NONSEMISIMPLE FUSION ALGEBRAS AND THE VERLINDE FORMULA

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ABSTRACT. We find a nonsemisimple fusion algebra $\mathfrak{F}_p$ associated with each $(1, p)$ Virasoro model. We present a nonsemisimple generalization of the Verlinde formula which allows us to derive $\mathfrak{F}_p$ from modular transformations of characters.

1. INTRODUCTION

Fusion algebras \[1, 2, 3, 4, 5\] describe basis-independent aspects of operator products and thus provide essential information about conformal field theory. They can in principle be found by calculating coinvariants, but the most practical derivation, which at the same time is of fundamental importance, is from the modular transformation properties of characters, via the Verlinde formula \[1\]. The relation between fusion and modular properties can be considered a basic principle underlying consistency of CFT.

A fusion algebra $\mathfrak{F}$ is a unital commutative associative algebra over $\mathbb{C}$ with a distinguished basis (the one corresponding to the “sectors,” or primary fields, of the model) in which the structure constants are nonnegative integers (we refer to this basis as the canonical basis of $\mathfrak{F}$ in what follows).

For rational CFTs, which possess semisimple fusion algebras, the Verlinde formula is often formulated as the motto that “the matrix $S$ diagonalizes the fusion rules.” This involves two statements at least. The first is merely a lemma of linear algebra and can be stated as follows: there exists a matrix $P$ that relates the canonical basis in the fusion algebra to the basis of primitive idempotents. This property is not specific to fusion algebras originating from conformal field theories, and in fact applies to any association scheme \[6\]; we borrow the terminology from \[6\] and call $P$ the eigenmatrix. The second, nontrivial, statement contained in the Verlinde formula is that the eigenmatrix $P$ is related to the matrix $S$ that represents the modular group element $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on the characters of the chiral algebra; this relation is given by $P = S K_{\text{diag}}$, where $K_{\text{diag}}$ (the denominator in the Verlinde formula) is a diagonal matrix whose entries are the inverse of the distinguished row of the $S$
matrix. With $P$ expressed this way, we arrive at the statement that the matrices of the regular representation of the fusion algebra are diagonalized by the $S$ matrix.

This cannot apply to nonsemisimple fusion algebras, however, for which the regular representation matrices cannot be diagonalized. The relation between modular transformations and the structure of nonsemisimple fusion algebras is therefore more difficult to identify, which considerably complicates attempts to build a nonsemisimple Verlinde formula.

Nonsemisimple fusion algebras are expected to arise in logarithmic models of conformal field theory [7, 8, 9, 10, 11, 12, 13, 14, 15], where irreducible representations of the chiral algebra allow nontrivial (indecomposable) extensions. In what follows, we generalize the Verlinde formula and derive nonsemisimple fusion algebras for the series of $(1, p)$ Virasoro models with integer $p \geq 2$.

The $(1, p)$ models provide an excellent illustration of complications involved in generalizing the Verlinde formula to the nonsemisimple case. Unlike the $(p', p)$ models with coprime $p', p \geq 2$, the $(1, p)$ model is defined not as the cohomology, but as the kernel of a screening, and the first question that must be answered in constructing its fusion, as well as the fusion beyond minimal models in general, is:

**Q1:** How to organize the Virasoro representations into a finite number of families? That is, which chiral algebra, extending the Virasoro algebra, is to be used to classify representations?

Assuming that such an algebra has been chosen, the fusion algebra can in principle be derived using different means, e.g., by directly finding coinvariants (if, against expectations, this is feasible). Another possibility is via a Verlinde-like formula, starting with characters of representations of the chosen chiral algebra. Compared to the semisimple case, the basic problems with constructing an analogue of the Verlinde formula are then as follows.

**Q2:** The matrices implementing modular transformations of the characters of chiral algebra representations involve the modular parameter $\tau$ and do not therefore generate a finite-dimensional representation of $SL(2, \mathbb{Z})$. How to extract a $\tau$-independent matrix $S$ representing $S \in SL(2, \mathbb{Z})$ on characters?

**Q3:** With fusion matrices that are not simultaneously diagonalizable, it is not a priori known which “special” (instead of diagonal) matrix form is to be used in a Verlinde-like formula. In other words, how to define the eigenmatrix $P$ that performs the transformation to a “special” form in a nonsemisimple fusion algebra?

**Q4:** Assuming the matrix $S$ is known and the matrix $P$ that performs the transformation to the chosen “special” basis has been selected, how are $S$ and $P$ related?
The most nontrivial part of the nonsemisimple generalization of the Verlinde formula is the answer to Q4. We also note that with a chiral algebra chosen in Q1 we face yet another problem of a “nonsemisimple” nature, originating in the structure of the category of representations of the chosen chiral algebra:

Q5: With indecomposable representations of the chiral algebra involved, how many generators should the fusion algebra have? More specifically, whenever there is a nonsplittable exact sequence $0 \to \mathcal{V}_0 \to \mathcal{R} \to \mathcal{V}_1 \to 0$, should there be fusion algebra generators corresponding to one (i.e., \(\mathcal{R}\)), two (i.e., \(\mathcal{V}_0\) and \(\mathcal{V}_1\)), or three representations? (This becomes critical, e.g., when \(\mathcal{V}_0\) corresponds to the unit element of the fusion algebra, cf. [9]).

We also note the following complications that are already apparent in nonunitary semisimple fusion (see the relevant remarks in [16]), but become more acute for nonsemisimple fusion:

R1: Whenever the S matrix is not symmetric, the sought generalization of the Verlinde formula is sensitive to the choice between S and $S^t$. This is essential, in particular, in selecting the distinguished row/column of S whose elements determine eigenvalues of the fusion matrices (the denominator of the Verlinde formula).

R2: The sector with the minimal conformal weight is different from the vacuum sector. It is therefore necessary to decide which of these two distinguished sectors is to play the “reference” role in the Verlinde formula. (That is, as a continuation of the previous question, the distinguished row of the S matrix to be used in the denominator of the Verlinde formula must be identified properly).

In answering Q5 one must be aware that fusion algebras only provide a “coarse-grained” description of conformal field theory, and there can be several degrees of neglecting the details. The concept of nonsemisimple fusion advocated in [8, 9] aims at accounting for the “fine” structure given by the different ways in which simple (irreducible) modules can be arranged into indecomposable representations. (Such a detailed fusion will be needed, e.g., in studying boundary conditions in conformal field theory models and for a proper interpretation of modular invariants.) In that setting, a natural basis in the fusion algebra would be given by all indecomposable representations (the irreducible ones included). A coarser description

\[1\] In addition, it becomes essential whether a representation or an antirepresentation of $SL(2, \mathbb{Z})$ is considered as the modular group action (in most of the known semisimple examples, this point can safely be ignored).

\[2\] The $p = 2$ fusion in [8, 9] is “intermediate” in that not all of the indecomposable representations are taken into account. But it is certainly sufficient for extracting the coarser, “$K_0$”-fusion that follows from Theorem 5.7 below for $p = 2$. 
is to think of the fusion algebra as the Grothendieck ring of the representation category of the chiral algebra, i.e., as the $K_0$ functor, not distinguishing between different compositions of the same subquotients. \textit{This fusion is sufficient for the construction of the generalized Verlinde formula.} Indeed, the appropriately generalized Verlinde formula should relate the matrix $S$ that represents $S \in SL(2, \mathbb{Z})$ on a collection of characters of the chiral algebra to the fusion algebra structure constants. But the character of an indecomposable representation $\mathcal{R}$ is \textit{the sum of the characters} of its simple subquotients, independently of how the algebra action “glues” them into $\mathcal{R}$. Therefore, for the fusion functor defined for the purpose of constructing the generalized Verlinde formula, an indecomposable representation $\mathcal{R}$ as in Q5 is indistinguishable from the direct sum of $\mathcal{V}_0$ and $\mathcal{V}_1$ (as well as from $\mathcal{R}'$ in $0 \to \mathcal{V}_1 \to \mathcal{R}' \to \mathcal{V}_0 \to 0$). In other words, the element of the fusion algebra corresponding to $\mathcal{R}$ is the sum of those corresponding to $\mathcal{V}_0$ and $\mathcal{V}_1$. In this paper, we only deal with this particular concept of fusion that corresponds to the $K_0$ functor.

Thus, the number of elements in a basis of the fusion algebra associated with a collection $\{\mathcal{V}_j, \mathcal{R}_i\}$ of chiral algebra representations must be given by the number of all \textit{simple subquotients} of all the indecomposable representations $\mathcal{R}_i$ and all simple $\mathcal{V}_j$ (with each irreducible representation occurring just once). But the fact that no linearly independent elements of the fusion algebra correspond to indecomposable representations does \textit{not} mean that “nonsemisimple effects” are neglected: the existence of a nontrivial extension of any two representations $\mathcal{V}_0$ and $\mathcal{V}_1$ already makes the fusion algebra nonsemisimple, giving rise to all of the problems listed above.

The answer to Q1 can be extracted from the literature [17, 9]: we take the maximal local subalgebra in the (nonlocal) chiral algebra that is naturally present in the $(1, p)$ model. This $W$ algebra, denoted by $W(p)$ for brevity, has $2p$ irreducible representations in the $(1, p)$ model.

As regards Q2, the answer amounts to the use of \textit{matrix} automorphy “factors,” as explained below (cf. [18]). The answer to Q3 is related to the structure of associative algebras [19] and, once a canonical basis is fixed, to nonsemisimple generalizations of some notions from the theory of association schemes [6]. Any finitely generated associative algebra $\mathfrak{A}$ (with a unit) is the vector-space sum of a distinguished ideal $\mathfrak{R}$, called the \textit{radical} (the intersection of all maximal left ideals, or equivalently, of all maximal right ideals), and a semisimple algebra (necessarily isomorphic to a direct sum of matrix algebras over division algebras over the base field) [19]. This implies that in any commutative associative algebra, there is a
basis\[ (e_A, w_\alpha), \quad A = 1, \ldots, n', \quad \alpha = 1, \ldots, n'' \]
(with \( n' + n'' = n = \dim \mathfrak{F} \)), composed of primitive idempotents \( e_A \) and elements \( w_\alpha \) in the radical. In the semisimple case, the radical is zero, and “diagonalization of the fusion” can equivalently be stated as the transformation to the basis \( (\lambda_1 e_1, \ldots, \lambda_n e_n) \) of “rescaled idempotents,” where \( \lambda_\alpha \) are scalars read off from the distinguished row of the \( S \) matrix (the row corresponding to the vacuum representation). Let \( X_I, \quad I = 1, \ldots, n, \) denote the elements of the canonical basis in the fusion algebra. Even for semisimple algebras, it is useful to distinguish between the \( S \) matrix that transforms the canonical basis \( X_\bullet \) to the basis \( (\lambda_1 e_1, \ldots, \lambda_n e_n) \) and the matrix \( P \) that transforms the canonical basis to the basis of primitive idempotents, even though \( S \) and \( P \) are related by multiplication with a diagonal matrix. In the nonsemisimple case, the \textit{eigenmatrix} \( P \) that maps the canonical basis to the basis consisting of primitive idempotents and elements in the radical,
\[
\begin{pmatrix}
X_1 \\
\vdots \\
X_n
\end{pmatrix} = P \begin{pmatrix} e_A \\
w_\alpha \end{pmatrix},
\]
is related to the \( S \) matrix in a more complicated way. The resolution of \( Q4 \), which is the heart of the nonsemisimple Verlinde formula, is the construction, from the entries of \( S \), of a (nondiagonal) \textit{interpolating matrix} \( K \) (which plays the role of the denominator in the Verlinde formula) such that
\[ P = S K. \]

The points raised in \( R2 \) and \( R1 \) can be clarified as follows. The rows and columns of \( S \) are labeled by chiral algebra representations in the model under consideration. The \( S \) matrix has a distinguished row that corresponds to the vacuum representation and a distinguished column that corresponds to the minimum-dimension representation of the chiral algebra (the entries in this column are related to the asymptotic form of the characters labeled by the respective rows of \( S \)). The columns of the \( P \) matrix are labeled by the elements \( (e_A, w_\alpha) \) of the basis consisting of primitive idempotents and elements in the radical, and its rows correspond to elements of the canonical basis in the fusion algebra; the distinguished row of \( P \) then corresponds to the unit element of the algebra. (The choice of rows vs. columns in \( P, S, \) and other matrices is of course conventional, but the replacement rows \( \leftrightarrow \) columns must be made consistently with other transpositions and change of the order in matrix multiplication.)
We now summarize our strategy to construct the \((1, p)\) fusion via a nonsemisimple generalization of the Verlinde formula and also describe the contents of the paper:

1. In the \((1, p)\) model, we identify the maximal local algebra \(\mathcal{W}(p)\) as the chiral algebra of the model. There then exist only \(2p\) irreducible \(\mathcal{W}(p)\) representations in the model, which solves Q1. (The algebra is introduced in Sec. 2.2, and its category of representations is described in Secs. 2.3 and 2.4.)

