REFLECTION GROUPS ACTING ON THEIR HYPERPLANES

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Abstract. After having established elementary results on the relationship between a finite complex (pseudo-)reflection group $W \subset \text{GL}(V)$ and its reflection arrangement $\mathcal{A}$, we prove that the action of $W$ on $\mathcal{A}$ is canonically related with other natural representations of $W$, through a ‘periodic’ family of representations of its braid group. We also prove that, when $W$ is irreducible, then the squares of defining linear forms for $\mathcal{A}$ span the quadratic forms on $V$, which imply $|\mathcal{A}| \geq n(n+1)/2$ for $n = \dim V$, and relate the $W$-equivariance of the corresponding map with the period of our family.

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1. Introduction

Let $V$ be finite-dimensional $\mathbb{C}$-vector space, $W \subset \text{GL}(V)$ be a finite (pseudo-)reflection group with corresponding hyperplane arrangement $\mathcal{A}$. We assume that $\mathcal{A}$ is essential, meaning that $\bigcap \mathcal{A} = \{0\}$ and denote $n = \dim V$ the rank of $W$. We recall that an arrangement $\mathcal{A}$ is called irreducible if it cannot be written as $\mathcal{A}_1 \times \mathcal{A}_2$, and that $W$ is called irreducible if it acts irreducibly on $V$. A basic result can be written as follows

(0) $\mathcal{A}$ is irreducible iff $W$ is irreducible.

Steinberg showed that the exterior powers of $V$ are irreducible. His proof is based on the encryption of irreducibility in the connectedness of certain graphs. From this approach, the following is easily deduced

(1) If $W$ is irreducible, then it contains an irreducible parabolic subgroup.

Although this result is probably well-known to experts and easily checked, it does not seem to appear in print, and is a key tool for the sequel.

We then consider the permutation $W$-module $\mathbb{C}\mathcal{A}$. A choice of linear maps $\alpha_H \in V^*$ with kernel $H \in \mathcal{A}$ defines a linear map $\Phi : \mathbb{C}\mathcal{A} \rightarrow S^2V^*$ through $\alpha_H \mapsto \alpha_H^2$. This map can be chosen to be a morphism of $W$-modules when $W$ is a Coxeter group. We prove

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(2) $\Phi$ is onto iff $W$ is irreducible

meaning that each quadratic form on $V$ is a linear combination of the quadratic forms $\alpha^2_H$, as soon as $W$ is irreducible. As a corollary, we get

(3) The cardinality of $A$ is at least $n(n + 1)/2$.

This lower bound is better than the usual $|A| \geq n/2$ of [OT], cor. 6.98, and is sharp, as $|A| = n(n + 1)/2$ when $W$ is a Coxeter group of type $A_n$.

We denote $d_H$ the order of the (cyclic) fixer in $W$ of $H \in A$, and define the distinguished reflection $s \in W$ to be the reflection in $W$ with $H = \text{Ker}(s - 1)$ and additional eigenvalue $\zeta_H = \exp(2i\pi/d_H)$. We let $d : A \to \mathbb{Z}$ denote $H \mapsto d_H$. We did not find the following in the standard textbooks:

(4) The data $(A, d)$ determines $W$.

Letting $B$ denote the braid group associated to $W$, we show that $\mathbb{C}A$, considered as a linear representation of $B$, can be deformed through a path in $\text{Hom}(B, \text{GL}(V))$ which canonically connects $\mathbb{C}A$ to other representations of $W$. This turns out to provide a natural generalization of the action of Weyl groups on their positive roots to arbitrary reflection groups.

Finally, we prove that this path $h \mapsto R_h$ is periodic, namely that $R_{h + \kappa(W)} \simeq R_h$ for some integer $\kappa(W)$, with $\kappa(W) = 2$ when $W$ is a Coxeter group. Moreover, $\kappa(W) = 2$ if and only if the morphism $\Phi$ above can be chosen to be a morphism of $W$-modules. In particular, we get

(5) If $\kappa(W) = 2$ then the $W$-module $S^2 V^*$ is a quotient of $\mathbb{C}A$.

We emphasize the fact that the proofs presented here are elementary in the sense that, except for one of the last results, no use is made either of the Shephard-Todd classification of pseudo-reflection groups, nor of the invariants theory of these groups.

2. REFLECTION GROUPS AND REFLECTION ARRANGEMENTS

We recall from [OT] the following basic notions about reflection groups and hyperplane arrangements. An endomorphism $s \in \text{GL}(V)$ is called a (pseudo-)reflection if it has finite order and $\text{Ker}(s - 1)$ is an hyperplane of $V$. A finite subgroup $W$ of some $\text{GL}(V)$ which is generated by reflections is called a (complex) (pseudo-)reflection group. The hyperplane arrangement associated to it is the collection $A$ of the reflecting hyperplanes $\text{Ker}(s - 1)$ for $s$ a reflection of $W$. There is a natural function $d : A \to \mathbb{Z}$, $H \mapsto d_H$ which associates to each $H \in A$ the order of the subgroup of $W$ fixing $H$. We let $\zeta_H = \exp(2i\pi/d_H)$, and call a reflection $s$ distinguished if its nontrivial eigenvalue is $\zeta_H$, with $\text{Ker}(s - 1) = H$.

A nontrivial subgroup $W_0$ of $W$ is called parabolic if it is the fixer of some linear subspace of $V$. By a fundamental result of Steinberg, this linear subspace lies inside some intersection of reflecting hyperplanes, and $W_0$ is also a reflection group in $\text{GL}(V)$.

In general, a (central) hyperplane $A$ arrangement is a finite collection of linear hyperplanes in $V$. When $A$ originates from a reflection group $W$, then $A$ is called a reflection arrangement. An arrangement $A$ is called essential if $\bigcap A = \{0\}$; for two arrangements $A_1, A_2$ in $V_1, V_2$, the arrangement $A$ in $V = V_1 \times V_2$ is defined as $\{H \oplus V_2; H \in A_1\} \cup \{V_1 \oplus H; H \in A_2\}$; two
arrangements in $V$ are isomorphic if they are deduced one from the other by some element of $\text{GL}(V)$; an essential arrangement $\mathcal{A}$ is called irreducible if it is not isomorphic to some nontrivial $\mathcal{A}_1 \times \mathcal{A}_2$.

The following lemma shows that, when $\mathcal{A}$ is a reflection arrangement, the arrangement $\mathcal{A}$ together with the order of the reflections determines the reflection group. In particular, there is at most one reflection group with reflections of order 2 admitting a given reflection arrangement. Notice that $\mathcal{A}$ can be assumed to be essential, as the action of $W$ on $\bigcap \mathcal{A}$ is necessarily trivial. Although basic, this fact does not appear in standard textbooks. The proof given here has been found in common with François Digne and Jean Michel.

**Proposition 2.1.** Let $\mathcal{A}$ be an essential hyperplane arrangement in $V$.

1. If $P \in \text{GL}(V)$ satisfies $P(H) \subset H$ for all $H \in \mathcal{A}$, then $P$ is semisimple.
2. If $\mathcal{A}$ is a reflection arrangement associated to a complex reflection group $W \subset \text{GL}(V)$, then $(\mathcal{A}, d)$ determines $W$.

