STABLE MINIMAL HYPERSURFACES IN $\mathbb{R}^4$

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ABSTRACT. We prove that a complete, two-sided, stable minimal immersed hypersurface in $\mathbb{R}^4$ is flat.

1. INTRODUCTION

A complete, two-sided, immersed minimal hypersurface $M^n \to \mathbb{R}^{n+1}$ is stable if
\[ \int_M |A_M|^2 f^2 \leq \int_M |\nabla f|^2 \] (1)
for any $f \in C_0^\infty(M)$. We prove here the following result.

**Theorem 1.** A complete, connected, two-sided, stable minimal immersion $M^3 \to \mathbb{R}^4$ is a flat $\mathbb{R}^3 \subset \mathbb{R}^4$.

This resolves a well-known conjecture of Schoen (cf. [19, Conjecture 2.12]). The corresponding result for $M^2 \to \mathbb{R}^3$ was proven by Fischer-Colbrie–Schoen, do Carmo–Peng, and Pogorelov [26, 23, 43] in 1979. Theorem 1 (and higher dimensional analogues) has been established under natural cubic volume growth assumptions by Schoen–Simon–Yau [44] (see also [52, 47]). Furthermore, in the special case that $M^n \subset \mathbb{R}^{n+1}$ is a minimal graph (implying (1) and volume growth bounds) flatness of $M$ is known as the Bernstein problem, see [27, 22, 51] concerning stability in related contexts.

It is well-known (cf. [58, Lecture 3]) that a result along the lines of Theorem 1 yields curvature estimates for minimal hypersurfaces in $\mathbb{R}^4$.

**Theorem 2.** There exists $C < \infty$ such that if $M^3 \to \mathbb{R}^4$ is a two-sided, stable minimal immersion, then
\[ |A_M(p)|d_M(p, \partial M) \leq C. \]
More generally, we recall that a minimal immersion $M^3 \to (N^4, g)$ is stable if
\[ \int_M (|A_M|^2 + \text{Ric}_N(\nu, \nu)) f^2 \leq \int_M |\nabla f|^2 \]
for all $f \in C_0^\infty(M \setminus \partial M)$.

**Theorem 3.** Suppose $(N^4, g)$ is a closed Riemannian manifold. There exists $C = C(N, g)$ such that if $M^3 \to N^4$ is a two-sided, stable minimal immersion, then
\[ |A_M(p)| \min\{1, d_M(p, \partial M)\} \leq C. \]
We have recently generalized Theorem 3 to hold in (non-compact) ambient \((N^4, g)\) with bounded sectional curvature in a joint work with Stryker \([15, Corollary 2.5]\), which resolves \([19, Conjecture 2.13]\).

Theorems 2 and 3 are the four-dimensional analogue of the well-known curvature estimate of Schoen \([46]\) for minimal surfaces in three-dimensions. Note that by the work of Schoen–Simon–Yau \([44]\), such an estimate was previously known to hold where \(C\) depended on an upper bound for volume of \(M^3\) in small balls.

Remark 4. There have been several interesting developments since the first version of this paper was posted. The authors have discovered \([14]\) a new proof of Theorem 1 that can be localized to obtain (interior) volume estimates (in the spirit of Pogorelov \([43]\); cf. \([17]\)). This new proof is related to the study of uniformly positive scalar curvature, while the current paper is related to the study of non-negative scalar curvature. Subsequently, Catino–Mastrolia–Roncoroni have discovered \([28]\) a completely different proof of Theorem 1 related to the study of Bakry–Émery–Ricci curvature. Interestingly, the dimension restriction \(n + 1 = 4\) enters each proof in a different way.

A slight modification of the proof of Theorem 1 yields a structure theorem for finite index minimal hypersurfaces in \(\mathbb{R}^4\), analogous to the well-known results of Gulliver, Fischer–Colbrie, and Osserman \([25, 29, 41]\). Recall that a complete, two-sided, immersed minimal hypersurface \(M^n \to \mathbb{R}^{n+1}\) has finite Morse index if

\[
\text{index}(M) := \sup \{ \dim V : V \subset C_0^\infty(M), Q(f, f) < 0 \text{ for all } f \in V \setminus \{0\} \} < \infty,
\]

where \(Q(f, f) = \int_M |\nabla f|^2 - \int_M |A_M|^2 f^2\).

**Theorem 5.** A complete, two-sided, minimal immersion \(M^3 \to \mathbb{R}^4\) has finite Morse index if and only if it has finite total curvature \(\int_M |A_M|^3 < \infty\).

We remark that Tysk \([57]\) proved the same statement for a complete, two-sided, minimal immersion \(M^n \to \mathbb{R}^{n+1}\) (for \(3 \leq n \leq 6\)) under the assumption that \(M\) has Euclidean volume growth.

Theorem 5 has strong consequences on the structure of \(M\) near infinity. We recall the following definition.

**Definition 6** (\([50, Section 2]\)). Suppose \(n \geq 3\), \(M^n \to \mathbb{R}^{n+1}\) is a complete minimal immersion. An end \(E\) of \(M\) is regular at infinity if it is the graph of a function \(w\) over a hyperplane \(\Pi\) with the asymptotics

\[
w(x) = b + a|x|^{2-n} + \sum_{j=1}^n c_j x_j |x|^{-n} + O\left(|x|^{-n}\right),
\]

for some constants \(a, b, c_j\), where \(x_1, \ldots, x_n\) are the coordinates in \(\Pi\).

By \([4, 50, 51, 57]\) (see also \([31, Appendix A]\)), if \(M^n \to \mathbb{R}^{n+1}\) is a minimal immersion with finite total curvature, then each end of \(M\) is regular at infinity. Moreover, by \([33]\) a two-sided minimal immersion with finite Morse index has finitely many ends. Combined with Theorem 5 this yields the following result.
Corollary 7. Suppose $M^3 \to \mathbb{R}^4$ is a complete, two-sided, minimal immersion with finite Morse index. Then $M$ has finitely many ends, each of which is regular at infinity. In particular, $M$ has cubic volume growth, i.e.

$$\sup_{r \to \infty} \frac{\text{Vol}(B_r(0) \cap M)}{r^3} < \infty.$$ 

In fact, by [31], we can bound the volume growth rate linearly in terms of the Morse index

$$\sup_{r \to \infty} \frac{\text{Vol}(B_r(0) \cap M)}{\frac{2}{3} \pi r^3} \leq 6 (\text{index}(M) + 1).$$

These results should be relevant to the study of minimal hypersurfaces with bounded index in a four-manifold (cf. [13, 53]).

Remark 8. We also note that Theorems 1 and 2 hold if (1) is weakened to

$$\left(1 - \delta_s\right) \int_M |A_M|^2 f^2 \leq \int_M |\nabla f|^2$$

for a constant $\delta_s \in [0, \frac{1}{3}]$. This notion (called $\delta_s$-stability) was introduced by Colding–Minicozzi [18] (cf. [17]) as a technical tool to quantify the well-known phenomenon that smooth convergence with multiplicity yields stable minimal hypersurfaces. (Note that Tam-Zhou [55] proved that the 3-dimensional catenoid is $\frac{2}{3}$-stable.) We discuss this further in Appendix E.

1.1. Idea of the proof of Theorem 1. Consider a non-flat, complete, two-sided stable minimal hypersurface $M^3 \to \mathbb{R}^4$. It turns out to be no loss of generality to assume that $M$ is simply connected and of bounded curvature.

It is well-known (see, e.g. [32]) that $M$ admits a positive Green’s function of the Laplacian operator denoted here by $u$. (In other words, $M$ is non-parabolic.) The central quantity considered in the proof of Theorem 1 is

$$F(t) := \int_{\Sigma_t} |\nabla u|^2$$

where $\Sigma_t = u^{-1}(t)$. (Quantities of this type have been considered in several contexts including [40, 16, 20, 21, 37, 2, 1].) A simple computation shows that $F(t) = 4\pi t^2$ when $M = \mathbb{R}^3$.

For general $M$, we can view any estimate of $F(t)$ as $t \to 0$ as certain kind of control on the growth rate of $M$ near infinity. It is straightforward to show that $F(t)$ satisfies the a priori estimate $F(t) = O(t)$ as $t \to 0$ (see Lemma 15). An important observation is that if we can upgrade this to the sharp estimate $F(t) = O(t^2)$, then we can conclude that $M$ is flat.

At a heuristic level, if we were to pretend that $u \sim r^{-1}$ and $|\nabla u| \sim r^{-2}$ (this holds on $\mathbb{R}^3$), we would have $F(t) \sim t^4 |\partial B_{t^{-1}}| = r^{-4} |\partial B_r|$. Thus $F(t) = O(t^2)$ can be thought of as the estimate $|\partial B_r| = O(r^2)$ which would suffice to show $M$ is flat by work of Schoen–Simon–Yau [44]. In fact, a closely related argument actually works: we can consider $f = \varphi \circ u$ in the $L^3$-version of the Schoen–Simon–Yau estimates (see Proposition 27) and use the co-area formula combined with a log-cutoff near the pole and the end to see that $F(t) = O(t^2)$ implies that $M$ is flat.
It thus remains to explain how to show \( F(t) = O(t^2) \). Our strategy is to find two (competing) estimates relating \( F(t) \) and the quantity

\[
A(t) = \int_{\Sigma_t} |A_M|^2.
\]

One can view \( A(t) \) as measuring the bending of \( M \) near infinity in a certain sense.

Our first estimate follows by extending a recent argument of Munteanu–Wang who obtained \([37]\) a sharp monotonicity formula for the quantity \( F(t) \) on a 3-manifold with non-negative scalar curvature by a clever application of Stern’s rearrangement of the Bochner formula \([54]\) (allowing them to leverage Gauss–Bonnet on the level sets \( \Sigma_t \)).

In the present situation, \( M \) actually has non-positive scalar curvature \( R_M = -|A_M|^2 \), so this comes in with a bad sign when applying their method. Moreover, our \textit{a priori} bound \( F(t) = O(t) \) causes complications with a crucial step in their argument. However, keeping track of this bad sign and developing a delicate regularization procedure to control this limit appropriately (cf. Remark 19), we find

\[
F(t) \leq O(t^2) + \frac{1}{4} t \int_0^t A(s) \, ds + \frac{1}{4} t^3 \int_1^t s^{-2} A(s) \, ds
\]

(see Proposition 18). To obtain an estimate for \( F(t) \) we thus need to estimate \( A(t) \) appropriately. We achieve this by considering the stability inequality \([1]\) with \( f = |\nabla u|^2 \varphi(u) \). Using Stern’s rearrangement of the Bochner formula and the co-area formula, we obtain

\[
\int_0^\infty \varphi(s)^2 A(s) \, ds \leq \frac{8\pi}{3} \int_0^\infty \varphi(s)^2 \, ds + \frac{4}{3} \int_0^\infty \varphi'(s)^2 F(s) \, ds,
\]

which yields

\[
t \int_0^t A(s) \, ds + t^3 \int_1^t s^{-2} A(s) \, ds \leq O(t^2) + \frac{4}{3} t^3 \int_1^t s^{-4} F(s) \, ds
\]

after a judicious choice of the test function (see Proposition 13 and Corollary 16).

Putting these inequalities together we obtain

\[
F(t) \leq O(t^2) + \frac{1}{3} t^3 \int_1^t s^{-4} F(s) \, ds.
\]

Because the constant \( \frac{1}{3} \) is \(< 1 \), it is possible to “absorb” the integral into the left-hand side, yielding \( F(t) = O(t^2) \). As explained above, this implies that \( M \) is flat.

