ADDITION OF DIVISORS ON TRIGONAL CURVES
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Abstract. In this paper we present a realisation of addition of divisors on the basis of uniformisation of Jacobi varieties of plain algebraic curves. Addition is realised through consecutive reduction of a degree $g + 1$ divisor to a divisor of degree $g$. Representation of these divisors does not go beyond the functions which give a solution of the Jacobi inversion problem — the functions of the smallest order. We focus on the simplest non-hyperelliptic family which is the family of trigonal curves. The method can be generalised to a curve of arbitrary gonality.

1. Introduction

We start with several definitions.

Definition 1. A divisor of degree 0 is called a reduced divisor if it has the form: $\hat{D} - g\infty$ and $\deg \hat{D} = g$.

Definition 2. A divisor $D$ of degree 0 on a curve is called principal when it arises as a divisor of a rational function on the curve. That is, with a rational function $R$ on the curve let $D_z$ be the divisor of zeros of the function, and $D_p$ be the divisor of poles of $R$. Then divisor $D_z - D_p$ is a principal divisor on the curve. All principle divisors on the curve are equivalent. And $D_z$ is equivalent to $D_p$. Here sign $\sim$ will denote the equivalence relation.

It follows from the Riemann-Roch theorem that any non-special divisor of the form $D - (\deg D)\infty$ is equivalent to a reduced divisor. An accurate explanation of what we call a non-special divisor is given in subsection 2.5.

Reduction Problem. Given a non-special divisor $D - (\deg D)\infty$ of degree $\deg D = g + m$ with $m > 0$ on an algebraic curve of genus $g$, find the corresponding reduced divisor $\hat{D}$ such that $D - (g + m)\infty \sim \hat{D} - g\infty$.

Addition Problem. Given two non-special divisors $D_1$ and $D_2$ of degrees $g + m_1$ and $g + m_2$, $m_1, m_2 \geq 0$, respectively, find a reduced divisor $\hat{D}$ such that $D_1 + D_2 - (2g + m_1 + m_2)\infty \sim \hat{D} - g\infty$.

Evidently, solving the reduction problem will solve the addition problem. Indeed, we can assume that $D_1, D_2$ together compose a non-special divisor $D$ of degree $2g + m_1 + m_2$, and then come to the reduction problem for the new divisor $D$. On the other hand, the standard addition problem arises when the both divisors $D_1, D_2$ are firstly reduced to divisors $\hat{D}_1, \hat{D}_2$ of degree $g$ each — this setup is used in addition law.

The addition problem earned interest both in the theory of Abelian functions on algebraic curves and in cryptology. In the theory of Abelian functions, the problem is formulated as addition law on Jacobian varieties and represented in terms of Abelian functions. At the same time, the theory of constructing addition laws, see [2], is helpful in solving the addition problem as it stated in the present paper. The

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theory of addition laws in [1] is developed in general for a plain algebraic curve, and applied to hyperelliptic curves where the problem is solved completely and explicitly. Also in [1] another technique, which is based on polylinear relations, is presented. This technique is used to obtain the addition law for simple trigonal curves in [2, Sect. 5.2]. Duplication formulas in terms of sigma function for simple trigonal curves are obtained by the technique of determinant expressions in [3, 4]. A geometric interpretation of the group law for Jacobian varieties is given in [5].

In cryptology interest to the addition problem arose when it was realised that Jacobians of hyperelliptic curves can be used as a source for cryptosystems [6]. The discrete logarithm problem in Jacobians of higher genus curves is difficult to solve, more difficult comparatively to the elliptic case, thus fast algorithms of adding divisors in Jacobians produce effective and secure public cryptosystems for public key cryptography. In [6] fast arithmetics on Jacobians of hyperelliptic curves in \( g = 2, 3 \) are proposed, and for higher genus hyperelliptic curves in [7]. Many improvements of the fast addition algorithm were suggested mostly in genera 2 and 3, for example [8, 9, 10]. The authors also proposed a modification of the addition problem setup and its solution in [11]. This modification suggests to consider addition of pointwise defined divisors as a reduction problem which is solved by means of reducing a degree \( g + 1 \) divisor on each step.

While much work was done in the case of hyperelliptic curves not much is known on fast additions in non-hyperelliptic cases. Some papers and conference talks are devoted to fast addition algorithms in Jacobians of non-hyperelliptic curves of genus 3. These are Picard curves [12, 13, 14, 15], sometimes called superelliptic cubics, which are cyclic trigonal curves of genus 3, and more general trigonal curves \( C_{3,4} \) [16, 17, 18, 19]. The approach of [14, 16] is developed from the study on rationality of the intersection points of a rational line and a smooth plane quartic over a finite field. This approach is oriented to curves of genus 3 and can not be generalised to higher genera or gonality. An interesting approach is given in [19] which is an implementation of the theory presented in earlier papers [20, 21] and developed for an arbitrary algebraic curve. This approach considers a Jacobian as an embedding into a projective space constructed with the help of a very ample line bundle on a curve, and divisors are described by sections of line bundles which ultimately are what we call rational functions. The final algorithms involve only linear algebra, and are claimed to be the fastest.

In this paper we propose what we believe is the first simple addition algorithm for \((3, s)\)-curves. The authors are not aware of any addition algorithms on Jacobians of curves of genera higher than 3. Though high genera are considered as non-secure and have a limited application in cryptology, they are still interesting. One application of such algorithms can be finding conditions on torsion points that exist in Jacobians similar to the mentioned in [22, 23, 24]. Those conditions were used in [25] to give new algorithms to compute Falting's invariants for hyperelliptic curves. Similar results should be available if a corresponding theory of addition algorithms is developed for arbitrary curves.

This paper is organised as follows. In the Preliminaries we recall the notion of \((n, s)\)-curve and define trigonal curves as well as Abel's map on them and solution of the Jacobi inversion problem. In Section 3 we analyse rational functions on trigonal curves, and construct functions which we use as representation of divisors in what follows. Next, in Section 4 all steps of the future reduction algorithm are described in detail. Finally, in Section 5 the reduction algorithm is presented.

Our setup and solution of the addition problem are similar to the approach in [11]. On the first step we suppose that initial divisors are given in terms of points. From this setup a representation in terms of rational functions is constructed. On
each step a degree $g+1$ divisor is reduced to a degree $g$ divisor, then the next point is added. Though such a setup is not typical in cryptology, it is highly effective. Starting from points it is much easier to perform addition. And the person who would realize a cryptography algorithm always can choose points over a desired finite field which produce rational functions with coefficients over the same field.

2. Preliminaries

In the present paper we deal with trigonal curves representing them by $(n, s)$-forms with $n = 3$. So, firstly, we recall the notion of $(n, s)$-curves and then classify trigonal curves with the help of this form.

2.1. $(n, s)$-Curves. An $(n, s)$-curve, introduced in [26], is defined by equation

$$0 = f(x, y) = -y^n + x^s + \sum_{j=0}^{n-1} \sum_{i=0}^{s-1} \lambda_{ns-sj-ni} y^j x^i, \quad \lambda_{k<0} = 0,$$

with co-prime integers $n$ and $s$, and $(x, y) \in \mathbb{C}^2$. Function $f$ serves as the unfolding of Pham singularity $y^n = x^s$ with the minimal number of parameters $\lambda_k \in \mathbb{C}$. This type of curves serves as a kind of canonical forms among plane algebraic curves, like Weierstrass form serves as the canonical form of elliptic curves. In other words, any plane algebraic curve is mapped by a Möbius transformation into an $(n, s)$-curve, possibly with double points. For the purpose of more generality we define $(n, s)$-curve by equation

$$(1a) \quad 0 = f(x, y) = -y^n + x^s + \sum_{j=0}^{n-1} \sum_{i=0}^{s-1} \lambda_{ns-sj-ni} y^j x^i,$$

$$(1b) \quad \lambda_{k<0} = 0,$$

with the full number of parameters. In what follows we consider the curve over the field of complex numbers with complex parameters $\lambda$. As seen from (1) the $(n, s)$-curve is constructed in such a way that infinity serves as a branch point connecting all $n$ sheets of the curve, this is a Weierstrass point and serves as the base point.

Genus of an $(n, s)$-curve is computed by the formula

$$g = \frac{1}{2}(n-1)(s-1),$$

which is guaranteed by condition (1b). The curve (1) is supposed to be non-degenerate, that is its genus equals $g$, and is never less that $g$. In the case of less genus the corresponding non-degenerate $(n, s)$-form should be used.

We also introduce the notion of Sato weight, which is respected by the theory of $(n, s)$-curves. Sato weight equals the opposite to the exponent of the leading term in the expansion near infinity. Actually, parametrisation of (1) near infinity is

$$(3) \quad x = \xi^{-n}, \quad y = \xi^{-s}(1 + O(\lambda))$$

where $\xi$ serves as a local parameter. Thus, Sato weights of $x$ and $y$ are $\text{wgt } x = n$, and $\text{wgt } y = s$. The weight is also assigned to every function, for example $f$ has weight $ns$. Sato weights are used as indices of parameters $\lambda_k$, namely: $\text{wgt } \lambda_k = k$. Parameters $\lambda_k$ in the equation of an $(n, s)$-curve have only positive Sato weights, parameters $\lambda_{k<0}$ of negative weights are supposed to be zero. With the help of Sato weights the order relation is introduced in the space of monomials $y^j x^i$, which are used to construct the equation of a curve and also rational functions on it. Weierstrass gap sequence $\mathbf{w}$ is obtained as the complement to the following sequence in the set of natural numbers

$$\mathbf{w}^* = an + bs, \quad a, b = \{0\} \cup \mathbb{N}.$$
Thus \( \mathbb{w} = \mathbb{N} \setminus \mathbb{m}^* \).

