ENTROPY THEORY OF GEODESIC FLOWS

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Abstract. In this paper we prove the upper semicontinuity of the entropy map for (non-compact) systems satisfying a 'simplified entropy formula' and whose ergodic measures are 'weak entropy dense'. We apply our results to the geodesic flow on a pinched negatively curved manifold. We relate the escape of mass with a critical exponent that takes into consideration the geometry at infinity of the manifold. We also prove the upper semicontinuity of the entropy map for the geodesic flow on a non-positively curved rank one manifold when restricting to a big subset of the nonwandering set.

1. Introduction

In the theory of dynamical systems the concept of entropy is of fundamental importance, we want to measure how complicated is our system and entropy theories provide tools on how to do that. In this paper we will investigate regularity properties of the measure theoretic entropy, more precisely, we will prove that under certain circumstances the entropy map is upper semicontinuous. We emphasize that we will be mostly interested in the non-compact case, there are many criterions for the upper semicontinuity of the entropy map for compact spaces. For instance, it is a classical result of Bowen [Bo1] that the entropy map of an \( h \)-expansive map on a compact metric space is upper semicontinuous. The same property holds for asymptotically \( h \)-expansive maps (a class of dynamical systems introduced by Misiurewicz [Mi]). It worth mentioning that Buzzi [Bu], using the techniques developed by Yomdin [Y], proved that a \( C^\infty \) diffeomorphisms on a compact manifold is asymptotically \( h \)-expansive (the upper semicontinuity of the entropy map was previously proved by Newhouse [N]). In the non-compact case there are no partitions with arbitrarily small diameter (for most of the metrics we want to consider in our space). This simple fact makes that many of the classical arguments do not work anymore. As an example, consider the full shift in a countable alphabet \((\mathbb{N}^\infty, \sigma)\), this dynamical system is expansive and the entropy map is not upper semicontinuous at any measure of finite entropy. This suggests that in the non-compact setting the upper semicontinuity of the entropy map can be much more intricate.

One of the main goals of this paper is to describe the ergodic theory of the geodesic flow on a (non-compact) negatively curved manifold. Since the geodesic flow on a negatively curved manifold is an Anosov flow, in the compact case, we have at our disposal a (compact) symbolic model (see [Bo2] and [Ra]). In particular its ergodic theory and thermodynamical formalism are well understood, for instance see [BR]. Even though in the non-compact case it is not know if the geodesic can be coded as a suspension flow over a topological Markov shift, a very complete theory about its thermodynamical formalism was recently developed by Paulin, Pollicot

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A. VELOZO and Schapira [PPS]. In particular they proved that for Hölder potentials there is at most one equilibrium state, and that it has a very concrete form. A very good complement to this work is the recent paper of Pit and Schapira [PS] where they provided strong criteria for the existence of equilibrium states for Hölder potentials, in particular for the existence of measures of maximal entropy for the geodesic flow. As a consequence of our results we can prove the existence of measures of maximal entropy when a critical gap is satisfied. We will also prove the existence of equilibrium states of a large family of continuous potentials, in this case the theory developed in [PPS] do not apply (see Section 8). A quantity that measures the complexity of the dynamics, say $T$, at the ends of our phase space is the so called entropy at infinity, which is defined as

$$h_{\infty}(T) = \sup \limsup_{(\mu_n) \mu_n \to 0} h_{\mu_n}(T),$$

where the supremum runs over sequences of $T$-invariant probability measures converging vaguely to the zero measure. This quantity was defined in [IRV] and computed for the geodesic flow on extended Schottky manifolds via symbolic methods. It was later computed for the geodesic flow on geometrically finite manifolds [RV]. In this paper we will prove that the entropy at infinity corresponds to a critical exponent that takes into consideration the geometry at the ends of our negatively curved manifold. We now proceed to state our results in more detail.

1.1. **Statement of results.** We start with some definitions that are needed to make sense of our results. Let $(X, d)$ be a metric space and $T : X \to X$, a continuous map. For $n \in \mathbb{N}$ we define the dynamical metric $d_n$ as

$$d_n(x, y) = \max_{0 \leq k \leq n-1} d(T^k x, T^k y).$$

A ball of $d_n$-radius $\epsilon$ is called a $(n, \epsilon)$-dynamical ball. Given $\epsilon > 0$ and $\delta \in (0, 1)$ we define $N_{\mu}(n, \epsilon, \delta)$ as the minimum number of $(n, \epsilon)$-dynamical balls needed to cover a set of measure strictly bigger than $1 - \delta$. Katok proved in [Ka, Theorem 1.1] that if $X$ is compact, then for every ergodic measure $\mu$ the following formula holds

$$h_{\mu}(T) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_{\mu}(n, \epsilon, \delta),$$

where $h_{\mu}(T)$ is the measure theoretic entropy of $\mu$. It was proven by Riquelme [Ri] that the same formula holds for Lipschitz maps on topological manifolds. The space of $T$-invariant probability measures is denoted by $\mathcal{M}(X, T)$ and the set of ergodic $T$-invariant probability measures by $\mathcal{M}_e(X, T)$.

**Definition 1.1 (Simplified entropy formula).** We say that $(X, d, T)$ satisfies a simplified entropy formula if for every $\epsilon > 0$ sufficiently small, $\mu \in \mathcal{M}_e(X, T)$ and $\delta \in (0, 1)$ we have

$$h_{\mu}(T) = \limsup_{n \to \infty} \frac{1}{n} \log N_{\mu}(n, \epsilon, \delta).$$

We say that $(X, d, T)$ satisfies a simplified entropy inequality if for every $\epsilon > 0$ sufficiently small, $\mu \in \mathcal{M}_e(X, T)$ and $\delta \in (0, 1)$ we have

$$h_{\mu}(T) \leq \limsup_{n \to \infty} \frac{1}{n} \log N_{\mu}(n, \epsilon, \delta).$$
**Definition 1.2** (Weak entropy dense). We say that \( \mathcal{M}_e(X, T) \) is weak entropy dense in \( \mathcal{M}(X, T) \) if the following holds. For every \( \mu \in \mathcal{M}(X, T) \) and \( \eta > 0 \) there exists a sequence \((\mu_n)_{n \in \mathbb{N}}\) of ergodic measures satisfying

1. \( \lim_{n \to \infty} \mu_n = \mu \).
2. For every \( n \in \mathbb{N} \) we have \( h_{\mu_n}(T) > h_\mu(T) - \eta \).

We also refer to this property by saying that \((X, d, T)\) is weak entropy dense.

1.2. Abstract result. Let \((X, d)\) be a non-compact topological manifold. Let \( T : X \to X \) be an \( L \)-Lipschitz homeomorphism, i.e., a homeomorphism satisfying \( d(Tx, Ty) \leq Ld(x, y) \), for every \((x, y) \in X \times X\). Let \( K \) be a compact subset of \( X \).

Define

\[ K(n) = K \cap \bigcap_{i=1}^{n-2} T^{-i}K \cap T^{-(n-1)}K. \]

Given a point \( x \in K \) and \( r > 0 \), we define \( C(x, n, r) \) as the number of \((n, r)\)-dynamical balls needed to cover \( B(x, r) \cap K(n) \). We define the critical exponent of \( K \) at scale \( r \) by the formula

\[ \delta_x(K, r) = \limsup_{n \to \infty} \frac{1}{n} \log \sup_{z \in K} C(x, n, r). \]

**Definition 1.3.** The critical exponent at infinity of \((X, d, T)\) is the quantity

\[ \delta_x = \inf_{\{K_n\}} \liminf_{n \to \infty} \inf_{r > 0} \delta_x(K_n, r), \]

where the infimum in front runs over sequences \( \{K_n\}_{n \in \mathbb{N}} \) where each \( K_n \) is compact, \( K_n \subset K_{n+1} \) and \( X = \bigcup_{n \geq 1} K_n \).

**Theorem 1.4.** Let \((X, d)\) be a non-compact topological manifold and \( T \) an \( L \)-Lipschitz homeomorphism with finite topological entropy. Assume that \((X, d, T)\) satisfies a simplified entropy formula and that its ergodic measures are weak entropy dense. Let \((\mu_n)_{n \geq 1}\) be a sequence of \( T \)-invariant probability measures converging to \( \mu \) in the vague topology. Then

\[ \limsup_{n \to \infty} h_{\mu_n}(T) \leq |\mu|h_{\mu/|\mu|}(T) + (1 - |\mu|)\delta_x. \]

If the sequence \((\mu_n)_{n \geq 1}\) converges to the zero measure the RHS of the inequality is understood as \(\delta_x\).

**Remark 1.5.** The finite entropy assumption is required to ensure that \(\delta_x\) is finite, which is important to make sense of our formula.

1.3. Geodesic flow on negatively curved manifolds. Let \((M, g)\) be a non-compact complete negatively curved manifold. We will assume there exist constants \(a, b > 0\) such that \(-a < K_g < -b\), where \(K_g\) is the sectional curvature of \((M, g)\).

The geodesic flow on \(M\) is denoted by \((T^1M, (g_t)_{t \in \mathbb{R}})\). In this situation the geodesic flow satisfies a simplified entropy inequality (after a mild modification in the definition of the dynamical balls) and the set of ergodic measures is weak entropy dense (see Section 6). We can then apply Theorem 1.14 and get the following result.

**Theorem 1.6.** Let \((M, g)\) be a pinched negatively curved manifold. Let \((\mu_n)_{n \geq 1}\) be a sequence of \(g_1\)-invariant probability measures converging to \( \mu \) in the vague topology. Then

\[ \limsup_{n \to \infty} h_{\mu_n}(g) \leq |\mu|h_{\mu}(g) + (1 - |\mu|)\delta_x. \]
If the sequence converges vaguely to zero, then the RHS is understood as \( \delta_\infty \). Moreover, the entropy at infinity of the geodesic flow on \( M \) is exactly \( \delta_\infty \).

In Theorem 1.6 the definition of \( \delta_\infty \) is slightly different from the one in the previous subsection. For completeness we will define the critical exponent at infinity of the geodesic flow. Let \( Q \subset M \) be a compact subset, where \( \tilde{M} \) is the universal cover of \( M \), and let \( \Gamma = \pi_1(M) \). Given \( n \in \mathbb{N} \) we define

\[
\Gamma_Q(n) = \{ \gamma \in \Gamma : \exists x \in T^1Q \text{ such that } g_k(x) \in T^1Q, \text{ for } k \in [1,n-2] \text{ and } g_{n-1}(x) \in T^1\gamma Q \}.
\]

Then define

\[
\delta_\infty(Q) = \limsup_{n \to \infty} \frac{1}{n} \log \# \Gamma_Q(n).
\]

Let \( P \) be a Dirichlet fundamental domain of \( M \) in \( \tilde{M} \).

**Definition 1.7.** The critical exponent at infinity of \( M \) is the number

\[
\delta_\infty = \inf_{(P_k)} \liminf_{k \to \infty} \delta_\infty(P_k),
\]

where the infimum runs over compact exhaustions of \( P \), in other words, increasing sequences \( (P_k)_{k \geq 1} \) such that each \( P_k \) is compact and \( \bigcup_{n \geq 1} P_k = P \).

A key ingredient in the proof of the simplified entropy inequality is the existence of a Gibbs measure for the geodesic flow. For this we will use results in [PPS] and [PS]. We will also present another proof of this result that does not require the existence of a Gibbs measure, and which also applies to the geodesic flow on non-positive curvature (see Proposition 6.4).

1.4. Geodesic flow on nonpositively curved rank one manifolds. Now assume \((M, g)\) is a complete Riemannian manifold with non-positive sectional curvature. We will moreover assume that the sectional curvature is bounded below. We will prove the following result.

**Theorem 1.8.** Let \((M, g)\) be a non-positively curved manifold with curvature bounded below. Let \((\mu_n)_{n \in \mathbb{N}}\) a sequence of ergodic probability measures converging to \( \mu \) in the weak-* topology. Then

\[
\limsup_{n \to \infty} h_{\mu_n}(g) \leq h_\mu(g).
\]

If we further assume that \((M, g)\) has rank one we can restrict the geodesic flow to a subset \( \Omega_{\text{reg}} \) of the nonwandering set of \( T^1M \) where the entropy map is upper semicontinuous. The set \( \Omega_{\text{reg}} \) is the space of rank one recurrent vectors in \( T^1M \), in particular it contains all rank one closed geodesics of \( M \), and we refer to it as the regular part of the geodesic flow.

