Complex Trajectories and Dynamical Origin of Quantum Probability

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Abstract

Complex quantum trajectories, which were first obtained from a modified de Broglie-Bohm quantum mechanics, demonstrate that Born’s probability axiom in quantum mechanics originates from dynamics itself. We show that a normalisable probability density can be defined for the entire complex plane, though there may be regions where the probability is not locally conserved. Examining this for some simple examples such as the harmonic oscillator, we also find why there is no appreciable complex extended motion in the classical regime.

1 Introduction

Complex quantum trajectories were first obtained and drawn for the case of some fundamental problems, such as the harmonic oscillator, potential step, wave packets etc., by modifying the de Broglie-Bohm (dBB) approach to quantum mechanics [1]. For obtaining this representation, we shall substitute $\Psi = e^{i S/\hbar}$ in the Schrödinger equation to obtain the quantum Hamilton-Jacobi equation (QHJE) [2] and then postulate an equation of motion similar to that of de Broglie:

$$m \ddot{x} \equiv \frac{\partial S}{\partial x} = \frac{\hbar}{i} \frac{1}{\Psi} \frac{\partial \Psi}{\partial x},$$

for the particle. The trajectories $x(t)$ are obtained by integrating this equation with respect to time and they will lie in a complex $x$-plane. It was observed that the above identification $\Psi = e^{i S/\hbar}$ helps to utilize all the information contained in $\Psi$ while obtaining the trajectory.
We thus consider $x \equiv x_r + ix_i$ as a complex variable and restrict ourselves to single particles in one dimension. The complex eigentrajectories in the free particle, harmonic oscillator and potential step problems and complex trajectories for a wave packet solution were obtained in [1]. As an example, complex trajectories in the $n = 1$ harmonic oscillator is shown in figure 1. These are the famous Cassinian ovals. The Jacobi lemniscate, that passes through $x_r = 0, x_i = 0$ is the special case of these ovals.

![Figure 1](image)

Figure 1:

We may note that even for an eigenstate, the particle can be in any one of its infinitely many possible quantum trajectories, depending on its initial position in the complex plane. Therefore, the expectation values of dynamical variables are to be evaluated over an ensemble of particles in all possible trajectories. It was postulated that the average of a dynamical variable $O$ can be obtained using the measure $\Psi^*\Psi$ as $\langle O \rangle = \int_{-\infty}^{\infty} O\Psi^*\Psi \, dx$, where the integral is to be taken along the real axis [1]. Also it was noted that in this form, there is no need to make the conventional operator replacements. The above postulate is equivalent to the Born’s probability axiom for observables such as position, momentum, energy, etc., and one can show that $\langle O \rangle$ coincides with the corresponding quantum mechanical expectation values. This makes the new scheme equivalent to standard quantum mechanics when averages of dynamical variables are computed.

However, one of the challenges before this complex quantum trajectory representation is to explain the quantum probability axiom. In a recent work which explores the connection between probability and complex quantum trajectories [3], the probability density to find the particle around some point on the real axis $x = x_r 0$ was proposed to be
\[ \Psi^* \Psi(x_{r0}, 0) \equiv P(x_{r0}) = \mathcal{N} \exp \left( -\frac{2m}{\hbar} \int_{x_{r0}}^{x_r} \dot{x}_i dx_r \right), \quad (2) \]

where the integral is taken along the real line. Since it is defined and used only along the real axis, the continuity equation for probability in the standard quantum mechanics is valid here also, without any modifications. This possibility of regaining the quantum probability distribution from the velocity field is a unique feature of the complex trajectory formulation. For instance, in the de Broglie approach, the velocity fields for all bound eigenstates are zero everywhere and it is not possible to obtain a relation between velocity and probability.

At the same time, since we have the complex paths, it would be natural to consider the probability for the particle to be in a particular path. In addition, we may consider the probability to find the particle around different points in the same path, which can also be different. Thus it is desirable to extend the probability axiom to the \(x_r x_i\)-plane and look for the probability of a particle to be in an area \(dx_r dx_i\) around some point \((x_r, x_i)\) in the complex plane. Let this quantity be denoted as \(\rho(x_r, x_i)dx_r dx_i\).

It is natural to expect that such an extended probability density agrees with Born’s rule along the real line. We impose such a boundary condition for \(\rho(x_r, x_i)\). Also, it is necessary to see whether probability conservation holds everywhere in the \(x_r x_i\)-plane. It is ideal if we have an expression for \(\rho(x_r, x_i)\), which satisfies these two conditions.

Such an extended probability density was proposed in [3]. It was postulated that if \(\rho_0\), the extended probability density at some point \((x_{r0}, x_{i0})\) is given, then \(\rho(x_r, x_i)\) at another point that lies on the trajectory which passes through \((x_{r0}, x_{i0})\), is

\[ \rho(x_r, x_i) = \rho_0 \exp \left[ -\frac{4}{\hbar} \int_{t_0}^{t} Im \left( \frac{1}{2} m \dot{x}^2 + V(x) \right) dt' \right]. \quad (3) \]

Here, the integral is taken along the trajectory \([x_r(t'), x_i(t')]\). One can show that the desired continuity equation for the particle, as it moves along, follows from it.

