Yetter-Drinfeld modules for Hom-bialgebras

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Abstract

The aim of this paper is to define and study Yetter-Drinfeld modules over Hom-bialgebras, a generalized version of bialgebras obtained by modifying the algebra and coalgebra structures by a homomorphism. Yetter-Drinfeld modules over a Hom-bialgebra with bijective structure map provide solutions of the Hom-Yang-Baxter equation. The category $H^H YD$ of Yetter-Drinfeld modules with bijective structure maps over a Hom-bialgebra $H$ with bijective structure map can be organized, in two different ways, as a quasi-braided pre-tensor category. If $H$ is quasitriangular (respectively coquasitriangular) the first (respectively second) quasi-braided pre-tensor category $H^H YD$ contains, as a quasi-braided pre-tensor subcategory, the category of modules (respectively comodules) with bijective structure maps over $H$.

Introduction

The first examples of Hom-type algebras were related to $q$-deformations of Witt and Virasoro algebras, which play an important rôle in Physics, mainly in conformal field theory. In a theory with conformal symmetry, the Witt algebra $W$ is a part of the complexified Lie algebra $\text{Vect}^C(S) \times \text{Vect}^C(S)$, where $S$ is the unit circle, belonging to the classical conformal symmetry. The central extensions of $W$ by $\mathbb{C}$ become important for the quantization process. The $q$-deformations of Witt and Virasoro algebras are obtained when the derivation is replaced by a $\sigma$-derivation. It was observed in the pioneering works [1, 7, 8, 9, 10, 11, 13, 18, 19, 25] that they are no longer Lie algebras. Motivated by these examples and their generalization, Hartwig, Larson and Silvestrov in [17, 21, 22, 23] introduced the notion of Hom-Lie algebra as a deformation of Lie algebras in which the Jacobi identity is twisted by a homomorphism. The associative-type objects corresponding to Hom-Lie algebras, called Hom-associative algebras, have been introduced in [27]. Usual functors between the categories of Lie algebras and associative algebras have been extended to the Hom-setting. It was shown in [27] that a commutator of a Hom-associative algebra gives rise to a Hom-Lie algebra; the construction of the free Hom-associative

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algebra and the enveloping algebra of a Hom-Lie algebra have been provided in [32]. Since then, Hom-analogues of various classical structures and results have been introduced and discussed by many authors. For instance, representation theory, cohomology and deformation theory for Hom-associative algebras and Hom-Lie algebras have been developed in [2, 28, 31]. See also [15, 16] for other properties of Hom-associative algebras. All these generalizations coincide with the usual definitions when the structure map equals the identity.

The dual concept of Hom-associative algebras, called Hom-coassociative coalgebras, as well as Hom-bialgebras and Hom-Hopf algebras, have been introduced in [29, 30] and also studied in [6, 34]. As expected, the enveloping Hom-associative algebra of a Hom-Lie algebra is naturally a Hom-bialgebra. A twisted version of module algebras called module Hom-algebras has been studied in [33], where $q$-deformations of the $\mathfrak{sl}(2)$-action on the affine plane were provided. Objects admitting coactions by Hom-bialgebras have been studied first in [34]. A matrix Hom-associative algebra was endowed with a Hom-bialgebra structure $H$ and examples of $H$-comodule Hom-algebra structures on the Hom-affine plane $A_2$ have been provided. In [39, 40, 41], various generalizations of Yang-Baxter equations and related algebraic structures have been studied. D. Yau provided solutions of HYBE, a twisted version of the Yang-Baxter equation called the Hom-Yang-Baxter equation, from Hom-Lie algebras, quantum enveloping algebra of $\mathfrak{sl}(2)$, the Jones-Conway polynomial, Drinfeld’s (co)quasitriangular bialgebras and Yetter-Drinfeld modules (over bialgebras). It was shown that solutions of HYBE can be extended to operators that satisfy the braid relations, which can then be used to construct representations of the braid group, in case an invertibility condition holds. Moreover, a generalization of the classical Yang-Baxter equation and its connection to Hom-Lie bialgebras have been explored. See also [5] for other results related to Hom-Lie bialgebras. In the series of papers [36, 37, 38], D. Yau studied Hom-type generalizations of (co)quasitriangular bialgebras, quantum groups and the quantum Yang-Baxter equation (QYBE). It was shown that quasitriangular and coquasitriangular Hom-bialgebras come equipped with a solution of the quantum Hom-Yang-Baxter equation (QHYBE) or operator quantum Hom-Yang-Baxter equation (OQHYBE). Examples of quasitriangular Hom-bialgebras have been given, including Drinfeld’s quantum enveloping algebra $U_q(\mathfrak{g})$ of a semi-simple Lie algebra or a Kac-Moody algebra $\mathfrak{g}$ and anyonic quantum groups. In [14], Hom-quasi-bialgebras have been introduced and concepts like gauge transformation and Drinfeld twist generalized. Moreover, an example of a twisted quantum double was provided. One of the main tools to construct examples is the ”twisting principle” introduced by D. Yau for Hom-associative algebras and since then extended to various Hom-type algebras. It allows to construct a Hom-type algebra starting from a classical-type algebra and an algebra homomorphism.

The aim of this paper is to introduce and study Yetter-Drinfeld modules over a Hom-bialgebra $H$, as objects $(M, \alpha_M)$ such that $(M, \alpha_M)$ is both a left $H$-module and a left $H$-comodule and a certain compatibility condition between the two structures holds. This condition was chosen in such a way that if $(M, \alpha_M)$ is a left module over a quasitriangular Hom-bialgebra or a left comodule over a coquasitriangular Hom-bialgebra then $M$ becomes a Yetter-Drinfeld module over that Hom-bialgebra. We will denote by $H^H_{H^H}YD$ the category whose objects are Yetter-Drinfeld modules $(M, \alpha_M)$ with $\alpha_M$ bijective over a Hom-bialgebra $H$ with bijective structure map $\alpha_H$.

The paper is organized as follows. In Section 1, we review the main definitions and properties of pre-tensor categories, Hom-associative algebras, Hom-bialgebras and related structures. In Section 2, we introduce Yetter-Drinfeld modules and discuss some elementary aspects. We extend the twisting principle to Yetter-Drinfeld modules over bialgebras and we show that Yetter-Drinfeld modules over a Hom-bialgebra with bijective structure map give rise to solutions of the HYBE. In Section 3, we prove that $H^H_{H^H}YD$ can be organized as a quasi-braided pre-tensor
category (with nontrivial associators), for which the quasi-braiding satisfies the usual braid relation (besides the dodecagonal braid relation involving the associators). It turns out that, if $H$ is a quasitriangular Hom-bialgebra, the category of left $H$-modules with bijective structure maps is a quasi-braided pre-tensor subcategory of $\text{Hom}YD$. In Section 4, we find another quasi-braided pre-tensor category structure on $\text{Hom}YD$, with the property that if $H$ is a coquasitriangular Hom-bialgebra then $\text{Hom}YD$ contains the category of left $H$-comodules with bijective structure maps as a quasi-braided pre-tensor category.

1 Preliminaries

We work over a base field $k$. All algebras, linear spaces etc. will be over $k$; unadorned $\otimes$ means $\otimes_k$. For a comultiplication $\Delta : C \to C \otimes C$ on a vector space $C$ we use a Sweedler-type notation $\Delta(c) = c_1 \otimes c_2$, for $c \in C$. Unless otherwise specified, the (co)algebras ((co)associative or not) that will appear in what follows are not supposed to be (co)unital, and a multiplication $\mu : V \otimes V \to V$ on a linear space $V$ is denoted by juxtaposition: $\mu(v \otimes v') = vv'$.

We recall now several concepts and results, fixing thus the terminology to be used in the rest of the paper.

**Definition 1.1** ([24]) A pre-tensor category is a category satisfying all the axioms of a tensor category in [20] except for the fact that we do not require the existence of a unit object and of left and right unit constraints. If $(C, \otimes, a)$ is a pre-tensor category, a quasi-braiding $c$ in $C$ is a family of natural morphisms $\Delta(c) = c_1 \otimes c_2$, for $c \in C$. Unless otherwise specified, the (co)algebras ((co)associative or not) that will appear in what follows are not supposed to be (co)unital, and a multiplication $\mu : V \otimes V \to V$ on a linear space $V$ is denoted by juxtaposition: $\mu(v \otimes v') = vv'$.

Exactly as for usual braided categories, a quasi-braiding on a pre-tensor category satisfies the dodecagonal braid relation in [20], p. 317.

**Definition 1.2** Let $H$ be a bialgebra and $M$ a linear space which is a left $H$-module with action $H \otimes M \to M$, $h \otimes m \mapsto h \cdot m$ and a left $H$-comodule with coaction $M \to H \otimes M$, $m \mapsto m(-1) \otimes m(0)$. Then $M$ is called a (left-left) Yetter-Drinfeld module over $H$ if the following compatibility condition holds, for all $h \in H$, $m \in M$:

\[
(h_1 \cdot m)(-1)h_2 \otimes (h_1 \cdot m)(0) = h_1 m(-1) \otimes h_2 \cdot m(0).
\]

(1.1)

We summarize several definitions and properties about Hom-type structures. Since various authors use different terminology, some caution is necessary. In what follows, we use terminology as in Yau’s paper [33], which is different from the original terminology in [27], [28] (where no extra assumption on the linear map $\alpha$ is made) and also different from Yau’s paper [34], where for instance the multiplicativity of the map $\alpha$ is emphasized by calling ”multiplicative Hom-associative algebra” what we will call for simplicity in what follows ”Hom-associative algebra”.