**Theorem 1.9.** Let \((M, g)\) be a non-positively curved rank one manifold with curvature bounded below. Let \((\mu_n)_{n \in \mathbb{N}}\) a sequence of \( g_1 \)-invariant probability measures supported on the regular part of the geodesic flow converging vaguely to \( \mu \). Then

\[
\limsup_{n \to \infty} h_{\mu_n}(g_1) \leq |\mu| h_{|\mu|}(g_1) + (1 - |\mu|) \delta_\infty.
\]

In particular the entropy map of the dynamical system \((\Omega_{\text{reg}}, g_1)\) is upper semicontinuous.
1.5. Organization of the paper. The paper is organized as follows. In Section 2 we recall some basics facts about measure theory and the thermodynamical formalism of the geodesic flow. In Section 3 we prove an inequality which is the key ingredient in the proof of Theorem 1.4. In Section 4 we prove our abstract result Theorem 1.4. In Section 5 we prove that the geodesic flow on a negatively curved manifold is weak entropy dense and that it satisfies a simplified entropy inequality. In Section 6 we prove Theorem 1.6, Theorem 6.6 and Theorem 1.9. In Section 7 we prove that our critical exponent \( \delta \) is the entropy at infinity of the geodesic flow. In Section 8 we state several applications of the results obtained in previous sections.

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2. Preliminaries

In this paper we will constantly use the notation \( A^c \) to denote the complement of \( A \) when the total space is clear.

2.1. Measure theory. Let \( (X, d) \) be a locally compact metric space and \( T \) a continuous map. We denote by \( \mathcal{M}(X, T) \) to the space of borelian \( T \)-invariant probability measures on \( X \) and \( \mathcal{M}_{\leq 1}(X, T) \) the space of borelian \( T \)-invariant measures of total mass in \([0, 1]\). Clearly \( \mathcal{M}(X, T) \subseteq \mathcal{M}_{\leq 1}(X, T) \). The space of ergodic probability measures is denoted by \( \mathcal{M}_e(X, T) \). We endow the space \( C_b(X) \) (resp. \( C_c(X) \)) of bounded (resp. compactly supported) continuous functions with the uniform norm \( ||f|| = \sup_{x \in X} |f(x)| \). We endow \( \mathcal{M}(X, T) \) with the weak-* topology, i.e. a sequence \( (\mu_n)_{n \geq 1} \) converges weakly to \( \mu \) if for every \( f \in C_b(X) \) we have

\[
\lim_{n \to \infty} \int f d\mu_n = \int f d\mu.
\]

In a similar way we endow \( \mathcal{M}_{\leq 1}(X, T) \) with the vague topology, i.e. a sequence \( (\mu_n)_{n \geq 1} \) converges vaguely to \( \mu \) if for every \( f \in C_c(X) \) we have

\[
\lim_{n \to \infty} \int f d\mu_n = \int f d\mu.
\]

If there exists a compact exhaustion of \( X \), i.e. an increasing sequence \( (K_n)_{n \geq 1} \) of compact sets such that \( X = \bigcup_{n \geq 1} K_n \), then the space \( C_c(X) \) is separable. In this case consider a dense subset \( (f_n)_{n \geq 0} \) of the unit ball of \( C_c(X) \). We define a metric \( d \) on \( \mathcal{M}_{\leq 1}(X, T) \) as

\[
d(\mu_1, \mu_2) = \sum_{n \geq 1} \frac{1}{2^n} \left| \int f d\mu_1 - \int f d\mu_2 \right|.
\]

This metric is compatible with the vague topology. By Banach-Alaoglu theorem we know that \( \mathcal{M}_{\leq 1}(X, T) \) is a compact metrizable space. We remark that the space \( \mathcal{M}(X, T) \) is still compact, but not metrizable (it can not be sequentially compact because of the escape of mass phenomenon). If \( X \) is a non-compact manifold, then \( X \) admits a compact exhaustion, and therefore this discussion apply to that case.
Remark 2.1. In the definition of the weak-* topology it is enough to check the convergence for uniformly continuous functions (even Lipschitz). In this paper we will always consider uniformly continuous functions as test functions for the weak-* topology (for the vague topology this follows from the compact support).

2.2. Geodesic flows.
Let $(M, g)$ be a complete Riemannian manifold. We define the unit tangent bundle of $M$ as $T^1M = \{v \in TM : ||v||_g = 1\}$. Since $(M, g)$ is complete, the geodesic flow on $T^1M$ is well defined for all times, we denote it by $(g_t)_{t \in \mathbb{R}}$. The Riemannian metric $g$ makes $M$ into a metric space, the induced distance function (shortest path distance) is denoted by $d$. Let $\pi : T^1M \to M$ be the canonical projection. We define a metric on $T^1M$ which we still denote by $d$ in the following way

$$d(x, y) = \max_{t \in [0,1]} d(\pi g_t(x), \pi g_t(y)),$$

for every $x, y \in T^1M$. We emphasize that this is the metric used in all the statement about the geodesic flow. The same notation is used to denote the metric and distance function on $\tilde{M}$, the universal cover of $M$. Analogously we define a metric on $T^1\tilde{M}$, so we can talk about $(n, \epsilon)$-dynamical balls in $T^1\tilde{M}$.

From now on we will assume that $(M, g)$ is a complete Riemannian manifold of negative sectional curvature. We will moreover assume that $K_g \in [-a, -b]$, for some $a, b > 0$, where $K_g$ is the sectional curvature of $M$. As before, the universal cover of $M$ is denoted by $\tilde{M}$. We denote by $\partial_x \tilde{M}$ to the visual or Gromov boundary of $\tilde{M}$. The fundamental group $\Gamma = \pi_1(M)$ acts isometrically, freely and discontinuously on $\tilde{M}$. The nonwandering set of the geodesic flow will be denoted by $\Omega \subset T^1M$.

The action of any isometry on $\tilde{M}$ extends to an homeomorphism of $\partial_x \tilde{M}$. The limit set of $\Gamma$ is the smallest closed subset of $\partial_x \tilde{M}$ invariant by the action of $\Gamma$, this set is denoted by $L(\Gamma)$. Given two points $x, y \in \tilde{M}$, we denote by $[x, y]$ to the oriented geodesic segment starting at $x$ and ending at $y$. For a function $G : T^1\tilde{M} \to \mathbb{R}$, we use the notation $\int_x^y G$ to represent the integral of $G$ over the vectors tangent to the path $[x, y]$. Given a function $F : T^1M \to \mathbb{R}$, we denote by $\tilde{F} : T^1\tilde{M} \to \mathbb{R}$ to the function $\tilde{F} = F \circ p$, where $p$ is the canonical projection $p : T^1\tilde{M} \to T^1M$. Recently F. Paulin, M. Pollicott and B. Schapira in [PPS] developed a very complete theory about thermodynamical formalism for negatively curved manifolds. They introduced the following important definition (also compare with the definitions in [Con]).

Definition 2.2. Let $F : T^1M \to \mathbb{R}$ be a continuous function and $\tilde{F}$ its lift to $T^1\tilde{M}$. Define the Poincaré series associated to $(\Gamma, F)$ based at $z \in \tilde{M}$ as

$$P(s, F) = \sum_{\gamma \in \Gamma} \exp\left(\int_z^\gamma (\tilde{F} - s)\right).$$

The critical exponent of $(\Gamma, F)$ is

$$\delta^F = \inf\{s \mid P(s, F) \text{ is finite}\}.$$

We say that the pair $(\Gamma, F)$ is of convergence type if $P(\delta^F, F) < \infty$, in other words the Poincaré series converges at its critical exponent. Otherwise we say $(\Gamma, F)$ is of divergence type.
We use the notation $\delta_\Gamma$ when referring to the critical exponent of $(\Gamma, 0)$, where 0 is the zero function. If $F$ is Hölder continuous, then the critical exponent does not depend on the base point $z$. Observe that if $F$ is bounded, then $\delta_\Gamma^F$ is finite. This observation is one of the main reasons why in most of the statements of this paper we assume $F$ to be bounded, in general the hypothesis $\delta_\Gamma^F < \infty$ will be enough. As usual, the pressure of a potential $F$ is defined by

$$P(F) = \sup_{\mu \in \mathcal{M}(g)} \{ h_\mu(g) + \int F d\mu : \int F d\mu > -\infty \}.$$ 

We say that a measure $\mu \in \mathcal{M}(g)$ is an equilibrium state for $F$ if

$$P(F) = h_\mu(g) + \int F d\mu.$$

A general procedure due to S. Patterson and D. Sullivan associates to $\Gamma$ a family of conformal measures on $\partial_x \tilde{M}$ of exponent $\delta_\Gamma$, the so-called Patterson-Sullivan conformal measures of $\Gamma$. Using this family of conformal measures one can canonically construct a $(g_t)$-invariant measure, we refer to this measure as the Bowen-Margulis measure (for its precise construction we refer the reader to [OP]). Similar to the construction of the Patterson-Sullivan conformal measures, we can associate to the pair $(\Gamma, F)$ a family of conformal measures. We briefly recall this construction.

A Patterson density of dimension $\delta$ for $(\Gamma, F)$ is a family of finite Borel measures $(\sigma_x)_{x \in \tilde{M}}$ on $\partial_x \tilde{M}$, such that, for every $\gamma \in \Gamma$, for all $x, y \in \tilde{M}$ and for every $\xi \in \partial_x \tilde{M}$ we have

$$\gamma_* \sigma_x = \sigma_{\gamma x} \quad \text{and} \quad \frac{d\sigma_x}{d\sigma_y}(\xi) = e^{\gamma_F - \gamma_F(x,y)},$$

where $\gamma_F(x,y)$ is the Gibbs cocycle defined as

$$C_{\gamma_F}(x,y) = \lim_{t \to \infty} \int_y^x F \, F - \int_x^\xi F,$$

for any geodesic ray $t \to \xi_t$ ending at $\xi$. Note that the limit in the definition of the Gibbs cocycle always exists since the manifold has negative curvature and the potential is Hölder continuous. If $\gamma_F < \infty$, then there exists at least one Patterson density of dimension $\gamma_F$ for $(\Gamma, F)$, which support lies in the limit set $L(\Gamma)$ of $\Gamma$. If $(\Gamma, F)$ is of divergence type then there is an unique Patterson-Sullivan density of dimension $\gamma_F$. Using this family of Patterson densities one can construct a $(g_t)$-invariant measure $m_F$. We refer to this measure as the Gibbs measure associated to the potential $F$ and it is denoted by $m_F$. A fundamental property of $m_F$ is that, whenever finite, it is the unique equilibrium measure for the potential $F$.

**Theorem 2.3.** [PPS, Theorem 2.3] Let $F : T^1 X \to \mathbb{R}$ be a bounded Hölder potential. Then

$$P(F) = \delta_F^\Gamma.$$ 

Moreover, if there exists a finite Gibbs measure $m_F$ for $(\Gamma, F)$, then $m_F/||m_F||$ is the unique equilibrium state of $F$. Otherwise there is not equilibrium measure.

We remark that this result was obtained by Otal and Peigne in the case $F = 0$ (see [OP]). The proof of Theorem 2.3 follows very closely the proof in [OP].
3. The Key Inequality

**Theorem 3.1.** Let \((X,d)\) be a non compact topological manifold and \(T\) a \(L\)-Lipschitz homeomorphism with finite topological entropy. Assume that \((X,d,T)\) satisfies a simplified entropy formula. Let \((\mu_n)_{n \geq 1}\) be a sequence of ergodic measures converging to \(\mu\) in the vague topology. Suppose there exists a compact set \(K\) such that \(\mu(\partial K) = 0\), and \(\mu_n(K) > 0\), for every \(n \in \mathbb{N}\). Then

\[
\limsup_{n \to \infty} h_{\mu_n}(T) \leq |\mu| h_{\mu/|\mu|}(T) + (1 - \mu(A(K))) \delta_x(K, r),
\]

where \(A(K) = \{x \in X : T^k x \in K, \text{ for some } k \geq 0\text{ and some } k \leq 0\}\).

