INTEGRABLE 1+1 DIMENSIONAL GRAVITY MODELS

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Integrable models of dilaton gravity coupled to electromagnetic and scalar matter fields in dimensions 1+1 and 0+1 are reviewed. The 1+1 dimensional integrable models are either solved in terms of explicit quadratures or reduced to the classically integrable Liouvillian equation. The 0+1 dimensional integrable models emerge as sectors in generally non integrable 1+1 dimensional models and can be solved in terms of explicit quadratures. The Hamiltonian formulation and the problem of quantizing are briefly discussed. Applications to gravity in any space - time dimension are outlined and a generalization of the so called ‘no - hair’ theorem is proven using local properties of the Lagrange equations for a rather general 1+1 dimensional dilaton gravity coupled to matter.

1. Introduction

Many exact solutions of the Einstein - Hilbert (EH) gravity equations were discovered and studied in the past. However, it has been only recently realized that these solutions may be viewed as exact solutions of some integrable Hamiltonian systems having much less degrees of freedom than the original EH field theory (for examples and further references see ref. 1 - ref. 4). Typically, 1+1 dimensional field theories or finite systems (0+1 dimensional field theories) emerge in this way (similar models are produced by string theories as well as by topological gauge theories). The Hamiltonian integrability is very useful in classical theory and it is vital for quantizing. The idea is that we first exactly quantize the integrable Hamiltonian subsystem of the full (presumably nonintegrable) system and then apply a perturbation theory to quantize it. Here we concentrate on the first step of this program - constructing integrable Hamiltonian subsystems of the EH gravity coupled to matter.

We will consider here a rather general 1+1 dimensional field theory which we call dilaton - gravity - matter model (DGM). The gravitational variables in this model are the metric tensor $g_{ij}$ and the scalar dilaton field $\phi$. The matter fields are the electromagnetic field tensor $F_{ij}$ and the scalar field $\psi$. The DGM Lagrangian is

$$L = \sqrt{-g} \left[ UR(g) + V + W g^{ij} \phi_i \phi_j + X F_{ij} F^{ij} + Y + Z g^{ij} \psi_i \psi_j \right],$$

(1)
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where $U - Z$ are functions of $\phi$ and $Y$ may in addition depend on $\psi$; $R$ is the scalar curvature; the lower letter indices denote partial derivatives ($\phi_i = \partial_i \phi$, etc.), except when used in $g_{ij}$ or $F_{ij}$. The model may be extended by including many scalar fields and nonabelian gauge fields; we will not do this exercise here.

With arbitrary functions $U - Z$, this DGM model is certainly not integrable. However, if $Y = Z = 0$, it is integrable due to local symmetries discussed below; we will call this important special case the dilaton gravity (DG) though the Lagrangian includes coupling to the electromagnetic field. Moreover, the general DG can be exactly reduced (at least, classically) to a finite-dimensional Hamiltonian system with one constraint. For the general DGM this is impossible. Other special case $X = 0$ will be called the dilaton-gravity-scalar model (DGS). DGS models are generally not integrable. A new class of integrable 1+1 DGS models as well as DGM theories having an integrable 0+1 dimensional sectors have been recently found and will be described below.

The most familiar example of DGM is derived by a dimensional reduction of the spherically symmetric sector of the $d$-dimensional EH gravity coupled to scalar and electromagnetic fields having the same symmetry. Following ref. 5, we get the effective Lagrangian in the form (1) with the potentials

$$U = e^{-2\phi}, \quad V = 2e^{-2\phi}(\alpha e^{4\nu\phi} + \Lambda), \quad W = 4(1 - \nu)e^{-2\phi},$$

$$X = -e^{-2\phi}/\beta, \quad Y = -\gamma y(\psi)e^{-2\phi}, \quad Z = -\gamma e^{-2\phi},$$

where $d \equiv n + 2$, $\nu \equiv 1/n$. Here the parameters $\alpha, \beta, \gamma$ may depend on $\nu$; $\alpha$ is proportional to the curvature of the sphere, $\beta$ and $\gamma$ are normalization constants. We also have added to the $d$-dimensional action the cosmological term (by $R \mapsto R + 2\Lambda$). As was pointed out in ref. 5, the parameter $\nu$ may formally assume any real value. In particular, for $\nu = 0$ and $X = Y = Z = 0$ we obtain the well-known CGHS dilaton gravity. Note that to get the CGHS model coupled to the scalar field we have to take the constant potential $Z = -\gamma_0 \neq 0$ and choose $y(\psi) = 0$.

