Abstract

We consider nonlocal minimal surfaces in a cylinder with prescribed datum given by the complement of a slab. We show that when the width of the slab is large the minimizers are disconnected and when the width of the slab is small the minimizers are connected. This feature is in agreement with the classical case of the minimal surfaces.

Nevertheless, we show that when the width of the slab is large the minimizers are not flat discs, as it happens in the classical setting, and, in particular, in dimension 2 we provide a quantitative bound on the stickiness property exhibited by the minimizers.

Moreover, differently from the classical case, we show that when the width of the slab is small then the minimizers completely adhere to the side of the cylinder, thus providing a further example of stickiness phenomenon.

1 Introduction

Nonlocal minimal surfaces were introduced in [8] and constitute one of the most fascinating, and challenging, research topics in the realm of fractional equations. Roughly speaking, the problem is that of minimizing an energy functional built by the pointwise interaction of a set versus its complement (this energy functional can also be conveniently “localized” in a given domain by taking into account the interactions in which at least one point lies in the domain). The prototype interaction taken into account is scaling and translation invariant and with polynomial decay (but we mention that other versions of the problem considered also interactions via integrable kernels, see [28, 29, 13]).

The nonlocal minimal surfaces constructed by this minimization procedure have relevant features in terms of differential geometry and geometric measure theory, since their energy functional can be considered as a nonlocal approximation of the classical perimeter functional and the nonlocal minimal surfaces as a fractional variant of the classical minimal surfaces, see [3, 17, 30, 10, 2, 11]. Critical points of the nonlocal perimeter energy functional satisfy an integral relation that can be seen as a vanishing nonlocal mean curvature prescription (see [8, 11, 19, 12]) and accordingly the study of volume prescribed minimizers leads to the analysis of surfaces with constant nonlocal mean curvature (see [18, 6, 7, 14]). Moreover, nonlocal minimal surfaces arise as the large-scale limit of long-range phase coexistence models (see [31]), as discrete iterations of fractional heat equations (see [9]), and as continuous approximations of interfaces of long-range Ising models (see [16]).

Given the importance of nonlocal minimal surfaces from all these perspectives, it is desirable to develop some intuition about their basic geometric features. For this, since it is very rare to

---

*Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Hwy, Crawley WA 6009, Australia. E-mail: serena.dipierro@uwa.edu.au
†Scuola Normale Superiore, Piazza dei Cavalieri 7, Pisa 56126, Italy. E-mail: fumihiko.onoue@sns.it
‡E-mail: enrico.valdinoci@uwa.edu.au
have explicit solutions and precise formulas which entirely describe nonlocal minimal surfaces, it is often convenient to focus on some simplified cases in which the reference domain and the external datum possess some special characteristics which lead to a deep understanding of at least some cardinal aspects of the object under investigation.

This note follows precisely in this line of research, namely we will consider a very simple domain, that is a vertical cylinder in $\mathbb{R}^n$, and a very special external datum, that is the complement of a horizontal slab, and detect how the minimizers of the nonlocal perimeter functional change when the width of the slab varies.

On the one hand, when the width of the slab is large, we will show that these minimizers are disconnected, and this is somehow the nonlocal counterpart of the fact that the classical perimeter gets minimized by far-away parallel and co-axial discs.

On the other hand, when the width of the slab is small, we will show that these minimizers become connected. This change of topology is in agreement with the classical case, since perimeter minimizers constrained to two nearby parallel and co-axial circumferences are connected necks of catenoids. Nonetheless, the specific geometry exhibited in this case by nonlocal minimal surfaces is rather different from that of catenoids, since we will additionally show that when the width of the slab is small the nonlocal minimal surface obtained with this procedure actually coincides inside the cylinder with the cylinder itself.

More precisely, and in further detail, the mathematical framework that we use in this paper is the one introduced in [8] and can be summarized as follows. Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be an open subset with Lipschitz boundary. Then we define the nonlocal perimeter or $s$-perimeter $P_s(E; \Omega)$ for a measurable set $E \subset \mathbb{R}^n$ by

$$P_s(E; \Omega) := \int_{E \cap \Omega} \int_{E^c \cap \Omega^c} \frac{dx dy}{|x-y|^{n+s}} + \int_{E \cap \Omega^c} \int_{\Omega \cap E^c} \frac{dx dy}{|x-y|^{n+s}}, \quad (1.1)$$

where we denote by $E^c$ the complement of a set $E$. We say that a set $E \subset \mathbb{R}^n$ is a $s$-minimizer or $s$-minimal set in $\Omega$ if it holds that $P_s(E; \Omega') \leq P_s(F; \Omega')$ for any open, bounded, and Lipschitz set $\Omega'$ contained in $\Omega$ and any $F \subset \mathbb{R}^n$ with $F \setminus \Omega' = E \setminus \Omega'$. See also [27] for additional details regarding the minimization procedure in bounded or unbounded domains.

