ON THE EXISTENCE OF THREE-DIMENSIONAL HYDROSTATIC AND MAGNETOSTATIC EQUILIBRIA OF SELF-GRAVITATING FLUID BODIES

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ABSTRACT

We develop an analytical spectral method to solve the equations of equilibrium for a self-gravitating, magnetized fluid body, under the only hypotheses that (a) the equation of state is isothermal, (b) the configuration is scale-free, and (c) the body is electrically neutral. All physical variables are represented as series of scalar and vector spherical harmonics of degree \(l\) and order \(m\), and the equilibrium equations are reduced to a set of coupled quadratic algebraic equations for the expansion coefficients of the density and the magnetic vector potential. The method is general, and allows to recover previously known hydrostatic and magnetostatic solutions possessing axial symmetry. A linear perturbation analysis of the equations in spectral form show that these basic axisymmetric states, considered as a continuous sequence with the relative amount of magnetic support as control parameter, have in general no neighboring nonaxisymmetric equilibria. This result lends credence to a conjecture originally made by H. Grad and extends early results obtained by E. Parker to the case of self-gravitating magnetized bodies. The only allowed bifurcations of this sequence of axisymmetric equilibria are represented by distortions with dipole-like angular dependence \((l = 1)\) that can be continued into the nonlinear regime. These new configurations are either (i) azimuthally asymmetric \((m = \pm 1)\) or (ii) azimuthally symmetric but without reflection symmetry with respect to the equatorial plane \((m = 0)\). It is likely that these configurations are not physically acceptable solutions for isolated systems, but represent instead the manifestation of a general gauge freedom of self-similar isothermal systems. To the extent that interstellar clouds can be represented as isolated magnetostatic equilibria, the results of this study suggest that the observed triaxial shapes of molecular cloud cores can be interpreted in terms of weakly damped Alfvén oscillations about an equilibrium state.

Subject headings: Hydrodynamics, Magnetohydrodynamics, Molecular Clouds
1. Introduction

Much of what we know of magnetostatic (MS) equilibria in plasma physics and astrophysics has come from studying systems with one ignorable coordinate: axisymmetric systems are one example. The symmetry associated to with one ignorable coordinate conveniently allows MS problems to be reduced to neat tractable forms. In the three-dimensional case, no one has ever been able to find nonaxisymmetric solutions of the MS equations for a self-gravitating fluid, and their actual existence has remained dubious (see Sect. 1.2). The goal of this paper is to explore the the existence of configurations of equilibrium of self-gravitating fluid bodies in the presence of a large-scale magnetic field without symmetry restrictions, with an application in mind to the study of the densest parts of molecular clouds (the so-called cloud cores), the sites of star formation.

The motivation for this work is the following. The dense cores of interstellar molecular clouds are observed to be close to a state of virial equilibrium (balance of gravitational, thermal and magnetic energy (see e.g. Myers 1999), yet their shapes are distinctly non-spherical: the best fit to the projected axial ratio distribution is obtained with triaxial ellipsoids (Jones & Basu 2002; Goodwin, Ward-Thompson, & Whitworth 2003). The question is then to consider the possible role of the anisotropic forces associated to the magnetic fields present in these cores (see e.g. Crutcher 2001) to allow the existence of shapes of non-trivial topology.

The paper is organized as follows: in Sect. 2 we summarize the available results on the equilibrium of ideal (i.e. non viscous and non resistive) plasmas, discussing in particular the case of non-selfgravitating configurations; in Sect. 3 we formulate mathematically the problem and introduce non-dimensional self-similar variables; in Sect. 4 we describe the method of solution, based on the expansion of the angular part of the variables in scalar and vector spherical harmonics; in Sect. 5 and 6 we show that our method of solution recovers previously known axially symmetric hydro- and magnetostatic equilibria, respectively; in Sect. 7 we linearise the equations around these basic equilibria, and look for neutral modes (neighbouring equilibria) possessing a different symmetry than the basic states; finally, in Sect. 8 we summarize our results and discuss their implications for the observed shapes of molecular cloud cores.

2. Ideal magnetostatic equilibria

Consider a magnetized, isothermal, self-gravitating fluid body satisfying the ideal MS equations

\[ c_s^2 \nabla \rho + \rho \nabla V = \frac{1}{4\pi} (\nabla \times B) \times B, \]

\[ \nabla \cdot B = 0, \]

\[ \nabla^2 V = 4\pi G \rho, \]

where \( B \) is the magnetic field, \( \rho \) is the density, \( V \) is the gravitational potential, \( G \) is the constant of gravitation and \( c_s \) is the sound speed. With applications to interstellar molecular clouds in mind,
we have assumed here an isothermal equation of state. All known solutions of the set of equations (1)–(3) are characterized by a coordinate symmetry that reduces the number of variables from three to two or one. The symmetry associated with an ignorable coordinate allows MS problems to be reduced to a second-order elliptic partial differential equation (Dungey 1953), conventionally called the \textit{Grad-Shafranov equilibrium equation}. These symmetric equilibria may be grouped into (i) \textit{axisymmetric} ($\partial/\partial \varphi = 0$), (ii) \textit{cylindrically symmetric} ($\partial/\partial z = 0$), and (iii) \textit{helically symmetric} systems ($\partial/\partial \varphi = k \partial/\partial z$). As shown by Edenstrasser (1980), helical symmetry represents the most general admissible invariance property of the MS equations, with rotational and translational invariance as limiting cases.

Applications of eq. (1)–(3) to the study of interstellar molecular clouds have largely focused on axisymmetric magnetic configurations (e.g. Mouschovias 1976; Nakano 1979; Mestel & Ray 1985; Tomisaka, Ikeuchi, & Nakamura 1988; Barker & Mestel 1990). Cylindrically symmetric equilibria have also been extensively studied, originally in connection with the stability of galactic spiral arms (Chandrasekhar & Fermi 1953; Stodólkiewicz 1963), and later as models of filamentary clouds (Nagasawa 1987; Nakamura, Hanawa, & Nakano 1995). In the latter context, magnetic configurations possessing helical symmetry have also been explored (Nakamura, Hanawa, & Nakano 1993, Fiege & Pudritz 2000a, b). An excellent introduction to the subject can be found in the monography by Mestel (1999).

### 2.1. Parker’s theorem

For non-selfgravitating plasmas, the fundamental question as to the existence of MS equilibria of a more general symmetry, or with no symmetry at all (3-D MS equilibria), has been formulated several times (e.g. Low 1980, Degtyarev et al. 1985) but never properly answered. Grad (1967) conjectured that, with $V = 0$, only “highly symmetric” solutions of the system (1)–(2) should be expected, in order to balance the highly anisotropic Lorentz force with with pressure gradients and gravity, which are forces involving scalar potentials. In a fundamental paper, Parker (1972) proved rigorously the non existence of 3-D MS equilibria that are small perturbations of 2-D equilibria having translational symmetry. This result is conventionally referred to as \textit{Parker’s theorem}.

On the basis of this result, Parker (1979) argued that realistic magnetic fields with no well defined symmetries must evolve in a genuinely time-dependent way, until all non-symmetric components of the field are destroyed by dissipation and reconnection and the topology becomes symmetric, a process known as \textit{topological non-equilibrium} of configurations lacking high degrees of symmetry (see also Tsinganos, Distler, & Rosner 1984). The numerical simulations of Vainshtein et al. (2000) provide a striking illustration of this process. The problem with interstellar clouds is that ohmic dissipation times are of the order of $\sim 10^{15}$ yr, and therefore the kind of monotonic relaxation to equilibrium envisaged by Parker (1979) can be ruled out. For a non-dissipative plasma, Moffatt (1985, 1986) has shown that \textit{stable} non-axisymmetric magnetostatic equilibria of non-trivial topology do exist, but may require the presence of tangential field discontinuities (current sheets).
The mathematical method adopted in this paper, based on analytical functions and regular perturbation expansions, cannot address the question of the existence of Moffatt-type equilibria in self-gravitating ideal plasmas. However, given the general character of Moffatt’s conclusions, the question is considered again at the light of our results in Sect. 8.

Does the presence of fluid motions modify this picture? Tsinganos (1982) found that Parker’s theorem remains valid for steady dynamical ($v \neq 0$) configurations possessing translational invariance, and stressed the analogy between this result and the familiar Taylor-Proudman theorem of hydrodynamics. However, Galli et al. (2001) found that a class of two-dimensional axisymmetric MHD equilibria (rotating, magnetized, self-gravitating singular isothermal disks) does have neighboring non-axisymmetric equilibrium states, provided the rotation speed becomes sufficiently high (supermagnetosonic). Thus, not only the presence of fluid motions, but also the geometry of the system, seems to play a crucial role in breaking the symmetry of the equilibrium state.

