A new geometric condition equivalent to the maximum angle condition for tetrahedrons

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Abstract

For a tetrahedron, suppose that all internal angles of faces and all dihedral angles are less than a fixed constant $C$ that is smaller than $\pi$. Then, it is said to satisfy the maximum angle condition with the constant $C$. The maximum angle condition is important in the error analysis of Lagrange interpolation on tetrahedrons. This condition ensures that we can obtain an error estimation, even on certain kinds of anisotropic tetrahedrons. In this paper, using two quantities that represent the geometry of tetrahedrons, we present an equivalent geometric condition to the maximum angle condition for tetrahedrons.

Keywords: Lagrange interpolation, tetrahedrons, maximum angle condition, finite element

2010 MSC: 65D05, 65N30

1. Introduction

Lagrange interpolation on tetrahedrons and the associated error analysis are important subjects in numerical analysis. They are particularly crucial for the mathematical theory of finite element methods. Let $T$ be an arbitrary tetrahedron and $h_T := \text{diam} T$. Let $k$ be a positive integer, and $P_k(T)$ be the set of all polynomials defined on $T$ whose degree is at most $k$. Let $T^k v \in P_k(T)$
be the Lagrange interpolant of $v \in W^{k+1,p}(T)$, where $W^{k+1,p}(T)$ is a usual Sobolev space with $1 \leq p \leq \infty$ defined on $T$ (see \cite{12} for the definition of Lagrange interpolation). To obtain an error estimate such as

$$
|v - T^k_T v|_{W^{m,p}(T)} \leq C h_T^{k+1-m} |v|_{W^{k+1,p}(T)}, \quad \forall v \in W^{k+1,p}(T)
$$

for an integer $m$ ($0 \leq m \leq k$), we need to impose a certain geometric condition on $T$. The constant $C$ usually depends on this geometric condition. Two well-known geometric conditions for tetrahedrons are listed below.

**Condition 1 (Shape-regularity condition).** Let $\rho_T$ be the diameter of the maximum ball inscribed in $T$. If there exists a fixed constant $\sigma$ such that

$$
\frac{h_T}{\rho_T} \leq \sigma,
$$

then $T$ is said to satisfy the **shape-regularity condition** with $\sigma$. This condition is also called the **inscribed-ball condition**.

The shape-regularity condition requires that $T$ is not too “flat” or degenerated. If a tetrahedron satisfies the shape-regularity condition, it is said to be **isotropic**. If $T$ satisfies the shape-regularity condition, then the estimate (1) holds with $C = C(\sigma)$ for $k = 1$ and $3/2 < p < \infty$ or $k \geq 2$ and $1 \leq p \leq \infty$ (see \cite[Theorem 4.4]{6} and \cite[Theorem 3.1.5]{7}). In \cite{4}, conditions that are equivalent to the shape-regularity condition are discussed.

Besides the shape-regularity condition, the following condition is known for the geometry of tetrahedrons.

**Condition 2 (Maximum angle condition).** For a tetrahedron $T$, suppose that there exists a constant $\gamma_{\text{max}} \in [\pi/3, \pi)$, such that all internal angles of faces and all dihedral angles between faces are less than or equal to $\gamma_{\text{max}}$.

The maximum angle condition for tetrahedrons was introduced by Křížek \cite{14}. Under the maximum angle condition, tetrahedrons may be “flat” or degenerated in a certain way. Such tetrahedrons are said to be **anisotropic**. Estimate (1) can be proved under the maximum angle condition with $C = C(\gamma_{\text{max}})$ for
For the finite element error analysis on anisotropic meshes, readers are refereed to [1, 2]. A useful survey on the geometric conditions associated with triangles and tetrahedrons is given in [5].

Recently, a new error estimation of Lagrange interpolations on tetrahedrons was presented [12, 10]. Let $h_i \ (i=1, \cdots, 6)$ be the edge lengths of $T$ with $h_1 \leq h_2 \leq \cdots \leq h_6 = h_T$. The volume of $T$ is denoted by $|T|$. Let $R_T$, which represents the geometry of tetrahedrons, be defined by

$$R_T := \frac{h_1 h_2 h_T}{|T|} h_T.$$  \hspace{1cm} (2)

Then, for $v \in W^{k+1,p}(T)$, we have

$$|v - T_T^k v|_{W^{m,p}(T)} \leq C(m,k,p) \left( \frac{R_T}{h_T} \right)^m h_T^{k+1-m} |v|_{W^{k+1,p}(T)} \hspace{1cm} (3)$$

with

$$\begin{cases}
2 < p \leq \infty & \text{if } k - m = 0, \\
\frac{3}{2} < p \leq \infty & \text{if } k = 1, m = 0, \\
1 \leq p \leq \infty & \text{if } k \geq 2 \text{ and } k - m \geq 1.
\end{cases} \hspace{1cm} (4)$$

Note that the constant $C(m,k,p)$ depends only on $m$, $k$, $p$ (and the space dimension 3). Therefore, we can apply estimation (3) to arbitrary tetrahedrons.

