The real non-attractive fixed point conjecture and beyond

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Abstract

Multipliers are the local character of any rational map. For simple fixed point, it behaves like a linear map, according to value of multiplier it contracts, rotates, stretches for the sufficiently small neighborhood, for multiple fixed points it behaves like a monomial for a sufficiently small neighborhood of that fixed points. In this paper we are defining family of polynomial have a unique distribution and although how multipliers are globally distributed in complex plane.

1 Introduction

Throughout this article, \( \hat{\mathbb{C}} \) denotes the Riemann sphere \( \mathbb{C} \cup \{ \infty \} \). A map \( R: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is called rational, if \( R(z) = \frac{P(z)}{Q(z)} \), where \( P \) and \( Q \) are polynomials with complex coefficients and with no common factor. Degree of \( R \) is defined as the maximum of the degrees of \( P \) and \( Q \). Each rational map is analytic in \( \hat{\mathbb{C}} \). It is important to note that every analytic function from \( \hat{\mathbb{C}} \) onto \( \hat{\mathbb{C}} \) is a rational map \[2\]. Here the notion of analyticity at \( \infty \) is as follows. If \( f(\infty) = \infty \), we say \( f \) is analytic at \( \infty \) if \( \frac{1}{f(z)} \) is analytic at 0. If \( f(\infty) \in \mathbb{C} \), we say \( f \) is analytic at \( \infty \) if \( f(\frac{1}{z}) \) is analytic at 0. If \( f(z_0) = \infty \) for some \( z_0 \in \mathbb{C} \), we say \( f \) is analytic at \( z_0 \) if \( \frac{1}{f(z)} \) is analytic at \( z_0 \). A rational map is a polynomial if and only if it has a single pole and that is at infinity. Fixed points of polynomials and the distribution of their multipliers in the plane are the object of our interest.

A point \( z_0 \), is called a periodic point of a rational function \( R \), if \( R^p(z_0) = z_0 \) for some \( p \geq 1 \). The smallest such \( p \) is called the period of \( z_0 \). The number \( \lambda = (R^p)'(z_0) \) is called the multiplier of \( z_0 \). It is this \( \lambda \) that usually controls the local iterative behaviour of \( R \) at \( z_0 \). If \( z_0 \) is a periodic point of \( R \) of period \( p \) then \( z_0 \) is a fixed point of \( R^p \). A periodic point of period 1 is called a fixed point. If \( R(\infty) = \infty \) then the multiplier of \( \infty \) is defined as \( h'(0) \) where \( h(z) = \frac{1}{R(z)} \). Fixed points are classified according to the value of their multipliers. A fixed point \( z_0 \) is called attracting if \( |\lambda| < 1 \) (super-attracting if \( \lambda = 0 \)). It is called

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indifferent if $|\lambda| = 1$. Further, it is called rationally indifferent if $\lambda^m = 1$ for some $m \in \mathbb{N}$ and irrationally indifferent if $\lambda = e^{2\pi i \alpha}, \alpha \in \mathbb{Q}$. If $|\lambda| > 1$ then the fixed point $z_0$ is called repelling. A fixed point whose multiplier is 1 or with modulus bigger than 1 is called weakly repelling. It is known that every rational map has at least one weakly repelling fixed point (See Corollary 2.1). Weakly repelling fixed points are important in the iteration theory of rational maps. More details can be found in [1, 5]. This article is concerned with a special type of weakly repelling fixed points, namely those the real part of whose multiplier is bigger than or equal to 1.

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ where $a_i \in \mathbb{C}$ for all $i$, $a_n \neq 0$ and $n \geq 2$. Then $P(\infty) = \infty$ and the multiplier of $\infty$ is given by $h'(0)$ where $h(z) = \frac{1}{P(\frac{1}{z})} = \frac{z^n}{a_n + a_{n-1}z + \cdots + a_0 z^n}$. Since $h'(0) = 0$, $\infty$ is a super-attracting fixed point for $P$. Finite fixed points of a polynomial are our concern. This article is motivated by the following question raised by Coehlo et al. in [3].

Every polynomial with degree at least two has a fixed point, the real part of whose multiplier is bigger than or equal to 1.

The authors refer it as the real non-attractive fixed point conjecture. They have settled it affirmatively for quadratic and cubic polynomials. We have observed that the conjecture is true for all higher degree polynomials.

The assumption that the degree of the polynomial is at least two is essential. In fact, the conjecture is true for an affine map $z \mapsto az + b$ if and only if $|a - \frac{1}{2}| \leq \frac{1}{2}$ or $\Re(a) \geq 1$. To see it, note that $\infty$ is the only fixed point when $a = 1$ and its multiplier is 1. For $a \neq 1$, the fixed points are $\infty$ and $\frac{b}{1-a}$ with multipliers $\frac{1}{a}$ and $a$ respectively.

