On cubic Berwald spaces

Nicoleta Brinzei
Transilvania University, Brasov, Romania

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Abstract

We show that, for Finsler spaces with cubic metric, Landsberg spaces are Berwaldian. Also, for decomposable metrics, we determine specific conditions for a space with cubic metric to be of Berwald type, thus refining the result in [6].

1 Introduction

Spaces with cubic metric are studied by Matsumoto and Numata, [6], [7]. They are Finsler spaces in a wider sense, [9].

An interesting problem related to m-th root metric spaces is the following: is any Landsberg space with m-th root metric Berwaldian?

A partial answer for spaces with cubic metric with fundamental function $F = \sqrt[3]{c_1 \alpha^2 \beta + c_2 \beta^3}$ (where $\alpha^2$ is a pseudo-Riemannian metric and $\beta$ is a 1-form) is given by Lee and Jun, [5]. In what follows, we generalize this result: namely, for all cubic Finsler spaces $(M, F)$, $F = \sqrt[3]{a_{ijk}(x)y^i y^j y^k}$ with $a_{ijk}$ differentiable, if $(M, F)$ is of Landsberg type, then it is of Berwald type.

Also, for spaces whose fundamental function is decomposable as a product of two factors $\bar{F}^3 = a \cdot b$, between a Riemannian metric $a$ and a 1-form $b$ on $M$, we show that $(M, F)$ is of Berwald type if and only if the 1-form $b$ is parallelly transported with respect to the Levi-Civita connection of $a$. An analogous result is proven by Z. Shen for spaces with $(\alpha, \beta)$-metrics of the form $F = \alpha \phi(\frac{\beta}{\alpha})$, [11].

The techniques we used mainly rely on expressing the involved geometrical objects in terms of the third power $T = F^3$ of the fundamental function, which is a polynomial function of the directional variables $y^i$.

2 Spaces with cubic metric

Let $M^n$ be a differentiable manifold of dimension $n$ and class $C^\infty$, $TM$ its tangent bundle and $(x^i, y^i)$ the coordinates in a local chart on $TM$. Let $F$ be
the following function on $M$, :

$$F = \sqrt[3]{a_{ijk}(x)y^iy^jy^k}. \quad (1)$$

(with $a_{ijk}$ symmetric in all its indices) and

$$T = F^3 = a_{ijk}(x)y^iy^jy^k. \quad (2)$$

In the following, for a function $f = f(x, y)$, we shall denote by ”” and ”” the partial derivatives w.r.t. $x$ and $y$, respectively. Also, if $N$ is a nonlinear connection on $TM$, we denote by ”” its associate covariant derivative

$$f;_i = \frac{\delta f}{\delta x^i} = \frac{\partial f}{\partial x^i} - N^r_i \frac{\partial f}{\partial y^r}, \ f \in F(TM)$$

and we denote by null index transvection by $y$ (for instance, $T_{t0} = T_{ij}y^j$).

**Remark 1** [3] If $F = T^{1/m}$ is a Finslerian fundamental function on $M$, then the Hessian $[T_{ij}]$ is an invertible matrix, its inverse has the entries:

$$T^{ij} = \frac{1}{m(m-1)F^{m-2}} \{(m-1)g^{ij} - (m-2)l^i l^j\},$$

where $g^{ij}$ denotes the contravariant version of the usual Finslerian metric tensor attached to $F$ and $l^i = y^i / F$.

Hence, $T^{ij}$ and $T_{ij}$ can be used for raising and lowering indices of tensors. Moreover, $T_{ij}$ are polynomial functions of $y$, and $T^{ij}$ are rational functions of $y$.

### 3 Geodesics and canonical spray

In the following, we shall express the equations of geodesics of a cubic metric space and the related geometric objects in terms of $T = F^3$ of the fundamental function and of its derivatives.

Unit speed geodesics of $(M, F)$ are described by the Euler-Lagrange equation:

$$\frac{\partial F}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial F}{\partial y^i}\right) = 0.$$

Taking into account the fact that, along such curves, $F(x, \dot{x}) = 1$, the above is equivalent to:

$$\frac{\partial T}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial T}{\partial y^i}\right) = 0.$$

An easy computation leads to:

$$\frac{dy^i}{dt} + T^{ih}(T_{h,k}y^k - T_{h,k}) = 0, \quad y^i = \dot{x}^i. \quad (3)$$

Consequently,
Proposition 2

1. In spaces with cubic metric the coefficients of the canonical spray, \([1], [3]\), are rational functions of \((y^i)\), given by

\[
2G^i = T^{ih}(T_{h,k}y^k - T_{h}).
\]  \(\text{(4)}\)

2. The canonical nonlinear connection has the coefficients:

\[
N^i_{j} = G^i_{j} = \frac{1}{2}\{T^{ih}(T_{h,k}y^k - T_{h}) + T^{ik}(T_{h,j,k}y^k + T_{h,j} - T_{j,k})\}.
\]

We denote in the following by \(B\Gamma\) the Berwald connection, \([1], [2]\) determined by \(F = \sqrt{T}\) and by \(G^i_{jk}\) its coefficients. According to \(\text{(4)}\), for \(m\)-th root metric spaces, \(G^i_{jk}\) are rational functions of \(y\).