**Definition 1.3** (i) A Hom-associative algebra is a triple $(A, \mu, \alpha)$, in which $A$ is a linear space, $\alpha : A \to A$ and $\mu : A \otimes A \to A$ are linear maps, with notation $\mu(a \otimes a') = aa'$, satisfying the following conditions, for all $a, a', a'' \in A$:

\[
\alpha(aa') = \alpha(a)\alpha(a'), \quad \text{multiplicativity}
\]

\[
\alpha(a)(a'a'') = (aa')\alpha(a''). \quad \text{(Hom – associativity)}
\]
We call \( \alpha \) the structure map of \( A \).

A morphism \( f : (A, \mu_A, \alpha_A) \rightarrow (B, \mu_B, \alpha_B) \) of Hom-associative algebras is a linear map \( f : A \rightarrow B \) such that \( \alpha_B \circ f = f \circ \alpha_A \) and \( f \circ \mu_A = \mu_B \circ (f \otimes f) \).

(ii) A Hom-coassociative coalgebra is a triple \((C, \Delta, \alpha)\), in which \( C \) is a linear space, \( \alpha : C \rightarrow C \) and \( \Delta : C \rightarrow C \otimes C \) are linear maps, satisfying the following conditions:

\[
(\alpha \otimes \alpha) \circ \Delta = \Delta \circ \alpha, \quad \text{(comultiplicativity)}
\]
\[
(\Delta \otimes \alpha) \circ \Delta = (\alpha \otimes \Delta) \circ \Delta. \quad \text{(Hom-coassociativity)}
\]

A morphism \( g : (C, \Delta_C, \alpha_C) \rightarrow (D, \Delta_D, \alpha_D) \) of Hom-coassociative coalgebras is a linear map \( g : C \rightarrow D \) such that \( \alpha_D \circ g = g \circ \alpha_C \) and \( (g \otimes g) \circ \Delta_C = \Delta_D \circ g \).

Remark 1.4 Assume that \((A, \mu_A, \alpha_A)\) and \((B, \mu_B, \alpha_B)\) are two Hom-associative algebras; then \((A \otimes B, \mu_{A \otimes B}, \alpha_A \otimes \alpha_B)\) is a Hom-associative algebra (called the tensor product of \( A \) and \( B \)), where \( \mu_{A \otimes B} \) is the usual multiplication: \((a \otimes b)'(a' \otimes b') = aa' \otimes bb'\).

Definition 1.5 (\( [22, 23] \))

(i) Let \( (A, \mu_A, \alpha_A) \) be a Hom-associative algebra, \( M \) a linear space and \( \alpha_M : M \rightarrow M \) a linear map. A left \( A \)-module structure on \((M, \alpha_M)\) consists of a linear map \( A \otimes M \rightarrow M \), \( a \otimes m \mapsto a \cdot m \), satisfying the conditions:

\[
\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m), \quad (1.2)
\]
\[
\alpha_A(a) \cdot (a' \cdot m) = (aa') \cdot \alpha_M(m), \quad (1.3)
\]

for all \( a, a' \in A \) and \( m \in M \). If \((M, \alpha_M)\) and \((N, \alpha_N)\) are left \( A \)-modules (both \( A \)-actions denoted by \( \cdot \)), a morphism of left \( A \)-modules \( f : M \rightarrow N \) is a linear map satisfying the conditions \( \alpha_N \circ f = f \circ \alpha_M \) and \( f(a \cdot m) = a \cdot f(m) \), for all \( a \in A \) and \( m \in M \).

(ii) Let \( (C, \Delta_C, \alpha_C) \) be a Hom-coassociative coalgebra, \( M \) a linear space and \( \alpha_M : M \rightarrow M \) a linear map. A left \( C \)-comodule structure on \((M, \alpha_M)\) consists of a linear map \( \lambda : M \rightarrow C \otimes M \) (usually denoted by \( \lambda(m) = m_{(-1)} \otimes m_{(0)} \)) satisfying the following conditions:

\[
(\alpha_C \otimes \alpha_M) \circ \lambda = \lambda \circ \alpha_M, \quad (1.4)
\]
\[
(\Delta_C \otimes \alpha_M) \circ \lambda = (\alpha_C \otimes \lambda) \circ \lambda. \quad (1.5)
\]

If \((M, \alpha_M)\) and \((N, \alpha_N)\) are left \( C \)-comodules, with structures \( \lambda_M : M \rightarrow C \otimes M \) and \( \lambda_N : N \rightarrow C \otimes N \), a morphism of left \( C \)-comodules \( g : M \rightarrow N \) is a linear map satisfying the conditions \( \alpha_N \circ g = g \circ \alpha_M \) and \((id_C \otimes g) \circ \lambda_M = \lambda_N \circ g \).

Definition 1.6 (\( [22, 30] \))

A Hom-bialgebra is a quadruple \((H, \mu, \Delta, \alpha)\), in which \((H, \mu, \alpha)\) is a Hom-associative algebra, \((H, \Delta, \alpha)\) is a Hom-coassociative coalgebra and moreover \( \Delta \) is a morphism of Hom-associative algebras.

In other words, a Hom-bialgebra is a Hom-associative algebra \((H, \mu, \alpha)\) endowed with a linear map \( \Delta : H \rightarrow H \otimes H \), with notation \( \Delta(h) = h_1 \otimes h_2 \), such that the following conditions are satisfied, for all \( h, h' \in H \):

\[
\Delta(h_1) \otimes \alpha(h_2) = \alpha(h_1) \otimes \Delta(h_2), \quad (1.6)
\]
\[
\Delta(hh') = h_1 h_1' \otimes h_2 h_2', \quad (1.7)
\]
\[
\Delta(\alpha(h)) = \alpha(h_1) \otimes \alpha(h_2). \quad (1.8)
\]

The following result provides a way to construct examples of Hom-associative algebras, Hom-coassociative coalgebras or Hom-bialgebras. It is called the ”twisting principle” or sometimes a composition method.
Proposition 1.7 (30, 32) (i) Let \((A, \mu)\) be an associative algebra and \(\alpha : A \to A\) an algebra endomorphism. Define a new multiplication \(\mu_\alpha := \alpha \circ \mu : A \otimes A \to A\). Then \((A, \mu_\alpha, \alpha)\) is a Hom-associative algebra, denoted by \(A_\alpha\).

(ii) Let \((C, \Delta)\) be a coassociative coalgebra and \(\alpha : C \to C\) a coalgebra endomorphism. Define a new comultiplication \(\Delta_\alpha := \Delta \circ \alpha : C \to C \otimes C\). Then \((C, \Delta_\alpha, \alpha)\) is a Hom-coassociative coalgebra, denoted by \(C_\alpha\).

(iii) Let \((H, \mu, \Delta)\) be a bialgebra and \(\alpha : H \to H\) a bialgebra endomorphism. If we define \(\mu_\alpha\) and \(\Delta_\alpha\) as in (i) and (ii), then \(H_\alpha = (H, \mu_\alpha, \Delta_\alpha, \alpha)\) is a Hom-bialgebra.

Proposition 1.8 (33) Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra.

(i) If \((M, \alpha_M)\) and \((N, \alpha_N)\) are left \(H\)-modules, then \((M \otimes N, \alpha_M \otimes \alpha_N)\) is also a left \(H\)-module, with \(H\)-action defined by \(H \otimes (M \otimes N) \to M \otimes N, h \otimes (m \otimes n) \mapsto h \cdot (m \otimes n) := h_1 \cdot m \otimes h_2 \cdot n\).

(ii) If \((M, \alpha_M)\) and \((N, \alpha_N)\) are left \(H\)-comodules, with coactions denoted by \(M \to H \otimes M, m \mapsto m_{(-1)} \otimes m_{(0)}\) and \(N \to H \otimes N, n \mapsto n_{(-1)} \otimes n_{(0)}\), then \((M \otimes N, \alpha_M \otimes \alpha_N)\) is also a left \(H\)-comodule, with \(H\)-coaction \(M \otimes N \to H \otimes (M \otimes N), m \otimes n \mapsto m_{(-1)} n_{(-1)} \otimes (m_{(0)} \otimes n_{(0)})\).

Definition 1.9 (33) (i) Let \((A, \mu_A)\) be an associative algebra, \(\alpha_A : A \to A\) an algebra endomorphism, \(M\) a left \(A\)-module with action \(A \otimes M \to M, a \otimes m \mapsto a \cdot m\), and \(A \otimes M : M \to M\) a linear map satisfying the condition \(\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m)\), for all \(a \in A, m \in M\). Then \((M, \alpha_M)\) becomes a left module over the Hom-associative algebra \(A_{\alpha_A}\), with action \(A_{\alpha_A} \otimes M \to M, a \otimes m \mapsto a \cdot m := \alpha_A(a) \cdot \alpha_M(m)\).

