Gaussian Mean Field Regularizes by Limiting Learned Information

Julius Kunze $^1$ Louis Kirsch $^{1,2}$ Hippolyt Ritter $^1$ David Barber $^{1,3}$

Abstract

Variational inference with a factorized Gaussian posterior estimate is a widely used approach for learning parameters and hidden variables. Empirically, a regularizing effect can be observed that is poorly understood. In this work, we show how mean field inference improves generalization by limiting mutual information between learned parameters and the data through noise. We quantify a maximum capacity when the posterior variance is either fixed or learned and connect it to generalization error, even when the KL-divergence in the objective is rescaled. Our experiments demonstrate that bounding information between parameters and data effectively regularizes neural networks on both supervised and unsupervised tasks.

1. Introduction

Bayesian machine learning is a popular framework for dealing with uncertainty in a principled way by integrating over model parameters rather than finding point estimates (Bishop, 2006; Barber, 2012; Ghahramani, 2015). Unfortunately, exact inference is usually not feasible due to the intractable normalization constant of the posterior. A popular alternative is variational inference (Wainwright et al., 2008), where a tractable approximate distribution is optimized to resemble the true posterior as closely as possible. Due to its amenability to stochastic gradient descent (Hoffman et al., 2013; Kingma & Welling, 2013; Titsias & Lázaro-Gredilla, 2014; Rezende & Mohamed, 2015), variational inference is scalable to large models and datasets.

The most common choice for the variational posterior is a factorized Gaussian. Outside of Bayesian inference, parameter noise has been found to be an effective regularizer (Graves et al., 2013; Plappert et al., 2018; Fortunato et al., 2018), e.g. for training neural networks. In combination with $L2$-regularization, additive Gaussian parameter noise corresponds to variational inference with a Gaussian approximate posterior with fixed variance. Interestingly, it has been observed that flexible posteriors can perform worse than simple ones (Turner & Sahani, 2011; Trippe & Turner, 2018; Braithwaite & Kleijn, 2018; Shu et al., 2018).

Variational inference follows the Minimum Description Length (MDL) principle (Rissanen, 1978; 1983; Hinton & van Camp, 1993), a formalization of Occam’s Razor. Loosely speaking, it states that of two models describing the data equally well, the ‘simpler’ one should be preferred. However, MDL is only an objective for compressing the training data and the model, and makes no formal statement about generalization to unseen data. Yet, generalization to new data is a key property of a machine learning algorithm.

Recent work (Xu & Raginsky, 2017; Bu et al., 2019; Bassily et al., 2018; Russo & Zou, 2015) has proposed upper bounds on the generalization error as a function of the mutual information between model parameters and training data. It states that the gap between train and test error can be reduced by limiting the mutual information. However, to the best of our knowledge these bounds and specific inference methods have so far not been linked.

In this work, we show that Gaussian mean field inference in models with Gaussian priors can be reinterpreted as point estimation in corresponding noisy models. This leads to an upper bound on the mutual information between model parameters and data through the data processing inequality. Our result holds for both supervised and unsupervised models. We discuss the connection to generalization bounds from Xu & Raginsky (2017) and Bu et al. (2019), suggesting that Gaussian mean field aids generalization. In our experiments, we show that limiting model capacity via mutual information is an effective measure of regularization, further supporting our theoretical framework.

2. Regularization through Mean Field

In our derivation, we denote a generic model as $p(\theta, D) = p(\theta)p(D \mid \theta)$ with unobserved variables $\theta$ and data $D$. We refer to $\theta$ as the model parameters, however in latent variable models $\theta$ can also include the per-data point latents. The model consists of a prior $p(\theta)$ and a likelihood $p(D \mid \theta)$. Ideally, one would like to find the posterior $p(\theta \mid D)$.
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\[ \text{Inference model} \quad \text{Generative model} \]

\[ \begin{aligned}
\sigma^2 & \quad \sigma^2 \\
\mu & \quad \theta \\
\theta & \quad \theta \\
D & \quad D \\
\end{aligned} \]

(a) Original model

\[ \begin{aligned}
\sigma^2 & \quad \sigma^2 \\
\mu & \quad \theta \\
\theta & \quad \theta \\
D & \quad D \\
\end{aligned} \]

(b) Noisy model: Fixed variance

\[ \begin{aligned}
\sigma^2 & \quad \sigma^2 \\
\mu & \quad \theta \\
\theta & \quad \theta \\
D & \quad D \\
\end{aligned} \]

(c) Noisy model: Learned variance

Figure 1. Gaussian mean field inference on model parameters \( \theta \) with a Gaussian prior (a) can be reinterpreted as optimizing a point estimate on a model with injected noise both when variance is fixed (b) and learned (c). For the latter case, we show this for the more general case where the complexity term in the objective is fixed (b) and learned (c). For the latter case, we show this for the same objective as mean-field variational inference in the original model.

