Modified dispersionless Veselov–Novikov equations and corresponding hydrodynamic chains

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Abstract
Various links connecting well-known hydrodynamic chains and corresponding 2+1 nonlinear equations are described.

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1 Introduction

Theory of integrable hydrodynamic chains started with famous paper [1] (see also, [2], [6], [7], [8], [9], [10], [12], [20]). Recent investigations are obtained in [3] and in [15] (see also, [5], [4]).

This paper is devoted to a description of various links associated with the integrable hydrodynamic chain

\[ B_{t}^k = B_{t}^{k+1} + \frac{1}{2} B^{0} B_{t}^{k} + (k + 1) B^{k} B_{t}^{0}, \quad k \in \mathbb{Z}. \quad (1) \]

which is a particular case of extended Kupershmidt hydrodynamic chains (see [11], [16])

\[ B_{t}^k = B_{t}^{k+1} + \frac{1}{\beta} B^{0} B_{t}^{k} + (k + \gamma) B^{k} B_{t}^{0}, \quad k \in \mathbb{Z}. \]

This hydrodynamic chain can be written in the Hamiltonian form

\[ B_{t}^k = [(2k + 1) B^{k+n} D_{y} + (2n + 1) D_{y} B^{k+n}] \frac{H_1}{\delta B^n}. \]
where the Hamiltonian is given by $H = \int \left[ B^1/3 + (B^0)^2/4 \right] dt^0$. The extended Kupershmidt hydrodynamic chains possess an infinite series of conservation laws. All positive conservation law densities $h_n$ are polynomial with respect to positive moments $B^k$. Negative conservation law densities $h_{-n}$ are more complicated. For instance, $h_{-1} = -2 \exp(B^{-1/2})$, $h_{-2} = -B^{-2} \exp(3B^{-1/2})/3$. Thus, the above hydrodynamic chain possesses an infinite series of commuting flows. For instance, two first negative flows are

$$B_{t^{-1}}^k = e^{B^{-1/2}}[B_{t^0}^{k-1} - kB^{k-1}B_{t^0}^{-1}], \quad (2)$$

$$B_{t^{-2}}^k = e^{3B^{-1/2}} \left[ B_{t^0}^{k-2} + B^{-2}_{t^0} B_{t^0}^{k-1} - \left( \frac{3}{2} kB^{k-1}B^{-2} + (k-1)B^{-2} - kB^{-1}B_{t^0}^{-2} \right) \right], \quad (3)$$

2 Dispersionless Veselov–Novikov equations

The first negative commuting flow (2) is invariant (see [16]) under a transformation of independent variables $t^0 \leftrightarrow t^{-1}$ and the substitution

$$\tilde{B}^k = B^{-2} e^{(k+1)B^{-1}}.$$

All other commuting flows are invariant under a transformation of independent variables $t^{k-1} \leftrightarrow t^{-k}$. For instance, the second commuting flow (3) transforms into (1), while (1) transforms into (3).

Let us combine these commuting flows, i.e. we have new integrable hydrodynamic chain

$$B_t^k = B_{t^0}^{k+1} + \frac{B^0}{2} B_{t^0}^k + (k+1)B^k B_{t^0}^0$$

$$+ e^{3B^{-1/2}} \left[ B_{t^0}^{k-2} + \frac{B^{-2}}{2} B_{t^0}^{k-1} - \left( \frac{3}{2} kB^{k-1}B^{-2} + (k-1)B^{-2} - kB^{-1}B_{t^0}^{-2} \right) B_{t^0}^{-1} - kB^{-1}B_{t^0}^{-2} \right], \quad (4)$$

determined by the mixed Hamiltonian $H \equiv H_1 + H_{-2} = \int [B^1/3 + (B^0)^2/4 - B^{-2} \exp(3B^{-1/2})/3] dt^0$ (see (1) and (3)).

