2-TRACK ALGEBRAS AND THE ADAMS SPECTRAL SEQUENCE

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Dedicated to Ronnie Brown on the occasion of his eightieth birthday.

Abstract. In previous work of the first author and Jibladze, the $E_3$-term of the Adams spectral sequence was described as a secondary derived functor, defined via secondary chain complexes in a groupoid-enriched category. This led to computations of the $E_3$-term using the algebra of secondary cohomology operations. In work with Blanc, an analogous description was provided for all higher terms $E_r$. In this paper, we introduce 2-track algebras and tertiary chain complexes, and we show that the $E_4$-term of the Adams spectral sequence is a tertiary Ext group in this sense. This extends the work with Jibladze, while specializing the work with Blanc in a way that should be more amenable to computations.

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1. Introduction

A major problem in algebraic topology consists of computing homotopy classes of maps between spaces or spectra, notably the stable homotopy groups of spheres $\pi^S_*(S^0)$. One of the most useful tools for such computations is the Adams spectral sequence $\textbf{1}$ (and its unstable analogues $\textbf{2}$), based on ordinary mod $p$ cohomology. Given finite spectra $X$ and $Y$, Adams constructed a spectral sequence of the form:

$$E^{s,t}_2 = \text{Ext}_{H^*}^{s,t}(H^*(Y; \mathbb{F}_p), H^*(X; \mathbb{F}_p)) \Rightarrow [\Sigma^{t-s} X, Y^\wedge]$$
where $\mathfrak{A}$ is the mod $p$ Steenrod algebra, consisting of primary stable mod $p$ cohomology operations, and $Y_p^\wedge$ denotes the $p$-completion of $Y$. In particular, taking sphere spectra $X = Y = S^0$, one obtains a spectral sequence

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\mathfrak{A}}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi^S_{t-s}(S^0)_p$$

abutting to the $p$-completion of the stable homotopy groups of spheres. The differential $d_r$ is determined by $r^{th}$ order cohomology operations \cite{14}. In particular, secondary cohomology operations determine the differential $d_2$ and thus the $E_3$-term. The algebra of secondary operations was studied in \cite{2}. In \cite{3}, the first author and Jibladze developed secondary chain complexes and secondary derived functors, and showed that the Adams $E_3$-term is given by secondary Ext groups of the secondary cohomology of $X$ and $Y$. They used this in \cite{5}, along with the algebra of secondary operations, to construct an algorithm that computes the differential $d_2$.

Primary operations in mod $p$ cohomology are encoded by the homotopy category $\operatorname{Ho}(\mathcal{K})$ of the Eilenberg-MacLane mapping theory $\mathcal{K}$, consisting of finite products of Eilenberg-MacLane spectra of the form $\Sigma^{n_1}H\mathbb{F}_p \times \cdots \times \Sigma^{n_k}H\mathbb{F}_p$. More generally, the $n^{th}$ Postnikov truncation $P_n\mathcal{K}$ of the Eilenberg-MacLane mapping theory encodes operations of order up to $n+1$, which in turn determine the Adams differential $d_{n+1}$ and thus the $E_{n+2}$-term \cite{4}. However, $P_n\mathcal{K}$ contains too much information for practical purposes. In \cite{6}, the first author and Blanc extracted from $P_n\mathcal{K}$ the information needed in order to compute the Adams differential $d_{n+1}$.

The resulting algebraic-combinatorial structure is called an *algebra of left $n$-cubical balls*.

In this paper, we specialize the work of \cite{6} to the case $n = 2$. Our goal is to provide an alternate structure which encodes an algebra of left 2-cubical balls, but which is more algebraic in nature and better suited for computations. The combinatorial difficulties in an algebra of left $n$-cubical balls arise from triangulations of the sphere $S^{n-1} = \partial D^n$. In the special case $n = 2$, triangulations of the circle $S^1$ are easily described, unlike in the case $n > 2$. Our approach also extends the work in \cite{3} from secondary chain complexes to tertiary chain complexes.

**Organization and main results.** We define the notion of 2-track algebra (Definition 5.1) and show that each 2-track algebra naturally determines an algebra of left 2-cubical balls (Theorem 7.3). Building on \cite{6}, we show that higher order resolutions always exist in a 2-track algebra (Theorem 8.7). We show that a suitable 2-track algebra related to the Eilenberg-MacLane mapping theory recovers the Adams spectral sequence up to the $E_4$-term (Theorem 7.3). We show that the spectral sequence only depends on the weak equivalence class of the 2-track algebra (Theorem 7.5).

**Remark 1.1.** This last point is important in view of the strictification result for secondary cohomology operations: these can be encoded by a graded pair algebra $B_*$ over $\mathbb{Z}/p^2$ \cite[§5.5]{2}. The secondary Ext groups of the $E_3$-term turn out to be the usual Ext groups over $B_*$ \cite[Theorem 3.1.1]{5}, a key fact for computations. We conjecture that a similar strictification result holds for tertiary operations, i.e., in the case $n = 2$.

Appendix \[A\] explains why 2-track groupoids are not models for homotopy 2-types, and how to extract the underlying 2-track groupoid from a bigroupoid or a double groupoid.

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2. Cubes and tracks in a space

In this section, we fix some notation and terminology regarding cubes and groupoids.

**Definition 2.1.** Let \( X \) be a topological space.

An \( n \)-cube in \( X \) is a map \( a: I^n \to X \), where \( I = [0,1] \) is the unit interval. For example, a 0-cube in \( X \) is a point of \( X \), and a 1-cube in \( X \) is a path in \( X \).

An \( n \)-cube can be restricted to \( (n-1) \)-cubes along the \( 2n \) faces of \( I^n \). For \( 1 \leq i \leq n \), denote:

\[
d_i^0(a) = a \text{ restricted to } I \times I \times \cdots \times \{0\} \times \cdots \times I
\]

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\]

An \( n \)-track in \( X \) is a homotopy class, relative to the boundary \( \partial I^n \), of an \( n \)-cube. If \( a: I^n \to X \) is an \( n \)-cube in \( X \), denote by \( \{a\} \) the corresponding \( n \)-track in \( X \), namely the homotopy class of \( a \) rel \( \partial I^n \).

In particular, for \( n = 1 \), a 1-track \( \{a\} \) is a path homotopy class, i.e., a morphism in the fundamental groupoid of \( X \) from \( a(0) \) to \( a(1) \). Let us fix our notation regarding groupoids. In this paper, we consider only small groupoids.

**Notation 2.2.** A **groupoid** is a (small) category in which every morphism is invertible. Denote the data of a groupoid by \( G = (G_0, G_1, \delta_0, \delta_1, \text{id}_x^\square, \Box, (-)\Box) \), where:

- \( G_0 = \text{Ob}(G) \) is the set of objects of \( G \).
- \( G_1 = \text{Hom}(G) \) is the set of morphisms of \( G \). The set of morphisms from \( x \) to \( y \) is denoted \( G(x,y) \). We write \( x \in G \) and \( \text{deg}(x) = 0 \) for \( x \in G_0 \), and \( \text{deg}(x) = 1 \) for \( x \in G_1 \).
- \( \delta_0: G_1 \to G_0 \) is the source map.
- \( \delta_1: G_1 \to G_0 \) is the target map.
- \( \text{id}_x^\square: G_0 \to G_1 \) sends each object \( x \) to its corresponding identity morphism \( \text{id}_x^\square \).
- \( \Box: G_1 \times_{G_0} G_1 \to G_1 \) is composition in \( G \).
- \( f\Box: y \to x \) is the inverse of the morphism \( f: x \to y \).

Groupoids form a category \( \text{Gpd} \), where morphisms are functors between groupoids.

For any object \( x \in G_0 \), denote by \( \text{Aut}_G(x) = G(x,x) \) the automorphism group of \( x \).

Denote by \( \text{Comp}(G) = \pi_0(G) \) the components of \( G \), i.e., the set of isomorphism classes of objects \( G_0/\sim \).

Denote the fundamental groupoid of a topological space \( X \) by \( \Pi_1(X) \).

**Definition 2.3.** Let \( X \) be a pointed space, with basepoint \( 0 \in X \). The constant map \( 0: I^n \to X \) with value \( 0 \in X \) is called the **trivial** \( n \)-cube.

A **left** 1-cube or **left path** in \( X \) is a map \( a: I \to X \) satisfying \( a(1) = 0 \), that is, \( d_1^1(a) = 0 \), the trivial 0-cube. In other words, \( a \) is a path in \( X \) from a point \( a(0) \) to the basepoint \( 0 \). We denote \( \delta a = a(0) \).

A **left** 2-cube in \( X \) is a map \( \alpha: I^2 \to X \) satisfying \( \alpha(1,t) = \alpha(t,1) = 0 \) for all \( t \in I \), that is, \( d_1^1(\alpha) = d_2^1(\alpha) = 0 \), the trivial 1-cube.

More generally, a **left** \( n \)-cube in \( X \) is a map \( \alpha: I^n \to X \) satisfying \( \alpha(t_1, \ldots, t_n) = 0 \) whenever some coordinate satisfies \( t_i = 1 \). In other words, for all \( 1 \leq i \leq n \) we have \( d_i^1(\alpha) = 0 \), the trivial \((n-1)\)-cube.
A left $n$-track in $X$ is a homotopy class, relative to the boundary $\partial I^n$, of a left $n$-cube.

The equality $I^{m+n} = I^m \times I^n$ allows us to define an operation on cubes.

**Definition 2.4.** Let $\mu: X \times X' \to X''$ be a map, for example a composition map in a topologically enriched category $\mathcal{C}$. For $m, n \geq 0$, consider cubes

\begin{align*}
a &: I^m \to X \\
b &: I^n \to X'.
\end{align*}

The \textbf{⊗-composition} of $a$ and $b$ is the $(m + n)$-cube $a \otimes b$ defined as the composite

\[
a \otimes b: I^{m+n} = I^m \times I^n \xrightarrow{a \times b} X \times X' \xrightarrow{\mu} X''.
\]

For $m = n$, the \textbf{pointwise composition} of $a$ and $b$ is the $n$-cube defined as the composite

\[
ab: I^n \xrightarrow{(a \circ b)} X \times X' \xrightarrow{\mu} X''.
\]

The pointwise composition is the restriction of the $\otimes$-composition along the diagonal:

\[
I^n \xrightarrow{\Delta} I^n \times I^n \xrightarrow{a \otimes b} X''.
\]

**Remark 2.5.** For $m = n = 0$, the 0-cube $x \otimes y = xy$ is the pointwise composition, which is the composition in the underlying category. For higher dimensions, there are still relations between the $\otimes$-composition and the pointwise composition. In suggestive formulas, pointwise composition of paths is given by $(ab)(t) = a(t)b(t)$ for all $t \in I$, whereas the $\otimes$-composition of paths is the 2-cube given by $(a \otimes b)(s, t) = a(s)b(t)$.