We now show that maximizing a lower bound on the log joint probability of the noisy model results in an identical objective as for variational inference in the clean model

\[ \log p'(D, \theta) \]

\[ = \log \int p'(D \mid \hat{\theta}) p'\hat{\theta} \mid \theta) d\hat{\theta} + \log p'(\theta) \]

\[ \geq \mathbb{E}_{\tilde{\theta} \sim \mathcal{N}(\theta, \sigma^2 I)} \log p'(D \mid \tilde{\theta}) - \frac{1}{2\sigma^2} \sum_i \theta_i^2 + \text{const.} \]

\[ = \mathbb{E}_{\tilde{\theta} \sim \mathcal{N}(\mu, \sigma^2 I)} \log p'(D \mid \tilde{\theta}) - \frac{1}{2\sigma^2} \sum_i \mu_i^2 + \text{const.} \]

where Equation 5 follows from Jensen’s inequality as in Equation 1. In the final equation we have replaced \( \theta \) with \( \mu \) (which is simply a change of names since we are maximizing the objective over this free variable) to emphasize that the objective functions are identical.

Since \( D \) is independent of \( \theta \) given \( \tilde{\theta} \), the joint \( p(\theta, \tilde{\theta}, D) \) forms a Markov chain and the data processing inequality (Cover & Thomas, 2012) limits the mutual information \( I(D, \theta) \) between learned parameters and data through

\[ I(D, \theta) \leq I(\tilde{\theta}, \theta) \]

The upper bound is given by

\[ I(\tilde{\theta}, \theta) = H(\tilde{\theta}) - H(\tilde{\theta} \mid \theta) = \frac{K}{2} \log \left( 1 + \frac{\sigma^2}{\sigma^2} \right) \]

where \( K \) denotes the number of parameters. Here, we exploit that \( \theta \) and \( \theta \mid \theta \) are Gaussian with \( H(\tilde{\theta}) = \frac{K}{2} \log 2\pi e (\sigma^2 + \sigma^2) \) and \( H(\tilde{\theta} \mid \theta) = \frac{K}{2} \log 2\pi e \sigma^2 \). This quantity is known as the capacity of channels with Gaussian noise in signal processing (Cover & Thomas, 2012).
Intuitively, a high prior variance $\sigma_p^2$ corresponds to a large capacity, while a high noise variance $\sigma^2$ reduces it. Any desired capacity can be achieved by simply adjusting the signal-to-noise ratio $\sigma_p^2 / \sigma^2$.

### 2.2. Generalization Error vs. Limited Information

Intuitively, we characterize overfitting as learning too much information about the training data, suggesting that limiting the amount of information extracted from the training data into the hypothesis should improve generalization. This idea has recently been formalized by Xu & Raginsky (2017); Bu et al. (2019); Bassily et al. (2018); Russo & Zou (2017); Bu et al. (2019); Bassily et al. (2018); Russo & Zou (2017); Bu et al. (2019). Therefore, we are optimistic that the assumption of exact inference aids generalization over this sampling process. Bu et al. (2019) relax the condition on the loss and prove applicability to a simple estimation algorithm involving L2-loss.

Exact Bayesian inference returns the true posterior $p(\theta \mid D)$ on a model $p(\theta, D)$. The theorem then states that a bound on $I(D, \theta)$ limits the expected generalization error as described in Bu et al. (2019) if the model captures the nature of the generating process in the marginal $p(D) = \int d\theta p(\theta) p(D \mid \theta)$. This is a common assumption necessary to justify any (variational) Bayesian approach.

Exact inference is intractable on deep models, and instead, one typically learns variational or point estimates for the posterior. That is also true for the objective on the noisy model above, where we only used a point estimate as given by Equation 6. Therefore, the assumption of exact inference is straight-forward to adapt the derivation to the general case.

