[Continued fractions and Bessel functions...]
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Continued fractions and Bessel functions

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Abstract

Elementary transformations of equations $A\psi = \lambda \psi$ are considered. The invertibility condition (Theorem 1) is established and similar transformations of Riccati equations in the case of second order differential operator $A$ are constructed (Theorem 2). Applications to continuous fractions for Bessel functions and Chebyshev polynomials are established. It is shown particularly that the elementary solutions of Bessel equations are related to a fixed point transformations of Riccati equations.

Keywords: Bessel functions, invertible Darboux transforms, continued fractions, Euler operator, Riccati equation.

1 Introduction

Let $A$ be a differential operator $A$ of order $n$

$$A = a_0(x)D_x^n + a_1(x)D_x^{n-1} + \ldots + a_n(x), \quad D_x = \frac{d}{dx}. \quad (1)$$

We consider transformations of this operator defined by substitutions of the form $\hat{\psi} = (b_0D_x + b_1)\psi$ and their superpositions. In the case $b_0 = 0$ the transformation is invertible and the operator $A$ in the considered equation transforms in the operator $\hat{A}$ as follows

$$\hat{A} = b_1 \circ A \circ b_1^{-1}.$$ 

The following theorem [1] holds true in the general case.
Theorem 1 (on eigenfunctions). The equation for eigenfunctions $A\psi = \lambda \psi$, $\lambda \neq 0$ admits an invertible substitution\(^1\)

$$\hat{\psi} = (D_x - g)\psi, \quad g = (\log \varphi)_x = \frac{\varphi_x}{\varphi}, \quad A\varphi = 0. \quad (2)$$

First, we prove the following lemma.

Lemma 1. The differential operator $A$ of order $n > 1$ is right divisible by the first order operator $A_1 = D_x - g$ iff $g = (\log \varphi)_x$, where $\varphi \in \ker A$.

◮ Let $\varphi \in \ker A$ and $g = (\log \varphi)_x$. The formula (1)

$$A = \sum_{j=0}^{n} a_j(x)D_x^{n-j} = \tilde{A}(D_x - f), \quad \tilde{A} = \sum_{j=0}^{n-1} \tilde{a}_j(x)D_x^{n-(j+1)}, \quad g = (\log \varphi)_x,$$

implies that $A(\varphi) = 0$ because $(D_x - f)(\varphi) = 0$. Then by the substitution $y = \varphi \hat{\varphi}$ we obtain an operator $\tilde{A}$ with the zero coefficient $a_n = 0$. Consequently, this polynomial is divisible by $D_x$ iff the initial operator is divisible by $D_x - f$.

◮

Now we prove the theorem on eigenfunctions.

◮ Note that $A(\varphi) = 0$ and operator $A$ takes the form (1):

$$A = \sum_{j=0}^{n} a_j(x)D_x^{n-j}.$$

By substituting $A$ in $A\psi = \lambda \psi$ we find

$$\tilde{A}\hat{\psi} = \lambda \hat{\psi} \quad (3)$$

From (3) we have

$$\lambda \hat{\psi} = a_0\hat{\psi}^{n-1} + a_1\hat{\psi}^{n-2} + ... + a_n\hat{\psi}. \quad (4)$$

Then by the substitution (2) from (4) we obtain

$$\lambda \hat{\psi} = \frac{d}{dx}[a_0\hat{\psi}^{n-1} + a_1\hat{\psi}^{n-2} + ... + a_n\hat{\psi}] \quad (5)$$

If $\lambda \neq 0$ then the equation (5) and original equation $A\psi = \lambda \psi$ have the same order, but coefficients in (5) are different.

Hence, we have proved that the equation $A\psi = \lambda \psi$, $\lambda \neq 0$ admits a substitution $\hat{\psi} = (D_x - g)\psi$ if $g = (\log \varphi)_x$ and this substitution is invertible.

◮

From this point on, we consider applications of Theorem 1 in the case when $A$ is Euler operator.

\(^1\)Note that the replacement (2) is invertible and its inverse is written by the formula (4).
Definition 1. Euler operator has the form

\[ A = e^{mt} k(D_t) \]

where \( k(D_t) \) is a polynomial in \( D_t = \frac{d}{dt} \) with constant coefficients.

Lemma 2. If \( A = e^{mt} k(D_t) \), \( B = e^{nt} z(D_t) \) then a superposition of Euler operators \( A \) and \( B \) takes the form:

\[ A \circ B = e^{(m+n)t} c(D_t), \quad c(D_t) = k(D_t + n)z(D_t). \]

In this case the substitution \( \hat{\psi} = (b_0 D_x + b_1)\psi \) becomes an Euler operator of the first order

\[ \hat{\psi} = e^t(D_t + c)\psi, \quad c \in \mathbb{C}. \] (6)

Indeed, \( x = e^{-t}, \, dx = -e^{-t}dt \), therefore \( D_x = -e^t D_t \).