What is needed to prove the above conjecture, is the Rational fixed point theorem. This is a well-known result. For completeness we include an elaborated proof and a detailed discussion following Minor [4]. Section 2 discusses the Rational fixed point theorem with special emphasis on polynomials. The real non-attractive fixed point conjecture is shown to be true in this section. This conjecture is shown to be true for all rational maps with at least one super-attracting fixed point. For rational maps without any super-attracting fixed point, the conjecture is sometimes, but not always true. Both types of examples are given. It also follows that every polynomial has a multiple fixed point or has a fixed point whose multiplier is non-negative. This article also deals with other issues on distribution of multipliers. The extreme situation when all multipliers have real part 1 is considered. All quadratic and cubic polynomials, all multipliers (of fixed points) of which have real part 1 are characterized in
Theorem 3.1 and Theorem 3.2 respectively. Theorem 3.3 and 3.4 giving a necessary and sufficient condition for all multipliers to be equidistant from 1 are proved for polynomials with degree upto four. Finally some further directions of investigation are suggested. By a fixed point of a polynomial, we mean a finite fixed point of the polynomial throughout the article.

2 Rational fixed point theorem

A fixed point of a rational map $R$ is a root of $R(z) - z$. The local degree of this root is important in our deliberations.

**Definition 2.1.** Let $R: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and $R(\hat{z}) = \hat{z}$ and the Taylor series of $R(z) - z$ about $\hat{z}$ be $a_m(z - \hat{z})^m + a_{m+1}(z - \hat{z})^{m+1} + \cdots$ with $a_m \neq 0$. Then the unique natural number $m$ is called the multiplicity of $\hat{z}$. If $m = 1$, $\hat{z}$ is called a simple fixed point of $R$. Otherwise $\hat{z}$ is called multiple.

The multiplicity of a fixed point $\hat{z}$ of $R$ is the order of $\hat{z}$ considered as a root of $R(z) - z$.

Following lemma taken from [4] reveals the relation between the multiplier and the multiplicity of a fixed point.

**Lemma 1.** A fixed point is multiple if and only if its multiplier is 1.

**Proof.** Let $R(\hat{z}) = \hat{z}$ and $R(z) - z = a_m(z - \hat{z})^m + a_{m+1}(z - \hat{z})^{m+1} + \cdots$. Differentiating both sides with respect to $z$, we get $R'(z) - 1 = ma_m(z - \hat{z})^{m-1} + (m - 1)a_{m-1}(z - \hat{z})^{m-2} + \cdots$. Therefore $R'(\hat{z}) = 1$ if and only if $m \geq 2$.

Attracting, repelling and irrationally indifferent fixed points are always simple. But a rationally indifferent fixed point is simple only when the multiplier is different from 1. For example, 0 is a rationally indifferent fixed point of $P(z) = iz + z^2$ which is simple.

In order to understand all the fixed points together, it is necessary to know the number of fixed points of a rational map. Following lemma [4] reveals exactly that.

**Lemma 2.** If $R$ is a rational map of degree $d \geq 1$, then it has $d + 1$ fixed points counting multiplicity.

**Proof.** Suppose that $R = \frac{P}{Q}$ is a rational map with degree $d \geq 1$, then the fixed points of $R$ are precisely the roots of the equation $P(z) - zQ(z) = 0$. 

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Suppose that the degree of $P$ is greater than that of $Q$. Then $R(\infty) = \infty$, and the degree of $P$ is $d$. Since $P(z) - zQ(z)$ is a polynomial of degree $d$, it has $d$ many finite roots which are fixed points of $R$ counting multiplicity.

Suppose that the degree of $P$ is less than or equal to that of $Q$. Then $P(z) - zQ(z) = 0$ is a polynomial of degree $d + 1$, and it has $d + 1$ many finite roots counting multiplicities. Note that $\infty$ is not a fixed point of $R$ in this case. This implies that $R$ has $d + 1$ fixed points counting multiplicity.

There is an important quantity associated with a fixed point of a rational map.

**Definition 2.2.** Given a rational map $R$, the residue fixed point index $\iota(R, \hat{z})$ of a fixed point $\hat{z}$ of $R$, is defined to be the complex number

$$\frac{1}{2\pi i} \oint \frac{dz}{z - R(z)},$$

where the integration is over a small simple positively oriented loop around $\hat{z}$ such that it does not enclose any other fixed point of $R$.

The word “small simple positively oriented loop” in the above definition can be any circle centered at $\hat{z}$ with radius $0 < r < \min_{R(z) = \hat{z}} |z - \hat{z}|$. For any such $r$, the integral has the same value by the Cauchy theorem. In fact, $\hat{z}$ is a pole of $\frac{1}{z - R(z)}$ and $\iota(R, \hat{z})$ is its residue.

