Evaluation and Optimization of Gaussian Approximation for Polar Codes

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Abstract—Gaussian approximation (GA) is widely used to construct polar codes. However when the code length is relatively long, the subchannel selection inaccuracy due to the calculation error of conventional approximate GA (AGA), which uses two-segment approximation function, results in a catastrophic performance loss. In this paper, new principles to design GA approximation functions are proposed. First, we introduce the concepts of polarization violation set (PVS) and polarization reversal set (PRS) to explain the essential reason that conventional AGA can not work well. In the AGA process, the subchannels belonging to these two sets lead to the reliability sorting error among the subsequent subchannels, which violates Arıkan’s fundamental polarization principle. Moreover, we propose a new metric, named cumulative-logarithmic error (CLE), to quantitatively evaluate the remainder approximation error of AGA in logarithm. We derive the upper bound of CLE to simplify its calculation. Guided by PVS, PRS and CLE bound, we further propose new rules to design multi-segment GA approximation functions, which obviously improve the calculation accuracy of AGA so as to guarantee the excellent performance of polar codes. Numerical results show that the new proposed GA approximation functions are critical to construct high-performance polar codes with long code lengths.

Index Terms—Polar codes, Gaussian approximation (GA), polarization violation set (PVS), polarization reversal set (PRS), cumulative-logarithmic error (CLE).

I. INTRODUCTION

Polar codes proposed by Arıkan [1] have been proved to achieve the capacity of binary input symmetric discrete memoryless channels (B-DMCs) under a successive cancellation (SC) decoder as the code length goes to infinity. To construct polar codes, the channel reliabilities can be calculated efficiently using the symmetric capacities of subchannels or Bhattacharyya parameters for binary-input erasure channels (BECs). For channels other than BECs, Mori et al. viewed the construction of polarized subchannels as an instance of density evolution (DE) [2]. Considering its high computational complexity, Tal and Vardy et al. devised two approximation methods to simplify the calculation of DE, by which one can get the upper and lower bounds on the error probability of each bit-channel efficiently [3]–[5]. Afterwards, Gaussian approximation (GA) was proposed to further reduce the computational complexity of DE [6] without much sacrifice in accuracy.

In GA, the log-likelihood ratio (LLR) of each subchannel is assumed to obey Gaussian distribution. So in the construction of polar codes, the iterative evaluation of variable and check nodes is only involved with the mean update of LLRs. However, the LLR updates in check nodes require complex integration. Consequently, the computational complexity of exact GA (denoted by EGA) grows exponentially with the polarization levels. In practical implementation, like GA utilized in LDPC codes, the well known approximate version of GA (denoted by AGA) based on two-segment approximation function is used to speed up the calculations [7]–[10]. Nevertheless, in the recursion process of polar codes, the calculation error of AGA versus EGA will be cumulated and amplified. This phenomenon causes inaccurate subchannel selection and leads to a catastrophic loss of block error ratio (BLER) performance for long code lengths.

Our aim in this paper is to provide new principles to design multi-segment GA approximation functions so as to improve the calculation accuracy of AGA and guarantee the excellent performance of polar codes. The main contributions of this paper can be summarized in the following three aspects: (1) First, we introduce the concepts of polarization violation set (PVS) and polarization reversal set (PRS). In AGA process, if the subchannel’s LLR mean belongs to the two sets, it will lead to the reliability sorting error among the following subchannels, which violates Arıkan’s fundamental polarization relationship. These two sets reveal the essential reason that polar codes constructed by conventional AGA present a catastrophic performance loss at long code lengths. (2) Second, after eliminating PVS and PRS, we further propose a new metric, named cumulative-logarithmic error (CLE) of channel polarization, to quantitatively evaluate the remainder calculation error between AGA and EGA. We also derived the upper bound of CLE to simplify its calculation. With this bound, the performance of different versions of AGA can be easily evaluated by analytic calculation rather than redundant Monte-Carlo simulation. (3) Finally, guided by PVS, PRS and CLE bound, we propose new rules to design multi-segment GA approximation functions for AGA. In this way, a complete framework is established to design high accuracy AGA scheme for polar codes at any code length. Following the proposed rules, two new AGA schemes are also given to improve the conventional AGA so as to guarantee the excellent performance of polar codes.

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The remainder of the paper is organized as follows. The preliminaries of polar coding are described in Section II. Then the conventional GA is introduced in Section III. Section IV makes detailed evaluation of GA, in which the concepts of PVS, PRS and CLE are proposed. And the new design rules of GA approximation functions are also addressed in Section IV. Then different versions of AGA are compared with the help of CLE bound in Section V, where the simulation results are also analyzed. Finally, Section VI concludes this paper.

II. PRELIMINARIES

A. Notation Conventions

In this paper, we use calligraphic characters, such as $\mathcal{X}$, to denote sets. Let $|\mathcal{X}|$ denote cardinality of $\mathcal{X}$. We write lowercase letters (e.g., $x$) to denote scalars. We use notation $v_i^N$ to denote a vector $(v_1, v_2, \ldots, v_N)$ and $u_i^j$ to denote a subvector $(v_1, v_{i+1}, \ldots, v_j)$. The sets of binary and real field are denoted by $\mathbb{B}$ and $\mathbb{R}$, respectively. Specially, let $\mathcal{N}(a, b)$ denote Gaussian distribution, where $a$ and $b$ represent the mean and the variance respectively.

Only square matrices are involved in this paper, and they are denoted by bold letters. The subscript of a matrix indicates its size, e.g. $\mathbf{F}_N$ represents an $N \times N$ matrix $\mathbf{F}$. The Kronecker product of two matrices $\mathbf{F}$ and $\mathbf{G}$ is expressed as $\mathbf{F} \otimes \mathbf{G}$, and the $n$-fold Kronecker power of $\mathbf{F}$ is denoted by $\mathbf{F}^\otimes n$.

Throughout this paper, $\log (\cdot)$ means “logarithm to base 2”, and $\ln (\cdot)$ stands for the natural logarithm.

B. Polar Codes and SC Decoding

Let $W: \mathcal{X} \rightarrow \mathcal{Y}$ denote a B-DMC with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$. The channel transition probabilities are given by $W(y|x)$, $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Given code length $N = 2^n$, $n = 1, 2, \ldots$, information length $K$, and code rate $R = K/N$, the polar coding over $W$ is described as follows [1]. After channel combining and splitting operations on $N$ independent duplicates of $W$, we obtain $N$ successive uses of synthesized binary input channels $W^{(j)}_N$, $j = 1, 2, \ldots, N$, with transition probabilities $W^{(j)}_N(y_1^N, u_1^{j-1}, u_j)$. The information bits can be assigned to the channels with indices in the information set $A$, which are the more reliable subchannels. The complementary set $A^c$ denotes the frozen bit set and the frozen bits $u_{A^c}$ can be set as fixed bit values, such as all zeros, for the symmetric channels.

To put it in another way, polar coding is performed on the constraint $x_1 = u_1 G_N$, where $G_N$ is the generator matrix and $u_1, x_1 \in \{0, 1\}^N$ are the source and code block respectively. The source block $u_1^N$ consists of information bits $u_{A}$ and frozen bits $u_{A^c}$. The generator matrix can be defined as $G_N = B_N F_2^\otimes n$, where $B_N$ is the bit-reversal permutation matrix and $F_2 = \{1, 0\}$.