**Proof.** To prove (1), we choose linear forms $\alpha_H \in V^*$ with kernel $H \in \mathcal{A}$. Since $\mathcal{A}$ is essential, $V^*$ is generated by the $\alpha_H$, hence admits a basis made out some of them. The assumption then states that the $\alpha_H$ are eigenvectors for $tP \in \text{GL}(V^*)$, hence $tP$ is semisimple and so is $P$. Now we prove (2), assuming that $W_1, W_2 \subset \text{GL}(V)$ are two reflection groups with the same data $(\mathcal{A}, d)$. Let $H \in \mathcal{A}$ and $s_i \in W_i$ the distinguished reflection with $\text{Ker}(s_i - 1) = H$. Then $x = s_1s_2^{-1}$ fixes $H$ and acts by 1 on $V/H$, hence is unipotent. The endomorphism $x \in \text{GL}(V)$ clearly permutes the hyperplanes. Since $\mathcal{A}$ is finite, some power of $x$ setwise stabilizes every $H \in \mathcal{A}$, hence is semisimple by (1). Since it is also unipotent this power of $x$ is the identity, hence $x = \text{Id}$ because $x$ is unipotent. It follows that $s_1 = s_2$ hence $W_1 = W_2$. □

### 3. A consequence of Steinberg lemma

Let $W \subset \text{GL}(V)$ be a reflection group and $\mathcal{A}$ the corresponding reflection arrangement. A basic fact is that the notions of irreducibility for $W$ and $\mathcal{A}$ coincide and can be checked combinatorially on some graph. After recalling a proof of this, we notice a useful consequence.

We endow $V$ with a $W$-invariant hermitian scalar product. Call $v \in V$ a **root** if it is an eigenvector of a reflection $s \in V$ such that $s.v \neq v$. For $L$ a finite set of linearly independent roots we let $V_L$ denote the subspace of $V$ spanned by $L$, and $\Gamma_L$ the graph on $L$ connecting $v_1$ and $v_2$ if and only if $v_1$ and $v_2$ are not orthogonal. Notice that, if $s \in W$ is a reflection with root $v \in V$, the following properties hold: if $v \in V_L$ then $s(V_L) \subset V_L$, because $V_L = (\mathbb{C}v) \oplus (\text{Ker}(s - 1) \cap V_L)$; if $v \in V_L^\perp$ then $V_L \subset (\mathbb{C}v)^\perp$ is pointwise stabilized by $s$.

The following proposition is basic. We provide a proof of $(1) \iff (2)$ for the convenience of the reader, because of a lack of reference. $(1) \iff (3)$ is due to Steinberg.

**Proposition 3.1.** The following are equivalent, for an essential reflection arrangement $\mathcal{A}$.
(1) $W$ acts irreducibly on $V$.
(2) $\mathcal{A}$ is an irreducible hyperplane arrangement.
(3) $V$ admits a basis $L$ of roots such that $\Gamma_L$ is connected.

Proof. In the direction (2) $\Rightarrow$ (1), if $V = V_1 \oplus V_2$ with the $V_i$ being $W$-stable subspaces, then we define $\mathcal{A}_i = \{ H \in \mathcal{A} \mid (s_H)_{|V_i} \neq \text{Id} \}$ with $s_H$ the distinguished reflection w.r.t. $H \in \mathcal{A}$, and we have $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. In the direction (1) $\Rightarrow$ (2), we let $V = V_1 \oplus V_2$ be the decomposition of $V$ corresponding to $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. We choose a collection of roots for $\mathcal{A}$. Let $s_1, s_2$ be two distinguished reflections associated to $H_1 \in \mathcal{A}_1, H_2 \in \mathcal{A}_2$, respectively, and let $H = H_1 \oplus H_2 \subset V$. Consider some reflection $s \in W$ such that $\text{Ker}(s) \supset H$. If $\text{Ker}(s - 1)$ can be written as $H_0 \oplus V_2$ with $H_0$ some hyperplane of $V_1$, then $H_0 \oplus V_2 \supset H_1 \oplus H_2$ implies $H_0 \supset H_1$, hence $H_0 = H_1$ by equality of dimensions, meaning that $s$ is some power of $s_1$. Similarly, if $\text{Ker}(s - 1)$ can be written as $V_1 \oplus H_0$ with $H_0$ some hyperplane of $V_2$, then $s$ is a power of $s_2$. Considering the reflection $s_2 s_1 s_2^{-1}$, which fixes $H$ and has reflecting hyperplane $s_2$. $\text{Ker}(s - 1)$, since $s_1 \neq s_2$ it follows that $s_2 s_1 s_2^{-1}$ is a power of $s_1$. Then $s_2 \text{Ker}(s - 1) = \text{Ker}(s - 1)$ hence $s_1, s_2$ commute and have orthogonal roots. The subspace $V_1^0$ spanned by all roots aring from $\mathcal{A}_1$ is thus setwise stabilized by all reflections of $W$, hence $V_1^0 = V$. On the other hand, the hermitian scalar product induces an isomorphism between $V_1^0$ and $V_1^{*}$ (because $A_1$, like $A$, is essential), hence $V_2 \neq \{0\} \Rightarrow V_1^0 \neq V$, a contradiction.

We now prove (1) $\iff$ (3). Let $L_0$ be of maximal size among the sets $L$ of linearly independent roots with connected $\Gamma_L$. We prove that $|L| = \dim V$ if $W$ is irreducible. Indeed, since $W$ is irreducible generated by reflections and $V_{L_0} \subset V$, there would otherwise exist a reflection $s$ such that $s(V_{L_0}) \not\subset V_{L_0}$. Letting $v \in V$ be a root of $s$, we have $v \not\in V_{L_0}$ and $v \not\in (V_{L_0})^\perp$. This proves that $L = L_0 \cup \{v\}$ is made out linearly independant roots and that $\Gamma_L$ is connected, since $v \not\in (V_{L_0})^\perp$ cannot be orthogonal to all roots spanning $L_0$ and $L_0$ is already connected. From this contradiction it follows that $L_0$ has cardinality $\dim V$. Conversely, if $V$ admits a basis $L$ of roots such that $\Gamma_L$ is connected, then $W$ is irreducible, for otherwise $V = V_1 \oplus V_2$ with $V_1, V_2$ nontrivial orthogonal $W$-stable subspaces, and $L = L_1 \cup L_2$ with $L_i = \{ x \in L \mid x \in U_i \}$. Then $\Gamma_L = \Gamma_{L_1} \cup \Gamma_{L_2}$, contradicting the connectedness of $\Gamma_L$.

Corollary 3.2. If $W \subset \text{GL}(V)$ is an irreducible reflection group then it admits an irreducible parabolic subgroup of rank $\dim V - 1$.

Proof. Considering a set $L$ of linearly independent roots such that $\Gamma_L$ is connected, as given by the proposition, there exists $L_0 \subset L$ with $L = L_0 \cup \{v\}$ such that $\Gamma_{L_0}$ is still connected. Then $V_{L_0}$ has dimension $\dim V - 1$, and its orthogonal is spanned by some $v' \in V$. Letting $W_0$ denote the parabolic subgroup fixing $v'$, it has rank $\dim V - 1$, admits for roots all elements of $L_0$, hence is irreducible since $\Gamma_{L_0}$ is connected. □

4. Quadratic forms on $V$

Let $\mathcal{A}$ be an essential hyperplane arrangement in $V$. The integer $n = \dim V$ is the rank $\text{rk}\mathcal{A}$ of $\mathcal{A}$. For each $H \in \mathcal{A}$ we let $\alpha_H \in V^*$ denote
some linear form with kernel $H$. For a field $k$, let $kA$ denote a vector space with basis $v_H, H \in A$, and define a linear map $\Phi : C A \to S^2 V^*$ by $\Phi(v_H) = \alpha_2^2$.

For $\Phi$ to be onto, it is necessary that $A$ is irreducible. Indeed, if $A = A_1 \times A_2$ corresponds to some direct sum decomposition $V = V_1 \oplus V_2$, then choosing two nonzero linear forms $\varphi_i \in V_i^*$ defines a quadratic form $\varphi_1 \varphi_2 \in S^2 V^*$ which does not belong to $\text{Im} \Phi$. This condition is also sufficient in rank 2.