Introducing new field variables

$$v = \frac{B^0}{2}, \quad u = e^{B^{-1/2}}, \quad w = \frac{B^{-2}}{2} e^{B^{-1}},$$

two conservation laws (see (2))

$$\partial_{t^{-1}} \frac{B^0}{2} = \partial_{t^0} e^{B^{-1/2}}, \quad \partial_{t^{-1}} e^{B^{-1/2}} = \partial_{t^0} \left( \frac{B^{-2}}{2} e^{B^{-1}} \right)$$

can be written in Egorov’s form (see [19], [15])

$$v_{t^{-1}} = u_{t^0}, \quad u_{t^{-1}} = w_{t^0}.$$
These conservation laws together with the conservation law (see (4))

\[ u_t = (uv)_t + (uw)_{t-1} \]

determine a dispersionless limit of the remarkable symmetric Veselov–Novikov equation

\[ \Omega_{\phi t-1} = (\Omega_{\phi t-1} \Omega_{\phi} + \Omega_{\phi t-1} \Omega_{t-1} - 1)_{t-1}, \]  

(5)

where

\[ v = \Omega_{\phi t}, \quad u = \Omega_{\phi t-1}, \quad w = \Omega_{t-1}. \]

Another 2+1 quasilinear system of the first order can be derived from the Kupershmidt hydrodynamic chain (1) and its first higher commuting flow

\[ B_{t0}^k = B_{t0}^{k+2} + 3 \frac{B_0^0 B_{t0}^{k+1} + (B_1^1 + 5(B_0^0)^2) B_{t0}^k + (k+2)B_{t0}^{k+1}B_0^0 + (k+1)B_0^1 B_{t0}^{k+1} + 5B_0^0}{2} \]

determined by the Hamiltonian

\[ H_2 = \int [B^2/5 + B^0 B^1/2 + 5(B^0)^3/24] dt. \]

First two conservation laws of the Kupershmidt hydrodynamic chain (1)

\[ B_{t1}^0 = \partial_{t0} \left( B^1 + \frac{3}{4}(B^0)^2 \right), \quad \partial_{t1} \left( B^1 + \frac{3}{4}(B^0)^2 \right) = \partial_{t0} \left( B^2 + 2B^0 B^1 + \frac{3}{4}(B^0)^3 \right) \]

and first conservation law of its above higher commuting flow

\[ B_{t2}^0 = \partial_{t0} \left( B^2 + \frac{5}{2}B^0 B^1 + \frac{25}{24}(B^0)^3 \right) \]

can be written together as a dispersionless limit of non-symmetric Veselov–Novikov equation

\[ \Omega_{t1t1} = \Omega_{\phi t2} - \Omega_{\phi t1} \Omega_{t1} + \frac{1}{3} \Omega_{\phi t0}, \]

(6)

where

\[ \Omega_{t1t1} = \frac{B^1}{2} + \frac{3}{8}(B^0)^2, \quad \Omega_{t1t2} = \frac{B^2}{2} + \frac{5}{4}B^0 B^1 + \frac{25}{48}(B^0)^3. \]

**Remark:** The symmetric dispersionless Veselov–Novikov equation (5) can be obtained from the compatibility condition \( \partial_{t1}(\partial_{t-1} p) = \partial_{t-1}(\partial_{t1} p) \), where the generating function of conservation law densities \( p \) satisfies

\[ p_{t-1} = -\partial_{t0} \frac{u}{p}, \quad p_t = \partial_{t0} \left( \frac{p^3}{3} + vp - \frac{uw}{p} - \frac{u^3}{3p^3} \right), \]

while the non-symmetric dispersionless Veselov–Novikov equation (6) can be obtained from the compatibility condition \( \partial_{t1}(\partial_{t2} p) = \partial_{t2}(\partial_{t1} p) \), where the generating function of conservation law densities \( p \) satisfies

\[ p_{t1} = \partial_{t0} \left( \frac{p^3}{3} + \Omega_{\phi t0} p \right), \quad p_{t2} = \partial_{t0} \left[ \frac{p^5}{5} + \Omega_{\phi t0} p^3 + (\Omega_{t1t1} + \Omega_{t0t0}) p \right]. \]
Corresponding Gibbons equations (see [6], [17]) associated with the above hydrodynamic chains are

