High dimensional regression for regenerative time-series: an application to road traffic modeling

Mohammed Bouchouia And François Portier

 Télécom Paris, Institut Polytechnique de Paris

Abstract

This paper investigates statistical models for road traffic modeling. The proposed methodology considers road traffic as a (i) high-dimensional time-series for which (ii) regeneration occurs at the end of each day. Since (ii), prediction is based on a daily modeling of the road traffic using a vector autoregressive model that combines linearly the past observations of the day. Considering (i), the learning algorithm follows from an $\ell_1$-penalization of the regression coefficients. Excess risk bounds are established under the high-dimensional framework in which the number of road sections goes to infinity with the number of observed days. Considering floating car data observed in an urban area, the approach is compared to state-of-the-art methods including neural networks. In addition of being very competitive in terms of prediction, it enables to identify the most determinant sections of the road network.

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1 Introduction

Due to the side effects of traffic congestion, including for instance, pollution and economically ineffective transportation (Bull, 2003; Pellicer et al., 2013; Harrison and Donnelly, 2011), achieving smart mobility has become one of the leading challenges of emerging cities (Washburn et al., 2009). To set up effective solutions, such as developing intelligent transportation management systems for urban planners, or extending the road network efficiently, smart cities need to understand road traffic precisely. This paper investigates interpretable predictive models estimated from floating car data, allowing to identify the determinants of road traffic.

The proposed methodology addresses two important issues related to floating car data. First, a particular point contrasting with traditional time series analysis is that the number of vehicles using the network almost vanishes during each night. Hence in terms of probabilistic dependency, the road traffic between the different days is assumed to be independent. Such a phenomenon is referred to as "regeneration" in the Markov chains literature (Meyn and Tweedie, 2012), and we say that the road network "regenerates" at the beginning of each new day. Second, the size of the road network, especially in urban areas, can be relatively large compared to the number of observed days. This implies that the algorithms employed must be robust to the well-known high-dimensional regression setting in which the number of features is large.

For any time $1 \leq t \leq T$ of day $1 \leq i \leq n$, denote by $W_{t}^{(i)} \in \mathbb{R}^p$ the vector of speeds registered in the road network. Hence $p$, $n$ and $T + 1$ stand for the number of sections in the network, the number of days in the study, and the number of time instant within each day. Inspired by time series analysis, the proposed model is similar to the popular vector auto-regressive (VAR) model, as described in the econometric literature (Brockwell et al., 1991; Hamilton, 1994), with the difference that it only applies within each day due to the regeneration property. We therefore consider the following model, that we called regenerative VAR,

$$W_{t+1}^{(i)} \simeq b_t + AW_{t}^{(i)}, \quad \text{for each } t \in \{0, \ldots, T-1\}, \ i \in \{1, \ldots, n\},$$

(1)
with $A \in \mathbb{R}^{p \times p}$ and $b_t \in \mathbb{R}^p$. The parameter $A$ encodes for the influence between different road sections and the parameters $(b_t)_{t=0, \ldots, T-1}$ account for the daily (seasonal) variations.

The approach taken for estimating the parameters of the regenerative VAR, $(b_t)_{t=0, \ldots, T-1}$ and $A$, follows from applying ordinary least-squares (OLS) while penalizing the coefficients of $A$ using the $\ell_1$-norm just as in the popular Lasso procedure [Tibshirani, 1996; Hastie et al., 2015]. The estimator is computed by minimizing over $(b_t)_{t=0, \ldots, T-1} \in \mathbb{R}^{p \times T}$ and $A \in \mathbb{R}^{p \times p}$ the following objective function

$$
\sum_{t=0}^{T-1} \sum_{i=1}^{n} \|W_{t+1}^{(i)} - b_t - AW_t^{(i)}\|_F^2 + 2\lambda \|A\|_1.
$$

(2)

Whereas the estimation of VAR models is well-understood for decades (Brockwell et al., 1991; Hamilton, 1994), only recently penalization has been introduced to estimate the model coefficients, see among others (Valdés-Sosa et al., 2005; Wang et al., 2007; Haufe et al., 2010; Song and Bickel, 2011; Kock and Caliskan, 2015; Basu et al., 2015; Baek et al., 2017). The previous references advocate for the use of the Lasso or some of its variants in time-series prediction when the dimension of the time series is relatively large. For instance, in Song and Bickel (2011), the penalization of the variables’ own lags is different from others’ lags. In particular, the others’ lags penalization is based on the group Lasso procedure (Yuan and Lin, 2006). Note in passing that the group Lasso has been used successfully in multi-task learning problems (with independent data) when a common sparsity structure is shared among the tasks (Obozinski et al., 2011; Lounici et al., 2009). Different penalization procedure such as the Ridge, the Lasso, the group Lasso are investigated in Haufe et al. (2010). Other approaches achieving selection in VAR models but without using the Lasso are proposed in Davis et al. (2016).

From a theoretical standpoint, the aim is to provide guarantees on the prediction performance of the procedure described in (2) when $p$ is large compared to $n$. We first establish a bound on the predictive risk, defined as the normalized $L_2$-error between the true value and the prediction, for the case $\lambda = 0$. This situation corresponds to ordinary least-squares. The order is in $p^2/n$ and therefore deteriorates when the value of $p$ is getting large. Moreover, the conditions for its validity are rather strong because they require the smallest eigenvalue of the Gram matrix to be sufficiently large. Then under a sparsity assumption on the matrix $A$, claiming that only a set of $s$ (rather than $p^2$) are useful to predict, we study the regularized case when $\lambda > 0$. In this case, even when $p$ is much larger than $n$, we obtain a bound order in $s/n$ (up to a logarithmic factor), and the eigenvalue condition is fairly alleviated as it concerns only the eigenvalues restricted to the active variables. One of the difficulties of the present approach relies on controlling the eigenvalues of the design matrix which is random. We rely on recent results developed in (Tropp et al., 2015). Because of the regeneration property, the approach taken here is fairly different than the ones of time-series analysis but rather close to random design high-dimensional regression studies including for instance (Bunea et al., 2007; Van De Geer and Bühlmann, 2009).

From a practical perspective, the regenerative VAR, in virtue of its simplicity, contrasts with the past approaches, mostly based on deep neural networks, that have been used to handle road traffic data. For instance, in (Lv et al., 2014), autoencoders are used to extract spatial and temporal information from the data before prediction. In (Dai et al., 2017), multilayer perceptrons (MLP) and long short-term memory (LSTM) are combined together to analyze time-variant trends in the data. Finally, LSTM with spatial graph convolution are employed in (Lv et al., 2018; Li et al., 2017). We now point out three advantages of the proposed approach:

(i) It enables to consider very large road networks including typically not only the main roads but also the primary roads. The previous neural networks methods either use data collected from fixed sensors on the main roads (Dai et al., 2017; Lv et al., 2014) or, when using floating car data, restrict the network to the main roads ignoring primary roads (Epelbaum et al., 2017). Even if this prevents from overfitting, as it avoids a large number of features compared to the sample size, there might be some loss of information in reducing the data to an arbitrary subset.

(ii) The estimated coefficients are easily interpretable due to the linear model and the Lasso selection procedure which shrinks to zero irrelevant sections. This very last point provides data-driven graphical representations of the dependency within the network that could be useful for road maintenance. Once again, this is contrasting with complex deep learning models in which interpretation is known to be difficult.
Figure 1: (a) Rennes road network with representatives sections, (b) section with cyclic speed behavior, (c) high speed section, (d) low speed section, (e) average behavior of all sections in the road network.

(iii) Changes in the distribution of road traffic during the day might be handled easily using a regime switching approach. It actually consists in a simple extension of the initial regenerative VAR proposed in (1) in which the matrix $A$ is allowed to change during time.

To demonstrate the practical interest of the proposal, the data used is concerned with the urban area of a French city, Rennes, made of $n = 144$ days and $p = 556$ road sections (see Figure 1). Among all the considered methods, including classical baseline from time series analysis as well as the most recent neural network architecture, this is the regime switching model that has the best performance.

The outline is as follows. In section 2, the probabilistic framework is introduced. The optimal linear predictor is characterized, and the main assumptions are discussed. In section 3, we present the main theoretical results of the paper that are bounds on the prediction error of (2). Section 4 investigates the regime switching variant. A comparative study of the practical behavior of different methods is proposed in section 5. All the proofs of the stated results are given in the Appendix.

2 Probabilistic framework

Consider a set of $p \geq 1$ real-valued variables of interest that evolve during a time period $\mathcal{T} = \{0, \ldots, T\}$. These variables are gathered in a matrix of size $p \times (T+1)$ that we denote $W = (W_{k,t})_{k \in \mathcal{S}, t \in \mathcal{T}} \in \mathbb{R}^{p \times (T+1)}$ with $\mathcal{S} = \{1, \ldots, p\}$. In the running example of road traffic, $k \in \mathcal{S}$ stands for the section index of the road network, $t \in \mathcal{T}$ for the time instants within the day, and $W_{k,t}$ is the speed recorded at section $k$ and time $t$. The following probabilistic framework will be adopted.

**Assumption 1** (probabilistic framework). Let $(\Omega, \mathcal{F}, P)$ be a probability space. The matrix $W = (W_{\cdot,0}, \ldots, W_{\cdot,T}) \in \mathbb{R}^{p \times (T+1)}$ is a random element defined on $(\Omega, \mathcal{F})$ and valued in $(\mathbb{R}^{p \times (T+1)}, \mathcal{B}(\mathbb{R}^{p \times (T+1)}))$ with distribution $P$. For all $k \in \mathcal{S} = \{1, \ldots, p\}$ and $t \in \mathcal{T} = \{0, \ldots, T\}$, we have $E[W_{k,t}^2] < +\infty$ ($E$ stands for the expectation with respect to $P$).

The task of interest is to predict the state variable at time $t$, $W_{\cdot,t}$, using the information available at time $t-1$, $W_{\cdot,t-1}$. For a matrix $A \in \mathbb{R}^{p \times p}$ and an intercept collection $b = (b_{\cdot,1}, \ldots, b_{\cdot,T}) \in \mathbb{R}^{p \times T}$, define the linear predictor $L$, given by

$$L(W_{\cdot,t-1}) = b_t + AW_{\cdot,t-1}, \quad \forall t \in \{1, \ldots, T\}.$$
Proposition 1. Suppose that Assumption 1 is fulfilled. There exists a unique minimizer $L^* \in \text{argmin}_{L \in \mathcal{L}} R(L)$. Moreover $L = L^*$ if and only if
\[
\begin{align*}
E[Y - L(X)] &= 0, \\
E[(Y - L(X))X^T] &= 0.
\end{align*}
\]

The following decomposition of the risk underlines the prediction loss associated to the predictor $L$ by comparing it with the best predictor $L^*$ in $\mathcal{L}$. Define the excess risk by
\[
\mathcal{E}(L) = R(L) - R(L^*), \quad L \in \mathcal{L}.
\]

Proposition 2. Suppose that Assumption 2 is fulfilled. It holds:
\[
\forall L \in \mathcal{L}, \quad \mathcal{E}(L) = E[\|L(X) - L^*(X)\|_F^2].
\]

Assumption 2 (daily regeneration). The sequence $(W^{(i)})_{1 \leq i \leq n}$ is an independent and identically distributed sequence of random variables defined on $(\Omega, \mathcal{F}, P)$.