As mentioned in [1], polar codes can be decoded by successive cancellation (SC) decoding algorithm. Let $\hat{u}_i^N$ denote an estimate of source block $u_i^N$. After receiving $y_1^N$, the bits $\hat{u}_j$ are successively determined with index from 1 to $N$ in the following way:

$$
\hat{u}_j = \begin{cases} 
   h_j(y_1^N, \hat{u}_{j-1}^N) & j \in A, \\
   u_j & j \in A^c, 
\end{cases}
$$

where

$$
   h_j(y_1^N, \hat{u}_{j-1}^N) = \begin{cases} 
   0 & W_j^{(j)}(y_1^N, \hat{u}_{j-1}^N, 0) \geq 1, \\
   1 & \text{otherwise}. 
\end{cases}
$$

Given a polar code with code length $N$, information length $K$, and selected channels indices $A$, the BLER under SC decoding algorithm is upper bounded by

$$
P_e(N, K, A) \leq \sum_{j \in A} P_e(W_j^{(j)}),
$$

where $P_e(W_j^{(j)})$ denotes the error probability of the $j$-th subchannel. This BLER upper bound is named as SC bound.

However, the finite-length performance of polar codes under SC decoding is not competitive. Later, some improved decoders, such as successive cancellation list (SCL) decoding [9], [11] and successive cancellation stack (SCS) decoding [12], were introduced to approach the performance of maximum likelihood (ML) decoder with an acceptable complexity. Recently, cyclic redundancy check aided (CRC-aided) SCL/SCS (CA-SCL/SCS) decoding [13]–[15] were reported to outperform the performance of advanced channel codes, such as, LDPC and turbo codes. These results represent the breakthrough of polar codes in practical applications.

III. GAUSSIAN APPROXIMATION FOR POLAR CODES

In this section, we use the code tree to describe the process of channel polarization. Based on the tree structure, we present and analyze the basic procedure of GA.

A. Code Tree

The channel polarization process can be expressed on a code tree. For a polar code with code length $N = 2^n$, the corresponding code tree $\mathcal{T}$ is a perfect binary tree. Specifically, $\mathcal{T}$ can be represented as a 2-tuple $(\mathcal{V}, \mathcal{B})$, where $\mathcal{V}$ and $\mathcal{B}$ denote the set of nodes and the set of edges, respectively.

The depth of a node is the length of the path from the root to this node. The set of all the nodes at a given depth $i$ is denoted by $\mathcal{V}_i$, $i = 0, 1, 2, \ldots, n$. The root node has a depth of zero. Let $v_i^{(j)}$, $j = 1, 2, \ldots, 2^i$, denote the $j$-th node from left to right in $\mathcal{V}_i$. Except for the nodes at the $n$-th depth, each $v_i^{(j)} \in \mathcal{V}_i$ has two descendants in $\mathcal{V}_{i+1}$, and the two corresponding edges are labeled as 0 and 1, respectively. The nodes $v_{n-1}^{(j)} \in \mathcal{V}_n$ are called leaf nodes. Let $\mathcal{T}(v_i^{(j)})$ denote a subtree with a root node $v_i^{(j)}$. The depth of this subtree can be defined as $n - i$ which indicates the difference between the depth of the leaf node and that of the root node. In addition, the node $v_i^{(j)}$ has two subtrees, that is, the left subtree $\mathcal{T}_{\text{left}} = \mathcal{T}(v_{i+1}^{(j-1)})$ and the right subtree $\mathcal{T}_{\text{right}} = \mathcal{T}(v_{i+1}^{(2^i-j)})$.

All the edges in the set $\mathcal{B}$ are partitioned into $n$ levels $\mathcal{B}_l$, $l = 1, 2, \ldots, n$. Each edge in the $l$-th level $\mathcal{B}_l$ is incident to two nodes: one at depth $l - 1$ and the other at depth $l$. An $i$-depth node is corresponding to a path $(b_i, b_2, \ldots, b_1)$ which consists of $i$ edges, with $b_l \in \mathcal{B}_l$, $l = 1, 2, \ldots, i$. A vector $b_i$ is used to depict the above path.
B. Gaussian Approximation for Polar Codes

For binary input AWGN (BI-AWGN) channels with noise variance \( \sigma^2 \), source bits are modulated using binary phase shift keying (BPSK). The transition probability \( W(y|x) \) is written as

\[
W(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(y+1-2x))^2}{2\sigma^2}},
\]

where \( x \in \mathbb{B} \) and \( y \in \mathbb{R} \). The LLR of each received symbol \( y \) is denoted by

\[
L(y) = \ln \frac{W(y|0)}{W(y|1)} = \frac{2y}{\sigma^2}.
\]

Without loss of generality, we assume that all-zero codeword is transmitted. One can check \( L(y) \sim N\left(\frac{y}{\sigma}, \frac{4}{\sigma^2}\right) \).

**GA assumption:** The LLR of each subchannel obeys a consistent Gaussian distribution in which the mean is half of the variance \( [6] - [10] \).

According to the GA assumption, the only issue needed to be dealt with is the mean. To obtain the reliability of each subchannel, we trace their LLR mean. This recursive calculation process can be performed on the code tree. The set of LLRs corresponding to the nodes at depth \( i \) is denoted by \( L_i \), \( i = 0, 1, 2, \ldots, n \). Let \( L^{(j)}_i \) \( j = 1, 2, \ldots, 2^i \), denote the \( j \)-th element in \( L_i \). We write \( m^{(j)}_i \) as the mean of \( L^{(j)}_i \). So the mean of LLR from the channel information can be written as \( m^0_0 = \frac{1}{2} \), and under GA assumption we have

\[
L^{(j)}_i \sim N\left(m^{(j)}_i, 2m^{(j)}_i\right).
\]

Here, \( m^{(j)}_i \) can be computed recursively as

\[
m^{(2j-1)}_{i+1} = f_1(m^{(j)}_i), \quad m^{(2j)}_{i+1} = f_2(m^{(j)}_i),
\]

where the functions \( f_1(x) \) and \( f_2(x) \) are used for check nodes and variable nodes, respectively. And they are given by

\[
\begin{cases} 
  f_1(x) = \phi^{-1} \left(1 - (1 - (1 - \phi(x))^2\right), \\
  f_2(x) = 2x.
\end{cases}
\]

In EGA, \( \phi(x) \) is written as

\[
\phi(x) = \begin{cases} 
  1 - \frac{1}{\sqrt{4\pi\sigma^2}} \int_{-\infty}^{\infty} \tanh\left(\frac{u}{2}\right) e^{-\frac{(u-x)^2}{4\sigma^2}} du & x > 0, \\
  0 & x = 0,
\end{cases}
\]

where \( \tanh(\cdot) \) denotes hyperbolic tangent function \([6]\). It is easy to check that \( \phi(x) \) is continuous and monotonically decreasing on \([0, +\infty)\), with \( \phi(0) = 1 \) and \( \phi(+\infty) = 0 \).

As an illustration, Fig. 1 shows a toy example of code tree with \( N = 16 \), which includes 4 levels. The red bold edge in Fig. 1 depicts the recursive calculation process of \( m^0_4 \), whose corresponding path is \((b_1, b_2, b_3, b_4) = (0, 1, 1, 1)\).

