ADIABATIC LIMITS AND NONCOMMUTATIVE WEYL FORMULA

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Abstract. We discuss asymptotic behavior of the eigenvalue distribution of the differential form Laplacian on a Riemannian foliated manifold when the metric on the ambient manifold is blown up in directions normal to the leaves (in the adiabatic limit). Motivated by analogies with semiclassical spectral asymptotics, we use ideas and notions of noncommutative geometry to suggest a conjectural formula for the eigenvalue distribution in the adiabatic limit, which we call noncommutative Weyl formula. We review known results and discuss the correctness of the noncommutative Weyl formula in each case.

INTRODUCTION

In this paper we discuss a particular asymptotic spectral problem for the Laplace operator on a closed Riemannian foliated manifold, namely, the asymptotic behavior of its eigenvalue distribution in the adiabatic limit.

Let $(M, F)$ be a closed foliated manifold, $\dim M = n, \dim F = p, p + q = n$, endowed with a Riemannian metric $g$. Then we have a decomposition of the tangent bundle to $M$ into a direct sum $TM = F \oplus H$, where $F = TF$ is the tangent bundle to $F$ and $H = F^\perp$ is the orthogonal complement of $F$, and the corresponding decomposition $g = g_F + g_H$ of the metric into the sum of tangential and transverse components. Define a one-parameter family $g_\varepsilon$ of Riemannian metrics on $M$ by

\begin{equation}
    g_\varepsilon = g_F + \varepsilon^{-2} g_H, \quad \varepsilon > 0.
\end{equation}

By the adiabatic limit, we mean the asymptotic behavior of Riemannian manifolds $(M, g_\varepsilon)$ as $\varepsilon \to 0$. In this form, the notion of the adiabatic limit was introduced by Witten [19] in the study of the global anomaly. We refer the reader to a survey paper [14] for some history and references.

For any $\varepsilon > 0$, consider the Laplace operator $\Delta_\varepsilon$ on differential forms defined by the metric $g_\varepsilon$. It is a second order elliptic differential operator, which is self-adjoint in the Hilbert space $L^2(M, \Lambda^* M, g_\varepsilon)$ of square integrable differential forms on $M$ endowed with the inner product induced by $g_\varepsilon$ and has a complete orthonormal system of eigenforms. Denote by $0 \leq \lambda_0(\varepsilon) \leq \lambda_1(\varepsilon) \leq \lambda_2(\varepsilon) \leq \cdots$ the eigenvalues of $\Delta_\varepsilon$, taking multiplicities into account. We will discuss the asymptotic behavior as $\varepsilon \to 0$ of the trace of $f(\Delta_\varepsilon)$:

\[ \text{tr} f(\Delta_\varepsilon) = \sum_{i=0}^{+\infty} f(\lambda_i(\varepsilon)), \]

for any sufficiently nice function $f$, for instance, for $f \in S(\mathbb{R})$.

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In order to explain what kind of spectral problems we have, we first transfer the operators \( \Delta_\varepsilon \) to the fixed Hilbert space

\[
L^2\Omega = L^2(M, \Lambda T^*M, g),
\]

using an isomorphism \( \Theta_\varepsilon \) from \( L^2(M, \Lambda T^*M, g_\varepsilon) \) to \( L^2\Omega \) defined as follows. With respect to a bigrading on \( \Lambda T \) we have, for \( L \) set up on \( \Lambda T \),

\[
\Lambda^{i,j}T^*M = \Lambda^i H^* \otimes \Lambda^j F^*, \quad 0 \leq i \leq q, \quad 0 \leq j \leq p,
\]

we have, for \( u \in L^2(M, \Lambda^{i,j}T^*M, g_\varepsilon) \),

\[
\Theta_\varepsilon u = \varepsilon^i u.
\]

The operator \( \Delta_\varepsilon \) in \( L^2(M, \Lambda T^*M, g_\varepsilon) \) corresponds under the isometry \( \Theta_\varepsilon \) to the operator \( L_\varepsilon = \Theta_\varepsilon \Delta_\varepsilon \Theta_\varepsilon^{-1} \) in \( L^2\Omega \).

With respect to the above bigrading of \( \Lambda T^*M \), the de Rham differential \( d \) can be written as

\[
d = d_F + d_H + \theta,
\]

where

1. \( d_F = d_{0,1} : C^\infty(M, \Lambda^{i,j}T^*M) \to C^\infty(M, \Lambda^{i,j+1}T^*M) \) is the tangential de Rham differential, which is a first order tangentially elliptic operator, independent of the choice of \( g \);
2. \( d_H = d_{1,0} : C^\infty(M, \Lambda^{i,j}T^*M) \to C^\infty(M, \Lambda^{i+1,j}T^*M) \) is the transversal de Rham differential, which is a first order transversally elliptic operator;
3. \( \theta = d_{2,1} : C^\infty(M, \Lambda^{i,j}T^*M) \to C^\infty(M, \Lambda^{i+2,j-1}T^*M) \) is a zeroth order differential operator.

One can show that

\[
d_\varepsilon = \Theta_\varepsilon d\Theta_\varepsilon^{-1} = d_F + \varepsilon d_H + \epsilon^2 \theta,
\]

and the adjoint of \( d_\varepsilon \) in \( L^2\Omega \) is

\[
\delta_\varepsilon = \Theta_\varepsilon \delta \Theta_\varepsilon^{-1} = \delta_F + \varepsilon \delta_H + \varepsilon^2 \theta^*.
\]

Therefore, for the operator

\[
L_\varepsilon = \Theta_\varepsilon \Delta_\varepsilon \Theta_\varepsilon^{-1} = d_\varepsilon \delta_\varepsilon + \delta_\varepsilon d_\varepsilon,
\]

one has

\[
L_\varepsilon = \Delta_F + \epsilon^2 \Delta_H + \varepsilon^4 \Delta_\theta + \varepsilon K_1 + \epsilon^2 K_2 + \epsilon^3 K_3,
\]

where

- \( \Delta_F = d_F d_F^* + d_F^* d_F \) is the tangential Laplacian;
- \( \Delta_H = d_H d_H^* + d_H^* d_H \) is the transverse Laplacian;
- \( \Delta_\theta = \theta \theta^* + \theta^* \theta \) and \( K_2 = d_F \theta^* + \theta^* d_F + \delta_F \theta + \theta \delta_F \) are of zero order;
- \( K_1 = d_F \delta_H d_F + \delta_H d_F d_H + d_H \delta_F \) and \( K_3 = d_H \theta \theta^* + \theta^* d_H + \delta_H \theta + \theta \delta_H \) are first order differential operators.

Differential (or pseudodifferential) operators with a small parameter are usually set up on \( \mathbb{R}^n \), within the Weyl calculus, under the name of \( h \)-admissible or semiclassical operators. Recall that a semiclassical differential operator on \( \mathbb{R}^n \) is the differential operator \( B_h \), depending on a parameter \( h > 0 \), of the form

\[
B_h = \sum_{j=0}^k h^j B_j(x, hD_x), \quad x \in \mathbb{R}^n,
\]
where each $B_j(x, \xi)$ is a polynomial in $\xi$:

$$B_j(x, \xi) = \sum_{|\alpha| \leq k} b_{\alpha}^{(j)} \xi^\alpha,$$

and

$$B_j(x, hD_x) = \sum_{|\alpha| \leq k} b_{\alpha}^{(j)} D_x^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is a multi-index, and we use standard notation: $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, $D_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j}$.

The leading term $B_0(x, \xi)$ is called the principal symbol of $B_h$.

On an arbitrary smooth manifold, the definitions of semiclassical differential operators and their principal symbols are much more delicate [15]. Nevertheless, there is one particular important case, when this can be done rather easily, the case of Schrödinger operator.

Let $(X, g)$ be a compact Riemannian manifold, $\dim X = n$. The Schrödinger operator on $X$ is the self-adjoint second order differential operator $H_h$ acting in $C^\infty(X)$ by the formula

$$H_h = -h^2 \Delta + V,$$

where $h > 0$ is the semiclassical parameter, $\Delta$ is the Laplace-Beltrami operator associated with the Riemannian metric $g$, and $V \in C^\infty(X)$ is a real-valued smooth function on $X$ (a potential), which is identified with the corresponding multiplication operator. The principal symbol $\sigma(H_h) \in C^\infty(T^*X)$ of the semiclassical operator $H_h$ is given by

$$\sigma(H_h)(x, \xi) = g_{T^*X}^T(\xi, \xi) + V(x), \quad (x, \xi) \in T^*X,$$

where $g_{T^*X}^T$ is the induced metric in $T^*X$.

For any $f \in S(\mathbb{R})$, the operator $f(H_h)$ is of trace class, and its asymptotic behavior as $h \to 0$ is described by the semiclassical Weyl formula:

$$\text{tr} f(H_h) = \frac{1}{(2\pi h)^n} \int_{T^*X} f(\sigma(H_h)(x, \xi)) \, dx \, d\xi + o(h^{-n}), \quad h \to 0+,$$

where $dx \, d\xi$ is the canonical Liouville measure on $T^*X$.

In our case, in a local foliated chart with coordinates $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$, where the leaves of the foliation are given by the level sets $y = \text{const}$, the operator $L_\varepsilon$ has the form

$$L_\varepsilon = \sum_{j=0}^k \varepsilon^j A_j(x, y, D_x, \varepsilon D_y), \quad x \in \mathbb{R}^p, \quad y \in \mathbb{R}^q.$$

In other words, the parameter $\varepsilon$ enters the coefficients of derivatives with respect to $y$, and there is no $\varepsilon$ in the coefficients of the derivatives with respect to $y$.

Asymptotic spectral problems of this kind appeared for the first time in molecular quantum mechanics in a paper by Born and Oppenheimer in 1927. Born and Oppenheimer studied molecular bound states, that is, roughly speaking, eigenfunctions of the Schrödinger operator

$$H_h = -\frac{h^2}{2m} \Delta_x - \frac{h^2}{2M} \Delta_y + V(x, y)$$

in $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$, where the $x$ variables describe the electron motion, the $y$ variables describe the nuclear motion, $m$ is the electron mass, $M$ is the nuclei mass and $V$ is
the interaction potential. They suggested an approximation for molecular bound states based on disparity between the electron mass \(m\) and the nuclei mass \(M\). This approximation makes use of asymptotic expansions in the small parameter \(\varepsilon\), where \(\varepsilon^4 = m/M\).

