Analysis of the properties of new forms of transitional sections of railway and highway curves

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Abstract. The properties of new forms of transitional sections of curves of railways and highways are largely determined by the specific features of the design and functioning of the systems “railway line-rolling stock” and “driver-car-road”. Taking them into account had a significant impact on the formalization of targets and criteria for the quality of the functioning of these systems, as well as on the mathematical means and methods of their achievement. The results obtained in this case significantly differ from the traditional ones. The geometric differences between the transitional sections of railways are in the nonlinear regularities of strictly monotonic, G4 smooth changes in the curvature of their line axis and G3 smooth cross slope, which are described by non-identical functions with variable properties. At the same time, the quality of movement, assessed at the design level of the rolling stock, is ensured by the harmonization of the length of the transition section and the variable properties of its functions with the given characteristics of other properties of the system. The differences between the transitional sections of highways consist in the inclusion of the calculated speed and the speed of its change directly into the principle of the trajectory curvature of vehicles. This ensures the safety and convenience of curvilinear movement of cars on them both at constant and variable speeds. The theoretically substantiated potential for the improvement of the quality of functioning of these systems with the proposed solutions determines the reasonability of their comprehensive testing in practice and further improvement of the theory, norms and rules for the design of transitional sections of curved rail and highways.

1. Introduction

The physical principles of the movement of vehicles on railways and roads have significant differences. Despite this, the provisions of the current paradigms are based on the general principles of the design of their routes with identical principles in the properties of their curving elements. The paradox of this situation is largely reasoned by the significant simplification of models and criteria for the assessment of the safety and efficiency of these systems on these sections of the routes. Mostly they are focused on the established mode of curvilinear movement of rolling stock and cars on a section with a constant curvature $k$, speed $v$ and cross-slope of track $i$ or the surface of the roadway.

At the same time, the main indicators of the safety and convenience of this process on circular sections of curvature of railways and highways are the non-reduced side $a$ and the dimensionless side force coefficient $\mu = a/g$ (where $g = 9.81 \text{ m/s}^2$) directly dependent on it. The value of the non-reduced side acceleration $a$ is calculated from the conditions of its action at the level of the top of the rail heads or the roadway surface according to the formula common for both modes of transport.
\[ a = k \cdot v^2 + g \cdot i \]  

(1)

This equation allows establishing such a combination of design speed \( v \), curvature \( k \) and side slope \( i \), at which the balance \( a \leq a_{\text{max}} \) or \( \mu \leq \mu_{\text{max}} \) required for reasons of safety and convenience of movement will be provided. In order to change the mode of movement from a permanently rectilinear to a permanently curvilinear one, a transition section is arranged with a linearly varying curvature of its axis and a cross slope. Automatically it is assumed that the side acceleration acting on such transitional sections can not exceed its maximum on the circular part of the rounding. Therefore, the only controllable factor that can affect the efficiency and quality of the systems functioning on this fragment of the design of their rounding is only its length \( L \). The provisions of the current paradigms for tracing railways and highways provide the limitation of only the minimum lengths of transition sections in terms of the maximum rate of change in the non-reduced side acceleration \( \frac{da}{dt} \) and along the length of the section of the curve or retraction of the elevation of the outer rail of the line over the inner one.

In some cases, they are unreasonably limited by the value of the radius of curvature \( R \) or the value of the shift of the circular curve \( p \).

The simplicity of this design of the transition section and the corresponding “simplicity” of standardizing its parameters significantly limits the possibilities of the improvement of the quality of the systems functioning on the curves and even adjacent sections of the routes. As the speed of movement increases and the requirements for its quality increase, the negative influence of both the structures of the transition sections and simplified criteria for the assessment of their effectiveness is increasingly manifested. The limited list of their universal indicators does not take into account important features of the structure and functioning of each of these systems. For the elimination of these shortcomings, various authors proposed many alternative forms of transition sections with the laws of curvature and cross slope that are different from linear ones [1].

However, the empirical nature of the search for these solutions by the method of trial and error did not provide the formalization of the aim pursued by them and the justification of the criteria for its achievement. The positive and negative properties of these solutions were investigated using multifactorial deterministic kinematic models [2]. The obtained results allowed substantiating the goals of the improvement of the forms of transitional sections of curves of rail and highways, mathematical means and methods for their achievement.

