Unified Bernoulli-Euler polynomials of Apostol type

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Abstract The object of this paper is to introduce and study the properties of unified Apostol-Bernoulli and Apostol-Euler polynomials noted by \( \{V_n(x; \lambda; \mu)\}_{n \geq 0} \). We study some arithmetic properties of \( \{V_n(x; \lambda; \mu)\}_{n \geq 0} \) as their connection to Apostol-Euler polynomials and Apostol-Bernoulli polynomials. Also, we give derivation and integration representations of \( \{V_n(x; \lambda; \mu)\}_{n \geq 0} \). Finally, we use the umbral calculus approach to deduce symmetric identities.

Keywords Euler polynomials · Bernoulli polynomials · Apostol-Bernoulli and Apostol-Euler polynomials · generating function

Mathematics Subject Classification 11B68 · 11B83 · 11C08 · 11C20

1 Introduction

The Bernoulli \( \{B_n(x)\}_{n \geq 0} \) and the Euler \( \{E_n(x)\}_{n \geq 0} \) polynomials respectively are generated by the following power series (see [4,6]):

\[
\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi)
\]

and

\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (|t| < \pi).
\]

As a particular case, for \( x = 0 \), we denote \( B_n := B_n(0) \) and \( E_n := E_n(0) \), which are called the Bernoulli and the Euler numbers, respectively. They have numerous important applications in various fields of mathematics, like number theory, analysis and combinatorics.

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Apostol [1] introduced and investigated the extended form of the classical Bernoulli polynomials and numbers, known as the Apostol-Bernoulli polynomials and numbers. The Apostol-Euler and the Apostol-Genocchi polynomials were introduced by Srivastava [12]. Belbachir et al. [2, 3] proposed a new family of polynomials called Euler-Genocchi polynomials and studied their properties like linear recurrences and difference equations using a determinantal approach and generating function.

2 Determinantal representation of the Bernoulli-Euler polynomials of Apostol type

According to [9], the Apostol-Bernoulli polynomials \{\mathcal{B}_n(x; \lambda)\}_{n\geq 0} and the Apostol-Euler polynomials \{\mathcal{E}_n(x; \lambda)\}_{n\geq 0} are generated by the following power series:

\[
\frac{t}{\lambda e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{t^n}{n!} \quad (|t + \ln \lambda| < 2\pi, \quad \lambda \in \mathbb{R}^*_+)
\]

and

\[
\frac{2}{\lambda e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n(x; \lambda) \frac{t^n}{n!} \quad (|t + \ln \lambda| < \pi, \quad \lambda \in \mathbb{R}^*_+).
\]

The Apostol-Bernoulli numbers \(\mathcal{B}_n(\lambda)\) and the Apostol-Euler numbers \(\mathcal{E}_n(\lambda)\) are given by \(\mathcal{B}_n(\lambda) = \mathcal{B}_n(0; \lambda)\) and \(\mathcal{E}_n(\lambda) = \mathcal{E}_n(0; \lambda)\).

Letting

\[
T(x, \lambda, t) = \frac{2}{\lambda e^t + 1} e^{xt} - \frac{t}{\lambda e^t - 1} e^{xt} = -\frac{2t}{\lambda^2 e^{2t} - 1} e^{2xt}.
\]

Taking into account the right hand side of (1) and (2), a direct computation gives

\[
\lambda^2 T(x + 1, \lambda, t) - T(x, \lambda, t) = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k} \left[ \lambda^2 \mathcal{B}_{n-k}(x, \lambda) \mathcal{E}_k(x, \lambda) - \mathcal{B}_{n-k}(x, \lambda) \mathcal{E}_k(x, \lambda) \right] \right\} \frac{t^n}{n!}.
\]

On the other hand, we have

\[
\lambda^2 T(x + 1, \lambda, t) - T(x, \lambda, t) = \lambda^2 \frac{2t}{\lambda^2 e^{2t} - 1} e^{2xt} - \frac{2t}{\lambda^2 e^{2t} - 1} e^{2xt} = 2t e^{2xt} = \sum_{n=0}^{\infty} n 2^n \frac{x^n t^n}{n!}.
\]

Comparing the two expansions of \(\lambda^2 T(x + 1, \lambda, t) - T(x, \lambda, t)\), we formulate the next result.