**Proposition 4.1.** If $A$ is essential of rank 2, then $\Phi$ is onto if and only if $A$ is irreducible.

**Proof.** Since $A$ is essential, $A$ contains at least two hyperplanes $H_1, H_2$. We denote $\alpha_i = \alpha_{H_i}$ the corresponding (linearly independent) linear forms. If $A = \{H_1, H_2\}$, then $A$ is obviously reducible, so we may assume that $A$ contains at least another hyperplane. Let $\beta$ denote the corresponding linear form. It can be written as $\beta = \lambda_1 \alpha_1 + \lambda_2 \alpha_2$ with $\lambda_1 \neq 0, \lambda_2 \neq 0$. Since $\beta^2 = \lambda_1^2 \alpha_1^2 + 2\lambda_1 \lambda_2 \alpha_1 \alpha_2 + \lambda_2^2 \alpha_2^2$, we get that $\alpha_1, \alpha_2, \beta^2 \in \text{Im} \Phi$.

This condition is not sufficient in rank 3, as shows the following example. Consider in $C^3$ the central arrangement of polynomial $xyz(x-y)(y-z)$. The morphism $\Phi$ is obviously not surjective, as $\dim C A = 5$ and $\dim S^2 V^* = 6$. However, $A$ is irreducible, because its Poincaré polynomial is $P_A(t) = (1 + t)(1 + 4t + 4t^2)$, which is not divisible by $(1 + t)^2$ — recall from [OT] that $P_{A_1 \times A_2} = P_{A_1} P_{A_2}$ and that $P_A(t)$ is divisible by $1 + t$ whenever $A$ is central.

It is however sufficient when $A$ is a reflection arrangement.

**Theorem 4.2.** Let $A$ be a (essential) reflection arrangement. Then $\Phi$ is surjective if and only if $A$ is irreducible.

**Proof.** We assume that $A$ is irreducible, and prove that $\Phi$ is surjective by induction on $\text{rk} A$. If $\text{rk} A \leq 2$, this is a consequence of the above proposition, so we can assume $\text{rk} A \geq 3$. We denote $W$ the corresponding (pseudo-)reflection group, and endow $V$ with a $W$-invariant hermitian scalar product. By corollary 3.2 there exists an irreducible maximal parabolic subgroup $W_0 \subset W$, defined by $W_0 = \{w \in W \mid w v = v\}$ for some $v \in V \setminus \{0\}$. We let $H_0 = (C v)^{^\perp}$. By Steinberg theorem $W_0$ is a reflection group, whose (pseudo-)reflections are the reflections of $W$ contained in $W_0$. Let $A_0 \subset A$ denote the arrangement in $V$ corresponding to $W_0$. Since $v \in H$ for all $H \in A_0$, by the induction hypothesis we have $Q = S^2 H_0^* \subset S^2 V^*$ is induced by $H^* \subset V^*$, letting $\gamma \in H_0^*$ act on $H_0^*$ by 0. Let $\alpha \in V^* \setminus \{0\}$ such that $H_0 = \text{Ker} \alpha$. We have $S^2 V^* = S^2 H_0^* \oplus \alpha H_0^* \oplus \overline{\alpha} H_0^*$. Since $A$ is irreducible, there exists $H \in A$ such that $\alpha_H \notin C \alpha$ and $\alpha_H \notin S^2 H_0^*$. Such a linear form can be written $\lambda(\alpha + \beta)$ with $\lambda \in C \setminus \{0\}$ and $\beta \in S^2 H_0^* \setminus \{0\}$. Then $(\alpha + \beta)^2 \in Q$ and $\beta^2 \in Q$, so we have $\alpha^2 + 2 \alpha \beta \beta \in Q$. We make $W$ act on $V^*$ by $w. \gamma(x) = \gamma(w \gamma(x))$, for $x \in V, \gamma \in V^*$. Of course this action can be restricted to a $W_0$-action on $H_0^* \subset V^*$. Then $w. (\alpha + \beta) \in Q$ for all $w \in W$, and since $w \alpha = \alpha$ whenever $w \in W_0$, we get $\alpha^2 + 2 \alpha (w \beta) \beta \in Q$ for all $w \in W_0$. Consider now the subspace $U$ of $H^*$ spanned by the $w_1, \beta - w_2, \beta$ for $w_1, w_2 \in W_0$. It is a $W_0$-stable
Corollary 4.3. If $Q \supset \mathbb{C} \beta$, both $W_\alpha$ and $W_\beta$ are irreducible under the action of $W_\alpha$. If $U = \{0\}$ then $w.\beta = \beta$ for all $w \in W_0$, hence $H_0 = \mathbb{C} \beta$ and $\dim V = 2$, which has been excluded. Thus $U \neq \{0\}$ hence $U = H_0^*$. By $2\alpha(w_1.\beta - w_2.\beta) = (\alpha^2 + 2\alpha(w_1.\beta)) - (\alpha^2 + 2\alpha(w_2.\beta))$ we thus get $\alpha H_0^* \subset Q$. Then $(\alpha + \beta)^2 \in \alpha^2 + \alpha H_0^* + S^2 H_0^* \subset \alpha^2 + Q$ implies $\alpha^2 \in Q$. It follows that $Q \supset S^2 V^*$ which concludes the proof. \hfill $\Box$

**Corollary 4.3.** If $A$ is an irreducible reflection arrangement of rank $n$, then $|A| \geq n(n + 1)/2$.

Notice that the above lower bound is sharp, as it is reached for Coxeter type $A_n$.

When $A$ is a reflection arrangement with corresponding reflection group $W$, both $\mathbb{C}A$ and $S^2 V^*$ can be endowed by natural $W$-actions, where the action on $\mathbb{C}A$ is defined by $w.\nu H = \nu w(H)$. It is thus natural to ask whether the linear forms $\alpha_H$ can be chosen such that $\Phi$ is a morphism of $W$-modules.

**Proposition 4.4.** If $A$ is a complexified real reflection arrangement (in particular $W$ is a finite Coxeter group), then the linear forms $\alpha_H$ can be chosen such that $\Phi$ is a morphism of $W$-modules.

**Proof.** We choose a $W$-invariant scalar product on the original real form $V_0$ of $V$ and extend it to a $W$-invariant hermitian scalar product on $V$. For every $H \in A$ we choose $x_H \in V_0$ orthogonal to $H$ with norm 1, and define $\alpha_H : y \mapsto (x_H|y)$, our notion on hermitian scalar products being that they are linear on the right. Then, for any $w \in W$, $w.x_H \in V_0$ is orthogonal to $w(H)$ of norm 1, hence $w.x_H = \pm x_{w(H)}$. Since $w.\alpha_H$ maps $y$ to $(w.x_H|y)$ we have $(w.\alpha_H)^2 = \alpha^2_{w(H)}$, which shows that $\Phi$ is a morphism of $W$-modules. \hfill $\Box$

When $W$ is not a Coxeter group, the $W$-modules $\mathbb{C}A$ and $S^2 V^*$ are generally unrelated. However, this property is not a characterization of Coxeter groups, as there is at least one example of a (non-Coxeter) complex reflection group for which $\Phi$ can be a morphism of $W$-module. This is the group labelled $G_{12}$ in the Shephard-Todd classification. Notice that, in such a case, one must have $\sum \alpha_H^2 = 0$, otherwise this sum would provide a copy of the trivial representation inside $S^2 V^*$, forcing $W$ to be a real reflection group.