For our purposes, we will often denote coordinates in $\mathbb{R}^n$ by $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and we will focus here on the case of "cylindrical" domains of the form

$$\Omega := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x'| < 1\}. \quad (1.2)$$

We are interested in sets $E$ whose exterior prescription outside $\Omega$ is the complement of a strip. Namely, given $M > 0$, we define

$$E_0 := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x_n| > M\} \quad (1.3)$$

and we consider $s$-minimal sets in $\Omega$ such that $E \setminus \Omega = E_0 \setminus \Omega$. See e.g. [27] Theorem 0.2.5] for existence results for this type of $s$-minimal sets.

Our main concern in this note is how the variation of the parameter $M$ affects the topological property of the $s$-minimizer and we will show that for small values of $M$ the $s$-minimizer is connected while for large values it is disconnected.

Furthermore, we will show that for small values of $M$ the $s$-minimizer in $\Omega$ coincides with $\Omega$ itself, and this is an interesting difference with respect to the classical case of minimal surfaces. Indeed, when $n \geq 3$ minimal surfaces in a cylinder do not coincide with the cylinder itself and, when connected, they develop a "neck" inside the cylinder, as exhibited by the classical example of the catenoid (as a matter of fact, when $n \geq 3$ the cylinder does not have vanishing mean curvature, hence it cannot be a minimizer for the classical perimeter functional).
Figure 1: The minimizers in Theorem 1.1 (left) and Theorem 1.2 (right).

Therefore, our construction of nonlocal minimal surfaces that coincide with the cylinder in their free domain heavily relies on the nonlocal character of the problem taken into consideration and can be seen as a new example of the stickiness theory for nonlocal minimal surfaces which was introduced in [22] and developed in [21, 5, 23, 24]. See also [25, 20] for surveys on nonlocal minimal surfaces discussing, among other topics, the stickiness phenomenon (and, for instance [26] to appreciate the structural differences with respect to the classical case).

In further detail, the precise result that we have concerning the connectedness of the $s$-minimizer and its stickiness properties for small values of $M$ goes as follows:

**Theorem 1.1.** Let $\Omega$ be as in (1.2) and let $E_0$ be defined by (1.3). Then, there exists $M_0 \in (0,1)$, depending only on $n$ and $s$, such that, for any $M \in (0, M_0)$, the minimizer $E_M$ in $\Omega$ of $P_s$ coincides with $\Omega$. In particular, $E_M$ is connected.

The minimizer described in Theorem 1.1 is depicted in Figure 1a. As a counterpart of Theorem 1.1 the disconnectedness result for large values of $M$ is the following:

**Theorem 1.2.** Let $\Omega$ be as in (1.2) and let $E_0$ be defined by (1.3). Then, there exists $M_0 > 1$, depending only on $n$ and $s$, such that, for any $M > M_0$, the minimizer $E_M$ in $\Omega$ of $P_s$ is disconnected.

To favor the intuition, a sketch on how we believe the minimizer in Theorem 1.2 looks like is given in Figure 1b.

Interestingly, the situation described in Theorem 1.2 is similar, but structurally different from the one exhibited by classical minimal surfaces. Indeed, the analogy with the classical case is given by the disconnectedness of the minimizers. The difference in the pattern is that classical minimal surfaces in the framework of Theorem 1.2 are just flat disc, and this is not the case for their corresponding nonlocal counterpart (as we will make precise in Proposition 4.1).
The forthcoming Sections 2 and 3 contain the proofs of Theorems 1.1 and 1.2 respectively. In Section 4 we will present further similarities and differences with respect to the classical case in the framework of large $M$ given by Theorem 1.2.

2 Proof of Theorem 1.1

Let $E_M$ be the minimizer selected in Theorem 1.1, see Figure 2 (at this stage of the proof, we do not really know how this minimizer looks like, so the one depicted in Figure 2 will not be the “real” minimizer after all).

By [8, Corollary 5.3], we know that
\[
\{x_n > M\} \cup \{x_n < -M\} \subset E_M.
\] (2.1)

Given $t \in \mathbb{R}$ and $r \in (0, 1)$, we consider the ball of radius $r$ with center $te_n$, where $e_n = (0, \ldots, 0, 1)$. By (2.1), we have that $B_r(te_n) \subset E_M$ for every $t > M + 1$. Hence, we can slide such a ball downwards till it touches $\partial E_M$ inside $\Omega$. The content of Theorem 1.1 is precisely that this touching does not occur, hence, by contradiction, we suppose instead that there exist $t_0 \in \mathbb{R}$ and $r_0 \in (0, 1)$ such that
\[
B_{r_0}(te_n) \subset E_M \quad \text{for all } t > t_0
\] (2.2)

with
\[
\partial B_{r_0}(t_0e_n) \cap \partial E_M \neq \emptyset.
\] (2.3)

Then, setting $z := t_0e_n$, we can choose a point $q = (q', q_n) \in \partial B_{r_0}(z) \cap \partial E_M$.

