LOCAL TERMS FOR THE CATEGORICAL TRACE

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Abstract. In this paper we introduce the categorical “true local terms” maps for Artin stacks and show that they are additive and commute with proper pushforwards, smooth pullbacks and specializations. In particular, we generalize results of [Va2] to this setting.

As an application, we supply proofs of two theorems stated in [AGKRRV1]. Namely, we show that the “true local terms” of the Frobenius endomorphism coincide with the “naive local terms” and that the “naive local terms” commute with !-pushforwards. The latter result is a categorical version of the classical Grothendieck–Lefschetz trace formula.

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Introduction

0.1. Let k be an algebraically closed field.

(a) Let X be an Artin stack of finite presentation over k. To X we can associate two DG categories: category Shv(X) of \(\bar{Q}_\ell\)-adic sheaves on X (see [AGKRRV1 Appendix F]) and category Shv(X)\text{ren} := \text{Ind} \text{Shv}(X)\text{constr} of ind-constructible sheaves (see [AGKRRV1 Section F.5]). We have a natural fully faithful renormalization functor

\[ \text{ren}_X : \text{Shv}(X) \to \text{Shv}(X)\text{ren}, \]

which has a continuous right adjoint unren\(_X\).

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(b) Let \( c = (c_l, c_r) : C \to X \times X \) be a morphism of Artin stacks of finite presentation over \( k \), which we call a correspondence. A correspondence \( c \) gives rise to continuous endofunctors

\[
(c_l)^\bullet \circ c_r^! : \text{Shv}(X) \to \text{Shv}(X) \quad \text{and} \quad (c_l)_\ast \circ c_r^! : \text{Shv}(X)^\text{ren} \to \text{Shv}(X)^\text{ren},
\]

where \((-)^\bullet\) denote the renormalized pushforward (see Section [1.2]), and we denote both of these endofunctors by \([c]\).

(c) Both DG categories \( \text{Shv}(X) \) and \( \text{Shv}(X)^\text{ren} \) are compactly generated, thus dualizable, hence one can consider traces \( \text{Tr}(\text{Shv}(X), [c]) \) and \( \text{Tr}(\text{Shv}(X)^\text{ren}, [c]) \), which are objects of the \( \infty \)-category \( \text{Vect} \) of \( \mathbb{Q}_\ell \)-vector spaces. Furthermore, by the trace formalism (see [GKRV Section 3.2]) the renormalization functor \( \text{ren} \) gives rise to a morphism between traces

\[
\text{Tr}(\text{ren}_X, [c]) : \text{Tr}(\text{Shv}(X), [c]) \to \text{Tr}(\text{Shv}(X)^\text{ren}, [c]).
\]

0.2. True local terms. Let \( c = (c_l, c_r) : C \to X \times X \) be a correspondence between Artin stacks of finite presentation over \( k \), and let \( A \in \text{Shv}(X)^\text{constr} \) be a constructible sheaf.

(a) Extending the construction of [Va2 Section 1.2.2], to this data one can associate the trace map

\[
\mathcal{T}_{r,c,A} : \text{Hom}_{\text{Shv}(X)^\text{ren}}(A, [c](A)) \to \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}),
\]

where

\[
\bullet \text{ Fix}(c) := C \times_{X \times X} X \text{ is the Artin stack of fixed points of } c,
\]

\[
\bullet \omega_Y \text{ denotes the dualizing sheaf of a stack } Y, \text{ and}
\]

\[
\bullet \Gamma(Y, -) : \text{Shv}(Y)^\text{ren} \to \text{Vect} \text{ denotes the functor of global sections.}
\]

(b) On the other hand, using functoriality of trace maps (see [GKRV Section 3.5.4]) one associates to this data the Chern character map

\[
\text{ch}_{c,A} : \text{Hom}_{\text{Shv}(X)^\text{ren}}(A, [c](A)) \to \text{Tr}(\text{Shv}(X)^\text{ren}, [c]).
\]

(c) The first goal of the paper is to associate to a correspondence \( c \) a true local terms map

\[
\text{LT}_{c}^{\text{true}} : \text{Tr}(\text{Shv}(X)^\text{ren}, [c]) \to \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)})
\]

such that the map of part (a) decomposes as

\[
\mathcal{T}_{r,c,A} \simeq \text{LT}_{c}^{\text{true}} \circ \text{ch}_{c,A}.
\]

(d) Slightly abusing the notation, we also denote the composition

\[
\text{Tr}(\text{Shv}(X), [c]) \xrightarrow{\text{Tr}(\text{ren}_X, [c])} \text{Tr}(\text{Shv}(X)^\text{ren}, [c]) \xrightarrow{\text{LT}_{c}^{\text{true}}} \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)})
\]

by \( \text{LT}_{c}^{\text{true}} \) and call it the true local terms map as well.

0.3. Functoriality.

The main technical result of the paper asserts that the true local term maps commute with proper pushforwards, smooth pullbacks and restrictions to closed subschemes \( Z \subseteq X \) such that \( c \) is contracting near \( Z \), which we are going to formulate now (see Section 5 for a slightly more general assertion and more details): Consider the commutative diagram

\[
\begin{array}{cc}
C & \xrightarrow{c} & X \times X \\
\downarrow g & & \downarrow f \times f \\
D & \xrightarrow{d} & Y \times Y
\end{array}
\]
of Artin stacks of finite presentation over $k$, and denote by $g_{\Delta} : \text{Fix}(c) \to \text{Fix}(d)$ the induced map.

**Theorem 0.4.**

(a) Every commutative diagram (0.1) such that morphisms $f$ and $g$ are proper and safe\(^1\) gives rise to a homotopy commutative diagram

$$
\begin{align*}
\text{Tr}(\text{Shv}(X)^{\text{ren}}, \{c\}) & \xrightarrow{\text{LT}_{\text{true}} c} \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
\text{Tr}((f^*)!) & \downarrow (g_{\Delta})! \\
\text{Tr}(\text{Shv}(Y)^{\text{ren}}, \{c\}) & \xrightarrow{\text{LT}_{\text{true}} d} \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}).
\end{align*}
$$

(b) Every commutative diagram (0.1) such that either

(i) morphisms $f$ and $g$ are smooth of the same relative dimension and $g_{\Delta}$ is étale or

(ii) $f$ is a closed embedding, the diagram is Cartesian and $d$ is contracting near $f(X) \subseteq Y$

gives rise to a homotopy commutative diagram

$$
\begin{align*}
\text{Tr}(\text{Shv}(X)^{\text{ren}}, \{c\}) & \xrightarrow{\text{LT}_{\text{true}} c} \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
\text{Tr}((f^*)!) & \downarrow g_{\Delta} \\
\text{Tr}(\text{Shv}(Y)^{\text{ren}}, \{c\}) & \xrightarrow{\text{LT}_{\text{true}} d} \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}).
\end{align*}
$$

**0.5. Remarks.**

(a) As in [Va2], in order to show Theorem 0.4(b)(ii), we show that the true local terms commute with specializations, and we apply this in the case of a specialization to the normal cone. Furthermore, for further reference, we divide the commutation with specialization assertion into two and prove the commutation with nearby cycles and with extensions of scalars.

(b) Using observations of Section 0.2(c), the commutation of true local terms with proper pushforwards and specializations generalize the corresponding results of [Va2]. On the other hand, the commutation with smooth pullbacks seems to be new even in the classical setting of [Va2].

**0.6. The case of the Frobenius endomorphism.**

(a) Assume from now that $k = \overline{F}_q$, but that $X$ is defined over $F_q$, so that it carries the geometric *Frobenius* endomorphism, denoted $\text{Fr}$. Then we can associate to $X$ the groupoid $X(F_q)$, and hence the (classical) vector space $\text{funct}(X(F_q), \mathbb{Q}_\ell)$.

In addition, the endomorphism $\text{Fr}$ induces continuous endofunctors

$$
\text{Fr}_X = \text{Fr}_* : \text{Shv}(X) \to \text{Shv}(X) \text{ and } \text{Fr}_* : \text{Shv}(X)^{\text{ren}} \to \text{Shv}(X)^{\text{ren}},
$$

hence we can form traces $\text{Tr}(\text{Shv}(X), \text{Fr}_*), \text{Tr}(\text{Shv}(X)^{\text{ren}}, \text{Fr}_*) \in \text{Vect}$.

(b) Let $X$ and $Y$ be a pair of Artin stacks as above, and let $f : X \to Y$ be a morphism between them defined over $F_q$. Then $f$ gives rise to a map of groupoids $f(F_q) : X(F_q) \to Y(F_q)$, and hence to a map

$$
f(F_q) : \text{funct}(X(F_q), \mathbb{Q}_\ell) \to \text{funct}(Y(F_q), \mathbb{Q}_\ell)
$$

\(^1\)See Section 1.3 what safe morphism means.

\(^2\)Recently, the commutation of true local terms with smooth pullbacks in the classical setting appeared in [FYZ].
of $\overline{\mathbb{Q}}_\ell$-vector spaces, given by the summation along the fibers.

Also $f$ gives rise to a functor $f_! : \text{Shv}(X) \to \text{Shv}(Y)$, admitting a continuous right adjoint, given by $f^!$, and interchanging the Frobenius actions.

Therefore by the trace formalism (see [GKRV, Section 3.2]) functor $f_!$ induces a map

$$\text{Tr}(f_!, Fr_*): \text{Tr}(\text{Shv}(X), Fr_*) \to \text{Tr}(\text{Shv}(Y), Fr_*)$$

inVect. Moreover, the assignment $f_! \mapsto \text{Tr}(f_!, Fr_*): \text{Shv}(X) \to \text{Shv}(Y)$, admitting a continuous right adjoint, given by $f^!$, and hence induces a map between traces

$$\text{Tr}(f, Fr_*): \text{Tr}(\text{Shv}(X)^{\text{ren}}, Fr_*) \to \text{Tr}(\text{Shv}(Y)^{\text{ren}}, Fr_*)$$

in Vect.

### 0.7. Local terms and the sheaf–function correspondence.

Let again $X$ be as above (an Artin stack over $\mathbb{F}_q$, but defined over $\mathbb{F}_q$).

(a) Then one can associate to $X$ two pairs

$$\text{LT}^\text{naive}_X, \text{LT}^\text{true}_X : \text{Tr}(\text{Shv}(X), Fr_*) \to \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$$

and

$$\text{LT}^\text{naive}_X, \text{LT}^\text{true}_X : \text{Tr}(\text{Shv}(X)^{\text{ren}}, Fr_*) \to \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$$

of naturally defined maps called the naive local term and true local term maps, respectively.

Namely, the true local terms maps $\text{LT}^\text{true}_X$ are simply the maps $\text{LT}^\text{true}_X c$ (see Section 0.2), corresponding to the correspondence $(Fr, \text{Id}) : X \to X \times X$, while the naive local terms map $\text{LT}^\text{naive}_X$ is characterized by the condition that for every point $x \in X(\mathbb{F}_q)$ corresponding to the morphism $\eta_x : \text{pt} := \text{Spec} \mathbb{F}_q \to X$ the composition

$$\text{Tr}(\text{Shv}(X), Fr_*) \xrightarrow{\text{LT}^\text{naive}_X} \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) \xrightarrow{\text{ev}_x} \overline{\mathbb{Q}}_\ell$$

is equal to the map of traces

$$\text{Tr}(\eta^*_x, Fr_*) : \text{Tr}(\text{Shv}(X), Fr_*) \to \text{Tr}((\text{pt}, Fr_*) \simeq \overline{\mathbb{Q}}_\ell,$$

induced by the pullback $\eta^*_x : \text{Shv}(X) \to \text{Shv}(\text{pt}) = \text{Vect}$ (and similarly for $\text{Shv}(X)^{\text{ren}}$).

(b) Notice that for every $A \in \text{Shv}(X)^{\text{constr}}$ we have $Fr_*(A) \in \text{Shv}(X)^{\text{constr}}$, so the Chern character map of Section 0.2(b) has the form

$$\text{ch}_{X,A} : \mathcal{H}om_{\text{Shv}(X)^{\text{constr}}}(A, Fr_*(A)) \to \text{Tr}(\text{Shv}(X)^{\text{ren}}, Fr_*)$$

Furthermore, the composition

$$\text{LT}^\text{naive}_X \circ \text{ch}_{X,A} : \mathcal{H}om_{\text{Shv}(X)^{\text{constr}}}(A, Fr_*(A)) \to \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$$

recovers the Grothendieck “sheaf-function correspondence”.

As an application of Theorem 0.4 we prove the following result, stated as [AGKRRV1, Theorems 22.1.9 and 22.2.8].

### Theorem 0.8.

(a) For every $X$ as above, we have natural homotopies of morphisms

$$\text{LT}^\text{naive}_X \simeq \text{LT}^\text{true}_X : \text{Tr}(\text{Shv}(X), Fr_*) \to \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$$
and
\[ \text{LT}^{\text{true}}_X \simeq \text{LT}^{\text{true}}_X : \text{Tr}(\text{Shv}(X)^{\text{ren}}, \text{Fr}_*) \to \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_l) \]

(b) The naive local term functor
\[ \text{LT}^{\text{naive}} : \text{Tr}(\text{Shv}(-), \text{Fr}) \to \text{funct}(-, \mathbb{F}_q) \] (resp. \( \text{LT}^{\text{naive}} : \text{Tr}(\text{Shv}(-)^{\text{ren}}, \text{Fr}) \to \text{funct}(-, \overline{\mathbb{Q}}_l) \)) commutes with all \(!\)-pushforwards (resp. \(!\)-pushforwards with respect to the safe morphisms).

Namely, for every morphism \( f : X \to Y \), the following diagram commutes up to a canonical homotopy:

\[ \begin{array}{ccc}
\text{Tr}(\text{Shv}(X), \text{Fr}) & \xrightarrow{\text{LT}^{\text{naive}}_X} & \text{funct}(X(F_q), \overline{\mathbb{Q}}_l) \\
\text{Tr}(f_!, \text{Fr}) & \downarrow & \downarrow f(F_q) \\
\text{Tr}(\text{Shv}(Y), \text{Fr}) & \xrightarrow{\text{LT}^{\text{naive}}_Y} & \text{funct}(Y(F_q), \overline{\mathbb{Q}}_l),
\end{array} \]

and the corresponding result for \( \text{Shv}(-)^{\text{ren}} \) holds when \( f \) is safe.

Combining Theorem 0.8(a) and Section 0.7(b), we get the following consequence used in [AGKRRV3].

**Corollary 0.9.** For every Artin stack \( X \) over \( \mathbb{F}_q \), defined over \( \mathbb{F}_q \) and every \( A \in \text{Shv}(X)^{\text{constr}} \) the composition
\[ \text{LT}^{\text{true}}_X \circ \text{ch}_{X, A} : \text{Hom}_{\text{Shv}(X)^{\text{constr}}}(A, \text{Fr}_*(A)) \to \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_l) \]
equals the Grothendieck “sheaf-function correspondence” map.

**0.10. Strategy of the proof.** Let us explain how to deduce Theorem 0.8 from Theorem 0.4.

(i) Notice that the correspondence \((\text{Fr}, \text{Id})\) is contracting near every closed substack \( Z \subseteq X \) defined over \( \mathbb{F}_q \). Therefore it follows from Theorem 0.4 that the true local terms maps \( \text{LT}^{\text{true}}_X \) commute with \(!\)-pushforwards with respect to proper safe morphisms and \(*\)-pullbacks with respect to morphisms, which are either smooth or closed embeddings.

(ii) Next we observe that in order to show Theorem 0.8(a), it suffices to show that true local terms commute with \(*\)-pullbacks for every \( \eta_x : \text{pt} \to X \) with \( x \in X(\mathbb{F}_q) \). Let \( G_x := \text{Aut}_X(x) \) be the group of automorphisms of \( x \). Then \( \eta_x \) decomposes as a composition of morphisms
\[ \eta_x : \text{pt} \to \text{pt} / (G_x)_{\text{red}} \to \text{pt} / (G_x) \to X, \]
the first of which is smooth, the second one is a proper universal homomorphism, while the last one is a composition of an open and a closed embedding. Therefore the commutation of \( \text{LT}^{\text{true}}_X \) with \( \eta_x^* \) follows from the observations of (i).

(iii) Note that it suffices to show the assertion of Theorem 0.8(b) under an assumption that \( Y = \text{pt} \). Indeed, using base change the assertion of Theorem 0.8(b) for \( f = \text{pt} \) is equivalent to the corresponding assertions for every fiber \( f^{-1}(x) \to \text{pt} \) of \( f \). Moreover, by additivity of traces and Noetherian induction, we can replace \( X \) by its open non-empty substack. Thus we can assume that \( f \) decomposes as \( X \to X' \to \text{pt} \), where \( X' \) is an affine scheme, and \( f' : X \to X' \) is a gerbe, hence all fibers of \( f' \) are classifying stacks \( BG \). Thus it suffices to show the assertion when \( X \) is either an affine scheme or a classifying stack \( BG \).

(iv) In the affine scheme case, by additivity we can assume that \( X \) is projective, in which case the assertion follows part (a) and the fact that \( \text{LT}^{\text{true}}_X \) commute with \(!\)-pushforwards with respect...
to proper safe morphisms. In the case, when \( X = BG \), we can separately consider the case when \( G \) is connected and \( G \) is finite. In the first case, the assertion follows from the \( X = G \) case and Lang’s theorem. In the second case, we can imbed \( G \) into \( GL_n \), and deduce the assertion from the \( B(GL_n) \)-case and the scheme case.

0.11. **Plan of the paper.** The paper is organized as follows:

In Section 1 we recall basic properties of DG categories of sheaves on Artin stacks, mainly recalling results from \([AGKRRV1, AGKRRV2]\), and formulate standard properties of safe stacks and safe morphisms, whose proofs are recalled in Appendix A.

In Section 2 we introduce correspondences and discuss properties of functors, induced by correspondences.

In Section 3 we introduce the true local terms map and its refined version.

In Section 4 we formulate properties of true local terms and deduce generalizations of \([Va2]\).

In Section 5 we deduce Theorem 0.8 from results formulated in Section 4.

In Section 6 we state the result that true local terms commute with nearby cycles and extensions of scalars, and deduce the assertion about contracting correspondences, formulated in Section 4, using deformation to the normal cone.

In Section 7 we provide proof of Propositions 3.5 and 3.7.

In Sections 8 and 9 we provide proofs of functorial properties of true local terms, formulated in Sections 3, 4 and 6.

Finally, in Appendix B we review properties of quasi-smooth maps, used earlier.

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1. **Sheaves on Artin stacks**

Let \( k \) be an algebraically closed field. All Artin stacks will be assumed to be of finite presentation over \( k \).

1.1. **Sheaves on Artin stacks.**

(a) As in \([AGKRRV1\) Appendix F] to every Artin stack \( X \) one associates a \( \overline{Q}_p \)-linear stable \( \infty \)-category \( \text{Shv}(X) \) of ind-constructible \( \overline{Q}_p \)-sheaves. This category is compactly generated, thus dualizable, and we denote by \( \text{Shv}(X)^c \subseteq \text{Shv}(X)^{\text{constr}} \subseteq \text{Shv}(X) \) the full subcategories of compact objects and of constructible sheaves, respectively.

(b) To every morphism \( f : X \to Y \) of Artin stacks, one can associate two adjoint pairs \( (f_!, f^!) \) and \( (f^*, f_* ) \) of functors between \( \text{Shv}(X) \) and \( \text{Shv}(Y) \). The functors \( f_! , f^! \) and \( f^* \) are automatically continuous.

1.2. **Renormalized \( * \)-pushforward.**

(a) As in \([AGKRRV2\) Section A.2.3], to every morphism \( f : X \to Y \) of Artin stacks one associates the renormalized \( * \)-pushforward functor

\[
 f^* : \text{Shv}(X) \to \text{Shv}(Y),
\]

defined as the unique continuous functor, whose restriction to \( \text{Shv}(X)^c \) is \( f_*|_{\text{Shv}(X)^c} \).
(b) By definition, we have a natural morphism of functors
\[\text{can}_f : f^! \to f_* ,\]
whose restriction to \(\text{Shv}(X)^c\) is an isomorphism. Moreover, \(\text{can}_f\) is an isomorphism if and only if \(f_*\) is continuous (or, equivalently, when \(f^*\) preserves compact objects).

(c) For a composition \(X \xrightarrow{f} Y \xrightarrow{g} Z\) of Artin stacks, we have a canonical morphism
\[(1.1) \quad g^! \circ f^! \to (g \circ f)^! \]
of functors \(\text{Shv}(X) \to \text{Shv}(Z)\), whose restriction to \(\text{Shv}(X)^c\) is the morphism
\[g^! \circ f^!|_{\text{Shv}(X)^c} = g^! \circ f_*|_{\text{Shv}(X)^c} \xrightarrow{\text{can}_g \circ \text{can}_f} g_* \circ f_*|_{\text{Shv}(X)^c} \simeq (g \circ f)_*|_{\text{Shv}(X)^c} = (g \circ f)^!|_{\text{Shv}(X)^c} .\]
In particular, morphism (1.1) is automatically an isomorphism, if \(f_*|_{\text{Shv}(X)^c} \subseteq \text{Shv}(Y)^c\).

(c)' By construction, morphism (1.1) can be characterized as the unique morphism making the following diagram homotopy commutative:
\[\begin{array}{ccc}
g^! \circ f^! & \xrightarrow{(1.1)} & (g \circ f)^! \\
\text{can}_{g^! \circ f^!} \downarrow & & \downarrow \text{can}_{g \circ f^!} \\
g_* \circ f_* & \xrightarrow{\sim} & (g \circ f)_*.
\end{array}\]

(d) For every Cartesian diagram of Artin stacks
\[\begin{array}{ccc}
A & \xrightarrow{a} & C \\
g \downarrow & & \downarrow f \\
B & \xrightarrow{b} & D,
\end{array}\]
we have a canonical morphism
\[(1.2) \quad g^! \circ a^! \to b^! \circ f^! \]
of functors \(\text{Shv}(C) \to \text{Shv}(B)\), whose restriction to \(\text{Shv}(C)^c\) is the composition
\[g^! \circ a^!|_{\text{Shv}(C)^c} \xrightarrow{\text{can}_g} g_* \circ a_*|_{\text{Shv}(C)^c} \xrightarrow{\text{base change}} b^! \circ f_*|_{\text{Shv}(C)^c} = b^! \circ f^!|_{\text{Shv}(C)^c} .\]
In particular, morphism (1.2) is automatically an isomorphism, if \(a^!(\text{Shv}(C)^c) \subseteq \text{Shv}(A)^c\).

(d)' By construction, morphism (1.2) can be characterized as the unique morphism making the following diagram homotopy commutative:
\[\begin{array}{ccc}
g^! \circ a^! & \xrightarrow{(1.2)} & b^! \circ f^! \\
\text{can}_{g^! \circ a^!} \downarrow & & \downarrow \text{can}_f \\
g_* \circ a_* & \xrightarrow{\sim} & b^! \circ f_*.
\end{array}\]

(e) For every Artin stack \(X\), we denote by \(p_X : X \to \text{pt} := \text{Spec } k\) the projection, and write \(\Gamma(X, -) : \text{Shv}(X) \to \text{Vect}\) instead of \((p_X)_*\), and \(\Gamma^!(X, -) : \text{Shv}(X) \to \text{Vect}\) instead of \((p_X)^!\). By part (b), we have a canonical morphism \(\Gamma^!(X, -) \to \Gamma(X, -)\) of functors \(\text{Shv}(X) \to \text{Vect} \).
1.3. Safe morphisms. Let $f : X \to Y$ be a morphism of Artin stacks.

(a) To every geometric point $x$ of $X$, one associates the automorphism group
$$\text{Aut}_f(x) := \text{Aut}_{f^{-1}(f(x))}(x)$$
of $x$, viewed as a point of the Artin stack $f^{-1}(f(x))$.

(b) Following [DG, Definition 10.2.2], we say that a morphism $f$ is safe if for every geometric point $x$ of $X$, the connected component of the reduced part of $\text{Aut}_f(x)$ is unipotent. We will say that $X$ is safe if and only if the projection $X \to \text{pt}$ is safe.

(c) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of morphisms of Artin stacks such that $g$ is safe. Then morphism $f$ is safe if and only if $g \circ f$ is safe.

(d) By Proposition 1.4 below, a morphism $f$ is safe if and only if the functor $f^*$ is continuous.

(e) Notice that every representable morphism is safe, and every geometric Frobenius morphism $\text{Fr} : X \to X$ is safe.