We emphasize that both of the competing inequalities described above rely on Stern’s Bochner formula, and thus use the three-dimensionality of \( M \) in a strong way. In particular, Stern’s version of the Bochner formula as used in the stability inequality preserves more of the factor in front of \( A(s) \) than the “standard” Bochner formula would (basically, this is due to the use of Gauss–Bonnet). Extra loss at this step would make the constant \( \frac{1}{3} \) larger and could cause issues with the “absorption” step.

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1See also their followup work \([38]\) which appeared after the first version of this article was posted.

2This part of the argument is inspired by the work of Schoen–Yau \([18]\) who considered \( f = |\nabla u|^2 \varphi(u) \) and used the standard Bochner formula in \([1]\) to prove that \( M \) does not admit nonconstant finite energy harmonic functions.
1.2. Organization of the paper. In Section 2 we collect basic properties of the Green’s function on a stable minimal hypersurface. In Section 3 we combine Stern’s Bochner formula with the stability inequality and in Section 4 we extend the work of Munteanu–Wang and combine this with the previous section to prove that $F(t) = O(t^2)$. In Section 5 we use this to prove Theorems 1, 2, and 3. Appendix A recalls the one-ended (finite-ended) property of stable (finite index) minimal hypersurfaces. Appendix B recalls Yau’s Harnack and Michael–Simon Sobolev inequality in this setting. Appendix C recalls properties of the nodal/critical set of a harmonic function, and Appendix D proves a $L^3$ version of the Schoen–Simon–Yau estimates. Appendix E contains an extension of Theorem 1 to almost stable minimal immersions.

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2. The Green’s Function on a Stable Minimal Hypersurface

It is well-known (cf. [32, Theorem 10.1]) that when $n > 2$ a stable minimal hypersurface in $\mathbb{R}^{n+1}$ is non-parabolic (i.e., admits a positive Green’s function). We recall this fact below as well as establishing several useful facts about the behavior of the level sets of such a Green’s function.

For $n \geq 3$, recall that the Euclidean Green’s function on $\mathbb{R}^n$ (at a point $p$) is $C_n|x-p|^2-n$, for $C_n^{-1} = (n-2)\omega_{n-1}$ (so $C_3^{-1} = 4\pi$); here, $\omega_{n-1}$ denotes the $(n-1)$-volume of the unit sphere in $\mathbb{R}^n$.

Proposition 9. For $n > 2$, suppose that $M^n \to \mathbb{R}^{n+1}$ is a complete, connected, simply connected, two-sided, stable minimal immersion with uniformly bounded curvature $|A_M| \leq K$. Then, for $p \in M$, there exists $u \in C^\infty_{\text{loc}}(M \setminus \{p\})$ with the following properties:

(1) $u$ is harmonic, $\Delta u = 0$, on $M \setminus \{p\}$,
(2) $u > 0$ on $M \setminus \{p\}$ and $\inf u = 0$,
(3) $u(x) = C_n(1 + o(1))d_M(x,p)^{2-n}$, $|\nabla u|(x) = (n-2)C_n(1 + o(1))d_M(x,p)^{-n}$, as well as $\langle \nabla^2 u, \nabla^2 u \rangle = (n-1)(n-2)C_n(1 + o(1))d_M(x,p)^{-n}$ as $x \to p$,
(4) if $K$ is a compact set containing $p$, then $\int_{M \setminus K} |\nabla u|^2 < \infty$,
(5) $u(x) \to 0$ as $d_M(x,p) \to \infty$,
(6) for all $s \in (0, \infty)$, the set $\Omega_s := \{u \geq s\} \cup \{p\}$ is compact, and
(7) if $s \in (0, \infty)$ is a regular value of $u$ then $\partial \Omega_s = u^{-1}(s) := \Sigma_s$ is a closed connected hypersurface in $M$.

Proof. If $M$ is a flat $\mathbb{R}^n$, then we can take $u$ to be the standard Green’s function on $\mathbb{R}^n$. As such, we assume below that $M$ is not flat.

We construct a sequence of approximations to $u$ following [32, §2]. Choose an exhaustion $\Omega_1 \subset \Omega_2 \subset \cdots \subset M$ with $\Omega_i$ a pre-compact open set, $p \in \Omega_i$, and $\partial \Omega_i$ smooth. There exists $u_i \in C^\infty_{\text{loc}}(\bar{\Omega}_i \setminus \{p\})$ so that

(1') $\Delta u_i = 0$ on $\Omega_i \setminus \{p\}$,
(2') \( u_i > 0 \) on \( \Omega_i \) and \( u_i = 0 \) on \( \partial \Omega_i \), and 
(3') \( u_i(x) = c_n (1 + o(1)) d_M(x, p)^{2-n} \), \( \left| \nabla u_i \right|(x) = (n - 2) c_n (1 + o(1)) d_M(x, p)^{1-n} \), as well as \( \left( \nabla \left| \nabla u_i \right|, \frac{\nabla u_i}{\left| \nabla u_i \right|} \right) = (n - 1) (n - 2) c_n (1 + o(1)) d_M(x, p)^{-n} \) as \( x \to p \).

This is simply the statement of existence of a Green’s function on a compact manifold with boundary.

Note that \( u_i(x) \leq u_j(x) \) for \( i < j \) and \( x \in \Omega_i \setminus \{ p \} \). Indeed, by (2'), for any \( \delta > 0 \), it holds that \( u_i(x) \leq (1 + \delta) u_j(x) \) for \( x \in \partial B_\varepsilon(p) \) for all \( \varepsilon > 0 \) sufficiently small (depending on \( \delta \)). Since \( 0 = u_i(x) < u_j(x) \) for \( x \in \partial \Omega_i \), we thus find \( u_i \leq (1 + \delta) u_j \) on \( \Omega_i \setminus B_\varepsilon(p) \). Sending \( \delta, \varepsilon \to 0 \), the inequality \( u_i(x) \leq u_j(x) \) follows.

Set 

\[ \mu_i := \max_{x \in \partial \Omega_i} u_i(x). \]

The maximum principle implies that

\[ u_1 \leq u_i \leq u_1 + \mu_i \tag{2} \]
on \( \Omega_1 \). We have seen that \( \{\mu_i\}_{i \in \mathbb{N}} \) is increasing.

**Claim** (cf. [32, Theorem 2.3]). The sequence \( \{\mu_i\}_{i \in \mathbb{N}} \) is bounded above.

**Proof of the claim.** Assume that \( \mu_i \to \infty \). The Harnack inequality (cf. Proposition [24]) and interior estimates implies that up to passing to a subsequence, \( \mu_i^{-1} u_i \) converges in \( C^\infty_{\text{loc}}(M \setminus \{p\}) \) to some non-negative harmonic function \( u \in C^\infty_{\text{loc}}(M \setminus \{p\}) \).

Because we have assumed that \( \mu_i \to \infty \), then we find that \( 0 \leq u \leq 1 \) on \( \Omega_1 \). On the other hand, the maximum principle (and Dirichlet boundary conditions for \( u_i \)) implies that

\[ \max_{\Omega_i \setminus \Omega_1} u_i = \mu_i, \]

so \( u \leq 1 \) on \( M \setminus \Omega_1 \). The maximum principle then implies that \( u \equiv 1 \) on \( M \setminus \{p\} \).

Choose \( f_i \in C^{0,1}(M) \cap C^\infty(\Omega_i \setminus \Omega_1) \) so that

\[
\begin{aligned}
\Delta f_i &= 0 \quad \text{in } \Omega_i \setminus \Omega_1 \\
f_i &= 1 \quad \text{in } \Omega_1 \\
f_i &= 0 \quad \text{in } M \setminus \Omega_i.
\end{aligned}
\]

The maximum principle implies that \( \mu_i^{-1} u_i \leq f \) on \( \Omega_i \setminus \Omega_1 \), so we see that \( f_i \to 1 \) in \( C^{0,1}_{\text{loc}}(M) \cap C^\infty_{\text{loc}}(M \setminus \Omega_1) \). Taking \( f_i \) in the stability inequality [11] we find (cf. [32, Theorem 10.1])

\[
\int_M |A_M|^2 f_i^2 \leq \int_{\Omega_i \setminus \Omega_1} |\nabla f_i|^2 = -\int_{\partial \Omega_1} D_\nu f_i
\]

for \( \nu \) the outwards pointing unit normal to \( \Omega_1 \). Letting \( i \to \infty \), since \( f_i \) converges to 1, we find that

\[
\int_M |A_M|^2 = 0,
\]
a contradiction since we assumed that \( M \) was not flat. \(\square\)
Write $\mu_i \to \mu_\infty$. The claim, the Harnack inequality (cf. Proposition [21]), and interior estimates allow us to pass to a subsequence so that $u_i$ converges to a harmonic function $u$ in $C^\infty_{\text{loc}}(M \setminus \{p\})$. Note that $u > 0$ by the maximum principle.

By (2) we have

$$u_1 \leq u \leq u_1 + \mu_\infty.$$ 

After shifting $u$ so that $\inf u = 0$, we see that we have established (1)-(3). (The derivative estimates in (3) follow because $u - u_1$ is bounded near $p$ and thus extends across $p$.)

We now establish property (4). Consider the solution to the following problem

$$\begin{cases}
\Delta w_i = 0 & \text{in } \Omega_i \setminus \Omega_1 \\
w_i = 0 & \text{on } \partial \Omega_i \\
w_i = u & \text{on } \partial \Omega_1
\end{cases}$$

By the maximum principle (and since $u_i$ is increasing to $u$), $u_i \leq w_i \leq u$ on $\Omega_i \setminus \Omega_1$. Thus, we find that $w_i$ converges to $u$ in $C^\infty_{\text{loc}}(M \setminus \Omega_1)$. Furthermore, we have that

$$\int_{\Omega_i} |\nabla w_i|^2 \leq \int_{\Omega_1} |\nabla w_1|^2$$

for $i \in \mathbb{N}$. Passing this inequality to the limit, we have proven (4).

We now consider property (5) (cf. [40, Remark 2.4(d)]). We claim that for any (intrinsically) diverging sequence $\{x_i\}_{i \in \mathbb{N}} \subset M$ it holds that $u(x_i) \to 0$. Choose $\varphi \in C^\infty(M)$ so that $\varphi \equiv 0$ in $\Omega_2$ and $\varphi \equiv 1$ on $M \setminus \Omega_3$. Consider the function $\varphi w_i$ where $w_i$ is as in (3). Note that $\varphi w_i \in C^{0,1}_c(M)$. By the Michael–Simon–Sobolev inequality (cf. Proposition [25]) we have

$$\left( \int_M (\varphi w_i)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C \int_M \varphi^2|\nabla w_i|^2 + w_i^2|\nabla \varphi|^2.$$

Because $w_i$ has uniformly bounded Dirichlet energy and $\nabla \varphi$ is compactly supported we can let $i \to \infty$ (recalling that $w_i$ converges to $u$ in $C^\infty_{\text{loc}}(M \setminus \Omega_1)$ to find

$$\left( \int_M (\varphi u)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} < \infty.$$ 

In particular,

$$\lim_{i \to \infty} \left( \int_{M \setminus \Omega_i} u^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} = 0.$$ 

(4)

We now assume (for contradiction) that

$$\lim_{i \to \infty} \sup_{M \setminus \Omega_i} u = a \in (0, \infty).$$

If this holds, we can find a (intrinsically) diverging sequence $\{x_i\}_{i \in \mathbb{N}} \subset M$ with $u(x_i) \to a$ as $i \to \infty$. Passing to a subsequence we can assume that

$$d_M(x_i, \Omega_i) \to \infty$$

as $i \to \infty$. Because $M$ has uniformly bounded curvature, Schauder estimates yield bounds on all derivatives of curvature. Thus, we can pass $(M, g_M, x_i)$ to the limit
(\tilde{M}, \tilde{g}, \tilde{x}) in the pointed Cheeger–Gromov sense (where \( g_M \) is the induced metric on \( M \)). Note that by the Harnack inequality (cf. Proposition 24) and interior estimates we can also pass the harmonic function \( u \) to a pointed limit \( \tilde{u} \) (passing to a further subsequence). Note that \( \tilde{u} \) is harmonic, \( \tilde{u}(\tilde{x}) = a \) and \( \tilde{u} \leq a \). Thus, the maximum principle implies that \( \tilde{u} \equiv a \). On the other hand, (4) implies that \( \tilde{u} \equiv 0 \). This is a contradiction, completing the proof of property (5).