The first rigorous mathematical paper on Born-Oppenheimer approximation was published by Seiler in 1973. Mathematical investigations of Born-Oppenheimer approximation continue very actively to present day (see, for instance, survey papers \([2, 7]\) and references there). Later, similar problems were also studied in many areas of mathematical and theoretical physics (see, for instance, \([2]\)).

Observe that an operator of the form \((3)\) can be considered as a semiclassical differential operator in \(\mathbb{R}^q\) with an operator-valued symbol. In particular, the asymptotic formula for the trace of \(f(L_\varepsilon)\) for such an operator \(L_\varepsilon\) in \(\mathbb{R}^n\) can be written in a form, similar to the semiclassical Weyl formula \((2)\). For instance, let us consider the Schrödinger operator

\[
H_\varepsilon = \Delta_X + \varepsilon^2 \Delta_Y + V
\]

in \(M = X \times Y\), where \(X\) and \(Y\) are closed Riemannian manifolds, \(\Delta_X\) and \(\Delta_Y\) are the Laplace-Beltrami operators on \(X\) and \(Y\) respectively, \(V \in C^\infty(M)\). The principal symbol of \(H_\varepsilon\) is a smooth function \(\sigma(H_\varepsilon)\) on \(T^*Y\), whose values are differential operators on \(X\):

\[
\sigma(H_\varepsilon)(y, \eta) = g^T_X(\eta, \eta) + \Delta_X + V_y, \quad (y, \eta) \in T^*Y,
\]

where \(V_y \in C^\infty(X)\) is the restriction of \(V\) to \(X \times \{y\}\).

For any \(f \in S(\mathbb{R})\) and \((y, \eta) \in T^*Y\), the operator \(f(\sigma(H_\varepsilon)(y, \eta))\) is a smoothing operator on \(X\), so it is a trace class operator. Moreover, one can show that

\[
\int_{T^*Y} \text{tr} f(\sigma(H_\varepsilon)(y, \eta)) \, dy \, d\eta < \infty.
\]

Then we have

\[
\text{tr} f(H_\varepsilon) = \frac{1}{(2\pi\varepsilon)^q} \int_{T^*Y} \text{tr} f(\sigma(H_\varepsilon)(y, \eta)) \, dy \, d\eta + o(\varepsilon^{-q}), \quad \varepsilon \to 0 + .
\]

This formula was proved, for instance, in \([1]\) for the case of \(\mathbb{R}^n\), using machinery of pseudodifferential operators with operator-valued symbols.

In the foliation case, such a nice picture holds only locally. The global structure of a foliation may be very complicated. There may be no base manifold, leaves may be non-compact, and elliptic operators along leaves have, in general, continuous spectrum. Therefore, it is even unclear how a general asymptotic formula for \(\text{tr} f(D_\varepsilon)\) could look like. Nevertheless, one can write such a conjectural formula, which we call noncommutative Weyl formula, using ideas and notions of noncommutative geometry developed by A. Connes. (For the basic information on noncommutative geometry of foliations, we refer the reader to \([12, 13]\) and references therein.)

First, observe that, using the results of \([11]\), one can show that, for any \(f \in S(\mathbb{R})\), we have the estimate

\[
|\text{tr} f(D_\varepsilon)| = |\text{tr} f(L_\varepsilon)| < C_1\varepsilon^{-q}, \quad 0 < \varepsilon \leq 1,
\]

with some \(C_1 > 0\). Moreover, it is easy to show that the terms \(\varepsilon^2 K_2, \varepsilon^3 K_3\) and \(\varepsilon^4 \Delta_\theta\) are of lower order in some sense, and, therefore, they don’t contribute to the leading term of the formula. More precisely, if we denote

\[
\bar{L}_\varepsilon = \Delta_F + \varepsilon K_1 + \varepsilon^2 \Delta_H,
\]

where \(\Delta_F = \Delta_X + \varepsilon^2 (\Delta_Y + V_y)\), then

\[
\text{tr} f(\bar{L}_\varepsilon) = \frac{1}{(2\pi\varepsilon)^q} \int_{T^*Y} \text{tr} f(\sigma(\bar{L}_\varepsilon)(y, \eta)) \, dy \, d\eta + o(\varepsilon^{-q}), \quad \varepsilon \to 0 + .
\]
then, for any \( f \in S(\mathbb{R}) \), we have (with some \( C_2 > 0 \)):

\[
| \text{tr} f(L_\varepsilon) - \text{tr} f(\bar{L}_\varepsilon) | < C_2 \varepsilon^{1-q}, \quad 0 < \varepsilon \leq 1,
\]

Therefore, we can restrict our considerations by the operator \( \bar{L}_\varepsilon \).

Now we have certain difficulties mentioned above with an appropriate definition of the principal symbol of \( \bar{L}_\varepsilon \), first of all, with a definition of the principal symbol of the term \( \varepsilon K_1 \). We were able find a solution in the case when \( \varepsilon K_1 \) is a lower order operator in some sense (this happens for Riemannian foliations) or when \( K_1 = 0 \). In this case, our considerations are reduced to the operator

\[
\bar{L}_\varepsilon = \Delta_F + \varepsilon^2 \Delta_H,
\]

Here we observe that the operator \( \bar{L}_\varepsilon \) has the form of a Schrödinger operator on the leaf space \( M/F \), where \( \Delta_H \) plays the role of the Laplace operator, and \( \Delta_F \) the role of the operator-valued potential on \( M/F \). Using this analogy, we define the principal symbol of this operator as a second order differential operator \( \sigma(\Delta_\varepsilon) \) on the conormal bundle \( N^*F \) of \( F \), which is tangentially elliptic with respect to a natural foliation \( F_N \) on \( N^*F \). We will consider the operator \( \sigma(\Delta_\varepsilon) \) as a family of self-adjoint elliptic operators along the leaves of \( F_N \). Then, for any \( f \in S(\mathbb{R}) \), one can define \( f(\sigma(\Delta_\varepsilon)) \) as an element of the twisted \( C^* \)-algebra \( C^*(N^*F, F_N, \pi^*\Lambda T^*M) \) associated with the foliation \( (N^*F, F_N) \). The leaf space \( N^*F/F_N \) can be considered as the cotangent bundle to \( M/F \), and the algebra \( C^*(N^*F, F_N, \pi^*\Lambda T^*M) \) can be viewed as the noncommutative analogue of the algebra of continuous vector-valued differential forms on this singular space.

The foliation \( F_N \) has a natural transverse symplectic structure. The corresponding canonical transverse Liouville measure is holonomy invariant and, by noncommutative integration theory [6], determines the trace \( \text{tr}_{F_N} \) on the \( C^* \)-algebra \( C^*(N^*F, F_N, \pi^*\Lambda T^*M) \). The trace \( \text{tr}_{F_N} \) is the noncommutative analogue of the integral over the leaf space \( N^*F/F_N \) with respect to the transverse Liouville measure.

Replacing in the formula (53) the integration over \( T^*X \) and the operator trace \( \text{Tr} \) by the trace \( \text{tr}_{F_N} \) and the principal symbol \( \sigma(H_\varepsilon) \) by the principal \( \sigma(\Delta_\varepsilon) \), we suggest the following noncommutative analogue of this formula.

**Conjecture 0.1 (The noncommutative Weyl formula).** For any \( f \in S(\mathbb{R}) \), the following asymptotic formula holds:

\[
\text{tr} f(\Delta_\varepsilon) = \frac{1}{(2\pi \varepsilon)^q} \text{tr}_{F_N} f(\sigma(\Delta_\varepsilon)) + o(\varepsilon^{-q}), \quad \varepsilon \to 0.
\]

It is a remarkable fact that such a noncommutative Weyl formula was rigorously proved for Riemannian foliations in [11] (see also [11]). We will explain the details and give two concrete examples in Section 1.

It is very little known about the problem in question for non-Riemannian foliations. There are only two computations in some particular cases [21, 22, 23]. In all these cases, we have \( K_1 = 0 \), so it is natural to ask whether the noncommutative Weyl formula (6) holds. We show that the formula obtained in [21] can be written in the form (6), while the formula obtained in [22, 23] seems to be not compatible with the formula (6). These are main new results of the paper. They will be discussed in Section 2.
1. RIEMANNIAN FOLIATIONS

1.1. General results. Recall (see, for instance, [15]) that a foliation $\mathcal{F}$ is called Riemannian, if there exists a Riemannian metric $g$ on $M$ such that the induced metric $g_\tau$ on the normal bundle $\tau = TM/F$ is holonomy invariant, or, equivalently, in any foliated chart $\phi : U \to I^p \times I^q$ with local coordinates $(x, y)$, the restriction $g_H$ of $g$ to $H = F^\perp$ is written in the form

$$g_H = \sum_{\alpha, \beta = 1} g_{\alpha \beta}(y) \theta^\alpha \theta^\beta,$$

where $\theta^\alpha \in H^*$ is the 1-form, corresponding to the form $dy^\alpha$ under the isomorphism $H^* \cong T^* \mathbb{R}^q$, and $g_{\alpha \beta}(y)$ depend only on the transverse variables $y \in \mathbb{R}^q$. Such a Riemannian metric is called bundle-like.

Suppose that $\mathcal{F}$ is a Riemannian foliation and $g$ is a bundle-like metric. Then the key observation is that, in this case, the first order differential operator $\Delta_k$ can be written in the form (6). We start with a definition of the principal symbol of $\Delta_k$.