Also, let

\[
L^i_{jk} = T^{ih}\left(\frac{\delta T_{kj}}{\delta x^h} + \frac{\delta T_{hk}}{\delta x^j} - \frac{\delta T_{jk}}{\delta x^h}\right),
\]

\[
T^i_{jk} = T^{ih}\left(\frac{\partial T_{kj}}{\partial y^h} + \frac{\partial T_{hk}}{\partial y^j} - \frac{\partial T_{jk}}{\partial y^h}\right) = \frac{T^{ih}}{2}T_{hjk},
\]

denote the coefficients of the canonical metrical connection \(CT\) attached to the Lagrange-type metric \(T_{ij}\), \([3]\).

4 Specific Landsberg&Berwald conditions for \(m\)-th root metrics

There are a lot of alternative definitions of Landsberg and Berwald-type Finsler spaces, \([1], [4]\). In the present paper, we shall use the following:

A Finsler space \((M, F)\) is a Landsberg space if: (1) the Cartan tensor \(C_{ijk}\) satisfies \(C_{ijk|0} = 0\), where the covariant derivative is taken with respect to the Berwald connection \(B\Gamma\), or (2): the Berwald connection \(B\Gamma\) is metrical.

In Landsberg spaces, the horizontal coefficients of the Cartan connection \(F^i_{jk}\) coincide with those of the Berwald connection: \(F^i_{jk} = G^i_{jk}\).

A Finsler space is called a Berwald space if: (1) with respect to \(B\Gamma(N)\), there holds \(C_{ijk|0} = 0\) or (2) the coefficients \(G^i_{jk}\) of the Berwald connection are functions of \(x^i\) alone: \(G^i_{jk} = G^i_{jk}(x)\).

The last statement is equivalent to the fact that the coefficients \(G^i\) of the canonical spray are homogeneous polynomial functions of degree 2 in \(y^i\). There hold the inclusions:

\[
\text{Riemann spaces} \subset \text{Berwald spaces} \subset \text{Landsberg spaces}.
\]

For Finsler spaces with \(m\)-th root metric \((M, F)\), we get more convenient such characterizations by using the third order derivatives \(T_{ijk}\) (where \(T = F^m\)) instead of the Cartan tensor \(C_{ijk}\).

Using the results in \([10]\), we have proven in \([3]\), that
Proposition 3 The horizontal coefficients $L^i_{jk}$ of the canonical metrical connection $C^i_{\Gamma jk}$ attached to the Hessian $T_{ij}$ coincide with those of the Cartan connection of $(M, F)$. Hence, in Landsberg $m$-th root metric spaces, we have $L^i_{jk} = F^i_{jk} = G^i_{jk}$.

Corollary 4 An $m$-th root metric space $(M, F)$ is a Berwald space (resp. Landsberg space) if and only if, w.r.t. the canonical metrical connection $C_{\Gamma}(N)$, we have $T_{ijk|l} = 0$ (resp. $T_{ijk|0} = 0$).

5 Landsberg-Berwald equivalence

In the following, we show that Landsberg spaces with cubic metrics are Berwaldian.

Let

$$T = F^3 = a_{ijk}(x)y^i y^j y^k,$$

with $a_{ijk} = a_{ijk}(x)$ of class at least 1, define a Landsberg space; according to the results in the previous section, this means

$$T_{ijk|0} = 0.$$

For a cubic metric, the third derivatives $T_{ijk}$ depend only on $x$, which entails

$$\frac{\delta T_{ijk}}{\delta x^l} = \frac{\partial T_{ijk}}{\partial x^l}.$$

Then,

$$T_{ijk|l} = T_{ijk,l} - L^h_{il}T_{hjk} - L^h_{jl}T_{ihk} - L^h_{kl}T_{ijh}. \quad (5)$$

