Internal structure of non-Abelian black holes
and nature of singularity

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Abstract

Recent results concerning the internal structure of static spherically-symmetric non-Abelian black holes in the Einstein–Yang–Mills (EYM) theory and its generalizations including scalar fields are reviewed and discussed with an emphasis on the problem of a generic singularity in black holes. It is argued that in the theories admitting a violation of the naive no-hair conjecture the structure of singularity is essentially affected by the “hair roots”. This invalidates an image of a non-Abelian black hole as a Schwarzschild black hole sitting inside the soliton. We give an analytic description of the generic oscillatory approach to the singularity in the pure EYM theory in terms of a divergent discrete sequence and show that the mass function is exponentially growing “in average”. The second type of a generic approach to the singularity in hairy black holes is a “power-law mass inflation” which is realized in the theories including scalar fields. Both singularities are spacelike and no Cauchy horizons are met in the full interior region in conformity with the Strong Cosmic Censorship conjecture. Black holes violating this conjecture exist only for certain discrete values of the event horizon radius thus forming a subset of zero measure.

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1 Introduction

The no-hair conjecture [1] has played an important role in understanding the nature of black holes. As a result of the early investigation of various field theories coupled to gravity it was generally believed that the only fields which may extend outside the event horizon (apart from external ones not related to the black hole itself) are those associated with the conserved or topological charges carried by the hole.

Later a notable counter example to the naive no-hair conjecture was found in the framework of the Einstein–Yang–Mills theory. Although no classical glueballs may exist in the flat spacetime because of purely repulsive nature of the constituent vector fields, such particle-like objects become possible when gravitational attraction is taken into account (Bartnik–McKinnon (BK) solutions [2]). It was shown [3] that the lower mass BK particle topologically is similar to the sphaleron of the Weinberg–Salam theory, with a substantial difference, however, due to existence in the BK case of the gravitational negative mode. Historically, just the gravitational instability of the BK solutions was discovered first [4] and later invoked as an argument favoring the sphaleron interpretation of the BK solutions [5]. More detailed analysis showed, however, that the gravitational (even parity) negative modes of the BK particles had nothing to do with an expected sphaleron picture. Genuine sphaleronic features of BK solutions are manifest via the odd-parity negative modes along the vacuum-to-vacuum direction in the functional space [6]. Physically the lower mass BK solution may be seen as interpolating between the neighboring topologically distinct YM vacua in the EYM theory. It carries neither electric nor magnetic charge, while the Chern–Simons number $1/2$ is naturally associated with it [3, 7].

It was soon realized that the EYM theory also admits the black hole counterparts to the BK solutions [8, 9]. The $SU(2)$ EYM black holes form a two-parameter family labeled by a continuously varying radius of the event horizon $r_h$ and the number $n$ of oscillations of the YM field in the strip $[-1, 1]$ bounded by the neighboring YM vacua values. Since no asymptotic charges are present, these black holes apparently violate the no-hair principle. Other solutions sharing the same property were found in the non-Abelian gauge theories including scalar fields [10, 11, 12] as well as in the gravity coupled Skyrme model. Meanwhile, a common feeling reasonably persisted that this kind of hair should be distinguished from the “allowed” hair associated with
the global charges. It was suggested [13] to interpret a non-Abelian field structure outside the event horizon as a “wig” rather than a genuine hair. Indeed, the EYM black hole can lose its YM hair as a result of the gravitational instability leaving a bare Schwarzschild black hole. As far as the black holes inside the magnetic monopoles are concerned, they were often thought of (at least for the horizon radius much smaller than the monopole size) as tiny Schwarzschild black holes sitting inside the solitons [14]. If so, the inside singularity should remain unaffected by the wig.

However, little was known until recently about the realistic internal structure of non-Abelian black holes (apart from some qualitative discussion in [9]). Closer investigation [15] has shown that the idea of a Schwarzschild black hole sitting inside regular solitons is essentially incorrect. The interior structure and the character of the singularity inside the EYM black hole with an arbitrary (continuously varying) radius of the horizon are strikingly different from those described by the Schwarzschild geometry. Both the Schwarzschild and Reissner–Nordström type singularities (predicted in [9]) are encountered indeed, but only for some discrete values of the horizon radius. This “second quantization” in non-Abelian black holes follows mathematically from the same kind of a non-linear boundary value problem (now in the interior region) as the “first quantization” of the BK particle mass. In this sense the picture of a Schwarzschild black hole inside the BK particle covers only the set of solutions of zero measure in the parameter space. Although it could be anticipated by continuity that the non-Abelian hair should penetrate inside the event horizon, an essential role of hair in the formation of the singularity seems not to have been clearly realized before. A generic singularity in the spherically symmetric \( t \)-independent EYM coupled system turns out to be spacelike and of the oscillatory nature as was often suggested in view of the Bianchi IX analysis by Belinskii, Khalatnikov and Lifshitz (BKL) [16]. In the black hole context, however, the cosmological counterpart is given by the (non-Bianchi) closed Kantowski–Sachs cosmology, and the nature of oscillations observed is rather different from that in the BKL case.

