Resolving by a free action linear category and applications to Hochschild-Mitchell (co)homology

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Abstract

Let $G$ be a group acting on a small category $C$ over a field $k$, that is $C$ is a $G$-$k$-category. We first obtain an unexpected result: $C$ is resolvable by a category which is $G$-$k$-equivalent to it, on which $G$ acts freely on objects.

This resolving category enables to show that if the coinvariants and the invariants functors are exact, then the coinvariants and invariants of the Hochschild-Mitchell (co)homology of $C$ are isomorphic to the trivial component of the Hochschild-Mitchell (co)homology of the skew category $C[G]$.

If the action of $G$ is free on objects, then there is a canonical decomposition of the Hochschild-Mitchell (co)homology of the quotient category $C/G$ along the conjugacy classes of $G$. This way we provide a general frame for monomorphisms which have been described previously in low degrees.

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1
1 Introduction

Let $k$ be a field. A $k$-category $C$ is a small category enhanced over the category of $k$-vector spaces. B. Mitchell in [25] called these categories "algebras with several objects". Indeed, a $k$-category $C$ with only a finite set of objects provides a $k$-algebra $\mathcal{a}(C)$ through the direct sum of all its morphisms. B. Mitchell introduced Hochschild-Mitchell (co)homology of $C$, see also [7, 13, 20, 23]. If the number of objects of $C$ is finite, then the Hochschild-Mitchell (co)homology of $C$ is isomorphic to the Hochschild (co)homology of $\mathcal{a}(C)$, see for instance [7].

Let $G$ be a group. A $G$-$k$-category is a $k$-category $C$ with an action of $G$ on $C$. More precisely there is a group homomorphism $G \to \text{Aut}_k C$, where $\text{Aut}_k C$ is the group of $k$-functors $C \to C$ which are isomorphisms. In this situation there exists a skew category $C[G]$. If the number of objects of $C$ is finite, then $\mathcal{a}(C[G])$ is isomorphic to the usual skew group algebra $\mathcal{a}(C)[G]$, see [6].

The action of $G$ on $C$ is called free if the action on the objects of $C$ is free. In this case the quotient $k$-category $C/G$ exists, see for instance [7]. Moreover the functor $C \to C/G$ is a Galois covering. This construction has several applications in representation theory, see for instance [3, 28, 12, 1, 19].

A central result obtained in [6] is that if the action of $G$ is free, then $C/G$ and $C[G]$ are equivalent $k$-categories. However $C[G]$ exists for any action, while a free action is essential for defining $C/G$. Hence for a non free action $C[G]$ is a substitute of $C/G$.

In the same vein, in this paper we first introduce in Section 2 a resolving main result. Namely let $C$ be a $G$-$k$-category. There exists a $G$-$k$-category $M_G(C)$ - called the resolving category - which has a free action of $G$ and which is $G$-$k$-equivalent to $C$. The resolving category is related to the infinite matrix algebra considered by J. Cornick in [9], which in turn is linked with Cohen-Montgomery duality in [5], see also [8].

In Sections 3 and 4 we will use the resolving category for Hochschild-Mitchell homology and cohomology. Indeed, for a $G$-$k$-category $C$ the resolving category enables to relate the Hochschild-Mitchell (co)homologies of $C$, of $C/G$, and of $C[G]$. In doing so, we underline that we do not change the bimodules of
coefficients, in contrast with the Cartan-Leray type spectral sequence obtained in [7], see also [17] and [26].

In Subsection 3.1 we show that there is a direct sum decomposition of the Hochschild-Mitchell homology of a \( G \)-graded \( k \)-category along the set of conjugacy classes of \( G \), as in the case of \( G \)-graded algebras, see [9, 22, 30].

For instance the homology of the graded category \( C[G] \) decomposes as above. Suppose that the action is free and the coinvariants functor is exact. In Subsection 3.2 we prove that there is an isomorphism between the trivial conjugacy class direct summand of the homology of \( C[G] \), and the coinvariants of the homology of \( C \).

The resolving category and results of E. Herscovich in [16] enables then to prove that the isomorphism quoted above also holds when the action is not free.

In Subsection 3.3 we infer the analog results for a Galois covering.

For Hochschild-Mitchell cohomology, the defining cochains are direct products of vector spaces while for homology the chains are direct sums. Nevertheless, for a graded \( k \)-category, we get to show in Subsection 4.1 that the complex of cochains is also a direct product along the conjugacy classes of \( G \). Moreover, the subcomplex associated to the trivial conjugacy class is a subdifferential graded algebra of it.

If the action is free and if the invariants functor is exact, we show that there is an isomorphism of algebras between the trivial component of the cohomology of \( C[G] \) and the invariants of the cohomology of \( C \). Note that the proof is quite different than the one for homology. The resolving category enables then to extend the result for a non free action.

In Subsection 4.3 we translate these results into the Galois covering setting. An immediate consequence is that the invariants of the cohomology of \( C \) are a canonical direct summand of the cohomology of \( C/G \). We recover this way the monomorphism obtained in [24] which is made explicit in low degrees in [15].

In Section 5 we restrict our results to the case of a \( k \)-algebra with a finite group acting by automorphism, and its Hochschild cohomology. We underline that the proof relies in an essential way on the existence of the resolving category.

When specialising to algebras, the results of this work are related with explicit computations made for instance in [11, 14, 29, 27] for Hochschild (co)homology of specific skew group algebras, in particular for the symmetric algebra over a finite dimensional vector space \( V \) over a field \( k \), with \( G \) a finite subgroup of \( GL(V) \) which order is invertible in \( k \).

Finally we observe that the results obtained in this work rely on the standard complexes of chains and of cochains which compute the Hochschild-Mitchell (co)homology of a \( k \)-category. It is well known that these complexes are generally not suitable for effective computations. A similar work on reduced or normalised resolutions will allow to consider concrete situations where our theoretical results may apply. This could be a subject of another paper.

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2 The resolving category with free action

Let $k$ be a field. A $k$-category is a small category $C$ enriched over the category of $k$-vector spaces. In other words the objects of $C$ are a set denoted $C_0$, for any pair of objects $x, y \in C_0$ the space of morphisms $yC_x$ from $x$ to $y$ has a structure of vector space, the composition in $C$ is $k$-bilinear, and the image of the canonical inclusion $k \hookrightarrow yC_x$ is central in $xC_x$ for all $x \in C_0$. In particular $xC_x$ is a $k$-algebra for any $x \in C_0$.

We often write $yf_x$ for a morphism $f$ from $x$ to $y$, that is belonging to $yC_x$.

Definition 2.1 Let $G$ be a group. A $G$-$k$-category is a $k$-category $C$ with an action of $G$ by $k$-isomorphisms of $C$.

Remark 2.2 Equivalently, a $G$-$k$-category $C$ is a $k$-category with an action of $G$ on the set of objects $C_0$ such that for all $s \in G$ and $yf_x \in yC_x$ there is an element $sy(sf)_{sx} \in syC_{sx}$. Moreover, the map $yC_x \to syC_{sx}$ given by $f \mapsto sf$ is $k$-linear. For all $s, t \in G$ and for any morphism $f$ we have $t(sf) = (ts)f$. Finally for any object $x$ and any $s \in G$, we have $s(x1_x) = sx1_{sx}$.

Example 2.3 Let $\Lambda$ be a $k$-algebra and let $\Lambda_1$ be the single object $k$-category where the endomorphism algebra of the object is $\Lambda$. Let $G$ be a group acting by algebra automorphisms of $\Lambda$. Then $\Lambda_1$ is a $G$-$k$-category.

Definition 2.4 A $G$-$k$-category $C$ is a category with a free action of $G$ if the action of $G$ on its objects is free. That is if $s \in G$ and $x \in C_0$, then $sx = x$ only holds if $s = 1$.

Remark 2.5 Except when $G$ is trivial, there is no free action of $G$ on the $G$-$k$-category $\Lambda_1$ of Example 2.3.

Recall that for a finite object $k$-category $C$, its $k$-algebra is $a(C) = \bigoplus_{x,y \in C_0} yC_x$ with product given by matrix multiplication combined with the composition of $C$. The identity element of $a(C)$ is the sum of the identities of the objects.