2.2. Trigonal curves. On the base of \((n, s)\)-forms with \(n = 3\) trigonal curves are classified into two families, see also \([27]\): \((3, 3m + 1)\)-curves of genera \(3m\) denoted by \(C^1\), and \((3m + 2)\)-curves of genera \(3m + 1\) denoted by \(C^2\). Such a trigonal curve is defined by equation of the form

\[
0 = f(x, y) = -y^3 + y^2 \mathcal{T}(x) + yQ(x) + P(x)
\]

where

\[
(4b) \quad C^1: \quad P(x) = x^{3m+1} + \sum_{i=0}^{3m} \lambda_{9m+3-3i}x^i,
Q(x) = \sum_{i=0}^{2m} \lambda_{6m+2-3i}x^i, \quad \mathcal{T}(x) = \sum_{i=0}^{m} \lambda_{3m+1-3i}x^i;
\]

\[
(4c) \quad C^2: \quad P(x) = x^{3m+2} + \sum_{i=0}^{3m+1} \lambda_{9m+6-3i}x^i,
Q(x) = \sum_{i=0}^{2m+1} \lambda_{6m+4-3i}x^i, \quad \mathcal{T}(x) = \sum_{i=0}^{m} \lambda_{3m+2-3i}x^i.
\]

All curves are supposed non-degenerate.

The simplest trigonal curve is \((3, 4)\)-curve of genus \(g = 3\), namely:

\[
(5) \quad 0 = f(x, y) = -y^3 + y^2(\lambda_1x + \lambda_4) + y(\lambda_2x^2 + \lambda_5x + \lambda_8)
+ x^4 + \lambda_3x^3 + \lambda_6x^2 + \lambda_9x + \lambda_{12}.
\]

Another simple trigonal curve is \((3, 5)\)-curve of genus \(g = 4\)

\[
(6) \quad 0 = f(x, y) = -y^3 + y^2(\lambda_2x + \lambda_5) + y(\lambda_1x^3 + \lambda_4x^2 + \lambda_7x + \lambda_{10})
+ x^5 + \lambda_3x^4 + \lambda_6x^3 + \lambda_9x^2 + \lambda_{12}x + \lambda_{15}.
\]

Curves of families \(C^1\) and \(C^2\) represent all trigonal curves up to Möbius transformations. A curve with leading terms \(y^3\) and \(x^{3m}\) and without double points transforms into a curve from the family \(C^2\) of genus \(3(m - 1) + 1\). We distinguish between \(C^1\) and \(C^2\) because they have some differences in gap sequences and the order of monomials. The gap sequence is

\[
C^1: \quad \mathbb{w} = \{3k - 2 \mid k = 1, \ldots, m\} \cup \{3k - 1 \mid k = 1, \ldots, 2m\};
C^2: \quad \mathbb{w} = \{3k - 1 \mid k = 1, \ldots, m\} \cup \{3k - 2 \mid k = 1, \ldots, 2m + 1\}.
\]

And the ordered set of monomials is the following

\[
(7a) \quad C^1: \quad \mathfrak{M} = \{1, x, \ldots, x^{m-1}, x^m, y, y^{m+1}, yx, \ldots, x^{2m-1}, yx^{m-1},
\times x^{2m}, yx^m, y^2, x^{2m+1}, yx^{m+1}, y^2x, \ldots\};
\]

\[
(7b) \quad C^2: \quad \mathfrak{M} = \{1, x, \ldots, x^{m-1}, x^m, y, y^{m+1}, yx, \ldots, x^{2m-1}, yx^{m-1}, x^{2m},
yx^m, x^{2m+1}, y^2, yx^{m+1}, x^{2m+2}, y^2x, \ldots\}.
\]

The first \(g\) monomials appear as numerators of holomorphic differentials.
2.3. Jacobian variety and Abel’s map. Each curve is related to a Jacobian variety with coordinates $u = (u_{w_1}, u_{w_2}, \ldots, u_{w_g})$ where $\{w_k\}_{k=1}^g$ is the gap sequence defined above, and $\deg u_{w_k} = -w_k$. Holomorphic differentials are introduced as follows

\begin{align}
(8a) \quad C^1: \quad & du_{3k-1} = \frac{x^{2m-k}dx}{\partial_y f}, \quad k = 1, \ldots, 2m, \\
& du_{3k-2} = \frac{yx^{m-k}dx}{\partial_y f}, \quad k = 1, \ldots, m;
\end{align}

\begin{align}
(8b) \quad C^2: \quad & du_{3k-2} = \frac{x^{2m+1-k}dx}{\partial_y f}, \quad k = 1, \ldots, 2m + 1, \\
& du_{3k-1} = \frac{yx^{m-k}dx}{\partial_y f}, \quad k = 1, \ldots, m.
\end{align}

Abel’s map is defined by

\begin{align}
(9) \quad & \mathcal{A}(P) = \int_{\infty}^{P} du,
\end{align}

where $du = (du_{w_1}, du_{w_2}, \ldots, du_{w_g})^t$, and on a divisor $D_g = \sum_{k=1}^g (x_k, y_k)$ by

\begin{align}
(10) \quad & \mathcal{A}(D_g) = \sum_{k=1}^g \mathcal{A}(x_k, y_k).
\end{align}

A curve is uniformised by $\wp$-functions which are multiply periodic and defined through $\sigma$ function, namely:

\begin{align}
(11a) \quad & \wp_{i,j}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u), \\
(11b) \quad & \wp_{i,j,k}(u) = -\frac{\partial^3}{\partial u_i \partial u_j \partial u_k} \log \sigma(u), \ldots
\end{align}

for more details about $\sigma$ function see [11].

2.4. Jacobi inversion problem. The Jacobi inversion problem for $u = \mathcal{A}(D_g)$ with a degree $g$ divisor $D_g = \sum_{k=1}^g (x_k, y_k)$ on a trigonal curve is solved by the system

\begin{align}
(12) \quad & \mathcal{R}_{2g}(x, y; u) = 0, \quad \mathcal{R}_{2g+1}(x, y; u) = 0,
\end{align}

where the entire rational functions $\mathcal{R}_{2g}$ and $\mathcal{R}_{2g+1}$ of orders $2g$ and $2g + 1$ have $D_g$ as the common roots. Explicit form of the functions is obtained from the Klein formula. On trigonal curves the functions have the following form

\begin{align}
(13a) \quad C^1: \quad & \mathcal{R}_{2g}(x, y; u) = x^{2m} - y \sum_{k=1}^m x^{m-k} \wp_{1,3k-2}(u) - \sum_{k=1}^{2m} x^{2m-k} \wp_{1,3k-1}(u), \\
& \mathcal{R}_{2g+1}(x, y; u) = 2yx^{m} - \lambda_1 x^{2m} - y \sum_{k=1}^m x^{m-k} (\wp_{2,3k-2}(u) - \wp_{1,3k-2}(u)) \\
& \quad - \sum_{k=1}^{2m} x^{2m-k} (\wp_{2,3k-1}(u) - \wp_{1,3k-1}(u));
\end{align}

\begin{align}
(13b) \quad C^2: \quad & \mathcal{R}_{2g}(x, y; u) = yx^{m} - y \sum_{k=1}^m x^{m-k} \wp_{1,3k-1}(u) - \sum_{k=1}^{2m+1} x^{2m+1-k} \wp_{1,3k-2}(u), \\
& \mathcal{R}_{2g+1}(x, y; u) = 2yx^{m} - \lambda_1 x^{2m} - y \sum_{k=1}^m x^{m-k} (\wp_{2,3k-2}(u) - \wp_{1,3k-2}(u)) \\
& \quad - \sum_{k=1}^{2m+1} x^{2m+1-k} (\wp_{2,3k-1}(u) - \wp_{1,3k-1}(u)).
\end{align}
$$R_{2g+1}(x, y; u) = 2x^{2m+1} + \lambda_1yx^m - y \sum_{k=1}^{m} x^{m-k} (\varphi_{2,3k-2}(u) - \varphi_{1,1,3k-2}(u))$$

$$+ \lambda_3x^{2m} - \sum_{k=1}^{2m+1} x^{2m-k} (\varphi_{2,3k-1}(u) - \varphi_{1,1,3k-1}(u)),$$

see also [27] Theorem 2.6. Note that $R_{2g}$ is even with respect to $u$ since $\varphi_{l,j}(u)$ is even.

