CENTRAL LIMIT THEOREM FOR HOTELLING’S $T^2$ STATISTIC UNDER LARGE DIMENSION

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Abstract. In this paper, we prove the central limit theorem for Hotelling’s $T^2$ statistic when the dimension of the random vectors is proportional to the sample size.

1. Introduction and main results

Since the famous Marcenko and Pastur law was found in [16], the theory of large sample covariance matrices has been further developed. Among others, we mention Jonsson [14], Yin [24], Silverstein [18], Watcher [23], Yin, Bai and Krishanaiah [25]. Lately, Johnstone [13] discovered the law of the largest eigenvalue of the Wishart matrix, Bai and Silverstein [5] established the central limit theorems (CLT) of linear spectral statistics, and Bai, Miao and Pan [2] derived CLT for functionals of the eigenvalues and eigenvectors. We also refer to [12], [22], [9] for CLT on linear statistics of eigenvalues of other classes of random matrices.

The sample covariance matrix is defined by

$$S = \frac{1}{n} \sum_{j=1}^{n} (s_j - \bar{s})(s_j - \bar{s})^T,$$

where $\bar{s} = n^{-1} \sum_{j=1}^{n} s_j$ and $s_j = (X_{1j}, \ldots, X_{pj})^T$. Here $\{X_{ij}\}$, $i, j = \cdots$, is a double array of independent and identically distributed (i.i.d.) real r.v.’s with $EX_{11} = 0$ and $EX_{11}^2 = 1$. However, in the large random matrices theory

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(RMT), the commonly used sample covariance matrix is

\[ S = \frac{1}{n} \sum_{j=1}^{n} s_j s_j^T = \frac{1}{n} X_n X_n^T, \]

where \( X_n = (s_1, \ldots, s_n) \).

Note that \( S = S - \bar{s}\bar{s}^T \) and thus by the rank inequality there is no difference when one is only concerned with the limiting empirical spectral distribution (ESD) of the eigenvalues in large random matrices. Therefore the limiting ESD of \( S \) is Marcenko and Pastur’s law \( F_c(x) \) (see [16] and [14]), which has a density function

\[ p_c(x) = \begin{cases} 
(2\pi c x)^{-1/2} \sqrt{(b-x)(x-a)} & \text{if } a \leq x \leq b, \\
0 & \text{otherwise}
\end{cases} \]

and has point mass \( 1 - c^{-1} \) at the origin if \( c > 1 \), where \( a = (1 - \sqrt{c})^2 \) and \( b = (1 + \sqrt{c})^2 \). The Stieljes transform \( m(z) \) of \( F_c(x) \) satisfies the equation (see [21])

\[ m(z) = \frac{1}{1 - c - czm(z) - z}, \]

where the Stieljes transform for any function \( G(x) \) is defined by

\[ m_G(z) = \int \frac{1}{\lambda - z} dG(\lambda), \quad z \in \mathbb{C}^+ = \{ z \in \mathbb{C}, \quad v = \Re z > 0 \}. \]

Observe that the spectra of \( n^{-1}X_n X_n^T \) and \( n^{-1}X_n^T X_n \) are identical except for zero eigenvalues. This leads to the equality

\[ m_n^S(z) = -\frac{1 - p/n}{z} + \frac{p}{n} m_n(z), \]

and therefore

\[ z = -\frac{1}{m(z)} + \frac{c}{1 + m(z)}, \]

where \( m_n^S(z) \) and \( m_n^S(z) \) denote, respectively, the Stieljes transform of the ESD of \( n^{-1}X_n X_n^T \) and \( n^{-1}X_n^T X_n \), and, correspondingly, \( m(z) \) is the limit of \( m_n^S(z) \).

Sample covariance matrices are also of essential importance in multivariate statistical analysis because many test statistics involve their eigenvalues and/or eigenvectors. The typical example is \( T^2 \) statistic, which was proposed by Hotelling [10]. We refer to [1] and [15] for various uses of the \( T^2 \) statistic.

The \( T^2 \) statistic, which is the origin of multivariate linear hypothesis tests and the associated confidence sets, is defined by

\[ T^2 = n(\bar{s} - \mu_0)^T S^{-1}(\bar{s} - \mu_0), \]
whose distribution is invariant under the transformation $s_j' = \Sigma^{1/2}s_j, j = 1, 2, \cdots, n$ with $\Sigma$ any non-singular $p$ by $p$ matrix when $\mu_0 = 0$. If $\{s_1, \cdots, s_n\}$ is a sample from the $p$-dimensional population $N(\mu, \Sigma)$, then $[T^2/(n-1)]\left[(n-p)/p\right]$ follows a noncentral $F$ distribution and moreover, the $F$ distribution is central if $\mu = \mu_0$. When $p$ is fixed, the limiting distribution of $T^2$ for $\mu = \mu_0$ is the $\chi^2$ distribution even if the parent distribution is not normal.

In recent three or four decades, in many research areas, including signal processing, network security, image processing, genetics, stock marketing and other economic problems, people are interested in the case where $p$ is quite large or proportional to the sample size. Thus it will be desirable if one can obtain the asymptotic distribution of the famous Hotelling's $T^2$ statistic when the dimension of the random vectors is proportional to the sample size. It is the aim of this work. In addition, we would like to point out that some discussions about the two-sample $T^2$ statistic under the assumption that the underlying r.v.'s are normal were presented in [3].

Before stating the results, let us introduce some notation. Let $m(z) = \int(x - z)^{-1}dF_c(x)$ and $m_n(z) = \int(x - z)^{-1}dF_{c_n}(x)$, where $c_n = p/n$ and $F_{c_n}(x)$ denotes $F_c(x)$ by substituting $c_n$ for $c$.

The main results are then presented in the following theorems.

**Theorem 1.** Suppose that:
(1) For each $n X_{ij} = X^n_{ij}, i, j = 1, 2, \cdots$, are i.i.d. real r.v.'s with $EX_{11} = \mu, EX_{11}^2 = 1$ and $EX_{11}^4 < \infty$.
(2) $p \leq n, c_n = p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$.
Then, when $\mu = \mu_0 = (\mu, \cdots, \mu)$,
$$\sqrt{n}\frac{\sqrt{n}}{\sqrt{2c_n\int x^{-2}dF_{c_n}(x)}}[T^2_n - c_n\int \frac{dF_{c_n}(x)}{x}] \rightarrow D N(0, 1).$$

We will prove Theorem 1 by establishing Theorem 2 which presents asymptotic distributions of random quadratic forms involving sample means and sample covariance matrices.

For any analytic function $f(\cdot)$, define
$$f(\mathcal{S}) = U^T diag(f(\lambda_1), \cdots, f(\lambda_p))U,$$
where $U^T diag(\lambda_1, \cdots, \lambda_p)U$ denotes the spectral decomposition of the matrix $\mathcal{S}$.

**Theorem 2.** In addition to the assumption (1) of Theorem 1, suppose that $c_n = p/n \rightarrow c > 0, EX_{11} = 0, g(x)$ is a function with a continuous first derivative in a neighborhood of $c$, and $f(x)$ is analytic on an open region containing
the interval
\[(1.5) [I_{(0,1)}(c)(1 - \sqrt{c})^2, (1 + \sqrt{c})^2].\]

Then,
\[
\sqrt{n} \left[ \frac{s^T f(S)s}{\|s\|^2} - \int f(x) dF_{c_n}(x) \right], \sqrt{n} (g(s^T s) - g(c_n)) \xrightarrow{D} (X, Y),
\]
where \(Y \sim N(0, 2c(g'(c))^2)\), which is independent of \(X\), a Gaussian r.v. with \(EX = 0\) and
\[(1.6) \text{Var}(X) = \frac{2}{c} \left( \int f^2(x) dF_c(x) - \left( \int f(x) dF_c(x) \right)^2 \right).
\]

**Remark 1.** Let \(X_n = (x_{n1}, \ldots, x_{np})^T \in \mathbb{R}^p, \|x_n\| = 1\) where \(\| \cdot \|\) denotes the Euclidean norm. Note that, when \(\max x_{ni} \to 0\) (see (1.16) in [17] or [20]),
\[(1.7) \sqrt{n} [x_n^T f(S)x_n - \int f(x) dF_{c_n}(x)] \xrightarrow{D} X.
\]

Therefore, \(\bar{s}/\|s\|\) can be treated as a fixed unit vector \(x_n\) when dealing with \(s^T f(S)s/\|s\|^2\).

Theorem 2 relies on Lemma 1 below, which deals with the asymptotic joint distribution of
\[X_n(z) = \sqrt{n} \left[ \frac{s^T (S - zI)^{-1}s}{\|s\|^2} - m_n(z) \right], \quad Y_n = \sqrt{n} (g(s^T s) - g(c_n)).\]

The stochastic process \(X_n(z)\) is defined on a contour \(C\), given as below. Let \(v_0 > 0\) be arbitrary and set \(C_u = \{u + iv_0, u \in [u_l, u_r]\}\), where \(u_l\) is any negative number if the left endpoint of (1.5) is zero, otherwise \(u_l\) is any positive number smaller than the left end-point of (1.5), and \(u_r\) any number larger than the right end-point of (1.5). Then define
\[C^+ = \{u_l + iv : v \in [0, v_0]\} \cup C_u \cup \{u_r + iv : v \in [0, v_0]\}\]
and let \(C^-\) be the symmetric part of \(C^+\) about the real axis. Then \(C = C^+ \cup C^-\).

We further define \(X_n(z)\), a truncated version of \(X_n(z)\), as in [3]. Select a sequence of positive numbers \(\rho_n\) satisfying for some \(\beta \in (0, 1),
\[(1.8) \rho_n \downarrow 0, \quad \rho_n \geq n^{-\beta}.
\]

Let
\[C_l = \begin{cases} \{u_l + iv : v \in [n^{-1}\rho_n, v_0]\}, & \text{if } u_l > 0 \medskip \{u_l + iv : v \in [0, v_0]\}, & \text{if } u_l < 0 \end{cases}
\]
and
\[C_r = \{u_r + iv : v \in [n^{-1}\rho_n, v_0]\}.
\]
Write $C_{+}^{z} = C_{\ell} \cup C_{u} \cup C_{r}$. We can now define the truncated process for $z = u + iv \in C$ by

\[(1.9)\]

\[
\hat{X}_{n}(z) = \begin{cases} 
X_{n}(z), & \text{if } z \in C_{+}^{z} \cup C_{-}^{z} \\
\frac{nu}{2\rho_{n}}X_{n}(z_{r1}) + \frac{nu}{2\rho_{n}}X_{n}(z_{r2}), & \text{if } u = u_{r}, v \in [-n^{-1}\rho_{n}, n^{-1}\rho_{n}], \\
\frac{nu}{2\rho_{n}}X_{n}(z_{l1}) + \frac{nu}{2\rho_{n}}X_{n}(z_{l2}), & \text{if } u = u_{l} > 0, v \in [-n^{-1}\rho_{n}, n^{-1}\rho_{n}],
\end{cases}
\]

where $z_{r1} = u_{r} + in^{-1}\rho_{n}, z_{r2} = u_{r} - in^{-1}\rho_{n}, z_{l1} = u_{l} + in^{-1}\rho_{n}, z_{l2} = u_{l} - in^{-1}\rho_{n}$ and $C_{-}^{z}$ denotes the symmetric part of $C_{+}^{z}$ about the real axis. Then $\hat{X}_{n}(z)$ may be viewed as a random element in the metric space $C(C, \mathbb{R}^{2})$ of continuous functions from $C$ to $\mathbb{R}^{2}$. We are now in a position to state Lemma [1].

**Lemma 1.** Under the assumptions of Theorem [2], we have for $z \in C$,

\[
\left( \hat{X}_{n}(z), Y_{n} \right) \xrightarrow{D} (X(z), Y),
\]

where $Y \sim N(0, 2c(g'(c))^{2})$, which is independent of $X(z)$, a Gaussian stochastic process with mean zero and covariance function $\text{Cov}(X(z_{1}), X(z_{2}))$ equal to

\[(1.10)\]

\[
\frac{2}{cz_{1}z_{2}(1 + m(z_{1}))(1 + m(z_{2}))} - \frac{2m(z_{1})m(z_{2})}{c}.
\]

**Remark 2.** Also, note that $X(z)$ is exactly the weak limit of the stochastic process $\sqrt{n}(x_{n}^{T}(S - zI)^{-1})x_{n} - m_{n}(z)$ when $\max_{i} x_{ni} \to 0$, whose covariance function is,

\[
\text{Cov}(X(z_{1}), X(z_{2})) = \frac{2(z_{1}m(z_{2}) - z_{1}m(z_{1}))^{2}}{c^{2}z_{1}z_{2}(z_{1} - z_{2})(m(z_{1}) - m(z_{2}))},
\]

(see [2] and [17]).

We conclude this section by presenting the structure of this work. To transfer Lemma [1] to Theorem [2], we introduce a new empirical distribution function

\[(1.11)\]

\[
F_{2}^{S}(x) = \sum_{i=1}^{p} t_{i}^{2}I(\lambda_{i} \leq x),
\]

where $t = (t_{1}, \cdots, t_{n})^{T} = Us/\|s\|$ and $U$ is the eigenvector matrix of $S$. It turns out that $F_{2}^{S}(x)$ and the ESD of $S$ have the same limit. That is, $F_{2}^{S}(x) \xrightarrow{i.p.} F_{c}(x)$. Thus, $s^{T}f(S)s/\|s\|^{2}$ in Theorem [2] is transferred to the Stieljes transform of $F_{2}^{S}(x)$. Moreover, note that

\[(1.12)\]

\[
\frac{s^{T}A^{-1}(z)s}{1 - s^{T}A^{-1}(z)s} = s^{T}(S - zI)^{-1}s,
\]
where $A^{-1}(z) = (S - zI)^{-1}$. Indeed, this is from the identity

$$(1.13) \quad r^T(B + arr^T)^{-1} = \frac{r^TB^{-1}}{1 + ar^TB^{-1}r},$$

where $B$ and $B + arr^T$ are both invertible, $r \in \mathbb{R}^p$ and $a \in \mathbb{R}$. The stochastic process $X_n(z)$ in Lemma 1 is then transferred to the stochastic process $M_n(z)$, where

$$M_n(z) = \sqrt{n}(s^TA^{-1}(z)s - \frac{c_nm_n(z)}{1 + c_nm_n(z)}).$$

The convergence of the stochastic process $M_n(z)$ is given in the next two sections. The proofs of Theorems and Lemma and Remark are included in section 4. The last section picks up the truncation of the underlying r.v.’s.

Throughout this paper, to save notation, $M$ may denote different constants on different occasions.