As for the effect of gravity, Field (1982, quoted by Tsinganos et al. 1984) objected that the neglect of the constraining effect exerted by the plasma’s self-gravity may severely limit the domain of existence of MS equilibria. In a series of papers (Low 1985; Bogdan & Low 1986; Low 1991), Low and collaborators have elaborated a general method to solve the equations of MS in the presence of an external gravitational field, but the problem presents considerable mathematical difficulties. In agreement with Grad’s conjecture, Low concluded that applying some form of symmetry to the magnetic field is probably essential for the existence of equilibrium, in order to balance the highly anisotropic Lorentz force with pressure gradients and gravity, which are forces involving scalar potentials. In the special case of an imposed gravitational field, either uniform or due to a point mass, Low was able to find families of three-dimensional MS solutions. For a self-gravitating gas, Low (1991) showed that the problem can be reduced to the solution of two coupled partial differential equations for two unknown functions, but did not proceed further.

3. The equations of the problem

In the following we specialize to scale-free equilibria. Working in spherical coordinates ($r, \theta, \varphi$), we assume that every physical quantity of the problem can be factorized in one function (power-law or logarithmic) of $r$ times a function of $\theta$ and $\varphi$ only. This assumption allows a considerable simplification of the equations of the problem. We then show that the magnetic field must be poloidal in the Stratton-Chandrasekhar classification (Sect. 3.1), and we derive the governing set of non-dimensional equations (Sect. 3.2). Finally, we obtain the equations of the problem in the special case of axial symmetry (Sect. 3.3).

The assumption of a power-law (or logarithmic) dependence on radius for the variables of the problem can be justified on observational and theoretical grounds. The observed density profiles of cloud cores approach in general a $r^{-2}$ power-law behaviour in the outer parts, but are flatter near the cloud’s centre (see e.g. Ward-Thompson, Motte, & André 1999). Idealised models that
assume a power-law behaviour of the density (and the intensity of the magnetic field, etc.) over all radii are justified by our expectation that the evolution of the observed cloud cores tends to a singular configuration. The driving process responsible for this evolution has been identified in the ambipolar diffusion of a weakly ionised gas, as originally proposed by Mestel & Spitzer (1956). Several numerical calculations of cloud evolution driven by ambipolar diffusion show that the density profile steepens to $r^{-2}$ to arbitrary number of decades in radius as the singular state is approached (Fiedler & Mouschovias 1993; Ciolek & Mouschovias 1993, 1994; Basu & Mouschovias 1994, 1995). Thus, scale-free (singular) configurations provide realistic models for molecular cloud cores in the so-called “pivotal” state, i.e. on the verge of gravitational collapse. The price to pay for this simplifying assumption, is, of course, the introduction of an artificial singularity at the origin, where the density and the magnetic field diverge.

3.1. Toroidal and poloidal fields

Any solenoidal vector field $\mathbf{B}$ can be expressed as a linear combination of a certain basic toroidal ($\mathbf{B}_t$) and poloidal ($\mathbf{B}_p$) fields (Stratton 1941, Chandrasekhar 1961), given by

$$\mathbf{T} = \nabla \left( \frac{\Theta}{r} \right) \times \mathbf{r}, \quad (4)$$

and

$$\mathbf{S} = \nabla \times \left[ \nabla \left( \frac{\Psi}{r} \right) \times \mathbf{r} \right], \quad (5)$$

where $\Theta$ and $\Psi$ are arbitrary scalar functions of position. In addition to have zero divergence, the toroidal and poloidal fields $\mathbf{T}$ and $\mathbf{S}$ are characterized by vanishing radial component ($T_r = 0$) and vanishing radial component of the curl ($[\nabla \times \mathbf{S}]_r = 0$), respectively.\(^1\)

For a self-similar problem, if there is no radial current at one $r$, then the same is true for all $r$. We want it to be true at large $r$ because otherwise there would be a flow of charge to infinity, and the cloud would become electrically charged. Thus, in our problem, the curl of the magnetic field has zero radial component, and is therefore poloidal in the Stratton-Chandrasekhar terminology,

$$\mathbf{B} = \mathbf{S} = \nabla \times \left[ \nabla \left( \frac{\Psi}{r} \right) \times \mathbf{r} \right]. \quad (6)$$

In axial symmetry, we recover the condition $\mathbf{B} \cdot \nabla \Phi = 0$, where $\Phi \equiv -2\pi \sin \theta \partial \Phi / \partial \theta$ is the usual magnetic flux function.

\(^1\)Notice that this terminology is different from the one commonly adopted in astrophysics, where is customary to call poloidal a field with components only along $r$ and $\theta$, and toroidal a field with only a component along $\varphi$. 


3.2. Nondimensional self-similar variables

It is easy to see from eq. (1)–(3) and eq. (6) that if \( \rho, V \) and \( \Psi \) are chosen to have appropriate power-law dependences in \( r \) one can eliminate all \( r \)-dependences in the MS equations for an isothermal gas. Following Li & Shu (1996, hereafter LS96) we adopt the scaling

\[
\rho = \frac{c_s^2}{2\pi Gr^2} R(\theta, \varphi),
\]

\[
V = 2c_s^2 [(1 + H_0) \ln r + V(\theta, \varphi)],
\]

\[
\Psi = 2c_s^2 r \sqrt{G_F}(\theta, \varphi),
\]

where the quantity \( H_0 \) in the expression of the gravitational potential eq. (8) is a dimensionless constant to be specified. We also define the angular parts of the \( \nabla \) and \( \nabla^2 \) operators,

\[
\nabla_\Omega \equiv \hat{\theta} \frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{\sin \theta} \frac{\partial}{\partial \varphi}, \quad \nabla^2_\Omega \equiv \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.
\]

Inserting eq. (9) into eq. (6) we obtain the expression for the magnetic field

\[
B = \frac{2c_s^2}{\sqrt{Gr}} [A \hat{r} + \nabla_\Omega F],
\]

where we have defined

\[
A \equiv -\nabla^2_\Omega F.
\]

Finally, the Lorentz force is given by

\[
\frac{1}{4\pi} (\nabla \times B) \times B = \frac{c_s^4}{\pi Gr^3} [(\nabla_\Omega F \cdot \nabla_\Omega A) \hat{r} - A \nabla_\Omega A].
\]

Notice that a magnetic field with no radial component of its curl and proportional to \( r^{-1} \), as assumed here, cannot be force-free.

Inserting expressions (7)–(9) into eq. (1)–(3), we rewrite the condition of force balance in the radial direction as

\[
H_0 R = \nabla_\Omega F \cdot \nabla_\Omega A,
\]

and the condition of force balance in the tangential direction as

\[
\frac{1}{2} \nabla_\Omega R + R \nabla_\Omega V + A \nabla_\Omega A = 0.
\]

Finally, Poisson’s equation becomes

\[
\nabla^2_\Omega V = R - (1 + H_0).
\]

We now show that eq. (14)–(16) generalize to the set of equations derived by LS96 for axisymmetric isothermal scale-free equilibria.
3.3. The axisymmetric case

Assuming $\partial / \partial \varphi = 0$, and defining $\phi(\theta) \equiv -\sin \theta F'$, we obtain

$$A(\theta) = \frac{\phi'}{\sin \theta},$$

(17)

where a prime indicates derivation with respect to $\theta$. With this definition, eq. (14) reduces to eq. (13) of LS96,

$$\frac{d}{d\theta} \left( \frac{\phi'}{\sin \theta} \right) = -H_0 R \frac{\phi}{\phi} \sin \theta.$$

(18)

Poisson’s eq. (16) reads

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dV}{d\theta} \right) = R - 1 - H_0.$$

(19)

Eliminating $dV/d\theta$ using eq. (15), and simplifying the result with the help of eq. (18), we obtain eq. (12) of LS96,

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta \left( 2H_0 \frac{\phi'}{\phi} - \frac{R'}{R} \right) \right] = 2(R - 1 - H_0),$$

(20)

that completes the set of equations governing axisymmetric, scale-free, isothermal equilibria.

Pure hydrostatic equilibria are described by the single equation

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{R'}{R} \right) = 2(1 - R),$$

(21)

obtained by setting $\phi = 0$ and $H_0 = 0$ in eq. (20). Medvedev & Narayan (2000, hereafter MN00) have found an analytical solution of this equation,

$$R(\theta) = \frac{1 - e^2}{(1 \pm e \cos \theta)^2},$$

(22)

where iso-density contours (described by $r(\theta) \propto \sqrt{R(\theta)}$ according to eq. [7]) are confocal ellipsoids of eccentricity $e$, with $0 < e < 1$. For $e = 0$, this solution reduces trivially to the singular isothermal sphere, with $R = 1$. We will return to this family of equilibria in Sect. 5.