\[
\begin{align*}
\lambda_t - (p^2 + v) \lambda_{\varphi} &= \frac{\partial \lambda}{\partial p} \left[ p_t - \partial_{\varphi} \left( \frac{p^3}{3} + vp \right) \right], \\
\lambda_{t-2} - \left( \frac{uw}{p^2} + \frac{u^2}{p^4} \right) \lambda_{\varphi} &= \frac{\partial \lambda}{\partial p} \left[ p_{t-2} + \partial_{\varphi} \left( \frac{uw}{p} + \frac{u^3}{3p^3} \right) \right], \\
\lambda_t - \left( p^2 + v + \frac{uw}{p^2} + \frac{u^2}{p^4} \right) \lambda_{\varphi} &= \frac{\partial \lambda}{\partial p} \left[ p_t - \partial_{\varphi} \left( \frac{p^3}{3} + vp - \frac{uw}{p} - \frac{u^3}{3p^3} \right) \right], \\
\lambda_{t-1} - \frac{u}{p^2} \lambda_{\varphi} &= \frac{\partial \lambda}{\partial p} \left( p_{t-1} + \partial_{\varphi} \frac{u}{p} \right), \\
\lambda_{t-2} - (p^4 + 3vp^2 + v^2 + s) \lambda_{\varphi} &= \frac{\partial \lambda}{\partial p} \left[ p_{t-2} - \partial_{\varphi} \left( \frac{p^5}{5} + vp^3 + (v^2 + s)p \right) \right],
\end{align*}
\]

where \( v = \Omega_{\varphi \varphi}, \ s = \Omega_{\varphi t}, \) and (6) can be written as 2+1 quasilinear system of the first order

\[
\begin{align*}
\partial_t v &= \partial_{\varphi} s, \\
\partial_t s &= \partial_{t\varphi} \left( \frac{v^3}{3} - vs \right).
\end{align*}
\]

**Remark:** Since the function \( \Omega \) is unique for both Veselov–Novikov equations, both of them possess the same set of hydrodynamic reductions determined by the Gibbons–Tsarev system (see [8], [16]).

### 3 Modified VN equation

The generating function of conservation laws and commuting flows (see [16]) is given by

\[
\partial_{\tau(\zeta)} p(\lambda) = \frac{1}{2} \partial_{\varphi \varphi} \ln \frac{p(\lambda) - p(\zeta)}{p(\lambda) + p(\zeta)},
\]

where \( p(\lambda) \) can be found due to the Bürmann–Lagrange series (see, for instance, [13]) from the Riemann mappings

\[
\lambda|_{p \to \infty} = p^2 \exp \left( \sum_{k=0}^{\infty} \frac{B^k}{p^{2(k+1)}} \right), \quad \lambda|_{p \to 0} = p^{-2} \exp \left( \sum_{k=1}^{\infty} B^{-k} p^{2(k-1)} \right).
\]

In this section we consider another expansion

\[
p(\lambda) = \tilde{h}_0 + \lambda \tilde{h}_1 + \lambda^2 \tilde{h}_2 + \ldots, \quad \lambda \to 0,
\]

where \( \lambda \) is another local parameter. In such a case, (7) leads to an infinite series of generating functions

\[
\partial_{\varphi \varphi} p(\lambda) = \frac{1}{2} \partial_{\varphi \varphi} \ln \frac{p(\lambda) - \tilde{h}_0}{p(\lambda) + \tilde{h}_0}, \quad \partial_{\varphi \varphi} p(\lambda) = -\partial_{\varphi \varphi} \frac{\tilde{h}_1 p(\lambda)}{p^2(\lambda) - \tilde{h}_0^2}, \ldots,
\]

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where
\[ \partial_{\tau(\lambda)} = \partial_{\tau^0} + \lambda \partial_{\tau^1} + \lambda^2 \partial_{\tau^2} + \ldots \]

Introducing a new generating function of conservation law densities
\[ \tilde{p}(\lambda) = \ln \frac{p(\lambda) + \tilde{h}_0}{p(\lambda) - \tilde{h}_0}, \]
and eliminating \( \tau^0 \), both above generating functions can be written in the form
\[ \partial_{\tau^1} \tilde{p}(\lambda) = \partial_{\tau^0} \left( \frac{\tilde{h}_1}{\tilde{h}_0} \sinh \tilde{p}(\lambda) \right) \]
which is a first coefficient from (7) (see (9) and [15]).

