CONVEXITY OF WEAKLY REGULAR SURFACES OF DISTRIBUTIONAL NONNEGATIVE INTRINSIC CURVATURE

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Abstract. We prove that the image of an isometric embedding into $\mathbb{R}^3$ of a two dimensional complete Riemannian manifold $(\Sigma, g)$ without boundary is a convex surface, provided that, first, both the embedding and the metric $g$ enjoy a $C^{1,\alpha}$ regularity for some $\alpha > 2/3$, and second, the distributional Gaussian curvature of $g$ is nonnegative and nonzero. The analysis must pass through some key observations regarding solutions to the very weak Monge-Ampère equation.

1. Introduction.

1.1. Background. It is a well-known fact of differential geometry that any $C^2$-smooth complete surface in $\mathbb{R}^3$, whose Riemannian metric has non-negative and non-zero intrinsic Gaussian curvature, is a convex surface, i.e. it is the boundary of a convex region. This statement is one of the consequences of the fact that, by Theorema Egregium, the intrinsic and extrinsic notions of curvatures coincide for a $C^2$ surface.

One could directly work with the notion of extrinsic curvature to obtain similar results. Pogorelov generalized the above statement on convexity of surfaces to the case of $C^1$ surfaces with bounded extrinsic curvature, see [33, Theorem 2, p. 615]. On the other hand, by the celebrated results of Nash and Kuiper [30, 23], there exist $C^1$ non-convex surfaces which are isometrically equivalent with the unit sphere. The Riemannian metric of these surfaces is that of the sphere whose intrinsic curvature is positive and constant; yet they are not surfaces of bounded extrinsic curvature. The following question can hence be put forward: Under which regularity assumptions on a surface and its metric, the intrinsic and extrinsic notions of the Gaussian curvature coincide?

The above question is a variant of the many problems on the flexibility vs. rigidity dichotomy regarding the solutions to nonlinear PDEs; (see [11] for a background survey on this aspect of our problem). On one hand, it can be shown that the identity of the two curvatures fails to be true in general for surfaces of $C^{1,\alpha}$ regularity with $\alpha < 1/5$ [9, 5]. On the other hand, when $\alpha > 2/3$ and $\beta > 0$, positive answers to the question was provided in [1] and in [6] (see also [7]) for $C^{1,\alpha}$-isometric embeddings of 2d complete surfaces, whose Riemannian metrics are $C^{2,\beta}$-regular and of positive Gaussian curvature.

In this article, we generalize the main results of [1, 6] as follows: For the same exponent regime $\alpha > 2/3$, we assume only a $C^{1,\alpha}$ regularity for the Riemannian metric, and we relax the positivity condition of the curvature to the condition that the now distributionally defined intrinsic curvature is non-zero and nonnegative. The main ingredient of the proof is to show that provided a Riemannian metric $g \in C^{1,\alpha}$ on a domain $\Omega \subset \mathbb{R}^2$, whose distributional Gaussian curvature is nonnegative, and any isometric embedding $u \in C^{1,\alpha}$ of $(\Omega, g)$ into $\mathbb{R}^3$, the image $u(\Omega)$ is of bounded

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extrinsic curvature. Our analysis uses the ideas of Conti, De Lellis, and Székelyhidi in [6, 7] but cannot directly follow their methodology: The best conceivable result from their approach, if only the nonnegativity of the curvature is assumed, is that the degree of the Gauss map \( \vec{n} \) at any point in its image is nonnegative, which is not enough to conclude with a bound on the total variation of the areas of images of \( \vec{n} \) (a necessary step in bounding the extrinsic curvature).

In order to pass to the analysis of the nonnegative curvature case, the image surface close to a point is considered as the graph of a function \( v \). Writing the curvature in terms of \( v \) leads to a version of the Monge-Ampère equation with nonnegative data. A perturbation argument, originally due to Kirchheim [20], shows that if \( v \) satisfies \( \det(\nabla^2 v) \geq 0 \), then \( \nabla v \) can be uniformly approximated with deformations having positive Jacobian determinants, yielding that the positivity of the topological degree of \( \nabla v \) on its image points. This latter conclusion is the key for proving the bounded extrinsic property of the graph of \( v \). The main technical difficulty is to apply this approach in a weak regularity setting, specially in a context where nonlinear expressions involving distributions in negative order function spaces enter into the picture.

To put our results in a more general context, our results must be compared with the statements in [30, 23, 32, 1, 2, 33, 6, 7, 17, 9, 5, 10, 26] on isometric immersions, in [20, 29, 36, 24] on the rigidity of Sobolev solutions to the Monge-Ampère equation, and in [3, 21, 22, 14, 27, 26] on geometric or topological properties of weakly regular deformations.

1.2. Main results. We adapt the terminology of [13]. We say \( f \geq h \) for two distributions \( f, h \in \mathcal{D}'(\Omega) \), when

\[
\forall \varphi \in C^\infty_c(\Omega) \quad \varphi \geq 0 \implies f[\varphi] \geq h[\varphi].
\]

We remark that by [13, Theorem 1.4.2(ii)] any nonnegative distribution \( f \in \mathcal{D}'(\Omega) \) can be extended as bounded linear operator on \( C_c(\Omega) \) and hence induces a Radon measure \( \mu_f \in \mathcal{M}^+(\Omega) \), satisfying

\[
\forall \varphi \in C^\infty_c(\Omega) \quad \int_\Omega \varphi \, d\mu_f = f[\varphi].
\]

Conversely, any Radon measure induces a nonnegative distribution. Throughout the paper, we will hence use the notions of Radon measures and nonnegative distributions interchangeably. For \( \mu, \nu \in \mathcal{M}(\Omega) \) we say \( \mu \geq \nu \) if and only if for all Borel sets \( A \subset \Omega \) \( \mu(A) \geq \nu(A) \). This is an equivalent condition to \( f \geq h \) as distributions, when \( \mu \) and \( \nu \) are respectively induced by \( f \) and \( h \).

Here we also motivate and provide the definition of the distributional Gaussian curvature - which generalizes the standard notion to Riemannian metrics of lower regularity. Let \( \Omega \subset \mathbb{R}^2 \) be any open set. We recall that the Christoffel symbols associated with a Riemannian metric \( g = [g_{ij}] \in C^2(\Omega, \mathbb{R}^{2 \times 2}_{sym, pos}) \) are given by

\[
\Gamma^i_{jk}(g) = \frac{1}{2} g^{im} (\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk}),
\]

where the Einstein summation convention is used. In view of the formula for the \((0, 4)\)-Riemann curvature tensor [16, Equation (2.1.2)]

\[
R_{ijkl} = g_{lm} (\partial_k \Gamma^m_{ij} - \partial_j \Gamma^m_{ik} + \Gamma^m_{ks} \Gamma^s_{ij} - \Gamma^m_{js} \Gamma^s_{ik}),
\]
we have:
\[
R_{1212}(g) = g_{1m}(\partial_1 \Gamma_{22}^m - \partial_2 \Gamma_{21}^m + \Gamma_{1s}^m \Gamma_{22}^s - \Gamma_{2s}^m \Gamma_{21}^s)
\]
\[
= \partial_1 (g_{1m} \Gamma_{22}^m) - \partial_2 (g_{1m} \Gamma_{21}^m) + \partial_1 g_{1m} \Gamma_{22}^m + \partial_2 g_{1m} \Gamma_{21}^m + g_{1m} (\Gamma_{1s}^m \Gamma_{22}^s - \Gamma_{2s}^m \Gamma_{21}^s)
\]
\[
= 2\partial_1 g_{12} - \partial_1 g_{22} - \partial_2 g_{11} - \partial_1 g_{1m} \Gamma_{22}^m + \partial_2 g_{1m} \Gamma_{21}^m + g_{1m} (\Gamma_{1s}^m \Gamma_{22}^s - \Gamma_{2s}^m \Gamma_{21}^s)
\]
\[
= -\operatorname{curl} \operatorname{curl} g - \partial_1 g_{1m} \Gamma_{22}^m + \partial_2 g_{1m} \Gamma_{21}^m + g_{1m} (\Gamma_{1s}^m \Gamma_{22}^s - \Gamma_{2s}^m \Gamma_{21}^s).
\]

Moreover, by the Gauss equation [16, Equation (2.1.7)], we have for the Gaussian curvature of \( g \):
\[
\kappa(g) := \frac{R_{1212}}{\det g}.
\]

The above calculations motivate the following definition:

**Definition 1.1.** Assume \( U \subset \mathbb{R}^2 \) is an open set, and \( g \in C^1(U, \mathbb{R}^{2 \times 2}_{sym, pos}) \). Let
\[
L(g) := \frac{1}{\det g} \left( -\partial_1 g_{1m} \Gamma_{22}^m + \partial_2 g_{1m} \Gamma_{21}^m + g_{1m} (\Gamma_{1s}^m \Gamma_{22}^s - \Gamma_{2s}^m \Gamma_{21}^s) \right) \in C(U).
\]

The distributional Gaussian curvature \( \kappa_g \) of \( g \) is defined by
\[
\forall \varphi \in C_0^\infty(U) \quad \kappa_g[\varphi] := \int_U \frac{1}{\det g} (\operatorname{curl} g) \cdot \nabla^\perp \varphi + (\operatorname{curl} g) \cdot \nabla^\perp (\frac{1}{\det g}) \varphi + L(g) \varphi,
\]
which can be extended to a bounded operator on \( W_{0,1}^1(U) \) if \( g \in C^1(\bar{U}) \).