In our running example, the “daily regeneration” assumption means that at the end of each day, the network vanishes and then regenerates the next day with the same distribution. This is in accordance with the practical use of the road network where at night only a few people use the network so it can regenerate and hence forgets its past. The fact that each day has the same distribution means essentially that a similar type of day is considered in the data. For instance, this assumption might not hold when mixing weekend days and workdays. This assumption also implies that the network structure remains the same during the study.

Alternative models Two alternative modeling approach might have been considered at the price of additional notation and minor changes in the proofs of the results presented in the next section. The first model is obtained by imposing the matrix $A$ to be diagonal. In this case, each 1-dimensional vector coordinates is fitted using an auto-regressive model based on one single lag. Other types of restrictions might be considered. The second class is to use more than one lag, say $H \geq 1$, to predict the next coming instance. This is done by enlarging the matrix $A$ to $(A_1, \ldots, A_H)$ where each $A_h \in \mathbb{R}^{p \times p}$ and by stacking in $X$ the $H$ previous lags. These types of variations though interesting, are not presented in the paper for the sake of readability. Another alternative modeling approach which will be addressed in section 4 consists in allowing the matrix $A$ to change across time and to detect those changes from the data.
3 Empirical risk minimization

Given a sequence \((W(i))_{1 \leq i \leq n} \subset \mathbb{R}^{p \times (T+1)}\), define \((b_n, A_n)\) as a minimizer of

\[
\sum_{i=1}^{n} \|Y(i) - b - AX(i)\|_F^2 + 2\lambda\|A\|_1, \tag{4}
\]

with \(Y(i) = (W(i)_{1,1}, \ldots, W(i)_{1,T})\), \(X(i) = (W(i)_{2,1}, \ldots, W(i)_{2,T-1})\), \(\lambda \geq 0\). For any matrix \(A \in \mathbb{R}^{p \times p}\), the \(\ell_q\)-norm \((q \geq 1)\) is defined as \(\|A\|_q^q = \sum_{1 \leq k, \ell \leq p} |A_{k,\ell}|^q\). When \(\lambda = 0\), referred to as OLS, it consists basically of applying the empirical risk minimization paradigm with a quadratic loss. When \(\lambda > 0\), referred to as Lasso, the empirical risk is regularized with an \(\ell_1\)-penalty on the matrix \(A\). Both estimators are treated using a common decomposition of the excess risk which is presented in the subsequent section. Next, we shall focus on the excess risk of the OLS and the Lasso.

3.1 A useful decomposition

The prediction at point \(X \in \mathbb{R}^{p \times T}\) is given by \(L_n(X) = b_n + A_nX\). As the intercept in classical regression, the matrix parameter \(b_n\) is only a centering term. Indeed, when minimizing \(\ell_2\) with respect to \(b\) and with \(A\) fixed, we find \(b = n^{-1}\sum_{i=1}^{n} Y(i) - A_nX(i)\). The value of \(A_n\) can thus be obtained by solving the least-squares problem \(\ell_2\) without intercept and with empirically centered variables. Given \(A_n\), the matrix \(b_n\) can be recovered using the simple formula \(b_n = n^{-1}\sum_{i=1}^{n} Y(i) - A_nX(i)\). Consequently, the prediction at point \(X\) may be written as

\[ L_n(X) = \overline{Y}^n + A_n(X - \overline{X}^n), \]

where, generically, \(\overline{M}^n = n^{-1}\sum_{i=1}^{n} M(i)\). Note that, in a similar way, one has \(L^\star(X) = E[Y] + A^\star(X - E(X))\). The previous two expressions emphasize that the excess risk might be decomposed according to 2 terms: the one dealing with the error relative to the estimation of \(A^\star\) and the one relative to the error on the averages \(E[Y]\) and \(E[X]\). This will be the key to control the excess risk of both the OLS predictor and the Lasso predictor. We now state this decomposition. Define

\[ \Sigma = E[(X - E(X))(X - E(X))^T]. \]

**Proposition 3.** Suppose that \((b_n, A_n)\) minimizes \(\ell_p\) with \(\lambda \geq 0\). Under Assumption 4, it holds that

\[
\mathcal{E}(L_n) \leq \|(A^\star - A_n)\Sigma^{1/2}\|_F^2 + 2\|E(Y) - \overline{Y}^n\|_F^2 \\
+ 2\|A_n(E(X) - \overline{X}^n)\|_F^2.
\]

3.2 Ordinary least-squares

The OLS estimate \((b_n, A_n^{ols})\) is defined as any minimizer of \(\ell_2\) when \(\lambda = 0\). Denote by \(L_n^{(ols)}\) the associated predictor. The aim is to establish a bound on the excess risk \(\mathcal{E}(L_n^{(ols)})\) in an asymptotic regime where both \(n\) and \(p\) go to infinity. Though theoretic, this regime is of practical interest as it permits to analyze cases where the number of sections \(p\) is relatively large (in particular large before \(n\)). Hence in the assumptions, we clarify the dependence in \(p\) of the introduced quantities. The following assumption claims that the covariates are bounded uniformly in the number of sections \(p\).

**Assumption 3** (bounded variables). With probability 1,

\[
\lim_{p \to \infty} \max_{k \in \mathcal{S}, t \in T} |X_{k,t} - E[X_{k,t}]| < \infty.
\]

The following invertibility condition on \(\Sigma\) can be seen as an identification condition as the matrix \(A^\star\) is unique under this hypothesis.

**Assumption 4.** For any \(p \geq 1\), the matrix \(\Sigma\) is invertible. Denote by \(\gamma > 0\) (resp. \(\Gamma\)) its smallest (resp. greatest) eigenvalue.
A certain control on the smallest eigenvalue $\gamma$ compared to $p$ and $n$ will actually be required to derive the upper bound on the excess risk. Finally, introduce the noise level $\sigma^2 > 0$ which consists in a bound on the conditional variance of the residual variables

$$\epsilon_{k,t} = \{Y - L^*(X)\}_{k,t},$$

given the covariates $X_t$. Formally, $\sigma^2$ is the smallest positive real number such that, with probability $1$,

$$\max_{k \in S, \ell \in T} E[\epsilon_{k,t}^2 | X_{\ell,t}] \leq \sigma^2.$$

Note that $\sigma^2$ might depend on $p$. The fact that the bound does not depend on $X$ stresses that the underlying regression model is homoscedastic.

The following result provides a bound on the excess risk in terms of the OLS procedure and valid with probability going to $0$. The result bound depends explicitly on the quantities of interest $n$ and $p$ as well as on the underlying probabilistic model through $\sigma$, $\Sigma$ and $A^*$. The asymptotic framework we consider is with respect to $n \to \infty$ and we allow the dimension $p$ to go to infinity (with a certain restriction).

The parameters that are particularly affected by $p$ are $\gamma$, which is expected to be small when $p$ becomes large, $\sigma^2$ which decreases as $p$ grows and naturally $\|A^*\|_F$. A discussion is provided below the proposition.

**Proposition 4.** Suppose that Assumptions 1, 2, 3 and 4 are fulfilled. Suppose that $n \to \infty$ and $p \to \infty$ such that $p \log(p)/(n \gamma) \to 0$, we have

$$\mathcal{E}(L_n^{(ols)}) = O_p\left(\frac{p^2 \sigma^2 + p + \Gamma \|A^*\|_F^2}{n}\right).$$

When the number of parameters in $A$, $p^2$, is large compared to the number of examples $n$, the OLS estimator encounters the two classical problems: (i) a deterioration of the statistical performance and (ii) some numerical instability. Point (i) is illustrated by Proposition 4 the bound on the excess risk of the $L_n^{(ols)}$ is badly affected by the parameter $p$. First of all, the condition $p \log(p)/(n \gamma) \to 0$ implies that $p \ll n$ which restrict the application of the OLS method. Note that if some linearities are observed between the covariates, the previous condition might be even more difficult to realize. Second, one may ask whether the excess risk goes to $0$ or not. As the excess risk is a sum of quadratic risks, each corresponding to a particular section of the network, it is suitable for this discussion to normalize its value by $p$. The two leading terms are $p \sigma^2/n$ and $\Gamma \|A^*\|_F^2/p n$. The first one underlines a balance between the number of covariates and the information they carry to predict the output. For the second term, suppose for simplicity that $\Gamma = 1$. When $A^* = I$, it equals $1/n$ and becomes negligible. In contrast, when $A = (1)_{k,t}$, i.e., all the covariates are used to predict each output, it equals $p/n$ and becomes the leading term. In between, we have the situation where each line of $A$ possesses only a few non-zero coefficients, say $s = o(p)$. Then the order of this term is $s/n$.

### 3.3 Regularized least-squares

The regularized least-squares approach is when solving (3) with $\lambda > 0$. This approach, which is similar to the Lasso approach, is introduced to remedy the difficulties of the OLS, (i) and (ii), previously discussed. The aim of such a procedure is to enforce the matrix $A_n^{(lasso)}$ to be “sparse”, i.e., to have only a few non-zero coefficients. This permits to reduce the number of parameters to estimate when $n$ is small compared to $p$ and of course to take advantage of any sparsity structure in the matrix $A^*$.

Let $\lambda > 0$ and define $(b_n, A_n^{(lasso)})$ as any minimizer of (3). Because the Frobenius norm writes as the sum of the $\ell_2$-norms over the lines of the matrix, problem (3) might be written in terms of $p$ sub-problems each having $T$ outputs. This allows to select different $\lambda$ in each sub-problem. For the sake of presentation, we prefer to keep fixed the value of $\lambda$ during the theoretical analysis.