Properties (3) and (5) imply property (6).

We now consider property (7). For \( s \) a regular value, \( \Omega_s = \{u \geq s\} \) is a compact set with smooth boundary \( \Sigma_s = u^{-1}(s) \) a closed (regular) hypersurface.

**Claim.** For \( s \in (s_0, \infty) \) a regular value of \( u \), \( M \setminus \Omega_s \) has exactly one component and it is unbounded.

**Proof of the claim.** Assume that \( M \setminus \Omega_s \) has a bounded component \( \Gamma \). Note that \( 0 < u < s \) on \( \Gamma \) and \( u = s \) on \( \partial \Gamma \). Thus, \( u \) attains its minimum on the interior of \( \Gamma \). This is a contradiction since \( u \) is not constant. This shows that \( M \setminus \Omega_s \) is disconnected then \( M \) has at least two ends. This would contradict the one-ended result of Cao–Shen–Zhu (cf. Proposition 22). \( \Box \)

To finish the proof of (7), if \( \Sigma_s \) were disconnected, then by the claim we can construct a loop with non-trivial algebraic intersection with one of the components of \( \Sigma_s \) (concatenate a path connecting two of the components of \( \Sigma_s \) inside of \( \Omega_s \) with one outside of \( \Omega_s \)). This contradicts the assumed simple connectivity of \( M \). \( \Box \)

### 2.1. Integration over level sets of the Green’s function.

Given \( M^n \to \mathbb{R}^{n+1} \) a complete, non-compact, connected, simply connected two-sided stable minimal immersion with uniformly bounded curvature. For \( p \in M \) fixed, consider the Green’s function \( u \in C^\infty_{\text{loc}}(M \setminus \{p\}) \) constructed in Proposition 9. Set \( R \) the regular values of \( u \) and \( S \) the singular values of \( u \). Recall that we have defined \( \Sigma_s := u^{-1}(s) \). Below we will use this notation even when \( s \in S \) is a singular value.

**Lemma 10.** The set of regular values \( R \) is open and dense in \((0, \infty)\).

**Proof.** Proposition 9 implies that \( u : M \setminus \{p\} \to (0, \infty) \) is proper. This easily is seen to imply openness of \( R \). Density follows from Sard’s theorem as usual. \( \Box \)

For any \( s \in (0, \infty) \) we set

\[
\Sigma_s^* = \{x \in M : u(x) = s, |\nabla u|(x) > 0\}.
\]

Note that \( \Sigma_s^* \) is a smooth (possibly incomplete) hypersurface in \( M \). By Proposition 26 \( \dim_{\text{Haus}}(\Sigma_s \setminus \Sigma_s^*) \leq n - 2 \). In particular, we see that \( \mathcal{H}^{n-1}(\Sigma_s) < \infty \) and for any function \( f \in C^\infty_{\text{loc}}(M) \),

\[
\int f d\mathcal{H}^{n-1}[\Sigma_s] = \int_{\Sigma_s^*} f
\]

where the right hand side is taken with respect to the induced Riemannian volume form on \( \Sigma_s^* \).

**Lemma 11.** For \( f \in C^0_{\text{loc}}(M \setminus \{p\}) \), the function \( s \mapsto \int_{\Sigma_s^*} f \) is continuous.
Proof. Cover $\Sigma_s \setminus \Sigma_s^*$ by balls $B_{r_i}(x_i)$ with $\sum_i r_i^{n-2+\eta} < \varepsilon$ and $r_i \leq 1$. It is clear that
\[
\delta \mapsto \int_{\Sigma_s^+ \cup \cup_i B_{r_i}(x_i)} f
\]
is continuous at $\delta = 0$. Furthermore, by Proposition 26, we have
\[
\int_{\Sigma_s^+ \cap \cup_i B_{r_i}(x_i)} f \leq C \sum_i \mathcal{H}^{n-1}(B_{r_i}(x_i)) \leq C \sum_i r_i^{n-1} \leq C \varepsilon,
\]
where the constant $C$ is independent of $\delta$ small. Putting these facts together, the assertion follows. $\square$

We now consider the continuous functions
\[
F(s) := \int_{\Sigma_s^*} |\nabla u|^2 \quad (5)
\]
\[
A(s) := \int_{\Sigma_s^*} |A_M|^2, \quad (6)
\]
defined for $s \in (0, \infty)$.

**Lemma 12.** The function $F(s)$ is locally Lipschitz on $(0, \infty)$.

**Proof.** For a fixed compact subset $K \subset (0, \infty)$, consider $s < t$ regular values of $u$ with $s, t \in K$. Below we will allow the Lipschitz constant to depend on $K$.

Consider the region $\Omega_{s,t} = u^{-1}([s, t])$. Observe that $\Omega_{s,t}$ is a compact region with smooth boundary $\Sigma_s \cup \Sigma_t$. Write $\eta$ for the outwards pointing unit normal to $\Omega_{s,t}$ and note that $\eta = -\frac{\nabla u}{|\nabla u|}$ along $\Sigma_s$ and $\eta = \frac{\nabla u}{|\nabla u|}$. We have
\[
F(s) - F(t) = \int_{\partial \Omega_{s,t}} \langle |\nabla u| \nabla u, \eta \rangle
\]
\[
= \lim_{\delta \to 0} \int_{\partial \Omega_{s,t}} \left\langle (|\nabla u|^2 + \delta)^{\frac{1}{2}} \nabla u, \eta \right\rangle
\]
\[
= \lim_{\delta \to 0} \int_{\Omega_{s,t}} \text{div} \left( (|\nabla u|^2 + \delta)^{\frac{1}{2}} \nabla u \right)
\]
\[
= \lim_{\delta \to 0} \int_{\Omega_{s,t}} -\frac{1}{2} (|\nabla u|^2 + \delta)^{-\frac{1}{2}} \left\langle \nabla |\nabla u|^2, \nabla u \right\rangle.
\]

Note that
\[
(|\nabla u|^2 + \delta)^{-\frac{1}{2}} \left\langle \nabla |\nabla u|^2, \nabla u \right\rangle \leq |\nabla |\nabla u|^2| \leq 2|\nabla u| D^2 u|,
\]
using the Kato inequality. From this, we find
\[
|F(s) - F(t)| \leq \int_{\Omega_{s,t}} |\nabla u| D^2 u| \leq C \int_s^t \mathcal{H}^{n-1}(\Sigma_\tau) d\tau,
\]
using the co-area formula (where $C = C(M, g, u, K)$). Using Proposition 26 to bound the volume of $\Sigma_\tau$, the assertion follows (the Lipschitz estimate for $s, t$ possibly singular follows from the above estimate and continuity of $F(t)$ proven in Lemma 11). $\square$
3. Stern’s Bochner Formula and Stability

Recently, Stern has discovered [5] that one can combine the Bochner formula with the Schoen–Yau rearrangement [49] of the Gauss equation to relate the scalar curvature of a three-manifold to the behavior of a harmonic function on that manifold. In this section we consider this idea in the context of the stability operator for a stable minimal 3-dimensional hypersurface (cf. [15]).

We consider $M^3 \to \mathbb{R}^4$ a complete, connected, simply connected, two-sided, stable minimal immersion with uniformly bounded curvature. For $p \in M$ fixed, consider the Greens’ function $u \in C^\infty_{\text{loc}}(M \setminus \{p\})$ constructed in Proposition 11. Recall the definition of $F(s), A(s)$ in (4) and (5). By Lemma 11 $F(s), A(s)$ are continuous on $(0, \infty)$.

**Proposition 13.** For any function $\varphi \in C^0_{\text{loc}}((0, \infty))$ it holds

$$
\int_0^\infty \varphi(s)^2 A(s) \, ds \leq \frac{8\pi}{3} \int_0^\infty \varphi(s)^2 \, ds + \frac{4}{3} \int_0^\infty \varphi'(s)^2 \, ds.
$$

**Proof.** Consider the following regularization of the square root of the gradient of $u$

$$
e_\delta = (|\nabla u|^2 + \delta) \frac{1}{2}.
$$

Note that $e_\delta \in C^\infty_{\text{loc}}(M \setminus \{p\})$ and

$$
\nabla e_\delta = \frac{1}{2} e_\delta^{-3} \nabla |\nabla u|^2,
$$

as well as

$$
\Delta e_\delta = \frac{1}{4} e_\delta^{-3} \Delta |\nabla u|^2 - \frac{3}{16} e_\delta^{-7} |\nabla |\nabla u|^2|^2.
$$

Applying the Bochner formula, we find

$$
\Delta e_\delta = \frac{1}{4} e_\delta^{-3} (|\nabla u|^2 + \text{Ric}_M(\nabla u, \nabla u)) - \frac{3}{16} e_\delta^{-7} |\nabla |\nabla u|^2|^2
$$

$$
= \frac{1}{4} e_\delta^{-3} (|\nabla u|^2 - \frac{3}{8} e_\delta^{-4} |\nabla |\nabla u|^2|^2 + \text{Ric}_M(\nabla u, \nabla u)).
$$

We now consider $\psi \in C^\infty_{\text{loc}}(M \setminus \{p\})$ and then take $f = e_\delta \psi$ in the stability inequality (1) yielding

$$
\int_M |A_M|^2 e_\delta^2 \psi^2 \leq \int_M |\nabla e_\delta|^2 f^2 + \frac{1}{2} \langle \nabla e_\delta^2, \nabla \psi^2 \rangle + e_\delta^2 |\nabla \psi|^2
$$

$$
= \int_M |\nabla e_\delta|^2 f^2 - \frac{1}{2} \psi^2 \Delta e_\delta^2 + e_\delta^2 |\nabla \psi|^2
$$

$$
= \int_M -e_\delta \psi^2 \Delta e_\delta + e_\delta^2 |\nabla \psi|^2
$$

$$
= \int_M -\frac{1}{4} e_\delta^{-2} (|D^2 u|^2 - \frac{3}{8} e_\delta^{-4} |\nabla |\nabla u|^2|^2 + \text{Ric}_M(\nabla u, \nabla u)) \psi^2 + e_\delta^2 |\nabla \psi|^2.
$$

Rearranging this yields

$$
\int_M (|A_M|^2 e_\delta^2 + \frac{1}{2} e_\delta^{-2} (|D^2 u|^2 - \frac{3}{8} e_\delta^{-4} |\nabla |\nabla u|^2|^2 + \text{Ric}_M(\nabla u, \nabla u)) \psi^2 \leq \int_M e_\delta^2 |\nabla \psi|^2.
$$

Note that $\text{Ric}_M \geq -|A_M|^2$ by the Gauss equations. Furthermore, the improved Kato inequality yields

$$
\frac{3}{8} |\nabla |\nabla u|^2|^2 \leq |\nabla u|^2 |D^2 u|^2 \leq e_\delta^2 |D^2 u|^2.
$$
Thus, we find that
\[ |A_M|^2 e_\delta^2 + \frac{1}{2} e_\delta^{-2} (|D^2 u|^2 - \frac{3}{8} e_\delta^{-4} |\nabla|\nabla u|^2|^2 + \text{Ric}_M (\nabla u, \nabla u)) \geq 0. \]

Let \( \mathfrak{B} \) be an open subset of \((0, \infty)\) containing all singular values of \( u \), and \( \mathfrak{A} = (0, \infty) \setminus \mathfrak{B} \).

