TIME-DEPENDENT PERPENDICULAR TRANSPORT OF FAST CHARGED PARTICLES IN A TURBULENT MAGNETIC FIELD

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Received 2011 January 3; accepted 2011 April 4; published 2011 May 27

ABSTRACT

We present an analytic derivation of the temporal dependence of the perpendicular transport coefficient of charged particles in magnetostatic turbulence, for times smaller than the time needed for charged particles to travel the turbulence correlation length. This time window is left unexplored in most transport models. In our analysis all magnetic scales are taken to be much larger than the particle gyroradius, so that perpendicular transport is assumed to be dominated by the guiding center motion. Particle drift from the local magnetic field lines (MFLs) and magnetic field line random walk are evaluated separately for slab and three-dimensional (3D) isotropic turbulence. Contributions of wavelength scales shorter and longer than the turbulence coherence length are compared. In contrast to the slab case, particles in 3D isotropic turbulence unexpectedly diffuse from local MFLs; this result questions the common assumption that particle magnetization is independent of turbulence geometry. Extensions of this model will allow for a study of solar wind anisotropies.

Key words: cosmic rays – ISM: magnetic fields – turbulence

1. INTRODUCTION

The behavior of individual fast charged particles in magnetic turbulence is relevant to a number of problems in plasma astrophysics, from the solar wind (e.g., Bruno & Carbone 2005) to the interstellar medium (e.g., Elmegreen & Scalo 2004) and cosmic rays at highest energy (e.g., Fraschetti 2008). However, in contrast to cosmic rays with energies beyond the GeV scale, a thorough understanding of the particle transport properties can be attained only in interplanetary space, where in situ measurements of both the magnetic turbulence energy spectrum and particle energy are possible. Diffusion theory (e.g., Jokipii 1966), as the main tool to study charged particle propagation in magnetic turbulence, yields a statistical description of a population of particles and relies on the approximation (Jokipii 1972) that a characteristic time $T$ exists that is much larger than the correlation time $t_c$ of the magnetic field fluctuations (as seen by the particle) but also much smaller than the timescale of both the variation of these fluctuations and of the average distribution function. The Vlasov–Boltzmann equation for the charged particles’ phase-space distribution function can be therefore considerably simplified to terms of the second-order moments of the magnetic field fluctuations. In this scenario higher-order moments are not necessary to determine the particles’ motion as the process is Markovian; diffusion is governed by the central limit theorem (Chandrasekhar 1943).

Observational constraints posed by the heliospheric environment on perpendicular diffusion across the average magnetic field involving, e.g., jovian electrons (Chenette et al. 1977), have not yet been included in a first-principles unified theoretical picture. Perpendicular diffusion occurring in the ecliptic plane is invoked as a plausible explanation of the time delays in solar energetic particle events detected by Helios (Wibberenz & Cane 2006). A more remarkable longitudinal separation in the combined electron observations by Stereo A/B and SOHO from the 2010 January 17 event suggests a strong diffusion perpendicular to the mean magnetic field. The access of high-energy particles to high-latitude heliospheric regions observed by Ulysses (Malandraki et al. 2009) is dominated by particle propagation along the mean magnetic field lines (MFLs), although cross-field diffusion cannot be excluded.

Perpendicular transport in a magnetic field depending on two or fewer space coordinates originates only from the meandering of the MFLs (Jokipii et al. 1993) whereas in an arbitrary three-dimensional (3D) turbulence it also emerges as a general property of the particle motion (Giacalone & Jokipii 1994). A kinetic approach has been applied to perpendicular scattering of strongly magnetized charged particles by using a model for the collision integral (Chuvilgin & Ptuskin 1993). Recent numerical simulations (Minnie & Jokipii 1993) investigated the common assumption that the charged-particle gyrocenter follows the MFLs: the approach of the guiding center cross-field motion to the transverse field line random walk for various parallel mean free paths is studied for a solar-wind-like turbulence. However, we note that the assumption that the guiding center follows the MFLs does not result directly from the equation of motion; therefore it may be realized only for particular turbulence models.

An approximate diffusive perpendicular transport model based on the guiding center motion (nonlinear guiding center, NLGC) has been put forward in Matthaeus et al. (2003); this provided a method to compute magnetic fluctuations along a perturbed particle trajectory. However, the NLGC’s assumption that the probability density of perpendicular displacement is diffusive at all times is a limitation of this model. The subdiffusive nature of perpendicular transport in a slab turbulence was not recovered, contrary to the expectation from the conservation of canonical momentum of ignorable coordinates (Jokipii et al. 1993) and to the findings from test particle numerical simulations in turbulence having a dominant slab component (Qiu et al. 2002). NLGC has been extended to the early phase of perpendicular scattering (Le Roux et al. 2010), where the probability distribution function of the waiting time between two scatterings decays more slowly than exponential, therefore...
including memory effects (non-Markovian process). For a review of other improvements of NLGC see the references in le Roux et al. (2010).