We briefly describe this example. The group $G_{12}$ can be described in $GL_2(\mathbb{C})$ by 3 generators $a, b, c$ of order 2, satisfying the relation $abca = bcab = cabc$. We choose the following model:

$$a = \begin{pmatrix} 1 & 1 + \sqrt{-2} \\ 0 & -1 \end{pmatrix} \quad b = \begin{pmatrix} -1 & 0 \\ 1 - \sqrt{-2} & 1 \end{pmatrix} \quad c = \begin{pmatrix} \sqrt{-2} & -1 + \sqrt{-2} \\ -1 - \sqrt{-2} & -\sqrt{-2} \end{pmatrix}$$

We define a collection of vectors $e_H \in V$, such that $w.e_H = \pm e_{w(H)}$. Letting $\alpha_H : x \mapsto (e_H|x)$, the associated $\Phi : \mathbb{C}A \to S^2 V^*$ is then a morphism of $W$-modules. A $W$-invariant hermitian scalar product is given on this matrix model by $(X|Y) = ^t X A Y$ with

$$A = \begin{pmatrix} 2 & 1 + \sqrt{-2} \\ 1 - \sqrt{-2} & 2 \end{pmatrix}$$
We choose for $e_H$ the 12 following vectors, which are fixed by the corresponding reflection $s$.

\[
\begin{array}{c|c|c|c}
 s & \text{babab} & a & b \\
 e_H & (1 + \sqrt{-2}, -2) & (1,0) & (0,1) \\
 s & \text{ababa} & \text{bcb} & c \\
 e_H & (-2, 1 - \sqrt{-2}) & (1, \sqrt{-2}) & (1, -1) \\
 s & \text{acaca} & \text{cbc} & \text{aba} \\
 e_H & (1 - \sqrt{-2}, 1 + \sqrt{-2}) & (-1 + \sqrt{-2}, -\sqrt{-2}) & (-1 - \sqrt{-2}, 1) \\
 s & \text{bab} & \text{cac} & \text{aca} \\
 e_H & (-1, 1 - \sqrt{-2}) & (-\sqrt{-2}, 1 + \sqrt{-2}) & (-\sqrt{-2}, 1)
\end{array}
\]

It can be checked that the reflections $a, b, c$ act on these vectors by monomial matrices, with nonzero entries in $\{\pm 1\}$ (hence factors through the hyperoctahedral group of rank 12). On this example, $S^2V^*$ is a selfdual $W$-module.

We make the following remark.

**Proposition 4.5.** For $\Phi$ to be a morphism of $W$-modules it is necessary that $\kappa(W) \leq 2$, where

\[
\kappa(W) = \min\{n \in \mathbb{Z}_{>0} \mid \forall w \in W \forall H \in \mathcal{A} \ w.\alpha_H = \zeta \alpha_H \Rightarrow \zeta^n = 1\}
\]

Using the Shephard-Todd classification, we will show in section 6 that this condition is actually sufficient when $W$ is irreducible.

### 5. A path between representations

In this section we define a natural connection between the action of $W$ on $\mathbb{C}^\mathcal{A}$ and more surprising representations of $W$. For this we need to introduce the space $X = V \setminus \bigcup \mathcal{A}$ of regular vectors, on which $W$ acts freely, and its quotient (orbit) space $X/W$. We choose a base point $\underline{z} \in X$. The fundamental groups $B = \pi_1(X/W)$ and $P = \pi_1(X)$ are known as the braid group and pure braid group associated to $W$, respectively. There is a natural morphism $\pi : B \to W$ with kernel $P$. We first construct a deformation of $W \to \text{GL}(\mathbb{C}^\mathcal{A})$ as a linear representation of the braid group.

This deformation should not be confused with the one described in [Ma07] when $W$ is a 2-reflection group.

#### 5.1. A representation of the braid group

To each $H \in \mathcal{A}$ is canonically associated a differential form $\omega_H = \frac{d\alpha_H}{\alpha_H}$, using some arbitrary linear form $\alpha_H$ with kernel $\alpha_H$. We introduce idempotents $p_H \in \text{End}(\mathcal{C}^\mathcal{A})$ defined by $p_{H_1.\nu H_2} = v_{H_2}$ if $H_1 = H_2$, $p_{H_1.\nu H_2} = 0$ otherwise. Choosing $h \in \mathbb{C}$, the 1-form

\[
\omega = h \sum_{H \in \mathcal{A}} p_H \omega_H \in \Omega^1(X) \otimes \mathfrak{gl}(\mathcal{C}^\mathcal{A})
\]

satisfies $\omega \wedge \omega = 0$, hence defines a flat connection on the trivial vector bundle $X \times \mathbb{C}^\mathcal{A} \to X$, which is clearly $W$-equivariant for the diagonal action on $X \times \mathcal{C}^\mathcal{A}$. Dividing out by $W$, the corresponding flat bundle over $X/W$ thus defines by monodromy a linear representation of $B$ in $\mathcal{C}^\mathcal{A}$. Letting $\gamma$ denote a representative loop of $\sigma \in B = \pi_1(X/W)$, we can lift it to a path $\tilde{\gamma}$ in $X$ with endpoints $\underline{z}$ and $\pi(\sigma).\underline{z}$, where $\underline{z}$ is the chosen basepoint
in $X$. The 1-forms $\tilde{\gamma}^*\omega_H$ can be written as $\gamma_H(t)\,\text{d}t$ for some function $\gamma_H$ on $[0,1]$, and the differential equation $\text{d}f = (\gamma^*\omega)f$ to consider is then $f'(t) = h(\sum_{H \in A} \gamma_H(t)p_H)f(t)$, with $f(0) = \text{Id} \in \text{End}(\mathbb{C}A)$. Since the $p_H$ commute one to the other, the solution is easy to compute:

$$f(t) = \prod_{H \in A} \exp\left(\hbar p_H \int_0^t \gamma_H(u)\,\text{d}u\right)$$

and the monodromy representation is given by

$$\sigma \mapsto R_h(\sigma) = \pi(\sigma) \prod_{H \in A} \exp(h\hbar \int_\gamma \omega_H)$$

where we identified $w \in W$ with $R_0(w) \in \text{End}(\mathbb{C}A)$. In particular, the image of $P$ is commutative. More precisely, if $\gamma_0$ is a loop in $X$ around a single hyperplane $H$, the class $[\gamma_0] \in P$ is mapped to $\exp(2\pi h\hbar)$. Since $P$ is generated by such classes, it follows that $R_n(P) = \{\text{Id}\}$ hence $R_n$ factors through a representation of $W$ whenever $n \in \mathbb{Z}$.

We recall that $B$ is generated by so-called braided reflections (‘generators-of-the-monodromy’ in [BMR]), which are defined as follows. For a distinguished reflection $s \in W$, an element $\sigma \in B$ with $\pi(\sigma) = s$ is called a braided reflection if it admits as representative a path $\gamma$ from $z$ to $s.z$ which is a composite $(s.\gamma_0)^{-1} \ast \gamma_1 \ast \gamma_0$ of paths with the following properties. Here $\gamma_0 : z \rightsquigarrow z_0$, $\gamma_1 : z_0 \rightsquigarrow s.z_0$ and $(s.\gamma_0)^{-1} : s.z_0 \rightsquigarrow s.z$ is the reverse path of $s.\gamma_0$, and $\gamma_1(t) = \varepsilon \exp(2\pi i/t/d_H)z_0^{-} + z_0^{+}$ where $z_0^{+}$ and $z_0^{-}$ are the orthogonal projection on $H$ and $H^\perp$, respectively, for $\varepsilon > 0$ small enough and $z_0$ sufficiently close to $H$ so that the homotopy class of this path does not vary when $\varepsilon$ decreases and $z_0^{+} \notin H'$ for $H' \in A \setminus \{H\}$.