Remark. In [12], $R_T$ was defined as the projected circumradius of $T$ and (3) was proved. In [10], $R_T$ is redefined as (2), making the proof of (3) much simpler. It is conjectured that $R_T$ defined by (2) and the projected circumradius of $T$ are equivalent.

For Lagrange interpolation on triangles, a similar estimation to (3) holds by setting $R_T$ as the circumradius of $T$ [11]. From the law of sines, we realize that

\[1\text{See also [Erratum Corollary 2D] in https://arxiv.org/abs/2002.09721.}\]
a triangle $T$ satisfies the maximum angle condition (which is defined in a similar manner) if and only if there exists a fixed constant $D$ such that
\[
\frac{R_T}{h_T} = \frac{1}{2 \sin \theta_T} \leq D,
\]
where $\theta_T$ is the maximum internal angle of $T$. If the above inequality holds for a triangle $T$ with a fixed constant $D$, then the triangle is said to satisfy the \textit{semiregularity} condition with $D$.

The aim of this paper is to prove the following theorem, which claims that a similar situation holds for tetrahedrons.

**Theorem 3 (Main theorem).** Let $T \subset \mathbb{R}^3$ be an arbitrary tetrahedron and $R_T$ be defined by (2). Then, $T$ satisfies the maximum angle condition with $\gamma_{\text{max}} \in [\pi/3, \pi)$, if and only if there exists a fixed constant $D = D(\gamma_{\text{max}})$ such that
\[
\frac{R_T}{h_T} \leq D.
\]

**Corollary 4.** Let $k$ be a positive integer. Suppose that a tetrahedron $T$ satisfies the maximum angle condition with $\gamma_{\text{max}} \in [\pi/3, \pi)$. Then, for the Lagrange interpolation $I^k_T$ on $T$, the following estimate holds:
\[
|v - I^k_T v|_{W^{m,p}(T)} \leq Ch_T^{k+1-m} |v|_{W^{k+1,p}(T)}, \quad \forall v \in W^{k+1,p}(T),
\]
where $m, p$ are taken as in (4) and the constant $C$ depends on $m, k, p,$ and $\gamma_{\text{max}}$.

The above mentioned estimations (1), (3), and Corollary 4 give upper bounds of the interpolation errors on a single tetrahedron $T$. Error estimations of the global Lagrange interpolation $I^k_{T_h}$ defined on a simplicial mesh $T_h$ of a bounded polyhedral domain $\Omega$ can be obtained as
\[
|v - I^k_{T_h} v|_{m,p,\Omega} = \left( \sum_{T \in T_h} |v - I^k_T v|_{m,p,T}^{p} \right)^{1/p}, \quad \forall v \in W^{k+1,p}(\Omega).
\]
Hence, from (3), we have
\[
|v - T_h^k v|_{m,p,\Omega} \leq C_1 \max_{T \in T_h} \left( R_T h_T^{k+1-2m} \right) |v|_{k+1,p,\Omega}
\]
\[
\leq C_1 R h^{k+1-2m} |v|_{k+1,p,\Omega}, \quad \forall v \in W^{k+1,p}(\Omega),
\]
where \( h := \max_{T \in T_h} h_T, R := \max_{T \in T_h} R_T, \) and \( C_1 = C_1(m,k,p) \) is independent of the geometry of the tetrahedrons in \( T_h \). If all \( T \in T_h \) satisfy the maximum angle condition with a fixed constant \( \gamma_{\text{max}} \), we have, from Corollary 4,
\[
|v - T_h^k v|_{m,p,\Omega} \leq C_2 h^{k+1-m} |v|_{k+1,p,\Omega}, \quad \forall v \in W^{k+1,p}(\Omega),
\]
where \( C_2 = C_2(m,k,p,\gamma_{\text{max}}) \).

