OTHER QUANTUM RELATIVES OF THE ALEXANDER POLYNOMIAL THROUGH THE LINKS-GOULD INVARIANTS

BEN-MICHAEL KOHLI AND BERTRAND PATUREAU-MIRAND

Abstract. Oleg Viro studied in [14] two interpretations of the (multivariable) Alexander polynomial as a quantum link invariant: either by considering the quasi triangular Hopf algebra associated to $U_q\mathfrak{sl}(2)$ at fourth roots of unity, or by considering the super Hopf algebra $U_q\mathfrak{gl}(1|1)$. In this paper, we show these Hopf algebras share properties with the $-1$ specialization of $U_q\mathfrak{gl}(n|1)$ leading to the proof of a conjecture of David De Wit, Atsushi Ishii and Jon Links on the Links-Gould invariants.

1. Introduction

The Links-Gould invariants of links $LG^{n,m}$ are two variable quantum link invariants. They are derived from super Hopf Algebras $U_q\mathfrak{gl}(n|m)$. David De Wit, Atsushi Ishii and Jon Links conjectured [3] that for any link $L$

$$LG^{n,m}(L; \tau, e^{i\pi/m}) = \Delta_L(\tau^{2m})^n,$$

where $\Delta_L$ is the Alexander-Conway polynomial of $L$. They proved the conjecture when $(n,m) = (1,m)$ and when $(n,m) = (2,1)$ for a particular class of braids. A complete proof of the $(n,1)$ case for $n = 2,3$ is given in [3]. However this is achieved by studying the invariants at hand at the level of representations, which requires computation of an explicit $R$-matrix for each $LG^{n,1}$, making that method hard to implement as $n$ grows.

Here we prove the $(n,1)$ case of the conjecture for any $n$:

$$LG^{n,1}(L; \tau, -1) = \Delta_L(\tau^2)^n.$$

To do so we study the structure of the universal objects directly, and in particular the (super) Hopf algebras and universal R-matrices that are involved.

However, the strong version of the conjecture is still open.

2. Hopf algebras for the Alexander polynomial

We first define a Hopf algebra $U$ which is an essential ingredient for the quantum relatives of the Alexander polynomial. Unfortunately this algebra is only braided in a weak sense. Then we recall two quantum groups which can be seen as central extensions of $U$. One was first used by Murakami [10], both were studied by Viro in [14]. Finally we compare the braidings of the two Hopf algebras.

2.1. A braided Hopf algebra $U$. The following Hopf algebra $U$ is a version of quantum $\mathfrak{sl}(2)$ when the quantum parameter $q$ is a fourth root $i$ of 1. The complex algebra $U$ is presented by generators $k^\pm, e, f$ and relations

$$ke + ek = kf + fk = e^2 = f^2 = 0 \quad \text{and} \quad ef - fe = k - k^{-1}.$$
The coproduct, counit and antipode of $U$ are given by
\[
\Delta(e) = 1 \otimes e + e \otimes k, \quad \varepsilon(e) = 0, \quad S(e) = -ek^{-1}, \\
\Delta(f) = k^{-1} \otimes f + f \otimes 1, \quad \varepsilon(f) = 0, \quad S(f) = -kf, \\
\Delta(k) = k \otimes k, \quad \varepsilon(k) = 1, \quad S(k) = k^{-1}.
\]

This Hopf algebra can be seen in a sense as a “double” of Bodo Pareigis’ Hopf algebra $\langle k, f \rangle$ with our notations. A pivotal structure is a group-like element $\phi$ whose conjugation is equal to the square of the antipode. There is non obviously a better choice which is given by $\phi = k^{-1}$.

Let $\tau : x \otimes y \mapsto y \otimes x$ be the switch of factors. Hopf algebra $U$ is not quasi-triangular but it is braided in the sense of [12]: there exists an (outer) algebra automorphism $\mathcal{R}$ of $U \otimes U \to U \otimes U$ different from $\tau$ that satisfies
\[
\mathcal{R} \circ \Delta = \tau \circ \Delta, \\
\Delta_1 \circ \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23}, \\
\Delta_2 \circ \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12}.
\]

Automorphism $\mathcal{R}$ admits a regular splitting (see [12]) $\mathcal{R} = D \circ \text{Ad} \bar{R}$ where $\text{Ad} \bar{R}$ is the conjugation by the invertible element
\[
\bar{R} = 1 + e \otimes f
\]
and $D$ is an outer automorphism satisfying equations similar to (2) and (3) and defined by:
\[
\mathcal{D} \circ \tau = \tau \circ \mathcal{D}, \quad \mathcal{D}(e \otimes 1) = e \otimes k, \quad \mathcal{D}(f \otimes 1) = f \otimes k^{-1} \quad \text{and} \quad \mathcal{D}(k \otimes 1) = k \otimes 1.
\]
The elements $k^{\pm 1}$ generate a central sub-Hopf algebra and for any $g \in \mathbb{C} \setminus \{0, 1\}$, the quotient $U/(k^2 - g)$ is a 8-dimensional semi-simple Hopf algebra with two isomorphism classes of irreducible representations $V_{\pm a}$ where $a^2 = g$. The representation $V_a$ is 2-dimensional and can be written in a certain basis $(e_0, e_1)$
\[
k = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ a^{-1} & 0 \end{pmatrix}.
\]

The central element $ef + fe$ acts by $(a - a^{-1})I_2$.

2.2. The $\mathfrak{sl}(2)$ model and the Alexander polynomial. From [13] the $\mathfrak{sl}(2)$ model is the unrolled version of quantum $\mathfrak{sl}(2)$ at $q = i = \exp(i\pi/2)$. It is an algebra $U_i^H \mathfrak{sl}(2)$ generated by $K^{\pm 1}, E, F, H$. Its presentation is obtained from that of $U \ (U_0 = \langle K^{\pm 1}, E, F \rangle \simeq U)$ by adding the generator $H$ and the following relations:
\[
[H, K] = 0, \quad [H, E] = 2E, \quad [H, F] = -2F.
\]
We will consider the category $\mathcal{C}$ of weight modules: finite dimensional vector spaces where element $H$ acts diagonally and
\[
K = i^H = \exp(i\pi H/2).
\]
The pivotal Hopf algebra structure $U$ is extended to $U_i^H \mathfrak{sl}(2)$ by the following relations:
\[
\Delta(H) = 1 \otimes H + H \otimes 1, \quad \varepsilon(H) = 0, \quad S(H) = -H.
\]

\[\text{1} \text{Compared to Viro, we use the opposite coproduct here.}\]
As in $U$, the pivotal element is $\Phi = K^{-1}$ so that $S^2(\cdot) = \Phi \cdot \Phi^{-1}$. With this pivotal structure, category $\mathcal{C}$ is ribbon with braiding given by the switch $\tau : x \otimes y \mapsto y \otimes x$ composed with the action of the universal $R$-matrix:
\[
R^H = i^{H \otimes H/2} (1 + E \otimes F).
\]

