Incompressible Limits of the Patlak-Keller-Segel Model and Its Stationary State

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Abstract
We complete previous results about the incompressible limit of both the $n$-dimensional ($n \geq 3$) compressible Patlak-Keller-Segel (PKS) model and its stationary state. As in previous works, in this limit, we derive the weak form of a geometric free boundary problem of Hele-Shaw type, also called congested flow. In particular, we are able to take into account the unsaturated zone, and establish the complementarity relation which describes the limit pressure by a degenerate elliptic equation. Not only our analysis uses a completely different framework than previous approaches, but we also establish two novel uniform estimates in $L^3$ of the pressure gradient and in $L^1$ for the time derivative of the pressure. We also prove regularity à la Aronson-Bénilan. Furthermore, for the Hele-Shaw problem, we prove the uniqueness of solutions, meaning that the incompressible limit of the PKS model is unique. In addition, we establish the corresponding incompressible limit of the stationary state for the PKS model with a given mass, where, different from the case of PKS model, we obtain the uniform bound of pressure and the uniformly bounded support of density.

Keywords Keller-Segel system · Incompressible limit · Aronson-Bénilan estimate · Complementarity relation · Free boundary · Hele-Shaw problem

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1 Introduction

The Patlak-Keller-Segel (PKS) model can be used to describe the collective dynamics of a large number of individual agents interacting through a diffusive signal. For instance, it appears for the chemotaxis phenomena of various types of cells, aggregation dynamics of crowds or to describe the gravitational collapse, see [15, 50, 57], and the references therein. With a source term, it is used as a mechanical description of tumor growth [29, 47]. Including nonlinear diffusivity and Newtonian interactions, the PKS model is written

\begin{align}
\begin{aligned}
\partial_t \rho_m &= \Delta \rho_m + \nabla \cdot (\rho_m \nabla \mathcal{N} \ast \rho_m) & \text{for} & (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \quad n \geq 3, \\
\rho_m(x,0) &= \rho_{m,0}(x) \geq 0 & \text{for} & x \in \mathbb{R}^n, \quad m > 2 - 2/n, \quad \text{(subcritical case)}.
\end{aligned}
\end{align}

(1.1)

For chemotaxis, \(\rho_m(x,t) \geq 0\) represents the cell density and \(\mathcal{N} \ast \rho_m\) represents the chemical substance concentration obtained by convolution with the Newtonian potential

\[
\mathcal{N}(x) = \frac{-1}{n(n-2)\alpha_n|x|^{n-2}} \quad \text{for} \quad x \in \mathbb{R}^n \setminus \{0\}, \quad \Delta \mathcal{N} = \delta,
\]

with \(\alpha_n > 0\) being the volume of the \(n\)-dimensional unit ball and \(\delta\) the Dirac measure. The conservation of mass for the Cauchy problem Eq. (1.1) holds

\[
\int_{\mathbb{R}^n} \rho_m(x,t)dx = \int_{\mathbb{R}^n} \rho_{m,0}dx := M, \quad \forall \ t \geq 0.
\]

For solutions of Eq. (1.1), the pressure denotes a power of the density (Darcy’s law) as

\[
P_m := \frac{m}{m-1} \rho_m^{m-1}, \quad P_{m,0} := \frac{m}{m-1} \rho_{m,0}^{m-1}.
\]

(1.2)

We can rewrite Eq. (1.1) for the density \(\rho_m\) and pressure \(P_m\) in terms of the transport equation with the effective velocity \(u_m\) as

\[
\partial_t \rho_m = \nabla \cdot (\rho_m u_m), \quad u_m := \nabla P_m + \nabla \mathcal{N} \ast \rho_m.
\]

(1.3)

By a direct computation, the pressure satisfies the equation

\[
\partial_t P_m = (m - 1) P_m (\Delta P_m + \rho_m) + \nabla P_m \cdot (\nabla P_m + \nabla \mathcal{N} \ast \rho_m).
\]

(1.4)

The competition between the degenerate diffusion and the nonlocal aggregation is the main characteristic of Eq. (1.1) or Eq. (1.3). This is well represented by the free energy functional

\[
F_m(\rho_m) = \frac{1}{m - 1} \int_{\mathbb{R}^n} \rho_m^m dx - \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \mathcal{N} \ast \rho_m|^2 dx.
\]

(1.5)

It satisfies the energy identity, which shows that \(F_m(\rho_m)\) is non-increasing with time,

\[
\frac{dF_m(\rho_m)}{dt} + \int_{\mathbb{R}^n} \rho_m |\nabla (P_m + \mathcal{N} \ast \rho_m)|^2 dx = 0.
\]

(1.6)

Since \(\delta F_m(\rho_m) = P_m + \mathcal{N} \ast \rho_m\) represents the chemical potential, there exists a gradient flow structure for the PKS model,

\[
\frac{dF_m(\rho_m)}{dt} + \int_{\mathbb{R}^n} \rho_m |\nabla \frac{\delta F_m(\rho_m)}{\delta \rho_m}|^2 dx = 0.
\]

(1.7)
Solutions \( \rho_{m,s} \) of the stationary PKS system (SPKS) satisfy that the free energy \( F_m(\rho_m(t)) \) is constant in time and thus are determined by

\[
\nabla \rho_{m,s}^m + \rho_{m,s} \nabla N^* \rho_{m,s} = 0 \quad \text{for} \ x \in \mathbb{R}^n, \quad P_{m,s} = \frac{m}{m-1} \rho_{m,s}^{m-1}.
\]

(1.8)

Since a decade, and the paper [54] motivated by tumor growth, a large literature has been devoted to studying the incompressible (Hele-Shaw) limit, which means the limit as \( m \to \infty \), for several variants of the porous medium equations (see below). In particular, establishing this limit with Newtonian interactions, as in Eq. (1.1)–(1.8), has been a long standing question. It was solved by Craig, Kim and Yao in [20] based both on optimal transportation methods and viscosity solutions. Compared to optimal transport method, our approach has the advantage to be extended to growth terms in the right hand side of Eq. (1.1), which is important for some applications, including tumor growth. Compared to viscosity solutions methods, it uses a totally different background in terms of initial data, assumptions and methods.

**Incompressible Limit** Our purpose is to complete the understanding, from [20], of the incompressible (Hele-Shaw) limit for the PKS model Eq. (1.1) in various directions. Firstly, we introduce a third approach based on weak solutions as described below. In particular, our assumptions on the initial data are more general (not necessarily patches data), and the method can easily be extended to source terms when mass varies. Secondly, we prove new regularity results: an \( L^3 \) estimate on \( \nabla P_m \) and regularity à la Aronson-Bénilan showing bounds on the second derivatives of the pressure \( P_m \). Thirdly, we can prove directly an estimate on the time derivative of the pressure based on a new idea since a direct approach would not work. Finally, we prove a new uniqueness theorem for the limiting Hele-Shaw problem.

Following [20], the Hele-Shaw limit system writes

\[
\left\{
\begin{array}{l}
\partial_t \rho_\infty = \Delta P_\infty + \nabla \cdot (\rho_\infty \nabla N^* \rho_\infty), \quad \text{in} \ D'(\mathbb{R}^n \times \mathbb{R}_+), \\
(1 - \rho_\infty) P_\infty = 0, \quad 0 \leq \rho_\infty \leq 1, \quad \text{a.e.} \ (x,t) \in \mathbb{R}^n \times \mathbb{R}_+.
\end{array}
\right.
\]

(1.9)

This is a weak version of the geometric Hele-Shaw problem including chemotaxis. We also prove the complementarity relation (in distributional sense)

\[
P_\infty(\Delta P_\infty + \rho_\infty) = 0.
\]

(1.10)

It describes the limit pressure by a degenerate elliptic equation once we know the regularity of the set \( \{ \rho_\infty > 0 \} \), which is a major challenge for the Hele-Shaw problem, see [12, 33, 52] and reference therein. Furthermore, with Eqs. (1.9)–(1.10) at hand, the limiting free energy functional easily follows,

\[
\left\{
\begin{array}{l}
F_\infty(\rho_\infty) = \frac{1}{2} \int_{\mathbb{R}^n} \rho_\infty N^* \rho_\infty dx, \quad 0 \leq \rho_\infty \leq 1, \\
\frac{dF_\infty(\rho_\infty(t))}{dt} + \int_{\mathbb{R}^n} \rho_\infty(t) |\nabla (P_\infty(t) + N^* \rho_\infty(t))|^2 dx = 0.
\end{array}
\right.
\]

(1.11)

Compared with the free energy (1.5), the diffusive effect is replaced by the height constraint \( \rho_\infty \leq 1 \). In the end, we extend the uniqueness [4, 7] of solution to the PKS model Eq. (1.1) to the uniqueness of solution to the Hele-Shaw limit system Eq. (1.9).
In the stationary case, the incompressible (Hele-Shaw) limit from the SPKS model Eq. (1.8) as \( m \to \infty \), is represented as
\[
\begin{align*}
\nabla P_{\infty,s} + \rho_{\infty,s} \nabla N^* \rho_{\infty,s} &= 0, & \text{in } D'(\mathbb{R}^n), \\
(1 - \rho_{\infty,s}) P_{\infty,s} &= 0, & 0 \leq \rho_{\infty,s} \leq 1, & \text{a.e. } x \in \mathbb{R}^n.
\end{align*}
\]
(1.12)
As before, this corresponds to vanishing dissipation for the free energy \( F_\infty(\rho_\infty(t)) \).

The limits (1.9)–(1.11) can be formally derived from the PKS model Eq. (1.1). Indeed, taking the limit as \( m \to \infty \) in Eq. (1.1), we formally obtain the first equation in (1.9). Since we can prove that the limit pressure \( P_\infty \) is bounded, from (1.2) we recover (1.9)2. Also, we can formally attain the complementarity relation Eq. (1.10) thanks to a direct calculation of Eq. (1.4) as \( m \to \infty \). In addition, from (1.6), we can formally obtain the limit energy functional (1.11). It should be emphasized that the structure of gradient flow as in (1.7) is still present in a weak form as in the optimal transportation approach, cf. [18, 51]. Similarly, the incompressible limit Eq. (1.12) is formally derived from the SPKS model Eq. (1.8) as \( m \to \infty \). As it is well-known, establishing rigorously these limits faces deep difficulties due to the nonlinearities and weak regularity; the limit \( \rho_\infty \) is discontinuous in space and \( P_\infty \) can undergo discontinuities in time.

**Review of Literature** As mentioned earlier, several approaches are possible to overcome the above mentioned difficulties. Optimal transportation methods are used in conservative cases, and the incompressible limit is the so-called *congested flows*. This method was initiated in [49], and is well adapted for the transitions from discrete to continuous models [50]. It was extended to the two-species case in [40]. The case of Newtonian drift, and the limit \( m \to \infty \) was proved in [20].

We already mentioned viscosity solutions, see for instance [37, 39], for an external drift see [2, 38] and [19, 20] for Newtonian drifts. In particular, this approach can handle source terms as initiated in [39]. It has the advantage of handling specifically the free boundary in the limit with minimal assumptions for this purpose.

Our approach is by weak solutions as defined below (see Def. 1.1) and is motivated by tumor growth models of the form
\[
\partial_t \rho_m = \Delta \rho_m^m + \rho_m G(P_m) \quad \text{for } m > 1,
\]
(1.13)
where \( G(P) \) is a given decreasing function satisfying \( G(P_M) = 0 \) for some threshold \( P_M > 0 \). This problem was first solved in [54] using regularity as introduced by Aronson and Bénilan [3] and \( BV \) estimates. The method was extended to include a drift, see [23], to replace Darcy’s law by Brinkman’s law [25, 26] and to a system with nutrients in [22] using a new estimate in \( L^4_{t,x} \) for \( \nabla P_m \). Recently, multispecies problems were handled in [11, 34], and a major improvement for compactness followed by [46, 55], see also [36] and the most advanced version in [21].

Furthermore, let us recall that for the porous medium equation (PME), i.e., when \( G \equiv 0 \), the problem leads to the so-called *mesa problem* and was also treated in a large literature, see for instance [6, 31, 32, 35] and references therein. The weak formulation and the variational formulation (using the so-called Baiocchi variable), of Hele-Shaw type were first introduced in [28, 30] respectively.

Concerning the Keller-Segel model, with \( m \) fixed, very much is known and methods are nowadays well established. Important progresses have been made recently on global existence, large time behaviors, critical mass and finite time blow-up for the multi-dimensional
PKS model. In particular, the solutions with different diffusion exponent exhibit different behaviors. For diffusion exponent $1 \leq m < 2 - 2/n$ (supercritical case), the diffusion is dominant at the parts of low density and the aggregation is dominant at the parts of high density, then the solution to Eq. (1.1) exhibits finite time blow-up for large mass and global existence in time for small mass, cf. [14, 41, 56, 60, 61]. For $m = 2 - 2/n$ (critical case), there exists a critical mass $M_c > 0$ such that the solution blows up in a finite time for the initial mass $M > M_c$, [14, 41, 60], and exists globally in time for the initial mass $M < M_c$, see [9, 15] and reference therein. And for diffusion exponent $m > 2 - 2/n$ (subcritical case), the diffusion dominates at the parts of high density, the solution to this model is uniformly bounded and exists globally in time without any restriction on the size of the initial data, cf. [8, 13, 56, 61]. In addition, the large time behaviors have been investigated extensively, one can refer to [18, 37, 42, 48, 62] and references therein.

The SPKS model Eq. (1.8) has also been widely studied. For existence of solutions, see [16, 45, 59], for uniqueness see [5, 17, 27], and for radial symmetry see [17, 18, 35]. Critical points of free energy $F_m(\rho_m)$ in (1.5) have been studied, see [16, 45, 53, 59] and references therein. For the multi-dimensional SPKS model with more general attractive potential, the authors in [18] proved that the solution is radially decreasing symmetric up to a translation obtained by the method of continuous Steiner symmetrization, then it was proved in [27] uniqueness ($m \geq 2$) and non-uniqueness ($1 < m < 2$) of the solution to the SPKS with general attractive potential. Before that, the authors in [59] proved that all compactly supported solutions to the 3-dimensional SPKS model Eq. (1.8) with $m > 4/3$ must be radially symmetric up to a translation, hence obtaining uniqueness of the solution among compactly supported functions. Furthermore, for the same case, the authors in [44] proved, in 3 dimensions, uniqueness of the solution among radial functions for a given mass, and their method can handle general potential when $m > 2 - 2/n$. Similar results were obtained in [17] for 2-dimensional case with $m > 1$ by an adapted moving plane technique. Carrillo et al. in [16] showed the existence and compact support property of the radially symmetric solutions using dynamical system arguments.

Difficulties and Novelties However, it should be emphasized that the arguments for passing to incompressible limit in [22, 23, 54] cannot be applied directly to Eq. (1.1). This is due to the Newtonian drift in the PKS model, even though it is of lower order than the diffusion term. Its singularity gives rise to new and essential challenges for rigorously establishing the incompressible limits (1.9)–(1.11). Indeed, for the models of tumor growth as Eq. (1.13), the source term $\rho_m G(P_m)$ helps the authors to obtain a uniform $L^1$ estimate for the time derivative of both the density and the pressure by Kato’s inequality. But, for Eq. (1.1), on the one hand, the nonlocal Newtonian interaction leads to the absence of comparison principle, which means that it is impossible to get a uniform bound for the pressure. On the other hand, one of main challenges is to obtain a uniform $L^1$ estimate for the time derivative of pressure without the help of the source term, despite the effect of nonlocal interaction. Thus, it is difficult to gain the desired compactness on not only the density but also the pressure for the PKS model. Besides, using the weak formulation approach for the incompressible limit for the SPKS model Eq. (1.8) is a new and interesting topic for the diffusion-aggregation equations by the method of weak solutions, see [20] for viscosity solution methods.

Therefore, to achieve our goals, we develop new estimates and strategies as follows:

- We obtain the complementarity relation Eq. (1.10) for the PKS model Eq. (1.1). We first derive a uniform $L^3$ estimate on the pressure gradient in the spirit of [1, 22].
- In addition, we establish a new uniform $L^1$ estimate for the time derivative of pressure. To our knowledge, this is the first time such an estimate is obtained for the high-dimensional
porous medium equation (Darcy’s law) with a nonlocal attractive interaction since working directly on the pressure is not sufficient.

- Then, we establish the uniform Aronson-Bénilian (AB) estimates in $L^3 \cap L^1$ as initiated in [34]. In particular, we show a decay rate for the AB estimate in $L^3$ under the form

$$\| \min \{ \Delta P_m + \rho_m, 0 \} \|^3_{L^3(Q_T)} \leq \frac{C(T)}{m}.$$ 

- To prove the uniqueness of the solution to the Hele-Shaw limit system Eq. (1.9), the key is to show that the limit pressure is somehow monotone to the limit density. Suppose that $P_i \rho_i = P_i$ and $0 \leq \rho_i \leq 1$ for $i = 1, 2$ hold, we find $(P_1 - P_2)(\rho_1 - \rho_2) \geq 0$.

To establish the incompressible limit of the SPKS model with a given mass, we gain the uniform bound of the pressure and the uniformly bounded support of the density.

**Notations** We use the following notations and definitions.

**Notation 1.1** We set

- $Q_T = \mathbb{R}^n \times (0, T)$, $Q = \mathbb{R}^n \times (0, \infty)$.
- $B_R := \{ x : |x| \leq R \}$, $R > 0$.
- $|f(x)|_+ = \max\{f(x), 0\}$, $|f(x)|_- = -\min\{f(x), 0\}$.
- $\nabla^2 f : \nabla^2 g := \sum_{i,j=1}^n \partial^2_{ij} f \partial^2_{ij} g$, $(\nabla^2 f)^2 := \nabla^2 f : \nabla^2 f = \sum_{i,j=1}^n (\partial^2_{ij} f)^2$.

Also, we use $C$ as a generic constant independent of time $t$ and diffusion exponent $m$, $C(T)$ or $C(T, R)$ denote generic constants only depending on the time $T$ or on $T$ and $R > 0$.

**Definition 1.1 (Weak solution)** The weak solutions of the PKS model Eq. (1.1) and the SPKS model Eq. (1.8) are defined as follows:

- We recall that a weak solution to Eq. (1.1) means that for all $T > 0$ and all test function $\varphi \in C_0^\infty(Q_T)$, such that $\varphi(T) = 0$, it holds

$$\int_{Q_T} \left[ \rho_m \partial_t \varphi + \rho_m^m \Delta \varphi - \rho_m \nabla \varphi \cdot \nabla \mathcal{N} \ast \rho_m \right] dx \, dt = \int_{\mathbb{R}^n} \rho_{m, 0} \varphi(0) dx.$$

For that $\rho_m$, $\rho_m^m$ and $\rho_m \nabla \mathcal{N} \ast \rho_m$ are supposed to be integrable.

- A weak solution to Eq. (1.8) is defined for all test function $\varphi \in C_0^\infty(\mathbb{R}^n)$ as

$$\int_{\mathbb{R}^n} \left[ \nabla \rho_{m, s} \cdot \nabla \varphi + \rho_{m, s} \nabla \mathcal{N} \ast \rho_{m, s} \cdot \nabla \varphi \right] dx = 0,$$

where $\nabla \rho_{m, s}$ and $\rho_{m, s} \nabla \mathcal{N} \ast \rho_{m, s}$ are supposed to be integrable.

**2 Main Results**

To state our main results on the incompressible limit of PKS model, we need assumptions on the initial data $\rho_{m, 0}$. Firstly, for $\rho_{m, 0}$, we assume

$$\begin{cases}
\int_{\mathbb{R}^n} \rho_{m, 0}(x) dx =: M < \infty, \\
\| \rho_{m, 0}^{m+1} \|_{L^1(\mathbb{R}^n)} \leq C, \\
\| \rho_{m, 0} \|_{L^\infty(\mathbb{R}^n)} < \infty, \\
\| \rho_{m, 0} - \rho_{\infty, 0} \|_{L^1(\mathbb{R}^n)} \to 0, \quad as \ m \to \infty, \\
\text{supp}(\rho_{m, 0}) \subset B_{R_m} \text{ for some constant } R_m > 1.
\end{cases} \tag{2.1}$$
Secondly, for some results, in particular the Aronson-Bénilan estimate, we also need additional regularity assumptions on the initial data,

\[
\| P_{m,0} \|_{L^2(\mathbb{R}^n)} + \| \nabla P_{m,0} \|_{L^2(\mathbb{R}^n)} \leq C, \quad (2.2)
\]

\[
\| \Delta P_{m,0} + \rho_{m,0} \rho_{m,0} \|_{L^1(\mathbb{R}^n)} \leq C. \quad (2.3)
\]

Furthermore, a compatibility condition is also needed for obtaining the \( L^1 \) estimate of the time derivative for the pressure,

\[
\| (m+1) \rho_{m,0}^{m+1} (\Delta P_{m,0} + \rho_{m,0}) + \nabla \rho_{m,0}^{m+1} (\nabla P_{m,0} + \nabla N * \rho_{m,0}) \|_{L^1(\mathbb{R}^n)} \leq C. \quad (2.4)
\]

Finally, to show the compact support of the solution of the Hele-Shaw limit system Eq. (1.9), we need an additional uniform support assumption

\[
\text{supp}(\rho_{m,0}) \subset B_{R_0} \text{ for a fixed constant } R_0 > 0. \quad (2.5)
\]

**Remark 2.1** Let \( \varphi \in C_0^\infty(\mathbb{R}^n) \) and \( 0 \leq \varphi \leq \frac{1}{2} \), one immediately verifies that the initial data \( \rho_{m,0} = \varphi \) satisfies the assumptions (2.1)–(2.4).

Assumption (2.1) guarantees global existence of solutions to the Cauchy problem (1.1) because \( m > 2 - 2/n \), as mentioned earlier. We also recall in Appendices A and B that solutions satisfy, for some \( R_m(T) \),

\[
\| \rho_m \|_{L^\infty(Q_T)} \leq C(m, T), \quad \text{supp}(\rho_m(T)) \subset B_{R_m(T)}, \forall T > 0.
\]

We now gather several uniform regularity estimates, and then establish the stiff limit of the PKS model as \( m \to \infty \).

