Stein’s Method and Minimum Parsimony Distance after Shuffles

Running head: Stein’s Method and Minimum Parsimony

Version of 10/25/04

By Jason Fulman (fulman@math.pitt.edu)

University of Pittsburgh Math Department, 414 Thackeray Hall
Pittsburgh, PA 15260

Abstract: Motivated by Bourque and Pevzner’s simulation study of the parsimony method for studying genome rearrangement, Berestycki and Durrett used techniques from random graph theory to prove that the minimum parsimony distance after iterating the random transposition shuffle undergoes a transition from Poisson to normal behavior. This paper establishes an analogous result for minimum parsimony distance after iterates of riffle shuffles or iterates of riffle shuffles and cuts. The analysis is elegant and uses different tools: Stein’s method and generating functions. A useful technique which emerges is that of making a problem more tractable by adding extra symmetry, then using Stein’s method to exploit the symmetry in the modified problem, and from this deducing information about the original problem.

2000 Mathematics Subject Classification: Primary 60F05; Secondary 60C05.

Key words and phrases: Stein’s method, minimum parsimony distance, shuffle, Poisson approximation.

1. Introduction

In the study of genome rearrangement, one often views genomes as signed permutations, where each integer corresponds to a unique gene/marker and the sign corresponds to its orientation. For unichromosomal genomes, the most frequent rearrangements are reversals. A reversal $\rho(i, j)$ applied to a permutation $\pi = \pi_1 \cdots \pi_{i-1} \pi_i \cdots \pi_{j+1} \cdots \pi_n$ reverses the segment $\pi_i \cdots \pi_j$ to obtain a new permutation $\pi_1 \cdots \pi_{i-1} - \pi_j - \pi_{j-1} \cdots - \pi_i \pi_{j+1} \cdots \pi_n$. For instance the reversal $\rho(2, 4)$ would send $4 \ 1 \ 5 \ 2 \ 3$ to $4 \ 2 \ 5 \ 1 \ 3$. In this context, the minimum parsimony distance of a signed permutation $\pi$ is defined as the minimum number of reversals needed to bring the identity permutation to $\pi$. Hannenhalli and Pevzner [HaPe] found an exact combinatorial formula for the minimum parsimony distance of a signed permutation; for this and other results, see the book [2].

A fundamental problem, emphasized in [BoPe] and the survey [Du], is to understand the distribution of the minimum parsimony distance after a given number of reversals has occurred. More generally for any shuffling technique on permutations (which from now on we assume are unsigned so that we are dealing with the symmetric group $S_n$), one can define the minimum parsimony distance of a permutation $\pi$ as the number of shuffles needed to
bring the identity to $\pi$. And it is natural to study the distribution of the minimum parsimony distance after a given number of shuffles has occurred.

An exciting recent work along this lines is the paper [BeDu], which studied the random transposition walk on the symmetric group on $n$ symbols. Then the minimum parsimony distance of $\pi$ is simply $n - \text{number of cycles of } \pi$. Letting $D^t$ be the minimum parsimony distance after $t$ iterations of the random transposition walk, they showed that $D_{cn/2} \sim u(c)n$, where $u$ is an explicit function satisfying $u(c) = c/2$ for $c \leq 1$ and $u(c) < c/2$ for $c > 1$. They also described the fluctuation of $D_{cn/2}$ about its mean in each of three regimes (subcritical where the fluctuations are Poisson, critical, and supercritical where the fluctuations are normal). They exploit a connection between the transposition walk and random graphs (about which an enormous amount is known).

In the current paper we examine minimum parsimony distance for a more vigorous shuffling method, the Gilbert-Shannon-Reeds riffle shuffle. While we do not know if this is of biological interest, the mathematical ubiquity of riffle shuffles (see the survey [Di] for an overview of connections to dynamical systems, Lie theory and much else) as well as possible applications in casinos more than justifies the question. Riffle shuffling proceeds as follows. Given a deck of $n$ cards, one cuts it into 2 packets with probability of respective pile sizes $j, n - j$ given by $\frac{\binom{n}{j}}{2^n}$. Then cards are dropped from the packets with probability proportional to the packet size at a given time; thus if the current packet sizes are $A_1, A_2$, the next card is dropped from packet $i$ with probability $A_i / (A_1 + A_2)$. Bayer and Diaconis [BayDi] prove the fundamental result that after $r$ riffle shuffles, the probability of obtaining the permutation $\pi^{-1}$ is $\binom{n + 2r - d(\pi)}{2^n}$. Here $d(\pi)$ denotes the number of descents of $\pi$, that is $|\{i : 1 \leq i \leq n - 1, \pi(i) > \pi(i + 1)\}|$. For instance the permutation $3 1 4 2 5$ has two descents. From the Bayer-Diaconis formula it is clear that the minimum parsimony distance of a permutation $\pi^{-1}$ is $\lceil \log_2 (d(\pi) + 1) \rceil$. Thus the study of minimum parsimony distance for riffle shuffles is the study of the distribution of $d(\pi)$ under the measure $\binom{n + 2r - d(\pi)}{2^n}$.

More generally, for $k$ and $n$ integers, we let $R_{k,n}$ denote the measure on $S_n$ which chooses $\pi$ with probability $\frac{\binom{n + k - d(\pi)}{k}}{2^n}$ and study the number of descents. First let us review what is known. As $k \to \infty$, the distribution $R_{k,n}$ tends to the uniform distribution on $S_n$. It is well known ([CKSS], [T]) that for $n \geq 2$ the number of descents has mean $\frac{n-1}{2}$ and variance $\frac{n+1}{12}$ and that $\frac{d(\pi) - (n-1)/2}{\sqrt{(n+1)/12}}$ is asymptotically normal. Aldous [A] proved that $\frac{2}{\sqrt{n}} \log_2(n)$ riffle shuffles are necessary and suffice to be close to the uniform distribution on $S_n$. Bayer and Diaconis [BayDi] give more refined asymptotics, proving
that for $k = 2^c n^{3/2}$ with $c$ a real number,

$$\frac{1}{2} \sum_{\pi \in S_n} |R_{k,n}(\pi) - \frac{1}{n!}| = 1 - 2\Phi\left(-\frac{2^{-c}}{4\sqrt{3}}\right) + O(n^{-1/4})$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$ 

Motivated by this result, Mann \cite{Ma} proved that if $k = an^{3/2}$, with $a$ fixed, then the number of descents has mean $\frac{n-1}{2} - \frac{n^{1/2}}{12a} + O(1)$ and variance $\frac{n}{12} + O(n^{1/2})$, and is asymptotically normal as $n \to \infty$. He deduces this from Tanny’s local limit theorem for $d(\pi)$ under the uniform distribution \cite{T} and from the formula for $R_{k,n}$.

We prove two new results concerning the distribution of $d(\pi)$ under the measure $R_{k,n}$. First, we complement the above results on normal approximation by using Stein’s method to upper bound the total variation distance between the distribution of $k - 1 - d(\pi)$ and a Poisson variable with mean $\frac{k}{n}$. Our bound shows the approximation to be good when $\frac{k}{n}$ is small. Second, we use generating functions to give very precise asymptotic estimates for the mean and variance of $d(\pi)$ when $\frac{k}{n} > \frac{1}{2\pi}$. For instance if $k = \alpha n$ with $\alpha > \frac{1}{2\pi}$, we show that

$$\left| \mathbb{E}_{R_{k,n-1}}(d + 1) - n \left( \alpha - \frac{1}{e^{1/\alpha}} - 1 \right) - \left( \frac{e^{1/\alpha}(-2\alpha e^{1/\alpha} + 2\alpha + e^{1/\alpha} + 1)}{2\alpha^2(e^{1/\alpha} - 1)^3} \right) \right|$$

is at most $C_\alpha n$ and that

$$\left| \text{Var}_{R_{k,n-1}}(d) - n \left( \frac{e^{1/\alpha}(2\alpha e^{2/\alpha} - 2\alpha^2 e^{1/\alpha} - e^{1/\alpha})}{\alpha^2(e^{1/\alpha} - 1)^4} \right) \right| \leq A_\alpha$$

where $C_\alpha, A_\alpha$ are constants depending on $\alpha$ (and are independent of $\alpha$ for $\alpha \geq 1$). Mann \cite{Ma} had exact expressions for the mean and variance (which we derive another way) but only obtained asymptotics when $k = an^{3/2}$ with $a$ fixed. The main technical point and effort of the paper \cite{SGO} on information loss in card shuffling was to obtain asymptotics for the mean and variance of $d(\pi)$ under the measure $R_{k,n}$. By a clever application of the method of indicator variables, they obtain bounds, but ours are much better.

