FIBERING FLAT MANIFOLDS OF DIAGONAL TYPE AND THEIR FUNDAMENTAL GROUPS

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Abstract. An $n$-dimensional closed flat manifold is said to be of diagonal type if the standard representation of its holonomy group $G$ is diagonal. An $n$-dimensional Bieberbach group of diagonal type is the fundamental group of such a manifold. We introduce the diagonal Vasquez invariant of $G$ as the least integer $n_d(G)$ such that every flat manifold of diagonal type with holonomy $G$ fibers over a flat manifold of dimension at most $n_d(G)$ with flat torus fibers. Using a combinatorial description of Bieberbach groups of diagonal type, we give both upper and lower bounds for this invariant. We show that the lower bounds are exact when $G$ has low rank. We apply this to analyze diffuseness properties of Bieberbach groups of diagonal type. This leads to a complete classification of Bieberbach groups of diagonal type with Klein four-group holonomy and to an application to Kaplansky’s Unit Conjecture.

1. Introduction

Closed flat Riemannian manifolds of diagonal type have received considerable attention in recent years due to their special properties. Subclasses of these manifolds include generalized Hantzsche-Wendt manifolds [20], diagonal flat Kähler manifolds $M_d\Gamma$ [17] and real Bott manifolds. For instance, Popko and Szczepański in [19] showed that Hantzsche-Wendt manifolds are cohomologically rigid. In [15], it was shown that, in contrast to the case of real Bott manifolds, there are flat manifolds of diagonal type for which the existence of a spin structure may not be detected by its finite proper covers. In [17], Miatello and Podestá, constructed an infinite family of Dirac isospectral flat Kähler manifolds $M_d\Gamma$ which are pairwise non-homeomorphic to each other, thus, giving another counterexample to Mark Kac’s problem of “Can one hear the shape of a drum?” More recently, in [8], Gardam showed that the fundamental group of didicosm, which is a 3-dimensional non-diffuse Hantzsche-Wendt group, is a counterexample to the Kaplansky’s Unit Conjecture.

An $n$-dimensional crystallographic group $\Gamma$ is a discrete and cocompact subgroup of the group of isometries Isom($\mathbb{E}^n) = O(n) \ltimes \mathbb{R}^n$ of the $n$-dimensional Euclidean space $\mathbb{E}^n$. A torsion-free crystallographic group is called a Bieberbach group. It is well known that the...
fundamental group of any closed flat Riemannian manifold is a Bieberbach group and every such manifold arises as a quotient of an Euclidean space by an isometric action of a Bieberbach group (see [4, Chapter II]). By the First Bieberbach Theorem (see Section 2), there is a short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \to G \to 1$$

where \(\mathbb{Z}^n \cong \Gamma \cap \mathbb{R}^n\) is a maximal abelian subgroup of \(\Gamma\) called the lattice subgroup and \(G\) is a finite group called the holonomy group of \(\Gamma\). Given such a short exact sequence, it induces a representation \(\rho : G \to \text{GL}_n(\mathbb{Z})\) called the holonomy representation of \(\Gamma\). It is well-known that \(\rho\) is a faithful representation (see [21, Chapter 2]).

A crystallographic group \(\Gamma\) is said to be of diagonal type if it is isomorphic to a crystallographic group whose holonomy representation is diagonal. As an immediate consequence of diagonality of the holonomy representation, it follows that the holonomy group of a crystallographic group of diagonal type is isomorphic to an elementary abelian 2-group \(C_2^k\) for some \(k \geq 1\). A closed flat manifold is said to be of diagonal type if its fundamental group is of diagonal type.

In Section 2, we give a detailed introduction to crystallographic groups of diagonal type. In Section 3, we introduce the key combinatorial method for studying such groups. Given a crystallographic group \(\Gamma\) of diagonal type, we associate to it a characteristic matrix \(A_\Gamma\) with entries in the Klein four-group \(D\). Studying certain additive operations over \(D\) in \(A_\Gamma\), allows us to give a complete combinatorial characterisation of crystallographic groups of diagonal type. This is a variation of the method introduced in [19] and in [15].

In Section 4, we study the Vasquez invariant and introduce its analog for flat manifolds of diagonal type. Vasquez invariant allows one to determine whether a given closed flat Riemannian manifold fibers over a lower dimensional flat Riemannian manifold with fibers flat tori [24].

Let \(M\) be a closed flat Riemannian manifold with the fundamental group \(\pi_1(M) = \Gamma\). Let \(T^k = \mathbb{R}^n/\mathbb{Z}^n\) be a flat torus where \(\Gamma\) acts on it by isometries. Then \(\Gamma\) also acts on the space \(\widetilde{M} \times T^k\) by isometries, where \(\widetilde{M}\) is the universal cover of \(M\). It is easy to show that the space \((\widetilde{M} \times T^k)/\Gamma\) is a flat manifold (see [24, Section 2]). \((\widetilde{M} \times T^k)/\Gamma\) is called the flat toral extension of the manifolds \(M\). We shall make the convention that a point is the 0-dimensional torus, and hence any flat manifold can be a flat toral extension of itself.

We first recall the definition of the Vasquez invariant introduced by A. T. Vasquez in [24].

**Theorem 1.1** (Vasquez, [24, Theorem 3.6]). For any finite group \(G\), there exists a number \(n(G) \in \mathbb{N}\) minimal with the property that if \(\Gamma\) is a Bieberbach group with holonomy group
isomorphic to $G$, then the lattice subgroup $L \subseteq \Gamma$ contains a subgroup $N < \Gamma$ such that $\Gamma/N$ is a Bieberbach group of dimension at most $n(G)$.

The Vasquez invariant has an equivalent geometric reformulation.

**Theorem 1.2** (Vasquez, [24, Theorem 4.1]). For any finite group $G$, there exists a number $n(G) \in \mathbb{N}$ minimal with the property that if $M$ is any compact flat Riemannian manifolds with holonomy group isomorphic to $G$, then $M$ is a flat toral extension of some compact flat Riemannian manifolds of dimension at most $n(G)$.

The integer $n(G)$ is called the **Vasquez invariant or Vasquez number** of $G$. In Section 4, we define the diagonal Vasquez invariant by modifying the definition of Vasquez invariant.

**Definition 1.3.** Let $G$ be an elementary abelian $2$-group. Define $n_d(G)$ to be the smallest integer with the property that if $\Gamma$ is a Bieberbach group of diagonal type with holonomy group isomorphic to $G$, then its lattice subgroup contains a normal subgroup $N$ such that $\Gamma/N$ is a Bieberbach group of diagonal type of dimension at most $n_d(G)$.

Equivalently, $n_d(G)$ can be defined as the smallest integer with the property that any flat manifold of diagonal type with holonomy group $G$, fibers over a flat manifold of diagonal type of dimension at most $n_d(G)$ with flat torus fibers. We call $n_d(G)$ the **diagonal Vasquez invariant** of $G$.

Theorem 4.5 ensures that $n_d(G) < \infty$.

Let $R$ be a ring. Recall that a **characteristic algebra** of $M$ is the subalgebra of $H^*(M,R)$ generated by the characteristic classes, e.g. Stiefel-Whitney classes, $R = \mathbb{F}_2$ or Pontryagin classes, $R = \mathbb{Z}$. Analogously to [24, Corollary 2.8], we obtain the following immediate application of the diagonal Vasquez invariant.

**Corollary 1.4.** Let $M$ be an $n$-dimensional closed flat manifold of diagonal type with holonomy group $G$. Then the characteristic algebra of $M$ vanishes in dimensions greater than $n_d(G)$.

In Section 4, we give a detailed introduction to the diagonal Vasquez invariant of finite groups. In Section 5, we estimate the diagonal Vasquez invariant by using the characteristic matrix $A_\Gamma$ constructed in Section 3 and obtain
Theorem A. For \(k \geq 2\), we have
\[
5 \cdot 2^{k-3} + 1 \geq n_d(C_2^k) \geq \begin{cases}
\frac{k(k+1)}{2} & \text{if } k \geq 2 \text{ is even,} \\
\frac{k(k+1)}{2} - 1 & \text{if } k \geq 3 \text{ is odd.}
\end{cases}
\]

Theorem B. For \(k \in \{1,2,3,4\}\), we have
\[
n_d(C_2^k) = \begin{cases}
1 & \text{if } k = 1, \\
3 & \text{if } k = 2, \\
5 & \text{if } k = 3, \\
10 & \text{if } k = 4.
\end{cases}
\]

By Remark 4.6, we know that \(n_d(G) \leq n(G)\). By [5, Section 2], \(n(G) \leq \sum_{C \in \mathcal{X}} |G : C|\) where \(\mathcal{X}\) is the set of conjugacy classes of subgroups of \(G\) of prime order. Moreover, \(n(G) = \sum_{C \in \mathcal{X}} |G : C|\) if \(G\) is a \(p\)-group (see [5, Theorem 2]). So for example, \(n(C_2^2) = 6\) whereas by Theorem 13 we have \(n_d(C_2^2) = 3\). Thus, any closed flat manifold with holonomy group \(C_2^2\) of diagonal type is a flat toral extension of a compact flat 3-manifold and hence its characteristic algebra vanishes in dimensions greater than three.

Question 1.5. We can view the diagonal Vasquez invariant of \(C_2^k\) as a function of the rank
\[
f : \mathbb{N} \to \mathbb{N} : k \mapsto n_d(C_2^k).
\]
By Theorem A we know that \(f\) is not linear. Is \(f\) quadratic?

In Section 6 we turn our attention to the diffuseness property of Bieberbach groups. Diffuseness was introduced by B. Bowditch in [1]. It is an important property of a group which is related to the Kaplansky’s Zero Divisor Conjecture and the connectivity property of the group \(C^*\)-algebra. The Zero Divisor Conjecture states that if a group \(\Gamma\) is torsion-free and \(R\) is an integral domain, then the group ring \(RG\) has no zero divisors. B. Bowditch discovered that the conjecture is true if the group \(G\) is diffuse (see [1, Proposition 1.1]). For the definition of diffuseness we refer to Section 6. Instead, we state equivalent conditions for Bieberbach groups which is a combination of several results.

Theorem 1.6 ([7, 11, 12]). Let \(\Gamma\) be a Bieberbach group. Then the following are equivalent.

(i) \(\Gamma\) is diffuse.

(ii) The kernel of the trivial representation \(C^*(G) \to \mathbb{C}\) is a connective \(C^*\)-algebra.

(iii) \(\Gamma\) is a poly-\(\mathbb{Z}\) group.
Every nontrivial subgroup of \( \Gamma \) has a nontrivial center.

Every nontrivial subgroup of \( \Gamma \) has a nontrivial first Betti number.

Evidently, diffuse Bieberbach groups are well-understood. A group is said to be non-diffuse if it is not diffuse. We use the diagonal Vasquez invariant to characterise non-diffuse Bieberbach groups of diagonal type. The below two theorems are our main results in this direction.

**Theorem C.** Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy group isomorphic to \( C_2^2 \). Then \( \Gamma \) is non-diffuse if and only if

\[
\Gamma \cong Z(\Gamma) \oplus (\mathbb{Z}^{n-k-3} \rtimes \Delta_P)
\]

where \( k = b_1(\Gamma) \) and \( \Delta_P \) is the 3-dimensional non-diffuse Hantzsche-Wendt group (also known as the Promislow group or Passman group).

We immediately obtain the following complete characterisation.

**Corollary 1.7.** Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy group isomorphic to \( C_2^2 \). Then either \( \Gamma \) is poly-\( \mathbb{Z} \) or \( \Gamma \cong Z(\Gamma) \oplus (\mathbb{Z}^{n-k-3} \rtimes \Delta_P) \).

For general diagonal holonomy, we have

**Theorem D.** Let \( \Gamma \) be a non-diffuse Bieberbach group of diagonal type. Then there exists \( \Gamma' \leq \Gamma \) and a normal poly-\( \mathbb{Z} \) subgroup \( N \leq \Gamma' \), such that \( \Delta_P \cong \Gamma'/N \). In addition, if \( \Gamma \) is a non-diffuse generalized Hantzsche-Wendt group, then \( \Delta_P \leq \Gamma \).

\( \Delta_P \) is the fundamental group of the flat 3-manifold called didicosm by John Conway [6]. It is isomorphic to the Fibonacci group \( F(2, 6) \). Recently, in [8], Giles Gardam showed that \( \Delta_P \) is a counterexample to the Kaplansky’s Unit Conjecture; namely, the group ring of \( \Delta_P \) over the field of two elements \( \mathbb{F}_2 \) contains a nontrivial unit. Since the Unit Conjecture is known to hold for all poly-\( \mathbb{Z} \) groups [10], combining Gardam’s counterexample with Corollary 1.7 and Theorem D, one immediately obtains the following corollary.

**Corollary 1.8.** Let \( G \) be either a Hantzsche-Wendt group or a Bieberbach group of diagonal type with holonomy group isomorphic to \( C_2^2 \). Then \( G \) satisfies the Unit Conjecture over the field \( \mathbb{F}_2 \) if and only if \( G \) is poly-\( \mathbb{Z} \).

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2. Preliminaries on Bieberbach groups

We begin this section by stating the first and third Bieberbach Theorems.

**Theorem 2.1** (First Bieberbach Theorem, [21]). Let \( \Gamma \) be an \( n \)-dimensional crystallographic group. Then \( \Gamma \cap \mathbb{R}^n \) is a torsion-free and finitely generated abelian group of rank \( n \), and is a maximal abelian, normal subgroup of finite index.

**Theorem 2.2** (Third Bieberbach Theorem, [21]). Let \( \Gamma_1 \) and \( \Gamma_2 \) be \( n \)-dimensional crystallographic groups. Then \( \Gamma_1 \) is isomorphic to \( \Gamma_2 \) if and only if there exists \( \alpha \in \text{GL}_n(\mathbb{R}) \rtimes \mathbb{R}^n \) such that \( \alpha \Gamma_1 \alpha^{-1} = \Gamma_2 \).

**Remark 2.3.** Let \( \gamma \) be an element in \( \text{Isom}(\mathbb{E}^n) \cong O(n) \rtimes \mathbb{R}^n \) defined by \( x \mapsto Ax + a \) where \( x \in \mathbb{R}^n \), \( A \in O(n) \) and \( a \in \mathbb{R}^n \). We can view \( \gamma \) either as a tuple \((A,a)\) or as an \((n+1) \times (n+1)\)-matrix \(
\begin{pmatrix}
A & a \\
0 & 1
\end{pmatrix}
\).

Let \( \Gamma \) be an \( n \)-dimensional crystallographic group of diagonal type. By The First Bieberbach Theorem, \( \Gamma \) fits into the following short exact sequence

\[
0 \longrightarrow \mathbb{Z}^n \overset{\iota}{\longrightarrow} \Gamma \overset{p}{\longrightarrow} G \longrightarrow 1
\]

where \( G \) is a finite group, \( \iota : \mathbb{Z}^n \hookrightarrow \Gamma \) is the inclusion map defined by \( e_i \mapsto (I_n, e_i) \) where \( e_1, \ldots, e_n \) are the standard basis of \( \mathbb{Z}^n \) and \( p : \Gamma \to G \) is the projection map defined by \((A,a) \mapsto A \). Given such a short exact sequence, it induces a representation \( \rho : G \to \text{GL}_n(\mathbb{Z}) \) given by \( \rho(g)x = \bar{g}\iota(x)\bar{g}^{-1} \), where \( x \in \mathbb{Z}^n \) and \( p(\bar{g}) = g \). Thus, we can view \( \mathbb{Z}^n \) as a \( \mathbb{Z}G \)-module.