Remark 2.6 Notice that if $C$ and $D$ are finite object $k$-categories, a $k$-functor $F : C \to D$ does not provides in general a multiplicative $k$-morphism between $a(C)$ and $a(D)$.

The following result is a straightforward generalisation of Example 2.3.

Lemma 2.7 Let $C$ be a $G$-$k$-category with a finite set of objects. The group $G$ acts on $a(C)$ by algebra automorphisms.
Next we will show that a $G$-$k$-category $C$ is resolvable by a $G$-$k$-category with a free action of $G$. That is the resolving category is $G$-$k$-equivalent to $C$.

**Definition 2.8** (see also [9], [10]) Let $C$ be a $G$-$k$-category. The objects of the resolving $k$-category $M_G(C)$ are $G \times C_0$. The $k$-vector space of morphisms of $M_G(C)$ from $(s, x)$ to $(t, y)$ is

$$(t, y)(M_G(C))(s, x) = yC_x.$$ 

The composition is given in the evident way by the composition of $C$. The action of $G$ on $M_G(C)$ is defined as follows: for $r \in G$, let $r(s, x) = (rs, rx)$, and for $f \in (t, y)(M_G(C))(s, x)$ let

$$rf \in yC_x, \\ f \in (t, y)(M_G(C))(s, x) = yC_x \text{ let } r f \in yC_x = (rt, ry)(M_G(C))(rs, rx).$$

Notice that the above action of $G$ is free on the objects of $M_G(C)$.

**Example 2.9** Let $\Lambda$ be a $k$-algebra with a group acting by automorphisms, and let $\Lambda_1$ be its single object $G$-$k$-category.

- The resolving category $M_G(\Lambda_1)$ has set of objects $G$.
- Each vector space of morphisms of $M_G(\Lambda_1)$ is a copy of $\Lambda$.
- The composition of $M_G(\Lambda_1)$ is given by the product of $\Lambda$.
- The action of $G$ on the objects is the product of $G$. On morphisms it is given by the action on objects followed by the action on $\Lambda$.

Next we describe the resolving category of $C$ as a tensor product of categories.

**Definition 2.10** Let $C$ and $D$ be $k$-categories. Their tensor product $C \otimes D$ has set of objects $C_0 \times D_0$. Its morphisms are given by:

$$(c', d')(C \otimes D)(c, d) = c'C_c \otimes d'D_d$$

with the obvious composition.

If $C$ and $D$ are $G$-$k$-categories then $C \otimes D$ is a $G$-$k$-category through the diagonal action of $G$.

**Remark 2.11** Let $C$ be a $G$-$k$-category. Let $k_1$ be the $G$-$k$-trivial category, with one object whose endomorphisms are given by $k$ and trivial $G$-action. We have $M_G(C) = M_G(k_1) \otimes C$.

**Theorem 2.12** Let $C$ be a $G$-$k$-category. There is an equivalence of $G$-$k$-categories $L : M_G(C) \rightarrow C$. 

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Proof. Let $L : M_G(C) \rightarrow C$ be the functor defined on the objects by $L(s, x) = x$, while on morphisms $L$ is given by the suitable identity maps. Hence $L$ is a fully faithful $G$-functor which is surjective on the objects, so it is an equivalence of $G$-$k$-categories.

We end this section by describing the algebra associated to a resolving category.

**Proposition 2.13** Let $\Lambda$ be a $k$-algebra with an action of a finite group $G$ by automorphisms of $\Lambda$. Let $M_G(\Lambda_1)$ be the resolving category of $\Lambda_1$.

The $k$-algebra $a(M_G(\Lambda_1))$ is isomorphic to the matrix algebra $M_G(\Lambda)$ with columns and rows indexed by $G$. The action of $G$ on a matrix is the combination of the action of $G$ on $\Lambda$ with the action of $G$ on the indices of the rows and of the columns.

The proof relies on the description given at example 2.9.

**Remark 2.14** Let $E$ be the set of diagonal idempotents of the algebra above $M_G(\Lambda)$, each one is given by $1 \in \Lambda$ on a certain spot of the diagonal and zeros elsewhere. The number of elements of $E$ is $|G|$.

The group $G$ acts freely on $E$.

## 3 Hochschild-Mitchell homology

### 3.1 $G$-$k$-categories and graded $k$-categories

We begin this section by recalling the definition of the Hochschild-Mitchell homology of a $k$-category, see for instance [25, 20, 23, 7]. Next we give properties of it for $G$-$k$-categories.

Moreover we provide the decomposition of the homology of a graded category along the conjugacy classes of $G$. This will be used in Subsection 3.2 for skew categories.

**Definition 3.1** Let $C$ be a $k$-category and let $C_\ast(C)$ be the chain complex given by:

$$C_n(C) = \bigoplus_{x_0, x_1, \ldots, x_n \in C_0} x_0 C_{x_0} \otimes \cdots \otimes x_2 C_{x_1} \otimes x_1 C_{x_0},$$

with boundary map $d$ given by the usual formulas used to compute the Hochschild homology of an algebra, see for instance [21, 32, 4]. Note that $C_0(C) = \bigoplus_{x \in C_0} x C_x$.

The Hochschild-Mitchell homology $HH_\ast(C)$ of $C$ is the homology of the complex above.
3.1.1 \( G \)-\( k \)-categories

**Proposition 3.2** Let \( C \) be a \( G \)-\( k \)-category. The group \( G \) acts on the chain complex \( C_\bullet(C) \) by automorphims.

**Proof.** For \( s \in G \) and \((f_n \otimes \cdots \otimes f_1 \otimes f_0) \in C_n(C)\) we define
\[
s f = (s f_n \otimes \cdots \otimes s f_1 \otimes s f_0)
\]
and we have \( d(sf) = sd(f) \). For simplicity we provide the verification for \( n = 2 \), the general case follows the same kind of computations.
\[
d(sf) = sx_0(s(f_2 f_1))_{sx_1} \otimes sx_1(s(f_0)_{sx_0} - sx_0(s(f_2)_{sx_2} \otimes sx_2(s(f_1)_{sx_1} +
\]
and
\[
sd(f) = sx_0(s(f_2 f_1))_{sx_1} \otimes sx_1(s(f_0)_{sx_0} - sx_0(s(f_2)_{sx_2} \otimes sx_2(s(f_1)_{sx_1} +
\]

\( \Box \)

**Corollary 3.3** Let \( C \) be a \( G \)-\( k \)-category. The Hochschild-Mitchell homology \( HH_*\)(\( C \)) is a \( kG \)-module.

Recall that for a \( kG \)-module \( M \), the \( kG \)-module of coinvariants of \( M \) is
\[
M_G = M/ \langle sm - m \rangle
\]
where the denominator is the sub \( kG \)-module of \( M \) generated by \( \{ sm - m \mid m \in M, s \in G \} \). The module of coinvariants is the largest quotient of \( M \) with trivial action of \( G \). Considering \( M \) as a \( kG \)-bimodule with trivial action on the right, we have \( M_G = kG \otimes_{kG \otimes (kG)^{op}} M \).

If \( G \) is of finite order invertible in \( k \), then the coinvariants functor is exact. Moreover, \( M_G \) is canonically isomorphic to the invariants
\[
M^G = \{ m \in M \mid sm = m \text{ for all } s \in G \}
\]
through the morphism \( m \mapsto \frac{1}{|G|} \sum_{s \in G} sm \).

3.1.2 Graded categories

**Definition 3.4** Let \( G \) be a group. A \( k \)-category \( B \) is \( G \)-graded if for all \( x, y \in B_0 \) there is a direct sum decomposition of vector spaces
\[
yB_x = \bigoplus_{s \in G} yB^s_x
\]
such that \( yB^s_y \subset yB^t_y \) for all objects \( x, y, z \in B_0 \) and for every \( s, t \in G \). A morphism \( f \in yB^s_x \) is called homogeneous of degree \( s \) from \( x \) to \( y \), we often write it \( yf^s_x \) instead of \( f \).
Next we provide a decomposition of the chain complex of a graded category. This corresponds to M. Lorenz [22] decomposition for the Hochschild homology of a \( G \)-graded \( k \)-algebra (see also [31]).

