improve the bound of $\lambda(G)$. Here we remark that by the same reason noted in Section 1.2 and Remark 4.1 in [1], the constructed graph $G$ cannot be a Ramanujan graph which has the optimal second largest eigenvalue.

Acknowledgement

We appreciate Masato Mimura and Hyungrok Jo for their helpful comments. S. Satake has been supported by Grant-in-Aid for JSPS Fellows 20J00469 of the Japan Society for the Promotion of Science.

References

[1] N. Alon, S. Ganguly, N. Srivastava, High-girth near-Ramanujan graphs with localized eigenvectors, To appear in Israel J. Math. arXiv:1908.03694.
[2] R. C. Baker, G. Harman, J. Pintz, The difference between consecutive primes. II, Proc. London Math. Soc., 83 (2001), 532–562.
[3] S. Brooks, E. Lindenstrauss, Non-localization of eigenfunctions on large regular graphs, Israel J. Math., 193 (2013), 1–14.
[4] S. M. Cioabă, M. R. Murty, Expander graphs and gaps between primes, Forum Math., 20 (2008), 745–756.
[5] S. M. Cioabă, D. A. Gregory, W. H. Haemers, Matchings in regular graphs from eigenvalues, J. Combin. Theory Ser. B, 99 (2009), 287–297.
[6] G. Davidoff, P. Sarnak, A. Valette, Elementary Number Theory, Group theory, and Ramanujan Graphs, Cambridge University Press, 2003.
[7] R. Guy, Unsolved Problems in Number Theory, Third edition, Springer, 2004.
[8] S. Ganguly, N. Srivastava, On non-localization of eigenvectors of high girth graphs, Int. Math. Res. Not., available online. doi:10.1093/imrn/rna008.
[9] R. Heath-Brown, The differences between consecutive primes, V, Int. Math. Res. Not., available online, doi:10.1093/imrn/rna295.
[10] R. A. Horn, C. R. Johnson, Matrix Analysis, Cambridge University Press, 1990.
[11] A. Lubotzky, R. Phillips, P. Sarnak, Ramanujan Graphs, Combinatorica, 8 (1988), 261–277.
[12] M. R. Murty, Ramanujan graphs, J. Ramanujan Math. Soc., 18 (2001), 1–20.
[13] J. Nagura, On the interval containing at least one prime number, Proc. Japan Acad., 28 (1952), 177–181.

Faculty of Advanced Science and Technology, Kumamoto University, 2-39-1, Kurokami, Chu, Kumamoto, 860-8555, Japan
Email address: satakecomb@gmail.com