As an immediate consequence of diagonality of the holonomy representation, it follows that \( G \cong C_2^k \) for some \( k \geq 1 \). Thus, we have

\[
0 \longrightarrow \mathbb{Z}^n \overset{\iota}{\longrightarrow} \Gamma \overset{p}{\longrightarrow} C_2^k \longrightarrow 1
\]

(1)

Since \( C_2^k \) is acting diagonally on \( \mathbb{Z}^n \), invoking the third Bieberbach Theorem and by conjugating \( \Gamma \) with a suitable element in \( \text{GL}_n(\mathbb{R}) \rtimes \mathbb{R}^n \), we can assume \( \rho(g) \) is a diagonal matrix with all diagonal entries equal to \( \pm 1 \) for all \( g \in C_2^k \). In other words, we can assume
\{e_1, \ldots, e_n\} to be a set of basis of \(\mathbb{Z}^n\) such that \(g \cdot e_i = \pm e_i\) for all \(i \in \{1, \ldots, n\}\) and for all \(g \in C_2^k\).

Throughout the rest of this paper, if \(\Gamma\) is a crystallographic group of diagonal type, we will assume the holonomy group acts diagonally on the lattice subgroup. We denote

\[
\text{diag}(a_1, \ldots, a_n) = \begin{pmatrix}
    a_1 & 0 & \cdots & 0 \\
    0 & \ddots & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots \\
    0 & 0 & \cdots & a_n
\end{pmatrix}
\]

Let \(\omega \in H^2(C_2^k; \mathbb{Z}^n)\) be the cohomology class defining the short exact sequence (1). Since \(C_2^k\) acts diagonally on \(\mathbb{Z}^n\), we have the isomorphism

\[
H^2(C_2^k; \mathbb{Z}^n) \cong H^2(C_2^k; M_1 \oplus \cdots \oplus M_n) \cong H^2(C_2^k; M_1) \oplus \cdots \oplus H^2(C_2^k; M_n)
\]

where \(M_j \cong \mathbb{Z}\) and we can identify \(\omega\) with \((\omega_1, \ldots, \omega_n)\), where \(\omega_j \in H^2(C_2^k; M_j)\) for \(j = 1, \ldots, n\). The action \(C_2^k\) on \(M_j\) extends to \(\mathbb{R}\) and we obtain the following short exact sequence of \(C_2^k\)-modules

\[
0 \to M_j \to \mathbb{R} \to \mathbb{R}/M_j \to 0
\]

By the corresponding long exact sequence in cohomology, we have

\[
H^2(C_2^k; M_j) \cong H^1(C_2^k; \mathbb{R}/M_j)
\]

where under the isomorphism \(\omega_j\) is identified with an element

\[
[\alpha_j] \in H^1(C_2^k; \mathbb{R}/M_j) \cong \text{Der}(C_2^k, \mathbb{R}/M_j)/\text{P}(C_2^k, \mathbb{R}/M_j)
\]

where for each \(j = 1, \ldots, n\),

\[
(3) \quad \text{Der}(C_2^k, \mathbb{R}/M_j) = \{f : C_2^k \to \mathbb{R}/M_j \mid \forall x, y \in C_2^k, f(xy) = x \cdot f(y) + f(x)\}
\]

and

\[
(4) \quad \text{P}(C_2^k, \mathbb{R}/M_j) = \{f : C_2^k \to \mathbb{R}/M_j \mid \exists m \in \mathbb{R}/\mathbb{Z}, \forall x \in C_2^k, f(x) = x \cdot m - m\}
\]

**Lemma 2.4.** Using the same notation as above, for any \(j \in \{1, \ldots, n\}\), there is a representative \(\beta_j \in [\alpha_j]\) such that \(\beta_j(g) \in \{[0], [\frac{1}{2}]\}\) for all \(g \in C_2^k\).

**Proof.** Since \(\alpha_j\) is a derivation, we have \(\alpha_j(1) = 0\), for all \(j = 1, \ldots, n\). First, we assume \(C_2^k\) acts trivially on \(\mathbb{R}/M_j\). It follows that \(\text{P}(C_2^k, \mathbb{R}/M_j)\) is trivial. For any \(g \in C_2^k\), by (3), we have

\[
0 = \alpha_j(1) = \alpha_j(gg) = g \cdot \alpha_j(g) + \alpha_j(g) = 2\alpha_j(g)
\]

It follows that for all representatives \(\beta_j \in [\alpha_j]\), we have \(\beta_j(g) \in \{[0], [\frac{1}{2}]\}\) for all \(g \in C_2^k\). Next, we assume \(C_2^k\) acts non-trivially on \(\mathbb{R}/M_j\). Let \(g_1, \ldots, g_k\) be generators of \(C_2^k\) and assume without loss of generality that \(g_1\) acts non-trivially on \(\mathbb{R}/M_j\) and \(g_i\) acts trivially
on $\mathbb{R}/M_j$ for all $i = 2, \ldots, k$. Let $\beta_j \in \text{Der}(C^k_2, \mathbb{R}/M_j)$ be a derivation defined by $\beta_j(g_1) = 0$ and $\beta_j(g) = \alpha_j(g)$ for all $g \in \langle g_2, \ldots, g_k \rangle$. We check that $\beta_j$ is indeed a derivation, since for all $g \in \langle g_2, \ldots, g_k \rangle$, we have
\[
0 = \beta_j(1) = \beta_j(gg) = g \cdot \beta_j(g) + \beta_j(g) = 2\beta_j(g)
\]
and
\[
\beta_j(g_1g) = g_1 \cdot \beta_j(g) + \beta_j(g_1) = -\beta_j(g)
\]
Since $C^k_2 = g\langle g_2, \ldots, g_k \rangle \sqcup \langle g_2, \ldots, g_k \rangle$, it follows that $\beta_j(g) \in \{[0], [1]_2\}$ for all $g \in C^k_2$. It remains to show that $\beta_j$ and $\alpha_j$ are in the same cohomology class. For all $g \in \langle g_2, \ldots, g_k \rangle$, we have
\[
\beta_j(g) = \alpha_j(g)
\]
and
\[
\beta_j(g_1g) - \alpha_j(g_1g) = g_1 \cdot \beta_j(g) + \beta_j(g_1) - g_1 \cdot \alpha_j(g) - \alpha_j(g_1)
\]
\[
= -\beta_j(g) + \beta_j(g_1) + \alpha_j(g) - \alpha_j(g_1)
\]
\[
= -\alpha_j(g_1)
\]
Thus we have
\[
(\beta_j - \alpha_j)(g) = \begin{cases} 0 & \text{if } g \text{ acts trivially on } \mathbb{R}/M_j, \\ -\alpha_j(g_1) & \text{if } g \text{ acts non-trivially on } \mathbb{R}/M_j. \end{cases}
\]
Hence we have $(\beta_j - \alpha_j)(g) = g \cdot \left(\frac{\alpha_j(g_1)}{2}\right) - \frac{\alpha_j(g_1)}{2}$ for all $g \in C^k_2$. It follows that $\beta_j - \alpha_j \in P(C^k_2, \mathbb{R}/M_j)$, which finishes the claim. \qed

**Remark 2.5.** By the above lemma, we may assume $\alpha_j(g) \in \{[0], [1]_2\}$ for all $j \in \{1, \ldots, n\}$ and for all $g \in C^k_2$ and thus $\alpha_j$ has order 1 or 2. By a slight abuse of notation, we consider $\alpha_j(g) \in \{0, 1\}$ under the identification of $\{[0], [1]_2\}$ with its lift $\{0, 1\} \subseteq \mathbb{Q}$.

Let $q : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the natural projection. We say $s : C^k_2 \to \mathbb{R}^n$ is a *vector system* for $\Gamma$ if $q \circ s : C^k_2 \to \mathbb{R}^n/\mathbb{Z}^n$ is a derivation. For a vector system $s$ for $\Gamma$, by [16, Section 3], there are isomorphisms
\[
\Gamma \cong \left\{ \begin{pmatrix} \rho(g) & s(g) + z \\ 0 & 1 \end{pmatrix} \right\} \quad g \in C^k_2, \ z \in \mathbb{Z}^n
\]
and
\[
\Gamma \cong \left\{ \begin{pmatrix} \rho(g) & s(g) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} I_n & e_i \\ 0 & 1 \end{pmatrix} \right\} \quad g \in C^k_2, \ i \in \{1, \ldots, n\}.
\]
where $I_n$ is the $n$-dimensional identity matrix and $e_i$ is the $i^{th}$ column of $I_n$ and the image of $\alpha_j$ is \{0, $\frac{1}{2}$\} for all $j = 1, \ldots, n$. Thus, we may take
\[
\alpha = (\alpha_1, \ldots, \alpha_n) : C^k_2 \to \{0, \frac{1}{2}\}^n
\]
to be a vector system for $\Gamma$. So any $\gamma \in \Gamma$ can be expressed as
\[
\gamma = (\text{diag}(a_1, \ldots, a_n), (x_1, \ldots, x_n))
\]
where $a_1, \ldots, a_n \in \{-1, 1\}$ and $x_1, \ldots, x_n \in \{\frac{a}{2} \mid a \in \mathbb{Z}\}$. Moreover, \{\ell(e_1), \ldots, \ell(e_n), \gamma_1, \ldots, \gamma_k\} is a generating set for $\Gamma$ where $\gamma_i = \begin{pmatrix} \rho(g_i) & \alpha(g_i) \\ 0 & 1 \end{pmatrix}$. We call \{\gamma_1, \ldots, \gamma_k\} the set of non-lattice generators for $\Gamma$.

3. Combinatorics of Bieberbach groups of diagonal type

In this section, for each $n$-dimensional crystallographic group of diagonal type with holonomy group $C^k_2$, we define a $(2^k - 1) \times n$-matrix which gives a complete combinatorial description of the crystallographic group of diagonal type.

Let $S^1$ be the unit circle in $\mathbb{C}$. We consider the elements $g_i \in \text{Aut}(S^1)$ given by
\[
g_0(z) = z, \quad g_1(z) = -z, \quad g_2(z) = \bar{z}, \quad g_3(z) = -\bar{z}
\]
for all $z \in S^1$.

Equivalently, we can identify $S^1$ with $\mathbb{R}/\mathbb{Z}$. For any $[t] \in \mathbb{R}/\mathbb{Z}$, we have
\[
g_0([t]) = [t], \quad g_1([t]) = [t + \frac{1}{2}], \quad g_2([t]) = [-t], \quad g_3([t]) = [-t + \frac{1}{2}]
\]
Let $\mathcal{D} = \langle g_i \mid i = 0, 1, 2, 3 \rangle$. It is easy to check that
\[
g_3 = g_1g_2, \quad g_i^2 = g_0 \quad \text{and} \quad g_2g_0 = g_0g_2 = g_i
\]
for $i = 1, 2, 3$. Notice that $\mathcal{D}$ is isomorphic to the Klein four-group. We define an action of $\mathcal{D}^n$ on $T^n$ by
\[
(t_1, \ldots, t_n)(z_1, \ldots, z_n) = (t_1z_1, \ldots, t_nz_n)
\]
for $(t_1, \ldots, t_n) \in \mathcal{D}^n$ and $(z_1, \ldots, z_n) \in T^n = S^1 \times \cdots \times S^1$. Any subgroup $\mathbb{Z}_2 \subseteq \mathcal{D}^n$ defines a $1 \times n$-row vector with entries in $\mathcal{D}$. We define a row vector with entries in the set $\{0, 1, 2, 3\}$ under the identification $i \leftrightarrow g_i$ for $0 \leq i \leq 3$.

Let $\Gamma$ be an $n$-dimensional crystallographic group of diagonal type and let $\omega \in H^2(C^k_2; \mathbb{Z}^n)$ be the cohomology class corresponding to standard extension of $\Gamma$. As mentioned at Section 2 the class $\omega$ corresponds to
\[
[\alpha] = [(\alpha_1, \ldots, \alpha_n)] \in H^1(C^k_2; \mathbb{R}/M_1) \oplus \cdots \oplus H^1(C^k_2; \mathbb{R}/M_n)
\]
where $[\alpha] \in H^1(C^k_2; \mathbb{R}^n/\mathbb{Z}^n)$ and $M_j \cong \mathbb{Z}$ for $j = 1, \ldots, n$. Let $g \in C^k_2$ be a non-identity element and $\rho : C^k_2 \to \mathbb{Z}^n$ be the holonomy representation of $\Gamma$. We have $\rho(g) = diag(X_1, \ldots, X_n)$ and $\alpha(g) = (\alpha_1(g), \ldots, \alpha_n(g))^T = (x_1, \ldots, x_n)^T$ where $X_j \in \{1, -1\}$ and $x_j \in \{0, \frac{1}{2}\}$ for $j = 1, \ldots, n$. The corresponding element of $D^n$ is an $n$-tuple $(t_1, \ldots, t_n) \in D^n$ defined by

$$t_j([t]) = [X_j t + x_j]$$

where $t \in \mathbb{R}$ and $j \in \{1, \ldots, n\}$. We define $A_\Gamma(g, M_j) = t_j \in \{0, 1, 2, 3\}$ where $j \in \{1, \ldots, n\}$ under the identification $i \leftrightarrow g$, for $0 \leq i \leq 3$. In other words, we have

$$(7) \quad A_\Gamma(g, M_j) = \begin{cases} 
0 & \text{if } \rho(g)_{j,j} = 1 \text{ and } \alpha_j(g) = 0, \\
1 & \text{if } \rho(g)_{j,j} = 1 \text{ and } \alpha_j(g) = \frac{1}{2}, \\
2 & \text{if } \rho(g)_{j,j} = -1 \text{ and } \alpha_j(g) = 0, \\
3 & \text{if } \rho(g)_{j,j} = -1 \text{ and } \alpha_j(g) = \frac{1}{2}.
\end{cases}$$

Denote by $h_1, \ldots, h_{2^k-1}$ all the non-identity elements of $C^k_2$. We define a $(2^n - 1) \times n$-dimensional matrix $A_\Gamma$ as $(A_\Gamma)_{i,j} = A_\Gamma(h_i, M_j)$.

Note that given $\Gamma$, the matrix $A_\Gamma$ is not unique since we could re-index the holonomy group elements $h_i$ and the modules $M_j$. We say that $A_\Gamma$ is a characteristic matrix of $\Gamma$.

Let $r_1 = (a_1 \cdots a_n)$ and $r_2 = (b_1 \cdots b_n)$ be two rows of $A_\Gamma$. We denote by $\star$ the “sum” corresponding to the multiplication $[\{]$. In other words, we define $a \star b = c$ if $g_a g_b = g_c$ and define $r_1 \star r_2 = (a_1 \cdot b_1 \cdots a_n \cdot b_n)$.

