On topological groups with an approximate fixed point property

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Abstract. A topological group $G$ has the Approximate Fixed Point (AFP) property on a bounded convex subset $C$ of a locally convex space if every continuous affine action of $G$ on $C$ admits a net $(x_i)$, $x_i \in C$, such that $x_i - gx_i \to 0$ for all $g \in G$. In this work, we study the relationship between this property and amenability.

Keywords: Amenable groups, AFP property, Følner net, Polish groups.

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1. INTRODUCTION

One of the most useful known characterizations of amenability is stated in terms of a fixed point property. A classical theorem of (Day, 1961) says that a topological group $G$ is amenable if and only if every continuous affine action of $G$ on a compact convex subset $C$ of a locally convex space has a fixed point, that is, a point $x \in C$ with $g \cdot x = x$ for all $g \in G$. This result generalizes earlier theorems of (Kakutani, 1938) and (Markov, 1936) obtained for abelian acting groups.

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An active branch of current research is devoted to the study of existence of the approximate fixed points for single maps, in a case where a convex set $C$ is no longer assumed compact. Apparently this was first exploited in (Scarf, 1967), where a constructive method for computing fixed points of continuous mappings of a simplex into itself was described. More recent results include, for instance, (Hazewinkel and van de Vel, 1978), Hadžić (Hadžić, 1996), (Idzik, 1988), (Park, 1972), (Barroso, 2009; Barroso and Lin, 2010; Barroso et al., 2012, 2013).

Let us mention just one representative result.

**Theorem 1.1** (Proposition 2.5(i) in (Barroso et al., 2012)). Let $X$ be a topological vector space, $C \subset X$ a nonempty bounded convex set, and let $f : C \rightarrow C$ be an affine selfmap. Then the $f$ has an approximate fixed point sequence, that is, a sequence $(u_k)$ in $C$ such that $u_k - f(u_k) \rightarrow 0$.

In this note, we study the existence of common approximate fixed points for a set of transformations forming a topological group. Not surprisingly, the approximate fixed point property for an acting group $G$ is also closely related to amenability of $G$, however the relationship appears to be more complex.

We show that a discrete group $G$ is amenable if and only if every continuous affine action of $G$ on a bounded convex subset $C$ of a locally convex space (LCS) $X$ admits approximate fixed points. For a locally compact group, a similar result holds if we consider actions on bounded convex sets $C$ which are complete in the additive uniformity, while in general we can only prove that $G$ admits weakly approximate fixed points. This criterion of amenability is no longer true in the more general case of a Polish group, even if amenability of Polish groups can be expressed in terms of the approximate fixed point property on bounded convex subsets of the Hilbert space.

We view our investigation as only the first step, and so we close the article with a discussion of open problems for further research.

2. **Amenability**

Here is a brief reminder of some facts about amenable topological groups. For a more detailed treatment, see e.g. (Paterson, 1988). All the topologies considered here are assumed to be Hausdorff.

Let $G$ be a topological group. The right uniform structure on $G$ has as basic entourages of the diagonal the sets of the form $U_V = \{(g,h) \in G \times G \mid hg^{-1} \in V\}$, where $V$ is a neighbourhood of the identity $e$ in $G$. This structure is invariant under right translations. Accordingly, a function $f : G \rightarrow \mathbb{R}$ is right uniformly continuous if for all $\varepsilon > 0$, there exists a neighbourhood $V$ of $e$ in $G$ such that
$xy^{-1} \in V$ implies $|f(x) - f(y)| < \varepsilon$ for every $x, y \in G$. Let $RUCB(G)$ denote the space of all right uniformly continuous functions equipped with the uniform norm. The group $G$ acts on $RUCB(G)$ on the left continuously by isometries: for all $g \in G$ and $f \in RUCB(G)$, $g.f = g^{-1}f$ where $g.f(x) = f(g^{-1}x)$ for all $x \in G$.

**Definition 2.1.** A topological group $G$ is **amenable** if it admits an invariant mean on $RUCB(G)$, that is, a positive linear functional $m$ with $m(1) = 1$, invariant under the left translations.