**Example 1.** On a $(3,4)$-curve of the form (5) introduce first and second kind differentials $du = (du_1, du_2, du_5)^t$ and $dr = (dr_1, dr_2, dr_5)^t$ as follows, cf. [28] Eq (2.22),

\[(14) \quad du = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \frac{dx}{\partial y}, \quad dr = \begin{pmatrix} x^2 \\ 2xy - \lambda_1x^2 \\ 5x^2y - \frac{5}{18} \lambda_1x^3 + \frac{5}{18} \lambda_2y^2 + \rho_5(x, y) \end{pmatrix} \frac{dx}{\partial y},\]

\[\rho_5(x, y) = xy(\lambda_1\lambda_2 + 3\lambda_3) + x^2(\frac{4}{9} \lambda_2^2 - \lambda_1\lambda_3 - 2\lambda_4)
\]
\[+ y\left(\frac{5}{18} \lambda_2^2 + \frac{5}{18} \lambda_2\lambda_4 + \frac{5}{18} \lambda_1\lambda_4 + \lambda_6\right)
\]
\[+ x\left(\frac{5}{18} \lambda_2^2 + \frac{5}{18} \lambda_2\lambda_5 + \frac{5}{18} \lambda_1\lambda_5 - \lambda_3\lambda_4\right).\]

Note that $wgt\, du_n = -n$ and $wgt\, dr_n = n$.

Then the functions $R_{2g}$ and $R_{2g+1}$ giving a solution of the Jacobi inversion problem have the form, cf. [25] Eq (5.7)–(5.8)

\[(15a) \quad R_{2g}(x, y; u) = x^2 - y\varphi_{1,1}(u) - x\varphi_{1,2}(u) - \varphi_{1,5}(u),
\]
\[(15b) \quad R_{2g+1}(x, y; u) = 2xy - \lambda_1x^2 - y(\varphi_{1,2}(u) - \varphi_{1,1}(u))
\]
\[-x(\varphi_{2,2}(u) - \varphi_{1,1,2}(u)) - (\varphi_{2,5}(u) - \varphi_{1,1,5}(u)).\]

**Example 2.** On a $(3,5)$-curve of the form (6) introduce first and second kind differentials $du = (du_1, du_2, du_4, du_7)^t$ and $dr = (dr_1, dr_2, dr_4, dr_7)^t$ as follows

\[(16) \quad du = \begin{pmatrix} 1 \\ x \\ y \\ x^2 \end{pmatrix} \frac{dx}{\partial y}, \quad dr = \begin{pmatrix} xy \\ 2x^3 + \lambda_1xy + \lambda_3x^2 \\ 4x^2y + x^3(\frac{4}{9} \lambda_2^2 - 2\lambda_2) + \rho_4(x, y) \\ 7x^3y - \frac{7}{27} \lambda_2x^4 + \rho_7(x, y) \end{pmatrix} \frac{dx}{\partial y},\]

\[\rho_4(x, y) = xy(\frac{1}{3} \lambda_1\lambda_2 + 2\lambda_3) + x^2(\frac{2}{9} \lambda_1^3 + \frac{2}{9} \lambda_1\lambda_4 - \lambda_2\lambda_3 - \lambda_5) + \frac{1}{9} \lambda_1\lambda_5y,
\]
\[\rho_7(x, y) = x^2y(-\frac{1}{9} \lambda_1^3 + \frac{1}{9} \lambda_1\lambda_4 + 5\lambda_3) + x^3(-\frac{1}{9} \lambda_2^2 + \frac{1}{9} \lambda_1^2\lambda_2 - \frac{1}{9} \lambda_1^2\lambda_3 + \frac{1}{9} \lambda_1\lambda_2^2)
\]
\[+ 2\lambda_1\lambda_4 - \frac{1}{3} \lambda_2\lambda_3 - 3\lambda_5) + xy(-\frac{1}{9} \lambda_2^2 + \frac{1}{9} \lambda_1\lambda_4 + \frac{1}{9} \lambda_1\lambda_2^2)
\]
\[+ \frac{3}{9} \lambda_1^2 + \lambda_2\lambda_4 + 3\lambda_6) + x^2(-\frac{1}{9} \lambda_1^2\lambda_2\lambda_4 - \frac{1}{9} \lambda_1^2\lambda_6 + \frac{1}{9} \lambda_1\lambda_7 + \frac{1}{9} \lambda_1\lambda_4
\]
\[+ \lambda_2\lambda_6 + \frac{2}{9} \lambda_2^2 + 2\lambda_3\lambda_5) + y(-\frac{1}{9} \lambda_2^2 + \lambda_2\lambda_5 - \frac{1}{9} \lambda_2\lambda_7 + \frac{2}{9} \lambda_2^2)
\]
\[+ \frac{1}{9} \lambda_2\lambda_7 + \frac{1}{9} \lambda_1\lambda_5 + \lambda_9) + x(-\frac{1}{9} \lambda_1^2\lambda_2\lambda_7 - \frac{8}{9} \lambda_1^2\lambda_9 + \frac{3}{9} \lambda_1\lambda_{10}
\]
\[+ \frac{1}{9} \lambda_2^2\lambda_7 - \frac{1}{9} \lambda_2\lambda_9 + \frac{4}{9} \lambda_4\lambda_7 - \lambda_5\lambda_6).\]

And the functions $R_{2g}$ and $R_{2g+1}$ giving a solution of the Jacobi inversion problem have the form

\[(17a) \quad R_{2g}(x, y; u) = xy - x^2\varphi_{1,1}(u) - y\varphi_{1,2}(u) - x\varphi_{1,4}(u) - \varphi_{1,7}(u),
\]
\[(17b) \quad R_{2g+1}(x, y; u) = 2x^3 + \lambda_1xy - x^2(\varphi_{1,2}(u) - \varphi_{1,1,1}(u) - \lambda_3)
\]
\[-y(\varphi_{2,2}(u) - \varphi_{1,1,2}(u)) - x(\varphi_{2,4}(u) - \varphi_{1,1,4}(u))
\]- (\varphi_{2,7}(u) - \varphi_{1,1,7}(u)).
2.5. Non-special divisors. Recall that involution on a trigonal curve connects three points with equal \( x \)-coordinates when they are not singular or ramification points.

We call a divisor non-special if it has degree not less than the genus of a curve and contains no three points connected by involution. We also introduce the notion of a strictly non-special divisor which contains no two points connected by involution.

If a divisor contains three points connected by involution, then we suggest to eliminate these three points and replace the divisor by its truncated version. So a divisor is non-special if after eliminating all collections of three points in involution the degree of divisor is not less than \( g \). Otherwise, a divisor is special.

3. Divisors in terms of rational functions on a trigonal curve

3.1. Rational functions on a curve. Rational functions on a curve are constructed as linear combinations of monomials from \( \mathcal{R} \). Each function is characterised by its roots on the curve. We say that function with \( N \) roots has order \( N \). The order of a function, which is the number of roots, is shown by Sato weight of the function. A rational function of order \( 2g + p \) with \( p \geq 0 \) is constructed from the first \( g + p + 1 \) elements from the list \( \mathcal{R} \) of monomials ordered in ascending Sato weights. In the case of trigonal curve the list \( \mathcal{R} \) is defined by (7).

Now we illustrate how to construct rational functions on a trigonal curve. Given a strictly non-special divisor \( D_g = \sum_{k=1}^{g} (x_k, y_k) \) with all distinct points one obtains the function of order \( 2g \) vanishing on \( D_g \) as \( \det \mathcal{R}_{2g} \) with matrix \( \mathcal{R}_{2g} \) based on the first \( g + 1 \) monomials from (7). In the case of \( (3, 3m+1) \)-curve \( 3m + 1 \) monomials are taken:

\[(18a) \quad \mathcal{R}_{2g} = \begin{vmatrix}
1 & x & \ldots & x^{m-1} & y & x^{m+1} & yx & \ldots & yx^{m-1} & x^{2m} & yx^{2m} & \ldots & yx^{2m} \times_{k=1}^{g} y_k \times_{k=1}^{g} x_k
\end{vmatrix},
\]

and \( 3m + 2 \) monomials on a \( (3, 3m+2) \)-curve:

\[(18b) \quad \mathcal{R}_{2g} = \begin{vmatrix}
1 & x & \ldots & x^{m-1} & y & x^{m+1} & yx & \ldots & yx^{m-1} & x^{2m} & yx^{2m} & \ldots & yx^{2m} \times_{k=1}^{g} y_k \times_{k=1}^{g} x_k
\end{vmatrix}.
\]

We denote the function of order \( 2g \) by \( I \), it has the form

\[(19) \quad I(x, y) = y \alpha_y(x) + \alpha_x(x),
\]

\( C^1 : \deg \alpha_y = m - 1, \quad \deg \alpha_x = 2m, \)

\( C^2 : \deg \alpha_y = m, \quad \deg \alpha_x = 2m, \)

and coefficients of \( \alpha_y \) and \( \alpha_x \) are computed from coordinates of \( D_g \).