2. Preliminaries

2.1. Notation

Let \( T \) be a tetrahedron in \( \mathbb{R}^3 \) with vertices \( P_1, P_2, P_3, \) and \( P_4 \). The edge connecting \( P_i \) and \( P_j \) and its length are denoted by \( P_i P_j \) and \( |P_i P_j| \), respectively \((i,j = 1,2,3,4, i \neq j)\).

We introduce the following notation convention on \( T \). Let \( F_i \) be the face of \( T \) opposite to \( P_i \). We denote the dihedral angle between the faces \( F_i \) and \( F_j \) by \( \psi^{i,j} \). Note that \( \psi^{j,i} = \psi^{i,j} \). Furthermore, we denote the internal angle at \( P_j \) on \( F_i \) by \( \theta_j^i \), and the angle between \( F_i \) and \( P_i P_j \) by \( \phi_j^i \).

| \( P_i \) | the vertices of \( T \). |
| \( F_i \) | the face opposite to \( P_i \). |
| \( \psi^{i,j} \) | the dihedral angle between \( F_i \) and \( F_j \). |
| \( \theta_j^i \) | the internal angle of \( F_i \) at \( P_j \). |
| \( \phi_j^i \) | the angle between \( F_i \) and \( P_i P_j \). |
Let $A$ and $B$ be the feet of perpendicular lines from $P_j$ to $F_j$ and from $P_j$ to $P_n P_k$, respectively (see Figure 1). Then, we have

$$|P_j P_n| \sin \phi_{j,n} = |P_j A| = |P_j B| \sin \psi_{j,m} = |P_j P_n| \sin \theta_{m,n} \sin \psi_{j,m}.$$ 

A similar equation holds for $\phi_{j,n}$, $\theta_{k,n}$, and $\psi_{k,j}$. Therefore,

$$\sin \phi_{j,n} = \sin \theta_{k,n} \sin \psi_{k,j} = \sin \theta_{m,n} \sin \psi_{m,j},$$

$$j = 1, 2, 3, 4, \quad m, n, k \in \{1, 2, 3, 4\} \backslash \{j\}. \quad (6)$$

### 2.2. Classification of tetrahedrons into two types

As noted in [1][10][12], to deal with arbitrary tetrahedrons (including anisotropic ones) uniformly, we need to classify tetrahedrons into two types. Let $e_2$ be the shortest edge of $T$ and $e_1$ be the longest edge connected to $e_2$. We assume that $P_1$ and $P_2$ are the endpoints of $e_1$, and that $e_2$ is an edge of the face $F_4 = \triangle P_1 P_2 P_3$.

Consider the plane that is perpendicular to $e_1$ and intersects $e_1$ at its midpoint. Then, $\mathbb{R}^3$ is divided by this plane into two half-spaces. In this situation, we have two cases, and tetrahedrons are classified as either Type 1 or Type 2 accordingly:
• **Case 1.** If vertices $P_3$ and $P_4$ belong to the same half-space, then $T$ is classified as Type 1.

• **Case 2.** If vertices $P_3$ and $P_4$ belong to different half-spaces, then $T$ is classified as Type 2.

If $P_3$ or $P_4$ is on the plane, we suppose that $P_3$ and $P_4$ belong to the same half-space.

We now introduce the following assignment of the vertices for each case.

• If $T$ is Type 1, the endpoints of $e_2$ are $P_1$ and $P_3$, and $\alpha_2 := |P_1P_3|$. 

• If $T$ is Type 2, the endpoints of $e_2$ are $P_2$ and $P_3$, and $\alpha_2 := |P_2P_3|$. 

Define $\alpha_1 := |P_1P_2|$ and $\alpha_3 := |P_1P_4|$ for both cases. Note that we have implicitly assumed that $P_1$ and $P_4$ belong to the same half-space for both cases.

![Tetrahedrons of Type 1 (left) and Type 2 (right).](image)

2.3. Another quantity that represents the geometry of $T$

For a tetrahedron $T$, we define $H_T$, which also represents the geometry of $T$, by

$$H_T := \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} h_T. \quad (7)$$

The values $H_T$ and $R_T$, defined by [2], have the following equivalence [10, Lemma 3].
Lemma 5. For an arbitrary tetrahedron $T$, $H_T$ and $R_T$ are equivalent:

$$\frac{1}{2} H_T \leq R_T \leq 2 H_T.$$ 

Therefore, to prove Theorem 3 we may use $H_T$ instead of $R_T$.

