Double Copy Relation for AdS

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We present a double copy relation in AdS, which relates tree-level four-point amplitudes of supergravity, super Yang-Mills and bi-adjoint scalars.

INTRODUCTION

Scattering amplitudes in flat space exhibit surprising properties which encode deep lessons for quantum field theories and gravity. While we believe many curved-spacetime generalizations exist, explicit realizations are far from obvious to find. Recently, there has been a lot of activity trying to extend two remarkable flat-space properties, color-kinematic duality [1] and the double copy relation [2], to the simplest curved background – AdS space [3][7][8]. The flat-space relations gauge theory and gravity amplitudes, and have numerous applications in modern amplitude research [9]. Since AdS/CFT maps AdS amplitudes to CFT correlators, generalizations to AdS are especially interesting. While color-kinematic duality has been observed for four points [5][7], AdS double copy so far has only worked for three-point functions [3][4]. In fact, it was not clear if the flat-space relation has to be drastically modified at higher points. In this paper, we present an AdS generalization which realizes the double copy construction in four-point amplitudes for the first time. We relate tree-level amplitudes in AdS$_5 \times S^5$ IIB supergravity, AdS$_5 \times S^3$ SYM, and nonsupersymmetric AdS$_5 \times S^1$ bi-adjoint scalars, in a simple way that mirrors the flat space relation. Moreover, our AdS relation works for all amplitudes in these theories, applying to massless and massive particles alike.

We will use the Mellin representation for CFT correlators [10][11]. AdS amplitudes become Mellin amplitudes and enjoy simple analytic structure resembling the flat-space one. Tree-level Mellin amplitudes of AdS supergravity and super gauge theories in various spacetime dimensions were systematically studied in [7][12][14], and a Mellin color-kinematic relation similar to the flat-space one was pointed out in [7]. Unfortunately, applying the flat-space double copy prescription led to no sensible amplitudes. In this paper, we revisit these results. We will focus on AdS$_5$ and take advantage of supersymmetry, which allows us to reduce the Mellin amplitudes to simpler reduced Mellin amplitudes. We find that it is in these reduced objects that color-kinematic duality and double copy relation are naturally realized.

Schematically, we will write the reduced amplitude of AdS$_5$ super gluons with $N = 2$ superconformal symmetry as a finite sum labelled by integers $i, j$

$$\tilde{\mathcal{M}} \sim \sum_{i,j} \frac{n_s^{i,j} c_s}{s - s_{i,j}} + \frac{n_t^{i,j} c_t}{t - t_{i,j}} + \frac{n_u^{i,j} c_u}{u - u_{i,j}},$$

where the number of terms is determined by the external masses. $c_{s,t,u}$ are standard color factors satisfying $c_s + c_t + c_u = 0$. The kinematic factors $n_{s,t,u}^{i,j}$ turn out to obey the same relation $n_s^{i,j} + n_t^{i,j} + n_u^{i,j} = 0$, giving rise to an AdS color-kinematic duality. Replacing $c_{s,t,u}$ with $n_{s,t,u}^{i,j}$, we recover precisely super graviton reduced amplitudes of AdS$_5 \times S^5$ IIB supergravity [12][13]. On the other hand, replacing $n_{s,t,u}^{i,j}$ by $c_{s,t,u}$ leads to Mellin amplitudes of conformally coupled bi-adjoint scalars on AdS$_5 \times S^1$, which were not studied in the literature. We will prove it by direct calculation. The AdS$_5$ double copy relation presented here relates theories with varying $N = 0, 2, 4$ superconformal symmetry. However, it also implies that purely bosonic theories of Einstein gravity, YM, and bi-adjoint scalars on AdS$_5$ should be related by double copy, as we will briefly discuss at the end.