![Fig. 1. An example of code tree for N = 16, n = 4. The red bold edge shows the recursive calculation process of m_4^0.](image)

Obviously, the exact calculation of LLR mean in check nodes requires complex integration, which results in a high computational complexity. Therefore, Chung et al. give the well-known two-segment approximation function of \( \phi(x) \), denoted by \( \varphi(x) \), for the analysis of LDPC codes in \([7]\).

\[
\varphi(x) = \begin{cases} 
  e^{-0.4527x^{0.86}+0.0218} & 0 < x < 10, \\
  \sqrt{2} e^{-\frac{x}{4}} (1 - \frac{4x}{\sigma^2}) & x \geq 10.
\end{cases}
\]

**Proposition 1.** Under GA assumption, each subchannel’s symmetric capacity monotonically increases with its LLR mean.

**Proof:** In GA process, each subchannel is approximated by a BI-AWGN channel \( W \) with LLR mean \( m \). With GA assumption, the variance of corresponding additive white noise is denoted by \( \sigma^2 \). Gaussian Approximation for Polar Codes is also widely used in the construction of polar codes \([8]\). Its corresponding AGA algorithm is denoted by “Chung”. By the GA assumption of \([6]\), the error probabilities of polarized subchannel \( P_e(W^{(j)}_N) \) can be written by

\[
P_e(W^{(j)}_N) = Q\left(\frac{m^{(j)}_N}{\sqrt{2m^{(j)}_N}}\right) = Q\left(\sqrt{\frac{m^{(j)}_N}{2}}\right),
\]

where \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{1}{2} t^2} dt \). Thus, the SC bound can be written as

\[
P_e(N, K, \mathcal{A}) \leq \sum_{j \in \mathcal{A}} Q\left(\sqrt{\frac{m^{(j)}_N}{2}}\right).
\]

Since \( Q(x) \) is a monotone decreasing function, the subchannel \( W^{(j)}_N \) with a larger mean \( m^{(j)}_N \) has higher reliability. The construction of polar codes corresponds to the selection of best \( K \) subchannels among \( N \) as information set \( \mathcal{A} \) in terms of the LLR means \( m^{(j)}_N \), where \( j = 1, 2, \ldots, N \).

**IV. ERROR ANALYSIS OF GAUSSIAN APPROXIMATION**

In this section, we first introduce the concepts of polarization violation set (PVS) and polarization reversal set (PRS). By calculating the two sets, we demonstrate the intrinsic reason that polar codes constructed by conventional AGA suffer from catastrophic performance loss at long code lengths. In order to quantitatively evaluate the remainder calculation error between AGA and EGA, we further propose the concept of cumulative-logarithmic error (CLE) of channel polarization and give a bound to simplify its calculation. Based on the CLE bound, we can efficiently evaluate the performance of different approximation functions in AGA.

**A. PVS and PRS**

**Proposition 1.** Under GA assumption, each subchannel’s symmetric capacity monotonically increases with its LLR mean.

**Proof:** In GA process, each subchannel is approximated by a BI-AWGN channel \( W \) with LLR mean \( m \). With GA assumption, the variance of corresponding additive white noise is denoted by \( \sigma^2 \). Gaussian Approximation for Polar Codes is also widely used in the construction of polar codes \([8]\). Its corresponding AGA algorithm is denoted by “Chung”. By the GA assumption of \([6]\), the error probabilities of polarized subchannel \( P_e(W^{(j)}_N) \) can be written by

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P_e(W^{(j)}_N) = Q\left(\frac{m^{(j)}_N}{\sqrt{2m^{(j)}_N}}\right) = Q\left(\sqrt{\frac{m^{(j)}_N}{2}}\right),
\]

where \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{1}{2} t^2} dt \). Thus, the SC bound can be written as

\[
P_e(N, K, \mathcal{A}) \leq \sum_{j \in \mathcal{A}} Q\left(\sqrt{\frac{m^{(j)}_N}{2}}\right).
\]

Since \( Q(x) \) is a monotone decreasing function, the subchannel \( W^{(j)}_N \) with a larger mean \( m^{(j)}_N \) has higher reliability. The construction of polar codes corresponds to the selection of best \( K \) subchannels among \( N \) as information set \( \mathcal{A} \) in terms of the LLR means \( m^{(j)}_N \), where \( j = 1, 2, \ldots, N \).
Gaussian noise is $\sigma^2 = \frac{1}{m}$. The BI-AWGN channels capacity $I(W)$ is given by

$$I(W) = h(\sigma^2)$$

$\Delta = \frac{1}{2} \sum_{x \in S} \int_{\mathbb{R}} W(y|x) \log \left( \frac{2W(y|x)}{W(y|0) + W(y|1)} \right) dy,$ \hspace{1cm} (13)

where $W(y|x)$ is given as (4). Since the function $h(\sigma^2)$ monotonically decreases with $\sigma^2$, the symmetric capacity $I(W)$ monotonically increases with its LLR mean $m$. In addition, we have

$$\lim_{m \to 0} I(W) = 0, \lim_{m \to +\infty} I(W) = 1.$$ \hspace{1cm} (14)

**Proposition 2.** For size-two channel polarization transform, suppose $(W, W') \to (W^{(1)}, W^{(2)})$. Under GA assumption, the LLR means corresponding to $W, W^{(1)}$ and $W^{(2)}$ are represented as $m, m^{(1)}$ and $m^{(2)}$, respectively. Then, the LLR means should satisfy

$$m^{(1)} \leq m \leq m^{(2)}$$ \hspace{1cm} (15)

with equality if and only if $m = 0$ or $m = +\infty$.

This result follows Proposition 4 in [1]. Combining with Proposition 7 the proof of (15) is immediate. It can be seen from (15), the reliability of original channel $W$ is redistributed. Based on this interpretation, we may say that after one step polarization transform, a “bad” channel $W^{(1)}$ and a “good” channel $W^{(2)}$ have been created.

**Theorem 1.** In AGA construction of polar codes, the approximation function $\Omega(x)$ of $\phi(x)$ should satisfy

$$0 < \Omega(x) < 1.$$ \hspace{1cm} (16)

**Proof:** For GA construction of polar codes, the calculation of EGA is a computationally intensive task. Therefore, AGA based on approximation function will be employed to simplify the calculations. However, different from LDPC codes, the approximation function $\Omega(x)$ for polar codes should guarantee that the relationship of (15) holds. The well-known two-segment approximation function $\varphi(x)$ in (10) is a specific form of $\Omega(x)$. Therefore, in size-two channel polarization transform, for $x \in (0, +\infty)$, $\Omega(x)$ should satisfy

$$\Omega^{-1}\left(1 - (1 - \Omega(x))^2\right) < x < 2x,$$ \hspace{1cm} (17)

which follows Proposition 2. Assuming $\Omega(x)$ monotonically decreases on $(0, +\infty)$, the left inequality in (17) can therefore be simplified as

$$1 - (1 - \Omega(x))^2 > \Omega(x) \Rightarrow 0 < \Omega(x) < 1,$$ \hspace{1cm} (18)

which is the necessary and sufficient condition for $\Omega(x)$ to satisfy Proposition 2.

If $\Omega(x)$ can not meet (16), its approximation error with respect to exact $\phi(x)$ will result in the following two types of reliability sorting error:

Type 1: In size-two polarization, suppose $\Omega(x)$ leading to $m < m^{(2)}$, this error indicates that the reliabilities of subchannels are partially violated, which is named as “polarization violation” phenomenon.