**Proposition 3.5** Let \( B \) be a \( G \)-graded \( k \)-category. Let \( D \) be a conjugacy class of \( G \). The following is a subcomplex of \( C_\bullet(B) \):

\[
C^D_n(B) = \bigoplus_{s_n, \ldots, s_0 \in D} x_0 B^s_{x_n} \otimes \cdots \otimes x_2 B^s_{x_1} \otimes x_0 B^s_{x_0}.
\]

Let \( Cl_G \) be the set of conjugacy classes of \( G \). There is a decomposition

\[
C_\bullet(B) = \bigoplus_{D \in Cl_G} C^D_\bullet(B).
\]

**Proof.** We provide the verification for \( n = 2 \), since it already gives the whole track to prove the result for all \( n \). This check has the advantage of an easier reading, it presents the main ideas of the calculations, without too many technical details.

Let \( x_0 f_2 x_2 \otimes x_2 f_1 x_1 \otimes x_1 f_0 x_0 \in C^D_2(B) \), with \( s_2 s_1 s_0 \in D \). We have

\[
d(f_2 \otimes f_1 \otimes f_0) = f_2 f_1 \otimes f_0 - f_2 \otimes f_1 f_0 + f_0 f_2 \otimes f_1 \in (x_0 B^s_{x_1} \otimes x_1 B^s_{x_0}) \oplus (x_0 B^s_{x_2} \otimes x_2 B^s_{x_0}) \oplus (x_1 B^s_{x_2} \otimes x_2 B^s_{x_1}).
\]

Note that for the last summand \( s_0 s_2 s_1 = s_0 (s_2 s_1 s_0)^{-1} \in D \).

\( \diamond \)

**Theorem 3.6** Let \( B \) be a \( G \)-graded \( k \)-category and let \( HH^D_\bullet(B) = H_\bullet(C^D_\bullet(B)) \). There is a decomposition

\[
HH_\bullet(B) = \bigoplus_{D \in Cl_G} HH^D_\bullet(B).
\]

### 3.2 Skew categories

In this section we compare the coinvariants of the Hochschild-Mitchell homology of a \( G \)-\( k \)-category \( C \) with the trivial component of the homology of the skew category \( C[G] \). Indeed \( C[G] \) (defined below) is graded, hence we will use Theorem 3.6.

**Definition 3.7** [6] Let \( C \) be a \( G \)-\( k \)-category. The skew category \( C[G] \) has the same set of objects than \( C \). Let \( y^C[G]_x = y^C_{ax} \). The morphisms of \( C[G] \) from \( x \) to \( y \) are

\[
y^C[G]_x = \bigoplus_{s \in G} y^C[G]^s_x.
\]

(3.1)
The composition is defined through adjusting the first morphism in order to make it possible to compose it in \( C \) with the second one, as follows. If 
\[
yf_{sx} \in yC_{sx} \subseteq yC[G]_x \quad \text{and} \quad zg_{ty} \in zC_{ty} \subseteq zC[G]_y,
\]
then 
\[
(zg_{ty})(yf_{sx}) = z(g \circ tf)_{tsx} \in zC[G]_x,
\]
where \( \circ \) denotes the composition of \( C \).

**Remark 3.8**

1. By definition, the direct summands of (3.1) are in one to one correspondence with elements of \( G \). However some of the summands may be zero.

2. If \( C \) is a single object \( G \)-\( k \)-category with endomorphism algebra \( \Lambda \), it is shown in [6] that the endomorphism algebra of the single object \( k \)-category \( C[G] \) is the usual skew group algebra. Namely as a vector space, \( \Lambda[G] = \Lambda \otimes kG \). For \( a, b \in \Lambda \) and \( s, t \in G \), the product is given by 
\[
(a \otimes s)(b \otimes t) = as(b) \otimes st.
\]

**Lemma 3.9** The category \( C[G] \) is graded.

**Proof.** We have
\[
zC[G]_y^t \cdot yC[G]_x^s \subseteq zC[G]_x^{ts}.
\]

Hence the decomposition of Theorem 3.6 holds for the homology of \( C[G] \).

### 3.2.1 Free action

To prove the next result we first need to restrict ourselves to a free action. Later we will be able to consider the general case by using the resolving category of Section 2.

**Theorem 3.10** Let \( C \) be a \( G \)-\( k \)-category with free action of \( G \). Let \( \{1\} \) be the trivial conjugacy class of \( G \). There is an isomorphism
\[
HH^{(1)}_*((C[G])) \simeq H_*(((C_{\bullet}(C))_G).
\]

If the coinvariants functor is exact then
\[
HH^{(1)}_*((C[G])) \simeq H_*(C)_G.
\]

Next we provide properties of a skew category that we will need for proving Theorem 3.10. The following result is proved in [6], we give a proof for completeness.
Lemma 3.11 Let \( C \) be a \( G \)-\( k \)-category. Let \( x \) and \( y \) be objects in the same orbit of the action of \( G \). They are isomorphic in \( C[G] \).

Proof. Let \( t \in G \) such that \( y = tx \). Recall that \( t_* C[G]_x = \bigoplus_{s \in G} t_* C_{sx} \). Let \( a = t_* 1_{tx} \in t_* C[G]_x \) and \( b = x_1 \in x C[G]_{tx} \). Let \( a = tx_1 = t_* C_{tx} \) and \( b = x_1 \in x C[G]_{tx} \).

Using the composition defined in \( C[G] \), we obtain that \( a \) and \( b \) are mutual inverses in the skew category.

\[ \therefore \]

Definition 3.12 Let \( G \) be a group acting on a set \( E \). A transversal \( T \) of the action is a subset of \( E \) consisting of exactly one element in each orbit of the action.

Equivalently \( T \subseteq E \) is a transversal if for each \( x \in E \) there exists a unique \( u(x) \in T \) such that there exists some \( s \in G \) with \( x = su(x) \).

Note that the action is free if and only if for each \( x \in E \), there exists a unique \( s \in G \) such that \( x = su(x) \).

Lemma 3.13 Let \( C \) be a \( G \)-\( k \)-category, let \( T \subseteq C_0 \) be a transversal of the action of \( G \) on \( C_0 \), and let \( C_T[G] \) be the full subcategory of \( C[G] \) with set of objects \( T \). For each conjugacy class \( D \) of \( G \) we have

\[ HH^D_s (C_T[G]) = H^D_s (C[G]). \]

Proof. In [10] E. Herscovich (see also [2, 18]) proved that if \( C \) and \( D \) are \( k \)-categories and if \( F : C \to D \) is a \( k \)-equivalence, then \( F \) induces a quasi-isomorphism \( C_\bullet(C) \to C_\bullet(D) \). Moreover, if \( C \) and \( D \) are \( G \)-graded and if \( F \) is homogeneous, then the induced quasi-isomorphism clearly preserves the decomposition along conjugacy classes of \( G \). Since \( T \) is a transversal, the above Lemma 3.11 shows that the inclusion functor \( C_T[G] \subseteq C[G] \) is dense. Moreover it corresponds to a full subcategory, hence it is full and faithful, in addition of being homogeneous. The induced quasi-isomorphism provides then the result.

\[ \therefore \]

Proof of Theorem 3.10 As previously, we provide the proof for \( n = 2 \). This avoids lengthy and uninformative computations.