We finally need to provide the following definition on surfaces of bounded extrinsic curvature, which follows [33, p. 590].

**Definition 1.2.** Let \( S \subset \mathbb{R}^3 \) be a \( C^1 \) surface. We say that \( S \) has the bounded extrinsic curvature property whenever, the set function \( H^2(\bar{n}(F)) \), defined for \( F \subset S \) and induced by the Gauss map \( \bar{n} : S \to \mathbb{S}^2 \), is of bounded total variation over \( S \), i.e., there exists a constant \( C = C(S) < \infty \) such that for any finite collection \( \{F_i\}_{i=1}^k \) of pairwise disjoint closed subsets of \( S \)
\[
\sum_{i=1}^k H^2(\bar{n}(F_i)) < C.
\]

If a surface \( S \) has the bounded extrinsic curvature property, the absolute curvature of an open set \( O \) in \( S \) is defined to be the supremum of \( \sum_{i=1}^k H^2(\bar{n}(F_i)) \), over all finite collections \( \{F_i\}_{i=1}^k \) of closed subsets of \( O \) which are pairwise disjoint. The absolute curvature of any subset \( A \) of \( S \) is defined as the infimum of its absolute curvatures on open subsets containing \( A \). By [33, Theorem 2] (see also [33, p. 573, Theorem 1]), the absolute curvature is a \( \sigma \)-additive set function on the \( \sigma \)-algebra of Borel subsets of \( S \).

For a surface of bounded extrinsic curvature \( S \), we will refer to [33, p. 591] for the definitions of regular, elliptic, parabolic, hyperbolic, and flat points \( p \in S \). In particular a point \( p \in S \) is defined to be a regular point if the tangent plane at \( p \) is not parallel to any other tangent plane in a sufficiently small neighbourhood
\[
(1.1) \quad \exists O \subset S \text{ open neighborhood of } p \quad \forall q \in O \setminus \{p\} \quad \bar{n}(q) \neq \bar{n}(p).
\]

For every set \( A \subset S \), the positive (resp. negative) curvature of \( A \) consists in the absolute curvature of the subset of \( S \) formed by elliptic (resp. hyperbolic) points of \( A \). The total curvature of \( A \) is defined to be the difference between its positive and negative curvatures, and \( S \) is a surface of nonnegative curvature if the negative curvature of \( S \) equals zero [33, p. 611].
Our main result is the following.

**Theorem 1.** Assume that \( \Omega \subset \mathbb{R}^2 \) is an open set and that \( \alpha > 2/3 \). Let \( g \) be a \( C^{1,\alpha} \) Riemannian metric on \( \Omega \) for which the distributional Gaussian curvature is nonnegative (and hence is a Radon measure over \( \Omega \)). If \( u \in C^{1,\alpha}((\Omega, g), \mathbb{R}^3) \) is an isometric embedding, \( S = u(\Omega) \), and \( S' \) is a surface compactly contained in \( S \), then \( S' \) has the bounded extrinsic curvature property, and is of nonnegative curvature. If \( \kappa_g \) is, moreover, nonzero, then there exists \( S' \) as above such that the positive curvature of \( S' \) is non-zero.

**Remark 1.3.** One can potentially replace definition 1.2 with a local variant: i.e. to define surfaces of locally bounded extrinsic curvature where the finiteness of the total variation is only assumed to hold true on relatively compact open subsets of \( S \). In that case a surface of locally bounded extrinsic curvature of nonnegative or nonzero curvature would be also well-defined, and under the assumptions of Theorem 1, one can show that \( S = u(\Omega) \) is a surface of locally bounded extrinsic curvature, of nonnegative curvature, which is of nonzero curvature provided \( \kappa_g \) is nonzero. We prefer to stay with the terminology of \([33]\) to avoid any confusion, even though the local variant is implicitly assumed in e.g. \([33, \text{Theorem 2, p. 615}]\).

As we shall see in Proposition 4.4, the distributional Gaussian curvature is invariant under a natural change of coordinate formula for smooth enough changes of coordinates, establishing that its definition can be extended to \( C^1 \) metrics over smooth manifolds of two dimensions. The following corollary of Theorem 1 on surfaces of nonnegative distributional curvature is immediately obtained by \([33, \text{Theorem 2, p. 615}]\), generalizing the results of \([1]\), and \([7, \text{Corollary B}]\). Note that an examination of the proof of \([33, \text{Theorem 2, p. 615}]\) shows that it uses the bounded extrinsic curvature property assumption only on bounded sections of the possibly unbounded surface. We leave the verification of this detail to the reader. As a consequence, Theorem 1 can be applied to reach the following conclusion.

**Corollary 2.** Let \( \alpha > 2/3 \), and \( \Sigma \) be a complete 2-dimensional \( C^2 \) manifold with no boundary. Assume moreover that \( g \) is a \( C^{1,\alpha} \) Riemannian metric on \( \Sigma \) for which the distributional Gaussian curvature is nonnegative and nonzero. If \( u \in C^{1,\alpha}((\Sigma, g), \mathbb{R}^3) \) is an isometric embedding, then, \( u(\Sigma) \) is either a closed convex surface, or an unbounded convex surface with no boundary.

**Remark 1.4.** The above conclusions are not true if \( \alpha < 1/5 \), even for smooth metrics with positive curvature \([9, 5]\). The question remains open for the range \( 1/5 \leq \alpha \leq 2/3 \). Gromov \([15, \text{Section 3.5.5.C, Open Problem 34-36}]\) conjectures the critical exponent to be \( \alpha = 1/2 \), see also \([14]\).

To achieve the above goals, we will have to study the very weak Monge-Ampère equation in an open domain \( \Omega \subset \mathbb{R}^2 \):

\[
(1.2) \quad \text{Det} D^2 v := -\frac{1}{2} \text{curl} \text{curl} (\nabla v \otimes \nabla v) = f \in D'(\Omega)
\]

See \([18, 12]\) for an introduction to the very weak Hessian determinant. We will in particular consider the \( C^{1,\alpha} \)-solution \( v \) to (1.2) for which the given distribution \( f \geq 0 \) under the assumption that \( \alpha > 2/3 \). Let us recall that, on the other hand, \( v \) is called an Alexandrov solution to \( \text{det} D^2 v = \mu \) on \( \Omega \) when, for any convex subset \( U \subset \Omega \), \( v \) is convex on \( U \), \( \mu_v(A) = |\partial v(A)| \) defines a Borel measure on \( U \), and \( \mu_v = \mu \). In this line another minor consequence of our analysis is the following statement.

\(^1\) Note that the solutions are not assumed to be convex.
Theorem 3. Assume that $2/3 < \alpha < 1$ and that $f \in \mathcal{D}'(\mathbb{R}^2)$ is a nonnegative nonzero distribution. If $v \in C^{1,\alpha}(\mathbb{R}^2)$ is a solution to $\text{Det} \mathcal{D}^2 v = f$ in $\mathcal{D}'(\mathbb{R}^2)$, then modulo a sign the function $v$ is convex over $\mathbb{R}^2$ and is an Alexandrov solution to $\text{det} \nabla^2 v = \mu f$ on $\mathbb{R}^2$.

Remark 1.5. Note that since $v \in C^1$, we have $|\partial v(A)| = |\nabla v(A)|$ for all Borel sets $A \subseteq U$.

Remark 1.6. A similar result for solutions on a general domain $\Omega \subset \mathbb{R}^2$ was announced by Lewicka and the author in [25, Theorem 1.4]. The proof was supposed to appear in a forthcoming paper, but a gap was found. The obstacle is not overcome to this day and the problem for arbitrary domains remains open.

As it is suggested in Remark 1.4, the main challenging problem is to extend the results of this paper to the case $\alpha > 1/2$. Apart from partial results regarding isometric extensions [8,4] for this regime, the problem is still largely open and cannot be answered using the current methods. Here, we conjecture that slight improvements are possible regarding the $2/3$ regime, in the line of analysis in [24,26].

Conjecture 4. The above results can be generalized with some extra technical maneuvers to Besov regularity regime $B_{2,s,\infty}^{1+s}$, for $s > 2/3$.

The paper is organized as follows: In Section 2, we will recall some known statements about the very weak Hessian determinant and its properties regarding the topological degree and the change of variable formula. In Section 3, we define and establish the bounded extrinsic property for the graphs of $C^{1,\alpha}$ functions whose very weak Hessian determinants are nonnegative. We will prove Theorem 1 in Section 4. Finally, Section 5 is dedicated to a short proof of Theorem 3.

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2. Properties of the very weak Hessian in higher regularity regimes

In this section we gather some statements on the properties of $\text{Det} \mathcal{D}^2 v$ in our Hölder regime of regularity and draw some useful conclusions. The first statement is in the same line of the known results on the compensated regularity of the distributional Jacobians dating back to Wente [38] and spanning many important contributions, in particular [35,3] concerning fractional regularity. But it is rather formulated for the very weak Hessian determinant operator and the proof uses its structure. For definitions and statements regarding Besov spaces $B_{p,q}^s$, we refer to [34,37].