Introduce the active set $S_k^\lambda$, the set of non-zero coefficients of the $k$-th line of $A^*$, i.e., for each $k \in \{1, \ldots, p\}$,

$$S_k^\lambda = \{ \ell \in \{1, \ldots, p\} : A_{k,\ell}^\lambda \neq 0 \}.$$

The whole sparsity level is denoted by $s = \sum_{k=1}^p |S_k^\lambda|$. The following additional assumption is related to the approach taken in the proof in which we rely on the Bernstein concentration inequality.
Assumption 5. With probability 1, \( \limsup_{p \to \infty} \max_{k \in S, t \in T} |\epsilon_{k,t}| < \infty \) and \( \liminf_{p \to \infty} \sigma > 0 \).

The following assumption is a relaxation of Assumption 4 which was needed to study the OLS approach. For any set \( S \subset \mathcal{S} \), introduce the restricted covariates \( X_{\{S,\}} = (X_{k,t})_{k \in S, t \in T} \) and the associated Gram matrix \( \Sigma_S = E[X_{\{S,\}} X_{\{S,\}}^T] \).

Assumption 6. There exists \( \gamma^* > 0 \) such that for all \( p \geq 1 \) and all \( k \in \{1, \ldots, p\} \), we have
\[
\gamma^* \|u\|_2, \quad \forall u \in \mathbb{R}^{S_k^c}.
\]

Assumption 7. \( \limsup_{p \to \infty} \max_{k \in S} |S_k| < \infty \).

Proposition 5. Suppose that Assumptions 1, 2, 3, 5, 6 and 7 are fulfilled. Suppose that \( n \to \infty \) and \( p \to \infty \) such that \( \log(p)/n \to 0 \), we have
\[
\mathcal{E}(L_{\text{Lasso}}) = O_p \left( \frac{s \sigma^2 \log(p) + p + \Gamma^* \|A^*\|^2_2}{n} \right),
\]
provided that \( \lambda = C \sigma^2 \log(p) \), for some constant \( C > 0 \).

The parameter \( p^2 \) which influences badly the bound of the OLS is now replaced by \( s \), the number of active variables. This shows that without any knowledge on the active variables, the Lasso approach enables to recover the accuracy (at the price of a logarithmic factor) of an “oracle” OLS estimator that would use only the active variables. Another notable advantage is that the assumptions for the validity of the bound have been reduced to \( \log(p)/n \to 0 \) compared to the OLS which needs \( p \log(p)/\gamma n \to 0 \).

Because of the regeneration property, the approach taken in the proofs is simpler than the ones of Song and Bickel (2011) where the error is supposed to be Gaussian. Second, regarding the distribution of the error, it is assumed to be independent of the covariates with a diagonal covariance matrix or with Kock and Callot (2015) where the error is supposed to be Gaussian. Second, regarding the distribution of \( \{W_t\}_{t=0,\ldots,T} \), there are no stationarity issues and the spectral decomposition of the matrix \( A \) is not subjected to any constraint as is usual in time series analysis.

4 Regime switching

In our framework of road traffic, regime switching is when the optimal matrix \( A^* \), which is used to predict the next speed configuration of the road network, changes during the day \( T \). This certainly occurs when the conditional distribution of road traffic given the past changes after some time in the day. For instance, it might happen that a certain matrix \( A^* \) is suitable to model the morning traffic while another matrix is needed to fit conveniently the afternoon behavior. In this approach, the set of predictors is given by
\[
\tilde{\mathcal{L}} = \{ b + (A_1 X_1, \ldots, A_T X_T) : b \in \mathbb{R}^{p \times T}, A_t \in \mathbb{R}^{p \times p}, \forall t = 1, \ldots, T \}
\]

The optimal predictor \( \tilde{L}^* = \arg\min_{L \in \tilde{\mathcal{L}}} R(L) \) satisfies some normal equations that can be obtained by applying Proposition 1 with \( T = \{ t - 1, t \} \) for each \( t = 1, \ldots, T \). Also a similar decomposition as the one in Proposition 2 is valid with \( \tilde{\mathcal{L}} \) in place of \( \mathcal{L} \). We have, for all \( L \in \tilde{\mathcal{L}} \),
\[
R(L) = R(\tilde{L}^*) + E[\|\tilde{L}^*(X_t) - L(X_t)\|^2_2].
\]
Because \( \mathcal{L} \subset \tilde{\mathcal{L}} \), one may choose \( L = L^* \) in the previous equation to express the gap in terms of risk between the time-homogeneous linear model \( \mathcal{L} \) and the linear model \( \tilde{\mathcal{L}} \). It is equal to 0 as soon as \( (A_t^*)_{t=1,\ldots,T} \), which characterizes \( L^* \), does not depend on \( t \). Hence the risk decomposition advocates (as usual) for the use of the largest model \( \tilde{\mathcal{L}} \). The drawback of considering a large model often comes from the estimation variance which increases with the number of parameters.
The proposed estimation approach realizes a trade-off between the two previously discussed strategies, the one based on $L$ (small variance) and the one based on $\tilde{L}$ (small bias). The regime switching is determined by the data in a greedy way. At each step of the procedure, we decide, using cross-validation, if a regime switching is beneficial. If it does, we continue. If it does not, we stop.

To be more specific, suppose that $T \geq 2$ and for each $t \in \{1, \ldots, T-1\}$, define $U_t = \{1, \ldots, t\}$ and $\tilde{U}_t = \{t+1, \ldots, T\}$. Let $\mathcal{F}$ be a partition of $\{1, \ldots, n\}$. Each element of $\mathcal{F}$ is called a fold. For each $F \in \mathcal{F}$, solve both optimization problems

$$\min_{b \in \mathbb{R}^{p \times 1}, A \in \mathbb{R}^{p \times p}} \sum_{i \in F} \|Y^{(i)}_{\{1, U_t\}} - b - AX^{(i)}_{\{U_t\}}\|^2_F + 2\lambda\|A\|_1,$$

$$\min_{b \in \mathbb{R}^{p \times (T-1)}, A \in \mathbb{R}^{p \times p}} \sum_{i \in F} \|Y^{(i)}_{\{1, \tilde{U}_t\}} - b - AX^{(i)}_{\{\tilde{U}_t\}}\|^2_F + 2\lambda\|A\|_1,$$

and compute the risk over the complementary of $F$ in $\{1, \ldots, n\}$. The resulting estimated risk is then the average of the previously computed risks over the folds $F \in \mathcal{F}$. Choose $t^*$ as the minimum over all the estimated risks. This $t^*$ represents the best instant from which a different (linear) model shall be used. Finally compare the risk associated to $t^*$ with the one without regime switching. If $t^*$ is selected then iterate the previous process with $t \in \{1, \ldots, T-1\} \setminus t^*$. If not, the model selected is the one that estimate a single matrix $A$ for the whole day.

5 Real data analysis

In this section, we conduct a comparative study of different methods. We first expose the real world data used for this purpose. Then, we present the different methods in competition and analyze the obtained results. Finally, we show graphical interpretations of the proposed Lasso approach.

5.1 Dataset

The initial dataset contains the speed (km/h) and the localization of each car using Coyote navigation system (Floating car data). Data is collected in the Rennes road network, over the period from 1st of December 2018 until the 9th of July 2019 every 30 seconds. Signals were received from 113577 different cars.

Some pre-processing has been done to correct sensor errors and structure the data in a convenient format. First, we map-match the locations to the OpenStreetMap road network dataset [Greenfeld 2002], so that we get a section for each car location. Then, to obtain proper sections data, we aggregate the observed speed values of cars into 15-minute intervals and average those for each section. In some cases, we do not get data for a given section and time interval in a given day, so, we impute these values using the historical average value for each time interval and each section. In this study, we consider only the 15-minute time intervals in the period from 3pm to 8pm (local time zone) of each day (thus $|\mathcal{T}| = 20$) on weekdays only. The resulting dataset corresponds to our $W^{(i)}$, $i = 1, \ldots, n$, matrices where $p = 556$ and $n = 144$ (Figure [1]). We also remove the days where no data is found anywhere in the whole road network for at least one $t \in \mathcal{T}$ (this was due to some data acquisition problem). It is worth noting that, this dataset comes from a low frequency collection, meaning that, for a given time and section, the average number of observations on the selected sections of the network is about 6 logs per 15-minutes per section, this is due to the fact that, we only observe a portion of the vehicles in the network (those equipped with Coyote GPS devices), and that suburban sections have low traffic. Thus, going for better precision (less than 15 minutes) would increase the number of missing values.

We divide the data into 3 subsets: train 63% (1710 examples), validation 27% (741 examples), test 10% (285 examples), splits were made such that days are not cut in the middle and all subsets contains sequential full days data.

5.2 Comparative study

5.2.1 Methods

The methods used in this study can be divided into 2 groups reflecting the information that is used to predict. Methods of the first group, called baselines, predicts the speed of a given section using only
data coming from this section. Methods of the second group might predict using data collected from the whole network. This includes the linear predictors that have been studied previously and also certain neural network predictors that we introduce for the sake of completeness.

Baselines

- **Historical average** (HA): for each section \( s \) and time interval \( t \), the prediction is given by the average speed at time interval \( t \) and section \( s \). The averaged speed is computed on the training dataset.
- **Previous observation** (PO): for each section \( s \) and time interval \( t \), the prediction is given by the observed speed at \( t - 1 \) and section \( s \).
- **Autoregressive** (AR): for each section \( s \) and time interval \( t \), the prediction is given by a linear combination of the \( \alpha \) previous observed time speed \( t - 1, \ldots, t - \alpha \) in section \( s \). The coefficient are estimated using ordinary least-squares on the training dataset. In our study, we use the implementation from the "Statsmodels" package and vary the order \( \alpha \) from 1 to 5.

Linear predictors

- **Ordinary least-squares** (OLS): we first compute the linear predictor for each section separately. It corresponds to solving independently \( p \)-problems using OLS. Then we stack the solutions \((\beta_k)_{1 \leq k \leq p} \subset \mathbb{R}^p\) to form the solution \( A^T = (\beta_1, \ldots, \beta_p) \in \mathbb{R}^{p \times p} \) of the main problem.
- **Lasso**: We compute the Lasso predictor the same way we compute the OLS one with the addition that we regularize each sub-solution coefficients separately by adding the \( \ell_1 \)-penalty. We search for the optimal regularization coefficients using 5-fold cross-validation from the sklearn package on our training data. Constraining the regularization rate to be equal for all sections produce a model with equivalent coefficients with same performance.
- **Time-specific Lasso** (TS-Lasso): We subdivide the problem into smaller time-specific problems that we solve separately. The model find a different coefficients matrix \( A \) for each time \( t \in \{1, \ldots, T\}\). We compute each of these predictors in the same way as the Lasso.
- **Regime switching Lasso** (RS-Lasso): We build The RS-Lasso model by exhaustively searching the time-space \( \{1, \ldots, T\} \) for the best switch time \( t^* \). In practice, for each time switch \( t_{\text{switch}} \in \{1, \ldots, T\} \) the data is split in two partitions \( \{1, \ldots, t_{\text{switch}}\} \) and \( \{t_{\text{switch}}, \ldots, T\} \). On each partition, we fit a Lasso model as we explained above.
- **Group Lasso** (Grp-Lasso): In group Lasso, we penalize the norm of each column of the coefficients matrix \( A \). This approach filters out sections that are found to be irrelevant to predict the complete network and hence is useful when the same sparsity structure is shared among the different sections.

