On Gorenstein homological dimension of groups

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Abstract

Let $G$ be a group and $R$ be a ring. We define the Gorenstein homological dimension of $G$ over $R$, denoted by $\text{Ghd}_RG$, as the Gorenstein flat dimension of trivial $RG$-module $R$. It is proved that $\text{Ghd}_SG \leq \text{Ghd}_RG$ for any flat extension of commutative rings $R \to S$; in particular, $\text{Ghd}_RG$ is a refinement of $\text{Ghd}_ZG$ if $R$ is $\mathbb{Z}$-torsion-free. We show a Gorenstein homological version of Serre’s theorem, i.e. $\text{Ghd}_RG = \text{Ghd}_RH$ for any subgroup $H$ of $G$ with finite index. As an application, $G$ is a finite group if and only if $\text{Ghd}_RG = 0$; this is different from the fact that the homological dimension of any non-trivial finite group is infinity.

Key Words: Gorenstein homological dimension, group ring, Gorenstein flat.

2010 MSC: 20J05, 18G20, 16S34.

1. Introduction

In group theory, it is a long history issue to study groups through their (co)homological properties, which arose from both topological and algebraic sources. For any group $G$, the cohomological dimension $\text{cd}_ZG$ and the homological dimension $\text{hd}_ZG$, is defined as the projective dimension and the flat dimension of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, respectively.

Enochs, Jenda and Torrecillas introduced the concepts of Gorenstein projective, Gorenstein injective and Gorenstein flat modules, and developed Gorenstein homological algebra [12], which has its origin dated back to the study of G-dimension by Auslander and Bridger [3] in 1960s. As counterparts in Gorenstein homological algebra, the Gorenstein (co)homological dimensions of groups are extensively studied, see for example [1, 2, 4, 6, 11, 17, 25].

In [2, Definition 4.5], the Gorenstein homological dimension $\text{Ghd}_ZG$ of any group $G$ is defined to be the Gorenstein flat dimension of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. Analogously, we may define Gorenstein homological dimension of group $G$ over any coefficient ring $R$, denoted by $\text{Ghd}_RG$, to be the Gorenstein flat dimension of the trivial $RG$-module $R$. The Gorenstein homological dimension of groups generalizes the notion of homological dimension of groups, in the sense that $\text{Ghd}_RG = \text{hd}_RG$ if $\text{hd}_RG$ is finite.

First, we intend to compare $\text{Ghd}_RG$ with $\text{Ghd}_ZG$. More general, we concern the Gorenstein homological dimensions under the extension of coefficient rings. If $R \to S$ is a flat extension
of commutative rings, we show that for any group $G$, $\operatorname{Ghd}_S G \leq \operatorname{Ghd}_R G$ holds; see Theorem 2.6. This implies that if $R$ is $\mathbb{Z}$-torsion-free, then $\operatorname{Ghd}_R G$ is a refinement of $\operatorname{Ghd}_S G$; especially, $\operatorname{Ghd}_S G \leq \operatorname{Ghd}_Z G$ (see Corollary 2.7). A specific case leading to the equality $\operatorname{Ghd}_R G = \operatorname{Ghd}_S G$ is given in Proposition 2.8 where $G$ is a countable group and $S = R[x]/(x^n)$.

It is natural to consider the behavior of Gorenstein homological dimensions under extensions of groups. If a subgroup $H$ of $G$ is assumed to be of finite index, it is proved in [2, Proposition 4.11] that $\operatorname{Ghd}_Z H \leq \operatorname{Ghd}_Z G$. Moreover, the equality $\operatorname{Ghd}_Z H = \operatorname{Ghd}_Z G$ holds provided that the supremum of injective length (dimension) of flat $\mathbb{Z}$-modules, denoted by $\operatorname{silf}(\mathbb{Z}G)$, is finite; see [2, Theorem 4.18].

Our result strengthens and extends [2, Proposition 4.11] and [2, Theorem 4.18]. We show in Theorem 3.2 that if $H$ is a subgroup of $G$ with finite index, then $\operatorname{Ghd}_R H = \operatorname{Ghd}_R G$ for any commutative ring $R$, where the assumption for the finiteness of $\operatorname{silf}(\mathbb{Z}G)$ is removed. Recall that Serre’s Theorem establishes an equality between cohomology dimensions of a torsion-free group and its subgroup of finite index; see details in [7, Theorem VIII 3.1]. In this sense, the result can also be regarded as a Gorenstein homological version of Serre’s Theorem.

