Intensinist Social Welfare and Ordinal Intensity-Efficient Allocations

Georgios Gerasimou*
University of Glasgow

This (interim) draft: January 23, 2025
First draft: November 9, 2020

Abstract

This paper studies social welfare and allocation efficiency in situations where, in addition to having ordinal preferences, agents also have ordinal intensities: they can make comparisons such as “I prefer a to b more than I prefer c to d” without necessarily being able to quantify them. In this new informational environment for social choice, the paper first introduces a rank-based criterion for interpersonal comparisons of such ordinal intensities. Building on it, the “intensinist” social welfare function is introduced. This maps profiles of ordinal intensities to weak orders over social alternatives using a scoring method which generalizes that of the classic Borda count in a way that allows for differences in agents’ intensities to be reflected in preference aggregation more accurately and in a much richer class of situations. Building on the same comparability criterion, the paper also studies the classic assignment problem by defining an allocation to be “intensity-efficient” if it is Pareto efficient with respect to the preferences induced by the agents’ intensities and also such that, when another allocation assigns the same pairs of items to the same pairs of agents but in a “flipped” way, the former allocation assigns the commonly preferred item in every such pair to the agent who prefers it more. Some first results on the (non-)existence of such allocations are presented without imposing restrictions on preferences or intensities other than strictness, and the relation—or lack thereof—is studied between intensity-efficient and classical utilitarian allocations.

*Georgios.Gerasimou@glasgow.ac.uk. Earlier versions were presented in October 2023 at the Universities of Melbourne, Queensland, Glasgow and, subsequently, at SSCW 2024 (Paris), CMID 2024 (Budapest), EEA-ESEM 2024 (Rotterdam) and the Hurwicz Workshop on Mechanism Design 2024 (Warsaw). The main concepts in Section 5 were first presented as part of other work at the June 2019 conference on Risk, Uncertainty and Decision. Acknowledgments to be added.
“The problem I have with utilitarianism is not that it is excessively rational, but that the epistemological foundations are weak. My problem is: What are those objects we are adding up? I have no objection to adding them up if there’s something to add.”

Kenneth J. Arrow (1987)

“Suppose I am left with a ticket to a Mozart concert I am unable to attend and decide to give it to one of my closest friends. Which friend should I actually give it to? One thing I will surely consider in deciding this is which friend of mine would enjoy the concert most.”

John C. Harsanyi (1990)

1 Introduction

Given a set of choice alternatives that are relevant to a society’s agents, the domain of interest to the welfare or voting analyst has typically comprised either profiles of preference relations on that set, in the tradition of Arrow (1951), or, in the cardinal-welfarist tradition of Harsanyi (1953, 1955), profiles of cardinally unique and interpersonally comparable utility functions. The former approach allows the analyst to incorporate only ordinal information of the agents’ preferences into society’s preference aggregation problem, and is associated with Arrow’s well-known impossibility theorem. The latter approach, by contrast, enables the analyst to potentially interpret the real numbers (and differences thereof) that capture agents’ utilities (respectively, utility differences) at distinct alternatives as conveying information about their preference intensities, and to incorporate this information into the preference-aggregation problem. In this paper we consider the novel third domain that instead comprises preference-intensity profiles, i.e. collections of quaternary relations that reflect the agents’ preference intensity orderings. More specifically, similar to the way in which preferences are elicited in various matching markets, we assume that information about agents’ intensities can be obtained by asking them to respond to simple questions such as “Do you prefer a to b more than you prefer c to d?”, and that these responses are internally consistent in a way that we make precise. Crucially, unlike existing work on intensity-inclusive social choice or allocation problems that follow the cardinal-welfarist tradition, we do not assume that those comparisons are necessarily quantifiable/(pseudo-)cardinalizable, although such more refined perceptions are indeed encompassed as special cases in the class of intensity relations that we consider.

Operating within this new informational environment for social choice and welfare, we are interested in two questions:

1. How might a social welfare function be defined over preference intensity profiles so as to map a weak ordering over social alternatives to each such profile in a way that is informed by agents’ generally differing ordinal intensities?

2. If alternatives are indivisible and must be allocated to agents one-for-one, how might Pareto efficiency be improved upon in a distributively just way if monetary transfers are infeasible and agents do not have preferences over lotteries over allocations?

The informational environment within which these questions are raised, in particular, rules out agents’ willingness to pay for the different items from being a potential source of information about their generally differing preference intensities, and also the possibility of

1Quoted in Ellingsen (1994)
their attitudes towards risk acting as confounds of their preference intensities (Arrow, 1963; Ellingsen, 1994; Schoemaker, 1982; Sen, 2017/1970). This non-cardinal, non-utilitarian framework where preference and preference-intensity information is nevertheless available to the social planner/matching-platform designer naturally raises the question of how this information might be used to arrive at some normatively appealing preference-aggregation method (Question 1) or refinement of Pareto efficiency (Question 2) that would also reflect differences in the agents’ preference intensities. Similar to utilitarian or other cardinal-utility notions of efficiency, such a refinement requires that one be able to make some kind of interpersonal comparisons (Baccelli, 2023; Echenique et al., 2023; Fleurbaey and Hammond, 2004). Unlike those notions, however, in our framework interpersonal comparisons cannot be based on the agents’ utilities and must rely on the information contained in the above ordinal intensity rankings.

To make progress against this analytical challenge we assume that such comparisons can be made when it is possible to contrast the rank-order position of pairs of alternatives \((a, b)\) in the different agents’ intensity orderings. In particular, when both agents \(i\) and \(j\) prefer \(a\) to \(b\) but the pair \((a, b)\) lies higher in \(i\)’s intensity ranking than in \(j\)’s, and these rankings have the same length, then we assume that \(i\) prefers it more. We refer to this postulate, formalized and defended in Section 3, as Interpersonal Comparisons of Ordinal Intensities.

Towards answering our first question, in Section 4 we build upon this postulate and introduce the intensinist social welfare function. This maps ordinal intensity profiles to weak orderings over social alternatives following a “scoring” procedure. More specifically, alternative \(a\) is considered socially preferred to \(b\) under this method if the total –across all agents– “intensity rank” of \(a\) exceeds that of \(b\), where by the intensity rank of \(a\) for some agent we mean the highest-ranked difference in preference between \(a\) and the worst possible alternative after \(a\) in that agent’s preference ordering. Thus, this number reflects information both about how many alternatives are worse than \(a\) in the agent’s preference ranking and the intensity of preference between \(a\) and its inferior alternatives in that ranking. We show that the intensinist social welfare function coincides with the classic Borda rule in the special case of “linear” intensity profiles whereby all agents’ intensity orderings feature intensity equivalence between pairs of alternatives that are ranked consecutively in their strict preference orderings. More importantly, we also show how the intensinist rule produces more intuitive predictions than Borda in the more general case of non-linear intensity profiles.

In Section 5 we again build on the above comparability postulate to study the problem described in our second question. More specifically, assuming here that all agents’ intensity comparisons are strict in the sense that they contain no intensity-equivalences, we define an allocation \(x\) as intensity-efficient if it is Pareto efficient and also such that, whenever another allocation \(y\) assigns the same pairs of objects to the same pairs of agents but in a “flipped” way, i.e. when \((x_i, x_j) = (y_j, y_i) = (a, b)\) for agents \(i, j\) and alternatives \(a, b\), then \(x\) assigns the commonly preferred alternative in each such pair to the agent who prefers it more. We show that, without any further restrictions, an intensity-efficient allocation exists for all strict intensity profiles when there are three agents and alternatives. Yet we also show that with four or more the existence of intensity-efficient allocations is not guaranteed because the underlying dominance relation may be cyclic if additional restrictions are not imposed. We also study the connection –or lack thereof– between intensity-efficient and classical utilitarian allocations in the special informational environment where the latter are well-defined. Perhaps surprisingly, classical utilitarian allocations are not always intensity-efficient. However, we identify an intuitive “balancedness” sufficient condition on utility profiles under
which this is indeed the case. An implication of this fact is that, in every situation where
the analyst implicitly or explicitly assumes balanced profiles of interpersonally comparable
cardinal utility functions whose utility-difference rankings define strict intensity orderings,
intensity-efficient allocations always exist.

2 Decision-Theoretic Foundations

A = \{a_1, \ldots, a_m\} is the finite set of social alternatives. The preference intensity relation
of agent \( l \leq n \) on \( A \) is denoted by \( \succsim_l \), with \( \succ_l \) and \( \sim_l \) its asymmetric and symmetric parts. These are quaternary relations on \( A \) or, equivalently, binary relations on \( A \times A \). The statement \((a, b) \succsim_l (c, d)\) will be interpreted as “\( a \) is preferred to \( b \) at least as much as \( c \) is to \( d \)” when the first option in each pair is the (possibly weakly) preferred one at that pair, and as “\( b \) is preferred to \( a \) no more than \( d \) is preferred to \( c \)” when the converse is true.\(^2\) In line with these interpretations, agent \( l \)'s binary preference relation \( \succsim_l \) on \( A \) is derived from their intensity relation \( \succsim_l \) in the following way:

\[
(a, b) \succsim_l (c, d) \iff (a, b) \succsim_l (b, a)
\]

We will refer to \( \succsim_l \) as the preference relation that is induced by \( \succsim_l \). As usual, the asymmetric and symmetric parts of \( \succsim_l \) are denoted by \( \succ_l \) and \( \sim_l \). Under the structure that \( \succsim_l \) will be endowed with below, both that relation and its induced \( \succsim_l \) are weak orders on their respective domains.

For \( a, b \in A \), the intensity-equivalence class of \((a, b)\) under \( \succsim_l \) is defined by

\[
[a, b] := \{(a', b') \in A \times A : (a', b') \sim_l (a, b)\}
\]

Furthermore, the quotient set of \( A \times A \) under \( \sim_l \) is defined by

\[
(A \times A) \sim_l := \{[a, b] : [a, b] \text{ is an intensity-equivalence class under } \sim_l\}
\]

Because \( A \) is finite and \( \succsim_l \) weakly orders \( A \times A \), with a mild abuse of notation one can order the finitely many intensity-equivalence classes in the quotient set \((A \times A) \sim_l\) by

\[
[a, a']_1 \succ_l [b, b']_2 \succ_l \cdots \succ_l [z, z']_l,
\]

where \([p, p']_i \succ_l [q, q']_j\) means \((a, b) \succ_l (c, d)\) for every \((a, b) \in [p, p']_i\) and \((c, d) \in [q, q']_j\).

In the definitions that follow we omit existential quantifiers, yet they should be read with the understanding that the imposed conditions apply to all objects in the respective domains.