### 2.3. Learned-Variance Gaussian Mean Field Inference

The variance in Gaussian mean field inference is typically learned for each parameter (Kingma et al., 2015; Rezende & Mohamed, 2015; Blundell et al., 2015). Similar to when the variance in the approximate posterior is fixed, one can obtain a capacity constraint. This is the case even for a generalization of the objective from Equation 1 where the KL-term is scaled by some factor $\beta > 0$. In the following, we quantify a general capacity depending on $\beta$, where $\beta = 1$ recovers the standard variational objective. For notational simplicity, we here assume a prior variance of $\sigma_p^2 = 1$. It is straightforward to adapt the derivation to the general case.

In this case, the objective can be written as

$$
\begin{align*}
\mathbb{E}_{\theta \sim \mathcal{N}(\mu, \sigma^2)} & \log p(D \mid \theta) \\
+ & \frac{\beta}{2} \sum_i \left( \log \sigma_i^2 - \sigma_i^2 - \mu_i^2 - 1 \right)
\end{align*}
$$

where now both $\mu$ and $\sigma^2$ represent learned vectors, and $\mathcal{N}(\mu, \sigma^2)$ denotes a variable composed of pairwise independent Gaussian components with means and variances given by the elements of $\mu$ and $\sigma^2$.

Similar to the previous section, we show a lower bound on the log-joint of a new noisy model to be identical to Equation 9. Specifically, we define the noisy model $p'(\theta, \sigma^2, \hat{D}, D) = p'(\theta)p'(\sigma^2)p'(\hat{\theta} \mid \theta, \sigma^2)p'(D \mid \hat{\theta})$ (Figure 1c), with independent priors $\theta_i \sim \mathcal{N}\left(0, \frac{1}{\beta^2}\right)$ and $\sigma_i^2 \sim \Gamma\left(\frac{\beta}{2} + 1, \frac{\beta}{2}\right)$ where $\Gamma(\cdot, \cdot)$ denotes the Gamma distribution. As previously done subsection 2.1, we define the noise-injected parameters as $\hat{\theta} \sim \mathcal{N}(\theta, \sigma^2)$ and likelihood as $p'(D \mid \hat{\theta}) = p(D \mid \theta)$.

The priors are chosen so that with Jensen’s inequality, we find a lower bound on the log-joint probability of this model that recovers the objective from Equation 9

$$
\begin{align*}
& \log p'(D, \theta, \sigma^2) \\
= & \log \int p'(D \mid \hat{\theta})p'(\hat{\theta} \mid \theta, \sigma^2) d\hat{\theta} + \log p'(\theta) + \log p'(\sigma^2) \\
\ge & \mathbb{E}_{\hat{\theta} \sim \mathcal{N}(\theta, \sigma^2)} \log p'(D \mid \hat{\theta}) + \sum_i \left( \log p'(\theta_i) + \log p'(\sigma_i^2) \right) \\
= & \mathbb{E}_{\hat{\theta} \sim \mathcal{N}(\theta, \sigma^2)} \log p'(D \mid \hat{\theta}) \\
+ & \frac{\beta}{2} \sum_i \left( \log \sigma_i^2 - \sigma_i^2 - \mu_i^2 + \text{const.} \right)
\end{align*}
$$

In the noisy model, the data processing inequality and the

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1 Higgins et al. (2017) propose using $\beta > 1$ to learn ‘disentangled’ representations in variational autoencoders. Further, $\beta$ is commonly annealed from 0 to 1 for expressive models (e.g. Bowman et al. (2015); Blundell et al. (2015); Sønderby et al. (2016)).
The objective function for variational inference is maximized when the approximate posterior is equal to the true one. This motivates the development of flexible families of posterior distributions (Rezende & Mohamed, 2015; Kingma et al., 2016; Salimans et al., 2015; Ranganath et al., 2016; Huszár, 2017; Chen et al., 2018; Vertes & Sahani, 2018; Burda et al., 2015; Cremer et al., 2017). In the case of exact inference, a bound on generalization as discussed in subsection 2.2 only applies if the model itself has finite mutual information between data and parameters. However, estimating mutual information is generally a hard problem, particularly in high-dimensional, non-linear models. This makes it hard to state a generic bound, which is why we focus on the case of Gaussian mean field inference.