Examples of such groups include finite groups, solvable topological groups (including nilpotent, in particular abelian topological groups) and compact groups. Here are some more examples:

1. The unitary group $U(\ell^2)$, equipped with the strong operator topology (de la Harpe, 1973).
2. The infinite symmetric group $S_{\infty}$ with its unique Polish topology.
3. The group $\mathcal{J}(k)$ of all formal power series in a variable $x$ that have the form $f(x) = x + \alpha_1x^2 + \alpha_2x^3 + \ldots$, $\alpha_n \in k$, where $k$ is a commutative unital ring (Babenko and Bogatyi, 2011).

Let us also mention some examples of non-amenable groups:

1. The free discrete group $F_2$ of two generators. More generally, every locally compact group containing $F_2$ as a closed subgroup.
2. The unitary group $U(\ell^2)$, with the uniform operator topology (de la Harpe, 1979).
3. The group $\text{Aut}(X, \mu)$ of all measure-preserving automorphisms of a standard Borel measure space $(X, \mu)$, with the uniform topology, i.e. the topology determined by the metric $d(\tau, \sigma) = \mu\{x \in X : \tau(x) \neq \sigma(x)\}$ (Giordano and Pestov, 2002).

The following is one of the main criteria of amenability in the locally compact case.

**Theorem 2.2** (Følner’s condition). Let $G$ be a locally compact group and denote $\lambda$ the left invariant Haar measure. Then $G$ is amenable if and only if $G$ satisfies the Følner condition: for every compact set $F \subseteq G$ and $\varepsilon > 0$, there is a Borel set $U \subseteq G$ of positive finite Haar measure $\lambda(U)$ such that $\frac{\lambda(xU \triangle U)}{\lambda(U)} < \varepsilon$ for each $x \in F$.

Recall that a **Polish group** is a topological group whose topology is Polish, i.e., separable and completely metrizable.

**Proposition 2.3** (See e.g. [Al-Gadid et al., 2011], Proposition 3.7). A Polish group $G$ is amenable if and only if every continuous affine
action of $G$ on a convex, compact and metrizable subset $K$ of a LCS $X$ admits a fixed point.

For the most interesting recent survey about the history of amenable groups, see (Grigorchuk and de la Harpe, 2014).

3. Groups with Approximate Fixed Point Property

**Definition 3.1.** Let $C$ be a convex bounded subset of a topological vector space $X$. Say that a topological group $G$ has the approximate fixed point (AFP) property on $C$ if every continuous affine action of $G$ on $C$ admits an approximate fixed point net, that is, a net $(x_i) \subseteq C$ such that for every $g \in G$, $x_i - gx_i \longrightarrow 0$.

We will analyse the AFP property of various classes of amenable topological groups.

3.1. Case of discrete groups.

**Theorem 3.2.** The following properties are equivalent for a discrete group $G$:

1. $G$ is amenable.
2. $G$ has the AFP property on every convex bounded subset of a locally convex space.

**Proof.** (1) $\Rightarrow$ (2). Let $G$ a discrete amenable group acting by continuous affine maps on a bounded convex subset $C$ of a locally convex space $X$. Choose a Følner’s $(F_i)_{i \in I}$ net, that is, a net of finite subsets of $G$ such that