**Theorem 2.1** (Uniform bounds and compactness) Assume (2.1), then the global solution \( \rho_m \) to the Cauchy problem Eq. (1.1) in the sense of Def. 1.1 satisfies for any \( T > 0 \),

\[
\sup_{0 \leq t \leq T} \| \rho_m(t) \|_{L^q(\mathbb{R}^n)} + \| \rho_m^m \|_{L^2(Q_T)} + \| \nabla \rho_{m}^m \|_{L^2(Q_T)} \leq C(T), \quad \forall q \in [1, m+1],
\]

\[
\int_{Q_T} \nabla \rho_m^m \cdot \nabla \rho_m^{p-1} dxdt \leq C(T, p), \quad 1 < p \leq 2,
\]

\[
\| \rho_m \|_{L^{2+\frac{2}{h}(\frac{3}{m}+\frac{3}{2})}}(Q_T) + \| \nabla P_m \|_{L^2(Q_T)} \leq C(T),
\]

\[
\sup_{0 \leq t \leq T} \| \rho_m(t) \|_{L^2(\mathbb{R}^n)} - 1 + \| \rho_m \|_{L^2(\mathbb{R}^n)} \leq \frac{C(T)}{\sqrt{m}},
\]

\[
\| \partial_t \nabla N * \rho_m \|_{L^2(Q_T)} + \| \nabla N * \rho_m \|_{L^\infty(Q_T)} + \sup_{0 \leq t \leq T} \| \nabla N * \rho_m(t) \|_{L^2(\mathbb{R}^n)} \leq C(T),
\]

\[
\sup_{0 \leq t \leq T} \| \nabla^2 N * \rho_m(t) \|_{L^q(\mathbb{R}^n)} \leq C(T, q),
\]

where \( C(T, q) \sim \frac{1}{q-1} \) for \( 0 < q - 1 \ll 1 \) and \( C(T, q) \sim q \) for \( q \gg 1 \) and \( m > n - 1 \).
Theorem 2.2 (Stiff limit) With assumption (2.1), this limit, \((\rho_\infty, P_\infty)\) satisfies the Hele-Shaw limit system in the sense of Def. 1.1 as

\[
\begin{align*}
\partial_t \rho_\infty - \Delta P_\infty &= \nabla \cdot (\rho_\infty \nabla N^\ast \rho_\infty), & \text{in } \mathcal{D}'(Q_T), \quad (2.6) \\
(1 - \rho_\infty) P_\infty &= 0, & 0 \leq \rho_\infty \leq 1, & \text{a.e. in } Q_T. \quad (2.7)
\end{align*}
\]

Then, using the additional assumptions (2.2)–(2.3) on the initial data, we obtain the higher regularity estimates on the pressure. We can establish the

Theorem 2.3 (Complementarity relation and semi-harmonicity) Assume \(m > \max\{n - 1, \frac{5n - 2}{n + 2}\}\) and that the initial data satisfies (2.1)–(2.4), then the global weak solution \(\rho_m\) to (1.1) satisfies the additional regularity estimates

\[
\begin{align*}
\|\sqrt{P_m} \nabla P_m\|_{L^2(Q_T)} &+ \sup_{0 \leq t \leq T} \|\nabla P_m(t)\|_{L^2(\mathbb{R}^n)} \leq C(T), \\
\|\nabla P_m\|_{L^3(Q_T)} &+ \|\sqrt{P_m} \nabla^2 P_m\|_{L^3(Q_T)} \leq C(T), \\
\|\sqrt{P_m} \omega_m\|_{L^2(Q_T)}^2 &+ \|\|\omega_m\|_\infty\|_{L^3(Q_T)}^3 \leq \frac{C(T)}{m}, & \omega_m := \Delta P_m + \rho_m, \\
\sup_{0 \leq t \leq T} \|\|\omega_m\|_\infty\|_{L^2(Q_T)} &+ \|\nabla P_m(t)\|_{L^1(\mathbb{R}^n)} + \|\partial_t P_m\|_{L^1(Q_T)} \leq C(T).
\end{align*}
\]

Furthermore, after the extraction of subsequences, as \(m \to \infty\), \(\nabla P_m\) converges strongly in \(L^2_{nloc}(Q_T)\) to \(\nabla P_\infty \in L^3(Q_T) \cap L^\infty(0, T; L^2(\mathbb{R}^n))\), and the complementarity relation and semi-harmonicity hold

\[
P_\infty(\Delta P_\infty + \rho_\infty) = 0, \quad \Delta P_\infty + \rho_\infty \geq 0, & \text{in } \mathcal{D}'(Q_T). \quad (2.8)
\]

It follows that

\[(1 - \rho_\infty) \nabla P_\infty = 0, & \text{a.e. in } Q_T.\]

For the Hele-Shaw system Eqs. (2.6)–(2.7), the weak solution to the Cauchy problem is unique.

Theorem 2.4 (Uniqueness) Being given two global weak solutions \(\rho_i\) for \(i = 1, 2\) to the Hele-Shaw system (2.6)–(2.7) with the initial assumption \(\rho_i(x, 0) = \rho_2(x, 0) \in \dot{H}^{-1}(\mathbb{R}^n)\), we have

\[
\rho_1 = \rho_2, \quad P_1 = P_2, & \text{a.e. in } Q.
\]

Next, we establish that the limit free energy functional \(F_\infty(\rho_\infty(t))\), with \(0 \leq \rho_\infty \leq 1\), is non-increasing as time increases.
Theorem 2.5 (Compact support and limit energy functional) Under the initial assumptions (2.1)–(2.5), the limit \((\rho_\infty, P_\infty)\), as in Theorems 2.2–2.3, are compactly supported for any finite time. For some \(R_0 \geq \max(R_0, \sqrt{4n + \frac{4n^2}{n-2}})\), we have

\[
\text{supp}(P_\infty(t)) \subset \text{supp}(\rho_\infty(t)) \subset B_{R(t)},
\]

\(\mathcal{R}(t) := (R_0 + n\|\nabla \ast \rho_\infty\|_{L^\infty(Q)})e^\frac{t}{2} - n\|\nabla \ast \rho_\infty\|_{L^\infty(Q)}\).

Furthermore, the limit energy dissipation holds for a.e. \(t \in [0, \infty)\),

\[
\frac{dF_\infty(\rho_\infty(t))}{dt} + \int_{\mathbb{R}^n} \rho_\infty(t)|\nabla (P_\infty(t) + \mathcal{N} \ast \rho_\infty(t))|^2 dx = 0 \quad \text{with} \quad 0 \leq \rho_\infty \leq 1.
\]

From \([16, 18, 27]\), we know that the solution to the SPKS model Eq. (1.8) are radially decreasing symmetric up to a translation and compactly supported. This allows us to gather some useful a priori estimates in order to prove the compactness for \(\rho_{m,s}, P_{m,s}\), and \(\mathcal{N} \ast \rho_{m,s}\). Then, we can derive the incompressible limit of the SPKS model Eq. (1.8).

Theorem 2.6 (Incompressible limit for stationary state) Let \(m \geq 3\), \(\rho_{m,s}\) be a weak solution to the SPKS model Eq. (1.8) in the sense of Def. 1.1 with a given mass \(\|\rho_{m,s}\|_{L^1(\mathbb{R}^n)} = M > 0\), \(\int_{\mathbb{R}^n} x\rho_{m,s}(x)dx = 0\). We define \(R_m(M) > 0\) satisfying \(B_{R_m(M)} := \text{supp}(\rho_{m,s})\) and \(\alpha_m := \rho_{m,s}(0) = \|\rho_{m,s}\|_{L^\infty(\mathbb{R}^n)}\), then the following regularity estimates hold,

\[
\alpha_m^{m-1} \leq \alpha_m + \frac{2M}{n(n-2)\omega_n} \leq \frac{1 + \sqrt{1 + \frac{8M}{n(n-2)\omega_n}}}{2} + \frac{2M}{n(n-2)\omega_n},
\]

\(R_m(M) \leq R_s(M), \|\rho_{m,s}\|_{L^1 \cap L^\infty(\mathbb{R}^n)} \leq C,\)

\(\|\nabla \mathcal{N} \ast \rho_{m,s}\|_{L^\infty(\mathbb{R}^n)} \leq C, \|\nabla^2 \mathcal{N} \ast \rho_{m,s}\|_{L^p(\mathbb{R}^n)} \leq C(M, p),\)

\(\|\omega_{m,s} - \|\omega_{m,s}\|_{L^3(\mathbb{R}^n)}\|_{L^1(\mathbb{R}^n)} \leq C, \omega_{m,s} = \Delta P_{m,s} + \rho_{m,s},\)

\(\|P_{m,s}\|_{L^1 \cap L^\infty(\mathbb{R}^n)} + \|\nabla P_{m,s}\|_{L^1 \cap L^\infty(\mathbb{R}^n)} + \|\Delta P_{m,s}\|_{L^1(\mathbb{R}^n)} \leq C,\)

where \(R_s(M) = \log \left(1 + \exp \left[2n(n-1)\left(\frac{1+\sqrt{1+\frac{8M}{n(n-2)\omega_n}}}{2} + \frac{2M}{n(n-2)\omega_n}\right)^{1/2}\right]\right)\), \(C(M, p) \sim \frac{1}{p-1}\) for \(0 < p - 1 \ll 1\) and \(C(M, p) \sim p\) for \(p \gg 1\).

Furthermore, after extracting subsequences, as \(m \to \infty\), \(P_{m,s}\) converges strongly in \(L^\infty(\mathbb{R}^n)\) to \(P_{\infty,s} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\), \(\rho_{m,s}\) converges weakly in \(L^p(\mathbb{R}^n)\) for \(1 < p < \infty\) to \(\rho_{\infty,s} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\), \(\nabla P_{m,s}\) converges strongly in \(L^p(\mathbb{R}^n)\) for \(1 \leq p < \infty\) to \(\nabla P_{\infty,s} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\), \(\nabla \mathcal{N} \ast \rho_{m,s}\) locally converges strongly in \(L^p(\mathbb{R}^n)\) for \(1 \leq p < \infty\) to \(\nabla \mathcal{N} \ast \rho_{\infty,s} \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap W^{1,q}(\mathbb{R}^n)\) for \(1 < q < \infty\). Therefore, the incompressible (Hele-Shaw) limit of the SPKS model Eq. (1.8) satisfies

\[
\|\rho_{\infty,s}\|_{L^1(\mathbb{R}^n)} = M, \int_{\mathbb{R}^n} x\rho_{\infty,s}dx = 0, 0 \leq \rho_{\infty,s} \leq \rho_{\infty,s} = 0, \quad a.e. \text{in } \mathbb{R}^n,
\]
\[\nabla P_{\infty,s} + \rho_{\infty,s} \nabla N * \rho_{\infty,s} = 0, \quad \text{a.e. in } \mathbb{R}^n,\]
\[\Delta P_{\infty,s} + \rho_{\infty,s} \geq 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^n).\]

Moreover, it holds for \(R(M) > 0\) satisfying \(|B_{R(M)}|_n = M\) that
\[\rho_{\infty,s} = \chi_{\{P_{\infty,s} > 0\}} = \chi_{B_{R(M)}}, \quad \text{a.e. in } \mathbb{R}^n.\]

**Remark 2.2** The results in Theorem 2.6 show that the incompressible limit of the SPKS model Eq. (1.8) is the stationary state of the Hele-Shaw problem Eqs. (2.6)–(2.8).

### 3 Bounds, Compactness and Stiff Limit

We establish the estimates in Theorem 2.1 and then the stiff limit in Theorem 2.2.

We begin with the a priori regularity results on the density \(\rho_m\), and then treat the nonlocal term.

**Lemma 3.1 (Regularity estimate on density and pressure)** Assume that the initial data satisfies (2.1). Let \(\rho_m\) be the weak solution to Eq. (1.1) in the sense of Def. 1.1, then it follows
\[\sup_{0 \leq t \leq T} \|\rho_m(t)\|_{L^{m+1}(\mathbb{R}^n)} + \|\nabla \rho_m^m\|_{L^2(Q_T)} \leq C(T), \quad (3.1)\]
\[\sup_{0 \leq t \leq T} \|\rho_m(t)\|_{L^p(\mathbb{R}^n)} + \int_{Q_T} \nabla \rho_m^m \cdot \nabla \rho_m^p dx \leq C(T), \quad 1 < p \leq 2, \quad (3.2)\]
\[\|\rho_m\|_{L^{2+2/n}(\mathbb{R}^n)} + \|\rho_m \nabla P_m\|_{L^2(Q_T)} \leq C(T), \quad (3.3)\]
\[\|P_m\|_{L^2(Q_T)} + \|\rho_m^m\|_{L^2(Q_T)} + \|\nabla P_m\|_{L^2(Q_T)} \leq C(T). \quad (3.4)\]

**Proof** For (3.1), we multiply Eq. (1.1) by \(\rho_m^m\) and integrate by parts on \(\mathbb{R}^n\), we find
\[\frac{1}{m+1} \frac{d}{dt} \int_{\mathbb{R}^n} \rho_m^{m+1} dx + \int_{\mathbb{R}^n} |\nabla \rho_m^m|^2 dx \leq \frac{m}{m+1} \int_{\mathbb{R}^n} \rho_m^{m+2} dx \]
\[\leq \left[\|\rho_m\|_{L^1(\mathbb{R}^n)} \|\rho_m^m\|_{L^{2/m}(\mathbb{R}^n)}\right]^{m+2} \leq \|\rho_m\|_{L^1(\mathbb{R}^n)} \|\rho_m^m\|_{L^{2/m}(\mathbb{R}^n)}^{m+2}, \]

where we have used interpolation inequality with \(\frac{1}{m+2} = 1 - \theta + \theta \frac{n-2}{2mn}\). We notice that \((1 - \theta)(m + 2) \leq 1\). Using this and Sobolev’s inequality (Theorem C.3), we get, using \(\theta \frac{m+2}{m} < 2\) for \(m > 2 - \frac{2}{n}\),
\[\frac{1}{m+1} \frac{d}{dt} \int_{\mathbb{R}^n} \rho_m^{m+1} dx + \int_{\mathbb{R}^n} |\nabla \rho_m^m|^2 dx \leq \max(1, \|\rho_m\|_{L^1(\mathbb{R}^n)}) \|\nabla \rho_m^m\|_{L^{2/m}(\mathbb{R}^n)}^{m+2} \]
\[\leq C + \frac{1}{2} \|\nabla \rho_m^m\|_{L^2(\mathbb{R}^n)}^2, \]

After time integration, we get the inequality
\[\int_{\mathbb{R}^n} \rho_m(t)^{m+1} dx + \frac{m+1}{2} \int_0^t \|\nabla \rho_m^m\|_{L^2(\mathbb{R}^n)}^2 dx + C t (m+1), \quad (3.5)\]
which implies (3.1) in Lemma 3.1;

\[ \| \rho_m(t) \|_{L^{m+1}(\mathbb{R}^n)} \leq (C + C \tau (m + 1))^{\frac{1}{m+1}} \leq C(T). \]

A similar calculation gives (3.2) and we have

\[ \frac{1}{p} \int_{\mathbb{R}^n} \rho_m(t)^p dx + \frac{4m(p-1)}{(m+p-1)^2} \int_{Q_T} |\nabla \rho_m|^{\frac{m+p-1}{2}} dx \frac{ds}{ds} \]

\[ = \frac{1}{p} \int_{\mathbb{R}^n} \rho_m^p dx + \frac{p-1}{p} \int_{Q_T} \rho_m^{p+1} dx. \]

Interpolating between \( L^{m+1}(\mathbb{R}^n) \) and \( L^1(\mathbb{R}^n) \), we know that the terms on the right hand side are controlled and thus the gradient term is under control. It remains to notice that

\[ \frac{4m(p-1)}{(m+p-1)^2} \int_{Q_T} |\nabla \rho_m|^{\frac{m+p-1}{2}} dx = m(p-1) \int_{Q_T} \rho_m^{m+p-3} |\nabla \rho_m|^2 dx = \int_{Q_T} \nabla \rho_m^m, \nabla \rho_m^{p-1} dx, \]

and (3.2) is proved.

We turn to (3.3). Thanks to the interpolation inequality, we have, for \( \alpha \geq 0 \) and \( t \leq T \),

\[ \int_{\mathbb{R}^n} \rho_m(t)^{2m+\alpha} dx \leq \| \rho_m(t) \|_{L^{m+1}(\mathbb{R}^n)}^{(1-\theta)(2m+\alpha)} \| \rho_m(t) \|_{L^{2m+\alpha}(\mathbb{R}^n)}^{\theta(2m+\alpha)} \]

\[ = \| \rho_m(t) \|_{L^{m+1}(\mathbb{R}^n)}^{(1-\theta)(2m+\alpha)} \| \rho_m(t) \|_{L^{2m+\alpha}(\mathbb{R}^n)}^{\theta(2m+\alpha)} \]

with

\[ \frac{1}{2m+\alpha} = \frac{1}{m+1} + \frac{\theta(n-2)}{2mn}, \quad 0 \leq \theta \leq 1. \]

By Sobolev’s inequality and the estimate in \( L^{m+1} \), we obtain

\[ \int_{\mathbb{R}^n} \rho_m(t)^{2m+\alpha} dx \leq C(T)^{(1-\theta)(2m+\alpha)} \| \nabla \rho_m^m \|_{L^{2m+\alpha}(\mathbb{R}^n)}^{\theta(2m+\alpha)} \]

It remains to choose \( \alpha \) such that \( \theta \frac{2m+\alpha}{m} = 2 \) and we find, integrating in time,

\[ \int_{Q_T} \rho_m(t)^{2m+\alpha} dx \leq C(T)^{(1-\theta)(2m+\alpha)} \int_{Q_T} |\nabla \rho_m|^2 dx. \]

To compute the value of \( \alpha \), we write the condition successively as

\[ 2m = (2m+\alpha)\theta = (2m+\alpha) \left[ \frac{1}{m+1} - \frac{1}{2m+\alpha} \right] \left[ \frac{1}{m+1} - \frac{n-2}{2mn} \right]^{-1}, \]

\[ 2m = [2m+\alpha -(m+1)] \left[ 1 - \frac{(n-2)(m+1)}{2mn} \right]^{-1}, \]

\[ 2m - \frac{(n-2)(m+1)}{n} = 2m + \alpha - (m+1), \quad \alpha = \frac{2}{n}(m+1). \]
This gives the first statement of (3.3). Then, since $\nabla \rho_m = \rho_m \nabla P_m$, we have from (3.1) $\| \rho_m \nabla P_m \|_{L^2(Q_T)} \leq C(T)$ and (3.3) is proved.

The first estimate of (3.4) is obtained by interpolation and Sobolev’s inequality for gradient (Theorem C.3) between two estimates in (3.1), for $\gamma = 0, 1$.

$$
\int_{Q_T} \rho_m^{2m-2\gamma} \, dx \, dt \leq \left( \int_{Q_T} \rho_m^{2m+1} \, dx \, dt \right)^{2n-2\gamma/\gamma m} \left( \int_{Q_T} \rho_m \, dx \, dt \right)^{2\gamma+1/\gamma m}
\leq C(T) \int_0^T \left( \int_{\mathbb{R}^n} (\rho_m^n) \, dx \right)^{\frac{n-2}{n}} \left( \int_{\mathbb{R}^n} \rho_m^\gamma \, dx \right)^{\frac{2}{\gamma}} \, dt
\leq C(T) \int_{Q_T} \| \nabla \rho_m \|^2_{L^2} \, dx \, dt \leq C(T).
$$

We prove the second estimate of (3.4) by means of the estimates (3.1)$_{2\text{nd}}$–(3.2)$_{2\text{nd}}$

$$
\int_{Q_T} |\nabla P_m|^2 \, dx \, dt = m^2 \int_{Q_T} \rho_m^{2m-2} |\nabla \rho_m|^2 \, dx \, dt
\leq \int_{Q_T} \left( \frac{2m^2}{m-1} \rho_m^{m-1} + \frac{m^2(m-3)}{m-1} \rho_m^{m-2} \right) |\nabla \rho_m|^2 \, dx \, dt
= \int_{Q_T} \left( \frac{2m}{m-1} \nabla \rho_m \cdot \nabla \rho_m + \frac{m^2}{m-1} \nabla \rho_m^m \right) \, dx \, dt \leq C(T).
$$

Now, we turn to the nonlocal term.

**Lemma 3.2** (Regularity of the nonlocal term) Assume $m > n - 1$ and (2.1), let $\rho_m$ be a weak solution to Eq. (1.1) in the sense of Def. 1.1, then it holds for any $T > 0$ that

$$
\| \nabla \mathcal{N} * \rho_m \|_{L^\infty(Q_T)} \leq C(T), \quad \| \partial_t \nabla \mathcal{N} * \rho_m(t) \|_{L^2(Q_T)} \leq C(T),
$$

$$
\sup_{0 \leq t \leq T} \| \nabla^2 \mathcal{N} * \rho_m(t) \|_{L^2(\mathbb{R}^n)} \leq C(T, q), \quad \sup_{0 \leq t \leq T} \| \nabla \mathcal{N} * \rho_m(t) \|_{L^q(\mathbb{R}^n)} \leq C(T).
$$

where $C(T, q) \sim \frac{1}{q^{-1}}$ for $0 < q - 1 \ll 1$ and $C(T, q) \sim q$ for $q \gg 1$. Furthermore, after extraction, it holds that

$$
\nabla \mathcal{N} * \rho_m \to \nabla \mathcal{N} * \rho_\infty, \text{ strongly in } L^2_{\text{loc}}(Q_T), \text{ as } m \to \infty.
$$

**Proof** For the first estimate of (3.6), by means of Lemma 3.1, we obtain the $L^\infty$ estimate for $\nabla \mathcal{N} * \rho_m(t)$ with $m > n - 1$ because

$$
|\nabla \mathcal{N} * \rho_m(t)| \leq C \int_{\mathbb{R}^n} \frac{\rho_m(y, t)}{|x-y|^{n-1}} \, dy \leq C \int_{|x-y| \leq 1} \frac{\rho_m(y, t)}{|x-y|^{n-1}} \, dy + C \int_{|x-y| > 1} \frac{\rho_m(y, t)}{|x-y|^{n-1}} \, dy
\leq C \left( \int_{|x-y| \leq 1} \frac{1}{|x-y|^{n-1}} \, dy \right)^{\frac{n-1+c}{n}} \left( \int_{|x-y| > 1} \rho_m^{n+c} \, dy \right)^{\frac{1}{n+c}} + C
\leq C(T) \quad \forall t \in [0, T] \text{ and } 0 < \varepsilon \ll 1.
$$

Let the Laplace inverse operator $\Delta^{-1} = \mathcal{N}^\ast$ act on Eq. (1.1), we get a new equation

$$
\partial_t \mathcal{N} * \rho_m = \rho_m^m + \nabla \cdot \mathcal{N} * (\rho_m \nabla \mathcal{N} * \rho_m).
$$
Then, using the singular integral theory for Newtonian potential (Lemma C.1), (3.8), and Lemma 3.1, we obtain

\[ \int_{\mathbb{R}^n} |\nabla \nabla \cdot N \ast (\rho_m \nabla N \ast \rho_m)|^2 \, dx \leq C \int_{\mathbb{R}^n} |\rho_m \nabla N \ast \rho_m|^2 \, dx \]

\[ \leq C \|\nabla N \ast \rho_m\|_{L^\infty(Q_T)}^2 \int_{\mathbb{R}^n} \rho_m^2 \, dx \leq C(T). \tag{3.10} \]

Due to Eq. (3.9), we use (3.10) and Lemma 3.1, then it follows

\[ \|\partial_t \nabla N \ast \rho_m\|_{L^2(Q_T)} \leq \|\nabla \rho_m\|_{L^2(Q_T)} + \|\nabla \nabla \cdot N \ast (\rho_m \nabla N \ast \rho_m)\|_{L^2(Q_T)} \leq C(T) \]

and the second bound of (3.6) is proved.

For the first estimate of (3.7), we again use the singular integral theory for Newtonian potential (Lemma C.1), and we have for all \( t \in [0, T] \),

\[ \|\nabla^2 N \ast \rho_m(t)\|_{L^q(\mathbb{R}^n)} \leq C(q) \|\rho_m\|_{L^q(\mathbb{R}^n)} \leq C(T, q), \tag{3.11} \]

where \( C(T, q) \sim \frac{1}{q-1} \) for \( 0 < q - 1 \ll 1 \) and \( C(T, q) \sim q \) for \( q \gg 1 \).