There is a homological characterization for finite groups immediately; see Corollary 3.3. That is, $G$ is a finite group if and only if $\operatorname{Ghd}_R G = 0$ for any commutative ring $R$; this generalizes [2, Proposition 4.12]. It is worth to compare this with a well-known fact: for any non-trivial finite group $G$ one always has $\operatorname{hd}_Z G = \infty$, since the finiteness of $\operatorname{hd}_Z G$ implies the group $G$ is necessarily torsion-free, while every finite group is torsion. We may also compare this result with [11, Corollary 2.3], which concerns Gorenstein cohomological dimension of finite groups.

2. Gorenstein homological dimension of groups

Let $A$ be a ring, $M$ be a (left) $A$-module. An acyclic complex

$$
\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots
$$

is called a totally acyclic complex of flat modules, provided that each $F_i$ is a flat (left) $A$-module, and for any injective (right) $A$-module $I$, the complex remains acyclic after applying $I \otimes_A -$. The module $M$ is said to be Gorenstein flat if there exists a totally acyclic complex of flat modules such that $M \cong \operatorname{Ker}(F_0 \rightarrow F_{-1})$; see [12, 13].

It is proved in [15, Theorem 3.7] that for a right coherent ring $A$, the class of Gorenstein flat left $A$-modules is closed under extensions and direct summands. This result is extended and generalized to any ring recently by the work [24]. The following is immediate from [24, Corollary 4.12].

**Lemma 2.1.** Let $A$ be a ring. Then the class of Gorenstein flat $A$-modules is closed under extensions and direct summands.

The Gorenstein flat dimension of modules is defined in the standard way by using resolutions. Let $M$ be any left $A$-module. The Gorenstein flat dimension of $M$, denoted by $\operatorname{Gfd}_A M$, is defined
by declaring that \( \text{Gfd}_A M \leq n \) if and only if \( M \) has a Gorenstein flat resolution \( 0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0 \) of length \( n \); see for example [12, 15].

**Remark 2.2.** It is not easy to show that the class of Gorenstein flat modules is closed under extensions; however, this basic property is crucial for studying homological properties of Gorenstein flat modules. For this reason, the rings are assumed to be coherent in [12]; as a generalization, the notion of GF-closed rings is introduced in [7]. Thanks to [24, Corollary 4.12], now we can remove the assumptions of coherent rings and GF-closed rings in many situations, see for example [15, Theorem 3.14] and the main theorem of [26], when dealing with Gorenstein flat modules and Gorenstein flat dimension of modules.

Recall that for any group \( G \), the Gorenstein homological dimension of \( G \) is defined to be the Gorenstein flat dimension of the \( \mathbb{Z}G \)-module \( \mathbb{Z} \) with the trivial group action; see [2, Definition 4.5]. Analogously, we have the following.

**Definition 2.3.** Let \( R \) be a ring. For any group \( G \), the Gorenstein homological dimension of \( G \) over \( R \), denoted by \( \text{Ghd}_R G \), is defined to be the Gorenstein flat dimension of the trivial \( RG \)-module \( R \).

Let \( R \) be a, \( G \) be any group. Recall that the homological dimension of a group \( G \) over \( R \), denoted by \( \text{hd}_R G \), is defined to be the flat dimension \( \text{id}_{RG} \) of the trivial \( RG \)-module \( R \). Since flat modules are necessarily Gorenstein flat, and the flat dimension of any Gorenstein flat module is either zero or infinity, it follows that \( \text{Ghd}_R G \leq \text{hd}_R G \) with the equality if \( \text{hd}_R G \) is finite.

First, we intend to compare \( \text{Ghd}_R G \) with \( \text{Ghd}_\mathbb{Z} G \). More general, we consider Gorenstein homological dimensions of the group under extensions of coefficient rings; see Theorem 2.6.

**Lemma 2.4.** Let \( R \to S \) be a flat extension of commutative rings, i.e. \( S \) is flat as an \( R \)-module. Then, for any group \( G \), \( RG \to SG \) is also a flat extension.

