Modeling flow in anisotropic porous medium with full permeability tensor

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Abstract. The flow in anisotropic porous medium is significant for the modeling of subsurface fluids transportation. The subsurface porous medium is usually both heterogeneous and anisotropic, caused by the compaction and sedimentations effects on the formation. Full permeability tensor is therefore needed in modeling flow in anisotropic medium. In this research, two widely used finite volume schemes, Two-Point Flux Approximation (TPFA) and Multi-Point Flux Approximation (MPFA), are applied to solve the flow model with a full permeability tensor. The results verified that ignoring anisotropy of the porous medium results in overestimation of the total flux. The TPFA methods have high computational efficiency, but failed to represent the anisotropy. The MPFA scheme takes more CPU time than TPFA for same grid block resolution, but incorporates the anisotropy using a full tensor. The comparison between the results from two methods indicates that ignoring anisotropy results in significant errors in determined flux.

1. Introduction

Modeling the flow in subsurface porous medium is an interesting and significant topic for underground fluids flow, subsurface contaminant transport etc. The porous medium like the subsurface formation is usually anisotropic, with a higher permeability along the formation bedding ($K_H$) than the permeability perpendicular to the formation bedding ($K_V$). This is usually the case for most reservoirs as the formation affected by sedimentation along geological time. The two permeability values are usually called the principal permeability, and their corresponding directions are the principal directions. The porous medium is called isotropic if the two permeabilities are equal while anisotropic if the two permeabilities are different. In general, the permeability tensor can be used to describe the porous medium permeability. It is symmetrical and positive definite second order tensor, depending on the coordinate system chosen [1][2]. The tensor is diagonal when the chosen coordinate axes are along the principal directions while the off-diagonal terms exist if the axes are not along the principal directions.

The conservation law for the flow problem in porous medium without source/sink terms can be described as

$$\nabla \cdot (-\lambda \nabla p) = q. \quad (1)$$
where \( \lambda = \frac{K}{\mu} \), \( K \) is the permeability tensor, \( \mu \) is the fluid viscosity, \( p \) is potential or pressure if gravity is ignored for simplicity, \( q \) the density flux term. The flux in the porous medium, based on Darcy’s law, is described by the gradient law, where

\[
u = -\lambda \nabla p. \tag{2}
\]

The flow problem in reservoirs is usually solved assuming that the permeability tensor is diagonal, which, however, may result in significant error in flow rate and orientation [3]. To achieve high accuracy of the modeling, a full tensor needs to be applied.

In a general case, the flow equation using full permeability tensor becomes

\[
\begin{bmatrix}
\dot{u}_x \\
\dot{u}_y \\
\dot{u}_z
\end{bmatrix} = -\frac{1}{\mu}
\begin{bmatrix}
k_{xx} & k_{xy} & k_{xz} \\
k_{xy} & k_{yy} & k_{yz} \\
k_{xz} & k_{yz} & k_{zz}
\end{bmatrix}
\begin{bmatrix}
p_x \\
p_y \\
p_z
\end{bmatrix}, \tag{3}
\]

where the pressure gradient in \( x, y, z \) directions are simplified as \( p_x, p_y \) and \( p_z \), respectively. Here, Cartesian coordinates are applied. The equivalent full tensor and associated conservation law in Cylindrical coordinates, usually used for near-well domain, can be found in [4]. Compared with the simplified case of isotropic flow, the full tensor case is more complicated and computational expensive. The conservation equations are usually solved numerically in reservoir simulation. Though the anisotropy results in more complicated model and more expensive computation process, the anisotropy effects cannot be ignored in numerical simulation of flow in subsurface porous medium [5].

When the formation deviated from the horizontal plane, the Cartesian coordinates deviates from the principal directions of the permeability field. Hence, the structured grids can be orthogonal, however, not K-orthogonal. For those cases, numerical methods available includes the Two-Point Flux Approximation method [6], Multi-Point Flux Approximation method [7], and the lower order mixed finite element method [8] etc. Special attentions have been paid to the applicability of various numerical methods to reservoir anisotropy, as well as their computational efficiency. In general, the compromised choice will be made between the computational efficiency and numerical accuracy in choosing a proper numerical scheme. The focus of this paper is to test the criteria of reservoir anisotropy and formation deviation in choosing high computational efficiency method, and verify the error interval with inappropriate choice.

In this paper, we apply two types of the widely used finite volume methods, Two-Point Flux Approximation (TPFA) and Multi-Point Flux Approximation (MPFA) methods, to model the flow in anisotropic porous medium with full permeability tensor. The comparisons between the two methods verified that the TPFA are more computational efficiency than MPFA method; however, it cannot well represent the flow problem with a full tensor, i.e. in the non-orthogonal cases. The calculation examples also demonstrate that ignoring the off-diagonal terms in the full permeability tensor results in over estimation of the total flux and significant errors in determining the pressure and flux. The paper is organized in the following fashion. First, the full permeability tensor representation is presented for anisotropic porous medium. Then, the conservation equation is constructed with the full permeability tensor. With the two numerical methods, the incompressible single phase flow problem is solved numerically using TPFA and MPFA methods. The computation examples and discussions are then followed toward the final conclusions.