(f) Note that if $f$ is a separated morphism, then all automorphism groups $\text{Aut}_f(x)$ are proper. Therefore a separated morphism $f$ is safe if and only if all automorphism groups $\text{Aut}_f(x)$ are finite. For example, a separated morphism between Artin stacks with affine diagonals is automatically safe.

The following assertion is an analog of [DG, Theorem 10.2.4 and Corollary 10.2.7]. For completeness, we provide its proof in Appendix A.

**Proposition 1.4.** (a) The following properties of an Artin stack $X$ are equivalent:

(i) $X$ is safe;
(ii) the constant sheaf $\mathbb{Q}_\ell \in \text{Shv}(X)$ is compact;
(iii) every constructible sheaf $A \in \text{Shv}(X)_{\text{constr}}$ is compact.

(b) The following properties of a morphism $f : X \to Y$ of Artin stacks are equivalent:

(i) $f$ is safe;
(ii) the functor $f_* : \text{Shv}(X) \to \text{Shv}(Y)$ is continuous;
(iii) the functor $f_! : \text{Shv}(X) \to \text{Shv}(Y)$ satisfies $f_!(\text{Shv}(X)_{\text{constr}}) \subseteq \text{Shv}(X)_{\text{constr}}$.

The proof of the following result will be given in Appendix A as well.

**Corollary 1.5.** For every proper safe morphism $f : X \to Y$ between Artin stacks, we have a natural isomorphism $f^! \simeq f_*$ of functors $\text{Shv}(X) \to \text{Shv}(Y)$.

1.6. Renormalized category of sheaves.

(a) Let $\text{Shv}(X)^{\text{ren}} := \text{Ind} \text{Shv}(X)_{\text{constr}}$ be the ind-completion of $\text{Shv}(X)_{\text{constr}}$. Notice that we have a pair of continuous adjoint functors $(\text{ren}_X, \text{unren}_X)$, where
$$\text{ren}_X : \text{Shv}(X) \to \text{Shv}(X)^{\text{ren}}$$
be the ind-completion of the inclusion $\text{Shv}(X)^c \hookrightarrow \text{Shv}(X)_{\text{constr}}$, and
$$\text{unren}_X : \text{Shv}(X)^{\text{ren}} \to \text{Shv}(X)$$
is characterised by the condition that $\text{unren}_X |_{\text{Shv}(X)_{\text{constr}}}$ is the inclusion $\text{Shv}(X)_{\text{constr}} \hookrightarrow \text{Shv}(X)$.
(b) As it is explained in [AGKRRV1, Section F.5.2], both categories $\text{Shv}(X)$ and $\text{Shv}(X)^{\text{ren}}$ are equipped with perverse $t$-structures, and functor $\text{unren}_X$ induces an equivalence

$$(\text{Shv}(X)^{\text{ren}})^{\geq -n} \to \text{Shv}(X)^{\geq -n}$$

for all $n$.

**1.7. Remark.** Notice that the functor $\text{ren}_X : \text{Shv}(X) \to \text{Shv}(X)^{\text{ren}}$ is an equivalence of categories if and only if the inclusion $\text{Shv}(X)^c \hookrightarrow \text{Shv}(X)^{\text{constr}}$ is an equivalence. Thus, by Proposition 1.4(a), this happens if and only if $X$ is safe.

**1.8. Functors between renormalized categories.** Let $f : X \to Y$ be a morphism of Artin stacks.

(a) Note that both pullbacks $f^*, f^! : \text{Shv}(Y) \to \text{Shv}(X)$ map $\text{Shv}(Y)^{\text{constr}}$ to $\text{Shv}(X)^{\text{constr}}$, thus give rise to unique continuous functors

$$(f^*)^{\text{ren}}, (f^!)^{\text{ren}} : \text{Shv}(Y)^{\text{ren}} \to \text{Shv}(X)^{\text{ren}},$$

extending $f^*|_{\text{Shv}(Y)^{\text{constr}}}, f^!|_{\text{Shv}(Y)^{\text{constr}}} : \text{Shv}(Y)^{\text{constr}} \to \text{Shv}(X)^{\text{constr}}$.

To simplify the notation, we will denote functors $(f^*)^{\text{ren}}$ and $(f^!)^{\text{ren}}$ by $f^*$ and $f^!$, respectively.

(b) By construction, the functor $f^* : \text{Shv}(Y)^{\text{ren}} \to \text{Shv}(X)^{\text{ren}}$ from part (a) maps compact objects to compact objects, thus has a continuous right adjoint $f_* : \text{Shv}(X)^{\text{ren}} \to \text{Shv}(Y)^{\text{ren}}$.

(c) Note that the functor $f^! : \text{Shv}(Y)^{\text{ren}} \to \text{Shv}(X)^{\text{ren}}$ from part (a) has a left adjoint $(f^!)^{\text{ren}}$ if and only if functor $f_! : \text{Shv}(X) \to \text{Shv}(Y)$ preserves constructible objects. Using Proposition 1.4(b), this happens if and only if $f$ is safe. In this case, $(f^!)^{\text{ren}}$ is equal to the unique continuous extension of $f_!|_{\text{Shv}(X)^{\text{constr}}} : \text{Shv}(X)^{\text{constr}} \to \text{Shv}(Y)^{\text{constr}}$ and will be denoted simply by $f_!$.

**Lemma 1.9.** Let $f : X \to Y$ be a morphism of Artin stacks.

(a) We have natural isomorphisms

$$f^* \circ \text{unren}_Y \simeq \text{unren}_X \circ f^* \text{ and } f^! \circ \text{unren}_Y \simeq \text{unren}_X \circ f^!$$

of functors $\text{Shv}(Y)^{\text{ren}} \to \text{Shv}(X)$.

(b) We have a natural morphism

$$\text{unren}_Y \circ f_* \to f_* \circ \text{unren}_X$$

of functors $\text{Shv}(X)^{\text{ren}} \to \text{Shv}(Y)$, whose restriction to $\text{Shv}(X)^{\text{constr}}$ is an isomorphism.

(c) We have a natural isomorphism

$$f_\triangle \simeq \text{unren}_Y \circ f_* \circ \text{ren}_X$$

of functors $\text{Shv}(X) \to \text{Shv}(Y)$.

(d) We have natural morphisms

$$\text{ren}_X \circ f^* \to f^* \circ \text{ren}_Y, \text{ren}_X \circ f^! \to f^! \circ \text{ren}_Y, (\text{resp. } \text{ren}_Y \circ f_\triangle \to f_* \circ \text{ren}_X)$$

of functors $\text{Shv}(Y) \to \text{Shv}(X)^{\text{ren}}$ (resp. $\text{Shv}(X) \to \text{Shv}(Y)^{\text{ren}}$).
Proof. (a) By continuity, it suffices to show isomorphisms between the corresponding functors \( \text{Shv}(Y)^{\text{constr}} \to \text{Shv}(X) \), which follow immediately from definitions.

(b) The morphism \( \text{unren}_Y \circ f_* \to f_* \circ \text{unren}_X \) is obtained by adjunction from the (iso)morphism \( f^* \circ \text{unren}_Y \to \text{unren}_X \circ f^* \) from part (a). To show the isomorphism assertion, we have to show that for every \( A \in \text{Shv}(X)^{\text{constr}} \) and \( B \in \text{Shv}(Y)^c \) the natural morphism

\[
\text{Hom}_{\text{Shv}(Y)}(B, (\text{unren}_Y \circ f_*)(A)) \to \text{Hom}_{\text{Shv}(Y)}(B, (f_* \circ \text{unren}_X)(A))
\]

is an isomorphism. By adjunction, the above morphism is the composition of isomorphisms

\[
\text{Hom}_{\text{Shv}(Y)}((f^* \circ \text{ren}_Y)(B), A) \overset{\text{unren}_X}{\to} \text{Hom}_{\text{Shv}(X)}((\text{unren}_X \circ f^* \circ \text{ren}_Y)(B), \text{unren}_X(A)) \overset{\text{unren}_X}{\to} \text{Hom}_{\text{Shv}(X)}(f^*(B), \text{unren}_X(A))
\]

where the first map is isomorphism because \( \text{unren}_X_{\text{Shv}(X)^{\text{constr}}} \) is fully faithful, and the second isomorphism is induced by the isomorphism

\[
\text{unren}_X \circ f^* \circ \text{ren}_Y \overset{(a)}{\cong} f^* \circ \text{unren}_Y \circ \text{ren}_Y \overset{\text{unit}}{\cong} f^*;
\]

where we recall that functor \( \text{ren}_Y \) is fully faithful.

(c) By continuity, it suffices to construct an isomorphism between the corresponding functors \( \text{Shv}(X)^c \to \text{Shv}(Y) \), and we define the corresponding isomorphism to be the composition

\[
\text{unren}_Y \circ f_* \circ \text{ren}_X_{\text{Shv}(X)^c} \overset{(b)}{\cong} f_* \circ \text{unren}_X \circ \text{ren}_X_{\text{Shv}(X)^c} \overset{\text{unit}}{\cong} f_* \circ \text{Shv}(X)^c \overset{\text{Id}}{\cong} f_* \circ \text{Shv}(X)^c.
\]

(d) follows from parts (a),(c), adjunction and isomorphism \( \text{unren}_Y \circ \text{ren}_Y \cong \text{Id} \).

\[\square\]

Corollary 1.10. For every Cartesian diagram of Artin stacks

\[
\begin{array}{ccc}
A & \to & C \\
\downarrow^s & & \downarrow^f \\
B & \to & D,
\end{array}
\]

(a) we have a canonical (base change) isomorphism

\[
b^1 \circ f_* \cong g_* \circ a^1 : \text{Shv}(C)^{\text{ren}} \to \text{Shv}(B)^{\text{ren}};
\]

(b) the following diagram of functors \( \text{Shv}(C) \to \text{Shv}(B)^{\text{ren}} \) is homotopy commutative

\[
\begin{array}{ccc}
\text{ren}_B \circ g_\triangledown \circ a^1 & \overset{\text{1.1.9 (d)}}{\to} & g_* \circ \text{ren}_A \circ a^1 \\
\downarrow^{1.2 \text{d}} & & \downarrow^{1.1.9 \text{d}} \\
\text{ren}_B \circ b^1 \circ f_\triangledown & \overset{\text{1.1.9 (d)}}{\to} & b^1 \circ \text{ren}_D \circ f_\triangledown
\end{array}
\]

Proof. (a) By continuity, it suffices to construct a canonical isomorphism

\[
b^1 \circ f_*(\mathcal{F}) \cong g_* \circ a^1(\mathcal{F})
\]

for every \( \mathcal{F} \in \text{Shv}(C)^{\text{constr}} \). In this case, both \( b^1 \circ f_*(\mathcal{F}) \) and \( g_* \circ a^1(\mathcal{F}) \) lie in \( \text{Shv}(B)^{\text{ren}} \geq n \) for some \( n \). Hence, by Section 1.10 (b), it suffices to construct an isomorphism

\[
\text{unren}_B \circ b^1 \circ f_*(\mathcal{F}) \cong \text{unren}_B \circ g_* \circ a^1(\mathcal{F}).
\]
Since \( F \in \text{Shv}(C)^{\text{constr}} \), we get \( a^!(F) \in \text{Shv}(A)^{\text{constr}} \). It therefore follows from Lemma 1.9(a),(b) that it suffices to construct an isomorphism \( b^! \circ f_* \simeq g_* \circ a^! \) of functors \( \text{Shv}(C)^{\text{constr}} \to \text{Shv}(B) \), which is well-known.

(b) Using definitions of morphisms in Lemma 1.9(d), it suffices to show that the following diagram of functors \( \text{Shv}(C) \to \text{Shv}(B) \) is homotopy commutative:

\[
\begin{array}{ccc}
\text{unren}_B \circ g_* \circ \text{ren}_A \circ a^! & \xrightarrow{\sim} & \text{unren}_B \circ g_* \circ a^! \circ \text{ren}_C \\
\downarrow & & \downarrow \sim (a) \\
\text{unren}_B \circ b^! \circ f_* \circ \text{ren}_C & \xrightarrow{\sim} & \text{unren}_B \circ b^! \circ f_* \circ \text{ren}_C.
\end{array}
\]

For this, it suffices to evaluate all functors on objects of \( \text{Shv}(C)^c \), in which case the assertion follows from Section 1.2(d) by unwinding definitions of all morphisms involved. \( \square \)

1.11. Application to safe morphisms.

(a) Note that if \( f : X \to Y \) is a safe morphism of Artin stacks, then the morphism

\[
\text{ren}_X \circ f^* \to f^* \circ \text{ren}_Y
\]

of functors \( \text{Shv}(Y) \to \text{Shv}(X)^\text{ren} \) from Lemma 1.9(d) is an isomorphism. Indeed, since \( f^*(\text{Shv}(Y)^c) \subseteq \text{Shv}(X)^c \) (because \( f \) is safe), the restrictions of morphism \( f^* \) to \( \text{Shv}(Y)^c \) is the identity endomorphism of the composition \( \text{Shv}(Y)^c \xrightarrow{f^*} \text{Shv}(X)^{\text{constr}} \subseteq \text{Shv}(X)^{\text{ren}} \).

(b) For every commutative diagram of Artin stacks

\[
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
\downarrow g & & \downarrow f \\
B & \xrightarrow{b} & D,
\end{array}
\]

such that \( g \) is safe we have a canonical morphism

\[
f^* \circ b^! \to a^! \circ g^*
\]

of functors \( \text{Shv}(B) \to \text{Shv}(C) \), defined to be as a composition

\[
\begin{align*}
f^* \circ b^! & \xRightarrow{\sim} f^* \circ \text{unren}_D \circ b_* \circ \text{ren}_B \\
& \xRightarrow{\sim a^!} \text{unren}_C \circ f^* \circ b_* \circ \text{ren}_B \\
& \xrightarrow{\text{base change}} \text{unren}_C \circ a_* \circ \text{ren}_A \circ g^! \\
& \xRightarrow{\sim} \text{unren}_C \circ a_* \circ \text{ren}_A \circ g^*
\end{align*}
\]

(b)' Unwinding definitions, morphism of part (b) can be characterized as the unique morphism making the following diagram homotopy commutative:

\[
\begin{array}{ccc}
f^* \circ b^! & \xrightarrow{(b)} & a^! \circ g^* \\
\downarrow \text{can}_b & & \downarrow \text{can}_a \\
f^* \circ b_* & \xrightarrow{\text{base change}} & a_* \circ g^*.
\end{array}
\]
1.12. The dualizing sheaf. We denote by $\omega_{\text{ren}}^X \in \text{Shv}(X)_{\text{ren}}$ the image of the dualizing sheaf $\omega_X \in \text{Shv}(X)_{\text{constr}}$ under the embedding $\text{Shv}(X)_{\text{constr}} \hookrightarrow \text{Shv}(X)_{\text{ren}}$. Then it follows from Lemma 1.9(b) that we have a canonical isomorphism

$$\Gamma(X, \omega_X) := (p_X)_*(\omega_X) \simeq (p_X)_*(\omega^\text{ren}_X) =: \Gamma(X, \omega^\text{ren}_X)$$

between objects of

$$\text{Shv}(\text{pt}) \simeq \text{Vect} \simeq \text{Shv}(\text{pt})_{\text{ren}}.$$  

2. Generalized base change morphisms

2.1. Set-up. Note that every commutative diagram of Artin stacks

$$\begin{array}{ccc}
A & \xrightarrow{a} & C \\
g \downarrow & & \downarrow f \\
B & \xrightarrow{b} & D
\end{array}$$

(2.1)

decomposes as

$$\begin{array}{ccc}
A & \xrightarrow{p} & B \times_D C & \xrightarrow{\tilde{b}} & C \\
g \downarrow & & \downarrow f \\
B & \xrightarrow{b} & D
\end{array}$$

(2.2)

2.2. Pushable squares.

(a) Motivated by [FYZ, Definition 3.1.1], we call a commutative diagram (2.1) pushable, if the morphism $p$ from diagram (2.2) is proper and safe.

(b) In the situation of part (a), we have a canonical morphism

$$f_1 \circ a \xrightarrow{=} b \circ g!$$

of functors $\text{Shv}(A) \to \text{Shv}(D)$, defined to be the composition

$$f_1 \circ a \simeq f_1 \circ \tilde{b} \circ p \xrightarrow{=} b \circ \tilde{f} \circ p \simeq b \circ g!,$$

where

- the first morphism is induced by the inverse of the isomorphism

$$\tilde{b} \circ p \xrightarrow{=} (\tilde{b} \circ p|_\Delta) = a|_\Delta,$$

from Section 1.2(c), which is an isomorphism because $p$ is proper and safe that thus by Corollary 1.5 we have

$$p_*((\text{Shv}(A)^c) = p_*((\text{Shv}(A)^c) \subseteq \text{Shv}(B \times_D C)^c;$$

- the second morphism is induced by the base change morphism

$$g! \circ \tilde{b} \to b \circ \tilde{f},$$

obtained by adjunction from the composition

$$\tilde{b} \xrightarrow{\text{unit}} \tilde{b} \circ \tilde{f} \circ f^! \circ \tilde{f} \circ b \circ \tilde{f},$$
• the last morphism is induced by the isomorphism

\[
\tilde{f}_1 \circ p_\bullet \cong f_1 \circ p_\bullet \cong \tilde{f}_1 \circ p_1 \cong f_1.
\]

(c) In the situation of part (a), assume that morphism \( f \) is safe. Hence morphism \( g \) is safe as well (see Section 1.3(c)). Replacing \((-)_\bullet\) by \((-)_\ast\), in part (b) in all places, we have a canonical morphism

\[
f_1 \circ a_\ast \to b_\ast \circ g_1
\]
of functors \( \mathbf{Shv}(A)_{\text{ren}} \to \mathbf{Shv}(D)_{\text{ren}} \).

(d) Assume that morphism \( f \) is proper and safe. Then, by Section 1.3(c), a diagram (2.1) is pushable if and only if morphism \( g \) is proper and safe.

Moreover, it is not difficult to see that in this case, the morphism of part (b) decomposes as

\[
f_1 \circ a_\bullet \cong f_\bullet \circ a_\bullet \cong (f \circ a)_\bullet \cong (b \circ g)_\bullet \cong b_\bullet \circ g_1,
\]

while the morphism of part (c) decomposes as

\[
f_1 \circ a_\ast \cong f_\ast \circ a_\ast \cong (f \circ a)_\ast \cong (b \circ g)_\ast \cong b_\ast \circ g_1.
\]

2.3. Quasi-smooth morphisms and Gysin maps.

(a) For an Artin stack \( X \), an object \( A \in \mathbf{Shv}(X) \) and \( n \in \mathbb{Z} \), we set \( A(n) := A(n)[2n] \). More generally, for a locally constant function \( n : X \to \mathbb{Z} \), we denote by \( A(n) \) an object of \( \mathbf{Shv}(X) \) such that for every connected component \( X' \subseteq X \), we have \( A(n)|_{X'} = A(n)|_{X'} \).

(b) To every Artin stack \( X \) we associate the dimension function

\[
\dim_X : X \to \mathbb{Z},
\]
given by the formula \( \dim_X(x) = \dim_x(X) \) for every \( x \in X \).

To every morphism \( f : X \to Y \) of Artin stacks, we associate the dimension function

\[
\dim_f : X \to \mathbb{Z},
\]
given by the formula \( \dim_f = \dim_X - f(\dim_Y) \), where \( f \) denotes pullback of functions.

For every composition \( X \xrightarrow{f} Y \xrightarrow{g} Z \), we have an equality

\[
\dim_{g\circ f} = \dim_f + f(\dim_g).
\]

In particular, \( f \) is of relative dimension zero, that is, \( \dim_f = 0 \), if and only if \( \dim_{g\circ f} = f(\dim_g) \).

(c) Note that if \( f : X \to Y \) is quasi-smooth (also called lci), then the cotangent complex \( T^*(X/Y) \) is perfect, and the dimension function \( \dim_f \) is locally constant and equals the Euler-characteristic of \( T^*(X/Y) \).

Also for every composition \( X \xrightarrow{f} Y \xrightarrow{g} Z \) of Artin stacks such that \( g \) is smooth, the morphism \( f \) is quasi-smooth if and only if \( g \circ f \) is quasi-smooth.

(d) To every quasi-smooth morphism \( f : X \to Y \) of Artin stacks (and more generally of derived Artin stacks) one can associate the relative fundamental class map

\[
\text{cl}_f : f^*(\overline{Q}_\mathbb{Z}) \to f'(\overline{Q}_\mathbb{Z})(-\dim_f)
\]

(see, for example, [Kh] Construction 3.6)), hence the Gysin map

\[
\text{Gys}_f : f^* \to f'(-\dim_f)
\]
of functors $\text{Shv}(Y) \to \text{Shv}(X)$, defined as a composition
\[
f^* = f^* \otimes f^*(\mathbb{Q}_\ell) \xrightarrow{\text{Id} \otimes \text{cl}_f} f^* \otimes f^!(\mathbb{Q}_\ell)(-\dim f) \xrightarrow{\text{can}} f^!(\mathbb{Q}_\ell)(-\dim f),
\]
where $\text{can}$ is the canonical map $f^*(A) \otimes f!(B) \to f!(A \otimes B)$.

(e) By construction the map $\text{Gys}_f$ from part (d) is a canonical isomorphism
\[
f^* \sim f^!(-\dim f)
\]
when $f$ is smooth, and Gysin maps are compatible with compositions (by [Kh, Theorem 3.12]).

(f) Moreover, for every homotopy Cartesian diagram of Artin stacks (2.1) such that $f$ is quasi-smooth, the morphism $g$ is quasi-smooth, satisfies $\dim g = a \cdot (\dim f)$, and the following diagram is homotopy commutative:
\[
\begin{array}{ccc}
a^* \circ f^* & \xrightarrow{\text{Gys}_f} & (a^* \circ f^!)(-a \cdot \dim f) \\
\downarrow & & \downarrow \text{base change} \\
g^* \circ b^* & \xrightarrow{\text{Gys}_f} & (g^! \circ b^!)(-\dim p)
\end{array}
\]
(by [Kh, Theorem 3.13]).

(g) Clearly, the Gysin map $\text{Gys}_f$ from part (d) can be viewed as a morphism of functors $\text{Shv}(Y) \xrightarrow{\text{constr}} \text{Shv}(X) \xrightarrow{\text{constr}}$, and hence as a morphism of functors $\text{Shv}(Y) \xrightarrow{\text{ren}} \text{Shv}(X) \xrightarrow{\text{ren}}$.

2.4. Pullable squares.

(a) Slightly modifying [FYZ, Definition 3.1.1], we call a commutative diagram (2.1) pullable, if the morphism $p$ from diagram (2.2) is quasi-smooth.

(b) In the situation of part (a), we have a canonical morphism
\[
(2.3) \quad g^* \circ b^! \to (a^! \circ f^*)(-\dim p)
\]
of functors $\text{Shv}(D) \to \text{Shv}(A)$ and $\text{Shv}(D) \xrightarrow{\text{ren}} \text{Shv}(A) \xrightarrow{\text{ren}}$ defined to be the composition
\[
g^* \circ b^! \simeq p^* \circ f^* \circ b^! \xrightarrow{\text{base change}} p^* \circ b^! \circ f^* \xrightarrow{\text{Gys}_p} p^!(\mathbb{Q}_\ell) \circ b^! \circ f^* \simeq a^! \circ f^*(-\dim p),
\]
where $\text{Gys}_p : p^* \to p^!(\mathbb{Q}_\ell)$ is the Gysin map (see Section 2.3(d)).

(c) Assume that morphism $f$ is smooth. Then, by Sections 2.3(c), a diagram (2.1) is pullable if and only if morphism $g$ is quasi-smooth. Moreover, using Sections 2.3(c),(b), it is not difficult to see that morphism (2.3) decomposes in this case as
\[
g^* \circ b^! \xrightarrow{\text{Gys}_p} (g^! \circ b^!)(-\dim p) \simeq (a^! \circ f^!)(-\dim p) \xrightarrow{\text{Gys}_f} (a^! \circ f^*(-\dim p) + a^!(\dim f)) \simeq a^! \circ f^*(-\dim p).
\]

2.5. Remarks. Though our notion of a pullable square is motivated by the corresponding notion of [FYZ, Definition 3.1.1], the two notions are not equivalent. Namely, in [FYZ] the authors consider commutative diagram of derived Artin stacks and require that the induced map $A \to B \times^h_C$ to the homotopy fiber product is quasi-smooth.

