Weak Convergence (IIA) - Functional and Random Aspects of the Univariate Extreme Value Theory

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Dedicatory.

To my wife Mbaye Ndaw Fall who is accompanying for decades
with love and patience

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Abstract. The univariate extreme value theory deals with the convergence in type of powers of elements of sequences of cumulative distribution functions on the real line when the power index gets infinite. In terms of convergence of random variables, this amounts to the weak convergence, in the sense of probability measures weak convergence, of the partial maxima of a sequence of independent and identically distributed random variables. In this monograph, this theory is comprehensively studied in the broad frame of weak convergence of random vectors as exposed in Lo et al. (2016). It has two main parts. The first is devoted to its nice mathematical foundation. Most of the materials of this part is taken from the most essential Loève(1936,177) and Haan (1970), based on the stunning theory of regular, pi or gamma variation. To prepare the statistical applications, a number contributions I made in my PhD and my Doctorate of Sciences are added in the last chapter of the last chapter of that part. Our real concern is to put these materials together with others, among them those of the authors from his PhD dissertations and Science doctorate thesis, in a way to have an almost full coverage of the theory on the real line that may serve as a master course of one semester in our universities. As well, it will help the second part of the monograph. This second part will deal with statistical estimations problems related to extreme values. It addresses various estimation questions and should be considered as the beginning of a survey study to be updated progressively. Research questions are tackled therein. Many results of the author, either unpublished or not sufficiently known, are stated and/or updated therein.

Keywords. .

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Résumé. La théorie des valeurs extrêmes univariées traite de la convergence en type de puissances d’éléments de suites de fonctions de répartition sur l’ensemble des nombres réels lorsque la puissance devient infinie. En termes de convergence des variables aléatoires, cela se reformule par la convergence faible, dans le sens de celle de mesures de probabilité, des maxima partiels d’une suite de variables aléatoires réelles indépendantes et identiquement réparties. Dans ce monographe, cette théorie est étudiée de manière exhaustive, et placée dans le cadre général de la faible convergence des vecteurs aléatoires exposée dans Lo et al. (2016). Il comporte deux parties principales. La première est consacrée à sa très belle fondation mathématique autour des notions variation régulière et de variation pi and gamma. La plupart des matériaux de cette partie proviennent de Loève(1963, 1977) and de Haan (1970). Afin de préparer les applications statistiques, nous avons introduit certaines contribution la thorie dans le dernier chapitre de cette partie. Notre véritable préoccupation, ici, est de mettre ces matériaux ensemble avec d’autres, parmi eux ceux de l’auteur issus des ses thèses de doctorat, de manière à aboutir à une couverture unifiée et presque complète de la théorie sur l’ensemble des nombres réels dans un forme d’un texte pouvant servir pour un cours doctoral d’un semestre dans nos universités. De plus, cela aidera pour la deuxième partie. Justement, cette deuxième partie traitera de problèmes d’estimation statistiques liés aux valeurs extrêmes. Elle aborde diverses questions d’estimation et doit être considérée comme le début d’une revue de résultats du domaine qui devra être mise à jour régulièrement. Des questions de recherche y sont abordées. De nombreux résultats de l’auteur, non publiés ou pas suffisamment connus, y sont énoncés et / ou mis à jour.
jappa té bagna bayi
Doh té bagna taxaw
Yokka té bagna wagni
Weak Convergence (IIA) - Functional and Random Aspects of the Univariate Extreme Value Theory
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General Preface

This textbook is the first of series whose ambition is to cover broad part of Probability Theory and Statistics. These textbooks are intended to help learners and readers, of all levels, to train themselves.

As well, they may constitute helpful documents for professors and teachers for both courses and exercises. For more ambitious people, they are only starting points towards more advanced and personalized books. So, these textbooks are kindly put at the disposal of professors and learners.

Our textbooks are classified into categories.

A series of introductory books for beginners. Books of this series are usually accessible to student of first year in universities. They do not require advanced mathematics. Books on elementary probability theory and descriptive statistics are to be put in that category. Books of that kind are usually introductions to more advanced and mathematical versions of the same theory. The first prepare the applications of the second.

A series of books oriented to applications. Students or researchers in very related disciplines such as Health studies, Hydrology, Finance, Economics, etc. may be in need of Probability Theory or Statistics. They are not interested by these disciplines by themselves. Rather, the need to apply their findings as tools to solve their specific problems. So adapted books on Probability Theory and Statistics may be composed to on the applications of such fields. A perfect example concerns the need of mathematical statistics for economists who do not necessarily have a good background in Measure Theory.

A series of specialized books on Probability theory and Statistics of high level. This series begin with a book on Measure Theory, its counterpart of probability theory, and an introductory book on topology. On that basis, we will have, as much as possible, a coherent
presentation of branches of Probability theory and Statistics. We will try to have a self-contained, as much as possible, so that anything we need will be in the series.

Finally, **research monographs** close this architecture. The architecture should be so large and deep that the readers of monographs booklets will find all needed theories and inputs in it.

We conclude by saying that, with only an undergraduate level, the reader will open the door of anything in Probability theory and statistics with **Measure Theory and integration**. Once this course validated, eventually combined with two solid courses on topology and functional analysis, he will have all the means to get specialized in any branch in these disciplines.

Our collaborators and former students are invited to make live this trend and to develop it so that the center of Saint-Louis becomes or continues to be a reknown mathematical school, especially in Probability Theory and Statistics.
General Preface of Our Series of Weak Convergence

The series Weak convergence is an open project with three categories.

The special series Weak convergence I consists of texts devoted to the core theory of weak convergence, each of them concentrated on the handling of one specific class of objects. The texts will have labels A, B, etc. Here are some examples.

(1) Weak convergence of Random Vectors (IA).
(2) Weak convergence of stochastic processes and empirical processes (IB).
(3) Weak convergence of random measures (IC).
(4) Etc.

The special series Weak convergence II consists of textbooks related to the theory of weak convergence, each of them concentrated on one specialized field using weak convergence. Usually, these sub-fields are treated apart in the literature. Here, we want to put them in our general frame as continuations of the Weak Convergence Series I. Some examples are the following.

(1) Weak laws of sums of random variables.
(3) Univariate Extreme value Theory.
(4) Multivariate Extreme value Theory.
(5) Etc.

The special series Weak convergence III consists of textbooks focusing on statistical applications of Parts of the Weak Convergence Series I and Weak Convergence Series II. Examples:

(1) A handbook of Gaussian Asymptotic Distribution Using the Functional Empirical Process.

(2) A handbook of Statistical Estimation of the Extreme Value index.

(1) etc.
Part 1

The portal
In this chapter, we give a global picture and basic notation in univariate extreme value theory in the independent and identical distribution setting. Theoretically, readers solely interested in the statistical estimation might just read this portal and go Part II reserved to Statistical aspects.

We will guide the reader how to use these components if he wishes to expertise himself in this theory. But he will find all what he needs to go with us in the statistical and probability aspects we will be dealing with in the first part.

As this monograph is part of our Probability and Statistics series, it should be read after the basic introduction of weak convergence theory that was fully and broadly exposed in [Lo (2016c)]. This monograph is cited all along this one. For example, the broad theory of weak convergence is rounded up the first section, with a special focus on specific tools.

First of all, we will have to deal with a sequence $X_1, X_2, \ldots$ of independent copies (s.i.c) of a real random variable ($rv$) $X$ with $df$ $F(x) = \mathbb{P}(X \leq x)$, all of them being defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For each $n \geq 1$, the order statistics associated with the sample $X_1, X_2, \ldots, X_n$ is denoted as $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$.

The support of the distribution function $F$ is $[\text{lep}(F), \text{uep}(F)]$ where

\[
\text{lep}(F) = \inf\{x \in \mathbb{R}, F(x) > 0\}
\]

is called the lower endpoint of $F$ (denoted $\text{lep}$) and

\[
\text{uep}(F) = \sup\{x \in \mathbb{R}, F(x) < 1\}
\]

is its upper endpoint (denoted $\text{uep}$).
In the Univariate Extreme Value Theory (UEVT), particularly in the extreme value index estimation, we often have to use the logarithm transformation \( Y = \log^+ X \), with \( \log^+ x = (\log x) \mathbb{1}_{(x>0)} \) where \( \mathbb{1}_A \) stands for the indicator function of the set \( A \).

It would be better and simpler to work with the logarithm function in place of \( \log^+ \). So we may add an additional assumption based on the following arguments. Indeed, we are mainly concerned by extreme values of the observations near \( \text{uep}(F) \). If \( \text{uep}(F) \leq 0 \), we translate all the data by a real \( t > 0 \) large enough to ensure that \( \text{uep}(F) + t > 0 \). The translated data are associated to the distribution function \( F(\circ - t) \) with upper endpoint \( \text{uep}(F) + t > 0 \). In both cases \( \text{uep}(F) \leq 0 \) and \( \text{uep}(F) > 0 \), we may assume that, eventually at the cost translating the data, that the extreme values of the observations are positive. Based on these facts, we may and do assume from now that the random variable is positive that is \( X > 0 \) and we simply write \( Y = \log X \).

Hence \( Y_1, Y_2, \ldots \) will be a sequence of independent copies of \( Y \) with \( df \ G(y) = F(e^y), \ y \in \mathbb{R} \) and \( Y_{1,n} \leq Y_{2,n} \leq \ldots \leq Y_{n,n} \) are the order statistics associated with \( Y_1, Y_2, \ldots, Y_n \).

1.Weak convergence

Extreme value theory in \( \mathbb{R} \) begins with the knowledge of the asymptotic law of the sequence of the maxima \( X_{n,n} \) when \( n \) tends to infinity under the hypothesis that the observations are independent and identically distributed. This means that the extreme value theory with this respect is part of the weak convergence theory.

The weak convergence theory in \( \mathbb{R}^k, \ k \geq 1 \) is exposed in details in Chapter 2 in [Lo (2016c)], and more specifically in Chapter 4 of the same textbook, for real random variables. All the needs of the UEVT regarding weak convergence are to be found there. In this particular case, the link of this theory with generalized inverses in \( \mathbb{R} \) is described details in Chapter 4, [Lo (2016c)] refcv.R, along with other useful and specific tools.

The general of weak convergence in metric spaces is provided in Chapter 2 in [Lo (2016c)], this chapter being largely inspired by the book of [Billingsley (1968)] and that of [van der Vaart(1996)].
Here, we are going only to give the main tools we have need to ensure a linear reading of the book.

To introduce the weak convergence in \( \mathbb{R} \), consider a sequence of real random variables \( Z_1, Z_2, \ldots \) with distribution functions \( H_1, H_2, \ldots \)

**Definition 1.** The sequence of real random variables \( Z_1, Z_2, \ldots \) with distribution functions \( H_1, H_2, \ldots \) converges weakly to a real random variable \( Z \) with distribution function \( H \) iff and only if one of these equivalent assertions holds.

(i) For any continuous and bounded functions \( f : \mathbb{R} \rightarrow \mathbb{R} \),
\[
\lim_{n \to +\infty} \int f d\mathbb{P}_{Z_n} = \int f d\mathbb{P}_Z.
\]

(ii) For any continuity point \( x \) of \( H \),
\[
\lim_{n \to +\infty} H_n(x) = H(x).
\]

(iii) For any \( u \in \mathbb{R} \),
\[
\lim_{n \to +\infty} \int e^{iux} d\mathbb{P}_{Z_n}(x) = \int e^{iux} f d\mathbb{P}_Z.
\]

When \( (Z_n)_{n \geq 1} \) weakly converges to \( Z \), we mainly use the notation
\[
Z_n \rightsquigarrow Z \text{ or } H_n \rightsquigarrow H
\]

and we may also use \( Z_n \rightarrow^w Z \) (\( w \) standing for weakly) or \( Z_n \rightarrow^d Z \) (\( d \) standing for \( : \) in distribution). We also shift to the distribution functions and say : \( (H_n)_{n \geq 1} \) weakly converges to \( H \).

Point (i) is the main definition. Points (ii) and (iii) are parts of what is called the Portmanteau Theorem (See Theorem 2 for the general case and Theorem 3 for the particular case of \( \mathbb{R}^k \), Chapter 2, [Lo (2016c)]).

An interesting property is that the convergence of the distribution functions is uniform when \( H \) is continuous, in the particular case of \( \mathbb{R} \). This gives :
Proposition 1. The sequence of real random variables \( Z_1, Z_2, \ldots \) with distributions functions \( H_1, H_2, \ldots \) converges weakly to a real random variable \( Z \) with a continuous distribution function \( H \) iff and only iff
\[
\lim_{n \to +\infty} \sup_{x \in \mathbb{R}} |H_n(x) - H(x)| = 0.
\]
The proof of this is given in Point (5) in Chapter 4, [Lo (2016c)].

When all the \( Z_1, Z_2, \ldots \) and \( Z \) are absolutely continuous with respect to a \( \sigma \)-finite measure in \( \mathbb{R} \), with Radon-Nikodym derivatives, denoted by \( f_{Z_n}, n \geq 1 \) and \( f_Z \), we have the following result.

Theorem 1. (Schéffé). If for any \( x \in \mathbb{R} \),
\[
\lim_{n \to +\infty} f_{Z_n}(x) = f_Z(x), \text{ v.a.e,}
\]
then the sequence \( (Z_n)_{n \geq 1} \) weakly converges to \( Z \) and
\[
\sup_{B \in \mathcal{B}(\mathbb{R})} |\mathbb{P}_{Z_n}(B) - \mathbb{P}_Z(B)| = \frac{1}{2} \int |f_{Z_n} - f_Z| d\nu \to 0 \text{ as } n \to \infty.
\]

Proof. See proof of Theorem 4 in Chapter 2, in Lo [Lo (2016c)], or [Billingsley (1968)].

This theorem of Schéffé is very handy when one needs to find rate of convergence with presence of probability densities.

But weak convergence in \( \mathbb{R} \) may be done entirely with inverses functions. Define the inverse function of the distribution function \( H \) by
\[
H^{-1}(u) = \inf\{x \in \mathbb{R}, F(x) \geq u\}, 0 \leq u \leq 1.
\]
If \( H \) is right-continuous and nondecreasing, \( H^{-1} \) is also nondecreasing but left-continuous, as proved in Point 3 in Chapter 4 in [Lo (2016c)].
Let us point right now these two equivalence formulas:

\[
H^{-1}(u) \leq t \iff u \leq F(t)
\]
and
\[
H^{-1}(u) > t \iff u > F(t).
\]
In Chapter 4 in \( \text{[Lo (2016c)]} \), we expose and prove a long list of properties of generalized function of nondecreasing functions not necessarily with values in \([0,1]\). We will discover these properties when needed. Right now we need the Points 4 and 5 in just mentioned section. The first is

**Lemma 1.** Let \( (H_n)_{n \geq 1} \) weakly converges to \( H \), then \( (H_n^{-1})_{n \geq 1} \) weakly converges to \( H^{-1} \), that is for any continuity point of \( H^{-1} \),

\[ H_n^{-1}(x) \to H(x) \text{ as } n \to +\infty. \]

The second is

**Lemma 2.** \( (H_n)_{n \geq 1} \) weakly converges to a continuous distribution function \( H \) of ans only if

\[ \sup_{x \in \mathbb{R}} |H_n(x) - H(x)| \to 0 \text{ as } n \to +\infty. \]

The generalized inverse transform or the quantile transform is instrumental in all parts of extreme value theory and in its statistical branch.

In the frame we are constructing in this chapter, the logarithm transform \( Y = \log X \) plays an important role, specially in the statistical part. It implies

\[ G^{-1}(u) = \log F^{-1}(u), 0 \leq u \leq 1. \]

Throughout the text, the \( G \) stands for the distribution function of \( Y \).

### 2. The three nondegenerated extreme values distributions and the generalized extreme value distribution

#### 2.1. Convergence in type

**Definition 2.** The sequence of real random variables \( Z_1, Z_2, ... \) with distributions functions \( H_1, H_2, ... \) converges in type to a real random variable \( Z \) with distribution function \( H \) iff and only if there exists a sequence positive real numbers \( (a_n > 0)_{n \geq 0} \) and a sequence of real numbers \( (b_n)_{n \geq 0} \) such that one of these assertions hold:

(i) We have

\[ \frac{Z_n - b_n}{a_n} \overset{\text{d}}{\to} Z \]

(ii) For any continuity point \( x \) of \( H \),

\[ H_n(a_n x + b_n) \to H(x). \]
By Lemma 1 above, we have the following

**Lemma 3.** The sequences of real random variables $Z_1, Z_2, \ldots$ with distributions functions $H_1, H_2, \ldots$ converges in type to $Z$ with distribution function $H$, that is there exist a sequence positive real numbers $(a_n > 0)_{n \geq 0}$ and a sequence of real numbers $(b_n)_{n \geq 0}(ii)$ such that for any continuity point $x$ of $H$,

$$H_n(a_n x + b_n) \longrightarrow H(x).$$

Then for any continuity point $H^{-1}$, we also have

$$\frac{H_n^{-1}(x) - b_n}{a_n} \longrightarrow H^{-1}(x).$$

The next lemma will allow to define the convergence in type, that is used in UEVT.

**Lemma 4.** Let $(H_n)_{n \geq 0}$ be a sequence of probability distribution functions. Suppose there exist sequences $(a_n > 0)_{n \geq 0}$, $(\alpha_n > 0)_{n \geq 0}$, $(b_n)_{n \geq 0}$ and $(\beta_n)_{n \geq 0}$, and probability distributions functions $H_1$ and $H_2$ such that

$$\lim_{n \to \infty} F_n(a_n x + b_n) = H_1(x), \ x \in C(H_1)$$

and

$$\lim_{n \to \infty} F_n(\alpha_n x + \beta_n) = H_2(x), \ x \in C(H_2).$$

Then there exist reals numbers $A > 0$ and $B$ such that, as $n \to \infty$,

$$\lim_{n \to \infty} \frac{\alpha_n/a_n}{(\beta_n - b_n)/a_n} \to A \text{ and } (\beta_n - b_n)/a_n \to B,$$

as $n \to +\infty$, and for any $x \in \mathbb{R}$,

$$H_2(x) = H_1(Ax + B).$$

Inversely, If (2.1) and (2.3) hold both, then (2.2) is true, where $H_2$ defined in 2.4.
Formula (2.4) defines an equivalence class in the class of all real probability distribution functions. Let us denote this equivalence relation by $R_{\text{type}}$. And we say that $H_1$ and $H_2$ are of the same type if one is obtained from the other by affine transformation of the argument. The lemma says that the limit of convergence is unique in type, meaning that all the possible limits in type are of the same type.

The proof of the lemma is given is Lemma 2 in Chapter 2 following the lines of [Resnick (1987)].

Now, we are going to apply this to the special case of the sequences of the maxima $X_{n,n}$ of the samples.

3. The three nondegenerated extreme values distributions

Let us begin by the important Gnedenko theorem which actually covered more than the stability of the maximum but also concerned that of the sums.

3.1. Gnedenko Theorem. Before we state the theorem, recall that random variable is degenerated if and only it is concentrated on a single point almost surely.

**Theorem 2. (Gnedenko)** Let $X_1, X_2, ...$ be independent copies (s.i.c) of a real random variable (rv) $X$ with df $F(x) = \mathbb{P}(X \leq x)$, all of them being defined on the same probability space $(\Omega, A, \mathbb{P})$. Define, for each $n \geq 1$, $X_{n,n} = \max(X_1, X_2, ..., X_n)$.

Then $X_{n,n}$ in type to nondegenerated random variable $Z$ with distribution function $H$, that is: there exist sequences $(a_n > 0)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ such that

\[ \frac{X_{n,n} - b_n}{a_n} \rightarrow Z \]

or equivalently, for any continuity point of $H$,

\[ \lim_{n \to \infty} F^n(a_n x + b_n) = H(x), \]

if and only $H$ is one of the three types of distributions:
The type of Gumbel:
(3.3) \( \Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R} \)

The type of Fréchet of parameter \( \alpha > 0 \):
(3.4) \( \varphi_\alpha(x) = \exp(-x^{-\alpha})1_{(x \geq 0)} \).

The type of Weibull of parameter \( \beta > 0 \):
(3.5) \( \psi_\beta(x) = \exp((-x)^\beta)1_{(x < 0)} + 1_{(x \geq 0)} \).

Throughout this text, random variables respectively associated with the distribution functions given in (3.3), (3.4) and (3.5) will be denoted as \( \Lambda, F_\alpha(\alpha), \alpha > 0 \) and \( W(\beta), \alpha > 0 \), in the same order.

Proof. The full proof is given in that of Theorem 4 in Chapter 2. The proof is a direct rephrasing of the one given in [Resnick (1987)], which itself is rather a classical one. But we complete the proofs by giving full details of the solutions the Hamel equations along with the principles of Littlewood in Chapter 4.

When (3.1) or (3.2) hold, it is said that the maxima \( X_{n,n} \) attracted to \( Z \) or \( F \) is attracted to \( H \) or \( F \) is in the attraction domain of \( H \) denoted \( F \in D(H) \).

To avoid confusion with the attraction domains concerning infinitely divisible laws using sums of independent random variables, we use the term of extreme domain of attraction. We have this simple result that does not need to be proved.

**Proposition 2.** Two distributions of the same type have the same extreme domain of attraction.

So the Theorem of Gnedenko says that the only three nondegenerated extreme domains of attractions are those of (3.3), (3.4) and (3.5). We notice that these distributions functions are continuous. This implies, in view of Lemma 2 above, that the convergence (3.2) holds uniformly in \( x \) on \( \mathbb{R} \). And this gives:
Proposition 3. \( F \in D(H) \) where \( H \in \{ \Lambda, \varphi_\alpha, \psi_\beta \} \) if and only if there exist sequences \((a_n > 0)_{n \geq 0}\) and \((b_n)_{n \geq 0}\) such that equivalently, for any continuity point of \( H \),

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |F^n(a_n x + b_n) - H(x)| = 0.
\]

3.2. Immediate and simple examples. We are going to use the convergence of the probability distribution functions. Since the three possible limits the Gnedenko’s Theorem are continuous, we have to check the convergence at any point of \( \mathbb{R} \).

Let us consider three simple examples.

(a) \( X \sim \mathcal{E} \),

that is \( X \) is standard exponential random variable with probability distribution function

\[
F(x) = (1 - \exp(-x))1_{x \geq 0}, \quad x \in \mathbb{R}.
\]

By using the distribution functions, we want to prove that

\[ M_n - \log n \overset{d}{\to} \Lambda \text{ as } x \to +\infty. \]

Indeed, we have

\[
P(M_n - \log n \leq x) = P(M_n \leq x + \log n) = F(M_n \leq x + \log n)^n.
\]

But for any \( x \in \mathbb{R}, \ x + \log n \geq 0 \) for \( n \geq \exp(-x) \). Then for large values of \( n \), \( P(M_n \leq x + \log n) = (1 - \exp(-x - \log n)) \) and next for any \( x \in \mathbb{R} \) and for \( n \) large enough,

\[
P(M_n - \log n \leq x) = (1 - \frac{e^{-x}}{n}) \to e^{-e^{-x}} = \Lambda(x).
\]

We conclude that \( X \in D(\Lambda) \).

(b) \( X \sim \mathcal{P}ar(\alpha), \alpha > 0, \)

that is \( X \) is a Pareto random variable with parameter \( \alpha > 0 \) with probability distribution function

\[
F(x) = (1 - x^{-\alpha})1_{(x \geq 1)}, \quad x \in \mathbb{R}
\]
We have to prove that

\[ n^{-1/\alpha} M_n \xrightarrow{d} C(\alpha) \text{ as } x \to +\infty. \]

The observation \( X_i \) are nonnegative since the support of a \( \mathcal{P}ar \) law is \( \mathbb{R}_+ \). So the maxima \( M_n \) are nonnegative for any \( n \geq 1 \). We may discuss two cases.

Case \( x \leq 0 \). In this case, we have

\[ P(n^{-1/\alpha} M_n \leq 0) = 0 = \varphi_\alpha(x), \]

and then 3.2 holds.

Case \( x > 0 \). In this case

\[ P(n^{-1/\alpha} M_n \leq x) = P(M_n \leq n^{1/\alpha} x). \]

For large values of \( n \), we have \( n^{1/\alpha} x > 1 \) (take \( n \geq (1/x)^{-\alpha} \), to ensure that) and for these values,

\[
\begin{align*}
P(n^{-1/\alpha} M_n \leq x) &= F(n^{1/\alpha} x)^n = (1 - (n^{1/\alpha} x)^{-\alpha})^n \\
&= (1 - \frac{x^{-\alpha}}{n})^n \to \exp(-x^{-\alpha}) \\
&= \varphi_\alpha(x).
\end{align*}
\]

So 3.2 also for this case. Then it holds for any \( x \in \mathbb{R} \).

Conclusion : \( X \in D(\mathcal{F}R(\alpha)) \).

(c) \( X \sim \mathcal{U}(0, 1) \),

that is \( X \) is uniformly distributed on \( (0, 1) \) with probability distribution function :

\[ F(x) = x 1_{(0 \leq x \leq 1)} + 1_{(x \geq 1)}, \; x \in \mathbb{R}. \]

We want to prove that

\[ n(M_n - 1) \xrightarrow{d} W(1) \text{ as } x \to +\infty. \]

We have

\[ P(n(M_n - 1) \leq x) = F(1 + \frac{x}{n})^n. \]
We have two cases.

Case $x \geq 0$. We see that $1 + x/n$ is nonnegative $n \geq 1$ et

$$P(n(M_n - 1) \leq x) = F(1 + \frac{x}{n})^n = 1 = \psi_1(x)$$

and we see that holds.

Case $x < 0$. For large values of $n$, we have $0 \leq 1 + x/n \leq 1$ (take $x \geq -n$ i.e. $n \geq -(x) \geq 0$, to get it) and for these values of $n$,

$$P(n(M_n - 1) \leq x) = (1 + \frac{x}{n})^n \rightarrow e^x = \psi_1(x).$$

We get that for any $x \in \mathbb{R}$,

$$P(n(M_n - 1) \leq x) \rightarrow \psi_1(x).$$

Conclusion : $W \in D(W(1))$.

### 3.3. The Generalized Extreme Value (GEV) Distribution.

Now we know the three types of extreme value distributions from Theorem 1, let us try to gather them into one form with the help of Lemma 2.

Let us change our sequences $(a_n > 0)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ by $(\alpha_n > 0)_{n \geq 1}$ and $(\beta_n)_{n \geq 1}$.

For the case of $H_1 = \varphi_\alpha$, take $A = \gamma = 1/\alpha$ and $B = 1$. We get $\gamma > 0$ and

$$H_2(x) = G_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma})1_{(1+\gamma x \geq 0)}.$$  

For $H_1 = \psi_\beta$, choose $A = -\gamma = -1/\beta$ and $V = -1$. We get $\gamma < 0$ and

$$H_2(x) = G_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma})1_{(1+\gamma x \geq 0)} + 1_{(1+\gamma x < 0)}.$$  

Finally, for $H_1(x) = \exp(-e^{0-x})$, we may use the following the limit

$$\Lambda(x) = \lim_{\gamma \to 0} \exp(- (1 + \gamma x)^{-1/\gamma})$$

and define $\Lambda(x)$ as an extension of $\exp(-(1 + \gamma x)^{-1/\gamma})$ by continuity at $\gamma = 0$, and write
\( \Lambda(x) = G_0(x), x \in \mathbb{R}. \)

We are now able to gather the whole extreme domain of attraction by one parameterized extreme value distribution

\[ G_{\gamma} = \exp(- (1 + \gamma x)^{-1/\gamma}), \quad 1 + \gamma x \geq 0, \]

where \( \gamma < 0 \) corresponds to the Frechet domain, \( \gamma = 0 \) to the Gumbel domain and \( \gamma > 0 \) to Weibull domain. Throughout the text, we will use these compact form.

3.4. Malmquist and uniform representation. We are introducing important representation tools that can greatly help to handle the extremes.

A - Uniform representations.

We already introduced the generalized inverse (see (1.1) above). We are going to use it for studied sequence of real random variables \( X_1, X_2, \ldots \). By (1.2) above, we have this equivalence

\[
(3.7) \quad \forall (t, s) \in \mathbb{R} \times [0, 1], \quad (F^{-1}(s) \leq t) \iff (s \leq F(t)).
\]

Now, let \( U \sim U(0,1) \) following a uniform law on \((0,1)\) defined on \((\Omega, \mathcal{A}, \mathbb{P})\). It is easy to see that \( V = 1 - U \) follows also a uniform law on \((0,1)\). By (3.7), we have

\[
\forall (x \in \mathbb{R}), \quad \mathbb{P}(F^{-1}(V) \leq x) = \mathbb{P}(V \leq F(x)) = F(x).
\]

So, by the characterization of the law of a real random variable by its distribution function, we see that \( X \) and \( F^{-1}(1 - U) \) have the same distribution, denoted as

\[ X =_d F^{-1}(1 - U). \]

This equality in laws enables to replace the whole sequence \( X_1, X_2, \ldots \) that is studied in this text by in this text by a sequence \( F^{-1}(1 - U_1), F^{-1}(1 - U_2), \ldots \), where \( U_1, U_2, \ldots \) are independent and uniform random variables on \((0,1)\). And we have the equalities in distribution of stochastic processes :

\[
\{X_j, j \geq 1\} =_d \{F^{-1}(1 - U_j), j \geq 1\},
\]
\[ \{ \{ X_{1,n}, X_{2,n}, \ldots, X_{n,n} \} : n \geq 1 \} \]

\[ =_d \{ \{ F^{-1}(1 - U_{n}), F^{-1}(1 - U_{n-1,n}), \ldots, F^{-1}(1 - U_{1,n}) \} : n \geq 1 \}, \]

\[ \{ Y_j, j \geq 1 \} =_d \{ \log F^{-1}(1 - U_j), j \geq 1 \}, \]

and

\[ \{ \{ X_{1,n}, X_{2,n}, \ldots, X_{n,n} \} : n \geq 1 \} \]

\[ =_d \{ \{ \log F^{-1}(1 - U_{n}), \log F^{-1}(1 - U_{n-1,n}), \ldots, \log F^{-1}(1 - U_{1,n}) \} : n \geq 1 \}. \]

With these representations that preserve the laws of the concerned stochastic processes, all the exact laws and asymptotic laws using solely the probability laws are true for both processes \( \{ X_j, j \geq 1 \} \) and \( \{ F^{-1}(1 - U_j), j \geq 1 \} \).

To these elements, we add the Malmquist representation.

**B - Malmquist representation.**

As announced, we shall use the following Malmquist representation.

\[ (3.8) \quad \{ \log \left( \frac{U_{j+1,n}}{U_{j,n}} \right)^j, j = 1, \ldots, n \} =_d \{ E_{1,n}, \ldots, E_{n,n} \}, \]

where \( E_{1,n}, \ldots, E_{n,n} \) is an array of independent standard exponential random variables. We write \( E_i \) instead of \( E_{i,n} \) for simplicity sake.

Details and proof in Proposition 31, Chapter 4, [Lo (2016c)].

3.5. **Regular variation and \( \pi \)-variation.** The notions of regular variation and \( \pi \)-variation are very useful tools in extreme value theory and its applications. We may treat these notion in the neighbourhood of zero or in that of \( +\infty \). Both approaches are equivalent and one may move from one to the other by the inverse transform. So let

\[ S : \quad (a_0, +\infty] \quad \longmapsto \quad R \]

\[ x \quad \longmapsto \quad S(x) \]
be a measurable function, that is integrable on compact sets, where \( a_0 > 0 \), is some positive real number.

### 3.5.1. Regular Variation

**Definition 3.** The function \( S \) is said to be regularly varying with exponent \( \rho \in \mathbb{R} \) at infinity, and we denote \( S \in RV(\rho, +\infty) \) if and only if for all \( \mu \in \mathbb{R}_+ \)

\[
\lim_{x \to +\infty} \frac{S(\mu x)}{S(x)} = \mu^\rho.
\]

If \( \rho = 0 \), it is said that \( S \) is slowly varying at infinity and we write \( S \in SV(+\infty) \).

In this formula, only the final values of \( x \) matter and these values are in \((a_0, +\infty]\) for large values of \( x \).

In Chapter 2, Section 3, a complete theory of regularly varying functions is exposed there, at least all we want on them here.

Before we proceed further, suppose that we denote \( s(u) = S(1/u) \), \( 0 < u < \min(1, 1/a_0) \) We have the regularly variation in the neighborhood at 0, from (3.9), as follows

\[
\lim_{x \to +\infty} \frac{s(\mu x)}{s(x)} = \mu^{-\rho},
\]

and we write \( s \in RV(-\rho, 0) \) and \( s \in SV(0) \) of \( \rho = 0 \).

Among interesting results, we will have these points

**Karamata representation.** The function \( S \) is a regularly varying function at \(+\infty\) with exponent \( \rho \) is and only if there exists a constant \( c > 0 \) and there exist functions \( p(u) \) and \( b(u) \) of \( u \to (a_0, +\infty[ \) satisfying

\[
(p(x), b(x)) \to (0, 0) \text{ as } x \to +\infty,
\]

such that \( S \) admits the following representation of Karamata

\[
S(x) = c(1 + p(x))x^\rho \exp\left( \int_x^{+\infty} \frac{b(y)}{y} \, dy \right).
\]

\( \pi \)-variation.
This variation is more adapted for the neighborhood of zero.

Suppose that $s$ is slowly varying at infinity, which is equivalent to saying that $s(o)$ is slowly varying at zero. Define

$$U(t) = s(t) + \int_{0}^{u_0} \frac{s(u)}{du}, \quad 0 \leq t \leq u_0.$$ 

By using the Karamata representation, one may readily prove, for any $\mu > 0$ and for any $x > 0, x \neq 1$, that

$$\lim_{u \to 0} \frac{U(\mu u) - U(u)}{s(u)} = \log \mu$$  \hspace{2cm} (3.10)

and

$$\lim_{u \to 0} \frac{U(\lambda u) - U(u)}{s(xu) - s(u)} = \frac{\log \mu}{\log x}.$$ \hspace{2cm} (3.11)

This has been established in the proof of Theorem 6, Section 3, Chapter 2, in the part $(c) \Rightarrow (d)$.

Both of these formulas are particular cases of this definition.

**Definition 4.** (Definition-Theorem) A function $T(u)$ of $u \in (0, 1)$ is of $\pi$-variation if and only of one these two propositions holds.

(a) There exists a slowly varying function on $(0, 1)$ at zero such that $\mu > 0$ and for any $\kappa > 0$,

$$\lim_{u \to 0} \frac{T(\mu u) - T(\kappa u)}{s(u)} = \log(\mu/\kappa).$$

(b) For any $\mu > 0$ and for any $\kappa > 0, \kappa \neq 1$,

$$\frac{s(\lambda u) - s(u)}{s(\kappa u) - s(u)} = \frac{\log \mu}{\log \kappa}.$$ \hspace{2cm} (3.12)

Slowly varying functions have interesting uniform convergence properties, both in deterministic and random frames. We have:
PROPOSITION 4. Let $S(u)$ be a function $u \in (0,1)$ that is slowly varying at zero. We have the following uniform convergence in deterministic and random versions.

(a) Let $A(h)$ and $B(h)$ two functions of $h \in (0, +\infty[$ such that for each $h \in (0, +\infty[$, we have $0 < A(h) \leq B(h) < +\infty$ and $(A(h), B(h)) \to (0,0)$ as $h \to 0$. Suppose that there exist two real numbers $A$ and $B$ satisfying $0 < C < D < +\infty$ such that

$$C < \lim_{h \to +\infty} \inf_{0 < u, v \leq B(h)} \frac{S(u)}{S(v)} - 1 = 0.$$  \hfill (3.13)

Then, we have

$$\lim_{h \to +\infty} \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| = 0.$$

(b) Let $A(h)$ and $B(h)$ two families, indexed by $h \in (0, +\infty[$, of real-valued applications defined on a probability space $(\Omega, A, \mathbb{P})$ such that for each $h \in (0, +\infty[$, we have $0 < A(h) \leq B(h) < +\infty$. Suppose that there exist two families $A^*(h)$ and $B^*(h)$, indexed by $h \in (0, +\infty[$, of measurable real-valued applications defined on $(\Omega, A, \mathbb{P})$ such that for each $h \in (0, +\infty[$, $A^*(h) \leq A(h) \leq B(h) \leq B^*(h)$, and such that

$$\lim_{h \to +\infty} \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| = 0.$$  \hfill (3.14)

and

$$\lim_{h \to +\infty} \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| = 0.$$  \hfill (3.15)

We say that the family $\{B^*(h), h \in (0, +\infty]\}$ is asymptotically bounded in probability against $+\infty$ and the family $\{B^*(h), h \in (0, +\infty]\}$ is asymptotically bounded in probability against $0$ and accordingly, we say that the family $\{B(h), h \in (0, +\infty]\}$ is asymptotically bounded in outer probability against $+\infty$ and the family $\{A(h), h \in (0, +\infty]\}$ is asymptotically bounded in outer probability against $0$.

Then any $\eta > 0$, for any $\delta > 0$, there exists a measurable subset $\Delta(\delta)$ of such that

$$\left( \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| > \eta \right) \subset \Delta(\delta),$$
with
\[ \mathbb{P}(\Delta(\delta)) \leq \delta. \]

Consequently, if the quantities
\[ \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| > \eta \]
are measurable for \( h \in h \in (0, +\infty) \), we have that
\[ \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| \xrightarrow{p} as \ h \to +\infty. \]

**Proof.** See Lemma 17 in Chapter 2

### 3.6. Theorem of Karamata and Theorem of de Haan

How to link extreme domains to regular or slowly variation? We have these three characterizations.

**Proposition 5.** We have the following characterizations for the three extremal domains.

(a) \( F \in D(H_{\gamma}), \ \gamma > 0 \), if and only if there exist a constant \( c \) and functions \( a(u) \) and \( \ell(u) \) of \( u \to u \in [0, 1] \) satisfying
\[
(a(u), \ell(u)) \to (0, 0) \text{ as } u \to +\infty,
\]
such that \( F^{-1} \) admits the following representation of Karamata
\[
(3.16) \quad F^{-1}(1 - u) = c(1 + a(u))u^{-\gamma} \exp\left(\int_{u}^{1} \frac{\ell(t)}{t} dt\right).
\]

(b) \( F \in D(H_{\gamma}), \ \gamma < 0 \), if and only if \( \text{uep}(F) < +\infty \) and there exist a constant \( c \) and functions \( a(u) \) and \( \ell(u) \) of \( u \in [0, 1] \) satisfying
\[
(a(u), \ell(u)) \to (0, 0) \text{ as } u \to +\infty,
\]
such that \( F^{-1} \) admit the following representation of Karamata
\[
(3.17) \quad \text{uep}(F) - F^{-1}(1 - u) = c(1 + a(u))u^{-\gamma} \exp\left(\int_{u}^{1} \frac{\ell(t)}{t} dt\right).
\]

(c) \( F \in D(H_{0}) \) if and only if there exist a constant \( d \) and a slowly varying function \( s(u) \) such that
\[
(3.18) \quad F^{-1}(1 - u) = d + s(u) + \int_{u}^{1} \frac{s(t)}{t} dt, \ 0 < u < 1,
\]
and there exist a constant $c$ and functions $a(u)$ and $\ell(u)$ of $u \to u \in [0, 1]$ satisfying

$$(a(u), \ell(u)) \to (0, 0) \text{ as } u \to +\infty,$$

such that $s$ admits the representation

$$s(u) = c(1 + a(u)) \exp\left(\int_u^1 \frac{\ell(t)}{t} dt\right).$$

Moreover, if $F^{-1}(1-u)$ is differentiable for small values of $s$ such that $r(s) = -s(F^{-1}(1-s))' = udF^{-1}(1-s)/ds$ is slow varying at zero, then 3.18 may be replaced by

$$F^{-1}(1-u) = d + \int_u^{u_0} \frac{r(t)}{t} dt, 0 < u < u_0 < 1,$$

which will be called a reduced de Haan representation of $F^{-1}$.

These representations are important. Proofs of them are to be found in the proofs of Proposition 12, Section 3 on Chapter 2 in the lines of proofs in [Loève (1963-1977)] or in [de Haan (1970)].

But we rather use the same representations but on $G$.

3.6.1. Representations implied by the logarithm transformation. Since we deal with both $F$ and $G$ using the logarithm transform, we should have the exact relation between them relatively to their belonging to extreme domains of attraction. From there, we derive representations for $G^{-1}$.

Define the first asymptotic moment

$$R(x, F) = \int_x^{\text{uep}(F)} \frac{1 - F(t)}{1 - F(x)} dt, x < \text{uep}(F).$$

We have this result.

**Proposition 6.** We have the following equivalences.

(a) Let $\gamma > 0$. Then $F \in D(H_\gamma) \iff G \in D(H_0)$ and $R(x, G) \to \gamma$ as $x \to \text{uep}(G)$. 

(b) $F \in D(H_0) \iff G \in D(H_0)$ and $R(x, G) \to 0$ as $x \to \text{uep}(G)$.

