In this study, a new partial randomized response model (RRM) has been proposed for estimating the population mean of two quantitative sensitive variables simultaneously. The utility of proposed model under stratification is also explored. The efficiency comparisons of the proposed model under simple and stratified random sampling are carried out numerically. A real data set was collected through direct questioning, proposed partial RRM and competitor randomized device from the students of statistics and animal sciences departments of Quaid-I-Azam University Islamabad, Pakistan. The performance of the proposed partial RRM is better than competitor RRM under simple and stratified random sampling.

1. Introduction

The social survey is one of leading mechanisms to obtain reliable data on attitudes, behaviors, and opinions of the human population. Sometimes, the facts about the individuals are inaccessible to the investigators due to social stigma, such facts are considered as sensitive information. When asked directly, respondents may consciously or unconsciously provide incorrect information on stigmatizing characteristics. To reduce the bias and to procure reliable data, Warner [1] developed a randomized response model (RRM) to estimate the population proportion of a sensitive attribute. In Warner’s [1]’s model, a randomly selected proportion of respondents are asked the sensitive question and the remaining proportion of respondents are asked complement of the sensitive question. The researcher does not know whether the respondents answered the sensitive or insensitive question. Greenberg et al. [2] extended the Warner’s idea for mean estimation of quantitative sensitive variables. Some other developments for the estimation of mean are due to Eichhorn and Hayre [3]; Bar-Lev et al. [4]; Gupta et al. [5]; Gupta et al. [6] Hussain et al. [7]; Singh et al. [8]; Singh and Suman [9]; Lee and Hong [10]; Narjis and Shabbir [11, 12]; and Muneer et al. [13].

Scrambled randomized response models are built on the idea of obtaining masked rather than actual responses. Masking can be achieved by adding to, subtracting from, or multiplying a random component to actual responses. Scrambled response models may be categorized as full, partial, and optional models. In full RRM, all respondents are requested to provide the scrambled response, whereas, in partial RRM, a randomly selected group of respondents are requested to provide the truthful response and remaining are requested to provide the scrambled response. In optional RRM, respondents are requested to provide the scrambled response if he/she considers the question sensitive, and truthful response if he/she considers the question to be non-sensitive. Mangat and Singh [14] and Gupta et al. [15] introduced the partial RRM and the optional RRM, respectively. The purpose of all RRMs are to protect privacy and increase cooperation.

Researchers in the field of social, medical, and environmental sciences have well documented the situations where they may be interested to estimate the two dependent
Suppose a sample and 1.2. non-negative integers with known mean and variance, which we wish to estimate. Let \( y \) be two quantitative sensitive variables of interest with uncorrelated RR model for simultaneous estimation of actual income and the income reported in their tax return. Consider a finite population of both quantitative sensitive variables and with each other, the distribution of scrambling variables are known.

Let \( E(S_1) = \bar{y}_1 = e_{10}, \ E(S_2) = \bar{y}_2 = e_{01}, \ V(S_1) = \delta_{20}, \ V(S_2) = \delta_{02}, \ Cov(S_1, S_2) = e_{02}, \ E(Y_1) = \mu_{y_1}, \ E(Y_2) = \mu_{y_2}, \ V(Y_1) = \sigma_{y_1}^2, \ V(Y_2) = \sigma_{y_2}^2, \) and \( \sigma_{y_1 y_2} = E[Y_{1i} - \mu_{y_1}][Y_{2i} - \mu_{y_2}], \) where, \( a \) and \( b \) are non-negative integers.

**Lemma 1.** The moments of order four or less of scrambling variables \( S_1 \) and \( S_2 \), the expression of \( E(S_1^4 S_2^3), a \) and \( b \) being non-negative integers with \((a + b) \leq 4\), are given by

\[
E(S_1^4) = \delta_{40} + 4\delta_{30}\bar{y}_1 + 6\delta_{20}\bar{y}_1^2 + \bar{y}_1^4 = e_{40},
\]

(3)

\[
E(S_2^4) = \theta_1^4 + \delta_{02} = e_{02},
\]

(4)

\[
E(S_1^3 S_2) = \delta_{03} + 3\delta_{02}\bar{y}_2 + \bar{y}_2^3 = e_{03},
\]

(5)

\[
E(S_1^2 S_2^2) = \delta_{04} + 4\delta_{03}\bar{y}_2 + 6\delta_{02}\bar{y}_2^2 + \bar{y}_2^4 = e_{04},
\]

(6)

\[
E(S_1 S_2^3) = \theta_1\bar{y}_2 = e_{11},
\]

(7)

\[
E(S_1^2 S_2^2) = (\theta_1^2 + \delta_{02})\bar{y}_2 = e_{21},
\]

(8)

\[
E(S_1^3 S_2) = \theta_1(\theta_1^2 + \delta_{02}) = e_{12},
\]

(9)

\[
E(S_1^2 S_2^2) = (\theta_1^2 + \delta_{02})^2 = e_{22},
\]

(10)

\[
E(S_1^4 S_2) = (\delta_{03} + 3\delta_{02}\bar{y}_2 + \bar{y}_2^3)\theta_1 = e_{31},
\]

(11)

\[
E(S_1^3 S_2^2) = \theta_1(\delta_{03} + 3\delta_{02}\bar{y}_2 + \bar{y}_2^3) = e_{13},
\]

(12)

**1.2. Notations and Terminology under Stratified Random Sampling.** Consider a finite population \( \Omega = \{1, 2, 3, \ldots, N\} \) which are partitioned into \( L \) homogeneous subgroups called strata, such that the \( h \)th stratum consists of \( N_h \) units, where \( h = 1, 2, \ldots, L \) and \( \sum_{h=1}^L N_h = N \). A sample of size \( n_h \) from \( N_h \) stratum is drawn by using SRSWR such that \( \sum_{h=1}^L n_h = n \). Let \( Y_{1h} \) and \( Y_{2h} \) be the \( h \)th population values of two quantitative sensitive variables in the \( h \)th stratum, \( i = 1, 2, \ldots, N_h \) and \( W_h = N_h/N \) is the known proportion of population units falling in the \( h \)th stratum. Similarly, \( S_{1h} \) and \( S_{2h} \) are two independent scrambling variables with known means and variances.

Let \( E(S_{1h}) = \theta_{1h} = e_{10h}, E(S_{2h}) = \theta_{2h} = e_{01h}, V(S_{1h}) = \delta_{20h}, V(S_{2h}) = \delta_{02h}, Cov(S_{1h}, S_{2h}) = 0, \delta_{abh} = E[S_{1h} - \theta_{1h}][S_{2h} - \theta_{2h}], \sigma_{y_{1h} y_{2h}} = E[Y_{1h} - \mu_{y_{1h}}][Y_{2h} - \mu_{y_{2h}}], \) where, \( a \) and \( b \) are non-negative integers.