(ii) Let \((C, \Delta_C)\) be a coassociative coalgebra, \(\alpha_C : C \to C\) a coalgebra endomorphism, \(M\) a left \(C\)-comodule with structure \(M \to C \otimes M, m \mapsto m_{(-1)} \otimes m_{(0)}\), and \(A \otimes M : M \to M\) a linear map satisfying the condition \(\alpha_M(m_{(-1)}) \otimes \alpha_M(m_{(0)}) = \alpha_C(m_{(-1)}) \otimes \alpha_M(m_{(0)})\), for all \(m \in M\). Then \((M, \alpha_M)\) becomes a left comodule over the Hom-coassociative coalgebra \(C_{\alpha_C}\), with coaction \(M \to C_{\alpha_C} \otimes M, m \mapsto m_{<1>} \otimes m_{<0>} := \alpha_M(m_{(-1)} \otimes \alpha_M(m_{(0)}) = \alpha_C(m_{(-1)}) \otimes \alpha_M(m_{(0)})\).

Definition 1.10 (36, 37) Let \((H, \mu, \Delta, \alpha)\) be a Hom-bialgebra and \(R \in H \otimes H\) an element, with Sweedler-type notation \(R = R^1 \otimes R^2 = r^1 \otimes r^2\). We call \((H, \mu, \Delta, \alpha, R)\) a quasitriangular Hom-bialgebra if the following axioms are satisfied:

\[
(\Delta \otimes \alpha)(R) = \alpha(R^1) \otimes \alpha(r^1) \otimes R^2 r^2, \tag{1.9}
\]

\[
(\alpha \otimes \Delta)(R) = R^1 r^1 \otimes \alpha(r^2) \otimes \alpha(R^2), \tag{1.10}
\]

\[
\Delta^{\text{cop}}(h)R = R\Delta(h), \tag{1.11}
\]

for all \(h \in H\), where we denoted as usual \(\Delta^{\text{cop}}(h) = h_2 \otimes h_1\).

Definition 1.11 (37) Let \((H, \mu, \Delta, \alpha)\) be a Hom-bialgebra and \(\sigma : H \otimes H \to k\) a linear map. We call \((H, \mu, \Delta, \alpha, \sigma)\) a coquasitriangular Hom-bialgebra if, for all \(x, y, z \in H\), we have:

\[
\sigma(xy \otimes \alpha(z)) = \sigma(x) \otimes \sigma(y) \otimes \sigma(z), \tag{1.12}
\]

\[
\sigma(x \otimes yz) = \sigma(x_1 \otimes \alpha(z)) \sigma(x_2 \otimes \alpha(y)), \tag{1.13}
\]

\[
y_1 x_1 \sigma(x_2 \otimes y_2) = \sigma(x_1 \otimes y_1) x_2 y_2. \tag{1.14}
\]

2 Yetter-Drinfeld modules

We introduce in this section the concept of Yetter-Drinfeld module over a Hom-bialgebra. We study the category of Yetter-Drinfeld modules for which the structure map is bijective and such that the Hom-bialgebra structure map is bijective as well.
Definition 2.1 Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra, \(M\) a linear space and \(\alpha_M : M \to M\) a linear map such that \((M, \alpha_M)\) is a left \(H\)-module with action \(H \otimes M \to M, \ h \otimes m \mapsto h \cdot m\) and a left \(H\)-comodule with coaction \(M \to H \otimes M, \ m \mapsto m_{(-1)} \otimes m_{(0)}\). Then \((M, \alpha_M)\) is called a (left-left) Yetter-Drinfeld module over \(H\) if the following identity holds, for all \(h \in H, \ m \in M\):

\[
(h \cdot m)_{(-1)} \alpha_H^2(h_2) \otimes (h_1 \cdot m)_{(0)} = \alpha_H^2(h_1) \alpha_H(m_{(-1)}) \otimes \alpha_H(h_2) \cdot m_{(0)}.
\]  

Definition 2.2 Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra such that \(\alpha_H\) is bijective. We denote by \(H^YD\) the category whose objects are Yetter-Drinfeld modules \((M, \alpha_M)\) over \(H\), with \(\alpha_M\) bijective; the morphisms in the category are morphisms of left \(H\)-modules and left \(H\)-comodules.

The choice of the compatibility condition \((2.1)\) is motivated by the following result:

Proposition 2.3 Let \((H, \mu_H, \Delta_H)\) be a bialgebra, \(\alpha_H : H \to H\) a bialgebra endomorphism, \(M\) a Yetter-Drinfeld module over \(H\) with notation as in Definition 1.2, \(\alpha_M : M \to M\) a linear map satisfying the conditions in Proposition 1.3 (both in (i) and (ii)), so we can consider \((M, \alpha_M)\) both as a left \(H\alpha_H\)-module with action \(\triangleright\) and as a left \(\alpha_H\)-comodule with coaction \(m \mapsto m_{<1>} \otimes m_{<0>}\), as in Proposition 1.9. Then \((M, \alpha_M)\) with these structures is a Yetter-Drinfeld module over the Hom-bialgebra \(H\alpha_H\).

Proof. We only need to check the Yetter-Drinfeld compatibility condition \((2.1)\), which in this case reads:

\[
(h_{(1)} \triangleright m)_{<1>} \ast \alpha_H^2(h_{(2)}) \otimes (h_{(1)} \triangleright m)_{<0>} = \alpha_H^2(h_{(1)}) \ast \alpha_H(m_{<1>}) \otimes \alpha_H(h_{(2)}) \triangleright m_{<0>},
\]

where we denoted by \(h \ast h' = \alpha_H(hh')\) and \(h_{(1)} \otimes h_{(2)} = \alpha_H(h_1) \otimes \alpha_H(h_2)\) the multiplication and comultiplication of \(H\alpha_H\). Now we compute:

\[
(h_{(1)} \triangleright m)_{<1>} \ast \alpha_H^2(h_{(2)}) \otimes (h_{(1)} \triangleright m)_{<0>}
\]

\[
= \alpha_H((\alpha_H(h_1) \triangleright m)_{<1>}) \alpha_H^2(h_{(2)}) \otimes (\alpha_H(h_1) \triangleright m)_{<0>}
\]

\[
= \alpha_H(\alpha_H((\alpha_H(h_1) \triangleright m)_{<1>})) \alpha_H^2(h_{(2)}) \otimes \alpha_H(m_{<1>}) \alpha_H(h_{(2)}) \triangleright m_{<0>}
\]

\[
= \alpha_H(\alpha_H^2(h_{(1)}) \alpha_H(m_{<1>})) \alpha_H(h_{(2)}) \triangleright m_{<0>}
\]

finishing the proof.

Proposition 2.4 Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra with \(\alpha_H\) bijective, \((M, \alpha_M)\), \((N, \alpha_N)\) two Yetter-Drinfeld modules over \(H\), with notation as above, and define the linear map

\[
B_{M,N} : M \otimes N \to N \otimes M, \quad B_{M,N}(m \otimes n) = \alpha_H^{-1}(m_{(-1)}) \cdot n \otimes m_{(0)}.
\]

Then, we have \((\alpha_N \circ \alpha_M) \circ B_{M,N} = B_{M,N} \circ (\alpha_M \circ \alpha_N)\) and, if \((P, \alpha_P)\) is another Yetter-Drinfeld module over \(H\), the maps \(B_{\_\_\_}\) satisfy the Hom-Yang-Baxter equation (HYBE):

\[
(\alpha_P \otimes B_{M,N}) \circ (B_{M,P} \otimes \alpha_N) \circ (B_{N,M} \otimes \alpha_P) = (B_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes B_{M,P}) \circ (B_{M,N} \otimes \alpha_P).
\]
Proof. The condition \((\alpha_N \otimes \alpha_M) \circ B_{M,N} = B_{M,N} \circ (\alpha_M \otimes \alpha_N)\) is very easy to prove and is left to the reader. Now we compute:

\[
\begin{align*}
& (\alpha_P \otimes B_{M,N}) \circ (B_{M,P} \otimes \alpha_N) \circ (\alpha_M \otimes B_{N,P}) (m \otimes n \otimes p) \\
& = ((\alpha_P \otimes B_{M,N}) \circ (B_{M,P} \otimes \alpha_N))(\alpha_M(m) \otimes \alpha_H^{-1}(n_{(-1)}) \cdot p \otimes n(0)) \\
& = (\alpha_P \otimes B_{M,N})(\alpha_H^{-1}(\alpha_M(m)(-1)) \cdot (\alpha_H^{-1}(n_{(-1)}) \cdot p) \otimes \alpha_M(m)(0) \otimes \alpha_N(n)(0)) \\
& \overset{1.4}{=} (\alpha_P \otimes B_{M,N})(m_{(-1)} \cdot (\alpha_H^{-1}(n_{(-1)}) \cdot p) \otimes \alpha_M(m)(0) \otimes \alpha_N(n)(0)) \\
& \overset{1.3}{=} (\alpha_P \otimes B_{M,N})(\alpha_H^{-1}(m_{(-1)}n_{(-1)}) \cdot \alpha_P(p) \otimes \alpha_M(m)(0) \otimes \alpha_N(n)(0)) \\
& \overset{1.2}{=} (m_{(-1)}n_{(-1)}) \cdot \alpha_P^2(p) \otimes \alpha_H^{-1}(\alpha_M(m)(0))^{-1} \cdot \alpha_N(n)(0) \otimes \alpha_M(m)(0) \\
& \overset{1.4}{=} (m_{(-1)}n_{(-1)}) \cdot \alpha_P^2(p) \otimes m_{(0)(-1)} \cdot \alpha_N(n)(0) \otimes \alpha_M(m)(0),
\end{align*}
\]