Many other models (describing black holes, strings, cosmologies) can be written in the form (1). The approach of our paper may also be used for coupling of DG to many scalar fields, nonabelian gauge fields, spinor fields, etc. To compare different models considered in a rather extensive literature on this subject, one has to keep in mind that, classically, different parametrizations of the potentials in terms of $\phi$ and the Weyl transformations of (1) can be used. For example, for positive definite $U$ we may use the representation $U \equiv e^{-2\phi}$ or $U \equiv \phi^2$. For simplicity, we will mainly use $U \equiv \phi$. In classical theory, it is always possible. In quantum theory, this parametrization is not necessarily equivalent to the exponential one or to $U \equiv \phi^2$, etc. (see e.g ref. 7). The Weyl transformation $g_{ij} = \Omega(\phi)\bar{g}_{ij}$ is even more dangerous. However, in the classical framework, it can also be used to compare differently looking models. If two Lagrangians can be identified by using Weyl rescaling and a different choice for $U$ in terms of $\phi$, they are classically equivalent.

2. Equations of Motion
The equations of motion can be derived by varying $\mathcal{L}$ with respect to the variables $g_{ij}$, $\phi$, $\psi$ and $A_i$ (the electromagnetic potential). Then, using their general covariance, it is convenient to rewrite them in the conformal flat, light-like metric $ds^2 = -4f(u,v)du dv$, in which

$$R = f^{-1}(\log f)_{uv}. \quad (4)$$

To further simplify the equations we define the function $w(\phi)$, $w' / w \equiv W / U'$ (the prime always denotes the derivative of a function depending on one variable, thus $U' \equiv dU / d\phi$, etc.), and write them in the form that is invariant under the Weyl rescaling, $g_{ij} = \Omega(\phi)\bar{g}_{ij}$, i.e. $\bar{f} = \Omega^{-1}f$. The Lagrangian (1) is invariant if $\bar{w} = \Omega w$ while $U, Z, f w \equiv \bar{f}, V / w \equiv \bar{V}, X w = \bar{X}, X F_{ij} F^{ij} / w$ and $Y / w \equiv \bar{Y}$ are invariant. Using the fact that the electric charge $Q = X F_{uv} / 2f$ is locally conserved (i.e. $Q_u = Q_v = 0$) and $X F_{ij} F^{ij} = -2Q^2 / X$, we may include the electromagnetic coupling into the potential $V$ and then forget about the electromagnetic field. More precisely, we only have to add the term $\bar{V}_{em} \equiv 2Q^2 / \bar{X}$ to the potential $\bar{V}$.

With all these conventions, the equations can be written in the form

$$\bar{f}(U_i / \bar{f})_i = Z\psi^2_u, \quad (i = u, v); \quad (5)$$

$$U_{uv} + \bar{f}(\bar{V} + \bar{Y}) = 0; \quad (6)$$

$$U' (\log \bar{f})_{uv} + \bar{f}(\bar{V'} + \bar{Y}_\phi) = Z' \psi_u \psi_v; \quad (7)$$

$$(Z\psi_u)_v + (Z\psi_v)_u + \bar{f}\bar{Y}_\psi = 0, \quad \bar{Y}_\phi \equiv \partial_\phi \bar{Y}, \quad \bar{Y}_\psi \equiv \partial_\psi \bar{Y}. \quad (8)$$

Not all these equations are independent. Thus, eqs. (5) - (7) imply (8) if $\psi_u^2 + \psi_v^2 \neq 0$. Similarly, eqs. (5), (6), (8) imply eq. (7) if $\phi_u^2 + \phi_v^2 \neq 0$. The statements are easy to prove by considering $(Z\psi_u)_v$ and $(Z\psi_v)_u$. A useful corollary is the following. If $Z' = 0$, we only need to solve eqs. (5) and (6) and then define $\psi^2_i$ by eqs. (5); eq. (8) is then automatically satisfied. These simple observations are very useful for analyzing integrability. Note that (7) is often neglected but it is the key equation in our approach to integrable DGS models.