Substituting the Taylor expansion (8) into the above generating functions, one can obtain new 2+1 nonlinear equations
\[ \Omega_{\tau^0,\tau^0} = \frac{1}{2} \ln \frac{\Omega_{\tau^0,\tau^1}}{\Omega_{\tau^0,\tau^0}}, \quad \Omega_{\tau^1,\tau^1} = \Omega_{\tau^0,\tau^2} - \Omega_{\tau^0,\tau^1}^2 + \frac{1}{2} \exp(4\Omega_{\tau^0,\tau^0}), \]
where \( \tilde{h}_k \equiv \Omega_{\tau^0,\tau^k} \), because \( p(\lambda) \equiv \Omega_{\tau^0,\tau(\lambda)} \) (the last formula is a consequence of (7), if \( p \to \infty \)).

Remark: The second above equation was found in [15]. This is a nontrivial generalization of a continuum limit of discrete KP hierarchy (see, for instance, [14]). Thus, we described links between different well-known 2+1 integrable equations in this paper.

Theorem: The hydrodynamic chain
\[ A^k_{\tau^1} = A^{k+1}_{\tau^0} - \left( (k+2)A^{k+1} - kA^{k-1} \right) A^0_{\tau^0}, \quad k = 0, 1, 2, \ldots \]
satisfies the Gibbons equation
\[ \lambda_{\tau^1} - \frac{\cosh \tilde{p}}{A^0} \cdot \lambda_{\tau^0} = \lambda_{\tilde{p}} \left( \tilde{p}_{\tau^1} - \partial_{\tau^0} \sinh \tilde{p} \right), \]
where the equation of the Riemann surface is given by
\[ \lambda = \sum_{n=0}^{\infty} \frac{A^n}{\cosh^{n+1} \tilde{p}}. \]

Proof: The substitution (11) and (13) in (12) leads to an identity.

Theorem: The above hydrodynamic chain (11) is Hamiltonian
\[ A^k_{\tau^1} = \left[ [(k+2)A^{k+1} - kA^{k-1}] D_{\tau^0} + A^{k+1}_{\tau^0} \right] \frac{\partial h}{\partial A^0}, \]
where the Hamiltonian density \( h = \ln A^0 \).

**Proof:** Integrable hydrodynamic chains associated with more general Hamiltonian structure were considered in [18].

**Remark:** Simplest hydrodynamic reductions

\[
a^i_{\tau_1} = \partial_{\tau_0} \frac{\sinh a^i}{A^0}
\]

are given by the moment decomposition

\[
A^k = \frac{1}{k+1} \sum_{i=1}^{N} \epsilon_i (\cosh a^i)^{k+1}, \quad k = 0, 1, 2, \ldots, \quad \sum_{i=1}^{N} \epsilon_i = 0.
\]

**Theorem:** 2+1 nonlinear equation

\[
\Omega_{\tau_1 \tau_1} = \Omega_{\tau_0 \tau_2} - \frac{1}{2} \Omega_{\tau_0 \tau_0}^2 + \epsilon \exp(-2\Omega_{\tau_0 \tau_0})
\]

can be determined from the compatibility condition \( \partial_{\tau_1}(\partial_{\tau_2}p) = \partial_{\tau_2}(\partial_{\tau_1}p) \), where the generating functions of conservation laws are given by

\[
p_{\tau_1} = \partial_{\tau_0} [e^{-U}(e^p + \epsilon e^{-p})], \quad p_{\tau_2} = \partial_{\tau_0} \left[ \frac{e^{2p} - \epsilon^2 e^{-2p}}{2} e^{-2U} - V(e^p - \epsilon e^{-p}) e^{-U} \right],
\]

where \( U = \Omega_{\tau_0 \tau_0}, V = \Omega_{\tau_0 \tau_1} \).