Next let us describe the technique we use to study the distribution of $d(\pi)$ under $R_{k,n}$, as we believe this to be as interesting as the result itself. To apply Stein’s method to study a statistic $W$, one often uses an exchangeable pair $(W, W')$ of random variables (this means that the distribution of $(W, W')$ is the same as that of $(W', W)$) such that the conditional expectation $\mathbb{E}(W'|W)$ is approximately $(1 - \lambda)W$. Typically to construct such a pair one would use a Markov chain on $S_n$ which is reversible with respect to the measure $R_{k,n}$, choose $\pi$ from $R_{k,n}$, let $\pi'$ be obtained from $\pi$ from
one step in the chain, and finally set \((W, W') = (W(\pi), W(\pi'))\). For the problem at hand this does not seem easy. Thus we modify the problem; instead of considering the measure \(R_{k,n}\), we consider a measure \(C_{k,n}\) which chooses a permutation \(\pi\) with probability \(\binom{n+k-c(\pi)-1}{n-1}\). Here \(c(\pi)\) is the number of cyclic descents of \(\pi\), defined as \(d(\pi)\) if \(\pi(n) < \pi(1)\) and as \(d(\pi) + 1\) if \(\pi(n) > \pi(1)\). This probability measure was introduced in \([Fu1]\) and \(C_{2r,n}(\pi)\) gives the chance of obtaining \(\pi^{-1}\) after first cutting the deck at a uniformly chosen random position and then performing \(r\) iterations of a riffle shuffle. The advantage of working with \(C_{k,n}\) is that it has a natural symmetry which leaves it invariant, since performing two consecutive cuts at random positions is the same as performing a single cut. As will be explained in Section 2 this symmetry leads to an exchangeable pair \((d, d')\) with the very convenient property that \(E_C(d'|\pi)\) is approximately \((1 - \frac{1}{n})d\). We obtain a Poisson approximation theorem for \(d\) under the measure \(C_{k,n}\). Although the measures \(R_{k,n}\) and \(C_{k,n}\) are not close when \(k/n\) is small (a main result of \([Fu3]\) is that that the total variation distance between them is roughly \(\frac{1}{4n}\) for \(k \geq n\)), we show that the distribution of \(k-d\) under \(C_{k,n}\) is close to the distribution of \(k-1-d\) under \(R_{k,n-1}\). This implies a Poisson approximation theorem for the original problem of interest.

Incidentally, it is proved in \([Fu1]\) that \(r\) iterations of “cut and then riffle shuffle” yields exactly the same distribution as performing a single cut and then iterating \(r\) riffle shuffles. Thus the chance of \(\pi^{-1}\) after \(r\) iterations of “cut and then riffle shuffle” is \(C_{2r,n}(\pi)\), which implies that the minimum parsimony distance of the “cut and then riffle shuffle” process is \([\log_2(c(\pi))].\) Thus the study of minimum parsimony distance for the “cut and then riffle shuffle” procedure is equivalent to the study of \(c\) under the distribution \(C_{k,n}\). But as mentioned in the last paragraph, we will prove that this is the same as the distribution of \(d+1\) under \(R_{k,n-1}\). Hence the theory of minimum parsimony distance for “cut and then riffle shuffle” is equivalent to the theory for riffle shuffles, and we shall say nothing more about it.

The reader may wonder why we don’t apply our exchangeable pair for normal approximation. Most theorems for Stein’s method for normal approximation assume that the pair \((W, W')\) satisfies the property \(E(W'|W) = (1 - \lambda)W\) for some \(\lambda\). In our case this only approximately holds, that is \(E(W'|W) = (1 - \lambda)W + G(W)\) where \(G(W)\) is small. There are normal approximation results in the literature \([RR], [Ch]\) for dealing with this situation, but they require that \(\frac{E(G(W))}{\lambda}\) goes to 0. Using interesting properties of Eulerian numbers, we show that even for the uniform distribution (the \(k \to \infty\) limit of \(C_{k,n}\)) the quantity \(\frac{E(G(W))}{\lambda}\) is bounded away from 0. Finding a version of Stein’s method which allows normal approximation for our exchangeable pair (even for the uniform distribution) is an important open problem. Incidentally, for the special case of the uniform distribution,
it is possible to prove a central limit theorem for $d$ by Stein’s method \cite{Fu2}, using a different exchangeable pair.

Having described the main motivations and ideas of the paper, we describe its organization. Section 2 defines an exchangeable pair to be used in the study of $d(\pi)$ under the measure $C_{k,n}$, and develops a number of properties of it. It also gives closed formulas (but not asymptotics) for the mean and variance of $d(\pi)$, by relating them to the mean and variance of $c(\pi)$, and computing the latter using generating functions. Section 3 uses the exchangeable pair of Section 2 to prove a Poisson approximation theorem for $k - d(\pi)$ under the measure $C_{k,n}$ (the bounds are valid for all integer values of $k$ and $n$ but informative only when $\frac{k}{n}$ is small). It then shows how to deduce from this a Poisson approximation theorem for $k - 1 - d(\pi)$ under the measure $R_{k,n}$. Section 4 gives asymptotics for the mean and variance for $c(\pi)$ under $C_{k,n}$ for $\frac{k}{n} > \frac{1}{2\pi}$ (and so also for $d(\pi)$ under $C_{k,n}$ and $R_{k,n}$). It then explores further properties of the exchangeable pair which are related to normal approximation. Finally, it gives a quick algorithm for sampling from $R_{k,n}$, which should be useful in empirically investigating the nature of the transition from Poisson to normal behavior.

2. The Exchangeable Pair, Mean, and Variance

This section constructs an exchangeable pair $(d, d')$ for the measure $C_{k,n}$ and develops some of its properties. Throughout we let $E_C$ denote expectation with respect to $C_{k,n}$. We relate $E_C(d)$ and $E_C(d^2)$ to $E_C(c)$ and $E_C(c^2)$, and then use generating functions to find expressions (whose asymptotics will be studied later) for $E_C(c)$ and $E_C(c^2)$.

To begin let us construct an exchangeable pair $(d, d')$. We represent permutations $\pi$ in two line form. Thus the permutation represented by

| $i$  | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|
| $\pi(i)$ | 6 | 4 | 1 | 5 | 3 | 2 | 7 |

sends 1 to 6, 2 to 4, and so on. One constructs a permutation $\pi'$ by choosing uniformly at random one of the $n$ cyclic shifts of the symbols in the bottom row of the two line form of $\pi$. For instance with probability 1/7 one obtains the permutation $\pi'$ which is represented by

| $i$  | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|
| $\pi'(i)$ | 5 | 3 | 2 | 7 | 6 | 4 | 1 |

An essential point is that if $\pi$ is chosen from the measure $C_{k,n}$, then so is $\pi'$; note that this would not be so for the measure $R_{k,n}$. Thus if one chooses $\pi$ from $C_{k,n}$, defines $\pi'$ as above, and sets $(d, d') = (d(\pi), d(\pi'))$, it follows that $(d, d')$ is exchangeable with respect to the measure $C_{k,n}$. Observe also that $d' - d \in \{0, \pm 1\}$.

Recall that $\pi$ is said to have a cyclic descent at position $j$ if either $1 \leq j \leq n - 1$ and $\pi(j) > \pi(j + 1)$ or $j = n$ and $\pi(n) > \pi(1)$. It is helpful to define random variables $\chi_j(\pi)$ ($1 \leq j \leq n$) where $\chi_j(\pi) = 1$ if $\pi$ has a cyclic...
descent at position \( j \) and \( \chi_j(\pi) = 0 \) if \( \pi \) does not have a cyclic descent at position \( j \). We let \( I \) denote the indicator function of an event. We also use the standard notion that if \( Y \) is a random variable, \( \mathbb{E}(Y|A) \) is the conditional expectation of \( Y \) given \( A \).

**Lemma 2.1.**

\[
\mathbb{E}_C(d' - d|\pi) = -\frac{d}{n} + \frac{n - 1}{n} I_{\chi_n(\pi) = 0}.
\]

**Proof.** Note that \( d' = d + 1 \) occurs only if \( \pi \) has a cyclic descent at \( n \) and then it occurs with probability \( \frac{n - 1 - d}{n} \). Note also that \( d' = d - 1 \) occurs only if \( \pi \) does not have a cyclic descent at \( n \), and that it then occurs with probability \( \frac{d}{n} \). To summarize,

\[
\mathbb{E}_C(d' - d|\pi) = -\frac{d}{n} I_{\chi_n(\pi) = 0} + \frac{n - 1 - d}{n} I_{\chi_n(\pi) = 1} = -\frac{d}{n} + \frac{n - 1}{n} I_{\chi_n(\pi) = 1}.
\]

□

As a corollary, we obtain \( \mathbb{E}_C(d) \) in terms of \( \mathbb{E}_C(c) \).