**Proof.** Since \( S \) is a flat \( R \)-module, by Lazard’s theorem we have \( S \cong \lim P_i \), where \( P_i \) are finitely generated projective \( R \)-modules. Then, the isomorphism \( SG \cong RG \otimes_R S \cong \lim (RG \otimes_R P_i) \) implies that \( SG \) is a flat \( RG \)-module, that is, \( RG \to SG \) is a flat extension, as expected. \( \square \)

**Lemma 2.5.** Let \( R \to S \) be a flat extension of commutative rings, and \( G \) be any group. For any Gorenstein flat \( RG \)-module \( M \), the induced \( SG \)-module \( S \otimes_R M \) is also Gorenstein flat.

**Proof.** Let \( M \) and \( N \) be \( RG \)-modules. By using the anti-automorphism \( g \to g^{-1} \) of \( G \), we can set \( mg = g^{-1}m \) for any \( g \in G \) and \( m \in M \). Note that \( M \otimes_R N \) is obtained from \( M \otimes_R N \) by introducing the relations \( g^{-1}m \otimes n = mg \otimes n = m \otimes gn \). Then, we replace \( m \) by \( gm \) and obtain \( m \otimes n = gm \otimes gn \). Hence, we see that \( M \otimes_R N = (M \otimes_R N)_G \), where \( G \) acts diagonally on \( M \otimes_R N \), and the group of co-invariants \((M \otimes_R N)_G \) is defined to be the largest quotient of \( M \otimes_R N \) on which \( G \) acts trivially.

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Let \( F \) be any flat left \( RG \)-module, which can be restricted to be a flat \( R \)-module. By considering \( S \) as an \( SG \)-\( R \)-bimodule, we have an induced \( SG \)-module \( S \otimes_R F \), where the \( G \)-action on \( S \) is trivial. Let \( M \) be any right \( SG \)-module, which has a natural \( RG \)-module structure by the ring extension \( RG \to SG \). We have

\[
M \otimes_S (S \otimes_R F) = (M \otimes_S (S \otimes_R F))_G \cong (M \otimes_R F)_G = M \otimes_{RG} F.
\]

This implies that the functor \( - \otimes_S (S \otimes_R F) \) is exact since the \( RG \)-module \( F \) is flat and the functor \( - \otimes_{RG} F \) is exact. Hence, the induced \( SG \)-module \( S \otimes_R F \) is flat.

Now assume that \( M \) is a Gorenstein flat \( RG \)-module. Then \( M \) admits a totally acyclic complex of flat \( RG \)-modules \( F = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots \) such that \( M \cong \text{Ker}(F_0 \to F_{-1}) \). Since \( S \) is a flat \( R \)-module, we obtain an acyclic complex of flat \( SG \)-modules

\[
S \otimes_R F = \cdots \to S \otimes_R F_1 \to S \otimes_R F_0 \to S \otimes_R F_{-1} \to \cdots
\]

by applying the functor \( S \otimes_R - \).

For any injective right \( SG \)-module \( I \), we have an isomorphism \( I \otimes_S (S \otimes_R F) \cong I \otimes_{RG} F \). We claim that \( I \) is restricted to be an injective right \( RG \)-module. There are natural equivalences of functors

\[
\text{Hom}_{RG}(-, I) \cong \text{Hom}_{RG}(-, \text{Hom}_{SG}(SG, I)) \cong \text{Hom}_{SG}(- \otimes_{RG} SG, I),
\]

where the second one is from the standard adjunction. Since \( R \to S \) is assumed to be a flat extension, it follows from Lemma \([2, \text{Lemma 2.3}]\) that \( SG \) is a flat \( RG \)-module. Moreover, by noting \( I \) is an injective right \( SG \)-module, we infer that the functor \( \text{Hom}_{SG}(- \otimes_{RG} SG, I) \) is exact. Hence, \( \text{Hom}_{RG}(-, I) \) is an exact functor and \( I \) is an injective right \( RG \)-module, as desired.

