DECOHERENCE, CHAOS, AND THE SECOND LAW

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ABSTRACT: We investigate implications of decoherence for quantum systems which are classically chaotic. We show that, in open systems, the rate of von Neumann entropy production quickly reaches an asymptotic value which is: (i) independent of the system-environment coupling, (ii) dictated by the dynamics of the system, and (iii) dominated by the largest Lyapunov exponent. These results shed a new light on the correspondence between quantum and classical dynamics as well as on the origins of the “arrow of time.”

PACS: 03.65.Bz, 05.45.+b, 05.40+j
The relation between classical and quantum chaos has been always somewhat unclear and, at times, even strained. The cause of the difficulties can be traced to the fact that the defining characteristic of classical chaos – *sensitive dependence on initial conditions* – has no quantum counterpart: it is defined through the behavior of neighboring trajectories, a concept which is essentially alien to quantum mechanics. Moreover, when the natural language of quantum mechanics of closed systems is adopted, an analogue of the exponential divergence cannot be found. This is not to deny that many interesting insights into quantum mechanics have been arrived at by studying quantized versions of classically chaotic systems. These insights have typically much to do with the energy spectra, and leave the issue of the relationship between the quantum and the classical largely open.

The aim of this paper is to investigate implications of the process of decoherence for quantum chaos. Decoherence is caused by the loss of phase coherence between the set of preferred quantum states in the Hilbert space of the system due to the interaction with the environment. Preferred states are singled out by their stability (measured, for example, by the rate of predictability loss – the rate of entropy increase) under the joint influence of the environment and the self–Hamiltonian. Thus, the strength and nature of the coupling with the environment play a crucial role in selecting preferred states, which – given the distance–dependent nature of typical interactions – explains the special function of the position observable. Coupling with the environment also sets the decoherence timescale – the time on which quantum interference between preferred states disappear. Classicality is then an emergent property of an open quantum system. It is caused by the incessant monitoring by the environment, the state of which keeps a “running record” of the preferred observables of the evolving quantum system. For simple quantum systems the programme sketched above can be carried out rigorously, and yields
intuitively appealing results\(^5,^9\). For example, preferred states of an underdamped harmonic oscillator turn out to be its coherent states\(^8\).

If decoherence does induce a transition from quantum to classical, then it should be possible to utilize it in the context of quantum chaos to establish a more straightforward correspondence between the behavior of classically chaotic systems and their quantum counterparts. With this goal in mind we will consider a classically chaotic system, characterized by a potential \(V(x)\), coupled to an external environment. A master equation for the density operator of an open quantum system can be derived under a variety of reasonable assumptions\(^10\). Here, we shall focus our considerations on the simplest special case, the high temperature limit of an ohmic environment, leaving the discussion of other cases (low temperature regime, non–ohmic environments, etc) for later publications. In this case, the Wigner function of the system evolves according to\(^10\):

\[
\dot{W} = \{H, W\}_PB + \sum_{n \geq 1} \frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!} \partial_x^{2n+1}V \partial_p^{2n+1}W + 2\gamma \partial_p(pW) + D \partial_{pp}^2W \quad (1)
\]

where \(\gamma\) is the relaxation rate, the diffusion coefficient is \(D = 2\gamma mk_B T\) (\(T\) is the temperature of the environment and \(m\) is the mass of the system). The first term is the Poisson bracket, which generates the ordinary Liouville flow. Both the Poisson bracket and the higher derivative terms result from an expansion of the Moyal bracket, \(\{H, W\}_{MB} = -i\sin(i\hbar \{H, W\}_PB)/\hbar\), which generates evolution in phase space of a closed system (this expansion is valid when \(V(x)\) is an analytic function).

The last two terms in (1) arise due to the interaction with the environment. The first of them produces relaxation – gradual loss of energy to the reservoir – and the last one diffusion (this diffusion term is responsible for the \textit{decoherence} process).