Type 2: Furthermore, if $\Omega(x)$ leads to $m < m^{(2)} \leq m^{(1)}$, this error indicates the reliabilities of subchannels are wrongly reversed, which is named as “polarization reversal” phenomenon.

**Definition 1.** Given the approximation function $\Omega(x)$, polarization violation set (PVS) $S_{\text{PVS}}$ is defined as

$$S_{\text{PVS}} = \left\{ x \mid \Omega^{-1}\left(1 - (1 - \Omega(x))^2\right) < 2x \right\},$$ \hspace{1cm} (19)

where $x \in (0, +\infty)$.

Obviously, in size-two channel polarization transform, for any LLR mean $m$ belonging to $S_{\text{PVS}}$, $\Omega(x)$ will certainly lead to $m < m^{(2)}$, which violates the basic order in Proposition 2. Therefore, for AGA algorithm with $\Omega(x)$, if $S_{\text{PVS}} \neq \emptyset$, any subchannel whose LLR mean belongs to $S_{\text{PVS}}$ will create inaccurately two “good” channel in size-two polarization transform, which will lead to obvious approximation error in AGA process.

**Definition 2.** Given the approximation function $\Omega(x)$, polarization reversal set (PRS) $S_{\text{PRS}}$ is defined as

$$S_{\text{PRS}} = \left\{ x \mid \Omega^{-1}\left(1 - (1 - \Omega(x))^2\right) \geq 2x \right\},$$ \hspace{1cm} (20)

where $x \in (0, +\infty)$.

Interestingly, in size-two channel polarization transform, for any LLR mean $m$ belonging to $S_{\text{PRS}}$, $\Omega(x)$ will result in $m < m^{(2)} \leq m^{(1)}$. In other words, the split “good” channel and “bad” channel swap their roles due to the calculation error of $\Omega(x)$, which yields severe error in size-two polarization transform. This phenomenon then leads to substantial error in reliability sorting.

**Proposition 3.** The relationship between PVS and PRS is expressed as

$$S_{\text{PRS}} \neq \emptyset \Rightarrow S_{\text{PVS}} \neq \emptyset.$$ \hspace{1cm} (21)

**Proof:** Note that when $x \in (0, +\infty)$, for any $x \in S_{\text{PRS}}$, it leads to $\Omega^{-1}(1 - (1 - \Omega(x))^2) \geq 2x$ by Definition 2 which will inevitably result in $\Omega^{-1}(1 - (1 - \Omega(x))^2) \geq x$. Therefore, according to Definition 7, the proof of Proposition 2 is immediate. Namely, $S_{\text{PRS}} \neq \emptyset$ is the sufficient condition of $S_{\text{PVS}} \neq \emptyset$. On the contrary, if $S_{\text{PVS}} = \emptyset$, we have $S_{\text{PRS}} = \emptyset$.

Suppose $\Omega(x)$ monotonically decreases on $(0, +\infty)$, the left inequality in (19) can therefore be simplified as

$$\Omega(x) \geq 1 - (1 - \Omega(x))^2 \Rightarrow \Omega(x) \geq 1.$$ \hspace{1cm} (22)

Analogously, the inequality in (20) will also be simplified as

$$1 - (1 - \Omega(x))^2 \leq \Omega(2x) \Rightarrow 2\Omega(x) - \Omega(x)^2 \leq \Omega(2x).$$ \hspace{1cm} (23)

Recall that in Chung’s conventional AGA scheme, the two-segment approximation function $\varphi(x)$ is a specific form of $\Omega(x)$. Since $\varphi(0) = e^{0.0218} > 1$, $\varphi(x)$ can not satisfy
For \( \varphi(x) \), its corresponding PVS and PRS are denoted by \( S_{\text{PVS}} = (a_1, a_2) \) and \( S_{\text{PRS}} = (0, a_1) \), respectively. The boundary points \( a_1 \) and \( a_2 \) are given in the following equations

\[
\begin{align*}
&\begin{cases} 
2 \varphi(a_1) - \varphi(a_1)^2 = \varphi(2a_1), \quad \Rightarrow \quad a_1 = 0.01476, \\
&\varphi(a_2) = 1, \\
&a_2 = 0.02939,
\end{cases} \\
&\text{which follows from } (22) \text{ and } (23). \quad \text{Hence for } \varphi(x) \text{ we have}
\end{align*}
\]

\[
S_{\text{PVS}} = (0.01476, 0.02939), \quad S_{\text{PRS}} = (0, 0.01476),
\]

which are denoted in Fig. 2.

**Theorem 2.** For \( N \)-channel polarization transform, where \( N = 2^n, n \geq 1 \), suppose that the original channel has two configurations of the LLR mean, which are denoted by \( \tilde{m}_0^{(1)} \) and \( \tilde{m}_0^{(1)} \). And they satisfy \( \tilde{m}_0^{(1)} \geq \tilde{m}_0^{(1)} \), then under GA assumption, for \( j = 1, 2, \cdots, 2^k \), we have

\[
\tilde{m}_{n}^{(j)} \geq \tilde{m}_{n}^{(j)}.
\]

**Proof:** This result will be proved by mathematical induction. Under GA assumption, according to the basic property of \( \phi(x) \), one can easily check that \( f_1(x) \) and \( f_2(x) \) in (8) monotonously increase on \([0, +\infty)\).

Suppose \( N = 2^k \), \( k = 1 \), if the two LLR mean configurations of original channel satisfy \( \tilde{m}_0^{(1)} \geq \tilde{m}_0^{(1)} \), then

\[
\begin{align*}
&\begin{cases} 
f_1(\tilde{m}_0^{(1)}) \geq f_1(\tilde{m}_0^{(1)}), \\
f_2(\tilde{m}_0^{(1)}) \geq f_2(\tilde{m}_0^{(1)}),
\end{cases} \\
&\Rightarrow \begin{cases} 
\tilde{m}_1^{(1)} \geq \tilde{m}_1^{(1)}, \\
\tilde{m}_1^{(2)} \geq \tilde{m}_1^{(2)}.
\end{cases}
\end{align*}
\]

(27)

With the increased LLR mean of original channel \( P \), it can be seen that the LLR means of two polarized subchannels will uniformly increase.

Next, suppose \( N = 2^{k+1} \), if we have \( \tilde{m}_0^{(1)} \geq \tilde{m}_0^{(1)} \), then for \( j = 1, 2, \cdots, 2^k \), \( \tilde{m}_k^{(j)} \geq \tilde{m}_k^{(j)} \) holds. Thus, when \( N = 2^{k+1} \), one can check

\[
\begin{align*}
&\begin{cases} 
f_1(\tilde{m}_k^{(j)}) \geq f_1(\tilde{m}_k^{(j)}), \\
f_2(\tilde{m}_k^{(j)}) \geq f_2(\tilde{m}_k^{(j)}),
\end{cases} \\
&\Rightarrow \begin{cases} 
\tilde{m}_{k+1}^{(j)} \geq \tilde{m}_{k+1}^{(j)}, \\
\tilde{m}_{k+1}^{(j)} \geq \tilde{m}_{k+1}^{(j)}.
\end{cases}
\end{align*}
\]

(28)

In other words, for \( j = 1, 2, \cdots, 2^{k+1} \), \( \tilde{m}_{k+1}^{(j)} \geq \tilde{m}_{k+1}^{(j)} \). From above analysis, the proof of (26) is finished.