### 3. Integrability of General DG Theory

If $Y = 0$ and $Z = 0$, we only need to solve the equations (5) and (6). It is easy to show that they are completely integrable with arbitrary potential $\bar{V}$. In fact, we can transform them into very simple linear equations by a Bäcklund-like transformation

$$M = N(U) + U_u U_v / \bar{f}, \quad \Phi_u = \bar{f} / U_v, \quad \Phi_v = \bar{f} / U_u. \quad (9)$$

The new variables $M$ and $\Phi$ satisfy the linear equations

$$\Phi_{uv} = 0, \quad M_u = 0, \quad M_v = 0. \quad (10)$$
The D’Alembert equation gives the constraint equations (3), while the local conservation of $M$ gives the main ‘dynamical’ equation (4). The solution of the equations (4) is obvious,

$$\Phi = a(u) + b(v) \equiv \tau, \quad M = \text{const}.$$  \hfill (11)

To find $U$ and $\tilde{f}$ in terms of $M$ and $\tau$ we have first to express $U$ as a function of $\Phi$ and of $M$, $U = \hat{U}(\Phi, M)$. It is not difficult to infer from eqs. (4) that $\hat{U}$ has to satisfy the equation $\hat{U}\Phi = M - N(\hat{U})$ which we are free to supplement by the ‘initial’ condition $\hat{U}(0, M) = 0$. Thus $\Phi = \Phi(\hat{U}, M)$ is defined by one quadrature and resolving this equation with respect to $\hat{U}$ we find $U$ as a function of $\tau$ and $M$.

Then we have

$$\tilde{f} = \hat{U}_\Phi \Phi_u \Phi_v = [M - N(\hat{U}(\Phi, M))] \Phi_u \Phi_v,$$ \hfill (12)

which completes the solution ($\tilde{h} \equiv hw$),

$$\tilde{f} = [M - N(\hat{U}(\tau, M))] a'(u)b'(v) \equiv \tilde{h}(\tau, M) a'(u)b'(v).$$ \hfill (13)

Thus, $\phi$ depends on one coordinate $\tau$. Choosing $a$ and $b$ as the new coordinates, we find that the metric also depends on one coordinate,

$$ds^2 = -4fdu dv = -4hdadb.$$ \hfill (14)

We see that the solution of the equations of motion essentially depend on one variable $\tau = a + b$. It is called ‘static’ because it is a generalization of the static Schwarzschild solution in general relativity (black hole). Then zeroes of $h(\phi) \equiv h(U(\phi))$ for finite values of $\phi$ are called the horizons; with one horizon, we have a Schwarzschild - like black hole solution and $M$ is the mass of the black hole.

We will not further elaborate the physics interpretation of the solution and will not give the Hamiltonian formulation of DG in dimension 1+1. We may simply use the fact that the solutions of the DG equations are essentially 0+1 dimensional and write the corresponding constrained Hamiltonian equations. This will be shown for the static sectors of the DGM models which, in general, are not integrable and can not be completely reduced to 0+1 dimension.

4. Integrable DGS Models

The 1+1 dimensional integrable models discussed below classically reduce to the Liouville equation (to understand why, please look at the expression for the two dimensional scalar curvature (4)):

$$G_{uv} + 2g \exp G = 0.$$ \hfill (15)

Its general solution may be found by simple Bäcklund transformations of the Liouville equation to the D’Alembert equation $\Phi_{uv} = 0$, for example,

$$\exp G = -g^{-1} \Phi_u \Phi_v / \Phi^2.$$ \hfill (16)
This and other representations of $G$ in terms of the solutions of the D'Alembert equation may be obtained from the following simple observation (known to Liouville!): if $G^0(u,v)$ is a solution of (15) then

$$G^1(u,v) = G^0(a(u),b(v)) + \log(a'(u)b'(v))$$

(17)

satisfy the same equation for arbitrary functions $a(u)$ and $b(v)$. Now, taking $G^0$ defined by $\exp G^0 = \pm g^{-1}(u \mp v)^{-2}$, where $\pm g > 0$, we find that $G^1$ can be represented in the form $\exp G = \pm g^{-1}G_uG_v/\sin^2 \Phi$. (18)