**Definition 1.**

\( \succsim_l \) is utility-difference representable if there is a function \( u_l : A \to \mathbb{R} \) such that

\[
u_l(a) - u_l(b) \geq u_l(c) - u_l(d) \iff (a, b) \succsim_l (c, d)\]

\(^2\)Furthermore, with the structure it will be endowed with, \( \succsim_l \) will always rank \((a, b)\) above \((c, d)\) when \( a \) is preferred to \( b \) and \( d \) is to \( c \), and vice versa when the converse is true.
Utility-difference representations are more general than neoclassical cardinal utility ones and, either in the latter or in their more general form, have been the subject of extensive study by economists, logicians, philosophers, psychologists and historians of economic thought (Lange, 1934; Alt, 1936, 1971; Samuelson, 1938; Suppes and Winet, 1955; Scott and Suppes, 1958; Scott, 1964; Adams, 1965; Luce and Suppes, 1965; Krantz et al., 1971; Shapley, 1975; Roberts, 1979; Basu, 1982; Fuhrken and Richter, 1991; Ellingsen, 1994; Köberling, 2006; Baccelli and Mongin, 2016; Echenique and Saito, 2017; Moscati, 2018; Baccelli, 2024; Pearce, forthcoming). When $w_l$ in (3) is not cardinal in the sense of being unique up to a positive affine transformation, it has the complex uniqueness property of additive representations on finite sets. This encompasses all cardinal transformations but also some ordinal ones, the latter being relation-dependent and not easy to pin down.

Completing the analysis that was initiated in Scott and Suppes (1958), Scott (1964) characterized the existence of a utility-difference representation over a finite set of alternatives by means of the following three axioms:

1. $\succsim_l(a, b)$ or $\succsim_l(c, d)$ implies $\succsim_l(a, b) \lor \succsim_l(c, d)$. (1$_D$)

2. $(b_i, c_i) \succsim_l(b_{\pi(i)}, c_{\sigma(i)})$ for $i < n$, $i > 0$, implies $(b_{\pi(0)}, c_{\sigma(0)}) \succsim_l(b_0, c_0)$ for all sequences $b_0, b_1, \ldots, b_{n-1}, c_0, \ldots, c_{n-1} \in A$ and all permutations $\pi, \sigma$ of $\{0, \ldots, n-1\}$, where $n > 0$. (2$_D$)

3. $(a, b) \succsim_l(c, d)$ implies $(d, c) \succsim_l(b, a)$. (3$_D$)

The first imposes completeness on $\succsim_l$. (3$_D$) is a symmetry requirement, essential for $\succsim_l$ to be interpreted as a preference intensity relation rather than as an arbitrary relation over pairs of alternatives. (2$_D$) is known as a cancellation condition, described thus in Luce and Suppes (1965) (p. 277): “What Axiom Scheme [(2$_D$)] really comprises is all possible cancellation laws [...] The difficulty with this axiom from a psychological standpoint is that there seems to be no simple way of summarizing what it says about choice behavior, but this we take to be an inherent complexity of the structural relations that must hold between elements of any finite set in order to guarantee the existence of a utility function that preserves the order of utility differences.” Scott (1964) explained (p. 244) how (2$_D$) implies transitivity of $\succsim_l$ but also remarked (p. 245) that “(2$_D$) is an infinite bundle of conditions (for each $n > 0$, each $\pi, \sigma$), and it was shown in Scott and Suppes (1958) that no finite number of them could be sufficient”. Of course, once the finite number $m$ of alternatives in $A$ is fixed, then there are finitely many such conditions to be satisfied. Scott’s remark, however, clarifies that it is impossible to reduce this condition to a finite number of statements (axioms) that, together with (1$_D$) and (3$_D$), would be sufficient to characterize (3) for any finite $m$.

An alternative and more general approach toward representing intensity relations numerically was proposed recently by this author in Gerasimou (2021, 2022).

**Definition 2.** $\succsim_l$ is preference-intensity representable if there is a function $s_l : A \times A \rightarrow \mathbb{R}$ such that

\[
(a, b) \succsim_l(c, d) \iff s_l(a, b) \geq s_l(c, d) \quad (4)
\]

\[
(a, b) \succsim_l(b, a) \iff s_l(a, b) \geq s_l(e, e) \geq s_l(b, a) \quad (5)
\]

\[
\min\{s_l(a, b), s_l(b, c)\} \geq (\succ) s_l(e, e) \implies s_l(a, b) \geq (\succ) \max\{s_l(a, b), s_l(b, c)\} \quad (6)
\]

A function $s_l$ with this property is preference intensity function for agent $l$ and is unique up to a strictly increasing transformation.
In words, \( s_l \) is a numerical function over pairs of alternatives whose values preserve the agent’s intensity ordering ([4]), represents the preferences induced by it ([5]), and ensures that these preferences are also ordered and that intensities are monotonically increasing in that ordering ([6]).

**Lemma 1.**
A preference-intensity representable \( \succsim_l \) is utility-difference representable if and only if it can be represented by a preference intensity function \( s_l \) that satisfies

\[
s_l(a, c) = s_l(a, b) + s_l(b, c)
\]  

(7)

**Proof of Lemma 1.**
Upon viewing \( s_l \) as a general function of two variables, (7) is known as Sincov’s functional equation, whose solution is a unique \( f : A \rightarrow \mathbb{R} \) such that \( s_l(a, b) \equiv f(a) - f(b) \) (Aczél, 1966). This establishes the “if” part of the lemma. The “only if” part is immediate upon defining \( s_l(a, b) := u_l(a) - u_l(b) \) for the postulated function \( u_l \) with the requisite property. ■

The fact that neoclassical cardinal utility representations can be re-formulated by means of bivariate functions that satisfy (7) was first observed in Samuelson (1938). The “additive intensities” requirement imposed by (7) in this re-formulation was relaxed with the ordinal “lateral consistency” property (6) in Gerasimou (2021, 2022), leading to the above model of ordinal preference intensity measurement. The class of intensity relations that admit such a representation was shown there to comprise those that satisfy (3D) and the following two conditions:

(1′D). \( (a, b) \succsim_l (c, d) \) or \( (c, d) \succsim_l (a, b) \); \( (a, b) \succsim_l (c, d) \) \( \succsim_l (e, f) \) implies \( (a, b) \succsim_l (e, f) \).

(4D). \( (a, c) \succsim_l (b, c) \) if and only if \( (a, b) \succsim_l (b, a) \).

The former is the standard weak order condition and strengthens (1D) by explicitly imposing transitivity on \( \succsim_l \). (4D) is a translocation consistency condition, requiring that \( a \) is preferred to \( b \) if and only if \( a \) is preferred to any third alternative \( c \) more than \( b \) is preferred to \( c \).

The normalization of \( s_l \) introduced next is important for the analysis that follows.

**Definition 3.**
A preference intensity function \( s_l \) represents \( \succsim_l \) canonically if

\[
s_l(A \times A) = \{-k, -k + 1, \ldots, -1, 0, 1, \ldots, k - 1, k\}
\]  

(8)

where \( k \) is the number of intensity equivalence classes \([a_i, a_j] \in (A \times A) \succsim_l \) such that \( a_i \succsim_l a_j \).

In words, \( s_l \) is a canonical preference intensity function if it is onto a set of consecutive integers that is symmetric around zero. The usefulness of such a representation lies in the fact that the integer it associates with a pair of alternatives reveals the rank position of that pair in the agent’s intensity ordering.

**Lemma 2.**
Any preference-intensity representable \( \succsim_l \) admits a canonical representation.

**Proof of Lemma 2.**
Let \( s_l \) be an arbitrary preference intensity function for \( \sim_l \). Define the level set of \( s_l \) at \((a, b) \in A \times A\) by \( s_l[a, b] := \{(a', b') \in A \times A : s_l(a, b) = s_l(a', b')\}\). Next, define \( r_l : A \times A \to \mathbb{R} \) by

\[
\begin{align*}
    r_l(a, b) &:= \begin{cases} 
        |\{s_l[a', b'] : s_l(a, b) > s_l(a', b') \geq s_l(e, e)\}|, & \text{if } s_l(a, b) > s_l(e, e) \\
        s_l(a, b), & \text{if } s_l(a, b) = s_l(e, e) \\
        -r_l(b, a), & \text{if } s_l(a, b) < s_l(e, e)
    \end{cases}
\end{align*}
\]

By construction, \( r_l \) is canonical and \( r_l(a, b) \geq r_l(c, d) \iff s_l(a, b) \geq s_l(c, d) \).

3 Interpersonal Comparisons of Ordinal Intensities

3.1 Notation

By \( \mathcal{R} \) and \( \mathcal{P} \) we denote the collections of all weak and linear orders on \( A \), respectively. By \( \mathcal{I} \), moreover, we denote the collection of all intensity relations on \( A \) such that the preferences induced by them are strict. Relations in this class rule out indifferences but do allow for non-trivial intensity equivalences such as \((a, b) \sim_I (c, d)\) for \( a \succ_I b \) and \( c \succ_I d \). Of course, \( \mathcal{I} \) also includes all strict intensity relations, i.e. those where intensity-equivalence between distinct pairs is only possible for pairs on the diagonal of \( A \times A \). We will assume this more restricted domain in Section 5. The finite set of agents is \( V = \{1, \ldots, n\} \). By \( \mathcal{P} = (\succ_1, \ldots, \succ_n) \) we denote a (strict) preference profile and by \( \mathcal{P}^V \) the collection of all preference profiles. Furthermore, by \( S = (\succ_1, \ldots, \succ_n) \) we denote a preference intensity profile and by \( S \subset \mathcal{I}^V \) the collection of all such profiles with the property that every agent has the same number of intensity-equivalence classes in any given profile. We assume this domain in the present section and the next. If \( S = (\succ_1, \ldots, \succ_n) \) is an intensity profile in \( S \), then its induced preference profile \( P_S = (\succ_1, \ldots, \succ_n) \in \mathcal{P}^V \) is defined by the property that \( \succ_l \) is the preference relation (in view of \( \mathcal{I} \), strict) that is induced by \( \succ_l \) for \( l \leq n \). For profile \( S \in S \) we write \( s = (s_1^S, \ldots, s_n^S) \) for the profile of canonical preference intensity functions that represent the agents’ preferences and intensities as specified in \( S \).

3.2 The Central Postulate

We are now in position to state formally the following:

**Interpersonal Comparisons of Ordinal Intensities**

Given an intensity profile \( S \in S \) and the corresponding canonical profile \( s \), the statement

\[
s_l^S(a, b) > s_l^S(c, d) > 0
\]

is interpreted as implying that agent \( l \) prefers \( a \) to \( b \) more than agent \( t \) prefers \( c \) to \( d \).