Remark 1. The function of order \( 2g \) vanishing on \( D_g \) can be constructed also if \( D_g \) contains duplicated points. Say \( (x_g, y_g) = (x_1, y_1) \), then the last row in \( \mathcal{R}_{2g} \) corresponding to point \( (x_g, y_g) \) should be replaced by the limit as \( (x_g, y_g) \to (x_1, y_1) \) of the total derivative with respect to \( x_g \), that is in (18a)

\[\{0, 1, \ldots, mx_1^{m-1}, y_1', (m+1)x_1^m, y_1 + x_1y_1', \ldots, 2mx_1^{2m-1}, mx_1^{m-1} y_1 + x_1 y_1'y_1'_{k=1}^g\}.
\]

where \( y_1' = \lim_{(x, y) \to (x_1, y_1)} dy / dx_g \). In such a way one can replace each row corresponding to a duplicated point. If multiplicity of a point is greater than \( 2 \), say \( n \), then \( n - 1 \) rows are replaced by total derivatives of orders up to \( n - 1 \).

Similarly, the function of order \( 2g + 1 \) vanishing on a given strictly non-special divisor \( D_{g+1} = \sum_{k=1}^{g+1} (x_k, y_k) \) with all distinct points is obtained as \( \det \mathcal{R}_{2g+1} \) with
matrix $\mathcal{R}_{2g+1}$ based on the first $g + 2$ monomials from (17). With $3m + 2$ monomials on a $(3, 3m + 1)$-curve the following matrix is constructed

$$
(20a) \quad \mathcal{R}_{2g+1} = \begin{vmatrix}
1 & x & \ldots & x^m & y & x^{m+1} & yx & \ldots & x^{2m} & yx^m \\
1 & x_k & \ldots & x_k^m & yk & x_k^{m+1} & ykx_k & \ldots & x_k^{2m} & ykx_k^m \end{vmatrix},
$$

and with $3m + 3$ monomials in the case of $(3, 3m + 2)$-curve

$$
(20b) \quad \mathcal{R}_{2g+1} = \begin{vmatrix}
1 & x & \ldots & x^m & y & x^{m+1} & yx & \ldots & x^{2m+1} & x^{2m+2} \\
1 & x_k & \ldots & x_k^m & yk & x_k^{m+1} & ykx_k & \ldots & x_k^{2m+1} & x_k^{2m+2} \end{vmatrix}.
$$

We denote the function of order $2g + 1$ by $G$, it has the form

$$
(21) \quad G(x, y) = y\gamma_y(x) + \gamma_x(x),
$$

$$
C^1: \quad \deg \gamma_y = m, \quad \deg \gamma_x = 2m,
$$

$$
C^2: \quad \deg \gamma_y = m, \quad \deg \gamma_x = 2m + 1,
$$

and coefficients of $\gamma_y$ and $\gamma_x$ are computed from coordinates of $D_{g+1}$.

Given a divisor $D_{g+p} = \sum_{k=1}^{g+p}(x_k, y_k)$ a matrix $\mathcal{R}_{2g+p}$ can be constructed on the base of the first $g + p + 1$ monomials from (17). Then $\det \mathcal{R}_{2g+p}$ defines the function $B_0$ of order $2g + p$ vanishing on the given divisor $D_{g+p}$. The function with $p = 0$ we denote by $I$ and with $p = 1$ by $G$.

**Remark 2.** Note that rational functions $I$ and $G$ of weights $2g$ and $2g + 1$ are linear in $y$, since $y^2$ has weight $2g + 2$. Thus, the functions $I$ and $G$ are improper to characterise non-strictly non-special divisors, which contain points in involution.

Above we considered rational functions of order not less than $2g$ and saw that $g$ of their roots never appear in the process of construction. Now we describe rational functions from a more general point of view, taking into consideration all their roots. Let $D$ be a divisor whose support is formed by all roots of a rational function $R$ on a curve $f(x, y) = 0$, so

$$
(22) \quad D - (\deg D)\infty \sim 0,
$$

and

$$
(23) \quad A(D - (\deg D)\infty) = 0.
$$

The left hand side of (23) is a vector with $g$ components depending on $\deg D$ points of the curve, which form the support of $D$. Thus, not all points of $D$ can be chosen arbitrarily. When constructing $R$, for example by the determinant expression as shown above, one chooses $\deg D - g$ arbitrary roots on the curve, while the remaining $g$ roots are determined completely from the arbitrary roots. We call the latter dependent roots.

Next, we analyse the relation between the roots $D$ of a rational function $R$ and its coefficients at monomials. If $\deg D \geq 2g$ then all points in the determinant expression can be taken arbitrarily, since there are $g$ gaps in the sequence of weights corresponding to the list of monomials. Then the number of coefficients in $R$ up to a constant multiple coincides with the number of arbitrary roots of $R$. Evidently, the same divisor $D$ serves as roots for many rational functions which are constructed from any selection of $\deg D - g$ points from $D$. At the same time each rational function $R$ is uniquely defined by a selection of $\deg D - g$ arbitrary roots.

If $g < \deg D < 2g$ then the determinant expression also can be used for constructing $R$, but only $\deg D - g$ points in the expression are independent. Thus, coefficients of $R$ are connected by a number of relations, and can not be chosen arbitrarily. For example, a function $y - b$ has weight $s = g + 1$ on a trigonal curve, and its roots consist of points $(a^{(i)}, b)$, where $(a^{(i)})_{i=1}^{g+1}$ are solutions of $f(x, b) = 0$. 

If \( \deg D \leq g \) then only special collections of points can form roots of the rational function \( \mathcal{R} \). For example, a function of the minimal weight has the form \( x - a \), and on a trinodal curve it has three roots: \( (a, b^{(1)}), (a, b^{(2)}), (a, b^{(3)}) \) such that \( b^{(1)}, b^{(2)}, b^{(3)} \) are three solutions of \( f(a, y) = 0 \) for \( y \). We call these points related by involution.

### 3.2. Rational function of order \( 2g \).

**Theorem 1.** Let \( \mathcal{I} \) be a rational function of order \( 2g \) on a curve \( \mathcal{L} \), that is it has \( 2g \) roots which are solutions of the system

\[
\mathcal{I}(x, y) = 0, \quad f(x, y) = 0.
\]

If there is no points in involution among the roots of \( \mathcal{I} \), system \((24)\) is equivalent to the following

\[
\mathcal{Z}_{2g}(x) = 0, \quad y = -\frac{\alpha_x(x)}{\alpha_y(x)},
\]

where

\[
\mathcal{Z}_{2g}(x) = \alpha_y(x)^3 f(x, -\alpha_x(x)/\alpha_y(x)) \]

\[
= \alpha_x(x)^3 + \alpha_x(x)^2 \alpha_y(x) T(x) - \alpha_x(x) \alpha_y(x)^2 Q(x) + \alpha_y(x)^3 P(x).
\]

Among the roots only \( g \) points are taken independently, they define the function \( \mathcal{I} \) completely. The remaining \( g \) points are computed from the system \((24)\) or \((24)\).

**Proof.** Taking into account that a rational function \( \mathcal{I} \) of order \( 2g \) has the form \((19)\), solve \( \mathcal{I}(x, y) = 0 \) for \( y \), and substitute into \((1)\). Then a new system \((25)\) is obtained. Polynomial \( \mathcal{Z}_{2g} \) has degree \( 2g \), since \( \deg \alpha_x^3 = 6m \) on a \((3, 3m + 1)\)-curve, and \( \deg \alpha_y^3 P = 6m + 2 \) on a \((3, 3m + 2)\)-curve, which is easily computed from \((1)\) and \((1)\). Evidently, the new system \((25)\) has \( 2g \) roots, whose \( x \)-coordinates are roots of \( \mathcal{Z}_{2g} \), and the corresponding \( y \)-coordinates are computed from the second equation. Since \( \mathcal{I} \) is linear in \( y \), all solutions of \((25)\) are distinct points. Thus, points in involution among roots of \( \mathcal{I} \) are inadmissible when \((24)\) is equivalent to \((24)\). This condition guarantees that \( \alpha_y(x) \neq 0 \).

Multiple roots are possible, they require a specific construction of \( \mathcal{I} \) as described in Remark \((1)\).