Let \( T \) be a transversal of the free action of \( G \) on \( C_0 \). In order to define an isomorphism of chain complexes \( A : C_\bullet(C)_G \to C_\bullet(C_T^{(1)}[G]) \), let \( x_0 f_2 x_2 \otimes x_2 f_1 x_1 \otimes x_1 f_0 x_0 \) be a chain of \( C_2(C) \). In \( (C_2(C))_G \), we begin by modifying the chain in order that the starting (and hence the ending) object \( x_0 \) belongs to \( T \). More precisely, there exists a unique \( s \in G \) such that \( sx_0 = u_0 \in T \). Then

\[ f_2 \otimes f_1 \otimes f_0 \equiv s(f_2 \otimes f_1 \otimes f_0) = u_0(s f_2)_{x_2} \otimes s_{x_2}(s f_1)_{x_1} \otimes s_{x_1}(s f_0)_{u_0}. \]
In other words we can assume that the chain is of the form

\[ u_0(f_2)x_2 \otimes x_2(f_1)x_1 \otimes x_1(f_0)u_0 \text{ for } u_0 \in T. \]

For \( i = 1, 2 \), let \( u_i = u(x_i) \) be the unique element of \( T \) which is in the orbit of \( x_i \). Moreover let \( s_i \) be the unique element of \( G \) such that \( x_i = s_iu_i \). We define

\[
A(u_0(f_2)s_2u_2 \otimes s_2u_2(f_1)s_1u_1 \otimes s_1u_1(f_0)u_0) = u_0(f_2)s_2u_2 \otimes u_2(s_2^{-1}f_1)s_2^{-1}s_1u_1 \otimes u_1(s_1^{-1}f_0)s_1^{-1}u_0 = u_0[f_2]^{s_2} \otimes u_2[s_2^{-1}f_1]^{s_2^{-1}s_1} \otimes u_1[s_1^{-1}f_0]^{s_1^{-1}}u_0.
\]

This chain belongs to \( C_2^{(1)}(C_T[G]) \) since \( s_2(s_2^{-1}s_1)s_1^{-1} = 1 \). For a 3-chain the formula defining \( A \) is

\[
A(u_0f_3s_3u_3 \otimes s_3u_3f_2s_2u_2 \otimes s_2u_2f_1s_1u_1 \otimes s_1u_1f_0u_0) = u_0[f_3]^{s_3} \otimes u_3[s_3^{-1}(f_2)]^{s_3^{-1}s_2} \otimes u_2[(s_2^{-1}f_1)]^{s_2^{-1}s_1} \otimes u_1[s_1^{-1}f_0]^{s_1^{-1}}u_0.
\]

Next we verify that \( A \) is a chain map.

\[
dA(f_2 \otimes f_1 \otimes f_0) = [f_2]^{s_2}[s_2^{-1}f_1]^{s_2^{-1}s_1} \otimes [s_1^{-1}f_0]^{s_1^{-1}} - [f_2]^{s_2} \otimes [s_2^{-1}f_1]^{s_2^{-1}s_1}[s_1^{-1}f_0]^{s_1^{-1}} + [s_1^{-1}f_0]^{s_1^{-1}}[f_2]^{s_2} \otimes [s_2^{-1}f_1]^{s_2^{-1}s_1}
\]

\[
= [f_2f_1]^{s_1} \otimes [s_1^{-1}f_0]^{s_1^{-1}} - [f_2]^{s_2} \otimes [(s_2^{-1}f_1)(s_1^{-1}f_0)]^{s_1^{-1}} + [(s_1^{-1}f_0)(s_1^{-1}f_2)]^{s_1^{-1}s_2} \otimes [s_2^{-1}f_1]^{s_2^{-1}s_1}.
\]

Recall that

\[
d(f_2 \otimes f_1 \otimes f_0) = u_0(f_2f_1)s_1u_1 \otimes s_1u_1(f_0)u_0 - u_0(f_2)s_2u_2 \otimes s_2u_2(f_1f_0)u_0 + s_1u_1(f_0f_2)s_2u_2 \otimes s_2u_2(f_1)s_1u_1.
\]

In order to compute \( Ad \), notice that up to the action, that is in the coinvariants, the last term of the previous sum can be rewritten:

\[
s_1u_1(f_0f_2)s_2u_2 \otimes s_2u_2(f_1)s_1u_1 \equiv u_1(s^{-1}(f_0f_2))s_1^{-1}s_2u_2 \otimes s_1^{-1}s_2u_2(s_1^{-1}f_1)u_1.
\]

This way the last summand of \( d(f_2 \otimes f_1 \otimes f_0) \) starts and ends at \( u_1 \in T \), which is required in order to apply \( A \). Hence

\[
Ad(f_2 \otimes f_1 \otimes f_0) = A(u_0(f_2f_1)s_1u_1 \otimes s_1u_1(f_0)u_0 - u_0(f_2)s_2u_2 \otimes s_2u_2(f_1f_0)u_0 + s_1u_1(f_0f_2)s_2u_2 \otimes s_2u_2(f_1)s_1u_1)
\]

\[
= ([f_2f_1]^{s_1} \otimes [s_1^{-1}f_0]^{s_1^{-1}} - [f_2]^{s_2} \otimes [s_2^{-1}f_1]^{s_2^{-1}s_1})^{s_1^{-1}s_2} \otimes ((s_1^{-1}s_2)^{-1}s_1^{-1}f_1)^{(s_1^{-1}s_2)^{-1}}.
\]
shows that $Ad = dA$.

Let $g_2 \otimes g_1 \otimes g_0 \in C^{(1)}_2(C_T[G])$, that is

$$g_2 \otimes g_1 \otimes g_0 = u_0[g_2]^s_{u_2} \otimes u_2[g_1]^s_{u_1} \otimes u_1[g_0]^s_{u_0} = u_0 g_2 s_{2u_2} \otimes u_2 g_1 s_{1u_1} \otimes u_1 g_0 s_{0u_0}$$

with $s_2 s_1 s_0 = 1$. Let

$$B : C^{(1)}_\bullet(C_T[G]) \to (C_\bullet(C))_G$$

be defined by

$$B(g_2 \otimes g_1 \otimes g_0) = u_0(g_2) s_{2u_2} \otimes s_{2u_2}(g_2 s_{1u_1} \otimes s_{2s_1u_1})(s_2 s_1 g_0) s_{2s_1s_0u_0}.$$  

We observe that since $s_2 s_1 s_0 = 1$, we have that $s_2 s_1 s_0 u_0 = u_0$. Next we will show that $A$ and $B$ are mutual inverses. This will imply that $B$ is a chain map, since $A$ is a chain map.

$$AB(g_2 \otimes g_1 \otimes g_0) = A(u_0 g_2 s_{2u_2} \otimes s_{2u_2}(g_2 s_{1u_1} \otimes s_{2s_1u_1})(s_2 s_1 g_0) u_0) = u_0 g_2 s_{2u_2} \otimes u_2(s_2^{-1} g_1 s_{1u_1} \otimes u_1((s_2 s_1)^{-1}(s_2 s_1) g_0)(s_2 s_1)^{-1} u_0).$$

Since $s_2 s_1 s_0 = 1$, we obtain

$$u_0 g_2 s_{2u_2} \otimes u_2 g_1 s_{1u_1} \otimes u_1 g_0 s_{0u_0} = u_0[g_2]^s_{u_2} \otimes u_2[g_1]^s_{u_1} \otimes u_1[g_0]^s_{u_0}.$$  

$$BA(u_0 g_2 s_{2u_2} \otimes s_{2u_2} f s_{1u_1} \otimes s_{1u_1} f_0 u_0) =$$

$$B(u_0[g_2]^s_{u_2} \otimes u_2[s_2^{-1} f_1]^s_{u_1} \otimes u_1[s_1^{-1} f_0]^s_{u_0}) =$$

$$u_0 g_2 s_{2u_2} \otimes u_2 g_1 s_{1u_1} \otimes s_{2s_2^{-1} s_1 u_1}((s_2 s_1)^{-1} f_0)(s_2 s_1)^{-1} s_1^{-1} u_0 =$$

$$u_0 g_2 s_{2u_2} \otimes u_2 g_1 s_{1u_1} \otimes s_{1u_1} f_0 u_0.$$  

\[\Box\]

### 3.2.2 General case

The next Lemma enables to generalise Theorem 3.10 to a $G$-$k$-category where the action of $G$ is not necessarily free. Recall that $M_G(C)$ is the resolving category of a $G$-$k$-category, see Definition 2.8.

**Lemma 3.14** Let $C$ be a $G$-$k$-category. The chain complexes $C_\bullet(M_G(C))$ and $C_\bullet(C)$ are $kG$-quasi-isomorphic.