**Lemma 2.1.** For any two representations $V, W \in \mathcal{C}$, the conjugation in $V \otimes W$ by $D^H := i^{H \otimes H/2}$ induces an automorphism $\mathcal{D}^H$ of $\text{End}_\mathcal{C}(V \otimes W)$ which satisfies
\[
\rho_{V \otimes W} \circ \mathcal{D} = \mathcal{D}^H \circ \rho_{V \otimes W} : U \otimes U \rightarrow \text{End}_\mathcal{C}(V \otimes W).
\]

**Proof.** This is an easy consequence of Equation (5). More generally, if $x, y \in U$ satisfy
\[
[H, x] = 2mx \text{ and } [H, y] = 2m'y, \text{ then } H \otimes H, x \otimes y = x \otimes y.(H + 2m) \otimes (H + 2m')
\]
\[
\text{so } i^{\rho_{V \otimes W}(H \otimes H/2)} \rho_{V \otimes W}(x \otimes y) = \rho_{V \otimes W}(x \otimes y) i^{\rho_{V \otimes W}((H+2m) \otimes (H+2m')/2)} = \rho_{V \otimes W}((x \otimes K^m)(K^{m'} \otimes y)) i^{\rho_{V \otimes W}(H \otimes H/2)} = \rho_{V \otimes W}(\mathcal{D}(x \otimes y)) i^{\rho_{V \otimes W}(H \otimes H/2)}.
\]

For each complex number $\alpha$ which is not an odd integer, $U^H sl(2)$ possesses, up to isomorphism, a unique two dimensional irreducible representation $V_\alpha$ with $\text{Spec}(H) = \{\alpha + 1, \alpha - 1\}$. Its restriction to $U$ is representation $V_\alpha$ where $a = i^{\alpha+1}$ and the action of $H$ is given by $H = \begin{pmatrix} \alpha + 1 & 0 \\ 0 & \alpha - 1 \end{pmatrix}$.

In the representation $V_\alpha \otimes V_\beta$, with respect to basis $(e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1)$ the braiding is:
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

In the case where $\alpha = \beta$, the $R$-matrix then takes the particular form
\[
\tau R^H = i^{\frac{\alpha^2 - 1}{2}} \begin{pmatrix}
-t^{-1/2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & (t^{-1/2} - t^{1/2}) & 0 \\
0 & 0 & 0 & -t^{1/2}
\end{pmatrix}
\]

where we set $t^{1/2} = i^{-\alpha-1}$.

The ribbon category we consider here allows us to apply the Reshetikhin-Turaev theory \cite{R-T} to construct a framed link isotopy invariant in $S^3$. It becomes an unframed link isotopy invariant if one divides the above $R$-matrix on $V_\alpha \otimes V_\alpha$ by the value of the twist $\theta_\alpha = i^{2/\alpha}$. In this particular case, the invariant we find is the Conway normalization of the classical Alexander polynomial, see \cite{R-T}. The Links-Gould invariants $LG^n$ that will interest us in the following are obtained by the same general construction using other Hopf algebras. Explicitly, the Reshetikhin-Turaev functor gives representations of braid groups $B_\ell$
\[
\Psi_{V_\alpha^\otimes \ell} : B_\ell \rightarrow GL(V_\alpha^\otimes \ell) \\
\sigma_i \mapsto \text{Id}_{V_\alpha^\otimes (\ell-1)} \otimes \theta_\alpha^{-1} \tau R^H \otimes \text{Id}_{V_\alpha^\otimes (\ell-1)},
\]

where $\sigma_i$ is the $i$th standard Artin generator of braid group $B_\ell$.

Let $L$ be an oriented link in $S^3$ obtained as closure of a braid in $\ell$ strands $b \in B_\ell$. Then:
1) There exists a scalar $c$ such that $\text{trace}_{\ell, 2, \ldots, \ell}((\text{Id}_{V_\alpha} \otimes (K^{-1})^\otimes (\ell-1)) \circ \Psi_{V_\otimes \ell}(b)) = c \cdot \text{Id}_{V_\alpha}$,
2) $L \mapsto c$ is a link invariant and is equal to the Alexander polynomial of $L$, $\Delta_L(t)$. 


The relations $H$ identified with the category of $\mathfrak{gl}(1|1)$, the partial trace operator is defined by $\text{trace}_{2,3,\ldots} (f_1 \otimes \ldots \otimes f_4) := \text{trace}(f_2) \text{trace}(f_3) \ldots \text{trace}(f_4)f_1 \in \text{End}_C(V_\alpha)$ for any $f_1, \ldots, f_4 \in \text{End}_C(V_\alpha)$.

2.3. An example of bosonization: the $\mathfrak{gl}(1|1)$ model.

2.3.1. Bosonization. Here we recall Majid’s trick \([\mathbb{I}]\) to transform a super Hopf algebra into an ordinary one.

Let $H$ be a pivotal super Hopf algebra and $\mathcal{C}$ be its even monoidal category of representations (morphisms are formed by even $H$-linear maps). Let $H^\sigma$ be the bosonization of $H$: as an algebra, $H^\sigma$ is the semi-direct product of $H$ with $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$ where the action of $\sigma$ or equivalently the commutation relations in $H^\sigma$ are given by

$$\forall x \in H, \sigma x = (-1)^{|x|} x \sigma.$$  

The coproduct $\Delta^\sigma$ on $H^\sigma$ is given by $\Delta^\sigma \sigma = \sigma \otimes \sigma$ and

$$\forall x \in H, \Delta^\sigma(x) = \sum_i x_i \sigma^{|x|} \otimes x_i^\prime \text{ where } \Delta(x) = \sum_i x_i \otimes x_i^\prime.$$  

If $R = \sum_i R_i^{(1)} \otimes R_i^{(2)}$ is the universal $R$-matrix in $H$, then the following formula defines a universal $R$-matrix in $H^\sigma$:

$$R^\sigma = R_1 \sum_i R_i^{(1)} (\sigma^{|R_i^{(2)}|} \otimes R_i^{(2)}), \text{ where } R_1 = \frac{1}{2} (1 \otimes 1 + \sigma \otimes 1 + 1 \otimes \sigma - \sigma \otimes \sigma).$$  

Given a super representation $V = V_\sigma \oplus V_{\sigma}^\perp$ of $H$ we get a representation of $H^\sigma$ by setting $\sigma_{|V} = \text{Id}_{V_\sigma} - \text{Id}_{V_{\sigma}^\perp}$. Reciprocally, since $\sigma^2 = 1$, every $H^\sigma$-module inherits a natural $\mathbb{Z}/2\mathbb{Z}$ grading: $W$ splits into $W = W_\sigma \oplus W_{\sigma}^\perp$ where we define $W_\sigma = \ker(\sigma - 1)$ and $W_{\sigma}^\perp = \ker(\sigma + 1)$.

