1. Introduction

This paper provides a combinatorial dictionary between three sets of objects: multisegments, multipartitions, and the irreducible modules of the affine Hecke algebra $H_n$ (for generic $q$). The dictionary is dictated by Grojnowski’s Theorem 14.3, [3], (repeated here as Theorems 10.1 and 10.2) in which he constructs the crystal graph with nodes given by the irreducible modules of $H_n$ and proves that graph is $B(\infty)$ (Kashiwara’s crystal graph associated to $U(\eta^{-})$). He also shows the subgraph with nodes the irreducible modules of $H_n^\lambda$ is $B(\lambda)$, the crystal graph associated to the irreducible highest weight representation of $\mathfrak{gl}_\infty$ with highest weight $\lambda$.

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One can also construct purely combinatorial crystal graphs whose nodes are Bernstein-Zelevinsky multisegments, or are Kleshchev multipartitions, and these crystal graphs are abstractly isomorphic to $B(\infty)$ and $B(\Lambda)$, respectively. In this paper we give explicit isomorphisms. In particular, we compute the action of the crystal operator $\tilde{e}_i$ on an irreducible module both in terms of its parameterization by multisegments (theorem 3.1, rule 1) and by multipartitions (theorem 3.5, rule 3).

The results below contain new representation-theoretic content. This is because, following [BZ], we directly define a map from multisegments to irreducible modules $\Delta \mapsto \text{cosoc Ind}\Delta$ (the notation is explained in section 2 below). We then prove that this map intertwines the action of $\tilde{e}_i$ on multisegments (rule 1 of section 2.3) with the action of $\tilde{e}_i$ on modules (section 2.2). Similarly, we define a map from $\lambda$-colored Kleshchev multipartitions (section 2.4) to irreducible modules $(\mu^{(1)}, \ldots, \mu^{(r)}) \mapsto \text{cosoc Ind}\, N_{\mu^{(1)}, i_1} \boxtimes \cdots \boxtimes N_{\mu^{(r)}, i_r}$ and prove that this map also intertwines the action of $\tilde{e}_i$. This is new.

Grojnowski’s Theorem 14.3 suffices to parameterize the modules in terms of paths to the highest weight node, i.e. by a sequence of operators cosoc Ind. The advantage of the approach of this paper is that we only have to take cosoc once. However, it is not true in general that cosoc commutes with Ind, and this is the representation-theoretic difficulty we overcome.

Another byproduct of Theorem 3.1 below is the determination of which multisegments parameterize modules of the cyclotomic Hecke algebra $H_n^\lambda$ (Theorem 3.2). The theorems also partially explain why the rule for computing $\tilde{e}_i$ mirrors the rule we know for that on a tensor product of crystal graphs.

The proofs given here do not rely on [G] Theorem 14.3, but do use several results from [G, GV] (and of course from [BZ] and [Z]). One reason for this is that the proofs are relatively straightforward. Another is that several steps in the proofs given here are necessary ingredients for proving Theorems 3.4 and 3.5.

There are related results in the literature in [A2, AM, G2]. Both our proofs and construction are different from theirs, and we comment on this at appropriate points in the text.

This introduction and section 10 interpret the main theorems of this paper into the language of crystal graphs. However, the rest of the sections avoid that language and simply deal with algebras and modules. No background in crystal graphs is necessary to understand the proofs, (but is helpful in appreciating them).

We begin with some necessary definitions and notation, then immediately state the main results. However, the reader may want to read the more extensive definitions is section 4 before section 3. In sections 8 and 9 the proofs are given. In section 10 some results and proofs that appear in [BZ, Z] are given, recast in this notation, and included for convenience to the reader.

We point out that these proofs require $q$ to be generic, whereas the theorems of [G] hold for all $q \in R^\times$.

2. Background notations and results

In the following subsections, we define three sets of objects which will be the nodes of three different crystal graphs. These are the irreducible modules of $H_n$ in
the subcategory $\text{Rep}_q$, multisegments, and colored multipartitions. We define the action of an operator $e_i$ on each of these three sets, i.e. we describe the edges of the crystal graphs. In section 2 are the theorems showing the three crystal graphs are isomorphic.

2.1. $H_n^{\text{fin}}$, $H_n$ and $H_n^{\text{aff}}$. Throughout the paper we fix an algebraically closed field $R$, and an invertible element $q \in R$, such that $q^\ell = 1$ implies $\ell = 0$. In this situation $\{q^i\}_{i \in \mathbb{Z}}$ is infinite and we say $q$ is generic.

The finite Hecke algebra, $H_n^{\text{fin}}$ is the $R$-algebra with generators

$$T_1, \ldots, T_{n-1}$$

and relations

1. braid relations  \[ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i, \quad |i - j| > 1 \]

2. quadratic relations  \[ (T_i + 1)(T_i - q) = 0. \]

The braid relations imply that if $w = s_{i_1} \cdots s_{i_k}$ and $\ell(w) = k$, then $T_{i_1} \cdots T_{i_k}$ depends only on $w \in S_n$. It is denoted $T_w$, and the $\{ T_w \mid w \in S_n \}$ form a basis of $H_n^{\text{fin}}$ over $R$.

The affine Hecke algebra $H_n$ (or $H_n^{\text{aff}}$) is the $R$-algebra, which as an $R$-module is isomorphic to

$$H_n^{\text{fin}} \otimes_R R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}].$$

The algebra structure is given by requiring that $H_n^{\text{fin}}$ and $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ are subalgebras, and that

$$T_i X_j T_i = q X_j.$$  

Denote by $\text{Rep}_q H_n$ the finite dimensional modules $M$ for $H_n$ such that the only eigenvalues of the $X_j$ on $M$ are powers of $q$. Let $\text{Rep}_q = \bigoplus_{n \geq 0} \text{Rep}_q H_n$.

Remark 1. It follows from the computation in [G], section 6.2, or is explained in [AK], or many other places, that to understand $\text{Rep}_q H_n$ it is enough to understand $\text{Rep}_q H_n$.

The cyclotomic Hecke algebra or Ariki-Koike algebra $H_n^{\lambda}$ is the quotient $H_n/I_{\lambda}$, where $I_{\lambda}$ is the ideal generated by the polynomial in $X_1$: $\prod (X_1 - q^i)^{m_i}$, and $\lambda = \sum m_i \Lambda_i$ is a weight of $gl_{\infty}$ (where $\{ \Lambda_i \mid i \in \mathbb{Z} \}$ denote the fundamental weights). We may also write $\lambda = \sum_{k=1}^r \Lambda_{i_k}$ and so $I_{\lambda}$ is generated by $\prod_{k=1}^r (X_1 - q^{i_k})$.

Remark 2. Observe that any irreducible module in $\text{Rep}_q$ is an irreducible $H_n^{\lambda}$-module if we take $\lambda$ large enough. Just let $\prod_{i=1}^r (X_1 - q^{i})^{m_i}$, be the characteristic polynomial of $X_1$ acting on the module. Conversely, we identify any irreducible $H_n^{\lambda}$-module as an irreducible $H_n$-module (on which $(X_1 - q^i)^{m_i}$ vanishes).

2.2. $e_i$, $\overline{e}_i$, and $\varepsilon_i$. Given an irreducible $H_n$-module $M$, consider $\text{Res}_n^{H_n} M = \text{Res}^{H_n} M$. As $X_n - q^i \subseteq Z(H_n \otimes H_1)$, left multiplication by $(X_n - q^i)^m$ induces an endomorphism of $\text{Res}^{H_n} M$, and its kernel for $m \gg 0$ (we can take $m \leq \dim M$) is the generalized eigenspace of $X_n - q^i$. Define $e_i M$ to be the restriction of that generalized eigenspace to $H_n-1$. Because restriction and taking generalized eigenspaces are exact functors, we have the following claim.

Claim 1. $e_i$ is an exact functor $\text{Rep}_q H_n \rightarrow \text{Rep}_q H_{n-1}$ and $\text{Res}^{H_n}_{H_{n-1}} M = \bigoplus_{i \in \mathbb{Z}} e_i M$. 

Theorem 2.1 (GV Theorem B) is zero or is irreducible, and cosocle of $M$. We also define trivial representation of $H$ and that the cosocle of $N$, denoted $\text{cosoc}(N)$, is its largest semisimple quotient.

We refer to [GV] Theorem B.

**Theorem 2.1 (GV Theorem B).** Let $M$ be an irreducible $H_n$-module. Then $\widehat{e}_i M$ is zero or is irreducible, and $\text{cosoc} e_i M$ is isomorphic to $\widehat{e}_i M$.

Similarly, given $M \in \text{Rep}_q H_n$, define $\widehat{e}_i M$ to be the $X_1 - q^i$ generalized eigenspace of $\text{Res}^n_{n-1} M$, further restricted to $H_{n-1}$. (Technically, one should then re-index the $T_k$ and $X_k$ to $T_{k-1}$ and $X_{k-1}$.) If $M$ is irreducible, let $\widehat{e}_i M = \text{soc} \widehat{e}_i M$ and $\widehat{e}_i (M) = \max\{ m \geq 0 \mid \widehat{e}_i^m M \neq 0 \}$. Analogous to theorem 2.1, if $M$ is irreducible, then $\widehat{e}_i M$ is irreducible or zero.

2.3. **Multisegments.** Write $\Delta_{(i,j)} = (q^i q^{i+1} \cdots q^j)$, $i \leq j$ for the one-dimensional trivial representation of $H_{j-i+1}$ on which each $T_k - q$ and $X_k - q^{k+i-1}$ vanishes. We refer to $\Delta_{(i,j)}$ as a segment. Adopt the convention that $\Delta_{(j,j-1)} = 1$, the one dimensional $H_0$-module, or that $\{ \Delta_{(j,j-1)} \} = \emptyset$.

We introduce the symbol $0$, not to be confused with $\emptyset$, which will stand for the zero module.

Call a multiset of segments a multisegment. Theorem 2.2 below is the result from [BZ 2] that multisegments classify the irreducible modules of $\text{Rep}_q$.

In the following theorems, we will consider two total orderings on segments. Examples will be given in section 3.

**Right order:** Order segments so $\Delta_{(i_1,j_1)} > \Delta_{(i_2,j_2)}$ if $i_1 > i_2$ or if $i_1 = i_2$ and $j_2 > j_1$.

**Left order:** Order segments so $\Delta_{(i_1,j_1)} > \Delta_{(i_2,j_2)}$ if $j_1 > j_2$ or if $j_1 = j_2$ and $i_2 > i_1$.

Here we define a function from multisegments to multisegments $\cup \{ 0 \}$

$$\Delta \mapsto \overline{e}_j \Delta.$$

**Rule 1.** Given a multisegment $\Delta$, first put $\Delta$ in right order, so that $\Delta = \{ \Delta_{(a_1,b_1)}, \Delta_{(a_2,b_2)}, \ldots, \Delta_{(a_m,b_m)} \}$ with $a_1 \geq a_2 \geq \cdots \geq a_m$. We write its "j-signature" as follows. Assign a blank to any segment such that $b_k \neq j, j-1$, assign $-$ if $b_k = j$, and $+$ if $b_k = j-1$. In the corresponding word $\pm \pm \cdots \pm$ (interspersed with blanks), cancel any adjacent $- +$, continuing, ignoring all previously cancelled symbols until what is left uncanceled has the form

$$+ + \cdots + - \cdots .$$

Suppose that the leftmost uncanceled $-$ (if it exists) is in position $i$, i.e. that it came from $\Delta_{(a_i,j)}$. Define

$$\overline{e}_j \Delta = \Delta \cup \{ \Delta_{(a_i,j-1)} \} \setminus \{ \Delta_{(a_i,j)} \},$$

and $\overline{e}_j \Delta = 0$ if no such $-$ exists.
Observe that $\pm$ word corresponding to $\tilde{e}_j\Delta$, if $\tilde{e}_j\Delta \neq 0$, is exactly that of $\Delta$ except that the leftmost uncanceled $-$ has been changed to $+$. Further, if $\varepsilon$ is the total number of uncanceled $-$ signs, then $\tilde{e}_j^{\varepsilon+1}\Delta = 0$.

Define a second function from multisegments to multisegments $\cup \{0\}$

$$\Delta \mapsto \tilde{e}_i\Delta.$$ 

**Rule 2.** Given a multisegment $\Delta$, we define $\tilde{e}_i\Delta$ as follows: put $\Delta$ into left order. For each $\Delta(i,z)$ write $-$, for each $\Delta(i+1,z')$ write $+$, and for all other segments write a blank. Now we ignore all $+-$ pairs, similar to rule 1, which will leave uncanceled symbols $\ldots \cdot \cdot \cdot - \cdot \cdot \cdot +$. Then, if $\Delta(i,z)$ corresponds to the rightmost $-$, we replace that segment with $\Delta(i+1,z)$. If no such uncanceled $-$ exists, then $\tilde{e}_i\Delta = 0$.

Given a multisegment and a choice of ordering of its segments $\Delta = \{\Delta(i_1,j_1), \ldots, \Delta(i_m,j_m)\}$ let

$$\text{Ind}\Delta$$

denote $\text{Ind}_{n_1,n_2,\ldots,n_m}^{\mathbb{Z}} \Delta(i_1,j_1) \cdot \cdots \cdot \Delta(i_m,j_m)$, where $n_k = j_k - i_k + 1$ and $n = \sum_k n_k$.

Define $n(\Delta) = n$ and $m(\Delta) = m$. (Observe $n(\tilde{e}_i\Delta) = n(\Delta) - 1$ if $\tilde{e}_i\Delta \neq 0$.)

Multisegments are by definition unordered. However, it is necessary when discussing induced modules to have an order understood. From now on, unless stated otherwise, given a multisegment $\Delta$, let $\Delta$ denote the multisegment in right order, and $\hat{\Delta}$ denote it in left order.

Define

$$M_\Delta = \text{cosoc} \text{Ind}\Delta.$$ 

Set $M_0 = 0$. Again, the symbol 0 is not to be confused with the empty segment $\emptyset$ for which $M_0 = 1$, the one dimensional module of $H_0 = R$.

The following theorem of [BZ, Z] show that multisegments parameterize the irreducible modules in $\text{Rep}_q H^n_m$, $n \geq 0$. A proof will be given in section 6.3.

**Theorem 2.2 ([BZ, Z]).** Let $\Delta$ be a multisegment with $n = n(\Delta)$. Then

1. $M_\Delta := \text{cosoc} \text{Ind}\Delta$ is an irreducible $H_n$-module.
2. If $\Delta \neq \Delta'$ then $M_\Delta \nmid M_{\Delta'}$.
3. Given any irreducible $M \in \text{Rep}_q H_n$, there exists a multisegment $\Delta$ (with $n(\Delta) = n$) such that $M = M_\Delta$.

The main theorem of section 3 proved in section 4, is that $\tilde{e}_j M_\Delta = M_{\tilde{e}_j\Delta}$ (and $\tilde{e}_i M_\Delta = M_{\tilde{e}_i\Delta}$).

2.4. **Partitions.** Recall that $\mu$ is a partition of $n$ with length $k$ if $\mu = (\mu_1, \ldots, \mu_k)$ such that $\sum_m \mu_m = n$ and $\mu_1 \geq \cdots \geq \mu_k > 0$. We write $|\mu| = n$.

**Definition 1.** For a partition $\mu$ of length $k$ and for $i \in \mathbb{Z}$, let $\Delta(\mu, i) = \{\Delta(i,i+\mu_1-1), \ldots, \Delta(i-k+1,i-k+\mu_k)\}$. Observe $|\mu| = n(\Delta)$.