Assume moreover that $\mu$ satisfies

\[
\mu(x, 0) = \mu(0, x') = 0
\]

for the basepoints $0 \in X, 0 \in X', 0 \in X''$. For example, $\mu$ could be the composition map in a category $\mathcal{C}$ enriched in $(\text{Top}_*, \wedge)$, the category of pointed topological spaces with the smash product as monoidal structure. If $a$ and $b$ are left cubes, then $a \otimes b$ and $ab$ are also left cubes.

### 3. 2-Track Groupoids

We now focus on left 2-tracks in a pointed space $X$, and observe that they form a groupoid. Define the groupoid $\Pi_{(2)}(X)$ with object set:

\[
\Pi_{(2)}(X)_0 = \text{set of left 1-cubes in } X
\]

and morphism set:

\[
\Pi_{(2)}(X)_1 = \text{set of left 2-tracks in } X
\]

where the source $\delta_0$ and target $\delta_1$ of a left 2-track $\alpha: I \times I \to X$ are given by restrictions

\[
\delta_0(\alpha) = d^0_1(\alpha) \\
\delta_1(\alpha) = d^0_2(\alpha)
\]
and note in particular $\delta\delta_0(\alpha) = \delta\delta_1(\alpha) = \alpha(0, 0)$. In other words, a morphism $\alpha$ from $a$ to $b$ looks like this:

\[
\begin{array}{c}
\delta a = \delta b \\
\begin{array}{c}
\alpha \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\delta a = \delta b \\
\begin{array}{c}
\alpha \\
\end{array}
\end{array}
\]

**Remark 3.1.** Up to reparametrization, a left 2-track $\alpha: a \Rightarrow b$ corresponds to a path homotopy from $a$ to $b$, which can be visualized in a globular picture:

\[
\delta a = \delta b \quad \blacklozenge \quad 0.
\]

However, the $\otimes$-composition will play an important role in this paper, which is why we adopt a cubical approach, rather than globular or simplicial.

Composition $\beta \boxtimes \alpha$ of left 2-tracks is described by the following picture:

(3.1)

\[
\begin{array}{c}
\delta a = \delta b \\
\begin{array}{c}
\alpha \\
\end{array}
\end{array}
\]

**Remark 3.2.** To make this definition precise, let $\alpha: a \Rightarrow b$ and $\beta: b \Rightarrow c$ be left 2-tracks in $X$, i.e., composable morphisms in $\Pi(2)(X)$. Choose representative maps $\tilde{\alpha}, \tilde{\beta}: I^2 \to X$. Consider the map $f_{\alpha,\beta}: [0, 1] \times [-1, 1] \to X$ pictured in (3.1). That is, define

\[
f(s, t) = \begin{cases} 
\tilde{\alpha}(s, t) & \text{if } 0 \leq t \leq 1 \\
\tilde{\beta}(-t, s) & \text{if } -1 \leq t \leq 0.
\end{cases}
\]

Now consider the reparametrization map \( w: I^2 \to [0, 1] \times [-1, 1] \) illustrated in this picture:

Explicitly, the restriction \( w|_{\partial I^2} \) to the boundary is the piecewise linear map satisfying

\[
\begin{align*}
  w(0, 0) &= (0, 0) \\
  w(0, 1) &= (0, 1) \\
  w(\frac{1}{2}, 1) &= (1, 1) \\
  w(1, 1) &= (1, 0) \\
  w(1, \frac{1}{2}) &= (1, -1) \\
  w(1, 0) &= (0, -1)
\end{align*}
\]

and \( w(x) \) is defined for points \( x \in I^2 \) in the interior as follows. Write \( x = p(0, 0) + qy \) as a unique convex combination of \((0, 0)\) and a point \( y \) on the boundary \( \partial I^2 \). Then define \( w(x) = pw(0, 0) + qw(y) = qw(y) \). Finally, the composition \( \beta \Box \alpha: a \Rightarrow c \) is \( \{f_{\alpha, \beta} \circ w\} \), the homotopy class of the composite

\[
I^2 \xrightarrow{w} [0, 1] \times [-1, 1] \xrightarrow{f_{\alpha, \beta}} X
\]

relative to the boundary \( \partial I^2 \).

In other notation, we have inclusions \( d_2^0: I^1 \hookrightarrow I^2 \) as the bottom edge \( I \times \{0\} \) and \( d_1^0: I^1 \hookrightarrow I^2 \) as the left edge \( \{0\} \times I \), our \( w \) is a map \( w: I^2 \to I^2 \cup_{I^1} I^2 \), and \( \beta \Box \alpha \) is the homotopy class of the composite

\[
I^2 \xrightarrow{w} I^2 \cup_{I^1} I^2 \xrightarrow{[\alpha \beta]} X.
\]

Given a left path \( a \) in \( X \), the identity of \( a \) in the groupoid \( \Pi_{(2)}(X) \) is the left 2-track is pictured here:
More precisely, for points \( x \in I^2 \) in the interior, write \( x = p(0,0) + qy \) as a unique convex combination of \((0,0)\) and a point \( y \) on the boundary \( \partial I^2 \). Then define \( \text{id}_a^\square(x) = a(q) \).

The inverse \( \alpha \bowtie b \Rightarrow a \) of a left 2-track \( \alpha : a \Rightarrow b \) is the homotopy class of the composite \( \alpha \circ T \), where \( T : I^2 \to I^2 \) is the map swapping the two coordinates: \( T(x,y) = (y,x) \).

**Lemma 3.3.** Given a pointed topological space \( X \), the structure described above makes \( \Pi(2)(X) \) into a groupoid, called the **groupoid of left 2-tracks** in \( X \).

**Proof.** Standard. \( \square \)

**Definition 3.4.** A groupoid \( G \) is **abelian** if the group \( \text{Aut}_G(x) \) is abelian for every object \( x \in G_0 \). The groupoid \( G \) is **strictly abelian** if it is pointed (with basepoint \( 0 \in G_0 \)), and is equipped with a family of isomorphisms

\[
\psi_x : \text{Aut}_G(x) \xrightarrow{\sim} \text{Aut}_G(0)
\]

indexed by all objects \( x \in G_0 \), such that the diagram

\[
\begin{array}{ccc}
\text{Aut}_G(y) & \xrightarrow{\varphi^f} & \text{Aut}_G(x) \\
\downarrow{\psi_y} & & \downarrow{\psi_x} \\
\text{Aut}_G(0) & & \text{Aut}_G(0)
\end{array}
\]

commutes for every map \( f : x \to y \) in \( G \), where \( \varphi^f \) denotes the “change of basepoint” isomorphism

\[
\varphi^f : \text{Aut}_G(y) \xrightarrow{\sim} \text{Aut}_G(x) \\
\alpha \mapsto \varphi^f(\alpha) = f\square\alpha\square f.
\]

**Remark 3.5.** A strictly abelian groupoid is automatically abelian. Indeed, the compatibility condition (3.2) applied to automorphisms \( f : 0 \to 0 \) implies that conjugation \( \varphi^f : \text{Aut}_G(0) \to \text{Aut}_G(0) \) is the identity.

**Definition 3.6.** A groupoid \( G \) is **pointed** if it has a chosen basepoint, i.e., an object \( 0 \in G_0 \). Here \( 0 \) is an abuse of notation: the basepoint is not assumed to be an initial object for \( G \).

The **star** of a pointed groupoid \( G \) is the set of all morphisms to the basepoint \( 0 \), denoted by:

\[
\text{Star}(G) = \{ f \in G_1 \mid \delta_0(f) = 0 \}.
\]

For a morphism \( f : x \to 0 \) in \( \text{Star}(G) \), we write \( \delta f = \delta_0 f = x \).

If \( G \) has a basepoint \( 0 \in G_0 \), then we take \( \text{id}_0^\square \in G_1 \) as basepoint for the set of morphisms \( G_1 \) and for \( \text{Star}(G) \subseteq G_1 \); we sometimes write \( 0 = \text{id}_0^\square \). Moreover, we take the component of the basepoint \( 0 \) as basepoint for \( \text{Comp}(G) \), the set of components of \( G \).

**Proposition 3.7.** \( \Pi(2)(X) \) is a strictly abelian groupoid, and it satisfies \( \text{Comp} \Pi(2)(X) \simeq \text{Star} \Pi(3)(X) \).
Proof. Let \( a \in \Pi_{(2)}(X)_0 \) be a left path in \( X \). To any automorphism \( \alpha : 0 \Rightarrow 0 \) in \( \Pi_{(2)}(X) \), one can associate the well-defined left 2-track indicated by the picture

\[
\begin{array}{c}
\text{Diagram (3.3)}
\end{array}
\]

which is a morphism \( a \Rightarrow a \). This assignment defines a map \( \text{Aut}_{\Pi_{(2)}(X)}(0) \rightarrow \text{Aut}_{\Pi_{(2)}(X)}(a) \) and is readily seen to be a group isomorphism, whose inverse we denote \( \psi_a \). One readily checks that the family \( \psi_a \) is compatible with change-of-basepoint isomorphisms.

The set \( \text{Comp}_{\Pi_{(2)}}(X) \) is the set of left paths in \( X \) quotiented by the relation of being connected by a left 2-track. The set \( \text{Star}_{\Pi_{(1)}}(X) \) is the set of left paths in \( X \) quotiented by the relation of path homotopy. But two left paths are path-homotopic if and only if they are connected by a left 2-track. \( \square \)

The bijection \( \text{Comp}_{\Pi_{(2)}}(X) \simeq \text{Star}_{\Pi_{(1)}}(X) \) is induced by taking the homotopy class of left 1-cubes. Consider the function \( q : \Pi_{(2)}(X)_0 \rightarrow \Pi_{(1)}(X)_1 \) which sends a left 1-cube to its left 1-track \( q(a) = \{a\} \). Then the image of \( q \) is \( \text{Star}_{\Pi_{(1)}}(X) \subseteq \Pi_{(1)}(X)_1 \) and \( q \) is constant on the components of \( \Pi_{(2)}(X)_0 \). We now introduce a definition based on those features of \( \Pi_{(2)}(X) \).

**Definition 3.8.** A 2-track groupoid \( G = (G_{(1)}, G_{(2)}) \) consists of:

- Pointed groupoids \( G_{(1)} \) and \( G_{(2)} \), with \( G_{(2)} \) strictly abelian.
- A pointed function \( q : G_{(2)}_0 \rightarrow \text{Star} G_{(1)} \) which is constant on the components of \( G_{(2)} \), and such that the induced function \( q : \text{Comp} G_{(2)} \xrightarrow{\approx} \text{Star} G_{(1)} \) is bijective.

We assign degrees to the following elements:

\[
\deg(x) = \begin{cases} 
0 & \text{if } x \in G_{(1)}_0 \\
1 & \text{if } x \in G_{(2)}_0 \\
2 & \text{if } x \in G_{(2)}_1 
\end{cases}
\]

and we write \( x \in G \) in each case.