3. Lemmas

In this section, we prepare some useful lemmas. In the following, we abbreviate “maximum angle condition” as MAC.

Lemma 6 (Cosine rules on tetrahedrons). Let $T \subset \mathbb{R}^3$ be a tetrahedron. Let $j = 1, 2, 3, 4$ and $\{k, m, n\} = \{1, 2, 3, 4\} \setminus \{j\}$. Then, we have

$$\cos \theta_{kj} = \cos \theta_{jm} \cos \theta_{jn} + \sin \theta_{jm} \sin \theta_{jn} \cos \psi_{mn},$$

$$\cos \psi_{mn} = \sin \psi_{mk} \sin \psi_{nk} \cos \theta_{kj} - \cos \psi_{mk} \cos \psi_{nk}.$$  \hspace{1cm} (8)

Proof. See [9, 16]. \hspace{1cm} □

Lemma 7. Let $T \subset \mathbb{R}^2$ be a triangle and let $\theta_i$ ($i = 1, 2, 3$) be the internal angles of $T$ with $\theta_1 \leq \theta_2 \leq \theta_3$. If there exists $\gamma_{\text{max}} \in [\pi/3, \pi)$ such that $\theta_3 \leq \gamma_{\text{max}}$, then we have

$$\sin \theta_2, \sin \theta_3 \geq \min \left\{ \sin \left( \frac{\pi - \gamma_{\text{max}}}{2} \right), \sin \gamma_{\text{max}} \right\}.$$ \hspace{1cm} (9)

Proof. Because $\theta_1 + \theta_2 + \theta_3 = \pi$, the assumptions yield

$$2\theta_2 \geq \theta_1 + \theta_2 = \pi - \theta_3 \geq \pi - \gamma_{\text{max}} \quad \text{and} \quad \frac{\pi - \gamma_{\text{max}}}{2} \leq \theta_2 \leq \theta_3 \leq \gamma_{\text{max}},$$

which implies (9). \hspace{1cm} □

Lemma 8. For $\gamma \in [\pi/3, \pi)$, we have

$$0 < \frac{\cos \gamma + 1}{\sin \frac{\gamma}{2} + 1} \leq 1.$$ 

Proof. This lemma can be proved immediately from

$$\frac{\cos \gamma + 1}{\sin \frac{\gamma}{2} + 1} = 2 \left( 1 - \sin \frac{\gamma}{2} \right), \quad \frac{\pi}{6} \leq \frac{\gamma}{2} < \frac{\pi}{2}, \quad \frac{1}{2} \leq \sin \frac{\gamma}{2} < 1. \hspace{1cm} □$$
Lemma 9. Let $T \subset \mathbb{R}^3$ be a tetrahedron. Suppose that $T$ satisfies the MAC with $\gamma_{\text{max}} \in [\pi/3, \pi)$. Additionally, assume that $\theta^j_n$ is not the minimum angle of face $F_j = \triangle P_m P_n P_k$, and $\theta^j_n < \pi/2$, where $j = 1, 2, 3, 4$ and $\{m,n,k\} = \{1,2,3,4\} \setminus \{j\}$. Then, setting $\delta$ to

$$\sin \delta = \left( \frac{\cos \gamma_{\text{max}} + 1}{\sin \frac{\gamma_{\text{max}}}{2} + 1} \right)^{1/2}, \quad 0 < \delta \leq \frac{\pi}{2},$$

we have either

$$\psi^{m,j} \geq \delta, \quad \text{or} \quad \psi^{k,j} \geq \delta. \quad (10)$$

Proof. From Lemma 8 we have

$$0 < \frac{\cos \gamma_{\text{max}} + 1}{\sin \frac{\gamma_{\text{max}}}{2} + 1} < 1,$$

and we confirm that $\delta$ is well-defined.

The proof is by contradiction. Suppose that

$$0 < \psi^{m,j} < \delta \quad \text{and} \quad 0 < \psi^{k,j} < \delta.$$

Then, we have $0 < \sin \psi^{m,j} \sin \psi^{k,j} < \sin^2 \delta$ and $1 > \cos \psi^{m,j} \cos \psi^{k,j} > \cos^2 \delta$.