In this paper, we explore the transition to the diffusion regime in a magnetostatic turbulence by using the first-order orbit theory as proposed by Rossi & Olbert (1970), which is based on two assumptions: (1) the particle gyroradius is much smaller than any variation length scale of magnetic field and (2) the turbulent magnetic energy is much smaller than the average magnetic field energy. This allows us to disentangle the cross-field particle motion into two separate components: field lines meandering and gradient/curvature drift from the local field line. In the present paper, the drift is meant to be the transverse gyroperiod-averaged motion of the guiding center away from the local field line (Rossi & Olbert 1970). Field line meandering has been recognized in early solar turbulence studies to be the main contribution to motion perpendicular to average field direction (Jokipii 1966; Jokipii & Parker 1968); the diffusive nature of the field line spread has been related to the turbulence power spectrum power-law index at low wavenumber (Ragot 1999). On the other hand, individual particle gradient/curvature drift on scales smaller than the correlation scale does not seem to have been the object of theoretical investigation (see however Schlickeiser & Jenko 2010), on the basis of the common belief that particle magnetization does not depend on the particular turbulence geometry, in contrast to recent numerical findings (Tautz & Scalici 2010). Therefore, the knowledge of particle trajectory along and across local field lines has remained completely undetermined. In this paper, we shed light on this distinction by analyzing cases propedentially relevant to the solar wind propagating cosmic rays and astrophysical blast waves of supernova remnants.

2. TRANSVERSE GUIDING CENTER DRIFT

We consider a process of propagation of a charged particle in magnetic turbulence which is statistically homogeneous in time, i.e., the velocity correlation depends only on time difference along the orbit. The instantaneous mean square displacement along the space coordinate $x$ after a time $\Delta t = t - t'$ for a particle propagating in an arbitrary medium can be defined as (Taylor 1922; Green 1951; Kubo 1957)

$$\langle (\Delta x)^2 \rangle = 2 \int_0^{\Delta t} d\xi (\Delta t - \xi) \langle v_x(t') + \xi v_x(t') \rangle,$$  

(1)

where $t'$ is an arbitrary initial time, $\xi$ the time lag, and the ensemble average $\langle \cdots \rangle$ is meant to be the average over a population of particles and over an ensemble of turbulence realizations. We note that Equation (1) applies for any value of $\Delta t$. We define

$$d_{xs}(t) = \frac{1}{2} \frac{d}{dt} \langle (\Delta x)^2 \rangle = \int_0^t d\xi \langle v_x(t' + \xi) v_x(t') \rangle.$$  

(2)

The standard perpendicular coefficient of diffusion can then be defined as

$$\kappa_{xs} = \lim_{t \to \infty} d_{xs}(t).$$  

(3)

We here investigate the transverse motion of a low-rigidity particle for a time smaller than $t_s$ in general 3D magnetostatic turbulence. The diffusion approximation therefore may not be valid.

We consider a spatially homogeneous, fluctuating, time-independent magnetic field. The amplitude of the fluctuation $\delta B$ is assumed to be much smaller than the average field magnitude $B_0$. We represent such a magnetic field as $B(x) = B_0 + \delta B(x)$, with an average component $B_0 = B_0 e$, and $\delta B(x) = 0$ and $\delta B(x)/B_0 \ll 1$. This approximation is known to be valid in several turbulent media, such as the solar wind, where the propagation of the magnetic fluctuation is much smaller than the velocity of the bulk ionized fluid. We will make use of the first-order orbit theory (Rossi & Olbert 1970): the particle gyroradius $r_g$ is much smaller than the length scale of any magnetic field variation:

$$r_g \ll \min_{i,j=1,3} \left| \frac{B_i}{\partial_j B_i} \right|,$$  

(4)

where $B_i$ is the $i$th component of the perturbed field $B(x)$. No further assumption is made on the spatial dependence of $\delta B$ or geometry. In this approximation, we consider the guiding center motion. In a spatially varying magnetic field, the guiding center may drift significantly from the average field direction due to the action of the field gradient on the particle magnetic moment. We therefore consider non-zero gradient and curvature drifts. We estimate that drift and resulting displacement after a time shorter than the correlation time. The guiding position $X(t) = (X, Y, Z)$ for a particle of mass $m$, charge $Ze$, and momentum $p$ having coordinate $x(t) = (x, y, z)$ is described, in c.g.s. units, by

$$X(t) = x(t) - \frac{c}{Ze} \frac{B \times p(t)}{B^2}.$$  

(5)

If the scales of magnetic fluctuation are much larger than the gyroradius $r_g$, the guiding center motion defined in Equation (5) has the role of effective gyroperiod-averaged motion. Therefore, Equation (5) can well describe the motion perpendicular to the local magnetic field. In the case of “finite Larmor radius,” the gyroradius only represents the typical scale of particle motion and Equation (5) provides the instantaneous guiding center position and gyroperiod average becomes meaningless. In the magnetostatic field described above, the guiding center velocity transverse to the field $B(x)$ is given to first order in $\delta B(x)/B_0$ by the gyroperiod average (Rossi & Olbert 1970)

$$V_{\perp G}(t) = \frac{v p c}{Ze B_0} \left[ 1 + \frac{\mu^2}{2} \frac{B \times \nabla B + \mu^2 B (\nabla \times B)_{\perp}}{B^2} \right].$$  

(6)

where $\alpha$ is the particle pitch angle and $\mu = \cos \alpha$. Equation (6) gives the most general first-order expression of the guiding center velocity orthogonal to the local magnetic field direction (Balescu 1988). Here, the variation of $\alpha$ is assumed to be negligible over a gyroperiod. As $V_{\perp G}(t)$ is a gyroperiod average, magnetic field can be computed at the guiding center position during that gyroperiod. In contrast to Matthaeus et al. (2003), the transverse motion of the guiding center from the field line is not parameterized in the present paper through some constants to be inferred from numerical simulations, but is described directly from the equation of motion of the guiding center. The finite-time average square transverse displacement of the particle from the direction of local $B$ due to drift $d_{D_0}(t)$ can then be written in this approximation using Equation (2):

$$d_{D_0}(t) = \int_0^t d\xi \langle V_{\perp G}(t') V_{\perp G}(t' + \xi) \rangle.$$  