**Proof.** By Theorem 2.12 there is an equivalence of categories $L : M_G(C) \to C$. As for Lemma 3.13 we infer from [16] that there is an induced quasi-isomorphism $C_\bullet(M_G(C)) \to C_\bullet(C).$  

\[\Box\]

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Theorem 3.15 Let \( \mathcal{C} \) be a \( G \)-k-category.

\[
HH^*_+(\mathcal{C}[G]) = H_*(C_*(\mathcal{C})_G).
\]

If the coinvariants functor is exact

\[
HH^*_+(\mathcal{C}[G]) = HH_*(\mathcal{C}).
\]

Proof. Due to Theorem 3.10 the result holds for the \( G \)-k-category \( M_G(\mathcal{C}) \). The equivalence of \( G \)-k-categories \( L : M_G(\mathcal{C}) \to \mathcal{C} \) provides a homogeneous equivalence of \( G \)-graded \( k \)-categories

\[
L[G] : M_G(\mathcal{C})[G] \to \mathcal{C}[G]
\]

which gives a quasi-isomorphism

\[
C_*(M_G(\mathcal{C})[G]) \to C_*(\mathcal{C}[G])
\]

preserving the decomposition of chain complexes along the conjugacy classes of \( G \). Hence

\[
HH^*_+(M_G(\mathcal{C})[G]) = HH^*_+(\mathcal{C}[G]).
\]

By the above proposition

\[
H_*(C_*(M_G(\mathcal{C}))) = H_*(C_*(\mathcal{C})).
\]

If the coinvariants functor is exact then

\[
HH_*(M_G(\mathcal{C}))_G = (HH_*(\mathcal{C})).
\]

3.3 Galois coverings

In this section we reformulate Theorem 3.10 in terms of Galois coverings. First we recall the definition of a quotient category.

Definition 3.16 (see [3, 28] and also [5]) Let \( \mathcal{C} \) be a \( G \)-k-category with a free action of \( G \). The quotient category \( \mathcal{C}/G \) has set of objects the set of orbits \( \mathcal{C}_0/G \). Let \( \alpha \) and \( \beta \) be orbits. The vector space of morphisms from \( \alpha \) to \( \beta \) is

\[
\beta(\mathcal{C}/G)_\alpha = \left( \bigoplus_{x \in \alpha} \bigoplus_{g \in G} \mathcal{C}_x \right).
\]

Let \( \gamma, \beta, \alpha \) be orbits. Let

\[
g \in \beta \mathcal{C}_y \text{ and } f \in y \mathcal{C}_x,
\]
where \( z \in \gamma \), \( y \) and \( y' \in \beta \), and \( x \in \alpha \). Let \( s \) be the unique element of \( G \) such that \( sy = y' \), then \( f \equiv sf \) in the coinvariants. The composition \( gf \) in \( C/G \) is

\[
gf = zy' syf sx \in \gamma(C/G)_{\alpha}
\]

There is no difficulty in verifying that this is a well defined associative composition.

**Definition 3.17** A Galois covering of \( k \)-categories is a functor \( C \to C/G \), where \( C \) is a \( G \)-\( k \)-category with free action, and where the functor is the canonical projection functor.

If \( C_0 \) is finite we infer from a Galois covering as above a map \( a(C) \to a(C/G) \). Note that \( a(C) \) is not an algebra map in general, see Remark 2.6.

Let \( C \to C/G \) be a Galois covering and let \( T \) be a transversal of the action of \( G \) on \( C_0 \). For each orbit \( \alpha \), let \( u_\alpha \in T \) be the unique element of \( T \) which belongs to \( \alpha \).

It is shown in Lemma 2.2 of [6] that through a canonical identification we have

\[
\beta(C/G)_{\alpha} = \bigoplus_{s \in G} u_\beta Cs_{u_\alpha},
\]

This provides that \( C/G \) is graded by \( G \). Indeed let \( \beta(C/G)_{\alpha} = u_\beta Cs_{u_\alpha}, \) and notice that

\[
(u_\beta g_{u_\beta})(u_\gamma f_{u_\alpha}) = (u_\gamma g_{u_\beta})(u_\gamma tf_{u_\alpha}) = u_\gamma (g(t)f) t_{u_\alpha} \in \gamma(C/G)_{\alpha}.
\]

The following result can be deduced from [6]. We provide a proof for completeness.

**Proposition 3.18** Let \( C \to C/G \) be a Galois covering. Let \( T \subset C_0 \) be a transversal of the free action of \( G \), and consider the \( G \)-grading of \( C/G \) determined by \( T \).

The graded categories \( C/G \) and \( C_T[G] \) are isomorphic by a homogeneous functor, that is by a functor which is homogeneous on morphisms and graded with respect to composition of morphisms.

**Proof.** There is a bijection between the objects of \( C/G \) and those of \( C_T[G] \). The previous considerations shows that the morphisms of both categories are identified in a homogeneous manner. Moreover we have already used that the inclusion \( C_T[G] \subset C[G] \) provides an equivalence of categories, and this equivalence is homogeneous. \( \Box \)

The above analysis provides the translation of Theorem 3.10 in terms of Galois coverings, as follows.
Theorem 3.19 Let $\mathcal{C} \to \mathcal{C}/G$ be a Galois covering.

$$HH_*^{(1)}(\mathcal{C}/G) = H_*(C_*(\mathcal{C})_G).$$

If the coinvariants functors is exact

$$HH_*^{(1)}(\mathcal{C}/G) = HH_*(\mathcal{C}).$$

4 Hochschild-Mitchell cohomology

4.1 Graded categories

As mentioned in the Introduction, Hochschild-Mitchell cohomology is more intricate than homology since it makes use of direct products. We begin this section by recalling its definition. If the category is graded, we provide the decomposition along conjugacy classes.

Definition 4.1 Let $\mathcal{C}$ be a $k$-category. Let $C^\bullet(\mathcal{C})$ be the complex of cochains given by:

$$C^0(\mathcal{C}) = \prod_{x \in \mathcal{C}_0} xC_x,$$

$$C^n(\mathcal{C}) = \prod_{x_{n+1},...,x_1 \in \mathcal{C}_0} C_{x_{n+1},...,x_1}$$

for $n > 0$

where

$$C_{x_{n+1},...,x_1} = \text{Hom}_k(x_{n+1}C_{x_n} \otimes \cdots \otimes x_2C_{x_1}, x_{n+1}C_{x_1})$$

The coboundary $d$ is given by the formulas below which are the usual ones for computing Hochschild cohomology, see for instance [4, 32].

Let $\varphi$ be a cochain of degree $n$, that is a family of $k$-morphisms $\varphi = \{\varphi(x_{n+1},...,x_1)\}$. Its coboundary $d\varphi$ is the family $\{(d\varphi)(x_{n+2},...,x_1)\}$ given by

$$(d\varphi)(x_{n+2},...,x_1)(f_{n+1}x_{n+1} \otimes \cdots \otimes x_2(f_1)x_1) =$$

$$(-1)^{n+1} f_{n+1}\varphi(x_{n+1},...,x_1)(f_n \otimes \cdots \otimes f_1) +$$

$$\sum_{i=1}^n (-1)^{i+1} \varphi(x_{n+1},...,x_{n+2},x_{i+1},...,x_1)(f_{n+1} \otimes \cdots \otimes f_{i+1}f_i \otimes \cdots \otimes f_1) +$$

$$\varphi(x_{n+2},...,x_2)(f_{n+1} \otimes \cdots \otimes f_2)f_1. \quad (4.1)$$

Note that $d$ is well defined since the spaces of cochains are direct products. The Hochschild-Mitchell cohomology of $\mathcal{C}$ is $HH^*(\mathcal{C}) = H^*(C^\bullet(\mathcal{C}))$.

Remark 4.2 The compatibility with the Hochschild cohomology of a $k$-algebra arises from the following. Let $\mathcal{C}$ and $\mathcal{D}$ be $k$-categories, let $\mathcal{C} \otimes_k \mathcal{B} = \text{their tensor product}$ and let $\text{Fun}_k(\mathcal{C}, \mathcal{D})$ be the $k$-functors from $\mathcal{C}$ to $\mathcal{D}$. Then for $n \geq 0$ we have

$$C^n(\mathcal{C}) = \text{Fun}_k(C^{\otimes k^n}, \mathcal{C}).$$

Note that $C^{\otimes k^0}$ is the following category. The objects are the same than those of $\mathcal{C}$. The space of morphisms between any pair of different objects is zero. The endomorphisms of each object is the $k$-algebra $k$. 