Then, for the totally acyclic complex of flat \( RG \)-modules \( F \), the complex \( I \otimes_{RG} F \) remains acyclic. Consequently, the complex \( I \otimes_S (S \otimes_R F) \) is acyclic, i.e. \( S \otimes_R F \) is a totally acyclic complex of flat \( SG \)-modules, and moreover, \( S \otimes_R M \cong \text{Ker}(S \otimes_R F_0 \to S \otimes_R F_{-1}) \) is a Gorenstein flat \( SG \)-module. This completes the proof.

\[\square\]

Now, we are in a position to state the following.

**Theorem 2.6.** Let \( R \to S \) be a flat extension of commutative rings. For any group \( G \), we have

\[
\text{Ghd}_S G \leq \text{Ghd}_R G.
\]

**Proof.** There is nothing to prove if \( \text{Ghd}_R G = \infty \). Then, it only suffices to consider the case where \( \text{Ghd}_R G = n \) is finite. By \([15, \text{Theorem 3.17}]\), there exists an exact sequence of \( RG \)-modules \( 0 \to K \to M \to R \to 0 \), where \( M \) is Gorenstein flat, \( \text{id}_{RG} K = n - 1 \). By applying the functor \( S \otimes_R - \), we obtain an exact sequence of induced \( SG \)-modules

\[
0 \to S \otimes_R K \to S \otimes_R M \to S \to 0.
\]
By Lemma 2.5, $S \otimes_R M$ is a Gorenstein flat $SG$-module. Remark that for any flat $RG$-module $F$, the induced $SG$-module $S \otimes_R F$ is also flat. Then, $\text{fd}_{SG}(S \otimes_R K) = n - 1$, and we infer from the above exact sequence that $Ghd_S G = \text{Gfd}_{SG} S \leq n$. This completes the proof. □

The following is immediate, which implies that under a quite mild condition, the Gorenstein homological dimension $Ghd_R G$ of a group $G$ over any commutative ring $R$ is a refinement of $Ghd_\mathbb{Z} G$, the one introduced in [2] over the ring of integers $\mathbb{Z}$.

**Corollary 2.7.** Let $R$ be a commutative ring, $G$ a group. If $R$ is a $\mathbb{Z}$-flat module ($\mathbb{Z}$-torsion-free), then $Ghd_R G \leq Ghd_{\mathbb{Z}} G$. In particular, $Ghd_R G \leq Ghd_{\mathbb{Z}} G$.

**Proposition 2.8.** Let $R$ be a commutative ring, $G$ be a countable group. Let $S = R[x]/(x^n)$ be the quotient of the polynomial ring, where $n > 1$ is an integer, and $x$ is a variable which is supposed to commute with all the elements of $G$. Then we have $Ghd_R G = Ghd_S G$.

**Proof.** Note that $R \to S$ is a flat extension, and then $Ghd_S G \leq Ghd_R G$ follows by Theorem 2.6. We consider $R$ and $S$ as $RG$-modules with trivial $G$-actions, then $R$ is a direct summand of $S = R[x]/(x^n)$, and $Ghd_R G = \text{Gfd}_{RG} R \leq \text{Gfd}_{RG} S$. It remains to prove the inequality $\text{Gfd}_{RG} S \leq Ghd_S G = \text{Gfd}_{SG} S$.

Observe that $SG = RG[x]/(x^n)$, which is easily seen from the equation

$$\sum_{j=0}^{n-1} \sum_{i \in \mathbb{N}} r_{ij} g_i x^j = \sum_{i \in \mathbb{N}} \left( \sum_{j=0}^{n-1} r_{ij} x^j \right) g_i, \forall r_{ij} \in R, g_i \in G.$$ 

Let $M$ be a Gorenstein flat $SG$-module and $\mathbb{F} = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ be a totally acyclic complex of flat $SG$-modules such that $M \cong \text{Ker}(F_0 \to F_{-1})$. Let $I$ be any injective right $RG$-module. There are isomorphisms $I \otimes_{RG} SG \cong \text{Hom}_{RG}(SG, I)$ and $I \otimes_{RG} \mathbb{F} \cong I \otimes_{RG} SG \otimes_{SG} \mathbb{F}$. We imply that the induced right $SG$-module $I \otimes_{RG} SG$ is injective, and the complex $I \otimes_{RG} \mathbb{F}$ is acyclic. That is, by restriction we can obtain a totally acyclic complex $\mathbb{F}$ of flat $RG$-modules, and hence $M$ is restricted to be a Gorenstein flat $RG$-module.