2. WEAK CONVERGENCE OF THE FINITE DIMENSIONAL DISTRIBUTIONS

For $z \in C_n$, let $M_n(z) = M_n^{(1)}(z) + M_n^{(2)}(z)$, where

$$M_n^{(1)}(z) = \sqrt{n}(s^TA^{-1}(z)s - Es^TA^{-1}(z)s),$$

and

$$M_n^{(2)}(z) = \sqrt{n}(Es^TA^{-1}(z)s - \frac{c_nm_n(z)}{1 + c_nm_n(z)}).$$

In this section, the aim is to prove that for any positive integer $r$ and complex numbers $a_1, \cdots, a_r$,

$$\sum_{i=1}^r a_iM_n^{(1)}(z_i), \Im z_i \neq 0$$

converges in distribution to a Gaussian r.v., and to derive the asymptotic covariance function. Before proceeding, r.v.’s need to be truncated. However, we shall postpone the truncation of r.v.’s until the last section. As a consequence of Lemma 7, we assume that the underlying r.v.’s satisfy

$$(2.1) \quad |X_{ij}| \leq \varepsilon_n\sqrt{n}, \quad EX_{11} = 0, \quad E|X_{11}|^2 = 1, \quad E|X_{11}|^4 < \infty,$$

where $\varepsilon_n$ is a positive sequence which converges to zero as $n$ goes to infinity. We begin with a list of notation, mathematical tools and estimates.
2.1. Notation, mathematical tools and estimates. We first introduce some notation. Let $A_j(z) = A(z) - n^{-1}s_js_j^T$, $A_{ij}(z) = A(z) - n^{-1}s_is_i^T - n^{-1}s_j s_j^T$, $\tilde{s}_j = \tilde{s} - n^{-1}s_j$, $D_j(z) = A_j^{-1}(z)\tilde{s}_j s_j^T A_j^{-1}(z)$,

$$\beta_j(z) = \frac{1}{1 + \frac{1}{n}s_j^TA_j^{-1}(z)s_j}, \quad \beta_{ij}^{tr}(z) = \frac{1}{1 + \frac{1}{n}trA_j^{-1}(z)}, \quad b_1(z) = \frac{1}{1 + \frac{1}{n}trA_j^{-1}(z)},$$

$$\gamma_j(z) = \frac{1}{n}s_j^TA_j^{-1}(z)s_j - \frac{1}{n}trA_j^{-1}(z), \quad \xi_j(z) = \frac{1}{n}s_j^TA_j^{-1}(z)s_j - \frac{1}{n}trA_j^{-1}(z),$$

$$\alpha_j(z) = \frac{1}{n}s_i^TA_j^{-1}(z)s_j s_j^TA_j^{-1}(z)s_j - \frac{1}{n}s_i^TA_j^{-2}(z)s_j, \quad \tilde{A}_j(z) = A_j^{-1}(z)s_j s_j^TA_j^{-1}(z),$$

$$\beta_{ij}(z) = \frac{1}{1 + \frac{1}{n}s_i^TA_{ij}^{-1}(z)s_i}, \quad \beta_{ij}^{tr}(z) = \frac{1}{1 + \frac{1}{n}trA_{ij}^{-1}(z)}, \quad b_{12}(z) = \frac{1}{1 + \frac{1}{n}trA_{12}^{-1}(z)},$$

and

$$\xi_{ij}(z) = \frac{1}{n}s_i^TA_{ij}^{-1}(z)s_i - \frac{1}{n}trA_{12}^{-1}(z), \quad \gamma_{ij}(z) = \frac{1}{n}s_i^TA_{ij}^{-1}(z)s_i - \frac{1}{n}trA_{ij}^{-1}(z).$$

Next we list some results which will be frequently used below.

**Lemma 2.** (Burkholder (1973)) Let $\{Y_i\}$ be a complex martingale difference sequence with respect to the increasing $\sigma$-field $\{F_i\}$. Then for $k \geq 2$

$$E \left| \sum_i Y_i \right|^k \leq M_k E \left( \sum_i E(|Y_i|^2|F_{i-1}) \right)^{k/2} + M_k E \left( \sum_i |Y_i|^k \right).$$

**Lemma 3.** (Theorem 35.12 of Billingsley (1995)) Suppose for each $n$, $Y_{n,1}, Y_{n,2}, \ldots, Y_{n,r_n}$ is a real martingale difference sequence with respect to the increasing $\sigma$-field $\{F_{n,j}\}$ having second moments. If as $n \to \infty$

(i) $\sum_{j=1}^{r_n} E(Y_{n,j}^2|F_{n,j-1}) \xrightarrow{ip} \sigma^2$

(ii) $\sum_{j=1}^{r_n} E(Y_{n,j}^2I(|Y_{n,j}| \geq \varepsilon)) \to 0$

where $\sigma^2$ is a positive constant and $\varepsilon$ is an arbitrary positive number, then

$$\sum_{j=1}^{r_n} Y_{n,j} \xrightarrow{D} N(0, \sigma^2).$$

**Lemma 4.** (Lemma 2.7 in [1]) Let $Y = (Y_1, \ldots, Y_p)^T$, where $Y_i$'s are i.i.d. real r.v.'s with mean 0 and variance 1. Let $B = (b_{ij})_{p \times p}$, a deterministic complex matrix. Then for any $k \geq 2$, we have

$$E|Y^TB^*Y - trB|^k \leq M_k(E|Y|^4 trBB^*)^{k/2} + M_k E(Y_1)^{2k} tr(BB^*)^{k/2},$$
where $B^*$ denotes the complex conjugate transpose of $B$.

**Lemma 5.** Let $C = (c_{ij})_{p \times p}$ be a complex matrix with $c_{jj} = 0$ and $Y = (Y_1, \cdots, Y_p)^T$, defined in Lemma 4. Then for any $k \geq 2$,

$$E|Y^T CY|^k \leq M_k |E|Y_1|^k |trCC^*|^{k/2}. \quad (2.2)$$

Lemma 5 directly follows from the argument of Lemma A.1 of [4].

A direct calculation indicates that the following equalities are true:

$$E(s_1^T A s_1 - trA)(s_1^T B s_1 - trB) \quad (2.3)$$

$$E[(s_1^T A s_1 - trA)s_1^T B r] = EX_{11}^3 \sum_{i=1}^{p} a_i e_i^T B r, \quad (2.4)$$

where $B = (b_{ij})_{p \times p}$ and $A = (a_{ij})_{p \times p}$ are deterministic complex matrices and $r$ is a deterministic vector. Here $e_i$ is the vector with the $i$-th element being 1 and zero otherwise.

In what follows, to facilitate the analysis in the subsequent subsections, we shall assume $v = \exists z > 0$. Note that $\beta_j(z)$, $\beta_j^s(z)$, $\beta_{ij}(z)$, $b_1(z)$, $b_2(z)$ are bounded in absolute value by $|z|/v$ (see (3.4) of [4]). From (1.13) we have

$$A^{-1}(z) - A^{-1}_j(z) = A^{-1}(z)(A_j(z) - A(z))A_j^{-1}(z) = -\frac{1}{n} \hat{A}_j(z) \beta_j(z), \quad (2.5)$$

and from Lemma 2.10 of [4] for any matrix $B$

$$|tr(A^{-1}(z) - A^{-1}_j(z))B| \leq \frac{||B||}{v}, \quad (2.6)$$

where $|| \cdot ||$ denote the spectral norm of a matrix. Moreover, Section 4 in [4] shows that

$$n^{-k}E|tr(A^{-1}_1(z) - Etr(A^{-1}_1(z))|^k = O(n^{-k/2}), \quad k \geq 2. \quad (2.7)$$

One should note that (2.7) is still true when $A^{-1}_1(z)$ is replaced by $A^{-1}_{12}(z)$.

To simplify the statements, assume that the spectral norms of $B$, $B_i$, $A_i$, $C$ involved in the equalities (2.8)–(2.10) are all bounded above by a constant. For $k \geq 2$, it follows from Lemma 4 (2.1) and (2.7) that

$$n^{-k}E|s_1^T B s_1 - trB|^k = O(\varepsilon_n^{2k-4}n^{-1}), \quad E|\xi_1(z)|^k = O(\varepsilon_n^{2k-4}n^{-1}), \quad (2.8)$$

and that

$$n^{-k}E|s_1^T B e_j^T C s_1|^k \leq M n^{-k}[E|s_1^T B e_j e_j^T C s_1 - trB e_j e_j^T C|^k + E|e_j^T C B e_i|^k] = O(\varepsilon_n^{2k-4}n^{-2}). \quad (2.9)$$
We shall establish the estimates (2.10)-(2.12) below:

(2.10) \( E|s_i^T B s_1|^k = O(n^{k-2} \varepsilon_n^{-k+4}) \), \( k \geq 4 \), \( E|\alpha_1(z)|^k = O(n^{-2} \varepsilon_n^{2k-4}) \), \( k \geq 2 \),

(2.11) \( E|s_i^T B s_2|^k = O(n^{k-2} \varepsilon_n^{-k+4}) \), \( k \geq 4 \),

and for \( m \geq 0 \), \( q \geq 1 \), \( 0 \leq r \leq 2 \),

(2.12) \( E \left| \prod_{i=1}^{m} \frac{1}{n} s_i^T A_i s_1 \prod_{j=1}^{q} \frac{1}{n} (s_i^T B_j s_1 - tr B_j) (s_i^T C_i s_1)^r \right| = O(n^{-\frac{1}{2}} \varepsilon_n^{(q-2)v_0}) \).

One should note that (2.10) and (2.11) also give the estimates for \( k = 2 \). For example

(2.13) \( E|s_i^T B s_1|^2 \leq (E|s_i^T B s_1|^4)^{1/2} = O(1) \).

In addition, from (2.8) and (2.11) we also conclude that

(2.14) \( E|n^{-1} s_i^T B s_1 s^T C s_2|^4 \leq ME|n^{-1} (s_i^T B s_1 - tr B) s_i^T C s_2|^4 + ME|s_i^T C s_2|^4 = O(n^{5/2}) \).

Consider (2.10) first. Note that for \( k \geq 4 \)

(2.15) \( E|s_i^T s_i|^k \leq \frac{M}{n^{2k}} \left[ E \left| \sum_{i=2}^{n} s_i^T s_i \right|^k + E \left| \sum_{i_1 \neq i_2, i_1 > 1, i_2 > 1} s_i^T s_i \right|^k \right] = O(1) \).

Applying Lemma 2 twice to the second expectation in (2.15) gives

\[
E \left| \sum_{i=2}^{n} s_i^T s_i \right|^k \leq M E \left| \sum_{i=2}^{n} (s_i^T s_i - E(s_i^T s_i)) \right|^k + M \left| \sum_{i=2}^{n} E(s_i^T s_i) \right|^k
\]

\[
\leq M \left( \sum_{i=2}^{n} E(s_i^T s_i - E(s_i^T s_i))^2 \right)^{k/2} + M \sum_{i=2}^{n} E|s_i^T s_i - E(s_i^T s_i)|^k + M n^{2k}
\]

\[
\leq M n^k + M n \left[ \sum_{m=1}^{p} E|X_{m2}^2 - 1|^2 \right]^{k/2} + \sum_{m=1}^{p} E|X_{m2}^2 - 1|^k + M n^{2k}
\]

\[
\leq M n^k.
\]

The third expectation in (2.15) can be estimated by using Lemma 2 three times,

\[
E \left| \sum_{i_1 \neq i_2, i_1 > 1, i_2 > 1} s_i^T s_i \right|^k \leq n^{k-1} \sum_{i_1 > 1} E \left| \sum_{i_2 > 1, i_1 \neq i_2} s_i^T s_i \right|^k \leq n^k E \left| \sum_{i=3}^{n} s_i^T s_i \right|^k
\]

\[
\leq M n^k E \left| \sum_{i=3}^{n} E[(s_i^T s_i)^2|G_i = 1] \right|^{k/2} + M n^k \sum_{i=3}^{n} E|s_i^T s_i|^k
\]

\[
\leq M [n^k E|s_3^T s_2 - E s_3^T s_2|^k + M n^{2k} + n^{k+1} E|s_2^T s_3|^k] = O(n^{2k}),
\]
where $\mathcal{G}_i = \sigma(s_2, \cdots, s_i)$. It follows from (2.13) that for $k \geq 4$

\begin{equation}
E|s_i^T \mathbf{B} s_1|^k = E\|s_i^T \mathbf{B} s_1\|^k \leq E(\|s_i^T\|\|\mathbf{B}\|\|s_1\|)^k \leq M E\|s_i^T \mathbf{B} s_1\|^k \leq M,
\end{equation}

where $\|\cdot\|$ denotes the spectral norm of a matrix. This, together with Lemma 4 ensures that for $k \geq 4$

\[
E|s_i^T \mathbf{B} s_1|^k = E|s_i^T \mathbf{B} s_1|^{\frac{k}{2}} \leq ME|s_i^T \mathbf{B} s_1 - \bar{s}_i^T \mathbf{B} \bar{s}_1|^{\frac{k}{2}} + ME|\bar{s}_i^T \mathbf{B} \bar{s}_1|^{\frac{k}{2}} \leq [Mn^{\frac{k}{2}} - 2\varepsilon_n^{k-4} + M]E|s_i^T \mathbf{B} s_1|^{\frac{k}{2}} + M \leq Mn^{\frac{k}{2}} - 2\varepsilon_n^{k-4},
\]

which gives (2.10) as well as the order of $E|\alpha_1(z)|^k$.

Second, consider (2.11). Let $y = (y_1, \cdots, y_p)^T = \mathbf{B} s_2$ and then by lemma 2 and (2.8), for $k \geq 4$,

\[
E|s_1^T y|^k \leq ME\left(\sum_{m=1}^p|y_m|^2\right)^{k/2} + M\sum_{m=1}^p E|X_m|^k E|y_m|^k \leq ME|\mathbf{y}^* \mathbf{y}|^{k/2} + Mn^{\frac{k}{2}} - 2\varepsilon_n^{k-4}E|\mathbf{y}^* \mathbf{y}|^{k/2} \leq (M + Mn^{\frac{k}{2}} - 2\varepsilon_n^{k-4})E|s_2^T \mathbf{B}^* \mathbf{B} s_2 - tr \mathbf{B}^* \mathbf{B}|^{k/2} + Mn^{k/2} + Mn^{k-2}\varepsilon_n^{k-4} = O(n^{k-2}\varepsilon_n^{k-4}),
\]

where we also use the fact that for $k \geq 4$

\[
\sum_{m=1}^p|y_m|^k \leq \left(\sum_{m=1}^p|y_m|^2\right)^{k/2}.
\]

As for (2.12), if $m = 0$ and $r = 0$, then (2.12) directly follows from (2.8) and the Hölder inequality. If $m \geq 1$ and $r = 0$, then by induction on $m$ we have

\[
E \left| \prod_{i=1}^{m} \frac{1}{n} s_i^T \mathbf{A}_i s_i \prod_{j=1}^{q} \frac{1}{n} (s_i^T \mathbf{B}_j s_1 - tr \mathbf{B}_j) \right| \leq E \left| \prod_{i=1}^{m-1} \frac{1}{n} s_i^T \mathbf{A}_i s_i \frac{1}{n} (s_i^T \mathbf{A}_m s_1 - tr \mathbf{A}_m) \prod_{j=1}^{q} \frac{1}{n} (s_i^T \mathbf{B}_j s_1 - tr \mathbf{B}_j) \right| + pM \left( \prod_{i=1}^{m-1} \frac{1}{n} tr \mathbf{A}_i \prod_{j=1}^{q} \frac{1}{n} (s_i^T \mathbf{B}_j s_1 - tr \mathbf{B}_j) \right) = O(n^{-\frac{1}{2}}\varepsilon_n^{(q-2)\nu_0}).
\]
Repeating the argument above gives

\[ E \left| \prod_{i=1}^{m} \frac{1}{n} s_i^T A_i s_1 \prod_{j=1}^{q} \frac{1}{n} (s_1^T B_j s_1 - tr B_j) \right|^2 = O(n^{-1} \varepsilon_n^{(2q-4)\vee 0}), \]

\((m = 0 \text{ by (2.8)} \text{ and } m \geq 1 \text{ by induction}). \text{ Thus, for the case } m \geq 1 \text{ and } r \geq 1, \text{ by (2.10) we obtain}

\[ E \left| \prod_{i=1}^{m} \frac{1}{n} s_i^T A_i s_1 \prod_{j=1}^{q} \frac{1}{n} (s_1^T B_j s_1 - tr B_j) (s_1^T C_1 \bar{s}_1)^r \right|^2 \leq \frac{1}{2} E \left| \prod_{i=1}^{m} \frac{1}{n} s_i^T A_i s_1 \prod_{j=1}^{q} \frac{1}{n} (s_1^T B_j s_1 - tr B_j) \right|^2 E |s_1^T C_1 \bar{s}_1|^{2r} \]

\[ = O(n^{-1} \varepsilon_n^{(q-2)\vee 0}). \]

When \(m = 0\) and \(r \geq 1\), (2.12) can be obtained similarly. Thus we have proved (2.12).