However, if one restricts to commutative diagrams of classical Artin stacks, then our notion is more general. Indeed, every pullable square in the sense of [FYZ] is also pullable in our sense (by Lemma B.3 below), but the converse is false. For example, if a commutative diagram is Cartesian, but not homotopy Cartesian, then it is pullable in our sense but not in the sense of [FYZ] (by Section B.11(b) below).
3. Functoriality of correspondences and traces

3.1. Correspondences.
(a) By a correspondence \( c \) on \( X \), we will mean a morphism of Artin stacks \( c = (c_l, c_r) : C \to X \times X \), i.e., a diagram
\[
\begin{array}{ccc}
X & \xleftarrow{c_l} & C \\
\downarrow & & \downarrow \tilde{c} \\
X & \xrightarrow{c_r} & X \times X,
\end{array}
\]
(b) A correspondence \( c : C \to X \times X \) gives rise to a Cartesian diagram
\[
\begin{array}{ccc}
\text{Fix}(c) & \xrightarrow{\Delta} & C \\
\downarrow & & \downarrow c \\
X & \xrightarrow{\Delta_X} & X \times X,
\end{array}
\]
where \( \Delta_X : X \to X \times X \) is the diagonal morphism on \( X \). We will refer to \( \text{Fix}(c) \) as the stack of fixed points of \( c \).
(c) A correspondence \( c : C \to X \times X \) induces continuous functors
\[
[c] := (c_l)^\triangle \circ c_r^! : \text{Shv}(X) \to \text{Shv}(X) \quad \text{and} \quad \tilde{c} := (c_l)_* \circ c_r^! : \text{Shv}(X)^\text{ren} \to \text{Shv}(X)^\text{ren}.
\]
Since DG categories \( \text{Shv}(X) \) and \( \text{Shv}(X)^\text{ren} \) are compactly generated thus dualizable, the trace formalism (see \cite{GKRV} Section 3) applies. In particular, one can associate to \( c \) the vector spaces
\[
\text{Tr}(\text{Shv}(X), [c]), \text{Tr}(\text{Shv}(X)^\text{ren}, [c]) \in \text{Vect},
\]
where we remind that \( \text{Vect} \) denotes the stable \( \infty \)-category of \( \mathbb{Q}_l \)-vector spaces.
(d) Furthermore, a correspondence \( c : C \to X \times X \) gives rise to a lax-commutative diagram
\[
\begin{array}{ccc}
\text{Shv}(X) & \xrightarrow{\text{ren}_X} & \text{Shv}(X)^\text{ren} \\
\downarrow \tilde{c} & & \downarrow \tilde{c} \\
\text{Shv}(X) & \xrightarrow{\text{ren}_X} & \text{Shv}(X)^\text{ren},
\end{array}
\]
where \( \alpha \) is the composition
\[
\text{ren}_X \circ [c] = \text{ren}_X \circ (c_l)^\triangle \circ c_r^! \circ (c_l)_* \circ \text{ren}_C \circ (c_r)_* \circ c_r^! \circ \text{ren}_X = [c] \circ \text{ren}_X
\]
of morphisms from Lemma \[L9\](d).
Moreover, since functor \( \text{ren}_X \) has a continuous right adjoint (given by \( \text{unren}_X \)) it induces a morphism of traces
\[
\text{Tr}(\text{ren}_X, [c]) : \text{Tr}(\text{Shv}(X), [c]) \to \text{Tr}(\text{Shv}(X)^\text{ren}, [c]).
\]
3.2. Chern character.
Following \cite{GKRV} Section 3.5.4], to a correspondence \( c : C \to X \times X \) and a constructible sheaf \( \mathcal{A} \in \text{Shv}(X)^\text{constr} \) one associates the Chern character map
\[
\text{ch}_{c, \mathcal{A}} : \mathcal{H}(\text{Shv}(X)^\text{ren})(\mathcal{A}, [c](\mathcal{A})) \to \text{Tr}(\text{Shv}(X)^\text{ren}, [c]).
\]
Namely, every point \( u \in \Hom_{\Shv(X)\ren}(A, [c](A)) \) gives rise to a lax-commutative diagram

\[
\begin{array}{c}
\text{Vect} \ar[r]^A & \Shv(X)\ren \\
\ar[ur]_u & & & \ar[d]_{[c]} \\
\text{Vect} \ar[r]_A & \Shv(X)\ren.
\end{array}
\]

Moreover, since \( A \in \Shv(X)\constr \) is a compact object in \( \Shv(X)\ren \), the corresponding morphism \( A : \Vect \to \Shv(X)\ren \) has a right adjoint. Thus diagram (3.2) induces a morphism of traces

\[
Q_\ell = \Tr(\Vect, \Id) \to \Tr(\Shv(X)\ren, [c]),
\]

hence defines a point of \( \Tr(\Shv(X)\ren, [c]) \).

3.3. Morphisms of correspondences. Let \( c : C \to X \times X \) and \( d : D \to Y \times Y \) be correspondences.

By a morphism of correspondences \( c \to d \), we mean a pair of morphisms \([f, g]\) = \((f, g)\), making the following diagram commute:

\[
\begin{array}{c}
X \ar[l]^{c_l} \ar[d]_f & C \ar[r]^{c_r} \ar[d]^g & X \\
Y \ar[l]_{d_l} & D \ar[r]^{d_r} & Y.
\end{array}
\]

3.4. Pushforward. Let \([f] : c \to d\) be a morphism of correspondences (see Section 3.3) such that the left inner square of diagram (3.3) is pushable (see Section 2.2).

Notice that this condition is satisfied if either

(i) morphisms \( f \) and \( g \) are proper and safe (see Section 2.2(d))

or

(ii) the left inner square of diagram (3.3) is Cartesian.

(a) In this case we have a natural morphism

\[
[f] ! : f_! \circ [c] \to [d] \circ f_!
\]

of functors \( \Shv(X) \to \Shv(Y) \), defined as a composition

\[
f_! \circ [c] = f_! \circ (c_l)_* \circ c_r^! \to (d_l)_* \circ g_! \circ c_r^! \to (d_l)_* \circ d_r^! \circ f_! = [d] \circ f_!,
\]

where

- the first morphism is induced by a canonical morphism of functors

\[
f_! \circ (c_l)_* \to (d_l)_* \circ g_!
\]

from Section 2.2(b), corresponding to the left inner square of diagram (3.3):

- the second morphism is induced by the base change morphism \( g_! \circ c_r^! \to d_r^! \circ f_! \), corresponding to the right inner square of diagram (3.3).

Moreover, since functor \( f_! : \Shv(X) \to \Shv(Y) \) has a continuous right adjoint (given by \( f^! \)), the morphism \([f] !\) induces a map of traces

\[
\Tr([f] !) : \Tr(\Shv(X), [c]) \to \Tr(\Shv(Y), [d]).
\]
(b) Assume in addition that morphism $f$ is safe. Then by a version of the arguments of part (a) (which are much simpler now), we see that $[f]$ induces a canonical morphism

$$[f]! : f_! \circ [c] \to [d] \circ f_!$$

of functors $\text{Shv}(X)^{\text{ren}} \to \text{Shv}(Y)^{\text{ren}}$, hence map of traces

$$\text{Tr}([f]!) : \text{Tr}(\text{Shv}(X)^{\text{ren}}, [c]) \to \text{Tr}(\text{Shv}(Y)^{\text{ren}}, [d]).$$

(c) In the situation of part (b), for every $A \in \text{Shv}(X)^{\text{constr}}$ we have $f_!(A) \in \text{Shv}(Y)^{\text{constr}}$ (by Proposition 3.5). Next, the morphism $[f]!$ of part (b) gives rise to a map

$$\mathcal{H}\text{om}_{\text{Shv}(X)^{\text{ren}}}(A, [c](A)) \xrightarrow{f_!} \mathcal{H}\text{om}_{\text{Shv}(Y)^{\text{ren}}}(f_!(A), f_!(c)(A)) \xrightarrow{[f]!} \mathcal{H}\text{om}_{\text{Shv}(Y)^{\text{ren}}}(f_!(A), [d](f_!(A))),$$

which we denote again by $[f]!$.

Moreover, unwinding definitions, it follows from compatibility of trace maps with compositions that the following diagram is homotopy commutative:

$$
\begin{array}{ccc}
\mathcal{H}\text{om}_{\text{Shv}(X)^{\text{ren}}}(A, [c](A)) & \xrightarrow{\text{ch}_{c,A}} & \text{Tr}(\text{Shv}(X)^{\text{ren}}, [c]) \\
[f]! & \downarrow & \text{Tr}([f]!)
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{H}\text{om}_{\text{Shv}(X)^{\text{ren}}}(f_!(A), [d](f_!(A))) & \xrightarrow{\text{ch}_{d,f_!(A)}} & \text{Tr}(\text{Shv}(Y)^{\text{ren}}, [d])
\end{array}
$$

(3.5)

The following result, whose proof will be given in Section 8, asserts that morphisms from Sections 3.4(a) and 3.4(b) are compatible with renormalization functors.

**Proposition 3.5.** In the situation of Section 3.4(b), we have a homotopy commutative diagram

$$
\begin{array}{ccc}
\text{Tr}(\text{Shv}(X), [c]) & \xrightarrow{\text{Tr}([f]!)} & \text{Tr}(\text{Shv}(Y), [d]) \\
\text{Tr}(\text{ren}X, [c]) & \downarrow & \text{Tr}(\text{ren}Y, [d])
\end{array}
$$

(3.6)

$$
\begin{array}{ccc}
\text{Tr}(\text{Shv}(X)^{\text{ren}}, [c]) & \xrightarrow{\text{Tr}([f]!)} & \text{Tr}(\text{Shv}(Y)^{\text{ren}}, [d]).
\end{array}
$$

**3.6. Pullback.** Let $[f] : c \to d$ be a morphism of correspondences (see Section 3.3) such that such that the right inner square of diagram (3.3) is pullable in the sense of Section 2.4.

Notice that this condition is satisfied if either

(i) $f$ is smooth, $g$ is quasi-smooth such that $\dim g = c^* (\dim f)$ (see Section 2.4(c))

or

(ii) the right inner square of diagram (3.3) is Cartesian.

(a) Assume in addition that morphisms $f$ and $g$ are safe. In this case we have a natural morphism

$$[f]^* : f^* \circ [d] \to [c] \circ f^*$$

of functors $\text{Shv}(Y) \to \text{Shv}(X)$, defined as a composition

$$f^* \circ [d] = f^* \circ (d_! \triangleleft \circ d_r) \to (c_!) \triangleleft \circ g^* \circ d_r \to (c_! \triangleleft \circ c_r) \circ f^* = [c] \circ f^*,$$

where

- the first morphism is induced by the morphism

$$f^* \circ (d_! \triangleleft) \to (c_!) \triangleleft \circ g^*$$

(see Section 1.11(b)), which is defined because $g$ is safe;
\begin{itemize}
  \item the second morphism is induced by a canonical morphism
    \begin{equation}
    g^* \circ d'_c \rightarrow c'_f \circ f^*,
    \end{equation}
  from Section 2.4(b).

  Moreover, since functor $f^*$ has a continuous right adjoint (because $f$ is safe), morphism $[f]^*$ induces a map of traces
  \begin{equation}
  \Tr([f]^*) : \Tr(\Shv(Y), [d]) \rightarrow \Tr(\Shv(X), [c]).
  \end{equation}

  (b) Even without the assumptions that morphisms $f$ and $g$ are safe one can show (slightly modifying the arguments of part (a)) that morphism $[f]^*$ induces a canonical morphism
  \begin{equation}
  [f]^* : f^* \circ [d] \rightarrow [c] \circ f^*
  \end{equation}
  of functors $\Shv(Y)^{\ren} \to \Shv(X)^{\ren}$, hence a map of traces
  \begin{equation}
  \Tr([f]^*) : \Tr(\Shv(Y)^{\ren}, [d]) \rightarrow \Tr(\Shv(X)^{\ren}, [c]).
  \end{equation}

  (c) In the situation of part (b), for every $A \in \Shv(Y)^{\constr}$ we have $f^*(A) \in \Shv(Y)^{\constr}$. Then the morphism $[f]^*$ of part (b) gives rise to a map
  \begin{equation}
  \Hom_{\Shv(Y)^{\ren}}(A, [d](A)) \xrightarrow{f^*} \Hom_{\Shv(X)^{\ren}}(f^*(A), f^*([d](A))) \xrightarrow{[f]^*} \Hom_{\Shv(Y)^{\ren}}(f^*(A), [c](f^*(A))),
  \end{equation}
  which we denote again by $[f]^*$.

  Moreover, unwinding definitions, it follows from compatibility of trace maps with compositions that the following diagram is homotopy commutative
  \begin{equation}
  \begin{array}{ccc}
  \Hom_{\Shv(Y)^{\ren}}(A, [d](A)) & \xrightarrow{\text{ch}_{c,A}} & \Tr(\Shv(Y)^{\ren}, [d]) \\
  [f]^* \downarrow & & \downarrow \Tr([f]^*) \\
  \Hom_{\Shv(X)^{\ren}}(f^*(A), [c](f^*(A))) & \xrightarrow{\text{ch}_{c,f^*(A)}} & \Tr(\Shv(X)^{\ren}, [c]).
  \end{array}
  \end{equation}

  The following result, whose proof will be given in Section \ref{sec:homotopy-diagrams}, asserts that morphisms from Sections 3.6(a) and 3.6(b) are compatible with functors $\ren_X$ and $\ren_Y$.

  \textbf{Proposition 3.7.} \textit{In the situation of Section 3.6(a), we have a homotopy commutative diagram:
  \begin{equation}
  \begin{array}{ccc}
  \Tr(\Shv(Y), [d]) & \xrightarrow{\Tr([f]^*)} & \Tr(\Shv(X), [c]) \\
  \Tr(\ren_Y, [d]) \downarrow & & \downarrow \Tr(\ren_X, [c]) \\
  \Tr(\Shv(Y)^{\ren}, [d]) & \xrightarrow{\Tr([f]^*)} & \Tr(\Shv(X)^{\ren}, [c]).
  \end{array}
  \end{equation}
  }

  \textbf{3.8. Restriction to open and closed substacks.} Let $c : C \to X \times X$ be a correspondence, $Z \subseteq X$ a closed substack and $U := X \setminus Z \subseteq X$ the complementary open substack.

  (a) We denote by $|c|_Z : c^{-1}(Z \times Z) \to Z \times Z$ and $|c|_U : c^{-1}(U \times U) \to U \times U$ the restrictions of $c$ to $Z$ and $U$, respectively. Then the pair of inclusions $[i_c] = (Z \hookrightarrow X, c^{-1}(Z \times Z) \hookrightarrow C)$ defines a morphism of correspondences $c|_Z \to c$, and $[j_c] = (U \hookrightarrow X, c^{-1}(U \times U) \hookrightarrow C)$ defines a morphism of correspondences $c|_U \to c$.

  (b) Note that $[i_c]$ satisfies assumptions of Section 3.6(b), while $[j_c]$ satisfies assumptions of Section 3.6(a). Therefore we have maps of traces
  \begin{equation}
  \Tr([i_c]) : \Tr(\Shv(Z), [c|_Z]) \to \Tr(\Shv(X), [c]), \quad \Tr(\Shv(Z)^{\ren}, [c|_Z]) \to \Tr(\Shv(X)^{\ren}, [c])
  \end{equation}
Lemma 3.9. Let \( c \) be a closed \( c \)-invariant substack, and in the situation of part (c) isomorphisms \( i^* \circ i_! \simeq \text{Id}_Z \) and \( j^* \circ j_! \simeq \text{Id}_U \) induce canonical homotopies

\[
\text{Tr}(j_{c!}^*) \circ \text{Tr}(i_{c!}) \simeq \text{Id} \quad \text{and} \quad \text{Tr}(j_{c!}^*) \circ \text{Tr}(i_{c!}) \simeq \text{Id}.
\]

(c) We say that a closed substack \( Z \subseteq X \) is \( c \)-invariant, if we have an equality of schematic preimages \( c_{r!}^{-1}(Z) = c^{-1}(Z \times Z) \), that is, an inclusion \( c_{r!}^{-1}(Z) \subseteq c_{l!}^{-1}(Z) \). In this case, morphism \( [i_c] \) satisfies assumptions of Section 3.6(a), thus it induces maps of traces

\[
\text{Tr}([i_c]^*) : \text{Tr}(\text{Shv}(X), [c]) \to \text{Tr}(\text{Shv}(U), [c|U]), \quad \text{Tr}(\text{Shv}(X)^{\text{ren}}, [c]) \to \text{Tr}(\text{Shv}(U)^{\text{ren}}, [c|U]).
\]

(d) In the situation of part (c), the open substack \( U := X \setminus Z \subseteq X \) satisfies \( c_{l!}^{-1}(U) = c_{r!}^{-1}(U \times U) \). In this case, morphism \( [j_c] \) satisfies assumptions of Section 3.4(b), thus it induces maps of traces

\[
\text{Tr}([j_c]) : \text{Tr}(\text{Shv}(U), [c|U]) \to \text{Tr}(\text{Shv}(X), [c]), \quad \text{Tr}(\text{Shv}(U)^{\text{ren}}, [c|U]) \to \text{Tr}(\text{Shv}(X)^{\text{ren}}, [c]).
\]

(e) By a straightforward verification, in the situation of part (c) isomorphisms \( i^* \circ i_! \simeq \text{Id}_Z \) and \( j^* \circ j_! \simeq \text{Id}_U \) induce canonical homotopies

\[
\text{Tr}([i_c]) \circ \text{Tr}(i_{c!}) \simeq \text{Id} \quad \text{and} \quad \text{Tr}([j_c]) \circ \text{Tr}(j_{c!}) \simeq \text{Id}.
\]
4. True local terms maps

4.1. Goals of this section.

(a) To a correspondence $c : C \to X \times X$ we are going to associate a canonically defined map

$$\text{LT}^\text{true}_c : \text{Tr}(\text{Shv}(X)^\text{ren}[,c]) \to \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)})$$

called the true local terms map. Then we are going to consider the composition

$$\text{Tr}(\text{Shv}(X), [c]) \xrightarrow{\text{Tr}(\text{ren}_X, [c])} \text{Tr}(\text{Shv}(X)^\text{ren}, [c]) \xrightarrow{\text{LT}^\text{true}_c} \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)})$$
denote it again by $\text{LT}^\text{true}_c$ and call it the true local terms map as well.

(b) Assume now that Artin stacks $X$, $C$ and $\text{Fix}(c)$ are Verdier-compatible (see Section 4.10 below). In this case, we are going to associate a canonically defined map

$$\text{LT}^\text{true}_{c, \triangledown} : \text{Tr}(\text{Shv}(X), [c]) \to \Gamma_{\triangledown}(\text{Fix}(c), \omega_{\text{Fix}(c)})$$

which we call the refined true local terms map such that the following diagram is homotopy commutative:

$$\begin{array}{ccc}
\text{Tr}(\text{Shv}(X), [c]) & \xrightarrow{\text{LT}^\text{true}_{c, \triangledown}} & \Gamma_{\triangledown}(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
\text{Tr}(\text{Shv}(X)^\text{ren}, [c]) & \xrightarrow{\text{LT}^\text{true}_c} & \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}).
\end{array}$$

(4.1)

4.2. Remark on terminology. Our terminology slightly differs from that of [AGKRRV1]. Namely, in [AGKRRV1] only $\text{LT}^\text{true}_c$ is considered and it is called true local terms map there. The reason why we call the map refined is because the (usual) true local terms map $\text{LT}^\text{true}_c$ is defined for an arbitrary correspondence, while in the situation of Section 4.1(b), the true local terms map $\text{LT}^\text{true}_c$ has a natural lift to a map $\text{LT}^\text{true}_{c, \triangledown}$.

4.3. We start with the following few observations: Let $X$ be an Artin stack of finite presentation over $k$.

(a) The category $\text{Shv}(X)^\text{ren}$ is canonically self dual, and the equivalence of categories

$$\text{Shv}(X)^\text{ren} \to (\text{Shv}(X)^\text{ren})^\vee = \text{Funct}_{\text{cont}}(\text{Shv}(X)^\text{ren}, \text{Vect}),$$
is given by the formula

$$\mathcal{B} \mapsto \mathcal{H}om_{\text{Shv}(X)^\text{ren}}(\mathcal{D}_X(\mathcal{B}), -), \quad \mathcal{B} \in \text{Shv}(X)^{\text{constr}},$$
where $\mathcal{D}_X$ denotes the Verdier duality functor $(\text{Shv}(X)^{\text{constr}})^{\text{op}} \to \text{Shv}(X)^{\text{constr}}$.

The above equivalence corresponds to the pairing

$$\text{ev}_X : \text{Shv}(X)^\text{ren} \otimes \text{Shv}(X)^\text{ren} \to \text{Vect} : \mathcal{A} \otimes \mathcal{B} \mapsto \Gamma(X, \mathcal{A} \otimes \mathcal{B}).$$

(b) Since the DG category $\text{Shv}(X)^\text{ren}$ is dualizable, for every Artin stack $Y$ the map

$$\text{Shv}(Y)^\text{ren} \otimes (\text{Shv}(X)^\text{ren})^\vee \to \text{Funct}_{\text{cont}}(\text{Shv}(X)^\text{ren}, \text{Shv}(Y)^\text{ren}),$$
given by the formula $\mathcal{A} \otimes \mathcal{F} \mapsto \mathcal{A} \otimes (\mathcal{F}(-))$, is an equivalence.

(c) We denote by

$$\varphi : \text{Shv}(Y)^\text{ren} \otimes \text{Shv}(X)^\text{ren} \to \text{Shv}(Y)^\text{ren} \otimes (\text{Shv}(X)^\text{ren})^\vee \to \text{Funct}_{\text{cont}}(\text{Shv}(X)^\text{ren}, \text{Shv}(Y)^\text{ren})$$
the composition of the equivalences from parts (a) and (b). Explicitly, for every $A \in \text{Shv}(Y)^{\text{constr}}$ and $B \in \text{Shv}(X)^{\text{constr}}$, we have

$$\varphi(A \otimes B) = A \otimes \mathcal{H}om_{\text{Shv}(X)^{\text{ren}}}(\mathbb{D}_X(B), -).$$

(d) Consider the functor

$$\otimes : \text{Shv}(Y)^{\text{ren}} \otimes \text{Shv}(X)^{\text{ren}} \rightarrow \text{Shv}(Y \times X)^{\text{ren}}$$

such that for every $A \in \text{Shv}(Y)^{\text{constr}}$ and $B \in \text{Shv}(X)^{\text{constr}}$ we have $\otimes(A \otimes B) = A \otimes B$. This functor preserves compactness, hence admits a continuous right adjoint, to be denoted $\otimes^R$.

4.4. In what follows, for $\mathcal{X} \in \text{Shv}(Y \times X)^{\text{ren}}$ we define its action by right functors to be

$$\mathcal{X}(-) := (p_Y)_*(\mathcal{X} \otimes p_X^!(\mathbb{D}_X(B))).$$

Lemma 4.5. The right adjoint $\otimes^R : \text{Shv}(Y \times X)^{\text{ren}} \rightarrow \text{Shv}(Y)^{\text{ren}} \otimes \text{Shv}(X)^{\text{ren}}$ is characterized by the property that

$$\varphi \circ \otimes^R : \text{Shv}(Y \times X)^{\text{ren}} \rightarrow \text{Shv}(Y)^{\text{ren}} \otimes \text{Shv}(X)^{\text{ren}} \rightarrow \text{Funct}_{\text{cont}}(\text{Shv}(X)^{\text{ren}}, \text{Shv}(Y)^{\text{ren}})$$

is the action by right functors, i.e., the map $\mathcal{X} \mapsto \mathcal{X}(-)$ from Section 4.4

Proof. Unwinding the definitions, we need to show that for $A \in \text{Shv}(Y)^{\text{constr}}$ and $B \in \text{Shv}(X)^{\text{constr}}$, we have a canonical identification

$$\mathcal{H}om_{\text{Shv}(Y \times X)^{\text{ren}}}(A \otimes B, \mathcal{X}) \simeq \mathcal{H}om_{\text{Shv}(Y)^{\text{ren}}}(A, (p_Y)_*(\mathcal{X} \otimes p_X^!(\mathbb{D}_X(B))))).$$

Now isomorphism (4.2) follows from the adjunction between

$$p_Y^*(-) \otimes p_X^!(B) \simeq - \otimes B$$

and $(p_Y)_*(- \otimes p_X^!(\mathbb{D}_X(B)))$.

4.6. Let $u_X^{\text{ren}} = u_{\text{Shv}(X)^{\text{ren}}}$ be the unit object of $\text{Shv}(X)^{\text{ren}} \otimes \text{Shv}(X)^{\text{ren}}$, i.e., the object so that the functor $\varphi(u_X) : \text{Shv}(X)^{\text{ren}} \rightarrow \text{Shv}(X)^{\text{ren}}$ is the identity, and let $\omega_X^{\text{ren}} \in \text{Shv}(X)^{\text{ren}}$ be the dualizing sheaf.

