Contractibility of Maximal Ideal Spaces of Certain Algebras of Almost Periodic Functions

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Abstract

We study some topological properties of maximal ideal spaces of certain algebras of almost periodic functions. Our main result is that such spaces are contractible. We present several analytic corollaries of this result.

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1. Introduction.

1.1. This paper is devoted to the study of some topological properties of maximal ideal spaces of certain algebras of almost periodic functions. We apply our results to the solution of a matrix version of the corona problem for such algebras. To formulate these results we first introduce some notation and basic definitions.

The algebra $AP$ of (uniformly) almost periodic functions is, by definition, the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions $e_\lambda := e^{i\lambda x}$, $\lambda \in \mathbb{R}$. A finite sum $p = \sum r_j e_{\lambda_j} \in AP$ with $r_j \in \mathbb{C}$ and $\lambda_j \in \mathbb{R}$ is called an almost periodic polynomial. The set of almost periodic polynomials is denoted by $AP^0$.

The Bohr-Fourier spectrum $\Omega(a)$ of $a \in AP$ is the subset of $\mathbb{R}$ defined by the formula

$$\Omega(a) := \{ \lambda \in \mathbb{R} : \lim_{N \to \infty} \frac{1}{2N} \int_{-N}^{N} a(x) e^{-\lambda x} dx \neq 0 \} .$$

It is well known that $\Omega(a)$ is at most countable, see, e.g., [BKS, Sect. 2.7]. For instance, for $p = \sum_{j=1}^{k} r_j e_{\lambda_j} \in AP^0$ with all $r_j$ nonzero, $\Omega(p) = \{ \lambda_1, \ldots, \lambda_k \}$.

Let $\Sigma \subset [0, \infty)$ be an additive semigroup (i.e., $\lambda, \mu \in \Sigma$ implies that $\lambda + \mu \in \Sigma$) and suppose $0 \in \Sigma$. Put

$$AP^0_\Sigma := \{ a \in AP^0 : \Omega(a) \subset \Sigma \} .$$

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Let $AP_\Sigma$ be the closure of $AP^0_\Sigma$ in $AP$. Corollary 7.6 of [BKS] shows that

$$AP_\Sigma = \{a \in AP : \Omega(a) \subset \Sigma\}.$$ 

Since $\Sigma$ is a semigroup containing 0, $AP_\Sigma$ is a Banach algebra. Let $M(AP_\Sigma)$ be the maximal ideal space of $AP_\Sigma$, that is the space of nonzero homomorphisms $AP_\Sigma \to \mathbb{C}$ equipped with the weak* topology (i.e., the Gelfand topology). The analytic structure of $M(AP_\Sigma)$ is described by the Arens-Singer theorem, see, e.g., [BKS, Theorem 12.4]. In contrast, our main result describes the topological structure of $M(AP_\Sigma)$.

**Theorem 1.1** $M(AP_\Sigma)$ is contractible.

**Corollary 1.2** All Čech cohomology groups $H^m(M(AP_\Sigma), \mathbb{Z})$, $m \geq 1$, are trivial.

1.2. Note that every function $a \in AP_\Sigma$ can be extended to a holomorphic function $\hat{a}$ in the upper half-plane $\mathbb{H}_+$ by the Poisson integral formula. Obviously,

$$\sup_{\mathbb{H}_+} |\hat{a}| = \sup_{\mathbb{R}} |a|.$$

This produces a natural embedding $\mathbb{H}_+ \hookrightarrow M(AP_\Sigma)$, $z \mapsto \{\text{evaluation at } z\}$. The *corona problem* for $AP_\Sigma$ is to decide whether $\mathbb{H}_+$ is dense in $M(AP_\Sigma)$ (in the Gelfand topology). The density is equivalent to the following statement (see, e.g., [BKS, Sect. 12.1]):

Given a collection $f_1, \ldots, f_n$ of functions from $AP_\Sigma$ satisfying

$$\inf_{z \in \mathbb{H}_+} \sum_{j=1}^n |\hat{f}_j(z)| > 0,$$

there are $g_1, \ldots, g_n \in AP_\Sigma$ such that

$$\hat{f}_1(z)\hat{g}_1(z) + \ldots + \hat{f}_n(z)\hat{g}_n(z) = 1 \quad \text{for all } z \in \mathbb{H}_+.$$

In [BKS, Theorem 12.7] a necessary and sufficient condition for $\Sigma$ is given to conclude that $\mathbb{H}_+$ is dense in $M(AP_\Sigma)$. For instance, it is true if $\Sigma = \Delta \cap [0, \infty)$ where $\Delta \subset \mathbb{R}$ is an additive group. (It is not true, e.g., for $\Sigma = \{k + l\sqrt{2} : k, l \in \mathbb{Z}_+\}$, see [BKS, Example 12.9].)

In this paper we consider the following matrix version of the corona problem.

Let $A = (a_{ij})$ be an $n \times k$ matrix, $k < n$, with entries in $AP_\Sigma$. Assume that the family of determinants of the submatrices of $A$ of order $k$ satisfies condition (1.1). Is there an $n \times n$ matrix $\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} \in AP_\Sigma$, so that $\tilde{a}_{ij} = a_{ij}$ for $1 \leq j \leq k$, $1 \leq i \leq n$, and $\det(\tilde{A}) = 1$?

