Horizons and the Wave Function of Planckian Quantum black holes

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Abstract

At the Planck scale the distinction between elementary particles and black holes becomes fuzzy. The very definition of a "quantum black hole" (QBH) is an open issue. Starting from the idea that, at the Planck scale, the radius of the event horizon undergoes quantum oscillations, we introduce a black hole mass-radius Generalized Uncertainty Principle (GUP) and derive a corresponding gravitational wavelength. Next we recover a GUP encoding effective geometry. This semi-classical gravitational description admits black hole configurations only for masses higher than the Planck mass. Quantum corrections lead to a vanishing Hawking temperature when the Planck mass is approached from above. Finally we replace our semi-classical model by a relativistic wave equation for the "horizon wave function". The solution admits a discrete mass spectrum which is bounded from below by a stable ground state with energy close to the Planck mass. Interestingly higher angular momentum states fit onto Regge trajectories indicating their stringy nature.

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1. Introduction

The "point-particle" is the simplest way to represent microscopic objects. It corresponds to a mathematical point endowed with physical properties like mass, charge, etc. Such an idealized representation of elementary particles is to be taken with due caution. True physical objects cannot have zero volume nor infinite density. The experimental side- the term point-particle is justified for objects whose linear size is below the resolution power of the instrument used to probe them.

The very existence of fundamental (dimensional) constants like $c$-speed of light, $G_N$-Newton constant, $\hbar = \hbar/2\pi$ -(reduced) Planck constant, enables one to combine them into two fundamental length scales. By conveniently setting $c = 1$ one defines

- $r_g \equiv m G_N$, which is the gravitational length scale. In this range of length gravitation is "strong" and closed trapping surfaces, colloquially referred to as "horizons" \[5\], can appear. Whenever a physical falls through a "hoop" of radius $r_s$ it will unavoidably collapse under its own weight into a black hole. This is the so-called hoop conjecture introduced by K. Thorne \[1\].

- $\lambda_C \equiv \hbar/m$ is the Compton wavelength. This is the length scale where quantum effects are dominant and a particle can only be described in terms of its wave function.

The above length scales identify two distinct mass regions:

- Black hole sector: $r_g > \lambda_C$, with $m > m_P$
- Particle sector: $\lambda_C > r_g$, with $m < m_P$

The boundary between the two regimes is determined by the Planck mass $m_P$ and Planck length $l_P$:

$$m_P^2 = \frac{\hbar}{G_N} = \frac{l_P^2}{G_N^2},$$

(1)

(2)

\[5\]It is common practice to define $r_G$ as $r_G = 2m G_N$, is the Schwarzschild radius of a "classical" black hole. We shall show that, near the Planck scale, the QBH radius is actually one half of the Schwarzschild radius.
The Planck scale is the regime where gravity merges with quantum mechanics. An object of mass \( m < m_P \) is an ordinary quantum particle obeying known quantum mechanical rules. When its energy is above \( m_P \) a quantum particle develops an event horizon shielding it from an asymptotic observer and becomes quantum black hole QBH. The behavior and properties of these kind of microscopic objects deserve a thorough investigation which is still underway.

This paper is organized as follows. In Section (2) we introduce a new Mass-Horizon Uncertainty Principle and recover the quantum wavelength \( \lambda_G \) for a Planckian object. In Section (3) we identify the two roots of \( \lambda_G \) with the radii of the inner (Cauchy) and outer (Killing) horizon of an effective geometry. A black hole configuration is admitted only if the mass is heavier than the Planck mass. We study the thermodynamical properties of this object and find that the Hawking temperature and the entropy both vanish at the Planck mass. This behavior is a signal of a breakdown of the effective geometric description and a full quantum approach is needed. In Section (4) we solve the horizon wave equation and recover the discrete mass spectrum of a QBH. Section (5) is devoted to a brief summary of the main results obtained.

2. The wavelength of QBH

At the Planck scale the distinction between particle and black hole becomes murky. In quantum mechanics the position and momentum of a particle are conjugate variables which cannot be simultaneously measured with arbitrary precision. Accordingly, one expects a similar behavior for a QBH. The effects of gravity lead to the Generalized Uncertainty Principle (GUP) whose purpose is to put quantum particles and QBHs on the same footing [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

The GUP can be written in a most general form as [14]

\[
\Delta x \Delta p \geq \frac{\hbar}{2} + \alpha G_N (\Delta p)^2 + \beta \frac{(\Delta x)^2}{G_N},
\]

where \( \alpha, \beta \) are numerical constants.