The important point is that the solutions of the Liouville equation essentially depend on one variable $\Phi = a(u) + b(v)$ and that, in general, they have singularities in this variable. This poses serious problems in their geometric interpretation as well as in quantizing. To the best of my knowledge, these problems are not yet solved. Note that the class of the solutions without singularities may be represented in the form

$$\exp G = g^{-1}G_uG_v/\cosh^2 \Phi,$$

(19)

which can be derived by substituting tan $a$ and tan $b$ for $a$ and $b$. One may try to approach the mentioned problems by studying first this non-singular sector of the Liouville equation.

Now, let us write integrable DGS models with $Z' = 0$ (due to their relation to string theory they are often called string motivated dilaton gravity models). Without restricting the generality of our consideration, we will choose the parametrization $U = \phi$ (then $U' \equiv 1$). Adding eq.(6) multiplied by constants $\pm g_1$ to eq.(7), we find

$$F^{\pm} = \log (\tilde{f}e^{\mp g_1 \phi}) \exp F^{\pm} = 0,$$

(20)

where $F^{\pm} = \log (\tilde{f}e^{\mp g_1 \phi})$ and $\epsilon$ is the sign of $\tilde{f}$. Now, if we choose $\tilde{V}$ so that the expression in the square brackets is constant, $2g_{\pm}$, these equations coincide with the Liouville equation. The most general potential satisfying this requirement is

$$\tilde{V} = (g_+e^{g_1 \phi} - g_-e^{-g_1 \phi})/g_1.$$

(21)

Using the above formulas we may reduce eqs.(20) to two D'Alembert equations $\Phi^{\pm}_{uv} = 0$ and thus solve them in terms of four arbitrary functions $a^{\pm}(u), b^{\mp}(v)$.

A simpler integrable model is defined by the potential

$$\tilde{V} = g_3 + 2g_2 \phi \quad g_2 \equiv g_+,$$

(22)

which can be obtained from (21) with $g_- \equiv g_+ - g_3 g_1$ and $g_1 \to 0$. Then (21) is the Liouville equation for the metric while (3) is the linear equation for $\phi$ defining scattering of the dilaton field on the metric. Both equations can be explicitly solved. Note that there is no scattering on nonsingular metrics defined by the solutions of
the Liouville equation (19). To see this, define the new coordinates \( r = a + b \) and \( t = a - b \). Then the equation for the dilaton field is the Klein-Gordon equation with the reflectionless potential \( 2/\cosh^2(r) \). A detailed discussion of this and other solutions will be presented elsewhere.

The obtained integrable DGS models significantly generalize the CGHS model, in which the curvature \( R \) is zero. In our model (22), it is constant while in the most general integrable model (21) it is not constant. Even on the classical level, constructing global solutions of these models is an unsolved problem. I think that the singularities signal a necessity of considering topologically nontrivial space-time surfaces. In quantum theory, some formal operator solutions of the Liouville equation are known. Whether these solutions allow a Hilbert space realization and can give a complete quantum theory of the integrable DGS models is at the moment unclear.

5. Integrable 0+1 Dimensional Sectors of DGM Models

All DGM model obtained by the dimensional reduction of the EH gravity coupled to scalar fields have \( Z' \neq 0 \). However, I do not know any integrable DGS model with \( Z' \neq 0 \) in dimension 1+1. Nevertheless, for the spherically symmetric sector of the EH gravity coupled to scalar fields several exact solutions are known for long time. In our language, these solutions generate integrable 0+1 dimensional constrained systems which form certain sectors in 1+1 dimensional DGM. We will call these sectors ‘static’. Let us define what is the static solution of DGM not using these intuitive argument and in more precise terms. We call the solution of eqs. \((\ref{eq:5}) - (\ref{eq:8})\) static if there exist functions \( a(u) \) and \( b(v) \) such that \( \phi, h \equiv f/a'(u)b'(v) \) and \( \psi \) depend only on \( \tau \equiv a + b \) (in view of the above considerations, this definition is quite natural). In fact, if \( \phi = \phi(\tau) \) and \( \psi_a + \psi_b \neq 0 \), the equations \((\ref{eq:5}), (\ref{eq:6})\) tell us that \( \psi = \psi(\tau), h = h(\tau) \) and thus the solution is static. The set of all static solutions forms the static sector of a given DGS model.