Note that $\int_{s.\gamma_0} \omega_{H'} = \int_{\gamma_0} \omega_{s(H')} = 0$ for all $H' \in A$, hence $\int_\gamma \omega_H = \int_{\gamma_1} \omega_H = (2\pi)/d_H$. In particular, for such a braided reflection $\sigma$ we get

$$R_h(\sigma).v_H = \pi(\sigma) \exp(h\hbar \int_\gamma \omega_H) v_H = \exp(2\pi h/d_H) v_H.$$  

Moreover, if $H$ and $H'$ have orthogonal roots, then again $\int_\gamma \omega_{H'} = \int_{\gamma_1} \omega_{H'}$. But in this case $\alpha_{H'}(\gamma_1(t))$ is constant hence $\int_\gamma \omega_{H'} = 0$. An immediate consequence of this is that we can restrict ourselves to irreducible groups, namely

**Proposition 5.1.** If $W = W_1 \times \cdots \times W_r$ is a decomposition of $W$ in irreducible components, with corresponding decompositions $B = B_1 \times \cdots \times B_k$ and $A = A^1 \times \cdots \times A^r$, then $R_h = R_h^{(1)} \times \cdots \times R_h^{(r)}$ with $R_h^{(k)} : W_k \rightarrow \text{GL}(\mathbb{C}A_k)$.

From the formulas above follows that, under the action of $R_h$, $\mathbb{C}A$ is the direct sum of the stable subspaces $\mathbb{C}A_k$, where $A = A^1 \sqcup \cdots \sqcup A^r$ is the decomposition of $A$ in orbits under the action of $W$. We let $R_h^{(k)} : B \rightarrow \text{GL}(\mathbb{C}A_k)$, so that $R_h = R_h^{(1)} \oplus \cdots \oplus R_h^{(r)}$.

**Proposition 5.2.** If $h \notin \mathbb{Z}$, then $R_h^{(k)}$ is irreducible for each $1 \leq k \leq r$.

**Proof.** For each $H \in A_k$ we choose a loop $\gamma_H$ based at $z$ around the hyperplane $H$. We have $\int_{\gamma_H} \omega_H = 2\pi$ and $\int_{\gamma_H} \omega_{H'} = 0$ for $H \neq H'$. Letting $Q_H$
denote the class of $\gamma_H$ in $P = \pi_1(X, z)$ we thus have $R_h^k(Q_H) = \exp(2i\pi h p_H)$, hence $R_h^k(Q_H) - \text{Id}$ is a nonzero multiple of $p_H$ if $h \notin \mathbb{Z}$. It follows that the elements $R_h^k(Q_H)$ generate the commutative algebra of diagonal matrices in $\text{End}(C_k\mathbb{A}_k)$. Let $G_k$ be the oriented graph on the $v_H, H \in \mathbb{A}_k$ with an edge $(v_{H_1}, v_{H_2})$ if there exists $x \in B$ such that the matrix $R_h^k(x)$ has nonzero entry at $(v_{H_1}, v_{H_2})$. If $G_k$ is connected, then $R_h^k$ is irreducible (see e.g. [Ma04] prop. 3 cor. 2). Choosing for each distinguished reflection $s \in W$ a braided reflection $\sigma$, $R_h^k(\sigma)$ has nonzero entries in $(v_{H}, v_{s(H)})$ and $(v_{s(H)}, v_H)$ for each $H \in \mathbb{A}$. Since $\mathbb{A}_k$ is an orbit under $W$ and $W$ is generated by distinguished reflections, it follows that $G_k$ is connected, concluding the proof.

Since $R_h$ factors through $W$ when $h \in \mathbb{Z}$, this has the following consequence.

**Corollary 5.3.** For all $h \in \mathbb{C}$, the representation $R_h$ of $B$ is semisimple.

We choose a collection of roots $e_H, H \in \mathbb{A}$. Notice that, for $w \in W$, $w(H) = H$ implies $w.e_H = e^i\theta e_H$ for some $\theta \in \mathbb{R}$.

**Lemma 5.4.** If $\gamma : z \sim w.z$ is a path in $X$ with $w \in W$ such that $w.e_H = e^i\theta e_H$, then $\int_\gamma \omega \in i\theta + 2i\pi \mathbb{Z}$.

**Proof.** We can assume $-\pi < \theta \leq \pi$. Since $\int_\gamma \omega_H$ is independent of the choice of $\alpha_H$, we can choose $\alpha_H : x \mapsto (e_H|x)$ with $(e_H|e_H) = 1$. We have $\alpha_H(w.x) = e^i\alpha(x)$. We write $\gamma(t) = \gamma_H(t) + \gamma_0(t)e_H$ with $\gamma_0 : [0, 1] \to \mathbb{C}$ and $\gamma_H : [0, 1] \to H$. Then $\alpha_H(\gamma(t)) = \gamma_0(t)$ and $\int_\gamma \omega_H = \int_{\gamma_0} \frac{d\gamma}{\gamma}$. Letting $x = \alpha_H(z) \in \mathbb{C}^X$, we have $\gamma_0 : x \sim e^i\theta x$. If $\gamma_1 : x \mapsto e^i\theta x$ is an arbitrary path in $\mathbb{C}^X$, then $\gamma_0 \ast \gamma_1^{-1}$ is a loop in $\mathbb{C}^X$, hence $\int_\gamma \frac{d\gamma}{\gamma} = \int_{\gamma_0} \frac{d\gamma}{\gamma}$ is a multiple of $2i\pi$. If $e^i\theta = 1$ this concludes the proof. If $e^i\theta = -1$ we consider $\gamma_1(t) = xe^{i\pi t}$, for which $\int_{\gamma_1} \frac{d\gamma}{\gamma} = i\pi t$. If $e^i\theta = \zeta \not\in \{1, -1\}$ we consider $\gamma_1(t) = (1 - t)x + te^i\theta x$ and $\int_{\gamma_1} \frac{d\gamma}{\gamma} = \log((1 + (e^i\theta - 1)t)^1_0)$, where log denotes the natural determination of the logarithm over $\mathbb{C} \setminus \mathbb{R}^-$. It follows that $\int_{\gamma_1} \frac{d\gamma}{\gamma} = log e^i\theta = i\theta$, and the conclusion follows.

We recall from section 4 the definition of $\kappa(W)$.

$$\kappa = \kappa(W) = \min\{n \in \mathbb{Z}_{>0} | \forall w \in W \forall H \in \mathbb{A} \text{ w.e}_H = \zeta e_H \Rightarrow \zeta^n = 1\}$$

**Theorem 5.5.** For all $h \in \mathbb{C}$, $R_{h+\kappa}$ is isomorphic to $R_h$. Moreover, $\kappa$ is the smallest positive real number such that $R_{h} \simeq R_0$.

**Proof.** Recall from corollary 5.3 that, for all $h \in \mathbb{C}$, $R_h$ is semisimple. Letting $\chi_h$ denote the character of $R_h$ on $B$, it is thus sufficient to prove $\chi_h = \chi_{h+\kappa}$ for all $h \in \mathbb{C}$ in order to get $R_{h+\kappa} \simeq R_h$. Let $g \in B$ with $w = \pi(g)$, and $\gamma : z \sim w.z$ a representing path. By the explicit formulas above, we have

$$\chi_h(g) = \sum_{w(H) = H} \exp(h \int_\gamma \omega_H)$$

and $R_{h+\kappa} \simeq R_h$ follows by lemma 5.4. We now show that $\kappa$ is minimal with this property. Assuming otherwise, we let $0 < h < \kappa$ such that $\chi_h = \chi_0$. By definition of $\kappa$ there exists $w \in W$, $H \in \mathbb{A}$ such that $w.e_H = e^i\theta e_H$ with
\[ e^{i\theta h} \neq 1. \] Letting \( g \in B \) with \( \pi(g) = w \) and \( \gamma : z \sim w, z \) a representing path, we have \( \int_{\gamma} \omega_H \in i\theta + 2i\pi\mathbb{Z} \), hence \( \exp(h \int_{\gamma} \omega_H) \neq 1 \). It follows that \( |\chi_0(g)| < \chi_0(g) \) hence a contradiction.

\[ \Box \]

**Proposition 5.6.** For any \( H \in A \) and \( h \in \mathbb{C} \), if \( \sigma \) is a braided reflection around \( H \), then \( R_h(\sigma) \) is conjugated to \( R_0(\sigma) \exp(h(2\pi/d_H)p_H) \).