A heuristic derivation of the GUP consists in including gravity in the "Heisenberg microscope" ideal experiment where the particle position is determined by observing a scattered photon and applying simple optical principles [2, 15]. While for a particle this is a legitimate approach, for a QBH the situation is more subtle. A photon reaching the QBH center cannot be scattered back...
into the microscope. Therefore the QBH center is not directly observable being shielded by the horizon. In principle one can by-pass the problem by reconstructing the location of the QBH center through the angular distribution of outgoing thermal photons during Hawking radiation. We will show in Section(3) that a QBH has a vanishing temperature as its mass approaches \( m_P \) and the argument advocating the Hawking radiation fails.

Another derivation of GUP [16] considers the disturbance induced in the metric by the measuring procedure to obtain

\[
\Delta x \geq \frac{\hbar}{2\Delta E} + 2G_N \Delta E
\]

(4)

In all formulations \( \Delta x \) refers to the uncertainty in the position of the QBH center, exactly as in the case of an ordinary point-particle. The existence of the horizon is not explicitly encoded into this form of GUP. As a matter of fact, when considering the evolution of a BH one is not referring to its motion in space, but to the mass and radius variations. However, at the Planck scale the horizon cannot be described as a static, geometric boundary. It is subject to quantum fluctuations making its position uncertain. A preliminary analysis of both classical and quantum dynamics of an oscillating horizon can be found in [17].

On the quantum mechanical side, let us recall the fundamental lesson from the hydrogen atom example: the dynamics of that system is determined by the relative motion of the electron with respect to the common center of mass, and not by the (trivial) motion of the center of mass itself. In the same way the dynamics of a QBH is uniquely referred to by its horizon fluctuations and not by its global motion through space. Thus one needs to start from a version of GUP explicitly taking into account these fluctuations rather than the uncertainty in the position of the QBH center.

Furthermore the GUP has to encode the essential difference between a particle and QBH as discussed in Section(1) and described by the condition \( m \geq m_P \).

For all these reasons, we propose a version of (3) entitled Horizon Uncertainty Principle (HUP), where \( \Delta x \) is replaced by \( \Delta r_H \) and \( \Delta p \) by \( \Delta m \), with \( \alpha = 0 \) and \( \beta \) a free parameter. The result is:

\[
\Delta r_H \Delta m \geq \frac{\hbar}{2} + \beta \frac{(\Delta r_H)^2}{G_N}
\]

(5)

Relation (5) imposes the lower bound condition:
\[ \Delta m \geq \sqrt{\frac{\hbar \beta}{G_N}} = \sqrt{\beta m_P} \]  

(6)

which is in agreement with the appearance of the horizon only above the Planck mass.

In our case, the so-called Compton-Schwarzschild wavelength \( \lambda_G \) introduced in [6] takes the form

\[ \lambda_G = \frac{\hbar}{2m} + \beta \frac{\lambda_G^2}{G_N m} \]  

(7)

As we are dealing with gravity, it is useful to express our results in terms of \( l_P \) and \( m_P \) as fundamental constants using relations

\[ G_N = \frac{l_P}{m_P}, \]  

\[ \hbar = m_P^2 G_N = m_P l_P \]  

(8)

Equation (7) is a quadratic relation between \( m \) and \( \lambda_G \). Solving the equation we find

\[ \frac{\lambda_G^\pm}{l_P} = \frac{m}{2\beta m_P} \left( 1 \pm \sqrt{1 - 2\beta \frac{m_P^2}{m^2}} \right) \]  

(10)

Again, we notice that \( \lambda_G^\pm \) are real only for \( m \geq \sqrt{2\beta m_P} \) as required. If we assume that the transition from particles to QBHs to occurs precisely at the Planck scale, then \( \beta = 1/2 \).

For \( m = m_P \) the two roots coincide \( \lambda_G^+ = \lambda_G^- = l_P \). It is interesting to consider the large mass limit of \( \lambda_G^\pm \). Far above the Planck mass we find

\[ m >> m_P, \]  

\[ \lambda_G^+ \sim 2 G_N m = r_s, \]  

\[ \lambda_G^- = \frac{\lambda_C}{2}, \]  

\[ \lambda_C = \frac{m_P}{m} l_P \]  

(11-14)

The resulting picture, presented graphically in Figure [1], shows that by increasing the mass of a particle its Compton wavelength approaches the
Figure 1: Plot of the Compton wavelength and the two horizons by varying $m$. The three curves meet at the bifurcation point $m = m_P$. Each of them approaches a different asymptote. Note that $r_+$ differs from the classical Schwarzschild radius $r_s$ as $m \to m_P$. We remark that the region of lengths smaller than $l_P$ has no physical meaning and only the branch above 1 of the plotted functions is relevant.

*bifurcation point* $m = m_P$ where $\lambda_C$ splits into $\lambda^+_G$ and $\lambda^-_G$. With further increase of $m$, $\lambda^+_G$ approaches the classical Schartzschild radius while $\lambda^-_G$ decreases towards zero and merges with $\lambda_C/2$ in the unphysical region below the Planck length. The presence of a bifurcation point indicates the transition between particles and black holes. Also notice that $\lambda^-_G$ for any $m > m_P$ is below the Planck length and has no physical meaning. As a matter of fact, only $\lambda^+_G$ is physically relevant.