We will say $\mu$ is colored by $i$ if we associate it with $\Delta(\mu, i)$. We picture it as follows. Consider the Young diagram associated to $\mu$, which consists of $\mu_1$ boxes in
the first row, \( \mu_2 \) boxes in the second, etc. Fill the \((x,y)\) box with \(i + x - y\).

\[
\begin{array}{cccc}
  i & i+1 & \cdots & b_1 \\
  i-1 & i & \cdots & b_2 \\
  \vdots & & & \vdots \\
  i-k+1 & \cdots & b_k \\
\end{array}
\]

Then the row fillings correspond to the segments of \( \Delta(\mu,i) \), with \( b_m = i + \mu_m - m \). Then we say \( \mu, i \) has a removable \( j \)-box if you can remove a \( j \)-filled box from the diagram such that the result is again the diagram of a partition (colored by \( i \)). We say \( \mu, i \) has an addable \( j \)-box if you can add a \( j \)-filled box to the diagram such that the result is again the diagram of a partition (colored by \( i \)).

We will call an \( r \)-tuple of partitions \( \mu = (\mu^{(1)}, \ldots, \mu^{(r)}) \) a multipartition of \( n \) if \( n = \sum_m [\mu^{(m)}] \).

Likewise, given a multipartition \( \mu \) and a weight \( \lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_r} \) of level \( r \), with \( i_1 \leq i_2 \leq \cdots \leq i_r \), we will say \( \mu \) is colored by \( \lambda \) if \( \mu^{(m)} \) is colored by \( i_m \).

If \( \mu, \lambda \) satisfy the condition
\[
\mu^{(t)}_{i_t-i_{t+1}+x} \leq \mu^{(t+1)}_x \quad \text{for all } x \geq 1, 1 \leq t \leq r-1
\]
then we will say it is a Kleshchev multipartition. The only multipartitions we will consider in Theorems 3.4 and 3.5 will be Kleshchev multipartitions. See corollary 3.7 for a crystal-theoretic interpretation of Kleshchev multipartitions.

**Definition 2.** If \( \mu \) is colored by \( \lambda \), set \( \Delta(\mu, \lambda) = \bigcup_{j=1}^r \Delta(\mu^{(j)}, i_j) \).

**Rule 3.** Given a multipartition \( \mu \) and weight \( \lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_r} \) with \( i_1 \leq i_2 \leq \cdots \leq i_r \), we define \( \overline{e}_j(\mu, \lambda) \) as follows. If \( \mu^{(k)} \) has a removable \( j \)-box when colored by \( i_k \), write the symbol \(-\). If \( \mu^{(k)} \) has an addable \( j \)-box when colored by \( i_k \), write \(+\). Otherwise, write a blank. In the resulting word of length \( r \), cancel all occurrences of \(+\). In the remaining uncanceled symbols \(+\cdots +\cdots\), we remove the \( j \)-box from the \( \mu^{(k)} \) that corresponds to the leftmost \(-\) symbol, if one exists. The resulting colored multipartition is \( \overline{e}_j(\mu, \lambda) \). Otherwise, \( \overline{e}_j(\mu, \lambda) = \emptyset \).

Again, we note that \( \overline{e}_j \) “changes” the leftmost \(-\) to a \(+\).

**Definition 3.** Define \( N_{\mu,\lambda} = M_{\Delta(\mu,i)} \). Let the symbol \( 0 \) be such that \( \Delta(0,i) = 0 \) and \( N_{0,i} = 0 \) (not to be confused with the empty partition \( \mu = \emptyset \), in which case \( N_{\emptyset,i} = 1 \)). Define \( N_{\mu,\lambda} = \cosoc \text{Ind} N_{\mu^{(1)}, i_1} \oplus \cdots \oplus N_{\mu^{(r)}, i_r} \).

In Theorem 3.4 we will see the modules \( N_{\mu,\lambda} \) are irreducible when \( \mu, \lambda \) is a Kleshchev multipartition. In section 3, we will also see the rule for \( \overline{e}_j \) on colored multipartition is compatible with that for the corresponding multisegment and with that of \( \overline{e}_j \) on the corresponding irreducible module.

### 3. Main results

Below are the theorems that give the action of \( \overline{e}_j \) on irreducible modules of \( \text{Rep}_q \) both in terms of parameterization by multisegments and by colored multipartitions.
The action of $\tilde{e}_j$ on irreducibles corresponds to the action of $\tilde{e}_j$ on multisegments and on Kleshchev multipartitions. In other words, of the three crystal graphs $B_{BZ}$, $B_{aff}$, $B^{\lambda}_\lambda$ with edges described by

$$
\begin{align*}
\tilde{e}_1 \Delta & \to \Delta \\
\tilde{e}_1 M & \to M \\
\tilde{e}_1(\mu, \lambda) & \to \mu, \lambda
\end{align*}
$$

the first two are isomorphic with isomorphism given by $\Delta \mapsto M_\Delta$.

The third is a connected component of the graph of all $\lambda$-colored multipartitions, and is isomorphic to the subgraph $B_\lambda$ of $B_{aff}$ corresponding to $\text{Rep} \ H_n^\lambda$ for $n \geq 0$. That embedding is

$$
\mu, \lambda \mapsto N_{\mu, \lambda} \text{ or } \mu, \lambda \mapsto \Delta(\mu, \lambda).
$$

Proofs are postponed until sections 8 and 9.

Theorem 3.1. Let $\Delta$ be any multisegment. Then $\tilde{e}_j M_\Delta = M_{\tilde{e}_j \Delta}$.

We have the analogous theorem

Theorem 3.2. Let $\Delta$ be any multisegment. Then $\tilde{e}_i M_\Delta = M_{\tilde{e}_i \Delta}$.

As a corollary, we determine which $M_\Delta$ are in $\text{Rep} \ H_n^\lambda$.

Corollary 3.3. Let $\Delta$ be a multisegment with $n = n(\Delta)$. Let $\lambda = \sum m_i \Lambda_i$. Then $M_\Delta \in \text{Rep} \ H_n^\lambda$ if and only if $\hat{e}_i \Delta$ for all $i$, in the $\pm$ word computed as in Rule 2 to calculate $\hat{e}_i \Delta$, there are $m_i$ uncanceled $-$ signs. In other words, $\hat{e}_i \Delta = 0$.

Proof. This follows from theorem 3.2 and also relies on Theorem 9.13 of [G]: that $\hat{e}_i (M)$ is the maximal size of the Jordan block with eigenvalue $q'$ for $X_1$ acting on an irreducible module $M$.

Theorem 3.3 describes a different isomorphism $B_{BZ} \to B_{aff}$ than Theorem 3.1, and hence an automorphism of $B_{aff}$. This automorphism is also given by $M \mapsto \text{rev}^* M$ where $\text{rev} : H_n \to H_n$ is the algebra automorphism $\text{rev}(X_k) = X_{n+1-k}$, $\text{rev}(T_k) = -(T_{n-k} + 1 - q)$. In fact, it is this involution we really use to deduce Theorem 3.1 from Theorem 3.3. Because $\text{rev}$ exchanges $X_1$ and $X_n$, any statement made regarding $e_i$ and an irreducible module $M$ can be made for $\tilde{e}_i$ and $\text{rev}^* M$.

Theorem 3.4. Let $M_\Delta$ be an $H_n^\lambda$-module where $\lambda = \Lambda_{i_1} + \Lambda_{i_2} + \cdots + \Lambda_{i_r}$, $i_1 \geq \cdots \geq i_r$. Then there exists an $r$-tuple of partitions $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ satisfying

$$
\mu_{i_t - n + 1 + x}^{(t)} \leq \mu_x^{(t+1)} \text{ for all } x \geq 1, 1 \leq t \leq r - 1
$$

such that $M_\Delta = N_{\mu, \lambda}$ and $\Delta = \Delta(\mu, \lambda)$.

Conversely, if $N_{\mu, \lambda}$ satisfies (4), then $N_{\mu, \lambda}$ is isomorphic to $M_\Delta$ for $M_\Delta = \Delta(\mu, \lambda)$.

Recall that we call an $r$-tuple of partitions satisfying condition (4) a Kleshchev multipartition.

Theorem 3.5. For a colored multipartition satisfying (4), $\tilde{e}_j N_{\mu, \lambda} = N_{\tilde{e}_j (\mu, \lambda)}$.

Further, $\tilde{e}_j (\Delta(\mu, \lambda)) = \Delta(\tilde{e}_j (\mu, \lambda))$. 
Corollary 3.6. For fixed $\lambda$, and for $\mu$ satisfying (4), the modules $N_{\mu, \lambda} = \text{cosoc Ind} N_{\mu^{(1)}, i_1} \boxtimes \cdots \boxtimes N_{\mu^{(r)}, i_r}$

are irreducible, distinct, and range over all irreducible $H^n_\lambda$-modules. In other words, Kleshchev multipartitions parameterize the irreducible $H^n_\lambda$-modules.

It is instructive to compare this result and the following result with similar results of [AM, A]. Both their construction of the modules $N_{\mu, \lambda}$ and the proof that they are nonzero are different from ours.

In section 10, we explain the following corollary, which interprets the preceding theorems in terms of the crystals $B(\lambda), B(\infty)$ and $B(\Lambda_{i_1}) \otimes \cdots \otimes B(\Lambda_{i_r})$.

Corollary 3.7. The crystal graph $B'_\lambda$ is isomorphic to $B_\lambda$ via $\mu, \lambda \mapsto N_{\mu, \lambda}$ and is isomorphic to the connected component of the unique node of weight $\lambda$ in $B(\Lambda_{i_1}) \otimes \cdots \otimes B(\Lambda_{i_r})$ via $\mu, \lambda \mapsto \mu^{(r)} \otimes \cdots \otimes \mu^{(1)}$.

4. Further definitions

We need a few more definitions to proceed. Here they are.

If $A$ is an $R$-algebra, we write $\text{Rep} A$ for the category of left $A$-modules which are finite dimensional as $R$-modules. We recall that the socle of a module $M$, denoted $\text{soc}(M)$, is the largest semisimple submodule of $M$, and that the cosocle of $M$, denoted $\text{cosoc}(M)$, is its largest semisimple quotient. We also write $Z(A)$ for the center of the algebra $A$.

If $A$ and $A'$ are two $R$-algebras with modules $M$ and $M'$ respectively, let $M \boxtimes M'$ denote the $A \otimes A'$-module which is isomorphic to $M \otimes_R M'$ as an $R$-module and has $A \otimes A'$ action given by $(a \otimes a') \cdot (m \otimes m') = am \otimes a'm'$.

Because $R$ is algebraically closed, if $M$ is an irreducible $A$-module and $M'$ is an irreducible $A'$-module, then $M \boxtimes M'$ will be an irreducible representation of $A \otimes A'$, and all such are of this form.

4.1. Induction and Restriction. Recall that if $A \subset B$ are $R$-algebras such that $B$ is finitely generated both as a left and right $A$-module, the exact functor of restriction $\text{Res}_A^B : \text{Rep} B \to \text{Rep} A$

has left and right adjoints, $\text{Ind}$ and $\hat{\text{Ind}}$ defined by

$\text{Ind}_A^B : \text{Rep} A \to \text{Rep} B \quad M \mapsto B \otimes_A M$

$\hat{\text{Ind}}_A^B : \text{Rep} A \to \text{Rep} B \quad M \mapsto \text{Hom}_A(B, M)$;

i.e.

$\text{Hom}_B(\text{Ind}_A M, N) = \text{Hom}_A(M, \text{Res} N) \quad \text{Hom}_B(N, \hat{\text{Ind}}_A M) = \text{Hom}_A(\text{Res} N, M)$.

If $B$ is a free $A$-module, then $\text{Ind}$ and $\hat{\text{Ind}}$ are exact functors also. Further, if $A \subset B \subset C$ are inclusions of $R$-algebras, we have transitivity of induction and restriction:

$\text{Res}_A^B \text{Res}_B^C = \text{Res}_A^C, \quad \text{Ind}_A^C \text{Ind}_B^A = \text{Ind}_A^C, \quad \hat{\text{Ind}}_A^C \hat{\text{Ind}}_B^A = \hat{\text{Ind}}_A^C$.
Now apply these remarks to the affine Hecke algebra. Given a sequence $P = (a_1, \ldots, a_k)$ of non-negative integers summing to $n$, write $H_P = H_{a_1} \otimes \cdots \otimes H_{a_k}$. We have an obvious embedding

$$H_P = H_{a_1} \otimes \cdots \otimes H_{a_k} \hookrightarrow H_n$$

which makes $H_n$ a free $H_P$-module. Applying the previous remarks we get exact functors $\text{Res}, \text{Ind}, \hat{\text{Ind}}$. When unambiguous from context, we just write $\text{Res}_P$ or $\text{Res}$ for $\text{Res}^{H_n}_P$, and similarly for $\text{Ind}$ to lighten notation. These functors depend on the order $(a_1, \ldots, a_k)$ and not just on the underlying set.

4.2. characters. Write $S = R[X_1^\pm, \ldots, X_n^\pm]$ and let $M \in \text{Rep}_q H_n$.

Define the generalized $S$-eigenspace

$$M[\gamma] = \{v \in M \mid (X_i - \gamma_i)^m v = 0, \ m \gg 0, \ \text{for all} \ 1 \leq i \leq n\}$$

where $\gamma = (\gamma_1, \ldots, \gamma_n) \in R^n$.

Because the $X_i$ commute, we can decompose $M$ into a direct sum of generalized eigenspaces

$$\text{Res}_S^{H_n} M = \bigoplus_{\gamma \in (q^n)} M[\gamma].$$

We define the character of $M$ to be the formal sum

$$\text{ch} M = \sum_{\gamma \in (R^n)} (\dim M[\gamma]) \gamma.$$

Since $S = R[X_1^\pm, \ldots, X_n^\pm] = R[X_1^\pm] \otimes_R \cdots \otimes_R R[X_n^\pm] = H_{(1,1,\ldots,1)}$, we will also write $\gamma = \gamma_1 \boxtimes \gamma_2 \boxtimes \cdots \boxtimes \gamma_n$ for the 1-dimensional $S$-module with character $\gamma = (\gamma_1, \ldots, \gamma_n)$. For ease, we will also write $\gamma = \gamma_1 \gamma_2 \cdots \gamma_n$ for a term in the character of an $H_n$-module.

5. Useful propositions

In this section, we collect useful results that we will need later. The results in this section are well-known and most can be found in [B], [G], [GV], [Ka], [V].

**Proposition 1.** If $M$ is an irreducible $H_n$-module, then $M$ is finite dimensional with $\dim_R M \leq n!$.

Consequently, $\text{Rep} H_n$ includes all irreducible $H_n$-modules.

**Theorem 5.1** (Kato’s Theorem). Let $q^i K_n$ denote $\text{Ind}^n_{1,1,\ldots,1} q^i \boxtimes \cdots \boxtimes q^i$. Then $q^i K_n$ is irreducible, and is the unique $H_n$-irreducible module with $q^i q^{i+1} \cdots q^j$ occurring in its character.

From [B] Proposition 12.1 we have:

**Proposition 2** (Serre relations). As operators on the Grothendieck group $\bigoplus_n K(\text{Rep}_q H_n)$

1. if $|i - j| > 1$, then $e_i e_j = e_j e_i$.
2. $e_i^2 e_{i+1} + e_{i+1} e_i^2 = 2e_i e_{i+1} e_i$.

**Remark 3.** We can reinterpret proposition [B] that, first, if $q^i \neq q^{i+1}$, then for any finite dimensional module $M$, if $\gamma_1 \gamma_2 \cdots q^i q^{i+1} \cdots \gamma_n$ occurs in $\text{ch} M$ then so does $\gamma_1 \gamma_2 \cdots q^i q^{i} \cdots \gamma_n$. (Specifically: $\dim M[\gamma_1, \gamma_2, \ldots, q^i q^{i+1}, \ldots, \gamma_n] = \dim M[\gamma_1, \gamma_2, \ldots, q^i q^{i}, \ldots, \gamma_n]$.)


Secondly, $\gamma_1 \cdots q^i q^{i+1} \cdots \gamma_n$ occurs in ch $M$ if and only if $\gamma_1 \cdots q^i q^{i+1} \cdots \gamma_n$ or $\gamma_1 \cdots q^{i+1} q^i \cdots \gamma_n$ occur. (Furthermore, the dimensions of the generalized $S$-eigenspaces spaces satisfy that twice the first is the sum of the next two.)