Our results [13] concerning the EYM generic black hole interior solutions were confirmed by Breitenlohner et al. [17] who also attempted to extend the analysis to monopole black holes in the EYM-Higgs (EYMH) theory with the triplet Higgs. In the latter case no oscillations were observed numerically, and the behavior of the mass function near the singularity was found to be monotonous. This behavior was interpreted in [17] as exhibit-
ing an exponential “mass inflation”. Soon after we have found an analytical solution for the mass function \[18\] in the \(SU(2)\) EYMH theories both with the (real) triplet and (complex) doublet Higgs using a consistent truncation of the system of equations near the singularity. Such a truncation is possible also in the EYM–dilaton (EYMD) theory \[19\], in both cases the behavior of the metric near the singularity is dominated by the kinetic (gradient) contribution of the scalar field. In terms of the radial dependence of the mass function one finds a power-law divergence towards the singularity. Therefore no mass-inflation (in the usual sense \[20\]) is detected inside the EYMH black holes where the singularity is dominated by the scalar field. A similar scalar dominated asymptotic behavior was found earlier in the framework of the Kantowski–Sachs cosmology \[21\] driven by a non-linear scalar field (in the Abelian model). In this latter case no black hole counterpart may exist because of the no-hair theorem for a non-linear one-component scalar field, but locally the solution is the same. Our formula for the power-law mass function in the scalar-dominated regime was reproduced later in \[22\] (see this volume) though without necessary restrictions on the power index following from the consistency of the truncation. It is worth noting that the domains of variation of the power index (depending on the horizon radius of the black hole) are different in the EYMH and EYMD cases, so in the Kantowski–Sachs interpretation the corresponding singularities look differently. The scalar-dominated singularity of the “power-law mass-inflationary” type inside spherical black holes in various field models including scalar fields seems to be a rather general phenomenon. This singularity is spacelike in conformity with the Strong Cosmic Censorship. Both oscillatory and the power-law singularities in hairy black holes therefore support this principle by the genericity argument.

The plan of the talk is as follows. First we discuss the local solutions near the singularity which may be seen as dressed Schwarzschild and Reissner–Nordström singularities. Then (Sec. 3) we give an analytic description of the oscillatory approach to the singularity in the generic EYM black holes. In Sec. 4 it is shown that no exponential mass inflation develops inside the EYMH and EYMD black holes where the typical power-law divergence of the mass function is met. We conclude with a brief discussion of a new role of the black hole hair which was revealed in the investigation of black hole interiors.
2 Hairy Schwarzschild and Reissner–Nordström singularities

We start with the pure $SU(2)$ EYM system

$$S = \frac{1}{16\pi} \int \{-R + F^2\} \sqrt{-g}d^4x,$$  \hspace{1cm} (2.1)

where $F$ is the $SU(2)$ field, assuming the metric to be invariant under the time translations and the $SO(3)$ spatial rotations:

$$ds^2 = \frac{\Delta \sigma^2}{r^2}dt^2 - \frac{r^2}{\Delta}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$  \hspace{1cm} (2.2)

Here the metric functions $\Delta$, $\sigma$ depend only on $r$. This parameterization of spacetime is suitable until a turning point of $r$ is reached. It happens that for asymptotically flat solutions there are no such points neither in exterior, nor in interior regions up to the curvature singularity at $r = 0$. Such points exist, however, for asymptotically non-flat solutions, in which case another chart should be used [17]. Here we will not be dealing with $a priori$ asymptotically non-flat solutions, so the coordinates (2.2) are good both in the exterior and interior regions. Of course one should keep in mind that in the region where $\Delta < 0$ the radial coordinate $r$ is timelike, while $t$ is spacelike, so that $t$-independence means spatial homogeneity of the spacetime rather than staticity.

As usual, we choose the $t$-independent spherically symmetric magnetic ansatz for the YM potential

$$A = (W(r) - 1)(T_\varphi d\theta - T_\theta \sin \theta d\varphi),$$  \hspace{1cm} (2.3)

where $T_{\varphi,\theta}$ are spherical projections of the $SU(2)$ generators. The value $W = 1$ is a trivial YM vacuum, while $W = -1$ corresponds to the vacuum of the next topological sector (this state is related to the trivial vacuum by a “large” gauge transformation).