From Lemma 7 and the assumption, we have

$$\frac{\pi - \gamma_{\text{max}}}{2} \leq \theta^j_n < \frac{\pi}{2}, \quad 0 < \cos \theta^j_n \leq \cos \left( \frac{\pi - \gamma_{\text{max}}}{2} \right) = \sin \frac{\gamma_{\text{max}}}{2}.$$

Thus, we obtain

$$\sin \psi^{m,j} \sin \psi^{k,j} \cos \theta^j_n < \sin^2 \delta \sin \frac{\gamma_{\text{max}}}{2}.$$

The cosine rule and the above inequalities yield

$$\cos \psi^{m,k} = \sin \psi^{m,j} \sin \psi^{k,j} \cos \theta^j_n - \cos \psi^{m,j} \cos \psi^{k,j}$$

$$< \sin^2 \delta \sin \frac{\gamma_{\text{max}}}{2} - (1 - \sin^2 \delta)$$

$$= \frac{\cos \gamma_{\text{max}} + 1}{\sin \frac{\gamma_{\text{max}}}{2} + 1} \left( \sin \frac{\gamma_{\text{max}}}{2} + 1 \right) - 1 = \cos \gamma_{\text{max}},$$

which contradicts the MAC: $\psi^{m,k} \leq \gamma_{\text{max}}$. □
Corollary 10. Under the assumptions of Lemma \[9\], we have

\[
\sin \psi^{m,j} \geq C_0, \quad \text{or} \quad \sin \psi^{k,j} \geq C_0, \quad C_0 := \min\{\sin \delta, \sin \gamma_{\max}\}.
\]

Lemma 11. For \( j = 1, 2, 3, 4 \), let \( \{m, n, k\} = \{1, 2, 3, 4\}\setminus\{j\} \). Let \( p \in \{m, n, k\} \), and \( \{q, r\} = \{m, n, k\}\setminus\{p\} \). Suppose that there exists a positive constant \( M \) with \( 0 < M < 1 \) such that \( \sin \phi_j^p \sin \theta_j^n \geq M \). Then, setting \( \gamma(M) := \pi - \sin^{-1} M \left( \frac{\pi}{2} < \gamma(M) < \pi \right) \), the MAC with \( \gamma(M) \) is satisfied on faces \( F_j, F_q, F_r \), and \( \psi^{j,q}, \psi^{j,r} \leq \gamma(M) \).

Proof. From the assumption, we have

\[
M \leq \sin \phi_j^p \sin \theta_j^n \leq \sin \theta_j^n \quad \text{and} \quad M \leq \sin \phi_j^p.
\]

Hence, the definition of \( \gamma(M) \) yields \( \pi - \gamma(M) \leq \theta_j^n \leq \gamma(M) \). Because \( \theta_j^n + \theta_j^m + \theta_j^k = \pi \), we see that \( \theta_j^m, \theta_j^k < \theta_j^n \leq \gamma(M) \). That is, the MAC with \( \gamma(M) \) is satisfied on face \( F_j = \Delta P_m P_n P_k \).

Moreover, it follows from (10) that

\[
M \leq \sin \phi_j^p = \sin \theta_j^p \sin \psi^{q,j} = \sin \theta_j^p \sin \psi^{r,j} \\
\leq \sin \theta_j^p, \sin \theta_j^p, \sin \psi^{r,j}, \sin \psi^{q,j}
\]

By the same reasoning, we find that the MAC with \( \gamma(M) \) is satisfied on faces \( F_q \) and \( F_r \), and \( \psi^{j,q}, \psi^{j,r} \leq \gamma(M) \). □

4. Proof of Theorem \[3\]

In this section, we prove Theorem \[3\]. As explained in Section \[2.3\], we may and will use \( H_T \) instead of \( R_T \) in the proof. We divide the proof into four cases.

4.1. Type 1: Proof of “MAC implies \[3\]”

First, we suppose that \( T \) is of Type 1 and satisfies the MAC with \( \gamma_{\max} \), \( \pi/3 \leq \gamma_{\max} < \pi \). Because \( |T| = \frac{1}{2} \alpha_1 \alpha_2 \alpha_3 \sin \theta_1^4 \sin \phi_1^4 \), we have

\[
\frac{H_T}{h_T} = \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} = \frac{6}{\sin \theta_1^4 \sin \phi_1^4}.
\]
From the definition of Type 1, we realize that $\theta_2^4 \leq \theta_1^4 \leq \theta_3^4$, that is, $\theta_3^4$ and $\theta_2^4$ are the maximum and minimum angles of face $F_4 = \triangle P_1 P_2 P_3$, respectively. Thus, it follows from Lemma 7 that

$$\frac{\pi - \gamma_{\text{max}}}{2} \leq \theta_1^4 \leq \gamma_{\text{max}}, \quad \sin \theta_1^4 \geq \min\left\{\sin \frac{\pi - \gamma_{\text{max}}}{2}, \sin \gamma_{\text{max}}\right\} =: C_1.$$  