**Theorem 1** Let \(x\) be a real number and \(n\) an integer. Then

\[
x^n = \sum_{k=0}^{n+1} \Lambda_{n,k} \times \Delta_{n+1-k,k}(x, \lambda),
\]

where \(\Lambda_{n,k} = \frac{1}{2^{n+1}(n+1)} \binom{n+1}{k}\) and \(\Delta_{n,k}(x, \lambda) = \begin{vmatrix} \lambda \mathcal{B}_n(x+1, \lambda) & \mathcal{E}_k(x, \lambda) \\ \mathcal{B}_n(x, \lambda) & \lambda \mathcal{E}_k(x+1, \lambda) \end{vmatrix}\).

In particular, taking \(\lambda = 1\) in (3), we get the following result in terms of the Bernoulli and the Euler polynomials.

**Corollary 1** [3] Let \(x\) be a real number and an integer \(n \geq 0\), we have

\[
x^n = \frac{1}{2^{n+1}(n+1)} \sum_{k=0}^{n+1} \binom{n+1}{k} \begin{vmatrix} B_{n-(k-1)}(x+1) & E_k(x) \\ B_{n-(k-1)}(x) & E_k(x+1) \end{vmatrix}.
\]
3 Unified Bernoulli-Euler polynomials of Apostol type

In this section, we define the unified Bernoulli-Euler polynomials of Apostol type and study their properties using power series.

**Definition 1** Let \( \lambda \in \mathbb{R}_+ \) and \( \mu \in \mathbb{R}_+ - \{1\} \), we define the unified Bernoulli-Euler polynomials of Apostol type \( V_n(x; \lambda; \mu) \) by the following power series:

\[
\frac{2 - \mu + \frac{\mu t}{2}}{\lambda e^t + (1 - \mu)} e^{xt} = \sum_{n \geq 0} V_n(x; \lambda; \mu) \frac{t^n}{n!},
\]

(5)

where

\[
\begin{align*}
&\left\{ \ln \left( \frac{\lambda}{1 - \mu} \right) + t \right\} < 2\pi, \text{ for } 0 \leq \mu < 1; \\
&\left\{ \ln \left( \frac{\lambda}{\mu - 1} \right) + t \right\} < \pi, \text{ otherwise.}
\end{align*}
\]

Furthermore, the unified Bernoulli-Euler numbers of Apostol type, denoted \( \mathcal{V}_n(\lambda; \mu) \), are given by

\[
\mathcal{V}_n(\lambda; \mu) := V_n(0; \lambda; \mu).
\]

(6)

We summarize in the following table some special polynomials related to this extension.

| Parameters | Generating functions | Polynomials |
|------------|---------------------|-------------|
| \( \mu = 0, \lambda = 1 \) | \( \frac{2}{e^t + 1} e^{xt} = \sum_{n \geq 0} E_n(x) \frac{t^n}{n!}, \ |t| < \pi \) | Euler polynomials |
| \( \mu = 2, \lambda = 1 \) | \( \frac{t}{e^t - 1} e^{xt} = \sum_{n \geq 0} B_n(x) \frac{t^n}{n!}, \ |t| < 2\pi \) | Bernoulli polynomials |
| \( \mu = 2 \) | \( \frac{t}{\lambda e^t - 1} e^{xt} = \sum_{n \geq 0} \mathcal{V}_n(x; \lambda) \frac{t^n}{n!}, \ |t + \ln \lambda| < 2\pi \) | Apostol-Bernoulli polynomials |
| \( \mu = 0 \) | \( \frac{2}{\lambda e^t + 1} e^{xt} = \sum_{n \geq 0} \mathcal{E}_n(x; \lambda) \frac{t^n}{n!}, \ |t + \ln \lambda| < \pi \) | Apostol-Euler polynomials |

We list some properties of the unified Bernoulli-Euler polynomials of Apostol type using generating function approach.