Kato’s Theorem implies that if $\gamma_1 \cdots \underbrace{q^i \cdots q^i}_{k} \cdots \gamma_n$ occurs in ch $M$, it does so with multiplicity at least $k!$.

**Proposition 3** ([GV], Lemma 3.5, part 3). For any irreducible module $N$

$$M = \text{cosoc} \text{Ind}N \boxtimes q^i K_k$$

is irreducible with $\varepsilon_i(M) = \varepsilon_i(N) + k$, and all other composition factors of $\text{Ind}N \boxtimes q^i K_k$ have strictly smaller $\varepsilon_i$.

**Proposition 4** ([GV], Corollary 3.6). Let $M$ be an irreducible $H_n$-module.

1. Let $\varepsilon = \varepsilon_i(M)$ and $N = \overline{c}_i M$. Then for $0 \leq k \leq \varepsilon$

$$\overline{c}_i^k M = \text{cosoc} \text{Ind}N \boxtimes q^i K_{\varepsilon-k}.$$  

2. $\overline{c}_i^\varepsilon M \simeq \text{cosoc} \varepsilon_i M$.

3. If $L$ is also irreducible and $\overline{c}_i M \simeq \overline{c}_i L \neq 0$ then $M \simeq L$.

**Corollary 5.2.** Let $M$ be an irreducible $H_n$-module and let $\varepsilon = \varepsilon_i(M)$. Then $\varepsilon_i^\varepsilon M$ is the direct sum of $\varepsilon!$ copies of $\varepsilon_i^1 M$.

**Proof.** From proposition [3], if $N = \overline{c}_i^1 M$, then $M = \text{cosoc} \text{Ind}^n_{n-\varepsilon,1 \cdots 1} N \boxtimes q^\ell \boxtimes \cdots \boxtimes q^\ell = \text{cosoc} \text{Ind}^n_{n-\varepsilon,1 \cdots 1} N \boxtimes q^i K_{\overline{c}_i}$ and $\varepsilon_i(N) = 0$. Applying the exact functor $\varepsilon_i^\varepsilon$ (and using lemma 3) yields a surjection

$$\bigoplus_{i=1}^{\varepsilon} N \twoheadrightarrow \varepsilon_i^\varepsilon M.$$  

But by Frobenius reciprocity, we have a map $N \boxtimes q^i K_{\varepsilon} \rightarrow \text{Res}^n_{n-\varepsilon,\varepsilon} M$, which must be an injection by Kato’s theorem. Since restriction is exact, this shows $\varepsilon_i^\varepsilon M$ must be an isomorphism.

The next proposition follows from proposition 3 and Proposition 10.4.

**Proposition 5.** Let $M$ be an irreducible $H_n$-module such that $\varepsilon_i(M) > 0$. Then

1. $\varepsilon_i(\overline{c}_i M) = \varepsilon_i(M) - 1$.

2. $\varepsilon_{i+1}(\overline{c}_i M) = \begin{cases} \varepsilon_{i+1}(M) & \text{or} \\ \varepsilon_{i+1}(M) + 1 \end{cases}$

**5.1.** The shuffle lemma. We write $S_n$ for the symmetric group on $n$ letters, $s_i = (i \ i+1)$, $1 \leq i \leq n-1$, for the simple transpositions. Denote length by $\ell(w)$.

If $P = (a_1, \ldots, a_k)$ is an ordered tuple, $a_i \in \mathbb{Z}_{>0}$, $\sum a_i = n$, it is convenient to denote $S_P = S_{a_1} \times \cdots \times S_{a_k} \subseteq S_n$. For example, $S_{(n)} = S_n$, $S_{(1, \ldots, 1)} = \{1\}$.

For such $P$, we write $W^P_{\text{min}}$ for the set of minimal length left coset representatives of $S_P \subseteq S_n$.

The following lemmas are special cases of the “Mackey formula” relating the composite of induction from $H_P$ to $H_n$ with the restriction from $H_n$ to $H_P$, for various $P$ and $P’$. In particular, in lemma $P = (a,b)$, $P’ = (n-1,1)$ (but then
we further restrict to $H_{n-1}$; in lemma 3 below, $P = P' = (1, 1, \ldots , 1)$ so that $H_P = S$; and finally in lemma 3, $P = (m, n)$ and $P' = (1, 1, \ldots , 1)$.

Below we compute the action of $\epsilon_i$ on an induced module. We refer the reader to [G] or [GV] for a proof.

Lemma 1. Let $A$ be an irreducible $H_a$-module and $B$ be an irreducible $H_b$-module. Let $n = a + b$. The following is an exact sequence:

$$0 \rightarrow \text{Ind}^{n-1}_{a-1,b} e_i A \boxtimes B \rightarrow \epsilon_i(\text{Ind}^n_{a,b} A \boxtimes B) \rightarrow \text{Ind}^{n-1}_{a,b-1} A \boxtimes \epsilon_i B \rightarrow 0$$

Theorems 3.1 and 3.5 describe that for very special $A$ and $B$, we can determine exactly when

$$\tilde{\epsilon}_i(\text{cosoc Ind} A \boxtimes B) = \begin{cases} \text{cosoc(Ind} \epsilon_i A \boxtimes B) \\ \text{cosoc(Ind} A \boxtimes \epsilon_i B) \end{cases}$$

and that the cosocles above are irreducible. The first case happens when $\epsilon_i(A) > \varphi_i(B)$, and the second when $\epsilon_i(A) \leq \varphi_i(B)$. We will define $\varphi_i$ in section 11.

Compare this to equation (17) in section 11 concerning a tensor product of crystals.

Lemma 2. Let $\gamma = (\gamma_1, \ldots , \gamma_n) \in \{q^i\}^n$. Then

$$\text{ch(Ind}^{H_n}_{a} \gamma_1 \boxtimes \cdots \boxtimes \gamma_n) = \sum_{w \in S_n} w \cdot \gamma.$$ 

We now describe the character of $\text{Ind}^n_{a,b} M$ in terms of $\text{ch} M$. We say a string $\gamma = \gamma_1 \gamma_2 \cdots \gamma_k$ is a shuffle of $t$ and $u$ if $t$ is a subword of $\gamma$ and $u$ is its complementary subword. The shuffle of $t$ and $u$, denoted $t \cup u$, is then the formal sum of all shuffles of $t$ and $u$, with multiplicity.

The permutations of $\{1, 2, \ldots , n+m\}$ that keep $1, \cdots , m$ in order and $m + 1, \cdots , m+n$ in order (i.e. their shuffles) are exactly the minimal length left coset representatives $W_{\text{min}}^{(m,n)}$. It follows that if we write $tu$ for the concatenation of $t$ and $u$, which have length $m$ and $n$ respectively, then

$$t \cup u = \sum_{w \in W_{\text{min}}^{(m,n)}} w \cdot (tu).$$

We extend $\cup$ linearly to sums of words.

Lemma 3. Let $M \in \text{Rep} H_m$, $N \in \text{Rep} H_n$. Then

$$\text{ch Ind}^{m+n}_{(m,n)} M \boxtimes N = \sum_{\gamma \text{ is a shuffle of } t \text{ and } u} (\dim M[t] \dim N[u]) \gamma = \text{ch} M \cup \text{ch} N.$$ 

5.2. Induced modules.

Proposition 6. Let $A$ be an irreducible $H_a$-module and $B$ be an irreducible $H_b$-module. Let $n = a + b$. Then $\text{Ind}^n_{a,b} A \boxtimes B \simeq \text{Ind}^n_{b,a} B \boxtimes A$, where $n = a + b$.

Proof. A full proof can be found in Proposition 3.5 of [V]. We will outline the construction of an explicit isomorphism.

Let $W = W_{\text{min}}^{(a,b)} = \{w\}$ be minimal length left coset representatives for $S_a \times S_b \leq S_n$. Let $W' = \{x_0 w^{-1}\}_{w \in W} = \{w'\}$ be minimal length right coset representatives for $S_b \times S_a \leq S_n$, where $x_0$ is the longest element of $W$. If $\{u \cup v\}$ is a basis of $A \boxtimes B$, then $\{v \otimes u\}$ is a basis of $B \boxtimes A$, so $\{T_w \otimes (u \otimes v)\}_{w \in W}$ is a basis of
\[ \text{Ind}^{n}_{a,b} A \boxtimes B, \text{ and } \{ \phi_{w',u \boxtimes v} \}_{w' \in W'} \text{ is a basis of } \hat{\text{Ind}}^{n}_{a,b} B \boxtimes A = \text{Hom}_{H_{(b,a)}} (H_n, B \boxtimes A), \]

where \( \phi_{w',u \boxtimes v}(T_{x'}) = \delta_{w',x'} v \otimes u \) for \( x' \in W' \) (and extend to a homomorphism on all of \( H_{(b,a)} \)).

Write \( \phi_{u \boxtimes v} \) for \( \phi_{x_{a},u \boxtimes v} \). Define the isomorphism

\[ \psi : \text{Ind}^{n}_{a,b} A \boxtimes B \rightarrow \hat{\text{Ind}}^{n}_{a,b} B \boxtimes A \]

\[ h \otimes (u \otimes v) \mapsto \psi h \phi_{u \boxtimes v} \quad \text{for } h \in H_n. \]

To check that \( \psi \) is well-defined, it suffices by Frobenius reciprocity to check that the map \( u \otimes v \mapsto \phi_{u \boxtimes v} \) is an \( H_a \otimes H_b \)-map. Then one must show \( \psi \) is surjective, yielding it is an isomorphism by a dimension count. We leave these details to the reader. \( \square \)

**Proposition 7.** Let \( N = \text{cosoc} \text{Ind}^{n}_{a,b} A \boxtimes B \) where \( n = a + b \). Then \( N \simeq \text{soc} \text{Ind}^{n}_{a,b} B \boxtimes A \simeq \text{soc} \text{Ind}^{n}_{a,b} A \boxtimes B \).

**Proof.** It is well-known that there exists an involution \( D : \text{Rep}_q \rightarrow \text{Rep}_q \) that takes irreducibles to themselves and

\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \iff 0 \rightarrow D(C) \rightarrow D(B) \rightarrow D(A) \rightarrow 0. \]

This is an unpublished result of Bernstein that has since entered the literature. The functor also has the property that

\[ D(\text{Ind} M) = \hat{\text{Ind}} D(M). \]

In other words, there is an even stronger relationship between \( \text{Ind} \) and \( \hat{\text{Ind}} \): for \( M \) irreducible, the factors in the socle series of \( \text{Ind} M \) coincide with the factors in the cosocle series of \( \text{Ind} M \) in order. \( \square \)

We remark that where we use proposition in this paper, it is possible to use alternate arguments using characters, but they are far less elegant.

**Proposition 8.** Let \( P = (n_1, \ldots, n_k) \) and \( n = \sum n_i \). Let \( \gamma_i \) be a one-dimensional \( H_{n_i} \)-module, and so we can write \( \gamma_i \) for its character as well. (In section 4.3 we took \( n_i = 1 \).) If \( Q \) is any quotient of \( \text{Ind}^n_{P} \gamma_1 \boxtimes \cdots \boxtimes \gamma_k \), then \( \text{ch} Q \) contains the concatenation \( \gamma_1 \gamma_2 \cdots \gamma_k \). If \( L \) is any submodule of \( \text{Ind}^n_{P} \gamma_1 \boxtimes \cdots \boxtimes \gamma_k \) then \( \text{ch} L \) contains the term \( \gamma_k \cdots \gamma_2 \gamma_1 \).

**Proof.** Frobenius reciprocity gives \( \text{Hom}_{H_n} (\text{Ind}^n_{P} \gamma_1 \boxtimes \cdots \boxtimes \gamma_k, Q) = \text{Hom}_{H_P} (\gamma_1 \boxtimes \cdots \boxtimes \gamma_k, \text{Res}^n_{P} Q) \), from which the first statement is immediate once we restrict from \( H_P \) to \( S = H_{(1, \ldots, 1)} \).

Similarly, the second follows by proposition since \( \text{Hom}_{H_n} (L, \text{Ind}^n_{P} \gamma_1 \boxtimes \cdots \boxtimes \gamma_k) = \text{Hom}_{H_n} (L, \hat{\text{Ind}}^n_{P} \gamma_1 \boxtimes \cdots \boxtimes \gamma_2 \boxtimes \gamma_1) = \text{Hom}_{H_P} (\text{Res}^n_{P} L, \gamma_k \boxtimes \cdots \boxtimes \gamma_2 \boxtimes \gamma_1) \). \( \square \)

**Proposition 9.** Let \( M \) be an irreducible \( H_P \)-module. Suppose \( \gamma \) occurs as a term in \( \text{ch}(\text{Ind}^n_{P} M) \) with multiplicity \( m \) and also that \( \gamma \) occurs with multiplicity \( m \) in the character of any quotient of \( \text{Ind}^n_{P} M \). Then \( \text{cosoc} \text{Ind}^n_{P} M \) is irreducible and occurs with multiplicity one as a composition factor of \( \text{Ind}^n_{P} M \). If in addition \( \gamma \) occurs in \( \text{ch}(\text{soc} \text{Ind}^n_{P} M) \) then \( \text{Ind}^n_{P} M \) is irreducible.
Proof. The first statement follows since the map

\[ \text{Ind}M \to \text{cosoc Ind}M \]

is also a homomorphism of \( S = R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \)-modules. Restricting to \( S \), the \( \gamma \)-isotypic component (generalized eigenspace) of \( \text{Ind}M \) must surject to that of \( \text{cosoc Ind}M \) because taking generalized eigenspaces is exact. Hence \( \gamma \) can occur with multiplicity at most \( m \) in the cosocle. By hypothesis, each irreducible component of \( \text{cosoc Ind}M \) accounts for \( m \) copies of \( \gamma \), and so the cosocle must be irreducible. A similar counting argument shows it must occur with multiplicity one as a composition factor of \( \text{Ind}M \).

Further, since the cosocle accounts for all \( m \) copies of \( \gamma \), if \( \gamma \) also occurs in \( \text{ch}(\text{soc Ind}M) \), then \( \text{soc Ind}M \supseteq \text{cosoc Ind}M \). Hence, \( \text{cosoc Ind}M \) is a submodule and we can consider the quotient \( \text{Ind}M/\text{cosoc Ind}M \) whose character cannot have any copies of \( \gamma \) and so must be zero. Thus \( \text{Ind}M = \text{cosoc Ind}M \) is irreducible (by the first statement).

Remark 4. The same argument as above also shows that if \( \text{cosoc Ind}M \) is irreducible, occurs with multiplicity one as a composition factor of \( \text{Ind}M \), and also occurs in the socle, then \( \text{Ind}M \) is irreducible.

6. More theorems on multisegments

This section is devoted to recalling results from [BZ, Z] on multisegments. For convenience to the reader, we give complete proofs. We end by giving the significance of these results in the language of crystal graphs.

Recall if \( i \leq j \) then \( \Delta(i,j) = (q^i q^{i+1} \cdots q^j) \) is the one dimensional \( H_{j-i+1} \)-module on which all \( T_k - q \) and all \( X_k - q^{k+i-1} \) vanish.

Observe that \( e_j \Delta(i,j) = \tilde{e}_j \Delta(i,j) = \Delta(i,j-1) \), (with the convention \( \Delta(i,i-1) = 1 \)), \( e_i 1 = 0 \), and if \( k \neq j \) then \( e_k \Delta(i,j) = \tilde{e}_k \Delta(i,j) = 0 \).

6.1. Linking Rule.

Lemma 4 ([BZ, Z] The linking rule). (i) \( \text{Ind}\Delta(i,j) \boxtimes \Delta(k,l) \) is irreducible if \( j + 1 < k \).

(ii) \( \text{Ind}\Delta(i,l) \boxtimes \Delta(j,k) \) is irreducible if \( i \leq j \leq k \leq l \).