(c) Let $\gamma > 0$. Then $F \in D(H_\gamma) \iff F \in D(H_\gamma)$.

**Proof.** See proof of Proposition ZZZ...

Now based on this proposition, we expose the quantile representations for $G^{-1}$.

**Proposition 7.** We have the following characterizations for the three extremal domains.

(a) $F \in D(H_\gamma)$, $\gamma > 0$, if and only if there exist a constant $c$ and functions $p(u)$ and $b(u)$ of $u \to u \in ]0, 1]$ satisfying
\[
(p(u), b(u)) \to (0, 0) \text{ as } u \to +\infty,
\]
such that $xF^{-1}$ admit the following representation :
\[
G^{-1}(1 - u) = c + \log(1 + a(u)) - \gamma \log u + \int_u^1 \frac{b(t)}{t} dt.
\]

(b) $F \in D(H_\gamma)$, $\gamma < 0$, if and only if $\text{uep}(G) < +\infty$ and there exist a constant $c$ and functions $p(u)$ and $b(u)$ of $u \in ]0, 1]$ satisfying
\[
(p(u), b(u)) \to (0, 0) \text{ as } u \to +\infty,
\]
such that $G^{-1}$ admit the following representation :
\[
\text{uep}(G) - G^{-1}(1 - u) = c(1 + a(u)) u^{-\gamma} \exp(\int_u^1 \frac{b(t)}{t} dt).
\]

(c) $F \in D(H_0)$ if and only if there exist a constant $d$ and a slowly varying function $s(u)$ such that
\[
G^{-1}(1 - u) = d + s(u) + \int_u^1 \frac{s(t)}{t} dt, \ 0 < u < 1,
\]
and there exist a constant $c$ and functions $p(u)$ and $b(u)$ of $u \to u \in ]0, 1]$ satisfying
\[
(p(u), b(u)) \to (0, 0) \text{ as } u \to +\infty,
\]
such that $s$ admits the representation
\[ s(u) = c(1 + p(u)) \exp\left(\int_u^1 \frac{b(t)}{t} dt\right). \]

### 3.7. General normalizing and centering sequences.
From the representations of the quantile functions of distribution functions in the extreme domain of attraction, we are able to find general expressions of normalizing and centering coefficients as given below.

**Proposition 8.** We have

(a) If $F \in D(G_{\gamma})$, $\gamma > 0$, then
\[ \frac{X_{n,n}}{F^{-1}(1 - 1/n)} \rightsquigarrow Fr(1/\gamma). \]

(b) If $F \in D(G_{\gamma})$, $\gamma < 0$, then $uep(F) < +\infty$ and
\[ \frac{X_{n,n} - uep(F)}{uep(F) - F^{-1}(1 - 1/n)} \rightsquigarrow W(-1/\gamma). \]

(c) If $F \in D(G_0)$, then
\[ \frac{X_{n,n} - F^{-1}(1 - 1/n)}{F^{-1}(1 - 1/(n\ell)) - F^{-1}(1 - 1/n)} \rightsquigarrow \Lambda. \]

**Proof.** Let us proceed case by case.

**Case** $F \in D(G_{\gamma})$, $\gamma > 0$. By (3.16), we have for $n \geq 1$,

\[ X_{n,n} = F^{-1}(1 - U_{1,n}) = c (1 + a(U_{1,n})) (U_{1,n})^{-\gamma} \exp\left(\int_{U_{1,n}}^{1/n} \frac{u^{-\gamma}}{u} du\right). \]

and

\[ F^{-1}(1 - 1/n) = c (1 + a(1/n)) n^{-\gamma} \exp\left(\int_{1/n}^{1} \frac{\ell(u)}{u} du\right). \]

We have

\[ X_{n,n}/F^{-1}(1 - 1/n) = \frac{1 + a(U_{1,n})}{1 + a(1/n)} (nU_{1,n})^{-\gamma} \exp\left(\int_{U_{1,n}}^{1/n} \frac{\ell(u)}{u} du\right), n \geq 1. \]
Put
\[ \ell_n = \sup \{ |b(t)| ; t \leq \max (U_{1,n}, 1/n) \}, \]
\[ a_n = \sup \{ |a(t)| ; t \leq \max (U_{1,n}, 1/n) \}. \]

Since \( U_{1,n} \to P 0 \) as \( n \to +\infty \), we have
\[ (a_n \ell_n) \to P (0, 0) \] as \( n \to +\infty \).

Now
\[ \left| \frac{1 + a(U_{1,n})}{1 + a(1/n)} - 1 \right| \leq \frac{|a(1/n) + a(U_{1,n})|}{1 + a(1/n)} \equiv A_n, \]
and obviously, \( A_n \to P 0 \) as \( n \to +\infty \). We also have
\[ \left| \exp \left( \int_{U_{1,n}}^{1/n} \frac{\ell(u)}{u} \, du \right) - 1 \right| \leq \left| \int_{U_{1,n}}^{1/n} \frac{\ell(u)}{u} \, du \right| \]
\[ \leq \ell_n \int_{\min(U_{1,n},1/n)}^{\max(U_{1,n},1/n)} \, \frac{du}{u} \]
\[ \leq \ell_n \log \left[ \frac{\max(U_{1,n},1/n)}{\min(U_{1,n},1/n)} \right] \]
\[ \leq \ell_n |\log nU_{1,n}|. \]

We know that \( nU_{1,n} \) weakly converges to a standard exponential random variable \( E(1) \) and by the continuous mapping theorem (see Proposition 6, Chapter 2, [Lo (2016c)]), \( |\log (nU_{1,n})| \) weakly converges to \( |\Lambda| = |\log E(1)|. \) This leads to \( |\log (nU_{1,n})| = O_P(1). \) Then
\[ \exp \left( \int_{U_{1,n}}^{1/n} \frac{\ell(u)}{u} \, du \right) - 1 = O_P(\ell_n) = o_P(1) \to 0 \]
and
\[ X_{n,n}/F^{-1}(1 - 1/n) = (nU_{1,n})^{-\gamma} (1 + o_P(1)) (1 + O(A_n)). \]

But, for \( x \geq 0, \)
\[ \mathbb{P}((nU_{1,n})^{-\gamma} \leq x) = \mathbb{P}((n(1 - U_{n,n})^{-\gamma} \leq x) \]
\[ = \mathbb{P} \left( U_{n,n} \leq \left( 1 - \frac{x^{-1/\gamma}}{n} \right) \right) \]
\[ = \left( 1 - \frac{x^{-1/\gamma}}{n} \right) \] (for large values of \( n \))
\[ \to \exp(-x^{-1/\gamma}) = \varphi_{1/\gamma}(x) \]
Since $\mathbb{P}((nU_{1,n})^{-\gamma} \leq x) = 0$ for $x \leq 0$,\n
$$(nU_{1,n})^{-\gamma} \rightsquigarrow Fr(1/\gamma).$$

and then

$$X_{n,n}/F^{-1}(1 - 1/n) \rightsquigarrow Fr(1/\gamma).$$

**Cas de Weibull** $\gamma < 0$. By (3.17), we have for $n \geq 1$,

$$x_0 (F) - F^{-1} (1 - U_{1,n}) = c(1 + a(U_{1,n})) (U_{1,n})^{-\gamma} \exp \left( \int_{U_{1,n}}^{1/n} \frac{\ell(u)}{u} \, du \right).$$

and

$$x_0 (F) - F^{-1} (1 - U_{1,n}) = c(1 + a(1/n)) n^\gamma \exp \left( \int_{U_{1,n}}^{1/n} \frac{\ell(u)}{u} \, du \right).$$

This leads, for $n \geq 1$, to

$$\frac{X_{n,n} - x_0 (F)}{x_0 (F) - F^{-1} (1 - U_{1,n})} = -(1 + O(A_n)) (nU_{1,n})^{-\gamma} \exp \left( \int_{U_{1,n}}^{1/n} \frac{\ell(u)}{u} \, du \right) (1 + O(A_n))$$

$$= -(nU_{1,n})^{-\gamma} (1 + O(A_n)) (1 + O(\ell_n))$$

$$= -(nU_{1,n})^{-\gamma} (1 + o_P(1))$$

by the computations done before. And for $x \in \mathbb{R}$,

$$\mathbb{P}(- (nU_{1,n})^{-\gamma} \leq x) = \mathbb{P}(-(n(1 - U_{n,n})^{-\gamma} \leq x)$$

$$= \mathbb{P} \left( U_{n,n} \leq \left( 1 - \left( -\frac{x}{n} \right)^{-1/\gamma} \right) \right).$$

If $x \geq 0$, $\mathbb{P}(U_{n,n} \leq (1 - \left( -\frac{x}{n} \right)^{1/\gamma})) = 1$. If $x \leq 0$, we have for large values of $n$,

$$\mathbb{P}(U_{n,n} \leq (1 - \left( -\frac{x}{n} \right)^{1/\gamma})) = \left( 1 - \left( -\frac{x}{n} \right)^{-1/\gamma} \right)^n \rightarrow \exp(-(-x)^{-1/\gamma}).$$

Then

$$\frac{X_{n,n} - x_0 (F)}{x_0 (F) - F^{-1} (1 - U_{1,n})} = -(nU_{1,n})^{-\gamma} (1 + o_P(1))$$

$$\rightsquigarrow W(-1/\gamma).$$

**Case of Gumbel** $\gamma = 0$. By (3.18), we have for $n \geq 1$

$$X_{n,n} = c - s(U_{1,n}) + \int_{U_{1,n}}^{1} \frac{s(t)}{t} \, dt.$$
and
\[ F^{-1}(1 - 1/n) = c - s(1/n) + \int_{1/n}^{1} \frac{s(t)}{t} dt. \]

Then for \( n \geq 1 \),
\[
\frac{X_{n,n} - F^{-1}(1 - 1/n)}{s(1/n)} = \frac{s(1/n) - s(U_{1,n})}{s(1/n)} + \int_{U_{1,n}}^{1/n} \frac{s(t)}{s(1/n)} \times \frac{1}{t} dt.
\]

By Proposition 4, we have
\[
\sup \left\{ \left| \frac{s(t)}{s(1/n)} - 1 \right| , U_{1,n} \wedge 1/n \leq t \leq U_{1,n} \vee 1/n \right\} = d_n \to_p 0.
\]
and
\[
\sup \left\{ \left| \frac{s(t)}{s(1/n)} - 1 \right| , 1/(ne) \leq t \leq 1/n \right\} = c_n \to 0.
\]

Then,
\[
\left| \frac{s(1/n) - s(U_{1,n})}{s(1/n)} \right| \leq d_n \to_p 0
\]
and
\[
\left| \int_{U_{1,n}}^{1/n} \left( \frac{s(t)}{s(1/n)} - 1 \right) \frac{dt}{t} \right| \leq d_n |\log(nU_{1,n})| \to 0.
\]

We conclude that
\[
\frac{X_{n,n} - F^{-1}(1 - 1/n)}{s(1/n)} = o_p(1) + \int_{U_{1,n}}^{1/n} \frac{1}{t} dt
= \log(nU_{1,n}) + o_p(1)
\approx \Lambda.
\]

We also have
\[
\frac{X_{n,n} - b_n}{a_n} = o_p(1) + \log(nU_{1,n}) \sim \Lambda.
\]
further
\[ \frac{F^{-1}(1 - 1/(ne)) - F^{-1}(1 - 1/n)}{s(1/n)} = \frac{s(1/n) - s(1/(ne))}{s(1/n)} + \int_{1/ne}^{1/n} \frac{s(t)}{s(1/n)} \frac{dt}{t} \]
\[ = \frac{s(1/n) - s(1/(ne))}{s(1/n)} + \int_{1/(ne)}^{1/n} \left( \frac{s(t)}{s(1/n)} - 1 \right) \frac{dt}{t} \]
\[ + \int_{1/(ne)}^{1/n} \frac{dt}{t} \]
\[ = O(c_n) + \int_{1/(ne)}^{1/n} \frac{dt}{t} \]
\[ = o(1) + 1. \]

By Lemma 4,
\[ \frac{X_{n,n} - F^{-1}(1 - 1/n)}{s(1/n)} \overset{d}{\to} \Lambda. \]

We easily see that if (3.20) holds, similar but less heavy computations also lead to
\[ \frac{X_{n,n} - F^{-1}(1 - 1/n)}{r(1/n)} \overset{d}{\to} \Lambda. \]

Let us give a few number of applications.

3.8. Examples of more complex normalizing and centering coefficients.

3.8.1. Upper extreme extreme of a standard Gaussian random variable. Suppose that \( X \sim N(0,1) \), with probability density function
\[ \phi_d(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, -\infty < t < +\infty \]
and distribution function
\[ \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \int_{-\infty}^{x} \phi_d(t) dt. \]
Let us give some useful expansion of functions related to this law, all of them demonstrated in Section 6 of Chapter 2. From these formalae, we will be able to clearly expose the law of of \( X_{n,n} \). First, \( 1 - \phi(x) \) admits the expansion for \( x > 0 \),
\[ (3.21) \quad C \left\{ \frac{1}{x} - \frac{1}{x^2} \right\} e^{-x^2/2} \leq 1 - \phi(x) \leq \frac{Ce^{-x^2/2}}{x}. \]
where $C = 1/\sqrt{2\pi}$. From this, the quantile $\phi^{-1}(1 - s)$ is expanded for $s \downarrow 0$, as

\begin{equation}
\phi^{-1}(1-s) = \left\{ (2 \log(1/s))^{1/2} - \frac{\log 4\pi + \log \log(1/s)}{2(2 \log(1/s))^{1/2}} + O((\log \log(1/s)^2(\log 1/s)^{-1/2})) \right\}.
\end{equation}

The derivative of $Q(1-s) = \phi^{-1}(1-s)$ is, as $s \downarrow 0$,

\begin{align}
\phi^{-1}(1-s) &= (2 \log(1/s))^{1/2} - \frac{\log 4\pi + \log \log(1/s)}{2(2 \log(1/s))^{1/2}} \\
&\quad + O((\log \log(1/s)^2(\log 1/s)^{-1/2})).
\end{align}

These three formulae respectively correspond to (6.1), (6.3) and to (6.4) in Section 6 of Chapter 2. Now, we see that $r(s) = -s(Q(1-s))' = (2 \log 1/s)^{1/2}(1 + o(1))$ is a slowly varying function at zero. By Proposition 5, we have the representation (3.20). So by (3.7), we have

\[
\frac{X_{n,n} - Q(1 - 1/n)}{r(1/n)} \Rightarrow \Lambda,
\]

that is,

\[
\frac{X_{n,n} - Q(1 - 1/n)}{(2 \log n)^{1/2}} \Rightarrow \Lambda.
\]

Since, we know that $\phi \in D(\Lambda)$, we have that

\[
\frac{X_{n,n} - Q(1 - 1/n)}{Q(1 - 1/(ne)) - Q(1 - 1/n)} \Rightarrow \Lambda.
\]

Set, for $n \geq 2$,

\[
b_n = (2 \log n)^{1/2} - \frac{\log 4\pi + \log \log n}{2(2 \log n)^{1/2}}.
\]

It is easily got from (3.23) that

\[
\frac{Q(1 - 1/n) - b_n}{(2 \log n)^{1/2}} \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

we conclude that

\[
\frac{X_{n,n} - (2 \log n)^{1/2} - \left( \frac{\log 4\pi + \log \log n}{2(2 \log n)^{1/2}} \right)}{(2 \log n)^{1/2}} \Rightarrow \Lambda.
\]
We get the following by-result, as a consequence of Lemma 4
\[
\frac{Q(1 - 1/(ne)) - Q(1 - 1/n)}{(2 \log n)^{1/2}} \to 1, \text{ as } n \to \infty.
\]
However, a direct proof of this result is given in (6.6) in Section 6 of Chapter 2.

4. The Csörgő et al. space

We already mentioned in Subsection 3.4, that we will replace the sequences of random variables \(X_1, X_2, \ldots\) by their uniform representations \(X_1 = F^{-1}(U_1), X_2 = F^{-1}(U_2), \ldots\) In many situations, we will suppose that the sequence \(U_1, U_2, \ldots\) are defined on the Csörgő-Csörgő-Horváth-Mason space. This space is described in lemma 3 below. Before we state it, let us introduce some notation.

For a sequence of a sequence \(U_1, U_2, \ldots\) of independent random variables uniformly distributed on \((0,1)\), we define, for each \(n \geq 1\), the empirical quantile function
\[
U_n(s) = \frac{j}{n} U_{1,n} \leq s < U_{j+1,n}, j = 1, \ldots, n
\]
and zero elsewhere, with the convention that \(U_{0,n} = 0\) and \(U_{n+1,n} = 1\). The uniform quantile process is defined for \(n \geq 1\) by
\[
V_n(s) = \begin{cases} 
U_{j,n} & \text{for } \frac{j-1}{n} < s \leq \frac{j}{n}, j = 1, \ldots, n \\
U_{1,n} & \text{for } s = 0.
\end{cases}
\]
The uniform empirical process is given for for \(n \geq 1\)
\[
\mathbb{G}_n(s) = \{\sqrt{n}(U_n(s) - s), 0 \leq s \leq 1\}
\]
and the quantile process is
\[
\mathbb{V}_n(s) = \{\sqrt{n}(U_n(s) - s), 0 \leq s \leq 1\}.
\]
Finally a standard brownian bridge \(\{B(s), 0 \leq s \leq 1\}\) is a centered continuous Gaussian process with variance and co-variance function
\[
\mathbb{E}(B(s)B(t)) = \min(s,t) - st, \ (s,t) \in [0,1]^2.
\]
Now, here is the so-important theorem of Csörgő-Csörgő-Horváth-Mason (1986).
Theorem 3. (See [Csörgö (1986)]) There exists a probability space holding a sequence of independent uniform random variables $U_1, U_2, \ldots$ and a sequence of Brownian bridges $B_1, B_2, \ldots$ such that for each $0 < \nu < 1/4$, as $n \to \infty$,

\[
\sup_{1/n \leq s \leq 1-1/n} \frac{\sqrt{n}(U_n(s) - s) - B_n(s)}{(s(1-s))^{1/2-\nu}} = O_p(n^{-\nu})
\]

and

\[
\sup_{1/n \leq s \leq 1-1/n} \frac{|B_n(s) - \sqrt{n}(s - V_n(s))|}{(s(1-s))^{1/2-\nu}} = O_p(n^{-\nu}),
\]

where for each $n \geq 1$, $U_n(s) = j/n$ for $U_{j,n} \leq s < U_{j+1,n}$ is the uniform empirical df and $V_n(s) = U_{j,n}$ for $(j-1)/n < s \leq j/n$, and $V_n(0) = U_{1,n}$, is the uniform quantile function and, finally, $U_{1,n} \leq \ldots \leq U_{n,n}$ are the order statistics of $U_1, \ldots, U_n$ with by convention $U_{0,n} = 0 = 1 - U_{n+1,n}$. 
Part 2

Functional Aspects of Univariate Extreme Value Theory
CHAPTER 2

Mathematical Background of Extreme Value Theory

In this paper we will expose the very beginning of Extreme value Theory, that is the extreme values domains of attraction for independent and identically distributed randoms variables (iid).

We begin this chapter by defining the notion of convergence in type, which is a sub-domain of weak convergence.

1. Convergence in type

Basic and main notations concerning the general theory of weak convergence are provided in the monograph [Lo (2016c)] of this series. In particular, in this text, the most used definition of weak convergence concerns the distribution functions in a broad sense: a real distribution function is a function $F$ from $\mathbb{R}$ to $\mathbb{R}$, right-continuous and non-decreasing. If $F(-\infty) = 0$ and $F(+\infty) = 1$, it becomes a probability distribution function. Next by the Probability Foundation Theorem of Kolmogorov, there exists a probability space holding a real random variable $X$ such that on that probability space, we have $F(x) = P(X \leq x)$, $x \in \mathbb{R}$.

We remind the definition of weak convergence just to make the reader ready to go through. In case he wants more details, he will be free to go back to [Lo (2016c)] which is entirely devoted to that theory.

**Definition 5.** The sequence of random vectors $X_n : (\Omega_n, A_n, P_n) \mapsto (\mathbb{R}, B(\mathbb{R}))$ weakly converges to $X : (\Omega_\infty, A_\infty, P_\infty) \mapsto (\mathbb{R}, B(\mathbb{R}^k))$ if and only if for any point $x \in \mathbb{R}$ such that $P(X = x) = 0$, that is $x$ is a continuity point of $F_X(x) = P(X \leq x)$,

\begin{equation}
F_{X_n}(x) = P(X_n \leq x) \rightarrow F_X(x) = P(X \leq x) \text{ as } n \rightarrow +\infty.
\end{equation}

**Remark 1.** By this definition, we see that the convergence of distribution concerns only the distribution. If you replace in (1.1), $(X_n)_{n \geq 0}$
by another sequence \((\tilde{X}_n)_{n \geq 0}\) such that \(X_n = d \tilde{X}_n\) for each \(n \geq 0\), it remains true. Also, the weak limit is unique in distribution because any random variable is fully determined by its probability distribution function.

The theory of extreme value deals with a particular case of weak convergence named as convergence in type and defined below.

**Definition 6.** A sequence of probability distribution functions \((F_n)_{n \geq 0}\) converges in type to the probability distribution functions \(H\) if and only if there exist a sequence of positive real numbers \((a_n > 0)_{n \geq 0}\) and a sequence of real numbers \((b_n)_{n \geq 0}\) such that the sequence of distributions \(F_n(a_n x + b_n)\) weakly converges to a probability distribution function \(H(x)\), that is, for any continuity point of \(H\),

\[
\lim_{n \to \infty} F_n(a_n x + b_n) = H(x) .
\]

We may rephrase this definition into a different version. We may use the Kolmogorov Theorem to place ourselves in a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) holding a sequence a random variable \(Z\) and a sequence of random variables \((X_n)_{n \geq 0}\) such that \(X\) has the probability distribution function \(H\) and each \(X_n\) has the probability distribution function \(F_n\). With this representation, \((1.2)\) is equivalent to

\[
\frac{X_n - b_n}{a_n} \overset{d}{\to} Z \iff \frac{X_n - b_n}{a_n} \to Z, \text{ as } n \to +\infty,
\]

where the symbols respectively denote the weak convergence and the convergence in distribution. These two convergences being equivalent, except when we deal with non-measurability.

In the definition, the sequences \((a_n > 0)_{n \geq 0}\) and \((b_n)_{n \geq 0}\) are not unique, nor is \(H\). But one can only change them by satisfying the relations below.

In the sequel, we will use this notation: \(C(H)\) denotes the set of continuity points of distribution function \(H\).

**Lemma 5.** Let \((F_n)_{n \geq 0}\) be a sequence of probability distribution functions. Suppose there exist sequences \((a_n > 0)_{n \geq 0}\), \((\alpha_n > 0)_{n \geq 0}\), \((b_n)_{n \geq 0}\) and \((\beta_n)_{n \geq 0}\), probability distribution functions \(H_1\) and \(H_2\) such that
1. CONVERGENCE IN TYPE

\[(1.3) \quad \lim_{n \to \infty} F_n(a_n x + b_n) = H_1(x), \quad x \in C(H_1)\]

and

\[(1.4) \quad \lim_{n \to \infty} F_n(\alpha_n x + \beta_n) = H_2(x), \quad x \in C(H_2).\]

Then there exist reals numbers \(A > 0\) and \(B\) such that, as \(n \to +\infty\),

\[(1.5) \quad \alpha_n/a_n \to A \text{ and } (\beta_n - b_n)/a_n \to B, \quad \text{as } n \to +\infty.\]

and for any \(x \in \mathbb{R}\)

\[(1.6) \quad H_2(x) = H_1(Ax + B).\]

Reversely, If (1.3) and (1.5) hold both, then (1.4) is true, where \(H_2\) defined in (1.6).

Formula (1.6) defines an equivalence class in the class of all real probability distribution functions. Let us denote this equivalence relation by \(R_{\text{type}}.\) And we say that \(H_1\) and \(H_2\) are of the same type if one is obtained from the other by a non-constant affine transformation of the argument. The lemma says that the limit of convergence is unique in type, meaning that all the possible limits in type are of the same type.

**Proof.** We will use the Skorohod representation theorem for weak limits on \(\mathbb{R}\) (see Theorem 11, Section 4, [Lo (2016c)], in this series).

First assume that (1.3), (1.4) and (1.5) hold. We are going to prove (1.6). Placing ourselves in the right space with the help of Kolmogorov Theorem, we say that \(F_n\) is the probability distribution function of \(X_n,\) \(H_1\) is the probability distribution function of \(Z_1\) and \(H_2\) is the probability distribution function of \(Z_2,\) all of these random variables being defined on the same probability space and as \(n \to +\infty,\)

\[
U_n = \frac{X_n - b_n}{a_n} \leadsto Z_1 \text{ and } V_n = \frac{X_n - \beta_n}{\alpha_n} \leadsto Z_2.
\]

By the Skorohod Theorem we mentioned at the opening of the proof, there is also a probability space holding random variables \(\tilde{Z}_1, \tilde{Z}_2, \tilde{U}_n, \tilde{V}_n,\) \(n \geq 0,\) such that we have the following equalities in distribution

\[
\tilde{Z}_1 =_d Z_1, \quad \tilde{Z}_2 =_d Z_2, \quad \tilde{U}_n =_d U_n, \quad \tilde{V}_n =_d V_n, \quad \text{for } n \geq 0
\]
and, as \( n \to +\infty \),
\[
\tilde{U}_n \to \tilde{Z}_1 \text{ a.s. and } \tilde{V}_n = \frac{X_n - \beta_n}{\alpha_n} \to Z_2 \text{ a.s.}
\]
Here we only need the convergence of probability, that is, as \( n \to +\infty \),
\[
\tilde{U}_n \to P \tilde{Z}_1 \text{ and } \tilde{V}_n = \frac{X_n - \beta_n}{\alpha_n} \to P Z_2.
\]
It is evident that for any \( n \geq 1 \),
\[
a_n \tilde{U}_n + b_n = d \alpha_n \tilde{V}_n + \beta_n.
\]
Denote, for each \( n \geq 1 \)
\[
\tilde{X}_n(1) = a_n \tilde{U}_n + b_n = X_n(2) = \alpha_n \tilde{V}_n + \beta_n = d X_n.
\]
Then for each \( n \geq 1 \),
\[
\tilde{X}_n(1) = d \tilde{X}_n(2),
\]
and, as \( n \to +\infty \),
\[
\frac{\tilde{X}_n(1) - b_n}{\alpha_n} \to P Z_1 \text{ and } V_n = \frac{\tilde{X}_n(2) - \beta_n}{\alpha_n} \to P Z_2.
\]
This leads to
\[
(\tilde{X}_n(1) - \beta_n)/\alpha_n = d \left\{(\tilde{X}_n(2) - b_n)/\alpha_n \right\} (\alpha_n/\alpha_n) + \{(b_n - b_n)/\alpha_n \} (\alpha_n/\alpha_n).
\]
The right member converges to \( Z_2 \) in probability, then in distribution by Proposition 12, Section 6 in [Lo (2016c)]. The second member converges in probability to \( A(Z_1 - B) \), then also in distribution. By Remark 1 above, we have
\[
Z_2 = A(Z_1 - B)
\]
and this implies (1.6) and we have the first part of the proof.

To complete the proof, suppose that (1.3) and (1.4) hold. We are going to prove that 1.5 and (1.6) also holds when both \( H_1 \) and \( H_2 \) are non-degenerated. If so, their generalized inverses \( H_1^{-1} \) and \( H_2^{-1} \) are also non degenerated. Remind the definition of \( H_1^{-1} \)
\[
H_1^{-1}(u) = \inf x \in \mathbb{R}, \quad H_1(x) \geq u, \quad u \in [0, 1],
\]
which is a left-continuous and non-decreasing function. Properties of generalized inverses are fully exposed in Chapter 4 in [Lo (2016c)]. By using Point 8 in the mentioned section, we may find to continuity points \( u_1 \) and \( u_2 \) of both \( H_1^{-1} \) and \( H_2^{-1} \) such that
\[
u_1 < u_2, \quad H_1^{-1}(u_1) < H_1^{-1}(u_2), \quad H_1^{-1}(u_1) < H_1^{-1}(u_2).
\]
Next, also by Point 4 in Chapter 4 in [Lo (2016c)], the weak convergence of $L_n$, defined by $L_n(x) = F_n(a_n x + b_n), x \in \mathbb{R}$, to $H_1$ in 1.3 implies convergence of their generalized inverses: $L_n^{-1} \rightsquigarrow H_1^{-1}$, that for any $u \in C(H_1)$,

$$
\frac{F_n^{-1}(u) - b_n}{a_n} \to H_1^{-1}(u).
$$

Applying this to $u_i, i = 1, 2$. We get, as $n \to +\infty$,

(1.7) $$
\frac{F_n^{-1}(u_i) - b_n}{a_n} \to H_1^{-1}(u_i), i = 1, 2.
$$

By taking the difference in each of the two formulas for $u_1$ and $u_2$, we have as $n \to +\infty$,

(1.8) $$
\frac{F_n^{-1}(u_2) - F_n^{-1}(u_1)}{a_n} \to H_1^{-1}(u_2) - H_1^{-1}(u_1)
$$

and

$$
\frac{F_n^{-1}(u_2) - F_n^{-1}(u_1)}{a_n} \to H_2^{-1}(u_2) - H_2^{-1}(u_1).
$$

By taking the ratio of the two last formula, we arrive at, as $n \to +\infty$,

$$
\frac{\alpha_n/a_n}{a_n} \to \frac{H_2^{-1}(u_2) - H_2^{-1}(u_1)}{H_1^{-1}(u_2) - H_1^{-1}(u_1)} : A > 0.
$$

From

$$
\frac{F_n^{-1}(u_1) - b_n}{a_n} \to H_1^{-1}(u_1)
$$

and

$$
\frac{F_n^{-1}(u_1) - \beta_n}{a_n} = \frac{F_n^{-1}(u_1) - \beta_n}{\alpha_n} \frac{\alpha_n/a_n}{a_n} \to AH_2^{-1}(u_1)
$$

and by taking their difference, we get as $n \to +\infty$,

$$
\frac{\beta_n - b_n}{a_n} \to H_1^{-1}(u_1) - AH_2^{-1}(u_1) = B.
$$

The proof is complete.

At this stage, we want to make two remarks.

**Remark 2.** $H$ is degenerated if and only $H^{-1}$ is degenerated, that is they have only one increase points at the same time. Next if (1.6)
holds, $H_1$ is degenerated if and only if $H_2$ is. To see that, suppose that $H_1$ is degenerated to the constant $a$, that is

$$H_1(x) = \begin{cases} 1 & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases}.$$ 

Then

$$H_2(x) = H_1\left(\frac{x-B}{A}\right) = \begin{cases} 1 & \text{if } x \geq (a-B)/A \\ 0 & \text{if } x < (a-B)/A. \end{cases}$$

Thus $H_2$ is also degenerated. In this reasoning, we may exchange the roles of $H_1$ and $H_2$ and consider the inverse relation $H_2(x) = H_1\left(\frac{x-B}{A}\right)$,

Remark 3. Once we have the convergence in type, say (1.3), we may change the sequences by setting first $\gamma_n = F_n^{-1}(u_2) - F_n^{-1}(u_1)$ so that by (1.8),

$$\gamma_n/\alpha_n \to C = H_1^{-1}(u_2) - H_1^{-1}(u_1) > 0.$$ 

Next, set $\delta_n = F_n^{-1}(u_1)$. By (1.7), we get

$$\delta_n - b_n/a_n \to H_1^{-1}(u_1) = D.$$ 

By applying the theorem, we have for $x \in C(H_1)$

$$F_n(\gamma_n x + \delta_n) \to H(Cx + D).$$

We will keep in mind this choice for the extreme value theory:

$$\begin{cases} \gamma_n = F_n^{-1}(u_2) - F_n^{-1}(u_1) \\ \delta_n = F_n^{-1}(u_1) \end{cases}$$

We have also

Lemma 6. Let $F$ be a nondegenerated probability distribution function. If for any $x \in \mathbb{R}$, we have $F(ax+b) = F(cx+d)$, for real numbers $a > 0$, $c > 0$, $b$ and $d$, then $a = c$ and $b = d$.

Proof of Lemma 6. Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$, where $\lambda$ the standard Lebesgue measure which is a probability. The canonical injection for $[0,1]$ to $\mathbb{R}$ follows the uniform standard law. Then $X = F^{-1}(U)$ has the probability distribution function $F$. Also $Z_1 = (X - b)/a$ has probability distribution function $F(ax + b)$ and $Z_2 = (X - d)/c$ the probability distribution function $F(cx + d)$. The equality $F(ax + b) = F(cx + d)$, $x \in \mathbb{R}$, implies for any $u \in [0,1]$,
Then
\[ \lambda(Z_1 = Z_2) = \lambda(u \in [0, 1], (F^{-1}u - b)/a = (F^{-1}u - b)/c) = 1. \]
Hence \( Z_1 = Z_2 \) a.s which leads do \( aZ_1 - cZ_1 = c - b \) a.s If \( a \neq c \), \( Z_1 = (c - d)/(a - c) = A \) a.s and next, \( X = aA + b \) a.s. So, unless \( X \) is degenerated, we have \( a = c \) and next the equality \( c = d \) easily follows.

2. The different non-degenerated type limits in extreme value theory

Let \( X_1, X_2, \ldots \) a sequence of independent and identically distributed (iid) random variables (rv's) defined on a probability space \( (\Omega, A, \mathbb{P}) \), of common probability distribution function \( F \). The univariate and classical extreme Value Theory is related on the characterization of the possible limits in type of the sequence of maxima
\[ M_n = \max(X_1, ..., X_n), \ n \geq 1. \]
If the sequence \( (M_n)_{n \geq 0} \) converges in type to a random variable \( Z \) of probability distribution function \( H \), we say that the sequence is attracted to \( Z \). This notion is more precise if we use the probability distribution functions, to say that \( F \) is attracted to \( H \), in other terms: \( F \) lies in the extreme value domain of attraction of \( H \), denoted \( F \in D(H) \), if and only there exist sequences \( (a_n > 0)_{n \geq 0} \) and \( (b_n)_{n \geq 0} \) such that
\[
\frac{M_n - b_n}{a_n} \xrightarrow{a.s.} Z \text{ as } n \to +\infty,
\]
that is, for any \( x \in C(H) \)
\[
F^n(a_n x + b_n) \to H(x) \text{ as } n \to +\infty().
\]
Right now, we say that by the pioneering work of [?] and others who will be cited that there are only three types non-degenerated possible limits. This is one the most beautiful result of that theory.

Theorem 4. (Fisher-Tippet (1928), Gnedenko (1943)). Suppose that \( F \in D(H) \), where \( H \) is nondegenerated. Then only three nondegenerated possible types of \( H \) are the following.

**The Gumbel type of distribution function:**
\[
\Lambda(x) = \exp(-\exp(-x)), \ x \in \mathbb{R},
\]
The Fréchet type df of parameter $\gamma > 0$:

\[(2.3) \quad \phi_\gamma(x) = \exp(-x^{-\gamma})I_{[0, +\infty]}(x), \ x \in \mathbb{R}\]

The Weibull type df of parameter $\gamma < 0$:

\[(2.4) \quad \psi_\gamma(x) = \exp(-(x)^{-\gamma})I_{(-\infty, 0]}(x) + (1 - 1_{(-\infty, 0]}(x)), \ x \in \mathbb{R}, \]

where $I_A$ denotes the indicator function of the set $A$.

We may the following notation in the sequel. $D(\phi) = \cup_{\gamma > 0} D(\phi_\gamma), D(\psi) = \cup_{\gamma > 0} D(\psi_\gamma)$, and $\Gamma = D(\phi) \cup D(\psi) \cup D(\Lambda)$.

**Proof of Theorem 4.** Suppose that (2.1) holds. For a fixed $t > 0$, we apply that formula for the indices $[nt]$ where $[.]$ stands for the integer part of a real number, that is the greatest integer less or equal to that number. We get sequences of real numbers $(a_n(t) > 0)_{n \geq 0}$ and $(b_n(t))_{n \geq 0}$ such that

\[(2.5) \quad F_{[nt]}(a_n(t)x + b_n(t)) \to H(x)\]

for any $x \in C(H)$. Right here, we remark that the function $a$ which maps $t$ to $a_{[nt]}$ and the function $b$ which associates $t$ to $b_{[nt]}$ are measurable, since for exemple for the first case, for any Borel set $B$ of $\mathbb{R}$,

\[(a \in B) = \sum_{a_k \in B} \{t, [nt] = k\} = \sum_{a_k \in B} \left[ \frac{k}{n}, \frac{k + 1}{n} \right],\]

which is a Borel set. Further, (2.5) implies for $x \in C(H)$

\[(2.6) \quad F_{[nt]}(a_n x + b_n(t)) = (F^n(a_n x + b_n(t)))^{[nt/n]} \to H^t(x).\]

By Lemma 2 and by comparing (2.5) and (2.6) there exists real numbers $\alpha(t) > 0$ and $\beta(t)$, such that

\[(2.7) \quad \frac{a_{[nt]}}{a_n} \to \alpha(t) \text{ and } \frac{b_n - b_{[nt]}}{a_{[nt]}} \to \beta(t)\]

and for any $x \in \mathbb{R}$

\[(2.8) \quad H(\alpha(t)x + \beta(t)) = H^t(x).\]

Let us apply this latter to the product $st$ to have

\[H^{st}(x) = H(\alpha(st)x + \beta(st))\]
But also
\[(2.9)\]
\[H^{st}(x) = (H^s(x))^t = H(\alpha(s)x + \beta(s))^t.\]
We apply (2.8) to \(x = \alpha(s)x + \beta(s)\) to get
\[(2.10)\]
\[H^{st}(x) = H(\alpha(t)\{\alpha(s)x + \beta(s)\} + \beta(t)).\]
We arrive at the equality for any \(x \in \mathbb{R}\),
\[H(\alpha(st)x + \beta(st)) = H(\alpha(s)\alpha(t)x + \alpha(t)\beta(s) + \beta(t)).\]
From this, and since \(H\) is nondegenerated, we get by Lemma 6,
\[(2.11)\]
\[\forall(s,t) \in (\mathbb{R}_+ \setminus 0)^2, \alpha(st) = \alpha(s)\alpha(t)\]
and
\[(2.12)\]
\[\forall(s,t) \in (\mathbb{R}_+ \setminus 0)^2, \beta(st) = \alpha(t)\beta(s) + \beta(t).\]
Formula (2.11) is a Hamel-Cauchy equation. The solutions of such equations are given in Chapter 4. By Corrolary 3.1 of that chapter, we get that there exists \(\rho \in \mathbb{R}\) such that
\[\alpha(t) = t^\rho, t > 0.\]
Let us consider the three cases corresponding to three signs of \(\rho\).

**Case** \(\rho = 0\).

Then \(\alpha(t) = 1\) and (2.11) implies
\[\forall(s,t) \in (\mathbb{R}_+ \setminus 0)^2, \beta(st) = \beta(s) + \beta(t).\]
By 3.1 of Chapter 4, for \(c = -\beta(e)\), we have
\[\beta(st) = -c\log t, t > 0.\]
Formula (2.8) gives
\[(2.13)\]
\[H^t(x) = H(x - c\log t), t > 0.\]
We are going to make a number of remarks and implications of this fact. First, for a fixed \(x \in \mathbb{R}\), \(H^t(x)\) in non-increasing in \(t > 0\), and since \(H\) is non-decreasing, we necessarily have that \(c > 0\). Next, we are going to see that \(H\) has an unbounded above support, that is \(H(x) < 1\), for all \(x \in \mathbb{R}\). Otherwise, there exists \(x_0\) such that \(H(x_0) = 1\). This would imply, from (2.13),
\[1 = H(x_0 - c\log t), t > 0.\]
But \{x_0 - c \log t, t > 0\} = \mathbb{R} so that we would get \(H(x) = 1\) for all \(x \in \mathbb{R}\) and then by right continuity \(H(-\infty) = 1\). This is a contradiction. So \(H(x) < 1\), for all \(x \in \mathbb{R}\). By the same argument \(H(0) = 0\) implies that \(H(x) = 0\) for all \(x \in \mathbb{R}\). Now let
\[
p = -\log(-\log H(0)),
\]
that is
\[
H(0) = \exp(-e^{-p}).
\]
Next from \(2.13\), we get
\[
H^t(0) = H(-c \log t), t > 0.
\]
From there, make the change of variance \(x = -c \log t\), that is \(t = \exp(-x/c)\) and next
\[
H(x) = (H(0))^{\exp(-x/c)} = \exp(-\exp(-x/c) \exp(-p)) = \exp(-\exp(-\frac{x + cp}{c})), x \in \mathbb{R}.
\]
By letting \(a = c > 0\) and \(b = -cp\), we have
\[
H(x) = \exp(-\exp(-\frac{x - b}{a})), x \in \mathbb{R}
\]
which is of type of
\[
\Lambda(x) = \exp(-e^{-x}), x \in \mathbb{R}.
\]
**Case** \(\rho < 0\).

