SEQUENTIALLY COHEN-MACAULAY BIPARTITE GRAPHS: VERTEX DECOMPOSABILITY AND REGULARITY

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Abstract. Let $G$ be a bipartite graph with edge ideal $I(G)$ whose quotient ring $R/I(G)$ is sequentially Cohen-Macaulay. We prove: (1) the independence complex of $G$ must be vertex decomposable, and (2) the Castelnuovo-Mumford regularity of $R/I(G)$ can be determined from the invariants of $G$.

1. Introduction

Let $G = (V_G, E_G)$ denote a finite simple graph with vertices $V_G = \{x_1, \ldots, x_n\}$ and edge set $E_G$. By identifying the vertices with the variables in the polynomial ring $R = k[x_1, \ldots, x_n]$, we can associate to each simple graph $G$ a monomial ideal $I(G) = \langle \{x_ix_j \mid \{x_i, x_j\} \in E_G\} \rangle$. The ideal $I(G)$ is the edge ideal of $G$ and was first introduced by Villarreal [21]. Also associated to $G$ is a simplicial complex $\Delta(G)$, called the independence complex, whose faces are the independent sets of the graph $G$. That is, $F \in \Delta(G)$ if and only if the set $F$ is an independent set of vertices. The independence complex is the simplicial complex associated to $I(G)$ via the Stanley-Reisner correspondence.

We call a graph $G$ a sequentially Cohen-Macaulay graph if the corresponding ring $R/I(G)$ is sequentially Cohen-Macaulay (SCM). In this paper we consider SCM graphs $G$ that are also bipartite, that is, we can partition $V_G$ as $V_G = V_1 \cup V_2$ so that every edge $e \in E_G$ has one endpoint in $V_1$ and the other endpoint in $V_2$. SCM bipartite graphs, which includes the set of Cohen-Macaulay bipartite graphs, have been studied in [1, 5, 6, 7, 9, 12, 13, 20]. With the additional assumption that $G$ is bipartite, these papers have shown that the algebraic property of being SCM (or CM) is really a combinatorial property.

In this short note we present two new results that further highlight the fact that being bipartite and SCM is really a combinatorial property. Our first main theorem (Theorem 2.10) shows that $G$ is SCM if and only if $\Delta(G)$ is a vertex decomposable simplicial complex. The notion of a vertex decomposable simplicial complex was independently introduced to the study of edge ideals by Dochtermann and Engström [4] and Woodroofe [24]. Our result gives a new proof that $\Delta(G)$ must also be shellable (as first proved by the author and Villarreal [20]). Our second result (Theorem 3.3) is a formula for the Castelnuovo-Mumford regularity in terms of the number of 3-disjoint edges in a graph. We recover Zheng’s [25] formula for the regularity of the edge ideals of trees as corollary.

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Although not discussed directly in this paper, unmixed bipartite graphs (graphs whose minimal vertex covers all have the same cardinality) were studied in [11, 15, 16, 23]. In this situation, the algebraic properties of the ring \( R/I(G) \) are again highly connected with the invariants of the graph. Note that the intersection of the set of unmixed bipartite graphs and the set of SCM graphs is precisely the set of CM graphs. However, little appears to be known about the edge ideals of bipartite graphs that are neither unmixed or SCM; this appears to be an area that requires further exploration. In fact, at the end of the paper, we raise a question about the regularity of these ideals.

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2. SCM and Vertex Decomposable Graphs

In this section we show that sequentially Cohen-Macaulay bipartite graphs have independence complexes that are vertex decomposable. A simplicial complex \( \Delta \) on \( V = \{x_1, \ldots, x_n\} \) is a collection of subsets of \( V \) such that: (1) \( \{x_i\} \in \Delta \) for \( i = 1, \ldots, n \), and (2) if \( F \in \Delta \) and \( G \subseteq F \), then \( G \in \Delta \). Elements of \( \Delta \) are called the faces of \( \Delta \), and the maximal elements, with respect to inclusion, are called the facets. A simplicial complex is pure if all its facets have the same cardinality.

Vertex decomposability was first introduced by Provan and Billera [17] in the pure case, and extended to the non-pure case by Björner and Wachs [2, 3]. It is defined in terms of the deletion and link; if \( F \in \Delta \) is a face, then the link of \( F \) is the simplicial complex
\[
\text{lk}_\Delta(F) = \{ H \in \Delta \mid H \cap F = \emptyset \text{ and } G \cup F \in \Delta \},
\]
while the deletion of \( F \) is the simplicial complex
\[
\text{del}_\Delta(F) = \{ H \in \Delta \mid H \cap F = \emptyset \}.
\]
When \( F = \{x\} \) is a single vertex, we abuse notation and write \( \text{del}_\Delta(x) \) and \( \text{lk}_\Delta(x) \).

