Further discussion of Tomboulis’ approach to the confinement problem

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Abstract

We discuss in some detail certain gaps and open problems in the recent paper by E. T. Tomboulis, which claims to give a rigorous proof of quark confinement in 4D lattice Yang-Mills theory for all values of the bare coupling. We also discuss what would be needed to fill the gaps in his proof.

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1 Introduction

A paper by E. T. Tomboulis that was posted last year on the arXiv [1] claims to present a rigorous proof of quark confinement [2] in 4D lattice gauge theory. This problem has been fascinating many physicists since the last century and still remains an important open problem; in fact, its resolution probably would represent a first step towards the solution of one of the millennium problems posed by the Clay Mathematics Institute [3]. In a comment posted on the arXiv [4] we pointed out that in our opinion the proof in [1] depended on a claim which remains to be proved. The purpose of this public comment was to prevent confusion in the community about the question whether confinement on the lattice was proven or not. Tomboulis then posted a reply [5] on the arXiv stating that the presumed gap found by us was simply due to a misunderstanding of his argument.

We appreciate that Tomboulis gave a concise and clear exposition of the logic of his argument in his reply. But we still think that our original criticism stands and in this note we try to give a more detailed explanation of the main open problems we see in Tomboulis’ approach. We also discuss possible ways to complete his proof.

2 Tomboulis’ claimed theorem

To make this note self-contained, we summarize Tomboulis’ notation and his arguments with some simplifications:

1. $\Lambda \subset \mathbb{Z}^4$ is a box in $\mathbb{Z}^4$ of size $L_1 \times \ldots \times L_4$ with center at the origin, where $L_i = ab^{n_i}$, $n_i \gg 1$, $i = 1, \ldots, 4$ and $a, b$ are positive integers larger than 1.

2. $\Lambda^{(k)} \subset \mathbb{Z}^4$ is the box in $\mathbb{Z}^4$ of each side length $L_i b^{-k}$ obtained from $\Lambda$ by $k$ steps of the renormalization transformation of Migdal-Kadanoff type ($\Lambda = \Lambda^{(0)}$). Periodic boundary conditions are employed in all directions of $\mu = 1, \ldots, 4$. (actually periodic boundary conditions in the directions of $\mu = 3, 4$ would be enough.)

3. $f(\{c_j\}, U) = 1 + \sum_{j \neq 0} c_j d_j \chi_j(U)$ where $d_j$ is the dimension of the representation $\chi_j$, and we assume $U \in G = SU(2)$. Moreover $U =$
\[ U_p = \prod_{b \in \partial p} U_b \] for a plaquette \( p \subset \Lambda \) which consists of bonds \( b \in \partial p \) oriented positively with respect to \( p \).

We start with

\[
\exp \left[ \frac{\beta}{2} \chi_{1/2}(U) \right] = F_0(0)f(\{c_j\}, U) \tag{2.1a}
\]

\[
f(\{c_j\}, U) = 1 + \sum_{j \neq 0} c_j d_j \chi_j(U) \tag{2.1b}
\]

\[
F_0(0) = \int \exp \left[ \frac{\beta}{2} \chi_{1/2}(U) \right] dU \tag{2.1c}
\]

where \( dU \) is the Haar measure of \( G = SU(2) \). We then apply the renormalization group (RG) recursion formulas of Migdal-Kadanoff (MK) type \[6\] with some modifying parameters. If there are no such parameters (the standard recursion formula), we have

\[
f^{(n-1)}(U) = f(\{c_j(n-1)\}, U) = 1 + \sum_{j \neq 0} c_j(n-1) d_j \chi_j(U) \rightarrow f^{(n)}(U) = f(\{c_j(n)\}, U)
\]

where

\[
f^{(n)}(U) = \frac{1}{F_0(n)} \int \left[ f^{(n-1)}(U_{b_1}) f^{(n-1)}(U_{b_2}^{-1}) \cdots f^{(n-1)}(U_{b_2}^{-1}) \right] dU_k \tag{2.2a}
\]

\[
F_0(n) = \left( \int [f^{(n-1)}(U)] dU \right)^{b^2} \tag{2.2b}
\]

or equivalently

\[
c_j(n) = (c_j(n))^{b^2} = \frac{F_j(n)}{F_0(n)} \tag{2.3a}
\]

\[
F_j(n) = \left( \int [f^{(n-1)}(U)]^{b^2} \chi_j(U) dU \right)^{b^2} \tag{2.3b}
\]

in terms of the coefficients of the character expansions. Then \[7\]:

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Theorem 2.1 For $D \leq 4$ and for $G = SU(N)$ or $G = U(N)$,

$$\lim_{n \to \infty} c_j(n) = 0 \quad \text{for } j \neq 0$$

These recursions are just approximate and yield upper bounds for the partition functions \[1\]. Tomboulis \[1\] then introduces two parameters $\alpha \in (0, 1]$ and $t > 0$ and a function $h(\alpha, t) \in (0, 1]$, designed so that this transformation becomes numerically exact for the partition function:

$$Z = \int dU_\Lambda \prod_{p \subset \Lambda} f(\{c_j\}, U_p)$$

$$= [F_0(1)^{h(\alpha, t)\Lambda(1)}] \int dU_{\Lambda(1)} \prod_{p \subset \Lambda(1)} f(\{\tilde{c}_j(\alpha)\}, U_p)$$

$$= \tilde{Z}_1(\tilde{c}(\alpha), t)$$

where $dU_\Lambda = \prod_{b \in \Lambda} dU_b$,

$$\tilde{Z}_1(\tilde{c}(\alpha), t) = [F_0(1)^{h(\alpha, t)\Lambda(1)}]Z_1$$

$$Z_1 = \int dU_{\Lambda(1)} \prod_{p \subset \Lambda(1)} f(\{\tilde{c}_j(\alpha)\}, U_p)$$

and as usual, $U_p = \prod_{b \in \partial p} U_b$ are plaquette actions defined as the product of group elements $U_b = U_{(x,x+e_\mu)} \in G$ attached to the (oriented) bonds $b \in \Lambda$. Moreover

$$h(\alpha, t) = \exp[-t(1 - \alpha)/\alpha]$$

$$\tilde{c}_j(\alpha) = \tilde{c}_j^{(1)}(\alpha) = \alpha c_j(1)$$

$$c_j(1) = \frac{F_j(1)}{F_0(1)}$$

( where without loss of generality we have chosen one particular form of the function $h$, as suggested in \[1\]). Given $t > 0$, $\alpha \in [0, 1]$ is chosen so that the above relation becomes exact; thus $\alpha$ has to be considered as a function of $t$.

The author of \[1\] then introduces a vortex sheet $V = \{v \subset \Lambda\}$ \[8, 9\] which is a collection of plaquettes $\{v = \{x_0 + n_3e_3 + n_4e_4, x_0 + e_1 + n_3e_3 + n_4e_4, x_0 + n_3e_3 + n_4e_4, x_0 + \}$
\[ e_2 + n_3 e_3 + n_4 e_4, x_0 + e_1 + e_2 + n_3 e_3 + n_4 e_4, \{ n_i = 0, 1, \ldots, L_i, (i = 3, 4) \}, \]

i.e. a plaquette \( p = (x_0, x_0 + e_1, x_0 + e_1 + e_2, x_0 + e_2) \) in an \( x_1 - x_2 \) plane and its translates along the axis normal to \( p \) (say, 3rd and 4th axis). Following [1] we define

\[
Z^{(-)} = \int dU_{\Lambda} \prod_{p \subset \Lambda} f(\{ c_j \}, (-1)^{\nu(p)} U_p) \tag{2.7}
\]

where

\[
\nu(p) = \begin{cases} 
0 & \text{if } p \notin V \\
1 & \text{if } p \in V 
\end{cases}
\tag{2.8}
\]

and then

\[
Z^{(-)} = \int dU_{\Lambda} \prod_{p \subset \Lambda \setminus V} (1 + \sum_{j \neq 0} c_j d_j \chi_j(U_p)) \\
\times \prod_{q \subset V} (1 + \sum_{j \neq 0} (-1)^{2j} c_j d_j \chi_j(U_q)) \tag{2.9}
\]

Note that the position of the vortex sheet \( V \subset \Lambda \) can be freely moved in the \( x_1 - x_2 \) plane by gauge invariance.

First of all, note that the coefficients of \( \chi_j(U) \) in \( \prod_p (1 + \sum_{j \neq 0} c_j d_j \chi_j(U_p)) \) and

\[
\prod_p (1 + \sum_{j \neq 0} c_j d_j \chi_j(U_p)) + \prod_p (1 + \sum_{j \neq 0} c_j d_j \chi_j((-1)^{\nu(p)} U_p))
\]

are nonnegative. The main claims in [1] are:

Claim 2.1 (1) There exist \( t, t^+ > 0 \) and functions \( \alpha \) and \( \alpha^+ \) such that

\[
\frac{Z_{\omega}^{(-)}(\{ c_j \})}{Z_{\omega}(\{ c_j \})} = \frac{Z_{\omega}^{(-)}(\{ \tilde{c}_j^{(n)}(\alpha^+(t^+)) \})}{Z_{\omega}(\{ \tilde{c}_j^{(n)}(\alpha(t)) \})}
\]

where

\[
\tilde{c}_j^{(n)}(\alpha(t)) = \alpha(t)c_j(n).
\]