2. Specific features of the device and functioning of the “railway line-rolling stock” system

One of the important features of railways is the construction of their line from rails with a certain curving stiffness. Therefore, polylines representing projections of their 3D models in the horizontal and vertical projection planes must provide at least the G3\textsuperscript{th} order of smoothness at the junction points of the elements that make up the polyline. This requirement is observed when the values of both the mathematical functions of adjacent elements and their first, second and third derivatives change continuously at these points. The higher the order of geometric smoothness of the polyline, the more accurately it describes the model of the corresponding projection of the rail and the less it is noticeable in movement along it that the project of its model was composed of dissimilar geometric elements. The accuracy of the mathematical model of the line increases the reliability of the forecast of its functional indicators and contributes to the increase in the accuracy of its physical embodiment on the ground. The compliance with the minimum G3\textsuperscript{th} order of smoothness is especially important for the so-called continuous rail. In this case, the traditional project of the line model on the rounding section with clothoid transition sections will consist of polylines with G0\textsuperscript{th} and G2\textsuperscript{nd} orders of smoothness that does not meet this requirement. As a result of the mechanical properties inherent in the rails, they will inevitably curve out of design. The negative consequences of this will lead to shocks and swaying of the rolling stock, which will increase the rate of breakdown of the geometry of the line, increase the wear of rails, wheels and its other elements.

The second important feature is reasoned by the fact that as a rule the elevation \( H \) of the center of mass of the rolling stock or any of its other functionally significant design points above the level of the top of the rail heads exceeds the distance \( S \) between their axes. With the inconsistent effect of the
principles of the retraction of the curvature of the line and its cross slope, the kinematics of the design point deteriorates significantly. This has the most negative effect on the curvature of the horizontal projection of the trajectory of its movement, which determines the value of the centrifugal acceleration. As a result, the non-reduced side acceleration $a$ acting at the level of the design point, will significantly deviate from the value predicted by the formula (1). The additional action of the side force on the line rails due to this acceleration will also increase in proportion to the $H/S$ ratio.

The third important feature is reasoned by the influence on the functional indicators of the system that the curvature of the profile of the track axis $k_V$, as well as the vertical curvature of $k_L$ and $k_R$ of each of the strands of its rail have. Their values depend on the order of vertical smoothness of a symmetric or asymmetric non-linear elevation retraction, as well as on its maximum value $D$ and the length $L$ of the transition section, which is used in the project.

3. Studied parameters of the “railway line-rolling stock” system

The influence of the above mentioned factors on the forces of interaction between the rails and the wheels of the rolling stock are determined by the analytical dependences of the laws of physics, which are the basis of the multifactorial deterministic kinematic model shown in Figure 1 a), b) and c) fragments of the design scheme. The coordinate method for the description of the kinematics of the design point of the rolling stock $M$ is of great importance in ensuring its adequacy. It is determined taking into account its elevation $H$ above the level of the top of the rail heads and the geometric properties of one form or another of the transition section (Fig. 1 d)).

![Figure 1. Gravitational (a) and centrifugal (b) components of accelerations and forces (c) of the interactions of the elements of the system and the coordinate method for the description of the kinematics of the design point of a vehicle M (d)](image-url)
In accordance with this model, the coordinates $x_M(l)$ and $y_M(l)$ of the horizontal projection of the trajectory of the design point can be described by the following parametric equations:

\[ x_M(l) = x_0(l) - H \cdot i(l) \cdot \sin \beta(l) \]  
\[ y_M(l) = y_0(l) + H \cdot i(l) \cdot \cos \beta(l) \]

where $x_0(l)$ and $y_0(l)$ – current coordinates of the line axis, calculated in accordance with the regularity of the angle of the tangent $\beta(l)$ to the line axis; $\beta(l)$ – current angle of the tangent to the line axis, calculated in accordance with its principle and the value of the parameter $U$; $i(l)$ – current cross slope calculated in accordance with its principle and the value of the parameter $Z$; $H$ – elevation of the design point $M$ above the level of the top of the rail heads.

Taking into account this model, the principle of curvature $k_H(l)$ of the horizontal projection of the trajectory of any point $M$ located at the design level of the crew $H \geq 0$ should be described by the differential equation

\[ \frac{d^2 x_M}{dl^2} - \frac{d^2 y_M}{dl^2} = \left( \frac{dx_M}{dl} \right)^2 + \left( \frac{dy_M}{dl} \right)^2 \]  

These and other dependencies of the model allow predicting the performance of the system with one form or another of the transitional rounding sections, taking into account the factors predetermined and controlled at the design stage. Their addition with the parameter $U$ of the principle of the tangent angle $\beta(l)$ to the line axis of its transition sections and the parameter $Z$ of the principle of the properties of the retraction of their cross slope $i(l)$ expands the possibilities of the provision of the required quality level of the project of its model. The analysis of many different combinations of the principles of the properties of the transition sections showed that the quality of the system functioning depends on how harmoniously the values of the parameters $L$, $U$, and $Z$ are combined with the values of the design speed $V$, the curvature radius $R$, the design elevation of the outer rail above the inner one $D$ and also the distance between the line axes $S$ and the elevation $H$ of the design point above the level of the top of rail heads.