Note that the analytic bijections of the Riemann sphere are precisely the mobius maps $z \mapsto \frac{az+b}{cz+d}, ad - bc \neq 0$. These are important for our purpose. Two rational maps $R$ and $S$ are called conformally conjugate if there is a mobius map $g$ such that $S(z) = g(R(g^{-1})(z))$ for all $z \in \hat{C}$. Important normalizations are to be realized via this conjugacy. One such is used in the proof of following lemma giving a formula for residue index [4].

**Lemma 3.** If the multiplier $\lambda$ of a fixed point is not equal to 1 then its residue fixed point index is given by $\frac{1}{1-\lambda}$.

**Proof.** Without loss of generality, assume that 0 is a fixed point of a rational map $R$ with multiplier $\lambda$. If $R(\hat{z}) = \hat{z}$ and $\lambda = R'(\hat{z})$ then consider $R(z + \hat{z}) - \hat{z}$ and note that this fixes origin and the multiplier is $\lambda$.) Expanding $R$ as its Taylor series around 0, we get $R(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$. Since $\lambda \neq 1$, $z - R(z) = (1 - \lambda) z - a_2 z^2 - a_3 z^3 + \cdots$. This gives that $\frac{1}{z - R(z)} = \frac{1}{(1 - \lambda) z (b_1 + b_2 z + b_3 z^2 + \cdots)}$ for some $b_1, b_2, b_3, \cdots$. Consequently, $\frac{1}{z - R(z)} = \frac{1 + O(\infty)}{(1-\lambda)z}$ where $O(\infty)$ is a bounded analytic function in a neighborhood of the origin. It gives that $\iota(R, 0) = \frac{1}{2\pi i} \oint \left(\frac{1}{(1-\lambda)z} + \frac{O(z)}{(1-\lambda)z}\right)dz$, where the integration is over a small positively oriented
circle about the origin. The residue index is nothing but \( \frac{1}{1-\lambda} \) as the second term is 0 by the Cauchy theorem.

**Remark 1.** For \( \lambda = 1 \), the residue index \( \iota(f, \hat{z}) \) is still well defined and is finite.

Following lemma proves that the residue index is invariant under conformal conjugacy.

**Lemma 4.** Let \( R \) and \( S \) be conformally conjugate, i.e., for a mobius map \( g \), \( S(z) = g(R(g^{-1}))(z) \) for all \( z \in \hat{\mathbb{C}} \). If \( R(\hat{z}) = \hat{z} \) then \( S(g(\hat{z})) = g(\hat{z}) \) and, \( \iota(R, \hat{z}) = \iota(S, g(\hat{z})) \).

**Proof.** Note that \( S(g(\hat{z})) = g(\hat{z}) \) if and only if \( R(\hat{z}) = \hat{z} \). Further, \( R'(\hat{z}) = S'(g(\hat{z})) \). If \( \hat{z} \) is simple (i.e., the multiplier \( \lambda = R'(\hat{z}) \) is different from 1), then the residue index \( \iota(R, \hat{z}) \) is \( \frac{1}{1-\lambda} \), which is nothing but \( \iota(S, g(\hat{z})) \). Let \( \hat{z} \) be a multiple fixed point of \( R \). Then \( R'(\hat{z}) = 1 \) and, therefore \( S'(\hat{w}) = 1 \) where \( g(\hat{z}) = \hat{w} \). Consider a perturbed map \( S_\alpha(z) = S(z) + \alpha \). Also, consider \( D_\epsilon = \{ z : 0 < |z - \hat{w}| < \epsilon \} \) such that \( S'(w) \neq 1 \) for any \( w \in D_\epsilon \) and \( S_\alpha \) has no fixed point on the boundary of \( D_\epsilon \). This is possible as the zeros of a non-constant analytic function are isolated. Further, by decreasing \( \epsilon \) if necessary so that \( g^{-1} \) has no pole on the boundary of \( D_\epsilon \), we can assume that the boundary of \( g^{-1}(D_\epsilon) \) is a circle. All the fixed points \( a_1, a_2, \ldots, a_m \) of \( S_\alpha \) lying in \( D_\epsilon \) are simple and, \( \sum_{i=1}^{m} \iota(S_\alpha, a_i) = \oint_{\partial D_\epsilon} \frac{dz}{z - S_\alpha(z)} \). Since the integral tends to \( \iota(S, \hat{w}) \) as \( \alpha \to 0 \),

\[
\lim_{\alpha \to 0} \sum_{i=1}^{m} \iota(S_\alpha, a_i) = \iota(S, \hat{w})
\]  \hspace{1cm} (1)