$$\frac{|gF_i \triangle F_i|}{|F_i|} \longrightarrow 0 \quad \forall g \in G.$$ 

Now, let $\gamma \in G$, fix $x \in C$ and define $x_i = \frac{1}{|F_i|} \sum_{g \in F_i} gx$. Since $C$ is convex, $x_i \in C$ for all $i \in I$. Notice that $|F_i \setminus \gamma F_i| = |\gamma F_i \setminus F_i| =
\[ \frac{1}{2} |\gamma \Phi_i \triangle \Phi_i| \text{ for all } i \in I. \] Therefore we have
\[
x_i - \gamma x_i = \frac{1}{|\Phi_i|} \left[ \sum_{g \in \Phi_i} gx - \sum_{g \in \gamma \Phi_i} \gamma gx \right]
\]
\[
= \frac{1}{|\Phi_i|} \left[ \sum_{g \in \Phi_i} gx - \sum_{h \in \gamma \Phi_i} hx \right]
\]
\[
= \frac{1}{|\Phi_i|} \left[ \sum_{g \in (\Phi_i \setminus \gamma \Phi_i)} gx - \sum_{g \in (\gamma \Phi_i \setminus \Phi_i)} gx \right]
\]
\[
= \frac{|\gamma \Phi_i \triangle \Phi_i|}{2|\Phi_i|} \left[ \frac{1}{|\Phi_i \setminus \gamma \Phi_i|} \sum_{g \in (\Phi_i \setminus \gamma \Phi_i)} gx - \frac{1}{|\gamma \Phi_i \setminus \Phi_i|} \sum_{g \in (\gamma \Phi_i \setminus \Phi_i)} gx \right]
\]
Thus \( x_i - \gamma x_i \in \frac{|\gamma \Phi_i \triangle \Phi_i|}{2|\Phi_i|} (C - C) \) and hence \( x_i - \gamma x_i \to 0 \) since \( C \) is bounded.

(2) \( \Rightarrow \) (1). Let \( G \) be a discrete group acting continuously and by affine transformations on a nonempty compact and convex set \( K \) in a topological vector space \( X \). By hypothesis, there is a net \( (x_i) \subseteq K \) such that \( \forall g \in G, \ x_i - gx_i \to 0 \). By compactness of \( K \), this net has accumulation points in \( K \). Since \( \forall g \in G, \ x_i - gx_i \to 0 \), this insures invariance of accumulation points and shows the existence of a fixed point in \( K \). Therefore \( G \) is an amenable group by Day’s fixed point theorem mentioned in the Introduction.

3.2. Case of locally compact groups. Recall from [Bourbaki, 1963] the following notion of integration of functions with range in a locally convex space.

Let \( F \) be a locally convex vector space on \( \mathbb{R} \) or \( \mathbb{C} \). \( F' \) denotes the dual space of \( F \), \( F'' \) the double dual of \( F \) and \( F^{*} \) the algebraic dual of \( F' \). We identify as usual \( F \) (seen as a vector space without topology) with a subspace of \( F'' \) by associating to any \( z \in F \) the linear form \( F' \ni z' \mapsto \langle z, z' \rangle \in \mathbb{R} \).

Let \( T \) be a locally compact space and let \( \mu \) a positive measure on \( T \). A map \( f : T \to F \) is essentially \( \mu \)-integrable if for every element \( z' \in F' \), \( \langle z', f \rangle \) is essentially \( \mu \)-integrable. If \( f : T \to F \) is essentially \( \mu \)-integrable, then \( z' \mapsto \int_T \langle z', f \rangle \, d\mu \) is a linear map on \( F' \), i.e. an
element of $F^*$. The integral of $f$ is the element of $F^*$ denoted $\int_T f \, d\mu$ and defined by the condition: $\langle z', \int_T f \, d\mu \rangle = \int_T \langle z', f \rangle \, d\mu$ for every $z' \in F'$.

Note that, in general we don’t have $\int_T f \, d\mu \in F$. But we have the following.

**Proposition 3.3** ([Bourbaki, 1963], chap. 3, Proposition 7). Let $T$ be a locally compact space, $E$ a LCS and $f : T \to E$ a function with compact support. If $f(T)$ is contained in a complete (with regard to the additive uniformity) convex subset of $E$, then $\int_T f \, d\mu \in E$.

**Theorem 3.4.** The following are equivalent for a locally compact group $G$:

1. $G$ is amenable,
2. $G$ has the AFP property on every complete, convex, and bounded subset of a locally convex space.