Since $E_M$ is a local minimizer of $P_s$ in $\Omega$, we obtain, by using the Euler-Lagrange equation in the viscosity sense shown in [8, Theorem 5.1] (see also [5, Theorem B.9]), that
\[
\int_{\mathbb{R}^n} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y-q|^n + s} \, dy \geq 0.
\] (2.4)

Our goal is now to produce a contradiction with (2.4) by showing that the left hand side is strictly negative. To this end, we let
\[
S_M := \mathbb{R}^{n-1} \times [q_n - 2M, q_n + 2M].
\]
We remark that

\[ E_M^c \subset S_M \setminus B_{r_0}(z). \]  \hspace{1cm} (2.5)

Indeed, by (2.1) we know that \( q_n \in [-M, M] \) and \( E_M^c \subset \{ x_n \in [-M, M] \} \), whence \( E_M^c \subset S_M \).

This and (2.2) give (2.5).

We also observe that \( S_M \supset \{ |x_n| \leq M \} \), and therefore, in light of (2.1),

\[ S_M^c \subset E_M. \]  \hspace{1cm} (2.6)

Moreover, using the change of variable \( y \mapsto y + q \),

\[ \int_{S_M^c} \frac{dy}{|y - q|^{n+s}} = \int_{\mathbb{R}^{n-1} \times ((-\infty, -2M) \cup (2M, +\infty))} \frac{dy}{|y|^{n+s}} \geq \int_{B_M(3M\epsilon_n)} \frac{dy}{|y|^{n+s}} \geq cM^{-s}, \]  \hspace{1cm} (2.7)

for a constant \( c > 0 \) depending only on \( n \).

Now we set \( \overline{q} := z + 2(q - z) \) and we consider the symmetric ball \( B_{r_0}(\overline{q}) \) with respect to \( q \), see Figure 3. Moreover, we take a free parameter \( \Lambda \geq 4, \) to be chosen conveniently large in what follows and we observe that, by symmetry,

\[ \int_{S_M \cap B_{\Lambda M}(q) \cap B_{r_0}(z)} \frac{dy}{|y - q|^{n+s}} = \int_{S_M \cap B_{\Lambda M}(q) \cap B_{r_0}(\overline{q})} \frac{dy}{|y - q|^{n+s}}. \]

Also, by (2.5),

\[ \int_{S_M \cap B_{\Lambda M}(q) \cap B_{r_0}(z)} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} dy \leq -\int_{S_M \cap B_{\Lambda M}(q) \cap B_{r_0}(z)} \frac{dy}{|y - q|^{n+s}}, \]

and consequently

\[ \int_{S_M \cap B_{\Lambda M}(q) \cap B_{r_0}(z)} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} dy + \int_{S_M \cap B_{\Lambda M}(q) \cap B_{r_0}(\overline{q})} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} dy \leq -\int_{S_M \cap B_{\Lambda M}(q) \cap B_{r_0}(z)} \frac{dy}{|y - q|^{n+s}} + \int_{S_M \cap B_{\Lambda M}(q) \cap B_{r_0}(\overline{q})} \frac{dy}{|y - q|^{n+s}} = 0. \]
Therefore,

\[
\int_{S_M \cap B_{\lambda M}(q)} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy
\]

\[
= \int_{S_M \cap B_{\lambda M}(q) \cap B_{\alpha}(z)} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy + \int_{S_M \cap B_{\lambda M}(q) \cap B_{\alpha}(z) \setminus B_{\alpha}(z)} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy
\]

\[
+ \int_{S_M \cap (B_{\lambda M}(q) \setminus (B_{\alpha}(z) \cup B_{\alpha}(\bar{z})))} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy
\]

\[
\leq \int_{S_M \cap (B_{\lambda M}(q) \setminus (B_{\alpha}(z) \cup B_{\alpha}(\bar{z})))} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy
\]

\[
\leq \int_{B_{\lambda M}(q) \setminus (B_{\alpha}(z) \cup B_{\alpha}(\bar{z}))} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy
\]

\[
\leq C \Lambda^{1-s} M^{1-s},
\]

for some \( C > 0 \) depending only on \( n \) and \( s \), where [24] Lemma 3.1 has been used in the last inequality (here with \( R := 1 \) and \( \lambda := \Lambda M \)).