**Proof.** Let \( \sigma \) be a braided reflection with corresponding paths \( \gamma_0, \gamma_1 \) as above. Since \( \gamma_0 \) and \( s_0 \gamma_0 \) represent the same path in \( X/W \), \( R_h(\sigma) \) is conjugated to the monodromy along the loop \( \gamma_1 \) in \( X/W \), so that we can assume \( z = z_0, \gamma = \gamma_1 \). In view of the formulas above, we thus only need to show that \( \int_{\gamma_1} \omega_{H'} = 0 \) for \( H' \neq H \). This can be done by direct computation, as \( \alpha_{H'}(\gamma_1(t)) = \varepsilon \exp(2i\pi t/d_H)\alpha_{H'}(z_0^\pm) + \alpha_{H'}(z_0^-) \) with \( \alpha_{H'}(z_0^-) \neq 0 \), and \( \int_{\gamma_1} \omega_{H'} \) is constant when \( \varepsilon \to 0 \). Since \( \int_{\gamma_1} \omega_{H'} \to 0 \) when \( \varepsilon \to 0 \) we get \( \int_{\gamma_1} \omega_{H'} = 0 \) and the conclusion.

\[ \Box \]

5.2. New representations of \( W \). When \( n \in \mathbb{Z} \), the representation \( R_n \) of \( B \) factorizes through \( W \). In case \( W \) is irreducible, the action of the center is easy to describe.

**Lemma 5.7.** If \( w \in W \) acts by \( \lambda \in \mathbb{C}^x \) on \( V \), then \( R_n(w) = \lambda^n \)\( \text{Id} \) if \( n \in \mathbb{Z} \).

More generally, if there exists \( v \in X \) such that \( w.v = \lambda v \) for some \( \lambda \in \mathbb{C}^x \), then \( R_n(w) \) is conjugated to \( \lambda^n R_0(w) \)

**Proof.** We first assume that \( w \) acts on \( V \) by \( \lambda \). We can write \( \lambda = \exp(i\theta) \) with \( 0 < \theta \leq 2\pi \). We consider the loop \( \gamma(t) = e^{i\theta t} \zeta \) in \( X/W \), whose image in \( W \) is \( w \). By direct calculation we have \( \int_{\gamma} \omega_H = i\theta \) for all \( H \in A \) and the conclusion follows from the general formula for \( R_1 \). Now assume \( w.v = \lambda v \) for some \( \lambda = \exp(i\theta) \) with \( 0 < \theta \leq 2\pi \). Up to conjugation, we can assume \( v = \zeta \), the loop \( \gamma(t) = e^{i\theta t} \zeta \) in \( X/W \) has image \( w \) in \( W \) and we conclude as before.

\[ \Box \]

More involved tools prove the following.

**Proposition 5.8.** If \( W_0 \) is a parabolic subgroup of \( W \) with hyperplane arrangement \( A \) and \( n \in \mathbb{Z} \), then the restriction of \( R_n \) to \( W_0 \) is isomorphic to the direct sum of the representation \( R_n \) of \( W_0 \) and the permutation representation of \( W_0 \) on \( \mathbb{C}(A \setminus A_0) \).

**Proof.** We let \( R_h^0 \) denote the representation \( R_h \) for \( W_0 \) acting on \( \mathbb{C} A_0 \), and \( S_h \) the direct sum of \( R_h^0 \) and the permutation representation of \( W_0 \) on \( A \setminus A_0 \). We can embed the braid group \( B_0 \) of \( W_0 \) inside \( B \) such that, as representations over \( \mathbb{C}[[h]] \), the restriction to \( B_0 \) of \( R_h \) is isomorphic to \( S_h \) (see [Ma07], theorem 2.9). In particular, for all \( g \in B_0 \), the traces of \( R_h(g) \) and \( S_h(g) \) are equal, as formal series in \( h \). Since these traces are holomorphic functions in \( h \), it follows that they are equal for all \( h \in \mathbb{C} \). This means that the semisimple representations of \( B_0 \) associated to the restriction of \( R_h \) and to \( S_h \) are isomorphic. Since the restriction of \( R_n \) and \( S_n \) are semisimple for all \( n \in \mathbb{Z} \) the conclusion follows.

\[ \Box \]

The determination of the action of the center enables us to prove that, contrary to \( R_0 \), \( R_1 \) is faithful in general.
Proposition 5.11.

(1) $R_0$ has kernel $Z(W)$.

(2) $R_1$ is faithful on $W$.

(3) $\text{Ker } R_n = \{w \in Z(W) \mid w^n = 1\}$

Proof. Without loss of generality (because of proposition 5.1) we may assume that $W$ is irreducible. Obviously (3) $\Rightarrow$ (2). Although (1) is also a special case of (3), we prove it separately. If $|A| = 1$ the statement is obvious, so we assume $|A| \geq 2$. Clearly $Z(W) \subset \text{Ker } R_0$, as $\text{Ker } (wgw^{-1} - 1) = w.\text{Ker } (g - 1)$ for all $g, w \in W$. Let $w \in W$ such that $R_0(w) = \text{Id}$, that is $w(H) = H$ for all $H \in A$. Let $s \in W$ be a distinguished reflection with reflection hyperplane $H$. Then $wsw^{-1}$ is a reflection with $\text{Ker } (wsw^{-1} - 1) = H$ which has the same nontrivial eigenvalue as $s$, hence $wsw^{-1} = s$. It follows that $w$ commutes to all distinguished reflections of $W$, hence $w \in Z(W)$ since $W$ is generated by such elements.

We now prove (3). Let $w \in \text{Ker } R_n$. Since $R_1(w) = R_0(w)D$ for some diagonal matrix $D$, the nonzero entries of $R_n(w)$ determine the permutation matrix $R_0(w)$, hence $w \in Z(W)$. Since $W$ is irreducible, $w$ acts on $V$ by some scalar $\lambda \in \mathbb{C}^\times$, hence $R_n(w) = \lambda^n = 1$ by lemma 5.7, hence $w^n = 1$. The converse inclusion is obvious by lemma 5.7.

Corollary 5.10. The exponent of $Z(W)$ divides $\kappa(W)$. If $W$ is irreducible then $|Z(W)|$ divides $\kappa(W)$.

Proof. By the proposition, the period of the sequence $\text{Ker } R_n$ is the exponent of $Z(W)$. Since $\text{Ker } R_n$ is $\kappa(W)$-periodic the conclusion follows. If $W$ is irreducible then $Z(W)$ is cyclic hence its order equals its exponent.

In the proof of theorem 5.5, we computed the character $\chi_n$ of $R_n$. We recall the result here:

Proposition 5.11. For any $w \in W$ and $n \in \mathbb{Z}$ we have

$$\chi_n(w) = \sum_{w.e_H = \zeta \in H} \zeta^n$$

If $\tilde{K} = \mathbb{Q}(\zeta_d)$ is a cyclotomic field containing all eigenvalues of $R_1(W)$, then letting $c_n \in \text{Gal}(\tilde{K}|\mathbb{Q})$ for $n$ and $d = 1$ be defined by $c_n(\zeta_d) = \zeta_d^n$ we get from this proposition that $\chi_n = c_n \circ \chi_1$ for all $n$ prime to $d$.