\[
\begin{align*}
& (B_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes B_{M,P}) \circ (B_{M,N} \otimes \alpha_P) (m \otimes n \otimes p) \\
& = ((B_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes B_{M,P}))(\alpha_H^{-1}(m_{(-1)}) \cdot n \otimes m(0) \otimes \alpha_P(p)) \\
& = (B_{N,P} \otimes \alpha_M)(\alpha_N(\alpha_H^{-1}(m_{(-1)}) \cdot n) \otimes \alpha_H^{-1}(m_{(0)(-1)}) \cdot \alpha_P(p) \otimes m(0)) \\
& \overset{1.2}{=} (B_{N,P} \otimes \alpha_M)(m_{(-1)} \cdot \alpha_N(n) \otimes \alpha_H^{-1}(m_{(0)(-1)}) \cdot \alpha_P(p)) \\
& \overset{1.3}{=} [\alpha_H^{-2}((m_{(-1)} \cdot \alpha_N(n)(-1)) \alpha_H^{-1}(m_{(0)(-1)})] \cdot \alpha_P^2(p) \\
& \overset{1.5}{=} \alpha_H^{-2}(\alpha_H^{-1}(m_{(-1)}(1) \cdot \alpha_N(n)(-1)) \alpha_H(m_{(-1)}(2)) \cdot \alpha_P^2(p)) \\
& \otimes \alpha_H^{-1}(m_{(-1)} \cdot \alpha_N(n)(0) \otimes \alpha_M(m)(0)) \\
& \overset{1.8}{=} \alpha_H^{-2}(\alpha_H^{-1}(m_{(-1)}(1) \cdot \alpha_N(n)(-1)) \alpha_H^{-2}(\alpha_H^{-1}(m_{(-1)}(2)) \cdot \alpha_P^2(p)) \\
& \otimes \alpha_H^{-1}(m_{(-1)}(1) \cdot \alpha_N(n)(0) \otimes \alpha_M(m)(0)) \\
& \overset{2.1}{=} \alpha_H^{-2}(\alpha_H^{-2}(\alpha_H^{-1}(m_{(-1)}(1) \cdot \alpha_N(n)(-1)) \cdot \alpha_H(m_{(-1)}(2)) \cdot \alpha_P^2(p)) \\
& \otimes \alpha_H^{-1}(m_{(-1)}(1) \alpha_N(n)(0) \otimes \alpha_M(m)(0)) \\
& \overset{1.8}{=} (\alpha_H^{-1}(m_{(-1)}(1) \cdot \alpha_N(n)(-1))) \cdot \alpha_P^2(p) \otimes m(1) \cdot \alpha_N(n)(0) \otimes \alpha_M(m)(0)) \\
& \overset{1.4}{=} (m_{(-1)}n_{(-1)}) \cdot \alpha_P^2(p) \otimes m_{(0)(-1)} \cdot \alpha_N(n)(0) \otimes \alpha_M(m)(0),
\end{align*}
\]

and the two terms are obviously equal. \(\square\)

Let now \((H, \mu_H, \Delta_H)\) be a bialgebra, \(M\) a Yetter-Drinfeld module over \(H\) with notation as in Definition 1.12 and \(\alpha_M : M \to M\) a morphism of left \(H\)-modules and left \(H\)-comodules. If we consider the map \(\alpha_H := \text{id}_H\), then one can easily see that the hypotheses of Proposition 2.3 are satisfied. So, \((M, \alpha_M)\) is a Yetter-Drinfeld module over the Hom-bialgebra \(H_{\text{id}_H}\) (which is actually the bialgebra \(H\)). For this Yetter-Drinfeld module \((M, \alpha_M)\) we apply Proposition 2.4 it follows that the linear map \(B : M \otimes M \to M \otimes M, B(m \otimes m') = m_{(-1)} \cdot m' \otimes m(0)\), for \(m, m' \in M\),
satisfies the HYBE \((\alpha_M \otimes B) \circ (B \otimes \alpha_M) \circ (\alpha_M \otimes B) = (B \otimes \alpha_M) \circ (\alpha_M \otimes B) \circ (B \otimes \alpha_M)\). This is exactly the content of Theorem 4.1 in [40], which may thus be seen as a particular case of Proposition 2.4.

3 The quasi-braided pre-tensor category \((H \mathcal{Y}D, \hat{\otimes}, a, c)\)

We show, in this section, that over a Hom-bialgebra with bijective structure map, the category of Yetter-Drinfeld modules with bijective structure maps is a quasi-braided pre-tensor category. It comes with solutions to the braid relation and the HYBE.

**Proposition 3.1** Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra with \(\alpha_H\) bijective, \((M, \alpha_M), (N, \alpha_N)\) two Yetter-Drinfeld modules over \(H\), with notation as above, and define the linear maps

\[ H \otimes (M \otimes N) \to M \otimes N, \quad h \otimes (m \otimes n) \mapsto h_1 \cdot m \otimes h_2 \cdot n, \]

\[ M \otimes N \to H \otimes (M \otimes N), \quad m \otimes n \mapsto \alpha_H^{-2}(m_{(-1)}n_{(-1)}) \otimes (m_{(0)} \otimes n_{(0)}). \]

Then \((M \otimes N, \alpha_M \otimes \alpha_N)\) with these structures becomes a Yetter-Drinfeld module over \(H\), denoted in what follows by \(M \hat{\otimes} N\).

**Proof.** We know from Proposition 2.4 that \(M \hat{\otimes} N\) is a left \(H\)-module. A similar and straightforward computation shows that \(M \hat{\otimes} N\) is also a left \(H\)-comodule. So we only have to prove the Yetter-Drinfeld compatibility condition (2.1). We compute:

\[
(h_1 \cdot (m \otimes n))_{(-1)} \alpha_H^2(h_2) \otimes (h_1 \cdot (m \otimes n))_{(0)}
\]

\[= ((h_1)_1 \cdot m \otimes (h_1)_2 \cdot n)_{(-1)} \alpha_H^2(h_2) \otimes ((h_1)_1 \cdot m \otimes (h_1)_2 \cdot n)_{(0)}\]

\[= \alpha_H^{-2}((\alpha_H(h_1) \cdot m)_{(-1)}((h_2)_1 \cdot n)_{(-1)} \alpha_H(h_2)_2 \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes ((h_2)_1 \cdot n)_{(0)})\]

\[= \alpha_H^{-2}((\alpha_H(h_1) \cdot m)_{(-1)}((h_2)_1 \cdot n)_{(-1)} \alpha_H^2((h_2)_2) \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes ((h_2)_1 \cdot n)_{(0)})\]

\[= \alpha_H^{-2}(\alpha_H((\alpha_H(h_1) \cdot m)_{(-1)}})((h_2)_1 \cdot n)_{(-1)} \alpha_H^2((h_2)_2) \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes ((h_2)_1 \cdot n)_{(0)}\]

\[= \alpha_H^{-2}(\alpha_H((\alpha_H(h_1) \cdot m)_{(-1)})(\alpha_H(h_2))_{(2)} \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes (\alpha_H(h_2))_{(2)} \cdot n_{(0)})\]

\[= \alpha_H^{-2}(\alpha_H(h_1)(\alpha_H(h_2))_{(2)} \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes (\alpha_H(h_2))_{(2)} \cdot n_{(0)})\]

\[= \alpha_H^{-2}((\alpha_H(h_1))_1 \cdot m \otimes (\alpha_H(h_2))_2 \otimes n_{(-1)}) \otimes ((h_1)_1 \cdot m)_{(0)} \otimes (\alpha_H(h_2))_{(2)} \cdot n_{(0)})\]

\[= \alpha_H^{-2}((\alpha_H(h_1))_1 \cdot m \otimes (\alpha_H(h_2))_2 \otimes n_{(-1)}) \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes (\alpha_H(h_2))_{(2)} \cdot n_{(0)})\]

\[= \alpha_H^{-2}((\alpha_H(h_1) \cdot m)_{(-1)} \alpha_H^2((h_2)_2) \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes (\alpha_H(h_2))_{(2)} \cdot n_{(0)})\]

\[= \alpha_H^{-2}((\alpha_H(h_1) \cdot m)_{(-1)} \alpha_H^2((h_2)_2) \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes (\alpha_H(h_2))_{(2)} \cdot n_{(0)})\]

\[= \alpha_H^{-2}((\alpha_H(h_1) \cdot m)_{(-1)} \alpha_H^2((h_2)_2) \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes (\alpha_H(h_2))_{(2)} \cdot n_{(0)})\]

\[= \alpha_H^{-2}((\alpha_H(h_1) \cdot m)_{(-1)} \alpha_H^2((h_2)_2) \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes (\alpha_H(h_2))_{(2)} \cdot n_{(0)})\]

finishing the proof. \(\square\)