**Theorem 2** indeed indicates that under GA assumption, if the LLR mean of original channel \( W \) increases, the polarized subchannel’s LLR mean will uniformly increase.

In AGA construction of polar codes, with \( \Omega(x) \), if \( S_{\text{PRS}} \neq \emptyset \), then for any LLR mean \( m_i^{(j)} \in S_{\text{PRS}} \), we have \( m_{i+1}^{(j)} \geq m_{i+1}^{(j)} \) by the definition of \( S_{\text{PRS}} \) in the recursive calculation of GA. However, \( \phi(x) \) in EGA can guarantee \( S_{\text{PRS}} = \emptyset \) and \( S_{\text{PVS}} = \emptyset \), which claims \( m_{i+1}^{(j)} < m_{i+1}^{(j)} \) for any \( m_i^{(j)} > 0 \). The above analysis indicates that in AGA process, as presented in the code tree, due to \( S_{\text{PRS}} \neq \emptyset \), the condition \( m_i^{(j)} \in S_{\text{PRS}} \) leads to the reliability sorting error among the leaf nodes which belong to \( T_{\text{left}} = T(v_i^{(1)}) \) and \( T_{\text{right}} = T(v_i^{(2)}) \), following Theorem 2, one can check

\[
\begin{align*}
m_i^{(1)} &\geq m_i^{(5)}, \\
m_i^{(2)} &\geq m_i^{(6)}, \\
m_i^{(3)} &\geq m_i^{(7)}, \\
m_i^{(4)} &\geq m_i^{(8)}.
\end{align*}
\]

(30)

Similarly, since \( m_i^{(5)} \in S_{\text{PVS}} \), we have \( m_i^{(9)} \geq m_i^{(5)} \) whereas in EGA, it should be \( m_i^{(9)} < m_i^{(5)} \), and all the "\( \geq \)" in (30) should be "<". Thus, compared to EGA, the approximation error of AGA results in obvious sorting error within the 16 leaf nodes.

The above analysis indicates that in AGA construction of polar codes, if \( S_{\text{PRS}} \neq \emptyset \), its approximation error leads to polarization reversal. Considering its definition, \( S_{\text{PRS}} \) lies in
the vicinity of 0. As stated in [1], when \( N \) tends to infinity, the symmetric capacity terms \( \{ I(W_n^{(j)}) \} \) cluster around 0 and 1, and the corresponding LLR means cluster around 0 and \( +\infty \). Therefore, when the code length becomes longer, there are more LLR means falling in \( S_{PRS} \) during the AGA recursive computation process. This is the essential reason that polar codes constructed by some AGA suffer from catastrophic performance loss with long code lengths.

During the recursive process of AGA, assuming the number of code tree nodes belonging to the two sets \( \emptyset \) are denoted by \( \mu_{PVS} \) and \( \mu_{PRS} \), the corresponding ratios with respect to all nodes are written as

\[
\theta_{PVS} = \frac{\mu_{PVS}}{\sum_{k=0}^{n-1} 2^k}, \quad \theta_{PRS} = \frac{\mu_{PRS}}{\sum_{k=0}^{n-1} 2^k}.
\]

For Chung’s conventional AGA, Table I gives the distribution of \( \mu_{PVS} \) and \( \mu_{PRS} \) with different polarization levels \( n \). The corresponding \( \theta_{PVS} \) and \( \theta_{PRS} \) are also listed in Table II, where \( E_b/N_0 = 1\text{dB} \) (\( \sigma^2 = 1.1915 \)). From Table I we observe that with the increase of code length, there are more and more nodes falling in PRS, and its corresponding ratio also becomes larger. Therefore, the polar codes constructed by Chung’s conventional AGA suffer from catastrophic jitter in performance when the code length is long.

Fig. IV demonstrates the BLER performance of polar codes constructed by Chung’s conventional AGA and EGA under BI-AWGN channels. The polar codes are constructed depending on the signal-to-noise ratio (SNR) one by one, and all the schemes have code rate \( R = 1/3 \) with SC decoding. The code length \( N \) is set to be \( 2^{12}, 2^{14} \) and \( 2^{18} \). We observe that for long code lengths Chung’s scheme obviously presents catastrophic performance loss. It is consist with the analysis of Table II.

**Remark 2:** Recall that Arıkan in [16] suggested a heuristic BEC approximation method to construct polar codes for arbitrary binary-input channels, which has also yielded good results in experiments. The above PVS and PRS analyses also give interpretation about this “good results”. One can check BEC approximation has strict order preserving property in size-two polarization transform, which shows \( I(W_2^{(1)}) < I(W) < I(W_2^{(2)}) \). Heuristic BEC approximation will not lead to polarization violation and polarization reversal in this sense. Therefore, this heuristic method will just bring some moderate performance loss rather than catastrophic jitter.

**Remark 3:** Due to the high computation complexity of EGA, AGA is essentially used to perform GA construction in practice. However, compared with EGA, if \( S_{PVS} \neq \emptyset \) and \( S_{PRS} \neq \emptyset \), the approximation function \( \Omega(x) \) of AGA will lead to catastrophic performance loss.

### B. CLE of Channel Polarization

Polarization violation and polarization reversal reveal the essential reason that AGA can not work well for long code lengths. Besides these two sets, in this subsection, we further propose a new metric, named cumulative-logarithmic error (CLE) of channel polarization, to quantitatively evaluate the remainder approximation error between AGA and EGA. Then CLE is utilized to guide the design of GA approximation function \( \Omega(x) \).

For \( \Omega(x) \) in AGA, suppose its \( S_{PVS} = \emptyset \) and \( S_{PRS} = \emptyset \). CLE will play a crucial role in evaluating the performance of AGA. We concern about the subchannel’s capacity, which is a function of LLR mean under GA assumption. The difference between \( \Omega(x) \) and \( \phi(x) \) brings in calculation error in subchannel capacity evaluation. The original absolute error of capacity calculation between AGA and EGA is denoted by \( \Delta(x) \), which is a function of LLR mean \( \mu \). Without ambiguity, \( \Delta(x) \) will be abbreviated to \( \Delta \) in this paper.

Assume \( \Delta \) occurs after \( r \) recursions, denoted by \( \Delta_r \). Thus \( \Delta_r \) is accumulated as final error after \( n-r \) polarization levels.

**TABLE I**

| \( n \) | \( \mu_{PVS} \) | \( \theta_{PVS} \) | \( \mu_{PRS} \) | \( \theta_{PRS} \) |
|-------|---------|---------|---------|---------|
| 10    | 40      | 3.910%  | 64      | 7.326%  |
| 11    | 89      | 4.438%  | 88      | 4.299%  |
| 12    | 191     | 4.664%  | 225     | 5.495%  |
| 13    | 394     | 4.810%  | 549     | 6.702%  |
| 14    | 803     | 4.901%  | 1297    | 7.917%  |
| 15    | 1617    | 4.935%  | 3003    | 9.165%  |
| 16    | 3280    | 5.005%  | 6820    | 10.414% |
| 17    | 6540    | 4.837%  | 15240   | 11.836% |
| 18    | 12528   | 4.779%  | 29098   | 14.804% |
| 19    | 24550   | 4.683%  | 69064   | 14.612% |
| 20    | 48036   | 4.581%  | 159132  | 15.183% |

**Remark 1:** For AWGN channels, compared with accurate DE algorithm, EGA is also found to well approximate the actual polarized subchannels. Note that EGA has strict order preserving property, which follows Proposition 2. This order preserving property of EGA indeed gives reasonable interpretation about its “good results” versus DE. Therefore, in general, the error between EGA and DE is so small that it can be ignored.