Lemma 2.1. Let $V \subset \mathbb{R}^2$ be an open smooth bounded domain, $\alpha \in (1/2,1)$ and $v \in C^{1,\alpha}(\overline{V})$. Then

\begin{equation}
\text{Det} \mathcal{D}^2 v \in B_{2,\infty,\infty}^{2\alpha - 2}(V).
\end{equation}

Proof. We can extend $v \in C^{1,\alpha}(\overline{V})$ to $\tilde{v} \in C^{1,\alpha}(\mathbb{R}^n)$ such that

$\|\nabla \tilde{v}\|_{0,\alpha} \leq C\|v\|_{1,\alpha,V}$.
For a standard 2d mollifier \( \psi \in C_0^\infty(B_1(0)) \) and the sequence \( \psi_\varepsilon(x) = \varepsilon^{-2}\psi(x/\varepsilon) \), we define \( \hat{v}_\varepsilon := \hat{v} \ast \psi_\varepsilon \) and \((\nabla \hat{v} \otimes \nabla \hat{v})_\varepsilon := (\nabla \hat{v} \otimes \nabla \hat{v}) \ast \psi_\varepsilon \) on \( V \). We calculate for any \( 0 < \varepsilon \leq 1 \)

\[
\| (\text{Det}D^2 \hat{v}) \ast \psi_\varepsilon \|_0 \leq \| \text{curl} \text{curl}(\nabla \hat{v} \otimes \nabla \hat{v})_\varepsilon \|_0 \\
\leq \| \text{curl} \text{curl}(\nabla \hat{v} \otimes \nabla \hat{v})_\varepsilon - \text{curl} \text{curl}(\nabla \hat{v}_\varepsilon \otimes \nabla \hat{v}_\varepsilon) \|_0 + \| \text{curl} \text{curl}(\nabla \hat{v}_\varepsilon \otimes \nabla \hat{v}_\varepsilon) \|_0 \\
\leq C(\varepsilon^{2\alpha-2}\|\nabla \hat{v}\|_0^{2\alpha} + \|\det \nabla^2 \hat{v}_\varepsilon\|_0) \leq C\|v\|_1^{2\alpha}\varepsilon^{2\alpha-2},
\]

where according to [6, Lemma 1] (see also [25, Lemma 4.3]) we used the commutator estimate

\[
(2.2) \quad \| (\nabla \hat{v} \otimes \nabla \hat{v})_\varepsilon - (\nabla \hat{v}_\varepsilon \otimes \nabla \hat{v}_\varepsilon) \|_k \leq C\varepsilon^{2\alpha-2}\|\nabla \hat{v}\|_0^{2\alpha}
\]

and the estimate

\[
\| \nabla^2 \hat{v}_\varepsilon \|_0 \leq C\varepsilon^{\alpha-1}\|\nabla \hat{v}\|_0^{\alpha}.
\]

Hence we obtain from [37, Corollary 1.12] that \( \text{Det}D^2 \hat{v} \in B^{2\alpha-2}_\infty(\mathbb{R}^n) \). Restricting this distribution to \( V \) yields \( \text{Det}D^2 \hat{v} \in \mathcal{D}'(V) \) according to [34, Definition 2.4.1/2].

Now, note that when \( \alpha > 2/3, C_{c,0}^\alpha(V) \) densely embeds into \( W_0^{2-2\alpha,1}(V) \), since \( 2-2\alpha < \alpha \), and that by [34, Proposition 2.1.5]

\[
(2.3) \quad B^{2\alpha-2}_\infty(\mathbb{R}^n) = (B^{2-2\alpha}_1(\mathbb{R}^n))^\prime = (W^{2-2\alpha,1}(\mathbb{R}^n))^\prime.
\]

As a consequence, by an extension argument, \( B^{2\alpha-2}_\infty(V) \to (C_{c,0}^\alpha(V))^\prime \) and the action of \( \text{Det}D^2 \hat{v} \)

any element of \( C_{c,0}^\alpha(V) \) is defined. Remember that for a continuous function \( u : \mathcal{U} \to \mathbb{R}^2 \), \( \deg(u, U, y) \) is its Brouwer degree at a point \( y \in \mathbb{R}^2 \setminus u(\partial U) \). We now state a slightly more general formulation of the degree formula from [25].

**Proposition 2.2.** Let \( \Omega \subset \mathbb{R}^2 \) be an open domain. Assume that \( 2/3 < \alpha < 1, v \in C^{1,\alpha}(\Omega) \). For \( \delta \in \mathbb{R} \), \( x = (x_1, x_2) \in \Omega \) set \( u^\delta(x_1, x_2) := \nabla v(x_1, x_2) + \delta(-x_2, x_1) \). Let \( U \Subset \Omega \) be an open set. Then for every \( g \in C_{c}^\infty(\mathbb{R}^2 \setminus u^\delta(\partial U)) \) the following formula holds true:

\[
(2.4) \quad \int_{\mathbb{R}^2} g(y) \deg(u^\delta, U, y) \, dy = \text{Det}D^2 v[g \circ u^\delta] + \delta^2 \int_{U} (g \circ u^\delta) \, dx.
\]

**Proof.** Note that when \( u \in C^{0,\alpha}(\Omega, \mathbb{R}^2) \), \( \alpha > 1/2 \), the distributional Jacobian derivative \( J(u) := \text{Det}(\nabla u) \) is well-defined as an element of \( D^\prime(\Omega) \) and \( J(u_k) \to J(u) \) if \( u_k \to u \) in \( C^{0,\alpha} \)-norm [3, Corollary 1]. It can hence be shown by approximation that for \( v \in C^{1,\alpha}(\Omega) \) in the same regime of regularity we have

\[
J(u^\delta) = \text{Det}D^2 v + \delta^2.
\]

As a consequence, when \( v \in C^{1,\alpha} \) and \( \alpha > 2/3 \), (2.3) can be also deduced from similar degree formulas for the distributional Jacobian, see [26, Lemma 3.1] and also [28, 14]. We leave the details to the reader.

**Corollary 5.** Let \( \Omega \subset \mathbb{R}^2 \) be any open domain, \( U \Subset \Omega \) be an open set. Assume that \( \alpha, v \) are as in Proposition 2.2 and that

\[
f := \text{Det}D^2 v \geq 0 \quad \text{in} \ D^\prime(\Omega).
\]

Then

\[
\text{(i) } \deg(\nabla v, U, \cdot) \geq \chi_{\nabla v(U) \setminus \nabla v(\partial U)}
\]

\footnote{An approximation argument through Dynkin’s \( \pi\lambda \) theorem yields the same result for \( g \in L^\infty(\mathbb{R}^2) \) with \( \text{supp } g \subset \mathbb{R}^2 \setminus \nabla v(\partial U) \). We will not need this fact.}
(ii) $\text{deg}(\nabla v, U, \cdot) \in L^1(\mathbb{R}^2 \setminus \nabla v(\partial U))$ and:

$$\int_{\mathbb{R}^2 \setminus \nabla v(\partial U)} \text{deg}(\nabla v, U, y) \, dy \leq \mu_f(U).$$

**Proof.** Let $u^\delta$ be defined as in Proposition 2.2. We recall that $\text{deg}(u^\delta, U, \cdot)$ is well-defined and constant on each connected component $\{W_i\}_{i=0}^\infty$ of $\mathbb{R}^2 \setminus u^\delta(\partial U)$, and that it equals 0 on $\mathbb{R}^2 \setminus u^\delta(U)$. It follows from (2.4) and the nonnegativity of $f$ that for any $\delta > 0$ we have

$$\text{deg}(u^\delta, U, \cdot) \geq \chi_{u^\delta(U) \setminus u^\delta(U)}.$$ 

Now, let $\nabla v(x) = y \in \nabla v(U) \setminus \nabla v(\partial U)$ for some $x \in U$. Choose $r > 0$ such that $B(y, 2r) \subset \mathbb{R}^2 \setminus \nabla v(\partial U)$ and $\delta_0$ small enough for which $\|u^\delta - \nabla v\|_0 < r$ for all $\delta < \delta_0$. For all such $\delta$ we have $B(y, r) \subset \mathbb{R}^2 \setminus u^\delta(\partial U)$, which implies that $\text{deg}(u^\delta, U, \cdot)$ is constant on $B(y, r)$. But $|u^\delta(x) - \nabla v(x)| < r$, i.e. $u^\delta(x) \in B(y, r)$, and hence

$$\text{deg}(u^\delta, U, y) = \text{deg}(u^\delta, U, u^\delta(x)) \geq 1.$$ 

Therefore, passing to the limit as $\delta \to 0$ and using [19, Proposition 2.1] we conclude with (i).