**Proposition 3.2** Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra such that \(\alpha_H\) is bijective and assume that \((M, \alpha_M), (N, \alpha_N), (P, \alpha_P)\) are three Yetter-Drinfeld modules over \(H\), with notation as above, such that \(\alpha_M, \alpha_N, \alpha_P\) are bijective; define the linear map

\[a_{M,N,P} : (M \hat{\otimes} N) \hat{\otimes} P \to M \hat{\otimes} (N \hat{\otimes} P), \quad a_{M,N,P}(m \otimes n \otimes p) = \alpha_M^{-1}(m) \otimes (n \otimes \alpha_P(p)).\]

Then \(a_{M,N,P}\) is an isomorphism of left \(H\)-modules and left \(H\)-comodules.
Proof. It is obvious that $a_{M,N,P}$ is bijective and satisfies the relation $(\alpha_M \otimes \alpha_N \otimes \alpha_P) \circ a_{M,N,P} = a_{M,N,P} \circ (\alpha_M \otimes \alpha_N \otimes \alpha_P)$. The $H$-linearity of $a_{M,N,P}$ follows from the computation performed in [6], proof of Proposition 2.6, but we include a proof here for reader’s convenience:

$$a_{M,N,P}(h \cdot ((m \otimes n) \otimes p)) = a_{M,N,P}((h_1) \cdot m \otimes (h_2) \cdot n \otimes h_2 \cdot p) = \alpha_M^{-1}((h_1) \cdot m) \otimes ((h_2) \cdot n \otimes \alpha_P(h_2 \cdot p))$$

1.2

$$= \alpha_H^{-1}((h_1)_1 \cdot m) \otimes ((h_2)_2 \cdot n \otimes \alpha_H(h_2) \cdot \alpha_P(p)) 
= h_1 \cdot \alpha_M^{-1}(m) \otimes ((h_2)_1 \cdot n \otimes (h_2)_2 \cdot \alpha_P(p))
= h_1 \cdot \alpha_M^{-1}(m) \otimes h_2 \cdot (n \otimes \alpha_P(p))
= h \cdot a_{M,N,P}((m \otimes n) \otimes p), \text{ q.e.d.}$$

Now we prove the $H$-colinearity of $a_{M,N,P}$ (denoting by $\lambda_X$ the left $H$-comodule structure of a Yetter-Drinfeld module $X$):

$$(id_H \otimes a_{M,N,P}) \circ \lambda_{(M \otimes N) \otimes P}((m \otimes n) \otimes p)$$

$$= (id_H \otimes a_{M,N,P})(\alpha_H^{-2}((m \otimes n) \otimes ((-1)\lambda(1) \otimes n) \otimes p) \otimes (m \otimes n) \otimes (0) \otimes (0))$$

$$= \alpha_H^{-2}(\alpha_H^{-2}(m(-1) \otimes n(-1)) \otimes a_{M,N,P}((m_0 \otimes n_0) \otimes p) \otimes (m_0 \otimes n_0 \otimes \alpha_P(p)(0)))$$

$$(\lambda_{(M \otimes N) \otimes P} \circ a_{M,N,P})(((m \otimes n) \otimes p)$$

$$= \lambda_{(M \otimes N) \otimes P}(\alpha_M^{-1}(m) \otimes (n \otimes \alpha_P(p)))$$

$$= \alpha_H^{-2}(\alpha_H^{-2}(m(-1) \otimes n(-1)) \otimes \alpha_M^{-1}(m(0)) \otimes (n \otimes \alpha_P(p))(0))$$

$$= \alpha_H^{-2}(\alpha_H^{-2}(m(-1) \otimes n(-1)) \otimes \alpha_M^{-1}(m(0)) \otimes (n \otimes \alpha_P(p)(0)))$$

1.4

$$= \alpha_H^{-4}(\alpha_H(m(-1)) \alpha_H(p(-1))) \otimes \alpha_M^{-1}(m(0)) \otimes (n \otimes \alpha_P(p)(0))$$

$$(\lambda_{(M \otimes N) \otimes P} \circ a_{M,N,P})(((m \otimes n) \otimes p))$$

and the two terms are obviously equal.

\[ \square \]

Proposition 3.3 Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra such that $\alpha_H$ is bijective, let $(M, \alpha_M)$ and $(N, \alpha_N)$ be two Yetter-Drinfeld modules over $H$, with notation as above, such that $\alpha_M$ and $\alpha_N$ are bijective, and define the linear map

$$c_{M,N} : M \otimes N \rightarrow N \otimes M, \quad c_{M,N}(m \otimes n) = \alpha_N^{-1}(\alpha_H^{-1}(m(-1)) \cdot n) \otimes \alpha_M^{-1}(m(0)). \quad (3.1)$$

Then $c_{M,N}$ is a morphism of left $H$-modules and left $H$-comodules.

Proof. The relation $(\alpha_N \otimes \alpha_M) \circ c_{M,N} = c_{M,N} \circ (\alpha_M \otimes \alpha_N)$ follows by an easy computation using (1.4) and (1.2). We prove now the $H$-linearity of $c_{M,N}$:

$$c_{M,N}(h \cdot (m \otimes n)) = c_{M,N}(h_1 \cdot m \otimes h_2 \cdot n)$$

$$= \alpha_N^{-1}(\alpha_H^{-1}((h_1 \cdot m)(-1)) \cdot (h_2 \cdot n)) \otimes \alpha_M^{-1}((h_1 \cdot m)(0))$$

1.6
\[
\begin{align*}
\alpha_N^{-1} & (\alpha_H^{-2}((h_1 \cdot m)(-1))h_2) \cdot \alpha_N(n) \otimes \alpha_M^{-1}((h_1 \cdot m)(0)) \\
= & \alpha_N^{-1}(\alpha_H^{-2}((h_1 \cdot m)(-1))\alpha_H^n(h_2)) \cdot \alpha_N(n) \otimes \alpha_M^{-1}((h_1 \cdot m)(0)) \\
\end{align*}
\]

\[
\begin{align*}
\alpha_N^{-1} & (\alpha_H^{-2}(\alpha_H^n(h_1)\alpha_H(m(-1)))) \cdot \alpha_N(n) \otimes \alpha_M^{-1}(\alpha_H(h_2) \cdot m(0)) \\
\end{align*}
\]

\[
\begin{align*}
\alpha_N^{-1} & (\alpha_H(h_1) \cdot (\alpha_H^{-1}(m(-1)) \cdot n)) \otimes \alpha_M^{-1}(\alpha_H(h_2) \cdot m(0)) \\
= & h \cdot c_{M,N}(m \otimes n), \; q.e.d.
\end{align*}
\]

Now we prove the \( H \)-colinearity of \( c_{M,N} \) (we denote by \( \lambda_{M\otimes N} \) and \( \lambda_{N\otimes M} \) the left \( H \)-comodule structures of \( M \hat{\otimes} N \) and respectively \( N \hat{\otimes} M \)):

\[
(\lambda_{N\otimes M} \circ c_{M,N})(m \otimes n)
\]

\[
= \lambda_{N\otimes M}(\alpha_N^{-1}(\alpha_H^{-1}(m(-1)) \cdot n) \otimes \alpha_M^{-1}(m(0)))
\]

\[
= \alpha_H^{-2}(\alpha_N^{-1}(\alpha_H^{-1}(m(-1)) \cdot n) \otimes \alpha_M^{-1}(m(0))) \otimes \alpha_N^{-1}(\alpha_H^{-1}(m(-1)) \cdot n) \otimes \alpha_M^{-1}(m(0))
\]

\[
= \alpha_N^{-1}(\alpha_H^{-2}(m(-1)) \cdot n) \otimes \alpha_H^{-1}(m(-1)) \cdot n) \otimes \alpha_M^{-1}(m(0))
\]

\[
= \alpha_N^{-1}(\alpha_H^{-2}(m(-1)) \cdot n) \otimes \alpha_H^{-1}(m(-1)) \cdot n) \otimes \alpha_M^{-1}(m(0))
\]

\[
= \alpha_H^{-2}(\alpha_N^{-1}(\alpha_H^{-1}(m(-1)) \cdot n) \otimes \alpha_M^{-1}(m(0))
\]

\[
= \alpha_H^{-2}(\alpha_N^{-1}(\alpha_H^{-1}(m(-1)) \cdot n) \otimes \alpha_M^{-1}(m(0))
\]

\[
= (id_H \otimes c_{M,N})(\alpha_H^{-2}(m(-1)) \cdot n) \otimes (m(0) \otimes n(0))
\]

\[
= (id_H \otimes c_{M,N})(\alpha_H^{-2}(m(-1)) \cdot n) \otimes (m(0) \otimes n(0))
\]

\[
= (id_H \otimes c_{M,N}) \circ \lambda_{M\hat{\otimes}N}(m \otimes n),
\]
finishing the proof.

\[\Box\]

**Theorem 3.4** Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra such that \( \alpha_H \) is bijective. Then \( H \hat{\otimes} Y \hat{\otimes} D \) is a quasi-braided pre-tensor category, with tensor product \( \hat{\otimes} \), associativity constraints \( a_{M,N,P} \) and quasi-braiding \( c_{M,N} \) defined in Propositions 3.1, 3.2 and 3.3 respectively.

