Solvable Limit For SU(N) Kondo Model

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We study a single channel one dimensional Kondo Model where the impurity spin is replaced by an su(n) spin. Using Abelian bosonization and canonical transformation we explicitly show that this system has an exactly solvable point. The calculation also shows that there are $n$ collective excitation modes in the system, one charged and $n-1$ neutral spin excitation modes.

I. INTRODUCTION

The Kondo problem [1] and its subsequent multi-channel generalization [2] is a classic problem of condensed matter physics. Different approaches [2, 3, 4, 5, 6, 7] have been used to address this problem ranging from an early application of the renormalization group to exact solution by Bethe Ansatz. With the advancement of new methods in micro-fabrication and other experimental techniques Kondo physics has been observed in semiconductor quantum dots and carbon nanotube quantum dots [8, 9]. The conventional Kondo problem has a spin rotation or su(2) symmetry but in nano-structures other higher symmetries are also possible. In particular there is growing interest in su(4) symmetry, the case relevant to carbon nanotubes. In this paper we study the single channel su(n) Kondo model. It was discovered by Toulouse [10] that the conventional su(2) Kondo model has a simple solvable limit. The su(2) Toulouse solution was subsequently extended to provide useful insights into the multichannel and Kondo lattice problems [6, 11]. Here we demonstrate that the single channel su(n) Kondo model has a solvable limit, generalizing the conventional su(2) result.

II. THE SU(N) KONDO MODEL

The model we consider is a single channel wire where electrons in the lead are assumed to be non-interacting. We place a magnetic impurity at the center of the wire so that the electrons interact with the impurity via exchange coupling. The electron is assumed to have $n$ internal degrees of freedom. The case $n=2$ corresponds to an electron with spin. Higher $n$ values result if the electronic state is labeled by a sub-band index, as in a nanotube, or by a valley index, as in silicon. The Hamiltonian of the system has the form

$$H = H_0 + H_{Kondo},$$

where the kinetic energy is given by

$$H_0 = \sum_{\alpha=1}^{n} \int_{-\infty}^{\infty} \psi_{\alpha}^{\dagger}(x)(-i\partial_x)\psi_{\alpha}(x)dx$$

and the exchange term has the form

$$H_{Kondo} = \sum_{\nu=1}^{n^2-1} J_{\nu} \bar{S}^\nu \tau^\nu.$$  

Here we working in units of $\hbar=\nu_F=1$, where $\nu_F$ is the Fermi velocity. $\bar{S}$ is the su(n) impurity “spin” and

$$\bar{S} = \sum_{\alpha,\beta=1}^{n} \psi_{\alpha}^{\dagger}(0)\Sigma_{\alpha\beta}\psi_{\beta}(0)$$

is the su(n) “spin” density of the conduction electrons at the origin. $J_{\nu}$ is the coupling, which we assume to be independent of energy and $\Sigma$ are the $n \times n$ traceless Hermitian matrices that represent the su(n) “spin” operators. They are a set of $n^2-1$ matrices that constitute a basis for the set of $n \times n$ traceless hermitian matrices. Evidently $n-1$ of them are diagonal. They satisfy the “orthogonality” condition

$$Tr(\Sigma_{\alpha}\Sigma_{\beta}) = 2\delta_{\alpha\beta}.$$  

The $\Sigma$ matrices are called the Pauli matrices in su(2) case, the Gell-mann matrices for su(3), etc.