Denote by $\pi : T^* \mathcal{F} \to M$ the conormal bundle to $\mathcal{F}$. It is a well-known fact in foliation theory (cf., for instance, [15, 12]) that $T^* \mathcal{F}$ carries a natural foliation $\mathcal{F}_{\pi}$, which we will call the linearized foliation. The leaves of $\mathcal{F}_{\pi}$ can be described as follows. First of all, observe that, for any piecewise smooth leafwise path $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$, sliding along the leaves of the foliation determines a linear map $dh_\gamma : N^*_y \mathcal{F} \to N^*_x \mathcal{F}$, called the linear holonomy map. Then the leaf through $\nu \in N^* \mathcal{F}$ consists of all $dh_\gamma(\nu) \in N^* \mathcal{F}$ with $\gamma : [0, 1] \to M$ such that $\gamma(1) = \pi(\nu)$.

The leaves of $\mathcal{F}_{\pi}$ are transverse to the fibers of $\pi$, and the restriction of $\pi$ to any leaf of $\mathcal{F}_{\pi}$ is a covering over a leaf of $\mathcal{F}$. Therefore, the tangential Laplacian $\Delta_{\mathcal{F}}$ can be lifted to a tangentially elliptic (relative to $\mathcal{F}_{\pi}$) operator $\Delta_{\mathcal{F}_{\pi}}$ in $C^\infty(N^* \mathcal{F}, \pi^* \Lambda T^* M)$. Let $g^N \in C^\infty(N^* \mathcal{F})$ be the fiberwise Riemannian metric on $N^* \mathcal{F}$ induced by the metric on $M$. Denote by $g^N$ the multiplication operator in $C^\infty(N^* \mathcal{F}, \pi^* \Lambda T^* M)$ by $g^N$. The principal symbol of $\Delta_{\mathcal{F}}$ is a tangentially elliptic operator in $C^\infty(N^* \mathcal{F}, \pi^* \Lambda T^* M)$ given by (cf. [11])

$$\sigma(\Delta_{\mathcal{F}}) = g^N(\eta, \eta) + \Delta_{\mathcal{F}_{\pi}}.$$

To define the trace of $f(\sigma(\Delta_{\mathcal{F}}))$, we will use noncommutative integration theory developed by A. Connes in [6].

Let $G$ be the holonomy groupoid of $\mathcal{F}$. Let us briefly recall its definition. Denote by $\sim_h$ the equivalence relation on the set of piecewise smooth leafwise paths $\gamma : [0, 1] \to M$, setting $\gamma_1 \sim_h \gamma_2$ if $\gamma_1$ and $\gamma_2$ have the same initial and final points and the same holonomy maps. The holonomy groupoid $G$ is the set of $\sim_h$ equivalence classes of leafwise paths. The set of units of $G$ is $G^{(0)} = M$. $G$ is equipped with the
source and the range maps \( s, r : G \to M \) defined by \( s(\gamma) = \gamma(0) \) and \( r(\gamma) = \gamma(1) \). The multiplication \( \gamma_1\gamma_2 \in G \) of \( \gamma_1 \in G \) and \( \gamma_2 \in G \) is defined by concatenation of paths. It is defined if and only if \( s(\gamma_1) = r(\gamma_2) \). Recall also that, for any \( x \in M \), the set \( G^x = \{ \gamma \in G : r(\gamma) = x \} \) is the covering of the leaf \( L_x \) through the point \( x \), associated with the holonomy group of the leaf. We will identify any \( x \in G^{(0)} = M \) with the element of \( G \) given by the constant path \( \gamma(t) = x \), \( t \in [0, 1] \).

The holonomy groupoid \( G_{FN} \) of the linearized foliation \( F_N \) can be described as the set of all \( (\gamma, \nu) \in G \times N^sF \) such that \( r(\gamma) = \pi(\nu) \). The source map \( s_N : G_{FN} \to N^sF \) and the range map \( r_N : G_{FN} \to N^sF \) are defined as \( s_N(\gamma, \nu) = dh^*_\gamma(\nu) \) and \( r_N(\gamma, \nu) = \nu \). The product of \( (\gamma_1, \nu_1) \in G_{FN} \) and \( (\gamma_2, \nu_2) \in G_{FN} \) is defined if \( \nu_2 = dh^*_\gamma(\nu_1) \), and, under this condition, it is given by \( (\gamma_1, \nu_1)(\gamma_2, \nu_2) = (\gamma_1\gamma_2, \nu_2) \).

Denote by \( L(\pi^*\Lambda^*M) \) the vector bundle on \( G_{FN} \), whose fiber at a point \( (\gamma, \nu) \in G_{FN} \) is the space of linear maps \( (\pi^*\Lambda^*M)_{s_N(\gamma, \nu)} \to (\pi^*\Lambda^*M)_{r_N(\gamma, \nu)} \). There is a standard way (due to Connes [6]) to introduce the structure of involutive algebra on the space \( C^\infty_c(G_{FN}, L(\pi^*\Lambda^*M)) \) of smooth, compactly supported sections of \( L(\pi^*\Lambda^*M) \). For any \( \nu \in N^sF \), this algebra has a natural representation \( R_\nu \) in the Hilbert space \( L^2(G_{FN}, s_N(\pi^*\Lambda^*M)) \) that determines its embedding to the \( C^* \)-algebra of all bounded operators in \( L^2(G_{FN}, s_N(\pi^*\Lambda^*M)) \).

Let \( \lambda_L \) denote the Riemannian volume form on a leaf \( L \) given by the induced metric, and, for any \( x \in M \), let \( \lambda^x \) denote the lift of \( \lambda_{L_x} \) via the holonomy covering map \( s : G^x \to L_x \). For any \( k \in C^\infty_c(G_{FN}, L(\pi^*\Lambda^*M)) \), the action of \( R_\nu(k) \) on \( \zeta \in L^2(G_{FN}, s_N(\pi^*\Lambda^*M)) \) is given by

\[
R_\nu(k)\zeta(\gamma, \nu) = \int_{G^x} k((\gamma, \nu)^{-1}(\gamma_1, \nu))\zeta(\gamma_1, \nu)d\lambda^x(\gamma_1), \quad r(\gamma) = x.
\]

Taking the closure of the image of this embedding, we get a \( C^* \)-algebra, called the twisted foliation \( C^* \)-algebra and denoted by \( C^*(N^sF, F_N, \pi^*\Lambda^*M) \).

The foliation \( F_N \) has a natural transverse symplectic structure, which can be described as follows. Consider a foliated chart \( \varsigma : U \subset M \to I^p \times I^q \) on \( M \) with coordinates \( (x, y) \in I^p \times I^q \) (\( I \) is the open interval \( (0, 1) \)) such that the restriction of \( F \) to \( U \) is given by the sets \( y = \text{const} \). One has the corresponding coordinate chart in \( T^*M \) with coordinates denoted by \( (x, y, \xi, \eta) \in I^p \times I^q \times \mathbb{R}^p \times \mathbb{R}^q \). In these coordinates, the restriction of the conormal bundle \( N^*F \) to \( U \) is given by the equation \( \xi = 0 \). So we have a coordinate chart \( \varsigma_n : U_1 \subset N^*F \to I^p \times I^q \times \mathbb{R}^q \) on \( N^*F \) with the coordinates \( (x, y, \eta) \in I^p \times I^q \times \mathbb{R}^q \). Indeed, the coordinate chart \( \varsigma_n \) is a foliated coordinate chart for \( F_N \), and the restriction of \( F_N \) to \( U_1 \) is given by the level sets \( y = \text{const} \) on \( N^*F \). The transverse symplectic structure for \( F_N \) is given by the transverse two-form \( \sum_j dy_j \wedge d\eta_j \).

The corresponding canonical transverse Liouville measure \( dy \, d\eta \) is holonomy invariant and, by noncommutative integration theory [6], defines the trace \( \text{tr}_{F_N} \) on the \( C^* \)-algebra \( C^*(N^sF, F_N, \pi^*\Lambda^*M) \). Combining the Riemannian volume forms along the leaves of \( F_N \) and the transverse Liouville measure \( dy \, d\eta \), we get a volume form \( d\nu \) on \( N^*F \). For any \( k \in C^\infty_c(G_{FN}, L(\pi^*\Lambda^*M)) \), its trace is given by the formula

\[
\text{tr}_{F_N}(k) = \int_{N^*F} k(\nu)d\nu.
\]

Consider \( \sigma(\Delta_\varsigma) \) as a family of elliptic operators along the leaves of the foliation \( F_N \) and lift these operators to the holonomy coverings of the leaves. For
any $\nu \in N^* F$, we get a formally self-adjoint uniformly elliptic operator $\sigma(\Delta_{\nu})$ in $C^\infty(G_{\mathcal{F}_N}^\nu, s_N^\nu(\pi^*\Lambda^* M))$, which is essentially self-adjoint in the Hilbert space $L^2(G_{\mathcal{F}_N}^\nu, s_N^\nu(\pi^*\Lambda^* M))$. For any $f \in S(\mathbb{R})$, the family $\{f(\sigma(\Delta_{\nu})), \nu \in N^* F\}$ defines an element $f(\sigma(\Delta_{\nu}))$ of the $C^*$-algebra $C^*(N^* F, \mathcal{F}_N, \pi^*\Lambda^* M)$:

$$f(\sigma(\Delta_{\nu})) = R_{\nu}(f(\sigma(\Delta_{\nu}))), \quad \nu \in N^* F.$$ 

One can show that

$$\text{tr}_{\mathcal{F}_N} f(\sigma(\Delta_{\nu})) < \infty.$$ 

**Theorem 1.1** \cite{10}. Let $(M, \mathcal{F})$ be a Riemannian foliation equipped with a bundle-like Riemannian metric $g$. For any $f \in S(\mathbb{R})$, the asymptotic formula holds:

$$\text{tr} f(\Delta_{\varepsilon}) = \frac{1}{(2\pi\varepsilon)^d} \text{tr}_{\mathcal{F}_N} f(\sigma(\Delta_{\varepsilon})) + o(\varepsilon^{-q}), \quad \varepsilon \to 0.$$ 

It should be noted that the formula \cite{10} (or, equally, \cite{9}) makes sense for an arbitrary foliation $\mathcal{F}$, not necessarily Riemannian. The only difference is that in general the groupoid $G_{\mathcal{F}_N}$ does not coincide with the holonomy groupoid of $\mathcal{F}_N$, but we can use it to write the formula. A very important feature of the case of Riemannian foliation is the fact that, in this case, the terms $g_N(\eta, \eta)$ and $\Delta_{\mathcal{F}_N}$ commute.