3. Related Work

Regularization in Neural Networks Gaussian mean field is intimately related with other popular regularization approaches in deep learning: As apparent from Equation 6, fixed-variance Gaussian mean field applied to training neural network weights is equivalent to L2-regularization (weight decay) combined with Gaussian parameter noise (Graves et al., 2013; Plappert et al., 2018; Fortunato et al., 2018) on all network weights. Molchanov et al. (2017) shows that additive parameter noise results in multiplicative noise on the unit activations. The resulting dependencies between noise components on the layer output can be ignored without significantly changing empirical results (Wang & Manning, 2013). This is in turn equivalent to scaled Gaussian dropout (Kingma et al., 2015).

Information Bottlenecks The Information Bottleneck principle by Tishby et al. (2000); Shamir et al. (2010) aims to find a representation Z of some input X that is most useful to predict an output Y. For this purpose, the objective is to maximize the amount of information I(Y, Z) the representation contains about the output under a bounded amount of information I(X, Z) about the input

$$\max_{I(X,Z)<C} I(Y,Z)$$ (12)

They describe a training procedure using the softly constrained objective

$$\min L_{IB} = \min I(X,Z) - \beta I(Y,Z)$$ (13)

where $\beta > 0$ controls the trade-off.

Alemi et al. (2016) suggest a variational approximation for this objective. For the task of reconstruction, where labels

$$\min E_{X\sim p_x} D_{KL}(q_{\phi}(z|x) \| p(z)) + \beta \mathbb{E}_{z\sim p(z)} D_{KL}(p_{\theta}(x|z) \| q_{\phi}(x|z))$$ (14)
are identical to inputs $X$, this results exactly in the $\beta$-VAE objective (Achille & Soatto, 2017; Alemi et al., 2018). This is in accordance with our result from subsection 2.3 that there is a maximum capacity per latent dimension that decreases for higher $\beta$. Setting $\beta > 1$, as suggested by Higgins et al. (2017) for obtaining disentangled representations, corresponds to lower capacity per latent component than achieved by standard variational inference.

Both Tishby et al. (2000) and Higgins et al. (2017) introduce $\beta$ as a trade-off parameter without a quantitative interpretation. With our information-theoretic perspective, we quantify the implied capacity and provide a link to the generalization error. Further, both methods are concerned with the information in the latent representation. They do not limit the mutual information with the model parameters, leaving them vulnerable to model overfitting under our theoretical assumptions. We experimentally validate this vulnerability and explore the effect of filling this gap by applying Gaussian mean field inference to the model parameters.

### Information Estimation with Neural Networks

Multiple recent techniques (Belghazi et al., 2018; van den Oord et al., 2018; Hjelm et al., 2018) propose the use of neural networks for obtaining a lower bound on the mutual information. This is useful in settings when we want to maximize mutual information, e.g. between the data and a lower-dimensional representation. In contrast, we show that Gaussian variational inference on variables with a Gaussian prior implicitly places an upper bound on the mutual information between these variables and the data, and explore its regularizing effect.

### 4. Experiments

In this section, we analyze the implications of applying Gaussian mean field inference of fixed scale to the model parameters in the supervised and unsupervised context. Our theoretical results suggest that varying the capacity will affect the generalization capability and we show this effect on small data regimes and how it changes with the training set size. Furthermore, we investigate whether capacity is the only predictor for generalization or whether varying priors and architectures also have an effect. Finally, we demonstrate qualitatively how the capacity bounds are reflected in fashion MNIST reconstruction.

#### 4.1. Supervised Learning

We begin with a supervised classification task on the CIFAR10 dataset, training only on a subset of the first 5000 samples. We use 6 3x3 convolutional layers with 128 channels each followed by a ReLU activation function, every second of which implements striding 2 to reduce the input dimensionality. Finally, the last layer is a linear projection which parameterizes a categorical distribution. The capacity of each parameter in this network is set to specific values given by Equation 8.

Figure 3 shows that decreasing the model capacity per dimension (by increasing the noise) reduces the training loglikelihood and increases the test loglikelihood until both of them meet at an optimal capacity. One can observe that very small capacities lead to a signal that is too noisy and good predictions are no longer possible. In short, regimes of underfitting and overfitting are generated depending on the capacity.

#### 4.2. Unsupervised Learning

We now evaluate the regularizing effect of fixed-scale Gaussian mean field inference in an unsupervised setting for MNIST image reconstruction. Therefore, we use a VAE (Kingma & Welling, 2013) with 2 latent dimensions and a 3-layer neural network parameterizing the conditional factorized Gaussian distribution. As usual, it is trained using the free energy objective, but different from the original work, we also use Gaussian mean field inference for the model parameters. Again, we use a small training set of 200 examples for the following experiments if not denoted otherwise.