**Proof.** The only nontrivial things left to prove are the pentagon axiom for \( a_{M,N,P} \) and the two hexagonal relations for \( c_{M,N} \). The pentagon axiom for \( a_{M,N,P} \) follows by a straightforward computation that shows the equality

\[
((id_M \otimes a_{N,P,Q}) \circ (a_{M,N,P} \otimes id_Q))((m \otimes n) \otimes p) \otimes q)
\]

\[
= (a_{M,N,P} \otimes a_{M,N,P,Q})((m \otimes n) \otimes p) \otimes q)
\]

\[
= \alpha_M^{-2}(m) \otimes \alpha_N^{-1}(n) \otimes a_P(p) \otimes \alpha_Q^{-2}(q),
\]

\]

10
for any objects \((M, \alpha_M), (N, \alpha_N), (P, \alpha_P), (Q, \alpha_Q) \in \mathcal{H}_D\).

We prove the first hexagonal relation for \(c_{M,N}\). Let \((M, \alpha_M), (N, \alpha_N), (P, \alpha_P) \in \mathcal{H}_D\); we compute:

\[
(a_{N,P,M} \circ c_{M,N} \circ P \circ a_{M,N,P})(m \otimes n \otimes p)
= (a_{N,P,M} \circ c_{M,N} \circ P \circ a_{M,N,P})(\alpha_M^{-1}(m) \otimes (n \otimes \alpha_P(p)))
= a_{N,P,M}((\alpha_N^{-1} \otimes \alpha_P^{-1})(\alpha_H^{-1}(\alpha_M^{-1}(m)) \cdot (n \otimes \alpha_P(p))) \otimes \alpha_M^{-1}(\alpha_M^{-1}(m)))
\]

\[\text{(1.4)}\]

\[
a_{N,P,M}((\alpha_N^{-1}(\alpha_H^{-2}(m) \cdot n) \otimes \alpha_P^{-1}(\alpha_H^{-2}(m) \cdot \alpha_P(p))) \otimes \alpha_M^{-1}(m))
= \alpha_N^{-2}(\alpha_H^{-2}(m) \cdot n) \otimes \alpha_P^{-1}(\alpha_H^{-2}(m) \cdot \alpha_P(p)) \otimes \alpha_M^{-1}(m),
\]

and the two terms are obviously equal.

Now we prove the second hexagonal relation for \(c_{M,N}^{-1}\):

\[
(a_{P,M,N}^{-1} \circ c_{M,N} \circ P \circ a_{M,N,P}^{-1})(m \otimes (n \otimes p))
= (a_{P,M,N}^{-1} \circ c_{M,N} \circ P \circ a_{M,N,P}^{-1})(\alpha_P^{-1}(\alpha_M^{-1}(m) \otimes n) \otimes \alpha_P^{-1}(p))
\]

\[\text{(1.4)}\]

\[
a_{P,M,N}^{-1}(\alpha_P^{-1}(\alpha_M^{-1}(m) \otimes n) \otimes \alpha_P^{-1}(p)) \otimes (\alpha_M^{-1} \otimes \alpha_N^{-1})(\alpha_M^{-1}(m) \otimes n))
= a_{P,M,N}^{-1}(\alpha_P^{-1}(\alpha_M^{-1}(m) \otimes n) \otimes \alpha_P^{-1}(p)) \otimes (\alpha_M^{-1} \otimes \alpha_N^{-1}(n))
\]

\[\text{(1.4)}\]

\[
a_{P,M,N}^{-1}(\alpha_P^{-1}(\alpha_M^{-1}(m) \otimes n) \otimes \alpha_P^{-1}(p)) \otimes \alpha_N^{-1}(n)
= \alpha_P^{-3}(\alpha_H^{-1}(m) \otimes n) \cdot \alpha_P^{-1}(p) \otimes \alpha_N^{-1}(n),
\]

finishing the proof.
Theorem 3.5 Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra such that \(\alpha_H\) is bijective and \((M, \alpha_M), (N, \alpha_N), (P, \alpha_P)\) three objects in \(\mathcal{H}\). Then the quasi-braiding \(c\) satisfies the braid relation

\[
(id_P \otimes c_{M,N}) \circ (c_{M,P} \otimes id_N) \circ (id_M \otimes c_{N,P}) = (c_{N,P} \otimes id_M) \circ (id_N \otimes c_{M,P}) \circ (c_{M,N} \otimes id_P).
\] (3.2)

Proof. Since \(c\) is the quasi-braiding of the pre-tensor category \(\mathcal{H}\), whose associators are nontrivial, it follows that \(c\) satisfies the dodecahedral braid relation (see [20], p. 317)

\[
(id_P \otimes c_{M,N}) \circ a_{P,M,N} \circ (c_{M,P} \otimes id_N) \circ a_{M,N,P}^{-1} \circ (id_M \otimes c_{N,P}) \circ a_{M,N,P} \circ (id_N \otimes c_{P,M}) = a_{P,N,M} \circ (c_{N,P} \otimes id_M) \circ a_{N,P,M}^{-1} \circ (id_N \otimes c_{M,P}) \circ a_{N,P,M} \circ (id_M \otimes c_{N,P}).
\]

which, by using the formulae of the associators, becomes

\[
(id_P \otimes c_{M,N}) \circ (\alpha_P^{-1} \otimes id_M \otimes \alpha_N) \circ (c_{M,P} \otimes id_N) \circ (\alpha_M \otimes id_P \otimes \alpha_N^{-1}) \circ (id_M \otimes c_{N,P}) \circ (\alpha_M^{-1} \otimes id_N \otimes \alpha_P) = (\alpha_P^{-1} \otimes id_N \otimes \alpha_M) \circ (\alpha_N \otimes id_P \otimes \alpha_M^{-1}) \circ (id_N \otimes c_{M,P}) \circ (\alpha_N^{-1} \otimes id_M \otimes \alpha_P) \circ (c_{M,N} \otimes id_P).
\]

In the left hand side of this relation, \(\alpha_N\) and \(\alpha_N^{-1}\) cancel each other, as well as \(\alpha_M\) and \(\alpha_M^{-1}\). Similarly, in the right hand side, \(\alpha_M\) and \(\alpha_M^{-1}\) and also \(\alpha_N\) and \(\alpha_N^{-1}\) cancel each other. So, this relation becomes:

\[
(id_P \otimes c_{M,N}) \circ (\alpha_P^{-1} \otimes id_M \otimes id_N) \circ (c_{M,P} \otimes id_N) \circ (id_M \otimes c_{N,P}) \circ (id_M \otimes id_N \otimes \alpha_P) = (\alpha_P^{-1} \otimes id_N \otimes id_M) \circ (c_{N,P} \otimes id_M) \circ (id_N \otimes c_{M,P}) \circ (id_N \otimes id_M \otimes \alpha_P) \circ (c_{M,N} \otimes id_P),
\]

which may be written as

\[
(\alpha_P^{-1} \otimes id_N \otimes id_M) \circ (id_P \otimes c_{M,N}) \circ (c_{M,P} \otimes id_N) \circ (id_M \otimes c_{N,P}) \circ (id_M \otimes id_N \otimes \alpha_P) = (\alpha_P^{-1} \otimes id_N \otimes id_M) \circ (c_{N,P} \otimes id_M) \circ (id_N \otimes c_{M,P}) \circ (c_{M,N} \otimes id_P) \circ (id_M \otimes id_N \otimes \alpha_P),
\]

which is obviously equivalent to (3.2). \(\square\)

Note that, for objects \((M, \alpha_M), (N, \alpha_N) \in \mathcal{H}\), i.e. Yetter-Drinfeld modules with bijective \(\alpha_M\) and \(\alpha_N\), the maps \(B_{M,N}\) defined in (2.2) and the maps \(c_{M,N}\) defined in (3.1) are related by the formula \(B_{M,N} = (\alpha_N \otimes \alpha_M) \circ c_{M,N}\). Our next result shows that, in this case, the fact that the maps \(B_{M,N}\) satisfy the HYBE is a consequence of the fact that the maps \(c_{M,N}\) satisfy the braid relation. One may call the HYBE equation "Hom-braid relation", since it is a twisting of the braid relation.