The formula \cite{10} can be rewritten in terms of the spectral data of leafwise Laplace operators. We will formulate the corresponding result for the spectrum distribution function

$$N_{\varepsilon}(\lambda) = \sharp \{i : \lambda_i(\varepsilon) < \lambda\}.$$ 

Restricting the tangential Laplace operator $\Delta_{\mathcal{F}}$ to the leaves of $\mathcal{F}$ and lifting the restrictions to the holonomy coverings of leaves, we get the Laplacian $\Delta_x$ acting in $C^\infty_c(G^x, s^x\Lambda^* M)$. Using the assumption that $\mathcal{F}$ is Riemannian, it can be checked that, for any $x \in M$, $\Delta_x$ is formally self-adjoint in $L^2(G^x, s^x\Lambda^* M)$, that, in turn, implies its essential self-adjointness in this Hilbert space (with initial domain $C^\infty_c(G^x, s^x\Lambda^* M)$). For each $\lambda \in \mathbb{R}$, let $E_{\varepsilon}(\lambda)$ be the spectral projection of $\Delta_x$, corresponding to the semi-axis $(-\infty, \lambda]$. The Schwartz kernels of the operators $E_{\varepsilon}(\lambda)$ define a leafwise smooth section $e_{\lambda}$ of the bundle $\mathcal{L}(\Lambda^* M)$ over $G$.

We introduce the spectrum distribution function $N_{\mathcal{F}}(\lambda)$ of the operator $\Delta_{\mathcal{F}}$ by

$$N_{\mathcal{F}}(\lambda) = \int_M \text{Tr} e_{\lambda}(x) \, dx, \quad \lambda \in \mathbb{R},$$ 

where $dx$ denotes the Riemannian volume form on $M$. By \cite{9}, for any $\lambda \in \mathbb{R}$, the function $\text{Tr} e_{\lambda}$ is a bounded measurable function on $M$, therefore, the spectrum distribution function $N_{\mathcal{F}}(\lambda)$ is well-defined and takes finite values.

As above, one can show that the family $\{E_{\varepsilon}(\lambda) : x \in M\}$ defines an element $E(\lambda)$ of the twisted von Neumann foliation algebra $W^*(G, \Lambda^* M)$, the holonomy invariant transverse Riemannian volume form for $\mathcal{F}$ defines a trace $\text{tr}_{\mathcal{F}}$ on $W^*(G, \Lambda^* M)$, and the right hand side of the last formula can be interpreted as the value of this trace on $E(\lambda)$.

**Theorem 1.2** \cite{10}. Let $(M, \mathcal{F})$ be a Riemannian foliation, equipped with a bundle-like Riemannian metric $g$. Then we have

$$N_{\varepsilon}(\lambda) = \varepsilon^{-q} \frac{(4\pi)^{-q/2}}{\Gamma((q/2) + 1)} \int_{-\infty}^{\lambda} (\lambda - \tau)^{q/2} d\tau N_{\mathcal{F}}(\tau) + o(\varepsilon^{-q}), \quad \varepsilon \to 0.$$
1.2. Riemannian submersions. Suppose that the foliation $F$ is given by the fibers of a fibration $p : M \to B$ over a compact manifold $B$. Then, for any $x \in M$, $N_x^* F$ coincides with the image of the cotangent map $dp(x)^* : T^*_x B \to T^*_x M$. The inverse maps $(dp(x))^*^{-1} : N^*_x F \to T^*_{p(x)} B$ determine a fibration $N^* F \to T^* B$ whose fibers are the leaves of the linearized foliation $F_N$. Thus, $N^* F$ is diffeomorphic to the fiber product

$$M \times_B T^* B = \{(x, \eta) \in M \times T^* B : p(x) = y = \pi_B(y, \eta)\}$$

with a diffeomorphism $M \times_B T^* B \cong N^* F$, given by

$$p(x) \in M \times_B T^* B \mapsto dp(x)^*(\eta) \in N^*_x F.$$  

A Riemannian metric $g_M$ on $M$ is bundle-like if and only if there exists a Riemannian metric $g_B$ on $B$ such that, for any $x \in M$, the restriction of the tangent map $dp(x) : T_x M \to T_{p(x)} B$ to $H_x \subset T_x M$ induces an isometry from $(H_x, g_H)$ to $(T_{p(x)} B, g_B)$, or, equivalently, $p : (M, g_M) \to (B, g_B)$ is a Riemannian submersion. The holonomy groupoid $G$ of $F$ is the fiber product

$$M \times_B M = \{(x_1, x_2) \in M \times M : p(x_1) = p(x_2)\},$$

where $s(x_1, x_2) = x_2, r(x_1, x_2) = x_1$. Similarly, the holonomy groupoid $G_{F_N}$ is the fiber product $N^* F \times_{T^* B} N^* F$, which consists of all $(\nu_{x_1}, \nu_{x_2}) \in N^*_x F \times N^*_x F$ such that $(x_1, x_2) \in M \times M$ and $(dp(x_1))^*^{-1}(\nu_{x_1}) = (dp(x_2))^*^{-1}(\nu_{x_2})$, with $s_N(\nu_{x_1}, \nu_{x_2}) = \nu_{x_2}, r_N(\nu_{x_1}, \nu_{x_2}) = \nu_{x_1}$. On the other hand, $N^* F \times_{T^* B} N^* F$ is also diffeomorphic to the fiber product

$$G \times_B T^* B = \{(x_1, x_2, \eta) \in M \times M \times T^* B : p(x_1) = p(x_2) = y = \pi_B(y, \eta)\}.$$  

A diffeomorphism $G \times_B T^* B \cong N^* F \times_{T^* B} N^* F$ can be defined as

$$(x_1, x_2, \eta) \in G \times_B T^* B \mapsto (dp(x_1)^*(\eta), dp(x_2)^*(\eta)) \in N^*_x F \times_{T^* B} N^*_x F.$$  

For simplicity, we will consider scalar operators. For any $(y, \eta) \in T^* B$, let $\Psi^{-\infty}((N^* F)_{(y, \eta)})$ be the involutive algebra of all smoothing operators, acting on $C^\infty((N^* F)_{(y, \eta)})$, where $(N^* F)_{(y, \eta)}$ is the fiber of the fibration $N^* F \to T^* B$ at $(y, \eta)$. Consider a field $\Psi^{-\infty}(N^* F)$ of involutive algebras on $T^* B$ whose fiber at $\eta \in T^* B$ is $\Psi^{-\infty}((N^* F)_{(y, \eta)})$. For any section $K$ of the field $\Psi^{-\infty}(N^* F)$, the Schwartz kernels of the operators $K_{(y, \eta)}$ in $C^\infty((N^* F)_{(y, \eta)})$ determine a well-defined function $\sigma_K$ on $G_{F_N} \cong N^* F \times_{T^* B} N^* F$. We say that the section $K$ is smooth, if the corresponding function $\sigma_K$ is smooth. This defines an involutive algebra isomorphism of $C^\infty(B, \Psi^{-\infty}(N^* F))$ with $C^\infty(G_{F_N})$, where the structure of involutive algebra on $C^\infty(G_{F_N})$ is given by the fiberwise composition and the fiberwise adjoint. Finally, for any smooth section $K$ of $\Psi^{-\infty}(N^* F)$, its trace is given by the formula

$$\text{tr}_{F_N}(K) = \int_{T^* B} \text{tr} K_{(y, \eta)} \, dy \, d\eta.$$  

The principal symbol of the operator $\Delta_c$ on functions is a tangentially elliptic operator in $C^\infty(N^* F) \cong C^\infty(M \times_B T^* B)$ given by

$$\sigma(\Delta_c) = g^{T^* B} + \Delta_{M/B} \otimes \text{Id},$$

where the vertical Laplace operator $\Delta_{M/B}$ is given by the smooth family of Laplace operators along the fibers of $p$. 

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**DIABATIC LIMITS AND NONCOMMUTATIVE WEYL FORMULA**

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In this case, the operator $\Delta_\varepsilon$ can be represented, at least, locally over the base, as a differential operator on the base with operator-valued coefficients, and it looks likely that in this case the formula (10) can be proved, using suitable pseudodifferential calculus, for instance, pseudodifferential operators with operator-valued coefficients [1] or adiabatic pseudodifferential calculus of Melrose [16, 17].

1.3. Linear foliations on the torus. Here we discuss the example of a linear foliation on the 2-torus. So consider the two-dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the coordinates $(u, v) \in \mathbb{R}^2$, taken modulo integer translations, endowed with the Euclidean metric $g = du^2 + dv^2$.

Let $\tilde{X}$ be the vector field on $\mathbb{R}^2$ given by

$$\tilde{X} = \frac{\partial}{\partial u} + \alpha \frac{\partial}{\partial v},$$

where $\alpha \in \mathbb{R}$. Since $\tilde{X}$ is translation invariant, it determines a vector field $X$ on $T^2$. The orbits of $X$ define a one-dimensional foliation $F$ on $T^2$. The leaves of $F$ are the images of the parallel lines $\tilde{L}(u_0, v_0) = \{(u_0 + t, v_0 + t\alpha) : t \in \mathbb{R}\}$, parameterized by $(u_0, v_0) \in \mathbb{R}^2$, under the projection $\mathbb{R}^2 \to T^2$.