4. Method of solution

At first sight, the problem appears to be overconstrained, since there are four unknown functions ($R, V, A$ and $F$) and four equations (eq. [12], [14], [15] and [16]), one of which (eq. [15]) has two components. However, it is easy to see that the vectors $\nabla \Omega R$, $\nabla \Omega V$ and $\nabla \Omega A$ in eq. (15) are parallel, thus the condition of force balance in the tangential direction reduces to one constraint only. To see this, first take the curl of eq. (15), obtaining $\nabla \Omega R \times \nabla \Omega V = 0$, a condition implying that isodensity and equipotential surfaces are coincident. Then take the vector product of eq. (15) with $\nabla \Omega R$, obtaining $\nabla \Omega R \times \nabla \Omega A = 0$. It follows then that $\nabla \Omega V \times \nabla \Omega A = 0$ (CVD). The meaning
of the condition $\nabla_\Omega V \times \nabla_\Omega A = 0$ can be understood by writing the expression for the electric current $j$ from Ampère’s law using eq. (11),

$$j = \frac{c}{4\pi} \nabla \times B = -\frac{c^2\gamma}{2\pi\sqrt{Gr}} \hat{r} \times \nabla_\Omega A.$$  (23)

Eq. (23) shows that the electric current $j$ is perpendicular to $\nabla_\Omega A$, and therefore implies that the current flows over equipotential (or isodensity) surfaces.

The method adopted in this work is based on the expansion of the vector variables of the problem in vector spherical harmonics. Any vector quantity is represented by a convergent infinite sequence of complex coefficients, and the problem is formulated in the Hilbert space. The solution of the nonlinear partial differential equations of the problem is thus reduced to the solution of infinite set of nonlinear algebraic equations. The coupling coefficients are expressed in terms of Wigner $3j$ symbols, and are evaluated by the Racah’s formula (see Appendix A).

4.1. Multipole expansion

Vector spherical harmonics (see e.g. Arfken 1985) provide the natural basis for a multipole expansion of eq. (14)–(16). Here we follow Morse & Feshbach (1953) defining

$$P_{lm} = Y_{lm} \hat{r}, \quad B_{lm} = \frac{1}{\sqrt{l(l+1)}} \nabla_\Omega Y_{lm}, \quad C_{lm} = -\hat{r} \times B_{lm}.$$  (24)

As we will see, the properties of the product of vector spherical harmonics allow non-linear terms to be dealt with in a systematic way by Wigner $3j$ symbols (or Clebsch-Gordan coefficients).

Clearly, an infinite number of orthonormal bases can be generated from the $P_{lm}, B_{lm}, C_{lm}$ basis by the application of unitary transformation. The set (24) is especially convenient for situations where a preferred radial direction is present, like in our case, since the $P_{lm}$ are radial whereas the $B_{lm}$ and $C_{lm}$ are tangential to the unit sphere. Thus, if we expand $R, V, A$ and $F$ in spherical harmonics, and we eliminate $F$ and $V$ by using eq. (12) and (16), the remaining equations of the problems for $R$ and $A$ are naturally expressed in the basis (24), with $\nabla_\Omega A$ and $\nabla_\Omega R$ expanded in series of $B_{lm}$ and $j$ in series of $C_{lm}$. For the reasons explained in Sect. 4, we expect therefore the expansion along $C_{lm}$ of the equation of force balance (eq. [1]) to be trivially satisfied.

We expand the functions $R, V, A$ and $F$ in spherical harmonics,

$$R(\theta, \varphi) = 1 + H_0 + \sum_{lm} R_{lm} Y_{lm}(\theta, \varphi),$$  (25)

$$V(\theta, \varphi) = \sum_{lm} V_{lm} Y_{lm}(\theta, \varphi),$$  (26)

$$A(\theta, \varphi) = \sum_{lm} A_{lm} Y_{lm}(\theta, \varphi),$$  (27)
\[ F(\theta, \varphi) = \sum_{lm} F_{lm} Y_{lm}(\theta, \varphi), \quad (28) \]

where the sum is for \( l \geq 1 \) and \(-l \leq m \leq l\). In general, \( R_{lm}, V_{lm}, A_{lm} \) and \( F_{lm} \) are complex coefficients. Since \( R, V, A \) and \( F \) are real functions, we have to require that

\[ R_{l-m} = (-1)^m R^{*}_{lm}, \quad \text{etc.} \quad (29) \]

The constant factor \( 1 + H_0 \) in the expansion of \( R(\theta, \varphi) \) is chosen to simplify Poisson's equation (16). Notice that with this choice

\[ \oint R \, d\Omega = 4\pi(1 + H_0), \quad (30) \]

a condition equivalent to the “integral constraint” of LS96 (their eq. [18]). Therefore, as in LS96, \( H_0 \) measures the fractional increase in the mean density that arises because the magnetic field contributes to support the cloud against self-gravity.

It is straightforward to compute the spherical mass-to-flux ratio, or the ratio of the mass contained in a sphere centered on the origin to the magnetic flux through a circle of the same radius in the equatorial plane. The mass enclosed in a sphere of radius \( r \) is

\[ M = \frac{c^2 s}{2\pi G} \oint R \, d\Omega = \frac{2c^2 s}{G} (1 + H_0). \quad (31) \]

The magnetic flux through a circle of the radius \( r \) in the equatorial plane is equal by Gauss theorem to the magnetic flux through a semisphere of radius \( r \),

\[ \Phi = \frac{2c^2 s}{\sqrt{Gr}} \int \mathbf{B} \cdot \mathbf{\hat{r}} \, dS = \frac{2c^2 s r}{\sqrt{G}} \oint d\varphi \int_0^{\pi/2} A(\theta, \varphi) \sin \theta \, d\theta, \quad (32) \]

where we have used the expression of \( B \) given by eq. (11). With the expansion (27) we obtain

\[ \Phi = \frac{4\pi c^2 s}{\sqrt{G}} \sum_l A_{l0} \int_0^{\pi/2} Y_{l0} \sin \theta \, d\theta = \frac{\pi c^2 s r}{\sqrt{G}} \sum_l \phi_l A_{l0}, \quad (33) \]

where

\[ \phi_l = \frac{\sqrt{l+1}}{\Gamma(l - \frac{1}{2}) \Gamma\left(\frac{3}{2} + \frac{l}{2}\right)}. \quad (34) \]

Thus, in non-dimensional units, the spherical mass-to-flux ratio results

\[ \lambda_r \equiv 2\pi\sqrt{G} \frac{M}{\Phi} = \frac{4(1 + H_0)}{\sum_l \phi_l A_{l0}}. \quad (35) \]

Using known properties of the vector spherical harmonics (see e.g. Varshalovich, Moskalev, & Khersonskii 1988), vector quantities like \( \nabla_{\Omega} R, \nabla_{\Omega} V \), etc. are immediately expressed as

\[ \nabla_{\Omega} R = \sum_{lm} \sqrt{l(l+1)} R_{lm} \mathbf{B}_{lm}, \quad \nabla_{\Omega} V = \sum_{lm} \sqrt{l(l+1)} V_{lm} \mathbf{B}_{lm}, \quad \text{etc.} \quad (36) \]
In addition, the expansion in spherical harmonics presents the advantage that it makes possible to solve immediately Poisson’s equation (eq. [16]) and the relation between \( A \) and \( F \) (eq. [12]), as both equations involve the angular part of the Laplacian operator. Since

\[
\nabla^2 Y_{lm} = -l(l+1)Y_{lm},
\]

(37)
eq (eq. (12)) gives the relation between the coefficients \( F_{lm} \) and \( A_{lm} \),

\[
A_{lm} = l(l+1)F_{lm},
\]

(38)
whereas Poisson’s equation (eq. 16) gives the relation between the coefficients \( V_{lm} \) and \( R_{lm} \),

\[
R_{lm} = -l(l+1)V_{lm}.
\]

(39)
Inserting the expansions eq. (25)–(27) in eq. (11) and eq. (23), and eliminating the coefficients \( F_{lm} \) using eq. (38), we obtain the expansion in vector spherical harmonics of the magnetic field and the electric current,

\[
B = \frac{2c^2}{\sqrt{Gr}} \sum_{lm} A_{lm} \left[ P_{lm} + \frac{B_{lm}}{\sqrt{l(l+1)}} \right],
\]

(40)
and

\[
j = \frac{c^2}{2\pi \sqrt{Gr}} \sum_{lm} \sqrt{l(l+1)} A_{lm} C_{lm}.
\]

(41)
Using the vector relation

\[
\nabla \times C_{lm} = \frac{1}{r} \left[ \sqrt{l(l+1)} P_{lm} + B_{lm} \right],
\]

(42)
we immediately recognize in the expression (40) for \( B \) the curl of a vector potential \( A \) given by

\[
A = \frac{2c^2}{\sqrt{G}} \sum_{lm} \frac{A_{lm}}{\sqrt{l(l+1)}} C_{lm}.
\]

(43)

4.2. Spectral form of the equations

We first consider the radial component of the equation of force balance, eq. (14). In terms of the vector spherical harmonics this equation can be written

\[
\sqrt{4\pi} H_0 (1 + H_0) Y_{00} + H_0 \sum_{l'm'} R_{l'm'} Y_{l'm'} = \sum_{l'm'} \sum_{l''m''} \left[ \frac{l'(l'+1)}{l''(l''+1)} \right]^{1/2} A_{l'm'} A_{l''m''} B_{l'm'} \cdot B_{l''m''},
\]