**Proof.** (1) $\implies$ (2). Let $G$ be a locally compact amenable group acting continuously by affine maps on a complete, bounded, convex subset $C$ of a locally convex space $X$. Again, select a Følner net $(F_i)_{i \in I}$ of compact subsets of $G$ such that $\frac{\lambda(gF_i \triangle F_i)}{\lambda(F_i)} \to 0$ $\forall g \in G$. Fix $x \in C$ and let $\eta_x : G \ni g \mapsto gx \in C$ be the corresponding orbit map. Define $x_i = \frac{1}{\lambda(F_i)} \int_{F_i} \eta_x(g) \, d\lambda(g)$. By the above, this is an element of $C$; the barycenter of the push-forward measure $(\eta_x)_*(\lambda|_{F_i})$ on $X$. We have, just like in the discrete case:

$$x_i - \gamma x_i = \frac{1}{\lambda(F_i)} \left[ \int_{F_i} \eta_x(g) \, d\lambda(g) - \gamma \int_{F_i} \eta_x(g) \, d\lambda(g) \right]$$

$$= \frac{1}{\lambda(F_i)} \left[ \int_{F_i} \eta_x(g) \, d\lambda(g) - \int_{F_i} \eta_x(\gamma g) \, d\lambda(g) \right]$$

$$= \frac{1}{\lambda(F_i)} \left[ \int_{F_i} [\eta_x(g) - \eta_x(\gamma g)] \, d\lambda(g) \right]$$

$$= \frac{1}{\lambda(F_i)} \left[ \int_{\gamma F_i \triangle F_i} \eta_x(v) \, d\lambda(v) \right].$$
Now, let $q$ be any continuous seminorm on $C$. We have:

$$q(x_i - \gamma x_i) \leq \frac{\lambda(\gamma F_i \Delta F_i)}{\lambda(F_i)} K$$

where $K = \sup_{v \in G} q \circ \eta_x(v) < \infty$ since $C$ is bounded. Thus $x_i - \gamma x_i \to 0$.

(2) $\Rightarrow$ (1). Same argument as in the case of discrete groups. $\square$

The assumption of completeness of $C$ does not look natural in the context of approximate fixed points, but we do not know if it can be removed. It depends on the answer to the following.

**Question 3.5.** Let $f$ be an affine homeomorphism of a bounded convex subset $C$ of a locally convex space $X$. Can $f$ be extended to a continuous map (hence, a homeomorphism) of the closure of $C$ in $X$?

Nevertheless, we can prove the following.

**Theorem 3.6.** Every amenable locally compact group $G$ has a weak approximate fixed point property on each bounded convex subset $C$ of a locally convex space $X$.

**Proof.** In the notation of the proof of Theorem 3.4, let $\mu_i = (\eta_x)_*(\lambda|_{F_i})$ denote the push-forward of the measure $\lambda_i = \lambda \upharpoonright F_i$ along the orbit map $\eta_x : G \ni g \mapsto gx \in C$. Let $x_i$ be the barycenter of $\mu_i$. This time, $x_i$ need not belong to $C$ itself, but will belong to the completion $\hat{C}$ of $C$ (the closure of $C$ in the locally convex vector space completion $\hat{X}$).

For every $g \in G$, denote $z^g_i$ the barycenter of the measure $g_\ast \mu_i = (\eta_x)_*(g\lambda_i)$. Just like in the proof of Theorem 3.4, for every $g$ we have $x_i - z^g_i \to 0$.

Now select a net $\nu_j$ of measures with finite support on $G$, converging to $\lambda_i$ in the vague topology (Bourbaki, 1963). Denote $y_j$ the barycenter of the push-forward measure $(\eta_x)_*(\nu_j)$. Then $y_j \Rightarrow x_i$ in the vague topology on the space of finite measures on the compact space $F_i, x$. Clearly, $y_j \in C$, and so $g \cdot y_j$ is well-defined and $g \cdot y_j \Rightarrow z^g_i$ for every $g \in \Phi$. It follows that $gy_j - y_j$ weakly converges to 0 for every $g \in G$. $\square$

**Remark 3.7.** Clearly the weak AFP property implies amenability of $G$ as well.

3.3. **Case of Polish groups.** The above criteria do not generalize beyond the locally compact case in the ways one might expect: not every amenable non-locally compact Polish group has the AFP property, even on a bounded convex subset of a Banach space.
Proposition 3.8. The infinite symmetric group $S_\infty$ equipped with its natural Polish topology does not have the AFP property on closed convex bounded subsets of $\ell^1$.