The Riemannian metric $g_\varepsilon$ on $T^2$ defined by (1) is given by

$$g_\varepsilon = -\frac{1}{1 + \alpha^2} (du + \alpha dv)^2 - \frac{\varepsilon^2}{1 + \alpha^2} (-\alpha du + dv)^2.$$ 

The Laplace-Beltrami operator $\Delta_\varepsilon$ defined by $g_\varepsilon$ has the form

$$\Delta_\varepsilon = -\frac{1}{1 + \alpha^2} \left( \frac{\partial}{\partial u} + \alpha \frac{\partial}{\partial v} \right)^2 - \frac{\varepsilon^2}{1 + \alpha^2} \left( -\alpha \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right)^2.$$ 

Remark 1. The operator $\Delta_\varepsilon$ has the complete orthogonal system of eigenfunctions $\{f_{kl} \in C^\infty(T^2) : (k, l) \in \mathbb{Z}^2\}$ given by

$$f_{kl}(u, v) = e^{2\pi i (ku + lv)}, \quad (u, v) \in T^2,$$

with the corresponding eigenvalues

$$\lambda_{kl}(\varepsilon) = (2\pi)^2 \left( \frac{1}{1 + \alpha^2} (k + \alpha l)^2 + \frac{\varepsilon^2}{1 + \alpha^2} (-\alpha k + l)^2 \right).$$

The eigenvalue distribution function of $\Delta_\varepsilon$ has the form

$$N_\varepsilon(\lambda) = \# \{(k, l) \in \mathbb{Z}^2 : (2\pi)^2 \left( \frac{1}{1 + \alpha^2} (k + \alpha l)^2 + \frac{\varepsilon^2}{1 + \alpha^2} (-\alpha k + l)^2 \right) < \lambda \}.$$ 

Thus we see that $N_\varepsilon(\lambda)$ equals the number of integer points in the ellipse

$$\{(\xi, \eta) \in \mathbb{R}^2 : (2\pi)^2 \left( \frac{1}{1 + \alpha^2} (\xi + \alpha \eta)^2 + \frac{\varepsilon^2}{1 + \alpha^2} (-\alpha \xi + \eta)^2 \right) < \lambda \}.$$ 

This ellipse is centered at the origin. Its semi-axes are equal to

$$a = \frac{(1 + \alpha^2) \sqrt{\lambda}}{2\pi}, \quad b = \frac{(1 + \alpha^2) \sqrt{\lambda}}{2\pi \varepsilon}.$$ 

Thus, $a$ is independent of $\varepsilon$, and $b \to \infty$ as $\varepsilon \to 0$.

So we see that, in this case, our problem is related with some lattice point distribution problems.

In this case, by a direct calculation, one can show the following result.
Theorem 1.3 ([20]). The following asymptotic formula for the eigenvalue distribution function $N_\varepsilon(\lambda)$ of the operator $\Delta_\varepsilon$ for a fixed $\lambda \in \mathbb{R}$ holds:

1. For $\alpha \notin \mathbb{Q}$,
   $$N_\varepsilon(\lambda) = \frac{1}{4\pi} \varepsilon^{-1} \lambda + o(\varepsilon^{-1}), \quad \varepsilon \to 0.$$

2. For $\alpha \in \mathbb{Q}$ of the form $\alpha = \frac{p}{q}$, where $p$ and $q$ are coprime,
   $$N_\varepsilon(\lambda) = \varepsilon^{-1} \sum_{k \in \mathbb{Z}} \frac{1}{\pi \sqrt{p^2 + q^2}} (\lambda - \frac{4\pi^2}{p^2 + q^2} k^2)^{1/2} + o(\varepsilon^{-1}), \quad \varepsilon \to 0.$$

One can also derive the asymptotic formulae of Theorem 1.3 from Theorem 1.2 (see [20] for details). By Theorem 1.3, we have, for $\alpha \notin \mathbb{Q}$,

$$\text{tr} e^{-t\Delta_\varepsilon} = \int_0^{+\infty} e^{-t\lambda} dN_\varepsilon(\lambda) = \frac{1}{4\pi} \varepsilon^{-1} \int_0^{+\infty} e^{-t\lambda} d\lambda + o(\varepsilon^{-1}) = \frac{1}{4\pi t} \varepsilon^{-1} + o(\varepsilon^{-1}), \quad \varepsilon \to 0.$$

As an illustration, let us show that, for $\alpha \notin \mathbb{Q}$, we have

$$\text{tr}_{F_\alpha} e^{-t\sigma(\Delta_\varepsilon)} = \frac{1}{2t}, \quad t > 0,$$

that agrees with [10]. It is easy to see that $F = \text{span}(U_1), H = \text{span}(U_2)$, where

$$U_1 = \frac{1}{\sqrt{1 + \alpha^2}} \left( \frac{\partial}{\partial u} + \alpha \frac{\partial}{\partial v} \right), \quad U_2 = \frac{1}{\sqrt{1 + \alpha^2}} \left( -\alpha \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right).$$

Therefore, we have $F^* = \text{span}(\omega_1), H^* = \text{span}(\omega_2)$, where

$$\omega_1 = \frac{1}{\sqrt{1 + \alpha^2}} (du + adv), \quad \omega_2 = \frac{1}{\sqrt{1 + \alpha^2}} (-adu + dv).$$

The conormal bundle $N^*F$ is canonically isomorphic to $H^*$ and, therefore, $N^*F$ is diffeomorphic to $T^2 \times \mathbb{R}$ by means of the diffeomorphism

$$(u, v, p_2) \in T^2 \times \mathbb{R} \mapsto p_2\omega_2(u, v) \in N^*F.$$

The leaves of the lifted foliation $F_N$ coincide with the orbits of the induced flow on $N^*F \cong T^2 \times \mathbb{R}$ given by

$$T_\tau(u, v, p_2) = (u + \tau, v + \alpha \tau, p_2), \quad (u, v, p_2) \in T^2 \times \mathbb{R}.$$

The foliation has no holonomy, and there is a natural identification $\tilde{L}_{(u,v,p_2)} \rightarrow L_{(u,v)}$ given by

$$(u + \tau, v + \alpha \tau, p_2) \mapsto (u + \tau, v + \alpha \tau), \quad \tau \in \mathbb{R}.$$

Therefore, the lift $\Delta_{F_N}$ of the tangential Laplacian $\Delta_F$ to a tangentially elliptic (relative to $F_N$) operator in $C^\infty(N^*F)$ is given by

$$\Delta_{F_N} = -\frac{1}{1 + \alpha^2} \left( \frac{\partial}{\partial u} + \alpha \frac{\partial}{\partial v} \right)^2.$$

The induced metric $g_N$ in the fibers of $N^*F$ is given by

$$g_{(u,v)}^N(p_2) = p_2^2, \quad (u, v, p_2) \in N^*F.$$

Theorem 1.3 (20). The following asymptotic formula for the eigenvalue distribution function $N_\varepsilon(\lambda)$ of the operator $\Delta_\varepsilon$ for a fixed $\lambda \in \mathbb{R}$ holds:

1. For $\alpha \notin \mathbb{Q}$,
   $$N_\varepsilon(\lambda) = \frac{1}{4\pi} \varepsilon^{-1} \lambda + o(\varepsilon^{-1}), \quad \varepsilon \to 0.$$

2. For $\alpha \in \mathbb{Q}$ of the form $\alpha = \frac{p}{q}$, where $p$ and $q$ are coprime,
   $$N_\varepsilon(\lambda) = \varepsilon^{-1} \sum_{k \in \mathbb{Z}} \frac{1}{\pi \sqrt{p^2 + q^2}} (\lambda - \frac{4\pi^2}{p^2 + q^2} k^2)^{1/2} + o(\varepsilon^{-1}), \quad \varepsilon \to 0.$$
Thus, the principal symbol of $\Delta_{e}$ is a tangentially elliptic operator in $C^\infty(N^*F)$ given by

$$\sigma(\Delta_{e}) = -\frac{1}{1 + \alpha^2} \left( \frac{\partial}{\partial u} + \alpha \frac{\partial}{\partial v} \right)^2 + p^2.$$ 

The foliation $F$ is given by the orbits of a free action of $\mathbb{R}$ on $T^2$, and its holonomy groupoid is described as follows: $G = T^2 \times \mathbb{R}$, $G^{(0)} = T^2$, $s(u,v,\tau) = (u - \tau, v - \alpha \tau)$, $r(u,v,\tau) = (u,v,\tau)$, $(u,v) \in T^2$, etc., and the product of $(u_1,v_1,\tau_1)$ and $(u_2,v_2,\tau_2)$ is defined if $u_2 = u_1 - \tau_1, v_2 = v_1 - \alpha \tau_1$ and equals

$$(u_1,v_1,\tau_1)(u_2,v_2,\tau_2) = (u_1,v_1,\tau_1 + \tau_2).$$

The holonomy groupoid $G_{F_N}$ is described as follows: $G_{F_N} = T^2 \times \mathbb{R}$, $G^{(0)}_{F_N} = T^2 \times \mathbb{R}$, $s_N(u,v,p_0,\tau) = (u - \tau, v - \alpha \tau, p_0)$, $r_N(u,v,p_0,\tau) = (u,v,p_0) \in T^2 \times \mathbb{R}$, and the product of $(u_1,v_1,p_{0,1},\tau_1)$ and $(u_2,v_2,p_{0,2},\tau_2)$ is defined if $u_2 = u_1 - \tau_1, v_2 = v_1 - \alpha \tau_1, p_{0,2} = p_{0,1}$, and equals

$$(u_1,v_1,p_{0,1},\tau_1)(u_2,v_2,p_{0,2},\tau_2) = (u_1,v_1,p_{0,1},\tau_1 + \tau_2).$$

The restriction of the operator $\sigma(\Delta_{e})$ to $L^2(u,v,p_2) \cong \mathbb{R}$ is the second order elliptic differential operator in the space $L^2(\mathbb{R}, \sqrt{1 + \alpha^2} d\tau)$:

$$\sigma(\Delta_{e})(u,v,p_2) = -\frac{1}{1 + \alpha^2} \frac{\partial^2}{\partial \tau^2} + p^2.$$ 

The change of variables $\sigma = \tau \sqrt{1 + \alpha^2}$ moves this operator to the space $L^2(\mathbb{R}, d\sigma)$, getting

$$\sigma(\Delta_{e})(u,v,p_2) = -\frac{\partial^2}{\partial \sigma^2} + p^2.$$ 

So the heat kernel of the operator $\sigma(\Delta_{e})(u,v,p_2)$ in $L^2(\mathbb{R}, \sqrt{1 + \alpha^2} d\tau)$ is given by

$$K_t(\tau_1,\tau_2) = (4\pi t)^{-1/2} e^{-\tau^2_{2t}} \exp \left( -\frac{(\tau_1 - \tau_2)^2}{4t(1 + \alpha^2)} \right).$$