**Proof.** We will explain how to construct some good partitions. Define \(Y = A(K)\) as the set of points in \(X\) that enter to \(K\) under positive and negative iterates of \(T\). Let \(A_k\) be the set of points in \(K\) that have their first return to \(K\) at time \(k\). Given \(x \in Y\) define \(n_2(x)\) as the smallest non-negative number such that \(T^{n_2(x)} x \in K\) and \(n_1(x)\) as the smallest non-negative number such that there exists \(y \in K\) satisfying \(T^{n_1(x)}(y) = x\). For \(x \in Y\) define \(n(x) = n_1(x) + n_2(x)\), and declare \(n(x) = \infty\) whenever \(x \in Y^c\). For \(n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}\) define

\[
C_n = \{x \in X : n(x) = n\}.
\]

By the ergodicity of \(\mu_n\) and the hypothesis \(\mu_n(K) > 0\) we have that \(\mu_n(C_{\infty}) = 0\). Moreover, the fact \(x \in C_n\) means that \(x\) is in the orbit of a point in \(A_n\). Since \(T\) is Lipschitz and \(\sup_{x \in K} d(x,Tx)\) is finite, we conclude that \(\bigcup_{n=0}^M C_n\) is bounded, and therefore relatively compact, for each \(L \in \mathbb{N}\). Define

\[
\alpha_{N,M} = \bigcup_{n > N} C_n, \text{ and } \alpha_N = \bigcup_{n > N} C_n.
\]

It worth mentioning that \(\partial \alpha_N \subseteq \bigcup_{k \in \mathbb{Z}} T^{-k}\partial K\) and the same holds for \(\alpha_{N,M}\). This implies that \(\mu(\partial \alpha_{N,M}) = \mu(\partial \alpha_N) = 0\). By the definition of \(\delta_x(K, r)\) we know that given \(\epsilon > 0\), there exists \(N_0 = N_0(\epsilon)\) such that the following holds. For every \(n \geq N_0\) and \(x \in K(n) = K \cap \bigcap_{i=0}^{n-1} \bigcap_{k} \cup_{i=0}^{n-1} T^{-k}K \cap T^{-1}(n-1)K\), we have that \(B(x,r) \cap K(n)\) can be covered by at most \(e^{\alpha_n(\delta_x(K, r) + \epsilon)}\) \((n,r)\)-dynamical balls.

Choose natural numbers \(k \geq 2\) and \(N \geq N_0(\epsilon)\). Define \(\beta_{k,N} = \{\alpha_k \cap N, \alpha_{N,k} \cap N, Q_N^1, \ldots, Q_N^s, C_{\infty}\}\), where \(Q_N^1\) are balls of diameter less than \(r/L^{k+2}\) covering \(\bigcup_{n=0}^N C_n\) (which is relatively compact) and let \(\beta'_{k,N} = \{Q_N^1, \ldots, Q_N^s\}\). We choose this covering such that \(\mu(\partial Q_N^i) = 0\) for every \(i\). In particular we know \(\mu(\partial \beta_{k,N}) = 0\).

Given \(Q \in \beta_{k,n} \cap \alpha_N\) we say that \([r,s) \subseteq [0,n)\) is an excursion into \(\alpha_N\) if \(T^tQ \subseteq \alpha_N\) for every \(t \in [r,s)\), \(T^{s-1}Q \subseteq \alpha_N^r\) and \(T^sQ \subseteq \alpha_N^s\). Define \(m_{k,N}(Q)\) as the number of excursions into \(\alpha_N\) that contain at least one excursion into \(\alpha_{k,N}\) and let \(|E_{N,n}| = \#\{k \in [0,n) : T^kQ \subseteq \alpha_N\}\).

**Remark 3.2.**

1. Observe that if \(x \in K^c\) then \(n_2(Tx) = n_2(x) - 1\), and that \(n_1(Tx) \leq n_1(x) + 1\). In other words if \(x \in K^c\), then \(n(Tx) \leq n(x)\). This implies that if \(x \in C_n \cap K^c\), then \(Tx \in C_t\), for some \(t \leq n\).
Let \( [r,s) \) be an excursion of \( Q \) into \( \alpha_N \). Suppose there exists \( x \in T^{r-1}Q \cap K' \), then by (1) we have \( n(x) \geq n(Tx) > N \). In particular \( x \in \alpha_N \), which contradicts that \( T^{r-1}Q \subset \alpha_N' \). We conclude that \( T^{r-1}Q \subset K \).

(3) If \( x \in Q \), where \( Q \subset \alpha_N' \), then \( T^t x \in K \), for some \( 0 \leq t \leq N \). If \( x \in \bigcup_{n=0}^{N} C_n \), then by definition \( n_2(x) \leq N \) and the conclusion follows.

We claim that an atom \( Q \in \beta_{k,N}^n \) such that \( Q \subset K \cap T^{-(n-1)}K \), can be well covered by \((n,r)\)-dynamical balls. More precisely

**Proposition 3.3.** Let \( \beta_{k,N} \) as above. Then an atom \( Q \in \beta_{k,N}^n \) such that \( Q \subset K \cap T^{-(n-1)}K \) can be covered by no more than

\[
C_0 e^{\|E_{N,n}(Q)(\delta_x(K,r)+\epsilon)\|} e^{m_{k,N,n}(Q)N(\delta_x(K,r)+\epsilon)}
\]

\((n,r)\)-dynamical balls, where \( C_0 = C_0(m,q,N,k) \).

To simplify notation we will forget the subindex \( N \) and \( k \), but we remember that \( N \) is related to the error term \( \epsilon \) in the exponents. We remark that since \( C_{\infty} \) satisfies \( TC_{\infty} \subset C_{\infty} \), the assumption \( T^{n-1}Q \subset K \) rules out the possibility that \( Q \) entered to \( C_{\infty} \) before the \((n-1)\)th iterate. The proof is inductive. First decompose

\([0,n-1] = W_1 \cup V_1 \cup \ldots \cup V_s \cup W_{s+1},\]

according to the excursions into \( \alpha_N \) that contain at least one excursion into \( \alpha_{k,N} \). It worth mentioning that by Remark 3.2 each excursion into \( \alpha_N \) can contain at most one excursion into \( \alpha_{k,N} \). More precisely, let \( V_i = [n_i, n_i + h_i) \) and \( W_i = [l_i, l_i + L_i) \) with \( l_i + L_i = n_i \) and \( n_i + h_i = l_i+1 \). The segment \( V_i \) denotes an excursion into \( \alpha_N \) that contains an excursion into \( \alpha_{k,N} \).

**Step 0:** Cover \( K \) and \( \bigcup_{n=0}^{N} C_n \) by balls of diameter \( r/L^{kN+2} \). We do this covering such that the boundary of the partition has zero \( \mu \)-measure. We denote the number of balls required for this covering as \( C_0 = C_0(K,r,N,k) \).

**Step 1:** Assume we have covered \( Q \) by

\[C_0 e^{\|E_{N,n}(Q)(\delta_x(K,r)+\epsilon)\|} e^{m_{k,N,n}(Q)N(\delta_x(K,r)+\epsilon)} \]

\((l_i + 1, r)\)-dynamical balls. We claim the same number of balls suffices to cover \( Q \) with \([l_i+L_i, r)\)-dynamical balls. Observe that by hypothesis \( T^k Q \subset \alpha_N' \), therefore \( \text{diam } T^k Q \leq r/L^{kN+2} \). Since the balls used to cover \( \beta' \) have all diameter smaller than \( r/L^{kN+2} \) the same hold if \( Q \) spends some extra time in \( \beta' \). If \( Q \) has an excursion into \( \alpha_N \) that does not enter to \( \alpha_{k,N} \), then by definition it must comes back to \( \beta' \) before \( kN \) iterates (see Remark 3.2 (3)). In particular if the excursion into \( \alpha_N \) is \([p_i, p_i + q_i) \), then \( q_i \leq kN \). Observe that \( \text{diam } T^{p_i-1}Q \leq r/L^{kN+2} \) implies that \( \text{diam } T^{p_i+1}Q \leq r \) for every \( t \in [0,kN] \). In particular the same holds for \( t \in [0,q_i] \).

But now we have entered to \( \beta' \) again and we can repeat this process until we find an excursion into \( \alpha_{k,N} \), in that case we proceed as in Step 2.

**Step 2:** Assume we have covered \( Q \) by

\[C_0 e^{\|E_{N,n}(Q)(\delta_x(K,r)+\epsilon)\|} e^{m_{k,N,n}(Q)N(\delta_x(K,r)+\epsilon)} \]

\((n_i, r)\)-dynamical balls. To get a covering with \((n_i + h_i + 1, r)\)-dynamical balls we will cover each \((n_i, r)\)-dynamical ball in the given covering by \((n_i + h_i + 1, r)\)-dynamical balls. Let \( x \in Q \) be the center of one of the \((n_i, r)\)-dynamical balls (if
the center of the ball is not in $Q$ one takes a point in the ball that do belong to $Q$ and change $r$ by $2r$ in our next argument, for simplicity we assume that $x \in Q$ is the center of the dynamical ball). By definition $T^i x \in \alpha_N$ for $i \in [n_i, n_i + h_i)$, $T^{n_i-1} x \in \alpha_N$ and $T^{n_i+h_i} x \in \alpha_N$. Let $s(x) \geq 0$ be the smallest number such that $T^{n_i+h_i+s(x)}(x) \in K$. Notice that by Remark 3.2(2) we have $T^{n_i-1}(x) \in K$, and by Remark 3.2(3) we know that $s(x) \leq N$. Since $T^{n_i-1}B_n (x, r) \subset B(T^{n_i-1} x, r)$, we can just focus on covering $B = B(T^{n_i-1} x, r) \cap K(h_i + s(x) + 1)$ with $(h_i + 1, r)$ dynamical balls. By the definition of $\delta_x(K, r)$ we know that $B$ can be covered with at most $e^{(\delta_x(K, r) + \epsilon)(h_i + s(x))}(h_i + s(x) + 1, r)$-dynamical balls. We conclude that $B$ can be covered by at most $e^{(\delta_x(K, r) + \epsilon)(h_i + N)}(h_i + 1, r)$-dynamical balls. This proves that the number of $(n_i + h_i + 1, r)$-dynamical balls needed to cover $Q$ is at most the number of balls we had at the beginning of Step 2 times $e^{(\delta_x(K, r) + \epsilon)h_i \epsilon e^{(\delta_x(K, r) + \epsilon)N}}.

We conclude that $Q$ can be covered with at most

$$C_0 e^{(|E_{N,n}(Q)|) (\delta_x(K, r) + \epsilon) \epsilon e^{m_k, N, n}(Q)N (\delta_x(K, r) + \epsilon)},$$

$(n, r)$-dynamical balls, where $C_0 = C_0(m, q, N, k)$. We remark that $C_0$ is a constant independent of $n$. We also remark that the term $|E_{N,n}(Q)|$ is a very rough bound, we can actually use the time spent in excursions into $\alpha_N$ containing excursions into $\alpha_kN$.