(iii) \( N = \text{cosoc Ind}\Delta(j,l) \boxtimes \Delta(i,k) \) is irreducible if \( i < j, k < l, j \leq k + 1 \), and the following

\[
0 \to \text{Ind}\Delta(j,k) \boxtimes \Delta(i,l) \to \text{Ind}\Delta(j,i) \boxtimes \Delta(i,k) \to N \to 0
\]

is exact. In this case, we say \( \Delta(i,k) \) and \( \Delta(j,l) \) are linked or are a linked pair.

If \( j = k + 1 \) then \( \Delta(i,k) \) and \( \Delta(j,i) \) are adjacent.

Proof. (i) This is the case that the intervals \([ij]\) and \([kl]\) are far apart, so that their union is not again an interval.
Let $M = \text{Ind}_{j-i+1,l-k+1}^n \triangle_{(i,j)} \boxtimes \triangle_{(k,l)}$ where $n = j + l - i - k + 2$. By the shuffle lemma,

\begin{equation}
\text{ch } M = \sum_{w \in \mathcal{W}_{\min}(j-i+1,l-k+1)} w \cdot (q^i \cdots q^j q^k \cdots q^l).
\end{equation}

Frobenius reciprocity implies that the term $q^i \cdots q^j q^k \cdots q^l$ must occur in any quotient of $M$, but by remark 3 all of the $(\begin{array}{c}n \\ j-i+1\end{array})$ terms in (7) must occur as well, yielding $M$ is irreducible.

(ii) This is the case that one interval $[jk] \subseteq [il]$ is contained in the other.

Let $n = l - i + k - j + 2$ and $L = \text{Ind}_{l-i+1,k-j+1}^n \triangle_{(i,l)} \boxtimes \triangle_{(j,k)}$ where $i \leq j \leq k \leq l$. By Frobenius reciprocity the term $q^i q^i+1 \cdots q^j q^j+1 \cdots q^k$ must occur in ch(cosoc $L$). The term $q^i \cdots q^i q^i+1 q^j+1 \cdots q^k q^k \cdots q^l$ must also occur in ch(cosoc $L$) by remark 3 along with the shuffle lemma, and Kato’s theorem (theorem 5.1) implies it occurs with multiplicity at least $2^{(k-j+1)}$. On the other hand, the shuffle lemma shows this is its multiplicity in ch $L$. Similar reasoning shows this term occurs in ch(soc $L$). By proposition 3, $L = \text{Ind} \triangle_{(i,l)} \boxtimes \triangle_{(j,k)}$ is irreducible.

(iii) This is the case that the union of the intervals $[jl] \cup [ik]$ is a longer interval (in $\mathbb{Z}$). Let $n = l - j + k - i + 2$ and $N = \text{cosoc Ind} \triangle_{(j,l)} \boxtimes \triangle_{(i,k)}$ where $i < j, k < l, j \leq k + 1$.

First we’ll consider the case that $j = k + 1$, that is $\triangle_{(j,l)}$ and $\triangle_{(i,k)}$ are an adjacent pair.
We want to show
\[ 0 \to \triangle_{(i,l)} \to \text{Ind} \triangle_{(k+1,l)} \boxtimes \triangle_{(i,k)} \to N \to 0. \] (8)
is exact and that \( N \) is irreducible. By Frobenius reciprocity, because \( \text{Res}_{k,i-1}^n \triangle_{(i,l)} = \triangle_{(i,k)} \boxtimes \triangle_{(k+1,l)} \) there is a nonzero map \( \triangle_{(i,l)} \to \text{Ind}_{k,i-1}^n \triangle_{(i,k)} \boxtimes \triangle_{(k+1,l)} \cong \text{Ind}_{k,k-i+1}^n \triangle_{(j,k)} \boxtimes \triangle_{(i,k)}. \) Further as \( q^i \cdots q^j \) occurs only once in \( \chi \text{Ind}_{k,i-1}^n \triangle_{(j,k)}, \) the socle of the induced module (in \( 8 \)) must be \( \triangle_{(i,l)}. \) (Note that Frobenius reciprocity (proposition 8) shows this one-dimensional submodule cannot be in the cosocle.) The quotient by this submodule has character \( \sum_{w \in W_{\min}^{[k-i+1,j-k+1]}} w. \)

Using (9), the exactness of induction we have an injection
\[ 0 \to \text{Ind} \triangle_{(j,k)} \boxtimes \triangle_{(i,l)} \to \text{Ind} \triangle_{(j,k)} \boxtimes \triangle_{(k+1,l)} \boxtimes \triangle_{(i,k)} \] (10)and from (8) a surjection
\[ \text{Ind} \triangle_{(j,k)} \boxtimes \triangle_{(k+1,l)} \boxtimes \triangle_{(i,k)} \to \text{Ind} \triangle_{(j,l)} \boxtimes \triangle_{(i,k)} \to 0. \] (11)
From the shuffle lemma and the fact \( i < j, k < l, \) the term \( q^i \cdots q^k q^i \cdots q^k \cdots q^l \) occurs in each of the characters of \( \text{Ind} \triangle_{(j,k)} \boxtimes \triangle_{(i,l)}, \text{Ind} \triangle_{(j,k)} \boxtimes \triangle_{(k+1,l)} \boxtimes \triangle_{(i,k)}, \)
and $\text{Ind}\Delta_{(j,l)} \boxtimes \Delta_{(i,k)}$ with multiplicity one. Because $\alpha$ is injective and $\beta$ is surjective this implies that $\beta \circ \alpha$ is nonzero on the $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ eigenvector with eigenvalue $(q^1, \ldots, q^i, q^k, \ldots, q^l)$. By part (ii) $\text{Ind}\Delta_{(j,k)} \boxtimes \Delta_{(i,l)} = \text{Ind}q^i \cdots q^k \boxtimes (q^1 \cdots q^i \cdots q^k \cdots q^l)$ is irreducible, and so $\beta \circ \alpha$ is an injection.

Observe that the term $q^1 \cdots q^i \cdots q^k$ occurs with multiplicity one in $\text{ch}(\text{Ind}\Delta_{(j,l)} \boxtimes \Delta_{(i,k)})$ so that its cosocle $N$ is irreducible by proposition 3. Furthermore $N$ cannot intersect $\text{Ind}\Delta_{(j,k)} \boxtimes \Delta_{(i,l)}$ because this term is not in its character. Thus we have $\text{Ind}\Delta_{(j,k)} \boxtimes \Delta_{(i,l)} \subseteq B$ where $B$ is the kernel of the map

$$(12) \quad 0 \to B \to \text{Ind}\Delta_{(j,l)} \boxtimes \Delta_{(i,k)} \to N \to 0.$$ 

We want to show this inclusion is an equality.

Let $A$ be any composition factor of $\text{Ind}\Delta_{(j,l)} \boxtimes \Delta_{(i,k)}$. We will show $\varepsilon_k(A) = 1$. First, the shuffle lemma implies $\varepsilon_k(A) \leq 1$. Let $w \cdot (q^i \cdots q^j q^1 \cdots q^l)$ be a term in $\text{ch}A$ for some $w \in W_{\text{min}}^{(l-j+1,k-i+1)}$. If $w(n) = n$ we are done.

If $w(n) \neq n - l + k$, then remark 3 implies the term $s_{w(n)} \cdots s_{w(1) + 1} s_{w(1)} w \cdot (q^i \cdots q^j q^1 \cdots q^l)$ also occurs in $\text{ch}A$ and this permutation does fix $n$. In other words, the irreducible module $\text{Ind}q^i \cdots q^j q^1 \cdots q^l \boxtimes (q^1 \cdots q^i \cdots q^k \cdots q^l) \boxtimes \Delta_{(j,k)}$ has $\varepsilon_k = 1$, and this module occurs in the restriction $\text{Res}_{w(n),n-w(n)}^{w(n),n-w(n)} \text{Res}_{n-w(n),n-w(n)}^n A$.

In the case $w(n) = n - l + k$, let $a$ be such that $w(n-l+k) = n-l+k+1+a$. Then $w \cdot (q^i \cdots q^j q^1 \cdots q^l) = (\cdots q^k \cdots q^i \cdots q^k-1 q^k q^{k+1} \cdots q^l)$. Repeated application of remark 3 allows us to “slide” the leftmost $q^k$ over so that $\cdots q^{k-1} q^k \cdots q^l q^i$ also occurs in $\text{ch}A$, yielding $\varepsilon_k(A) = 1$.

Now we induct on $k-j+1$, the case $k-j+1 = 0$ already completed for adjacent pairs above. Assume $k-j+1 > 0$. Starting with (12), we apply the exact functor $e_k$ yielding

$$(13) \quad 0 \to e_k B \to \text{Ind}\Delta_{(j,l)} \boxtimes \Delta_{(i,k-1)} \to e_k N \to 0.$$ 

The middle term is $e_k \text{Ind}\Delta_{(j,l)} \boxtimes \Delta_{(i,k)}$ by lemma 4. Because $k-j+1 > 0$, the pair $\Delta_{(j,l)}$, $\Delta_{(i,k-1)}$ is still linked, and by the inductive hypothesis we know its two composition factors. Because $\varepsilon_k(N) = 1$, proposition 3 implies $e_k N = e_k \Delta_{(i,k-1)}$ is irreducible, and by the inductive hypothesis it must be equal to $\text{cosoc} \text{Ind}\Delta_{(j,l)} \boxtimes \Delta_{(i,k-1)}$. From the inductive hypothesis and the exactness of (13), we must have that $e_k B = \text{Ind}\Delta_{(j,k-1)} \boxtimes \Delta_{(i,l)}$, which is irreducible by part (ii) of this lemma. If $B$ were reducible, then because all composition factors of $B$ have $\varepsilon_k \neq 0$, $e_k B$ would also be reducible. Hence we must have $B = \text{Ind}\Delta_{(j,k)} \Delta_{(i,l)}$. 

6.2. Classification. Recall a multiset of segments is a multiset. Theorem 2.4 below is the result from [BZ, Z] that multisegments parameterize the irreducible modules of $\text{Rep}_q H_n^{\text{aff}}$.

We also remind the reader of the two orderings introduced in section 2 and include examples.

right order: $\Delta_{(i_1,j_1)} > \Delta_{(i_2,j_2)}$ if $i_1 > i_2$ or if $i_1 = i_2$ and $j_2 > j_1$. The following picture shows the segments $\{\Delta_{(5,6)}, \Delta_{(5,7)}, \Delta_{(4,7)}, \Delta_{(3,3)}, \Delta_{(3,6)}, \Delta_{(1,7)}, \Delta_{(3,7)}, \Delta_{(2,6)}, \Delta_{(2,7)}, \Delta_{(2,9)}, \Delta_{(-1,7)}, \Delta_{(-1,3)}, \Delta_{(-2,3)}\}$ in right order, weakly decreasing from left to right.
left order: $\Delta_{(i_1,j_1)} > \Delta_{(i_2,j_2)}$ if $j_1 > j_2$ or if $j_1 = j_2$ and $i_2 > i_1$. The following picture shows the same segments as above in left order, weakly decreasing from left to right. In other words: $\{\Delta_{(2,9)}, \Delta_{(-1,7)}, \Delta_{(2,7)}, \Delta_{(3,7)}, \Delta_{(3,7)}, \Delta_{(4,7)}, \Delta_{(5,7)}, \Delta_{(2,6)}, \Delta_{(3,6)}, \Delta_{(3,6)}, \Delta_{(5,6)}, \Delta_{(3,3)}, \Delta_{(-2,2)}, \Delta_{(-1,1)}\}$. (Notice that if one rotates a picture of segments in right order by $180^\circ$, one gets a picture of different segments in left order. However, this observation is useful in seeing why theorems for $e_i$ also hold for $\hat{\bar{e}}_i$. In fact, if we think of the above as a picture of $\text{ch}\, \text{Ind}\Delta$, then the rotated picture (with arrows pointing down) would be of $\text{ch}(\text{rev}^*M)$.)

If we say $\Delta$ is in one of the orders above, we mean its segments are weakly decreasing from left to right.

Given a multisegment $\Delta$, let $\Delta$ denote the multisegment in right order, and $\bar{\Delta}$ denote it in left order. We use right order when computing $\hat{\bar{e}}_i$ and left order when computing $\tilde{e}_i$.

Given a multisegment and a choice of ordering of its segments $\Delta = \{\Delta_{(i_1,j_1)}, \ldots, \Delta_{(i_m,j_m)}\}$, recall that we let

$$\text{Ind}\Delta$$

denote $\text{Ind}^{n_{i_1,n_2,\ldots,n_m}}_{n_1,n_2,\ldots,n_m} \Delta_{(i_1,j_1)} \oplus \cdots \oplus \Delta_{(i_m,j_m)}$, where $n_k = j_k - i_k + 1$ and $n = \sum_i n_k$.

Also $n(\Delta) = n$ and $m(\Delta) = m$.

Observe that different orderings of a multisegment $\Delta$ can give non-isomorphic $\text{Ind}\Delta$. For instance, let $\Delta = \{\Delta_{(0,0)}, \Delta_{(1,1)}\}$. We know that $\text{Ind}\Delta = \text{Ind}^1_{1,1} 1 \oplus q \not\cong \text{Ind}^1_{1,1} q \oplus 1 = \text{Ind}\Delta$, although they do (always) have the same composition factors.

The corollary below says The modules $\text{Ind}\Delta$ are isomorphic to each other for the two orderings above.

**Corollary 6.1.** Let $\Delta$ be a multisegment. Then $\text{Ind}\Delta \simeq \text{Ind}\Delta$.
Proof. From $\Delta$ in right order, consider a segment $\Delta_{(i_0,j_0)}$ with largest left order, i.e. with largest $j$, and among those with smallest $i$. In terms of a picture, that means one (not necessarily unique) that reaches highest, and is longest among those. (In the example above, that would be the eleventh segment $\Delta_{(2,9)}$.) Because $j_0$ is largest, all segments $\Delta_{(i,j)}$ to the left of $\Delta_{(i_0,j_0)}$ have $j \leq j_0$; and because $\Delta$ is in right order, we also have $i \geq i_0$. In other words $i_0 \leq i \leq j \leq j_0$, so by part (ii) of the linking lemma $\text{Ind}\Delta_{(i,j)} \boxplus \Delta_{(i_0,j_0)} \simeq \text{Ind}\Delta_{(i_0,j_0)} \boxplus \Delta_{(i,j)}$. By transitivity of induction, and repeating this argument, the module $\text{Ind}\Delta$ does not change if we slide $\Delta_{(i_0,j_0)}$ all the way over to the left, where it belongs with respect to left order. (One can see $\Delta_{(2,9)}$ is indeed leftmost in the example of left order above.) We can repeat this process with the remaining segments, eventually yielding $\text{Ind}\Delta \simeq \text{Ind}\Delta$. \qed

Theorem 2.2 stated in section 3 shows that multisegments parameterize the irreducible modules in $\text{Rep}_q H_n$, $n \geq 0$. In other words, given a multisegment $\Delta$, the module $M_\Delta = \text{cosoc Ind}\Delta$ is irreducible, and as $\Delta$ ranges over all multisegments, $M_\Delta$ ranges over all irreducibles in $\text{Rep}_q$.

We will include the proof of theorem 2.2 here, starting with the following lemma.

Lemma 5. Suppose $\Delta = \{\Delta_{(i_1,j)}, \Delta_{(i_2,j)}, \ldots, \Delta_{(i_m,j)}\}$. Then $\text{Ind}\Delta$ is irreducible.

Proof. Let $\Delta' = \{\Delta_{(i_1,j-1)}, \Delta_{(i_2,j-1)} \}$ (entirely omitting segments for which $i_k = j$). By induction, we may assume $\text{Ind}\Delta'$ is irreducible. Observe that the base case of this lemma is given by Kato’s Theorem, which asserts $\text{Ind}q^l \boxtimes \cdots \boxtimes q^l$ is irreducible.