Neural networks predictors

- **Multilayer perceptron** (MLP): 2 hidden layers fully connected, with batch normalization layer between hidden layers and \( \ell_1 \) regularization in all layers (hidden and output).
- **Fully connected long short-term memory** (FC-LSTM): composed of one LSTM Layer followed by a fully connected layer (output layer) with 2-time steps. We use \( \ell_1 \) regularization in both layers (note that we do not regularize the recurrent step in the LSTM). We do not use batch normalization in this approach.

We build these neural networks using tanh as the activation function except for the output layer, which we leave without activation. We scale the input data values to the \([-1, 1]\) interval. We train our models using ADAM optimizer and *mean squared error loss*. Because of the high dimensionality of the model \( (p = 556) \), all non-regularized approaches overfit the training dataset. To assess this problem, we rely on different regularization techniques.

**Earlystopping** [Caruana et al.](2001) is used to stop model training when the model starts overfitting.
• $\ell_1$-regularization is conducted by adding a penalization term to the loss function that enforces a sparse structure of the parameters.

Combining $\ell_1$-regularization with earlystopping gave us the best performance for all neural network models. Other configurations have been tried, such as using the sigmoid, the ReLU and its variants as the activation function. Log-scaling values, scaling to $[0, 1]$ interval, keeping raw values, varying the number of layers/neurons, different regularization techniques, increasing the number of lags. We present here the ones that give the best performance. The models are built using the package Keras with TensorFlow backend in python.

5.2.2 Results

To compare the performance of the different methods, we compute the Mean Squared Error and the Mean Absolute Error on the test subset of data. We also present graphics with different aggregations of the error (over time and days).

Performance on the whole road network

| Model       | AR 1 | AR 3 | AR 5 | FC-2LSTM | HA | Lasso | MLP | OLS | PO | RS-Lasso | TS-Lasso | Grp-Lasso |
|-------------|------|------|------|----------|----|-------|-----|-----|----|----------|----------|-----------|
| MAE         | 7.25 | 7.21 | 7.14 | 7.26     | 9.20| 9.38  | 7.13| 7.34|    | 109.04   | 115.37   | 113.81    |
| MSE         | 113.85| 113.22| 113.22| 111.44  | 113.57| 102.79| 183.38| 109.04 |    | 113.81   |          |           |

Table 1: Performance comparison on MSE, MAE for all models.

![Figure 2](image.png)

Figure 2: The prediction error (a) over days, (b) over time.

Table 1 shows the performance of all models over the whole network. We see that, as expected, the simple OLS model does not generalize well (in fact, it overfits the training data), together with PO prediction, they give the worst performances. Other baselines performances are variables with AR giving a better performance than the historical average. Note that increasing the number of lags does not lead to better results. The neural networks give close results to those of the AR with the LSTM performing slightly better than the MLP model. The different Lasso approaches also gave variant results, the model with regime-switching (RS-Lasso with time switch at 6:15pm) performs better than all other models and is just slightly better than the Lasso. Using a different Lasso predictor for each time (TS-Lasso) does not improve the error and gave worse performance than the general one (Lasso). The same observation on Grp-Lasso suggests that entirely filtering some sections causes a loss in performance. In Figure 2 we present the error observed over time, and days (for readability, we only present a subset of the models), the figure shows the good performance of Lasso and RS-Lasso over days and time and confirms that they outperform all other models. The two approaches alternate performance over time, we notice that the RS-Lasso performs better at first and last 1.5 hours, and is worse than the Lasso in the middle
hours. As this cut-off is happening about the RS-Lasso switch (6:15pm), it might be possible that the RS-Lasso parts (before and after switching) focus on modeling the regime of the corresponding periods and overlooked the middle period since the switch cuts it.

Performance on highly variable sections

To further understand the results, we compare the models on highly variable sections. To select these sections we compute descriptive statistics (Min, Max, Mean, standard deviation, Quartiles) on each section for each time, then run Kmeans algorithm on this data to search for 3 clusters out of which we pick the cluster with the most variable behavior (the cluster represents \( \sim 30\% \) of the sections).

| Model     | AR 1 | AR 3 | AR 5 | FC-2LSTM | HA   | Lasso | MLP   | OLS   | PO   | RS-Lasso | TS-Lasso | Grp-Lasso |
|-----------|------|------|------|----------|------|-------|-------|-------|------|----------|----------|-----------|
| MAE       | 9.27 | 9.26 | 9.27 | 9.11     | 10.18| 9.01  | 9.17  | 11.50 | 11.81| 8.94     | 9.20     | 9.17      |
| MSE       | 172.59| 171.94| 171.98| 163.45   | 207.18| 161.29| 165.39| 238.91| 274.92| 159.66   | 168.01   | 165.28    |

Table 2: Performance comparison on MSE, MAE for all models on highly variable sections.

Figure 3: The prediction error (a) over days, (b) over time.

Table 2 shows the performance over highly variable sections only. All model performances deteriorate compared to the performances on the full road network. Similar observations as those on the full road network can be made on these results with the exceptions that, both neural networks and time-specific Lasso (TS-Lasso) performs better than the autoregressive models and that the performance gap between the models is wider, the RS-Lasso is clearly better than the Lasso even though the alternation of performance over time is still observable.

5.3 Graphical visualization of the Lasso model

The Lasso coefficients are indicators of the traffic flow dependencies between the sections of the road network. To visualize these connections, we draw on top of a map some active sets \( S_k^* \) as arcs going from one section to another.
Figure 4: Model coefficients on some sections (arcs color and width are scaled by the values of the coefficients).

Figure 4 shows connections inferred by the model for multiple subsets of sections closely localized on the map. Figure 4(a, b) represents sections from the Rennes ring roads, we see that these sections connect strongly to adjacent sections in the ring and also to the closest highways, we also observe this kind of behavior on the highways (Figure 4(c)) where we observe higher dependence on adjacent section of the highway. Figure 4(d, e) shows sections that are part of link roads, we see that naturally, these sections have a high number of connections mainly on the ring roads, we observe similar aspects on sections in the center of Rennes (Figure 4(f)) in the sense that these sections mostly link to the ring roads. We can see that, using the Lasso helps to capture the main features of the architecture of the road network, by (i) finding only the most informative connections between roads, and (ii) weighting these connections accordingly.

Another meaningful visualization provided by the model consists in representing the influence of each section on the overall network using the values of its regression coefficients. For section \( l \), the criterion is given by \( \sum_{1\leq k\leq p, A_{kl}>0} A_{kl} \). Taking into account negative coefficients implies an inverse relationship.
between sections, meaning that when a section is getting congested, another is getting less traffic. This may happen when there are incidents on the road but does not characterize the average influence between sections. Thus we only consider positive coefficients in building the criterion (consisting of 93% of the coefficients).

Figure 5: Influence of each section.

For each section in the network, the influence criterion is represented with color scaled with respect to its values in Figure 5. It shows that the prediction is mainly influenced by the sections of the road ring of Rennes and the main highways, while most sections in the center of Rennes, and those in the rural and suburban areas around Rennes have little influence on the prediction.

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Appendix A  Proofs of the stated results

A.1  Proof of Proposition 1

Denote by $L_2(P)$ the Hilbert space composed of the random elements $W$ defined on $(\Omega, \mathcal{F}, P)$ such that $E|W|_F^2 < +\infty$. The underlying scalar product between $X$ and $Y$ is $\text{tr}(E[X^T Y])$. Hence $\inf_{L \in \mathcal{L}} E[||Y - L(X)||_F^2]$ is a distance between the element $Y$ of $L_2(P)$ and the linear subspace of $L_2(P)$ made of $L(X)$, $L \in \mathcal{L}$. The Hilbert projection theorem ensures that the infimum is uniquely achieved and that $L^*(X)$ is the argument of the minimum if and only if $\text{tr}(E[(Y - L^*(X))^T L(X)]) = 0$ for all $L \in \mathcal{L}$. Taking $A = 0$, in $L(X) = b + AX$, gives the first set of normal equations. Taking $b = 0$ and noting that $\text{tr}(E[(Y - L^*(X))^T L(X)]) = \text{tr}(E[X(Y - L^*(X))^T]A) = 0$ for all $A \in \mathbb{R}^{p \times p}$ is equivalent to $E[(Y - L^*(X))X^T] = 0$, which is the second set of normal equations.

A.2  Proof of Proposition 2

The statement follows from developing $R(L) = E[||Y - L^*(X)) + (L^*(X) - L(X))||_F^2]$ and then using that $\text{tr}(E[(Y - L^*(X))(L(X) - L^*(X))^T]) = 0$ which is a consequence of the normal equations given in Proposition 1.