Theorem 2.3 (Majid Theorem 4.2). The even category of super $H$-modules can be identified with the category of $H^\sigma$-modules.

Remark that the antipode of $H^\sigma$ is given by $x \mapsto \sigma^{|x|} S(x)$ and if $H$ as a pivot $\phi$ then one can choose $\phi^\sigma = \sigma \phi$ as a pivot in $H^\sigma$.

2.3.2. The $\mathfrak{gl}(1|1)$ model. Using the same notations as Viro: $U_q\mathfrak{gl}(1|1)$ is the pivotal super Hopf algebra generated by two odd generators $X, Y$, two even generators $I, G$ satisfying the relations

$$XY + YX = \frac{C - C^{-1}}{q - q^{-1}}, \quad X^2 = Y^2 = 0,$$

$$[I, X] = [I, Y] = [I, G] = 0,$$

$$[G, X] = X, \quad [G, Y] = -Y,$$

where $C = q^I$, with coproduct

$$\Delta(I) = 1 \otimes I + I \otimes 1, \quad \Delta(G) = 1 \otimes G + G \otimes 1,$$

$$\Delta(X) = X \otimes C^{-1} + 1 \otimes X, \quad \Delta(Y) = Y \otimes 1 + C \otimes Y,$$

counit

$$\varepsilon(X) = \varepsilon(Y) = \varepsilon(I) = \varepsilon(G) = 0,$$

antipode

$$S(I) = -I, \quad S(G) = -G, \quad S(X) = -XC, \quad S(Y) = -YC^{-1},$$

pivot

$$\phi = K.$$
and universal $R$-matrix

$$R = (1 + (q - q^{-1})(X \otimes Y)(C \otimes C^{-1}))q^{-i \otimes G - G \otimes i}.$$  

Its bosonization $U_q \mathfrak{gl}(1|1)^\sigma$ contains a sub-Hopf algebra $U_1$ isomorphic to $U$ given by

$$e = (q - q^{-1})X\sigma, \quad f = Y \quad \text{and} \quad k = C^{-1}\sigma.$$  

Indeed, these elements satisfy the following:

$$ef - fe = (q - q^{-1})(X\sigma Y - YX\sigma) = (q - q^{-1})(-XY - YX)\sigma = k - k^{-1},$$

$$ke + ek = kf +fk = 0,$$

$$\Delta^\sigma(e) = (q - q^{-1})\Delta^\sigma(X\sigma) = (q - q^{-1})(X \otimes C^{-1} + \sigma \otimes X)(\sigma \otimes \sigma) = e \otimes k + k^{-1} \otimes e,$$

$$\Delta^\sigma(f) = \Delta^\sigma(Y) = Y \otimes 1 + C\sigma \otimes Y = f \otimes 1 + k^{-1} \otimes f,$$

$$\Delta^\sigma(k) = k \otimes k.$$  

In the bosonization, the universal $R$-matrix is

$$R^\sigma = R_1 q^{-i \otimes G + G \otimes i}(1 + e \otimes f), \quad \text{where} \quad R_1 = \frac{1}{2}(1 \otimes 1 + \sigma \otimes 1 + 1 \otimes \sigma - \sigma \otimes \sigma).$$

**Lemma 2.4.** Denoting $D' = q^{-i \otimes G - G \otimes i}$ and $D^\sigma = R_1 D'$ we have, for any $x, y \in U = U_1$:

$$R_1(x \otimes y)R_1^{-1} = \sigma^{|x|,|y|}(x \otimes y|x|), \quad D'(x \otimes y)(D')^{-1} = xC^{-d_G(y)} \otimes y C^{-d_G(x)},$$

$$D^\sigma(x \otimes y)(D^\sigma)^{-1} = (C^{-1} \sigma)^{d_G(y)} x \otimes y (C^{-1} \sigma)^{d_G(x)} = \mathcal{D}(x \otimes y),$$

where $d_G(x) \in \mathbb{Z}$ is defined by $[G, x] = d_G(x) x.$

**Remark 2.5.** For a homogeneous $a \in U_0$, $|a| = d_G(a)$ modulo 2.

Let us recall a family of 2-dimensional $U_q \mathfrak{gl}(1|1)^\sigma$-modules. This family is parametrized by two complex numbers $(j, f)$ and $\varepsilon \in \{0, 1\}$, see [14]. It extends the representation $V_a$ of $U_1$ where $a = (-1)^\varepsilon q^{-2}$. Written in matrix form,

$$I = \begin{pmatrix} 2j & 0 \\ 0 & 2\bar{j} \end{pmatrix}, \quad G = \begin{pmatrix} \frac{j+1}{2} & 0 \\ 0 & \frac{j-1}{2} \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & \sqrt{q^2 - q^{-2}} \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} (-1)^\varepsilon & 0 \\ 0 & -(1)^\varepsilon \end{pmatrix}.$$  

**2.4. Comparing the actions of $R^\sigma$ and $R^H$.** $U_0 \subset U_1^{H} \mathfrak{sl}(2)$ and $U_1 \subset U_q \mathfrak{gl}(1|1)^\sigma$ are two isomorphic Hopf algebras. The goal of this paragraph is to show the action of

$$R^H = i^{H \otimes H/2}(1 + E \otimes F) \in U_1^{H} \mathfrak{sl}(2) \otimes U_1^{H} \mathfrak{sl}(2)$$

and that of

$$R^\sigma = R_1 q^{-i \otimes G + G \otimes i}(1 + e \otimes f) \in U_q \mathfrak{gl}(1|1)^\sigma \otimes U_q \mathfrak{gl}(1|1)^\sigma$$

on two representations $V_1^{H} \otimes V_2^{H}$ of $U_1^{H} \mathfrak{sl}(2)$ and $V_1^{\sigma} \otimes V_2^{\sigma}$ of $U_q \mathfrak{gl}(1|1)^\sigma$ are identical up to a scalar multiple of the identity, when $V_1^{H}$ and $V_1^{\sigma}$ have the same underlying $U_0 = U_1$-module structure.

We recall conjugations by the elements $D^H$ on one side and $D^\sigma$ on the other side both induce the same automorphism $\mathcal{D}$ of $U \otimes U$.