Corollary 4.7. We have a canonical isomorphism $u_X^{\text{ren}} \simeq \otimes^R((\Delta_X)_*(\omega_X^{\text{ren}}))$.

Proof. The assertion follows from the fact that the action of $(\Delta_X)_*(\omega_X^{\text{ren}}) \in \text{Shv}(X \times X)^{\text{ren}}$ by right functors is the identity functor on $\text{Shv}(X)^{\text{ren}}$.

4.8. Construction.

(a) Consider a lax commutative diagram

$$(3.4)$$

where
\[ \alpha_t \text{ corresponds to the morphism } \Xi(u^{\text{ren}}_X) \to (\Delta_X)_*(\omega^{\text{ren}}_X), \text{ obtained by adjunction from the isomorphism } u^{\text{ren}}_X \cong \Xi H(\Delta_X, (\omega^{\text{ren}}_X) \text{ of Corollary 4.7.} \]

- \( \alpha_m \) is the canonical isomorphism \( \Xi o ([c] \otimes \text{Id}) \cong [c \times \text{Id}] o \Xi \);

- \( \alpha_r \) is the tautological isomorphism \( \Xi \cong \Delta_X \circ \Xi \).

(b) By definition, the composition of the top arrow in the lax-commutative diagram (4.3) is \( \text{Tr}(\text{Shv}(X)_{\text{ren}}, [c]) \), while the composition of the bottom arrow is \( \Gamma(X, \Delta_X^! [c \times \text{Id}]((\Delta_X)_*(\omega^{\text{ren}}_X))) \).

We denote by \( \text{LT}_c^{\text{true}} \) the composition

\[ \text{Tr}(\text{Shv}(X)_{\text{ren}}, [c]) \xrightarrow{(4.3)} \Gamma(X, \Delta_X^! [c \times \text{Id}]((\Delta_X)_*(\omega^{\text{ren}}_X))) \cong \Gamma(\text{Fix}(c), \omega^{\text{ren}}_{\text{Fix}(c)}) \cong \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}), \]

where the first map is induced by the lax commutative diagram (4.3), the second map is the base change isomorphism, and the last isomorphism is the one from Section 1.1.2.

4.9. Relation with the trace maps of [Va2].

For every constructible sheaf \( A \in \text{Shv}(X)_{\text{const}} \), we denote by \( \mathcal{T}_{c,A} \) the composition

\[ \mathcal{H}\text{om}_{\text{Shv}(X)_{\text{ren}}}(A, [c](A)) \xrightarrow{\text{ch}_{c,A}} \text{Tr}(\text{Shv}(X)_{\text{ren}}, [c]) \xrightarrow{\text{LT}_c^{\text{true}}} \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}). \]

Unwinding the definition, the last map can be written as a composition

\[ \mathcal{H}\text{om}_{\text{Shv}(X)_{\text{ren}}}(A, [c](A)) \cong \Gamma(X, \Delta_X^! (\mathcal{D}_X(A) \boxtimes [c](A))) \cong \Gamma(X, \Delta_X^! [c \times \text{Id}]((\mathcal{D}_X(A) \boxtimes A)) \to \Gamma(X, \Delta_X^! [c \times \text{Id}]((\Delta_X)_*(\omega^{\text{ren}}_X))) \cong \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}), \]

where

- the first two isomorphisms are standard;
- the third map is induced by the canonical morphism \( \mathcal{D}_X(A) \boxtimes A \to (\Delta_X)_*(\omega^{\text{ren}}_X) \), obtained by adjointness from the evaluation map \( \mathcal{D}_X(A) \boxtimes A \to \omega_X \);
- the last isomorphism is the one from Section 4.3 (b).

In particular, the map \( \mathcal{T}_{c,A} \) is an extension to stacks and complexes of the trace map of [Va2, Section 1.2.2], which in its turn is motivated by the Verdier pairing of Illusie [Il].

4.10. Verdier-compatible stacks.

(a) Following [AGKRRV1, Section 2.6], we say that an Artin stack \( X \) is Verdier-compatible, if the Verdier duality preserves the subcategory of compact objects \( \text{Shv}(X)^c \subseteq \text{Shv}(X)_{\text{const}} \).

(b) As it is shown in [AGKRRV1, Theorem F.2.8] any Artin stack \( X \) that can be covered by open substacks of the form \( S/G \), where \( S \) is an algebraic space of finite type over \( k \) and \( G \) is an algebraic group (of finite type) over \( k \) is Verdier-compatible.\(^3\)

Note that this class of Artin stacks is closed under finite products and fiber products.

\(^3\)Though the assertion is only shown under an assumptions that \( S \) is a scheme and \( G \) is affine, these assumptions are not needed. Namely, the assumption that \( S \) is a scheme was never used, and the only place, where the assumption on \( G \) was used, was to establish an isomorphism [AGKRRV1, Formula (F.4)] asserting that if \( \pi_{pt} : pt \to BG \) is the projection, then \( (\pi_{pt})_*(\Xi_d) \cong (\pi_{pt})_!(\Xi_d) \) for some \( d \). However, this assertion for a general \( G \) is a formal consequence of the assertion for the affine \( G \). Namely, by Chevalley theorem, there exists a (unique) normal closed connected subgroup \( H \) of \( G \) such that \( H \) is affine and \( G/H \) is proper. Therefore the projection \( pt \to BG \) decomposes as a composition \( pt \to BH \to BG \), where the second morphism is proper. Therefore isomorphism [AGKRRV1, Formula (F.4)] for \( G \) formally follows from that for \( H \).
(c) Note that for every morphism \( f : X \rightarrow Y \) between Verdier-compatible Artin stacks the functor \( f_* \) preserves compact objects (because \( f_! \) does), and \( f^! \) preserves compact objects if \( f \) is safe (because \( f^* \) does).

(d) Combining part (c) with Section \([12](c),(d)\), we see that the morphism \([11]\) is an isomorphism if \( X \) and \( Y \) are Verdier-compatible, and that the base change morphism \([12]\) is an isomorphism if \( A \) and \( C \) are Verdier-compatible and \( a \) is safe (compare \([AGKRRV2\ Section A.3]\)).

Until end of this section we assume that \( X \) is Verdier-compatible.

4.11. Construction.

(a) Since \( X \) is assumed to be Verdier-compatible, the DG category \( \text{Shv}(X) \) is canonically self dual, and the equivalence of categories

\[
\text{Shv}(X) \rightarrow \text{Shv}(X)^\vee = \text{Funct}_{\text{cont}}(\text{Shv}(X), \text{Vect})
\]

is given by the formula

\[
\mathcal{B} \mapsto \mathcal{H}om_{\text{Shv}(X)}(\mathcal{D}_X(\mathcal{B}), -), \quad \mathcal{B} \in \text{Shv}(X)^c,
\]

where \( \mathcal{D}_X \) denotes the Verdier duality functor \( (\text{Shv}(X)^c)^{\text{op}} \rightarrow \text{Shv}(X)^c \).

The above equivalence corresponds to the pairing

\[
ev_X : \text{Shv}(X) \otimes \text{Shv}(X) \rightarrow \text{Vect} : A \otimes B \mapsto \Gamma_\bullet(X, A \otimes \mathcal{B}).
\]

(b) Let \( u_X = u_{\text{Shv}(X)} \) be the unit object of \( \text{Shv}(X) \otimes \text{Shv}(X) \). Then, as it was shown in \([AGKRRV1\ Section 22.2.4]\), we have a canonical isomorphism \( u_X \simeq \mathbb{R}[((\Delta_X)_*(\omega_X))] \), where \( \mathbb{R} : \text{Shv}(X) \otimes \text{Shv}(X) \rightarrow \text{Shv}(X \times X) \) is the exterior product functor, and \( \mathbb{R}^! \) is its right adjoint.

(c) Consider a lax commutative diagram

\[
\begin{array}{ccc}
\text{Vect} & \xrightarrow{\alpha_i} & \text{Shv}(X) \otimes \text{Shv}(X) \\
\downarrow \text{id} & & \downarrow \mathbb{R} \\
\text{Vect} & \xrightarrow{\alpha_m} & \text{Shv}(X) \otimes \text{Shv}(X) \\
\downarrow \gamma & & \downarrow \mathbb{R}^! \\
\text{Vect} & \xrightarrow{\alpha_r} & \text{Shv}(X) \otimes \text{Shv}(X) \\
\downarrow \text{id} & & \downarrow \mathbb{R}^! \\
\text{Vect} & \xrightarrow{\ev_X} & \text{Vect}
\end{array}
\]

where the 2-morphisms are defined as in Section \([18\ (a)\).

(d) By definition, the composition of the top arrow in diagram \([4.4]\) is \( \text{Tr}(\text{Shv}(X), [c]) \), while the composition of the bottom arrow is \( \Gamma_\bullet(X, \Delta_X'[c \times \text{Id}]((\Delta_X)_*(\omega_X))) \). In particular, the lax commutative diagram \([4.4]\) gives rise to the morphism

\[
\text{Tr}(\text{Shv}(X), [c]) \rightarrow \Gamma_\bullet(X, \Delta_X'[c \times \text{Id}]((\Delta_X)_*(\omega_X))).
\]

\[\text{ALTER}\] Alternatively, it can be shown by modifying the argument of Corollary \([4.7]\). Namely, observations \([13]\) and the formulation of Lemma \([4.5]\) will continue to hold without any changes if one replaces \( \text{Shv}(-)^{\text{cont}} \) by \( \text{Shv}(-) \), \( \text{Shv}(-)^{\text{constr}} \) by \( \text{Shv}(-)^c \), \( \Gamma \) by \( \Gamma_\bullet \) and \( (p_Y)_* \) by \( (p_Y)_! \) in all places, so it remains to show the modified version of formula \([4.2]\). For this we observe that since \( A \) and \( A \boxtimes \mathcal{B} \) are compact, both sides in the above modified formula \([4.2]\) commute with colimits in \( K \), and hence we can assume that \( K \) is compact. In this case, \( K \boxtimes \mathcal{B} \) is compact as well (see \([AGKRRV1\ Lemma F.4.4]\)), thus \( (p_Y)_! \) can be replaced by \( (p_Y)_* \), and we finish the proof as in Lemma \([4.5]\).
(e) Assume in addition that Artin stacks $C$ and $\text{Fix}(c)$ are Verdier-compatible. In this case, using Section 4.10(d), we have a canonical (base change) isomorphism
\[ \Gamma_\bullet(X, \Delta_X^1[c \times Id]((\Delta_X)_*(\omega_X))) \cong \Gamma_\bullet(X, \Delta_X^1(c_\bullet(\omega_C))) \cong \Gamma_\bullet(\text{Fix}(c), \omega_{\text{Fix}(c)}). \]

(f) We denote by $\text{LT}^{\text{true}}$ the composition
\[ \text{Tr}(\text{Shv}(X), [c]) \xrightarrow{(d)} \Gamma_\bullet(X, \Delta_X^1[c \times Id]((\Delta_X)_*(\omega_X))) \xrightarrow{(e)} \Gamma_\bullet(\text{Fix}(c), \omega_{\text{Fix}(c)}). \]

The proof of the following result is obtained by unwinding the definitions and will be given in Section 9.

**Proposition 4.12.** The diagram (4.1) is homotopy commutative.

### 5. Functoriality of true local terms

#### 5.1. Notation.

(a) Every morphism $[f]: c \to d$ of correspondences (see Section 3.3) induces a morphism $g_\Delta: \text{Fix}(c) \to \text{Fix}(d)$ between stacks of fixed points.

(b) Assume that the induced morphism $g_\Delta: \text{Fix}(c) \to \text{Fix}(d)$ is proper and safe. Then it follows from Corollary 1.5 and Sections 1.2(c), 1.3(d) that we have natural maps
\[ (g_\Delta)_!: \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \cong \Gamma(\text{Fix}(d), (g_\Delta)_!(\omega_{\text{Fix}(c)})) \to \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}) \]
and
\[ (g_\Delta)_!: \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \cong \Gamma(\text{Fix}(d), (g_\Delta)_!(\omega_{\text{Fix}(c)})) \to \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}). \]

Furthermore, these maps extend to a homotopy commutative diagram:
\[ \begin{array}{ccc}
\Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) & \xrightarrow{(g_\Delta)_!} & \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}) \\
\downarrow_{(1.2\, c)} & & \downarrow_{(1.2\, c)} \\
\Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) & \xrightarrow{(g_\Delta)_!} & \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}). 
\end{array} \]

(c) Assume now that morphism $g_\Delta$ is safe and quasi-smooth of relative dimension zero, e.g. étale. Then we have a natural map
\[ g_\Delta^! : \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}) \to \Gamma(\text{Fix}(c), g_\Delta^!(\omega_{\text{Fix}(d)})) \xrightarrow{\text{Gys}_g} \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}), \]
where the first morphism is induced by the composition
\[ \Gamma_\bullet(\text{Fix}(d), -) \xrightarrow{\text{unit}} \Gamma_\bullet(\text{Fix}(d), -) \circ (g_\Delta)_! \circ g_\Delta^! \cong \Gamma_\bullet(\text{Fix}(d), -) \circ (g_\Delta)_! \circ g_\Delta^! \cong \Gamma_\bullet(\text{Fix}(c), -) \circ g_\Delta^!, \]
and $\text{Gys}_{g_\Delta} : g_\Delta^* \to g_\Delta^!$ is the Gysin map (see Section 2.3). Moreover, even without the assumption that $g_\Delta$ is safe, we have a natural map
\[ g_\Delta^! : \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}) \to \Gamma(\text{Fix}(c), g_\Delta^!(\omega_{\text{Fix}(d)})) \xrightarrow{\text{Gys}_g} \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}). \]
Furthermore, these maps extend to a homotopy commutative diagram:

$$
\begin{array}{ccc}
\Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}) & \xrightarrow{g_\Delta^*} & \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
\downarrow & & \downarrow \\
\Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}) & \xrightarrow{g_\Delta^*} & \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)})
\end{array}
$$

The following result, whose proof will be given in Sections 9 and 10, asserts that the true local terms commute with proper pushforwards and smooth pullbacks.

**Theorem 5.2.**

(a) In the situation of Section 3.4(i), the following diagram commutes up to a canonical homotopy:

$$
\begin{array}{ccc}
\text{Tr}((\text{Shv}(X)^{\text{ren}}, [c]) & \xrightarrow{\text{LT}^{\text{true}}_{\text{true}}} & \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
\downarrow & & \downarrow (g_\Delta) \\
\text{Tr}((\text{Shv}(Y)^{\text{ren}}, [d]) & \xrightarrow{\text{LT}^{\text{true}}_{\text{true}}} & \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)})
\end{array}
$$

(b) In the situation of Section 3.6(i), assume that morphism $g_\Delta$ is quasi-smooth of relative dimension zero. Then the following diagram commutes up to a canonical homotopy:

$$
\begin{array}{ccc}
\text{Tr}((\text{Shv}(Y)^{\text{ren}}, [d]) & \xrightarrow{\text{LT}^{\text{true}}_{\text{true}}} & \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}) \\
\downarrow & & \downarrow (g_\Delta) \\
\text{Tr}((\text{Shv}(X)^{\text{ren}}, [c]) & \xrightarrow{\text{LT}^{\text{true}}_{\text{true}}} & \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)})
\end{array}
$$

5.3. Contracting correspondences. Let $c: C \to X \times X$ be a correspondence, and let $Z \subseteq X$ be a $c$-invariant closed substack (see Section 3.3).

(a) Since $Z \subseteq X$ is $c$-invariant, we have $c(I_Z) \subseteq c(I_Z)$, where $I_Z \subseteq O_X$ is the sheaf of ideals of $Z$, and $c$ and $c$ denote pullbacks of sheaves.

(b) Following [Va2, Section 2.1.1], we say that $c$ is contracting near $Z$, if there exists $n \in \mathbb{N}$ such that $c(I_Z)^n \subseteq c(I_Z)^{n+1}$.

(c) It was proved in [Va2, Theorem 2.1.3(a)] that if $c$ contracting near $Z$, then the inclusion $i_\Delta: \text{Fix}(c|Z)_{\text{red}} \hookrightarrow \text{Fix}(c)_{\text{red}}$ is an open embedding, so we have restriction functors (see Section 5.1(c))

and

$$
i_\Delta^*: \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \to \Gamma(\text{Fix}(c|Z), \omega_{\text{Fix}(c|Z)})
$$

(d) Example. Assume that $X$ is defined over $\mathbb{F}_q$, and let $c$ be the correspondence

$$(\text{Fr}, \text{Id}): X \to X \times X.$$

Then $c$ is contracting near every closed substack $Z \subseteq X$ defined over $\mathbb{F}_q$.

The proof of the following result will be given in Section 7.
Theorem 5.4. Assume that a correspondence \( c : C \to X \times X \) is contracting near \( Z \subseteq X \). Then the true local terms map commute with \(*\)-restriction with respect to the embedding \( i : Z \hookrightarrow X \). In other words, in the notation of Sections 3.8(c) and 5.3(c), the following diagram commutes up to a canonical homotopy:

\[
\begin{array}{ccc}
\text{Tr}(\text{Shv}(X)^{\text{ren}}, [c]) & \xrightarrow{\text{LT}_{\text{true}}^c} & \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
\text{Tr}([i, i]^c) & & \downarrow \imath^c \\
\text{Tr}(\text{Shv}(Z)^{\text{ren}}, [c|Z]) & \xrightarrow{\text{LT}_{\text{true}}^c} & \Gamma(\text{Fix}(c|Z), \omega_{\text{Fix}(c|Z)}).
\end{array}
\]

5.5. Remark. We expect that one can show that Theorem 5.4 continues to hold if one replaces the assumption that \( c \) is contracting near \( Z \subseteq X \) by a weaker assumption that \( c \) has no almost fixed points in the punctured tubular neighborhood of \( Z \) (see \([Va3\text{, Definition 4.4 and Theorem 4.10}]\)).

Combining Theorems 5.2 and 5.4 with the commutativity of diagram (3.5) and (3.9), we get the following corollary:

Corollary 5.6.
(a) In the situation of Theorem 5.2(a), for every constructible sheaf \( A \in \text{Shv}(X)^{\text{constr}} \), the following diagram commutes up to a canonical homotopy:

\[
\begin{array}{ccc}
\mathcal{H}om_{\text{Shv}(X)^\text{ren}}(A, [c](A)) & \xrightarrow{\mathcal{H}om_{\text{Shv}(X)^\text{ren}}(f_! \mathcal{H}om(A), [d](f_! \mathcal{H}om(A)))} & \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
[f]^c & & \downarrow \delta^c \\
\mathcal{H}om_{\text{Shv}(X)^\text{ren}}(f^*(A), [d_*(f^*(A))) & \xrightarrow{\mathcal{H}om_{\text{Shv}(X)^\text{ren}}(f^*(A), [d_*(f^*(A)))} & \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}).
\end{array}
\]

(b) In the situation of Theorem 5.2(b), for every constructible sheaf \( A \in \text{Shv}(Y)^{\text{constr}} \), the following diagram commutes up to a canonical homotopy:

\[
\begin{array}{ccc}
\mathcal{H}om_{\text{Shv}(Y)^\text{ren}}(A, [d](A)) & \xrightarrow{\mathcal{H}om_{\text{Shv}(Y)^\text{ren}}(f_! \mathcal{H}om(A), [d](f_! \mathcal{H}om(A)))} & \Gamma(\text{Fix}(d), \omega_{\text{Fix}(d)}) \\
[f]^* & & \downarrow \delta^c \\
\mathcal{H}om_{\text{Shv}(X)^\text{ren}}(f^*(A), [c](f^*(A))) & \xrightarrow{\mathcal{H}om_{\text{Shv}(X)^\text{ren}}(f^*(A), [c](f^*(A)))} & \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}).
\end{array}
\]

(c) In the situation of Theorem 5.4 for every constructible sheaf \( A \in \text{Shv}(X)^{\text{constr}} \), the following diagram commutes up to a canonical homotopy:

\[
\begin{array}{ccc}
\mathcal{H}om_{\text{Shv}(X)^\text{ren}}(A, [c](A)) & \xrightarrow{\mathcal{H}om_{\text{Shv}(X)^\text{ren}}(f_! \mathcal{H}om(A), [d](f_! \mathcal{H}om(A)))} & \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
[c]^* & & \downarrow \delta^c \\
\mathcal{H}om_{\text{Shv}(Z)^\text{ren}}(A|Z, [c|Z](A|Z)) & \xrightarrow{\mathcal{H}om_{\text{Shv}(Z)^\text{ren}}(A|Z, [c|Z](A|Z))} & \Gamma(\text{Fix}(c|Z), \omega_{\text{Fix}(c|Z)}).
\end{array}
\]

5.7. Remark. Notice that while parts (a) and (c) of Corollary 5.6 are extensions to stacks and complexes of the corresponding results of \([Va2\text{, part b}]\), part (b) seems to be completely new.

\footnote{Recently, a derived version of part (b) also appeared in \([FYZ\text{, Proposition 4.5.4}]\).}
The following result, whose proof will be given in Sections 9 and 10, asserts that Theorem 5.2 has an analog for refined true local terms maps. This result is not needed for Theorem 0.8 and is only included for completeness and future references.

**Theorem 5.8.** Assume that Artin stacks \( X, Y, C, D, \text{Fix}(c) \) and \( \text{Fix}(d) \) are Verdier-compatible.

(a) In the situation of Section 3.4(i), the following diagram commutes up to a canonical homotopy:

\[
\begin{array}{ccc}
\text{Tr}(\text{Shv}(X), [c]) & \xrightarrow{LT_{\text{true}}^\Delta} & \Gamma_{\Delta}(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
\text{Tr}([f]) & \downarrow & (g_{\Delta}) \\
\text{Tr}(\text{Shv}(Y), [d]) & \xrightarrow{LT_{\text{true}}^\Delta} & \Gamma_{\Delta}(\text{Fix}(d), \omega_{\text{Fix}(d)})
\end{array}
\]

(b) In the situation of Section 3.6(i)(b), assume that morphism \( g_{\Delta} \) satisfies the assumptions of Section 5.1(c). Then the following diagram commutes up to a canonical homotopy:

\[
\begin{array}{ccc}
\text{Tr}(\text{Shv}(Y), [d]) & \xrightarrow{LT_{\text{true}}^\Delta} & \Gamma_{\Delta}(\text{Fix}(d), \omega_{\text{Fix}(d)}) \\
\text{Tr}([f]^{\ast}) & \downarrow & g_{\Delta} \\
\text{Tr}(\text{Shv}(X), [c]) & \xrightarrow{LT_{\text{true}}^\Delta} & \Gamma_{\Delta}(\text{Fix}(c), \omega_{\text{Fix}(c)})
\end{array}
\]

For completeness we also state a result (without proof) asserting that Theorem 5.2 and 5.8 have a common refinement.

**Theorem 5.9.**

(a) In the situation of Theorem 5.8(a), we have a natural homotopy commutative cube:

\[
\begin{array}{ccc}
\text{Tr}(\text{Shv}(X), [c]) & \xrightarrow{LT_{\text{true}}^\Delta} & \Gamma_{\Delta}(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
\text{Tr}(\text{Shv}(Y), [d]) & \xrightarrow{LT_{\text{true}}^\Delta} & \Gamma_{\Delta}(\text{Fix}(d), \omega_{\text{Fix}(d)}) \\
\end{array}
\]

whose boundary consists of homotopy commutative squares of Theorems 5.2(a) and 5.8(a), diagram (4.1), Proposition 3.5 and Section 5.1(b).

(b) In the situation of Theorem 5.8(b), we have a natural homotopy commutative cube (similar to part (a)), whose boundary consists of homotopy commutative squares of Theorems 5.2(b) and 5.8(b), diagram (4.1), Proposition 3.5 and Section 5.1(c).
We finish this section by stating another result without a proof asserting that Theorem 5.4 also have a refinement in which refined true local terms maps are taken into an account.

**Theorem 5.10.** In the situation of Theorem 5.4, assume that $X$ and $C$ satisfy the assumption of Section 4.10(b). Then the following diagram commutes up to a canonical homotopy:

$$
\begin{array}{ccc}
\text{Tr}(\text{Shv}(X), [c]) & \xrightarrow{\text{LT}^{\text{true}}_{c}} & \Gamma_{\Delta}(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
\downarrow_{\text{Tr}(i_c^*')} & & \downarrow_{i_{\Delta}} \\
\text{Tr}(\text{Shv}(Z), [c|Z]) & \xrightarrow{\text{LT}^{\text{true}}_{c|Z}} & \Gamma_{\Delta}(\text{Fix}(c|Z), \omega_{\text{Fix}(c|Z)}).
\end{array}
$$

Moreover, the above diagram together with that of Theorem 5.4, diagram (4.1), Proposition 3.7 and Section 5.1(c) naturally extends to a homotopy commutative cube (as in Theorem 5.9(a)).