**Proposition 3.4.** Let $\beta_{k,N}$ the partition defined in Proposition 3.3. Let $\mu \in \mathcal{M}_e(X, T)$ satisfying $\mu(K) > 0$. Then

$$h_\mu(T) \leq h_\mu(T, \beta_{k,N}) + \mu(\alpha_N)(\delta_x(K, r) + \epsilon) + \frac{1}{k}(\delta_x(K, r) + \epsilon).$$

**Proof.** Recall that by Definition 1.1 we know that for every ergodic measures $\mu$ we have

$$h_\mu(T) = \lim_{n \to \infty} \frac{1}{n} \log N_\mu(n, r, \delta).$$

Using the ergodicity of $\mu$ and the assumption $\mu(K) > 0$ we can find an increasing sequence $(n_i)_{i \in \mathbb{N}}$ such that

$$\mu(K \cap T^{-n_i}K) > \delta_1,$$

for every $i \in \mathbb{N}$, where $\delta_1$ is sufficiently small but positive (and independent of $n_i$). By Shannon-McMillan-Breiman theorem the set

$$A_{\epsilon_1, N} = \{x \in X : \forall n \geq N, \mu(\beta^n(x)) \geq \exp(-n(h_\mu(T, \beta) + \epsilon_1))\},$$

has measure converging to 1 as $N$ goes to $\infty$, for every $\epsilon_1 > 0$. Fix $\epsilon_1 > 0$ small. By Birkhoff ergodic theorem there exists a set $W_{\epsilon_1}$ such that

$$\frac{1}{n} \sum_{i=0}^{n-1} 1_{A_{\epsilon_1, N}}(T^i x) < \mu(\alpha_N) + \epsilon_1,$$

and $\mu(W_{\epsilon_1}) > 1 - \frac{\delta_1}{4}$, for all $x \in W_{\epsilon_1}$ and $n \geq n(\epsilon_1)$. We finally define

$$X_i = W_{\epsilon_1} \cap A_{\epsilon_1, n_i} \cap K \cap T^{-n_i}K.$$

By construction, for $i$ sufficiently large, we have $\mu(X_i) > \frac{\delta_1}{4}$. From now on we will always assume $i$ is sufficiently large. Our goal is to cover $X_i$ by $(n_i, r)$-dynamical balls. By definition of $A_{\epsilon_1, n_i}$ we know $X_i$ can be covered by $\exp(n_i(h_\mu(T, \beta) + \epsilon_1))$ many elements of $\beta^n$. We will use Proposition 3.3 to cover efficiently each of those
atoms by dynamical balls. Let \( Q \in \beta^{m_i} \) be an atom intersecting \( X_i \), in particular \( Q \in K \cap T^{-(n-1)K} \). By the choice of \( \nu_{i_1} \) we have
\[
|E_{N,n_i}(Q)| < (\mu(\alpha_N) + \epsilon_1)n_i.
\]
We claim that \( m_{k,N,n_i}(Q) \leq \frac{1}{T^2} |Q| \). This follows from Remark 3.2. Let \([p, p + q]\) be an excursion of \( Q \) into \( \alpha_N \) that contain an excursion into \( \alpha_kN \). There exists a smallest \( h \geq 0 \) such that \( T^{p+q+k}Q \subseteq K \). By definition of \( \alpha_kN \) we have \( q + h + 1 \geq kN \). Moreover \( T^kQ \subseteq \alpha_N^c \) for every \( k \in [p + q + 1, p + q + h] \). In particular each excursion into \( \alpha_kN \) generates an interval of length at least \( kN \) where no other excursion into \( \alpha_N \) can occur. Putting all together, we get that \( N(n_i, r, 1 - \frac{4K}{T}) \) is bounded from above by
\[
 e^{n_i \log(\mu(\alpha_N) + \epsilon_1)} C_0 e^{n_i \log(\mu(\alpha_N) + \epsilon_1)} e^{\frac{1}{T}N(\delta_\mu(K, r) + \epsilon)}.
\]
Finally we obtained
\[
 h_\mu(T) \leq h_\mu(T, \beta_kN) + \epsilon_1 + (\delta_\mu(K, r) + \epsilon)(\mu(\alpha_N) + \epsilon_1) + \frac{1}{k}(\delta_\mu(K, r) + \epsilon).
\]
Since \( \epsilon_1 > 0 \) was arbitrary we are done. \( \square \)

We now explain how to get Theorem 3.1 from Proposition 3.4. First assume \( \mu(X) > 0 \), and fix \( \epsilon_0 > 0 \). We remark that by construction \( \mu(\beta\beta_{k,N}) = 0 \). To simplify notation we use \( \beta \) instead of \( \beta_{k,N} \) to denote our partition. Choose \( m \) sufficiently large such that
\[
 h_{\beta^m}(T) + \varepsilon_0 > \frac{1}{m} H_{\beta^m}(\beta^m), \quad 2^{e-1} < \frac{\varepsilon_0}{2},
\]
and \( -1/m \log \mu(X) < \varepsilon_0 \). Then
\[
 |\mu| h_{\beta^m}(T) + 2\varepsilon_0 > \frac{1}{m} \sum_{P \in \beta^m} \mu(P) \log \mu(P).
\]
Define \( A = \bigcap_{i=0}^{m-1} T^{-i} \alpha_kN \) and observe that by the definition of the vague convergence
\[
 \lim_{n \to \infty} \sum_{Q \in \beta^m \setminus \{A\}} \mu_n(Q) \log \mu_n(Q) = \sum_{Q \in \beta^m \setminus \{A\}} \mu(Q) \log \mu(Q).
\]
Choosing \( n \) sufficiently large we get the inequality
\[
 |\mu| h_{\beta^m}(T) + 3\varepsilon_0 > \frac{1}{m} H_{\mu_n}(\beta^m).
\]
Finally using Proposition 3.4 we get
\[
 \mu(X) h_{\beta^m}(T) + 3\varepsilon > \frac{1}{m} H_{\mu_n}(\beta^m) \geq h_{\mu_n}(T, \beta)
\]
\[
 > h_{\mu_n}(T) - (\delta_\mu(K, r) + \epsilon)\mu_n(\alpha_N) - \frac{1}{k}(\delta_\mu(K, r) + \epsilon).
\]
Observe that \( \bigcup_{n=0}^N C_n \) is relatively compact and that \( \mu_n(\alpha_N) = 1 - \mu_n(\bigcup_{n=0}^N C_n) \). We remark that by construction \( \mu(\beta\alpha_N) = 0 \). Therefore
\[
 \limsup_{n \to \infty} h_{\mu_n}(T) \leq \mu(X) h_{\beta^m}(T) + (\delta_\mu(K, r) + \epsilon)(1 - \mu(\bigcup_{n=0}^N C_n)) + \frac{1}{k}(\delta_\mu(K, r) + \epsilon).
\]
Observe that by construction we can send $\epsilon$ to zero as $N$ goes to infinity. Finally take $k \to \infty$ and $N \to \infty$. We obtained the desired inequality
\[ \limsup_{n \to \infty} h_{\mu_n}(T) \leq |\mu| h_{\mu_\nu}(T) + (1 - |\mu|) \delta_\infty(K, r). \]

The case when $\mu(X) = 0$ follows directly from Proposition 3.4 since $h_{\mu_n}(g_\beta) \to 0$ and $\mu_n(\alpha_N) = 1 - \mu_n(\bigcup_{s=1}^N C_s) \to 1$ as $n$ tends to $\infty$. \hfill \square

**Remark 3.5.** Theorem 3.1 still holds if the dynamical system satisfies a simplified entropy inequality (see Definition 1.1). For our applications to the geodesic flow we will verify that they satisfy a simplified entropy inequality after some mild modification in the definition of the dynamical balls.

4. Proof of the main result

We have finally all the ingredients to prove Theorem 1.4.

**Theorem 4.1.** Let $(X,d)$ be a non compact topological manifold and $T$ a $L$-Lipschitz homeomorphism with finite topological entropy. Assume that $(X,d,T)$ satisfies a simplified entropy formula and that its ergodic measures are weak entropy dense. Let $(\mu_n)_{n \geq 1}$ be a sequence of $T$-invariant probability measures converging to $\mu$ in the vague topology. Then
\[ \limsup_{n \to \infty} h_{\mu_n}(T) \leq |\mu| h_{\mu_\nu}(T) + (1 - |\mu|) \delta_\infty. \]

*Proof.* We will prove that
\[ (1) \quad \limsup_{n \to \infty} h_{\mu_n}(T) \leq |\mu| h_{\mu_\nu}(T) + (1 - \mu(A(K))) \delta_\infty(K, r), \]

for every sufficiently large compact set $K$ and $r > 0$. Let $\mu_0$ be a $T$-invariant measure that gives positive measure to $K$ (which exists because $K$ is sufficiently large) and define $\mu_\nu' = (1 - \frac{1}{n})\mu_n + \frac{1}{n}\mu_0$. By hypothesis, the space of ergodic measures is weak entropy dense (see Definition 1.2), therefore we can find an ergodic measure $\nu_n$ arbitrarily close to $\mu_\nu'$ such that $h_{\nu_n}(T) > h_{\mu_n}(T) - \frac{1}{n}$. In particular we can assume $\nu_n(K) > 0$ and that $\lim_{n \to \infty} \nu_n = \nu$. We can now use Theorem 3.1 to the sequence $\{\nu_n\}_{n \in \mathbb{N}}$ and get
\[ \limsup_{n \to \infty} h_{\nu_n}(T) \leq |\mu| h_{\mu_\nu}(T) + (1 - \mu(A(K))) \delta_\infty(K, r). \]

By construction we have
\[ h_{\nu_n}(T) > h_{\mu_\nu'}(T) - \frac{1}{n} = (1 - \frac{1}{n}) h_{\mu_n} + \frac{1}{n} h_{\mu_0}(T) - \frac{1}{n}, \]

and therefore
\[ \limsup_{n \to \infty} h_{\nu_n}(T) \geq \limsup_{n \to \infty} h_{\mu_n}(T), \]

which implies the inequality (2). Take an increasing sequence $(K_i)_{i \in \mathbb{N}}$ of compact sets such that $\lim_{i \to \infty} \delta_\infty(K_i) = \delta_\infty$. Now observe that $A(K_i) \subset A(K_{i+1})$ and $\bigcup_{i \geq 1} A(K_i) = X$. This implies that $\lim_{i \to \infty} \mu(A(K_i)) = \mu(X)$ and therefore inequality (2) implies the result. \hfill \square
5. Properties of the geodesic flow

We refer the reader to subsection 2.2 for notation and basic results about the geodesic flow. In this section we will prove that the geodesic flow on a pinched negatively curved manifold satisfies a mild modification of the simplified entropy inequality and that its ergodic measures are weak entropy dense.

5.1. Entropy density for geodesic flow. In this subsection we will check that the proof of Theorem B in [EKW] extends to the non-compact case for the geodesic flow on a negatively curved manifold. Our next result together with the upper semicontinuity of the entropy map implies that the set of ergodic measures in \((X, T)\) is entropy dense (see Theorem 8.9). We will need the following definitions.

**Definition 5.1** (Closing lemma). We say that a flow \((\varphi_t)_{t \in \mathbb{R}}\) on a metric space \((X, d)\) satisfy the closing lemma if for all \(x \in X\), there exists a neighborhood \(W_x\) of \(x\) such that the following holds. Given \(\epsilon > 0\), there exists \(\delta > 0\) such that for all \(y \in W_x\) and \(t \geq t_0\), if \(d(y, \varphi_t x) < \delta\) and \(\varphi_t y \in W_x\), then there exists \(y'\) and \(s > 0\) such that \(|t - s| < \epsilon\), \(\varphi_s y' = y'\) and \(d(\varphi_h y, \varphi_h y') < \epsilon\), for \(h \in (0, \min\{t, s\})\).

Define the following sets
\[
W^s_\epsilon(x) = \{y \in X : d(\varphi_t(x), \varphi_t(y)) \leq \epsilon, \text{ for all } t \geq 0\},
\]
\[
W^u_\epsilon(x) = \{y \in X : d(\varphi_t(x), \varphi_t(y)) \leq \epsilon, \text{ for all } t \leq 0\}.
\]

**Definition 5.2** (Local product structure). We say that the flow \((\varphi_t)_{t \in \mathbb{R}}\) admits a local product structure if for all \(x \in X\), there exists a neighborhood \(V_x\) of \(x\) such that the following holds. Given \(\epsilon > 0\), there exists \(\delta > 0\) such that for all \(y, z \in V_x\) satisfying \(d(x, y) < \delta\), there exists a point \(w \in y, z \in X\) and a real number \(t \in (-\epsilon, \epsilon)\) so that \(<y, z \in W^u_\epsilon(\varphi_t(x)) \cap W^s_\epsilon(y)\).

**Remark 5.3.** It is a standard fact (see for instance [Ba]) that the geodesic flow on a pinched negatively curved manifold is transitive, satisfies the closing lemma and admits local product structure. This properties are important for us because of the following result.

**Proposition 5.4.** Let \((X, d)\) be a metric space where closed balls are compact. Let \((\varphi_t)_{t \in \mathbb{R}}\) be a continuous flow which is transitive, admits local product structure and satisfies the closing lemma. Then for every measure \(\mu \in \mathcal{M}(X, \varphi_1)\) and \(\epsilon > 0\), there exists an ergodic measure \(\mu_e\) arbitrarily close to \(\mu\) (in the weak-* topology) such that \(h_{\mu_e}(\varphi_1) > h_\mu(\varphi_1) - \epsilon\). We can moreover assume that \(\text{supp}\mu_e\) is compact.