A morphism of 2-track groupoids \( F : G \rightarrow G' \) consists of a pair of pointed functors

\[
F_{(1)} : G_{(1)} \rightarrow G'_{(1)} \\
F_{(2)} : G_{(2)} \rightarrow G'_{(2)}
\]

satisfying the following two conditions.
(1) \textit{(Structural isomorphisms)} For every object \(a \in G_{(2)0}\), the diagram
\[
\begin{array}{ccc}
\text{Aut}_{G_{(2)}}(a) & \xrightarrow{F_{(2)}} & \text{Aut}_{G'_{(2)}}(F_{(2)}a) \\
\downarrow \psi_a & & \downarrow \psi_{F_{(2)}a} \\
\text{Aut}_{G_{(2)}}(0) & \xrightarrow{F_{(2)}} & \text{Aut}_{G'_{(2)}}(0')
\end{array}
\]
commutes.

(2) \textit{(Quotient functions)} The diagram
\[
\begin{array}{ccc}
G_{(2)0} & \xrightarrow{F_{(2)}} & G'_{(2)0} \\
\downarrow q & & \downarrow q' \\
\text{Star} G_{(1)} & \xrightarrow{F_{(1)}} & \text{Star} G'_{(1)}
\end{array}
\]
commutes.

Let \(\mathbf{Gpd}_{(1,2)}\) denote the category of 2-track groupoids.

\textbf{Remark 3.9.} If \(\alpha : a \Rightarrow b\) is a left 2-track in a space, then the left paths \(a\) and \(b\) have the same starting point \(\delta a = \delta b\). This condition is encoded in the definition of 2-track groupoid. Indeed, if \(\alpha : a \Rightarrow b\) is a morphism in \(G_{(2)}\), then \(a, b \in G_{(2)0}\) belong to the same component of \(G_{(2)}\). Thus, we have \(q(a) = q(b) \in \text{Star} G_{(1)}\) and in particular \(\delta q(a) = \delta q(b) \in G_{(1)0}\).

\textbf{Definition 3.10.} The \textbf{fundamental 2-track groupoid} of a pointed space \(X\) is
\[
\Pi_{(1,2)}(X) := (\Pi_{(1)}(X), \Pi_{(2)}(X)).
\]
This construction defines a functor \(\Pi_{(1,2)} : \text{Top} \to \mathbf{Gpd}_{(1,2)}\).

\textbf{Remark 3.11.} The grading on \(\Pi_{(1,2)}(X)\) defined in 3.8 corresponds to the dimension of the cubes. For \(x \in \Pi_{(1,2)}(X)\), we have \(\text{deg}(x) = 0\) if \(x\) is a point in \(X\), \(\text{deg}(x) = 1\) if \(x\) is a left path in \(X\), and \(\text{deg}(x) = 2\) if \(x\) is a left 2-track in \(X\). This 2-graded set is the left 2-cubical set \(\text{Nul}_2(X)\) [6 Definition 1.9].

\textbf{Definition 3.12.} Given a 2-track groupoid \(G\), its \textbf{homotopy groups} are
\[
\pi_0 G = \text{Comp} G_{(1)} \\
\pi_1 G = \text{Aut} G_{(1)}(0) \\
\pi_2 G = \text{Aut} G_{(2)}(0).
\]
Note that \(\pi_0 G\) is a priori only a pointed set, \(\pi_1 G\) is a group, and \(\pi_2 G\) is an abelian group.

A morphism \(F : G \to G'\) of 2-track groupoids is a \textbf{weak equivalence} if it induces an isomorphism on homotopy groups.

\textbf{Remark 3.13.} Let \(X\) be a topological space with basepoint \(x_0 \in X\). Then the homotopy groups of its fundamental 2-track groupoid \(G = \Pi_{(1,2)}(X, x_0)\) are the homotopy groups of the space \(\pi_i G = \pi_i(X, x_0)\) for \(i = 0, 1, 2\).

The following two lemmas are straightforward.
Lemma 3.14. \( \text{Gpd}_{(1,2)} \) has products, given by \( G \times G' = \left( G_{(1)} \times G'_{(1)}, G_{(2)} \times G'_{(2)} \right) \), and where the structural isomorphisms

\[
\psi_{(x,x')} \colon \text{Aut}_{G(2) \times G'(2)} ((x,x')) \xrightarrow{\sim} \text{Aut}_{G(2) \times G'(2)} ((0,0'))
\]

are given by \( \psi_x \times \psi_{x'} \), and the quotient function

\[
(G \times G')_{(2)0} = G_{(2)0} \times G'_{(2)0}
\]

\[
\text{Star}(G \times G')_{(1)} = \text{Star} G_{(1)} \times \text{Star} G'_{(1)}
\]
is the product of the quotient functions for \( G \) and \( G' \).

Lemma 3.15. The fundamental 2-track groupoid preserves products:

\[
\Pi_{(1,2)}(X \times Y) \cong \Pi_{(1,2)}(X) \times \Pi_{(1,2)}(Y).
\]

4. 2-TRACKS IN A TOPOLOGICALLY ENRICHED CATEGORY

Throughout this section, let \( C \) be a category enriched in \((\text{Top}_*, \wedge)\). Explicitly:

- For any objects \( A \) and \( B \) of \( C \), there is a morphism space \( C(A, B) \) with basepoint denoted \( 0 \in C(A, B) \).
- For any objects \( A, B, \) and \( C \), there is a composition map

\[
\mu \colon C(B, C) \times C(A, B) \to C(A, C)
\]

which is associative and unital.

- Composition satisfies

\[
\mu(x, 0) = \mu(0, y) = 0
\]

for all \( x \) and \( y \).

We write \( x \in C \) if \( x \in C(A, B) \) for some objects \( A \) and \( B \). For \( x, y \in C \), we write \( xy = \mu(x, y) \) when \( x \) and \( y \) are composable, i.e., when the target of \( y \) is the source of \( x \). From now on, whenever an expression such as \( xy \) or \( x \otimes y \) appears, it is understood that \( x \) and \( y \) must be composable.

By Definition 2.4 we have the \( \otimes \)-composition \( x \otimes y \) for \( x, y \in \Pi_{(1)} C \) and \( \deg(x) + \deg(y) \leq 1 \). For \( \deg(a) = \deg(b) = 1 \), we have:

\[
ab = (a \otimes \delta_1 b) \Box (\delta_0 a \otimes b)
\]

\[
=(\delta_1 a \otimes b) \Box (a \otimes \delta_0 b).
\]

This equation holds in any category enriched in groupoids, where \( ab \) denotes the (pointwise) composition. Note that for paths \( \tilde{a} \) and \( \tilde{b} \) representing \( a \) and \( b \), the boundary of the 2-cube \( \tilde{a} \otimes \tilde{b} \) corresponds to the equation.

Conversely, the \( \otimes \)-composition in \( \Pi_{(1)} C \) is determined by the pointwise composition. For \( \deg(x) = \deg(y) = 0 \) and \( \deg(a) = 1 \), we have:

\[
\begin{align*}
x \otimes y &= xy \\
x \otimes a &= \text{id}_x^y a \\
a \otimes x &= \text{id}_x^a.
\end{align*}
\]
We now consider the 2-track groupoids $\Pi^{(1,2)}C(A, B)$ of morphism spaces in $C$, and we write $x \in \Pi^{(1,2)}C$ if $x \in \Pi^{(1,2)}C(A, B)$ for some objects $A, B$ of $C$. By Definition 2.4, composition in $C$ induces the $\otimes$-composition:

$$x \otimes y \in \Pi^{(1,2)}C$$

if $x$ and $y$ satisfy $\text{deg}(x) + \text{deg}(y) \leq 2$. For $\text{deg}(x) = \text{deg}(y) = 1$, $x$ and $y$ are left paths, hence $x \otimes y$ is well-defined. The $\otimes$-composition satisfies:

$$\text{deg}(x \otimes y) = \text{deg}(x) + \text{deg}(y).$$

The $\otimes$-composition is associative, since composition in $C$ is associative. The identity elements $1_A \in C(A, A)$ for $C$ provide identity elements $1 = 1_A \in \Pi^{(1,2)}C(A, A)$, with $\text{deg}(1_A) = 0$, and $x \otimes 1 = x = 1 \otimes x$.

Let us describe the $\otimes$-composition of left paths more explicitly. Given left paths $a$ and $b$, then $a \otimes b$ is a 2-track from $\delta_0(a \otimes b) = (\delta a) \otimes b$ to $\delta_1(a \otimes b) = a \otimes (\delta b)$, as illustrated here:

\[
\begin{array}{ccc}
0 & \otimes & 0 \\
\delta_0(a \otimes b) = (\delta a) \otimes b & \otimes & a \otimes b \\
\delta_1(a \otimes b) = a \otimes (\delta b)
\end{array}
\]

**Definition 4.1.** The 2-track algebra associated to $C$, denoted $(\Pi^{(1)}C, \Pi^{(1,2)}C, \Box, \otimes)$, consists of the following data.

- $\Pi^{(1)}C$ is the category enriched in pointed groupoids given by the fundamental groupoids $(\Pi^{(1)}C(A, B), \Box)$ of morphism spaces in $C$, along with the $\otimes$-composition, which determines (and is determined by) the composition in $\Pi^{(1)}C$.
- $\Pi^{(1,2)}C$ is given by the collection of fundamental 2-track groupoids $(\Pi^{(1,2)}C(A, B), \Box)$ together with the $\otimes$-composition $x \otimes y$ for $x, y \in \Pi^{(1,2)}C$ satisfying $\text{deg}(x) + \text{deg}(y) \leq 2$.

**Proposition 4.2.** Let $x, \alpha, \beta \in \Pi^{(1,2)}C$ with $\text{deg}(x) = 0$ and $\text{deg}(\alpha) = \text{deg}(\beta) = 2$. Then the following equations hold:

$$\begin{cases}
x \otimes (\beta \Box \alpha) = (x \otimes \beta) \Box (x \otimes \alpha) \\
(\beta \Box \alpha) \otimes x = (\beta \otimes x) \Box (\alpha \otimes x).
\end{cases}$$

**Proof.** This follows from functoriality of $\Pi^{(2)}$ applied to the composition maps $\mu(x, -) : C(A, B) \to C(A, C)$ and $\mu(-, x) : C(B, C) \to C(A, C)$.

**Proposition 4.3.** Let $c, \alpha \in \Pi^{(1,2)}C$ with $\text{deg}(c) = 1$ and $\text{deg}(\alpha) = 2$. Then the following equations hold:

$$\begin{cases}
\delta_1 \alpha \otimes c = (\alpha \otimes \delta c) \Box (\delta_0 \alpha \otimes c) \\
c \otimes \delta_0 \alpha = (c \otimes \delta_1 \alpha) \Box (\delta c \otimes \alpha).
\end{cases}$$
Proof. Write $a = \delta_0 \alpha$ and $b = \delta_1 \alpha$, i.e., $\alpha$ is a left 2-track from $a$ to $b$:

and note in particular $\delta a = \delta b$. Let $\bar{\alpha}$ be a left 2-cube that represents $\alpha$ and consider the left 3-cube $\bar{\alpha} \otimes c$:

Its boundary exhibits the equality of 2-tracks:

- top face $\square$ right face = front face
- $(\alpha \otimes \delta c) \square (a \otimes c) = b \otimes c$
- $(\alpha \otimes \delta c) \square (\delta_0 \alpha \otimes c) = \delta_1 \alpha \otimes c$. 