2.2. The simplification of \(M_n^{(1)}(z)\). Define the \(\sigma\)-field \(\mathcal{F}_j = \sigma(s_1, \ldots, s_j)\), and let \(E_j(\cdot) = E(\cdot | \mathcal{F}_j)\) and \(E_0(\cdot)\) be the unconditional expectation. Now write

\[ M_n^{(1)}(z) = \sqrt{n} \sum_{j=1}^{n} E_j(s^T A^{-1}(z) \bar{s}) - E_{j-1}(s^T A^{-1}(z) \bar{s}) \]

\[ = \sqrt{n} \sum_{j=1}^{n} E_j(s^T A^{-1}(z) \bar{s} - s^T_j A^{-1}_j(z) \bar{s}_j) \]

\[ - E_{j-1}(s^T A^{-1}(z) \bar{s} - s^T_j A^{-1}_j(z) \bar{s}_j) \]

\[ = \sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})(a_{n1} + a_{n2} + a_{n3}), \]

where

\[ a_{n1} = (\bar{s} - \bar{s}_j)^T A^{-1}(z) \bar{s}, \]

\[ a_{n2} = \bar{s}_j^T (A^{-1}(z) - A^{-1}_j(z)) \bar{s}, \]

\[ a_{n3} = \bar{s}_j^T A^{-1}_j(z) (\bar{s} - \bar{s}_j). \]

The above first two terms will be further simplified one by one below. One should note that \(M_n^{(1)}(z)\) is now a sum of martingale difference sequences.

First, splitting \(A^{-1}(z)\) into the sum of \(A^{-1}(z) - A^{-1}_j(z)\) and \(A^{-1}_j(z)\) and splitting \(\bar{s}\) into the sum of \(\bar{s}_j\) and \(\bar{s}_j/n\), we have

\[ a_{n1} = a_{n1}^{(1)} + a_{n1}^{(2)} + a_{n1}^{(3)} + a_{n1}^{(4)}, \]
where
\[ a_{n1}^{(1)} = -\frac{1}{n^3}(s_j^T A_j^{-1}(z)s_j)^2 \beta_j(z), \quad a_{n1}^{(2)} = -\frac{1}{n^2}s_j^T \tilde{A}_j(z)s_j \beta_j(z), \]
and
\[ a_{n1}^{(3)} = \frac{1}{n^2}s_j^T A_j^{-1}(z)s_j, \quad a_{n1}^{(4)} = \frac{1}{n}s_j^T A_j^{-1}(z)s_j. \]

Using (2.12), shows that
\[ \beta_j(z) = \beta_j^{tr}(z) - \beta_j(z) \beta_j^{tr}(z) \gamma_j(z), \]
we have
\[ (E_j - E_{j-1})(a_{n1}^{(1)}) = (E_j - E_{j-1})[\frac{1}{n^3}(s_j^T A_j^{-1}(z)s_j)^2 \beta_j^{tr}(z)] - \zeta_n \]
\[ = (E_j - E_{j-1})[\frac{1}{n^2}\gamma_j(z) \beta_j^{tr}(z)] + (E_j - E_{j-1})[\frac{2}{n^2}\gamma_j(z) \beta_j^{tr}(z) tr A_j^{-1}(z)] - \zeta_n, \]
where \( \zeta_n = (E_j - E_{j-1})[\frac{1}{n^2}(s_j^T A_j^{-1}(z)s_j)^2 \beta_j(z) \beta_j^{tr}(z) \gamma_j(z)]. \) This, together with (2.12), shows that
\[ E|\sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})(a_{n1}^{(1)})|^2 = n \sum_{j=1}^{n} E|(E_j - E_{j-1})(a_{n1}^{(1)})|^2 \]
\[ \leq ME|\gamma_1(z)|^4 + ME|\gamma_1(z)|^2 + ME|\gamma_1(z)| \frac{1}{n^2}(s_j^T A_j^{-1}(z)s_j)^2 = O(n^{-\frac{3}{2}}), \]
which gives
\[ \sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})(a_{n1}^{(1)}) \xrightarrow{i.p.} 0. \]

By (2.8) it is a simple matter to verify that
\[ \sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})(a_{n1}^{(3)}) \xrightarrow{i.p.} 0. \]

Appealing to (2.12) we have
\[ E|\sum_{j=1}^{n} (E_j - E_{j-1}) \gamma_j(z) \frac{1}{\sqrt{n}} s_j^T A_j^{-1}(z)s_j \beta_j^{tr}(z)|^2 = O(n^{-1/2}) \]
and
\[ E|\sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n^2}s_j^T \tilde{A}_j(z)s_j \beta_j(z) \beta_j^{tr}(z) \gamma_j(z)\beta_j^{tr}(z)|^2 = O(n^{-1/2}), \]
which, together with (2.20), leads to
\[ \sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})(a_{n1}^{(2)}) = - \sum_{j=1}^{n} E_j[(1 - \beta_j^{tr}(z)) \frac{1}{\sqrt{n}} s_j^T A_j^{-1}(z)s_j] + o_p(1). \]
This ensures that

\[
\sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})(a_{n1}) = \sum_{j=1}^{n} E_j(\beta_j^{tr}(z) - \frac{1}{\sqrt{n}} s_j^T A_j^{-1}(z)s_j) + o_p(1)
\]

(2.21)

\[
= -zm(z) \sum_{j=1}^{n} E_j(- \frac{1}{\sqrt{n}} s_j^T A_j^{-1}(z)s_j) + o_p(1),
\]

because, by (2.17) in [5], (2.7) and (2.10),

\[
E[(\beta_j^{tr}(z) + zm(z))s_j^T A_j^{-1}(z)s_j] = o(1).
\]

(2.22)

Secondly, splitting \( \bar{s} \) into the sum of \( \bar{s}_j \) and \( s_j/n \) further gives

\[
a_{n2} = -\frac{1}{n^2} s_j^T \bar{A}_j(z)s_j \beta_j(z) - \frac{1}{n} s_j^T \bar{A}_j(z)s_j \beta_j(z)
\]

and thus, as in treating \( a_{n1}^{(2)} \), we have

\[
\sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})(a_{n2}) = -\sum_{j=1}^{n} (E_j - E_{j-1})[(1 - \beta_j^{tr}(z)) - \frac{1}{\sqrt{n}} s_j^T A_j^{-1}(z)s_j]
\]

\[
- \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (E_j - E_{j-1})[s_j^T \bar{A}_j(z)\bar{s}_j \beta_j^{tr}(z)] + o_p(1)
\]

\[
= -(1 + zm(z)) \sum_{j=1}^{n} E_j(- \frac{1}{\sqrt{n}} s_j^T A_j^{-1}(z)s_j) + zm(z) \sum_{j=1}^{n} \sqrt{n}E_j(\alpha_j(z)) + o_p(1),
\]

where in the last step we also use the estimate

\[
E[(\beta_j^{tr}(z) + zm(z))\alpha_j(z)]^2 = E[(\beta_j^{tr}(z) + zm(z))\alpha_j(z)]^2 | \sigma(s_i, i \neq j)]
\]

\[
= E[(\beta_j^{tr}(z) + zm(z)]^2 E(|\alpha_j(z)|^2 | \sigma(s_i, i \neq j))] = o(n^{-2})
\]

which is from (2.17) in [5], (2.7) and (2.10).

Consequently, for finite dimension convergence, we need consider only the sum

\[
\sum_{i=1}^{r} a_i \sum_{j=1}^{n} Y_j(z_i) = \sum_{j=1}^{n} \sum_{i=1}^{r} a_i Y_j(z_i),
\]

(2.23)

where

\[
Y_j(z) = -2zm(z)E_j(- \frac{1}{\sqrt{n}} s_j^T A_j^{-1}(z)s_j) + zm(z) \sqrt{n}E_j(\alpha_j(z)).
\]

Recalling \( D_j(z) = A_j^{-1}(z)\bar{s}_j s_j^T A_j^{-1}(z) \) write

\[
\alpha_j(z) = \alpha_j^{(1)}(z) + \alpha_j^{(2)}(z) + \alpha_j^{(3)}(z),
\]

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where
\[
\alpha_j^{(3)}(z) = \frac{1}{n} \sum_{h \neq l} e^T_{1} D_j(z) e_l X_{h j} X_{l j},
\]
and
\[
\alpha_j^{(2)}(z) = \frac{1}{n} \sum_{h=1}^{p} e^T_{h} D_j(z) e_h [X^2_{h j} I(|X_{h j}| \leq \log n) - EX^2_{h j} I(|X_{h j}| \leq \log n)]
\]
and
\[
\alpha_j^{(1)}(z) = \frac{1}{n} \sum_{h=1}^{p} e^T_{h} D_j(z) e_h [X^2_{h j} I(|X_{h j}| > \log n) - EX^2_{h j} I(|X_{h j}| > \log n)].
\]
Lemma 5 and (2.16) show that \( E|\alpha_j^{(3)}(z)|^4 = O(n^{-4}) \). Lemma 2 and (2.16) give \( E|\alpha_j^{(2)}(z)|^4 = O(n^{-4} \log n)^4 \) because
\[
\sum_{j=1}^{n} |\tilde{e}^T_{h} \tilde{A}_j(z) e_h|^k \leq \sum_{j=1}^{n} e^T_{h} A_j^{-1}(z) s_j s_j^T A_j^{-1}(z) e_h |e_h|^k = (s_j^T A_j^{-1}(z) A_j^{-1}(z) s_j)^k,
\]
where \( k = 2 \) or \( 4 \) and \( A_j^{-1}(z) \) denotes the complex conjugate of \( A_j^{-1}(z) \). Since \( EX_{11}^4 I(|X_{11}| > \log n) \rightarrow 0 \) we have \( E|\alpha_j^{(1)}(z)|^2 = o(n^{-2}) \). Therefore we obtain
\[
\sum_{j=1}^{n} E \left| \sum_{i=1}^{r} a_i Y_j(z_i) \right|^2 I(| \sum_{i=1}^{r} a_i Y_j(z_i) | \geq \varepsilon)
\]
\[
\leq 4 \sum_{j=1}^{n} \sum_{h=1}^{4} E |a_i Y_j^{(h)}(z_i)|^2 I(| \sum_{i=1}^{r} a_i Y_j^{(h)}(z_i) | \geq \varepsilon/4)
\]
\[
\leq \frac{M}{\varepsilon^2} \sum_{j=1}^{n} \sum_{h=2}^{4} E |a_i Y_j^{(h)}(z_i)|^4 + 4 \sum_{j=1}^{n} E \left| \sum_{i=1}^{r} a_i Y_j^{(1)}(z_i) \right|^2 \rightarrow 0,
\]
where \( Y_j^{(h)}(z) = zm(z) \sqrt{n} E_j(\alpha_j^{(h)}(z)), h = 1, 2, 3 \) and \( Y_j^{(4)}(z) = -2zm(z) \times E_j (\frac{1}{\sqrt{n}} s_j^T A_j^{-1}(z) s_j) \). Here we also use \( E|Y_j^{(4)}(z)|^4 = O(n^{-2}) \) by (2.10). Thus the condition (ii) of Lemma 3 is satisfied. Hence, the next task is to find, for \( z_1, z_2 \in \mathbb{C} \setminus \mathbb{R} \), the limit in probability of
\[
\sum_{j=1}^{n} E_{j-1}(Y_j(z_1) Y_j(z_2)).
\]
To this end, it is enough to find the limits in probability for the following:

\[(2.25) \quad \frac{1}{n} \sum_{j=1}^{n} E_{j-1}[E_j(s_j^T A_{ij}^{-1}(z_1) \bar{s}_j) E_j(s_j^T A_{ij}^{-1}(z_2) s_j)],\]

\[(2.26) \quad \sum_{j=1}^{n} E_{j-1}[E_j(s_j^T A_{ij}^{-1}(z_1) \bar{s}_j) E_j(\alpha_j(z_2))],\]

\[(2.27) \quad n \sum_{j=1}^{n} E_{j-1}[E_j(\alpha_j(z_1)) E_j(\alpha_j(z_2))].\]

The limits of \[(2.25), (2.26), (2.27)\] and finally \[(2.24)\] will be determined in the subsequent subsections.

2.3. The limit of \[(2.25)\]. Introduce \(A_j^{-1}(z)\) and \(\bar{s}_j\) like \(A_{ij}^{-1}(z)\) and \(s_j\), respectively, but \(A_{ij}^{-1}(z)\) and \(\bar{s}_j\) are now defined by \(s_1, \ldots, s_{j-1}, \bar{s}_j, s_{j+1}, \ldots, s_n\) instead of \(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n\). Here \(\{s_{j+1}, \ldots, s_n\}\) are i.i.d copies of \(s_1\) and independent of \(\{s_j, j = 1, \ldots, n\}\). Therefore \[(2.25)\] is equal to

\[
\frac{1}{n} \sum_{j=1}^{n} tr[E_j(A_{ij}^{-1}(z_1) s_j) E_j(s_j^T A_{ij}^{-1}(z_2))]= \frac{1}{n} \sum_{j=1}^{n} E_j[s_j^T A_{ij}^{-1}(z_2) A_{ij}^{-1}(z_1) \bar{s}_j].
\]

Applying \(\bar{s}_j = \frac{1}{n} \sum_{i \neq j} s_i\) and \[(1.13)\] further gives

\[(2.28) \quad E_j[s_j^T A_{ij}^{-1}(z_2) A_{ij}^{-1}(z_1) \bar{s}_j] = \frac{1}{n} \sum_{i \neq j} E_j[\beta_{ij}(z_2) s_i^T A_{ij}^{-1}(z_2) A_{ij}^{-1}(z_1) \bar{s}_j].\]

The next aim is to replace \(\beta_{ij}(z_2)\) in the equality above by \(\beta_{ij}^T(z_2)\). To this end, consider the case \(i > j\) first. By \[(2.12)\]

\[(2.29) \quad E[|E_j[(\beta_{ij}(z_2) - \beta_{ij}^T(z_2)) s_i^T A_{ij}^{-1}(z_2) A_{ij}^{-1}(z_1) \bar{s}_j]|] = O(n^{-1/2}).\]