If we think of $S_\infty$ as the group of all self-bijections of the natural numbers $\mathbb{N}$, then the natural (and only) Polish topology on $S_\infty$ is induced from the embedding of $S_\infty$ into the Tychonoff power $\mathbb{N}^\mathbb{N}$, where $\mathbb{N}$ carried the discrete topology.

We will use the following well-known criterion of amenability for locally compact groups.

Theorem 3.9 (Reiter’s condition). Let $p$ be any real number with $1 \leq p < \infty$. A locally compact group $G$ is amenable if and only if for any compact set $C \subseteq G$ and $\varepsilon > 0$, there exists $f \in L^p(G)$, $f \geq 0$, $\|f\|_p = 1$, such that: $\|g.f - f\| < \varepsilon$ for all $g \in C$.

Proof of Proposition 3.8. Denote $\text{prob}(\mathbb{N})$ the set of all Borel probability measures on $\mathbb{N}$, in other words, the set of positive functions $b : \mathbb{N} \rightarrow [0, 1]$ such that $\sum_{n \in \mathbb{N}} b(n) = 1$. This is the intersection of the unit sphere of $\ell^1$ with the cone of positive elements, a closed convex bounded subset of $\ell^1$. The Polish group $S_\infty$ acts canonically on $\ell^1$ by permuting the coordinates:

$$S_\infty \times \ell^1(\mathbb{N}) \ni (\sigma, (x_n)_n) \mapsto (\sigma. (x_n)_n) = (x_{\sigma^{-1}(n)})_n \in \ell^1(\mathbb{N}).$$

Clearly, $\text{prob}(\mathbb{N})$ is invariant and the restricted action is affine and continuous. We will show that the action of $S_\infty$ on $\text{prob}(\mathbb{N})$ admits no approximate fixed point sequence.

Assume the contrary. Make the free group $\mathbb{F}_2$ act on itself by left multiplication and identify $\mathbb{F}_2$ with $\mathbb{N}$. In this way we embed $\mathbb{F}_2$ into $S_\infty$ as a closed discrete subgroup. This means that the action of $\mathbb{F}_2$ by left regular representation on $\text{prob}(\mathbb{N}) \cong \text{prob}(\mathbb{F}_2)$ also has almost fixed points, and $\ell^1(\mathbb{F}_2)$, with regard to the left regular representation of $\mathbb{F}_2$, has almost invariant vectors. But this is the Reiter’s condition ($p = 1$) for $\mathbb{F}_2$, a contradiction with non-amenability of this group. \qed

However, it is still possible to characterize amenability of Polish groups in terms of the AFP property.

Theorem 3.10. The following are equivalent for a Polish group $G$:

(1) $G$ is amenable,

(2) $G$ has the AFP property on every bounded, closed and convex subset of the Hilbert space.
Proof. (1) ⇒ (2). It is enough to show that a norm-continuous affine action of $G$ on a bounded closed convex subset $C$ of $\ell^2$ is continuous with regard to the weak topology, because then there will be a fixed point in $C$ by Day’s theorem.

Let $x \in C$ and $g \in G$ be any, and let $V$ be a weak neighborhood of $g.x$ in $C$. The weak topology on the weakly compact set $C$ coincides with the $\sigma(\text{span } C, \ell^2)$ topology, hence one can choose $x_1, x_2, ..., x_n \in C$ and $\varepsilon > 0$ so that $y \in V$ whenever $|\langle x_i, y - gx \rangle| < \varepsilon$ for all $i$.

