On Associativity Equations in Dispersionless Integrable Hierarchies

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We discuss the origin of the associativity (WDVV) equations in the context of quasiclassical or Whitham hierarchies. The associativity equations are shown to be encoded in the dispersionless limit of the Hirota equations for KP and Toda hierarchies. We show, therefore, that any tau-function of dispersionless KP or Toda hierarchy provides a solution to associativity equations. In general, they depend on infinitely many variables. We also discuss the particular solution to the dispersionless Toda hierarchy that describes conformal mappings and construct a family of new solutions to the WDVV equations depending on finite number of variables.

1 Introduction

The Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations were originally found in the context of 2D topological theories as an associativity condition of the operator algebra of primary fields. This is a highly overdetermined system of nonlinear differential equations
satisfied by third-order derivatives of the “free energy” function. In particular, solutions of the WDVV equations relevant to the simplest topological Landau-Ginzburg models are of the form $F = \log \tau$, where $\tau$ is the dispersionless limit of the tau-function for a reductions of the KP (Kadomtsev-Petviashvili) hierarchy of integrable equations \[^2\]. Thus one may hope that a vast class of solutions can be constructed in a more general context of Whitham integrable hierarchies \[^3\]. Solutions to WDVV equations were also found to be related to so-called Frobenius manifolds \[^4\].

It turned out later that the associativity equations is a more general phenomenon. For example, some new classes of solutions to the associativity equations were found \[^3\] in the Seiberg-Witten theory, closely related to Whitham hierarchies (see also \[^3, 7\] for more recent progress in this direction), as well as in a more general context of integrable hierarchies \[^8, 9\]. The origin of the WDVV equations in most of these cases is still a pending problem.

In this paper, we discuss a particular aspect of the WDVV equations related to dispersionless integrable hierarchies. Below we address the following two questions:

- Does any tau-function of the dispersionless KP (dKP) or Toda (dToda) hierarchies provide a solution to WDVV equations?
- What are the conditions for these tau-functions (which are functions of an infinite number of variables) to obey a finite system of WDVV equations?

Concerning the first question, we show that all dispersionless tau-functions do satisfy the WDVV equations or, in general, an infinite system of equations of the WDVV type. In other words, we show that the associativity equations can be considered as an intrinsic feature of integrable hierarchies and furthermore they can be derived from the dispersionless limit of Hirota relations by elementary manipulations. This result applies, in particular, to the new class of dispersionless tau-functions, constructed recently in connection with conformal maps \[^10, 11\]. However, in generic situation the system of WDVV equations obeyed by $F = \log \tau$ is infinite, i.e. both the number of equations and variables are infinite. (In the language of topological theories this means that the small phase space is not finite-dimensional).

Concerning the second question, we demonstrate the following: A) Imposing a reduction of KP hierarchy that leaves only finite number of independent fields, it is straightforward to show that the system of WDVV equations becomes finite (a well known example is the $N$-reduced dKP hierarchy relevant to topological Landau-Ginzburg theories); B) A new class of finite-dimensional solutions to WDVV is obtained by a restriction of the tau-function for conformal mappings \[^10\] to finite-dimensional spaces of times related to conformal maps which are Laurent polynomials of a fixed degree.

In Sec. 2 we introduce notations and recall some standard definitions from the literature on WDVV equations. In Sec. 3 we show that any solution of the dKP hierarchy obeys the infinite system of WDVV equations. In Sec. 3.3 we obtain the corresponding infinite-dimensional algebra and the residue formula for third order derivatives of the tau-function directly from the Hirota equation. In Sec. 3.4 the reduction to their finite version is discussed. Sec. 4.1 is devoted to the associativity equations in the dToda hierarchy. In Sec. 4.2 we construct some finite-dimensional solutions to the WDVV equations related to conformal maps.
2 Associativity equations: the basic notation

Let $\mathcal{F}$ be a differentiable function of variables $t_1, t_2, t_3, \ldots$ ("times"). Set

$$\mathcal{F}_{ij} \equiv \frac{\partial^2 \mathcal{F}}{\partial t_i \partial t_j}, \quad \mathcal{F}_{ijk} \equiv \frac{\partial^3 \mathcal{F}}{\partial t_i \partial t_j \partial t_k},$$

for brevity. Let us choose one of these times, say $t_1$, and assume that the matrix $\eta_{ij} = \mathcal{F}_{ij1}$, called metric, is non-degenerate. Then one can pass from the set of variables $t_i$ to the set of variables $\mathcal{F}_{j1}$. The matrix $C_{ij} = \partial \mathcal{F}_{ij} / \partial \mathcal{F}_{1l}$ connects $\mathcal{F}_{ijk}$ and $\eta_{ij} = \mathcal{F}_{ij1}$:

$$\mathcal{F}_{ijk} = \sum_l C_{ij}^l \mathcal{F}_{kl1}. \tag{2}$$

This equality can be understood as a definition of $C_{ij}^l$: $C_{ij}^l = \sum_k \mathcal{F}_{ijk} (\eta^{-1})^{kl}$, where $\eta^{-1}$ is the inverse matrix to the $\eta$.