2The leaf nodes whose LLR mean falls into the two sets are not counted in, because it will not lead to sorting error among the descendants.
and this process can be represented on a subtree with a depth \( n - r \). To evaluate the calculation error of AGA, we focus on the difference of subchannel’s capacity calculated by AGA and EGA in logarithmic domain. The capacities calculated by EGA can form a set \( \mathcal{I} \) defined on this code subtree with the following properties:

On this subtree, the set of capacities corresponding to the nodes at a given depth \( d \) is denoted by \( \mathcal{I}_d \), \( d = r, r+1, \ldots, n \). Let \( I^{(k)}_d \), \( k = 1, 2, \ldots, 2^{d-r} \), denote the \( k \)-th element in \( \mathcal{I}_d \). For each \( I^{(k)}_d \in \mathcal{I}_d \), \( I^{(k)}_d \) takes value on \([0,1]\). And for \( d > r \), \( I^{(k)}_d \) is a function of vector \( b^{r+1}_i \), which is actually the binary expansion of \( k - 1 \). Therefore, at the root node \( v_r^{(j)} \), \( \Delta_r \) can be written as

\[
\Delta_r = \tilde{I}^{(1)}_r - I^{(1)}_r = h \left( \frac{2}{m_r^{(j)}} \right) - h \left( \frac{2}{m^{(j)}} \right),
\]

where \( \tilde{I}^{(1)}_r, I^{(1)}_r \) and \( m_r^{(j)}, m^{(j)} \) stand for the initial symmetric capacities and LLR means calculated by AGA and EGA respectively, and the formula of \( h(\cdot) \) is written in [13].

As stated in Remark 2, without much sacrifice in accuracy, BEC approximation will act as faithful surrogate for GA in error analysis. Due to the iteration structure in the subtree, we have

\[
\begin{align*}
I^{(2k-1)}_{d+1} &= I^{(k)}_d \quad \text{when } b_{d+1} = 0, \\
I^{(2k)}_{d+1} &= 2I^{(k)}_d - I^{(k)}_d \quad \text{when } b_{d+1} = 1.
\end{align*}
\]

Furthermore, when \( b_{d+1} = 1 \), we have \( I^{(2k)}_{d+1} \leq 2I^{(k)}_d \). Thus, in logarithmic domain, we can get

\[
\begin{align*}
\log I^{(2k-1)}_{d+1} &= 2\log I^{(k)}_d \quad \text{when } b_{d+1} = 0, \\
\log I^{(2k)}_{d+1} &\leq \log I^{(2k)}_d + 1 \quad \text{when } b_{d+1} = 1.
\end{align*}
\]

Define \( \tilde{I}^{(k)}_d = I^{(k)}_d + \Delta^{(k)}_d \), where \( \tilde{I}^{(k)}_d \) denotes the capacity corresponding to AGA, \( \Delta^{(k)}_d \) represents the absolute error of capacity calculation between AGA and EGA. For \( r < d \leq n \), \( \tilde{I}^{(k)}_d \) and \( I^{(k)}_d \) represent the capacities, which are calculated by BEC approximation, of AGA and EGA respectively. In this paper, We only analyze the error between AGA and EGA, rather than the error of GA itself or the error brought in by heuristic BEC approximation. Therefore, \( \Delta_r^{(1)} = \Delta_r \). Let \( \rho^{(k)}_d = \Delta^{(k)}_d / \tilde{I}^{(k)}_d \) denote the relative error, and \( e^{(k)}_d = \log \tilde{I}^{(k)}_d - \log I^{(k)}_d = \log \left( 1 + \rho^{(k)}_d \right) \) denote the capacity calculation error in logarithmic domain. Hence, the partial cumulative-logarithmic error (PCLE) can be written as

\[
C_{r:n} = \sum_{k=1}^{2^{n-r}} \left| e^{(k)}_n \right|.
\]

The cumulative-logarithmic error (CLE) will be \( C = \sum_r C_{r:n} \).

C. CLE Bound

The precise calculation of CLE is too complicated to be analyzed using recursive relation (33). In this subsection, we propose a upper bound on CLE to simplify its calculation.

**Proposition 4.** For the \( k \)-th leaf node corresponding to a path \( b^n_{r+1} \), suppose \(|\{r+1 \leq i \leq n : b_i = 0\}| = \alpha\) and \(|\{r+1 \leq i \leq n : b_i = 1\}| = n - r - \alpha\).

**Theorem 3.** The logarithmic error \( e^{(k)}_n \) can be bounded by

\[
\left| e^{(k)}_n \right| \leq 2^\alpha \log \left( 1 + \rho^{(k)}_n \right) = 2^\alpha \log \left( 1 + \frac{\Delta^{(1)}_n}{\tilde{I}^{(1)}_n} \right).
\]

**Proof:** Let \( \tilde{e}^{(k)}_d \) denote the bound of \( e^{(k)}_d \). Then, \( \tilde{e}^{(k)}_n \) is determined by specifying \( \tilde{e}^{(1)}_n = e^{(1)}_n = \log \left( 1 + \rho^{(1)}_n \right) \) and

\[
\begin{align*}
\tilde{e}^{(2k-1)}_{d+1} &= D \left( \tilde{e}^{(k)}_d \right) \quad \text{when } b_{d+1} = 0, \\
\tilde{e}^{(2k)}_{d+1} &= E \left( \tilde{e}^{(k)}_d \right) \quad \text{when } b_{d+1} = 1,
\end{align*}
\]

where \( E : \mathbb{R} \to \mathbb{R}, E(x) = x \) denotes equality, and \( D : \mathbb{R} \to \mathbb{R}, D(x) = 2x \) denotes doubling.

If \( \Delta^{(1)}_n \geq 0 \), it claims that \( \Delta^{(k)}_d \geq 0 \) holds by **Theorem 2**. Note that during the iteration, when \( b_{d+1} = 0 \),

\[
0 \leq \tilde{e}^{(2k-1)}_{d+1} = \tilde{e}^{(k)}_d + \Delta^{(k)}_d = \tilde{e}^{(2k)}_{d+1}.
\]

And when \( b_{d+1} = 1 \), it can be proved that \( \tilde{e}^{(2k)}_{d+1} \leq \tilde{e}^{(2k)}_{d+1} \). According to the first equation of (33), it is easy to get that

\[
\tilde{I}^{(2k)}_{d+1} = 2\tilde{I}^{(k)}_d - \left( \tilde{I}^{(k)}_d \right)^2 \\
= 2\left( I^{(k)}_d + \Delta^{(k)}_d \right) - \left( I^{(k)}_d + \Delta^{(k)}_d \right)^2 \\
= 2I^{(k)}_d - \left( I^{(k)}_d \right)^2 + 2\Delta^{(k)}_d - 2\Delta^{(k)}_d \Delta^{(k)}_d - \left( \Delta^{(k)}_d \right)^2.
\]