FOUR-POINT CORRELATORS

No supersymmetry. Let us start with the non-supersymmetric case. We consider the correlator of four scalar operators $O_k$, with conformal dimensions $k_i$ [20]

$$G_{k_1 k_2 k_3 k_4} = \langle O_{k_1} O_{k_2} O_{k_3} O_{k_4} \rangle .$$

(1)

In Mellin space correlators are represented as [10][11]

$$G_{k_1 k_2 k_3 k_4} = \int_{-\infty}^{\infty} [dsdt] \mathcal{K}(x_{ij}^2; s, t, u) \mathcal{M}_{k_1 k_2 k_3 k_4} \Gamma_{\{k_i\}}(s, t, u) .$$

(2)

where $[dsdt] = \frac{dsdt}{(2\pi)^2}$, and $\mathcal{K}(x_{ij}^2; s, t, u)$ is a factor containing all spacetime dependence

$$\mathcal{K}(x_{ij}^2; s, t, u) = (\frac{x_{12}^2}{x_{12}^2 - x_{24}^2} - \frac{x_{12}^2}{x_{12}^2 - x_{14}^2} - \frac{x_{12}^2}{x_{12}^2 - x_{24}^2} - \frac{x_{12}^2}{x_{12}^2 - x_{23}^2} - \frac{x_{12}^2}{x_{12}^2 - x_{13}^2} ) .$$

Here $x_{ij} = x_i - x_j$, and $s, t, u$ are Mandelstam variables satisfying $s + t + u = \sum_{i=1}^4 k_i = \Sigma$ [21]. We have also extracted a factor of Gamma functions

$$\Gamma_{\{k_i\}}(s, t, u) = \Gamma[k_i + k_j - \Sigma/2] \Gamma[k_i + k_j - s] \Gamma[k_i + k_j - t] \Gamma[k_i + k_j - t] \Gamma[k_i + k_j - u] .$$

(3)

which captures the contribution of double-trace operators universally present in the holographic limit [11]. All dynamic information is contained in $\mathcal{M}_{k_1 k_2 k_3 k_4}$, known as the Mellin amplitude. The four-point function $G_{k_1 k_2 k_3 k_4}$
obeys Bose symmetry which permutes operators. Bose symmetry acts on the Mellin amplitude by interchanging $k_i$, as well as permuting the Mandelstam variables $s, t, u$ in the same way it acts on a flat-space amplitude.

$\mathcal{N} = 2$ superconformal symmetry. We now consider CFTs with $\mathcal{N} = 2$ superconformal symmetry, focusing on the 1/2-BPS operators. These operators are of the form $\mathcal{O}^{a_1 \ldots a_k}_{k}$, where $a_i = 1, 2$ are indices of the R-symmetry group $SU(2)_R$ [22]. The operator $\mathcal{O}^{a_1 \ldots a_k}_{k}$ transforms in the spin $j_R = \frac{k}{2}$ representation of $SU(2)_R$, and has conformal dimensions $k = 2, 3, \ldots$. To conveniently keep track of the $SU(2)_R$ indices, we contract them with auxiliary two-component spinors $\psi^a$.

\[ \mathcal{O}_k(x, v) = \mathcal{O}^{a_1 \ldots a_k}_{k} \psi^{a_1} \ldots \psi^{a_k} \epsilon_{a_1 b_1} \ldots \epsilon_{a_k b_k} . \] (4)

We then consider their four-point functions (1), and define the Mellin amplitude via the reduced correlator

\[ \mathcal{M}^{\mathcal{N} = 2}_{k_1 k_2 k_3 k_4} = \mathbb{R}^{(2)} \circ \tilde{\mathcal{M}}^{\mathcal{N} = 2}_{k_1 k_2 k_3 k_4} . \] (9)

The factor $\mathbb{R}^{(2)}$ now becomes a difference operator $\mathbb{R}^{(2)}$ [7]. To obtain it, we interpret each monomial $U^m V^n$ in

\[ \mathbb{R}^{(2)} = \frac{(v_1 \cdot v_2)^2 (v_3 \cdot v_4)^2 x_{13}^2 x_{24}^2}{(v_1^2 + v_2^2) (v_3^2 + v_4^2) (x_{13}^2 + x_{24}^2)} \] (10)
as a difference operator $U^m V^n \rightarrow \mathbb{O}^{(2)} m, n$ which acts on functions $f(s, t)$ according to

\[ \mathbb{O}^{(N_m, n)} f(s, t) = \Gamma(s, t) \frac{\Gamma_k(1 - s - t + 2m + 2n)}{\Gamma_k(1 - s - t)} \times f(s - 2m, t - 2n) \] (11)