We thus find
\[
\int_{\mathfrak{A}} \left( |A_M|^2 e_\delta^2 + \frac{1}{2} e_\delta^{-2} (|D^2 u|^2 - \frac{3}{8} e_\delta^{-4} |\nabla|\nabla u|^2|^2 + \text{Ric}_M (\nabla u, \nabla u)) \right) \psi^2 \leq \int_{\mathfrak{A}} e_\delta^2 |\nabla \psi|^2 + \int_{\mathfrak{B}} e_\delta^2 |\nabla \psi|^2.
\]

We can send \( \delta \to 0 \) and apply Fatou’s lemma to find
\[
\int_{u^{-1}(\mathfrak{A})} \left( |A_M|^2 |\nabla u| + \frac{1}{2} |\nabla u|^{-1} (|D^2 u|^2 - \frac{3}{2} |\nabla|\nabla u|^2 + \text{Ric}_M (\nabla u, \nabla u)) \right) \psi^2 \leq \int_{u^{-1}(\mathfrak{A})} |\nabla u||\nabla \psi|^2 + \int_{u^{-1}(\mathfrak{B})} |\nabla u||\nabla \psi|^2.
\]

By the co-area formula, we find
\[
\int_{\mathfrak{A}} \left( \int_{\Sigma_s} \left( |A_M|^2 e_\delta^2 + \frac{1}{2} e_\delta^{-2} (|D^2 u|^2 - \frac{3}{8} e_\delta^{-4} |\nabla|\nabla u|^2|^2 + \text{Ric}_M (\nabla u, \nabla u)) \right) \psi^2 \right) ds \leq \int_{\mathfrak{A}} \left( \int_{\Sigma_s} |\nabla \psi|^2 \right) ds + \int_{u^{-1}(\mathfrak{B})} |\nabla u||\nabla \psi|^2.
\]

For \( \varphi \in C^\infty_c((0, \infty)) \), take \( \psi = \varphi(u) \) to yield
\[
\int_{\mathfrak{A}} \varphi(s)^2 \left( \int_{\Sigma_s} \left( |A_M|^2 e_\delta^2 + \frac{1}{2} e_\delta^{-2} (|D^2 u|^2 - \frac{3}{8} e_\delta^{-4} |\nabla|\nabla u|^2|^2 + \text{Ric}_M (\nabla u, \nabla u)) \right) \right) ds \leq \int_{\mathfrak{A}} \varphi'(s)^2 \left( \int_{\Sigma_s} |\nabla u|^2 \right) ds + \int_{u^{-1}(\mathfrak{B})} \varphi'(u)^2 |\nabla u|^3
\]

If \( s \in \mathfrak{A} \), note that \( \nu = \frac{\nabla u}{|\nabla u|} \) is a unit normal to \( \Sigma_s \). We now follow the ideas used in [54, Theorem 1]. The traced Gauss equations yield
\[
2 \text{Ric}_M (\nu, \nu) = R_M - 2 K_{\Sigma_s} - |A_{\Sigma_s}|^2 + H_{\Sigma_s}^2.
\]

Similarly, using the Gauss equations for \( M^3 \to \mathbb{R}^4 \) we have \( R_M = -|A_M|^2 \). The scalar second fundamental form of \( \Sigma_s \) satisfies
\[
A_{\Sigma_s} = \left| \nabla u \right|^{-1} D^2 u |\Sigma_s|
\]
so
\[
|\nabla u|^2 |A_{\Sigma_s}|^2 = |D^2 u|^2 - 2 |\nabla|\nabla u||^2 + D^2 u(\nu, \nu)^2
\]
and (because \( u \) is harmonic)
\[
|\nabla u|^2 H_{\Sigma_s}^2 = D^2 u(\nu, \nu)^2.
\]

Thus,
\[
\text{Ric}_M (\nabla u, \nabla u) = -\frac{1}{2} |\nabla u|^2 |A_M|^2 - |\nabla u|^2 K_{\Sigma_s} - \frac{1}{2} |D^2 u|^2 + |\nabla|\nabla u||^2.
\]
along $\Sigma_s$. Thus,
\[
\int_\mathcal{A} \varphi(s)^2 \left( \int_{\Sigma_s} \frac{3}{4} |A_M|^2 + \frac{1}{4} |\nabla u|^{-1} (|D^2 u|^2 - |\nabla \nabla u|^2) \right) ds \\
\leq \int_\mathcal{A} \varphi(s)^2 \left( \int_{\Sigma_s} \frac{1}{2} K_{\Sigma_s} \right) ds + \int_\mathcal{A} \varphi'(s)^2 \left( \int_{\Sigma_s} |\nabla u|^2 \right) ds + \int_{u^{-1}(\mathcal{B})} \varphi'(u)^2 |\nabla u|^3 ds
\]

By (6) in Proposition 9 for $s \in \mathcal{R}$, $\Sigma_s$ is connected, so $\int_{\Sigma_s} K_{\Sigma_s} \leq 4\pi$ (by Gauss–Bonnet). Using this, the Kato inequality, and the definition of $F(s)$, $\mathcal{A}(s)$, we find
\[
\int_\mathcal{A} \varphi(s)^2 \mathcal{A}(s) ds \leq \frac{8\pi}{3} \int_\mathcal{A} \varphi(s)^2 ds + \frac{4}{3} \int_\mathcal{A} \varphi'(s)^2 F(s) ds + \frac{4}{3} \int_{u^{-1}(\mathcal{B})} \varphi'(u)^2 |\nabla u|^3 ds
\]

Since $\varphi'(u)^2 |\nabla u|^3$ is uniformly bounded, we can send $|\mathcal{B} \cap \text{supp } \varphi| \to 0$ and conclude
\[
\int_0^\infty \varphi(s)^2 \mathcal{A}(s) ds \leq \frac{8\pi}{3} \int_0^\infty \varphi(s)^2 ds + \frac{4}{3} \int_0^\infty \varphi'(s)^2 F(s) ds.
\]

A standard approximation argument for $\varphi$ completes the proof. \ \qed

**Remark 14.** By using the improved Kato inequality it is easy to generalize the previous argument to prove that
\[
\int_0^\infty \varphi(s)^2 \left( \int_{\Sigma_s} \frac{3}{4} |A_M|^2 + \frac{1}{4} |\nabla u|^{-1} |A_{\Sigma_s}|^2 - K_{\Sigma_s} \right) ds \leq \int_0^\infty \varphi'(s)^2 F(s) ds,
\]

where $A_{\Sigma_s}$ is the second fundamental form of $\Sigma_s$ in $M$. However, we will not need this expression in the sequel.

We thus see that the behavior of $F(s)$ near 0 and $\infty$ determines the potential test functions $\varphi$ that can be used in the inequality from Proposition 13. The behavior of $F(t)$ as $t \to \infty$ is independent of the geometry of $M$ ($F(t)$ behaves like the Green’s function on $\mathbb{R}^3$). However, the behavior as $t \to 0$ is of crucial importance to our argument. We begin with an *a priori* bound that will be improved in the sequel.

**Lemma 15.** It holds that $F(t) = O(t)$ as $t \to 0$ and $F(t) = (1 + o(1))4\pi t^2$ as $t \to \infty$.

**Proof.** The bound as $t \to \infty$ follows in a straightforward manner from the asymptotics in (3) in Proposition 9.

For the other assertion, by integrating $\Delta u = 0$ over $\{t \leq u \leq \tau\}$ (where $t, \tau \in \mathcal{R}$) we find
\[
\int_{\Sigma_t} |\nabla u| = \int_{\Sigma_\tau} |\nabla u|.
\]

By Lemma 11 $t \mapsto \int_{\Sigma_t} |\nabla u|$ is constant. Because $M$ has uniformly bounded Ricci curvature, we can apply the Harnack inequality (Proposition 21) to obtain
\[
F(t) = \int_{\Sigma_t} |\nabla u|^2 \leq Ct \int_{\Sigma_t} |\nabla u|.
\]

(for $t$ bounded away from $\infty$). This proves the assertion. \ \qed
Corollary 16. We have
\[
\limsup_{\ell \to 0} \int_{t}^{\ell} A(s) \, ds + t^2 \int_{t}^{1} s^{-2} A(s) \, ds \leq O(t) + \frac{4}{3} \int_{t}^{1} t^2 s^{-4} F(s) \, ds
\]
as \(t \to 0\).

Proof. For \(\varepsilon \in (0, 1)\) and \(\ell < t\), we consider the following function
\[
\varphi_\varepsilon(s) = \begin{cases} 
0 & s \in (0, \varepsilon \ell) \\
1 - \frac{\log s - \log \ell}{\log \varepsilon} & s \in [\varepsilon \ell, \ell) \\
1 & s \in [\ell, t) \\
t s^{-1} & s \in [t, 1) \\
t(2 - s) & s \in [1, 2) \\
0 & s \in (2, \infty).
\end{cases}
\]
We have that
\[
\int_{0}^{1} \varphi_\varepsilon(s)^2 \, ds = O(t).
\]
Using \(F(t) = O(t)\) as \(t \to 0\) from Lemma 15, we can further estimate
\[
\int_{0}^{\ell} \varphi_\varepsilon'(s)^2 F(s) \, ds = O(|\log \varepsilon|^{-2}) \int_{\varepsilon \ell}^{\ell} s^{-1} \, ds = O(|\log \varepsilon|^{-1}).
\]
As such, taking \(\varphi_\varepsilon\) in Proposition 13 and sending \(\varepsilon \to 0\) we find
\[
\int_{t}^{\ell} A(s) \, ds + t^2 \int_{t}^{1} s^{-2} A(s) \, ds \leq O(t) + \frac{4}{3} \int_{t}^{1} t^2 s^{-4} F(s) \, ds,
\]
where the \(O(t)\) is bounded independently of \(\ell > 0\). This completes the proof. \(\square\)

4. An extension of Munteanu–Wang’s monotonicity formula

Munteanu–Wang have recently established [37] a sharp monotonicity formula for the quantity \(F(s)\) defined via a Green’s function on a non-parabolic three-manifold with non-negative scalar curvature. In this section we refine this estimate in two ways to apply in the present situation. First of all, the minimal hypersurface does not have non-negative scalar curvature so we must keep track of the resulting defect term. Second, we have to regularize their argument to account for our relatively weak \(a \text{ priori}\) bounds for \(F(s)\) as \(s \to 0\).