1.1 Second order equations

Let us consider second order equations and application of Theorem 1 in this case. An operator \( A \) can be described as follows

\[ A = a_0(x)D^2 + a_1(x)D + a_2(x) \] (7)

Using the substitution \( \psi = e^x \hat{\psi} \) and assuming that a coefficient of \( D \) is equal to zero, we can obtain that a coefficient of \( D^2 \) is equal to 1, i.e.

\[ A = D^2 + q(x). \] (8)

Then (8) takes the form:

\[ A = (D - g)(D + g) \] (9)

Indeed,

\[ A\psi = (D^2 + q(x))\psi = \psi'' + q(x)\psi. \]

On the other hand,

\[ A\psi = (D - g)(D + g)\psi = \psi'' + (-g' + g^2)\psi = \psi'' + q(x)\psi, \]

where \( q(x) + g' + g^2 = 0 \).
Definition 2. A Riccati equation associated with the equation \( A\psi = \lambda \psi \) is the following equation for the logarithmic derivative \( f = \frac{\psi'}{\psi} \):

\[
a_0(f' + f^2) + a_1 f + a_2 = \lambda. \tag{10}
\]

In the particular case that an operator \( A \) is of the form (9) the equation (10) can be written as follows:

\[
f' + f^2 + q(x) = \lambda, \quad f = \frac{\psi'}{\psi}.
\]

2 Bessel equations

Suppose that an operator \( A \) is given by

\[
A = D^2 + \frac{1}{x} D - \frac{\beta^2}{x^2}. \tag{11}
\]

Then by the substitution \( x = e^{-t} \):

\[
D_x = \frac{d}{dx} = -e^t \frac{d}{dt} = -e^t D_t,
\]

one can rewrite the equation \( A\psi = \lambda \psi \) in the following form:

\[
A\psi = e^{2t}(D^2_t - \beta^2)\psi = e^t(D_t - \beta - 1) \circ e^t(D_t + \beta)\psi = \lambda \psi. \tag{12}
\]

Note that

\[
(D^2_x = (-e^t D_t) \circ (-e^t D_t) = e^{2t}(D_t + D^2_t).
\]

The equation for eigenfunctions of an operator \( A \) is \( A\psi = \lambda \psi \). Here an operator \( A \) is of the form (11). The equation considered here is called Bessel equation.

By applying Theorem 1 and Lemma 2 to equation (12) we obtain

\[
e^t(D_t + \beta)\psi = \hat{\psi}, \quad e^t(D_t - \beta - 1)\hat{\psi} = \lambda \hat{\psi}. \tag{13}
\]

As a result, we have that the equation \( A\psi = \lambda \psi \) takes the form

\[
e^{2t}(D^2_t - \hat{\beta}^2)\hat{\psi} = \lambda \hat{\psi}, \quad \hat{\beta} \overset{\text{def}}{=} \beta + 1. \tag{14}
\]

Rewriting now equations (13) in terms of \( f_\beta = (\log \psi)_t \) and \( f_{\hat{\beta}} = (\log \hat{\psi})_t \) one obtains (see Definition 2):

\[
\hat{\psi} = e^t(f_{\beta} + \beta)\psi, \quad \lambda \psi = e^t(f_{\hat{\beta}} - \hat{\beta}), \quad (f_{\beta} + \beta)(f_{\hat{\beta}} - \hat{\beta}) = \lambda \cdot e^{-2t} = \lambda \cdot x^2 \tag{15}
\]

Without loss of generality we put \( \lambda = 1 \) in the last equation and prove the main theorem.
Theorem 2. Let \( f = f_\beta \) be a solution of the Riccati equation \( f_t + f^2 = \beta^2 + x^2 \) and the function \( \hat{f} = f_\hat{\beta} \) be defined by the following equation

\[
(f + \beta)(\hat{f} - \hat{\beta}) = x^2, \quad \hat{\beta} = \beta + 1,
\]
then this equation states the equivalence of two Riccati equations

\[
f_t + f^2 = \beta^2 + x^2 \iff \hat{f}_t + \hat{f}^2 = \hat{\beta}^2 + x^2.
\]

\( \Box \)

Let the function \( \mu = \mu(t) \) satisfies the differential equation \( \mu_t = -2\mu \) and \( f_\beta \) be a solution of the Riccati equation \( f_t + f^2 = \beta^2 + \mu(t) \). Then the function \( f_\hat{\beta} \) is defined by the formula for \( \hat{f} \) as follows

\[
\hat{f} = \frac{\mu}{f + \beta} + \hat{\beta}.
\]

(17)

By differentiating (17) with respect to \( t \):

\[
\hat{f}_t = -2\frac{\mu(f + \beta) - \mu f_\mu}{(f + \beta)^2} = -2\frac{\mu}{f + \beta} - f_t\frac{\mu}{(f + \beta)^2}.
\]