Therefore, \( e^{(2k)}_{d+1} \) can be written as

\[
e^{(2k)}_{d+1} = \log \left( 1 + \rho^{(2k)}_{d+1} \right) \\
= \log \left( 1 + \frac{\Delta^{(k)}_d - 2\Delta^{(k)}_d \Delta^{(k)}_d - \left( \Delta^{(k)}_d \right)^2}{2\Delta^{(k)}_d - \left( I^{(k)}_d \right)^2} \right).
\]

And we have

\[
\tilde{e}^{(2k)}_{d+1} = \log \left( 1 + \rho^{(2k)}_{d+1} \right) = e^{(k)}_d = \log \left( 1 + \frac{\Delta^{(k)}_d}{I^{(k)}_d} \right).
\]

Then we can check

\[
\rho^{(2k)}_{d+1} - \rho^{(2k)}_{d+1} \geq \frac{\Delta^{(k)}_d \left( I^{(k)}_d + \Delta^{(k)}_d \right)}{I^{(k)}_d \left( 2 - I^{(k)}_d \right)} \geq 0.
\]

It can be inferred that \( \tilde{e}^{(k)}_d \geq e^{(k)}_d \geq 0 \) holds.

Recall **Proposition 4** during the iterative calculation of \( e^{(k)}_n \), we count doubling \( \alpha \) times and equality \( n-r-\alpha \) times. Hence, we have

\[
0 \leq e^{(k)}_n \leq e^{(k)}_n = E^{n-r-\alpha} D^{\alpha} \left( e^{(1)}_r \right) = 2^\alpha e^{(1)}_r.
\]

Analogously, if \( \Delta^{(1)}_n < 0 \), we have \( \Delta^{(k)}_d < 0 \). From (37) and (41), we can get

\[
0 > e^{(k)}_n > e^{(k)}_n = E^{n-r-\alpha} D^{\alpha} \left( e^{(1)}_r \right) = 2^\alpha e^{(1)}_r.
\]
Combing (42) and (43), we prove the theorem. ■

**Theorem 4.** PCLE $C_{r,n}$ can be upper bounded by

$$C_{r,n} \leq 3^{n-r} \left| \log \left( 1 + \rho_r^{(1)} \right) \right| = 3^{n-r} \left| \log \left( 1 + \frac{\Delta_r^{(1)}}{I_r^{(1)}} \right) \right|.$$  

**Proof:** For any $k \in \{1, 2, \ldots, 2^{n-r}\}$, the number of vectors $b_{r+1}^n$ satisfying Proposition [4] is $\binom{n-r}{a}$, where $b_{r+1}^n$ is the binary expansion of $k - 1$. Combined with definition (35) and Theorem [3], $C_{r,n}$ satisfies the following constraint

$$C_{r,n} \leq \sum_{k=1}^{2^{n-r}} |e_r^{(k)}| = \sum_{a=0}^{n-r} \binom{n-r}{a} 2^a |e_r^{(1)}| = 3^{n-r} |e_r^{(1)}|. \quad (44)$$

The last equation in (44) uses binomial theorem. Therefore, CLE $C$ can be upper bounded by $\sum_r 3^{n-r} \left| \log_2 \left( 1 + \rho_r^{(1)} \right) \right|$, and the exponent $n - r$ stands for polarization levels. ■

**V. IMPROVED GAUSSIAN APPROXIMATION**

In this section, guided by the previous PVS, PRS and CLE analyses, we propose new rules to design AGA for polar codes. Then we give two specific forms of the approximation function in AGA, which have advantages in both complexity and performance.

**A. Design Rules of AGA**

For AWGN channels, AGA is widely used to construct polar codes. However, in practical implementation, the accuracy of $\Omega(x)$ will greatly affect the construction of polar code especially when the code length is long. According to Theorem [4], the initial error will be exponentially amplified with the increase of polarization levels. Note that PCLE bound is mainly affected by two factors: the first term $3^{n-r}$ is relevant to polarization levels, and the second term is dependent on the original relative error $\rho_r^{(1)} = \Delta_r^{(1)}/I_r^{(1)}$. In general, the absolute error $\Delta_r^{(1)}$ is tiny. Hence, for the good channels whose capacities $I_r^{(1)}$ approach 1, their original relative errors $\rho_r^{(1)}$ are so small that they can be ignored. However, for the bad channels whose capacities $I_r^{(1)}$ approach 0, their original relative errors are not negligible. Subsequently, given a fixed $\Delta_r^{(1)}$, the more $I_r^{(1)}$ is close to 0, the larger the original relative error $\rho_r^{(1)}$ becomes.

The above analysis indicates that CLE bound is mainly affected by the terms $C_{r,n}$ with small initial capacity $I_r^{(1)}$, which corresponds to bad channels. Due to the error, some frozen subchannels will be wrongly identified as information-carrying ones (role flipping), which results in performance degradation. Note that the capacity $I_r^{(1)}$ monotonically increases with LLR mean $x$ under GA assumption. In addition, according to (14), we have

$$\lim_{x \to 0} I_r^{(1)} = 0, \quad \lim_{x \to +\infty} I_r^{(1)} = 1. \quad (45)$$

Guided by above analysis, the approximation function design in AGA is composed of three rules:

**Rule 1)** PVS and PRS eliminating: $\Omega(x)$ should guarantee $S_{\text{PVS}} = \emptyset$ and $S_{\text{PRS}} = \emptyset$. According to Proposition 4 and its converse-negative proposition, if $S_{\text{PVS}} = \emptyset$, we have $S_{\text{PRS}} = \emptyset$. Hence, in order to empty PVS and PRS, we should ensure $0 < \Omega(x) < 1$ for any $x \in (0, +\infty)$.

**Rule 2)** Low SNR design: When $x$ comes close to 0, we must guarantee $\lim_{x \to +\infty} \Omega(x) = 1$ to reduce approximation error. Since CLE bound is amplified exponentially with the growth of polarization levels, the only way to reduce CLE bound is to lower the initial relative error $\rho_r^{(1)}$. Therefore, $\Omega(x)$ needs to be divided into more segments when $x$ approaches 0. This rule can reduce the original absolute error $\Delta_r^{(1)}$ in the vicinity of 0 so as to lower $\rho_r^{(1)}$.

**Rule 3)** High SNR design: When $x$ stays away from 0, thanks to the relatively large $I_r^{(1)}$, CLE bound can tolerate a more obvious absolute error $\Delta_r^{(1)}$. Therefore, $\Omega(x)$ can be selected with some simpler forms to reduce the computational complexity.

Besides, $\Omega(x)$ should keep continuity between the adjacent two segments, which can mitigate the jitter of CLE bound by keeping smooth of initial error. Among these three rules, Rule 1 is the most crucial, which can prevent the corresponding AGA constructing scheme from catastrophic performance loss. Then the importance of Rule 2 is second, which reduces the remainder approximation error between AGA and EGA. Rule 3 plays a less important role, which helps to further reduce the computational complexity of AGA.

**B. Improved GA Approximation Function**

Guided by above rules, we design the following new two-segment approximation function, denoted by $\gamma(x)$,

$$\gamma(x) = \begin{cases} 
    e^{0.0116x^2 - 0.4212x} & 0 < x \leq a, \\
    e^{-0.2944x} - 0.3169 & a < x, 
\end{cases} \quad (46)$$

where the boundary point $a = 7.0633$. The corresponding AGA algorithm is denoted by AGA-2. For constructing polar codes, compared with Chung’s scheme, AGA-2 will not lead to catastrophic performance loss at a long code length, and it has lower complexity.