**Varying Model Capacity and Priors** In our first experiment, we analyze generalization by inspecting the test ELBO when varying the model capacity which can be seen in Figure 4a. Similar to the supervised case, we can observe that there is a certain model capacity range that explains the data very well while less or more capacity results in noise drowning and overfitting respectively. In the same fig-
Figure 4. MNIST test reconstruction with a VAE training on 200 samples for various priors and capacities.

(a) The test ELBO is not invariant when varying the prior on the model parameters. Nevertheless, the first increasing and then decreasing trend when changing the capacity remains.

(b) Using an improper prior, similar to just using Gaussian dropout on the weights, leads to an accelerated decrease of generalization for smaller noise scales.

Figure 5a shows how limiting the capacity affects the test ELBO for varying amounts of training data. Models with very small capacity extract less information from the data into the model, thus yielding a good test ELBO somewhat independent of the dataset size. This is visible as a graph that ascends very little with more training data (e.g. total model capacity 330 kbits). Note that we here report the capacity of the entire model, which is the sum of the capacities for each parameter. In order to improve the test ELBO, more information from the data has to be extracted into the model. But clearly, this leads to non-generalizing information being extracted when the dataset is small, leading to overfitting. Only for larger datasets the extracted information generalizes. This is visible as a strongly ascending test ELBO with larger dataset sizes and bad generalization for small datasets. We can therefore conclude that the information bottleneck needs to be chosen based on the amount of data that is available. Intuitively, when more information is available, more information should be extracted into the model.

Varying Model Size Furthermore, we inspected how the size of the model (here in terms of number of layers) affects generalization in Figure 5b. Similar to varying the prior distribution, we are interested in how well the total capacity predicts generalization and the role the architecture plays. It can be observed that larger networks are more resilient to larger total capacities before they start overfitting. This indicates that the total capacity is less important than the individual capacity (i.e. noise) per parameter. Nevertheless, larger networks are more prone to overfitting for very large model capacities. This makes sense as their functional form is less constrained, an aspect that is not captured by our theory.

Qualitative Reconstruction Finally, we plot test reconstruction means for the binarized fashion MNIST dataset under the same setup for various capacities in Figure 6. In accordance with previous experiments, we observe that if the capacity is chosen too small, the model is not learning
Figure 5. MNIST test reconstruction with a VAE; training on varying dataset sizes, architectures, and model capacities.

(a) Varying the number of samples. Depending on the size of the dataset higher capacities of the model are required to fit all the datapoints.

(b) Varying architecture. Overfitting is not getting worse for more layers if capacity is low enough. More layers do overfit only for higher capacities.

5. Discussion

In this section, we discuss how the capacity can be set, as well as the effect of model architecture and learning dynamics.

5.1. Choosing the Capacity

We have obtained a new trade-off parameter, the capacity, that has a simple quantitative interpretation: It determines how many bits to maximally extract from the training set. In contrast, for the $\beta$ parameter introduced in Tishby et al. (2000) and Higgins et al. (2017) a clear interpretation is not known. Yet, it may still be hard to set the capacity optimally. Simple mechanisms such as evaluation on a validation set to determine its value may be used. We expect that more theoretically rigorous methods could be developed.

Furthermore, in this paper, we have focused on the regularization that Gaussian mean field inference implies on the model parameters. The same concept is valid for data-dependent latent variables, for instance in VAEs, as discussed in subsection 2.4. In VAEs, Gaussian mean field inference on the latents leads to a restricted latent capacity, but leaves the capacity of the model unbounded. This leaves VAEs vulnerable to model overfitting, as demonstrated in the experiments, and setting $\beta$ as done in Higgins et al. (2017) is not sufficient to control complexity. This motivates the limitation of capacity between the data and both per-datapoint latents and model parameters. The interaction between the two is an interesting future research direction.

5.2. Role of Learning Dynamics and Architecture

As discussed in subsection 2.2, it is necessary to perform exact inference in the noisy model for the bounds on the generalization error to hold. However, this assumption is not met. In practice $p_{\theta}(\theta | D)$ encodes the complete learning algorithm, which in deep learning typically includes parameter initialization and dynamics of the stochastic gradient descent optimization.