The corresponding element $k_t$ of $C^\infty(G_{F_N})$ such that

$$e^{-\sigma(\Delta_{e})(v,p_2)} = R_{(u,v,p_2)}(k_t)$$

is related with $K_t$ by

$$K_t(\tau_1,\tau_2) = k_t(\{u,v,p_2,\tau_1\}^{-1}(u,v,p_2,\tau_2)),$$

where we consider $(u,v,p_2,\tau_1)$ and $(u,v,p_2,\tau_2)$ as elements of $G_{F_N}$ (cf. $\mathbb{S}$). Therefore, it is given by

$$k_t(u,v,p_0,\tau) = K_t(0,\tau)$$

$$= (4\pi t)^{-1/2} e^{-\tau^2_{2t}} \exp \left( -\frac{\tau^2}{4t(1 + \alpha^2)} \right), \quad (u,v,p_0,\tau) \in G_{F_N}.$$ 

Putting $\tau = 0$, we get a well-defined function $k_t$ on $N^*F$, the restriction to the set of units $G^{(0)}_{F_N} = N^*F \cong T^2 \times \mathbb{R}$:

$$k_t(u,v,p_2) = (4\pi t)^{-1/2} e^{-\tau^2_{2t}}, \quad (u,v,p_2) \in T^2 \times \mathbb{R}.$$ 

Finally, by $\mathbb{S}$, we obtain

$$\text{tr}_{F_N} e^{-t\sigma(\Delta_{e})} = \int_{T^2 \times \mathbb{R}} k_t(u,v,w,p_2) du dv dp_2 = \frac{1}{2t}.$$
2. SOME EXAMPLES OF NON-RIEMANNIAN FOLIATIONS

In this Section we consider two examples of non-Riemannian foliations. We start with one-dimensional foliations given by the orbits of invariant flows on the three-dimensional Heisenberg manifold.

2.1. RIEMANNIAN HEISENBERG MANIFOLDS. Recall that the real three-dimensional Heisenberg group $H_3$ is the Lie subgroup of $\text{GL}(3, \mathbb{R})$ consisting of all matrices of the form

$$\gamma(u, v, w) = \begin{bmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}, \quad u, v, w \in \mathbb{R}.$$ 

Its Lie algebra $h_3$ is the Lie subalgebra of $\text{gl}(3, \mathbb{R})$ consisting of all matrices of the form

$$X(u, v, w) = \begin{bmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix}, \quad u, v, w \in \mathbb{R}.$$ 

Denote by $U = X(1, 0, 0)$, $V = X(0, 1, 0)$, $W = X(0, 0, 1)$ the standard basis in the Lie algebra $h_3$. The corresponding left-invariant vector fields on $H_3$ are given by

$$U = \frac{\partial}{\partial u}, \quad V = \frac{\partial}{\partial v} + u \frac{\partial}{\partial w}, \quad W = \frac{\partial}{\partial w}, \quad \gamma(u, v, w) \in H.$$ 

The dual basis of left-invariant differential one-forms is given by

$$U^* = du, \quad V^* = dv, \quad W^* = dw - u dv.$$ 

Consider the uniform discrete subgroup $\Gamma$ of $H_3$ defined as

$$\Gamma = \{ \gamma(u, v, w) : u, v, w \in \mathbb{Z} \}.$$ 

A Riemannian Heisenberg manifold $M$ is defined to be a pair $(\Gamma \backslash H_3, g)$, where $g$ is a Riemannian metric on $\Gamma \backslash H_3$ whose lift to $H_3$ is left $H$-invariant.

It is easy to see that $g$ is uniquely determined by the value of its lift to $H_3$ at the identity $\gamma(0, 0, 0)$, that is, by a symmetric positive definite $3 \times 3$-matrix.

Let $\alpha \in \mathbb{R}$. Consider the left-invariant vector field on $H_3$ associated with $X(1, \alpha, 0)$.

$$X(1, \alpha, 0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} \in h_3.$$ 

Since $X(1, \alpha, 0)$ is a left-invariant vector field, it determines a vector field on $M = \Gamma \backslash H_3$. The orbits of this vector field define a one-dimensional foliation $\mathcal{F}$ on $M$. The leaf through a point $\Gamma \gamma(u, v, w) \in M$ is described as

$$L_{\Gamma \gamma(u, v, w)} = \{ \gamma(u + \tau, v + \alpha \tau, w + \alpha \tau u + \alpha \tau^2/2) \in \Gamma \backslash H_3 : \tau \in \mathbb{R} \}.$$ 

Let us assume that $g$ corresponds to the identity $3 \times 3$-matrix. Consider the adiabatic limit associated with the Riemannian Heisenberg manifold $(\Gamma \backslash H_3, g)$ and
the one-dimensional foliation $\mathcal{F}$. The corresponding Laplace-Beltrami operator $\Delta_\epsilon$ on the group $H_3$ has the form:

$$\Delta_\epsilon = \frac{1}{1 + \alpha^2} \left( \frac{\partial}{\partial u} + \alpha \left( \frac{\partial}{\partial v} + u \frac{\partial}{\partial w} \right) \right)^2 + \epsilon^2 \left[ \frac{1}{1 + \alpha^2} \left( -\alpha \frac{\partial}{\partial u} + \frac{\partial}{\partial v} + v \frac{\partial}{\partial w} \right)^2 + \frac{\partial^2}{\partial w^2} \right].$$

Using an explicit computation of the heat kernel on the Heisenberg group, one can show the following asymptotic formula.

**Theorem 2.1** ([21]). For any $t > 0$, we have as $\epsilon \to 0$

$$\text{tr} e^{-t\Delta_\epsilon} = \frac{1}{8\pi^2 \epsilon^2} \int_{-\infty}^{+\infty} \frac{\eta}{\sinh(t\eta)} e^{-t\eta^2} d\eta + o(\epsilon^{-2}).$$

Now we show that the formula (12) can be written in the form (6).

**Theorem 2.2.** Under current assumptions, for any $t > 0$, we have

$$\text{tr} e^{-t\Delta_\epsilon} = \frac{1}{4\pi^2 \epsilon^2} \text{tr}_{\mathcal{F}_N} e^{-t\sigma(\Delta_\epsilon)} + o(\epsilon^{-2}), \quad \epsilon \to 0,$$

where $\sigma(\Delta_\epsilon)$ is defined by (7).

**Proof.** We have $F = \text{span}(U_1), H = \text{span}(U_2, U_3)$, where

$$U_1 = \frac{1}{\sqrt{1 + \alpha^2}} (U + \alpha V) = \frac{1}{\sqrt{1 + \alpha^2}} \left[ \frac{\partial}{\partial u} + \alpha \left( \frac{\partial}{\partial v} + u \frac{\partial}{\partial w} \right) \right],$$

$$U_2 = \frac{1}{\sqrt{1 + \alpha^2}} (-\alpha U + V) = \frac{1}{\sqrt{1 + \alpha^2}} \left[ -\alpha \frac{\partial}{\partial u} + \frac{\partial}{\partial v} + \frac{\partial}{\partial w} \right],$$

$$U_3 = W = \frac{\partial}{\partial w}. $$

We also have $F^* = \text{span}(\omega_1), H^* = \text{span}(\omega_2, \omega_3)$, where

$$\omega_1 = \frac{1}{\sqrt{1 + \alpha^2}} (U^* + \alpha V^*) = \frac{1}{\sqrt{1 + \alpha^2}} (du + adv),$$

$$\omega_2 = \frac{1}{\sqrt{1 + \alpha^2}} (-\alpha U^* + V^*) = \frac{1}{\sqrt{1 + \alpha^2}} (-adu + dv),$$

$$\omega_3 = dw - u dv.$$

The conormal bundle $N^*\mathcal{F}$ is canonically isomorphic to $H^*$ and, therefore, its fiber at $\Gamma\gamma(u, v, w)$ consists of all $p_2\omega_2(u, v, w) + p_2\omega_3(u, v, w) \in T^*_\gamma(u, v, w)H_1$ with $p_2, p_3 \in \mathbb{R}$. So $N^*\mathcal{F}$ is the quotient of $H_3 \times \mathbb{R}^2$ by the induced action of $\Gamma$. The leaves of the lifted foliation $\mathcal{F}_N$ coincide with the orbits of the induced flow on $N^*\mathcal{F}$, which given by:

$$T_\gamma(u, v, w, p_2, p_3) = (u + \tau, v + \alpha \tau, w + \alpha \tau u + \frac{\alpha \tau^2}{2}, p_2 - \tau \sqrt{1 + \alpha^2} p_3).$$

The foliation has no holonomy, and there is a natural identification of the leaf $L^*_\gamma(u, v, w, p_2, p_3)$ of $\mathcal{F}_N$ through $\Gamma_\gamma(u, v, w, p_2, p_3) \in N^*\mathcal{F}$ with the leaf $L_\gamma(u, v, w)$ of $\mathcal{F}$ through $\Gamma_\gamma(u, v, w) \in M$ given by

$$(u + \tau, v + \alpha \tau, w + \alpha \tau u + \frac{\alpha \tau^2}{2}, p_2 - \tau \sqrt{1 + \alpha^2} p_3, p_3) \mapsto (u + \tau, v + \alpha \tau, w + \alpha \tau u + \frac{\alpha \tau^2}{2}).$$
Therefore, the lift $\Delta_{F,\ell}$ of the tangential Laplacian $\Delta_F$ to a tangentially elliptic (relative to $F$) operator in $C^\infty(N^*F)$ is given by

$$\Delta_{F,\ell} = -\frac{1}{1+\alpha^2} \left( \frac{\partial}{\partial u} + \alpha \left( \frac{\partial}{\partial v} + u \frac{\partial}{\partial w} \right) - \sqrt{1+\alpha^2} p_3 \frac{\partial}{\partial p_2} \right)^2.$$