3. Effective geometry

In the previous section we introduced a QBH wavelength which smoothly interpolates between the Compton wavelength and the Schartzschild radius. On the other hand, people are more familiar with the geometric description of BHs in terms of the space-time metric. A simple way to introduce an effective geometry is to identify $\lambda^+_G$ with the radii of the inner and outer
To obtain the corresponding metric, we rearrange HUP (5) in the form

\[ 1 - \frac{2G_N \Delta m}{\Delta r_H} + \frac{G_N m_P^2}{(\Delta r_H)^2} = 0 \]

which is nothing but a horizon equation in geometric description. It is now easy to reproduce the corresponding line element

\[ ds^2 = -f(r) \, dt^2 - f^{-1}(r) \, dr^2 + r^2 \, d\Omega^2 \]

\[ f(r) = 1 - \frac{2mG_N}{r} + \frac{\hbar G_N}{r^2} \]

\[ \hbar = G_N m_P^2 \]

The result is a Schwarzschild-like geometry with a quantum gravity correction \( \hbar G_N/r^2 \). Contrary to the standard geometry, the line element (18) describes a black hole only for \( m \geq m_P \) as it is required for a QBH. A closer look at (18) shows that it has the same form as Reissner-Nordström BH geometry, but with the fixed value of the “charge” \( Q^2 = \hbar G_N = G_N^2 m_P^2 \).

It is interesting to see the impact of quantum corrections on the BH thermodynamics.

In geometric units, \( c = G_N = h = k_B = 1 \) ( \( k_B \) is the Boltzmann constant ), the Hawking temperature

\[ T_H = \frac{1}{4\pi} \left( \frac{df}{dr} \right)_{r=r_+} = \frac{1}{4\pi r_+} \left( 1 - \frac{l_P^2}{r_+^2} \right) \]

\[ T_H (r_+ = \sqrt{3} l_P) \equiv T_{max} = \frac{1}{6\pi \sqrt{3} l_P} \]

\[ T_H (r_+ = r_- = l_P) = 0 \]
Initially, the temperature increases towards the maximum $T_{\text{max}}$ and then quickly drops to zero for $m = m_P$. Contrary to the general expectation, $T_H$ does not diverge at the Planck scale. Rather the black hole freezes to the extreme degenerate configuration:

$$r_{\text{ext}} = G_N m_P = l_P,$$
$$m_0 = m_P,$$
$$T_H = 0$$

In accordance with the Third Law of Thermodynamics one expects the entropy to vanish for $T_H = 0$. In fact by integrating the First Law of Thermodynamics one obtains

$$dm = T_H dS_H \rightarrow S_H = \frac{2\pi}{G_N} \int_{r_{\text{ext}}}^{r_+} dx = \frac{\pi}{G_N} (r_+^2 - r_{\text{ext}}^2)$$

For large BHs with $r_+ >> l_P$ one recovers the standard form of the Area Law

$$S_H \rightarrow \frac{\pi}{G_N} r_+^2 = \frac{A_H}{4G_N}$$

while, for $r_+ \to l_P$ one obtains zero entropy.

Important feature of any quantum theory of gravity is the Holographic Principle (HP) [18, 19] which implies that the BH dynamics is confined solely on the horizon and not in the bulk. It follows that BH entropy is stored on its surface, with one bit of information per Planck cell. Zero entropy is achieved once the horizon surface shrinks to a single cell at zero temperature. The results obtained in this section refer to a neutral, non-rotating BH, but their validity is general. In fact, in case of rotating and charged BHs, the Hawking radiation depletes both charge and angular momentum leading finally to a Schwarzschild-like QBH.

4. The horizon wave function.

Classical models of evolving QBHs were introduced many years ago in the framework of the theory of relativistic, self-gravitating membranes [20, 21, 22, 23, 24]. Recently, particle-like models of QBHs have been considered [25, 26, 27, 28]. In accordance with the latter, we provided [17] a model where
the horizon dynamics is translated into the motion of a relativistic point-particle trapped in self-consistent gravitational potential. In this framework the horizon oscillations are effectively described by the periodic motion of its particle-like analogue. The dynamics of this representative ” particle ” is encoded in a ” relativistic Hamiltonian ” given by

\[ \mathcal{H} ≡ \vec{p}_H^2 + m_H^2 \] (27)

where \( \vec{p}_H \) is the momentum, and the particle mass \( m_H \) is expressed in terms of the horizon radius \( r_+ \). For the metric (4) this function is determined by the horizon condition \( f (r_H) = 0 \)

\[ m_H(r_+) = \frac{r_+}{2G_N} \left( 1 + \frac{G_N^2 m_P^2}{r_+^2} \right) \] (28)