2. We then evaluate the \(2p\) characters \(\chi\) of these representations and find \((\tau\text{-dependent}) 2p \times 2p\) matrices \(J(\gamma, \tau)\) such that \(\chi(\gamma \tau) = J(\gamma, \tau) \chi(\tau)\) for \(\gamma \in SL(2, \mathbb{Z})\). (The characters are evaluated in Sec. 3.1 and their modular transformation properties are derived in Sec. 3.3.)

3. Next, we find a \(2p \times 2p\) automorphy “factor” \(j(\gamma, \tau)\), satisfying the cocycle condition, such that \(\gamma \mapsto \rho(\gamma) = j(\gamma, \tau)J(\gamma, \tau)\) is a representation of \(SL(2, \mathbb{Z})\). This solves Q2 (Secs. 4.1 and 4.2) and gives the \(S\) matrix (Sec. 4.3).

4. From the entries of the distinguished row of \(S = \rho(S)\), we build the interpolating matrix \(\mathcal{K}\) and use it to construct the eigenmatrix \(P\) of the fusion algebra as \(P = SK\). This solves Q4 (Sec. 5.4).

5. From the eigenmatrix \(P\), we uniquely reconstruct the fusion algebra \(\mathcal{F}_p\) in the canonical basis whose elements are labeled by the rows of \(P\), via a recipe that involves answering Q3 (Sec. 5.6).

For the impatient, we here present the answer for the structure constants expressed through the entries of the \(S\) matrix: arranged into matrices \(N_I\) in the standard way, the structure constants of the fusion algebra are given by \(N_I = SO_1S^{-1}\), where \(S = \rho(S)\) acts in a finite-dimensional (in \((1, p)\) models, \(2p\)-dimensional) representation of \(SL(2, \mathbb{Z})\) and \(O_I = O_{I0} \oplus O_{I1} \oplus \ldots \oplus O_{Ip-1}\) are block-diagonal matrices with the \(2 \times 2\) blocks given by \(O_{I0} = \text{diag}(S^1_{I\Omega}, S^2_{I\Omega})\) and

\[
O_{IJ} = \frac{1}{S^2_{I\Omega} - S^2_{J\Omega}} \times \\
\begin{pmatrix}
S^2_{I\Omega} + S^2_{J\Omega} - 2S^2_{I\Omega}S^2_{J\Omega} & S^2_{I\Omega}S^2_{J\Omega} + S^2_{I\Omega}S^2_{J\Omega} & 0 \\
S^2_{I\Omega}S^2_{J\Omega} & S^2_{I\Omega} + S^2_{J\Omega} - 2S^2_{I\Omega}S^2_{J\Omega} & 0 \\
0 & 0 & S^2_{I\Omega} + S^2_{J\Omega} - 2S^2_{I\Omega}S^2_{J\Omega}
\end{pmatrix},
\]

where \(j = 1, \ldots, p-1\) and \(S^I_{J\Omega}\) with \(I, J = 1, 2, \ldots, 2p\) are entries of the \(S\) matrix, with \(S^\bullet_{I\Omega}\) being its row corresponding to the vacuum representation. Thus written, these formulas may seem messy (and the labeling of \(S^I_{J\Omega}\) involves a convention on ordering the representations in accordance with their linkage classes), but they in fact have a clear structure (Eqs. (5.16), (5.13) – (5.14), and (5.8) – (5.10)), to be
explained in what follows. The resulting \((1, p)\) fusion algebra is given in Theorem\[5.7\]. A posteriori, it turns out to have positive integral coefficients, although we do not derive the proposed recipe for the generalized Verlinde formula from first principles such that this property would be guaranteed in advance.

2. The maximal local \(W\) algebra in the \((1, p)\) model

2.1. Energy-momentum tensor, screening operators, and resolutions. For the \((p', p)\) minimal Virasoro models with coprime \(p', p \geq 2\), the Kac table of size \((p'−1) \times (p−1)\) (after suitable identifications of boxes) contains those modules that do not admit nontrivial extensions among themselves. The extended Kac table of size \(p' \times p\) then corresponds to a logarithmic extension. The Kac table is selected as the cohomology, and the extended Kac table as the kernel, of an appropriate screening. We consider the models with \(p' = 1\), where the Kac table is empty, while the extended Kac table has size \(1 \times p\), with its boxes corresponding to Virasoro representations \(V_s, s = 1, \ldots, p\). Similarly to the logarithmically extended \((p', p)\) models, the \((1, p)\) model is also defined as the kernel of the corresponding screening operator (this does not automatically yield its chiral algebra, however, which has then to be found, see below).

The conformal dimensions (weights) of the primary fields corresponding to the irreducible modules \(V_s, s = 1, \ldots, p\), are given by \(\Delta(1, s)\), where for future use we define

\[
\Delta(r, s) := \frac{r^2 - 1}{4p} + \frac{s^2 - 1}{4p} + \frac{1 - rs}{2}.
\]

In the free-field realization through a scalar field \(\varphi(z)\) with the OPE

\[
\varphi(z) \varphi(w) = \log(z - w),
\]

the corresponding primary fields are represented by the vertex operators \(e^{j(1,s)\varphi}\), where

\[
j(r, s) := \frac{1 - r}{2} \alpha_+ + \frac{1 - s}{2} \alpha_-
\]

with

\[
\alpha_+ = \sqrt{2p}, \quad \alpha_- = -\sqrt{\frac{2}{p}}.
\]

Because \(p\alpha_- = -\alpha_+\), we have \(j(r, s + np) = j(r - n, s), n \in \mathbb{Z}\). The energy-momentum tensor is given by

\[
T = \frac{1}{2} \partial \varphi \partial \varphi + \frac{\alpha_0}{2} \partial^2 \varphi
\]

(here and in similar formulas below, normal ordering is implied in the products), where \(\alpha_0 = \alpha_+ + \alpha_-\), and the central charge is \(c = 13 - 6(p + \frac{1}{p})\). There then exist
two screening operators

\[ S_+ = \oint e^{\alpha_+ \varphi}, \quad S_- = \oint e^{\alpha_- \varphi}, \]

satisfying \([S_\pm, T(z)] = 0\).

Let \(F_{j(r,s)}\) denote the Fock module generated from (the state corresponding to) the vertex operator \(e^{j(r,s) \varphi}\) by elements of the Heisenberg algebra generated by the modes of the current \(\partial \varphi\). Set \(F_s = F_{j(1,s)}\), and let the corresponding Feigin–Fuchs module over the Virasoro algebra (2.1) be denoted by the same symbol. For each \(1 \leq s \leq p - 1\), \(F_s\) is included into the acyclic Felder complex

\[ \ldots \leftarrow F_{[s-2p]} \overset{S^{p-s}}{\leftarrow} F_{[s]} \overset{S^s}{\leftarrow} F_{[-s]} \overset{S^{p-s}}{\leftarrow} F_{[-s+2p]} \overset{S^s}{\leftarrow} F_{[s+2p]} \leftarrow \ldots, \]

where \(F_{[s+2np]} = F_{j(1-2n, s)}\).

We define a (nonlocal) algebra \(A(p)\) as the kernel \(A(p) := \text{Ker} S_- |_F\) of the \(S_-\) screening on the direct sum

\[ F := \bigoplus_{r \in \mathbb{Z}} F_{j(r,s)} \]

of Fock modules. The algebra \(A(p)\) is generated by

\[ a^- := e^{-\alpha_+ \varphi} \quad \text{and} \quad a^+ := [S_+, a^-] \]

and is therefore determined by the lattice \(\frac{1}{2} \alpha_+ \mathbb{Z}\). It is slightly nonlocal: the scalar products of lattice vectors are in \(\frac{1}{2} \mathbb{Z}\).

2.2. The maximal local algebra. We next consider the W algebra that is the maximal local subalgebra in \(A(p)\) and use the notation \(W(p)\) for it for brevity. It is generated by the three currents \(W^-, W^0\), and \(W^+\) given by

\[ W^-(z) := e^{-\alpha_+ \varphi}(z), \quad W^0(z) := [S_+, W^-(z)], \quad W^+(z) := [S_+, W^0(z)]. \]

We note that \(W^0\) is a (free-field) descendant of the identity operator, while \(W^+\) is a descendant of \(e^{\alpha_+ \varphi}\). The fields \(W^-, W^0\), and \(W^+\) are Virasoro primaries; their conformal dimensions are given by \(2p - 1\).

2.2.1. Example. For \(p = 3\), the \(W(3)\) generators are given by

\[ W^0 = \frac{1}{2} \partial^3 \varphi \partial^2 \varphi + \frac{1}{4} \partial^4 \varphi \partial \varphi + \frac{3}{2} \sqrt{\frac{3}{2}} \partial^2 \varphi \partial^2 \varphi \partial \varphi + \frac{3}{2} \partial^3 \varphi \partial \varphi \partial \varphi \]

\[ + 3 \partial^2 \varphi \partial \varphi \partial \varphi \partial \varphi + \frac{3}{5} \sqrt{\frac{3}{2}} \partial \varphi \partial \varphi \partial \varphi \partial \varphi \partial \varphi + \frac{1}{20 \sqrt{6}} \partial^5 \varphi, \]
and
\[ W^+ = \left( -\frac{\sqrt{3}}{2} \partial^4 \varphi - 39 \partial^2 \varphi \partial^2 \varphi + 18 \partial^3 \varphi \partial \varphi + 12\sqrt{6} \partial^2 \varphi \partial \varphi - 18 \partial \varphi \partial \varphi \partial \varphi \partial \varphi \right) e^{\sqrt{3} \varphi} \]

(in the last formula, despite the brackets introduced for compactness of notation, the nested normal ordering is from right to left, e.g., \( \partial^2 \varphi(\partial \varphi(\partial \varphi(e^{\sqrt{3} \varphi}))) \)).