The field equations consist of a coupled system for $W$, $\Delta$

$$\Delta U' + \left(1 - \frac{V^2}{r^2}\right)W' = \frac{WV}{r},$$  \hspace{1cm} (2.4)

$$\left(\frac{\Delta}{r}\right)' + 2\Delta U^2 = 1 - \frac{V^2}{r^2},$$  \hspace{1cm} (2.5)
where $V = (W^2 - 1)$, $U = W'/r$, and a decoupled equation for $\sigma$:

$$\frac{d}{dr^2} \ln \sigma = U^2. \quad (2.6)$$

These equations admit black hole solutions in the domain $r \geq r_h$ for any radius of the event horizon $r_h$ \[8\]. The solutions for $W$ outside the horizon lie within the strip $-1 < W < 1$ approaching $\pm 1$ asymptotically. They are specified by the number $n$ of nodes of $W$ thus forming a discrete set for each $r_h$. Local solutions in the vicinity of the regular event horizon of a given radius contain one free parameter $W(r_h)$ \[9\]. A “quantization” of $W(r_h)$ results from an imposition of the boundary conditions at infinity. Each asymptotically flat exterior solution starts with some uniquely determined value $W_n(r_h)$ at the horizon. All other $W(r_h) \neq W_n(r_h)$ generate asymptotically non-flat solutions. This quantization does not lead to the quantization of the black hole mass since the radius of the horizon still remains an arbitrary continuously varying parameter.

To start the analysis of the internal structure of non-Abelian black holes, which is an essentially numerical problem, one has to explore first local solutions of the field equations near the singularity. Two such local branches were found earlier \[9\]. The first one is the Schwarzschild–like (S) solution, it corresponds to the vacuum value of the YM field $|W(0)| = 1$. Introducing the mass function $m(r)$, $\Delta = r^2 - 2mr$, one has

$$W = -1 + br^2 + b^2(3 - 8b)r^5/(30m_0) + O(r^6),$$
$$m = m_0(1 - 4b^2r^2 + 8b^4r^4) + 2b^2r^3 + O(r^5), \quad (2.7)$$

where $m_0$, $b$ are (the only) free parameters. This local branch is non-generic already by counting free parameters (a generic solution of the system (2.4, 2.5) should have three free parameters).

The second is the Reissner–Nordström type branch which can be found assuming the leading term of $\Delta$ to be a positive constant (related to the charge parameter $P^2 = \Delta(0)$). This requires $W(0) = W_0 \neq \pm 1$ and gives \[9\]

$$W = W_0 - W_0r^2/(2V_0) + cr^3/(2V_0) + O(r^4),$$
$$\Delta = V_0^2 - 2m_0r + r^2 + 2W_0(c + m_0W_0/V_0^2)r^3 + O(r^4), \quad (2.8)$$

what corresponds to the RN metric of the mass $m_0$ and the (magnetic) charge $P^2 = V_0^2$, $V_0 = V(W_0)$. The expansion contains three free parameters $W_0$, $m_0$, $c$ (i.e., is locally generic).
These two local solutions may be regarded as describing “hairy” Schwarzschild and Reissner–Nordström singularities. Their essential distinction from the usual Schwarzschild and Reissner–Nordström singularities consists in presence of the additional free parameters (\(b\) in the first case and \(c\) in the second) responsible for hair degrees of freedom.

We have also found the third local power series solution \([15]\) assuming a negative value for \(\Delta(0)\) (i.e., imaginary \(P\)):

\[
\begin{align*}
W &= W_0 \pm r - W_0 r^2/(2V_0) + O(r^3), \\
\Delta &= -V_0^2 \mp 4W_0V_0r + O(r^2), \\
\sigma &= \sigma_1(r^2 \mp 4W_0r^3/V_0) + O(r^3).
\end{align*}
\] (2.9)

Here there is only one free parameter (\(W_0\)) for \(W, \Delta\). The corresponding space-time near the singularity is conformal to \(R^2 \times S^2\): after a time rescaling one obtains

\[
\begin{align*}
ds^2 &= r^2(dr^2 - dt^2 - d\theta^2 - \sin^2 \theta d\phi^2).
\end{align*}
\] (2.10)

This geometry was encountered in the previous study of black hole interiors in the framework of the perturbed Einstein–Maxwell theory \([23, 24]\) and called homogeneous mass-inflation model (HMI).

It is easy to realize that neither of these asymptotics may correspond to a generic black hole. Imposing boundary conditions in the singularity, we obtain the same kind of the singular boundary value problem as one encountered in the exterior problem where a similar role is played by the asymptotic flatness condition. This interior boundary value problem leads to the second quantization condition, now for the event horizon radius \(r_h\). Therefore, the EYM black holes with the S and RN type interiors may constitute only the zero measure set in the whole EYM black hole solution space.