From the linking lemma, $\text{Ind}\Delta_{(i_1,j-1)} \boxtimes \Delta_{(i_2,j)} \boxtimes \Delta_{(j,j)} \simeq \text{Ind}\Delta_{(i_1,j-1)} \boxtimes \Delta_{(j,j)} \boxtimes \Delta_{(i_2,j)}$, which surjects to $\text{Ind}\Delta_{(i_1,j)} \boxtimes \Delta_{(i_2,j)}$. Iterating this argument, we get a surjection

$$\text{Ind}\Delta \boxtimes q^l \boxtimes q^l \boxtimes \cdots \boxtimes q^l \twoheadrightarrow \text{Ind}\Delta \twoheadrightarrow \text{cosoc Ind}\Delta.$$  

Because $\text{Ind}\Delta$ is irreducible by the inductive hypothesis, proposition 3 shows that $\text{Ind}\Delta \boxtimes q^l \boxtimes \cdots \boxtimes q^l$ has irreducible cosocle $M$. Therefore we must also have that $\text{cosoc Ind}\Delta = M$. Furthermore, $M$ occurs with multiplicity one and is the unique composition factor of $\text{Ind}\Delta \boxtimes q^l \boxtimes \cdots \boxtimes q^l$, and thus of $\text{Ind}\Delta$, which has $\varepsilon_j(M) = m$ (also by proposition 3). On the other hand, the linking lemma shows $\text{Ind}\Delta$ does not depend on the order of its segments, and so by proposition 3, $M = \text{soc Ind}\Delta$ as well. Then by the remark following proposition 3, $\text{Ind}\Delta = M$ is irreducible. \qed

Observe that for $\Delta$ as in the lemma above, $\varepsilon_j(\text{Ind}\Delta) = m$, and then by Corollary 2.2 and Lemma 1, $\varepsilon_j^n(\text{Ind}\Delta) = \text{Ind}\Delta$.

Now we complete the proof of theorem 2.2.

Proof. In light of corollary 2.3, we may prove the theorem for $M_\Delta = \text{cosoc Ind}\Delta$, which we will do so inducting on $n$ and making use of the functor $e_i$.

We will show $M_\Delta$ is irreducible. First, we can write $\Delta$ as a disjoint union of multisegments $\Delta = \bigcup_j \Delta^{(j)}$ where $\Delta^{(j)} = \{\Delta_{(i,k)} \in \Delta \mid k = j\}$. The lemma above showed $\text{Ind}\Delta^{(j)}$ is irreducible.

For $\Delta^{(j)} = \{\Delta_{(i_1,j)}, \ldots, \Delta_{(i_m,j)}\}$ with $i_1 \leq i_2 \leq \cdots \leq i_m$, define $\beta = \beta(\Delta^{(j)})$ to be the partition conjugate to $(j - i_1 + 1, j - i_2 + 1, \ldots, j - i_m + 1)$. 
Let \( Q(\Delta^{(j)}) \) be the term \( q^{i_1}q^{i_1+1}\cdots q^{i_2}q^{i_2+1}\cdots q^{i_3}q^{i_3+1}\cdots q^{i_t}q^{i_t+1} \) and observe that \( Q(\Delta^{(j)}) \) occurs with multiplicity \( \beta! = \beta_1!\beta_2!\cdots \beta_{j-i+1}! \) in \( \text{ch} \text{Ind}\Delta^{(j)} \). Also notice that the proof of the previous lemma, the observation made following its proof, and induction show

\[
(14) \quad 1 = M_\emptyset = e_{i_1}^{\beta_1-1+i_1+1}\cdots e_{i_j-1}^{\beta_j} e_j (\text{Ind}\Delta^{(j)}).
\]

Now return to \( \Delta = \Delta^{(j)} \cup \cdots \cup \Delta^{(j_t)} \) with \( j_1 > j_2 > \cdots > j_t \). Because \( j \) is strictly bigger than any exponent occurring in the segments of \( \Delta^{(j_0)} \) if \( j > j_0 \), from the shuffle lemma we see that the concatenation \( Q(\Delta^{(j_1)})\cdots Q(\Delta^{(j_t)}) \) occurs as a term in \( \text{ch} \text{Ind}\Delta \) with multiplicity \( \beta(\Delta^{(j_1)})!\cdots \beta(\Delta^{(j_t)})! \). This is its multiplicity in \( \Delta \) as well. The irreducibility of \( \text{Ind}\Delta^{(j_0)} \) and Frobenius reciprocity show this term occurs in any quotient of \( \text{Ind}\Delta \) with that same multiplicity. Therefore by proposition 5, \( M_\Delta = \text{cosoc} \text{Ind}\Delta \) must be irreducible (and furthermore occurs with multiplicity one in \( \text{Ind}\Delta \)).

For parts 2 and 3 of the theorem, we give only a brief sketch of the argument from [BZ2, 2]. We will see later that it also follows from the computation for \( e_i\text{M}_\Delta \).

Next, we will show irreducibles corresponding to distinct multisegments are distinct. Given an irreducible module \( M_\Delta \), let \( j \) be the smallest integer for which \( \varepsilon_j(M_\Delta) > 0 \). Write \( \varepsilon = \varepsilon_j(M_\Delta) \). Let \( N = e_j^\varepsilon \text{M}_\Delta \). From lemma 2 and the fact that we may assume \( \Delta \) is in left order, we must have \( \varepsilon = m(\Delta^{(j)}) = m \). Using lemma 3 and corollary 5.4, we must have that \( N = M_\Gamma \) where

\[
\Gamma = (\Delta^{(j)})^\circ \cup \bigcup_{k \neq j} \Delta^{(k)}.
\]

Notice all the \( k \) we union over are bigger than \( j \) by our choice of \( j \). Suppose \( M_\Delta \simeq M_{\Delta'} \). Then also \( N = e_j^\varepsilon \text{M}_{\Delta'} = M_\Gamma \) for \( \Gamma' \) obtained from \( \Delta' \) in the same way as above. By induction \( \Gamma = \Gamma' \). Lemma 1 gives us a well-defined and reversible way to go from \( \Delta \) to \( \Gamma \). By the properties of left order, if \( \Gamma \) contains \( k \) segments of the form \( \Delta_{i,j-1} \), each one must have come from a \( \Delta_{i,j} \) in \( \Delta \), and further \( \Delta \) must contain \( \varepsilon - k \) segments \( \Delta_{i,j} \). This holds as well for \( \Delta' \), so that \( M_\Delta \simeq M_{\Delta'} \Rightarrow \Delta = \Delta' \).

Finally, for part 3, we will show every irreducible module in \( \text{Rep}_q \) is some \( M_\Delta \).

Let \( M \) be any irreducible module in \( \text{Rep}_q \), and let \( N = e_j^\varepsilon_j(M) M \), where again \( j \) is smallest possible. By induction there is some multisegment \( \Gamma \) such that \( N = M_\Gamma \).

We construct a new multisegment \( \Delta \) from \( \Gamma \) by replacing each \( \Delta_{i,j-1} \in \Gamma \) by \( \Delta_{i,j} \), and also adding in \( \varepsilon_j(M) - \varepsilon_{j-1}(N) \) many segments \( \Delta_{i,j} \). This number is nonnegative because \( \varepsilon_{j-1}(M) = 0 \) by choice of \( j \) and proposition 3 shows that \( \varepsilon_{j-1}(M) + k \geq \varepsilon_{j-1}(e_j^k(M)) \). Then \( \text{Ind}\Delta \otimes q^1 \otimes \cdots \otimes q^1 \) has cosocle \( M \), but by an
argument similar to that in the proof of lemma \( \overline{4} \), it also surjects to \( \text{Ind}^\Delta \), which has cosocle \( M^\Delta \). Thus \( M = M^\Delta \).

**Corollary 6.2.** \( M^\Delta \) occurs with multiplicity one as a composition factor of \( \text{Ind}^\Delta \).

**Remark 5.** Because \( M^\Delta = \text{cosoc Ind}^\Delta \), if \( \triangle_{(i,j)} \in \Delta \) is smallest in left order, then \( j \) is the smallest integer such that \( \varepsilon_j(M^\Delta) \neq 0 \) and furthermore

\[
\varepsilon_j(M^\Delta) = | \{ \triangle_{(i,j)} \in \Delta \mid i \in \mathbb{Z} \} | = m(\Delta^{(j)}).
\]

For other \( j \) the cardinality of this set is merely an upper bound on \( \varepsilon_j(M^\Delta) \).

In the proof of parts 2 and 3 of the above theorem, we saw that lemma \( \overline{4} \) allowed us to compute the multisegment \( \Gamma \) for which \( e_j^{\varepsilon_j(M^\Delta)} M^\Delta = M^\Gamma \), where \( j \) was the smallest integer for which \( \varepsilon_j(M^\Delta) > 0 \). However, at this stage we have not proved that Rule \( \overline{4} \) describes \( e_i M^\Delta \) in terms of multisegments for \( i \neq j \), nor even \( e_k M^\Delta \) when \( k \neq \varepsilon_j(M^\Delta) \). This will be proved below. However having done so, it will be clear theorem \( \overline{3} \) also gives a proof of parts 2 and 3 of theorem \( \overline{2} \).

**Remark 6.** One could now appeal to Grojnowski’s Theorem 14.3: that the graph whose nodes are indexed by multisegments \( \Delta \) and whose edges are given by \( \Gamma \overset{\varepsilon}{\rightarrow} \Delta \) if

\[
\overline{e}_i M^\Delta = M^\Gamma
\]

is the crystal graph \( B(\infty) \).

Given a multisegment \( \Delta \), we can iterate the rule that we always take \( e_j^{\varepsilon_j} \) for the smallest \( j \) such that \( \varepsilon_j \neq 0 \). Combinatorially, we are just iterating the replacement of \( \Delta^{(j)} \) with \( (\Delta^{(j)})^{-} \), (if we wrote \( \Delta = \bigcup_k \Delta^{(k)} \)). This constructs a distinguished path on the crystal graph from \( \Delta \) back to the empty multisegment \( \emptyset \). Now that we have determined where on the crystal graph the node \( \Delta \) sits, the determination of a single edge

\[
? \overset{i}{\rightarrow} \Delta
\]

is purely combinatorial. And conversely, we can determine the distinguished path leading from \( ? \) back to \( \emptyset \), and thus re-express \( \overline{e}_i \Delta \) as a multisegment. This is exactly given by rule \( \overline{4} \).

Theorem \( \overline{3} \) and rule \( \overline{4} \) give the combinatoric that Grojnowski’s theorem dictates, but the proof here is module-theoretic. Admittedly, a proof appealing to the known crystal structure described in Theorem 14.3 of \( \overline{3} \) is much slicker.

7. Example of computing \( \overline{e}_i \)

Here we give an example of computing \( \overline{e}_i \Delta \).

Let \( \Delta = \{ \Delta_{(5,6)}, \Delta_{(5,7)}, \Delta_{(4,7)}, \Delta_{(3,3)}, \Delta_{(3,6)}, \Delta_{(3,7)}, \Delta_{(7,7)}, \Delta_{(2,6)}, \Delta_{(2,7)}, \Delta_{(2,9)}, \Delta_{(-1,7)}, \Delta_{(-1,1)}, \Delta_{(-2,2)} \} \) as in the example for right order in section \( \overline{6.2} \).
We will compute $\tilde{e}_7^2 \Delta$. First, we compute the word $+-+-++++++--$ and cancel as $+(-(-)+)-(-)+--$ leaving $+-+-+$ uncanceled. Thus $e_7(M_\Delta) = 3$ and $\tilde{e}_7^2 \Delta = 0$.

As an exercise, the reader can consider $\Delta$ and compute $\tilde{e}_{-1}(\Delta) = 2$, $\tilde{e}_2 = 2$, $\tilde{e}_3 = 4$, $\tilde{e}_4 = 1$, $\tilde{e}_5 = 2$.

8. Proof of theorem 3.1

Remark 7. In this section and the next, we will repeatedly use the fact that if $\text{cosoc Ind} A \boxtimes B$ is irreducible, then it coincides with $\text{cosoc(Ind(cosoc A) \boxtimes B)}$ and $\text{cosoc(Ind A \boxtimes (cosoc B))}$. 
Lemma 6 (sliding lemma). 1. Let $\Delta = \{\Delta(b,j), \Delta(a,j-1)\}$. Then if $a < b \leq c$

$$\text{Ind}M_\Delta \boxtimes \Delta(c,j) \simeq \text{Ind}\Delta(c,j) \boxtimes M_\Delta$$

is irreducible.

2. Let $\Gamma = \{\Delta(a,z_0), \Delta(a-1,z_1), \ldots, \Delta(a-k,z_k)\}$, where $z_0 > z_1 > \cdots > z_k \geq z$.

Then

$$\text{Ind}M_\Gamma \boxtimes \Delta(a-k,z) \simeq \text{Ind}\Delta(a-k,z) \boxtimes M_\Gamma$$

is irreducible.

Proof. Write $M = M_\Delta$. Let $N = \text{cosec} \text{Ind}\Delta(c,j) \boxtimes M$. Then $\Delta(c,j), \Delta(b,j), \Delta(a,j-1)$

are in order, so that by the remark \textcircled{3} above, $N = M_\Gamma\{\Delta(c,j), \Delta(b,j), \Delta(a,j-1)\}$.

First, we claim $\varepsilon_j M = 0$ and $\varepsilon_j N = 1$. We have an exact sequence

$$0 \to \text{Ind}\Delta(b,j-1) \boxtimes \Delta(a,j) \to \text{Ind}\Delta(b,j) \boxtimes \Delta(a,j-1) \to M \to 0$$

by the linking lemma. Applying the exact functor $e_j$ and using lemma \textcircled{3} yields

$$0 \to \text{Ind}\Delta(b,j-1) \boxtimes \Delta(a,j-1) \to \text{Ind}\Delta(b,j-1) \boxtimes \Delta(a,j-1) \to e_j M \to 0.$$

Because the first two terms are irreducible and isomorphic by the linking lemma, the last term must be zero and thus $\varepsilon_j M = 0$.

By the shuffle lemma $\varepsilon_j N \leq 1$. On the other hand, since $\text{Ind}\Delta(c,j) \boxtimes \Delta(b,j) = \text{Ind}\Delta(b,j) \boxtimes \Delta(c,j)$, we know by proposition \textcircled{3} that ch $N$ has a term $q^b q^{b+1} \cdots q^c q^{c+1} q^{c+1} \cdots q^j q^a \cdots q^j$ and therefore a term $q^{j+1} q^{j+1} q^{j+1} \cdots q^j$, so finally $q^b \cdots q^j q^j q^{j+1} q^j$ by remark \textcircled{3}. Thus $\varepsilon_j N \geq 1$.

Because $\varepsilon_j N = 1$, $e_j N = \bar{\varepsilon}_j N$ is irreducible (and nonzero) and we have by lemma \textcircled{4}

$$\text{Ind}\Delta(c,j-1) \boxtimes M \to e_j N = \bar{\varepsilon}_j N,$$

which also shows that $\bar{\varepsilon}_j N = M_\Gamma\{\Delta(c,j-1), \Delta(b,j), \Delta(a,j-1)\}$ since these segments are in order.

Let $L = \text{cosec} \text{Ind}M \boxtimes \Delta(c,j)$. Then as above we get

$$\text{Ind}M \boxtimes \Delta(c,j-1) \to e_j L = \bar{\varepsilon}_j L.$$

Now $\Delta(b,j), \Delta(a,j-1), \Delta(c,j-1)$ are in left order, so by theorem 2.2 and corollary 6.1, $\bar{\varepsilon}_j L \simeq M_\Gamma\{\Delta(b,j), \Delta(a,j-1), \Delta(c,j-1)\} \simeq M_\Gamma\{\Delta(c,j-1), \Delta(b,j), \Delta(a,j-1)\} \simeq \bar{\varepsilon}_j N$. (In particular, this shows $L$ was irreducible, since any summands of it must have $e_j = 1$ as well.)

Therefore, from proposition \textcircled{3} we know $L \simeq N$.