Let \( R_\alpha(z) = g^{-1}(g(R) + \alpha) \) be the perturbed map of \( R \). Then \( g^{-1}(a_i) \) is a simple fixed point of \( R_\alpha \) with multiplier \( R_\alpha'(g^{-1}(a_i)) \), which is same as \( S'_\alpha(a_i) \). This gives that \( \iota(R_\alpha, g^{-1}(a_i)) = \iota(S_\alpha, a_i) \) for each \( i \). Consequently, \( \sum_{i=1}^{m} \iota(S_\alpha, a_i) = \sum_{i=1}^{m} \iota(R_\alpha, g^{-1}(a_i)) \). The right hand side is \( \oint_{\gamma} \frac{dz}{z - R_\alpha(z)} \) where \( \gamma \) is the boundary of \( g^{-1}(D_\epsilon) \). Note that \( \gamma \) is a circle not surrounding any root of \( R_\alpha \) other than \( g^{-1}(a_i) \)'s, and there is no fixed point of \( R_\alpha \) on it. Further, the point \( \hat{z} \) is the only fixed point of \( R \) surrounded by \( \gamma \). Since \( R_\alpha \to R \) as \( \alpha \to 0 \), \( \lim_{\alpha \to 0} \sum_{i=1}^{m} \iota(R_\alpha, g^{-1}(a_i)) = \iota(R, \hat{z}) \). This along with Equation(1) gives that \( \iota(R, \hat{z}) = \iota(S, \hat{w}) \). \hfill \Box

The following is known as Rational fixed point theorem. Though the proof is available, for example in [4], we present it elaborately for the sake of completeness.

**Theorem 2.1.** For every non-identity and non-constant rational map \( R : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \), \( \sum_{z=R(z)} \iota(R, z) = 1 \).
Proof. Conjugating \( R \) by a mobius map \( \mu \), if necessary, we may assume without loss of generality that \( R(\infty) \neq 0, \infty \). This is possible by Lemma 3. For this, \( \mu \) can be chosen such that \( R(\mu^{-1}(\infty)) \notin \{\mu^{-1}(0), \mu^{-1}(\infty)\} \) and then consider \( \mu R \mu^{-1} \) instead of \( R \). If \( R(\infty) = 0 \) then \( \mu(z) = z + 1 \) can be chosen. If \( R(\infty) = \infty \) then consider \( \mu(z) = \frac{1}{z - z_0} \) where \( R(z_0) \neq 0 \) and \( R(z_0) \neq \infty \). Such a \( z_0 \) can be always found for every non-constant and non-identity rational map.

Now \( \lim_{z \to \infty} R(z) = l \neq 0 \) means that for every \( \epsilon > 0 \), there is a positive number \( M \) such that \( |R(z) - l| < \epsilon \) whenever \( |z| > M \). A bigger \( M \) can be chosen, if necessary, to additionally ensure that \( |R(z)| < \frac{|l|}{2} \) and all fixed points of \( R \) have modulus less than \( M \). Thus it follows that \( |z - R(z)| \geq |z| - |R(z)| = \frac{|l|}{2} + \frac{|l|}{2} - |R(z)| > \frac{|l|}{2} \) for all \( z \) with \( |z| > M \). Note that there is a \( K > 0 \) such that \( |R(z)| < K \) for all \( z, |z| > M \). Consequently, \( \left| \frac{R(z)}{z(z-R(z))} \right| < \frac{2K}{|z|^2} \) whenever \( |z| > M \). Let \( r > M \) and \( \partial D_r \) be the simple positively oriented loop \( |z| = r \). We have

\[
\left| \frac{1}{2\pi i} \oint_{\partial D_r} \frac{R(z)dz}{z(z-R(z))} \right| \leq \frac{1}{2\pi} \oint_{\partial D_r} \frac{|R(z)||dz|}{|z(z-R(z))|} \leq \frac{1}{2\pi} \oint_{\partial D_r} \frac{2K|dz|}{|z|^2} = \frac{2K}{r}.
\]

As \( r \to \infty \), the right hand side goes to zero giving that \( \lim_{r \to \infty} \frac{1}{2\pi i} \oint_{\partial D_r} \frac{R(z)dz}{z(z-R(z))} = 0 \). Since

\[
\lim_{r \to \infty} \frac{1}{2\pi i} \oint_{\partial D_r} \frac{dz}{z-R(z)} = \lim_{r \to \infty} \frac{1}{2\pi i} \oint_{\partial D_r} \frac{dz}{z} = 1
\]

(2)

Here, both the integrals are independent of \( r \), for sufficiently large \( r \) and the value of the right hand side is 1. Further,

\[
\sum_{z=R(z)} \tau(R, z) = \sum_{z=R(z)} \frac{1}{2\pi i} \oint_{\partial D} \frac{dz}{z-R(z)}
\]

(3)

where \( \partial D \) is a small positively oriented loop that surrounds only one fixed point of \( R \) and the sum is taken over all fixed points of \( R \). Since all the fixed points of \( R \) have modulus less than \( r \), it follows from the Cauchy theorem that the integral in Equation (3) is equal to \( \frac{1}{2\pi i} \oint_{\partial D_r} \frac{dz}{z-R(z)} \). Now, it follows from Equation (1) and (2) that \( \sum_{z=R(z)} \tau(R, z) = 1 \). \( \Box \)

Remark 2. Since residue index of each polynomial \( P \), with degree at least 2, at \( \infty \) is 1, we have

\[
\sum_{P(z)=z; z \in \mathbb{C}} \tau(P, z) = 0
\]

(4)

Here is a relation between the type of a simple fixed point and its residue index.
Lemma 5. A fixed point with multiplier $\lambda \neq 1$ is attracting if and only if the real part of its residue index is bigger than $\frac{1}{2}$.