**Proof.** As in the proof of the entropy density in the compact case we start with the following general fact.

**Lemma 5.5.** [EKW, Proposition 6.1] Let \((X, d)\) be a metric space and \(T\) a continuous transformation. Given \(\mu \in \mathcal{M}(X, T)\), \(\alpha > 0\), \(\beta > 0\) and \(f_1, ..., f_l \in C_c(X)\), there exists \(n_0\) and \(\gamma > 0\) such that for all \(n \geq n_0\) there exists a \((n, \gamma)\) separated set \(S \subset X\) such that
\[
(1) \quad |S| \geq \exp(n(h_\mu(T) - \alpha)).
\]
\[
(2) \quad \left| \frac{1}{l} \sum_{i=0}^{l-1} f_j(T^k x) \right| < \beta, \text{ for all } x \in S \text{ and } j \in \{1, ..., l\}.
\]

Let \(K \subset X\) be a measurable set satisfying \(\mu(K) > 3/4\). Then we can moreover assume that \(S \subset K \cap T^{-(n+k_0)} K\). We can choose \(\gamma\) so that it does not depend on \(K\) and \(k_0\).
We only need to justify the last part of the proposition since points (1) and (2) are taken without modification from \([EKW]\). We follow the notation in the proof of \([EKW, \text{Proposition 6.1}]\) and modify the definition of \(F_n\) by the formula

\[
F_n = E_n \cap K \cap T^{-(n+k_0)}K \cap \{ x \in X : \frac{1}{n} \sum_{k=0}^{n-1} \chi_V(T^k x) \leq 2\delta \}.
\]

Since \(\mu(K \cap T^{-(n+k_0)}K) > \frac{1}{2}\), and the measure of the other two sets involved in the definition of \(F_n\), by Birkhoff ergodic theorem, tends to 1 as \(n\) goes to infinity, we conclude that for \(n\) sufficiently large we have \(\mu(F_n) > \frac{1}{2}\). With this definition of \(F_n\) the rest of the proof follows without modification the proof of \([EKW, \text{Proposition 6.1}]\).

We will use the notation \(T = \varphi_1\). We want to prove that given \(\mu \in \mathcal{M}(X,T)\), \(\epsilon > 0, \eta > 0\), and \(f_1, \ldots, f_l \in C_c(X)\), there exists an ergodic measure \(\mu_\epsilon \in V(f_1, \ldots, f_l; \mu, \epsilon)\) such that \(h_{\mu_\epsilon}(T) > h_\mu - \eta\). As in the proof of Theorem B \([EKW]\) we can reduce the problem to the case \(\mu = \frac{1}{N} \sum_{k=1}^{N} \mu_k\), where \(\{ \mu_k \}_{k=1}^{N}\) is a collection of ergodic measures. Fix \(K\) a compact set such that \(\mu_i(K) > 3/4\), for all \(i\). By the uniform continuity of \(f_i\)'s, there exists \(\epsilon_0 > 0\) such that if \(d(x,y) < \epsilon_0\), then \(|f_i(x) - f_i(y)| < \frac{\epsilon}{4}\). Choose \(n\) big enough and \(\gamma > 0\) such that there exist \((n, \gamma)\)-separated sets \(S_i \subset K \cap \varphi_{-\eta}K\) as in Lemma 5.5 for the measure \(\mu_i, \beta = \epsilon/4\) and \(\alpha = \eta/2\). By the definition of the local product structure and the compactness of \(K\), given \(\epsilon_0 > 0\) there exists \(\delta_0 > 0\) such that if \(x, y \in K\) satisfy \(d(x,y) < \delta_0\), then there exists a point \(<x, y>\) such that \(<x, y> \in W^u(\varphi_{\gamma}(x)) \cap W^s_{\epsilon_0}(y)\) and \(|t| < \epsilon_0\).

We will moreover assume that \(\epsilon_0\) satisfy \(\gamma > \epsilon_0/4\). By the transitivity of the flow and the compactness of \(K\), given \(\delta_0 > 0\) there exists a constant \(N = N(\delta_0)\) such that for every \(x, y \in K\), there exists \(z \in X\) such that \(d(x,z) < \delta_0\), and \(d(\varphi_p z, y) < \delta_0\) for some \(p \in [0, N]\). In particular if \(x, y \in \bigcup_{k=1}^{N} S_k\), there exists \(z = z(x, y) \in X\) such that \(d(\varphi_n x, z) < \delta_0\), and \(d(\varphi_p z, y) < \delta_0\), for some \(p = p(x,y) \in [0, N]\). By choosing \(n\) sufficiently large we can assume \(N(\delta_0)/n\) is very small to be determined later. Choose \((x_1, x_2, \ldots, x_{MN}) \in (\bigcup_{i=1}^{N} S_i)^M\). As in the proof of \([\text{CS, Proposition 3.2}]\), using the closing lemma and the local product structure, we can construct a periodic orbit that \(\epsilon_0\)-shadows the broken orbit

\[
W = O_0^n(x_1) \cup O_0^n(x_1, x_2) \cup O_0^n(x_2) \cup \ldots \cup O_0^n(x_n) \cup O_0^n(x_{MN}, x_1).
\]

We think of \(W\) is a parametrized map which represent the sequence of segments above. We denote this periodic orbit by \(w = w(x_1, \ldots, x_{MN})\). The period of \(w\) is approximately the domain of \(W\). By the choice of \(\epsilon_0\) (in terms of the sequence of functions \( \{f_i\}_{i=1}^l \)) and because \(N(\delta_0)/n\) is sufficiently small, one easily verifies that the periodic measure \(\mu_w\) associated to \(w\) belongs to \(V(f_1, \ldots, f_l; \mu, \frac{\epsilon}{4})\). Define \(S = \bigcup_{M \geq 1} (\bigcap_{i=1}^{N} S_i)^M\). Denote by \(O(w) \subset X\) to the orbit of \(w\). Fix some \(x_0 \in K\). There exists a constant \(L = L(n, N(\delta_0))\) such that \(O(w(x)) \subset B(x_0, L)\) for every \(x \in S\), where \(w(x)\) is the periodic orbit obtained from the construction above. In particular the set

\[
\Psi = \bigcup_{x \in S} O(w(x)),
\]

is relatively compact in \(X\) and \((\varphi_t)\)-invariant. This implies that \(\Psi_0 = \overline{\Psi}\) is compact and \((\varphi_1)\)-invariant. It is easy to see that because of the choice of \(\epsilon_0, \delta_0\) and the sets
(S_i)_{i=1}^N we have

\[ \Psi_0 \subset \{ x \in X : |\frac{1}{nN} \int_0^{nN-1} f_i(\varphi_t x) \, dt - \int f_id\mu| \leq \epsilon \}. \]

Since \( \Psi_0 \) is (\( \varphi_1 \))-invariant we get

\[ \Psi_0 \subset \{ x \in X : |\frac{1}{nN} \int_0^{nN-1} f_i(\varphi_{t+s} x) \, dt - \int f_id\mu| \leq \epsilon, \forall s \geq 0 \} \]

\[ \subset \{ x \in X : |\frac{1}{nNk} \int_0^{nNk-1} f_i(\varphi_t x) \, dt - \int f_id\mu| \leq \epsilon, \forall k \geq 1 \}. \]

This implies that every ergodic measure supported in \( \Psi_0 \) belongs to \( V(f_1, \ldots, f_i; \mu, \epsilon) \) (just take a generic point for the ergodic measure). Recall that \( \gamma > \epsilon_0/4 \). By construction if \( x, y \in (\prod_{i=1}^N S_i)^M \) and \( x \neq y \), then

\[ d_{M(n+N(\delta_0))}(w(x), w(y)) > \epsilon_0/2. \]

In other words \( \Psi_0 \) contains a \( (M(n + N(\delta_0)), \epsilon_0/2) \)-separated set of cardinality

\[ \exp(nM(\frac{1}{n} \sum_{k=1}^N h_{\mu_k}(\varphi_1) - \frac{\eta}{2})). \]

Then

\[ h_{top}(\Psi_0) \geq \limsup_{M \to \infty} \frac{nM(h_{\mu}(\varphi_1) - \frac{\eta}{2})}{M(n + N(\delta_0))}. \]

By choosing \( N(\delta_0)/n \) sufficiently small we can make the LHS in the last inequality strictly bigger than \( h_{\mu}(\varphi_1) - \eta \). Finally take an ergodic measure \( \mu_e \) supported in \( \Psi_0 \) with entropy at least \( h_{\mu}(\varphi_1) - \eta \), since we had already proved that \( \mu_e \in V(f_1, \ldots, f_i; \mu, \epsilon) \), this finishes the proof. \( \square \)

5.2. **Simplified entropy formula for geodesic flows.** We saw in the proof of Theorem 3.1 that a simplified entropy formula helps to understand the behaviour of the entropy map. In fact we only require the inequality

\[ h_\mu(T) \leq \limsup_{n \to \infty} \frac{1}{n} \log N_\mu(n, \epsilon, \delta), \]

to be able to prove Theorem 3.1. We will now proceed to prove that the geodesic flow on a negatively curved manifold satisfies this inequality after a small modification in the definition of the dynamical balls. In this section \( (M, g) \) will always stand for a complete negatively curved Riemannian manifold. We will moreover assume that the sectional curvature is pinched between two negative constants and that the derivative of the sectional curvature is uniformly bounded. For short we will say that \( (M, g) \) is negatively curved, but we emphasize those hypothesis are always assumed. Let \( p : T^1 \tilde{M} \to T^1 M \) be the canonical projection, where \( \tilde{M} \) is the universal cover of \( M \).

**Definition 5.6.** Let \( y \in T^1 \tilde{M} \) and \( x = p(y) \), we define \( B_p^n(x, r) \) as the image under \( p \) of the \( (n, r) \)-dynamical ball in \( T^1 \tilde{M} \) centered at \( y \). We say that \( B_p^n(x, r) \) is the \( p(n, \epsilon) \)-dynamical ball centered at \( x \), where \( p \) stands for projection.

We recall that \( N_\mu(n, \epsilon, \delta) \) is the minimum number of \( (n, \epsilon) \)-dynamical balls needed to cover a set of measure strictly bigger than \( 1 - \delta \). We define \( N_p^n(n, \epsilon, \delta) \)
as the minimum number of $p(n, \epsilon)$-dynamical balls needed to cover a set of measure strictly bigger than $1 - \delta$. We also use the notation $N^p(C, n, \epsilon)$ to denote the minimum number of $p(n, \epsilon)$-dynamical balls needed to cover the set $C \subset T^1 M$. To simplify notation we define $X = T^1 M$ (and to be consistent with the notation before). To prove the main result of this subsection we will need the following lemma, that follows easily from the criterion given in [PS].

**Lemma 5.7.** Let $(M, g)$ be a negatively curved manifold. Then there exists a Hölder continuous function $\varphi : T^1 M \to \mathbb{R}$ which admits an equilibrium state.

**Proposition 5.8.** Let $(M, g)$ be a negatively curved manifold. Then for every ergodic measure $\mu$ have

$$h_\mu(g_1) \leq \liminf_{n \to \infty} \frac{1}{n} \log N^p_\mu(n, r, \delta),$$

where $\delta \in (0, 1)$ and $r > 0$.

**Proof.** Fix $\delta \in (0, 1)$, $r > 0$ and $r' \in (0, r)$.

Choose $m \in \mathbb{N}$ such that $1 - \delta > \frac{1}{m}$. Let $E_n \subset X$ be a set satisfying $N^p_\mu(n, r, 1 - \frac{1}{m}) = N^p(F_n, r, 1 - \frac{1}{m})$ and $\mu(F_n) > \frac{1}{m}$. By Birkhoff ergodic theorem there exists $F' \subset X$ and $N_0 > 0$ such that $\mu(F') > 1 - \frac{1}{8m}$, and $\left| \frac{1}{n} \sum_{t=0}^{n-1} \varphi(g_t x) dt - \int \varphi d\mu \right| < \epsilon$, for every $x \in F'$ and $n \geq N_0$. From now on we will assume $n \geq N_0$. Let $K$ be a compact subset such that $\mu(K) > 1 - \frac{1}{8m}$. We will need the following fact, which follows directly from the formula $\mu(A_1 \cup \ldots \cup A_t) = \sum_{i=1}^{n-1} (-1)^{i+1} \sum \mu(A_{i_1} \cap \ldots \cap A_{i_t})$.