Proposition 3.6 Let \(M, N, P\) be linear spaces and \(\alpha_M : M \rightarrow M\), \(\alpha_N : N \rightarrow N\), \(\alpha_P : P \rightarrow P\) linear maps satisfying the following conditions:

\[
(\alpha_N \otimes \alpha_M) \circ c_{M,N} = c_{M,N} \circ (\alpha_M \otimes \alpha_N),
\] (3.3)

\[
(\alpha_P \otimes \alpha_M) \circ c_{M,P} = c_{M,P} \circ (\alpha_M \otimes \alpha_P),
\] (3.4)

\[
(\alpha_P \otimes \alpha_N) \circ c_{N,P} = c_{N,P} \circ (\alpha_N \otimes \alpha_P),
\] (3.5)

\[
(id_P \otimes c_{M,N}) \circ (c_{M,P} \otimes id_N) \circ (id_M \otimes c_{N,P}) = (c_{N,P} \otimes id_M) \circ (id_N \otimes c_{M,P}) \circ (c_{M,N} \otimes id_P).
\] (3.6)
Define the maps $B_{M,N} := (\alpha_N \circ \alpha_M) \circ c_{M,N}$, $B_{M,P} := (\alpha_P \circ \alpha_M) \circ c_{M,P}$, $B_{N,P} := (\alpha_P \circ \alpha_N) \circ c_{N,P}$.

Then the following relations hold:

\[
\begin{align*}
(\alpha_N \circ \alpha_M) \circ B_{M,N} &= B_{M,N} \circ (\alpha_M \circ \alpha_N), \quad (3.7) \\
(\alpha_P \circ \alpha_M) \circ B_{M,P} &= B_{M,P} \circ (\alpha_M \circ \alpha_P), \quad (3.8) \\
(\alpha_P \circ \alpha_N) \circ B_{N,P} &= B_{N,P} \circ (\alpha_N \circ \alpha_P), \quad (3.9) \\
(\alpha_P \circ B_{M,N}) \circ (B_{M,P} \circ \alpha_N) &\circ (\alpha_M \circ B_{N,P}) \\
&= (B_{N,P} \circ \alpha_M) \circ (\alpha_N \circ B_{M,P}) \circ (B_{M,N} \circ \alpha_P). \quad (3.10)
\end{align*}
\]

**Proof.** The first three relations are obvious, because of (3.3)-(3.5). We prove (3.10): 

\[
\begin{align*}
(\alpha_P \circ B_{M,N}) \circ (B_{M,P} \circ \alpha_N) &\circ (\alpha_M \circ B_{N,P}) \\
&= (\alpha_P \circ \alpha_N \circ \alpha_M) \circ (id_P \circ c_{M,N}) \circ (\alpha_P \circ \alpha_M \circ \alpha_N) \circ (c_{M,P} \circ id_N) \\
&\circ (\alpha_M \circ \alpha_P \circ \alpha_N) \circ (id_M \circ c_{N,P}) \quad (3.3) = (3.5) \\
&\circ (\alpha_P \circ \alpha_N \circ \alpha_M) \circ (\alpha_P \circ \alpha_N \circ \alpha_M) \circ (id_P \circ c_{M,N}) \circ (c_{M,P} \circ id_N) \\
&\circ (\alpha_M \circ \alpha_P \circ \alpha_N) \circ (c_{M,N} \circ id_P) \quad (3.6) \\
&\circ (\alpha_P \circ \alpha_N \circ \alpha_M) \circ (c_{N,P} \circ id_M) \circ (\alpha_N \circ \alpha_P \circ \alpha_M) \circ (id_N \circ c_{M,P}) \\
&\circ (\alpha_N \circ \alpha_M \circ \alpha_P) \circ (c_{M,N} \circ id_P) \quad (3.3) = (3.5) \\
&= (B_{N,P} \circ \alpha_M) \circ (\alpha_N \circ B_{M,P}) \circ (B_{M,N} \circ \alpha_P),
\end{align*}
\]

finishing the proof. \(\square\)

A particular case of Proposition 3.6 is the following result given in 40:

**Corollary 3.7** Let $V$ be a linear space, $c : V \otimes V \rightarrow V \otimes V$ a linear map satisfying the braid relation $(id_V \otimes c) \circ (c \otimes id_V) \circ (id_V \otimes c) = (c \otimes id_V) \circ (id_V \otimes c) \circ (c \otimes id_V)$ and $\alpha : V \rightarrow V$ a linear map such that $(\alpha \otimes \alpha) \circ c = c \circ (\alpha \otimes \alpha)$. Then the linear map $B := (\alpha \otimes \alpha) \circ c : V \otimes V \rightarrow V \otimes V$ satisfies the relations $(\alpha \otimes \alpha) \circ B = B \circ (\alpha \otimes \alpha)$ and $(\alpha \otimes B) \circ (B \otimes \alpha) \circ (\alpha \otimes B) = (B \otimes \alpha) \circ (\alpha \otimes B) \circ (B \otimes \alpha)$.

We can make now the connection between Yetter-Drinfeld modules and modules over quasitriangular Hom-bialgebras.

**Proposition 3.8** Let $(H,\mu_H,\Delta_H,\alpha_H, R)$ be a quasitriangular Hom-bialgebra such that

\[
(\alpha_H \circ \alpha_H)(R) = R. \quad (3.11)
\]

(i) Let $(M,\alpha_M)$ be a left $H$-module with action $H \otimes M \rightarrow M$, $h \otimes m \mapsto h \cdot m$. Define the linear map $\lambda_M : M \rightarrow H \otimes M$, $\lambda_M(m) = m_{(-1)} \otimes m_{(0)} := \alpha_H(\mathbb{R}_1^2) \otimes R^1 \cdot m$. Then $(M,\alpha_M)$ with these structures is a Yetter-Drinfeld module over $H$.

(ii) Assume that $\alpha_H$ is bijective. Let $(N,\alpha_N)$ be another left $H$-module with action $H \otimes N \rightarrow N$, $h \otimes n \mapsto h \cdot n$, regarded as a Yetter-Drinfeld module as in (i), via the map $\lambda_N : N \rightarrow H \otimes M$, $\lambda_N(n) = n_{(-1)} \otimes n_{(0)} := \alpha_H(\mathbb{R}_2^2) \otimes r^1 \cdot n$. We regard $(M \otimes N,\alpha_M \circ \alpha_N)$ as a left $H$-module via the standard action $h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n$ and then we regard $(M \otimes N,\alpha_M \circ \alpha_N)$ as a Yetter-Drinfeld module as in (i). Then this Yetter-Drinfeld module $(M \otimes N,\alpha_M \circ \alpha_N)$ coincides with the Yetter-Drinfeld module $M \otimes N$ defined as in Proposition 3.7.
Proof. (i) First we have to prove that \((M, \alpha_M)\) is a left \(H\)-comodule; \((1.4)\) is easy and left to the reader, we check \((1.5)\):

\[
(\Delta_H \otimes \alpha_M)(\lambda_M(m)) = \Delta_H(\alpha_H(R^2)) \otimes \alpha_M(R^1 \cdot m)
\]

\[
\overset{1.8, 1.2}{=} \alpha_H(R^1_1) \otimes \alpha_H(R^2_2) \otimes \alpha_H(R^1) \cdot \alpha_M(m)
\]

\[
\overset{1.10}{=} \alpha_H^2(r^2) \otimes \alpha_H^2(R^2) \otimes (R^1 \cdot r^1) \cdot \alpha_M(m)
\]

\[
\overset{1.3}{=} \alpha_H^2(r^2) \otimes \alpha_H^2(R^2) \otimes \alpha_H(R^1) \cdot (r^1 \cdot m)
\]

\[
\overset{3.11}{=} \alpha_H^2(r^2) \otimes \alpha_H(R^2) \otimes R^1 \cdot (r^1 \cdot m)
\]

\[
= \alpha_H^2(r^2) \otimes \lambda_M(r^1 \cdot m)
\]

\[
= (\alpha_H \otimes \lambda_M)(\lambda_M(m)), \text{ q.e.d.}
\]

Now we check the Yetter-Drinfeld condition \((2.1)\):

\[
(h_1 \cdot m)_{(-1)} \alpha_H^2(h_2) \otimes (h_1 \cdot m)_{(0)} = \alpha_H(R^2) \alpha_H^2(h_2) \otimes R^1 \cdot (h_1 \cdot m)
\]

\[
\overset{3.11}{=} \alpha_H^2(R^2) \alpha_H^2(h_2) \otimes \alpha_H(R^1) \cdot (h_1 \cdot m)
\]

\[
\overset{1.3}{=} \alpha_H^2(R^2 h_2) \otimes (R^1 h_1) \cdot \alpha_M(m)
\]

\[
\overset{1.11}{=} \alpha_H^2(h_1 R^2) \otimes (h_2 R^1) \cdot \alpha_M(m)
\]

\[
\overset{1.3}{=} \alpha_H^2(h_1) \alpha_H^2(R^2) \otimes \alpha_H(h_2) \cdot (R^1 \cdot m)
\]

\[
= \alpha_H^2(h_1) \alpha_H^2(h_2) \otimes \alpha_M(h_2) \cdot m_{(0)}, \text{ q.e.d.}
\]