Proof. A fixed point $\hat{z}$ with multiplier $\lambda$ is attracting if $|\lambda| < 1$. Note that the function $\frac{1}{1-z}$ maps the unit disk $\{z : |z| < 1\}$ onto the half plane $H = \{z : \Re(z) > \frac{1}{2}\}$. This is clear since the function $z \mapsto 1 - z$ maps the unit disk onto the disk $D = \{z : |z - 1| < 1\}$, and the disk $D$ is mapped onto $H$ by $\frac{1}{z}$. The converse follows easily. \hfill \Box

Remark 3. From the above lemma and its proof it follows that the residue index of an indifferent fixed point has real part equal to $\frac{1}{2}$, and that of a repelling fixed point has real part less than $\frac{1}{2}$.

Here is an important consequence of the Rational fixed point theorem. This is already known [4].

Corollary 2.1. Every rational map of degree $d \geq 2$ has a weakly repelling fixed point or a rationally indifferent fixed point with multiplier 1 or both.

Proof. If possible, for a rational map $R$, suppose that there is neither a fixed point with multiplier 1 nor a repelling fixed point. Since there is no fixed point of multiplier 1, all fixed points are simple. The real part of their residue indices is bigger or equal to $\frac{1}{2}$ by Remark [3]. By the Rational fixed point Theorem, $\sum_{z=R(z)} t(R, z) = 1$. There are $d + 1$ simple fixed points and hence $d + 1$ terms in the sum. Consequently, the right hand side has real part bigger than or equal to $\frac{d+1}{2}$. However this is not possible since $d \geq 2$. Therefore, each rational map always has a rationally indifferent fixed point with multiplier 1 or has a repelling fixed point. \hfill \Box

Following theorem settles the Real non-attractive fixed point (RNFP) conjecture in its full generality.

Theorem 2.2. Every polynomial with degree at least two has a fixed point, the real part of whose multiplier is bigger than or equal to 1.

Proof. If a polynomial has a multiple fixed point then its multiplier is 1 and we are done. Let $P$ be a polynomial, all of whose fixed points are simple. If $P$ is of degree $d$ then it has $d + 1$ fixed points including $\infty$. Suppose that $\lambda_i = P'(z_i)$ where $P(z_i) = z_i$ for $i = 1, 2, 3, \ldots, d + 1$. Therefore, each rational map always has a rationally indifferent fixed point with multiplier 1 or has a repelling fixed point.
By the Rational fixed point theorem, \( \sum_{i=1}^{d+1} \frac{1}{1-\lambda_i} = 1 \). However, the point at \( \infty \) is a super-attracting fixed point. Letting \( z_{d+1} = \infty \), we have \( \lambda_{d+1} = 0 \). Therefore,

\[
\sum_{i=1}^{d} \frac{1}{1-\lambda_i} = 0 \tag{5}
\]

Comparing the real part, we get that

\[
\sum_{i=1}^{d} \Re(1-\bar{\lambda}_i) \left| 1-\lambda_i \right|^2 = 0 \tag{6}
\]

Since \( \Re(1-\bar{\lambda}_i) = 1-\Re(\lambda_i) \) and \( |1-\lambda_i|^2 > 0 \) for each \( i \), it is not possible to have \( \Re(\lambda_i) < 1 \) for all \( i \). Therefore, there exists \( j \in \{1, 2, \cdots, d\} \) such that \( \Re(\lambda_j) \geq 1 \).

A number of useful remarks follow.

**Remark 4.**

1. If \( \Re(\lambda_i) > 1 \) for each \( i \) then each term in the left hand side of Equation (6) is negative leading to a contradiction. Therefore every polynomial with degree at least two has a fixed point, the real part of whose multiplier is less than or equal to 1. Further, if all fixed points are simple and some fixed points have their multipliers with real part bigger than 1 then there is a fixed point whose multiplier has real part less than 1.

2. Comparing the imaginary parts in Equation (5), it is observed that \( P \) has a fixed point with multiplier 1 or has a fixed point with imaginary part at least 0. In other words, if all the fixed points of a polynomial are simple then it has a fixed point, the imaginary part of whose multiplier is non-negative.