(44)
where we have used eq. (38) to eliminate the coefficients \( F_{lm} \). Then, we multiply each term on both sides by \( Y^*_{lm} \), and integrate over solid angle \( d\Omega \), using the known orthonormality properties of spherical harmonics. It is convenient to introduce the coupling coefficients \( \alpha_{ll'm'm''} \), defined as

\[
\alpha_{ll'm'm''} \equiv \frac{1}{\sqrt{l(l+1)}} \int B_{l'm'} \cdot B_{l''m''} Y^*_{lm} \, d\Omega,
\]

(45)
involving the product of three spherical harmonics (or their derivatives). The expressions for these coupling coefficients are given in Appendix A. We obtain from eq. (44) the condition

\[ \sqrt{4\pi} H_0 (1 + H_0) \delta_{l,0} + H_0 R_{lm} = \sum_{l'm'} \sum_{l''m''} \left[ \frac{l'(l' + 1)}{l''(l'' + 1)} \right]^{1/2} \alpha_{l'l''m'm''}^{m'm'} A_{l'm'} A_{l''m''}. \]  

The value of \( \sqrt{l(l+1)} \alpha_{l'l''m'm''}^{m'm'} \) for \( l = 0 \) can be easily obtained from the formulae in Appendix A.

\[ \sqrt{l(l+1)} \alpha_{l'l''m'm''}^{m'm'} \bigg|_{l=0} = \frac{(-1)^{m'}}{4\pi} \delta_{l',l''} \delta_{m'-m''}, \]  

and using the relation (29) between coefficients of opposite \( m \), eq. (46) for \( l = 0 \) simplifies to

\[ \sum_{lm} |A_{lm}|^2 = 4\pi H_0 (1 + H_0), \]  

whereas, for \( l \geq 1 \), it reads

\[ \frac{H_0}{\sqrt{l(l+1)}} R_{lm} = \sum_{l'm'} \sum_{l''m''} \left[ \frac{l'(l' + 1)}{l''(l'' + 1)} \right]^{1/2} \alpha_{l'l''m'm''}^{m'm'} A_{l'm'} A_{l''m''}. \]  

We now expand the equation for the tangential force, eq. (15) along the two orthogonal sets of vectors \( \mathbf{B}_{lm} \) and \( \mathbf{C}_{lm} \). With the expansions given above, eq. (15) becomes

\[ \sum_{l'm'} l'(l' + 1) - 2(1 + H_0) \frac{R_{l'm'}}{2 \sqrt{l'(l' + 1)}} B_{l'm'} + \sum_{l'm'} \sum_{l''m''} \sqrt{l''(l'' + 1)} R_{l'm'} V_{l''m''} Y_{l'm'} B_{l''m''} \]

\[ + \sum_{l'm'} \sum_{l''m''} \sqrt{l''(l'' + 1)} A_{l'm'} A_{l''m''} Y_{l'm'} B_{l''m''} = 0. \]

We take the scalar product of this equation with \( \mathbf{B}_{lm}^\ast \) and integrate over solid angle \( d\Omega \), using the relations eq. (39) and (38) and defining the coupling coefficients \( \beta_{l'l''m'm''}^{m'm'} \) (see Appendix A),

\[ \beta_{l'l''m'm''}^{m'm'} = \frac{1}{\sqrt{l'(l' + 1)}} \int Y_{l'm'} \mathbf{B}_{l''m''} \cdot \mathbf{B}_{lm}^\ast \, d\Omega. \]  

The result is

\[ \frac{l(l+1) - 2(1 + H_0)}{2 \sqrt{l(l+1)}} R_{lm} - \sum_{l'm'} \sum_{l''m''} \left[ \frac{l'(l' + 1)}{l''(l'' + 1)} \right]^{1/2} \beta_{l'l''m'm''}^{m'm'} R_{l'm'} R_{l''m''} \]

\[ + \sum_{l'm'} \sum_{l''m''} [l'(l' + 1) l''(l'' + 1)]^{1/2} \beta_{l'l''m'm''}^{m'm'} A_{l'm'} A_{l''m''} = 0. \]  

We have noticed in Sect. 4 that the tangential component of the equation of force balance has no component in the direction of electric-current lines. Since we see from eq. (41) that \( \mathbf{j} \) is expressed...
as a series containing the harmonics \( C_{lm} \), we expect therefore each coefficient of the expansion of eq. (50) in terms of \( C_{lm} \) vectors to be zero. This is verified in Appendix B.

Summarizing, the equations of the problem are:

\[
\frac{H_0}{\sqrt{l(l+1)}} R_{lm} = \sum_{l' \nu} \sum_{l'' \nu''} \left[ \frac{l'(l'+1)}{l''(l''+1)} \right]^{1/2} \alpha_{l'\nu'}^{m'm''} A_{l'\nu'} A_{l''\nu''},
\]

from the condition of force balance in the radial direction;

\[
\frac{l(l+1) - 2(1 + H_0)}{2\sqrt{l(l+1)}} R_{lm} = -\sum_{l' \nu} \sum_{l'' \nu''} \left[ \frac{l'(l'+1)}{l''(l''+1)} \right]^{1/2} \beta_{l'\nu'}^{m'm''} A_{l'\nu'} A_{l''\nu''}
\]

\[+ \sum_{l' \nu} \sum_{l'' \nu''} \left[ \frac{l'(l'+1)}{l''(l''+1)} \right]^{1/2} \beta_{l'\nu''}^{m'm''} R_{l'\nu'} R_{l''\nu''}, \]

from the condition of force balance of force in the tangential direction; and

\[
\sum_{lm} |A_{lm}|^2 = 4\pi H_0(1 + H_0),
\]

defining the amount of support provided by the magnetic field (monopole component of the equation of force balance in the radial direction). In this way, the representation of the physical variables has been transferred from function space, in terms of \( \theta \) and \( \varphi \), to the infinite-dimensional Hilbert space, each vector component now being the coefficient of the corresponding harmonic. The process is analogous to transforming from the Schrödinger to the Heisenberg description in quantum mechanics (it is incomplete, in that the radial dependence is still represented in function space).

The procedure for solving the equations of MS in spectral form is to select a finite set of coefficients by truncating the series expansion to some \( l = l_{\text{max}} \), setting to zero all remaining coefficients. As a test of the method, in the next two subsections we solve the MS equations with \( m = 0 \) for various values of \( l_{\text{max}} \), and compare the results with the axisymmetric solutions obtained by MN00 and LS96 for the hydrostatic and magnetostatic case, respectively.

### 5. Hydrostatic equilibria

Setting \( H_0 = 0 \) and \( A_{lm} = 0 \) for any \((l, m)\), we obtain the set of equations governing hydrostatic equilibria,

\[
\frac{l(l+1) - 2}{2\sqrt{l(l+1)}} R_{lm} = \sum_{l'} \sum_{l''} \left[ \frac{l'(l'+1)}{l''(l''+1)} \right]^{1/2} \beta_{l'l''}^{m'm''} R_{l'\nu'} R_{l''\nu''}.
\]

Remarkably, for \( l = 1 \) both the LHS and RHS of eq. (56) are zero. To see this, observe that for \( l = 1 \) the triangular relation imply that all coefficients \( \beta_{l'l''}^{m'm''} \) vanish unless \( l' = l'' \), or \( l' = l'' \pm 1 \).

Thus, the RHS contains terms like

\[
\beta_{l'l''}^{m'm''} R_{l'\nu'}^2,
\]

(57)
or like
\[
\left\{ \left[ \frac{l(l+1)}{(l'+1)(l'+2)} \right]^{1/2} \beta_{l'l'+1}^{\text{nn}m'm'} + \left[ \frac{(l'+1)(l'+2)}{l'(l+1)} \right]^{1/2} \beta_{l'l'+1}^{\text{nn}m'0} \right\} R_{l'l'} R_{l'+1}^{m'm'}. \tag{58}
\]
Terms like (57) are zero because \(\alpha\) and \(\beta\) coefficients are zero when \(l+l'\) is odd (see Appendix A); terms like (58) are zero because
\[
\beta_{l'l'+1}^{\text{nn}m'm'} = \frac{l'+2}{l'} \beta_{l'l'+1}^{\text{nn}m'm'}, \tag{59}
\]
as shown in Appendix A. Thus, all terms allowed by the triangular conditions in eq. (56) with \(l=1\), are identically zero. This implies that the coefficient \(R_{1m}\) (or \(V_{1m}\)) is undefined, an intrinsic degree of freedom of the problem that we refer to as the “dipole gauge”. The expansion of the density function \(R\) contains dipole terms proportional to \(Y_{1m}(\theta, \phi)\) with \(m=0, \pm 1\),
\[
R(\theta, \phi) = 1 + \sum_{m=-1,0,1} R_{1m} Y_{1m}(\theta, \phi) + \ldots = 1 + (c_1 \cos \theta + c_2 \cos \theta \cos \varphi + c_3 \cos \theta \sin \varphi) + \ldots, \tag{60}
\]
where \(c_1\), \(c_2\) and \(c_3\) are real coefficients. The last three terms represent an eccentric distortion of the basic spherically symmetric equilibrium \((R = 1, \text{the singular isothermal sphere})\) along three perpendicular axes, confirming the result of MN00 that the singular isothermal sphere is neutrally stable with respect to dipole-like density perturbations. Computing additional terms in the series (60), the series converges to the function \(R(\theta, \phi)\) for each value of \(R_{1m}\). Without loss of generality we set \(m=0\), equivalent to assuming that the symmetry axis of the configuration lies in the \(z\) direction, and for better clarity we omit the index 0 in the expansion terms and coupling coefficients. For \(l_{\text{max}} = 2\), we have to solve for \(R_2\) as function of \(R_1\) the quadratic equation
\[
2R_2 = \sqrt{6} \beta_{111} R_1^2 + (\sqrt{2} \beta_{212} + 3\sqrt{2} \beta_{221}) R_1 R_2 + \sqrt{6} R_2^2, \tag{61}
\]
with solution
\[
R_2 = \frac{14\pi}{5} \left[ 1 - \sqrt{1 - \left( \frac{3}{28\pi} \right) R_1^2} \right] \approx \frac{3}{20} R_1^2 \quad \text{if } R_1 \ll 1. \tag{62}
\]
The fast decrease of the coefficients with increasing \(l\), at least for small \(R_1\), suggests a rapid convergence of the series expansion. For \(l_{\text{max}} \geq 2\), the system of truncated equations can be easily solved by iteration. The procedure usually converges to a solution with a relative accuracy of less than \(10^{-5}\) in 4 iterations. The series solution obtained in this way corresponds to the ellipsoidal solutions parametrized by the ellipticity \(e\) (see Sect. 3.3), found by MN00. We show in Fig. 1 the solution obtained assuming \(R_1 = 2\) for \(l_{\text{max}} = 2, 3\) and 4. As anticipated, the terms of the series expansion decrease exponentially with increasing \(l\). For comparison we also show the corresponding analytical solution of MN00 (eq. [22], with \(e = 0.466\))