**Lemma 3.1.** Let $\Gamma$ be an $n$-dimensional Bieberbach group of diagonal type. Let

$$[\alpha] = [(\alpha_1, \ldots, \alpha_n)] \in H^1(C^k_2; \mathbb{R}/M_1) \oplus \cdots \oplus H^1(C^k_2; \mathbb{R}/M_n)$$

where $M_j \cong \mathbb{Z}$ for all $j = 1, \ldots, n$ be the cohomology class corresponding to standard extension of $\Gamma$ and let $h_1, h_2, h_3 \in C^k_2$. If $h_1 = h_2 h_3$, then for any $j \in \{1, \ldots, n\}$, we have

$$A_\Gamma(h_1, M_j) = A_\Gamma(h_2, M_j) \star A_\Gamma(h_3, M_j)$$

**Proof.** Let $\rho : C^k_2 \to GL_n(\mathbb{R})$ be the holonomy representation of $\Gamma$. Let $\rho(h_2) = diag(X_1, \ldots, X_n)$, $\alpha(h_2) = (x_1, \ldots, x_n)$, $\rho(h_3) = diag(Y_1, \ldots, Y_n)$ and $\alpha(h_3) = (y_1, \ldots, y_n)$. We have

$$\begin{pmatrix}
X_1 & 0 & x_1 \\
\vdots & \ddots & \vdots \\
0 & X_n & x_n \\
0 & \cdots & 0 & 1
\end{pmatrix} \begin{pmatrix}
Y_1 & 0 & y_1 \\
\vdots & \ddots & \vdots \\
0 & Y_n & y_n \\
0 & \cdots & 0 & 1
\end{pmatrix} = \begin{pmatrix}
X_1 Y_1 & 0 & X_1 y_1 + x_1 \\
\vdots & \ddots & \vdots \\
0 & X_n Y_n & X_n y_n + x_n \\
0 & \cdots & 0 & 1
\end{pmatrix}$$
Fix $j \in \{1, ..., n\}$, the entry $A_{\Gamma}(h_1, M_j)$ is defined by

$$g[t] = [X_jY_jt + X_jy_j + x_j] = [X_j(Y_jt + y_j) + x_j]$$

where $t \in \mathbb{R}$ and $g \in D$. Therefore, we have

$$A_{\Gamma}(h_1, M_j) = A_{\Gamma}(h_2, M_j) \ast A_{\Gamma}(h_3, M_j).$$

□

**Remark 3.2.** Using the same notation as above, assume $C^k_2$ is generated by the elements $h_1, ..., h_k$. There is a matrix $A$ obtained from $A_{\Gamma}$ by deleting some of its rows. More specifically, define a $k \times n$-submatrix $A$ of $A_{\Gamma}$ by $A_i,j = A_{\Gamma}(h_i, M_j)$ where $1 \leq i \leq k$ and $1 \leq j \leq n$. By Lemma 3.1, the matrix $A_{\Gamma}$ is the matrix generated by the rows of $A$ using the $\ast$ operation. The matrix $A$ is the same as the matrix constructed in [15, Section 2].

Next, we would like to reverse the above construction. Namely, given a $k \times n$-dimensional matrix $A$ with entries in $\{0, 1, 2, 3\}$, we are going to define an $n$-dimensional crystallographic group $\widetilde{\Gamma}_A$ of diagonal type.

Let $\widetilde{A}$ be a $(2^k - 1) \times n$-dimensional characteristic matrix generated by rows of $A$. Denote by $g_1, ..., g_{2^k - 1}$ all non-identity elements of $C^k_2$ and assume $g_1, ..., g_k$ are the generators of $C^k_2$. First, we need to define a representation $\rho : C^k_2 \to \text{GL}_n(\mathbb{Z})$. For any $1 \leq i \leq k$, we define $\rho(g_i) = \text{diag}(X_1, ..., X_n)$ where

$$X_j = \begin{cases} 1 & \text{if } (\widetilde{A})_{ij} \in \{0, 1\}, \\ -1 & \text{if } (\widetilde{A})_{ij} \in \{2, 3\}, \end{cases}$$

for all $1 \leq j \leq n$. Next, we are going to define a cohomology class

$$[\alpha] \in H^1(C^k_2; (\mathbb{R}/\mathbb{Z})^n)$$

where the $C^k_2$-module structure on $(\mathbb{R}/\mathbb{Z})^n$ is induced by $\rho$. For any $1 \leq i \leq k$, we define $\alpha(g_i) = (s_1, ..., s_n)$ where

$$s_j = \begin{cases} 0 & \text{if } (\widetilde{A})_{ij} \in \{0, 2\}, \\ \frac{1}{2} & \text{if } (\widetilde{A})_{ij} \in \{1, 3\}, \end{cases}$$

for all $1 \leq j \leq n$. We define an $n$-dimensional crystallographic group of diagonal type $\widetilde{\Gamma}_A$ corresponding to the cohomology class $[\alpha]$. By [5], $\widetilde{\Gamma}_A$ is generated by $\{\iota(e_1), ..., \iota(e_n), \gamma_1, ..., \gamma_k\}$ where

$$\iota(e_j) = \begin{pmatrix} I_n & e_j \\ 0 & 1 \end{pmatrix}$$
and $e_j$ is the $j^{th}$ column of the $n$-dimensional identity matrix for $1 \leq j \leq n$ and

$$
\gamma_i = \begin{pmatrix} \rho(g_i) & \alpha(g_i) \\ 0 & 1 \end{pmatrix}
$$

for $1 \leq i \leq k$. By the construction, we can see that the characteristic matrix of $\Gamma_A$ equals to $\tilde{A}$. Notice that the holonomy group of $\Gamma_{\tilde{A}}$ is not necessary isomorphic to $C_2^k$ because the representation $\rho : C_2^k \to \text{GL}_n(\mathbb{Z})$ is not necessary faithful. Conversely, again by construction, we can see that $\Gamma_{A'} \cong \Gamma$.

If two characteristic matrices define isomorphic crystallographic groups, we will say that they are equivalent. In particular, the matrix obtained from swapping rows or columns of $A_{\Gamma}$ is equivalent to $A_{\Gamma}$.

**Example 3.3.** Let $\Gamma$ be the Bieberbach group enumerated in CARAT as “min.19.1.1.7”. Let

$$
\gamma_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

be non-lattice generators of $\Gamma$. The holonomy group of $\Gamma$ is $C_2^2$ which is generated by $h_1 = p(\gamma_1)$ and $h_2 = p(\gamma_2)$ where $p : \Gamma \to C_2^2$ be the projection defined in (1). Using the same notations as in Remark 3.2, we have

$$
A = \begin{pmatrix} 2 & 2 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}
$$

We obtain the third row of $A_{\Gamma}$ by using Lemma 3.1

$$
\left( \begin{array}{cccc} 2 & 2 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{array} \right) \ast \left( \begin{array}{cccc} 1 & 0 & 2 & 2 \end{array} \right) = \left( \begin{array}{c} 3 \\ 2 \\ 3 \\ 1 \end{array} \right)
$$

Thus

$$
A_{\Gamma} = \begin{pmatrix} 2 & 2 & 1 & 3 \\ 1 & 0 & 2 & 2 \\ 3 & 2 & 3 & 1 \end{pmatrix}
$$

A simple calculation shows that

$$
\gamma_1 \gamma_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & \frac{1}{2} - 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

Therefore we verify that the third row of $A_{\Gamma}$ is indeed equal to $(3 \ 2 \ 3 \ 1)$. 
Example 3.4. Let
\[ \tilde{A} = \begin{pmatrix} 2 & 1 & 2 & 3 \\ 1 & 2 & 0 & 2 \\ 3 & 3 & 2 & 1 \end{pmatrix} \]
be a matrix generated by its first two rows. We are going to define its associated 4-dimensional crystallographic group of diagonal type. We can assume \( g_1 \) and \( g_2 \) generate \( C_2^2 \). First, we define a representation \( \rho : C_2^2 \to \text{GL}_4(\mathbb{Z}) \) where
\[ \rho(g_1) = \text{diag}(-1, 1, -1, -1) \quad \text{and} \quad \rho(g_2) = \text{diag}(1, -1, 1, -1) \]
Next, we define the cohomology class \([\alpha] \in H^1(C_2^2, \mathbb{R}/\mathbb{Z})^4\) by
\[ \alpha(g_1) = (0, \frac{1}{2}, 0, \frac{1}{2}) \quad \text{and} \quad \alpha(g_2) = (\frac{1}{2}, 0, 0, 0) \]
Thus the characteristic matrix \( \tilde{A} \) defines a 4-dimensional crystallographic group \( \Gamma_{\tilde{A}} \) where its non-lattice generators are
\[ \gamma'_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma'_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]
Comparing the group \( \Gamma_{\tilde{A}} \) with \( \Gamma \) defined in Example 3.3, observe that we have
\[ \gamma'_i = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_i = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]
for \( i = 1, 2 \) and \( \gamma_i \) are the elements defined in Example 3.3. So \( \Gamma \cong \Gamma_{\tilde{A}} \) and hence \( \tilde{A} \) is equivalent to \( A_{\Gamma} \). This result is not surprising because \( \tilde{A} \) can be obtained by swapping the second and the third columns of \( A_{\Gamma} \).

Next, we derive some properties of the matrix \( A_{\Gamma} \).

Lemma 3.5. Let \( \Gamma \) be a crystallographic group of diagonal type and let
\[ [(\alpha_1, \ldots, \alpha_n)] \in \bigoplus_{1 \leq i \leq n} H^1(C_2^k, \mathbb{R}/M_i) \]
be the cohomology class corresponding to the standard extension of \( \Gamma \) where \( M_i \cong \mathbb{Z} \) for all \( i = 1, \ldots, n \). Then \( A_{\Gamma}(h, M_j) = 1 \) if and only if \( 0 \neq \text{res}_{[h]}[\alpha_j] \in H^1(C_2; \mathbb{R}/M_j) \).
Proof. Let $\Gamma'$ be the group corresponding to $\text{res}_{(h)}[\alpha_j] \in H^1(C_2; \mathbb{R}/M_j)$. It then fits into the exact sequence

$$0 \to M_j \cong \mathbb{Z} \to \Gamma' \xrightarrow{p} \langle h \rangle \cong C_2.$$  

Observe that $\text{res}_{(h)}[\alpha_j] \not= [0]$ if and only if $\Gamma' \cong \mathbb{Z}$.

Let $\rho : C_2^k \to \text{GL}(\mathbb{Z}^n)$ be the holonomy representation of $\Gamma$ and recall that $(\alpha_1, \ldots, \alpha_n) : C_2^k \to \{0, \frac{1}{2}\}^n$ is the chosen vector system for $\Gamma$. Since $\Gamma$ is a crystallographic group of diagonal type, we can decompose $\rho$ as $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ such that $\rho(h) = \text{diag}(\rho_1(h), \ldots, \rho_n(h))$. By (5), we have

$$\Gamma' \cong \left\langle \begin{pmatrix} \rho_j(h) & \alpha_j(h) \\ 0 & \alpha_j(h) \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle.$$  

If $(\rho_j(h), \alpha_j(h)) = (1, \frac{1}{2})$, then clearly $\Gamma' \cong \mathbb{Z}$.

Suppose now that $\Gamma' \cong \mathbb{Z}$, but $(\rho_j(h), \alpha_j(h)) \not= (1, \frac{1}{2})$. Then, $(\rho_j(h), \alpha_j(h)) = (1, 0)$. There is $q \in \{1, \ldots, n\}$, such that $h$ acts nontrivially on $M_q$. Since $\text{res}_{(h)}[\alpha_j] \not= [0]$, it follows that $\text{res}_{(h)}[\alpha_j \oplus \alpha_q] \not= [0]$. By (5), the crystallographic group $\Gamma''$ corresponding to $\text{res}_{(h)}[\alpha_j \oplus \alpha_q]$, contains the element

$$\begin{pmatrix} \rho_q(h) & 0 & \alpha_q(h) \\ 0 & \rho_j(h) & \alpha_j(h) \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & \alpha_q(h) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is of order 2. This implies that $\text{res}_{(h)}[\alpha_j \oplus \alpha_q] = [0]$, which is a contradiction.

Hence, we conclude that $\Gamma' \cong \mathbb{Z}$ if and only if $\text{A}_\Gamma(h, M_j) = 1$. \hfill \Box

Lemma 3.6. Let $A$ be a $k \times n$-dimensional matrix with entries in $\{0, 1, 2, 3\}$ and $\tilde{A}$ be the $(2^k - 1) \times n$-dimensional matrix generated by $A$. Then

(i) Then $\Gamma_{\tilde{A}}$ has a torsion element if and only if $\tilde{A}$ has a row where all entries are not equals to 1.

(ii) The holonomy group of $\Gamma_{\tilde{A}}$ is not isomorphic to $C_2^k$ if and only if $\tilde{A}$ has a row where every entry is either equal to 0 or 1.

Proof. By construction, $\tilde{A}$ defines a cohomology class

$$[\alpha] = [(\alpha_1, \ldots, \alpha_n)] \in \bigoplus_{1 \leq j \leq n} H^1(C_2^k; \mathbb{R}/M_j).$$
where $M_j \cong \mathbb{Z}$ for all $j = 1, ..., n$. By [21, Theorem 3.1], $\Gamma$ has a torsion element if and only if there exists $g \in C^k_2$ such that

$$res(g)[\alpha] = res(g)[\alpha_1] \oplus \cdots \oplus res(g)[\alpha_n] = 0.$$ 

Hence, $\Gamma_{\tilde{A}}$ has a torsion element if and only if $res(g)[\alpha_j] = 0$ for all $j = 1, ..., n$. By Lemma 3.5, we can conclude that $\Gamma_{\tilde{A}}$ has a torsion element if and only if $(\tilde{A})_{ij} \neq 1$ for all $j = 1, ..., n$, where the $i$-th row corresponds to the element $g$.

Next, we note that the holonomy group of $\Gamma_{\tilde{A}}$ is not isomorphic to $C^k_2$ if and only if there exists $g \in C^k_2$ such that $g$ acts trivially on $\mathbb{R}^n/\mathbb{Z}^n$. By construction, $g$ acts trivially on $\mathbb{R}^n/\mathbb{Z}^n$ if and only if $(\tilde{A})_{ij} \in \{0, 1\}$ for all $j = 1, ..., n$, where the $i$-th row corresponds to the element $g$. □

**Proposition 3.7.** Let $\Gamma$ be an $n$-dimensional Bieberbach group of diagonal type with holonomy group $C^k_2$. Then every row of $A_\Gamma$ has an entry equal to 1 and an entry equal to either 2 or 3.