The system \([2.4, 2.3]\) was integrated numerically in the region \(0 < r < r_h\) using an adaptive step size Runge–Kutta method for various \(r_h = 10^{-8}, ..., 10^6\). The integration started at the left vicinity of the event horizon \(r_h\) where the local power series solution contains one free parameter \(W_h = W(r_h)\) satisfying the inequalities \(|W_h| < 1\) and \(1 - W_h^2 < r_h\) which are the necessary conditions for the asymptotic flatness \([9]\). For given \(r_h\), the interior solutions meeting the expansions \([2.7, 2.3]\) may exist only for some discrete \(W_h\). A numerical strategy used to find such \(W(r_h)\) consisted in detecting the change of sign of the derivative \(W'\). In the S-case we found the curve \(W(r_h)\) which
Table 1: S– and RN–type solutions.

|                      | S–type, n = 1 | RN–type, n = 2 | RN–type, n = 3 |
|----------------------|---------------|----------------|----------------|
| $r_h$                | 0.613861419   | 1.273791       | 1.0318420      |
| $W(r_h)$             | −0.8478649145| −0.113763994   | −0.10185163    |
| $r_-$                | —             | 0.02171654     | 0.08948446     |
| $W(0)$               | −1            | −1.212296124   | −1.3566052     |
| $\sigma(0)$         | 0.2263801     | 5.991210 × 10^{-3} | 1.751928 × 10^{-3} |
| Mass                 | 0.8807931     | 1.018002       | 1.000277       |

starts at $-1$ as $r_h \to 0$ and approaches $-0.1424125$ for large $r_h$ (Fig. 1) (without loss of generality we choose $W_h < 0$). Our S-curve intersects the $n = 1$ branch of the family of trajectories $W^n(r_h)$ corresponding to the set of external asymptotically flat solutions. Parameters of this black hole are shown in Tab. 1, its global behavior is depicted in Fig. 2 (for higher $n$ S-solutions see [17]).

Interior solutions of the RN-type, meeting the expansions (2.8) in the singularity, were found for $r_h > r^*_h = 0.990288617$. The corresponding curve $W(r_h)$ (also shown in Fig. 2) intersects the trajectories $W^n(r_h)$ for all $n \geq 2$. These solutions possess an inner Cauchy horizon at some $r_0 < r_h$ with $|W(r_0)| > 1$ (Fig. 3).

Solutions of the third type (2.9) were studied numerically starting from the vicinity of the origin. The unique solution has been found for the horizon data subject to the necessary conditions for an asymptotic flatness $|W_h| < 1$, $1 - W_h^2 < r_h$ for the upper sign in (2.9) and $W(0) = -0.9330656$, corresponding to $r_h = 1.889088$. This solution, however, does not meet any value $W^n(r_h)$ and thus does not represent a black hole. Thus a pure HMI interior can not be attributed to the EYM black holes. But it turns out to be an unstable fixed point of an asymptotic (truncated) dynamical system [15]. Deviations from this fixed point correspond to generic EYM interior solutions sharing the same property to possess no Cauchy horizons.

More general families of internal solutions not restricted by the asymptotical flatness condition were found in [17] (in particular, other internal solutions of the third type). They do not have, however, a direct significance for the EYM black holes.
Figure 1: $W(r_h)$ for the S– and RN–type interior solutions. Dashed lines — $W^n(r_h)$ for $n = 1, 2$ (higher–$n$ curves lie between the $n = 2$ one and the boundary $r_h = 1 - W_h^2$, dotted line). Note that S and RN curves $W(r_h)$ do not merge.
Figure 2: The $n = 1$ EYM black hole (S-type). EH — events horizon.
Figure 3: The $n = 2$ EYM black hole (RN-type). EH — events horizon, CH — Cauchy horizon.
3 Oscillatory approach to singularity

Since both hairy Schwarzschild and hairy Reissner–Nordström singularities are encountered only for certain discrete values of $r_h$ (and hence the black hole masses), while the external solutions exist for continuously varying $r_h$, we must look for an alternative regime of approach to the singularity for generic $r_h$. It is generically observed during numerical integration from the event horizon towards the origin that sooner or later (depending on $r_h$ and $W_h$) the metric function $\Delta$ starts to oscillate in the negative region with a very fast growing amplitude [15]. Because of huge numbers encountered in these oscillations the accuracy of the computation can hardly be maintained, so one has to seek appropriate truncations of the system of equations to describe the asymptotic regime analytically. Numerically it is observed that, when oscillation progress, the right hand side of the Eq. (2.4) becomes small with respect to terms at the left hand side. Neglecting it one obtains the following approximate first integral of the system:

$$Z = \Delta U \sigma / r = \text{const}, \quad (3.1)$$

which relates oscillations of the mass function to the evolution of $\sigma$. Numerical experiments also show that while the YM function $W$ remains almost constant up to $r = 0$, its derivative is still non-zero and is rapidly changing on some very small intervals of $r$. The function $U$ exhibits a step-like behavior being constant with high accuracy during almost all the chosen oscillation cycle (Fig. 4) and then jumping to a greater absolute value corresponding to the next cycle. It is clear from (2.6) that $\sigma$ is exponentially falling down with decreasing $r$ while $U \approx \text{const}$, whereas in the tiny intervals of $U$–jumps $\sigma$ remains almost unchanged. So $\sigma$ tends to zero through an infinite sequence of exponential falls with increasing powers in the exponentially decreasing intervals. Combining this with (3.1) and the above mentioned properties of $U$ one can deduce rather detailed description of the metric behavior.