**Lemma 2.** The moments of order four or less of scrambling variables \( S_{1h} \) and \( S_{2h} \), the expression of \( E(S_{1h}^4 S_{2h}^3), a \) and \( b \) being non-negative integers with \((a + b) \leq 4\), are given by

\[
E(S_{1h}^4) = \delta_{40h} + 4\delta_{30h}\theta_{1h} + 6\delta_{20h}\theta_{1h}^2 + \theta_{1h}^4 = e_{40h},
\]

(13)

\[
E(S_{2h}^4) = \theta_{1h}^4 + \delta_{02h} = e_{02h},
\]

(14)

\[
E(S_{1h}^3 S_{2h}) = \delta_{03h} + 3\delta_{02h}\theta_{1h} + \theta_{1h}^3 = e_{03h},
\]

(15)

\[
E(S_{1h}^2 S_{2h}^2) = \delta_{04h} + 4\delta_{03h}\theta_{1h} + 6\delta_{02h}\theta_{1h}^2 + \theta_{1h}^4 = e_{04h},
\]

(16)

\[
E(S_{1h}^3 S_{2h}) = \theta_{1h}(\theta_{1h}^2 + \delta_{02h}) = e_{11h},
\]

(17)
\[ E(S'_{2a}) = \delta_{0a} + 4\delta_{0a}\theta_{2a} + 6\delta_{2a}\theta_{2a}^2 + \theta_{2a}^3 = e_{0a}, \]  
\[ E(S_{1a}S_{2a}) = \theta_{1a}\theta_{2a} = e_{11a}, \]  
\[ E(S'_{1a}S_{2a}) = (\theta_{1a}^2 + \delta_{0a})\theta_{2a} = e_{21a}, \]  
\[ E(S_{1a}S'_{2a}) = \theta_{1a}\left(\theta_{2a}^2 + \delta_{0a}\right) = e_{13a}, \]  
\[ E(S'_1S'_2) = (\theta_{1a}^2 + \delta_{2a})(\theta_{2a}^2 + \delta_{0a}) = e_{22a}, \]  
\[ E(S'_{1a}S'_{2a}) = \theta_{1a}\left(\theta_{2a}^3 + 3\delta_{2a}\theta_{2a} + \theta_{2a}^4\right) = e_{31a}, \]  
\[ E(S_{1a}S'_2) = \theta_{1a}\left(\theta_{2a}^2 + 3\delta_{0a}\theta_{2a} + \theta_{2a}^3\right) = e_{13a}, \]  
\[ \text{The interest of researchers is always to investigate the true response from a population. To attain this desire, Mangat and Singh [14] proposed an ingenious partial RRT model by injecting an element of truthful responses into the Warner [1]'s model. Gupta and Thornton [24] described the partial RRT model for quantitative sensitive variables and many others also give the improvement in this area. It is important to note that almost all such types of RRT models can only estimate one quantitative sensitive variable at a time. So, in this article keeping in mind the desire of researchers, we propose an additive partial RRT model to estimate two quantitative sensitive variables simultaneously. The proposed model is an extension of Ahmed et al. [22]'s model under simple and stratified random sampling. The basic purpose of this study is to obtain truthful responses from some proportion of people and increase efficiency.} \]

This paper is organized as follows: In Section 2, we give some existing RRT models. In Section 3, we introduce a partial randomized response model under SRSWR and numerically compare it with Ahmed et al. [22] model. In Section 4, we present a partial randomized response model under stratification and numerically compare it with stratified model of Ahmed et al. [22]. In Section 5, an application of real life data is given and comparison of proposed partial RRM is made with Ahmed et al. [22] model on basis of direct response technique. Finally, Section 6 provides a conclusion.

\[ V(\tilde{\mu}_{y_{a1}}) = \frac{(Pe_{11} + (1 - P)e_{02})^2a_{Z_{A_1}}^2 + e_{01}^2a_{Z_{a1}}^2 - 2(Pe_{11} + (1 - P)e_{02})e_{01}a_{Z_{a1}}a_{Z_{a2}}}{n[(1 - P)a_{Z_{02}}e_{01} - Pa_{Z_{02}}e_{01}]} \]  
and

\[ V(\tilde{\mu}_{y_{a2}}) = \frac{e_{10}^2a_{Z_{a2}} + (Pe_{20} + (1 - P)e_{11})^2a_{Z_{a2}} - 2e_{10}(Pe_{20} + (1 - P)e_{11})a_{Z_{a1}}a_{Z_{a2}}}{n[(1 - P)a_{Z_{02}}e_{01} - Pa_{Z_{02}}e_{01}]} \]
where
\[ \sigma_{A_1}^2 = (\mu_{y_1}^2 + \sigma_{y_1}^2) \delta_{20} + (\mu_{y_2}^2 + \sigma_{y_2}^2) \delta_{02} + \sigma_{y_1}^2 e_{10}^2 + \sigma_{y_2}^2 e_{01}^2 + 2 \sigma_{y_1 y_2} e_{11}, \] (32)
\[ \sigma_{Z_{A_2}}^2 = (\mu_{y_1}^2 + \sigma_{y_1}^2) \left( P e_{40} + (1 - P) e_{21} \right) + (\mu_{y_2}^2 + \sigma_{y_2}^2) \left( P e_{22} + (1 - P) e_{04} \right) + 2 \left( \mu_{y_1} \mu_{y_2} + \sigma_{y_1 y_2} \right) \left( P e_{31} + (1 - P) e_{13} \right) - \left[ \left( P e_{20} + (1 - P) e_{11} \right) \mu_{y_1} + \left( P e_{11} + (1 - P) e_{02} \right) \mu_{y_2} \right]^2, \] (33)
and
\[ \sigma_{Z_{A_1} Z_{A_2}} = (\mu_{y_1}^2 + \sigma_{y_1}^2) \left( P e_{30} + (1 - P) e_{21} \right) + (\mu_{y_2}^2 + \sigma_{y_2}^2) \left( P e_{12} + (1 - P) e_{03} \right) + 2 \left( \mu_{y_1} \mu_{y_2} + \sigma_{y_1 y_2} \right) \left( P e_{21} + (1 - P) e_{12} \right) - \left[ \mu_{y_1} e_{10} + \mu_{y_2} e_{01} \right] \left[ \left( P e_{20} + (1 - P) e_{11} \right) \mu_{y_1} + \left( P e_{11} + (1 - P) e_{02} \right) \mu_{y_2} \right]. \] (34)

2.2. Model under Stratified Random Sampling. From Ahmed et al. [22], in stratified random sampling, the scrambled response from the \( i \)th respondent of the \( h \)th stratum is obtained as
\[ Z_{A_{ih}} = S_{ih} Y_{1ih} + S_{ih} Y_{2ih}. \] (35)
The second response from the \( i \)th respondent of the \( h \)th stratum is obtained as
\[ Z_{hi} = \begin{cases} S_{ih} & \text{with probability } P_{ih}, \\ S_{ih} & \text{with probability } (1 - P_{ih}). \end{cases} \] (36)
The unbiased estimators of the population means \( \mu_{y_1} \) and \( \mu_{y_2} \) are given by
\[ \bar{\mu}_{y_{A_1}} = \frac{\sum_{h=1}^{L} W_h \left( P_{ih} e_{11ih} + (1 - P_{ih}) e_{02ih} \right) Z_{A_{ih}} - e_{01ih} Z_{A_{ih}}}{(1 - P_{ih}) \delta_{02ih} e_{10ih} - P_{ih} \delta_{20ih} e_{01ih}}, \] (37)
\[ \bar{\mu}_{y_{A_2}} = \sum_{h=1}^{L} W_h \left( P_{ih} e_{11ih} + (1 - P_{ih}) e_{02ih} \right) Z_{A_{ih}} - e_{01ih} Z_{A_{ih}} \] (38)