(7)
where \(i\) stands for any transverse coordinate, \(X\) or \(Y\). The average square displacement is computed from the following expression, to lowest order in \(\delta B / B_0\):

\[
d_{D_{\nu}}(t) \simeq \left( \frac{vpc}{ZeB_0^2} \right)^2 \int_0^t dx \left[ \frac{1 - \mu^2}{2} \partial_j \delta B_3 + \mu^2 \partial_3 \delta B_j \right] |x(t')| \int_0^t dx \left[ \frac{1 - \mu^2}{2} \partial_j \delta B_3 + \mu^2 \partial_3 \delta B_j \right] |x(t' + \xi)|,
\]

(8)

where \((i, j) = (1, 2)\) or \((2, 1)\) and the fields are evaluated at the perturbed particle position \(x(t)\). The average square of the displacement \(d_{D_{\nu}}(t)\) does not depend on the sign of the electric charge \(Ze\), at variance with the drift velocity in Equation (6). We note that in the first-order orbit approximation the particular case of two-dimensional (2D) turbulence defined by \(\delta B(x, y) = (\delta B_x, \delta B_y, 0)\) provides a zero transverse velocity drift; this is because, on the right-hand side of Equation (8), this form of turbulence has \(\delta B_3 = 0\) and also \(\delta B_j\) does not depend on the \(z\)-coordinate. Thus, this analytic method cannot be applied to the composite slab/2D solar wind model of Bieber et al. (1996), a very useful but empirical description of the MHD-scale turbulence in a slow solar wind. Moreover, the slab/2D model could be incomplete as the non-wave (“2D”) turbulence might have an additional component along \(B_0\). This implies, due to the sub-diffusive nature of the perpendicular particle transport in slab turbulence, drifts from the local field line found in numerical simulations for the composite model (Minnie et al. 2009) are second-order contributions. Different anisotropies may be compatible with large-scale solar wind observations; in this paper we indicate a possible alternative method.

We may simplify the derivation by using the Fourier representation of \(\delta B(x)\):

\[
\delta B(x) = \Re(\int_{-\infty}^{\infty} d^3k \delta B(k) e^{ik \cdot x(t)}),
\]

(9)

where \(\Re(\cdot)\) stands for the real part and \(x(t)\) is the particle position at time \(t\). Therefore, the average displacement in Equation (8) contains terms of type

\[
\partial_l \delta B_j(x) = \Re(\int_{-\infty}^{\infty} d^3k \delta B_j(k) \bar{v}_l) e^{ik \cdot x(t)}
\]

(10)

with \(l, j = 1, 2, 3\). We compute the particle position in Equation (10) along the local magnetic field: \(x(t) = x_0(t) + x_{MFL}(z(t))\), where the unperturbed particle orbit is \(x_0(t) = (\sin \theta \sqrt{1 - \mu^2 / \Omega^2} - \cos \phi \sqrt{1 - \mu^2 / \Omega^2}, v_t(t))\); here \(v_t\) is the unperturbed particle velocity along \(z\), \(\phi\) is the particle azimuth angle in the plane orthogonal to \(B_0\), and \(\Omega = ZeB_0/(m\gamma c)\) is the particle gyrofrequency in the background field containing the Lorentz factor \(\gamma\); \(x_{MFL}(z(t)) = (x_{MFL}, y_{MFL}, z(t))\) is the offset in the plane orthogonal to \(B_0\) due to the magnetic field line random walk (MFLRW) at \(z = z(t)\). The assumption of ballistic motion along \(B_0\), i.e., \(z = v_t t\), relies on the choice \(\delta B \ll B_0\), at times smaller than the correlation time of the perpendicular fluctuation, a fortiori we cannot assume parallel diffusion. At the small length scales considered here, parallel and perpendicular motions can be disentangled and any non-Markovian parallel motion, e.g., the memory effect of a particle tracing back its trajectory, is not expected to interfere with the perpendicular transport, in contrast to the case of compound diffusion. We can write

\[
e^{ik \cdot x(t)} \simeq e^{ik \cdot x_0(t)} e^{i k \cdot x_{MFL}(z(t))}.
\]

(11)

The MFLs are defined by \(dx_{MFL} \times B = 0\). This implies that a finite distance \(\Delta x_{MFL}\) in the ballistic approximation is a first-order term in \(\delta B\): \(\Delta x_{MFL} \simeq (\delta B / B_0) v_t t\). Therefore, the exponential \(e^{i k \cdot x_{MFL}(z(t))}\) contributes only at zero order in Equation (11) and the fluctuation in Equation (10) can be computed along the unperturbed trajectory: \(e^{i k \cdot x(t)} \simeq e^{i k \cdot x_0(t)}\), which is equivalent to the quasi-linear approximation. To first order in Equation (10) we can replace the exponential as

\[
e^{i k \cdot x_0(t)} \simeq e^{i (W \sin(\phi - \phi) + k_l v_t t)}.
\]

(12)

where \(\psi = \psi_1 - \psi_2, k_1 = k_z, k_\perp = \sqrt{k_x^2 + k_y^2}, \) and \(W = k_\perp \sqrt{1 - \mu^2 / \Omega^2} = k_\perp r_s\).

