Corrigendum

Corrigendum: Fourth order superintegrable systems separating in polar coordinates. I. Exotic potentials (J. Phys. A: Math. Theor. 50 495206)

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The left-hand side of the original equation (27), page 9, must be multiplied by $6r^3$. The right-hand side is unchanged. The correct formula is the following:

\[
6r^2 \Theta(\theta) = 288 \left[ B_1 \sin 2\theta - B_2 \cos 2\theta - 8D_1 \sin 4\theta + 8D_2 \cos 4\theta \right] S \\
+ 120 \left[ 20D_1 \cos 4\theta + 20D_2 \sin 4\theta - B_1 \cos 2\theta - B_2 \sin 2\theta \right] S' \\
+ 60 \left[ B_1 \sin 2\theta - B_2 \cos 2\theta + 14D_1 \sin 4\theta - 14D_2 \cos 4\theta \right] S'' \\
- 30 \left[ B_1 \cos 2\theta + B_2 \sin 2\theta + 4D_1 \cos 4\theta + 4D_2 \sin 4\theta \right] S^{(3)} \\
- 3 \left[ B_1 \sin 2\theta - B_2 \cos 2\theta + 2D_1 \sin 4\theta - 2D_2 \cos 4\theta \right] S^{(4)} \\
- r \left[ 6 \left( A_4 \sin \theta + A_3 \cos \theta - 9C_1 \sin 3\theta - 9C_2 \cos 3\theta \right) S \\
- 120 \left( A_4 \cos \theta - A_3 \sin \theta + 9C_2 \sin 3\theta - 9C_1 \cos 3\theta \right) S' \\
+ 480 \left( C_1 \sin 3\theta + C_2 \cos 3\theta \right) S'' \\
+ 30 \left( A_3 \sin \theta - A_4 \cos \theta + 3C_2 \sin 3\theta - 3C_1 \cos 3\theta \right) S^{(3)} \\
- 6 \left( A_4 \sin \theta + 3A_3 \cos \theta + 3C_1 \sin 3\theta + 3C_2 \cos 3\theta \right) S^{(4)} \right] \\
+ 576r^2 \left[ D_2 \cos 4\theta - D_1 \sin 4\theta \right] R \\
+ 6r^3 \left[ 6r^3 \left( A_2 \sin \theta + A_1 \cos \theta \right) - 4r^2 \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) \\
- r \left( A_4 \sin \theta + A_3 \cos \theta + 9C_1 \sin 3\theta + 9C_2 \cos 3\theta \right) \\
- 3B_1 \sin 2\theta + 3B_2 \cos 2\theta + 66D_1 \sin 4\theta - 66D_2 \cos 4\theta \right] R' \\
+ 6r^4 \left[ 6r^3 \left( A_2 \sin \theta + A_1 \cos \theta \right) - 4r^2 \left( B_3 \sin 2\theta - B_4 \cos 2\theta \right) \\
+ r \left( A_4 \sin \theta + A_3 \cos \theta + 9C_1 \sin 3\theta + 9C_2 \cos 3\theta \right) \\
+ 3 \left( B_1 \sin 2\theta - B_2 \cos 2\theta - 6D_1 \sin 4\theta + 6D_2 \cos 4\theta \right) \right] R'' \\
+ 3r^5 \left[ 2r^3 \left( A_2 \sin \theta + A_1 \cos \theta \right) + 4r^2 \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) \\
+ 2r \left( A_4 \sin \theta + A_3 \cos \theta - 3C_1 \sin 3\theta - 3C_2 \cos 3\theta \right) \\
- 2 \left( B_1 \sin 2\theta - B_2 \cos 2\theta + 2D_2 \cos 4\theta - 2D_1 \sin 4\theta \right) \right] R^{(3)}. \tag{1} \]

All the results and conclusions of the paper remain unchanged. Dividing both sides of the above formula by $6r^3$ and then differentiating it with respect to $r$ we obtain the equation (22), page 7, presented in *J. Phys. A: Math. Theor.* 51 455202.

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Fourth order superintegrable systems separating in polar coordinates. I. Exotic potentials

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Abstract
We present all real quantum mechanical potentials in a two-dimensional Euclidean space that have the following properties: 1. They allow separation of variables of the Schrödinger equation in polar coordinates, 2. They allow an independent fourth order integral of motion, 3. Their angular dependent part $S(\theta)$ satisfies a nonlinear ODE that has the Painlevé property and its solutions can be expressed in terms of the Painlevé transcendent $P_6$. We also study the corresponding classical analogs of these potentials. The polynomial algebra of the integrals of motion is constructed in the classical case.

Keywords: superintegrability, Painlevé property, separation of variables, exotic potentials

1. Introduction

This article is part of a series devoted to a study of classical and quantum superintegrable systems. Roughly speaking, a Hamiltonian with $n$ degrees of freedom is integrable if it allows $n$ independent well defined integrals of motion in involution. It is minimally superintegrable if it allows $n+1$ such integrals, maximally superintegrable if it allows $2n-1$ integrals (only subsets of $n$ integrals among them can be in involution).

The best known superintegrable systems are the harmonic oscillator with its $su(n+1)$ algebra of integrals, and the Kepler–Coulomb system with its $o(n+1)$ algebra (when restricted to fixed bound state energy values).
A recent review article gives more precise definitions, a general setting and motivation for studying superintegrable systems [54]. It follows from Bertrand’s theorem [7] that $\omega r^2$ and $\alpha/r$ are the only maximally superintegrable spherically symmetrical potentials in Euclidean real space $E_n$. Systematic searches for superintegrable classical and quantum systems in $E_2$ and $E_3$ established a connection between second order superintegrability and multiseparability in the Schrödinger or Hamilton–Jacobi equation [9, 10, 27, 28, 45, 48].

An extensive literature exists on second order superintegrability in spaces of 2, 3 and $n$ dimensions, Riemannian and pseudo-Riemannian, real or complex, [39–43, 47].

A systematic study of higher order integrability is more recent. Pioneering work is due to Drach [22, 23]. For more recent work see [6, 11–13, 26, 33, 34, 36, 37, 50–52, 58, 60–63, 64].

The Painlevé transcendents were first introduced in a purely mathematical study by Painlevé [56] and Gambier [30] and were popularized in books e.g. by Ince [38] and Davis [20]. They are characterized by the fact that they are solutions of second order nonlinear ODEs that are single valued about any movable singularity of the ODE (movable means that the position of the singularity depends on the initial conditions). We shall call this the Painlevé property. Painlevé and Gambier also classified all ODEs with the Painlevé property of the form $y'' = R(x, y, y')$ with $R$ rational in $y$ and $y'$ and analytical in $x$ into 50 equivalence classes under the action of the group preserving the Painlevé property. Of these 50 six give rise to the famous irreducible Painlevé transcendents. The others can either be reduced to one of these six, or integrated in terms of already known functions like elliptic functions, or solutions of linear equations. Linear ODEs have the Painlevé property by default: all the singularities of their solutions are fixed, i.e. they can only occur where the coefficients of the ODEs are themselves singular.

In this sense we can say that the nonlinear ODEs with the Painlevé property are the closest ones to linear ODEs. Nonlinear equations with the Painlevé property became important in applications after the discovery of the inverse scattering theory by Kruskal et al [31] and more generally of soliton theory (for review see e.g. [1, 14, 55] and references therein). A Painlevé test was proposed [2–4], a simple algorithmic test the passing of which is a necessary condition for an ODE to have the Painlevé property. A Painlevé conjecture was formulated [12], namely that a necessary condition for a PDE to be integrable by inverse scattering techniques is that all of the ODEs obtained as reductions of the PDE should have the Painlevé property.

A systematic search for analytical solutions of many of the PDEs of hydrodynamics, plasma physics, and nonlinear optics lead to various Painlevé transcendents. Painlevé transcendents to our knowledge appeared for the first time in quantum mechanics in articles by Fushchych and Nikitin [29] and by Doebner and Zhdanov [21]. A systematic search for superintegrable systems in $E_2$ with one integral of motion of order $N \geq 3$ and two others of order $N \leq 2$ was started in [32] and [33] (for $N = 3$). Exotic potentials, by definition not satisfying any linear ODE, were obtained. It turned out that they could always be expressed in terms of the Painlevé transcendents $P_1, P_2, P_4$ or elliptic functions. The lower order integrals were chosen to be of Cartesian type that is they forced the potential to allow separation of variables in Cartesian coordinates. A similar study for $N = 3$ was conducted for third order integrals of polar type [62]. Exotic potentials appeared again and this time they were expressed in terms of $P_0$. For $N = 4$ the situation is similar [52], namely, exotic potentials appear in the Cartesian case, expressed in terms of $P_1, ..., P_5$. For $N = 4$, the polar case is the present article, and as we shall see below exotic potentials exist. Unlike the case $N = 3$, they are expressed in terms of the completely general $P_0$ transcendent. Specific results have also been obtained for $N = 5$ in the Cartesian case [5]. New features appear here, namely potentials expressed in terms of solutions of higher order ODEs with the Painlevé property. We conjecture that for all $N \geq 3$ exotic potentials will exist and be solutions of ODEs with the Painlevé property.
The present article is a contribution to a series [32, 33, 49, 52, 57, 59, 62] devoted to superintegrable systems in \( E_2 \) with one integral of order \( n \geq 3 \) and one of order \( n \leq 2 \). In particular, it is a generalization of a paper [62] devoted to the case of a fourth order integral \( Y \).

In this article we restrict ourselves to the space \( E_2 \). The Hamiltonian has the form

\[
H = \frac{1}{2}(p_x^2 + p_y^2) + V(x,y),
\]

(1)

in classical mechanics \( p_x \) and \( p_y \) are the momenta conjugate to the Cartesian coordinates \( x \) and \( y \). In quantum mechanics they are the corresponding operators \( p_x = -i\hbar \frac{\partial}{\partial x} \) \( p_y = -i\hbar \frac{\partial}{\partial y} \). In polar coordinates \( (x,y) \equiv (r \cos \theta, r \sin \theta) \), the classical Hamiltonian reads

\[
H = \frac{1}{2} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) + V(r, \theta), \quad V(r, \theta) = R(r) + \frac{1}{r^2} S(\theta),
\]

(2)

here \( p_r \) and \( p_\theta \) are the associated canonical momenta. The corresponding quantum operator takes the form

\[
H = -\frac{k^2}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) + V(r, \theta).
\]

(3)

In this article we concentrate on quantum superintegrability and on ‘exotic’ potentials, namely those that do not satisfy any linear differential equations. In all equations we keep the Planck constant \( \hbar \) explicitly. Classical exotic potentials will be obtained in the limit \( \hbar \to 0 \). We emphasize that this limit is singular: highest order terms in the equation which defines the potential in (1) vanish, so the classical and quantum cases can differ greatly.

In addition to the Hamiltonian \( H \), we have two more conserved quantities which are

\[
X = p_\theta^2 + 2S(\theta),
\]

(4)

\[
Y = \sum_{i,j,k=4} A_{ijk} \{ L_i^-, p_j^-, p_k^+ \} + \{ g_1(x,y), p_1^2 \} + \{ g_2(x,y), p_x p_y \}
\]

\[
+ \{ g_3(x,y), p_y^2 \} + g_4(x,y),
\]

(5)

here \( p_\theta = xp_y - yp_x = -i\hbar \frac{\partial}{\partial \theta} \). The bracket \( \{ \cdot, \cdot \} \) denotes an anticommutator, the set \( \{ A_{ijk} \} \) are real constants and \( R(r), S(\theta), g_{1,2,3,4}(x,y) \) are real functions such that the commutators

\[
[H, Y] = [H, X] = 0.
\]

(6)

The operator \( Y \) in (5) is given in Cartesian coordinates for brevity. Putting

\[
p_x = -i \hbar (\cos \theta \frac{\partial}{\partial \theta} - \frac{\sin \theta}{r} \frac{\partial}{\partial r}), \quad p_y = -i \hbar (\sin \theta \frac{\partial}{\partial \theta} + \frac{\cos \theta}{r} \frac{\partial}{\partial r}),
\]

we obtain the corresponding expression in polar coordinates. It’s leading terms are given explicitly below in (9) and used throughout this article.

We have \( [Y, X] = C \neq 0 \) where \( C \) is in general a fifth order linear operator. In general, we thus obtain a finitely generated polynomial algebra of integrals of motion [18, 19, 39, 40, 44, 46, 51, 53]. We are looking for fourth-order superintegrable systems, so at least one of \( A_{ijk} \) is different from zero. The operator \( Y \) is the most general polynomial expression for a fourth-order Hermitian operator of the required form. The commutator \([H, Y]\) contains derivatives of order up to three.

Before calculating the commutator \([H, Y]\) we note that three ‘trivial’ fourth order integrals exist, namely \( X^2, H^2 \) and \( \{X, H\} \). Each of these is a scalar (invariant) under \( O(2) \) rotations.

\[
\]
By linear combinations of the form $Y + u_1 X^2 + u_2 H^2 + u_3 X H$, where the $u_i$ are constants, we can eliminate 3 parameters among the $\{A_{ij}\}$ and consequently three terms in $Y$. Now, we introduce a more convenient set of parameters defined by the relations

$$A_{004} = \frac{1}{2}(D_1 - B_1), \quad A_{040} = \frac{1}{2}(B_1 + D_1), \quad A_{400} = 0,$$

$$A_{022} = -3D_1, \quad A_{013} = B_2 - 2D_2, \quad A_{031} = B_2 + 2D_2,$$

$$A_{103} = A_4 - C_1, \quad A_{301} = A_2, \quad A_{220} = B_3,$$

$$A_{121} = 3C_1 + A_4, \quad A_{112} = A_3 - 3C_2, \quad A_{310} = A_1,$$

$$A_{211} = 2B_4, \quad A_{130} = C_2 + A_3, \quad A_{202} = -B_3. \quad (7)$$

With the above parameters the fourth-order integral (5) takes the following form:

$$Y = A_1 \{L_1^3, p_x\} + A_2 \{L_1^3, p_y\} + A_3 \{L_1^3, p_z\} (p_x^2 + p_y^2\} + A_4 \{L_1^3, p_z\} (p_x^2 + p_y^2\} + B_1 (p_z^4 - p_x^4 + 2B_2 p_x p_y (p_x^2 + p_y^2) + B_3 \{L_2^3, p_z - p_x^2\} + 2B_4 \{L_2^3, p_x p_y\}
+ C_1 \{L_3, 3p_x p_y - p_z^3\} + C_2 \{L_3, 3p_2 p_x + 3p_2 p_y + D_1 (p_x^4 + p_y^4 - 6p_x^2 p_y^2)
+ 4D_2 p_x p_y (p_x^2 - p_y^2) + lower \text{ order \ terms}. \quad (8)$$