Likewise, for second equation, consider the left 3-cube \( c \otimes \tilde{\alpha} \):

\[
\begin{array}{c}
\begin{array}{c}
\delta c \otimes b \\
\delta c \otimes a \\
\delta c \otimes \tilde{\alpha}
\end{array}
\end{array}
\]

Its boundary exhibits the equality of 2-tracks:

\[
top \text{ face } \square \text{ right face } = \text{ front face}
\]

\[
(c \otimes b) \square (\delta c \otimes \alpha) = c \otimes a
\]

\[
(c \otimes \delta_1 \alpha) \square (\delta c \otimes \alpha) = c \otimes \delta_0 \alpha.
\]

5. 2-TRACK ALGEBRAS

We now collect the structure found in \( (\Pi(1)C, \Pi(1,2)C, \square, \otimes) \) into the following definition.

**Definition 5.1.** A **2-track algebra** \( \mathcal{A} = (\mathcal{A}_{(1)}, \mathcal{A}_{(1,2)}, \square, \otimes) \) consists of the following data.

1. A category \( \mathcal{A}_{(1)} \) enriched in pointed groupoids, with the \( \otimes \)-composition determined by Equation (4.1).
2. A collection \( \mathcal{A}_{(1,2)} \) of 2-track groupoids \((\mathcal{A}_{(1,2)}(A, B), \square)\) for all objects \( A, B \) of \( \mathcal{A}_{(1)} \), such that the first groupoid in \( \mathcal{A}_{(1,2)}(A, B) \) is equal to the pointed groupoid \( \mathcal{A}_{(1)}(A, B) \).
3. For \( x, y \in \mathcal{A}_{(1,2)} \), the \( \otimes \)-composition \( x \otimes y \in \mathcal{A}_{(1,2)} \) is defined. For \( \deg(x) = 0 \) and \( \deg(y) = 1 \), the following equations hold in \( \mathcal{A}_{(1)} \):

\[
\begin{cases}
q(x \otimes y) = x \otimes q(y) \\
q(y \otimes x) = q(y) \otimes x.
\end{cases}
\]

The following equations are required to hold.

1. **(Associativity)** \( \otimes \) is associative: \( (x \otimes y) \otimes z = x \otimes (y \otimes z) \).
2. **(Units)** The units \( 1 \in \mathcal{A}_{(1)} \), with \( \deg(1_A) = 0 \), serve as units for \( \otimes \), i.e., satisfy \( x \otimes 1 = x = 1 \otimes x \) for all \( x \in \mathcal{A}_{(1,2)} \).
3. **(Pointedness)** \( \otimes \) satisfies \( x \otimes 0 = 0 \) and \( 0 \otimes y = 0 \).
4. For \( x, y, \alpha, \beta \in \mathcal{A}_{(1,2)} \) with \( \deg(x) = \deg(y) = 0 \) and \( \deg(\alpha) = \deg(\beta) = 2 \), we have:

\[
\begin{align*}
\delta_i(x \otimes \alpha \otimes y) &= x \otimes (\delta_i \alpha) \otimes y \quad \text{for } i = 0, 1 \\
x \otimes (\beta \square \alpha) \otimes y &= (x \otimes \beta \otimes y) \square (x \otimes \alpha \otimes y)
\end{align*}
\]
(5) For \( a, b \in \mathcal{A}_{(1,2)} \) with \( \deg(a) = \deg(b) = 1 \), we have:
\[
\begin{cases}
\delta_0(a \otimes b) = \delta a \otimes b \\
\delta_1(a \otimes b) = a \otimes \delta b.
\end{cases}
\]

(6) For \( c, \alpha \in \mathcal{A}_{(1,2)} \) with \( \deg(c) = 1 \) and \( \deg(\alpha) = 2 \), we have:
\[
\begin{cases}
\delta_1 \alpha \otimes c = (\alpha \otimes \delta c) \square (\delta_0 \alpha \otimes c) \\
c \otimes \delta_0 \alpha = (c \otimes \delta_1 \alpha) \square (\delta c \otimes \alpha).
\end{cases}
\]

Definition 5.2. A morphism of 2-track algebras \( F: \mathcal{A} \to \mathcal{B} \) consists of the following.

1. A functor \( F_{(1)}: \mathcal{A}_{(1)} \to \mathcal{B}_{(1)} \) enriched in pointed groupoids.
2. A collection \( F_{(1,2)} \) of morphisms of 2-track groupoids
\[
F_{(1,2)}(A, B): \mathcal{A}_{(1,2)}(A, B) \to \mathcal{B}_{(1,2)}(FA, FB)
\]
for all objects \( A, B \) of \( \mathcal{A} \), such that \( F_{(1,2)}(A, B) \) restricted to the first groupoid in \( \mathcal{A}_{(1,2)}(A, B) \) is the functor \( F_{(1)}(A, B): \mathcal{A}_{(1)}(A, B) \to \mathcal{B}_{(1)}(FA, FB) \).
3. (Compatibility with \( \otimes \)) \( F \) commutes with \( \otimes \):
\[
F(x \otimes y) = Fx \otimes Fy.
\]

Denote by \( \text{Alg}_{(1,2)} \) the category of 2-track algebras.

Definition 5.3. Let \( \mathcal{A} \) be a 2-track algebra. The underlying homotopy category of \( \mathcal{A} \) is the homotopy category of the underlying track category \( \mathcal{A}_{(1)} \), denoted
\[
\pi_0 \mathcal{A} := \pi_0 \mathcal{A}_{(1)} = \text{Comp} \mathcal{A}_{(1)}.
\]
We say that \( \mathcal{A} \) is based on the category \( \pi_0 \mathcal{A} \).

Definition 5.4. A morphism of 2-track algebras \( F: \mathcal{A} \to \mathcal{B} \) is a weak equivalence (or Dwyer-Kan equivalence) if the following conditions hold:

1. For all objects \( A \) and \( B \) of \( \mathcal{A} \), the morphism
\[
F_{(1,2)}: \mathcal{A}_{(1,2)}(A, B) \to \mathcal{B}_{(1,2)}(FA, FB)
\]
is a weak equivalence of 2-track groupoids (Definition 3.12).
2. The induced functor \( \pi_0 F: \pi_0 \mathcal{A} \to \pi_0 \mathcal{B} \) is an equivalence of categories.

6. Higher order chain complexes

In this section, we construct tertiary chain complexes, extending the work of [3] on secondary chain complexes. We will follow the treatment therein.

Definition 6.1. A chain complex \( (A, d) \) in a pointed category \( \mathcal{A} \) is a sequence of objects and morphisms
\[
\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots
\]
in \( \mathcal{A} \) satisfying \( d_{n-1}d_n = 0 \) for all \( n \in \mathbb{Z} \). The map \( d \) is called the differential.
A chain map \( f: (A, d) \to (A', d') \) between chain complexes is a sequence of morphisms \( f_n: A_n \to A'_n \) commuting with the differentials:

\[
\cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots
\]

\[
\cdots \to A'_{n+1} \xrightarrow{d'_{n+1}} A'_n \xrightarrow{d'_n} A'_{n-1} \xrightarrow{d'_{n-1}} \cdots
\]

i.e., satisfying \( f_n d_n = d'_n f_{n+1} \) for all \( n \in \mathbb{Z} \).

**Definition 6.2.** [3, Definition 2.6] Let \( B \) be a category enriched in pointed groupoids. A secondary pre-chain complex \( (A, d, \gamma) \) in \( B \) is a diagram of the form:

\[
\cdots \to A_{n+2} \xrightarrow{d_{n+1}} A_{n+1} \xrightarrow{d_n} A_n \xrightarrow{d_{n-1}} A_{n-1} \xrightarrow{d_{n-2}} \cdots
\]

More precisely, the data consists of a sequence of objects \( A_n \) and maps \( d_n: A_{n+1} \to A_n \), together with left tracks \( \gamma_n: d_n d_{n+1} \Rightarrow 0 \) for all \( n \in \mathbb{Z} \).

\( (A, d, \gamma) \) is a secondary chain complex if moreover for each \( n \in \mathbb{Z} \), the tracks

\[
d_{n-1} d_n d_{n+1} \Rightarrow d_{n-1}0 \Rightarrow 0
\]

and

\[
d_{n-1} d_n d_{n+1} \Rightarrow 0 d_{n+1} \Rightarrow 0
\]

coincide. In other words, the track

\[
\mathcal{O}(\gamma_{n-1}, \gamma_n) := (\gamma_{n-1} \otimes d_{n+1}) \square (d_{n-1} \otimes \gamma_n) : 0 \Rightarrow 0
\]
in the groupoid \( B(A_{n+2}, A_{n-1}) \) is the identity track of 0.

We say that the secondary pre-chain complex \( (A, d, \gamma) \) is based on the chain complex \( (A, \{d\}) \) in the homotopy category \( \pi_0 B \).

**Remark 6.3.** One can show that the notion of secondary (pre-)chain complex in \( B \) coincides with the notion of 1st order (pre-)chain complex in \( \text{Nul}_1 B \) described in [3, §4, c.f. Example 12.3].

**Definition 6.4.** A tertiary pre-chain complex \( (A, d, \delta, \xi) \) in a 2-track algebra \( A \) is a sequence of objects \( A_n \) and maps \( d_n: A_{n+1} \to A_n \) in the category \( A_{(1)0} \), together with left
paths $\gamma_n \cdot d_n d_{n+1} \to 0$ in $A_{(1,2)}$, as illustrated in the diagram

\[ \xymatrix{ & 0 \ar[dr] & & \cdots \ar[r] & A_{n+3} \ar[r]^{d_{n+2}} & A_{n+2} \ar[r]^{d_{n+1}} & A_{n+1} \ar[r]^{d_{n}} & A_{n} \ar[r]^{d_{n-1}} & A_{n-1} \ar[r] & \cdots \ar[ul] } \]

along with left 2-tracks $\xi_n : \gamma_n \otimes d_{n+2} \Rightarrow d_n \otimes \gamma_{n+1}$ in $A_{(1,2)}$, for all $n \in \mathbb{Z}$.

$(A, d, \gamma, \xi)$ is a tertiary chain complex if moreover for each $n \in \mathbb{Z}$, the left 2-track:

$$d_{n-1} \otimes \gamma_n \otimes d_{n+2} \xrightarrow{d_{n-1} \otimes \xi_n} d_{n-1} d_n \otimes \gamma_n \otimes d_{n+2} \xrightarrow{\gamma_n \otimes d_{n+2}} d_{n-1} \otimes \gamma_n \otimes d_{n+2}$$

is the identity of $d_{n-1} \otimes \gamma_n \otimes d_{n+2}$ in the groupoid $A_{(2)}(A_{n+3}, A_{n-1})$. In other words, the element:

$$O(\xi_{n-1}, \xi_n) := \psi_{d_{n-1} \otimes \gamma_n \otimes d_{n+2}}((\xi_{n-1} \otimes d_{n+2}) \square (\gamma_{n-1} \otimes \gamma_{n+1}) \square (d_{n-1} \otimes \xi_n))$$

$$\in \pi_2 A_{(1,2)}(A_{n+3}, A_{n-1})$$

is trivial. Here, $\psi$ is the structural isomorphism in the 2-track groupoid $A_{(1,2)}(A_{n+3}, A_{n-1})$, as in Definitions 3.4 and 3.8.