By using the Bessel function identities (see Abramowitz & Stegun 1964, Equation (9.1.41))

\[
e^{i \pi \sin \phi} = \sum_{n = -\infty}^{\infty} J_n(z) e^{i n \phi}.
\]

(13)

Equation (11) is rewritten as (see also Schlickeiser 2002, Section 12.2.1)

\[
e^{i k \cdot x_0(t)} = \sum_{|n| = \infty}^{\infty} J_n(W) e^{i k_3 \psi + i n \phi}.
\]

(14)

The magnetic fluctuation space derivative in Equation (10) can then be written in the following way:

\[
\delta_\nu \delta B_j(x) = \Re(\sum_{|n| = \infty}^{\infty} \int_{-\infty}^{\infty} d^3k \delta B_j(k)(ik_n) J_n(W)\times e^{i k_3 \psi + i n \phi} \cdot \partial_p \delta B^*_p[x(t' + \xi)]).
\]

(15)

with \(l, j = 1, 2, 3\). The typical term in Equation (8) is of type

\[
\Re(\left( \frac{vpc}{ZeB_0^2} \right)^2 F(\mu^2) \int_0^t dx \left[ \partial_l \delta B_j(x(t')) \cdot \partial_p \delta B^*_p(x(t' + \xi)) \right];
\]

(16)

here \(F(\mu^2)\) represents various \(\mu\) factors resulting from the expansion of Equation (8). Using Equation (15), we obtain for Equation (16)

\[
\Re(\left( \frac{vpc}{ZeB_0^2} \right)^2 F(\mu^2) \times \left[ \sum_{|n| = \infty}^{\infty} \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d^3k' \delta B_j(k) \times \delta B^*_p(k') J_n(W)(ik_n) J_n(W)(-ik'_p) \times e^{i(k_3 - k'_3) \psi + i(n \phi + \mu 2 \Omega \xi)} \right])
\]

(17)

We assume the standard inertial range magnetic turbulence power spectrum which is uncorrelated at different wavenumber vectors:

\[
\langle \delta B_j(k) \delta B^*_p(k') \rangle = \delta(k - k') P_{\Omega}(k).
\]

(18)
Thus, Equation (17) reduces to
\[ \Re \left( \frac{v_{pc}}{\gamma e B_0^2} \right)^2 \int_{-\infty}^{\infty} d^3k R(k, t) P_{eq}(k) \hat{k}_i \hat{k}_j f^2_n(W) \]
whose time dependence is entirely contained in
\[ R(k, t) \equiv \int_0^t d\xi e^{-i(k_i v_i + n\Omega \xi)} = e^{-i(k_i v_i + n\Omega t)} - \frac{1}{i(k_i v_i + n\Omega)}. \]
(20)

Since \( J_n(W) = 0 \), where \( J_n \) stands for imaginary part, we may consider \( \Re R(k, t) \):
\[ \Re R(k, t) = \frac{\sin[(k_i v_i + n\Omega t)]}{k_i v_i + n\Omega}. \]
(21)

The orthogonal scale \( 1/k_\perp \) can be estimated as \(|B_i/\hat{\theta}, B_j|\); thus Equation (4) states
\[ W \sim k_\perp v/\Omega \lesssim 1. \]
(22)

For \( W \ll 1, J_0(W) \gg J_n(W) \) for \( n \geq 1 \); moreover, \( \Re R(k, t) \sim 1/n \) for large \( n \). We may therefore approximate the sum in Equation (19) by its term with \( n = 0 \). Therefore Equation (19) yields, using Equation (21), four terms of type
\[ \left( \frac{v_{pc}}{\gamma e B_0^2} \right)^2 \int_{-\infty}^{\infty} d^3k P_{eq}(k) \hat{k}_i \hat{k}_j \sin[(k_i v_i + n\Omega t)]/k_i v_i, \]
(23)
with indices \((r, q, l, p) = (3, 3, 2, 2), (3, 2, 2, 3), (2, 3, 3, 2), (2, 2, 3, 3)\) for \( d_{MFL}^r \) and \((r, q, l, p) = (3, 3, 1, 1), (3, 1, 1, 3), (1, 3, 1, 3), (1, 1, 3, 1)\) for \( d_{MFL}^p \). Equation (23) represents the general term contributing to the first-order transverse drift coefficient of a particle in a static first-order perturbed magnetic field.

3. MAGNETIC FIELD LINE RANDOM WALK

In this section, we compute the contribution to the time-dependent particle transverse transport due to MFLRW. If the correlation function of the magnetic fluctuation is homogeneous in space, the mean square displacement of the MFL orthogonal to the z-axis can be defined, in analogy to Equation (2), as
\[ d_{MFL}(z) = \frac{1}{2} \frac{d}{dz} \langle (\Delta x_{MFL})^2 \rangle(z) \]
\[ = \frac{1}{B_0^2} \int_0^z dz' \langle \Delta B_x(x_z) \Delta B_x(x_0) \rangle. \]
(24)

We compute the magnetic turbulence \( \delta B(x) \) in Equation (24) along the unperturbed trajectory of a particle traveling with zero pitch-angle, as in Equations (9) and (12). The motion along the average field is then ballistic, i.e., \( z = v_0 t \). In these approximations, the transverse displacement of an MFL corresponding to a distance \( v_0 t \) along \( B_0 \) traveled by a low-rigidity particle can be written as
\[ d_{MFL}(t) = \frac{1}{B_0^2} \int_0^\infty d^3k P_{eq}(k) \sin[(k_i v_i + n\Omega t)/k_i]. \]
(25)

Equation (25) is in agreement with Equation (17) of Shalchi (2005) derived for pure slab turbulence. The MFL coefficient of diffusion describing the random walk of the field lines can then be defined as
\[ \kappa_{MFL} = \lim_{t \to \infty} \frac{d_{MFL}(t)}{t}. \]
(26)

In our approach, MFL diffusion cannot be assumed because traveled distance \( z \) smaller than parallel correlation lengths is considered; nevertheless, a simple ballistic motion along the z-axis allows us to recover the standard result of MFL perpendicular diffusion in quasi-linear theory (QLT). The discussion of the previous two sections implies that the instantaneous coefficient of diffusion perpendicular to the average magnetic field \( B_0 \) is given, in the presence of weak turbulence and neglecting parallel scattering, by two contributions, i.e., the random walk of the field line and the guiding center drift from the field line:
\[ d(t) = d_p(t) + v_1 d_{MFL}(t), \]
\[ \kappa = \kappa_D + v_1 \kappa_{MFL} = \lim_{t \to \infty} [d_p(t) + v_1 d_{MFL}(t)]. \]
(27)

In the next section the previous results are applied to the slab and 3D isotropic turbulences.