By \((2.12)\) and by the symmetry of the roles of \(s\) and \(t\),
\[
\alpha(t)\beta(s) + \beta(t) = \alpha(s)\beta(t) + \beta(s),
\]
which implies
\[
\frac{\beta(s)}{1 - \alpha(s)} = \frac{\beta(s)}{1 - \alpha(s)},
\]
for any \(s > 0\) and \(t > 0\) with \(s \neq 1\) and \(t \neq 1\). Then \(\beta(t)/(1 - \alpha(t)) = c\) is a constant \(c\) for \(t \neq 1\) and next
\[
\beta(t) = c(1 - t^\rho), t > 0, t \neq 1,
\]
and \(2.8\) becomes for \(t > 0, t \neq 1\)
\[
H^t(x) = H(t^\rho(x - c) + c), x \in \mathbb{R},
\]
that is for \(t > 0, t \neq 1\)
\[
H^t(x + c) = H(x t^\rho + c), x \in \mathbb{R}.
\]
Let us change $H(x)$ into the type $H_1(x) = H(x + c)$ which is of the same type of $H$ to get

\[(2.14) \quad H_1'(x) = H_1(x t^\rho), x \in \mathbb{R}.\]

As for the precedent case, let us see that the fact $H_1(0) = 1$ is impossible. Let us assume that $H_1(0) = 1$ and set

$$a = \inf\{x < 0, H_1(x) = 1\}$$

If $a = -\infty$, we would get that $H_1 = 1$ on the whole real line, which is absurd. Then $a$ is finite. If we have $H_1(x) = 0$ for all $x < a$, $H_1$ would be degenerated. Finally, it would exists $x_0 < a$ such that $H_1(x_0) > 0$. By the definition of $a$, we would have $H_1(x_0) < 1$ and hence, $0 < H_1(x_0) < 1$. By (2.14), we would have

$$H_1'(x_0) = H(x_0 t^\rho), t > 0, t \neq 1.$$ 

is impossible since, as $t$ increases (while avoiding the value 1), the left-side member of this equality is decreasing to zero while the right-side member is nondecreasing. Then we exclude the fact that $H_1(x) = 0$. Now, application of 2.14 to $x = 0$ gives

$$H_1(0) = H_1(0), t > 0, t \neq 1$$

Since $0 \leq H_1(0) \leq 1$, we have only two possibilities $H_1(0) = 1$ or $H_1(0) = 0$. The case $H_1(0) = 1$ has already been excluded. We keep $H_1(0) = 0$. Now we are going to see that $H_1(1)$ is different of 0 and of 1. Indeed, $H_1(1) = 0$ implies through (2.14) that

$$0 = H_1(t^\rho), t > 0, t \neq 1.$$ 

Then for $t$ small enough, $H_1(t^\rho) = 0$ and $H_1(x) = 0$ for $x \leq t^\rho$. By letting $t \downarrow 0$, we have $H_1(x) = 0$ for $x \leq t^\rho \nearrow +\infty$ so that $H_1(x) = 0, x \in \mathbb{R}$. This is impossible. Next $H_1(1) = 1$ implies

$$1 = H_1(t^\rho), t > 0, t \neq 1.$$ 

Then for $t$ large enough,

$$H_1(x) \text{ for } x \geq t^\rho \searrow 0 \text{ as } t \nearrow +\infty.$$ 

Then

$$H_1(x) = 1, x \geq 0.$$ 

By combining this with the fact that $H_1(0) = 0$, we arrive at the conclusion that $H_1$ concentrated at zero, which is in contraction with our assumption. Then we have

$$H_1(0) = 0 \text{ and } 0 < H_1(1) < 1.$$
Finally, (2.14) yields for $t > 0, t \neq 1$,

$$H'_1(1) = H_1(t^\rho)$$

and by change of variables $x = t^\rho > 0$, $x \neq 1$

$$H_1(x) = H_1(1)^{1/\rho} = \exp(-x^{1/\rho}(-\log H_1(1))) = \exp(-ax^{-\gamma}),$$

where $a = -\log H_1(1) > 0$, $\gamma = -1/\rho$. By putting together all what precedes and by using right continuity to handle the point $x = 1$, we have

$$H_1(x) = \left\{ \begin{array}{ll}
\exp(-ax^{-\gamma}) & \text{if } x \geq 0 \\
0 & \text{if } x < 0,
\end{array} \right.$$

which is of the type of

$$\Phi_\gamma(x) = \left\{ \begin{array}{ll}
\exp(-x^{-\gamma}) & \text{if } x \geq 0 \\
0 & \text{if } x < 0.
\end{array} \right.$$  

**Case** $\rho > 0$.

In the proof of Case $\rho < 0$, the part from the beginning and to Formula (2.12) does not depend on the value of $\rho \neq 0$. So we may start this proof from 2.14. We are going to prove the upper endpoint of $H_1(x)$ is nonpositive, that is $uep(H_1) \leq 0$. Suppose that $uep(H_1) > 0$. If we have $H_1(x) = 0$ for all $x < uep(H_1)$, $H_1$ would be degenerated. Hence, it would exists $x_1 < uep(H_1)$ such that $H_1(x_1) > 0$. Of we choose $x_0$ such that $\max(0, x_1) < x_0 < uep(H_1)$, we would get $0 < H_1(x_0) < 1$. We would finally have

(2.15) \hspace{1cm} H'_1(x_0) = H_1(x_0 t^\rho), t > 0, t \neq 1.

The left-side member decreases in $t$ while the right-side is nondecreasing in $t$. This impossibility combined with the first one and the right-continuity at zero, allows to conclude that $uep(H_1) \leq 0$ and then

$$H_1(x) = 1, x \geq 0.$$  

As well, since $H_1$ is nondegenerated, there exists $x_1 < 0$ such that $0 < H_1(x_1) < 1$ and (2.14) implies

(2.16) \hspace{1cm} H'_1(x_1) = H_1(x_1 t^\rho), t > 0, t \neq 1.

Set $x = x_1 t^\rho < 0$ to get
\[ H_1(x) = H_1(x_1)^{(x/x_1)^{1/\rho}} \]
\[ = \exp((x/x_1)^{1/\rho} \log H_1(x_1)) = \exp(-b(-x)^\gamma), \quad x \neq -1, \]
where \( b = -(-x)^{1/\rho} \log H_1(x_1) > 0 \). Put together all what precedes and use the right-continuity at \(-1\) to get that
\[ H_1(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
\exp(-b(-x)^\gamma) & \text{if } x < 0,
\end{cases} \]
which is of the type of
\[ \Psi_\gamma(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
\exp(-b(-x)^\gamma) & \text{if } x < 0, \quad \gamma > 0.
\end{cases} \]

**Remark 4.** Theorem 4 does not say that these three limits in type effectively occur. It only says that if \( H \) is non-degenerated and is limit of type of \( F^n \), then it is necessarily of the these three types.

In the portal chapter 1, in Subsection 3.2, we already gave simple examples leading to these three types.

We want to finish with a generalization of the result in the lines just after Remark 3. This result is a by-product of the proof of Lemma. It allows to find arbitrary normalizing and centering coefficients once we know that a sequence of distribution functions \( F_n \) converges in type to a non-degenerated distribution \( H \). Is is stated as follows.

**Lemma 7.** Let \( F_n \) be a sequence of probability distribution functions weakly converging to a non-degenerated distribution function \( H \). Consider any other distribution of type of \( H \) of the form \( G(x) = H(Ax+B) \), \( x \in \mathbb{R} \) with \( A > 0 \). Choose \( 0 < u_1 < u_2 < 1 \) such that \( u_1 \) and \( u_2 \) are continuity points of \( H^{-1} \) and \( G^{-1} \). Set for \( n \geq 1 \)
\[ \gamma_n = F_n^{-1}(u_2) - F_n^{-1}(u_1), \]
\[ \delta_n = F_n^{-1}(u_1), \]
and
\[ a = \frac{G^{-1}(u_2) - G^{-1}(u_1)}{H^{-1}(u_2) - H^{-1}(u_1)} \]
and
\[ b = H^{-1}(u_1) - a^{-1}(u_1). \]
Then $F_n(\gamma_n x + \delta_n) \to H(ax + b)$ for any continuity point of $H$.

We are going to state a useful continuous version of that result, which is very important for the characterization of the Gumbel extreme domain of attraction.

**Lemma 8.** Let $H$ be a strictly increasing and continuous distribution function on $S(H) = [\text{lep}(H), \text{uep}(H)]$. Let $(F_n)_{n \geq 1}$ be a sequence of probability distribution which weakly converges to $H$. Choose two real numbers $u_1$ and $u_2$ such that $0 < u_1 < u_2 < 1$. Let $(u_1(n))_{n \geq 1}$ and $(u_2(n))_{n \geq 1}$ be two sequences of numbers in $]0, 1[$ such that $u_1(n) \rightarrow u_1$ and $u_1(n) \rightarrow u_2$ as $n \rightarrow +\infty$. Denote

$$
\begin{cases}
    b = H^{-1}(u_1) \\
    a = H^{-1}(u_2) - H^{-1}(u_1)
\end{cases}
$$

and define the sequences

$$
\begin{cases}
    b_n = F_n^{-1}(u_1(n)) \\
    a_n = F_n^{-1}(u_2(n)) - F_n^{-1}(u_1(n))
\end{cases}, n \geq 1.
$$

Put $A = a - b, B = b$ and $A_n = a_n - b_n, B_n = b_n, n \geq 1$.

We have

(I) $A > 0, A_n > 0$ for large values of $n$, and for any $x \in S(H)$,

$$F_n(A_n x + B_n) \to H(Ax + B) \text{ as } n \to \infty.$$

(II) If for all $x \in D$ such that

$$F_n(c_n x + d_n) \to H(x),$$

for sequences $(c_n > 0)_{n \geq 1}$ and $(d_n)_{n \geq 1}$, then we have

$$
\frac{F_n^{-1}(u_2(n)) - F_n^{-1}(u_1(n))}{c_n} \to H^{-1}(a) - H^{-1}(b) \text{ as } n \to +\infty.
$$

and

$$
\frac{F_n^{-1}(u_1(n)) - d_n}{c_n} \to 0 \text{ as } n \to +\infty.
$$
Proof. First, we fix $\delta > 0$ such that $\delta < (u_2 - u_1)/2$. So for large values of $n$ we have
\begin{equation}
0 < u_1 - \delta/2 < u_1(n) < u_1 + \delta/2 < u_2 - \delta/2 < u_2(n) < u_2 + \delta/2 < 1.
\end{equation}
By continuity of $H$, there exist $\varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_0$,
\[
\min\{H(y + \varepsilon) - H(y), H(y) - H(y - \varepsilon), y \in \{a, b\}\} < \delta/2.
\]
This implies, for $0 < \varepsilon < \varepsilon_0$, that
\[
H(y + \varepsilon) - \delta/2 < H(y) < H(y + \varepsilon) + \delta/2.
\]
Since $F_n \rightarrow H$, there exist sequences of real numbers $(c_n > 0)_{n \geq 1}$ and $(d_n)_{n \geq 1}$ such that for each $x \in D$, $F_n(c_n x + d_n) \rightarrow H(x)$. Then for $y \in \{a, b\}$, we have
\[
F_n(c_n(y - \varepsilon) + d_n) \rightarrow H(y - \varepsilon) \text{ and } F_n(c_n(y + \varepsilon) + d_n) \rightarrow H(y + \varepsilon).
\]
Then for $n$ large enough
\begin{equation}
F_n(c_n(y - \varepsilon) + d_n) < H(y - \varepsilon) + \delta
\end{equation}
and
\begin{equation}
F_n(c_n(y + \varepsilon) + d_n) > H(y + \varepsilon) - \delta.
\end{equation}
From this point, we handle the cases $y = a$ and $y = b$ one after the other. We have for large values of $n$,
\[
b_n = F_n^{-1}(u_1(n)) \implies F_n(b_n-) < u_1(n) \leq F_n(b_n)
\]
Fix $n_0$ such that (2.17), (2.18) and (2.19) hold for $n \geq n_0$. For $n \geq n_0$, we have
\[
F_n(b_n) \geq u_2(n) > u_1 - \delta/2 = H(b) - \delta > H(b + \varepsilon) - \delta > F_n(c_n(b + \varepsilon) + d_n).
\]
And $F_n(b_n) > F_n(c_n(y + \varepsilon) + d_n)$ for $n \geq n_0$ implies
\[
b_n > c_n(y + \varepsilon) + d_n, \text{ for } n \geq n_0.
\]
As well, for $n \geq n_0$,
\[
F_n(b_n-) < u_1(n) < u_1 + \delta = H(b) + \delta/2 < H(y + \varepsilon) + \delta < F_n(c_n(b + \varepsilon) + d_n)
\]
which implies that
\[
F_n(b_n-) < F_n(c_n(b + \varepsilon) + d_n).
\]
By the definition of the left limit, for each $n \geq n_0$, there exists $h > 0$ such that
\[ F_n(b_n - h) \leq F_n(b_n - h) < F_n(c_n(b + \varepsilon) + d_n), \]

which ensures that

\[ b_n - h < c_n(b + \varepsilon) + d_n. \]

By combining all these results, we have for \( n \geq n_0 \) and for \( 0 < \varepsilon < \varepsilon_0 \),

\[ c_n(b - \varepsilon) + d_n < b_n < c_n(b + \varepsilon) + d_n, \tag{2.20} \]

that is

\[ \left| \frac{b_n - d_n}{c_n} \right| < \varepsilon. \]

Taking the limit superior and next letting \( \varepsilon \downarrow 0 \) together give

\[ \frac{b_n - d_n}{c_n} \to 0 \text{ as } n \to \infty. \]

As well, similar considerations that led to (2.20) yield: for \( n \geq n_0 \) and for \( 0 < \varepsilon < \varepsilon_0 \),

\[ c_n(a - \varepsilon) + d_n < a_n < c_n(a + \varepsilon) + d_n. \tag{2.21} \]

Formulas (2.20) and (2.21) together ensure that for \( n \geq n_0 \) and for \( 0 < \varepsilon < \varepsilon_0 \),

\[ c_n(a - b - 2\varepsilon) < a_n - b_n < c_n(a - b + 2\varepsilon), \]

that is for \( n \geq n_0 \) (recall that \( a_n \geq b_n \) for all \( n \geq 1 \) and \( a > b \)).

\[ \left| \frac{a_n - b_n}{c_n} - (a - b) \right| < 2\varepsilon. \]

Similarly, taking the limit superior and next letting \( \varepsilon \downarrow 0 \) gives

\[ \frac{a_n - b_n}{c_n} \to (a - b) \text{ as } n \to \infty. \]

Part (II) of the lemma is the summary of the formulas proved by the computations above. Part (I) is the result of the application of Lemma 2 to these formula.

Before we begin the characterizations of distribution functions in the extreme value domain, we have to take a serious and deep tour of the important classes of functions that are in the heart of univariate extreme value theory.
3. Regularly and $\pi$-variation

This section is devoted to representations of functions involved in Extreme Value Theory, in particular to representations of Regularly varying functions and $\pi$-variation functions. The Karamata Representation Theorem and that of de Haan will be our special guests.

Let us begin by the concept of regular variation.

(A) Regular and Slow Variation.

Throughout this section, we will deal with functions $U : \mathbb{R}_+ \to \mathbb{R}_+$ that are measurable and Lebesgue locally integrable and not vanishing in the neighborhood of $+\infty$, that is, $(\forall 0 \leq a \leq b < +\infty, U \in L([a,b], \lambda)$. This will ensure in particular that $U$ is continuous a.e. and the formula

$$
\left( \int_0^x U(t) d\lambda(t) \right)' = U(x), \text{ } \lambda - \text{a.e. on } [0, +\infty[.
$$

holds.

3.1. Definitions. We begin to define the regular and the slow variation.

**Definition 7.** The function $U$ is regularly varying at $+\infty$ with exponent $\rho \in \mathbb{R}$ if and only for any $\gamma > 0$,

$$
\lim_{x \to +\infty} U(\gamma x)/U(x) = \gamma^\rho.
$$

When (3.1) holds, we use the notation

$$
U \in RV(\rho, +\infty).
$$

If $\rho = 0$, we say that $U$ is slowly varying at $+\infty$, denoted as

$$
U \in U \in RV(0, +\infty) \text{ or } U \in SV(+\infty).
$$

We are going to work with limits at $+\infty$. We also frequently work in neighborhoods of zero. So, we have to adapt the definition above for functions $u : [0, u_0] \to \mathbb{R}_+$, where $u_0 > 0$, defined in a right-neighborhood of zero. Using the transform
\[ u(s) = U(1/s), \quad s \in [0, u_0], \]

allows to transfer the definition in the following way.

**Definition 8.** The function \( u : [0, u_0] \to \mathbb{R}_+ \), where \( u_0 > 0 \), is regularly varying at zero with exponent \( \rho \in \mathbb{R} \) if and only for any \( \gamma > 0 \)

\[ \lim_{s \to 0} u(\gamma s)/u(s) = \gamma^\rho. \tag{3.2} \]

When (3.1), we use the notation

\[ u \in RV(\rho, 0). \]

If \( \rho = 0 \), we say that \( u \) is slowly varying at zero, denoted as

\[ u \in RV(0, 0) \text{ or } u \in SV(0). \]

We will be able to move from one of these two versions to the other, simply by remarking that: for \( u(s) = U(1/s) \), \( u \in RV(\rho, 0) \) if and only if \( U \in RV(-\rho, +\infty) \). So, the theory will be made for one the version and translated to the other if needed.

At the very beginning, let us notice these immediate properties.

**Lemma 9.** Let \( U : \mathbb{R}_+ \to \mathbb{R}_+ \) be a measurable function Lebesgue locally integrable and not vanishing in the neighborhood of \( +\infty \), that is, \( \forall \ (0 \leq a \leq b < +\infty), \ U \in L([a,b], \lambda) \) where \( \lambda \) is the Lebesgue measure. The function \( U \) is regularly varying if and only if for any \( x > 0 \),

\[ \lim_{x \to +\infty} U(\gamma x)/U(x) \text{ exists in } \mathbb{R}_+. \tag{3.3} \]

**Proof.** It is clear that (3.3) holds, if \( U \) is regularly varying. Now, suppose now that (3.3). Denote, for \( x > 0 \),

\[ \lim_{t \to +\infty} U(tx)/U(t) = h(x). \]

Now for any \( x > 0, y > 0 \),

\[ h(xy) = \lim_{t \to +\infty} U(txy)/U(t) \]

\[ = \lim_{t \to +\infty} \left\{ \frac{U(txy)}{U(tx)} \right\} \left\{ \frac{U(tx)}{U(t)} \right\} = h(x)h(y). \]
The function \( h : [0, +\infty[ \) is measurable and satisfies the Hamel Equation: \( h(xy) = h(x)h(y), \ x > 0, y > 0 \). By Corollary in Chapter 4, the unique solution is 
\[
    h(x) = x^{h(e)}, \ x > 0.
\]
So \( U \in RV(h(e), +\infty) \).

The next lemma gives some algebras on regular varying functions.

**Lemma 10.** If \( U \in RV(\rho_1, +\infty) \) and \( V \in RV(\rho_2, +\infty) \) then

1. \( UV \in RV(\rho_1 + \rho_2, +\infty) \).
2. \( UV^{-1} \in RV(\rho_1 - \rho_2, +\infty) \).

**Proof.** The proofs are immediate.

Regular variation is used in Extreme value Theory by the so important Karamata representation we are going to introduce in the next subsection.

### 3.2. Karamata Representation Theorem

Let us begin by these two lemmas.

**Lemma 11.** (de Haan, 1970). Let \( U : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be a measurable function Lebesgue locally integrable and not vanishing in the neighborhood of \(+\infty\), that is, \( \forall (0 \leq a \leq b < +\infty), U \in L([a,b], \lambda) \) where \( \lambda \) is the Lebesgue measure. The following assertions holds.

\[
(A) \text{ If } U \in RV(\rho, +\infty), \rho > -1, \ U^*(x) = \int_0^x U(t)d\lambda(t) \text{ satisfies } U^*(+\infty) = +\infty \text{ and } U^* \in RV(\rho + 1, +\infty). \\
(B) \text{ If } U \in RV(\rho, +\infty), \rho < -1, \ U_*(x) = \int_0^x U(t)d\lambda(t) \text{ satisfies } U_*(+\infty) < +\infty \text{ and } U_* \in RV(\rho + 1, +\infty). \\
(C) \text{ If } U \in RV(-1, +\infty), \text{then } U^* \in RV(0, +\infty). \text{ If further } U_*(+\infty) < +\infty, \text{ then } U_* \in RV(1, +\infty). 
\]

**Proof of Lemma 11.**
Part (A). Let \( U \in RV(\rho, +\infty), \rho > -1 \). We have to prove that \( U^*(+\infty) = +\infty \). Let \( c > 1 \). We have \( U(ct)/U(t) \to c^\rho > c^{-1} \). Let \( \varepsilon > 0 \) such that \( \rho - \varepsilon > -1 \). Then, there exists \( x_0 > 1 \), such that for \( t \geq x_0 \),
\[
U(ct)/U(t) > c^{-1+\varepsilon}.
\]
For \( n \) such that \( c^n \geq x_0 \),
\[
\int_{c^n}^{c^{n+1}} U(t)d\lambda(t) = \int_{c^n}^{c^{n+1}} cU(ct)d\lambda(t) > c^\varepsilon \int_{c^n}^{c^{n+1}} U(t)d\lambda(t).
\]
Now let \( n_0 \) the first integer such that \( c^{n_0} \geq x_0 \) and \( \int_{c^{n_0}}^{c^{n_0+1}} U(t)d\lambda(t) \neq 0 \). For any \( n > n_0 \), we have
\[
\int_{c^n}^{c^{n+1}} U(t)d\lambda(t) > c^\varepsilon \int_{c^n}^{c^{n+1}} U(t)d\lambda(t) > c^{2\varepsilon} \int_{c^{n_0}}^{c^{n_0+1}} U(t)d\lambda(t)
\]
\[
\ldots > c^{(n-n_0)\varepsilon} \int_{c^{n_0}}^{c^{n_0+1}} U(t)d\lambda(t).
\]
Next
\[
\int_0^{+\infty} U(t)d\lambda(t) \geq \int_{x_0}^{+\infty} U(t)d\lambda(t) \geq \sum_{n \geq n_0} \int_{c^n}^{c^{n+1}} U(t)d\lambda(t)
\]
\[
\geq \left\{ \int_{c^{n_0}}^{c^{n_0+1}} U(t)d\lambda(t) \right\} \left\{ \sum_{n \geq n_0} c^{(n-n_0)\varepsilon} \right\} = +\infty.
\]
To complete the proof, we have to check that we can find \( n_0 \) such that \( c^{n_0} \geq x_0 \) and \( \int_{c^{n_0}}^{c^{n_0+1}} U(t)d\lambda(t) \neq 0 \). If we cannot, then \( U = 0 \) a.e. on \([N_0, +\infty[\) where \( N_0 \) is the first integer such that \( c^{N_0} \geq 1 \). This is impossible because of the assumption. Thus
\[
U_*(+\infty) = +\infty.
\]
For \( \gamma > 0 \), we have
\[
\frac{U^*(\gamma x)}{U^*(x)} = \frac{\int_0^{\gamma x} U(t)d\lambda(t)}{\int_0^x U(t)d\lambda(t)} = \frac{\gamma \int_0^x U(t)d\lambda(t)}{\int_0^x U(t)d\lambda(t)} = \gamma.
\]
Now we may apply Part (A) of Lemma 18, for \( f(t) = U(\gamma t) \) and \( g(t) = U(t) \) to get that
\[
\lim_{x \to +\infty} \frac{U^*(\gamma x)}{U^*(x)} = \gamma \lim_{x \to +\infty} \frac{U(\gamma x)}{U(x)} = \gamma^{\rho+1}.
\]
Part (B). Let $U \in RV(\rho, +\infty), \rho < -1$. Let us prove that $U_* (+\infty) < \infty$. Let $c > 1$. We have $U(ct)/U(t) \to c^\rho < c^{-1}$. Let $\varepsilon > 0$ such that $\rho + \varepsilon < -1$. Then, there exists $x_0 > 1$, such that for $t \geq x_0$,

$$U(ct)/U(t) < c^{-1-\varepsilon} = \delta < 1.$$ 

For $n$ such that $c^n \geq x_0$,

$$\int_{c^n}^{c^{n+1}} U(t) d\lambda(t) = \int_{c^{n-1}}^{c^n} cU(ct) d\lambda(t) < \delta \int_{c^{n-1}}^{c^n} U(t) d\lambda(t).$$

Now let $n_0$ the first integer such that $c^{n_0} \geq x_0$. For any $n > n_0$, we have

$$\int_{c^n}^{c^{n+1}} U(t) d\lambda(t) < \delta \int_{c^{n-1}}^{c^n} U(t) d\lambda(t) > \delta^n \int_{c^{n-2}}^{c^{n-1}} U(t) d\lambda(t)$$

$$\ldots < \delta^n U_{c^{n_0}} U_{c^{n_0+1}} U_{c^{n_0-1}} U_{c^{n_0-2}} \ldots \int_{c^{n_0}}^{c^{n_0+1}} U(t) d\lambda(t).$$

Next

$$\int_{c^{n_0}}^{+\infty} U(t) d\lambda(t) = \sum_{n \geq n_0} \int_{c^n}^{c^{n+1}} U(t) d\lambda(t)$$

$$\leq \left\{ \int_{c^{n_0}}^{c^{n_0+1}} U(t) d\lambda(t) \right\} \left\{ \sum_{n \geq n_0} \delta^{n-n_0} \right\}$$

$$= (1 - \delta)^{-1} \left\{ \int_{c^{n_0}}^{c^{n_0+1}} U(t) d\lambda(t) \right\} < +\infty.$$ 

Next

$$\int_{0}^{+\infty} U(t) d\lambda(t) = \int_{0}^{c^{n_0}} U(t) d\lambda(t) + \int_{c^{n_0}}^{+\infty} U(t) d\lambda(t) < +\infty.$$ 

For $\gamma > 0$, we have

$$\frac{U_*(\gamma x)}{U_* (\gamma x)} = \frac{\int_{\gamma x}^{+\infty} U(t) d\lambda(t)}{\int_{x}^{+\infty} U(t) d\lambda(t)} = \frac{\gamma \int_{x}^{+\infty} U(\gamma t) d\lambda(t)}{\int_{x}^{+\infty} U(t) d\lambda(t)}.$$ 

Now we may apply Part (B) of Lemma 18, for $f(t) = U(\gamma t)$ and $g(t) = U(t)$ to get that

$$\lim_{x \to +\infty} \frac{U_*(\gamma x)}{U_*(x)} = \gamma \lim_{x \to +\infty} \frac{U(\gamma x)}{U(x)} = \gamma^{\rho + 1}.$$ 

(3.5)
Part (C). Let \( U \in RV(0, +\infty) \). As for \( U^* \), either \( U^*(+\infty) = +\infty \) and we use again the proof of Part (A) above from (3.4) that needed only that \( U_\ast \) is infinite at \( +\infty \). Either \( U^*(+\infty) \) is infinite at \( +\infty \).

As for \( U_\ast \), the condition \( U_\ast(+\infty) < +\infty \) leads to (3.5) [ in the proof of Part 2] to get the same conclusion. In both case, \( U_\ast \in RV(0, +\infty) \).

The next lemma is the main Karamata Theorem. We expose it as a lemma and next give the induced representation as the theorem in this book.

**Lemma 12.** We have the following assertions.

(A) If \( U \in RV(\rho, +\infty), \rho \geq -1 \), then

\[
b(x) = \frac{xU(x)}{\int_0^x U(t)d\lambda(t)} \to \rho + 1 \text{ as } x \to +\infty
\]

and there is a constant \( c > 0 \), for \( x \geq 0 \),

(3.6) \[ U(x) = cx^{-1}b(x) \exp \left( -\int_1^x t^{-1}b(t)dt \right) \]

(B) \( U \in RV(\rho, +\infty), \rho < -1 \). We have

\[
B(x) = \frac{xU(x)}{\int_x^\infty U(t)d\lambda(t)} \to -(\rho + 1) \text{ as } x \to +\infty,
\]

and there is a constant \( c > 0 \), for \( x \geq 0 \),

(3.7) \[ U(x) = cx^{-1}B(x) \exp \left( -\int_1^x t^{-1}B(t)dt \right). \]

The functions \( b \) and \( B \) are bounded.

**Proof of Lemma 12.**

Part (A). We begin to remark that if \( U \in RV(\rho, +\infty) \) with \( \rho \geq -1 \), then \( xU(x) \in RV(\rho + 1, +\infty) \) and \( \int_0^x U(t)d\lambda(t) \in RV(\rho + 1, +\infty) \) by Lemma 11. Applying Lemma 10,

\[
b(\omega) \in VR(0, +\infty).
\]
Now, by Lemma 11, $b(\circ) \in RV(\rho + 1)$. And
\[
x^{-1}b(x) = \frac{U(x)}{\int_0^x U(t) d\lambda(t)} = \left( \int_0^x U(t) d\lambda(t) \right)' / \left( \int_0^x U(t) d\lambda(t) \right) \\
= \left( \log \int_0^x U(t) d\lambda(t) \right)'.
\]
This gives, for $x > 0$,
\[
\log \int_0^x U(t) d\lambda(t) = \int_1^x t^{-1}b(t) dt + c_1, \text{ a.e.}
\]
where $c_1$ is some constant. Then, for some constant $c > 0$, for for $x > 0$,
\[(3.8) \quad \int_0^x U(t) d\lambda(t) = c \exp \left( \int_1^x t^{-1}b(t) dt \right), \text{ a.e.,}\]
and next, for $x >$,
\[(3.9) \quad U(x) = cx^{-1}b(x) \exp \left( \int_1^x t^{-1}b(t) dt \right), \text{ a.e.,}\]

By making the change of variable $s = tx$, we have
\[
\int_0^x U(s) d\lambda(s) = \int_0^1 xU(tx) d\lambda(t).
\]
Using this and the Fatou-Lebesgue Theorem, we have
\[
\liminf_{x \to \infty} \frac{b(x)^{-1}}{x} = \liminf_{x \to \infty} \frac{\int_0^x U(t) d\lambda(t)}{xU(x)} \\
= \liminf_{x \to \infty} \int_0^1 U(tx) d\lambda(t) \cdot \frac{1}{U(x)} \\
\geq \int_0^1 \liminf_{x \to \infty} \left\{ \frac{U(tx)}{U(x)} \right\} d\lambda(t) \\
= \int_0^1 t^\rho d\lambda(t) = (1 + \rho)^{-1}.
\]
If $\rho = -1$, we have $\liminf_{x \to \infty} b(x)^{-1} = +\infty$. Hence $\limsup_{x \to +\infty} b(x) = 0$ and then
\[
b(x) \to 1 + \rho \text{ for } \rho = -1.
\]
What happens for $\rho > -1$? We recall that
\[
\liminf_{x \to \infty} b(x)^{-1} = \liminf_{t \to +\infty} \inf \{b(t)^{-1}, t \geq x\}.
\]
Since \( \inf \{b(t)^{-1}, t \geq x\} \nearrow 1+\rho > 0 \), then for some \( x_0 > 0 \), \( \inf \{b(t)^{-1}, t \geq x\} > (1+\rho)/2 \). This implies that \( b(t)^{-1} > (1+\rho)/2 = M > 0 \) for all \( t \geq x_0 \). It comes that \( b(\circ) \) is bounded on \([x_0, +\infty]\). Thus, for \( x \geq x_0 \), for \( t \geq 1 \), for \( s \geq 1 \),

\[
\frac{|b(tx) - b(x)|}{t} \leq Mt^{-1} \in L([1, s], \lambda).
\]

Further, since \( b(\circ) \in RV(0, +\infty) \), we have for any fixed \( t \in [1, s] \),

\[
\frac{b(tx) - b(x)}{t} \to 0 \text{ as } x \to \infty.
\]

We may apply the Dominated Lebesgue Theorem to get

\[
\int_1^s \frac{b(tx) - b(x)}{t} d\lambda(t) \to 0.
\]

\( \forall \) \((0 \leq a \leq b < +\infty), U \in L([a, b], \lambda)\)

But, as \( x \in +\infty \),

(3.10) \[
\int_1^s \frac{b(tx) - b(x)}{t} d\lambda(t) = \int_1^s t^{-1} b(tx) d\lambda(t) - b(x) \log s \to 0.
\]

Now from (3.8) and from Part (A) of Lemma 11, \( h(x) = \exp \left( \int_1^x t^{-1} b(t) dt \right) \in RV(\rho + 1, +\infty) \). Then

\[
\frac{h(sx)}{h(x)} = \exp(\int_x^{xs} t^{-1} b(t) dt) = \exp(\int_1^s u^{-1} b(xu) du).
\]

Then

(3.11) \[
\int_1^s t^{-1} b(tx) dt = \log \left\{ \frac{h(sx)}{h(x)} \right\} \to \log s^{(1+\rho)} \text{ as } x \to \infty.
\]

By comparing (3.10) and (3.11), we get

\[
\int_1^s t^{-1} b(tx) dt - b(x) \log s \to 0 \text{ and } \int_1^s t^{-1} b(tx) dt - (1 + \rho) \log s \to 0.
\]

This leads to

\[
b(x) \to 1 + \rho \text{ as } x \to +\infty.
\]

This finishes the proof of Part A.
Part (B). As in Part (A), if $U \in RV(\rho, +\infty)$ with $\rho < -1$, then $xU(x) \in RV(\rho + 1, +\infty)$ and $\int_x^{+\infty} U(t)d\lambda(t) \in RV(\rho + 1, +\infty)$ by Lemma 11. By applying Lemma 10, we get

$$B(\circ) \in VR(0, +\infty).$$

Since $A = \int_0^\infty U(t)d\lambda(t)$ is finite by Lemma 11, we have

$$x^{-1}B(x) = \frac{U(x)}{A - \int_0^x U(t)d\lambda(t)} = -\left( A - \int_0^x U(t)d\lambda(t) \right)' \left/ \left( A - \int_0^x U(t)d\lambda(t) \right) \right., \text{ a.e.,}$$

This gives, for $x > 0$,

$$\log \left( A - \int_0^x U(t)d\lambda(t) \right) = -\int_1^x t^{-1}B(t)dt + c_1, \text{ a.e.}$$

where $c_1$ is some constant. Then, for some constant $c > 0$, for $x > 0$,

$$\int_x^{+\infty} U(t)d\lambda(t) = A - \int_0^x U(t)d\lambda(t) = c \exp \left( -\int_1^x t^{-1}B(t)dt \right), \text{ a.e.}$$

and next, for $x > 0$,

$$U(x) = cx^{-1}B(x) \exp \left( -\int_1^x t^{-1}B(t)dt \right).$$

From here, we will be able to conclude by re-conducting the same methods as in Part (A), if we prove the analogues of (3.10) and (3.11) for $B(\circ)$, that is, as $x \to +\infty$,

$$\int_1^x t^{-1}B(tx)dt - B(x) \log s \to 0,$$

and

$$\int_1^s t^{-1}B(tx)dt + (1 + \rho) \log s \to 0.$$
To get (3.14), we may get it as we did for 3.10 only by showing that $B(\infty)$ is ultimately bounded. To get this, we also use the Fatou-Lebesgue Theorem to have

$$\liminf_{x \to \infty} b(x)^{-1} = \liminf_{x \to \infty} \frac{\int_{x}^{+\infty} U(t) d\lambda(t)}{xU(x)}$$

$$= \liminf_{x \to \infty} \frac{\int_{1}^{+\infty} U(tx) d\lambda(t)}{U(x)}$$

$$\geq \int_{1}^{+\infty} \liminf_{x \to \infty} \left\{ \frac{U(tx)}{U(x)} \right\} d\lambda(t)$$

$$= \int_{0}^{1} t^\rho d\lambda(t) = -(1 + \rho)^{-1},$$

where we took into account that $1 + \rho < 0$. As in Part (B), this ensures that $B(\infty)$ and this ensures (3.14).

To establish (3.15), we may see from (3.12) and from Part (B) of Lemma 11 that

$$h(x) = \exp \left( - \int_{1}^{x} t^{-1} B(t) dt \right) \in RV(\rho + 1, +\infty).$$

Then

$$\frac{h(sx)}{h(x)} = \exp(- \int_{x}^{sx} t^{-1} B(t) dt) = \exp(- \int_{1}^{s} t^{-1} B(xt) dt)$$

and

$$(3.16) \quad \int_{1}^{s} t^{-1} B(tx) dt = \log \left\{ \frac{h(sx)}{h(x)} \right\} \to -(1 + \rho) \log s \text{ as } x \to \infty,$$

and this implies

$$\int_{1}^{s} t^{-1}(B(tx) + (1 + \rho)) dt \to 0 \text{ as } x \to +\infty,$$

which is (3.15). Besides, it is clear from the details of the proof that the functions $b$ and $c$ and $B$ are bounded.

The proof of this important theorem is now complete.

The following representation result is a key tool in Extreme value Theory.
Theorem 5. (Karamata’s Theorem) Let $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a measurable function Lebesgue locally integrable and not vanishing in the neighborhood of $+\infty$, that is, $\forall (0 \leq a \leq b < +\infty), U \in L([a,b], \lambda)$ where $\lambda$ is the Lebesgue measure.

Then $U \in RV(\rho, +\infty)$ if and only there exist two measurable functions $a(x)$ and $\ell(x)$ of $x \in \mathbb{R}$ and a constant $c > 0$ such that $(p(x), \ell(x)) \rightarrow (0, 0)$ as $x \rightarrow \infty$, for any $x \geq 0$

$$U(x) = c(1 + a(x)) \exp(\int_1^x (1 - \rho) t^{-1} \ell(t) d\lambda(t)).$$

Besides, the functions $a(x)$ and $\ell(x)$ of $x \in \mathbb{R}$ are bounded in neighborhood of $+\infty$.

Proof. Let $U \in RV(\rho, +\infty)$. Let first $\rho \neq -1$. Use Lemma 12. Take $m = b$ in Part (A) and $m = -B$ in Part (B) so that $\ell(x) = m(x) - (1 - \rho) \rightarrow 0$ as $x \rightarrow \infty$ and next

$$U(x) = c |1 + \rho| (1 + p(x)) x^{-1} \exp(\int_1^x (1 - \rho) t^{-1} d\lambda(t)) \exp(\int_1^x t^{-1} \ell(t) d\lambda(t)),$$

where $p(x) \rightarrow 0$ as $x \rightarrow \infty$. For $C = c |1 + \rho|$, we get for $x > 0$,

$$U(x) = C(1 + p(x)) x^\rho \exp(\int_1^x t^{-1} \ell(t) d\lambda(t)).$$

If $\rho = -1$, then $xU(x) \in RV(0, +\infty)$. Then $xU(x)$ admits a representation

$$xU(x) = C(1 + p(x)) \exp(\int_1^x t^{-1} \ell(t) d\lambda(t))$$

which implies

$$U(x) = C(1 + p(x)) x^{-1} \exp(\int_1^x t^{-1} \ell(t) d\lambda(t)).$$

The reverse direction is quite straightforward. Suppose that (3.17) holds. Then for any fixed $\gamma > 0$, we have

$$U(\gamma x)/U(x) = \frac{1 + p(\gamma x)}{1 + p(x)} \gamma^\rho \exp(\int_x^{\gamma x} t^{-1} \ell(t) d\lambda(t)).$$

For any $\varepsilon > 0$, we have for $x > 0$, large enough, $0 < 1 - \varepsilon \leq 1 + p(\gamma x), 1 + p(x) \leq 1 + \varepsilon$ and max$\{|b(t)|, |t| \leq \max(x, \gamma x)| \leq \varepsilon$ and then

$$\frac{1 - \varepsilon}{1 + \varepsilon} \gamma^{\rho - \varepsilon} \leq \frac{U(\gamma x)}{U(x)} \leq \frac{1 + \varepsilon}{1 - \varepsilon} \gamma^{\rho + \varepsilon}.$$
Hence
\[
\frac{1 - \varepsilon \gamma^{\rho - \varepsilon}}{1 + \varepsilon \gamma^{\rho}} \leq \liminf_{x \to +\infty} \frac{U(\gamma x)}{U(x)} \leq \limsup_{x \to +\infty} \frac{U(\gamma x)}{U(x)} \leq \frac{1 + \varepsilon \gamma^{\rho + \varepsilon}}{1 - \varepsilon}.
\]

Letting \( \varepsilon \downarrow 0 \) leads to
\[
\lim_{x \to +\infty} \frac{U(\gamma x)}{U(x)} = \gamma^\rho.
\]

The measurability and the boundedness of \( p \) and \( \ell \) come from that of \( m \), that is that of \( b \) and \( B \).

The proof is now complete.