**Lemma 5.9.** Let $F$ be a measurable set satisfying $\mu(F) > \frac{1}{2}$. Then for every $h \in \mathbb{Z}$ there exists $k \in [h, h + 2s)$ such that $\mu(F \cap g^{-k} F) > \frac{1}{2m}$.

Define $S_n = F_n \cap K \cap F'$ and observe that by construction $\mu(S_n) > \frac{1}{2m}$. Then there exists $k_n \in (-4m, 0]$ such that

$$\mu(S_n \cap g^{-(n-1+k_n)} S_n) > \frac{1}{24m}.$$  

Define $A_n = S_n \cap g^{-(n-1+k_n)} S_n$. In particular we get

$$N^p_\mu(n, r', 1 - \frac{1}{24m}) \leq N^p(A_n, n, r').$$

Consider a $p(n, r)$-covering with minimal cardinality of $A_n$ and denote by $R$ the set of centers of such dynamical balls. For each $x \in R$, let $E_x$ be a $p(n, r')$-separated set of maximal cardinality in $B^n_p(x, r)$. We can moreover assume that $E_x \subset A_n$. By definition, the $p(n, r'/2)$-dynamical balls with centers in $E_x$ are disjoint. Moreover, since $\#E_x$ is maximal, the collection of $p(n, r')$-dynamical balls having centers in $E_x$ is a $p(n, r')$-covering of $B^n_p(x, r)$. Define $Y = \bigcup_{y \in E_x} g_y K$. Since $K$ is compact, the same holds for $Y$. Let $\varphi : T^1 M \to \mathbb{R}$ be a Hölder continuous function admitting an equilibrium state. Let $m$ be its (unique) equilibrium state. Therefore

$$\sum_{y \in E_x} m(B^n_p(y, r'/2)) = m \left( \bigcup_{y \in E_x} B^n_p(y, r'/2) \right) \leq m(B^n_p(x, r + r')),$$

and so

$$\#E_x \leq \frac{m(B^n_p(x, r + r'))}{\min_{y \in E_x} m(B^n_p(y, r'/2))}.$$
Observe that by construction if $x \in A_n$, then $g_{n-1+k_n}(x) \in S_n$, which implies $g_n(x) \in Y$. In particular $A_n \subset Y \cap g_{-n}Y$. Recall that $m$ satisfies the Gibbs property, i.e. there exists a constant $C = C(Y, r + r', r'/2)$ such that

$$C^{-1} \leq \frac{m(B^n_g(y, r_0))}{\exp(\int_0^{n-1} \phi(g_{ny})dt - np(\phi))} \leq C,$$

for every $y \in Y \cap g_{-n}Y$ and $r_0 \in \{r + r', r'/2\}$. Using the notation above this implies the bound

$$\#E_x \leq C^2 \exp\left(\int_0^{n-1} \phi(g_{ny})dt - \min_{y \in E_x} \int_0^{n-1} \phi(g_{ny})dt\right).$$

Therefore, by the definition of $F'$, we have

$$\#E_x \leq C' \exp(2n\epsilon).$$

Observe that

$$N^p_\mu(n, r', 1 - \frac{1}{24m}) \leq N^p(A_n, n, r') \leq C' \exp(2n\epsilon)\#R = C' \exp(2n\epsilon)N^p(A_n, n, r)$$

$$\leq C' \exp(2n\epsilon)N^p(F_n, n, r) = C' \exp(2n\epsilon)N^p_\mu(n, 1 - \frac{1}{m})$$

$$\leq C' \exp(2n\epsilon)N^p_\mu(n, r, \delta).$$

We remark that $C'$ is independent of $n$, it depends on $r$, $m$ and $K$. Then

$$\liminf_{n \to \infty} \frac{1}{n} \log N^p_\mu(n, r', 1 - \frac{1}{24m}) \leq \liminf_{n \to \infty} \frac{1}{n} \log N^p_\mu(n, r, \delta) + 2\epsilon,$$

But $\epsilon > 0$ was arbitrary. Observe that

$$h(\mu) = \lim_{r' \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N^p_\mu(n, r', 1 - \frac{1}{24m}) \leq \liminf_{n \to \infty} \frac{1}{n} \log N^p_\mu(n, r', 1 - \frac{1}{24m}).$$

Therefore

$$h(\mu) \leq \liminf_{n \to \infty} \frac{1}{n} \log N^p_\mu(n, r, \delta).$$

\[\square\]

6. Entropy map for geodesic flows

In this section we will discuss how Theorem 1.4 applies to the geodesic flow on a non-positively curved manifold. We will first focus on the negatively curved case.

6.1. Geodesic flow on negatively curved manifold. Let $(M, g)$ be a complete Riemannian manifold satisfying $-a < K_g < -b < 0$, where $K_g$ is the sectional curvature of $(M, g)$. We will moreover assume that the curvature has uniformly bounded derivative (this ensure that the horospherical foliations are Hölder). As mentioned in Remark 5.3, the geodesic flow on a negatively curved manifold satisfy the hypothesis of Proposition 5.4. In particular we know that the geodesic flow is weak entropy dense. It was proved in Proposition 5.8 that for every measure $\mu$ in $\mathcal{M}_e(T^1M, \phi_1)$ the following inequality holds

$$h(\mu) = \lim_{r' \to 0} \frac{1}{n} \log N^p_\mu(n, r, \delta),$$

where $r > 0$ and $\delta \in (0, 1)$. If one follows the proof of Theorem 3.1 one can easily verify that all the arguments run just the same when using $p(n, r)$-dynamical balls instead of $(n, r)$-dynamical balls. It is important to see that the definition of $\delta_\infty$ also
slightly changes. For completeness we will define the critical exponent at infinity of the geodesic flow. Let $K$ be a compact set of $T^1M$. Define

$$K(n) = K \cap \bigcap_{i=1}^{n-2} g_{-i}K^c \cap g_{-(n-1)}K.$$  

Given $x \in K$ and $r > 0$ we define $C^p(x, n, r)$ as the number $p(n, r)$-dynamical balls needed to cover $B(x, r) \cap K(n)$ (see Definition 5.6 for the meaning of a $p(n, r)$-dynamical ball). We emphasize that we are covering this set with $p(n, r)$-dynamical balls, and not with $(n, r)$-dynamical balls. We then define

$$\delta_X(K, r) = \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in K} C^p(x, n, r).$$

**Definition 6.1.** The critical exponent at infinity of $M$ is the number

$$\delta_X = \inf \liminf_{\{K_n\}, n \to \infty} \inf_{r > 0} \delta_X(K_n, r),$$  

where the infimum runs over sequences $\{K_n\}_{n \in \mathbb{N}}$, where each $K_n$ is compact, $K_n \subset K_{n+1}$ and $\Omega = \bigcup_{n \geq 1} K_n$.

**Remark 6.2.** As mentioned in subsection 1.3, Definition 6.1 is equivalent to Definition 1.7. This can be deduced from Lemma 6.5 below, so it also holds for the geodesic flow on a non-positively curved manifold.

Using Proposition 5.8 we can follow the proof of Theorem 3.1 and obtain the result for the critical exponent defined in this section. Then combining Proposition 5.4 with the proof of Theorem 4.1 we obtain one of the main results of this paper.

**Theorem 6.3.** Let $(M, g)$ be a pinched negatively curved manifold. Let $(\mu_n)_{n \geq 1}$ be a sequence of $g_1$-invariant probability measures converging to $\mu$ in the vague topology. Then

$$\limsup_{n \to \infty} h_{\mu_n}(g_1) \leq |\mu|h_{\mu}(g_1) + (1 - |\mu|)\delta_X.$$  

If the sequence converges vaguely to zero, then the RHS is understood as $\delta_X$.

We emphasize that $\delta_X \leq \delta_1$, which is finite because of the bounded curvature. We remark that a very similar statement was proved in [RV, Theorem 1.1] when $(M, g)$ is a geometrically finite manifold. In this case it is easy to verify that our critical exponent $\delta_X$ corresponds exactly with $\delta_P$, the maximum of the critical exponent of the parabolic subgroups of $\pi_1(M)$. We will see in Section 7 that the constant $\delta_X$ is sharp.

**6.2. Geodesic flow on rank one non-positively curved manifolds.** Let $(M, g)$ be a complete Riemannian manifold with non-positive sectional curvature. We will assume that the sectional curvature is bounded below. A key ingredient used in the proof of the inequality (2) is the existence of a Gibbs measure. If we only assume that $(M, g)$ is non-positively curved it is not known the existence of Gibbs measures. In this situation we will argue differently. We will use the same notation as in previous sections.
Proposition 6.4. Let \((M, g)\) be a non-positively curved manifold with sectional curvature bounded below. Then

\[
h_\mu(g_1) \leq \liminf_{n \to \infty} \frac{1}{n} \log N_\mu^p(n, r, \delta),
\]

for every \(r > 0\) and \(\delta \in (0, 1)\).

Proof. As before we denote by \(\widetilde{M}\) to the universal cover of \(M\). We will use the notation \(X = T^1M\) and \(\widetilde{X} = T^1\widetilde{M}\). We start with the following lemma about the geometry of \(\widetilde{X}\).

Lemma 6.5. Given \(r, r' > 0\), there exists \(L(r, r')\) such that any ball of radius \(r\) in \(T^1\widetilde{M}\) can be cover with less than \(L(r, r')\) \((n, r')\)-dynamical balls.

Proof. We will make use of the following two important facts.

Fact 1: The non-positive curvature and the curvature bounded below has the following implication. Given \(r_0 > 0\), there exists \(\epsilon(r_0)\) such that the following holds. For every pair of points \(x, y \in \widetilde{X}\) such that \(d(x, y) < \epsilon(r_0)\) and \(d(g_{n-1}(x), g_{n-1}(y)) < \epsilon(r_0)\), then \(d_n(x, y) < r_0\).

Fact 2: The bounded geometry of \((\widetilde{M}, g)\) implies that given \(r_1, r_2 > 0\), there exists a constant \(C(r_1, r_2)\) such that for every point \(x \in \widetilde{X}\) we have that \(B(x, r_1)\) can be covered with less than \(C(r_1, r_2)\) balls of radius \(r_2\).

This two facts imply the lemma. Pick a point \(x \in \widetilde{X}\) and cover \(B(x, r)\) and \(B(g_{n-1}(x), r)\) by balls of radius \(\epsilon(r'/2)\). We can cover each of them with less than \(C(r, \epsilon(r'/2))\) balls. Denote by \(\{B_i\}\) and \(\{B'_i\}\) the families of balls of radius \(\epsilon(r'/2)\) covering \(B(x, r)\) and \(B(g_{n-1}(x), r)\) respectively. The covering \(\mathcal{U} = \{B_i \cap g_{-(n-1)}B'_j\}\) of \(B(x, r)\) has at most \(C(r, \epsilon(r'/2))^2\) elements. By the definition of \(\epsilon(r'/2)\) it follows that each element in \(\mathcal{U}\) is contained in a \((n, r')\)-dynamical ball of \(\widetilde{X}\). It follows that \(B(x, r)\) can be covered by at most \(C(r, \epsilon(r'/2))^2\) \((n, r')\)-dynamical balls, and that this is independent of \(x\). \(\square\)

By definition of the \(p(n, r)\)-dynamical balls, there is a one to one correspondence between dynamical balls in \(\widetilde{X}\) and \(p(n, r)\)-dynamical balls. In particular, using Lemma 6.5 we know that a \(p(n, r)\)-dynamical ball can be covered by \(C(r, r')\) \((n, r')\)-dynamical balls. This implies that for \(r' \in (0, r]\) we have

\[
\liminf_{n \to \infty} \frac{1}{n} \log N_\mu^p(n, r', \delta) \leq \liminf_{n \to \infty} \frac{1}{n} \log N_\mu^p(n, r, \delta).
\]

Finally be obtain

\[
h_\mu(g_1) = \lim_{r' \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N_\mu(n, r', \delta) \leq \lim_{r' \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N_\mu^p(n, r', \delta) \leq \lim_{n \to \infty} \frac{1}{n} \log N_\mu(n, r, \delta).
\]

We remark that the topological entropy of the geodesic flow on \(M\) is bounded above by \(\delta_1\), in particular, because of the bounded curvature, is finite. If \(M\) is compact, then the topological entropy of the geodesic flow is \(\delta_1\) by a result of Manning [M]. A direct application of Proposition 6.4 is the following result.
Corollary 6.6. Let \((M, g)\) be a non-positively curved manifold with curvature bounded below. Let \((\mu_n)_{n \in \mathbb{N}}\) a sequence of ergodic probability measures converging vaguely to \(\mu\). Assume \(\mu\) is not the zero measure. Then
\[
\limsup_{n \to \infty} h_{\mu_n}(g_1) \leq |\mu|h_{\mu/(\mu)}(g_1) + (1 - |\mu|)\delta_{x, K}.
\]

Proof. If \(K\) is big enough, then \(\mu(K) > 0\), in particular \(\mu_n(K) > 0\) for every \(n\) sufficiently large. Using Proposition 6.9 and Theorem 3.1 we get
\[
\limsup_{n \to \infty} h_{\mu_n}(g) \leq |\mu|h_{\mu/(\mu)}(g) + (1 - \mu(A(K)))\delta_{x, K}(K, r).
\]
Since \(\bigcup_{K \text{ compact}} A(K) = X\), by making \(K\) growth we get the desired inequality. \(\square\)

Remark 6.7. We emphasize that in Corollary 6.6 the critical exponent \(\delta_{x, K}\) is the quantity defined in Definition 1.7 (see also Remark 6.2).