4. THE TURBULENCE POWER SPECTRUM

In this section, we apply the approach developed in previous sections to derive the instantaneous transverse particle transport coefficients of both the guiding center drift from the local MFL (see Equation (23)) and the MFL from the average field direction in Equation (25) by using the coherence length of the turbulence to disentangle small- from large-scale contributions to perpendicular diffusion. We will consider two cases: (1) slab turbulence, introduced (Jokipii 1966) to represent the static limit of the solar wind magnetic fluctuations and studied extensively with Monte Carlo numerical simulations; we will compare the result with the QLT limit; and (2) 3D isotropic turbulence, the idealized case likely to provide an unperturbed model for anisotropies observed in the solar wind.

4.1. Slab

We consider slab turbulence, the static limit of transversely and longitudinally propagating Alfvén waves: \( \delta B = e B(z) \) and \( \delta B(x) \cdot e_z = 0 \). In this case, from Equation (8), \( d_{MFL}^p(t) = d_{MFL}^p(t) = d_{MFL}^p(t) \). The turbulence wavenumber is aligned to the average magnetic field; thus we adopt the following form of the power spectrum: \( P_{eq}(k) = G(k) \delta(k/L_\parallel)/L_\parallel^2 \delta \) with \( r, q = 1, 2, \) and \( P_{eq}(k) = 0 \) with \( i = 1, 2, 3 \). The one-dimensional (1D) spectrum is assumed to be of Kolmogorov type. Observations of electron-density fluctuations inferred from scintillation measurements exhibit a Kolmogorov power law, with index approximately equal to 5/3, over five orders of magnitude (Armstrong et al. 1995). Several other observations of magnetic turbulent media, from Earth’s magnetosphere to galaxy clusters, validate the Kolmogorov power spectrum up to a range of 12 orders of magnitude. We mention that solar wind observations show that at scales smaller than the ion thermal gyroradius \((-10^7 \text{ cm around the Earth})\), much smaller than the scales considered in this paper, the magnetic turbulence spectrum deviates from Kolmogorov, having an index of \(-2.12 \) (Bale et al. 2005). At length scales larger than the coherence length the measured interplanetary magnetic turbulence is well described by a flattening power spectrum (Hedgcock 1975; Bieber et al. 1994). On the other hand, a consistent comparison with the quasi-linear limit requires the power spectrum to be defined at scales larger than coherence length, i.e., for \( k_i < k_i^{\min} \), up to the physical scale of the system \( 2\pi/k_i^{\min} \); we will adopt here a simplified form:
\[ G(k) = \begin{cases} G_0 k_i^{-q} & \text{if } k_i^{\min} < k_i < k_i^{\max} \\ G_0(k_i^{\min})^{-q} & \text{if } k_i^{\min} < k_i < k_i^{\min} \end{cases} \]
(28)
where \( k_{\text{max}} \) corresponds to the scale where the dissipation rate of the turbulence overcomes the energy cascade rate. The choice of a constant power spectrum at large scales instead of a function smoothly connected to the inertial range already used in the literature is dictated merely by easier mathematical tractability. Here, \( q = 5/3 \) and the constant \( G_0 \) is determined from the normalization
\[
(\delta B)^2 = \int_{-\infty}^{\infty} d^3k (P_{11} + P_{22} + P_{33}),
\]
(29)
implying, using cylindrical coordinates \( (d^3k = dk || k_\perp d k_\perp d \psi) \),
\[
G_0 = \frac{(\delta B)^2(q - 1)}{4\pi q (k_{\text{max}})^{q-1}},
\]
(30)
with the assumption \( k_0^0 \ll k_{\text{min}}^0 \ll k_{\text{max}}^0 \).