We continue to consider \(M^3 \to \mathbb{R}^4\) a complete, connected, simply connected, two-sided stable minimal immersion with uniformly bounded curvature. For \(p \in M\) fixed, consider the Greens’ function \(u \in C^\infty_{\text{loc}}(M \setminus \{p\})\) constructed in Proposition 9. Recall the definition of \(F(s), A(s)\) in [5] and [9]. By Lemma 14 \(F(s), A(s)\) are continuous on \((0, \infty)\). Furthermore, by Lemma 12 \(F(s)\) is locally absolutely continuous. Finally, if \(R\) is the open and dense (by Lemma 10) set of regular values of \(u\), we see that \(F(s)\) is differentiable at all \(s \in R\).

We consider \(\alpha \mapsto \lambda(\alpha)\) where
\[
\lambda(\alpha) := \alpha + 1 - \sqrt{(\alpha + 1)(1 - \frac{1}{3}\alpha)}.
\]
Note that $\lambda(\alpha) \in \mathbb{R}$ for $\alpha \in [-1, 3]$.

**Lemma 17.** There is $\alpha_0 \in (1, 2)$ so that if $\alpha \in (\alpha_0, 2)$ then $\alpha - \frac{3}{2} \lambda(\alpha) + 1 > 0$.

**Proof.** Taylor expanding around $\alpha = 2$ yields

$$\alpha - \frac{3}{2} \lambda(\alpha) + 1 = -(\alpha - 2) + O((\alpha - 2)^2).$$

This proves the assertion. \qed

**Proposition 18 (cf. [37, Theorem 3.1]).** There is $C > 0$ so that for $t \in (0, 1)$, it holds that

$$F(t) \leq C t^3 + 4 \pi t^2 + \frac{1}{4} t \liminf_{\ell \searrow 0} \int_{\ell} A(s) \, ds + \frac{1}{4} t^3 \int_{t}^{1} s^{-2} A(s) \, ds.$$  

**Proof.** We begin by considering $t \in \mathcal{R}$. Let $\nu = \frac{\nabla u}{|\nabla u|}$ (note that $\nu$ is inwards pointing for the set $\Omega_s = \{ u \geq s \}$). The family $t \mapsto \Sigma_t$ has normal velocity $|\nabla u|^{-1} \nu$. Furthermore, the mean curvature of $\Sigma_s$ satisfies

$$H = -|\nabla u|^{-1} \langle \nabla |\nabla u|, \nu \rangle.$$  

With this convention, we have

$$F'(t) = \int_{\Sigma_t} |\nabla u|^{-1} \langle \nabla |\nabla u|^2, \nu \rangle + |\nabla u|^{-1} H |\nabla u|^2 = \int_{\Sigma_t} \langle \nabla |\nabla u|, \nu \rangle \quad (8)$$

Fix $\alpha \in (\alpha_0, 2)$ as in Lemma 17 and write $\lambda = \lambda(\alpha)$ (defined in (7)). Note that

$$t^{-\alpha} F'(t) + \alpha t^{-\alpha - 1} F(t) = \int_{\Sigma_t} u^{-\alpha} \langle \nabla |\nabla u|, \nu \rangle + \alpha u^{-\alpha - 1} |\nabla u|^2$$

$$= \int_{\Sigma_t} u^{-\alpha} \langle \nabla |\nabla u|, \nu \rangle - |\nabla u| \langle \nabla u^{-\alpha}, \nu \rangle.$$  

Set

$$j_\delta = (|\nabla u|^2 + \delta)^{\frac{1}{2}}$$

so that

$$t^{-\alpha} F'(t) + \alpha t^{-\alpha - 1} F(t) = \lim_{\delta \searrow 0} \int_{\Sigma_t} u^{-\alpha} \langle \nabla j_\delta, \nu \rangle - j_\delta \langle \nabla u^{-\alpha}, \nu \rangle.$$  

Now, if $0 < t < \tau \leq 1$, $t, \tau \in \mathcal{R}$, Green’s second identity on $\Omega_{t, \tau} := \{ t \leq u \leq \tau \}$ thus yields

$$\int_{\Sigma_t} u^{-\alpha} \langle \nabla j_\delta, \nu \rangle - j_\delta \langle \nabla u^{-\alpha}, \nu \rangle = \int_{\Sigma_{\tau}} u^{-\alpha} \langle \nabla j_\delta, \nu \rangle - j_\delta \langle \nabla u^{-\alpha}, \nu \rangle$$

$$= - \int_{\Omega_{t, \tau}} u^{-\alpha} \Delta j_\delta - j_\delta \Delta u^{-\alpha}.$$  

(Recall that $\nu$ is inwards pointing for $\Omega_s$.) Because $u$ is harmonic

$$\Delta u^{-\alpha} = \alpha(\alpha + 1) u^{-\alpha - 2} |\nabla u|^2.$$  

Furthermore, the Bochner formula yields

$$\Delta j_\delta = j_\delta^{-1} (|D^2 u|^2 - \frac{1}{4} j_\delta^{-2} |\nabla |\nabla u|^2|^2) + j_\delta^{-1} \text{Ric}_M(\nabla u, \nabla u).$$
Thus, we find
\[
\begin{align*}
\int_{\Sigma_t} u^{-\alpha} \langle \nabla j_\delta, \nu \rangle - j_\delta \langle \nabla u^{-\alpha}, \nu \rangle - \int_{\Sigma_r} u^{-\alpha} \langle \nabla j_\delta, \nu \rangle - j_\delta \langle \nabla u^{-\alpha}, \nu \rangle \\
= - \int_{\Omega_{t,r}} u^{-\alpha} j_\delta^{-1} (|D^2 u|^2 - \frac{1}{4} j_\delta^{-1} |\nabla |\nabla u|^2|^2) + u^{-\alpha} j_\delta^{-1} \text{Ric}_M(\nabla u, \nabla u) \\
+ \int_{\Omega_{t,r}} \alpha (\alpha + 1) u^{-\alpha - 2} j_\delta |\nabla u|^2.
\end{align*}
\]

Note that the Kato inequality yields
\[
|D^2 u|^2 - \frac{1}{4} j_\delta^{-1} |\nabla |\nabla u|^2|^2 \geq 0.
\]

Let $\mathcal{B}$ be an open subset of $(0, \infty)$ containing all singular values of $u$, and $\mathcal{A} = (0, \infty) \setminus \mathcal{B}$. We thus find
\[
\begin{align*}
\int_{\Sigma_t} u^{-\alpha} \langle \nabla j_\delta, \nu \rangle - j_\delta \langle \nabla u^{-\alpha}, \nu \rangle - \int_{\Sigma_r} u^{-\alpha} \langle \nabla j_\delta, \nu \rangle - j_\delta \langle \nabla u^{-\alpha}, \nu \rangle \\
\leq - \int_{(t, r) \cap \mathcal{A}} \left( \int_{\Sigma_s} u^{-\alpha} |\nabla u|^{-1} j_\delta^{-1} (|D^2 u|^2 - \frac{1}{4} j_\delta^{-1} |\nabla |\nabla u|^2|^2) \right) ds \\
- \int_{(t, r) \cap \mathcal{A}} \left( \int_{\Sigma_s} u^{-\alpha} |\nabla u|^{-1} j_\delta^{-1} \text{Ric}_M(\nabla u, \nabla u) \right) ds \\
+ \int_{(t, r) \cap \mathcal{A}} \left( \int_{\Sigma_s} \alpha (\alpha + 1) u^{-\alpha - 2} j_\delta |\nabla u|^2 \right) ds \\
+ \int_{\Omega_{t,r} \cap u^{-1}(\mathcal{B})} \alpha (\alpha + 1) u^{-\alpha - 2} j_\delta |\nabla u|^2 - u^{-\alpha} j_\delta^{-1} \text{Ric}_M(\nabla u, \nabla u)
\end{align*}
\]

Because $\alpha (\alpha + 1) u^{-\alpha - 2} j_\delta |\nabla u|^2 - u^{-\alpha} j_\delta^{-1} \text{Ric}_M(\nabla u, \nabla u)$ is uniformly bounded in $L^\infty(\Omega_{t,r})$ as $\delta \to 0$, we can send $\delta \to 0$ and then $|\mathcal{B}| \to 0$ to find
\[
(t^{-\alpha} F'(t) + \alpha t^{-\alpha - 1} F(t)) - (\tau^{-\alpha} F'(\tau) + \alpha \tau^{-\alpha - 1} F(\tau)) \\
\leq - \int_{t}^{\tau} \left( \int_{\Sigma_s} u^{-\alpha} |\nabla u|^{-2} (|D^2 u|^2 - |\nabla |\nabla u|^2|^2) \right) ds \\
- \int_{t}^{\tau} \left( \int_{\Sigma_s} u^{-\alpha} |\nabla u|^{-2} \text{Ric}_M(\nabla u, \nabla u) \right) ds \\
+ \int_{t}^{\tau} \left( \int_{\Sigma_s} \alpha (\alpha + 1) u^{-\alpha - 2} |\nabla u|^2 \right) ds.
\]

Using Stern’s rearrangement \cite{Stern} of the Bochner terms as in the proof of Proposition \cite{Lee} we can write (along $\Sigma_s$, $s \in \mathcal{R}$)
\[
|\nabla u|^{-2} \text{Ric}_M(\nabla u, \nabla u) = -\frac{1}{2} |A_M|^2 - K_{\Sigma_s} + |\nabla u|^{-2} |\nabla |\nabla u|^2|^2 - \frac{1}{2} |\nabla u|^{-2} |D^2 u|^2
\]
Hölder inequalities on $(8)$, we find

$$
( t^{-\alpha}F'''(t) + \alpha t^{-\alpha-1}F(t)) - (\tau^{-\alpha}F'(\tau) + \alpha\tau^{-\alpha-1}F(\tau))
$$

$$
\leq \frac{1}{2} \int_{t}^{\tau} s^{-\alpha} \left( \int_{\Sigma_s} |A_M|^2 \right) + \int_{t}^{\tau} s^{-\alpha} \left( \int_{\Sigma_s} K_{\Sigma_s} \right) \, ds
$$

$$
- \frac{1}{2} \int_{t}^{\tau} s^{-\alpha} \left( \int_{\Sigma_s} |\nabla u|^2 |D^2 u|^2 \right) \, ds + \int_{t}^{\tau} \alpha(\alpha + 1)s^{-\alpha-2}F(s) \, ds.
$$

By Gauss–Bonnet and Proposition 9,

$$
\int_{t}^{\tau} s^{-\alpha} \left( \int_{\Sigma_s} K_{\Sigma_s} \right) \, ds \leq \frac{1}{\alpha - 1} 4\pi(t^{1-\alpha} - \tau^{1-\alpha}).
$$

Furthermore, using the improved Kato inequality, as well as the Cauchy–Schwarz and Hölder inequalities on $(8)$, we find

$$
\int_{\Sigma_s} |\nabla u|^2 |D^2 u|^2 \geq \int_{\Sigma_s} \frac{3}{2} |\nabla u|^2 |\nabla \nabla u|^2
$$

$$
\geq \int_{\Sigma_s} \frac{3}{2} |\nabla u|^2 (\nabla |\nabla u|, \nu)^2
$$

$$
\geq \frac{3}{2} F(s)^{-1} F'(s)^2
$$

for $s \in \mathcal{R}$. Putting this together we find

$$
( t^{-\alpha}F'(t) + \alpha t^{-\alpha-1}F(t)) - (\tau^{-\alpha}F'(\tau) + \alpha\tau^{-\alpha-1}F(\tau))
$$

$$
\leq \frac{1}{2} \int_{t}^{\tau} s^{-\alpha} A(s) \, ds + \frac{1}{\alpha - 1} 4\pi(t^{1-\alpha} - \tau^{1-\alpha})
$$

$$
+ \int_{t}^{\tau} (-\frac{3}{4} s^{-\alpha} F(s)^{-1} F'(s)^2 + \alpha(\alpha + 1)s^{-\alpha-2}F(s)) \, ds
$$