We now focus on the su(4) case to find the solvable limit. Later we will generalize the exact solution to the su(n) case. The $\Sigma$ matrices are a set of fifteen $4 \times 4$ traceless matrices in this case. We choose the diagonal
\[ D_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ D_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ D_3 = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \]

The twelve off-diagonal matrices are selected from the matrices \( O(\alpha, \beta) \) and \( \tilde{O}(\alpha, \beta) \) where
\[ O(\alpha, \beta)_{ij} = \delta_{\alpha i} \delta_{\beta j} + \delta_{\alpha j} \delta_{\beta i}, \]
\[ \tilde{O}(\alpha, \beta)_{ij} = -i(\delta_{\alpha i} \delta_{\beta j} - \delta_{\alpha j} \delta_{\beta i}). \]

These matrices are the generalizations of the Pauli matrices \( \sigma_x \) and \( \sigma_y \) and we denote them by \( O_i \) where \( i = 1, \ldots, 12 \) and
\[ O_1 = O(1, 2), \quad O_2 = O(1, 3), \quad O_3 = O(1, 4) \]
\[ O_4 = O(2, 3), \quad O_5 = O(2, 4), \quad O_6 = O(3, 4) \]
\[ O_7 = \tilde{O}(1, 2), \quad O_8 = \tilde{O}(1, 3), \quad O_9 = \tilde{O}(1, 4) \]
\[ O_{10} = \tilde{O}(2, 3), \quad O_{11} = \tilde{O}(2, 4), \quad O_{12} = \tilde{O}(3, 4). \]

If the exchange coupling \( J_\nu \) in eq. 18 is independent of \( \nu \) the Kondo model has a full \( su(n) \) symmetry. Here we consider an anisotropic case for the exchange coupling where \( J_\nu \) takes either \( J_\parallel \) or \( J_\perp \). This reduces the interaction part of the Hamiltonian into a parallel and perpendicular part,
\[ H_{Kondo} = H_{Kondo}^{\parallel} + H_{Kondo}^{\perp} \]
where
\[ H_{Kondo}^{\parallel} = J_\parallel \sum_{\alpha, \beta = 1}^{4} \sum_{\nu = 1}^{3} \tau_{\nu}^\alpha \psi_\alpha^\dagger(0)(O_\nu)_{\alpha \beta} \psi_\beta(0) \]
and
\[ H_{Kondo}^{\perp} = J_\perp \sum_{\alpha, \beta = 1}^{4} \sum_{\nu = 1}^{3} \tau_{\nu, \parallel}^\alpha \psi_\alpha^\dagger(0)(O_\nu)_{\alpha \beta} \psi_\beta. \]

### III. BOSONIZATION AND UNITARY TRANSFORMATION

The Hamiltonian of the system can take the form of a free Hamiltonian by bosonising the fermionic operators and then make a canonical transformation. Since the spin dynamics of the system depends only on the algebra that the spin operators satisfy we can always work on the canonically transformed operators. The bosonization procedure can be done using the Mandelstam formula where we can write chiral fermionic fields \( \psi_\alpha \)’s in terms of the bosonic fields \( \phi_\alpha \)’s as
\[ \psi_\alpha(x) = \frac{1}{\sqrt{2\pi}\epsilon} e^{-i\phi_\alpha(x)}, \]
where
\[ \phi_\alpha(x) = \sqrt{\pi} \left[ \int_{-\infty}^{x} dy \Pi_\alpha(y) + \phi_\alpha(x) \right]. \]

Here \( \epsilon \) is the cut off and \( \Pi_\alpha(x) \) is the conjugate momentum of \( \phi_\alpha(x) \) which satisfies the commutation relations
\[ [\phi_\alpha(x), \Pi_\alpha(y)] = i\delta_{\alpha \beta}(x - y). \]

For convenience we define the following excitations, which we call them spin(s), flavor(f), spin-flavor(sf) and charge(c) excitations, as
\[ \phi_\alpha^c = e^{\frac{i}{\sqrt{2}}}(\phi_\alpha - \phi_\beta) \]
\[ \phi_\alpha^f = \sqrt{6}(\phi_\alpha + \phi_\beta - 2\phi_\gamma) \]
\[ \phi_\alpha^{sf} = \sqrt{6}(\phi_\alpha + \phi_\beta + \phi_\gamma - 3\phi_\delta) \]
\[ \phi_\alpha^c = 2(\phi_\alpha + \phi_\beta + \phi_\gamma + \phi_\delta). \]