The variance of the proposed estimators \( \bar{\mu}_{y_{A_1}} \) and \( \bar{\mu}_{y_{A_2}} \) are given by
\[ V \left( \bar{\mu}_{y_{A_1}} \right) = \frac{\sum_{h=1}^{L} W_h^2 \sigma_{A_{1h}}^2}{n_h}, \] (39)
\[ V \left( \bar{\mu}_{y_{A_2}} \right) = \frac{\sum_{h=1}^{L} W_h^2 \sigma_{A_{2h}}^2}{n_h}. \] (40)

where
\[ \sigma_{A_{1h}}^2 = \frac{\left( P_{ih} e_{11ih} + (1 - P_{ih}) e_{02ih} \right)^2 \sigma_{Z_{A_{ih}}}^2 + e_{01ih}^2 \sigma_{Z_{A_{ih}}}^2 - 2 \left( P_{ih} e_{11ih} + (1 - P_{ih}) e_{02ih} \right) e_{01ih} \sigma_{Z_{A_{ih}}} Z_{A_{ih}}}{\left[ (1 - P_{ih}) \delta_{02ih} e_{10ih} - P_{ih} \delta_{20ih} e_{01ih} \right]^2}, \] (41)
\[ \sigma_{A_{2h}}^2 = \frac{e_{10ih}^2 \sigma_{Z_{A_{ih}}}^2 + \left( P_{ih} e_{20ih} + (1 - P_{ih}) e_{11ih} \right)^2 \sigma_{Z_{A_{ih}}}^2 - 2 e_{01ih} \left( P_{ih} e_{20ih} + (1 - P_{ih}) e_{11ih} \right) \sigma_{Z_{A_{ih}}} Z_{A_{ih}}}{\left[ (1 - P_{ih}) \delta_{02ih} e_{10ih} - P_{ih} \delta_{20ih} e_{01ih} \right]^2}, \] (42)
\[ \sigma_{Z_{A_{1h}}}^2 = \left( \mu_{y_1}^2 + \sigma_{y_1}^2 \right) \delta_{20ih} + \left( \mu_{y_2}^2 + \sigma_{y_2}^2 \right) \delta_{02ih} + \sigma_{y_1}^2 e_{10ih}^2 + \sigma_{y_2}^2 e_{01ih}^2 + 2 \sigma_{y_1 y_2} e_{11ih}^2, \] (43)
\[ \sigma_{Z_{A_{2h}}}^2 = \left( \mu_{y_1}^2 + \sigma_{y_1}^2 \right) \left( P_{ih} e_{40ih} + (1 - P_{ih}) e_{21ih} \right) + \left( \mu_{y_2}^2 + \sigma_{y_2}^2 \right) \left( P_{ih} e_{22ih} + (1 - P_{ih}) e_{04ih} \right) + 2 \left( \mu_{y_1} \mu_{y_2} + \sigma_{y_1 y_2} \right) \left( P_{ih} e_{31ih} + (1 - P_{ih}) e_{13ih} \right) - \left[ \left( P_{ih} e_{20ih} + (1 - P_{ih}) e_{11ih} \right) \mu_{y_1} + \left( P_{ih} e_{11ih} + (1 - P_{ih}) e_{02ih} \right) \mu_{y_2} \right]^2, \] (44)
and
\[ \sigma_{Z_{A_{1h}} Z_{A_{2h}}} = \left( \mu_{y_1}^2 + \sigma_{y_1}^2 \right) \left( P_{ih} e_{30ih} + (1 - P_{ih}) e_{21ih} \right) + \left( \mu_{y_2}^2 + \sigma_{y_2}^2 \right) \left( P_{ih} e_{12ih} + (1 - P_{ih}) e_{03ih} \right) + 2 \left( \mu_{y_1} \mu_{y_2} + \sigma_{y_1 y_2} \right) \left( P_{ih} e_{21ih} + (1 - P_{ih}) e_{12ih} \right) - \left[ \mu_{y_1} e_{10ih} + \mu_{y_2} e_{01ih} \right] \left[ \left( P_{ih} e_{20ih} + (1 - P_{ih}) e_{11ih} \right) \mu_{y_1} + \left( P_{ih} e_{11ih} + (1 - P_{ih}) e_{02ih} \right) \mu_{y_2} \right]. \] (45)
3. Proposed Partial RRM under Simple Random Sampling

In this section, we propose a partial randomized response model for simultaneous estimation of means of two quantitative sensitive variables. In the proposed partial RRM, each respondent selected in the \( i \)th sample is requested for two responses by using two randomized response (RR) devices. The RR Device I provides the scramble and true response of sensitive variables, whereas the RR Device II provides the fake response, that is free from the sensitive variables.

The RR Device I, bearing two types of statements:

(i) Report the additive true value of both sensitive variables, say \((Y_{1i} + Y_{2i})\) with probability \(T\) and

(ii) Report the scrambled response as, \((S_1Y_{1i} + S_2Y_{2i})\) with probability \((1 - T)\). Mathematically, each respondent is requested to report the response \(Z_{1i}\) as

\[
Z_{1i} = \begin{cases} 
Y_{1i} + Y_{2i} & \text{with probability } T, \\
S_1Y_{1i} + S_2Y_{2i} & \text{with probability } (1 - T).
\end{cases} \tag{46}
\]

The partial randomized response \(Z_{1i}\) in the \( i \)th sample is given by

\[
Z_{1i} = \beta_iY_{1i} + Y_{2i} + (1 - \beta_i)(S_1Y_{1i} + S_2Y_{2i}). \tag{47}
\]

For the second response, each respondent is requested to use RR Device 2 which is same as equation (26); thus, the fake response \(Z_i\) from equation (26) in the \( i \)th sample is given by

\[
Z_i = y_iS_{1i} + (1 - y_i)S_{2i}, \tag{48}
\]

where \(y_i\) and \(\beta_i\) are Bernoulli random variables with means \(P\) and \(T\), respectively, which are the known parameters.

From equations (47) and (48), we generate response \(Z_{2i}\) as follows:

\[
Z_{2i} = Z_iZ_{1i} = \begin{cases} 
S_1Y_{1i} + S_1Y_{2i} & \text{with probability } PT, \\
S_1^2Y_{1i} + S_1S_2Y_{2i} & \text{with probability } (1 - T)P, \\
S_2Y_{1i} + S_2Y_{2i} & \text{with probability } (1 - T)(1 - P), \\
S_1S_2Y_{1i} + S_1^2Y_{2i} & \text{with probability } (1 - P)(1 - T).
\end{cases} \tag{49}
\]

The generated response \(Z_{2i}\) in the \( i \)th sample is given by

\[
Z_{2i} = y_i\beta_i(S_1Y_{1i} + S_1Y_{2i}) + (1 - \beta_i)y_i(S_1^2Y_{1i} + S_1S_2Y_{2i}) \\
+ (1 - y_i)\beta_i(S_2Y_{1i} + S_2Y_{2i}) \\
+ (1 - y_i)(1 - \beta_i)(S_1S_2Y_{1i} + S_1^2Y_{2i}). \tag{50}
\]

Taking expected values on both sides of equations (47) and (50), we have

\[
E(Z_{1i}) = T(\mu_{Y_1} + \mu_{Y_2}) + (1 - T)(e_{10}\mu_{Y_1} + e_{01}\mu_{Y_2}), \tag{51}
\]

\[
E(Z_{2i}) = PT[e_{10}\mu_{Y_1} + e_{10}\mu_{Y_2}] \\
+ (1 - T)[P[e_{20}\mu_{Y_1} + e_{11}\mu_{Y_2}] \\
+ (1 - P)[(1 - T)e_{11}\mu_{Y_1} + e_{02}\mu_{Y_2}]. \tag{52}
\]