**Proof.** The proof follows immediately from Lemma 3.6 and the discussion after Remark 3.2. □

Let $\omega \in H^2(C^k_2; \mathbb{Z})$. It will be convenient for computations to define

$$\mathcal{R}(\omega) = \{ g \in C^k_2 \mid res(g)(\omega) \neq 0 \}.$$ 

**Remark 3.8.** Let $\Gamma$ be an $n$-dimensional crystallographic group with holonomy $C^k_2$. Let $\omega \in H^2(C^k_2; \mathbb{Z}^n)$ be the cohomology class corresponding to standard extension of $\Gamma$. Note that $\omega = (\omega_1, \ldots, \omega_n)$ where $\omega_j \in H^2(C^k_2; M_j)$ and $M_j \cong \mathbb{Z}$ for all $j = 1, ..., n$. By [22] and Lemma 3.5 for any $j \in \{1, ..., n\}$, we have $A_\Gamma(g, M_j) = 1$ if and only if $g \in \mathcal{R}(\omega_j)$.

**Proposition 3.9.** Let $0 \neq \omega \in H^2(C^k_2; \mathbb{Z})$ where $C^k_2$ acts trivially on $\mathbb{Z}$. Then we have $|\mathcal{R}(\omega)| = 2^{k-1}$.

**Proof.** Under the isomorphism discussed in Section 2, $\omega$ corresponds to $[\alpha] \in H^1(C^k_2; \mathbb{R}/\mathbb{Z})$. By Lemma 2.4, we can assume $\alpha(g) \in \{0, \frac{1}{2}\}$ for all $g \in C^k_2$. Thus

$$|\mathcal{R}(\omega)| = \left| \left\{ g \in C^k_2 \mid \alpha(g) = \frac{1}{2} \right\} \right|.$$ 

Since $[\alpha] \neq 0$, there exists $g \in C^k_2$ such that $\alpha(g) = \frac{1}{2}$. Let $C^k_2 = H \sqcup gH$ where $H \leq C^k_2$ and $H \cong C^{k-1}_2$. For any $h \in H$, we have

$$\alpha(gh) = \alpha(g) + \alpha(h) = \begin{cases} 0 & \text{if } \alpha(h) = \frac{1}{2}, \\ \frac{1}{2} & \text{if } \alpha(h) = 0. \end{cases}$$
Proposition 3.12. Let \( |R| \neq 0 \leq H^2(C_2^k; \mathbb{Z}) \) where \( C_2^k \) acts non-trivially on \( \mathbb{Z} \) via \( \rho : C_2^k \to \text{GL}(\mathbb{Z}) \). Then \( |R(\beta)| = 2^{k-2} \).

Proof. Define \( \omega = \text{res}_{\ker(\rho)}(\beta) \in H^2(\ker(\rho); \mathbb{Z}) \). Since \( \beta \neq 0 \) and \( H^2((g); \mathbb{Z}) = 0 \) for all \( g \notin \ker(\rho) \), it follows that \( \omega \neq 0 \) and
\[
|\mathcal{R}(\beta)| = |\{ g \in C_2^k | \text{res}(g)(\beta) \neq 0 \}| = |\{ h \in \ker(\rho) \cong C_2^{k-1} | \text{res}(h)(\omega) \neq 0 \}| = |\mathcal{R}(\omega)|
\]

By Proposition 3.9 we have \( |\mathcal{R}(\beta)| = 2^{k-2} \).

Remark 3.11. Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy \( C_2^k \). Let
\[
\omega_1 \oplus \cdots \oplus \omega_n \in \bigoplus_{1 \leq j \leq n} H^2(C_2^k; M_j)
\]
be the cohomology class corresponding to standard extension of \( \Gamma \), where \( M_j \cong \mathbb{Z} \) for all \( j = 1, \ldots, n \). By Proposition 3.9 and Proposition 3.10, for any \( j \in \{1, \ldots, n\} \), we have \( |\mathcal{R}(\omega_j)| = 2^{k-2} \) or \( 2^{k-1} \). By Remark 3.8 we can conclude that in every column of \( A_\Gamma \), there exist at least \( 2^{k-2} \) entries equal to 1.

Proposition 3.12. Let \( \omega \in H^2(C_2^k; \mathbb{Z}) \) where \( C_2^k \) acts non-trivially on \( \mathbb{Z} \) via \( \rho : C_2^k \to \text{GL}(\mathbb{Z}) \). If \( \mathcal{T} \subseteq \mathcal{R}(\omega) \) with \( |\mathcal{T}| \geq 2^{k-3} + 1 \), then \( \langle \mathcal{T} \rangle = \langle \mathcal{R}(\omega) \rangle = \ker(\rho) \).

Proof. Since \( \mathcal{T} \subseteq \mathcal{R}(\omega) \subseteq \ker(\rho) \), we have \( \langle \mathcal{T} \rangle \leq \ker(\rho) \). We assume by contradiction that \( \langle \mathcal{T} \rangle \subsetneq \ker(\rho) \). Since \( |\mathcal{T}| \geq 2^{k-3} + 1 \), we have \( \langle \mathcal{T} \rangle \cong C_2^{k-2} \). Consider \( \theta = \text{res}(\mathcal{T})(\omega) \in H^2(\langle \mathcal{T} \rangle; \mathbb{Z}) \). Recall that \( \mathcal{R}(\theta) = \{ h \in \langle \mathcal{T} \rangle | \text{res}(h)(\theta) \neq 0 \} \). By Proposition 3.9 we have \( |\mathcal{R}(\theta)| = 2^{k-3} \). Since \( \mathcal{T} \subseteq \mathcal{R}(\theta) \), we have
\[
2^{k-3} + 1 \leq |\mathcal{T}| \leq |\mathcal{R}(\theta)| = 2^{k-3}
\]
which is a contradiction.

Corollary 3.13. Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy \( C_2^k \). Let
\[
\omega_1 \oplus \cdots \oplus \omega_n \in H^2(C_2^k; M_1) \oplus \cdots \oplus H^2(C_2^k; M_n)
\]
be the cohomology class corresponding to standard extension of \( \Gamma \) where \( M_z \cong \mathbb{Z} \) for \( z = 1, \ldots, n \). Let \( \rho_z : C_2^k \to \text{GL}(M_z) \) be the representations given by the \( C_2^k \)-action on \( M_z \) for all \( z = 1, \ldots, n \). If there exists \( i, j \in \{1, \ldots, n\} \) such that \( \rho_i \) and \( \rho_j \) are non-trivial representations and there exists a subset \( \mathcal{T} \subseteq \mathcal{R}(\omega_i) \cap \mathcal{R}(\omega_j) \) such that \( |\mathcal{T}| \geq 2^{k-3} + 1 \), then \( \mathcal{R}(\omega_i) = \mathcal{R}(\omega_j) \).
Lemma 4.3. Let \( i, j \in \{1, \ldots, n\} \) such that \( \rho_i \) and \( \rho_j \) are non-trivial representation and there exists a subset \( \mathcal{T} \subseteq \mathcal{R}(\omega_i) \cap \mathcal{R}(\omega_j) \) such that \(|\mathcal{T}| \geq 2^k + 1\). By Proposition 3.12, we have \( \ker(\rho_i) = \langle \mathcal{T} \rangle = \ker(\rho_j) \). Since \( \mathcal{R}(\omega_i) \subseteq \ker(\rho_i) = \langle \mathcal{T} \rangle \) and \( \mathcal{R}(\omega_j) \subseteq \ker(\rho_j) = \langle \mathcal{T} \rangle \), every element belongs to \( \mathcal{R}(\omega_i) \cup \mathcal{R}(\omega_j) \) can be expressed as a combination of elements of \( \mathcal{T} \).

Proof. Let \( x \) be a \( G \)-lattice where \( \mathcal{R}(\omega_i) \cap \mathcal{R}(\omega_j) \) such that \(|\mathcal{T}| \geq 2^k + 1\). By Proposition 3.12, we have \( \ker(\rho_i) = \langle \mathcal{T} \rangle = \ker(\rho_j) \). Since \( \mathcal{R}(\omega_i) \subseteq \ker(\rho_i) = \langle \mathcal{T} \rangle \) and \( \mathcal{R}(\omega_j) \subseteq \ker(\rho_j) = \langle \mathcal{T} \rangle \), every element belongs to \( \mathcal{R}(\omega_i) \cup \mathcal{R}(\omega_j) \) can be expressed as a combination of elements of \( \mathcal{T} \).

4. Vasquez invariant of diagonal type

In this section, we give an alternative definition of the Vasquez invariant for Bieberbach groups of diagonal type which is more suited for our purposes.

Definition 4.1. A \( G \)-lattice is any \( \mathbb{Z}G \)-module whose underlying abelian group is isomorphic to a free abelian group \( \mathbb{Z}^n \) for some \( n \geq 1 \). Let \( M \) be a \( G \)-lattice where \( \{e_1, \ldots, e_n\} \) is a generating set of \( M \). We say \( M \) is a diagonal \( G \)-lattice if \( g \cdot e_i = \pm e_i \) for all \( g \in G \) and \( i \in \{1, \ldots, n\} \). We say \( M \) is a faithful \( G \)-lattice if \( G \) acts faithfully on \( M \).

By [22] Theorem 3, there is a different way to define the Vasquez invariant of finite groups.

Definition 4.2. Let \( G \) be a finite group and let \( L \) be a \( G \)-lattice. An element \( \omega \in H^2(G; L) \) is said to be special if its extension defines a Bieberbach group. The \( G \)-lattice \( L \) is said to have property \( S \) if for any \( G \)-lattice \( M \) and any special element \( \omega \in H^2(G; M) \), there exists a \( G \)-homomorphism \( f : M \to L \) such that \( f^* : H^2(G; M) \to H^2(G; L) \) maps \( \omega \) to another special element \( f^*(\omega) \in H^2(G; L) \). The Vasquez invariant of a finite group \( G \) is then

\[
\text{val}(G) = \text{min}\{\text{rank}_\mathbb{Z}(L) \mid L \text{ is } G\text{-lattice with property } S\}.
\]

Lemma 4.3. Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type and let \( \omega \in H^2(G; \mathbb{Z}^n) \) be the cohomology class corresponding to the standard extension of \( \Gamma \). Let \( f : \mathbb{Z}^n \to M \) be a \( G \)-epimorphism such that \( f^*(\omega) \) is special. Then \( f^*(\omega) \) defines a Bieberbach group of diagonal type.

Proof. Since \( \Gamma \) is a Bieberbach group of diagonal type, we can assume \( \mathbb{Z}^n \) is a diagonal \( G \)-lattice where \( \{e_1, \ldots, e_n\} \) is a basis of \( \mathbb{Z}^n \) such that \( g \cdot e_i = \pm e_i \) for all \( g \in G \) and \( i = 1, \ldots, n \). Let \( M_\mathbb{R} = M \otimes \mathbb{R} \). Then \( \{e_1, \ldots, e_n\} \) form a basis for \( \mathbb{R}^n \). Denote by \( f_\mathbb{R} : \mathbb{R}^n \to M_\mathbb{R} \) the epimorphism induced by \( f \). Since the set \( \{f_\mathbb{R}(e_1), \ldots, f_\mathbb{R}(e_n)\} \) spans \( M_\mathbb{R} \), there exists
a subset \( \{ f_\mathbb{R}(e_{i_1}), ..., f_\mathbb{R}(e_{i_n}) \} \) that forms a basis for \( M_\mathbb{R} \). The holonomy representation of the Bieberbach group defined by \( f^*(\omega) \in H^2(G; M) \) is given by the \( G \)-action on \( M \). Thus, it suffices to check that \( G \) acts diagonally on \( \{ f_\mathbb{R}(e_{i_1}), ..., f_\mathbb{R}(e_{i_n}) \} \), for in this case the holonomy representation of \( G \) on \( M \) is conjugate in \( \text{GL}_d(\mathbb{R}) \) to a diagonal one. Since \( f_\mathbb{R} \) is a module homomorphism, for all \( g \in G \) and for all \( i \in \{1, ..., n\} \) we have
\[
g \cdot f_\mathbb{R}(e_i) = f_\mathbb{R}(g \cdot e_i) = f_\mathbb{R}(\pm e_i) = \pm f_\mathbb{R}(e_i)
\]
Hence, \( f^*(\omega) \) defines a Bieberbach group of diagonal type. \( \square \)

As an immediate application we obtain the following.

**Proposition 4.4.** Let \( \Gamma \) be a Bieberbach group of diagonal type and let \( N \) be normal subgroup of its lattice subgroup. If \( \Gamma/N \) is a Bieberbach group, then it is of diagonal type.

**Theorem 4.5.** Let \( G \) be an elementary abelian 2-group. If \( \Gamma \) is a Bieberbach group of diagonal type where its holonomy group is isomorphic to \( G \), then the lattice subgroup \( L \subseteq \Gamma \) contains a normal subgroup \( N \) such that \( \Gamma/N \) is a Bieberbach group of diagonal type of dimension at most \( n_d(G) \).

**Proof.** The result follows from Proposition 4.4 and Theorem 1.1. \( \square \)

**Remark 4.6.** Combining Proposition 4.4 and Theorem 1.1, we have that \( n_d(G) \leq n(G) \).

**Remark 4.7.** Let \( \Gamma \) be a Bieberbach group of diagonal type. Let \( \omega \in H^2(G; L) \) be the cohomology class that defines \( \Gamma \) where \( G \) is an elementary abelian 2-group and \( L \) is a diagonal faithful \( G \)-lattice. By Theorem 4.5 there exists a normal subgroup \( N \subseteq L \) such that \( \Gamma/N \) is a Bieberbach group of diagonal type with dimensional at most \( n_d(G) \). In other words, we can define a \( G \)-homomorphism \( f : L \rightarrow L/N \) such that \( f^* : H^2(G; L) \rightarrow H^2(G; L/N) \) maps \( \omega \) to another special element \( f^*(\omega) \) defining a Bieberbach group of diagonal type of dimension at most \( n_d(G) \).

**Definition 4.8.** Let \( G \) be an elementary abelian 2-group and let \( L \) be a diagonal \( G \)-lattice. An element \( \omega \in H^2(G; L) \) is said to be a diagonal special element if its extension defines a Bieberbach group of diagonal type. We say a diagonal \( G \)-lattice \( L \) has property \( S_d \) if for any diagonal \( G \)-lattice \( M \) and any diagonal special element \( \omega \in H^2(G; M) \), there exists a \( G \)-homomorphism \( f : M \rightarrow L \) such that \( f^* : H^2(G; M) \rightarrow H^2(G; L) \) maps \( \omega \) to another diagonal special element \( f^*(\omega) \).
Theorem 4.9. Let $G$ be an elementary abelian 2-group. Define 
\[ n'_d(G) = \min \{ \text{rank}_Z(L) \mid L \text{ is a diagonal } \mathbb{Z}G\text{-lattice with property } S_d \} \]
Then we have $n_d(G) = n'_d(G)$.

Proof. By definition, it is clear that $n_d(G) \leq n'_d(G)$. Now we want to prove that $n'_d(G) \leq n_d(G)$. Let $L$ be a diagonal $G$-lattice of minimal rank with property $S_d$. In other words, $\text{rank}_Z(L) = n'_d(G)$. Let $M$ be any diagonal $G$-lattice and $\omega \in H^2(G; M)$ be any diagonal special element. Since $L$ has property $S_d$ and by Definition 4.8 there exists a $G$-homomorphism $g : M \to L$ such that $g^*(\omega) \in H^2(G; L)$ is a diagonal special element.