**Remark.** We may see in this proof that the representation (3.17) is true, whenever \( b(x) \to \lambda = \rho + 1 \in ]0, +\infty[ \) or \( B(x) \to \lambda = - (\rho + 1) \in ]0, +\infty[ \) and (3.17) ensures that \( U \in RV(\rho, +\infty) \). We then get this inverse lemma to Lemma 12

**Lemma 13.** Let \( U : \mathbb{R}_+^+ \to \mathbb{R}_+ \) be a measurable function Lebesque locally integrable and not vanishing in the neighborhood of \(+\infty\) that is, \( \forall 0 \leq a \leq b < +\infty, U \in L([a, b], \lambda) \) where \( \lambda \) is the Lebesgue measure. If, for some \( \rho \in \mathbb{R} \),
\[
b(x) = \frac{x U(x)}{\int_{0}^{+\infty} U(t) d\lambda(t)} \to \lambda \in ]0, +\infty[ \text{ as } x \to +\infty,
\]
then \( U \in RV(\lambda - 1, +\infty) \).

If \( U \) in integrable over subsets of the form \([a, +\infty[\), \( 0 < a \in \mathbb{R} \) and
\[
b(x) = \frac{x U(x)}{\int_{x}^{+\infty} U(t) d\lambda(t)} \to \lambda \in ]0, +\infty[ \text{ as } x \to +\infty,
\]
then \( U \in RV(-\lambda - 1, +\infty) \).

**3.3. Weak conditions for regular variation.** It may be handy to have weaker conditions for establishing that some function is regularly varying. This subsection provides such weaker conditions.

**Lemma 14.** Let \( U : \mathbb{R}_+^+ \to \mathbb{R}_+ \) be a monotone function not vanishing in the neighborhood of \(+\infty\). Then \( U \) is regularly varying if and only if there exist two sequences \((\lambda_n)_{n \geq 1}\) and \((a_n)_{n \geq 1}\) such that
\[
\lim_{n \to +\infty} \lambda_{n+1}/\lambda_n = 1 \text{ and } \lim_{n \to +\infty} a_n = +\infty
\]
and such that for all $x > 0$, \( \lambda_n U(a_n x) \) admits a limit in \( \mathbb{R}_+ \setminus \{0\} \), denoted by

\[
\lim_{n \to +\infty} \lambda_n U(a_n x) = \ell(x).
\]

Moreover, the function \( \ell(\cdot) \) is regularly varying with the same exponent as \( U(\cdot) \).

**Proof.** Let us prove it for \( U \) non-increasing first. Then

\[
\inf U = U(+\infty) = \lim_{x \uparrow +\infty} U(x) \leq U(0).
\]

**Case** \( U(+\infty) > 0 \). Then \( U(0+) \in ]0, U(0)[ \). We have for any \( x > 0 \),

\[
\lim_{t \to +\infty} \frac{U(tx)}{U(t)} = \frac{U(+\infty)}{U(+\infty)} = 1,
\]

that is \( U \in RV(0, +\infty) \). At the same time, we have, by taking \( \lambda_n = 1 \) and \( a_n = n \), that

\[
\lim_{n \to +\infty} \lambda_n U(a_n x) = U(+\infty) = \ell(x) \text{ positive and finite},
\]

which is also in \( RV(0, +\infty) \). So the equivalence holds for \( U(0+) > 0 \).

**Case** \( U(+\infty) = 0 \).

**Part (A).** Suppose that there exist two sequences \( (\lambda_n)_{n \geq 1} \) and \( (a_n)_{n \geq 1} \) such that

\[
\lim_{n \to +\infty} \lambda_{n+1}/\lambda_n = 1 \text{ and } \lim_{n \to +\infty} a_n = +\infty.
\]

Define for \( t > 0 \),

\[
n(t) = \inf\{n \geq 0, a_{n+1} > t\},
\]

which implies for any \( t > 0 \),

\[
a_{n(t)} \leq t < a_{n(t)+1}.
\]

It is clear that \( n(t) \to \infty \) as \( t \to +\infty \). Using the monotonicity of \( U \), we have

\[
\frac{U(tx)}{U(t)} \leq \frac{U(xa_{n(t)+1})}{U(a_{n(t)})} = \frac{\lambda_{n(t)+1} U(xa_{n(t)+1})}{\lambda_n U(a_n)} \left\{ \frac{\lambda_{n(t)}}{\lambda_{n(t)+1}} \right\} \to \ell(x)/\ell(1).
\]

Also
\[
\frac{U(tx)}{U(t)} \geq \frac{U(xa_n(t))}{U(a_n(t)+1)} = \frac{\lambda_n(t)U(xa_n(t))}{\lambda_{n(t)+1}U(a_n(t)+1)} \left\{ \frac{\lambda_{n(t)+1}}{\lambda_n(t)} \right\} \to \frac{\ell(x)}{\ell(1)}.
\]

Then
\[
\frac{U(tx)}{U(t)} \to h(x) = \frac{\ell(x)}{\ell(1)} \text{ finite.}
\]

By Lemma 9, we conclude that \( U \in RV(\ell(e)/\ell(1), +\infty) \).

**Part (B).** Now, let us suppose that \( U \) is regularly varying. Recall that \( U \) is non-increasing. We may define the generalized inverse
\[
V(y) = \inf \{ x > 0, U(x) \leq y \}.
\]

Let us show that for any \( y > 0 \),
\[
(3.18) \quad U(V(y)+) \leq y \leq U(V(y)-) y > 0.
\]

By definition of the infimum, there exists a sequence \( (x_n)_{n \geq 1} \) such that
\[
\forall (n \geq 1), U(x_n) \leq y \text{ and } x_n \downarrow V(y) \text{ as } n \uparrow +\infty.
\]

Then by letting \( n \uparrow +\infty \) in \( U(x_n) \leq y \) we get \( U(V(y)+) \leq y \). Next, for any \( \eta > 0 \), \( V(y) - \eta \) cannot satisfy \( U(V(y)-\eta) \leq y \), otherwise \( V(y) \) would not be the infimum of \( \{ x > 0, U(x) \leq y \} \). So \( U(V(y)-\eta) \geq y \) for all \( \eta > 0 \). By letting \( \eta \downarrow 0 \), we get \( U(V(y)-) \geq y \). Since \( U(\cdot) \) does not take the null value in the neighborhood of zero and since \( U(+\infty) = 0 \), it becomes clear that
\[
(3.19) \quad \lim_{y \uparrow +\infty} V(y) = +\infty.
\]

Suppose now that \( U \in RV(\rho, +\infty) \). Let us also show that
\[
(3.20) \quad \lim_{t \to +\infty} \frac{U(t-)}{U(t)} = \lim_{t \to +\infty} \frac{U(t+)}{U(t)} = 1.
\]

To see this, let \( 0 < x < 1 \). For any \( t > 0 \) fixed, for any \( s \) such that \( tx < s < t \), \( U(t) \leq U(s) \leq U(tx) \). Then, we have
\[
1 \leq \frac{U(s)}{U(t)} \leq \frac{U(tx)}{U(t)}.
\]

By letting \( s \uparrow 0 \) and next \( t \uparrow +\infty \), we get
\[
(3.21) \quad 1 \leq \limsup_{t \to +\infty} \frac{U(t-)}{U(t)} \leq \lim_{t \to +\infty} \frac{U(tx)}{U(t)} = x^\rho.
\]
If $\rho = 0$ we have

(3.22) \[ \lim_{t \to +\infty} \frac{U(t-)}{U(t)} = 1. \]

If $\rho \neq 0$, we let $x \uparrow 1$ in (3.21) to get (3.22). This proves the first limit in (3.20). To get the second limit, consider $x > 1, t > 0, t < s < tx$, to get

\[ \frac{U(tx)}{U(t)} \leq \frac{U(s)}{U(t)} \leq 1. \]

We will able to conclude by letting successively $s \downarrow t, t \uparrow +\infty$ and $x \downarrow 1$.

Combining (3.18), (3.19) and (3.20) leads to

\[ \lim_{y \to +\infty} U(V(y))/y = 1. \]

Finally take $\lambda_n = n$ and $a_n = V(1/n)$ for $n \geq 1$. We get

\[ \lim_{n \to +\infty} \lambda_n U(a_n x) = \lim_{n \to +\infty} \{ \lambda_n U(a_n x) \} \left\{ \frac{U(a_n x)}{U(a_n)} \right\} = \lim_{n \to +\infty} \left\{ \frac{U(V(1/n))}{(1/n)} \right\} \left\{ \frac{U(a_n x)}{U(a_n)} \right\} = x^\rho. \]

The proof of the lemma is complete for $U$ non-increasing. To extend this a non-increasing fonction $U$, we use the transform $V = 1/U$ and take into account the three equivalences :

(1) : ($U$ non-increasing, finite and nonzero in the neighborhood of $+\infty$) $\iff$ ($1/U$ non-decreasing finite and nonzero in the neighborhood of $+\infty$).

(2) : ($U \in RV(\rho, +\infty), \rho \neq 0$) $\iff$ ($1/U \in RV(-\rho, +\infty), \rho \neq 0$).

(3) There exist two sequences $(\lambda_n)_{n \geq 1}$ and $(a_n)_{n \geq 1}$ such that

\[ \lim_{n \to +\infty} \lambda_{n+1}/\lambda_n = 1 \text{ and } \lim_{n \to +\infty} a_n = +\infty, \]

and such that for all $x > 0$

\[ \lim_{n \to +\infty} \lambda_n U(a_n x) = \ell(x) \]

is equivalent to :
there exist two sequences \((\lambda'_n)_{n \geq 1}\) and \((a'_n)_{n \geq 1}\) such that
\[
\lim_{n \to +\infty} \frac{\lambda'_{n+1}}{\lambda'_n} = 1 \quad \text{and} \quad \lim_{n \to +\infty} a'_n = +\infty,
\]
and such that for all \(x > 0\),
\[
\lim_{n \to +\infty} \lambda_n U(a_n x) = \ell'(x)
\]
with
\[
\lambda'_n = 1/\lambda'_n, \quad a'_n = a_n, \quad \ell'(x) = 1/\ell(x), \quad x > 0.
\]
The proof is now complete with these remarks.

Inspired by the proof of Theorem 1.1.2 in [de Haan (1970)], we get this result, which is more simpler.

**Lemma 15.** Let \(U : \mathbb{R}_+ \to \mathbb{R}_+\) be a monotone function not vanishing in the neighborhood of \(+\infty\). Then \(U \in RV(\rho, +\infty)\) if and only if for any integer \(p > 0\),
\[
(3.23) \quad \lim_{t \to +\infty} U(pt)/U(t) = p^\rho.
\]

**Proof.** We have to prove that (3.23) implies that \(U \in RV(\rho, +\infty)\). So assume that (3.23) holds and at the first place that \(U\) is non-decreasing. Then we necessarily have \(\rho \geq 0\). Let us begin to prove that
\[
(3.24) \quad \lim_{n \to +1} U(n+1)/U(n) = 1.
\]
Let \(c > 1\). For large enough value of \(n\), we have \((n+1)/n < c\), and for such values of \(n\),
\[
1 \leq \frac{U(n+1)}{U(n)} \leq \frac{U(cn)}{U(n)}
\]
and
\[
1 \leq \lim_{n \to +\infty} \sup_{n \to +\infty} \frac{U(n+1)}{U(n)} \leq \lim_{n \to +\infty} \frac{U(cn)}{U(n)} = c^\rho.
\]
Letting \(c \downarrow 1\) proves (3.24). Now let \(x > 0\) fixed. We want to extend (3.24) to \(p = x\mathbb{R}_+\). Let \(\varepsilon > 0\). By density of the set \(\mathbb{Q}_+\) of rational positive numbers in \(\mathbb{R}_+\), we may find \(r'\) and \(r''\) such that \(x - \varepsilon < r' < x < r'' < x + \varepsilon\). By writing \(r'\) and \(r''\) with a common denominator, we get that there exists non-negative integers \(p, q\) and \(r \neq 0\) such that
\[
x - \varepsilon < (p/r) < x < (q/r) < x + \varepsilon.
\]
Take for any $t > 0$, $n(t) = \lfloor t/r \rfloor$, that is $n(t)$ is integer and

$$n(t)r \leq t < r(n(t) + 1).$$

We easily check that $n(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. By combining the last two double inequalities, we have

$$pn(t) \leq tx \leq q(n(t) + 1)$$

By non-decreasingness of $U$, we have

$$\frac{U(tx)}{U(t)} \leq \frac{U(q(n(t) + 1))}{U(rn(t))}$$

$$= \left\{ \frac{U(q(n(t) + 1))}{U(n(t) + 1)} \right\} \left\{ \frac{U(n(t))}{U(rn(t))} \right\} \left\{ \frac{U(n(t) + 1)}{U(n(t))} \right\}$$

$$\rightarrow q^\rho r^\rho = (q/r)^\rho \leq (x + \varepsilon)^\rho,$$

since $\rho \geq 0$. Likely, we have

$$\frac{U(tx)}{U(t)} \geq \frac{U(p(n(t) + 1))}{U(rn(t))}$$

$$= \left\{ \frac{U(p(n(t) + 1))}{U(n(t) + 1)} \right\} \left\{ \frac{U(n(t))}{U(rn(t))} \right\} \left\{ \frac{U(n(t) + 1)}{U(n(t))} \right\}$$

$$\rightarrow p^\rho r^\rho = (p/r)^\rho \leq (x - \varepsilon)^\rho.$$

$$(x - \varepsilon)^\rho \leq \lim_{x+\infty} \inf U(tx) U(t) \leq \lim_{x+\infty} \sup U(tx) U(t) \leq (x + \varepsilon)^\rho,$$

for any $\varepsilon > 0$. We get the searched result by letting $\varepsilon \downarrow 0$.

To finish, we have to give the proof for $U$ non-increasing. We easily get the proof by using the transform $1/U$.

3.3.1. Regular variation and generalized inverses. For the needs of Extremen value Theory for example, we frequently use the quantile function which is a generalized inverse of the distribution function. This lemma will greatly help.

**Proposition 9.** Let $U : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a a non-constant and monotone function not vanishing in the neighborhood of $+\infty$ such that $U(+\infty) = +\infty$. If $U \in RV(\rho, +\infty)$, $\rho \neq 0$, then the generalized inverse of $U$ defined when $U$ is non-decreasing by

$$V(y) = \inf \{x > 0, U(x) \geq y\}$$

and when $U$ is non-increasing by
(3.26) \[ V(y) = \inf\{x > 0, U(x) \leq x\} \]
is in \(RV(1/\rho, +\infty)\).

To prove this, we need this lemma.

**Lemma 16.** Let \(U : \mathbb{R}_+ \longrightarrow \mathbb{R}_+\) \(\rho\)-regularly varying. Let \((a_n)_{n \geq 0}\) and \((b_n)_{n \geq 0}\) be two sequences of positive real numbers such that \(a_n/b_n \rightarrow c\) as \(n \rightarrow +\infty\). If \(c \in \mathbb{R}_+ \setminus \{0\}\), then

\[
(3.27) \quad \lim_{x \rightarrow +\infty} \frac{U(a_n)}{U(b_n)} = c^\rho.
\]

If \(\rho \neq 0\), then for any \(c \in \mathbb{R}_+ \cup \{+\infty\}\), (3.27) holds.

**Proof of 16.** First, let \(c \in \mathbb{R}_+ \setminus \{0\}\). We are going to use of Karamata representation in Theorem 5 to write

\[
U(x) = c(1 + p(x))x^\rho \exp \left( \int_1^x t^{-1} \ell(t) d\lambda(t) \right),
\]

with \((p(x), \ell(x)) \rightarrow (0, 0)\) as \(x \rightarrow +\infty\). We have

\[
(3.28) \quad \frac{U(a_n)}{U(b_n)} = \left\{ \frac{1 + p(a_n)}{1 + p(b_n)} \right\} \left( \frac{a_n}{b_n} \right)^\rho \exp \left( \int_{b_n}^{a_n} t^{-1} \ell(t) d\lambda(t) \right)
\]

\[= (1 + o(1)) \left( \frac{a_n}{b_n} \right)^\rho \exp \left( \int_{b_n}^{a_n} t^{-1} \ell(t) d\lambda(t) \right).\]

Set

\[\varepsilon_n = \sup\{|b(t)|, 0 \leq t \leq \max(a_n, b_n)\}\]

and remark that, \(a_n/b_n \rightarrow c\) finite and positive, as \(n \rightarrow +\infty\). Hence, for any fixed \(\eta > 0\) with \(\eta < c\), there exists \(n_0\) such that for any \(n \geq n_0\), \(b_n(c - \eta) \leq a_n \leq b_n(c + \eta)\). We get from all this that

\[
\left| \int_{b_n}^{a_n} t^{-1} \ell(t) d\lambda(t) \right| \leq \varepsilon_n \log \left\{ \frac{\max(a_n, b_n)}{\max(a_n, b_n)} \right\} \leq \varepsilon_n \log \frac{c + \eta}{c - \eta} = A_n \rightarrow 0.
\]

Then

\[
\frac{U(a_n)}{U(b_n)} = \left\{ \frac{1 + p(a_n)}{1 + p(b_n)} \right\} \left( \frac{a_n}{b_n} \right)^\rho (1 + O(A_n)) \rightarrow c^\rho.
\]

So (3.27) is true for \(c \in \mathbb{R}_+ \setminus \{0\}\).

From there, we prove the second statement of the lemma for \(\rho > 0\). If the statement holds for \(\rho > 0\) and if \(U \in RV(-\rho, +\infty)\), we apply it to
1/\(U\) which is in \(U \in RV(\rho, +\infty)\) and get if for \(U\).

**So, in our second step**, we let \(\rho > 0\) and suppose that \(c = 0\). We may write \(a_n = b_n r_n\) where \(r_n = a_n/b_n \to 0\). We surely have \(a_n < b_n\) for large values. For those values,

\[
\left| \int_{b_n}^{a_n} t^{-1} \ell(t) d\lambda(t) \right| \leq \varepsilon_n \log \frac{b_n}{a_n} = -\varepsilon_n \log r_n = r_n^{-\varepsilon_n}.
\]

Plugging this in (3.28) leads to

\[
(1 + o(1)) r_n^{\rho + \varepsilon_n} \leq \frac{U(a_n)}{U(b_n)} \leq (1 + o(1)) r_n^{\rho - \varepsilon_n},
\]

which implies for \(n\) large so that \(\rho/2 \leq \rho - r_n \leq 3\rho/2\) and \(0 < r_n < 1\),

\[
(1 + o(1)) r_n^{\rho/2} \leq \frac{U(a_n)}{U(b_n)} \leq (1 + o(1)) r_n^{3\rho/2},
\]

and this ensures that

\[
\frac{U(a_n)}{U(b_n)} \to 0 = 0^\rho \text{ as } n \to +\infty.
\]

**In a third and last step**, let \(\rho > 0\) and \(c = +\infty\) Use the notation of the second step where \(r_n \to +\infty\). We have

\[
\frac{U(b_n)}{U(a_n)} = (1 + o(1)) r_n^{-\rho} \exp(\int_{a_n}^{b_n} t^{-1} \ell(t) d\lambda(t)),
\]

where, for large values of \(n\),

\[
\left| \int_{a_n}^{b_n} t^{-1} \ell(t) d\lambda(t) \right| \leq \varepsilon_n \log(a_n/b_n) = r_n^{\varepsilon_n}.
\]

This leads to, for large values of \(n\),

\[
(1 + o(1)) r_n^{-\rho + \varepsilon_n} \leq \frac{U(b_n)}{U(a_n)} \leq (1 + o(1)) r_n^{-\rho + \varepsilon_n}.
\]

Since \(r_n \to +\infty\), \(\varepsilon_n \to 0\) and \(\rho > 0\), it comes that

\[
\frac{U(b_n)}{U(a_n)} \to 0 \implies \frac{U(a_n)}{U(b_n)} \to +\infty = (+\infty)^\rho.
\]

The proof is now complete.
Now, we are able to prove Proposition 9, following the lines of [de Haan (1970)], who quoted a personal communication of W. Vervaart.

**Proof of Proposition 9.** We are going to give the proof only for $U$ non-decreasing and $\rho > 0$. The other case of $U$ non-increasing and $\rho < 0$ is obtained by using the transform $1/U$. Let us assume the conditions and the notations of the proposition. We are going to prove

$$
\forall (x > 0), \lim_{t \to \infty} \frac{V(tx)}{V(t)} = x^\rho.
$$

Suppose (3.29) false. It is easy to see that $V(+\infty) = +\infty$ and $V$ is also non-decreasing. Besides, an analogue to Formula (3.18) which we established for the generalized function of a non-increasing can be easily derived, in a very similar way, for the generalized inverse of our non-decreasing function, in the form

$$
(3.30) \quad U(V(y)-) \leq y \leq U(V(y)+), \quad y > 0.
$$

Formula (3.29) is false if and only for some $x_0 > 0$, either

$$
\lim \inf_{t \to \infty} \frac{V(tx_0)}{V(t)} = c \neq x_0^\rho
$$

or

$$
\lim \sup_{t \to \infty} \frac{V(tx_0)}{V(t)} = c \neq x_0^\rho.
$$

In both cases, there exists a sequence $(t_n)_{n \geq 1}$ such that $t_n \to \infty$ and

$$
(3.31) \quad \lim_{t \to \infty} \frac{V(t_n x_0)}{V(t_n)} = c \neq x_0^\rho.
$$

Let us consider the three following cases.

**Case 1.** $c$ is finite and positive. We get, by using Inequality (3.30) , the non-decreasingness and the definitions of right and left limits :

$$
(3.32) \quad x_0 = \frac{t_n x_0}{t_n} \leq \frac{U(V(t_n x_0)+)}{U(V(t_n)-)} \leq \frac{U(V(t_n x_0)+1)}{U(V(t_n)-1)}.
$$

Let us take $a_n = V(t_n x_0) + 1$ and $b_n = V(t_n) - 1$. Since $a_n/b_n \to c$ finite and positive by (3.31), and $(a_n, b_n) \to (+\infty, +\infty)$, we are in a position to apply Lemma 9 and let $n \to +\infty$ in (3.32) to obtain

$$
x_0 \leq c^\rho.
$$
In a similar way, we also have
\begin{equation}
(3.33) \quad x_0 = \frac{t_n x_0}{t_n} \geq \frac{U(V(t_n x_0) -)}{U(V(t_n) +)} \leq \frac{U(V(t_n x_0) - 1)}{U(V(t_n) + 1)}.
\end{equation}

But the same method, we also get \( x_0 \geq c \rho \). The final conclusion is \( x_0 = c \rho \), which contradicts our supposition. Hence (3.29) is true for \( c \) finite and positive.

**Case 2**: \( c = 0 \). Use \( a_n = V(t_n x_0) + 1 \) and \( b_n = V(t_n) - 1 \). Since \( a_n/b_n \to 0 \) and since \( \rho > 0 \) in the present case, we let \( n \to +\infty \) in (3.32) to get \( x_0 = 0 \).

**Case 3**: \( c = +\infty \). Use \( a_n = V(t_n x_0) - 1 \) and \( b_n = V(t_n) + 1 \). Since \( a_n/b_n \to +\infty \) and since \( \rho > 0 \) in the present case, we let \( n \to +\infty \) in (3.33) to get \( x_0 = +\infty \).

When we put all this together, we say that if (3.29) is false, then (3.31) holds for some number \( c \) and some finite and positive \( x_0 \). The cases 2 and 3 above showed that \( c \) is necessarily positive and finite. And the case 1 showed the last \( c \) cannot be positive and finite. The conclusion is that (3.29) is true.

Before, we close the door, let us show a nice uniform convergence property of slowly varying functions.

**Lemma 17.** Let \( S(u) \) be a function \( u \in (0, 1) \) that is slowly varying at zero. We have the following uniform convergence in deterministic and random versions.

(a) Let \( A(h) \) and \( B(h) \) two functions of \( h \in (0, +\infty] \) such that for each \( h \in (0, +\infty] \), we have \( 0 < A(h) \leq B(h) < +\infty \) and \( (A(h), B(h)) \to (0, 0) \) as \( h \to 0 \). Suppose that there exist two real numbers \( 0 < C < D < +\infty \) such that
\begin{equation}
(3.34) \quad C < \liminf_{h \to +\infty} A(h)/B(h), \quad \limsup_{h \to +\infty} B(h)/A(h) < B.
\end{equation}

Then, we have
\[
\lim_{h \to +\infty} \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| = 0.
\]
(b) Let $A(h)$ and $B(h)$ be two families, indexed by $h \in (0, +\infty[$, of real-valued applications defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that for each $h \in (0, +\infty[$, we have $0 < A(h) \leq B(h) < +\infty$. Suppose that there exist two families $A^*(h)$ and $B^*(h)$, indexed by $h \in (0, +\infty[$, of measurable real-valued applications defined on $(\Omega, \mathcal{A}, \mathbb{P})$ such that for each $h \in (0, +\infty[$, $A^*(h) \leq A(h) \leq B(h) \leq B^*(h)$, and such that

$$(3.35) \quad \lim_{h \to +\infty} \sup_{\lambda \to +\infty} \inf \mathbb{P}(B^*(h)/A(h) > \lambda) = 0.$$ 

and

$$(3.36) \quad \lim_{h \to +\infty} \sup_{\lambda \to +\infty} \inf \mathbb{P}(A^*(h)/B^*(h) < 1/\lambda) = 0.$$ 

We say that the family $\{B^*(h), h \in (0, +\infty]\}$ is asymptotically bounded in probability against $+\infty$ and the family $\{B^*(h), h \in (0, +\infty]\}$ is asymptotically bounded in probability against $0$ and accordingly, we say that the family $\{B(h), h \in (0, +\infty]\}$ is asymptotically bounded in outer probability against $+\infty$ and the family $\{A(h), h \in (0, +\infty]\}$ is asymptotically bounded in outer probability against $0$.

Then any $\eta > 0$, for any $\delta > 0$, there exists a measurable subset $\Delta(\delta)$ of such that

$$\left( \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| > \eta \right) \subset \Delta(\delta),$$

with

$$\mathbb{P}(\Delta(\delta)) \leq \delta.$$ 

Consequently, if the quantities

$$\sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| > \eta$$

are measurable for $h \in (0, +\infty[$, we have that

$$\sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| \to \mathbb{P} \text{ as } h \to +\infty.$$ 

**Proof.** Let us use the Kamarata Representation Theorem 5 of $S$ : there exist a constant $c$ and functions $a(u)$ and $b(u)$ of $u \in [0, 1]$ satisfying

$$(a(u), b(u)) \to (0, 0) \text{ as } u \to +\infty,$$
such that $S$ is written as

\begin{equation}
S(u) = c(1 + a(u)) \exp \left( \int_u^1 \frac{b(t)}{t} dt \right). \tag{3.37}
\end{equation}

**Proof of Point (a).** Suppose that Condition (3.34) holds. Let $\varepsilon > 0$ such that $\varepsilon < 1$. Then two functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ of $\varepsilon \in ]0,1[$,

$$
h_1(\varepsilon) = C^\varepsilon \frac{1 - \varepsilon}{1 - \varepsilon} \quad \text{and} \quad h_2(\varepsilon) = D^\varepsilon \frac{1 + \varepsilon}{1 - \varepsilon}
$$

both converge to 0 as $\varepsilon \downarrow 0$. So for any $\eta > 0$, there exist $\varepsilon_0$, $0 < \varepsilon < \varepsilon_0 < 1$, such that

\begin{equation}
1 - \eta \leq h_1(\varepsilon), h_2(\varepsilon) \leq 1 + \eta. \tag{3.38}
\end{equation}

So, let $\eta > 0$ and let $\varepsilon_0 < 1$ such that (3.38) holds. Fix $\varepsilon$, $0 < \varepsilon < \varepsilon_0$. Now, by the assumptions on the functions $b$ and $p$, there exists $t_0$ such that for $0 \leq t \leq t_0$

$$
\max(|p(t)|, |b(t)|) \leq \varepsilon.
$$

Since $B(h) \to 0$ as $h \to +\infty$, and since (3.34) holds, there is a value $h_0 > 0$ such that $h \geq h_0$ implies that $0 \leq B(h) \leq t_0$ and

$$
C \leq A(h)/B(h) \quad \text{and} \quad B(h)/A(h) \leq D.
$$

Then for $h \geq h_0$ and $(u, v) \in [A(h), B(h)]^2$, we have

\begin{equation}
\frac{S(u)}{S(v)} = \frac{1 + p(u)}{1 + p(v)} \exp \left( \int_u^v \frac{b(t)}{t} dt \right) \tag{3.39}
\end{equation}

\begin{equation}
\leq \frac{1 + \varepsilon}{1 - \varepsilon} \exp \left( \sup_{0 \leq t \leq t_0} |b(t)| \int_u^v \frac{dt}{t} \right) \tag{3.40}
\end{equation}

\begin{equation}
\leq \frac{1 + \varepsilon}{1 - \varepsilon} \exp \left( \varepsilon \log \left\{ \frac{\max(u, v)}{\min(u, v)} \right\} \right) \cdot
\end{equation}

Thus for $h \geq h_0$ and $(u, v) \in [A(h), B(h)]^2$, we have

$$
\frac{S(u)}{S(v)} \leq D^\varepsilon \frac{1 + \varepsilon}{1 - \varepsilon}.
$$

By using lower bounds on place of upper bounds in (3.40), we also have $h \geq h_0$ and $(u, v) \in [A(h), B(h)]^2$,

$$
\frac{S(u)}{S(v)} \geq C^\varepsilon \frac{1 - \varepsilon}{1 - \varepsilon}.
$$

By putting together the previous facts, we have, for $h \geq h_0$ and $(u, v) \in [A(h), B(h)]^2$

\begin{equation}
1 - \eta \leq \frac{S(u)}{S(v)} \leq 1 + \eta. \tag{3.41}
\end{equation}
This implies that for any \( \eta > 0 \), we have found \( h_0 \) such that for \( h \geq h_0 \) and \( (u, v) \in [A(h), B(h)]^2 \), we have

\[
\sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| \leq \eta.
\]

Thus

\[
\lim_{h \to +\infty} \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| = \lim_{h \to +\infty} \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| = 0.
\]

**Proof of Point (b).** Suppose that the conditions of this point hold. For any \( \delta > 0 \), there exist a real number \( h_1 > 0 \), and a number \( \lambda > 0 \) \( (\lambda = \lambda(\delta)) \) and a real number \( h_1 = h_1(\delta) \) both depend on \( \delta \) such that

\[
\mathbb{P}(\lambda^{-1} \leq A^*(h)/B^*(h), B^*(h)/A^*(h) \leq \lambda) \geq 1 - \delta/2.
\]

If the latter property holds for a number \( \lambda > 0 \), it also holds for any greater number. So, we may and do choose \( \lambda > 1 \). Put \( C = \lambda^{-1} \) and \( D = \lambda \). From here, we follow partially use the proof of Point (a). Let \( \eta > 0 \) and consider \( \varepsilon_0, 0 < \varepsilon_0 < 1 \) such that for any \( 0 < \varepsilon < \varepsilon_0 < 1 \), we have

\[
(3.42) \quad 1 - \eta \leq h_1(\varepsilon), h_2(\varepsilon) \leq 1 + \eta.
\]

And let \( t_0 > 0 \) such that for any \( 0 \leq t \leq t_0 \)

\[
\max(|p(t)|, |b(t)|) \leq \varepsilon.
\]

Since \( B^*(h) \to_P 0 \) as \( h \to 0 \), there exists a value \( h_2 > 0 \) such that for any \( h \geq h_2 \), we have

\[
\mathbb{P}(B^*(h) > t_0) < \delta/2.
\]

Denote \( h_0 = \max(h_1, h_2) \). The conditions under which (3.41) was proved are satisfied on the event \((\lambda^{-1} \leq A^*(h)/B^*(h), B^*(h)/A^*(h) \leq \lambda) \cap (B^*(h) < t_0), h \geq h_0 \). Hence, we have on \((\lambda^{-1} \leq A^*(h)/B^*(h), B^*(h)/A^*(h) \leq \lambda) \cap (B^*(h) < t_0)\), for \( h \geq h_0 \)

\[
\sup_{A^*(h) \leq u, v \leq B^*(h)} \left| \frac{S(u)}{S(v)} - 1 \right| \leq \eta.
\]

Let us denote

\[
\Delta(\delta, h) = (\lambda^{-1} \leq A^*(h)/B^*(h), B^*(h)/A^*(h) \leq \lambda) \cap (B^*(h) \leq t_0).
\]

We have for \( h \geq h_0 \),

\[
\mathbb{P}(\Delta(\delta, h)) \leq \mathbb{P}(\Delta(\lambda^{-1} \leq A^*(h)/B^*(h), B^*(h)/A^*(h) \leq \lambda)) + \mathbb{P}(B^*(h) < t_0)
\]

\[
\leq \delta/2 + \delta/2 = \delta.
\]
We also have
\[ \Delta(\delta)^c \subset \left( \sup_{A^*(h) \leq u, v \leq B^*(h)} \left| \frac{S(u)}{S(v)} - 1 \right| \leq \eta \right) \]
This gives for \( h \geq h_0 \),
\[ \left( \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| > \eta \right) \subset \left( \sup_{A(h) \leq u, v \leq B^*(h)} \left| \frac{S(u)}{S(v)} - 1 \right| > \eta \right) \subset \Delta(\delta) \]
Thus for any \( \eta > 0 \), for any \( \delta > 0 \), we have found \( h_0 > 0 \) such that for \( h \geq h_0 \),
\[ \left( \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| > \eta \right) \subset \Delta(\delta), \]
with \( \mathbb{P}(\Delta(\delta)) \leq \delta \).

The proof is complete \( \blacksquare \)
When dealing with $\pi$-varying functions, we adopt the former assumptions, that is we work on functions $U: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that are measurable and Lebesgue locally integrable and not vanishing in the neighborhood of $+\infty$, that is, $\forall (0 \leq a \leq b < +\infty)$, $U \in L([a, b], \lambda)$. But we add the conditions:

(PV1) $U$ is non-decreasing.

(PV2) For all $t > 0$, for any $x > 0, x \neq 1$, there exist $y_0(t, x)$ such that
\[ (t \geq y_0(t, x)) \implies (|U(tx) - U(x)| > 0). \]

Condition (PV2) is automatically implied by the increasingness of $U$ in a neighborhood of $+\infty$.

We will see $\pi$-variation is strongly linked to slowly variation. That we are able to define the $\pi$-variation.

**Definition 9.** A function $U: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying (PV1) and (PV2) is $\pi$-varying at infinity, denoted $U \in \Pi(+\infty)$ if and only if
\[ \forall (y > 0), \forall (x > 0, x \neq 1), \lim_{t \rightarrow +\infty} \frac{U(ty) - U(t)}{U(tx) - U(t)} = \frac{\log y}{\log x}. \]

**Remark.** We understand now why the condition (PV2) is imposed as a necessary condition to develop the theory of $\pi$-variation.

Before giving interesting properties of $\pi$-variation in connection of slow variation, let us state the important representation of de Haan (1970).

**3.4. Theorem of de Haan.** Before we proceed any further, let give this simple and useful identity.
Proposition 10. Let \( U : \mathbb{R}_+ \to \mathbb{R}_+ \) be a measurable function Lebesgue locally integrable on \([0, +\infty[\). Then for
\[
g(x) = U(x) - \frac{1}{x} \int_1^x U(t)dt,
\]
we have
\[
U(x) = g(x) + \int_1^x t^{-1}g(t)dt, \text{ a.e.}
\]

Proof. Define \( g(\circ) \) as in the statement. Then
\[
\int_1^x t^{-1}g(t)dt = \int_1^x t^{-1}U(t)dt - \int_1^x \left( \int_1^t U(s)ds \right) t^{-2}dt.
\]
By integration by parts, we have for all \( x > 0 \),
\[
\int_1^x \left( \int_1^t U(s)ds \right) t^{-2}dt = \int_1^x \left( \int_1^t U(s)ds \right) d(-t^{-1})
\]
\[
= \left[ -t^{-1} \left( \int_1^t U(s)ds \right) \right]_1^x + \int_1^x t^{-1}U(t)dt
\]
\[
= \frac{1}{x} \int_1^x U(t)dt + \int_1^x t^{-1}U(t)dt.
\]
By combining this (3.43), we get
\[
\int_1^x t^{-1}g(t)dt = \frac{1}{x} \int_1^x U(t)dt,
\]
which, by definition of \( g(\circ) \), leads to
\[
\int_1^x t^{-1}g(t)dt = U(x) - g(x).
\]
This puts an end to the proof.

Let us give the Theorem of de Haan(1970) which includes the representation we need.

Theorem 6. Let \( U : \mathbb{R}_+ \to \mathbb{R}_+ \) be a function that is measurable and Lebesgue locally integrable and not vanishing in the neighborhood of \(+\infty\), that is, \( \forall 0 \leq a \leq b < +\infty, U \in L([a, b], \lambda) \) and such that (PV1) and (PV2) hold. Then the following assertions are equivalent.

(a) \( U \in \Pi(+\infty) \).
(b) The function
\[ g(x) = U(x) - \frac{1}{x} \int_1^x U(t) dt, \]
is slowly varying at \(+\infty\).

(c)
\[ \lim_{x \to +\infty} x \int_1^x U(t) dt - x \int_1^x U(t) dt - 2 \int_1^x \int_1^y U(y) dy dt x^2 U(x) - x \int_1^x U(t) dt \]
\[ = \frac{1}{2}. \]
There exist a slowly varying function \( g(\circ) \) and a real constant \( c \) such that, for \( x > 0 \),
\[ U(x) = c + g(x) + \int_1^x t^{-1} g(t) dt. \]

(e) There exists a positive slowly varying \( g(x) \) of \( x > 0 \) such that for each \( x > 0 \),
\[ \lim_{t \to +\infty} \frac{U(tx) - U(t)}{g(t)} = \log x. \]

Proof. We begin to prove that (a) \( \implies \) (b). Assume (a) holds. Fix \( x > 0 \) and \( x \neq 1 \). Let us show that the function in \( t > 0 \),
\[ h(t) = U(t) - U(tx) \]
is slowly varying. Remark that by (PV1), \( h \) is non-negative. Indeed, we have
\[ \frac{h(ty)}{h(t)} = \frac{U(ty) - U(tx)}{U(t) - U(tx)} = \frac{U(t) - U(tx)}{U(t) - U(tx)} \cdot \frac{U(t) - U(ty)}{U(t) - U(tx)}. \]
\[ \rightarrow \frac{\log xy}{\log x} - \frac{\log y}{\log x} = 1. \]
Next, by Lemma 12, we have
\[ th(t)/ \int_0^t h(s) ds \to 1, \]
which implies by taking the inverses \( 1/\circ \), as \( t \to +\infty\),
\[ \frac{\int_0^t U(s) ds - \int_0^t U(xs) ds}{t(U(tx) - U(t))} \to 1 \text{ as } t \to +\infty. \]
Check that we may use integrals on \([1, +\infty]\) instead of integrals on \([0, +\infty]\) in this formula. Next use right change of variables to have
Let us explain the coming computations. We want to prove that $g(\circ)$ is slowly varying, that is for $y > 0$ fixed, $g(ty)/g(t) \to 1$ which equivalent to

$$\frac{g(ty) - g(t)}{g(t)} \to 0 \quad \text{as} \quad t \to +\infty.$$ 

All the coming details are intended to express the quantity

$$\frac{g(ty) - g(t)}{g(t)}$$

and to have it bounded by a quantity that tends to zero. To this purpose, remark that

$$C(t) - 1 = C(t) - \frac{U(t) - U(tx)}{U(t) - U(tx)} \to 0.$$ 

But the left member becomes, by putting the terms together,

$$= - \left( \frac{U(t) - \frac{1}{t} \int_1^t U(xs) \, ds}{U(t) - U(tx)} \right) - \left( \frac{U(tx) - \frac{1}{tx} \int_1^{tx} U(s) \, ds}{U(t) - U(tx)} \right)$$

$$= \frac{g(tx) - g(t)}{U(t) - U(tx)} \to 0.$$ 

We get for any $0 < x < 1$,

$$g(tx) - g(t) \to 0 \quad \text{as} \quad t \to \infty.$$ 

We are almost done if we are able to replace $U(t) - U(tx)$ by $g(t)$. So we are going to compare $U(t) - U(tx)$ with $g(t)$. First, remark that, by a change of variable,

$$\frac{1}{t} \int_1^t U(s) \, ds = \int_{1/t}^1 U(ts) \, ds$$
and next,
\[
\frac{g(t)}{U(t) - U(tx)} = \frac{U(t) - \int_{1/t}^{1} U(ts)ds}{U(t) - U(tx)} = \frac{U(t) - \left( \int_{1/t}^{1/2} U(ts)ds + \int_{1/2}^{1} U(ts)ds \right)}{U(t) - U(tx)} = \frac{1/2U(t) - \frac{1}{2}U(t) - \int_{1/2}^{1} U(ts)ds}{U(t) - U(tx)}.
\]
Since
\[
\frac{1}{2}U(t) = \int_{1/2}^{1} U(t)ds,
\]
we arrive at
\[
\frac{g(t)}{U(t) - U(tx)} = \frac{1}{2}U(t) - \int_{1/2}^{1} U(ts)ds + \int_{1/2}^{1} U(t) - U(ts)ds.
\]
Next, using the non-decreasingness of \( U \) leads to, for \( t \geq 1 \),
\[
\frac{1/2U(t) - \int_{1/t}^{1/2} U(ts)ds}{U(t) - U(tx)} \geq \frac{1/2U(t) - \int_{1/2}^{1/2} U(t/2)ds}{U(t) - U(tx)} = \frac{1/2U(t) - (1/2 - 1/t)U(t/2)}{U(t) - U(tx)} \geq \frac{1U(t) - U(t/2)}{2U(t) - U(tx)}.
\]
We arrive at
\[
\frac{g(t)}{U(t) - U(tx)} \geq \frac{1}{2} \frac{U(t) - U(t/2)}{U(t) - U(tx)} + \int_{1/2}^{1} U(t) - U(ts)ds.
\]
Remark that the function in the integral of the second term in the left member is non-negative on \([0, 1]\). Apply Point (a) and Fatou-Lebesgue theorem (in the last integral) to get
\[
\liminf_{t \to \infty} \frac{g(t)}{U(t) - U(tx)} \geq \frac{1}{2} \left( -\log \frac{2}{\log x} \right) + \int_{1/2}^{1} \left( \log s \right) \frac{ds}{\log x} \geq \frac{1}{2} \left( -\log \frac{2}{\log x} \right) + \frac{1}{\log x} \left[ s \log s - s \right]_{1/2}^{1} = \frac{1}{2} \left( -\log \frac{2}{\log x} \right) + \frac{-1 - (-\log 2)/2 + 1/2}{\log x} = \frac{1}{2} \left( -\log \frac{2}{\log x} \right) > 0.
\]
This implies that
\[ \limsup_{t \to +\infty} \frac{U(t) - U(tx)}{g(t)} \leq 2(-\log x). \]

By combining this with (3.44), we get
\[ \lim_{t \to +\infty} \frac{|g(tx) - g(t)|}{g(t)} \leq \frac{|g(tx) - g(t)|}{U(t) - U(tx)} \times \frac{U(t) - U(tx)}{g(t)} \]

and next, since \( \log(1/x) \) is finite and nonzero,
\[ (3.45) \quad \lim_{t \to +\infty} \frac{|g(tx) - g(t)|}{g(t)} = 0, \]
for any \( x > 0, x \neq 1 \). To finish the proof, remark that (3.45) is obvious for \( x = 1 \) and for \( x > 1 \), we have for \( y = 1/x \in ]0,1[ \),
\[ \lim_{t \to +\infty} \frac{g(tx) - g(t)}{g(t)} = \lim_{s \to +\infty} \frac{g(s) - g(sy)}{g(sy)} \times \frac{g(y)}{g(sy)} = 0, \]
since by (3.45), \( (g(s) - g(sy))/g(y) \to 0 \) and \( g(y)/g(sy) \to 1 \) as \( s \to \infty \).