**Proposition 2.6.** Set for $i = 1, 2$ $V_i^{H}$ a representation of $U_1^{H} \mathfrak{sl}(2)$ and $V_i^{\sigma}$ a representation of $U_q \mathfrak{gl}(1|1)^\sigma$ which both restrict to the same irreducible representation of $U = U_0 = U_1$. Then $D^H(D^\sigma)^{-1} \in \text{End}_\mathbb{C}(V_1 \otimes V_2)$ is a scalar multiple of the identity.
Denote the representations at hand \( \rho_{V_i} \), \( \rho_{V_j} \) for \( i = 1, 2 \). We supposed
\[
\rho_{V_i]|_U = \rho_{V_j}|_U.
\]
So if \( \rho_H = \rho_{V_{i}} \otimes \rho_{V_{j}} \) and \( \rho_{\sigma} = \rho_{V_{i}} \otimes \rho_{V_{j}} \) we define \( \rho := \rho_{H}|_{U \otimes U} = \rho_{\sigma}|_{U \otimes U} \). Using Lemma 2.1 and Lemma 2.4, for any \( x, y \in U \):
\[
\rho_H(D^H) \rho(x \otimes y) \rho_H((D^H)^{-1}) = \rho(D(x \otimes y)) = \rho_{\sigma}(D^\sigma) \rho(x \otimes y) \rho_{\sigma}((D^\sigma)^{-1}).
\]
Which means
\[
\rho_H(D^H)^{-1} \rho_{\sigma}(D^\sigma) \rho(x \otimes y) = \rho(x \otimes y) \rho_H(D^H)^{-1} \rho_{\sigma}(D^\sigma).
\]
Using the density theorem, \( \rho_H(D^H)^{-1} \rho_{\sigma}(D^\sigma) \) commutes with any element in \( \text{End}_{\mathbb{C}}(V_1) \otimes \text{End}_{\mathbb{C}}(V_2) = \text{End}_{\mathbb{C}}(V_1 \otimes V_2) \). So this linear map is a scalar multiple of the identity.

From now on, we consider Hopf algebra \( A = U_1^H \mathfrak{sl}(2) \otimes U_q \mathfrak{gl}(1|1)^\sigma \). A contains both algebras \( U_1^H \mathfrak{sl}(2) \) and \( U_q \mathfrak{gl}(1|1)^\sigma \).

Formally, setting \( q = e^h \), \( q^T := e^{h^T} \) and \( \iota^\sigma = e^{i \frac{x^T}{2}} \), we also consider that
\[
I^H = k = q^{-1} \sigma
\]
which means that we will only study representations of \( A \) that satisfy this relation. Recall from Equations (2.6) the representation of \( U \) with parameter \( a \). We can look for the representations of \( A \) that simultaneously extend to the representations of \( U_1^H \mathfrak{sl}(2) \) and \( U_q \mathfrak{gl}(1|1)^\sigma \) we already described. If \( \varepsilon \in \{0, 1\} \) is the degree of the first vector \( e_0 \) of the basis \( (e_0, e_1) \) we choose, direct computation of such a representation \( V(a, a, 2j, \varepsilon, J) \) shows it is well defined if and only if:

\[
\begin{cases}
(-1)^\varepsilon q^{-2j} = a \\
a = e^{i \frac{x^T(a+1)}{2}} = i^{a+1}
\end{cases}
\]

Setting \( s = q^{i \frac{a-\varepsilon-\varepsilon'}{2}} = \pm 1 \), we can compute the coefficient \( R^H / R^\sigma = D^H / D^\sigma \) given by Proposition 2.6 in our case.

**Proposition 2.7.** \( R^H / R^\sigma = D^H / D^\sigma = ss'(-1)^{\varepsilon \varepsilon'} i^{\varepsilon + \varepsilon'} i^{\frac{a \varepsilon' - 1}{2}} q^{j J' + j' J} \).

**Proof.** Using representation \( V \otimes V' = V(a, a, 2j, \varepsilon, J) \otimes V(a', a', 2j', \varepsilon', J') \) in basis \( (e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1) \), we can write:
\[
D^H = i^{a \varepsilon' / 2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & i^{-\alpha + \alpha' - 1} & 0 & 0 \\
0 & 0 & i^{-\alpha + \alpha' - 1} & 0 \\
0 & 0 & 0 & i^{-\alpha - \alpha' + 1} 
\end{pmatrix}.
\]
Moreover, \( D^\sigma = R_1 D' \) and
\[
R_1 = (-1)^{\varepsilon \varepsilon'} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & (-1)^{\varepsilon} & 0 & 0 \\
0 & 0 & (-1)^{\varepsilon'} & 0 \\
0 & 0 & 0 & (-1)^{\varepsilon + \varepsilon' + 1}
\end{pmatrix},
\]
The superalgebra has 

\[ (q^{-j-j'} - 0 0 0) 
\begin{pmatrix}
  0 & q^{-j-j'} & 0 \\
  0 & 0 & q^{-j+j'} \\
  0 & 0 & 0 
\end{pmatrix}. \]

Since \( a = i^q = (-1)^{q+1}i a^{-2j} \), the formulas make appear two square roots of \( a \):

\[ \sqrt[a]{a} = i^q \quad \text{and} \quad \sqrt[a]{a} = i^q \sqrt[2q-j]{s} \sqrt[a]{a}. \]

That way, computing any of the diagonal coefficients of \( D^H(D^q)^{-1} \) we find the announced element.

### 3. An integral form of \( U_q \mathfrak{gl}(n|1) \) and its specialization

#### 3.1. Quasitriangular Hopf superalgebra \( U_q \mathfrak{gl}(n|1) \)

Here we define the \( h \)-adic quasitriangular Hopf superalgebra \( U_q \mathfrak{gl}(n|1) \) that we will use to construct the Links-Gould invariant \( LG^{n-1} \). The conventions we use for generators and relations are those chosen by Zhang and De Wit in [13] [2]. \( I = \{1, 2, \ldots, n + 1\} \) will be the set of indices. We introduce a grading \( [a] \in \mathbb{Z}/2\mathbb{Z} \) for any \( a \in I \) by setting

\[ [a] = 0 \quad \text{if} \quad a < n \quad \text{and} \quad [a] = 1 \quad \text{when} \quad a = n + 1. \]

The superalgebra has \( (n+1)^2 \) generators divided into three families. There are \( n + 1 \) even Cartan generators \( E^a_a \). There are \( \frac{1}{2} n(n+1) \) lowering generators \( E^a_b \) parametrized by \( a < b \). Finally there are \( \frac{1}{2} n(n+1) \) raising generators \( E^b_a \), with \( a < b \). The degree of \( E^a_b \) is given by \( [a] + [b] \).