Let us denote by $r_k$ the value of radial coordinate where $\Delta$ has $k$-th local maximum. Soon after passing this point, the function $U$ stabilizes at some value $U_k$ approximately equal to the doubled value at the point of local maximum (similarly, $U$ increases by about a factor of two when approaching the local maximum, whereas $\Delta$ is almost stationary). Then, according to
Figure 4: The first oscillation cycle for an EYM solution. The functions $\Delta$ for EYM (solid line) and EYMD (dashed) are shown in lower half-plane. In upper half-plane — mass functions $m(r)$ (analogously) and the function $U$ for EYM (dotted). All functions are power rescaled with the power index $1/10$. Here $r_h = 4$; $W_h = -0.283993$ for EYM, $W_h = -0.298357$, $\phi_h = 0.05623$ for EYMD (asymptotically flat solutions with one node of $W$.)
\[ \sigma(r) \approx \sigma(r_k) \exp \left[ U_k^2 (r^2 - r_k^2) \right], \quad (3.2) \]

From (3.1) one finds that, while \( U_k \approx \text{const} \),
\[ \Delta(r) = \frac{\Delta(r_k)}{r_k} r \exp \left[ U_k^2 (r_k^2 - r^2) \right]. \quad (3.3) \]

This function falls down with decreasing \( r \) until it reaches a local minimum at
\[ R_k = \frac{1}{\sqrt{2|U_k|}} \approx \frac{\sqrt{|\Delta(r_k)|}}{2|V(r_k)|^{1/2} r_k}. \quad (3.4) \]

In what follows, in view of the observed fact that in the course of oscillations of \( \Delta \) the YM function \( W \) changes insignificantly, we will put \( V = \text{const} \).

Therefore, the mass function is inflating exponentially while \( r \) decreases from \( r_k \) to \( R_k \). After passing \( R_k \), an exponential in (3.3) becomes of the order of unity, hence \( \Delta \) starts to grow linearly, and the mass function \( m(r) \) stabilizes at the value \( M_k = m(R_k) \). Such a behavior holds until the point of local maximum of \( \Delta/r^2 \) is reached; this takes place when \( \Delta \approx -V^2 \) at the point
\[ r_k^* \approx \frac{V^2}{|\Delta(r_k)|} r_k \exp \left[ -(U_k r_k)^2 \right]. \quad (3.5) \]

After this a rapid fall of \( |\Delta| \) is observed causing a violent rise of \( |U| \). Then the term \( 2\Delta U^2 \) in the Eq. (2.5) becomes negligible and consequently at this stage
\[ U \Delta \approx -V^2 U_k, \quad (3.6) \]

while \( r \) practically stops. This implies that very soon \( \Delta \) reaches the next local maximum at the point \( r_{k+1} \approx r_k^* \), while \( m(r) \) rapidly falls down to \( m_{k+1} \). At the point of local maximum of \( \Delta \) one has in the Eq. (2.4) \( |\Delta| \ll V^2 \), then in view of the smallness of \( r \) we find
\[ |U(r_k)| \approx \frac{|V|}{\sqrt{2|\Delta(r_k)| r_k}}. \quad (3.7) \]

To obtain the estimates by the order of magnitude we will neglect all numerical coefficients elsewhere except for the power indices in exponentials,
in particular putting $U(r_k) = U_k$, and omitting also (quasi-constant) factors $V$. With this accuracy one obtains from (3.3)–(3.7):

$$r_{k+1} = M_k^{-1}, \quad r_{k+1}^2 = R_k R_{k+1}, \quad M_k = \frac{R_k^2}{r_k^3} \exp \left( \frac{r_k^2}{2R_k^2} \right), \quad (3.8)$$

$$|\Delta(r_k)| = \left( \frac{R_k}{r_k} \right)^2, \quad \frac{r_{k+1}}{r_k} = \frac{r_k^2}{R_k^2} \exp \left[ -\left( \frac{r_k^2}{2R_k^2} \right) \right]. \quad (3.9)$$

Thus, introducing a variable $x_k = (r_k/R_k)^2 (\gg 1)$, we can derive the following recurrent formula

$$x_{k+1} = x_k^{-3} e^{x_k}, \quad (3.10)$$

which shows that $x_k$ is an exponentially diverging sequence. In terms of $x_k$ one has

$$\frac{r_{k+1}}{r_k} = x_k e^{-x_k/2}, \quad (3.11)$$

this relation can also be understood as a ratio of the neighboring oscillation periods since $r_k \gg r_{k+1}$. Values of the function $|\Delta|$ at the points $r_k$ rapidly tend to zero:

$$|\Delta(r_k)| = x_k^{-1}, \quad (3.12)$$

so we deal with an infinite sequence of “almost” Cauchy horizons as $r \to 0$. At the same time the values of $|\Delta|$ at the points $R_k$ grow rapidly:

$$|\Delta(R_k)| = x_k^{-3/2} e^{x_k/2}, \quad (3.13)$$

and the values of the mass function grow correspondingly as

$$\frac{M_k}{M_{k-1}} = x_k^{-1} e^{x_k/2}. \quad (3.14)$$

While $r$ decreases form $r_k$ to $R_k$, the function $\sigma$ rapidly falls down to the value $\sigma_k = \sigma(R_k)$, which then remains unchanged until the point $r_{k+1}$. As the singularity is approached, the sequence $\sigma_k$ decreases according to

$$\frac{\sigma_{k+1}}{\sigma_k} = e^{-x_k/2}. \quad (3.15)$$

Therefore for a generic (continuously varying) $r_h$ one observes a rather unexpected approach to the singularity through an infinite sequence of oscillations with an exponentially growing amplitude.
Apart from the discrete picture given above one can find a truncated two-dimensional dynamical system clearly showing an infinitely oscillating nature of the generic solution. As we have already noted, the oscillation region starts with an exponential fall of $\Delta$ which typically occurs after passing a local maximum $r^{\text{max}}$ (Fig. 5), and the right hand side of (2.4) becomes comparatively small with respect to other terms. Another important feature is that the YM function $W$ (contrary to $\Delta$) possesses a finite limit $W_0$ in the singularity. Omitting the right hand side of (2.4), replacing $W$ by its limiting value $W_0$, and neglecting 1 as compared with $V^2/r^2$ one can derive from the system (2.4) the following two-dimensional dynamical system

\[
\begin{align*}
\dot{q} &= p, \\
\dot{p} &= (3e^{-q} - 1)p + 2e^{-2q} - 1/2,
\end{align*}
\] (3.16)

where $\Delta = -(V_0^2/2)\exp(q)$, and a dot stands for derivatives with respect to $\tau = 2\ln(r_h/r)$. This system has one (focal) fixed point $(p = 0, q = \ln 2)$, corresponding to some imaginary charge RN-like local solution of the type (2.9), with eigenvalues $\lambda = (1 \pm i\sqrt{15})/4$. Its phase portrait is shown in Fig. 6 together with an invariant set $p = -e^{-q} - 1/2$ corresponding to the RN-type local solution (2.8). The generic oscillating solutions lie above this curve. The phase motion in this region is unbounded, and there are no limiting circles. One can easily show that the rotation in the phase plane never stops and the limit $q = -\infty$ ($\Delta = 0$) can not be reached. The metric function $\Delta$ remains negative valued as $r \to 0$ and passes an infinite sequence of local maxima and minima. The class of oscillating solutions is stable in a sense that a small deviation from one such solution moves us to another oscillating solution. Thus the existence of the consistent two-dimensional truncation of the full set of equations proves that the oscillating approach to the singularity is generic for the EYM black holes. The dynamical system (3.16) was derived in our paper [15] and reproduced in a slightly different notation in [17].

4 Power-law “mass-inflationary” singularity

In non-Abelian theories including scalar fields, such as EYMH and EYMD, another regime of the generic approach to the singularity is realized. In this case the scalar kinetic term becomes dominant soon after entering an
Figure 5: The beginning of Δ–oscillations for $n = 1$ EYM BH solution, $r_h = 2.4$, $W(r_h) = -0.31652531$. EH — events horizon.
Figure 6: Phase portrait of the dynamical system (3.16), RN — an invariant set, corresponding to the RN–type solution (dashed — zero slope lines).
asymptotic regime what results in a monotonous behavior of the metric. Consider first the \(SU(2)\) EYMH theory
\[
S = \frac{1}{16\pi} \int \left\{ -R - F^2 + 2|D\Phi|^2 - \frac{\lambda}{2}(|\Phi|^2 - \eta^2)^2 \right\} \sqrt{-g} d^4x,
\]
where \(\Phi\) is a Higgs field in either vector (real triplet) or fundamental (complex doublet) representations, \(D\Phi\) is the corresponding YM covariant derivative (in the doublet case \(|D\Phi|^2 = (D\Phi)^\dagger D\Phi\), \(|\Phi|^2 = \Phi^\dagger \Phi\)) and without loss of generality both the Planck mass and the gauge coupling constant are set to unity. In the flat space-time the triplet version of the theory gives rise to regular magnetic monopoles, the doublet version — to sphalerons. New physically interesting configurations emerge when gravity is coupled in a self-consistent way, in particular, static spherical black holes exist in both cases: monopole [11] and sphaleron [12].

Static spherically symmetric configurations of the YM fields are still given by the ansatz (2.3), while the Higgs field is \(\Phi^a T_a = \phi(r) T_r\) in the triplet case, and \(\Phi = \phi(r) v\) in the doublet one, where \(v\) is some (here irrelevant) spinor depending only on the angle variables. In both cases \(\phi(r)\) is the only real scalar function of the radial variable.