To show (ii), we apply (2.4) for $\delta = 0$ to an increasing sequence of test functions $g_k \in C_c(\mathbb{R}^2 \setminus \nabla v(\partial U))$ that converge pointwise to $\chi_{\mathbb{R}^2 \setminus \nabla v(\partial U)}$. In view of the monotone convergence theorem both sides of (2.4) converge by (i), from which we obtain in the limit as $k \to \infty$:

$$\int_{\mathbb{R}^2 \setminus \nabla v(\partial U)} \text{deg}(\nabla v, U, y) \, dy = \mu_f\left(U \setminus (\nabla v)^{-1}(\nabla v(\partial U))\right) \leq \mu_f(U).$$

**Proposition 2.3.** Let $\Omega \subset \mathbb{R}^2$ be an open domain, $\alpha \in (1/2, 1)$, $v \in C^{1,\alpha}(\Omega)$. Assume that

$$\text{Det}D^2 v = f \geq 0 \quad \text{in} \quad D'(\Omega),$$

and that $U \Subset \Omega$ is piecewise smooth. Then

$$\int_{\mathbb{R}^2} \text{deg}(\nabla v, U, y) \, dy = \mu_f(U).$$

**Remark 2.4.** The statement is independent of Corollary 3.1(ii). Here, the seemingly stronger (2.6) is merely stated for a piecewise smooth domain $U$, while it is valid for a better range $\alpha > 1/2$. The inequality in Corollary 3.1(ii) is established for all open sets $U$, but only for $\alpha > 2/3$, as its proof uses the localized version of the degree formula (2.4). Indeed it is an open problem whether (2.4) and Corollary 3.1(ii) hold for $\alpha > 1/2$, even when $\delta = 0$ and $\text{Det}D^2 v$ is regular enough for the trilinear expression $\text{Det}D^2 v[g \circ \nabla v]$ to be well-defined.

Also note that when $\partial U$ is piecewise smooth, and $\alpha > 1/2$, $\nabla v(\partial U)$ is of measure zero [6, Lemma 4], and $\text{deg}(\nabla v, U, y) \in L^1(\mathbb{R}^2)$ [28]. This is no more valid when $U$ is arbitrary, and hence we shall remove $\nabla v(\partial U)$ from the domain of integration in Corollary 3.1(ii).

**Proof.** The proposition is a consequence of the similar degree formula for the distributional Jacobian $J(\nabla v) = \text{Det}D^2 v$, when $v \in C^{1,\alpha}$ and $\alpha > 1/2$, see [14] Theorem 3 and its proof. 


3. The graph \( S = \xi_v(\Omega) \) as a surface of bounded extrinsic curvature.

We denote by \( S = \xi_v(\Omega) \) the graph of \( v \in C^{1,\alpha}(\Omega) \), that is the \( C^{1,\alpha} \) surface that is the image of \( \xi_v(x) = (x, v(x)) \). Let \( N = \vec{n} \circ \xi_v \in C^{0,\alpha}(\Omega, \mathbb{S}^2) \), and we note that up to the choice of the orientation of \( \vec{n} \), and for \( \eta : \mathbb{R}^2 \to \mathbb{S}^2 \), defined by:

\[
\eta(x) = \frac{1}{\sqrt{1 + |x|^2}}(x, -1),
\]

we have \( N = \eta \circ \nabla v \).

**Proposition 3.1.** Let \( \Omega \subset \mathbb{R}^2 \) be an arbitrary open set, and \( \alpha, v \) be as in Proposition 2.2. Assume that

\[
f := \text{Det} \mathcal{D}^2 v \geq 0.
\]

Then for every finite collection \( \{E_i\}_{i=1}^k \) of closed subsets of \( \Omega \) which are pairwise disjoint:

\[
\sum_{i=1}^k |\nabla v(E_i)| \leq \mu_f(\Omega).
\]

Moreover, if \( \mu_f(\Omega) < \infty \), the surface \( S = \xi_v(\Omega) \) is a surface of bounded extrinsic curvature, and is of nonnegative curvature. Moreover, if \( f \neq 0 \), then the positive curvature of \( S \) is non-zero.

**Proof.** Let \( \{E_i\}_{i=1}^k \) be a finite collection of pairwise disjoint closed subsets of \( \Omega \). Approximating each \( E_i \) from within with an increasing sequence of compact sets, we note that the value of \( |\nabla v(E_i)| \) is the limit of evaluations on the given sequence of compact subsets. Therefore, it is sufficient to assume that each \( E_i \) is compact in order to establish (3.2) by a passing to the limit argument for the more general case.

If each \( E_i \) is compact, there are pairwise disjoint open smooth sets \( U_i \supset E_i \), compactly contained in \( \Omega \), for \( i = 1 \ldots k \). Noting the fact that since \( U_i \) are smooth and \( \alpha > 1/2 \), we have \( |\bigcup_{i=1}^k \nabla v(\partial U_i)| = 0 \) [6] Lemma 4]. Corollary [5] yields

\[
\sum_{i=1}^k |\nabla v(E_i)| \leq \sum_{i=1}^k \int_{\mathbb{R}^2} \chi_{\nabla v(U_i)} = \sum_{i=1}^k \int_{\mathbb{R}^2 \setminus \nabla v(\partial U_i)} \chi_{\nabla v(U_i) \setminus \nabla v(\partial U_i)} \\
\leq \sum_{i=1}^k \int_{\mathbb{R}^2 \setminus \nabla v(\partial U_i)} \deg(\nabla v, U_i, y) \ dy \leq \sum_{i=1}^k \mu_f(U_i) \leq \mu_f(\Omega),
\]

where we used the nonnegativity of \( \mu_f \) in the last inequality. Hence (3.2) is established.

Since the map \( \eta \) in (3.1) is smooth, and \( |\partial_1 \eta \times \partial_2 \eta| \leq 1 \), (3.2) implies \( \sum_{i=1}^k \mathcal{H}^2(N(E_i)) \leq \mu_f(\Omega) \) for any collection \( \{E_i\}_{i=1}^k \) as specified. As a straightforward conclusion, \( S \) has the bounded extrinsic curvature property if \( \mu_f(\Omega) < \infty \) and the absolute curvature of \( S \) is bounded by \( \mu_f(\Omega) \).

By [33] Lemma, p. 594], the topological index of \( \vec{n} \) at an elliptic, parabolic or hyperbolic point \( p = \xi_v(x) \in S \), which we denote by \( i(p) \), equals, respectively, \( +1, 0, -1 \), and the index of a flat point is less than \(-1\). Consider the contour \( \gamma \) lying on the surface \( S \) and encircling the point \( p \) as described on [33] p. 595], and its projection on \( \Omega \), encircling the point \( x \), which is parameterized as the simple closed curve \( \Gamma : \mathbb{S}^1 \to \mathbb{R}^2 \). Since \( p \) is a regular point, we can fix \( r > 0 \) and choose \( \gamma \) such that the image of \( \Gamma \) lies in \( B_r(x) \), when \( O := \xi_v(B_r(x)) \) satisfies (1.1). A careful examination of the geometric definition of the index on [33] p. 595] in our setting, considering the relationship
\( \bar{n} \circ \xi_v = \eta \circ \nabla v \), shows that the index of \( \bar{n} \) at a point \( p \) equals the total change in the angle induced by the mapping

\[
z : S^1 \to S^1 \quad a \to z(a) := \frac{\nabla v(\Gamma(a)) - \nabla v(x)}{\| \nabla v(\Gamma(a)) - \nabla v(x) \|}
\]
divided by \( 2\pi \), when \( a \) makes one counter-clockwise turn over the circle \( S^1 \). In other words,

\[
i(p) = \frac{1}{2\pi}(\theta(1) - \theta(0)),
\]
when \( \theta : [0, 1] \to \mathbb{R} \) is the lifting of the map \( \tilde{z}(t) := z(e^{2\pi it}) \) to \( \mathbb{R} \), satisfying \( \tilde{z}(t) = e^{i\theta(t)} \). It is a well known fact that the latter quantity is the topological degree of the mapping \( z \) as a continuous mapping of the unit circle into itself (see e.g. [31, Exercic 6, p. 113]), therefore \( i(p) = \text{deg}(z) \).

Now we approximate \( \Gamma \) uniformly with a sequence of regular simple closed curves \( \Gamma_k : S^2 \to \mathbb{R}^2 \) with images inside \( B_r(x) \), and define

\[
z_k : S^1 \to S^1 \quad a \to z_k(a) := \frac{\nabla v(\Gamma_k(a)) - \nabla v(x)}{\| \nabla v(\Gamma_k(a)) - \nabla v(x) \|}.
\]

As \( z_k \) converges uniformly to \( z \), then \( \text{deg}(z_k) \) converges to \( \text{deg}(z) \), and degree being an integer value, we obtain that \( i(p) = \text{deg}(z_k) \) for \( k \) large enough. Fix such \( k \), and let \( X = \Gamma_k(S^1) \). Then \( X \subset B_r(x) \) is a 1-dimensional simple smooth curve, with \( X = \partial D \), and \( x \in D \subset B_r(x) \). Since \( \Gamma_k \) is a diffeomorphism between \( S^1 \) and \( X \), \( \text{deg}(z_k) \) equals to the degree of the mapping

\[
w : X \to S^1 \quad y \to w(y) := \frac{\nabla v(y) - \nabla v(x)}{\| \nabla v(y) - \nabla v(x) \|}
\]
defined on \( X \), i.e. \( i(p) = \text{deg}(w) \). However, by [31] Proposition IV.4.5, we have

\[
\text{deg}(w) = \text{deg}(\nabla v, D, \nabla v(x)).
\]

Since \( \xi_v(B_r(x)) \) satisfies (1.1), for all \( y \in B_r(x) \), \( \nabla v(y) \neq \nabla v(x) \) for \( y \neq x \), which finally yields

\[
i(p) = \text{deg}(\nabla v, D, \nabla v(x)) = \text{deg}(\nabla v, B_{r/2}(x), \nabla v(x)),
\]

since \( \nabla v(x) \notin \nabla v(D \setminus B_{r/2}(x)) \) [19, Corollaire 2.4]. It follows from Corollary [31(i)] that any regular point \( p \in S \) is elliptic. This implies that \( S \) is a surface of nonnegative curvature.