(2) For large \( n \) such that \( \{ c_j(n) \} \) are sufficiently small, there exists a \( t_* \geq 0 \) (the same for numerator and denominator!) such that

\[
\frac{Z_{\omega}^{(-)}(\{ c_j \})}{Z_{\omega}(\{ c_j \})} = \frac{Z_{\omega}^{(-)}(\{ \tilde{c}_j^{(n)}(\alpha(t_*)) \})}{Z_{\omega}(\{ \tilde{c}_j^{(n)}(\alpha(t_*)) \})}
\]

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If this were correct, the following would be true:
\[
\frac{Z^{(-)}_\Lambda(\{c_j\})}{Z_\Lambda(\{c_j\})} = \frac{Z^{(-)}_{\Lambda(\alpha)}(\{\tilde{c}^{(n)}_j(\alpha(t))\})}{Z_{\Lambda(\alpha)}(\{\tilde{c}^{(n)}_j(\alpha(t))\})}
\]

Since for the (unmodified) Migdal-Kadanoff RG described above, \(\{c^{(n)}_j \geq 0\}\) tends to the high temperature fixed point (i.e. \(\{c^{(n)}_j(n) \to 0\}\) as \(n \to \infty\) if the dimension is \(\leq 4\), whether \(G\) is abelian or non-abelian (see the remark below and [7]), this would mean strict positivity of 't Hooft’s string tension and thus establish permanent confinement of quarks in the sense of Wilson at least for all values of the bare coupling constant [9, 10] in 4 dimensional lattice gauge theory, thereby solving part of a longstanding problem in modern physics (the missing part, as Tomboulis makes clear, would still be the proof that there is a continuum limit in which confinement persists).

As pointed out in our earlier note [4], this cannot possibly be correct, however, since there exists a deconfining transition in 4D lattice gauge theory based on abelian gauge groups. But it is meaningful for the future study of the confinement problem to ask what is going wrong in reaching this conclusion.

The original idea to study vortex free energies in order to understand confinement was formulated by Mack and Petkova in [9]. They fix boundary variables \(G_{\partial \Lambda} \equiv \{g_b \in G; b = (x, x + e_\mu) \in \partial \Lambda\}\) and take the maximum of \(Z^{(-)}_\Lambda/Z_\Lambda\) over \(G_{\partial \Lambda}\). On the other hand 't Hooft’s string tension is defined using (twisted) periodic boundary conditions at \(\partial \Lambda\), see [10]. The following are well-known facts:

**Theorem 2.2** Define
\[
F \equiv -\log \frac{Z^{(-)}_\Lambda}{Z_\Lambda}
\]

(1) Let \(G = U(1)\) or \(G = SU(2)\). Then there is a \(\beta_1 > 0\) such that for \(\beta < \beta_1\)
\[
\lim_{L_3L_4 \to \infty} \frac{1}{L_3L_4} F \leq \text{const} \exp[-\sigma L_1L_2]
\]
(area decay law). \(\sigma > 0\) is, if chosen maximally, is 't Hooft’s string tension.

(2) Let \(G = U(1)\). Then there is a \(\beta_0 < \infty\) such that for \(\beta > \beta_0\)
\[
\lim_{L_3L_4 \to \infty} \frac{1}{L_3L_4} F \geq \text{const} \frac{1}{L_1L_2}
\]
(deconfinement).
(1) is a standard result of the convergent high-temperature expansion (see for instance [14]). The analogous statement for $SU(N)$ holds with $Z(-)$ replaced by $Z^\omega$, the partition function twisted by an element $\omega$ of the center of $SU(N)$.

(2) is a variation of the results of [11] and [12] (see [13]).

Remarks 2.1 (1) The introduction of $0 < \alpha \leq 1$ into $1 + \sum_{j \neq 0} c_j d_j \chi_j(U)$ does not violate conditions (positivity, analyticity, class functions etc.) on $f^{(n)}(v)$ in [7] since

$$1 + \sum_{j \neq 0} \alpha c_j d_j \chi_j(U) = (1 - \alpha) + \alpha \left( 1 + \sum_{j \neq 0} c_j d_j \chi_j(U) \right)$$

and $0 < \alpha \leq 1$. Then \{c_j(n) \geq 0\} tends to 0 as $n \to \infty$ (if $r = 1$, see Section 3.3).

(2) The conjecture raised in [9] is proved rigorously in [10]. Namely 't Hooft’s string tension is smaller than or equal to Wilson’s string tension.

(3) The center of $G = U(1)$ is again $U(1)$. Then we consider $U(p) = \cos(\theta_p)$ and $U_\omega(p) = \cos(\theta_p + \omega)$ for $p \in V$. $U_\omega(p) = -U(p)$ for $\omega = \pi$.

(4) The argument in [11] uses the Fourier transformation of $e^{\beta \cos \theta}$ and Poincaré’s lemma. Since $\Lambda$ is a torus, Poincaré’s lemma has to be replaced by the Hodge decomposition in order to adapt the arguments in [11] or [12]. See [13] and [14].

