Synchrotron radiation of dissipative solitons in optical fiber cavities

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New resonant emission of dispersive waves by oscillating solitary structures is considered analytically and numerically. The resonance condition for the radiation is derived and it is demonstrated that the predicted resonances match the spectral lines observed in numerical simulations perfectly. The complex recoil of the radiation on the soliton dynamics is discussed.

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Since the pioneering work by Akhmediev and Karlsson the resonant radiation of optical solitons propagating in fibers with high order dispersion has been attracting much of attention because of both the fundamental interest and practical importance. The condition of the resonance is the equality of the soliton velocity to the phase velocity of a dispersive wave and in this way the radiation is similar to the effect of Cherenkov radiation of charged particles. A similar effect of transitional radiation of Bragg solitons was later discovered in periodic media.

The aim of the present Letter is to address the resonant radiation of oscillating optical solitons and to reveal the analogy of this radiation to the synchrotron radiation of charges appearing when the velocity of relativistic charges varies periodically in time.

It should be noted here that from the point of view of the field theory the synchrotron radiation can be interpreted as an excitation of resonant waves by an oscillating localized source moving at a relativistic velocity. The periodical variations of the velocity of a relativistic charge is just a way to obtain the right driving force in the equation for the electromagnetic waves.

Aiming to develop the analogy between synchrotron radiations of the charges and solitons it is much simpler and more instructive to consider one-dimensional oscillating solitons rather than two-dimensional nonlinear localized structures relativistically moving along one direction and wobbling in the transverse direction (though the latter is also possible).

The effects of Cherenkov and synchrotron radiation are widely used in vacuum electronics, in particular in forward- and backward waves oscillators. The optical counterparts of Cherenkov effects can play an important role in the generation of new optical frequencies by the solitons propagating in fibers with high order dispersion. Indeed, since 1995 when Cherenkov radiation of optical solitons was reported for the first time this effect has been actively studied both experimentally and theoretically, especially in the context of the generation of optical supercontinuum. Recently a number of papers were published on Cherenkov emission shed by shock waves and ultra-short pulses.

All these effects are important for the explanation of the dynamics of solitons and domain walls in wave-guiding systems. The effect of synchrotron emission of oscillating solitons can also affect the dynamics of the solitons strongly and thus is of interest from both fundamental and practical points of view.

Optical solitons in fibers seem to be the best candidates for experimental observation of the discussed effect of synchrotron emission, however similar effects can take place in polariton condensates, plasmas, hydrodynamics and other systems of different physical origins.

The resonant emission is possible not only in the conservative but also in weakly dissipative systems. In this Letter we consider the radiation in annular dissipative optical cavities with external pump. The advantage of dissipative systems is that the resonant radiation can occur there in a stationary regime when it is not disguised by any transitional processes.

Our choice is also motivated by the fact that these systems are not only a convenient test bench for investigation of synchrotron radiation but are of interest on their own and have been actively investigated in recent years theoretically and experimentally.

Propagation of oscillating solitons in the presence of high order dispersion was studied experimentally and numerically and very rich dynamics of the solitons was reported. Cherenkov radiation of oscillating cavity solitons was recently considered but the effect of the synchrotron radiation was out of the scope of that paper. In the present Letter we focus on synchrotron radiations specific for oscillating solitons.

To describe the cavity we adopt a well known Lugiato-Lefever model in the form as it is written in.

\[ \partial_t A = -(1 + i\theta)A + |A|^2A + i\frac{\partial^2}{\partial x^2}A + d_3\frac{\partial^4}{\partial x^4}A + P \quad (1) \]

where \( \theta \) is the detuning between the cavity resonance and the frequency of the pump, \( P \) is the amplitude of the pump and \( d_3 \) is the third order dispersion. To make the resonance spectral lines well pronounced we choose the detuning to be equal to a realistic value \( \theta = 15 \) which is just slightly greater than the values used in. The boundary conditions for annular cavities are periodic but it is checked that for the for large cavities considered in the present Letter the boundary conditions do not affect the dynamics of the solitons.