In principle, the sequence of equilibria defined by the parameter \(R_1\), with \(0 < R_1 < \infty\), can bifurcate into non-axisymmetric configurations of equilibrium. A linearization of eq. (56) gives the condition
\[
\frac{l(l+1) - 2}{2\sqrt{l(l+1)}} - \left\{ \left[ \frac{2}{l(l+1)} \right]^{1/2} \beta_{l11}^{\text{nn}m'm'} + \left[ \frac{l(l+1)}{2} \right]^{1/2} \beta_{l11}^{\text{nn}m0} \right\} R_1 = 0, \tag{63}
\]
for the occurrence of bifurcations with angular dependence defined by harmonics with general \( l, m \) along the \( l = 1, m = 0 \) sequence. However, \( \beta_{ll1m}^{m0} = \beta_{ll1m}^{m0d} = 0 \) since \( 2l + 1 \) is odd (see Appendix A), and therefore no azimuthally asymmetric bifurcation occurs along the sequence of hydrostatic equilibria generated by the \( R_1 \) term.

6. Magnetostatic equilibria with azimuthal and equatorial symmetry

In this section we look for highly symmetrical solutions, possessing both azimuthal and equatorial symmetry. We thus set \( m = 0 \), and we impose the existence of a plane of symmetry at \( \theta = \pi/2 \) (equatorial plane). The density \( R \), the gravitational potential \( V \), the components \( B_\theta \) and \( B_\varphi \) of the magnetic field are symmetric with respect to the equatorial plane, whereas the radial component of the magnetic field, \( B_r \), is anti-symmetric. This implies that the expansion of \( V \) and \( R \) contains multipole terms with \( l \) even, whereas the expansion of \( A \) (and \( F \)) contains multipole terms with \( l \) odd. Since \( m = 0 \), all expansion coefficients are real. As in the previous section, we omit the index \( m \) in the expansion terms and coupling coefficients.

We truncate the infinite system of nonlinear algebraic equations to some index \( l_{\text{max}} \) and we set all coefficient of the expansion equal to zero for \( l > l_{\text{max}} \). At the lowest level of approximation, \( l_{\text{max}} = 2 \), we have to solve two equations for the two coefficients \( A_1 \) and \( R_2 \),

\[
\frac{H_0}{\sqrt{6}} R_2 = \alpha_{211} A_1^2, \tag{64}
\]

\[
A_1^2 = 4\pi H_0 (1 + H_0), \tag{65}
\]

where \( \alpha_{211} = -1/\sqrt{120\pi} \). The solution is

\[
R_2 = -2\sqrt{\frac{\pi}{5}} (1 + H_0), \quad A_1 = \sqrt{4\pi H_0 (1 + H_0)}, \tag{66}
\]

The coefficients \( R_2 \) and \( A_1 \) determined in this way constitute the lowest-order terms of a series expansion for the the functions \( R(\theta) \) and \( \phi(\theta) \),

\[
R(\theta) \approx \frac{3}{2} (1 + H_0) \sin^2 \theta, \quad \phi(\theta) \approx \frac{1}{2} \sqrt{3H_0 (1 + H_0)} \sin^2 \theta, \tag{67}
\]

and the spherical mass-to-flux ratio, from eq. (35),

\[
\lambda_r \approx \frac{4(1 + H_0)}{\phi_1 A_1} = 2 \sqrt{\frac{1 + H_0}{3H_0}}. \tag{68}
\]

This series solution converges to the solution determined by LS96 (singular isothermal toroids). As in the case of the hydrostatic equilibria discussed in the previous section, the convergence of the series solution is rapid: the approximated expression of \( \lambda_r \), obtained with only the first term in the expansion of \( R(\theta) \) and \( \phi(\theta) \), is in good agreement with the numerical values calculated by LS96 as function of \( H_0 \), the largest discrepancy being \( \sim 15\% \) for \( H_0 \gg 1 \).
To obtain a more accurate series solution, for \( l_{\text{max}} > 2 \) we solve numerically the equations of MS in spectral form with Newton’s method, assuming as a first guess the analytical solution eq. (66). The procedure converges to a solution with a relative accuracy of \( 10^{-8} \) in 4 iterations. In Table 1 we list the values of the coefficients \( A_l \) and \( R_l \), and the spherical mass-to-flux ratio for \( H_0 = 0.5 \) and \( l_{\text{max}} = 2, 3, 4, 5 \) and 6. The comparison with the exact numerical solution of LS96 is shown in Fig. 2 and 3.

7. Three-dimensional magnetostatic equilibria

Useful information about the existence of solutions of the MS equations with lower degrees of symmetry than those considered in Sect. 5 and 6, can be obtained by linearization of the spectral equations. The occurrence of neutrally stable configurations along the sequence of equilibria controlled by the parameter \( H_0 \) is indicated by the vanishing of the determinant of the linearized system for some value of the control parameter (in our case the parameter \( H_0 \) measuring the relative amount of support provided by the magnetic field). The zeroes of the determinant signal the presence of neutrally stable equilibria or bifurcations points in the sense of Poincaré (see e.g. Galli et al. 2001).

In this way, we show in Sect. 7.1 and 7.2 that the degree of freedom represented by the “dipole gauge” affecting hydrostatic equilibria (Sect. 5), is also present in the case of MS equilibria independently on the degree of support provided by the magnetic field. The magnetic field however removes the degeneracy of the purely hydrostatic case, where the density distortion with \( l = 1 \) and \( m = 0, \pm 1 \) gives origin to three orthogonal orientations of the same configuration. Magnetized equilibria instead are split by the “dipole gauge” into two families: one possessing an equatorial plane of symmetry but azimuthally asymmetric (with \( m = \pm 1 \)) and elongated in two orthogonal directions; and one without a plane of symmetry but azimuthally symmetric (with \( m = 0 \)). In addition, the dipolar distortion of the density is coupled to a quadrupolar distortion of the vector potential, with the same \( m \).

Finally, in Sect. 7.3 we address the question of the existence of neighboring equilibria to the basic axisymmetric solutions determined in Sect. 6, limiting the analysis to perturbations in the density described by sectorial harmonics (with \( l = m \)). We find that the only allowed perturbation of this kind has \( l = m = 1 \), as anticipated in Sect. 7.2.

7.1. Case \( l = 1, m = 0 \): equatorially asymmetric density distortions

First we consider axisymmetric distortions that introduce an asymmetry of the configuration with respect to the equatorial plane. These are represented by the small coefficients \( R_{10} \) for the
density and \( A_{20} \) for the vector potential, and are governed by the linearized equations

\[
\sqrt{3} H_0 R_{10} - \sqrt{2} (\alpha^{000}_{112} + 3 \alpha^{000}_{121}) \tilde{A}_{10} A_{20} = 0, \tag{69}
\]

and

\[
\sqrt{3} H_0 R_{10} + \sqrt{2} (\beta^{000}_{112} + 3 \beta^{000}_{121}) \tilde{R}_{20} R_{10} - 6 \sqrt{2} (\beta^{000}_{112} + \beta^{000}_{121}) \tilde{A}_{10} A_{20} = 0, \tag{70}
\]

where a tilde (\( \tilde{\cdot} \)) indicates the coefficients of the axisymmetric solution.