$\mathcal{N} = 4$ superconformal symmetry. The kinematics of $\mathcal{N} = 4$ is similar. The 1/2-BPS operator, labelled by an integer $k = 2, 3, \ldots$, transforms in the rank-k symmetric traceless representation of the $SO(6)_R$ R-symmetry group, and has dimension $k$. We keep track of the R-symmetry indices by using null $SO(6)$ vectors $\tilde{t}^a$ [24].

\[ \mathcal{O}_k(x, t) = \mathcal{O}^{r_1 \ldots r_k}(x) t^{r_1} \ldots t^{r_k} , \quad r_i = 1, \ldots, 6 \] (12)

where $t \cdot t = 0$. The $\mathcal{N} = 4$ superconformal symmetry dictates that the four-point function is of the form [22, 23]

\[ \mathcal{G}^{\mathcal{N} = 4}_{k_1 k_2 k_3 k_4} = \mathcal{G}^{\mathcal{N} = 4}_{0, k_1 k_2 k_3 k_4} + \mathbb{R}^{(4)} \mathcal{H}^{\mathcal{N} = 4}_{k_1 k_2 k_3 k_4} \] (13)

where $\mathcal{G}^{\mathcal{N} = 4}_{0, k_1 k_2 k_3 k_4}$ is the protected part, and $\mathcal{H}^{\mathcal{N} = 4}_{k_1 k_2 k_3 k_4}$ is the reduced correlator. Note that the reduced correlator also has shifted quantum numbers, with dimensions $k_1 + 2$ and $SO(6)$ spin $k_1 - 2$ for each operator. The factor $\mathbb{R}^{(4)}$ is determined by supersymmetry

\[ \mathbb{R}^{(4)} = \left( \frac{t_{12} t_{23} t_{34}^2}{t_{12} t_{23} t_{34}} \right)^2 (1 - z)(1 - \tilde{z})(1 - z\tilde{z}) \right) , \] (14)

doubles the $\mathcal{N} = 2$ factor [6]. Here $t_{ij} = t_i \cdot t_j$, and

\[ \alpha \tilde{\alpha} = \frac{t_{13} t_{24}}{t_{12} t_{34}} = \sigma , \quad (1 - \alpha)(1 - \tilde{\alpha}) = \frac{t_{14} t_{23}}{t_{12} t_{34}} = \tau . \] (15)

The full correlator $\mathcal{G}^{\mathcal{N} = 4}_{k_1 k_2 k_3 k_4}$ gives rise to the full amplitude $\mathcal{M}^{\mathcal{N} = 4}_{k_1 k_2 k_3 k_4}$ via [2]. The $\mathcal{N} = 4$ reduced amplitude is similarly given by

\[ \mathbb{H}^{\mathcal{N} = 4}_{k_1 k_2 k_3 k_4} = \int_{-\infty}^{\infty} [ds dt] K(x_{ij}^2; s, t, \tilde{u}) \tilde{\mathcal{M}}^{\mathcal{N} = 4}_{k_1 k_2 k_3 k_4} \Gamma(k_i)(s, t, \tilde{u}) . \]

But note here that the shift in $\tilde{u}$ is by 4, i.e., $\tilde{u} = u - 4$. The greater shift is due to the higher conformal weights of $\mathbb{R}^{(4)}$. Bose symmetry again permutes $s, t, u$ in the tree-level regime the protected part $\mathbb{G}^{\mathcal{N} = 2}_{0, k_1 k_2 k_3 k_4}$ does not contribute to the Mellin amplitude [2]. Rather it is generated by a contour pinching mechanism described in [13]. Therefore full amplitudes are completely determined by reduced amplitudes, with the precise relation given by translating both sides of (5) into Mellin space.
\( \mathcal{M}_{k_1 k_2 k_3}^{N=4} \), and \( s, t, u \) in \( \widehat{\mathcal{M}}_{k_1 k_2 k_3}^{N=4} \). At AdS tree level, the protected part again does not contribute to the Mellin amplitude \( \mathcal{M}_{k_1 k_2 k_3}^{N=4} \). Therefore the full amplitudes are determined by the reduced amplitudes via

\[
\mathcal{M}_{k_1 k_2 k_3}^{N=4} = \mathbb{R}^{(4)} \circ \widehat{\mathcal{M}}_{k_1 k_2 k_3}^{N=4}
\]

where we have promoted \( \mathbb{R}^{(4)} \) into a difference operator \( R^{(4)} \) \( \mathbb{R}^{(4)} = R^{(4)}(t_{34})^{2}t_{13}^{2}t_{24}^{2} \). The action of each monomial \( u^n v^n \) in \( R^{(4)} \) is given by \( (1) \) with \( N = 4 \).