Additionally, we may apply Lemma 9 to $\theta_1^4$ and $F_4$, and find that either $\psi_{2,4} \geq \delta$ or $\psi_{3,4} \geq \delta$, where $\delta = \delta(\gamma_{\text{max}}), 0 < \delta \leq \pi/2$ is defined as

$$\sin \delta = \left(\frac{\cos \gamma_{\text{max}} + 1}{\sin \frac{\gamma_{\text{max}}}{2} + 1}\right)^{1/2}. \quad (11)$$

Suppose that $\psi_{2,4} \geq \delta$. By Corollary 10 and (6), we have

$$\sin \phi_1^4 = \sin \theta_1^4 \sin \psi_{2,4} \geq C_0 \sin \theta_1^2,$$

where $C_0$ is the constant defined in Corollary 10. By the definition of Type 1, $\theta_1^2$ is not the minimum angle of $F_2 = \triangle P_1 P_3 P_4$, and therefore, we have

$$\frac{\pi - \gamma_{\text{max}}}{2} \leq \theta_1^2 \leq \gamma_{\text{max}}, \quad \sin \theta_1^2 \geq C_1.$$  

Thus, we obtain $\sin \phi_1^4 \geq C_0 C_1$.

Next, suppose that $\psi_{3,4} \geq \delta$. Replacing $\psi_{2,4}$, $\theta_1^4$, and $F_2$ with $\psi_{3,4}$, $\theta_1^3$, and $F_3$ in the above argument, we obtain $\sin \phi_1^4 \geq C_0 C_1$ in the same manner.

Gathering the above results, we conclude that

$$\frac{H_T}{h_T} = \frac{6}{\sin \theta_1^4 \sin \phi_1^4} \leq \frac{6}{C_0 C_1^2} =: D$$

in both cases, that is, (5) holds.

4.2. Type 1: Proof of “(5) implies MAC”

Now, we suppose that $T$ is of Type 1 and

$$\frac{H_T}{h_T} = \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} = \frac{6}{\sin \theta_1^4 \sin \phi_1^4} \leq D.$$  

Because $\theta_1^4 < \pi/2$ and $\sin \theta_1^4 \sin \phi_1^4 < 1$, we have

$$\sin \theta_1^4 \sin \phi_1^4 \geq \frac{6}{D} =: M, \quad 0 < M < 1.$$
By Lemma \[\text{Lemma 11}\] with \(j = 4\) and \(p = 1\), setting \(\gamma(M) := \pi - \sin^{-1} M\), we have \(\frac{\pi}{2} < \gamma(M) < \pi\), and the MAC with \(\gamma(M)\) is satisfied on \(F_2\), \(F_3\), \(F_4\), and \(\psi^{3,4} \leq \gamma(M)\).

Note that \(|T| = \frac{1}{6} \alpha_1 \alpha_2 \alpha_3 \sin \theta_3^1 \sin \phi_3^1\), and we have

\[
\frac{H_T}{h_T} = \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} = \frac{6}{\sin \theta_3^1 \sin \phi_3^1} \leq D.
\]

Thus, by Lemma \[\text{Lemma 11}\] with \(j = 3\) and \(p = 1\), we find that \(\psi^{2,3} \leq \gamma(M)\).

Because \(|P_3 P_4| < |P_1 P_4| + |P_1 P_3| \leq 2 \alpha_3\) on \(F_2 = \Delta P_1 P_3 P_4\) and \(|P_2 P_3| \leq \alpha_1\), we note that

\[
|T| = \frac{1}{6} \alpha_2 |P_2 P_3| |P_3 P_4| \sin \theta_3^1 \sin \phi_3^1 \leq \frac{1}{3} \alpha_1 \alpha_2 \alpha_3 \sin \theta_3^1 \sin \phi_3^1.
\]

Thus, we have

\[
D \geq \frac{H_T}{h_T} > \frac{3}{\sin \theta_3^1 \sin \phi_3^1} \quad \text{and} \quad \sin \theta_3^1 \sin \phi_3^1 > \frac{3}{D} = M/2.
\]

From Lemma \[\text{Lemma 11}\] setting \(\gamma(M/2) := \pi - \sin^{-1}(M/2)\), we have \(\frac{\pi}{2} < \gamma(M/2) < \pi\) and MAC with \(\gamma(M/2)\) is satisfied on \(F_1\), and \(\psi^{2,1}, \psi^{4,1} \leq \gamma(M/2)\).