3. The RNFP conjecture is true for some, but not for all rational maps. Using the above proof, the conjecture can be seen to be true for all rational maps with at least one super-attracting fixed point. However, nothing can be said for a rational map without any super-attracting fixed point. For example, the two rational functions \( \frac{1}{z^d-1}, d > 1 \) and \( \frac{kz}{z^2+z+1}, k \neq 0 \) do not have any super-attracting fixed point. Multiplier of each fixed point of the first function is \( 1-d \) which is negative for \( d > 2 \) giving that RNFP conjecture is false for \( \frac{1}{z^d-1}, d > 2 \). On the other hand, 0 is a fixed point of \( \frac{kz}{z^2+z+1} \) with multiplier \( k \). The RNFP conjecture is true for \( \frac{kz}{z^2+z+1} \) whenever \( \Re(k) \geq 1 \).

### 3 Beyond the real non-attractive fixed point conjecture

We now deal with some issues related to the real non-attractive fixed point conjecture.
3.1 Multipliers with real part one

When all the finite fixed points of a polynomial have real part 1? In some sense, this is an extreme situation of the RNFP conjecture. It is not difficult to construct polynomials for which the answer is yes. Let \( P(z) = z + ik(z - a_1)(z - a_2)\cdots(z - a_m)(z - b_1)^{p_1}(z - b_2)^{p_2}\cdots(z - b_n)^{p_n} \) where \( a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_n, k \in \mathbb{R}\setminus\{0\} \) and \( p_1, p_2, \cdots, p_n \) are natural numbers bigger than 1. Then each \( a_i \) and \( b_j \) are fixed points of \( P \), and the real part of each multiplier is 1.

We now look at the general case of quadratic polynomials. Since every quadratic polynomial is conformally conjugate to \( z^2 + c \) for some \( c \) and multipliers are preserved under conformal conjugacy, it is enough to consider \( z^2 + c \).

**Theorem 3.1.** Let \( P(z) = z^2 + c \). If \( c = \frac{1}{4} \) then \( P \) has a single fixed point and its multiplier is 1. If \( c \neq \frac{1}{4} \) then \( P \) has two simple fixed points. Further, the multipliers have real part equal to 1 if and only if \( c > \frac{1}{4} \). In other words, the multiplier of each fixed point of \( P \) has real part equal to 1 if and only if \( c \geq \frac{1}{4} \).

**Proof.** Finite fixed points of \( P(z) = z^2 + c \) are \( \frac{1 \pm \sqrt{1 - 4c}}{2} \) with multipliers \( 1 \pm \sqrt{1 - 4c} \).

There is only one fixed point if and only if \( c = \frac{1}{4} \). In this case, the multiplier is 1.

If \( c \neq \frac{1}{4} \) then \( \frac{1 + \sqrt{1 - 4c}}{2} \) and \( \frac{1 - \sqrt{1 - 4c}}{2} \) are two distinct fixed points with distinct multipliers. The real part of these multipliers is 1 if and only if \( 1 - 4c < 0 \) which is nothing but \( c > \frac{1}{4} \). □

For dealing with general cubic polynomials, we first need to prove a simple case.

**Lemma 6.** Suppose that a cubic polynomial \( Q \) has three distinct fixed points 0, 1 and \( \alpha \). Then all multipliers have real part 1 if and only if \( \alpha \) is real and \( Q'(0) - 1 \) is purely imaginary.

**Proof.** If a cubic polynomial \( Q \) fixes three distinct points 0, 1 and \( \alpha \) then \( Q(z) = z + k(z - 1)(z - \alpha) \) for some non-zero \( k \). Further, the multipliers are \( 1 + k\alpha, 1 + k - k\alpha \) and \( 1 + k\alpha^2 - k\alpha \).

Each has real part 1 if and only if \( k\alpha, k - k\alpha \) and \( k\alpha^2 - k\alpha \) are purely imaginary. This is true if and only if \( k \) is purely imaginary and \( \alpha \) is real. The proof completes by observing that \( k = \frac{Q'(0) - 1}{\alpha} \). □

**Theorem 3.2.** Suppose that a cubic polynomial has three distinct fixed points. Then the real part of all multipliers is 1 if and only if all the fixed points are collinear and the real part of any one multiplier is 1.

**Proof.** If \( a, b, c \) are the fixed points of a cubic polynomial \( P \) then \( Q = \phi P \phi^{-1} \) fixes 0 and 1 where \( \phi(z) = \frac{z-a}{b-a} \). The other fixed point of \( Q \) is \( \frac{c-a}{b-a} \), which is clearly different from 0 and 1.
Note that \( P'(a) = Q'(0), P'(b) = Q'(1) \) and \( P'(c) = Q'\left(\frac{c-a}{b-a}\right) \). Further, \( a, b, c \) are collinear if and only if \( 0, 1, \frac{c-a}{b-a} \) are so. These follow from two useful properties of an affine map, namely inverse of an affine map is affine, and an affine map takes lines onto lines. Letting \( \alpha = \frac{c-a}{b-a} \), observe that \( Q(z) = z + \{kz(z-1)(z-\alpha)\} \) for some non-zero \( k \).