While evaluating \(\rho\) with the help of (3) above, one needs to know \(\rho_0\) at \((x_{r0}, x_{i0})\) and if we choose this point as \((x_{r0}, 0)\), the point of crossing of the trajectory on the real line, then \(\rho_0\) may take the value \(P(x_{r0})\) and may be found using (2).
On the other hand, if we solve the continuity equation for time-independent problems with the given boundary condition, it is possible to show that the two methods give identical results. The important property that the characteristic curves for the continuity equation are identical to the complex paths of particles in the present quantum trajectory representation can also be demonstrated.

As mentioned above, Eq. (3) gives a conserved probability along any trajectory in the $x_r x_i$-plane. But we have required that the extended probability agrees with $\Psi^* \Psi$ probability along the real line. It is seen that such an agreement is not possible for those trajectories which do not enclose the poles of $\dot{x}$. In the context of solving the conservation equation, it is easy to see that this is due to the boundary condition overdetermining the problem. In the trajectory integral approach, one can explain it as a disagreement of the values of $\rho$ at two consecutive points of crossing of the trajectory on the real axis, with that prescribed by Born’s rule. Put in other words, as a particle trajectory is traversed, if the probability $\rho$ at one point of crossing $x_{r0}$ on the real axis agrees with $P(x_{r0})$, then at the other point, say the point $x_{r1}$ where the trajectory again crosses the real line, the probability calculated according to (3) will be different from that of $P(x_{r1})$.

Given this situation, one can ask whether it is possible to find a trajectory integral definition for $\rho$ that can agree with the Born’s rule (on the real line) in this region, even if it is not conserved in the extended region. Such a definition was found [4] similar to that in equation (3):

$$\rho'(x_r, x_i) \propto P(x_{r0}) \exp \left[ -\frac{4}{\hbar} \int_{t_0}^t \text{Im} \left( \frac{1}{2} m \dot{x}^2 \right) dt' \right],$$

(4)

The difference with the definition (3) is the absence of the potential term.
$V(x)$ in the integrand. This trajectory integral approach can be seen to give
the same $\Psi^*(x)\Psi(x)$, the extended probability considered in [5, 6].

The extended probability density for the $n = 1$ harmonic oscillator in
the region inside the lemniscate (for $x_r > 0$), computed using the trajectory
integral approach in (4) is shown overlapped with the ‘leaf-shaped’ surface
$\Psi^*\Psi$ in this case, in Fig. (3). We can show that the total extended probability
can be normalized for the $n = 1$ harmonic oscillator.

![Figure 3:](image)

Summarising, it is seen that both the Born’s probability along the real
line and the extended probability in the $x_rx_i$-plane are obtainable in terms
of certain line integrals. In the extended case, a conserved probability, which
agrees with the boundary condition (Born’s rule) along the real axis, is found
to exist in most regions. The trajectory integral in this case is over the
imaginary part of $(1/2)m\dot{x}^2 + V(x)$. For other regions (such as the region
inside the lemniscate in the $n = 1$ harmonic oscillator case), an alternative
definition for probability satisfies the boundary condition on the real axis,
though it is not a conserved one. The integrand here is simply $(1/2)m\dot{x}^2$.
Since the latter is defined only for the subnets which do not enclose the
poles of $\dot{x}$, there is no difficulty in normalising the combined probability for
the entire extended plane.

Another observation we make is regarding the classical limit of this com-
plex trajectory formulation. First we note that the probability in the region
inside the lemniscate is substantial; $43.25\%$ of the total probability lies inside
this region for the $n = 1$ harmonic oscillator. The maximum value of $x_i$ for
the lemniscate (a measure of its width) in this case is $x_i^{max} = X_{x_r^{max}}^{max}/\alpha = 0.4858/\alpha$. This explains how classical particles are confined close to the real
line. For instance, consider a classical harmonic oscillator of mass $m \sim 1$ kg
and frequency $\sim 1$ Hz in the $n = 1$ state. The maximum value of $x_i$ for its
lemniscate is \( x^\text{max}_i \approx 10^{-17}/\sqrt{m\omega_0} \) in units of metres. Thus 43.25\% of the total probability in this case lies within the lemniscate of width \( \sim 10^{-17} \) m. It may also be noted that since the extended probability outside the lemniscate decreases fast for larger ovals, most of the probability outside also lies close to the real line.

For the higher energy eigenstates of the harmonic oscillator, the lemniscates are seen to be narrower than that of the low energy ones. It can be assured that for a classical oscillator of any energy and having mass 1 kg., the width of the region, where the large part of probability lies, is less than or of the order of \( 10^{-17}/\sqrt{\omega_0} \) m. This indicates that the probable imaginary part of position for classical particles are of extremely small size. However, we may also see that an electron executing harmonic oscillation with frequency \( \omega_0 \) has to accommodate relatively large values for \( x_i \), approximately equal to \( 0.01/\sqrt{\omega_0} \) m. In summary, the probability axiom in the modified de Broglie quantum mechanics helps to distinguish the classical limit of quantum harmonic oscillator as one in which the oscillator is probable to be found only very close to the real axis. This result is very important for complex quantum trajectories, for it explains why the complex extension is not observable even indirectly in the classical limit. We anticipate that this property is generally true.

References

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