Furthermore,

\[
\int_{S_M \setminus B_{\lambda M}(q)} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \leq \int_{S_M \cup B_{\lambda M}(q) \setminus B_{\alpha}(z)} \frac{dy}{|y - q|^{n+s}}
\]

\[
= \int_{(\mathbb{R}^{n-1} \times [-2M,2M]) \setminus B_{\lambda M}} \frac{dy}{|y|^{n+s}} \leq \int_{(\mathbb{R}^{n-1} \times [-2M,2M]) \setminus B_{\lambda M}} \frac{dy}{|y|^{n+s}} = \frac{C_0}{\Lambda^{1+s} M^s},
\]

for some \( C_0 > 0 \) depending only on \( n \) and \( s \).

Hence, combining this information with (2.8),

\[
\int_{S_M} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \leq C \Lambda^{1-s} M^{1-s} + \frac{C_0}{\Lambda^{1+s} M^s}.
\]

This, (2.4) and (2.7) lead to

\[
\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy
\]

\[
= - \int_{S_M} \frac{dy}{|y - q|^{n+s}} + \int_{S_M} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy
\]

\[
\leq - c M^{-s} + C \Lambda^{1-s} M^{1-s} + \frac{C_0}{\Lambda^{1+s} M^s}
\]

\[
= - c M^{-s} \left( 1 - \frac{C \Lambda^{1-s} M}{c} - \frac{C_0}{c \Lambda^{1+s}} \right).
\]

Now we choose \( \Lambda := \max \left\{ 4, \left( \frac{2C_0}{c} \right)^{1+s} \right\} \) and we thus obtain that

\[
\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \leq - c M^{-s} \left( \frac{1}{2} - \frac{C \Lambda^{1-s} M}{c} \right).
\]

Taking now \( M \) conveniently small, we conclude that

\[
\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \leq - \frac{c M^{-s}}{4} < 0,
\]

which produces the desired contradiction with [24].
Figure 4: The touching between the ball $B_{\sqrt{M}}(z)$ and the symmetric ball $B_{\sqrt{M}}(z)$ at the point $q$.

3 Proof of Theorem 1.2

We let $M > 1$ to be chosen conveniently large. Given $t \in \mathbb{R}$, we consider the ball $B_{\sqrt{M}}(te_1)$, where $e_1 = (1,0,\ldots,0)$, and we slide it from left to right till it touches $\partial E_M$. Notice indeed that $B_{\sqrt{M}}(te_1) \subset E'_0$ when $t < -\sqrt{M}$ and, to prove Theorem 1.2, we suppose by contradiction that there exists $t_0 \in \mathbb{R}$ such that $B_{\sqrt{M}}(te_1) \subset E'_M$ for all $t < t_0$ with $\partial B_{\sqrt{M}}(t_0e_1) \cap \partial E_M \neq \emptyset$.

We set $z := t_0e_1$ and we pick a point $q = (q', q_n) \in \partial B_{\sqrt{M}}(z) \cap \partial E_M$. By the Euler-Lagrange equation in the viscosity sense (see [8, Theorem 5.1] and [5, Theorem B.9]), we know that

$$\int_{\mathbb{R}^n} \frac{\chi_{E'_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \leq 0. \quad (3.1)$$

We consider the symmetric ball with respect to $q$, by defining $\overline{z} := z + 2(q - z)$ and taking into account the ball $B_{\sqrt{M}}(\overline{z})$, see Figure 4.

We define

$$S := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } |x' - q'| \leq 3\}.$$

By symmetry,

$$\int_{S \cap B_{\sqrt{M}}(z)} \frac{dy}{|y - q|^{n+s}} = \int_{S \cap B_{\sqrt{M}}(\overline{z})} \frac{dy}{|y - q|^{n+s}}.$$
and therefore
\[
\int_S \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \\
= \int_{S \cap B_{\sqrt[n]{|z|}}(z)} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy + \int_{S \cap B_{\sqrt[n]{|z|}}(z)} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \\
+ \int_{S \cap B_{\sqrt[n]{|z|}}(z)} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \\
\geq \int_{S \cap B_{\sqrt[n]{|z|}}(z)} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \\
+ \int_{S \cap B_{\sqrt[n]{|z|}}(z)} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \\
\geq - \int_{S \cap B_{\sqrt[n]{|z|}}(z)} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy.
\]

(3.2)

Now, in view of [21], Lemma 3.1, used here with \( R := \sqrt{M} \) and \( \lambda := 1/\sqrt{M} \), we know that
\[
\int_{B \setminus \sqrt[n]{|z|}}(B \setminus \sqrt[n]{|z|}) \frac{dy}{|y - q|^{n+s}} \leq CM^{-\frac{1+n}{s}},
\]
for some \( C > 0 \) depending only on \( n \) and \( s \). As a result,
\[
\int_{S \cap \{y - q|^{n+s} \leq \frac{dy}{|y - q|^{n+s}} \} + \int_{S \cap \{y - q|^{n+s} \} \leq \frac{dy}{|y - q|^{n+s}} \} + \int_{S \cap \{y - q|^{n+s} \} \leq \frac{dy}{|y - q|^{n+s}} \} \leq CM^{-\frac{1+n}{s}} + C_1M^{-\frac{n}{s}} \leq C_2 M^{-\frac{n}{s}},
\]
for some \( C_1 > 0 \) depending only on \( n \) and \( s \), with \( C_2 := C + C_1 \).