Taking into account that our space is a Landsberg one (i.e., $L^h_{il} = G^h_{il}$ etc.), we have

$$T_{ijk|0} = T_{ijk,1}y^l - N^h_{il}T_{hjk} - N^h_{jl}T_{ihk} - N^h_{kl}T_{ijh} = 0.$$

Deriving by $y^l$ and taking into account that $T_{ijk}$ depend only on $x$, we get

$$T_{ijk,l} - L^h_{il}T_{hjk} - L^h_{jl}T_{ihk} - L^h_{kl}T_{ijh} = 0,$$

which is nothing but $T_{ijk|l} = 0$. We have thus obtained

Proposition 5 Let $(M, F)$ be a space with cubic metric $F = \sqrt[a_{ijk}(x)]{y^i y^j y^k}$.
If the functions $a_{ijk}$ are of class at least one, then there holds the implication:

$(M, F)$ is a Landsberg space $\Rightarrow$ $(M, F)$ is a Berwald space.

Further, for spaces with cubic metric, the inclusion Riemannian spaces $\subset$ Berwald spaces is strict. Namely, the Berwald-Moor conformal space with

$$T = F^3 = e^\sigma(x)y^1 y^2 y^3,$$

where $\sigma(x)$ is a differentiable function, provides an example of Berwald cubic space, which is non-Riemannian.
6 Decomposable cubic metrics

Let us consider a space \((M, F = \sqrt[3]{T})\), where \(T\) decomposes as a product
\[
T = a \cdot b
\]  
where \(a = \gamma_{ij}(x)y^i y^j\) is a Riemannian metric and \(b = b_i(x)\) is a 1-form, such that:
\[
\|b\|^2 = \gamma_{ij} b_i b_j = 1.
\]

For cubic spaces with \(T = F^3\) as in (6), it is proven in [6] that the space is a Berwald one if and only if there exists some 1-form \(f \in \mathcal{X}^* M\) such that
\[
\gamma_{ij,k} = f_k(x) \gamma_{ij}; \quad b_{i,k} = -f_k(x) b_i,
\]
where the covariant derivative is taken with respect to the Berwald connection determined by the "whole" fundamental function \(F = \sqrt[3]{ab}\).

In the following, we shall find the relation between \(a\) and \(b\) such that the space \((M, F = \sqrt[3]{ab})\) is Berwaldian; more precisely, we shall take into consideration the covariant derivatives
\[
\nabla_i b_j,
\]
where \(\nabla\) denotes the Levi-Civita connection attached to \(\gamma_{ij}\).

By direct computation, we get

Lemma 6 If \(a = \gamma_{ij}(x)y^i y^j\) is a Riemannian metric and \(b = b_i(x)\) is a 1-form with \(\gamma^{ij} b_i b_j = 1\), then:

1. The Hessian matrix \([T_{ij}]\) is invertible iff
\[
\Delta := 4b^2 - a
\]
does not vanish.

2. The inverse matrix has the entries
\[
T^{ij} = \frac{1}{2b\Delta}(\Delta \gamma^{ij} - 2bb^i y^j - 2bb^j y^i + ab^i b^j + y^i y^j),
\]
where the indices of \(b\) were raised by \(\gamma^{ih}: b^i = \gamma^{ih} b_h\).

Further, in [1], p. 110-111, it is proven the following result:

Lemma 7 [1]: If \((M, F)\) and \((M, \bar{F})\) are two Finsler spaces on the same underlying manifold, then the local coefficients of the corresponding canonical sprays are related by
\[
2\bar{G}^i = 2G^i + \frac{\bar{F}_i y^i}{F} - \bar{F} g^{ij} r_j(\bar{F}),
\]
where \(|\) denotes Berwald covariant derivative determined by \(F\) and
\[
r_j(S) = S_{ij} - y^r S_{r,j}, \quad \forall S \in \mathcal{F}(TM).
\]
In the following, we shall express the above in terms of the m-th power of $\bar{F}$, $m \geq 2$; hence, let for the moment

$$T = \bar{F}^m.$$ 

Then, there hold the relations:

- $$\bar{F}y^i = \frac{1}{m} T y^i.$$ 

- The contravariant Finslerian metric tensor $\bar{g}^{ij}$ is expressed in terms of $T$ as

$$\bar{g}^{ij} = \frac{T - \frac{2}{m}}{m - 1} (Tm(m - 1)T^{ij} + (m - 2)y^iy^j).$$

- $$r_j(\bar{F}) = \frac{1}{m} T^{1 - \frac{2}{m}} \left( Tr_j(T) + \frac{m - 1}{m} T_j T|_0 \right);$$