The static sector is a 0+1 dimensional theory and we will show that \( \phi(\tau), \psi(\tau) \) and \( h(\tau) \) are coordinates of a constrained Hamiltonian system (the constraint is the Hamiltonian \( H \) itself, i.e. \( H = 0 \)). This system is integrable if there exist two more integrals depending on the coordinates \( h, \phi, \psi \) and velocities \( \dot{h}, \dot{\phi}, \dot{\psi} \) (or momenta introduced below) and the system of the first-order equations defined by these integrals is explicitly integrable (recall that the omitted equation for the electromagnetic field is always integrable). Then we call the DGM model \( s\)-integrable.

If \( \dot{Y} = 0 \), the equation \((\ref{eq:5})\), that now is simply \( (Z\ddot{\psi}) = 0 \), gives the integral \( C_0 = Z\dot{\psi} \). Thus the problem is to find one more integral. Keeping this in mind, let us rewrite the remaining equations \((\ref{eq:5}) - (\ref{eq:7})\) for the static sector. Then we have the following ordinary differential equations (we now take \( U \equiv \phi \) and return to the Weyl-noninvariant notation):

\[
\begin{align*}
\dot{\psi} - W\dddot{\phi}^2 - \dot{\phi}F &= Z\dddot{\psi}^2; \quad (23) \\
\dddot{\psi} + hV &= 0; \quad (24)
\end{align*}
\]
2W\ddot{\phi} + W'\dot{\phi}^2 + \dot{F} + hV' = Z\dot{\psi}^2 \tag{25}

(for convenience, we introduce a temporal notation \( F \equiv \dot{h}/h \)).

The Hamiltonian constraint can be directly obtained from (23) and (24)

\[ L \equiv W\dot{\phi}^2 + \dot{\phi}F + hV + Z\dot{\psi}^2 = 0, \quad F \equiv \dot{h}/h. \tag{26} \]

As was shown in ref.\textsuperscript{4}, an additional integral can be found in the following two cases. If \( Z \) and \( \tilde{V} \) satisfy the relation (recall that \( N'(\phi) \equiv \tilde{V} \)):

\[ Z = (g_2 + g_1 N(\phi))\tilde{V}^{-1}, \tag{27} \]

where \( g_1, g_2 \) are real constants, we have the integral

\[ ZF + (ZW - g_1)\dot{\phi} = C_1. \tag{28} \]

The three available integrals \( C_0, C_1 \) and \( L \) allow us to find the general solution to all the equations in terms of explicit quadratures.

The second s-integrable case is given by two relations for the potentials

\[ V = W(\bar{g}_4 w^2 - \bar{g}_1), \quad Z^{-1} = W(\bar{g}_3 + \bar{g}_2 \log w), \tag{29} \]

where \( \bar{g}_1 - \bar{g}_4 \) are arbitrary real constants. The additional integral of motion is

\[ (F/W)^2 + 4\bar{g}_1 h + 2\bar{g}_2 C_0^2 \log h = \bar{C}_1. \tag{30} \]

As in the first case, this integral allows to write the solution of all the equation in terms of explicit quadratures.

6. Hamiltonian Formulation of 0+1 Dimensional Models

Let us now give the Hamiltonian formulation of the ‘static’ equations of DGM. It is not difficult to show that Eqs. (23) - (25) as well as the constraint (26) and the omitted equation for \( \psi \) can be derived by varying the Lagrangian

\[ L^{(s)} = (L - hV)/l - lhV \]

in all variables including the Lagrangian multiplier \( l(\tau) \). Then the constraint equation (26) is reproduced for \( l = 1 \), which is nothing but a choice of a gauge. The necessity of this gauge fixing is related to the fact that we are using the conformal flat metric depending on one function \( f(u, v) \). If we would use a more general metric, e.g. \( ds^2 = \alpha(r)dr^2 - \beta(r)dt^2 \), a Lagrangian multiplier will emerge automatically.