Equation (1) has bright soliton solutions that lose stability through Andronov-Hopf bifurcation leading to the formation of an oscillating soliton for the pumps exceeding a threshold value. For the chosen para-
It is convenient to write an equation for \( u \) in vector form \( \vec{u} = (Re \, u, Im \, u)^T \)

\[
\partial_t \vec{u} + \vec{L} \vec{u} = \vec{f}.
\]  

(2)

The linear operator \( \vec{L} \) has the form

\[
\vec{L} = \begin{pmatrix}
2I - Re \, A_0^2 - \theta + \partial_x^2 & -2I - Re \, A_0^2 + \theta - \partial_x^2 \\
-2I - Re \, A_0^2 + \theta - \partial_x^2 & 1 - Im \, A_0^2 + d_3 \partial_x^2
\end{pmatrix},
\]

where \( I = |A_0|^2 \). The right hand side is given by \( \vec{f} = (Re \, f_0, Im \, f_0)^T \) with

\[
f_0 = P - (1 + i \theta - i|A_0|^2 - i \partial_x^2 - d_3 \partial_x^2) A_0 - \partial_t A_0. \tag{3}
\]

The left hand side of equation (2) describes propagation of waves on the background hosting the soliton. In the conservative case the resonance of the right hand side \( \vec{f} \) with a delocalized eigenmode of the medium results in the formation of a continuous wave propagating away from the soliton. In the dissipative case the length of the radiation tail generated by the resonance is, of course, always finite but can be very large for low losses.

The oscillations of the soliton make the coefficients in \( \vec{L} \) to be periodic functions of time. It means that the eigenfunctions of the operator \( \vec{L} \) are Bloch functions in time. The important fact, however, is that far from the soliton the problem becomes uniform in both space and time and thus the asymptotic of the delocalized Bloch modes \( \xi_\omega \) at \( x \to \pm \infty \) are just plane waves \( \xi_\omega \sim e^{ikt - \omega x} \) parametrized by the frequency \( \omega \) and the wave vector \( k \). So the dispersion characteristics if the delocalized eigenmodes is nothing else but the dispersion characteristics of plane waves \( \omega = \omega \pm |k| \) on the background

\[
\omega_\pm = i + d_3 k^2 \pm \sqrt{(\theta + k^2 - I_0)(\theta + k^2 - 3I_0)},
\]  

(4)

where \( I_0 = I[|\pm \infty|] \) is the intensity of the background.

To proceed let us represent the right hand side of (2) in the form \( \vec{f} = \sum_l \vec{U}_l[x - vt] \cdot e^{i \omega_0 t} \) where \( \omega_0 = 2\pi/T \) is the fundamental frequency of the soliton. Then each of the term \( \vec{U}_l \) in the sum can be represented as series over the eigenfunctions of the operator \( \vec{L} \). From the fact that all functions \( \vec{U}_l[x - vt] \) are moving without changing their shapes it follows that the frequencies of the driving force components depend on their wave vectors as \( \omega = kv + l\omega_0 \).

The emission of radiation takes place when a harmonic of the right hand side \( \vec{f} \) is in resonance with an eigenwave of the medium. This gives the condition of the phase synchronism

\[
\omega_\pm(k_r) = k_r v + l\omega_0.
\]  

(5)

For \( l = 0 \) this is the condition of Cherenkov radiation. If the condition is satisfied for \( l \neq 0 \) then the radiated wave is excited by \( l \)-th temporal harmonic of the oscillating soliton.

Let us now discuss how the derived resonance conditions explain the results of the numerical experiments. To
do this we study the spectrum and the resonance conditions at low $k$ of the oscillating (thicker black line) and non-oscillating solitons (thinner blue line). The spectrum in the vicinity of Cherenkov resonance is shown in panel (b). The graphical solutions of the resonance conditions are shown in panels (c) and (d), the green line corresponds to the Cherenkov resonance and the red ones to the resonances with $l \neq 0$. Resonances marked as $1_+$, $1_-$ on panel (a) and by 1 on panel (b) are given by different crossings of the same red line with the dispersion characteristic of the linear excitations on the soliton background; the same is for the resonances marked by 2. The third order dispersion is $d_3 = 0.02$; the pump is $P = 8$.