- $$y^j r_j(T) = (1 - m)T|_0.$$ 

Then, the last term in (8) is

$$\bar{F} \bar{g}^{ij} r_j(\bar{F}) = \frac{1}{m} T^{1 - \frac{2}{m}} \left( Tr_j(T) + \frac{m - 1}{m} T_j T|_0 \right) = \frac{T - \frac{2}{m}}{m - 1} (Tm(m - 1)T^{ij} + (m - 2)y^iy^j) \cdot$$

Replacing into (8) and taking (9) into account, we get

**Lemma 8** If $(M, F)$ and $(M, \bar{F})$ are two Finsler spaces on the same underlying manifold, then the coefficients of the corresponding canonical sprays are related by

$$2\bar{G}^i = 2G^i - T^{ij} r_j(T),$$

where $\mid$ denotes Berwald covariant derivative determined by $F$ and

$$T = \bar{F}^m, \quad m \geq 2, \quad r_j(T) = T|_j - y^i T|_{r,j}. $$
We shall also use the following relations, which can be deduced by direct computation:

\[ r_j(b) = (\nabla jb_r - \nabla_r bj)y^r; \]
\[ y^r r_j(b) = 0; \]  \hspace{1cm} (11)
\[ T^{ij} b_j = \frac{1}{2\Delta} (2bb^i - y^i); \]
\[ T^{ij} a_j = \frac{1}{\Delta} (2by^i - b^i a); \]
\[ \|b\| = 1 \Rightarrow b^i \nabla_j b_i = 0. \]

Let now \( G^i \) be determined by the Riemannian metric \( \gamma_{ij}(x) \), where \( a = \gamma_{ij}(x)y^i y^j \), and \( \bar{G}^i \), by \( T = T^3 = a \cdot b \) as above. Then, \( |i| = \nabla_i \), and

\[ r_j(T) = \nabla_j(ab) - y^r \frac{\partial}{\partial y^j} \nabla_r(ab), \]

and taking into account that \( \nabla_j a = 0 \), we get

\[ r_j(T) = ar_j(b) - a_j \nabla_0 b, \]

where \( \nabla_0 b = y^r \nabla_r b \).

The cubic space \((M, \bar{F})\) is a Berwald one if and only if the functions \(2\bar{G}^i \) are polynomial in \( y^i \). This is equivalent to the fact that the difference

\[ 2B^i := 2\bar{G}^i - 2G^i = -T^{ij} r_j(T) \]

is a polynomial function of degree 2 in \( y \). There holds

**Theorem 9** The space \((M, F = \sqrt[3]{T})\), where \( T \) decomposes as a product

\[ T = a \cdot b \]  \hspace{1cm} (12)

where \( a = \gamma_{ij}(x)y^i y^j \) is a Riemannian metric and \( b = b_i(x) \) is a 1-form, such that:

\[ \|b\|^2 = \gamma_{ij} b_i b_j = 1 \]

1. is of Berwald type, if and only if \( b \) is parallel with respect to \( a \):

\[ \nabla_i b_j = 0, \quad \forall i, j = 1, \ldots, n. \]

**Proof:**

Let us suppose that \((M, \bar{F} = \sqrt[3]{ab})\) is Berwaldian and let us fix some arbitrary \( x \in M \). Then \( 2B^i \) are polynomials of degree 2 and hence, so are \( 2B^i b_i \). By (11), we have

\[ T^{ij} b_j = \frac{1}{2\Delta} (2bb^i - y^i), \]

consequently,

\[ -2B^i b_i = \frac{1}{2\Delta} (2bb^i - y^i) r_j(T) = \frac{1}{2\Delta} (2bb^i - y^i)(ar_j(b) - a_j \nabla_0 b) = \]

\[ = \frac{1}{\Delta} (ab b_j r_j(b) - 2b^2 \nabla_0 b + a \nabla_0 b). \]
But, \( a - 2b^2 = 2b^2 - \Delta \), so we can write

\[
-2B^i_b = \frac{1}{\Delta} \{abb^i r_j(b) + (2b^2 - \Delta)\nabla_0 b\} = -\nabla_0 b + \frac{1}{\Delta} \{abb^i r_j(b) + 2b^2 \nabla_0 b\}.
\]

Since the latter is a polynomial, \( \Delta \) divides the polynomial \( abb^i r_j(b) + 2b^2 \nabla_0 b = b(ab^i r_j(b) + 2b \nabla_0 b) \). Since \( a \) does not decompose in factors, \( a \) and \( b \) have no common factors; we notice that, in this case, \( b \) and \( \Delta \) are also relatively prime, hence

\[
\Delta \mid ab^i r_j(b) + 2b \nabla_0 b.
\]