The system of equations following from (4.1) with this ansatz may be presented as a set of three coupled equations for \(W, \phi\), and \(\Delta = r^2 - 2mr\):
\[
\left( \frac{\Delta}{r^2} W' \right)' + \frac{\Delta}{r} W' \phi'^2 = \frac{1}{2} \frac{\partial V}{\partial W} - Q \frac{W'}{r}, \quad (4.2)
\]
\[
\left( \frac{\Delta}{r} \right)' + \Delta \phi'^2 = 1 - 2V - Q, \quad (4.3)
\]
\[
(\Delta \phi')' + \Delta r \phi'^3 = \frac{\partial V}{\partial \phi} - Q r \phi', \quad (4.4)
\]
where \(Q = 2\Delta W'^2/r^2\), and
\[
V \equiv V(W, \phi, r) = \frac{V^2}{2r^2} + \frac{\lambda r^2}{8}(\phi^2 - \eta^2)^2 + P^2, \quad (4.5)
\]
with \(P = W \phi\) in the triplet case and \(P = (W + 1) \phi\) in the doublet one. An equation for \(\sigma\) now becomes
\[
(\ln \sigma)' = \frac{2}{r} W' r^2 + r \phi'^2, \quad (4.6)
\]
and can be easily integrated once $W, \phi$ are found.

Local expansions of three types listed in the Sec. 2 can be easily generalized to include Higgs \cite{17}, and the same arguments can be used to prove that neither of the corresponding global solutions is generic. Meanwhile, the forth local branch can be found analytically in this case \cite{18}. It can be derived using the consistent truncation of the non-linear systems of equations. An appropriate truncation consists in omitting from the equations all matter terms except for those related to the gradient of the scalar field. This reduces to dropping the right hand sides in Eqs. (4.2–4.4). The resulting system can be easily disentangled leading to the following decoupled equations:

\begin{align}
W'' - \frac{W'}{r} &= 0, \\
\phi'' + \frac{\phi'}{r} &= 0,
\end{align}

which can be solved as follows:

\begin{align}
W &= W_0 + br^2, \\
\phi &= \phi_0 + k \ln r.
\end{align}

Here $W_0, \phi_0, b, k$ are free parameters. Note that contrary to the expansions discussed in the Sec. 2, this is not a power series solution. Higgs field is logarithmically divergent, so that its derivative diverges as $r^{-1}$, this is why the corresponding terms become dominant for sufficiently small $r$. Once $W$ and $\phi$ are found, the metric function $\Delta$ can be obtained by integrating the simple equation

\begin{equation}
\frac{\Delta'}{\Delta} = \frac{1}{r} - r\phi'^2,
\end{equation}

what gives

\begin{equation}
\Delta = -2m_0r^{(1-k^2)}
\end{equation}

with the fifth (positive) constant $m_0$. Hence, by counting free parameters, this is a generic solution with non-positive $\Delta$. Now, to find whether our truncation, one has to substitute the solution into the Eqs. (4.2–4.4) and to check whether the right hand side terms are indeed comparatively small. One finds the following condition: $k^2 > 1$. This means that the metric function
\( \Delta \) is divergent at the singularity. The corresponding mass-function is also divergent according to the power-law
\[
m = \frac{m_0}{r^{k^2}}. \tag{4.13}
\]
The asymptotic behavior of \( \sigma \) dominated by the scalar term then reads
\[
\sigma = \sigma_0 r^{k^2}, \tag{4.14}
\]
with (positive) constant \( \sigma_0 \).

This local solution can be interpreted as exhibiting a “power-law mass inflation”. As a matter of fact this regime has nothing to do with the usual (exponential) mass-inflation which presumably takes place once a Cauchy horizon is approached.

The second example of the scalar-dominated singularity is given by the EYMD theory:

\[
S = \frac{1}{16\pi} \int \left\{ -R + 2(\nabla \phi)^2 - e^{-2\phi} F^2 \right\} \sqrt{-g} d^4x, \tag{4.15}
\]
The equations of motion for \( W, \Delta, \phi \) now take the form
\[
\Delta U' - 2\Delta U \phi' = WV/r - F W', \tag{4.16}
\]
\[
(\Delta/r)' + \Delta \phi'^2 = F - 2\Delta U^2 e^{-2\phi}, \tag{4.17}
\]
\[
(\Delta \phi')' + \Delta r \phi'^3 = F - 2\Delta (\phi/r + 1)U^2 e^{-2\phi} - 1, \tag{4.18}
\]
where \( F = 1 - V^2 e^{-2\phi} r^{-2}, \quad V = W^2 - 1 \). The remaining equation for \( \sigma \) reads
\[
(ln \sigma)' = r \left( \phi'^2 + 2U^2 e^{-2\phi} \right). \tag{4.19}
\]

Similarly to the EYMH case for sufficiently small \( r \) the right sides of the Eqs. (4.16)–(4.18) become small in comparison with the left hand side terms and one gets the following truncated system
\[
(ln U)' - 2\phi' = 0, \quad [ln(\Delta/r)]' = [ln(\Delta \phi')]' = -r \phi'^2. \tag{4.20}
\]
Its integration gives the following five-parameter (i.e., generic) family of solutions
\[
W = W_0 + br^{2(1-\lambda)}, \quad \Delta = -2\mu r^{(1-\lambda^2)}, \quad \phi = c + \ln \left( r^{-\lambda} \right), \tag{4.21}
\]
\[20\]
with constant $W_0$, $b$, $c$, $\mu$, $\lambda$. The validity of the truncated equations (4.20) now can be checked by substituting the asymptotic solution (4.21) into the full system (4.16)–(4.18). For consistency it is sufficient that the following inequalities hold:

$$\sqrt{2} - 1 < \lambda < 1,$$

which is in agreement with the numerical data.