First, we assume $L$ is a faithful diagonal $G$-lattice. Since $L$ is faithful, by Remark 4.7 there exists a diagonal $G$-lattice $K$ with $\text{rank}_Z(K) \leq n_d(G)$ and a $G$-homomorphism $h : L \to K$ such that $h^*(g^*(\omega))$ is a diagonal special element. Hence $K$ is a diagonal $G$-lattice with property $S_d$. It follows that $n'_d(G) \leq \text{rank}_Z(K)$. Therefore we have $n'_d(G) \leq \text{rank}_Z(K) \leq n_d(G)$.

Now assume $L$ is not a faithful diagonal $G$-lattice. Let $P$ be a faithful diagonal $G$-lattice. Consider the faithful diagonal $G$-lattice $L \oplus P$. We have $(g^*(\omega), 0) \in H^2(G; L) \oplus H^2(G; P)$ is a diagonal special element. By Remark 4.7 there exists a diagonal $G$-lattice $N$ with $\text{rank}_Z(N) \leq n_d(G)$ and a $G$-homomorphism $f : L \oplus P \to N$ such that $f^*((g^*(\omega), 0)) \in H^2(G; N)$ is a special element. Since $\text{Hom}_G(L \oplus P, N) \cong \text{Hom}_G(L, N) \oplus \text{Hom}_G(P, N)$, we can let $f = f_1 \oplus f_2$ where $f_1 \in \text{Hom}_G(L, N)$ and $f_2 \in \text{Hom}_G(P, N)$. Therefore, $f^*((g^*(\omega), 0)) = (f_1)^*(g^*(\omega))$. Thus $N$ is a diagonal $G$-lattice with property $S_d$. It follows that $n'_d(G) \leq \text{rank}_Z(N)$. Hence, we have $n'_d(G) \leq n_d(G)$. \qed

5. Proofs of Theorems A and B

In this section, given a Bieberbach group $\Gamma$ of diagonal type, we analyse the characteristic matrix $A_\Gamma$ to determine whether there exists a normal subgroup such that the quotient is still a Bieberbach group.

Definition 5.1. Let $\Gamma$ be an $n$-dimensional Bieberbach group of diagonal type. The characteristic matrix $A_\Gamma$ is said to be col-reducible (by $i^{th}$ column) if after removing a column ($i^{th}$ column) from $A_\Gamma$, there still exists an entry equal to 1 in every row. We say $A_\Gamma$ is col-irreducible if it is not col-reducible.

Lemma 5.2. Let $f : N \to M$ be a $C_2^N$-homomorphism where $N = M_1 \oplus \cdots \oplus M_n$ and $M_1, \ldots, M_n$ are all $C_2^k$-lattices of rank one. Let $e_i$ be a generator of $M_i$ and $\rho_i : C_2^k \to$
GL($M_i$) be the representation defining the $C^k_2$-action on $M_i$ for all $1 \leq i \leq n$. For $i \in \{1, ..., n\}$, if there exists $b \neq 0$, $t \geq 2$, and a linearly independent set $\{f(e_{i1}), \ldots, f(e_{it})\}$ such that

$$bf(e_i) = a_{i1}f(e_{i1}) + \cdots + a_{it}f(e_{it})$$

where $a_{i1}, ..., a_{it} \neq 0$, then $\ker(\rho_i) = \cdots = \ker(\rho_i)$.

**Proof.** Assuming the hypothesis, for any $g \in C^k_2$ that acts trivially on $M_i$, we have

$$a_{i1}f(e_{i1}) + \cdots + a_{it}f(e_{it}) = bf(e_i) = bf(g \cdot e_i) = g \cdot bf(e_i) = \sum_{z=1}^{t} a_{iz}g \cdot f(e_{iz}) = \sum_{z=1}^{t} a_{iz}f(g \cdot e_{iz}).$$

Since $g \cdot f(e_{iz}) = \pm f(e_{iz})$, this shows that $f(g \cdot e_{iz}) = f(e_{iz})$ for all $z \in \{1, ..., t\}$. It follows that $g \in \ker(\rho_{iz})$ for all $z \in \{1, ..., t\}$. For each $h \in C^k_2$ that acts non-trivially on $M_i$, by similar calculation, we get

$$-a_{i1}f(e_{i1}) - \cdots - a_{it}f(e_{it}) = -bf(e_i) = f(h \cdot be_i) = h \cdot bf(e_i) = \sum_{z=1}^{t} a_{iz}h \cdot f(e_{iz}) = \sum_{z=1}^{t} a_{iz}f(h \cdot e_{iz}).$$

It follows that $f(h \cdot e_{iz}) = f(-e_{iz})$ for all $z \in \{1, ..., t\}$. Therefore $h \not\in \ker(\rho_{iz})$ for all $z \in \{1, ..., t\}$. Hence we can conclude that $\ker(\rho_{i1}) = \cdots = \ker(\rho_{it})$. □

**Corollary 5.3.** Let $\Gamma$ be an $n$-dimensional Bieberbach group of diagonal type with holonomy $C^k_2$ and let

$$\omega = \omega_1 \oplus \cdots \oplus \omega_n \in H^2(C^k_2; M_i) \oplus \cdots \oplus H^2(C^k_2; M_n)$$

be the corresponding cohomology class where $M_i \cong \mathbb{Z}$ for all $i = 1, ..., n$. Let $\rho_i : C^k_2 \to \text{GL}(M_i)$ be the representation given by the $C^k_2$-action on $M_i$ for all $1 \leq i \leq n$. If $\ker(\rho_i) \neq \ker(\rho_j)$ for all $i \neq j$ and $A_{\Gamma}$ is col-irreducible, then there does not exist a $C^k_2$-homomorphism $f : \mathbb{Z}^n \to \mathbb{Z}^s$ where $s < n$ such that $f^*(\omega) \in H^2(G; \mathbb{Z}^s)$ is special.

**Proof.** Assume by contradiction that there exists a $C^k_2$-homomorphism $f : \mathbb{Z}^n \to \mathbb{Z}^s$ where $s < n$ such that $f^*(\omega) = f^*(\omega_1) \oplus \cdots \oplus f^*(\omega_n)$ is special. Let $e_i$ be the generator of $M_i$ for each $1 \leq i \leq n$. Since $\{f(e_1), \ldots, f(e_n)\}$ spans $f(\mathbb{Z}^n)$, there is a linear independent subset $\{f(e_{i1}), \ldots, f(e_{it})\}$ that spans a finite index sublattice in $f(\mathbb{Z}^n)$.

Suppose $f(e_i) = 0$ for some $i$. Then $f$ factors through the projection $\pi_i : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ mapping $e_i$ to 0. Denote the resulting homomorphism $f_i : \mathbb{Z}^{n-1} \to \mathbb{Z}^s$. Let $\Gamma_i$ be the group defined by $\pi_i^*(\omega) \in H^2(G; \mathbb{Z}^{n-1})$. Let $\Gamma'$ be the Bieberbach group defined by $f^*(\omega)$. Note that the homomorphism from $\Gamma$ to $\Gamma'$ defined by $f$ factors through the homomorphism $\psi_i : \Gamma_i \to \Gamma'$ defined by $f_i$. Since $\Gamma'$ is torsion-free and $\ker(\psi_i) = \ker(f_i)$, it follows that $\Gamma_i$ is torsion-free and hence a Bieberbach group (of diagonal type). By Proposition 3.7, there
exists an entry equal to 1 in every row of $A_{\Gamma_i}$. But $A_{\Gamma_i}$ is the matrix obtained from $A_\Gamma$ by deleting the $i$-th column. This is a contradiction to the fact that $A_\Gamma$ is col-irreducible.

It follows that for each $i$, $f(e_i) \neq 0$. So, there exists $b_i \neq 0$ and nonzero $a_{i_1}, \ldots, a_{i_t} \in \mathbb{Z}$ such that $b_i f(e_i) = a_{i_1} f(e_{s_1}) + \cdots + a_{i_t} f(e_{s_t})$. By Lemma 5.2, if $t \geq 2$, we have $\ker(\rho_{s_1}) = \cdots = \ker(\rho_{s_t})$, which contradicts the fact that the kernels of $\rho_i$ are all distinct for $1 \leq i \leq n$. It follows that for each $1 \leq i \leq n$, there exists $b_i \neq 0$, such that $b_i f(e_i) = a_{i'} f(e_{i'}) \neq 0$ for some $i' \in \{s_1, \ldots, s_t\}$ and $a_{i'} \neq 0$. Since $t \leq s < n$, there exists $1 \leq i \leq n$, so that $i \neq i'$. For any $g \in G$, we have

$$\pm b_i f(e_i) = b_i f(g \cdot e_i) = a_{i'} f(g \cdot e_{i'}) = \pm a_{i'} f(e_{i'}).$$

This shows that $\ker(\rho_i) = \ker(\rho_{i'})$, which is a contradiction. \hfill \Box

**Definition 5.4.** Let $M$ be an $m \times n$-matrix with entries in $\{0, 1, 2, 3\}$. Define the map $\phi : \{0, 1, 2, 3\} \to \{p, q\}$ such that $\phi(0) = \phi(1) = p$ and $\phi(2) = \phi(3) = q$. Define $\Phi(M)$ to be the $m \times n$-matrix such that $[\Phi(M)]_{ij} = \phi(M_{ij})$.

**Lemma 5.5.** Let $A$ be a $k \times n$-dimensional matrix with entries in $\{0, 1, 2, 3\}$. Let $\Gamma_A$ be the crystallographic group of diagonal type obtained by $A$. Let

$$\omega_1 \oplus \cdots \oplus \omega_n \in H^2(C_2^k; M_1) \oplus \cdots \oplus H^2(C_2^k; M_n)$$

be the cohomology class corresponding to standard extension of $\Gamma_A$. Let $C_2^k$ acts on $M_\Gamma \equiv C_2 \to GL(\mathbb{Z})$ for $1 \leq z \leq n$. Then the $i^{th}$ column of $\Phi(A)$ is equal to the $j^{th}$ column of $\Phi(A)$ if and only if $\ker(\rho_i) = \ker(\rho_j)$.

**Proof.** Let $g_1, \ldots, g_k$ be the generators of $C_2^k$. Assume without loss of generality, we have $A_{ij} = A_\Gamma(g_i, M_j)$. If $\ker(\rho_i) \neq \ker(\rho_j)$, then there exists $r \in \{1, \ldots, k\}$ such that $g_r$ acts differently on $M_i$ from $M_j$. We may assume $g_r$ acts trivially on $M_i$ and acts non-trivially on $M_j$. By equation (7), we have $A_\Gamma(g_r, M_i) \in \{0, 1\}$ and $A_\Gamma(g_r, M_j) \in \{2, 3\}$. Thus by definition, we get $\Phi(A)_{ri} = \Phi(A_\Gamma(g_r, M_i)) = p$ and $\Phi(A)_{rj} = \Phi(A_\Gamma(g_r, M_j)) = q$. Therefore the $i^{th}$ column of $\Phi(A)$ is not equal to the $j^{th}$ column of $\Phi(A)$.

Now, we assume the the $s^{th}$ column of $\Phi(A)$ is not equal to the $i^{th}$ column of $\Phi(A)$. Without loss of generality, we may assume there exists $r \in \{1, \ldots, k\}$ such that $\Phi(A)_{rs} = \Phi(A_\Gamma(g_r, M_s)) = p$ and $\Phi(A)_{ri} = \Phi(A_\Gamma(g_r, M_i)) = q$. Hence $A_\Gamma(g_r, M_i) \in \{0, 1\}$ and $A_\Gamma(g_r, M_s) \in \{2, 3\}$. By equation (7), the holonomy group element $g_r$ acts trivially on $M_s$ and non-trivially on $M_i$. Thus we have $\ker(\rho_s) \neq \ker(\rho_i)$. \hfill \Box

**Proposition 5.6.** Let $\Gamma$ be an $n$-dimensional Bieberbach group of diagonal type with holonomy $C_2^k$ and let $A_{\Gamma}$ be its characteristic matrix. The matrix $A_{\Gamma}$ is col-irreducible if and
only if \( A_{\Gamma} \) can be transformed to a matrix \( Y = \begin{pmatrix} X & N \end{pmatrix} \) by swapping rows and columns of \( A_{\Gamma} \), where \( X \) is an \( n \times n \)-matrix with all diagonal entries equal to 1 and all other entries not equal to 1.

**Proof.** First, we assume \( A_{\Gamma} \) can be transformed to \( Y \) and we want to prove that \( A_{\Gamma} \) is col-irreducible. It is sufficient to show that \( Y \) is col-irreducible. For any \( i \in \{1, \ldots, n\} \), if we remove the \( i^{th} \) column of \( Y \), then the \( i^{th} \) row of the new matrix will not have an entry equal to 1. Hence, we conclude that \( Y \) is col-irreducible.

Next, we assume \( A_{\Gamma} \) is col-irreducible. For any \( i \in \{1, \ldots, n\} \), we consider the \( i^{th} \) column of \( A_{\Gamma} \). By definition of col-irreducibility, if we remove the \( i^{th} \) column of \( A_{\Gamma} \), there is \( r_i \in \{1, \ldots, 2^k - 1\} \) such that the \( r_i^{th} \) row of the new matrix does not have entries equal to 1. By Proposition 3.7, the \( r_i^{th} \) row of \( A_{\Gamma} \) has at least one entry equal to 1. Therefore we can conclude that \( (A_{\Gamma})^{r_i}_{,i} = 1 \) and \( (A_{\Gamma})^{r_i}_{,s} \neq 1 \) for all \( s \neq i \). Notice that we have \( r_i \neq r_j \) for all \( i \neq j \). We obtain a new matrix \( A'_{\Gamma} \) by swapping the rows of \( A_{\Gamma} \) such that the \( i^{th} \) row of \( A'_{\Gamma} \) equals to the \( r_i^{th} \) row of \( A_{\Gamma} \). Hence \( A'_{\Gamma} \) can be transformed to \( Y \). \( \square \)

**Lemma 5.7.** Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy \( C^k_2 \) and let \( \Gamma \cap \mathbb{R}^n = \langle e_1, \ldots, e_n \rangle \cong \mathbb{Z}^n \). If \( A_{\Gamma} \) is col-reducible by \( i^{th} \) column, then \( \Gamma / \langle e_i \rangle \) is a Bieberbach group of diagonal type.