**Corollary 2.2.**

\[
\mathbb{E}_C(d) = \frac{n - 1}{n} \mathbb{E}_C(c).
\]

**Proof.** Since \( (d, d') \) is an exchangeable pair, \( \mathbb{E}_C(d' - d) = 0 \). It follows that \( \mathbb{E}_C(\mathbb{E}_C(d' - d|\pi)) = 0 \). So from Lemma 2.1,

\[
\mathbb{E}_C(d) = (n - 1) \mathbb{E}_C(I_{\chi_n(\pi) = 1}).
\]

Since the variables \( \chi_1(\pi), \ldots, \chi_n(\pi) \) have the same distribution under \( C_{k,n} \), and \( c = \chi_1(\pi) + \cdots + \chi_n(\pi) \), the result follows. □

Lemma 2.3 will be helpful at several points in this paper.

**Lemma 2.3.**

\[
\mathbb{E}_C(d I_{\chi_n(\pi) = 1}) = \mathbb{E}_C\left(\frac{c(c - 1)}{n}\right).
\]

**Proof.** Observe that

\[
\mathbb{E}_C(d I_{\chi_n(\pi) = 1}) = \mathbb{E}_C((c - 1) I_{\chi_n(\pi) = 1})
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_C((c - 1) I_{\chi_i(\pi) = 1})
\]

\[
= \frac{1}{n} \mathbb{E}_C(c(c - 1)).
\]

□

As a consequence, we obtain \( \mathbb{E}_C(d^2) \) in terms of \( \mathbb{E}_C(c) \) and \( \mathbb{E}_C(c^2) \).
Corollary 2.4.

\[ \mathbb{E}_C(d^2) = (1 - \frac{2}{n})\mathbb{E}_C(c^2) + \frac{1}{n}\mathbb{E}_C(c). \]

Proof.

\[ \mathbb{E}_C(c^2) = \mathbb{E}_C(d + 1\chi_n(\pi)=1)^2 = \mathbb{E}_C(d^2) + 2\mathbb{E}_C(d\chi_n(\pi)=1) + \mathbb{E}_C(1\chi_n(\pi)=1) \]

\[ = \mathbb{E}_C(d^2) + \frac{2}{n}\mathbb{E}_C(c^2 - c) + \frac{1}{n}\mathbb{E}_C(c), \]

where the final equality is Lemma 2.3. This is equivalent to the statement of the corollary. \(\square\)

Next we use generating functions to compute \(\mathbb{E}_C(c)\) and \(\mathbb{E}_C(c^2)\). For this some lemmas are useful.

Lemma 2.5. \(\text{(Ful)}\) For \(n > 1\), the number of elements in \(S_n\) with \(i\) cyclic descents is equal to \(n\) multiplied by the number of elements in \(S_{n-1}\) with \(i-1\) descents.

Lemma 2.6.

\[ \sum_{\pi \in S_n} \frac{t^{c(\pi)}}{(1-t)^n} = n \sum_{m \geq 0} m^{n-1} t^m. \]

Proof. Given Lemma 2.5, the result now follows from the well known generating function for descents (e.g. [FoS])

\[ \sum_{\pi \in S_n} \frac{t^{d(\pi)+1}}{(1-t)^{n+1}} = \sum_{m \geq 0} m^n t^m. \]

\(\square\)

Proposition 2.7 gives a closed formula for \(\mathbb{E}_C(c)\).

Proposition 2.7. For \(n > 1\),

\[ \mathbb{E}_C(c) = k - \sum_{j=1}^{k-1} j^{n-1}. \]

Proof. Multiplying the equation of Lemma 2.6 by \((1-t)^n\) and then differentiating with respect to \(t\), one obtains the equation is

\[ \sum_{\pi \in S_n} c(\pi)t^{c(\pi)-1} = n(1-t)^n \sum_{m \geq 0} m^n t^{m-1} - n^2(1-t)^{n-1} \sum_{m \geq 0} m^{n-1} t^m. \]

Multiplying both sides by \(\frac{t}{nk^{n-1}(1-t)^n}\) gives the equation

\[ \sum_{\pi \in S_n} \frac{c(\pi)t^{c(\pi)}}{nk^{n-1}(1-t)^n} = \frac{1}{k^{n-1}} \sum_{m \geq 0} m^n t^m - \frac{n}{k^{n-1}(1-t)} \sum_{m \geq 0} m^{n-1} t^{m+1}. \]
The coefficient of \( t^k \) on the left hand side is precisely the expected value of \( c \) under the measure \( C_{k,n} \). The proposition now follows by computing the coefficient of \( t^k \) on the right hand side. \( \square \)

By a similar argument, one obtains an exact expression for \( \mathbb{E}_C(c^2) \).

**Proposition 2.8.** For \( n > 1 \),

\[
\mathbb{E}_C(c^2) = k^2 - \frac{n(n+1)}{k^{n-1}} \sum_{j=1}^{k-1} j^n + \frac{n(nk-n-k)}{k^{n-1}} \sum_{j=1}^{k-1} j^{n-1}.
\]

**Proof.** From the proof of Proposition 2.7, we know that

\[
\sum_{\pi \in S_n} c(\pi) t^c(\pi) = n(1-t)^n \sum_{m \geq 0} m^n t^m - n^2 (1-t)^{n-1} \sum_{m \geq 0} m^{n-1} t^{m+1}.
\]

Differentiate with respect to \( t \), multiply both sides by \( \frac{t}{nk^{n-1}(1-t)^2} \), and take the coefficient of \( t^k \). On the left hand side one gets \( \mathbb{E}_C(c^2) \). On the right hand side one obtains the coefficient of \( t^k \) in

\[
\frac{1}{k^{n-1}} \sum_{m \geq 0} m^{n+1} t^m - \frac{2n}{k^{n-1}(1-t)} \sum_{m \geq 0} m^n t^{m+1} - \frac{n}{k^{n-1}(1-t)} \sum_{m \geq 0} m^{n-1} t^{m+1} + \frac{n(n-1)}{k^{n-1}(1-t)^2} \sum_{m \geq 0} m^{n-1} t^{m+2}.
\]

After elementary simplifications the result follows. \( \square \)

3. **Poisson regime**

A main result of this section is a Stein’s method proof that for \( k \) much smaller than \( n \), the random variable \( X(\pi) := k - d(\pi) \) under the measure \( C_{k,n} \) is approximately Poisson with mean \( \lambda := \frac{k}{n} \). Then we show how this can be used to deduce Poisson limits for \( k - c(\pi) \) under the measure \( C_{k,n} \) and for \( k - 1 - d(\pi) \) under the measure \( R_{k,n} \).

To begin we recall Stein’s method for Poisson approximation. A book length treatment of Stein’s method for Poisson approximation is [BarHJ], but that book emphasizes the coupling approach. We prefer to work from first principles along the lines of Stein’s original formulation as presented in [St].

Throughout we use the exchangeable pair \((X, X')\), where \( \pi \) and \( \pi' \) are as in Section 2, \( X(\pi) = k - d(\pi) \), \( X' = X(\pi') \), and the underlying probability measure is \( C_{k,n} \). Let \( \mathbb{P}_\lambda \) denote probability under the Poisson distribution of mean \( \lambda \), and as usual let \( \mathbb{P}_C \) denote probability with respect to the measure \( C_{k,n} \). Let \( A \) be subset of \( \mathbb{Z}^+ \), the set of non-negative integers. Stein’s method is based on the following “Stein’s equation”

\[
\mathbb{P}_C(X \in A) - \mathbb{P}_\lambda(A) = \mathbb{E}_C(iT_\lambda - T\alpha)g_{\lambda,A}.
\]

Let us specify the terms on the right hand side of the equation, in the special case of interest to us.
(1) The function \( g = g_{\lambda,A} : \mathbb{Z}^+ \to \mathbb{R} \) is constructed to solve the equation
\[
\lambda g(j + 1) - jg(j) = 1_{j \in A} - \mathbb{P}_{\lambda}(A), \quad j \geq 0
\]
where \( g(0) \) is taken to be 0. We also need the following lemma which bounds certain quantities related to \( g \).

**Lemma 3.1.** \(^{(\text{BarHL}, \text{Lemma 1.1.1})} \)

(a) Let \( ||g|| \) denote \( \sup_j |g_{\lambda,A}(j)| \). Then \( ||g|| \leq 1 \) for all \( A \).

(b) Let \( \Delta(g) \) denote \( \sup_j |g_{\lambda,A}(j + 1) - g_{\lambda,A}(j)| \). Then \( \Delta(g) \leq 1 \) for all \( A \).

(2) The map \( T_\lambda \) sends real valued functions on \( \mathbb{Z}^+ \) to real valued functions on \( \mathbb{Z}^+ \) and is defined by \( T_\lambda(f)[j] = \lambda f(j + 1) - jf(j) \).

(3) The map \( i \) sends real valued functions on \( \mathbb{Z}^+ \) to real valued functions on \( S_n \), the symmetric group. It is defined by \( (if)[\pi] = f(X(\pi)). \)

(4) The map \( T \) is a map from the set of real valued antisymmetric functions on \( S_n \times S_n \) to the set of real valued functions on \( S_n \). It is defined by \( T(f)[\pi] = EC(f(\pi, \pi'))/\pi. \) (Since the pair \( (\pi, \pi') \) is exchangeable, \( EC(Tf) = 0 \), which is crucial for the proof of the Stein equation).