(ii) We only need to prove that the two comodule structures on \(M \otimes N\) coincide, that is, for all \(m \in M, n \in N\),

\[
\alpha_H^{-2}(m_{(-1)} n_{(-1)}) \otimes (m_{(0)} \otimes n_{(0)}) = \alpha_H(R^2) \otimes R^1 \cdot (m \otimes n),
\]

that is

\[
\alpha_H^{-2}(\alpha_H(R^2) \alpha_H(r^2)) \otimes (R^1 \cdot m \otimes r^1 \cdot n) = \alpha_H(R^2) \otimes (R^1 \cdot m \otimes R^2 \cdot n),
\]

which, because of \((3.11)\), is equivalent to

\[
R^2 r^2 \otimes (\alpha_H(R^1) \cdot m \otimes \alpha_H(r^1) \cdot n) = \alpha_H(R^2) \otimes (R^1 \cdot m \otimes R^2 \cdot n),
\]

and this is an obvious consequence of \((1.9)\). \(\square\)

As a consequence of various results obtained so far, we also obtain the following:

**Theorem 3.9** Let \((H, \mu_H, \Delta_H, \alpha_H, R)\) be a quasitriangular Hom-bialgebra with \(\alpha_H\) bijective and \((\alpha_H \otimes \alpha_H)(R) = R\). Denote by \(H \mathcal{M}\) the category whose objects are left \(H\)-modules \((M, \alpha_M)\) with \(\alpha_M\) bijective and morphisms are morphisms of left \(H\)-modules. Then \(H \mathcal{M}\) is a quasi-braided pre-tensor subcategory of \(H \mathcal{YD}\), with tensor product defined as in Proposition \(1.5\) (i), associativity constraints defined by the formula \(a_{M,N,P}(m \otimes n) \otimes p = \alpha_M^{-1}(m) \otimes (n \otimes \alpha_P(p))\), for \(M, N, P \in H \mathcal{M}\), and quasi-braiding \(c_{M,N} : M \otimes N \to N \otimes M, c_{M,N}(m \otimes n) = \alpha_N^{-1}(R^2 \cdot n) \otimes \alpha_M^{-1}(R^1 \cdot m)\), for all \(M, N \in H \mathcal{M}\).
We recall the following result ([36], Theorem 4.4):

**Proposition 3.10** Let \((H, \mu_H, \Delta_H, \alpha_H, R)\) be a quasitriangular Hom-bialgebra such that \((\alpha_H \otimes \alpha_H)(R) = R\) and \((M, \alpha_M)\) a left \(H\)-module. Then the linear map \(B : M \otimes M \to M \otimes M, B(m \otimes m') = R^2 \cdot m' \otimes R^1 \cdot m\) is a solution of the HYBE for \((M, \alpha_M)\).

It turns out that the particular case of this result in which \(\alpha_H\) is bijective is a particular case of Proposition 2.4 via Proposition 3.8.

### 4 The quasi-braided pre-tensor category \((\mathcal{YD}^H, \otimes, a, c)\)

We have seen in the previous section that modules over quasitriangular Hom-bialgebras become Yetter-Drinfeld modules. Similarly, comodules over coquasitriangular Hom-bialgebras become Yetter-Drinfeld modules; inspired by this, we can introduce a second quasi-braided pre-tensor category structure on \(\mathcal{YD}^H\). We include these facts here for completeness. Each of the next results is the analogue of a result in the previous section; their proofs are similar to those of their analogues and are left to the reader.

**Theorem 4.1** Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra such that \(\alpha_H\) is bijective.

(i) Let \((M, \alpha_M)\) and \((N, \alpha_N)\) be two Yetter-Drinfeld modules over \(H\), with notation as above, and define the linear maps
\[
H \otimes (M \otimes N) \to M \otimes N, \quad h \otimes (m \otimes n) \mapsto \alpha_H^{-2}(h_1) \cdot m \otimes \alpha_H^2(h_2) \cdot n,
\]
\[
M \otimes N \to H \otimes (M \otimes N), \quad m \otimes n \mapsto m_{(-1)}n_{(-1)} \otimes (m_{(0)} \otimes n_{(0)}).
\]

Then \((M \otimes N, \alpha_M \otimes \alpha_N)\) with these structures becomes a Yetter-Drinfeld module over \(H\), denoted in what follows by \(M \otimes N\).

(ii) \(\mathcal{YD}^H\) is a quasi-braided pre-tensor category, with tensor product \(\tilde{\otimes}\) as in (i) and associativity constraints \(a_{M,N,P}\) and quasi-braiding \(c_{M,N}\) defined as follows:
\[
a_{M,N,P} : (M \otimes N) \tilde{\otimes} P \to M \tilde{\otimes} (N \otimes P), \quad a_{M,N,P}(m \otimes n) \otimes p) = \alpha_M(m) \otimes (n \otimes \alpha_P^{-1}(p)),
\]
\[
c_{M,N} : M \tilde{\otimes} N \to N \otimes M, \quad c_{M,N}(m \otimes n) = \alpha_N^{-1}(\alpha_H^{-1}(m_{(-1)}) \cdot n) \otimes \alpha_M^{-1}(m_{(0)}).
\]

**Proposition 4.2** Let \((H, \mu_H, \Delta_H, \alpha_H, \sigma)\) be a coquasitriangular Hom-bialgebra satisfying the condition \(\sigma = \sigma \circ (\alpha_H \otimes \alpha_H)\).

(i) Let \((M, \alpha_M)\) be a left \(H\)-comodule with coaction \(M \to H \otimes M, m \mapsto m_{(-1)} \otimes m_{(0)}\). Define the linear map \(H \otimes M \to M, h \otimes m \mapsto h \cdot m := \sigma(m_{(-1)} \otimes \alpha_H(h))m_{(0)}\). Then \((M, \alpha_M)\) with these structures is a Yetter-Drinfeld module over \(H\).

(ii) Assume that \(\alpha_H\) is bijective. Let \((N, \alpha_N)\) be another left \(H\)-comodule with coaction \(N \to H \otimes N, n \mapsto n_{(-1)} \otimes n_{(0)}\), regarded as a Yetter-Drinfeld module as in (i), via the map \(H \otimes N \to N, h \otimes n \mapsto h \cdot n := \sigma(n_{(-1)} \otimes \alpha_H(h))n_{(0)}\). We regard \((M \otimes N, \alpha_M \otimes \alpha_N)\) as a left \(H\)-comodule via the standard coaction \(M \otimes N \to H \otimes (M \otimes N), m \otimes n \mapsto m_{(-1)}n_{(-1)} \otimes (m_{(0)} \otimes n_{(0)})\) and then we regard \((M \otimes N, \alpha_M \otimes \alpha_N)\) as a Yetter-Drinfeld module as in (i). Then this Yetter-Drinfeld module coincides with the Yetter-Drinfeld module \(M \hat{\otimes} N\) defined in Theorem 4.7.

**Theorem 4.3** Let \((H, \mu_H, \Delta_H, \alpha_H, \sigma)\) be a coquasitriangular Hom-bialgebra with \(\alpha_H\) bijective and \(\sigma = \sigma \circ (\alpha_H \otimes \alpha_H)\). Denote by \(\mathcal{M}^H\) the category whose objects are left \(H\)-comodules \((M, \alpha_M)\) with \(\alpha_M\) bijective and morphisms are morphisms of left \(H\)-comodules. Then \(\mathcal{M}^H\) is a quasi-braided pre-tensor subcategory of \(\mathcal{YD}^H\), with tensor product defined as in Proposition 4.4.
associativity constraints defined by the formula \( a_{M,N,P}(m \otimes n \otimes p) = \alpha_M(m) \otimes (n \otimes \alpha_{P}^1(p)) \), for \( M, N, P \in H \mathcal{M} \), and quasi-braiding \( c_{M,N} : M \otimes N \to N \otimes M, c_{M,N}(m \otimes n) = \sigma(n_{(-1)} \otimes m_{(-1)})\alpha_{N}^{-1}(n_{(0)}) \otimes \alpha_{M}^{-1}(m_{(0)}) \), for all \( M, N \in H \mathcal{M} \).

We recall the following result ([37], Theorem 7.4):

**Proposition 4.4** Let \((H, \mu_H, \Delta_H, \alpha_H, \sigma)\) be a coquasitriangular Hom-bialgebra such that \( \sigma = \sigma \circ (\alpha_H \otimes \alpha_H) \). If \((M, \alpha_M), (N, \alpha_N)\) are left \( H \)-comodules, we define the linear map

\[
B_{M,N} : M \otimes N \to N \otimes M, \quad B_{M,N}(m \otimes n) = \sigma(n_{(-1)} \otimes m_{(-1)})n_{(0)} \otimes m_{(0)}.
\]

Then \( (\alpha_N \otimes \alpha_M) \circ B_{M,N} = B_{M,N} \circ (\alpha_M \otimes \alpha_N) \) and, if \((P, \alpha_P)\) is another left \( H \)-comodule, the maps \( B_{-,-} \) satisfy the HYBE

\[
(\alpha_P \otimes B_{M,N}) \circ (B_{M,P} \otimes \alpha_N) \circ (\alpha_M \otimes B_{N,P}) = (B_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes B_{M,P}) \circ (B_{M,N} \otimes \alpha_P).
\]

It turns out that the particular case of this result in which \( \alpha_H \) is bijective is a particular case of Proposition [2.4] via Proposition [4.2]

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