SUPER GLUON AMPLITUDES

We are now ready to discuss holographic correlators in specific theories. We start with tensors in AdS5 preserving \( N = 2 \) superconformal symmetry, which can be realized as D3 branes probing F theory singularities \( [20] [27] \), or as \( N = 4 \) SYM with probe flavor D7 branes \( [25] \). In both cases, there is an AdS5 \( \times S^5 \) subspace in the holographic description, on which live localized degrees of freedom transforming in the adjoint representation of a color group \( G_F \). These degrees of freedom form a vector multiplet, and its Kaluza-Klein reduction gives infinite towers of \( \frac{1}{2} \)-BPS superconformal multiplets. We refer to the \( \frac{1}{2} \)-BPS superprimaries as super gluons. At large central charge, gravity decouples and one has only a spin-1 gauge theory. Note \( S^3 \) has isometry \( SO(4) = SU(2)_R \times SU(2)_L \). The first factor is identified with the \( N = 2 \) R-symmetry group, while the second \( SU(2)_L \) is a global symmetry suppressed in the above discussion. The operator \( \mathcal{O}_{a} \) has spin \( \frac{k_2}{2} \) under \( SU(2)_L \). We can similarly contract the indices with \( k - 2 \) \( SU(2)_L \) spinors \( \tilde{v}^a, \tilde{u} = 1, 2 \). In reduced correlators, \( v \) and \( \bar{v} \) further recombine into null vectors of \( \mathbb{SO}(4) \) via Pauli matrices, and appear only as polynomials of \( t_{ij} \).

\[
t' = \sigma_{i a} v^a \bar{v}^\alpha, \quad r' = 1, \ldots, 4, \quad t \cdot t = 0. \tag{17}
\]

To write down the super gluons amplitudes, let us choose, without loss of generality, the ordering \( k_1 \leq k_2 \leq k_3 \leq k_4 \), and distinguish two cases

\[
k_1 + k_4 \geq k_2 + k_3 \quad (\text{case I}) \quad k_1 + k_4 < k_2 + k_3 \quad (\text{case II}).
\]

To measure the deviation from the equal weight case \( k_i = \Sigma \), it is useful to introduce the following parameters

\[
\kappa_s = |k_3 + k_4 - k_1 - k_2|, \quad \kappa_t = |k_1 + k_4 - k_2 - k_3|, \quad \kappa_u = |k_1 + k_3 - k_2 - k_4|.
\]

The reduced Mellin amplitudes are given by \( \mathcal{M}_{k_1 k_2 k_3}^{N=2} \), and

\[
\sum_{\substack{i+j+k = \Sigma-2 \to 1 \leq i,j,k \leq \Sigma-2 \to \Sigma-2}} \frac{i! j! k!}{(2 + \kappa_a)(2 + \kappa_t)(2 + \kappa_u)!} \times \left[ \frac{n_{t_i}^{ij}}{s - s_M + 2k} + \frac{n_{t_i}^{ij}}{t - t_M + 2j} + \frac{n_{t_i}^{ij}}{u - u_M + 2l} \right] \times \mathcal{I}(t_{ab})
\]

which has been rewritten to manifest Bose symmetry. Let us unpack this expression a bit. Here

\[
\mathcal{E} = \frac{k_1 + k_2 + k_3 - k_4}{2} \quad (\text{case I}), \quad \mathcal{E} = k_1 \quad (\text{case II})
\]

is the extremality, which determines the complexity of the amplitude. After extracting a factor in \( t_{ab} \)

\[
\mathcal{I}(t_{ab}) = t_{34}^{\mathcal{E}} t_{24}^{\mathcal{E}} (t_{12} t_{34})^{-\mathcal{E}+2} \times \left( \frac{t_{23}^\mathcal{E}}{t_{23}^{\mathcal{E}}} \right) \quad (\text{case I}), \quad \mathcal{I}(t_{ab}) = t_{23}^{\mathcal{E}} (t_{12} t_{34})^{-\mathcal{E}+2} \times \left( \frac{t_{23}^\mathcal{E}}{t_{23}^{\mathcal{E}}} \right) \quad (\text{case II}),
\]