We say that the tertiary pre-chain complex $(A, d, \gamma, \xi)$ is based on the chain complex $(A, \{d\})$ in the homotopy category $\pi_0 A$.

**Toda brackets of length 3 and 4.** Let $\mathcal{C}$ be a category enriched in $(\text{Top}, \wedge)$. Let $\pi_0 \mathcal{C}$ be the category of path components of $\mathcal{C}$ (applied to each mapping space) and let

$$Y_0 \xrightarrow{y_1} Y_1 \xrightarrow{y_2} Y_2 \xrightarrow{y_3} Y_3 \xrightarrow{y_4} Y_4$$

be a diagram in $\pi_0 \mathcal{C}$ satisfying $y_1 y_2 = 0$, $y_2 y_3 = 0$, and $y_3 y_4 = 0$. Choose maps $x_i$ in $\mathcal{C}$ representing $y_i$. Then there exist left 1-cubes $a$, $b$, $c$ as in the diagram

\[ \xymatrix{ & 0 \ar[dr] & & 0 \ar[dr] & & \cdots \ar[r] & Y_0 \ar[r]^{x_1} & Y_1 \ar[r]^{x_2} & Y_2 \ar[r]^{x_3} & Y_3 \ar[r]^{x_4} & Y_4. \ar[ul] } \]

**Definition 6.5.** The Toda bracket of length 3, denoted $\langle y_1, y_2, y_3 \rangle \subseteq \pi_1 \mathcal{C}(Y_3, Y_0)$, is the set of all elements in $\text{Aut}(0) = \pi_1 \mathcal{C}(Y_3, Y_0)$ of the form

$$O(a, b) := (a \otimes x_3) \square (x_1 \otimes b)$$

as above.

Assume now that we can choose left 2-tracks $\alpha : a \otimes x_3 \Rightarrow x_1 \otimes b$ and $\beta : b \otimes x_4 \Rightarrow x_2 \otimes c$ in $\Pi_{(1,2)} \mathcal{C}$. Then the composite of left 2-tracks

$$(\alpha \otimes x_4) \square (a \otimes c) \square (x_1 \otimes \beta)$$
is an element of \( \text{Aut}(x_1 \otimes b \otimes x_4) \), to which we apply the structural isomorphism

\[
\psi_{x_1 \otimes b \otimes x_4} : \text{Aut}(x_1 \otimes b \otimes x_4) \xrightarrow{\cong} \pi_2 \mathcal{C}(Y_4, Y_0).
\]

The set of all such elements is the Toda bracket of length 4, denoted \( \langle y_1, y_2, y_3, y_4 \rangle \subseteq \pi_2 \mathcal{C}(Y_4, Y_0) \).

Note that the existence of \( \alpha \), resp. \( \beta \), implies that the bracket \( \langle y_1, y_2, y_3 \rangle \), resp. \( \langle y_2, y_3, y_4 \rangle \) contains the zero element.

**Remark 6.6.** For a secondary pre-chain complex \((A, d, \gamma)\), we have

\[
\mathcal{O}(\gamma_{n-1}, \gamma_n) \in \langle d_{n-1}, d_n, d_{n+1} \rangle
\]

for every \( n \in \mathbb{Z} \). Likewise, for a tertiary pre-chain complex \((A, d, \gamma, \xi)\), we have

\[
\mathcal{O}(\xi_{n-1}, \xi_n) \in \langle d_{n-1}, d_n, d_{n+1}, d_{n+2} \rangle
\]

for every \( n \in \mathbb{Z} \). However, the vanishing of these Toda brackets does not guarantee the existence of a tertiary chain complex based on the chain complex \((A, \{d\})\). In a secondary chain complex \((A, d, \gamma)\), these Toda brackets vanish in a compatible way, that is, the equations \( \mathcal{O}(\gamma_{n-1}, \gamma_n) = 0 \) and \( \mathcal{O}(\gamma_n, \gamma_{n+1}) = 0 \) involve the same left track \( \gamma_n : d_n d_{n+1} \Rightarrow 0 \).

#### 7. The Adams differential \( d_3 \)

Let \( \text{Spec} \) denote the topologically enriched category of spectra and mapping spaces between them. More precisely, start from a simplicial (or topological) model category of spectra, like that of Bousfield–Friedlander [9, §2], or symmetric spectra or orthogonal spectra [13], and take \( \text{Spec} \) to be the full subcategory of fibrant-cofibrant objects; c.f. [6, Example 7.3].

Let \( H := H F_p \) be the Eilenberg-MacLane spectrum for the prime \( p \) and let \( \mathfrak{A} = H^* H \) denote the mod \( p \) Steenrod algebra. Consider the collection \( \text{EM} \) of all mod \( p \) generalized Eilenberg-MacLane spectra that are bounded below and of finite type, i.e., degree-wise finite products \( A = \prod_i \Sigma^{n_i} H \) with \( n_i \in \mathbb{Z} \) and \( n_i \geq N \) for some integer \( N \) for all \( i \). Since the product is degree-wise finite, the natural map \( \bigvee_i \Sigma^{n_i} H \to \prod_i \Sigma^{n_i} H \) is an equivalence, so that the mod \( p \) cohomology \( H^* A \) is a free \( \mathfrak{A} \)-module. Moreover, the cohomology functor restricted to the full subcategory of \( \text{Spec} \) with objects \( \text{EM} \) yields an equivalence of categories in the diagram:

\[
\begin{array}{ccc}
\pi_0 \text{Spec}^{\text{op}} & \xrightarrow{H^*} & \text{Mod}_{\mathfrak{A}} \\
\downarrow & & \downarrow \\
\pi_0 \text{EM}^{\text{op}} & \xrightarrow{H^*} & \text{Mod}_{\mathfrak{A}}^{\text{fin}}
\end{array}
\]

where \( \text{Mod}_{\mathfrak{A}}^{\text{fin}} \) denotes the full subcategory consisting of free \( \mathfrak{A} \)-modules which are bounded below and of finite type.

Given spectra \( Y \) and \( X \), consider the Adams spectral sequence:

\[
E_2^{s,t} = \text{Ext}_{\mathfrak{A}}^{s,t}(H^* X, H^* Y) \Rightarrow [\Sigma^{t-s} Y, X_p^*].
\]

Assume that \( Y \) is a finite spectrum and \( X \) is a connective spectrum of finite type, i.e., \( X \) is equivalent to a CW-spectrum with finitely many cells in each dimension and no cells below a certain dimension. Then the mod \( p \) cohomology \( H^* X \) is an \( \mathfrak{A} \)-module which is bounded
below and degreewise finitely generated (as an $\mathcal{A}$-module, or equivalently, as an $\mathbb{F}_p$-vector space). Choose a free resolution of $H^*X$ as an $\mathcal{A}$-module:

$$
\cdots \longrightarrow F_2 \xrightarrow{e_1} F_1 \xrightarrow{e_0} F_0 \xrightarrow{\lambda} H^*X
$$

where each $F_i$ is a free $\mathcal{A}$-module of finite type and bounded below. This diagram can be realized as the cohomology of a diagram in the stable homotopy category $\pi_0\text{Spec}$:

$$
\cdots \longrightarrow A_2 \xrightarrow{d_1} A_1 \xrightarrow{d_0} A_0 \xrightarrow{\epsilon} A_{-1} = X
$$

with each $A_i$ in $\text{EM}$ (for $i \geq 0$) and satisfying $H^*A_i \cong F_i$. We consider this diagram as a diagram in the opposite category $\pi_0\text{Spec}^{\text{op}}$ of the form:

$$
\cdots \longrightarrow A_2 \xrightarrow{d_1} A_1 \xrightarrow{d_0} A_0 \xrightarrow{\epsilon} A_{-1} = X
$$

Since $A_\bullet \rightarrow X$ is an $\text{EM}$-resolution of $X$ in $\pi_0\text{Spec}^{\text{op}}$, there exists a tertiary chain complex $(A, d, \gamma, \xi)$ in $\Pi_{(1,2)}\text{Spec}^{\text{op}}$ based on the resolution $A_\bullet \rightarrow X$, by Theorem 8.7.

**Notation 7.1.** Given spectra $X$ and $Y$, let $\text{EM}\{X, Y\}$ denote the topologically enriched subcategory of $\text{Spec}$ consisting of all spectra in $\text{EM}$ and mapping spaces between them, along with the objects $X$ and $Y$, with the mapping spaces $\text{Spec}(X, A)$ and $\text{Spec}(Y, A)$ for all $A$ in $\text{EM}$; c.f. [3, Remark 4.3] [6, Remark 7.5]. We consider the 2-track algebra $\Pi_{(1,2)}\text{EM}\{X, Y\}^{\text{op}}$, or any 2-track algebra $A$ weakly equivalent to it. In the following construction, everything will take place within $\Pi_{(1,2)}\text{EM}\{X, Y\}^{\text{op}}$, but we will write $\Pi_{(1,2)}\text{Spec}^{\text{op}}$ for notational convenience.

Start with a class in the $E_2$-term:

$$
x \in E^{s,t}_2 = \text{Ext}_{\mathcal{A}}^{s,t}(H^*X, H^*Y) = \text{Ext}_{\mathcal{A}}^{s,0}(H^*X, \Sigma^t H^*Y)
$$

represented by a cocycle $x': F_s \rightarrow \Sigma^t H^*Y$, i.e., a map of $\mathcal{A}$-modules satisfying $x'd_s = 0$. Realize $x'$ as the cohomology of a map $x'': A_s \rightarrow \Sigma^t Y$ in $\text{Spec}^{\text{op}}$. The equation $x'd_s = 0$ means that $x''d_s$ is null-homotopic; let $\gamma: x''d_s \rightarrow 0$ be a null-homotopy. Consider the diagram in $\text{Spec}^{\text{op}}$:

$$
\cdots \longrightarrow A_{s+2} \xrightarrow{d_{s+1}} A_{s+1} \xrightarrow{d_s} A_s \xrightarrow{d_{s-1}} A_{s-1} \cdots \longrightarrow A_0 \xrightarrow{\epsilon} X
$$

Now consider the underlying secondary pre-chain complex in $\Pi_{(1)}\text{Spec}^{\text{op}}$:

$$
(7.1) \quad \cdots \longrightarrow A_{s+3} \xrightarrow{d_{s+2}} A_{s+2} \xrightarrow{d_{s+1}} A_{s+1} \xrightarrow{d_s} A_s \xrightarrow{x''} \Sigma^t Y
$$

in which the obstructions $O(\gamma_i, \gamma_{i+1})$ are trivial, for $i \geq s$. 
Theorem 7.2. The obstruction $O(\gamma, \gamma_s) \in \pi_1 \text{Spec}^{op}(A_{s+2}, \Sigma Y) = \pi_0 \text{Spec}^{op}(A_{s+2}, \Sigma^{t+1} Y)$ is a (co)cycle and does not depend on the choices, up to (co)boundaries, and thus defines an element:

$$d_2(x) \in \text{Ext}_{\mathbb{A}}^{s+2, t+1}(H^*X, H^*Y).$$

Moreover, this function

$$d_2: \text{Ext}_{\mathbb{A}}^{s,t}(H^*X, H^*Y) \to \text{Ext}_{\mathbb{A}}^{s+2, t+1}(H^*X, H^*Y)$$

is the Adams differential $d_2$.