If all multipliers (of fixed points) of \( P \) have real part 1 then the same is true for \( Q \). However, \( Q'(0) = 1 + k\alpha, Q'(1) = 1 + k - k\alpha, Q'(\alpha) = 1 + k\alpha^2 - k\alpha \), and it follows from Lemma 6 that \( \alpha \) is real and \( k \) is purely imaginary. In other words, the arguments of \( c-a \) and \( b-a \) are same or differ by \( \pi \). Therefore \( a, b, c \) are collinear. Since \( k \) is purely imaginary by previous lemma, \( \Re(P'(a)) = \Re(Q'(0)) = 1 \).

Conversely, let the fixed points \( a, b, c \) of \( P \) be collinear and one multiplier have real part 1. Let the fixed point of \( P \) whose multiplier has real part 1 be denoted by \( a \). Then the fixed points \( 0, 1, \alpha = \frac{c-a}{b-a} \) of \( Q \) are collinear, giving that \( \frac{c-a}{b-a} \) is real. Since the real part of \( P'(a) \) is 1, the real part of \( Q'(0) = 1 + k\alpha \) is 1. In other words, \( k \) is purely imaginary. It is non-zero since \( Q'(0) \neq 1 \). Now, by Lemma 6, the real part of each multiplier of \( Q \) is 1. Therefore, all multipliers of \( P \) have real part equal to 1.

\[ \square \]

Remark 5. Upto conformal conjugacy, the only cubic polynomial (with only simple fixed points) whose all multipliers have real part 1 is \( kz^3 - (k + k\alpha)z^2 + (k\alpha + 1)z \) for some non-zero and purely imaginary \( k \) and a real number \( \alpha \) different from 0 and 1.

3.2 Multipliers equidistant from 1

Consider \( P(z) = z + (z-a_1)^{p_1}(z-a_2)^{p_2} \cdots (z-a_k)^{p_k} \) where \( a_1, a_2, \ldots, a_k \in \mathbb{C} \) and \( p_1, p_2, \ldots, p_k \) are positive integers bigger than 1. Then \( a_1, a_2, \ldots, a_k \) are multiple fixed points of \( P \). In this case, all finite fixed points have same multiplier, namely 1, and all the multipliers are obviously equidistant from 1. However, for polynomials with a multiple as well as a simple fixed point, all multipliers can never be equidistant from 1. Thus, the question of all multipliers being equidistant from 1 makes sense only when all the fixed points are simple. The importance of this issue is underlined by Equation (6) which gives that \( \sum_{i=1}^{d} \Re(1 - \lambda_i) = 0 \) whenever all multipliers are equidistant from 1. In this situation, the average of real parts of all multipliers is 1. In particular, all the multipliers cannot have real part strictly bigger than 1.

The case is again straightforward for quadratic polynomials. If each fixed point of a quadratic polynomial is simple then both multipliers, \( \lambda_1, \lambda_2 \) are equidistant from 1. In fact, it follows from the Rational fixed point theorem that \( \lambda_1 + \lambda_2 = 2 \), which gives that \( |\lambda_1 - 1| = \)
\(|\lambda_2 - 1|\). Now we consider polynomials, with degree at least three, all of whose fixed points are simple. For a natural number \(n\), a regular \(n\)-gon is a (convex) polygon with vertices at \(v_k = a + re^{i(\theta + \frac{2\pi k}{n})}\), \(k = 0, 1, 2, \cdots, (n - 1)\) for some \(a \in \mathbb{C}, r > 0\) and \(\theta \in (0, 2\pi]\). A regular \(n\)-gon is completely determined by \(a, r, \) and \(\theta\). We say it is centered at \(a\) and with radius \(r\).

**Theorem 3.3.** Let \(P\) be a polynomial with degree \(n, n \geq 3\) and each of its fixed points is simple. If all the fixed points are on the vertices of a regular \(n\)-gon then all the multipliers are equidistant from 1. Moreover, its multipliers are the vertices of a regular \(n\)-gon centered at 1.

**Proof.** If all the fixed points of a polynomial \(P\) with degree \(n\) are simple, and are on the vertices of a regular \(n\)-gon centered at \(a\) and with radius \(r\) then \(P(z) = z + M((z - a)^n - (re^{i\theta})^n)\) for some non-zero complex number \(M\). Note that each \(v_k = a + re^{i(\theta + \frac{2\pi k}{n})}\) is a fixed point of \(P\) and \(P'(v_k) = 1 + Mn(re^{i(\theta + \frac{2\pi k}{n})})^{n-1}\) which is nothing but \(1 + Mn(re^{i\theta})^{n-1}e^{-\frac{2\pi ik}{n}}\). Note that each \(P'(v_k)\) is equidistant from 1. Further, these multipliers represent the vertices of a regular \(n\)-gon with centre at 1 and radius \(|M|nr^{n-1}\).