**Theorem 2** Let \( n \) be a nonnegative integer, we have

\[
\mathcal{V}_n(x + y; \lambda; \mu) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{V}_k(x; \lambda; \mu) y^{n-k}.
\]

(7)

In particular, for \( x := 0 \) and \( y := x \), the above relation becomes

\[
\mathcal{V}_n(x; \lambda; \mu) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{V}_k(\lambda; \mu) x^{n-k}.
\]

(8)

**Proof** We establish the power series defined in (5) for \( \mathcal{V}_n(x + y; \lambda; \mu) \), we have

\[
\sum_{n \geq 0} \mathcal{V}_n(x + y; \lambda; \mu) \frac{t^n}{n!} = \left( \frac{2 - \mu + \frac{\mu t}{2}}{\lambda e^t + (1 - \mu)} \right) e^{(x+y)t} = \sum_{n \geq 0} \sum_{k \geq 0} \mathcal{V}_n(x; \lambda; \mu) y^k \frac{t^{n+k}}{n! k!}.
\]

Applying the product series and then comparing the coefficients of \( t^n \) on both sides, we obtain Identity (7). \( \square \)
Remark 1 Expression (7) allows us to obtain $\mathcal{V}_n(\lambda; \mu)$ the unified Bernoulli-Euler numbers of Apostol type in terms of the unified Bernoulli-Euler polynomials of Apostol type. Indeed, it suffices to replace $y$ by $-x$ in Formula (7), we get the following expression:

$$
\mathcal{V}_n(\lambda; \mu) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \mathcal{V}_k(x; \lambda; \mu)x^{n-k}.
$$

As a first consequence of Theorem 1, we show that the unified Bernoulli-Euler polynomials of Apostol type, $\{\mathcal{V}_n(x, \lambda; \mu)\}_{n \geq 0}$ given by the power series in (5), can be expressed in terms of the Apostol-Bernoulli and the Apostol-Euler polynomials. That is, by a straightforward calculation, the substitution of $x^n$ given by (3) in Expression (8) allows us to obtain the following formula:

**Proposition 1** Let $n$, $k$ and $j$ be three integers, it holds that

$$
\mathcal{V}_n(x; \lambda; \mu) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{V}_{n-k}(\lambda; \mu) \sum_{j=0}^{k+1} \Delta_{k,j} \times \Delta_{k+1-j, j}(x, \lambda).
$$

4 Generalized Raabe’s Theorem

In this section, we give an extension of Raabe’s Theorem for the unified Bernoulli-Euler polynomials of Apostol type.

**Theorem 3** Let $r$ and $m$ be a nonnegative integers with $m$ odd, for $\lambda = 1 - \mu$ and $\mu \neq 1$, we have

$$
\sum_{k=0}^{m-1} (-1)^k \mathcal{V}_n \left( \frac{x + k}{m}; 1 - \mu; \mu \right) = \frac{1 - m}{m!^{n+1}} \left( \frac{\mu - 2}{2(\mu - 1)} \right) E_n(x) + \frac{1}{m^{n+1}} \mathcal{V}_n(x; 1 - \mu; \mu).
$$

**Proof** It follows from (5) that

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{n-1} (-1)^k \mathcal{V}_n \left( \frac{x + k}{m}; 1 - \mu; \mu \right) \frac{t^n}{n!} = \frac{1 - m}{m!^{n+1}} \left( \frac{\mu - 2}{2(\mu - 1)} \right) E_n(x) + \frac{1}{m^{n+1}} \mathcal{V}_n(x; 1 - \mu; \mu).
$$