Cauchy–Schwarz yields

$$
2s^{-\alpha}F'(s)F(s) \leq \lambda^{-1}F'(s)^2 + \lambda s^{-2}F(s)^2,
$$

(where $\lambda = \lambda(\alpha)$ as defined in (7)). Rearranging, we find

$$
-\frac{3}{4} s^{-\alpha} F(s)^{-1} F'(s)^2 \leq -\frac{3}{2} \lambda s^{-\alpha-1} F'(s)^2 + \frac{3}{4} \lambda^2 s^{-\alpha-2} F(s).
$$

Using this above we find

$$
( t^{-\alpha}F'(t) + \alpha t^{-\alpha-1}F(t)) - (\tau^{-\alpha}F'(\tau) + \alpha\tau^{-\alpha-1}F(\tau))
$$

$$
\leq \frac{1}{2} \int_{t}^{\tau} s^{-\alpha} A(s) \, ds + \frac{1}{\alpha - 1} 4\pi(t^{1-\alpha} - \tau^{1-\alpha})
$$

$$
+ \int_{t}^{\tau} (-\frac{3}{2} \lambda s^{-\alpha-1} F'(s)^2 + (\frac{3}{4} \lambda^2 + \alpha(\alpha + 1))s^{-\alpha-2}F(s)) \, ds
$$

$$
= \frac{1}{2} \int_{t}^{\tau} s^{-\alpha} A(s) \, ds + \frac{1}{\alpha - 1} 4\pi(t^{1-\alpha} - \tau^{1-\alpha})
$$

$$
+ \int_{t}^{\tau} (\frac{3}{4} \lambda^2 - \frac{3}{2} \lambda(\alpha + 1) + \alpha(\alpha + 1))s^{-\alpha-2}F(s) \, ds
$$

$$
- \frac{3}{2} \lambda^{-\alpha-1} F(\tau) + \frac{3}{2} \lambda^{-\alpha-1} F(t),
$$

so (also using that $u$ is constant along its level sets)
where we integrated by parts in the last step. Observe that
\[ \frac{3}{4} \lambda^2 - \frac{3}{2} \lambda (\alpha + 1) + \alpha (\alpha + 1) = 0 \]
by (7), so we can rearrange this to read
\[
(t^{-\alpha} F'(t) + (\alpha - \frac{3}{2} \lambda) t^{-\alpha - 1} F(t)) - (\tau^{-\alpha} F'(\tau) + (\alpha - \frac{3}{2} \lambda) \tau^{-\alpha - 1} F(\tau))
\leq \frac{1}{2} \int_{\ell}^{\tau} s^{-\alpha} A(s) \, ds + \frac{1}{\alpha - 1} 4\pi (t^{1-\alpha} - \tau^{1-\alpha})
\leq \frac{1}{2} \int_{\ell}^{\tau} s^{-\alpha} A(s) \, ds + \frac{1}{\alpha - 1} 4\pi t^{1-\alpha}.
\]
We thus find
\[
(t^{-\alpha} F(t))' \leq C t^{2\alpha - \frac{3}{2} \lambda} + \frac{1}{\alpha - 1} 4\pi t^{2\alpha - \frac{3}{2} \lambda + 1} + \frac{1}{2} t^{2\alpha - \frac{3}{2} \lambda} \int_{\ell}^{\tau} s^{-\alpha} A(s) \, ds
\]
where \( C = C(\tau) \) is bounded uniformly for \( \alpha \in (\alpha_0, 2) \). By Lemma \[12\] we can integrate this on \((\ell, t)\) for \( \ell, t \in \mathcal{R}, \ell < t < \tau \), yielding
\[
t^{\alpha - \frac{3}{2} \lambda} F(t) \leq \ell^{\alpha - \frac{3}{2} \lambda} F(\ell) + \frac{C}{2\alpha - \frac{3}{2} \lambda + 1} t^{2\alpha - \frac{3}{2} \lambda + 1} + \frac{1}{(\alpha - 1)(\alpha - \frac{3}{2} \lambda + 2)} 4\pi t^{2\alpha - \frac{3}{2} \lambda + 2}
\]
\[
+ \frac{1}{2} \int_{\ell}^{t} \int_{\sigma}^{\tau} \sigma^{2\alpha - \frac{3}{2} \lambda} s^{-\alpha} A(s) \, ds \, d\sigma
\]
(where we have dropped several negative terms evaluated at \( \ell \), cf. Lemma \[17\]. We apply Fubini’s theorem to write
\[
\int_{\ell}^{t} \int_{\sigma}^{\tau} \sigma^{2\alpha - \frac{3}{2} \lambda} s^{-\alpha} A(s) \, ds \, d\sigma
\]
\[
= \int_{\ell}^{t} \int_{\ell}^{s} \sigma^{2\alpha - \frac{3}{2} \lambda} s^{-\alpha} A(s) \, ds \, d\sigma + \int_{t}^{\tau} \int_{\ell}^{t} \sigma^{2\alpha - \frac{3}{2} \lambda} s^{-\alpha} A(s) \, ds \, d\sigma
\]
\[
\leq \frac{1}{2\alpha - \frac{3}{2} \lambda + 1} \int_{\ell}^{t} s^{\alpha - \frac{3}{2} \lambda + 1} A(s) \, ds + \frac{1}{2\alpha - \frac{3}{2} \lambda + 1} t^{2\alpha - \frac{3}{2} \lambda + 1} \int_{t}^{\tau} s^{-\alpha} A(s) \, ds.
\]
By Lemma \[17\] \( s^{\alpha - \frac{3}{2} \lambda + 1} \leq 1 \) for \( s \in (0, 1) \). Because \( t < 1 \) we can thus estimate
\[
\int_{\ell}^{t} \int_{\sigma}^{\tau} \sigma^{2\alpha - \frac{3}{2} \lambda} s^{-\alpha} A(s) \, ds \, d\sigma
\]
\[
\leq \frac{1}{2\alpha - \frac{3}{2} \lambda + 1} \int_{\ell}^{t} A(s) \, ds + \frac{1}{2\alpha - \frac{3}{2} \lambda + 1} t^{2\alpha - \frac{3}{2} \lambda + 1} \int_{t}^{\tau} s^{-\alpha} A(s) \, ds.
\]
On the other hand, by Lemmas \[15\] and \[17\] we have that
\[
\ell^{\alpha - \frac{3}{2} \lambda} F(\ell) = O(\ell^{\alpha - \frac{3}{2} \lambda + 1}) = o(1)
\]
as $\ell \to 0$. Thus, we can pass to the limit as $\ell \searrow 0$ to obtain
\[
\ell^{\alpha - \frac{3}{2} \lambda} F(t) \leq C \ell^{2\alpha \lambda - \frac{3}{2}} + \frac{1}{(\alpha - 1)(\alpha - \frac{3}{2} \lambda + 2)} 4\pi t^{\alpha - \frac{3}{2} \lambda + 2} \lim \inf_{\ell \searrow 0} \int_{\ell}^{t} A(s) \, ds \\
+ \frac{1}{2(2\alpha - \frac{3}{2} \lambda + 1)} t^{2\alpha - \frac{3}{2} \lambda + 1} \int_t^r s^{-\alpha} A(s) \, ds.
\]

Because $\alpha \mapsto \lambda(\alpha)$ is continuous at $\alpha = 2$ and $\lambda(2) = 2$ we can then send $\alpha \nearrow 2$ to find
\[
t^{-1} F(t) \leq Ct^2 + 4\pi t + \frac{1}{4} \lim \inf_{\ell \searrow 0} \int_{\ell}^{t} A(s) \, ds + \frac{1}{4} t^2 \int_t^r s^{-2} A(s) \, ds
\]
Because $\tau \leq 1$, this yields the assertion. \(\square\)

**Remark 19.** In the first version of this article, we suggested that the method of proof of Proposition 18 should be capable of weakening the hypothesis $\lim \inf_{x \to \infty} \text{Ric} \geq 0$ in [37, Corollary 3.2] to $\lim \inf_{x \to \infty} \text{Ric} > -\infty$. This has recently been carried out in a more general context in [10].

Finally, we are able to obtain the following sharp decay estimate.

**Corollary 20.** We have $F(t) = O(t^2)$ as $t \to 0$.

**Proof.** Combining Proposition 18 with Corollary 19 we obtain
\[
F(t) \leq O(t^2) + \frac{1}{3} \int_1^t s^{-3} F(s) \, ds
\]
as $t \to 0$. Assume for contradiction that $\limsup_{t \searrow 0} F(t)t^{-2} = \infty$. If this held, then we could choose $\{t_j\}_{j \in \mathbb{N}} \subset (0, 1)$ so that
\[
F(t_j)t_j^{-2} = \max_{[t_j, 1]} F(s)s^{-2} \to \infty.
\]
Using (9) at $t = t_j$ we find
\[
F(t_j) \leq O(t_j^2) + \frac{1}{3} t_j^3 \int_{t_j}^1 s^{-2}(s^{-2} F(s)) \, ds
\leq O(t_j^2) + \frac{1}{3} t_j F(t_j) \int_{t_j}^1 s^{-2} \, ds
= O(t_j^2) + \frac{1}{3} F(t_j)(1 - t_j).
\]
Rearranging this we find
\[
(1 - \frac{1}{3}(1 - o(1))) F(t_j) \leq O(t_j^2).
\]
This is a contradiction, completing the proof. \(\square\)
5. Proof of Theorems 1, 2, and 3

Theorem 1 follows from Theorem 2 and the fact that there are no compact minimal surfaces in $\mathbb{R}^{n+1}$. To prove Theorems 2 and 3, we briefly recall a standard point-picking argument (cf. [58, Lecture 3]) as follows: if Theorem 2 (or Theorem 3) was not true, then we could find a sequence of two-sided, stable minimal immersed hypersurfaces $M_i$ in $\mathbb{R}^4$ (or in $(N^4, g)$) and $p_i \in M_i$ such that

$$|A_{M_i}(p_i)|d_{M_i}(p_i, \partial M_i) = R_i \rightarrow \infty.$$ 

Here the distance is intrinsic on $M_i$. By considering an appropriate subset of $M_i$ we can assume that $M_i$ is compact and smooth up to its boundary. This allows us to assume that $p_i$ maximizes $|A_i(x)|d_M(x, \partial M_i)$. By translating and rescaling (we still denote the surfaces by $M_i$), we can ensure that $p_i = 0$ and $|A_i(0)| = 1$. Then, for any $r < R_i$ and $x \in M_i$ with $d_{M_i}(0, x) \leq r$ we find

$$|A_{M_i}(x)| \leq \frac{R_i}{d_{M_i}(x, \partial M_i)} \leq \frac{R_i}{R_i - r},$$

and thus for each $r > 0$,

$$\sup_{d_{M_i}(x,0) \leq r} |A_{M_i}(x)| \leq \frac{R_i}{R_i - r} \rightarrow 1.$$

Therefore $M_i$ subsequentially converges smoothly to a complete, two-sided, stable minimal immersion $M^3 \rightarrow \mathbb{R}^4$ with $|A_M(0)| = 1$ and $|A_M(x)| \leq 1$ for all $x \in M$ (this last condition was not assumed a priori in the statement of Theorem 1).