On the other hand, $\text{cosec} \text{Ind}\Delta(c,j) \boxtimes M = N \simeq L = \text{soc} \text{Ind}\Delta(c,j) \boxtimes M$ by proposition \textcircled{3}. However by corollary 6.2, $N$ occurs with multiplicity one in $\text{Ind}\Delta(c,j) \boxtimes \Delta(b,j) \boxtimes \Delta(a,j-1)$ and therefore in $\text{Ind}\Delta(c,j) \boxtimes M$. From remark following proposition \textcircled{3}, $\text{Ind}\Delta(c,j) \boxtimes M$ is irreducible, and thus also isomorphic to $\text{Ind}M \boxtimes \Delta(c,j)$.

This proves part 1. The proof of part 2 is quite similar, but we use $\bar{\varepsilon}_i$ instead of $e_i$. Notationally, it is slightly messier, but it is more useful in this form for use in future theorems.

Now renotate, letting $M = M_\Gamma$ and let $N = \text{cosec} \text{Ind}M_\Gamma \boxtimes \Delta(a-k,z)$. Write $\Gamma' = \{\Delta(a,z_0), \ldots, \Delta(a-k,z_k), \Delta(a-k,z)\}$. Then $\Gamma'$ is in left order, so that $N = M_{\Gamma'}$ is irreducible.

The same argument as in part 1 shows $\bar{\varepsilon}_{a-i}(M_{\Gamma'}) = 0$ if $i > 0$, and so $\bar{\varepsilon}_{a-k}(N) \leq 1$. An argument similar to that in part 1 also shows that $\text{Ind}M_\Gamma\{\Delta(a-k+1,z_{k-1}), \Delta(a-k,z_k)\} \boxtimes \Delta(a-k,z) \simeq \text{Ind}\Delta(a-k,z) \boxtimes M_\Gamma\{\Delta(a-k+1,z_{k-1}), \Delta(a-k,z_k)\}$. Together with remark \textcircled{3} and proposition \textcircled{3} this shows $\bar{\varepsilon}_{a-k}(N) = 1$. 


Let $L = \text{cosoc Ind}_\Delta(a_{-k}, z) \otimes M_T$. Observe that $\text{Ind}_M \otimes \Delta(a, z) = \text{Ind}_\Delta(a, z) \otimes M_T$ by corollary \ref{corollary1} if $a \leq z$. Let $b = \min\{a - 1, z\}$, so that $\tilde{e}_b \tilde{e}_{a + 1} \cdots \tilde{e}_{a - k} \Delta(a_{-k}, z) = \Delta(a, z)$, where we interpret $\Delta(a_{-}, z) = \Delta_0 = 1$ if $a > z$. (If $a > z$, then $b = z$ and interpreting $\Delta(a_{-}, z) = 1$, the observation still holds.) By repeated application of the exact functor $\tilde{e}_i$ and the analogue to lemma \ref{lemma6}, the cosocle of the above module is equal to each of the following: $\tilde{e}_b \cdots \tilde{e}_{a - k + 1} \Delta(a_{-k}, z) = \tilde{e}_b \cdots \tilde{e}_{a - k} L = \text{cosoc Ind}_\Delta(a_{-k}, z) \otimes M_T$.

Then by the sliding lemma, $a \in \mathcal{R}$ implies that $L \simeq N$. A repeat of the argument that concludes part 1 shows that $\text{Ind}_M \otimes \Delta(a_{-k}, z) \simeq \text{Ind}_\Delta(a_{-k}, z) \otimes M_T$ is irreducible.

The previous lemma is the final step toward computing $\tilde{e}_j M_\Delta$ where $\Delta = \Delta^{(j)} \cup \Delta^{(j-1)}$, after which we can compute $\tilde{e}_j M_\Delta$ for any $M_\Delta$. Then we shall show it coincides with rule \ref{rule5}.

**Lemma 7.** For $\Delta = \Delta^{(j)} \cup \Delta^{(j-1)}$,

$$\tilde{e}_j M_\Delta = M_{\varepsilon_j \Delta}$$

and $\tilde{e}_j M_\Delta = 0$ if $\tilde{e}_j \Delta = 0$. In particular, $\varepsilon_j(M_\Delta)$ is the number of $-$ signs left after all cancelling is done as in rule \ref{rule5}.

**Proof.** First we prove the second statement. We have $M_\Delta = \text{cosoc Ind}_\Delta = \text{cosoc Ind}_\Delta(a_{1, \pm}) \otimes \cdots \otimes \Delta(a_m, \pm)$, writing $-$ for $j$ and $+$ for $j - 1$. The $a_{1, \pm}$ weakly decrease, since we take $\Delta$ in right order. Recall from the proof of the sliding lemma \ref{lemma6} that $\varepsilon_j(M_{(\Delta_{a_{-1}, j}, \Delta_{a_{1, j}})}) = 0$ if $a_{j - 1} > a_1$, in other words, if the pair is linked. So, using remark \ref{remark6} whenever you see an adjacent $\Delta_{(a_{-1}, -)} \otimes \Delta_{(a_{1, +})}$, it does not contribute to $\varepsilon_j(M_\Delta)$. In other words, we can replace each $-+$ by $M_1 = M_{(\Delta_{a_{-1}, j}, \Delta_{a_{1, j}})}$, yielding $M_\Delta = \text{cosoc Ind}_\Delta(a_{1, \pm}) \otimes \cdots \otimes M_1 \otimes \cdots \otimes \Delta(a_m, \pm)$.

Repeat. Then by the sliding lemma, a $\Delta_{(a_{-k}, \cdot)}$ to the left of $M_1$ can slide past it, leaving the cosocle of the induced module unchanged.

Then we may repeat the first step, which in effect cancels any adjacent $-+$ pairs, and then by the sliding lemma continues to cancel newly created $-+$ pairs from the remaining symbols, and so on. In the end we are left with something that looks like

$$M_\Delta = \text{cosoc Ind}_\Delta(a_{1, +}) \cdots \otimes M_1 \cdots \otimes \Delta(a_{k, +}) \otimes \Delta(a_{k+1, -}) \otimes \cdots \otimes \Delta(a_m, -)$$

where $A$ is an induced module consisting solely of $\Delta(a_{1, +})$’s and $M_i$’s (which have the form $M_{(\Delta_{a_{-i}, \cdot}, \Delta_{a_{i, \cdot}})}), B$ is induced from $\Delta(a_{1, -})$’s. Thus $\varepsilon_j(A) = 0$ and $\varepsilon_j(B) = \varepsilon_j(M_\Delta)$ is the number of uncancelled $-$ signs remaining from the procedure. If there are no uncancelled $-$ signs left, then it is clear $\varepsilon_j(M_\Delta) = 0$, so that $\tilde{e}_j M_\Delta = 0 = M_0 = M_{\varepsilon_j \Delta}$. This proves the second statement. Notice that it was very important that $\Delta$ was in right order (and not left order) to start with.

However, we have also shown that if we alter right order by sliding segments from left to right past linked pairs, uncancelled $\pm$ symbols will be unchanged.

Now we can determine $\tilde{e}_j M_\Delta$. Let $\varepsilon = \varepsilon_j(M_\Delta)$. We know $M_\Delta = \text{cosoc Ind}_A \otimes \Delta(a_{1, j}) \otimes \cdots \otimes \Delta(a_{\varepsilon, j})$, and $\varepsilon_j(A) = 0$. Lemma \ref{lemma6} tells us that $\varepsilon_j(\text{Ind}_A \otimes \Delta(a_{1, j}) \otimes \cdots \otimes \Delta(a_{\varepsilon, j}))$ is filtered by modules whose successive quotients have the form

$$\text{Ind}_A \otimes \Delta(a_{1, j}) \otimes \cdots \otimes \Delta(a_{\varepsilon, j-1}) \otimes \cdots \otimes \Delta(a_{1, j}).$$


for \( s = 1, \ldots, \varepsilon \). Thus \( \bar{e}_j M_\Delta \) is in the cosocle of one of these. We will show below that each of these induced modules of the form \( \Delta(a_{i,j-1}) \) has irreducible cosocle and so the above procedure lets us compute \( \varepsilon_j \) of those cosocles.

We will show the module in \( \Delta(a_{i,j-1}) \) is a quotient of \( \text{Ind}\Delta' \), where \( \Delta' = \Delta \cup \{ \Delta(a_{i,j-1}) \} \) (in right order), and so its cosocle is \( M_\Delta' \), which is irreducible. Then we may apply the first part of the proof to compute \( \varepsilon_j(M_\Delta') \).

Starting with \( \text{Ind}\Delta' \), which we know has irreducible cosocle, we will apply the same moves we performed on \( \Delta \), i.e. ignore \(-+\) pairs by forming \( M_t \) to end with a module as in \( \Delta(a_{i,j-1}) \) as a quotient of \( \text{Ind}\Delta' \). We justify being able to perform these moves as follows. First, by the linking lemma, \( \text{Ind}\Delta' \) is the same whether \( \Delta' \) is in right order, or in the order inherited from \( \Delta \) in right order with \( \Delta(a_{i,j}) \) replaced by \( \Delta(a_{i,j-1}) \). So let us start with \( \Delta' \) in the latter order. Observe that when \( c < b \leq a \), two applications of the linking lemma yield \( \text{Ind}\delta_{(b,j)} \otimes \delta_{(c,j-1)} \otimes \delta_{(a,j-1)} \simeq \text{Ind}\delta_{(a,j-1)} \otimes \delta_{(b,j)} \otimes \delta_{(c,j-1)} \), the first of which is in left order and so has irreducible cosocle. Therefore by remark 3,

\[
\text{cosoc Ind}M\{\delta_{(b,j)}, \delta_{(c,j-1)}\} \otimes \delta_{(a,j-1)} \simeq \text{cosoc Ind}\delta_{(a,j-1)} \otimes M\{\delta_{(b,j)}, \delta_{(c,j-1)}\}.
\]

If we compare this to the sliding lemma (lemma 3), it says any time we slid \( \delta_{(a,j)} \) past an \( M_t \) in the above process for \( \Delta \) (cancelling a \(-+\)), we would also have been allowed to slide the segment past if it were replaced by \( \delta_{(a,j-1)} \). So, starting from \( \Delta' \) we form the same \( M_t \) and do the same slidings to end with the module in equation 1 as a quotient of \( \text{Ind}\Delta' \). Hence

\[
M_{\Delta'} = \text{cosoc Ind} \Delta \otimes \delta_{(a_{1,j-1})} \otimes \cdots \otimes \delta_{(a_{i-1,j-1})} \otimes \cdots \otimes \delta_{(a_{s,j-1})}.
\]

In particular, the cosocle is irreducible.

Now we may apply what we have already proved to compute \( \varepsilon_j(M_{\Delta'}) \leq \varepsilon - 2 \) if \( s \neq 1 \).

However, we know from proposition 3 that \( \varepsilon_j(\bar{e}_j M_\Delta) = \varepsilon - 1 \), so we must have \( \bar{e}_j M_\Delta \) occurring only as the cosocle of the induced module as in \( \Delta(a_{i,j-1}) \) with \( s = 1 \). In other words, it is the leftmost remaining (uncanceled) \(-+\) sign that signifies which segment to alter in computing \( \bar{e}_j \Delta \). This process matches rule 1, and so proves the lemma. (The only difference in \( \Delta \) and rule 1 is that we slide \(-+\) (i.e. linked) pairs from the right to the left of some \(-+\) signs. However it is the uncanceled symbols and the segments they correspond to that are used in both computations, identically.)

We will now use lemma 2 to complete the proof of theorem 3.1, that \( \bar{e}_j M_\Delta = M_{ \bar{e}_j \Delta } \).

**Proof.** From corollary 6.4, we know \( M_\Delta = \text{cosoc Ind} \Delta = \text{cosoc Ind} \Delta(k_1) \otimes \cdots \otimes \Delta(j) \otimes \Delta(k_2) \otimes \cdots \otimes \Delta(k_i) \). Let \( j = k_t, j-1 = k_{t+1} \). Let \( A = \text{cosoc Ind} \Delta(k_1) \otimes \cdots \otimes \Delta(k_{t-1}) \) and \( B = \text{cosoc Ind} \Delta(k_{t+2}) \otimes \cdots \otimes \Delta(k_i) \), so that by remark 3

\[
M_\Delta = \text{cosoc Ind} A \otimes \Delta(j) \otimes \Delta(j-1) \otimes B.
\]

Clearly \( \varepsilon_j(A) = 0 \) and \( \varepsilon_j(B) = 0 \) by the shuffle lemma. Further, all terms in \( \text{ch} B \) have support consisting of \( q^k \) where \( k < j - 1 \). Then remark 3 implies \( \text{Ind} q^j \otimes B \simeq \text{Ind} B \otimes q^j \) (which furthermore is irreducible). In other words, \( \varepsilon_j(M_\Delta) = \varepsilon_j(\text{cosoc Ind} \Delta(j) \otimes \Delta(j-1)) \) and we have reduced theorem 3.1 to lemma 2, where the same argument holds. To compute \( \bar{e}_j \) we now consider \( \Delta(j) \cup \Delta(j-1) \) in right order, and indeed this coincides with computing the \(-+\) rule on all of \( \Delta \) in right order.
order because the other segments in $\Delta^{(k)}$ only contribute blanks. Using remark 3 and the fact from the proof of lemma 2 that $\tilde{e}_j M_{\Delta^{(j)} \cup \Delta^{(j-1)}}$ is the only composition factor of $e_j M_{\Delta^{(j)} \cup \Delta^{(j-1)}}$ with $\varepsilon_j = \varepsilon_j(M_\Delta) - 1$, we get

$$\tilde{e}_j M_\Delta = \cosoc \text{Ind} A \boxtimes (\tilde{e}_j M_{\Delta^{(j)} \cup \Delta^{(j-1)}}) \boxtimes B = \cosoc \text{Ind}(\tilde{e}_j \Delta) = M_{\delta_j \Delta}.$$ 

The proof of theorem 3.2, that $\tilde{e}_i M_\Delta = M_{\delta_i \Delta}$, is analogous to the proof of theorem 3.3, and so will not be included.

As a result, we also get corollary 3.3, which uses rule 2 to determine which $M_\Delta$ are in $\text{Rep} H_n^\lambda$. In the next section we will study the case $\lambda = \Lambda_i$.

9. Multipartitions

In this section, we will combine remark 3 with lemma 6 to build up $M_\Delta$ from irreducible $H_n^\lambda$-modules.

We provide a dictionary between multisegments and certain colored multipartitions, “Kleshchev multipartitions”. The reverse map is obvious, but the forward one is more subtle. Only certain colored multipartitions can arise. In section 10 we explain those colored multipartitions correspond to nodes in a connected component of a tensor product of level 1 crystals.

We can use theorem 3.2 to identify for which $\Delta$ is $M_\Delta \in \text{Rep} H_n^\lambda$.

**Theorem 9.1.** The $\Delta$ such that $M_\Delta \in \text{Rep} H_n^\lambda$ are of the form $\Delta = \{\Delta_{(i,b_1)}, \Delta_{(i-1,b_2)}, \ldots, \Delta_{(i-k+1,b_k)}\}$, where $b_1 > b_2 > \cdots > b_k$, or $\Delta = \emptyset$.

In other words, we can associate to $\Delta$ the $i$-colored partition $\mu(\Delta) = (b_1 - i + 1, b_2 - i + 2, \ldots, b_k - i + k)$, pictured as

$$\begin{array}{ccccccc}
  & & & & i & & b_1 \\
  & & & i+1 & & & \\
  & & i-1 & & i & & b_2 \\
  & & & & & & \\
  & & & & & & \\
  & & & & & & \\
  i-k+i & & & & & & b_k \\
\end{array}$$

In this case, $\Delta(\mu, i) = \Delta$.

**Proof.** To be in $\text{Rep} H_n^\lambda$, means that $\tilde{e}_i(M_\Delta) \leq 1$ and $\tilde{e}_j(M_\Delta) = 0$ if $j \neq i$. Rule 2 for computing $\tilde{e}_j(M_\Delta)$ shows that if the segment $\Delta_{(j,b)}$ occurs in $\Delta$ and $j \neq i$, then it must be preceded in left order by $\Delta_{(j+1,b')}$, which forces $b < b'$. Further, if $\Delta \neq \emptyset$ then some segment $\Delta_{(i,b_1)}$ must occur. Only the $\Delta$ listed above satisfy these requirements. 