(5) Finally (and this is where one has to make a careful choice), the map \( \alpha \) is a map from real valued functions on \( \mathbb{Z}^+ \) to antisymmetric real valued functions on \( S_n \times S_n \). In the proof of Theorem 3.3 we will specify which \( \alpha \) we use.

In order to approximate \( X \) by a Poisson(\( \lambda \)) random variable, it will be useful to approximate the mean of the random variable \( c(\pi) \) by \( \lambda \). This is accomplished in the next lemma, the second part of which is not needed in the sequel.

**Lemma 3.2.** Let \( \lambda = \frac{k}{n} \), where \( k, n \) are positive integers.

(1) \[
\left| \frac{EC(c)}{n} - \lambda \right| \leq k(1 - \frac{1}{k})^n.
\]

(2) \[
\left| \frac{EC(c)}{n} - \lambda \right| \leq \frac{k}{n}.
\]

**Proof.** For the first assertion, note that by Proposition 2.7
\[
\left| \frac{EC(c)}{n} - \lambda \right| = \frac{1}{k^{n-1}} \sum_{j=1}^{k-1} j^{n-1} \leq \frac{(k - 1)^n}{k^{n-1}}.
\]

The second assertion follows since the formula for \( C_{k,n} \) forces \( c \leq k \) with probability 1. \( \square \)

**Theorem 3.3.** Let \( \lambda = \frac{k}{n} \) where \( k, n \) are positive integers. Then for any \( A \subseteq \mathbb{Z}^+ \),
\[
|\mathbb{P}_C(k - d(\pi) \in A) - \mathbb{P}_\lambda(A)| \leq \left( \frac{k}{n} \right)^2 + k(n + 1)(1 - \frac{1}{k})^n.
\]
Proof. As above, let $X(\pi) = k - d(\pi)$ and $X'(\pi) = k - d(\pi')$. Since $A, \lambda$ are fixed, throughout the proof the function $g_{\lambda,A}$ is denoted by $g$. We specify the map $\alpha$ to be used in the “Stein equation”

$$P_C(X \in A) - P_{\lambda}\{A\} = E_C(iT_\lambda - T\alpha)g.$$ 

Given a real valued function $f$ on $\mathbb{Z}^+$, we define $\alpha f$ by

$$\alpha f[p_1, p_2] = f(X(p_2))I_{X(p_2)=X(p_1)+1} - f(X(p_1))I_{X(p_1)=X(p_2)+1}.$$

Note that as required this is an antisymmetric function on $S_n \times S_n$.

Then one computes that $T\alpha g$ is the function on $S_n$ defined by

$$T\alpha g(p) = \mathbb{E}_C(\alpha g(p, p')|p) = \mathbb{E}_C(g(X')I_{X'=X+1} - g(X)I_{X=X'+1}|p).$$

Thus by the reasoning of Lemma 2.1

$$T\alpha g(p) = g(X + 1)I_{\chi_n(p)=0} - g(X)I_{\chi_n(p)=1}(1 - \frac{c(p)}{n}).$$

Since $iT_\lambda g(p) = \lambda g(X + 1) - Xg(X)$ and $I_{\chi_n(p)=0} = 1 - I_{\chi_n(p)=1}$, one concludes that

$$\begin{align*}
(iT_\lambda g - T\alpha g)[p] &= \left[\left(\lambda - \frac{c(p)}{n}\right)g(X + 1)\right] + \left[(1 - \chi_n(p)=1)g(X)\right] \\
&\quad + \left[\frac{c(p)}{n}I_{\chi_n(p)=1}(g(X + 1) - g(X))\right] \\
&= \left[\left(\lambda - \frac{c(p)}{n}\right)g(X + 1)\right] + \left[(c(p) - k)g(X)\right] \\
&\quad + \left[\frac{c(p)}{n}I_{\chi_n(p)=1}(g(X + 1) - g(X))\right].
\end{align*}$$

Thus to complete the proof, for each of the three terms in square brackets, we bound the expectation under the measure $C_{k,n}$. Lemma 3.1 and part 1 of Lemma 3.2 give that

$$\left|\mathbb{E}_C\left(g(X + 1)(\lambda - \frac{c(p)}{n})\right)\right| \leq ||g||\mathbb{E}_C(|\lambda - \frac{c(p)}{n}|) \leq \mathbb{E}_C(|\lambda - \frac{c(p)}{n}|) \leq k(1 - \frac{1}{k})^n.$$ 

For the second term in square brackets, one argues as for the first term in square brackets to get an upper bound of $nk(1 - \frac{1}{k})^n$. 

To bound the expectation of the third term in square brackets, note that the nonnegativity of $c(\pi)\mathbb{I}_{\chi_n(\pi)=1}$ and Lemma 3.1 imply that

$$\left| \mathbb{E}_C \left[ \frac{c(\pi)}{n} \mathbb{I}_{\chi_n(\pi)=1} (g(X + 1) - g(X)) \right] \right|$$

$$\leq \Delta(g) \mathbb{E}_C \left( \frac{c(\pi)}{n} \mathbb{I}_{\chi_n(\pi)=1} \right)$$

$$\leq \mathbb{E}_C \left( \frac{c(\pi)}{n^2} \sum_{i=1}^{\infty} \mathbb{I}_{\chi_i(\pi)=1} \right)$$

$$= \mathbb{E}_C \left( \frac{c(\pi)^2}{n^2} \right).$$

By the explicit formula for $C_{k,n}$, it follows that $c(\pi) \leq k$ with probability 1. Hence this is at most $(\frac{k}{n})^2$.

To conclude this section, we show how Theorem 3.3 can be used to deduce Poisson approximations for the two statistics we really care about: $k - c(\pi)$ under the measure $C_{k,n}$ and $k - 1 - d(\pi)$ under the measure $R_{k,n}$.

**Proposition 3.4.** For all $A \subseteq \mathbb{Z}^+$,

$$|\mathbb{P}_C(k - d(\pi) \in A) - \mathbb{P}_C(k - c(\pi) \in A)| \leq \frac{2k}{n}. $$

**Proof.** Observe that for any $l \geq 0$,

$$\mathbb{P}_C(d = l)$$

$$= \mathbb{P}_C(d' = l)$$

$$= \mathbb{P}_C(d' = l, c = l) + \mathbb{P}_C(d' = l, c = l + 1)$$

$$= \mathbb{P}_C(c = l) \mathbb{P}_C(d' = l|c = l) + \mathbb{P}_C(c = l + 1) \mathbb{P}_C(d' = l|c = l + 1)$$

$$= \mathbb{P}_C(c = l) \frac{n - l}{n} + \mathbb{P}_C(c = l + 1) \frac{l + 1}{n}.$$ 

Thus

$$\mathbb{P}_C(d = l) - \mathbb{P}_C(c = l) = -\frac{l}{n} \mathbb{P}_C(c = l) + \frac{l + 1}{n} \mathbb{P}_C(c = l + 1)$$

which implies that

$$|\mathbb{P}_C(d = l) - \mathbb{P}_C(c = l)| \leq \frac{l}{n} \mathbb{P}_C(c = l) + \frac{l + 1}{n} \mathbb{P}_C(c = l + 1).$$
Summing over $l \geq 0$ gives that
\[
\sum_{l \geq 0} |\mathbb{P}_C(d = l) - \mathbb{P}_C(c = l)| \leq \sum_{l \geq 0} \left( \frac{l}{n} \mathbb{P}_C(c = l) + \frac{l+1}{n} \mathbb{P}_C(c = l+1) \right)
= \frac{2}{n} \sum_{l \geq 0} l \mathbb{P}_C(c = l)
= \frac{2}{n} \mathbb{E}_C(c)
\leq \frac{2k}{n},
\]
where the final inequality is Proposition 2.7 (or also since $c \leq k$ with probability 1). The result follows. \qed

**Corollary 3.5.** Let $\lambda = \frac{k}{n}$ where $k, n$ are positive integers. Then for any $A \subseteq \mathbb{Z}^+$,
\[
|\mathbb{P}_C(k - c(\pi) \in A) - \mathbb{P}_A(A)| \leq \left( \frac{k}{n} \right)^2 + \frac{2k}{n} + k(n+1)(1 - \frac{1}{k})^n.
\]

*Proof.* This is immediate from Theorem 3.3 and Proposition 3.4. \qed

Proposition 3.6 shows that the distribution of $d(\pi) + 1$ under the measure $R_{k,n}$ is exactly the same as the distribution of $c(\pi)$ under the measure $C_{k,n+1}$. **Proposition 3.6.** For any $r \geq 0$,
\[
\mathbb{P}_{R_{k,n}}(d(\pi) = r) = \mathbb{P}_{C_{k,n+1}}(c(\pi) = r + 1).
\]