We get the result by simple manipulations and equating the coefficients of $t^n$ on both sides. \qed
As a consequence of Theorem 3, for $\mu = 2$ and $\mu = 0$ respectively, we have a multiplication Theorem for Euler and Bernoulli polynomials proved by Raabe in [10], as specified by Kargin and Kurt [8]. They are given as follows:

$$\sum_{k=0}^{m-1} B_n \left( \frac{x + k}{m} \right) = \frac{1}{m^{n-1}} B_n(x)$$

and

$$\sum_{k=0}^{m-1} (-1)^k E_n \left( \frac{x + k}{m} \right) = \frac{1}{m^n} E_n(x).$$

5 Some explicit formulas

In this section, we give some explicit formulas of the unified Bernoulli-Euler polynomials of Apostol type.

**Theorem 4** For $\lambda \in \mathbb{R}^+ \setminus \{ 0 \}$ and $\mu \in \mathbb{R}^+ - \{ 1 \}$, it holds that

$$\mathfrak{B}_n(x; \lambda; \mu) = \frac{1}{2(\mu - 1)} \left[ (\mu - 2) \mathfrak{B}_n(x; \frac{\lambda}{1 - \mu}) - \frac{\mu n}{2} \mathfrak{D}_{n-1}(x; \frac{\lambda}{1 - \mu}) \right] \quad (n \in \mathbb{N}).$$

**Proof** We can reformulate (5) as follows

$$\sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda; \mu) \frac{t^n}{n!} = \left( \frac{1}{2(\mu - 1)} \right) (\mu - 2) - \frac{\mu n}{2} \frac{t^n}{n!} e^{\lambda t} \left( \frac{1}{1 + \frac{1}{\mu} e^t} \right) e^{\lambda t}$$

$$= \frac{1}{2(\mu - 1)} \sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda; \mu) \frac{t^n}{n!} - \frac{\mu n}{2} \sum_{n=1}^{\infty} \mathfrak{D}_{n-1}(x; \lambda; \mu) \frac{t^{n+1}}{n!}$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides, we obtain Identity (9). \qed

Here, we give an explicit formula as a dual convex combination of classical Bernoulli and Euler polynomials of Apostol type.

**Theorem 5** Let $n$ be a nonnegative integer and a real number $\mu \neq 1$, we have

$$\mathfrak{B}_n(x; \lambda; \mu) = \frac{1}{1 - \mu} \left[ \left( 1 - \frac{\mu}{2} \right) \mathfrak{B}_n\left( x; \frac{\lambda}{1 - \mu} \right) - \frac{\mu n}{2} \mathfrak{D}_n\left( x; \frac{\lambda}{\mu - 1} \right) \right].$$

**Proof** From (5), we have

$$\sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda; \mu) \frac{t^n}{n!} = \left( 2 - \mu + \frac{\mu}{2} t \right) \frac{t^n}{\lambda e^t + (1 - \mu) e^t} e^{\lambda t} = \left( 2 - \mu + \frac{\mu}{2} t \right) \frac{2 - \mu + \frac{\mu}{2} t}{2(\mu - 1)} \frac{2}{\lambda e^t + 1} \frac{t^n}{e^t} + \left( 2 - \mu + \frac{\mu}{2} t \right) \frac{2}{\lambda e^t + 1} \frac{t^n}{e^t}.$$  

Using (1) and (2) leads to get (10). \qed

**Theorem 6** For $\mu \neq 1$ and $n \geq 1$, the following formula holds:

$$(\mu - 1) \sum_{k=1}^{n} \binom{n}{k} \mathfrak{Y}_{n,k,0} \left( \frac{x}{2}, \lambda, \mu \right) + \left( \frac{\mu}{2} - 1 \right) \Delta_{n,0,0} \left( \frac{x}{2}, \frac{\lambda}{1 - \mu} \right) = n(\mu - 2)x^{n-1} - n(n - 1)\mu x^{n-2},$$

where $\mathfrak{Y}_{n,k}(x, \lambda, \mu) = \left| \frac{1}{1 - \mu} \mathfrak{B}_k(x + 1; \lambda) \mathfrak{B}_k(x; \lambda; \mu) - \frac{1}{1 - \mu} \mathfrak{B}_k(x; \lambda; \mu) \right|$.  