Proof. This is [3, Theorems 4.2 and 7.3], or the case $n = 1, m = 3$ of [6, Theorem 15.11]. Here we used the natural isomorphism:

$$\text{Ext}^{i,j}_{\pi_0 \text{EM}^{op}}(H^*X, H^*Y) \cong \text{Ext}^{i,j}_{\mathbb{A}}(H^*X, H^*Y).$$

where the left-hand side is defined as in Example 8.4. Using the equivalence of categories $H^*: \pi_0 \text{EM}^{op} \cong \text{Mod}_{\mathbb{A}}^{\text{fin}}$, this natural isomorphism follows from the natural isomorphisms:

$$\pi_0 \text{Spec}^{op}(A_{s+2}, \Sigma^{t+1} Y) = \text{Hom}_{\mathbb{A}}(F_{s+2}, H^*\Sigma^{t+1} Y) = \text{Hom}_{\mathbb{A}}(F_{s+2}, \Sigma^{t+1} H^* Y).$$

Cocycles modulo coboundaries in this group are precisely $\text{Ext}_{\mathbb{A}}^{s+2, t+1}(H^*X, H^*Y)$. □

Now assume that $d_2(x) = 0$ holds, so that $x$ survives to the $E_3$-term. Since the obstruction

$$O(\gamma, \gamma_s) = (\gamma \otimes d_{s+1}) \Box (x'' \otimes \gamma_s)$$

vanishes, one can choose a left 2-track $\xi: \gamma \otimes d_{s+1} \Rightarrow x'' \otimes \gamma_s$, which makes (7.1) into a tertiary pre-chain complex in $\Pi_{(1,2)} \text{Spec}^{op}$. Since $(A, d, \gamma, \xi)$ was a tertiary chain complex to begin with, the obstructions $O(\xi_i, \xi_{i+1})$ are trivial, for $i \geq s$.

Theorem 7.3. The obstruction $O(\xi, \xi_s) \in \pi_2 \text{Spec}^{op}(A_{s+3}, \Sigma Y) = \pi_0 \text{Spec}^{op}(A_{s+3}, \Sigma^{t+2} Y)$ is a (co)cycle and does not depend on the choices up to (co)boundaries, and thus defines an element:

$$d_3(x) \in E_3^{s+3, t+2}(X, Y).$$

Moreover, this function

$$d_3: E_3^{s,t}(X, Y) \to E_3^{s+3, t+2}(X, Y)$$

is the Adams differential $d_3$.

Proof. This is the case $n = 2, m = 4$ of [6, Theorem 15.11]. More precisely, by Theorem 9.3 the tertiary chain complex $(A, d, \gamma, \xi)$ in $\Pi_{(1,2)} \text{Spec}^{op}$ yields a 2nd order chain complex in $\text{Nu}_2 \text{Spec}^{op}$ based on the same EM-resolution $A_{\bullet} \to X$ in $\pi_0 \text{Spec}^{op}$. The construction of $d_3$ above corresponds to the construction $d_3$ in [6, Definition 15.8]. □

Remark 7.4. The groups $E_3^{s,t}(X, Y)$ are an instance of the secondary Ext groups defined in [3, §4]. Likewise, the next term $E_4^{s,t}(X, Y) = \ker d_3/\im d_3$ is a higher order Ext group as defined in [6, §15].
Theorem 7.5. A weak equivalence of 2-track algebras induces an isomorphism of higher Ext groups, compatible with the differential $d_{(3)}$. More precisely, let $F: \mathcal{A} \to \mathcal{A}'$ be a weak equivalence between 2-track algebras $\mathcal{A}$ and $\mathcal{A}'$ which are weakly equivalent to $\Pi_{(1,2)}EM\{X,Y\}^{\text{op}}$. Then $F$ induces isomorphisms $E_{3,\mathcal{A}}^{s,t}(X,Y) \cong E_{3,\mathcal{A}'}^{s,t}(FX,FY)$ making the diagram

$$\egin{array}{ccc}
E_{3,\mathcal{A}}^{s,t}(X,Y) & \xrightarrow{d_{(3),\mathcal{A}}} & E_{3,\mathcal{A}}^{s+3,t+2}(X,Y) \\
\cong & & \cong \\
E_{3,\mathcal{A}'}^{s,t}(FX,FY) & \xrightarrow{d_{(3),\mathcal{A}'}} & E_{3,\mathcal{A}'}^{s+3,t+2}(FX,FY)
\end{array}$$

commute. Here the additional subscript $\mathcal{A}$ or $\mathcal{A}'$ denotes the ambient 2-track category in which the secondary Ext groups and the differential are defined.

Proof. This follows from the case $n = 2$ of [6, Theorem 15.9], or an adaptation of the proof of [3, Theorem 5.1]. \qed

8. Higher order resolutions

In this section, we specialize some results of [6] about higher order resolutions to the case $n = 2$. We use the fact that a 2-track algebra has an underlying algebra of left 2-cubical balls, which is the topic of Section 9.

First, we recall some background on relative homological algebra; more details can be found in [3, §1].

Definition 8.1. Let $\mathcal{A}$ be an additive category and $\mathfrak{a} \subseteq \mathcal{A}$ a full additive subcategory.

1. A chain complex $(A,d)$ is $\mathfrak{a}$-exact if for every object $X$ of $\mathfrak{a}$ the chain complex $\text{Hom}_\mathcal{A}(X,A)$ is an exact sequence of abelian groups.
2. A chain map $f: (A,d) \to (A',d')$ is an $\mathfrak{a}$-equivalence if for every object $X$ of $\mathfrak{a}$, the chain map $\text{Hom}_\mathcal{A}(X,f)$ is a quasi-isomorphism.
3. For an object $A$ of $\mathcal{A}$, an $A$-augmented chain complex $A_\bullet$ is a chain complex of the form

$$\cdots \to A_1 \xrightarrow{d_0} A_0 \xrightarrow{\epsilon} A \to 0 \to \cdots$$

i.e., with $A_{-1} = A$ and $A_n = 0$ for $n < -1$. Such a complex can be viewed as a chain map $\epsilon: A_\bullet \to A$ where $A$ is a chain complex concentrated in degree 0. The map $\epsilon = d_{-1}$ is called the augmentation.
4. An $\mathfrak{a}$-resolution of $A$ is an $A$-augmented chain complex $A_\bullet$ which is $\mathfrak{a}$-exact and such that for all $n \geq 0$, the object $A_n$ belongs to $\mathfrak{a}$. In other words, an $\mathfrak{a}$-resolution of $A$ is a chain complex $A_\bullet$ in $\mathfrak{a}$ together with an $\mathfrak{a}$-equivalence $\epsilon: A_\bullet \to A$.

Example 8.2. Consider the category $\mathcal{A} = \text{Mod}_R$ of $R$-modules for some ring $R$, and the subcategory $\mathfrak{a}$ of free (or projective) $R$-modules. This recovers the usual homological algebra of $R$-modules.

Definition 8.3. Let $\mathcal{A}$ be an abelian category and $F: \mathcal{A} \to \mathcal{A}$ an additive functor. The $\mathfrak{a}$-relative left derived functors of $F$ are the functors $L^n_\mathfrak{a}F: \mathcal{A} \to \mathcal{A}$ for $n \geq 0$ defined by

$$(L^n_\mathfrak{a}F)A = H_n(F(A_\bullet))$$

where $A_\bullet \to A$ is any $\mathfrak{a}$-resolution of $A$. 
Likewise, if $F: \mathbf{A}^{\text{op}} \to \mathbf{A}$ is a contravariant additive functor, its \textbf{a-relative right derived functors} of $F$ are defined by

$$(R^n_a F)A = H^n (F(A_a)) \, .$$

\textbf{Example 8.4.} The a-relative Ext groups are given by

$$\text{Ext}^n_a(A, B) := (R^n_a \text{Hom}_A(-, B))(A) = H^n \text{Hom}_A(A_{\bullet}, B).$$

\textbf{Proposition 8.5} (Correction of 1-tracks). Let $\mathbf{B}$ be a category enriched in pointed groupoids, such that its homotopy category $\pi_0 \mathbf{B}$ is additive. Let $a \subseteq \pi_0 \mathbf{B}$ be a full additive subcategory. Then there exists a secondary pre-chain complex in $\mathbf{B}$ based on an a-resolution $A_{\bullet} \to X$ of an object $X$ in $\pi_0 \mathbf{B}$. Then there exists a secondary chain complex $(A, d, \gamma')$ in $\mathbf{B}$ with the same objects $A_i$ and differentials $d_i$. In particular $(A, d, \gamma')$ is also based on the a-resolution $A_{\bullet} \to X$.

\textbf{Proof.} This follows from an adaptation of the proof of \cite[Lemma 2.14]{B}, or the case $n = 1$ of \cite[Theorem 13.2]{B}.

\textbf{Proposition 8.6} (Correction of 2-tracks). Let $\mathbf{A}$ be a 2-track algebra such that its homotopy category $\pi_0 \mathbf{A}$ is additive. Let $a \subseteq \pi_0 \mathbf{A}$ be a full additive subcategory. Then there exists a tertiary chain complex $(A, d, \gamma, \xi')$ in $\mathbf{A}$ with the same objects $A_i$, differentials $d_i$, and left paths $\gamma_i$. In particular, $(A, d, \gamma, \xi')$ is also based on the a-resolution $A_{\bullet} \to X$.

\textbf{Proof.} This follows from the case $n = 2$ of \cite[Theorem 13.2]{B}.

\textbf{Theorem 8.7} (Resolution Theorem). Let $\mathbf{A}$ be a 2-track algebra such that its homotopy category $\pi_0 \mathbf{A}$ is additive. Let $a \subseteq \pi_0 \mathbf{A}$ be a full additive subcategory. Then there exists a tertiary chain complex in $\mathbf{A}$ based on the resolution $A_{\bullet} \to X$.

\textbf{Proof.} This follows from the resolution theorems \cite[Theorems 8.2 and 14.5]{B}.

\section{9. \textbf{Algebras of left 2-cubical balls}}

\textbf{Proposition 9.1.} Every left cubical ball of dimension 2 is equivalent to $C_k$ for some $k \geq 2$, where $C_k = B_1 \cup \cdots \cup B_k$ is the left cubical ball of dimension 2 consisting of $k$ closed 2-cells going cyclically around the vertex 0, with one common 1-cell $e_i$ between successive 2-cells $B_i$ and $B_{i+1}$, where by convention $B_{k+1} := B_1$.

See Figure 3, which is taken from \cite[Figure 3]{B}.