*Proof.* Note from the formula for $R_{k,n}$ that for any $r$, the probability of $r$ descents under the measure $R_{k,n}$ is $\frac{\binom{k+r-1}{n}}{k^n}$ multiplied by the number of permutations in $S_n$ with $r$ descents. Similarly, from the formula for $C_{k,n+1}$ one sees that for any $r$, the probability of $r + 1$ cyclic descents under the measure $C_{k,n+1}$ is $\frac{\binom{n+k+r-1}{n}}{k^n}$ multiplied by the number of permutations in $S_{n+1}$ with $r + 1$ cyclic descents. The result follows from Lemma 2.5. \qed

**Corollary 3.7.** Let $\lambda = \frac{k}{n+1}$ where $k, n$ are positive integers. Then for any $A \subseteq \mathbb{Z}^+$,
\[
|\mathbb{P}_{R_{k,n}}(k - 1 - d(\pi) \in A) - \mathbb{P}_A(A)| \leq \left( \frac{k}{n+1} \right)^2 + \frac{2k}{n+1} + k(n+2)(1 - \frac{1}{k})^{n+1}.
\]

*Proof.* This is immediate Corollary 3.5 and Proposition 3.6. \qed

### 4. Other regimes

This section is organized into three subsections. Subsection 4.1 gives good asymptotic results for the mean and variance of $c(\pi)$ under $C_{k,n}$ (and so also for $d(\pi)$ under $D_{k,n}$). Subsection 4.2 develops further properties of the exchangeable pair $d, d'$ under $C_{k,n}$ which are relevant to normal approximation.
Subsection 4.3 gives a speedy algorithm for sampling from the measures $C_{k,n}$ and $R_{k,n}$.

4.1. Asymptotics of mean and variance. This subsection derives sharp estimates for the mean and variance of $c(\pi)$ under the measure $C_{k,n}$ when $\frac{k}{n} \geq \frac{1}{2\pi}$. Since by Proposition 3.6 the distribution of $d(\pi)$ under $R_{k,n}$ is the same as the distribution of $c(\pi) - 1$ under $C_{k,n+1}$, one immediately obtains results (which we stated in the introduction) for the mean and variance of $d(\pi)$ under $R_{k,n}$. We also remark that Corollaries 2.2 and 2.4 imply results for $d(\pi)$ under $C_{k,n}$.

Throughout we will use information about the Bernoulli numbers $B_n$. They are defined by the generating function $f(z) = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = z e^z - 1$ so that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$ and $B_i = 0$ if $i \geq 3$ is odd. The zero at 0 in the denominator of $f(z)$ cancels with the zero of $z$, so $f(z)$ is analytic for $|z| < 2\pi$ but has first order poles at $z = \pm 2\pi i, \pm 4\pi i, \cdots$.

We also use the notation that $\binom{n}{t}$ denotes $n(n-1) \cdots (n-t+1)$ and that $\binom{n}{0} = 1$.

To see the connection with Bernoulli numbers, Lemma 4.1 shows how to write $\mathbb{E}_{C}(c)$ in terms of them.

Lemma 4.1.

$$\mathbb{E}_{C}(c) = -k \sum_{t=1}^{n-1} \frac{B_t(n)_t}{t! k^t}.$$ 

Proof. This follows from Proposition 2.7 and by the expansion of partial power sums in [GR]:

$$\sum_{r=0}^{a-1} r^n = \frac{a^n}{n+1} \left( a + \sum_{t=0}^{n-1} \frac{B_{t+1}(n+1)t+1}{(t+1)!a^t} \right).$$

Lemma 4.2 and 4.3 give two elementary estimates.

Lemma 4.2. For $0 \leq t \leq n$,

$$\left| \left(1 - \frac{(n)_t}{n^t}\right) - \frac{\left(\frac{t}{2}\right)^2}{2n^2}\right| \leq \frac{\left(\frac{t}{2}\right)^2}{2n^2}.$$

Proof. For $t = 0, 1$ the result is clear, so suppose that $t \geq 2$. We show that

$$\frac{\left(\frac{t}{2}\right)^2}{2n^2} \leq 1 - \frac{(n)_t}{n^t} \leq \frac{\left(\frac{t}{2}\right)^2}{2n^2}.$$

To see this write $1 - \frac{(n)_t}{n^t} = 1 - (1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{t-1}{n})$. One proves by induction that if $0 < x_1, \cdots, x_n < 1$, then $\prod_{j}(1 - x_j) \geq 1 - \sum_{j} x_j$. Thus

$$(1 - \frac{1}{n}) \cdots (1 - \frac{t-1}{n}) \geq 1 - \sum_{j=1}^{t-1} \frac{j}{n} = 1 - \frac{\left(\frac{t}{2}\right)}{n}.$$
which proves the upper bound. For the lower bound note that
\[
(1 - \frac{1}{n}) \cdots (1 - \frac{t-1}{n}) = e^{\log(1-1/n) \cdots (1-(t-1)/n)} \leq e^{-\left(\frac{t}{n} + \cdots + \frac{1}{n}\right)} = e^{-\frac{(t)}{n}} \leq 1 - \frac{(t)}{n} + \frac{(t)^2}{2n^2}.
\]

The last inequality is on page 103 of \[HLP\].

**Lemma 4.3.** Suppose that \( \alpha > \frac{1}{2\pi} \). Then for \( n \geq 1 \),
\[
\sum_{t=n}^{\infty} \frac{|B_t|a^t}{\alpha^t t!} \leq \frac{C_{l,\alpha} n^l}{(2\pi\alpha)^n}
\]
where \( C_{l,\alpha} \) is a constant depending on \( l \) and \( \alpha \) (and if \( \alpha \geq 1 \) the constant depends only on \( l \)).

**Proof.** Recall that \( B_t \) vanishes for \( t \geq 3 \) odd and that there is a bound
\[
|B_{2t}| \leq 8\sqrt{\pi t} \left( \frac{t}{e} \right)^{2t}
\]
for \( t \geq 1 \) \[Fe\]. Combining this with Stirling’s bound
\[
t! \geq \sqrt{2\pi t} \left( \frac{t}{e} \right)^t e^{t/12t + 1}
\]
one concludes that
\[
\sum_{t=n}^{\infty} \frac{|B_t|a^t}{\alpha^t t!} \leq C \sum_{t=n}^{\infty} \frac{(2\pi\alpha)^{-t}}{t!} \sum_{t=0}^{\infty} \frac{(t+n)^l}{(2\pi\alpha)^n} \sum_{t=0}^{\infty} \frac{(1+t)^t}{(2\pi\alpha)^t}
\]
where \( C \) is a universal constant. The ratio test shows that \( \sum_{t=0}^{\infty} \frac{(1+t)^t}{(2\pi\alpha)^t} \) converges for \( 2\pi\alpha > 1 \) (and moreover is at most a constant depending on \( l \) if \( \alpha \geq 1 \)).

We also require a result about Bernoulli numbers.

**Lemma 4.4.** Suppose that \( \alpha > \frac{1}{2\pi} \). Then
\[
\begin{align*}
(1) \quad & \sum_{t=0}^{\infty} \frac{B_t}{t! \alpha^t} = \frac{1}{\alpha(e^{1/\alpha} - 1)}. \\
(2) \quad & \sum_{t=0}^{\infty} \frac{B_t}{t! \alpha^t} = \frac{e^{1/\alpha}(-2\alpha e^{1/\alpha} + 2\alpha e^{1/\alpha} + 1)}{2\alpha^3(e^{1/\alpha} - 1)^3}.
\end{align*}
\]
\[
\begin{align*}
(3) \quad & \sum_{t=0}^{\infty} \frac{B_t t!}{t! \alpha^t} = \frac{\alpha e^{1/\alpha} - e^{1/\alpha} - \alpha}{\alpha(e^{1/\alpha} - 1)^3}.
\end{align*}
\]
\[
\begin{align*}
(4) \quad & \sum_{t=0}^{\infty} \frac{B_t t! (t+1)!}{t! \alpha^t} = \frac{\alpha e^{1/\alpha}(-3\alpha e^{2/\alpha} - 2\alpha e^{1/\alpha} - 3\alpha - 1)}{2\alpha^3(e^{1/\alpha} - 1)^4}.
\end{align*}
\]

**Proof.** We use the generating function \( f(z) = \sum_{t=0}^{\infty} \frac{B_t}{t!} z^t = \frac{z}{e^z - 1} \) for the Bernoulli numbers, which as mentioned earlier is analytic for \( |z| < 2\pi \). For the first assertion simply set \( z = 1/\alpha \). For the second assertion, one computes \( \frac{z^2}{2} \frac{d}{dz} f(z) \) and evaluates it at \( z = 1/\alpha \). For the third equation one differentiates \( f(z) \) with respect to \( z \) and then sets \( z = 1/\alpha \). For the fourth equation one differentiates \( f(z) \) three times with respect to \( z \), then multiplies by \( \frac{z^2}{2} \) and sets \( z = 1/\alpha \).
Next we give an estimate for $E_C(c)$.