The inequality $\text{Gfd}_{RG} S \leq Ghd_S G$ is obviously true if $Ghd_S G = \infty$. Now we assume that $Ghd_S G = n$ is finite. There is an exact sequence of $SG$-modules

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to S \to 0,$$

where all $M_i$ are Gorenstein flat. By the above argument, we may consider $M_i$ as Gorenstein flat $RG$-modules, and hence, $\text{Gfd}_{RG} S \leq n$. This completes the proof. □

### 3. A version of Serre’s theorem

In this section, we consider the behavior of Gorenstein homological dimension of subgroups. Let $H$ be a subgroup of $G$ with finite index. Recall that there is an inequality $Ghd_\mathbb{Z} H \leq Ghd_\mathbb{Z} G$, and moreover, the equality $Ghd_\mathbb{Z} H = Ghd_\mathbb{Z} G$ holds provided the supremum of injective length
(dimension) of flat $\mathbb{Z}G$-modules, denoted by $\text{sif}(\mathbb{Z}G)$, is finite; see [2, Proposition 4.11] and [2, Theorem 4.18] respectively.

We have the following result, where the finiteness of “sif” is not necessarily needed; see Theorem 3.2. Hence, it strengthens and extends [2, Proposition 4.11] and [2, Theorem 4.18]. By Serre’s Theorem, there is an equality between cohomology dimensions of any torsion-free group and its subgroup of finite index; see details in [3, Theorem VIII 3.1]. In this sense, the result can also be regarded as a Gorenstein homological version of Serre’s Theorem.

Let $H$ be any subgroup of $G$. There exist simultaneously an induction functor $\text{Ind}_H^G = RG \otimes_{RH} -$ and a coinduction functor $\text{Coind}_H^G = \text{Hom}_{RH}(RG, -)$ from the category of $RH$-modules $\text{Mod}(RH)$ to the category of $RG$-modules $\text{Mod}(RG)$. We denote by $\text{Res}_H^G$ the standard restriction functor, which sends every $RG$-module to be an $RH$-module.

We have the following observations.

**Lemma 3.1.** Let $G$ be a group, and $H$ be any subgroup of $G$ with finite index. For any $RG$-module $M$, the following hold.

1. If $M$ is a Gorenstein flat module, then the restricted $RH$-module $\text{Res}_H^G M$ is also Gorenstein flat.

2. The $RH$-module $\text{Res}_H^G M$ is Gorenstein flat, if and only if the induced $RG$-module $\text{Ind}_H^G \text{Res}_H^G M$ is also Gorenstein flat.

**Proof.** (1) For the Gorenstein flat $RG$-module $M$, let $F = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots$ be a totally acyclic complex of flat $RG$-modules such that $M \cong \text{Ker}(F_0 \rightarrow F_{-1})$. By restriction, we obtain an acyclic complex $\text{Res}_H^G F$ of flat $RH$-modules. Since the index $[G : H]$ is finite, there is an equivalence of functors $\text{Ind}_H^G \cong \text{Coind}_H^G$; see for example [4, Proposition III 5.9]. For any injective right $RH$-module $I$, the $RG$-module $\text{Ind}_H^G I \cong \text{Coind}_H^G I$ is injective, and then the complex $\text{Ind}_H^G I \otimes_{RG} F$ is acyclic. We infer from the isomorphism $I \otimes_{RH} \text{Res}_H^G F \cong \text{Ind}_H^G I \otimes_{RG} F$ that the complex $I \otimes_{RH} \text{Res}_H^G F$ is acyclic, and then $\text{Res}_H^G F$ is a totally acyclic complex of flat $RH$-modules. Consequently, $M$ is restricted to be a Gorenstein flat $RH$-module, as claimed.