A.2.1  Proof of Proposition 3

Let $X_c = X - E[X]$. Using Proposition 2 and that $L_n(X) = b_n + A_n X$ and $L^*(X) = E[Y] + A^* X_c$. We have

$$\mathcal{E}(L_n) = E_X [||(A^* - A_n) X_c + E(Y) - A_n E(X) - b_n||_F^2].$$

Because, $X_c$ has mean 0, we get

$$\mathcal{E}(L_n) = E_X [||(A^* - A_n) X_c||_F^2] + ||E(Y) - A_n E(X) - b_n||_F^2..$$

Finally, as $E_X [||A^* - A^{(ols)}_n(X - E(X))||_F^2] = ||A^* - A^{(ols)}_n|| \Sigma_1^{1/2} ||_F^2$, and using the triangle inequality, we find

$$\mathcal{E}(L_n) \leq ||(A^* - A_n) \Sigma_1^{1/2}||_F^2 + 2||E(Y) - \overline{Y}||_F^2 + 2||A_n(E(X) - \overline{X})||_F^2.$$

A.2.2  Proof of Proposition 4

The inequality of Proposition 3 applied to the OLS estimate gives

$$\mathcal{E}(L_n^{(ols)}) \leq ||(A^* - A^{(ols)}_n) \Sigma_1^{1/2}||_F^2 + 2||\overline{Y} - E(Y)||_F^2 + 2||A^{(ols)}_n \Sigma_1^{1/2} \overline{Z}^n||_F^2.$$

Using that $||A^{(ols)}_n \Sigma_1^{1/2} \overline{Z}^n||_F \leq ||(A^{(ols)}_n - A^*) \Sigma_1^{1/2}||_F ||\overline{Z}^n||_F + ||A^* \Sigma_1^{1/2} \overline{Z}^n||_F$, we get

$$\mathcal{E}(L_n^{(ols)}) \leq ||(A^* - A^{(ols)}_n) \Sigma_1^{1/2}||_F^2 + 2||\overline{Y} - E(Y)||_F^2 + 4||A^* \Sigma_1^{1/2} \overline{Z}^n||_F^2.$$

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Applying Lemma 6 we deduce that
\[
\mathcal{E}(L_n^{(ols)}) = \| (A^* - A_n^{(ols)}) \Sigma^{1/2} \|_F^2 \left( 1 + \mathcal{O}_p \left( \frac{p}{n} \right) \right) + \mathcal{O}_p \left( \frac{p + \| A^* \Sigma^{1/2} \|_F^2}{n} \right) \\
= \| (A^* - A_n^{(ols)}) \Sigma^{1/2} \|_F^2 \mathcal{O}_p(1) + \mathcal{O}_p \left( \frac{p + \| A^* \Sigma^{1/2} \|_F^2}{n} \right) \\
= \| (A^* - A_n^{(ols)}) \Sigma^{1/2} \|_F^2 \mathcal{O}_p(1) + \mathcal{O}_p \left( \frac{p + \Gamma \| A^* \|_F^2}{n} \right),
\]
where we just used that \( p/n \to 0 \) and \( \| A^* \Sigma^{1/2} \|_F^2 = \sum_{k=1}^p \| A_{(k, \cdot)}^T \Sigma^{1/2} \|_2^2 \leq \Gamma \| A^* \|_F^2 \). The fact that \( p/n \to 0 \) is a consequence of Assumption 3 which implies that \( \| X_n \|_2^2 \leq pC \) where \( C \) is a positive constant independent of \( p \). Using that \( p \gamma \leq \text{tr}(Z) = E[\| X_n \|_2^2] \), we can deduce that \( p/n \leq C p \log(p)/(\gamma n) \to 0 \). Hence the proof of Proposition 4 will be completed if it is shown that
\[
\| (A_n^{(ols)} - A^*) \Sigma^{1/2} \|_F^2 = \mathcal{O}_p \left( \frac{p^2 \gamma^2}{n} \right).
\]
By definition of \( A_n^{(ols)} \), it holds that
\[
\sum_{i=1}^n \| \tilde{Y}_c(i) - A_n^{(ols)} \Sigma^{1/2} \tilde{Z}_c(i) \|_F^2 \leq \sum_{i=1}^n \| \tilde{Y}_c(i) - A^* \Sigma^{1/2} \tilde{Z}_c(i) \|_F^2,
\]
where
\[
\tilde{Z}_c(i) = \Sigma^{-1/2}(X(i) - \bar{X}) \quad \text{and} \quad \tilde{Y}_c(i) = Y(i) - \bar{Y}.
\]
Introducing \( \Delta = (A^* - A_n^{(ols)}) \Sigma^{1/2} \) and developing the squared-norm in the left-hand side, we obtain
\[
\sum_{i=1}^n \| \Delta \tilde{Z}(i) \|_F^2 \leq 2 \sum_{i=1}^n \| \tilde{Y}_c(i) - A^* \Sigma^{1/2} \Delta \tilde{Z}(i) \|_F \\
= 2 \sum_{i=1}^n \| (\epsilon(i) - \bar{\epsilon}, \Delta \tilde{Z}(i)) \|_F. \tag{6}
\]
For the left-hand side of (6), we provide the following lower bound. Define
\[
\Pi_n = \Sigma^{1/2} \Sigma_n \Sigma_S^{1/2}.
\]
By definition, we have \( n^{-1} \sum_{i=1}^n \| \Delta \tilde{Z}(i) \|_F^2 = \text{tr}(\Delta \Pi_n \Delta^T) \), and because \( \Pi_n \) is symmetric we can write \( \Pi_n = U D U^T \) with \( U^T U = I_p \) (\( I_p \) is the identity matrix of size \( p \times p \)) and \( D = \text{diag}(d_1, \ldots, d_p) \). Hence, defining \( \Delta = (\Delta_{(\cdot, 1)}, \ldots, \Delta_{(\cdot, p)}) = \Delta U \), it holds that
\[
n^{-1} \sum_{i=1}^n \| \Delta \tilde{Z}(i) \|_F^2 = \text{tr} \left( \sum_{k=1}^p d_k \Delta_{(k, \cdot)} \Delta_{(\cdot, k)}^T \right) \geq \gamma_n \sum_{k=1}^p \| \Delta_{(k, \cdot)} \|_2^2 = \gamma_n \| \Delta \|_F^2.
\]
For the right-hand side of (6), we provide the following upper bound. Using the Cauchy-Schwarz inequality, we have
\[
\left| \sum_{i=1}^n (\epsilon(i) - \bar{\epsilon}, \Delta \tilde{Z}(i)) \right| = \left| \sum_{i=1}^n (\epsilon(i) \tilde{Z}(i)^T, \Delta) \right| \\
= \left| \sum_{i=1}^n (\epsilon(i) - \bar{\epsilon} \tilde{Z}(i)^T, \Delta) \right| \\
\leq \| \sum_{i=1}^n (\epsilon(i) - \bar{\epsilon} \tilde{Z}(i)^T) \|_F \| \Delta \|_F.
\]
Together with (6), the two previous bounds imply that
\[ \gamma_n \| \Delta \|_F \leq 2n^{-1} \sum_{i=1}^{n} (\epsilon(i) - \bar{v}_n) \bar{Z}_{(i)T} \|_F. \]

Note that, for all \( u \in \mathbb{R}^p \) with \( \|u\|_2 = 1 \),
\[ u^T \Pi_n u = 1 + u^T (\Pi_n - I) u \geq 1 - \| \Pi_n - I \|, \]
where \( \| \cdot \| \) stands for the spectral norm. Hence we have that
\[ (1 - \| I - \Pi_n \|) \| \Delta \|_F \leq 2n^{-1} \sum_{i=1}^{n} (\epsilon(i) - \bar{v}_n) \bar{Z}_{(i)T} \|_F. \]

Because the term in the right-hand side is \( O_P(p^2 \sigma^2 / n) \) in virtue of Lemma 7, we just have to show that \( \| \Pi_n - I \| \to 0 \) in probability to conclude the proof. For that purpose we use Lemma 10 with \( S = \{1, \ldots, p\} \) and \( B_S \) given by the following inequality (which holds using Assumptions 3 and 4),
\[ \text{tr} \left( X_n^T \Sigma^{-1} X_n \right) = \sum_{t=1}^{T} (X_{(t)} - E(X_{(t)})^T \Sigma^{-1} (X_{(t)} - E(X_{(t)})) \]
\[ \leq \sum_{t=1}^{T} \gamma^{-1} \| X_{(t)} - E(X_{(t)}) \|_2^2 \]
\[ \leq T p U^2 / \gamma, \]
where, almost surely, \( U = \limsup_{p \to \infty} \max_{k \in S, t \in T} \| X_{k,t} - E(X_{k,t}) \| \). Because \( p \log(p) / (n \gamma) \to 0 \), the upper bound given in Lemma 7 goes to 0 and it holds that \( \| \Pi_n - I \| \to 0 \), in probability.

\[ \square \]

### A.2.3 Proof of Proposition 5

For any set \( S \subset \{1, \ldots, p\} \), define
\[ \Sigma_{n,S} = n^{-1} \sum_{i=1}^{n} (X_{(S,i)} - \bar{X}_{(S,i)})(X_{(S,i)} - \bar{X}_{(S,i)})^T \]
\[ \Pi_{n,S} = \Sigma_{n,S}^{1/2} \Sigma_{n,S}^{1/2}. \]

Denote by \( s_{\delta}^* = \limsup_{p \to \infty} \max_{k \in S} \| S_k \| \) and let \( U \) be such that, almost surely,
\[ \limsup_{p \to \infty} \max_{k \in S, t \in T} \| X_{k,t} - E(X_{k,t}) \| \leq U \quad \text{and} \quad \limsup_{p \to \infty} \max_{k \in S, t \in T} \| \epsilon_{k,t} \| \leq U. \]

Set
\[ \lambda = 8 \sqrt{T^2 U^2 \sigma^2 n \log(p^3)} \]
\[ \Delta = (A^* - A_n^{(\text{Lasso})}). \]

Intermediate results, that are useful in the subsequent development, are now claimed and will be proved at the end of the proof. Let \( 0 < \delta < 1 \). For \( n \) large enough, we have with probability \( 1 - \delta/2 \), for all \( k \in \{1, \ldots, p\} \),
\[ n \| \Sigma_{n,S}^{1/2} \Delta_{(k,)} \|_2^2 \leq \lambda (3 \| \Delta_{(k,s_k^*)} \|_1 - \| \Delta_{(k,\delta S_k^*)} \|_1), \quad (7) \]
Moreover, for \( n \) large enough, with probability \( 1 - \delta/2 \),
\[ \max_{k=1, \ldots, p} \| \Pi_{n,S_k^*} - I_{|S_k^*|} \| \leq 1/2, \quad (8) \]
\[ 16 \| \Sigma_n - \Sigma_{\text{asso}}(s_{\delta}^* / \gamma^*) \| \leq 1/2. \quad (9) \]
In the following, we assume that (7), (8) and (9) are satisfied. This occurs with probability $1 - \delta$. A similar decomposition to (6) is valid for the Lasso. We have

$$E(I_n^{(\text{Lasso})}) \leq \|\Delta \Sigma^{1/2}\|_F^2 O_p(1) + O_p\left(\frac{p + \Gamma \|A^*\|_2^2}{n}\right).$$

Hence, we only have to show that

$$\|\Delta \Sigma^{1/2}\|_F^2 = O_p((\sigma^2/n)\log(p)).$$

The minimization of (4), can be done by solving $p$ minimization problems where each of them gives one of the lines of $A_n^{(\text{Lasso})} = (A_{n,1}, \ldots, A_{n,p})^T$. More formally, we have

$$A_{n,k}^{(\text{Lasso})} \in \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^n \|Y_k^{(i)} - \sum_{k}^n - (X^{(i)} - \bar{X})^T \beta\|_2^2 + 2\lambda \|\beta\|_1.$$