If one wants $C_{ij}^l$ to be structure constants of an associative algebra $\phi_i \cdot \phi_j = \sum_l C_{ij}^k \phi_k$, the conditions $\sum_l C_{ij}^l C_{lk}^m = \sum_l C_{ik}^l C_{lj}^m$ should be imposed. Equivalently

$$\sum_l C_{ij}^l \mathcal{F}_{lkn} = \sum_l C_{ik}^l \mathcal{F}_{ljn}. \tag{3}$$

In other words

$$X_{ijkn} \equiv \sum_l C_{ij}^l \mathcal{F}_{lkn} \tag{4}$$

is symmetric with respect to permutations of any indices. The system (3) is called associativity equations or WDVV equations (WDVV for short).

Assuming that eqs. (3) hold, one may choose any other index to define the metric $\eta(a)_{ij} = \mathcal{F}_{aij}$ (as long as it is non-degenerate) and structure constants $\mathcal{F}_{ijk} = \sum_l C_{ij}^l (a) \mathcal{F}_{kla}$. They obey the same associativity relations (3). Introducing matrices $F_i$ with matrix elements $(F_i)_{jk} = \mathcal{F}_{ijk}$, one may write the full set of WDVV equations in the form (3):

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i \quad \text{for all } i, j, k. \tag{5}$$

Eqs. (3) with $C_{ij}^l$ defined via (2) are rather restrictive. They can be viewed as a set of non-linear equations for the function $\mathcal{F}$ expressed through its third order derivatives. In the context of 2D topological theories and Seiberg-Witten theories one is usually interested in solutions to WDVV with finite number of variables, i.e. the sum in (3) is finite(1). In this paper we allow the number of variables to be infinite and do not assume any special properties of solutions. The only assumption is that the series $\sum_k z^{-k} \mathcal{F}_{tk}$ defines a function holomorphic and univalent in some domain which includes infinity.

In the case of infinitely many variables it is convenient to use generating functions for second and third order derivatives of $\mathcal{F}$. Introduce the operator

$$D(z) = \sum_{k=1}^{\infty} z^{-k} \frac{\partial}{\partial t_k}. \tag{6}$$

1Besides, two additional requirements on solutions are usually imposed: a) The metric is constant, i.e. $\eta_{ij}$ does not depend on $t_k$; b) $\mathcal{F}$ is a quasihomogeneous function. Sometimes these requirements are included into the definition of the WDVV system.
The functions
\[ D_1 D_2 F \equiv D(z_1) D(z_2) F = \sum_{k,m=1}^{\infty} \frac{z_1^{-k} z_2^{-m}}{k} F_{km}, \]
\[ D_1 D_2 D_3 F \equiv D(z_1) D(z_2) D(z_3) F = \sum_{k,m,n=1}^{\infty} \frac{z_1^{-k} z_2^{-m} z_3^{-n}}{k} F_{kmn}, \]
generate the sets of \( F_{km} \) and \( F_{kmn} \). We also introduce generating functions for the structure constants
\[ C^l(z_1, z_2) = \sum_{i,j=1}^{\infty} C^{ij}_{kl} \frac{z_1^{-i} z_2^{-j}}{i} \]
and for the \( X_{ijkmn} \)
\[ X(z_1, z_2, z_3, z_4) \equiv \sum_{i,j,k,n=1}^{\infty} \frac{z_1^{-i} z_2^{-j} z_3^{-k} z_4^{-n}}{k} X_{ijkmn}. \]
The infinite WDVV equations (3) are then equivalent to the symmetry of the \( X(z_1, z_2, z_3, z_4) \) under permutations of \( z_1, z_2, z_3, z_4 \).