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In degree zero we have

\[ HH^0(C) = \{ (xf)x \in C_0 \mid yg_x \cdot f_x = yf_y \cdot yg_x \text{ for all } yg_x \in yC_x \}. \]

As for Hochschild cohomology of algebras, the cup product is defined at the cochain level as follows. Let \( \varphi \in C_{x_{n+1}, \ldots, x_1} \) and \( \psi \in C_{y_{m+1}, \ldots, y_1} \). If \( x_{n+1} \neq y_1 \) the cup product \( \psi \cdot \varphi \) is zero. Otherwise the cup product \( \psi \cdot \varphi \in C_{y_{m+1}, \ldots, y_1, x_n, \ldots, x_1} \) is

\[ (\psi \cdot \varphi)(f_{n+m} \otimes \cdots \otimes f_1) = \psi(f_{n+m} \otimes \cdots \otimes f_{n+1})\varphi(f_n \otimes \cdots \otimes f_1). \]

The cup product verifies the graded Leibniz rule, and it provides a graded commutative \( k \)-algebra structure on \( HH^*(C) \). In particular \( HH^0(C) \) is a commutative \( k \)-algebra which is the center of the category.

**Proposition 4.3** Let \( B \) be a \( G \)-graded category, and let \( Cl(G) \) be the set of conjugacy classes of \( G \). There is a decomposition

\[ HH^*(B) = \prod_{D \in Cl(G)} HH^*_D(B) \]

where \( HH^*_D(B) \) is a subalgebra.

**Proof.** For \( D \in Cl(G) \) we provide a subcomplex of cochains \( C^*_D(B) \) of \( C^*(B) \) as follows. Let \( \varphi \) be a cochain of degree \( n \). We say that \( \varphi \) is homogeneous of type \((s_n, \ldots, s_1, s_0)\) if:

1. Each component of \( \varphi \) has its image contained in the set of homogeneous morphisms of degree \( s_0 \).

2. For \((s'_n, \ldots, s'_1) \neq (s_n, \ldots, s_1)\), each component of \( \varphi \) restricted to tensors of homogeneous morphisms degree \((s'_n, \ldots, s'_1)\) is zero.

The formula (4.1) which defines the coboundary \( d \) has \( n+2 \) summands. Let \( d_{n+1} \) be the first one, let \( d_0 \) be the last one, and let \( d_i \) denote the in between summands indexed according to the appearance of the composition “\( f_{i+1}f_i \)” for \( i = n, \ldots, 1 \).

Let \( \varphi \) be homogeneous of type \((s_n, \ldots, s_1, s_0)\). We observe the following:

- \( d_{n+1} \varphi \) is a sum of homogeneous cochains of types \((s, s_n, \ldots, s_1, s_0)\) for \( s \in G \).

- \( d_i \varphi \) is a sum of homogeneous cochains of types

\[ (s_n, \ldots, s_{i+1}, s''_{i+1}, s'_1, s_{i-1}, \ldots, s_1, s_0) \]

for \( s'', s' \in G \) with \( s''s' = s_i \).
• $d_0\varphi$ is a sum of homogeneous cochains of types $(s_n, \ldots, s_1, s, s_0s)$ for $s \in G$.

Let us call the class of $(s_n, \ldots, s_1, s_0)$ the product $s_n \ldots s_1 s_0^{-1} \in G$. The above considerations show that if $\varphi$ is homogeneous of type $(s_n, \ldots, s_1, s_0)$, that is of class $c = s_n \ldots s_1 s_0^{-1}$, then $d\varphi$ is a sum of homogeneous cochains. Note that these cochains are in general of different sorts, but their classes are all conjugated to $c$.

Let $D$ be a conjugacy class and let $C^s_D(C)$ be the homogeneous cochains which classes of types are in $D$. We have showed that $C^s_D(C)$ is a cochains subcomplex of $C^s(C)$. Moreover

$$C^s(C) = \prod_{D \in Cl(G)} C^s_D(C).$$

Clearly, if $\varphi$ and $\psi$ are homogeneous cochains which classes of types are both 1, then $\psi \sim \varphi$ is also of class type 1.

\section*{4.2 Skew categories}

We recall that the skew category is graded, so results of the Subsection 4.1 are in force. As for homology, at a first glance we are only able to provide a result if the action is free. The resolving category of Section 2 enables to extend the result for any $G$-$k$-category.

Firstly we provide some properties of the cohomology of a $G$-$k$-category that we need in the sequel.

If $C$ is a $G$-$k$-category then $C^n(C)$ is a $kG$-module as follows. For $n > 0$, let $\varphi = \{ \varphi(x_{n+1}, \ldots, x_1) \} \in C^n(C)$ be a cochain, where

$$\varphi(x_{n+1}, \ldots, x_1) : x_{n+1}C_{x_{n+1}} \otimes \cdots \otimes x_1C_{x_1} \to x_{n+1}C_{x_1}.$$

Let $s \in G$ and let

$$s[\varphi(x_{n+1}, \ldots, x_1)] : sx_{n+1}C_{sx_{n+1}} \otimes \cdots \otimes sx_1C_{sx_1} \to sx_{n+1}C_{sx_1}$$

be defined by

$$s[\varphi(x_{n+1}, \ldots, x_1)](sf_n \otimes \cdots \otimes f_1) = s[\varphi(x_{n+1}, \ldots, x_1)](s^{-1}f_n \otimes \cdots \otimes s^{-1}f_1).$$

We set $s\varphi = \{ s[\varphi(x_{n+1}, \ldots, x_1)] \}$.

For $n = 0$, let $\varphi = \{ \varphi_x \} \in C^0(C)$, where $\varphi_x \in _xC_x$. For $s \in G$ we have $s\varphi_x \in _sxC_{sx}$. We set $s\varphi = \{ s\varphi_x \}$.

\begin{remark}
Let $(\_)_G^G$ be the invariants functor. Then $\varphi \in (C^n(C))^G$ if and only if for all $s \in G$ and for any sequence of objects $x_{n+1}, \ldots, x_1$ we have that

$$\varphi(sx_{n+1}, \ldots, sx_1)(sf_n \otimes \cdots \otimes sf_1) = s[\varphi(x_{n+1}, \ldots, x_1)](f_n \otimes \cdots \otimes f_1).$$

\end{remark}
Clearly the action of $G$ commutes with the coboundary of $C^\bullet(C)$. Moreover the action of $G$ is by automorphisms of the algebra structure given by the cup product. In other words $C^\bullet(C)$ is a graded differential algebra with an action of $G$ by automorphisms of its structure.

In particular $(C^\bullet(C))^G$ is a graded differential algebra. Moreover the inferred action of $G$ on $HH^*(C)$ is by automorphisms of the algebra. If the invariants functor is exact, then $(HH^*(C))^G = H^*(C^\bullet(C))^G$ as $k$-algebras.

4.2.1 Free action

**Theorem 4.5** Let $C$ be a $G$-$k$-category with a free action of $G$, and let $C[G]$ be the $G$-graded skew category. There is an isomorphism of $k$-algebras

$$HH^*_{\{1\}}(C[G]) \simeq H^* \left( (C^\bullet(C))^G \right).$$

If the invariants functor is exact, we infer an isomorphism of $k$-algebras

$$HH^*_{\{1\}}(C[G]) \simeq HH^* (C)^G.$$

**Proof.** Let $T$ be a transversal of the action of $G$ on $C_0$ and let $C_T[G]$ be the full subcategory of $C[G]$ with set of objects $T$. Let $D$ be a conjugacy class of $G$. We assert that

$$HH^*_D(C_T[G]) = HH^*_D(C[G]).$$

Indeed the equivalence of categories given by the inclusion $C_T[G] \subseteq C[G]$ induces a quasi-isomorphism of the complexes of cochains which preserves the decomposition along the conjugacy classes of $G$.

Moreover for the trivial conjugacy class the quasi-isomorphism is a morphism of differential graded algebras. Hence

$$HH^*_{\{1\}}(C_T[G]) = HH^*_{\{1\}}(C[G])$$

as $k$-algebras.

In what follows we will prove that there are morphisms of graded differential algebras

$$(C^\bullet(C))^G \xrightarrow{A} C^\bullet_{\{1\}}(C_T[G])$$

which are mutual inverses.

We just give the proof for $n = 3$, which is easier to read than the general case. Moreover, the main idea of the calculation is already highlighted – the general case would unnecessarily lengthen the text.

Let $\psi \in (C^3(C))^G$ and let $u_4, u_3, u_2, u_1 \in T$. We will define $(A\psi)_{(u_4, u_3, u_2, u_1)}$ on each homogeneous component.