Definition 2.1. Let \( \Delta \) be a simplicial complex on the vertex set \( V = \{x_1, \ldots, x_n\} \). Then \( \Delta \) is vertex decomposable if either:

(i) The only facet of \( \Delta \) is \( \{x_1, \ldots, x_n\} \), i.e., \( \Delta \) is a simplex, or \( \Delta = \emptyset \).

(ii) there exists an \( x \in V \) such that \( \text{del}_\Delta(x) \) and \( \text{lk}_\Delta(x) \) are vertex decomposable, and such that every facet of \( \text{del}_\Delta(x) \) is a facet of \( \Delta \).

If \( \Delta \) is pure, we call \( \Delta \) pure vertex decomposable.

Let \( G = (V_G, E_G) \) be a graph; an independent set of \( G \) is a subset \( F \subseteq V_G \) such that \( e \not\in F \) for every \( e \in E_G \). The independence complex of \( G \), denoted \( \Delta(G) \), is the simplicial complex on \( V_G \) with face set \( \Delta(G) = \{ F \subseteq V_G \mid F \text{ is an independent set of } G \} \). Since we want to know when \( \Delta(G) \) is vertex decomposable, we introduce the terminology:

Definition 2.2. A finite simple graph \( G \) is a vertex decomposable graph (or simply, vertex decomposable) if the independence complex \( \Delta(G) \) is vertex decomposable.
To determine if a graph is vertex decomposable, we can always make the assumption that the graph is connected. The next lemma is Lemma 20 in [24]:

**Lemma 2.3.** Let $G_1$ and $G_2$ be two graphs such that $V_{G_1} \cap V_{G_2} = \emptyset$, and set $G = G_1 \cup G_2$. Then $G$ is vertex decomposable if and only if $G_1$ and $G_2$ are vertex decomposable.

For any subset $S \subseteq V_G$ in $G$, we let $G \setminus S$ denote the graph obtained by removing all the vertices of $S$ from $G$, and any edge which has at least one of its endpoints in $S$. When $S = \{x\}$ we abuse notation and write $G \setminus x$. Also, we let $N(x) := \{y \in V_G \mid \{x, y\} \in E_G\}$ be the set of neighbors of $x$. To prove our main result, we will proceed by induction; the following lemma (see [4, Lemma 4.2]) will facilitate this induction:

**Lemma 2.4.** Let $G$ be a graph, and suppose that $x, y \in V_G$ are two vertices such that $\{x\} \cup N(x) \subseteq \{y\} \cup N(y)$. If $G \setminus y$ and $G \setminus (\{y\} \cup N(y))$ are both vertex decomposable, then $G$ is vertex decomposable.

Vertex decomposability was introduced, in part, as a tool to study the shellability of a simplicial complex. We review the relevant connections.

**Definition 2.5.** A simplicial complex $\Delta$ is **shellable** if the facets of $\Delta$ can be ordered, say $F_1, \ldots, F_s$, such that for all $1 \leq i < j \leq s$, there exists some $x \in F_j \setminus F_i$ and some $\ell \in \{1, \ldots, j-1\}$ with $F_j \setminus F_\ell = \{x\}$. If $\Delta$ is pure, we call $\Delta$ **pure shellable**.

As in [20], we call a graph $G$ a **shellable graph** if $\Delta(G)$ is a shellable simplicial complex. A vertex decomposable graph is then shellable because of the following more general result (see [3, Theorem 11.3]):

**Theorem 2.6.** If $\Delta$ is a vertex decomposable simplicial complex, then $\Delta$ is also shellable.

A graded $R$-module $M$ is called **sequentially Cohen-Macaulay** (over $k$) if there exists a finite filtration of graded $R$-modules $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that each quotient $M_i/M_{i-1}$ is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing: $\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1})$. As first shown by Stanley [18], shellability implies sequentially Cohen-Macaulayness.

**Theorem 2.7.** Let $\Delta$ be a simplicial complex, and suppose that $R/I_\Delta$ is the associated Stanley-Reisner ring. If $\Delta$ is shellable, then $R/I_\Delta$ is sequentially Cohen-Macaulay.