This interpretational postulate is central in the paper and, to our knowledge, new in the literature. Towards motivating it, let us recall that preference intensities at the level of the individual decision maker are not assumed here to be quantifiable with any precision beyond
the level of an ordinal ranking. In particular, no assumption is made of a reliable common scale—like Edgeworth’s imaginary hedonometer—on which the different agents’ pleasure or pain from (not) receiving certain alternatives can be measured.

Yet some information about the generally different hedonic effects of having a or b vs c or d is available here for all agents. Let us first consider the case of individual l. Suppose they prefer a to b to c to d, and the first over the second more than the third over the fourth. This is equivalent to the statement $s^S_l(a, b) > s^S_l(c, d) > 0$: namely, the pair (a, b) lies higher than (c, d) in l’s intensity ranking. Now consider agent t and suppose for simplicity that the same applies to them too: $s^S_t(a, b) > s^S_t(c, d)$. Now recall that, by construction of the collection of intensity profiles $S$, agents l and t have exactly the same number of possible rank positions in their respective intensity orderings that pairs (a, b) and (c, d) can occupy. Considering this fact, and that both have preferences and intensities over the same set of alternatives, how might inequality $s^S_l(a, b) > s^S_t(c, d)$ be interpreted?

It is factually correct to say that the difference in preference within pair (a, b) is ranked higher—and in the same scale—by agent l than the difference in preference within pair (c, d) is ranked by agent t. But the question emerges now: Should all agents’ intensity orderings be treated equally by the social planner? While equal treatment may or not be an appropriate approach to follow in practice, depending on what else is known about agents and the problem at hand, since the intensity orderings are taken here to encode all available welfare-relevant information, treating them in any way other than equal would call for a justification that appears elusive. Against this background, our suggested interpretation of the inequality in question as one that portrays agent l preferring a to b more than agent t prefers c to d might be thought of as a reasonable starting point for interpersonal comparisons in such an informational and analytical environment.

## 4 Intensinist Social Welfare

We start this analysis by drawing a formal distinction between the standard Arrovian social welfare environment and the more enriched one that we will be working with.

**Definition 4.**
A mapping $W : P^V \rightarrow R$ is a classic social welfare function.
A mapping $W : S \rightarrow R$ is an extended social welfare function.

Using this terminology, let us now recall the famous Borda ranking method (Borda, 1781; Young, 1974; Nitzan and Rubinstein, 1981; Maskin, 2024).

**Definition 5.**
The Borda classic social welfare function $B_0 : P^V \rightarrow R$ is defined by

$$a_i \ B_0(P) \ a_j \iff \sum_{l=1}^n c^P_l(a_i) \geq \sum_{l=1}^n c^P_l(a_j),$$  \hspace{1cm} (9)$$

where

$$c^P_l(a_i) := \left| \left\{ a_k \in A : a_i \succ^P_l a_k \right\} \right|.$$
That is, when strict preference profiles act as inputs to the social welfare function, $a$ is socially preferred to $b$ under $\text{Bo}$ if the total –across all agents– number of alternatives that are considered inferior to $a$ is larger than the total number corresponding to $b$.

We now proceed to introducing the main new concept of this section.

**Definition 6.**

The *intensinist* extended social welfare function $\text{In} : S \rightarrow \mathcal{R}$ is defined by

\[
a_i \text{ In}(S) a_j \iff \sum_{l=1}^{n} r^S_l(a_i) \geq \sum_{l=1}^{n} r^S_l(a_j),
\]

where

\[
r^S_l(a_i) := \max_{a_k \in A} s^S_l(a_i, a_k)
\]

and $s^S_l$ is agent $l$’s canonical preference intensity function at profile $S$.

There are notable similarities and differences between the intensinist and Borda social welfare functions. Both are so-called “scoring” rules in the sense that the way in which they rank alternatives comes about as the outcome of summing their “scores” under some criterion. However, $\text{Bo}$ is defined over preference profiles and takes cardinality of agents’ lower preference-contour set as its scoring criterion. As such, it is oblivious to differences in strength of preference between agents. In contrast, $\text{In}$ takes as its input profiles of intensity relations so as to account for such differences. The specific way in which it does so, moreover, is by identifying for every agent the *intensity rank* of each alternative, and then summing across agents each alternative’s intensity rank to determine its relative standing in the resulting social ranking. In view of (11), $a$’s intensity rank for agent $l$ is formally defined as the highest position in $l$’s intensity ordering where $a$ finds itself being preferred to another item in $A$. What does that mean intuitively? By the lateral consistency property of preference intensity functions, the intensity rank of $a$ is the highest-ranked difference in preference between $a$ and the worst possible alternative after $a$. Thus, this number reflects information both about how many alternatives are worse than $a$ in $l$’s preference ranking and the intensity of preference between $a$ and its inferior alternatives.

An alternative’s intensity rank and Borda score are generally distinct but subtly related concepts. Towards clarifying their connection we introduce an additional notion.

**Definition 7.**

An intensity relation $\sim_l$ is *linear* if, whenever $a_{\pi(1)} \succ_l a_{\pi(2)} \succ_l \cdots \succ_l a_{\pi(m)}$ is true under some permutation $\pi : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\}$, then

\[
(a_{\pi(i)}, a_{\pi(i+1)}) \sim_l (a_{\pi(i+1)}, a_{\pi(i+2)})
\]

holds for all $i = 1, \ldots, m - 2$.\footnote{Evidently, this notion of linearity for quaternary relations is unrelated to the notion of linear order for binary relations.} A profile $S \in S$ is *linear* if $\sim_l$ is linear for every $l \in V$.  

---

\[\text{8}\]
If $\succsim_l$ is linear and admits a utility-difference representation with some $u_l : A \to \mathbb{R}$, then $u_l(a_{\pi(1)}) - u_l(a_{\pi(2)}) = \cdots = u_l(a_{\pi(m-1)}) - u_l(a_{\pi(m)}) = c > 0$. In other words, utility—as measured within the numerical scale defined by $u_l$—increases by a constant increment in any transition from some alternative to the one that lies just above it in $l$’s preference ranking.

As we show next, the social ordering prescribed by the intensinist social welfare function at any linear intensity profile coincides with the social ordering prescribed by the Borda function at the corresponding induced preference profile.

**Theorem 1.**
If $S \in \mathcal{S}$ is a linear intensity profile, then $\mathbf{In}(S) = \mathbf{Bo}(P_S)$.

**Proof of Theorem 1.**
To ease notation we write $r_l$ instead of $r^S_l$ in much of the proof. We start with the following lemma, which is valid regardless of whether the intensity relation $\succsim_l$ is linear or induces a strict or weak preference relation $\succeq_l$:

**Lemma 3.**
The function $r_l : A \to \mathbb{R}$ defined in (11) represents the $\succsim_l$-induced preference relation $\succeq_l$ ordinally.

**Proof of Lemma 3.**
By definition,

$$r_l(a_i) = \max_{a_k \in A} s_l(a_i, a_k)$$

$$\geq r(a_j) = \max_{a_k \in A} s_l(a_j, a_k)$$

$$\iff s_l(a_i, a_k) \geq s_l(a_j, a_k) \forall a_k \in A \iff (a_i, a_k) \succsim_l (a_j, a_k) \forall a_k \in A \iff (a_i, a_j) \succeq_l (a_j, a_i) \iff a_i \succeq_l a_j,$$

where the penultimate equivalence immediately follows from (4D), which in turn is implied by the postulate that $s_l$ represents $\succsim_l$ in profile $S \in \mathcal{S}$.

**Lemma 4.**
If $\succsim_l \in \mathcal{I}$ is linear, then the function $r_l : A \to \mathbb{R}$ defined in (11) satisfies

$$r_l(A) = \{0, 1, \ldots, m - 1\}$$

(13)

**Proof of Lemma 4.**
Consider a linear $\succsim_l$. Since $\succsim_l \in \mathcal{I}$, it is true by assumption that the $\succsim_l$-induced $\succeq_l$ is a linear order. To ease notation, assume without loss that the $m$ elements in $A$ are $\succ_l$-ranked by every agent $l \in \mathcal{V}$ thus:

$$a_1 \succ_l a_2 \succ_l a_3 \succ_l \cdots \succ_l a_{m-1} \succ_l a_m$$
We will show that, because the \( \preceq_l \) which induces this \( \succeq_l \) is linear, the “upper half” of that ordering—in which the left item in each pair is (weakly) preferred to its right counterpart—exhibits the following pattern:

\[
\begin{align*}
(a_1, a_m) & \succ_l (a_1, a_{m-1}) \sim_l (a_2, a_m) \\
& \vdots \\
(a_1, a_3) & \sim_l (a_2, a_4) \sim_l \cdots \sim_l (a_{m-2}, a_m) \\
& \succ_l (a_1, a_2) \sim_l (a_3, a_4) \sim_l \cdots \sim_l (a_{m-1}, a_m) \\
& \succ_l (a_1, a_1) \sim_l (a_2, a_2) \sim_l (a_3, a_3) \sim_l \cdots \sim_l (a_{m-1}, a_{m-1}) \sim_l (a_m, a_m)
\end{align*}
\]

The bottom two lines are true by reflexivity of \( \preceq_l \) and by the postulated linearity of \( \succeq_l \), respectively, while the \( \succ_l \) sign between them is implied by \( a_i \succ_l a_{i+1} \) for all \( i \leq m - 1 \). Consider now the preceding line. We know from the above that \( s_l(a_i, a_{i+1}) = s_l(a_{i+1}, a_{i+2}) = s_l(a_{i+2}, a_{i+3}) > 0 \). Hence, by (6), \( s_l(a_i, a_{i+2}) = s_l(a_{i+1}, a_{i+2}) \) is also true. Since \( s_l(a_{i+1}, a_{i+2}) = s_l(a_{i+2}, a_{i+3}) \) holds too, (6) further implies \( s_l(a_{i+1}, a_{i+3}) = s_l(a_{i+1}, a_{i+2}) \). Combined, these yield \( s_l(a_i, a_{i+2}) = s_l(a_{i+1}, a_{i+3}) \) or, equivalently, \( (a_i, a_{i+2}) \sim_l (a_{i+1}, a_{i+3}) \). This proves that the intensity-equivalence statements in that line are valid too, while the \( \sim_l \) sign following it is a direct consequence of (6). By finiteness, proceeding upwards in this fashion and applying the same argument repeatedly establishes the claimed pattern in the structure of \( \succeq_l \). Equivalently, this pyramidal scheme shows how the top-half intensity-equivalence classes of \( \succeq_l \) are formed and ordered by \( \succ_l \).