**Remark:** Substituting the transformation

\[
\mu = e^{-U}(e^p + \epsilon e^{-p}) - V
\]

into the above generating functions and eliminating \( \tau_0 \), one can obtain the generating function of conservation laws

\[
\mu_{\tau_2} = \partial_{\tau_1} \left[ \frac{\mu + V}{2} \sqrt{(\mu + V)^2 - 4\epsilon e^{-2U} - V(\mu + V) - W} \right]
\]

generalizing (\( \epsilon = 0 \)) the generating function of conservation laws for the Benney hydrodynamic chain

\[
\mu_{\tau_2} = \partial_{\tau_1} \left[ \frac{\mu^2}{2} \frac{V^2}{2} - W \right],
\]

where \( W = \Omega_{\tau_0 \tau_2} \). Such a transformation for \( \epsilon = 0 \) was found first by Yuji Kodama (see, for instance, [14]).

Let us introduce new notations \( U = -C^{-1}, V = -C^0 \) and \( q = p - U \).

**Theorem:** The integrable hydrodynamic chain

\[
C_{\tau_1}^{-1} = C_{\tau_0}^0, \quad C_{\tau_1}^0 = \partial_{\tau_0} \left( C^1 - \epsilon e^{-2C^{-1}} \right),
\]

\[
C_{\tau_1}^k = C_{\tau_0}^{k+1} + k C_{\tau_0}^0 - \epsilon e^{-2C^{-1}} [C_{\tau_0}^{k-1} + 2(k-1)C_{\tau_0}^{k-2}C_{\tau_0}^{-1}], \quad k = 1, 2, \ldots
\]
satisfies the Gibbons equation (see (10))

\[
\lambda_{\tau^1} - (e^q - e\epsilon e^{-2C^{-1}-q})\lambda_{\tau^0} = \frac{\partial \lambda}{\partial q} \left[ q_{\tau^1} - \partial_{\tau^0} \left( e^q + C^0 + e\epsilon e^{-2C^{-1}-q} \right) \right], \tag{15}
\]

where the Riemann mapping is determined by

\[
\lambda = e^q + \sum_{n=0}^{\infty} C^n e^{-nq}. \tag{16}
\]

Proof: Indeed, a substitution the above series and the hydrodynamic chain (14) into the Gibbons equation (15) leads to an identity.

This Riemann mapping (16) is the same as for a continuum limit of two-dimensional Toda lattice. This hydrodynamic chain (14) is a member of an integrable hierarchy generalizing (\(\epsilon = 0\)) an integrable hierarchy containing a continuum limit of two-dimensional Toda lattice.

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References

[1] D.J. Benney, Some properties of long non-linear waves, Stud. Appl. Math., 52 (1973) 45-50.

[2] B.A. Dubrovin, Hamiltonian formalism of Whitham-type hierarchies and topological Landau-Ginsburg models, Comm. Math. Phys., 145 (1992) 195-207. B.A. Dubrovin, Geometry of 2D topological field theories, Lecture Notes in Math. 1620, Springer-Verlag (1996) 120-348.

[3] E.V. Ferapontov, K.R. Khusnutdinova, On integrability of (2+1)-dimensional quasi-linear systems, Comm. Math. Phys., 248 (2004) 187–206, E.V. Ferapontov, K.R. Khusnutdinova, The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, J. Phys. A: Math. Gen., 37 No. 8 (2004) 2949–2963.

[4] E.V. Ferapontov, K.R. Khusnutdinova, M.V. Pavlov, Classification of integrable (2+1)-dimensional quasilinear hierarchies, Theor. Math. Phys. 144 (2005) 35-43.

[5] E.V. Ferapontov, D.G. Marshall, Differential-geometric approach to the integrability of hydrodynamic chains: the Haantjes tensor, arXiv:nlin.SI/0505013.
[6] J. Gibbons, Collisionless Boltzmann equations and integrable moment equations, Physica D, 3 (1981) 503-511.

[7] J. Gibbons, Yu. Kodama, Solving dispersionless Lax equations. In N. Ercolani et al., editor, Singular limits of dispersive waves, v. 320 of NATO ASI Series B, page 61. Plenum (1994) New York.