As is clear from (1), as a consequence of the quantum correction terms that contain higher derivatives, the Wigner function of an isolated non–linear system
does not follow the classical Liouville flow. In a classically chaotic system these non-classical corrections rapidly gain importance, as can be seen by the following argument: When a chaotic flow is investigated locally in the phase space, the evolution operator can be expanded in coordinates “comoving” with a reference trajectory. The pattern of flow of the neighboring trajectories is then generated by the Jacobian of the transformation. Eigenvalues of this Jacobian are known as local Lyapunov exponents $\lambda_i$, which must sum to zero, since the transformation preserves phase space volume. Eigenvectors define directions in the phase space along which the neighboring trajectories either only expand ($\lambda_i > 0$) or only contract ($\lambda_i < 0$) with respect to the fiducial trajectory at a rate given by the corresponding Lyapunov exponent. The exponential contraction rapidly generates small scale structure in the Wigner function. Thus, the high derivative terms are $\partial_p^n W \propto \sigma_p^{-n} W$ with $\sigma_p \propto \sigma_p(0) \exp(\lambda t)$ where $\lambda$ is a Lyapunov exponent. Hence, non classical corrections will become important after a characteristic crossover time $t_\chi$ which can be estimated by comparing the magnitude of the nonlinear corrections with the contribution of the Poisson bracket in equation (1). Defining a characteristic length for the non-linear terms in the potential as $\chi_n \propto (\partial_x V/\partial_x^{n+1} V)^{1/n}$ we obtain that the $n$-th order term in (1) becomes comparable with the Poisson bracket at a time $t_\chi^{(n)}$ given by

$$t_\chi^{(n)} \propto \lambda^{-1} \log(\chi_n \sigma_p(0)/\hbar)$$

One of the points we want to make here is that, in the presence of decoherence, the regime in which the quantum correction become important is easily avoided. The main reason for this is the existence of diffusive effects which put a lower bound on the small scale structure which can be produced by the chaotic evolution. As a result, $\sigma_p(t)$ can never become sufficiently small to result in large corrections:
Poisson bracket is an excellent approximation of the quantum Moyal bracket for smooth Wigner functions. This line of reasoning, as we will argue below, has also important consequences concerning the rate of entropy production.

To simplify our analysis, based on equation (1), we will neglect the relaxation term which, as pointed out in the literature\(^9\), can be made arbitrarily small without decreasing the effectiveness of the decoherence process (e.g. by letting \(\gamma\) approach zero while keeping \(D\) constant). In this way, we will focus in the important reversible classical limit\(^6,9,11\). We will not limit ourselves to models leading to Eq. (1) which, as they break the symmetry between \(x\) and \(p\) coupling with the environment through position, have momentum diffusion only. It is convenient (especially in the context of quantum optics, where the “rotating wave approximation” can be invoked) to use a symmetric coupling (of the form \(a^\dagger b + ab^\dagger\), where \(a\) and \(b\) are the annihilation operators of the system and the mode of the environment\(^12\)). The corresponding equation differs from (1) in the form of the diffusion which is now symmetric, \(\propto D(\partial^2_{pp} + \partial^2_{xx})W\). We shall alternate between using this symmetric diffusion and the more exact diffusion operator of equation (1) in the discussion below.

Our objective here is to study the interplay between the evolution which classically results in an exponential divergence of neighboring trajectories (the characteristic feature of chaos) and the destruction of quantum coherence between a preferred set of states in the Hilbert space of an open system (a defining feature of decoherence). We shall do this by using a simple unstable system that still captures the essential features we want to consider. In general, a chaotic geodesic flow pattern is locally analogous to the one occurring near a saddle point, with stable and unstable directions defined in an obvious manner. The simplest example of
such a saddle point is afforded by an *unstable harmonic oscillator*. We shall use it as a “generic” model of locally chaotic phase space dynamics in our considerations below. In this case, the potential is $V(x) = -\lambda x^2/2$ ($\lambda$ is the Lyapunov exponent) and equation (1) reduces to a very simple form. We will use the lessons learned from this simple unstable system to argue that quantum corrections, which in the case of the unstable oscillator vanish identically, can be neglected whenever decoherence is effective. Our analysis will also show that three different stages of the evolution can be identified: (i) decoherence, (ii) approximately reversible Liouville flow and (iii) irreversible diffusion–dominated evolution.

To analyze in detail the unstable oscillator it is convenient to use contracting and expanding coordinates defined by $u^v = (p \mp m\lambda x)$. Evolution generated by Eq. (1) causes exponential expansion in $v$ and, without the diffusive term, it would also cause an exponential contraction in $u$, so that the volume in the phase space (as well as entropy) would be constant (see Fig. 1). Expansion in $v$ would also result in an exponential decrease of gradients in that direction. Thus, after a sufficient number of e-foldings the equation governing evolution of $W$ would be dominated by the expression:

$$
\dot{W} = \lambda \left( u \partial_u - v \partial_v + \frac{1}{2} \sigma_c^2 \partial_{uu} \right) W.
$$

The characteristic dispersion, which will play an important role below, is

$$
\sigma_c^2 = \frac{2D}{\lambda}.
$$

We can now easily examine the fate of a generic initial quantum state. The general solution of equation (3) can be found by noticing that the eigenfunctions of the operator appearing in its right hand side are $v^n F_m(u/\sigma_c)$ where $F_m(x) = \exp(-x^2/2)H_{m-1}(x/\sqrt{2})$ and $H_m(x)$ are Hermite polynomials. Expanding $W(u,v,t)$
in terms of these eigenfunctions (whose eigenvalue is simply \(-(n + m)\)) we obtain

$$W(u, v, t) = \sum_{n \geq 0} a_{nm}(ve^{-\lambda t})^n F_m(u)e^{-m\omega t} \quad (5)$$