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Proof From Theorem 4, we have
\[
\sum_{k=1}^{n} \binom{n}{k} T_{n-k,k}(x, \lambda, \mu) = \frac{1}{2(\mu-1)} \left( (\mu-2) \sum_{k=1}^{n} \binom{n}{k} \Delta_{n-k,k}(x, \frac{\lambda}{1-\mu}) - \mu \sum_{k=1}^{n} \binom{n}{k} \Delta_{n-k,k-1}(x, \frac{\lambda}{1-\mu}) \right).
\]
Apply Theorem 1 and a straightforward computation, we obtain
\[
\sum_{k=1}^{n} \binom{n}{k} T_{n-k,k}(x, \lambda, \mu) = \frac{1}{\mu-1} \left\{ n(\mu-2)(2x)^{n-1} - n(n-1)\mu(2x)^{n-2} \right\} - \frac{\mu-2}{2(\mu-1)} \Delta_{n,0}(x, \frac{\lambda}{1-\mu}).
\]
We get the desired identity by multiplying both sides by ($\mu - 1$).

6 Derivation and integration representations of unified Bernoulli-Euler polynomials of Apostol type

In this section, we present derivation and integration representations for the unified Bernoulli-Euler polynomials of Apostol type.

Theorem 7 Let $l, n$ be two nonnegative integers. Then
\[
\frac{d^l}{dx^l} \mathcal{V}_n(x; \lambda; \mu) = (n)! \mathcal{V}_{n-l}(x; \lambda; \mu),
\]
\[
\int_{x}^{y} \mathcal{V}_n(z; \lambda; \mu) dz = \frac{1}{(n+1)} \left( \mathcal{V}_{n+1}(y; \lambda; \mu) - \mathcal{V}_{n+1}(x; \lambda; \mu) \right),
\]
where $(x)_n := x(x-1) \cdots (x-n+1)$ with $(x)_0 = 1$.

Proof The assertion (11) follows from (5) by successive differentiation with respect to $x$ and then uses the induction principle on $l$. Furthermore, taking $l = 1$ in (11) and integrating both sides of the resulting equation with respect to $z$ over the interval $[x, y]$, $(y > x)$, we obtain the Integral Formula (12).

Remark 2 Setting $\lambda = 1, \mu = 2$ in (11) and (12), we obtain known results due to Luo et al. [9].

Corollary 2 Let $n$ be a nonnegative integer. Then
\[
\int_{x}^{x+y} \mathcal{V}_n(z; \lambda; \mu) dz = \frac{1}{(n+1)} \sum_{k=0}^{n} \binom{n}{k} \mathcal{V}_k(x; \lambda; \mu) y^{n-(k-1)}.
\]

Proof Replacing $y$ by $x + y$ in the Integral Formula (12) and using Formula (7), by successive calculations, we obtain the Integral Formula (13).

Theorem 8 For $\mu \in \mathbb{R}_+^\ast \setminus \{1, 2\}$ and $n$ nonnegative integer, the following formula holds:
\[
\mathcal{V}_{n+1}(x; \lambda; \mu) - x \mathcal{V}_{n}(x; \lambda; \mu) = \frac{1}{2-\mu} \sum_{i=0}^{n} \binom{n}{i} (n-i)! \
\times \left( \frac{\mu}{2(\mu-2)} \right)^{n-i} \left[ \frac{\mu}{2} \mathcal{V}_i(x; \lambda; \mu) - \lambda \sum_{k=0}^{i} \binom{i}{k} \mathcal{V}_k(\lambda; \mu) \mathcal{V}_{i-k}(x+1; \lambda; \mu) \right].
\]
Proof Differentiating both sides of (5) with respect to \( t \), we express the factors \( \left( 1 + \frac{\mu}{2(\mu-1)} t \right)^{-1} \) in series form for \( |t| < \frac{2}{\mu} |2 - \mu| \), and using Formulas (5) and (6), we obtain