In terms of the language of nature communication, the desired level of this quality means the comfort of passenger movement and the positive dynamics of the force interaction of the system elements. However, the criteria of the prior technical level are not suitable for an unambiguous numerical assessment of its achievement. It is mainly because of their extremely mediated and indirect connection with those indicators that can characterize the desired quality of the system functioning. Thus, for example, the so-called elevating speed of wheel lifting along the elevation of the outer rail with such indicators of the quality of the system functioning as with the non-reduced side acceleration $a$ at the level $H$ or with the rate of its change $\psi = da/dt$ at the same level is not always obvious and even contradictory.

Therefore, as a formalized target of the harmonization of the properties of the system, an integral indicator was proposed, by the numerical value of which it is possible to determine the degree of achievement of the required quality of its functioning [1]. The value of this indicator characterizes the variance of the oscillation amplitudes of the principle of the derivative of the change ratio of the non-reduced side acceleration acting at the calculated level $H$. In this case, the highest level of harmonization quality of the system properties is achieved at the minimum variance of the functional $d/dt = d^2 a/dt^2$, estimated at the central segment of the transitional section between points $\delta_h$ and $\delta_e$ (Fig. 2).
Figure 2. Sample of the shape of the diagram of the functional $\frac{du/dt}{N}$ with a minimum of the variance of the oscillations amplitudes of its values in the central section $\delta_0 - \delta_e$, which is the integral target of the harmonization of the properties of the system (red).

If we meet the criterion for the achievement of this integral target, each of the 4 generally recognized local quality targets of the transitional section of the railway is achieved (Fig. 3).

**Target № 1**  
Strictly monotonic curvature $k_0$ of the line axis at the level of the top of the rail heads

**Target № 2**  
Quasilinear plot of NPU $a(l)$ at the design level $H$

**Target № 3**  
Continuity of change in NPU $\Psi(l) = da/dt$  
With minimal speed $\Psi_{max} \rightarrow \text{MIN}$

**Target № 4**  
Strictly monotonic and smooth change in difference of reaction force of rails

$\Delta F(l) = FL - FR$

**Figure 3.** Local targets of the geometric and functional quality of the system, achieved by harmonizing the parameters L, U and Z at a minimum $du/dt = \frac{a^2}{dt^2}$

4. **Comparison of the properties of new forms of transitional sections of curves of railways with properties of forms of the prior technical level**

The analysis of the results of harmonization of more than a thousand forms of transitional sections confirmed the high quality of their properties, which were achieved with a minimum dispersion of the oscillation amplitudes of $du/dt$ functional, which varied in the range from $10^{-8}$ to $10^{-5}$ m/s$^4$. Each of the forms of these transition sections corresponded to a unique variant of the combination of its predetermined properties, varying in the range of speeds $V$ from 100 km/h to 400 km/h with such combinations of radii $R$ and elevations $D$, which for each of the standard line widths $S = 1520$ mm and $S = 1435$ mm provided the maximum values of the non-reduced side acceleration $a$ in the range from 0.1 m/s$^2$ to 1.0 m/s$^2$ at the design levels $1800 \leq H \leq 2500$ mm.

The data selection from the results of this research at a design speed of $V = 400$ km/h allows comparing the lengths $L$ of the harmonized transition sections with the lengths of a number of other forms. In accordance with the standard recommended for their use [4], their lengths are set equal to the
length of the equivalent clothoid, calculated taking into account the type of used coefficient. Taking this fact into account, for the purposes of this comparison, the lengths of only basic clothoids were calculated, which ensured the speed of the wheel elevation along the elevation of the outer rail \( v_d = 28 \) mm/s. According to the results of these calculations, the graphs of the dependence of the lengths of new and clothoid forms on the calculated values of the non-reduced side acceleration \( a \) and other parameters of curve were created (Fig. 4).

![Figure 4. Graphs of the dependence of the lengths \( L \) of new and clothoid forms of the transition sections on the calculated value of the non-reduced side acceleration \( a \) and other parameters of the options for curve of the studied railways at \( V = 400 \) km/h, \( H = 2200 \) mm and \( S = 1520 \) mm](image)

In all the research cases, the functional quality of the harmonized forms exceeded the quality of the transition sections of the forms of the prior technical level and the values of their lengths \( L \) were strictly ordered according to the values of the parameter \( a \) in the form of nonlinear graphs of implicit functions \( L = f(a) \). The results of the calculation of the unique coordinates \( a \) and \( L \) of these graphs varied depending on the options for the values of other parameters \( H \) and \( S \), as well as on the values of the parameters \( U \) and \( Z \) found at the time of the determination of each of these lengths. In contrast to this, the values of the basic lengths of the linear graphs of clothoid forms did not change, since the provisions of the current paradigm did not provide these parameters of the “railway line-rolling stock” system.