And for the second bound of (3.7), thanks to the Hardy-Littlewood-Sobolev inequality (Theorem C.1) and Lemma 3.1, we get for all \( t \in [0, T] \),

\[ \int_{\mathbb{R}^n} |\nabla N \ast \rho_m(t)|^2 \, dx = -\int_{\mathbb{R}^n} (\Delta N \ast \rho_m)N \ast \rho_m \, dx = -\int_{\mathbb{R}^n} \rho_m N \ast \rho_m \, dx \leq C(n) \|\rho_m\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^2 \leq C(T). \tag{3.12} \]

The last statement of Lemma 3.2 follows from Sobolev’s compactness embeddings. \( \square \)

In order to obtain convergence rate on \(|\rho_m - 1|_+\) in Theorem 2.1, it remains to establish the

**Lemma 3.3 (convergence rate on \(|\rho_m - 1|_+\))** Under the initial assumptions (2.1), let \( \rho_m \) be the weak solution to the Cauchy problem Eq. (1.1) in the sense of Def. 1.1 with \( m > n - 1 \), then

\[ \sup_{0 \leq t \leq T} \|\rho_m(t) - 1\|_{L^2(\mathbb{R}^n)} \leq \frac{C(T)}{\sqrt{m}}. \]

**Proof** Since, for \( m > n - 1 \geq 2 \) and \( \rho_m \geq 1 \), we have

\[ \rho_m^{m+1} \geq \frac{m(m+1)}{2} (\rho_m - 1)^2, \]

we conclude

\[ \text{sgn}(|\rho_m - 1|_+) \rho_m^{m+1} \geq \frac{m(m+1)}{2} |\rho_m - 1|_+^2. \]

From (3.5), we obtain

\[ \int_{\mathbb{R}^n} |\rho_m(t) - 1|_+^2 \, dx \leq \frac{2}{m(m+1)} \int_{\mathbb{R}^n} \rho_m^{m+1}(t) \, dx \leq \frac{C(T)}{m} \quad \text{for all } 0 \leq t \leq T. \]
Remark 3.1 The result of Lemma 3.3 implies that larger diffusion exponent means stronger diffusive effect on the zone of high density.

In the following, with the regularity estimates in Lemmas 3.1–3.3 in hand, we prove the stiff limit statements in Theorem 2.2.

We recall that, thanks to the a priori regularity estimates in Lemmas 3.1–3.3, $\rho_m$ has a weak limit $\rho_\infty$ with $\rho_\infty \leq 1$ in $L^p(Q_T)$ for $1 < p < \infty$, $P_m$ has a weak limit $P_\infty$ in $L^2(Q_T)$, and we have locally strong convergence of $\nabla N^* \rho_m$ to $\nabla N^* \rho_\infty$ in $L^2(Q_T)$.

Proof of Eq. (2.6). The stiff limit equation (2.6) in Theorem 2.2 follows immediately with these weak limits and the definition of weak solutions in Def. 1.1, where the nonlinear term $\rho_m \nabla N^* \rho_m$ can pass to the limit $\rho_\infty \nabla N^* \rho_\infty$ by weak-strong convergence, and the other nonlinear term $\rho_m^m = \left( m - \frac{1}{m} \right) \rho_m P_m$ weakly converges to $P_\infty$ in $L^2(Q_T)$ from (3.13).

Proof of Eq. (2.7), $\rho_\infty P_\infty = P_\infty$, a.e. $(x, t) \in Q_T$. For the case of tumor growth model in [54], the proof is obtained because $\rho_m$ converges strongly, which is not available here. Therefore, we argue in two steps. We firstly prove that after extraction,

$$\rho_m P_m \rightharpoonup P_\infty,$$

weakly in $L^2(Q_T)$, as $m \to \infty$. (3.13)

For that, thanks to the relation $\rho_m P_m = \left( m - \frac{1}{m} \right) \rho_m^m \leq 2 \rho_m^m$, it follows from Lemma 3.1 that $\rho_m P_m$ is bounded in $L^2(Q_T)$ and thus, after extraction, has a weak limit in $L^2(Q_T)$, which we call $Q_\infty$.

Due to Young’s inequality, we have

$$P_m = \frac{m}{m - 1} \rho_m^{m-1} \leq \frac{m}{m - 1} \left( \frac{m - 1}{m} \rho_m^m + \frac{1}{m} \right) = \frac{m - 1}{m} \rho_m P_m + \frac{1}{m - 1}.$$

In the weak limit, we obtain

$$P_\infty \leq Q_\infty.$$

For the reverse inequality, we consider $A > 1$ and $m$ sufficiently large. Then, we have

$$\rho_m P_m = \rho_m \min\{A, P_m\} + \rho_m |P_m - A|_+ \leq \left( \frac{m - 1}{m} A \right)^{\frac{1}{m-1}} P_m + \rho_m |P_m - A|_+. \quad (3.14)$$

We can estimate the last term by

$$\rho_m |P_m - A|_+ \leq \chi_{\{P_m > A\}} \frac{m}{m - 1} \rho_m^m = \chi_{\{P_m > A\}} \frac{m}{m - 1} \rho_m^m \leq \left( \frac{m - 1}{m} A \right)^{\frac{1}{m-1}},$$

and thus, for any non-negative smooth test function $\varphi \in C^\infty_0(Q_T)$, we conclude

$$\limsup_{m \to \infty} \int_{Q_T} \rho_m |P_m - A|_+ \varphi \, dx \, dt \leq \limsup_{m \to \infty} \int_{Q_T} \frac{\rho_m^{2m}}{\left( \frac{m - 1}{m} A \right)^{m/(m-1)}} \, dx \, dt \leq \frac{C}{A}.$$

On the other hand, $\left( \frac{m - 1}{m} A \right)^{\frac{1}{m-1}}$ converges strongly to 1. Therefore by weak-strong convergence $\left( \frac{m - 1}{m} A \right)^{\frac{1}{m-1}} P_m$ weakly converges to $P_\infty$. Passing to the weak limit in (3.14), we conclude that, for all $A > 1$

$$Q_\infty \leq P_\infty + \frac{C}{A}.$$

We may take $A \to \infty$ and find the desired result, namely (3.13).
Secondly, we prove that \( \rho_m P_m \rightharpoonup \rho_\infty P_\infty \). For any smooth test function \( \varphi \in C_0^\infty (Q_T) \), we have, recalling the strong convergence proved in Lemma 3.2,

\[
\lim_{m \to \infty} \int_{Q_T} \rho_m P_m \varphi \, dx \, dt = \lim_{m \to \infty} \int_{Q_T} \Delta N \ast \rho_m \varphi \, dx \, dt
\]

\[
= - \lim_{m \to \infty} \int_{Q_T} \nabla N \ast \rho_m \cdot \nabla P_m \varphi \, dx \, dt - \lim_{m \to \infty} \int_{Q_T} \nabla N \ast \varphi P_m \, dx \, dt
\]

\[
= - \int_{Q_T} \nabla N \ast \rho_\infty \cdot \nabla P_\infty \varphi \, dx \, dt - \int_{Q_T} \nabla N \ast \rho_\infty \cdot \nabla \varphi P_\infty \, dx \, dt
\]

\[
= \int_{Q_T} \rho_\infty P_\infty \varphi \, dx \, dt.
\]

This means that \( \rho_m P_m \rightharpoonup \rho_\infty P_\infty \) and we have obtained the result.

**Proof of Eq. (2.7)**, \( 0 \leq \rho_\infty \leq 1 \), a.e. in \( Q_T \) It is directly obtained by Lemma 3.3 and the inequality \( \| \rho_\infty \|_{L^2(Q_T)} \leq \liminf_{m \to \infty} \| \rho_m \|_{L^2(Q_T)} \leq C(T) \). □

### 4 Additional Regularity Estimates for Pressure

The classical Aronson-Bénilan(AB) estimate [3, 54] provides regularity for the pressure \( P_m \). But the nonlocal interaction results in the absence of comparison principle and the \( L^\infty \) bound from below cannot be established. Therefore, we prove uniform AB-type estimates in \( L^3 \& L^1 \) versions, adapting the method in [22, 47]. Such regularity is interesting by itself and is used to establish the complementarity relation Eq. (2.8) since it gives strong compactness in \( L^2_{loc}(Q_T) \) of the sequence \( \{ \nabla P_m \}_{m>1} \).

In this section, we need to further assume \( m > \max\{n - 1, \frac{5n - 2}{n + 2} \} \) because of inequality (4.30).

#### 4.1 The \( L^3 \) Estimate for \( \nabla p \)

We are going to establish the uniform \( L^3 \) estimate for the pressure gradient. Recently, David and Perthame [22, Theorem 3.2] proved a uniform sharp \( L^4 \) estimate for the pressure gradient and a similar result is established by Alazard and Bresch in [1]. In contrast, we obtain here a uniform \( L^3 \) estimate for the pressure gradient by adapting their proof to take into account that the nonlocal interaction term resulting in the absence of a uniform bound for the pressure.

**Theorem 4.1 (\( L^3 \) estimate for pressure gradient)** Under the initial assumptions (2.1)–(2.2), let \( \rho_m \) be a weak solution to the Cauchy problem Eq. (1.1) in the sense of Def. 1.1, then it holds for any given \( T > 0 \) and \( m > n - 1 \) that

\[
\sup_{0 \leq t \leq T} \| \nabla P_m(t) \|_{L^2(\mathbb{R}^n)} \leq C(T), \quad \| \sqrt{P_m} (\Delta P_m + \rho_m) \|_{L^2(Q_T)} \leq \frac{C(T)}{\sqrt{m}}, \quad (4.1)
\]

\[
\| \sqrt{P_m} \nabla^2 P_m \|_{L^2(Q_T)} \leq C(T), \quad \| \nabla P_m \|_{L^1(Q_T)} \leq C(T). \quad (4.2)
\]

We begin with preliminary lemmas.
Lemma 4.1 Under the initial assumptions (2.1)–(2.2), let ρm be a weak solution to the Cauchy problem Eq. (1.1) in the sense of Def. 1.1, then it holds
\[ \|\sqrt{P_m} \nabla P_m\|_{L^2(\Omega_T)} \leq C(T) \quad \forall T > 0. \] (4.3)

**Proof** Multiplying Eq. (1.4) by \( P_m \) and integrating on \( \mathbb{R}^n \), then we have
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} P_m^2 dx + (2m-3) \int_{\mathbb{R}^n} P_m |\nabla P_m|^2 dx = (m - \frac{3}{2}) \int_{\mathbb{R}^n} P_m^2 \rho_m dx. \] (4.4)

Due to the Sobolev inequality for gradient (Theorem C.3), Holder’s inequality, and Lemma 3.1, we obtain
\[ (m - \frac{3}{2}) \int_{\mathbb{R}^n} P_m^2 \rho_m dx \leq (m - \frac{3}{2})(\int_{\mathbb{R}^n} P_m^{\frac{2m}{(m-1)}} dx)^{\frac{n-2}{n}} \left( \int_{\mathbb{R}^n} \rho_m^\frac{n}{(n-2)} dx \right)^\frac{2}{n} \]
\[ \leq (m - \frac{3}{2})C(n)C(T) \int_{\mathbb{R}^n} |\nabla P_m|^2 dx. \] (4.5)

Taking (4.5) into (4.4) and integrating (4.4) on \([0, T]\) gives (4.3). □

**Proof of Theorem 4.1** We multiply the pressure Eq. (1.4) by \(-\Delta P_m + \rho_m\) and integrate that on \( \mathbb{R}^n \), then it follows
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla P_m|^2 dx - \partial_t \int_{\mathbb{R}^n} \rho_m^m dx + (m - 1) \int_{\mathbb{R}^n} P_m (\Delta P_m + \rho_m)^2 dx \\
+ \int_{\mathbb{R}^n} |\nabla P_m|^2 \Delta P_m dx + \int_{\mathbb{R}^n} |\nabla P_m|^2 \rho_m dx + \int_{\mathbb{R}^n} \nabla P_m \cdot \nabla * \rho_m \Delta P_m dx \\
+ \int_{\mathbb{R}^n} \rho_m \nabla P_m \cdot \nabla * \rho_m dx = 0. \] (4.6)

Integrating by parts, we have
\[ \int_{\mathbb{R}^n} |\nabla P_m|^2 \Delta P_m dx = \int_{\mathbb{R}^n} P_m \Delta (|\nabla P_m|^2) dx \\
= 2 \int_{\mathbb{R}^n} P_m \nabla P_m \cdot \nabla (\Delta P_m) dx + 2 \int_{\mathbb{R}^n} P_m (\nabla^2 P_m)^2 dx \\
= -2 \int_{\mathbb{R}^n} P_m |\Delta P_m|^2 dx - 2 \int_{\mathbb{R}^n} |\nabla P_m|^2 \Delta P_m dx + 2 \int_{\mathbb{R}^n} P_m (\nabla^2 P_m)^2 dx. \]

Hence, it holds
\[ \int_{\mathbb{R}^n} |\nabla P_m|^2 \Delta P_m dx = -\frac{2}{3} \int_{\mathbb{R}^n} P_m |\Delta P_m|^2 dx + \frac{2}{3} \int_{\mathbb{R}^n} P_m (\nabla^2 P_m)^2 dx. \] (4.7)
Similarly, integrating by parts, we obtain

\[
\int_{\mathbb{R}^n} \nabla P_m \cdot \nabla N^* \rho_m \Delta P_m \, dx
\]

\[
= - \sum_{i,j} \int_{\mathbb{R}^n} \partial_i P_m \partial_i N^* \rho_m - \sum_{i,j} \int_{\mathbb{R}^n} \partial_i P_m \partial_i j N^* \rho_m \partial_j P_m \, dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^n} |\nabla P_m|^2 \rho_m \, dx + \int_{\mathbb{R}^n} P_m \nabla \rho_m : \nabla P_m \, dx + \int_{\mathbb{R}^n} P_m \nabla^2 N^* \rho_m : \nabla^2 P_m \, dx
\]

\[
= \frac{3m - 1}{2m - 2} \int_{\mathbb{R}^n} P_m \cdot \nabla \rho_m \, dx + \int_{\mathbb{R}^n} P_m \nabla^2 N^* \rho_m : \nabla^2 P_m \, dx.
\]

Thus, inserting both (4.7) and (4.8) into (4.6), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla P_m|^2 \, dx - \frac{d}{dt} \int_{\mathbb{R}^n} \rho_m^m \, dx + (m - 1) \int_{\mathbb{R}^n} P_m (\Delta P_m + \rho_m^m) \, dx
\]

\[
+ \frac{2}{3} \int_{\mathbb{R}^n} P_m (\nabla^2 P_m)^2 \, dx + \frac{3m - 1}{2m - 2} \int_{\mathbb{R}^n} \nabla P_m \cdot \nabla \rho_m^m \, dx
\]

\[
\leq \frac{2}{3} \int_{\mathbb{R}^n} P_m |\Delta P_m|^2 \, dx - \int_{\mathbb{R}^n} P_m \nabla^2 P_m : \nabla^2 N^* \rho_m \, dx
\]

\[
\leq \frac{2}{3} \int_{\mathbb{R}^n} P_m |\Delta P_m|^2 \, dx + \frac{1}{3} \int_{\mathbb{R}^n} P_m (\nabla^2 P_m)^2 \, dx + \frac{3}{4} \int_{\mathbb{R}^n} P_m (\nabla^2 N^* \rho_m)^2 \, dx,
\]

where the last inequality follows from

\[
| - \int_{\mathbb{R}^n} \nabla^2 P_m : \nabla^2 N^* \rho_m \, dx | \leq \frac{1}{3} \int_{\mathbb{R}^n} P_m (\nabla^2 P_m)^2 \, dx + \frac{3}{4} \int_{\mathbb{R}^n} P_m (\nabla^2 N^* \rho_m)^2 \, dx.
\]

It easily follows from Lemma 3.1 and Sobolev’s inequality that

\[
\int_{\mathbb{R}^n} \rho_m^m \, dx \leq \int_{\mathbb{R}^n} P_m (\nabla P_m)^2 \, dx + C(T).
\]

Similarly, thanks to Lemma 3.2, the singular integral theory for Newtonian potential (Lemma C.1), Holder’s inequality and Young’s inequality, then we have

\[
\frac{3}{4} \int_{\mathbb{R}^n} P_m (\nabla^2 N^* \rho_m)^2 \, dx \leq \frac{3}{4} \sum_{i,j} \left( \int_{\mathbb{R}^n} P_m \frac{2m}{n+2} \, dx \right)^{\frac{n-1}{2n}} \left( \int_{\mathbb{R}^n} \partial_i^2 N^* \rho_m \, dx \right)^{\frac{n+2}{2n}}
\]

\[
\leq \frac{3}{4} \sum_{i,j} \left( C(n) \int_{\mathbb{R}^n} |\nabla P_m|^2 \, dx \right)^{\frac{1}{2}} \left( C(n) \int_{\mathbb{R}^n} \rho_m \frac{4n}{n+2} \, dx \right)^{\frac{n+2}{2n}}
\]

\[
\leq \frac{1}{8} \int_{\mathbb{R}^n} |\nabla P_m|^2 \, dx + \frac{9}{8} C(n)n^2 \left( C(n) \int_{\mathbb{R}^n} \rho_m \frac{4n}{n+2} \, dx \right)^{\frac{n+2}{2n}}
\]

\[
\leq \frac{1}{8} \int_{\mathbb{R}^n} |\nabla P_m|^2 \, dx + C(T).
\]
Integrating (4.9) on [0, t] for any \( t \in (0, T] \) and using both (4.10) and (4.11), then we obtain

\[
\frac{1}{4} \int_{\mathbb{R}^n} |\nabla P_m(t)|^2 \, dx + \int_{\mathbb{R}^n} \rho_m^m \, dx + (m - \frac{7}{3}) \int_0^t \int_{\mathbb{R}^n} P_m(\Delta P_m + \rho_m)^2 \, dx \, ds \\
+ \frac{1}{3} \int_0^t \int_{\mathbb{R}^n} P_m(\nabla^2 P_m)^2 \, dx \, ds + \frac{3m - 1}{2m - 2} \int_0^t \int_{\mathbb{R}^n} \nabla P_m \cdot \nabla \rho_m^m \, dx \, ds \\
\leq \frac{4}{3} \int_0^t \int_{\mathbb{R}^n} P_m \rho_m^m \, dx \, ds + \frac{1}{8} \int_0^t \int_{\mathbb{R}^n} |\nabla P_m|^2 \, dx \, ds + C(T)t \\
+ \frac{1}{2} \int_{\mathbb{R}^n} |\nabla P_m,0|^2 \, dx,
\]

(4.12)

where \( \frac{2}{3} \int_0^t \int_{\mathbb{R}^n} P_m(\Delta P_m)^2 \, dx \, ds \leq \frac{4}{3} \int_0^t \int_{\mathbb{R}^n} P_m \rho_m^2 \, dx \, ds + \frac{4}{3} \int_0^t \int_{\mathbb{R}^n} P_m(\Delta P_m + \rho_m)^2 \, dx \, ds \) is used. It easily follows from Lemma 3.1 and Sobolev’s inequality that

\[
\frac{4}{3} \int_0^t \int_{\mathbb{R}^n} P_m \rho_m^2 \, dx \, ds \leq \frac{1}{3} \int_0^t \int_{\mathbb{R}^n} |\nabla P_m|^2 \, dx \, ds + C(T) \leq C(T).
\]

Inserting this into (4.12) and by virtue of Lemma 3.1, we have, for all \( t \in [0, T] \),

\[
\frac{1}{4} \int_{\mathbb{R}^n} |\nabla P_m(t)|^2 \, dx + (m - \frac{7}{3}) \int_0^t \int_{\mathbb{R}^n} P_m(\Delta P_m + \rho_m)^2 \, dx \, ds \\
+ \frac{1}{3} \int_0^t \int_{\mathbb{R}^n} P_m(\nabla^2 P_m)^2 \, dx \, ds \leq C(T).
\]

(4.13)

Therefore, it follows from (4.13) that

\[
\sup_{0 \leq t \leq T} \|\nabla P_m(t)\|_{L^2(Q_T)} + m \int_{Q_T} P_m(\Delta P_m + \rho_m)^2 \, dx \, dt + \int_{Q_T} P_m(\nabla^2 P_m)^2 \, dx \, dt \leq C(T),
\]

and thus (4.1) and the first estimate of (4.2) are obtained.

For the second bound of (4.2), the above inequality and Lemma 4.1 lead to

\[
\int_{Q_T} |\partial_t P_m|^3 \, dx \, dt = \int_{Q_T} \partial_t P_m \partial_t P_m |\partial_t P_m| \, dx \, dt \leq 2 \int_{Q_T} P_m |\partial_t P_m| |\partial_t P_m| \, dx \, dt \\
\leq \int_{Q_T} P_m(\partial_i P_m)^2 \, dx \, dt + \int_{Q_T} P_m(\partial_i P_m)^2 \, dx \, dt \leq C(T).
\]

Therefore, the second estimate of (4.2) holds and the proof of Theorem 4.1 is completed. □

4.2 The Uniform Aronson-Bénilan Estimate

Next, our goal is to establish the uniform AB estimate which uses the new unknown

\[
\omega_m := \Delta P_m + \rho_m.
\]

(4.14)
Theorem 4.2 (The Aronson-Bénilan estimate) Assume that $\rho_{m,0}$ satisfies (2.1)–(2.3), let $\rho_m$ be a weak solution to the Cauchy problem Eq. (1.1) in the sense of Def. 1.1, then

$$\sup_{0 \leq t \leq T} \|\omega_m(t)\|_{L^1(\mathbb{R}^n)} \leq C(T), \quad \sup_{0 \leq t \leq T} \|\Delta P_m(t)\|_{L^1(\mathbb{R}^n)} \leq C(T).$$

$$\sup_{0 \leq t \leq T} \|\omega_m(t)\|_{L^2(\mathbb{R}^n)} \leq C(T), \quad \|\omega_m\|_{L^3(\mathbb{R}^n)}^3 \leq \frac{C(T)}{m}. \quad (4.15)$$

Proof We first write an inequality for $\omega_m$. The equation of the density and pressure are

$$\partial_t \rho_m = \Delta \rho_m + \nabla \cdot (\rho_m \nabla N^* \rho_m)$$
$$= \rho_m (\Delta P_m + \rho_m) + \nabla \rho_m \cdot (\nabla P_m + \nabla N^* \rho_m) \quad (4.17)$$
$$= \rho_m \omega_m + \nabla \rho_m \cdot (\nabla P_m + \nabla N^* \rho_m),$$

$$\partial_t P_m = (m - 1) P_m \omega_m + \nabla P_m \cdot \nabla P_m + \nabla P_m \cdot \nabla N^* \rho_m.$$

Then, we compute

$$\partial_t P_m = (m - 1) \Delta (P_m \omega_m) + \nabla (\Delta P_m) \cdot (\nabla N^* \rho_m + \nabla P_m)$$
$$+ \nabla P_m \cdot \nabla \omega_m + 2 \nabla^2 P_m : (\nabla^2 P_m + \nabla^2 N^* \rho_m). \quad (4.18)$$

Combining (4.17) and (4.18), the equation of $\omega_m$ is

$$\partial_t \omega_m = (m - 1) \Delta (P_m \omega_m) + \nabla \omega_m \cdot \nabla N^* \rho_m + 2 \nabla^2 P_m : (\nabla^2 P_m + \nabla^2 N^* \rho_m)$$
$$+ \rho_m \omega_m + 2 \nabla P_m \cdot \nabla \omega_m.$$

Thus, we have

$$\partial_t \omega_m \geq (m - 1) \Delta (P_m \omega_m) + \nabla \omega_m \cdot \nabla N^* \rho_m - \frac{1}{2} (\nabla^2 N^* \rho_m)^2$$
$$+ 2 \nabla P_m \cdot \nabla \omega_m + \rho_m \omega_m. \quad (4.19)$$

Next, we may turn to the $L^1$ estimate. We multiply (4.19) by $-\text{sgn}(\omega_m\omega)$ and get

$$\partial_t |\omega_m\omega| \leq (m - 1) \Delta (P_m |\omega_m\omega|) + \nabla |\omega_m\omega| \cdot \nabla N^* \rho_m + \text{sgn}(|\omega_m\omega|) \frac{1}{2} (\nabla^2 N^* \rho_m)^2$$
$$+ 2 \nabla P_m \cdot \nabla |\omega_m\omega| + \rho_m |\omega_m\omega|.$$

After integration on $\mathbb{R}^n$, using the singular integral theory for Newtonian potential (Lemma C.1), and Holder’s inequality, we attain

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\omega_m\omega| dx \leq 2 \int_{\mathbb{R}^n} |\omega_m\omega| \rho_m dx + 2 \int_{\mathbb{R}^n} |\omega_m\omega|^2 dx + \int_{\mathbb{R}^n} \frac{1}{2} (\nabla^2 N^* \rho_m)^2 dx$$
$$\leq \int_{\mathbb{R}^n} \rho_m^2 dx + 3 \int_{\mathbb{R}^n} |\omega_m\omega|^2 dx + C(n) \int_{\mathbb{R}^n} \rho_m^2 dx \leq C(T).$$
Integrating in $t$ as before, we conclude the first estimate of (4.15) for Theorem 4.2, namely

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} |\omega_m(t)|_- \, dx \leq C(T). \quad (4.20)$$

Finally, since $|\Delta P_m| \leq |\Delta P_m + \rho_m| + \rho_m = \Delta P_m + \rho_m + 2|\omega_m|_- + \rho_m$, we find

$$\int_{\mathbb{R}^n} |\Delta P_m| \, dx \leq \int_{\mathbb{R}^n} \Delta P_m + 2\rho_m \, dx + 2 \int_{\mathbb{R}^n} |\omega_m|_- \, dx \leq 2C + 2 \int_{\mathbb{R}^n} |\omega_m|_- \, dx.$$

Therefore, the second bound of (4.15) follows from (4.20).