It is natural to ask when the converse of the above result is true. For cubic and quartic polynomials, we do have the answers.

**Theorem 3.4.** Let \(P\) be a polynomial with degree \(n\) and each of its fixed points is simple. If all the multipliers are equidistant from 1 then all the fixed points are on the vertices of an equilateral triangle or a rectangle when \(n = 3\) or \(4\) respectively.

**Proof.** Let all the fixed points of a polynomial with degree \(n\) be simple. Then

\[
P(z) = z + k(z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \cdots (z - \alpha_n)
\]

for some distinct complex numbers \(\alpha_1, \alpha_2, \cdots, \alpha_n\) and a non-zero complex number \(k\). Differentiating \((7)\), we get

\[
P'(z) = 1 + k \sum_{i=1}^{n} L_i(z)
\]

where

\[
L_i(z) = \prod_{j=1, j \neq i}^{n} (z - \alpha_j).
\]

Note that \(L_i(\alpha_j) = 0\) for \(i \neq j\) and it follows that

\[
\lambda_i = P'(\alpha_i) = 1 + kL_i(\alpha_i)
\]
Now, $|1 - \lambda_i| = |1 - \lambda_j|$ for each $i \neq j$ if and only if $|L_i(\alpha_i)| = |L_j(\alpha_j)|$ for all $i, j$. Note that $|L_i(\alpha_i)|$ is nothing but the product of distances of all other fixed points from $\alpha_i$.

If $n = 3$ then the assumption of this theorem means that $|(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)| = |(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)| = |(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)|$. In other words, $|\alpha_1 - \alpha_2| = |\alpha_2 - \alpha_3| = |\alpha_3 - \alpha_1|$. Hence the fixed points are on the vertices of an equilateral triangle.

Let $n = 4$ and $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$ be the fixed points of $P$. Then, denoting the distance $|\alpha_i - \alpha_j|$ by $\alpha_{ij}$, it is observed that $\alpha_{12}\alpha_{13}\alpha_{14} = \alpha_{21}\alpha_{23}\alpha_{24}$, $\alpha_{21}\alpha_{23}\alpha_{24} = \alpha_{31}\alpha_{32}\alpha_{34}$, $\alpha_{31}\alpha_{32}\alpha_{34} = \alpha_{41}\alpha_{42}\alpha_{43}$ and $\alpha_{41}\alpha_{42}\alpha_{43} = \alpha_{12}\alpha_{13}\alpha_{14}$. The first two equations give that $\alpha_{12}\alpha_{14} = \alpha_{32}\alpha_{34}$ whereas it follows from the second and third that $\alpha_{21}\alpha_{23} = \alpha_{41}\alpha_{43}$. Since $\alpha_{ij} > 0$ and $\alpha_{ij} = \alpha_{ji}$, these equalities give that $\alpha_{14} = \alpha_{23}$ and consequently $\alpha_{12} = \alpha_{34}$. Now, the fixed points $\alpha_i, i = 1, 2, 3, 4$ can be thought of as vertices of a quadrilateral, whose opposite sides have same length. In otherwords, this is a parallelogram. Further, it also follows from $\alpha_{12}\alpha_{13}\alpha_{14} = \alpha_{21}\alpha_{23}\alpha_{24}$ that $\alpha_{13} = \alpha_{24}$, i.e., the length of diagonals of this parallelogram are same giving that it is a rectangle.

![Figure 1: Rectangle with vertices $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$](image)

For a polynomial with at least two multiple fixed points, the corresponding multipliers are 1. It is natural to ask when two simple finite fixed points have same multipliers. This is never true for quadratic polynomials. Polynomials of degree bigger than two can be found with simple fixed points having the same multiplier. For example, the two fixed points 0 and 1 of $P(z) = z^4 - 2z^2$ have same multiplier 0. However a general condition ensuring this may be found.

The remark following Theorem 2.2 gives that some rational maps satisfy the RNFP conjecture whereas some do not. A complete characterization of rational maps for which RNFP conjecture holds good is worth knowing. The RNFP conjecture can be asked for periodic points instead of fixed points. Since a $p$-periodic point of a polynomial $P$ is a fixed point of
$P^p$, it follows from Theorem 2.2 that there is a fixed point of $P^p$ whose multiplier has real part at least 1. However, it is not trivial to decide whether this fixed point of $P^p$ is actually a periodic point of $P$. This is of course a periodic point with period $q$ where $q$ divides $p$.

Dynamics of polynomials, stated in the Remark 5 may be studied. In particular, it is worth knowing whether they exhibit any common dynamical property.

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