Furthermore, we introduce a new interval in the vicinity of 0 and propose a new piecewise function $\lambda(x)$ with three segments, that is

$$\lambda(x) = \begin{cases} 
    e^{0.06725x^2 - 0.4908x} & 0 < x \leq a, \\
    e^{-0.4527x^2 + 0.0218} & a < x \leq b, \\
    e^{-0.2832x} - 0.4254 & b < x, 
\end{cases} \quad (47)$$

where the boundary points $a = 0.6357$ and $b = 9.2254$. Its corresponding AGA algorithm is denoted by AGA-3. Note that AGA-3 is specially designed for polar codes especially with long code lengths, which follows the proposed rules.

Firstly, one can check that for both $\gamma(x)$ and $\lambda(x)$, they monotonically decrease for $x \in (0, +\infty)$. And we have

$$\begin{align*} 
    \lim_{x \to 0} \gamma(x) &= 1, \quad \lim_{x \to +\infty} \gamma(x) = 0, \\
    \lim_{x \to 0} \lambda(x) &= 1, \quad \lim_{x \to +\infty} \lambda(x) = 0. \quad (48)
\end{align*}$$

Therefore, both $\gamma(x)$ and $\lambda(x)$ satisfy Rule 1. Correspondingly, $S_{\text{PVS}} = \emptyset$ and $S_{\text{PRS}} = \emptyset$. 
Secondly, according to the left side of (48), we can verify that both \( \gamma(x) \) and \( \lambda(x) \) satisfy Rule 2. Besides, compared with the two-segment \( \gamma(x) \), the three-segment \( \lambda(x) \) has better accuracy in the vicinity of 0.

In addition, guided by Rule 3, for \( f_1(x) \) in (49), when \( x \) is away from 0, \( 1 - \phi(x) \) tends to 1, which indicates

\[
(1 - \phi(x))^2 \approx 1 - \phi(x) \Rightarrow f_1(x) \approx x.
\]

Followed by Proposition 2, \( f_1(x) \) should satisfy \( f_1(x) < x \). Thus, when \( x \) stays away from 0, the complex \( f_1(x) \) can be further approximated as

\[
f_1(x) = x - \varepsilon,
\]

where \( \varepsilon \) denotes the offset. Then in terms of (8), when \( \phi(x) \) tends to 0, one can check

\[
\phi(x - \varepsilon) = 2\phi(x) - \phi(x)^2 \approx 2\phi(x).
\]

Therefore, for \( \lambda(x) \), in terms of (51), when \( x \gg 0 \) we have

\[
e^{-0.2832(x-\varepsilon)-0.4254} = e^{-0.2832x-0.4254+\ln 2}
\]

\[
\Rightarrow \varepsilon = 2.4476,
\]

where \( x-\varepsilon \) should locate in the third segment, namely \( x-\varepsilon > b \), which claims \( x > 11.673 \). For the entire AGA-3 scheme, its \( f_1(x) \) is denoted by

\[
f_1(x) = \begin{cases} 
\lambda^{-1} \left( 1 - (1 - \lambda(x))^2 \right) & 0 \leq x \leq \tau, \\
x - 2.4476 & x > \tau,
\end{cases}
\]

where the boundary point \( \tau = 11.673 \). Similarly, in AGA-2 scheme, \( f_1(x) \) can be written as

\[
f_1(x) = \begin{cases} 
\gamma^{-1} \left( 1 - (1 - \gamma(x))^2 \right) & 0 \leq x \leq \tau, \\
x - 2.3544 & x > \tau,
\end{cases}
\]

where the boundary point \( \tau = 9.4177 \). Compared with AGA-2, AGA-3 performs better at the cost of higher complexity.

The computation complexity of \( f_1(x) \) has been remarkably reduced thanks to Rule 3 in the above two proposed AGA schemes. Moreover, Rule 1 and Rule 2 help AGA to achieve excellent performance. Thus, for the general construction of polar codes, the proposed AGA-2 scheme is a good alternate to improve the Chung’s conventional AGA scheme. When the code length becomes longer, the proposed AGA-3 scheme will achieve better performance than AGA-2 scheme. Certainly, if the code length becomes extremely long, the three-segment approximation function in AGA-3 will also bring calculation error, which is stated in Rule 2. Nevertheless, by using Rule 1~3, we can still design AGA with multi-segment approximation function to keep the calculation accuracy. In this way, we provide a complete framework to design high accuracy AGA scheme for any code length.

VI. Numerical Results

In this section, as an evaluation tool, CLE bound is used to compare the performance of AGA-2, AGA-3 and Chung’s conventional AGA. The BLER performance is analyzed via simulation results over BI-AWGN channels.

Fig. 5. The precise results and corresponding upper bounds of CLE for various AGAs, where \( x \) stands for LLR mean and polarization level \( n = 8 \).

Fig. 6. CLE bounds comparison for different AGAs and polarization levels.

Fig. 7. BLER performance comparison of polar codes with the code length \( N = 2^n \) \( (n = 12, 14, 18) \) and code rate \( R = 1/3 \) in BI-AWGN channel.
Fig. 5 shows the precise results and corresponding upper bounds of CLE for various AGAs, where the polarization level $n = 8$ and $r = 0$ (CLE $C = C_{0.8}$). It can be seen that the CLE bound and the exact result coincide well. Therefore, the CLE bound can be used as an effective tool to evaluate the performance of AGA.

Fig. 6 shows the CLE bounds for different AGAs. It can be found that AGA-3 obviously outperforms that of AGA-2 when $x$ approaches 0. However, when the number of polarization levels increases from $n = 14$ to $n = 20$, their corresponding CLE bounds significantly increase no matter for AGA-2 or AGA-3. Therefore, when the code length is extremely long, the three-segment approximation function will also bring in obvious error so that $\Omega(x)$ needs more segments.

Fig. 7 depicts the BLER performance comparisons among different GAs under BI-AWGN channels. The corresponding polar codes are constructed depending on the SNR one by one. All the schemes have code rate $R = 1/3$ with SC decoding. The code length $N$ is set to be $2^{12}, 2^{14}$ and $2^{18}$, and SC bound is calculated under exact GA (EGA SC bound) according to [3]. For long code length, Chung’s scheme suffers from a dramatical performance loss. On the contrast, the polar codes constructed by AGA-3 perform well. When $N = 2^{18}$, AGA-2 shows a little performance loss which agrees with the analysis in last paragraph of section V. And all the results are consistent with the CLE analysis.

VII. CONCLUSION

In this paper, we introduced the concepts of PVS and PRS which claim the essential reason that polar codes constructed by conventional AGA present catastrophic performance loss. Then we proposed a new metric, named CLE, to quantitatively evaluate the remainder error of AGA. We further derived the upper bound of CLE to simplify its calculation. Guided by PVS, PRS and CLE bound, we proposed new rules to design AGA for polar codes. Simulation results show that the performance of all GA algorithms is consistent with CLE analysis. And when polarization levels increase, conventional AGA suffers from catastrophic performance jitter. By contrast, the multi-segment GA approximation function guided by the proposed rules stably guarantee the excellent performance of polar codes.

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