Second, when \(i < j\), break \(A_{ij}^{-1}(z_1)\) into the sum of \(A_{ij}^{-1}(z_1)\) and \(A_{ij}^{-1}(z_1) - A_{ij}^{-1}(z_1)\), \(\bar{s}_j\) into the sum of \(\bar{s}_j\) and \(\bar{s}_j - \bar{s}_{ij}\), where \(A_{ij}(z_1) = A_{ij}(z_1) - n^{-1} s_i s_i^T\) and \(\bar{s}_{ij} = \bar{s}_j - s_i/n\). Then, when \(i < j\), with notation

\[
\beta_{ij}(z) = \frac{1}{1 + \frac{1}{n} s_i^T A_{ij}^{-1}(z) s_i},
\]

we have

\[(2.30) \quad E_j[(\beta_{ij}(z_2) - \beta_{ij}^T(z_2)) s_i^T A_{ij}^{-1}(z_2) A_{ij}^{-1}(z_1) \bar{s}_j] = c_{n1} + c_{n2} + c_{n3} + c_{n4},\]

where

\[
c_{n1} = E_j[(\beta_{ij}(z_2) - \beta_{ij}^T(z_2)) s_i^T A_{ij}^{-1}(z_2) A_{ij}^{-1}(z_1) \bar{s}_{ij}],
\]

\[
c_{n2} = \frac{1}{n} E_j[(\beta_{ij}(z_2) - \beta_{ij}^T(z_2)) s_i^T A_{ij}^{-1}(z_2) A_{ij}^{-1}(z_1) s_i],
\]

\[
c_{n3} = \frac{1}{n} E_j[(\beta_{ij}(z_2) - \beta_{ij}^T(z_2)) s_i^T A_{ij}^{-1}(z_2) A_{ij}^{-1}(z_1) s_i],
\]

\[
c_{n4} = \frac{1}{n} E_j[(\beta_{ij}(z_2) - \beta_{ij}^T(z_2)) s_i^T A_{ij}^{-1}(z_2) A_{ij}^{-1}(z_1) \bar{s}_j].
\]
\[ c_{n3} = -\frac{1}{n} E_j[(\beta_{ij}(z_2) - \beta^*_{ij}(z_2)) s_i^T A_{ij}^{-1}(z_2) \overline{A}_{ij}^{-1}(z_1) s_i s_i^T A_{ij}^{-1}(z_1) \beta_{ij}(z_1) \overline{s}_{ij}], \]

and

\[ c_{n4} = -\frac{1}{n^2} E_j[(\beta_{ij}(z_2) - \beta^*_{ij}(z_2)) s_i^T A_{ij}^{-1}(z_2) \overline{A}_{ij}^{-1}(z_1) s_i s_i^T A_{ij}^{-1}(z_1) \beta_{ij}(z_1) s_i]. \]

It follows from (2.12) that \( E|c_{nij}| \leq M n^{-1/2}, j = 1, 2, 3, 4. \) Moreover, note that \( E_j[\beta^*_{ij}(z_2) s_i^T A_{ij}^{-1}(z_2) \overline{A}_{ij}^{-1}(z_1) \overline{s}_{ij}] = 0 \) when \( i > j. \) In what follows we use the notation \( o_{L_1}(1) \) to denote convergence to zero in \( L_1. \) This, together with (2.29) and (2.30), implies that

\[
E_j[s_i^T A_{ij}^{-1}(z_2) \overline{A}_{ij}^{-1}(z_1) \overline{s}_j] = \frac{1}{n} \sum_{i \neq j} E_j[\beta^*_{ij}(z_2) s_i^T A_{ij}^{-1}(z_2) \overline{A}_{ij}^{-1}(z_1) s_i \overline{A}_{ij}^{-1}(z_1) \beta_{ij}(z_1)],
\]

\[= \frac{1}{n} \sum_{i < j} E_j[\beta^*_{ij}(z_2) s_i^T A_{ij}^{-1}(z_2) \overline{A}_{ij}^{-1}(z_1) s_i \overline{A}_{ij}^{-1}(z_1) \beta_{ij}(z_1)] + o_{L_1}(1)
\]

(2.31)

where

\[ d_n_1 = \frac{1}{n^2} \sum_{i < j} E_j[\beta^*_{ij}(z_2) s_i^T A_{ij}^{-1}(z_2) \overline{A}_{ij}^{-1}(z_1) s_i \overline{A}_{ij}^{-1}(z_1) \beta_{ij}(z_1)],\]

\[ d_n_2 = \frac{1}{n} \sum_{i < j} E_j[\beta^*_{ij}(z_2) s_i^T A_{ij}^{-1}(z_2) \overline{A}_{ij}^{-1}(z_1) s_i \overline{A}_{ij}^{-1}(z_1) \beta_{ij}(z_1)] \]

and

\[ d_n_3 = -\frac{1}{n^2} \sum_{i < j} E_j[\beta^*_{ij}(z_2) s_i^T A_{ij}^{-1}(z_2) \overline{A}_{ij}^{-1}(z_1) s_i s_i^T A_{ij}^{-1}(z_1) \beta_{ij}(z_1) \overline{s}_{ij}], \]

Here in the last step we apply \( \overline{s}_j = s_j/n + \overline{s}_{ij} \) first, then use (1.13) and finally split \( \overline{A}_{ij}^{-1}(z_1) \) into two parts as before.

We claim that the terms \( d_{n2} \) and \( d_{n3} \) are both negligible. To see this, we first prove the following estimate

\[ E \left| \frac{1}{n} \sum_{i < j} s_i^T A_{ij}^{-1}(z_2) \overline{A}_{ij}^{-1}(z_1) \overline{s}_j \right|^2 = o(1). \]

(2.32)

Indeed, the left side of (2.32) may be expanded as

\[ \frac{1}{n^2} \sum_{i_1 < j_1, i_2 < j_2} E \left( s_i^T A_{i_1 j_1}^{-1}(z_2) \overline{A}_{i_1 j_1}^{-1}(z_1) \overline{s}_{i_2 j_2} s_i^T A_{i_1 j_1}^{-1}(z_2) \overline{A}_{i_1 j_1}^{-1}(z_1) \overline{s}_{i_2 j_2} \right). \]

(2.33)

From (2.10), the above term corresponding to \( i_1 = i_2 \) is bounded by

\[ \frac{1}{n^2} \sum_{i_1 < j} E \left| s_i^T A_{i_1 j}^{-1}(z_2) \overline{A}_{i_1 j}^{-1}(z_1) \overline{s}_{i_1 j} \right|^2 = O(1/n). \]
To treat the case $i_1 \neq i_2$, we need to further split $A_{i_1 i_2}^{-1}(z_2)$ as the sum of $A_{i_1 i_2}^{-1}(z_2)$ and $A_{i_1 i_2}^{-1}(z_2) - A_{i_1 i_2}^{-1}(z_2)$, where $A_{i_1 i_2}(z_2) = A_{i_1 i_2}(z_2) - n^{-1}s_i s_i^T$. Moreover, both $A_{i_1 i_2}^{-1}(z_1)$ and $\bar{s}_{i_1 j}$ are also needed to be similarly split. To simplify notation, define

$$\beta_{i_1 i_2}(z) = \frac{1}{1 + \frac{1}{n}s_{i_2}^T A_{i_1 i_2}^{-1}(z)s_{i_2}}, \quad \tilde{\beta}_{i_1 i_2}(z) = \frac{1}{1 + \frac{1}{n}s_{i_2}^T A_{i_1 i_2}^{-1}(z)s_{i_2}},$$

and

$$A_{i_1 i_2}(z) = A_{i_1 j}(z) - s_{i_2} s_{i_2}^T, \quad \bar{s}_{i_1 i_2} = \bar{s}_{i_1 j} - \frac{s_{i_2}}{n}, \quad \zeta_{i_2} = s_{i_2}^T A_{i_1 i_2}^{-1}(z) \bar{s}_{i_1 i_2}.$$

By (2.10), (2.11) and (2.14) we have

$$\frac{1}{n} \left| E \left( s_{i_1}^T A_{i_1 i_2}^{-1}(z_2) A_{i_1 j}^{-1}(z_1) s_{i_2} \zeta_{i_2 j} \right) \right|\leq \frac{M}{n} \left| E \left( s_{i_1}^T A_{i_1 i_2}^{-1}(z_2) A_{i_1 j}^{-1}(z_1) \beta_{i_1 i_2}(z_1) s_{i_2} \zeta_{i_2 j} \right) \right| + \frac{M}{n^2} \left| E \left( s_{i_1}^T A_{i_1 i_2}^{-1}(z_2) s_{i_2} s_{i_2}^T A_{i_1 i_2}^{-1}(z_2) \beta_{i_1 i_2}(z_2) A_{i_1 i_2}^{-1}(z_1) \beta_{i_1 i_2}(z_1) s_{i_2} \zeta_{i_2 j} \right) \right|$$

$$\leq \frac{M}{n} \left( E \left| s_{i_1}^T A_{i_1 i_2}^{-1}(z_2) s_{i_2} \right|^2 E \left| \zeta_{i_2 j} \right|^2 \right)^{1/2} + \frac{M}{n^2} \left( E \left| s_{i_1}^T A_{i_1 i_2}^{-1}(z_2) s_{i_2} s_{i_2}^T A_{i_1 i_2}^{-1}(z_2) A_{i_1 i_2}^{-1}(z_1) \bar{s}_{i_1 i_2} \right|^2 E \left| \zeta_{i_2 j} \right|^2 \right)^{1/2} = O(n^{-3/8});$$

$$\frac{1}{n} \left| E \left( s_{i_1}^T A_{i_1 i_2}^{-1}(z_2) A_{i_1 j}^{-1}(z_1) s_{i_2} \zeta_{i_2 j} \right) \right|\leq \frac{M}{n} \left( E \left| s_{i_1}^T A_{i_1 i_2}^{-1}(z_2) s_{i_2} \right|^4 E \left| s_{i_2}^T A_{i_1 i_2}^{-1}(z_2) A_{i_1 i_2}^{-1}(z_1) \bar{s}_{i_1 i_2} \right|^4 \right)^{1/4} \left( E \left| \zeta_{i_2 j} \right|^2 \right)^{1/2} = O(n^{-1/2});$$

$$\frac{1}{n} \left| E \left( s_{i_1}^T A_{i_1 i_2}^{-1}(z_2) A_{i_1 j}^{-1}(z_1) s_{i_2} \zeta_{i_2 j} \right) \right|\leq \frac{M}{n} \left( E \left| s_{i_1}^T A_{i_1 i_2}^{-1}(z_2) A_{i_1 i_2}^{-1}(z_1) \bar{s}_{i_1 i_2} \zeta_{i_2 j} \right|^4 \right)^{1/4} \left( E \left| \zeta_{i_2 j} \right|^2 \right)^{1/2} = O(n^{-1/2});$$
Therefore from (2.10) we obtain
\[
\frac{1}{n^2} E \left( s_{i_1}^T A_{i_1i_2j}^{-1}(z_2) s_{i_2} s_{i_2}^T A_{i_2j}^{-1}(z_2) A_{i_1i_2j}^{-1}(z_2) \right) \\
\times A_{i_1i_2j}^{-1}(z_1) s_{i_2} s_{i_2}^T A_{i_1i_2j}^{-1}(z_1) \beta_{i_1i_2j}(z_1) \bar{s}_{i_1i_2j} \zeta_{i_2j} \right) \\
\leq \frac{M}{n^2} (E |s_{i_1}^T A_{i_1i_2j}^{-1}(z_2) s_{i_2} s_{i_2}^T A_{i_1i_2j}^{-1}(z_1) A_{i_1i_2j}^{-1}(z_1) s_{i_2}|^4 E|s_{i_1} A_{i_1i_2j}^{-1}(z_1) \bar{s}_{i_1i_2j}|^4)^{1/4} \\
\times (E |\zeta_{i_2j}|^2)^{1/2} = O(n^{-3/8}),
\]

The above four estimates, together with the fact that
\[
E \left( s_{i_1}^T A_{i_1i_2j}^{-1}(z_2) A_{i_1i_2j}^{-1}(z_1) \times \bar{s}_{i_1i_2j} \zeta_{i_2j} \right) = 0, \ i_1 \neq i_2,
\]
imply that all terms in (2.33) corresponding to \(i_1 \neq i_2\) are bounded in absolute value by \(M n^{-3/8}\), which ensures (2.32).

Consider the term \(d_{n2}\) now. In view of (2.7) and (2.12) we may substitute \(b_{i_2}(z_2)\) for \(\beta_{i_2j}(z_2)\) in the term \(d_{n2}\) first and then applying (2.32) we conclude that \(E |d_{n2}| = o(1)\). As for the term \(d_{n3}\), it follows from (2.7) and (2.12) that \(\beta_{i_2j}(z_2), \beta_{i_2j}(z_1)\) and \(s_{i_1}^T A_{i_1i_2j}^{-1}(z_2) A_{i_1i_2j}^{-1}(z_1) s_{i_1}\), can be replaced by \(b_{i_2}(z_2), \bar{b}_{i_2}(z_1)\) and \(\frac{1}{n} tr A_{i_1i_2j}^{-1}(z_2) A_{i_1i_2j}^{-1}(z_1)\), respectively, where
\[
b_{i_2}(z) = \frac{1}{1 + \frac{1}{n} tr A_{i_1i_2j}^{-1}(z)},
\]
(note: \(\bar{b}_{i_2}(z) = b_{i_2}(z)\)). Moreover, by an inequality similar to (2.6) we have
\[
|E_j [s_{i_1}^T A_{i_1i_2j}^{-1}(z_1) s_{i_1}] \frac{1}{n} (tr A_{i_1i_2j}^{-1}(z_2) A_{i_1i_2j}^{-1}(z_1) - tr A_{i_1i_2j}^{-1}(z_2) A_{i_1i_2j}^{-1}(z_1))] \\
\leq \frac{M}{n} E_j [s_{i_1}^T A_{i_1i_2j}^{-1}(z_1) s_{i_1}] \\
\]
Therefore from (2.10) we obtain
\[
d_{n3} = -\frac{b_{i_2}(z_2) b_{i_2}(z_1)}{n^2} E_j [tr A_{i_1i_2j}^{-1}(z_2) A_{i_1i_2j}^{-1}(z_1) \sum_{i<j} s_{i}^T A_{i_1i_2j}^{-1}(z_1) \bar{s}_{i_2}] + o_{L_1}(1).
\]
As in (2.32) we may prove that (even simpler)
\[
E \left| \frac{1}{n} \sum_{i<j} s_{i}^T A_{i_1i_2j}^{-1}(z_1) \bar{s}_{i_2} \right|^2 = o(1),
\]
which then implies that \(E |d_{n3}| = o(1)\).
As for \( d_{n1} \), we conclude from (2.7), (2.12) and (2.6) that
\[
d_{n1} = \frac{b_{12}(z_2)b_{12}(z_1)}{n^2} \sum_{i<j} E_j[\text{tr} A_{ij}^{-1}(z_2)A_{ij}^{-1}(z_1)] + o_L(1)
\]
\[
= \frac{b_{12}(z_2)b_{12}(z_1)}{n^2} (j-1)E_j(\text{tr} A_j^{-1}(z_2))E_j(A_j^{-1}(z_1)) + o_L(1).
\]

Summarizing the above, we have thus proved that
\[
E_j(s_j^T A_j^{-1}(z_2))E_j(A_j^{-1}(z_1)s_j)
\]
\[
= \frac{j-1}{n^2} b_{12}(z_2)b_{12}(z_1)E_j(\text{tr} A_j^{-1}(z_2))E_j(A_j^{-1}(z_1)) + o_L(1)
\]
\[
(2.35) = \frac{j-1}{n^2} z_1 z_2 m(z_1)m(z_2) E_j(\text{tr} A_j^{-1}(z_2))E_j(A_j^{-1}(z_1)) + o_L(1),
\]
using the fact that, by (2.17) in [5] and (2.6),
\[
(2.36) b_{12}(z) \rightarrow -zm(z).
\]

2.4. The limit of (2.26). We first present a lemma below, which is necessary for finding the limit of (2.26), for the next subsection and Section 3.