Introducing the momenta

\[ p_h = \dot{\phi}/lh, \quad p_\phi = (2W\dot{\phi} + \dot{h}/h)/l, \quad p_\psi = 2\dot{\psi}/l, \tag{32} \]

we find the Hamiltonian \( H^{(s)} = lH \), where

\[ H = h p_h p_\phi - WH^2 p_h^2 + hV + p_\phi^2/AZ. \tag{33} \]
Now one may express the integrals of motion in terms of the canonical coordinates and momenta and check that their Poisson brackets with $H$ vanish when these variables satisfy the canonical equations of motion (including the constraint $H = 0$). Of course, the canonical equations are equivalent to the Lagrangian ones that were solved above (with $l = 1$).

Note that the form of the Hamiltonian (and of the Lagrangian) is not uniquely defined, due to a freedom in the choice of the Lagrangian multiplier $l(\tau)$. We may multiply $H$ by a function of the coordinates, $\lambda(h,\phi,\psi)$ (not having zeroes inside the domain of definition of the coordinates) and correspondingly divide $l(\tau)$ by $\lambda$. The new Hamiltonian $\bar{H}(s) \equiv \bar{l}H$ define the same equations of motion due to the constraint $H = 0$. Such a freedom is useful because the new Hamiltonian may have additional integrals of motion. This observation was used in our approach to quantizing black holes. In that case, the mass of the black hole is proportional to $M$ defined in (9). It is conserved when the scalar field is completely decoupled, i.e. for $C_0 = 0$. For the ‘static’ case, the mass function is simply

$$M = N(\phi) + \hbar p_{\psi}^2/w \equiv N(\phi) + \dot{\psi}^2/\hbar w.$$  \hspace{1cm} (34)

When $p_{\psi} \equiv 2C_0 \neq 0$, $M$ is not conserved but, for $s$-integrable models, there may exist other integrals of motion $\bar{C}_1$. In the $s$-integrable models, the integrals $C_1$ and $\bar{C}_1$ found in this paper play the role of $M$. Thus, it is not difficult to show that

$$C_1 = -g_1 \frac{w}{p_{h}} \left( M + \frac{g_2}{g_1} + \frac{P_{\psi}^2}{4g_1 \hbar w} \right).$$  \hspace{1cm} (35)

When $p_{\psi} \equiv 2C_0 = 0$, the factor $w/p_{f}$ in (34) becomes an additional integral of motion because $C_2 \equiv p_{h}/w = \dot{\phi}/\hbar w$ is independent of $\tau$ due to (23). A relation similar to (35) may be derived for $\bar{C}_1$. We write it only for $p_{\psi} = 0$:

$$\bar{C}_1 = (M^2 - 4\bar{g}_1 \bar{g}_4)/C_2^2.$$  \hspace{1cm} (36)

Note that the above Lagrangian and Hamiltonian formulations are valid in general, when additional integrals of motion are not known or even do not exist. Unfortunately, for nonlinear Hamiltonian systems with two or more independent coordinates the existence of such an integral is a rare event. Apparently simple systems with two coordinates are not integrable and thus exhibit complex phenomena known as dynamical chaos. A famous example is the Henon - Heiles system of two oscillators with cubic couplings between them. If one compares our general ‘static’ system (66) (with fixed $p_{\psi}$ and arbitrary functions $V$, $W$ and $Z$) to such well-studied nonintegrable systems, one may infer that it must be not integrable (one may show that, due to the Weyl invariance and the constraint $H = 0$, the general model essentially depends only on one arbitrary function but this does not help).