(2) For the “only if” part, note that $RG$ is a projective $RH$-module. For the Gorenstein flat $RH$-module $\text{Res}_H^G M$, there is a totally acyclic complex of flat $RH$-modules $F$. Then, we obtain an induced complex $\text{Ind}_H^G F$, which is acyclic with each item being flat $RG$-module. Let $E$ be any injective right $RG$-module. The restricted $RH$-module $\text{Res}_H^G E$ is injective, and we infer from the isomorphism $E \otimes_{RG} \text{Ind}_H^G F \cong \text{Res}_H^G E \otimes_{RH} F$ that the complex $\text{Ind}_H^G F$ is totally acyclic. Hence, the induced module $\text{Ind}_H^G \text{Res}_H^G M$ is a Gorenstein flat $RG$-module. Here, we do not need to assume that the index $[G : H]$ is finite.

For any $RG$-module $M$, there is a canonical $RG$-map

$$\text{Ind}_H^G \text{Res}_H^G M = RG \otimes_{RH} \text{Res}_H^G M \rightarrow M$$

given by $g \otimes m \mapsto gm$. This map is surjective; moreover, as an $RH$-map it is a split surjective. If the $RG$-module $\text{Ind}_H^G \text{Res}_H^G M$ is Gorenstein flat, we infer from (1) that $\text{Res}_H^G \text{Ind}_H^G \text{Res}_H^G M$ is
a Gorenstein flat $RH$-module. Consequently, by Lemma 2.1 its direct summand $\text{Res}_H^G M$ is a Gorenstein flat $RH$-module. This proves the “if” part. □

**Theorem 3.2.** Let $G$ be a group, $H$ be a subgroup of $G$ with finite index. There is an equality $\text{Ghd}_R G = \text{Ghd}_R H$.

**Proof.** To simplify the burden of notations, for any $RG$-module $M$, the restricted $RH$-module $\text{Res}_H^G M$ will be denoted by $M$ as well. It follows immediately from the first assertion of Lemma 3.1 that for any $RG$-module $N$, one has an inequality $\text{Gfd}_R H N \leq \text{Gfd}_R G N$. In particular, for the trivial $RG$-module $R$, we have $\text{Ghd}_R H \leq \text{Ghd}_R G$.

It remains to prove $\text{Ghd}_R G \leq \text{Ghd}_R H$. Since the inequality is trivial if $\text{Ghd}_R H = \infty$, it suffices to assume $\text{Ghd}_R H = n$ is finite. Take an exact sequence $0 \to N \to F_{n-1} \to \cdots \to F_0 \to R \to 0$ of $RG$-modules with each $F_i$ being flat. Since $F_i$ are restricted to be flat $RH$-modules, it follows from $\text{Ghd}_R H = n$ that as a restricted $RH$-module, $N$ is Gorenstein flat. For the required inequality, it suffices to show that $N$ is a Gorenstein flat $RG$-module.

Let $I$ be any injective right $RG$-module. There is a canonical map of $RG$-modules

$$I \to \text{Hom}_{RH}(RG, I) = \text{Coind}_H^G I$$

given by $x \mapsto (g \mapsto gx)$ for any $x \in I$ and any $g \in G$. This map is injective, and then $I$ is a direct summand of $\text{Coind}_H^G I$. For any $i > 0$, we have $\text{Tor}_i^{RG}(I, N) = 0$ since by restriction $N$ is a Gorenstein flat $RH$-module, and $I$ is an injective $RH$-module. Then, we infer from the isomorphism

$$\text{Tor}_i^{RG}(\text{Coind}_H^G I, N) \cong \text{Tor}_i^{RG}(\text{Ind}_H^G I, N) \cong \text{Tor}_i^{RH}(I, N)$$

that $\text{Tor}_i^{RG}(I, N) = 0$.

Let $\alpha : N \to \text{Ind}_H^G N$ be a composition of the canonical $RG$-map $N \to \text{Coind}_H^G N$, followed by the isomorphism $\text{Coind}_H^G N \to \text{Ind}_H^G N$. Then $\alpha$ is an $RG$-monic and is split as an $RH$-map.

Since $N$ is a Gorenstein flat $RH$-module, it follows from Lemma 3.1 that the induced module $\text{Ind}_H^G N$ is a Gorenstein flat $RG$-module. Hence, there is an exact sequence of $RG$-modules

$$0 \to \text{Ind}_H^G N \xrightarrow{\beta} F_0 \to \text{Coker} \beta \to 0$$

for which $F_0$ is flat and $\text{Coker} \beta$ is Gorenstein flat.