The induced metric $g^N$ in the fibers of $N^*F$ is given by

$$g^N_{(u,v,w)}(p_2,p_3) = p_2^2 + p_3^2, \quad (u,v,p_2,p_3) \in N^*F.$$

Thus, the principal symbol of $\Delta_{\varepsilon}$ is a tangentially elliptic operator in $C^\infty(N^*F)$ given by

$$\sigma(\Delta_{\varepsilon}) = -\frac{1}{1+\alpha^2} \left( \frac{\partial^2}{\partial t^2} + \sqrt{1+\alpha^2} \frac{\partial}{\partial t} \right) + (p_2 - \alpha t + \sqrt{1+\alpha^2} p_3)^2 + p_3^2.$$

The change of variable $\sigma = \tau\sqrt{1+\alpha^2}$ transfers this operator to the space $L^2(\mathbb{R},dt)$.

$$\sigma_{(u,v,w,p_2,p_3)} = -\frac{\partial^2}{\partial \sigma^2} + (p_2 - \sigma p_3)^2 + p_3^2.$$

Now we use the well-known Mehler formula for the heat kernel $H_t$ of the harmonic oscillator $H_\omega = -\frac{\partial^2}{\partial x^2} + \omega^2 x^2$ in $L^2(\mathbb{R},dx)$:

$$H_t(x,y) = (4\pi t)^{-1/2} \left( \frac{2\omega t}{\sinh(2\omega t)} \right)^{1/2} \exp \left( -\frac{2\omega t}{\sinh(2\omega t)} \left[ \cosh(2\omega t)(x^2 + y^2) - 2xy \right] \right).$$

So the heat kernel of the operator $\sigma_{(u,v,w,p_2,p_3)}$ in $L^2(\mathbb{R},\sqrt{1+\alpha^2}dt)$ is given by

$$K_t(\tau_1,\tau_2) = H_t^{p_3} \left( \tau_1 \sqrt{1+\alpha^2} - \frac{p_2}{p_3}, \tau_2 \sqrt{1+\alpha^2} - \frac{p_2}{p_3} \right).$$

The foliation $\mathcal{F}$ is given by the orbits of a free action of $\mathbb{R}$ on $M$, and its holonomy groupoid is described as follows: $G = M \times \mathbb{R}$, $G^{(0)} = M$, for any $(u,v,w,\tau) \in G$, $s(u,v,w,\tau) = (u-\tau,v-\alpha \tau,w-\alpha \tau u + \frac{\alpha^2}{2} \tau^2)$ and $r(u,v,w,\tau) = (u,v,w)$. The product of $(u_1,v_1,w_1,\tau_1)$ and $(u_2,v_2,w_2,\tau_2)$ is defined if $u_2 = u_1 - \tau_1, v_2 = v_1 - \alpha \tau_1, w_2 = w_1 - \alpha \tau_1 u_1 + \frac{\alpha^2}{2} \tau_1^2$ and equals $(u_1,v_1,w_1,\tau_1)(u_2,v_2,w_2,\tau_2) = (u_1,v_1,w_1,\tau_1 + \tau_2)$.

The holonomy groupoid $G_{\mathcal{F},\ell}$ is described as follows: $G_{\mathcal{F},\ell} = M \times \mathbb{R}^2 \times \mathbb{R}$, $G_{\mathcal{F},\ell}^{(0)} = M \times \mathbb{R}^2$, and for any $(u,v,w,p_2,p_3,\tau) \in G_{\mathcal{F},\ell}$, we have $s_N(u,v,w,p_2,p_3,\tau) = (u-\tau,v-\alpha \tau,w-\alpha \tau u + \frac{\alpha^2}{2} \tau^2,p_2 + \tau \sqrt{1+\alpha^2} p_3,p_3)$, and $r_N(u,v,w,p_2,p_3,\tau) = (u,v,w,p_2,p_3)$.

The corresponding element $k_t$ of $C^\infty(G_{\mathcal{F},\ell})$ is given by

$$k_t(u,v,w,p_2,p_3,\tau) = K_t(0,\tau) = H_t^{p_3} \left( -\frac{p_2}{p_3}, \tau \sqrt{1+\alpha^2} - \frac{p_2}{p_3} \right).$$
Putting $\tau = 0$, we get a well-defined function $k_t$ on $N^*F$, the restriction to the set of units $G^{(0)}_{F_N} = N^*F$:

$$k_t(u, v, w, p_2, p_3) = (4\pi t)^{-1/2} \left( \frac{2p_3t}{\sinh(2p_3t)} \right)^{1/2} e^{-p_3^2t} \exp \left( -\frac{\sinh(p_3t)}{\cosh(p_3t)} \frac{p_2^2}{p_3} \right).$$

To compute the Liouville transverse volume form, we choose a foliated chart defined by

$$x_1 = u, \quad y_1 = v - \alpha u, \quad y_2 = w - \frac{1}{2} \alpha u^2.$$

Thus, we have

$$dy_1 = dv - \alpha du = \sqrt{1 + \alpha^2} \omega_2, \quad dy_2 = dw - \alpha u du = \omega_3 + \sqrt{1 + \alpha^2} u \omega_2.$$

The identity

$$\eta_1 dy_1 + \eta_2 dy_2 = p_2 \omega_2 + p_3 \omega_3$$

implies a relation between the dual coordinates $(\eta_1, \eta_2)$ in this foliated chart and $(p_2, p_3)$:

$$\eta_1 = \frac{1}{\sqrt{1 + \alpha^2}} p_2 - up_3, \quad \eta_2 = p_3.$$

So we see that the combination of the transverse Liouville measure $|dy_1 \wedge dy_2 \wedge \eta_1 \wedge \eta_2|$ with the leafwise Riemannian volume density $|\omega_1|$ determines the measure $d\nu$ on $N^*F$ given by

$$d\nu = du dv dw dp_2 dp_3.$$

By (11), it follows that

$$\text{tr}_{F_N} e^{-t\sigma(\Delta_\ast)} = \int k_t(u, v, w, p_2, p_3) du dv dw dp_2 dp_3$$

$$= (4\pi t)^{-1/2} \int \left( \frac{2p_3t}{\sinh(2p_3t)} \right)^{1/2} e^{-p_3^2t} \exp \left( -\frac{\sinh(p_3t)}{\cosh(p_3t)} \frac{p_2^2}{p_3} \right) dp_2 dp_3.$$

Integrating with respect to $p_2$ in the last integral and using the formula

$$\int e^{-ax^2} dx = \frac{\pi^{1/2}}{a^{1/2}},$$

we obtain

$$\text{tr}_{F_N} e^{-t\sigma(\Delta_\ast)} = \frac{1}{2} \int \left( \frac{2p_3}{\sinh(2p_3t)} \right)^{1/2} \left( \frac{p_3 \cosh(p_3t)}{\sinh(p_3t)} \right)^{1/2} e^{-p_3^2t} dp_3$$

$$= \frac{1}{2} \int \frac{p_3}{\sinh(p_3t)} e^{-p_3^2t} dp_3.$$

Combined with (12), this concludes the proof. \(\square\)

2.2. Sol-manifolds. In this Section we consider another example of adiabatic limits associated with non-Riemannian foliations, namely, with one-dimensional foliations given by the orbits of invariant flows on Riemannian Sol-manifolds.

Let $A = \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right)$ be such that $|\text{tr} A| > 2$. A Sol-manifold can be defined as the quotient $M^2_A = (T^2 \times \mathbb{R})/\mathbb{Z}$, where the action of $\mathbb{Z}$ on $T^2 \times \mathbb{R}$ is defined by the diffeomorphism $T_A$ of $T^2 \times \mathbb{R}$ given by

$$T_A(x, y, z) = (a_{11}x + a_{12}y, a_{21}x + a_{22}y, z + 1), \quad (x, y) \in T^2 = \mathbb{R}^2/\mathbb{Z}^2, \quad z \in \mathbb{R}.$$
It is well known that $M^3_A$ is a compact manifold.

Denote by $\lambda$ and $\lambda^{-1}$ the eigenvalues of $A$ and suppose that $\lambda > 1$. Let $\{e_u, e_v\}$ be the corresponding positively oriented base of eigenvectors of $A$. We introduce another coordinate system $(u, v, w)$ on $\mathbb{R}^3$ by

$$(x, y) = ue_u + ve_v, \quad w = z \ln \lambda.$$ 

In the coordinates $(u, v, w)$ the map $T_A$ takes the following form:

$$T_A(u, v, w) = (\lambda u, \lambda^{-1} v, w + \ln \lambda).$$

Denote by $\widetilde{M}^3$ the Lie subgroup of the Lie group $GL(3, \mathbb{R})$ of matrices of the form

$$\gamma(u, v, w) = \begin{pmatrix} e^w & 0 & u \\ 0 & e^{-w} & v \\ 0 & 0 & 1 \end{pmatrix}, \quad u, v, w \in \mathbb{R}.$$ 

The corresponding Lie algebra $\mathfrak{m}^3$ of $\widetilde{M}^3$ is the Lie subalgebra of the Lie algebra $gl(3, \mathbb{R})$ of matrices of the form

$$X(u, v, w) = \begin{pmatrix} w & 0 & u \\ 0 & -w & v \\ 0 & 0 & 0 \end{pmatrix}, \quad u, v, w \in \mathbb{R}.$$ 

Denote by

$$U = X(1, 0, 0), \quad V = X(0, 1, 0), \quad W = X(0, 0, 1)$$

the standard basis in the Lie algebra $\mathfrak{m}^3$. The corresponding left-invariant vector fields on $\widetilde{M}^3$ are given by

$$U = e^w \frac{\partial}{\partial u}, \quad V = e^{-w} \frac{\partial}{\partial v}, \quad W = \frac{\partial}{\partial w}, \quad \gamma(u, v, w) \in \widetilde{M}^3.$$ 

The dual basis $U^*, V^*, W^*$ of left-invariant one-forms is given by

$$U^* = e^{-w} du, \quad V^* = e^w dv, \quad W^* = dw.$$ 

Observe that in the new coordinates two points $(u, v)$ and $(u', v')$ define the same point on $\mathbb{T}^2$ if and only if $(u - u', v - v') = k(c_1^1, c_2^1) + m(c_1^2, c_2^2)$, where $k, m \in \mathbb{Z}$ and $\{c_1 = (c_1^1, c_1^2), c_2 = (c_2^1, c_2^2)\}$ is the basis of the integer lattice $\Gamma$ of $\mathbb{T}^2$ defined by the equation

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix}^{-1} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix}.$$ 

Therefore, if we introduce a discrete subgroup $G_A$ of $\widetilde{M}^3$, which consists of all $\gamma(u, v, w) \in \widetilde{M}^3$ such that

$$(u, v) = k(c_1^1, c_2^1) + l(c_1^2, c_2^2) \in \Gamma, \quad w = m \ln \lambda, \quad k, l, m \in \mathbb{Z},$$

then it is easy to see that $M^3_A = G_A \backslash \widetilde{M}^3$.