It is no surprise that the multisegments above correspond to partitions. We note that for generic $q$, $H_n^\lambda \simeq RS_n$, the group algebra of the symmetric group. The irreducible modules of $S_n$ are parameterized by partitions. The irreducible module parameterized by $\mu$ is called the Specht module $S^\mu$ and it is isomorphic to $M_{\Delta(\mu, 0)}$. (The modules for $H_n^\lambda$ differ only in that $X_1$ acts by the scalar $q'$.)

Recall we had defined $\Delta(\mu, i) = \{\Delta_{(i,i+\mu_1-1)}, \ldots, \Delta_{(i-k+1,i-k+\mu_k)}\}$, where $k$ is the length of $\mu$. Hence, $\mu(\Delta(\mu, i)) = \mu$ undoes this operation. Theorem 3.4 explains how to associate a multipartition to an arbitrary multisegment.

Also recall that $N_{\mu, i} = M_{\Delta(\mu, i)}$, with the convention that $N_{0, i} = 0$. 

Lemma 8. 1. For all \( j \), \( \varepsilon_j(N_{\mu,i}) \leq 1 \).
2. \( \varepsilon_jN_{\mu,i} = N_{\varepsilon_j(\mu,\Lambda_i)} \), where, as in Rule 3, \( \varepsilon_j(\mu,\Lambda_i) \) denotes the removal of a removable \( j \) box from the diagram of \( \mu \) when colored by \( i \), or denotes 0 if no such removable \( j \) box exists.

\[ \text{Proof.} \]
In the multisegment \( \Delta = \Delta(\mu,i) = \{ \triangle(i,b_1), \ldots, \triangle(i-k+1,b_k) \} \) with \( b_1 > \cdots > b_k \), observe each \( b_j \) is distinct. Thus in our rule for computing \( \varepsilon_j(M_{\Delta}) \), there is at most one \( - \) symbol. This shows part 1 (using theorem 8).

In fact, if \( \varepsilon_j(N_{\mu,i}) = 1 \), it means we must see \(-\) but not \(-+\) in calculating as in rule 3. This corresponds to the \( i\)-colored diagram for \( \mu \) having a removable \( j\)-box. The \(-\) means some row of \( \mu \) ends in a \( j\)-box. But the absence of \(+\) means the rows below it must not end in a \((j-1)\)-box. That means the \( j\)-box has no box directly below it (else it would be filled with \( j-1 \)). Then it is clearly removable, and \( \Delta(\mu\backslash j,i) = \varepsilon_j\Delta \). The converse also holds, since if \( \varepsilon_j \) is removable, the row beneath it cannot end in \( j-1 \) but nor can any of the rows beneath it, since their fillings strictly decrease. Thus \( \varepsilon_jN_{\mu,i} = \varepsilon_jM_{\Delta} = M_{\Delta(\mu\backslash j,i)} = N_{\varepsilon_j(\mu,\Lambda_i)} \).

\[ \square \]

Theorem 9.2. 1. Let \( M_{\Delta} \) be an \( H_n^{\Lambda_i+\Lambda_j} \)-module. Suppose \( \mu \geq h \). Then there exist partitions \( \mu \) and \( \nu \) such that \( M_{\Delta} = \cosoc IndN_{\mu,i} \boxtimes N_{\nu,h} \).

Further, \( \Delta = \Delta(\mu,i) \cup \Delta(\nu,h) \) and \( b_{i-h+x} \leq c_x \), where \( \Delta(\mu,i) = \{ \triangle(i,b_1), \ldots, \triangle(i-m+1,b_m) \} \) and \( \Delta(\nu,h) = \{ \triangle(h,c_1), \ldots, \triangle(h-k+1,c_k) \} \).

2. Given \( \mu, \nu \) such that \( \mu_{i-h+x} \leq \nu_x \) for \( x \geq 1 \), \( \cosoc IndN_{\mu,i} \boxtimes N_{\nu,h} \) is irreducible and equal to \( M_{\Delta} \) where \( \Delta = \Delta(\mu,i) \cup \Delta(\nu,h) \).

\[ \text{Proof.} \]
Similar to lemma 8, \( \Delta \) must consist of two consecutive (with respect to the first “coordinate” of a segment) decreasing subsequences. Applying the sliding lemma will let us sort the segments comprising the subsequences into two pieces corresponding to \( \mu \) and \( \nu \).

Write \( \Delta = \{ \triangle(a_1,b_1), \ldots, \triangle(a_t,b_t) \} \) in right order. We must have \( \tilde{\varepsilon}_k(M_{\Delta}) = 0 \) if \( k \neq i, h, \) \( \tilde{\varepsilon}_i(M_{\Delta}) \leq 1 \) and \( \tilde{\varepsilon}_h(M_{\Delta}) \leq 1 \) (or \( \tilde{\varepsilon}_i(M_{\Delta}) \leq 2 \) if \( i = h \)). Theorem 8 shows that \( \{ a_1, \ldots, a_t \} \) is the union of two intervals \( \{ i, i-1, \ldots, i-s+1 \} \) and \( \{ h, h-1, \ldots, h-t+s+1 \} \) where their corresponding \( b \)‘s are also strictly decreasing.

If these intervals are disjoint, then the concatenation of the intervals is in right order and it is clear
\[
M_{\Delta} = \cosoc Ind \Delta
= \cosoc Ind \triangle(i,b_1) \boxtimes \triangle(i-1,b_2) \boxtimes \cdots \boxtimes \triangle(i-s+1,b_s) \boxtimes \triangle(h,b_{s+1}) \boxtimes \cdots \boxtimes \triangle(h-t+s+1,b_t)
= \cosoc Ind_{\Lambda,\nu} \triangle(i,b_1) \boxtimes \cdots \boxtimes \triangle(i-s+1,b_s) \boxtimes \cosoc Ind_{\Lambda,h} \triangle(h,b_{s+1}) \boxtimes \cdots \boxtimes \triangle(h-t+s+1,b_t)
= \cosoc Ind_{\Lambda,\nu} N_{\mu,i} \boxtimes N_{\nu,h},
\]
where \( \mu = (b_1-i+1, \ldots, b_s-i+s) \) and \( \nu = (b_{s+1}-h+1, \ldots, b_t-h+t-s) \). By construction \( \Delta = \Delta(\mu,i) \cup \Delta(\nu,h) \).

However, there may be overlap between the intervals. In that case, we partition \( \Delta = \Delta_1 \cup \Delta_2 \) as follows. If \( a_m \in \{ i, \ldots, i-s+1 \} \setminus \{ h, \ldots, h-t+s+1 \} \) then put \( \triangle(a_m,b_m) \) in \( \Delta_1 \). If \( a_m \in \{ h, \ldots, h-t+s+1 \} \setminus \{ i, \ldots, i-s+1 \} \) then put \( \triangle(a_m,b_m) \) in \( \Delta_2 \). For \( a \in \{ i, \ldots, i-s+1 \} \cap \{ h, \ldots, h-t+s+1 \} \), there exist \( c \geq b \) such that both of \( \triangle(a,b) \) and \( \triangle(a,c) \) are in the multisegment \( \Delta \). Put the shorter \( \triangle(a,b) \) in
\( \Delta_1 \) and the longer \( \triangle_{(a,c),i} \) in \( \Delta_2 \). Then \( \mu = \mu(\Delta_1) \) and \( \nu = \mu(\Delta_2) \) will be partitions and \( \Delta_1 = \Delta(\mu, i) \), \( \Delta_2 = \Delta(\nu, h) \).

Define \( \Gamma_1^2 = \{ \triangle_{(a,b),i} \in \Delta_1 \mid \alpha > a \} \) and \( \Gamma_2^2 = \{ \triangle_{(a,b),i} \in \Delta_2 \mid \alpha > a \} \). Suppose \( b \leq c \). Then it follows from part 2 of the sliding lemma \( \square \) that

\[
\cosoc \text{Ind} M_{\Gamma_1^2} \boxtimes M_{\Gamma_2^2} \boxtimes \triangle_{(a,b)} \triangle_{(a,c)} \triangle_{(a,b)} \triangle_{(a,c)}
= \cosoc \text{Ind} M_{\Gamma_1^2} \boxtimes \triangle_{(a,b)} \boxtimes M_{\Gamma_2^2} \boxtimes \{ \triangle_{(a,c)} \}
= \cosoc \text{Ind} M_{\Gamma_1^2 \cup \{ \triangle_{(a,c)} \}} \boxtimes M_{\Gamma_2^2 \cup \{ \triangle_{(a,c)} \}} \cosoc \text{Ind} M_{\Gamma_{1}^{2} - 1} \boxtimes M_{\Gamma_{2}^{2} - 1}.
\]

If \( a \) is such that both \( \triangle_{(a,b)}, \triangle_{(a,c)} \in \Delta \), then \( \Gamma_{1}^{a-1} = \Gamma_{1}^{2} \cup \{ \triangle_{(a,b)} \} \) and \( \Gamma_{2}^{a-1} = \Gamma_{2}^{2} \cup \{ \triangle_{(a,c)} \} \). Continuing as above, starting with \( \Delta \) in right order, yields \( M_{\Delta} = \cosoc \text{Ind} M_{\Delta_1} \boxtimes M_{\Delta_2} = \cosoc \text{Ind} N_{\mu,i} \boxtimes N_{\nu,h} \).

Conversely, given any \( \mu, \nu \) with the property that \( \mu_{x} = \nu_{x} \) for all \( x \geq a \) (with the convention \( \mu_{x} = 0 \) if \( x > \ell(\mu) \)), the above argument shows \( \cosoc \text{Ind} N_{\mu,i} \boxtimes N_{\nu,h} \) is irreducible and equal to \( M_{\Delta} \) for \( \Delta = \Delta(\mu, i) \cup \Delta(\nu, h) \).

Observe that the condition that \( \mu \) and \( \nu \) satisfy are equivalent to the multipartition \( (\mu, \nu) \) colored by \( \Lambda_{i}, + \Lambda_{h} \) being a Kleshchev multipartition.

Observe that \( n(\Delta) = |\mu| + |\nu| \). Also it is possible that \( \mu \) or \( \nu \) be the empty partition.

Theorem \( \square \) extends the same argument to \( \lambda \) of any level. We repeat its statement here for convenience.

**Theorem** (theorem \( \square \)). Let \( M_{\Delta} \) be an \( H_{\Lambda}^{\mu} \)-module where \( \lambda = \Lambda_{i_{1}} + \Lambda_{i_{2}} + \cdots + \Lambda_{i_{r}} \), \( i_{1} \geq \cdots \geq i_{r} \). Then there exists a Kleshchev multipartition \( \triangle \) such that \( M_{\Delta} = N_{\mu,\lambda} \) and \( \Delta = \Delta(\mu, \lambda) \).

Conversely, if \( \mu \) is a Kleshchev multipartition, then \( N_{\mu,\lambda} = \cosoc \text{Ind} N_{\mu^{(1)},i_{1}} \boxtimes N_{\mu^{(r)},i_{r}} \) is irreducible and isomorphic to \( M_{\Delta} \) for \( \Delta = \Delta(\mu, \lambda) \).

If \( \mu, \lambda \) is not a Kleshchev multipartition, then \( \cosoc \text{Ind} N_{\mu^{(1)},i_{1}} \boxtimes \cdots \boxtimes N_{\mu^{(r)},i_{r}} \) need not be irreducible. Furthermore, Theorem \( \square \) (repeated below) need not hold for arbitrary colored multipartitions.

The reader may recognize the condition

\[
\mu_{i_{x}}^{(1)} \leq \mu_{x}^{(1)}
\]

as the condition that the nodes in the connected component of highest weight \( \lambda \) satisfy in the tensor of crystal graphs \( B(\Lambda_{i_{x}}) \otimes \cdots B(\Lambda_{i_{1}}) \). See section \( \square \) and corollary \( \square \) for a description of these theorems in the language of crystal graphs.

We will now justifiy that the \( \pm \) rule \( \square \) given for \( \overline{e_{1}} \Delta \) is compatible with the one for multipartitions.

**Theorem** (theorem \( \square \)). Given a multisegment \( \Delta \) and a Kleshchev multipartition \( \mu, \lambda \) such that \( \Delta = \Delta(\mu, \lambda) \), then \( \overline{e_{1}} \Delta = \Delta(\overline{e_{1}} \mu, \lambda) \). In other words, \( \overline{e_{1}} N_{\mu,\lambda} = N_{\mu,\lambda} \).

**Proof.** When we compute \( \overline{e_{1}} \Delta \), we require that \( \Delta \) be in right order. However, in constructing \( \mu, \lambda \) in theorem \( \square \) that order is changed by repeated application of the sliding lemma.

We’ll consider all possible re-orderings that occur and show they do not change the canceled + and – symbols. There are 3 cases.

...
Case 1. In $\Delta$ we see $\Delta_{(a,j-1)}$ followed by $\Delta_{(a,j)}$. Then these two segments remain in that relative order. (Recall in partitioning $\Delta$ into two pieces that our rule would put $\Delta_{(a,j-1)}$ in $\Delta_1$ and $\Delta_{(a,j)}$ in $\Delta_2$.)

Case 2. In $\Delta$ we see $\Delta_{(b,j-1)}$ followed by $\Delta_{(a,j)}$ and $b > a$. Again, these two segments remain in that relative order.

Case 3. In $\Delta$ we see $\Delta_{(b,j)}$ followed by $\Delta_{(a,j-1)}$ and $b > a$. If these two segments do switch position, it would be by applying the sliding lemma—that $\text{Ind} M_\Gamma \boxtimes \Delta_{(a,j-1)} \simeq \text{Ind} \Delta_{(a,j-1)} \boxtimes M_\Gamma$ where $\Gamma \ni \Delta_{(b,j)}$ and the last segment in $\Gamma$ is $\Delta_{(a,c)}$ for some $j > c \geq j-1$. Thus, $c = j - 1$. Before sliding the pattern of symbols is $- + +$, whereas afterwards it is $+ - +$. However, all adjacent $- +$ get cancelled in either rule for $\tilde{e}_j$ and so we have not changed the uncanceled symbols (nor the segments they are attached to) at all; both configurations contribute just $+ a$ attached to the $\Delta_{(a,j-1)}$.

Applying the $\pm$ rule to compute $\tilde{e}_jM_\Delta = M_{\delta_j \Delta}$ is thus equivalent to that of computing $\tilde{e}_jN_{\omega,\lambda} = N_{\delta_{\mu} \omega,\lambda}$.

Above we have built up the partitions “row by row”, with each row corresponding to a segment. We could have chosen to build them “column by column” which would have meant using Steinberg modules in place of trivial modules. These are also one dimensional, but have character $(q^i q^r \cdots q^1)$, and all $T_k + 1$ vanish on them. In that case we would have a similar $+ -$ rule that would correspond to looking for addable/removable boxes at the ends of columns (not rows).

10. Discussion: Crystals

Now we can put the theorems of section 3 into the language of Kashiwara’s crystal graphs. We refer the reader to [3] for the definitions and properties of crystal graphs. A good reference in this context is [2], where he also includes when $q$ is a root of unity, in which case one must be more careful.

Fix $\lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_r}$ with $i_1 \leq i_2 \leq \cdots \leq i_r$, a weight of the Lie algebra $\mathfrak{gl}_\infty$. Let $L(\lambda)$ denote the irreducible integrable representation of $\mathfrak{gl}_\infty$ with highest weight $\lambda$ and let $B(\lambda)$ denote its crystal graph.

Let $B(\infty)$ denote the crystal graph associated to $U(\eta^-)$. We can think of each node of $B(\infty)$ as labeled by a multisegment $\Delta$. (Observe another interpretation of a multisegment is as a sum of positive roots of $\mathfrak{gl}_\infty$. A segment $\Delta_{(i,j)}$ corresponds to the positive root $\alpha_{ij} = \alpha_i + \cdots + \alpha_j$, and the multisegment $\Delta$ corresponds to their sum.)