For the $L^2$ estimate, we multiply Eq. (4.19) by $-2|\omega_m|_-$. Due to Kato’s inequality, we obtain

$$\partial_t |\omega_m|_-^2 \leq 2(m - 1) \Delta (P_m|\omega_m|_-)|\omega_m|_- + \nabla |\omega_m|_- \cdot \nabla \mathcal{N} * \rho_m + 2\nabla |\omega_m|_-^2 \cdot \nabla P_m + (\nabla^2 \mathcal{N} * \rho_m)^2|\omega_m|_- + 2\rho_m |\omega_m|_-^2. \quad (4.21)$$

Integrating (4.21) on $\mathbb{R}^n$ and integrating by parts, we find

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\omega_m|_-^2 \, dx \leq 2(m - 1) \int_{\mathbb{R}^n} \Delta (P_m|\omega_m|_-)|\omega_m|_- \, dx$$

$$- 2 \int_{\mathbb{R}^n} |\omega_m|_- \Delta (P_m + \rho_m) \, dx$$

$$+ 3 \int_{\mathbb{R}^n} |\omega_m|_-^2 \rho_m \, dx + \int_{\mathbb{R}^n} (\nabla^2 \mathcal{N} * \rho_m)^2|\omega_m|_- \, dx$$

$$= 2(m - 1) \int_{\mathbb{R}^n} \Delta (P_m|\omega_m|_-)|\omega_m|_- \, dx + 2 \int_{\mathbb{R}^n} |\omega_m|_-^3 \, dx$$

$$+ 3 \int_{\mathbb{R}^n} |\omega_m|_-^2 \rho_m \, dx + \int_{\mathbb{R}^n} (\nabla^2 \mathcal{N} * \rho_m)^2|\omega_m|_- \, dx. \quad (4.22)$$

Recalling the definition of $\omega_m$, we compute

$$2(m - 1) \int_{\mathbb{R}^n} \Delta (P_m|\omega_m|_-)|\omega_m|_- \, dx = -2(m - 1) \int_{\mathbb{R}^n} \nabla (P_m|\omega_m|_-) \nabla |\omega_m|_- \, dx$$

$$= - (m - 1) \int_{\mathbb{R}^n} \nabla P_m \cdot \nabla |\omega_m|_-^2 \, dx - 2(m - 1) \int_{\mathbb{R}^n} P_m |\nabla |\omega_m|_-^2 \, dx$$

$$= (m - 1) \int_{\mathbb{R}^n} \Delta P_m |\omega_m|_-^2 \, dx - 2(m - 1) \int_{\mathbb{R}^n} P_m |\nabla |\omega_m|_-^2 \, dx$$

$$= - (m - 1) \int_{\mathbb{R}^n} |\omega_m|^3 \, dx - (m - 1) \int_{\mathbb{R}^n} \rho_m |\omega_m|^2 \, dx$$

$$- 2(m - 1) \int_{\mathbb{R}^n} P_m |\nabla |\omega_m|_-^2 \, dx. \quad (4.23)$$
And, inserting this in (4.22), we get
\[
\frac{d}{dt} \int_{\mathbb{R}^n} |\omega_m|^2 \, dx \\
+ \int_{\mathbb{R}^n} \left[ 2(m-1) P_m |\nabla |\omega_m| |^2 + (m-4) \rho_m |\omega_m|^2 + (m-3) |\omega_m|^3 \right] \, dx \leq \int_{\mathbb{R}^n} (\nabla^2 N * \rho_m)^2 |\omega_m| \, dx.
\] (4.24)

Thanks to Young’s inequality, Lemma 3.1, and Lemma C.1, we have
\[
\int_{\mathbb{R}^n} (\nabla^2 N * \rho_m)^2 |\omega_m| \, dx \leq \sum_{ij} \frac{2n}{3^{3/2}} \int_{\mathbb{R}^n} |\partial^2_{ij} \nabla N | \rho_m^3 \, dx + \sum_{ij} \frac{1}{n^2} \int_{\mathbb{R}^n} |\omega_m|^3 \, dx \\
\leq \sum_{ij} \frac{2n}{3^{3/2}} C(n) \int_{\mathbb{R}^n} \rho_m^3 \, dx + \int_{\mathbb{R}^n} |\omega_m|^3 \, dx \\
\leq \int_{\mathbb{R}^n} |\omega_m|^3 \, dx + C(T).
\]

Inserting this into (4.24), we arrive at
\[
\frac{d}{dt} \int_{\mathbb{R}^n} |\omega_m|^2 \, dx + \int_{\mathbb{R}^n} \left[ 2(m-1) P_m |\nabla |\omega_m| |^2 + (m-4) \rho_m |\omega_m|^2 + (m-3) |\omega_m|^3 \right] \, dx \leq C(T).
\]

After time integration, we obtain
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} |\omega_m(t)|^2 \, dx + 2m \int_{\mathbb{R}^n} P_m |\nabla |\omega_m| |^2 \, dx + m \int_{\mathbb{R}^n} [\rho_m |\omega_m|^2 + |\omega_m|^3] \, dx dt \\
\leq C(T).
\]

This proves (4.16) and complete the proof of Theorem 4.2. □

Finally, we justify the $L^1$ estimate for the time derivative of the pressure. This estimate is useful to get locally strong compactness of the pressure gradient sequences $\{\nabla P_m\}_{m>1}$. We first give two useful preliminary lemmas.

**Lemma 4.2** Assume that the initial data $\rho_{m,0}$ satisfies the assumptions (2.1)–(2.2), let $\rho_m$ be a weak solution to the Cauchy problem Eq. (1.1) in the sense of Def. 1.1, then it follows
\[
\int_{Q_T} \left( |\nabla \rho_m|^2 + |\nabla \rho_m^{m+1}|^2 + |\nabla \rho_m^{m+2}|^2 \right) \, dx dt \leq C(T).
\]

**Proof** These estimates can be written as $L^2(Q_T)$ bounds on $\rho_m^3 \nabla P_m$, $\rho_m^2 \nabla P_m$, $\rho_m \nabla P_m$. They are obvious consequences of (3.4) and (4.3). □

**Lemma 4.3** Under the initial assumptions (2.1)–(2.3), let the pair $(P_m, \rho_m)$ be a weak solution to the PKS model Eq. (1.1) in the sense of Def. 1.1, then it holds
\[
\|\partial_t \rho_m - \nabla \rho_m \cdot \nabla N * \rho_m\|_{L^1(Q_T)} \leq C(T).
\]
Proof Since

$$\partial_t \rho_m - \nabla \rho_m \cdot \nabla \mathcal{N} \ast \rho_m \geq \rho_m \omega_m,$$

we have

$$|\partial_t \rho_m - \nabla \rho_m \cdot \nabla \mathcal{N} \ast \rho_m| \leq \rho_m |\omega_m| \leq \frac{1}{2} \left[ \rho_m^2 + |\omega_m|^2 \right].$$

Consequently, using mass conservation, Theorem 4.2 and Lemma 3.1,

$$\int_{Q_T} |\partial_t \rho_m - \nabla \rho_m \cdot \nabla \mathcal{N} \ast \rho_m| \, dx \, dt = \int_{Q_T} (\partial_t \rho_m - \nabla \rho_m \cdot \nabla \mathcal{N} \ast \rho_m) \, dx \, dt + 2 \int_{Q_T} |\partial_t \rho_m - \nabla \rho_m \cdot \nabla \mathcal{N} \ast \rho_m| \, dx \, dt \leq 2 \int_{Q_T} \rho_m^2 \, dx \, dt + \int_{Q_T} |\omega_m|^2 \, dx \, dt \leq C(T).$$

We are now ready to prove the $L^1$ estimate for the time derivative of pressure.

Theorem 4.3 ($L^1$ estimate of the time derivative of pressure) Under the initial assumptions (2.1)–(2.4), let $\rho_m$ be a weak solution to the Cauchy problem (1.1) for the PKS in the sense of Def. 1.1, then

$$\|\partial_t P_m\|_{L^1(Q_T)} \leq C(T).$$

Proof We cannot work directly on $\partial_t P_m$ because of the power arising in a remainder term, and thus we use $\partial_t \rho_m^{m+1}$. For this reason, we rewrite the cell density equation (1.1) with two formulas

$$\partial_t \rho_m = \rho_m (\Delta P_m + \rho_m) + \nabla \rho_m \cdot (\nabla P_m + \nabla \mathcal{N} \ast \rho_m), \quad (4.25)$$

$$\partial_t \rho_m = \Delta \rho_m^{m+1} + \rho_m^2 + \nabla \rho_m \cdot \nabla \mathcal{N} \ast \rho_m, \quad (4.26)$$

and we give two useful equations

$$\partial_t \rho_m^{m+1} = (m+1) \rho_m^{m+1} (\Delta P_m + \rho_m) + \nabla \rho_m^{m+1} \cdot (\nabla P_m + \nabla \mathcal{N} \ast \rho_m), \quad (4.27)$$

$$\Delta \rho_m^{m+1} = \frac{m+1}{m} (\rho_m \Delta \rho_m^m + \nabla \rho_m \cdot \nabla \rho_m^m). \quad (4.28)$$

With the help of Kato’s inequality, we differentiate Eq. (4.27) with respect to the time and multiply this by $-\text{sgn}(|\partial_t \rho_m|)$, then it holds

$$\partial_t |\partial_t \rho_m^{m+1}| \leq (m+1) \left[ |\partial_t \rho_m^{m+1}| \Delta P_m + \rho_m |\partial_t \rho_m| + \rho_m^{m+1} |\partial_t \rho_m| - \rho_m^{m+1} \Delta |\partial_t P_m| \right]$$

$$+ \nabla |\partial_t \rho_m^{m+1}| \cdot \nabla [P_m + \mathcal{N} \ast \rho_m] + \nabla \rho_m^{m+1} \cdot \nabla |\partial_t P_m| - \text{sgn}(\partial_t \rho_m) \nabla \rho_m^{m+1} \cdot \nabla \mathcal{N} \ast \partial_t \rho_m$$

$$\Box$$
and after integration by parts on \( \mathbb{R}^n \) and insertion of Eq. (4.28), we find
\[
\frac{d}{dt} \int_{\mathbb{R}^n} |\partial_t \rho_m^{m+1}| \, dx \leq \int_{\mathbb{R}^n} |\partial_t \rho_m^{m+1}| \, dx + (m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m^m| \, dx + m \int_{\mathbb{R}^n} |\partial_t \rho_m| \, dx
\]
\[
+ \int_{\mathbb{R}^n} \nabla \rho_m^{m+1} \cdot \nabla \rho_m \, dx
\]
\[
= (m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m^{m+1}| \, dx + (m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m^m| \, dx + (m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m| \, dx
\]
\[
+ \int_{\mathbb{R}^n} |\nabla \rho_m^{m+1}| \cdot \nabla \rho_m \, dx.
\]

We insert Eqs. (4.25)–(4.26) into this inequality, and use Eq. (3.9) for the last term.
\[
\frac{d}{dt} \int_{\mathbb{R}^n} |\partial_t \rho_m^{m+1}| \, dx \leq (m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m^{m+1}| \, dx + (m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m^m| \, dx + (m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m| \, dx
\]
\[
+ \int_{\mathbb{R}^n} \nabla \rho_m^{m+1} \cdot \nabla \rho_m \, dx + \int_{\mathbb{R}^n} \nabla \rho_m^m \cdot \nabla \rho_m \, dx + \int_{\mathbb{R}^n} \nabla \rho_m^{m+1} \cdot \nabla \rho_m \, dx + \int_{\mathbb{R}^n} \nabla \rho_m^m \cdot \nabla \rho_m \, dx
\]
\[
= (m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m^{m+1}| \, dx + (m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m^m| \, dx + (m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m| \, dx
\]
\[
+ \int_{\mathbb{R}^n} \nabla \rho_m^{m+1} \cdot \nabla \rho_m \, dx + \int_{\mathbb{R}^n} \nabla \rho_m^m \cdot \nabla \rho_m \, dx + \int_{\mathbb{R}^n} \nabla \rho_m^{m+1} \cdot \nabla \rho_m \, dx + \int_{\mathbb{R}^n} \nabla \rho_m^m \cdot \nabla \rho_m \, dx.
\]

The two terms with \((m+1)|\partial_t P_m| \, \nabla \rho_m \cdot \nabla \rho_m^m\) and \(-(m+1)|\partial_t \rho_m^m| \, \nabla \rho_m \cdot \nabla P_m\) cancel due to \(|\partial_t P_m| \, \nabla \rho_m \cdot \nabla \rho_m^m = |\partial_t \rho_m^m| \, \nabla \rho_m \cdot \nabla P_m\), then it holds
\[
\frac{d}{dt} \int_{\mathbb{R}^n} |\partial_t \rho_m^{m+1}| \, dx + (m-1) \int_{\mathbb{R}^n} |\partial_t \rho_m^{m+1}| \, dx
\]
\[
+ 2(m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m^m| \, dx
\]
\[
\leq A + \int_{\mathbb{R}^n} \nabla \rho_m^m \cdot \nabla \rho_m \, dx + B.
\]

Here \(A\) and \(B\), are defined and estimated as follows:
\[
A = 2(m+1) \int_{\mathbb{R}^n} \rho_m^{m-1} |\partial_t \rho_m| \, \nabla \rho_m \cdot \nabla \rho_m \, dx
\]
\[
\leq (m+1) \int_{\mathbb{R}^n} \rho_m^{m-1} |\partial_t \rho_m| \, dx + (m+1) \int_{\mathbb{R}^n} \rho_m^{m-1} \nabla \rho_m \cdot \nabla \rho_m \, dx
\]
\[
\leq (m+1) \int_{\mathbb{R}^n} |\partial_t \rho_m| \, dx + (m+1) \int_{\mathbb{R}^n} \rho_m \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \rho_m^{m+1}|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \rho_m^m|^2 \, dx
\]
\[
B \leq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \rho_m^{m+1}|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \rho_m^m|^{\frac{2}{n}} \, dx
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \rho_m^{m+1}|^2 \, dx + \frac{1}{2} C(n) \|\nabla \rho_m^m\|_{L^\infty(Q_T)}^2 \int_{\mathbb{R}^n} \rho_m^2 \, dx.
\]
From Lemma 3.1 in which we let \( p = 2 \), we control in \( L^1(Q_T) \) the term \( \nabla \rho_m^m \cdot \nabla \rho_m \). The terms in \( B \) are also controlled thanks to the bounds (in particular in Lemma 4.2), as well as the second term in final expression of \( A \). All together the known bounds reduce (4.29) to

\[
\frac{d}{dt} \int_{\mathbb{R}^n} |\partial_t \rho_m^{m+1}| \, dx + \frac{m^2 - 1}{m + 2} \int_{\mathbb{R}^n} |\partial_t \rho_m^{m+2}| \, dx + (m + 1) \int_{\mathbb{R}^n} |\partial_t \rho_m^m| \, |\partial_t \rho_m| \, dx \leq (m + 1) C(T).
\]

Therefore, it holds

\[
\int_{Q_T} |\partial_t \rho_m^{m+2}| \, dxdt + \int_{Q_T} |\partial_t \rho_m^m| \, |\partial_t \rho_m| \, dxdt \leq C(T). \tag{4.30}
\]

Taking account of Lemma 3.1 and Theorem 4.1, we use Sobolev’s inequality and obtain

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} \rho_m^{m+2} \, dx \leq C \sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} \rho_m^3 \, dx \leq C \sup_{0 \leq t \leq T} \|\nabla P_m(t)\|_{L^2(\mathbb{R}^n)} + C(T)
\]

\[
\leq C(T).
\]

Thus, combining the above inequality and (4.30), we get

\[
\int_{Q_T} |\partial_t \rho_m^{m+2}| \, dxdt \leq \int_{Q_T} |\partial_t \rho_m^{m+2}| \, dxdt + 2 \int_{Q_T} |\partial_t \rho_m^{m+2}| \, dxdt \leq \int_{\mathbb{R}^n} \rho_m^{m+2}(T) \, dx - \int_{\mathbb{R}^n} \rho_{m,0}^{m+2} \, dx + 2 \int_{Q_T} |\partial_t \rho_m^{m+2}| \, dxdt \leq C(T). \tag{4.31}
\]

Furthermore, combining (4.31), Lemma 3.2, and Lemma 4.2, we have

\[
\int_{Q_T} |\partial_t \rho_m^{m+2} - \nabla \rho_m^{m+2} \cdot \nabla N * \rho_m| \, dxdt \leq \int_{Q_T} \left[ |\partial_t \rho_m^{m+2}| + \frac{1}{2} |\nabla \rho_m^{m+2}|^2 + \frac{1}{2} |\nabla N * \rho_m|^2 \right] dxdt \leq C(T). \tag{4.32}
\]

By Lemma 4.3 and (4.32), we obtain

\[
\int_{Q_T} |\partial_t \rho_m^{m-1} - \nabla \rho_m^{m-1} \cdot \nabla N * \rho_m| \, dxdt
\]

\[
= \int_{Q_T} |\partial_t \rho_m^{m-1} - \nabla \rho_m^{m-1} \cdot \nabla N * \rho_m| \left[ \chi_{\{\rho_m \leq \frac{1}{2}\}} + \chi_{\{\rho_m > \frac{1}{2}\}} \right] \, dxdt
\]

\[
\leq (m - 1) \frac{1}{2m-2} \int_{Q_T} |\partial_t \rho_m - \nabla \rho_m \cdot \nabla N * \rho_m| \, dxdt
\]

\[
+ \frac{8(m - 1)}{m + 2} \int_{Q_T} |\partial_t \rho_m^{m+2} - \nabla \rho_m^{m+2} \cdot \nabla N * \rho_m| \, dxdt \leq C(T).
\]
Combining this with Lemma 3.1 and Lemma 3.2, we end up with
\[
\int_{Q_T} |\partial_t P_m| \, dx \, dt \leq \int_{Q_T} |\partial_t P_m - \nabla P_m \cdot \nabla \nabla^* \rho_m| \, dx \, dt + \int_{Q_T} |\nabla P_m \cdot \nabla \nabla^* \rho_m| \, dx \, dt
\]
\[
\leq \frac{m}{m-1} \int_{Q_T} |\partial_t \rho_m^{m-1} - \nabla \rho_m^{m-1} \cdot \nabla \nabla^* \rho_m| \, dx \, dt
\]
\[
+ \frac{1}{2} \int_{Q_T} \left[|\nabla P_m|^2 \, dx \, dt + |\nabla \nabla^* \rho_m|^2 \right] \, dx \, dt
\]
\[
\leq C(T),
\]
where the first inequality is the application of the triangle inequality and the second inequality is due to the Cauchy-Schwarz inequality. The proof is completed. □

Remark 4.1 It should be emphasized that the first step in the proof of Theorem 4.3 is to compute the time derivative of \( \rho_m^{m+1} \). We can also begin with \( \rho_m^m \) (not \( P_m \)), which requires to use, from the density dissipation formula (3.2) from Lemma 3.1,
\[
\int_{Q_T} |\nabla \rho_m^m \cdot \nabla \rho_m| \rho_m^{-1} \, dx \, dt \leq C(T).
\]
This requires a bound for the entropy \( \rho_m \log \rho_m \) and thus for the initial data.

5 Complementarity Relation and Semi-Harmonicity

Thanks to the a priori regularity estimates provided by Lemmas 3.1–3.3 and Theorems 4.1–4.3, we can obtain the strong compactness on the pressure gradient, which allows us to obtain the complementarity relation. Moreover, the semi-harmonicity follows from the AB estimate (Theorem 4.2).

Theorem 5.1 (Complementarity relation and semi-harmonicity) Under the initial assumptions (2.1)–(2.4), then, the complementarity relation and the semi-harmonicity property, see Eq. (2.8), hold.

Proof From Lemma 3.1 and Theorems 4.3, we have
\[
\| \nabla P_m \|_{L^2(Q_T)} \leq C(T), \quad \| \partial_t P_m \|_{L^1(Q_T)} \leq C(T).
\]
Then, after the extraction of subsequences, we obtain
\[
P_m \rightarrow P_\infty, \text{ strongly in } L^1_{loc}(Q_T), \text{ as } m \rightarrow \infty,
\]
with the help of Sobolev’s compactness embedding. From Theorem 4.1, we obtain the weak compactness of the pressure gradient, up to a subsequence, we have
\[
\nabla P_m \rightharpoonup \nabla P_\infty, \text{ weakly in } L^3(Q_T), \text{ as } m \rightarrow \infty.
\]
Consider a smooth cutoff function $0 \leq \varphi \leq 1$, $\varphi(x) = 1$, for $|x| \leq 1$, $\varphi(x) = 0$ for $|x| \geq 2$, and for $R > 0$, we let $\varphi_R(x) = \varphi(\frac{x}{R})$ and $P_{m,R} = \varphi_R(x)P_m$. By direct computations, we obtain
\[
\|\partial_t P_{m,R}\|_{L^1(\Omega_T)} \leq C(T, R), \quad \|\nabla P_{m,R}\|_{L^3(\Omega_T)} \leq C(T, R), \\
\|\Delta P_{m,R}\|_{L^1(\Omega_T)} \leq C(T, R).
\] (5.1)

For the sake of the above three estimates (5.1), inspired by [22, Theorem 6.1], we can establish
\[
\nabla P_{m,R} \to \nabla P_{\infty,R}, \text{ strongly in } L^1(\Omega_T), \text{ as } m \to \infty.
\]

In other words, we can extract a subsequence such that
\[
\nabla P_m \to \nabla P_{\infty}, \text{ strongly in } L^1_{loc}(\Omega_T), \text{ as } m \to \infty.
\]

Then, using the uniform $L^3$ bound for the pressure gradient in Theorem 4.1, we have
\[
\nabla P_m \to \nabla P_{\infty}, \text{ strongly in } L^q_{loc}(\Omega_T), \text{ for } 1 \leq q < 3.
\]

Hence, in particular, the case $q = 2$ is selected to achieve our goal.