Since there are \( m \) alternatives in \( A \), the above shows that there are \( m - 1 \) intensity-equivalence classes \( [a_i, a_j] \in (A \times A)_{\sim_l} \) such that \( a_i \succ_l a_j \). From (11), we have

\[
r_l(a_i) := \max_{a_k \in A} s_l(a_i, a_k),
\]

where \( s_l \) is a canonical preference intensity function. This and the above-postulated \( \succ_l \) ordering on \( A \) imply

\[
\begin{align*}
s_l(a_1, a_m) &= m - 1 \\
s_l(a_2, a_m) &= s_l(a_1, a_{m-1}) = m - 2 \\
& \vdots \\
s_l(a_1, a_2) &= \cdots = s_l(a_{m-1}, a_m) = 1 \\
s_l(a_1, a_1) &= \cdots = s_l(a_m, a_m) = 0
\end{align*}
\]
Thus, in line with Lemma 3, we have $a_i \succ_l a_j \iff r_l(a_i) > r_l(a_j)$ and, in addition,

$$r_l(a_i) = m - i \quad \text{for } i \leq m,$$

from which the claim follows.

Now consider the induced profile $P_S \in \mathcal{P}^V$. From (10) we have

$$c^P_S(a_i) := |\{a_k \in A : a_i \succ_l a_k\}|$$

This and the given $\succ_l$-ordering on $A$ readily imply

$$c^P_S(a_i) = m - i$$

Therefore, for every linear intensity profile $S \in \mathcal{S}$, every agent $l \leq n$, and every alternative $a_i \in A$,

$$r^S_l(a_i) = c^P_S(a_i) = m - i$$

By (10) and (9) it now follows that $\text{In}(S) = \text{Bo}(P_S)$ for every linear profile $S \in \mathcal{S}$. $lacksquare$

This result clarifies that the intensinist rule in the extended social welfare domain that comprises ordinal intensity profiles coincides for linear such profiles with the well-understood Borda rule when the definition of that rule is suitably extended to this richer domain. The Borda rule, however, produces identical recommendations for any two intensity profiles that induce the same preference profile. The added value and potential usefulness of the intensitist rule lie precisely in the fact that it does not do so. Instead, it generally treats agents and alternatives differently when intensity profiles do not take the simple—and very special—linear form, favouring in its produced ranking those alternatives that are preferred relatively more intensely than others. The example that follows clarifies this point.

### 4.1 Example: Intensinism vs Borda at a Non-Linear Profile

Let $n = 4$ and consider the intensity profile $S \in \mathcal{S}$ that is represented canonically thus:

| $s_1(a, d)$ | $s_2(b, d)$ | $s_3(c, d)$ | $s_4(d, a)$ |
|-------------|-------------|-------------|-------------|
| $s_4(b, d)$ | $s_4(a, d)$ | $s_4(a, d)$ | $s_4(c, a)$ |
| $s_1(a, c)$ | $s_2(b, c)$ | $s_3(b, d)$ | $s_4(b, a)$ |
| $s_1(b, c)$ | $s_2(a, c)$ | $s_3(c, b)$ | $s_4(d, b)$ |
| $s_1(c, d)$ | $s_2(c, d)$ | $s_3(a, b)$ | $s_4(c, b)$ |
| $s_1(a, b)$ | $s_2(b, a)$ | $s_3(c, a)$ | $s_4(d, c)$ |
Clearly, $S$ is not linear and defines the following $r^S_i$ and $c^{Ps}_i$ functions:

$$
\begin{align*}
  r^S_1(a) &= 6 & r^S_2(b) &= 6 & r^S_3(c) &= 6 & r^S_4(d) &= 6 \\
  r^S_1(b) &= 5 & r^S_2(a) &= 5 & r^S_3(a) &= 5 & r^S_4(c) &= 5 \\
  r^S_1(c) &= 2 & r^S_2(c) &= 2 & r^S_3(b) &= 4 & r^S_4(b) &= 4 \\
  r^S_1(d) &= 0 & r^S_2(d) &= 0 & r^S_3(d) &= 0 & r^S_4(a) &= 0
\end{align*}
$$

$$
\begin{align*}
  c^{Ps}_1(a) &= 3 & c^{Ps}_2(b) &= 3 & c^{Ps}_3(c) &= 3 & c^{Ps}_4(d) &= 3 \\
  c^{Ps}_1(b) &= 2 & c^{Ps}_2(a) &= 2 & c^{Ps}_3(a) &= 2 & c^{Ps}_4(c) &= 2 \\
  c^{Ps}_1(c) &= 1 & c^{Ps}_2(c) &= 1 & c^{Ps}_3(b) &= 1 & c^{Ps}_4(b) &= 1 \\
  c^{Ps}_1(d) &= 0 & c^{Ps}_2(d) &= 0 & c^{Ps}_3(d) &= 0 & c^{Ps}_4(a) &= 0
\end{align*}
$$

Letting $c^{Ps}(a_k) := \sum_{l=1}^{4} c^{Ps}_l(a_k)$ and $r^S(a_k) := \sum_{l=1}^{4} r^S_l(a_k)$ for $a_k \in A$, these lead to

$$
\begin{align*}
  r^S(a) &= 16, & c^{Ps}(a) &= 7 \\
  r^S(b) &= 19, & c^{Ps}(b) &= 7 \\
  r^S(c) &= 15, & c^{Ps}(c) &= 7 \\
  r^S(d) &= 6, & c^{Ps}(d) &= 3
\end{align*}
$$

Hence, the distinct social welfare orderings $\text{In}(S)$ and $\text{Bo}(P_s)$ are

$$
\begin{align*}
  b & \succ_{\text{In}} a \succ_{\text{In}} c \succ_{\text{In}} d \\
  a & \sim_{\text{Bo}} b \sim_{\text{Bo}} c \succ_{\text{Bo}} d
\end{align*}
$$

Each agent in this example has a different most preferred option, while three of them rank $d$ as their worst. It is not surprising, therefore, that both $\text{In}$ and $\text{Bo}$ place $d$ at the bottom of the respective social welfare orderings. Why is option $b$ favoured more than $a$ and $c$ in $\text{In}$ while $\text{Bo}$ ranks them equally? One observes that $a$ is ranked twice as second and once as fourth, whereas $b$ and $c$ are each ranked once as second and twice as third. Borda’s preference-positional criterion balances out these trade-offs and produces a tie. The intensity-rank positional criterion of the intensinist solution on the other hand is sensitive to the fact that, in both instances where $b$ and $c$ are ranked third, $b$ is preferred to the fourth option more than $c$ is. Hence, since $b$ and $c$ are otherwise symmetric, $\text{In}$ naturally favours $b$ over $c$. Why does it rank $b$ above $a$? Upon noticing the offsetting symmetry between them that occurs in orders $\succ_1$ and $\succ_2$, it is sufficient to focus on $\succ_3$ and $\succ_4$. There, we observe that while $a \succ_3 b$ and $b \succ_4 a$, the 4th agent prefers $b$ to $a$ –the worst option in that ranking– more than the 3rd agent prefers $a$ to $b$, even though the two alternatives are ranked consecutively in $\succ_3$ and $\succ_4$. This fact, in turn, translates into the intensity-rank difference $r^S_4(b) - r^S_4(a) > r^S_3(a) - r^S_3(b)$, which ultimately places $b$ above $a$ in that social welfare ranking.

\section{5. Intensity-Efficient Allocations}

\subsection{5.1 Strict Intensities}

Operating in the same informational environment to the one introduced and studied earlier, the purpose of this section is to propose and study a notion of distributively just assignments of $m = n$ indivisible items to $n$ agents when monetary transfers are infeasible and agents
are not assumed to have preferences over lotteries over allocations. For this problem we will assume the following additional property on the intensity relation of every agent \( l \leq n \):

\[(5_D). \ (a, b) \sim_l (c, d) \text{ if and only if } (a, b) = (c, d) \text{ or } a = b \land c = d.\]

This is a strictness condition that rules out all non-trivial intensity equivalences. As such, it is analogous to the preference-strictness postulate which, in fact, it implies. We denote by \( \hat{\mathcal{I}} \subset \mathcal{I} \) the class of intensity relations that have this additional property. We further denote by \( \hat{\mathcal{S}} := \mathcal{I}^V \) the collection of all such intensity profiles. Under the above strictness condition the canonical preference intensity function of every agent is onto the set \( \{-k, \ldots, -2, -1, 0, 1, 2, \ldots, k\} \), where the integer \( k \) that dictates the cardinality of this range reduces now to the number of distinct pairs of distinct alternatives in \( A \), i.e. \( k \equiv \left( \frac{n}{2} \right) - n \).

Table 1 enumerates the number of distinct utility-difference and preference-intensity representable intensity relations on small finite sets when \( (5_D) \) is assumed, and juxtaposes them to the corresponding number of strict preference relations on those sets.

Table 1: Distinct ordinal utility, utility-difference and ordinal preference-intensity representations in small domains that are possible under the strictness condition \((5_D)\).

| \( n \) | Strict ordinal utility representations | Strict utility-difference representations | Strict ordinal preference intensity representations |
|-------|-------------------------------------|------------------------------------------|-------------------------------------------------|
| 3     | 6                                   | 12                                       | 12                                              |
| 4     | 24                                  | 240                                      | 384                                             |
| 5     | 120                                 | 13,680                                   | 92,160                                          |

Source for 3rd & 4th columns: MiniZinc\textsuperscript{4}computations with the Gecode solver.

This table clarifies that, with 3 alternatives in the domain, the two models coincide under strictness. It also clarifies, however, that the more general ordinal model can account for considerably more preference intensity comparisons than the utility-difference one when there are more than 3 alternatives, with explanatory gains that are increasing in that number (60% and 573% when \( n = 3 \) and \( n = 4 \), respectively).

### 5.2 Allocations, Intensity-Dominance and Efficiency

An allocation of the \( n \) goods is a permutation on \( A \). The set of all allocations is denoted by \( \mathcal{A} \). We are interested in allocations that, for every possible strict intensity profile, assign the \( n \) objects in \( A \) to the \( n \) agents in \( V \) in a way that is not only Pareto efficient but also just once the agents’ ordinal intensities information is accounted for.

To this end, building on the comparability assumption of Section 3, we proceed with introducing the following novel notions of dominance and efficiency.

**Definition 8.**

Let \( S = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n) \in \hat{\mathcal{S}} \) be a strict intensity profile and \( s \) its canonical representation. Given allocations \( x \) and \( y \), the former is said to intensity-dominate the latter if, for every pair of agents \((i, j)\) such that

\[(x_i, x_j) = (y_j, y_i),\]

\(\text{4See Nethercote et al. (2007).}\)
it holds that
\[ s_i(x_i, x_j) \geq s_j(y_j, y_i), \]
and there is at least one pair \((i, j)\) where this inequality is strict.

**Definition 9.**
An allocation is *intensity-efficient* at profile \(S \in \hat{S}\) if it is Pareto efficient with respect to the induced preference profile \(P_S \in P^V\) and is not intensity-dominated.