\textbf{Proof.} Let $B$ be a left cubical ball of dimension 2. For each closed 2-cell $B_i$, equipped with its homeomorphism $h_i: I^2 \cong B_i$, the faces $\partial^i_1 B_i$ and $\partial^i_2 B_i$ are required to be 1-cells of the boundary $\partial B \cong S^1$, while the faces $\partial^i_0 B_i$ and $\partial^i_2 B_i$ are not in $\partial B$, and therefore must be faces of some other 2-cells. In other words, we have $\partial^i_0 B_i = \partial^i_1 B_j$ or $\partial^i_1 B_i = \partial^i_2 B_j$ for some other 2-cell $B_j$, in fact a unique $B_j$, because $B$ is homeomorphic to a 2-disk.

Pick any 2-cell of $B$ and call it $B_1$. Then the face $e_1 := \partial^2_1 B_1$ appears as a face of exactly one 2-cell, which we call $B_2$. The remaining face $e_2$ of $B_2$ appears as a face of exactly one other 2-cell, which we call $B_3$. Repeating this process, we list distinct 2-cells $B_1, \ldots, B_k$, and $B_{k+1}$ is one of the previously labeled 2-cells. Then $B_{k+1}$ must be $B_1$, with $e_k = \partial^2_1 B_1$, since
a 1-cell cannot appear as a common face of three 2-cells. Finally, this process exhausts all 2-cells, because all 2-cells share the common vertex 0, which has a neighborhood homeomorphic to an open 2-disk. □

**Figure 1.** The left cubical balls $C_2$, $C_3$, and $C_4$.

**Proposition 9.2.** A left 2-cubical ball ([6, Definition 10.1]) in a pointed space $X$ corresponds to a circular chain of composable left 2-tracks:

$$a = a_0 \overset{\alpha_1}{\rightarrow} a_1 \overset{\alpha_2}{\rightarrow} \cdots \overset{\alpha_k}{\rightarrow} a_{k-1} \overset{\alpha_k}{\rightarrow} a_k = a$$

where the sign $\epsilon_i = \pm 1$ is the orientation of the 2-cells in the left cubical ball ([6, Definition 10.8]). Moreover, such an expression $(\alpha_1, \ldots, \alpha_k)$ of a left 2-cubical ball is unique up to cyclic permutation of the $k$ left 2-tracks $\alpha_i$. For example, $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $(\alpha_2, \ldots, \alpha_k, \alpha_1)$ represent the same left 2-cubical ball. See Figure 2.

**Proof.** By our convention for the $\square$-composition, a left 2-track $\alpha$ defines a morphism between left paths $\alpha: d^0_i \alpha \Rightarrow d^2_i \alpha$. The gluing condition for a left 2-cubical ball $(\alpha_1, \ldots, \alpha_k)$ based on a left cubical ball $B = B_1 \cup \cdots \cup B_k$ as in Proposition 9.1 is that the restrictions $\alpha_i|_{e_i}$ and $\alpha_{i+1}|_{e_i}$ agree on the common edge $e_i \subset B_i \cap B_{i+1}$. This is the composability condition for $\alpha_{i+1} \square \alpha_i$. Indeed, up to a global sign, the sign of $B_i$ is:

$$\epsilon_i = \begin{cases} +1 & \text{if } e_i = \partial^2_i B_i \\ -1 & \text{if } e_i = \partial^0_i B_i \end{cases}$$

so that we have $\alpha_i^{\epsilon_i}: \alpha_i|_{e_i-1} \Rightarrow \alpha_i|_{e_i}$ and we may take $a_i = \alpha_i|_{e_i}$.

**Figure 2.** A left 2-cubical ball.
Theorem 9.3.  
(1) A 2-track algebra $\mathcal{A}$ yields an algebra of left 2-cubical balls ([6 Definition 11.1]) in the following way. Consider the system $\Theta(\mathcal{A}) := ((\mathcal{A}_{(1,2)}, \otimes), \pi_0\mathcal{A}, D, \mathcal{O})$, where:
- $(\mathcal{A}_{(1,2)}, \otimes)$ is the underlying 2-graded category of $\mathcal{T}$ (described in Definition 5.1).
- $\pi_0\mathcal{A}$ is the homotopy category of $\mathcal{A}$.
- $q: (A)^0 = \mathcal{A}_{(1)0} \to \pi_0\mathcal{A}$ is the canonical quotient functor.
- $D: (\pi_0\mathcal{A})^{op} \times \pi_0\mathcal{A} \to \text{Ab}$ is the functor defined by $D(A, B) = \pi_2\mathcal{A}_{(1,2)}(A, B)$.
- The obstruction operator $\mathcal{O}$ is obtained by concatenating the corresponding left 2-tracks and using the structural isomorphisms $\psi$ of the mapping 2-track groupoid:
$$\mathcal{O}_B(\alpha_1, \alpha_2, \ldots, \alpha_k) = \psi_a(\alpha^0_1 \square \cdots \square \alpha^0_i \square \cdots \square \alpha^0_k \square \alpha^0_l) \in \text{Aut}_{\mathcal{A}_{(2)}(A, B)}(0) = \pi_2\mathcal{A}_{(1,2)}(A, B)$$
where we denoted $a = \delta_0 \alpha_1 = \delta_1 \alpha_k$.
(2) Given a category $\mathcal{C}$ enriched in pointed spaces, $\Theta(\Pi_{(1,2)}\mathcal{C})$ is the algebra of left 2-cubical balls
$$(\text{Nul}_2 \mathcal{C}, \pi_0\mathcal{C}, \pi_2\mathcal{C}(-, -), \mathcal{O})$$
described in [6 §11].
(3) The construction $\Theta$ sends a tertiary pre-chain complex $(A, d, \delta, \xi)$ in $\mathcal{A}$ to a 2nd order 2-track category of $\Theta(\mathcal{A})$, in the sense of [6 Definition 11.4]. Moreover, $(A, d, \delta, \xi)$ is a tertiary chain complex if and only if the corresponding 2nd order 2-track pre-chain complex in $\Theta(\mathcal{A})$ is a 2nd order 2-track chain complex.

Proof. Let us check that the obstruction operator $\mathcal{O}$ is well-defined. By 9.2 the only ambiguity is the starting left 1-cube $a_i$ in the composition. Two such compositions are conjugate in the groupoid $\mathcal{A}_{(2)}(A, B)$:

$$\alpha_{i-1}^0 \square \cdots \square \alpha_i^0 \square \alpha_1^0 \square \cdots \square \alpha_{i+1}^0 \square \alpha_i^0$$

$$= (\alpha_{i-1}^0 \square \cdots \square \alpha_i^0) \square \alpha_k^0 \square \cdots \square \alpha_{i+1}^0 \square \alpha_i^0 \square \cdots \square \alpha_i^0 \square (\alpha_{i-1}^0 \square \cdots \square \alpha_i^0) \square$$

$$= \beta^0 \square \alpha_k^0 \square \cdots \square \alpha_i^0 \square \beta$$

with $\beta = (\alpha_{i-1}^0 \square \cdots \square \alpha_i^0) \Rightarrow: a_i \Rightarrow a_0$. Since $\mathcal{A}_{(2)}(A, B)$ is a strictly abelian groupoid, we have the commutative diagram:

$$\begin{array}{ccc}
\text{Aut}(a_0) & \xrightarrow{\psi_a} & \text{Aut}(a_i) \\
\downarrow{\psi_{a_0}} & & \downarrow{\psi_{a_i}} \\
\text{Aut}(0) & & \\
\end{array}$$

so that $\mathcal{O}_B(\alpha_1, \ldots, \alpha_k)$ is well-defined.

The remaining properties listed in [6 Definition 11.1] are straightforward verifications. \(\square\)

Appendix A. Models for Homotopy 2-types

Recall that the left $n$-cubical set $\text{Nul}_n(X)$ of a pointed space $X$ depends only on the $n$-type $P_nX$ of $X$ [6 §1]. In particular the fundamental 2-track groupoid $\Pi_{(1,2)}(X)$ depends only on the 2-type $P_2X$ of $X$. There are various algebraic models for homotopy 2-types in the literature, using 2-dimensional categorical structures. Let us mention the weak 2-groupoids of [15], the bigroupoids of [12], the double groupoids of [10], the two-typical double groupoids of [7], and the double groupoids with filling condition of [11].
In contrast, 2-track groupoids are not models for homotopy 2-types, not even of connected homotopy 2-types. In the application we are pursuing, the functor $\Pi(1,2)$ will be applied to topological abelian groups, hence products of Eilenberg-MacLane spaces. We are not trying to encode the homotopy 2-type of the Eilenberg-MacLane mapping theory, but rather as little information as needed in order to compute the Adams differential $d_3$.

The fundamental 2-track groupoid $\Pi(1,2)(X)$ encodes the 1-type of $X$, via the fundamental groupoid $\Pi(1)(X)$. Moreover, as noted in Remark 3.13 it also encodes the homotopy group $\pi_2(X)$. However, it fails to encode the $\pi_1(X)$-action on $\pi_2(X)$, as we will show below.

A.1. Connected 2-track groupoids. Recall that a category $\mathcal{C}$ is called skeletal if any isomorphic objects are equal. A skeleton of $\mathcal{C}$ is a full subcategory on a collection consisting of one representative object in each isomorphism class of objects of $\mathcal{C}$. Every groupoid is equivalent to a disjoint union of groups, that is, a coproduct of single-object groupoids. The inclusion $\text{sk}G \sim \rightarrow G$ of a skeleton of $G$ provides such an equivalence. A similar construction yields the following statement for 2-track groupoids.

**Lemma A.1.** Let $G = (G(1), G(2))$ be a 2-track groupoid.

1. There is a weak equivalence of 2-track groupoids $\text{sk}(1)G \sim \rightarrow G$ where the first groupoid of $\text{sk}(1)G$ is skeletal.

2. If $G$ is connected and $G(1)$ is skeletal, then there is a weak equivalence of 2-track groupoids $\text{sk}(2)G \sim \rightarrow G$ where both groupoids of $\text{sk}(2)G$ are skeletal.

In particular, if $G$ is connected, then $\text{sk}(2)\text{sk}(1)G \sim \rightarrow \text{sk}(1)G \sim \rightarrow G$ is a weak equivalence between $G$ and a 2-track groupoid whose constituent groupoids are both skeletal.

**Lemma A.2.** Let $G$ and $G'$ be connected 2-track groupoids whose constituent groupoids are skeletal. If there are isomorphisms of homotopy groups $\varphi_1 : \pi_1 G \simeq \pi_1 G'$ and $\varphi_2 : \pi_2 G \simeq \pi_2 G'$, then there is a weak equivalence $\varphi : G \sim \rightarrow G'$.

**Proof.** Since $G(1)$ and $G'(1)$ are skeletal, they are in fact groups, and the group isomorphism $\varphi_1$ is an isomorphism of groupoids $\varphi_1 : G(1) \sim \rightarrow G'(1)$.