A bifurcation can occur if the determinant of the system is zero. Substituting the values of the coefficients, we see that the determinant vanishes for any value of \( H_0 \). For these linearized equilibria, the relation between \( R_{10} \) and \( A_{20} \) is

\[
H_0 \sqrt{\frac{5\pi}{2}} R_{10} = \tilde{A}_{10} A_{20}. \tag{71}
\]

An example of these azimuthally symmetric equilibria lacking equatorial symmetry is shown in Fig. 4 (left panel).

### 7.2. Case \( l = 1, m = \pm 1 \): azimuthally asymmetric density distortions

Next, we consider density perturbations azimuthally asymmetric but conserving the symmetry of the original equilibrium state with respect to the equatorial plane. These are characterized by expansion coefficients \( R_{11} \) and \( A_{21} \) for the density and the vector potential, respectively, and are governed by the linearized equations

\[
\sqrt{3} H_0 R_{11} - \sqrt{2} (\alpha^{101}_{112} + 3 \alpha^{110}_{121}) \tilde{A}_{10} A_{21} = 0, \tag{72}
\]

and

\[
\sqrt{3} H_0 R_{11} + \sqrt{2} (\beta^{101}_{112} + 3 \beta^{110}_{121}) \tilde{R}_{20} R_{11} - 6 \sqrt{2} (\beta^{101}_{112} + \beta^{110}_{121}) \tilde{A}_{10} A_{21} = 0. \tag{73}
\]

As before, the coefficients \( \tilde{R}_{20} \) and \( \tilde{A}_{10} \) are those of the axisymmetric solution.

A bifurcation can occur if the determinant of the system is zero for some value of \( H_0 \). Substituting the values of the coefficients, we see that the determinant vanishes for any value of \( H_0 \). For these linearized equilibria, the relation between \( R_{11} \) and \( A_{21} \) is

\[
H_0 \sqrt{\frac{5\pi}{3}} R_{11} = \tilde{A}_{10} A_{21}. \tag{74}
\]

An example of these azimuthally asymmetric equilibria with equatorial symmetry is shown in Fig. 4 (right panel).
7.3. Sectorial density distortions: linear analysis

To assess the validity of Parker’s theorem for axisymmetric self-gravitating equilibria, we check whether the class of axisymmetric solutions obtained in Sect. 6 (converging to the singular isothermal toroids of LS96) allows neighbouring 3-D equilibria. To this goal, we perform a linearization of the set of nonlinear algebraic equations near the axisymmetric solution and we consider for simplicity distortions of the density function with arbitrary $l$, assuming for simplicity $m = l$ (sectorial distortions). As before, the expansion coefficients corresponding to the axisymmetric solution are indicated by a tilde ($\tilde{\cdot}$).

The linearized equations for small distortions characterized by expansion coefficients $R_{\ell l}$ and $A_{\ell + 1 l}$, with arbitrary $l$, are

$$
\frac{H_0}{\sqrt{l(l+1)}} R_{\ell l} = \left( \sqrt{\frac{2}{(l+1)(l+2)}} \alpha_{\ell l}^{l+1} + \sqrt{\frac{(l+1)(l+2)}{2}} \alpha_{\ell l}^{l+11} \right) \tilde{A}_{10} A_{\ell + 1 l}, \quad (75)
$$

$$
l(l+1) - 2(1+H_0) \frac{R_{\ell l}}{2\sqrt{l(l+1)}} = \left( \sqrt{\frac{6}{l(l+1)}} \beta_{\ell l}^{l+1} + \sqrt{\frac{2(l+1)}{6}} \beta_{\ell l}^{l+11} \right) \tilde{R}_{20} R_{\ell l} - \sqrt{2(l+1)(l+2)}(\beta_{\ell l}^{l+1} + \beta_{\ell l}^{l+11}) \tilde{A}_{10} A_{\ell + 1 l} \quad (76)
$$

The determinant $\Delta_{\ell l}$ of this systems of equations is

$$
\Delta_{\ell l} / \tilde{A}_{10} = H_0 \sqrt{\frac{2(l+2)}{l}} (\beta_{\ell l}^{l+1} + \beta_{\ell l}^{l+11}) + \left( \sqrt{\frac{2}{(l+1)(l+2)}} \alpha_{\ell l}^{l+1} + \sqrt{\frac{(l+1)(l+2)}{2}} \alpha_{\ell l}^{l+11} \right)
$$

$$
\left[ \frac{l(l+1) - 2(1+H_0)}{2\sqrt{l(l+1)}} \right] - \left( \sqrt{\frac{6}{l(l+1)}} \beta_{\ell l}^{l+1} + \sqrt{\frac{2(l+1)}{6}} \beta_{\ell l}^{l+11} \right) \tilde{R}_{20}, \quad (77)
$$

and vanishes for $l = 1, m = 1$, as anticipated in Sect. 5 and 7.2. For $l \neq 1$, we have evaluated the determinant numerically as function of $H_0$ for $l = 2, 3, 4, 5$ and 6, and $0 < H_0 < 2$. The results are shown in Fig. 5. At least in this part of parameter space, the determinant is always positive, its value monotonically increasing with $H_0$ and $l$, and shows no sign of having a zero for particular values of $H_0$. We can then safely conclude that the axisymmetric MS equilibria determined by LS96 and discussed in Sect. 6, have no other neighboring equilibria than those allowed for all values of $H_0$ by the “dipole gauge” discussed in Sect. 6, 7.1 and 7.2.

In this sense, these non-symmetric equilibria represent the only exceptions (within the assumption of the present study) to a generalized version of Parker’s theorem, originally formulated for systems with translational symmetry, extended to self-gravitating equilibria with axial symmetry. In the next section we consider the equilibria originated by the “dipole gauge” from a physical point of view, and we conclude that all these solutions are probably not force-free at the origin, a singular point for a scale-free configuration, and therefore they cannot represent realistic models of equilibrium of isolated cosmic bodies.
8. Summary

The results obtained in this paper show that previously known axisymmetric solutions of the MS equations for an isothermal self-gravitating gas, under the hypothesis of scale invariance and global neutrality, allow only neighbouring equilibria characterized by a $l = 1, m = 0, \pm 1$ angular dependence of the density distortion for any value of the degree of magnetic support (including zero and infinite). For $m = 0$, the original axisymmetric equilibrium is distorted by a “bending” of the isodensity contours with respect to the equatorial plane, preserving azimuthal symmetry; for $m = \pm 1$, the equilibrium is distorted by a “stretching” of the isodensity contours along one of two orthogonal directions in the equatorial plane, preserving up-down reflection symmetry. In the absence of magnetic fields (for $H_0 = 0$) these two classes of distorted equilibria reduce to the ellipsoidal equilibria found by MN00. In the limit of vanishing thermal support ($H_0 \to \infty$) the configuration of equilibrium reduces to a thin disk supported only by magnetic tension against its self-gravity, and the $m = 1$ density distortion corresponds to the elliptical disklike equilibria found by Galli et al. (2001). What is the significance of the neutral stability of these configurations to density distortions characterized by a dipolar angular dependence?

According to MN00, the ellipsoidal hydrostatic equilibria are not force-free at the origin, where the gravity of all the matter of the configuration produces a non-vanishing force trying to restore symmetry with respect to the equatorial plane. According to Cai & Shu (2004), the same conclusion holds for the azimuthally asymmetric solutions found by Galli et al. (2001) for MS equilibria in the thin-disk limit. These two classes of equilibria stem from the singular isothermal sphere and the singular isothermal disk, respectively, that in turn are the limiting cases (for $H_0 = 0$ and $H_0 = \infty$) of the family of singular isothermal toroids of LS96. It is therefore tempting to conclude, that all singular isothermal equilibria, characterized by a distortion of the isodensity (or equipotential) surfaces proportional to the $l = 1$ harmonics, are not force-free at the origin for any value of $H_0$. If this is the case, their relevance to represent realistic equilibria is doubtful.

The same conclusion probably hold for configurations rotating with spatially uniform velocity $u_\phi$ (the only rotation law compatible with isothermality and spatial self-similarity). Galli et al. (2001) found that rotating singular isothermal disks are neutrally stable to $m = 1$ perturbations for any value of the rotation rate. In a similar vein, MN00 found that the the azimuthally and equatorially symmetric rotating models of Toomre (1982) and Hayashi et al. (1982) can be “continued”, for any value of the rotation velocity, into a sequence of axisymmetric equilibria lacking equatorial symmetry (in our language, originated by a $l = 1, m = 0$ density distortion). At the light of the results described in Sect 7.1 and 7.2, it is natural to expect that the sequence of Toomre-Hayashi models also possess non-axisymmetric counterparts with a dominant $l = 1, m = 1$ asymmetry, although solutions of this kind are not known. Likely, also these hypothetical rotating asymmetric equilibria are not force-free at the origin. The $l = 1, m = 0, \pm 1$ distortion, the allowed distortion of axisymmetric equilibria found in the present study, may then represent a gauge freedom that creeps somehow into general self-similar isothermal equilibria, irrespectively of the presence of magnetic fields, or rotation, or else. The appearance of this gauge freedom in self-similar isothermal systems,
and its physical significance, deserves further scrutiny. As a counterexample, it should be easy to show that, assuming a non-isothermal (e.g. polytropic) equation of state, the analogues of the Li-Shu magnetized equilibria studied by Galli et al. (1999), (or the polytropic analogues of the Toomre-Hayashi models) are not affected by this gauge freedom.