Under rotations around the z-axis, each of the six pairs of parameters $(A_1, A_2), (A_3, A_4), (B_1, B_2), (B_3, B_4), (C_1, C_2), (D_1, D_2)$, in (8) forms a doublet (all $O(2)$ singlets have been removed). Under rotations through the angle $\theta$ the doublets $A_1, B_1, C_1, D_1$ rotate through $\theta, 2\theta, 3\theta$ and $4\theta$, respectively. In particular, the doublets $(A_1, A_2)$ and $(B_1, B_2)$ will play a central role in the main equations of the present paper. Explicitly, in polar coordinates, the leading terms of the integral $Y$ are

$$\frac{Y}{\hbar^4} = (B_1 \cos \theta + B_2 \sin 2\theta + D_1 \cos 4\theta + D_2 \sin 4\theta) \partial_\theta^2 \frac{1}{r^2} \left[ D_2 \sin 4\theta \right.
+ D_1 \cos 4\theta - 2r (A_1 r^2 + A_4) \sin \theta - (B_2 + 2B_4 r^2) \sin 2\theta + 2r (A_2 r^2 + A_4) \cos \theta
- (B_1 + 2B_3 r^2) \cos 2\theta - 2r (C_1 \cos 3\theta - C_2 \sin 3\theta) \right] \partial_\theta^2 \frac{1}{r^2}
- \left. \frac{2}{r^2} \right) \left[ 3(D_1 \cos 4\theta + D_2 \sin 4\theta) - r^2(B_3 \cos 2\theta + B_4 \sin 2\theta) \right.
+ r(A_3 \sin \theta - A_4 \cos \theta - 3(C_1 \cos 3\theta - C_2 \sin 3\theta)) \right] \partial_\theta^2 \frac{1}{r^2}
- \left. \frac{2}{r^2} \right) \left[ B_1 \sin 2\theta - B_2 \cos 2\theta + 2(D_1 \sin 4\theta - D_2 \cos 4\theta) - r(C_1 \sin 3\theta + C_2 \cos 3\theta
+ A_3 \cos \theta + A_4 \sin \theta \right] \partial_\theta^2 \frac{1}{r^2} - \left. \frac{2}{r^3} \right) \left[ B_1 \sin 2\theta - B_2 \cos 2\theta \right.
- 2(D_1 \sin 4\theta + D_2 \cos 4\theta) - r(A_3 \cos \theta + A_4 \sin \theta - 3(C_1 \sin 3\theta + C_2 \cos 3\theta) \right)
+ \left. 2r^2(B_3 \sin 2\theta - B_4 \cos 2\theta) - r^3(A_1 \cos \theta + A_2 \sin \theta) \right] \partial_\theta^2 \frac{1}{r^2} + \text{lower \ order \ terms}. \quad (9)$$

We introduce the functions

$$G_1(r, \theta) = g_1 \cos^2 \theta + g_2 \sin^2 \theta + g_3 \cos \theta \sin \theta,$$
Then, the quadratic and zero order terms in the integral \( Y \) can now be written in polar coordinates as

\[
\begin{align*}
\{g_1(x,y), p_4^2\} + \{g_2(x,y), p_4 p_5\} + \{g_3(x,y), p_3^2\} + g_4(x,y) &= -\hbar^2 \left( \{G_1(r, \theta), \partial^2_\theta\} + \{G_3(r, \theta), \partial_\theta\} + \{G_2(r, \theta), \partial^2_\theta\} \right) + G_4(r, \theta).
\end{align*}
\]  

(11)

The structure of this article is as follows. In section 2.1 we derive the determining equations that govern the existence and form of the fourth-order integral \( Y \). In section 2.2 we present a linear compatibility condition that must be satisfied by the potential \( V(r, \theta) \) in order for a fourth order integral \( Y \) to exist. In general this is a fourth order PDE. In section 3 we turn to the question of superintegrability. The existence of the second order integral \( X \) guarantees that the potential \( V(r, \theta) \) has the form given in (2). In section 4 we consider the case \( A_1 \) and \( B_3 \) and \( B_4 \) respectively. From section 4 on we restrict to exotic potentials which by definition do not satisfy any linear equation. The function \( R(r) \) is already determined and is not exotic. The function \( S(\theta) \) satisfies a linear equation which must be satisfied identically for the exotic potentials. This requires that all coefficients in \( Y \) vanish except \( A_1, A_2, B_3 \) and \( B_4 \). In section 4 we consider the case \( R(r) = 0 \), i.e. a nonconfining potential (with no bound states). Section 4 is devoted to confining potentials \( R(r) = \frac{b r^2}{2} \) and \( R(r) = \frac{a}{r} \). In all cases the function \( S(\theta) \) is expressed in terms of the Painlevé transcendent \( P_6(\gamma_1, \gamma_2, \gamma_3, \gamma_4; \gamma) \) where \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) are arbitrary constants. Section 6 is devoted to classical potentials obtained in the (singular) limit \( \hbar \to 0 \). The fourth order compatibility condition reduces to a second order non-linear ODE which, interestingly, does not have the Painlevé property. The polynomial algebra generated by the integrals of motion is presented in section 7. The main results are summed up as theorems in the final section 8.

2. Determining equations for a fourth order integral

2.1. Commutator \([H, Y]\)

The commutator between the Hamiltonian \( H \) (3) and the fourth order integral \( Y \), written in polar coordinates, is a third order differential operator given by

\[
[H, Y] = A_{rr} \frac{\partial^3}{\partial r^3} + A_{r\theta} \frac{\partial^3}{\partial r^2 \partial \theta} + A_{r\theta} \frac{\partial^3}{\partial r \partial \theta^2} + A_{\theta\theta} \frac{\partial^3}{\partial \theta^3} + B_{rr} \frac{\partial^2}{\partial r^2} + B_{r\theta} \frac{\partial^2}{\partial r \partial \theta} + B_{\theta\theta} \frac{\partial^2}{\partial \theta^2} + C_r \frac{\partial}{\partial r} + C_0 = 0,
\]  

(12)
where the coefficients $A_{rr}, A_{r\theta}, \ldots$ are real functions of $r$ and $\theta$. Terms multiplying derivatives of order five and four vanish identically (they are already accounted for in the form of $Y$ in (5)). In order for $Y$ to be an integral of motion all ten coefficients must vanish simultaneously. The odd order terms in (12) provide us with useful information. The even order terms in (5)) In order for $Y$ to be an integral of motion all ten coefficients must vanish simultaneously. The odd order terms in (12) provide us with useful information. The even order terms in (5))

Vanishing of the coefficients of the third order terms $A_{rr}, A_{r\theta}, A_{r\theta}$ in (12) yields, respectively, the following relations:

$$G_1^{(1,0)} = F_1(\theta) V^{(1,0)} + F_2(r, \theta) V^{(0,1)},$$

$$\frac{1}{r^2} \left( G_2^{(0,1)} + \frac{1}{r} G_3 \right) = F_3(r, \theta) V^{(1,0)} + F_4(r, \theta) V^{(0,1)},$$

$$\frac{1}{r^2} G_2^{(1,0)} + G_3^{(1,0)} = 3 F_2(r, \theta) V^{(1,0)} + F_3(r, \theta) V^{(0,1)},$$

$$\frac{2}{r^2} G_1 + G_2^{(1,0)} + \frac{1}{r^2} G_3^{(1,0)} = F_3(r, \theta) V^{(1,0)} + 3 F_3(r, \theta) V^{(0,1)}.$$  

From the two equations $C_r = 0$ and $C_\theta = 0$, we obtain

$$2 G_1 V^{(1,0)} + G_3 V^{(0,1)} - \frac{1}{2} G_4^{(1,0)}$$

$$+ \hbar^2 \left[ \frac{G_1^{(1,2)}}{r^2} - \frac{2 G_2^{(2,0)}}{r^2} + \frac{G_3^{(2,0)}}{r^2} - \frac{3 G_4^{(3,0)}}{r^2} - \frac{3}{2} G_2^{(1,0)} \right.$$  

$$+ \frac{1}{2} G_2^{(1,2)} - \frac{1}{2} G_2^{(2,0)} - \frac{G_4^{(3,0)}}{4 r^2} - \frac{3 G_4^{(3,0)}}{4 r^2} + \frac{5 G_3^{(1,1)}}{4 r} + \frac{3}{4} G_3^{(2,1)} \right]$$

$$= \hbar^2 \left[ F_1 V^{(3,0)} + F_3 V^{(0,3)} + F_4 V^{(1,2)} + F_5 V^{(2,1)} + F_{10} V^{(2,0)} + 2 F_7 V^{(1,1)} + F_8 V^{(0,2)} + F_{11} V^{(1,0)} + F_{12} V^{(0,1)} \right],$$

$$2 G_2 V^{(0,1)} + G_3 V^{(1,0)} - \frac{G_4^{(0,1)}}{2 r^2}$$

$$+ \hbar^2 \left[ \frac{G_1^{(1,1)}}{r^2} + \frac{G_2^{(2,1)}}{2 r^2} - \frac{G_3^{(1,0)}}{2 r^2} + \frac{G_4^{(0,1)}}{2 r^2} + \frac{G_2^{(1,0)}}{2 r^2} \right.$$  

$$+ \frac{1}{2} G_2^{(2,1)} + \frac{G_3}{4 r^2} + \frac{5 G_3^{(0,2)}}{4 r^2} - \frac{G_4^{(1,0)}}{4 r^2} + \frac{3 G_4^{(1,2)}}{4 r^2} + \frac{G_3^{(0,2)}}{2 r} + \frac{1}{4} G_3^{(3,0)} \right]$$

$$= \hbar^2 \left[ F_2 V^{(3,0)} + 3 F_3 V^{(1,2)} + F_4 V^{(0,3)} + F_5 V^{(2,1)} + F_7 V^{(2,0)} + F_9 V^{(0,2)} + 2 F_{11} V^{(1,1)} + F_{12} V^{(1,0)} + F_{13} V^{(0,1)} \right],$$

respectively, where we define $V^{(i,j)} = \partial_r^i \partial_\theta^j V(r, \theta)$.

The Planck constant $\hbar$ is present in the lowest order coefficients $C_r$ and $C_\theta$ only (see (17) and (18)).

The functions $F_1, \ldots, F_{13}$ are completely determined by the constants $A, B, C, D$ figuring in the leading part of the integral $Y$. They are given in appendix A.
2.2. The linear compatibility condition

The system (13)–(16) viewed as a system of 4 PDE for $G_1$, $G_2$ and $G_3$ is overdetermined and for general potential $V(r, \theta)$ has no solutions. The first step towards finding solutions of this system is to establish a necessary linear compatibility condition involving $V(r, \theta)$ alone. Such an equation will exist as a consequence of the equality of all mixed derivatives of analytical functions. To obtain the compatibility condition we denote the l.h.s. of the equations (13)–(16) as $E_1, \ldots, E_4$, respectively, and take partial derivatives of these terms (up to third order). The following linear combination of the derivatives vanishes identically

$$0 = r^6 E_2(1,0) + 12 r^5 E_2(2,0) + 36 r^4 E_2(2,1) - r^4 E_4(2,1) - r^3 E_3(2,0) - 6 r^3 E_4(1,1) + 24 r^2 E_2 - 3 r^2 E_3(1,0) - 6 r^2 E_4(0,1) + r^2 E_3(1,2) + 2 r E_3(0,2) + 3 r E_4(1,1) - E_4(0,1) - E_4(0,3)$$

hence the same combination of the r.h.s. of (13)–(16) must vanish too and we obtain the compatibility condition:

$$0 = r^6 F_3 V^{(4,0)} + r^5 (F_3 r^2 - F_5) V^{(3,1)} - 3 r^2 (F_3 r^2 - F_2) V^{(2,2)} + (F_3 r^2 - F_1) V^{(1,3)} - F_2 V^{(0,4)} + r^3 (3 r^3 F_3(1,0) + 12 r^2 F_3 - r F_3(0,1) - 3 F_2) V^{(3,0)} + r (3 r^5 F_4(1,0) + 12 r^4 F_4 - 3 r^3 F_3(0,1) - 2 r^3 F_5(1,0) - 7 r^2 F_5 + 6 r F_2(0,1) + 3 F_1) V^{(2,1)} + (3 r^2 F_2(1,0) - 6 r^2 F_3(1,0) - 18 r^2 F_3 + 2 r^2 F_5(0,1) + 9 r F_2 - 3 F_1) V^{(1,2)} + (r^2 F_5(1,0) + 2 r F_3 - 3 F_2(0,1)) V^{(0,3)} + r (3 r^5 F_3(2,0) + 24 r^4 F_5(1,0) + 36 r^3 F_3 - 2 r F_3(1,1) - 6 r^2 F_5(0,1) - 6 r^2 F_5(1,0) + 3 r F_2(0,2) - 9 r F_2 + 3 F_1) V^{(2,0)} + (3 r^6 F_4(2,0) + 24 r^5 F_4(1,0) - 6 r^4 F_3(1,1) + 36 r^3 F_4 - r^3 F_5(2,0) - 18 r^3 F_3(0,1) - 8 r^3 F_5(1,0) + 6 r^2 F_2(1,1) + r^2 F_5(0,2) - 9 r^2 F_5 + 15 r F_2(0,1) - F_1 - 3 F_4(0,1) + (3 r^4 F_3(2,0) + 18 r^3 F_3(1,0) + 18 r^2 F_3 - 2 r^2 F_3(1,1) - 3 r F_2(1,0) - 4 r F_5(0,1) + F_2 + 3 F_2(0,2)) V^{(0,2)} + r^6 F_3(3,0) + 12 r^5 F_3(2,0) + 36 r^4 F_3(1,0) - r^4 F_3(2,1) - 3 r^3 F_2(2,0) + 24 r^2 F_3 - 6 r^2 F_3(1,1) - 9 r^2 F_2(1,0) + 3 r^2 F_2(1,2) - 6 r^2 F_5(0,1) + 6 r F_2(0,2) - F_1 - F_3(3) V^{(1,0)} + r^6 F_4(3,0) + 12 r^5 F_4(2,0) - 3 r^4 F_3(2,1) + 36 r^4 F_4(1,0) + 24 r^3 F_4 - 18 r^3 F_3(1,1) - r^3 F_3(2,0) - 18 r^2 F_3(0,1) - 3 r^2 F_5(1,0) + r^2 F_5(1,2) + 2 r F_5(0,2) + 3 r F_2(1,1) - F_2(0,1) - F_2(0,3)) V^{(0,1)}.$$  

Relation (20) is a fourth order linear PDE for the potential and is a necessary (but not sufficient) condition for the existence of the fourth order integral $Y$ of the form (5). This relation
3. Superintegrability: separation in polar coordinates

3.1. The determining equations

Vanishing of the commutator $[H, X] = 0$ implies that the potential has the separable form of $V(r, \theta)$ in (2) and thus allows separation of variables in polar coordinates in the Schrödinger equation (and in the Hamilton–Jacobi equation).