The fact that only \( g \) roots are independent follows immediately from the determinant formula with matrix \((18)\). \( \square \)

The model of function \( \mathcal{I} \) of order \( 2g \) presented by \((11)\) and \((18)\) allows to analyse the relation between its form and the structure of divisor \( D_y \) which gives rise to the function \( \mathcal{I} \). Here we use normalisation by the determinant \( W_y \) of Vandermonde matrix constructed from \( \{x_k\}_{k=1}^g \) which are \( x \)-coordinates of the support of \( D_y \). Let \( \partial \mathcal{I} = \deg \alpha_y \), then

\[
W_y^{-1} \alpha_y(x) = \frac{1}{\partial \mathcal{I}^2} \sum_{k_1,\ldots,k_{2g+1}}^{g} \left( \prod_{l=1}^{\partial \mathcal{I}} y_{k_l} \right) L_{k_1\ldots,k_{2g}}^{(g)}(x),
\]

\[
W_y^{-1} \alpha_x(x) = \frac{-1}{(\partial \mathcal{I} + 1)!} \sum_{k_1,\ldots,k_{2g+1}}^{g} \left( \prod_{l=1}^{\partial \mathcal{I} + 1} y_{k_l} \right) M_{k_1\ldots,k_{2g+1}}^{(g)}(x),
\]

where

\[
L_{k_1\ldots,k_g}^{(g)}(x) = \prod_{l=1}^{g} \frac{(x - x_{k_l})}{\prod_{i \neq k_1\ldots,k_g} (x - x_i)},
\]
Suppose $D_y$ contains two points $(x_k, y_k^{(i)})$, $i = 1, 2$, related by involution on the curve, and $\mathcal{I}$ is formally constructed from $D_y$ by the determinant formula with matrix $[15]$. In this case $W_y$ vanishes, and in $\alpha_y$ and $\alpha_x$ only terms which contain one of $y_k^{(i)}$ do not vanish, so $\alpha_y$ and $\alpha_x$ have the common multiple $(y_k^{(1)} - y_k^{(2)})(x - x_k)$. Thus, $\mathcal{I}$ and $Z_{2g}$ factorise as follows

\begin{equation}
\mathcal{I}(x, y) = (y_k^{(1)} - y_k^{(2)})(x - x_k)(\hat{\alpha}_y(x)y + \hat{\alpha}_x(x)),
\end{equation}

\begin{equation}
Z_{2g}(x) = (y_k^{(1)} - y_k^{(2)})^2(x - x_k)^3\left(\hat{\alpha}_x(x)^3 + \hat{\alpha}_x(x)^2\hat{\alpha}_y(x)\mathcal{T}(x) - \hat{\alpha}_x(x)\hat{\alpha}_y(x)^2\mathcal{Q}(x) + \hat{\alpha}_y(x)^3\mathcal{P}(x)\right),
\end{equation}

\[\deg \hat{\alpha}_y = \deg \alpha_y - 1, \quad \deg \hat{\alpha}_x = \deg \alpha_x - 1.\]

This means that points in involution arise among the roots of $\mathcal{I}$ all together: $(x_k, y_k^{(i)})$, $i = 1, 2, 3$. If two of them occur among the arbitrary roots, then the third is located among the dependent roots. In this case $\mathcal{I}$ is not equivalent to $[24]$, because $y$-coordinates of the points in involution are computed only from $f(x, y) = 0$. If the number of pairs in involution among the roots of $\mathcal{I}$ is greater than $\deg \alpha_y$, polynomial $\alpha_y$ vanishes, and $Z_{2g}(x) = \alpha_x(x)^3$. Thus, the whole collection of $2g$ roots of $\mathcal{I}$ decomposes into $2m$ collections of three points in involution whose $x$-coordinates are roots of $\alpha_x$. When $D_y$ which is used to construct the function $\mathcal{I}$ contains all three points $(x_k, y_k^{(i)})$, $i = 1, 2, 3$, related by involution, only zero function $\mathcal{I}$ is obtained. The above consideration gives the reason for eliminating points in involution from divisors.

In what follows we suppose that the collection of roots of $\mathcal{I}$ does not contain points in involution. This means, that $\mathcal{I}$ does not admit arbitrary coefficients in polynomials $\alpha_y$ and $\alpha_x$. A situation when $\mathcal{I}$ factorises is inadmissible since this is the necessary condition of the presence of points in involution among the roots of $\mathcal{I}$. The sufficient condition consists in verification if the roots computed by $[25]$ are located on the curve $f(x, y) = 0$.

3.3. Rational function of order $2g + 1$.

Theorem 2. Let $G$ be a rational function of order $2g + 1$ on a curve $[14]$, that is it has $2g + 1$ roots, which are solutions of the system

\begin{equation}
G(x, y) = 0, \quad f(x, y) = 0.
\end{equation}

If there is no points in involution among the roots of $G$, system $[25]$ is equivalent to the following

\begin{equation}
Z_{2g+1}(x) = 0, \quad y = -\gamma_x(x)\gamma_y(x),
\end{equation}

where

\begin{equation}
Z_{2g+1}(x) = \gamma_y(x)^3f(x, -\gamma_x(x)/\gamma_y(x))
= \gamma_x(x)^3 + \gamma_x(x)^2\gamma_y(x)\mathcal{T}(x) - \gamma_x(x)\gamma_y(x)^2\mathcal{Q}(x) + \gamma_y(x)^3\mathcal{P}(x).
\end{equation}

Among the roots only $g+1$ points are taken independently, they define the function $G$ completely. The remaining $g$ points are computed from the system $[24]$ or $[25]$.

Proof. Taking into account that $G$ has the form $[21]$, solve $G(x, y) = 0$ for $y$, and substitute into $[14]$. Then system $[25]$ is obtained. Polynomial $Z_{2g+1}$ has degree $2g + 1$, since $\deg \gamma_y\mathcal{P} = 6m + 1$ on a $(3, 3m + 1)$-curve, and $\deg \gamma_x = 6m + 3$. 

\[\mathcal{I}_2(x, y) = (y_1(x) - y_2(x))(x - x_1)(\hat{\alpha}_y(x)y + \hat{\alpha}_x(x)), \quad \mathcal{I}_3(x, y) = (y_1(x) - y_2(x))(x - x_1)(y_2(x) - y_3(x))(x - x_2)(\hat{\alpha}_y(x)y + \hat{\alpha}_x(x)).\]
on a \((3, 3m + 2)\)-curve, which is easily computed from (11) and (21). Note that \(\gamma_y(x) \neq 0\), if there is no points in involution among the roots of \(G\). In this case (23) is equivalent to (25). Then \(x\)-coordinates of \(2g + 1\) roots of \(G\) are roots of \(Z_{2g+1}\), and the corresponding \(y\)-coordinates are computed from the second equation (24).

Multiple roots are possible, they require a specific construction of \(G\) as described in Remark 11.

The fact that only \(g + 1\) roots are independent follows immediately from the determinant formula with matrix (20).

Now analyse the model of function \(G\) of order \(2g + 1\) presented by (21) and (20). Let \(D_{g+1}\) be a divisor which gives rise to the function \(G\), and \(W_{g+1}\) is the determinant of Vandermonde matrix constructed from \(\{x_k\}_{k=1}^{g+1}\) which are \(x\)-coordinates of the support of \(D_{g+1}\). Here normalisation by \(W_{g+1}\) is used. Let \(d_G = \deg \gamma_y\), then

\[
W_{g+1}^{-1} \gamma_y(x) = \frac{1}{d_G!} \sum_{k_1, \ldots, k_{g+1} = 1}^{g+1} \prod_{l=1}^{d_G} (\prod_{k \neq k_{g+1}} y_k) L_{k_1, \ldots, k_{g+1}}^{(g+1)}(x),
\]

(30a)

\[
W_{g+1}^{-1} \gamma_x(x) = \frac{-1}{(d_G + 1)!} \sum_{k_1, \ldots, k_{g+1} = 1}^{g+1} \prod_{k \neq k_{g+1}} y_k M_{k_1, \ldots, k_{g+1}}^{(g+1)}(x),
\]

(30b)

where \(L_{k_1, \ldots, k_3}^{(g)}\) and \(M_{k_1, \ldots, k_3}^{(g)}\) are defined by (27).

If \(D_{g+1}\) contains two points \((x_k, y_k^{(i)}), i = 1, 2\), related by involution on the curve, then \(G\) and \(Z_{2g+1}\) factorise as follows

\[
G(x, y) = (y_k^{(1)} - y_k^{(2)})(x - x_k)(\gamma_y(x)y + \gamma_x(x)),
\]

\[
Z_{2g+1}(x) = (y_k^{(1)} - y_k^{(2)})^3(x - x_k)^3(\gamma_x(x)^3 + \gamma_y(x)^2 \gamma_y(x)T(x) - \gamma_x(x) \gamma_y(x) T^2(x) + \gamma_y(x)^3 P(x)),
\]

\[
\deg \gamma_y = \deg \gamma_y - 1, \quad \deg \gamma_x = \deg \gamma_x - 1.
\]

So if \(D_{g+1}\) contains two points in involution, then the third point is among the remaining roots of \(G\). If the number of pairs in involution among the roots of \(G\) is greater than \(\deg \gamma_y\), polynomial \(\gamma_y\) vanishes, and \(Z_{2g+1}(x) = \gamma_x(x)^3\). Thus, the whole collection of \(2g + 1\) roots of \(G\) decomposes into \(\deg \gamma_x\) sets of three points in involution whose \(x\)-coordinates are roots of \(\gamma_x\). If \(D_{g+1}\) contains all three points \((x_k, y_k^{(i)}), i = 1, 2, 3\), connected by involution, the function \(G\) vanishes identically.

In what follows we suppose that there is no points in involution among the roots of \(G\). That \(G\) does not factorise, which is the necessary condition. And it is sufficient to verify if the roots computed by (20) are located on the curve \(f(x, y) = 0\).