the reduced Mellin amplitudes are degree-(\( \mathcal{E} - 2 \)) polynomials in \( \sigma \) and \( \tau \) defined in \( \mathcal{M}_{k_1 k_2 k_3}^{N=2} \). The color dependence is captured by the color factors

\[
c_s = f^{t_1 t_2} f^{t_3 t_4}, \quad c_t = f^{t_1 t_4} f^{j_1 j_2}, \quad c_u = f^{t_1 t_2} f^{j_1 j_2}.
\]

where \( f^{j_1 j_2} \) are the structure constants of the color group \( G_F \). Thanks to the Jacobi identity, they satisfy \( c_s + c_t + c_u = 0 \). The kinematic factors \( n_{s,t,u}^{ij} \) are given by

\[
\begin{align*}
n_{s}^{ij} &= \frac{1}{t - t_M + 2j} \left( \frac{1}{s - s_M + 2k} \right) \left( \frac{1}{u - u_M + 2l} \right), \\
n_{t}^{ij} &= \frac{1}{\tilde{u} - u_M + 2l} \left( \frac{1}{s - s_M + 2k} \right) \left( \frac{1}{t - t_M + 2j} \right), \\
n_{u}^{ij} &= \frac{1}{s - s_M + 2k} \left( \frac{1}{\tilde{u} - u_M + 2l} \right) \left( \frac{1}{t - t_M + 2j} \right).
\end{align*}
\]

The non-locality of these expressions is only superficial, and should not raise any alarm. In fact, a similar phenomenon occurs in flat space \( [20] \). Evidently, \( n_{s,t,u}^{ij} \) obey

\[
n_{s}^{ij} + n_{t}^{ij} + n_{u}^{ij} = 0, \tag{21}
\]

which gives rise to a realization of the color-kinematic duality \( [2] \) in AdS. In contrast to the duality pointed out in \( [7] \), this new realization has the same form for both massless \( (k_i = 2) \) and massive \( (k_i > 2) \) super gluons. Finally, the remaining parameters are given by

\[
s_M = \min\{k_1 + k_2, k_3 + k_4\} - 2, \quad t_M = \min\{k_1 + k_4, k_2 + k_3\} - 2, \quad u_M = \min\{k_1 + k_3, k_2 + k_4\} - 2.
\]

SUPER GRAVITON AMPLITUDES

Let us now take a further step with the color-kinematic duality \( [21] \), and replace color factors \( c_{s,t,u} \) in each monomial \( \sigma^\mathcal{E} \tau^\mathcal{E} \) by kinematic factors \( n_{s,t,u}^{ij} \). The result is

\[
\widehat{\mathcal{M}}_{k_1 k_2 k_3 k_4}^{N=2} = \sum_{\substack{i+j+k = \Sigma-2 \to 1 \leq i,j,k \leq \Sigma-2 \to \Sigma-2}} \frac{i! j! k!}{(2 + \kappa_a)(2 + \kappa_t)(2 + \kappa_u)!} \times \left[ \frac{n_{t_i}^{ij}}{s - s_M + 2k} + \frac{n_{t_i}^{ij}}{t - t_M + 2j} + \frac{n_{t_i}^{ij}}{u - u_M + 2l} \right] \times \mathcal{I}(t_{ab})
\]

\[
\times \frac{-9 \mathcal{I}(t_{ab})}{(s - s_M + 2k)(t - t_M + 2j)(u - u_M + 2l)}.
\]
To interpret it as $N = 4$ reduced amplitudes, we need to replace the $\tilde{u}$ variable with the $N = 4$ one, as required by Bose symmetry of $M_{k_1k_2k_3k_4}^{N=4}$. Furthermore, we replace the $SO(4)$ vectors $t'$ by $SO(6)$ null vectors \cite{31}. Remarkably, it gives all the super graviton reduced Mellin amplitudes of IIB supergravity on $\mathrm{AdS}_5 \times S^5$ \cite{12,13}

$$M_{k_1k_2k_3k_4}^{N=4} = \sqrt{k_1k_2k_3k_4} \times \tilde{M}_{k_1k_2k_3k_4}^{N=2 \otimes N=2},$$  

(23) up to an overall factor \cite{32}. This generalizes the double copy relation \cite{2} into AdS space for four-point functions \cite{33}. In fact, redefining the supergravitons by $O_k \rightarrow O_k/\sqrt{k}$ gets rid of the normalization factor, and gives the super graviiton three-point functions also as the square of the super gluon ones \cite{34}.