3 Tomboulis’ Proof revisited

We now analyze the arguments in [11] in more detail. First, using the fact that the partition function $Z = Z_\Lambda$ increases by the Migidal-Kadanoff (MK) recursion formula [1], Tomboulis introduces two interpolation parameters $\alpha$ ($Z$ increases as $\alpha \nearrow 1$) and $t$ (the factor $|F_0(n)|^{h(\alpha,t)}$ decreases as $t$ increases).

Then he claims that there exist functions $\alpha(t)$ and $\alpha^+(t)$ such that

$$Z_\Lambda(\{c_j\}) = [F_0(1)]^{h(\alpha(t),t)\Lambda(\alpha t)} Z_{\Lambda(\alpha t)}(\{\tilde{c}_j(\alpha(t))\})$$  \hspace{1cm} (3.1a)

$$Z_+^\Lambda(\{c_j\}) = [F_0(1)]^{h(\alpha^+(t),t)\Lambda(\alpha^+ t)} Z_{\Lambda(\alpha^+ t)}(\{\tilde{c}_j(\alpha^+(t))\})$$  \hspace{1cm} (3.1b)

where

$$Z^+ = \frac{1}{2}(Z + Z(-))$$
and the right hand sides are manifestly independent of $t$.

The author of [1] then claims is that there exists a $t_\ast > 0$ such that

$$\alpha(t_\ast) = \alpha^+(t_\ast)$$

which yields

$$\frac{Z^+_A(\{c_j\})}{Z_A(\{c_j\})} = \frac{Z^+_A(\{\tilde{c}_j\})}{Z_A(\{\tilde{c}_j\})}$$

where

$$\tilde{c}_j = \tilde{c}_j(\alpha(t_\ast)).$$

The procedure is then iterated to produce an analogous relation involving the $n$–th iteration of the MK RG transformation.

Two arguments are given in [1] for this proof.

large $\beta$ region: p.24–p.26, based on the arguments in pages 14,15,22, 23 and Appendix B.

small $\beta$ region: p.26–p.27 and Appendix C.

The first argument is used to drive the system to the high-temperature (small $\beta$) region by the MK recursion formulas, and the second argument depends on the implicit function theorem to derive the main claim which yields strict positivity of ’t Hooft’s string tension.

### 3.1 Outline of the Proof in [1]

To understand the proof in in [1], we streamline the arguments in [1] by extracting key parts from [1] (some equations are added below by the present authors). But before doing so, we have to stress that Tomboulis is modifying the MK recursion by introducing a further parameter $r \in (0, 1]$ (Eq. (2.19) in [1]), which is the source of some serious problems, because any choice $r < 1$ changes the MK RG flow for large $\beta$ drastically (see below).

$r$ is introduced by replacing (2.3a) with $c_j(n) = \hat{c}_j(n)^{b^2r}$ (this amounts to replacing the $b^2$ convolutions in Eq.(2.2a) by $b^2r$ convolutions).
\[ \frac{\partial \alpha_{\Lambda,h}(t,r)}{\partial t} = v(\alpha_{\Lambda,h}(t,r), t,r) \] (3.25)

where

\[ v(\alpha, t, r) \equiv -\frac{\partial h(\alpha, t) / \partial t}{\partial h(\alpha, t) / \partial \alpha} + A_{\Lambda(m)}(\alpha, r) \] (3.26)

with \( \partial h(\alpha, t) / \partial t = -[(1 - \alpha) / \alpha]h(\alpha, t) < 0 \) and

\[ \frac{\partial}{\partial t} \alpha_{\Lambda,h}(t,r) > \eta_1(\delta) > 0 \] (3.32)

\[ \delta' < \alpha_{\Lambda,h}(t,r) < 1 - \delta \] (3.30)

... by choosing \( r \) to vary, if necessary, away from unity in the domain

\[ 1 > r > 1 - \varepsilon \] (3.31)

where \( 0 < \varepsilon << 1 \) with \( \varepsilon \) independent of \( |\Lambda^{(m)}| \). With (3.30) in place, (3.25) and (3.29) imply (Appendix B) that

\[ \frac{\partial}{\partial t} \alpha_{\Lambda,h}^+(t,r) > \eta_1^+(\delta^+) > 0 \] (4.25)

holds ...
Quotation (II): [page 25] ... (5.4), taken at general $r$, implies that...

$$\left| \alpha_{A,h}^+(t,r) - \alpha_{A,h}(t,r) \right| \leq O\left( \frac{1}{|A|} \right)$$ (5.6)

... For any given $t_1^+$, choose $t_1$ in $\tilde{Z}_{A,h}(\beta, \alpha_{A,h}(t_1), t_1)$ so that

$$h(\alpha_{A,h}(t_1), t_1) = h(\alpha_{A,h}^+(t_1^+), t_1^+)$$ (5.7)

This is clearly always possible by (3.32) and (4.25), and by (5.6).