**Definition 10.**
Two intensity-efficient allocations \(x\) and \(y\) are *equivalent* if \(s_i(x_i, x_j) = s_j(y_j, y_i)\) for all pairs of agents \((i, j)\) such that \((x_i, x_j) = (y_j, y_i)\), and \(x_k = y_k\) for every other agent \(k \leq n\).

If \(x\) and \(y\) are Pareto efficient allocations and \(x\) intensity-dominates \(y\), then in every pair of agents that is “flipped” by \(x\) and \(y\) in the sense that both these allocations assign the same two alternatives \(a\) and \(b\) to the two agents in that pair but do so in opposite ways, the agent receiving \(a\) (which, under the postulated Pareto efficiency, is the mutually preferred one) under \(x\) prefers it to \(b\) weakly more than the agent receiving it under \(y\) and, in at least one case, strictly more. An allocation that is both Pareto efficient and not intensity-dominated is intensity-efficient. Two intensity-efficient allocations \(x\) and \(y\) are equivalent if the agents in every “flipping” pair \((i, j)\) who are assigned \(a\) and \(b\) by \(x\) and \(y\) but in opposite ways have the same preference intensity for \(a\) over \(b\) in the sense that the difference in preference between these two alternatives is ranked the same way in the two agents’ intensity orderings, while every agent who is not in such a pair is treated identically by \(x\) and \(y\).

Intensity efficiency thus defined appears to be the first refinement of Pareto efficiency that is operational in an environment where neither the agents’ utilities are required to be inter- and intra-personally comparable nor monetary transfers between agents are assumed to be feasible. Its status in turn as a normative concept that conforms with intuitive principles of distributive justice in the present analytical environment rests on the appropriateness of the interpersonal comparability assumption that was introduced and defended in Section 3.

### 5.3 Intensity-Efficient and Utilitarian Allocations Are Distinct

For the illustrative purposes of this and the next subsection only, let us temporarily invoke the *Cardinal Unit Comparability* (Baccelli, 2023) or *Independence of Individual Origins of Utility* (Echenique et al., 2023) informational basis to social welfare, adapted to the present environment where the strictness condition \((5_D)\) is also in place. That is, assume that:

(i) each agent \(l\) is associated with a utility function \(u_l : A \rightarrow \mathbb{R}\) that is unique up to a positive affine transformation and features utility differences that define an intensity relation which abides with \((5_D)\);

(ii) interpersonal utility comparisons are possible under such transformations of any utility profile \(U = (u_1, \ldots, u_n)\) where each \(u_l\) is of this kind, as long as the scale/multiplicative parameter thereof is the same for all \(u_l\).

As is well-known, this informational basis is sufficient for the classical utilitarian social
welfare functional—which is recalled below—to be meaningful. Furthermore, each \( u_i \) defines the intensity relation \( \succsim_l \) by \((a, b) \succsim_l (c, d) \iff u_i(a) - u_i(b) \geq u_i(c) - u_i(d) \) which, by assumption (i) above, satisfies (5D). That is, profile \( U \) is strict. As was remarked earlier in the discussion of (3) and (7), such intensity relations, irrespective of how the utility indices underpinning them were constructed, are special cases of the class that we analyse in this paper. Hence, they too admit canonical intensity representations in the sense of (8).

Fixing now a utility profile \( U \) with the above properties, define \( x_{CU}(U) \) as a classical utilitarian allocation under that profile if it holds that

\[
x_{CU}(U) \in \arg \max_{y \in A} \sum_{l=1}^{n} u_i(y_l)
\]

We are now in position to state the following:

**Proposition 1.**

There exist utility profiles with intensity-efficient allocations that are not classical utilitarian.

**Proof of Proposition 1.**

Let \( n = 3 \) and consider the following—assumed cardinally comparable—utility profile over items in \( A = \{a, b, c\} \): \( U = ((53, 38, 9), (46, 33, 21), (20, 30, 50)) \).\(^5\) The two Pareto efficient allocations here are \((a, b, c)\) and \((b, a, c)\). Of these, the intensity-efficient one is the latter, which can be verified upon constructing the agents’ intensity relations by ordering the three relevant pairs according to the ordering of their utility differences: \((a, c) \succsim_1 (b, c) \succsim_1 (a, b)\) and \((a, c) \succsim_2 (a, b) \succsim_2 (b, c)\). Yet the first allocation is the classical utilitarian one \((136 > 134)\).

This potential divergence between intensity efficiency and the hallmark cardinal-welfarist allocation criterion is important because it shows that the hereby proposed notion is indeed distinct even under the richest possible informational assumptions. Furthermore, while perhaps not obvious ex ante, as is clarified by the preceding example, the reason why such a divergence may occur is that the classic utilitarian criterion is guided by the levels of utilities and their differences across agents. By contrast, in this special environment where intensity orderings are effectively defined by the ordering of utility differences, the Interpersonal Comparisons of Ordinal Intensities postulate upon which the intensity efficiency criterion is built compares the relative ranking of those differences across agents’ intensity orderings.

Naturally, one may inquire about the conditions under which classical utilitarian allocations are intensity-efficient. To this end, and motivated by the preceding remarks, we introduce the following condition that a utility profile may satisfy.

**Definition 11.**

A utility profile \( U = (u_1, \ldots, u_n) \) is balanced if, for any two agents \( i, j \) and alternatives \( a, b \),

\[
u_i(a) - u_i(b) > u_j(a) - u_j(b) \iff s^U_i(a, b) > s^U_j(a, b),
\]

where \( s^U = (s^U_1, \ldots, s^U_n) \) is the profile of canonical preference intensity functions that represent

\(^5\)In the spirit of online platforms that implement algorithms to allocate items to agents by asking them to distribute across all items a fixed number of points that are implicitly interpreted as absolutely measurable utility values, all agents’ utilities in this profile have been constructed so that they add up to 100.
sent the profile $S = (\tilde{\succsim}_{1}, \ldots, \tilde{\succsim}_{n}) \in \hat{S}$ that is defined by $U$.

In words, an interpersonally comparable cardinal utility profile is balanced if interpersonal differences in the levels of utility differences are one-to-one aligned with interpersonal differences in their rank orders.

**Proposition 2.**

Utilitarian allocations of strict and balanced utility profiles are intensity-efficient.

Under the postulated conditions, suppose to the contrary that $x(U)$ is a utilitarian but not intensity-efficient allocation. Then, given that agent $l$’s intensity relation $\tilde{\succsim}_{l}$ is defined by $(a, b) \tilde{\succsim}_{l} (c, d) \iff u_{l}(a) - u_{l}(b) \geq u_{l}(c) - u_{l}(d)$, from the definition of intensity-dominance and intensity-efficiency there exist at least one pair of agents $(i, j)$ and of objects $(a, b)$ such that $x_{i}(U) = a, x_{j}(U) = b$ and

$$s_{j}(a, b) > s_{i}(a, b) \quad (16)$$

By balancedness, this implies

$$u_{j}(a) - u_{j}(b) > u_{i}(a) - u_{i}(b) \quad (17)$$

Consider allocation $x'$ that is identical to $x(U)$ except that $x'_{i} = b$ and $x'_{j} = a$. From (17),

$$u_{j}(a) - u_{i}(a) > u_{j}(b) - u_{i}(b)$$

This shows that the utilitarian benefit from the $i \leftrightarrow j$ swap in the transition from $x(U)$ to $x'$, $u_{j}(a) - u_{i}(a)$, exceeds the corresponding loss, $u_{j}(b) - u_{i}(b)$. It follows that

$$\sum_{l=1}^{n} u_{l}(x'_{l}) > \sum_{l=1}^{n} u_{l}(x_{l}(U)),$$

which contradicts the postulated utilitarian optimality of $x(U)$. ■

It is well-known that classical utilitarianism refines the set of Pareto efficient allocations. Proposition 1 shows that, when well-defined, this benchmark criterion has the additional normative property whereby the Pareto efficient allocations it selects are, in fact, intensity-efficient. It also implies the following existence result that is perhaps worth stating explicitly:

**Corollary 1.**

An intensity-efficient allocation exists for any intensity profile that is induced by an interpersonally comparable, strict and balanced cardinal utility profile.

### 5.4 (Non-)Existence in the General Case

We now return to the general environment in which intensity profiles come from the class of relations that were specified in Section 5.1. Without imposing any further restrictions
such as cardinal or even utility-difference representability of the individuals’ intensities, we establish the following new result in this setting.

**Theorem 2.**

An intensity-efficient allocation exists for every intensity profile in $\hat{S}$ when $n = 3$ but not in general when $n \geq 4$.

**Proof of Theorem 2.**

From Table 1, there are $12^3 = 1,728$ unique preference intensity profiles under Strictness ($5_D$). The argument proceeds by considering the possible ways in which an arbitrary such profile might generate a sequence of distinct Pareto-efficient allocations that are implicated in an intensity-dominance cycle. To this end, let $D$ be the intensity-dominance relation that is introduced in Definition 8. Suppose to the contrary that

$$w^1 D w^2 D \ldots D w^k D w^1$$

for Pareto efficient allocations $w^1, \ldots, w^k$ on $X := \{a, b, c\}$.

**Observation 1.**

The $n = 3$ postulate implies that for any two allocations $w^i, w^{i+1}$ such that $w^i D w^{i+1}$ it must be that $w^l_i = w^l_{i+1}$ for exactly one agent $l \in \{1, 2, 3\}$ and $(w^j_i, w^k_i) = (w^j_{i+1}, w^k_{i+1})$ for $j, k \neq l$.

**Observation 2.**

The $n = 3$ postulate implies $k \leq 6$.

**Observation 3.**

Pareto efficiency of $w^i = (a', b', c')$ and Strictness together imply

$$s_2(a', b') > 0 \implies s_1(a', b') > 0,$$

$$s_3(b', c') > 0 \implies s_2(b', c') > 0,$$

$$s_3(a', c') > 0 \implies s_1(a', c') > 0.$$ (19) (20) (21)

**Observation 4.**

Strictness and the canonicality assumption for intensify function profile $s$ imply $s_1(a', b') = s_i(c', d') > 0 \iff (a', b') = (c', d')$ and $s_i(a', b') > 0 \iff s_i(a', b') \in \{1, 2, 3\}$.

Notice that (18) is impossible for $k = 2$ because $D$ is asymmetric by construction. Suppose $k = 3$. Without loss of generality, write $w^1 := (a, b, c)$ and $w^2 := (b, a, c)$. Then, In view of Observation 1, either $w^3 = (b, c, a)$ or $w^3 = (c, a, b)$ must be true. Since, in both cases, $w^1$ and $w^3$ are $D$-incomparable by construction, the $w^3 D w^1$ postulate in (18) is contradicted.