Denote $K$ the diameter of $C$. Because the action is norm-continuous, we can find $U \ni e$ in $G$ so that $\|u^{-1}x_i - x_i\| < \varepsilon/2K$ for all $i$. The set $Ug$ is a neighborhood of $g$ in $G$.

As a weak neighborhood of $x$, take the set $W$ formed by all $\zeta \in C$ with $|\langle g^{-1}x_i, \zeta - x \rangle| < \varepsilon/2$ for all $i$. Equivalently, the condition on $\zeta$ can be stated $|\langle x_i, g\zeta - gx \rangle| < \varepsilon/2$ for all $i$.

If now $u \in U$ and $\zeta \in W$, one has

$$|\langle x_i, (ug) \cdot \zeta - gx \rangle| = |\langle u^{-1}x_i, g\zeta \rangle - \langle x_i, gx \rangle|$$

$$= |\langle u^{-1}x_i, g\zeta \rangle - \langle x_i, g\zeta \rangle| + |\langle x_i, g\zeta \rangle - \langle x_i, gx \rangle|$$

$$\leq \|ux_i - x_i\| \cdot K + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

This shows that $(ug) \cdot \zeta \in V$, and so the action of $G$ on $C$ is continuous with regard to the weak topology.

(2) ⇒ (1). Suppose that $G$ acts continuously and by affine transformations on a compact convex and metrizable subset $Q$ of a LCS $E$. If $C(Q)$ is equipped with the usual norm topology, then $G$ acts continuously by affine transformations on the subspace $A(Q)$ of $C(Q)$ consisting of affine continuous functions on $Q$. Since $Q$ is a metrizable compact set, the space $C(Q)$ is separable, so is the space $A(Q)$. Fix a dense countable subgroup $H$ of $G$, and let $F = \{f_n : n \in \mathbb{N}\}$ be a dense subset of the closed unit ball of $A(Q)$ which is $H$-invariant. The map $T : Q \ni x \mapsto \left(\frac{1}{n}f_n(x)\right) \in \ell^2$ is an affine homeomorphism of $Q$ onto a convex compact subset of $\ell^2$. The subgroup $H$ acts continuously and by affine transformations on the affine topological copy $T(Q)$ of $Q$ by the obvious rule $h.T(x) = T(h.x)$. The action of $H$ extends by continuity to a continuous affine action of $G$ on $T(Q)$. By hypothesis, $G$ admits an approximative fixed point sequence in $T(Q)$, and every accumulation point of this sequence is a fixed point since $Q$ is compact. Therefore $G$ is amenable.

\qed
4. Discussion and concluding remarks

4.1. Approximately fixed sequences. As we have already noted, we do not know if every locally compact amenable group has the AFP property on all convex bounded subsets of locally convex spaces. Another interesting problem is to determine when does an acting group possess not merely an approximately fixed net, but an approximate fixed sequence.

Recall that a topological group \( G \) is \( \sigma \)-compact if it is a union of countably many compact subsets. It is easy to see that if an amenable locally compact group \( G \) is \( \sigma \)-compact, then it admits an approximate fixed sequence for every continuous action by affine maps on a closed bounded convex set.

**Question 4.1.** Let \( G \) be a metrisable separable group acting continuously and affinely on a convex bounded subset \( C \) of a metrisable and locally convex space. If the action has an approximate fixed net, does there necessarily exist an approximate fixed sequence?

This is the case, for example if \( G \) is the union of a countable chain of amenable locally compact (in particular, compact) subgroups, and the convex set \( C \) is complete.

Recall in this connection that amenability (and thus Day’s fixed point property) is preserved by passing to the completion of a topological group. At the same time, the AFP property is not preserved by completions. Indeed, the group \( S_{\infty}^{Fin} \) of all permutations of integers with finite support is amenable as a discrete group, and so, equipped with any group topology, will have the AFP property on every bounded convex subset of a locally convex space. However, its completion with regard to the pointwise topology is the Polish group \( S_{\infty} \) which, as we have seen, fails the AFP property on a bounded convex subset of \( \ell^1 \).