Let $u \in \mathbb{R}^{|S_k^*|}$. We have

$$\|\Sigma_{S_k^*}^{1/2} u\|_2^2 \leq \|\Sigma_{n,S_k^*}^{1/2} u\|_2^2 + \|u^T (\Sigma_{n,S_k^*} - \Sigma_{S_k^*}) u\|_2 \leq \|\Sigma_{n,S_k^*}^{1/2} u\|_2^2 + \|u^T \Sigma_{S_k^*}^{1/2}(\Pi_n S_k^* - I_{|S_k^*|}) \Sigma_{S_k^*}^{1/2} u\|_2^2 \leq \|\Sigma_{n,S_k^*}^{1/2} u\|_2^2 + \|\Pi_n S_k^* - I_{|S_k^*|}\| \|\Sigma_{S_k^*}^{1/2} u\|_2^2$$

Hence, in virtue of (8), it holds that

$$\|\Sigma_{S_k^*}^{1/2} u\|_2^2 \leq 2\|\Sigma_{n,S_k^*}^{1/2} u\|_2^2, \quad \forall u \in \mathbb{R}^{|S_k^*|}.$$

Injecting this into the right-hand side of (7) and using that $\|u\|_2 \leq \sqrt{|S_k^*|} \|u\|_2 \leq \sqrt{|S_k^*|} \|\Sigma_{S_k^*}^{1/2} u\|_2$, $\forall u \in \mathbb{R}^{|S_k^*|},$ we obtain

$$n \|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2^2 \leq 3\lambda \sqrt{|S_k^*|} \|\Sigma_{S_k^*}^{1/2} \Delta_{(k, S_k^*)}\|_2 \leq 6\lambda \sqrt{|S_k^*|} \|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2,$$

which implies that

$$n \|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2 \leq 6\lambda \sqrt{|S_k^*|}.$$

From (7), we have $\|\Delta_k\|_1 = \|\Delta_{k,S_k^*}\|_1 + \|\Delta_{k,S_k^*}\|_1 \leq 4\|\Delta_{k,S_k^*}\|_1$ and it follows that

$$\|\Sigma_{i}^{1/2} \Delta_{(k, \cdot)}\|_2^2 \leq \|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2^2 + \|\Delta_{(k, \cdot)}^T (\Sigma_{n} - \Sigma) \Delta_{(k, \cdot)}\|_2 \leq \|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2^2 + \|\Sigma_{n} - \Sigma\|_2 \|\Delta_{(k, S_k^*)}\|_2^2 \leq \|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2^2 + \|\Sigma_{n} - \Sigma\|_2 \|\Sigma_{S_k^*}^{1/2} \Delta_{(k, S_k^*)}\|_2^2 \leq \|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2^2 + \|\Sigma_{n} - \Sigma\|_2 \|\Sigma_{S_k^*}^{1/2} \Delta_{(k, S_k^*)}\|_2^2 \leq \|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2^2 + \|\Sigma_{n} - \Sigma\|_2 \|\Sigma_{S_k^*}^{1/2} \Delta_{(k, S_k^*)}\|_2^2 \leq 2\|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2^2 \leq 72(\lambda/n) \frac{|S_k^*|}{\gamma^*}.$$

We have just used Jensen inequality and the fact $\gamma^*$ is a lower bound on the smallest eigenvalue of $\Sigma_{S_k^*}$. Hence, using (9), we have that

$$\|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2 \leq 2\|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2 \leq 72(\lambda/n) \frac{|S_k^*|}{\gamma^*}.$$

We conclude the proof noting that

$$\|\Delta_{n}^{1/2}\|_F^2 = \sum_{k=1}^{p} \|\Sigma_{n}^{1/2} \Delta_{(k, \cdot)}\|_2^2 \leq \frac{72\lambda^2}{n^2 \gamma^*} \sum_{k=1}^{p} |S_k^*| = \frac{C\sigma^2 \log(p)}{n},$$

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for some $C > 0$ that does not depend on $(n, p)$, and recalling that the previous happens with probability $1 - \delta$ with $\delta$ arbitrarily small.

**Proof of Equation [7], [8] and [9].** We first give some basic inequalities that will be useful in the proof. As $\log(p)/n \to \infty$ and $\liminf_{p \to \infty} \sigma^2 > 0$, we have, for $n$ large enough,

\[
\sqrt{n} \geq 16^2(s_\infty^*/\gamma^*)TU^2\sqrt{\log(4p^2/\delta)} \\
\sqrt{n} \geq 16\sqrt{s_\infty^*U^2\log(16s_\infty^*TP/\delta)}/\gamma^* \\
n \geq 4(U^2/\sigma^2)\log(6p^2/\delta) \geq 4\log(4p^2/\delta).
\]

Note that the last inequality implies that we can apply Lemma 8 and 9.

**Proof of Equation [7].** Note that for $p$ large enough,

\[
\lambda \geq \lambda_\delta = 8\sqrt{T^2U^2\sigma^2n\log(12p^2/\delta)}.
\]

It follows from Lemma 8 that with probability $1 - \delta/2$:

\[
\lambda \geq 2\left\| \sum_{i=1}^n (e^{(i)} - \bar{e}^{(i)}) (X^{(i)} - \bar{X}^n)^T \right\|_\infty.
\] \hspace{1cm} (10)

Let $k \in \{1, \ldots, p\}$. Note that because of (10), it holds that

\[
\lambda \geq 2\left\| \sum_{i=1}^n (e^{(i)}_{(k, \cdot)} - \bar{e}^{(i)}_{(k, \cdot)}) (X^{(i)} - \bar{X}^n)^T \Delta_{(k, \cdot)} \right\|_\infty.
\]

We have

\[
\frac{1}{2}\sum_{i=1}^n \left( (X^{(i)} - \bar{X}^n)^T \Delta_{(k, \cdot)} \right)^2 \\
\leq \left\| \sum_{i=1}^n (e^{(i)}_{(k, \cdot)} - \bar{e}^{(i)}_{(k, \cdot)}) (X^{(i)} - \bar{X}^n)^T \Delta_{(k, \cdot)} \right\|_1 + \lambda(\|A^*_k\|_1 - \|A_{n,k}\|_1).
\]

First consider the scalar product term of the right-hand side. We have

\[
\left| \sum_{i=1}^n (e^{(i)}_{(k, \cdot)} - \bar{e}^{(i)}_{(k, \cdot)}) (X^{(i)} - \bar{X}^n)^T \Delta_{(k, \cdot)} \right| = \left| \sum_{i=1}^n (X^{(i)} - \bar{X}^n)(e^{(i)}_{(k, \cdot)} - \bar{e}^{(i)}_{(k, \cdot)}) \Delta_{(k, \cdot)} \right|_F \\
\leq \left\| \sum_{i=1}^n (X^{(i)} - \bar{X}^n)(e^{(i)}_{k} - \bar{e}^{(i)}_{k}) \right\|_1 \|\Delta_{(k, \cdot)}\|_1 \\
\leq \left\| \sum_{i=1}^n (e^{(i)}_{(k, \cdot)} - \bar{e}^{(i)}_{(k, \cdot)}) (X^{(i)} - \bar{X}^n)^T \right\|_\infty \|\Delta_{(k, \cdot)}\|_1 \\
\leq (\lambda/2)\|\Delta_{(k, \cdot)}\|_1
\]

Now we deal with $\|A^*_k\|_1 - \|A_{n,k}\|_1$. Note that, by the triangle inequality,

\[
\|A_{n,k}\|_1 = \|A^*_k - \Delta_{(k, \cdot)}\|_1 \geq \|A^*_k\|_1 - \|\Delta_{(k, \cdot)}\|_1 \geq \|A^*_k\|_1 - \|\Delta_{(k, \cdot)}\|_1.
\]

Using the previous, restricted to the active set $S_\delta^k$, we obtain

\[
\|A^*_k\|_1 - \|A_{n,k}\|_1 = \|A^*_k\|_1 - \|A_{n,k,S_\delta^k}\|_1 - \|A_{n,k,S_\delta^k}\|_1 \\
\leq \|A^*_k\|_1 - (\|A_{S_\delta^k}\|_1 - \|\Delta_{(k,S_\delta^k)}\|_1) - \|A_{n,k,S_\delta^k}\|_1 \\
= \|\Delta_{(k,S_\delta^k)}\|_1 - \|\Delta_{(k,S_\delta^k)}\|_1.
\]
All this together gives
\[
\frac{1}{2} \sum_{i=1}^{n} \|(X^{(i)} - \overline{X}^{(n)})^T \Delta_{(k, \cdot)}\|^2 \leq (\lambda/2)(\|\Delta_{(k, \cdot)}\|_1 + 2\|\Delta_{(k, S_k^c)}\|_1 - 2\|\Delta_{(k, S_k^c)}\|_1) = (\lambda/2)(3\|\Delta_{(k, S_k^c)}\|_1 - \|\Delta_{(k, S_k^c)}\|_1),
\]
and \(7\) follows from remarking that
\[
\sum_{i=1}^{n} \|\Delta_{(k, \cdot)}^T (X^{(i)} - \overline{X}^{(n)})\|_2^2 = n\|\Sigma_n^{1/2} \Delta_{(k, \cdot)}\|_2^2.
\]

Proof of Equation \(8\). The result follows from applying Lemma \(10\). Under the stipulated assumptions, we have that, for each \(k = 1, \ldots, p\), \(B_{S_k} \leq 4s_{\infty}^* U^2/\gamma^*\). Hence, for each \(k = 1, \ldots, p\), we have with probability \(\delta/(4p)\),
\[
\|\Pi_{n, S_k^c} - I_{|S_k^c|}\| \leq 4 \sqrt{4s_{\infty}^* U^2 \log(16s_{\infty}^* Tp/\delta)} \leq 1/2.
\]
From the union bound, we deduce that, with probability \(1 - \delta/4\),
\[
\max_{k = 1, \ldots, p} \|\Pi_{n, S_k^c} - I_{|S_k^c|}\| \leq 4 \sqrt{4s_{\infty}^* U^2 \log(4s_{\infty}^* Tp/\delta)}.
\]

Appendix B Intermediate results

Define the standardized predictors \(Z = \Sigma^{-1/2} (X - E(X))\), and for any \(i = 1, \ldots, n\), \(Z^{(i)} = \Sigma^{-1/2} (X^{(i)} - E(X))\).