In particular, this means that the general DGM theory is not integrable. This does not mean that some other examples of integrable DGM models can not be discovered in future. On the other hand, chaotic phenomena in classical nonintegrable
DGM models might be of significant physics interest. In quantum framework, non-integrable models may still be useful if they can be treated perturbatively on some explicitly integrable background (like our s-integrable models).

7. An Example and a Theorem

Here we give a very simple static solution for DGM that shows why the scalar field can not be treated as a perturbation. We also will argue that the static metric for general DGM (integrable or not) have no horizons. This is a new version of the so called ’no-hair’ theorem.

The model described by the potentials (29) with $\bar{g}_1 = \bar{g}_2 = 0$ is very easy to integrate. By solving the equations of motion we find

$$h_0 w = (\bar{g}_4/C_1)^{-\frac{1}{2}} (1 + 2\delta) |h/h_0|^\delta |1 + \epsilon|h/h_0|^{1+2\delta}|^{-1},$$

(37)

where $h_0$ is the integration constant, $\epsilon$ is the sign of $h$ and

$$2\delta = \sqrt{1 - 4\bar{g}_3C_2^2/C_1} - 1.$$  

(38)

If $h \to 0$, then $w \to 0$ or $w \to \infty$. For the dimensionally reduced gravity defined by (2) (with $\Lambda = 0$) the coupling constant $\bar{g}_3$ is negative and thus $w \to 0, \phi \to 0$. The metric has no zeroes for finite values of $\phi$, i.e. no horizons. When $C_0^2 = 0$ and thus $\delta = 0$, the scalar field disappears and the horizon reappears. However, there is no analytic transition $\delta \to 0$. This means that the scalar field can not be treated as a perturbation.

Now we will show that the absence of zeroes in $h$ for finite values of $\phi$ is a general property of DGM. To prove this, rewrite (23) and (24) as follows

$$\dot{\phi} = \chi, \quad \dot{\chi} = -hV, \quad \dot{h} = -h\chi^{-1}(C_0^2Z^{-1} + hV).$$

(39)

As the right-hand sides of these equations are independent of $\tau$, we can reduce these three equations to two equations for $\chi$ and $\phi$ as functions of $h$. Let us consider the behavior of $\chi$ and $\phi$ for $h \to 0$ assuming that $\phi \to \phi_0 \neq \infty$. Suppose also that, in this limit, $V \to V_0 \neq \infty$ and $Z \to Z_0 \neq 0, \infty$. Then it is easy to find that

$$\frac{d\log \chi}{dh} \to \frac{Z_0V_0}{C_0^2},$$

(40)

which means that $\chi \to \chi_0 \neq 0, \infty$. Now one can show that

$$\frac{d\phi}{d\log h} \to -\frac{\chi_0^2Z_0}{C_0^2}.$$  

(41)

It follows that $|\phi| \to \infty$. This contradicts our assumption that $\phi_0 \neq \infty$.

In the picturesque language used in the gravity literature, this means that static black hole solutions of the Einstein gravity have no scalar ‘hair’. The precise formulation of the theorem we have just proven is: if $Z \neq 0$, the static solutions
of the general DGM theory have no horizons (in our proof we supposed also that \(Y = 0\)). Further generalizations and refinements of this theorem will be published elsewhere. Note that, in this formulation, the ‘no-hair’ theorem is a local property of the differential equations of motion.

8. Summary

We have studied a general dilaton gravity coupled to electromagnetic and scalar fields. For the general \(U - X\) model (DG) we presented some known results in a more compact and simple form emphasizing its integrability. For the \(U - Z\) models (DGS) we concentrated on a somewhat simpler case \(Y = 0\) and obtained a class of integrable theories with the constant potential \(Z\), thus generalizing the CGHS dilaton gravity. Note that there exist integrable models with \(Y \neq 0\) which we will present elsewhere. For the DGM theories with arbitrary \(Z\), we found two classes of s-integrable (and explicitly soluble) systems. A special case of the s-integrable DGM gives new exact solutions of the Einstein gravity coupled to matter in any space - time dimension. We also pointed out a generalization of the ‘no - hair’ theorem. By constructing the Hamiltonian formulation of the s-integrable systems we hopefully paved a way to their quantizing. Further applications to gravity will be presented elsewhere.

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