Let

$$f_3 \otimes f_2 \otimes f_1 \in u_4 C_T[G]^{u_3_3} \otimes u_3 C_T[G]^{u_2_2} \otimes u_2 C_T[G]^{u_1_1}.$$
Recall that by the definition of the morphisms of $C[G]$ we have that $f_i \in u_{i+1} C_{i+1}$ for $i = 1, 2, 3$. Let

$$(A\psi)_{(u_4,u_3,u_2,u_1)}(f_3 \otimes f_2 \otimes f_1) = \psi_{(u_4,s_3 u_3,s_3 s_2 u_2,s_3 s_2 s_1 u_1)}(f_3 \otimes s_3 f_2 \otimes s_3 s_2 f_1).$$

We observe that this definition makes sense since

$$f_3 \otimes s_3 f_2 \otimes s_3 s_2 f_1 \in u_4 C_{s_3 u_3} \otimes s_3 u_3 C_{s_3 s_2 u_2} \otimes s_3 s_2 u_2 C_{s_3 s_2 s_1 u_1}.$$ 

Moreover

$$A\psi(f_3 \otimes f_2 \otimes f_1) \in u_4 C_{s_3 s_2 s_1 u_1} = u_4 C_T[G]_{u_1}^{s_3 s_2 s_1},$$

that is we have indeed defined a homogeneous cochain of type $(s_3, s_2, s_1, s_3 s_2 s_1)$, which is of class $\{1\}$.

The verification that $dA = Ad$ is straightforward, it uses in a crucial way that $\psi$ is an invariant; the formulas defining the composition in $C[G]$ are required as well. Analogously, it is easy to verify that $A(\psi' \otimes \psi) = A(\psi') \sim A(\psi)$.

Let $\varphi \in C^0(C)$. In order to define $(B\varphi)_{(x_4,x_3,x_2,x_1)}$ we first observe that since the action of $G$ on $C_0$ is free, there exist $s_4, s_3, s_2, s_1 \in G$ which are unique such that $x_i = s_i u_i$ for $i = 1, 2, 3, 4$.

Let $g_3 \otimes g_2 \otimes g_1 \in s_4 u_4 C_{s_3 u_3} \otimes s_3 u_3 C_{s_3 s_2 u_2} \otimes s_3 s_2 u_2 C_{s_3 s_2 s_1 u_1}$. We define $(B\varphi)_{(x_4,x_3,x_2,x_1)}$ as follows:

$$(B\varphi)(g_3 \otimes g_2 \otimes g_1) = s_4 \varphi_{(u_4,u_3,u_2,u_1)}(s_4^{-1} g_3 \otimes s_3^{-1} g_2 \otimes s_2^{-1} g_1).$$

In order to verify that this is well defined, note first that

$$s_4^{-1} g_3 \otimes s_3^{-1} g_2 \otimes s_2^{-1} g_1 \in u_4 C_{s_4^{-1} s_3 u_3} \otimes u_3 C_{s_3^{-1} s_2 u_2} \otimes u_2 C_{s_2^{-1} s_1 u_1} = u_4 C[G]_{u_3}^{s_4^{-1} s_3} \otimes u_3 C[G]_{u_2}^{s_3^{-1} s_2} \otimes u_2 C[G]_{u_1}^{s_2^{-1} s_1}.$$ 

Secondly, using that $\varphi$ is a cochain for the trivial conjugacy class, we obtain

$$\varphi_{(u_4,u_3,u_2,u_1)}(s_4^{-1} g_3, s_3^{-1} g_2, s_2^{-1} g_1) \in u_4 C[G]_{u_1}^{s_4^{-1} s_3 s_3^{-1} s_2 s_2^{-1} s_1} = u_4 C[G]_{u_1}^{s_4^{-1} s_1} = C_{s_4^{-1} s_1 u_1}.$$ 

Hence $(B\varphi)(g_3 \otimes g_2 \otimes g_1) \in s_4 u_4 C_{s_4 u_1}$, therefore $B\varphi \in C^3(C)$. Next we check that $B\varphi$ is an invariant cochain. Let $t \in G$, we assert that

$$t(B\varphi)_{(s_4 u_4,s_3 u_3,s_2 u_2,s_1 u_1)}(g_3 \otimes g_2 \otimes g_1) = B\varphi_{(ts_4 u_4,ts_3 u_3,ts_2 u_2,ts_1 u_1)}(tg_3 \otimes tg_2 \otimes tg_1).$$

Indeed, the second term is by definition

$$ts_4 \varphi_{(u_4,u_3,u_2,u_1)}((ts_4)^{-1} t g_3 \otimes (ts_3)^{-1} t g_2 \otimes (ts_2)^{-1} t g_1),$$

which equals the first term.
Let $\psi \in C^3(G)^G$, we assert that $BA\psi = \psi$. Recall that if 
\[
f_2 \otimes f_2 \otimes f_1 \in u_4 C[G]_{u_3}^{s_3} \otimes u_3 C[G]_{u_2}^{s_2} \otimes u_2 C[G]_{u_1}^{s_1},
\]
then 
\[
(A\psi)_{u_4,u_3,u_2,u_1}(f_2 \otimes f_2 \otimes f_1) = \psi(f_3 \otimes t_3 f_2 \otimes t_3 t_2 f_1).
\]
Let $g_3 \otimes g_2 \otimes g_1 \in s_4 u_4 C_{s_3 u_3} \otimes s_3 u_3 C_{s_2 u_2} \otimes s_2 u_2 C_{s_1 u_1}$. Then 
\[
BA\psi(g_3 \otimes g_2 \otimes g_1) = s_4 A\psi(s_4^{-1} g_3 \otimes s_3^{-1} g_2 \otimes s_2^{-1} g_1)
\]
where 
\[
s_4^{-1} g_3 \otimes s_3^{-1} g_2 \otimes s_2^{-1} g_1 \in u_4 C[G]_{u_3}^{s_3} \otimes u_3 C[G]_{u_2}^{s_2} \otimes u_2 C[G]_{u_1}^{s_1}.
\]
Hence 
\[
BA\psi(g_3 \otimes g_2 \otimes g_1) = s_4 \psi(s_4^{-1} g_3 \otimes (s_3^{-1} s_3^{-1}) g_2 \otimes (s_4^{-1} s_3 s_3^{-1} s_2) s_2^{-1} g_1) = s_4 \psi(s_4^{-1} g_3 \otimes s_4^{-1} g_2 \otimes s_4^{-1} g_1).
\]
Since $\psi$ is invariant, the later equals $\psi(g_3 \otimes g_2 \otimes g_1)$.

Let $\varphi \in C_{\{1\}}(C_T[G])$, next we will show $AB\varphi = \varphi$. Consider 
\[
g_3 \otimes g_2 \otimes g_1 \in t_4 u_4 C_{t_3 u_3} \otimes t_3 u_3 C_{t_2 u_2} \otimes t_2 u_2 C_{t_1 u_1}.
\]
We have 
\[
B\varphi(g_3 \otimes g_2 \otimes g_1) = t_4 \varphi(t_4^{-1} g_3 \otimes t_3^{-1} g_2 \otimes t_2^{-1} g_1).
\]
Let 
\[
f_3 \otimes f_2 \otimes f_1 \in u_4 C[G]_{u_3}^{s_3} \otimes u_3 C[G]_{u_2}^{s_2} \otimes u_2 C[G]_{u_1}^{s_1}.
\]
Then 
\[
AB\varphi(f_3 \otimes f_2 \otimes f_1) = (B\varphi)(f_3 \otimes s_3 f_2 \otimes s_3 s_3 f_1)
\]
where 
\[
f_3 \otimes s_3 f_2 \otimes s_3 s_3 f_1 \in u_4 C_{s_3 u_3} \otimes s_3 u_3 C_{s_2 u_2} \otimes s_3 s_2 u_2 C_{s_3 s_2 s_1 u_1}.
\]
Hence 
\[
AB\varphi(f_3 \otimes f_2 \otimes f_1) = \varphi(f_3 \otimes s_3^{-1} s_3 f_2 \otimes (s_3 s_3)^{-1} s_3 s_3 f_1) = \varphi(f_3 \otimes f_2 \otimes f_1).
\]
4.2.2 General case

Our next aim is to show that the isomorphism of Theorem 4.5 remains valid when the action of the group is not necessarily free. The following result has been proved in [2, 16], see also [17].