As an illustration of this section, we do the example of $W$ of type $G_4$ generated by

$$s = \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix}, \quad t = \frac{1}{3} \begin{pmatrix} 1 + 2j & j - 1 \\ 2j - 2 & j + 2 \end{pmatrix}.$$ 

It is a reflection group of order 24, with two generators $s, t$ of order 3 satisfying $sts = tst$, and center of order 2. It admits 3 one-dimensional (irreducible) representations $S_\alpha : s, t \mapsto \alpha$, 3 two-dimensional representations $A_\alpha$ with $\text{tr } A_\alpha(s) = -\alpha$ for $\alpha \in \{1, j, j^2\}$ with $j = \exp(2\pi i / 3)$ and a 3-dimensional one that we denote $U$. The reflection representation is $A_{j_2}$, and $\kappa(W) = 6$. From the character table of $W$ one gets

$$
\begin{align*}
R_0 &= S_1 + U \\
R_1 &= A_1 + A_{j_2} \\
R_2 &= S_{j_2} + U \\
R_3 &= A_{j} + A_{j_2} \\
R_4 &= S_j + U' \\
R_5 &= A_1 + A_{j_2}
\end{align*}
$$
5.3. The case of Coxeter groups. If \( W \) is a Coxeter group, we get a simpler form of this representation. Recall that, in this case, \( \mathcal{A} \) is the complexification of some real arrangement \( \mathcal{A}_0 \) in \( V_0 \), where \( V_0 \) is a real form of \( V \); moreover, choosing some connected component \( C \) of \( V_0 \setminus \bigcup \mathcal{A}_0 \), called a Weyl chamber, determines \( n \) hyperplanes \( H_1, \ldots, H_n \) called the walls of \( C \), and the corresponding \( n \) reflections \( s_1, \ldots, s_n \) are called the simple reflections associated to \( C \). If \( z \in C \), there is also a special set of generators for \( B \), namely the braided reflections \( \sigma_i \) around \( H_i \) such that \( \gamma_0 \) is a straight (real) segment orthogonal to \( H_i \). These are called the Artin generators of \( B \) (associated to a choice of Weyl chamber).

**Proposition 5.12.** If \( W \) is a Coxeter group with simple reflections \( s_1, \ldots, s_n \), then \( \sigma_i \mapsto R_0(s_i) \exp(i \pi h p_{H_i}) \) defines a representation of \( B \) which is equivalent to \( \mathcal{R}_h \). In particular, \( R_1 \) is equivalent to a representation of \( W \) on \( \mathbb{C} \mathcal{A} \) for which \( s_iv_H = v_{s_i(H)} \) is \( H \neq H_i \), \( s_i.v_{H_i} = -v_{H_i} \), and \( R_h+2 \) is equivalent to \( R_h \) for any \( h \in \mathbb{C} \), while \( R_1 \neq R_0 \).

**Proof.** We introduce the Weyl chamber \( C \subset V_0 \) with respect to the simple reflections \( s_1, \ldots, s_n \), with walls \( H_i = \text{Ker}(s_i - 1) \), \( 1 \leq i \leq n \). Up to conjugacy the base point \( z \) can be chosen inside the Weyl chamber, and we define roots \( e_H \in V_0 \) of norm 1 such that \( \mathbb{C} e_H = \text{Ker}(s - 1)^+ \) and \( (e_H|z) > 0 \) for \( z \in C \). We choose for \( \alpha_H \) the linear form \( x \mapsto (e_H|x) \). Let us denote \( \log^+ \) the complex logarithm on \( \mathbb{C} \backslash i \mathbb{R} \), and define

\[
D_h = \prod_{H \in \mathcal{A}} \exp(i \pi p_H \log^+(e_H|z))
\]

We consider a simple reflection \( s_i \) around a wall \( H_i \). Then the path \( \gamma \) representing \( \sigma_i \) can be chosen with \( \varepsilon \) small enough so that \( (e_H|\gamma(t)) \) has positive real part for each \( t \in [0,1] \) and \( H \neq H_i \). It follows that \( t \mapsto \log^+(e_H|\gamma(t)) \) has differential \( \gamma^* \omega_H \) and \( R_h(\sigma_i) \) equals

\[
R_0(s_i) \prod_{H \in \mathcal{A}} \exp(h p_H \int_{\gamma} \omega_H) = R_0(s_i) \prod_{H \in \mathcal{A}} \exp(h p_H(\log^+(e_{s_i}|z) - \log^+(e_H|z)))
\]

(see [Ma07], lemma 7.10). Moreover, \( (e_H|s_i|z) = (s_i.e_{s_i}|z) = (e_{s_i(H)}|z) \) if \( H \neq H_i \) (see e.g. [Ma07], lemma 7.9) and \( (e_{H_i}|s_i|z) = -(e_{H_i}|z) \). It follows that

\[
R_h(\sigma_i) = s_i \exp(i \pi h p_{H_i}) \prod_{H \in \mathcal{A}\setminus\{H_i\}} \exp(h p_H(\log^+(e_{s_0(H)}|z) - \log^+(e_H|z)))
\]

namely

\[
R_h(\sigma_i) = D_h s_i \exp(i \pi h p_{H_i}) D_h^{-1}
\]

for all \( i \in [1, n] \), which concludes the proof. \( R_1 \neq R_0 \) because \( \text{tr} R_1(s_1) = \text{tr} R_0(s_1) - 1 \). \( \square \)

The representation of \( W \) described in this proposition for \( h = 1 \) is natural in the realm of root systems. Indeed, if a set \( \mathcal{P} \) of roots for \( \mathcal{A}_0 \) is chosen, such that \( \mathcal{P} \) satisfies the axioms \( (SR)_1 \) and \( (SR)_2 \) of a root system (see [BG]), and \( \mathcal{P} \) is subdivided in positive and negative roots \( \mathcal{P}^+ \), \( \mathcal{P}^- \) according to the chosen Weyl chamber, where \( \mathcal{P}^+ = \{ e_H, H \in \mathcal{A} \} \), then the representation described here is isomorphic to one on \( \mathbb{C} \mathcal{P}^+ \) described by \( w.f_H = f_{w(H)} \) if
we have \( w.e_H \in \mathcal{P}^+ \) and \( w.f_H = -f_{w(H)} \) if \( w.e_H \in \mathcal{P}^- \), where \( f_H \) denotes the basis element of \( \mathbb{C}\mathcal{P}^+ \) corresponding to \( e_H \in \mathcal{P}^+ \).

Finally, we notice that, when \( W \) is a Coxeter group, then the representation \( R_h \) for arbitrary \( h \) factorizes through the extended Coxeter group \( B/(P,P) \) introduced by J. Tits in [Ti].

We give in the following table the decomposition in irreducibles of \( R_0 \) for the classical Coxeter groups of type \( A_n, B_n, D_n \). We label as usual irreducible representations of \( \mathfrak{S}_n \) by partitions of size \( n \) (with the convention that \( [n] \) is the trivial representation), of \( W \) of type \( B_n \) by couples of partitions \((\lambda, \mu)\) of total size \( n \), and denote \( \{\lambda, \mu\} \) the restriction of \((\lambda, \mu)\) to the usual index-2 subgroup of \( W \) of type \( D_n \). Recall that \( \{\lambda, \mu\} = \{\mu, \lambda\} \) is irreducible if and only if \( \lambda \neq \mu \).

| \ | \( R_0 \) |
|---|---|
| \( A_n, n \geq 3 \) | \([n - 1, 1] + [n, 1] + [n + 1] \) |
| \( B_n, n \geq 4 \) | \((n - 2, 2], \emptyset) + ([n - 2, 2]) + 2([n - 1, 1], \emptyset) + 2([n], \emptyset) \) |
| \( B_3 \) | \((1], [2]) + 2([1, 1], \emptyset) + 2([3], \emptyset) \) |
| \( D_n, n \geq 4 \) | \{[n - 2, 2], \emptyset\} + {([n - 2], [2]}) + {([n - 1, 1], \emptyset) + ([n], \emptyset)\} |