This and (3.2) lead to
\[
\int_{\mathbb{R}^n} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \\
= \int_S \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy + \int_{S^c} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \\
\geq -C_2 M^{-\frac{n}{s}} + \int_{S^c} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \\
\geq -C_2 M^{-\frac{n}{s}} + \int_{\{y - q|^{n+s} \geq M\}} \frac{dy}{|y - q|^{n+s}} + \int_{\{y - q|^{n+s} \} < M\}} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \\
\geq -C_2 M^{-\frac{n}{s}} + \int_{\{y - q|^{n+s} \} < M\}} \frac{dy}{|y - q|^{n+s}} \\
= -C_2 M^{-\frac{n}{s}} - C_3 M^{-\frac{s}{s}} + \int_{S \cap \{y - q|^{n+s} \} < M\}} \frac{dy}{|y - q|^{n+s}}.
\]
for some \( C_3 > 0 \) depending only on \( n \) and \( s \).

Thus, since \( S^c \cap \{y - q| < M\} \supset B_1(q + 5e_1) \), letting \( C_4 := C_2 + C_3 \) we have
\[
\int_{\mathbb{R}^n} \frac{\chi_{E_M}(y) - \chi_{E_M}(y)}{|y - q|^{n+s}} \, dy \geq -C_4 M^{-\frac{n}{s}} + \int_{B_1(q + 5e_1)} \frac{dy}{|y - q|^{n+s}} \\
= -C_4 M^{-\frac{n}{s}} + \int_{B_1(q + 5e_1)} \frac{dy}{|y|^{n+s}} = -C_4 M^{-\frac{n}{s}} + c,
\]
for some \( c > 0 \) depending only on \( n \) and \( s \). In particular, if \( M \) is sufficiently large, we deduce that the left-hand side of (3.1) is strictly positive, thus reaching a contradiction with (3.1).
4 Further remarks on Theorem 1.2

The goal of this section is to stress that the result in Theorem 1.2 is, on the one hand, related to the classical case of minimal surfaces, since both the classical and the nonlocal regimes exhibit disconnected minimizers for large values of $M$, but, on the other hand, presents some significant structural differences with respect to the classical scenario.

More precisely, differently from the classical case, the minimizer constructed in Theorem 1.2 exhibits the features listed below:

**Proposition 4.1.** Let $M$ and $E_M$ be as in Theorem 1.2. Then,

$$E_M \supset \{x_n > M\} \cup \{x_n < -M\}. \quad (4.1)$$

Moreover,

$$E_M \supset B_{cM^{-s}}(0, \ldots, 0, -M) \cup B_{cM^{-s}}(0, \ldots, 0, M), \quad (4.2)$$

for some $c > 0$ depending only on $n$ and $s$.

In addition, if $n = 2$, given any $\epsilon_0 > 0$ there exists $c_* > 0$, depending only on $s$ and $\epsilon_0$, such that

$$E_M \supset \left((-1, 1) \times (-\infty, -M + c_* M^{-\frac{(3\epsilon_0 + s)}{2}})\right) \cup \left((-1, 1) \times (M - c_* M^{-\frac{(3\epsilon_0 + s)}{2}}, +\infty)\right). \quad (4.3)$$

We remark that (4.2) and (4.3) are quantitative versions of (4.1) and a sketch of an argument used in the proof of Proposition 4.1 is depicted in Figure 5. Though (4.2) and (4.3) provide a stronger result than (4.1), we give an independent proof of (4.1) based on a simple symmetry argument, while the proofs of (4.2) and (4.3) rely on finer quantitative arguments. We also point out that (4.3) provides an explicit quantitative bound on the stickiness property in dimension 2.
Proof of Proposition [4.4]. To prove (4.1), we need to show that the inclusion in (2.1) is strict. For this, we argue by contradiction and suppose that \( E_M = \{x_n > M\} \cup \{x_n < -M\} \). Then we can use the Euler-Lagrange equation in the viscosity sense shown in [8, Theorem 5.1] at the point \( q := (0, \ldots, 0, -M) \in \partial E_M \), thus finding that

\[
0 = \int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E_M}(y)}{|y-q|^{n+s}} \, dy = \int_{\{y_n < M\}} \frac{dy}{|y-q|^{n+s}} - \int_{\{y_n \geq M\}} \frac{dy}{|y-q|^{n+s}}.
\]