For arbitrary \( l = m \neq 1 \), a perturbation analysis shows that the symmetric magnetized solutions found by LS96 do not have neighbouring non-axisymmetric equilibria. This results is analogous to Parker’s theorem for systems with translational invariance, and suggests that the validity of Parker’s theorem can be extended to self-gravitating axisymmetric equilibria. Combining this result with the findings of Galli et al. (2001), one is led to the conclusion that the presence of fluid motions (specifically, super-magnetosonic rotation) is a crucial ingredient for the occurrence of symmetry break-ups in MHD equilibria.

The results of this paper imply that, under the assumed conditions, very few (and possibly not physically meaningful) non-axisymmetric solutions of the steady MHD equations do exist. Of the assumed conditions, probably the most severe are the assumptions that (i) the magnetic field is analytic everywhere, and that (ii) the new, non-symmetric solutions are accessible from the basic states by regular perturbations (i.e. small-parameter expansions). As for the former assumption, Moffatt (1985) has shown that magnetostatic equilibria of non-trivial topology in a perfectly conducting fluid may generally contain tangential discontinuities (current sheets), i.e. they cannot be described in terms of analytic functions; in contrast, equilibrium fields that are analytic functions of space are subject to severe structural constraints, as shown by Arnol’d (1965, 1966). As for the latter assumption, our results show that equilibria of non-trivial topology, if they exist, cannot in general be reached starting from axisymmetric states by regular perturbation. This is consistent with the conclusions of Rosner & Knobloch (1982), that the response of stationary solutions of nonlinear equations to finite-amplitude, symmetry-breaking perturbations may not in general be obtained in terms of small-parameter expansions of the variables. These limitations should be kept in mind in interpreting the results of this paper.

8.1. Implications for molecular cloud cores

Recent statistical studies (Jones, Basu, & Dubinski 2001; Jones & Basu 2002; Goodwin, Ward-Thompson, & Whitworth 2003) based on available catalogues of molecular cloud cores and Bok globules (typical size \( L = 0.1 \) pc, sound speed \( c_s = 0.2 \) km s\(^{-1}\), average density \( n(\text{H}_2) = 10^4 \) cm\(^{-3}\), and typical magnetic field strength \( B = 10 \) \( \mu \)G), do not support the possibility that cores are axisymmetric configurations. A good fit to the observed axial distribution is generally found assuming instead that cores are triaxial ellipsoids. The best-fit axial ratios \( a : b : c \) determined statistically for molecular cloud cores (1 : 0.9 \( \pm \) 0.1 : 0.5 \( \pm \) 0.1 according to Jones & Basu 2002; 1 : 0.8 \( \pm \) 0.2 : 0.4 \( \pm \) 0.2 according to Goodwin, Ward-Thompson, & Whitworth 2003), suggest that cores are preferentially flattened in one direction and nearly oblate \( (a \approx b > c) \), and imply that they may not be particularly far from conditions of equilibrium.
Taking these observational results at face value, one is led to consider the fate of a cosmic cloud with its frozen-in magnetic field formed by whatever process in a configuration lacking a high degree of symmetry and presumably not in an exact equilibrium state. As discussed in Sect. 2.2, Parker (1979) argued that realistic magnetic fields with no well defined symmetries must evolve in a genuinely time-dependent way, until all non-symmetric components of the field are destroyed by dissipation and reconnection and the topology becomes symmetric. This process can hardly be of any relevance for the interstellar gas, where ohmic dissipation times are larger than the age of the Universe. Ambipolar diffusion on the other hand can only redistribute the mass inside flux tubes and drive a stable equilibrium to the threshold of dynamical instability, but cannot dissipate the magnetic energy stored in the field.

To the extent that interstellar clouds can be represented as isolated MS equilibria, these systems must instead undergo Alfvèn oscillations (weakly damped by ambipolar diffusion) around the equilibrium state with period (Woltjer 1962)

\[
\tau \approx \frac{L}{(c_s^2 + v_A^2)^{1/2}},
\]

where \(L\) is the size of the system, \(c_s\) is the sound speed and \(v_A\) is the Alfvèn speed. This kind of behaviour is evident in the numerical and analytical calculations of Hennebelle (2003) relative to the homologous evolution of prolate and oblate magnetized isothermal spheroids. In the non-selfgravitating case, the stability of magnetostatic equilibria of arbitrary complex topology was studied by Moffatt (1986) through construction of the second variations of the magnetic and kinetic energies with respect to a virtual displacement about the equilibrium configuration. Moffatt's (1986) results show that a general class of space-periodic magnetostatic equilibria is stable to disturbances of arbitrary lengthscale. If perturbed in some way, the fluid responds executing oscillations about this equilibrium, that are eventually damped if due account is taken of viscosity. If we adopt the deviation from axial symmetry of the shapes of cloud cores as rough measure of their nonequilibrium, a simple harmonic oscillator analogy provides an estimate of the average velocity \(\langle v^2 \rangle^{1/2}\) of pulsation,

\[
\frac{\langle v^2 \rangle}{(c_s^2 + v_A^2)} \approx \frac{(a - b)^2}{a^2},
\]

that, for \(a : b = 1 : 0.8\) gives \(\langle v^2 \rangle^{1/2} \approx 0.2 (c_s^2 + v_A^2)^{1/2}\). For typical conditions of molecular cloud cores, this implies coherent pulsations with period \(\tau \approx 4 \times 10^5\) yr, average velocity \(\langle v^2 \rangle^{1/2} \approx 0.05\) km s\(^{-1}\), and maximum velocity \(\sim 0.1\) km s\(^{-1}\). These pulsational motions may have already been detected in the isolated globule B68 (Lada et al. 2003).

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A. The coupling coefficients

The coupling integrals of eq. (45) and (51) can be expressed in terms of integrals of three spherical harmonics using standard recurrence formulae to eliminate the derivatives. The resulting integrals can be evaluated with the Gaunt formula,

$$\int Y_{l''m''}^{*} Y_{l'm'} Y_{l''m''} \, d\Omega = [l(l+1)l'(l'+1)l''(l''+1)]^{1/2} N_{ll'l''} G_{ll'l''}^{mm'm''},$$  \hspace{1cm} (A1)

where

$$N_{ll'l''} \equiv \frac{1}{2} \left[ \frac{(2l+1)(2l'+1)(2l''+1)}{4\pi l(l+1)l'(l'+1)l''(l''+1)} \right]^{1/2},$$  \hspace{1cm} (A2)

and

$$G_{ll'l''}^{mm'm''} \equiv (-1)^m \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ -m & m' & m'' \end{pmatrix}.$$  \hspace{1cm} (A3)

The $3j$ symbols are algebraically defined by e.g. the Racah formula (see e.g. Landau & Lifshitz 1997, Varshalovich, Moskalev & Khersonskii 1998). The evaluation of the coupling coefficients is best achieved by group-theoretical methods and the use of Wigner 6j and 3j symbols. Here we quote only the results, details on the procedure can be found in Jones (1985). In terms of the quantities $N_{ll'l''}$ and $G_{ll'l''}^{mm'm''}$, we have

$$\alpha_{ll'l''}^{mm'm''} = \Lambda_l N_{ll'l''} G_{ll'l''}^{mm'm''}, \quad \beta_{ll'l''}^{mm'm''} = \Lambda_{l'} N_{ll'l''} G_{ll'l''}^{mm'm''},$$  \hspace{1cm} (A4)

where

$$\Lambda_l = l''(l''+1) + l'(l'+1) - l(l+1), \quad \Lambda_{l'} = l''(l''+1) + l(l+1) - l'(l'+1).$$  \hspace{1cm} (A5)

The coupling coefficients $\alpha_{ll'l''}^{mm'm''}$ and $\beta_{ll'l''}^{mm'm''}$ are real, and are equal to zero when the following (triangular) conditions on the Wigner 3j symbols are not all satisfied:

$$l \leq l' + l'', \quad l' \leq l + l'', \quad l'' \leq l + l', \quad l + l' + l'' \text{ even},$$  \hspace{1cm} (A6)

and

$$m = m' + m''.$$  \hspace{1cm} (A7)

We recall that a 3j symbol is invariant under even permutation of columns and is multiplied by the phase factor $(-1)^{(l+l'+l'')}$ for odd permutations. Thus $G_{ll'l''}^{mm'm''}$ being the product of two 3j symbols, is invariant for any permutation. Using this property, one obtains, for example,

$$\beta_{ll+l''}^{mm'm''} = -\frac{l+2}{l} \beta_{ll'+l''}^{mm'm'}.$$  \hspace{1cm} (A8)

The coefficients $\alpha_{ll'l''}^{mm'm''}$ and $\beta_{ll'l''}^{mm'm''}$ are real, whereas $\gamma_{ll'l''}^{mm'm''}$ is imaginary:

$$\gamma_{ll'l''}^{mm'm''} = -i \Gamma_l N_{ll'l''} H_{ll'l''}^{mm'm''},$$  \hspace{1cm} (A9)
we obtain

\[ H_{ll''m''}^{mm'm''} = (-1)^m \begin{pmatrix} l & l'' & l' - 1 \\ 0 & 0 & 0 \\ -m & m' & m'' \end{pmatrix}, \]

and

\[ \Gamma_l = [(l' + l'' - l)(l' + l - l'')(-l' + l + l'' + 1)(l'' + l' + l + 1)]^{1/2}. \]

Notice that all the dependence of the coupling coefficients on the azimuthal number \( m \) is contained in the Wigner 3\( j \) symbols, whereas the remaining factors depend on \( l \) only (the Wigner-Eckart theorem).