### 2. Full permeability tensor representation for the flow in anisotropic medium

In most subsurface cases, the porous medium is laterally isotropic but vertically anisotropic. Suppose the coordinates axes \( (x_o, y_o, z_o) \) are along the principal permeability direction, in which case the permeability tensor can be expressed as

\[
K_o = \begin{bmatrix}
K_H & 0 & 0 \\
0 & K_H & 0 \\
0 & 0 & K_V
\end{bmatrix}. \tag{4}
\]
However, the formation bedding is usually not perfectly horizontal, which forms an angle (denote as $\theta$) with the horizontal plane. If reservoir model is constructed along the horizontal and vertical directions, we need to apply a rotation $R_1(\theta)$ on the original permeability tensor $K_0$, i.e.

$$R_1 = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}. \quad (5)$$

Then, the permeability tensor in the constructed coordinates $(x_1, y_1, z_1)$ becomes

$$K_1 = R_1^T K_0 R_1 = \begin{bmatrix} K_{xx} & 0 & K_{xz} \\ 0 & K_H & 0 \\ K_{xz} & 0 & K_{zz} \end{bmatrix}, \quad (6)$$

where

$$K_{xx} = K_H \cos^2 \theta + K_V \sin^2 \theta$$
$$K_{xz} = (K_H - K_V) \sin \theta \cos \theta$$
$$K_{zz} = K_H \sin^2 \theta + K_V \cos^2 \theta. \quad (7)$$

This is the permeability tensor used for solving the conversation equation with numerical methods in this research.

3. Finite volume methods for the mass conservation equation

Different from the finite difference methods using differential discrete point-values in the domain, the finite volume methods are derived from the conservation over the considering domains. Hence, the approximation in finite volume methods is based on averaged quantities and therefore physically more rational. Two types of finite volume methods are used and briefly summarized in this section. For more details of the basic numerical formulation, one can refer the general descriptions, for example in [9].

The flux in TPFA method is approximated using average values in a grid block. For the flux between adjacent grid blocks $i$ and $j$, the flux is determined by

$$u_{i,j} = -\int_{\gamma_{ij}} \lambda \nabla P \cdot n \, dv, \quad (8)$$

where $\gamma_{ij}$ is the interface between adjacent grid blocks $i$ and $j$, $n$ is the perpendicular direction to the adjacent surface. The conservation equation using TPFA method can then be written as

$$\sum_j t_{ij} (p_i - p_j) = \int_{\Omega_i} q \, dx, \forall \Omega_i \in \Omega \quad (9)$$

where $t_{ij}$ is the transmissibility using distance-weighted harmonic average, $\Omega_i$ is the control volumes (grid blocks) in the concerning domain $\Omega$. More specifically, consider a regular hexahedral grid block aligned with the principal direction and the adjacent grid blocks $i$ and $j$ are in the $x$-direction. Then, we have $n_{ij} = (1,0,0)^T$ and

$$t_{ij} = 2 |\gamma_{ij}| \left( \frac{\Delta x_i}{\lambda_i} + \frac{\Delta x_j}{\lambda_j} \right)^{-1}. \quad (10)$$

The mass conservation equation for each adjacent grid blocks $\Omega_i \in \Omega$ can finally be gathered to complete the TPFA method.

The TPFA scheme is simple in formulation and has a superior computation speed. However, the TPFA method converges only when using the K-orthogonal grid blocks, which means that the permeability tensor must be diagonal and the grid blocks are parallelepiped. This may be problematic for full tensor cases, which will be demonstrated using various calculation examples in the following section.

To overcome the convergence issues of the TPFA method on grid blocks that are not K-orthogonal, a class of MPFA schemes are developed. As the name indicated, MPFA method determines the flux with multi points, for example in [10]. One method in this class is the MPFA O-method [11][12]. For a full tensor case of the permeability, assume that the adjacent grid blocks are also in $x$-direction, then the flux across the adjacent interface $\gamma_{ij}$ can be represented by
\[
\int_{\gamma_{ij}} \mathbf{u} \cdot \mathbf{n}_{i,j} \, dv = \int_{\gamma_{ij}} -(k_{xx}p_x + k_{xy}p_y + k_{xz}p_z) \, dv,
\] (11)

which means that the flux is contributed from three directional pressure gradient contributions. This indicates that to construct the interfacial fluxes for the grids, the partial derivatives along the coordinate directions, are needed. Hence, more point values or volume averages are needed, resulting in the multi grid averages in the MPFA method.

As a demonstration, the flux determination is presented for two dimensional case as shown in Figure 1. New pressures at the centre of the edges are introduced, noted as \(P_n, P_w, P_s, P_e\). Then, the new flux in the sub grids (the right figure in Figure 1.) are then determined by

\[
u_{14} = -t^{xy}_{12}(p_1 - p_3) - t^{xy}_{14}(p_1 - p_4)
= -t^{xy}_{12}(p_3 - p_2) - t^{xy}_{14}(p_4 - p_s),
\] (12)

where the superscript indicates the permeability coefficients (as in Equation 3) used in transmissibility. Similarly, each flux is determined together with two adjacent grid blocks in two directions as above. The pressure at the centre of the edges are determined based on the assumption that pressure changes linearly. Local mass conservation is also applied to determine extra pressure points introduced. The conservation equations, for single phase flow, results in a linear system with pressures being the unknowns.