In this case the determining equations (13)–(18), coming from the condition $[H, Y] = 0$, take the form

$$G_1^{(1,0)} = F_1 R' - \frac{2F_1}{r^3} S + \frac{F_3}{r^2} S',$$

(21)

$$\frac{1}{r^3} \left(G_2^{(0,1)} + \frac{1}{r} G_3\right) = F_3 R' - \frac{2F_3}{r^3} S + \frac{F_5}{r^2} S',$$

(22)

$$\frac{1}{r^3} G_1^{(0,1)} + G_3^{(1,0)} = 3F_2 R' - \frac{6F_2}{r^3} S + \frac{F_5}{r^2} S',$$

(23)

$$\frac{2}{r^3} G_1 + G_2^{(1,0)} + \frac{1}{r^2} G_3 = F_3 R' - \frac{2F_3}{r^3} S + \frac{3F_3}{r^2} S',$$

(24)

$$G_3 \frac{1}{r^3} S' + 2G_1 \left(R' - \frac{2}{r^3} S\right) - \frac{1}{2} G_4^{(1,0)}$$

$$+ \hbar^2 \left[\frac{G_1^{(0,2)}}{2r^3} - \frac{3G_3^{(0,1)}}{4r^2} + \frac{G_3^{(0,3)}}{4r^2} - \frac{G_1^{(1,0)}}{r^3} - \frac{3}{2} G_2^{(1,0)} + \frac{5G_4^{(1,1)}}{4r}\right]$$

$$+ \frac{G_4^{(1,2)}}{2r^3} + \frac{1}{2} G_2^{(1,2)} + \frac{2G_1^{(2,0)}}{r} - \frac{1}{2} r G_3^{(2,0)} + \frac{3}{4} G_4^{(2,1)} + G_4^{(3,0)}$$

$$= \hbar^2 \left(\frac{6F_3}{r^3} - \frac{24F_1}{r^5} - \frac{2F_{11}}{r^3}\right) S + \left(\frac{6F_{10}}{r^4} - \frac{4F_7}{r^3} + \frac{F_{12}}{r^2}\right) S'$$

$$+ \left(\frac{F_3}{r^3} - \frac{2F_3}{r^3}\right) S'' + \frac{F_3}{r^2} S''' + F_{11} R' + F_6 R'' + F_1 R'''\right],$$

(25)

$$G_3 \left(R' - \frac{2}{r^3} S\right) + 2G_2 \frac{1}{r^3} S' - \frac{G_4^{(0,1)}}{2r^2}$$

$$+ \hbar^2 \left[\frac{G_3^{(1,0)}}{4r^3} - \frac{G_2^{(1,1)}}{2r^3} - \frac{3G_3^{(1,2)}}{4r^2} + \frac{G_1^{(2,0)}}{2r} + \frac{1}{2} G_2^{(2,1)} + \frac{1}{4} G_3^{(3,0)}\right]$$

$$= \hbar^2 \left(\frac{6F_3}{r^3} - \frac{24F_2}{r^5} - \frac{2F_{12}}{r^3}\right) S + \left(\frac{6F_3}{r^4} - \frac{4F_8}{r^3} + \frac{F_{13}}{r^2}\right) S'$$

$$+ \left(\frac{F_3}{r^3} - \frac{6F_3}{r^3}\right) S'' + \frac{F_3}{r^2} S''' + F_{12} R' + F_7 R'' + F_3 R'''\right].$$

(26)
3.2. The linear compatibility condition

Substituting the separable form (2) of the potential into the compatibility condition (20), and integrating once over \( r \), we obtain

\[
\Theta(\theta) = 288 \left[ B_1 \sin 2 \theta - B_2 \cos 2 \theta - 8D_1 \sin 4 \theta + 8D_2 \cos 4 \theta \right] S
+ 120 \left[ 20D_1 \cos 4 \theta + 20D_2 \sin 4 \theta - B_1 \cos 2 \theta - B_2 \sin 2 \theta \right] S'
+ 60 \left[ B_1 \sin 2 \theta - B_2 \cos 2 \theta + 14D_1 \sin 4 \theta - 14D_2 \cos 4 \theta \right] S''
- 30 \left[ B_1 \cos 2 \theta + B_2 \sin 2 \theta + 4D_1 \cos 4 \theta + 4D_2 \sin 4 \theta \right] S^{(3)}
- 3 \left[ B_1 \sin 2 \theta - B_2 \cos 2 \theta + 2D_1 \sin 4 \theta - 2D_2 \cos 4 \theta \right] S^{(4)}
- r \left[ 96 \left[ A_4 \sin \theta + A_1 \cos \theta - 9C_1 \sin 3 \theta - 9C_2 \cos 3 \theta \right] S
- 120 \left[ A_4 \cos \theta - A_3 \sin \theta + 9C_2 \sin 3 \theta - 9C_1 \cos 3 \theta \right] S'
+ 480 \left[ C_1 \sin 3 \theta + C_2 \cos 3 \theta \right] S''
+ 30 \left[ A_3 \sin \theta - A_4 \cos \theta + 3C_2 \sin 3 \theta - 3C_1 \cos 3 \theta \right] S^{(3)}
- 6 \left[ A_4 \sin \theta + A_3 \cos \theta + C_1 \sin 3 \theta + C_2 \cos 3 \theta \right] S^{(4)} \right)
+ 576 r^2 \left[ D_2 \cos 4 \theta - D_1 \sin 4 \theta \right] R
+ 6 r^3 \left[ 6r^3 \left( A_2 \sin \theta + A_1 \cos \theta \right) - 4r^2 \left( B_4 \cos 2 \theta - B_3 \sin 2 \theta \right) \right]
- r \left( A_4 \sin \theta + A_3 \cos \theta + 9C_1 \sin 3 \theta + 9C_2 \cos 3 \theta \right)
- 3B_1 \sin 2 \theta + 3B_2 \cos 2 \theta + 66D_1 \sin 4 \theta - 66D_2 \cos 4 \theta \right] R'
+ 6 r^4 \left[ 6r^3 \left( A_2 \sin \theta + A_1 \cos \theta \right) - 4r^2 \left( B_3 \sin 2 \theta - B_4 \cos 2 \theta \right) \right]
+ r \left( A_4 \sin \theta + A_3 \cos \theta + 9C_1 \sin 3 \theta + 9C_2 \cos 3 \theta \right)
+ 3 \left( B_1 \sin 2 \theta - B_2 \cos 2 \theta - 6D_1 \sin 4 \theta + 6D_2 \cos 4 \theta \right) \right] R''
+ 3 r^5 \left[ 2r^3 \left( A_2 \sin \theta + A_1 \cos \theta \right) + 4r^2 \left( B_4 \cos 2 \theta - B_3 \sin 2 \theta \right) \right]
+ 2 r \left( A_4 \sin \theta + A_3 \cos \theta - 3C_1 \sin 3 \theta - 3C_2 \cos 3 \theta \right)
- 2 \left( B_1 \sin 2 \theta - B_2 \cos 2 \theta + 2D_2 \cos 4 \theta - 2D_1 \sin 4 \theta \right) \right] R^{(3)}.
\]

(27)

where \( \Theta(\theta) \) is an arbitrary function of \( \theta \). Since \( S(\theta) \) and \( R(r) \) are functions of one variable only, (27) is no longer a PDE. We will obtain several ODEs from it. We differentiate (27) twice with respect to \( r \). This eliminates \( S(\theta) \) and \( \Theta(\theta) \) from the equation. We then expand in a basis of linearly independent trigonometric functions \( \sin \theta, \sin 2\theta, \sin 3\theta, \sin 4\theta, \cos \theta, \cos 2\theta, \cos 3\theta, \cos 4\theta \) and obtain the following set of 8 equations that \( R(r) \) must satisfy simultaneously:

\[
12 \left( A_3 - 15 r^2 A_1 \right) R' - 12 r \left( A_3 + 27 r^2 A_1 \right) R'' - r^2 \left( 39 A_3 + 146 r^2 A_1 \right) R^{(3)}
- r^3 \left( 13 A_3 + 22 r^2 A_1 \right) R^{(4)} - r^4 \left( A_3 + r^2 A_1 \right) R^{(5)} = 0.
\]

(28a)
\[2 \left(9B_2 - 40 r^2 B_3 \right) R' - 2 r \left(9B_2 - 40 r^2 B_3 \right) R'' - r^2 \left(B_2 - 128 r^2 B_3 \right) R^{(3)} + r^3 \left(7B_2 + 32 r^2 B_4 \right) R^{(4)} + r^4 \left(B_2 + 2 r^2 B_3 \right) R^{(5)} = 0, \quad (28b)\]

\[C_2 \left(36 R' - 36 r R'' + 3 r^2 R^{(3)} + 9 r^3 R^{(4)} + r^4 R^{(5)} \right) = 0, \quad (28c)\]

\[D_2 \left(96 R - 6 r R' - 42 r^2 R'' + 19 r^3 R^{(3)} - r^4 R^{(4)} - r^5 R^{(5)} \right) = 0, \quad (28d)\]

\[24 \left(A_4 - 15 r^2 A_2 \right) R' - 24r \left(A_4 + 27 r^2 A_2 \right) R'' - r^2 \left(78 A_4 + 292 r^2 A_2 \right) R^{(3)} - r^3 \left(26 A_4 + 44 r^2 A_2 \right) R^{(4)} - 2 r^4 \left(A_4 + r^2 A_2 \right) R^{(5)} = 0, \quad (28e)\]

\[2 \left(9B_1 - 40 r^2 B_3 \right) R' - 2 r \left(9B_1 - 40 r^2 B_3 \right) R'' - r^2 \left(B_1 - 128 r^2 B_3 \right) R^{(3)} + r^3 \left(7B_1 + 32 r^2 B_3 \right) R^{(4)} + r^4 \left(B_1 + 2 r^2 B_3 \right) R^{(5)} = 0, \quad (28f)\]

\[C_1 \left(36 R' - 36 r R'' + 3 r^2 R^{(3)} + 9 r^3 R^{(4)} + r^4 R^{(5)} \right) = 0, \quad (28g)\]

\[D_1 \left(96 R - 6 r R' - 42 r^2 R'' + 19 r^3 R^{(3)} - r^4 R^{(4)} - r^5 R^{(5)} \right) = 0. \quad (28h)\]

Taking linear combinations of equations \((28a)\)–\((28h)\), we get the following Euler–Cauchy type differential equations

\[\left(A_1 A_4 - A_2 A_3 \right) \left[180 R' + 324r R'' + 146 r^2 R^{(3)} + 22 r^3 R^{(4)} + r^4 R^{(5)} \right] = 0, \quad (29a)\]

\[\left(B_1 B_4 - B_2 B_3 \right) \left[40 R' - 40 r R'' - 64 r^2 R^{(3)} - 16 r^3 R^{(4)} - r^4 R^{(5)} \right] = 0, \quad (29b)\]

\[\left(C_1 + C_2 \right) \left[36 R' - 36 r R'' + 3 r^2 R^{(3)} + 9 r^3 R^{(4)} + r^4 R^{(5)} \right] = 0, \quad (29c)\]

\[\left(D_1^2 + D_2^2 \right) \left[96 R - 6 r R' - 42 r^2 R'' + 19 r^3 R^{(3)} - r^4 R^{(4)} - r^5 R^{(5)} \right] = 0. \quad (29d)\]

In particular, the above equations have solutions

\[R(r) = \frac{\alpha_1}{r^3} + \frac{\alpha_2}{r^2} + \frac{\alpha_3}{r} + \frac{\alpha_4}{r^4} + \alpha_5, \quad (A_1 A_4 - A_2 A_3) \neq 0, \quad (30a)\]

\[R(r) = \frac{\alpha_1}{r^4} + \frac{\alpha_2}{r^2} + \frac{\alpha_3}{r} + r^2 \alpha_4 + \alpha_5, \quad (B_1 B_4 - B_2 B_3) \neq 0, \quad (30b)\]

\[R(r) = \frac{\alpha_1}{r^3} + \frac{\alpha_2}{r} + r^2 \alpha_3 + r^4 \alpha_4 + \alpha_5, \quad C_1 + C_2 \neq 0, \quad (30c)\]

\[R(r) = \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r} + r^2 \alpha_3 + r^4 \alpha_4 + r^6 \alpha_5, \quad D_1^2 + D_2^2 \neq 0, \quad (30d)\]

respectively. Otherwise, for
\[(A_1 A_4 - A_2 A_3) = 0, \quad (B_1 B_4 - B_2 B_3) = 0, \quad C_1^2 + C_2^2 = 0, \quad D_1^2 + D_2^2 = 0,\]

the equations (29a)–(29d) are trivially satisfied with arbitrary \(R(r)\), but then (28a)–(28h) are satisfied only when all parameters \(A_i, B_i, C_i\) and \(D_i\) vanish (so that no fourth order integral \(Y\) exists).

The compatibility between common solutions of (28a)–(29d) and the determining equations (21)–(26), shows that when (27) is satisfied trivially then the most general form of the radial part \(R(r)\) of the potential \(V(r, \theta)\) is

- \(R(r) = b r^2\); in this case all parameters are zero except \(B_1, B_4\).
- \(R(r) = \frac{\beta}{r}\); all parameters are zero except \(A_1, A_2, B_3, B_4\).
- \(R(r) = 0\); all parameters are zero except \(A_1, A_2, B_3, B_4\).

Exotic potentials \(V(r, \theta)\) with radial part \(R(r) = b r^2, \frac{\beta}{r}\) or 0 are also the only ones that could allow a third order integral [62].

4. Nonconfining potential \(V(r, \theta) = \frac{S(\theta)}{r^2}\).

This potential corresponds to \(R(r) = 0\). In this article we are interested in exotic potentials. Equation (27) is linear, so it must be satisfied trivially. Hence all parameters in (8) vanish except \(A_1, A_2, B_3, B_4\). It is worth mentioning that the singular potentials of the form \(V(r, \theta) = \frac{S(\theta)}{r^2}\) require a renormalization scheme in order to obtain a well defined problem with a discrete spectrum [8, 24, 35].