From equations (51) and (52), by the method of moments, we have

\[
A_{11}\tilde{\mu}_{Y_1} + A_{12}\tilde{\mu}_{Y_2} = Z_1, \tag{53}
\]

and

\[
A_{21}\tilde{\mu}_{Y_1} + A_{22}\tilde{\mu}_{Y_2} = Z_2, \tag{54}
\]

where

\[
A_{11} = T + (1 - T)e_{10} \\
A_{12} = T + (1 - T)e_{01} \\
A_{21} = PTe_{10} + (1 - T)Pe_{20} + (1 - P)Te_{01} + (1 - P)(1 - T)e_{11} \\
A_{22} = PTe_{10} + (1 - T)Pe_{11} + (1 - P)Te_{02} + (1 - P)(1 - T)e_{02}.
\]

Solving equations (53) and (54) by using Cramer’s rule, we have unbiased estimators of \(\mu_{Y_1}\) and \(\mu_{Y_2}\), respectively, given by

\[
\tilde{\mu}_{Y_1} = \frac{A_{22}Z_1 - A_{12}Z_2}{(1 - T)[(1 - P)\delta_{02}A_{11} - P\delta_{20}A_{12}]}, \tag{55}
\]

and

\[
\tilde{\mu}_{Y_2} = \frac{A_{21}Z_1 - A_{22}Z_2}{(1 - T)[(1 - P)\delta_{02}A_{11} - P\delta_{20}A_{12}]]. \tag{56}
\]

**Theorem 1.** The variance of proposed estimators \(\tilde{\mu}_{Y_1}\) and \(\tilde{\mu}_{Y_2}\) is, respectively, given by

\[
V(\tilde{\mu}_{Y_1}) = \frac{A_{22}^2\sigma_{Z_1}^2 + A_{12}^2\sigma_{Z_2}^2 - 2A_{12}A_{22}\sigma_{Z_1}Z_2}{n(1 - T)^2[(1 - P)\delta_{02}A_{11} - P\delta_{20}A_{12}]^2}, \tag{57}
\]

and

\[
V(\tilde{\mu}_{Y_2}) = \frac{A_{11}^2\sigma_{Z_1}^2 + A_{12}^2\sigma_{Z_2}^2 - 2A_{12}A_{21}\sigma_{Z_1}Z_2}{n(1 - T)^2[(1 - P)\delta_{02}A_{11} - P\delta_{20}A_{12}]^2} \tag{58}
\]

where

\[
\sigma_{Z_1}^2 = (\mu_y^2 + \sigma_y^2)\{ T + (1 - T)e_{20} \} + (\mu_y^2 + \sigma_y^2)\{ T + (1 - T)e_{02} \}
+ 2(\mu_y^2 + \sigma_y^2)\{ T + (1 - T)e_{11} \} - \left[ A_{11}\mu_1 + A_{12}\mu_y \right]^2,
\]

(59)

\[
\sigma_{Z_2}^2 = (\mu_y^2 + \sigma_y^2)\{ PTE_{20} + (1 - T)Pe_{40} + (1 - P)T_e_{02} + (1 - P)(1 - T)e_{22} \}
+ (\mu_y^2 + \sigma_y^2)\{ PTE_{20} + (1 - T)Pe_{22} + (1 - P)T_e_{02} + (1 - P)(1 - T)e_{04} \}
+ 2(\mu_y^2 + \sigma_y^2)\{ PTE_{20} + (1 - T)Pe_{31} + (1 - P)T_e_{02} + (1 - P)(1 - T)e_{13} \}
- \left[ A_{21}\mu_1 + A_{22}\mu_y \right]^2,
\]

(60)

and

\[
\sigma_{Z_1, Z_2}^2 = (\mu_y^2 + \sigma_y^2)\{ PTE_{10} + (1 - P)T_e_{01} + (1 - T)Pe_{30} + (1 - P)(1 - T)e_{21} \}
+ (\mu_y^2 + \sigma_y^2)\{ PTE_{10} + (1 - T)Pe_{12} + (1 - P)(1 - T)e_{03} \}
+ 2(\mu_y^2 + \sigma_y^2)\{ PTE_{10} + (1 - P)T_e_{21} + (1 - P)(1 - T)e_{12} \}
- \left[ A_{11}\mu_1 + A_{12}\mu_y \right]\left[ A_{21}\mu_1 + A_{22}\mu_y \right].
\]

(61)

Proof. Note that the variance expressions for two estimators \( \hat{\mu}_{Y_1} \) and \( \hat{\mu}_{Y_2} \) can be obtained through the formula

\[ \text{Var}(A \pm B) = \text{Var}(A) + \text{Var}(B) + 2\text{Cov}(A, B). \]

Now, the variance \( \sigma_{Z_1}^2 \) is given by

\[
\sigma_{Z_1}^2 = E\{ Z_{1i}^2 \} - [E\{ Z_{1i} \}]^2
= T(E(Y_{1i} + Y_{2i}))^2 + (1 - T)E(S_{11} + S_{22})^2
- [E(\beta_i(Y_{1i} + Y_{2i}))^2
+ (1 - \beta)(S_{11} + S_22)]^2,
\]

(62)

or

\[
\sigma_{Z_2}^2 = E\{ Z_{2i}^2 \} - [E\{ Z_{2i} \}]^2
= PTE(S_{11} + S_{12})^2 + (1 - T)PE(S_{11} + S_{21})^2 + (1 - P)TE(S_{11} + S_{21})^2 + (1 - P)(1 - T)E(S_{11} + S_{21})^2
- [E(\beta_i(S_{11} + S_{12}) + (1 - \beta)S_{11} + S_{22} + (1 - \gamma)(S_{11} + S_{22} + (1 - \beta)(S_{11} + S_{22})]^2.
\]

(64)

or

\[
\sigma_{Z_1}^2 = (\mu_y^2 + \sigma_y^2)\{ PTE(S_{11}^2) + (1 - T)PE(S_{11}^2) + (1 - P)TE(S_{11}^2) \}
+ (1 - P)(1 - T)E(S_{11}^2)^2 \}
+ (\mu_y^2 + \sigma_y^2)\{ PTE(S_{11}^2) + (1 - T)PE(S_{11}^2) \}
+ (1 - P)TE(S_{11}^2) + (1 - P)(1 - T)E(S_{11}^2) \}
- [A_{11}\mu_1 + A_{12}\mu_y]^2.
\]

(65)

On substituting the values of \( E(S_{11}^2), E(S_{12}^2), E(S_{13}^2), E(S_{14}^2), E(S_{15}^2), \) and \( E(S_{16}^2) \) in equation (65), we have equation (60).
The covariance $\sigma_{Z_iZ_j}$ between $Z_{1i}$ and $Z_{2i}$ is given by

$$\sigma_{Z_iZ_j} = \text{Cov}(Z_{1i}, Z_{2i}) = E(Z_{1i}Z_{2i}) - E(Z_{1i})E(Z_{2i})$$

$$= E(y_1\beta_1\beta_2[(Y_{1i} + Y_{2i})(S_{1i}Y_{1i} + S_{2i}Y_{2i})] + (1 - \beta_1)(S_{1i}Y_{1i} + S_{2i}Y_{2i})(S_{1i}Y_{1i} + S_{2i}Y_{2i})]$$

$$+ (1 - \beta_2)y_1\beta_1[(Y_{1i} + Y_{2i})(S_{2i}Y_{1i} + S_{1i}Y_{2i})] + (1 - \beta_2)(S_{2i}Y_{1i} + S_{1i}Y_{2i})(S_{2i}Y_{1i} + S_{1i}Y_{2i})]$$