From this expression we see that the Wigner function depends on \(v\) only through the combination \(v_0 = v \exp(-\lambda t)\), which is the comoving coordinate. That is, along this direction, the Wigner function just expands. Moreover, after a few dynamical times the most important contribution to (5) will always come from the \(m = 1\) term. Thus, in the contracting direction the Wigner function approaches a Gaussian with a critical width and has the form:

$$W(u, v, t) \approx \frac{1}{\sqrt{2\pi\sigma_c^2}} e^{-u^2/2\sigma_c^2} e^{-\omega t} \int_{-\infty}^{\infty} du W(u, v_0, t = 0). \quad (6)$$

The existence of the critical width \(\sigma_c\) is a consequence of the interplay between the exponential divergence of trajectories and diffusion. In effect, a competition between the chaotic evolution (which attempts to “squeeze” the wavepacket in the contracting direction), and the diffusion (which has the opposite tendency) leads to a compromise steady state which results in the Gaussian written above.

How do these general considerations imply the existence of the three distinguishable phases of the evolution? The analysis of decoherence, responsible for the quantum to classical transition, follows simply. Non–classical states possessing a rapidly oscillating non–positive \(W\) quickly evolve towards a mixture of localized states eventually resulting in a positive Wigner function. For example, if the initial state is a superposition of two coherent states separated by a distance \(L\) (along \(u\)), the ratio between the wavelength of the interference fringes \(\ell \propto 1/L\) and \(\sigma_c\) is \(\sigma_c^2/\ell^2 = (\gamma/\lambda)(L/\lambda_{dB})^2\) where \(\lambda_{dB}\) is the thermal de Broglie wavelength \(^{9}\). Therefore, the decoherence time is

$$\tau_{dec} = \gamma^{-1}(\frac{\lambda_{dB}}{L})^2 \quad (7)$$
which, for macroscopic scales, is much smaller than the dynamical times even for very weakly dissipative systems. Our analysis of the decoherence period is still incomplete since equation (6) suggests that the negativity of the Wigner function may persist along the expanding direction (in that equation the initial state is not changed along $v$ but just “stretched” by the geodesic flow). However, equation (3) was obtained by neglecting gradients along $v$. When a diffusion term $D\partial^2_{vv}W$ is added to the right hand side of (3), the eigenfunctions change in such a way that the powers $(ve^{-\lambda t})^n$ are replaced by Hermite polynomials $H_n(v/\sqrt{2}\sigma_c)e^{-n\lambda t}$. If we reexpress the solution in terms of $v_0$ we notice that, for times of the order of $1/\lambda$ only the highest power in $H_n(v/\sqrt{2}\sigma_c)$ survives. This implies that the asymptotic form of $W$ is no longer given by (6), which just contains the initial state expanded along the unstable direction. The correct expression simply contains a smoothed version of the initial condition in which the details smaller than $\sigma_c$ are washed out along the unstable direction. Again, oscillations with wavelength $\ell \propto 1/L$ are typically destroyed after the decoherence time (7).

The analysis of the reversible and irreversible stages can be illustrated by following the evolution of a Gaussian $W$. Here, the existence of $\sigma_c$ is again very important. The von Neumann entropy $\mathcal{H}$ of a Gaussian state can be easily related to the area $A$ enclosed by a $1-\sigma$ contour of the Wigner function. Thus, $\mathcal{H}$ is a monotonic function of $A$ which, when $A \gg h = 2\pi\hbar$ approaches $\mathcal{H} \simeq k_B \ln A/h$. Using the above equations, one can show that the rate of entropy production is

$$\dot{\mathcal{H}} = \frac{\dot{A}}{A} = \lambda \frac{\sigma_c^2}{\sigma_p^2(t)}$$

(8)

where $\sigma_p(t)$ is the width of the Gaussian along the direction of $p$. Consequently when the width of the Gaussian approaches the critical value, the entropy growth becomes equal to the positive Lyapunov exponent $\lambda$. The evolution of $\dot{\mathcal{H}}$ can be
approximately analyzed as follows: one can use (1) to show that the ratio $R \equiv (\sigma_p(t)/\sigma_c)^2$ evolves according to the equation $\dot{R} = 2\lambda (1 - \beta R)$ where $\beta(t)$, which is related to the degree of squeezing of the state, approaches unity exponentially fast.