Now suppose $k = 4$. By (18) and the above implications, we may take $w^1, w^2, w^3$ to be as in the $k = 3$ case, from which it then follows that allocation $w^4$ must satisfy either $w^4 = (c, b, a)$ or $w^4 = (a, c, b)$. Notice that each of these possibilities is compatible with $w^3 = (b, c, a)$ and with $w^3 = (c, a, b)$. We therefore have the following 4 cases to consider:

[Remark: in what follows we make repeated use, often without explicit reference, of the Pareto efficiency implications (19)–(21), the skew-symmetry property $s_i(a, b) = -s_i(b, a)$ of canonical preference intensity representations and, whenever exact values of the $s_i$ functions
are asserted, of the lateral-consistency property (6) together with the assumption that every 
$s_i$ is canonical and strict (cf Observations 3–4).]

Case 1. \( w^3 = (b, c, a) \), \( w^4 = (c, b, a) \).

By definition,

\[
\begin{align*}
\text{Case 2. } w^3 & = (b, c, a), \ w^4 = (c, b, a). \\
\text{By definition,} \\
& w^3 Dw^2 \implies s_1(a, b) > s_2(a, b), \\
& w^2 Dw^3 \implies s_2(a, c) > s_3(a, c), \\
& w^3 Dw^4 \implies s_1(b, c) > s_2(b, c), \\
& w^4 Dw^1 \implies s_3(a, c) > s_1(a, c).
\end{align*}
\]

Therefore,

\[ s_2(a, c) > s_3(a, c) > s_1(a, c). \quad (22) \]

By assumption, \( w^i \) is Pareto efficient for \( i \leq 4 \). So, it follows from (19)–(21) that there are 
4 subcases to consider:

**Subcase 1a.** \( s_1(a, b) > s_2(a, b) > 0 \) and \( s_1(b, c) > s_2(b, c) > 0 \). By (6) and the fact that \( s_1, s_2 \) 
are canonical, this implies \( s_1(a, c) = 3 \), which contradicts (22).

**Subcase 1β.** \( s_1(a, b) > s_2(a, b) > 0 \) and \( s_2(c, b) > s_1(c, b) > 0 \). Suppose \( s_1(a, c) > 0 \) is also 
true. Then, by (6), (22) and the fact that \( s_1 \) is canonical, \( s_1(a, c) = 1 \) and \( s_1(a, b) = 3 \). If 
\( s_2(a, c) > 0 \) is also true, then \( s_2(a, b) = 3 \). This contradicts \( s_1(a, b) > s_2(a, b) \). So, it must be 
that \( s_2(c, a) > 0 \) instead. But in this case \( s_2(c, a) > 0, s_2(a, b) > 0 \) and (6) together imply 
\( s_2(c, b) = 3 \). This contradicts \( s_2(a, c) = 3 \) which is now implied by (22) and the fact that the 
profile \( s \) is canonical. Thus, it must be that \( s_1(c, a) > 0 \) instead. So now we have \( s_1(c, a) > 0, \\
1(a, b) = 0 \), which implies \( s_1(c, b) = 3 \). But since, by assumption, \( s_2(c, b) > s_1(c, b) \) and \( s_2 \) 
is canonical, this is a contradiction.

**Subcase 1γ.** \( s_2(b, a) > s_1(b, a) > 0 \) and \( s_1(b, c) > s_2(b, c) > 0 \). Suppose first that \( s_2(a, c) > 0 \) 
is also true. Then, \( s_2(b, a) > 0 \) and \( s_2(a, c) > 0 \) implies \( s_2(b, c) = 3 \). If \( s_1(a, c) > 0 \) is also true, 
then (22) and the fact that \( s \) is canonical together imply \( s_2(a, c) = s_2(b, c) \), which contradicts 
(6) and Strictness. So, it must be that \( s_1(c, a) > 0 \). From \( s_1(b, c) > 0 \) and \( s_1(c, a) > 0 \) we 
now get \( s_1(b, a) = 3 \). In view of \( s \) being canonical, this contradicts \( s_2(b, a) > s_1(b, a) \).

**Subcase 1δ.** \( s_2(b, a) > s_1(b, a) > 0 \) and \( s_2(c, b) > s_1(c, b) > 0 \). Because \( s \) is canonical, this and 
(6) readily imply \( s_1(c, a) = s_2(c, a) = 3 \). But since (22) is equivalent to \( s_1(c, a) > s_2(c, a) > \\
s_2(c, a) \), this is a contradiction.

Hence, \( w^4 Dw^1 \) is impossible for such \( w^3 \) and \( w^4 \).

Case 2. \( w^3 = (c, a, b), \ w^4 = (c, b, a) \).

By definition,

\[
\begin{align*}
\text{Case 2. } w^3 & = (c, a, b), \ w^4 = (c, b, a). \\
\text{By definition,} \\
& w^3 Dw^2 \implies s_1(a, b) > s_2(a, b), \\
& w^2 Dw^3 \implies s_1(b, c) > s_3(b, c), \\
& w^3 Dw^4 \implies s_2(a, b) > s_3(a, b), \\
& w^4 Dw^1 \implies s_3(a, c) > s_1(a, c).
\end{align*}
\]
Therefore,

\[ s_1(a, b) > s_2(a, b) > s_3(a, b). \]  

(27)

In view of (19)–(21), we can now consider the following 4 exhaustive and mutually exclusive subcases:

**Subcase 2α.** \( s_1(a, b) > s_2(a, b) > 0 \) and \( s_1(b, c) > s_3(b, c) > 0 \). By (27) and (19)–(21), the former postulate implies \( s_3(a, b) > 0 \). Since \( s \) is canonical, this further implies \( s_3(a, b) = 1 \), \( s_2(a, b) = 2 \) and \( s_1(a, b) = 3 \). This, together with \( s_1(b, c) > s_3(b, c) > 0 \) also implies \( s_3(a, b) = 1 \), \( s_2(a, b) = 2 \) and \( s_1(a, b) = 3 \), which contradicts Strictness.

**Subcase 2β.** \( s_1(a, b) > s_2(a, b) > 0 \) and \( s_3(c, b) > s_1(c, b) > 0 \). For the same reasons as in 2α, we have \( s_3(a, b) = 1 \), \( s_2(a, b) = 2 \) and \( s_1(a, b) = 3 \). This, together with canonicality of \( s \) and \( s_3(c, b) > s_1(c, b) > 0 \), further implies \( s_1(c, b) = 1 \). Hence, it also follows that either \( s_1(a, c) = 2 \) or \( s_1(c, a) = 2 \). The latter possibility cannot be valid, for (6) and \( s_1(c, a) > 0 \), \( s_1(a, b) > 0 \) would then imply \( s_1(c, b) = 3 \), which contradicts \( s_1(c, b) = 1 \). Consider then the case of \( s_1(c, a) = 2 \). This, together with (26) and canonicality of \( s \), implies \( s_3(a, c) = 3 \). Thus, we have \( s_3(a, c) = 3 \), \( s_3(a, b) = 1 \) and, from \( s_3(c, b) > s_1(c, b) > 0 \) and canonicality, \( s_3(c, b) = 2 \). But, by (6) and canonicality, \( s_3(a, c) > 0 \) and \( s_3(c, b) > 0 \) implies \( s_3(a, b) = 3 \), a contradiction.

**Subcase 2γ.** \( s_2(b, a) > s_1(b, a) > 0 \) and \( s_1(b, c) > s_3(b, c) > 0 \). The former postulate, together with (27) and canonicality, implies \( s_3(b, a) = 3 \), \( s_2(b, a) = 2 \), \( s_1(b, a) = 1 \). By (25), either \( s_3(a, c) > s_1(c, a) > 0 \) or \( s_1(c, a) > s_3(c, a) > 0 \) also holds. Consider the first possibility. From \( s_1(b, a) = 1 \), \( s_1(a, c) > 0 \), (6) and canonicality we get \( s_1(a, c) = 2 \). This and (26) implies \( s_3(a, c) = 3 \). Since \( s_3(b, a) = 3 \) is also true, this contradicts Strictness. Hence, it must be that \( s_1(c, a) > s_3(c, a) > 0 \). But in this case \( s_1(b, c) > 0 \), \( s_1(c, a) > 0 \), (6) and canonicality imply \( s_1(b, a) = 3 \), which contradicts (27).

**Subcase 2δ.** \( s_2(b, a) > s_1(b, a) > 0 \) and \( s_3(c, b) > s_1(c, b) > 0 \). As in 2γ, we have \( s_3(b, a) = 3 \), \( s_2(b, a) = 2 \), \( s_1(b, a) = 1 \). But \( s_3(c, b) > 0 \), \( s_3(b, a) > 0 \) and (6) imply \( s_3(c, a) > s_3(b, a) = 3 \) which, by canonicality, is impossible.

Hence, \( w^4 Dw^1 \) is impossible for such \( w^3 \) and \( w^4 \) too.

**Case 3.** \( w^3 = (b, c, a) \), \( w^4 = (a, c, b) \).

By definition:

\[ w^1 Dw^2 \implies s_1(a, b) > s_2(a, b); \]  

(28)
\[ w^2 Dw^3 \implies s_1(b, c) > s_3(b, c); \]  

(29)
\[ w^3 Dw^4 \implies s_3(a, b) > s_1(a, b); \]  

(30)
\[ w^4 Dw^1 \implies s_3(b, c) > s_2(b, c). \]  

(31)

It follows that

\[ s_3(a, b) > s_1(a, b) > s_2(a, b). \]  

(32)

We proceed by considering the following 4 mutually exclusive and exhaustive subcases:

**Subcase 3α.** \( s_1(a, b) > s_2(a, b) > 0 \) and \( s_1(b, c) > s_3(b, c) > 0 \). The latter, together with (19)–(21), (31) and canonicality, implies \( s_3(a, b) = 1 \). But canonicality, (19)–(21) and (21) also implies \( s_3(a, b) = 3 \), which contradicts Strictness.

**Subcase 3β.** \( s_1(a, b) > s_2(a, b) > 0 \) and \( s_3(c, b) > s_1(c, b) > 0 \). The first postulate and
(32), together with canonicality, implies \(s_3(a, b) = 3\), \(s_1(a, b) = 2\) and \(s_2(a, b) = 1\). Since \(s_3(c, b) > s_1(c, b) > 0\) is also assumed, this and canonicality further imply \(s_3(c, b) = 2\). Now, because \(s_3(a, b) > 0\) and \(s_3(c, b) > 0\), it follows from (6) that \(s_3(c, a) > 0\). But (6) in this case further implies \(s_3(c, a) > s_3(a, b) = 3\), which is impossible.