It is instructive to compare the radiation of oscillating and not oscillating solitons. The blue curves in panels (a) and (b) show the spectrum of a non oscillating soliton forming in the case of a weaker pump $P = 6$. The velocities of the solitons are practically the same for $P = 6$ and $P = 8$ and so the Cherenkov spectral lines of the oscillating and non-oscillating solitons practically coincide. However it is clearly seen that the non-oscillating soliton emits neither cyclotron no synchrotron radiation.

In the case of third order dispersion the synchrotron and Cherenkov resonances always appear together, but it is important to remark that there are dispersion characteristics allowing for synchrotron resonances in the absence of the Cherenkov one.

To shed more light on the resonant emission we consider the shapes of the radiation fields. Panel (a) of Fig. shows stationary Cherenkov radiation of non-oscillating soliton pumped by $P = 6$ when third order dispersion is equal to $d_3 = 0.02$. The radiation contains only one mode, see the spectrum in panel (b) of Fig. and decays exponentially with the rate $1/|v_p - v|$.

For stronger pump $P = 8$ the soliton develops oscillations and, as discussed above, the cyclotron radiation appears in front and behind the soliton. This radiation is clearly seen in panel (b). The radiations having small group velocity decays along $x$ quickly, so that this radiation is seen only in the vicinity of the soliton.

Cherenkov and synchrotron radiations with large $k$ have bigger group velocities and so decay slower than cyclotron radiation. This radiation marked by SR is weak and barely visible for $d_3 = 0.02$ but becomes much stronger for $d_3 = 0.06$, see panels (b) and (c).

A very important fact is that the radiation of an oscillating soliton is pulsing whereas the radiation of a
≈ the radiation pulses is $2$ is the group velocity of the radiation. Then the length of
be estimated as $\Delta$ non-oscillating soliton is a monotonically decaying con-
panels (b) and (c) for $d_3 = 0.06$ and $d_3 = 0.06$. The syn-
chrotron and cyclotron radiations are marked by "SR" and "CR" correspondingly.

FIG. 3: (Color online) Radiation field of the non-oscillating soliton is shown in panel (a), $P = 6$ $d_3 = 0.02$. The radiations of the oscillating solitons ($P = 8$) are illustrated in panels (b) and (c) for $d_3 = 0.02$ and $d_3 = 0.06$. The syn-
chrotron and cyclotron radiations are marked by "SR" and "CR" correspondingly.

non-oscillating soliton is a monotonically decaying continuous wave. The pulsations can be explained by the inter-
ference of synchrotron emissions at close frequencies. The distances between the synchrotron resonances can be估算 as $\Delta k = \omega_0/|v_{gs} - v|$ where $v_{gs} = \partial \omega(k_r)$ is the group velocity of the radiation. Then the length of the radiation pulses is $2\pi v_{gs}/\Delta \omega$ giving a good estimate of $\approx 15$ for $d_3 = 0.06$.

Now we turn our attention to a recoil from the radiation on the oscillating solitons. From numerics we extract the dependence of maximum intensity of the soliton $I_{max}[t]$ on time and calculate its temporal derivative $\partial_t I_{max}[t]$. These dependencies parametrically define the curves in $I_{max}[t]$, $\partial_t I_{max}[t]$ plane, see (a) of Fig. 4. For a periodic motion this curve is a closed loop and this is the case so for small $d_3 = 0.02$. However for large $d_3 = 0.06$ the motion looks very complicated. The temporal evolution of the spectrum is also not periodic for $d_3 = 0.06$, see panels (b) and (c).

This behaviour can be explained by the fact that the pulsations of the synchrotron radiation causes pulsating recoil changing the intensity and velocity of the soliton. The variations of the soliton parameters in their turn affect the instantaneous frequencies of the radiation. For large $d_3$ the interplay of these effects leads to a complex dynamics of the solitons. Possibly this explains the complicated dynamics of cavity solitons discussed in [25].

Now we summarize the main results of the paper. It is shown that oscillating solitons can emit resonant radiation analogous to the synchrotron and cyclotron radiations of moving charges. The resonance condition for the radiation is derived and it is demonstrated that the predicted resonances match the observed spectral lines very precisely. The structure of the emitted radiation and the effect of the recoil of the radiation on the solitons are discussed. The reported results are of general physical nature and thus can be observed not only in fiber cavities but in a wide class of physical systems supporting oscillating solitons.

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