2.3. \( \mathcal{W}(p) \) representations. The \( \mathcal{W}(p) \) generators change the “momentum” \( x \) of a vertex \( e^{x \varphi} \) by \( n\alpha_+ \) with integer \( n \), which corresponds to changing \( r \) in \( e^{j(r,s)\varphi} \) by an even integer. It therefore follows that for each fixed \( s = 1, \ldots, p \), the sum
$$ F(s) := \bigoplus_{r \in \mathbb{Z}} F_j(r,s) $$

of Fock modules contains two \( \mathcal{W}(p) \) modules, to be denoted by \( \Lambda(s) \) and \( \Pi(s) \), where \( \Lambda(s) \) is the \( \mathcal{W}(p) \) representation generated from \( e^{j(1,s)\varphi} \) (the highest-weight vector in \( F_j(1,s) \)), while \( \Pi(s) \) is the \( \mathcal{W}(p) \) representation generated from \( e^{j(2,s)\varphi} \) (the highest-weight vector in \( F_j(2,s) \)), see Fig. 1. The dimensions of the corresponding highest-weight vectors are given by
\[
\Delta_{\Lambda(s)} - \frac{c}{24} = \frac{(p-s)^2}{4p} - \frac{1}{24}, \quad \Delta_{\Pi(s)} - \frac{c}{24} = \frac{(2p-s)^2}{4p} - \frac{1}{24}. \tag{2.3}
\]

A somewhat more involved analysis shows that the corresponding kernel of the screening \( S_- \),
\[ \mathcal{K}(s) := \text{Ker} S_- |_{F_j(s)}, \quad s = 1, \ldots, p, \]

is precisely the direct sum
\[ \mathcal{K}(s) = \Lambda(s) \oplus \Pi(s). \]

2.4. Extensions among the representations. We next describe the nontrivial extensions allowed by the \( \mathcal{W}(p) \) representations. The category of representations of \( \mathcal{W}(p) \) in the \( (1, p) \) model decomposes into \textit{linkage classes} of representations, which are full subcategories of the representation category.\(^3\)

The representation category of the algebra \( \mathcal{W}(p) \) associated with the \( (1, p) \) model has \( p + 1 \) linkage classes; we denote them as \( \mathcal{L}\mathcal{C}, \mathcal{L}\mathcal{C}', \) and \( \mathcal{L}\mathcal{C}(s) \) with \( 1 \leq s \leq p-1 \). The indecomposable representations in each linkage class are as follows. The classes \( \mathcal{L}\mathcal{C} \) and \( \mathcal{L}\mathcal{C}' \) contain only a single indecomposable (hence, irreducible)

\(^3\)The term “linkage class” is borrowed from the theory of finite-dimensional Lie algebras. The linkage classes of an additive category \( \mathcal{C} \) are additive full subcategories \( \mathcal{C}_i \) such that there are no (nonzero) morphisms between objects in two distinct linkage classes, every object of \( \mathcal{C} \) is a direct sum of objects of the linkage classes, and none of the \( \mathcal{C}_i \) can be split further in the same manner.
representation each, $\Lambda(p)$ and $\Pi(p)$ respectively. For $1 \leq s \leq p-1$, the linkage class $\mathcal{L}(s)$ contains two irreducible representations $\Lambda(s)$ and $\Pi(p-s)$, as well as the following set of other indecomposable representations:

$$\mathcal{N}_0^\pm(s), \quad \mathcal{N}_1(s), \quad \mathcal{R}_0(s), \quad \mathcal{R}_1(s).$$

There are nontrivial extensions

$$0 \to \Lambda(s) \to \mathcal{N}_0^\pm(s) \to \Pi(p-s) \to 0,$$

$$0 \to \Pi(p-s) \to \mathcal{N}_1(s) \to \Lambda(s) \to 0,$$

and in addition,

$$0 \to \Lambda(s) \to \mathcal{N}_0(s) \to \Pi(p-s) \oplus \Pi(p-s) \to 0,$$

$$0 \to \Pi(p-s) \to \mathcal{N}_1(s) \to \Lambda(s) \oplus \Lambda(s) \to 0.$$
We note that $L_0$ is diagonalizable in each of these representations. The “logarithmic” modules (those with a nondiagonalizable action of $L_0$) appear in the extensions

\[ 0 \to N_0(s) \to R_0(s) \to \Lambda(s) \to 0, \quad 0 \to N_1(s) \to R_1(s) \to \Pi(p-s) \to 0. \]

It follows that $N_0^\pm(s) \cap N_0^\mp(s) = \Lambda(s)$ and $N_1^\pm(s) \cap N_1^\mp(s) = \Pi(p-s)$. Thus we have towers of indecomposable representations given by

\[ R_0(s) \supset N_0(s) \supset N_0^\pm(s) \supset \Lambda(s), \quad R_1(s) \supset N_1(s) \supset N_1^\pm(s) \supset \Pi(p-s) \]

for each $s = 1, \ldots, p - 1$. The detailed structure of these representations will be considered elsewhere (see more comments in the Conclusions, however).

\[ 2.4.1. \text{Example.} \] For $p = 2$, the four irreducible representations are $V_{-1} = \Lambda(2)$, $V_{2} = \Pi(2)$, $V_0 = A(1)$, and $V_1 = \Pi(1)$. The “logarithmic” modules are $R_0 = R_0(1)$ and $R_1 = R_1(1)$. In addition, there are six other indecomposable representations $N_0^\pm$, $N_0$, $N_1^\pm$, and $N_1$.

### 3. Modular transformations of the $\mathcal{W}(p)$ characters

In this section, we evaluate the characters of the $\mathcal{W}(p)$ representations introduced above and find their modular transformation properties.

**3.1. Calculation of the $\mathcal{W}(p)$ characters.** The route from representations to fusion starts with the characters of $\mathcal{W}(p)$ representations. We write $\chi_{\Xi,s,p}$ with $\Xi \in \{A, \Pi\}$ for the character of $\Xi(s)$ in the $(1, p)$ model,

\[ \chi_{\Xi,s,p}(q) = \langle q^{L_0 - \frac{s}{2p}} \rangle_{\Xi(s)}. \]

**3.2. Proposition.** The $\mathcal{W}(p)$ characters are given by

\[ \chi_{s,p}^A(q) = \frac{1}{\eta(q)} \left( \frac{s}{p} \theta_{p-s,p}(q) + 2 \theta'_{p-s,p}(q) \right), \]

\[ \chi_{s,p}^\Pi(q) = \frac{1}{\eta(q)} \left( \frac{s}{p} \theta_{s,p}(q) - 2 \theta'_{s,p}(q) \right), \quad 1 \leq s \leq p. \]

Here, we use the eta function

\[ \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \]

and the theta functions

\[ \theta_{s,p}(q, z) = \sum_{j \in \mathbb{Z} + \frac{s}{p}} q^{pj} z^j, \quad |q| < 1, \quad z \in \mathbb{C}, \]

and set $\theta_{s,p}(q) := \theta_{s,p}(q, 1)$ and $\theta'_{s,p}(q) := \frac{\partial}{\partial z} \theta_{s,p}(q, z) \big|_{z=1}$. 

}\]
Proof. Formulas (3.1) (which also have been suggested in [20]) can be derived by standard calculations, which we outline here for completeness.

The characters of $\Lambda(s)$ and $\Pi(s)$ are found by summing the characters of the kernels of $S_-$ on the appropriate Fock modules,

$$\chi^A_{s,p} = \text{char } \mathcal{K}(1, s) + 2 \sum_{a \geq 1} \text{char } \mathcal{K}(2a+1, s),$$

$$\chi^{\Pi}_{s,p} = 2 \sum_{a \geq 1} \text{char } \mathcal{K}(2a, s),$$

where

$$\mathcal{K}(r, s) := \text{Ker } S_-|_{\mathcal{F}_{j(r,s)}}.$$ The character of $\mathcal{K}(r, s)$, in turn, is easily calculated from a “half” of the complex (2.2), i.e., from the one-sided resolution, as either the kernel or the image of the corresponding differential, which amounts to taking the alternating sum of characters of the modules in the left or right part of the complex. A standard calculation (with some care to be taken in rearranging double sums) then gives the formulas in the proposition. □

3.3. $S$ and $T$ transformations of the characters. With the characters of $\Lambda(s)$ and $\Pi(s)$ expressed through theta functions, it is straightforward to find their modular properties. We resort to the standard abuse of notation by writing $\theta_{s,p}(\tau)$ for $\theta_{s,p}(e^{2i\pi \tau})$, for $\tau$ in the upper complex half-plane $\mathfrak{h}$.

3.4. Proposition. Under the $S$ transformation of $\tau$, the $W(p)$ characters transform as

$$\chi^A_{s,p}(-\frac{1}{\tau}) = \frac{1}{\sqrt{2p}} \left( \frac{s}{p} \chi^A_{p,p}(\tau) + (-1)^{p-s} \chi^{\Pi}_{p,p}(\tau) 
+ 2 \sum_{\ell=1}^{p-1} \cos\left( \frac{\ell(p-s)}{p} \right) \left( \chi^A_{p-\ell,p}(\tau) + \chi^{\Pi}_{\ell,p}(\tau) \right) 
- 2i\tau \sum_{\ell=1}^{p-1} \sin\left( \frac{\ell(p-s)}{p} \right) \left( \frac{\ell}{p} \chi^A_{p-\ell,p}(\tau) - \frac{p-\ell}{p} \chi^{\Pi}_{\ell,p}(\tau) \right) \right)$$

and

$$\chi^{\Pi}_{s,p}(-\frac{1}{\tau}) = \frac{1}{\sqrt{2p}} \left( \frac{s}{p} \chi^A_{p,p}(\tau) + (-1)^{s} \chi^{\Pi}_{p,p}(\tau) 
+ 2 \sum_{\ell=1}^{p-1} \cos\left( \frac{\ell s}{p} \right) \left( \chi^A_{p-\ell,p}(\tau) + \chi^{\Pi}_{\ell,p}(\tau) \right) 
+ 2i\tau \sum_{\ell=1}^{p-1} \sin\left( \frac{\ell s}{p} \right) \left( \frac{\ell}{p} \chi^A_{p-\ell,p}(\tau) - \frac{p-\ell}{p} \chi^{\Pi}_{\ell,p}(\tau) \right) \right)$$

(with $i = \sqrt{-1}$).
The functions $\theta_{s,p}$ and $\theta'_{s,p}$ are modular forms of different weights ($\frac{1}{2}$ and $\frac{3}{2}$ respectively) and do not therefore mix in modular transformations. In contrast, the characters are linear combinations of modular forms of weights 0 and 1 and hence involve explicit occurrences of $\tau$ in their $S$ transformation.