To prove that the entropy map is upper semicontinuous we need to be able to push this result to non-ergodic measures. We will explain how to partially solve this problem in the rank one situation. We say that a vector \(v\) has rank one, if the parallel Jacobi fields along the geodesic with initial condition \(v\) are proportional to the flow direction. We say that \(M\) is rank one if there exists a rank one vector in \(T^1M\). Rank one non-positively curved manifolds are in many senses similar to negatively curved manifolds. For a complete discussion about the dynamics of the geodesic flow on rank one manifolds we refer the reader to [Kn2]. From now on we will assume that \(M\) is rank one non-positively curved. As before the geodesic flow on \(T^1M\) is denoted by \((g_t)_{t \in \mathbb{R}}\). We denote by \(\widehat{M}\) the universal cover of \(M\). The geodesic flow on \(T^1\widehat{M}\) is still denoted by \((g_t)\). Similar to the definition of the sets \(W^{ss}_{x}(\phi)\) and \(W^{su}(x)\) (see Section 5.1) we define the sets
\[
W^{ss}(x) = \{y \in X : \lim_{t \to \infty} d(\phi_t(x), \phi_t(y)) = 0\},
\]
\[
W^{su}(x) = \{y \in X : \lim_{t \to \infty} d(\phi_t(x), \phi_t(y)) = 0\}.
\]

Following [CS] we define the following subset of the nonwandering set of the geodesic flow of \(M\).

Definition 6.8. Let \(v\) be a non-wandering rank one vector in \(T^1M\) and \(\overline{v}\) a lift of \(v\) to \(T^1\widehat{M}\). We say that \(v\) belongs to \(\Omega^+_1\) if the following holds. For every \(\overline{w} \in T^1\widehat{M}\) such that \(d(g_t(\overline{w}), g_t(\overline{v}))\) is uniformly bounded for \(t \geq 0\), then there exists \(t_0 \in \mathbb{R}\) such that \(g_{t_0}(\overline{w}) = W^{ss}(\overline{v})\). The set \(\Omega^-_1\) is defined analogously when replacing the condition \(t \geq 0\) by \(t \leq 0\), and \(W^{ss}(\overline{v})\) by \(W^{su}(\overline{v})\). We finally define
\[
\Omega_1 = \Omega^+_1 \cap \Omega^-_1.
\]

The following result was proved in [CS, Proposition 5.1] and [CS, Proposition 5.2].

Proposition 6.9. Let \((M, g)\) be a rank one non-positively curved manifold with sectorial curvature bounded below. Then the restriction of the geodesic flow to \(\Omega_1\) is transitive, admits a local product structure and satisfies the closing lemma.

Definition 6.10. Let \(\Omega\) be the non-wandering set of the geodesic flow on \(M\). We define the regular part of \(\Omega\) as the space of recurrent rank one vectors and it is denoted by \(\Omega_{reg}\). The singular part of \(\Omega\) is \(\Omega_{sing} = \Omega \setminus \Omega_{reg}\).
It follows from [Kn2, Proposition 4.4, Section 5] that $\Omega_{\text{reg}} \subset \Omega_1$. We say that $\mu \in \mathcal{M}(g_1)$ is supported in the regular part of the geodesic flow if $\mu(\Omega_{\text{reg}}) = 1$. As a corollary we obtain that $(\Omega_{\text{reg}}(g_t))_{t \in \mathbb{R}}$ satisfies the hypothesis of Proposition 5.4.

In particular $(\Omega_{\text{reg}}(g_t))_{t \in \mathbb{R}}$ is weak entropy dense. This, together with Corollary 6.6, implies the following result.

**Theorem 6.11.** Let $(M, g)$ be a non-positively curved rank one manifold with curvature bounded below. Let $(\mu_n)_{n \in \mathbb{N}}$ a sequence of $g_1$-invariant probability measures supported on the regular part of the geodesic flow converging vaguely to $\mu$. Then

$$\limsup_{n \to \infty} h_{\mu_n}(g_1) \leq |\mu|h_{\mu_1}(g_1) + (1 - |\mu|)\delta_\infty.$$

In particular the entropy map of the dynamical system $(\Omega_{\text{reg}}, g_1)$ is upper semicontinuous.

**Remark 6.12.** We say that a closed geodesic is regular if it is generated by a rank one vector. It follows from the definition of the regular part of the geodesic flow that all regular closed geodesics belong to $\Omega_{\text{reg}}$. It is proven in [Kn] that if $M$ is compact, then the topological entropy of $\Omega_{\text{reg}}$ is the topological entropy of the geodesic flow. The uniqueness of the measure of maximal entropy (also proved in [Kn]) implies that set of singular vectors (not rank one) form a closed subset of $T^1M$ with strictly less topological entropy.

7. Entropy at infinity for negatively curved manifolds

We start with the following definition.

**Definition 7.1.** We denote by $C_0(T^1M)$ to the completion of $C_c(T^1M)$, i.e. the space of continuous functions that vanish at infinity.

To verify that the entropy at infinity is exactly $\delta_\infty$ we will use our next lemma. Recall that if $Q \subset \widehat{M}$ is a compact subset we defined

$$\Gamma_Q(n) = \{\gamma \in \Gamma : \exists x \in T^1Q \text{ such that } g_k(x) \in T^1Q^c, \text{ for } k \in [1, n-2] \text{ and } g_{n-1}(x) \in T^1\gamma Q\},$$

and then defined

$$\delta_\infty(Q) = \limsup_{n \to \infty} \frac{1}{n} \log \#\Gamma_Q(n).$$

**Lemma 7.2.** Let $\varphi : T^1M \to \mathbb{R}$ be a Hölder continuous function in $C_0(T^1M)$. Then

$$P(\varphi) \geq \delta_\infty.$$

**Proof.** Let $N$ be a Dirichlet fundamental domain of $M$ in $\widehat{M}$. Given $\epsilon > 0$, there exists a compact subset $K = K(\epsilon)$ of $M$ such that $\varphi(x) \in [-\epsilon, \epsilon]$, for every $x \in T^1K^c$. Let $Q = Q(\epsilon)$ be the subset of $N$ corresponding to $K$ and $M = \sup_{x \in T^1M} |\varphi(x)|$.

Choose a reference point $z \in T^1Q$.

$$\sum_{\gamma \in \Gamma} \exp(\int_z^{\gamma z} \varphi - s) \geq \sum_{n \geq 1} \sum_{\gamma \in \Gamma_Q(n)} \exp(\int_z^{\gamma z} \varphi - s) \geq e^{-2M} \sum_{n \geq 1} \sum_{\gamma \in \Gamma_Q(n)} e^{-\epsilon(n-3)} \exp(-sd(z, \gamma z)).$$
Let $C = C(Q)$ be the diameter of $Q$. By the definition of $\Gamma_Q(n)$ we have that $d(z, \gamma z) < n + 2C$, for every $\gamma \in \Gamma_Q(n)$. This implies that
\[
\sum_{\gamma \in \Gamma} \exp(\int z \tilde{\varphi} - s) \geq e^{-2M} \sum_{n \geq 1} \sum_{\gamma \in \Gamma_Q(n)} e^{-\epsilon(n-3)} \exp(-sd(z, \gamma z)) \\
\geq e^{-2M}e^{-2s\epsilon} \sum_{n \geq 1} \#\Gamma_Q(n) \exp(-n(s + \epsilon)).
\]
By definition if $\delta_\varphi(Q)$ there exists an increasing sequence $(n_i)_{i \in \mathbb{N}}$ such that $\#\Gamma_Q(n_i) \geq \exp(n_i(\delta_\varphi(Q) - \epsilon))$. In particular we get
\[
\sum_{\gamma \in \Gamma} \exp(\int z \tilde{\varphi} - s) \geq e^{-2M}e^{-2s\epsilon} \sum_{i \geq 1} \exp(n_i(-s + \delta_\varphi(Q) - 2\epsilon)).
\]
Observe that if $s < \delta_\varphi(Q) - 2\epsilon$, then the RHS diverges, so does the LHS. In particular we obtain that
\[
P(\varphi) \geq \delta_\varphi(Q) - 2\epsilon.
\]
Taking an increasing sequence $(Q_k)$ such that $\delta_\varphi(Q_k)$ converges to $\delta_\varphi$ (and such that $Q_k$ contains $Q$) we obtain $P(\varphi) \geq \delta_\varphi + \epsilon$, but $\epsilon$ was arbitrary.

**Proposition 7.3.** Let $M$ be a negatively curved manifold. Then the entropy at infinity of the geodesic flow on $M$ is $\delta_\varphi$.

*Proof.* If follows from Theorem 6.3 that if a sequence $(\mu_n)_{n \in \mathbb{N}}$ converges vaguely to the zero measure, then
\[
\limsup_{n \to \infty} h_{\mu_n}(g_1) \leq \delta_\varphi.
\]
This implies that the entropy at infinity of the geodesic flow is at most $\delta_\varphi$. To prove the converse inequality we start with a strictly positive Hölder continuous function $\varphi \in C_0(T^1M)$. The variational principle implies the existence of $g_1$-invariant probability measures $(\mu_n)_{n \in \mathbb{N}}$ such that
\[
P(-n\varphi) - \frac{1}{n} \leq h_{\mu_n}(g_1) - n \int \varphi d\mu_n.
\]
Using Lemma 7.2 we get
\[
n \int \varphi d\mu_n \leq h_{\text{top}}(g_1) + \frac{1}{n} - \delta_\varphi.
\]
Let $K$ be a compact set in $T^1 M$ and set $m = \min_{x \in K} |\varphi(x)|$. Therefore
\[
\mu_n(K) \leq \frac{C}{nm},
\]
which implies that $(\mu_n)_{n \in \mathbb{N}}$ converges vaguely to zero. Finally observe that inequality (3) implies
\[
\delta_\varphi \leq h_{\mu_n}(g_1) + \frac{1}{n},
\]
and therefore
\[
\delta_\varphi \leq \limsup_{n \to \infty} h_{\mu_n}(g_1).
\]
This gives us that the entropy at infinite is at least $\delta_\varphi$, which finishes the proof. $\square$
8. Some consequences

In this section we will state several consequences of the results proved in previous sections. We start with the following definition.

**Definition 8.1.** We say that a non-positively curved manifold $(M, g)$ satisfies a critical gap condition if $\delta_x < \delta_\Gamma$ (for the definition of $\delta_x$ see Definition 1.7).

This definition is also relevant in the recent work of Schapira and Tapie [ST]. They proved that for manifolds satisfying the critical gap condition, the topological entropy varies $C^1$ in the $C^1$ topology on the space of metrics.

**Theorem 8.2** (Criterion for existence of measure of maximal entropy). Let $(M, g)$ be a pinched negatively curved manifold satisfying a critical gap condition. Then the geodesic flow on $M$ admits a measure of maximal entropy.