[top of page 26] We may now iterate this procedure $(n-1)$ decimation steps ..., at each step choosing $t_m$ and $t_m^+$ such that

$$h(\alpha_{A,h}^{(m)}(t_m), t_m) = h(\alpha_{A,h}^{+(m)}(t_m^+), t_m^+)$$ (5.9)

Carrying out ... one obtains

$$\left( 1 + \frac{Z_A^{(-)}}{Z_A} \right) = \frac{2\tilde{Z}_{A^{(n)}}^+(\beta, h, \alpha_{A,h}^{+(n)}(t^+, t^+))}{\tilde{Z}_{A^{(n)}}(\beta, h, \alpha_{A,h}^{(n)}(t), t)}$$ (5.10)

Quotation (III): [middle of page 26] Next, consider (5.10) rewritten as

$$\left( 1 + \frac{Z_A^{(-)}}{Z_A} \right) = \left( \frac{Z_{A^{(n-1)}}^+(\beta, h, \alpha_{A,h}^{(n)}(t), t)}{\tilde{Z}_{A^{(n-1)}}^+(\beta, h, \alpha_{A,h}^{(n)}(t), t)} \right) \left( 1 + \frac{Z_{A^{(n)}}^{(-)}(\{\tilde{c}_j(\ldots)\})}{\tilde{Z}_{A^{(n)}}(\{\tilde{c}_j(\ldots)\})} \right)$$ (5.13)

... It is then natural to ask whether there exists a value $t = t_{A,h}^{(n)}$ such that

$$\tilde{Z}_{A^{(n-1)}}^+(\beta, h, \alpha_{A,h}^{(n)}(t), t) = Z_{A^{(n-1)}}^+$$ (5.14)

Note that the graphs of $\alpha_{A,h}^{(n)}(t)$ and $\alpha_{A,h}^{+(n)}(t)$ must intersect at $t_{A,h}^{(n)}$. A unique solution to (5.14) indeed exists as shown in Appendix C provided

$$A_{A^{(n)}}(\alpha, r) \geq A_{A^{(n)}}^+(\alpha, r)$$ (5.15)
Consider $n$ successive decimation steps performed according the scheme (4.28). Assume that there is an $n_0$ such that the upper bound coefficients $c^U_j(n)$ become sufficiently small for $n > n_0$.

With the notation

$$\Phi^+(\alpha) \equiv \frac{1}{\ln F^U_0(n)} \frac{1}{|\Lambda^{(n)}|} \ln Z^+_{\Lambda^n}(\{\tilde{c}_j(n, \alpha)\})$$

we now define

$$\Psi(\lambda, t) \equiv h(\alpha(t), t) + (1 - \lambda)\Phi^+_{\Lambda^n}(\alpha_h(t)) + \lambda\Phi^+_{\Lambda^n}(\alpha_h(t)) - \Phi^+_{\Lambda^{n-1}}$$

and consider the equation

$$\Psi(\lambda, t) = 0$$

At $\lambda = 0$, eq.(C.7) is solved by setting $t = t_0$ . . . . By the implicit function theorem, if grad $\Psi$ is continuous and $\partial \Psi / \partial t \neq 0$, there exists a branch $t(\lambda)$ through $(0, t_0)$ on a . . . . Thus, from (C.8), $t(\lambda)$ . . . extends to the solution $t(1) > t_0$.

Quotation (III) is the main result and quotation (II) is used as the bridge to reach the main conclusion. The summary of the argument in [1] is:

(I) If $r < 1 - \varepsilon$ then $\alpha < 1 - \delta < 1$ and $\alpha^+ < 1 - \delta < 1$. $(\delta > 0)$. Thus $v = \partial \alpha(t, r) / \partial t$ and $v^+ = \partial \alpha^+(t, r) / \partial t$ are bounded from below by strictly positive constants $\eta_1 > 0$ and $\eta_2 > 0$.

(II) Since $|\alpha^+(t^+) - \alpha(t)| = O(|\Lambda_1|^{-1})$, the existence of $t_1$ such that

$$h(\alpha_{\Lambda, h}(t_1), t_1) = h(\alpha^+_{\Lambda, h}(t^+_1), t^+_1)$$

follows by a shift of $t$. Moreover it says that one can find $t = t_m$ successively to obtain $Z^+_{\Lambda(m)}(\alpha^+(m)(t^+)) / Z_{\Lambda(m)}(\alpha(m)(t))$. This proves Claim 2.1 (1).