Again, we have \( a = 4b^2 - \Delta \), and we get that \( \Delta \mid 4b^2b^i r_j(b) + 2b \nabla_0 b = 2b(2bb^i r_j(b) + \nabla_0 b), \) that is,

\[
\Delta \mid (2bb^i r_j(b) + \nabla_0 b).
\]

Both hand sides of the above are polynomials of degree 2 in \( y^i \), hence there exists some \( f = f(x) \) such that

\[
(2bb^i r_j(b) + \nabla_0 b) = f(x)\Delta.
\]

(13)

By identifying the coefficients in the above relation and taking into account that, by \( b^i \nabla_j b_l = 0 \), we get

\[
2b^i b^j \nabla_j b_r + 2b^i b^j \nabla_j b_i + \nabla_i b_i + \nabla_i b_r = f(x)(8b_i b_r - 2\gamma_{ir}).
\]

Contracting with \( b^j \) and taking into account that \( b^i b_i = 1 \), the above leads to

\[
b^i \nabla_i b_r = 2b_r f(x),
\]

(14)

which yields

\[
b^i r_j(b) = b^i (\nabla_j b_r - \nabla_r b_j) y^r = b^i \nabla_j b_o = 2b f(x).
\]

(15)

Replacing into \( (13) \), we have \( 4b^2 f(x) + \nabla_0 b = f(x)\Delta = f(x)(4b^2 - a) \); we obtain that

\[
\nabla_0 b = -af(x).
\]

(16)

Let us come back now to the expression of \( 2B^i \):

\[
-2B^i = T^{ij}(ar_j(b) - a_j \nabla_0 b)
\]

The last term, \( T^{ij} a_j \nabla_0 b \) is

\[
T^{ij} a_j \nabla_0 b = \frac{1}{\Delta} (2by^i - b^i a) \nabla_0 b = \frac{-a}{\Delta} (2by^i - b^i a) f(x).
\]

The first one, \( T^{ij} ar_j(b) \), is

\[
T^{ij} ar_j(b) = \frac{a}{2b\Delta} (\Delta y^{ij} - 2bb^i y^j - 2bb^j y^i + ab^i b^j + y^i y^j)r_j(b) = \\
= \frac{a}{2b\Delta} (\Delta y^{ij} r_j(b) - 0 - 4b^2 y^i f(x) + 2abb^i f(x) + 0).
\]
Then,

$$-2B^i = \frac{a}{2b\Delta} \{ \Delta \gamma^{ij} r_j(b) - 4b^2 y^i f(x) + 2abf(x) \} + \frac{2ab}{2b\Delta} (2by^i - b^i a) f(x)$$

The common denominator $2b\Delta$ has to divide the numerator. In particular, $b$ has to divide the numerator. The only term which does not contain $b$ explicitly as a factor is

$$a\Delta \gamma^{ij} r_j(b).$$

Since $b$ has no common factors neither with $a$, nor with $\Delta$, $b$ has to divide the polynomial $\gamma^{ij} r_j(b)$ (of degree 1). That is, there exists some $\phi = \phi(x)$ such that $\gamma^{ij} r_j(b) = \phi^i(x)b$. Lowering the indices,

$$r_j(b) = \phi_j(x)b.$$

But, since $y^i r_j(b) = 0$, we get $0 = y^i r_j(b) = (y^i \phi_j(x))b$. Together with $b \neq 0$, this yields $y^i \phi_j(x) = 0$, or

$$\phi_j = 0,$$

which is nothing but $r_j(b) = 0$. The latter means actually

$$\nabla_r b_i - \nabla_i b_r = 0. \quad (17)$$

Let’s now look at relation (14):

$$b^i \nabla_i b_r = 3b_r f(x), \quad (18)$$

By (17), it is equivalent to

$$b^i \nabla_i b_i = 3b_r f(x).$$

According to (11), we have $b^i \nabla_i b_i = 0$; the left hand side of the above is 0, hence

$$f(x) = 0,$$

which yields, together with (10),

$$\nabla_0 b = \nabla_j b_i y^j y^i = 0.$$

The latter, together with (17), leads to

$$\nabla_r b_i = 0,$$

q.e.d.

The converse statement is obvious.

**Remark 10** If $(M, \vec{F})$ is of Berwald type, then

$$2B^i := 2\bar{G}^i - 2G^i = -T^{ij} r_j(T) = 0,$$

consequently, it has the same geodesics as the Riemannian space $(M, a = \gamma_{ij}(x)y^i y^j)$. 

9
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