**Let us prove that** \( (b) \iff (c) \). It is easy to see that \( g(\circ) \in RV(0,+\infty) \iff q(x) = xg(x) \in RV(1,+\infty) \) which, by Lemma 13 is equivalent to
\[ (3.46) \quad \frac{xq(x)}{\int_1^x q(t)dt} \to 2 \text{ as } x \to \infty. \]

But, by partial integration,
\[ \int_1^x tU(t)dt = \int_1^x t \left( \int_1^t U(s)ds \right)' dt \]
\[ = \left[ t \int_1^t U(s)ds \right]_1^x - \int_1^x \int_1^t U(s)dsdt. \]
\[ = x \int_1^x U(s)ds - \int_1^x \int_1^t U(s)dsdt. \]

Then
\[ \int_1^x q(t)dt = \int_1^x tU(t)dt - \int_1^x \left( \int_1^t U(s)ds \right) dt \]
\[ = x \int_1^x U(s)ds - 2 \int_1^x \int_1^t U(s)dsdt. \]
and, by \((3.46)\),
\[
\frac{x \int_1^x U(s) ds - 2 \int_1^x \left( \int_1^s U(t) dt \right) ds}{x^2 U(x) - x \int_1^x U(t) dt} \to \frac{1}{2}.
\]
Then (b) is equivalent to
\[
\lim_{x \to +\infty} \frac{x \int_1^x U(s) ds - 2 \int_1^x \left( \int_1^s U(t) dt \right) ds}{x^2 U(x) - x \int_1^x U(t) dt} = \frac{1}{2}.
\]
\textbf{Let us prove that (c) \(\implies\) (d).} Assume (c) holds. Then (b) is true and \(g\) is slowly varying. By Lemma 10, the relation
\[
g(x) = U(x) - \frac{1}{x} \int_1^x U(t) dt,
\]
leads to
\[
U(x) = g(x) + \int_1^x t^{-1} g(t) dt, \text{ a.e.}
\]
and Point (d) is true.

\textbf{Let us prove that (d) \(\implies\) (e).} Suppose (d) holds. Then for any \(t > 0\) and \(x > 0\)
\[
\frac{U(tx) - U(t)}{g(t)} = \left\{ \frac{g(tx)}{g(t)} - 1 \right\} + \int_x^{tx} s^{-1} \left\{ \frac{g(s)}{g(t)} \right\} ds
\]
\[
= \int_x^{tx} s^{-1} ds + \left\{ \frac{g(tx)}{g(t)} - 1 \right\} + \int_x^{tx} \left\{ \frac{g(s)}{g(t)} - 1 \right\} ds
\]
Since \(g(\circ)\) is slowly varying, by Lemma 17 above, implies that,
\[
\sup_{s \in [\min(1,t), \max(1,t)]} \left| \frac{g(s)}{g(t)} - 1 \right| = \varepsilon(x, t) \to 0 \text{ as } x \to \infty.
\]
Then we have
\[
\left| \frac{U(tx) - U(t)}{g(t)} - \log x \right| \leq \varepsilon(x, t)(1 + |\log x|) \to 0 \text{ as } x \to +\infty.
\]
\textbf{Let us prove that (e) \(\implies\) (a).} Suppose that (e) holds. Then for \(y > 0, x > 0, x \neq 1,\)
\[
\frac{U(ty) - U(t)}{U(tx) - U(t)} = \frac{U(ty) - U(t)}{g(t)} \times \frac{g(t)}{U(tx) - U(t)} \to \frac{\log y}{\log x} \text{ as } x \to \infty.
\]
And (a) holds.
4. CHARACTERIZATIONS OF THE EXTREME DOMAIN

4. Characterizations of distribution functions in the extreme domain of attraction

4.1. Characterization of the Frechet domain. The Frechet domain is characterized by this theorem.

**Theorem 7.** Let $F$ be a distribution function.

(a) We have the following equivalence:

(a1) $F$ is in the domain of attraction of the Frechet type distribution function of parameter $\alpha > 0$,

$$\varphi_\alpha(x) = \exp(-x^{-\alpha})1_{(x \geq 0)},$$

that is

$$F \in D(\varphi_\alpha)$$

if and only if

(a2) the survival function $F^\circ = 1 - F$ is $(-\alpha)$-regularly varying at $+\infty$, that is

$$\forall (\gamma > 0), \lim_{x \to +\infty} \frac{1 - F(\gamma x)}{1 - F(x)} = \gamma^{-\alpha}.$$  

(b) Besides, if $F$ is the domain of attraction of a Frechet distribution, then the upper endpoint is infinite, that is

$$\operatorname{uep}(F) = +\infty.$$  

and for

$$a_n = F^{-1}(1 - 1/n), n \geq 1.$$  

we have for any $x \in \mathbb{R}$,

$$F^n(a_n x) \to \varphi_\alpha(x).$$

**Proof.** Suppose that $1 - F(\circ) \in RV(-\alpha, +\infty)$. In Proposition 8, we already showed that $F \in D(\varphi_\alpha)$ if we have the representation

$$F^{-1}(1 - s) = c(1 + p(x))x^{-1/\alpha} \exp(\int s^{-1} \ell(s) d\lambda(s)),$$

with $(p(s), \ell(s)) \to (0, 0)$ as $x \to +\infty$ and $F^n(a_n x) \to \varphi_\alpha(x)$ for $a_n = F^{-1}(1 - 1/n), n \geq 1$. But if $1 - F(\circ) \in RV(-\alpha, +\infty)$, we have by Proposition 9 that $(1 - F)^{-1}(1/x) = F^{-1}(1 - 1/x) \in RV(1/\alpha, +\infty)$. 
We may write the Karamata representation for $F^{-1}(1 - 1/x)$ which gives (4.5) for $s = 1/x$.

The new thing to prove is that $F \in D(\varphi_\alpha)$ implies that $1 - F(o) \in RV(-\alpha, +\infty)$. Now suppose that $F \in D(\varphi_\alpha)$. We are going to put ourselves in a position to use Lemma 14. Since $F \in D(\varphi_\alpha)$, there exist two sequences $(a_n > 0)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ such that for any $x \in \mathbb{R}$,

$$F^n(a_n x + b_n) \rightarrow \varphi_\alpha(x) \text{ as } n \rightarrow +\infty.$$ 

Let $s > 1$. We have, as $n \rightarrow +\infty$,

$$F^{[ns]}(a_{[ns]} x + b_{[ns]}) = \left\{ F^n(a_{[ns]} x + b_{[ns]}) \right\}^{[ns]/n} \rightarrow \varphi_\alpha(x),$$

and next

$$F^n(a_{[ns]} x + b_{[ns]}) \rightarrow \varphi_\alpha(x)^{1/s} = \varphi_\alpha((1/s)^{-1/\alpha} x) \text{ as } n \rightarrow +\infty.$$ 

By Lemma 2, we have

$$\frac{b_{[ns]} - b_n}{a_n} \rightarrow 0 \text{ and } \frac{a_{[ns]}}{a_n} \rightarrow (1/s)^{-1/\alpha} = \rho > 1 \text{ as } n \rightarrow +\infty.$$ 

Set $n(1) = [(s/(s - 1))]$ and $n(i + 1) = [n(i)s], \ i \geq 1$. We remark that

$$n(i) \geq n(i - 1)s \geq ... \geq s^{-i}n(1) \rightarrow +\infty \text{ as } i \rightarrow \infty$$

and

$$\frac{n(i + 1)}{n(i)} = \frac{[n(i)s]}{n(i)} \rightarrow s \text{ as } i \rightarrow +\infty.$$ 

Replacing $n$ by $n(i)$ is Formula (4.8) gives

$$\frac{b_{n(i+1)} - b_{n(i)}}{a_{n(i)}} \rightarrow 0 \text{ and } \frac{a_{n(i+1)}}{a_{n(i)}} \rightarrow \rho > 1 \text{ as } i \rightarrow +\infty$$

which implies that $a_{n(i)} \rightarrow +\infty \text{ as } i \rightarrow +\infty$. We have to prove also that

$$\frac{b_{n(i)}}{a_{n(i)}} \rightarrow 0 \text{ as } i \rightarrow +\infty.$$
To see that, put \( c_n(i) = b_n(i+1) - b_n(i), \) \( i \geq 0 \) with the convention that \( b_n(0) = 0 \). We have
\[
\frac{c_n(i)}{a_n(i)} \to 0 \text{ as } i \to +\infty
\]

Let \( \varepsilon > 0 \) such \( r = (1/\rho) + \varepsilon < 1 \). There exists \( i_0 \) such that for \( i \geq i_0 \),
\[
\left| \frac{c_n(i)}{a_n(i)} \right| \leq \varepsilon \quad \text{and} \quad \frac{a_n(i)}{a_n(i+1)} \leq r.
\]

We have for \( i > i_0 \)
\[
\frac{b_n(i+1)}{a_n(i)} = \frac{1}{a_n(i)} \sum_{k=1}^{i} c_n(k) \leq \frac{1}{a_n(i)} \sum_{k=1}^{i_0} c_n(k) + \sum_{i_0}^{i} \frac{c_n(k)}{a_n(i)} \frac{a_n(i)}{a_n(i)} \frac{a_n(i+1)}{a_n(i+1)} \leq \varepsilon r^{i_0-k} \sum_{p=0}^{i_0} \rho^p \leq \frac{\varepsilon r^{i_0-k}}{1-r}.
\]

So, for any \( \varepsilon > 0 \),
\[
\limsup_{i \to +\infty} \frac{b_n(i+1)}{a_n(i)} \leq \frac{\varepsilon r^{i_0-k}}{1-r}.
\]
which, by letting \( \varepsilon \downarrow 0 \), leads to
\[
\limsup_{i \to +\infty} \frac{b_{n(i+1)}}{a_{n(i+1)}} = \limsup_{i \to +\infty} \left\{ \frac{b_{n(i+1)}}{a_{n(i)}} \right\} \left\{ \frac{a_{n(i)}}{a_{n(i+1)}} \right\} = 0.
\]
This gives (4.9). This combined with the application of (4.6) for \( n = n(i - 1) \), that is
\[
F^{n(i)}(a_{n(i)}x + b_{n(i)}) \to \varphi_\alpha(x) \text{ as } i \to +\infty.
\]
ensures, via Lemma 2, that
\[
F^n(a_{n(i)}x) \to \varphi_\alpha(x) \text{ as } i \to +\infty.
\]
This implies that for any \( x > 0 \)
\[
n(i) \log F(a_{n(i)}x) \to \log \varphi_\alpha((1/s)^{-1/\alpha}x) = -x^{-\alpha}/s \text{ as } i \to +\infty.
\]
This is possible if and only if \( F(a_{n(i)}x) \to 1 \text{ as } i \to +\infty \) and
\[
n(i)(1 - F(a_{n(i)}x) \to x^{-\alpha} \text{ as } i \to +\infty.
\]
By Lemma 14, we may conclude that \( 1 - F \in D(\varphi_\alpha) \).

It remains to prove (4.3) in Point (b). Suppose that that \( uep(F) \) is finite. Consider \( x > \inf(0, uep(F)) \). Formula (4.10) would implies for \( i \) large enough
\[
n(i)(1 - F(a_{n(i)}x) = 0 \to x^{-\alpha},
\]
which is absurd.

### 4.2. Characterization of the Weibull domain.

The Weibull domain is characterized by this theorem.

**Theorem 8.** Let \( F \) be a probability distribution function.

(a) We have the following equivalence.

(a.1) \( F \) is in the domain of attraction of a Weibull type of distribution function of parameter \( \beta > 0 \),
\[
\psi_\beta(x) = \exp(-(-x)^\beta)1_{(x \geq 0)} + 1_{(x \geq 0)},
\]
that is
\[
F \in D(\psi_\beta)
\]
if and only if

\[(4.12)\quad \text{uep}(F) < +\infty\]

and

\[(4.13)\quad x \mapsto F^*(x) = F(\text{uep}(F) - \frac{1}{x})\]

is \((-\beta)\)-regularly varying, that is \(F^* \in D(\varphi_\beta)\):

\[(4.14)\quad \forall (\gamma > 0), \lim_{x \to +\infty} \frac{1 - F^*(\gamma x)}{1 - F^*(x)} = \gamma^{-\beta}.

(b) Besides, if \(F\) is the domain of attraction of a Weibull distribution of parameter \(\beta > 0\), for

\[a_n = \text{uep}(F) - F^{-1}(1 - 1/n), n \geq 1,\]

we have for any \(x \in \mathbb{R},\)

\[(4.15)\quad F^n(a_n x + \text{uep}(F)) \to \psi_\beta(x).\]

**Proof.** As in the Frechet’s case, we are going to focus the implication of (4.13) and (4.14) by (4.11). The other reverse implications are sufficiently handled in Chapter 1. We begin similarly as in the case of Frechet’s domain. Suppose that \(F \in D(\psi_\beta)\). We are going to put ourselves in a position to use again Lemma 14. There exist sequences \((a_n > 0)_{n \geq 0}\) and \((b_n)_{n \geq 0}\) such that for any \(x \in \mathbb{R},\)

\[F^n(a_n x + b_n) \to \psi_\beta(x).\]

Let \(s > 1\). We obtain

\[(4.16)\quad F^n(a_{[ns]} x + b_{[ns]}) \to \psi_\beta(x)^{1/s} = \psi_\beta((1/s)^{1/\beta} x).\]

By Lemma 2, we have

\[(4.17)\quad \frac{b_{[ns]} - b_n}{a_n} \to 0 \quad \text{and} \quad \frac{a_{[ns]}}{a_n} \to (1/s)^{1/\beta} = \rho < 1.\]

We are going to re-use the sequence \(n(i)_{i \geq 1}\) introduced above to establish that

\[b_{n(i)} \to \text{uep}(F) \quad \text{and} \quad \frac{a_{n(i+1)}}{a_{n(i)}} \to \rho < 1,\]

which implies that \(a_{n(i)} \to 0\) as \(i \to +\infty\). We have to prove also that
(4.18) \[ \frac{b_{n(i+1)} - b_{n(i)}}{a_{n(i)}} \to 0 \to \text{uep}(F) \] and \[ \frac{\text{uep}(F) - b_{n(i)}}{a_{n(i)}} \to 0. \]

Let us prove that \((b_{n(i)})_{i \geq 1}\) is a Cauchy sequence. Put again \(c_n(i) = b_{n(i+1)} - b_{n(i)}, i \geq 0\) with the convention that \(b_{n(0)} = 0\). We have

\[ \frac{c_{n(i)}}{a_{n(i)}} \to 0 \text{ as } i \to +\infty. \]

From these limits, we may find, for any \(\varepsilon > 0\) such \(r = \rho + \varepsilon < 1\), an integer \(i_0\) such that for \(i \geq i_0\),

\[ \left| \frac{c_{n(i)}}{a_{n(i)}} \right| \leq \varepsilon \text{ and } \frac{a_{n(i+1)}}{a_{n(i)}} \leq r. \]

This ensures that for any \(i > i_0\)

\[ a_{n(i)} \leq ra_{n(i-1)} \leq r^2a_{n(i-2)} \leq \ldots \leq r^{i-i_0}a_{n(i_0)} \]

We have for \(i_0 < i < j\),

\[ b_{n(j+1)} - b_{n(i)} = \sum_{k=i}^{j} c_n(k) = \sum_{k=i}^{j} \left\{ \frac{c_n(k)}{a_n(k)} \right\} a_n(k) = \varepsilon \sum_{k=i}^{j} a_n(k) \leq \varepsilon r^{-i_0}a_{n(i_0)} \sum_{k=i}^{j} r^k \leq \frac{r^{-i_0}a_{n(i_0)}}{1 - r} \varepsilon. \]

Hence, we obtain

(4.19) \[ \limsup_{i \to +\infty, j \to +\infty} b_{n(j+1)} - b_{n(i)} \leq \frac{r^{-i_0}a_{n(i_0)}}{1 - r} \varepsilon. \]

By letting \(\varepsilon \downarrow 0\), we get that \((b_{n(i)})_{i \geq 1}\) is a Cauchy sequence and hence, there exist a real number \(b\) such that

\[ b_{n(i)} \to b. \]

Let us use again these formulas to see that
\[
\frac{b_{n(j+1)} - b_{n(i)}}{a_{n(i)}} = \sum_{k=i}^{j} \left\{ \frac{c_n(k)}{a_n(k)} \right\} \left\{ \frac{a_n(k)}{a_n(i)} \right\} \\
\leq \varepsilon \sum_{k=i}^{j} r^{k-i} = \varepsilon r^{-i} \sum_{k=i}^{j} r^k.
\]

When \( j \to \infty \), we get
\[
\frac{b - b_{n(i)}}{a_{n(i)}} \leq \varepsilon r^{-i} \sum_{k=i}^{+\infty} r^k = \frac{\varepsilon}{1 - r}
\]
for any \( \varepsilon \downarrow 0 \). This is enough to prove that
\[
\frac{b - b_{n(i)}}{a_{n(i)}} \to 0.
\]

By Lemma 2, we have for any \( x \in \mathbb{R} \)
\[
(4.20) \quad F_n(i)(a_{n(i)}x + b) \to \psi_\beta(x), \text{ as } i \to +\infty.
\]
For \( x = 0 \), (4.20) implies that
\[
F(b) = 1.
\]

Let \( h > 0 \). Since \( a_{n(i)} \to 0 \), we have for large values of \( i \), \( b - h \leq b - a_{n(i)} \) and
\[
F_n(i)(b - h) \leq F_n(i)(b - a_{n(i)}) \to \psi_\beta(-1) = 1/e \text{ as } i \to +\infty.
\]
This implies for large values of \( i \),
\[
F(b - h) \leq \left\{ \frac{1,01}{e} \right\}^{1/n(i)} < 1.
\]
Hence
\[
b = \text{uep}(F).
\]
Now we have for \( x > 0 \),
\[
F_n(i)(b - a_{n(i)}x) \to \psi_\beta(-x) = \exp(x^\beta),
\]
which implies that
\[
n(i)(1 - F(b - a_{n(i)}x)) \to x^\beta,
\]
and by change of variable \( x \to 1/x \),
\[ n(i)(1 - F(b - \frac{a_n(i)}{x})) \to x^{-\beta} \]

which is exactly
\[ n(i)(1 - F^*(b - a_n^*(i)x)) \to x^{-\beta}, \]
where \( a_n^*(i) = 1/a_n(i) \to +\infty \). Then by Lemma 14, we conclude that \( 1 - F^* \in RV(-\beta) \).

### 4.3. Characterization of the Gumbel domain.

We give the main characterization of \( D(\Lambda) \) here. Before we give the characterization, let us highlight the following implication. Suppose that some distribution function \( F \) satisfies:

\[ \forall (0 < s, t \neq 1 < 1), \lim_{u \downarrow 0} \frac{F^{-1}(1 - su) - F^{-1}(1 - u)}{F^{-1}(1 - tu) - F^{-1}(1 - u)} = \frac{\log s}{\log t}, \]

that is \( F^{-1}(1-\circ) \) is \( \pi \)-varying at zero. Put \( U(x) = F^{-1}(1-1/x), x > 1 \). Then \( U \) is \( \pi \)-varying at \( +\infty \). We may use the de Haan representation given in Theorem 6: there exists a slowly varying function \( g \) and a constant \( c \) such that

\[ U(x) = d + g(x) + \int_1^x t^{-1} g(t) dt, x > 1. \]

Put \( s(u) = g(1/u), 0 < u < 1 \). We get the representation

\[ (4.21) \quad F^{-1}(1 - u) = d + s(u) + \int_u^1 t^{-1} s(t) dt, 0 < u < 1, \]

where \( s(\circ) \) is still slowly varying at zero. We proved in Point (c) of Proposition 8 in Chapter 1 that \( F \in D(\Lambda) \) whenever the latter representation (12) holds. The next theorem says that the reverse implication is also true.

**Proposition 11.** Let \( F \) be a probability distribution function. The following propositions are equivalent.

(a) \( F \in D(\Lambda) \).

(b) For any \( 0 < s, t \neq 1 < 1 \),

\[ \lim_{u \downarrow 0} \frac{F^{-1}(1 - su) - F^{-1}(1 - u)}{F^{-1}(1 - tu) - F^{-1}(1 - u)} = \frac{\log s}{\log t}. \]
(c) There exist a slowly varying function \( s \) and a constant \( d \) such that for \( 0 < u < 1 \),

\[
(4.23) \quad F^{-1}(1-u) = d + s(u) + \int_u^1 t^{-1} s(t)dt.
\]

**Proof.** Based on the arguments given in the introduction and Theorem 6, we see that we only have to prove \((a) \implies (b)\). So, suppose \((a)\) holds that is : there exist sequences \((c_n > 0)_{n \geq 1}\) and \((d_n)_{n \geq 1}\) such that for any \( x \in \mathbb{R} \),

\[
F^n(c_n x + d_n) \to \exp(-e^{-x}).
\]

We are going to apply Lemma 8 since \( H(x) = \exp(-e^{-x}), x \in \mathbb{R} \), is strictly increasing on its support which is \( \mathbb{R} \). Consider for \( x > 1 \),

\[
\begin{cases} 
  u_1(n) = (1 - 1/n)^n & \to u_1 = e^{-1}, \\
  u_2(n) = (1 - (nz))^{1/n} & \to u_2 = e^{-1/(nz)}.
\end{cases}
\]

Remark that for any \( 0 < s < 1 \), \((F^n)^{-1}(s) = F^{-1}(s^{1/n})\) and \( H^{-1}(e^{-1/x}) = \log x \) and \( H^{-1}(e^{-1}) = 0 \). Applying Part (II) of Lemma 8 gives

\[
\frac{F^{-1}(1 - 1/(nx)) - F^{-1}(1 - 1/n)}{c_n} \to H^{-1}(e^{-1/x}) - H^{-1}(e^{-1}) \log z.
\]

By doing the same with \( y > 1 \), we get a similar formula in \( y \) and by dividing the to two formulas leads to : For any \( x > 1, y > 1 \),

\[
(4.24) \quad \frac{F^{-1}(1 - 1/(nx)) - F^{-1}(1 - 1/n)}{F^{-1}(1 - 1/(ny)) - F^{-1}(1 - 1/n)} \to \frac{\log x}{\log y}.
\]

Put \( U(x) = F^{-1}(1 - 1/x), x > 1 \). We are going to prove that for any \( x > 1 \),

\[
(4.25) \quad \frac{U(n + 1) - U(n)}{U(nx) - U(n)} \to 0,
\]

and use it to conclude. Let \( y > 1 \). We have \( n + 1 \leq yn \) and for \( n \) large enough, and since \( U \) is non-decreasing,

\[
0 \leq \frac{U(n + 1) - U(n)}{U(nx) - U(n)} \leq \frac{U(ny) - U(n)}{U(nx) - U(n)},
\]

and then, by (4.24),

\[
0 \leq \limsup_{n \to +\infty} \frac{U(n + 1) - U(n)}{U(nx) - U(n)} \leq \frac{\log y}{\log x}.
\]
We get the result by letting \( y \downarrow 1 \). Now fix \( x > 1 \). We are going to use that for any \( \varepsilon > 0 \), we have \( \lfloor z + 1 \rfloor / [z] < 1 + \varepsilon \) for \( t \) large enough. Fix \( \varepsilon > 0 \) such that \((1 + \varepsilon)x > 1\). We have for \( s \) large enough, 

\[
1 - \frac{U([z] + 1) - U([z])}{U([tx]) - U([z])} = \frac{U([z]x) - U([z] + 1)}{U([z]x) - U([z])} \leq \frac{U(\lfloor z \rfloor x) - U(z)}{U(\lfloor z \rfloor x) - U([z])} \leq \frac{U((\lfloor z \rfloor + 1)x) - U([z])}{U([z]x) - U([z])} \leq \frac{U([z](1 + \varepsilon)x) - U([z])}{U([z]x) - U([z])}.
\]

Taking the limits of the extreme members as \( t \to +\infty \) and applying (4.24) and (4.25), we get 

\[
1 \leq \limsup \frac{U(zx) - U(z)}{U([z]x) - U([z])} \leq \frac{\log x + \log(1 + \varepsilon)}{\log x}.
\]

By letting \( \varepsilon \downarrow 1 \), we get for any \( x > 0 \),

\[
\frac{U(zx) - U(z)}{U([z]x) - U([z])} \to 1.
\]

By doing the same with \( y > 1 \), we get a similar formula and by dividing the two formulas and by applying (4.24), which uses limits over the integers, we get : \( x > 1, y > 1 \),

\[
\frac{U(zx) - U(z)}{U([z]y) - U([z])} = \frac{U(zx) - U(z)}{U([z]x) - U([z])} \times \frac{U([z]y) - U([z])}{U([z]y) - U([z])} \times \frac{U([z]y) - U([z])}{U([z]x) - U([z])} \to 1 \times 1 \times \frac{\log x}{\log y}.
\]

Finally setting \( s = 1/x \) and \( t = 1/y \) and \( u = 1/z \), we arrive at (4.22), which was our target.
4.4. Quantile representations. We are going to particularize the representations we obtained above to probability distributions functions $F$. Next for the needs of statistical estimation on the extreme domain of attraction, we will also give representations for the functions $G(x) = F(e^x), x \in \mathbb{R}$.

4.4.1. Direct representation of the quantile of a distribution in the extreme domain of attraction.

Let us work case by case.

**Case** $F \in D(\varphi_\alpha), \alpha > 0$.

By Theorem 7, we have that $1 - F \in RV(\alpha, +\infty)$. Then $U = 1/(1 - F) \in RV(\alpha, +\infty)$ with $U(+\infty) = U(+\infty)$. By Proposition 9, $U^{-1} \in RV(1/\alpha, +\infty)$. But $U^{-1}(x) = F^{-1}(1 - 1/x), x > 1$. By Theorem 5, we have the representation

$$F^{-1}(1 - 1/x) = c(1 + a_1(x))x^{1/\alpha} \exp\left(\int_1^x t^{-1}\ell_1(t)dt\right), x > 1$$

where $(a_1(x), \ell_1(x)) \to (0, 0)$ as $x \to +\infty$. Put $s = 1/x, 0 < s < 1, a(s) = a_1(1/s), \ell(s) = -\ell_1(1/s)$ to get

$$F^{-1}(1 - s) = c(1 + a(s))s^{-1/\alpha} \exp\left(\int_s^1 t^{-1}\ell(t)dt\right), x > 1, 0 < s < 1,$$

where $(a(s), \ell(s)) \to (0, 0)$ as $s \to 0$.

**Case** $F \in D(\psi_\beta), \beta > 0$.

By Theorem 7, $u_{ep}(F) < +\infty$ and $F^* \in RV(-\beta)$ where $F^*(x) = F(u_{ep}(F) - 1/x), x \in \mathbb{R}$. By using the above case, We have the representation

$$(F^*)^{-1}(1 - s) = c_1(1 + a_1(s))s^{-1/\beta} \exp\left(\int_s^1 t^{-1}\ell_1(t)dt\right), x > 1, 0 < s < 1,$$

where $(a_1(s), \ell_1(s)) \to (0, 0)$ as $s \to 0$. But it is easy to see that $(F^*)^{-1}(1 - s) = 1/(u_{ep}(F) - F^{-1}(1 - s)), 0 < s < 1$, with gives

$$u_{ep}(F) - F^{-1}(1 - s)) = c(1 + a(s))s^{1/\beta} \exp\left(\int_s^1 t^{-1}\ell(t)dt\right), x > 1, 0 < s < 1,$$

where $c = 1/c_1, (a(s), \ell(s)) = (a_1(s)/(1 + a_1(s)), -\ell(s)) \to (0, 0)$ as $s \to 0$.

**Case** $F \in D(\Lambda)$. 

we already saw in Lemma 11 that if $F \in D(\Lambda)$, there exist a slowly varying function $s$ and a constant $d$ such that for $0 < u < 1$,

$$(4.26) \quad F^{-1}(1-u) = d - s(u) + \int_u^1 t^{-1}s(t)dt.$$  

We conclude by this:

**Proposition 12.** We have the following characterizations for the three extremal domains.

(a) $F \in D(H_\gamma)$, $\gamma > 0$, if and only if there exist a constant $c$ and functions $a(u)$ and $\ell(u)$ of $u \in ]0, 1]$ satisfying

$$(a(u), \ell(u)) \to (0, 0) \text{ as } u \to +\infty,$$

such that $F^{-1}$ admits the following representation of Karamata

$$F^{-1}(1-u) = c(1 + a(u))u^{-\gamma}\exp\left(\int_u^1 \frac{\ell(u)}{u}du\right).$$

(b) $F \in D(H_\gamma)$, $\gamma < 0$, if and only if $uep(F) < +\infty$ and there exist a constant $c$ and functions $a(u)$ and $\ell(u)$ of $u \in ]0, 1]$ satisfying

$$(a(u), \ell(u)) \to (0, 0) \text{ as } u \to +\infty,$$

such that $F^{-1}$ admit the following representation of Karamata

$$uep(F) - F^{-1}(1-u) = c(1 + a(u))u^{-\gamma}\exp\left(\int_u^1 \frac{\ell(u)}{u}du\right).$$

(c) $F \in D(H_0)$ if and only if there exist a constant $d$ and a slowly varying function $s(u)$ such that

$$F^{-1}(1-u) = d + s(u) + \int_u^1 \frac{s(u)}{u}du, 0 < u < 1,$$

and there exist a constant $c$ and functions $a(u)$ and $\ell(u)$ of $u \to u \in ]0, 1]$ satisfying

$$(a(u), \ell(u)) \to (0, 0) \text{ as } u \to +\infty,$$

such that $s$ admits the representation

$$s(u) = c(1 + a(u))u^{-\gamma}\exp\left(\int_u^1 \frac{\ell(u)}{u}du\right).$$
As to the distribution function \( G \), defined earlier by \( G(y) = F(e^y), y \in \mathbb{R} \) with \( G^{-1}(1 - s) = \log F^{-1}(1 - s), 0 < s < 1 \), its representations come from the combination of the just given ones and the points of Lemma ZZZ which ensure that

\[
F \in D(G_0) \Rightarrow G \in D(D(G_0))
\]

and

\[
(F \in D(G_\gamma), \gamma < 0) \Rightarrow (F \in D(G_\gamma), \gamma < 0).
\]

This gives:

**Proposition 13.** We have the following characterizations for the three extremal domains.

(a) \( F \in D(H_\gamma), \gamma > 0 \), if and only if there exist a constant \( c \) and functions \( p(u) \) and \( b(u) \) of \( u \rightarrow u \in ]0, 1] \) satisfying

\[
(p(u), b(u)) \rightarrow (0, 0) \quad \text{as} \quad u \rightarrow +\infty,
\]

such that \( G^{-1} \) admit the following representation of Karamata

\[
G^{-1}(1 - u) = c + \log(1 + a(u)) - \gamma \log u + \int_u^1 \frac{b(u)}{u} \, du.
\]

(b) \( F \in D(H_\gamma), \gamma < 0 \), if and only if \( u_{ep}(F) < +\infty \) and there exist a constant \( c \) and functions \( p(u) \) and \( b(u) \) of \( u \in ]0, 1] \) satisfying

\[
(p(u), b(u)) \rightarrow (0, 0) \quad \text{as} \quad u \rightarrow +\infty,
\]

such that \( G^{-1} \) admit the following representation of Karamata

\[
u_{ep}(G) - G^{-1}(1 - u) = c(1 + a(u)) \, u^{-\gamma} \exp(\int_u^1 \frac{b(u)}{u} \, du).
\]

(c) \( F \in D(H_0) \) if and only if there exist a constant \( d \) and a slowly varying function \( s(u) \) such that

\[
G^{-1}(1 - u) = d + s(u) + \int_u^1 \frac{s(u)}{u} \, du, 0 < u < 1,
\]

and there exist a constant \( c \) and functions \( p(u) \) and \( b(u) \) of \( u \rightarrow u \in ]0, 1] \) satisfying

\[
(p(u), b(u)) \rightarrow (0, 0) \quad \text{as} \quad u \rightarrow +\infty,
\]
such that s admits the representation

\[ s(u) = c(1 + p(u)) \ u^{-\gamma} \exp\left(\int_u^1 \frac{b(u)}{u} \, du\right). \]

5. EVT Criteria Based of the Derivatives of the Distribution Function

6. Other Technical computations

6.1. Gaussian Extremes. This subsection is devoted to a number of expansions results for the distribution function \( \Phi(x) \), \( x > 0 \) and the quantile function \( \Phi^{-1}(s) \), \( 0 < s < 1 \) of a standard normal random variable. These expressions are useful to describe the extremes of samples from Gaussian variables. The results exposed here are used for example in Subsection 3.8 of Chapter 1.

6.1.1. Gaussian tails. Let

\[ \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} \, dt = \int_{-\infty}^x \phi_d(t) \, dt \]

where

\[ \phi_d(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, -\infty < t < +\infty. \]

Put \( C = 1/\sqrt{2\pi} \). We have for \( x > 0 \),

\[
C^{-1}(1 - \phi(x)) = \int_x^{+\infty} t^{-1} \ d(-e^{-t^2/2})
\]

\[
= \left[ -t^{-1} e^{-t^2/2} \right]_{t=x}^{t=+\infty} - \int_x^{+\infty} t^{-2} e^{-t^2/2} \, dt
\]

\[
= x^{-1} e^{-x^2/2} - \int_x^{+\infty} t^{-2} \, te^{-t^2/2} \, dt \leq x^{-1}.
\]

But, also

\[
\int_x^{+\infty} t^{-2} \, te^{-t^2/2} \, dt = \int_x^{+\infty} t^{-2} \ d(-e^{-t^2/2})
\]

\[
= \left[ -t^{-2} e^{-t^2/2} \right]_{t=x}^{t=+\infty} - \int_x^{+\infty} t^{-3} e^{-t^2/2} \, dt
\]

\[
= x^{-2} e^{-x^2/2} - \int_x^{+\infty} t^{-3} e^{-t^2/2} \, dt.
\]
This, combined with the first formulae, gives
\[ C^{-1}(1 - \phi(x)) = x^{-1} - \int_{x}^{+\infty} t^{-2} e^{-t^2/2} dt \]
\[ \geq e^{-x^2/2}(x^{-1} - x^{-2}). \]

We arrive at, for \( x > 0 \),
\[ C \left\{ \frac{1}{x} - \frac{1}{x^2} \right\} e^{-x^2/2} \leq 1 - \phi(x) \leq \frac{Ce^{-x^2/2}}{x}, \]
and next,
\[ 1 - \phi(x) = \frac{Ce^{-x^2/2}}{x}(1 + O(1/x)), \]
as \( x \to \infty \).