**Proof.** The formulas in the proposition are shown by directly applying the well-known relations

\[
\theta_{s,p}(-\frac{1}{\tau}) = \sqrt{-\frac{i}{2p}} \left( \theta_{0,p}(\tau) + (-1)^s \theta_{p,p}(\tau) + 2 \sum_{\ell=1}^{p-1} \cos\left(\pi\frac{\ell s}{p}\right) \theta_{\ell,p}(\tau) \right),
\]

\[
\theta'_{s,p}(-\frac{1}{\tau}) = -2i \sqrt{-\frac{i}{2p}} \sum_{\ell=1}^{p-1} \sin\left(\pi\frac{\ell s}{p}\right) \theta'_{\ell,p}(\tau).
\]

\[
\square
\]

3.4.1. The $S_p(\tau)$ matrix. We now write the $S$ transformation in a matrix form. To this end, we order the representations as

\[
(3.2) \quad A(p), \quad II(p), \quad A(1), \quad II(p-1), \ldots, \quad A(p-1), \quad II(1),
\]

and arrange the characters into a column vector $\chi_p$,

\[
\chi_p = (\chi_{p,p}^A, \chi_{p,p}^N, \chi_{1,p}^A, \chi_{p-1,p}^N, \ldots, \chi_{p-1,p}^A, \chi_{1,p}^N).
\]

This order is chosen such that representations in the same linkage class are placed next to each other; it is one of the ingredients that make the block structure explicit in what follows. The above $S$ transformation formulas then become

\[
(3.3) \quad \chi_p(-\frac{1}{\tau}) = S_p(\tau) \chi_p(\tau),
\]

where $S_p(\tau)$ is most conveniently written using the $2 \times 2$ block notation

\[
(3.4) \quad S_p(\tau) = \begin{pmatrix}
A_{0,0} & A_{0,1} & \cdots & A_{0,p-1} \\
A_{1,0} & A_{1,1} & \cdots & A_{1,p-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{p-1,0} & A_{p-1,1} & \cdots & A_{p-1,p-1}
\end{pmatrix}
\]

with

\[
A_{0,0} = \frac{1}{\sqrt{2p}} \begin{pmatrix} 1 & 1 \\ 1 & (-1)^p \end{pmatrix}, \quad A_{0,j} = \frac{2}{\sqrt{2p}} \begin{pmatrix} 1 & 1 \\ (-1)^{p-j} & (-1)^{p-j} \end{pmatrix},
\]

\[
A_{s,0} = \frac{1}{\sqrt{2p}} \begin{pmatrix} s/p & (-1)^{p+s} \\ p-s/p & (-1)^{p+s-p}\end{pmatrix}.
\]
and

\[ A_{s,j} = \sqrt{\frac{2}{p}} (-1)^{p+j+s} \times \]

\[ \begin{pmatrix}
\frac{s}{p} \cos \frac{\pi sj}{p} - i \frac{p-j}{p} \sin \frac{\pi sj}{p} & \frac{s}{p} \cos \frac{\pi sj}{p} + i \frac{j}{p} \sin \frac{\pi sj}{p} \\
\frac{p-s}{p} \cos \frac{\pi sj}{p} + i \frac{p-j}{p} \sin \frac{\pi sj}{p} & \frac{p-s}{p} \cos \frac{\pi sj}{p} - i \frac{j}{p} \sin \frac{\pi sj}{p}
\end{pmatrix} \]

for \(1 \leq s, j \leq p-1\).

### 3.4.2. The \(T_p\) matrix.

We next find the \(T\) transformation of the \(\mathcal{W}(p)\) characters. For the vector \(\chi_p\) introduced above, we have

\[ \chi_p(\tau+1) = T_p \chi_p(\tau), \tag{3.5} \]

where \(T_p\) is a block-diagonal matrix that can be compactly written as a direct sum of \(2 \times 2\) blocks,

\[ T_p = T_0 \oplus T_1 \oplus \cdots \oplus T_{p-1} \tag{3.6} \]

with

\[ T_0 = \begin{pmatrix}
e^{-i \frac{\pi}{p}} & 0 \\
0 & e^{i \pi (\frac{p-s}{p})}
\end{pmatrix}, \quad T_s = e^{i \pi (\frac{p-s}{p})} 1_{2 \times 2}, \quad s = 1, \ldots, p-1. \tag{3.7} \]

Starting from the \(\mathcal{W}(p)\) algebra in \((1, p)\) models, we thus arrived at the \(S_p(\tau)\) and \(T_p\) matrices that implement modular transformations on the characters. Problem Q1 in the Introduction has thus been solved. With the resulting \(S_p(\tau)\) involving a dependence on \(\tau\), we next face problem Q2 to be addressed in the next section.

### 4. A Finite-Dimensional \(SL(2, \mathbb{Z})\) Representation from Characters

#### 4.1. Matrix automorphy factors.

The modular group action on characters generated by \((3.3)\) and \((3.5)\) fits the following general scheme. It is well known (or easily checked) that

\[ (\gamma \cdot f)(\tau, \nu) := j(\gamma; \tau, \nu) f(\gamma \tau, \gamma \nu), \tag{4.1} \]

with \(j(\gamma; \tau, \nu)\) an \(n \times n\) -matrix satisfying the cocycle condition

\[ j(\gamma; \tau, \nu) = j(\gamma'; \tau, \nu) j(\gamma; \gamma' \tau, \gamma' \nu), \quad j(1; \tau, \nu) = 1_{n \times n}, \tag{4.2} \]

furnishes an action (actually, an \textit{anti}representation) of the modular group \(SL(2, \mathbb{Z})\) on the space of functions \(f: \mathfrak{h} \times \mathbb{C} \to \mathbb{C}^n\). We use the standard \(SL(2, \mathbb{Z})\) action
on $\mathfrak{h} \times \mathbb{C}$,
\[
  \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\tau, \nu) \mapsto (\gamma \tau, \gamma \nu) := \begin{pmatrix} \frac{a \tau + b}{c \tau + d} & \frac{\nu}{c \tau + d} \end{pmatrix}
\]
(the notation $\gamma \nu$ is somewhat loose, because this action depends on $\tau$). The matrix $j(\gamma; \tau, \nu)$ is called the (matrix) automorphy factor.

An example of a scalar automorphy factor is given by the following classic result in the theory of theta functions [21]: the Jacobi theta function $\vartheta(\tau, \nu)$ is invariant under the action of $\Gamma_{1,2} \subset SL(2, \mathbb{Z})$ (the subgroup of $SL(2, \mathbb{Z})$ matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ab$ and $cd$ even) on functions $f: \mathfrak{h} \times \mathbb{C} \to \mathbb{C}$ given by
\[
  (\gamma \cdot f)(\tau, \nu) = j\bigg(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \tau, \nu\bigg) f(\gamma \tau, \gamma \nu)
\]
with the automorphy factor
\[
  j\bigg(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \tau, \nu\bigg) = \zeta^{-1}_{c,d} (c \tau + d)^{-\frac{1}{2}} e^{-i \pi \frac{c d}{c \tau + d}},
\]
where $\zeta_{c,d}$ is an eighth root of unity (see [21]; its definition, which is far from trivial, ensures the cocycle condition for $j$).

4.2. Constructing a finite-dimensional $SL(2, \mathbb{Z})$ representation. The $\mathcal{W}(p)$ characters that we study here do not involve the $\nu$ dependence. Because $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $SL(2, \mathbb{Z})$, Eqs. (3.3) and (3.5) uniquely determine a $2p \times 2p$ matrix $J_p(\gamma, \tau)$ such that
\[
  \chi_p(\gamma \tau) = J_p(\gamma, \tau) \chi_p(\tau)
\]
for all $\gamma \in SL(2, \mathbb{Z})$. It then follows that $J_p$ satisfies the condition
\[
  J_p(\gamma \gamma', \tau) = J_p(\gamma, \gamma' \tau) J_p(\gamma', \tau), \quad \gamma, \gamma' \in SL(2, \mathbb{Z}).
\]

Given this $SL(2, \mathbb{Z})$ action, we now seek an $SL(2, \mathbb{Z})$ action on $\chi_p: \mathfrak{h} \to \mathbb{C}^{2p}$ with a $2p \times 2p$ matrix automorphy factor $j_p$,
\[
  \gamma \cdot \chi_p(\tau) = j_p(\gamma, \tau) \chi_p(\gamma \tau) = j_p(\gamma, \tau) J_p(\gamma, \tau) \chi_p(\tau),
\]
such that
\[
  \rho(\gamma) := j_p(\gamma, \tau) J_p(\gamma, \tau)
\]
is a finite-dimensional representation of $SL(2, \mathbb{Z})$ (in particular, the left-hand side must be independent of $\tau$). This condition is reformulated as the condition that $\rho$ and $j_p$ “strongly” commute, i.e., that
\[
  \rho(\gamma) j_p(\gamma', \tau) = j_p(\gamma', \tau) \rho(\gamma), \quad \gamma, \gamma' \in SL(2, \mathbb{Z}).
\]
It is easy to verify that for a given $J_p(\cdot, \cdot)$, each $j_p$ that satisfies both the commutation property (4.6) (with $\rho$ defined by (4.5)) and the cocycle condition (4.2) provides a (finite-dimensional) $SL(2, \mathbb{Z})$ representation $\rho$. Indeed,

$$
\rho(\gamma \gamma') = j_p(\gamma \gamma', \tau) J_p(\gamma \gamma', \tau) = j_p(\gamma', \tau) \rho(\gamma) J_p(\gamma', \tau) = \rho(\gamma) j_p(\gamma', \tau) J_p(\gamma', \tau) = \rho(\gamma) \rho(\gamma').
$$

4.3. $SL(2, \mathbb{Z})$ representation in $(1, p)$ models. We now find a matrix automorphy factor $j_p$ that “converts” the action in (3.3) – (3.6) into a representation. As noted above, $J_p(\cdot, \tau)$ is uniquely determined on all of $SL(2, \mathbb{Z})$ by Eqs. (4.4) from $J_p(T, \tau) = T_p(\tau)$ and $J_p(S, \tau) = S_p(\tau)$. With $S_p(\tau)$ and $T_p(\tau) = T_p$ given by (3.4) and (3.6), we define the automorphy factor $j_p(\cdot, \cdot)$ as a block-diagonal matrix consisting of $2 \times 2$ blocks that we compactly write as

$$
(4.7) \quad j_p(\gamma, \tau) = 1_{2 \times 2} \oplus B_1(\gamma, \tau) \oplus \cdots \oplus B_{p-1}(\gamma, \tau),
$$

where for $\gamma = S$,

$$
(4.8) \quad B_s(S, \tau) = \begin{pmatrix}
\frac{s}{p} + \frac{p-s}{\tau p} & \frac{s}{p} - \frac{s}{\tau p} \\
\frac{p-s}{p} - \frac{s}{\tau p} & \frac{p-s}{p} + \frac{s}{\tau p}
\end{pmatrix}, \quad s = 1, \ldots, p-1,
$$

and for $\gamma = T$,

$$
(4.9) \quad B_s(T, \tau) = \begin{pmatrix}
\frac{s}{p} + t \frac{p-s}{p} & \frac{s}{p} - \frac{s}{p} \\
\frac{p-s}{p} - t \frac{p-s}{p} & \frac{p-s}{p} + \frac{s}{p}
\end{pmatrix}, \quad s = 1, \ldots, p-1,
$$

with $t^3 = -i$ (we can set $t = i$). The structure in (4.8) is easily discernible by subjecting all matrices to the similarity transformation that relates the basis of characters to the basis provided by $\theta_{s,p}$ and $\theta'_{s,p}$. The automorphy factor is then diagonalized, as shown explicitly in the proof of the next proposition.