Lemma 6. Suppose that Assumptions \(2\), \(3\), \(4\) are fulfilled. It holds that
\[
\|\overline{Y}^n - E(Y)\|_F = O_p \left(\sqrt{p/n}\right).
\]
Moreover, for any real-valued matrix \(A\) with \(p\) columns, it holds
\[
\|A\overline{Z}^n\|_F = O_p \left(\sqrt{\|A\|_F^2/n}\right).
\]

Proof. If \(M_i\) are i.i.d. centered random matrices, we have that \(\mathbb{E}(\|\sum_{i=1}^{n} M_i\|_F^2) = n\mathbb{E}(\|M_1\|_F^2)\). It follows that
\[
E(\|\overline{Y}^n - E(Y)\|_F^2) = n^{-1} E[\|Y - E(Y)\|_F^2] \leq n^{-1} T p \max_{t \in T, k \in S} \text{Var}(Y_{k,t}).
\]
For the second assertion, because \(E[ZZ^T] = I_p\), we have
\[
E(\|A\overline{Z}^n\|_F^2) = n^{-1} \text{tr}(AEZZ^TA^T) = n^{-1} \|A\|_F^2.
\]

Lemma 7. Suppose that Assumptions \(2\), \(3\) and \(4\) are fulfilled. We have
\[
\|\sum_{i=1}^{n} e^{(i)}\|_F = O_p \left(\sqrt{n\sigma^2 p}\right)
\]
\[
\|\sum_{i=1}^{n} (e^{(i)} - \overline{e}^n)(Z^{(i)} - \overline{Z}^n)^T\|_F = O_p \left(\sqrt{n\sigma^2 p^2}\right).
\]
Proof. The first statement follows from
\[ E[\| \sum_{i=1}^{n} \epsilon^{(i)} \|_{F}^{2}] = nE[\epsilon^{2}] = n \sum_{i=1}^{T} \sum_{k=1}^{P} E[\epsilon_{i,k}^{2}] \leq nTp\sigma^{2}. \]
For the second statement, start by noting that
\[ \sum_{i=1}^{n} (\epsilon^{(i)} - \epsilon^{n})(Z^{(i)} - Z^{n})^{T} = \sum_{i=1}^{n} \epsilon^{(i)}Z^{(i)}T - n\epsilon^{n}(Z^{n})^{T}. \] (11)
Then use the triangle inequality to get
\[ \| \sum_{i=1}^{n} (\epsilon^{(i)} - \epsilon^{n})(Z^{(i)} - Z^{n})^{T} \|_{F} \leq \| \sum_{i=1}^{n} \epsilon^{(i)}Z^{(i)}T \|_{F} + n\| \epsilon^{n}(Z^{n})^{T} \|_{F}, \]
Using Lemma 6 and the first statement, we find that \( \|Z\|_{F} \sum_{i=1}^{n} \epsilon^{(i)} \|_{F} = O_{p}(\sigma p) \), Hence, showing that
\[ \| \sum_{i=1}^{n} \epsilon^{(i)}Z^{(i)}T \|_{F} = O_{p}(\sqrt{n}\sigma p) \]
will conclude the proof. From Assumption 2 and Proposition 1 \( E[\text{tr}(\epsilon^{(i)}Z^{(i)}T Z^{(j)}\epsilon^{(j)}T)] = 0 \) for all \( i \neq j \).
It follows that
\[ E[\| \sum_{i=1}^{n} \epsilon^{(i)}Z^{(i)}T \|_{F}^{2}] = nE[\epsilon^{(i)}Z^{(i)}T \|_{F}^{2}]. \]
Using the triangle inequality and then Jensen inequality, we get
\[ E[\| \sum_{i=1}^{n} \epsilon^{(i)}Z^{(i)}T \|_{F}^{2}] \leq nE \left[ \sum_{i=1}^{T} \| \epsilon_{(i,t)} Z^{T}_{(i,t)} \|_{F}^{2} \right] \]
\[ \leq nT \sum_{t=1}^{T} E[\| \epsilon_{(i,t)} Z^{T}_{(i,t)} \|_{F}^{2}] \]
\[ = nT \sum_{t=1}^{T} E[\text{tr}(\epsilon_{(i,t)} Z_{(i,t)}^{2} \epsilon_{(i,t)}^{T})] \]
\[ = nT \sum_{t=1}^{T} E[\| Z_{(i,t)} \|_{2}^{2}] \]
\[ \leq nTp\sigma^{2} \sum_{t=1}^{T} E[\| Z_{(i,t)} \|_{2}^{2}] = nT p^{2} \sigma^{2}. \]
The last inequality is a consequence of the definition of \( \sigma^{2} \).

Lemma 8. Suppose that Assumptions 2, 3, and 5 are fulfilled. Let \( U \) be such that \( \max_{k \in S, t \in T} |X_{k,t} - E(X_{k,t})| \leq U \) almost surely. If \( T \leq p \) and \( n \geq 4(U^{2}/\sigma^{2}) \log(6p^{2}/\delta) \), we have with probability \( 1 - \delta \):
\[ \left\| \sum_{i=1}^{n} (\epsilon^{(i)} - \epsilon^{n})(X^{(i)} - \overline{X}^{n})^{T} \right\|_{\infty} \leq 4 \sqrt{T^{2}U^{2}\sigma^{2}n \log(6p^{2}/\delta)}. \]
Proof. We have
\[
\left\| \sum_{i=1}^{n} (c^{(i)} - c^{n})(X^{(i)} - X^{n}) \right\|_{\infty}
= \left\| \sum_{i=1}^{n} c^{(i)}(X^{(i)} - X^{n}) \right\|_{\infty}
\leq \left\| \sum_{i=1}^{n} c^{(i)}(X^{(i)} - E(X))^T \right\|_{\infty} + n \left\| c^{n}(X^{n} - E(X))^T \right\|_{\infty}
\leq \left\| \sum_{i=1}^{n} c^{(i)}(X^{(i)} - E(X))^T \right\|_{\infty} + n \sum_{t=1}^{T} \left\| c^{(i)}(X_{t}^{(i)} - X_{t}^{n}) - E(X_{t}^{(i)})^T \right\|_{\infty}
\leq \left\| \sum_{i=1}^{n} c^{(i)}(X^{(i)} - E(X))^T \right\|_{\infty} + nT \max_{t=1,...,T} \left\| c^{(i)}(X_{t}^{(i)} - X_{t}^{n}) - E(X_{t}^{(i)})^T \right\|_{\infty}
\leq \left\| \sum_{i=1}^{n} c^{(i)}(X^{(i)} - E(X))^T \right\|_{\infty} + nT \max_{t=1,...,T} \left\| c^{(i)}(X_{t}^{(i)} - X_{t}^{n}) \right\|_{\infty} \left\| X_{t}^{n} - E(X) \right\|_{\infty}
\leq \left\| \sum_{i=1}^{n} c^{(i)}(X^{(i)} - E(X))^T \right\|_{\infty} + nT \left\| c^{n} \right\|_{\infty} \left\| X^{n} - E(X) \right\|_{\infty}
\]
**Lemma 9.** Suppose that Assumptions 1, 2 and 3 are fulfilled. If \( T \leq p \) and \( n \geq \log(4p^2/\delta) \) we have with probability \( 1 - \delta \):

\[
\|\Sigma_n - \Sigma\|_{\infty} \leq 8TU^2 \sqrt{\log(4p^2/\delta)/n},
\]

where \( U \) is such that \( \max_{k \in S, t \in T} |X_{k,t} - E(X_{k,t})| \leq U \) almost surely.

**Proof.** We have

\[
\Sigma_n - \Sigma = n^{-1} \sum_{i=1}^{n} \left\{ (X^{(i)} - E(X))(X^{(i)} - E(X))^T - \Sigma \right\} - (E(X) - \overline{X}^n)(E(X) - \overline{X}^n)^T.
\]

In the following, we bound each of the two previous terms under events of probability \( 1 - \delta/2 \), respectively. Apply Lemma 11 with \( A^{(i)} = (X^{(i)} - E(X))(X^{(i)} - E(X))^T - \Sigma \). Note that

\[
|A^{(i)}_{k,t}| \leq 2\|X^{(i)}_{k,\cdot} - E(X_{k,\cdot})\|_T (X^{(i)}_{\cdot,t} - E(X_{\cdot,t}))|
\]

\[
= 2 \left\{ \sum_{t=1}^{T} (X^{(i)}_{k,t} - E(X_{k,t})) (X^{(i)}_{\cdot,t} - E(X_{\cdot,t})) \right\}
\]

\[
\leq 2TU^2.
\]

and

\[
\Var(A^{(i)}_{k,t}) \leq E \left[ \left( (X^{(i)}_{k,\cdot} - E(X_{k,\cdot}))^T (X^{(i)}_{\cdot,t} - E(X_{\cdot,t})) \right)^2 \right] \leq (2TU^2)^2.
\]

Hence we apply Lemma 11 with \( v = (2TU^2)^2 \) and \( c = 2TU^2 \). We obtain, because \( 9n \geq 4 \log(4p^2/\delta) \), with probability \( 1 - \delta/2 \),

\[
\left\| n^{-1} \sum_{i=1}^{n} \left\{ (X^{(i)} - E(X))(X^{(i)} - E(X))^T - \Sigma \right\} \right\|_\infty \leq 4TU^2 \sqrt{\log(2p^2/(\delta/2))/n}.
\]

Note that

\[
((E(X) - \overline{X}^n)(E(X) - \overline{X}^n)^T)_{k,t} = \sum_{t=1}^{T} (E(X_{k,t}) - \overline{X}^n_{k,t}) (E(X_{\cdot,t}) - \overline{X}^n_{\cdot,t}).
\]

It follows that

\[
\left\| ((E(X) - \overline{X}^n)(E(X) - \overline{X}^n)^T)_{k,t} \right\| \leq T \max_{t=1,\ldots,T} \max_{k=1,\ldots,p} |E(X_{k,t}) - \overline{X}^n_{k,t}|^2.
\]

Applying Lemma 11 with \( A^{(i)} = X^{(i)} - E(X), \ c = U, \) and \( v = U^2 \) we get, because \( 9n \geq 4 \log(2pT/(\delta/2)) \), with probability \( 1 - \delta/2 \),

\[
\max_{t=1,\ldots,T} \max_{k=1,\ldots,p} |E(X_{k,t}) - \overline{X}^n_{k,t}| \leq \sqrt{4U^2 \log(2pT/(\delta/2))/n}.
\]

Hence we have shown that, with probability \( 1 - \delta \),

\[
\|\Sigma_n - \Sigma\|_{\infty} \leq 4TU^2 \sqrt{\log(4p^2/\delta)/n + 4TU^2 \log(4pT/\delta)/n} \leq 4TU^2 \sqrt{\log(4p^2/\delta)/n(1 + \sqrt{\log(4p^2/\delta)/n} )/n}
\]