From (4.21) it follows that the mass function diverges as $r \to 0$ according to the power law:

$$m(r) = \frac{\mu}{r^{\lambda^2}}.$$

The corresponding $\sigma$ tends to zero as

$$\sigma(r) = \sigma_1 r^{\lambda^2},$$

where $\sigma_1 = \text{const}$. A typical EYMD solution is shown in Fig. 4.

The only difference with the EYMH case is that the region of the power index (4.22) is different. This leads to different picture of the singularity in the Kantowski-Sachs interpretation: a point like singularity in the EYMH case, but a cigar singularity in the EYMD case [18].

5 Discussion

Non-Abelian black holes turned out to be a useful laboratory to explore the nature of the singularity inside “realistic” black holes. Unlike the gravity coupled Abelian field models such as Maxwell, Maxwell–dilaton–axion, or more general string-inspired systems, non-Abelian models have an advantage to give rise to several qualitatively different possibilities as far as the singularity is concerned. Moreover, in spite of the absence of exact analytic solutions, one can find a correspondence between the singularity structure and external parameters such as the black hole mass. Thus the relative weight of different type interior structures in the parameter space may be found. This opens a new way to probe the Strong Cosmic Censorship using the genericity argument. An advantage of this approach is that a non-trivial information may be extracted already at the level of static (“eternal”) black hole solutions. One can speculate that non-Abelian electroweak theory should replace the Maxwell electrodynamics at high energies which are reached in the course of
the mass-inflation inside a (perturbed) Reissner–Nordstrøm black hole due to the phase transition similar to that in cosmology. Therefore a further fate of the black hole should be in the scope of a non-Abelian theory. Although the full time-dependent picture is likely to be quite complicated, it is reassuring to feel that the static spherical non-Abelian black holes choose the spacelike singularity in conformity with the Strong Cosmic Censorship.

The new features observed in the interiors of non-Abelian black holes are due to the field components which are not connected with the conserved charges and are usually termed as a hair. Violation of the no-hair conjecture in black holes traditionally was attributed to their external appearance. New results clearly show that the hair is equally important for the internal structure of black holes. Hair “roots” penetrate up to the singularity supplying it with additional degrees of freedom. A tiny black hole inside a large magnetic monopole generically has a very different internal structure than the Schwarzschild black hole. Hairy black holes have hairy singularities.

Qualitatively speaking, the interiors of the EYM black holes look like the hair-perturbed Schwarzschild or Reissner–Nordstrøm interiors. So it is not surprising that one encounters an exponential growth of mass whenever the metric reaches an “almost” Cauchy horizon. This phenomenon, first noted in [13] and termed as “mass inflation” in [17], is, however, only a half of the story. In the oscillating regime one observes two qualitatively different “mass-inflations”: the first is the local inflation at some very short interval of each oscillation cycle, the second is associated with the exponential growth of mass from cycle to cycle [19] while the singularity is approached.

If scalar fields are present, the dynamics gets a new dimension. Once the scalar component is excited, it soon becomes a dominant factor of the evolution which changes drastically the approach to the singularity. In the scalar-dominated regime no mass-inflation is manifest, instead one observes a behavior which we call a “power-law mass-inflation”, i.e., a power-fashion divergence of the mass-function near the singularity. This behavior has a different origin as compared with the usual mass-inflation, in particular, it is not related to an approach to the Cauchy horizon. Rather it follows from the coupled Einstein–scalar field dynamics which is essentially Abelian. Indeed, the same behavior near the singularity was earlier observed in the Kantowski–Sachs cosmology with a non-linear (one-component) scalar field source. Still the non-Abelian nature of the model is substantial, otherwise the (asymptotically flat) static black holes are prohibited by the no-hair theorems.
The analysis given here is purely classical, a few words are in order about the relevance of the results to the full quantum theory. Vacuum polarization of the conformal scalar field on the HMI background was considered in [24]. It was found that the correction to the mass function diverges more strongly than in the classical case. Hence the singularity is not smoothened but rather intensified. In the oscillating regime there is no hope to compute quantum effects quasi-classically. Moreover, huge values of the mass function (in Planck’s units) encountered soon after entering such a regime indicate that the quantum behavior of the model should be considered nonperturbatively and may well be qualitatively different from the classical picture. However, the conclusion about the spacelike nature of the generic singularity is unlikely to be changed.

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