$$+ (1 - y_1)\beta_1\beta_2 [(Y_{1i} + Y_{2i})(S_{1i}Y_{1i} + S_{2i}Y_{2i})] + (1 - \beta_1)(S_{1i}Y_{1i} + S_{2i}Y_{2i})(S_{1i}Y_{1i} + S_{2i}Y_{2i})]$$

$$+ (1 - y_1)(1 - \beta_2)[y_1\beta_1[(Y_{1i} + Y_{2i})(S_{2i}Y_{1i} + S_{1i}Y_{2i})] + (1 - \beta_2)(S_{2i}Y_{1i} + S_{1i}Y_{2i})]$$

$$+ (1 - \gamma_1)(1 - \beta_2)[(S_{1i}Y_{1i} + S_{2i}Y_{2i})] + (1 - \beta_2)(S_{1i}Y_{1i} + S_{2i}Y_{2i})]$$

$$+ (1 - \gamma_1)(1 - \beta_2)[y_1\beta_1[(Y_{1i} + Y_{2i})(S_{2i}Y_{1i} + S_{1i}Y_{2i})] + (1 - \beta_2)(S_{2i}Y_{1i} + S_{1i}Y_{2i})]$$

or

$$\sigma_{Z_iZ_j} = \left(\mu^2_{\gamma_1} + \sigma^2_{\gamma_1}\right)[PTE(S_i) + (1 - P)PTE(S_j) + (1 - T)PTE(S_iS_j)] + (1 - P)TPE(S_iS_j)]$$

$$+ \left(\mu^2_{\gamma_1} + \sigma^2_{\gamma_1}\right)[PTE(S_i) + (1 - P)PTE(S_j) + (1 - T)PTE(S_iS_j)] + (1 - P)TPE(S_iS_j)]$$

$$+ 2(\mu_{\gamma_1}\mu_{\gamma_2} + \sigma_{\gamma_1\gamma_2})[PTE(S_i) + (1 - P)PTE(S_j) + (1 - T)PTE(S_iS_j)] + (1 - P)TPE(S_iS_j)]$$

$$- [A_{1i}\mu_{\gamma_1} + A_{2i}\mu_{\gamma_2} + A_{2i2i}\mu_{\gamma_1\gamma_2}].$$

On substituting the values of $E(S_i), E(S_j), E(S_i^2), E(S_j^2), E(S_iS_j), E(S_i^2S_j)$, and $E(S_iS_j^2)$ in equation (67), we have equation (61).

**Corollary 1.** The unbiased estimators for $V(\hat{\mu}_{\gamma_1})$ and $V(\hat{\mu}_{\gamma_2})$ are, respectively, given by

$$\hat{V}(\hat{\mu}_{\gamma_1}) = \frac{A_{1i}^{2}z_{i} + A_{11}^{2}z_{1} - 2A_{1i}A_{2i}z_{i}z_{1}}{n(1 - P)\delta_{1i}^{2} + \delta_{20}^{2}A_{i2}}$$

and

$$\hat{V}(\hat{\mu}_{\gamma_2}) = \frac{A_{1i}^{2}z_{i} + A_{11}^{2}z_{1} - 2A_{1i}A_{2i}z_{i}z_{1}}{n(1 - P)\delta_{1i}^{2} + \delta_{20}^{2}A_{i2}}.$$  

where $s^2_{Z_i} = (n - 1)^{-1}\sum_{i=1}^{n}(Z_{1i} - Z_{i})^2$ and $s^2_{Z_i}$ are unbiased estimators of $\sigma^2_{Z_i}, \sigma^2_{Z_j},$ and $\sigma^2_{Z_iZ_j}$, respectively.

**Remark 1.** When $T = 0$, the proposed partial RRM reduces to Ahmed et al. [22] model.

3.1. **Percent Relative Efficiency under SRSWR.** In this section, we compute the percent relative efficiency (PRE) of proposed estimators $\hat{\mu}_{\gamma_1}$ and $\hat{\mu}_{\gamma_2}$ over the estimators $\hat{\mu}_{\gamma_1}$ and $\hat{\mu}_{\gamma_2}$, respectively, as

$$\text{PRE}(1) = \frac{V(\hat{\mu}_{\gamma_1})}{V(\mu_{\gamma_1})} \times 100,$$

and

$$\text{PRE}(2) = \frac{V(\hat{\mu}_{\gamma_2})}{V(\mu_{\gamma_2})} \times 100.$$  

We performed a simulation study to verify the superiority of the proposed partial RRM through FORTRAN program and showed the situation where the proposed methods might be more efficient than the method of Ahmed et al. [22]. The simulation results give large number of situations where PRE(i), $i = 1, 2$ values of proposed partial RRM are more than 100. However, we presented only few values of PRE(1) and PRE(2) in Table 1, for different parameter values, $P = 0.5$, various values of $T$, $\mu_{\gamma_1} = 20$, $\mu_{\gamma_2} = 30$, two values of $\sigma_{\gamma_1}$ and $\sigma_{\gamma_2}$, $\theta_1 = 7.2, \theta_2 = 2$, $\delta_{10} = 4$, $\delta_{30} = -0.17$, $\delta_{13} = -1.65$, $\delta_{40} = 12.13$, and $\delta_{41} = 52.29$. The efficiency comparison on the values of scrambling variables that were earlier used in Ahmed et al. [22] model is also carried out, but our proposed model is less efficient on those values. Thus, we conclude that the efficiency of proposed partial RRM model can be increased or decreased by using different scrambling variables.

4. **Proposed Partial RRM under Stratified Random Sampling**

In this section, we present a partial randomized response model under stratification; a subsample in each stratum is drawn using a SRSWR sampling. Each sampled respondent in the $h^{th}$ stratum is requested for two responses by using two randomized response (RR) devices. The RR$_h$ Device I provides the scramble and true response of sensitive variables, whereas the RR$_r$ Device II provides the fake response, that is free from the sensitive variables.

The RR$_h$ Device I, bearing two types of statements: (i) Report the additive true value of both sensitive variables, say...
Table 1: PRE of proposed partial RR model vs Ahmed et al. [22] model.

| $\rho_{y_1,y_2}$ | $\sigma_{y_1}$ | $\sigma_{y_2}$ | $T = 0.1$ | $T = 0.3$ | $T = 0.5$ | $T = 0.7$ | $T = 0.9$ |
|-----------------|----------------|----------------|----------|----------|----------|----------|----------|
|                 |                |                | PRE(1)   | PRE(2)   | PRE(1)   | PRE(2)   | PRE(1)   |
| 2 2             | 112.87         | 110.15         | 145.35   | 133.00   | 189.95   | 157.43   | 246.90   | 172.20   | 239.90   | 114.70   |
| 2 6             | 112.87         | 110.15         | 145.37   | 133.02   | 190.01   | 157.47   | 247.05   | 172.31   | 240.38   | 114.93   |
| 6 2             | 112.87         | 110.15         | 145.34   | 133.00   | 189.92   | 157.40   | 246.81   | 172.14   | 239.61   | 114.56   |
| 6 6             | 112.87         | 110.15         | 145.36   | 133.02   | 190.00   | 157.46   | 247.02   | 172.29   | 240.27   | 114.88   |

|                 |                |                | PRE(1)   | PRE(2)   | PRE(1)   | PRE(2)   | PRE(1)   |
| 0.9             |                |                | 189.95   | 157.43   | 246.90   | 172.20   | 239.89   |
| 0.7             |                |                | 189.95   | 157.43   | 246.90   | 172.20   | 239.89   |
| 0.5             |                |                | 189.95   | 157.43   | 246.90   | 172.20   | 239.89   |
| 0.3             |                |                | 189.95   | 157.43   | 246.90   | 172.20   | 239.89   |
| 0.1             |                |                | 189.95   | 157.43   | 246.90   | 172.20   | 239.89   |
| 0.05            |                |                | 189.95   | 157.43   | 246.90   | 172.20   | 239.89   |
| 0.07            |                |                | 189.95   | 157.43   | 246.90   | 172.20   | 239.89   |
| 0.09            |                |                | 189.95   | 157.43   | 246.90   | 172.20   | 239.89   |