For \( a \in I \), \( a \neq n + 1 \), set \( K_a = q^{E^a_a} \), and set \( K_{n+1} = q^{-E^{n+1}_n} \). In the following \( [X, Y] \) denotes the super commutator \( [X, Y] = XY - (-1)^{|X||Y|}YX \).

Now let us present the relations there are between elements of \( U_q \mathfrak{gl}(n|1) \).

For any \( a, b \in I \) with \( |a - b| \geq 2 \) and for any \( c \) in the interval between \( a \) and \( b \),

\[ E^a_c = E^a_c E^c_b - q^{\text{sign}(a-b)} E^c_b E^a_c. \]

For any \( a, b \in I \),

\[ E^a_a E^b_b = E^b_b E^a_a, \quad E^a_a E^b_{b+1} = E^b_{b+1} \left( E^a_a + \delta^a_a - \delta^a_{a+1} \right) \]

\[ [E^a_{a+1}, E^b_{b+1}] = \delta^b_b \frac{K_a K_{a+1}^{-1} - K_{a+1}^{-1} K_a}{q - q^{-1}} \]

which generalizes for \( a < b \) to \( [E^a_b, E^b_a] = \frac{K_a K_{b+1}^{-1} - K_{b+1}^{-1} K_a}{q - q^{-1}} \),

\[ (E^a_{n+1})^2 = (E^a_n)^2 = 0, \quad \text{which implies} \quad (E^a_{n+1})^2 = (E^a_n)^2 = 0 \quad \text{for} \quad i < n + 1. \]

The Serre relations: for any \( a, b \in I \) with \( |a - b| \geq 2 \),

\[ E^a_{a+1} E^b_{b+1} = E^b_{b+1} E^a_{a+1}, \quad E^a_a E^b_{b+1} = E^b_{b+1} E^a_a, \]

and for \( a \leq n - 1 \),

\[ E^a_{a+1} E^a_{a+2} = q E^a_{a+2} E^a_{a+1}, \quad E^a_{a+1} E^a_{a+2} = q E^a_{a+2} E^a_{a+1}, \]

\[ E^a_{a+2} E^a_{a+3} = q E^a_{a+3} E^a_{a+2}, \quad E^a_{a+2} E^a_{a+3} = q E^a_{a+3} E^a_{a+2}. \]

These relations can be completed into a set of “quasi-commutation” relations indexed by pairs of root vectors (see [2] Lemma 1) where a reordering algorithm gives a constructive proof of the Poincaré-Birkhoff-Witt theorem) but these relations are redundant over the field \( \mathbb{C}(q) \).
We consider the Hopf algebra structure given by the coproduct
\[ \Delta(E_a) = E_a \otimes K_a K_a^{-1} + 1 \otimes E_a, \quad \Delta(E_a + 1) = K_a E_a + 1 + E_a \otimes 1 \]
\[ \Delta(K_a) = K_a \otimes K_a \quad \text{and} \quad \Delta(E_a) = E_a \otimes 1 + 1 \otimes E_a \]
which admits\(^2\) the universal \( R \)-matrix \( R^{\text{gl}} = D^{\text{gl}} \bar{R}^{\text{gl}} \) with \( D^{\text{gl}} = \sum_{i < a} E_i^i \otimes E_i^{n+1} + E_i^{n+1} \otimes E_i^i \)
and
\[ \bar{R}^{\text{gl}} = \prod_{i=1}^n \left( \prod_{j=i+1}^n e_q((q-q^{-1})E_i^j \otimes E_j^i) \right) e_q'(E_{n+1}^i \otimes E_{n+1}^{n+1}), \]
where \( e_q'(x) = (1-(q-q^{-1})x), e_q(x) = \sum_{k=0}^\infty \frac{x^k}{(q)_k}, (k)_q = \frac{1-q^k}{1-q} \) and \( (k)_q! = (1)_q(2)_q \ldots (k)_q \).
Remark that the order of the factors matters in \( \bar{R}^{\text{gl}} \).

3.2. Integral form and interesting subalgebras. We now give an integral form of \( U_q \mathfrak{gl}(n|1) \) which supports evaluation at \( q = -1 \). Let \( A_q \) be the \( \mathbb{Z}[q,q^{-1}] \)-subalgebra of \( U_q \mathfrak{gl}(n|1) \) generated by elements \( K_a, E_a^b := (q-q^{-1})E_a^b \) when \( a < b \) and \( E_a^b := E_a^b \) when \( a > b \). The relations of \( U_q \mathfrak{gl}(n|1) \)
\[ [E_a^b, E_a^c] = -\frac{K_a K_b^{-1} - K_b^{-1} K_a}{q-q^{-1}} \]
for \( a < b \), are replaced in algebra \( A_q \) by
\[ [E_a^b, E_a^c] = K_a K_b^{-1} - K_a^{-1} K_b. \]
Still, \( A_q \) admits a presentations similar to that of \( U_q \mathfrak{gl}(n|1) \). No additional relations are needed because the analog of the above commutation relations are enough to express any element in the Poincaré-Birkhoff-Witt basis.

In the bosonization \( A_q^\sigma \) of \( A_q \), define for \( i = 1, \ldots, n \) the algebra
\[ A_i = \langle e_i = -E_n^i, f_i = E_{n+1}^i, k_i = K_i K_i^{-1} \rangle \subset A_q^\sigma. \]

Proposition 3.1. Algebra \( A_i \) is isomorphic to \( U \). Indeed:
\[ e_i f_i - f_i e_i = k_i - k_i^{-1}, \]
\[ k_i e_i + e_i k_i = k_i f_i + f_i k_i = 0. \]
Proof. Direct computations from the defining relations of \( A_q \) and Lemma 1 of [2]. In particular, \( e_i f_i - f_i e_i = -E_{n+1}^i \sigma E_{n+1}^i + E_{n+1}^i \sigma E_{n+1}^i = [E_{n+1}^i, E_{n+1}^i] \sigma = k_i - k_i^{-1}. \]

Remark 3.2. However, \( A_i \) is not isomorphic to \( U \) as a Hopf algebra (except for \( A_n \)), which can be seen by looking at the coproduct of elements of \( A_i \) in \( A_q \). This will not be a problem for us.

Set \( 1 \leq i \neq j \leq n \). Using [2] Lemma 1 once again, we want to see at what conditions any \( x \in A_i \) and \( y \in A_j \) commute.

Lemma 3.3. We have the following commutations:
\[ e_i e_j = -q^{-1} e_j e_i, f_i f_j = -q^{-1} f_j f_i, k_i k_j = k_j k_i, \]
\[ \text{if } i < j, e_i f_j - f_j e_i = \sigma K_j K_{n+1}^{-1} E_j^i, \text{ otherwise } e_i f_j - f_j e_i = \sigma (q-q^{-1}) E_j^i K_{n+1} K_i^{-1}, \]
\[ k_j e_i = -q^{-1} e_i k_j, k_j f_i = -q f_i k_j. \]
\(^2\)we use here the coproduct and \( R \)-matrix of [2] conjugated by \( D^{\text{gl}} \).
Proof. The first two equalities correspond to \cite[Eq. (38) and (39)]{2} and the two brackets \([e_i, f_j]\) correspond to \cite[Eq. (36) (c) and (d)]{2}.