Invoking that \( n \geq \log(4p^2/\delta) \) we obtain the result. \( \square \)

Recall that, for any set \( S \subset \{1, \ldots, p\} \),

\[
\Sigma_{n,S} = n^{-1} \sum_{i=1}^{n} (X^{(i)}_{S,\cdot} - \overline{X}^n_{S,\cdot}) (X^{(i)}_{S,\cdot} - \overline{X}^n_{S,\cdot})^T
\]

\[
\Pi_{n,S} = \Sigma_S^{-1/2} \Sigma_{n,S} \Sigma_S^{1/2}.
\]

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Lemma 10. Suppose that Assumptions 4 and 5 are fulfilled. Let $S \subset \{1, \ldots, p\}$ be such that, almost surely,

$$B_S \geq \text{tr}((X_{(S^c)} - E(X_{(S^c)}))^T \Sigma_{S^{-1}}^{-1} (X_{(S^c)} - E(X_{(S^c)}))).$$

Then, for any $(n, \delta)$ such that $n \geq 4B_S \log(4|S|T/\delta)$, it holds, with probability $1 - \delta$:

$$\|\Pi_{n,S} - I_{|S|}\| \leq 4\sqrt{n^{-1}B_S \log(4|S|T/\delta)}.$$

Proof. Remark that

$$\Pi_{n,S} = \tilde{\Pi}_{n,S} - Z_{(S)}^{nT}$$

with

$$\tilde{\Pi}_{n,S} = n^{-1} \sum_{t=1}^{n} (X_{(S^c)} - E(X_{(S^c)}))(X_{(S^c)} - E(X_{(S^c)}))^T.$$  

Apply the triangle inequality to get

$$\|\Pi_{n,S} - I_{|S|}\| \leq \|\tilde{\Pi}_{n,S} - I_{|S|}\| + \|Z_{(S)}^{nT}\| = \|\tilde{\Pi}_{n,S} - I_{|S|}\| + \|Z_{(S)}^{nT}\|^2.$$

We shall apply Lemma 12 to control each of the two terms in the previous upper bound. First consider the case where $S^{(i)} = (Z_{(S)}^{(i)}Z_{(S)}^{(i)T} - I_{|S|})/n$. We need to specify the value for $L$ and $v$ that we can use. Note that

$$\|Z_{(S)}^{(i)}Z_{(S)}^{(i)T}\| = \|\sum_{t=1}^{T} Z_{(S)}^{(i)}Z_{(S)}^{(i)T}\| \leq \sum_{t=1}^{T} \|Z_{(S)}^{(i)}Z_{(S)}^{(i)T}\| \leq \sum_{t=1}^{T} \|Z_{(S)}^{(i)}\|^2 \leq B_S.$$  

Using the triangle inequality and Jensen inequality, we have, for any $i = 1, \ldots, n$,

$$\|S^{(i)}\| \leq n^{-1}(\|Z_{(S)}^{(i)}Z_{(S)}^{(i)T}\| + E[\|Z_{(S)}^{(i)}Z_{(S)}^{(i)T}\|]) \leq 2B_S n^{-1}.$$  

Consequently we can take $L = 2B_S n^{-1}$ for the value of $L$. Moreover, we have

$$\|\sum_{i=1}^{n} E[(Z_{(S)}^{(i)}Z_{(S)}^{(i)T} - I)^T (Z_{(S)}^{(i)}Z_{(S)}^{(i)T} - I)]\| = \|\sum_{i=1}^{n} (E[Z_{(S)}^{(i)}Z_{(S)}^{(i)T}Z_{(S)}^{(i)T} - I])\|$$

$$= n\|E[Z_{(S)}^{(i)}Z_{(S)}^{(i)T}Z_{(S)}^{(i)T}] - I\|

\leq n\|E[Z_{(S)}^{(i)}Z_{(S)}^{(i)T}Z_{(S)}^{(i)T}]\|

= n \sup_{\|u\| = 1} E[u^T Z_{(S)}^{(i)}Z_{(S)}^{(i)T}Z_{(S)}^{(i)T}u]$$

From the classic inequality $\|A^{1/2}u\|^2 \leq \|A\|\|u\|^2$, we deduce that $\|Au\|^2 \leq \|A\|\|A^{1/2}u\|^2$. Applying this with $A = Z_{(S)}^{(i)}Z_{(S)}^{(i)T}$ and using (12), we obtain

$$\|Z_{(S)}^{(i)}Z_{(S)}^{(i)T}u\| \leq \|Z_{(S)}^{(i)}Z_{(S)}^{(i)T}\|\|Z_{(S)}^{(i)}Z_{(S)}^{(i)T}\|^{1/2} \leq B_S \|Z_{(S)}^{(i)}Z_{(S)}^{(i)T}\|^{1/2}.$$  

Taking the expectation in the previous inequality, we get $E[\|Z_{(S)}^{(i)}Z_{(S)}^{(i)T}u\|^2] \leq B_S \|u\|^2$. Hence

$$\|\sum_{i=1}^{n} E[(Z_{(S)}^{(i)}Z_{(S)}^{(i)T} - I)^T (Z_{(S)}^{(i)}Z_{(S)}^{(i)T} - I)]\| \leq nB_S.$$  

Hence we can take $nTB_S$ for the value of $v$. Lemma 12 gives that

$$\mathbb{P}\left(\|\tilde{\Pi}_{n,S} - I_{|S|}\| > t\right) \leq 2|S| \exp\left(\frac{-t^2}{2n^{-1}B_S(1 + 2T/3)}\right).$$

Consequently, with probability $1 - \delta/2$, it holds that

$$\|\tilde{\Pi}_{n,S} - I_{|S|}\| \leq \sqrt{4n^{-1}B_S \log(4|S|/\delta)},$$
provided that $4n^{-1}B_S \log(4|S|/\delta) \leq 9/4$. Now we apply Lemma 12 with $S^{(i)} = Z^{(i)}_{(S,\cdot)}/n$ to provide a bound on $\|Z^{(i)}_{(S,\cdot)}\|$. We have, by (12), that

\[
\|Z^{(i)}_{(S,\cdot)}\| = \|Z^{(i)}_{(S,\cdot)} Z^T_{(S,\cdot)}\|^{1/2} \leq B_S^{1/2},
\]

max \{\|E[Z^{(i)}_{(S,\cdot)} Z^T_{(S,\cdot)}]\|, \|E[Z^T_{(S,\cdot)} Z^{(i)}_{(S,\cdot)}]\|\} = \max \{\|I_S\|, |S|\} = |S| \leq B_S.

It follows that, using that $4n^{-1}B_S \log(4|S|/\delta) \leq 9/4$, we have with probability smaller than

\[
P \left( \|Z^{(i)}_{(S,\cdot)}\|^n > t \right) \leq 2|S|T \exp \left( \frac{-t^2}{2(B_Sn^{-1} + tB_S^{1/2}n^{-1}/3)} \right).
\]

By taking $t = \sqrt{4n^{-1}B_S \log(4|S|/\delta)}$ we get that $\|Z^{(i)}_{(S,\cdot)}\|^n > t$ with probability smaller than

\[
2|S|T \exp \left( \frac{-(2 \log(4|S|/\delta))}{(1 + \sqrt{4n^{-1} \log(4|S|/\delta)/3})} \right) \leq \delta/2.
\]

provided that $4n^{-1} \log(4|S|/\delta) \leq 9$. Hence we have shown that with probability $1 - \delta$,

\[
\|\Delta\| + \|Z^{(i)}_{(S,\cdot)}\| \leq \sqrt{4n^{-1}B_S \log(4|S|/\delta)} + 4n^{-1}B_S \log(4|S|/\delta) \leq \sqrt{4n^{-1}B_S \log(4|S|/\delta)}(1 + \sqrt{4n^{-1}B_S \log(4|S|/\delta)})
\]

Use that $4n^{-1}B_S \log(4|S|/\delta) \leq 1$ to conclude. \hfill \Box

### Appendix C Auxiliary results

#### Lemma 11 (Bernstein inequality). Suppose that $(A^{(i)})_{i=1}^n$ is an iid sequence of centered random vectors valued in $\mathbb{R}^{p \times T}$. Suppose that $\|A^{(i)}\|_\infty \leq c$ a.s. and that $\text{Var}(A^{(i)}_{k,t}) \leq v$. For any $\delta \in (0, 1)$, $n \geq 1$ such that $9vn \geq 4c^2 \log(2pT/\delta)$, we have with probability $1 - \delta$:

\[
\left\| \sum_{i=1}^n A^{(i)} \right\|_\infty \leq \sqrt{4nv \log(2pT/\delta)}.
\]

**Proof.** By Bernstein inequality, it follows that

\[
P \left( \left\| \sum_{i=1}^n A^{(i)}_{k,t} \right\| > x \right) \leq 2 \exp \left( -\frac{x^2}{2(vn + cx/3)} \right).
\]

Choosing $x = \sqrt{4nv \log(2pT/\delta)}$ and because $vn \geq cx/3$, we get that

\[
P \left( \left\| \sum_{i=1}^n A^{(i)}_{k,t} \right\| > x \right) \leq \delta/(pT).
\]

Use the union bound to get that, for any $t > 0$,

\[
P \left( \left\| \sum_{i=1}^n A^{(i)} \right\|_\infty > x \right) \leq \sum_{1 \leq k,t \leq p} \P \left( \left\| \sum_{i=1}^n A^{(i)}_{k,t} \right\| > x \right) \leq \delta.
\]

\hfill \Box

#### Lemma 12 (Matrix Bernstein inequality). Let $S^{(i)}_1, \ldots, S^{(i)}_n$ be independent, centered random matrices with common dimension $d_1 \times d_2$, and assume that each one is uniformly bounded in spectral norm, i.e.,

\[
\E[S^{(i)}] = 0 \quad \text{and} \quad \|S^{(i)}\| \leq L \quad \text{for each} \quad i = 1, \ldots, n.
\]
Let $v > 0$ be such that
\[
\max \left\{ \left\| \sum_{i=1}^{n} E \left( S^{(i)} S^{(i)T} \right) \right\|, \left\| \sum_{i=1}^{n} E \left( S^{(i)} T S^{(i)} \right) \right\| \right\} \leq v.
\]

Then, for all $t \geq 0$,
\[
P \left( \left\| \sum_{i=1}^{n} S^{(i)} \right\| \geq t \right) \leq (d_1 + d_2) \exp \left( \frac{-t^2/2}{v + Lt/3} \right).
\]