### 6. Local terms and the Grothendieck–Lefschetz trace formula

In this section we will prove Theorem 0.8. Our ground field $k$ will be $\mathbb{F}_q$. However, all Artin stacks $X$ and all morphisms $f : X \to Y$ that will appear in this section will be assumed defined over $\mathbb{F}_q$, so that $X$ carries the geometric Frobenius endomorphism $F_r$ and $f$ intertwines endomorphisms $F_r$ on $X$ and $Y$.

#### 6.1. $*$-Pullbacks and the naive local terms for $\text{Shv}(\cdot)^{\text{ren}}$

(a) Notice that for every morphism $f : X \to Y$, the pullback functor $f^* : \text{Shv}(Y)^{\text{ren}} \to \text{Shv}(X)^{\text{ren}}$ induces a map of traces

$$
\text{Tr}(f^*, F_r) : \text{Tr}(\text{Shv}(Y)^{\text{ren}}, F_r) \to \text{Tr}(\text{Shv}(X)^{\text{ren}}, F_r),
$$

(see Section 3.6(ii)). Furthermore, the assignment $f \mapsto \text{Tr}(f^*, F_r)$ is compatible with compositions.

(b) For every $x \in X(\mathbb{F}_q)$, let $\text{Spec}(\mathbb{F}_q) := \text{pt}$ $\xrightarrow{i_x} X$ be the corresponding Frobenius-equivariant morphism. Hence, it gives rise to a map

$$
\text{Tr}(\eta_x^*, F_r) : \text{Tr}(\text{Shv}(X)^{\text{ren}}, F_r) \to \text{Tr}(\text{Shv}(\text{pt}), F_r) \simeq \overline{\mathbb{Q}_\ell}.
$$

(c) We define the naive local terms map

$$
\text{LT}_X^{\text{naive}} : \text{Tr}(\text{Shv}(X)^{\text{ren}}, F_r) \to \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})
$$

as the unique map such that

$$
\eta_x(\mathbb{F}_q)^* \circ \text{LT}_X^{\text{naive}} = \text{Tr}(\eta_x^*, F_r) \text{ for all } x \in X(\mathbb{F}_q).
$$

(d) By definition, naive local terms commute with $*$-pullbacks.

#### 6.2. The naive local terms for $\text{Shv}(\cdot)$

(a) By analogy with Section 4.1, we denote by $\text{LT}_X^{\text{naive}} : \text{Tr}(\text{Shv}(X), F_r) \to \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})$ the composition

$$
\text{Tr}(\text{Shv}(X), F_r) \xrightarrow{\text{Tr}(\text{ren}_X, F_r)} \text{Tr}(\text{Shv}(X)^{\text{ren}}, F_r) \xrightarrow{\text{LT}_X^{\text{naive}}} \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})
$$

and also call it the naive local terms map.
(b) Using Proposition 6.7 one can see that the map of part (a) can be also defined directly. Namely, the pullback map $\text{Tr}(f^*, \text{Fr})$ between the $\text{Shv}(-)$'s is defined for all safe morphisms, and the naive local terms map for $\text{Shv}(-)$ is characterized by the formula of Section 6.1(c) taking into account that every morphism $\eta_x$ is representable, hence safe.

(c) By part (b), the map of part (a) commutes with $\ast$-pullbacks with respect to safe morphisms.

6.3. The true local terms. Consider correspondence $c = (\text{Fr}, \text{Id}) : X \to X \times X$.

The induced endofunctors $[c]$ of both $\text{Shv}(X)$ and $\text{Shv}(X)^{\text{ren}}$ then coincide with $\text{Fr}_*$, so we have identifications

$$\text{Tr}(\text{Shv}(X)^{\text{ren}}, [c]) \simeq \text{Tr}(\text{Shv}(X)^{\text{ren}}, \text{Fr}_*)$$

and

$$\text{Tr}(\text{Shv}(X), [c]) \simeq \text{Tr}(\text{Shv}(X), \text{Fr}_*).$$

By Lemma 6.3 below, we can identify $\text{Fix}(c) \simeq X(\mathbb{F}_q)$, thus we get an identification

$$\Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \simeq \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell).$$

Therefore the maps $\text{LT}_{\text{true}}^c$ from Section 4.1(a) can be interpreted as maps

$$\text{Tr}(\text{Shv}(X)^{\text{ren}}, \text{Fr}) \to \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$$

and

$$\text{Tr}(\text{Shv}(X), \text{Fr}) \to \text{funct}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell),$$

are will be denoted by $\text{LT}_X^{\text{true}}$.

Lemma 6.4. The stack of fixed points $\text{Fix}(c)$ of the correspondence (6.7) is canonically identified with the discrete Deligne–Mumford stack, corresponding to a finite groupoid $X(\mathbb{F}_q)$.

Proof. Our argument is an almost a repetition of that of [Va1] Lemma 3.3, where the assertion was shown under a certain (unnecessary) assumption.

Note first that $Y := \text{Fix}(c)$ is an algebraic stack of finite presentation over $\overline{\mathbb{F}}_q$, defined over $\mathbb{F}_q$.

Claim 6.5. The natural maps

$$\pi : Y(\overline{\mathbb{F}}_q[t]/(t^2)) \to Y(\mathbb{F}_q) \text{ and } i : Y(\mathbb{F}_q) \to Y(\overline{\mathbb{F}}_q[t]/(t^2)),$$

corresponding to the homomorphisms of $\overline{\mathbb{F}}_q$-algebras $\pi : \mathbb{F}_q[t]/(t^2) \to \mathbb{F}_q$ and $i : \mathbb{F}_q \to \mathbb{F}_q[t]/(t^2)$, are equivalences of categories.

Proof. Since $\pi \circ i \simeq \text{Id}$, it suffices to show that the composition

$$i \circ \pi : Y(\mathbb{F}_q[t]/(t^2)) \to Y(\overline{\mathbb{F}}_q[t]/(t^2))$$

is isomorphic to the identity. But this follows from the observation that for every $\overline{x} \in X(\mathbb{F}_q[t]/(t^2))$, we have a canonical isomorphism

$$\text{Fr}(\overline{x}) \simeq i \circ \pi \circ \text{Fr}(\overline{x}).$$

Namely, this assertion is well-known when $X$ is a scheme, and the general case follows from it. \hfill \square

Let us now come back to the proof of the lemma. Using Claim 6.5 we conclude that the diagonal morphism $\Delta_Y : Y \to Y \times Y$ is unramified, hence $Y$ is a Deligne–Mumford stack (see [Stacks] Tag 06N3)). Using Claim 6.5 again we conclude that $Y$ is étale over $\overline{\mathbb{F}}_q$.

Note that we have a natural morphism $X(\mathbb{F}_q) \to Y$ of Deligne–Mumford stacks, where $X(\mathbb{F}_q)$ is viewed as a discrete Deligne–Mumford stack, and it suffices to show that the induced functor $\psi : X(\mathbb{F}_q) \to Y(\overline{\mathbb{F}}_q)$ on $\mathbb{F}_q$-points is an equivalence of categories.
The fact that $\psi$ is fully faithful follows from (and is actually equivalent to) the first axiom of a stack (the sheaf axiom for $\text{Isom}(x, y)$) applied to étale covers $\text{Spec} \mathbb{F}_q^m \to \text{Spec} \mathbb{F}_q$ for all $m \in \mathbb{N}$.

Thus, it suffices to show that functor $\psi$ is essentially surjective. Fix any object $y \in Y(\mathbb{F}_q)$. Then $y$ corresponds to a pair $(x, \phi)$, where $x \in X(\mathbb{F}_q)$ and $\phi \in \text{Isom}_{X(\mathbb{F}_q)}(\text{Fr}(x), x)$, and we want to show that there exists an object $x_0 \in X(\mathbb{F}_q)$ such that $\psi(x_0)$ is isomorphic to $(x, \phi)$.

Choose $m$ such that $x$ is (a pull-back of) an object of $X(\mathbb{F}_q^m)$, and $\phi \in \text{Isom}_{X(\mathbb{F}_q^m)}(\text{Fr}(x), x)$. Then the norm
\[
N_m(\phi) := \phi \circ \text{Fr}(\phi) \circ \ldots \circ \text{Fr}^{m-1}(\phi) : x = \text{Fr}^m(x) \cong x
\]
defines an $\mathbb{F}_q^m$-point of an algebraic group $H := \text{Isom}_X(x, x)$.

If $N_m(\phi) = 1$, then the existence of $x_0$ is equivalent to the second axiom of a stack applied to the étale covering $\text{Spec} \mathbb{F}_q^m \to \text{Spec} \mathbb{F}_q$. To show the general case, note that $H$ is of finite type over $\mathbb{F}_q$, hence an element $N_m(\phi) \in H(\mathbb{F}_q^m)$ is of finite order $d$, and thus $N_{dm}(\phi) = (N_m(\phi))^d = 1$. □

6.6. Remark. Assume that $X$ is Verdier-compatible.

(a) Since the stack of fixed points $\text{Fix}(c) \simeq X(\mathbb{F}_q)$ is safe, the canonical map $\Gamma^!_{\text{Fix}(c), \text{Fr}(\text{Fix}(c))} \to \Gamma(\text{Fix}(c), \text{Fr}(\text{Fix}(c)))$ of Section 5.2(e) is an isomorphism. Therefore it follows from Proposition 5.12 that the refined true local terms map $L_{\text{true}}$ coincides with $L_{\text{true}}^\text{true}$.

(b) Assume in addition that $X$ has finitely many isomorphism classes of $\mathbb{F}_q$-points. In this case, it follows from a combination of [AGKRRV1, Remark 22.2.6] and part (a) that the true local map $L_{\text{true}}^\text{true}$ is automatically an isomorphism.

The proof of Theorem 0.8 is based on the following corollary of Theorems 5.2 and 5.4.

Corollary 6.7. The true local term functor
\[
L_{\text{true}} : \text{Tr}(\text{Shv}(\text{ren}), \text{Fr}) \to \text{funct}(\mathbb{F}_q), \mathbb{Q}_{\ell}^!) \quad (\text{resp. } L_{\text{true}} : \text{Tr}(\text{Shv}(-), \text{Fr}) \to \text{funct}(\mathbb{F}_q), \mathbb{Q}_{\ell}^))
\]
commutes with:

(a) !-pushforwards with respect to proper safe morphisms;

(b) *-pullbacks with respect to smooth morphisms (resp. smooth safe morphisms);

(c) $\ast$-pullbacks with respect to closed embeddings.

Proof. The assertions (a) and (b) for $\text{Shv}(\text{ren})$ follows from Theorem 5.2, while assertion (c) follows from a combination of Example 5.3(d) and Theorem 5.4. Then the assertion for $\text{Shv}(-)$ follows from a combination of an assertion for $\text{Shv}(\text{ren})$ and Propositions 5.30 and 5.47. □

6.8. Proof of Theorem 0.8(a). By the definition of the naive local term map, we have to show that for every $x \in X(\mathbb{F}_q)$ there exists a canonically defined homotopy
\[
\eta_x(\mathbb{F}_q)^* \circ L_{\text{true}}^X \simeq \text{Tr}(\eta_x^*, \text{Fr}).
\]
In other words, we have to show that true local terms commute with $\ast$-pullback with respect to the morphism $\eta_x : \text{pt} \to X$.

Let $Y \subseteq X$ be the closure of $\eta_x(\text{pt})$, equipped with a reduced stack structure. Let $G_x := \text{Aut}_X(x)$ denote the group scheme of automorphisms of $x$, and let $G_{x, \text{red}}$ be the underlying reduced group scheme of $G_x$. Then $\eta_x$ factors as
\[
(6.2) \quad \eta_x : \text{pt} \xrightarrow{\eta_x} B(G_{x, \text{red}}) \xrightarrow{\text{pr}_2} B(G_x) \xrightarrow{\eta_x^*} Y \xrightarrow{i} X,
\]
and it suffices to show that true local terms commute with (*-pullbacks with respect to all the morphisms that appear in the diagram (6.3).

The required assertions follow from Corollary 6.7. Namely, the assertion for pr₁ follows by Corollary 6.7(b), the assertion for i follows by Corollary 6.7(c). The assertion for pr₂ follows by Corollary 6.7(a): indeed pr₂ is an equivalence of categories with inverse (pr₂)!, while pr₂ is proper. Finally, the assertion for \( \eta_x \colon B(G_x) \to Y \) follows again by Corollary 6.7(b), since \( \eta_x \) is an open embedding.

6.9. Proof of Theorem 6.8(b).

First, we record the following particular case of Lemma 3.9.

Corollary 6.10. Let \( Z \subseteq X \) be a closed substack, and let \( U \subseteq X \) be the complementary open. Then the functors \( i_1 : \text{Shv}(U) \to \text{Shv}(X) \) and \( j_1 : \text{Shv}(U) \to \text{Shv}(X) \) induce an isomorphism of traces

\[
\text{Tr}(j_1, Fr) \oplus \text{Tr}(i_1, Fr) : \text{Tr(Shv(U)}, Fr) \oplus \text{Tr(Shv}(Z), Fr) \to \text{Tr(Shv}(X), Fr).
\]

First we will show the assertion of Theorem 6.8(b) for Shv(−). We will carry out the proof in six steps.

Step 1. Note first that the assertion for morphisms \( g : Z \to X \) and \( f : X \to Y \) implies the assertion for \( f \circ g \). Conversely, if morphism Tr\((g, Fr)\) is isomorphism, then the assertion for morphisms \( g \) and \( f \circ g \) implies the assertion for \( f \).

Step 2. We claim that we can assume that \( Y = pt \). More precisely, the assertion for \( f : X \to Y \) is equivalent to the assertions for morphisms \( f_y : X_y \to pt \) for all \( y \in Y(\mathbb{F}_q) \), where \( X_y := X \times Y \{ y \} \).

Proof. For each \( y \in Y(\mathbb{F}_q) \), let \( \eta_y : pt \to Y \) and \( \tilde{\eta}_y : X_y \to X \) be the corresponding morphisms. We want to show that

\[
\eta_y(\mathbb{F}_q)^* \circ f(\mathbb{F}_q) \circ LT_{X_y}^\text{naive} = \eta_y(\mathbb{F}_q)^* \circ LT_{X}^\text{naive} \circ \text{Tr}(f, Fr).
\]

The LHS of formula (6.3) is homotopic to

\[
f_y(\mathbb{F}_q)^* \circ \tilde{\eta}_y(\mathbb{F}_q)^* \circ LT_{X}^\text{naive} \circ \text{Tr}(\tilde{\eta}_y, Fr).
\]

The RHS of formula (6.3) is homotopic to

\[
LT_{pt}^\text{naive} \circ \text{Tr}(\eta_y, Fr) \circ \text{Tr}(f, Fr) \simeq LT_{pt}^\text{naive} \circ \text{Tr}(f_y, Fr) \circ \text{Tr}(\tilde{\eta}_y, Fr),
\]

by the base change isomorphism \( \eta_y^* \circ f \simeq f_y \). Hence homotopy (6.3) would follow once we establish homotopy

\[
f_y(\mathbb{F}_q)^* \circ LT_{X}^\text{naive} \simeq LT_{pt}^\text{naive} \circ \text{Tr}(f_y, Fr).
\]

Step 3. Let \( Z \subseteq X \) be a closed substack, and let \( U \subseteq X \) be the complementary open. Then the assertion for morphism \( f \) is equivalent to the assertions for morphisms \( f|_U \) and \( f|_Z \).

Proof. Let \( g : U \sqcup Z \to X \) be the canonical map. By Corollary 6.10 the induced map

\[
\text{Tr}(g, Fr) : \text{Tr(Shv}(U \sqcup Z), Fr) \to \text{Tr(Shv}(X), Fr)
\]

is an isomorphism. Hence, by Step 1, it suffices to show the assertion for morphisms \( f \circ g \) and \( g \). Since \( g \) is a monomorphism, the assertion for \( g \) follows immediately from Step 2. Finally, the assertion for morphism \( f \circ g \) is equivalent to the assertions for morphisms \( f|_U \) and \( f|_Z \).
Step 4. The assertion holds for representable morphisms.

Proof. By Step 2, it is enough to treat the case when $X$ is an algebraic space and $Y = \text{pt}$. Then $X$ has an open subspace $U$, which is an affine scheme. Thus, by Step 3 and Noetherian induction, we can assume that $X$ is affine. In this case, $X$ has a compactification $\overline{X}$, hence using Step 3 again, we can assume that $X$ is projective. In this case, the assertion for the true local terms map follows from Corollary 6.7(a). Hence, the assertion for the naive local terms map follows from Theorem 0.8(a), which has already been proved. □

Step 5. The assertion holds, if $X$ is a classifying stack $BG$ for an algebraic group $G$ and $Y = \text{pt}$.

Proof. Note that the projection $\pi : \text{pt} \to BG$ is schematic. Therefore the assertion for $\pi$ follows from Step 4.

Assume first that $G$ is connected. In this case, the map $\pi(\mathbb{F}_q)$ is an isomorphism by Lang’s theorem. Since both $LT_{BG}^{\text{true}}$ and $LT_{BG}^{\text{true}}$ are isomorphisms by Remark 6.3(b), we conclude that $LT_{BG}^{\text{naive}}$ and $LT_{BG}^{\text{naive}}$ are isomorphisms by Theorem 6.8(a), hence the map $\text{Tr}(\pi, \text{Fr})$ is an isomorphism, because the assertion holds for $\pi$. Finally, since the assertion trivially holds for the composition $\text{pt} \to BG \to \text{pt}$, the assertion for the projection $BG \to \text{pt}$ now follows from Step 1.

Assume next that $G$ is finite. In this case, there exists an embedding $G \hookrightarrow GL_n$. Then the projection $BG \to \text{pt}$ factors as $BG \to B(GL_n) \to \text{pt}$. By Step 1, it suffices to prove the assertion for $BG \to B(GL_n)$ and $B(GL_n) \to \text{pt}$. The assertion for $B(GL_n) \to \text{pt}$ follows because $GL_n$ is connected. The assertion for $BG \to B(GL_n)$ follows from Step 4, because this map is schematic.

In the general case, let $G^0$ be the connected component of the identity of $G$, and let $\pi_0(G)$ be the group of connected components. Then the projection $BG \to \text{pt}$ factors as $BG \to B(\pi_0(G)) \to \text{pt}$, and the geometric fiber of the first map is isomorphic to $B(G^0)$. Therefore the assertion follows from Steps 1 and 2 and the particular cases, proven above. □

Step 6. The assertion holds in general.

Proof. By Step 2, we can assume that $Y = \text{pt}$. Also we can replace $X$ by $X_{\text{red}}$. Recall that for every reduced algebraic stack $X$ such that $I_X \to X$ is quasi-compact, there exists a (canonical) dense open substack $U \subseteq X$, which is a gerbe (see [Stacks, Tag 06RC]). Hence, using Step 3 and Noetherian induction, we can assume that $X$ is a gerbe over an algebraic space $X'$. Then the projection $X \to \text{pt}$ decomposes as $X \xrightarrow{\sim} X' \to \text{pt}$. Then, by Steps 1 and 2, the assertion for $X$ follows from that for $X'$ and geometric fibers of $\pi$. Since every geometric fiber of $\pi$ is isomorphism to $BG$ for some group scheme $G$, the assertion therefore follows from Steps 4 and 5. □

Finally, we show the assertion for $\text{Shv}(\mathbb{F}_q)^{\text{ren}}$. Arguing as in Step 2, it suffices to show the assertion under an assumption that $Y = \text{pt}$. Since $f$ was assumed to be safe, we conclude that $X$ is safe, thus, by Remark 1.7, the functor $\text{ren}_X : \text{Shv}(X) \to \text{Shv}(X)^{\text{ren}}$ is an equivalence. Therefore the assertion follows from the already shown assertion for $\text{Shv}(\mathbb{F}_q)^{\text{ren}}$.

7. Deformation to the normal cone, and proof of Theorem 5.4

As in [Va2], our proof is based on the observation that the true local term map commutes with specialization. With future applications in mind, we will work in a slightly more general set-up.
7.1. Specialization of correspondences. Let $\mathcal{D}$ be the spectrum of a DVR over $k$, and let $\eta$ and $s$ be the generic and the special points of $\mathcal{D}$, respectively. We will consider Artin stacks (and correspondences between them) of finite presentation over $\mathcal{D}$.

(a) Every correspondence $\tilde{c} = (\tilde{c}_l, \tilde{c}_r) : \tilde{C} \to \tilde{X} \times \tilde{X}$ over $\mathcal{D}$ gives rise to correspondences

$$\tilde{c}_\eta = (\tilde{c}_{\eta,l}, \tilde{c}_{\eta,r}) : \tilde{C}_\eta \to \tilde{X}_\eta \times \tilde{X}_\eta$$

and

$$\tilde{c}_s = (\tilde{c}_{s,l}, \tilde{c}_{s,r}) : \tilde{C}_s \to \tilde{X}_s \times \tilde{X}_s$$

(b) Recall that we have the nearby cycle functor $\Psi_\tilde{X} : \text{Shv}(\tilde{X}_\eta)_{\text{constr}} \to \text{Shv}(\tilde{X}_s)_{\text{constr}}$, which uniquely extends to a continuous functor $\Psi_\tilde{X} : \text{Shv}(\tilde{X}_\eta)^{\text{ren}} \to \text{Shv}(\tilde{X}_s)^{\text{ren}}$, and similarly for $\tilde{C}$.

(c) We have the base change morphism

$$\Psi_\tilde{X} \circ [\tilde{c}_\eta] = \Psi_\tilde{X} \circ (\tilde{c}_{\eta,l})_* \circ \tilde{c}_{\eta,r}^! \to (\tilde{c}_{s,l})_* \circ \Psi_\tilde{C} \circ \tilde{c}_{\eta,r}^! \to (\tilde{c}_{s,l})_* \circ \tilde{c}_{s,r}^! \circ \Psi_\tilde{X} = [\tilde{c}_s] \circ \Psi_\tilde{X}.$$  

Since $\Psi_\tilde{X}$ maps constructible sheaves to constructible sheaves, it has a continuous right adjoint, and therefore induces a map of traces

$$\text{Tr}(\Psi_\tilde{X}) : \text{Tr}(\text{Shv}(\tilde{X}_\eta)^{\text{ren}}, [\tilde{c}_\eta]) \to \text{Tr}(\text{Shv}(\tilde{X}_s)^{\text{ren}}, [\tilde{c}_s]).$$

(d) We also have the natural maps

$$\Psi_{\text{Fix}(\tilde{c})} : \Gamma(\text{Fix}(\tilde{c}_\eta), \omega_{\text{Fix}(\tilde{c}_\eta)}) \to \Gamma(\text{Fix}(\tilde{c}_s), \Psi_{\text{Fix}(\tilde{c})}(\omega_{\text{Fix}(\tilde{c}_\eta)})) \to \Gamma(\text{Fix}(\tilde{c}_s), \omega_{\text{Fix}(\tilde{c}_s)}).$$

7.2. Extension of scalars.

(a) Let $c : C \to X \times X$ be a correspondence, $k'/k$ a field extension. Let $c' : C' \to X' \times X'$ be the base change of $C$ to $k'$. Let $\pi$ denote any of the morphisms $X' \to X$, $C' \to C$, etc.

(b) Since $\pi^* : \text{Shv}(X) \to \text{Shv}(X')$ maps constructible objects to constructible objects, it extends to a continuous functor $\pi^* : \text{Shv}(X)^{\text{ren}} \to \text{Shv}(X')^{\text{ren}}$ admitting a continuous right adjoint.

(c) We have the base change morphism

$$\pi^* \circ [c] = \pi^* \circ (c)_* \circ c^!_s \simeq (c^!_l)_* \circ \pi^* \circ d^!_l \simeq (c^!_l)_* \circ c^!_s \circ \pi^* = [c'] \circ \pi^*.$$  

Hence, we obtain a map of traces

$$\text{Tr}([\pi c^*]) : \text{Tr}(\text{Shv}(X)^{\text{ren}}, [c]) \to \text{Tr}(\text{Shv}(X')^{\text{ren}}, [c']).$$

(d) We also have a canonically defined map

$$\pi^*_A : \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \to \Gamma(\text{Fix}(c'), \pi^* (\omega_{\text{Fix}(c')})) \simeq \Gamma(\text{Fix}(c'), \omega_{\text{Fix}(c')}).$$

The following result, whose proof will be given in Section 9, asserts that true local terms commute with nearby cycles and extension of scalars.