**Proposition 4.5.** Suppose that $k = \alpha n$ with $\alpha > \frac{1}{2\pi}$. Then

$$
\left| E_C(c(\pi)) - n \left( \alpha - \frac{1}{e^{1/\alpha} - 1} \right) - \left( \frac{e^{1/\alpha}(-2\alpha e^{1/\alpha} + 2\alpha + e^{1/\alpha} + 1)}{2\alpha^2(e^{1/\alpha} - 1)^3} \right) \right| < \frac{C_{15}}{n}
$$

where $C_{15}$ is a constant depending on $\alpha$ (and which is independent of $\alpha$ for $\alpha \geq 1$).

*Proof.* By Lemma 4.1,

$$
E_C(c) = k - k \sum_{t=0}^{n-1} \frac{B_t(n+1)}{t!k^t}.
$$

From Lemma 4.3, the error term is at most $\alpha n - \alpha n \sum_{t=0}^{n-1} \frac{B_t}{t!\alpha^t} + \alpha \sum_{t=0}^{n-1} \frac{B_t(t)}{t!\alpha^t}$.

From this and Lemma 4.2 it follows that

$$
\begin{align*}
\left| E_C(c) - \left( \alpha n - \alpha n \sum_{t=0}^{\infty} \frac{B_t}{t!\alpha^t} + \alpha \sum_{t=0}^{n-1} \frac{B_t(t)}{t!\alpha^t} \right) \right| & \\
\leq & \alpha n \sum_{t=n}^{\infty} \frac{|B_t|}{t!\alpha^t} + \frac{\alpha}{2n} \sum_{t=0}^{n-1} \frac{|B_t(t)|^2}{t!\alpha^t}.
\end{align*}
$$

Thus

$$
\begin{align*}
\left| E_C(c) - \left( \alpha n - \alpha n \sum_{t=0}^{\infty} \frac{B_t}{t!\alpha^t} + \alpha \sum_{t=0}^{\infty} \frac{B_t(t)}{t!\alpha^t} \right) \right| & \\
\leq & \alpha n \sum_{t=n}^{\infty} \frac{|B_t|}{t!\alpha^t} + \alpha \sum_{t=1}^{\infty} \frac{|B_t(t)|^2}{t!\alpha^t} + \frac{\alpha}{2n} \sum_{t=1}^{n-1} \frac{|B_t(t)|^2}{t!\alpha^t}.
\end{align*}
$$

From Lemma 4.3 the error term is at most $\frac{C_{15}}{n}$ where $C_{15}$ is a constant depending on $\alpha$ (and which is independent of $\alpha$ for $\alpha \geq 1$). The result now follows from parts 1 and 2 of Lemma 4.4. \(\square\)

**Proposition 4.6.** Suppose that $k = \alpha n$ with $\alpha > \frac{1}{2\pi}$. Then

$$
\left| Var_C(c(\pi)) - n \left( \frac{e^{1/\alpha}(\alpha^2 e^{2/\alpha} + \alpha^2 - 2\alpha^2 e^{1/\alpha} - e^{1/\alpha})}{\alpha^2(e^{1/\alpha} - 1)^4} \right) \right| \leq A_{15}
$$

where $A_{15}$ is a constant depending on $\alpha$ (and which is independent of $\alpha$ for $\alpha \geq 1$).
Proof. From Proposition 2.8 and the expansion of partial power sums in \( \text{GR} \):

\[
\sum_{r=0}^{a-1} r^n = \frac{a^n}{n + 1} \left( a + \sum_{t=0}^{n-1} \frac{B_{t+1}(n+1)t+1}{(t+1)!a^t} \right)
\]

it follows that

\[
\mathbb{E}_{C}(c^2) \quad = \quad k^2 - \frac{n(n+1)}{k^{n-1}} \sum_{j=1}^{k-1} j^n + \frac{n(nk - n - k)}{k^{n-1}} \sum_{j=1}^{k-1} j^{n-1}
\]

\[
= k^2 - nk \left( k + \sum_{t=0}^{n-1} \frac{B_{t+1}(n+1)t+1}{(t+1)!k^t} \right) + (nk - n - k) \left( \frac{nk - n - k}{k^{n-1}} \right)
\]

\[
= -nk - nk \sum_{t=0}^{n-1} \frac{B_{t+1}(n+1)t+1}{(t+1)!k^t} + \frac{n(nk - n - k)}{k^{n-1}} \sum_{t=0}^{n-2} \frac{B_{t+1}(n+1)t+1}{(t+1)!k^t}
\]

\[
= -nk - nk \sum_{t=0}^{n-1} \frac{B_{t+1}[(n+1)t+1 - (n)t+1]}{(t+1)!k^t} - (n + k) \sum_{t=0}^{n-1} \frac{B_{t+1}(n+1)t+1}{(t+1)!k^t}
\]

\[
- \frac{(nk - n - k)B_n}{k^{n-1}}.
\]

This simplifies to

\[
-nk - nk \sum_{t=0}^{n-1} \frac{B_{t+1}(n)t}{t!k^t} - (nk + k^2) \sum_{t=0}^{n-1} \frac{B_{t+1}(n)t}{(t+1)!k^{t+1}} + (nk - n - k) \frac{B_n}{k^{n-1}}
\]

\[
= k^2 - nk \sum_{t=0}^{n-1} \frac{B_{t+1}(n)t}{t!k^t} - (nk + k^2) \sum_{t=0}^{n-1} \frac{B_t(n)t}{t!k^t} - \frac{(nk - n - k)B_n}{k^{n-1}}
\]

\[
= \alpha^2 n^2 - \alpha n^2 \sum_{t=0}^{\frac{n-1}{\alpha}} \frac{B_{t+1}(n)t}{t!\alpha^t} - (\alpha n^2 + \alpha^2 n^2) \sum_{t=0}^{n} \frac{B_t}{t!\alpha^t}
\]

\[
+ \alpha^2 \sum_{t=0}^{\frac{n-1}{\alpha}} \frac{B_{t+1}(1 - \frac{n}{\alpha})t}{t!\alpha^t} + (\alpha n^2 + \alpha^2 n^2) \sum_{t=0}^{n} \frac{B_t(1 - \frac{n}{\alpha})t}{t!\alpha^t}
\]

\[
- \frac{(\alpha^2 - n - \alpha n)B_n}{(\alpha n)^{n-1}}.
\]
Lemma 4.2 implies that the absolute value of the difference between $E_C(c^2)$ and

$$\alpha^2 n^2 - \alpha n^2 \sum_{t=0}^\infty \frac{B_{t+1}}{t!\alpha^t} - (\alpha n^2 + \alpha^2 n^2) \sum_{t=0}^\infty \frac{B_t}{t!\alpha^t}$$

$$+ \alpha n \sum_{t=0}^{n-1} \frac{B_{t+1}(\frac{1}{2})}{t!\alpha^t} + (\alpha n + \alpha^2 n) \sum_{t=0}^n \frac{B_t(\frac{1}{2})}{t!\alpha^t}$$

is at most

$$\alpha^2 n^2 \sum_{t=n}^\infty \frac{|B_{t+1}|}{t!\alpha^t} + (\alpha n^2 + \alpha^2 n^2) \sum_{t=n+1}^\infty \frac{|B_t|}{t!\alpha^t} + \frac{\alpha}{2} \sum_{t=0}^{n-1} \frac{|B_{t+1}(\frac{1}{2})|^2}{t!\alpha^t}$$

$$+ \frac{\alpha}{2} \sum_{t=0}^{n-1} \frac{|B_{t+1}(\frac{1}{2})|^2}{t!\alpha^t} + \frac{|\alpha n^2 - n - \alpha n| |B_n|}{(\alpha n)^{n-1}}.$$ 

Thus the difference between $E_C(c^2)$ and

$$\alpha^2 n^2 - \alpha n^2 \sum_{t=0}^\infty \frac{B_{t+1}}{t!\alpha^t} - (\alpha n^2 + \alpha^2 n^2) \sum_{t=0}^\infty \frac{B_t}{t!\alpha^t}$$

$$+ \alpha n \sum_{t=0}^{n-1} \frac{B_{t+1}(\frac{1}{2})}{t!\alpha^t} + (\alpha n + \alpha^2 n) \sum_{t=0}^n \frac{B_t(\frac{1}{2})}{t!\alpha^t}$$

is upper bounded by the “error term”

$$\alpha^2 n^2 \sum_{t=n}^\infty \frac{|B_{t+1}|}{t!\alpha^t} \left(1 + \frac{\left(\frac{1}{2}\right)^t}{\alpha n}\right) + (\alpha n^2 + \alpha^2 n^2) \sum_{t=n+1}^\infty \frac{|B_t|}{t!\alpha^t} \left(1 + \frac{\left(\frac{1}{2}\right)^t}{\alpha n}\right)$$

$$+ \frac{\alpha}{2} \sum_{t=0}^{n-1} \frac{|B_{t+1}(\frac{1}{2})|^2}{t!\alpha^t} + \frac{\alpha + \alpha^2}{2} \sum_{t=0}^{n-1} \frac{|B_t(\frac{1}{2})|^2}{t!\alpha^t} + \frac{|\alpha n^2 - n - \alpha n| |B_n|}{(\alpha n)^{n-1}}.$$ 

Next it is necessary to bound the five summands in the error term. Lemma 4.3 shows that the first four summands in the error term are at most a constant depending on $\alpha$ (or a universal constant if $\alpha \geq 1$). Since $B_t$ vanishes for $t \geq 3$ odd, $|B_{2t}| \leq 8\sqrt{\pi t} (\frac{t}{e})^{2t}$ [Le], and $2\pi \alpha > 1$, the fifth summand in the error term goes to 0 much faster than a universal constant.