(III) After some steps, $\{\tilde{c}_j(n) \geq 0\}$ are sufficiently small for $j \neq 0$ and we can find $t = t_n$ such that $\alpha(n)(t) = \alpha^+(n)(t)$ by using the implicit function theorem. This leads us to Claim 2.1 (2).
3.2 Problematic parts in the proof

The previous proof may look perfect, but there must be something missing since this proof seems to work even for \( G = U(1) \). We mention some problematic points:

Problem 1. The existence of \( t_m \) satisfying (5.9) in [I] or in (II) is not established.

Problem 2. The existence of \( t^* \) satisfying
\[
\alpha^{+ (n)}_{\Lambda (n), \hat{h}}(t^*) = \alpha^{(n)}_{\Lambda (n), \hat{h}}(t^*)
\]
is not established.

Problem 3. The implicit function theorem is used to obtain \( t(\lambda) \) which satisfies \( \Psi(\lambda, t) = 0 \). It is not proven that \( t(\lambda) \) can be extended from \( t = 0 \) to \( t = 1 \) even if \( \Psi_t \neq 0 \).

First we remark that all these claims depend on the hypothesis that both \( \partial \alpha^{(n)}(t)/\partial t \) and \( \partial \alpha^{+ (n)}(t)/\partial t \) are bounded from below by strictly positive constants \( \eta_1 > 0 \) and \( \eta_2 > 0 \) respectively, uniformly in \( t \in \mathbb{R} \). But this cannot be correct without assuming \( \alpha(t) \leq 1 - \delta \) (\( \delta > 0 \)) for all \( t \) uniformly in \( n \) or \( \Lambda^{(n)} \) because of the form of \( h(\alpha, t) = \exp[-t(1 - \alpha)/\alpha] \). See (I). It is, however, very plausible that \( \alpha \) is very close to 1 since the original \( \alpha = 1 \) MK approximate formulas are very accurate. To ensure those lower bounds, the parameter \( r \) is introduced. We suspect that this is the origin of the contradictions as we explain in the next subsection.

Secondly we need to find \( t^* \) such that \( \alpha(t^*) = \alpha^+(t^*) \), otherwise we compare two different physical systems, which is absolutely meaningless. To do so, the author of [I] appeals to the implicit function theorem. But his claim does not follow from this theorem directly. We need that the map is contractive to ensure that this claim is correct.

3.3 The origin of the problems.

The author of [I] tries to ensure that \( \partial \alpha/\partial t = v \) and \( \partial \alpha^+/\partial t = v^+ \) are strictly bounded away from zero, and this is the reason for the introduction
of the parameter \( r \). The derivatives \( v \) and \( v^+ \) contain \( 1 - \alpha \) in the numerator and \( A_{\Lambda(m)} \) in the denominator which may make \( v \) and \( v^+ \) small. Here \( A_{\Lambda(m)} \) is (almost) equivalent to the derivative of the free energy by the (inverse) temperature and is similar to the specific heat, see (3.27) in Quotation (I) or in [1]. It is easily seen that \( A_{\Lambda(m)} > 0 \) is bounded from above since \( f((c_j), U) \geq 0 \). So the denominator is not dangerous.

Thus we have to worry about \( 1 - \alpha \to 0 \). Since the MK recursion formulas are very accurate for large \( \beta \), \( \alpha \) seems to be very close to 1 for large \( \beta \). (This is so even though the MK recursion formulas fail to describe the deconfining transition of the \( U(1) \) model even for \( \alpha = r = 1 \).) For this reason, the parameter \( r \in (0, 1] \) is introduced by eq.(2.19) in [1] so that \( \alpha < 1 - \delta \). But the price to pay for this is that \( c_j(n) \) may not converge to 0 (the strong coupling fixed point) as \( n \to \infty \! \) !

As remarked above, \( c_j(n) = \hat{c}_j(n)b^{2r} \) means that we replace \( b^2 \) convolutions in (2.2a) by \( b^{2r} \) convolutions. If \( r \) is chosen less than 1, the effective dimension of the MK recursion formula increases from \( D = 4 \) to \( D \geq 4: \)

\[
D = 4 \to D = 4 - \frac{2 \log r}{2 \log b + \log r}
\]

see [7]. The choice \( r = 1 \) corresponds to the original MK formula and \( D = 4 \) becomes the critical dimension in the sense that the number of convolutions (i.e. \( b^2 \)) is equal to the number of block plaquettes to be glued together (i.e. \( b^2 \)).

If \( r < 1 \), the recursion formula drives the system to the weak coupling fixed point if \( \beta \) is large and \( \alpha \) is close to 1. Namely if \( r < 1 \) and \( \beta \) is large, Theorem 2.1 does not hold and \( \lim_{n \to \infty} c_j(n) \neq 0 \) in general.