The following two theorems are essentially Grojnowski’s Theorem 14.3.

**Theorem 10.1.** [3] Let $B_\lambda$ be the graph whose nodes are the irreducible $H_n^{\lambda}$-modules for $n \geq 0$ with edges given by $\tilde{e}_i M \xrightarrow{i} M$ if $\tilde{e}_i M \neq 0$. Then $B_\lambda = B(\lambda)$.

**Theorem 10.2.** Let $B_{\text{aff}}$ be the graph whose nodes are the irreducible modules in $\text{Rep}_q$ with edges given by $\tilde{e}_i M \xrightarrow{i} M$ if $\tilde{e}_i M \neq 0$. Then $B_{\text{aff}} = B(\infty)$.

In this language, Theorems 3.1, 3.4, and 3.5 become

**Corollary 10.3.** $B_{\text{BZ}} = B_{\text{aff}}$ with crystal isomorphism $\Delta \mapsto M_\Delta$.

$B'_\lambda = B_\lambda$ with crystal isomorphism $\mu, \lambda \mapsto N_{\mu, \lambda}$.

Furthermore, $\mu, \lambda \mapsto \Delta(\mu, \lambda)$ coincides with the inclusion of $B_\lambda$ into $B_{\text{aff}}$. 


We can view $B_{\lambda}$ as a subgraph of $B_{\text{aff}}$ (or $B(\lambda) \subseteq B(\infty)$). Indeed the irreducible $H_n^\lambda$-modules are just a subset of the irreducibles of $\text{Rep}_q H_n$. Rule 2 for $\tilde{e}_i(M_\Delta)$ tells us exactly which subgraph $B_{\lambda}$ is, i.e. which multisegments comprise it, by requiring $\tilde{e}_i(M_\Delta)$ be less than or equal to the multiplicity of $\Lambda_i$ in $\lambda$.

On the other hand, $B(\lambda) \rightarrow B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_1})$. The nodes in the graph $B(\Lambda_i)$ correspond to all partitions $\mu$, colored by $i$. (In the case $i = 0$, $L(\Lambda_0)$ is the basic representation and the crystal graph $B(\Lambda_0)$ can be identified with Young’s lattice of partitions. For other $i \in \mathbb{Z}$ simply shift the edge labels by +$i$.) Thus, in the tensor product $B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_1})$, it is natural to label nodes by colored multipartitions $\mu, \lambda$. The nodes run over all $\lambda$-colored multipartitions, but this graph is not connected.

The connected component in $B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_1})$ of $\emptyset \otimes \emptyset \otimes \cdots \otimes \emptyset$, the unique node with weight $\lambda$, defines $B(\lambda)$ as a subgraph. These realizations of $B(\lambda)$ as a subgraph of two different crystals gives us a map from multisegments (those with $\tilde{e}_i$ bounded by $\lambda$) to colored multipartitions $\Delta \mapsto \mu^*(\Delta)$ which respects the action of $\tilde{e}_i$.

As $r$ goes to $\infty$ the domain ranges over all multisegments. However we never get all colored multipartitions in the image. Theorem 3.4 explains which ones we do get, and corollary 3.7 tells us how to interpret those in terms of a tensor product of crystal graphs.

There is a small technical point here which we must address. The map above is the “reverse” of the map described in theorem 3.4. One can simply think of $(\mu, \lambda)^*$ as $\mu, \lambda$ read from right to left. The conventional definition for tensoring crystals and the conventional definition for partitions requires that we introduce a reversal in order to be consistent. We choose to modify the convention for tensoring crystals instead of the convention of reading a partition from left to right. (One way we could have changed the definitions here to modify instead the convention for partitions would be if we had taken segments in increasing order and instead of taking cosoc $\text{Ind}$ everywhere we had taken soc $\text{Ind}$ (or cosoc $\hat{\text{Ind}}$), by proposition 3.) See remark 8 for further comments.)

This reversal motivates the following definition, which reverses the order of tensoring crystal graphs.

Let $B_1 \otimes^* B_2 = B_2 \otimes B_1$, and so $\tilde{e}_i$ acts via

$$
\tilde{e}_i(b_1 \otimes^* b_2) = \begin{cases} 
\tilde{e}_ib_1 \otimes^* b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2) \\
 b_1 \otimes^* \tilde{e}_ib_2 & \text{if } \varepsilon_i(b_1) \leq \varphi_i(b_2)
\end{cases}.
$$

(17)

The usual convention for the action of $\tilde{e}_i$ on $B_1 \otimes B_2$ is:

$$
\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{e}_ib_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\
 b_1 \otimes \tilde{e}_ib_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2)
\end{cases}.
$$

(18)

Either way is compatible with the following picture (in which all edges $\rightarrow$ are assumed labelled by $i$).
This is usually a picture of $B_2 \otimes B_1$, whereas here we think of it as $B_1 \otimes^* B_2$. And so the solid dot is $b_1 \otimes^* b_2 \in B_1 \otimes^* B_2$, or $b_2 \otimes b_1 \in B_2 \otimes B_1$, and the rule above tells you if an arrow leading to it approaches from the top or the left.

In general, the crystals $B_1 \otimes^* B_2$ and $B_1 \otimes B_2$ are not isomorphic; but in this setting, since the crystal graphs correspond to integrable highest weight representations of $gl_{\infty}$, they are isomorphic. Reversing the order here is preferable because then the node labelled by $N_{\mu_\lambda}$ in $B_\lambda$ will correspond to the node $\mu^{(1)} \otimes^* \mu^{(2)} \otimes^* \cdots \otimes^* \mu^{(r)}$ in $B(\Lambda_1) \otimes^* B(\Lambda_2) \otimes^* \cdots \otimes^* B(\Lambda_r)$ (read now from left to right). Those readers more familiar with partitions may prefer this order, while those more familiar with crystal graphs may find it annoying.

Compare the definition for $B_1 \otimes^* B_2$ to Theorems 3.1 and 3.3, which imply that for very special $A$ and $B$,

$$\bar{c}_i(\cosoc \text{Ind}_A \boxtimes B) = \begin{cases} \cosoc(\text{Ind}_A \boxtimes B) & \text{if } \epsilon_i(A) > \varphi_i(B) \\ \cosoc(\text{Ind}_A \boxtimes \bar{c}_i B) & \text{if } \epsilon_i(A) \leq \varphi_i(B) \end{cases}$$

and also that the cosocles above are irreducible.

We will now explain (19) in more detail and in the process prove corollary 3.7.

First we will define $\varphi_i$. We refer the reader to [3] to learn more about $\varphi_i$, which is more subtle than $\epsilon_i$. It depends on the crystal operator $\bar{f}_i$, mentioned only briefly here and used to simplify the proof of corollary 10.4 below.

The crystal operator $\bar{f}_i$ satisfies

$$\bar{c}_i M = N \iff M = \bar{f}_i N.$$ 

Module-theoretically, $\bar{f}_i N = \cosoc \text{Ind}_{n+1,1}^n N \boxtimes q^i = \cosoc \text{Ind} N \boxtimes \mathcal{D}_{(i,i)}(1)$. This gives a recipe of how to construct an irreducible $H_n$-module $M$ using the crystal graph $B_{\text{aff}}$, by following any path on the crystal from the node corresponding to $M$ back to the root, one step at a time. In other words, given $1 = \bar{c}_{j_n} \cdots \bar{c}_{j_1} M$, we have $M = \bar{f}_{j_n} \cdots \bar{f}_{j_1} \cdot 1$. As in remark 3.4, 3.5 already gave a distinguished such path. For more on the properties of $\bar{f}_i$, see [3].

For $B_\lambda$, $\varphi_i(M) = \max\{ m \geq 0 \mid \bar{f}_i^m M \in \text{Rep} H_n^\lambda \}$. For $B_{\text{aff}}$ it is slightly different. However, in this context, $\varphi_i$ is computed by counting uncanceled $+$ symbols as in Rule 3 or 4 just as $\epsilon_i$ counts uncanceled $-$ symbols.

If we decree that $\varphi_i$ counts uncanceled $+$ symbols, then we can see when Theorems 3.1 and 3.3 imply (19), as follows.

First, the $A$ and $B$ we consider are of the form $A = M_{\Gamma}$ and $B = M_{\Gamma'}$, where now $\Gamma$ is an initial segment of $\Delta$ and $\Gamma'$ is a final segment, with respect to right order, and $\Gamma \cup \Gamma' = \Delta$. (Note the different use of the words initial “segment” here is as initial subword with respect to right order.) Then $\text{Ind}\Delta = \text{Ind}\Gamma \boxtimes \Gamma'$. By
transitivity of induction and remark [9], $M_\Delta = \text{cosoc Ind} M_\Gamma \boxtimes M_\Gamma'$. Theorem [3.4] says that the $\pm$ rule works for all three of $M_\Delta$, $M_\Gamma$, and $M_\Gamma'$. The $\pm$ word for $\Delta$ is the concatenation of the $\pm$ words for $\Gamma$ and $\Gamma'$.

\[
\mp \cdots \mp \mp \pm \cdots + \mp
\]

If one then performs the requisite cancelling of $-+\pm$ pairs, the leftmost uncanceled $-$ for $\Gamma$ (which has a total of $\varepsilon_i(M_\Gamma)$ as yet uncanceled $-$) will remain such in $\Delta$ only if it is not followed by more than $\varepsilon_i(M_\Gamma)$ many as yet uncancled $+$, i.e. if $\varepsilon_i(M_\Gamma) > \varphi_i(M_\Gamma')$. But in the case $\varepsilon_i(M_\Gamma) \leq \varphi_i(M_\Gamma')$, all $-$ from $\Gamma$ will get cancelled by subsequent $+$, and it is now the leftmost uncanceled $-$ from $M_\Gamma'$ that becomes the leftmost uncanceled $-$ for $\Delta$. On the one hand, we are describing the rule for $\bar{e}_i M_\Delta$, and on the other hand, we are describing [49]. Notice at each stage, that all cosocles we take are irreducible (since the conditions of (19) ensure the concatenation $\bar{e}_i(\Gamma) \cup \Gamma'$ or $\Gamma \cup \bar{e}_i(\Gamma')$ is in right order).

Next, the argument for $A$ and $B$ of the form $N_{\mu, \lambda}$ is similar; but again, the multipartitions must be initial and final segments of a given multipartition, and all three must be Kleshchev multipartitions, i.e. colored multipartitions satisfying condition [1].

Consequently, we also get the following corollary, which describes how to build irreducible $H^\alpha_\lambda$-modules out of $H^\alpha_\mu$ and $H^\beta_\lambda$-modules when $\alpha + \beta = \lambda$.

**Corollary 10.4.** Suppose $\alpha$ and $\beta$ are weights with $\alpha + \beta = \lambda$ and $\alpha = \sum_{k=1}^a \Lambda_i$, $\beta = \sum_{k=1}^b \Lambda_k$, where $i_1 \leq i_2 \leq \cdots \leq i_r$. Suppose $\mu \otimes \nu \in B(\alpha) \otimes B(\beta)$ is in the connected component of the unique node with weight $\lambda$. Write $\mu \cdot \nu = (\mu^{(1)}, \ldots, \mu^{(a)}, \nu^{(1)}, \ldots, \nu^{(r-a)})$. Then the module

\[
N_{\mu, \nu, \lambda} := \text{cosoc Ind} N_{\mu, \alpha} \boxtimes N_{\nu, \beta}
\]

is irreducible and

\[
\bar{e}_i(N_{\mu, \nu, \lambda}) = \begin{cases} N_{\varepsilon_i(\mu) \otimes \nu, \lambda} & \text{if } \varepsilon_i(\mu_{\alpha, \lambda}) \geq \varphi_i(\nu_{\beta, \lambda}) \\ N_{\nu, \bar{e}_i(\nu), \lambda} & \text{if } \varepsilon_i(\mu_{\alpha, \lambda}) \leq \varphi_i(\nu_{\beta, \lambda}) \end{cases}
\]

**Proof.** The proof follows from the discussion after equation [1]. If we take the $\pm$ word corresponding to $\mu \cdot \nu, \lambda$ but then divide that word according to how $\alpha + \beta = \lambda$, the rule for computing $\bar{e}_i$ coincides with the rule for computing $\bar{e}_i(\mu \otimes \nu)$ in $B(\alpha) \otimes B(\beta)$.

We know by Theorem [3.4] that some sequence of $\bar{e}_i$ will bring $N_{\mu, \nu, \lambda}$ to $1 = N(\emptyset, \ldots, \emptyset, \lambda)$ if $\mu \cdot \nu$ satisfies condition [1]. The argument above shows the same sequence of $\bar{e}_i$ will bring $\mu \otimes \nu$ to $0 \otimes 0$. Thus Kleshchev multipartitions correspond to nodes in the connected component of $0 \otimes 0$.

To see that every node in that connected component corresponds to some Kleshchev multipartition, we need to verify that if $\bar{e}_i(\mu' \otimes \nu') = \mu \otimes \nu$ and $\mu, \nu, \lambda$ is Kleshchev, then so is $\mu', \nu', \lambda$.

This is less cumbersome to state if we make use of the crystal operator $\bar{f}_i$. We define $\bar{f}_i(\mu, \lambda)$ to add an $i$-box to the partition corresponding to the rightmost uncanceled $+$, if it exists, and otherwise it is $0$. Thus $\bar{e}_i(\mu', \lambda) = \mu', \lambda \iff \mu', \lambda = \bar{f}_i(\mu, \lambda)$. The argument about $\pm$ words above already shows that $\bar{f}_i(\mu, \nu, \lambda) = \mu', \nu', \lambda$.
if \( \tilde{f}_i(\mu \otimes \nu) = \mu' \otimes \nu' \), where

\[
(20) \quad \tilde{f}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2) \\
\tilde{f}_i b_1 \otimes \tilde{f}_i b_2 & \text{if } \varepsilon_i(b_1) < \varphi_i(b_2).
\end{cases}
\]

This definition is also compatible with the picture below (18), and “undoes” \( \tilde{e}_i \).

What we need to show is that if \( \mu, \nu, \lambda \) is Kleshchev, then so is \( \tilde{f}_i(\mu, \nu, \lambda) \), if it is nonzero. In fact, given the comment above, all we need show is that \( \mu, \lambda \) is Kleshchev \( \iff \tilde{f}_i(\mu, \lambda) \) is.

It is quite tedious to check this directly. Theorem 3.4 lets us show this indirectly. We point out that when \( \mu \) is an initial segment of \( \lambda \) which happens in the case \( \Gamma \) is an initial segment of \( \Delta \), then \( \tilde{e}_i \) is a root of unity, is that they are colored multipartitions corresponding to nodes in that connected component. For a condition parallel to (4) see [JM MO].

We make a final comment about the right/left disparity encountered. The usual tensoring of crystals is compatible with computing \( \tilde{e}_i \) and left order. Compare equation (18) to

\[
\tilde{e}_i(\text{cosoc Ind} M_\Gamma \boxtimes M_{\Gamma'}) = \begin{cases} 
\text{cosoc(Ind}\tilde{e}_i M_\Gamma \boxtimes M_{\Gamma'}) & \text{if } \tilde{\varphi}_i(M_\Gamma) \geq \tilde{\varphi}_i(M_{\Gamma'}) \\
\text{cosoc(Ind} M_\Gamma \boxtimes \tilde{e}_i M_{\Gamma'}) & \text{if } \tilde{\varphi}_i(M_\Gamma) < \tilde{\varphi}_i(M_{\Gamma'}).
\end{cases}
\]

which happens in the case \( \Gamma \) is an initial segment of \( \Delta \) and \( \Gamma' \) is a final segment, with respect to left order, and \( \Gamma \cup \Gamma' = \Delta \).

Just as \( \tilde{e}_i \) counted the number of uncanceled \(-\) signs when computing \( \tilde{e}_i \) in Rule 2, \( \tilde{\varphi}_i \) counts the number of uncanceled \( +\) signs.
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