**B. Proof that one component of the equation of force balance is trivially satisfied**

To show this, we take then the scalar product of eq. (50) with \( C_{lm}^* \) and integrate over solid angle, obtaining the condition

\[
\sum_{l''m''} \sum_{l'm'} \left[ l'(l' + 1) \right]^{1/2} \gamma_{ll''m''}^{mm'm''} R_{lm'} R_{l'm'm'} - \sum_{l''m''} \sum_{l'm'} \left[ l'(l' + 1) l''(l'' + 1) \right]^{1/2} \gamma_{ll''m''}^{mm'm''} A_{lm'} A_{l'm'm'} = 0, \quad (B1)
\]

where we have defined

\[
\gamma_{ll''m''}^{mm'm''} = \frac{1}{\sqrt{l'(l' + 1)}} \oint Y_{lm'} B_{l'm'} \cdot C_{lm}^* \, d\Omega. \quad (B2)
\]

The expressions of the coupling coefficients \( \gamma_{ll''m''}^{mm'm''} \) in terms of the Wigner 3\( j \) symbols is given in Appendix A. It is not difficult to see that, as expected, this equation is trivially satisfied, since the two terms in eq. (B1) are equal to the coefficients of the expansions of \( \nabla_\Omega R \times \nabla_\Omega V \) and \( \nabla_\Omega A \times \nabla_\Omega A \) in terms of \( P_{lm} \), both of which are zero.

We begin by expanding the vector product \( \nabla_\Omega R \times \nabla_\Omega V \),

\[
\nabla_\Omega R \times \nabla_\Omega V = \sum_{l''m''} \sum_{l'm'} R_{lm'} V_{l'm'} [l'(l' + 1) l''(l'' + 1)]^{1/2} (B_{l'm'} \times B_{l'm''}). \quad (B3)
\]

Eliminating the coefficients \( V_{l''m''} \) with eq. (39), and using the relation (see Jones 1985),

\[
B_{l'm'} \times B_{l''m''} = -(B_{l'm'} \cdot C_{l''m''}) \hat{r}, \quad (B4)
\]

we obtain

\[
\nabla_\Omega R \times \nabla_\Omega V = \sum_{l''m''} \sum_{l'm'} \left[ \frac{l'(l' + 1)}{l'(l'' + 1)} \right]^{1/2} R_{lm'} R_{l'm'm'} (B_{l'm'} \cdot C_{l''m''}) \hat{r}. \quad (B5)
\]

We now compute the expansion of this equation in series of \( P_{lm} \). For this, we need to evaluate the coefficients

\[
\oint Y_{lm}^* B_{l'm'} \cdot C_{l''m''} \, d\Omega = \oint [Y_{lm} B_{l'm'}^* \cdot C_{l''m''}]^* \, d\Omega
\]
\begin{align*}
&= (-1)^{m'} \int [Y_{lm} B_{l'-m'} \cdot C_{l'\nu \mu}^*] \, d\Omega = (-1)^{m'} \sqrt{l(l+1)} [\gamma_{l'\nu \mu}^{mm'}] \, d\Omega = \sqrt{l(l+1)} [\gamma_{l'\nu \mu}^{mm'}] \, d\Omega = \sqrt{l(l+1)} [\gamma_{l'\nu \mu}^{mm'}] \, d\Omega,
\end{align*}

where we have used the definition of \( \gamma_{l'\nu \mu}^{mm'} \) and the symmetry properties of Wigner 3\( j \) symbols. Thus, we finally obtain

\begin{align*}
\nabla_\Omega R \times \nabla_\Omega V &= \sum_{l,m} \sqrt{l(l+1)} P_{lm} \left\{ \sum_{l',m',l''} \sum_{\nu,\mu} \left[ l'(l' + 1) \right]^{1/2} \gamma_{l'\nu \mu}^{mm'} R_{l'm'} R_{l''m''} \right\}. \quad (B7)
\end{align*}

The same procedure, applied to the cross product \( \nabla_\Omega A \times \nabla_\Omega A \), shows that

\begin{align*}
\nabla_\Omega A \times \nabla_\Omega A &= - \sum_{l,m} \sqrt{l(l+1)} P_{lm} \left\{ \sum_{l',m',l'',\mu} \sum_{\nu} \left[ l'(l' + 1) l''(l'' + 1) \right]^{1/2} \gamma_{l'\nu \mu}^{mm'} A_{l'm'} A_{l''m''} \right\}. \quad (B8)
\end{align*}

Since \( \nabla_\Omega R \times \nabla_\Omega V = 0 \), as can be shown by taking the curl of eq. (15), and \( \nabla_\Omega A \times \nabla_\Omega A = 0 \), we conclude that for each \( (l, m) \) the quantities inside curly brackets in eq. (B7) and (B8) must be zero. Thus, each term of the expansion of the equation of force balance along \( C_{lm} \), eq. (B1), is zero.
Table 1. **Multipole Coefficients for $H_0 = 0$ and $m = 0$**

| $l_{\text{max}}$ | $R_1$ | $R_2$ | $R_3$ | $R_4$  |
|------------------|-------|-------|-------|-------|
| 2                | 2     | 0.7847|
| 3                | 2     | 0.7631| 0.2604|
| 4                | 2     | 0.7623| 0.2546| 0.0792|
| exact            | 2     | 0.7623| 0.2542| 0.0791|

Table 2. **Multipole Coefficients for $H_0 = 1/2$ and $m = 0$**

| $l_{\text{max}}$ | $A_1$  | $R_2$   | $A_3$  | $R_4$  | $A_5$  | $R_6$  | $\lambda_r$ |
|------------------|--------|---------|--------|--------|--------|--------|-------------|
| 2                | 3.0670 | −2.3780 |        |        |        |        | 2.002       |
| 3                | 3.0583 | −3.2861 | −0.2670|        |        |        | 1.943       |
| 4                | 3.0591 | −3.2573 | −0.2581| 0.6839 |        |        | 1.944       |
| 5                | 3.0593 | −3.2496 | −0.2532| 0.8388 | 0.0356 |        | 1.940       |
| 6                | 3.0593 | −3.2496 | −0.2534| 0.8249 | 0.0324 | −0.1523| 1.941       |
Fig. 1.— Iso-density contours for the axisymmetric hydrostatic ($H_0 = 0$) equilibrium with $R_1 = 2$ obtained with the method described in this paper (long-dashed curves, $l_{\text{max}} = 2$; short-dashed curves, $l_{\text{max}} = 3$; dotted curves $l_{\text{max}} = 4$). The analytical solution of MN00 with the same value of $R_1$ is shown by the solid curves.
Fig. 2.— Density function $R(\theta)$ and flux function $\phi(\theta)$ for the axisymmetric case with $H_0 = 0.5$. The solid curves are the numerical solutions of LS96. The solutions obtained with the method described in this paper for $m = 0$ are shown by long-dashed curves ($l_{\text{max}} = 2$), short-dashed curves ($l_{\text{max}} = 4$), and dotted curves ($l_{\text{max}} = 6$). The lower panels show the differences between the exact solution of LS96 and the series solution obtained in this work.
Fig. 3.— Axisymmetric magnetostatic equilibrium for $H_0 = 0.5$ (isodensity contours and magnetic field lines). The solution of LS96 is shown by *solid lines*, the solution obtained in this paper with $l_{\text{max}} = 6$ is shown by *dotted lines*. The agreement of the two solutions is very good.
Fig. 4.— The $l = 1$ gauge. For any value of $H_0$, magnetostatic equilibria with azimuthal and equatorial symmetry (dashed lines) possess neighboring equilibria with a non-zero $l = 1$ density component (solid lines). The two panels show examples of these equilibria with small density perturbations containing $l = 1, m = 0$ and $l = 1, m = \pm 1$ harmonics.
Fig. 5.— The determinant $\Delta_{ll}$ of the system of linearized equations describing sectorial ($l = m$) distortions of axisymmetric magnetostatic equilibria. The determinant is normalized to $\tilde{A}_{10}$, the value of the dipole term in the expansion of the vector potential for the magnetic field. Notice that the determinant is zero for any value of $H_0$ for $l = 1, m = \pm 1$ density distortions, as discussed in Sect. 7.2, and non-zero in all other cases.
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