4.4. Proposition. The matrix automorphy factor defined in (4.7) – (4.9) satisfies the cocycle condition (4.2).

Proof. The proof amounts to a direct verification of the formulas $(ST)^3 = (TS)^3 = S^2$ reformulated for $j_p(\gamma, \tau)$. That is, in proving that $j_p(S^2, \tau) = j_p((ST)^3, \tau)$, we have, in accordance with (4.2),

$$
(4.10) \quad j_p(S^2, \tau) = j_p(S, \tau) j_p(S, -\frac{1}{\tau}),
$$

$$
j_p((ST)^3, \tau) = j_p(ST, \tau) j_p(ST, -\frac{1}{\tau+1}) j_p(ST, -\frac{\tau-1}{\tau}).
$$
where in turn, \( j_p(ST, \tau) = j_p(T, \tau) j_p(S, \tau + 1) \). The calculation reduces to a separate computation for each of the \( 2 \times 2 \) blocks given above; further, each block can be diagonalized as

\[
B_s(\gamma, \tau) = L_s \begin{pmatrix} 1 & 0 \\ 0 & \zeta(\gamma) \alpha(\gamma, \tau) \end{pmatrix} L_s^{-1},
\]

where \( \zeta(\gamma) \) is the character of \( SL(2, \mathbb{Z}) \) defined by the relations

\[
(4.11) \quad \zeta(S) = i, \quad \zeta(T) = t, \quad t^3 = -i,
\]

and

\[
\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau = \frac{1}{ct + d}
\]

is already an automorphy factor \([21]\). Equations \((4.11)\) immediately imply that \( \zeta(S^2) = \zeta((ST)^3) \), and Eqs. \((4.10)\) are therefore proved. □

With this \( j_p \), we evaluate

\[
S(p) = j_p(S, \tau) S_p(\tau) = j_p(S, \tau) S_p(\tau) = S_p(i).
\]

That is, \( S(p) \) has a block form similar to that of \( S_p \) in Sec. 3.3 with the \( 2 \times 2 \) blocks \( S_{i,j} \) given by \( S_{0,0} = A_{0,0}, S_{0,j} = A_{0,j}, S_{s,0} = A_{s,0} \), and

\[
S_{s,j} = \sqrt{\frac{2}{p}} (-1)^{p + j + s} \begin{pmatrix} \frac{s p \cos \pi sj}{p} + \frac{p - j}{p} \sin \frac{\pi sj}{p} & \frac{s p \cos \pi sj}{p} - \frac{j}{p} \sin \frac{\pi sj}{p} \\ \frac{p - s}{p} \cos \frac{\pi sj}{p} - \frac{p - j}{p} \sin \frac{\pi sj}{p} & \frac{p - s}{p} \cos \frac{\pi sj}{p} + \frac{j}{p} \sin \frac{\pi sj}{p} \end{pmatrix}.
\]

Similarly,

\[
T(p) = j_p(T, \tau) T_p
\]

(where as we have seen, \( j_p(T, \tau) \) is actually independent of \( \tau \)). We do not write the blocks of \( T(p) \) explicitly because they are simply given by multiplication of the blocks in \((4.9)\) with matrices \((3.7)\).

4.5. Proposition. The matrices \( S(p) \) and \( T(p) \) generate a finite-dimensional representation of \( SL(2, \mathbb{Z}) \).

Proof. The proof consists in verifying \((4.6)\) for \( (\gamma, \gamma') \) being any of the pairs \( (S, T) \), \( (T, S) \), \( (S, S) \), and \( (T, T) \), which is straightforward. Together with the cocycle condition, this then implies that \( (S(p))^2 = (T(p)S(p))^3 = (S(p)T(p))^3 \). □

The above construction of the numeric (\( \tau \)-independent) matrix \( S(p) \) representing \( S \in SL(2, \mathbb{Z}) \) solves problem Q2 in the Introduction.
4.6. Some properties of the $S(p)$ matrix. The vacuum representation $\Lambda(1)$ is the third in the order of representations chosen in (3.2). This distinguishes the third row of the $S$ matrix; we let $\sigma_\Omega(p) \equiv \sigma_\Omega$ denote this distinguished row of $S(p)$. Explicitly, $\sigma_\Omega(p)$ is given by

\begin{equation}
\sigma_\Omega(p) = (-1)^p \frac{\sqrt{2}}{p^2} \left( \frac{-1}{2}, \frac{1}{2}, \right),
\end{equation}

\begin{align*}
&\cos \frac{\pi}{p} + (p-1) \sin \frac{\pi}{p}, \cos \frac{\pi}{p} - \sin \frac{\pi}{p}, \\
&- \cos \frac{2\pi}{p} - (p-2) \sin \frac{2\pi}{p}, \cos \frac{2\pi}{p} + 2 \sin \frac{2\pi}{p}, \\
&\ldots, \\
&(-1)^{j+1} \left( \cos \frac{j\pi}{p} + (p-j) \sin \frac{j\pi}{p} \right), (-1)^{j+1} \left( \cos \frac{j\pi}{p} - j \sin \frac{j\pi}{p} \right), \\
&\ldots, \\
&(-1)^p \left( \cos \frac{(p-1)\pi}{p} + \sin \frac{j\pi}{p} \right), (-1)^p \left( \cos \frac{(p-1)\pi}{p} - (p-1) \sin \frac{(p-1)\pi}{p} \right).
\end{align*}

Next, it follows from (3.3) that $(S(p))^2 \chi_p(i) = \chi_p(i)$. In fact, we have the following result.

4.7. Proposition.

\begin{equation}
(S(p))^2 = 1_{2p \times 2p}.
\end{equation}

Proof. Indeed, we evaluate $(S(p))^2$ as

\begin{align*}
\rho(S) \rho(S) &= \rho(S) j_p(S, \tau) J_p(S, \tau) \rho(S) J_p(S, \tau) \\
&= j_p(S, \tau) J_p(S, \tau) J_p(S, \tau) J_p(S, \tau) \rho(S) \rho(S).
\end{align*}

Next, $J_p(S^2, \tau) = 1_{2p \times 2p}$ because $S^2 \tau = \tau$. Finally we have $j_p(S, \tau) j_p(S, \tau) = 1_{2p \times 2p}$, which is obtained by a direct calculation similar to the one in the proof of Prop. 4.4. Equation (4.14) thus follows. \hfill \Box

4.8. Remark. With the explicit form of $S(p)$ given above, Prop. 4.7 can also be shown directly, which gives a good illustration of a typical calculation with the matrices encountered throughout this paper. Writing $C = (S(p))^2$ in the $2 \times 2$-block form

\begin{equation*}
C = \begin{pmatrix}
C_{0,0} & C_{0,1} & \ldots & C_{0,p-1} \\
C_{1,0} & C_{1,1} & \ldots & C_{1,p-1} \\
C_{p-1,0} & C_{p-1,1} & \ldots & C_{p-1,p-1}
\end{pmatrix},
\end{equation*}

Next, $J_p(S^2, \tau) = 1_{2p \times 2p}$ because $S^2 \tau = \tau$. Finally we have $j_p(S, \tau) j_p(S, \tau) = 1_{2p \times 2p}$, which is obtained by a direct calculation similar to the one in the proof of Prop. 4.4. Equation (4.14) thus follows. \hfill \Box

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\begin{equation*}
C = \begin{pmatrix}
C_{0,0} & C_{0,1} & \ldots & C_{0,p-1} \\
C_{1,0} & C_{1,1} & \ldots & C_{1,p-1} \\
C_{p-1,0} & C_{p-1,1} & \ldots & C_{p-1,p-1}
\end{pmatrix},
\end{equation*}
we concentrate on the more involved blocks $C_{s,j}$ with $0 < s, j < p$. Assuming that $p$ is odd for brevity (in order to avoid extra sign factors) we find that

$$C_{s,j} = \frac{2}{p^2} \times$$

$$\begin{pmatrix}
\sum_{\ell=[s+j]}^{p-1} \cos \pi \frac{\ell(p-j)}{p} \cos \pi \frac{\ell(p-s)}{p} \\
\sum_{\ell=[s+j]}^{p-1} \cos \pi \frac{\ell(p-j)}{p} \cos \pi \frac{\ell(p-s)}{p} \\
\sum_{\ell=[s+j]}^{p-1} \cos \pi \frac{\ell(p-j)}{p} \cos \pi \frac{\ell(p-s)}{p}
\end{pmatrix},$$

where $[a]_2 := a \mod 2$. Using elementary trigonometric rearrangements (expressing $\cos \alpha \sin \beta$ through the sine and cosine of $\alpha + \beta$ and $\alpha - \beta$), we see that all entries in the matrices above vanish, with the exception of the diagonal entries of $C_{s,s}$, which (for $0 < s \leq p$) are given by

$$\frac{2}{p^2} \left( s \sum_{\ell=0}^{p-1} \left( \cos \pi \frac{\ell(p-s)}{p} \right)^2 + (p-s) \sum_{\ell=1}^{p-1} \left( \sin \pi \frac{\ell(p-s)}{p} \right)^2 \right) = 1.$$

Together with similar (and in fact, simpler) calculations for the other blocks, this shows \(4.14\).

We also note that $S(p)$ is not symmetric, $S(p) \neq S(p)^t$. It admits a different symmetry

\[(4.15) \quad S(p)^\vee = S(p),\]

where for a matrix $r = (r_{ij})_{i,j=1,\ldots,2p}$ with $i$ and $j$ considered modulo $2p$, we define the involutive operation

$$(r^\vee)_{m,n} := (-1)^{p(1-\delta_{m,1}-\delta_{n,1})+(m+n+1)/2+mn} r_{2p-m+3,2p-n+3}.$$

For example, with $r = (r_{ij})_{i,j=1,\ldots,6}$, we have

$$r^\vee = \begin{pmatrix}
-r_{22} & r_{21} & -r_{26} & -r_{25} & r_{24} & r_{23} \\
-r_{12} & -r_{11} & r_{16} & r_{15} & -r_{14} & -r_{13} \\
-r_{62} & r_{61} & -r_{66} & -r_{65} & r_{64} & r_{63} \\
-r_{52} & r_{51} & -r_{56} & -r_{55} & r_{54} & r_{53} \\
r_{42} & -r_{41} & r_{46} & r_{45} & -r_{44} & -r_{43} \\
r_{32} & -r_{31} & r_{36} & r_{35} & -r_{34} & -r_{33}
\end{pmatrix}.$$

The symmetry \(4.15\) originates in the existence of a simple current, as we see below.
4.8.1. Example. For \( p = 2 \) and \( p = 3 \), the \( S(p) \) matrices can be evaluated as

\[
S(2) = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 1 & 1 \\
\frac{1}{2} & \frac{1}{2} & -1 & -1 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2}
\end{pmatrix},
\]

\[
S(3) = \begin{pmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\
\frac{1}{3\sqrt{6}} & \frac{1}{3\sqrt{6}} & \frac{-6+\sqrt{3}}{9\sqrt{2}} & \frac{3-\sqrt{3}}{9\sqrt{2}} & \frac{3-\sqrt{3}}{9\sqrt{2}} & \frac{-6+\sqrt{3}}{9\sqrt{2}} \\
\frac{1}{3\sqrt{6}} & \frac{1}{3\sqrt{6}} & \frac{\sqrt{2}(3-\sqrt{3})}{9\sqrt{2}} & \frac{3\sqrt{2}}{9\sqrt{2}} & \frac{-3\sqrt{2}}{9\sqrt{2}} & \frac{3\sqrt{2}}{9\sqrt{2}} \\
\frac{1}{3\sqrt{6}} & \frac{1}{3\sqrt{6}} & \frac{-6+\sqrt{3}}{9\sqrt{2}} & \frac{3-\sqrt{3}}{9\sqrt{2}} & \frac{-3\sqrt{2}}{9\sqrt{2}} & \frac{3\sqrt{2}}{9\sqrt{2}} \\
\frac{1}{3\sqrt{6}} & \frac{1}{3\sqrt{6}} & \frac{3\sqrt{2}(3-\sqrt{3})}{9\sqrt{2}} & \frac{-3\sqrt{2}}{9\sqrt{2}} & \frac{3\sqrt{2}}{9\sqrt{2}} & \frac{-3\sqrt{2}}{9\sqrt{2}}
\end{pmatrix}.
\]

5. Constructing the eigenmatrix \( P \) and the fusion

Having extracted a finite-dimensional \( SL(2, \mathbb{Z}) \) representation from the \( SL(2, \mathbb{Z}) \) action on characters, we now address problems \( Q3 \) and \( Q4 \) in the Introduction. We use the \( S(p) \) matrix found in the previous section in the construction of the eigenmatrix \( P \) of the fusion algebra. From the eigenmatrix, we then find the fusion. In Sec. 5.1, we first describe the role of the \( P \) matrix in a commutative associative algebra in a slightly more general setting than we actually need in \((1, p)\) models. In Sec. 5.4, we formulate the generalized Verlinde formula and use it to find the eigenmatrix \( P(p) \) in the \((1, p)\) model. In Sec. 5.6, we then obtain the fusion following the recipe in Sec. 5.1.