The final thing to prove is the MAC for \(\psi^{1,3}\). From the cosine rule \[\text{8}\], we have

\[
\cos \psi^{1,3} = \sin \psi^{3,4} \sin \psi^{4,1} \cos \theta_2^4 - \cos \psi^{3,4} \cos \psi^{4,1}.
\]

By the definition of Type 1, the angle \(\theta_2^4\) is the minimum angle of \(F_4 = \Delta P_1 P_2 P_3\), and therefore, we have

\[
\cos \theta_2^4 \geq \frac{1}{2}, \quad \sin \psi^{3,4} \sin \psi^{4,1} \cos \theta_2^4 > 0, \quad \text{and} \quad \cos \psi^{1,3} > -\cos \psi^{3,4} \cos \psi^{4,1}.
\]

From the above argument, we have \(\sin \psi^{3,4} > M, \sin \psi^{4,1} > M/2, \text{ and} \)

\[
\cos \psi^{3,4} \geq -\cos \psi^{3,4} \cos \psi^{4,1} \geq -|\cos \psi^{3,4}| |\cos \psi^{4,1}|
\]

\[
= -\sqrt{1 - \sin^2 \psi^{3,4}} \sqrt{1 - \sin^2 \psi^{4,1}} > -\sqrt{1 - M^2} \sqrt{1 - M^2} > -1.
\]

Therefore, we conclude that

\[
\psi^{1,3} < \cos^{-1} \left( -\sqrt{1 - M^2} \sqrt{1 - M^2} \right) < \pi,
\]

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and \( T \) satisfies the MAC with
\[
\gamma_{\text{max}} := \max \left\{ \gamma \left( \frac{M}{2} \right), \cos^{-1} \left( -\sqrt{1 - M^2} \sqrt{1 - \frac{M^2}{4}} \right) \right\}.
\]

4.3. Type 2: Proof of “MAC implies (5)”

First, we suppose that \( T \) is of Type 2 and satisfies the MAC with \( \gamma_{\text{max}} \in \left[ \frac{\pi}{3}, \pi \right) \).

The proof is very similar to that described in Section 4.1.

By the definition of Type 2, \( \alpha_3 = |P_1 P_4| \leq |P_2 P_4| \). Because
\[
|T| = \frac{1}{6} \alpha_1 \alpha_2 \alpha_3 \sin \theta_4^1 \sin \phi_4^1 = \frac{1}{6} \alpha_1 \alpha_2 |P_2 P_4| \sin \theta_4^1 \sin \phi_2^1,
\]
we have
\[
\frac{H_T}{h_T} = \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} = \frac{6}{\sin \theta_4^1 \sin \phi_4^1} \leq \frac{6}{\sin \theta_4^1 \sin \phi_2^1}. \tag{12}
\]

From the definition of Type 2, we realize that \( \theta_4^1 \leq \theta_4^2 \leq \theta_4^3 \) on \( F_4 \), \( \theta_2^3 \leq \theta_2^1 \leq \theta_2^3 \) on \( F_3 \), and \( \theta_1^2 \) is not the minimum angle of \( F_1 \). Thus, it follows from Lemma 7 that
\[
\frac{\pi - \gamma_{\text{max}}}{2} \leq \theta_2^1, \theta_3^1, \theta_4^3 \leq \gamma_{\text{max}}, \quad \sin \theta_4^1, \sin \theta_2^1, \sin \theta_2^1 \geq C_1.
\]

Additionally, we may apply Lemma 9 to \( \theta_4^1 \) and \( F_4 \), and find that either \( \psi^{1,4} \geq \delta \) or \( \psi^{3,4} \geq \delta \), where \( \delta = \delta(\gamma_{\text{max}}) \) is defined by (11).