For \( z \in S^2 \), and \( A \subset S \), let \( m_A(z) \) be the cardinality of the set \( A \cap \bar{n}^{-1}(z) \). By [33] p. 577, Lemma 3, the function \( m_A \) is measurable. It follows then from [33] p. 590, Theorem 3 that the absolute curvature of \( A \) equals \( \int_{S^2} m_A(z) \, ds(z) \), for every Borel subset \( A \) of \( S \). On the other hand, for any open set \( O \subset S \) and any finite collection of pairwise disjoint closed sets \( \{ F_i \}_{i=1}^k \) in \( O \), we have for \( E_i = \xi_v^{-1}(F_i) \):

\[
\sum_{i=1}^k \mathcal{H}^2(\bar{n}(F_i)) = \sum_{i=1}^k \mathcal{H}^2(N(E_i)) \leq \sum_{i=1}^k |\nabla v(E_i)| \leq \mu_f(\xi_v^{-1}(O)).
\]

Hence, by the definition of the absolute curvature, for any Borel set \( A \subset S \) we obtain

\[
\int_{S^2} m_A(z) \, ds(z) \leq \mu_f(\xi_v^{-1}(A)),
\]

which implies for \( A = S \) that

\[
(3.3) \quad \mu_f(\Omega) < \infty \implies \mathcal{H}^2(m_S^{-1}(+\infty)) = 0.
\]
If \( f \) is nonzero, then for some open disk \( U \subset \Omega \), \( \mu_f(U) > 0 \). We claim that \( m_S(N(x)) < \infty \) for some \( x \in U \). If, by contradiction, \( m_S(N(x)) = +\infty \) for every \( x \in U \), then \( N(U) \subset m_S^{-1}(+\infty) \). It follows from finiteness of \( \mu_f \) and \( \mu_f \) that \( \mathcal{H}^2(N(U)) \leq \mathcal{H}^2(m_S^{-1}(+\infty)) = 0 \), yielding that \( |\nabla v(U)| = 0 \), since the mapping \( \eta \) in (3.1) is a smooth diffeomorphism between \( \mathbb{R}^2 \) and \( \eta(\mathbb{R}^2) \subset \mathbb{S}^2 \). Thus, once again using the fact that \( |\nabla v(\partial U)| = 0 \) [8 Lemma 4], (2.6) yields:

\[
\mu_f(U) = \int_{\mathbb{R}^2 \setminus \nabla v(U)} \deg(\nabla v, U, y) \ dy = 0,
\]

contradicting the assumption on \( U \). Hence, there exists \( x \in U \) such that \( m_S(N(x)) < +\infty \), yielding that \( p = \xi_v(x) \) satisfies (1.1), and so it is a regular point. We already showed that any regular point must be elliptic. Since \( S \) contains elliptic points, [33 Theorem 12, p. 600] implies that the positive curvature of \( S \) is non-zero. \( \blacksquare \)

4. ISOMETRIC IMMERSIONS OF SURFACES OF NONNEGATIVE CURVATURE

4.1. Some properties of the distributional curvature. Throughout this section, \( \psi \in C^\infty_c(B_1(0)) \) is a standard 2d mollifier, with \( \psi \geq 0 \) and \( \int_{\mathbb{R}^2} \psi = 1 \). For the sequence \( \psi_\varepsilon(x) = \varepsilon^{-2} \psi(x/\varepsilon) \), and any function or mapping defined on an open set \( \Omega \subset \mathbb{R}^2 \), we denote, with an abuse of notation vis-à-vis \( \psi \) itself, the mollification \( f \ast \psi_\varepsilon \) by using the subscripted \( f_\varepsilon \). We will need the following consequence of [6 Lemma 1]:

**Lemma 4.1.** Let \( \Omega \subset \mathbb{R}^2 \) be an open set, \( V \subset \Omega \), and \( f, g, h \in C^{0,\alpha}(\Omega) \). Then

\[
\| (fgh)_\varepsilon - f_\varepsilon g_\varepsilon h_\varepsilon \|_{1;V} \leq C(\|f\|_{\alpha;\Omega}, \|g\|_{\alpha;\Omega}, \|h\|_{\alpha;\Omega}) \varepsilon^{2\alpha - 1}.
\]

**Proof.** The estimate is obtained by an iterated use of

\[
\|a\varepsilon\|_{0;V} \leq C \varepsilon^{\alpha - 1}, \quad \|(ab)\varepsilon - a_\varepsilon b_\varepsilon\|_{j;V} \leq C \varepsilon^{2\alpha - j}.
\]

for \( a, b \in C^{0,\alpha}(\Omega) \) and \( j = 0, 1 \), as proved in [6 Lemma 1]. The details are left to the reader. \( \blacksquare \)

**Lemma 4.2.** Let \( U \subset \mathbb{R}^2 \) be an open set and \( g_1, g_2 \in C^1(U, \mathbb{R}^{2\times 2}_{\text{sym}, \text{pos}}) \). Assume that \( V \subset U \) is an open bounded set and that \( \operatorname{det} g_i \geq \lambda > 0 \) on \( V \). Then for a constant \( C := C(\|g_1\|_{0;V}, \|g_2\|_{0;V}, \lambda) \) we have

\[
\forall \varphi \in W^{1,1}_0(V) \quad \|\kappa_{g_1}[\varphi] - \kappa_{g_2}[\varphi]\| \leq C\|\varphi\|_{W^{1,1}(V)} \sum_{j=0}^1 (\|g_1\|_{1;V} + \|g_2\|_{1;V})^{2-j} \|g_1 - g_2\|_{j;V}.
\]

The proof is obtained by straightforward calculations. We leave the details to the reader.

We now observe that the independence of the Gaussian curvature from the coordinate system is still valid in our weaker setting.

**Lemma 4.3.** Assume \( U \) and \( g \) as in Lemma 4.2. Let \( \xi : U' \to U \) be a \( C^2 \) smooth diffeomorphism. Let \( \xi^* g = (\nabla \xi)^T (g \circ \xi) \nabla \xi \), be the pull-back of the metric \( g \) by \( \xi \) on \( U' \). Then

\[
\kappa_{\xi^* g} = \kappa_g \circ \xi,
\]

where \( \kappa_g \circ \xi \) is defined through

\[
\forall \varphi \in C^\infty_c(U') \quad (\kappa_g \circ \xi)[\varphi] := \kappa_g[J(\xi)]^{-1} \varphi \circ \xi^{-1}.
\]
**Proof.** Since \( \xi \in C^2 \), we have \( \psi = |J(\xi)|^{-1} \varphi \circ \xi^{-1} \in W^{1,1}_0(V) \) with a suitable open set \( V \) with \( \text{supp} \psi \subset V \subset U \), and hence, as \( g \in C^1(\overline{V}) \), \( \kappa_g[|J(\xi)|^{-1} \varphi \circ \xi^{-1}] \) is well-defined, and the value is independent of the choice of \( V \).

Note the well known fact that the identity is valid for smooth \( g \in C^\infty(\overline{V}) \). Applying Lemma 4.2 to a regularizing sequence \( g_k \in C^\infty(\overline{V}) \) converging to \( g \) in \( C^1(\overline{V}) \), the conclusion is obtained by a straightforward passage to the limit.

The following proposition is an immediate corollary.

**Proposition 4.4.** Let \( \Sigma \) be a two dimensional \( C^2 \) manifold and let \( g \) be a \( C^1 \) Riemannian metric over \( \Sigma \). The distributional Gaussian curvature \( \kappa_g \) can be defined in each chart of \( \Sigma \) and is invariant by the admissible changes of coordinates through the formula (4.2), and hence it is a well-defined distribution on \( \Sigma \).

In what follows we discuss further properties of \( \kappa_g \) when \( g \) further enjoys some Hölder regularity.