In case that such a $\tilde{A}$ exists we say that $\tilde{A}$ completes $A$. As a corollary of Theorem 1.1 we obtain the following result.$^1$

**Theorem 1.3** Suppose that $\mathbb{H}_+$ is dense in $M(AP_\Sigma)$. Then for any $A$ satisfying the matrix version of the corona problem there is a matrix $\tilde{A}$ completing it.

Finally, we formulate one other application of Theorem 1.1.

**Theorem 1.4** Suppose that $\mathbb{H}_+$ is dense in $M(AP_\Sigma)$. Let $f \in AP_\Sigma$ satisfy (1.1) (with $n = 1$, $f_1 := f$). Then there is $g \in AP_\Sigma$ such that $f = e^g$.

$^1$Theorem 1.3 answers a question of L.Rodman [Ro].
2. Proofs.

Proof of Theorem 1.1. Let $X(\Sigma)$ be the vector space of complex-valued functions on $\Sigma$ equipped with the product topology. Consider the map $j : \mathbb{H}_+ \to X(\Sigma)$,

$$j(z)(\lambda) := e^{i\lambda z}, \quad z \in \mathbb{H}_+, \ \lambda \in \Sigma.$$  

Let $\{z_\lambda\}_{\lambda \in \Sigma}$ be the family of coordinate functionals on $X(\Sigma)$, i.e., $z_\lambda(f) := f(\lambda)$, $f \in X(\Sigma)$. By $P(\Sigma)$ we denote the algebra of polynomials on $X(\Sigma)$ in variables $\{z_\lambda\}$. By definition, any $p \in P(\Sigma)$ is a finite sum $\sum r_{i_1 \cdots i_k} z_{\lambda_{i_1}} \cdots z_{\lambda_{i_k}}$ with $r_{i_1 \cdots i_k} \in \mathbb{C}$. Let $H(\Sigma) \subset X(\Sigma)$ be the polynomial hull of $j(\mathbb{H}_+)$ with respect to $P(\Sigma)$, that is,

$$z \in H(\Sigma) \iff |p(z)| \leq \sup_{j(\mathbb{H}_+)} |p| \text{ for any } p \in P(\Sigma).$$

Then the direct limit construction of Royden [R] implies that $H(\Sigma)$ is homeomorphic to $M(AP\Sigma)$. From now on we identify these two objects.

Further, define the map $R : X(\Sigma) \times [0, 1] \to X(\Sigma)$ by the formula

$$R(f, t)(\lambda) := t^h f(\lambda), \quad f \in X(\Sigma), \ t \in [0, 1], \ \lambda \in \Sigma.$$  

Clearly this map is continuous if we consider $X(\Sigma)$ with the product topology and $[0, 1]$ with the usual one. So $R$ determines a contraction of $X(\Sigma)$ to 0. We will show that $R$ maps $M(AP\Sigma) \times [0, 1]$ to $M(AP\Sigma)$. This will complete the proof.

First, prove that $R$ maps $j(\mathbb{H}_+) \times (0, 1]$ to $j(\mathbb{H}_+)$. Indeed, for a fixed $t \in (0, 1]$ we have

$$R(j(z), t)(\lambda) := t^h e^{i\lambda z} = e^{i\lambda(-i \log t)} e^{i\lambda z} = e^{i\lambda(z+i \log(1/t))} = j(z + i \log(1/t))(\lambda).$$

We used here that $\log(1/t) \geq 0$. The above identity implies that $R(j(z), t) = j(z + i \log(1/t))$ as required. Note also that the closure $\overline{j(\mathbb{H}_+)}$ contains 0.

Now suppose that there is $v \in M(AP\Sigma)$ and $t \in (0, 1)$ such that $R(v, t) \not\in M(AP\Sigma)$. Then there is a polynomial $p(z_{\lambda_1}, \ldots, z_{\lambda_k}) \in P(\Sigma)$ such that

$$|p(t^{\lambda_1} v_{\lambda_1}, \ldots, t^{\lambda_k} v_{\lambda_k})| > \sup_{j(\mathbb{H}_+)} |p|$$

where $v_{\lambda_i} := z_{\lambda_i}(v)$. Consider the polynomial $q(z_{\lambda_1}, \ldots, z_{\lambda_k}) := p(t^{\lambda_1} z_{\lambda_1}, \ldots, t^{\lambda_k} z_{\lambda_k})$. Then the previous inequality implies that

$$|q(v)| > \sup_{j(\mathbb{H}_+)} |p| \geq \sup_{j(\mathbb{H}_+)} |p| = \sup_{j(\mathbb{H}_+)} |q|.$$

This contradicts the fact that $v \in M(AP\Sigma)$.  \[ \square \]

Proof of Corollary 1.2. The proof follows straightforwardly from the fact that $M(AP\Sigma)$ is homotopic to a point.  \[ \square \]

Proof of Theorem 1.3. Since $\mathbb{H}_+$ is dense in $M(AP\Sigma)$ and $M(AP\Sigma)$ is contractible, the proof is a direct consequence of Theorem 3 of Lin [L].  \[ \square \]

Proof of Theorem 1.4. The proof follows directly from the Arens-Royden theorem (see [R]), because in our case $H^1(M(AP\Sigma), \mathbb{Z}) = 0$.  \[ \square \]
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