We consider first the transverse drift in Equation (23). We average \( F(\mu^2) \) over an isotropic pitch angle distribution. Using cylindrical coordinates \( (d^3k = dk || k_\perp d k_\perp d \psi) \), we have
\[
d_p'(t) = \left( \frac{\nu_p c}{\pi \kappa_0^2} \right)^2 \frac{7}{5} \int_{k_0}^{k_{\text{max}}} d k_\perp k_\perp^2 G(k_\perp) \sin[k_\perp v_\perp t] k_\perp. \]
(31)
In units of the Bohm coefficient of diffusion \( (\kappa_B = (1/3)r_s v) \) and approximating \( r_s/v_\perp \sim \Omega^{-1} \), we obtain
\[
d_p'(t) = \frac{3}{20} \left( \frac{\delta B}{B_0} \right)^2 q - 1 q F(y^0, q), \]
(32)
where we defined
\[
F(y^0, q) = k_{\text{min}}^0 r_s \left[ I(2 - q, y^0) \right]^{2-q} + \frac{\sin y^0}{y^2_\perp} \]
(33)
where the time dependence is contained in the new variable \( y^0 = k_{\text{min}}^0 v_\perp t \sim k_{\text{min}}^0 r_s \Omega t \) (and \( y^0 = k_\perp^0 v_\perp t \sim k_\perp^0 r_s \Omega t \)) and we used
\[
I(a, u) = \int_0^\infty y^{a-1} \sin y dy
\]
\[
= \frac{i}{2} [e^{i\frac{\pi}{4} a} \Gamma(a, iu) - e^{i\frac{\pi}{4} a} \Gamma(a, -iu)],
\]
(34)
where \( \Gamma(a, z) \) is the incomplete gamma function (see Gradshteyn & Ryzhik 1973, Equation (3.761.2)). The time evolution of the drift coefficient \( d_p' \) is depicted in Figure 1. The first term in Equation (33), corresponding to scales smaller than the coherence scale \( 2\pi/k_{\text{max}}^0 \) (\( k > k_{\text{max}}^0 \)), dominates over the second term, corresponding to scales larger than \( 2\pi/k_{\text{min}}^0 \) (\( k < k_{\text{min}}^0 \)). The diffusive behavior can be found by using the approximation of \( \Gamma(a, z) \) for \( |z| = y^0 = k_{\text{min}}^0 v_\perp t \ll 1 \) (see Gradshteyn & Ryzhik 1973, Equation (8.354.2)) because \( y^0 \ll 1 \); since we assume a weak magnetic fluctuation, it is reasonable to assume that the perpendicular diffusion timescale is shorter than the parallel scattering timescale, i.e., \( 1/k_{\text{max}}^0 v_\perp \), or in other terms the diffusion limit is the dominant term in Equation (33) for large \( t \) and \( t < 1/k_{\text{max}}^0 v_\perp \): \( \Gamma(a, z) \sim z^{-a}/a \); therefore \( I(2 - q, y^0) \sim \sin(q\pi/2)\Gamma(2 - q) \) (dashed line in Figure 1).

In the diffusive regime, the second term in Equation (33), representing large scales (\( l > 2\pi/k_{\text{min}}^0 \) or \( k < k_{\text{min}}^0 \)), does not contribute significantly to the particle drift, as is manifest in Figure 1. The average transverse drift coefficient in units of \( \kappa_B \) (rescaled by a factor \( 10^3 \)) as a function of \( \Omega t \) for \( k_{\text{max}}^0 r_s = 10^{-3}, k_{\text{max}}^0/k_{\text{min}}^0 = 10^3, k_{\text{min}}^0/k_0^0 = 10^2, q = 5/3, \) and \( \delta B/B = 0.1 \). The diffusive approximation in Equation (32) is superposed (dashed line). The departure from the perturbed field line of the guiding center, which in the first-order orbit theory represents the real particle motion to a good approximation, decreases to zero. Thus, in a slab turbulence charged particles are tied to the weakly perturbed magnetic field lines.

Figure 1. We find that for slab turbulence, the transverse particle drift coefficient from the local MFL is given by
\[
d_p'(t) = \frac{3}{20} \left( \frac{\delta B}{B_0} \right)^2 q - 1 q \frac{1}{(k_{\text{min}}^0 r_s)^{1-q}} \Omega t^{2-q},
\]
(35)
and is thus sub-diffusive with behavior \( k_{\text{D}}^0(t) \sim t^{-2(q-1)} \) (depicted as the dashed line in Figure 1). Transverse sub-diffusion has also been found by considering timescales longer than the parallel scattering time and therefore allowing parallel scattering in Káta & Jokipii (2000). However, in that case particles are assumed to propagate back and forth along the MFL and to be tied to the MFL. We note that the time evolution of the drift coefficient in Equation (35) confirms that charged particles in a turbulence depending on fewer than three space coordinates remain confined within a gyroradius from the local field line (Jokipii et al. 1993; Jones et al. 1998). The time integration of Equation (35) up to \( t = L_\perp/\nu \sim 2\pi/(k_{\perp}^0 r_s \Omega) \) gives in the case of weak turbulence \( \delta B \ll B_0 \) the condition \( (\Delta x_\perp)^2 \ll r_z^2 \).

We note that the time integral of \( k_{\text{D}}^0 \sim t^{-q-1} \), which provides \( (\Delta x_\perp)^2 \sim t^{-q-1} \), is an increasing function of time for any observed physical value of \( q \); however, as shown above, this result does not contradict the theorem of reduced dimensionality. In summary, the present result has been obtained under three assumptions: (1) ballistic motion in the \( z \)-coordinate (\( z = v_\perp t \)), (2) average displacement transverse to the local field \( B \) due to first-order drift, and (3) a Kolmogorov power spectrum for magnetic fluctuations. Equations (32) and (33) represent the average transverse displacement computed in the first-order orbit approximation at any time smaller than the parallel scattering timescale, so that the approximation of ballistic motion parallel to the mean magnetic field holds.

From Equation (25), the MFLRW in units of magnetic coherence length \( L_\perp = 2\pi/k_{\text{min}}^0 \) is given by
\[
d_{\text{MFL}}'(t) = \left( \frac{\delta B}{B_0} \right)^2 q - 1 q H(y_\perp; q).
\]
(36)
where we defined
\[
H(y_\perp; q) \equiv \left( k_{\text{min}}^0 r_s \Omega \right)^q I(-q, y_\perp) + \cos(y_\perp; q),
\]
(37)
where $\text{Si}(x)$ is the Sine integral function. The first term in Equation (37), corresponding to scales smaller than the coherence scale $2\pi/k_{\text{min}}^*(k > k_{\text{min}})$, is dominated by the second term, corresponding to scales larger than $2\pi/k_{\text{min}}^*(k < k_{\text{min}})$. From Equations (36) and (37), the MFLRW diffusion coefficient is given by $\kappa_{\text{MFL}} = (\delta B/B_0)^2\pi(q - 1)/(4qk_{\text{min}}^*)$. In Figure 2, $\kappa_{\text{MFL}}^*$ is shown to recover the quasi-linear limit and is dominated by large wavelengths, given by the Sine integral term in Equation (37): $D_{\text{MFL}} = \pi^2 G(k_{\text{min}}) = 0)/B_0^2 = \kappa_{\text{MFL}}^*$, where the quasi-linear limit is expressed, as known, as a power spectrum at zero parallel wavenumber.