Suppose that \( \psi^{3,4} \geq \delta \). Using the same argument as in Section 4.1 we have
\[
\sin \phi_4^1 = \sin \theta_4^1 \sin \psi^{3,4} \geq C_0 \sin \theta_4^1 \geq C_0 C_1.
\]

Next, suppose that \( \psi^{1,4} \geq \delta \). We have
\[
\sin \phi_2^4 = \sin \theta_2^1 \sin \psi^{1,4} \geq C_0 \sin \theta_2^1 \geq C_0 C_1.
\]

Combining these results with (12), we obtain
\[
\frac{H_T}{h_T} \leq \frac{6}{C_0 C_1^2} =: D,
\]
that is, (5) holds.
4.4. Type 2: Proof of “(5) implies MAC”

Finally, we suppose that $T$ is of Type 2 and

$$
\frac{H_T}{h_T} = \frac{\alpha_1\alpha_2\alpha_3}{|T|} = \frac{6}{\sin \theta_2 \sin \phi_1} \leq D, \quad \sin \theta_2 \sin \phi_1 \geq \frac{6}{D} =: M.
$$

The proof is very similar to that described in Section 4.2. By Lemma 11 with $j = 4$ and $p = 1$, setting $\gamma(M) := \pi - \sin^{-1} M$, the MAC with $\gamma(M)$ is satisfied on $F_2, F_3, F_4$, and $\psi^{3,4}, \psi^{3,4} \leq \gamma(M)$.

Because $|P_2P_4| \leq \alpha_1$, we have

$$
|T| = \frac{1}{6} |P_2P_3||P_2P_4||P_1P_4| \sin \theta_2 \sin \phi_1 \leq \frac{1}{6} \alpha_1 \alpha_2 \alpha_3 \sin \theta_2 \sin \phi_1.
$$

This yields

$$
D \geq \frac{H_T}{h_T} \geq \frac{6}{\sin \theta_2 \sin \phi_1} \quad \text{and} \quad \sin \theta_2 \sin \phi_1 \geq \frac{6}{D} = M,
$$

and, by Lemma 11 with $j = 1$ and $p = 4$, we find that the MAC with $\gamma(M)$ is satisfied on $F_1$, and $\psi^{1,2}, \psi^{1,3} \leq \gamma(M)$.

The final thing to prove is the MAC for $\psi^{1,4}$ and $\psi^{2,3}$. By the cosine rule with $j = 2$, we have

$$
\cos \psi^{1,4} = \sin \psi^{1,3} \sin \psi^{3,4} \cos \theta_2^3 - \cos \psi^{1,3} \cos \psi^{3,4},
$$

$$
\cos \psi^{2,3} = \sin \psi^{2,4} \sin \psi^{3,4} \cos \theta_1^4 - \cos \psi^{2,4} \cos \psi^{3,4}.
$$

By the definition of Type 2, $\theta_2^3$ and $\theta_1^4$ are the minimum angles of $F_3$ and $F_4$, respectively. Therefore, we have $\cos \theta_2^3, \cos \theta_1^4 \geq \frac{1}{4}$ and thus

$$
\cos \psi^{1,4} > -\cos \psi^{1,3} \cos \psi^{3,4}, \quad \cos \psi^{2,3} > -\cos \psi^{2,4} \cos \psi^{3,4}.
$$

Because $\sin \psi^{1,3}, \sin \psi^{2,4}, \sin \psi^{3,4} > M$, we find that

$$
\cos \psi^{1,4} > -\cos \psi^{1,3} \cos \psi^{3,4} \geq -\sqrt{1 - \sin^2 \psi^{1,3}} \sqrt{1 - \sin^2 \psi^{3,4}} > M^2 - 1, \quad \cos \psi^{2,3} > M^2 - 1.
$$

Therefore, we conclude that $\psi^{1,4}, \psi^{2,3} < \cos^{-1}(M^2 - 1) < \pi$, and $T$ satisfies the MAC with

$$
\gamma_{\text{max}} := \max \{ \gamma(M), \cos^{-1}(M^2 - 1) \}.
$$
5. Concluding remarks

The equivalence between the maximum angle condition and the boundedness of $R_T/h_T$ (and $H_T/h_T$) has been established. Because the ratio $R_T/h_T$ appears in many error estimations, the equivalence relationship is very valuable in the mathematical theory of finite element methods. Additionally, because $h_T$ and $R_T$ can be easily computed, this will hopefully be very useful in many practical computations such as mesh generation and adaptive mesh refinement.

Mathematically, an interesting and challenging problem is to extend this result to the case of $d$-simplices ($d \geq 4$). To do this, we need to develop the theory of $d$-simplex geometry further to obtain deeper insights.

Acknowledgments The second and fourth authors were supported by JSPS KAKENHI Grant Number 20H01820. The authors thank the anonymous referees for their valuable comments.

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