To show that such an immersion does not exist, we first pass to the universal cover to arrange that $M$ is simply connected (two-sided stability passes to any covering space by [26, Theorem 1]). We can construct the Green’s function $u \in C_{\text{loc}}(M \setminus \{p\})$ as in Proposition 9 and conclude that

$$F(t) = \int_{\Sigma_t} |\nabla u|^2$$

satisfies $F(t) \leq Ct^2$ for all $t \in (0, \infty)$ by Lemma 15 and Corollary 20. Consider $f = \varphi \circ u$ for $\varphi \in C^{0,1}_c((0, \infty))$ in Proposition 27 and apply the co-area formula to write

$$\int_M |A_M|^3 \varphi(u)^3 \leq C \int_M |\nabla (\varphi \circ u)|^3$$

$$= C \int_M \varphi'(u)^3 |\nabla u|^3$$

$$= C \int_0^{\infty} \varphi'(s)^3 \left( \int_{\Sigma_s} |\nabla u|^2 \right) ds$$

$$= C \int_0^{\infty} \varphi'(s)^3 F(s) ds$$

$$\leq C \int_0^{\infty} \varphi'(s)^3 s^2 ds.$$
For $\rho \gg 0$ choose

$$
\varphi(t) = \begin{cases} 
0 & t \in [0, \rho^{-2}) \\
2 + \frac{\log t}{\log \rho} & t \in [\rho^{-2}, \rho^{-1}) \\
1 & t \in [\rho^{-1}, \rho) \\
2 - \frac{\log t}{\log \rho} & t \in [\rho, \rho^2) \\
0 & t \in [\rho^2, \infty).
\end{cases}
$$

We find

$$
\int_{\{\rho^{-1} \leq u \leq \rho\}} |A_M|^3 
\leq C \int_{\rho^{-2}}^{\rho^{-1}} s^2 \frac{s^2}{s^3 \log \rho} ds + C \int_\rho^{\rho^2} s^2 \frac{s^2}{s^3 \log \rho} ds = O(|\log \rho|^{-2}).
$$

Letting $\rho \to \infty$, we find that $A_M \equiv 0$. This completes the proof.

6. Proof of Theorem

By [57, §3], a minimal immersion $M^3 \to \mathbb{R}^4$ with finite total curvature $\int_M |A_M|^3 < \infty$ has finite index. Thus, it suffices to prove that finite index implies finite total curvature. Consider a complete, two-sided, minimal immersion $M^3 \to \mathbb{R}^4$ with finite index. By Proposition 23, $M$ has $k < \infty$ ends. Note that $M$ is oriented and has bounded curvature.

**Lemma 21.** Consider an exhaustion $\Omega_1 \subset \Omega_2 \subset \cdots \subset M$ by pre-compact regions with smooth boundary so that each component of $M \setminus \Omega_i$ is unbounded. Then for $i$ sufficiently large, $\partial \Omega_i$ has $k$ components.

**Proof.** Discarding finitely many terms, we can assume $M \setminus \Omega_i$ has $k$ components for all $i$. If the assertion fails, we can pass to a subsequence so that $\Omega_{i+1} \setminus \Omega_i$ has $k$ components and $\partial \Omega_i$ has at least two components in a fixed end $E$. This yields a sequence of surfaces $\Sigma_i \subset \partial \Omega_i$ that form a linearly independent set in $H_2(M)$. By Poincaré duality, this implies that $H^1_\partial(M; \mathbb{R})$ is infinite. By [2], Proposition 2.11] and Proposition 25, this further implies that the first $L^2$-Betti number is infinite, contradicting Proposition 23. □

Consider an end $E \subset M$. We can assume that $E$ has smooth boundary. Using the arguments in Proposition 23, we can construct a harmonic function $u \in C^\infty(E)$ with finite Dirichlet energy so that $u = 1$ on $\partial E$ and $u \to 0$ at infinity. By Lemma 21 we can choose $\tau_0 \in (0, 1)$ a regular value of $u$, so that for any other regular value $t \in (0, \tau_0)$, $\Sigma_t := u^{-1}(t)$ is connected and $\{u > \tau_0\}$ is stable.

The proof of Corollary 20 carries over to this situation to show that

$$
F(t) = \int_{\Sigma_t} |\nabla u|^2
$$

\[\text{********X********}\\
\]
satisfies $F(t) = O(t^2)$. We proceed essentially as in the proof of Theorems 1 and 2 and plug the test function $\varphi(u)$ into Proposition 27 where

$$
\varphi(t) = \begin{cases}
0 & t \in [0, \rho^{-2}) \\
2 + \frac{\log t}{\log \rho} & t \in [\rho^{-2}, \rho^{-1}) \\
1 & t \in [\rho^{-1}, \tau_0] \\
2 - \frac{2t}{\tau_0} & t \in [\tau_0, \tau_0].
\end{cases}
$$

Sending $\rho \to \infty$, we conclude that

$$
\int_E |A_M|^3 < \infty.
$$

Since there are finitely many ends (Proposition 23), this yields

$$
\int_M |A_M|^3 < \infty,
$$

as desired.

**Appendix A. Ends of stable (finite index) minimal hypersurfaces**

**Proposition 22** ([8]). For $n > 2$, if $M^n \to \mathbb{R}^{n+1}$ is a complete, connected, two-sided, stable minimal immersion then $M$ has only one end.

**Proposition 23** ([33]). For $n > 2$, if $M^n \to \mathbb{R}^{n+1}$ is a complete, connected, two-sided, stable minimal immersion with finite index, then $M$ has finitely many ends. Moreover, the space of $L^2$-harmonic 1-forms is finite.

**Appendix B. Harnack and Sobolev inequalities**

**Proposition 24** ([59], cf. [45, Theorem I.3.1]). Suppose that $(M^n, g)$ is a complete Riemannian manifold and $u$ is a positive harmonic function on $B_r(x)$. If $\text{Ric} \geq -K^2$ on $B_r(x)$ then

$$
|\nabla u|(x) \leq C(r^{-1} + K)u(x)
$$

for $C = C(n)$.

**Proposition 25** ([35]). For $n > 2$, suppose that $M^n \to \mathbb{R}^{n+1}$ is a complete minimal immersion. Then for any $w \in C_{c}^{0,1}(M)$, it holds that

$$
\left( \int_M w^{\frac{n}{n-2}} \right)^{\frac{n-2}{n}} \leq C \int_M |\nabla w|^2
$$

for $C = C(n)$.
Appendix C. Level sets of harmonic functions

**Proposition 26** ([30, 34, 11]). Fix a compact subset $K$ of a Riemannian $n$-manifold $(M^n, g)$, if $\Delta u = 0$ is a harmonic function on $(M, g)$ then

$$\mathcal{H}^{n-1}(B_\rho(x) \cap \{u = s\}) \leq C \rho^{n-1}$$

for any $s \in \mathbb{R}$, $x \in K$, $\rho \leq \rho_0 = \rho_0(M, g, K, u)$ where $C = C(M, g, K, u)$. Furthermore

$$\dim_{\text{Haus}}(\{u = s, |\nabla u| = 0\}) \leq n - 2$$

for any $s \in \mathbb{R}$.

Appendix D. The Schoen–Simon–Yau $L^3$ estimate

Schoen–Simon–Yau have obtained $L^p$ estimates for the second fundamental form of a stable minimal hypersurface $M^n \to \mathbb{R}^{n+1}$ in [44]. The following $L^3$ estimate follows from their arguments in a straightforward manner but we could not find it in the literature (the estimate from [44] is only stated for $p \geq 4$) so we include a proof.

**Proposition 27.** For $n < 8$, if $M^n \to \mathbb{R}^{n+1}$ a complete, connected, two-sided, stable minimal immersion then there is $C = C(n)$ so that

$$\int_M |A_M|^3 f^3 \leq C \int_M |\nabla f|^3.$$

for any $f \in C^{0,1}_0(M)$.

**Proof.** Set

$$a_\delta = (|A_M|^2 + \delta)^{\frac{1}{2}}.$$

For $f \in C_0^\infty(M)$ we can take $a_\delta f$ in the stability inequality to find

$$\int_M |A_M|^2 a_\delta^2 f^2 \leq \int_M |\nabla a_\delta|^2 f^2 + f \langle \nabla a_\delta^2, \nabla f \rangle + a_\delta^3 |\nabla f|^2$$

$$= \int_M \frac{1}{4} a_\delta^{-6} |\nabla |A_M|^2|^2 f^2 + \frac{1}{2} f a_\delta^{-2} \langle |A_M|^2, \nabla f \rangle + a_\delta^2 |\nabla f|^2$$

(10)

Multiplying Simons identity [52]

$$|\nabla A_M|^2 = \frac{1}{2} \Delta |A_M|^2 + |A_M|^4$$

by $a_\delta^{-2} f^2$ and integrating by parts we find

$$\int_M a_\delta^{-2} |\nabla A_M|^2 f^2$$

$$= \int_M \frac{1}{2} a_\delta^{-2} f^2 \Delta |A_M|^2 + a_\delta^{-2} |A_M|^4 f^2$$

$$= \int_M \frac{1}{2} a_\delta^{-6} |\nabla |A_M|^2|^2 f^2 - a_\delta^{-2} f \langle \nabla f, \nabla |A_M|^2 \rangle + a_\delta^{-2} |A_M|^4 f^2$$
\[
\int_M a_\delta^{-2} (\langle \nabla A_M \rangle^2 f^2 - \frac{1}{4} a_\delta^{-4} |\nabla A_M|^2 f^2) = \int_M -a_\delta^{-2} f \langle \nabla f, \nabla |A_M|^2 \rangle + a_\delta^{-2} |A_M|^4 f^2.
\]

Note that the improved Kato inequality reads
\[
\frac{1}{4} (1 + \frac{2}{n}) |\nabla |A_M|^2| \leq |A_M|^2 |\nabla A_M|^2 \leq a_\delta^4 |\nabla A_M|^2
\]
so we can rearrange this into
\[
\int_M \frac{1}{2n} a_\delta^{-6} |\nabla |A_M|^2| f^2 \leq \int_M -a_\delta^{-2} f \langle \nabla f, \nabla |A_M|^2 \rangle + a_\delta^{-2} |A_M|^4 f^2.
\]
Because \(a_\delta^{-2} \leq |A_M|^{-1}\) we find
\[
\int_M \frac{1}{2n} a_\delta^{-6} |\nabla |A_M|^2| f^2 \leq \int_M -a_\delta^{-2} f \langle \nabla f, \nabla |A_M|^2 \rangle + |A_M|^3 f^2,
\]
so we can combine this with stability to write
\[
\int_M \frac{8-n}{16n} a_\delta^{-6} |\nabla |A_M|^2| f^2 \leq \int_M -a_\delta^{-2} f \langle \nabla f, \nabla |A_M|^2 \rangle + a_\delta^3 |\nabla f|^2.
\]
Because \(n < 8\) we can use Cauchy–Schwarz to find \(C = C(n)\) so that
\[
\int_M a_\delta^{-6} |\nabla |A_M|^2| f^2 \leq C \int_M a_\delta^3 |\nabla f|^2 \tag{11}
\]
On the other hand, we can use Cauchy–Schwarz on (10) to find
\[
\int_M |A_M|^2 a_\delta^2 f^2 \leq \int_M \frac{17}{4n} a_\delta^{-6} |\nabla |A_M|^2|^2 f^2 + \frac{17}{4n} a_\delta^2 |\nabla f|^2. \tag{12}
\]
Combining (11) and (12) and taking the limit as \(\delta \to 0\) (using the dominated convergence theorem) we find
\[
\int_M |A_M|^3 f^2 \leq C \int_M |A_M| |\nabla f|^2.
\]
A standard argument shows that this holds for \(f \in C^{0,1}_c(M)\). We can then replace \(f\) by \(f^\frac{2}{3}\) and using Hölder’s inequality to conclude the proof. \(\square\)

**Appendix E. An extension to almost stable minimal hypersurfaces**

**Theorem 28.** Let \(\delta_s \in [0, \frac{3}{4}]\), and \(M^3 \to \mathbb{R}^4\) be a complete, connected, two-sided, \(\delta_s\)-stable minimal immersion, that is,
\[
\int_M (1 - \delta_s) |A_M|^2 f^2 \leq \int_M |\nabla f|^2
\]
for any \(f \in C^\infty_0(M)\). Then \(M\) is flat.
We do not know if the bound for $\delta_s$ assumed here (and in Proposition 29 below) is sharp. The proof of Theorem 28 closely follows that of Theorem 1. Note that the condition $\delta_s \leq \frac{1}{3}$ is required to carry out the Cao–Shen–Zhu proof of the one-endedness property for $\delta_s$-stable minimal hypersurfaces (Proposition 22), cf. [32, Lemma 10.2]. As such, by following the argument in Section 5 we can reduce the proof of Theorem 28 to the following result:

**Proposition 29.** Suppose that $M^3 \to \mathbb{R}^4$ is a complete, connected, simply connected, two-sided, one-ended, minimal immersion $M^3 \to \mathbb{R}^4$ that is $\delta_s$-stable for $\delta_s \in [0, \frac{1}{2})$. Then $M$ is flat.