3.4. Divisors in terms of rational functions.

**Theorem 3.** Suppose rational functions \(I\) and \(G\) of orders \(2g\) and \(2g + 1\) vanish on the same strictly non-special degree \(g\) divisor \(D_g\) on a curve \(C\), that is the system

\[
I(x, y) = 0, \quad G(x, y) = 0
\]

defines the divisor \(D_g\). Then \(D_g\) is equivalently defined by the system

\[
H(x) = 0, \quad I(x, y) = 0
\]

(31)

or

\[
H(x) = 0, \quad G(x, y) = 0
\]

(32)

(33)
where $\mathcal{H}$ is a polynomial of degree $g$ such that

\begin{equation}
\mathcal{H}(x) = \gamma_y(x)\mathcal{I}(x, y) - \alpha_y(x)\mathcal{G}(x, y)
\end{equation}

\begin{equation}
= \gamma_y(x)\alpha_x(x) - \alpha_y(x)\gamma_x(x).
\end{equation}

**Proof.** From [19] and [21] one easily finds that the degree of $\mathcal{H}$ equals $g$. Thus, equation $\mathcal{H}(x) = 0$ gives $g$ values of $x$-coordinate, and define $3g$ points on the curve, namely: $\{(x_k, y_k^{(i)}), i = 1, 2, 3\}_{k=1}^g$, where $y_k^{(1)}$, $y_k^{(2)}$, $y_k^{(3)}$ are solutions of $f(x_k, y) = 0$. Functions $\mathcal{I}$ and $\mathcal{G}$ are linear in $y$, thus allow to single the unique point $(x_k, y_k)$ which belongs to $D_g$ out from every three points $(x_k, y_k^{(i)})$, $i = 1, 2, 3$, connected by involution. \qed

We call the function $\mathcal{I}$ of order $2g$ vanishing on a degree $g$ divisor $D_g$ the **minimal rational function** which defines the divisor $D_g$.

**Corollary 1.** A strictly non-special divisor $D_g$ is uniquely defined by the system

\begin{equation}
\mathcal{H}(x) = 0, \quad \mathcal{I}(x, y) = 0,
\end{equation}

where $\mathcal{H}$ is a polynomial in $x$ of degree $g$, and $\mathcal{I}$ is a rational function linear in $y$, the both vanishing on $D_g$.

**Corollary 2.** A strictly non-special divisor $D_{g+1}$ is uniquely defined by the system

\begin{equation}
\mathcal{F}(x) = 0, \quad \mathcal{G}(x, y) = 0,
\end{equation}

where $\mathcal{F}$ is a polynomial in $x$ of degree $g+1$, and $\mathcal{G}$ is a rational function linear in $y$, the both vanishing on $D_{g+1}$.

### 3.5. Inverse divisors

Let $D$ be all roots of a function $\mathcal{R}$, and $D$ splits into two divisors $D_1$, $D_2$ such that $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$. Then [22] can be rewritten in the form

\[
D_1 - (\deg D_1)\infty \sim -(D_2 - (\deg D_2)\infty).
\]

With Abel’s map applied to the both sides, one finds

\[
\mathcal{A}(D_1 - (\deg D_1)\infty) = -\mathcal{A}(D_2 - (\deg D_2)\infty).
\]

So the left hand side and the right hand side map into opposite points on Jacobian of the curve, say points $u$ and $-u \in \text{Jac}$. This consideration provides a way of constructing **inverse divisors**. If deg $D < 2g$ then one of or both $D_1$ and $D_2$ have degree less than $g$ and so are special. It is also possible to construct the inverse to a special divisor from a divisor $D$ of deg $D \geq 2g$.

In what follows we suppose that $D$ has degree no less than $2g$, and $D_1$ and $D_2$ are composed of the arbitrary and the dependent roots of $\mathcal{R}$ respectively. This two divisor are called **inverse divisors** with respect to $\mathcal{R}$.

**Remark 3.** As shown in subsections [3.2] and [3.3] if the divisors $D_1$ of arbitrary roots of $\mathcal{I}$ or $\mathcal{G}$ contains two points in involution, say $(a, b^{(1)})$ and $(a, b^{(2)})$, then the third point $(a, b^{(3)})$ related to them by involution is located in the divisor $D_2$ of dependent roots, which is the inverse divisor. So $D$ contains all three points related by involution and $\mathcal{R}$ factorises as

\begin{equation}
\mathcal{R}(x, y) = (x-a)^3\tilde{\mathcal{R}}(x, y), \quad \text{ord } \tilde{\mathcal{R}} = \text{ord } \mathcal{R} - 9.
\end{equation}

This property does not extend to rational functions of order higher than $2g + 1$.

**Remark 4.** Let $D_g^*$ be the inverse divisor to a degree $g + 1$ divisor $D_{g+1}$. That is $D_{g+1}$ serves as the arbitrary roots of a function $\mathcal{G}$ of order $2g + 1$. On the other hand, let $\tilde{D}_g$ be the inverse to $D_g^*$ with respect to a function $\mathcal{I}$ of order $2g$. So divisor $\tilde{D}_g = g\infty$ is equivalent to $D_{g+1} - (g + 1)\infty$. This is the underlying idea of the reduction algorithm presented below.
Similarly, one can construct the inverse divisor $D_g^+$ to a divisor $D_{g+p}$ of degree $g + p$. These two divisors are inverse with respect to the rational function $I$ of order $2g + p$ constructed from $D_{g+p}$. Using the rational function $I$ of order $2g$ constructed from $D_g^+$, one finds a divisor $D_g$ such that $D_g - g\infty$ is equivalent to $D_{g+p} - (g + p)\infty$.

3.6. **Connection with the Jacobi inversion problem.** Functions $I$ and $G$ of orders $2g$ and $2g + 1$ are closely connected with the functions which give a solution of the Jacobi inversion problem. The function $I$ of order $2g$ constructed from a degree $g$ divisor $D_g$, and normalised by the multiple $c$ such that

$$ 1/c = (-1)^{g+2} \left\{ \frac{1}{x_k \ldots x_k^m y_k x_k \ldots y_k x_k^{m-1}} \right\} $$

in the case of $(3,3m+1)$-curve, and

$$ 1/c = (-1)^{g+2} \left\{ \frac{1}{x_k \ldots x_k^m y_k x_k \ldots y_k x_k^{2m+1}} \right\} $$

in the case of $(3,3m+2)$-curve, coincides with $R_{2g}$ in (13). From the equation

$$ cI(x, y) = R_{2g}(x, y; u) $$

one finds the relations between values of $\varphi_{i,j}$ functions at $u = A(D_g)$ and coefficients of $\alpha_g$, $\alpha_x$ expressed in terms of coordinates of points which form $D_g$.

Denote by $D_g^* = \sum_{k=1}^d (x_k, y_k)$ the inverse divisor with respect to $I$. Recall that $R_{2g}$ is even with respect to $u$. Thus, $R_{2g}(x, y; -u) = cI(x, y)$, and $-u = A(D_g^*)$, see (27) Theorem 2.6. The same function $I$, up to a constant multiple, is constructed from $D_g^*$. Moreover, the function $I$ of order $2g$ serves as the implementation of the inversion law according to (14), since the both divisors $D_g$ and $D_g^*$ are reduced and so uniquely represent points $u$ and $-u \in \text{Jac}$.

Suppose that the function $G$ of order $2g + 1$ is constructed from a degree $g + 1$ divisor $D_{g+1}$ and normalised by the multiple $c$ such that

$$ 2/c = (-1)^{g+3} \left\{ \frac{1}{x_k \ldots x_k^m y_k x_k \ldots y_k x_k^{m+1}} \right\} $$

in the case of $(3,3m+1)$-curve, and

$$ 2/c = (-1)^{g+3} \left\{ \frac{1}{x_k \ldots x_k^m y_k x_k \ldots y_k x_k^{2m+1}} \right\} $$

in the case of $(3,3m+2)$-curve. Suppose also that $g$ points of $D_{g+1}$ are the same as in $D_g$, and the remaining point is such that the coefficient of the term of weight $2g$ is the same as in (13). Thus,

$$ cG(x, y) = R_{2g+1}(x, y; u), $$

and coefficients of $\gamma_y$, $\gamma_x$ can be expressed in terms of $\varphi$ functions at $u = A(D_g)$.

**Remark 5.** Equalities (35) and (39) allow to compute any $\varphi$ function at a fixed point $u = A(D_g)$. Indeed, only $2g$ of $\varphi$ functions form a basis in the differential field of Abelian functions on the Jacobian of the curve. On the other hand, the functions $I$ and $G$ have $2g$ coefficients all together.

4. **THE PROCESS OF REDUCTION OF A DEGREE $g + 1$ DIVISOR**

We use the following notation for coefficients of the function $I$ of order $2g$

\begin{align*}
C^1 : \quad & I(x, y) = y \sum_{k=0}^{m-1} \alpha_{3m-1-3k} x^k + \sum_{k=0}^{2m} \alpha_{6m-3k} x^k, \\
C^2 : \quad & I(x, y) = y \sum_{k=0}^{m} \alpha_{3m-3k} x^k + \sum_{k=0}^{2m} \alpha_{6m+2-3k} x^k,
\end{align*}
and the function $G$ of order $2g + 1$

\[(41a) \quad \mathcal{C}^1 : \quad G(x, y) = y \sum_{k=0}^{m} \gamma_{3m-3k} x^k + \sum_{k=0}^{2m} \gamma_{6m+1-3k} x^k,\]

\[(41b) \quad \mathcal{C}^2 : \quad G(x, y) = y \sum_{k=0}^{m} \gamma_{3m+1-3k} x^k + \sum_{k=0}^{2m+1} \gamma_{6m+3-3k} x^k.\]

### 4.1. Inverse to degree $g+1$ divisor

Let $D_{g+1} = \sum_{k=1}^{g+1} (x_k, y_k)$ be a strictly non-special divisor which is defined by polynomial $F(x) = \prod_{k=1}^{g+1} (x - x_k)$ and rational function $G$ as in (41) vanishing on $D_{g+1}$, according to Corollary 2. In this subsection we find the divisor $D_g$ which is inverse to $D_{g+1}$.