**Lemma 6.** Under the assumptions of Theorem 1, as \( n \rightarrow \infty \),
\[
\max_i \sqrt{n} \left| E(e_i^T A_1^{-1}(z)s_1) \right| \rightarrow 0.
\]

**Proof.** We first prove that for \( i \neq j \), \( \sup_{i,j} \sqrt{n} \left| E(e_j^T A_1^{-1}(z)e_i) \right| \rightarrow 0 \). To this end, write
\[
A_1(z) + zI = \frac{1}{n} \sum_{m=2}^n s_m s_m^T.
\]
Multiplying by \( A_1^{-1}(z) \) from the right on both sides of the above equality gives
\[
I + zA_1^{-1}(z) = \frac{1}{n} \sum_{m=2}^n s_m s_m^T A_m^{-1}(z)\beta_m(z).
\]
Using
\[
(2.38) \beta_m(z) = b_{12}(z) - \beta_m(z)b_{12}(z)\xi_m(z)
\]
we obtain
\[
(2.39) I + zA_1^{-1}(z) = \frac{b_{12}(z)}{n} \sum_{m=2}^n s_m s_m^T A_m^{-1}(z) - \frac{b_{12}(z)}{n} \sum_{m=2}^n s_m s_m^T A_m^{-1}(z)\beta_m(z)\xi_m(z).
\]
It follows that for \( i \neq j \)
\[
(2.40) \quad z \sqrt{n} E(e_j^T A_{21}^{-1}(z) e_i) = b_{12}(z) \sqrt{n} \left( \sum_{m=2}^{n} E(e_j^T A_{m1}^{-1}(z) e_i) - \sum_{m=2}^{n} E(e_j^T s_m s_m^T A_{m1}^{-1}(z) \beta_m(z) \xi_m(z) e_i) \right)
\]
\[
= b_{12}(z) \sqrt{n} \left( E(e_j^T A_{21}^{-1}(z) e_i) - E(e_j^T s_2 s_2^T A_{21}^{-1}(z) \beta_{21}(z) \xi_{21}(z) e_i) \right) + O(n^{-1/2}),
\]
where we also use (2.9) and the fact that, as in (2.8), by Lemma 4 and (2.7),
\[
(2.41) \quad E|\xi_{21}(z)|^k = O(\varepsilon_{2k-4} n^{-1}), \quad k \geq 2.
\]
Here and in what follows (in this lemma) \( O(n^{-1/2}) \) and other bounds are independent of \( i \) and \( j \).

We conclude from (2.9) that
\[
b_{12}(z) \sqrt{n} E(e_j^T A_{21}^{-1}(z) e_i) = b_{12}(z) \sqrt{n} \left[ E(e_j^T A_{21}^{-1}(z) e_i) + E(e_j^T A_{21}^{-1}(z) s_2 s_2^T n A_{21}^{-1}(z) e_i \beta_{21}(z)) \right]
\]
\[
= b_{12}(z) \sqrt{n} E(e_j^T A_{11}^{-1}(z) e_i) + O(n^{-1/2}).
\]

For the second term in (2.40), first, by a martingale method similar to (2.18) and (2.9) we have, for \( e_l = e_i \) or \( e_j \),
\[
E|e_l^T A_{21}^{-1}(z_1) e_j - E(e_l^T A_{21}^{-1}(z_1) e_j)|^2
\]
\[
= E|\sum_{m=3}^{n} (E_m - E_{m-1})[e_l^T (A_{21}^{-1}(z_1) - A_{m21}^{-1}(z_1)) e_j]|^2
\]
\[
\leq \frac{M}{n^2} \sum_{m=3}^{n} E|s_m^T A_{m21}^{-1}(z_1) e_j e_l^T A_{m21}^{-1}(z_1) s_m|^2 = O(n^{-1}),
\]
\[
(2.42)
\]
This and (2.7) ensure that
\[
\frac{1}{n} E[e_j^T A_{21}^{-1}(z) e_i (tr A_{21}^{-1}(z) - E tr A_{21}^{-1}(z))]
\]
\[
= \frac{1}{n} E[(e_j^T A_{21}^{-1}(z) e_i - E e_j^T A_{21}^{-1}(z) e_i)(tr A_{21}^{-1}(z) - E tr A_{21}^{-1}(z))] \]
\[
\leq \frac{M}{n} (E|e_j^T A_{21}^{-1}(z) e_i - E e_j^T A_{21}^{-1}(z) e_i|^2 E|tr A_{21}^{-1}(z) - E tr A_{21}^{-1}(z)|^2)^{1/2} \leq \frac{M}{n}.
\]

Second, appealing to (2.3) gives
\[
E(e_j^T s_2 s_2^T A_{21}^{-1}(z) e_i \gamma_{21}(z)) = E(s_j^T A_{21}^{-1}(z) e_i e_j^T s_2 - e_j^T A_{21}^{-1}(z) e_i \gamma_{21}(z))
\]
\[
= \frac{EX_{11}^4}{n} E(e_j^T A_{21}^{-1}(z) e_i e_j^T A_{21}^{-1}(z) e_j) + \frac{2}{n} E(e_j^T A_{21}^{-1}(z) e_i e_j).
It follows that

\[ \sqrt{n}E(e_j^T s_2 s_2^T A_{21}^{-1}(z)e_1 \xi_{21}(z)) \]

\[ = \sqrt{n}E(e_j^T s_2 s_2^T A_{21}^{-1}(z)e_1 \gamma_{21}(z)) + \sqrt{n}E[e_j^T A_{21}^{-1}(z)e_1 \frac{1}{n}(tr A_{21}^{-1}(z) - Etr A_{21}^{-1}(z))] \]

\[ = O(n^{-1/2}). \]

On the other hand, in view of (2.9) and (2.41) we obtain

\[ \sqrt{n}E(e_j^T s_2 s_2^T A_{21}^{-1}(z)e_1 \beta_{21}(z)\xi_{21}^2(z)) = O(\varepsilon_n). \]

Therefore, by (2.38) we find

\[ \sqrt{n}E(e_j^T s_2 s_2^T A_{21}^{-1}(z)e_1 \beta_{21}(z)\xi_{21}(z)) \]

\[ = \sqrt{n}b_{12}(z) \left[ E(e_j^T s_2 s_2^T A_{21}^{-1}(z)e_1 \xi_{21}(z)) + E(e_j^T s_2 s_2^T A_{21}^{-1}(z)e_1 \beta_{21}(z)\xi_{21}^2(z)) \right] \]

\[ = O(\varepsilon_n). \]

Therefore, combining the above argument with (2.36), we have

(2.43) \[ \sup_{i \neq j} \left| \sqrt{n}E(e_j^T A_{11}^{-1}(z)e_i) \right| \to 0. \]

Next, applying (2.38) two times gives

\[ E(e_i^T A_{11}^{-1}(z_1)s_1) = \frac{1}{n} \sum_{m=2}^n E(e_i^T A_{m1}^{-1}(z_1)s_m \beta_{m1}(z_1)) \]

\[ = \frac{b_{12}(z_1)(n-1)}{n} \left[ - E(e_i^T A_{21}^{-1}(z_1)s_2 \xi_{21}(z_1)) + E(e_i^T A_{21}^{-1}(z_1)s_2 \beta_{21}(z_1)\xi_{21}^2(z_1)) \right]. \]

Obviously, we conclude from (2.41), (2.9) and Hölder’s inequality that

\[ \left| \frac{n-1}{n} E(e_i^T A_{21}^{-1}(z_1)s_2 \beta_{21}(z_1)\xi_{21}^2(z_1)) \right| = O(n^{-1/2} \varepsilon_n), \]

while (2.4), (2.6) and (2.43) yield

\[ \max_i \left| \frac{n-1}{n} E(e_i^T A_{21}^{-1}(z_1)s_2 \xi_{21}(z_1)) \right| \]

\[ = \max_i \left| \frac{EX_{11}^3(n-1)}{n^2} \sum_{j=1}^p E[e_i^T A_{21}^{-1}(z_1)e_j(A_{21}^{-1}(z_1))_{jj}] \right| \]

\[ \leq \frac{|EX_{11}^3|}{n} \max_i \sum_{j \neq i} \left| E[e_i^T A_{11}^{-1}(z_1)e_j(A_{11}^{-1}(z_1))_{jj}] \right| + \frac{M}{n} \]

\[ \leq M|EX_{11}^3| \frac{1}{n} tr A_{11}^{-1}(z_1) \max_{i \neq j} \left| E(e_i^T A_{11}^{-1}(z_1)e_j) \right| + \frac{M}{n} \]

\[ = o(n^{-1/2}). \]
Here we also use the estimate, via (2.7) and (2.42)
\[ E[(e_i^T A_i^{-1}(z_1)e_j - E(e_i^T A_i^{-1}(z_1)e_j))((A_i^{-1}(z_1)))_{jj} - E(A_i^{-1}(z_1))_{jj}] = O(n^{-1}). \]
Thus the proof of (2.37) is complete.

Consider (2.26) now. First, (2.4) indicates that
\[ (2.44) \quad (2.26) \mathcal{E}^3_{11} n \sum_{j=1}^n \sum_{i=1}^p [E_j(D_j(z_2))]_{ii} [E_j(e_i^T A_j^{-1}(z_1)s_j)]. \]
Next we shall prove that \( e_i^T A_j^{-1}(z_1)s_j \) above may be replaced by \( E(e_i^T A_j^{-1}(z_1)s_j) \). Using martingale decompositions as in (2.18) and the fact that \( e_i^T A_j^{-1}(z)s_j = s_j^T A_j^{-1}(z)e_i \), we obtain that
\[
\begin{align*}
\sum_{j=1}^n \sum_{i=1}^p [E_j(D_j(z_2))]_{ii} [E_j(e_i^T A_j^{-1}(z_1)s_j)]
&= \sum_{j=1}^n \sum_{i=1}^p [E_j(e_i^T A_j^{-1}(z_1)s_j)]
\end{align*}
\]
where
\[
g_{nm}(z) = \sum_{m \neq j} (E_m - E_{m-1})(\theta_{ijm}(z)), \quad \theta_{ijm}(z) = e_i^T A_j^{-1}(z)s_j - E(e_i^T A_j^{-1}(z)s_j),
\]
and
\[
\theta_{ijm}(z) = e_i^T A_j^{-1}(z)s_j - e_i^T A_{jm}(z)s_{jm} = \left[ -\frac{1}{n^2} e_i^T A_{jm}(z)s_m s_m^T A_{jm}^{-1}(z) s_{jm} + \frac{1}{n} e_i^T A_{jm}^{-1}(z)s_m \right].
\]
Here one should notice that \( \theta_{ij}(z) \) and \( g_{nm}(z) \) are the same.

As in (2.17), one can verify that
\[ (2.45) \quad E|n^{-1} e_i^T A_{jm}(z)s_m|^k = O(n^{-k}), \quad k = 2 or 4, \quad E|n^{-1} e_i^T A_{jm}(z)s_m|^8 = O(n^{-6}). \]
Thus, for \( k = 2 or 4 \), via (2.10),
\[ E\left|\frac{1}{n} e_i^T A_{jm}(z)s_m s_m^T A_{jm}^{-1}(z)s_{jm}\right|^k = O(n^{-2-\varepsilon_n k^{-2}}). \]
and, via (2.8),
\[ E\left|\frac{1}{n^2} e_i^T A_{jm}^{-1}(z_1)s_m s_m^T A_{jm}^{-1}(z)s_m\right|^k = O(n^{-\frac{3-k^2}{2}}). \]
These yield that \( E|\theta_{ijm}(z)|^2 = O(n^{-2}), \quad E|\theta_{ijm}(z)|^4 = O(n^{-2}\varepsilon_n) \) and then
\[ (2.46) \quad E|g_{nm}(z)|^2 = O(n^{-1}), \quad E|g_{nm}(z)|^4 = O(n^{-1}\varepsilon_n). \]
Therefore
\[(2.47) \quad [E \sum_{i=1}^{p} |[E_j(D_j(z_2))]_{ii} E_j(\theta_{ij}(z_1))|^2 \leq \sum_{i=1}^{p} E|e_i^T A_j^{-1}(z_2)s_j|^2 \sum_{i=1}^{p} E|\bar{s}_j^T A_j^{-1}(z_2)e_i E_j(\theta_{ij}(z_1))|^2 \leq M \sum_{i=1}^{p} \frac{|E|g_{nm_2}(z_2)|^4 E|g_{nm_1}(z_1)|^4|^{1/2}}{n} + M \sum_{i=1}^{p} |E(\bar{s}_j^T A_j^{-1}(z_2)e_i)|^2 E|g_{nm_1}(z_1)|^2 = O(\varepsilon_n).\]

Here by (2.16)
\[\sum_{i=1}^{p} |E(\bar{s}_j^T A_j^{-1}(z_2)e_i)|^2 E|g_{nm_2}(z_2)|^2 \leq \frac{M}{n} \sum_{i=1}^{p} |E(\bar{s}_j^T A_j^{-1}(z_2)e_i)|^2 \leq \frac{M}{n} E(\bar{s}_j^T A_j^{-1}(z_2)A_j^{-1}(z_2)s_j) \leq \frac{M}{n}.
\]

Thus, \(e_i^T A_j^{-1}(z_1)s_j\) may be replaced by \(E(e_i^T A_j^{-1}(z_1)s_j)\), as expected.

In addition, by (2.16) and (2.37)
\[E \sum_{i=1}^{p} |[E_j(D_j(z_2))]_{ii} E(e_i^T A_j^{-1}(z_1)s_j)| \leq E \sum_{i=1}^{p} |E_j(A_j^{-1}(z_2)s_j\bar{s}_j^T A_j^{-1}(z_2))|_{ii} |E(e_i^T A_j^{-1}(z_1)s_j)| \leq \max_i |E(e_i^T A_1^{-1}(z_1)s_1)| E(\bar{s}_j^T A_j^{-1}(z_2)A_j^{-1}(z_2)s_j) \to 0.
\]

It follows from (2.47) and (2.48) that
\[E \sum_{i=1}^{p} |[E_j(D_j(z_2))]_{ii} E_j(e_i^T A_j^{-1}(z_1)s_j)| \to 0,
\]
which then ensures that (2.26) converges to zero in probability.

2.5. The limit of (2.27). (2.3) shows that (2.27) is equal to (2.50)
\[\frac{E|X_{11}|^4}{n} - 3 \sum_{j=1}^{n} \sum_{i=1}^{p} E_j(D_j(z_1))_{ii} E_j(D_j(z_2))_{ii} + \frac{2}{n} \sum_{j=1}^{n} \text{tr} E_j(D_j(z_1)) E_j(D_j(z_2)).\]

As we shall see, the above first term converges to zero in probability and the second term has a close connection with (2.25).
Consider the second term of (2.50) first. Write
\[
\begin{align*}
tr E_j(D_j(z_1)) E_j(D_j(z_2)) &= E_j \left[ \tilde{s}_j^T A_j^{-1}(z_1) \tilde{A}_j^{-1}(z_2) \tilde{s}_j \tilde{s}_j^T A_j^{-1}(z_2) A_j^{-1}(z_1) \tilde{s}_j \right] \\
&\hspace{1em} = E_j \left[ \tilde{s}_j^T A_j^{-1}(z_1) \tilde{A}_j^{-1}(z_2) \tilde{s}_j \tilde{s}_j^T A_j^{-1}(z_2) E_j (A_j^{-1}(z_1) \tilde{s}_j) \right] + f_n
\end{align*}
\]
where
\[
f_n = E_j \left[ \tilde{s}_j^T A_j^{-1}(z_1) \tilde{A}_j^{-1}(z_2) \tilde{s}_j \tilde{s}_j^T A_j^{-1}(z_2) (A_j^{-1}(z_1) \tilde{s}_j - E_j (A_j^{-1}(z_1) \tilde{s}_j)) \right].
\]