Our experiments confirm the relevance of these other factors: L2-regularization works in practice, even though no noise is added to the parameters. This could be explained by the fact that noise is already implicitly added through stochastic gradient descent (Lei et al., 2018) or through the output distribution of the network. Similarly, Gaussian dropout (Graves et al., 2013; Plappert et al., 2018; Fortunato et al., 2018) without a prior on the parameters helps generalization. Again, early stopping combined with a finite reach of gradient descent steps effectively shapes a prior of finite variance in the parameter space. This could also formalize why the annealing schedule employed by Bowman et al. (2015); Blundell et al. (2015) and Sønderby et al. (2016) is effective.

This observed dependence on other factors suggests that quantifying mutual information $I_{\theta}(\theta, D)$ of the actual distribution created by the learning dynamics might be a promising approach to explain why neural networks often generalize well on their own. This idea is in accordance...
with recent work that links the learning dynamics of small neural networks to generalization behavior (Li & Liang, 2018).

On the other hand, the architecture choice also had an influence on generalization, which is expected by our theory since we only formulate a bound on mutual information that is completely agnostic to the actual model choice. Tightening this bound based on the model architecture and output distribution is usually hard, as discussed in subsection 2.5, but might be possible.

Another promising direction would be to approximately sample from the exact posterior on network parameters (i.e. as done by Marceau-Caron & Ollivier (2017)), on a capacity-limited architecture, instead of the usual approach of point estimation. In the limit of infinite training time, this would fully realize the discussed bound on the expected generalization error.

6. Conclusion

We have explained the regularizing effects observed in Gaussian mean field approaches from an information-theoretic perspective. The derivation features a capacity that can be naturally interpreted as a limit on the amount of information extracted about the given data by the inferred model. We validated its practicality for both supervised and unsupervised learning.

How this capacity should be set for parameters and latent variables depending on task and data is an interesting direction of research. We exploited a theoretical link of mutual information and generalization error. While this work is restricted to Gaussian mean field, incorporating the effect of learning dynamics on mutual information in future work might allow understanding why overparameterized neural networks still generalize well to unseen data.

A. Capacity in Learned-Variance Gaussian Mean Field Inference

The capacity per dimension for the model discussed in subsection 2.3 is given by

$$I(\hat{\theta}_i, (\theta_i, \sigma^2_i)) = H(\hat{\theta}_i) - H(\hat{\theta}_i | \theta_i, \sigma^2_i)$$

$$= -\int_{-\infty}^{\infty} p'(\hat{\theta}_i) \log p'(\hat{\theta}_i) d\hat{\theta}_i - \int_0^\infty p'(\sigma^2_i) \frac{1}{2} \log 2\pi e \sigma^2_i d\sigma^2_i$$

(14)

$$\tilde{\theta}_i | \theta_i, \sigma^2_i \sim N(\theta_i, \sigma^2_i) \text{ with } \theta_i \sim N\left(0, \frac{\beta}{2}\right) \text{ implies } \hat{\theta}_i | \sigma^2_i \sim N\left(0, \frac{\sigma^2 + \frac{1}{\beta}}{2}\right).$$

Together with $\sigma^2_i \sim \Gamma\left(\frac{\beta}{2} + 1, \frac{\beta}{2}\right)$, this implies

$$p'(\hat{\theta}_i) = \int_0^\infty d\sigma^2_i p'(\sigma^2_i) p'(\hat{\theta}_i | \sigma^2_i)$$

$$= \int_0^\infty d\sigma^2_i \frac{1}{\Gamma\left(\frac{\beta}{2}\right)} \left(\frac{\beta}{2}\sigma^2_i e^{-\sigma^2_i} \right)^{\frac{\beta}{2}}$$

$$\times \left(2\pi \left(\sigma^2_i + \frac{1}{\beta}\right)\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\sigma^2_i}$$

(15)

Numerical results for the capacity $I(\hat{\theta}_i, (\theta_i, \sigma^2_i))$ with varying $\beta$ are given below and plotted in Figure 2.

| $\beta$ | $I(\hat{\theta}_i, (\theta_i, \sigma^2_i))$ |
|--------|----------------------------------|
| 0.01   | 0.68 bits                        |
| 0.1    | 0.65 bits                        |
| 1      | 0.45 bits                        |
| 10     | 0.12 bits                        |
| 100    | 0.014 bits                       |
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