(4.4)

Also, by the transformation \((z', z_n) \mapsto (z', -z_n)\), we see that

\[
0 = -\int_{\{z_n \in (-2M, 0)\}} \frac{dz}{|z|^{n+s}} < 0.
\]

This contradiction proves (4.1), and we now deal with the proof of (4.2). To this end, we let \( \phi \in C_{0}^{\infty}(\mathbb{R}^{n-1}, [0, 1]) \) with \( \phi(x') = 1 \) if \( |x'| \leq 1/2 \) and \( \phi(x') = 0 \) if \( |x'| \geq 3/4 \). Given \( \eta > 0 \), we define

\[
F := \{x_n < \eta \phi(x')\}
\]

and we claim that, for every \( p \in \partial F \),

\[
\int_{\mathbb{R}^n} \frac{\chi_{F^c}(y) - \chi_F(y)}{|y-p|^{n+s}} \, dy \leq C_0 \eta,
\]

(4.5)

for some \( C_0 > 0 \) depending only on \( n, s \) and \( \phi \). To prove this, we let

\[
\Psi(x', x_n) := (x', x_n + \eta \phi(x')) \quad \text{and} \quad \Phi(x) := \Psi(x) - x = (0, \ldots, 0, \eta \phi(x'))
\]

Notice that \( F = \Psi(\{x_n < 0\}) \) and the Jacobian of \( \Phi \) is bounded by \( C\eta \), together with its derivatives, for some \( C > 0 \) depending only on \( n \) and \( \eta \). Furthermore, the inverse of \( \Psi \) is given by

\[
\Psi^{-1}(x) = (x', x_n - \eta \phi(x'))
\]

and, setting \( \Xi(x) := \Psi^{-1}(x) - x = -(0, \ldots, 0, \eta \phi(x')) \), we find that also the Jacobian of \( \Xi \) is bounded by \( C\eta \). Consequently, we are in the position of exploiting [15, Theorem 1.1] and deduce that

\[
\int_{\mathbb{R}^n} \frac{\chi_{F^c}(y) - \chi_F(y)}{|y-p|^{n+s}} \, dy \leq \int_{\mathbb{R}^n} \frac{\chi_{\{y_n > 0\}}(y) - \chi_{\{y_n < 0\}}(y)}{|y-\Psi^{-1}(p)|^{n+s}} \, dy + C_0 \eta = C_0 \eta,
\]

for some \( C_0 > 0 \) depending only on \( n, s \) and \( \phi \), thus completing the proof of (4.5).

Now we define

\[
G := F \cup \{x_n > 4M\},
\]

we point out that this union is disjoint for large \( M \) and small \( \eta \), and we claim that there exists \( c > 0 \), depending only on \( n, s \) and \( \phi \), such that if \( \eta \in (0, cM^{-s}) \) then, for every \( p \in \partial F \),

\[
\int_{\mathbb{R}^n} \frac{\chi_{G^c}(y) - \chi_G(y)}{|y-p|^{n+s}} \, dy < 0.
\]

(4.6)
Indeed, we have that \( \chi_G = \chi_F + \chi_{\{x_n > 4M\}} \), whence \( \chi_{G^c} = 1 - \chi_G = 1 - \chi_F - \chi_{\{x_n > 4M\}} = \chi_{F^c} - \chi_{\{x_n > 4M\}} \). Accordingly, we have that \( \chi_{G^c} - \chi_{F^c} = \chi_{F^c} - \chi_F - 2\chi_{\{x_n > 4M\}} \) and therefore, using (4.5),

\[
\int_{\mathbb{R}^n} \frac{\chi_{G^c}(y) - \chi_G(y)}{|y - p|^{n+s}} \, dy = \int_{\mathbb{R}^n} \frac{\chi_{F^c}(y) - \chi_F(y)}{|y - p|^{n+s}} \, dy - 2 \int_{\{y_n > 4M\}} \frac{dy}{|y - p|^{n+s}} \\
\leq C_0 \eta - 2 \int_{(-M,M)^n \times (4M,5M)} \frac{dy}{|y - p|^{n+s}} \leq C_0 \eta - c_0 M^{-s},
\]

for some \( c_0 > 0 \) depending only on \( n \) and \( s \), which plainly leads to (4.6).

By means of (4.6), we can thus use the set \( G \) as a sliding barrier from below with \( \eta := cM^{-s} \) (starting the sliding from a vertical translation of the set \( G \) equal to \( -2M \)) and find that \( E_M \supset \{x_n < -M + cM^{-s} \phi(x')\} \). In particular, we see that \( E_M \supset [-\frac{1}{2}, \frac{1}{2}]^{n-1} \times (-\infty, -M + cM^{-s}) \supset B_{cM^{-s}}(0, \ldots 0, -M) \).