**Proof.** We define a \( C^k_2 \)-homomorphism \( f : \mathbb{Z}^n \to \mathbb{Z}^{n-1} \) such that \( f(e_i) = 0 \) and \( f(e_j) = e_j \) for all \( j \neq i \). Let 

\[
\omega = \omega_1 \oplus \cdots \oplus \omega_n \in H^2(C^k_2; M_1) \oplus \cdots \oplus H^2(C^k_2; M_n)
\]

be the cohomology class corresponding to standard extension of \( \Gamma \) where \( M_i \cong \mathbb{Z} \) for all \( i = 1, \ldots, n \). We have 

\[
f^*(\omega) = \omega_1 \oplus \cdots \oplus \omega_{i-1} \oplus \omega_{i+1} \oplus \cdots \oplus \omega_n
\]

and \( f^*(\omega) \) defines the Bieberbach group \( \Gamma / \langle e_i \rangle \). The characteristic matrix corresponding to \( \Gamma / \langle e_i \rangle \) can be obtained by removing the \( i^{th} \) column of \( A_{\Gamma} \). Since \( A_{\Gamma} \) is col-reducible by \( i^{th} \) column, every row of \( A_{\Gamma / \langle e_i \rangle} \) has at least one entry equal to 1. By Lemma 3.6, \( \Gamma / \langle e_i \rangle \) is a Bieberbach group. By Lemma 4.3, \( \Gamma / \langle e_i \rangle \) is a Bieberbach group of diagonal type. \( \square \)

In the next two propositions, we obtain upper and lower bounds for the diagonal Vasquez number.

**Proposition 5.8.** Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy \( C^k_2 \). If \( n > 5 \cdot 2^{k-3} + 1 \) and \( k \geq 2 \), then \( A_{\Gamma} \) is col-reducible.
Proof. First, we consider the case where \( k = 2 \). Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy \( C_2^k \) and \( n \geq 4 \). Consider its characteristic matrix \( A_\Gamma \). The matrix \( A_\Gamma \) has 3 rows and at least 4 columns. Thus it cannot be col-irreducible by Proposition 5.6. It follows that \( A_\Gamma \) is col-reducible.

Now, assume \( k \geq 3 \) and let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy \( C_2^k \) and \( n \geq 5 \cdot 2^{k-3} + 2 \). Let

\[ \omega_1 \oplus \cdots \oplus \omega_n \in H^2(C_2^k; M_1) \oplus \cdots \oplus H^2(C_2^k; M_n) \]

be the cohomology class corresponding to the standard extension of \( \Gamma \), where \( M_i \cong \mathbb{Z} \) for all \( i = 1, \ldots, n \). Assume by contradiction that \( A_\Gamma \) is col-irreducible. By Proposition 5.6, we assume \( A_\Gamma = \begin{pmatrix} X \\ N \end{pmatrix} \) where \( X \) is an \( n \times n \)-matrix with all diagonal entries equal to 1 and all the others not equal to 1 and \( N \) is a matrix with \( 2^k - 1 - n \) rows. Since \( C_2^k \) acts faithfully on \( \mathbb{Z}^n \), there exists \( i, j \in \{1, \ldots, n\} \) such that \( C_2^k \) acts non-trivially on both \( M_i \) and \( M_j \) where \( i \neq j \). Consider the \( i^{th} \) and \( j^{th} \) columns of \( N \). By Proposition 3.10, the \( i^{th} \) and \( j^{th} \) columns of \( N \) each have exactly \( 2^{k-2} - 1 \) entries equal to 1. Since \( k \geq 3 \), this ensures that the \( i^{th} \) and \( j^{th} \) columns of \( N \) each have at least one entry equal to 1. Define

\[ z = |\{m \in \{1, \ldots, 2^k - 1 - n\} | N_{m,i} = N_{m,j} = 1\}| \]

and observe that we have

\[ 2^{k-2} - 1 - z = |\{m \in \{1, \ldots, 2^k - 1 - n\} | N_{m,i} = 1, N_{m,j} \neq 1\}| \]

and

\[ 2^{k-2} - 1 - z = |\{m \in \{1, \ldots, 2^k - 1 - n\} | N_{m,i} \neq 1, N_{m,j} = 1\}| \]

Since \( N \) has \( 2^k - 1 - n \) rows, we have

\[ 2(2^{k-2} - 1 - z) + z \leq 2^k - 1 - n \]

By rearranging the above inequality, we get

\[ n - 2^{k-1} - 1 \leq z \]

Since \( n \geq 5 \cdot 2^{k-3} + 2 \), it follows that

\[ z \geq 5 \cdot 2^{k-3} + 2 - 2^{k-1} - 1 = 2^{k-3} + 1 \]

Thus

\[ |\{g \in C_2^k | A_\Gamma(g, M_i) = A_\Gamma(g, M_j) = 1\}| \geq 2^{k-3} + 1 \]

By Corollary 3.13 we have \( \mathcal{R}(\omega_i) = \mathcal{R}(\omega_j) \) and hence by Remark 3.8

\[ \{g \in C_2^k | A_\Gamma(g, M_i) = 1\} = \{g \in C_2^k | A_\Gamma(g, M_j) = 1\}. \]

This is a contradiction to the fact that all non-diagonal entries in \( X \) are not equal to 1. \( \square \)
In the next two propositions, we obtain upper and lower bounds for the diagonal Vasquez number.

**Proposition 5.9.** For each \( k \geq 2 \), we have \( n_d(C^k_2) \leq 5 \cdot 2^{k-3} + 1 \).

*Proof.* We proceed by induction on \( k \). First, consider the base case where \( k = 2 \). Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy \( C^2_2 \) and \( n \geq 4 \). We claim that there exists a \( C^2_2 \)-homomorphism \( f : \mathbb{Z}^n \to \mathbb{Z}^s \) where \( s \leq 3 \) such that \( f^*(\omega) \) is a special element, where \( \omega \in H^2(C^2_2; \mathbb{Z}^n) \) is the cohomology class corresponding to standard extension of \( \Gamma \). If the claim is true, then \( n_d(C^2_2) \leq 3 \).

We proceed by induction on \( n \). First, assume \( \Gamma \) is 4-dimensional. By Proposition 5.8, its characteristic matrix \( A_\Gamma \) is col-reducible. Thus by Lemma 5.7, there exists a \( C^2_2 \)-homomorphism \( f : \mathbb{Z}^t \to \mathbb{Z}^{t-1} \) such that \( f^*(\omega) \) defines a Bieberbach group. If \( f^*(\omega) \) defines a Bieberbach group where its holonomy group is a proper subgroup of \( C^2_2 \). Since \( n_d(C_2) = 1 \), there must exist a \( C^2_2 \)-homomorphism \( g : \mathbb{Z}^t \to \mathbb{Z} \) such that \( g^*(f^*(\omega)) \) is a special element. Thus the base case where \( k = 2 \) holds.

We assume now that the statement is true for \( k \leq q - 1 \) and consider the case when \( k = q \geq 3 \). Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy \( C^q_2 \) and \( n > 5 \cdot 2^{q-3} + 1 \). By a similar induction method as above, we can show that there exists a \( C^q_2 \)-homomorphism \( f : \mathbb{Z}^n \to \mathbb{Z}^s \) where \( s \leq 5 \cdot 2^{q-3} + 1 \) such that \( f^*(\omega) \) defines a Bieberbach group of dimension at most \( 5 \cdot 2^{q-3} + 1 \), where \( \omega \in H^2(C^q_2; \mathbb{Z}^n) \) is the cohomology class corresponding to the standard extension of \( \Gamma \). Thus we have \( n_d(C^q_2) \leq 5 \cdot 2^{q-3} + 1 \). \( \square \)

**Proposition 5.10.** Set \( a = \frac{k(k-1)}{2} \). For each \( k \geq 2 \), we have

\[
\begin{cases} 
\quad \quad k + a & \text{if } k \text{ is even,} \\
\quad \quad k + a - 1 & \text{if } k \text{ is odd.} 
\end{cases}
\]

*Proof.* We will consider the cases where \( k \) is even and \( k \) is odd separately.

**Case:** \( k \) is even. First, we assume \( k \) is even. It suffices to construct a characteristic matrix \( A \) and show that it defines a \((k + a)\)-dimensional Bieberbach group \( \Gamma = \Gamma_A \) of diagonal
type with \( \omega \) the cohomology class corresponding to standard extension of \( \Gamma \) such that there does not exist a \( C^k_2 \)-homomorphism \( f \) with \( f^*(\omega) \) defining a smaller dimensional Bieberbach group.

To this end, define a \( k \times k \)-matrix \( Q \) such that
\[
Q_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
2 & \text{if } i \neq j,
\end{cases}
\]
for \( 1 \leq i \leq k \) and \( 1 \leq j \leq k \). Let \( S = \{(x, y) \in \{1, \ldots, k\} \times \{1, \ldots, k\} \mid x < y\} \). It is easy to see that \( |S| = a \). Enumerate the elements \( S \) by \( s_j = (s^{(1)}_j, s^{(2)}_j) \) for \( 1 \leq j \leq a \) using the lexicographic order on the pairs. Define a \( k \times a \)-matrix \( N \) such that
\[
N_{ij} = \begin{cases} 
2 & \text{if } i = s^{(1)}_j, \\
3 & \text{if } i = s^{(2)}_j, \\
0 & \text{otherwise},
\end{cases}
\]
where \( 1 \leq i \leq k \) and \( 1 \leq j \leq a \). In other words, fix \( j \in \{1, \ldots, a\} \) and consider the \( j^{th} \) column of \( N \). The \( (s^{(1)}_j)^{th} \) entry of the \( j^{th} \) column of \( N \) is equal to 2, the \( (s^{(2)}_j)^{th} \) entry of the \( j^{th} \) column of \( N \) is equal to 3 and all other entries of the \( j^{th} \) column of \( N \) is equal to 0.

Define a \( k \times (k + a) \)-matrix \( A = \left( Q \ N \right) \) by combining \( Q \) and \( N \) together. Let \( g_1, \ldots, g_k \) be generators of \( C^k_2 \) and \( M_z \cong \mathbb{Z} \) for all \( 1 \leq z \leq k + a \). By Section 3, the matrix \( A \) defines a \( (2^k - 1) \times (k + a) \)-matrix \( \hat{A} \) and a \( (k + a) \)-dimensional crystallographic group \( \Gamma = \Gamma_{\hat{A}} \) of diagonal type so that \( A_{\Gamma}(g_i, M_j) = \hat{A}_{i,j} \) for \( 1 \leq i \leq k \) and \( 1 \leq j \leq k + a \).

For example, if we assume \( k = 2 \), then \( S = \{(1, 2)\} \). We define \( Q \), \( N \) and \( A \) as below,

\[
Q = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix}
\]

and the third row of \( A_{\Gamma} \) can be calculated by Lemma \ref{g_3.1}. We get

\[
\begin{pmatrix} r_1 \\ r_2 \\ r_1 \ast r_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix} = A_{\Gamma}
\]

Going back to the general case, we denote the \( i^{th} \) row of \( A_{\Gamma} \) to be \( r_i \).

Next, we are going to show that \( \Gamma \) is in fact a Bieberbach group with holonomy group \( C^k_2 \), by using Lemma \ref{g_3.6}. Let \( r \) be an arbitrary row of \( A_{\Gamma} \). There exists \( m \in \{1, \ldots, k\} \) and
If \( m \) is odd, then pick \( j \). There exists \( k \) such that the row can be expressed as \( r = r_{i_1} \cdots r_{i_m} \). Notice that the \( j^{th} \) column of the row \( r \) equals to \( A_\Gamma(g_{i_1} \cdots g_{i_m}, M_j) \) and

\[
A_\Gamma(g_{i_1} \cdots g_{i_m}, M_j) = \bigstar_{1 \leq z \leq m} A_\Gamma(g_{i_z}, M_j).
\]

We claim that there exist \( c_1, c_2 \in \{1, \ldots, k + a\} \) such that \( A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{c_1}) = 1 \) and \( A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{c_2}) \in \{2, 3\} \). In other words, we claim that in row \( r \) there exists an entry equal to 1 and an entry equal to either 2 or 3. By Lemma 3.6, we can conclude that \( A_\Gamma \) defines a \((k + a)\)-dimensional Bieberbach group of diagonal type where its holonomy group is isomorphic to \( C_2^k \).

Let us now prove the claim. First, the claim clearly holds for \( m = 1 \). Assume \( 2 \leq m \leq k \). There exists \( j \in \{1, \ldots, a\} \) such that \( s_j = (i_1, i_2) \in S \). Then we have

\[
A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{k+j}) = \bigstar_{z=1}^m A_\Gamma(g_{i_z}, M_{k+j}) = 0 \cdots 2 \cdots 0 \cdots 0 \cdot 3 \cdots 0 \cdots 0 = 1
\]

It remains to show that there exists an entry equal to 2 or 3 in row \( r \) where \( 2 \leq m \leq k \). If \( m \) is odd, then pick \( i \in \{1, \ldots, k\} \setminus \{i_1, \ldots, i_m\} \). Thus we have

\[
A_\Gamma(g_{i_1} \cdots g_{i_m}, M_i) = \bigstar_{z=1}^m A_\Gamma(g_{i_z}, M_i) = 2 \cdots 2 = 2
\]

If \( m \) is even, then we have

\[
A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{i_1}) = \bigstar_{z=1}^m A_\Gamma(g_{i_z}, M_{i_1}) = 2 \cdots 2 \star 1 \star 2 \cdots 2 = 3
\]

This proves the claim.

By Lemma 3.6, the matrix \( A_\Gamma \) defines a \((k + a)\)-dimensional Bieberbach group of diagonal type with holonomy group isomorphic to \( C_2^k \). Next, notice that \( A_\Gamma \) is equivalent to

\[
\begin{pmatrix}
    r_1 \\
    \vdots \\
    r_k \\
    r_{s_1} \star r_{s_1}^{(2)} \\
    \vdots \\
    r_{s_a} \star r_{s_a}^{(2)} \\
    P
\end{pmatrix} = \begin{pmatrix} X \\ P \end{pmatrix}
\]

where \( X \) is a \((k + a) \times (k + a)\)-matrix such that the diagonal entries all equals to 1 and all other entries are not equal to 1. By Proposition 5.6, we can conclude that \( A_\Gamma \) is color-irreducible. Let \( \rho_i : C_2^k \to \text{GL}(M_i) \) be the representation given by the \( C_2^k \)-action on \( M_i \) for each \( 1 \leq i \leq k + a \). Observe that the columns of \( \Phi(A_\Gamma) \) are all distinct. So, by Lemma
we have \( \ker(\rho_i) \neq \ker(\rho_j) \) for all \( i \neq j \). By Corollary 5.3, there does not exist an \( C_2^k \)-homomorphism \( f \) such that \( f^*(\omega) \) defines a smaller dimensional Bieberbach group where \( \alpha \) is the cohomology class defining \( \Gamma \). Hence we have \( n_d(C_2^k) \geq k + a \) if \( k \) is even.

**Case: \( k \) is odd.** Now, we assume \( k \) is odd. We are going to construct a matrix \( A \) and show that it defines a \((k+a-1)\)-dimensional Bieberbach group \( \Gamma = \Gamma_A \) of diagonal type such that there does not exist a \( C_2^k \)-homomorphism \( f \) such that \( f^*(\omega) \) defines a smaller dimensional Bieberbach group where \( \omega \) is the cohomology class defining \( \Gamma \). First, we assume \( k \geq 5 \). We will deal with the case \( k = 3 \) afterwards.