Indeed, one may trace through the proof of Corollary 16 to show that the $\delta_s$-stability condition (with $\delta_s < \frac{3}{4}$) implies:

$$\limsup_{\ell \to 0} \int_\ell^1 A(s) \, ds + \int_\ell^1 s^{-2} A(s) \, ds \leq O(t) + \frac{4}{3 - 4\delta_s} \int_t^1 t^2 s^{-4} F(s) \, ds.$$

Since Proposition 18 does not depend on stability, we can thus follow along the proof of Theorem 1 to find

$$F(t) \leq O(t^2) + \frac{1}{3 - 4\delta_s} \int_t^1 s^{-4} F(s) \, ds.$$

The proof of $F(t) = O(t^2)$ given in Corollary 20 then carries over as long as $\delta_s < \frac{1}{7}$. Finally, the $L^3$-estimate in Proposition 27 holds for stable $M^n \to \mathbb{R}^{n+1}$ as long as $\delta_s < \frac{8-n}{8}$. Putting this together, Proposition 29 (and thus Theorem 28) follows.

**References**

[1] V. Agostiniani, L. Mazzieri, and F. Oronzio, *A Green’s function proof of the positive mass theorem*, [https://arxiv.org/abs/2108.08402](https://arxiv.org/abs/2108.08402) (2021).

[2] Virginia Agostiniani, Mattia Fogagnolo, and Lorenzo Mazzieri, *Sharp geometric inequalities for closed hypersurfaces in manifolds with nonnegative Ricci curvature*, Invent. Math. 222 (2020), no. 3, 1033–1101. MR 4169055

[3] F. J. Almgren, Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein’s theorem*, Ann. of Math. (2) 84 (1966), 277–292. MR 200816

[4] Michael Anderson, *The compactification of a minimal submanifold in euclidean space by the Gauss map*, [http://www.math.stonybrook.edu/~anderson/compactif.pdf](http://www.math.stonybrook.edu/~anderson/compactif.pdf) (1984).

[5] Pierre Bérard, *Remarques sur l’équation de J. Simons*, Differential geometry, Pitman Monogr. Surveys Pure Appl. Math., vol. 52, Longman Sci. Tech., Harlow, 1991, pp. 47–57. MR 1173032

[6] E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. 7 (1969), 243–268. MR 250205

[7] Xavier Cabré, Alessio Figalli, Xavier Ros-Oton, and Joaquim Serra, *Stable solutions to semilinear elliptic equations are smooth up to dimension 9*, Acta Math. 224 (2020), no. 2, 187–252. MR 4117051

[8] Huai-Dong Cao, Ying Shen, and Shunhui Zhu, *The structure of stable minimal hypersurfaces in $\mathbb{R}^{n+1}$*, Math. Res. Lett. 4 (1997), no. 5, 637–644. MR 1484695

[9] Gilles Carron, *$L^2$ harmonics forms on non compact manifolds*, [https://arxiv.org/abs/0704.3194](https://arxiv.org/abs/0704.3194) (2007).

[10] Pak-Yeung Chan, Jianchun Chu, Man-Chun Lee, and Tin-Yau Tsang, *Monotonicity of the $p$-green functions*, [https://arxiv.org/abs/2202.13832](https://arxiv.org/abs/2202.13832) (2022).

[11] Jeff Cheeger, Aaron Naber, and Daniele Valtorta, *Critical sets of elliptic equations*, Comm. Pure Appl. Math. 68 (2015), no. 2, 173–209. MR 3298662
[12] Qing Chen, Curvature estimates for stable minimal hypersurfaces in $\mathbb{R}^4$ and $\mathbb{R}^5$, Ann. Global Anal. Geom. 19 (2001), no. 2, 177–184. MR 1826400

[13] Otis Chodosh, Daniel Ketover, and Davi Maximo, Minimal hypersurfaces with bounded index, Invent. Math. 209 (2017), no. 3, 617–664. MR 3681392

[14] Otis Chodosh and Chao Li, Stable anisotropic minimal hypersurfaces in $\mathbb{R}^4$, Forum Math. Pi 11 (2023), Paper No. e3, 22. MR 4546104

[15] Otis Chodosh, Chao Li, and Douglas Stryker, Complete stable minimal hypersurfaces in positively curved 4-manifolds, https://arxiv.org/pdf/2202.07708 (2022).

[16] Tobias Colding, New monotonicity formulas for Ricci curvature and applications. I, Acta Mathematica 209 (2012), no. 2, 229–263.

[17] Tobias H. Colding and William P. Minicozzi, II, Estimates for parametric elliptic integrands, Int. Math. Res. Not. (2002), no. 6, 291–297. MR 1877004

[18] ———, The space of embedded minimal surfaces of fixed genus in a 3-manifold. II. Multi-valued graphs in disks, Ann. of Math. (2) 160 (2004), no. 2, 199–211. MR 2052896

[19] ———, Monotonicity and its analytic and geometric implications, Proc. Natl. Acad. Sci. USA 110 (2013), no. 48, 19233–19236. MR 3153951

[20] ———, Ricci curvature and monotonicity for harmonic functions, Calc. Var. Partial Differential Equations 49 (2014), no. 3-4, 1045–1059. MR 3168621

[21] Ennio De Giorgi, Una estensione del teorema di Bernstein, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 19 (1965), 79–85. MR 178385

[22] M. do Carmo and C. K. Peng, Stable complete minimal surfaces in $\mathbb{R}^3$ are planes, Bull. Amer. Math. Soc. (N.S.) 1 (1979), no. 6, 903–906. MR 546314 (80j:53012)

[23] Alessio Figalli and Joaquim Serra, On stable solutions for boundary reactions: a De Giorgi-type result in dimension 4 + 1, Invent. Math. 219 (2020), no. 1, 153–177. MR 4050103

[24] D. Fischer-Colbrie, On complete minimal surfaces with finite Morse index in three-manifolds, Invent. Math. 82 (1985), no. 1, 121–132. MR 808112

[25] Doris Fischer-Colbrie and Richard Schoen, The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature, Comm. Pure Appl. Math. 33 (1980), no. 2, 199–211. MR 562550 (81i:53044)

[26] Wendell H. Fleming, On the oriented Plateau problem, Rend. Circ. Mat. Palermo (2) 11 (1962), 69–90. MR 157263

[27] Alberto Roncoroni Giovanni Catino, Paolo Mastrolia, Two rigidity results for stable minimal hypersurfaces, https://arxiv.org/abs/2209.10500 (2022).

[28] Robert Gulliver, Index and total curvature of complete minimal surfaces, Geometric measure theory and the calculus of variations (Arcata, Calif., 1984), Proc. Sympos. Pure Math., vol. 44, Amer. Math. Soc., Providence, RI, 1986, pp. 207–211. MR 840274

[29] Robert Hardt and Leon Simon, Nodal sets for solutions of elliptic equations, J. Differential Geom. 30 (1989), no. 2, 505–522. MR 1010160

[30] Chao Li, Index and topology of minimal hypersurfaces in $\mathbb{R}^n$, Calc. Var. Partial Differential Equations 56 (2017), no. 6, Paper No. 180, 18. MR 3722074

[31] Peter Li, Lectures on harmonic functions, http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.77.1052&rep=rep1&type=pdf (2004).

[32] Peter Li and Jiaping Wang, Minimal hypersurfaces with finite index, Math. Res. Lett. 9 (2002), no. 1, 95–103. MR 1892316

[33] Fang-Hua Lin, Nodal sets of solutions of elliptic and parabolic equations, Comm. Pure Appl. Math. 44 (1991), no. 3, 287–308. MR 1090434

[34] J. H. Michael and L. M. Simon, Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$, Comm. Pure Appl. Math. 26 (1973), 361–379. MR 344978

[35] Reiko Miyaoka, $L^2$ harmonic 1-forms on a complete stable minimal hypersurface, Geometry and global analysis (Sendai, 1993), Tohoku Univ., Sendai, 1993, pp. 289–293. MR 1361194
Ovidiu Munteanu and Jiaping Wang, *Comparison theorems for three-dimensional manifolds with scalar curvature bound*, arXiv:2105.12103 (2021).

Barbara Nelli and Marc Soret, *Stably embedded minimal hypersurfaces*, Math. Z. 255 (2007), no. 3, 493–514. MR 2270286

Lei Ni, *Mean value theorems on manifolds*, Asian J. Math. 11 (2007), no. 2, 277–304. MR 2328895

Robert Osserman, *Global properties of minimal surfaces in $E^3$ and $E^n$*, Ann. of Math. (2) 80 (1964), 340–364. MR 179701

Bennett Palmer, *Stability of minimal hypersurfaces*, Comment. Math. Helv. 66 (1991), no. 2, 185–188. MR 1107838

Aleksei V. Pogorelov, *On the stability of minimal surfaces*, Soviet Math. Dokl. 24 (1981), 274–276.

Richard Schoen, L. Simon, and S. T. Yau, *Curvature estimates for minimal hypersurfaces*, Acta Math. 134 (1975), no. 3-4, 275–288. MR 423263

Richard Schoen and Shing Tung Yau, *Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative Ricci curvature*, Comment. Math. Helv. 51 (1976), no. 3, 333–341. MR 438388

Richard Schoen and Shing Tung Yau, *Existence of incompressible minimal surfaces in three-dimensional manifolds*, Seminar on minimal submanifolds, Ann. of Math. Stud., vol. 103, Princeton Univ. Press, Princeton, NJ, 1983, pp. 111–126. MR 795231

Richard Schoen and Leon Simon, *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. 34 (1981), no. 6, 741–797. MR 634285

Richard Schoen and Shing Tung Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 791–809 (1984). MR 730928

Richard M. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. 18 (1983), no. 4, 791–809 (1984). MR 730928

Richard M. Schoen, *Scalar curvature and harmonic maps to $S^1$*, J. Differential Geom. 122 (2022), no. 2, 259–269. MR 4516941

Shing Tung Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 201–228. MR 431040
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