The following pedagogical example makes this clear: Set \( f_0(U) = \exp[-\beta \theta^2/2] \), where \( \beta > 0 \) and \( \theta \in R \). This corresponds to (non-compact) abelian lattice QED. Then for the choice of \( r \in (0, 1] \), from (2.2a), we get

\[
f_0 \to f_n(\theta) = \exp \left[ -\frac{\beta}{2r^n} \theta^2 \right]
\]

which for \( r < 1 \) converges to the weak coupling fixed point (\( \delta(\theta) \) with suitable normalization). If \( r < 1 \), the same result will hold for compact \( U(N) \) and \( SU(N) \) lattice gauge theories for large \( \beta \). This is plausible because in this regime the Gaussian approximation is good; it has been confirmed also
numerically by us and [15]. As an illustration we show in the figure the results of numerical iteration for $G = SU(2)$ $r = 0.9$ and $\alpha = 1$. There is a ‘critical point’ at $\beta \approx 4.8$: for $\beta \geq 4.8$ the modified MK flow goes to the weak coupling fixed point, whereas for $\beta \leq 4.79$ it still flows to the strong coupling fixed point.

![Figure 1: Evolution of $c_j/c_0$ under Tomboulis’ modified MK RG with $r = 0.9$. $\beta = 4.80$ (left plot), $\beta = 4.79$ (right plot); lines drawn to guide the eye.](image)

It should not be difficult to prove this rigorously. These examples show that one has to be very careful if $r$ is chosen $< 1$: depending on $\beta$, $r$ has to be chosen close to 1, which in turn will mean that $\alpha$ is close to 1.

The author of [1] says that “choose $r < 1$ if necessary . . . ” in several places (e.g. 3rd, 16th lines on page 15, 8th line on page 18, 15th line on page 23, etc.) Though it is said that “$r < 1$ is actually irrelevant for $G = SU(2)$” in the 9th line on page 19, $r$ is chosen $< 1$ throughout the paper. On the other hand, to ensure $\lim_{n \to \infty} c_j(n) = 0$, $1 - r$ will have to be chosen very small for large $\beta$. But then the required upper bound $\alpha(r, t) < 1 - \delta$ becomes very subtle.

So we have two choices:

1. $r < 1$. The larger $\beta$, the closer to one $r$ will have to be chosen to make sure that the flow goes to the strong coupling fixed point. But then it
will be very difficult to ensure that $\alpha$ is sufficiently far from 1.

2. $r = 1$. All parameters depend on $\beta$ and $|\Lambda|$ in a very subtle way in this case. (But this is sure to fail for $G = U(N)$.)

These subtle points are not addressed in [1].

Finally there is a problem with proving the existence of a $t$ such that $\alpha(t) = \alpha^+(t)$ after sufficiently many iterations of the MK recursion formulas or when $\{c_j\}$ are small. To obtain the function $t(\lambda)$ satisfying $\Psi(\lambda, t) = 0$ with $t(0) = 0$, the author of [1] appeals to the implicit function theorem. Since

$$\log Z^+(\{c_j\}) = \log \left[ F_0(1)^{h(\alpha^+(t), t)}|\Lambda^{(1)}|Z_1^+(\{\tilde{c}(\alpha^+(t))\}) \right]$$

by the parametrization invariance ($t$-invariance) of the partition function (this is the definition of $\alpha^+$), we have

$$\Psi(\lambda = 0, t) = h(\alpha(t), t) - h(\alpha^+(t_I), t_I) \quad (3.5a)$$

$$\Psi(\lambda = 1, t) = \frac{1}{\log F_0(1)|\Lambda^{(1)}|} \times \left( \log \left[ F_0(1)^{h(\alpha(t), t)}|\Lambda^{(1)}|Z_1^+(\{\tilde{c}(\alpha(t))\}) \right] - \log Z^+(\{c_j\}) \right) \quad (3.5b)$$

We assume that the equation $\Psi(\lambda = 0, t) = 0$ is solved by $t = t_0$, and the equation $\Psi(\lambda, t) = 0$ is our required equation, and we want to know if the solution $t = t(\lambda)$ with $t(0) = t_0$ can be continued to $t(1)$. We have

$$t(\lambda) = \phi(t)(\lambda) \equiv t_0 + \int_0^\lambda F(s, t(s))ds \quad (3.6)$$

$$F(s, t(s)) = -\frac{\Psi_s(s, t(s))}{\Psi_t(s, t(s))} \quad (3.7)$$

where

$$\Psi_t(\lambda, t) = \left[ 1 - \frac{h_\alpha(\alpha, t) + \lambda A^+(\alpha)}{h_\alpha(\alpha, t) + A(\alpha)} \right] h_t(\alpha, t) \quad (3.8a)$$

$$\Psi_\lambda(\lambda, t) = \frac{1}{|\Lambda^{(1)}|\log F_0(1)} \times \left( \log Z_1^+(\{\tilde{c}(\alpha(t))\}) - \log Z^+(\{\tilde{c}(\alpha^+(t))\}) \right) \quad (3.8b)$$