Corollary 3.4. Setting \(q = -1\), in any quotient of \(A^\infty_{-1}\) such that for any \(1 \leq i < j \leq n\), \(E^j_i = 0\), the elements of two distinct \(A_i\) commute.

3.3. Highest weight representation \(V(0^n, \alpha)\). Let \(V(0^n, \alpha)\) be the highest weight irreducible \(2^n\)-dimensional representation of \(U_q\mathfrak{gl}(n|1)\) of weight \((0^n, \alpha)\), with \(\alpha \notin \mathbb{Z}\). So \(E^i_j\) is represented by 0, except for \(E^{n+1}_i\) that is represented by \(\alpha\). Set \(v_0\) a highest weight vector in \(V(0^n, \alpha)\) and let \(V_q(0^n, \alpha) = A_qv_0\). The Poincaré-Birkhoff-Witt theorem proves that 

\[
\left( \prod_{i=1}^n f_i^{m_i} v_0 \right)_{m_i \in \{0, 1\}}
\]

is a basis for vector space \(V(0^n, \alpha)\) and for the free \(\mathbb{Z}[q, q^{-1}]\)-module \(V_q(0^n, \alpha)\). Set \(A^\infty_{-1} = A^\infty_q \otimes_{q=-1} \mathbb{C}\) and \(V_{-1}(0^n, \alpha) = V_q(0^n, \alpha) \otimes_{q=-1} \mathbb{C}\)

**Proposition 3.5.** In the representation \(V_{-1}(0^n, \alpha)\), for any \(1 \leq i < j \leq n\), \(E^i_j v = 0\). So \(E^i_j\) belongs to the kernel \(I\) of the representation \(A^\infty_{-1} \to \text{End}(V_{-1}(0^n, \alpha))\). As a consequence, the following map is well defined:

\[
\Theta : \bigotimes_{i=1}^n A_i \otimes_x \mathbb{C} \to \frac{A^\infty_{-1}}{I} \otimes_x \mathbb{C}.
\]

Proof. We want to show that for any basis vector \(v \in V_{-1}(0^n, \alpha)\) and for \(1 \leq i < j \leq n\), \(E^i_j v = 0\). We can write \(v = f_1^{n_1} \cdots f_n^{n_n} v_0\) where \(n_k = 0, 1\).

Using \cite[Lemma 1 once more, if \(c < i\) then \(E^i_j, E^{n+1}_j = 0\) by \cite[Eq. (40)]{2}\]

\[
c < i \quad \text{then} \quad [E^i_j, E^{n+1}_j] = 0 \text{ by } \cite[Eq. (40)]{2}.
\]

\[
i < j \quad \text{then} \quad [E^i_j, E^{n+1}_j] = 0 \text{ by } \cite[Eq. (37), (40)]{2}.
\]

In all cases, \([E^i_j, f_c] = [E^i_j, E^{n+1}_j] = 0\) in \(A^\infty_{-1}\). So \(E^i_j v = f_1^{n_1} \cdots f_n^{n_n} (E^i_j v_0)\). But \(E^i_j\) is a raising generator, so \(E^i_j v_0 = 0\). Using Corollary 3.4 for \(i \neq j\) \(A_i\) and \(A_j\) commute in that representation.

3.4. \(\tilde{R}^g\) makes sense when \(q = -1\). Here we intend to show that the non diagonal part \(\tilde{R}^g\) of the universal \(R\)-matrix of \(U_q\mathfrak{gl}(n|1)\) supports evaluation at \(q = -1\), which is not obvious given the formula defining \(\tilde{R}^g\). In the bosonization \(U_q\mathfrak{gl}(n|1)\), the universal \(R\)-matrix is given by

\[
(R^g)^\sigma = D^g(\tilde{R}^g)^\sigma = D^g \prod_{i=1}^n \left( \prod_{j=i+1}^n c_q(E^i_j \otimes E^j_i) \right) (1 + e_i \otimes f_i).
\]

**Proposition 3.6.** For any \(1 \leq i < j \leq n\),

\[
\left( c_q(E^i_j \otimes E^j_i) - 1 \right) V_q(0^n, \alpha) \otimes V_q(0^n, \alpha) \subset (q + 1) \mathbb{Z}[q, q^{-1}]_{\text{loc}} V_q(0^n, \alpha) \otimes V_q(0^n, \alpha)
\]

where \(\mathbb{Z}[q, q^{-1}]_{\text{loc}}\) is the localization of \(\mathbb{Z}[q, q^{-1}]\) at \((q + 1)\). Hence \((R^g)^\sigma\) induces a well defined automorphism of \(V_{-1}(0^n, \alpha) \otimes V_{-1}(0^n, \alpha)\) where the action of \((R^g)^\sigma\) is given by

\[
(R^g)^\sigma = \prod_{i=1}^n (1 + e_i \otimes f_i).
\]
Proof. Define $V = \mathbb{Z}[q, q^{-1}]_{\omega} V_q(0^n, \alpha) \subset V(0^n, \alpha)$ so that $V_{-1}(0^n, \alpha) \cong V \otimes_{q=1} \mathbb{C}$. We wish to prove that for $1 \leq i < j \leq n$, in the representation $V \otimes V$, $e_q(\mathcal{E}_j^i \otimes \mathcal{E}_j^i) = 1$ mod $(q+1)$. Set $1 \leq i < j \leq n$. We show by induction on $k \geq 1$, that

$$\frac{\mathcal{E}_j^i}{(k)_q} V \subset (q+1)V.$$

For $k = 1$, it follows from $\mathcal{E}_j^i \in I$ (see Proposition 3.5). Now we suppose it holds for any $l \in \{1, \ldots, k-1\}$ and since $\frac{\mathcal{E}_j^i}{(k)_q} = \frac{\mathcal{E}_j^i}{(k-1)_q} \frac{\mathcal{E}_j^i}{(k)_q}$ it is enough to show that $\frac{\mathcal{E}_j^i}{(k)_q} V \subset V$.

We know that $\mathcal{E}_j^i V \subset (q+1)V$, so $\frac{\mathcal{E}_j^i}{(k)_q} V \subset \frac{q+1}{(k)_q} V$.