Note: \(Y_{1h} + Y_{2h}\) with probability \(T_h\), and (ii) Report the scrambled response as, \((S_1Y_{1h} + S_2Y_{2h})\) with probability \((1 - T_h)\). Mathematically, each \(h^{th}\) respondent of the \(i^{th}\) stratum is requested to report the response \(Z_{ih}\) as:

\[
Z_{ih} = \begin{cases} 
Y_{1h} + Y_{2h} & \text{with probability } T_h, \\
S_1Y_{1h} + S_2Y_{2h} & \text{with probability } (1 - T_h). 
\end{cases}
\]  

(72)

The partial randomized response \(Z_{ih}\) in the \(i^{th}\) sample of the \(h^{th}\) stratum is given by:

\[
Z_{ih} = \beta_{ih}(Y_{1h} + Y_{2h}) + (1 - \beta_{ih})(S_1Y_{1h} + S_2Y_{2h}),
\]  

(73)

and for the second response, each respondent is requested to use RR Device II which is same as equation (36); thus, the fake response \(Z_{ih}^{*}\) from the \(i^{th}\) sample of the \(h^{th}\) stratum is given by

\[
Z_{ih}^{*} = \begin{cases} 
S_1Y_{1h} + S_2Y_{2h} & \text{with probability } P_h T_h, \\
S_1^2Y_{1h} + S_2^2Y_{2h} & \text{with probability } ((1 - T_h) P_h), \\
S_1 Y_{1h} + S_2 Y_{2h} & \text{with probability } T_h (1 - P_h), \\
S_1 S_2 Y_{1h} + S_1 S_2 Y_{2h} & \text{with probability } (1 - P_h) (1 - T_h). 
\end{cases}
\]  

(75)
The generated response \( Z_{2n} \) in the \( ith \) sample of the \( hth \) stratum is given by
\[
Z_{2n} = y_{ln} \beta_{ln} (S_{1n} Y_{1n} + S_{hn} Y_{2n}) + (1 - \beta_{ln}) y_{hn} (S_{1n} Y_{1n} + S_{hn} Y_{2n}) \\
+ (1 - y_{hn}) \beta_{ln} (S_{2n} Y_{1n} + S_{hn} Y_{2n}) + (1 - y_{hn}) (1 - \beta_{ln}) (S_{1n} Y_{1n} + S_{hn} Y_{2n}).
\] (76)

Taking expected values on both sides of equations (73) and (76), we have
\[
E(Z_{1n}) = T_h \left( \mu_{y_{1n}} + \mu_{y_{2n}} \right) + (1 - T_h) \left( \epsilon_{1o1n} \mu_{y_{1n}} + \epsilon_{011n} \mu_{y_{2n}} \right),
\] (77)

\[
E(Z_{2n}) = P_h T_h \left( \epsilon_{101n} \mu_{y_{1n}} + \epsilon_{111n} \mu_{y_{2n}} \right) + (1 - T_h) P_h \left( \epsilon_{201n} \mu_{y_{1n}} + \epsilon_{111n} \mu_{y_{2n}} \right) \\
+ (1 - P_h) T_h \left( \epsilon_{111n} \mu_{y_{1n}} + \epsilon_{011n} \mu_{y_{2n}} \right) + (1 - P_h) (1 - T_h) \left( \epsilon_{111n} \mu_{y_{1n}} + \epsilon_{021n} \mu_{y_{2n}} \right).
\] (78)

From equations (77) and (78), by the method of moments, we have
\[
A_{11n} \bar{\mu}_{y_{1n}} + A_{12n} \bar{\mu}_{y_{2n}} = \bar{Z}_{1n},
\] (79)

and
\[
A_{21n} \bar{\mu}_{y_{1n}} + A_{22n} \bar{\mu}_{y_{2n}} = \bar{Z}_{2n},
\] (80)

where
\[
A_{11} = T_h (1 - T_h) \delta_{021} A_{111} - P_h \delta_{021} A_{121},
\]
\[
A_{12} = T_h (1 - T_h) \delta_{021} A_{111} - P_h \delta_{021} A_{121},
\]
\[
A_{21} = P_h T_h \delta_{021} A_{111} + (1 - P_h) \delta_{021} A_{111},
\]
\[
A_{22} = P_h T_h \delta_{021} A_{111} + (1 - P_h) \delta_{021} A_{111}.
\]

Solving equations (79) and (80) by using Cramer’s rule, we have unbiased estimators of \( \mu_{y_{1n}} \) and \( \mu_{y_{2n}} \), respectively, given by
\[
\bar{\mu}_{y_{1n}} = \frac{\sum_{h=1}^{L} W_h (A_{12n} \bar{Z}_{1n} - A_{21n} \bar{Z}_{2n})}{(1 - T_h) \delta_{021} A_{111} - P_h \delta_{021} A_{121}},
\] (81)

and
\[
\bar{\mu}_{y_{2n}} = \frac{\sum_{h=1}^{L} W_h (A_{21n} \bar{Z}_{1n} - A_{12n} \bar{Z}_{2n})}{(1 - T_h) \delta_{021} A_{111} - P_h \delta_{021} A_{121}}.
\]

**Theorem 2.** The variance of proposed estimators \( \bar{\mu}_{y_{1n}} \) and \( \bar{\mu}_{y_{2n}} \) is, respectively, given by
\[
V(\bar{\mu}_{y_{1n}}) = \sum_{h=1}^{L} W_h^2 \frac{\sigma^2_{1n}}{n_h},
\] (83)

and
\[
V(\bar{\mu}_{y_{2n}}) = \sum_{h=1}^{L} W_h^2 \frac{\sigma^2_{2n}}{n_h},
\] (84)

where
\[
\sigma^2_{1n} = \frac{A_{21n}^2 \sigma^2_{Z_{1n}} + A_{22n}^2 \sigma^2_{Z_{2n}} - 2 A_{11n} A_{21n} \sigma_{Z_{1n}Z_{2n}}}{(1 - T_h) [\delta_{021} A_{111} - P_h \delta_{021} A_{121}]}^2,
\] (85)

and
\[
\sigma^2_{2n} = \frac{A_{11n}^2 \sigma^2_{Z_{1n}} + A_{21n}^2 \sigma^2_{Z_{2n}} - 2 A_{11n} A_{21n} \sigma_{Z_{1n}Z_{2n}}}{(1 - T_h) [\delta_{021} A_{111} - P_h \delta_{021} A_{121}]}^2.
\] (86)
Corollary 2. The variance of proposed estimators \( \hat{\mu}_{y_i(at)} \) and \( \hat{\mu}_{y_j(at)} \) under different methods of sample allocation are as:

(i) Equal Allocation:

\[
V(\hat{\mu}_{y_i(at)})_{EA} = \frac{L}{n} \sum_{h=1}^{L} W_h^2 \sigma_{1h}^2,
\]

and

\[
V(\hat{\mu}_{y_j(at)})_{EA} = \frac{L}{n} \sum_{h=1}^{L} W_h^2 \sigma_{2h}^2.
\]

(ii) Proportional allocation:

\[
V(\hat{\mu}_{y_i(at)})_{PA} = \frac{1}{n} \sum_{h=1}^{L} W_h \sigma_{1h}^2,
\]

and

\[
V(\hat{\mu}_{y_j(at)})_{PA} = \frac{1}{n} \sum_{h=1}^{L} W_h \sigma_{2h}^2.
\]

(iii) Optimum Allocation:

\[
V(\hat{\mu}_{y_i(at)})_{OA} = \frac{1}{n} \left\{ \sum_{h=1}^{L} \frac{W_h \sigma_{1h}^2}{\eta \sigma_{1h}^2 + (1 - \eta) \sigma_{2h}^2 / C_h} \right\} \left\{ \sum_{h=1}^{L} W_h \sqrt{\eta \sigma_{1h}^2 + (1 - \eta) \sigma_{2h}^2 / C_h} \right\},
\]

and

\[
V(\hat{\mu}_{y_j(at)})_{OA} = \frac{1}{n} \left\{ \sum_{h=1}^{L} \frac{W_h \sigma_{2h}^2}{\eta \sigma_{1h}^2 + (1 - \eta) \sigma_{2h}^2 / C_h} \right\} \left\{ \sum_{h=1}^{L} W_h \sqrt{\eta \sigma_{1h}^2 + (1 - \eta) \sigma_{2h}^2 / C_h} \right\},
\]

where

\[
W_h = \eta \sigma_{1h}^2 + (1 - \eta) \sigma_{2h}^2 / C_h,
\]

which are estimated by using linear cost function such as \( C = C_0 + \sum_{h=1}^{L} n_h C_h \) (where \( C_0 \) is fixed cost, and \( C_h \) is the variable cost in each stratum) and the compensated variance as:

\[
V_{comp} = \sum_{h=1}^{L} W_h^2 (\eta \sigma_{1h}^2 + (1 - \eta) \sigma_{2h}^2 / n_h), \quad \text{whereas } \eta \text{ is known constant.}
\]

Proof is simple, so omitted.

Corollary 3. The unbiased estimators for \( V(\hat{\mu}_{y_i(at)}) \) and \( V(\hat{\mu}_{y_j(at)}) \) are, respectively, given by

\[
\bar{V}(\hat{\mu}_{y_i(at)}) = \frac{L}{n} \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma_{1h}^2,
\]

and

\[
\bar{V}(\hat{\mu}_{y_j(at)}) = \frac{L}{n} \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma_{2h}^2.
\]
where
\[ s_{1h}^2 = \frac{A_{2h}^2 z_{1h}^2 + A_{1h}^2 z_{2h}^2 - 2A_{12}A_{2h} s_{z_{1h}z_{2h}}}{(1 - \theta_h)^2[(1 - P_h)\delta_{1h}A_{1h} - P_h\delta_{2h}A_{12}]} \]  
(98)
and
\[ s_{2h}^2 = \frac{A_{1h}^2 z_{1h}^2 + A_{2h}^2 z_{2h}^2 - 2A_{12}A_{2h} s_{z_{1h}z_{2h}}}{(1 - \theta_h)^2[(1 - P_h)\delta_{1h}A_{1h} - P_h\delta_{2h}A_{12}]} \]  
(99)
\[ (n_h - 1)^{-1} \sum_{i=1}^{n_h} (Z_{1i} - Z_{1h})^2, s_{Z_{1h}Z_{2h}} = (n_h - 1)^{-1} \sum_{i=1}^{n_h} (Z_{1i} - Z_{1h})(Z_{2i} - Z_{2h}) \] are unbiased estimators of \( s_{Z_{1h}}, s_{Z_{2h}}, \) and \( s_{Z_{1h}Z_{2h}}, \) respectively.

4.1. Percent Relative Efficiency under Stratification. In this section, we compute the percent relative efficiency (PRE) of proposed partial RRM over Ahmed et al. [22] model under stratification using proportional allocation method. For numerical comparison, we use real data set that is taken from Rosner [25]; the childhood respiratory disease study of Boston. We consider \( Y_1 = AGE \) of a child and \( Y_2 = FEV \) (forced expiratory volume) both as sensitive variables as earlier used by Ahmed et al. [22]. The population is subdivided into two strata on the basis of gender. The PREs of proposed estimators \( \hat{\mu}_{y_{1}(\alpha)} \) and \( \hat{\mu}_{y_{2}(\alpha)} \) with respect to Ahmed et al. [22] estimators \( \tilde{\mu}_{y_{1}(\alpha)} \) and \( \tilde{\mu}_{y_{2}(\alpha)} \), respectively, are defined as:
\[ PRE(1) = \frac{V(\hat{\mu}_{y_{1}(\alpha)})}{V(\tilde{\mu}_{y_{1}(\alpha)})} \times 100 \]  
(100)
\[ \frac{\sum_{h=1}^{L} W_h \sigma_{\hat{\mu}_{y_{1}}}^2}{\sum_{h=1}^{L} W_h \sigma_{\tilde{\mu}_{y_{1}}}^2} \times 100, \] and
\[ PRE(2) = \frac{V(\hat{\mu}_{y_{2}(\alpha)})}{V(\tilde{\mu}_{y_{2}(\alpha)})} \times 100 \]  
(101)
\[ \frac{\sum_{h=1}^{L} W_h \sigma_{\hat{\mu}_{y_{2}}}^2}{\sum_{h=1}^{L} W_h \sigma_{\tilde{\mu}_{y_{2}}}^2} \times 100. \]

We also compute the PRE of proposed partial RRM under SRSWR over proposed partial RRM under stratification to observe the gain in efficiency due to stratification. For both estimators, the PRE is almost 100 at different values of design parameters. This is because the variation between strata is almost same, and the randomization devices are also identical.

In the next section, we consider an application of a real data set.

5. Application of Real Data Set

Hussain et al. [26] estimated the average total number of classes that were missed by the students and Gjestvang and Singh [27] considered the problem of estimating the average GPA of students. In this study, we simultaneously estimate the average total number of classes that were missed by the students and average GPA of the students, by using the proposed partial RRM under SRSWR.

We took a sample of 80 students from Stat 317, Stat 629, and Zoom 203 classes to estimate the average GPA and average total number of missed classes by the students, at Quaid-I-Azam University, Islamabad. We generated 20 random numbers of \( S_1 \) and \( S_2 \) separately from Poisson distribution with means 5 and 2, respectively. To collect the data through proposed RRM, two decks of cards were used: Deck I, a deck of yellow cards and Deck II, a deck of blue cards. A deck of yellow cards bearing two different types of statements by setting \( T = 0.5 \) to get the scrambled and true responses, whereas, deck of blue cards bearing values of two scramble variables by setting \( P = 0.7 \) to get fake responses.