We claim that
\[
(2.52) \quad E|f_n| = o(1).
\]
To see this, let \( E_{ij} = E(\cdot | s_1, \ldots, s_i, \tilde{s}_{j+1}, \ldots, \tilde{s}_n) \). Then, recalling the definitions of \( A_j^{-1}(z) \) and \( \tilde{s}_j \) we have
\[
\begin{align*}
\tilde{s}_j^T A_j^{-1}(z_2) (A_j^{-1}(z_1) \tilde{s}_j - E_{jj} (A_j^{-1}(z_1) \tilde{s}_j))
&= \sum_{i=j+1}^n E_{ij} \left[ \tilde{s}_j^T A_j^{-1}(z_2) A_j^{-1}(z_1) \tilde{s}_j \right] - E_{(i-1)j} \left[ \tilde{s}_j^T A_j^{-1}(z_2) A_j^{-1}(z_1) \tilde{s}_j \right] \\
&= \sum_{i=j+1}^n (E_{ij} - E_{(i-1)j}) \left[ \tilde{s}_j^T A_j^{-1}(z_2) A_j^{-1}(z_1) \tilde{s}_j - \tilde{s}_j^T A_j^{-1}(z_2) A_j^{-1}(z_1) \tilde{s}_{ij} \right] \\
&= f_{n1} + f_{n2}
\end{align*}
\]
where
\[
f_{n1} = \frac{1}{n} \sum_{i=j+1}^n (E_{ij} - E_{(i-1)j}) \left[ \tilde{s}_j^T A_j^{-1}(z_2) A_j^{-1}(z_1) s_i \beta_{ij}(z_1) \right]
\]
and
\[
f_{n2} = -\frac{1}{n} \sum_{i=j+1}^n (E_{ij} - E_{(i-1)j}) \left[ \tilde{s}_j^T A_j^{-1}(z_2) A_j^{-1}(z_1) s_i s_i^T A_j^{-1}(z_1) \tilde{s}_{ij} \beta_{ij}(z_1) \right].
\]
Note that \( \tilde{s}_j \) is independent of \( s_i \) for \( i > j \). Then applying (2.10) yields
\[
E|f_{n1}|^2 \leq \frac{M}{n^2} \sum_{i=j+1}^n E|\tilde{s}_j^T A_j^{-1}(z_2) A_j^{-1}(z_1) s_i|^2 = O \left( \frac{1}{n} \right).
\]
and

\[ E|f_{n2}|^2 \leq \frac{M}{n^2} \sum_{i=j+1}^{n} E|\tilde{s}_j^T \tilde{A}_j^{-1}(z_2) A_{ij}^{-1}(z_1) \tilde{s}_i \tilde{s}_j^T A_{ij}^{-1}(z_1) \tilde{s}_i|^2 \]

\[ \leq \frac{M}{n^2} \sum_{i=j+1}^{n} (E|\tilde{s}_j^T \tilde{A}_j^{-1}(z_2) A_{ij}^{-1}(z_1) \tilde{s}_i|^4 E|\tilde{s}_j^T A_{ij}^{-1}(z_1) \tilde{s}_i|^4)^{1/2} \]

\[ = O\left(\frac{1}{n}\right), \]

which ensures that

\[ E|\tilde{s}_j^T A_{j}^{-1}(z_2) (A_{j}^{-1}(z_1) \tilde{s}_j - E_{jj}(A_{j}^{-1}(z_1) \tilde{s}_j))|^2 = O\left(\frac{1}{n}\right). \]

So (2.52) follows from the above estimate and

\[ E|\tilde{s}_j^T A_{j}^{-1}(z_1) A_{j}^{-1}(z_2) \tilde{s}_j|^2 = O(1), \]

which may be obtained immediately by checking the argument of (2.16).

As in (2.52) we may also prove that

\[ E\left[ \left| E_j\left[ \tilde{s}_j^T A_{j}^{-1}(z_1) \tilde{s}_j \right] \tilde{s}_j^T A_{j}^{-1}(z_2) \tilde{s}_j \right] - E_j(\tilde{s}_j^T A_{j}^{-1}(z_2)) \right| \]

\[ E_j(A_{j}^{-1}(z_1) \tilde{s}_j) \right| = o(1). \]

Therefore, combining (2.51), (2.52), (2.53) with (2.35) we have

\[ tr E_j(D_j(z_1)) E_j(D_j(z_2)) \]

\[ = E_j\left[ \tilde{s}_j^T A_{j}^{-1}(z_1) A_{j}^{-1}(z_2) \tilde{s}_j \right] E_j(\tilde{s}_j^T A_{j}^{-1}(z_2)) E_j(A_{j}^{-1}(z_1) \tilde{s}_j) \]

\[ = E_j(\tilde{s}_j^T A_{j}^{-1}(z_1)) E_j(\tilde{s}_j^T A_{j}^{-1}(z_2) \tilde{s}_j) E_j(\tilde{s}_j^T A_{j}^{-1}(z_2)) E_j(\tilde{s}_j^T A_{j}^{-1}(z_1) \tilde{s}_j) \]

\[ + o_{L_1}(1) \]

\[ (2.54) = \frac{(j-1)^2}{n^4} z_j^2 z_j^2 m^2(z_1) m^2(z_2) [E_j(tr A_{j}^{-1}(z_2)) E_j(A_{j}^{-1}(z_1))]^2 + o_{L_1}(1). \]

We now turn to the first term in (2.50) and claim that

\[ (2.55) \]

\[ \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} E_j(D_j(z_1))_{ii} E_j(D_j(z_2))_{ii} \rightarrow^{i.p.} 0. \]

Indeed, recalling that \( \theta_{ij}(z) = e_i^T A_j^{-1}(z) \tilde{s}_j - E(e_i^T A_j^{-1}(z) \tilde{s}_j) \), we have

\[ E\left[ \sum_{i=1}^{p} E_j(D_j(z_2))_{ii} E_j \left( \theta_{ij}(z_1) \tilde{s}_j^T A_j^{-1}(z_1) e_i \right) \right] \leq \]

\[ \]
implies that
\[
\sum_{i=1}^{p} E\{E_j(D_j(z_2))_{ii}E_j(\theta_{ij}(z_1))^2\} + \sum_{i=1}^{p} E\{E_j(D_j(z_2))_{ii}E_j(\theta_{ij}(z_1))E(s_j^T A_j^{-1}(z_1)e_i)\}.
\]

The second term above is not greater than
\[
\max_i |E(s_j^T A_j^{-1}(z_1)e_i)| \sum_{i=1}^{p} E\{E_j(D_j(z_2))_{ii}E_j(\theta_{ij}(z_1))|,
\]
which converges to zero by (2.47) and (2.37). On the other hand, by (2.16) and (2.46)
\[
\left(\sum_{i=1}^{p} E|E_j(D_j(z_2))_{ii}E_j(\theta_{ij}(z_1))^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{p} E|E_j(D_j(z_2))_{ii}|^{\frac{1}{2}} \sum_{i=1}^{p} E|\theta_{ij}(z_1)|^{4}
\]
\[
\leq E\left(\sum_{i=1}^{p} s_j^T A_j^{-1}(z_1)e_i e_i^T A_j^{-1}(z_1)s_j\right) \sum_{i=1}^{p} E|g_{nm}(z_1)|^4 = O(\varepsilon_n).
\]
Moreover, it follows from Lemma 6 and (2.49) that
\[
E\left|\sum_{i=1}^{p} E_j(D_j(z_2))_{ii}E_j(e_i^T A_j^{-1}(z_1)s_j)E_j(s_j^T A_j^{-1}(z_1)e_i)\right|
\]
\[
\leq \max_i |E(e_i^T A_j^{-1}(z_1)s_j)| \sum_{i=1}^{p} E|E_j(D_j(z_2))_{ii}E_j(s_j^T A_j^{-1}(z_1)e_i)| \to 0.
\]
Consequently, the proof of (2.55) is complete.

2.6. The limit of (2.24). Note that
\[
tr E_j(A_j^{-1}(z_2))E_j(A_j^{-1}(z_1))\left[1 - \frac{(j - 1)p}{n^2} \frac{m_n(z_1)m_n(z_2)}{(1 + m_n(z_1))(1 + m_n(z_2))}\right]
\]
(2.56) = \frac{p}{z_1z_2(1 + m_n(z_1))(1 + m_n(z_2))} + l_n,
where \(E|l_n| \leq M\sqrt{n}\) (see (2.18) in [3]). Obviously, \(m_n(z) \to m(z)\). This implies that
\[
\frac{(j - 1)z_1z_2m(z_1)m(z_2)}{n^2} tr E_j(A_j^{-1}(z_2))E_j(A_j^{-1}(z_1))
\]
\[
= \frac{z_1z_2(1 + m(z_1))(1 + m(z_2))}{n^2} \sum_{i=1}^{p} E_j(A_j^{-1}(z_2))E_j(A_j^{-1}(z_1)) - 1 + o_{L}(1),
\]
which, together with (2.35) and (2.54), leads to
\[
4tr[E_j(A_j^{-1}(z_1)s_j)E_j(s_j^T A_j^{-1}(z_2))] + 2tr E_j(D_j(z_1))E_j(D_j(z_2))
\]
\[
= \frac{4(j - 1)z_1z_2m(z_1)m(z_2)}{n^2} E_j(tr A_j^{-1}(z_2))E_j(A_j^{-1}(z_1))
\]
\[+ \frac{2(j - 1)^2 z_i^2 z_j^2 m^2(z_1) m^2(z_2)}{n^4} [E_j(trA_j^{-1}(z_2))E_j(A_j^{-1}(z_1))]^2 + o_L(1)\]

\[= -2 + 2z_1^2 z_2^2 (1 + m(z_1))^2 (1 + m(z_2))^2 \frac{[trE_j(A_j^{-1}(z_1))E_j(A_j^{-1}(z_2))]^2}{p^2} + o_L(1).\]

Further, we conclude from (2.56) that

\[\frac{1}{np^2} \sum_{j=1}^{n} [trE_j(A_j^{-1}(z_1))E_j(A_j^{-1}(z_2))]^2\]

\[= \frac{1}{z_1^2 z_2^2 (1 + m(z_1))^2 (1 + m(z_2))^2} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{(1 - \frac{(j-1)p}{n^2} \frac{m(z_1)m(z_2)}{(1+m(z_1))(1+m(z_2))})^2} + o_p(1)\]

\[\text{i.p.,} \quad \frac{1}{z_1^2 z_2^2 (1 + m(z_1))^2 (1 + m(z_2))^2} \int_{0}^{1} \frac{dx}{(1 - \frac{cm(z_1)m(z_2)}{(1+m(z_1))(1+m(z_2))})^2}\]

\[= \frac{1}{z_1^2 z_2^2 (1 + m(z_1))(1 + m(z_2))[(1 + m(z_1))(1 + m(z_2)) - cm(z_1)m(z_2)]}.\]

It follows that

\[(2.24) = z_1 z_2 m(z_1) m(z_2) \frac{1}{n} \sum_{j=1}^{n} \left[ 4tr[E_j(A_j^{-1}(z_1)s_j)E_j(s_j^T A_j^{-1}(z_2))] \right.\]

\[\left. + 2trE_j(D_j(z_1))E_j(D_j(z_2)) \right] + o_p(1)\]

\[(2.57) \quad \text{i.p.,} \quad \frac{2cz_1 z_2 m^2(z_1) m^2(z_2)}{(1 + m(z_1))(1 + m(z_2)) - cm(z_1) m(z_2)}.\]

3. **Tightness of \( \tilde{M}_n^{(1)}(z) \) and Convergence of \( M_n^{(2)}(z) \)**

First, we proceed to prove the tightness of \( \tilde{M}_n^{(1)}(z) \) for \( z \in \mathcal{C} \), which is a truncated version of \( M_n(z) \) as in (1.9). By (2.10) we have

\[E\left| \sum_{i=1}^{m} a_i \sum_{j=1}^{n} Y_j(z_i) \right|^2 \leq \sum_{j=1}^{n} E\left| \sum_{i=1}^{m} a_i Y_j(z_i) \right|^2 \leq M, \quad v_0 = 3z_i,\]

which ensures that Condition (i) of Theorem 12.3 of [6] is satisfied, as pointed out in [5]. Here \( Y_j(z) \) is defined in (2.23). Condition (ii) of Theorem 12.3 of [6] will be verified if the following holds,

\[E \frac{|M_n^{(1)}(z_1) - M_n^{(1)}(z_2)|^2}{|z_1 - z_2|^2} \leq M, \quad \text{for } z_1, z_2 \in \mathcal{C}_n^+ \cup \mathcal{C}_n^- .\]
In the sequel, since $C_n^+$ and $C_n^-$ are symmetric we shall prove the above inequality on $C_n^+$ only. Throughout this section, all bounds including $O(\cdot)$ and $o(\cdot)$ expressions hold uniformly for $z \in C_n^+$.

In view of our truncation steps, (1.9a) and (1.9b) in \cite{5} apply to our case as well. That is, for any $\eta_1 > (1 + \sqrt{c})^2$, $0 < \eta_2 < I(0, 1)(c)(1 - \sqrt{c})^2$ and any positive $l$

\begin{equation}
(3.1) \quad P(\|S\| \geq \eta_1) = o(n^{-l}), \quad P(\lambda_{\min}(S) \leq \eta_2) = o(n^{-l})
\end{equation}

Note that when either $z \in C_n$ or $z \in C_t$ and $u_t < 0$, $\|A_j^{-1}(z)\|$ is bounded in $n$. But this is not the case for $z \in C_r$ or $z \in C_t$ and $u_t > 0$. In general, for $z \in C_n^+$, we have

\begin{equation}
(3.2) \quad \|A_j^{-1}(z)\| \leq M + v^{-1}I(\|A_j\| \geq h_r \text{ or } \lambda_{\min}(A_j) \leq h_l).
\end{equation}

Here $A_j = S - s_j s_j^T$, $h_r \in ((1 + \sqrt{c})^2, u_r)$ and $h_l \in (u_t, (1 - \sqrt{c})^2)$.

As in Section 2.1, now write

\begin{equation}
\frac{M_n^{(1)}(z_1) - M_n^{(1)}(z_2)}{z_1 - z_2} = \sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})[\bar{s}^T A^{-1}(z_1)A^{-1}(z_2)\bar{s} - \bar{s}_j^T A_j^{-1}(z_1)A_j^{-1}(z_2)\bar{s}_j].
\end{equation}

Moreover, expanding the above difference we get

$$\bar{s}^T A^{-1}(z_1)A^{-1}(z_2)\bar{s} - \bar{s}_j^T A_j^{-1}(z_1)A_j^{-1}(z_2)\bar{s}_j = q_{n1} + q_{n2} + q_{n3},$$

where

$$q_{n1} = (\bar{s} - \bar{s}_j)A^{-1}(z_1)A^{-1}(z_2)\bar{s}, \quad q_{n2} = \bar{s}_j^T (A^{-1}(z_1)A^{-1}(z_2) - A_j^{-1}(z_1)A_j^{-1}(z_2))\bar{s},$$

and

$$q_{n3} = \bar{s}_j^T A_j^{-1}(z_1)A_j^{-1}(z_2)(\bar{s} - \bar{s}_j).$$

It follows from (1.8), (2.10), (3.2) and (3.1) that

$$E|\sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})q_{n3}|^2 \leq \frac{1}{n} \sum_{j=1}^{n} E|\bar{s}_j^T A_j^{-1}(z_1)A_j^{-1}(z_2)\bar{s}_j|^2 \leq M + n^8 \rho_n^{-4}P(\|A_1\| \geq h_r \text{ or } \lambda_{\min}(A_1) \leq h_l) \leq M,$$

where we use, on the event ($\|A_j\| \geq h_r \text{ or } \lambda_{\min}(A_j) \leq h_l$),

\begin{equation}
|\bar{s}_j^T A_j^{-1}(z_1)A_j^{-1}(z_2)\bar{s}_j| \leq \|\bar{s}_j\|\|\bar{s}_j\|\|A_j^{-1}(z_1)A_j^{-1}(z_2)\| \leq Mv^{-2}n^2 \leq n^4 \rho_n^{-2},
\end{equation}

by (2.1).
For $q_{n2}$, expanding its difference term by term we have

$$q_{n2} = q_{n2}^{(1)} + \cdots + q_{n2}^{(6)},$$

where

$$q_{n2}^{(1)} = \frac{1}{n^2} \bar{s}_j^T \beta_j(z_1) \beta_j(z_2) \bar{A}_j(z_1) \bar{A}_j(z_2) \bar{s}_j,$$

$$q_{n2}^{(2)} = -\frac{1}{n} \bar{s}_j^T \beta_j(z_1) \bar{A}_j(z_1) \tilde{A}_j(z_2) \bar{s}_j,$$

$$q_{n2}^{(3)} = -\frac{1}{n} \bar{s}_j^T \beta_j(z_2) \bar{A}_j^{-1}(z_1) \bar{A}_j(z_2) \bar{s}_j,$$

$$q_{n2}^{(4)} = \frac{1}{n^3} \bar{s}_j^T \beta_j(z_1) \beta_j(z_2) \bar{A}_j(z_1) \bar{A}_j^{-1}(z_2) \bar{s}_j,$$

and

$$q_{n2}^{(5)} = -\frac{1}{n^2} \bar{s}_j^T \beta_j(z_1) \bar{A}_j(z_1) \bar{A}_j^{-1}(z_2) \bar{s}_j,$$

$$q_{n2}^{(6)} = -\frac{1}{n^2} \bar{s}_j^T \beta_j(z_2) \bar{A}_j^{-1}(z_1) \bar{A}_j(z_2) \bar{s}_j.$$

We conclude from (2.8), (2.10), (3.1), (3.2) and (3.4) that

$$E|\sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1}) q_{n2}^{(1)}|^2 \leq M + v^{-12} \sum (||S|| \geq h_r \text{ or } \lambda_{\text{min}}(A_1) \leq h_l) \leq M,$$

where we use, on the event $(||S|| \geq h_r \text{ or } \lambda_{\text{min}}(A_1) \leq h_l),

$$|\beta_j(z)| = |1 - n^{-1} s_j^T A^{-1}(z) s_j| \leq 1 + n^{-1} v^{-1} ||s_j||^2 \leq M v^{-1} n,$$

by (2.5). Obviously, this argument also works for $q_{n2}^{(j)}, j = 2, \cdots, 6$. Moreover, we may split $q_{n1}$ further and apply the above argument to conclude that

$$E|\sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1}) q_{n1}|^2 \leq M.$$

Here the details are skipped.