[8] J. Gibbons, S.P. Tsarev, Reductions of the Benney equations, Phys. Lett. A 211 (1996) 19-24. J. Gibbons, S.P. Tsarev, Conformal maps and reductions of the Benney equations, Phys. Lett. A 258 (1999) 263-271.

[9] Yu. Kodama, A method for solving the dispersionless KP equation and its exact solutions. Phys. Lett. A, 129 No. 4 (1988) 223-226. Yu. Kodama, A solution method for the dispersionless KP equation, Prog. Theor. Phys. Supplement. 94 (1988) 184.

[10] I.M. Krichever, The averaging method for two-dimensional ”integrable” equations, Funct. Anal. Appl. 22 No. 3 (1988) 200-213, I.M. Krichever, Spectral theory of two-dimensional periodic operators and its applications, Russian Math. Surveys 44 No. 2 (1989) 145-225. I.M. Krichever, The dispersionless equations and topological minimal models, Comm. Math. Phys., 143 No. 2 (1992) 415-429. I.M. Krichever, The τ-function of the universal Whitham hierarchy, matrix models and topological field theories, Comm. Pure Appl. Math. 47 (1994) 437-475.

[11] B.A. Kupershmidt, Deformations of integrable systems, Proc. Roy. Irish Acad. Sect. A 83 No. 1 (1983) 45-74. B.A. Kupershmidt, Normal and universal forms in integrable hydrodynamical systems, Proceedings of the Berkeley-Ames conference on nonlinear problems in control and fluid dynamics (Berkeley, Calif., 1983), in Lie Groups: Hist., Frontiers and Appl. Ser. B: Systems Inform. Control, II, Math Sci Press, Brookline, MA, (1984) 357-378.

[12] B.A. Kupershmidt, Yu.I. Manin, Long wave equation with free boundaries. I. Conservation laws. Func. Anal. Appl., 11 No. 3 (1977) 188–197. B.A. Kupershmidt, Yu.I. Manin, Long wave equations with a free surface. II. The Hamiltonian structure and the higher equations. Func. Anal. Appl., 12 No. 1 (1978) 20–29. D.R. Lebedev, Yu.I. Manin, Conservation laws and representation of Benney’s long wave equations, Phys. Lett. A, 74 No. 3,4 (1979) 154-156. D.R. Lebedev, Benney’s long wave equations: Hamiltonian formalism, Lett. Math. Phys., 3 (1979) 481–488.

[13] M.A. Lavrentiev, B.V. Shabat, Metody teorii funktsii kompleksnogo peremennogo(Russian) [Methods of the theory of functions of a complex variable] Third corrected edition Izdat. “Nauka”, Moscow (1965) 716 pp. P. Henrici, Metody teorii funktsii kompleksnogo peremennogo Topics in computational complex analysis. IV. The Lagrange-Bürmann formula for systems of formal power series. Computational aspects of complex analysis (Braunlage, 1982), 193–215, NATO Adv. Sci. Inst. Ser. C, Math. Phys. Sci., 102, Reidel, Dordrecht, 1983.

[14] L.A. Yu, Waterbag reductions of the dispersionless discrete KP hierarchy, J. Phys. A: Math. Gen., 33 (2000) 8127–8138.
[15] M.V. Pavlov, Classification of the Egorov hydrodynamic chains, Theor. Math. Phys., 138 No. 1 (2004) 55-71.

[16] M.V. Pavlov, The Kupershmidt hydrodynamic chains and lattices. to appear in IMRN.

[17] M.V. Pavlov, The Hamiltonian approach in the classification and the integrability of hydrodynamic chains. to appear in J. Math. Phys.

[18] M.V. Pavlov, Algebro-geometric approach in the theory of integrable hydrodynamic type systems. to appear in Comm. Math. Phys.

[19] M. V. Pavlov, S.P. Tsarev, Three-Hamiltonian structures of the Egorov hydrodynamic type systems, Funct. Anal. Appl., 37 No. 1 (2003) 32-45.

[20] V.E. Zakharov, Benney’s equations and quasi-classical approximation in the inverse problem method, Funct. Anal. Appl., 14 No. 2 (1980) 89-98. V.E. Zakharov, On the Benney’s Equations, Physica 3D (1981) 193-200.