**Proposition 4.6** Let $C$ and $D$ be $k$-categories and let $F : C \to D$ be an equivalence of $k$-categories. There is an induced map

$$C^*F : C^*(D) \to C^*(C)$$

which is a quasi-isomorphism.

**Remark 4.7** In the following the explicit definition of $C^*(F)$ will be useful, it is as follows. Let

$$\varphi = (\varphi_{y_{n+1}, \ldots, y_1}) \in C^n(D)$$

where

$$\varphi_{y_{n+1}, \ldots, y_1} : y_{n+1}Dy_n \otimes \cdots \otimes y_2Dy_1 \longrightarrow y_{n+1}Dy_1$$

is a $k$-morphism. The component $(x_{n+1}, \ldots, x_1)$ of $(C^*F)(\varphi)$ is given as follows. Let

$$f_{n+1} \otimes \cdots \otimes f_1 \in x_{n+1}C_{x_n} \otimes \cdots \otimes x_2C_{x_1}.$$ 

Then

$$[(C^*F)(\varphi)]_{x_{n+1}, \ldots, x_1}(f_{n+1} \otimes \cdots \otimes f_1) = (x_{n+1}F_{x_1})^{-1} \left( \varphi_{F(x_{n+1}), \ldots, F(x_1)}(F(f_{n+1}) \otimes \cdots \otimes F(f_1)) \right)$$

where

$$x_{n+1}F_{x_1} : x_{n+1}C_{x_1} \rightarrow F(x_{n+1})D_{F(x_1)}$$

is the $k$-isomorphism provided by the equivalence $F$.

Observe that in [16] the above Proposition is obtained in a more general setting, that is for a $D$-bimodule of coefficients $N$. In our case $N = D$. The restricted $C$-bimodule of coefficients is denoted $FN$ in [15], observe that $FD$ is isomorphic to $C$ via $F$. This later isomorphism explains that in our setting $x_{n+1}F_{x_1}^{-1}$ is required in the above formula while in [16] it is not needed since the bimodule of coefficients there is $FD$, not $C$.

**Theorem 4.8** Let $C$ and $D$ be $G$-$k$-categories and let $F : C \to D$ be a $G$-$k$-equivalence of categories. Then $F$ induces an isomorphism of $G$-$k$-algebras

$$HH^*(D) \rightarrow HH^*(C).$$
Proof. The explicit description of $C^*F$ given above enables to check without difficulty that it is multiplicative with respect to the cup product. Moreover, $C^*F$ commutes with the actions of $G$ on $C^*C$ and $C^*D$, that is $C^*F$ is a $kG$-morphism. Therefore the induced map in cohomology is an isomorphism of $G$-$k$-algebras.

We recall that if $C$ is a $G$-$k$-category, then $M_G(C)$ is a $G$-$k$-category where the action of $G$ on the objects of $M_G(C)$ is free, see Definition 2.8. Moreover there is a a $G$-$k$-functor $L : M_G(C) \to C$ which is an equivalence of categories.

**Theorem 4.9** Let $C$ be a $G$-$k$-category. Let $C[G]$ be the graded skew category, and let $\{1\}$ be the trivial conjugacy class of $G$. There is an isomorphism of $k$-algebras

$$HH^*_{\{1\}}(C[G]) \simeq H^*(C^*(C)^G).$$

If the invariants functor is exact, we have an isomorphism of $k$-algebras

$$HH^*_{\{1\}}(C[G]) \simeq HH^*(C^*(C)^G).$$

**Proof.** Let

$$L[G] : M_G(C)[G] \to C[G]$$

be the homogeneous equivalence of $G$-graded $k$-categories obtained from the $G$-$k$-equivalence of categories $L : M_G(C) \to C$ of Theorem 2.12.

We observe that if $B$ and $D$ are $G$-graded categories and $K : B \to D$ is a homogeneous equivalence, then the quasi-isomorphism

$$C^*(K) : C^*(B) \to C^*(D)$$

described in Remark 4.7 preserves the decomposition along the conjugacy classes of $G$. Hence $HH^*_{\{1\}}(B)$ and $HH^*_{\{1\}}(D)$ are isomorphic $k$-algebras.

4.3 Galois coverings

In this Subsection we will translate the results we have obtained for the cohomology of skew categories to a Galois coverings $C \to C/G$. Then we will provide a canonical monomorphism from the invariants of the cohomology of $C/G$ to the cohomology of $C$. This corresponds to the monomorphism of [24]. In low degrees it is described in [15].

The proof of the following result is along the same lines than the proof of Theorem 3.19.

**Theorem 4.10** Let $C \to C/G$ be a Galois covering.

$$HH^*_{\{1\}}(C/G) = H_*(C^*(C)^G).$$

If the invariants functors is exact, then

$$HH^*_{\{1\}}(C/G) = HH^*(C^*(C)^G).$$
Corollary 4.11 Let $C \to C/G$ be a Galois covering. If the invariants functor is exact, there is a canonical injective morphism

$$HH^\ast(C)^G \hookrightarrow HH^\ast(C/G)$$

which splits canonically.

**Proof.** The cohomology of $C/G$ has a direct sum decomposition along the conjugacy classes of $G$. The direct summand corresponding to the trivial conjugacy class is isomorphic to the invariants of the cohomology of $G$. $\diamond$

5 Hochschild cohomology of skew group algebras

In this section we will specialise Theorem 4.9 for $k$-algebras. Note that the proof of Theorem 4.9 requires the resolving category. We do not know a proof of Theorem 5.1 without using a resolving object which makes the action of the group free on a set. See Remark 5.2.

Let $\Lambda$ be a $k$-algebra, and let $G$ be a group acting by algebra automorphisms of $\Lambda$. Let $\Lambda[G]$ be the usual skew group algebra recalled in Remark 3.8. The Hochschild cohomology of a $k$-algebra $\Lambda$ is denoted $HH^\ast(\Lambda)$.

**Theorem 5.1** Let $G$ be a finite group whose order is invertible in $k$. Let $\Lambda$ be a $k$-algebra with an action of $G$ by algebra automorphisms. There is an isomorphism of algebras

$$HH^\ast_{\{1\}}(\Lambda[G]) \simeq HH^\ast(\Lambda)^G.$$

**Proof.** Let $\Lambda_1$ be the single object $G$-$k$-category of $\Lambda$ considered at Example 2.3. As noticed, the action of $G$ is not free on $\Lambda_1$ unless $G$ is trivial. By Theorem 4.9 we have an isomorphism of $k$-algebras

$$HH^\ast_{\{1\}}(\Lambda_1[G]) \simeq HH^\ast(\Lambda_1)^G. \quad (5.1)$$

Of course $a(\Lambda_1) = \Lambda$. Moreover we have that $HH^\ast(C) = HH^\ast(a(C))$, see for instance [7]. Hence the right hand side of (5.1) is isomorphic to $HH^\ast(\Lambda)^G$.

On the other hand, as quoted in the Introduction, if $G$ is finite and if $C$ is a $G$-$k$-category with a finite number of objects, then

$$a(C[G]) = a(C)[G].$$

Thus the left hand side of (5.1) is

$$HH^\ast_{\{1\}}(\Lambda_1[G]) = HH^\ast_{\{1\}}(a(\Lambda_1[G])) = HH^\ast_{\{1\}}(a(\Lambda_1)[G]) = HH^\ast_{\{1\}}(\Lambda[G]). \quad \diamond$$
Remark 5.2 The proof above of Theorem 5.1 relies on Theorem 4.9. Consequently, it is interesting to track the proof of Theorem 4.9 specifying it to an algebra. First, we have considered the matrix algebra $M_G(\Lambda)$ described in Proposition 2.13. Then the categorical proof translates into decomposing the cochains of $M_G(\Lambda)$ through the set $E$ of diagonal idempotents of Remark 2.14. The freeness of the action on $E$ enables to show that the module of invariants of the complex of cochains of $M_G(\Lambda)$ is isomorphic to the homogeneous cochains of the conjugacy class 1 of $M_G(\Lambda)[G]$. The final step consists in showing that the Hochschild cohomology of $\Lambda[G]$, as a $kG$-module, remains the same when considering the algebra $M_G(\Lambda)[G]$.

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