6.1.2. Tail quantile. Let
\[ s = 1 - \phi(x). \]
We get as \( s \to 0 \), which implies \( x \to \infty \),
\[ s = \frac{Ce^{-x^2/2}}{x}(1 + O(1/x)) \]
and
\[ \log s = \log C - x^2/2 - \log x + \log(1 + O(1/x)) \]
that is
\[ \log s = \log C - x^2/2 - \log x + O(1/x). \]
We get
\[ 2 \log(1/s)/x^2 = 1 + (2 \log C)/x^2 + (2 \log x)/x^2 + O(1/x^3). \]
Hence
\[ 2 \log(1/s)/x^2 = 1 + O(x^{-2} \log x) = 1 + o(1), \]
meaning, as \( s \to 0 \),
\[ x = \phi^{-1}(1 - s) \sim (2 \log 1/s)^{1/2}. \]
By going back to 6.2, we get
\[ x = (2 \log(1/s))^{1/2} \left\{ 1 + (2 \log C)/x^2 + (2 \log x)/x^2 + O(1/x^3) \right\}^{-1/2} \]
\[ = (2 \log(1/s))^{1/2} \left\{ 1 - \log C + \log x + O(x^{-2} \log x) \right\} \]
\[ = (2 \log(1/s))^{1/2} \left\{ 1 - \frac{2 \log C + 2 \log x + O(x^{-2} \log x)}{4 \log(1/s)} \right\} \]
\[ = (2 \log(1/s))^{1/2} \left\{ 1 - \frac{2 \log C + 2 \log x + O(x^{-2} \log x)}{4 \log(1/s)} \right\} \]
By denoting \( \varepsilon(s) = O((\log 1/s)^{-1} \log \log 1/s) \), we have

\[
= (2 \log(1/s))^{1/2} \left\{ 1 - \frac{\log 2 \pi + \log 2 + (\log \log 1/s) + \varepsilon(s)}{4 \log(1/s)} (1 + \varepsilon(s)) \right\}
\]

where \( \varepsilon(s) = O((\log 1/s)^{-1/2}) \). We also have

\[
x = (2 \log(1/s))^{1/2} \left\{ 1 - \frac{\log 4 \pi + \log \log(1/s) + \varepsilon(s)}{4 \log(1/s)} \right\}
\]

This leads to

\[
x = (2 \log(1/s))^{1/2} - \frac{\log 4 \pi + \log \log(1/s)}{2(2 \log(1/s))^{1/2}} + O((\log \log(1/s)^2(\log 1/s)^{-1}))
\]

We conclude by

\[
(6.3)
\]

\[
\phi^{-1}(1-s) = \left\{ (2 \log(1/s))^{1/2} - \frac{\log 4 \pi + \log \log(1/s)}{2(2 \log(1/s))^{1/2}} + O((\log \log(1/s)^2(\log 1/s)^{-1/2})) \right\}.
\]

6.1.3. Derivative at \(+\infty\). Set

\[
r(u) = Q(1-s) = \phi^{-1}(1-s), 0 < s < 1.
\]

We have

\[
(Q(1-s))' = -\phi_d(Q(1-s))^{-1}
\]

\[
= -\frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2} Q(1-s)^2\right).
\]

\[
= -\frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2} \left(2 \log(1/s)\right) \left\{ 1 - \frac{\log 4 \pi + \log \log(1/s) + \varepsilon(s)}{4 \log(1/s)} (1 + \varepsilon(s)) \right\}^2\right).
\]

Now, use the properties of \( \varepsilon(s) \), when \( s \) is near zero, to see that

\[
(Q(1-s))' = -\frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2} \left(2 \log(1/s) - \log 4 \pi - \log \log(1/s) + o(1)\right)\right)
\]

\[
= -\frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2} \left(2 \log(1/s)\right) - \log 4 \pi - \log \log(1/s) + o(1)\right)
\]

\[
= -\sqrt{2} s^{-1}(\log 1/s)^{1/2}(1 + o(1))
\]

\[
= -s^{-1}(2 \log 1/s)^{1/2}(1 + o(1)).
\]
We obtained, as $s \downarrow 0$,

\[(6.4) \quad (Q(1 - s))' = -s^{-1}(2 \log 1/s)^{1/2}(1 + o(1)).\]

6.1.4. Normalizing and centering coefficients. Denote $\varepsilon_n = (\log n)^{-1}(\log \log n)^2$. We get

\[(6.5) \quad \phi^{-1}(1 - 1/n) = \left\{ (2 \log n)^{1/2} - \frac{\log 4\pi + \log \log n}{2(2 \log n)^{1/2}} + O(\varepsilon_n) \right\},\]

and

\[\phi^{-1}(1 - 1/ne) = \left\{ (2 + 2 \log n)^{1/2} - \frac{\log 4\pi + \log(2 + \log n)}{2(2 + \log n)^{1/2}} + O(\varepsilon_n) \right\},\]

that is

\[a_n = \phi^{-1}(1 - 1/ne) - \phi^{-1}(1 - 1/n) = (2 + 2 \log n)^{1/2} - (2 \log n)^{1/2} + \frac{\log 4\pi}{2} \left\{ \frac{1}{(2 \log n)^{1/2}} - \frac{1}{(2 + 2 \log n)^{1/2}} + O(\varepsilon_n) \right\} + \frac{1}{2} \left\{ \frac{\log \log n}{(2 \log n)^{1/2}} - \frac{\log(1 + \log n)}{(2 + 2 \log n)^{1/2}} \right\} = A_n + B_n + C_n.\]

We surely have

\[B_n/\alpha_n = \frac{\log 4\pi}{2} \left\{ \frac{(2 \log n)^{1/2}}{(2 \log n)^{1/2}} - \frac{(2 \log n)^{1/2}}{(2 + 2 \log n)^{1/2}} + O((\log n)^{-1/2}(\log \log n)^2) \right\} \rightarrow 0.\]
Next

\[ C_n = \frac{1}{2} \left\{ \frac{\log \log n}{(2 \log n)^{1/2}} - \frac{\log (2 + \log n)}{(2 + 2 \log n)^{1/2}} \right\} \]
\[ = \frac{1}{2} \left\{ \frac{\log \log n}{(2 \log n)^{1/2}} - \frac{\log \log n(1 + 1/\log n)}{(2 \log n)^{1/2}(1 + 1/\log n)^{1/2}} \right\} \]
\[ = \frac{1}{2} \left\{ \frac{\log \log n}{(2 \log n)^{1/2}} - \frac{\log \log n + \log (1 + 1/\log n)}{(2 \log n)^{1/2}} (1 + 1/\log n)^{-1/2} \right\} \]
\[ = \frac{1}{2} \left( \frac{\log \log n + \log (1 + 1/\log n)}{(2 \log n)^{1/2}} \right) \left( -\frac{1}{2 \log n} + O(\log n)^{-2} \right). \]

Then

\[ C_n/\alpha_n = -\frac{1}{2} \log (1 + 1/\log n) \]
\[ -\frac{1}{2} \left( \log \log n + \log (1 + 1/\log n) \right) \left( -\frac{1}{2 \log n} + O(\log n)^{-2} \right) \]
\[ \to 0. \]

Finally

\[ A_n/\alpha_n = (2 \log n)(1 + 1/\log n)^{1/2} - (2 \log n) \]
\[ = (2 \log n)(1 + \frac{1}{2 \log n} + O((\log n)^{-2}) - (2 \log n) \]
\[ = 1 + O((\log n)^{-1}) \to 1. \]

Formally, we got

(6.6) \quad A_n/\alpha_n \to 1 \text{ as } n \to +\infty.

Finally, from (6.5),

\[ \frac{\beta_n - b_n}{\alpha_n} = O(\varepsilon_n/\alpha_n) = O((\log n)^{-1/2}(\log \log n)^2). \]

Hence

(6.7) \quad \frac{\beta_n - b_n}{\alpha_n} \to 0 \text{ as } n \to +\infty.
6. OTHER TECHNICAL COMPUTATIONS

6.2. L’Hospital Type Rules. We expose a result of de Haan (1970), that is useful to prove the Karamata representation theorem 5.

Lemma 18. Let $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be two measurable function Lebesgue locally integrable, that is, $\forall 0 \leq a \leq b < +\infty, f, g \in L([a, b], \lambda)$ where $\lambda$ is the Lebesgue measure.

(A) Suppose that
$$
\int_0^{+\infty} f(x) d\lambda(x) = \lim_{x \to +\infty} \int_0^x f(x) d\lambda(x) = +\infty \quad \text{and} \quad \int_0^{+\infty} g(x) d\lambda(x) = +\infty
$$
and
$$
\lim_{x \to +\infty} f(x)/g(x) = c \in [0, +\infty].
$$
Then
$$
\lim_{x \to +\infty} \left( \int_0^x g(t) d\lambda(t) \right) \left( \int_0^x g(t) d\lambda(t) \right)^{-1} = c.
$$

(B) Suppose that
$$
\int_0^{+\infty} f(x) d\lambda(x) < +\infty \quad \text{and} \quad \int_0^{+\infty} g(x) d\lambda(x) < +\infty
$$
and
$$
\lim_{x \to +\infty} f(x)/g(x) = c \in [0, +\infty].
$$
Then
$$
\lim_{x \to +\infty} \left( \int_x^{+\infty} g(t) d\lambda(t) \right) \left( \int_x^{+\infty} g(t) d\lambda(t) \right)^{-1} = c.
$$

Proof of (A). (de Haan, 1970). Suppose that
$$
\lim_{x \to +\infty} f(x)/g(x) = +\infty.
$$
Then for any $A > 0$, there exists $x_0 > 0$ such that for any $t \geq x_0$,
$$
f(t)/g(t) \geq A \implies f(t) \geq 2Ag(t).
$$
For $x_0$ fixed, $\int_{x_0}^{x} g(t) d\lambda(t) \to \infty$ as $x \to \infty$ and $\int_0^{x_0} g(t) d\lambda(t)$ is finite by locally integrability of $g$. So for some $x_1 > x_0$, we have for $x \geq x_1$,
$$
\int_0^{x_0} g(t) d\lambda(t)/\int_{x_0}^{x} g(t) d\lambda(t) \leq 1
$$
and next
\[
\frac{\int_0^x f(t) d\lambda(t)}{\int_0^x g(t) d\lambda(t)} = \frac{\int_0^{x_0} f(t) d\lambda(t) + \int_{x_0}^x f(t) d\lambda(t)}{\int_0^{x_0} g(t) d\lambda(t) + \int_{x_0}^x g(t) d\lambda(t)} \\
\geq \frac{\int_{x_0}^x f(t) d\lambda(t)}{\int_{x_0}^x g(t) d\lambda(t) + \int_{x_0}^x g(t) d\lambda(t)} \\
\geq \frac{2A \int_{x_0}^x g(t) d\lambda(t)}{\int_{x_0}^x g(t) d\lambda(t) + \int_{x_0}^x g(t) d\lambda(t)} \\
= 2A \left(1 + \int_0^{x_0} g(t) d\lambda(t) / \int_{x_0}^x g(t) d\lambda(t)\right)^{-1} \\
\geq A.
\]

This leads to
\[
\frac{\int_0^x f(t) d\lambda(t)}{\int_0^x g(t) d\lambda(t)} \to +\infty \text{ as } x \to +\infty.
\]

Suppose now that
\[
\lim_{x \to +\infty} f(x)/g(x) = c \in [0, +\infty[.
\]

Then for any \( \varepsilon > 0 \), there exists \( x_0 > 0 \) such that for any \( t \geq x_0 \),
\[
c - \varepsilon \leq f(t)/g(t) \leq c + \varepsilon \quad \implies (c - \varepsilon)g(t) \leq f(t) \leq (c + \varepsilon)g(t).
\]

Then
\[
(c - \varepsilon) = \frac{(c - \varepsilon) \int_{x_0}^x g(t) d\lambda(t)}{\int_{x_0}^x g(t) d\lambda(t)} \leq \frac{(c - \varepsilon) \int_{x_0}^x g(t) d\lambda(t)}{\int_{x_0}^x g(t) d\lambda(t)} \\
\leq \frac{\int_{x_0}^x f(t) d\lambda(t)}{\int_{x_0}^x g(t) d\lambda(t)} \leq \frac{(c + \varepsilon) \int_{x_0}^x g(t) d\lambda(t)}{\int_{x_0}^x g(t) d\lambda(t)} = (c + \varepsilon)
\]

For \( x_0 \) fixed, \( \int_{x_0}^x g(t) d\lambda(t) \to \infty \) and \( \int_{x_0}^x g(t) d\lambda(t) \to \infty \) as \( x \to \infty \). So for some \( x_1 > x_0 \), we have for \( x \geq x_1 \),
\[
0 \leq \int_0^{x_0} f(t) d\lambda(t) / \int_{x_0}^x f(t) d\lambda(t) \leq \varepsilon
\]
and
\[
\int_0^{x_0} g(t) d\lambda(t) / \int_{x_0}^x g(t) d\lambda(t) \leq \varepsilon.
\]

Then for \( x \geq x_1 \),
This leads to \( x \geq x_1 \)
\[
(c - \varepsilon)(1 + \varepsilon)^{-1} \leq \frac{\int_0^x f(t) d\lambda(t)}{\int_0^x g(t) d\lambda(t)} \leq (c + \varepsilon)(1 + \varepsilon).
\]

Hence for any \( \varepsilon > 0 \),
\[
(c - \varepsilon)(1 + \varepsilon)^{-1} \leq \liminf_{x \to +\infty} \frac{\int_0^x f(t) d\lambda(t)}{\int_0^x g(t) d\lambda(t)} \leq \limsup_{x \to +\infty} \frac{\int_0^x f(t) d\lambda(t)}{\int_0^x g(t) d\lambda(t)} \leq (c + \varepsilon)(1 + \varepsilon).
\]

By letting \( \varepsilon \downarrow 0 \), we conclude that
\[
\lim_{x \to +\infty} \frac{\int_0^x f(t) d\lambda(t)}{\int_0^x g(t) d\lambda(t)} = c.
\]

**Proof of (B).** We use very similar but easier ways. Suppose that
\[
\lim_{x \to +\infty} f(x)/g(x) = +\infty.
\]

Then for any \( A > 0 \), there exists \( x_0 > 0 \) such that for any \( t \geq x_0 \),
\[
f(t)/g(t) \geq A \implies f(t) \geq Ag(t).
\]

For \( x \geq x_0 \), \( \int_{x}^{+\infty} f(t) d\lambda(t) \) and \( \int_{x}^{+\infty} g(t) d\lambda(t) \) are finite and we may write and next
\[
\frac{\int_{x}^{+\infty} f(t) d\lambda(t)}{\int_{x}^{+\infty} g(t) d\lambda(t)} \geq \frac{\int_{x}^{+\infty} Ag(t) d\lambda(t)}{\int_{x}^{+\infty} g(t) d\lambda(t)} \geq A.
\]

This leads to
\[
\frac{\int_{x}^{+\infty} f(t) d\lambda(t)}{\int_{x}^{+\infty} g(t) d\lambda(t)} \to +\infty \text{ as } x \to +\infty.
\]

Suppose now that
\[
\lim_{x \to +\infty} f(x)/g(x) = c \in [0, +\infty[.
\]

Then for any \( \varepsilon > 0 \), there exists \( x_0 > 0 \) such that for any \( t \geq x_0 \),
\[
c - \varepsilon \leq f(t)/g(t) \leq c + \varepsilon \implies (c - \varepsilon)g(t) \leq f(t) \leq (c + \varepsilon)g(t).
\]
Then $x \geq x_0$, we have

\[
(c - \varepsilon) = (c - \varepsilon) \frac{\int_x^{+\infty} g(t)d\lambda(t)}{\int_x^{+\infty} g(t)d\lambda(t)} \leq \frac{\int_x^{+\infty} f(t)d\lambda(t)}{\int_x^{+\infty} g(t)d\lambda(t)} \\
\leq \frac{(c + \varepsilon) \int_x^{+\infty} g(t)d\lambda(t)}{\int_x^{+\infty} g(t)d\lambda(t)} = (c + \varepsilon)
\]

This leads to

\[
\frac{\int_x^{+\infty} f(t)d\lambda(t)}{\int_x^{+\infty} g(t)d\lambda(t)} \to c \quad \text{as} \quad x \to \infty.
\]
CHAPTER 3

Advanced Characterizations of the Univariate
Extreme Value Domain

1. Introduction

In this part, we suppose that the reader already knows the results of chapter 2 particularly, of Section 4 where the first and main characterization are given, alongside with the representation of Karamata for a distribution $F \in D(H_\gamma), \gamma \neq 0$ and the de Haan’s one for $F \in D(H_0)$ through the quantile transformation $F^{-1}(1-u), u \in (0, 1)$.

This part, as it is a free restitution of the reading of Sections 2.5 – 2.8 in [de Haan (1970)], will be based on a definition of the extreme domain using limit on $t \to \infty$ on $\mathbb{R}$, rather than on $n \to +\infty$ on $\mathbb{N}$.

We say that the probability distribution function $F$ is in the domain of attraction of $H$, in the sense of extreme value theory, if and only if :

\[(D_1)\] There exist two functions $a(t)$ and $b(t)$ of $t \in \mathbb{R}_+$ such that $a(t) \geq 0$ for all $t \geq 0$ and
\[
\forall x \in C(H), F^t(a(t)x + b(t)) \to H(x), ast \to \infty.
\]

Throughout this text, we will apply, without mentioning it, the discretization of limits for $t \to t_0$ in the following way :

Let $t_0$ be an adherent point of set $D \in \mathbb{R}$. A numerical function $f(t)$ of $t \in D$ converges to $a \in \mathbb{R}$ as $t \to t_0$ if and only if for any sequence $(t_n)_{n \geq 0} \subset D$ such that $t_n \to t_0$, as $n \to +\infty$, we have $f(t_n) \to a$ as $n \to +\infty$.

From this, let us quickly show that $(1.1)$ is equivalent to the classical definition.

\[(D_2)\] There exist two sequence of real numbers $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ such that
\((1.2)\) \(\forall x \in C(H), \ F^n(a_nx + b_nx) \to H(x), \) as \(t \to +\infty.\)

First, that \((1.1)\) implies \((1.2)\) is a simple consequence of the discretization principle of continuous limits.

Secondly, if \((1.2)\) holds, we have for any \(t \geq 1, \ a_\ell = a_t > 0 \) and \(b_\ell = b_t\) and for any \(x \in C(H)\)
\[F^t(a_t x + b_t) = (F^{[t]}(a[t] x + b[t]))^{\frac{t}{[t]}} \to H(x) \text{ as } t \to +\infty \text{ (as } [t] \to +\infty).\]

Based on this, we will work with Definition \((D_1)\). By using the discretization principle, we also have that \(\alpha(t) > 0, \ a(t) > 0, \ b(t)\) and \(\beta(t)\) are functions of \(t > 0\) such that, for any \(x \in C(H), \) as \(t \to +\infty, \)
\[(1.3) \ F^t(a_t x + b_t) \to H(x) \text{ as } t \to +\infty \]
and, as \(t \to +\infty, \)
\[(1.4) \ \frac{\alpha(t)}{a(t)} \to A > 0 \text{ and } \frac{\beta(t) - b(t)}{a(t)} \to B \in \mathbb{R}.\]

Then, for any \(x \in C(H_{A,B}), \) where \(H_{A,B}(x) = H(Ax + B)\) for all \(x \in \mathbb{R}, \)
\[F^t(a_t x + b_t) \to H(Ax + B) \text{ as } t \to +\infty.\]

As well if for any \(x \in C(H_1), \)
\[F^t(a_t x + b_t) \to H_1(x) \text{ as } t \to +\infty \]
and for any \(x \in C(H_2), \)
\[F^t(\alpha(t)x + \beta(t)) \to H_2(x) \text{ as } t \to +\infty.\]

Then, \((1.4)\) holds and
\[\forall x \in \mathbb{R}, \ H_2(x) = H_1(Ax + B).\]

**Remark R1.** This merely means that Lemma \((2)\) remains valid in the frame of definition \((D_2)\) above.

Now, it remains to introduce another version of expressing \((1.1). \) By taking the logarithm of the quantities involved in that limit, we have:

(a) \(\forall x \in \left[lep(H), uep(H)\right] \cap C(H) =: C^*(H),\)
\[ t \log F(a(t)x + b(t)) \rightarrow \log H(x). \]

(b) for any \( x < \text{lep}(H) \) and \( x > \text{uep}(F) \), \( x \) is a continuity point of \( H \) (\( H \) is constant on \( ] - \infty, \text{lep}(H) [ \) and equal to zero, \( H = 1 \) on \( ]\text{uep}(H), +\infty[ \), whenever these intervals are not empty). We also have on these intervals
\[ t \log F(a(t)x + b(t)) \rightarrow \log H(x). \]

In summary, for any \( x \in C(H) \),
\[ t \log F(a(t)x + b(t)) \rightarrow \log H(x). \]

For \( x \in C^*(H) \), since \( \log H(x) \in (-\infty, 0) \) and \( t \rightarrow +\infty \), it follows that \( \log F(a(t)x + b(t)) \rightarrow 0 \), that is
\[ F(a(t)x + b(t)) \rightarrow 1, \]
and then
\[ \log F(a(t)x + b(t)) = \log (1 + (F(a(t)x + b(t)) + 1)) \approx -(1 - F(a(t)x + b(t))) \]

Finally,
\[ t \log F(a(t)x + b(t)) \approx -(1 - F(a(t)x + b(t))) \rightarrow t \log H(x). \]

Then, for \( x \in C^*(H) \),
\[ t(1 - F(a(t)x + b(t)) \rightarrow -\log H(x). \]

Suppose (1.5) hold in turn. Then for \( x \in C^*(H) \),
\[ t(\log F(a(t)x + b(t)))(1 + o(1)) = (+ \log H(x)) + o(1), \]
which implies
\[ t \log F(a(t)x + b(t)) = \frac{1}{1 + o(1)}(+ \log H(x)) + o(1), \]
which implies
\[ \log F^{t}(a(t)x + b(t)) = + \log H(x) + o(1), \]

since \( \log H(x) \in (-\infty, 0) \). Hence, for \( x \in C^*(H) \),
\[ F^{t}(a(t) + b(t)) = H(x) \exp(o(1)) \rightarrow H(x). \]
Finally, we have the equivalence, as \( t \to +\infty \), \( \forall x \in C^*(H) \):
\[
F'(a(t) + b(t)) \to H(x) \iff \forall x \in C^*(H), \ t(1 - F(a(t)x + b(t))) \to H(x).
\]

In the special case where \( H(x) = H_0(x) = \exp(-e^{-x}) \), we get

**Lemma 19.** \( F \in D(H_0) \) and only if and only if: there exist two functions \( a(t) > 0 \) and \( b(t) \) of \( t > 0 \) such that:

\[
(1.6) \quad \forall x \in \mathbb{R}, \ t(1 - F(a(t)x + b(t))) \to e^{-x}, \text{ as } t \to +\infty.
\]

Now we may begin our round up of the chapter.
2. Γ-variation

**Definition 10.** We introduce the following definitions.

1. Let $F$ be a distribution function, $\alpha(t) > 0$ and $\beta(t)$ two function of $t < \text{u.ep}(F)$. We denote

$$\Gamma(F, x, \alpha(t), \beta(t)) = \frac{1 - F(\alpha(t)x + \beta(t))}{1 - F(t)}, \ t < \text{u.ep}(F).$$

2. A distribution function $F$ is said to be of Γ-variation if and only if there exist two $\alpha(t) > 0$ and $\beta(t)$ of $t < \text{u.ep}(F)$ such that

$$\forall x \in \mathbb{R}, \ \Gamma(F, x, \alpha(t), \beta(t)) \to \exp(-x) \text{ as } t \to \text{u.ep}(F)^-.$$ 

We have the following facts

**Lemma 20.** Let $F$ be distribution function such that (2.1) holds. We have

(A) There exist $t_0(x) < \text{u.ep}(F)$ such that

$$t_0(x) < t < \text{u.ep}(F) \implies \alpha(t)x + \beta(t) < \text{u.ep}(F).$$

Precisely, we have:

(a) If $\text{u.ep}(F) = +\infty$ the for any $x \in \mathbb{R}$,

$$\alpha(t)x + \beta(t) \to +\infty \text{ as } t \to \text{u.ep}(F)^-.$$

(b) If $\text{u.ep}(F) < \infty$, then for any $x \in \mathbb{R}$, there exists a real number $t_0(x) < \text{u.ep}(F)$ such that

$$t_0(x) < t < \text{u.ep}(F) \implies \alpha(t)x + \beta(t) < \text{u.ep}(F).$$

(a) If $\text{u.ep}(F) < +\infty$ the for any $x \in \mathbb{R}$,

$$\alpha(t)x + \beta(t) \to +\infty \text{ as } t \to \text{u.ep}(F)^-. $$

(B) If Formula 2.1 holds such that for $\beta(t) > 0$ for $t < \text{u.ep}(F)$ in the case where $\text{u.ep}(F) = +\infty$, and $\beta(t) < \text{u.ep}(F)$ for $t < \text{u.ep}(F)$ in the case where $\text{u.ep}(F) < +\infty$, then we have, as $t \to \text{u.ep}(F)^-$
(2.5) \[ \frac{\alpha(t)}{\beta(t)} \to 0, \]
when \( u_{ep}(F)^- = +\infty \) and

(2.6) \[ \frac{\alpha(t)}{u_{ep}(F) - \beta(t)} \to 0. \]
when \( u_{ep}(F) < +\infty. \)

(C) If the functions \( \alpha(t) > 0 \) and \( \beta(t) < u_{ep}(F) \) are finite and if Formulas 2.6 and 2.6 hold, then for all \( x \in \mathbb{R} \), there exists \( t(x) < u_{ep}(F) \) such that for \( t(x) \leq t < u_{ep}(F) \), we have \( \alpha(t)x + \beta(t) < u_{ep}(F). \)

**Proof of Lemma 20.** Let \( F \) be distribution function such that (2.1) holds.

**Proof of Part A.**

(a) Suppose that \( u_{ep}(F) = +\infty \) and fix \( x \in \mathbb{R} \). Let \( \ell \) be an arbitrary adherent point of \( \alpha(t)x + \beta(t) \), as \( t \to u_{ep}(F) \). So there exists a sequence \( (t_n)_{n \geq 1} \) such that

\[ t_n \to u_{ep}(F)^- \text{ and } \alpha(t_n)x + \beta(t_n) \to \ell \text{ as } n \to +\infty. \]

Put \( A = \ell + 1 \) if \( \ell \) is finite and consider an arbitrary real number \( A > 0 \) if \( \ell = -\infty \). So, by definition of the limit, we may find a value \( n_0 \geq 0 \) such that for any \( n \geq n_0 \), we have

\[ \alpha(t_n)x + \beta(t_n) \leq A. \]

It follows that for any \( n_0 \geq 0 \), we have

\[ \Gamma(F, x, \alpha(t_n), \beta(t_n)) \geq \frac{A}{1 - F(t_n)}. \]

Hence,

\[ \Gamma(F, x, \alpha(t), \beta(t)) + \infty \text{ as } n \to +\infty, \]

which contradicts (2.2). Thus, \( \ell = +\infty \) is the unique adherent point of \( \alpha(t)x + \beta(t) \), as \( t \to u_{ep}(F)^- \). Hence, Point (a) us proved.

(b) Let us suppose that \( u_{ep}(F) < +\infty \) and let \( x \in \mathbb{R} \) be fixed. If (2.4), we would be able to construct a sequence \( (t_n)_{n \geq 1} \) such that
2. Γ-VARIATION

\[ t_n \uparrow \text{uep}(F)^- \quad \text{and for all } n \geq 0, \quad \alpha(t_n)x + \beta(t_n) \geq \text{uep}(F). \]

We would have

\[ \Gamma(F, x, \alpha(t_n), \beta(t_n)) = \frac{0}{1 - F(t_n)} \rightarrow 0 \text{ as } n \rightarrow +\infty. \]

This would contradict (2.2). Thus (b) is proved.

(c). Suppose that \( \text{uep}(F) < +\infty \). By Point (b), \( \alpha(t)x + \beta(t) < \text{uep}(F) \) for \( t \) near enough \( \text{uep}(F) \). Suppose that \( \alpha(t)x + \beta(t) \) does not converge to \( \text{uep}(F) \) as \( t \rightarrow \text{uep}(F)^- \). Thus, there exist \( \varepsilon > 0 \) and \((t_n)_{n \geq 0}\) such that \( t_n \rightarrow \text{uep}(F)^- \) as \( n \rightarrow +\infty \) and for all \( n \geq 0 \), we have \( \alpha(t_n)x + \beta(t_n) < \text{uep}(F) - \varepsilon \). It follows that

\[ \liminf_{n \rightarrow +\infty} \Gamma(F, x, \alpha(t_n), \beta(t_n)) \geq \liminf_{n \rightarrow +\infty} \frac{1 - F(\text{uep}(F) - \varepsilon)}{1 - F(t_n)} = +\infty, \]

which is impossible since the right-hand is finite and equal to \( e^x \).

The three points (a), (b) and (c) are summarized in the statement of Point (A).

Proof of Part B.

Let us suppose that \( \text{uep}(F) = +\infty \). Let \( x < 0 \). Since \( \alpha(t)x + \beta(t) \rightarrow 0 \) as \( t \rightarrow +\infty \), there exists \( t_0 > 0 \) such that : \( t \geq t_0 \) implies that \( x\alpha(t)x + \beta(t) \), that is (since \( \beta(t) \geq 0 \)),

\[ \alpha(t)/\beta(t) \leq -1/x, \]

for all \( x < 0 \), for all \( t \geq t_0 \). We get the result by letting successively \( x \downarrow -\infty \) and \( t \rightarrow +\infty \).

Let us suppose that \( \text{uep}(F) < +\infty \). Let \( x > 0 \). Since \( \alpha(t)x + \beta(t) < \text{uep}(F) \) for \( t \) near \( \text{uep}(F) \), that is \( t_0 \leq < \text{uep}(F) \) for some \( t_0 \leq < \text{uep}(F) \), and since \( \beta(t) < \text{uep}(F) \) for \( t < \text{uep}(F) \), we have

\[ \alpha(t)/(\text{uep}(F) - \beta(t)) \leq 1/x, \]

for all \( t > 0 \), for all \( t_0 \leq t < 1 \). We get the result by letting successively \( x \uparrow +\infty \) and \( t \rightarrow \text{uep}(F)^- \).
**Proof of Part C.** There is nothing to do for \( \text{uep}(F) = +\infty \). Now, let us suppose now that \( \text{uep}(F) \) is finite. Let us begin with a positive \( x \). We have

\[
\alpha(t)x + \beta(t) = (\text{uep}(F) - \beta(t)) \left( x \frac{\alpha(t)}{\text{uep}(F) - \beta(t)} - 1 \right) + \text{uep}(F), \quad t < \text{uep}(F).
\]

Hence for any \( \varepsilon > 0 \) such that \( \varepsilon < 1 \), there exists \( t(x) \leq t < \text{uep}(F) \) such that for \( t_1(x) \leq t < \text{uep}(F) \), we have

\[
\alpha(t)x + \beta(t) < (\text{uep}(F) - \beta(t))(1 + \varepsilon) + \text{uep}(F) = \beta(t) + \varepsilon(\text{uep}(F) - \beta(t)) < \text{uep}(F).
\]

As well, the following fact will be useful for the sequel.

**Lemma 21.** Let \( a \) be finite or an infinite non-negative number. Let \( r(x) \) and \( b(x) \) be two functions of \( x < a \) and \( b(x) \to 0 \) as \( x \to a^- \). Suppose that we have for \( a = +\infty \),

\[
(2.7) \quad r(x) = c + \int_{x_1}^{x} b(t)dt
\]

where \( c \) is a constant and for \( a < +\infty \),

\[
(2.8) \quad r(x) = -\int_{x_1}^{a} b(t)dt.
\]

Then,

(a) we have, as \( t \to a^- \), \( r(t)/t \to 0 \) if \( a = +\infty \) and \( r(t)/(a-t) \to 0 \) if \( a < +\infty \)

and

(b) for all \( x \in \mathbb{R} \), \( xr(t) + t \to a \) as \( t \to a^- \) and there exists \( t(x) < a \) such that \( xr(t) + t < a \) for \( t(x) \leq t < a \).

These results still hold if we replace Conditions (3.30) and (3.31) by the following one : there exists \( x_0 < a \) such that the derivative function \( r'(x) \) exists for \( x_0 \leq x < a \), \( r'(x) \to 0 \) as \( x \to a^- \), and \( r(x) \to 0 \) as \( x \to a^- \).
Proof of Lemma 21.

Proof of Part (a). Let us prove from Formula 3.30, that

\[(2.9) \quad r(x)/x \to 0 \text{ as } x \to a, \]

when \(a = +\infty\). Indeed, for any \(\varepsilon > 0\), there exists \(x_2 > x_1\) such that \(|b(t)| \leq \varepsilon\) for \(x \leq x_2\). Thus

\[
0 \leq \limsup_{x \to +\infty, x > x_2} \frac{r(x)}{x} = \limsup_{x \to +\infty} \frac{1}{x} \left| \frac{r(x_1) + \int_{x_1}^{x} b(t)dt + \int_{x_2}^{x} b(t)dt}{x} \right| \leq \limsup_{x \to +\infty, x > x_2} \frac{1}{x} \left| \frac{r(x_1) + \int_{x_1}^{x_2} b(t)dt + \varepsilon(x - x_2)}{x} \right| \leq \varepsilon.
\]

We get the desired result by letting \(\varepsilon \downarrow 0\).

We have to prove, from Formula (3.31), that

\[(2.10) \quad r(x)/(uep(F) - x) \to 0 \text{ as } x \to a, \]

when \(a < +\infty\). But the proof is similar and still based on the fact that \(b(t) \to 0\) as \(x \to uep(F)\).

Proof of Part (a). Now, it is clear that for \(a = +\infty\), we have for all real \(x, r(t)x + t = t(1 + xr(t)/t) \to +\infty\) as \(t \to a = +\infty\). If \(a\) is finite, we have, as \(a \to a^-\),

\[
xr(t) + t = xr(t) + t - a + a = (a - t) \left( \frac{r(t)}{a - 1} - 1 \right) + a \to a.
\]

The last part of Part (b) follows exactly as in the proof of Part (C) in the above Lemma 20.

To finish, let us suppose that there exists \(x_0 < a\) such that the derivative function \(r'(x)\) exists for \(x_0 \leq x < a\), \(r'(x) \to 0\) as \(x \to a^-\), and \(r(x) \to 0\) as \(x \to a^-\) for \(a\) finite. Here, we put \(b(x) = r'(x)\) and Formula (3.30) holds in general. If \(a\) is finite, we may find the value of the constant by letting \(x \to a^-\) in Formula (3.30), and thus get Formula (3.31). From there, the proof above may be reproduced word by word. ■
Before we close this section, let us mention that the convergence in the $\Gamma$-variation is in fact continuous and locally compact. Precisely, we have

**Proposition 14.** Let $F \in D(H_0)$. Let $\alpha(t) > 0$ and $\beta(t)$ be two functions of $t < \text{uep}(F)$ such for all $x \in \mathbb{R}$, as $t \to \text{uep}(F)^-$, we have

$$\Gamma(F, x, \alpha(t), \beta(t)) \to \exp(-x). \quad (2.11)$$

Then for any $x \in \mathbb{R}$, for any function $x(t) > 0$ of $t < \text{uep}(F)$ such for all $x(t) \to x$ as $t \to \text{uep}(F)^-$, we have

$$\Gamma(F, x(t), \alpha(t), \beta(t)) \to \exp(-x),$$

as $t \to \text{uep}(F)^-$. Also, for any $-\infty < a < b < +\infty$, we have, $t \to \text{uep}(F)^-$,

$$\sup_{x \in [a, b]} |\Gamma(F, x, \alpha(t), \beta(t)) - \exp(-x)| \to 0. \quad (2.12)$$

**Proof of Proposition 14.** Assume the hypotheses of the proposition hold. Fix $x \in \mathbb{R}$. Let $x(t) > 0$ be a function of $t < \text{uep}(F)$ such for all $x(t) \to x$ as $t \to \text{uep}(F)^-$. For any $\varepsilon > 0$ fixed, there exists $t_0 < \text{uep}(F)$ such that

$$(t_0 \leq t < \text{uep}(F)) \Rightarrow x - \varepsilon \leq xx + \varepsilon.$$ 

Thus, we have, for $t_0 \leq t < \text{uep}(F)$, that

$$\Gamma(F, x + \varepsilon, \alpha(t), \beta(t)) \leq \Gamma(F, x + \varepsilon, \alpha(t), \beta(t)) \leq \Gamma(F, x - \varepsilon, \alpha(t), \beta(t)).$$

Then, as $t \to \text{uep}(F)^-$, we have

$$e^{-(x+\varepsilon)} = \Gamma(F, x + \varepsilon, \alpha(t), \beta(t)) \leq \liminf_{t \to \text{uep}(F)^-} \Gamma(F, x(t), \alpha(t), \beta(t)) \leq \limsup_{t \to \text{uep}(F)^-} \Gamma(F, x(t), \alpha(t), \beta(t)) \leq \limsup_{t \to \text{uep}(F)^-} \Gamma(F, x - \varepsilon, \alpha(t), \beta(t)) \leq e^{-(x-\varepsilon)}. \quad (2.17)$$

We may conclude by letting $\varepsilon \downarrow 0$ to get (2.11). The proves the first part.
To prove the second part, that is (2.12), we proceed by contradiction. Suppose that (2.12) is false. This suppose that

\[ \limsup_{t \to u_{\text{ep}}(F)^-} \sup_{x \in [a,b]} |\Gamma(F, x, \alpha(t), \beta(t)) - \exp(-x)| > 0. \]

This implies that there exists \( \varepsilon > 0 \) such that there exists a sequence \((t_n)_{n \geq 0}\) such that \( t_n \to u_{\text{ep}}(F)^- \) as \( n \to +\infty \) for which, for all \( n \geq 1, \)

\[ \sup_{x \in [a,b]} |\Gamma(F, x(t_n), \alpha(t_n), \beta(t_n)) - \exp(-x)| > \varepsilon. \]

Then for each \( n \geq 1, \) there exists \( x_n \) such that

\[ (2.18) \quad |\Gamma(F, x_n, \alpha(t_n), \beta(t_n)) - \exp(-x_n)| > \varepsilon. \]

Now, the the sequence \((x_n)_{n \geq 1}\) is \([a, b]\). By the Bolzano-Weierstrass, it admits a sub-sequence \((x_{n_k})_{k \geq 1}\) converging to a point \( x \in [a, b] \). By applying (2.18) to the sequence \((x_{n_k})_{k \geq 1}\), as \( n \to 0 \), together with (2.11) and the continuity of the exponential function at \( x \) lead the the contradiction that zero is greater than \( \varepsilon > 0 \). This proves that (2.12) holds. \( \blacksquare \)
3. Gumbel Extreme Value Domain and $\Gamma$-variation

In this section, we unveil the equivalence between a distribution $F$ lying in the Gumbel Extreme Value Domain and satisfying the $\Gamma$-variation. This equivalence will imply a number of interesting formulas and properties that will help in getting new characterizations in the Extreme Value domain of attraction. We begin with

PROPOSITION 15. Let $F$ be a cdf such that $F \in D(H_0)$. Then $F$ is of $\Gamma$-variation.

**Proof of 15.** Suppose that $F \in D(H_0)$. Then by definition, there exist two $a(t) > 0$ and $b(s)$ of $s \mathbb{R}$ such for all $x \in \mathbb{R}$, as $s \to +\infty$, we have $s \to +\infty$,

$$F^*(a(s), x + b(s)) \to H_0(x)$$

Now, by Formula (1.6), we have for all $x \in \mathbb{R}$, as $s \to +\infty$,

$$s(1 - F(a(s)x + b(s)) \to e^{-x}$$

Let us use this formula with

$$s(t) = 1/(1 - F(t))$$

so that $s(t) \to +\infty$ as $t \to u ep(F)^-$. Put

$$a(s(t)) = \alpha(t) \text{ and } b(s(t)) = \beta(t), \ t < u ep(F).$$

We get, for all $x \in \mathbb{R}$, as $t \to u ep(F) ^-$, we have

$$\Gamma(F, x, \alpha(t), \beta(t)) \to \exp(-x),$$

which establishes the $\Gamma$-variation of $F$. ■

The next propositions show how we could change the functions in the definition of $\Gamma$-variation.

PROPOSITION 16. Let $F$ be a cdf. Suppose that $F$ is $\Gamma$-varying such that for some function $\alpha(t) > 0$ of $t < u ep(F)$, we have, $x \in \mathbb{R}$, as $t \to u ep(F) ^-$,

$$\Gamma(F, x, \alpha(t), \beta(t)) \to \exp(-x),$$

Then $F \in D(H_0)$.
Precisely, if there exist two functions $\alpha(t) > 0$ and $\beta(t)$ of $t < \text{uep}(F)$ such that for all $x \in \mathbb{R}$, as $t \to \text{uep}(F)^-$, we have
$$
\Gamma(F, x, \alpha(t), \beta(t)) \to \exp(-x),
$$
then, by denoting $t(s) = (F^{-1}(1 - 1/s), a(s) = \alpha(t - s))$ and $b(s) = \beta(t(s))$ for $s > 1$, we have for all $x \in \mathbb{R}$,
$$
s(1 - F(a(s)x + b(s))) \to e^{-x}, \text{ as } s \to +\infty.
$$

**Proof of Proposition 16.** Assume that Formula (3.2) hold for a distribution $F$. We may use it for
$$
t(s) = F^{-1}(a - 1/s), \text{ } s > 1,
$$
so that $t(s) \to \text{uep}(F)^-$ as $s \to +\infty$. We get, for all $x \in \mathbb{R}$, as $s \to +\infty$,
$$
\frac{1 - F(\alpha(t(s))x + \beta(t(s)))}{1 - F(t(s))} \to e^{-x},
$$
with $a(s) = \alpha(t(s))$ and $b(s) = \beta(t(s)), s > 1$. Now, let us prove that
$$
s(1 - F(t(s))) \to 1 \text{ as } s \to +\infty.
$$

To this end, let us use the properties of generalized inverses functions (see for example, Chapter 4 in [Lo (2016c)] of the current series), in particular Point 9 in the above cited chapter, that is
$$
\forall u \in (0, 1), \text{ } F(F^{-1}(u) -) \leq u \leq F(F^{-1}(u)),
$$
since $F$ is right-continuous. Applying this to $u = 1 - 1/s, s > 1$, leads to
$$
\forall s > 1, \text{ } F(t(s) -) \leq 1 - 1/s \leq F(t(s)).
$$
Since $\alpha(t(s)) > 0$ for $t < \text{uep}(F)$, then for any $\varepsilon > 0$, for any $s > 0$, we have by the finition of the left-hand limit
$$
\forall s > 1, \text{ } F(t(s) - \varepsilon \alpha(t(s))) 1 - 1/s \leq F(t(s)),
$$
which my be rewritten as
$$
\forall s > 1, \text{ } 1 \leq \frac{1}{s(1 - F(t(s))} \leq \frac{1 - F(\alpha(t(s))\varepsilon x + t(s))}{1 - F(t(s))}.
$$
By applying Formula (3.2), we have for any $\varepsilon > 0$,  

(3.5) \[ 1 \leq \liminf_{s \to +\infty} s(1 - F(t(s))) \]
(3.6) \[ \leq \limsup_{s \to +\infty} s(1 - F(t(s))) \]
(3.7) \[ \leq \limsup_{s \to +\infty} \frac{1 - F(\alpha(t(s)) \epsilon x + t(s))}{1 - F(t(s))} \]
(3.8) \[ = \lim_{s \to +\infty} \frac{1 - F(\alpha(t(s)) \epsilon x + t(s))}{1 - F(t(s))} \]
(3.9) \[ = e^\epsilon. \]

By letting \( \epsilon \downarrow 0 \), we get (3.4). By plugging this in Formula (3.2) yields: for all \( x \in \mathbb{R} \), as \( s \to +\infty \),

\[ s(1 - F(a(s)x + b(s))) \to e^{-x}, \]

which, by the preliminary remarks of this chapter, implies that \( F \in D(H_0) \). \( \blacksquare \).

The following proposition shows how the limit is affected when the auxiliary functions \( \alpha(t) \) and \( \beta(t) \) of \( t < \text{uep}(F) \) are changed in a precise way.