Let $\zeta \in C_0^\infty(\Omega_T)$ be a test function, we multiply the pressure equation (1.4) by $\zeta$ and integrate on $\Omega_T$, then it follows
\[
-\frac{1}{m-1}\int_{\Omega_T} P_m \partial_t \zeta + |\nabla P_m|^2 \zeta + \nabla P_m \cdot \nabla N \star \rho_m \zeta \, dx \, dt \\
= \int_{\Omega_T} (-|\nabla P_m|^2 \zeta - P_m \nabla P_m \cdot \nabla \zeta + P_m \rho_m \zeta) \, dx \, dt.
\]

Hence, passing to limit as $m \to \infty$, we obtain the complementarity relation
\[
\int_{\Omega_T} (-|\nabla P_{\infty}|^2 \zeta - P_{\infty} \nabla P_{\infty} \cdot \nabla \zeta + P_{\infty} \rho_{\infty} \zeta) \, dx \, dt = 0,
\]

where $\rho_m P_m \rightharpoonup \rho_{\infty} P_{\infty}$, weakly in $L^2(\Omega_T)$, results from (3.15). This is equivalent to
\[
\int_{\Omega_T} P_{\infty}(\Delta P_{\infty} + \rho_{\infty}) \zeta \, dx \, dt = 0,
\]

which means that the complementarity relation of Eq. (2.8) holds.

From Theorem 4.2, we have $\int_{\Omega_T} |\Delta P_m + \rho_m|^3 \, dx \, dt \leq \frac{C(T)}{m}$. Let $\phi \in C_0^\infty(\Omega_T)$ be a non-negative test function in $\Omega_T$, then we attain
\[
\int_{\Omega_T} (\Delta P_{\infty} + \rho_{\infty}) \phi \, dx \, dt = \lim_{m \to \infty} \int_{\Omega_T} (\Delta P_m + \rho_m) \phi \, dx \, dt \\
\geq - \lim_{m \to \infty} \int_{\Omega_T} |\Delta P_m + \rho_m| \phi \, dx \, dt \\
\geq - \lim_{m \to \infty} \left( \int_{\Omega_T} |\Delta P_m + \rho_m|^3 \, dx \, dt \right)^{\frac{1}{3}} \left( \int_{\Omega_T} \phi^3 \, dx \, dt \right)^{\frac{2}{3}} \\
= 0.
\] (5.2)
Hence, we establish the second result (semi-harmonicity property) of Eq. (2.8) and complete the proof of Theorem (5.1).

□

Remark 5.1 This result tells us that, when enough regularity is available, we have
\[
\begin{aligned}
-\Delta P_\infty &= 1, \quad \text{in } \Omega(t) := \{x : P_\infty(x, t) > 0\}, \\
P_\infty &= 0 \quad \text{on } \partial \Omega(t).
\end{aligned}
\]

This is related to the geometric form of the Hele-Shaw free boundary problem while Eq. (2.6) is the weak form which determines the motion of the free boundary.

**Theorem 5.2** Under the initial assumptions (2.1)–(2.4), then we have
\[
\rho_\infty \nabla P_\infty = \nabla P_\infty, \quad \text{a.e. in } Q_T.
\]

**Proof** On the one hand, we have already proved in Theorem 2.2 that \(\rho_m \nabla P_m \to \nabla P_\infty\) weakly. On the other hand \(\rho_m \to \rho_\infty\) weakly in \(L^2(Q_T)\) and \(\nabla P_m \to \nabla P_\infty\) strongly in \(L^2_{\text{loc}}(Q_T)\), by weak-strong convergence we obtain that \(\rho_m \nabla P_m \to \rho_\infty \nabla P_\infty\) weakly and the result is proved.

□

6 Uniqueness, Compact Support and Energy Functional

In order to prove the uniqueness of the solution to the Hele-Shaw limit system (2.6)–(2.7), we use the lifting method in \(\dot{H}^{-1}\) as in [4, 7, 24] rather than the duality method [23, 54] or the entropy method [36]. The main new difficulty comes from the nonlocal interaction. The uniform upper bound for the limit density, and the property that the limit pressure is somehow monotone to the limit density, allow us to use the energy method to prove the uniqueness as in either [7, Theorem 2.4] or [4, Theorem 3] for an aggregation equation with degenerate diffusion.

**Proposition 6.1** (Uniqueness) Let \((\rho_1, P_1)\) and \((\rho_2, P_2)\) be two solutions to the Cauchy problem Eq. (2.6)–(2.7) with initial data satisfying \(\rho_1(x, 0) = \rho_2(x, 0) \in \dot{H}^{-1}(\mathbb{R}^n)\), then it follows
\[
\rho_1 = \rho_2, \quad P_1 = P_2, \quad \text{a.e. in } Q.
\]

**Proof** First of all, we state that the pressure is somehow monotone to the density. Since \(\rho_1 P_1 = P_1\) and \(\rho_2 P_2 = P_2\) hold, we have
\[
(\rho_1 - \rho_2)(P_1 - P_2) = \rho_1 P_1 + \rho_2 P_2 - \rho_1 P_2 - \rho_2 P_1
\]
\[
= (1 - \rho_2) P_1 + (1 - \rho_1) P_2
\]
\[
\geq 0.
\]

We estimate the difference of weak solutions in \(\dot{H}^{-1}(\mathbb{R}^n)\) motivated by the fact that the pressure is somehow monotone to the density as (6.1). Let \(\psi = \mathcal{N} * (\rho_1 - \rho_2)\), by the integrability and bound of \(\rho_1\) and \(\rho_2\), we have \(\psi \in L^\infty(Q_T) \cap C(0, T, \dot{H}^1(\mathbb{R}^n))\) and
$\nabla \psi \in L^\infty(0, T; L^p(\mathbb{R}^n)) \cap L^\infty(Q_T)$ for $2 \leq p < \infty$, and $\partial_t \psi$ solves

$$\Delta \partial_t \psi = \partial_t \rho_1 - \partial_t \rho_2.$$ 

Since $\|\rho_1(t) - \rho_2(t)\|_H^{-1}(\mathbb{R}^n) = \|\nabla \psi(t)\|_{L^2(\mathbb{R}^n)}$, we are going to show $\|\nabla \psi(t)\|_{L^2(\mathbb{R}^n)} = 0$ for all $t \geq 0$.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be smooth function satisfying $\varphi = 1$ in $B_1$ and $0 \leq \varphi \leq 1$ in $\mathbb{R}^n$. Set $\varphi_R(x) = \varphi(\frac{x}{R})$ for $x \in \mathbb{R}^n$ and $R > 1$, thanks to the regularity of $\psi$, possibly up to a set of measure zero, it holds

$$-\langle \partial_t \rho_1 - \partial_t \rho_2, \psi \rangle = \lim_{R \to \infty} -\langle \partial_t \rho_1 - \partial_t \rho_2, \nabla * [(\rho_1 - \rho_2)\varphi_R] \rangle > \lim_{R \to \infty} < \nabla \partial_t \psi, \nabla * [(\rho_1 - \rho_2)\varphi_R] > = < \nabla \psi, \nabla \partial_t \psi > = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla \psi|^2 dx.$$

Therefore, using the definition of weak solution in Theorem 2.2, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla \psi|^2 dx = \lim_{R \to \infty} -\int_{\mathbb{R}^n} (P_1 - P_2) \Delta (\psi \varphi_R) dx + \int_{\mathbb{R}^n} (\rho_1 \nabla * \rho_1 - \rho_2 \nabla * \rho_2) \cdot \nabla (\varphi_R \psi) dx

= \lim_{R \to \infty} \int_{\mathbb{R}^n} (P_1 - P_2) \Delta \psi dx + \int_{\mathbb{R}^n} (\rho_1 \nabla * \rho_1 - \rho_2 \nabla * \rho_2) \cdot \nabla \psi dx

= \lim_{R \to \infty} \int_{\mathbb{R}^n} (P_1 - P_2) (\rho_1 - \rho_2) dx + \int_{\mathbb{R}^n} (\rho_1 - \rho_2) \nabla * \rho_1 \cdot \nabla \psi dx

+ \int_{\mathbb{R}^n} \rho_2 \nabla * (\rho_1 - \rho_2) \cdot \nabla \psi dx

:= I_1 + I_2 + I_3.$$

From (6.1), we obtain

$$I_1 = -\int_{\mathbb{R}^n} (P_1 - P_2) (\rho_1 - \rho_2) dx \leq 0.$$

By integrating by parts, we have

$$I_2 = \lim_{R \to \infty} \int_{\mathbb{R}^n} \Delta \psi \nabla \nabla * \rho_1 \cdot \nabla (\psi \varphi_R) dx = \lim_{R \to \infty} \sum_{ij} \int_{\mathbb{R}^n} \partial_i \psi \partial_j (\psi \varphi_R) \partial_{ij} \nabla * \rho_1 dx

- \lim_{R \to \infty} \sum_{ij} \int_{\mathbb{R}^n} \partial_i \psi \partial_{ij} (\psi \varphi_R) \partial_{j} \nabla * \rho_1 dx

= -\sum_{ij} \int_{\mathbb{R}^n} \partial_i \psi \partial_j \psi \partial_{ij} \nabla * \rho_1 dx - \sum_{ij} \int_{\mathbb{R}^n} \partial_i \psi \partial_{ij} \psi \partial_{j} \nabla * \rho_1 dx.$$
Similarly, integrating by parts again, we get

\[
- \sum_{ij} \int_{\mathbb{R}^n} \partial_i \psi \partial_j \psi \partial_j N \ast \rho_1 dx = \lim_{R \to \infty} - \sum_{ij} \int_{\mathbb{R}^n} \partial_i \psi \partial_j \psi \partial_j N \ast \rho_1 \varphi_R dx
\]

\[
= \lim_{R \to \infty} \frac{1}{2} \sum_{ij} \int_{\mathbb{R}^n} (\partial_i \psi)^2 \partial_j (\partial_j N \ast \rho_1 \varphi_R) dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \psi|^2 \rho_1 dx,
\]

which together with (6.2) implies

\[
I_2 \leq \int_{\mathbb{R}^n} |\nabla \psi|^2 |\nabla^2 N \ast \rho_1| dx + \frac{1}{2} \|\nabla \psi\|_{L^2(\mathbb{R}^n)}^2.
\]

By Holder’s inequality and \( \nabla \psi \in L^\infty(Q_T), \) for \( p \geq 2, \) we have

\[
\int_{\mathbb{R}^n} |\nabla \psi|^2 |\nabla^2 N \ast \rho_1| dx \leq \|\nabla^2 N \ast \rho_1\|_{L^p(\mathbb{R}^n)}(\int_{\mathbb{R}^n} |\nabla \psi|^{2p}) \frac{p-1}{p} \leq p \|\rho_1\|_{L^p(\mathbb{R}^n)} \|\nabla \psi\|_{L^\infty(\mathbb{R}^n)}^2 (\int_{\mathbb{R}^n} |\nabla \psi|^2 dx)^{\frac{p-1}{p}} \tag{6.3}\]

\[
\lesssim p \int_{\mathbb{R}^n} |\nabla \psi|^2 dx \frac{p-1}{p},
\]

where the implicit constant depends only on the uniformly controlled \( L^p \) norms of \( \rho_1 \) and \( \rho_2 \) and the second step holds because of the singular integral theory (Lemma C.1).

As for \( I_3, \) we may directly justify

\[
I_3 = \int_{\mathbb{R}^n} \rho_2 \nabla N \ast (\rho_1 - \rho_2) \cdot \nabla \psi dx = \int_{\mathbb{R}^n} \rho_2 |\nabla \psi|^2 dx \lesssim \|\nabla \psi\|_{L^2(\mathbb{R}^n)}^2. \tag{6.4}
\]

Let \( \gamma(t) = \int_{\mathbb{R}^n} |\nabla \psi(t)|^2 dx, \) both (6.3) and (6.4) imply the differential inequality

\[
\frac{d}{dt} \gamma(t) \leq \hat{C} p \max\{\gamma(t)^{1-\frac{1}{p}}, \gamma(t)\},
\]

where \( \hat{C} \) depends only on the uniformly controlled \( L^p \) norm of \( \rho_1, \rho_2. \) All the solutions of this differential inequality are bounded from above by the maximal solution. Since \( \gamma(0) = 0 \) and \( \gamma(t) \) is continuous, there exists \( t^* > 0 \) such that \( 0 \leq \gamma(t) < 1, t \in [0, t^*], \) therefore

\[
\frac{d}{dt} \gamma(t) \leq \hat{C} p \gamma(t)^{1-\frac{1}{p}}, \quad \gamma(0) = 0,
\]

and \( \gamma(t) \) is a subsolution to the ordinary differential equation

\[
\frac{d}{dt} \tilde{\gamma}(t) = \hat{C} p \tilde{\gamma}(t)^{1-\frac{1}{p}}, \quad \tilde{\gamma}(0) = 0, \tag{6.5}
\]

and \( \tilde{\gamma}(t) = (\hat{C} t)^p \) is the unique solution to (6.5). Consequently, we obtain

\[
\gamma(t) \leq \tilde{\gamma}(t) \leq 2^{-p} < 1. \tag{6.6}
\]

For \( 0 < t < \frac{1}{4\hat{C}}. \) Therefore, we can extend \( t^* \) to be long enough such that \( t^* \) is more than \( \frac{1}{4\hat{C}}. \)
In Eq. (6.6), we may take $p \to \infty$ to deduce that $\gamma(t) = 0$ for $t \in [0, \frac{1}{4c}]$ and the proof of uniqueness is complete. □

In fact, we are able to prove the time continuity and initial trace for the Hele-Shaw limit system (2.6)–(2.7). So far the initial data is obtained in the weak sense of Def. 1.1. This means that the Hele-Shaw equation holds with the initial data $\rho_{\infty,0} = w - \lim \rho_{m,0}$. Notice that we know that $0 \leq \rho_{\infty,0} \leq 1$ because the argument of Lemma 3.3 still holds true.

We now prove an additional result, namely the initial density is also obtained by time continuity.

**Proposition 6.2** (Almost everywhere time continuity) Assume that initial data $\rho_{m,0}$ and $\rho_{\infty,0}$ satisfy the assumption (2.1). Then it holds

$$\lim_{t \to 0} \rho_{\infty}(t) := \rho_{\infty,0} \text{ a.e. in } \mathbb{R}^n.$$  

**Proof** Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. By a standard variant of the test function in Def. 1.1, we have for a.e. $t > 0$,

$$\int_{\mathbb{R}^n} (\rho_{m}(t) - \rho_{m,0})\varphi(x)dx = -\frac{m-1}{m} \int_{Q_t} \rho_{m} \nabla P_{m} \cdot \nabla \varphi dx - \int_{Q_t} \rho_{m} \nabla N^* \rho_{m} \cdot \nabla \varphi dx.$$  

Passing to limit, then it holds

$$\int_{\mathbb{R}^n} (\rho_{\infty}(t) - \rho_{\infty,0})\varphi(x)dx = -\int_{Q_t} \nabla P_{\infty} \cdot \nabla \varphi dx - \int_{Q_t} \rho_{\infty} \nabla N^* \rho_{\infty} \cdot \nabla \varphi dx.$$  

Multiplying (2.6) by $\varphi$ and integrating on $[0, t]$, we get

$$\int_{\mathbb{R}^n} (\rho_{\infty}(t) - \rho_{\infty,0})\varphi(x)dx = -\int_{Q_t} \nabla P_{\infty} \cdot \nabla \varphi dx - \int_{Q_t} \rho_{\infty} \nabla N^* \rho_{\infty} \cdot \nabla \varphi dx,$$  

therefore, we obtain that $\lim_{t \to 0} \rho_{\infty}(t) := \rho_{\infty,0}^0$ exists in weak-$L^2(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \rho_{\infty,0}^0 \varphi dx = \int_{\mathbb{R}^n} \rho_{\infty} \varphi dx,$$  

which supports our statement. □

Furthermore, we are about to show the compact support of the solution for the Hele-Shaw limit system (2.6)–(2.7). To study the support of the limit density or the limit pressure, the main difficulty comes from the nonlocal interaction which prevent bounds on $\rho_{m}$. However, we may follow [20, Lemma 3.8] and obtain uniformly control of the pressure, then we deduce that the speed of propagation for the limit density is finite.

Firstly, we give this approximate equation

$$\partial_t \xi_m = \Delta \xi_m^m + \nabla \cdot (\xi_m \nabla \Phi_{1/m}),$$  

where $\Phi_{1/m}(x, t) = \xi_{1/m} * (N^* \rho_{\infty})$ and $\rho_{\infty}$ is the unique limit density in Theorem 2.2. Define the corresponding pressure $P_m = \frac{m}{m-1} \xi_{m}^{m-1}$ that satisfies the following equation

$$\partial_t P_m = (m - 1) P_m \Delta P_m + |\nabla P_m|^2 + \nabla P_m \cdot \nabla \Phi_{1/m} + (m - 1) P_m \Delta \Phi_{1/m}.$$  

Similar to Theorems 2.1–2.2, we can get the following theorem with the same initial data.
Theorem 6.1 Let \( \rho_{m,0} \) and \( P_{m,0} \) be the initial data of the density \( \rho_m \) and the pressure \( P_m \), respectively satisfying (2.1) and \( \rho_\infty \) be the unique limit density in Theorem 2.2. Then, after the extraction of subsequences, \( \nabla \Phi_{1/m} \) converges for all \( T > 0 \) strongly in \( L^2(Q_T) \) as \( m \to \infty \) to limit \( \nabla N \ast \rho_\infty \), \( \rho_m \) and \( \rho_\infty P_m \) converges weakly for all \( T > 0 \) in \( L^2(Q_T) \) as \( m \to \infty \) to limits \( \rho_\infty \in L^\infty(0,T; L^1(\mathbb{R}^n)) \cap L^\infty(Q_T) \) and \( P_\infty \in L^2(0,T; H^1(\mathbb{R}^n)) \) respectively. Therefore, the following Hele-Shaw limit system for \( (P_\infty, \rho_\infty) \) holds as

\[
\begin{align*}
\partial_t \rho_\infty &= \Delta P_\infty + \nabla \cdot (\rho_\infty \nabla N \ast \rho_\infty), & \text{in } \mathcal{D}'(Q_T), \\
(1 - \rho_\infty)P_\infty &= 0, & \text{a.e. in } Q_T, \\
0 \leq \rho_\infty \leq 1, & \text{a.e. in } Q_T.
\end{align*}
\]

Proof We omit the detailed proof of this theorem, because its proof is similar to, but easier than, that of Theorems 2.1–2.2.

It is easy for us to prove that the Hele-Shaw limit system (2.6)–(2.7) and (6.9)–(6.11) have same solutions if we have the same initial assumptions. In other words, if we get a uniform support of \( \rho_m \) and \( P_m \), naturally, we can obtain the supports of \( \rho_\infty \) and \( P_\infty \).

Lemma 6.1 For \( (\rho_\infty, P_\infty) \) in Theorem 2.2 and for \( (\rho_\infty, P_\infty) \) in Theorem 6.1 with the initial assumption \( \rho_\infty(x,0) = \rho_\infty(x,0) \in H^{-1}(\mathbb{R}^n) \), then it follows

\[
\rho_\infty = \rho_\infty, \quad P_\infty = P_\infty, \quad \text{a.e. } (x, t) \in Q.
\]

Proof The proof of this lemma is similar to but easier than the proof of Proposition 6.1, hence, we omit the detailed processes.

Lemma 6.2 Let \( \rho_m \) be a nonnegative weak solution to Eq. (6.7) for any continuous, compactly supported initial data \( \rho_{m,0} \). Then the pressure variable \( P_m \) is a viscosity solution to (6.8).

Proof The result follows from [38, Corollary 3.11]

We now turn to the \( L^\infty \) estimate and the support of the solutions to (6.9), which are uniform on \( m \). The first ensures that if the initial data is bounded uniformly on \( m \), it remains uniformly bounded within any finite time. The second ensures that if the support of the initial data is bounded uniformly on \( m \), it likewise remains uniformly bounded within any finite time.

Lemma 6.3 \((L^\infty \text{ estimate and support of } P_m)\) Let \( P_m \) be a viscosity solution to Eq. (6.9) with continuous, compactly supported initial data \( P_m(\cdot, 0) \). Suppose that there exists \( R_0 \geq 1 \) sufficiently large so that \( \text{supp}(P_m(\cdot, 0)) \subseteq B_{R_0/2} \) and \( \mu_m(\cdot, 0) \leq \frac{R_0^2}{4n} \). Then there exist \( R(t) := (R_0 + n \| \nabla \rho_\infty \|_{L^\infty(Q)}) e^\frac{n}{2} - n \| \nabla \rho_\infty \|_{L^\infty(Q)} \) such that, for all \( t \geq 0, x \in \mathbb{R}^n \),

\[
\{ \rho_m(\cdot, t) > 0 \} \subseteq B_{R(t)}, \quad P_m(x, t) \leq \frac{R^2(t)}{2n}.
\]

Proof The result follows from [20, Lemma 3.8].
When the initial density $\rho_{\infty,0}$ satisfies $0 \leq \rho_{\infty,0} \leq 1$ and is compactly supported, we show that the solution $(\rho_{\infty}, P_{\infty})$ to (2.6)–(2.7) are bounded and compactly supported for all times.

**Theorem 6.2 (Compact support)** Suppose that there exists $R_0$ sufficiently large such that both $\text{supp}(\rho_{\infty}(\cdot,0)) \subseteq B_{R_0/2}$ and $1 + \frac{n}{n-2} \leq \frac{R_0^2}{4n}$, and also both $\rho_{\infty}(\cdot,0) \in L^1_+(\mathbb{R}^n)$ and $0 \leq \rho_{\infty}(\cdot,0) \leq 1$ hold. Then, for all $t > 0$, the solution $(\rho_{\infty}, P_{\infty})$ to the Cauchy problem for the Hele-Shaw limit system (2.6)–(2.7) satisfies, with $R(t)$ is given in Lemma 6.3,

$$\{\rho_{\infty}(\cdot, t) > 0\} \subseteq B_{R(t)}, \quad P_{\infty}(\cdot, t) \leq \frac{R^2(t)}{2n}.$$  

**Proof** We set $\rho_{m,0} = \rho_{\infty,0}$ for $m \geq 2n - 1$, so it holds $P_{m,0} = \rho_{m-1,0} \leq 1 + \frac{n}{n-2} \leq \frac{R_0^2}{4n}$.