For general irregular, quadrilateral grid blocks, sub-quadrilaterals are constructed using the midpoints of the quadrilateral grid interface, resulting in pressure solutions in the sub-quadrilaterals according the local mass conservations laws. Finally, discretised equations are solved using the automatic differentiation techniques [13] and the linear system is solved using MATLAB.

**Figure 1.** Grid block structure for MPFA in two dimensions.

4. **Numerical case studies and discussions**

In this section, the two numerical methods, TPFA and MPFA, are applied to solve the flow in anisotropic porous medium with the permeability tensor in Equation (6). For simplicity, we assume the parameters are unit values in SI units, which can then be treated dimensionless as divided by a unit value. This simplifies the unit distraction and reduces the focus on the units’ conversion. It has to be noted that the permeability values, usually in Darcy (D) or mili-Darcy (mD), converts to SI units (m²) using \(1D = 9.869 \times 10^{12} \text{m}^2\). Besides, assume the formation is \(\pi/6\) deviated from the horizontal surface. The anisotropy ratio, \(\sigma = K_v/K_h\), may vary from case to case. We construct a cube domain with unit length, on which we have constant unit pressure boundary in the left and right faces and no flow boundary for the rest faces. Also, assume that the flow is single phase incompressible flow.
4.1. Pressure solution

First, solve the flow in the cube domain using both TPFA and MPFA methods for isotropic medium and anisotropic medium. The pressure solutions for each case are shown in Figure 2.

![Pressure solutions of TPFA and MPFA](image)

**Figure 2.** Pressure solutions using TPFA and MPFA for both isotropic and anisotropic cases.

4.2. Effluent flux

Then, solve the total flux in the cube with unit boundary pressure for various anisotropy ratio. The flux depends on the anisotropy ratio in both methods, which for same ratio may result in different values. The flux results are shown in figure, as well as the relative difference defined by

\[ R_E = \frac{|u_{TPFA} - u_{MPFA}|}{u_{TPFA}} \times 100\% \]  

(13)

As shown in Figure 3, the TPFA methods have high computational efficiency, but failed to represent the anisotropy. The MPFA scheme takes more CPU time than TPFA for same grid block resolution. The results comparison between the two methods indicates that ignoring anisotropy results in significant errors in determined flux. For extreme anisotropic medium in which \( \sigma < 0.1 \), the relative error in flux can reach 30%-70%. The relative error is smaller than 10% if the anisotropy ratio is larger than 0.3. Hence, the TPFA method can only be acceptable for the medium which is not severely anisotropic.
The dimensionless effluent flux is investigated with varying deviation angle ($\theta$) but constant anisotropy ratio ($\sigma=0.1$), as shown in Figure 4a. The effluent flux from TPFA method is generally larger than that from MPFA method. They both result in unit flux at $\theta=0$ and $u=\sigma$ at $\theta=\pi/2$, indicating a horizontal formation and a vertical formation, respectively. The relative error of the TPFA method is increasing from 0 at $\theta=0$ to a peak value within $[\pi/4, 3\pi/8]$, and then decreasing to 0 at $\theta=\pi/2$.

Ultimately, the determination procedures are repeated for varying deviation angle and varying anisotropic ratio, resulting in a surface plot of the relative error. This, as shown in Figure 5, can be used to evaluate the qualification of TPFA method based on geological formation data. In general, the relative error can be acceptable (<5%) for slight and moderate anisotropy ($\sigma > 0.4$). This becomes significant as the anisotropy of porous media becomes severe, resulting in a sharp increase.
4.3. Computational efficiency

The codes files for both schemes are run in MATLAB on a workstation. The CPU time for various grid resolution scenarios are kept records for comparisons. The results are shown in Figure 6. The CPU time needed for MPFA method at same grid resolution are orders of magnitude higher than that for TPFA method. For large scale methods, MPFA method may result in more accurate but taking more computation time, and possibly fail to present results within limited time. Hence, the method choice of the two methods are actually the compromised balancing between accuracy and computation efficiency. Meanwhile, high efficiency algorithm considering full permeability tensor is of great demands.
5. Conclusions
In this research, the full permeability tensor is presented for the subsurface transport problem in anisotropic medium. The rotation transform is applied for cases that the coordinate is not along with the formation bedding. This results in a new permeability tensor that is not diagonal. Applying the full permeability tensor, the conservation of mass together with flux equations are solved using both TPFA and MPFA methods. The numerical results verified that the TPFA cannot well represent the full tensor of the porous medium and result in overestimation of the total flux. Meanwhile, the comparison indicates that ignoring anisotropy may result in significant errors in determined flux. This research presents a reference for choosing numerical methods given formation information, i.e. TPFA method are applicable in the slight and moderate anisotropy cases ($\sigma > 0.4$) but results in significant errors otherwise.

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