It is sufficient to define $D_g^*$ by the function $G$ and a polynomial $H$ in $x$ of degree $g$, as in Theorem 3. The polynomial $H$ is obtained from the system (29) which together with the form (40) we find the divisor $D_g^*$ which is inverse to $D_{g+1}$:

\[(42) \quad H(x) = \frac{Z_{2g+1}(x)}{F(x)} = \gamma_y(x)^3 f(x, -\gamma_x(x)/\gamma_y(x)).\]

where $Z_{2g+1}$ is divisible by $F$ due to $D_{g+1}$ is a subset of the roots of $G$.

**Example 3.** In the case of $(3,4)$-curve

$G(x, y) = y\gamma_y(x) + \gamma_x(x) = y(\gamma_0 x + \gamma_3) + \gamma_1 x^2 + \gamma_4 x + \gamma_7$,

and

\[\gamma_y(x) = \sum_{k=2}^{4} \frac{y_k (x - x_k)}{\prod_{j=1, j \neq k}^{4} (x_i - x_k)} = \sum_{k=1}^{4} y_k L_k^{(4)}(x),\]

\[\gamma_x(x) = \frac{1}{2} \sum_{k,j=1}^{4} \frac{y_k y_j \prod_{i=1, i \neq k,j} (x_i - x_k)(x_i - x_j)}{\prod_{i=1, i \neq k,j} (x_i - x_k)} = \frac{1}{2} \sum_{k,j=1}^{4} y_k y_j M_{k,j}^{(4)}(x).\]

Note that

\[\frac{1}{2} \sum_{j=1}^{4} M_{k,j}^{(4)}(x) = L_k^{(4)}(x),\]

\[\sum_{k=1}^{4} L_k(x) = 1.\]

### 4.2. Minimal function to define degree $g$ divisor

Given a strictly non-special divisor $D^*_g = \sum_{k=1}^{g} (x_k, y_k)$ defined by a polynomial $H$ of degree $g$ and a function $G$ as in (41) the both vanishing on $D^*_g$, find the minimal rational function $I$ of the form (40) which together with $H$ define $D_g^*$. In other words, we need to reduce the function $G$ of order $2g+1$ to the function $I$ of order $2g$.

Here we describe an algorithm of constructing $I$ from $H$ and $G$ by the example of a $(3,3m+1)$-curve. Let $H(x) = \sum_{i=1}^{3m} h_{3m-3i} x^i$ and recall that $G$ has the form (41a).

Define the required function $I$ as

\[(43) \quad I(x, y) = G(x, y)M(x) + H(x)N(x, y),\]

where

\[M(x) = \sum_{i=0}^{3m-1} a_{3m-3i} x^i,\]

\[N(x, y) = y \sum_{i=0}^{m-1} b_{3m-3i} x^i + \sum_{i=0}^{2m-1} c_{6m-2-3i} x^i \equiv y\nu_y(x) + \nu_x(x).\]
In the expression for \( I \) all coefficients at monomials \( y^x \) with \( i \geq m \) and \( x^i \) with \( i \geq 2m+1 \) should vanish. Thus, a system of \( 6m-1 \) linear equations in \( 6m \) unknown parameters \( p = \{ a_0, \ldots, a_{9m-3}, b_0, \ldots, b_{3m-3}, c_1, \ldots, c_{6m-2} \} \) arises, and its solution is the kernel of a linear transformation, its matrix is suitable to represent in the block form

\[
\begin{pmatrix}
\Gamma_a & H_a \\
\Gamma_b & H_b
\end{pmatrix}.
\]

Rows are divided into two blocks: corresponding to equations at monomials from \( y^x^{4m-1} \) to \( y^x^m \), and monomials from \( x^m \) to \( x^{2m+1} \). Columns are divided into parts corresponding to \( \{ a_0, \ldots, a_{9m-3} \} \) and \( \{ b_0, \ldots, b_{3m-3}, c_1, \ldots, c_{6m-2} \} \). So block \( \Gamma_a \) of size \( 3m \times 3m \) is lower triangular and \((m+1)\)-diagonal of the form

\[
\Gamma_a = \begin{pmatrix}
\gamma_0 & 0 & \ldots & 0 & \ldots & 0 \\
\gamma_3 & \gamma_0 & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \ddots & \vdots \\
3 & \gamma_3 & \gamma_0 & \ddots & \vdots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 & \vdots \\
\vdots & \ddots & \gamma_3 & \vdots & \ddots & \ddots \\
0 & \ddots & \gamma_0 & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & \gamma_3 & \gamma_0
\end{pmatrix},
\]

block \( \Gamma_b \) of size \((3m-1) \times 3m \) is lower triangular with a zero column on the right

\[
\Gamma_b = \begin{pmatrix}
\gamma_1 & 0 & \ldots & 0 \\
\gamma_4 & \gamma_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
\gamma_{6m+1} & \gamma_{6m-2} & \ldots & \gamma_1
\end{pmatrix},
\]

blocks \( H_a \) of size \( 3m \times 3m \) has \( 2m \) zero columns on the right, and \( H_b \) of size \((3m-1) \times 3m \) has \( m \) zero columns on the left, namely

\[
H_a = \begin{pmatrix}
h_0 & 0 & \ldots & 0 & 0 & 0 \\
h_3 & h_0 & \ddots & \vdots & 0 & \vdots \\
\vdots & \ddots & \ddots & 0 & \ddots & \ddots \\
h_{3m-3} & h_{3m-6} & \ddots & h_0 & 0 & \vdots \\
h_{3m} & h_{3m-3} & \ldots & h_3 & 0 & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
h_{9m-3} & h_{9m-6} & \ldots & h_{6m} & 0 & \vdots \\
0 & 0 & \ldots & 0 & h_0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & h_3 & h_0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & h_{6m-3} & h_{6m-6} & \ldots & h_0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & h_{9m-6} & h_{9m-9} & \ldots & h_{3m-3}
\end{pmatrix},
\]

The kernel of \( HH \) is one-dimensional, it is convenient to choose one of parameters \( \{ b_0, \ldots, b_{3m-3}, c_1, \ldots, c_{6m-2} \} \) as arbitrary and use it to avoid denominators in expressions for the unknowns.
In the case of (3, 4)-curve

\[ I(x, y) = \left( h_0 \gamma_0^2 x^2 + (h_3 \gamma_0^2 - h_0 \gamma_0 \gamma_3) x + \gamma_0^2 h_6 - h_3 \gamma_0 \gamma_3 + h_0 \gamma_3^2 \right) G(x, y) \]

\[ - \gamma_0 \left( \gamma_0^2 y + \gamma_0 \gamma_1 x + \gamma_0 \gamma_4 - \gamma_1 \gamma_3 \right) H(x). \]

Similarly, one can construct function \( I \) in the case of (3, 3m + 2)-curve.

**Remark 6.** In the expression for \( I \) function \( G \) is multiplied by polynomial \( M \) of degree \( q - 1 \), one less than the degree of \( H \). And polynomial \( H \) is multiplied by function \( N \) of order \( wgt G - wgt x \), that is \( \deg \nu_y = \deg \gamma_y - 1 \) and \( \deg \nu_x = \deg \gamma_x - 1 \).

### 4.3. Inverse to degree \( g \) divisor

Let \( D_g^* = \sum_{k=1}^{\mu} (x_k^*, y_k^*) \) be a strictly non-special divisor defined by a polynomial \( H \) in \( x \) of degree \( g \) and a rational function \( I \) of order \( 2g \) the both vanishing on \( D_g^{++} \). Find the divisor \( \hat{D}_g \) inverse to \( D_g^* \).

According to Corollary 1, we define the inverse divisor by the given function \( I \) and a polynomial \( \hat{H} \) in \( x \) of degree \( g \). Let \( \gamma \) be obtained from the system \( 43 \), namely:

\[ \hat{H}(x) = \frac{Z_{2g}(x)}{\hat{H}(x)} = \frac{\alpha(x)^2 f(x, \alpha(x)/\alpha(x))}{\hat{H}(x)} \]

where \( Z_{2g} \) is divisible by \( H \) due to \( D_g^* \) is a subset of the roots of \( I \).