For a proof it is enough to take a sequence $(\mu_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} h_{\mu_n}(g_1) = \delta_\Gamma$. The critical gap $\delta_x < \delta_\Gamma$ and Theorem 6.3 implies that the sequence does not lose mass, and that any weak-* limit must satisfy $h_\mu(g_1) = \delta_\Gamma$. We remark that by [OP] this measure is unique. The same argument gives us a slightly more general property.

**Proposition 8.3.** Let $(M, g)$ be a pinched negatively curved manifold.

1. Suppose that $M$ satisfies a critical gap condition. Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of $g_1$-invariant probability measures such that
   \[ \lim_{n \to \infty} h_{\mu_n}(g_1) = \delta_\Gamma. \]
   Then $(\mu_n)_{n \in \mathbb{N}}$ converges to the measure of maximal entropy.

2. Suppose that the geodesic flow on $M$ does not admit a measure of maximal entropy. Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of $g_1$-invariant probability measures such that $\lim_{n \to \infty} h_{\mu_n}(g_1) = \delta_\Gamma$. Then $(\mu_n)_{n \in \mathbb{N}}$ converges vaguely to zero. In this case we have $\delta_x = \delta_\Gamma$.

3. Suppose that the geodesic flow on $M$ admits a measure of maximal entropy. Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of $g_1$-invariant probability measures such that $\lim_{n \to \infty} h_{\mu_n}(g_1) = h_{\text{top}}(g_1)$. Then the accumulation points of $(\mu_n)_{n \in \mathbb{N}}$ lies in the set $\{ t \mu_{\text{max}} : t \in [0,1] \}$, where $\mu_{\text{max}}$ is the measure of maximal entropy.

The fact that a critical gap condition implies the existence of a measure of maximal entropy was first proven in [DOP] for geometrically finite manifolds. In this context the critical gap condition has a very explicit form. Define

\[ \overline{\delta}_P = \max_P \delta_P, \]

where the supremum runs over parabolic subgroups of $\pi_1(M)$. Since the non-compactness of the non-wandering set comes from the geometry at the cusps, and each of them corresponds to a maximal parabolic subgroup of $\pi_1(M)$, it is easy to check that $\delta_x = \overline{\delta}_P$. In [DOP] it is proven that $\overline{\delta}_P < \delta_\Gamma$ implies the existence of a measure of maximal entropy. This property was later generalized to incorporate Hölder potentials. It was proven in [PPS] that if $M$ is geometrically finite and

\[ \max_{P \text{ parabolic}} \delta_P^F < \delta_\Gamma^F, \]


then $F$ admits an equilibrium state. A final generalization was done in [PS] where Schapira and Pit were able to incorporate any pinched negatively curved manifold. We would like to emphasize that Theorem 8.2 also follows from [PS, Theorem 2]. To make sense of their result we start by recalling some notation used in [PS]. Given $\widehat{W} \subset \widehat{M}$ we define $\Gamma_{\widehat{W}}$ as the set of elements in $\Gamma$ such that there exists a geodesic starting at $\widehat{W}$ and finishing at $\gamma \widehat{W}$ that only meet $\Gamma_{\widehat{W}}$ at the beginning and at the end of its trajectory. Recall that if $F$ is a potential in $T^1M$ we denote by $\widehat{F}$ to its lift to $T^1\widehat{M}$ and that $\Omega$ is the non-wandering set of the geodesic flow. Let $\mathcal{P}$ be the set of closed geodesics and $n_W(p)$ the number of times a geodesic $p \in \mathcal{P}$ crosses $W$.

**Definition 8.4.** A potential $F : T^1M \to \mathbb{R}$ is said to be recurrent if there exists an open relatively compact subset $W \subset M$, such that $T^1W \cap \Omega \neq \emptyset$, and

$$\sum_{p \in \mathcal{P}} n_W(p) \exp(\int_p F - P(F)) = \infty.$$ 

**Definition 8.5.** We say that the pair $(\Gamma, \widehat{F})$ is positive recurrent with respect to $\widehat{W} \subset \widehat{M}$ if the following properties hold.

1. $T^1\widehat{W}$ has non-empty intersection with the lift of $\Omega$ to the universal cover.
2. $F$ is a recurrent potential.
3. There exists $x \in \widehat{M}$ such that $\sum_{\gamma \in \Gamma_{\widehat{W}}} d(x, \gamma x) \exp(\int_0^x \widehat{F} - P(F))$, is finite.

**Theorem 8.6.** [PS, Theorem 2] Let $M$ be a negatively curved manifold with pinched negative sectional curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder potential with finite pressure. Then

1. If $F$ is recurrent and $(\Gamma, \widehat{F})$ is positive recurrent with respect to some open relatively compact set $\widehat{W} \subset \widehat{M}$, then $m_F$ is finite.
2. If $m_F$ is finite, then $F$ is recurrent and $(\Gamma, \widehat{F})$ is positive recurrent with respect to any open relatively compact set $\widehat{W} \subset \widehat{M}$ meeting the projection to $\widehat{M}$ of the lift of $\Omega$ to $T^1\widehat{M}$.

If we specialize their result to $F = 0$ we obtain a very strong criterion for the existence of measure of maximal entropy. It is not hard to prove that the critical gap $\delta_X < \delta_1$, implies that $(\Gamma, 0)$ is positive recurrent. It worth mentioning that in the symbolic setting, potentials with critical gap are called strongly positive recurrent (in Sarig’s terminology [Sa2]). We remark that for locally constant potentials they were studied earlier by Gurevic and Savchenko [GS] by the name of stable-positive potentials. One of the main features of the family of strongly positive recurrent potentials is that it is very robust, it was proven in [CSa] that they form a dense and open subset of the space of weakly Hölder potentials (in many topologies, in particular with the $C^0$ norm). Potentials that are not strongly positive recurrent can be perturb (in topologies of perturbations of finite support) to transient potentials, see [CSa]. We recall that transient potentials are (weakly) Hölder potentials that are not recurrent, which is the meaning we also give to transient for the geodesic flow. To be coherent with the definition of critical gap in the case $F = 0$ and with the critical gap condition in the geometrically finite case we define

**Definition 8.7.** Given an open relatively compact subset $\widehat{W} \subset \widehat{M}$, such that $T^1\widehat{W}$ has non-empty intersection with the lift of $\Omega$ to the universal cover. Let $\delta^F_\infty(\widehat{W})$ be
the critical exponent of the Poincaré series

\[ P(s) = \sum_{\gamma \in \Gamma} \exp \left( \int_{x}^{\gamma x} \tilde{F} - s \right). \]

Then define \( \delta_{\Gamma, \infty}^F = \inf_{\tilde{W}} \delta_{\tilde{W}}^F (\tilde{W}) \). We say that \((\Gamma, \tilde{F})\) satisfy a critical gap condition if \( \delta_{\Gamma, \infty}^F < \delta_{\Gamma}^F \).

It is easy to verify that a potential with critical gap condition is positive recurrent, in particular by Theorem 8.6 it admits a unique equilibrium state. It follows from the definition that potentials with a critical gap condition are open in the \( C^0 \)-topology. It would be interesting to verify that the properties known in the symbolic case also hold in this situation, e.g. if potentials with a critical gap condition are dense in the space of Hölder potentials, or if any potential without critical gap condition can be approximated by transient potentials. Our next corollary follows directly from Theorem 6.3 and was the initial goal of this work.

**Theorem 8.8.** Let \((M, g)\) be a pinched negatively curved manifold. Let \((\mu_n)_{n \in \mathbb{N}}\) be a sequence of \( g_1 \)-invariant probability measures converging in the weak-* topology to \( \mu \). Then

\[ \limsup_{n \to \infty} h_{\mu_n}(g_1) \leq h_\mu(g_1). \]

In other words, the entropy map is upper semicontinuous.

Combining Proposition 5.4 and Theorem 8.8 we obtain

**Theorem 8.9.** Let \((M, g)\) be a pinched negatively curved manifold. Given a \( g_1 \)-invariant probability measure \( \mu \), there exists a sequence \((\mu_n)_{n \geq 1}\) of ergodic measures such that the following hold.

1. \( \lim_{n \to \infty} \mu_n = \mu \),
2. \( \lim_{n \to \infty} h_{\mu_n}(g_1) = h_\mu(g_1) \).
3. The support of \( \mu_n \) is compact.

If a dynamical system \((X, T)\) satisfies conditions (1) and (2) we say that the space of ergodic measures is *entropy dense* in \( \mathcal{M}(X, T) \). This property has several interesting consequences, one of them is treated in [IV].

### 8.1. Pressure.

Recall that \( C_0(T^1 M) \) is the space of continuous functions vanishing at infinity. Observe that if \( \varphi \leq 0 \) then the map \( \mu \mapsto \int \varphi d\mu \) is upper semicontinuous and that if \( \varphi \in C_0(T^1 M) \) then \( \mu \mapsto \int \varphi d\mu \) is continuous (both with respect to the vague topology). This observation immediately implies part (1) in the following result. For part (2) we refer the reader to [RV, Proposition 5.8].

**Theorem 8.10.** Let \((M, g)\) be a pinched negatively curved manifold.

1. Assume \( \varphi : T^1 M \to \mathbb{R} \) is a continuous function such that either \( \varphi \leq 0 \), or \( \varphi \in C_0(T^1 M) \). If \( P(\varphi) > \delta_{\infty} \), then \( \varphi \) admits at least one equilibrium measure.
2. Assume \( \varphi \in C_0(T^1 M) \) and that \( P(\varphi) > \delta_{\infty} \). Then the map \( t \mapsto P(\varphi) = P(t\varphi) \) is differentiable at \( t = 0 \), and

\[ P'_{\varphi}(0) = \int \varphi d\mu, \]

where \( \mu \) is any equilibrium state for \( \varphi \).
We emphasize that we require no further regularity on $\varphi$ than continuity. In particular the theory developed in [PPS] does not apply. A big difference with respect to more regular functions is the lack of uniqueness of equilibrium states for continuous potentials. It is proven in [IV] that one can slightly $C^0$-perturb any potential $\varphi \in C_b(T^1M)$ into a potential with uncountably many equilibrium states. A crucial ingredient for that result is Theorem 8.8, which allows us to identify tangential functionals of the pressure at $\varphi$ to its equilibrium states.

Remark 8.11. One can verify that if $F \in C_0(T^1M)$, then $\delta_F = \delta_{\varphi}$. In particular part (1) of Theorem 8.10 also follows from Theorem 8.6.

8.2. Non-positive curvature. Let $(M, g)$ be a non-positively curved rank one manifold. We say that a measure $\mu$ is of maximal entropy in the regular part of the geodesic flow if it is supported on $\Omega_{\text{reg}}$ and it verifies

$$h_{\mu}(g_1) = \sup_{\nu \in \mathcal{M}(\Omega_{\text{reg}}, g_1)} h_{\nu}(g_1),$$

where $\mathcal{M}(\Omega_{\text{reg}}, g_1)$ is the space of probability measures supported in $\Omega_{\text{reg}}$. We denote $\sup_{\nu \in \mathcal{M}(\Omega_{\text{reg}}, g_1)} h_{\nu}(g_1)$ as $h_{\text{top}}(\Omega_{\text{reg}})$. Theorem 6.11 provides a criterion for the existence of a measure of maximal entropy in $\Omega_{\text{reg}}$.

Theorem 8.12. Let $(M, g)$ be a non-positively curved rank one manifold with curvature bounded below. Assume that $\delta_{\varphi} < h_{\text{top}}(\Omega_{\text{reg}})$, then there exists a measure of maximal entropy in the regular part of the geodesic flow.

We remark that in some cases one can expect that $h_{\text{top}}(\Omega_{\text{reg}})$ is the topological entropy of the geodesic flow (here with topological entropy we just mean $\sup_{\nu \in \mathcal{M}(g_1)} h_{\nu}(g_1)$). In other words that the high entropy measures are concentrated in the rank one vectors. Here we do not attempt to prove or construct a counterexample where that property is violated. It worth mentioning that the topological entropy in the regular part is equal to the topological entropy when $M$ is compact, see [Kn]. In the non-positive curvature case very little is known, in particular, the uniqueness of the measure of maximal entropy is not known to hold in the non-compact setting, nor good criteria for the existence of a measure of maximal entropy.

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