**Proposition 4.5.** Let \( U \subset \mathbb{R}^2 \) be a bounded smooth domain. If \( g \in C^{1,\alpha}(\overline{U}) \) for \( 1/2 < \alpha' < \alpha < 1 \), then \( \kappa_g \) can be uniquely extended as a bounded linear operator over \( C^{0,\alpha'}_0(U) \), and

\[
\forall \varphi \in C^{0,\alpha'}_0(U) \quad |\kappa_g[\varphi]| \leq C(\|\varphi\|_{\alpha':U}).
\]

**Proof.** First note that since \( g \in C^{1,\alpha}(\overline{U}) = B^{1+\alpha}_{\infty,\infty}(U) \) \cite{34} Proposition 2.1.2, we have

\[
\text{curl curl } g \in B^{\alpha-1}_{\infty,\infty}(U)
\]

by \cite{34} Proposition 2.1.4/2. According to \cite{34} Theorem 4.6.1/2(i) the product

\[
B^{\alpha-1}_{\infty,\infty}(U) \cdot B^{\alpha}_{\infty,\infty}(U) \hookrightarrow B^{\alpha-1}_{\infty,\infty}(U),
\]

is continuous, which yields \( \kappa_g \in B^{\alpha-1}_{\infty,\infty}(U) \), since \( (\det g)^{-1} \in C^{0,\alpha}(\overline{U}) \), and \( L(g) \) is regular enough.

Now, since \( \alpha > 1/2 \), we can fix \( \sigma > 0 \) such that \( 1 - \alpha < 1/2 < \sigma < \alpha' \). By \cite{34} Theorems 2.4.4/1 and 2.4.4/4, we have the embedding

\[
B^{\alpha-1}_{\infty,\infty}(U) \hookrightarrow B^{\sigma}_{2,2}(U) = (\tilde{B}^\sigma_{2,2}(U))'.
\]

On the other hand, also by \cite{34} Theorem 2.4.4/1, \( C^{0,\alpha'}_0(U) \subset B^{\alpha'}_{\infty,\infty} \) embeds densely in \( \tilde{B}^\sigma_{2,2}(U) \), which yields the embedding

\[
(\tilde{B}^\sigma_{2,2}(U))' \hookrightarrow (C^{0,\alpha'}_0(U))'.
\]

Overall, we obtained

\[
\kappa_g \in B^{\alpha-1}_{\infty,\infty}(U) \hookrightarrow (C^{0,\alpha'}_0(U))'.
\]

as required.

**Corollary 6.** Assume \( U \), \( \alpha \), \( \alpha' \) and \( g \) as in Proposition 4.5. Let \( \xi : U' \to U \) be a \( C^{1,\alpha} \) smooth diffeomorphism. Then the composition \( \kappa_g \circ \xi \) is a well-defined distribution and for any sequence of diffeomorphisms \( \xi_k \to \xi \) converging in \( C^{1,\alpha'}(U') \) to \( \xi \) we have:

\[
\forall \varphi \in C^\infty_c(U') \quad \lim_{k \to \infty} (\kappa_{g_k} \circ \xi_k)[\varphi] = (\kappa_g \circ \xi)[\varphi].
\]
Proof. It is sufficient to observe that, since the diffeomorphism $\xi \in C^{1,\alpha}(\overline{U})$, we have

$$|J(\xi)|^{-1} \varphi \circ \xi^{-1} \in C^{0,\alpha'}(U')$$

and hence the action of $\kappa_g$ on $\frac{1}{|J(\xi)|\varphi \circ \xi^{-1}}$ is well-defined. The continuity follows from Proposition 4.3 by observing that

$$|J(\xi_k)|^{-1} \varphi \circ \xi_k^{-1} \xrightarrow{k \to \infty} |J(\xi)|^{-1} \varphi \circ \xi^{-1} \quad \text{in} \quad C^{0,\alpha'}(U').$$

\[ \blacksquare \]

In view of the above, we would have liked to claim that $\kappa_{\xi^*g} = \kappa_g \circ \xi$ when $\xi$ is C$^{1,\alpha}$ regularity, but our observations do not guarantee that the distributional Gaussian curvature of $\xi^*g$ is well-defined, as it might lack the assumed C$^1$ regularity. We will analyze hence the regularization of the pull-back metrics.

**Lemma 4.6.** Assume $U \subset \mathbb{R}^2$ is an open set and $\alpha > 1/2$, and $g \in C^{1,\alpha}(\overline{U}, \mathbb{R}^{2 \times 2}_{\text{sym,pos}})$. Let $\xi : U' \to U$ be a $C^{1,\alpha}$ diffeomorphism and

$$\xi^*g := (\nabla \xi)^T (g \circ \xi) \nabla \xi$$

be the pull-back metric. Let $(\xi^*g)_\varepsilon$ and $\xi_\varepsilon$ be the respective regularizations of $\xi^*g$ and $\xi$. If $x' \in U'$, there exists a disk $B' \Subset U'$ centered at $x'$ such that for all $\varepsilon$ small enough, $\xi_\varepsilon$ is a diffeomorphism on $\overline{B'}$,

$$\inf_{\varepsilon} \inf_{B'} \det(\xi_\varepsilon^*g) > 0,$$

$$\|(\xi^*g)_\varepsilon\|_{j;B'} + \|\xi_\varepsilon^*g\|_{j;B'} \leq C(\|\xi\|_{1,\alpha}, \|g\|_{1,\alpha})\varepsilon^{j(\alpha-1)}, \quad j \in \{0,1\}.$$  

and

$$\|(\xi^*g)_\varepsilon - \xi_\varepsilon^*g\|_{j;B'} \leq C(\|\xi\|_{1,\alpha}, \|g\|_{1,\alpha})\varepsilon^{(1+j)\alpha-j}, \quad j \in \{0,1\}.$$  

Proof. Noting the uniform convergence of $\nabla \xi_\varepsilon$ to $\nabla \xi$ on a neighborhood of $x'$, the inverse function theorem can be applied uniformly on a ball $B'$ centered at $x'$, to obtain that each $\xi_\varepsilon$ is a diffeomorphism on $B'$ and $\det(\xi_\varepsilon^*g)$ is uniformly bounded from below on $B'$. It is straightforward that $(\xi^*g)_\varepsilon$ and $\xi_\varepsilon^*g$ are uniformly C$^0$-bounded, and the estimates on their C$^1$ norms follow immediately from the C$^{1,\alpha}$ regularity of $\xi$ and $g$, via the first estimate in Lemma 4.4.

We now estimate, letting $F := \nabla \xi \in C^{0,\alpha}(\Omega)$:

$$\|(\xi^*g)_\varepsilon - \xi_\varepsilon^*g\|_{j;B'} \leq \| (\xi^*g)_\varepsilon - F^T_\varepsilon (g \circ \xi)_\varepsilon F_\varepsilon \|_{j;B'} + \| F^T_\varepsilon (g \circ \xi)_\varepsilon F_\varepsilon - F^T_\varepsilon (g \circ \xi_\varepsilon) F_\varepsilon \|_{j;B'}$$

The first term is estimated using Lemma 4.1:

$$\|I_1\|_{j;B'} = \left\| \left( F^T_\varepsilon (g \circ \xi) F_\varepsilon \right)_\varepsilon - F^T_\varepsilon (g \circ \xi_\varepsilon) F_\varepsilon \right\|_{j;B'} \leq C\varepsilon^{2\alpha-j} \leq C\varepsilon^{(1+j)\alpha-j}. $$

For the second term, we first establish an estimate on the C$^0$ norm:

$$\|I_2\|_{0;B'} \leq C\|(g \circ \xi)_\varepsilon - g \circ \xi_\varepsilon\|_{0;B'} \leq \|(g \circ \xi)_\varepsilon - g \circ \xi\|_{0;B'} + \|g \circ \xi_\varepsilon - g \circ \xi\|_{0;B'} \leq C\varepsilon \leq C\varepsilon^\alpha,$$
since both $\xi$ and $g$ are $C^1$. For the $C^0$ norm of the derivatives of $I_2$ we have the upper bound
\[ \|\nabla I_2\|_{0,B'} \leq C\|\nabla F\|_{0,B'} \|g \circ \xi\|_{\varepsilon} + C\|\nabla (g \circ \xi - g \circ \xi)\|_{0,B'} \]
\[ \leq C\varepsilon^{\alpha-1} + \|[\nabla (g \circ \xi)](\nabla g - \nabla g)\|_{0,B'} + \|\nabla (g \circ \xi) - \nabla (g \circ \xi)\|_{0,B'} \]
\[ \leq C\varepsilon^{2\alpha-1} + C\varepsilon^\alpha + \|[\nabla g \circ \xi] \nabla \xi - (\nabla g \circ \xi)\nabla \xi\|_{1,B'}, \]
where we used the fact that $g \circ \xi \in C^{1,\alpha}(U)$. It remains to estimate the last term in order to conclude. We have
\[ \|[\nabla g \circ \xi] \nabla \xi - (\nabla g \circ \xi)\nabla \xi\|_{1,B'} \leq \|[\nabla g \circ \xi - \nabla g \circ \xi] \nabla \xi\|_{1,B'} + \|[\nabla g \circ \xi - \nabla g \circ \xi] \nabla \xi\|_{1,B'} \]
\[ \leq C\|[\nabla g \circ \xi - \nabla g \circ \xi] \nabla \xi\|_{1,B'} + C\varepsilon \leq C\varepsilon^\alpha \leq C\varepsilon^{2\alpha-1}, \]
where once again we used the $\alpha$-Hölder continuity of $\nabla g$.