5.1. Fusion constants from the eigenmatrix. A fusion algebra is a finite-dimensional commutative associative algebra \( \mathfrak{H} \) over \( \mathbb{C} \) with a unit element \( 1 \), together with a canonical basis \( \{X_I\}, I = 1, \ldots, n = \dim \mathfrak{H} \) (containing 1), such that the structure constants \( N^K_{IJ} \) defined by

\[
X_I X_J = \sum_{K=1}^{n} N^K_{IJ} X_K
\]

are nonnegative integers. As any finitely generated associative algebra with a unit, \( \mathfrak{H} \) is a vector-space sum of the radical \( \mathfrak{R} \) and a semisimple algebra [19]. The algebra contains a set of primitive idempotents satisfying

\[
e_A e_B = \delta_{A,B} e_B
\]
and

\[(5.2) \quad \sum_{\text{all primitive idempotents}} e_A = 1.\]

The primitive idempotents characterize the semisimple quotient up to Morita equivalence. A *commutative* associative algebra has a basis given by the union of a basis in the radical and the primitive idempotents \(e_A\).

The primitive idempotents can be classified by the dimensions \(\nu_A\) of their images. For the purposes of \((1, p)\) models, we only need to consider the case where all \(\nu_A \leq 2\).\(^4\) The structure of the algebra \(\mathfrak{g}\) is then conveniently expressed by its quiver

\[
\begin{array}{c}
\bullet \\
\vdots \quad \vdots \\
\bullet \\
\bullet
\end{array}
\]

Here, the dots are in one-to-one correspondence with primitive idempotents. The quiver is disconnected because the algebra is commutative. A vertex \(e_A\) has a self-link if \(\nu_A = 2\), and has no links if \(\nu_A = 1\). Each link can be associated with an element in the radical, and moreover, these elements constitute a basis in the radical.

We let \(e_\alpha\) denote the primitive idempotents with \(\nu_\alpha = 2\) and let \(w_\alpha \in \mathfrak{g}\) be the corresponding element, defined modulo a nonzero factor, represented by the link of \(e_\alpha\) with itself. Then

\[(5.3) \quad e_\alpha w_\beta = \delta_{\alpha, \beta} w_\beta.\]

The other primitive idempotents, to be denoted by \(e_\alpha\), satisfy

\[(5.4) \quad e_\alpha w_\beta = 0.\]

The elements \(w_\alpha\) can be chosen such that they constitute a basis in the radical and satisfy

\[(5.5) \quad w_\alpha w_\beta = 0.\]

Let \(Y_\bullet\) be the basis consisting of \(e_\alpha, e_\alpha,\) and \(w_\alpha\); with \(r\) introduced in the quiver above (as \(r = \dim_{\mathbb{C}} \mathfrak{g}\)), we have \(a = 1, \ldots, n-2r\) and \(\alpha = n-2r+1, \ldots, n-r\). We

\(^4\)The fusion algebra for a general logarithmic conformal field theory can involve primitive idempotents with arbitrary \(\nu_A\). We restrict our attention to the particular case where \(\nu_A \leq 2\) because of the lack of instructive examples of higher-“rank” logarithmic theories; the definitions may need to be refined as further examples are worked out. When the set of idempotents with \(\nu_A = 2\) is empty, we recover the semisimple case \([16]\) (we do not impose conditions \(F2\) and \(F3\) in \([16]\) because they imply semisimplicity of the fusion algebra).
order the elements in this basis as

\[(5.6) \quad Y_1 = e_1, \ldots, Y_{n-2r} = e_{n-2r},
Y_{n-2r+1} = e_{n-2r+1}, Y_{n-2r+2} = w_{n-2r+1},
\ldots, \quad Y_{n-1} = e_{n-r}, Y_n = w_{n-r}.
\]

This ordering may seem inconvenient in that labeling of \( w_\alpha \) starts with \( w_{n-2r+1} \), but it is actually very useful in what follows, because it makes the \( 2 \times 2 \) block structure explicit by placing each element \( w_\alpha \) in the radical next to the primitive idempotent \( e_\alpha \) that satisfies \( e_\alpha w_\alpha = w_\alpha \); the primitive idempotents that annihilate the radical are given first. It may be useful to rewrite (5.6) as

\[
Y_I = \begin{cases} 
  e_I, & I = 1, 2, \ldots, n-2r, \\
  e_{(I+n+1)/2-r}, & I = n-2r+2i+1, 0 \leq i \leq r-1, \\
  w_{(I+n)/2-r}, & I = n-2r+2i, 1 \leq i \leq r.
\end{cases}
\]

The multiplication table of \( Y_\bullet \), Eqs. (5.1) – (5.5), defines an associative algebra. But it does not define a fusion algebra structure, because the latter involves a canonical basis. The canonical basis \( X_\bullet \) in \( \mathfrak{F} \) is specified by a nondegenerate \( n \times n \) matrix \( P \), called the eigenmatrix, that contains a row entirely consisting of 0 (\( r \) times) and 1 (\( n-r \geq r \) times). We let \( \pi_\Omega \) denote this row, and order the columns of \( P \) in accordance with (5.6), such that

\[
\pi_\Omega = \begin{pmatrix} 1, \ldots, 1 \\ n-2r \\ 1,0,1,0,\ldots,1,0 \\ 2r \end{pmatrix}.
\]

Elements of the canonical basis are given by

\[(5.7) \quad X_I = \sum_{J=1}^{n} P_I^J Y_J \]

and are therefore in one-to-one correspondence with the rows of \( P \); permuting the rows of \( P \) is equivalent to relabeling the elements of the canonical basis. The order of the columns of \( P \) is fixed by the assignments of \( Y_\bullet \) in (5.6), i.e., by the order chosen for the elements of the basis consisting of idempotents and elements in the radical, and is therefore conventional. Each column corresponding to an element in the radical (that is, containing zero in the intersection with the row \( \pi_\Omega \)) is defined up to a factor, because \( w_\alpha \) in the radical cannot be canonically normalized. In view of (5.2), it follows that \( X_\Omega = 1 \).
We now express the structure constants of the fusion algebra in the canonical basis through a given eigenmatrix $P$. We organize the structure constants into matrices $N_I$ with the entries

$$(N_I)^K_J := N^K_{IJ}.$$ 

Let $\pi_I = (P_I^1, \ldots, P_I^n)$ be the $I$th row of $P$. For each $I = 1, \ldots, n$, we define the $n \times n$ matrix

$$M_I := \begin{pmatrix}
P_I^1 & 0 & \cdots & 0 \\
0 & P_I^{n-2r+1} & & \\
& \ddots & \ddots & \ddots \\
0 & & & P_I^{n-2r+3} & P_I^{n-2r+4} \\
0 & \cdots & & 0 & P_I^{n-1} \\
0 & & & & P_I^n
\end{pmatrix},$$

which is the direct sum of a diagonal matrix and $r$ upper-triangular $2 \times 2$ matrices. These matrices relate the rows of $P$ as

$$\pi_I = \pi_\Omega M_I, \quad I = 1, \ldots, n.$$ 

They can be characterized as the upper-triangular $2 \times 2$-block-diagonal matrices that satisfy (5.9).

The next result answers the problem addressed in Q3.

5.2. Proposition. The structure constants are reconstructed from the eigenmatrix as

$$(5.10) \quad N_I = P M_I P^{-1}.$$ 

Proof. The regular representation $\lambda: \mathfrak{g} \to \text{End} F$ of the algebra $\mathfrak{g}$, where $F$ is the underlying vector space, is faithful because $1 \in \mathfrak{g}$; therefore, $\mathfrak{g}$ is completely determined by its regular representation. By definition, the matrices $N_I$ represent the elements $X_I \in \mathfrak{g}$ in the basis $X_*$:

$$\lambda(X_I) = N_I.$$ 

On the other hand, using relations (5.1) – (5.5), we calculate

$$X_I Y_A = \begin{cases}
P_I^J Y_J, & J = 1, 2, \ldots, n-2r, \\
P_I^J Y_J + P_I^{J+1} Y_{J+1}, & J = n-2r+2i+1, i \geq 0, \\
P_I^{J-1} Y_J, & J = n-2r+2i, i \geq 1
\end{cases}$$
(no summation over $J$). This implies that the matrices $M_I$ in (5.8) represent the elements $X_I \in \mathfrak{F}$ in the basis $Y_*$, and hence (5.10) follows.

5.3. Remark. The eigenmatrix $P$ of the fusion algebra is different from the modular transformation matrix $S$ even in the semisimple case. The most essential part of the semisimple Verlinde formula consists in the relation between the eigenmatrix $P$, which maps the canonical basis of the fusion algebra to primitive idempotents, and the matrix $S$, which represents $S \in SL(2, \mathbb{Z})$ on characters,

$$P = S K_{\text{diag}},$$

with $K_{\text{diag}}$ in turn expressed through the elements $(S_1, S_2, \ldots, S_n)$ of the vacuum row of $S$,

$$K_{\text{diag}} := \text{diag}(\frac{1}{S_1}, \frac{1}{S_2}, \ldots, \frac{1}{S_n}).$$

In the nonsemisimple case, a relation between $S$ and $P$ generalizing (5.11) – (5.12) gives the nontrivial part of the corresponding generalized Verlinde formula. This is studied in the next subsection.

5.4. From $S$ to $P$. We now construct $P$ from $S$ via a generalization of the Verlinde formula to nonsemisimple fusion algebras described in (5.1) – (5.5). The first step is to construct the interpolating matrix $K$ generalizing $K_{\text{diag}}$; the diagonal structure present in the semisimple case is replaced by a $2 \times 2$ block-diagonal structure. We recall that the rows and columns of $S$ are labeled by representations, and that the distinguished row

$$\sigma_\Omega = (S_1^1, S_2^2, \ldots, S_{2^{p-1}}^2, S_{2^p})$$

of $S(p)$ corresponds to the vacuum representation. Then $K$ is the block-diagonal matrix

$$K := K_0 \oplus K_1 \oplus \cdots \oplus K_{p-1}, \quad K_i \in \text{Mat}_2(\mathbb{C})$$

with

$$K_0 := \begin{pmatrix} 1 & 0 \\ \frac{1}{S_1} & 1 \end{pmatrix}, \quad K_j := \begin{pmatrix} 1 & -1 \\ \frac{S_{2j+1} - S_{2j+2}}{S_{2j+1} - S_{2j+2}} & S_{2j+1} \\ \frac{1}{S_{2j+1} - S_{2j+2}} & \frac{1}{S_{2j+1} - S_{2j+2}} \end{pmatrix}$$

for $j = 1, \ldots, p-1$. This matrix relates the distinguished rows of $P$ and $S$ as

$$\pi_\Omega = \sigma_\Omega K.$$

It can be characterized as the block-diagonal matrix of form (5.13), with diagonal $K_0$ and with each $K_i$, $i = 1, \ldots, p-1$, of the form $K_i = \begin{pmatrix} k_i & * \\ -k_i & * \end{pmatrix}$ defined up to a normalization of the second column, that satisfies (5.15). In (5.14), we chose the normalization such that $\det K_j = 1$; the freedom in this (nonzero) normalization
factor is related to the freedom in rescaling each element in the radical, and hence
rescaling the corresponding columns of \( P \).