The result now follows by combining the above observations with Lemma 4.4 and Proposition 4.5. □

4.2. Further properties of the exchangeable pair. This subsection develops further properties of the exchangeable pair $(d, d')$ from Section 2. Since we are interested in central limit theorems, it is natural to instead study $(W, W')$ where $W = \frac{d - E_C(d)}{\sqrt{\text{Var}_C(d)}}$ and $W' = \frac{d' - E_C(d)}{\sqrt{\text{Var}_C(d)}}$. Note that from Lemma 2.1, one knows $E_C(W' - W\mid \pi)$. In what follows we study
\( \mathbb{E}_C(W' - W|W) \), which is typically used in normal approximation by Stein’s method.

**Proposition 4.7.**

\[
\mathbb{E}_C(W' - W|d = r) = -\frac{W}{n} + \frac{1}{n\sqrt{\text{Var}(d)}} \left( \sum_{\pi : d(\pi) = r} \frac{\mathbb{P}_C(\pi)}{\mathbb{P}_C(d = r)} \left[ -\frac{r}{n} + \frac{n - 1}{n} \chi_{(\pi) = 1} \right] \right).
\]

**Proof.** Since \( d \) is a function of \( \pi \),

\[
\mathbb{E}_C(W' - W|d = r) = \sum_a \frac{a \mathbb{P}_C(W' - W = a, d = r)}{\mathbb{P}_C(d = r)}
= \sum_a \sum_{\pi : d(\pi) = r} \frac{\mathbb{P}_C(W' - W = a, \pi)}{\mathbb{P}_C(d = r)}
= \sum_{\pi : d(\pi) = r} \frac{\mathbb{P}_C(\pi)}{\mathbb{P}_C(d = r)} \sum_a \frac{\mathbb{P}_C(W' - W = a, \pi)}{\mathbb{P}_C(\pi)}
= \sum_{\pi : d(\pi) = r} \frac{\mathbb{P}_C(\pi)}{\mathbb{P}_C(d = r)} \mathbb{E}_C(W' - W|\pi).
\]

By Lemma 2.1, this is equal to

\[
-\frac{1}{\sqrt{\text{Var}(d)}} \left( \sum_{\pi : d(\pi) = r} \frac{\mathbb{P}_C(\pi)}{\mathbb{P}_C(d = r)} \left[ -\frac{r}{n} + \frac{n - 1}{n} \chi_{(\pi) = 1} \right] \right)
= -\frac{W}{n} + \frac{1}{n\sqrt{\text{Var}(d)}} \left( \frac{(n - 1) \sum_{\pi : d(\pi) = r} \mathbb{P}_C(\pi) \chi_{(\pi) = 1}}{\mathbb{P}_C(d = r)} - \mathbb{E}_C(d) \right)
= -\frac{W}{n} + \frac{1}{n\sqrt{\text{Var}(d)}} \left( \frac{\mathbb{P}_C(c = r + 1) (r + 1)(n - 1)}{\mathbb{P}_C(d = r)} - \mathbb{E}_C(d) \right).
\]

\( \square \)

In most examples of Stein’s method for normal approximation of a random variable \( W \), there is an exchangeable pair \( (W, W') \) such that \( \mathbb{E}(W'|W) = (1 - \lambda)W \). There are two recent papers ([RR], [Ch]) in which the Stein technique has been extended to handle the case where \( \mathbb{E}(W'|W) = (1 - \lambda)W + G(W) \) where \( G(W) \) is small. The bounds in these papers require that \( \frac{\mathbb{E}(G(W))}{\lambda} \) goes to 0. Proposition 4.9 uses interesting properties (Lemma 2.8) and asymptotics of Eulerian numbers to prove that for our exchangeable pair \( \mathbb{E}(G(W)) \) is bounded away from 0, even for \( C_{\infty,k} \) (the uniform distribution on the symmetric group), where we know that a central limit theorem holds.
Lemma 4.8. Let $A_{n,k}$ denote the number of permutations on $n$ symbols with $k-1$ descents.

1. If $n$ is odd and $0 \leq r \leq n-1$, then $(r+1)A_{n-1,r+1} \geq (n-r)A_{n-1,r}$ if and only if $0 \leq r \leq \frac{n-1}{2}$.
2. If $n$ is even and $0 \leq r \leq n-1$, then $(r+1)A_{n-1,r+1} \geq (n-r)A_{n-1,r}$ if and only if $0 \leq r \leq \frac{n}{2} - 1$.

Proof. Suppose first that $n$ is odd. For the if part, we proceed by reverse induction on $r$; thus the base case is $r = \frac{n-1}{2}$, and then the inequality is an equality since $A_{n-1,k} = A_{n-1,n-k}$ for all $k$. A result of Frobenius is that the polynomial $\sum_{k \geq 1} z^k A_{n-1,k}$ has only real roots. Thus an inequality of Newton (page 52 of [HLP]) implies that

$$\frac{(r+1)A_{n-1,r+1}}{(n-r)A_{n-1,r}} \geq \frac{A_{n-1,r+2}}{A_{n-1,r+1}} \frac{(r+2)(n-r-1)}{(n-r)(n-r-2)}.$$  

By the induction hypothesis the right hand side is at least $\frac{(n-r-1)^2}{(n-r)(n-r-1)} > 1$.

The only if part follows from the if part since $A_{n-1,k} = A_{n-1,n-k}$ for all $k$.

The case of $n$ even is similar. For the if part, we proceed by reverse induction on $r$. The induction step is the same but base case $r = \frac{n}{2} - 1$ is not automatic. However it follows using Newton’s inequality of the previous paragraph together with the symmetry property $A_{n-1,k} = A_{n-1,n-k}$:

$$(A_{n-1,k})^2 \geq (A_{n-1,k-1})(A_{n-1,k+1}) \frac{(n/2 + 1)}{(n/2 - 1)}$$

$$= (A_{n-1,k-1})^2 \frac{(n/2 + 1)}{(n/2 - 1)}$$

$$> (A_{n-1,k-1})^2 (1 + \frac{2}{n})^2.$$

Now take square roots. The only if part follows from the if part since $A_{n-1,k} = A_{n-1,n-k}$ for all $k$. \qed

Proposition 4.9. Let $U_n$ denote the uniform distribution $U_n$ on $S_n$. Let $(W,W')$ be the exchangeable pair of this subsection, so that by Proposition 4.7 $\mathbb{E}(W'|W) = (1 - \lambda)W + G(W)$ with $\lambda = \frac{1}{n}$. Then $\mathbb{E}_{U_n}(|G(W)|)$ is bounded away from 0 as $n \to \infty$.

Proof. It is elementary that $\mathbb{E}_{U_n}(d) = \frac{n-1}{2}$. Thus Proposition 4.7 implies that

$$\mathbb{E}_{U_n}(|G(W)|) = \frac{(n-1)}{n^2 \sqrt{\text{Var}_{U_n}(d)}} \sum_{r=0}^{n-1} \left| (r+1) \mathbb{P}_{U_n}(c = r+1) - \frac{n}{2} \mathbb{P}_{U_n}(d = r) \right|.$$  

Using the equality

$$\mathbb{P}_{U_n}(d = r) = \frac{(r+1)}{n} \mathbb{P}_{U_n}(c = r+1) + \frac{(n-r)}{n} \mathbb{P}_{U_n}(c = r),$$

we can simplify the above expression. \qed
and the symmetry $A$ where this simplifies to

$$
\frac{(n-1)}{2n^2 \sqrt{Var_{U_n}(d)}} \sum_{r=0}^{n-1} |(r+1)\mathbb{P}_{U_n}(c = r + 1) - (n-r)\mathbb{P}_{U_n}(c = r)|.
$$

By Lemma 2.5, this further simplifies to

$$
\frac{(n-1)}{(n-1)!2n^2 \sqrt{Var_{U_n}(d)}} \sum_{r=0}^{n-1} |(r+1)A_{n-1,r+1} - (n-r)A_{n-1,r}|,
$$

where $A_{n,k}$ denotes the number of permutations in $S_n$ with $k-1$ descents.