The equations (21)–(24) corresponding to the determining equations \(A_{rr} = A_{r\theta} = A_{\theta\theta} = A_{\theta\theta} = 0\), respectively, take the form

\[G_1^{(1,0)} = 0, \quad (31a)\]

\[\frac{1}{r^2} \left( G_2^{(0,1)} + \frac{G_3}{r} \right) \]

\[= -\frac{2}{r^2} \left( \begin{array}{c}
A_2 \sin \theta + A_1 \cos \theta + \frac{2}{r} (B_4 \cos 2\theta - B_3 \sin 2\theta) \\
+ \left( \frac{4}{r^2} (A_2 \cos \theta - A_1 \sin \theta) - \frac{4}{r^2} (B_3 \cos 2\theta + B_4 \sin 2\theta) \right) S',
\end{array} \right) \]

\[= \frac{1}{r^2} G_3^{(0,1)} + \frac{1}{r^2} G_1^{(0,1)} = \frac{2}{r^2} (B_3 \cos 2\theta + B_4 \sin 2\theta) S', \quad (31b)\]

\[G_3^{(1,0)} = \frac{1}{r^2} G_3^{(0,1)} + \frac{2}{r^2} G_1 = -\frac{4}{r^3} (B_3 \cos 2\theta - B_4 \sin 2\theta) S \]

\[+ \frac{3}{r^3} \left( A_2 \sin \theta + A_1 \cos \theta + \frac{2}{r} (B_4 \cos 2\theta - B_3 \sin 2\theta) \right) S'. \quad (31c)\]

In particular, the equations (31a), (31c) and (31d) define the \(r\) dependence of the functions \(G_{1,2,3}\). Indeed, from (31a) we obtain

\[G_1(r, \theta) = \beta_1(\theta). \quad (32)\]

Substituting (32) into equation (31c) and integrating we get:
\[ G_3(r, \theta) = -\frac{2}{r} (B_3 \cos 2\theta + B_4 \sin 2\theta) S' + \frac{1}{r} \beta''_1(\theta) + \beta_3(\theta). \]  

(33)

Substituting \( G_1, G_2 \) into equation (31d), we find

\[
G_2(r, \theta) = \frac{2}{r^2} (B_3 \cos 2\theta + B_4 \sin 2\theta) S' \]
\[ + \left[ -\frac{3}{r} \left( A_1 \cos \theta + A_2 \sin \theta \right) - \frac{5}{r^2} \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) \right] S' \]
\[ - \frac{1}{r} (B_3 \cos 2\theta + B_4 \sin 2\theta) S'' \]
\[ + \beta_2(\theta) + \frac{1}{r} \beta'_1(\theta) + \frac{1}{2r^2} \left( 2\beta_1(\theta) + \beta''_1(\theta) \right). \]  

(34)

Let us now determine the functions \( \beta_i(\theta) \). Substituting the above functions \( G_1, G_2, G_3 \) into (31b), i.e. into the determining equation \( A_{\theta \theta \theta} = 0 \), and collecting in powers of \( r \) one finds the following three equations which define the functions \( \beta_{1,2,3} \):

\[ \beta''_2(\theta) = 0, \]  

(35a)

\[ \beta''_3(\theta) + \beta_3(\theta) = -2 \left( A_1 \cos \theta + A_2 \sin \theta \right) S 
+ \frac{7}{2} \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) S' 
+ \frac{3}{2} \left( A_1 \cos \theta + A_2 \sin \theta \right) S'', \]  

(35b)

\[ \frac{1}{2} \beta''_1(\theta) + 2\beta'_1(\theta) = \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right) S'' 
+ \frac{7}{2} \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) S' 
- \frac{14}{2} \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right) S. \]  

(35c)

Equation (35a) implies that

\[ \beta_2(\theta) = c_{21}, \]  

(36)

where \( c_{21} \) is a constant.

Next, replacing

\[ S(\theta) = T'(\theta), \]

into (35c) and solving this equation we find the function \( \beta_1(\theta) \):

\[ \beta_1(\theta) = 2 \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) T + 2 \left( B_3 \cos 2\theta + B_3 \cos 2\theta \right) T' 
+ \frac{1}{2} c_{11} \sin 2\theta \]  

(37)

where the \( c \)'s are integration constants.

Similarly, the solution to equation (35b) provides the function \( \beta_3(\theta) \)
\[ \beta_3 (\theta) = c_{31} \cos \theta + c_{32} \sin \theta + (A_2 \cos \theta - A_1 \sin \theta) T + 3 (A_2 \sin \theta + A_1 \cos \theta) T'. \] (38)

Now let us turn to the equations (25)–(26). From the equation (25), \( C_r = 0 \), we find the function \( G_4(r, \theta) \):

\[
G_4(r, \theta) = \left[ -\frac{1}{2r} \left( 2 (A_1 \cos \theta + A_2 \sin \theta) S'' - 6 (A_2 \cos \theta - A_1 \sin \theta) S' \right) \right.
+ \left. \frac{1}{r^2} \left( \frac{1}{2} (B_4 \sin 2 \theta + B_5 \cos 2 \theta) S^{(4)} - 4 (B_3 \sin 2 \theta - B_4 \cos 2 \theta) S'' \right) \right.
+ \left. 10 (B_5 \cos 2 \theta + B_4 \sin 2 \theta) S' - 8 (B_4 \cos 2 \theta - B_3 \sin 2 \theta) S \right]
+ \left. \frac{1}{4} (\beta_4' - \beta_4) \right] \hbar^2
+ \beta_4 - \frac{2}{r} S' \beta_3 + \frac{1}{r^2} \left( 2 (B_4 \sin 2 \theta + B_3 \cos 2 \theta) S^2 - S' \beta_1' + 4 S \beta_1 \right). \] (39)

At this point, all eight coefficients \( A_{rrr}, A_{r\theta\phi}, A_{\theta\theta\phi}, B_{rr}, B_{r\theta}, B_{\theta\phi}, C_r \) in (12) vanish. In fact, the main equation to be solved is \( C_y = 0 \), presented in (26).

Substituting \( G_1, G_2, G_3, G_4 \) into the determining equation \( C_b = 0 \), (26), and collecting powers of \( r \) we get three equations that must be satisfied simultaneously in order for \( Y \) in (5) to be an integral of motion:

\[ 0 = \beta_4' - 4 c_{21} T'', \] (40)

\[ 0 = \left[ -4 (A_1 \cos \theta + A_2 \sin \theta) T' - 8 (A_1 \sin \theta - A_2 \cos \theta) T'' \right]
+ \left[ 6 (A_1 \cos \theta + A_2 \sin \theta) T^{(3)} + \frac{5}{2} (A_1 \sin \theta - A_2 \cos \theta) T^{(4)} \right]
+ \left[ -\frac{1}{2} (A_1 \cos \theta + A_2 \sin \theta) T^{(5)} \right] \hbar^2 - 12 (A_1 \cos \theta + A_2 \sin \theta) (T')^2
+ \left[ 24 (A_2 \cos \theta - A_1 \sin \theta) T'' + 6 (A_1 \cos \theta + A_2 \sin \theta) T^{(3)} \right]
+ \left[ 4 (A_1 \sin \theta - A_2 \cos \theta) T - 4 (c_{31} \cos \theta + c_{32} \sin \theta) \right] T'
+ \left[ 6 (A_1 \cos \theta + A_2 \sin \theta) (T'')^2 + 6 (A_1 \cos \theta + A_2 \sin \theta) T \right]
+ \left[ -6 (c_{31} \sin \theta - c_{32} \cos \theta) \right] T'' + \left[ 2 (A_2 \cos \theta - A_1 \sin \theta) T \right]
+ \left[ 2 (c_{31} \cos \theta + c_{32} \sin \theta) \right] T^{(3)}, \] (41)
0 = \left[ -32 \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) \frac{d^2 T}{d\theta^2} - 40 \left( B_4 \sin 2\theta + B_3 \cos 2\theta \right) \frac{d^3 T}{d\theta^3} \\
+ 20 \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) \frac{d^3 T}{d\theta^3} + 5 \left( B_4 \sin 2\theta + B_3 \cos 2\theta \right) \frac{d^4 T}{d\theta^4} \\
- \frac{1}{2} \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) \frac{d^5 T}{d\theta^5} \right] h^2 - 48 \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) (\frac{d^2 T}{d\theta^2})^2 \\
+ \left[ -48 \left( B_4 \sin 2\theta + B_3 \cos 2\theta \right) \frac{d^4 T}{d\theta^4} + 6 \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) \frac{d^5 T}{d\theta^5} \\
+ 32 \left( B_4 \sin 2\theta + B_3 \cos 2\theta \right) T - 8 \left( c_{11} \cos 2\theta + c_{12} \sin 2\theta \right) \right] \frac{d^2 T}{d\theta^2} \\
+ 6 \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right) \left( \frac{d^3 T}{d\theta^3} \right)^2 + 24 \left( B_3 \sin 2\theta - B_4 \cos 2\theta \right) T \\
- 6 \left( c_{11} \sin 2\theta - c_{12} \cos 2\theta \right) \frac{d^4 T}{d\theta^4} - 4 \left( B_4 \sin 2\theta + B_3 \cos 2\theta \right) T \\
- c_{11} \cos 2\theta - c_{12} \sin 2\theta \right] \frac{d^5 T}{d\theta^5} \right) . \tag{42} \]

At this stage we assume \( h \neq 0 \). We see that in the classical case \( (h \to 0) \) equations (41) and (42) simplify greatly. The above non-linear equations (41) and (42) will determine the angular part of the potential. They both pass the Painlevé test.

Equation (40) determines the function \( \beta_4 \)

\[
\beta_4(\theta) = 4 c_{21} S(\theta) + c_{41}, \tag{43}
\]

together with (36), this defines \( G_4 (r, \theta) \) of (39) completely in terms of \( S(\theta) \) and some constants.

The parameters \( c_{13}, c_{21} \) in (37) and (43) can be set equal to zero by linear combinations of \( H \) and \( X \). Moreover, \( c_{41} \) in (43) is simply a constant that commutes with \( H \) trivially. Therefore, without loss of generality we choose

\[
c_{13} = 0, \quad c_{21} = 0, \quad c_{41} = 0 .
\]

Equations (41) and (42) depend on mutually exclusive sets of parameters, namely \( (A_1, A_2, c_{31}, c_{32}) \) and \( (B_1, B_4, c_{11}, c_{12}) \), respectively. Moreover, for \( A_1 = A_2 = 0 \) (41) reduces to a linear equation, as does (42) for \( B_1 = B_4 = 0 \). Since we are looking for exotic potentials, all linear equations for \( T(\theta) \) must be satisfied identically. Hence we have 2 cases to consider

- **Case (I)**

\[
A_1^2 + A_2^2 \neq 0, \quad B_3 = B_4 = c_{11} = c_{12} = 0 ,
\]

By a rotation we can set \( A_1 = 0 \),

- **Case (II)**

\[
B_3^2 + B_4^2 \neq 0, \quad A_1 = A_2 = c_{31} = c_{32} = 0. \tag{44}
\]

By a rotation we can set \( B_4 = 0 \).

In case I and II one of the two equations (41) and (42) trivializes, so only one nonlinear equation must be solved. In both cases it already passed the Painlevé test.

- **Case (III)** \( A_1^2 + A_2^2 \neq 0 \), \( B_3^2 + B_4^2 \neq 0 \)

In this case the two nonlinear determining equation (41) and (42) remain. Thus they will either be incompatible or \( T(\theta) \) will be a very special case of the solutions obtained in case I and case II. We shall not investigate this case further since it cannot provide any new exotic potentials.
We also note that in the quantum case (41) and (42) are fifth order equations. In the classical limit \( h \to 0 \) they reduce to third order ones, to be considered in section 6.

4.1. Case I, \( A_2 \neq 0, A_1 = B_3 = B_4 = 0 \)

Equation (42) is linear and must be satisfied trivially, so we have \( c_{11} = c_{12} = 0 \). Equation (41) simplifies to

\[
A_2 \left[ 2 \sin \theta T' - 4 \cos \theta T'' - 3 \sin \theta T''' + \frac{5}{4} \cos \theta T^{(4)} + \frac{1}{4} \sin \theta T^{(5)} \right] h^2 + A_2 \left( 2 \cos \theta T' + 3 \sin \theta T'' - \cos \theta T''' \right) T + 6 \sin \theta A_2 T^2 \\
- \left( 12 A_2 \cos \theta T'' + 3 \sin \theta A_2 T''' - 2 (c_{31} \cos \theta + c_{32} \sin \theta) \right) T' - 3 A_2 \sin \theta (T'')^2 \\
- 3 (c_{32} \cos \theta - c_{31} \sin \theta) T'' - (c_{31} \cos \theta + c_{32} \sin \theta) T''' = 0 .
\]

(45)

This equation can be integrated once resulting in the 4th order equation

\[
A_2 \left[ 2 \cos \theta T' + 2 \sin \theta T'' - \cos \theta T''' - \frac{1}{4} \sin \theta T^{(4)} \right] h^2 + A_2 \left( \cos \theta T'' - 2 \sin \theta T' \right) T + 4 A_2 \cos \theta (T')^2 \\
+ \left( 3 A_2 \sin \theta T'' + 2 (c_{32} \cos \theta - c_{31} \sin \theta) \right) T' + (c_{31} \cos \theta + c_{32} \sin \theta) T''' + \frac{1}{2} K_1 = 0 ,
\]

where \( K_1 \) is an arbitrary integration constant. Transforming to the variable

\[
z = \tan \theta ,
\]

and dividing by \((1/4)(z^2 + 1)^2\), we get:

\[
A_2 \left[ 24 z^2 T' + 12 (3 z^2 + 2) z T'' + 4 (z^2 + 1) (3 z^2 + 1) T''' + (z^2 + 1)^2 z T^{(4)} \right] h^2 \\
- 4 A_2 T'' T - A_2 \left( 24 z^2 + 16 \right)(T')^2 - 12 (z^2 + 1) z A_2 T'' + 8 c_{32} T'' \\
- 4 (c_{32} z + c_{31}) T''' - 2 \frac{K_1}{(z^2 + 1)^{3/2}} = 0 .
\]

(46)

Putting \( c_{31} \to 2 A_2 c_{31}, c_{32} \to 2 A_2 c_{32}, K_1 \to 2 A_2 K_1 \), we integrate the above equation, using \( z \) as integrating factor to get the following third order non-linear differential equation

\[
\left[ 2 \left( 1 - 3 z^2 \right) T' - 2 z (3 z^2 + 1) (z^2 + 1) T'' - z^2 (z^2 + 1)^2 T''' \right] h^2 \\
- 2 T^2 + 4 (z T' - 2 c_{31}) T + 6 z^2 (z^2 + 1) \left( T' \right)^2 + 8 z (c_{32} z + c_{31}) T' \\
- 4 \frac{K_1}{\sqrt{z^2 + 1}} + K_2 = 0 ,
\]

(47)

here \( K_2 \) is another arbitrary integration constant. The transformation \((z, T(z)) \to (x, W(x))\):
\[
W(x) = \frac{x^2(x-1)^2}{4P_6(P_6-1)(P_6-x)} \left[ P_6^2 - \frac{P_6(P_6-1)}{x(x-1)} \right]^2 + \frac{1}{8} \left( 1 - \sqrt{2\gamma_1} \right)^2 (1 - 2P_6) + \frac{1}{4} \gamma_2 \left( 1 - \frac{2x}{P_6} \right) - \frac{1}{4} \gamma_3 \left( 1 - \frac{2(x-1)}{P_6} \right) + \left( \frac{1}{8} - \frac{\gamma_4}{4} \right) \left( 1 - \frac{2x(P_6-1)}{P_6-x} \right),
\]

and

\[
W'(x) = -\frac{x(x-1)}{4P_6(P_6-1)} \left[ P_6^2 - \sqrt{2\gamma_1} \frac{P_6(P_6-1)}{x(x-1)} \right]^2 - \frac{\gamma_2(P_6-x)}{2(x-1)P_6} \frac{\gamma_3(P_6-x)}{2(x(P_6-1))},
\]

where \( \sqrt{2\gamma_1} \) can take either sign and \( \gamma_1, \gamma_2, \gamma_3 \) and \( \gamma_4 \) are the arbitrary parameters that define the sixth Painlevé transcendent \( P_6 \) which satisfies the well known second order differential equation:

\[
P_6'' = \frac{1}{2} \left[ \frac{1}{P_6} + \frac{P_6-1}{P_6-x} \right] (P_6')^2 - \left[ \frac{1}{x} + \frac{1}{x-1} + \frac{1}{P_6-x} \right] P_6' + \frac{P_6(P_6-1)(P_6-x)}{x^2(x-1)^2} \left[ \gamma_1 + \frac{\gamma_2 x}{P_6} + \frac{\gamma_3 (x-1)}{(P_6-1)^2} + \frac{\gamma_4 x(x-1)}{(P_6-x)^2} \right].
\]

Thus, we have

\[
W = W(x; \gamma_1, \gamma_2, \gamma_3, \gamma_4).
\]
The parameters $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_4$ are related to the arbitrary constants of integration $c_{31}, c_{32}, K_1$ and $K_2$ through the relations

\[-4q_7 = \gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 - \sqrt{2\gamma_1} + 1,\]
\[-4q_8 = (\gamma_2 + \gamma_3)(\gamma_1 + \gamma_4 - \sqrt{2\gamma_1}),\]
\[-4q_9 = (\gamma_3 - \gamma_2)(\gamma_1 - \gamma_4 - \sqrt{2\gamma_1} + 1) + \frac{1}{4}(\gamma_1 - \gamma_2 - \gamma_3 + \gamma_4 - \sqrt{2\gamma_1})^2,\]
\[-4q_{10} = \frac{1}{4}(\gamma_3 - \gamma_2)(\gamma_1 + \gamma_4 - \sqrt{2\gamma_1})^2 + \frac{1}{4}(\gamma_2 + \gamma_3)^2(\gamma_1 - \gamma_4 - \sqrt{2\gamma_1} + 1). \tag{54}\]

In particular, (49) together with (54) imply that the constants $c_{31}$ and $c_{32}$ can be written in terms of the $\gamma$'s.