Applying the bozonizing procedure on the free part of the Hamiltonian we have
\[ H_0 = \frac{1}{2} \sum_{\alpha = 1}^{4} \int_{-\infty}^{\infty} dx \left[ (\partial_x \phi_\alpha(x))^2 + \Pi_\alpha^2(x) \right], \]
where \( \alpha = c, s, f \) and \( sf \). Similarly bozonization of the parallel part of the interaction Hamiltonian gives
\[ H_{Kondo}^{\parallel} = J_\parallel \sqrt{\frac{\tau_1^c}{\pi}} \left( \frac{\partial \phi^c_\alpha}{\partial x} + \tau_2^c \frac{\partial \phi^f_\alpha}{\partial x} + \tau_3^c \frac{\partial \phi^{sf}_\alpha}{\partial x} \right) \bigg|_{x=0}. \]

Bosonization of the perpendicular interaction term, \( H_{Kondo}^{\perp} \), leads to a more complicated form with interacting terms that couple pairs of \( \tau^\parallel \)'s. However, these couplings can be removed through a unitary transformation in the space of \( \tau^\parallel \). For an operator \( U \) we have
\[ U(t) = e^{i\mathcal{F}t} U(0) e^{-i\mathcal{F}t} \]
where \( t \) is a parameter and \( \mathcal{F} \) is the generator of the unitary transformation. We choose this generator to have the form
\[ \mathcal{F} = \left( D_1 \phi^c_\alpha + D_2 \phi^f_\alpha + D_3 \phi^{sf}_\alpha \right) \bigg|_{x=0}. \]
The canonical transformation completely decouples the different $\tau^\pm$’s in the perpendicular part of the interaction Hamiltonian for $t = \sqrt{4\pi}$; i.e. $H_{Kondo}^\perp$ takes the form

$$H_{Kondo}^\perp = \frac{J_1}{\pi e} \sum_{i=1}^6 \tau^\perp_{2i-1} \bigg|_{t=0}. \quad (20)$$

Upon applying the canonical transformation to the whole Hamiltonian for $t = \sqrt{4\pi}$, $H$ takes the form

$$H = \frac{1}{2} \sum_{\alpha} \int_{-\infty}^{\infty} dx \left[ (\partial_x \phi_\alpha(x))^2 + \Pi_\alpha^2(x) \right]$$
$$+ \left( \frac{J_\parallel}{\sqrt{\pi}} - t \right) \left( \tau^\parallel_1 \frac{\partial \phi_\alpha^\perp}{\partial x} + \tau^\parallel_2 \frac{\partial \phi_\beta^\parallel}{\partial x} + \tau^\parallel_3 \frac{\partial \phi_\gamma^\perp}{\partial x} \right) \bigg|_{x=0}$$
$$+ \frac{J_1}{\pi e} \sum_{i=1}^6 \tau^\perp_{2i-1} \bigg|_{t=0}. \quad (21)$$

Hence the solvable point for the problem is $J_\parallel = 2\pi$. For this value of $J_\parallel$, the terms in the middle line of Eq. (21) that couple the conduction electrons to the localized su(4) spin vanish.

IV. SU(N) GENERALIZATION

A direct generalization of the same procedure reveals that the su(n) single channel Kondo model has the same solvable limit as that of the su(4) model, i.e. $J_\parallel = 2\pi$. The su(n) generalization can be studied by bosonizing the Hamiltonian in eq.(1) and extending eq.(19) to get the generalized form of the generator of the rotation in the space of $\times n$ dimensional matrix spin space. The appropriate choice for the generator is

$$\mathcal{F} = \sum_{\alpha=1}^{n-1} D_\alpha \varphi^-_\alpha \bigg|_{x=0}, \quad (22)$$