(18)

Note that

\[
f_t = \beta^2 - f^2 + \mu.
\]

(19)

By substituting (19) in (18),

\[
\hat{f}_t = -2\frac{\mu}{f + \beta} + \frac{f - \beta}{f + \beta} - \frac{\mu^2}{(f + \beta)^2}
\]

\[
\frac{\mu^2}{(f + \beta)^2} = -\hat{f}_t - 2\frac{\mu}{f + \beta} + \frac{f - \beta}{f + \beta}.
\]

(20)

By squaring both sides of (17) we have

\[
\hat{f}^2 = \frac{\mu^2}{(f + \beta)^2} + 2\hat{\beta}\frac{\mu}{f + \beta} + \hat{\beta}^2.
\]

Indeed,

\[
\frac{\mu^2}{(f + \beta)^2} = \hat{f}^2 - 2\hat{\beta}\frac{\mu}{f + \beta} - \hat{\beta}^2.
\]

(22)

Henceforth, from equations (21) and (22) we can obtain:

\[
-\hat{f}_t - 2\frac{\mu}{f + \beta} + \frac{f - \beta}{f + \beta} = \hat{f}^2 - 2\hat{\beta}\frac{\mu}{f + \beta} - \hat{\beta}^2,
\]

\[
\hat{f}_t + \hat{f}^2 = \hat{\beta}^2 + \mu\frac{2 + f - \beta + 2(\beta + 1)}{f + \beta}.
\]

So,

\[
\hat{f}_t + \hat{f}^2 = \hat{\beta}^2 + \mu(t). \quad \Box
\]
Corollary of Theorem 2. The mapping $A \to \hat{A}$ defined in Theorem 2 has a fixed point:
\[
\hat{f} = f, \quad (\hat{\beta})^2 = \beta^2.
\] (23)

In the case (23), (16) we find by solving quadratic equations that
\[
f = f_{\pm} = \frac{1}{2} \pm x, \quad \beta = -\frac{1}{2}, \quad \hat{\beta} = \frac{1}{2},
\]
\[
f_t + f^2 = \frac{1}{4} + x^2, \quad (x = e^{-t}).
\] (24)

It easy as well to see that $\hat{f}_{\pm} = f_{\pm}$.

2.1 Recurrent relations

In the case of Chebyshev polynomials $T_n(x)$
\[
T_n(x) = \cos n\theta, \quad x = \cos \theta, \quad D \theta = -\sqrt{1-x^2} \ D_x
\] (25)
we have
\[
T_{n+1} + T_{n-1} = 2xT_n, \quad T_0 = 1, \quad T_1 = x.
\] (26)
This recurrent relation totally defines polynomials $T_n(x)$ in $x$-variable and rewriting (26) one obtains (Cf. [4]):
\[
(f_n - x)(f_{n+1} + x) = -1, \quad f_n = \frac{T_{n-1}}{T_n} + x, \quad f_1(x) = x + \frac{1}{x}
\] (27)
which looks very similar to (16). Generally speaking, recurrent relations (27) and (26) are equivalent. Moreover, by reversing in a certain sense Theorem 2 one may obtain from its proof and (27) the second order differential equation
\[
D_x^2T_n = \frac{x \cdot D_x}{1-x^2} T_n - \frac{n^2}{1-x^2} T_n
\]
for Chebyshev’s polynomials $T_n(x)$. In the case of Bessel functions we can choose, as a basic one, an analog of the linear recurrent relation (26) (see (15) and [3]), but the recurrent relation in the “Riccati” form (16) provides some advantages (Cf [2]) and yields the formulae (24) used below in order to obtain rational in $x$-variable solutions of the eq. (16). Introducing a numbering we denote (Cf. (16))
\[
\beta_1 = \frac{1}{2}, \quad \beta_{j+1} = \beta_j + 1, \quad j = 1, 2, \ldots .
\]
Proposition. Let $\beta_j = j - \frac{1}{2}$. Then the formula as follows

$$f_{j+1} = \beta_{j+1} + \frac{x^2}{f_j + \beta_j}, \quad f_1 = -x + \frac{1}{2},$$

provides rational solutions of the Riccati equation of Theorem 2

$$f_2 = \frac{3}{2} + \frac{x^2}{1 - x}.$$ 

Similarly determined $f_3, f_4, \ldots$

$$f_3 = \frac{5}{2} + \frac{x^2}{x^2 - 3x + 3}; \quad f_4 = \frac{7}{2} + \frac{x^2}{6x^2 - 15x + 15}.$$ 


Conclusion

Theorem 1 and equation (6) reduce the spectral problem $A\psi = \lambda\psi$ with Euler operator $A$ to an algebraic one. This allows us to investigate a generalization of the results of §2 for higher order Euler operators. Eigenfunctions in this case will provide higher order Bessel functions, but generalization of the continuous fraction approach is not known yet.

References

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