For a given \( S \), we set (restoring the explicit dependence on the parameter \( p \) that
specifies the model)

\[
(5.16) \quad P(p) := S(p) K(p).
\]

The above prescription for the interpolating matrix \( K \) and the resulting expres-
sion (5.16) for the eigenmatrix \( P \) solve problem Q4 in the Introduction.

5.5. Remark. Combining formulas (5.10) and (5.16), we can write the generalized
Verlinde formula as

\[
(5.17) \quad N_I = S(K \tilde{S}_I) S^{-1},
\]

where \( \tilde{S}_I := M_I K^{-1} \). In the semisimple case, this reduces to the ordinary Verlinde
formula written as \( N_I = S(K_{\text{diag}} \tilde{S}_{\text{diag},I}) S^{-1} \), with diagonal matrices \( K_{\text{diag}} \) given
by (5.11) and \( (\tilde{S}_{\text{diag},I})^K = S_I^K \delta^K_J \).

In the \((1, p)\) model, we use the \( S(p) \) matrix obtained in Sec. 4 and its distin-
guished row \((4.13)\) to derive

\[
K_0 = p \sqrt{2p} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{p+1} \end{pmatrix},
\]

\[
K_j = (-1)^{p+j} \sqrt{\frac{p}{2}} \begin{pmatrix} \frac{1}{\sin \frac{j\pi}{p}} & \frac{2}{p^2} \left( \cos \frac{j\pi}{p} - j \sin \frac{j\pi}{p} \right) \\ \frac{1}{\sin \frac{j\pi}{p}} & \frac{-2}{p^2} \left( \cos \frac{j\pi}{p} + (p-j) \sin \frac{j\pi}{p} \right) \end{pmatrix}, \quad j = 1, \ldots, p-1.
\]

A straightforward calculation then shows that

\[
(5.18) \quad P(p) = \begin{pmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,p-1} \\ P_{1,0} & P_{1,1} & \cdots & P_{1,p-1} \\ \cdots & \cdots & \cdots & \cdots \\ P_{p-1,0} & P_{p-1,1} & \cdots & P_{p-1,p-1} \end{pmatrix},
\]

with the \( 2 \times 2 \) blocks

\[
(5.19) \quad P_{0,0} = \begin{pmatrix} p & (-1)^{p+1} p \\ p & -p \end{pmatrix}, \quad P_{0,j} = \begin{pmatrix} 0 & -\frac{2}{p} \sin \frac{j\pi}{p} \\ -\frac{2}{p} \sin \frac{j\pi}{p} & 0 \end{pmatrix}, \quad j = 1, \ldots, p-1.
\]

\[
(5.20) \quad P_{s,0} = \begin{pmatrix} s & (-1)^{s+1} s \\ p-s & (-1)^{s+1} (p-s) \end{pmatrix},
\]
and

\[ P_{s,j} = (-1)^s \begin{pmatrix}
-\frac{\sin \frac{s\pi}{p}}{\sin \frac{j\pi}{p}} & \frac{2}{p^2} \left(-s \cos \frac{s\pi}{p} \sin \frac{j\pi}{p} + \sin \frac{s\pi}{p} \cos \frac{j\pi}{p}\right) \\
\frac{\sin \frac{s\pi}{p}}{\sin \frac{j\pi}{p}} & \frac{2}{p^2} \left(-(p-s) \cos \frac{s\pi}{p} \sin \frac{j\pi}{p} - \sin \frac{s\pi}{p} \cos \frac{j\pi}{p}\right)
\end{pmatrix} \]

for \(s, j = 1, \ldots, p-1\).

The first column of \(P(p)\) contains the quantum dimensions of all the irreducible representations in the model. They are given by

\[ (p, p, 1, p-1, 2, p-2, \ldots, p-1, 1) \]

listed in the order (3.2), i.e.,

\[ \text{qdim}(\Lambda(s)) = s = \text{qdim}(\Pi(s)), \quad s = 1, \ldots, p. \]

Remarkably, all these quantum dimensions are integral. This points to an underlying quantum-group structure, such that the quantum dimensions are the dimensions of certain quantum group modules. This quantum-group structure will be considered elsewhere (see more comments in the Conclusions, however).

As noted above, the normalization of each even column of \(P\) starting with the fourth can be changed arbitrarily because \(w_\alpha\) in the radical cannot be canonically normalized.

**5.5.1. Example.** For \(p = 2, 3, 4\), the eigenmatrices found above are evaluated as follows:

\[
P(2) = \begin{pmatrix}
2 & -2 & 0 & 1 \\
2 & -2 & 0 & -1 \\
1 & 1 & 1 & 0 \\
1 & 1 & -1 & 0
\end{pmatrix}, \quad P(3) = \begin{pmatrix}
3 & 3 & 0 & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\
3 & -3 & 0 & -\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\
1 & 1 & 1 & 0 & 1 & 0 \\
2 & 2 & -1 & \frac{1}{2\sqrt{3}} & -1 & \frac{1}{2\sqrt{3}} \\
2 & -2 & -1 & \frac{1}{2\sqrt{3}} & 1 & \frac{1}{2\sqrt{3}} \\
1 & -1 & 1 & 0 & -1 & 0
\end{pmatrix},
\]

\[
P(4) = \begin{pmatrix}
4 & -4 & 0 & \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2} & 0 & \frac{1}{2\sqrt{2}} \\
4 & -4 & 0 & -\frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2\sqrt{2}} \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
3 & 3 & -1 & \frac{1}{\sqrt{2}} & -1 & 0 & -1 & -\frac{1}{\sqrt{2}} \\
2 & -2 & -\sqrt{2} & \frac{1}{8\sqrt{2}} & 0 & \frac{1}{4} & \sqrt{2} & \frac{1}{8\sqrt{2}} \\
2 & -2 & \sqrt{2} & \frac{1}{8\sqrt{2}} & 0 & \frac{1}{4} & -\sqrt{2} & \frac{1}{8\sqrt{2}} \\
3 & 3 & 1 & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} \\
1 & 1 & -1 & 0 & 1 & 0 & -1 & 0
\end{pmatrix}.
\]
5.6. The fusion algebra \( \mathcal{F}_p \). From \( S_p(\tau) \) in (3.4), we have arrived at the eigenmatrix \( P(p) \) in (5.18) – (5.21). As we saw in Sec. 5.1, the fusion is reconstructed from the eigenmatrix. We now perform this reconstruction for the \((1, p)\) model.

5.7. Theorem. For each \( p \geq 2 \), the fusion algebra \( \mathcal{F}_p \) determined by the eigenmatrix \( P(p) \) is described by the following multiplication table of the \( 2p \) canonical basis elements \( \Lambda(p), \Pi(p), \Lambda(1), \Pi(p-1), \Lambda(2), \Pi(p-2), \ldots, \Lambda(p-1), \Pi(1) \):

\[
\begin{align*}
\Lambda(s) \otimes \Lambda(t) &= \sum_{\substack{r=|s-t|+1 \\
\text{step}=2}}^{s+t-1} \tilde{\Lambda}(r), \\
\Lambda(s) \otimes \Pi(t) &= \sum_{\substack{r=|s-t|+1 \\
\text{step}=2}}^{s+t-1} \tilde{\Pi}(r), \\
\Pi(s) \otimes \Pi(t) &= \sum_{\substack{r=|s-t|+1 \\
\text{step}=2}}^{s+t-1} \tilde{\Lambda}(r),
\end{align*}
\]

where

\[
\begin{align*}
\tilde{\Lambda}(r) &= \begin{cases} 
\Lambda(r), & 1 \leq r \leq p, \\
\Lambda(2p-r) + 2\Pi(r-p), & p+1 \leq r \leq 2p-1,
\end{cases} \\
\tilde{\Pi}(r) &= \begin{cases} 
\Pi(r), & 1 \leq r \leq p, \\
\Pi(2p-r) + 2\Lambda(r-p), & p+1 \leq r \leq 2p-1.
\end{cases}
\end{align*}
\]

Proof. We first evaluate the matrices \( M_I \) in accordance with (5.8). For each \( s = 0, \ldots, p-1 \), the matrix \( M_{2s+1} \) corresponds to the \((2s+1)\)th row of the eigenmatrix \( P(p) \), and hence to the representation \( \Lambda(s) \). For \( s = 1, \ldots, p-1 \), we have

\[
M_{2s+1} \equiv M(\Lambda(s)) =
\begin{pmatrix}
\vdots \\
(-1)^{s+1} s \\
\vdots \\
(-1)^{s+1} \frac{\sin \frac{sj\pi}{p}}{\sin \frac{\pi}{p}} \\
\vdots \\
(-1)^{s+1} \frac{2(-1)^s}{p^3} \left( \sin \frac{s \pi}{p} \cos \frac{j \pi}{p} - s \cos \frac{s \pi}{p} \sin \frac{j \pi}{p} \right) \\
0 \\
(-1)^{s+1} \frac{\sin \frac{s \pi}{p}}{\sin \frac{\pi}{p}} \\
\vdots
\end{pmatrix},
\]

where the dots denote the \( 2 \times 2 \) block of the indicated structure written \( p-1 \) times, for \( j = 1, \ldots, p-1 \). (In particular, \( M_3 = 1 \); the matrices \( M_1 \) and \( M_2 \) have a simple form and are not written here for brevity.) The matrices \( M_{2s+2} \), \( s = 0, \ldots, p-1 \),
have a similar structure, which can be written most compactly by first noting that

\[ M_4 \equiv M(\Pi(1)) = \begin{pmatrix} 1 \\ (-1)^p \\ \ddots \\ (-1)^{p+j} & 0 \\ 0 & (-1)^{p+j} \\ \ddots \end{pmatrix} \]

(where the block is again to be written \( p-1 \) times, for \( j = 1, \ldots, p-1 \)) and then

(5.23) \[ M_{2s+2} \equiv M(\Pi(s)) = M(\Pi(1)) M(\Lambda(s)). \]

With the \( M_I \) matrices thus found, we can reconstruct the structure constants from (5.10). But it is technically easier to find the same structure constants from the algebra satisfied by the matrices \( M_I \),

\[ M_I M_J = \sum_{K=1}^{2p} N_{IJ}^{K} M_K, \]

which (just by (5.10)) furnish an equivalent representation of the fusion algebra.

From (5.23), we conclude that \( \Pi(1) \otimes \Lambda(s) = \Pi(s) \); it immediately follows that \( \Pi(1) \otimes \Pi(s) = \Lambda(s) \), \( s = 1, \ldots, p \). By associativity, it therefore remains to prove only the \( \Lambda(s) \otimes \Lambda(t) \) fusion, that is, to show the matrix identities (assuming \( s \geq t \) for definiteness)

\[ M_{2s+1} M_{2t+1} = \sum_{a=0}^{t-1} M(\tilde{\Lambda}(s-t+1+2a)), \]

where we extend the mapping \( \Lambda(s) \mapsto M(\Lambda(s)), \Pi(s) \mapsto M(\Pi(s)) \) by linearity, such that

\[ M(\tilde{\Lambda}(r)) = M(\Lambda(2p-r)) + 2 M(\Pi(r-p)) = M_{2(2p-r)+1} + 2 M_{2(r-p)+2} \]

for \( r \geq p+1 \). But elementary calculations with the matrices explicitly given above show that

\[ M(\tilde{\Lambda}(r)) = M(\Lambda(r)) \]

(which may be rephrased by saying that \( M(\tilde{\Lambda}(r)) \) “continues” \( M(\Lambda(r)) \) to \( r \geq p+1 \)). Therefore, the statement of the theorem reduces to the matrix identity

\[ M_{2s+1} M_{2t+1} = \sum_{a=0}^{t-1} M_{2(s-t+1+2a)+1}, \]
which can be verified directly. For the upper-left $2 \times 2$ blocks, this is totally straightforward,

\[
\begin{pmatrix}
  s & 0 \\
  0 & (-1)^{s+1} s
\end{pmatrix}
\begin{pmatrix}
  t & 0 \\
  0 & (-1)^{t+1} t
\end{pmatrix}
= \sum_{a=0}^{t-1}
\begin{pmatrix}
  s - t + 1 + 2a & 0 \\
  (-1)^{s+t}(s - t + 1 + 2a) & 0
\end{pmatrix},
\]

and for the other blocks the calculation amounts to evaluating sums of the form

\[
\sum_{a=0}^{t-1} \sin \frac{r + 2a}{\alpha} = \frac{\sin \frac{t}{\alpha} \sin \frac{r + t - 1}{\alpha}}{\sin \frac{1}{\alpha}}
\]

and their derivatives. □

5.8. Remark. We see that $\Pi(1)$ is a simple current of order two, acting without fixed points; it underlies the symmetry (4.15). This simple current symmetry is analogous to the one present in rational CFTs. The permutations of the entries of $S(p)$ correspond to the action of the simple current $\Pi(1)$ by the fusion product, while the sign factors are exponentiated monodromy charges, which are combinations of conformal weights.