If $k$ is even, $(k)_q = (q+1)\left(\frac{k}{2}\right)_{q^2}$ with $\frac{k}{2} \equiv 1$ mod $(q+1)$ so $\frac{\mathcal{E}_j^i}{(k)_q} V \subset \frac{1}{\left(\frac{k}{2}\right)_{q^2}} V = V$.

If $k$ is odd, $(k)_q = 1$ mod $(q+1)$ and therefore $\frac{\mathcal{E}_j^i}{(k)_q} V \subset (q+1)V$. This concludes the proof. \[3.6\]

4. Links-Gould invariants and the conjecture

4.1. Links-Gould invariants $LG^{n,1}$. The Links-Gould invariants $LG^{n,1}$ are framed link invariants obtained by applying the modified (one has to use a modified trace, see [3]) Reshetikhin-Turaev construction to the ribbon Hopf algebras $U_q \mathfrak{gl}(n|1)^\sigma$ we just studied. Like in the Alexander case, the $R$-matrix can be divided by the value of the twist so that $LG^{n,1}$ becomes an unframed link invariant. Note that this definition and Viro’s work [13] show that the first LG invariant $LG^{1,1}$ coincides with the Alexander-Conway polynomial $\Delta$.

There are several sets of variables used in papers studying $LG$ invariants. Three of them appear regularly: $(t_0, t_1)$, $(\tau, q)$ and $(q^n, q)$. Each set can be expressed in terms of the others using the following defining relations:

$$t_0 = q^{-2\alpha}, \quad t_1 = q^{2\alpha + 2},$$

$$\tau = t_0^{1/2} = q^{-\alpha}.$$

In the case of $LG^{2,1}$, variables $(t_0, t_1)$ nicely lead to a symmetric Laurent polynomial that has all sorts of Alexander-type properties [6].

Here we are interested in what happens to $LG^{n,1}$ when you evaluate $q$ at $-1$, or in other words when you set $t_0 t_1 = 1$.

4.2. Proof of the conjecture. Our study of ribbon Hopf algebra $U_q \mathfrak{gl}(n|1)^\sigma$ allows us to prove the following, that was conjectured in [3]:

**Theorem 4.1.** For any link $L$ in $S^3$, $LG^{n,1}(L; \tau, -1) = \Delta_L(\tau^2)^n$. This can be translated in variables $(t_0, t_1)$:

$$LG^{n,1}(L; t_0, t_0^{-1}) = \Delta_L(t_0)^n.$$  

The rest of the section is devoted to proving this identity. First we identify $V_{-1}(0^n, \alpha)$ as a $\otimes_i A_i$-module as a $\otimes_i A_i$-module isomorphic to the 2-dimensional $U$-module $V_{q^{-\alpha}}$.

**Proposition 4.2.** Equipped with the action of $\otimes_i A_i$ induced by $\Theta : \otimes_i A_i \rightarrow \mathcal{A}_n^\sigma / I$, $V_{-1}(0^n, \alpha)$ is isomorphic to the irreducible representation $\otimes_i V^i$ where each $V^i$ is an $A_i$-module isomorphic to the 2-dimensional $U$-module $V_{q^{-\alpha}}$. 

Proof. By \( \otimes_i V^i \), we mean the representation
\[
\otimes_i \rho_i : \otimes_i A_i \to \otimes_i \text{End}_C(V^i) \cong \text{End}_C(\otimes_i V^i)
\]
where \( \rho_i : A_i \to \text{End}_C(V^i) \). Set \( a = q^{-a} \). For each \( i, k_i^2 \) acts by \( a^2 \) on \( V_{-1}(
abla^0, \alpha) \). Thus \( V_{-1}(
abla^0, \alpha) \) is a representation of the \( 8^n \)-dimensional semi-simple algebra \( \otimes_{i=1}^j (A_i/(k_i^2 - a^2)) \). But for each \( A_i \), \( v_0 \) is the highest weight vector of weight \( a \) so it belongs to a summand of the \( \otimes_{i=1}^j (A_i/(k_i^2 - a^2)) \)-module \( V_{-1}(
abla^0, \alpha) \) of the form \( \otimes_i V^i \). Comparing the dimensions which are equal to \( 2^n \) for both vector spaces, we have that \( V_{-1}(
abla^0, \alpha) \cong \otimes_i V^i \).

Now we study the action of the pivotal element of \( A_q^a \) in the representation at \( q = -1 \).

**Proposition 4.3.** If \( K_{2\rho}^v \) is the pivotal element of \( A_q^a \), in the representation \( V_{-1}(
abla^0, \alpha) \),
\[
K_{2\rho}^v = \Theta(\otimes_i \phi_i)
\]
where \( \phi_i = k_i^{-1} \in A_i \).

**Proof.** The antipode of \( U_q gl(n,1) \) satisfies \( S(E_i^{1+}) = -E_i^{1+}K_{i+1}K_i^{-1} \) and \( S(2E_i^{1+}) = K_iK_{i+1}E_i^{1+}K_{i+1}K_i^{-1} = K_{2\rho}E_i^{1+}K_{2\rho}^{-1} \). We can write \( K_{2\rho} \) in terms of Cartan generators:
\[
K_{2\rho} = K_n \prod_{k=1}^n K_i^{-2i}.
\]
Denoting \( (a,b) := \sum_{i=1}^n a_ib_i - a_{n+1}b_{n+1} \), and \( \rho \) the graded half sum of all positive roots, we find:
\[
2\rho = \sum_{i=1}^n (n - 2i)e_i + n\varepsilon_{n+1},
\]
where \( \varepsilon_i \) is the \( i \text{th} \) basis vector of \( \mathbb{C}^{n+1} \) and we write any vector \( x = \sum_{i=1}^{n+1} x_i \varepsilon_i \) in this basis. \( K_{2\rho} \) conjugates element \( e_i \in A_i \) as follows:
\[
K_{2\rho}e_i K_{2\rho}^{-1} = q^{(2\rho)(\varepsilon_i - \varepsilon_{n+1})} e_i
= q^{(n-2i+n)} e_i
= q^{2n-2i} e_i.
\]
So if \( q = -1 \),
\[
\sigma K_{2\rho}e_i K_{2\rho}^{-1} \sigma = -e_i
= \phi_i e_i \phi_i^{-1}
= \Theta(\otimes j \phi_j)(\varepsilon_i \Theta(\otimes j \phi_j^{-1})).
\]
Similarly to Proposition 2.6 we therefore can say that in the irreducible \( \otimes_i A_i \)-module \( V_{-1}(
abla^0, \alpha) \), \( K_{2\rho}^v \) is a scalar multiple of \( \Theta(\otimes j \phi_j) \). We call this element \( \lambda \). Since the two maps both act by \( q^{n\varepsilon} \) on the highest weight vector, we conclude that \( \lambda = 1 \).