The graphs from Figure 4 show that with the nonlinear nature of the length functions \( L \) of the harmonized forms of transitional sections, their values in the zone \( 0.4 \leq a \leq 0.7 \) m/s\(^2\) are 1.2 ÷ 1.5 times less than the basic lengths of the transitional sections of clothoid forms. As the smoothness of other forms of transition sections increases and the values of the corresponding coefficients increase, the difference between the lengths will increase even more. This indicates obvious problems of empirical methods of the assignment of the lengths of transition sections of traditional forms in proportion to the lengths of the basic clothoids. After all, the methods of calculating both the basic lengths of the clothoid forms of the transitional sections, or the coefficients of their proportions with the lengths of other half-sine forms, do not have any significant theoretical justification.

It is necessary to note that the shift of the circular curve \( p \) in the harmonized transition areas is significantly (several times!) less than in clothoid forms similar in other parameters. This creates the prerequisites for their harmonization within the existing curves with a possible increase in radii and elevations with minor shifts of their track from the original design position. Due to this, the level of operational quality of the existing transition areas can be significantly increased during the planned adjusting and tamping works.

5. Specific features of the device and functioning of the “driver-car-road” system
The safety and comfort of curved road traffic largely depends on the maintenance of a balance between the side grip coefficient \( \varphi \) and the side force coefficient \( \mu \) provided by their pavement. Its value depends on the part of the side acceleration \( a \) that is not reduced by the curve acting on a car. It is generally accepted that this balance is observed under the condition \( \varphi \geq \mu \). However, in real driving conditions, the values of the parameters \( \varphi \) and \( \mu \) are subject to significant deviation. It is reasoned by the combined effects of many meteorological, technical, situational and behavioral factors. A radical
reduction in the associated risks ensures compliance with the speed of driving curved sections of the line, which a driver must select in accordance with the current traffic conditions.

The justification of this choice and safety of its implementation largely depend on the length and regularities of the properties of the transition section of curve preceding its maximum curvature.

The most predictable and critical situation is in which a driver is forced or continues to reduce the speed of his vehicle in the transitional section of the curve. Therefore, its length $L$ must be enough to promptly and safely reduce the initial speed $v_b$ to the final speed $v_e$ with a comfortable deceleration $d$.

In this case, the current curvature $k$ of the transition section must be in line with the variable speed $v$ according to the balance condition $g\varphi \geq w$ between the acceleration depending on the values of the friction coefficient $\varphi$ and the vector sum of the side acceleration $e$ $kv2$-ge partially reduced by the curve slope $e$ with the algebraic sum of longitudinal accelerations from the slope $\pm g_i$ and vehicle deceleration $d$ (Fig. 5).

The required length of the transitional section of the curve $L$ is determined from the condition of a uniform change in speed $v = v_b + d \cdot t$ over time $t$, which is necessary for the driver's timely response to the beginning of the curvilinear trajectory of movement

$$L \geq 0.5 \cdot (v_e^2 - v_b^2) \cdot d^{-1} + v_b t$$  \hspace{1cm} (5)

**Figure 5.** Model for the assessment of the safety of curvilinear movement of a car with deceleration $d$ (a), principles of curvature $k$ (b) and the corresponding coordinates $x$ and $y$ of its trajectory (c)

The principle of the curvature of its axis $k$ is determined from the condition of uniform change in time $t$ in the general vector of vehicle accelerations $w = w_b + j \cdot t$. Taking this into account, its principle on a horizontal section of the road should correspond to following equition:

$$\left(\left( k_b v_b^2 \right)^2 + \partial^2 \right)^{\frac{1}{2}} = \left(\left( kv^2 \right)^2 + \partial^2 \right)^{\frac{1}{2}} + j t$$  \hspace{1cm} (6)

where

- $\partial$ – deceleration determining the law of curvature $k$, determined from the conditions of safety and comfort of movement with design deceleration $d$;
- $j$ – speed of uniform increase of the total acceleration vector $w$, determined without taking into account the superelevation $e$ and the longitudinal slope $i$ at the calculated value $\varphi$;
- $k_b$ – curvature at the beginning of the transition section of curve