According to Lemma 6.3, it is easy to complete the proof of Theorem 6.2. □

Finally, we will establish the limit energy functional for the Cauchy problem of the Hele-Shaw problem (2.6)–(2.8). For the PKS model Eq. (1.1) with the diffusion exponent $1 < m < \infty$, the energy functional is given by

$$\frac{dF_m(\rho_m)}{dt} + \int_{\mathbb{R}^n} \rho_m |\nabla P_m + \nabla N^* \rho_m|^2 dx = 0.$$  

The above equality shows that the free energy decreases as the time increases. Formally, as $m \to \infty$, the limit free energy satisfies

$$F_{\infty}(\rho_{\infty}) = \frac{1}{2} \int_{\mathbb{R}^n} \rho_{\infty} N^* \rho_{\infty} dx, \quad 0 \leq \rho_{\infty} \leq 1,$$

in which the diffusive effect is replaced by the height constraint of the limit density, and the limit energy functional is expressed as

$$\frac{dF_{\infty}(\rho_{\infty}(t))}{dt} + \int_{\mathbb{R}^n} \rho_{\infty}(t) |\nabla P_{\infty}(t) + \nabla N^* \rho_{\infty}(t)|^2 dx = 0, \quad 0 \leq \rho_{\infty} \leq 1. \quad (6.12)$$

In the following theorem, we show that the limit energy functional (6.12) holds.

**Theorem 6.3 (Limit energy functional)** Under the initial assumptions (2.1)–(2.4) and the uniform support assumption for the initial density (2.5), let $\rho_{\infty}$ and $P_{\infty}$ be the limit density and the limit pressure respectively as in Theorems 2.1–2.3. Then, for a.e. $t \in [0, \infty)$, the limit energy functional (6.12) holds.

**Proof** Under the initial assumptions (2.1)–(2.4) and the additional initial uniform support assumption (2.5), due to Theorem 6.2, it holds

$$\text{supp}(P_{\infty}(t)) \subset \text{supp}(\rho_{\infty}(t)) \subset B_{R(t)},$$

for $R(t) := 2(2R_0 + \frac{\|\nabla N^* \rho_{\infty}\|_{L^\infty(Q)}}{n})e^t - \frac{2\|\nabla N^* \rho_{\infty}\|_{L^\infty(Q)}}{n}$ with some $R_0 \geq R_0 > 0$.

From Theorems 2.1–2.3, for any $T > 0$, we have

$$\rho_{\infty} \in L^\infty(0, T; L^1_+(\mathbb{R}^n)) \cap L^\infty(Q_T), \quad P_{\infty} \in L^2(0, T; H^1(\mathbb{R}^n)), \quad \nabla P_{\infty} \in L^3(Q_T), \quad (6.13)$$
\[ N \ast \rho_\infty \in C(0, T; \dot{W}^{1,r}(\mathbb{R}^n)) \cap L^\infty(0, T; \dot{W}^{2,s}(\mathbb{R}^n)) \]
for \(2 \leq r \leq \infty\) and \(1 < s < \infty\).

Furthermore, we obtain
\[ \Delta P_\infty \in L^2(0, T; \dot{H}^{-1}(\mathbb{R}^n) \cap \dot{H}^{-2}(\mathbb{R}^n)), \quad \partial_t \rho_\infty \in L^2(0, T; \dot{W}^{2,s}(\mathbb{R}^n)). \]

Thanks to the complementarity relation (2.8) and Eq. (2.7), integrating (2.8) on \(\mathbb{R}^n\) and integrating by parts, then it follows from the regularities (6.13)–(6.15) that
\[ \int_{\mathbb{R}^n} P_\infty (\Delta P_\infty + \rho_\infty) dx = \int_{\mathbb{R}^n} P_\infty \rho_\infty - |\nabla P_\infty|^2 dx = 0. \]

Therefore, it follows
\[ \int_{\mathbb{R}^n} \rho_\infty |\nabla P_\infty + \nabla N \ast \rho_\infty|^2 dx \]
\[ = \int_{\mathbb{R}^n} \rho_\infty (|\nabla P_\infty|^2 + 2 \nabla P_\infty \cdot \nabla N \ast \rho_\infty + |\nabla N \ast \rho_\infty|^2) dx \]
\[ = \int_{\mathbb{R}^n} (|\nabla P_\infty|^2 - 2 P_\infty \rho_\infty + \rho_\infty |\nabla N \ast \rho_\infty|^2) dx \]
\[ = \int_{\mathbb{R}^n} (-P_\infty \rho_\infty + \rho_\infty |\nabla N \ast \rho_\infty|^2) dx, \]
where (6.16) is used in the last equality. Multiplying (2.6) by \(N \ast \rho_\infty\) and integrating on \(\mathbb{R}^n\), according to the symmetry of convolution operator, we have
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho_\infty N \ast \rho_\infty dx - \int_{\mathbb{R}^n} (\Delta P_\infty + \nabla \cdot (\rho_\infty \nabla N \ast \rho_\infty)) N \ast \rho_\infty dx = 0, \]
integrating by parts and using (6.17), then it holds due to (6.13)–(6.15) that
\[ - \int_{\mathbb{R}^n} (\Delta P_\infty + \nabla \cdot (\rho_\infty \nabla N \ast \rho_\infty)) N \ast \rho_\infty dx = \int_{\mathbb{R}^n} (-P_\infty \rho_\infty + \rho_\infty |\nabla N \ast \rho_\infty|^2) dx \]
\[ = \int_{\mathbb{R}^n} \rho_\infty |\nabla P_\infty + \nabla N \ast \rho_\infty|^2 dx. \]

Combining (6.18) and (6.19), for almost everywhere time \(t\), we obtain the limit energy functional
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho_\infty N \ast \rho_\infty dx + \int_{\mathbb{R}^n} \rho_\infty |\nabla P_\infty + \nabla N \ast \rho_\infty|^2 dx = 0, \quad 0 \leq \rho_\infty \leq 1. \]

**Remark 6.1** The result of Theorem 6.3 implies that the limit free energy \(F_\infty(\rho_\infty(t))\) is non-increasing as time \(t\) increases.

### 7 Incompressible Limit of Stationary State

This section is devoted to showing that the incompressible (Hele-Shaw) limit for the stationary state of Patlak-Keller-Segel (SPKS) model Eq. (1.8) is the stationary state Eq. (1.12)
of the Hele-Shaw problem Eq. (1.9)–(1.10). By direct computations, the equation of the corresponding pressure \( P_{m,s} = \frac{m}{m-1} P_{m,s}^{m-1} \) is expressed by
\[
(m - 1) P_{m,s} (\Delta P_{m,s} + \rho_{m,s}) + \nabla P_{m,s} \cdot (\nabla P_{m,s} + \nabla \mathcal{N} \ast \rho_{m,s}) = 0 \quad \text{for } x \in \mathbb{R}^n. \tag{7.1}
\]

The following preliminary lemma is combined and extracted from [16, 18, 27].

**Lemma 7.1** (Preliminary lemma) Assume that \( \rho_{m,s} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). Then the solution to the SPKS model Eq. (1.8) exists. Moreover, the solution to the SPKS model Eq. (1.8) is radially decreasing symmetric, unique up to a translation, and compactly supported.

In the rest of this section, we carry on the incompressible limit of the stationary state of PKS (SPKS) model Eq. (1.8) in the framework of radial symmetry, and \( C \) is a positive constant independent of the exponent \( m \).

For any given mass \( M > 0 \), we show that the solution to the SPKS model Eq. (1.8) is uniformly bounded on \( m \).

**Lemma 7.2** (Uniform bound of pressure) Let \( \rho_{m,s} \) be a weak solution to the SPKS model Eq. (1.8) in the sense of Def. 1.1 with \( \| \rho_{m,s} \|_{L^1(\mathbb{R}^n)} = M > 0, m \geq 3 \), and \( \int_{\mathbb{R}^n} x \rho_{m,s}(x) dx = 0 \). Set \( \alpha_m := \rho_{m,s}(0) = \| \rho_{m,s} \|_{L^\infty(\mathbb{R}^n)} \), then it holds
\[
\alpha_m \leq 1 + \frac{\sqrt{1 + \frac{8M}{n(n-2)\omega_n}}}{2}, \quad \alpha_m^{-1} \leq \frac{1 + \frac{8M}{n(n-2)\omega_n}}{2} + \frac{2M}{n(n-2)\omega_n}. \tag{7.5}
\]

**Proof** From the SPKS model Eq. (1.8), we have
\[
\rho_{m,s} (\nabla P_{m,s} + \nabla \mathcal{N} \ast \rho_{m,s}) = 0.
\]

Due to the radially decreasing symmetric property of \( \rho_{m,s} \), there exists a constant \( C > 0 \) such that
\[
P_{m,s} = (-\mathcal{N} \ast \rho_{m,s} - C)_+ \quad \text{for } x \in \mathbb{R}^n \tag{7.2}
\]
and
\[
C \leq \| -\mathcal{N} \ast \rho_{m,s} \|_{L^\infty(\mathbb{R}^n)}. \tag{7.3}
\]

Since \( \alpha_m = \| \rho_{m,s} \|_{L^\infty(\mathbb{R}^n)} = \rho_{m,s}(0) \), we have
\[
-\mathcal{N} \ast \rho_{m,s} = \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{\rho_{m,s}(x-y)}{|y|^{n-2}} dy
\leq \frac{1}{n(n-2)\omega_n} \int_{|y|>1} \frac{\rho_{m,s}(x-y)}{|y|^{n-2}} dy + \alpha_m \frac{1}{n(n-2)\omega_n} \int_{|y|\leq 1} \frac{1}{|y|^{n-2}} dy \tag{7.4}
\leq \frac{1}{n(n-2)\omega_n} M + \frac{\alpha_m}{2(n-2)} \frac{1}{n(n-2)\omega_n} M.
\]

Combining (7.2)–(7.4), we obtain
\[
\alpha_m^{-1} \leq \frac{m}{m-1} \alpha_m^{-1} \leq \alpha_m + \frac{2M}{n(n-2)\omega_n}, \tag{7.5}
\]
where \( 1 \leq \frac{m}{m-1} \leq 2 \) for \( m \geq 3 \) is used.
The positive solution to the quadratic equations with one unknown $y_2^2 = y_2 + \frac{2M}{n(n-2)\omega_n}$ is

$$y_2 = \frac{1 + \sqrt{1 + \frac{8M}{n(n-2)\omega_n}}}{2} > 1.$$  

Thus it follows from the property of algebraic equation that

$$0 \leq \alpha_m \leq y_2 \quad \text{for} \quad m \geq 3. \quad (7.6)$$

We combine (7.5)–(7.6) and obtain

$$\alpha_m^{m-1} \leq \alpha_m + \frac{2M}{n(n-2)\omega_n} \leq \frac{1 + \sqrt{1 + \frac{8M}{n(n-2)\omega_n}}}{2} + \frac{2M}{n(n-2)\omega_n}. \quad \square$$

**Remark 7.1** We point out that the conclusion of Lemma 7.2 still holds with the assumption $\rho_{m,s} \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, the radially symmetric property of solution is not necessary.

Next, we show uniformly bounded support of density, which can prevent the mass from escaping to infinity as $m \to \infty$. Let $\|\rho_{m,s}\|_{L^1(\mathbb{R}^n)} = M$ and $\text{supp}(\rho_{m,s}) = B_{R_m(M)}$. We show that there exists a constant $R_*(M)$ (only depending on $M$) such that

$$R_m(M) \leq R_*(M) \quad \text{for all} \quad m \geq 3.$$

Define $\Psi_m(r) = \Psi_m(|x|) = \rho_{m,s}^{m-1}(x)$ with $r = |x|$, then the SPKS model Eq. (1.8) as introduced in [16, Lemma 2.1] can be transformed as a dynamical system

$$\begin{align*}
\Psi_m''(r) + \frac{n-1}{r} \Psi_m'(r) &= -\frac{m-1}{m} \Psi_m^{1/(m-1)}(r) \quad \text{for all} \quad 0 < r < R_m(M), \\
\Psi_m(0) &= \alpha_m^{m-1}, \quad \Psi_m'(0) = 0,
\end{align*} \quad (7.7)$$

and the following conditions hold:

$$\begin{align*}
\Psi_m(r) > 0, & \quad \Psi_m'(r) < 0, \quad \text{on} \quad (0, R_m(M)), \\
\frac{\Psi_m(r)}{r} \to -\frac{1}{n} \Psi_m^{1/(m-1)}, & \quad \text{as} \quad r \to 0^+, \\
\Psi_m(r) \to 0, & \quad \text{as} \quad r \to R_m(M)^-.
\end{align*}$$

To show the uniform bound of $R_m(M)$, we build a plane autonomous system. Let

$$u_m(r) = -\frac{m}{m-1} \frac{r \Psi_m^{1/(m-1)}(r)}{\Psi_m'(r)} \quad \text{and} \quad v_m(r) = -\frac{r \Psi_m'(r)}{\Psi_m(r)},$$

by direct computations, there is a plane autonomous system of $(u_m, v_m)$ as

$$\begin{align*}
\frac{du_m}{dr} &= u_m(n - u_m - \frac{v_m}{m-1}), \\
\frac{dv_m}{dr} &= v_m\left(-(n-2) + u_m + v_m\right),
\end{align*} \quad (7.8)$$

for $r > 0$, with the initial data

$$\lim_{r \to 0^+} u_m(r) = n, \quad \lim_{r \to 0^+} v_m(r) = 0, \quad \text{and} \quad \lim_{r \to 0^+} \frac{v_m(r)}{r^2} = \frac{m-1}{mn} \alpha_m^{2-m}. \quad (7.9)$$
The strategy is to find \( R_s(M) < \infty \) satisfying
\[
\lim_{r \to R_s(M)^-} v_m(r) = +\infty,
\]
which means that \( R_m(M) < R_s(M) \) for any \( m \geq 3 \) holds.

**Lemma 7.3** Suppose that \((u_m,v_m)\) is a solution to the initial value problem Eqs. \((7.8)-(7.9)\), then it holds for \( r > 0 \) that
\[
n < u_m + \frac{v_m}{m-1}, \quad 0 < u_m < n, \quad \text{and} \quad 0 < v_m.
\]

**Proof** Lemma 7.3 is a direct result of [16, Lemma 2.2]. \(\square\)

**Lemma 7.4** (Uniform support of density) Suppose that \((u_m,v_m)\) is a solution to the initial value problem Eqs. \((7.8)-(7.9)\) with a given mass \( M > 0 \). Then, there exists \( R_s(M) := \log \left( \frac{1}{1+\exp \left[ 2n(n-1) \left( \frac{1+\sqrt{1+8M}}{2} + \frac{2M}{n(n-2)\omega_n} \right) \right]^{1/2} } \right) \) such that
\[
\lim_{r \to R_s(M)^-} v_m(r) = \infty \quad \text{for all} \quad m \geq 3.
\]

Furthermore, it holds
\[
\text{supp}(\rho_{m,s}) \subset B_{R_s(M)} \quad \text{for all} \quad m \geq 3.
\]

**Proof** From Lemma 7.3, we have
\[
u_m > u_m + \frac{v_m}{m-1} > n \quad \text{and} \quad u_m, v_m > 0,
\]
then it holds
\[
u_m + v_m > u_m + \frac{v_m}{m-1} > n.
\]

Combining the above inequality and \((7.8)_2-(7.9)_1\), we obtain
\[
\begin{cases}
  r \frac{dv_m}{dr} > 2v_m & \text{for } r > 0, \\
  \lim_{r \to 0^+} v_m(r) = 0, \quad \lim_{r \to 0^+} \frac{v_m(r)}{r^2} = \frac{m-1}{mn} \alpha_m^{2-m}.
\end{cases}
\] (7.10)

We give an ordinary differential equation
\[
\begin{cases}
  r \frac{dv}{dr} = 2v & \text{for } r > 0, \\
  \lim_{r \to 0^+} v(r) = 0, \quad \lim_{r \to 0^+} \frac{v(r)}{r^2} = \frac{m-1}{mn} \alpha_m^{2-m}.
\end{cases}
\] (7.11)

It is easy to obtain the unique solution to Eq. (7.11) as
\[
v(r) = \frac{m-1}{mn} \alpha_m^{2-m} r^2.
\] (7.12)

Since the solution \( v(r) \) (7.12) to Eq. (7.11) is a sub-solution to Eq. (7.10), then we have
\[
v_m(r) \geq v(r) = \frac{m-1}{mn} \alpha_m^{2-m} r^2 \quad \text{for all} \quad r > 0.
\] (7.13)
With the help of Lemma 7.2 for a given mass $M > 0$, a uniform lower bound independent of $m$ is obtained as

\[
v_m(r) \geq \frac{m - 1}{mn} a_m^{2-m} r^2 \geq \frac{m - 1}{mn} \left( a_m^{m-1} \right)^{(2-m)/(m-1)} r^2 \\
\geq \frac{m - 1}{mn} \left( 1 + \sqrt{1 + \frac{8M}{n(n-2)\omega_n}} \right) + \frac{2M}{n(n-2)\omega_n} \right)^{(2-m)/(m-1)} r^2 \\
\geq \frac{1}{2n} \left( 1 + \sqrt{1 + \frac{8M}{n(n-2)\omega_n}} \right) + \frac{2M}{n(n-2)\omega_n} \right)^{-1} r^2 \quad \text{for all } m \geq 3,
\]

(7.14)

where $\frac{m-1}{m} > \frac{1}{2}$ and $\frac{2-m}{m-1} > -1$ are used. There exists a positive constant $R'(M)$ (only depending on $M$ and $n$) such that

\[
\frac{1}{2n} \left( 1 + \sqrt{1 + \frac{8M}{n(n-2)\omega_n}} \right) + \frac{2M}{n(n-2)\omega_n} \right)^{-1} R'^2(M) = n - 1, 
\]

(7.15)

and $R'(M)$ can be precisely written like

\[
R'(M) = \left[ 2n(n-1) \left( 1 + \sqrt{1 + \frac{8M}{n(n-2)\omega_n}} \right) + \frac{2M}{n(n-2)\omega_n} \right]^{1/2} > 1.
\]

Then it follows from (7.13)–(7.15) that

\[
v_m(r) \geq n - 1 \quad \text{for all } m \geq 3 \text{ and } r \geq R'(M). 
\]

(7.16)

Combining (7.8)2 and (7.16), we have

\[
\frac{d v_m}{d r} \geq v_m \left( v_m - (n-2) \right) \geq \left( v_m - (n-2) \right)^2 \quad \text{for all } m \geq 3 \text{ and } r \geq R'(M). 
\]

(7.17)

Set $r := \log s$ with $R'(M) := \log S'(M)$, we define $v_m(r) = v_m(\log s) := v_m(s)$, then it follows from (7.17) that

\[
\frac{d v_m}{d s} \geq \left( v_m - (n-2) \right)^2 \quad \text{for all } m \geq 3 \text{ and } s \geq S'(M).
\]

On the other hand, we consider the following ordinary differential equation:

\[
\begin{cases}
\frac{d \omega}{d s} = \left( \omega - (n-2) \right)^2 & \text{for all } s \geq S'(M) , \\
\omega(S'(M)) = n - 1, 
\end{cases}
\]

Hence, it holds

\[
\omega(s) = \frac{\omega(S'(M)) - (n-2)}{1 - (s - S'(M)) \left( \omega(S'(M)) - (n-2) \right)} + n - 2.
\]
We find
\[ \omega(s) \to \infty \quad \text{as} \quad s \to \frac{1 + S'(M) \left( \omega(S'(M)) - (n - 2) \right)}{\omega(S'(M)) - (n - 2)} = 1 + S'(M) := S_s(M). \]

Since \( \omega(s) \) is a sub-function of \( v_m(s) \) for all \( s \geq S'(M) \) and \( m \geq 3 \), we have
\[ v_m(s) \to \infty, \quad \text{as} \quad s \to S_s(M). \]

Set
\[ R_s(M) = \log S_s(M) = \log(1 + e^{R'(M)}), \]
\[ = \log \left( 1 + \exp \left[ 2n(n - 1) \left( \frac{1}{2} + \frac{8M}{n(n - 2)\omega_n} \right) + \frac{2M}{n(n - 2)\omega_n} \right]^{1/2} \right). \]

then it follows
\[ v_m(r) \to \infty \quad \text{as} \quad r \to R_s(M) - . \quad (7.18) \]

We need to show \( \Psi_m(R_s(M)) = 0 \). If not, due to the monotonicity of \( \Psi_m \), we assume that
\[ \Psi_m(r) \geq \Psi_m(R_s(M)) > 0 \quad \text{for all} \quad r \in [0, R_s(M)]. \]

We multiply Eq. (7.7) by \( r^{n-1} \) and obtain
\[ [r^{n-1} \Psi_m'(r)]' = -\frac{m - 1}{m} r^{n-1} \Psi_m^{1/(m-1)}(r). \]

Integrating the above equation on \([0, r]\) for any \( r \in (0, R_s(M)] \), it holds
\[ r^{n-1} \Psi_m'(r) = -\frac{m - 1}{m} \int_0^r s^{n-1} \Psi_m^{1/(m-1)}(s)ds. \quad (7.19) \]

It follows from the definition of \( \Psi_m \) that
\[ \frac{m - 1}{m} \int_0^r s^{n-1} \Psi_m^{1/(m-1)}(s)ds = \frac{m - 1}{m} \int_0^r s^{n-1} \rho_{m,s}(s)ds \]
\[ = \frac{m - 1}{mn \omega_n} \int_{B_r} \rho_{m,s}dx \]
\[ \leq \frac{m - 1}{mn \omega_n} M, \]

which together with (7.19) implies that there exists a small \( \delta > 0 \) (may depending on \( m \)) such that
\[ |\Psi_m'(r)| < \infty \quad \text{for all} \quad r \in [\delta, R_s(M)]. \]

Therefore, we have
\[ |v_m(r)| = \left| -\frac{r \Psi_m'(r)}{\Psi_m(r)} \right| < \infty \quad \text{for all} \quad r \in [\delta, R_s(M)], \]
which is contradicted with (7.18). In this way, one can show that
\[ \Psi_m(R_s(M)) = 0, \]
and it holds
\[ \text{supp}(\rho_{m,s}) \subset B_{R_s(M)} \quad \text{for all } m \geq 3. \]

\[ \square \]

**Remark 7.2** It should be emphasized that
\[ R_s(M) = \log \left( 1 + \exp \left[ 2n(n-1) \left( \frac{1 + \sqrt{1 + \frac{8M}{n(n-2)\omega_n}}}{2} + \frac{2M}{n(n-2)\omega_n} \right) \right] \right)^{1/2} \]
is strictly increases with mass \( M > 0 \), which is consistent with the geometric induction that higher mass means larger support.

Next, we prove a regularity estimate on the convolution term. Indeed, to obtain the weak convergence of the nonlinear term \( \{\rho_{m,s} \nabla^2 \star \rho_{m,s}\}_{m>1} \), one way is to prove the strong convergence of \( \{\nabla \star \rho_{m,s}\}_{m>1} \) by means of the weak-strong convergence.

**Lemma 7.5** (Regularity estimate on convolution term) Let \( \rho_{m,s} \) be a weak solution to the SPKS model Eq. (1.8) in the sense of Def. 1.1 with \( \|\rho_{m,s}\|_{L^1(\mathbb{R}^n)} = M \) and \( m \geq 3 \), then
\[ \|\nabla \star \rho_{m,s}\|_{L^2 \cap L^\infty(\mathbb{R}^n)} \leq C, \quad \|\nabla^2 \star \rho_{m,s}\|_{L^p(\mathbb{R}^n)} \leq C(M, p), \]
where \( C(M, p) \sim \frac{1}{p-1} \) for \( 0 < p - 1 \ll 1 \) and \( C(M, p) \sim p \) for \( p \gg 1 \).