Define a \( k \times k \)-matrix \( Q \) where

\[
Q_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
2 & \text{if } i \neq j,
\end{cases}
\]

for \( 1 \leq i \leq k \) and \( 1 \leq j \leq k \). Let \( S = \{(x, y) \in \{1, \ldots, k\} \times \{1, \ldots, k\} | x < y \} \setminus \{(1, 2), (1, 3)\} \). It is easy to see that \( |S| = a - 2 \). Enumerate the elements of \( S \) by \( s_j = (s_j^{(1)}, s_j^{(2)}) \) for \( 1 \leq j \leq a - 2 \) using the lexicographic order on the pairs. Define a \((k \times (a - 2))\)-matrix \( N \) where

\[
N_{ij} = \begin{cases} 
2 & \text{if } i = s_j^{(1)}, \\
3 & \text{if } i = s_j^{(2)}, \\
0 & \text{otherwise},
\end{cases}
\]

Define a \( k \times (k + a - 1) \) matrix \( A \) such that

\[
A = \begin{pmatrix}
Q & N \\
2 \\
3 \\
3 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

Let \( g_1, \ldots, g_k \) be generators of \( C_2^k \) and \( M_z \cong \mathbb{Z} \) for \( 1 \leq z \leq k + a - 1 \). By Section 3 the matrix \( A \) defines a \((2^k - 1) \times (k + a - 1)\)-matrix \( A \) and a \((k + a - 1)\)-dimensional crystallographic group \( \Gamma = \Gamma_A \) of diagonal type so that \( A_{\Gamma}(g_i, M_j) = \tilde{A}_{i,j} \) for \( 1 \leq i \leq k \) and \( 1 \leq j \leq k + a - 1 \).
For example, if $k = 5$, we have

$$
A = \begin{pmatrix}
1 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
2 & 1 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 3 \\
2 & 2 & 1 & 2 & 2 & 0 & 0 & 3 & 0 & 0 & 2 & 0 \\
2 & 2 & 2 & 1 & 2 & 3 & 0 & 0 & 3 & 0 & 3 & 0 \\
2 & 2 & 2 & 2 & 1 & 0 & 3 & 0 & 0 & 3 & 3 & 0
\end{pmatrix}
$$

Going back to the general case, we denote the $i^{th}$ row of $A$ to be $r_i$.

Next, we are going to show that $\Gamma$ is a Bieberbach group with holonomy group $C_2^k$, by using Lemma 3.6. Let $r$ be an arbitrary row of $A_\Gamma$. There exists $m \in \{1, \ldots, k\}$ and $1 \leq i_1 < \ldots < i_m \leq k$ such that the row can be expressed as $r = r_{i_1} \ast \cdots \ast r_{i_m}$. Notice that the $j^{th}$ column of the row $r$ equals to $A_\Gamma(g_{i_1} \cdots g_{i_m}, M_j)$ and

$$
A_\Gamma(g_{i_1} \cdots g_{i_m}, M_j) = \bigstar_{1 \leq z \leq m} A_\Gamma(g_{i_z}, M_j)
$$

We claim that there exists $c_1, c_2 \in \{1, \ldots, k + a - 1\}$ such that $A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{c_1}) = 1$ and $A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{c_2}) \in \{2, 3\}$. In other words, we claim that on row $r$ there is an entry equal to 1 and an entry equal to either 2 or 3. By Lemma 3.6, we can conclude that $A_\Gamma$ defines a $(k + a - 1)$-dimensional Bieberbach group of diagonal type where its holonomy group is isomorphic to $C_2^k$.

First, it is clear that the claim is true for $m = 1$.

Next, we assume $2 \leq m \leq k - 1$. (i.e.,) $m \in \{(1, 2), (1, 3), (1, 2, 3)\}$, then $m = 2$ and we have

$$
A_\Gamma(g_{i_1}g_{i_2}, M_{k+a-1}) = A_\Gamma(g_{i_1}, M_{k+a-1}) \ast A_\Gamma(g_{i_2}, M_{k+a-1}) = 2 \ast 3 = 1
$$

If $(i_{m-1}, i_m) \notin \{(1, 2), (1, 3)\}$, then there exists $j \in \{1, \ldots, a - 2\}$ such that $s_j = (i_{m-1}, i_m) \in S$. Then we have

$$
A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{k+j}) = \bigstar_{z=1}^m A_\Gamma(g_{i_z}, M_{k+j}) = 0 \ast \cdots \ast 2 \ast 0 \ast \cdots \ast 0 \ast 3 \ast 0 \ast \cdots \ast 0 = 1
$$

Next, we are going to show there exists an entry equal to 2 or 3 in row $r$. If $m$ is even, we have

$$
A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{i_1}) = \bigstar_{z=1}^k A_\Gamma(g_{i_z}, M_{i_1}) = 2 \ast \cdots \ast 2 \ast 1 \ast 2 \ast \cdots \ast 2 = 3
$$

If $m = k$, then

$$
A_\Gamma(g_{i_1} \cdots g_k, M_{k+a-1}) = \bigstar_{z=1}^k A_\Gamma(g_{i_z}, M_{k+a-1}) = 2 \ast 3 \ast 3 \ast 0 \ast \cdots \ast 0 = 2
$$

If $m$ is odd and $m \neq k$, then pick $i \in \{1, \ldots, k\} - \{i_1, \ldots, i_m\}$ and we have

$$
A_\Gamma(g_{i_1} \cdots g_{i_m}, M_i) = \bigstar_{z=1}^k A_\Gamma(g_{i_z}, M_i) = 2 \ast \cdots \ast 2 = 2
$$
This finishes the claim. By Lemma 3.6, $A_{\Gamma}$ defines a $(k + a - 1)$-dimensional Bieberbach group of diagonal type $\Gamma$ where its holonomy group isomorphic to $C_2^k$.

Next, notice that $A_{\Gamma}$ is equivalent to

$$
\begin{pmatrix}
  r_1 \\
  \vdots \\
  r_k \\
  r_{s_0(1)} \star r_{s_0(2)} \\
  \vdots \\
  r_{s_{a-2}(1)} \star r_{s_{a-2}(2)} \\
  r_1 \star r_2 \\
  P
\end{pmatrix} = \begin{pmatrix} X \end{pmatrix}
$$

where $X$ is a $((k + a - 1) \times (k + a - 1))$ matrix such that the diagonal entries all equal to 1 and all other entries are not equal to 1. By Proposition 5.6, we can conclude that $A_{\Gamma}$ is col-irreducible. Let $\rho_i : C_2^k \to \text{GL}(M_i)$ be the representation given by the $C_2^k$-action on $M_i$ for all $1 \leq i \leq k + a - 1$. Observe that columns of $\Phi(A_{\Gamma})$ are all distinct.

For the case $k = 3$, we instead consider the 5-dimensional Bieberbach group $\Gamma$ and its characteristic matrix $A_{\Gamma}$ defined in Lemma 5.11 below. $A_{\Gamma}$ is col-irreducible and the columns of $\Phi(A_{\Gamma})$ are all distinct.

So, for each odd $k \geq 3$, by Lemma 5.5, we have $\rho_i \neq \rho_j$ for all $i \neq j$. By Corollary 5.3, there does not exist an $C_2^k$-homomorphism $f$ such that $f^*(\omega)$ defines a smaller dimensional Bieberbach group where $\alpha$ is the cohomology class defining $\Gamma$. Hence we have $n_d(C_2^k) \geq k + a - 1$ if $k$ is odd.

\textbf{Lemma 5.11}. \textit{There exists a 5-dimensional Bieberbach group of diagonal type $\Gamma$ where its holonomy is $C_2^5$ such that its characteristic matrix $A_{\Gamma}$ is col-irreducible and the columns of $\Phi(A_{\Gamma})$ are distinct.}

\textit{Proof}. Let $\Gamma$ be the Bieberbach group enumerated in CARAT as "min.72.1.1.502". Its non-lattice generators are

$$
\begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0 & 0 & 1/2 \\
  0 & 0 & -1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & -1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
  -1 & 0 & 0 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 1/2 \\
  0 & 0 & 0 & 1 & 0 & 1/2 \\
  0 & 0 & 0 & 0 & 1 & 1/2 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 1/2 \\
  0 & 1 & 0 & 0 & 0 & 1/2 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & -1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
$$
By easy calculation, we have

\[
A = \begin{pmatrix}
0 & 3 & 2 & 1 & 2 \\
2 & 2 & 1 & 1 & 1 \\
1 & 1 & 0 & 2 & 2 \\
2 & 1 & 3 & 0 & 3 \\
3 & 3 & 1 & 3 & 3 \\
3 & 0 & 3 & 2 & 1
\end{pmatrix}, \quad \Phi(A) = \begin{pmatrix}
p & q & q & p & q \\
q & q & p & p & p \\
p & p & p & q & q \\
p & p & q & p & q \\
q & q & p & q & q \\
q & p & q & q & p
\end{pmatrix}.
\]

By observing the matrix \(A\) and \(\Phi(A)\), we notice that \(A\) is col-irreducible and all columns of \(\Phi(A)\) are distinct.

By combining Proposition 5.9 and Proposition 5.10, we get Theorem A.

**Proof of Theorem B.**

The case \(k = 1\) is clear. Assume \(k = 2\). By Theorem A, we have \(3 \leq n_d(C_2^2) \leq \frac{7}{2}\). Thus we have \(n_d(C_2^2) = 3\).

Next, assume \(k = 3\). By Theorem A, we have \(5 \leq n_d(C_2^3) \leq 6\). It remains to show that if \(\Gamma\) is a 6-dimensional Bieberbach group of diagonal type with holonomy \(C_2^3\) and \(\omega \in H^2(C_2^3; \mathbb{Z}^6)\) is the cohomology class corresponding to standard extension of \(\Gamma\), then there exists \(f : \mathbb{Z}^6 \to \mathbb{Z}^5\) such that \(f^*(\omega)\) is special.

Assume by contradiction that there does not exist such \(f\) and hence we assume \(A\) is col-irreducible. By Proposition 5.6 we assume \(A = \begin{pmatrix} X \\ N \end{pmatrix}\) where the entries of \(X\) equal to 1 are exactly on the diagonal and \(N\) is a row matrix. By Remark 3.11 each column of \(A\) has at least two entries equal 1. This forces \(N\) to have all entries equal to 1. By Lemma 3.6 the holonomy group of \(\Gamma\) is not isomorphic to \(C_2^3\), which is a contradiction. We conclude that \(n_d(C_2^3) = 5\).

Now we assume \(k = 4\). By Theorem A, we have \(10 \leq n_d(C_2^4) \leq 11\). It remains to show that if \(\Gamma\) is a 11-dimensional Bieberbach group of diagonal type with holonomy \(C_2^4\) and \(\omega \in H^2(C_2^4; M_1 \oplus \cdots \oplus M_{11})\) where \(M_j \cong \mathbb{Z}\) for \(j = 1, \ldots, 11\) is the cohomology class corresponding to standard extension of \(\Gamma\), then there exists a \(C_2^4\)-homomorphism \(f : \mathbb{Z}^{11} \to \mathbb{Z}^{10}\) such that \(f^*(\omega)\) is special.

Assume by contradiction that there does not exist such \(f\). Then \(A\) is col-irreducible. By Proposition 5.6 we assume \(A = \begin{pmatrix} X \\ N \end{pmatrix}\) where the entries of \(X\) equal to 1 are exactly on the diagonal and \(N\) is matrix with four rows. By Remark 3.11 each column of \(A\) has
either 4 or 8 entries equal to 1. This implies that each column of $N$ has exactly three entries equal to 1. Since $N$ has eleven columns, there exist $i, j \in \{1, ..., 11\}$ such that $\left|\{g \in C^4_2 \mid A_\Gamma(g, M_i) = A_\Gamma(g, M_j) = 1\}\right| = 3$.

By Corollary 3.13 we have

$$\{g \in C^4_2 \mid A_\Gamma(g, M_i) = 1\} = \{g \in C^4_2 \mid A_\Gamma(g, M_j) = 1\}$$

It follows that $A_\Gamma$ is col-reducible, which is a contradiction. Hence $n_d(C^4_2) = 10$. □

6. DIFFUSENESS AND BIEBERBACH GROUPS OF DIAGONAL TYPE

In this section, we use the diagonal Vansquez invariant to characterise non-diffuse Bieberbach groups of diagonal type. First, we state some known results.

**Definition 6.1.** Let $G$ be a group and $A \subseteq G$ be a subset. An element $a \in A$ is an extremal point of $A$ if for all $g \in G/\{1\}$, either $ga \not\in A$ or $g^{-1}a \not\in A$. Define

$$\Delta(A) = \{a \in A \mid a \text{ is an extremal point}\}$$

The group $G$ is said to be diffuse if all finite subsets $A \subseteq G$ with $|A| \geq 2$ have $|\Delta(A)| \geq 2$. The group $G$ is said to be weakly diffuse if all finite non-empty subsets $A \subseteq G$ have $|\Delta(A)| \geq 1$.

The above definition was introduced by B. Bowditch in [1]. By [13, Proposition 6.2] of P. Linnell and D. W. Morris, a group is diffuse if and only if it is weakly diffuse.

**Remark 6.2.** Let $\Gamma$ be a group and let $N \leq \Gamma$ be a subgroup. Straight from the definition of diffuseness, it follows that if $N$ is non-diffuse, then $\Gamma$ is non-diffuse.

**Proposition 6.3** ([1, Theorem 1.2(1)]). Let $\Gamma$ be a torsion-free group and $N \trianglelefteq \Gamma$. If $N$ and $\Gamma/N$ are both diffuse, then $\Gamma$ is diffuse.

Recall that the first Betti number $b_1(\Gamma)$ of a group $\Gamma$ is defined as the (torsion-free) rank of the abelian group $H_1(\Gamma, \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$. Let $\Gamma$ be an $n$-dimensional Bieberbach group with holonomy group $G$. By [11 Corollary 1.3], we have $b_1(\Gamma) = rk((\mathbb{Z}^n)^G)$ where $G$ acts on $\mathbb{Z}^n$ via the holonomy representation. The following lemma is straightforward.

**Lemma 6.4.** Let $\Gamma$ be a group and $N \trianglelefteq \Gamma$. If $b_1(\Gamma) = 0$, then $b_1(\Gamma/N) = 0$.

In [3], using the computer programme CARAT, all Bieberbach groups of dimension at most 6 are computed and their standard presentations are given. Combining [12] and [14], gives a full list of all non-diffuse Bieberbach groups.
Definition 6.5. An $n$-dimensional closed flat manifold is called a generalized Hantzsche-Wendt (GHW) manifold if its holonomy group is isomorphic to $C_2^{n-1}$. An oriented GHW-manifold is called a Hantzsche-Wendt (HW) manifold. A fundamental group of a GHW-manifold $M$ is called a GWH-group and a HW-group if $M$ oriented.

Remark 6.6. By [20, Theorem 3.1], any generalized Hantzsche-Wendt group is a Bieberbach group of diagonal type.

Example 6.7. The Bieberbach group enumerated in CARAT as “group.32.1.1. 194” is a 4-dimensional diffuse GHW-group. Thus not all GHW-groups are non-diffuse.