Subcase 3γ. \(s_2(b, a) > s_1(b, a) > 0\) and \(s_1(b, c) > s_3(b, c) > 0\). The first postulate, together with (32) and canonicality, implies \(s_2(b, a) = 3\), \(s_1(b, a) = 2\), \(s_3(b, a) = 1\). The second postulate and \(s_1(b, a) = 2\), together with Strictness, implies \(s_1(b, c) = 3\). This in turn implies \(s_1(a, c) = 1\) or \(s_1(c, a) = 1\). If the latter is true, then \(s_1(b, c) > 0\), \(s_1(c, a) > 0\) and (6), together with canonicality, implies \(s_1(b, a) = 3\), a contradiction. Hence, it must be that \(s_1(a, c) = 1\).

We therefore have \(s_1(b, a) = 2\), \(s_1(a, c) = 1\) and, by (6) and canonicality, \(s_1(b, c) = 3\). From (29), (31), (19)–(21) and canonicality we also know that \(s_1(b, c) > s_3(b, c) > s_2(b, c) > 0\) implies \(s_3(b, c) = 2\) and \(s_2(b, c) = 1\). Thus, we have \(s_3(b, a) > 0\), \(s_3(b, c) > 0\) and, by (19)–(21) and \(s_1(a, c) > 0\), also \(s_3(a, c) > 0\). But \(s_3(b, a) > 0\), \(s_3(a, c) > 0\) together with (6) and canonicality implies \(s_3(b, c) = 3\), a contradiction.

Subcase 3δ. \(s_2(b, a) > s_1(b, a) > 0\) and \(s_3(c, b) > s_1(c, b) > 0\). These readily imply \(s_1(c, a) = 3\). As above, (32) implies \(s_2(b, a) = 3\), \(s_1(b, a) = 2\) and \(s_3(b, a) = 1\). By (6), \(s_1(c, a) = 3\) and \(s_1(a, b) = 2\) implies \(s_1(c, b) = 1\). From the above postulates and from (31), \(s_2(c, b) > s_3(c, b) > s_1(c, b)\) further implies \(s_2(c, b) = 3\), which contradicts \(s_2(b, a) = 3\) and Strictness.

Hence, \(w^4Dw^1\) is impossible for such \(w^3\) and \(w^4\) here as well.

Case 4. \(w^3 = (c, a, b)\), \(w^4 = (a, c, b)\).

By definition:

\[
\begin{align*}
w^3Dw^2 & \Rightarrow s_1(a, b) > s_2(a, b); \\
w^2Dw^3 & \Rightarrow s_1(b, c) > s_3(b, c); \\
w^3Dw^4 & \Rightarrow s_2(a, c) > s_1(a, c); \\
w^4Dw^1 & \Rightarrow s_3(b, c) > s_2(b, c).
\end{align*}
\]

Observe now that

\[
s_1(b, c) > s_3(b, c) > s_2(b, c).
\]

Suppose first that \(s_2(b, c) > 0\). Then, by (37) and (19)–(21), \(s_1(b, c) = 3\), \(s_2(b, c) = 1\) and \(s_3(b, c) = 2\). From \(s_1(b, c) = 3\) and (6) we also get \(s_1(b, a) > 0\) and \(s_1(a, c) > 0\). Hence, by (33), \(s_3(b, a) > s_1(b, a) > 0\) and, by (35), \(s_2(a, c) > s_1(a, c) > 0\). These inequalities and (6) together imply \(s_2(b, c) = 3\), which is a contradiction.

Now suppose instead that \(s_2(b, c) < 0\), i.e. \(s_2(c, b) > 0\). It follows from (37) that \(s_2(c, b) = 3\), \(s_3(c, b) = 2\) and \(s_1(c, b) = 1\). Suppose \(s_1(a, c) > 0\). From (35) and (19)–(21), \(s_2(a, c) > 0\). Since \(s_2(a, c) > 0\) and \(s_3(c, b) > 0\), by (6) we get \(s_2(a, b) > s_2(c, b) = 3\), which is impossible. Hence, \(s_1(c, a) > 0\) holds instead and, from (35) and (19)–(21), \(s_1(c, a) > s_2(c, a) > 0\) is also true. Suppose \(s_2(a, b) > 0\) holds too. By (33), \(s_1(a, b) > 0\). By (6) and \(s_1(c, a) > 0\), \(s_1(a, b) > 0\) we get \(s_1(c, b) = 3\), a contradiction. Hence, \(s_2(b, a) > 0\) must be true instead and, by (33), \(s_2(b, a) > s_1(b, a) > 0\) also. So, we have \(s_2(c, b) > 0\) and \(s_2(b, a) > 0\), which, by (6), implies \(s_2(c, a) > s_1(c, b) = 3\). This too is a contradiction.

Hence, \(w^4Dw^1\) is impossible for such \(w^3\) and \(w^4\) also.

Next, suppose \(k = 5\). Arguing as above, allocations \(w^1, \ldots, w^4\) in (18) must be as in one of the four cases considered previously. Combined with the fact that each \(w^r\) in sequence
(w^1, \ldots, w^5) must be distinct and the notational convention w^1 = (a, b, c) and w^2 = (b, a, c), this gives rise to the following possibilities:

\[ w^3 = (b, c, a), \quad w^4 = (c, b, a), \quad w^5 = (c, a, b); \]
\[ w^3 = (c, a, b), \quad w^4 = (c, b, a), \quad w^5 = (b, c, a); \]
\[ w^3 = (b, c, a), \quad w^4 = (a, c, b), \quad w^5 = (c, a, b); \]
\[ w^3 = (c, a, b), \quad w^4 = (a, c, b), \quad w^5 = (b, c, a). \]

Clearly, because either w^5 = (b, c, a) or w^5 = (c, a, b) must be true in all four cases, and recalling that w^1 = (a, b, c) by assumption, w^5 Dw^1 cannot happen.

Finally, suppose \( k = 6 \). With allocations \( w^1, \ldots, w^5 \) in (18) being as in the \( k = 5 \) case that was just considered above, \( w^6 \) can only coincide with allocation \( (a, c, b) \) in each of the four relevant cases. In view of the previous steps, these are as follows:

**Case 1:** \( w^1 = (a, b, c), w^2 = (b, a, c), w^3 = (b, c, a), w^4 = (c, b, a), w^5 = (c, a, b), w^6 = (a, c, b) \).

By definition:

\[ w^3 Dw^2 \implies s_1(a, b) > s_2(a, b), \]
\[ w^2 Dw^3 \implies s_2(a, c) > s_3(a, c), \]
\[ w^3 Dw^4 \implies s_1(b, c) > s_2(b, c), \]
\[ w^4 Dw^5 \implies s_3(a, b) > s_2(a, b), \]
\[ w^5 Dw^6 \implies s_2(a, c) > s_1(a, c), \]
\[ w^6 Dw^1 \implies s_3(b, c) > s_2(b, c). \]

It follows from the above that

\[ s_2(a, c) > s_1(a, c) > s_1(a, b) > s_2(a, b). \tag{38} \]

Suppose \( s_2(a, b) > 0 \). Then, (38) implies \( s_2(a, c) > 3 \), which contradicts canonicality. If \( s_2(b, a) > 0 \) instead, then (38) together with skew-symmetry of \( s_2 \) implies \( s_2(b, a) > 3 \) and results in the same contradiction.

**Case 2:** \( w^1 = (a, b, c), w^2 = (b, a, c), w^3 = (c, a, b), w^4 = (c, b, a), w^5 = (b, c, a), w^6 = (a, c, b) \).

Notice that the following postulated dominance implications

\[ w^3 Dw^2 \implies s_1(a, b) > s_2(a, b), \]
\[ w^3 Dw^4 \implies s_2(a, b) > s_3(a, b), \]
\[ w^5 Dw^6 \implies s_3(a, b) > s_1(a, b) \]

lead to \( s_1(a, b) > s_2(a, b) > s_3(a, b) > s_1(a, b) \), which is absurd.

**Case 3:** \( w^1 = (a, b, c), w^2 = (b, a, c), w^3 = (b, c, a), w^4 = (a, c, b), w^5 = (c, a, b), w^6 = (a, c, b) \).

Observe here that the postulated dominance implications

\[ w^4 Dw^5 \implies s_1(a, c) > s_2(a, c), \]
\[ w^5 Dw^6 \implies s_2(a, c) > s_1(a, c) \]

directly contradict each other.

**Case 4:** \( w^1 = (a, b, c), w^2 = (b, a, c), w^3 = (c, a, b), w^4 = (a, c, b), w^5 = (b, c, a), w^6 = (a, c, b) \).
Notice that, as in Case 2, the postulated dominance implications
\[ w^4Dw^5 =⇒ s_1(a, b) > s_3(a, b), \]
\[ w^5Dw^6 =⇒ s_3(a, b) > s_1(a, b) \]
result in the same contradiction.

It has been shown that \( D \) is acyclic when \( n = 3 \). This and the fact that allocations are finitely many jointly imply that an intensity-efficient allocation always exists in this case.

For the claimed potential non-existence when \( n \geq 4 \), consider the example strict intensity profile on \( A = \{a, b, c, d\} \) whose canonical representation is shown in the table below:

| \( s_i \) | \( i = \) | 1 | 2 | 3 | 4 |
|-----------|---------|---|---|---|---|
| 6         | (a, d)  | (a, d) | (a, d) | (a, d) |
| 5         | (b, d)  | (a, c) | (a, c) | (b, d) |
| 4         | (a, c)  | (b, d) | (b, d) | (a, c) |
| 3         | (b, c)  | (b, c) | (b, c) | (b, c) |
| 2         | (a, b)  | (c, d) | (c, d) | (a, b) |
| 1         | (c, d)  | (a, b) | (a, b) | (c, d) |

We start by observing that \( s_1 = s_4, s_2 = s_3 \) and
\[ a \succ_i b \succ_i c \succ_i d, \quad i = 1, \ldots, 4 \]
This implies that all 24 possible allocations, defined and listed below, are Pareto efficient.

\[
\begin{align*}
 x_1 &= (a, b, c, d) & x_2 &= (a, b, d, c) & x_3 &= (a, c, b, d) & x_4 &= (a, c, d, b) \\
 x_5 &= (a, d, b, c) & x_6 &= (a, d, c, b) & x_7 &= (b, a, c, d) & x_8 &= (b, a, d, c) \\
 x_9 &= (b, c, a, d) & x_{10} &= (b, c, d, a) & x_{11} &= (b, d, a, c) & x_{12} &= (b, d, c, a) \\
 x_{13} &= (c, a, b, d) & x_{14} &= (c, a, d, b) & x_{15} &= (c, b, a, d) & x_{16} &= (c, b, d, a) \\
 x_{17} &= (c, d, a, b) & x_{18} &= (c, d, b, a) & x_{19} &= (d, a, b, c) & x_{20} &= (d, a, c, b) \\
 x_{21} &= (d, b, a, c) & x_{22} &= (d, b, c, a) & x_{23} &= (d, c, a, b) & x_{24} &= (d, c, b, a) 
\end{align*}
\]