Furthermore, thanks to Sobolev’s embedding theorem, there exists \( \nabla \star \rho_{\infty,s} \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) such that
\[ \nabla \star \rho_{m,s} \rightarrow \nabla \star \rho_{\infty,s}, \quad \text{strongly in } L^p_{loc}(\mathbb{R}^n) \text{ for } 1 \leq p < \infty, \quad \text{as } m \rightarrow \infty. \]

**Proof** With the help of Lemma 7.2, we have
\[ \|\rho_{m,s}\|_{L^p(\mathbb{R}^n)} \leq \|\rho_{m,s}\|_{L^1(\mathbb{R}^n)}^{1/p} \|\rho_{m,s}\|_{L^\infty(\mathbb{R}^n)}^{(p-1)/p} \leq M^{1/p} \alpha_m^{(p-1)/p} \leq C, \quad (7.20) \]
where \( \alpha_m \leq \frac{1 + \sqrt{1 + \frac{8M}{n(n-2)\omega_n}}}{2} \) from Lemma 7.2 and \( C = \max\{M, \frac{1 + \sqrt{1 + \frac{8M}{n(n-2)\omega_n}}}{2}\} \).

It similarly follows from (3.12) that
\[ \int_{\mathbb{R}^n} \nabla \star \rho_{m,s} \cdot \nabla \star \rho_{m,s} \, dx \leq C. \]

The \( L^\infty \) estimate of \( \nabla \star \rho_{m,s} \) easily holds:
\[ |\nabla \star \rho_{m,s}| \leq C \int_{|x-y| \leq 1} \frac{\rho_{m,s}(y,t)}{|x-y|^{n-1}} \, dy + C \int_{|x-y| > 1} \frac{\rho_{m,s}(y)}{|x-y|^{n-1}} \, dy \]
\[ \leq C \alpha_m \int_{|x-y| \leq 1} \frac{1}{|x-y|^{n-1}} \, dy + C \int_{|x-y| > 1} \rho_{m,s} \, dy \]
\[ \leq C. \quad (7.21) \]
Similar to (3.11), it follows from (7.20) that
\[ \| \nabla^2 \mathcal{N} \ast \rho_{m,s} \|_{L^p(\mathbb{R}^n)} \leq C(p) \| \rho_{m,s} \|_{L^p(\mathbb{R}^n)} \leq C(M, p) \]
where \( C(p) \sim \frac{1}{p-1} \) for \( 0 < p - 1 \ll 1 \) and \( C(p) \sim p \) for \( p \gg 1 \).

Then, thanks to Sobolev’s embedding theorem, there exists \( \mathcal{N} \ast \rho_{\infty,s} = \mathcal{N} \ast \rho_{\infty,s} \) strongly in \( L^p_{loc}(\mathbb{R}^n) \) for \( 1 \leq p < \infty \), as \( m \rightarrow \infty \).

In the following, we establish the Aronson-Bénilan (AB) estimate corresponding to the stationary state and thus a second order spatial derivative estimate of the pressure. Similar to the case of PKS model, we use the notation
\[ \omega_{m,s} := \Delta P_{m,s} + \rho_{m,s}. \quad (7.22) \]

**Lemma 7.6 (Aronson-Bénilan estimate)** Let \( \rho_{m,s} \) be a weak solution to the SPKS model Eq. (1.8) in the sense of Def. 1.1 with \( \| \rho_{m,s} \|_{L^1(\mathbb{R}^n)} = M \), then, for all \( m \geq 3 \),
\[ \| | \omega_{m,s} | - \frac{3}{2} \|_{L^3(\mathbb{R}^n)} \leq C/m, \quad \| \omega_{m,s} \|_{L^1(\mathbb{R}^n)} \leq C/m^{1/3}, \quad \| \Delta P_{m,s} \|_{L^1(\mathbb{R}^n)} \leq C. \quad (7.23) \]

**Proof** To begin with, we rewrite the SPKS model Eq. (1.8) as
\[ 0 = \Delta \rho_{m,s} + \nabla \cdot (\rho_{m,s} \nabla \mathcal{N} \ast \rho_{m,s}) = \rho_{m,s} \omega_{m,s} + \nabla \rho_{m,s} \cdot (\nabla \rho_{m,s} + \nabla \mathcal{N} \ast \rho_{m,s}), \]
and the pressure equation Eq. (7.1) is
\[ (m - 1) P_{m,s} \omega_{m,s} + \nabla P_{m,s} \cdot \nabla \rho_{m,s} + \nabla P_{m,s} \cdot \nabla \mathcal{N} \ast \rho_{m,s} = 0. \]

Take the Laplace operator (\( \Delta \)) action on the above equation, then we have
\[ (m - 1) \Delta (P_{m,s} \omega_{m,s}) + \nabla (\Delta P_{m,s}) \cdot (\nabla \mathcal{N} \ast \rho_{m,s} + \nabla P_{m,s}) + \nabla P_{m,s} \cdot \nabla \omega_{m,s} + 2 \nabla^2 P_{m,s} : (\nabla^2 P_{m,s} + \nabla^2 \mathcal{N} \ast \rho_{m,s}) = 0. \]

Hence, the equation of \( \omega_{m,s} \) is as follows
\[ (m - 1) \Delta (P_{m,s} \omega_{m,s}) + \nabla \omega_{m,s} \cdot \nabla \mathcal{N} \ast \rho_{m,s} + 2 \nabla^2 P_{m,s} : (\nabla^2 P_{m,s} + \nabla^2 \mathcal{N} \ast \rho_{m,s}) + \rho_{m,s} \omega_{m,s} + 2 \nabla P_{m,s} \cdot \nabla \omega_{m,s} = 0. \]

Then, it follows from Kato’s inequality that
\[ 0 \leq - (m - 1) \Delta (P_{m,s} \omega_{m,s}) - \nabla \omega_{m,s} \cdot \nabla \mathcal{N} \ast \rho_{m,s} + \frac{1}{2} (\nabla^2 \mathcal{N} \ast \rho_{m,s})^2 - \rho_{m,s} \omega_{m,s} - 2 \nabla P_{m,s} \cdot \nabla \omega_{m,s}, \quad (7.24) \]
where we use the fact
\[ 2 \nabla^2 P_{m,s} : (\nabla^2 P_{m,s} + \nabla^2 \mathcal{N} \ast \rho_{m,s}) \geq - \frac{1}{2} (\nabla^2 \mathcal{N} \ast \rho_{m,s})^2. \]

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Multiplying (7.24) by \(|\omega_{m,s}|_-\) and thanks to Kato’s inequality, we have

\[
0 \leq (m - 1) \Delta (P_{m,s}|\omega_{m,s}|_-)|\omega_{m,s}|_- + \frac{1}{2} \nabla |\omega_{m}|_2^2 \cdot \nabla N \ast \rho_m + \nabla |\omega_{m}|_2 \cdot \nabla P_{m,s} \\
+ \frac{1}{2} (\nabla^2 N \ast \rho_{m,s})^2 |\omega_{m,s}|_- + \rho_{m,s} |\omega_{m,s}|^2_.
\]  

(7.25)

Similar to (4.23), it easily holds

\[
(m - 1) \int_{\mathbb{R}^n} \Delta (P_{m,s}|\omega_{m,s}|_-)|\omega_{m,s}|_- dx = - \frac{1}{2} (m - 1) \int_{\mathbb{R}^n} |\omega_{m,s}|^2 dx \\
- \frac{1}{2} (m - 1) \int_{\mathbb{R}^n} \rho_{m,s} |\omega_{m,s}|^2 dx - (m - 1) \int_{\mathbb{R}^n} P_{m,s} |\nabla |\omega_{m,s}|_-|^2 dx.
\]

Then, integrating (7.25) on \(\mathbb{R}^n\), we use the above inequality and obtain

\[
2(m - 1) \int_{\mathbb{R}^n} P_{m,s} |\nabla |\omega_{m,s}|_-|^2 dx + (m - 4) \int_{\mathbb{R}^n} \rho_{m,s} |\omega_{m,s}|^2 dx \\
+ (m - 3) \int_{\mathbb{R}^n} |\omega_{m,s}|^3 dx - \int_{\mathbb{R}^n} (\nabla^2 N \ast \rho_{m,s})^2 |\omega_{m,s}|_- dx \leq 0.
\]

(7.26)

By Young’s inequality, Lemma 7.2, and the singular integral theory for Newtonian potential (Lemma C.1), we have

\[
\int_{\mathbb{R}^n} (\nabla^2 N \ast \rho_{m,s})^2 |\omega_{m,s}|_- dx \\
\leq \sum_{ij} \frac{2n}{3^{3/2}} \int_{\mathbb{R}^n} |\partial_{ij} N \ast \rho_{m,s}|^3 dx + \sum_{ij} \frac{1}{n^2} \int_{\mathbb{R}^n} |\omega_{m,s}|^3 dx \\
\leq \sum_{ij} \frac{2n}{3^{3/2}} C \int_{\mathbb{R}^n} \rho_{m,s}^3 dx + \int_{\mathbb{R}^n} |\omega_{m,s}|^3 dx \\
\leq \int_{\mathbb{R}^n} |\omega_{m,s}|^3 dx + CM \alpha_m^2 \leq \int_{\mathbb{R}^n} |\omega_{m,s}|^3 dx + C.
\]

(7.27)

Taking (7.27) into (7.26), then we attain

\[
2(m - 1) \int_{\mathbb{R}^n} P_{m,s} |\nabla |\omega_{m,s}|_-|^2 dx + (m - 4) \int_{\mathbb{R}^n} \rho_{m,s} |\omega_{m,s}|^2 dx + (m - 4) \int_{\mathbb{R}^n} |\omega_{m,s}|^3 dx \leq C.
\]

Thus, the first estimate of (7.23) holds

\[
\int_{\mathbb{R}^n} |\omega_{m,s}|^3 dx \leq C / m.
\]

From Lemma 7.4 with a given mass \(M > 0\), there exists a positive constant \(R_s(M)\) (only depending on \(M\)) such that

\[
\text{supp}(|\omega_{m,s}|_-) \subset B_{R_s(M)},
\]
then we have

\[ \int_{\mathbb{R}^n} |\omega_{m,s}|^- dx = \int_{B_{R}(M)} |\omega_{m,s}|^- dx \leq |B_{R}(M)|^{2/3} \left( \int_{B_{R}(M)} |\omega_{m,s}|^3 dx \right)^{1/3} \leq C/m^{1/3}, \]

so the second estimate of (7.23) holds. For the last estimate of (7.23), by triangle inequality and direct computations, we obtain

\[ \int_{\mathbb{R}^n} |\Delta P_{m,s}| dx \leq \int_{\mathbb{R}^n} |\Delta P_{m,s} + \rho_{m,s}| dx + M \]

\[ = \int_{\mathbb{R}^n} (\Delta P_{m,s} + \rho_{m,s}) dx + 2 \int_{\mathbb{R}^n} |\omega_{m,s}|^- dx + M \]

\[ = 2 \int_{\mathbb{R}^n} |\omega_{m,s}|^- dx + 2M \]

\[ \leq C. \]

The proof is completed. \(\Box\)

Next, we turn to show the \(L^\infty\) estimate of the pressure gradient.

**Lemma 7.7 (\(L^\infty\) estimate on pressure gradient)** Let \(\rho_{m,s}\) be a weak solution to the SPKS model Eq. (1.8) in the sense of Def. 1.1 with a given mass \(M > 0\). Then it holds for all \(m \geq 3\) that

\[ \|\nabla P_{m,s}\|_{L^\infty \cap L^1(\mathbb{R}^n)} \leq C. \]

**Proof** Since

\[ \rho_{m,s} \nabla P_{m,s} = -\rho_{m,s} \nabla \mathcal{N} * \rho_{m,s}, \]

we have

\[ \nabla P_{m,s} = -\nabla \mathcal{N} * \rho_{m,s} \quad \text{for } x \in \text{supp}(P_{m,s}). \]

It follows from Lemma 7.5 that

\[ |\nabla P_{m,s}| \leq C \quad \text{for } x \in \text{supp}(P_{m,s}). \]

From Lemma 7.1, there exists \(R_m(M) \leq R_*(M)\) for all \(m \geq 3\) such that

\[ \text{supp}(P_{m,s}) = B_{R_m(M)} \subset B_{R_*(M)}, \]

which together with the radially symmetric property of \(P_{m,s}\) (Lemma 7.1) means

\[ \|\nabla P_{m,s}\|_{L^\infty \cap L^1(\mathbb{R}^n)} \leq C. \] \(\Box\)

In the end, with the regularity estimates on \(\rho_{m,s}, P_{m,s},\) and \(\mathcal{N} * \rho_{m,s},\) we are going to prove the incompressible (Hele-Shaw) limit of the SPKS model Eq. (1.8).
Lemma 7.8 (Incompressible limit) Let \( \rho_{m,s} \) be a weak solution to the SPKS Eq. (1.8) in the sense of Def. 1.1 with \( \int_{\mathbb{R}^n} x \rho_{m,s}(x) \, dx = 0 \), \( \| \rho_{m,s} \|_{L^1(\mathbb{R}^n)} = M \), and \( m \geq 3 \). Then, after extracting the subsequence, there exist \( P_{\infty,s} \), \( \nabla P_{\infty,s} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) such that

\[
\n \nabla P_{m,s} \to \nabla P_{\infty,s}, \quad \text{strongly in } L^r(\mathbb{R}^n) \text{ for } 1 \leq r < \infty, \quad \text{as } m \to \infty, \tag{7.28}
\]

\[
P_{m,s} \to P_{\infty,s}, \quad \text{strongly in } L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \quad \text{as } m \to \infty. \tag{7.29}
\]

Furthermore, there exists \( \rho_{\infty,s} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) such that

\[
\| \rho_{\infty,s} \|_{L^1(\mathbb{R}^n)} = M, \quad \int_{\mathbb{R}^n} x \rho_{\infty,s} \, dx = 0, \tag{7.30}
\]

\[
0 \leq \rho_{\infty,s} \leq 1, \quad \text{a.e. in } \mathbb{R}^n, \tag{7.31}
\]

\[
(1 - \rho_{\infty,s}) P_{\infty,s} = 0, \quad \text{a.e. in } \mathbb{R}^n, \tag{7.32}
\]

\[
(1 - \rho_{\infty,s}) \nabla P_{\infty,s} = 0, \quad \text{a.e. in } \mathbb{R}^n, \tag{7.33}
\]

\[
\Delta P_{\infty,s} + \rho_{\infty,s} \geq 0, \quad \text{in } D'(\mathbb{R}^n), \tag{7.34}
\]

\[
\n \nabla P_{\infty,s} + \rho_{\infty,s} \nabla \mathcal{N} \ast \rho_{\infty,s} = 0, \quad \text{in } D'(\mathbb{R}^n). \tag{7.35}
\]

Moreover, it holds for \( R(M) > 0 \) satisfying \( |B_R(M)|_n = M \) that

\[
\rho_{\infty,s} = \chi_{\{P_{\infty,s} > 0\}} = \chi_{B_R(M)} \quad \text{a.e. in } \mathbb{R}^n. \tag{7.36}
\]

**Proof** Since \( \| \nabla P_{m,s} \|_{L^1(\mathbb{R}^n)/L^\infty(\mathbb{R}^n)} \leq C \) (Lemma 7.7) and \( \text{supp}(P_{m,s}) \subset B_{R_s(M)} \) (Lemma 7.4), then there exist \( \nabla P_{\infty,s} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) and \( \text{supp}(P_{\infty,s}) \subset B_{R_s(M)} \) such that

\[
\n \nabla P_{m,s} \to \nabla P_{\infty,s}, \quad \text{weakly in } L^r(\mathbb{R}^n) \text{ for } 1 < r < \infty, \quad \text{as } m \to \infty. \tag{7.37}
\]

Thanks to \( \| \Delta P_{m,s} \|_{L^1(\mathbb{R}^n)} \leq C \) (Lemma 7.6) and \( \text{supp}(P_{m,s}) \subset B_{R(M)} \) (Lemma 7.4), then it follows from the compactness criterion in [10, (21)] and \( P_{m,s} \in W^{1,\infty}_0(B_{R_s(M)}) \) that

\[
\n \nabla P_{m,s} \to \nabla P_{\infty,s}, \quad \text{strongly in } L^r(\mathbb{R}^n) \text{ for } 1 < r < \infty, \quad \text{as } m \to \infty. \tag{7.38}
\]

By Sobolev’s inequality for gradient (Theorem C.3), we obtain

\[
\| P_{m,s} - P_{\infty,s} \|_{L^\infty(\mathbb{R}^n)} \leq \| \nabla P_{m,s} - \nabla P_{\infty,s} \|_{L^r(\mathbb{R}^n)} \to 0, \quad \text{as } m \to \infty.
\]

So Eq. (7.28)–(7.29) hold.

In addition, Eq. (7.30) follows from \( \| \rho_{m,s} \|_{L^1(\mathbb{R}^n)} = M, \int_{\mathbb{R}^n} x \rho_{m,s} \, dx = 0 \), and \( \text{supp}(\rho_{m,s}) \subset B_{R_s(M)} \) show that as \( m \to \infty \)

\[
\int_{\mathbb{R}^n} \rho_{m,s} \, dx \to \int_{\mathbb{R}^n} \rho_{\infty,s} \, dx = M, \quad \int_{\mathbb{R}^n} x \rho_{m,s} \, dx \to \int_{\mathbb{R}^n} x \rho_{\infty,s} \, dx = 0.
\]

Since

\[
\| \rho_{m,s} \|_{L^r(\mathbb{R}^n)} \leq \| \rho_{m,s} \|_{L^\infty(\mathbb{R}^n)}^{(r-1)/r} \| \rho_{m,s} \|_{L^1(\mathbb{R}^n)}^{1/r} \leq C_M^{(r-1)/r} M^{1/r} \leq C^{(r-1)/r} M^{1/r},
\]

there exists \( \rho_{\infty,s} \in L^r(\mathbb{R}^n) \) for \( 1 < r < \infty \) such that

\[
\rho_{m,s} \rightharpoonup \rho_{\infty,s}, \quad \text{weakly in } L^r(\mathbb{R}^n), \quad \text{as } m \to \infty.
\]
According to the weak semi-continuity of $L^r$ norm for $1 < r < \infty$, we have

$$\|\rho_{\infty,s}\|_{L^r(\mathbb{R}^n)} \leq \liminf_{m \to \infty} \|\rho_{m,s}\|_{L^r(\mathbb{R}^n)} \leq \liminf_{m \to \infty} (\alpha_m^{m-1})^{(r-1)/r(m-1)} M^{1/r}$$

$$\leq \liminf_{m \to \infty} \left(1 + \sqrt{1 + \frac{4M^2}{n^2(n-2)^2}}\right) + \frac{2M}{n(n-2)\omega_n} (r-1)/r(m-1) M^{1/r}$$

$$\leq M^{1/r}.$$  

We take $r \to \infty$ and obtain $\|\rho_{\infty,s}\|_{L^\infty(\mathbb{R}^n)} \leq 1$, which shows Eq. (7.31).

Let $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ be a smooth test function. Due to the definition of weak solution (1.1), we have

$$\int_{\mathbb{R}^n} \rho_{m,s} \nabla P_{m,s} \cdot \nabla \varphi dx + \int_{\mathbb{R}^n} \rho_{m,s} \nabla N * \rho_{m,s} \cdot \nabla \varphi dx = 0. \quad (7.39)$$

Passing (7.39) to limit, then we obtain

$$\int_{\mathbb{R}^n} \rho_{\infty,s} \nabla P_{\infty,s} \cdot \nabla \varphi dx + \int_{\mathbb{R}^n} \rho_{\infty,s} \nabla N * \rho_{\infty,s} \cdot \nabla \varphi dx = 0. \quad (7.40)$$

Since $\rho_{m,s} = (\frac{m-1}{m} P_{m,s})^{1/(m-1)}$, we have

$$\int_{\mathbb{R}^n} \rho_{\infty,s} P_{\infty,s} \varphi dx = \lim_{m \to \infty} \int_{\mathbb{R}^n} \rho_{m,s} P_{m,s} \varphi dx$$

$$= \lim_{m \to \infty} \int_{\mathbb{R}^n} (\frac{m-1}{m})^{1/(m-1)} (P_{m,s})^{m/(m-1)} \varphi dx = \int_{\mathbb{R}^n} P_{\infty,s} \varphi dx. \quad (7.41)$$

Similarly, it holds for $i = 1, \ldots, n$ that

$$\int_{\mathbb{R}^n} \rho_{\infty,s} \partial_i P_{\infty,s} \varphi dx = \lim_{m \to \infty} \int_{\mathbb{R}^n} \rho_{m,s} \partial_i P_{m,s} \varphi dx$$

$$= \lim_{m \to \infty} \int_{\mathbb{R}^n} (\frac{m-1}{m})^{m/(m-1)} (P_{m,s})^{m/(m-1)} \partial_i \varphi dx$$

$$= - \int_{\mathbb{R}^n} P_{\infty,s} \partial_i \varphi dx = \int_{\mathbb{R}^n} \partial_i P_{\infty,s} \varphi dx. \quad (7.42)$$

Using $\rho_{\infty,s}, P_{\infty,s}, \nabla P_{\infty,s} \in L^r(\mathbb{R}^n)$ for $1 < r < \infty$, Eqs. (7.32)–(7.33) follows from (7.41)–(7.42).

From Lemma 7.6, we obtain

$$\int_{\mathbb{R}^n} (\Delta P_{m,s} + \rho_{m,s}) \varphi dx \geq -\int_{\mathbb{R}^n} |\omega_{m,s}| - \varphi dx \geq -C \|\omega_{m,s}\|_{L^1(\mathbb{R}^n)} \geq -C/m^{1/3},$$

which means that, after taking the limit, Eq. (7.34) holds for a nonnegative smooth test function as

$$\int_{\mathbb{R}^n} (\Delta P_{\infty,s} + \rho_{\infty,s}) \varphi dx \geq 0.$$

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Combining (7.40) and f40, the last statement Eq. (7.35) is gotten as
\[ \nabla P_{\infty,s} + \rho_{\infty,s} \nabla N * \rho_{\infty,s} = 0, \quad \text{a.e. in } \mathbb{R}^n. \]

Since \( \rho_{m,s} \) is a solution to the SPKS model Eq. (1.8) with \( \int_{\mathbb{R}^n} x \rho_{m,s}(x) dx = 0 \), \( \rho_{m,s} \) and \( P_{m,s} \) are radially decreasing symmetric. Therefore, \( \rho_{\infty,s} \) and \( P_{\infty,s} \) are radially decreasing symmetric, and \( N * \rho_{\infty,s} \) is radially symmetric (Lemma 7.1). Since \( \rho_{\infty,s} P_{\infty,s} = P_{\infty,s} \), there exists \( R_1 \leq R_2 \) such that
\[ \text{supp}(P_{\infty,s}) = B_{R_1}, \quad \text{supp}(\rho_{\infty,s}) = B_{R_2}. \]

If \( R_1 < R_2 \), it follows from Lemma 7.8 that
\[ \rho_{\infty,s}(r) \frac{\partial}{\partial r} N * \rho_{\infty,s}(r) = 0 \quad \text{for } x \in (R_1, R_2). \]  \hspace{1cm} (7.43)

However, it follows from [37, (2.2)] that for \( r > 0 \)
\[ \frac{\partial}{\partial r} N * \rho_{\infty,s}(r) = 1 \frac{\int_{\partial B_r} |\nabla N * \rho_{\infty,s}| dS}{|\partial B_r|_{n-1}} = \frac{\int_{\partial B_r} \nabla N * \rho_{\infty,s} \cdot v dS}{|\partial B_r|_{n-1}} \]
\[ = \frac{1}{|\partial B_r|_{n-1}} \int_{B_r} \Delta N * \rho_{\infty,s} dx \frac{1}{|\partial B_r|_{n-1}} \int_{B_r} \rho_{\infty,s} dx > 0, \]
which contradicts (7.43). Thus, we have \( R_1 = R_2 \), and the last statement Eq. (7.36) is proved.