By [12], there is only one non-diffuse Bieberbach group of diagonal type of dimension at most three. We denote it by $\Delta_P$. It has a presentation $\Delta_P = \langle x, y \mid x^{-1}y^2yx^2 = y^{-1}x^2yx^2 = 1 \rangle$

$\Delta_P$ is the 3-dimensional Hantzsche-Wendt group (also known as Promislow group or Passman group).

Proposition 6.8. If $\Gamma$ be an $n$-dimensional Bieberbach group of diagonal type with holonomy $C_2^n$ and $b_1(\Gamma) = 0$, then $\Delta_P \leq \Gamma$.

Proof. Since $\Gamma$ is Bieberbach group of diagonal type and $b_1(\Gamma) = 0$, without loss of generality, we let

$$\alpha = (\text{diag}(X_1, ..., X_n), (x_1, ..., x_n)) \quad \text{and} \quad \beta = (\text{diag}(Y_1, ..., Y_n), (y_1, ..., y_n)),$$

$X_i, Y_i \in \{1, -1\}$ and $x_i, y_i \in \{0, \frac{1}{2}\}$ for all $i \in \{1, ..., n\}$ be the non-lattice generators of $\Gamma$. There exists $i, j \in \{1, ..., n\}$ such that $(X_i, x_i) = (1, \frac{1}{2})$ and $(Y_j, y_j) = (1, \frac{1}{2})$. Otherwise, $\alpha \in \Gamma$ or $\beta \in \Gamma$ is an element of order 2, which contradicts the fact that $\Gamma$ is torsion-free. There exists $k \in \{1, ..., n\}$ such that $X_k = Y_k = -1$ and $(x_k, y_k) \in \{(0, \frac{1}{2}), (\frac{1}{2}, 0)\}$, otherwise $\alpha \beta \in \Gamma$ has order 2. Since $b_1(\Gamma) = 0$, there does not exist $z \in \{1, ..., n\}$ such that $X_z = Y_z = 1$. By the third Bieberbach Theorem, we may assume

$$\alpha = \begin{pmatrix} I_s & 0 & 0 & a_1 \\ 0 & -I_p & 0 & b_1 \\ 0 & 0 & -I_q & c_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} -I_s & 0 & 0 & a_2 \\ 0 & I_p & 0 & b_2 \\ 0 & 0 & -I_q & c_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $s, p, q \in \mathbb{Z}^+$, $a_1, a_2 \in \{0, \frac{1}{2}\}^s$, $b_1, b_2 \in \{0, \frac{1}{2}\}^p$ and $c_1, c_2 \in \{0, \frac{1}{2}\}^q$. In addition, $a_1, b_2$, and $c_1 - c_2$ are non-zero.
By a simple calculation, we checked that $\alpha$ and $\beta$ satisfy the below relation
\[ \alpha^{-1} \beta^2 \alpha^2 = \beta^{-1} \alpha^2 \beta^2 = 1 \]
Since
\[ \Delta_P = \langle x, y \mid x^{-1} y^2 x y = y^{-1} x^2 y x^2 = 1 \rangle \]
there exists a normal subgroup $N \triangleleft \Delta_P$ such that $\langle \alpha, \beta \rangle \cong \Delta_P/N$. Let $\tilde{\Gamma} = \langle \alpha, \beta \rangle$. Since $\alpha^2$, $\beta^2$ and $(\alpha \beta)^2$ are three linearly independent elements inside the lattice $\tilde{\Gamma} \cap \mathbb{R}^n$ and thus $\dim(\tilde{\Gamma}) \geq 3$. This implies that $N$ has rank zero and is therefore trivial. Hence we have $\Delta_P \cong \langle \alpha, \beta \rangle \leq \Gamma$. \hfill \Box

**Remark 6.9.** The above proposition is a special case of Theorem 1 of \cite{9} which also covers the non-diagonal case. We should stress that our proof is different from \cite{9}.

**Lemma 6.10.** Let $\Gamma$ be an $n$-dimensional non-diffuse Bieberbach group of diagonal type with holonomy $C_2^n$. Then there exists $\mathbb{Z}^{n-3} \triangleleft \Gamma$ such that $\Gamma/\mathbb{Z}^{n-3} \cong \Delta_P$.

**Proof.** By Theorem 13 we have $n_d(C_2^n) = 3$. So, there exists $\mathbb{Z}^s \triangleleft \Gamma$ such that $\Gamma/\mathbb{Z}^s = \tilde{\Gamma}$ with $\dim(\tilde{\Gamma}) \leq 3$. By Proposition 6.3, $\tilde{\Gamma}$ is non-diffuse. Since $\Delta_P$ is the only non-diffuse Bieberbach group of dimension at most three, we can conclude that $s = n - 3$ and $\tilde{\Gamma} \cong \Delta_P$. \hfill \Box

**Proof of Theorem** Let $\{\alpha, \beta\}$ be a set of non-lattice generators of $\Gamma$. Since $b_1(\Gamma) = k$, without loss of generality, assume
\[ \alpha = (\text{diag}(X_1, \ldots, X_n), (x_1, \ldots, x_n)) \quad \text{and} \quad \beta = (\text{diag}(Y_1, \ldots, Y_n), (y_1, \ldots, y_n)) \]
where $x_j, y_j \in \{0, \frac{1}{2}\}$ for $j \in \{1, \ldots, n\}$, $(X_i, Y_i) \in \{(1, -1), (-1, 1), (-1, -1)\}$ for $i \in \{1, \ldots, n-k\}$ and $(X_i, Y_i) = (1, 1)$ for $i \in \{n-k+1, \ldots, n\}$.

Given an arbitrary element $\gamma = (\text{diag}(z_1, \ldots, z_{n-k}, 1, \ldots, 1), (s_1, \ldots, s_n)) \in \Gamma$ where $z_i \in \{1, -1\}$ for $i \in \{1, \ldots, n-k\}$ and $(s_1, \ldots, s_n) \in \mathbb{Q}^n$. By \cite{14}, there exists a homomorphism $f : \Gamma \to \mathbb{Z}^k$ which maps $\gamma$ to $(2s_{n-k+1}, \ldots, 2s_n) \in \mathbb{Z}^k$ and the kernel of the homomorphism is an $(n-k)$-dimensional Bieberbach group.

We claim that $x_i = 0$ for all $i \in \{n-k+1, \ldots, n\}$. Assume by contradiction that there exists $j \in \{n-k+1, \ldots, n\}$ such that $x_j \neq 0$. We have $\alpha \notin \ker(f)$. Then the holonomy group of $\ker(f)$ will either be identity or cyclic group of order two. By \cite{12} Theorem 3.5, $\ker(f)$ is diffuse. Since $\ker(f)$ and $\mathbb{Z}^k$ are both diffuse, by Proposition 6.3, $\Gamma$ is diffuse, which is a contradiction. Hence $x_i = 0$ for all $i \in \{n-k+1, \ldots, n\}$. By similar argument,
we get \( y_i = 0 \) for all \( i \in \{n - k + 1, \ldots, n\} \). Therefore \( \Gamma = Z(\Gamma) \oplus \bar{\Gamma} \), where \( Z(\Gamma) \) is the center of \( \Gamma \) and \( \bar{\Gamma} = ker(f) \). By Lemma 6.10 we have

\[
0 \longrightarrow \mathbb{Z}^{n-k-3} \xrightarrow{\ell} \Gamma \xrightarrow{\phi} \Delta_P \longrightarrow 1
\]

Notice that \( b_1(\bar{\Gamma}) = 0 \), otherwise \( b_1(\Gamma) > k \). By Proposition 6.8 we have \( \Delta_P \leq \bar{\Gamma} \). By restricting the domain of \( \phi \), we have

\[
0 \longrightarrow H \xrightarrow{\ell} \Delta_P \xrightarrow{\phi|_{\Delta_P}} G \longrightarrow 1
\]

where \( H \) is a subgroup of \( \mathbb{Z}^{n-k-3} \) and \( G \) is the image of the map \( \phi|_{\Delta_P} \). We claim that \( \phi|_{\Delta_P} \) is an isomorphism. Since \( H \) is a subgroup of \( \mathbb{Z}^{n-k-3} \), \( H \) is a diffuse group. By Proposition 6.8 \( G \) is non-diffuse, otherwise it contradicts that \( \Delta_P \) is a non-diffuse group. Beside, \( G \) is a quotient of \( \Delta_P \). Hence \( G \) is a non-diffuse Bieberbach group of dimension less than or equal to three. Thus \( G \cong \Delta_P \) and hence \( \phi|_{\Delta_P} \) is an isomorphism. Therefore (8) is a split short exact sequence.

Before proving Theorem D we need the below two propositions to consider the case when the Bieberbach group has trivial center.

**Proposition 6.11.** Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy \( C_2^k \) and \( b_1(\Gamma) = 0 \). Let \( p: \Gamma \to C_2^k \) be the standard projection. If \( n < 2^k - 1 \), then there exists \( C_2^{k-1} \leq C_2^k \) such that \( p^{-1}(C_2^{k-1}) \) is an \( n \)-dimensional Bieberbach group with holonomy group \( C_2^{k-1} \) and \( b_1(p^{-1}(C_2^{k-1})) = 0 \).

**Proof.** First note that there are \( 2^k - 1 \) subgroups in \( C_2^k \) isomorphic to \( C_2^{k-1} \). Let \( A_1, \ldots, A_{2^k-1} \) denote these subgroups. Assume by contradiction that \( p^{-1}(A_i) \) is Bieberbach group with non-trivial first Betti number for all \( i \in \{1, \ldots, 2^k - 1\} \). Hence we have \( (\mathbb{Z}^n)^{C_2^k} = 0 \) and \( (\mathbb{Z}^n)^{A_i} \) \( \neq 0 \) for all \( i \in \{1, \ldots, 2^k - 1\} \) where \( \mathbb{Z}^n \cong \Gamma \cap \mathbb{R}^n \). Fix \( i \in \{1, \ldots, 2^k - 1\} \), let \( z_i \in (\mathbb{Z}^n)^{A_i} \) and \( e_1, \ldots, e_n \) be the standard basis of \( \mathbb{Z}^n \) such that \( C_2^k \) acts diagonally on \( \{e_1, \ldots, e_n\} \). We can express \( z_i \) in term of the basis elements \( z_i = c_1 e_1 + \cdots + c_n e_n \). For all \( g \in A_i \), we have

\[
z_i = g \cdot z_i = c_1 (g \cdot e_1) + \cdots + c_n (g \cdot e_n)
\]

Thus if \( c_j \neq 0 \) then \( e_j \in (\mathbb{Z}^n)^{A_i} \) where \( j \in \{1, \ldots, n\} \). We conclude that for each \( i \in \{1, \ldots, 2^k - 1\} \), there exists \( t_i \in \{1, \ldots, n\} \) such that \( e_{t_i} \in (\mathbb{Z}^n)^{A_i} \). Notice that \( t_i \neq t_j \) for all \( i \neq j \), otherwise \( e_{t_i} \in (\mathbb{Z}^n)^{A_i} \cap (\mathbb{Z}^n)^{A_j} = (\mathbb{Z}^n)^{C_2^k} \), which contradicts that \( b_1(\Gamma) = 0 \). Thus we have \( n \geq 2^k - 1 \) which is a contradiction. \( \square \)
Recall that a group said to be \( poly-Z \) if it has a normal series with factor groups isomorphic to \( Z \).

**Proposition 6.12.** Let \( \Gamma \) be a Bieberbach group of diagonal type with \( b_1(\Gamma) = 0 \). Then there exists \( \Gamma' \leq \Gamma \) and a \( poly-Z \) subgroup \( N \leq \Gamma' \) such that \( \Gamma'/N \cong \Delta_P \).

**Proof.** Let \( C_2^p \) where \( p \geq 1 \) be the holonomy group of \( \Gamma \). We proceed by induction on \( p \). By Theorem [C], we know the statement holds for \( p = 2 \). Assume that it is true for all \( p \leq k - 1 \) and consider the case where \( p = k \). Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group of diagonal type with holonomy \( C_2^k \) and \( b_1(\Gamma) = 0 \). By Theorem [A], there exists a free abelian group \( \mathbb{Z}^t \) for some \( t \geq 0 \) such that \( \bar{\Gamma} = \Gamma/\mathbb{Z}^t \) is a Bieberbach group of diagonal type of dimension at most \( n_d(C_2^k) < 2^k - 1 \). By Lemma [6.11], we have \( b_1(\bar{\Gamma}) = 0 \). If \( hol(\bar{\Gamma}) \cong C_2^k \), then by Proposition [6.11], there exists \( \Gamma_1 \leq \bar{\Gamma} \) such that \( b_1(\Gamma_1) = 0 \) with \( hol(\Gamma_1) \cong C_2^{k-1} \). If \( hol(\bar{\Gamma}) \) is a proper subgroup of \( C_2^k \), then we take \( \Gamma_1 = \bar{\Gamma} \). In other words, \( \Gamma_1 \) is a Bieberbach group of diagonal type with \( b_1(\Gamma_1) = 0 \) and its holonomy group is \( C_2^s \) where \( s \leq k - 1 \). By induction hypothesis, there exists \( \Gamma_2 \leq \Gamma_1 \) and \( poly-Z \) subgroup \( A \leq \Gamma_2 \) such that \( \Gamma_2/A \cong \Delta_P \). Since \( A \leq \Gamma_2 \leq \bar{\Gamma} = \Gamma/\mathbb{Z}^t \), we have \( \Gamma_2 = \Gamma'/\mathbb{Z}^t \) and \( A = N/\mathbb{Z}^t \) where \( N \leq \Gamma' \leq \Gamma \). We get \( \Delta_P \cong \Gamma'/N \). \( \square \)

**Proof of Theorem [D]** Let \( \Gamma \) be an \( n \)-dimensional non-diffuse Bieberbach group of diagonal type. By Theorem [1.6], there exists a nontrivial subgroup \( \Gamma' \) of \( \Gamma \) such that \( b_1(\Gamma') = 0 \). By Proposition [6.12], there exists \( \Gamma'' \leq \Gamma' \) and a \( poly-Z \) subgroup \( N \leq \Gamma'' \) such that \( \Gamma''/N \cong \Delta_P \). Now, assume that \( \Gamma \) is a non-diffuse generalized Hantzsche-Wendt group. By [20] Theorem 3.1, \( \Gamma \) is a Bieberbach group of diagonal type. Let the holonomy group of \( \Gamma \) be \( C_2^p \). We proceed by induction on \( p \) to show that \( \Delta_P \leq \Gamma \). The base case \( p = 2 \) is clear. Assume that the statement is true for all \( p \leq k - 1 \) and consider \( p = k \). If \( b_1(\Gamma) = 0 \), then by [20] Proposition 8.2, we have \( \Delta_P \leq \Gamma \). Hence we may assume \( b_1(\Gamma) > 0 \). By [20] Proposition 4.1], there exists \( f: \Gamma \rightarrow \mathbb{Z} \) such that \( ker(f) \) is an \((n-1)\)-dimensional generalized Hantzsche-Wendt group. Since \( \Gamma \) is non-diffuse, by Proposition [6.3], \( ker(f) \) is non-diffuse. Hence by induction hypothesis, we have \( \Delta_P \leq ker(f) \leq \Gamma \). \( \square \)

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