Theorem 7.3.

(a) In the situation of Section 7.1, the following diagram commutes up to a canonical homotopy:

$$\text{Tr}(\text{Shv}(\tilde{X}_\eta)^{\text{ren}}, [\tilde{c}_\eta]) \xrightarrow{\text{LT}^{\text{true}}_{\tilde{c}_\eta}} \Gamma(\text{Fix}(\tilde{c}_\eta), \omega_{\text{Fix}(\tilde{c}_\eta)})$$

$$\text{Tr}(\Psi_\tilde{X}) \downarrow \quad \quad \downarrow \Psi_{\text{Fix}(\tilde{c})}$$

$$\text{Tr}(\text{Shv}(\tilde{X}_s)^{\text{ren}}, [\tilde{c}_s]) \xrightarrow{\text{LT}^{\text{true}}_{\tilde{c}_s}} \Gamma(\text{Fix}(\tilde{c}_s), \omega_{\text{Fix}(\tilde{c}_s)}).$$
(b) In the situation of Section 7.2, the following diagram commutes up to a canonical homotopy:

\[
\begin{array}{ccc}
\text{Tr}(\text{Shv}(X^{\text{ren}}, [c])) & \xrightarrow{\text{LT}^\text{true}_c} & \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \\
\downarrow & & \downarrow \pi^*_\Delta \\
\text{Tr}(\text{Shv}(X'^{\text{ren}}, [c'])) & \xrightarrow{\text{LT}^\text{true}_{c'}} & \Gamma(\text{Fix}(c'), \omega_{\text{Fix}(c')})
\end{array}
\]

We are now going to deduce Theorem 5.4 from Theorem 7.3.

7.4. Deformation to the normal cone. We set \( R := k[t](t) \) and \( D := \text{Spec}(R) \).

(a) Let \( Z \subseteq X \) be a closed substack. To a pair \((X, Z)\) one associates a morphism

\[
\phi : \tilde{X}_Z \to X_D := X \times_k D
\]

such that \( \phi_\eta \) is an isomorphism, and \( \phi_\Delta \) is the projection \( N_Z(X) \to Z \subseteq X \), where \( N_Z(X) \) is the (classical) normal cone of \( X \) to \( Z \). Namely, in the case of schemes this is the usual construction of deformation to the normal cone (see, for example, [Va2, Section 1.4]), and the extension to Artin stacks is immediate, because the assignment \((X, Z) \mapsto \tilde{X}_Z\) commutes with smooth (even flat) pullbacks.

(b) Let \( c : C \to X \times X \) be a correspondence such that \( Z \subseteq X \) is \( c \)-invariant. By the functoriality of the construction of part (a), we obtain a correspondence

\[
\tilde{c}_Z : \tilde{C}_{c^{-1}(Z)} \to \tilde{X}_Z \times \tilde{X}_Z
\]

over \( D \), whose generic fiber is the base change

\[
c_\eta : C_\eta \to X_\eta \times X_\eta
\]

of \( c \), and the special fiber is the induced correspondence

\[
N_Z(c) : N_{c^{-1}(Z)}(C) \to N_Z(X) \times N_Z(X)
\]

between the normal cones.

(c) We now use the following key observation (see [Va2 Remark 2.1.2]): the correspondence \( c \) is contracting near \( Z \) if and only if the set-theoretic image of the morphism

\[
N_Z(c_\eta) : N_{c^{-1}(Z)}(C) \to N_Z(X)
\]

lies in the zero section \( Z \subseteq N_Z(X) \).

7.5. Specialization to the normal cone. Let us be in the situation of Section 7.4.

(a) Denote by \( \text{sp}_Z : \text{Shv}(X)^{\text{ren}} \to \text{Shv}(N_Z(X))^{\text{ren}} \) the functor \( \mathcal{A} \mapsto \Psi_{\tilde{X}_Z}(A_\eta) \);

(b) Let \( i \) be the closed embedding \( Z \hookrightarrow X \), and let \( N_Z(i) \) denote the embedding \( Z \hookrightarrow N_Z(X) \) (whose image is the zero section of \( N_Z(X) \)). We have a natural transformation of functors

\[
N_Z(i)^* \circ \text{sp}_Z \to i^* : \text{Shv}(X)^{\text{ren}} \to \text{Shv}(Z)^{\text{ren}}
\]

and a theorem of Verdier (see [Ve] or [Va2 Proposition 1.4.2]) asserts that this transformation is an isomorphism.

(c) Combining the constructions of Sections 7.4 and 7.3 we obtain a map

\[
\text{Tr}(\text{sp}_Z(c)) : \text{Tr}(\text{Shv}(X)^{\text{ren}}, [c]) \to \text{Tr}(\text{Shv}(X_\eta)^{\text{ren}}, [c_\eta]) \to \text{Tr}(\text{Shv}(N_Z(X))^{\text{ren}}, [N_Z(c)])
\]
and a map
\[ \text{sp}_{\text{Fix}(c)} : \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \to \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \to \Gamma(\text{Fix}(N_Z(c)), \omega_{\text{Fix}(N_Z(c))}). \]

### 7.6. Proof of Theorem 5.4 assuming Theorem 7.3

The argument will essentially follow that of [Va2, Theorem 2.1.3].

Using the observation in Section 5.3(c) and replacing \( C \) by an open substack, we can assume that \( \text{Fix}(c)_{\text{red}} = \text{Fix}(c|Z)_{\text{red}}. \) In this case morphism
\[ \iota^*_\Delta : \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \to \Gamma(\text{Fix}(c|Z), \omega_{\text{Fix}(c|Z)}) \]
is an isomorphism, with inverse \( (i_\Delta)! \) (see Section 5.1). Therefore it suffices to show that
\[ LT_{\text{true}} \simeq (i_\Delta)! \circ LT_{c|Z} \circ \text{Tr}([i_c]^*). \]

Hence, by Theorem 5.2(a), it suffices to show that
\[ LT_{\text{true}} \simeq LT_{c|Z} \circ \text{Tr}([i_c]) \circ \text{Tr}([i_c]^*). \]

Furthermore, by Lemma 3.9 it suffices to show that
\[ LT_{c|Z} \circ \text{Tr}([j_{c|Z}]) \simeq 0. \]

It is shown in the proof of [Va2, Theorem 2.1.3] that the assumptions that \( c \) is contracting near \( Z \) and that \( \text{Fix}(c|Z)_{\text{red}} = \text{Fix}(c)_{\text{red}} \) imply that \( \text{Fix}(c|Z)_{\text{red}} \) is the constant family \( \text{Fix}(c)_{\Delta, \text{red}}. \)

Therefore, the specialization map
\[ \text{sp}_{\text{Fix}(c)} : \Gamma(\text{Fix}(c), \omega_{\text{Fix}(c)}) \to \Gamma(\text{Fix}(N_Z(c)), \omega_{\text{Fix}(N_Z(c))}) \]
is an isomorphism. Hence, it suffices to show that
\[ (\text{sp}_{\text{Fix}(c)} \circ \text{LT}_{\text{true}}) \circ \text{Tr}([j_{c|Z}]) \simeq 0. \]

Applying Theorem 7.3(b) to the morphism \( \eta \to \text{Spec}(k) \) and Theorem 7.3(a) to the correspondence \( \tilde{c}_Z, \) we obtain an isomorphism
\[ \text{sp}_{\text{Fix}(c)} \circ \text{LT}_{\text{true}} \simeq \text{LT}_{N_Z(c)} \circ \text{Tr}(\text{sp}_Z(c)). \]

Therefore it suffices to show that the composition
\[ \text{Tr}(\text{sp}_Z(c)) \circ \text{Tr}([j_{c|Z}]) \simeq \text{Tr}(\text{sp}_Z(c) \circ [j_{c|Z}]) : \text{Tr}(\text{Shv}(U)^{\text{ren}}, [c|Z]) \to \text{Tr}(\text{Shv}(N_Z(X))^{\text{ren}}, [N_Z(c)]) \]
is homotopic to zero.

By Lemma 7.7 below, it suffices to show that the composition
\[ (\text{sp}_Z \circ j|^R \circ [N_Z(c)]) : \text{Shv}(N_Z(X))^{\text{ren}} \to \text{Shv}(U)^{\text{ren}} \]
is homotopic to zero.

Since \( c \) is contracting near \( Z, \) the morphism \( N_Z(i)_*: N_{c|Z}(c|Z)_{\text{red}} \to N_Z(X) \) factors through \( N_Z(i) : Z \to N_Z(X) \) (see Section 7.3(c)). Since \( [N_Z(c)] = N_Z(c|Z)_* \circ N_Z(c|Z)^1, \) it suffices to show that
\[ (\text{sp}_Z \circ j|^R \circ N_Z(i)_* \simeq 0, \]

Passing to left adjoints, it suffices to show that the composite map \( N_Z(i)^* \circ \text{sp}_Z \circ j^{|R} \)
\[ \text{Shv}(U)^{\text{ren}} \to \text{Shv}(X)^{\text{ren}} \to \text{Shv}(N_Z(X))^{\text{ren}} \to \text{Shv}(Z)^{\text{ren}} \]
is homotopic to zero. However, by the theorem of Verdier (see Section 7.5(b)), this composition is homotopic to \( i^* \circ j \simeq 0. \) □
Lemma 7.7. Assume that we are given a lax commutative square

\[
\begin{array}{ccc}
C & \xrightarrow{F} & C \\
\downarrow{t} & & \downarrow{t} \\
D & \xrightarrow{G} & D
\end{array}
\]

such that 1-morphism \( t \) has a right adjoint \( t^R \) and \( t^R \circ G \simeq 0 \). Then the induced map between traces

\[ \text{Tr}(\alpha) : \text{Tr}(C, F) \to \text{Tr}(D, G) \]

is homotopic to zero.

Proof. By definition, \( \text{Tr}(\alpha) \) is the composition

\[ \text{Tr}(C, F) \to \text{Tr}(C, t^R \circ t \circ F) \xrightarrow{\alpha} \text{Tr}(C, t^R \circ G \circ t) \simeq \text{Tr}(D, G \circ t \circ t^R) \to \text{Tr}(D, G). \]

Therefore it is homotopic to zero, because \( t^R \circ G \simeq 0 \), thus \( \text{Tr}(C, t^R \circ G \circ t) \simeq 0 \). \( \square \)

8. Proof of Propositions 3.5 and 3.7

8.1. Proof of Proposition 3.5. Since trace maps are compatible with compositions, it suffices to show that the horizontal compositions of the lax-commutative diagrams

\[
\begin{array}{ccc}
\text{Shv}(X) & \xrightarrow{\text{ren}_X} & \text{Shv}(X) \text{ren} \\
\downarrow{[c]} & & \downarrow{[c]} \\
\text{Shv}(X) & \xrightarrow{\text{ren}_X} & \text{Shv}(X) \text{ren}
\end{array}
\quad
\begin{array}{ccc}
\text{Shv}(X) & \xrightarrow{\text{ren}_X} & \text{Shv}(X) \text{ren} \\
\downarrow{[f]} & & \downarrow{[d]} \\
\text{Shv}(X) & \xrightarrow{\text{ren}_X} & \text{Shv}(X) \text{ren}
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{Shv}(X) & \xrightarrow{f_1} & \text{Shv}(Y) \\
\downarrow{[c]} & & \downarrow{[d]} \\
\text{Shv}(X) & \xrightarrow{\text{ren}_X} & \text{Shv}(Y) \text{ren}
\end{array}
\quad
\begin{array}{ccc}
\text{Shv}(X) & \xrightarrow{f_1} & \text{Shv}(Y) \\
\downarrow{[f]} & & \downarrow{[d]} \\
\text{Shv}(X) & \xrightarrow{\text{ren}_X} & \text{Shv}(Y) \text{ren}
\end{array}
\]

are canonically isomorphic. In other words, we have to show that there is a canonical isomorphism

\[ f_1 \circ \text{ren}_X \simeq \text{ren}_Y \circ f_1 \]

of functors \( \text{Shv}(X) \to \text{Shv}(Y) \text{ren} \) making the following diagram homotopy commutative:

\[ f_1 \circ \text{ren}_X \circ [c] \xrightarrow{\alpha} f_1 \circ [c] \circ \text{ren}_X \xrightarrow{[f]} d \circ f_1 \circ \text{ren}_X \]

\[ \text{ren}_Y \circ f_1 \circ [c] \xrightarrow{[f]} \text{ren}_Y \circ [d] \circ f_1 \xrightarrow{\alpha} d \circ \text{ren}_Y \circ f_1, \]

\[ \text{8.1} \sim \text{8.1} \]
To get an isomorphism $\overset{(\text{8.1})}{\longrightarrow}$, we notice that for every $\mathcal{A} \in \operatorname{Shv}(X)^c$, both $(f_! \circ \operatorname{ren}_X)(\mathcal{A})$ and $(\operatorname{ren}_Y \circ f_!)(\mathcal{A})$ are simply $f_! (\mathcal{A}) \in \operatorname{Shv}(Y)^c \subseteq \operatorname{Shv}(Y)^\operatorname{constr} \subseteq \operatorname{Shv}(Y)^\operatorname{ren}$. Alternatively, it can be obtained from the isomorphism $\operatorname{unren}_X \circ f^! \sim f^! \circ \operatorname{unren}_Y$ from Lemma 1.9(a) by passing to left adjoints.

Next, unwinding the definitions, diagram 8.2 decomposes as
$$
\begin{array}{c}
\xymatrix{ f_! \circ \operatorname{ren}_X \circ (c_i)_\bullet \circ c_i' \ar[d]_{\sim}^{8.1} & f_! \circ (c_i)_* \circ \operatorname{ren}_C \circ c_i' \ar[d]_{\sim}^{8.4} & f_! \circ (c_i)_\bullet \circ c_i' \circ \operatorname{ren}_X \ar[d]_{8.4} \\
\operatorname{ren}_Y \circ f_! \circ (c_i)_\bullet \circ c_i' & (d_l)_* \circ g_l \circ \operatorname{ren}_C \circ c_i' \ar[d]_{\sim}^{8.4} & (d_l)_* \circ g_l \circ c_i' \circ \operatorname{ren}_X \\
\operatorname{ren}_Y \circ (d_l)_\bullet \circ g_l \circ c_i' \ar[d]_{\sim}^{8.4} & (d_l)_* \circ \operatorname{ren}_D \circ g_l \circ c_i' \ar[d]_{\sim}^{8.4} & (d_l)_* \circ \operatorname{ren}_Y \circ f_! \\
\operatorname{base change} & \operatorname{base change} & \operatorname{base change} \\
\operatorname{ren}_Y \circ (d_l)_\bullet \circ d'_l \circ f_l \ar[d]_{\sim}^{8.3} & (d_l)_* \circ \operatorname{ren}_D \circ d'_l \circ f_l \ar[d]_{\sim}^{8.3} & (d_l)_* \circ \operatorname{ren}_Y \circ f_!,
\end{array}
$$

where by 8.4, we denote the (easier) version of morphism 8.4, where $\bullet$ is replaced by $\circ$.

It remains to show that all inner diagrams of diagram 8.3 are homotopy commutative. This is clear for the top right and the bottom left inner diagrams. For the remaining ones, it remains to show that the following two diagrams are homotopy commutative:

$$
\begin{array}{c}
\xymatrix{ \operatorname{ren}_Y \circ f_! \circ (c_i)_\bullet \circ \operatorname{ren}_C \ar[d]_{\sim}^{8.1} & \operatorname{ren}_Y \circ (c_i)_\bullet \circ \operatorname{ren}_C \\
\operatorname{ren}_Y \circ (d_l)_\bullet \circ g_l \circ \operatorname{ren}_C \ar[d]_{\sim}^{8.1} & \operatorname{ren}_Y \circ (d_l)_\bullet \circ \operatorname{ren}_C \\
\operatorname{base change} & \operatorname{base change} \\
\operatorname{ren}_D \circ g_l \circ c_i' \ar[d]_{\sim}^{8.1} & \operatorname{ren}_D \circ c_i' \circ \operatorname{ren}_X \\
\operatorname{base change} & \operatorname{base change} \\
\operatorname{ren}_D \circ d'_l \circ f_l \ar[d]_{\sim}^{8.1} & \operatorname{ren}_Y \circ f_!
\end{array}
$$

By adjunction, the homotopy commutativity of diagram 8.5 follows from the fact that isomorphisms from Lemma 1.9(a) are compatible with compositions.

Next, decomposing the left inner square of diagram 8.3 as
$$
\begin{array}{c}
\xymatrix{ X \ar[d]_f \ar[r]^{\sim}_{\overset{\text{8.6}}{d_i}} & C_i := X \times_Y D \ar[r]^{p} & C \\
Y \ar[d]_{g} & \overset{\sim}{f} \ar[d]_{\overset{\text{8.6}}{f}} & \overset{\sim}{g} \\
D & D & D
\end{array}
$$
and unwinding the definition, diagram (8.4) decomposes as

\[
\begin{align*}
\text{ren}_{Y} \circ f_{i} \circ (c_{i})_{\Delta} & \xrightarrow{	ext{(8.4)}} f_{i} \circ \text{ren}_{X} \circ (c_{i})_{\Delta} \xrightarrow{1.3(d)} f_{i} \circ (c_{i})_{*} \circ \text{ren}_{C} \\
\sim & \xrightarrow{1.2(c)} f_{i} \circ (c_{i})_{*} \circ \text{ren}_{C} \\
\text{ren}_{Y} \circ f_{i} \circ (\widetilde{d}_{i})_{\Delta} \circ p_{\Delta} & \xrightarrow{\text{(8.5)}} f_{i} \circ \text{ren}_{X} \circ (\widetilde{d}_{i})_{\Delta} \circ p_{\Delta} \xrightarrow{1.3(d) \cdot 1.9(d) \cdot 1.3(d)} f_{i} \circ (\widetilde{d}_{i})_{*} \circ p_{*} \circ \text{ren}_{C} \\
\sim & \xrightarrow{1.3(d) \cdot 1.5} (d_{i})_{*} \circ \text{ren}_{D} \circ \widetilde{f}_{i} \circ p_{\Delta} \xrightarrow{1.3(d) \cdot 1.5} (d_{i})_{*} \circ \text{ren}_{D} \circ \text{ren}_{C} \\
\text{ren}_{Y} \circ (d_{i})_{\Delta} \circ g_{i} & \xrightarrow{\text{(8.5)}} (d_{i})_{*} \circ \text{ren}_{D} \circ g_{i} \xrightarrow{\text{(8.5)}} (d_{i})_{*} \circ g_{i} \circ \text{ren}_{C}.
\end{align*}
\]

(8.7)

Thus, it remains to show that all inner squares of diagram (8.7) are homotopy commutative. This is clear for the top left and the bottom left inner squares.

The assertion for the top right inner square follows from the fact that isomorphisms of Lemma 1.3(c) are compatible with compositions, while for the assertion for the bottom right square we also observe that the diagram

\[
\begin{align*}
\text{ren}_{C_{i}} \circ p_{i} & \xrightarrow{\text{(8.1)}} p_{i} \circ \text{ren}_{C} \\
\sim & \xrightarrow{1.3(d) \cdot 1.5} (d_{i})_{*} \circ \text{ren}_{D} \circ g_{i} \xrightarrow{1.3(d) \cdot 1.5} (d_{i})_{*} \circ \text{ren}_{C}
\end{align*}
\]

is homotopy commutative.

Finally, to see the commutativity of the middle inner square of diagram (8.7), we have to show the commutativity of the diagram

\[
\begin{align*}
\text{ren}_{X} \circ (\widetilde{d}_{i})_{\Delta} \circ \widetilde{f} & \xrightarrow{1.3(d)} (\widetilde{d}_{i})_{*} \circ \text{ren}_{C_{i}} \circ \widetilde{f} \xrightarrow{1.3(d)} (\widetilde{d}_{i})_{*} \circ \widetilde{f} \circ \text{ren}_{D} \\
\sim & \xrightarrow{1.3(d) \cdot 1.5} f_{i} \circ \text{ren}_{Y} \circ (d_{i})_{\Delta} \xrightarrow{1.3(d)} f_{i} \circ (d_{i})_{*} \circ \text{ren}_{D}
\end{align*}
\]

(8.8)

which follows from Corollary 1.10(b). □

8.2. Proof of Proposition 3.7 Arguing as in the proof of Proposition 3.3 but interchanging $X$ with $Y$, $C$ with $D$, $c$ with $d$, $f_{i}$ with $f^{*}$, $g_{i}$ with $g^{*}$, $[f]_{i}$ with $[f]^{*}$ and isomorphism (8.1) with (1.3) in all places, we end up showing the homotopy commutativity of diagrams
9.1. Proof of Theorem 5.8(a).

Recall that since \( f : X \to Y \) is proper and safe, the functor \( f_! : \text{Shv}(X) \to \text{Shv}(Y) \) is self-dual. Namely, under identifications

\[
\text{Shv}(X) \simeq \text{Shv}(X)^\vee \quad \text{and} \quad \text{Shv}(Y) \simeq \text{Shv}(Y)^\vee
\]

from Section 4.11(a), the dual functor \((f^!R)^\vee : \text{Shv}(X)^\vee \to \text{Shv}(Y)^\vee\) is naturally identified with \( f_! : \text{Shv}(X) \to \text{Shv}(Y) \). Indeed, this is equivalent to the assertion \((D_Y \circ f_! \circ D_X)(A) \simeq f_!(A)\) for all \( A \in \text{Shv}(X)^c\), and hence follows Corollary 1.10.

(b) Using part (a), one checks that the map

\[
\text{Tr}([f_!]) : \text{Tr}(\text{Shv}(X), [c]) \to \text{Tr}(\text{Shv}(Y), [d])
\]
is induced by the lax commutative square

\[
\begin{array}{cccc}
\text{Vect} & \overset{u_X}{\longrightarrow} & \text{Shv}(X) \otimes \text{Shv}(X) & \overset{\varepsilon} \longrightarrow & \text{Vect} \\
\text{id} & \overset{\alpha_l}{\longrightarrow} & f \circ f_l & \overset{\alpha_m}{\longrightarrow} & f \circ f_l \\
\text{Vect} & \overset{u_Y}{\longrightarrow} & \text{Shv}(Y) \otimes \text{Shv}(Y) & \overset{\varepsilon} \longrightarrow & \text{Vect},
\end{array}
\]

(9.1)

where

- $\alpha_l$ corresponds to the morphism
  \[(f \circ f_l)(u_X) \to u_Y \simeq \mathbb{R}((\Delta_Y)_*(\omega_Y)),\]

obtained by adjunction from the composition

\[\mathbb{R}((f \circ f_l)(u_X)) \simeq (f \times f_l)(\mathbb{R}(u_X)) \to (f \times f_l)((\Delta_X)_*(\omega_X)) \simeq (\Delta_Y)_*(f_l(\omega_X)) \to (\Delta_Y)_*(\omega_Y),\]

induced by natural maps $\mathbb{R}(u_X) \to (\Delta_X)_*(\omega_X)$ and $f_l(\omega_X) \to \omega_Y$;

- $\alpha_m$ corresponds to the morphism $[f] : f_l \circ [c] \to [d] \circ f_l$ from Section 3.4(a);

- $\alpha_r$ is the canonical morphism

\[\Gamma_{\bullet}(X, - \otimes -) \simeq \Gamma_{\bullet}(Y, f_l(- \otimes -)) \to \Gamma_{\bullet}(Y, f_l(-) \otimes f_l(-)),\]

induced by the canonical morphism $f_l(- \otimes -) \to f_l(-) \otimes f_l(-)$.