A superintegrable potential expressed in terms of the Painlevé transcendent $P_6$ was obtained earlier [62]. It allowed a third order integral and required a specific relation between the constants $\gamma_1, ..., \gamma_4$. Here we obtain the most general form of $P_6$.

From the inverse transformation $x \rightarrow z = \tan \theta$ in (48) we get

\[x_\pm = \frac{1}{2} \pm \frac{1}{2\sqrt{1 + z^2}} = \begin{cases} \sin^2 \left(\frac{\theta}{2}\right) \\ \cos^2 \left(\frac{\theta}{2}\right) \end{cases} \tag{55}\]

we obtain two solutions for $S(\theta) = T'(\theta)$. For case I we obtain two quantum potentials

\[V(r, \theta) = \frac{\partial^2 T(x_\pm)}{r^2} = \frac{\hbar^2}{r^2} \left[W'(x_\pm) + \frac{2 \cos \theta}{\sin \theta} W(x_\pm) + \frac{1}{2} \frac{1}{\sin^2 \theta} \Gamma \right]. \tag{56}\]

where $\Gamma = (\gamma_2 + \gamma_4 + \sqrt{2\gamma_1} - \gamma_1 - \gamma_3 - \frac{3}{8})$. Both $T$ and $W$ are completely defined through (48)–(54). The integral $Y$ in both cases is

\[Y = \hbar^2 \{\partial_\theta^2, \sin \theta \partial_\theta\} + \frac{\hbar^2}{r} \{\partial_\theta^3, \cos \theta \partial_\theta\} - \hbar^2 \{G_1(r, \theta), \partial_\theta^2\} - \hbar^2 \{G_2(r, \theta), \partial_\theta^3\} + G_4(r, \theta), \tag{57}\]

(A2 = 1) where

\[G_1(r, \theta) = 0,\]
\[G_2(r, \theta) = \frac{1}{r} \left(4 \cos \theta T' + 2c_{32} \cos \theta - (T + 2c_{31}) \sin \theta \right),\]
\[G_3(r, \theta) = 3 \sin \theta T' + (T + 2c_{31}) \cos \theta + 2c_{32} \sin \theta,\]
\[G_4(r, \theta) = \frac{1}{2r} \left(\sin \theta T^{(4)} + 4 \cos \theta T^{(3)} - 3 \sin \theta T'' - 2 \cos \theta T' \right)h^2 \]
\[- \frac{2}{r} \left(3 \sin \theta T' + \cos \theta T + 2c_{31} \cos \theta + 2c_{32} \sin \theta \right)T''. \tag{58}\]

here $T' = \partial_\theta T(x_\pm)$. The integral $Y$ and the corresponding potential $V(r, \theta)$ depend on the same constants, namely, the four parameters $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ in (54) which define the sixth Painlevé transcendent $P_6$. 

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4.2. Case II, \( B_3 \neq 0, A_1 = A_2 = B_4 = 0 \)

Equation (41) reduces to a linear one that must be satisfied trivially so we have to impose \( c_{31} = c_{32} = 0 \). Equation (42) simplifies to

\[
B_3 \left[ 16 \sin 2 \theta T' - 20 \cos 2 \theta T'' - 10 \sin 2 \theta T''' + \frac{5}{2} \cos 2 \theta T^{(4)} \right] \\
+ \frac{1}{4} \sin 2 \theta T^{(5)} + \Bigg( 16 \cos 2 \theta T' + 12 \sin 2 \theta T'' - 2 \cos 2 \theta T''' \Bigg) T \\
+ 24 B_3 \sin 2 \theta (T') T - \left( 24 B_3 \cos 2 \theta T'' + 3 B_3 \sin 2 \theta T''' + 4 \left( c_{11} \cos 2 \theta + c_{12} \sin 2 \theta \right) \right) T' \\
- 3 B_3 \sin 2 \theta (T'')^2 + 3 \left( c_{12} \cos 2 \theta - c_{11} \sin 2 \theta \right) T'' + \frac{1}{2} \left( c_{11} \cos 2 \theta + c_{12} \sin 2 \theta \right) T''' = 0. 
\]

(59)

This equation can be integrated once resulting in

\[
B_3 \left[ \frac{1}{4} \sin 2 \theta T^{(4)} + 2 \cos 2 \theta T^{(3)} - 6 \sin 2 \theta T''' - 8 \cos 2 \theta T' \right] h^2 \\
+ B_3 \left( 8 \sin 2 \theta T' - 2 \cos 2 \theta T'' \right) T - 8 B_3 \cos 2 \theta (T')^2 \\
+ \left( 2 \left( c_{12} \cos 2 \theta - c_{11} \sin 2 \theta \right) - 3 B_3 \sin 2 \theta T''' \right) T' + \frac{1}{2} \left( c_{11} \cos 2 \theta + c_{12} \sin 2 \theta \right) T'' + \frac{1}{2} K_1 = 0. 
\]

(60)

Putting \( z = \tan 2 \theta \), (and dividing by the common factor \( 4(z^2 + 1)^{3/2} \) we obtain

\[
B_3 \left( \frac{2}{2} \left( 6 z^2 + 1 \right) T' + \left( 36 z^3 + 26 z \right) T''' + 4 \left( z^2 + 1 \right) \left( 3 z^2 + 1 \right) T^{(3)} + \left( z^2 + 1 \right)^2 \left( z T^{(4)} \right) \right) h^2 \\
- 2 B_3 T'' T - B_3 \left( 12 z^2 + 8 \left( T' \right)^2 + \left( c_{12} - 6 \left( z^2 + 1 \right) z B_3 T''' \right) T' \\
+ \frac{1}{2} \left( c_{12} z + c_{11} \right) T''' + \frac{1}{8} \left( z^2 + 1 \right)^{3/2} \right) = 0. 
\]

(61)

We introduce \( c_{11} \to 2 B_3 c_{11}, c_{12} \to 2 B_3 c_{12} \) and \( K_1 = 2 B_3 K_1 \) and again integrate (61) to obtain

\[
\left( 2 \left( 3 z^4 + z^2 - 1 \right) T' + 2 z \left( 3 z^2 + 1 \right) \left( z^2 + 1 \right) T'' + z^2 \left( z^2 + 1 \right)^2 T^{(3)} \right) h^2 \\
+ T^2 - \left( c_{11} + 2 z T' \right) T - 3 z^2 \left( z^2 + 1 \right) \left( T' \right)^2 + z \left( c_{12} z + c_{11} \right) T'' \\
- \frac{1}{4} \frac{K_1}{\sqrt{z^2 + 1}} + K_2 = 0. 
\]

(62)

The transformation \((z, T(z)) \mapsto (x, W(x)):\)

\[
z = \frac{2 \sqrt{x} \sqrt{1 - x}}{1 - 2 x}, \quad T = \frac{1}{2} (1 - 2 x) \left( h^2 - c_{12} \right) + 2 h^2 W(x) + \frac{c_{11}}{2}. 
\]

(63)

maps (62) to an equation contained in the series of papers by Cosgrove on higher order Painlevé equations [15]. Equation (62) is mapped into the third order differential equation Chazy-I.a with parameters
\[ q_1 = q_4 = q_5 = q_6 = 0, \quad q_2 = -q_3 = 1, \quad q_7 = \frac{1}{16} \frac{c_{12}}{\hbar^2}, \quad q_8 = -\frac{K_1}{32 \hbar^2}, \quad q_9 = -\frac{c_{11}^2 + c_{12}^2 - K_1 - 4K_2 - \hbar^4}{64 \hbar^4}. \]  \hspace{1cm} (64)

The solution for the function \( W(x) \) is given in (51), however the independent variable is different, namely:

\[ x_{\pm} = \frac{1}{2} \pm \frac{1}{2 \sqrt{1 + z^2}} = \left\{ \begin{array}{ll} \sin^2 \theta \\ \cos^2 \theta. \end{array} \right. \]  \hspace{1cm} (65)

We obtain two solutions for \( S(\theta) = T'(\theta) \). By taking the derivative \( \partial_\theta T(x_{\pm}) \) we obtain the quantum potentials

\[ V(r, \theta) = \frac{\partial_\theta T(x_{\pm})}{r^2} = \frac{\hbar^2}{r^2} \left( 4 W'(x_{\pm}) \mp \frac{8 \cos 2\theta}{\sin^2 2\theta} W(x_{\pm}) + \frac{1}{\sin^2 2\theta} \Gamma \right), \]  \hspace{1cm} (66)

where \( \Gamma = 2(\gamma_2 + \gamma_4 + \sqrt{2\gamma_1 - \gamma_1 - \gamma_3 + \frac{1}{4}}) \). \( T \) is now defined through (63)–(64). These potentials correspond to the integral

\[ Y = \hbar^4 \{ \partial^2_{\theta}, \cos 2\theta \} \partial_r^2 - \frac{\hbar^4}{r^2} \{ \partial^2_\theta, \cos 2\theta \partial^2_r \} - \frac{2 \hbar^4}{r} \{ \partial^2_\theta, \sin 2\theta \partial_r \} \partial_r \\
+ \frac{2 \hbar^4}{r^2} \{ \partial^2_\theta, \sin 2\theta \partial_r \} - \frac{\hbar^4}{r} \{ \partial_\theta, \cos 2\theta \} \partial_r \\
- \hbar^2 \{ \{ G_1(r, \theta), \partial^2_r \} + \{ G_3(r, \theta), \partial_r \partial_\theta \} + \{ G_2(r, \theta), \partial^2_\theta \} \} + G_4(r, \theta), \]  \hspace{1cm} (67)

(with \( B_3 = 1 \)) where

\[ G_1 (r, \theta) = 2 \cos 2\theta T' - 2 \sin 2\theta T + c_{11} \sin 2\theta - c_{12} \cos 2\theta, \]

\[ G_2 (r, \theta) = \frac{1}{r^2} \left( 2 \sin 2\theta T - 4 \cos 2\theta T' - c_{11} \sin 2\theta + c_{12} \cos 2\theta \right), \]

\[ G_3 (r, \theta) = \frac{1}{r} \left( 2c_{11} \cos 2\theta T - 4 \cos 2\theta T' - 6 \sin 2\theta T' + 2c_{12} \sin 2\theta \right), \]

\[ G_4 (r, \theta) = \frac{1}{r} \left( -\frac{1}{2} \sin 2\theta T^{(4)} - 4 \cos 2\theta T^{(3)} + 10 \sin 2\theta T'' + 8 \cos 2\theta T' \right) \hbar^2 \\
+ \left( (4T - 2c_{11}) \cos 2\theta + (6T' - 2c_{12}) \sin 2\theta \right) \cos 2\theta T' + 8 \cos 2\theta T' \sin 2\theta \right). \]

\[ 5. \text{ Confining potentials} \]

\[ 5.1. \text{ Potential} \quad V(r, \theta) = br^2 + \frac{S(\theta)}{r} \]

In this case the compatibility condition (27) is satisfied trivially if all parameters are zero except \( B_3, B_2 \). Since we can rotate between these two terms we set \( B_4 = 0 \). The equation for \( S(\theta) \) corresponds to case II of section 4.
The only determining equation to solve is (59) and the solution for the function $S(\theta)$ will be the same as for $R(r) = 0$. The only difference with the case $R(r) = 0$ is reflected in the $G$ functions (10) which does not modify the form of the determining equation (59). The corresponding quantum potentials are

\[
V(r, \theta) = br^2 + \frac{\partial_b T(x_{\pm})}{r^2} = br^2 + \frac{\hbar^2}{r^2} \left( 4W'(x_{\pm}) + \frac{8 \cos 2\theta}{\sin^2 2\theta} W(x_{\pm}) + \frac{1}{\sin^2 2\theta} \Gamma \right),
\]

where $\Gamma = 2(\gamma_2 + \gamma_4 + \sqrt{2} \gamma_1 - \gamma_1 - \gamma_3 + \frac{1}{2})$. The function $T$ is defined through (63)-(64).

The integral of motion in this case is

\[
Y = \hbar^4 \left\{ \partial_{\theta}^2 \cos 2\theta \right\} \partial_r^2 - \frac{\hbar^4}{r^2} \left\{ \partial_{\theta}^2 \cos 2\theta \partial_{\theta} \right\} \partial_r - \frac{2\hbar^4}{r} \left\{ \partial_{\theta}^2 \sin 2\theta \partial_{\theta} \right\} \partial_r - \frac{2\hbar^4}{r} \left\{ \partial_{\theta}^2 \sin 2\theta \partial_{\theta} \right\} \partial_r - \frac{\hbar^4}{r^2} \left\{ \partial_{\theta}^2 \sin 2\theta \partial_{\theta} \right\} \partial_r + \frac{\hbar^4}{r^2} \left\{ \partial_{\theta}^2 \cos 2\theta \partial_{\theta} \right\} \partial_r
\]

\[
\left\{ G_1(r, \theta), \partial_r^2 \right\} + \left\{ G_2(r, \theta), \partial_r \partial_{\theta} \right\} + \left\{ G_3(r, \theta), \partial_{\theta} \right\} + \left\{ G_4(r, \theta), \partial_r \right\} + G_4(r, \theta),
\]

where

\[
G_1(r, \theta) = 2 \cos 2\theta T' - 2 \sin 2\theta T + c_{11} \sin 2\theta - c_{12} \cos 2\theta,
\]

\[
G_2(r, \theta) = \frac{1}{r} \left( 2 \sin 2\theta T - 4 \cos 2\theta T' - c_{11} \sin 2\theta + c_{12} \cos 2\theta \right) + 2b \cos 2\theta r^2,
\]

\[
G_3(r, \theta) = \frac{1}{r} \left( 2c_{11} \cos 2\theta T - 4 \cos 2\theta T' - 6 \sin 2\theta T' + 2c_{12} \sin 2\theta \right),
\]

\[
G_4(r, \theta) = \frac{1}{r} \left( \left( -\frac{1}{2} \sin 2\theta T^{(4)} - 4 \cos 2\theta T^{(3)} + 10 \sin 2\theta T'' + 8 \cos 2\theta T' \right) \hbar^2 + \left( (4T - 2c_{11}) \cos 2\theta + (6T' - 2c_{12}) \sin 2\theta \right) T'' + (8T' - 4c_{12}) T' \cos 2\theta - (8T - 4c_{11}) T' \sin 2\theta \right)
\]

\[
+ b \left[ -8 \sin 2\theta T + 8 \cos 2\theta T' + 4 \left( c_{11} \sin 2\theta - c_{12} \cos 2\theta \right) \right] r^2 - 8 \cos 2\theta b \hbar^2 r^2.
\]

5.2. Potential of the form $V(r, \theta) = \frac{a}{r} + \frac{S(\theta)}{r^2}$

In this case the compatibility condition (27) is satisfied trivially if all parameters are zero except $A_1, A_2, B_1, B_2$ (as in the case of $R(r) = 0$).