**Proposition 17.** Let \( F \) be a cdf. Suppose that there exist two functions \( \alpha(t) > 0 \) and \( \beta(t) \) of \( t < \text{uep}(F) \) such that for all \( x \in \mathbb{R} \), as \( t \to \text{uep}(F)^- \), we have

\[ \Gamma(F, x, a(t), b(t)) \to \exp(-x). \]

The following assertions hold.

(A) Suppose that there exist two functions \( \alpha(t) > 0 \) and \( \beta(t) \) of \( t < \text{uep}(F) \) such that \( s \to +\infty \),

\[ \alpha(t)/a(t) \to A > 0, \ A \in \mathbb{R} \text{ and } \frac{\beta(t) - b(t)}{a(t)} \to B \in \mathbb{R}, \]

then, for all \( x \in \mathbb{R} \), as \( t \to \text{uep}(F)^- \), we have

\[ \Gamma(F, x, \alpha(t), \beta(t)) \to \exp(-(Ax + B)). \]

(B) If Formula (3.13) holds on top of Formula 3.10 with \( A > 0 \) and \( B \in \mathbb{R} \), then for for \( t(s) = F^{-1}(1 - 1/s) \) with \( s > 1 \), we have, as \( s \to +\infty \) (which is equivalent to \( t(s) \to \text{uep}(F)^- \))
Proof of Proposition 17.

Proof of Part (A). Suppose that Formulas (3.10) and (3.11) hold. Then, there exists \( t_0 < \text{uprp}(F) \) such that for \( t_0 < t < \text{uep}(F) \),

\[
a(t)(A - \varepsilon) \leq \alpha(t) \leq a(t)(A + \varepsilon)
\]

and

\[
a(t)(B - \varepsilon) + b(t) \leq \beta(t) \leq a(t)(B + \varepsilon) + b(t).
\]

Now, for \( x \geq 0 \), by denoting \( x_1(\varepsilon) = (A - \varepsilon)x + (B - \varepsilon) \) and \( x_2(\varepsilon) = (A + \varepsilon)x + (B + \varepsilon) \), we have for \( t_0 < t < \text{uep}(F) \),

\[
(3.14) \quad a(t)x_1(\varepsilon) + b(t) \leq \alpha(t) + \beta(t) \leq a(t)x_2(\varepsilon) + b(t).
\]

and then \( t_0 < t < \text{uep}(F) \),

\[
\Gamma(F, x_2(\varepsilon), a(t), b(t)) \leq \Gamma(F, x, \alpha(t), \beta(t)) \leq \Gamma(F, x_1(\varepsilon), a(t), b(t))
\]

Now, by applying (3.10) and (3.13) in the latter formula, we get as \( t \to \text{uep}(F)^- \),

\[
e^{-((A+\varepsilon)x+B+\varepsilon)} \leq \lim_{t \to \text{uep}(F)^-} \inf \Gamma(F, x, \alpha(t), \beta(t)) \\
\leq \lim_{t \to \text{uep}(F)^-} \sup \Gamma(F, x, \alpha(t), \beta(t)) \\
\leq e^{-((A-\varepsilon)x+B-\varepsilon)}.
\]

Now by letting \( \varepsilon \uparrow 0 \), we get (3.13). For \( x < 0 \), we proceed by the same manner and by reversing the inequalities in Formula (3.14).

Proof of Part B. Let us suppose that Formula (evtp.11a). By denoting \( t(s) = (F^{-1}(1 - 1/s)) \), \( a(s) = \alpha(t - (s)) \) and \( b(s) = \beta(t(s)) \) for for \( s > 1 \), we have by Proposition 16, that for all \( x \in \mathbb{R} \)

\[
s(1 - F(a(s)x + b(s))) \to e^{-x}, \text{ as } s \to +\infty.
\]

Now, by Lemma 16, we get for all \( x \in \mathbb{R} \)
Then, by Remark (R1) in the preliminaries of this section, we have that Formula (3.13) holds. ■

But we want to go further and to show that Formula (3.13) in Lemma 17, actually holds for all \( t \) converging to \( \text{uep}(F)^- \) and not for the special case of \( t(s) \). Indeed, we have the final form of the lemma as in :

**Proposition 18.** Let \( F \) be a cdf. Suppose that there exist two functions \( \alpha(t) > 0 \) and \( \beta(t) \) of \( t < \text{uep}(F) \) such that for all \( x \in \mathbb{R} \), as \( t \to \text{uep}(F)^- \), we have

\[
(3.15) \quad \Gamma(F, x, a(t), b(t)) \to \exp(-x).
\]

The following assertions hold.

**(A)** Suppose that there exist two functions \( \alpha(t) > 0 \) and \( \beta(t) \) of \( t < \text{uep}(F) \) such that \( s \to +\infty \),

\[
(3.16) \quad \frac{\alpha(t)}{a(t)} \to A > 0, \quad A \in \mathbb{R} \quad \text{and} \quad \frac{\beta(t) - b(t)}{a(t)} \to B \in \mathbb{R},
\]

then, for all \( x \in \mathbb{R} \), as \( t \to \text{uep}(F)^- \), we have

\[
(3.17) \quad \Gamma(F, x, \alpha(t), \beta(t)) \to \exp(-(Ax + B)).
\]

**(B)** If Formula (3.17) holds on top of Formula 3.15 with \( A > 0 \) and \( B \in \mathbb{R} \), then Formula (3.16) holds.

**Proof of Proposition 18.** Here, we only need to prove the part B. The proof of Part B is based on a reasoning by contradiction, which is lengthy. We propose it in details in the Appendix Section 4.

The main door of the characterization is the following Theorem. Before we present it, let us introduce the asymptotic moments of \( F \). At the first order, we define :

\[
(3.18) \quad R(F, x) = (1 - F(x))^{-1} \int_{x}^{\text{uep}(F)} 1 - F(t) \, dt, \quad x < \text{uep}(F).
\]

The \( p \)-th moment of \( F \) is defined for \( x < \text{uep}(F) \) by
\begin{equation}
R_p(F, x) = \frac{1}{1 - F(x)} \int_x^{uep(F)} \int_{u_1}^{uep(F)} \cdots \int_{u_{p-1}}^{uep(F)} 1 - F(t) \, du_1 \cdots 1 - F(t) \, du_{p-1} \, dt.
\end{equation}

By convention, we usually write \( R_1(F, .) = R(F, .) \) and \( R_2(F, .) = W(F, .) \). When there is no reason of confusion, we drop the symbol \( F \).

We have:

**Theorem 9.** Let \( F \in D(H_0) \). Then, \( R(x) \) is finite for \( x < uep(F) \), and such for all \( x \in \mathbb{R} \), we have, as \( t \to uep(F)^- \),

\[ \Gamma(F, x, R(F, t), t) \to \exp(-x). \]

**Proof of 9.** To come soon.

This theorem is used with the following result as a main tool.

**Proposition 19.** Let \( F \in D(H_0) \). Then there exists \( x_0 \) such that

\begin{equation}
(3.20) \quad c = 1 - \int_{x_0}^{uep(F)} 1 - F(t) \, dt > 0.
\end{equation}

Then function defined by

\[ F_1(x) = \left( 1 - \int_x^{uep(F)} 1 - F(t) \, dt \right) 1_{(x \geq x_0)} \]

is a probability distribution function lying in \( D(H_0) \) such that

\[ R(F, t)/R(F_1, t) \to 1 \quad \text{as} \quad t \to uep(F)^-. \]

**Proof of 19.** By Part (B) of Lemma 20,

\[ A(x) = \int_x^{uep(F)} 1 - F(t) \, dt \]

is finite when \( x \) is near \( uep(F) \). Let \( x_1 \) a point \( x_1 < uep(F) \) such that \( A(x_1) \) finite. Then \( A(x) \) converges to zero, as \( x \to uep(F)^- \) by the Dominated Convergence Theorem. So there exists \( x_0 < uep(F) \) such that Formula (3.20) holds. Hence \( F_1 \) is well-defined as a probability distribution function. We are going to prove the points of the proposition by establishing that for all \( x \in \mathbb{R} \), as \( t \to uep(F)^- \),
(3.21) \[ \Gamma(F_1, x, R(F, t), t) \to e^{-x}. \]

Let \( y \) be an arbitrary real number. By Lemma 20, \( R(F, u)y + y \to e^{-y} \) for \( u \to \text{uep}(F)^- \). Then by Theorem 9, we have for all \( y \in \mathbb{R} \), for all \( x \in \mathbb{R} \), as \( u \to \text{uep}(F)^- \),

(3.22) \[ \Gamma(F, xR(F, R(F, u)y + u), R(F, u)y + u) \to e^{-x}, \]

meaning

(3.23) \[ \frac{1 - F(R(F, R(F, u)y + u)x + R(F, u)y + u)}{1 - F(R(F, u)y + u)} \to e^{-x}. \]

But, still by 9, we have, as \( u \to \text{uep}(F)^- \),

(3.24) \[ \frac{1 - F(R(F, u)y + u)}{1 - F(u)} \to e^{-y}. \]

By combining Formulas (3.22)-(3.24), we get for \( t(u) = R(F, u)y + u \), for all \( x \in \mathbb{R} \),

\[ \Gamma(F, x, R(F, t(u), t(u)) \to e^{-(x-y)}, \]

as \( t(u) \to \text{uep}(F)^- \). Hence by proposition 18,

\[ R(F, t(u))/R(F, u) \to 1 \text{ as } u \to \text{uep}(F)^-. \]

Now, remark that for \( x < \text{uep}(F) \) near enough \( \text{uep}(F) \), that is for some \( x_0 < \text{uep}(F) \), we have

\[ R(F, x) = \frac{1 - F_1(x)}{1 - F(x)}, \quad x_0 \leq x < \text{uep}(F). \]

Thus

\[ \left( \frac{1 - F_1(t(u))}{1 - F_1(u)} \right) \left( \frac{1 - F(t(u))}{1 - F(u)} \right)^{-1} \to 1 \text{ as } u \to \text{uep}(F)^-. \]

Hence, since \( t(u) = R(F, u)y + u \), we have

\[ \lim_{u \to \text{uep}(F)^-} \left( \frac{1 - F_1(t(u))}{1 - F_1(u)} \right) = e^{-y}. \]
It follows that $F_1$ is of $\Gamma$-variation with $\alpha(t) = R(F, t), t < \text{uep}(F)$. It follows that $F_1 \in D(H_0)$ and by the way, $\text{uep}(F) = \text{uep}(F_1)$.

The just discovered property is very important. Let us rephrase it again: $F_1$, which is a transformation of $F \in D(H_0)$, is also in $D(H_0)$ and is of $\Gamma$-variation. Then, there two functions $\alpha(t) > 0$ and $\beta(t)$ of $t < \text{uep}(F) = \text{uep}(F_1)$, and for all $x \in \mathbb{R}$, as $t \to \text{uep}(F)^-$, we have

$$\Gamma(F, x, \alpha(t), \beta(t)) \to \exp(-x),$$

(3.25) \quad $1 - F(x) = c(x) \exp \left( \int_{x_1}^{\text{uep}(F)} -\frac{a(t)}{r(t)} dt \right), \quad x_1 \geq x < \text{uep}(F)$,

where $c(x), a(x)$ and $r(x)$ are functions $x < \text{uep}(F)$ such that: $(c(x), a(x)) \to (c, 1)$ as $x \to \text{uep}(F)^-$, and $r(x)$ is positive and differentiable and $r'(x) \to 0$ as $x \to \text{uep}(F)^-$. Moreover, the function $r(x)$ of $x < \text{uep}(F)$ may be taken as: for $\text{uep}(F) = +\infty$,

$$r(x) = c_1 + \int_{x_1}^{x} b(t) dt, \quad x_1 \leq x < \text{uep}(F)$$

and for $\text{uep}(F) < +\infty$, and we may replace $\alpha(t)$ by $R(F_1, t)$ or by $R(F, t), t < \text{uep}(F)$. This means that $R(F_1, x)/R(F, x) \to 1$ as $x \to \text{uep}(F)^-$. But, by the definition of the $R(F,.)$ in Formula (3.18) and that of $F_1$, we have, for $x < \text{uep}(F)$,

(3.26) \quad $R(F_1, x) = \left( \int_{x}^{\text{uep}(F)} \int_{u}^{\text{uep}(F)} 1 - F(t) dt du \right) / \left( \int_{x}^{\text{uep}(F)} 1 - F(t) dt \right)$,

and next, for $x < \text{uep}(F)$,

(3.27) \quad $H(x, F) =: R(F_1, x) / R(F, x)$

By by Formula (3.19) and by the notation $W(F,.) = R_2(F,.)$, we have

(3.28) \quad $H(F, x) =: H(x) = W(F, x) / R(F, x)^2, \quad x < \text{uep}(F)$. 

In summary, the lines above yield the following result.

**Lemma 22.** Let $F$ be a cdf on $\mathbb{R}$, such that $F \in D(H_0)$. Then the function $W(F, x)R(F, x)^{-2}$ of $x < u_{ep}(F)$ is defined and finite and we have

$$W(F, x)R(F, x)^{-2} \rightarrow 1 \text{ as } x \rightarrow u_{ep}(F)^-.$$ 

Follows a characterization of $D(H_0)$ which reverses the result in Lemma 22 and contains a useful representation.

**Theorem 10.** Let $F$ be a cdf on $\mathbb{R}$. The following assertions are equivalent.

(a) $F \in D(H_0)$.

(b) The functions $R(F, x)$ and $W(F, x)R(F, x)^{-2}$ of $x < u_{ep}(F)$ is defined and finite and we have

$$W(F, x)R(F, x)^{-2} \rightarrow 1 \text{ as } x \rightarrow u_{ep}(F)^-.$$ 

is defined for $x < u_{ep}(F)$ and $H(x) \rightarrow 1 \text{ as } x \rightarrow u_{ep}(F)^-.$

(c) $F$ admits the following representation: there exists $x_1 < u_{ep}(F)$ and a constant $c > 0$ such that:

\begin{equation}
1 - F(x) = c(x) \exp \left( - \int_{x_1}^{x} \frac{a(t)}{r(t)} \, dt \right), \quad x_1 \leq x < u_{ep}(F),
\end{equation}

$c(x), a(x)$ and $r(x)$ are measurable functions such that $(c(x), a(x)) \rightarrow (c, 1)$ as $x \rightarrow u_{ep}(F)^-$ and $r(.)$ is positive and differentiable satisfying: $r'(x) \rightarrow 0 \text{ as } x \rightarrow u_{ep}(F)^-$ and $r(x) \rightarrow 0 \text{ as } x \rightarrow u_{ep}(F)^-$ if $u_{ep}(F)$ is finite.

Moreover, we may take for $u_{ep}(F) = +\infty$

$$r(x) = c_1 + \int_{x_1}^{x} b(t) \, dt, \quad x_1 \leq x < u_{ep}(F),$$

and $u_{ep}(F) < +\infty$

$$r(x) = - \int_{x}^{u_{ep}(F)} b(t) \, dt, \quad x_1 \leq x < u_{ep}(F),$$

where $c_1$ is a constant and $r'(x) = b(x) \rightarrow 0 \text{ as } x \rightarrow u_{ep}(F)^-.$
Proof of Theorem 10. Let us use a circular method by proving: 
(a) ⇒ (b), (b) ⇒ (c) and (c) ⇒ (a).

Proof of (a) ⇒ (b). This is exactly Lemma 22.

Proof of (b) ⇒ (c).

We begin to set \( b(x) = -1 + H(x) \), with \( H(x) = W(F, x)R(F, x)^{-2} \), \( x < \text{uep}(F) \). By the assumptions, \( b(x) \to 0 \) as \( x \to \text{uep}(F)^- \). Also, the function \( r(.) = R(F_1, .) \) (see Formula 3.26)

\[
r(x) = \int_x^{\text{uep}(F)} \int_u^{\text{uep}(F)} 1 - F(t)dt du, \quad x < \text{uep}(F),
\]

is finite and positive. A simple computation shows that \( L'(x) = b(x) \) a.e. Thus, for all \( x_1 < x < \text{uep}(F) \),

\[
(3.30) \quad r(x) = \int_{x_1}^{x} b(t)dt + r(x_1).
\]

If \( \text{uep}(F) < +\infty \), we have for \( x < \text{uep}(F) \),

\[
\int_x^{\text{uep}(F)} \int_u^{\text{uep}(F)} 1 - F(t)dt du \leq \int_x^{\text{uep}(F)} \int_x^{\text{uep}(F)} 1 - F(t)dt du \leq (\text{uep}(F) - x) \int_x^{\text{uep}(F)} 1 - F(t)dt du
\]

and next, \( 0 \leq r(x) \leq (\text{uep}(F) - x) \to 0 \) as \( x \to \text{uep}(F)^- \). By plugging this in the Formula 3.30, we get

\[
\int_{x_1}^{\text{uep}(F)} b(t)dt = -r(x_1)
\]

and next

\[
(3.31) \quad r(x) = -\int_x^{\text{uep}(F)} b(t)dt, \quad x < \text{uep}(F).
\]

By putting

\[
w(x) = \int_x^{\text{uep}(F)} \int_u^{\text{uep}(F)} 1 - F(t)dt du, \quad x < \text{uep}(F),
\]

we see that \( r^{-1}(x)dx = -dw(x)/w(x) \) and thus, for all \( x_1 < x < \text{uep}(F) \),
\[
\int_{x_1}^{x} r^{-1}(t)dt = -\log w(x) + \log w(x_1).
\]
which leads to
\[
w(x) = w(x_1)^{-1} \exp \left(-\int_{x_1}^{x} r^{-1}(t)dt\right).
\]
From there, we notice first that
\[
L(x) = r(x)/R(F,x) \to 1 \text{ as } x < \text{up}(F).
\]
and next
\[
(1 - F(x))L(x) = R(F,x)^{-2}w(x), \; x < \text{up}(F).
\]
It follows from the three last equations that
\[
1 - F(x) = d(x)w(x_1)^{-1}r(x)^{-2} \exp \left(-\int_{x_1}^{x} r^{-1}(t)dt\right), \; x < \text{up}(F).
\]
where \(d(x) \to 1\) as \(x \to \text{up}(F)\). Now, we have from both Formulas 3.31 and 3.30 that \(dr(x) = b(x)dx\), and thus for \(x_1 < x < \text{up}(F)\),
\[
\exp \left(\int_{x_1}^{x} \frac{b(t)}{r(t)}dt\right) = \exp \left(\int_{x_1}^{x} \frac{dr(t)}{r(t)}\right) = \exp (\log r(x) - \log r(x_1)) = r(x)/r(x_1).
\]
By combining the two last equations, we get
\[
1 - F(x) = d(x)w(x_1)^{-1}r(x_1)^2 \exp \left(-\int_{x_1}^{x} \frac{1 + 2b(t)}{r(t)}dt\right), \; x < \text{up}(F).
\]
By putting \(c(x) = d(x)w(x_1)^{-1}r(x_1)^2\), \(c = w(x_1)^{-1}r(x_1)^2\) and \(a(x) = 1 + 2b(x)\), we arrive at Formula 3.29. We still have a few number of points to check. First, we obviously have, from Formulas 3.31 and 3.30, that \(r'(x) = b(x) \to 0\) as \(x \to \text{up}(F)\).

Proof of (c) ⇒ (a). Suppose that (c) holds. Let us use (a) by showing that \(F\) is of \(\Gamma\)-variation. From the assumptions of (b), and by Lemma 21, we have, as \(t \to \text{up}(F)^-\), that \(r(t)/t \to 0\) if \(\text{up}(F) = +\infty\) and \(r(t)/(\text{up}(F) - t) \to 0\) if \(\text{up}(F) < +\infty\), and for all \(x \in \mathbb{R}\), \(x r(t) + t \to \text{up}(F)^-\) as \(t \to \text{up}(F)^-\) and there exists \(t(x) < \text{up}(F)\) such that
xr(t) + t < a for t(x) ≤ t < a. Then for x fixed, for t(x) ≤ t < a, we may apply Formula (3.29) to t and to xr(t) + t. Thus, we have

\[ c(xr(t) + t)/c(x) \to 1, \text{ as } t \to \text{uep}(F)⁻. \]

and then,

\[ \Gamma(F, x, r(t), t) = (1 + o(1)) \left( -\int_t^{xr(t)+t} \frac{a(s)}{r(s)} \, ds \right). \]

If x = 0, we get \( \Gamma(F, 0, r(t), t) = 1 = e^0 \) and there is nothing to prove.

Let us proceed with \( |x| > 0 \). For any \( |y| \leq |x| \), we have \( -|x|r(t) + t \leq yr(t) + t \leq |x|r(t) + t \), which implies that \( yr(t) + t \to \text{uep}(F)⁻ \) uniformly in \( |y| \leq |x| \). Since \( r'(t) \to 0 \) as \( t \to \text{uep}(F)⁻ \), we may find for an arbitrary \( \varepsilon > 0 \), a value \( t_0 \) such that for \( t_0 \leq s < \text{uep}(F) \),

\[ |r(yr(t) + t) - r(t)| = \int_t^{yr(t)+t} |r'(s)| \, ds \leq \varepsilon |x|r(t). \]

that is

\[ \sup_{|y| \leq |x|} \left| \frac{r(yr(t) + t)}{r(t)} - 1 \right| \leq \varepsilon |x|. \]

Hence

\[ A(t) = \sup_{|y| \leq |x|} \left| \frac{r(yr(t) + t)}{r(t)} - 1 \right| \to 0 \text{ as } t \to \text{uep}(F)⁻. \]

As well, we may see that

\[ \sup_{s \in [xr(t)+t \Delta t, (xr(t)+t)\vee t]} \left| a(s) - 1 \right| \to 0 \text{ as } t \to \text{uep}(F)⁻. \]

By combining all this, we are able to find, for an arbitrary \( \eta > 0 \), a value \( t_1 < \text{uep}(F) \) such that for \( t_1 \leq t < \text{uep}(F) \), we have

\[ B(t) = \sup_{s \in [(xr(t)+t)\Delta t, (xr(t)+t)\vee t]} \left| a(s) - 1 \right|, \quad |c(xr(t) + t)/c(x) - 1| \leq \eta, \]

and

\[ B(\sup_{|y| \leq |x|} \left| \frac{r(yr(t) + t)}{r(t)} - 1 \right|) \eta. \]

We finally get for \( t_1 \leq t < \text{uep}(F) \), from Formula 3.32, that
\[ (3.33) \quad \log \Gamma(F, x, r(t), t) = o(1) - \int_t^{x r(t) + t} \frac{a(s)}{r(s)} ds \]

\[ (3.34) \quad = \int_t^{x r(t) + t} \frac{1}{r(s)} ds + S(1, t), \]

where

\[ |S(1, t)| = \left| \int_t^{x r(t) + t} \frac{a(s) - 1}{r(s)} ds \right| \leq B(t) \int_{-|x r(t)| + t}^{x r(t) + t} \frac{1}{r(s)} ds. \]

Next, by change of variables \( u = (s - t)/r(t) \) in Formula 3.32,

\[ (3.35) \quad \int_t^{x r(t) + t} \frac{1}{r(s)} ds = \int_0^x \frac{r(u r(t) + 1)}{r(u)} du \]

\[ (3.36) \quad =: \int_0^x du + S(2, t) \]

\[ (3.37) \quad = x + S(2, t) \]

where

\[ |S(2, t)| = \left| \int_0^x \left( \frac{r(u r(t) + 1)}{r(u)} - 1 \right) du \right| \leq A(t)|x| as t \to u_{ep}(F)^-. \]

In the same manner

\[ |\int_{-|x r(t)| + t}^{x r(t) + t} \frac{1}{r(s)} ds| \leq 2A(t)|x| as t \to u_{ep}(F)^-. \]

In conclusion, the five last equations together yield that

\[ \Gamma(F, x, r(t), t) \to e^{-x} as t \to u_{ep}(F)^-, \]

for all \( x \in \mathbb{R} \), which was the target. \[ \blacksquare \]
4. Appendix

Proof of Part B of Lemma 18.

Let us suppose that Formulas 3.15 and 3.17 hold with $A > 0$ and $B \in \mathbb{B}$. Let us break the proof into two steps.

**Step 1.** Let $\ell_1$ be an adherent point of $\{\alpha(t)/a(t), t < u_{ep}(F)\}$ as $t \to u_{ep}(F)^-$. Then there exits a sequence $(t_n)_{n \geq 0}$ such that $t_n \to u_{ep}(F)^-$ and $\alpha(t_n)/a(t_n) \to \ell_1$ as $n \to +\infty$.

Now let $\ell_2$ be an adherent point of $\{(\beta(t_n) - b(t_n))/a(t_n), n \geq 0\}$ as $n \to +\infty$. Then there exists a sub-sequence $(t_{n_k})_{k \geq 0}$ of $(t_n)_{n \geq 0}$ such that $$(\beta(t_{n_k}) - b(t_{n_k}))/a(t_{n_k}) \to \ell_2$$ as $k \to +\infty$.

We also have that $\alpha(t_{n_k})/a(t_{n_k})$ converges to $\ell_1$ as a sub-sequence of $(\alpha(t_n)/a(t_n))_{n \geq 1}$.

We are going to see that we will necessarily have that $\ell_1 = A$ and $\ell_2 = B$. Let us prove but by excluding all the other possibilities. Let us give them into cases, we will show to be impossible. In the four first cases, we suppose that $\ell_1$ is infinite (equal to $+\infty$) or $\ell_2$ is. If this is impossible, we have that $\ell_1$ and $\ell_2$ are finite. Then we split the hypothesis $\ell_1 \neq A$ and $\ell_2 \neq B$ into fours cases which also are shown to be impossible. The conclusion will be that $\ell_1 = A$ and $\ell_2 = B$.

Case 1. $\ell_1 = +\infty$ and $\ell_2 = +\infty$. Fix $x_0 > 0$. It follows that for any $C > 0$, there exists $k_0$ such that for any $k \geq k_0$

$$t_{n_k}/a(t_{n_k}) \geq C \quad \text{and} \quad \beta(t_{n_k}) - b(t_{n_k}))/a(t_{n_k}) \geq C,$$

which implies that, for $k \geq k_0$, we have

$$\alpha(t_{n_k})x_0 + \beta(t_{n_k}) \geq Ca(t_{n_k})(x_0 + 1) + b(t_{n_k}),$$

which in turn imply for $k \geq k_0$,

$$\Gamma(F, x_0, \alpha(t_{n_k}), \beta(t_{n_k})) \leq \Gamma(F, C(x_0 + 1), a(t_{n_k}), b(t_{n_k})).$$

Now, by Formulas (3.15) and (3.17), we get, as $k \to +\infty$,

$$\exp(Ax_0 + B) \leq \exp(-C(x_0 + 1)).$$
for all $C > 0$. The conclusion is absurd since the left-hand member of the latter inequality is fixed while the right-hand one tends to zero as $C \to +\infty$.

**Remark.** In the next cases, we will use similar methods. So, we will skip some intermediate steps and go directly to comparison to the $\Gamma$ quantities and conclude.

Case 2. $\ell_1 = +\infty$ and $\ell_2 = -\infty$. Fix $x_0$ such that $-1 < x_0 < 0$. It follows that for any $C > 0$, there exists $k_0$ such that for any $k \geq k_0$

$$t_{n_k} / a(t_{n_k}) \geq C \quad \text{and} \quad \beta(t_{n_k}) - b(t_{n_k}) / a(t_{n_k}) \leq -C,$$

which implies that, for $k \geq k_0$, we have

$$\Gamma(F, x_0, \alpha(t_{n_k}), \beta(t_{n_k})) \geq \Gamma(F, C(x_0 - 1), a(t_{n_k}), b(t_{n_k})),\] which implies, as $k \to +\infty$,

$$\exp(Ax_0 + B) \leq \exp(-C(x_0 - 1)),$$

which implies, by letting $C \uparrow +\infty > 0$, that $\exp(Ax_0 + B) \geq +\infty$. This is absurd.

Case 3. $\ell_1 \in \mathbb{R}_+$ and $\ell_2 = +\infty$. Fix $x_0 > 0$. It follows that for any $C > 0$ and for any $\varepsilon > 0$, there exists $k_0$ such that for any $k \geq k_0$

$$t_{n_k} / a(t_{n_k}) \geq (\ell_1 - \varepsilon) \quad \text{and} \quad \beta(t_{n_k}) - b(t_{n_k}) / a(t_{n_k}) \geq C,$$

which implies that, for $k \geq k_0$, we have

$$\Gamma(F, x_0, \alpha(t_{n_k}), \beta(t_{n_k})) \leq \Gamma(F, ((\ell_1 - \varepsilon)x_0 + C), a(t_{n_k}), b(t_{n_k})),\] which in turn implies for $k \geq k_0$,

$$\exp(Ax_0 + B) \leq \exp(-(\ell_1 - \varepsilon)x_0 - C),$$

which is impossible since left-hand member of the latter inequality is fixed while the right-hand one tends to zero as $C \to +\infty$.

Case 4. $\ell_1 \in \mathbb{R}_+$ and $\ell_2 = -\infty$. Fix $x_0 < 0$. It follows that for any $C > 0$ and for any $\varepsilon > 0$, there exists $k_0$ such that for any $k \geq k_0$
which implies that, for $k \geq k_0$, we have

$$\Gamma(F, x_0, \alpha(t_{n_k}), \beta(t_{n_k})) \geq \Gamma(F, ((\ell_1 + \varepsilon)x_0 - C), a(t_{n_k}), b(t_{n_k}))$$

which in turn implies for $k \geq k_0$,

$$\exp(Ax_0 + B) \geq \exp(-(\ell_1 + \varepsilon)x_0 + C),$$

which is impossible since left-hand member of the latter inequality is fixed while the right-hand one tends to $+\infty$ as $C \to +\infty$.

Since the cases above are excluded, we necessarily have that $\ell_1$ and $\ell_2B$ are finite. Now let us consider the following cases.

Case 5. $\ell_1 \neq A$. Then, we have the sub-cases:

(a) $\ell_1 < A$ and $\ell_2 \leq A$. Fix $x_0 = 1$. Consider $\varepsilon > 0$ such that $\ell_1 < A - \varepsilon$. By the same, method, we get a value $k_0 \geq 0$ such that for $k \geq k_0$,

$$t_{n_k}/a(t_{n_k}) \geq (A - \varepsilon) \quad \text{and} \quad \beta(t_{n_k}) - b(t_{n_k})/a(t_{n_k}) \geq B + \varepsilon,$$

which leads, $k \geq k_0$, to

$$\Gamma(F, x_0, \alpha(t_{n_k}), \beta(t_{n_k})) \geq \Gamma(F, (Ax_0 + B - \varepsilon(x_0 + 1)), a(t_{n_k}), b(t_{n_k})),$$

which implies $1 \geq \exp(2\varepsilon)$, which is impossible.

(b) $\ell_1 < A$ and $\ell_2 \geq B$. Fix $x_0 = -1$. Consider $\varepsilon > 0$ such that $\ell_1 < A - \varepsilon$. By the same, method, we get a value $k_0 \geq 0$ such that for $k \geq k_0$,

$$\Gamma(F, x_0, \alpha(t_{n_k}), \beta(t_{n_k})) \leq \Gamma(F, (Ax_0 + B - \varepsilon(x_0 - 1)), a(t_{n_k}), b(t_{n_k}))$$

which implies $1 \leq \exp(-2\varepsilon)$, which is impossible.

(c) $\ell_1 > A$ and $\ell_2 \geq B$. Fix $x_0 = 1$. Consider $\varepsilon > 0$ such that $\ell_1 < A + \varepsilon$. By the same, method, we get a value $k_0 \geq 0$ such that for $k \geq k_0$,

$$\Gamma(F, x_0, \alpha(t_{n_k}), \beta(t_{n_k})) \leq \Gamma(F, (Ax_0 + B + \varepsilon(x_0 + 1)), a(t_{n_k}), b(t_{n_k}))$$

which implies $1 \leq \exp(-2\varepsilon)$, which is impossible.
which implies $1 \leq \exp(-2\varepsilon)$, which is impossible.

(d) $\ell_1 > A$ and $\ell_2 \leq B$. Fix $x_0 = -2$. Consider $\varepsilon > 0$ such that $\ell_1 < A + \varepsilon$. By the same method, we get a value $k_0 \geq 0$ such that for $k \geq k_0$,

$$\Gamma(F, x_0, \alpha(t_{n_k}), \beta(t_{n_k})) \geq \Gamma(F, (Ax_0 + B + \varepsilon(x_0 + 1)), a(t_{n_k}), b(t_{n_k}))$$

which implies $1 \geq \exp(\varepsilon)$, which is impossible.

At this step, we conclude that $\ell_1 = A$. Let us suppose that $\ell_2 \neq B$ in two cases.

Case 6. $\ell_2 \neq B$. Then, we have the sub-cases:

(a) $\ell_2 < B$. Fix $x_0 = 1/2$. Consider $\varepsilon > 0$ such that $\ell_2 < B - \varepsilon$. By the same method, we get a value $k_0 \geq 0$ such that for $k \geq k_0$,

$$\Gamma(F, x_0, \alpha(t_{n_k}), \beta(t_{n_k})) \geq \Gamma(F, (Ax_0 + B + \varepsilon(x_0 - 1)), a(t_{n_k}), b(t_{n_k}))$$

which implies $1 \geq \exp(\varepsilon/2)$, which is impossible.

(b) $\ell_2 > B$. Fix $x_0 = 1$. Consider $\varepsilon > 0$ such that $\ell_2 > B + \varepsilon$. By the same method, we get a value $k_0 \geq 0$ such that for $k \geq k_0$,

$$\Gamma(F, x_0, \alpha(t_{n_k}), \beta(t_{n_k})) \leq \Gamma(F, (Ax_0 + B + \varepsilon(x_0 + 1)), a(t_{n_k}), b(t_{n_k}))$$

which implies $1 \leq \exp(-2\varepsilon)$, which is impossible.

The conclusion of this first step is $\ell_1 = A$. Since all adherent points of $\{\alpha(t)/a(t), \ t < \text{uep}(F)\}$ as $t \to \text{uep}(F)^-$ are equal to $A$. Then $\alpha(t)/a(t) \to A$, as $t \to \text{uep}(F)^-$.

**Step 2.** Let $\ell_2$ be an adherent point of $\{(\beta(t) - b(t))/a(t), \ t < \text{uep}(F)\}$ as $t \to \text{uep}(F)^-$. Then there exists a sequence $(t_n)_{n \geq 0}$ such that $t_n \to \text{uep}(F)^-$ and

$$(\beta(t_n) - b(t_n))/a(t_n) \to \ell_2 \text{ as } n \to +\infty.$$  

But, by our partial conclusion, we also $\alpha(t_n)/a(t_n) \to \ell_1 = A$. By the first step, we have that $\ell_2 = B$.

This concludes the proof.■
Part 3

Appendix
CHAPTER 4

Elements of Theory of Functions and Real Analysis

1. Review on limits in $\mathbb{R}$. What should not be ignored on limits.

**Definition** $\ell \in \mathbb{R}$ is an accumulation point of a sequence $(x_n)_{n \geq 0}$ of real numbers finite or infinitie, in $\mathbb{R}$, if and only if there exists a subsequence $(x_{n(k)})_{k \geq 0}$ of $(x_n)_{n \geq 0}$ such that $x_{n(k)}$ converges to $\ell$, as $k \to +\infty$.

**Exercise 1**: Set $y_n = \inf_{p \geq n} x_p$ and $z_n = \sup_{p \geq n} x_p$ for all $n \geq 0$. Show that:

1. $\forall n \geq 0, y_n \leq x_n \leq z_n$

2. Justify the existence of the limit of $y_n$ called limit inferior of the sequence $(x_n)_{n \geq 0}$, denoted by $\liminf x_n$ or $\underline{\lim} x_n$, and that it is equal to the following

   $$\liminf x_n = \liminf_{n \to \infty} x_n = \sup_{n \geq 0} \inf_{p \geq n} x_p$$

3. Justify the existence of the limit of $z_n$ called limit superior of the sequence $(x_n)_{n \geq 0}$ denoted by $\limsup x_n$ or $\overline{\lim} x_n$, and that it is equal

   $$\limsup x_n = \limsup_{n \to \infty} x_n = \inf_{n \geq 0} \sup_{p \geq n} x_p$$

4. Establish that

   $$-\liminf x_n = \limsup(-x_n) \quad \text{and} \quad -\limsup x_n = \liminf(-x_n).$$
(5) Show that the limit superior is sub-additive and the limit inferior is super-additive, i.e. : for two sequences \((s_n)_{n \geq 0}\) and \((t_n)_{n \geq 0}\)
\[
\lim \sup (s_n + t_n) \leq \lim \sup s_n + \lim \sup t_n
\]
and
\[
\lim \inf (s_n + t_n) \leq \lim \inf s_n + \lim \inf t_n
\]
(6) Deduce from (1) that if
\[
\lim \inf x_n = \lim \sup x_n,
\]
then \((x_n)_{n \geq 0}\) has a limit and
\[
\lim x_n = \lim \inf x_n = \lim \sup x_n
\]

**Exercise 2.** Accumulation points of \((x_n)_{n \geq 0}\).

(a) Show that if \(\ell_1 = \lim \inf x_n\) and \(\ell_2 = \lim \sup x_n\) are accumulation points of \((x_n)_{n \geq 0}\). Show one case and deduce the second using point (3) of exercise 1.

(b) Show that \(\ell_1\) is the smallest accumulation point of \((x_n)_{n \geq 0}\) and \(\ell_2\) is the biggest. (Similarly, show one case and deduce the second using point (3) of exercise 1).

(c) Deduce from (a) that if \((x_n)_{n \geq 0}\) has a limit \(\ell\), then it is equal to the unique accumulation point and so,
\[
\ell = \lim x_n = \lim \sup x_n = \inf \limsup_{n \geq 0} x_p
\]

(d) Combine this result with point (6) of Exercise 1 to show that a sequence \((x_n)_{n \geq 0}\) of \(\mathbb{R}\) has a limit \(\ell\) in \(\mathbb{R}\) if and only if \(\liminf x_n = \limsup x_n\) and then
\[
\ell = \lim x_n = \lim \inf x_n = \lim \sup x_n
\]
Exercise 3. Let \((x_n)_{n \geq 0}\) be a non-decreasing sequence of \(\mathbb{R}\). Study its limit superior and its limit inferior and deduce that
\[
\lim x_n = \sup_{n \geq 0} x_n.
\]
Deduce that for a non-increasing sequence \((x_n)_{n \geq 0}\) of \(\mathbb{R}\),
\[
\lim x_n = \inf_{n \geq 0} x_n.
\]

Point 4. (Convergence criteria)

Prohorov Criterion Let \((x_n)_{n \geq 0}\) be a sequence of \(\mathbb{R}\) and a real number \(\ell \in \mathbb{R}\) such that: Every subsequence of \((x_n)_{n \geq 0}\) also has a subsequence ( that is a subssubsequence of \((x_n)_{n \geq 0}\) ) that converges to \(\ell\). Then, the limit of \((x_n)_{n \geq 0}\) exists and is equal \(\ell\).

Upcrossing or Dowcrossing Criterion. Upcrossings and downcrossings.

Let \((x_n)_{n \geq 0}\) be a sequence in \(\mathbb{R}\) and two real numbers \(a\) and \(b\) such that \(a < b\). We define
\[
\nu_1 = \left\{ \begin{array}{ll}
\inf & \{n \geq 0, x_n < a\} \\
+\infty & \text{if } (\forall n \geq 0, x_n \geq a)
\end{array} \right.
\]
If \(\nu_1\) is finite, let
\[
\nu_2 = \left\{ \begin{array}{ll}
\inf & \{n > \nu_1, x_n > b\} \\
+\infty & \text{if } (n > \nu_1, x_n \leq b)
\end{array} \right.
\]
As long as the \(\nu_j's\) are finite, we can define for \(\nu_{2k-2}(k \geq 2)\)
\[
\nu_{2k-1} = \left\{ \begin{array}{ll}
\inf & \{n > \nu_{2k-2}, x_n < a\} \\
+\infty & \text{if } (\forall n > \nu_{2k-2}, x_n \geq a)
\end{array} \right.
\]
and for \(\nu_{2k-1}\) finite,
\[
\nu_{2k} = \left\{ \begin{array}{ll}
\inf & \{n > \nu_{2k-1}, x_n > b\} \\
+\infty & \text{if } (n > \nu_{2k-1}, x_n \leq b)
\end{array} \right.
\]
We stop once one \(\nu_j\) is \(+\infty\). If \(\nu_{2j}\) is finite, then
\[
x_{\nu_{2j}} - x_{\nu_{2j-1}} > b - a.
\]
We then say: by that moving from \(x_{\nu_{2j-1}}\) to \(x_{\nu_{2j}}\), we have accomplished a crossing (toward the up) of the segment \([a, b]\) called \textit{up-crossings}. Similarly, if one \(\nu_{2j+1}\) is finite, then the segment \([x_{\nu_{2j}}, x_{\nu_{2j+1}}]\) is a crossing
downward (downcrossing) of the segment \([a, b]\). Let
\[D(a, b) = \text{number of upcrossings of the sequence of the segment } [a, b].\]

(a) What is the value of \(D(a, b)\) if \(\nu_{2k}\) is finite and \(\nu_{2k+1}\) infinite.