9.2. Consider the lax commutative square

\[
\begin{array}{cccc}
\text{Vect} & \overset{(\Delta_X)_* \omega_X}{\longrightarrow} & \text{Shv}(X \times X) & \overset{\varepsilon \circ \Delta'} \longrightarrow & \text{Vect} \\
\text{id} & \overset{\alpha_l}{\longrightarrow} & (f \times f_l)_! & \overset{\alpha_m}{\longrightarrow} & (f \times f_l)_! \\
\text{Vect} & \overset{(\Delta_Y)_* \omega_Y}{\longrightarrow} & \text{Shv}(Y \times Y) & \overset{\varepsilon \circ \Delta'} \longrightarrow & \text{Vect},
\end{array}
\]

(9.2)

where

- $\alpha_l$ corresponds to the morphism
  \[(f \times f)_!((\Delta_X)_* \omega_X) \simeq (\Delta_Y)_*(f_l(\omega_X)) \to (\Delta_Y)_*(\omega_Y)\]

(similar to the corresponding map in diagram (9.1));

- $\alpha_m$ is the morphism $(f \times f)_! \circ [c \times \text{Id}] \to [d \times \text{Id}] \circ (f \times f)_!$, given by the morphism of correspondences $(f \times f, g \times f) : c \times \text{Id}_X \to d \times \text{Id}_Y$ as in Section 3.4(a);

- $\alpha_r$ is the canonical morphism

\[\Gamma_{\bullet}(X, \Delta'_X (-)) \simeq \Gamma_{\bullet}(Y, (f_l \circ \Delta'_X)(-)) \to \Gamma_{\bullet}(Y, (\Delta'_Y \circ (f \times f)_!)(-)),\]

induced by the base change morphism $f_l \circ \Delta'_X \to \Delta'_Y \circ (f \times f)_!$. 
9.3. We can decompose the diagram of Theorem\ref{thm:local-terms-category-trace} \(9.3\) as
\[
\begin{array}{ccc}
\text{Tr}(\text{Shv}(X), [c]) & \xrightarrow{[\mathbf{c}]_{\mathcal{C}}} & \Gamma_{\square}(X, (\Delta^!_X \circ [c \times \text{Id}] \circ (\Delta^!_X)_{\ast})(\omega_X)) \\
\downarrow \text{Tr}([f]) & \downarrow [\mathbf{d}]_{\mathcal{C}} & \downarrow \text{base change} \\
\text{Tr}(\text{Shv}(Y), [d]) & \xrightarrow{[\mathbf{d}]_{\mathcal{C}}} & \Gamma_{\square}(Y, (\Delta^!_Y \circ [d \times \text{Id}] \circ (\Delta^!_Y)_{\ast})(\omega_Y)) \\
\end{array}
\]
where the top and the bottom arrows are the compositions from Section\ref{sect:local-terms-category-trace} \(9.2\) for \(c\) and \(d\), respectively, while the middle vertical arrow is induced by the lax commutative square \(9.2\). Therefore, it suffices to show that both inner squares of diagram \(9.3\) are canonically homotopy commutative.

9.4. In order to verify the commutativity of the left inner square of diagram \(9.3\), it suffices to show that the two maps
\[
\text{Tr}(\text{Shv}(X), [c]) \Rightarrow \Gamma_{\square}(Y, (\Delta^!_Y \circ [d \times \text{Id}] \circ (\Delta^!_Y)_{\ast})(\omega_Y))
\]
that arise from the following two lax commutative diagrams are canonically homotopic.
For this it suffices to establish that the vertical composition of the left (resp. middle, resp. right) inner square of diagram (9.4) is canonically homotopic to the vertical composition of the corresponding inner square of diagram (9.5). But all there three assertions are straightforward.

9.5. It remains to show the commutativity of the right inner square of diagram (9.3). This follows from the compatibility of base change morphisms with compositions (see [V2, proof of Proposition 1.2.5]).

9.6. Proof of Theorem 5.2(a). The proof is almost identical to that of Theorem 5.8(a), except we have to replace $\text{Shv}(\cdot)$, $\text{Shv}(\cdot)^c$, $\omega_-$ and $\Gamma_\Delta$ by $\text{Shv}(\cdot)^{\text{ren}}$, $\text{Shv}(\cdot)^{\text{constr}}$, $\omega_-^{\text{ren}}$ and $\Gamma$, respectively, in all places:

(i) Arguing as in Section 9.1 we see that the functor $f! : \text{Shv}(X)^{\text{ren}} \to \text{Shv}(Y)^{\text{ren}}$ is self-dual. Therefore the trace map

$$\text{Tr}(f!) : \text{Tr}(\text{Shv}(X)^{\text{ren}}, [\epsilon]) \to \text{Tr}(\text{Shv}(Y)^{\text{ren}}, [\epsilon])$$

is induced by a lax commutative square, obtained from diagram (9.1) by the replacement mentioned above.

(ii) Next, the diagram of Theorem 5.2(a) decomposes in a similar manner as diagram (9.3), where the top and the bottom arrows are the compositions from Section 4.8(b) for $c$ and $d$, respectively, while the middle vertical arrow is induced by a lax commutative square, obtained from diagram (9.2) by the replacement mentioned above. It remains to show that both inner squares are homotopy commutative.

(iii) Again, to show commutativity of the left inner square, it suffices to show that vertical compositions of the corresponding lax commutative squares are homotopic, which is a routine verification, while the commutativity of the right inner square follows from the fact that base change morphisms, are compatible with compositions.

9.7. Proof of Theorem 7.3 We will only discuss the proof of part (a), because the proof of part (b) is similar but easier.

(i) Since functor $\Psi X$ commutes with the Verdier duality on constructible objects, it is self-dual. Therefore the trace map

$$\text{Tr}(\Psi c) : \text{Tr}(\text{Shv}(\tilde{X}_\eta)^{\text{ren}}, [\tilde{\eta}]) \to \text{Tr}(\text{Shv}(\tilde{X}_s)^{\text{ren}}, [\tilde{s}])$$

is induced by a lax commutative square, similar to diagram (9.1), in which, in addition to replacements of Section 9.6 $X, c, Y, d$ and $f_! \otimes f_!$ are replaced by $\tilde{X}_\eta, \tilde{\eta}, \tilde{X}_s, \tilde{s}$ and $\Psi \tilde{X} \otimes \Psi \tilde{X}$, respectively (and 2-morphisms are modified appropriately).

(ii) Next, the diagram of Theorem 7.3(a) decomposes in a similar manner as diagram (9.3), where the middle vertical arrow is induced by a lax commutative square, similar to diagram (9.2), in which $(f \times f)_!$, are replaced by $\Psi \tilde{X} \times \tilde{X}$. It remains to show that both inner squares are homotopy commutative.

(iii) Again, to show commutativity of the left inner square, it suffices to show that vertical compositions of the corresponding lax commutative squares are homotopic. In this case, it is a routine verification, which uses the fact that nearby cycles commute with exterior products. Finally, the commutativity of the right inner square follows from the fact that base change morphisms, corresponding to Cartesian squares, commute with compositions (see [V2, proof of Proposition 1.3.5]).
9.8. Proof of Proposition 4.12

(i) Since \( \mathbb{D}_X \circ \text{ren}_X \circ \mathbb{D}_X (A) \cong \text{ren}_X (A) \) for every \( A \in \text{Shv}(X)^c \), we conclude that the functor \( \text{ren}_X : \text{Shv}(X) \to \text{Shv}(Y)^\text{ren} \) is self-dual. Therefore the trace map
\[
\text{Tr}(\text{ren}_X, [\cdot]) : \text{Tr}(\text{Shv}(X), [\cdot]) \to \text{Tr}(\text{Shv}(X)^\text{ren}, [\cdot])
\]
is induced by a lax commutative square, similar to diagram (9.1), in which \( \text{Shv}(Y), f, \omega_Y, [d] \) and \( \text{ev}_Y \) are replaced by \( \text{Shv}(X)^\text{ren}, \text{ren}_X, u_X^\text{ren}, [c] \) and \( \text{ev}_X \), respectively.

(ii) Next, diagram (4.1) of Proposition 4.12 decomposes in a similar manner as diagram (9.2), in which \( \text{Shv}(Y \times Y), (f \times f)_!, (\Delta_Y)_! (\omega_Y) \) and \( \Gamma_\bullet \circ \Delta_Y^! \) are replaced by \( \text{Shv}(X \times X)^\text{ren}, \text{ren}_{X \times X}, (\Delta_X)_! (\omega_X^\text{ren}) \) and \( \Gamma_\bullet \circ \Delta_X^! \), respectively. It remains to show that both inner squares are homotopy commutative.

(iii) Again, to show commutativity of the left inner square, it suffices to show that vertical compositions of the corresponding lax commutative squares are homotopic, which is a routine verification, which uses the fact renormalization functors commute with exterior products. Finally, the commutativity of the right inner square follows from Corollary 1.10(b) asserting that renormalization functors commute with base change morphisms.

10. Completion of proofs, II

In this section we will prove Theorems 5.2(b) and 5.8(b). Moreover, we will restrict ourselves with the proof of Theorem 5.8(b), because the proof of Theorem 5.2(b) can be obtained from it using the same modification as in Section 9.6. Although the overall strategy is similar to that of Theorem 5.8(a), some ingredients are new.

10.1. The \((\ast, !)-\)pullback. Consider the functor
\[
f^* \boxtimes f^! : \text{Shv}(Y \times Y) \to \text{Shv}(X \times X),
\]
defined as a composition of \( (f \times \text{Id})^* \) and \( (\text{Id} \times f)^! \) in either order. Namely, there is a canonical base change morphism of functors
\[
(f \times \text{Id}_X)^* \circ (\text{Id}_Y \times f)^! \to (\text{Id}_X \times f)^! \circ (f \times \text{Id}_Y)^*,
\]
which is an isomorphism because \( f \) is smooth.

10.2. Consider a lax commutative square
\[
\begin{array}{ccc}
\text{Vect} & \to & \text{Shv}(Y \times Y) \\
\downarrow \text{id} & & \downarrow [d \times \text{Id}] \\
\text{Vect} & \to & \text{Shv}(X \times X)
\end{array}
\]
\[
\begin{array}{ccc}
\text{Shv}(Y \times Y) & \to & \text{Shv}(Y \times Y) \\
\downarrow \alpha_m & & \downarrow \alpha_r \\
\text{Shv}(X \times X) & \to & \text{Shv}(X \times X)
\end{array}
\]
\[
\begin{array}{ccc}
\text{Vect} & \to & \text{Vect} \\
\downarrow \text{id} & & \downarrow \Gamma_\bullet \circ \Delta_Y^! \\
\text{Vect} & \to & \text{Vect},
\end{array}
\]
where
- \( \alpha_l \) corresponds to the morphism
\[
(\ast) \quad (f \times \text{Id}_X)^* (\text{Id}_Y \times f)^! ((\Delta_Y)_! (\omega_Y)) \cong (f \times \text{Id}_X)^* (\text{Id}_Y \times f)^! (\Delta_X)_! (\omega_X^\text{ren}) \xrightarrow{\text{counit}} (\Delta_X)_! (\omega_X^\text{ren}),
\]
obtained from the base change isomorphism \((\text{Id}_Y \times f)^! \circ (\Delta_Y)_* \simeq (f \times \text{Id}_X)_* \circ (\Delta_X)_* \circ f^!\), corresponding to the Cartesian diagram

\[
\begin{array}{ccc}
X & \xrightarrow{(f,\text{Id}_X)} & Y \times X \\
\downarrow f & & \downarrow \text{Id}_Y \times f \\
Y & \xrightarrow{\Delta_Y} & Y \times Y;
\end{array}
\]

\(\bullet \ \alpha_m\) is the morphism \((f^* \boxtimes f') \circ [d \times \text{Id}] \to [c \times \text{Id}] \circ (f^* \times f')\), induced by the morphism \((f \times \text{Id})^* \circ [d \times \text{Id}] \to [c \times \text{Id}] \circ (f \times \text{Id})^*\) (see Section 3.6(a));

\(\bullet \ \alpha_r\) is the canonical morphism

\[
\Gamma_{\bullet}(Y, \Delta_Y(-)) \xrightarrow{\mathcal{L}_{\bullet}} \Gamma_{\bullet}(X, (f^* \circ \Delta_Y)(-)) \to \Gamma_{\bullet}(X, (\Delta_X \circ (f^* \boxtimes f'))(-)),
\]
induced by the base change morphism

\[
f^* \circ \Delta_Y \to (\Delta_X \circ (\Delta_Y \circ f)) \simeq (\Delta_X \circ (f^* \boxtimes f')),
\]
corresponding to the Cartesian diagram

\[
\begin{array}{ccc}
X & \xrightarrow{(\text{Id}_X,f)} & X \times Y \\
\downarrow f & & \downarrow f \times \text{Id}_Y \\
Y & \xrightarrow{\Delta_Y} & Y \times Y.
\end{array}
\]

10.3. We have an identification \((\mathcal{D}_X \circ f^* \circ \mathcal{D}_Y)(\mathcal{A}) \simeq f^!(\mathcal{A})\) for every \(\mathcal{A} \in \text{Shv}(Y)^c\). Thus, unwinding the definitions, the trace map

\[
\text{Tr}([f]^*): \text{Tr}(\text{Shv}(Y), [d]) \to \text{Tr}(\text{Shv}(X), [c])
\]

is induced by a lax commutative square

\[
\begin{array}{ccc}
\text{Vect} & \xrightarrow{u_Y} & \text{Shv}(Y) \otimes \text{Shv}(Y) \\
\downarrow \text{id} & & \downarrow [d] \otimes \text{Id} \\
\text{Vect} & \xrightarrow{u_X} & \text{Shv}(X) \otimes \text{Shv}(X)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Vect} & \xrightarrow{\text{Shv}(Y) \otimes \text{Shv}(Y)} & \text{Shv}(Y) \otimes \text{Shv}(Y) \\
\downarrow \mathcal{L}_{\bullet} & & \downarrow \text{ev}_Y \\
\text{Vect} & \xrightarrow{\text{Shv}(X) \otimes \text{Shv}(X)} & \text{Shv}(X) \otimes \text{Shv}(X)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Vect} & \xrightarrow{\text{Shv}(X) \otimes \text{Shv}(X)} & \text{Shv}(X) \otimes \text{Shv}(X) \\
\downarrow \text{id} & & \downarrow \text{ev}_X \\
\text{Vect}
\end{array}
\]

where

\(\bullet \ \alpha_l\) corresponds to the morphism

\[
(f^* \circ f^!)(u_Y) \to u_X \simeq \otimes^R ((\Delta_X)_*(\omega_X)),
\]

obtained by adjunction from the composition

\[
\otimes((f^* \circ f^!)(u_Y)) \simeq (f^* \boxtimes f^!)(\otimes(u_Y)) \to (f^* \boxtimes f^!)(((\Delta_Y)_*(\omega_Y)) \xrightarrow{\text{10.2}} (\Delta_X)_*(\omega_X),
\]

induced by the natural map \(\otimes(u_Y) \to (\Delta_Y)_*(\omega_Y),\)

\(\bullet \ \alpha_m\) corresponds to the morphism \([f]^* : \mathcal{L}_{\bullet} \to \mathcal{L}_{\bullet} \circ [d] \circ f^*\) from Section 3.6(a);

\(\bullet \ \alpha_r\) is the composition
induced by the canonical morphism \( f^*(- \otimes -) \to f^*(- \otimes f^\dagger(-)). \)

10.4. (i) We can decompose the diagram of Theorem 5.8(b) as (10.4)

\[
\begin{array}{c}
\text{Tr(Shv(Y)}, [d]) \xrightarrow{(10.3)_{d}} \\
\text{Tr(Shv}(X), [c]) \xrightarrow{(10.3)_{c}} \\
\end{array}
\]

\[
\Gamma_{\Delta}(Y, (\Delta_Y^1 \circ [d \times Id] \circ (\Delta_Y)_*)(\omega_Y)) \xrightarrow{\text{base change}} \Gamma_{\Delta}(D, (\omega_{\text{Fix}(d)})) \xrightarrow{\text{base change}} \Gamma_{\Delta}(\text{Fix}(c), \omega_{\text{Fix}(c)}),
\]

where the top and the bottom arrows are the compositions from Section 4.11(b) for \( d \) and \( c \), respectively, while the middle vertical arrow is induced by the lax commutative square (10.1). Therefore, it suffices to show that both inner squares of diagram (10.3) are canonically homotopy commutative.

(ii) As in the proof of Theorem 5.8(a), all arrows in the left inner square of diagram (10.3) are horizontal compositions of the corresponding lax commutative squares. In particular, in order to show that the left inner square of diagram (10.3) is commutative, it suffices to show that vertical compositions of the corresponding lax commutative squares are homotopic. As in the proof of Theorem 5.8(a), it is a routine verification.

It remains to show the commutativity of the right inner square of diagram (10.4).

10.5. Set \( d^{op} := (d, d) : D \to Y \times Y \) and \( c^{op} := (c, c) : C \to X \times X \).

Note that the right inner square of diagram (10.3) decomposes as (10.5)

\[
\begin{array}{c}
\Gamma_{\Delta}(X, (\Delta_X^1 \circ [c \times Id] \circ (\Delta_X)_*)(\omega_X)) \xrightarrow{\text{BC}^3} \Gamma_{\Delta}(C, (c^{op})^1(\Delta_X)_*)(\omega_X)) \xrightarrow{\text{BC}^3} \Gamma_{\Delta}(\text{Fix}(c), \omega_{\text{Fix}(c)}),
\end{array}
\]

where

- morphism (1) is induced by the composition of morphisms defined in Section 10.2

\[
f^* \circ \Delta_Y^1 \circ [d \times Id] \to \Delta_X^1 \circ (f^* \otimes f^\dagger) \circ [d \times Id] \circ \Delta_Y^1 \circ [c \times Id] \circ (f^* \otimes f^\dagger);
\]

- morphism (2) is induced by the composition

\[
g^* \circ (d^{op})^1 \simeq g^* \circ (\text{Id}_D, d_1)^1 \circ (d_r \times \text{Id}_Y)^1 \xrightarrow{\text{base change}} (\text{Id}_X, f \circ c_1)^1 \circ (g \times \text{Id}_Y)^* \circ (d_r \times \text{Id}_Y)^1.
\]
induced by diagrams

\[
\begin{array}{ccc}
C & \xrightarrow{(\text{Id}_C, f \circ c_1)} & C \times Y \\
g \downarrow & & \downarrow g \times \text{Id}_Y \\
D & \xrightarrow{(\text{Id}_D, d_1)} & D \times Y
\end{array}
\]

\[
\begin{array}{ccc}
C \times Y & \xrightarrow{s \times \text{Id}_Y} & X \times Y \\
g \downarrow & & \downarrow f \times \text{Id}_Y \\
D \times Y & \xrightarrow{d_1 \times \text{Id}_Y} & Y \times Y
\end{array}
\]

• morphism \(BC_1\) is induced by the morphism

\[
\Delta_Y \circ [d \times \text{Id}] \rightarrow (d_1)_\bullet \circ (d^{\text{pp}})_1;
\]

obtained from the base change morphism \(\Delta_Y \circ (d_1 \times \text{Id})_\bullet \rightarrow (d_1)_\bullet \circ (\text{Id}_D, d_1)_1\) (the inverse of morphism \((1.2)\)), induced by the Cartesian diagram

\[
\begin{array}{ccc}
D & \xrightarrow{(\text{Id}_D, d_1)} & D \times Y \\
d_1 \downarrow & & d_1 \times \text{Id}_Y \\
Y & \xrightarrow{\Delta_Y} & Y \times Y
\end{array}
\]

Note that morphism \((1.2)\) is an isomorphism in our case, because morphism \(\Delta_Y\) (and hence its pullback \((\text{Id}_D, d_1))\) is representable, thus safe (see Section 4.10(d));

• morphism \(BC_2\) is induced by the composition of the base change morphisms

\[
f^* \circ \Delta_Y^1 \circ [d \times \text{Id}] \xrightarrow{(10.6)} f^* \circ (d_1)_\bullet \circ (d^{\text{pp}})_1 \xrightarrow{(1.11\ b)} (c_1)_\bullet \circ g^* \circ (d^{\text{pp}})_1;
\]

• morphism \(BC_3\) is induced by the morphism \(\Delta_X^1 \circ [c \times \text{Id}] \rightarrow (c_1)_\bullet \circ (c^{\text{pp}})_1\), defined similarly to \((10.6)\).

• morphism \(BC_4\) is induced by the base change morphism \((d^{\text{pp}})_1 \circ (\Delta_Y)_* \rightarrow (\Delta_d)_* \circ d^*_\Delta\), induced by the Cartesian diagram

\[
\begin{array}{ccc}
\text{Fix}(d) & \xrightarrow{\Delta_d} & D \\
d_\Delta \downarrow & & \downarrow d^{\text{pp}} \\
Y & \xrightarrow{\Delta_Y} & Y \times Y
\end{array}
\]

and we use the fact that morphism \(\Delta_Y\) (and hence its pullback \(\Delta_d\)) is representable, thus safe;

• morphism \(BC_6\) is induced by the base change morphism \((c^{\text{pp}})_1 \circ (\Delta_X)_* \rightarrow (\Delta_c)_* \circ c^*_\Delta\), defined similarly;

• morphism \(BC_5\) is induced by the composition of the base change morphisms

\[
g^* \circ (d^{\text{pp}})_1 \circ (\Delta_Y)_* \rightarrow g^* \circ (\Delta_d)_* \circ d^*_\Delta \rightarrow (\Delta_c)_* \circ g^*_\Delta \circ d^*_\Delta.
\]

It remains to show that all inner squares of diagram \((10.5)\) are homotopy commutative.

10.6. This clear for the bottom left square. Next, unwinding the definitions, the assertion for the top right inner square follows from the fact the functor of Section 1.2(c) is compatible with compositions, while the assertion for the top left square follows from this and Section 1.11(b)’.
Moreover, the assertion for the middle left inner square reduced to the homotopy commutativity of the diagram (10.7)

\[
\begin{array}{c}
\text{(10.7)} \\
f^* (d_l)_{\bullet} ((\text{Id}_D, d_l)^1) \{1.11\} \quad (c_l)_{\bullet} g^*(\text{Id}_D, d_l)^1 \{2.3\} \quad (c_l)_{\bullet} (\text{Id}_C, f \circ c_l)^1 (g \times \text{Id}_Y)^* \\
\end{array}
\]

\[
\begin{array}{c}
\{1.2\} \\
\{1.2\}
\end{array}
\]

\[
f^* \Delta^1_Y (d_l \times \text{Id}_Y)_{\bullet} \xrightarrow{\text{base change}} (\text{Id}_X, f)^1 (f \times \text{Id}_Y)^* (d_l \times \text{Id}_Y)_{\bullet} \{1.11\} \quad (\text{Id}_X, f)^1 (c_l \times \text{Id}_Y)_{\bullet} (g \times \text{Id}_Y)^*.
\]

Furthermore, it suffices to show the homotopy commutativity of the diagram, obtained from diagram (10.7) by replacing \((-)_\bullet\) by \((-)_\ast\) in all places and all morphisms by the base change morphisms. Finally, the homotopy commutativity of the corresponding diagram follows from the fact that base change morphisms are compatible with compositions.

Thus, it suffices to show that the bottom inner square of diagram (10.9) is homotopy commutative as well. In other words, it suffices to show the homotopy commutativity of the diagram (10.8)

\[
\begin{array}{c}
g^* \circ (d^{op})^1 \circ (\Delta_Y)_\ast \quad \text{base change} \quad g^* \circ (\Delta_d)_\ast \circ d^\dagger \quad \text{base change} \quad (\Delta_c)_\ast \circ g^\ast \circ d^\dagger \\
(2) \quad \downarrow \quad (\text{Id})_\ast \quad \downarrow \quad \text{Gys}_\Delta
\end{array}
\]

\[
\begin{array}{c}
\{10.2\} \\
\{10.2\}
\end{array}
\]

\[
\begin{array}{c}
(c^{op})^1 \circ f^! \circ (\Delta_Y)_\ast \quad \text{base change} \quad (c^{op})^1 \circ (\Delta_X)_\ast \circ f^! \circ (\text{Id})_\ast \circ c^\ast \circ f^!
\end{array}
\]

10.7. Recall that morphism \(f\) is a smooth, morphism \(g\) is quasi-smooth and \(\text{dim}_g = c_r (\text{dim}_f)\). By Section 1.11(b), using identifications \(f^! \simeq f^* (\text{dim}_f)\) and \(f^* \otimes f^! \simeq (f \times f)^* (\text{pr}_2^* (\text{dim}_f))\) diagram (10.9) can be rewritten as

\[
\begin{array}{c}
g^* \circ (d^{op})^1 \circ (\Delta_Y)_\ast \quad \text{base change} \quad g^* \circ (\Delta_d)_\ast \circ d^\dagger \quad \text{base change} \quad (\Delta_c)_\ast \circ g^\ast \circ d^\dagger \\
\{2.3\} \quad \downarrow \quad \{2.3\} \quad \downarrow \quad \{2.3\}
\end{array}
\]

\[
\begin{array}{c}
(c^{op})^1 \circ (f \times f)^* \circ (\Delta_Y)_\ast (d_g) \quad \text{base change} \quad (c^{op})^1 \circ (\Delta_X)_\ast \circ (f \times f)^* \circ (\text{Id})_\ast \circ c^\ast \circ f^! (d_g),
\end{array}
\]

where we set \(d_g := \text{dim}_g\). Thus it suffices to show that diagram (10.9) commutes.