The following comparisons, whose validity can be readily established by the reader, demonstrate that for each \( x_i, i = 1, \ldots, 24 \), in this set there is a distinct \( x_j \) that intensity-dominates \( x_i \) (a situation denoted by \( x_jDx_i \)).

\[
\begin{align*}
 x_6Dx_1 & \quad x_1Dx_2 & \quad x_4Dx_3 & \quad x_{13}Dx_{14} & \quad x_3Dx_5 & \quad x_{17}Dx_6 & \quad x_1Dx_7 & \quad x_2Dx_8 \\
 x_3Dx_9 & \quad x_8Dx_{10} & \quad x_9Dx_{11} & \quad x_{11}Dx_{12} & \quad x_{18}Dx_{13} & \quad x_{13}Dx_{14} & \quad x_{17}Dx_{15} & \quad x_{22}Dx_{16} \\
 x_{23}Dx_{17} & \quad x_{24}Dx_{18} & \quad x_8Dx_{19} & \quad x_{22}Dx_{20} & \quad x_{11}Dx_{23} & \quad x_{13}Dx_{22} & \quad x_{24}Dx_{23} & \quad x_{10}Dx_{24} 
\end{align*}
\]

Therefore, no intensity-efficient allocation exists in this profile. □

The somewhat surprising fact that a plausible dominance concept may by cyclic and prevent an optimal entity to emerge invites an informal analogy to be drawn between intensity-dominance cycles over allocations with at least 4
agents and Condorcet cycles over *alternatives* in majority-based preference aggregation with at least 3 agents (Sen, 2017/1970). Unlike that framework, however, although intensity-dominance cycles here may prevent refining the Pareto set, they do not lead to a “policy paralysis” problem because Pareto allocations always exist and, absent any distributionally juster suggestions, one of them might be promoted by the social planner.

5.5 Example where Intensity-Efficiency Refines Pareto ($n = 4$)

Let $A := \{a, b, c, d\}$ and consider the intensity profile $S = (\succsim_1, \succsim_2, \succsim_3, \succsim_4)$ that is represented canonically by

| $s_1(a, d)$ | $s_2(d, a)$ | $s_3(a, d)$ | $s_4(d, a)$ |
|------------|------------|------------|------------|
| 6          | 6          | 6          | 6          |
| $s_1(b, d)$ | $s_2(d, b)$ | $s_3(a, c)$ | $s_4(c, a)$ |
| 5          | 5          | 5          | 5          |
| $s_1(a, c)$ | $s_2(d, c)$ | $s_3(a, b)$ | $s_4(d, b)$ |
| 4          | 4          | 4          | 4          |
| $s_1(b, c)$ | $s_2(c, a)$ | $s_3(b, d)$ | $s_4(c, b)$ |
| 3          | 3          | 3          | 3          |
| $s_1(a, b)$ | $s_2(c, b)$ | $s_3(c, d)$ | $s_4(b, a)$ |
| 2          | 2          | 2          | 2          |
| $s_1(c, d)$ | $s_2(b, a)$ | $s_3(b, c)$ | $s_4(d, c)$ |
| 1          | 1          | 1          | 1          |

Notice first that the induced preference profile $P_S = (\succ_1, \succ_2, \succ_3, \succ_4)$ is such that

$\begin{align*}
    a & \succ_1 b \quad \succ_1 c \quad \succ_1 d \\
    d & \succ_2 c \quad \succ_2 b \quad \succ_2 a \\
    a & \succ_3 b \quad \succ_3 c \quad \succ_3 d \\
    d & \succ_4 c \quad \succ_4 b \quad \succ_4 a
\end{align*}$

Notice further that the set of Pareto efficient allocations corresponding to $P_S$ is

$$\{(a, c, b, d), (a, d, b, c), (b, c, a, d), (b, d, a, c)\}$$

Notice, finally, that $zDw$ holds here because $s_3(a, b) > s_1(a, b)$ and $s_4(c, d) > s_2(c, d)$; $zDw$ because $s_3(a, b) > s_1(a, b)$; and $zDy$ because $s_4(c, d) > s_2(c, d)$. Thus, $z$ is the unique intensity-efficient allocation.

References

ACZÉL, J. (1966): *Lectures on Functional Equations and their Applications*, New York: Academic Press.

ADAMS, E. W. (1965): “Elements of a Theory of Inexact Measurement,” *Philosophy of Science*, 32, 205–228.

ALT, F. (1936): “Über die Messbarkeit des Nutzens,” *Zeitschrift für Nationalökonomie*, 7, 161–169.
——— (1971): “On the Measurability of Utility,” in Preferences, Utility and Demand, ed. by J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein, New York: Harcourt Brace Jovanovich, chap. 20, (English translation of Alt (1936) by S. Schach).

ARROW, K. J. (1951): Social Choice and Individual Values, New York: Wiley.

——— (1963): Social Choice and Individual Values, New Haven: Yale University Press, 2nd ed.

ARROW, K. J. AND J. S. KELLY (1987): “An Interview with Kenneth J. Arrow,” Social Choice and Welfare, 4, 43–62.

BACCCELLI, J. (2023): “Interpersonal Comparisons of What?” Journal of Philosophy, 120, 5–41.

——— (2024): “Ordinal Utility Differences,” Social Choice and Welfare, 62, 275–287.

BACCCELLI, J. AND P. MONGIN (2016): “Choice-Based Cardinal Utility: A Tribute to Patrick Suppes,” Journal of Economic Methodology, 23, 268–288.

BASU, K. (1982): “Determinateness of the Utility Function: Revisiting a Controversy of the Thirties,” Review of Economic Studies, 49, 307–311.

BORDA, J.-C. (1781): “Mémoire sur les Élections au Scrutin,” Histoire de l’Academie Royale des Sciences, 657–665.

ECHENIQUE, F., N. IMMORLICA, AND V. V. VAZIRANI (2023): “Objectives,” in Online and Matching-Based Market Design, ed. by F. Echenique, N. Immorlica, and V. V. Vazirani, Cambridge: Cambridge University Press.

ECHENIQUE, F. AND K. SAITO (2017): “Response Time and Utility,” Journal of Economic Behavior & Organization, 139, 49–59.

ELLINGSEN, T. (1994): “Cardinal Utility: A History of Hedonimetry,” in Cardinalism, ed. by M. Allais and O. Hagen, Dordrecht: Kluwer, 105–165.

FLEURBAEY, M. AND P. HAMMOND (2004): “Interpersonally Comparable Utility,” in Handbook of Utility Theory, Volume 2: Extensions, ed. by S. Barbera, P. Hammond, and C. Seidl, Dordrecht: Kluwer, 1179–1285.

FUHRKEN, G. AND M. K. RICHTER (1991): “Additive Utility,” Economic Theory, 1, 83–105.

GERASIMOUG, G. (2021): “Simple Preference Intensity Comparisons,” Journal of Economic Theory, 192, 105199.

——— (2022): “Corrigendum to Gerasimou (2021) and Comment on Banerjee (2022),” Journal of Economic Theory, 205, 105542.

HARSANYI, J. C. (1953): “Cardinal Utility in Welfare Economics and in the Theory of Risk Taking,” Journal of Political Economy, 61, 434–435.

—— (1955): “Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility,” Journal of Political Economy, 63, 309–321.
KÖBBERING, V. (2006): “Strength of Preference and Cardinal Utility,” *Economic Theory*, 27, 375–391.

KRANTZ, D. H., R. D. LUCE, P. SUPPES, AND A. TVERSKY (1971): *Foundations of Measurement, Volume I*, New York: Wiley.

LANGE, O. (1934): “The Determinateness of the Utility Function,” *Review of Economic Studies*, 2, 218–225.

LUCE, R. D. AND P. SUPPES (1965): “Preference, Utility and Subjective Probability,” in *Handbook of Mathematical Psychology, Volume 3*, ed. by R. D. Luce, R. R. Bush, and E. H. Galanter, New York: Wiley, 249–410.

MASKIN, E. (2024): “Borda’s Rule and Arrow’s Independence Condition,” *Journal of Political Economy*, forthcoming.

MOSCATI, I. (2018): *Measuring Utility. From the Marginal Revolution to Behavioral Economics*, New York: Oxford University Press.

NETHERCOTE, N., P. STUCKEY, R. BECKET, S. BRAND, G. DUCK, AND G. TACK (2007): “MiniZinc: Towards a standard CP modelling language,” in *Proceedings of the 13th International Conference on Principles and Practice of Constraint Programming*, ed. by C. Bessiere, Springer, vol. 4741 of Lecture Notes in Computer Science, 529–543.

NITZAN, S. AND A. RUBINSTEIN (1981): “A Further Characterization of Borda Ranking Method,” *Public Choice*, 36, 153–158.

PEARCE, D. G. (forthcoming): “Individual and Social Welfare: A Bayesian Perspective,” in *Advances in Economics and Econometrics: 12th World Congress of the Econometric Society*, ed. by V. Chernozhukov, E. Farhi, J. Hörner, and E. L. Ferrara, Cambridge: Cambridge University Press.

ROBERTS, F. S. (1979): *Measurement Theory with Applications to Decisionmaking, Utility and the Social Sciences*, vol. 7 of *Encyclopedia of Mathematics and its Applications* (editor: Gian-Carlo Rota), Reading, MA: Addison-Wesley.

SAMUELSON, P. A. (1938): “The Numerical Representation of Ordered Classifications and the Concept of Utility,” *Review of Economic Studies*, 6, 65–70.

SCHOEMAKER, P. J. H. (1982): “The Expected Utility Model: Its Variants, Purposes, Evidence and Limitations,” *Journal of Economic Literature*, 20, 529–563.

SCOTT, D. (1964): “Measurement Structures and Linear Inequalities,” *Journal of Mathematical Psychology*, 1, 233–247.

SCOTT, D. AND P. SUPPES (1958): “Foundational Aspects of Theories of Measurement,” *Journal of Symbolic Logic*, 23, 113–128.

SEN, A. (2017/1970): *Collective Choice and Social Welfare*, UK: Penguin, (expanded edition of the 1970 original).
Shapley, L. S. (1975): “Cardinal Utility from Intensity Comparisons,” Technical Report R-1683-PR, The RAND Corporation.

Suppes, P. and M. Winet (1955): “An Axiomatization of Utility Based on the Notion of Utility Differences,” Management Science, 1, 259–270.

Young, H. P. (1974): “An Axiomatization of Borda’s Rule,” Journal of Economic Theory, 9, 43–52.