Before coming to our main result, we need two facts from [20] about SCM graphs.

**Theorem 2.8 ([20, Theorem 3.3]).** Let $x \in V_G$ be any vertex of $G$ and set $G' = G \setminus (\{x\} \cup N(x))$. If $G$ is SCM, then $G'$ is SCM.

**Lemma 2.9 ([20, Theorem 3.7]).** Let $G$ be a bipartite graph. If $G$ is SCM, then there exists a vertex $x \in V_G$ such that $\deg x = 1$.

We can now prove the main theorem of this section:

**Theorem 2.10.** Let $G$ be a bipartite graph. Then the following are equivalent:

(i) $G$ is SCM.
(ii) $G$ is shellable.
(iii) $G$ is vertex decomposable.

Proof. Note that (iii) $⇒$ (ii) $⇒$ (i) always holds for any graph $G$ by Theorems 2.6 and 2.7. It suffices to show that when $G$ is bipartite, then (i) $⇒$ (iii).

We do a proof by induction on $n$, the number of vertices. When $n = 2$, then $G$ consists of a single edge. This graph is SCM (in fact, CM), and the independence complex is a simplex, hence vertex decomposable. We therefore suppose that $n > 2$. By Lemma 2.9, we know that there is a vertex $x$ of degree 1; let $y$ denote the unique neighbour of $x$.

Set $G_1 = G \setminus \{x \cup N(x)\}$ and $G_2 = G \setminus \{y \cup N(y)\}$. By Theorem 2.8, both of these graphs are SCM, and thus by induction, $G_1$ and $G_2$ are vertex decomposable. Let $G'$ be the graph obtained by adding the isolated vertex $x$ to $G_1$. Because $x$ is only adjacent to $y$, the graph $G'_1$ is the same as the graph $G \setminus y$. Furthermore, since $G_1$ is vertex decomposable, then so is $G'_1$ by Lemma 2.3. So $G \setminus y$ and $G \setminus \{y \cup N(y)\}$ are vertex decomposable, and because $\{x \} \cup N(x) \subseteq \{y \} \cup N(y)$, Lemma 2.4 thus implies that $G$ is vertex decomposable. □

Remark 2.11. The equivalence of (i) and (ii) was first proved in [20]. Theorem 2.10 further highlights the combinatorial nature of SCM bipartite graphs.

The equivalence of (i) and (ii) in the corollary below was first proved in [5]. In the proof (and in the next section) we require the following notion: a subset $W \subseteq V_G$ is a vertex cover if for every $e \in E_G$, we have $W \cap e \neq \emptyset$. A minimal vertex cover is any vertex cover $W$ with the property that for every $x \in W$, $W \setminus \{x\}$ is not a vertex cover.

Corollary 2.12. Let $G$ be a bipartite graph. Then the following are equivalent:

(i) $G$ is Cohen-Macaulay.
(ii) $G$ is pure shellable.
(iii) $G$ is pure vertex decomposable.

Proof. A graph $G$ is Cohen-Macaulay if and only if $G$ is SCM and $I(G)$ is unmixed, i.e., all of its associated primes have the same height (see [9, Lemma 3.6]). But the associated primes of an edge ideal correspond to the minimal vertex covers of $G$ (see [22]). The complement of a vertex cover is an independent set, i.e., a face of $\Delta(G)$. Since all the minimal vertex covers have the same cardinality, so do the facets of $\Delta(G)$, i.e., $\Delta(G)$ is pure. So, (i) implies (iii) since $G$ is SCM, and thus $G$ is vertex decomposable by Theorem 2.10 and $\Delta(G)$ is pure. The implications (iii) $⇒$ (ii) $⇒$ (i) hold for any graph. □

3. The regularity of SCM bipartite graphs

In this section we will give a formula for the Castelnuovo-Mumford regularity of $R/I(G)$ when $G$ is SCM and bipartite in terms of the graph $G$.