We also note that the quantum dimensions (5.22) furnish a one-dimensional representation of the fusion algebra.

5.8.1. Example. For $p = 2$, the $\mathfrak{F}_2$ algebra coincides with the fusion obtained in [8], written in terms of linearly independent elements corresponding to the irreducible subquotients, as explained above.

For $p = 3$ and 4, we write the fusion algebras explicitly. To reduce the number of formulas, we note that for all $p$, $A(1)$ is the unit element and $\Pi(1)$ is an order-2 simple current that acts as

\[
\Pi(1) \otimes A(s) = \Pi(s), \quad \Pi(1) \otimes \Pi(s) = A(s).
\]

Further, $\Pi(s) \otimes \Pi(t) = A(s) \otimes A(t)$ and $A(s) \otimes \Pi(t) = A(t) \otimes A(s)$. The remaining relations are now written explicitly.

For $p = 3$, the remaining $\mathfrak{F}_3$ relations are given by

\[
\begin{align*}
A(2) \otimes A(2) &= A(1) + A(3), \\
A(2) \otimes \Pi(2) &= \Pi(1) + \Pi(3), \\
A(3) \otimes A(3) &= 2A(1) + 2\Pi(1), \\
A(3) \otimes \Pi(2) &= \Pi(3), \\
A(3) \otimes \Pi(3) &= 2A(2) + 2\Pi(1) + \Pi(3),
\end{align*}
\]
For \( p = 4 \), the remaining \( \mathcal{F}_4 \) relations are
\[
\Lambda(2) \otimes A(2) = A(1) + A(3), \quad A(2) \otimes A(3) = A(2) + A(4), \\
A(2) \otimes A(4) = 2 \Pi(1) + 2 A(3), \quad A(3) \otimes A(3) = A(1) + 2 A(3) + 2 \Pi(1), \\
A(3) \otimes A(4) = 2 A(2) + 2 \Pi(2) + A(4), \\
A(4) \otimes A(4) = 2 A(1) + 2 \Pi(3) + 2 A(3) + 2 \Pi(1), \\
A(2) \otimes \Pi(2) = \Pi(1) + \Pi(3), \quad A(2) \otimes \Pi(3) = \Pi(2) + \Pi(4), \\
A(2) \otimes \Pi(4) = 2 \Pi(3) + 2 A(1), \quad A(3) \otimes \Pi(3) = \Pi(1) + 2 \Pi(3) + 2 A(1), \\
A(3) \otimes \Pi(4) = 2 \Pi(2) + 2 A(2) + \Pi(4), \\
A(4) \otimes \Pi(4) = 2 \Pi(1) + 2 A(3) + 2 \Pi(3) + 2 A(1),
\]

6. Conclusions

To summarize, our proposal for a nonsemisimple generalization of the Verlinde formula is given by (5.16), with the interpolating matrix \( K \) built in accordance with (5.13) – (5.14) from \( S \) constructed in (4.12). From the matrix \( P \) that is provided by the generalized Verlinde formula (5.16), the structure constants of the fusion algebra are reconstructed via (5.8) and (5.10). In (1, \( p \)) models, this leads to the fusion in Theorem 5.7.

The rest of this concluding section is more a todo list than the conclusions to what has been done. First, we have used a generalization of the Verlinde formula to derive the fusion in (1, \( p \)) models, see Theorem 5.7, but we have not presented a systematic “first-principle” proof of the proposed recipe. The relevant first principles are the properly formulated axioms of chiral conformal field theory. The situation is thus reminiscent of the one with the ordinary (semisimple) Verlinde formula, whose proof could be attacked only after those axioms had been formulated [22] (see also [23, 24]) for rational conformal field theory. In the semisimple case, the structure constants are expressed through the defining data of the representation category, which is a modular tensor category, and thus through the matrices of the basic \( B \) and \( F \) operations of [22] as
\[
\sum_j S_{ij} \left( B \left[ \begin{array}{c} j+ k \\ j \\ k \end{array} \right] B \left[ \begin{array}{c} k \ j+ k+ \\ k \end{array} \right] \right)_{00} F_{kl} = N_{ikl},
\]
where
\[
F_k = F_{00} \left[ \begin{array}{c} k+ k \\ k \ k \end{array} \right].
\]
These formulas are to be related to the above construction of the fusion algebra constants expressed as
\[
N_K = S \mathcal{O}_K S,
\]
with the matrices $O_I = K M_I K^{-1}$ (already given in the Introduction) whose structure readily follows from Sec. 5. The necessary modifications of the RCFT axioms are then to lead to a block-diagonal structure, with nontrivial blocks being in one-to-one correspondence with the linkage classes, with the size of a block given by the number of irreducible representations in the relevant linkage class.

Another obvious task is to place the structures encountered here into their proper categorical context. For rational CFT, the representation category $\mathcal{C}$ of the chiral algebra—a rational conformal vertex algebra—is a modular tensor category, and can thus in particular be used to associate a three-dimensional topological field theory to the chiral CFT. For instance, the state spaces of the three-dimensional TFT are the spaces of chiral blocks of the CFT, and the modular $S$ matrix (or, to be precise, the symmetric matrix that diagonalizes the fusion rules) is, up to normalization, the invariant of the Hopf link in the three-dimensional TFT. Also, a full (nonchiral) CFT based on a given chiral CFT corresponds to a certain Frobenius algebra in the category $\mathcal{C}$, and the correlation functions of the full CFT can be determined by combining methods from three-dimensional TFT and from noncommutative algebra in monoidal categories [27, 28]. In the nonrational case, $\mathcal{C}$ is no longer modular, in particular not semisimple, but in any case it should still be an additive braided monoidal category. In addition, other properties of $\mathcal{C}$, as well as the relevance of noncommutative algebra in $\mathcal{C}$ to the construction of full from chiral CFT, can be expected to generalize from the rational to the nonrational case.

It is, however, an open (and complicated) problem to make this statement more precise. For instance, it is not known how to generalize the duality structure. (We note that the fusion rule algebra $\mathfrak{F}_p$ does not share the duality property familiar from rational fusion algebras: evaluation at the unit element does not furnish an involution of the algebra.) On the other hand, the fact that we are able to identify a finite-dimensional representation of the modular group in each of the $(1, p)$ models indicates that the chiral blocks of these models should nevertheless possess the basic covariance properties under the relevant mapping class group. This suggests, in turn, that they can still be interpreted as the state spaces of a suitable three-dimensional TFT. (For one proposal on how to associate a three-dimensional TFT to a nonrational CFT, see [29]. However, the $S$ matrix is generically not symmetric, which certainly complicates the relation to three-dimensional TFT.) Furthermore, we expect that this also applies to many other nonrational CFTs, at least to those for which $\mathcal{C}$ has a finite number of (isomorphism classes of) simple objects (and thus in particular finitely many linkage classes), with all of them having finite quantum dimensions.
A first step in developing the categorical context could consist in finding the “fine” fusion, where each indecomposable $\mathcal{W}(p)$ representation corresponds to a linearly independent generator in the fusion algebra. This fusion would define the monoidal structure of the category $\mathcal{C}$. It should therefore be important for finding modular invariants and possible boundary conditions in conformal field theory. For example, one can imagine that a boundary condition involves only an indecomposable representation, but not its subquotients (cf. \cite{25, 26}). A preliminary analysis shows that for $p = 2, 3$, invariants $\chi_p^\dagger H(p)\chi_p$ under the $SL(2, \mathbb{Z})$ action on the characters of irreducible $\mathcal{W}(p)$ representations are given by

$$H(2) = \begin{pmatrix}
\frac{1}{4}(h_1 + h_2) & 0 & 0 & 0 \\
0 & \frac{1}{4}(h_1 + h_2) & 0 & 0 \\
0 & 0 & h_1 & h_2 \\
0 & 0 & h_2 & h_1
\end{pmatrix}$$

and

$$H(3) = \begin{pmatrix}
\frac{1}{6}(h_1 + 2h_2) & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6}(h_1 + 2h_2) & 0 & 0 & 0 & 0 \\
0 & 0 & h_1 & h_2 & 0 & 0 \\
0 & 0 & h_2 & \frac{1}{2}(h_1 + h_2) & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}(h_1 + h_2) & h_2 \\
0 & 0 & 0 & 0 & h_2 & h_1
\end{pmatrix},$$

where in each case, the coefficients $h_{1,2}$ must be chosen such that the matrix entries are integers, for example, $h_1 = h_2 = 2$ for $p = 3$. The “fine” fusion is needed precisely here in order to correctly interpret the result. It allows distinguishing between inequivalent representations that possess identical characters and is therefore needed for interpreting the result for the modular invariant as a proper partition function not only at the level of characters, but also at the level of representations (or, rephrased in CFT terms, not just describing the dimensions of spaces of states of the full CFT, but completely telling which bulk fields result from combining the two chiral parts of the theory).

We also note that behind the scenes in Theorem 5.7 is a quantum group of dimension $2p^3$. Its representation category is equivalent to the category of $\mathcal{W}(p)$ representations described in Sec. 2.4 and the quantum dimensions (5.22) are the dimensions of its representations. The close relation between this quantum group and the fusion will be studied elsewhere.

Next, the structure of the indecomposable $\mathcal{W}(p)$ modules in Sec. 2.4 should be studied further. This can be done by traditional means, but a very useful approach is in the spirit of \cite{15} (which provides the required description for $p = 2$). The idea is to add extra modes to the algebra of $a^+$ and $a^-$ in Sec. 2.1 such that the
$W(p)$ action in the indecomposable modules is realized explicitly. With these extra
modes added, some states that are not singular vectors in the module in Fig. 1
become singular vectors built on new states, and the construction of these new
states can be rephrased as the “inversion” of singular vector operators, similarly to
how the operator of the simplest singular vector $L_{-1}$ was inverted in [15] (where
both the singular vector operator was the simplest possible and the $a^{\pm}$ operators
were actually fermions).

Finally, it is highly desirable, but apparently quite complicated, to extend the
analysis in this paper to logarithmic extensions of the $(p', p)$ models with coprime
$p', p \geq 2$. The extended Kac table of size $p' \times p$ is then selected as the kernel of
the appropriate screening operator. Already the $(2, 3)$ model (which is trivial in
its nonlogarithmic version) is of interest because of its possible relation to perco-
lation. However, it is not obvious how to describe the kernel of the screening in
reasonably explicit terms; in particular, we do not know good analogues of the
operators $a^+$ and $a^-$. 

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