**Proposition 4.4.** For any \( x \in A_i \otimes A_i \subset A_q \otimes A_q \), we have
\[
D^{\varepsilon_i} x (D^{\varepsilon_i})^{-1} = \mathcal{D}(x)
\]
where we identified \( A_i \otimes A_i \cong U \otimes U \).
The proof in this paper should adapt to show that

\[ \text{Let us sum up what we proved up to now to obtain 4.1. Let} \]

Proof of Theorem 4.1.

Proof. Let us consider the trace of the maps \( D^g \) and \( D^g \) as \( \tau \circ D^g = D^g \circ \tau \).

Proof of Theorem 4.2. Let us sum up what we proved up to now to obtain 4.1. Let \( V_1(0^n, \alpha) \simeq \bigotimes_{i=1}^n V^i \) be the isomorphic representations of Proposition 4.2. In the following we fix such an isomorphism. Let \( V^i \) be a \( U_{\text{sl}(2)} \)-module structure on \( V^i \) extending the representation of \( A_i \). We therefore obtain \( n \) commuting R-matrices \( R^g = D^g \beta \) in \( \text{End}_C(V^i \otimes V^j) \). Hence the braiding on \( \bigotimes_{i=1}^n V^i \) and on \( V_1(0^n, \alpha) \otimes V_1(0^n, \alpha) \) is proportional. Now the process of computing both the Links-Gould invariant and the Alexander polynomial, the R-matrices are rescaled by the inverse of their twist \( \theta^{-1} \) so that the invariants become framing independent:

\[ \text{trace}_g(\theta^{-1}(\text{Id} \otimes \phi) \tau R) = \text{Id}_{V_1(0^n, \alpha)} \]

(here \( \phi \) denotes any of the pivotal structures which are equal by Proposition 3.6). Hence the rescaled R-matrices \( R^g_{q=1} = \prod_i \chi_i(R_{V_i \otimes V_i}^H) \) and \( \bigotimes_{i=1}^n R_{V_i \otimes V_i}^H \) are equal up to reordering factors. Finally, for any braid \( \beta \in B_n \), the associated operators by the Reshetikhin-Turaev construction correspond up to reordering as well:

\[ \Psi_{V_1(0^n, \alpha) \otimes \ell}^g(\beta) = \left( \Psi_{V_1(0^n, \alpha) \otimes \ell}^{U_{\text{sl}(2)}}(\beta) \right)^{\otimes n} \]

At the end, if \( \text{trace}_g \cdots (\text{Id}_{V_1(0^n, \alpha)} \otimes \phi \otimes \cdots) \circ \Psi_{V_1(0^n, \alpha) \otimes \ell}^g(\beta) = d \cdot \text{Id}_{V_1(0^n, \alpha)} \) when \( \text{trace}_g \cdots (\text{Id}_{V_1(0^n, \alpha)} \otimes \phi \otimes \cdots) \circ \Psi_{V_1(0^n, \alpha) \otimes \ell}^{U_{\text{sl}(2)}}(\beta) = c \cdot \text{Id}_{V_1(0^n, \alpha)} \), we obtain

\[ d = c^n \]

by considering the trace of these two maps. Indeed, the trace is blind to reordering factors.

Remark 4.5. In [4], the \( LG \) invariant is extended to a multivariable link invariant \( M(L; q, q_1, \ldots, q_c) \) for links with \( c \geq 2 \) ordered components, taking its values in Laurent polynomials \( \mathbb{Z}[q^\pm, q_1^\pm, \ldots, q_c^\pm] \). It is shown in [4] that

\[ LG^{n,1}(\tau, q) = \left( \prod_{i=0}^{n-1} \frac{q_i - \tau}{q_i} \right) M(L; q, \tau^{-1}, \ldots, \tau^{-1}). \]

The proof in this paper should adapt to show that

\[ M(L; -1, q_1, \ldots, q_c) = \nabla(q_1, \ldots, q_c)^n \]

where \( \nabla \) is the Conway potential function, a version of the multivariable Alexander polynomial.
References

1. F. Costantino, N. Geer, B. Patureau-Mirand - Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories. J. Topology (2014) 7 (4) 1005–1053.

2. D. De Wit - Automatic construction of explicit R matrices for the one-parameter families of irreducible typical highest weight representations of $U_q[gl(m|n)]$. Comput. Phys. Commun. 145:2, 205-255.

3. D. De Wit, A. Ishii, J. Links - Infinitely many two-variable generalisations of the Alexander–Conway polynomial. Algebraic & Geometric Topology 5 (2005) 405–418.

4. N. Geer, B. Patureau-Mirand - Multivariable link invariants arising from Lie superalgebras of type I. J. Knot Theory Ramifications 19, Issue 1 (2010) 93–115.

5. N. Geer, B. Patureau-Mirand - On the Colored HOMFLY-PT, Multivariable and Kashaev Link Invariants. Commun. Contemp. Math. 10 (2008), no. 1 supp, 993-1011.

6. A. Ishii - The Links-Gould polynomial as a generalization of the Alexander-Conway polynomial, Pacific Journal of Mathematics, 06/2006; 225(2):273-285.

7. S.M. Khoroshkin, V.N. Tolstoy - Universal R-matrix for quantized (super)algebras. Comm. Math. Phys. 141 (1991), no. 3, 599–617.

8. B.M. Kohli - On the Links-Gould invariant and the square of the Alexander polynomial, J. Knot Theory Ramifications, Vol. 25, No. 02, 1650006 (2016).

9. S. Majid - Cross products by braided groups and bosonization, Journal of Algebra 163 (1), 165-190.

10. J. Murakami - The multi-variable Alexander polynomial and a one-parameter family of representations of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ at $q^2 = -1$. Quantum groups (Leningrad, 1990), 350–353, Lecture Notes in Math., 1510, Springer, Berlin, 1992.

11. B. Pareigis - A non-commutative non-cocommutative Hopf algebra in "nature". Journal of Algebra, Volume 70, Issue 2, June 1981, Pages 356-374.

12. N. Reshetikhin - Quasitriangularity of Quantum Groups at Roots of 1. Comm. Math. Phys. 170 (1995), 79–99.

13. N. Reshetikhin, V.G. Turaev - Ribbon graphs and their invariants derived from quantum groups. Comm. Math. Phys. 127 (1990), no. 1, 1–26.

14. O. Viro - Quantum relatives of the Alexander polynomial. Algebra i Analiz 18 (2006), no. 3, 63–157; translation in St. Petersburg Math. J. 18 (2007), no. 3, 391–457.

15. R.B. Zhang - Universal L operator and invariants of the quantum supergroup $U_q(gl(m|n))$, J. Math. Phys. 33 (1992), 1970–1979.