From the condition $[H, Y] = 0$ we obtain two fifth order non-linear equations equations in $T(\theta)$ that must be satisfied simultaneously, namely equation (42) and
\[ 0 = \left[ -4 (A_1 \cos \theta + A_2 \sin \theta) T' + 8 (A_2 \cos \theta - A_1 \sin \theta) T'' \\
+ 6 (A_1 \cos \theta + A_2 \sin \theta) T^{(3)} + \frac{5}{2} (A_1 \sin \theta - A_2 \cos \theta) T^{(4)} \\
- \frac{1}{2} (A_1 \cos \theta + A_2 \sin \theta) T^{(5)} + 15 a (B_3 \sin 2 \theta - B_4 \cos 2 \theta) \right] \hbar^2 \\
+ \left[ 4 (A_1 \sin \theta - A_2 \cos \theta) T' - 6 (A_1 \cos \theta + A_2 \sin \theta) T'' \\
+ 2 (A_2 \cos \theta - A_1 \sin \theta) T^{(3)} + 24 a (B_4 \sin 2 \theta + B_3 \cos 2 \theta) \right] T \\
- 12 (A_1 \cos \theta + A_2 \sin \theta) (T')^2 + \left[ 24 (A_2 \cos \theta - A_1 \sin \theta) T'' \\
+ 6 (A_2 \sin \theta + A_1 \cos \theta) T^{(3)} + 44 a (B_3 \sin 2 \theta - B_4 \cos 2 \theta) \\
- 4 c_{31} \cos \theta - 4 c_{32} \sin \theta \right] T' + 6 (A_2 \sin \theta + A_1 \cos \theta) (T'')^2 \\
- 6 (4 a B_4 \sin 2 \theta + 4 a B_3 \cos 2 \theta - c_{32} \cos \theta + c_{31} \sin \theta) T'' \\
- 2 (2 a B_4 \sin 2 \theta - 2 a B_3 \cos 2 \theta - c_{31} \cos \theta - c_{32} \sin \theta) T^{(3)} \\
- 6 a (c_{12} \sin 2 \theta + c_{11} \cos 2 \theta) \right]. \tag{71} \]

For \( a = 0 \) (71) coincides with (41).

**Case I.** \( B_3 = B_4 = c_{11} = c_{12} = 0, A_1 \) and \( A_2 \) arbitrary. The non-linear equation (42) is satisfied trivially, while (71) coincides with (45) and thus, in this case, we obtain the quantum potentials

\[ V(r, \theta) = \frac{a}{r} + \frac{\partial_T T(x_{\pm})}{r^2} = \frac{a}{r} + \frac{\hbar^2}{r^2} \left( W'(x_{\pm}) + \frac{2 \cos \theta}{\sin^2 \theta} W(x_{\pm}) + \frac{1}{2} \frac{\Gamma}{\sin^2 \theta} \right). \tag{72} \]

where \( \Gamma = (\gamma_2 + \gamma_4 + 2 \gamma_1 - \gamma_1 - \gamma_3 - \frac{1}{2}) \), and both \( T \) and \( W \) are completely defined through (48)–(54). These potentials correspond to the integral (\( A_2 = 1 \))

\[ Y = \hbar^2 \left\{ \partial_{r}^2, \sin \theta \partial_{\theta} \right\} + \frac{\hbar^2}{r} \left\{ \partial_{\theta}^2, \cos \theta \partial_{\theta} \right\} - \hbar^2 \left\{ G_1(r, \theta), \partial_{r}^2 \right\} - \hbar^2 \left\{ G_3(r, \theta), \partial_{r} \partial_{\theta} \right\} \\
- \hbar^2 \left\{ G_2(r, \theta), \partial_{\theta}^2 \right\} + G_4(r, \theta). \tag{73} \]

where

\[ G_1(r, \theta) = 0, \]

\[ G_2(r, \theta) = \frac{1}{r} \left( 4 \cos \theta \sin \theta T' + 2 c_{32} \sin \theta - (T + 2 c_{31}) \sin \theta \right) + a \cos \theta, \]

\[ G_3(r, \theta) = 3 \sin \theta T' + (T + 2 c_{31}) \cos \theta + 2 c_{32} \sin \theta, \]

\[ G_4(r, \theta) = \frac{1}{2r} \left( \sin \theta T^{(4)} + 4 \cos \theta \sin \theta T^{(3)} - 3 \sin \theta T'' - 2 \cos \theta T' \right) \hbar^2 \\
- \frac{2}{r} \left( 3 \sin \theta T' + \cos \theta T + 2 c_{31} \cos \theta + 2 c_{32} \sin \theta \right) T'' \\
- 2 a \sin \theta T + 4 a \cos \theta T' - 4 a (c_{31} \sin \theta - c_{32} \cos \theta) - a \hbar^2 \cos \theta, \tag{74} \]
here $T' = \partial_\theta T(x_{\pm})$.

**Case II.** For $A_1 = A_2 = c_{31} = c_{32} = 0$, $a \neq 0$ and $B_3, B_4$ arbitrary (71) reduces to a linear equation. For exotic potentials it must be satisfied identically. This implies $B_3 = B_4 = 0$, so no fourth order integral exists.

### 6. Classical potentials

The two determining equations (41) and (42) reduce to third order equations for $T'(\theta)$ once we impose the condition $\hbar \to 0$. The limit is singular and interestingly, the equations in this case do not pass the Painlevé test. The division into subcases (44) remains. We can always integrate (41) and (42) twice and we obtain a first order nonlinear equation of the form

$$Q_4(z) T'^2 + Q_1(z) TT' + T^2 + Q_2(z) T' + c T + Z(z) = 0,$$

where $Q_n(z)$ is a polynomial in $z$ of order $n$, $Z(z)$ is a rational function and $c$ is a constant. Using the transformation

$$T(z) = m(z) t(z) + n(z),$$

we can factorize (75) as follows

$$(t' - t'_0)(t' + t'_0) = 0,$$

where

$$t'_0 = \sqrt{(Q_1(mt + n) + Q_2)^2 - 4Q_4(Q_0(mt + n) + (mt + n)^2 + Z)},$$

and $m$ and $n$ satisfy

$$2Q_4(z) n'(z) + n(z) Q_1(z) + Q_2(z) = 0,$$

$$2Q_4(z) m'(z) + m(z) Q_1(z) = 0.$$

In general, explicit solutions to the equation $t' \pm t'_0 = 0$ are not known. However for special values of the parameters contained in the $Q_i$ and $Z$, the function $t'_0$ becomes linear in $t$ and explicit solutions can be constructed.

#### 6.1. Case $V(r, \theta) = S(\theta)$

**6.1.1. Case I.** The classical potential $S(\theta) = T'(\theta)$ satisfies (47) with $\hbar \to 0$. This limit is singular, the order of the equation (47) drops from three to one. The so obtained non-linear first order differential equation reads:

$$T^2 - 2(z T' - c_{31}) T - 3z^2 (z^2 + 1) (T')^2 - 4z (c_{32} z + c_{31}) T' + 2 \frac{K_1}{\sqrt{z^2 + 1}} - \frac{K_2}{2} = 0,$$

where $z = \tan \theta$. Factorization of the l.h.s in (76) in the form of a product of two factors of first order allows us to find particular solutions. These two factors become linear in $T$ for specific values of the parameters in (76) only. Namely, putting $K_1 = c_{32} = 0$ and $K_2 = -8c_{31}^2$ in (76) we obtain the equation
\[6 z^2 (1 + z^2) \left( T' + \frac{2 c_{31} + T}{3 z (1 + z^2)} \right) \left( T' + \frac{2 c_{31} + T}{3 z (1 + z^2)} \right) = 0,\]

from which we derive two particular solutions:

\[T_1 = -2 c_{31} + \alpha \frac{z^4 (3 z^2 + 2 \sqrt{3} z^2 + 4 + 5) \gamma}{(\sqrt{3} z^2 + 4 + 2)^\gamma},\]

\[T_2 = -2 c_{31} + \alpha \left( \frac{1 + z^2}{z^2} \right)^\gamma \frac{(2 + \sqrt{4 + 3 z^2}) \gamma}{(\sqrt{3} z^2 + 4 + 5)^\gamma},\]

where \(\alpha\) is an integration constant. By differentiating the preceding results (77) with respect to \(\theta\) we obtain the classical potentials:

\[V_1(\theta, \theta) = \frac{3 \alpha \sec^4 \theta \left[ 7 + 3 \sqrt{4 + 3 \tan^2 \theta} + \cos 2 \theta (1 + \sqrt{4 + 3 \tan^2 \theta}) \right]}{r^2 \tan^2 \theta \sqrt{4 + 3 \tan^2 \theta} \left( 2 + \sqrt{4 + 3 \tan^2 \theta} \right)} \left( 5 + 3 \tan^2 \theta + 2 \sqrt{4 + 3 \tan^2 \theta} \right)^\gamma,\]

and

\[V_2(\theta, \theta) = -\frac{\alpha \sec^4 \theta \left[ 47 + 17 \sqrt{4 + 3 \tan^2 \theta} + 18 \cot^2 \theta (2 + \sqrt{4 + 3 \tan^2 \theta}) + 3 \tan^2 \theta (5 + \sqrt{4 + 3 \tan^2 \theta}) \right]}{2 r^2 \sqrt{4 + 3 \tan^2 \theta} \left( 2 + \sqrt{4 + 3 \tan^2 \theta} \right) \left( 5 + 3 \tan^2 \theta + 2 \sqrt{4 + 3 \tan^2 \theta} \right)^\gamma}.\]

In general, the potentials \(V(r, \theta)\) associated with (76) possess the integral

\[Y = 2 \sin \theta p_\theta p_\theta + \frac{2}{r} \cos \theta p_\theta + 2 G_1(\theta, \theta) p_r^2 + 2 G_3(\theta, \theta) p_r p_\theta + 2 G_2(\theta, \theta) p_\theta^2 + G_4(\theta, \theta),\]

\((A_2 = 1)\) where

\[G_1(\theta, \theta) = 0,\]

\[G_2(\theta, \theta) = \frac{1}{r} \left( 4 \cos \theta T' + 2 c_{32} \cos \theta - (T + 2 c_{31}) \sin \theta \right),\]

\[G_3(\theta, \theta) = 3 \sin \theta T' + (T + 2 c_{31}) \cos \theta + 2 c_{32} \sin \theta,\]

\[G_4(\theta, \theta) = -\frac{2}{r} \left( 3 \sin \theta T' + \cos \theta T + 2 c_{31} \cos \theta + 2 c_{32} \sin \theta \right) T''.\]

6.1.2. Case II. The classical potential \(S(\theta)\) satisfies (62) with \(h \mapsto 0\). This limit is singular, the order of the equation (62) drops from three to one. The so obtained non-linear first order differential equation in \(T\) reads:

\[T^2 - \left( c_{11} + 2 z T' \right) T - 3 z^2 \left( z^2 + 1 \right) T' + \cos \theta T + 2 c_{31} \cos \theta + 2 c_{32} \sin \theta \]

\[= \frac{1}{4} K_1 + K_2 = 0,\]

(82)
Factorization of the l.h.s in (82) in the form of a product of two factors of first order allows us to find particular solutions again. The factors are linear in $T$ for specific values of the parameters in (82) only. These special values are $K_1 = c_{12} = 0$ and $K_2 = \frac{1}{4} c_{11}^2$. By substituting these values in (82) we derive two particular solutions

$$T_3 = \frac{c_{11}}{2} + \alpha \frac{z^2}{(\sqrt{3z^2 + 4} + 2)}^2 \left(3z^2 + 2\sqrt{3z^2 + 4} + 5\right)^{\frac{1}{2}},$$

$$T_4 = \frac{c_{11}}{2} + \alpha \frac{(1 + z^2)^{\frac{1}{2}} (2 + \sqrt{4 + 3 z^2})^2}{z(3z^2 + 2\sqrt{3z^2 + 4} + 5)^{\frac{1}{2}}},$$

(83)

where $\alpha$ is an integration constant. By differentiating the preceding results (83) with respect to $\theta$ we obtain the classical potentials:

$$V_3(r, \theta) = \frac{3\alpha \sec^2 2\theta \left[7 + 3\sqrt{4 + 3 \tan^2 2\theta} + \cos 4\theta(1 + \sqrt{4 + 3 \tan^2 2\theta})\right]}{r^2 \tan^2 2\theta \sqrt{4 + 3 \tan^2 2\theta} \left(2 + \sqrt{4 + 3 \tan^2 2\theta}\right)^2 \left(5 + 3 \tan^2 2\theta + 2\sqrt{4 + 3 \tan^2 2\theta}\right)^{\frac{1}{2}}},$$

(84)

and

$$V_4(r, \theta) = -\frac{\alpha \sec^2 2\theta \left[47 + 17\sqrt{4 + 3 \tan^2 2\theta} + 18 \cot^2 2\theta(2 + \sqrt{4 + 3 \tan^2 2\theta} + 3 \tan^2 2\theta(5 + \sqrt{4 + 3 \tan^2 2\theta})\right]}{2r^2 \sqrt{4 + 3 \tan^2 2\theta} \left(2 + \sqrt{4 + 3 \tan^2 2\theta}\right)^2 \left(5 + 3 \tan^2 2\theta + 2\sqrt{4 + 3 \tan^2 2\theta}\right)^{\frac{1}{2}}},$$

(85)

where $\alpha$ is a constant.