The indicator of movement comfort, traditionally estimated by the rate of increase of the centrifugal acceleration $C$, with variable curvature $k$ and variable speed $v$ at any point on the axis of the transition section should be calculated by the following formula:

$$C = \frac{dk}{dl} v^3 + 2 \cdot v \cdot k \cdot d$$  \hspace{1cm} (7)

At a variable speed $v_e$, the values of this indicator $C$ reach their functionally significant maximum only at the end of the transition section with curvature $k_e$. Therefore, the value of deceleration $\partial$, which determines the principle of the curvature of the axis of the transition section $k$, should be calculated taking into account the required convenience of movement $C_{max}$ according to the following equation:

$$j \cdot \left(1 + \partial^2 \cdot \left(k_e \cdot v_e^2 \right)^2 \right)^{\frac{1}{2}} + 2 \cdot v_e \cdot k_e \left( d - \partial \right) - C_{max} = 0$$  \hspace{1cm} (8)
As a rule, at $C_{\text{max}} \approx 50.4/V_e$ equation (8) is achieved at $\partial \leq d$. This ensures the observance of the balance $w \leq g \phi$ with some excess of the standard safety level of curvilinear movement with a variable speed. In Figure 6a the areas of such a excess near the VGV_Curve with a length of $L = 300$ m is blue. It is provided at $V_b = 120$ km/h, $V_e = 60$ km/h, $R = 150$ m, $e = 60$ %, $d = -1.4$ m/s$^2$, $C = 0.4$ m/s$^3$ and $\partial = -2.55$ m/s$^2$. The area of shortage is red, which increases the risk of road accidents on a clothoid transition section of the same length and radius.

Figure 6. Surplus and shortage of properties that ensure the safety of curvilinear traffic with variable speed along VGV_Kurve and clothoid (a), as well as a fragment of the project of exits to a traffic intersection with transition curves of variable speed

6. Conclusion
The geometric and functional properties of the transitional sections of the curves of railways and highways have significant differences due to the properties of the systems that include them. The presence and essence of these differences is illustrated by the characteristics of the alternative VGV_Kurve and VGV-41H-39 forms of the transition section of curve $R = 300$ m, which, in
traditional practice, could be implemented in the form of Clothoid with a linear elevation or superelevation (Table 1). The significance of these differences is also illustrated by the graphs of the curvature of the axes of these forms of transition sections (Fig. 7).

| Form          | Speed (km/h) | Comfort $\nu$ (m/s$^3$) | Length $L$ (m) | Shift $\rho$ (m) | Angle $\beta$ (g, m, s) | Additional data                                      |
|---------------|--------------|--------------------------|----------------|-----------------|------------------------|-----------------------------------------------------|
| Clothoid      | 90           | 0.58                     | 90             | 1.12            | 8°35'40''              |                                                     |
| VGV_Kurve     | from 140 to 90 | 0.6                      | 300            | 11.46           | 15°42'15''             | $d=\text{-}1.3 \text{ m/s}^2$, $\delta=\text{-}4.22 \text{ m/s}^2$ |
| VGV-41H-39    | 70           | 0.073                    | 100            | 0.44            | 9°32'57''              |                                                     |

The justified forecast of a significant increase in the operational quality of the functioning of these systems creates all the conditions for a comprehensive verification of the proposed solutions in practice and further improvement of the theory, current norms and rules for the design of transport infrastructure facilities with new, corresponding forms of transitional sections of their curves.

References
[1] Kufver B 1997 VTI rapport 420A *Mathematical description of railway alignments and some preliminary comparative studies* (Swedish National Road and Transport Research Institute) Available at: http://www.diva-portal.org/smash/record.jsf?pid=diva2%3A675179&dswid=4876
[2] Velichko G 2020 Quality analysis and evaluation technique of railway track + vehicle system performance at railway transition sections with various shape curves *Transport Means 2020: Proceedings of the 24th International Scientific Conference* Part II: 573-578 Available at: https://transportmeans.ktu.edu/wp-content/uploads/sites/307/2018/02/Transport-means-A4-II-dalis.pdf
[3] Velichko G 2020 Shape Harmonization of the Railway Track Transition Section & the Kinematics of Vehicle Body Design Point *Transport Means 2020: Proceedings of the 24th International Scientific Conference* Part II: 910-915 Available at: https://transportmeans.ktu.edu/wp-content/uploads/sites/307/2018/02/Transport-means-A4-II-dalis.pdf
[4] EN 13803-1:2010: Railway applications - Track-Track alignment design parameters - Track gauges 1435 mm and wider - Part 1: Plain line [Required by Directive 2008/57/EC]