4.2. **Proof of Theorem** \( \mathbb{I} \). Let $u : (\Omega, g) \to \mathbb{R}^3$ be given according to the assumptions of Theorem \( \mathbb{I} \) and $S'$ be given as a surface compactly contained in $S$. Consider a open set $\Omega''$, such that $S' \subset u(\Omega'') \subset S$, and let $\Omega'' := u(\Omega')$. For the first assertion, it is enough to show that for any $p \in \Omega''$, the area induced by the Gauss map $\tilde{n}$ on an open neighborhood of $p$ in $\Omega''$ has finite total variation. A covering argument will then imply that $S'$ is a surface of bounded extrinsic curvature. Without loss of generality, we can assume that $p = 0$, and that $\tilde{n}(p) = \tilde{e}_3$, so that the tangent plane $T_p S'$ is identified with $\mathbb{R}^2 \times \{0\}$.

Let $O$ be an open neighborhood of $p = 0$ in $\Omega''$, and let $\Omega_0 := u^{-1}(O)$, $x_0 = u^{-1}(p)$. We write $u := (\tilde{u}, u_3)$, where $\tilde{u} := (u_1, u_2) : \Omega_0 \to \mathbb{R}^2$. We note that
\[ \frac{\partial_1 u \times \partial_2 u}{|\partial_1 u \times \partial_2 u|}(x_0) = \tilde{n}(x_0) = \tilde{e}_3, \]
which yields that at $x_0$
\[ \det \nabla \tilde{u} = (\partial_1 \tilde{u})^\perp \cdot \partial_2 \tilde{u} = (\tilde{e}_3 \times \partial_1 u) \cdot \partial_2 u = (\partial_1 u \times \partial_2 u) \cdot \tilde{e}_3 > 0 \]
since $u$ is an immersion. Implicit function theorem implies that there exists open sets $U \subset \Omega_0$, and $U' = \tilde{u}(U) \supset 0$, such that $\tilde{u} : U \to U'$ is a $C^1$ diffeomorphism. Let $\xi \in C^1(U', U)$ be the inverse of $\tilde{u}$. If necessary by shrinking $U$ and $U'$, we can assume that $\tilde{u} \in C^{1,\alpha}(\overline{U}, \overline{U'})$ and $\det \nabla \tilde{u} > b_0 > 0$ in $\overline{U} \subset \Omega$.

**Lemma 4.7.** $\xi$ belongs to $C^{1,\alpha}(\overline{U'}, \mathbb{R}^2)$.

**Proof.** We estimate for any $x, y \in U$, $F = \nabla \tilde{u} \in C^{0,\alpha}(\overline{U}, \mathbb{R}^{2\times 2})$ and $b := \det \nabla \tilde{u} = C^{0,\alpha}(\overline{U})$
\[ \left|(\nabla \tilde{u})^{-1}(x) - (\nabla \tilde{u})^{-1}(y)\right| \leq \left|b^{-1}\text{Cof}(F)(x) - b^{-1}\text{Cof}(F)(y)\right| \]
\[ \leq \frac{\|F\|_{0,\alpha;U, \|b\|_{0,\alpha;U, b_0}}}{b_0^2} \|b(x) - b(y)\| + \frac{\|b\|_{0,\alpha;U, b_0}}{b_0} \|F(x) - F(y)\| \]
\[ \leq C(\|F\|_{0,\alpha;U}, \|b\|_{0,\alpha;U, b_0})|x - y|^{\alpha} \]
Hence for any \( x', y' \in U' \)
\[
| \nabla \xi(x') - \nabla \xi(y') | \leq \left| (\nabla \tilde{u})^{-1}(\xi(x')) - (\nabla \tilde{u})^{-1}(\xi(y')) \right|
\]
\[
\leq C(\|\tilde{u}\|_{1,\alpha,U}, b_0) |\xi(x') - \xi(y')|^{\alpha}
\]
\[
\leq C(\|\xi\|_{1,U}, \|\tilde{u}\|_{1,\alpha,U}, b_0) |x' - y'|^{\alpha}.
\]

To proceed we define
\[
v : U' \to \mathbb{R}, \quad v := u_3 \circ \xi.
\]
Note that \( v \in C^{1,\alpha}(\overline{U}') \), \( v(0) = 0 \), \( \nabla v(0) = 0 \in \mathbb{R}^2 \). Consider the graph mapping
\[
\xi_v : U' \to O, \quad \xi_v(x') := (x', v(x')),
\]
and note that
\[
\xi_v = u \circ \xi,
\]
and that \( \xi_v(U') = u(U) \subset O \subset S \) is the graph of \( v \) over \( U' \). We obtain hence the identity
\[
g' := \text{Id} + \nabla v \otimes \nabla v = (\nabla \xi_v)^T \nabla \xi_v = \nabla (u \circ \xi)^T \nabla (u \circ \xi) = \nabla \xi^T (g \circ \xi) \nabla \xi = \xi^* g,
\]
where we used the fact that \( u : (\Omega, g) \to \mathbb{R}^3 \) is an isometry, i.e.
\[
\nabla u^T \nabla u = g.
\]

From now on we use the fact that by assumption \( \alpha > 2/3 \). Fix \( 2/3 < \beta < \alpha \). By Lemma 2.1 we have
\[
\text{Det}D^2 v \in B^{2\beta-2}_{\infty,\infty}(U').
\]
Also note that \( (1 + |\nabla v|^2)^2 \in C^{0,\beta}(\overline{U}') = B^{\beta}_{\infty,\infty}(U') \). Since \( \beta + (2\beta - 2) > 0 \), we deduce that the distributional product
\[
\tilde{\kappa}_g' := (1 + |\nabla v|^2)^{-2}(\text{Det}D^2 v)
\]
is well defined as an element of \( B^{2\beta-2}_{\infty,\infty}(U') \) [34, Theorem 4.6.1/2(i)].

**Proposition 4.8.** \( \tilde{\kappa}_g' \) can be extended as a bounded linear operator on \( C_c^{0,\alpha}(U') \) and
\[
(4.4) \quad \forall \varphi \in C_c^{\infty}(U'), \quad \tilde{\kappa}_g'[\varphi] = \text{Det}D^2 v[(1 + |\nabla v|^2)^{-2} \varphi].
\]
Moreover, for any open set \( V' \subseteq U' \), let \( v_\varepsilon \) is the usual mollification of \( v \), defined on \( V' \) for \( \varepsilon \) small enough. Then
\[
(4.5) \quad \forall \varphi \in C_c^{\infty}(V'), \quad \int_{V'} (1 + |\nabla v_\varepsilon|^2)^{-2}(\text{Det}D^2 v_\varepsilon) \varphi \xrightarrow{\varepsilon \to 0} \tilde{\kappa}_g[\varphi].
\]

**Proof.** First note that \( C_c^{0,\alpha}(\overline{U}') \) densely embeds into \( W^{2-2\beta,1}_0(U') \), since \( 2 - 2\beta < \alpha \), which together with the duality (2.3), imply the first assertion through applying an extension argument.

To proceed, and before proving (4.4), we first show the last assertion (4.5). First we claim that
\[
\text{Det}D^2 v_\varepsilon \xrightarrow{\varepsilon \to 0} \text{Det}D^2 v \quad \text{in} \quad B^{2\beta-2}_{\infty,\infty}(V').
\]
Indeed we estimate, using (2.2) and interpolation
\[
\| \text{Det}D^2 v_\varepsilon - (\text{Det}D^2 v) * \psi_\varepsilon \|_{B^{2\beta-2}_{\infty,\infty}(V')} \leq \| \text{curl}(\nabla v_\varepsilon \otimes \nabla v_\varepsilon) - \text{curl}(\nabla v \otimes \nabla v_\varepsilon) \|_{0,2\beta-1;V'}
\]
\[
\leq \| \nabla v_\varepsilon \otimes \nabla v_\varepsilon - (\nabla v \otimes \nabla v_\varepsilon)_\varepsilon \|_{2;V'}^{2\beta-1} \| \nabla v_\varepsilon \otimes \nabla v_\varepsilon - (\nabla v \otimes \nabla v_\varepsilon) \|_{1;V'}^{1-(2\beta-1)}
\]
\[
\leq C_\varepsilon(2\alpha-2)(2\beta-1) + (2\alpha-1)(2-2\beta) = C_\varepsilon 2(\alpha-\beta) \xrightarrow{\varepsilon \to 0} 0,
\]
which yields our claim, since as $\text{Det} D^2 v \in B^{2 \alpha - 2}_{\infty, \infty}(U')$, $(\text{Det} D^2 v) \ast \psi_\varepsilon$ converges to $\text{Det} D^2 v$ in $B^{2 \beta - 2}_{\infty, \infty}(V')$. Further

$$(1 + |\nabla v|)^{-2} \overset{\varepsilon \to 0}{\longrightarrow} (1 + |\nabla v|)^{-2} \quad \text{in } B^{\beta}_{\infty, \infty}(V').$$

Now the continuity of the product

$$(4.6) \quad B^{2 \beta - 2}_{\infty, \infty}(V') \cdot B^{\beta}_{\infty, \infty}(V') \hookrightarrow B^{2 \beta - 2}_{\infty, \infty}(V'),$$

according to [34, Theorem 4.6.1/2(i)], establishes (4.3).