For (82) the potentials $V(r, \theta)$ possess the integral

$$Y = 2 p_r^2 p_\theta^2 \cos 2\theta - \frac{2}{r^2} p_\theta^4 \cos 2\theta - \frac{4}{r} p_r p_\theta^3 \sin 2\theta + 2G_1(r, \theta) p_r p_\theta + 2G_2(r, \theta) p_\theta + 2G_2(r, \theta) p_\theta + 2G_4(r, \theta),$$

(86)

(with $B_3 = 1$) where

$$G_1(r, \theta) = 2 \cos 2\theta T' - 2 \sin 2\theta T + c_{11} \sin 2\theta - c_{12} \cos 2\theta,$$

$$G_2(r, \theta) = \frac{1}{r^2} \left(2 \sin 2\theta T - 4 \cos 2\theta T' - c_{11} \sin 2\theta + c_{12} \cos 2\theta\right),$$

$$G_3(r, \theta) = \frac{1}{r} \left(2 c_{11} \cos 2\theta - 4 \cos 2\theta T - 6 \sin 2\theta T' + 2 c_{12} \sin 2\theta\right),$$

$$G_4(r, \theta) = \frac{1}{r} \left(\left((4T - 2c_{11}) \cos 2\theta + (6T' - 2c_{12}) \sin 2\theta\right) T'' + (8T' - 4c_{12}) T' \cos 2\theta - (8T - 4c_{11}) T' \sin 2\theta\right).$$

6.2. Potential $V(r, \theta) = br^2 + \frac{8\theta}{r^2}$

The classical potentials are given by

$$V(r, \theta) = br^2 + \frac{T'(\theta)}{r^2},$$

(87)

$T$ from (82), and they correspond to the integral
\[ Y = 2 p_r^2 p_\theta^2 \cos 2\theta - \frac{2}{r} p_r^2 p_\theta^2 \cos 2\theta - \frac{4}{r} p_r p_\theta^3 \sin 2\theta \]  
\[ + 2 G_1(r, \theta) p_r^2 + 2 G_3(r, \theta) p_r p_\theta + 2 G_2(r, \theta) p_\theta^2 + G_4(r, \theta). \]  
(88)

(with \( B_3 = 1 \)) where

\[ G_1(r, \theta) = 2 \cos 2\theta T' - 2 \sin 2\theta T + c_{11} \sin 2\theta - c_{12} \cos 2\theta, \]

\[ G_2(r, \theta) = \frac{1}{r'} \left( 2 \sin 2\theta T - 4 \cos 2\theta T' - c_{11} \sin 2\theta + c_{12} \cos 2\theta \right) + 2 b r^2 \cos 2\theta, \]

\[ G_3(r, \theta) = \frac{1}{r'} \left( 2 c_{11} \cos 2\theta - 4 \cos 2\theta T - 6 \sin 2\theta T' + 2 c_{12} \sin 2\theta \right), \]

\[ G_4(r, \theta) = \frac{1}{r'} \left( \left( 4 T - 2 c_{11} \right) \cos 2\theta + \left( 6 T' - 2 c_{12} \right) \sin 2\theta \right) T'' + \left( 8 T' - 4 c_{12} \right) T' \cos 2\theta - \left( 8 T - 4 c_{11} \right) T' \sin 2\theta \]  
\[ + 4 b r^2 \left( 2 \cos 2\theta T' - 2 \sin 2\theta T + c_{11} \sin 2\theta - c_{12} \cos 2\theta \right). \]  
(90)

6.3. Potential \( V(r, \theta) = \frac{2}{r} + \frac{g(\theta)}{r^2} \)

Similarly, the classical potentials are given by

\[ V(r, \theta) = \frac{a}{r} + \frac{T'(\theta)}{r^2}, \]  
(91)

\( T \) from (76), and they corresponds to the integral

\[ Y = 2 \sin \theta p_\theta^2 p_r + \frac{2}{r} \cos \theta p_\theta^2 + 2 G_1(r, \theta) p_r p_\theta + 2 G_3(r, \theta) p_\theta^2 + 2 G_2(r, \theta) p_\theta^2 + G_4(r, \theta), \]  
(92)

\( (A_2 = 1) \) where

\[ G_1(r, \theta) = 0, \]

\[ G_2(r, \theta) = \frac{1}{r} \left( 4 \cos \theta T' + 2 \cos \theta c_{32} - (T + 2 c_{31}) \sin \theta \right) + a \cos \theta, \]

\[ G_3(r, \theta) = \frac{3}{r} \sin \theta T' + (T + 2 c_{31}) \cos \theta + 2 \sin \theta c_{32}, \]

\[ G_4(r, \theta) = -\frac{2}{r} \left( 3 \sin \theta T' + \cos \theta T + 2 c_{31} \cos \theta + 2 c_{32} \sin \theta \right) T'' \]  
\[ + 2 a \left( 2 \cos \theta T' - \sin \theta T \right) - 4 a (c_{31} \sin \theta - c_{32} \cos \theta). \]  
(93)

7. Polynomial algebra

In this section we discuss the algebra of the integrals of motion in the classical case [18, 19, 25].

Take the second order integral \( X \) and the fourth order ones \( Y, (4) \) and \( (5) \) respectively. Let us define, via their Poisson bracket \( \{ \},_m \), the fifth order polynomial in momenta

\[ C \equiv \{ Y, X \}_m, \]  
(94)
which by construction is also an integral of motion. Now we study the algebra generated by the four quantities \( H, X, Y \) and \( C \). The relevant (non vanishing) Poisson brackets are \( \{X, C\}_m \) and \( \{Y, C\}_m \) only.

First we consider the case of the extended harmonic oscillator potential

\[
V(r, \theta) = br^2 + \frac{T'(\theta)}{r^2}.
\]

For the particular solutions (83), \( T_3 \) and \( T_4 \), the algebra generated by the integrals is given by

\[
\{X, C\}_m = 16XY,
\]

\[
\{Y, C\}_m = 8 \left[ 48H^2X^2 - 128X^3b - Y^2 + b\sigma_{3,4} \right],
\]

(95)

where \( \sigma_3 = \frac{512}{9} \alpha^3 \), \( \sigma_4 = -\frac{512}{3} \alpha^3 \) and \( \alpha \) a non zero constant, respectively. At \( b = 0 \) this algebra reduces to that of case II, \( R(r) = 0 \). For an arbitrary solution of (82), in order to the algebra to be closed the function \( T \) must satisfy a sixth order polynomial equation presented in the appendix B. Then the algebra takes the form

\[
\{X, C\}_m = 16XY - 32K_1H,
\]

\[
\{Y, C\}_m = 8 \left[ 48H^2X^2 - Y^2 - 128bX^3 - 64c_{12}H^2X \\
+ 16(c_{11} + c_{12} - 4K_2)H + 192b c_{12} X^2 \right] \\
- 512b (c_{11} + c_{12} - 4K_2)X - b\lambda,
\]

(96)

where \( \lambda \) is an arbitrary constant. It is a quartic polynomial algebra.

For the extended Coulomb potential

\[
V(r, \theta) = \frac{a}{r} + \frac{T'(\theta)}{r^2},
\]

with the particular solutions \( T_1 \) and \( T_2 \) we have that

\[
\{X, C\}_m = 4XY,
\]

\[
\{Y, C\}_m = 2 \left[ 32HX^3 + 12a^2X^2 - Y^2 + \sigma_{1,2}H \right],
\]

(97)

where \( \sigma_1 = -\frac{16}{9} \alpha^3 \) and \( \sigma_2 = \frac{16}{9} \alpha^3 \), respectively. At \( a = 0 \) this algebra corresponds to case I, \( R(r) = 0 \). Similarly, for a general solution of (76) the function \( T \) must also satisfy a sixth order polynomial equation and the corresponding algebra reads

\[
\{X, C\}_m = 4XY + 8aK_1,
\]

\[
\{Y, C\}_m = 2 \left[ 32HX^3 - Y^2 + 12a^2X^2 + 96c_{32}HX^2 + 64(c_{31} + c_{32} + K_2^2/8)HX + 32a^2c_{32}X \right] \\
- \lambda H + 32a^2 (c_{31} + c_{32} + K_2^2/8),
\]

(98)

In the classical case the algebra of \( H, X, Y \) and \( C \) is useful to obtain and classify the trajectories. In full generality, namely for general solutions of (76) and (82), an algebraic equation for the non-trivial part \( T(\theta) \) of the potential can be derived by requiring the algebra to be closed.
In the quantum case, once the functions \(G_1, ..., G_4\) figuring in the integral \(Y\) are calculated, it is possible to express the two commutators \([X, C]\) and \([Y, C]\) as polynomials in \(X, Y\) and \(H\). As a matter of fact, the condition that the algebra of the integrals of motion should close leads directly to the fifth order equations (45) and (59) for \(T\). Moreover, this closure also provides the integrals of these equations such as e.g. equation (50).

8. Conclusions

We studied superintegrability in a two-dimensional Euclidean space. Classical and quantum fourth-order superintegrable potentials separating in polar coordinates were derived. We can summarize the main results via the following theorems

**Theorem 1.** In quantum mechanics, the confining superintegrable systems correspond to

\[
V(r, \theta) = \frac{a}{r} + \frac{\hbar^2}{r^2} \left( W'(x_{\pm}) + \frac{2 \cos \theta}{\sin^2 \theta} W(x_{\pm}) + \frac{8(\gamma_2 + \gamma_4 + \sqrt{2} \gamma_1 - \gamma_1 - \gamma_3) - 3}{16 \sin^2 \theta} \right),
\]

where \(x_{\pm} = \sin^2 \left( \frac{\theta}{2} \right), \cos^2 \left( \frac{\theta}{2} \right)\) and

\[
V(r, \theta) = b r^2 + \frac{\hbar^2}{r^2} \left( 4 W'(x_{\pm}) \mp \frac{8 \cos 2\theta}{\sin^2 2\theta} W(x_{\pm}) + \frac{4(\gamma_2 + \gamma_4 + \sqrt{2} \gamma_1 - \gamma_1 + \gamma_3) - 3}{2 \sin^2 2\theta} \right).
\]

The non-confining potentials are given by (56) with integral (57), and (66) with integral (67).

The function

\[W = W(x, P_6(x); \gamma_1, \gamma_2, \gamma_3, \gamma_4)\]

is expressed in terms of the sixth Painlevé transcendent \(P_6\) in full generality. In the case of a third order superintegrable system, not all four \((\gamma_1, \gamma_2, \gamma_3, \gamma_4)\) but three constants in (53) are arbitrary only. Moreover, the third order system does not allow any confining potentials.

**Theorem 2.** In classical mechanics, the superintegrable confining systems correspond to

\[
V(r, \theta) = \frac{a}{r} + \frac{T'(\theta)}{r^2},
\]

where \(T'\) satisfies (76) and \(a\) is an arbitrary constant. The leading term of the integral \(Y\) in (8) is \(\{L_3^z, p_z\}\) and

\[
V(r, \theta) = b r^2 + \frac{T'(\theta)}{r^2},
\]

where \(T'\) satisfies (82), \(b\) is constant, and the leading term of \(Y\) is given by \(\{L_3^z, p_z^2 - p_z^4\}\).

Particular solutions of (76) and (82) were presented in (77) and (83), respectively.

The non-confining superintegrable systems are given by (78) and (79) with integral (80), and (84), (85) with integral (86), respectively.
Work is currently in progress on a continuation of this article. We will add a general investigation of the polynomial algebra generated by the integrals of motion in the classical and quantum cases. We also plan to present figures of the classical trajectories and to use the algebra of integrals to calculate the energy spectrum and the wave functions in the quantum case. Another part of the project is to determine all corresponding non-exotic potentials.

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Appendix A. Functions $F_i$

Explicitly, the functions $F_1, \ldots, F_{13}$ in (13)–(18) are given by

\begin{align}
F_1 &= 2 \left( B_1 \cos 2 \theta + B_2 \sin 2 \theta + D_1 \cos 4 \theta + D_2 \sin 4 \theta \right), \\
F_2 &= \frac{1}{r} \left( B_2 \cos 2 \theta - B_1 \sin 2 \theta - 2 D_1 \sin 4 \theta + 2 D_2 \cos 4 \theta \right) + A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3 \theta + C_2 \cos 3 \theta, \\
F_3 &= \frac{1}{r^2} \left( B_2 \cos 2 \theta - B_1 \sin 2 \theta + 2 D_1 \sin 4 \theta - 2 D_2 \cos 4 \theta \right) + \frac{1}{r} \left( A_4 \sin \theta + A_3 \cos \theta - 3 C_2 \cos 3 \theta - 3 C_1 \sin 3 \theta \right) - \frac{2}{r} \left( B_3 \sin 2 \theta - B_4 \cos 2 \theta \right) + A_2 \sin \theta + A_1 \cos \theta, \\
F_4 &= \frac{2}{r^2} \left( D_1 \cos 4 \theta + D_2 \sin 4 \theta - B_1 \cos 2 \theta - B_2 \sin 2 \theta \right) + \frac{4}{r} \left( A_4 \cos \theta - A_3 \sin \theta - C_1 \cos 3 \theta + C_2 \sin 3 \theta \right) - \frac{4}{r} \left( B_3 \cos 2 \theta + B_4 \sin 2 \theta \right) + \frac{4}{r} \left( A_2 \cos \theta - A_1 \sin \theta \right), \\
F_5 &= -\frac{6}{r^2} \left( D_1 \cos 4 \theta + D_2 \sin 4 \theta \right) + \frac{2}{r} \left( A_4 \cos \theta - A_3 \sin \theta + 3 (C_1 \cos 3 \theta - C_2 \sin 3 \theta) \right) + 2 \left( B_3 \cos 2 \theta + B_4 \sin 2 \theta \right), \\
F_6 &= \frac{3}{r} \left( B_2 \cos 2 \theta - B_1 \sin 2 \theta + 2 D_2 \cos 4 \theta - 2 D_1 \sin 4 \theta \right) + 3 \left( A_4 \sin \theta + A_3 \cos \theta + C_1 \sin 3 \theta + C_2 \cos 3 \theta \right),
\end{align}
\[ F_7 = \frac{12}{r^2} \left( D_1 \sin 4\theta - D_2 \cos 4\theta \right) + \frac{1}{2r} \left( A_4 \sin \theta + A_3 \cos \theta + 30 C_1 \sin 3\theta - 30 C_2 \cos 3\theta - 2 (B_3 \sin 2\theta - B_4 \cos 2\theta) \right), \]

\[ F_8 = \frac{3}{r^2} \left( B_1 \cos 2\theta + B_2 \sin 2\theta - 5 D_1 \cos 4\theta - 5 D_2 \sin 4\theta \right) + \frac{3}{2r^2} \left( A_4 \cos \theta - A_3 \sin \theta - 9 C_1 \cos 3\theta + 9 C_2 \sin 3\theta \right) - \frac{5}{r} \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right) + \frac{3}{2} \left( A_2 \cos \theta - A_1 \sin \theta \right), \]

\[ F_9 = \frac{9}{r^4} \left( B_1 \sin 2\theta - B_2 \cos 2\theta - 2 D_1 \sin 4\theta + 2 D_2 \cos 4\theta \right) + \frac{15}{2r^2} \left( 3 C_1 \sin 3\theta + 3 C_2 \cos 3\theta - A_3 \cos \theta - A_4 \sin \theta \right) + \frac{12}{r} \left( B_3 \sin 2\theta - B_4 \cos 2\theta \right) - \frac{9}{2r} \left( A_1 \cos \theta + A_2 \sin \theta \right), \]

\[ F_{10} = -\frac{9}{r} \left( D_1 \cos 4\theta + D_2 \sin 4\theta \right) + \frac{3}{2} \left( A_4 \cos \theta - A_3 \sin \theta + 3 C_1 \cos 3\theta - 3 C_2 \sin 3\theta \right), \]

\[ F_{11} = -\frac{3}{r^2} \left( B_1 \cos 2\theta + B_2 \sin 2\theta - 5 D_1 \cos 4\theta - 5 D_2 \sin 4\theta \right) - 4 \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right) + \frac{1}{r} \left( A_4 \cos \theta - A_3 \sin \theta - 9 C_1 \cos 3\theta + 9 C_2 \sin 3\theta \right), \]

\[ F_{12} = \frac{2}{r^3} \left( B_1 \sin 2\theta - B_2 \cos 2\theta - 14 D_1 \sin 4\theta + 14 D_2 \cos 4\theta \right) + \frac{3}{2r^2} \left( 11 C_1 \sin 3\theta + 11 C_2 \cos 3\theta - A_3 \cos \theta - A_4 \sin \theta \right) + \frac{5}{r} \left( B_3 \sin 2\theta - B_4 \cos 2\theta \right) - \frac{3}{2} \left( A_1 \cos \theta + A_2 \sin \theta \right), \]

\[ F_{13} = \frac{4}{r^2} \left( 2 B_1 \cos 2\theta + 2 B_2 \sin 2\theta - 11 D_1 \cos 4\theta - 11 D_2 \sin 4\theta \right) - \frac{2}{r^3} \left( A_4 \cos \theta - A_3 \sin \theta - 17 C_1 \cos 3\theta + 17 C_2 \sin 3\theta \right) + \frac{12}{r^2} \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right) - \frac{3}{r} \left( A_2 \cos \theta - A_1 \sin \theta \right). \]