Moreover, by adjunction, it suffices to show the homotopy commutativity of the diagram (10.10)

\[
\begin{array}{c}
\Delta^*_c \circ (d^{op})^1 \quad \{2.3\} \quad \Delta^*_c \circ (c^{op})^1 \circ (f \times f)^* (d'_g) \quad \text{base change} \quad c^\ast \circ \Delta^*_c \circ (f \times f)^* (d'_g) \\
\end{array}
\]

\[
\begin{array}{c}
g^\ast \circ d^\dagger \circ \Delta^*_c \circ (d^\dagger) \quad \{2.3\} \quad c^\ast \circ f^* \circ \Delta^*_c (d'_g),
\end{array}
\]

where we set \(d_g' := \Delta^*_c (\text{dim}_g) = c^\ast (\text{dim}_f)\).

10.8. Finally, to see the commutativity of diagram (10.10), it suffices to show that both horizontal compositions are naturally identified with morphism (2.3), corresponding to the commutative
diagram

\[
\begin{array}{ccc}
\text{Fix}(c) & \xrightarrow{\Delta} & X \\
g \circ \Delta_c & \downarrow & \Delta_Y \circ f \\
D & \xrightarrow{d^{op}} & Y \times Y.
\end{array}
\]

(10.11)

Note that diagram (10.11) has two decompositions

\[
\begin{array}{ccc}
\text{Fix}(c) & \xrightarrow{\Delta} & X \\
C & \xrightarrow{c^{op}} & X \times X \\
\text{Fix}(d) & \xrightarrow{\Delta} & Y \\
D & \xrightarrow{d^{op}} & Y \times Y.
\end{array}
\]

(10.12)

Using the right diagram of (10.12), whose bottom inner square is Cartesian, one sees that diagram (10.11) is pullable, and the induces morphism \( \tilde{p} : \text{Fix}(c) \to \text{Fix}(d) \times_Y X \simeq D \times_Y X \times Y, \)

is quasi-smooth of relative dimension \( \dim \Delta - c \Delta (\dim f) = -\Delta (\dim g). \)

Moreover, since base change morphisms are compatible with compositions, the bottom composition of diagram (10.10) naturally identifies with morphism (2.3), corresponding to diagram (10.11).

Next, the top inner square of the left diagram of (10.12) decomposes as

\[
\begin{array}{ccc}
\text{Fix}(c) & \xrightarrow{\tilde{p}} & D \times_Y X \times Y \\
\Delta_c & \downarrow & \Delta_X \\
C & \xrightarrow{p} & D \times_Y (X \times X) \xrightarrow{d^{op}} X \times X,
\end{array}
\]

and morphism \( p \) is quasi-smooth of relative dimension \( \dim g - c \cdot (2 \dim f) = -\dim c. \)

By Lemma B.2 below, the left inner square of (10.13) is homotopy Cartesian. Then the fact that the top composition of diagram (10.10) naturally identifies with morphism (2.3), corresponding to diagram (10.11) follows from Section 2.3(f).

**Appendix A. Proof of Proposition 1.4 and Corollary 1.5**

Though Proposition 1.4 can be showed by adapting proofs of the analogous assertions [DG, Theorem 10.2.4 and Corollary 10.2.7] for \( D \)-modules, we sketch the argument for completeness.

The following assertion is well-known to specialists.

**Lemma A.1.** Let \( G \) be a connected algebraic group over \( k \). The following are equivalent:

(i) \( G \) is unipotent;
(ii) The canonical morphism \( \mathcal{O}_G \to \Gamma(G, \mathcal{O}_G) \) is an isomorphism;
(ii)’ The cohomology groups \( H^i(G, \mathcal{O}_G) \) vanish for all \( i > 0; \)
(iii) The canonical morphism \( \mathcal{O}_G \to \Gamma(BG, \mathcal{O}_G) \) is an isomorphism;
(iii)’ The complex $\Gamma(BG, \overline{Q}_\ell) \in \text{Vect}$ is cohomologically bounded.

Proof. (i) $\implies$ (ii) is standard, while (ii) $\implies$ (iii)’ is clear.

(ii)’ $\implies$ (i) By Chevalley theorem, there exists an exact sequence of connected algebraic groups

$$1 \to G_1 \to G \overset{pr}{\to} G_2 \to 1,$$

where $G_1$ is affine and $G_2$ is an abelian variety. Then $\text{pr}_*(\overline{Q}_\ell) \in \text{Shv}(G_2)$ is a constant complex with value $\Gamma(G_1, \overline{Q}_\ell)$, thus we have

$$\Gamma(G, \overline{Q}_\ell) \simeq \Gamma(G_1, \overline{Q}_\ell) \otimes \Gamma(G_2, \overline{Q}_\ell).$$

Hence our assumption (ii)’ implies that

$$H^i(G_1, \overline{Q}_\ell) = H^i(G_2, \overline{Q}_\ell) = 0 \text{ for all } i > 0.$$

But it is well-known that these conditions imply that $G_2$ is trivial and $G_1$ is unipotent.

Remarks. For the rest of the proof, we equip $\text{Shv}(BG)$ with the usual, that is, non-perverse $t$-structure. Note that the composition

$$\text{pt} \overset{p}{\to} BG \overset{\pi}{\to} \text{pt}$$

is the identity. Since $G$ is connected and $\pi^* p_*(\overline{Q}_\ell) \simeq \Gamma(G, \overline{Q}_\ell)$, the pullback $\pi^*$ induces an equivalence $\text{Shv}(BG)^\circ \to \text{Shv}(\text{pt})^\circ$ on hearts, where the inverse functor is given by the composition

$$\mathcal{H}^0 \circ p_* : \text{Shv}(\text{pt})^\circ \to \text{Shv}(BG)^\circ.$$

In particular, every object of $\text{Shv}(BG)^\circ$ is a constant sheaf. Hence for every $i \in \mathbb{Z}$, the cohomology sheaf $\mathcal{H}^i(p_*(\overline{Q}_\ell))$ is a constant sheaf on $BG$ with value $H^i(G, \overline{Q}_\ell)$, thus we have an isomorphism

$$(A.1) \quad \pi_* \mathcal{H}^i(p_*(\overline{Q}_\ell)) \simeq H^i(G, \overline{Q}_\ell) \otimes \Gamma(BG, \overline{Q}_\ell).$$

Now we are ready to finish the proof of the lemma:

(ii) $\implies$ (iii): Assumption (ii) together with isomorphism $(A.1)$ imply that

$$\Gamma(BG, \overline{Q}_\ell) \simeq \pi_* p_*(\overline{Q}_\ell) \simeq \overline{Q}_\ell.$$

Since (iii) $\implies$ (iii)’ is clear, it suffices to show the implication (iii)’ $\implies$ (ii)’. We set

$$m := \max\{i \mid H^i(BG, \overline{Q}_\ell) \neq 0\} \text{ and } n := \max\{i \mid H^i(G, \overline{Q}_\ell) \neq 0\}.$$

Then we have a fiber sequence

$$\tau^{<n} p_*(\overline{Q}_\ell) \to p_*(\overline{Q}_\ell) \to \tau^{\geq n} p_*(\overline{Q}_\ell)$$

in $\text{Shv}(BG)$, hence a fiber sequence

$$(A.2) \quad \pi_* (\tau^{<n} p_*(\overline{Q}_\ell)) \to \overline{Q}_\ell \to \pi_* (\tau^{\geq n} p_*(\overline{Q}_\ell))$$

in $\text{Vect}$. Note that since $\tau^{<n} p_*(\overline{Q}_\ell)$ is an extension of $\mathcal{H}^i(p_*(\overline{Q}_\ell))[-i]$ with $0 \leq i \leq n-1$, we get from isomorphism $(A.1)$ that $\pi_* (\tau^{<n} p_*(\overline{Q}_\ell))$ is an extension of $H^i(G, \overline{Q}_\ell)[-i] \otimes \Gamma(BG, \overline{Q}_\ell)$, therefore

$$\pi_* (\tau^{<n} p_*(\overline{Q}_\ell)) \in \text{Vect}^{<n+m}.$$

Hence it follows from the fiber sequence $(A.2)$ that the induced map

$$\mathcal{H}^{m+n}(\overline{Q}_\ell) \to \mathcal{H}^{n+m}(\pi_* (\tau^{\geq n} p_*(\overline{Q}_\ell)))$$
between cohomologies is an isomorphism. On the other hand, using isomorphism (A.1) again we see that
\[ \pi_*(\tau \geq n(p_*(\overline{Q_{\ell}}))) \simeq H^n(G, \overline{Q_{\ell}})[-n] \otimes \Gamma(BG, \overline{Q_{\ell}}), \]
thus
\[ H^{n+m}(\pi_*(\tau \geq n(p_*(\overline{Q_{\ell}})))) \simeq H^n(G, \overline{Q_{\ell}}) \otimes H^m(BG, \overline{Q_{\ell}}) \neq 0. \]
Hence \( H^m+\pi \neq 0 \), hence \( n+m=0 \). Since \( n,m \geq 0 \), this implies that \( n=m=0 \). □

The following lemma is standard:

**Lemma A.2.** Let \( f : X \to Y \) be a surjective morphism between Artin stacks. Then the pullback functors \( f^*, f^! : \text{Shv}(Y) \to \text{Shv}(X) \) are conservative and have the property that \( A \in \text{Shv}(Y) \) is constructible if and only if \( f^*(A) \in \text{Shv}(X) \) (resp. \( f^!(A) \in \text{Shv}(X) \)) is constructible.

**Corollary A.3.** Consider Cartesian diagram of Artin stacks

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow a & & \downarrow b \\
X & \xrightarrow{f} & Y,
\end{array}
\]
where \( b \) is surjective. If morphism \( f' \) satisfies the property (ii) (resp. (iii)) of Proposition 1.4 (b), then so is \( f \).

**Proof.** Assume that functor \((f')_*\) is continuous. Then the composition \( b^! \circ f_* \simeq a^! \circ (f')_* \) is continuous. Since \( b^! \) is continuous and conservative (by Lemma A.2), we therefore conclude that \( f_* \) is continuous.

Next, if functor \((f')^!\) maps constructible sheaves to constructible, then for every \( A \in \text{Shv}(X)^{\text{constr}} \), the sheaf \( b^! \circ f_!(A) \simeq a^! \circ f^!(A) \) is constructible. Hence, the sheaf \( f_!(A) \) is constructible by Lemma A.2. □

**A.4.** Proof of Proposition 1.4 (a).

(ii) \( \Rightarrow \) (iii): Note that for every constructible \( A \in \text{Shv}(X)^{\text{constr}} \), the functor
\[ \mathcal{H}om_{\text{Shv}(X)}(A, -) \simeq \mathcal{H}om_{\text{Shv}(X)}(\overline{Q_{\ell}}, \mathcal{D}_X(A) \otimes -) \]
is continuous, because both functors \( \mathcal{D}_X(A) \otimes - \) and \( \mathcal{H}om_{\text{Shv}(X)}(\overline{Q_{\ell}}, -) \) are continuous.

(iii) \( \Rightarrow \) (i): Fix a geometric point \( x \) of \( X \) and set \( G : = (G_x)^0_{\text{red}} \). We want to show that \( G \) is unipotent. Note that the natural morphism \( i_x : BG \to X \) is schematic, thus the sheaf \( i_x^!(\overline{Q_{\ell}}) \in \text{Shv}(X) \) is constructible. Therefore the sheaf \( i_x^!(\overline{Q_{\ell}}) \) is compact (by assumption (iii)), hence the pushforward
\[ (p_X)(i_x^!(\overline{Q_{\ell}})) = (p_{BG})^!(\overline{Q_{\ell}}) \in \text{Shv}(pt) \]
is compact, thus cohomologically bounded. Then \( \Gamma(BG, \overline{Q_{\ell}}) \) is cohomologically bounded, so \( G \) is unipotent by Lemma A.1.

(i) \( \Rightarrow \) (ii): To make the proof more structural, we will divide it into five steps.
Step 1. Let $j : U \hookrightarrow X$ be an open embedding, and let $i : Z \hookrightarrow X$ be the complementary closed embedding. Using fiber sequence

$$j_i^*(\mathcal{Q}_\ell) \to \mathcal{Q}_\ell \to i^*j_i^*(\mathcal{Q}_\ell),$$

we see that $\mathcal{Q}_\ell \in \text{Shv}(X)$ is compact if and only if both $\mathcal{Q}_\ell \in \text{Shv}(U)$ and $\mathcal{Q}_\ell \in \text{Shv}(Z)$ are. Thus, by Noetherian induction, we can replace $X$ by an open non-empty substack.

Step 2. Combining Step 1 and [Stacks Tag 06RC], we can assume that $X$ is a gerbe over an algebraic space $Y$. Let $\pi : X \to Y$ be the projection. Since $\mathcal{Q}_\ell \in \text{Shv}(Y)$ is compact, it thus suffices to show that the projection $\pi_*$ is continuous. By [Stacks Tag 06QH], there exists a faithfully flat morphism $Y' \to Y$ such that $X \times_Y Y' \simeq Y'/G$ for some algebraic group space $G$ over $Y'$. Thus, by Corollary A.3, we can replace $\pi$ by its pullback to $Y'$, thus assuming that $\pi$ is the projection $Y/G \to Y$.

Step 3. Applying Step 1 again, we can replace $Y$ by this open subspace. Thus we can assume that $Y$ is a connected scheme, and $G$ is a group scheme. Furthermore, we can assume there exists a surjective morphism $Y' \to Y$ such that the reduced pullback $G' := (G \times_Y Y')_{\text{red}}$ satisfies the property that the projection $G' \to Y'$ is smooth, and all fibers of the connected component $(G')_0 \to Y$ are integral. Using Corollary A.3, we can replace $\pi$ by its pullback to $Y'$, thus assuming that $G$ is smooth over $Y$ and all geometric fibers of the projection $G_0 \to Y$ are irreducible. Furthermore, since $X/G$ is a safe stack, we conclude that all fibers of the projection $G_0 \to Y$ are unipotent.

Step 4. Consider the projection $p : Y \to Y/G$.

(a) Assume first that all geometric fibers of the projection $G \to Y$ are connected and unipotent. In this case, the projection $p : Y \to Y/G$ is smooth with unipotent geometric fibers, hence the pullback $p^* : \text{Shv}(Y/G) \to \text{Shv}(Y)$ is an equivalence of categories with inverse functor $p_*$. Therefore functor $\pi_* \simeq (p_*)^{-1} \simeq p^*$ is an equivalence of categories, thus it is continuous.

(b) Assume now that $G$ is a finite group. In this case, every object $A \in \text{Shv}(Y/G)$ is a direct factor of $p_*p^*(A)$. Thus for every $B \in \text{Shv}(Y)$ its pullback $\pi^*(B)$ is a direct factor of $p_*p^*(B) \simeq p_*(B) \simeq p(B)$.

Hence $p^*(B)$ is compact, as claimed.

Step 5. Now we are ready to show the assertion. Suppose that we are in the situation of Step 3. Then the morphism $\pi : Y/G \to Y$ decomposes as

$$Y/G \xrightarrow{\pi} Y/G^0 \xrightarrow{\pi^0} Y,$$

so it remains to show that both functors $\pi'_*\pi'_*$ and $\pi'_*\pi'_*$ are continuous. The assertion for $\pi'_*$ follows from Step 4(a). Next, by Corollary A.3, we can replace $\pi'_*$ with its pullback with respect to the projection $Y \to Y/G^0$. In other words, it suffices to show the continuity of $\pi_*$ when $\pi$ is the projection $Y/\pi_0(G) \to Y$. But this follows from Step 4(b). \qed

A.5. Proof of Proposition 1.4(b). First we claim that if $f$ is safe, then $f$ satisfies properties (ii) and (iii) of Proposition 1.4(b).

Consider Cartesian diagram A.3, where $b$ is a smooth covering and $Y'$ is a scheme of finite type over $k$. Then $f'$ is safe, and it follows from Corollary A.3 that the assertion for $f$ follows from that for $f'$. Thus we can assume that $Y$ is a scheme. In this case, the stack $X$ is safe, so by Proposition 1.4(a) every constructible $A \in \text{Shv}(X)$ is compact.
This implies both assertions: For property (ii) note that every \( A \in \text{Shv}(Y) \) is constructible, thus pullback \( f^*(A) \) is constructible, hence compact. For property (iii) notice that every constructible \( A \in \text{Shv}(X) \) is compact, thus \( f_!(A) \) is constructible.

It remains to show that if \( f \) satisfies either property (ii) or property (iii) of Proposition 1.4(b), then \( f \) is safe.

Fix a geometric point \( x \in X \), let \( G_x := \text{Aut}_f(x) \) be the stabilizer group, and let \( G := (G_x)_{\text{red}} \) be the connected component of its reduction. We want to show that \( G \) is unipotent.

Consider the commutative diagram

\[
\begin{array}{ccc}
BG & \xrightarrow{i_x} & X \\
\pi \downarrow & & \downarrow f \\
pt & \xrightarrow{\eta_f(x)} & Y.
\end{array}
\]

Then \( i_x \) and \( \eta_f(x) \) are safe, thus (by the proven above) functors \( (i_x)_\ast \) and \( (\eta_f(x))_\ast \) are continuous. If \( f_* \) is continuous, we conclude that \( f_\ast \circ (i_x)_\ast \simeq (\eta_f(x))_\ast \circ \pi_\ast \) is continuous. Thus \( \pi_\ast \) is continuous, because \( (\eta_f(x))_\ast \) is conservative, which implies that \( BG \) is safe (by Proposition 1.4(a)), thus \( G \) is unipotent.

Similarly, since \( i_x \) is safe, we conclude that the pushforward \( (i_x)_!(\mathbb{Q}_\ell) \in \text{Shv}(X) \) is constructible. If \( f_! \) preserves constructible sheaves, we conclude that

\[
f_!(i_x)_!(\mathbb{Q}_\ell) \simeq (\eta_f(x))_!(\pi_!(\mathbb{Q}_\ell))
\]

is constructible. Thus \( \pi_!(\mathbb{Q}_\ell) \) is cohomologically bounded, hence \( G \) is unipotent by Lemma A.1. □

A.6. Proof of Corollary 1.6 Since both \( f_* \) and \( f_! \) satisfy smooth base change, we can replace \( Y \) by its smooth covering, thus it suffices to show an isomorphism of functors \( f_! \simeq f_* \) when \( Y \) is a scheme of finite type over \( k \).

Since morphism \( f : X \to Y \) is supposed to be separated, it follows from a theorem of Olsson [Ol] Theorem 1.1) that there exists a proper surjective morphism \( p : \tilde{X} \to X \) from a scheme \( \tilde{X} \), which is quasi–projective over \( Y \). For every \( n \in \mathbb{N} \), we denote by \( \tilde{X}^{(n)} \) the \( (n+1) \)-times fiber product \( \tilde{X} \times_X \ldots \times_X \tilde{X} \) of \( \tilde{X} \) over \( X \), and let \( p^{(n)} : \tilde{X}^{(n)} \to X \) be the projection map.

Then both \( p^{(n)} : \tilde{X}^{(n)} \to Y \) and \( f \circ p^{(n)} : \tilde{X}^{(n)} \to Y \) are proper morphisms between algebraic spaces, so the assertion of the Corollary holds in these cases\(^6\) and thus we have a canonical isomorphism of functors

\[
(A.4) \quad f_! \circ (p^{(n)})_! \simeq (f \circ p^{(n)})_! \simeq (f \circ p^{(n)})_* \simeq f_* \circ (p^{(n)})_* \simeq f_* \circ (p^{(n)})_!.
\]

Since \( p \) is proper and surjective, the natural functor

\[
\text{colim}_{n \in \Delta^+} (p^{(n)})_! \colon (p^{(n)})_! \to \text{Id}_{\text{Shv}(X)}
\]

is an isomorphism\(^7\). Since both \( f_! \) and \( f_* \) are continuous, we thus get a natural isomorphism

\[
(A.4) \quad f_! \simeq f_! \circ \text{colim}_{n \in \Delta^+} (p^{(n)})_! \simeq \text{colim}_{n \in \Delta^+} f_! \circ (p^{(n)})_! \simeq \text{colim}_{n \in \Delta^+} f_! \circ (p^{(n)})_* \simeq f_* \circ (p^{(n)})_* \simeq f_* \circ (p^{(n)})_! \simeq f_!
\]

\(^6\)Alternatively, it can be deduced from the corresponding result for schemes by repeating the argument below.

\(^7\)Moreover, \( \text{Shv}(-) \) is a sheaf in the \( h \)-topology.
APPENDIX B. QUASI-SMOOTH MORPHISMS

B.1. Observations.

(a) For every derived Artin stack $X$, the cotangent complex $T^*(X_{cl}/X)$ lies in the cohomological degrees $\leq -2$. Indeed, passing to a smooth covering, we can assume that $\tilde{A}$ is an affine derived scheme. In this case, the assertion follows from [SAG Corollary 25.3.6.4].

(b) By part (a), the morphism $X_{cl} \to X$ is never quasi-smooth, if $X$ is not classical.

Lemma B.2. Let

$$
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
\downarrow g & & \downarrow f \\
B & \xrightarrow{b} & D
\end{array}
$$

(B.1)

be a Cartesian diagram of Artin stacks that morphisms $f$ and $g$ are quasi-smooth and satisfy $\dim_g = a(\dim_f)$. Then diagram (B.1) is homotopy Cartesian.

Proof. We have to show that the canonical morphism $p : A \to \tilde{A} := B \times^B C$ is an equivalence. Since the induced morphism $A \to \tilde{A}_{cl}$ is an isomorphism (since diagram (B.1) is Cartesian), it suffices to show that the cotangent complex $T^*(A/\tilde{A})$ vanishes (by [SAG Corollary 25.3.6.6]).

Note that a sequence of morphisms $A \to \tilde{A} \to B$ gives rise to a fiber sequence

$$p^*(T^*(\tilde{A}/B)) \to T^*(A/B) \to T^*(A/\tilde{A}).$$

Since morphisms $f$ and $g$ are quasi-smooth, we conclude that complexes $T^*(A/B)$ and $p^*(T^*(\tilde{A}/B)) \simeq a^*(T^*(C/D))$ are perfect of Tor-amplitude $\geq -1$ with Euler characteristic $\dim_f$ and $a^*(\dim_f)$, respectively. Therefore the cotangent complex $T^*(A/\tilde{A})$ is perfect of Tor-amplitude $\geq -2$ with Euler characteristic zero $\dim_g - a^*(\dim_f) = 0$.

On the other hand, since $A \simeq \tilde{A}_{cl}$, the cotangent complex $T^*(A/\tilde{A}) \simeq T^*(\tilde{A}_{cl}/\tilde{A})$ lies in the cohomological degrees $\leq -2$ (see Section B.1(a)). Thus $T^*(A/\tilde{A}) \simeq F[2]$ for certain locally free $O_{\tilde{A}}$-module $F$. Moreover, since the Euler characteristic of $T^*(A/\tilde{A}) \simeq F[2]$ is zero, we conclude that $T^*(A/\tilde{A})$ vanishes, as claimed. □

Lemma B.3. Let $f : X \to Y$ be a quasi-smooth morphism between derived Artin stacks such that $X$ is classical. Then the induced morphism $f_{cl} : X \to Y_{cl}$ is classical, the canonical map $T^*(X/Y_{cl}) \to T^*(X/Y)$ is an isomorphism, and we have an equality $\dim_{f_{cl}} = \dim_f$.

Proof. Let $\tilde{f} : \tilde{X} \to Y_{cl}$ be the homotopy pullback of $f$ under $Y_{cl} \to Y$. Then $\tilde{f}$ is quasi-smooth, and $f_{cl}$ is a retract of $\tilde{f}$, hence $f_{cl}$ is quasi-smooth. To show the second assertion, notice that sequence of morphisms $X \to Y_{cl} \to Y$ gives rise to a fiber sequence

$$f_{cl}^*T^*(Y_{cl}/Y) \to T^*(X/Y_{cl}) \to T^*(X/Y).$$

Moreover, since $T^*(Y_{cl}/Y)$ and hence also $f_{cl}^*T^*(Y_{cl}/Y)$ lies in cohomological degrees $\leq -2$, while $T^*(X/Y_{cl})$ and $T^*(X/Y)$ lie in cohomological degrees $\geq -1$, we get that $f_{cl}^*T^*(Y_{cl}/Y) = 0$, thus the map $T^*(X/Y_{cl}) \to T^*(X/Y)$ is an isomorphism. The third assertion follows immediately from the second one. □
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