Now suppose that $n$ is odd. Then the previous paragraph, Lemma 4.8, and the symmetry $A_{n-1,k} = A_{n-1,n-k}$ imply that $E_{U_n}(|G(W)|)$ is equal to

$$
\frac{(n-1)}{(n-1)!n^2 \sqrt{Var_{U_n}(d)}} \sum_{r=1}^{\frac{n}{2}} \left( rA_{n-1,r} - \sum_{r=1}^{\frac{n}{2}} (n-r)A_{n-1,r} \right)
$$

$$
= \frac{(n-1)}{(n-1)!n^2 \sqrt{Var_{U_n}(d)}} \left( \frac{n}{2}A_{n-1,\frac{n}{2}+1} + \sum_{r=1}^{\frac{n}{2}} (2r-n)A_{n-1,r} \right)
$$

$$
= \frac{(n-1)}{n^2 \sqrt{Var_{U_n}(d)}} \left( \frac{n+1}{2}A_{n-1,\frac{n}{2}+1} - \mathbb{E}_{U_{n-1}}[d - \mathbb{E}_{U_{n-1}}(d)] \right)
$$

$$
\geq \frac{(n-1)}{n^2 \sqrt{Var_{U_n}(d)}} \left( \frac{n+1}{2}A_{n-1,\frac{n}{2}+1} - \sqrt{Var_{U_{n-1}}(d)} \right).
$$

A similar argument for $n$ even shows that

$$
\mathbb{E}_{U_n}(|G(W)|) \geq \frac{(n-1)}{n^2 \sqrt{Var_{U_n}(d)}} \left( \frac{nA_{n-1,\frac{n}{2}}}{2(n-1)!} - \sqrt{Var_{U_{n-1}}(d)} \right).
$$

To conclude the argument, note that for $n \geq 2$, $Var_{U_n}(d) = \frac{n+1}{12}$, and that $\frac{A_{n-1,\frac{n}{2}}}{(n-1)!}$ is asymptotic to $\sqrt{\frac{6}{n\pi}}$. Thus $n\mathbb{E}_{U_n}(|G(W)|)$ is bounded away from 0, as desired. \hfill \Box

4.3. A Sampling Algorithm. To conclude the paper, we record a fast algorithm for drawing samples from the measure $R_{k,n}$. Since $C_{k,n}$ can be obtained by sampling from $R_{k,n}$ and then performing uniformly at random one of the $n$ cyclic rotations of the bottom row in the two line form of a permutation, we only give an algorithm for sampling from $R_{k,n}$. This algorithm should be quite useful for empirically studying the transition from Poisson to normal behavior.

We use the terminology that if $\pi$ is a permutation on $n$ symbols, the permutation $\tau$ on $n+1$ symbols obtained by inserting $n+1$ after position
Proposition 4.10. **Starting with the identity permutation in** \( S_1 \), **transition from an element** \( \pi \) **of** \( S_n \) **to an element** \( \tau \) **of** \( S_{n+1} \) **by inserting** \( n+1 \) **as described by the following 2 cases:**

(1) **If either** \( j = n \) **or** \( \pi(j) > \pi(j+1) \) **and** \( 1 \leq j \leq n-1 \), **the chance of inserting** \( n+1 \) **after** \( j \) **is** \( \frac{n+k-d(\pi)}{k(n+1)} \).

(2) **If either** \( j = 0 \) **or** \( \pi(j) < \pi(j+1) \) **and** \( 1 \leq j \leq n-1 \), **the chance of inserting** \( n+1 \) **after** \( j \) **is** \( \frac{k-d(\pi)-1}{k(n+1)} \).

Then after running the algorithm for \( n-1 \) steps, the distribution obtained on \( S_n \) is precisely \( R_{k,n} \).

**Proof.** First note that the transition probabilities of the algorithm sum to 1, since

\[
(d(\pi) + 1) \left( \frac{n+k-d(\pi)}{k(n+1)} \right) + (n-d(\pi)) \left( \frac{k-d(\pi)-1}{k(n+1)} \right) = 1.
\]

Now observe that if \( \tau \) is obtained from \( \pi \) by a Case 1 move, then \( d(\tau) = d(\pi) \). Thus from the formula for \( R_{k,n} \), \( \frac{R_{k,n+1}(\tau)}{R_{k,n}(\pi)} = \frac{n+k-d(\pi)}{k(n+1)} \). Similarly, if \( \tau \) is obtained from \( \pi \) by a Case 2 move, then \( d(\tau) = d(\pi) + 1 \). Thus \( \frac{R_{k,n+1}(\tau)}{R_{k,n}(\pi)} = \frac{k-d(\pi)-1}{k(n+1)} \).

\[\square\]

5. **Acknowledgements**

The author was partially supported by NSA grant MDA904-03-1-0049.

**References**

[A] Aldous, D., Random walks on finite groups and rapidly mixing Markov chains, in Springer Lecture Notes in Math. 986, Springer-Verlag, New York, 1983, pp. 243-297.

[BarHJ] Barbour, A., Holst, L., and Janson, S., *Poisson approximation*. Clarendon Press, Oxford, 1992.

[BayDi] Bayer, D. and Diaconis, P., Trailing the dovetail shuffle to its lair, *Annals Appl. Probab.* 2 (1992), 294-313.

[BeDu] Berestycki, N. and Durrett, R., A phase transition in the random transposition walk, paper [math.PR/0403259] at [http://xxx.lanl.gov](http://xxx.lanl.gov).

[BoPe] Bourque, G. and Pevzner, P., Genome-scale evolution: reconstructing gene orders in ancestral species, *Genome Research*. 12, 26-36.

[CKSS] Carlitz, L., Kurtz, D., Scoville, R., and Stackelberg, O., Asymptotic properties of Eulerian numbers, *Z. Wahrsch. Verw. Gebiete* 23 (1972), 47-54.

[Ch] Chatterjee, S., An abstract Berry-Esseen theorem, preprint.

[Di] Diaconis, P., Mathematical developments arising from the analysis of riffle shuffling, in “Groups, combinatorics, and geometry (Durham, 2001)”, World Sci. Publishing, NJ, 2003, 73-97.

[Du] Durrett, R., Genome rearrangement: recent progress and open problems, (2003), at [http://www.math.cornell.edu/~durrett](http://www.math.cornell.edu/~durrett).
[Fe] Feller, W., *An introduction to probability theory and its applications*, Second edition. John Wiley and Sons, 1957.

[FoS] Foata, D. and Schutzenberger, M., *Theorie geometrique des polynomes euleriens*, Springer Lecture Notes in Math. 138, Springer-Verlag, Berlin-New York, 1970.

[Fu1] Fulman, J., Affine shuffles, shuffles with cuts, the Whitehouse module, and patience sorting, *J. Algebra* 231 (2000), 614-639.

[Fu2] Fulman, J., Stein's method and nonreversible Markov chains, in “Stein’s method: expository lectures and applications”, IMS Lecture Notes Volume 46 (2004), pages 69-77.

[Fu3] Fulman, J., A card shuffling analysis of deformations of the Plancherel measure of the symmetric group, *Elec. J. Combin.* 11 (2004), Research paper 21, 15 pp. (electronic).

[Fr] Frobenius, Ueber die Bernoullischen zahlen und die Eulerschen polynome, *Sitz. Ber. Preuss. Akad. Wiss.* (1910), 808-847.

[GR] Gradshteyn, I. and Ryzhik, I., *Tables of integrals, series and products*, Sixth edition, Academic Press, San Diego, 2000.

[HaPe] Hannenhalli, S. and Pevzner, P., Transforming cabbage into turnip (polynomial algorithm for sorting signed permutations by reversals), *Journal of the ACM* 46 (1999), 1-27.

[HLP] Hardy, G.E., Littlewood, J.E., and Polya, G., *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 1952.

[Le] Leeming, D., The real zeros of the Bernoulli polynomials, *J. Approx. Theory* 58 (1989), 124-150.

[Ma] Mann, B., Shuffling n cards with cn^{3/2} hands gives asymptotically normal rising sequences, Unpublished, undated manuscript from mid 1990’s.

[P] Pevzner, P., *Computational molecular biology: an algorithmic approach*, MIT Press, 2000.

[RR] Rinott, Y. and Rotar, V., On coupling constructions with rates in the CLT for dependent summands with applications to the antiboter model and weighted U-statistics, *Annals Appl. Probab.* 7 (1997), 1080-1105.

[S GO] Stark, D., Ganesh, A., and O’Connell, N., Information loss in riffle shuffling, *Combin., Probab., and Computing* 11 (2002), 79-95.

[St] Stein, C., *Approximate computation of expectations*, Institute of Mathematical Statistics Lecture Notes, Volume 7, 1986.

[T] Tanny, S., A probabilistic interpretation of the Eulerian numbers, *Duke Math. J.* 40 (1973), 717-722. [Correction (1974) 41 689].