Therefore, divisor \( \hat{D}_g \) is defined by

\[ \hat{H}(x) = 0, \quad I(x, y) = 0. \]

### 4.4. Minimal function to define degree \( g + 1 \) divisor

Let divisor \( D_g^{++} \) be composed from \( D_g = \sum_{k=1}^{\mu} (x_k, y_k) \) and an additional point \( (\bar{x}, \bar{y}) \). Divisor \( \hat{D}_g \) is defined by a polynomial \( \hat{H} \) in \( x \) of degree \( g \) and a rational function \( I \) of order \( 2g \). Now define \( D_g^{++} \) according to Corollary 2 that is construct polynomial \( F \) in \( x \) of degree \( g + 1 \) and a rational function \( G \) of order \( 2g + 1 \). Polynomial \( F \) has the form

\[ F(x) = \hat{H}(x)(x - \bar{x}). \]

Function \( G \) is obtained from the known \( \hat{H} \) and \( I \) in the manner shown in subsection 12. The case of (3, 3m + 1)-curve is considered. Let \( \hat{H}(x) = \sum_{k=0}^{3m} h_{3m-3k} x^k \) and \( I \) has the form \( 46a \). Define

\[ G(x, y) = I(x, y) M(x) + \hat{H}(x) N(x, y), \]

where

\[ M(x) = \sum_{i=0}^{3m-1} a_{3m-3i} x^i, \]

\[ N(x, y) = y \sum_{i=0}^{m-2} b_{3m-3-3i} x^i + \sum_{i=0}^{2m-1} c_{6m-2-3i} x^i \equiv y \nu_y(x) + \nu_x(x). \]

In \( 46b \) equate coefficients at monomials \( y x^i \) with \( i \geq m + 1 \) and \( x^i \) with \( i \geq 2m + 1 \) to zero. The equations of number \( 6m - 2 \) are linear in \( 6m - 1 \) parameters \( p = \{ a_0, \ldots, a_{9m-3}, b_3, \ldots, b_{3m-3}, c_1, \ldots, c_{6m-2} \} \). Arrange the coefficients into block matrix of the form \( 46c \) such that rows are separated into two parts corresponding to monomials \( \{ y x^{4m-2}, \ldots, y x^{m+1} \} \) and \( \{ x^{4m-1}, \ldots, x^{2m+1} \} \), and columns are separated as \( \{ a_0, \ldots, a_{9m-3} \} \) and \( \{ b_3, \ldots, b_{3m-3}, c_1, \ldots, c_{6m-2} \} \). Block \( \Gamma_a \) of
size \((3m - 2) \times 3m\) is lower triangular and \(m\)-diagonal with two zero columns on the right, namely

\[
\Gamma_a = \begin{pmatrix}
\alpha_2 & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 \\
\alpha_5 & \alpha_2 & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots \\
& \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots \\
\alpha_{3m-1} & \alpha_5 & \alpha_2 & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & \alpha_{3m-1} & \ldots & \alpha_5 & \alpha_2 & 0 & 0
\end{pmatrix},
\]

block \(\Gamma_b\) of size \(3m \times 3m\) is lower triangular

\[
\Gamma_b = \begin{pmatrix}
\alpha_0 & 0 & \ldots & 0 & 0 \\
\alpha_3 & \alpha_0 & \ddots & \vdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
\alpha_{6m} & \ddots & \alpha_0 & 0 \\
0 & \alpha_{6m} & \ldots & \alpha_3 & \alpha_0
\end{pmatrix},
\]

block \(H_a\) of size \((3m - 2) \times (3m - 1)\) has 2\(m\) zero columns on the right, and \(H_b\) of size \(3m \times (3m - 1)\) has \(m - 1\) zero columns on the left

\[
H_a = \begin{pmatrix}
h_0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
h_3 & h_0 & \ddots & \vdots & 0 & \ddots & \ddots \\
& \ddots & \ddots & 0 & \ddots & \ddots & \ddots \\
h_{3m-6} & h_{3m-9} & \ddots & h_0 & 0 & \ldots & 0 \\
h_{3m-3} & h_{3m-6} & \ldots & h_3 & 0 & \ldots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
h_{9m-9} & h_{9m-12} & \ldots & h_{6m-3} & 0 & \ldots & 0
\end{pmatrix},
\]

\[
H_b = \begin{pmatrix}
0 & \ldots & 0 & h_0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & h_3 & h_0 & \ldots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & h_{6m-3} & h_{6m-6} & \ldots & h_0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & h_{9m-3} & h_{9m-6} & \ldots & h_{6m-6}
\end{pmatrix},
\]

Solution of the system is one-dimensional kernel of linear transformation defined by the block matrix. As an arbitrary parameter we choose \(a_{9m-3}\). Let \(\tilde{G}(x, y)\) be \(\text{LG}^3\) as \(a_{9m-3} = 0\). Then putting

\[
a_{9m-3} = \frac{\tilde{G}(\bar{x}, \bar{y})}{I(\bar{x}, \bar{y})},
\]

guarantees that \(G(x, y) = 0\). Factor \(c\) allows to avoid denominators in expressions for the unknowns.
In the case of (3,4)-curve situation is much simpler, and the required function has the form
\[ G(x, y) = \left( h_0x - \frac{h_0x \bar{I}(\bar{x}, \bar{y}) - \bar{H}(\bar{x})}{\bar{I}(\bar{x}, \bar{y})} \right) \bar{I}(\bar{x}, \bar{y}) - \tilde{H}(x). \]

Remark 7. In the expression for \( G \) function \( \bar{I} \) is multiplied by polynomial \( M \) of degree \( g - 1 \), which is \( \deg \tilde{H} - 1 \). The free term of \( M \) is used to introduce the additional point \((\bar{x}, \bar{y})\). Polynomial \( \tilde{H} \) is multiplied by a function \( N \) of order \( \text{wgt} I - \text{wgt} x \), that is \( \deg \nu_y = \deg \alpha_y - 1 \) and \( \deg \nu_x = \deg \alpha_x - 1 \).

5. Reduction Algorithm

In this section we present an algorithm which solves the reduction problem for a degree \( g + p \) divisor \( D_{g+p} \) with \( p > 0 \). In this direction the problem is solved uniquely. Conversely, a certain reduced divisor corresponds to a variety of divisors of degree greater than \( g \).

Start with an arbitrary collection of \( g + 1 \) points of divisor \( D_{g+p} \).

**Step 1** Given divisor \( D_{g+1} \) defined by
\[ F(x) = 0, \quad G(x, y) = 0, \]
where \( F \) is a polynomial in \( x \) of degree \( g + 1 \) and \( G \) is a function of order \( 2g + 1 \) on the curve, find degree \( g \) polynomial \( H \) by (42), for more details see subsection 4.1. In this way the complement divisor \( D_*^g \) is obtained, it is defined by the system
\[ H(x) = 0, \quad G(x, y) = 0. \]

**Step 2** Construct the minimal function \( \bar{I} \) of order \( 2g \) to define the divisor \( D_*^g \) from the previous step. The new definition of \( D_*^g \) is given by the system
\[ \bar{H}(x) = 0, \quad \bar{I}(x, y) = 0. \]

In detail the procedure of reducing \( G \) to \( \bar{I} \) is explained in subsection 4.2.

**Step 3** Find the inverse to \( D_*^g \), which is the reduced divisor \( \tilde{D}_g \) corresponding to \( D_{g+1} \) from Step 1. Divisor \( \tilde{D}_g \) is defined by the system
\[ \tilde{H}(x) = 0, \quad \bar{I}(x, y) = 0, \]
where degree \( g \) polynomial \( \tilde{H} \) is computed by (45). A detailed explanation is given in subsection 4.3.

**Step 4** Compose a new divisor \( D_{g+1} \) from \( \tilde{D}_g \) and point \((\bar{x}, \bar{y})\) taken from the initial divisor \( D_{g+p} \). The new divisor \( D_{g+1} \) is defined by the new system
\[ F(x) = \tilde{H}(x)(x - \bar{x}), \quad G(x, y) = 0, \]
where \( F(x) = \tilde{H}(x)(x - \bar{x}) \), and function \( G \) is constructed as explained in subsection 4.4. These new functions \( F \) and \( G \) are initial for the next iteration starting from Step 1.

6. Conclusion

In this paper we use rational functions arising in a solution of the Jacobi inversion problem to represent divisors. The Mumford representation in terms of two polynomials, which is widely used, we consider as improper since it leads to unnecessary growth of the order of functions in representation of divisors. Also the rational functions of orders \( 2g \) and \( 2g + 1 \) are found explicitly. Detailed analysis of these functions gave an instrument to analyse the structure of divisors and determine the presence of points in involution in divisors.
Addition is realised through the reduction algorithm with deals with divisors of degree not greater than \( g + 1 \) on each step, so the rational functions of orders \( 2g \) and \( 2g + 1 \) are sufficient to represent such divisors. The proposed algorithm is straightforward and quite elementary.

We see several applications of the result represented in the paper. The first is a generalisation of addition laws of Abelian functions whose study was initiated in [1]. The addition law [1] involves only non-special divisors of degree \( g \). Now the class of divisors involved into the addition law is extended to non-special divisors of arbitrary degree. The second is a solution of the generalised Jacobi inversion problem, which is stated for divisors of degree higher than \( g \). Indeed, the reduction algorithm produces the rational functions which represent the reduced divisor corresponding to the initial one of degree higher than \( g \). So values of all \( \wp \) functions at the initial divisor can be computed, recall Remark 5. It is also possible to obtain explicit expressions in terms of the support of the initial divisor, as it performed in [11] for hyperelliptic curves.

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