## 8 Conclusion, Extensions and Perspectives

In order to prove the incompressible limit of the Patlak-Keller-Segel system, and establish the corresponding Hele-Shaw free boundary equation, we have followed the same lines of proof as initiated in [54], with the gradient estimate as in [1, 22]. This has the advantage to also prove optimal second order estimates. A fundamental new ingredient is a uniform \( L^1 \) estimate on the time derivative of pressure. With this new estimate, another possible route to establish the complementarity condition, the hard part of the problem, would be to use the pure compactness method in [46, 55]. Still another possible route is through the obstacle problem, see [33] and the references therein. We have also established uniqueness, finite propagation speed, and limit energy functional of solutions to this Hele-Shaw type free boundary problem. In addition, we have studied the incompressible limit for the stationary state of the PKS model, which is new for the diffusion-aggregation equations.

We would like to point out that our analysis for the PKS model is compatible with growth terms, as they appear naturally when dealing with mechanical models of tumor growth, even though the technical details have to be checked. Also we treated dimension \( n \geq 3 \) to avoid technical issues with the Sobolev inequalities but we do not expect difficulties in two dimensions.

Several papers have treated of linear drift terms, see [20, 23, 38]. With our method, it is difficult to extend these cases or Newtonian potential to more general attractive potential because of the estimate for the time derivative of pressure which strongly depends on the structure of Newtonian potential. Among open problems, let us also mention the convergence rate with \( m \to \infty \), which has been obtained in few papers, [2, 24]. Finally the case
of systems is only treated without drift, see [11, 46]. Large time asymptotic of solutions to the Hele-Shaw system (2.6)–(2.7) is an interesting topic. [20] treated the 2-dimensional Hele-Shaw model when the initial density is a patch function. But that for the n-dimensional (n ≥ 3) Hele-Shaw case is still largely open. The regularity of free boundary for a Hele-Shaw problem of tumor growth was obtained in [52].

**Appendix A: \( L^\infty \) Bound**

Under the initial assumption (2.1), we show the solution to (1.1) is bounded in any finite time using Moser’s iteration technique. In order to justify the \( L^\infty \) estimate of density, we introduce the following approximate equations of the Cauchy problem (1.1):

\[
\begin{aligned}
\frac{\partial}{\partial t} \rho_\varepsilon &= \Delta \rho_\varepsilon^m + \varepsilon \Delta \rho_\varepsilon + \nabla \cdot \left( \rho_\varepsilon \nabla \left( J_\varepsilon \ast \mathcal{N}_\varepsilon \ast \rho_\varepsilon \right) \right), \\
\rho_\varepsilon(x, 0) &= \rho_{0, \varepsilon}(x) = J_\varepsilon \ast \rho_0, \\
\|\rho_\varepsilon(t)\|_{L^1(\mathbb{R}^n)} &= \|\rho_{0, \varepsilon}\|_{L^1(\mathbb{R}^n)} \leq \|\rho_0\|_{L^1(\mathbb{R}^n)},
\end{aligned}
\]

where \( J \in C^\infty(\mathbb{R}^n) \) with both \( \|J\|_{L^1(\mathbb{R}^n)} = 1 \) and \( \text{supp}(J) \subset B_1 \) is the standard mollifier and \( J_\varepsilon := \frac{1}{\varepsilon^n} J(\frac{x}{\varepsilon}) \).

**Lemma A.1** Let the initial assumption (2.1) hold. Then, the solution \( \rho_\varepsilon \) to the Cauchy problem Eq. (A.1) satisfies for any \( T > 0, r \in [1, \infty) \),

\[
\sup_{0 \leq t \leq T} \|\rho_\varepsilon(t)\|_{L^r(\mathbb{R}^n)} \leq C(M, T, m, n, r, \|\rho_0\|_{L^\infty(\mathbb{R}^n)}).
\]

**Proof** Multiplying Eq. (A.1) by \( \rho_\varepsilon^{r-1} \) and integrating on \( \mathbb{R}^n \), we find

\[
\frac{d}{dt} \|\rho_\varepsilon\|_{L^r}^r + \frac{4mr(r - 1)}{(r + m - 1)^2} \int_{\mathbb{R}^n} |\nabla \rho_\varepsilon^{\frac{r+m-1}{r}}|^2 \, dx \\
\leq -r \int_{\mathbb{R}^n} \rho_\varepsilon \nabla \mathcal{N} \ast (J_\varepsilon \ast \rho_\varepsilon) \cdot \nabla \rho_\varepsilon^{r-1} \, dx \\
\leq (r - 1) \int_{\mathbb{R}^n} \rho_\varepsilon \mathcal{N} \ast (J_\varepsilon \ast \rho_\varepsilon) \, dx \\
\leq (r - 1) \int_{\mathbb{R}^n} \rho_\varepsilon^{r+1} \, dx,
\]

because it follows from Holder’s inequality and convolution-type Young’s inequality that

\[
\int_{\mathbb{R}^n} \rho_\varepsilon \mathcal{N} \ast (J_\varepsilon \ast \rho_\varepsilon) \, dx \leq \left( \int_{\mathbb{R}^n} \rho_\varepsilon^{r+1} \, dx \right)^{r/(r+1)} \left( \int_{\mathbb{R}^n} (J_\varepsilon \ast \rho_\varepsilon)^{r+1} \, dx \right)^{1/(r+1)} \\
\leq \|\rho_\varepsilon\|_{L^{r+1}(\mathbb{R}^n)} \|J_\varepsilon\|_{L^1(\mathbb{R}^n)} \|\rho_\varepsilon\|_{L^{r+1}(\mathbb{R}^n)} \\
= \|\rho_\varepsilon\|_{L^{r+1}(\mathbb{R}^n)}.
\]
Thanks to the interpolation inequality (Theorem C.2) and Sobolev's inequality for gradient (Theorem C.3), we have

$$\| \rho_\varepsilon \|_{L^{r+1}(\mathbb{R}^n)} \leq \left( \int_{\mathbb{R}^n} \rho_\varepsilon \, dx \right)^{r + m + 2n - 2m + 1} \left( \int_{\mathbb{R}^n} \rho_\varepsilon \frac{(r + m + 2n - 2m + 1)}{r + m + 2n - 2m + 1} \, dx \right)^{r + (n - 1) \frac{2n - 2m + 1}{r + m + 2n - 2m + 1}}$$

$$\leq C(M, m, n, r) \left( \frac{4mr}{(r + m - 1)^2} \int_{\mathbb{R}^n} \| \nabla \rho_\varepsilon \|^{\frac{m + r - 1}{2}} \, dx \right)^{\frac{2n - 2m + 1}{r + m + 2n - 2m + 1}}$$

$$\leq C(M, m, n, r) + \frac{4mr}{(r + m - 1)^2} \int_{\mathbb{R}^n} \| \nabla \rho_\varepsilon \|^{\frac{m + r - 1}{2}} \, dx,$$

and it follows from (A.2) that

$$\frac{d}{dt} \| \rho_\varepsilon \|_{L^r} \leq C(M, m, n, r). \quad \text{(A.3)}$$

But the initial data is independent of $\varepsilon$ since for $r \geq 1$, we have

$$\| J_\varepsilon \ast \rho_0 \|_{L^r(\mathbb{R}^n)} \leq \| J_\varepsilon \|_{L^1(\mathbb{R}^n)} \| \rho_0 \|_{L^r(\mathbb{R}^n)} \leq \| \rho_0 \|_{L^1(\mathbb{R}^n)}^\frac{1}{r} \| \rho_0 \|_{L^\infty(\mathbb{R}^n)} \leq C.$$

Therefore, integrating (A.3) on $[0, t]$ the proof of the lemma is completed.

The $L^\infty$ estimate of the density $\rho_\varepsilon$ is obtained by adapting the proof in [60].

**Lemma A.2** Under the initial assumptions (2.1), the solution $\rho_\varepsilon$ to the Cauchy problem Eq. (A.1) satisfies for all $T > 0$

$$\sup_{0 \leq t \leq T} \| \rho_\varepsilon(t) \|_{L^\infty(\mathbb{R}^n)} \leq C(M, T, m, n, \| \rho_0 \|_{L^\infty(\mathbb{R}^n)}).$$

**Proof** Similar to (3.8), by Young’s inequality we have, for $t \in [0, T]$,

$$|\nabla (J_\varepsilon \ast N \ast \rho_\varepsilon(t))| \leq C(n) \left( \int_{\mathbb{R}^n} (J_\varepsilon \ast \rho_\varepsilon)^{2n-1}(y, t) \, dy \right)^{\frac{1}{2n-1}} + C(n) \| J_\varepsilon \ast \rho_\varepsilon(t) \|_{L^1(\mathbb{R}^n)}$$

$$\leq C(n) \| \rho_\varepsilon(t) \|_{L^{2n-1}(\mathbb{R}^n)} + \| \rho_\varepsilon(t) \|_{L^1(\mathbb{R}^n)}$$

$$\leq C(M, T, m, n, \| \rho_0 \|_{L^\infty(\mathbb{R}^n)}).$$

Multiplying Eq. (A.1) by $\rho_\varepsilon^{r-1}$ and integrating on $\mathbb{R}^n$, it follows that, for $r \geq m \geq 1$,

$$\frac{1}{r} \frac{d}{dt} \| \rho_\varepsilon \|_{L^r} + \frac{4m(r - 1)}{(r + m - 1)^2} \int_{\mathbb{R}^n} \| \nabla \rho_\varepsilon \|^{\frac{r + m - 1}{2}} \, dx \leq (r - 1) \int_{\mathbb{R}^n} \rho_\varepsilon^{r-1} \nabla \rho_\varepsilon \cdot \nabla (J_\varepsilon \ast N \ast \rho_\varepsilon) \, dx$$

$$\leq \frac{2m(r - 1)}{(r + m - 1)^2} \int_{\mathbb{R}^n} \| \nabla \rho_\varepsilon \|^{\frac{r + m - 1}{2}} \, dx + \frac{(r - 1)}{2m} \| \nabla (J_\varepsilon \ast N \ast \rho_\varepsilon(t)) \|_{L_\infty(\mathbb{R}^n)}^2 \int_{\mathbb{R}^n} \rho_\varepsilon^{r-1-m} \, dx.$$

Furthermore, it holds by Hölder’s and Young’s inequalities that

$$\frac{1}{r} \frac{d}{dt} \| \rho_\varepsilon \|_{L^r} + \frac{2m(r - 1)}{(r + m - 1)^2} \int_{\mathbb{R}^n} \| \nabla \rho_\varepsilon \|^{\frac{r + m - 1}{2}} \, dx$$

$$\leq (r - 1) C(M, T, m, n, \| \rho_0 \|_{L^\infty(\mathbb{R}^n)}) (\| \rho_\varepsilon \|_{L^1(\mathbb{R}^n)})^{\frac{m-1}{r-1}} (\| \rho_\varepsilon \|_{L^r(\mathbb{R}^n)})^{\frac{r-m}{r-1}} \quad \text{(A.4)}$$

$$\leq C'(M, T, m, n, \| \rho_0 \|_{L^\infty(\mathbb{R}^n)}) + r^2 \| \rho_\varepsilon \|_{L^r(\mathbb{R}^n)}^r, \quad \text{for } r + 1 \geq 2m.$$
By means of the interpolation inequality (Theorem C.2), Sobolev’s inequality for the gradient (Theorem C.3) and Young’s inequality, we also have

$$r^2 \| \rho_\epsilon \|_{L^r(\mathbb{R}^n)}^2 \leq r^2 \| \rho_\epsilon \|_{L^\infty(\mathbb{R}^n)} \| \rho_\epsilon \|_{L^{3r/(m+1)}(\mathbb{R}^n)}$$

$$= r^2 \| \rho_\epsilon \|_{L^\infty(\mathbb{R}^n)} \left( \int_{\mathbb{R}^n} |\nabla \rho_\epsilon| \frac{m+1}{2} \, dx \right)^{3r/(m+1)}$$

$$\leq \frac{m(r-1)}{(m+r-1)^2} \| \nabla \rho_\epsilon \|_{L^2(\mathbb{R}^n)}^2 + \frac{r+4(m-1)n}{(3n+1)r+4(m-1)n} \left( S_n^{-2} \frac{(r+m-1)^2}{mr(r-1)} \frac{3nr}{m(r-1)} \right)^{3n/(m+1)} \| \rho_\epsilon \|_{L^\infty(\mathbb{R}^n)}^2$$

$$\leq \frac{m(r-1)}{(m+r-1)^2} \| \nabla \rho_\epsilon \|_{L^2(\mathbb{R}^n)}^2 + r^N \| \rho_\epsilon \|_{L^\infty(\mathbb{R}^n)}^r + C^r(m,n), \quad (A.5)$$

where \( N := 9n+2, r \geq \max\{4,2m\} \), and in the last inequality we use,

$$\left( S_n^{-2} \frac{(r+m-1)^2}{mr(r-1)} \frac{3nr}{m(r-1)} \right)^{3n/(m+1)} \to \left( S_n^{-2} \frac{3n}{m(3n+1)} \right)^{3n/(m+1)}, \quad as \ r \to \infty.$$ 

Inserting (A.5) into (A.4), it holds

$$\frac{d}{dt} \| \rho_\epsilon \|_{L^r(\mathbb{R}^n)} \leq r C^r(m,n) + r^{N+1} \| \rho_\epsilon \|_{L^\infty(\mathbb{R}^n)}.$$

Integrating (A.6) on \([0, t]\) for any \( t \in (0, T] \), we have

$$\sup_{0 \leq s \leq t} \| \rho_\epsilon(t) \|_{L^r(\mathbb{R}^n)} \leq r^{N+1} \sup_{0 \leq s \leq t} \| \rho_\epsilon(t) \|_{L^\infty(\mathbb{R}^n)} + \| \rho_\epsilon,0 \|_{L^r(\mathbb{R}^n)} + r^r \quad C(m,n)$$

$$\leq r^{N+1} \sup_{0 \leq s \leq t} \| \rho_\epsilon(t) \|_{L^\infty(\mathbb{R}^n)} + \| \rho_\epsilon \|_{L^r(\mathbb{R}^n)} + r^r \quad C(m,n)$$

$$\leq r^{N+1} \max\{M, \| \rho_\epsilon \|_{L^\infty(\mathbb{R}^n)}, C(T, m,n), \sup_{0 \leq s \leq T} \| \rho_\epsilon(t) \|_{L^\infty(\mathbb{R}^n)} \}. \quad \text{Let } r = 4^p \text{ for } p \in \mathbb{N}^+, \text{ by iteration, we obtain}$$

$$\sup_{0 \leq t \leq T} \| \rho_\epsilon(t) \|_{L^{4^p}(\mathbb{R}^n)} \leq 4^{(p+1)/(4^p+1)} \max\{M, \| \rho_\epsilon \|_{L^\infty(\mathbb{R}^n)}, C(T, m,n), \sup_{0 \leq t \leq T} \| \rho_\epsilon \|_{L^{4^p}(\mathbb{R}^n)} \}$$

$$\leq \underbrace{4^{(N+1)(p+1)/(4^p+1)} + \cdots + \frac{2^p}{4^p+1}}_{\leq C(M, T, m,n, \| \rho_0 \|_{L^\infty(\mathbb{R}^n)})} \max\{M, \| \rho_0 \|_{L^\infty(\mathbb{R}^n)}, C(T, m,n), \sup_{0 \leq t \leq T} \| \rho_\epsilon(t) \|_{L^4(\mathbb{R}^n)} \}$$

$$\leq C(M, T, m,n, \| \rho_0 \|_{L^\infty(\mathbb{R}^n)}),$$

where \( \sup_{0 \leq t \leq T} \| \rho_\epsilon(t) \|_{L^{4^p}(\mathbb{R}^n)} \leq C(M, T, m,n, \| \rho_0 \|_{L^\infty(\mathbb{R}^n)}) \) for \( p^* := \inf\{p \in \mathbb{N}^+ : 4^p \geq \max\{4,2m\} \} \) holds from Lemma A.1. Taking \( p \to \infty \), we find the announced bound.

\( \square \)

**Appendix B: Compact Support Property**

We prove the \( m \) dependent compact support property.
Lemma B.1 Assume (2.1), then, the global weak solution to the Cauchy problem Eq. (1.1) in the sense of Def. 1.1 is compactly supported

\[
\text{supp}(\rho_m(t)) \subset B_{\mathcal{R}_m(T,t)},
\]

\[
\mathcal{R}_m(T,t) := (\mathcal{R}_m,0 + n\|\nabla N^* \rho_m\|_{L^\infty(Q_T)}) e^{\frac{\lambda_{m,T} t}{n}} - n\|\nabla N^* \rho_m\|_{L^\infty(Q_T)},
\]

with \(\mathcal{R}_{m,0} = 2 \max(R_m, \sqrt{n \rho_{m,0}})\) and \(A_{m,T} := \max(\|\rho_m\|_{L^\infty(Q_T)}, 1)\).

Set \(V_m(x,t) := \nabla N^* \rho_m\), it follows from the \(L^\infty\) bound in Lemma A.2 that

\[
\sup_{0 \leq t \leq T} \|V_m(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(M,T,m,n, \|\rho_{m,0}\|_{L^\infty(\mathbb{R}^n)}),
\]

\[
\sup_{0 \leq t \leq T} \|\nabla \cdot V_m(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(M,T,m,n, \|\rho_{m,0}\|_{L^\infty(\mathbb{R}^n)}),
\]

where the first inequality is obtained as (3.8). We rewrite Eq. (1.1) as

\[
\partial_t \rho_m = \Delta \rho_m + \nabla \cdot (\rho_m V_m),
\]

which is now considered as a porous medium equation with a given drift \(V_m\). Then, similar to [20, Lemma 3.8], we can construct a viscosity sup-solution with a uniform compact support in any finite time for the equation of pressure \(P_m = m - \frac{1}{\rho_m} \rho_m, \partial_t P_m = (m - 1)P_m (\Delta P_m + \nabla \cdot V_m) + |\nabla P_m|^2 + \nabla P_m \cdot V_m\). Hence, the solution \(\rho_m\) to the Cauchy problem (1.1) under the initial assumption (2.1) is compactly supported on \([0, T]\) for any \(T > 0\).

More precisely, let \(\mathcal{R}_m(T,t) := (\mathcal{R}_m,0 + n\|\nabla N^* \rho_m\|_{L^\infty(Q_T)}) e^{\frac{\lambda_{m,T} t}{n}} - n\|\nabla N^* \rho_m\|_{L^\infty(Q_T)}\) for any \(T > 0\) with \(\mathcal{R}_{m,0} \geq 1\) satisfying \(2R_m \leq \mathcal{R}_{m,0}\) and \(P_{m,0} \leq \frac{\pi^2 m_0}{4m}\), and \(A_{m,T} := \max(\|\rho_m\|_{L^\infty(Q_T)}, 1)\). Set \(\phi_{m,T}(x,t) := \frac{A_m T \mathcal{R}_m^2 (1 - \frac{|x|^2}{\pi^2 m_T})}{2n}\), it is easy to verify on the support of \(\phi_{m,T}\) that

\[
\partial_t \phi_{m,T} \geq (m - 1) \phi_{m,T} (\Delta \phi_{m,T} + \nabla \cdot V_m) + |\nabla \phi_{m,T}|^2 + \nabla \phi_{m,T} \cdot V_m \quad \text{for} \quad t \in [0, T]
\]

with \(P_{m,0} \leq \phi_{m,T}(x,0) \in \mathbb{R}^n\). Therefore, it holds by the comparison principle that

\[
P_m(x,t) \leq \phi_{m,T}(x,t), \quad \forall \ (x,t) \in \mathbb{R}^n \times [0, T],
\]

\[
\text{supp}(\rho_m(t)) \subset B_{\mathcal{R}_m(T,t)}, \quad \forall \ t \in [0, T].
\]

The proof of the compact support property is completed.

Appendix C: Some Functional Inequalities

Lemma C.1 (Singular integral for Newtonian potential) [4, 58] Let \(N\) be the Newtonian potential. For \(f(x) \in L^{p}(\mathbb{R}^n), 1 < p < \infty\), we have

\[
\|\nabla^2 N * f(x)\|_{L^p(\mathbb{R}^n)} \leq C(p,n) \|f(x)\|_{L^p(\mathbb{R}^n)},
\]

where \(C(p,n) \sim \frac{1}{p-1} \) for \(0 < p - 1 < 1\) and \(C(p,n) \sim p \) for \(p \gg 1\).
Theorem C.1 (Hardy-Littlewood-Sobolev inequality) [43] Let $p, r > 1$ and $0 < \lambda < n$ with $\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{r} = 2$. Let $f \in L^p(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. Then, we have

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x - y|^{-\lambda} h(y) dxdy \right| \leq C(n, \lambda, p) \| f \|_{L^p(\mathbb{R}^n)} \| h \|_{L^r(\mathbb{R}^n)},$$

(C.1)

where the constant satisfies $C(n, \lambda, p) = n \frac{\Gamma\left(\frac{n-\lambda}{n}\right)}{\Gamma\left(\frac{n+1}{2}\right)^{\frac{n-\lambda}{n}}}$. 

For $\lambda = n - 2$ and $p = r = \frac{2n}{n+2}$, we have $C(n, p, \lambda) = C(n) = \pi \frac{n}{\Gamma(\frac{n+1}{2})} \left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right)^{\frac{1}{2}}$.

Theorem C.2 (Interpolation inequality) Let $\Omega \subset \mathbb{R}^n$ be a measurable domain, let $f( x) \in L^p(\Omega) \cap L^q(\Omega)$, then, with $0 \leq \beta = \frac{q - r}{q - p} \leq 1$,

$$\int_{\Omega} |f'|^q dx \leq (\int_{\Omega} |f|^p dx)^{\beta} (\int_{\Omega} |f|^q dx)^{1-\beta}.$$  

Theorem C.3 (Sobolev’s inequality for gradient) [43] For $n \geq 3$, let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $\| \nabla f \|_{L^2} < \infty$. Then $f \in L^q(\mathbb{R}^n)$ and the following inequality holds:

$$S_n \| f \|_{L^q(\mathbb{R}^n)}^2 \leq \| \nabla f \|_{L^2(\mathbb{R}^n)}^2,$$

(C.2)

where $S_n := \frac{n(n-1)}{4} \pi^{\frac{1}{2}} \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)^{\frac{1}{n}}$ is the optimal constant for (C.2).

Theorem C.4 (Gagliardo-Nirenberg-Sobolev inequality) Assume $q, r$ satisfy $1 \leq q, r \leq \infty$ and $j, m \in \mathbb{Z}^+$ satisfy $0 \leq j < m$. For any $f(x) \in C_0^\infty(\mathbb{R}^n)$, then we have

$$\| D^j f(x) \|_{L^p(\mathbb{R}^n)} \leq C \| D^m f(x) \|_{L^r(\mathbb{R}^n)} \| u \|_{L^q(\mathbb{R}^n)}^{1-\alpha},$$

(C.3)

where $\frac{1}{p} - \frac{j}{n} = \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1 - \alpha)\frac{1}{q}$, $\frac{j}{m} \leq \alpha \leq 1$ and $C$ depends on $m, n, j, q, r, \alpha$. If $m - j - \frac{n}{r}$ is a nonnegative integer, then (C.3) is established for $\frac{j}{m} \leq \alpha < 1$.

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Declarations

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