Associated to an $R$-module $M$ is a minimal free graded resolution of the form:

$$0 \rightarrow \bigoplus_j R(-j)^{\beta_{p,j}(M)} \rightarrow \bigoplus_j R(-j)^{\beta_{p-1,j}(M)} \rightarrow \cdots \rightarrow \bigoplus_j R(-j)^{\beta_{0,j}(M)} \rightarrow M \rightarrow 0$$
where \( p \leq n \) and \( R(-j) \) is the \( R \)-module obtained by shifting the degrees of \( R \) by \( j \). The number \( \beta_{i,j}(M) \), the \( ij \)th graded Betti number of \( M \), equals the number of generators of degree \( j \) in the \( i \)th syzygy module. The Castelnuovo-Mumford regularity (or simply regularity) of \( M \) is

\[
\text{reg}(M) := \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.
\]

The projective dimension of \( M \) is the length of the minimal free resolution, that is,

\[
\text{pd}(M) := \max\{i \mid \beta_{i,j}(M) \neq 0 \text{ for some } j\}.
\]

As in [10], we say two edges \( \{x, y\} \) and \( \{w, z\} \) of \( G \) are 3-disjoint if the induced subgraph of \( G \) on \( \{x, y, w, z\} \) consists of exactly two disjoint edges. This condition is equivalent to saying that in the complement graph \( G^c \), i.e., the graph whose edges are precisely the non-edges of \( G \), the induced graph on \( \{x, y, w, z\} \) is an induced four-cycle. Zheng [25] (and also in [15]) called edges of this type disconnected. For any graph \( G \), we let \( a(G) \) denote the maximum number of pairwise 3-disjoint edges in \( G \).

For any graph \( G \), Katzman provided the following lower bound on \( \text{reg}(R/I(G)) \).

**Lemma 3.1 ([14] Lemma 2.2).** For any graph \( G \), \( \text{reg}(R/I(G)) \geq a(G) \).

There are examples of graphs which have \( \text{reg}(R/I(G)) > a(G) \). However, as shown below, we have an equality if \( G \) is SCM and bipartite.

Given a graph \( G \), we can associate to \( G \) another monomial ideal called the cover ideal:

\[
I(G)^\vee = \langle \{x_{i_1} \cdots x_{i_r} \mid W = \{x_{i_1}, \ldots, x_{i_r}\} \text{ is a minimal vertex cover of } G \} \rangle.
\]

The notation \( I(G)^\vee \) is used because \( I(G)^\vee \) is the Alexander Dual of the edge ideal. We require a result of Terai about the Alexander Dual of a square-free monomial ideal.

**Theorem 3.2 ([19]).** Let \( I \) be an square-free monomial ideal. Then \( \text{pd}(I^\vee) = \text{reg}(R/I) \).

We now prove the main result of this section.

**Theorem 3.3.** If \( G \) is a SCM bipartite graph with \( a(G) \) pairwise 3-disjoint edges, then

\[
\text{reg}(R/I(G)) = a(G).
\]

**Proof.** By Lemma 3.1, it suffices to show that \( a(G) \) is an upper bound. By Theorem 3.2, we have \( \text{reg}(R/I(G)) = \text{pd}(I(G)^\vee) \), so it is enough to prove that \( \text{pd}(I(G)^\vee) \leq a(G) \). We proceed by induction on \( n \). If \( n = 2 \), then \( G \) is single edge \( \{x, y\} \), and \( I(G)^\vee = (x, y) \) which has \( \text{pd}(I(G)^\vee) = 1 \leq 1 = a(G) \).

Suppose that \( n > 2 \). By Lemma 2.9, there exists a vertex \( x \) of degree one. Let \( y \) be the unique neighbour of \( x \), and let \( N(y) = \{x, x_{i_2}, \ldots, x_{i_r}\} \) be the neighbours of \( y \). Observe that if \( W \) is any minimal vertex cover of \( G \), then it cannot contain both \( x \) and \( y \), because if it did, then \( W \setminus \{x\} \) would still be a vertex cover. Also, if \( y \not\in W \), then \( N(y) \subseteq V \). Set \( G' = G \setminus \{y\} \cup N(y) \) and \( G'' = G \setminus (\{x\} \cup N(x)) \). Let \( I(G')^\vee \), respectively, \( I(G'')^\vee \), denote the cover ideal of \( G' \), respectively \( G'' \), but viewed as ideals of \( R = k[x_1, \ldots, x_n] \). We need two facts:

**Claim 1.** \( I(G)^\vee = xx_{i_2} \cdots x_{i_r} I(G')^\vee + yI(G'')^\vee \).
Proof of Claim. Let $m \in I(G)^\vee$ be a generator. Then $m$ corresponds to a minimal vertex cover of $G$. It contains either $x$ or $y$, but not both. If it contains $x$, then it contains $N(y) = \{x_i, \ldots, x_t\}$. So $m = xx_i \cdots x_t m'$ where $m'$ must be a cover of $G'$. So $m \in xx_i \cdots x_t I(G')^\vee$. If $x \nmid m$, then $m = ym'$, and $m'$ must be a cover of $G''$. So $m \in yI(G'')^\vee$, thus showing one containment. The reverse direction is proved similarly; each generator on the right hand side must be a cover of $G$, and so belong to $I(G)^\vee$. \hfill \Box

Claim 2. $xx_i \cdots x_t I(G)^\vee \cap yI(G'')^\vee = yxx_i \cdots x_t I(G')^\vee$.