Next, consider $M_{n}^{(2)}(z)$. By $\bar{s} = n^{-1} \sum_{i=1}^{n} s_i$, (1.13) and an equality similar to (2.38) we obtain

$$\sqrt{n} E(\bar{s}^T A^{-1}(z) \bar{s}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E(\beta_i(z)s_i^T A^{-1}(z) s_i)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E(\beta_i(z) s_i^T A^{-1}(z) \bar{s}_i) + \frac{1}{n^{3/2}} \sum_{i=1}^{n} E(\beta_i(z) s_i^T A^{-1}(z) s_i)$$

$$= \frac{b_1(z)}{n^{3/2}} \sum_{i=1}^{n} E(tr A^{-1}(z)) + t_{n1} + t_{n2},$$
where
\[
 t_{n1} = -b_1(z) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E(\beta(z)\xi_i(z) s_i^T A_i^{-1}(z)s_i), \\
 t_{n2} = -b_1(z) \frac{1}{n^{3/2}} \sum_{i=1}^{n} E(\beta(z)\xi_i(z) s_i^T A_i^{-1}(z)s_i).
\]

Again, using an equality similar to (2.38) gives
\[
 t_{n1} = t_{n1}^{(1)} + t_{n1}^{(2)}, \quad t_{n2} = t_{n2}^{(1)} + t_{n2}^{(2)},
\]
where
\[
 t_{n1}^{(1)} = -b_1^2(z) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E(\xi_i(z) s_i^T A_i^{-1}(z)s_i), \quad t_{n1}^{(2)} = b_1^2(z) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E(\beta(z)\xi_i(z) s_i^T A_i^{-1}(z)s_i)
\]
and
\[
 t_{n2}^{(1)} = -\frac{b_1^2(z)}{n^{3/2}} \sum_{i=1}^{n} E(\xi_i(z) s_i^T A_i^{-1}(z)s_i), \quad t_{n2}^{(2)} = \frac{b_1^2(z)}{n^{3/2}} \sum_{i=1}^{n} E(\beta(z)\xi_i(z) s_i^T A_i^{-1}(z)s_i).
\]

Note that \(|b_1(z)| \leq M\) for \(z \in \mathcal{C}_n\) (see three lines below (3.6) in [5]). It follows from (2.7), (2.8), (2.10), (3.1), (3.2) and (3.5) that
\[
 |t_{n1}^{(2)}| \leq M\varepsilon_n + M n^{10} \rho^{-4} P(\| S \| \geq h_r \text{ or } \lambda_{\min}(A_1) \leq h_l) \leq M\varepsilon_n,
\]
because \(|\beta(z)\xi^2(z) s_i^T A_i^{-1}(z)s_i| \leq n^5 v^{-4}\) on the event \((\| S \| \geq h_r \text{ or } \lambda_{\min}(A_1) \leq h_l)\). This argument clearly applies to \(t_{n2}^{(2)}\) as well, and so \(|t_{n2}^{(2)}| \leq M\varepsilon_n\). (2.4) shows that
\[
 |t_{n1}^{(1)}| = | -\frac{b_1^2(z)EX^3_{11}}{\sqrt{n}} \sum_{m=1}^{p} E(e_m^T A_1^{-1}(z)e_m e_m^T A_1^{-1}(z)s_1)|
\]
\[
 = | -\frac{b_1^2(z)EX^3_{11}}{\sqrt{n}} \sum_{m=1}^{p} E(e_m^T A_1^{-1}(z)s_1)| + o(1)
\]
\[
 \leq M|b_1^2(z)EX^3_{11} \frac{1}{n} Etr(A_1^{-1}(z))| \max_m \sqrt{n}|E(e_m^T A_1^{-1}(z)s_1)| + o(1)
\]
\[
 = o(1),
\]
where we make use of the facts that by (2.37), (3.1) and (3.2),
\[
 | \max_m \sqrt{n}|E(e_m^T A_1^{-1}(z)s_1)| = o(1),
\]
and that by (2.32), (2.46), (3.1) and (3.2),
\[
 E|e_m^T A_1^{-1}(z)e_m - E(e_m^T A_1^{-1}(z)e_m))(e_m^T A_1^{-1}(z)s_1 - E(e_m^T A_1^{-1}(z)s_1))| \leq Mn^{-1} + 4v^{-2}nP(\| A_1 \| \geq h_r \text{ or } \lambda_{\min}(A_1) \leq h_l) = O(n^{-1}).
\]
Note that \(E(\xi(z)s^TA^{-1}(z)s) = E\gamma^2(z) + n^{-2}E(trA^{-1}_i(z) - EtrA^{-1}_i(z))^2\) and then applying (2.7), (2.8), (3.1) and (3.2) gives \(t_{n2}^{(1)} = O(n^{-1/2})\).

Summarizing the above we obtain
\[
\sqrt{n}E(s^TA^{-1}(z)s) = \frac{b_1(z)}{n^{1/2}}E(trA^{-1}_i(z)) + o(1).
\]

Moreover, it is proven in Section 4 of [5] that \(n(EtrA^{-1}_i(z)/n - c_nm_n(z))\) is bounded for \(z \in C_n\). In addition, by (2.6), (3.1) and (3.2) we have
\[
\sqrt{n}\left|\frac{EtrA^{-1}_i(z)}{n} - \frac{EtrA^{-1}_i(z)}{n}\right| \leq \frac{M}{\sqrt{n}}.
\]

It follows that \(n(EtrA^{-1}_i(z)/n - c_nm_n(z))\) is bounded. This, together with the boundedness of \(b_1(z)\), shows that
\[
\sup_{z \in C_n} \sqrt{n}(E_s^TA^{-1}(z)s - \frac{c_nm_n(z)}{1 + c_nm_n(z)}) \to 0.
\]

4. THE PROOFS OF LEMMA 1, THEOREM 1 AND THEOREM 2

Proof of Lemma 1 To finish Lemma 1 \(s^T\bar{s} - c_n\) needs to be written as a sum of martingale difference sequence so that we can get a central limit theorem for \(s^T\bar{s} - c_n\) and, more importantly, obtain the asymptotic covariance between \(s^T\bar{s} - c_n\) and \(s^T\bar{A}^{-1}(z)s\).

Thus write
\[
\sqrt{n}(s^T\bar{s} - c_n) = \sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})(s^T\bar{s}) = \sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})(s^T\bar{s} - \bar{s}_j^T\bar{s}_j)
\]

\[(4.1) = \sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})(\frac{s^T_sj}{n} + \frac{s^T_sj}{n^2}) = \frac{2}{\sqrt{n}} \sum_{j=1}^{n} E_j(\bar{s}_j^T\bar{s}_j) + o_p(1),\]

because
\[
E|\sqrt{n} \sum_{j=1}^{n} (E_j - E_{j-1})(\frac{s^T_sj}{n^2})|^2 = \frac{1}{n^3} \sum_{j=1}^{n} E|E_j(\bar{s}_j^T\bar{s}_j) - p|^2 = O\left(\frac{1}{n}\right).
\]

From (2.10) we have
\[
\sum_{j=1}^{n} E\left|\frac{1}{\sqrt{n}}E_j(\bar{s}_j^T\bar{s}_j)\right|^2 I\left(\frac{1}{\sqrt{n}}E_j(\bar{s}_j^T\bar{s}_j) \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2} \sum_{j=1}^{n} E\left|\frac{1}{\sqrt{n}}E_j(\bar{s}_j^T\bar{s}_j)\right|^4 = O(n^{-1}),
\]
which implies Condition (ii) of Lemma 3. Look at Condition (i) of Lemma 3 next. It is easily seen that
\[ E_{j-1}[E_j(\bar{s}_j^T \bar{s}_j)]^2 = E_j(\bar{s}_j^T)E_j(\bar{s}_j) = \frac{1}{n^2} \sum_{k_1 < j, k_2 < j} \bar{s}_{k_1}^T \bar{s}_{k_2}. \]

Furthermore, for the term corresponding to \( k_1 = k_2 \), we have
\[ E\left\{ \frac{1}{n} \sum_{k_1 < j} [s_{k_1}^T s_{k_1} - E(s_{k_1}^T s_{k_1})] \right\}^2 = \frac{1}{n^4} \sum_{k_1 < j} E[s_{k_1}^T s_{k_1} - E(s_{k_1}^T s_{k_1})]^2 = O\left( \frac{1}{n^2} \right). \]

On the other hand, when \( k_1 \neq k_2 \)
\[ E\left\{ \frac{1}{n} \sum_{k_1 \neq k_2} s_{k_1}^T s_{k_2} \right\}^2 = \frac{1}{n^4} \sum_{k_1 \neq k_2} E[s_{k_1}^T s_{k_2} s_{k_1}^T s_{k_2}] = \frac{2}{n^4} \sum_{k_1 \neq k_2} E(s_{k_1}^T s_{k_2})^2 = O\left( \frac{1}{n} \right). \]

It follows that
\begin{equation}
\frac{4}{n} \sum_{j=1}^n E_{j-1}[E_j(\bar{s}_j^T \bar{s}_j)]^2 = \frac{4}{n} \sum_{j=1}^n c(j-1) \frac{1}{n} + o_p(1) \xrightarrow{i.p.} 4c \int_0^1 x dx = 2c.
\end{equation}

Therefore, by Lemma 3
\begin{equation}
\sqrt{n}(\bar{s}^T \bar{s} - c_n) \xrightarrow{D} N(0, 2c).
\end{equation}

We conclude from Section 2 and 3 that \( \hat{M}_n(z) \) converges weakly to a Gaussian process on \( \mathcal{C} \). Moreover, \( m_n(z) \rightarrow m(z) \) uniformly on \( \mathcal{C} \) by (4.2) in \[5\] and (1.2). These, together with (1.12), (4.12), (2.23) and (4.11), give, for any constants \( a_1 \) and \( a_2 \)
\begin{equation}
\frac{4}{n} \sum_{j=1}^n E_{j-1}[E_j(\bar{s}_j^T \bar{s}_j)]^2 \xrightarrow{d} 4c \int_0^1 x dx = 2c.
\end{equation}

We have
\begin{equation}
a_1 X_n(z) + a_2 \sqrt{n} \left( g(\|\bar{s}\|^2) - g(c_n) \right) = \tilde{a}_1(z) \sqrt{n} \left[ \bar{s}^T A^{-1}(z) \bar{s} - \frac{c_n m_n(z)}{1 + c_n m_n(z)} \right] + \tilde{a}_2(z) \sqrt{n} \left( \|\bar{s}\|^2 - c_n \right) + o_p(1)
\end{equation}
where \( \tilde{a}_1(z) = a_1 (1 + cm(z))^2 / c, \tilde{a}_2(z) = a_2 g'(c_n) - a_1 m(z) / c \), and
\begin{equation}
l_j(z) = \tilde{a}_1(z) Y_j(z) + \tilde{a}_2(z) \frac{2}{\sqrt{n}} E_j(\bar{s}_j^T \bar{s}_j).
\end{equation}

Here, the first \( o_p(1) \) denotes convergence in probability to zero in the \( C \) space and in the first step we use the fact that \( g(x) = g(c_n) + g'(a)(x - c_n) + o(|x - c_n|) \) as \( x \rightarrow c_n \). Thus, the tightness of \( \tilde{X}_n(z) \) is from the tightness of \( \hat{M}_n(z) \).

Since \( b_1(z) \rightarrow 1/(1 + cm(z)) \) and \( b_1(z) \rightarrow -zm(z) \) by (2.17) of \[5\], we have
\begin{equation}
1/(1 + cm(z)) = -zm(z).
\end{equation}
Moreover, we assume for the moment that

\begin{equation}
\sum_{j=1}^{n} E_{j-1} [Y_j(z) \frac{2}{\sqrt{n}} E_j(s_j^T s_j)] \overset{i.p.}{\rightarrow} \frac{2cm(z)}{(1 + cm(z))^2}.
\end{equation}

It follows from (2.57), (1.2), (4.6) and (4.5) that

\begin{align*}
\sum_{j=1}^{n} E_{j-1} [l_j(z_1)l_j(z_2)] &= \tilde{a}_1(z_1)\tilde{a}_1(z_2) \frac{2cz_1z_2m^2(z_1)m^2(z_2)}{(1 + m(z_1))(1 + m(z_2)) - cm(z_1)m(z_2)} \\
+ 2c\tilde{a}_2(z_1)\tilde{a}_2(z_2) + \tilde{a}_1(z_1)\tilde{a}_2(z_2) \frac{2cm(z_2)}{(1 + cm(z_2))^2} + \tilde{a}_1(z_2)\tilde{a}_2(z_1) \frac{2cm(z_2)}{(1 + cm(z_2))^2} + o_p(1) \\
&= a_1^2 \times (1.10) + a_2^2 \times 2c(g'(c))^2 + o_p(1).
\end{align*}

Thus Lemma 1 follows from the above argument, Lemma 3 and Cramer-Wold's device.