Equations (36) and (37) provide the MFLRW for distances $\Delta z$ along $B_0$ shorter than $L_1$. The guiding center perpendicular scattering in a slab is described as a series of bunches in the transverse drift which are asymptotically suppressed, confining the transverse motion to follow the MFL meandering as also found in low-rigidity test particle numerical simulations (Qin et al. 2002). Therefore, we confirm that slab transverse transport is due to the meandering of MFLs but we also model the transport across the MFL for first-order magnetic fluctuations not considered in previous treatments (Kôta & Jokipii 2000). For the slab transverse, the slab particle diffusion can be disentangled into two energetic contributions: drift coefficient is dominated by length scales smaller than the coherence length whereas the MFLRW is dominated by length scales larger than the coherence length.

4.2. Isotropic

We consider 3D isotropic turbulence, in which the turbulence $\delta B$ depends on all three space coordinates. We adopt the following power spectrum (Batchelor 1970):

$$P_{ij}(k) = \frac{G(k)}{8\pi k^2} \left[ \delta_{ij} - \frac{k_i k_j}{k^2} \right]$$

with

$$G(k) = \begin{cases} G_0 k^{-q} & \text{if } k_{\text{min}} < k < k_{\text{max}} \\ G_0 k_{\text{min}}^{-q} & \text{if } k_0 < k < k_{\text{min}} \end{cases}$$

where $k_{\text{max}}$ corresponds to the scale where the dissipation rate of the turbulence overcomes the energy cascade rate, the coherence length is given by $L = 2\pi/k_{\text{min}}^*$, and the physical scale of the system is given by $2\pi/k^0$ (for the spectrum at large scales, see the discussion in Section 4.1). The constant $G_0$ is fixed by normalization:

$$G_0 = \frac{(\delta B)^2(q - 1)}{q(k_{\text{min}})^{q-1}},$$

assuming $k_0 \ll k_{\text{min}} \ll k_{\text{max}}$. We use spherical coordinates for the wavenumber $k = (k_0, q, l, p)$ and 3D turbulence: $\delta B(x) = (\delta B_x, \delta B_y, \delta B_z)$. We first compute $d_i^{DXX}(t)$. In reference to Equation (23), the non-zero terms are $(r, q, l, p) = (3, 3, 2, 2), (r, q, l, p) = (2, 3, 2, 3)$, and $(r, q, l, p) = (2, 2, 3, 3)$. Combining the non-zero terms and averaging over an isotropic distribution in pitch angle, this gives

$$d_i^{D_i}(t) = \frac{3}{16} \left( \frac{\delta B_0}{B_0} \right)^2 \frac{q - 1}{q} (k_{\text{min}})^3,$$

with $y_m = k_{\text{min}} v_0 t \simeq k_{\text{min}} r_g \Omega t$ and $y_0 \simeq k_0 r_g \Omega t$; here, the term integrated over scales smaller than the coherence scale $2\pi/k_{\text{min}}^*(or k > k_{\text{min}})$ is recast as

$$F_i^{(1)}(y, q) = \frac{2}{15} \frac{y^{q-1}}{2 - q} (\cos y + y \text{Si}(y)) + \frac{6}{5} y^{2-q} \sin y$$

$$- \frac{14}{5} y^{-q-2} \sin y - \frac{2}{5} q^2 - 14 q + 18 q \left(1 - q, y \right)$$

$$+ \frac{14 y + q}{5} \left(1 - q, y \right),$$

and the term integrated over scales larger than the coherence scale $2\pi/k_{\text{min}}^*(or k < k_{\text{min}})$ is recast as

$$F_i^{(2)}(y, q) = \frac{y}{15} \cos y + y \text{Si}(y)$$

$$+ \sin y \left(\frac{21 - 5 y^2}{15 y^2} - \frac{7 \cos y}{5y} \right),$$

where we approximated $r_g/v_0 t \simeq \Omega t$ as in Equation (32) and used

$$I(a, u) = \int_u^{\infty} y^{a-1} \cos y d y$$

$$= \frac{1}{2} \left[ e^{-i \frac{u}{2} \Gamma(a, i u)} + e^{i \frac{u}{2} \Gamma(a, -i u)} \right].$$

from Gradshtey & Ryzhik (1973), Equation 3.761.7. The instantaneous transverse coefficient of diffusion in Equation (41) is represented in Figure 3. The dominant term in the diffusive limit, with the condition $y_m = k_{\text{min}} v_0 t < 1$, is given by

$$d_i^{D_i}(t) \rightarrow \pi \left( \frac{\delta B}{B_0} \right)^2 \frac{q - 1}{q - 2} (k_{\text{min}} r_g) = k_B^0 k_B$$

and represented in Figure 3. In contrast to the slab, the particle drift from the MFL does not depend only on the power spectrum at length scales smaller than $L_1$. As for the slab, transverse diffusion is axisymmetric: $d_i^{D_{\text{XX}}}(t) = d_i^{D_{\text{YY}}}(t) = d_i^{D_{\text{ZZ}}}(t)$. Statistically, charged particle motion is not tied to the local MFL in 3D isotropic turbulence. The theorem on
Integrating by parts and using Equation (4.421.1) of Gradshteyn & Ryzhik (1973), we find

\[ d \delta B(t) k_{min} = \frac{1}{4} \left( \frac{\Delta B}{B_0} \right)^2 \frac{q - 1}{q} \left[ H_1(y_m, q) + H_2(y_m, y_0, q) \right] \]