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and
\[ h_t(\alpha, t) = -\frac{1 - \alpha}{\alpha} h(\alpha, t), \quad h_\alpha(\alpha, t) = \frac{t}{\alpha^2} h(\alpha, t) \] (3.9)

The integral equation
\[ t(\lambda) = \phi(t)(\lambda) \]
can be solved analytically by the iterations if \( F(s, t) \) is bounded in the region. As Tomboulis pointed out in [1], \( A \geq A^+ \) if \( \{c_j \geq 0\} \) are small and the high-temperature expansion converges. But the condition \( \Psi_t(\lambda, t) \neq 0 \) does not guarantee that \( t(\lambda) \) can be defined for all \( \lambda \). The following example is given by Kanazawa [16]:
\[ \Psi(\lambda, t) = e^{-t} - 1 + 2\lambda \]

To justify the claim in [1], we need contractivity of the map \( \phi \) (in a suitable set of continuous functions) and to prove it is again not trivial at all.

4 Discussion

To sum up, the paper [1] contains several problematic points which remain to be proved or confirmed. It seems that the remaining problems are, however, not easy to solve.

If the conventional wisdom of quark confinement in 4D non-abelian lattice gauge theory is correct, the alleged theorem in [1] is certainly very plausible and may hold for \( G = SU(N) \). But it is again a very subtle problem to show the existence of \( t_* \) such that \( \alpha(t_*) = \alpha^+(t_*) \) for a value of \( r \) close enough to 1 to ensure convergence of the MK RG to the strong coupling fixed point.

The case of \( G = U(1) \) highlights this subtlety: For large \( \beta \) and \( r = 1 \) such a \( t_* \) does not exist, whereas for \( r < 1 \) \( t_* \) may exist, but does not imply confinement. We think in this note we made clear why it is so subtle to establish the existence of \( t_* \).

Though the MK RG recursion formulae cannot distinguish non-abelian groups from abelian ones, the velocities of the convergences of \( \{c_j(n)\}_j=1/2 \) to 0 as \( n \to \infty \) are very different. We are skeptical about the idea that the problem of quark confinement can be solved by soft analysis like this, but if the MK RG formulas should play a role in a rigorous proof of quark confinement in lattice gauge theory, these different velocities would certainly have to come into play. Presumably the dependence of \( r \) and hence \( \alpha \) on \( \beta \) and \( \Lambda \) must be clarified, but unfortunately it is very difficult to find out what the relationship is.
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References

[1] E. T. Tomboulis, Confinement for all values of the coupling in 4D SU(2) gauge theory, hep-th/0707.2179v1

[2] K. Wilson, Confinement of quarks, Phys. Rev. D10 (1975) 2445.

[3] A. Jaffe and E. Witten, Quantum Yang-Mills Theory, Clay Mathematics Institute, http://www.claymath.org/millennium/Yang-Mills_Theory

[4] K. R. Ito and E. Seiler, On the recent paper on quark confinement by Tomboulis, hep-th/0711.4930v1

[5] E. T. Tomboulis, Reply to arXiv:0711.4930[hep-th] by Ito and Seiler, hep-th/0712.2620v1

[6] A. A. Migdal, Recursion equations in gauge field theories, JETP 69 (1975) 810; L. Kadanoff, Notes on Migdal’s recursion formulas, Ann. Phys. 100 (1976) 359.

[7] K. R. Ito, Permanent quark confinement in 4D hierarchical lattice gauge theories of Migdal-Kadanoff type, Phys.Rev.Letters 55 (1985) 558.

[8] G. ’t Hooft, Nucl. Phys. B138 (1978) 1; B153 (1979) 141.

[9] G. Mack and V. B. Petkova, Sufficient condition for confinement of static quarks by a vortex condensation mechanism, Ann. of Physics 123 (1980) 117.

[10] C. Borgs and E. Seiler, Lattice Yang-Mills theory at non-zero temperature and the confinement problem, Commun.Math.Phys. 91 (1983) 329.
[11] J. Fröhlich and T. Spencer, Massless Phases and Symmetry Restorations in Abelian Gauge Theory and Spin System, Commun.Math.Phys. 83 (1982) 411.

[12] A. Guth, Existence Proof of a non-confining phase in 4D $U(1)$ lattice gauge theory, Phys. Rev. D21, (1980) 2291.

[13] K. R. Ito and E. Seiler, 't Hooft’s string tension in Lattice Quantum Electrodynamics, preprint in preparation.

[14] E. Seiler, Lattice Gauge Theory as a Problem of Constructive Quantum Field Theory and Statistical Mechanics, Lecture Notes in Physics (Springer Verlag, Heidelberg and New York 1982)

[15] H. Suzuki, Private communication to K. R. Ito.

[16] T. Kanazawa, private communication to K. R. Ito.