In Deck I or deck of yellow cards, 20 cards, on 10 cards, we wrote the statements:

\[ \text{PleaseReport:} \left( \begin{array}{c} \text{GPA} \\ \text{scored in} \\ \text{last semester} \end{array} \right) + \left( \begin{array}{c} \text{Total classes} \\ \text{missed during} \\ \text{last semester} \end{array} \right), \]  
(102)

and, on remaining 10 cards, we wrote the statements:

\[ \text{PleaseReport:} \left( \begin{array}{c} \text{GPA} \\ \text{scored in} \\ \text{last semester} \end{array} \right) \right) \left( \begin{array}{c} S_1 \\ \text{random} \\ \text{number} \end{array} \right) + \left( \begin{array}{c} \text{Total classes} \\ \text{missed during} \\ \text{last semester} \end{array} \right) \left( \begin{array}{c} S_2 \\ \text{random} \\ \text{number} \end{array} \right). \]  
(103)

In this process, 20 values of \( S_1, i = 1, 2 \) were written on each card.

The Deck II or deck of blue cards, consists of 20 cards, out of which 14 cards had the values of \( S_1 \) and the remaining six cards had values of \( S_2 \), with statement “kindly report one selected random number.”

Before starting the data collection process, we highlighted the importance of randomized response methods.
and explained the proposed partial RRM and Ahmed et al. [22] model to the students. Each student was requested to draw one card from the Deck I and report the response as requested on the card, on yellow color paper. Similarly, repeat the process for Deck II and report the response as requested on the card, on blue color paper. Then, each student was provided a pink color card written on Ahmed et al. [22] model which was similar to equation (103) and requested to provide the scramble response on pink color paper. After marking the three responses on three different color papers, the students were advised to staple these papers together and put them into the box lying on the table. Finally, all students wrote the actual GPA and true total sizes; thus, we suggest that a large scale sample survey should be conducted in future to reach more realistic outcomes. The estimates of randomized response models reflect that the students are more reluctant to admit the total number of missed classes which they had missed during last semester on white color paper without disclosing their identity. Table 3 presents the responses obtained from the students.

Table 4 presents results of the survey. The estimates of the means $\bar{\mu}_y$ and $\bar{\mu}_x$ of proposed partial RRM are close to the estimates based on true responses as compared to the estimates obtained from Ahmed et al. [22] model. We noted that the standard error values are large due to small sample sizes; thus, we suggest that a large scale sample survey should be conducted in future to reach more realistic outcomes. The estimates of randomized response models reflect that the students are more reluctant to admit the total number of missed classes which they had missed during last semester on white color paper without disclosing their identity. Table 3 presents the responses obtained from the students.

Table 2: PRE of proposed partial RRM with respect to Ahmed et al. [22] model under stratification.

| $(T_1, T_2)$ | $(P_1, P_2) = 0.2$ | $(P_1, P_2) = 0.4$ | $(P_1, P_2) = 0.6$ | $(P_1, P_2) = 0.8$ |
|-------------|-----------------|-----------------|-----------------|-----------------|
|             | PRE(1)         | PRE(2)         | PRE(1)         | PRE(2)         |
| 0.0         | 100            | 100            | 100            | 100            |
| 0.1         | 130            | 126            | 104            | 101            |
| 0.2         | 172            | 160            | 108            | 100            |
| 0.3         | 229            | 203            | 110            | 98             |
| 0.4         | 307            | 257            | 109            | 92             |
| 0.5         | 413            | 323            | 105            | 83             |
| 0.6         | 552            | 396            | 94             | 69             |
| 0.7         | 713            | 458            | 76             | 50             |
| 0.8         | 826            | 458            | 48             | 28             |
| 0.9         | 642            | 290            | 16             | 8              |

Table 3: Data obtained through Ahmed et al. [22]; proposed partial RRM and direct responses.

Data obtained through RR device I of Ahmed et al. [22] model

| $T$ | $\mu_1$ | $\mu_2$ |
|-----|---------|---------|
| 0.0 | 100     | 100     |
| 0.1 | 130     | 126     |
| 0.2 | 172     | 160     |
| 0.3 | 229     | 203     |
| 0.4 | 307     | 257     |
| 0.5 | 413     | 323     |
| 0.6 | 552     | 396     |
| 0.7 | 713     | 458     |
| 0.8 | 826     | 458     |
| 0.9 | 642     | 290     |

Data obtained through RR device I of proposed partial RRM by setting $T = 0.5$

| $T$ | $\mu_1$ | $\mu_2$ |
|-----|---------|---------|
| 0.0 | 100     | 100     |
| 0.1 | 130     | 126     |
| 0.2 | 172     | 160     |
| 0.3 | 229     | 203     |
| 0.4 | 307     | 257     |
| 0.5 | 413     | 323     |
| 0.6 | 552     | 396     |
| 0.7 | 713     | 458     |
| 0.8 | 826     | 458     |
| 0.9 | 642     | 290     |

Data obtained through RR device II of proposed partial RRM by setting $P = 0.7$

| $T$ | $\mu_1$ | $\mu_2$ |
|-----|---------|---------|
| 0.0 | 100     | 100     |
| 0.1 | 130     | 126     |
| 0.2 | 172     | 160     |
| 0.3 | 229     | 203     |
| 0.4 | 307     | 257     |
| 0.5 | 413     | 323     |
| 0.6 | 552     | 396     |
| 0.7 | 713     | 458     |
| 0.8 | 826     | 458     |
| 0.9 | 642     | 290     |

Data obtained through RR device II of Ahmed et al. [22]; proposed partial RRM and direct responses.

| $T$ | $\mu_1$ | $\mu_2$ |
|-----|---------|---------|
| 0.0 | 100     | 100     |
| 0.1 | 130     | 126     |
| 0.2 | 172     | 160     |
| 0.3 | 229     | 203     |
| 0.4 | 307     | 257     |
| 0.5 | 413     | 323     |
| 0.6 | 552     | 396     |
| 0.7 | 713     | 458     |
| 0.8 | 826     | 458     |
| 0.9 | 642     | 290     |

Table 4: Results of the survey.

| Model            | Estimated mean $\bar{\mu}_y$ | Standard error $\hat{\sigma}_y$ | 95% confidence interval $\bar{\mu}_y$ |
|------------------|------------------------------|---------------------------------|---------------------------------------|
| Proposed partial RRM       | 3.18                        | 6.17                           | [2.06, 4.30]                           |
| Ahmed et al. [22]          | 2.33                        | 7.53                           | [1.41, 3.25]                           |
| Direct questioning         | 3.19                        | 5.05                           | [3.07, 5.31]                           |

Total classes missed during last semester from direct response

6, 7, 2, 9, 4, 10, 5, 3, 6, 0, 0, 5, 12, 6, 0, 8, 4, 2, 5, 3, 3, 4, 3, 3, 2, 12, 12, 6, 6, 7, 2, 4, 5, 4, 0, 6, 10, 4, 3, 4, 3, 9, 4, 6, 5, 12, 2, 5, 5, 6, 6, 3, 3, 8, 5, 8, 4, 0, 6, 0, 0, 7, 3, 5, 10, 4, 7, 5, 5, 9, 6, 3, 10, 0, 5, 2, 6, 5, 6
6. Conclusion

In the present paper, we have suggested a partial randomized response model for estimating two population means simultaneously. Through simulation study and real life data application, it is observed that the proposed partial RRM performs better than the Ahmed et al. [22] model. The superiority of suggested partial RRM under stratification revealed through numerical comparison and it is observed that the proposed partial RRM under stratified random sampling performs better as compared to stratified model of Ahmed et al. [22]. Moreover, we also observed that design parameters play an important role in increasing or decreasing the efficiency of suggested models. The main advantage of proposed partial RRM is that it enables researchers to collect truthful responses at least from some proportion of people. Thus, the proposed partial randomized response model are therefore recommended for its use in practice as an alternative to Ahmed et al. [22] randomized response model.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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