Solving this equation approximately (using $\beta = 1$) one gets:

$$\dot{\mathcal{H}} = \lambda \left( 1 + \left( \frac{\sigma_p^2(0)}{\sigma_c^2} - 1 \right) \exp(-2\lambda t) \right)^{-1}$$

(9)

Evolution of a classical distribution (corresponding to a non-negative $W$, initially smooth on a scale much larger than $\sigma_c$ and spread over a regular patch with $A \gg h$) will typically proceed in two different stages (see Fig. 2). The first stage will be approximately area-preserving with the evolution dominated by the Liouville operator. It will last as long as each of the dimensions of the patch is much larger than the critical width. During this stage diffusion does little to alter the form of $W$. The Wigner function is merely “stretched” or “contracted” by the geodesic flow so that, with respect to the co-moving coordinates, “nothing happens” to $W$. By contrast, when the dimension of the patch becomes comparable with $\sigma_c$, diffusion will begin to dominate. Further contraction will be halted at $\sigma_c$ but the stretching will proceed at the rate set by the positive Lyapunov exponent. As a result, the area (or, more generally, the volume) in phase space will increase at the rate set by (9) with $\sigma_p = \sigma_c$. Using our approximate equation (9) one can estimate the time corresponding to the transition from reversible to irreversible evolution:

$$\tau_c = \lambda^{-1} \ln\left( \frac{\sigma_p(0)}{\sigma_c} \right)$$

(10)

An important condition must be valid in order to apply the above arguments, illustrated here in the simplest example, to more complicated non-linear systems: we need to assure that the Wigner function follows approximately the evolution generated by the Poisson bracket (i.e. that the effect of the higher derivative terms
As we pointed out above, in the absence of decoherence the quantum corrections to the evolution of $W$ become important at a crossover time $t_\chi$ given by equation (2). These quantum corrections will remain small if diffusion is strong enough to prevent the formation of small structure in $W$. As we argued above, the formation of small structure induced by the exponential contraction is stopped after a time $\tau_c$. Therefore, the condition for the Wigner function to evolve classically is

$$t_\chi \gg \tau_c,$$

(11)

which, taking into account equations (2) and (10), can be rewritten in the following suggestive way:

$$\chi_n \sigma_c \gg \hbar.$$  \hspace{1cm} (12)

This condition – a key criterion to assure the correspondence between quantum and classical dynamics – assures that $W$ follows the Liouville flow (albeit with diffusive contributions) and allows us to apply the conclusions of our previous analysis.

We have demonstrated that chaotic quantum systems can exhibit, in addition to the very rapid onset of decoherence, a nearly reversible phase of evolution which is necessarily followed by an irreversible stage in which the entropy increases linearly at the rate determined by the Lyapunov exponents. By contrast, open quantum systems with regular classical analogs continue to evolve with little entropy production (although possibly with a significant change in dynamics \(^{13}\)). This nearly constant rate of (von Neumann) entropy production, a consequence of the interplay between the chaotic dynamics of the system and its interaction with the environment, suggests not only a clear distinction between the integrable and chaotic systems, but also shows that increase of entropy in the context of quantum measurement \(^{14}\) and the dynamical aspects of the second law are intimately related and can be traced...
to the same cause: Impossibility of isolating macroscopic systems from their environments.

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Figure Captions

**Figure 1:** Three stages of the evolution of an initially Gaussian Wigner function shown in two different coordinate systems. Fig. (a)–(c) display $W$ in physical coordinates $(u, v$, with $v$ in logarithmic scale). For an open system which evolves according to equation (1) with $V = \lambda x^2 / 2$ (forefront), the width of the distribution reaches the asymptotic value $\sigma_c$. By contrast, when $W$ evolves unitarily (densely hatched and shown in the back), it continues to be squeezed in $u$. Fig. (a’)–(c’) shows the same three stages of the evolution, but now in co-moving coordinates $(\tilde{v} = v \exp(\lambda t), \tilde{u} = u \exp(-\lambda t))$. In the unitary case (cross-hatched, shown in the back) $W$ does not change. The interaction with the environment causes an exponential increase in the apparent width of $W$ (forefront). Since the width of the Gaussians in the expanding direction is approximately the same, the asymptotic regime of the diffusive evolution leads to an exponential increase of the area enclosed in $1-\sigma$ contour. Consequently, the entropy increases linearly at a rate determined by the Lyapunov exponent.

**Figure 2:** The rate of von Neumann entropy production for the quantum open system. The initial state is a Gaussian for which $\mathcal{H}(t = 0) \gg 1$ and with the initial width along the contracting direction much larger than $\sigma_c$. The (nearly) reversible and irreversible stages of the evolution are clearly distinguished.