Proof of Claim. We have $yxx_i \cdots x_t I(G')^\vee \subseteq xx_i \cdots x_t I(G')^\vee \subseteq yxx_i \cdots x_t I(G')^\vee$, then $m = yxm'$ where $m'$ is a cover of $G''$ so it is also in the second ideal, and thus in the intersection. For the other containment, if $m$ is the intersection, then $m = xx_i \cdots x_t m'$ with $m' \in I(G')^\vee$. But since $m \in yI(G'')^\vee$, we have that $y|m$, and this implies that $m' = ym''$. So $m = yxx_i \cdots x_t m''$. But $m''$ has to be in $I(G')^\vee$ because none of its generators are divisible by $y$. \hfill \Box

As a consequence of Claims 1 and 2, we have a short exact sequence

$0 \rightarrow yxx_i \cdots x_t I(G')^\vee \rightarrow xx_i \cdots x_t I(G')^\vee \oplus yI(G'')^\vee \rightarrow I(G)^\vee \rightarrow 0$.

In particular, the short exact sequences gives the following bound on $\text{pd}(I(G)^\vee)$:

$$\text{pd}(I(G)^\vee) \leq \max\{\text{pd}(yxx_i \cdots x_t I(G')^\vee) + 1, \text{pd}(xx_i \cdots x_t I(G')^\vee), \text{pd}(yI(G'')^\vee)\}.$$  

However, for any monomial ideal $I$ and monomial $m$ with the property that the support of $m$ is disjoint from the support of any generator of $I$, we have $\text{pd}(mI) = \text{pd}(I)$. Thus

$$\text{pd}(I(G)^\vee) \leq \max\{\text{pd}(I(G')^\vee) + 1, \text{pd}(I(G'')^\vee)\}.$$  

By induction, $\text{pd}(I(G')^\vee) + 1 \leq a(G') + 1$ and $\text{pd}(I(G'')^\vee) \leq a(G'')$. Now $a(G'') \leq a(G)$ is clear, and $a(G') + 1 \leq a(G)$ since we can take the $a(G')$ 3-disjoint edges of $G'$ along with the edge $\{x, y\}$ to form a set of $a(G') + 1$ 3-disjoint edges in $G$. \hfill \Box

As corollaries, we can recover a result of Zheng, who first linked the regularity to the invariant $a(G)$, and a result of the author with Francisco and Hà. The corollary is true because trees and CM bipartite graphs belong to the set of SCM bipartite graphs. Trees, which are always bipartite, were first shown to be SCM in [6]; alternative proofs were given in [1, 4, 20, 24].

**Corollary 3.4.** If $G$ is either a tree or a CM bipartite graph, then $\text{reg}(R/I(G)) = a(G)$.

Kummini recently showed that if $G$ is an unmixed bipartite graph, that is, all of its minimal vertex covers have the same cardinality, then $\text{reg}(R/I(G)) = a(G)$ also holds. Thus, for bipartite graphs, we would like to know an answer to the question:

**Question 3.5.** Suppose that $G$ is a bipartite graph that is not unmixed and not SCM. What is $\text{reg}(R/I(G))$?

The answer will not be $a(G)$ in general. As noted in [15], the cycle with eight vertices is a mixed bipartite graph with two 3-disjoint edges, but $\text{reg}(R/I(G)) = 3$. This graph is also not SCM by Lemma 2.3. On the other hand, Hà and the author proved that $\text{reg}(R/I(G)) \leq \alpha'(G)$ where $\alpha'(G)$ is the matching number, the largest set of pairwise
disjoint edges. For the same eight cycle, we have $\alpha'(G) = 4$, and thus $\text{reg}(R/I(G)) < \alpha'(G)$. It would be nice to determine a formula for $\text{reg}(R/I(G))$ for all bipartite graphs.

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