**Definition 2.3.** A Riemannian Sol-manifold is a pair $(M^3_A, g)$, where $g$ is a Riemannian metric on $M_A^3$ whose lift on $\widetilde{M}^3$ is left $\widetilde{M}^3$-invariant.

It is easy to see that $g$ is uniquely determined by its value at the identity $\gamma(0, 0, 0)$ of the Lie group $\widetilde{M}^3$, which is given by a symmetric positive definite $3 \times 3$-matrix.
Suppose that \( g \) corresponds to the identity matrix:
\[
g(G_A\gamma(0,0,0)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
In other words, the basis \( U, V, W \) is orthonormal. Then, in the coordinates \((u,v,w)\), the metric \( g \) has the form
\[
g(G_A\gamma(u,v,w)) = e^{-2w}du^2 + e^{2w}dv^2 + dw^2.
\]
Let \( \alpha \in \mathbb{R} \). Consider the left-invariant vector field on \( \widetilde{M}^3 \) associated with
\[
X(1,\alpha,0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} \in m^3.
\]
The orbits of the corresponding vector field on \( M_A^3 \) define a one-dimensional foliation \( \mathcal{F} \). The leaf \( L_{G_A\gamma(u,v,w)} \) of \( \mathcal{F} \) through \( G_A\gamma(u,v,w) \in M_A^3 \) has the form
\[
L_{G_A\gamma(u,v,w)} = \{ G_A\gamma(u+e^w t, v + \alpha e^{-w} t, w) : t \in \mathbb{R} \}.
\]
One can show that, if \( \alpha = 0 \), the metric \( g \) is bundle-like and the foliation \( \mathcal{F} \) is Riemannian. Otherwise, the metric \( g \) is not bundle-like.

Let us consider the adiabatic limit associated with the Riemannian Sol-manifold \((G_A, \widetilde{M}^3, g)\) and with the foliation \( \mathcal{F} \). Then the Riemannian metric \( g_\varepsilon \) on \( G_A\setminus\widetilde{M}^3 \) defined by (1) in the local coordinates \((u,v,w)\) has the form
\[
g_\varepsilon = \frac{1}{1 + \alpha^2} (e^{-w}du + \alpha e^{-w}dv)^2 + \varepsilon^{-2} \left[ \frac{1}{1 + \alpha^2} (-\alpha e^{-w}du + e^{-w}dv)^2 + dw^2 \right].
\]
For any \( \varepsilon > 0 \), consider the Laplace-Beltrami operator \( \Delta_\varepsilon \) defined by \( g_\varepsilon \):
\[
\Delta_\varepsilon = -\frac{1}{1 + \alpha^2} \left( e^{-w} \frac{\partial}{\partial u} + \alpha e^{-w} \frac{\partial}{\partial v} \right)^2 - \varepsilon^2 \left( \frac{1}{1 + \alpha^2} \left( -\alpha e^{-w} \frac{\partial}{\partial u} + e^{-w} \frac{\partial}{\partial v} \right)^2 + \frac{\partial^2}{\partial w^2} \right).
\]

**Theorem 2.4** ([22], [23]). For \( \lambda > 0 \) for the eigenvalue distribution function \( N_\varepsilon(\lambda) \) of \( \Delta_\varepsilon \), we have:
- For \( \alpha \neq 0 \)
  \[
  N_\varepsilon(\lambda) = \frac{1}{4\pi^2} \lambda^2 \varepsilon^{-2} + o(\varepsilon^{-2}).
  \]
- For \( \alpha = 0 \)
  \[
  N_\varepsilon(\lambda) = \frac{1}{6\pi^2} \lambda^2 \varepsilon^{-2} + o(\varepsilon^{-2}).
  \]

For \( \alpha \neq 0 \) the proof of this theorem makes use of the computation of the spectrum of the Laplace-Beltrami operator on Sol-manifolds given in [3] (see also [24]), which is a continuation of the study of geodesic flows on Riemannian Sol-manifolds (see [4], [5]), and semiclassical Weyl formula [8] for the modified Mathieu operator
\[
H_\varepsilon = -\varepsilon^2 \frac{d^2}{dx^2} + a \cosh(2\mu x), \quad x \in \mathbb{R}.
\]
For \( \alpha = 0 \) the result is a consequence of Theorem 1.1.

**Theorem 2.5.** For \( \alpha \neq 0 \), we have
\[
\text{tr}_{\mathcal{F}_N} e^{-t\sigma(\Delta_\varepsilon)} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{2\alpha}{1 + \alpha^2 \eta} e^{-\frac{2\alpha}{1 + \alpha^2} \eta} \sinh\left(\frac{2\alpha}{1 + \alpha^2} \eta\right) e^{-ty^2} \, d\eta.
\]
Proof. It is easy to see that \( F = \text{span}(U_1) \) and \( H = \text{span}(U_2, U_3) \), where
\[
U_1 = \frac{1}{\sqrt{1 + \alpha^2}} (U + \alpha V) = \frac{1}{\sqrt{1 + \alpha^2}} \left( e^w \frac{\partial}{\partial u} + \alpha e^{-w} \frac{\partial}{\partial v} \right),
\]
\[
U_2 = \frac{1}{\sqrt{1 + \alpha^2}} (\alpha U - V) = \frac{1}{\sqrt{1 + \alpha^2}} \left( e^{-w} \frac{\partial}{\partial u} - e^{-w} \frac{\partial}{\partial v} \right),
\]
\[
U_3 = W = \frac{\partial}{\partial w}.
\]

Therefore, we have \( F^* = \text{span}(\omega_1) \), \( H^* = \text{span}(\omega_2, \omega_3) \), where
\[
\omega_1 = \frac{1}{\sqrt{1 + \alpha^2}} (U^* + \alpha V^*) = \frac{1}{\sqrt{1 + \alpha^2}} (e^{-w} du + \alpha e^w dv),
\]
\[
\omega_2 = \frac{1}{\sqrt{1 + \alpha^2}} (-\alpha U^* + V^*) = \frac{1}{\sqrt{1 + \alpha^2}} (-\alpha e^{-w} du + e^w dv),
\]
\[
\omega_3 = W^* = dw.
\]

The conormal bundle \( N^*F \) is canonically isomorphic to \( H^* \) and, therefore, its fiber at \( \Gamma_{(u, v, w)} \) consists of all \( p_2\omega_2(u, v, w) + p_3\omega_3(u, v, w) \in T^{*}_{\gamma(u,v,w)}M^3 \) with \( p_2, p_3 \in \mathbb{R} \). So \( N^*F \) is the quotient of \( M^3 \times \mathbb{R}^2 \) by the induced action of \( \Gamma \). The leaves of the lifted foliation \( F_N \) coincide with the orbits of the induced flow on \( N^*F \), which given by:
\[
T_{\tau}(u, v, w, p_2, p_3) = (u + e^w \tau, v + \alpha e^{-w} \tau, w, p_2 + \frac{2\alpha}{\sqrt{1 + \alpha^2}} p_3, p_3).
\]

The foliation has no holonomy, and there is a natural identification \( L_{(u,v,w,p_2,p_3)} \rightarrow T_{(u,v,w)}F \) given by
\[
(u + e^w \tau, v + \alpha e^{-w} \tau, w, p_2 + \frac{2\alpha}{\sqrt{1 + \alpha^2}} p_3, p_3) \mapsto (u + e^w \tau, v + \alpha e^{-w} \tau, w).
\]

Therefore, the lift \( \Delta_{F_N} \) of the tangential Laplacian \( \Delta_F \) to a tangentially elliptic (relative to \( F_N \)) operator in \( C^\infty(N^*F) \) is given by
\[
\Delta_{F_N} = -\frac{1}{1 + \alpha^2} \left( e^w \frac{\partial}{\partial u} + \alpha e^{-w} \frac{\partial}{\partial v} + \frac{2\alpha}{\sqrt{1 + \alpha^2}} p_3 \frac{\partial}{\partial p_2} \right)^2.
\]

The induced metric \( g^N \) in the fibers of \( N^*F \) is given by
\[
g^N_{(u,v,w)}(p_2, p_3) = p_2^2 + p_3^2, \quad (u, v, w, p_2, p_3) \in N^*F.
\]

Thus, the principal symbol of \( \Delta_\varepsilon \) is a tangentially elliptic operator in \( C^\infty(N^*F) \) given by
\[
\sigma(\Delta_\varepsilon) = -\frac{1}{1 + \alpha^2} \left( e^w \frac{\partial}{\partial u} + \alpha e^{-w} \frac{\partial}{\partial v} + \frac{2\alpha}{\sqrt{1 + \alpha^2}} p_3 \frac{\partial}{\partial p_2} \right)^2 + p_2^2 + p_3^2.
\]

Thus, we see that the principal symbol has a form very similar to the case of Heisenberg manifolds (cf. (13)). So we can complete the proof, following the proof of Theorem (2.4).

By Theorem (2.4), we have
\[
\text{tr} e^{-\varepsilon \Delta_\varepsilon} = \int_0^{+\infty} e^{-\varepsilon \lambda} dN_\varepsilon(\lambda) = \frac{3}{8\pi^2 \varepsilon^2} \frac{\Gamma(\frac{4}{3})}{\varepsilon^{3/2}} + o(\varepsilon^{-2}), \quad \varepsilon \to 0.
\]

So we observe that in this case the formula (9) does not hold.
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