Similarly, one proves that \( E_M \supset B_{cM^{-s}}(0, \ldots 0, M) \), thus completing the proof of (4.2).

Now we suppose that \( n = 2 \) and we establish (4.3). For this, we fix \( \epsilon_0 > 0 \), we consider a suitably small \( \delta > 0 \) and we exploit [22, Corollary 7.2] to construct a set \( H \subset \mathbb{R}^2 \) such that

\[
H \subset \{x_2 < \delta\}, \\
H \cap \{x_1 < -1\} = (-\infty, -1) \times (-\infty, 0), \\
H \cap \{x_1 > 1\} = (1, +\infty) \times (-\infty, 0), \\
H \supset (-1, 1) \times (-\infty, \delta^{\frac{2+s}{1-s}}) \\
\text{and} \\
\int_{\mathbb{R}^2} \frac{\chi_{H^c}(y) - \chi_H(y)}{|y - p|^{2+s}} \, dy \leq \tilde{C} \delta
\]

for every \( p = (p_1, p_2) \in \partial H \) with \( |p_1| < 1 \), where \( \tilde{C} > 0 \) depends only on \( s \) and \( \epsilon_0 \).

We define

\[
L := H \cup \{x_2 > 4M\},
\]

and we see that \( \chi_{L^c} - \chi_L = \chi_{H^c} - \chi_H - 2\chi_{\{x_2 > 4M\}} \) and thus

\[
\int_{\mathbb{R}^2} \frac{\chi_{L^c}(y) - \chi_L(y)}{|y - p|^{2+s}} \, dy \leq \int_{\mathbb{R}^2} \frac{\chi_{H^c}(y) - \chi_H(y)}{|y - p|^{2+s}} \, dy - 2 \int_{\{y_2 > 4M\}} \frac{dy}{|y - p|^{2+s}} \\
\leq C \delta - 2 \int_{(-M,M) \times (4M,5M)} \frac{dy}{|y - p|^{2+s}} \leq C \delta - \tilde{c}M^{-s} < 0
\]

for every \( p = (p_1, p_2) \in \partial H \) with \( |p_1| < 1 \), where \( \tilde{c} > 0 \) depends only on \( s \), and \( \delta := \frac{\tilde{c}M^{-s}}{2C} \).

In this way, we can use \( L \) as sliding barrier from below (starting the sliding from a vertical translation of the set \( L \) equal to \( -2M \)) and deduce that

\[
E_M \cap \{|x_1| < 1\} \supset \left(-\infty, -M + \delta^{\frac{2+s}{1-s}}\right) = \left(-\infty, -M + c_* M^{-\frac{(2+s)\epsilon}{1-s}}\right)
\]

for some \( c_* > 0 \). Similarly, one finds that

\[
E_M \cap \{|x_1| < 1\} \supset \left(M - c_* M^{-\frac{(2+s)\epsilon}{1-s}}, +\infty\right).
\]

The proof of (4.3) is thereby complete. \( \square \)
Acknowledgments

Supported by the Australian Research Council DECRA DE180100957 “PDEs, free boundaries and applications” and the Australian Laureate Fellowship FL190100081 “Minimal surfaces, free boundaries and partial differential equations”.

The first and third authors are members of INdAM and AustMS.

This work has been carried out during a very fruitful visit of the second author to the University of Western Australia, whose warm hospitality is a pleasure to acknowledge.