For a non-confining potential \( R(r) = 0 \), the functions \( F_i \) (21)–(26) reduce to
\[ F_1 = 0, \]
\[ F_2 = 0, \]
\[ F_3 = A_1 \cos \theta + A_2 \sin \theta + \frac{2}{r} \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right), \]
\[ F_4 = \frac{4}{r} \left( A_2 \cos \theta - A_1 \sin \theta \right) - \frac{4}{r^2} \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right), \]
\[ F_5 = 2 \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right), \]
\[ F_6 = 0, \]
\[ F_7 = 2 \left( B_4 \cos 2\theta - B_3 \sin 2\theta \right), \]
\[ F_8 = \frac{3}{2} \left( A_2 \cos \theta - A_1 \sin \theta \right) - \frac{5}{r} \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right), \]
\[ F_9 = \frac{12}{r^2} \left( B_3 \sin 2\theta - B_4 \cos 2\theta \right) - \frac{9}{2r} \left( A_1 \cos \theta + A_2 \sin \theta \right), \]
\[ F_{10} = 0, \]
\[ F_{11} = -4 \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right), \]
\[ F_{12} = \frac{6}{r} \left( B_3 \sin 2\theta - B_4 \cos 2\theta \right) - \frac{3}{2} \left( A_1 \cos \theta + A_2 \sin \theta \right), \]
\[ F_{13} = \frac{12}{r^2} \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right) - \frac{3}{r} \left( A_2 \cos \theta - A_1 \sin \theta \right). \]

**Appendix B. Algebra of integrals of motion in classical limit**

An algebraic equation for the non-trivial part \( T(\theta) \) of the potential was derived by requiring the algebra generated by the integrals of motion \( H, X, Y \) and \( C \) to be closed. For the extended harmonic oscillator potential

\[ V(r, \theta) = \hbar r^2 + \frac{T'(\theta)}{r^2}, \]

the corresponding algebraic equation is a sixth order polynomial equation in \( T \) given by

\[ \tau_0(z) + \tau_1(z) T(z) + \tau_2(z) T^2(z) + \tau_3(z) T^3(z) + \tau_4(z) T^4(z) + \tau_5(z) T^5(z) + \tau_6(z) T^6(z) = 0, \]

(B.1)

where \( z = \tan 2\theta \) and

\[ \tau_0 = 786432 \left( c_{11}^1 ( K_1 - 4K_2 ) + 2c_{11}^2 ( K_1 - 4K_2 )^2 + 12c_{12}^1 + 48iK_2K_1^3 - 64K_1^3 - 12K_1^2K_2^2 \right) z^2 \]
\[ + 3072 c_{11} \left( -c_{11}^1 ( 256c_{12} ( 16K_2 - 3K_2 ) + \lambda ) - ( K_1 - 4K_2 ) ( 256c_{12} ( 16K_2 + 5K_1 ) + 9\lambda ) \right) z^3 \]
\[ + 3 \left( 512[-128c_{12}^1 (-128/2K_2K_1 + K_1^2) \right. \]
\[ - 6c_{11}^2 (-256c_{12}^1 ( K_1 - 8K_2 ) + c_{12} \lambda - 768K_1 ( K_1 - 4K_2 ) ) \]
\[ + 9c_{12} \lambda ( 8iK_2K_1 ) - 2048K_2 (-12K_2K_1 + 3K_1^2 + 32K_2^2 ) ) + 27\lambda \) \]
\[ \left. \right) z^4 \]
\[ - 1536c_{11} \left( -512c_{12} ( K_1 - 16K_2 ) + 2304c_{12} ( 8K_2K_1 ) + 6c_{12} \lambda + 27\lambda K_1 \right) \right) z^5 \]
\[ + 3 \left( 27 ( 65536c_{11}^2 \lambda ^2 ) - 1024 \left( c_{11}^1 \lambda + 1024c_{12}^1K_2 + 16384K_2^2 \right) ) \right) z^6. \]
\[
\tau_1 = 3145728 c_{11} (2c_{11}^2 (fK_1 - 4K_2) + c_{11}^4 - 8fK_1K_2 + K_1^2 + 16K_2^2) z^2 \\
- 6144c_{11}^2 (256c_{12} (11fK_1 + 16K_2) + 15\lambda + fK_1 (1024c_{12}K_2 - 9\lambda)) \\
+ 4K_2 (4096c_{12}K_2 + 9\lambda) - 1280c_{12}K_2^2 - 2048c_{12}c_{11}^4] z^3 \\
- 24576c_{11} (-256c_{11}^2 (3c_{11}^2 - 4fK_1 + 4K_2) \\
- 128 (c_{11}^2 (9fK_1 - 16K_2) + 12fK_1K_2 - 6K_1^2 + 64K_2^2) + 3c_{12}\lambda) z^4 \\
+ 3072 [4c_{11}^2 (256c_{12} (4c_{12}^2 + 9fK_1 - 32K_2) - 9\lambda) - 512c_{12}^2 (fK_1 - 16K_2) \\
+ 256c_{12} (72fK_1K_1 + 9K_1^2 - 128K_2^2) + 6c_{12}^2 + 9\lambda (3fK_1 - 8K_2)\lambda] z^5 \\
- 12288 c_{11} (9c_{12} + 4096c_{12}K_2 - 256c_{12} + 192 (9K_1^2 - 64K_2^2)) z^6, \\
\]
\[
\tau_2 = -3145728 (2c_{11}^2 (fK_1 - 4K_2) + c_{11}^4 - 8fK_1K_2 + K_1^2 + 16K_2^2) z^2 \\
+ 294912c_{11} (256c_{12} (-c_{11}^2 + fK_1 + 4K_2) + \lambda) z^3 \\
+ 24576 [3c_{12}\lambda - 12c_{11}^2 (22c_{12} + 33\lambdaK_1 + 16K_2) + 8c_{12} (K_2 - fK_1) \\
- 8c_{11}^2 + 12fK_1K_2 - 6K_1^2 + 64K_2^2]) z^4 + 12288 c_{11} (256c_{12} (16c_{11}^2 - 12c_{12}^2 - 27fK_1 + 96K_2) \\
+ 27\lambda) z^5 + 12288 [9c_{12} + 2048c_{12}^2 (c_{11}^2 + 2K_2) - 192 (64K_2 (c_{11}^2 + K_2) \\
- 9K_1^2) - 256c_{12}^2] z^6. \\
\]
\[
\tau_3 = -196608 (128c_{12} (-c_{11}^2 + fK_1 + 4K_2) + \lambda) z^3 \\
+ 50331648 c_{11} (-c_{11}^2 + 2c_{12}^2 + 3fK_1 + 4K_2) z^4 \\
- 24576 (256c_{12} (32c_{11}^2 - 4c_{12}^2 - 9fK_1 + 32K_2) + 9\lambda) z^5 \\
+ 50331648 c_{11} (c_{11}^2 - c_{12}^2 + 6K_2) z^6. \\
\]
\[
\tau_4 = -25165824 z^4 (c_{11}^2 (6z^2 - 1) - c_{12}^2 (z^2 - 2) - 10c_{12}c_{11}z + 3fK_1 + 2K_2 (3z^2 + 2)), \\
\tau_5 = 50331648 z^5 (3c_{11}z - 2c_{12}) . \\
\tau_6 = -50331648 z^6 .
\]

\( f = \sqrt{1 + z^2} \). For the special values \( K_1 = c_{12} = 0 \) and \( K_2 = \frac{1}{4}c_{11}^2 \), the algebraic equation (B.1) becomes
\[
26244 z^3 (c_{11} - 2T)^6 - 1024 \lambda (9z^2 + 8) (c_{11} - 2T)^3 - 27 \lambda^2 z (z^2 + 1) = 0 .
\]
solutions of which coincide with \( T_{3,4} \), as it should be.

For the extended Coulomb potential
\[
V(r, \theta) = \frac{a}{r} + \frac{T'(\theta)}{r^2},
\]
the corresponding algebraic equation is also a sixth order polynomial equation in \( T \) given by
\[
v_0(z) + v_1(z) T(z) + v_2(z) T^2(z) + v_3(z) T^3(z) + v_4(z) T^4(z) + v_5(z) T^5(z) + v_6(z) T^6(z) = 0 \text{,}
\]
(B.2)
where \( z = \tan \theta \) and

\[
\nu_0 = 128 \left( 64c_{33}^4 (K_2 - 4/K_1) + 16c_{31}^2 (K_2 - 4K_1) - 64fK_1^4 - 12fK_1K_2^2 + K_2^4 + 48K_1^2K_2 \right) z^2
\]
\[- 64c_{31} \left( 8c_{31}^4 (64c_{32} (3K_1 - K_2) + \lambda) + (4K_1 - fK_2) (64c_{32} (K_2 + 5K_1) - 9f\lambda) \right) z^3
\]  
\[+ \left( 64 \left[ 32c_{32}^2 \left( f^2K_2^2 + 10/K_2K_1 - 2K_1^2 \right) - 24c_{31}^2 (2fK_1 - K_2) + c_{32}^2 \left( fK_2 - 4K_1 \right) \right] - 9c_{32}^2 (fK_2 + 2K_1) + 4K_2 (-6fK_2K_1 + 24K_1^2 + fK_2^2) - 27\lambda^2 \right) z^4
\]  
\[- 128c_{31} \left( 256c_{32}^3 (K_2 - K_1) + 288c_{32}^2 (K_2 + 2K_1) + 12c_{32}^2 (2fK_1 - fK_2) + 27\lambda^2 \right) K_1 \right) z^5
\]  
\[+ \left( 27 \left( 4096c_{31}^2K_1^4 + \lambda^2 \right) + 128 \left( -4c_{31}^2 \lambda + 64c_{32}^2K_2 + K_2^2 \right) fK_2 \right) z^6,
\]

\[
\nu_1 = -1024f c_{31} \left( 16c_{31}^2 f (K_2 - 4/K_1) + 64c_{31}^2 f - 16K_1^2 - fK_2^2 - 8K_1K_2 \right) z^2
\]  
\[- 32f \left[ f^3K_2 (9\lambda - 64c_{32}K_2) - 4K_1 (16c_{32}K_2 + 9f\lambda) + 8c_{31}^4 (64c_{32} (K_2 + 11K_1) - 15f\lambda) \right] z^3
\]  
\[+ 1280f c_{32}^2 f (18K_2 (c_{31}^4 f) + 2fK_1K_2 + 4c_{32}^2 (-24c_{31} - 6fK_1 + K_2) + 96K_1 (3c_{31}^4 f + K_1) - 4K_2 (8c_{31}^4 f + K_1) - fK_2^2) z^4
\]  
\[- 32c_{32} (f (4c_{31}^2 f - 7K_1)) + 256c_{31}^2 f (K_2 - fK_1) - 64c_{32} K_1 (9\lambda + 4K_2) + 27\lambda K_1 \right) z^5
\]  
\[+ 512f c_{31} f (9c_{32} \lambda - 128c_{32}^2 + 216K_2^2 - 8K_2 (8c_{31}^2 f + 3K_1) - 6fK_2^2) z^6
\]
\[18432f c_{32} K_1 (16c_{31} - K_2) z^7,
\]

\[
\nu_2 = 256f \left( 16c_{31}^2 f (4K_1 - K_2) - 64c_{31}^2 f - 16K_1^2 - fK_2^2 + 9K_1K_2 \right) z^2
\]  
\[+ 3072f c_{31} (f (\lambda - 8c_{31}^2 (8c_{31}^2 f + 2f + 4/K_1)) z^3
\]  
\[+ 256f (3c_{31}^2 f + 32c_{31}^2 f (4c_{31}^2 f + 33fK_1 + K_2) + 16c_{31}^2 f (K_2 - 16K_1) + 512c_{31}^2 f + 96K_1^2 f
\]  
\[- 4fK_2^2 - 4fK_1K_2) z^4 + 128f c_{31} (64c_{32} (32c_{31}^2 f - 24c_{31} f - 6fK_2 + 15K_1) + 27\lambda) z^5
\]  
\[- 128f (9c_{32} \lambda + 64c_{31}^2 f (16c_{31} - K_2) - 24fK_1 (8c_{31} f - 216K_1^2 f - 6fK_2^2) z^6
\]
\[221184f c_{32} K_1 z^7,
\]

\[
\nu_3 = 512f (f (\lambda - 8c_{32} (8c_{31}^2 f + 2f + 4/K_1)) z^3 + 8192c_{31}^2 (8c_{31}^3 f - 16c_{31} + 12fK_1 + K_2) z^4
\]  
\[+ 64f (64c_{32} (-2f (4c_{31}^2 f + K_2) + 64c_{31}^2 f + 5K_1)) + 9\lambda) z^5
\]  
\[- 4096c_{31}^2 (-16c_{31} - 16c_{32}^2 - 3fK_2) z^6
\]  
\[+ 36864f c_{32} K_1 z^7,
\]

\[
\nu_4 = -512f^2 z^4 (16c_{31}^3 (6c_{31} - 1) + 16c_{31} (c_{31} - 2) - 160c_{32} c_{31} z - 24fK_1 - K_2 (3c_{31}^2 + 2)) z^5
\]  
\[- 4096c_{31} z (3c_{31} z - 2c_{32}),
\]

\[
\nu_5 = -1024 z^6.
\]

For the special values \( K_1 = c_{32} = 0 \) and \( K_2 = -8c_{31}^2 \), the algebraic equation (B.2) becomes

\[256c_{31}^3 (-36c_{31}^2 (2c_{31} + T) - 6 - 4c_{31}^3 (9c_{31}^2 + 8) (2c_{31} + T) + 3\alpha (z^3 + z)) = 0,
\]
in accordance with the particular solutions \( T_{1,2} \) (77).
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