To show (4.4), for all $\varphi \in C^\infty_c(U')$, we choose $V' \subseteq U'$ such that $\text{supp } \varphi \subseteq V'$. Applying (4.5) we obtain, as required,

$$\kappa g'_{\varepsilon}[\varphi] = \lim_{\varepsilon \to 0} \int_{V'} (1 + |\nabla v|)^{-2}(\text{Det} D^2 v) \varphi$$

$$= \lim_{\varepsilon \to 0} \text{Det} D^2 v[(1 + |\nabla v|)^{-2} \varphi]$$

$$= \text{Det} D^2 v[(1 + |\nabla v|)^{-2} \varphi],$$

where the last equality is obtained by an extension argument via the duality (2.3).

The following proposition reduces the problem to the case we studied in Section 2 regarding the very weak Monge-Ampère equation:

**Proposition 4.9.**

(i) Assume that $\kappa_g \geq 0$ as a distribution. Let $p, x_0, U'$ as above and let $V' \subseteq U'$ be any open set containing $x_0$. Then there exists an open disk $B' \subseteq V'$ containing $x_0$ such that $\text{Det} D^2 v \geq 0$ in $B'$.

(ii) If $\kappa_g \neq 0$, then, there exists $x_0 \in \Omega$, and an open disk $B'$ containing $x_0$, such that $\text{Det} D^2 v \neq 0$ in $B'$.

*Proof.* Let $v_\varepsilon$ be the mollifying sequence defined on $V'$ as above. Consider the sequence of Riemannian metrics

$$g'(\varepsilon) := (\nabla \xi_\varepsilon)^T \nabla \xi_\varepsilon = \text{Id} + \nabla v_\varepsilon \otimes \nabla v_\varepsilon,$$

for the graph parameterization $\xi_\varepsilon = (x, v_\varepsilon(x))$. It is a well known fact for smooth graphs that

$$\kappa_{g'(\varepsilon)} = (1 + |\nabla v|)^{-2}(\det \nabla^2 v) = (1 + |\nabla v|)^{-2}(\text{Det} D^2 v).$$

By the estimates (4.1)

$$\|g'(\varepsilon)\|_{1,V'} + \|g'(\varepsilon)\|_{1,V'} \leq C \varepsilon^{\alpha - 1},$$

and

$$\|g'(\varepsilon) - g(\varepsilon)\|_{j,V'} \leq \|(\nabla v \otimes \nabla v) - \nabla v_\varepsilon \otimes \nabla v_\varepsilon\|_{j,V'} \leq C \varepsilon^{2 \alpha - j}.$$
On the other hand, Lemmas 4.2 and 4.6 and (4.7) together imply in the same manner
\[ |\kappa_{g_\varepsilon}[\varphi] - \kappa_{g_\varepsilon}'[\varphi]| \leq C\varepsilon^{3\alpha-2} \xrightarrow{\varepsilon \to 0} 0. \]
Combining the last two convergences, and taking into account (4.4) and (4.5) we finally obtain for all \( \varphi \in C^\infty_c(B') \):
\[ \text{Det} D^2v[(1 + |\nabla v|^2)^{-2}\varphi] = \kappa_g'[\varphi] \]
\[ = \lim_{\varepsilon \to 0} \int_{V'} (1 + |\nabla v_\varepsilon|^2)^{-2}(\text{Det} D^2v_\varepsilon) \varphi \]
\[ = \lim_{\varepsilon \to 0} \kappa_{g_\varepsilon}'(\varphi) = \lim_{\varepsilon \to 0} \kappa_{g_\varepsilon}^\prime [\varphi] \]
\[ = \lim_{\varepsilon \to 0} (\kappa_g \circ \xi_\varepsilon)[\varphi] = (\kappa_g \circ \xi)[\varphi] \]
\[ = \kappa_g(|J(\xi)|^{-1} \varphi \circ \xi^{-1}), \]
where we used Lemma 4.3 Corollary 3 in the last two lines. (i) immediately follows by an approximation argument.

To see (ii), note that if \( \kappa_g \neq 0 \), then there is a point \( p = u(x_0) \in S = u(\Omega) \) for which for all \( r > 0 \), there exists \( \varphi \in C^\infty_c(B_r(x_0)) \) for which \( \kappa_g[\varphi] \neq 0 \).

Applying Proposition 3.1 it follows immediately that under the assumptions of Theorem 1 and for any \( p \in S'' \), \( \xi_\varepsilon(B') = u \circ \xi(\varepsilon B') \), which is an open neighborhood of \( p \) in \( S'' \), is a surface of bounded extrinsic curvature, of nonnegative curvature, with the absolute curvature of \( \xi_\varepsilon(B') \) bounded e.g. by the finite value \( |||J(\xi)|^{-1}(1 + |\nabla v|^2)^2||_{0,B'} \mu_{\kappa_g}(B') \).

Remember that \( S' \subset S'' \) is compact. Therefore the local property on \( S'' \) is sufficient to show via a covering argument and application of compactness property that \( S' \) is a surface of bounded extrinsic curvature and of nonnegative curvature. If the distributional curvature of \( g \) is nonzero, Proposition 3.1 once again implies that for a suitable choice of \( \Omega' \subset \Omega \), and \( p \in S' = u(\Omega') \), the positive curvature of a neighborhood of \( p \) in \( S' \) is nonzero, and hence, since the absolute curvature is \( \sigma \)-additive nonnegative set function, the positive curvature of \( S' \) is nonzero as required. The proof of Theorem 1 is complete.

5. PROOF OF THEOREM 3

By the assumptions of Theorem 3 \( v \in C^{1,\alpha}(\mathbb{R}^2) \) is a solution to \( \text{Det} D^2v = f \geq 0 \) in \( D'(\mathbb{R}^2) \), where \( 2/3 < \alpha < 1 \) and \( f \neq 0 \). Since \( \mu_f \) is nonzero, there exists \( x \in \mathbb{R}^2 \) for which \( \mu_f(U) > 0 \) for some open disk \( U \) containing \( x \). Apply Proposition 3.1 to increasing disks \( \Omega = B_R(x) \) as \( R \to \infty \), and deduce that the graph of \( v \) over \( \mathbb{R}^2 \) is a complete surface of nonnegative curvature with total nonzero curvature. 33 Theorem 2, p. 615] will now imply that \( S \) is an unbounded convex surface. As a consequence, \( v \) is either globally convex or concave.

It remains to be shown that \( v \) is an Alexandrov solution to \( \text{det} \nabla^2v = \mu_f \) in \( \mathbb{R}^2 \). We first claim that \( \nabla v \) is globally 1-1 on the set \( \mathbb{R}^2 \) of points \( x \in \mathbb{R}^2 \) for which \( \xi_\varepsilon(x) \) is regular. This is obvious since the convexity of \( v \) implies that if \( \nabla v(x) = \nabla v(z) \) for \( x \neq z \), then \( \nabla v \) is constant on the segment \( [x, z] \), so that \( \xi_\varepsilon(x) \) cannot be a regular point.

Now, consider any bounded piece-wise smooth open \( U \subset \mathbb{R}^2 \). Note that by 3 Lemma 4], \( |\nabla v(\partial U)| = 0 \), a fact that we will repeatedly use in what follows. It is sufficient to prove that
\( \mu_v(U) = \mu_f(U) \) to conclude the proof. Note that by [33] Theorems 1 and 4, p. 590-591, we have \( \mathcal{H}^2(N(\mathbb{R}^2 \setminus \mathbb{R}^2_r)) = 0 \), and thus \( |\nabla v(\mathbb{R}^2 \setminus \mathbb{R}^2_r)| = 0 \). Hence

(5.1) \[ |U'| = |\nabla v(U) \setminus \nabla(\partial U)|, \]

where \( U' := \nabla v(U) \setminus (\nabla(\partial U) \cup \nabla v(\mathbb{R}^2 \setminus \mathbb{R}^2_r)) \). We observe that for all \( y \in U' \), \( (\nabla v)^{-1}(y) \subset \mathbb{R}^2_r \), which, since \( \nabla v \) is injective on \( \mathbb{R}^2_r \), yields that for any such \( y \) that \( (\nabla v)^{-1}(y) = \{x\} \) for some \( x \in \mathbb{R}^2 \). We now choose \( r \) such that \( O := \xi_v(B_r(x)) \) is as in (1.1) and arguing as in Proposition 3.1 we note that \( \deg(\nabla v, B_{r/2}(x), y) = 1 \). For \( y \in U' \), there is no other solution to \( \nabla v(z) = y \) in \( U \) other than \( x \), which implies by the properties of the [19] Corollaire 2.4:

\[ \forall y \in U' \quad \deg(\nabla v, U, y) = \deg(\nabla v, B_{r/2}(x), y) = 1, \]

and thus (2.6), (5.1) and the fact that \( |\nabla v(\partial U)| = 0 \) imply

\[ \mu_v(U) = |\nabla v(U)| = |U'| = \int_{U'} \deg(\nabla v, U, y) \, dy = \int_{\nabla v(U) \setminus \nabla(\partial U)} \deg(\nabla v, U, y) \, dy = \mu_f(U). \]

This concludes the proof of Theorem 3.

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