References

[1] N. Abatangelo and E. Valdinoci, A notion of nonlocal curvature, Numer. Funct. Anal. Optim. 35 (2014), no. 7-9, 793-815.
[2] L. Ambrosio, G. De Philippis, and L. Martinazzi, Gamma-convergence of nonlocal perimeter functionals, Manuscripta Math. 134 (2011), no. 3-4, 377-403.
[3] J. Bourgain, H. Brézis, and P. Mironescu, Another look at Sobolev spaces, Optimal Control and Partial Differential Equations, pp. 439-455, IOS Press, Amsterdam, 2001.
[4] C. Bucur, S. Dipierro, L. Lombardini, and E. Valdinoci, Minimisers of a fractional seminorm and nonlocal minimal surfaces, to appear in Interfaces Free Bound.
[5] C. Bucur, L. Lombardini, and E. Valdinoci, Complete stickiness of nonlocal minimal surfaces for small values of the fractional parameter, Ann. Inst. H. Poincaré Anal. Non Linéaire, 36 (2019), no. 3, 655-703.
[6] X. Cabré, M. M. Fall, J. Solà-Morales, and T. Weth, Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay, J. Reine Angew. Math. 745 (2018), 253-280.
[7] X. Cabré, M. M. Fall, and T. Weth, Delaunay hypersurfaces with constant nonlocal mean curvature, J. Math. Pures Appl. (9) 110 (2018), 32-70.
[8] L. Caffarelli, J.M. Roquejoffre, and O. Savin, Nonlocal minimal surfaces, Comm. Pure Appl. Math., 63 (2010), no. 9, 1111-1144.
[9] L. Caffarelli and P. E. Souganidis, Convergence of nonlocal threshold dynamics approximations to front propagation, Arch. Ration. Mech. Anal. 195 (2010), no. 1, 1-23.
[10] L. Caffarelli and E. Valdinoci, Uniform estimates and limiting arguments for nonlocal minimal surfaces, Calc. Var. Partial Differential Equations 41 (2011), no. 1-2, 203-240.
[11] L. Caffarelli and E. Valdinoci, Regularity properties of nonlocal minimal surfaces via limiting arguments, Adv. Math. 248 (2013), 843-871.
[12] E. Cinti, J. Dávila, and M. del Pino, Solutions of the fractional Allen-Cahn equation which are invariant under screw motion, J. Lond. Math. Soc. (2) 94 (2016), no. 1, 295-313.
[13] E. Cinti, J. Serra, and E. Valdinoci, Quantitative flatness results and BV-estimates for stable nonlocal minimal surfaces, J. Differential Geom. 112 (2019), no. 3, 447-504.
[14] G. Ciraolo, A. Figalli, F. Maggi, and M. Novaga, Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature, J. Reine Angew. Math. 741 (2018), 275-294.
[15] M. Cozzi, On the variation of the fractional mean curvature under the effect of $C^{1,\alpha}$ perturbations, Discrete Contin. Dyn. Syst. 35 (2015), no. 12, 5769-5786.
[16] M. Cozzi, S. Dipierro, and E. Valdinoci, Planelike interfaces in long-range Ising models and connections with nonlocal minimal surfaces, J. Stat. Phys. 167 (2017), no. 6, 1401-1451.
[17] J. D´avila, *On an open question about functions of bounded variation*, Calc. Var. Partial Differential Equations **15** (2002), no. 4, 519-527.

[18] J. D´avila, M. del Pino, S. Dipierro, and E. Valdinoci, *Nonlocal Delaunay surfaces*, Nonlinear Anal. **137** (2016), 357-380.

[19] J. D´avila, M. del Pino, J. Wei, *Nonlocal s-minimal surfaces and Lawson cones*, J. Differential Geom. **109** (2018), no. 1, 111-175.

[20] S. Dipierro, A. Dzhugan, N. Forcillo, and E. Valdinoci, *Enhanced boundary regularity of planar nonlocal minimal graphs and a butterfly effect*, to appear in Bruno Pini Math. Anal. Semin.

[21] S. Dipierro, O. Savin, and E. Valdinoci, *Graph properties for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations, **55** (2016), no.4, 25pp.

[22] S. Dipierro, O. Savin, and E. Valdinoci, *Boundary behaviour of nonlocal minimal surfaces*, J. Funct. Anal., **272** (2017), no.5, 1791-1851.

[23] S. Dipierro, O. Savin, and E. Valdinoci, *Nonlocal minimal graphs in the plane are generically sticky*, Comm. Math. Phys., **376** (2020), no. 3, 2005-2063.

[24] S. Dipierro, O. Savin, and E. Valdinoci, *Boundary properties of fractional objects: Flexibility of linear equations and rigidity of minimal graphs*, to appear in J. Reine Angew. Math., DOI: 10.1515/crelle-2019-0045.

[25] S. Dipierro and E. Valdinoci, *Nonlocal minimal surfaces: interior regularity, quantitative estimates and boundary stickiness*, Recent developments in nonlocal theory, 165-209, De Gruyter, Berlin, 2018.

[26] R. Hardt and L. Simon, *Boundary regularity and embedded solutions for the oriented Plateau problem*, Ann. of Math., **110** (1979), no. 3, 439-486.

[27] L. Lombardini, *Minimization problems involving nonlocal functionals: nonlocal minimal surfaces and a free boundary*, PhD Thesis (2018), Università di Milano, https://arxiv.org/abs/1811.09746

[28] J. Mazón, J. D. Rossi, and J. J. Toledo, *Nonlocal perimeter, curvature and minimal surfaces for measurable sets*, Frontiers in Mathematics, Birkhäuser/Springer, Cham, 2019 xviii+123pp.

[29] J. Mazón, J. D. Rossi, and J. J. Toledo, *Nonlocal perimeter, curvature and minimal surfaces for measurable sets*, J. Anal. Math. **138** (2019), no. 1, 235-279.

[30] A. C. Ponce, *A new approach to Sobolev spaces and connections to Γ-convergence*, Calc. Var. Partial Differential Equations **19** (2004), no. 3, 229-255.

[31] O. Savin and E. Valdinoci, *Γ-convergence for nonlocal phase transitions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), no. 4, 479-500.