In order to keep the notation as simple as possible, we shell drop the suffix ”_+” from all the relevant quantities keeping in mind that all of them are referring to the outer BH horizon. Thus, the horizon wave equation [17]

\[ \mathcal{H}\Psi (\vec{r}) = E^2 \Psi (\vec{r}) \] (29)

The \( O(3) \) symmetry of the problem allows to express the angular dependence of the wave function in terms of spherical harmonics \( Y_l^m (\theta , \phi) \). The remaining part of the wave function \( \psi (r) \) satisfies the radial equation

\[ \left[ \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\psi}{dr} \right] + \left[ E^2 - \frac{r^2}{4G_N^2} \left( 1 + \frac{G_N^2 m_P^2}{r^2} \right)^2 - \frac{l(l + 1)}{r^2} \right] \psi (r) = 0 \] (30)

(30) looks like a relativistic wave equation for a ”particle” moving in some effective potential \( V_{eff} (r) \) given by

\[ V_{eff} (r) ≡ \frac{r^2}{4G_N} + \frac{\hbar^2}{r^2} + \frac{l(l + 1)}{r^2} \] (31)

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6In this section we essentially deal with quantum mechanics, thus it is convenient to use natural units: \( \hbar = 1 \), \( c = 1 \), \( G_N = l_P^2 = m_P^{-2} \).
The first term in (31) is an harmonic potential describing the vibrations of the horizon\(^7\). The second term is a GUP contribution \(\hbar = G_N m_P^2\) to the potential, while the third term is the usual centrifugal barrier. The wave equation admits an exact solution:

\[
\psi(r) = N_n \left( \frac{r^2}{2G_N} \right)^s e^{-r^2/4G_N} L_n^{2s+1/2} \left( \frac{r^2}{2G_N} \right)
\]  

(32)

where, \(L_n^{2s+1/2} \left( \frac{r^2}{2G_N} \right)\) is a generalized Laguerre polynomial; \(N_n\) is the normalization coefficient, and

\[
s = \frac{1}{4} \left[ \sqrt{G_N^2 m_P^4 + (2l + 1)^2} - 1 \right] = \frac{1}{4} \left[ \sqrt{1 + (2l + 1)^2} - 1 \right]
\]  

(33)

As the form of \(V_{\text{eff}}(r)\) suggests, one finds a discrete energy spectrum

\[
\frac{E_{n,l}^2}{m_P^2} = 2n + \sqrt{1 + (2l + 1)^2} + \frac{3}{2}, \quad n = 0, 1, 2, 3, \ldots
\]  

(34)

Thus, Hawking radiation at the Planck scale proceeds through single quantum jumps, similar to the decay of an excited atom. The minimal energy ground state is

\[
E_{0,0}^2 = \frac{3 + 2\sqrt{2}}{2} m_P^2
\]  

(35)

This is the stable remnant left by the QBH decay process. Finally, it is interesting to notice that, at high angular momentum \(l >> 1\), QBHs fit on a linear Regge trajectory

\[
l \simeq \alpha' E^2 + \alpha(0)
\]  

(36)

with a Regge slope \(\alpha' = 1/2m_P^2\) and a negative intercept \(\alpha(0) = -n - 3/4\). This behavior strongly indicates that the excited states of higher angular momentum are stringy in nature \[32, 33, 34, 35\].

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\(^7\)A slightly different approach to the horizon wave function has been proposed in \[29, 30, 31\]
5. Conclusions

We introduced a general, physically compelling criterion in order to distinguish between a quantum particle and a QBH. The separation is based on the ratio of the two basic length scales: the Compton wavelength and the gravitational radius. When their ratio is close to one we are in a genuine quantum gravity regime and the geometric, static Schwarzschild horizon does not provide a satisfactory description of a QBH. In order to provide a proper description, we introduced a Generalized Uncertainty Principle between the QBH mass and its horizon. Eventually GUP leads to an "effective" Scharzschild like-geometry with quantum gravity correction to the Newtonian potential. In spite of this simple looking form, the corresponding metric ensures that QBH can exist only above the Planck mass. On the thermodynamical side, QBH cannot evaporate completely, but rather ends up as zero temperature Planckian remnant.

Then we improved this semi-classical geometric description by allowing the horizon to undergo quantum fluctuations. A full quantum formulation, based on the GUP formulated at the beginning of the paper, is formulated. The wave equation looks like a Schrödinger-type equation for a particle subjected to an effective potential. The exact solution for the horizon wave equation is obtained and the mass-energy spectrum is found to be discrete and bounded from below. Interestingly enough excited states with high angular momentum correspond to linear Regge trajectory displaying a characteristic string-like behavior.

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