**BI-ADJOINT SCALAR AMPLITUDES**

In flat space one can also replace kinematic factors by color factors, and obtains amplitudes of bi-adjoint scalars. We show that the same happens in AdS, and it serves as a nontrivial check. Note that in the above example the superconformal factor $R^{(2)}$ was doubled to $R^{(4)}$ (c.f., \cite{6} and \cite{14}). Going in the opposite direction, we expect $R^{(0)} = 1$, i.e., the resulting theory has no supersymmetry. Moreover, since the internal spaces changed from $S^3$ to $S^5$, a reasonable guess is that this sequence starts with $S^4$, which will soon be confirmed. The symmetry groups are therefore $SO(N+2)$, and we recall that operators in the reduced amplitudes transform in the rank-$(k_i - 2)$ symmetric traceless representation.

Note that for $N = 0$, the null polarization vectors are two-component. Since we can rescale the null vectors, we are left with two inequivalent choices

$$t_{\pm} = \frac{1}{\sqrt{2}}(1, \pm 1).$$  

(24) The dimension $k$ operator $O_k^\pm \equiv O_k(x, t_{\pm})$ has $\pm(k-2)$ charges under $U(1) = SO(2)$, depending on the polarization chosen. Moreover, we assume the scalar interactions are only cubic. Then $U(1)$ charge conservation dictates that at least one of the $\kappa_s$, $\kappa_t$, $\kappa_u$ parameters in \cite{18} is zero. For the chosen ordering $k_1 \leq k_2 \leq k_3 \leq k_4$, we must impose the condition $k_t = 0$. This leaves

$$\langle O_{k_1}^+ O_{k_2}^- O_{k_3}^- O_{k_4}^- \rangle,$$

(25) which have identical amplitudes \cite{33}. Noting

$$\sigma = 1, \quad \tau = 0, \quad I(t_{ab}) = 1,$$  

(26) and replacing $n_{s,t,u}^{ij}$ with the color factors $c_{s,t,u}^{i}$ for another color group $G'_P$, we find

$$M_{k_1k_2k_3k_4}^{N=0} = \sum_{i+j+k+l-2} \frac{-2N_{k_1k_2k_3k_4}}{k!} \frac{(2i+\kappa_s)}{\sqrt{2}} \frac{(2j+\kappa_t)}{\sqrt{2}} \frac{(2k+\kappa_u)}{\sqrt{2}}$$

$$\times \left[ \begin{array}{c} c_s c'_s \\ c_t c'_t \\ c_u c'_u \end{array} \right] \left( \begin{array}{c} s - s_M + 2k \\ t - t_M \\ u - u_M + 2i \end{array} \right).$$  

(27) We dropped the tildes because non-supersymmetric theories have only full amplitudes \cite{2}, and there is no shift in the $\vec{u}$ variable. We also included a to-be-determined $k_i$-dependent normalization factor $-2N_{k_1k_2k_3k_4}$ as in the supergravity case. Remarkably, \cite{27} can be rewritten as the sum of three AdS scalar exchange diagrams

$$N_{k_1k_2k_3k_4} \left( \frac{c_s c'_s}{p_s - 1} S^{(s)} + \frac{c_t c'_t}{p_t - 1} S^{(t)} + \frac{c_u c'_u}{p_u - 1} S^{(u)} \right)$$

(28) where $S_p^{(s)}$ is the amplitude of exchanging a dimension-$p$ scalar in the s-channel, and similarly for the other two channels \cite{30}

$$S_p^{(s)} = \sum_{m=0}^{\infty} \frac{(-2)^m (2p-k_1-k_2)}{m!} \frac{(2p-k_1-k_3-k_4)}{m!}$$

$$\times \prod_{n=1}^{p} \frac{\Gamma[p]}{\Gamma[p-n+1]}.$$  

Moreover, the weights $p_s, p_t, p_u$ are precisely those selected by $U(1)$ charge conservation