Consider (4.6) now. Write

\begin{align*}
E_{j-1} [E_j(s_j^T A_j^{-1}(z)s_j) E_j(s_j^T s_j)] &= E_j(s_j^T) E_j(A_j^{-1}(z)s_j) \\
&= \frac{1}{n} \sum_{i<j} E_j(s_i^T A_{ij}^{-1}(z)\bar{s}_j \beta_{ij}(z)) \\
&= \frac{1}{n^2} \sum_{i<j} E_j(s_i^T A_{ij}^{-1}(z)s_i \beta_{ij}(z)) + \frac{1}{n} \sum_{i<j} E_j(s_i^T A_{ij}^{-1}(z)s_{ij} \beta_{ij}(z)),
\end{align*}

where we use \( \bar{s}_j = 1/n \sum_{i \neq j} s_i \) in the second step and \( \bar{s}_j = s_{ij} + s_i/n \) in the last step. By (2.7), (2.10) and (2.8)

\[ E \left[ \frac{1}{n} \sum_{i<j} E_j(s_i^T A_{ij}^{-1}(z)s_{ij}(\beta_{ij}(z)) - b_{12}(z)) \right] = O(\frac{1}{\sqrt{n}}), \]

which, together with (2.34), yields

\[ E \left[ \frac{1}{n} \sum_{i<j} E_j(s_i^T A_{ij}^{-1}(z)s_{ij} \beta_{ij}(z)) \right] = o(1). \]

On the other hand, appealing to (2.6), (2.7) and (2.8) ensures that

\[ \frac{1}{n^2} \sum_{i<j} E_j(s_i^T A_{ij}^{-1}(z)s_i \beta_{ij}(z)) = \frac{j - 1}{n} \frac{n^{-1} Etr A^{-1}(z)}{1 + n^{-1} Etr A^{-1}(z)} + o_L(1). \]
Therefore, we obtain
\[
\frac{1}{n} \sum_{j=1}^{n} E_{j-1}[E_j(s_j^T A_j^{-1}(z) s_j) E_j(s_j^T s_j)] = \frac{n^{-1} E tr A^{-1}(z)}{1 + n^{-1} E tr A^{-1}(z)} \left( \sum_{j=1}^{n} j - \frac{1}{n} \right) + o_L(1)
\]
(4.7)
\[\xrightarrow{i.p.} \frac{cm(z)}{2(1 + cm(z))}.\]

Next, by the Markov inequality and the Doob inequality
\[
P(\max_{i,j} \frac{1}{n} \sum_{k<j} v_{ik} | \geq \varepsilon) \leq \frac{\sum_{i=1}^{n} E(\max_{j} \frac{1}{n} \sum_{k<j} v_{ik})^4}{\varepsilon^4} \leq M \sum_{i=1}^{n} E(\frac{1}{n} \sum_{k<j} v_{ik})^4 \xrightarrow{\varepsilon^4} O(n^{-1}).
\]
which implies
\[\max_{i,j} \frac{1}{n} \sum_{k<j} v_{ik} \xrightarrow{i.p.} 0.
\]

It follows that
\[
\sum_{j=1}^{n} E_{j-1}[E_j(\alpha_j(z) E_j(s_j^T s_j))] = \frac{EX_{11}}{n} \sum_{j=1}^{n} \sum_{i=1}^{p} [E_j(D_j(z_2))]_{ii} [E_j(e_i^T s_j)]
\]
\[\leq \max_{i,j} \left| \frac{1}{n} \sum_{k<j} M \frac{n}{n} \sum_{j=1}^{p} [E_j(A_j^{-1}(z_2) s_j s_j^T A_j^{-1}(z_2))]_{ii}
\]
\[\leq \max_{i,j} \left| \frac{1}{n} \sum_{k<j} M \sum_{j=1}^{n} E_j(s_j^T A_j^{-1}(z_2) A_j^{-1}(z_2) s_j)
\]
(4.8) \[\xrightarrow{i.p.} 0,
\]
because (2.16) implies that \(n^{-1} \sum_{j=1}^{n} E_j(s_j^T A_j^{-1}(z_2) A_j^{-1}(z_2) s_j)\) is uniformly integrable. Based on (4.8) and (4.7) we have (4.6).

\[
\square
\]

Proof of Remark 2 By (1.3) we get
\[
\frac{m(z_1) - m(z_2)}{(z_1 - z_2)} = \frac{m(z_1) m(z_2)(1 + m(z_1))(1 + m(z_2))}{(1 + m(z_1))(1 + m(z_2)) - cm(z_1)m(z_2)}.
\]
Then
\[
2 \frac{cz_1z_2[(1 + m(z_1))(1 + m(z_2)) - cm(z_1)m(z_2)]}{2(m(z_1) - m(z_2))} = \frac{cz_1z_2(z_1 - z_2)[m(z_1)m(z_2)](1 + m(z_1))(1 + m(z_2))}{2(m(z_1) - m(z_2))}
\]
\[
= \frac{z_1z_2(z_1 - z_2)[1 + m(z_1)]^2(1 + m(z_2))^2}{2(m(z_1) - m(z_2))}
\]
\[
+ \frac{2(m(z_1) - m(z_2))}{c} \times \left[ \frac{m(z_1)m(z_2)}{(1 + m(z_1))(1 + m(z_2))} \right]
\]
\[
= \frac{2(m(z_1) - m(z_2))}{z_1z_2(z_1 - z_2)[1 + m(z_1)]^2(1 + m(z_2))^2} + \frac{2}{cz_1z_2(1 + m(z_1))(1 + m(z_2))}
\]
\[
+ \frac{2m(z_1)m(z_2)}{c}.
\]

where in the first step and the third step we use (4.9) and in the last step we use (4.5). On the other hand, via (1.3) one can verify that
\[
\frac{2(m(z_1) - m(z_2))}{z_1z_2(z_1 - z_2)[1 + m(z_1)]^2(1 + m(z_2))^2} = \frac{2(z_2m(z_2) - z_1m(z_1))^2}{c^2z_1z_2(z_1 - z_2)(m(z_1) - m(z_2))},
\]
which is exactly the covariance function in Lemma 2 of [2]. Therefore, Remark [2] holds.

**Proof of Theorem [2]** The idea from Lemma [1] to Theorem [2] is similar to that in [5]. First, by the Cauchy formula we have
\[
\int f(x)dG(x) = -\frac{1}{2\pi i} \oint f(z)m_G(z)dz,
\]
where the contour contains the support of \(G(x)\) on which \(f(x)\) is analytic. Then, with probability one, we have
\[
\int f(x)dG_n(x) = -\frac{1}{2\pi i} \oint f(z)X_n(z)dz,
\]
for all \(n\) large, where the complex integral is over \(C\) and
\[
G_n(x) = \sqrt{n}(F^S_2(x) - F_{\alpha n}(x)).
\]
Further,
\[
| \int f(z)(X_n(z) - \hat{X}_n(z))dz | \leq \frac{M\rho_n}{\sqrt{n}(u_r - \lambda_{\max}(S))} + \frac{M\rho_n}{\sqrt{n}(\lambda_{\min}(S) - u_l)} \xrightarrow{a.s.} 0,
\]
where, with probability one, $\lambda_{\text{max}}(S) \to (1 + \sqrt{c})^2$ by [11] and $\lambda_{\text{min}}(S) \to (1 - \sqrt{c})^2$ by [26]. Second, note that for any constants $a_1$ and $a_2$

$$(\tilde{X}_n(z), Y_n) \to a_1 \int f(z) \tilde{X}_n(z) dz + a_2 Y_n$$

is a continuous mapping. Therefore, the right side above converges in distribution by Lemma [11]. Moreover, Remark [2] shows that (1.9) follows from (1.12) and (1.15) in [2].

□

Proof of Theorem [11] By taking $f(x) = x^{-1}$ and $g(x) = x$ in Theorem [2] and noting that $c_n \to c$ as $n \to \infty$, we can complete the proof.

□

5. TRUNCATION OF THE UNDERLYING RANDOM VARIABLES

To guarantee the results holding under the fourth moment, it is necessary to truncate and centralize the underlying r.v.’s at an appropriate rate. As in (1.8) in [11] one may select a positive sequence $\varepsilon_n$ so that

$$(5.1) \quad \varepsilon_n \to 0 \quad \text{and} \quad \varepsilon_n^{-1} E|X_{ij}|^4 \geq 0.$$

Set $\check{X}_{ij} = X_{ij}I(|X_{ij}| \leq \varepsilon_n \sqrt{n}) - EX_{ij}I(|X_{ij}| \leq \varepsilon_n \sqrt{n})$ and $\check{X}_n = X_n - \check{X}_n = (\check{X}_{ij})$ with $\check{X}_n = (\check{X}_{ij})$. Let $\sigma_n = \sqrt{E|\check{X}_{11}|^2}$, $\tilde{S}_n = (n\sigma_n^2)^{-1} \check{X}_n \check{X}_n^T$, and $A^{-1}(z) = (\tilde{S}_n - z I)^{-1}$. Moreover, introduce $\tilde{s} = \frac{1}{n} \sum_{j=1}^n \tilde{s}_j$, where $\tilde{s}_j$ is the $j$-th column of the matrix $(\sigma_n)^{-1} \check{X}_n$.

Lemma 7. Assume that $X_{ij}, i = 1, \cdots, p, j = 1, \cdots, n$ are i.i.d. with $EX_{11} = 0, E|X_{11}|^2 = 1$ and $E|X_{11}|^4 < \infty$, for $z \in \mathbb{C}_n^+$, we have then

$$(5.2) \quad \sqrt{n}(\tilde{s}^T A^{-1}(z) \tilde{s} - \tilde{s}^T \check{A}^{-1}(z) \tilde{s}) \overset{i.p.}{\to} 0,$$

where the convergence in probability holds uniformly for $z \in \mathbb{C}_n^+$. Moreover,

$$(5.3) \quad \sqrt{n}(\tilde{s}^T \tilde{s} - \tilde{s}^T \tilde{s}) \overset{i.p.}{\to} 0.$$

Proof. Write

$$\sqrt{n}(\tilde{s}^T A^{-1}(z) \tilde{s} - \tilde{s}^T \check{A}^{-1}(z) \tilde{s}) = u_{n1} + u_{n2} + u_{n3}$$

where

$$u_{n1} = \sqrt{n}[\tilde{s}^T (A^{-1}(z) - \check{A}^{-1}(z)) \tilde{s}], \quad u_{n2} = \sqrt{n}[\tilde{S}^T (A^{-1}(z) - \check{A}^{-1}(z)) \tilde{s}]$$

and

$$u_{n3} = \sqrt{n}[\tilde{s}^T \check{A}^{-1}(z) (\tilde{s} - \tilde{s})].$$
Consider \( u_{n1} \) on the \( \mathcal{C}_u \) first. It is observed that
\[
|u_{n1}| \leq \sqrt{n} ||\bar{s} - \bar{s}||^T \|A^{-1}(z)||\bar{s}|| \leq \frac{\sqrt{n}}{v_0} ||(\bar{s} - \bar{s})^T||\bar{s}||
\]
\[
\leq \frac{\sqrt{n}}{v_0} |1 - \frac{1}{\sigma_n}||\bar{s}||^2 + \frac{\sqrt{n}}{v_0} \frac{1}{\sigma_n}||\bar{s}||\bar{s}||
\]
(5.4)
since \( \bar{s} - \bar{s} = (1 - \frac{1}{\sigma_n})\bar{s} + \frac{1}{\sigma_n}\bar{s} \) with \( \bar{s} = \sum_{j=1}^{n} \bar{s}_j/n \) and \( \bar{s}_j \) being the \( j \)-th column of \( \bar{X}_n \). Moreover, it follows from \((5.1)\) that
\[
1 - \sigma_n^2 \leq 2EX_{11}^2I(|X_{11}| \geq \varepsilon_n\sqrt{n}) \leq 2\varepsilon_n^{-2n}EX_{11}^4I(|X_{11}| \geq \varepsilon_n\sqrt{n}) = o(\varepsilon_n^2n^{-1}),
\]
which implies that
\[
\sqrt{n}(1 - 1/\sigma_n) = \sqrt{n}(\sigma_n^2 - 1)/[\sigma_n(1 + \sigma_n)] = o(n^{-1/2}).
\]
(5.5)
On the other hand
\[
E||\bar{s}||^2 = E[\sum_{i=1}^{p} \frac{1}{n} \sum_{j=1}^{n} \bar{X}_{ij}^2] = \frac{1}{n^2} \sum_{i=1}^{p} \sum_{j=1}^{n} E\bar{X}_{ij}^2 \leq \frac{M}{n\varepsilon_n^2} EX_{11}^4I(|X_{11}| \geq \varepsilon_n\sqrt{n}),
\]
which, via \((5.1)\), gives that
\[
\sqrt{n}||\bar{s}|| \overset{i.p.}{\to} 0.
\]
(5.6)
In addition, \( ||\bar{s}||^2 \) is uniformly integrable because \((2.15)\) remains true for \( k = 2 \) without truncation by a careful check on its argument. This, together with \((5.4)\) - \((5.6)\), ensures that \( u_{n1} \) converges in probability to zero uniformly on \( \mathcal{C}_u \).

Analyze \( u_{n2} \) next. Since \( X_n - \sigma_n^{-1}\bar{X}_n = (1 - \sigma_n^{-1})X_n + \sigma_n^{-1}\bar{X}_n \) we have
\[
|u_{n2}| \leq \sqrt{n}||\bar{s}||^{T}\|A^{-1}(z) - \bar{A}(z)||\bar{s}|| \leq \frac{\sqrt{n}}{v_0} ||\bar{s}||^{T}\|A(z) - \bar{A}(z)||\bar{s}||
\]
\[
\leq \frac{1}{v_0\sqrt{n}} ||\bar{s}||^{T}\|\bar{s}|| \left[\|X_n - \sigma_n^{-1}\bar{X}_n\||X_n^{T}|| + \|\sigma_n^{-1}\bar{X}_n\||X_n^{T}|| - \sigma_n^{-1}\bar{X}_n^{T}||\right]
\]
\[
\leq \frac{1}{v_0\sqrt{n}} ||\bar{s}||^{T}\|\bar{s}|| \left[\left(1 - \sigma_n^{-1}\right)||X_n\||X_n^{T}|| + \sigma_n^{-1}||\bar{X}_n\||X_n^{T}||
\]
\[
+ ||\sigma_n^{-1}\bar{X}_n||\left(1 - \sigma_n^{-1}\right)||X_n^{T}|| + \sigma_n^{-1}||\bar{X}_n||\bar{X}_n^{T}||\right].
\]
(5.7)
As before, \( ||\bar{s}|| \) and \( ||\bar{s}||^2 \) are uniformly integrable. Moreover, the spectral norms \( ||X_n^{T}||/\sqrt{n} \) and \( ||\sigma_n^{-1}\bar{X}_n||/\sqrt{n} \) both converge to \( (1 + \sqrt{c})^2 \) with probability one by \([25]\). In addition, \( ||\bar{X}_n^{T}||/\sqrt{nEX_{11}^2} \) converges to \( (1 + \sqrt{c})^2 \) with probability one. From \((5.1)\) we have
\[
nEX_{11}^2 \leq 2\varepsilon_n^{-2}EX_{11}^4I(|X_{11}| \geq \varepsilon_n\sqrt{n}) = O(\varepsilon_n^2),
\]
which, together with (5.5), yields that $u_{n2}$ converges in probability to zero uniformly on $C_u$.

Clearly, the argument for $u_{n1}$ works for $u_{n3}$ as well. Moreover, note that $\|A^{-1}(z)\|$ is bounded for $z \in C_l, u_l < 0$. As for $z \in C_l, u_l > 0$ or $z \in C_r$, by [25] we have

$$\lim_{n \to \infty} \min(u_r - \lambda_{\max}(A), \lambda_{\min}(A) - u_l) > 0, \text{ a.s.}$$

and

$$\lim_{n \to \infty} \min(u_r - \lambda_{\max}(\tilde{A}), \lambda_{\min}(\tilde{A}) - u_l) > 0, \text{ a.s.}$$

Therefore the above argument for $u_{nj}, j = 1, 2, 3$ for $z \in C_u$ of course applies to the cases (1). $z \in C_l, u_l < 0$; (2). $z \in C_l, u_l > 0$; (3) $z \in C_r$. Thus, (5.2) holds.

Finally, the above argument for (5.2) of course works for (5.3). We are done. \□

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