Consider the following diagram

$$\begin{array}{ccc}
0 & \rightarrow & N \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Ind}_H^G N
\end{array} \xrightarrow{\gamma} \begin{array}{ccc}
F_0 & \rightarrow & \text{Coker} \gamma \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
F_0 & \rightarrow & \text{Coker} \beta \rightarrow 0
\end{array}$$

Since $\alpha : N \to \text{Ind}_H^G N$ is a split $RH$-monic, the map $I \otimes_{RH} \alpha$ is injective. Since $\text{Coker} \beta$ is restricted to be a Gorenstein flat $RH$-module and $I$ is restricted to be an injective $RH$-module, we have $\text{Tor}_1^{RH}(I, \text{Coker} \beta) = 0$, which implies that the map $I \otimes_{RH} \beta$ is injective. Hence, the map
$I \otimes_{RH} \gamma$ is injective, as well. We infer from the exact sequence

$$0 \to \text{Tor}^1_{RH}(I, \text{Coker} \gamma) \to I \otimes_{RH} N \to I \otimes_{RH} F_0 \to I \otimes_{RH} \text{Coker} \gamma \to 0$$

that $\text{Tor}^1_{RH}(I, \text{Coker} \gamma) = 0$, and moreover, it yields by [15, Proposition 3.8] and [24, Corollary 4.12] that the restricted $RH$-module $\text{Coker} \gamma$ is Gorenstein flat. Analogous to the above argument, we have $\text{Tor}^1_{RG}(\text{Coind}^G_H I, \text{Coker} \gamma) \cong \text{Tor}^1_{RH}(I, \text{Coker} \gamma)$, and $I$ is a direct summand of $\text{Coind}^G_H I$ as $RG$-modules. Hence $\text{Tor}^1_{RG}(I, \text{Coker} \gamma) = 0$, and furthermore, the sequence

$$0 \to N \to F_0 \to \text{Coker} \gamma \to 0$$

remains exact after applying $I \otimes_{RG} -$.

Then, repeat the above argument for $\text{Coker} \gamma$, we will obtain inductively an acyclic complex

$$0 \to N \to F_0 \to F_{-1} \to F_{-2} \to \cdots$$

with each $F_i$ a flat $RG$-module, which remains acyclic after applying $I \otimes_{RG} -$ for any injective right $RG$-module $I$. By pasting this sequence with the flat resolution of $N$, we will obtain a totally acyclic complex of flat $RG$-modules for $N$, and then $N$ is a Gorenstein flat $RG$-module, as expected. This completes the proof.

We have the following immediately, which extends [2, Proposition 4.12].

**Corollary 3.3.** Let $G$ be a group. The following conditions are equivalent:

1. $G$ is a finite group.
2. For any commutative ring $R$, $\text{Ghd}_R G = 0$.
3. $\text{Ghd}_Z G = 0$.

**Proof.** (1)$\implies$(2). Let $H = \{e\}$ be the subgroup of $G$, which only contains the identity element of $G$. Since $G$ is a finite group, the index $[G : H]$ is finite. Hence, $\text{Ghd}_R G = \text{Ghd}_R H = 0$ follows immediately from Theorem 3.2.

(2)$\implies$(3) is trivial, and (3)$\iff$(1) is precisely [2, Proposition 4.12].

**Remark 3.4.** If $H$ is a subgroup of $G$ with finite index, then both $(\text{Ind}^G_H, \text{Res}^G_H)$ and $(\text{Res}^G_H, \text{Ind}^G_H)$ are adjoint pairs of functors. For any associative ring $R$, $RH \to RG$ is a Frobenius extension of group rings, and the pair of functors $(\text{Ind}^G_H, \text{Res}^G_H)$ is called a strongly adjoint pair by Morita [20], or a Frobenius pair by [8, Definition 1.1]. The Gorenstein homological properties of modules under Frobenius extension of rings and Frobenius pairs of functors were studied in [9, 16, 21–23, 24].

**Funding.** The second author is supported by the National Natural Science Foundation of China (No. 11871125) and Natural Science Foundation of Chongqing, China (No. cstc2018jcyjAX0541).
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