**Question 4.2.** Does every amenable group whose left and right uniformities are equivalent (a SIN group) have the AFP property on complete convex sets?

4.2. Distal actions. Let \( G \) be a topological group acting by homeomorphisms on a compact set \( Q \). The flow \((G, Q)\) is called distal if whenever \( \lim_{\alpha} s_\alpha \cdot x = \lim_{\alpha} s_\alpha \cdot y \) for some net \( s_\alpha \) in \( G \), then \( x = y \). A particular class of distal flows is given by equicontinuous flows, for which the collection of all maps \( x \mapsto g \cdot x \) forms an equicontinuous family on the compact space \( Q \). We have the following fixed point theorem:
Theorem 4.3 (Hahn 1967). If a compact affine flow \((G, Q)\) is distal, then there is a \(G\)-fixed point.

An earlier result by (Kakutani 1938) established the same for the class of equicontinuous flows.

Question 4.4. Is there any approximate fixed point analogue of the above results for distal or equicontinuous actions by a topological group on a (non-compact) bounded convex set \(Q\)?

4.3. Non-affine maps. Historically, Day’s theorem (and before that, the theorem of Markov and Kakutani) was inspired by the classical Brouwer fixed point theorem (Brouwer, 1911) and its later more general versions, first for Banach spaces (Schauder, 1930) and later for locally convex linear Hausdorff topological spaces (Tychonoff, 1935). (Recently it was extended to topological vector spaces (Cauty, 2001)). The Tychonoff fixed point theorem states the following. Let \(C\) be a nonvoid compact convex subset of a locally convex space and let \(f : C \to C\) be a continuous map. Then \(f\) has a fixed point in \(C\). The map \(f\) is not assumed to be affine here.

However, for a common fixed point of more than one function, the situation is completely different. Papers (Boyce, 1969) and (Hunek, 1969) contain independent examples of two commuting maps \(f, g : [0, 1] \to [0, 1]\) without a common fixed point. Hence if a common fixed point theorem were to hold, there should be further restrictions on the nature of transformations beyond amenability, and for Day’s theorem, this restriction is that the transformations are affine.

The Tychonoff fixed point theorem is being extended in the context of approximate fixed points. For instance, here is one recent elegant result.

Theorem 4.5 (Kalenda 2011). Let \(X\) be a Banach space. Then every nonempty, bounded, closed, convex subset \(C \subseteq X\) has the weak AFP property with regard to each continuous map \(f : C \to C\) if and only if \(X\) does not contain an isomorphic copy of \(\ell^1\).

We do not know if a similar program can be pursued for topological groups.

Question 4.6. Does there exist a non-trivial topological group \(G\) which has the approximate fixed point property with regard to every continuous action (not necessarily affine) on a bounded, closed convex subset of a locally convex space? of a Banach space?

Question 4.7. The same, for the weak AFP property.
Natural candidates are the \textit{extremely amenable groups}, see e.g. [Pestov (2006)]. A topological group is extremely amenable if every continuous action of $G$ on a compact space $K$ has a common fixed point. The action does not have to be affine, and $K$ is not supposed to be convex. This is a very strong nonlinear fixed point property.

Some of the most basic examples of extremely amenable Polish groups are:

1. The unitary group $U(ℓ^2)$ with the strong operator topology [Gromov and Milman, 1983].
2. The group $\text{Aut}(\mathbb{Q})$ of order-preserving bijections of the rational numbers with the topology induced from $S_\infty$ [Pestov, 1998].
3. The group $\text{Aut}(X, \mu)$ of measure preserving transformations of a standard Lebesgue measure space equipped with the weak topology. (Giordano and Pestov, 2002).

However, at least the group $\text{Aut}(\mathbb{Q})$ does not have the AFP property with regard to continuous actions on the Hilbert space. To see this, one can use the same construction as in Proposition 3.8, together with the well-known fact that $\text{Aut}(\mathbb{Q})$ contains a closed discrete copy of $\mathbb{F}_2$.

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