(46)

here the term integrated over scales smaller than the coherence scale 2π/k_{min} (or k > k_{min}) is recast as

\[ H_1(y, q) = \frac{1}{q y} (\cos y + y \sin y) + \frac{y - 1}{2 + q} \sin y \]

\[-2 \frac{q}{2 + q} + 2 \frac{q}{2 + q} + 1 \frac{q}{2 + q} \]

\[ \left( -1 - q, y \right) \]

(47)

and the term integrated over scales larger than the coherence scale 2π/k_{min} (or k < k_{min}) is recast as

\[ H_2(y_m, y_0, q) = \int_{y_0}^{y_m} dy \left( \frac{\sin y}{2} - \frac{\sin y}{2} - \frac{\cos y}{2} \right) \]

(48)

Figure 4 shows the diffusive behavior of the MFL for a magnetic turbulence with isotropic wavenumber spectrum. As for the slab case, scales larger than the coherence length (k < k_{min}) dominate the turbulent contribution (see Figure 4). The leading terms in the diffusive limit of H_{1}(y, q) are found in the integral and in the first term in parentheses in Equation (48). Integrating by parts and using Equation (4.421.1) of Gradshteyn & Ryzhik (1973), we find

\[ d \delta B(t) k_{min} = \frac{1}{4} \left( \frac{\Delta B}{B_0} \right)^2 \frac{q - 1}{q} \left[ \frac{\sin(y)}{y} \log(y) \right]^{y_m}_{y_0} + \frac{\pi}{2} \left( C + \frac{1}{2} \right) \]

(49)

where C = 0.577215 is the Euler constant and y_m = k_{min} v_t / l \approx k_{min} r_g \Omega (y \approx k_0 r_g \Omega). We find that the MFL of a 3D isotropic weakly turbulent magnetic field are superdiffusive according to Equation (49). This result does not imply the superdiffusion of the particles which propagate diffusively in a 3D isotropic turbulence (Giacalone & Jokipii 1999). Before being transported superdiffusively along a field line, a particle will undergo parallel scattering, not taken into account in this paper, and eventually decorrelate to another field line. These results might be qualitatively extended to MHD turbulence with a small k anisotropy so that the results in diffusion regime found here still apply. In this case, particle drift retains its diffusive character shown in Figure 3, even if only scales smaller than the coherence length (k > k_{min}, in dot-dashed) are taken into account. Therefore, at small length scales, the perturbative approach based on \( \delta B \ll B_0 \) could be applied to solar wind turbulence. At length scales much larger than the coherence scale (k \ll k_{min}), where the approximation \( \delta B \ll B_0 \) breaks down in the solar wind, these conclusions cannot be extended.

5. DISCUSSION AND CONCLUSION

We have described analytically the time evolution of individual charged particles drift and MFLRW across a static magnetic field to first order in the magnetic fluctuations. We consider the case where the motion perpendicular to the average magnetic field is dominated by guiding center motion which includes the meandering of the MFL and the drift from first-order orbit theory, in the approximation that the particle gyroradius is much smaller than the length scale of magnetic field variations. In contrast to previous models for the perpendicular transport, we do not assume diffusive scattering at all times; this allows us to treat consistently the slab turbulence perpendicular diffusion. Drift and MFL transverse transport are explicitly computed for both the slab and 3D isotropic cases. In the slab case,
the time evolution of the drift displacement shows how the particle diffusion from the MFL is suppressed. The instantaneous slab drift coefficient of diffusion transverse to the local field depends on the turbulence power-law spectral index; for a Kolmogorov turbulence it is found to decrease as $t^{-1/3}$, slower than compound diffusion displacement ($t^{-1/2}$), which is however computed transversely to the average field and not to the local magnetic field as in this paper. The recovery of the MFL coefficient of diffusion of QLT shows that this result does not depend on assuming MFL parallel diffusion. Second, we provide the analytical time dependence of drift and MFL coefficients of diffusion for a 3D isotropic turbulence. We found that for a 3D isotropic turbulence the particle drift from the local field line is diffusive, whereas the field line itself is superdiffusive. Previous numerical simulations are not contradicted by our result, which is obtained neglecting scattering parallel to the mean field. For the slab, we find that MFLRW is dominated by length scales larger than the coherence length whereas particle drift from the local field line is dominated by length scales smaller than the coherence length. This disentanglement does not hold for 3D isotropic turbulence: MFLRW is still dominated by large length scales whereas, as far as drift is concerned, turbulent energy contributes at all scales, both below and above the coherence length. The study carried out here provides a framework for particle transport in the solar wind and supernova remnant blast wave turbulence, and questions the common assumption that cosmic ray trajectories follow the MFL.

It is a pleasure to acknowledge the fruitful discussions with J. Giacalone, J. Kóta, and D. Ruffolo. We thank the anonymous referee for useful suggestions and comments. The work of F.F. was supported by NSF grant ATM0447354 and by NASA grants NNX07AH19G and NNX10AF24G; the work of J.R.J. was partially supported by NASA grant NNX08AH55G.

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