\( R_1 \)

| \ | \( R_1 \) |
|---|---|
| \( A_n, n \geq 3 \) | \([n - 1, 1, 1] + [n, 1] \) |
| \( B_n, n \geq 3 \) | \((n - 2, 1], [1]) + 2([n - 1], [1]) \) |
| \( D_n, n \geq 4 \) | \{([n - 2, 1], [1]) + {[n - 1, 1], [1]}\} |

We sketch a justification of this table. For small values of \( n \), we prove this by using the character table. Then we use induction with respect to a natural parabolic subgroup \( W_0 \) in the same series, for which the branching rule is well-known. Restrictions of \( R_0 \) and \( R_1 \) to this parabolic subgroup are then isomorphic to the sum of the corresponding representation \( R_0 \) or \( R_1 \) of the subgroup, plus the permutation action of the reflections in \( W \) which do not belong to \( W_0 \) (this is clear for \( R_0 \), and a consequence of proposition 5.8 for \( R_1 \)). The decomposition in irreducibles of this permutation representation is easy, namely \([n - 1, 1] + [n]\) for \( A_n \), \((n - 2, 1] + ([n - 2, 1], \emptyset) + 2([n - 1], \emptyset)\) for \( B_n \) and \{([n - 2], [1]) + {([n - 2, 1], \emptyset) + {[n - 1, 1], \emptyset}\} for \( D_n \). This provides the restrictions of \( R_0 \) and \( R_1 \) to \( W_0 \). From the combinatorial branching rule it is easy to check that, for say \( n \geq 5 \), only the given decompositions admit these restrictions.

6. TABLES FOR \( \kappa(W) \)

We compute here the value of \( \kappa(W) \) for all irreducible reflection groups \( W \). More precisely, we compute all \( d \in \mathbb{Z} \) such that there exists \( w \in W \) and \( H \in \mathcal{A} \) with \( w.e_H = \zeta e_H \) and \( \zeta \) of order \( d \). We call these integers the \( \mathcal{A} \)-indices of \( W \).

Recall that the group \( G(de, e, r) \) for \( r \geq 2 \) is defined as the set of \( r \times r \) monomial matrices with nonzero entries in \( \mu_{de}(\mathbb{C}) \), such that the product of these nonzero entries lie in \( \mu_d(\mathbb{C}) \).
Proposition 6.1. The $A$-indices of $W = G(de, e, r)$ are exactly the divisors of $\kappa(W)$. Moreover, $\kappa(W) = de$ if $d \neq 1$ or $r \geq 3$. If $W = G(e, e, 2)$ then $\kappa(W) = 2$.

Proof. Since $G(e, e, 2)$ is a Coxeter (dihedral) group, we can assume $d \neq 1$ or $r \geq 3$. First note that the standard hermitian scalar product on $\mathbb{C}^r$ is invariant under $W$. We introduce the hyperplane arrangement

$$A_{de,r}^0 = \{ z_i - \zeta z_j = 0 \mid \zeta \in \mu_{de}(\mathbb{C}) \}$$

We have $A_{de,r}^0 \subset A$, and the orthogonal to $H : z_i - \zeta z_j = 0$ is spanned by $e_H = e_i - \zeta^{-1} e_j$, if $e_1, \ldots, e_n$ denotes the canonical basis of $\mathbb{C}^r$. Let $w \in W$. Since $w$ is a monomial matrix, there exists $\lambda_1, \ldots, \lambda_r \in \mu_{de}(\mathbb{C})$ with $\lambda_i \in \mu_{de}(\mathbb{C})$, $\prod \lambda_i \in \mu_d(\mathbb{C})$, and $\sigma \in \Sigma_r$ such that $w.e_i = \lambda_i e_{\sigma(i)}$. Then $w.e_H = w.e_i - w.e_j$ iff $\lambda_1 e_{\sigma(i)} - \lambda_j e_{\sigma(j)} = \mu \lambda_i e_i + \mu \lambda_j e_j$. The two possibilities are $\mu = 1, \zeta = 1$ or $\mu \lambda_i = \lambda_j, \mu \lambda_i = \lambda_j \zeta^{-1}$, that is $\mu^2 = \zeta^{-1}, \mu = \lambda_i \lambda_j^{-1}$. It follows that $\mu \in \mu_{de}(\mathbb{C})$. Conversely, assume we choose $\mu \in \mu_{de}(\mathbb{C})$, and let $\zeta = \mu^{-1}$. If $r \geq 3$ we can define $w \in W$ by $\sigma = (1 2), \lambda_2 = 1, \lambda_1 = \mu$, $\lambda_3 = \mu^{-1}, \lambda_k = 1$ for $k \geq 4$, and $w.e_H = w.e_i$ for $H : z_1 - \zeta z_2 = 0$. We have $A = A_{de,r}^0$ when $d = 1$, so this settles this case and we can assume $d \neq 1$. In that case, $A = A_{de,r}^0 \cup A_{de,r}^+$, where $A_{de,r}^+$ is made out the hyperplanes $H_i : z_i = 0$, whose orthogonals are spanned by the $e_i$. If $w.e_i = \mu e_i$ for $w \in W$ we obviously have $\mu \in \mu_{de}(\mathbb{C})$, and conversely if $\mu \in \mu_{de}(\mathbb{C})$ we can define $w \in W$ by $w.e_1 = \mu e_1, w.e_2 = \mu^{-1} e_2$ and $w.e_i = e_i$ for $i \geq 3$. It follows that in this case too the set of $A$-indices is the set of divisors of $de$.

By noticing that $G(2,1,r), G(2,2,r)$ and $G(e, e, 2)$, are Coxeter groups, this gives the following.

Corollary 6.2. For $W = G(de, e, r)$, we have $\kappa(W) = 2$ iff $W$ is Coxeter group, if and only if $de = 2$ or $(d, r) = (1, 2)$.

By checking out the 34 exceptional reflection groups, we prove case by case the following.

Proposition 6.3. Let $W$ be an irreducible complex reflection group. The set of $A$-indices is exactly the set of divisors of $\kappa(W)$.

The following table gives the value of $\kappa(W)$, where $W$ an complex reflection group labelled by its Shephard-Todd number (ST).

| ST | $\kappa$ | ST | $\kappa$ | ST | $\kappa$ | ST | $\kappa$ | ST | $\kappa$ |
|----|----------|----|----------|----|----------|----|----------|----|----------|
| 4  | 6        | 6  | 10       | 12 | 16       | 10 | 22       | 4  | 28       | 2  | 34       | 6  | 4        |
| 5  | 6        | 11 | 14       | 20 | 23       | 30 | 30       | 30 | 2        | 36 | 2        | 7  | 13       | 9  | 30       | 14 | 12       | 20 | 20       | 24 | 24       | 24 | 24       | 2  | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30       | 30 | 30�

We remark that the only non-Coxeter irreducible reflection groups with $\kappa(W) = 2$ are $G_{12}$ and $G_{24}$. Like in the case of $G_{12}$, it is straightforward to check that it is possible to choose the 21 linear forms $\alpha_H$ such that the linear...
map \( \Phi : \mathbb{C}A \rightarrow S^2V^* \) is a morphism of \( W \)-modules. This phenomenon is reminiscent of the special properties of their “root systems” in the sense of \([Co]\). We refer to \([Sh]\) §2 and §4 for a detailed study of these special root systems of type \( G_{12} \) and \( G_{24} \). In particular, convenient linear forms for \( G_{24} \) are described in \([Sh]\), §4.1.

As a consequence of this case-by-case investigation, propositions 4.4 and 4.5 can be enhanced in the following

**Theorem 6.4.** Let \( W \) be an irreducible reflection group. The linear forms \( \alpha_H \) can be chosen such that \( \Phi \) is a morphism of \( W \)-modules if and only if \( \kappa(W) = 2 \). This is the case exactly when \( W \) is a Coxeter group or an exceptional reflection group of type \( G_{12} \) or \( G_{24} \).

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