(b) What is the value of \(D(a, b)\) if \(\nu_{2k+1}\) is finite and \(\nu_{2k+2}\) infinite.

(c) What is the value of \(D(a, b)\) if all the \(\nu_j's\) are finite.

(d) Show that \((x_n)_{n \geq 0}\) has a limit iff for all \(a < b, D(a, b) < \infty.\)

(e) Show that \((x_n)_{n \geq 0}\) has a limit iff for all \(a < b, (a, b) \in \mathbb{Q}^2, D(a, b) < \infty.\)

**Exercise 5.** (Cauchy Criterion). Let \((x_n)_{n \geq 0} \in \mathbb{R}\) be a sequence of (real numbers).

(a) Show that if \((x_n)_{n \geq 0}\) is Cauchy, then it has a unique accumulation point \(\ell \in \mathbb{R}\) which is its limit.

(b) Show that if a sequence \((x_n)_{n \geq 0} \subset \mathbb{R}\) converges to \(\ell \in \mathbb{R}\), then, it is Cauchy.

(c) Deduce the Cauchy criterion for sequences of real numbers.
SOLUTIONS

Exercise 1.

Question (1) : It is obvious that:
\[ \inf_{p \geq n} x_p \leq x_n \leq \sup_{p \geq n} x_p, \]
since \( x_n \) is an element of \( \{x_n, x_{n+1}, \ldots\} \) on which we take the supremum or the infimum.

Question (2) : Let \( y_n = \inf_{p \geq 0} x_p = \inf A_n \), where \( A_n = \{x_n, x_{n+1}, \ldots\} \) is a non-increasing sequence of sets: \( \forall n \geq 0, \)
\[ A_{n+1} \subset A_n. \]
So the infimum on \( A_n \) increases. If \( y_n \) increases in \( \mathbb{R} \), its limit is its upper bound, finite or infinite. So
\[ y_n \nearrow \lim x_n, \]
is a finite or infinite number.

Question (3) : We also show that \( z_n = \sup A_n \) decreases and \( z_n \downarrow \lim_{x_n} \).

Question (4) : We recall that
\[ -\sup \{x, x \in A\} = \inf \{-x, x \in A\}. \]
Which we write
\[ -\sup A = \inf -A. \]
Thus,
\[ -z_n = -\sup A_n = \inf -A_n = \inf \{-x_p, p \geq n\}. \]
The right hand term tends to \( -\lim x_n \) and the left hand to \( \lim -x_n \) and so
\[ -\lim x_n = \lim (-x_n). \]
Similarly, we show:
\[ -\lim (x_n) = \lim (-x_n). \]

Question (5). These properties come from the formulas, where \( A \subseteq \mathbb{R}, B \subseteq \mathbb{R} : \)
\[ \sup \{x + y, A \subseteq \mathbb{R}, B \subseteq \mathbb{R} \} \leq \sup A + \sup B. \]
In fact: \[ \forall x \in \mathbb{R}, x \leq \sup A \]
and \[ \forall y \in \mathbb{R}, y \leq \sup B. \]
Thus \[ x + y \leq \sup A + \sup B, \]
where \[ \sup_{x \in A, y \in B} x + y \leq \sup A + \sup B. \]
Similarly, \[ \inf(A + B) \geq \inf A + \inf B. \]
In fact:

\[ (x, y) \in A \times B, x \geq \inf A \text{ and } y \geq \inf B. \]
Thus \[ x + y \geq \inf A + \inf B. \]
Thus \[ \inf_{x \in A, y \in B} (x + y) \geq \inf A + \inf B \]

Application.

\[ \sup_{p \geq n} (x_p + y_p) \leq \sup_{p \geq n} x_p + \sup_{p \geq n} y_p. \]
All these sequences are non-increasing. Taking infimum, we obtain the limits superior:

\[ \liminf (x_n + y_n) \leq \liminf x_n + \liminf x_n. \]

Question (6): Set \[ \lim x_n = \lim x_n, \]
Since:

\[ \forall x \geq 1, y_n \leq x_n \leq z_n, \]
\[ y_n \to \lim x_n \]
and \[ z_n \to \lim x_n, \]
we apply Sandwich Theorem to conclude that the limit of \( x_n \) exists and:
\[
\lim x_n = \liminf x_n = \limsup x_n.
\]

**Exercice 2.**

**Question (a).**

Thanks to question (4) of exercise 1, it suffices to show this property for one of the limits. Consider the limit superior and the three cases:

**The case of a finite limit superior:**

\[\lim x_n = \ell \text{ finite}.\]

By definition,

\[z_n = \sup_{p \geq n} x_p \uparrow \ell.\]

So:

\[\forall \varepsilon > 0, \exists (N(\varepsilon) \geq 1), \forall p \geq N(\varepsilon), \ell - \varepsilon < x_p \leq \ell + \varepsilon.\]

Take less than that:

\[\forall \varepsilon > 0, \exists n_\varepsilon \geq 1 : \ell - \varepsilon < x_{n_\varepsilon} \leq \ell + \varepsilon.\]

We shall construct a subsequence converging to \(\ell\).

Let \(\varepsilon = 1:\)

\[\exists N_1 : \ell - 1 < x_{N_1} = \sup_{p \geq n} x_p \leq \ell + 1.\]

But if

\[(1.1) \quad z_{N_1} = \sup_{p \geq n} x_p > \ell - 1,\]

there surely exists an \(n_1 \geq N_1\) such that

\[x_{n_1} > \ell - 1.\]

if not we would have

\[(\forall p \geq N_1, x_p \leq \ell - 1 \implies \sup \{x_p, p \geq N_1\} = z_{N_1} \geq \ell - 1,\]

which is contradictory with (1.1). So, there exists \(n_1 \geq N_1\) such that

\[\ell - 1 < x_{n_1} \leq \sup_{p \geq N_1} x_p \leq \ell - 1.\]

i.e.

\[\ell - 1 < x_{n_1} \leq \ell + 1.\]
We move to step $\varepsilon = \frac{1}{2}$ and we consider the sequence $(z_n)_{n \geq n_1}$ whose limit remains $\ell$. So, there exists $N_2 > n_1$:

$$\ell - \frac{1}{2} < z_{N_2} \leq \ell - \frac{1}{2}.$$ 

We deduce like previously that $n_2 \geq N_2$ such that

$$\ell - \frac{1}{2} < x_{n_2} \leq \ell + \frac{1}{2}$$

with $n_2 \geq N_1 > n_1$.

Next, we set $\varepsilon = 1/3$, there will exist $N_3 > n_2$ such that

$$\ell - \frac{1}{3} < z_{N_3} \leq \ell - \frac{1}{3}$$

and we could find an $n_3 \geq N_3$ such that

$$\ell - \frac{1}{3} < x_{n_3} \leq \ell - \frac{1}{3}.$$ 

Step by step, we deduce the existence of $x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \ldots$ with $n_1 < n_2 < n_3 < \ldots < n_k < n_{k+1} < \ldots$ such that

$$\forall k \geq 1, \ell - \frac{1}{k} < x_{n_k} \leq \ell - \frac{1}{k},$$

i.e.

$$|\ell - x_{n_k}| \leq \frac{1}{k}.$$ 

Which will imply:

$$x_{n_k} \to \ell$$

Conclusion : $(x_{n_k})_{k \geq 1}$ is very well a subsequence since $n_k < n_{k+1}$ for all $k \geq 1$ and it converges to $\ell$, which is then an accumulation point.

**Case of the limit superior equal $+\infty$** :

$$\limsup x_n = +\infty.$$ 

Since $z_n \uparrow +\infty$, we have : $\forall k \geq 1, \exists N_k \geq 1$,

$$z_{N_k} \geq k + 1.$$ 

For $k = 1$, let $z_{N_1} = \inf_{p \geq N_1} x_p \geq 1 + 1 = 2$. So there exists

$$n_1 \geq N_1$$

such that :

$$x_{n_1} \geq 1.$$
For $k = 2$ : consider the sequence $(z_n)_{n \geq n_1 + 1}$. We find in the same manner
\[ n_2 \geq n_1 + 1 \]
and
\[ x_{n_2} \geq 2. \]
Step by step, we find for all $k \geq 3$, an $n_k \geq n_{k-1} + 1$ such that
\[ x_{n_k} \geq k. \]
Which leads to $x_{n_k} \to +\infty$ as $k \to +\infty$.

**Case of the limit superior equal $-\infty$**:

\[ \overline{\lim} x_n = -\infty. \]
This implies : $\forall k \geq 1, \exists N_k \geq 1$, such that
\[ z_{n_k} \leq -k. \]
For $k = 1, \exists n_1$ such that
\[ z_{n_1} \leq -1. \]
But
\[ x_{n_1} \leq z_{n_1} \leq -1 \]
Let $k = 2$. Consider $(z_n)_{n \geq n_1 + 1} \downarrow -\infty$. There will exist $n_2 \geq n_1 + 1$:
\[ x_{n_2} \leq z_{n_2} \leq -2 \]
Step by step, we find $n_{k_1} < n_{k+1}$ in such a way that $x_{n_k} \leq -k$ for all $k$ bigger that 1. So
\[ x_{n_k} \to +\infty \]

**Question (b).**

Let $\ell$ be an accumulation point of $(x_n)_{n \geq 1}$, the limit of one of its sub-sequences $(x_{n_k})_{k \geq 1}$. We have
\[ y_{n_k} = \inf_{p \geq n_k} x_p \leq x_{n_k} \leq \sup_{p \geq n_k} x_p = z_{n_k} \]
The left hand side term is a subsequence of $(y_n)$ tending to the limit inferior and the right hand side is a subsequence of $(z_n)$ tending to the limit superior. So we will have:
\[ \lim x_n \leq \ell \leq \overline{\lim} x_n, \]
which shows that $\overline{\lim} x_n$ is the smallest accumulation point and $\overline{\lim} x_n$ is the largest.
Question (c). If the sequence \((x_n)_{n \geq 1}\) has a limit \(\ell\), it is the limit of all its subsequences, so subsequences tending to the limits superior and inferior. Which answers question (b).

Question (d). We answer this question by combining point (d) of this exercise and point (6) of the exercise 1.

Exercise 3. Let \((x_n)_{n \geq 0}\) be a non-decreasing sequence, we have:
\[
z_n = \sup_{p \geq n} x_p = \sup_{p \geq 0} x_p, \forall n \geq 0.
\]
Why? Because by increasingness,
\[
\{x_p, p \geq 0\} = \{x_p, 0 \leq p \leq n - 1\} \cup \{x_p, p \geq n\}
\]
Since all the elements of \(\{x_p, 0 \leq p \leq n - 1\}\) are smaller than that of \(\{x_p, p \geq n\}\), the supremum is achieved on \(\{x_p, p \geq n\}\) and so
\[
\ell = \sup_{p \geq 0} x_p = \sup_{p \geq n} x_p = z_n
\]
Thus
\[
z_n = \ell \to \ell.
\]
We also have \(y_n = \inf \{x_p, 0 \leq p \leq n\} = x_n\) which is a non-decreasing sequence and so converges to \(\ell = \sup_{p \geq 0} x_p\).

Exercise 4.

Let \(\ell \in \overline{\mathbb{R}}\) having the indicated property. Let \(\ell'\) be a given accumulation point.
\[
(x_{n_k})_{k \geq 1} \subseteq (x_n)_{n \geq 0}\] such that \(x_{n_k} \to \ell'\).
By hypothesis this subsequence \((x_{n_k})\) has in turn a subsequence \((x_{n_{k(p)}})_{p \geq 1}\) such that \(x_{n_{k(p)}} \to \ell\) as \(p \to +\infty\).
But as a subsequence of \((x_{n_{(k)}})\),
\[
x_{n_{(k(\ell))}} \to \ell'.
\]
Thus
\[
\ell = \ell'.
\]
Applying that to the limit superior and limit inferior, we have:
\[
\lim \sup x_n = \lim \inf x_n = \ell.
\]
And so \( \lim x_n \) exists and equals \( \ell \).

**Exercise 5.**

**Question (a).** If \( \nu_{2k} \) finite and \( \nu_{2k+1} \) infinite, it then has exactly \( k \) up-crossings: \( [x_{\nu_{2j-1}}, x_{\nu_{2j}}], j = 1, \ldots, k : D(a, b) = k \).

**Question (b).** If \( \nu_{2k+1} \) finite and \( \nu_{2k+2} \) infinite, it then has exactly \( k \) up-crossings: \( [x_{\nu_{2j-1}}, x_{\nu_{2j}}], j = 1, \ldots, k : D(a, b) = k \).

**Question (c).** If all the \( \nu_j \)s are finite, then, there are an infinite number of up-crossings: \( [x_{\nu_{2j-1}}, x_{\nu_{2j}}], j \geq 1k : D(a, b) = +\infty \).

**Question (d).** Suppose that there exist \( a < b \) rationals such that \( D(a, b) = +\infty \). Then all the \( \nu_j \)s are finite. The subsequence \( x_{\nu_{2j-1}} \) is strictly below \( a \). So its limit inferior is below \( a \). This limit inferior is an accumulation point of the sequence \( (x_n)_{n \geq 1} \), so is more than \( \lim x_n \), which is below \( a \).

Similarly, the subsequence \( x_{\nu_{2j}} \) is strictly below \( b \). So the limit superior is above \( a \). This limit superior is an accumulation point of the sequence \( (x_n)_{n \geq 1} \), so it is below \( \lim x_n \), which is directly above \( b \). Which leads to:

\[
\lim x_n \leq a < b \leq \lim x_n.
\]

That implies that the limit of \( (x_n) \) does not exist. In contrary, we just proved that the limit of \( (x_n) \) exists, meanwhile for all the real numbers \( a \) and \( b \) such that \( a < b \), \( D(a, b) \) is finite.

Now, suppose that the limit of \( (x_n) \) does not exist. Then,

\[
\lim x_n < \lim x_n.
\]

We can then find two rationals \( a \) and \( b \) such that \( a < b \) and a number \( \epsilon \) such that \( 0 < \epsilon \), all the

\[
\lim x_n < a - \epsilon < a < b < b + \epsilon < \lim x_n.
\]

If \( \lim x_n < a - \epsilon \), we can return to question (a) of exercise 2 and construct a subsequence of \( (x_n) \) which tends to \( \lim x_n \) while remaining below \( a - \epsilon \). Similarly, if \( b + \epsilon < \lim x_n \), we can create a subsequence of \( (x_n) \) which tends to \( \lim x_n \) while staying above \( b + \epsilon \). It is evident with these two sequences that we could define with these two sequences all
\( \nu_j \) finite and so \( D(a, b) = +\infty \).

We have just shown by contradiction that if all the \( D(a, b) \) are finite for all rationals \( a \) and \( b \) such that \( a < b \), then, the limit of \( (x)n \) exists.

**Exercise 5.** Cauchy criterion in \( \mathbb{R} \).

Suppose that the sequence is Cauchy, i.e.,

\[
\lim_{(p,q)\to(+\infty,+\infty)} (x_p - x_q) = 0.
\]

Then let \( x_{n_{k,1}} \) and \( x_{n_{k,2}} \) be two subsequences converging respectively to \( \ell_1 = \lim x_n \) and \( \ell_2 = \overline{\lim} x_n \). So

\[
\lim_{(p,q)\to(+\infty,+\infty)} (x_{n_{p,1}} - x_{n_{q,2}}) = 0.
\]

, By first letting \( p \to +\infty \), we have

\[
\lim_{q\to+\infty} \ell_1 - x_{n_{q,2}} = 0.
\]

Which shows that \( \ell_1 \) is finite, else \( \ell_1 - x_{n_{q,2}} \) would remain infinite and would not tend to 0. By interchanging the roles of \( p \) and \( q \), we also have that \( \ell_2 \) is finite.

Finally, by letting \( q \to +\infty \), in the last equation, we obtain

\[
\ell_1 = \lim x_n = \overline{\lim} x_n = \ell_2.
\]

which proves the existence of the finite limit of the sequence \( (x_n) \).

Now suppose that the finite limit \( \ell \) of \( (x_n) \) exists. Then

\[
\lim_{(p,q)\to(+\infty,+\infty)} (x_p - x_q) = \ell - \ell = 0.
\]

Which shows that the sequence is Cauchy.
2. Topology and measure theory complements

In this section, we recall some relations on measurability of real-valued applications defined on intervals of $\mathbb{R}$ and their continuity. We begin by the seminal Egoroff result.

**Theorem 11.** (Egoroff’s Theorem) Let $E$ be a Borel subset of $\mathbb{R}$ such that $0 < \lambda(E) < +\infty$, where $\lambda$ is the Lebesgue measure on $\mathbb{R}$. Let $f_n : E \rightarrow \mathbb{R}$ be a sequence of finite and measurable functions with respect to the Borel $\sigma$-algebras $\mathcal{B}(E)$ and $\mathcal{B}(\mathbb{R})$.

Suppose that this sequence converges a.s to a finite function $f$. Let $\varepsilon > 0$ an arbitrary positive real number. Then, there exists a measurable subset $A$ of $E$ such that $\lambda(E \setminus A) < \varepsilon$ and $f_n$ converge uniformly to $f$ on $A$.

**Proof of Theorem 11.** By definition, $f_n$ converges to $f$ a.s if and only if

$$\lambda(\{x \in E, f_n(x) \text{ does not converges to } f(x)\}) = 0.$$ 

Since the $f_n$’s and $f$ are finite, we have

$$\{x \in E, f_n(x) \rightarrow f(x)\} = \bigcup_{\eta>0} \bigcap_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} \{x \in E, |f_m(x) - f(x)| \geq \eta\},$$

$\rightarrow$ stands for : does not converge to. This can be achieved on real numbers $\eta = 1/k, k \geq 1$ so that

$$\{x \in E, f_n(x) \rightarrow f(x)\} = \bigcap_{k>0} \bigcup_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} \{x \in E, |f_m(x) - f(x)| \geq 1/k\}.$$ 

So $f_n$ converges to $f$ a.s if and only if for any $k \geq 1$,

$$\lambda(\bigcap_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} \{x \in E, |f_m(x) - f(x)| \geq 1/k\}) = 0.$$ 

Put for any $k \geq 1$ and $n \geq 1$,

$$A_{k,n} = \bigcup_{m=n}^{+\infty} \{x \in E, |f_m(x) - f - x| \geq 1/k\}.$$ 

For a fixed $k \geq 1$, $(A_{k,n})_{n \geq 1}$ is a nondecreasing sequence (in $n$) of measurable sets with finite measures (their measures are bounded above by $\lambda(E)$) with limit
\[ A_k = \bigcap_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} \{ x \in E, |f_m(x) - f(x)| \geq 1/k \}. \]

Then, we have
\[(2.1) \quad \lambda(A_{k,n}) \downarrow \lambda(A_k) = 0.\]

Now let \( \varepsilon > 0 \) be fixed. For each fixed \( k \geq 1 \), we can use 2.1 and find an indice \( n(k) \) such that
\[ \lambda(A_{k,n(k)}) < 2^{-k} \varepsilon/2. \]

Set
\[ A = \bigcap_{k \geq 1} A_{k,n(k)}^c, \]
where the complements are taken in \( E \). Then we have
\[ E \setminus A = \bigcup_{k \geq 1} A_{k,n(k)}. \]

We surely have \( A \subset E \) and by the \( \sigma \)-subadditivity of the measure \( \lambda \), we see that
\[ \lambda(E \setminus A) = \lambda\left( \bigcup_{k \geq 1} A_{k,n(k)} \right) \leq \sum_{k=1}^{+\infty} \lambda(A_{k,n(k)}) \leq \varepsilon/2 < \varepsilon. \]

To finish, let us show that \( f_n \) converges to \( f \) uniformly on \( A \). Let \( \eta > 0 \) and consider a value \( k \geq 1 \) such that \( 1/k_0 < \eta \) and set \( N = n(k_0) \). Now any \( x \in A \) belongs to any of the \( A_{k,n(k)} \). Then for any \( x \in A, x \) belongs to \( A_{k_0,n(k_0)} \), and by this, we have
\[ m \geq N = n(k_0) \implies |f_m(x) - f(x)| < 1/k_0 < \eta. \]

We just proved this:
\[ \forall \eta > 0, \exists N \geq 0, \sup_{x \in A_{n \geq N}} |f_m(x) - f(x)| \leq 1/k_0 < \eta. \]

We conclude that \( f_n \) converges to \( f \) uniformly on \( A \).

We are moving now to the Lusin Theorem that will bring us back to outer measures.

**Theorem 12. (Lusin’s Theorem)** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a finite and measurable function with respect to the Borel \( \sigma \)-algebras \( B(\mathbb{R}) \) and \( \lambda \) be the Lebesgues measure on \( \mathbb{R} \). Let \( \varepsilon > 0 \) be an arbitrary positive real
number. Then, there exists a measurable set $E$ such that $\lambda(A^c) < \varepsilon$ and $f$ is continuous on $A$.

To prove the Lusin’s Theorem, we need the following. The first lemma will oblige us to go back the proof of Theorem of Caratheodory.

**Lemma 23.** Let be a Borel subset $A$ of $\mathbb{R}$ such that its Lebesgues measure is finite, that is $\lambda(E) < +\infty$. Then, for any $\varepsilon > 0$, there exists a finite union $K$ of bounded intervals such that $\lambda(E \Delta K) < \varepsilon$.

**Proof of Lemma 23.** We have to go back to the proof of the Theorem of Caratheodory which justified the existence which the Lebesgues measure.

Recall that the Lebesgues measure is uniquely defined on the class of intervals

$$\mathcal{I} = \{ [a, b], -\infty \leq a \leq b < +\infty \}$$

by

$$\lambda([a, b]) = b - a.$$  

This class $\mathcal{I}$ is a semi-algebra and the algebra $\mathcal{C}$ its generates is the class of finite sums of disjoints intervals. It is easily proved that $\lambda$ is additive on $\mathcal{I}$ and is readily extensible to a an additive application on $\mathcal{C}$, that we always denote by $\lambda$. The broad extension of $\lambda$ to a measure on a $\sigma$-algebra $\mathcal{A}^0$ including $\mathcal{C}$ may be done by the method of the outer measure, defined as follows, for any subset $A$ of $\mathbb{R}$:

$$\lambda^0(A) = \inf \left\{ \sum_{n=0}^{\infty} \lambda(A_n), A \subset \bigcup_{n=0}^{\infty} A_n, A_n \in \mathcal{C} \right\}.$$  

A subset of $\mathbb{R}$ is $\lambda^0$-mesurable if and only for any subset $D$ of $\mathbb{R}$, we have

$$\lambda^0(A) = \lambda^0(AD) + \lambda^0(AD^c).$$

By denoting $\mathcal{A}^0$ the set of $\lambda^0$-measurable subsets of $\mathbb{R}$, is proved that $(\Omega, \mathcal{A}^0, \lambda^0)$ is a measurable space and $\mathcal{C} \subset \mathcal{A}^0$. This measurable space, surely, includes the measurable space $(\Omega, \mathcal{B}(\mathbb{R}), \lambda^0)$ since $\mathcal{B}(R) = \sigma(\mathcal{C}) \subset \mathcal{A}^0$. The measure $\lambda^0$ is the unique extension of $\lambda$ to a measure on $\mathcal{B}(\mathbb{R})$, still denoted by $\lambda$.

By using 2.2 and the characterization of the infimum based on the fact that $\lambda(E)$ is finite, we conclude that for any $\varepsilon > 0$, there exists a union
\( \bigcup_{n \geq 1} A_n \), formed by elements \( A_n \) in \( C \) and covering \( E \), such that we have

\[
(2.3) \quad \sum_{n=0}^{\infty} \lambda(A_n) < \lambda(E) + \varepsilon/2.
\]

But for each \( n \geq 1 \), \( A_n \) is a finite sum of intervals - elements of \( \mathcal{I} \) - of the form

\[
A_n = \sum_{1 \leq j \leq p(n)} I_{n,j},
\]

with

\[
\lambda(A_n) = \sum_{1 \leq j \leq p(n)} \lambda(I_{n,j}),
\]

since \( \lambda \) is additive on \( C \). We may rephrase this by saying that for any \( \varepsilon > 0 \), there exists a union of intervals \( I_k \), covering \( E \), such that

\[
\sum_{k=1}^{\infty} \lambda(I_k) < \lambda(E) + \varepsilon/2.
\]

Here, we see that none of these intervals is unbounded. Otherwise, for one them, we would have \( \lambda(I_k) = +\infty \) and Formula (2.3) would be impossible. Since \( \sum_{k=1}^{\infty} \lambda(I_k) \) is finite, we may find \( k_0 \) such that

\[
(2.4) \quad 0 \leq \sum_{k=1}^{\infty} \lambda(I_k) - \sum_{k=1}^{k_0} \lambda(I_k) = \sum_{k=k_0+1}^{\infty} \lambda(I_k) < \varepsilon/2.
\]

Put \( K = \bigcup_{1 \leq j \leq k_0} I_j \). We finally have

\[
K \setminus E = KE^c \subset \left( \bigcup_{k=0}^{\infty} I_k \right)^c E^c = \left( \bigcup_{k=0}^{\infty} I_k \right) \setminus E
\]

with

\[
E \subset \left( \bigcup_{k=0}^{\infty} I_k \right) = \left( \bigcup_{n=0}^{\infty} A_n \right).
\]

Then, by 2.2, we have

\[
\lambda(K \setminus E) \leq \lambda \left( \bigcup_{k=0}^{\infty} I_k \right) - \lambda(E) \leq \left( \sum_{k=0}^{\infty} \lambda(A_k) \right) - \lambda(E) < \varepsilon/2.
\]

Next, by denoting \( J = \bigcup_{k=0}^{\infty} I_k \), we have \( E \subset J \) and

\[
K^c = \left( \bigcup_{k=0}^{\infty} I_k \setminus \bigcup_{k=0}^{k_0} I_k \right) + J^c
\]
and then
\[ E \setminus K = EK^c = E \left( \bigcup_{k=0}^{\infty} I_k \setminus \bigcup_{k=0}^{k_0} I_k \right) + EJ^c = E \left( \bigcup_{k=0}^{\infty} I_k \setminus \bigcup_{k=0}^{k_0} I_k \right) \subset \bigcup_{k_0+1}^{\infty} I_k \]
and
\[ \lambda(E \setminus K) \leq \lambda \left( \bigcup_{k_0+1}^{\infty} I_k \right) \leq \sum_{k \geq k_0+1} \lambda(I_k) < \varepsilon/2. \]

We conclude that
\[ \lambda(E \Delta K) < \varepsilon. \]

We are now able to prove Lusin’s Theorem.

**Proof of Theorem 12.**

Let \( \varepsilon > 0 \). Fix an arbitrary \( N \in \mathbb{Z} \) and put \( A_N = (N, N+1) \). Since \( \lambda(A_N) = 1 < \infty \), Egoroff’s theorem ensures we can find \( C_N \subset A_N \) such that such that \( \lambda(A_N \Delta C_N) < 2^{N+1} \varepsilon \) and
\[ f_n 1_{C_N} \to 1_{C_N} \]
uniformly. It is important that anything happens in the sets \( A_N \) in the sequel. In particular, the complements are meant within the sets \( A_N \). Since \( f_n 1_{C_N} \) is measurable, there exists a sequence of simple functions
\[ f_n 1_{C_N} = \sum_{j=1}^{m(n)} \alpha_{j,n} 1_{A_{n,j}} \]
such that \( f_n 1_{C_N} \to f_n 1_{C_N} \). We consider in (2.5) canonical simple functions, i.e., we have \( A_{n,1} + ... + A_{n,m(n)} = C_N \), and the values \( \alpha_{j,n} \) are finite and distinct between them.

By using Lemma 23, we may find for any \( n \geq 1 \) and for each \( 1 \leq j \leq 1 \), a finite union of closed bounded intervals \( K_{n,j} \subset A_N \) such that
\[ \lambda(A_{n,j} \Delta K_{n,j}) < 2^{N+n+1} \varepsilon/n. \]

Put
\[ \tilde{f}_n = \sum_{j=1}^{m(n)} \alpha_{j,n} 1_{K_{n,j}}. \]
Remark that
\[
\{ x \in C_N, \tilde{f}_n(x) = f_n(x) \} = \sum_{j=1}^{m(n)} \{ x \in C_N, \tilde{f}_n(x) = f_n(x) \} \cap A_{n,j}
\]
\[
= \sum_{j=1}^{m(n)} \{ x \in C_N, x \in A_{n,j} \cap K_{n,j} \} \cap A_{n,j}
\]
So, we have
\[
\{ x \in C_N, \tilde{f}_n(x) \neq f_n(x) \} \cap A_{n,j} = A_{n,j} \cap (A_{n,j} \cap K_{n,j})^c = A_{n,j} \setminus (A_{n,j} \cap K_{n,j}).
\]
By summing over \( j \), we get
\[
\{ x \in C_N, \tilde{f}_n(x) \neq f_n(x) \} = \bigcup_{j=1}^{m(n)} (A_{n,j} \setminus (A_{n,j} \cap K_{n,j})).
\]
So, we have on \( \tilde{f}_n = f_n \):
\[
B_N = \bigcap_{n=1}^{\infty} \bigcap_{j=1}^{m(n)} \{ (A_{n,j} \setminus (A_{n,j} \cap K_{n,j}))^c \} \cap C_N^c.
\]
with
\[
\lambda(B_N^c) \leq \sum_{n=1}^{\infty} \sum_{j=1}^{m(n)} \lambda((A_{n,j} \setminus (A_{n,j} \cap K_{n,j})) \cup C_N
\]
\[
\leq \sum_{n=1}^{\infty} \sum_{j=1}^{m(n)} \lambda((A_{n,j} \cap K_{n,j}) + \lambda(C_N
\]
\[
\leq 2^{N+1} \varepsilon + 2^{N+1} \varepsilon
\]
\[
= 2^N \varepsilon.
\]
We extend \( \tilde{f}_n \) to
\[
\left( \sum_{j=1}^{m(n)} K_{n,j} \right)^c
\]
by assigning the value zero to points outside of
\[
K_n = \sum_{j=1}^{m(n)} K_{n,j}.
\]
Next, each $K_{n,j}$ can be put into a sum of bounded intervals. Indeed, we have
\[ K_{n,j} = \bigcup_{1 \leq \ell \leq p(n,j)} K_{n,j,\ell} \]
\[ = K_{n,j,1} + K_{n,j,1}^c + \ldots + K_{n,j,2}^c \ldots K_{n,j,p-1(n,j)}^c \ldots K_{n,j,p(n,j)}. \]
The claim is true since each $K_{n,j,\ell}^c$ is also a finite sum of intervals and the class of finite sums of intervals is stable by intersection. Finally, $\tilde{f}_n$ may be expressed in the form
\[ \tilde{f}_n = \sum_{j=1}^{q(n)} \beta_{j,n} 1_{B_{n,j}} \]
where the $B_{n,j}$ are bounded intervals. The discontinuity points of the functions $\tilde{f}_n$ consists of some of the possibly closing ends of the bounded intervals $B_{n,j}$. The set $D$ of all discontinuity points of the $\tilde{f}_n$’s, $n \geq 1$, is atmost countable. And, we have on $B_N \cap D^c$, that the functions are continuous and converge uniformly to $f$ whereas
\[ \lambda((B_N \cap D^c)^c) \leq 2^{-N_\epsilon}. \]
We conclude that $f$ is continuous on $B_N^* = B_N \cap D^c$. Now
\[ B = \left( \bigcup_{N=1}^{\infty} B_N^* \right) \cap \mathbb{Z}. \]
Then $f$ is continuous on $B$ with
\[ \lambda(B) \leq \epsilon. \]
3. Hamel Equations

Lemma 24. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function such that
$$\forall (x, y) \in \mathbb{R}^2, h(x + y) = h(x) + h(y).$$
Then, we have
$$\forall x \in \mathbb{R}, h(x) = xh(1).$$

Proof of Lemma 3. We begin to prove that $h(0) = 0$. We have
$$h(0) = h(0 + 0) = h(0) + h(0) = 2h(0),$$
which implies that $h(0) = 0$. Next for any $x \in \mathbb{R}$
$$0 = h(0) = h(x + (-x)) = h(x) + h(-x)$$
so that for $x \in \mathbb{R}$
$$h(-x) = -h(x).$$
Based on this, we may and do concentrate on positive values of $x$. We may easily see, for $0 < p \in \mathbb{Z}$ and $0 < q \in \mathbb{Z}$, that
$$h(p) = h(1 + \ldots + 1) = ph(1)$$
and
$$h(1) = h((1/q) + \ldots + (1/q)) = qh(1/q)$$
so that
$$h(1/q) = 1/qh(1).$$
Then, for any nonnegative rational $x = p/q$, we have
$$h(p/q) = h(1/q + \ldots + 1/q) = ph(1/q) = (p/q)h(1).$$
Now, let $x$ be any nonnegative real number. There exists a sequence of nonnegative rational number $x_n > 0$ such that $x_n$ decreases to $x$ and a sequence of nonnegative rational number $y_n > 0$ such that $x_n$ increases.

If $h$ is right-continuous, we have
$$h(x_n) = x_nh(1) \rightarrow xh(1) \text{ and } h(x_n) \rightarrow h(x).$$
Now, if $h$ is left-continuous, we have
$$h(x_n) = y_nh(1) \rightarrow xh(1) \text{ and } h(y_n) \rightarrow h(x).$$
In both cases, since \( h(0) = 0 \), we have for any \( x \geq 0 \),
\[
h(x) = x h(1).
\]
For \( x < 0 \),
\[
h(x) = h(-(-x)) = -h(-x) = -x(-x h(1)) = x h(1).
\]
We are going to give more general solutions.

We have the first general result.

**Lemma 25.** Let \( h : \mathbb{R} \to \mathbb{R} \) be a function such that
\[
\forall (x, y) \in \mathbb{R}^2, h(x + y) = h(x) + h(y).
\]
If \( h \) is monotone or \( h \) is bounded in a neighborhood of zero, or if \( h \) is right-continuous or left-continuous at one point, then we have
\[
\forall x \in \mathbb{R}, h(x) = x h(1).
\]
This conclusion is obtained if \( h \) is only bounded in some right or left neighborhood of zero.

**Proof of lemma 25.** In the proof the previous lemma, we proved that if \( 3.1 \) holds, then we have any rational number \( r \),
\[
h(r) = r h(1).
\]
(a) Suppose that \( h \) is nondecreasing. For any \( x \in \mathbb{R} \), for any \( \delta > 0 \) there exist two rational numbers \( r_1 = r_1(\delta) \) and \( r_2 = r_2(\delta) \) such that
\[
r_1 < x < r_2 \text{ and } r_2 - r_1 < \delta.
\]
We have
\[
h(r_1) \leq h(x) \leq h(r_2) \text{ and } h(r_i) = r_i h(1).
\]
Then
\[
0 \leq h(r_2) - h(x) \leq h(r_2) - h(r_1),
\]
that is
\[
0 \leq r_2 h(1) - h(x) \leq \delta.
\]
As \( \delta \downarrow 0 \), \( r_2 = r_2(\delta \downarrow x) \) and then
\[
h(x) = x h(1).
\]
Formula \( 3.2 \) is proved for \( h \) non-decreasing. If \( h \) is non-increasing, then its opposite \( -h \) is non-decreasing and satisfies \( 3.1 \). Then \( 3.2 \) holds for \( -h \), and this formula is the same for \( h \) and for \( -h \). The proof
(b) Suppose that $h$ is bounded in some neighborhood of zero, say in $D = \{x, |x| \leq \delta_0\}$, $\delta_0 > 0$. Set, for $0 < \delta < \delta_0$,

$$A(\delta) = \sup_{|x| \leq \delta} |h(x)|.$$  

Fix $\delta$ such that $0 < \delta < \delta_0$. Then for any $n \geq 1$, for any $x$ such that $|x| \leq \delta/n$, we have

$$h(nx) = nh(x)$$

and then

$$|h(x)| = \left| \frac{h(nx)}{n} \right| \leq \frac{A(\delta)}{n}.$$

Now let $x$ be an arbitrary real number. For each $n \geq 1$, let $r_n$ a rational number such that

$$|x - r_n| \leq \frac{\delta}{n}.$$  

We have

$$|h(x) - h(1)x| = |h(x) - h(1)\{r + (x - r)\}|$$

$$= |h(x) - h(1)r + h(1)(x - r)|$$

$$= |h(x) - h(r) + (x - r)h(1)| \quad \text{(since } h(r) = h(1)r \text{)}$$

$$= |h(x - r) + (x - r)h(1)| \quad \text{(By Assumption 3.1 )}$$

$$\leq \frac{A(\delta)}{n} + |h(1)|\frac{\delta}{n}. \quad \text{(By (3.3) and (3.3)}$$

By Letting $n \to +\infty$, we get

$$h(x) = h(1)x.$$

(c) The proofs of (25) for a function $h$ satisfying (3.1) which is left-continuous or right-continuous at some point are very similar. We only give the proof for one case. Let $h$ be right-continuous at $x_0$. Let $x$ an arbitrary real number. For any sequence of real numbers $0 \leq r_n \downarrow 0$, as $n \uparrow +\infty$, we have

$$h(x_0 - x + r_n) = h(x_0 + r_n) + h(-x) \to h(x_0) + h(-x) = h(x_0 - x).$$
Then $h$ is right-continuous at all points $x_0 - x$, then at any point of $\mathbb{R}$. We conclude by applying Lemma through it right-continuity part.

The proof lemma (25) is complete. ■

Here are other versions of Hamel Equations.

**Corollary 1.** Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a measurable function such that
\[
\forall (s,t) \in \mathbb{R}^2_+, k(st) = k(s)k(t).
\]
Then we have
\[
\forall t \in \mathbb{R}, k(t) = t^{\log k(e)}.
\]

**Proof of Corollary 1.** Given the assumption holds, set $h(x) = \log k(e^x), x \in \mathbb{R}$. We have
\[
h(x + y) = \log k(e^{x+y}) = \log k(e^x) + \log k(e^y) = h(x) + h(y).
\]
Then by Lemma 3, we get for any $x \in \mathbb{R}$
\[
h(x) = \log k(e^x) = x \log k(e).
\]
By the transform $t = e^x$, we get for any $t \geq 0$,
\[
k(t) = t^{\log k(e)}
\]

**Corollary 2.** Let $\ell : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function such that
\[
\forall (s,t) \in \mathbb{R}^2, \ell(st) = \ell(s) + \ell(t).
\]
Then we have
\[
\forall t \in \mathbb{R}, \ell(t) = t^{\ell(e)}.
\]

**Proof of Corollary 2.** Given the assumption holds, set $h(y) = \ell(e^y), y > 0$. We have
\[
h(s + t) = \ell(e^{s+t}) = \ell(e^s) + \ell(e^t) = h(s) + h(t).
\]
Then
\[
h(y) = yh(1) = y\ell(1), y \in \mathbb{R}
\]
which implies
\[
\ell(e^y) = y\ell(1), y \in \mathbb{R}
\]
Put $x = e^y$ to get
\[
\ell(x) = \ell(e) \log x, x > 0
\]
4. Miscellaneous facts

FACT 1. For any $a \in \mathbb{R}$,

$$|e^{ia} - 1| = \sqrt{2(1 - \cos a)} \leq 2|\sin(a/2)| \leq 2|a/2|^{\delta}.$$  

This is easy for $|a/2| > 1$. Indeed for $\delta > 0$, $|a/2|^{\delta} > 0$ and

$$2|\sin(a/2)| \leq 2 \leq 2|a/2|^{\delta}$$

Now for $|a/2| > 1$, we have the expansion

$$2(1 - \cos a) = a^2 - \sum_{k=2}^{\infty} (-1)^2 \frac{a^{2k}}{(2k)!} = x^2 - 2 \sum_{k \geq 2, k \text{ even}} \frac{a^{2k}}{(2k)!} - \frac{a^{2(k+1)}}{(2(k+1))!}$$

$$= a^2 - 2x^{2(k+1)} \sum_{k \geq 2, k \text{ even}} \frac{1}{(2k)!} \left\{ \frac{1}{a^2} - \frac{1}{(2k + 1)((2k + 2)...(2k + k))} \right\}.$$ 

For each $k \geq 2$, for $|a/2| < 1$,

$$\left\{ \frac{1}{a^2} - \frac{1}{(2k + 1)((2k + 2)...(2k + k))} \right\} \geq \left\{ \frac{1}{4} - \frac{1}{(2k + 1)((2k + 2)...(2k + k))} \right\} \geq 0.$$ 

Hence

$$2(1 - \cos a) \leq a^2.$$ 

But for $|a/2|$, the function $\delta \mapsto |a/2|^\delta$ is non-increasing $\delta, 0 \leq \delta \leq 1$. Then

$$\sqrt{2(1 - \cos a)} \leq |a| = 2 |a/2|^{1} \leq 2 |a/2|^\delta.$$
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