where the $D_\alpha$’s are the $n$-1 diagonal matrices and $\varphi^-_\alpha$ are the $n$-1 different collective spin excitation modes, which are the generalizations of eq.(6) and eq.(15). A convenient choice of diagonal matrices is

$$[D_\alpha]_{ij} = \frac{d_\alpha(j)}{\sum_{j=1}^n d_\alpha(j)^2} \delta_{ij} \quad (23)$$

where

$$d_\alpha(j) = \begin{cases} 1 & \text{if } j < \alpha + 1 \\ -\alpha & \text{if } j = \alpha + 1 \\ 0 & \text{if } j > \alpha + 1 \end{cases} \quad (24)$$

Here $d(\alpha,j)$ are the $j^{th}$ matrix elements of the $\alpha^{th}$ diagonal matrix. The collective spin excitation modes $\varphi^-_\alpha$ can be written in terms of the left moving Bose fields as

$$\varphi^-_\alpha = \sum_{i=1}^n [D_\alpha]_{ii} \phi^-_i \quad (25)$$

and the charged mode is given by

$$\varphi^- = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^n \varphi^-_\alpha. \quad (26)$$

The off-diagonal matrices are given by extending the eq.(7) for $\times n$. The canonical transformation of the off-diagonal matrices $O$ and $\tilde{O}$ in the su(n) spin space is obtained from the equations

$$-i \frac{\partial O}{\partial t} = e^{i\mathcal{F}t}[\mathcal{F},O]e^{-i\mathcal{F}t}$$
$$-i \frac{\partial \tilde{O}}{\partial t} = e^{i\mathcal{F}t}[\mathcal{F},\tilde{O}]e^{-i\mathcal{F}t} \quad (27)$$

and the commutator between $\mathcal{F}$ and $O(\tilde{O})$ is given by

$$[O(\alpha,\beta),D_\gamma] = -i (d_\gamma(\alpha) - d_\gamma(\beta)) \tilde{O}(\alpha,\beta) \quad (28)$$
$$[\tilde{O}(\alpha,\beta),D_\gamma] = i (d_\gamma(\alpha) - d_\gamma(\beta)) O(\alpha,\beta) \quad (29)$$

where $O(\alpha,\beta)$ and $\tilde{O}(\alpha,\beta)$ are given by eq.(7).

The final form of the su(n) Hamiltonian, i.e eq.(1), after bozonization and taking the canonical transformation is

$$H = \frac{1}{2} \sum_{\alpha=1}^n \int_{-\infty}^{\infty} dx \left[ (\partial_x \varphi^-_\alpha(x))^2 + \Pi_\alpha^2(x) \right]$$
$$+ \left( \frac{J_\parallel}{\sqrt{\pi}} - t \right) \sum_{\alpha=1}^{n-1} \tau^\parallel_\alpha \frac{\partial \varphi^-_\alpha}{\partial x} \bigg|_{x=0}$$
$$+ \frac{J_1}{\pi e} \sum_{i=1}^6 \tau^\perp_{2i-1} \bigg|_{t=0}. \quad (30)$$

where again here we consider the spin independent anisotropic case of the exchange coupling, namely that $J_\nu$ is either $J_\parallel$ or $J_\perp$. Clearly eq. (30) shows that for the model we considered the solvable point is the same in the su(n) model.

V. SUMMARY AND CONCLUSION

In this article we have studied an su(n) Kondo spin in a one dimensional single channel wire with electrons in the lead assumed to be non-interacting. By Abelian bosonization of the chiral fermions and canonical transformation we have found a solvable point for the problem, which is the su(n) generalization of the Toulouse
limit [10]. This result may be used to test the large n approximation for the Kondo problem and a straightforward extension of this analysis can be applied to the su(n) many channel Kondo and Kondo lattice problems. Finally the exact solution obtained here may be used to compute the transport properties of nanostructures, a task to which we will return in future work.

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