$$p_s = k_2 - k_2 + 2, \quad p_t = k_1 + k_4 - 2, \quad p_u = k_3 - k_1 + 2.$$  

(29) Note that \cite{27} is equivalent to \cite{28} is highly nontrivial, and a priori does not need to happen. We can further fix the normalization $N_{k_1k_2k_3k_4}$ by noting $N_{k_1k_2k_3k_4}(p_1 - n) = 0$ etc, have the interpretation of products of three-point function coefficients $C_{k_1k_2p} C_{k_1k_3p_2}$. The solution, up to a $k_i$-independent overall factor, is

$$C_{k_1k_2k_3} = \frac{1}{V(k_1-1)(k_2-1)(k_3-1)}.$$  

(30) Finally, we confirm by direct calculation that the theory is conformally coupled scalars on $\mathrm{AdS}_5 \times S^5$. The conformal mass on this manifold is $M_{\text{conf}}^2 k_i = -4$ \cite{37}. Decomposing the scalar field $\phi$ into $S^1$ modes $\phi(z, \tau) = \sum \varphi_n(z) e^{i n \tau}$, we find each mode has mass $M_n^2 = n^2 - 4$. This translates into a conformal dimension $|n| + 2$, agreeing with our charge-dimension relation $n = \pm(k-2)$. We can further check three-point functions. A cubic vertex $\phi_3^\text{ad}$ in AdS space gives rise to infinitely many AdS cubic vertices $\sum \varphi_{n_1} \varphi_{n_2} \varphi_{n_3}$ where $\{n_i\}$ conserve the $U(1)$ charge. Using the result of \cite{38}, it is straightforward to show that three-point functions are precisely \cite{29}. Note that both $C_{k_1k_2k_3}$ and $N_{k_1k_2k_3k_4}$ can be set to one by redefining $O_k \rightarrow \sqrt{k} - 1 O_k$. Then the double copy relation also holds for three-point functions.

**DISCUSSIONS**

In this note we found an extension of the double copy relation in curved spacetimes which relates all tree-level four-point functions of $\mathrm{AdS}_5 \times S^5$ IIB supergravity, $\mathrm{AdS}_5 \times S^5$ SYM, and $\mathrm{AdS}_5 \times S^5$ bi-adjoint scalars. Although our result is supersymmetric, it has immediate
implications on bosonic Einstein gravity and YM theory in AdS$_5$ with no internal factor. Thanks to supersymmetry, four-graviton and four-gluon amplitudes can be obtained from the reduced correlators of $k_i = 2$ super gravitons and super gluons by action of differential operators 

[39]. At tree level these spinning correlators are identical to the ones in bosonic theories because the exchanged fields are the same [40]. Our result then indicates that the bosonic amplitudes should also be related by a double copy construction [41], of which the details we will leave to higher points, although more data of holographic correlators is needed [42]. While the focus here is AdS$_5$ amplitudes, double copy relations for other backgrounds are also worth exploring. In particular, the AdS$^7$ case [17] admits similar definitions of reduced amplitudes [43, 44]. Finally, it would be interesting to explore extensions at higher genus, where the relevant CFT techniques were developed in [45].

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[29] Here we have set the gauge coupling to a convenient value which does not affect the physics.

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\[ n_t = \frac{1}{3} (\frac{2}{3} - \frac{1}{3}), \quad n_u = \frac{1}{3} (\frac{2}{3} - \frac{1}{3}) \] are similarly non-local. However, this non-locality is merely an artifact of manifesting all the supersymmetry. We thank H. Johansson for comments on this.

[31] Note that the dimensionality of the SO group is invisible in scalar products \( t_{ab} \).

[32] The \( k_i \)-dependent normalization factor was fixed in \([49]\). We have also set the Newton constant to a convenient value.

[33] In flat space the four-point superamplitude of \( \mathcal{N} = 8 \) supergravity has the form \( \delta^8(Q)\delta^8(\tilde{Q})(-stu) \), where the doubled supercharge delta functions correspond to \( R^{(4)} \). Note the supergravity amplitude is also related to the flat space SYM amplitude in footnote \([30]\) by the double copy relation \( c_{s,t,u} \rightarrow n_{s,t,u} \).

[34] The new normalization makes the three-point function coefficients of super gravitons \([50]\) independent of \( k_i \), i.e., \( C_{k_1 \ldots k_3} \sim 1 \). The super gluons have exactly the same three-point functions \([7]\).

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