\[
\sum_{n=0}^{\infty} \mathcal{V}_n(x; \lambda; \mu) \frac{t^n}{n!} = x \sum_{n=0}^{\infty} \mathcal{V}_n(x; \lambda; \mu) \frac{t^n}{n!} + \frac{1}{(2 - \mu)} \left( \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \right) \left( \sum_{n=0}^{\infty} \frac{2^n (\mu - 2)^n}{n!} \right)
\]

Then taking into account the series product in (15) and equating the coefficients of \( t \), we get Identity (14). \( \Box \)

7 Identities inspired via umbral calculus

The umbral calculus approach is a useful tool to get and guess arithmetic and combinatorial identities, see for instance Gessel [7] on some applications of the classical umbral calculus, Di Crescenzo et al. [5] on umbral calculus. See also classical references as those of Roman and Rota [11].

Let \( B^n(\lambda; \mu) \) be the umbra defined by \( B^n(\lambda; \mu) := \mathcal{V}_n(\lambda; \mu) \) and \( (\mathcal{V}_n(x; \lambda; \mu))_{n \geq 0} \) defined by

\[
\sum_{n \geq 0} \mathcal{V}_n(x; \lambda; \mu) \frac{t^n}{n!} = F(t) e^{xt} = \exp ((B(\lambda; \mu) + x) t),
\]

where \( F(t) := \sum_{n \geq 0} \mathcal{V}_n(\lambda; \mu) \frac{t^n}{n!} = \exp (B(\lambda; \mu) t) \). So, \( \mathcal{V}_n(x; \lambda; \mu) \) admits the umbral representation

\[
\mathcal{V}_n(x; \lambda; \mu) = (B(\lambda; \mu) + x)^n.
\]

Theorem 9 Let \( n \) be a nonnegative integer. Then

\[
\mathcal{V}_n(x + 1; \lambda; \mu) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{V}_k(x; \lambda; \mu).
\]

Proof By the umbral representation \( \mathcal{V}_n(x; \lambda; \mu) = (B(\lambda; \mu) + x)^n \), we have

\[
\mathcal{V}_n(x + 1; \lambda; \mu) = (B(\lambda; \mu) + (x + 1))^n = \sum_{k=0}^{n} \binom{n}{k} (B(\lambda; \mu) + x)^k = \sum_{k=0}^{n} \binom{n}{k} \mathcal{V}_k(x; \lambda; \mu).
\]

\( \Box \)

Theorem 10 Let \( n, m \) be a nonnegative integers. Then

\[
\sum_{k=0}^{m} \binom{n}{k} y^{m-k} \mathcal{V}_{m+k}(x; \lambda; \mu) = \sum_{k=0}^{m} \binom{m}{k} (-y)^{m-k} \mathcal{V}_{n+k}(x+y; \lambda; \mu).
\]

Proof By the umbral representation \( \mathcal{V}_n(x; \lambda; \mu) = (B(\lambda; \mu) + x)^n \), on the one hand, we have

\[
(B(\lambda; \mu) + (x + y))^n (B(\lambda; \mu) + x)^m = (B(\lambda; \mu) + (x + y))^n (B(\lambda; \mu) + (x + y) - y)^m
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} (-y)^{m-k} (B(\lambda; \mu) + (x + y))^n
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} (-y)^{m-k} \mathcal{V}_{n+k}(x+y; \lambda; \mu),
\]

and on the other hand, we have

\[
(B(\lambda; \mu) + (x + y))^n (B(\lambda; \mu) + x)^m = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} (B(\lambda; \mu) + x)^{n+k}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} y^{n-k} \mathcal{V}_{m+k}(x; \lambda; \mu).
\]

Hence, the two expressions give the desired identity. \( \Box \)
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