Now we define a functor $\varphi_2 : G(2) \rightarrow G'(2)$. On objects, it is given by the composite

$$G_{(2)0} = \text{Comp} G_{(2)} \xrightarrow{q} \text{Star} G_{(1)} = G(1)(0,0) = \pi_1 G \xrightarrow{\varphi_1} \pi_1 G' = G'(1)(0,0) = \text{Star} G'(1) \xrightarrow{q} \text{Comp} G'_{(2)} = G'_{(2)0}$$

which is a bijection. On morphisms, $\varphi_2$ is defined as follows. We have $G_{(2)}(a,b) = \emptyset$ when $a \neq b$, so there is nothing to define then. On the automorphisms of an object $a \in G_{(2)0}$, define $\varphi_2$ as the composite

$$G_{(2)}(a,a) = \text{Aut}_{G(2)}(a) \xrightarrow{\psi_a} \text{Aut}_{G(2)}(0) = \pi_2 G \xrightarrow{\varphi_2} \pi_2 G' = \text{Aut}_{G'(2)}(0') \xrightarrow{\psi'_a} \text{Aut}_{G'(2)}(\varphi(a)) = G'_{(2)}(\varphi(a), \varphi(a)).$$
Then $\varphi(2)$ is a functor and commutes with the structural isomorphisms, by construction. Thus $\varphi = (\varphi(1), \varphi(2)) : G \to G'$ is a morphism of 2-track groupoids, and is moreover a weak equivalence.

**Corollary A.3.** Let $G$ and $G'$ be connected 2-track groupoids with isomorphic homotopy groups $\pi_i G \simeq \pi_i G'$ for $i = 1, 2$. Then $G$ and $G'$ are weakly equivalent, i.e., there is a zigzag of weak equivalences between them.

**Proof.** Consider the zigzag of weak equivalences

$$
\begin{array}{ccc}
G & \sim & G' \\
\downarrow & & \downarrow \\
\text{sk}(1)G & \sim & \text{sk}(1)G' \\
\downarrow & & \downarrow \\
\text{sk}(2)\text{sk}(1)G & \varphi & \text{sk}(2)\text{sk}(1)G'
\end{array}
$$

where the bottom morphism $\varphi$ is obtained from Lemma A.2. □

By Remark 3.13 the functor $\Pi_{(1,2)} : \text{Top}^* \to \text{Gpd}_{(1,2)}$ induces a functor

(A.1) \[ \Pi_{(1,2)} : \text{Ho} \left( \text{connected 2-Types} \right) \to \text{Ho} \left( \text{Gpd}_{(1,2)} \right) \]

where the left-hand side denotes the homotopy category of connected 2-types (localized with respect to weak homotopy equivalences), and the right-hand side denotes the localization with respect to weak equivalences, as in Definition 3.12.

**Proposition A.4.** The functor $\Pi_{(1,2)}$ in (A.1) is not an equivalence of categories.

**Proof.** Let $X$ and $Y$ be connected 2-types with isomorphic homotopy groups $\pi_1$ and $\pi_2$, but distinct $\pi_1$-actions on $\pi_2$. Then $X$ and $Y$ are not weakly equivalent, but $\Pi_{(1,2)}(X)$ and $\Pi_{(1,2)}(Y)$ are weakly equivalent, by Corollary A.3. □

A.2. **Comparison to bigroupoids.** Any algebraic model for (pointed) homotopy 2-types has an underlying 2-track groupoid. Using the globular description in Remark 3.1, the most direct comparison is to the bigroupoids of [12]. A pointed bigroupoid (resp. double groupoid) will mean one equipped with a chosen object, here denoted $x_0$ to emphasize that it is unrelated to the algebraic structure of the bigroupoid.

**Proposition A.5.** Let $\Pi_2^{\text{BiGpd}}(X)$ denote the homotopy bigroupoid of a space $X$ constructed in [12], where it was denoted $\Pi_2(X)$.

1. There is a forgetful functor $U$ from pointed bigroupoids to 2-track groupoids.
2. For a pointed space $X$, there is a natural isomorphism of 2-track groupoids $\Pi_{(1,2)}(X) \simeq U\Pi_2^{\text{BiGpd}}(X)$.

**Proof.** Let $B$ be a bigroupoid. We construct a 2-track groupoid $UB$ as follows. The first constituent groupoid of $UB$ is the underlying groupoid of $B$

$$UB_{(1)} := \pi_0 B$$
obtained by taking the components of each mapping groupoid $B(x, y)$. The second constituent groupoid of $UB$ is a coproduct of mapping groupoids

$$UB_{(2)} := \coprod_{x \in \text{Ob}(B)} B(x, x_0).$$

The quotient function $q: UB_{(2)0} \to \text{Star}UB_{(1)}$ is induced by the natural quotient maps $\text{Ob}(B(x, x_0)) \to \pi_0 B(x, x_0)$. To define the structural isomorphisms

$$\psi_a: \text{Aut}(a) \xrightarrow{\simeq} \text{Aut}(c_{x_0})$$

for objects $a \in UB_{(2)0}$, which are 1-morphisms to the basepoint $a: x \to x_0$, consider the diagram

\[
\begin{array}{ccc}
  x & \xrightarrow{\lambda} & x_0 \\
  \downarrow \text{id}^2_a & & \downarrow \alpha \\
  a & \xleftarrow{c_{x_0}} & c_{x_0}
\end{array}
\]

where $\lambda: c_{x_0} \circ a \Rightarrow a$ is the left identity coherence 2-isomorphism, $\circ$ denotes composition of 1-morphisms, and $c_{x_0}$ is the identity 1-morphism of the object $x_0$. (We kept our notation $\Box$ for composition of 2-morphisms.) The inverse $\psi_a^{-1}: \text{Aut}(c_{x_0}) \to \text{Aut}(a)$ is defined by going from top to bottom in the diagram, namely

$$\psi_a^{-1}(\alpha) = \lambda \Box (\alpha \cdot \text{id}^2_a) \Box \lambda.$$

One readily checks that $UB$ is a 2-track groupoid, that this construction $U$ is functorial, and that $U\Pi^\text{BiGpd}_2(X)$ is naturally isomorphic to $\Pi_{(1,2)}(X)$ as 2-track groupoids.

A.3. Comparison to double groupoids. The homotopy double groupoid $\rho^\square_2(X)$ from [10] is a cubical construction. Following the terminology therein, double groupoid will be shorthand for edge symmetric double groupoid with connection.

Let us recall the geometric idea behind $\rho^\square_2(X)$. A path $a: I \to X$ has an underlying semitrack $\langle a \rangle$, defined as its equivalence class with respect to thin homotopy rel $\partial I$. A semitrack $\langle a \rangle$ in turn has an underlying track $\{a\}$. A square $u: I^2 \to X$ has an underlying 2-track $\{u\}$. A 2-track $\{u\}$ in turn has an underlying equivalence class $\{u\}_T$ with respect to cubically thin homotopy, i.e., a homotopy whose restriction to the boundary $\partial I^2$ is thin (not necessarily stationary). The homotopy double groupoid $\rho^\square_2(X)$ encodes semitracks $\langle a \rangle$ in $X$ and 2-tracks $\{u\}_T$ up to cubically thin homotopy.

**Proposition A.6.** Let $\rho^\square_2(X)$ denote the homotopy double groupoid of a space $X$ constructed in [10].

1. There is a forgetful functor $U$ from pointed double groupoids to 2-track groupoids.
(2) For a pointed space $X$, there is a natural weak equivalence of 2-track groupoids
$$\Pi_{(1,2)}(X) \xrightarrow{\sim} U_{\rho_2^3}(X).$$

Proof. We adopt the notation of [10], including that compositions in a double groupoid are
written in diagrammatic order, i.e., $a + b$ denotes the composition $x \to y \to z$. However, we
keep our graphical convention for the two axes:

Let $D$ be a double groupoid, whose data is represented in the diagram of sets

![Diagram of a double groupoid](image)

along with connections $\Gamma^-, \Gamma^+: D_1 \to D_2$. Two 1-morphisms $a, b \in D_1$ with same endpoints
$\partial_1^{-}(a) = \partial_1^{-}(b) = x, \partial_1^{+}(a) = \partial_1^{+}(b) = y$ are called homotopic if there exists a 2-morphism
$u \in D_2$ satisfying $\partial_2^{-}(u) = a, \partial_2^{+}(u) = b, \partial_1^{-}(u) = \epsilon(x), \partial_2^{+}(u) = \epsilon(y)$. We write $a \sim b$ if $a$ and $b$ are homotopic.

We now define the underlying 2-track groupoid $UD$. The first constituent groupoid $UD_{(1)}$
has object set $D_0$ and morphism set $D_1/\sim$, with groupoid structure inherited from the
groupoid $(D_0, D_1)$. The second constituent groupoid $UD_{(2)}$ has object set
$$UD_{(2)} := \{a \in D_1 \mid \partial_1^{+}(a) = x_0\}.$$ 

A morphism in $UD_{(2)}$ from $a$ to $b$ is an element $u \in D_2$ satisfying $\partial_1^{-}(u) = a, \partial_2^{-}(u) = b,$
$\partial_1^{+}(u) = \epsilon(x_0), \partial_2^{+}(u) = \epsilon(x_0)$, as illustrated here:

![Diagram of a 2-track morphism](image)

Composition in $UD_{(2)}$ is defined as follows. Given 1-morphisms $a, b, c: x \to x_0$ in $D_1$ and
morphisms $u: a \Rightarrow b$ and $v: b \Rightarrow c$ in $UD_{(2)}$, their composition $v \Box u: a \Rightarrow c$ is defined by
$$v \Box u = (\Gamma^{+}(b) +_2 u) +_1 (v +_2 \circ_{x_0})$$
$$= (\Gamma^{+}(b) +_2 u) +_1 v$$
$$= (\Gamma^{+}(b) +_1 v) +_2 u$$
The identity morphisms in \( UD_2 \) are given by \( \text{id}^\Box_a = \Gamma^-(a) \). The inverse of \( u: a \Rightarrow b \) is given by

\[
 u^\Xi = \left( (-1) \Gamma^+(b) + (-1)u \right) + 1 \left( \epsilon_2(a) + 2 \Gamma^-(a) \right)
\]

\[
 = (-1) \Gamma^+(b) + 2 (-1)u + 1 \Gamma^-(a)
\]

as illustrated here:

The structural isomorphisms \( \psi_a^{-1}: \text{Aut}(\epsilon(x_0)) \rightarrow \text{Aut}(a) \) are defined by

\[
 \psi_a^{-1}(u) = \left( \Gamma^-(a) + 2 \odot x_0 \right) + 1 \left( \odot x_0 + 2 u \right)
\]

\[
 = \Gamma^-(a) + 1 u
\]

\[
 = \Gamma^-(a) + 2 u
\]

as illustrated here:
The quotient function $q: UD(2) \to \text{Star } UD(1)$ is induced by the quotient function $D_1 \to D_1/\sim$. One readily checks that $UD$ is a 2-track groupoid, and that this construction $U$ is functorial.

For a pointed space $X$, define a comparison map $\Pi_{(1,2)}(X) \to U\rho^2_2(X)$ which is an isomorphism on $\Pi_{(1)}(X)$, and which quotients out the thin homotopy relation between left paths in $X$ and the cubically thin homotopy relation between left 2-tracks. This defines a natural weak equivalence of 2-track groupoids.

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