3 Hirota’s relations for dKP hierarchy and associativity equations

3.1 The dKP hierarchy

Our starting point in this section is the bilinear identity for the tau-function [12] which we refer to as Hirota equation. Let \( F \) be the dispersionless limit of logarithm of the KP tau-function: \( F \equiv \log \tau \). In the dispersionless limit the Hirota equation encodes a set of relations for the second order derivatives \( F_{ij} \). In generating form they can be written as [13, 14]:
\[ (z_1 - z_2) \left( 1 - e^{D(z_1) D(z_2) F} \right) = \left( D(z_1) - D(z_2) \right) \partial_z F, \]
where we use the operator (3). The symmetric version of this equation is
\[ (z_1 - z_2) e^{D(z_1) D(z_2) F} + (z_2 - z_3) e^{D(z_2) D(z_3) F} + (z_3 - z_1) e^{D(z_3) D(z_1) F} = 0. \]
Note that one can obtain (11) from (12) in the limit \( z_3 \to \infty \). These equations should be understood as an infinite set of algebraic relations for \( F_{ij} \) obtained by expanding both sides as a power series in \( z_i \) and comparing the coefficients. These relations can be resolved with respect to \( F_{ij} \) with \( i, j \geq 2 \). Indeed, writing (11) as
\[ D(z_1) D(z_2) F = \log \frac{p(z_1) - p(z_2)}{z_1 - z_2}, \]
where
\[ p(z) = z - \sum_{k=1}^{\infty} \frac{z^{-k}}{k} F_{1k}, \]
we conclude that \( F_{ij} = P_{ij}(F_{11}, F_{12}, F_{13}, \ldots) \), with \( P_{ij} \) being polynomials. This representation of the dKP hierarchy goes back to [13] (see also [14] for more details).
Second order derivatives of the tau-function allow one to define a set of commuting flows with generators $H_k$ determined from the series

$$D(z_1)D(z_2)\mathcal{F} = -\log \left(1 - \frac{z_2}{z_1} \right) - \sum_{k=1}^{\infty} \frac{z_1^{-k}}{k} H_k(z_2). \quad (15)$$

Acting by $D(z_3)$ on both sides and interchanging $z_1$ and $z_3$ one finds that

$$\frac{\partial H_i(z)}{\partial t_j} = \frac{\partial H_j(z)}{\partial t_i}. \quad (16)$$

Note that $H_1(z) = p(z)$ (14). The relations (16) can be viewed as a hierarchy of evolution equations for the $p(z)$:

$$\frac{\partial p(z)}{\partial t_k} = \frac{\partial H_k(z)}{\partial t_1}. \quad (17)$$

Equations (17), being rewritten as evolution equations for the function $z(p)$, have the form of dispersionless Lax-Sato equations:

$$\frac{\partial z(p)}{\partial t_k} = \{H_k(p), z(p)\}_{KP}, \quad (18)$$

where the Poisson brackets are defined as $\{f, g\}_{KP} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial t_1} - \frac{\partial f}{\partial t_1} \frac{\partial g}{\partial p}$ and the derivatives in $t_i$ are taken at fixed $p$. Moreover, as it follows from (13), the $H_k$ turn out to be polynomials in $p$. On the other hand, (13) fixes $H_k$ to be of the form $H_k = z^k - D(z)\partial_k \mathcal{F}$, i.e. $H_k = z^k + O(z^{-1})$. Therefore

$$H_k = (z^k(p))_{\geq 0}, \quad (19)$$

where the symbol $(f(p))_{\geq 0}$ means the non-negative part of the Laurent series in $p$. This is the dKP hierarchy (see e.g. [13]). Given a Lax function $z(p) = p + O(z^{-1})$ and $H_k$ obtained from it by means of (19), one can reconstruct the second order derivatives $\mathcal{F}_{jk}$ via the formula

$$\mathcal{F}_{jk} = \frac{1}{j+k} \text{res}_{\infty} \left( \frac{dH_j dH_k}{d \log z} \right). \quad (20)$$

A remark is in order. Note that eqs. (13–18) hold for any function $\mathcal{F}$. We want to stress that this is not an integrable hierarchy yet, in spite of the fact that there are infinitely many commuting flows (16). The crucial relation, which really makes an integrable hierarchy out of this, is (19). The Hirota equation gives a relation between the generating function of the flows and $p$ and allows one to determine $H_k$ as functions $H_k(p)$ with certain analytic properties. In the dKP-case they are polynomials. From this point of view, it is the Hirota identity that encodes integrability of the system.

Plugging (13) in the r.h.s. of (15) and differentiating w.r.t. $p = p(z_2)$, one arrives at the relation (cf. [14])

$$\frac{1}{p(z_1) - p} = \sum_{k \geq 1} \frac{z_1^{-k}}{k} \frac{dH_k(p)}{dp}. \quad (21)$$

It is used below to obtain an explicit realization of the associative algebra.

Let us stress that all basic relations of the dKP hierarchy discussed in this section contain second order derivatives of $\mathcal{F}$ only. We are going to take an extra derivative and rearrange the resulting relations into the form of the associativity equations.
3.2 Proof of WDVV equations

An elementary manipulation with the Hirota equation \((11)\) allows one to bring it to the form \((2)\), which is the defining relation for the structure constants \(C_{ij}^l\). Apply \(D(z_3)\) to the both sides of \((11)\). This gives

\[
D_1 D_2 D_3 F = -\frac{1}{p_1 - p_2} (D_1 - D_2) D_3 \partial_{t_1} F,
\]

where \(p_i \equiv p(z_i)\).

We observe that eq. \((22)\), being written in modes (by means of \((1, 3, 4)\)) is equivalent to the infinite-dimensional version of \((2)\):

\[
F_{ijk} = \sum_{l=1}^{\infty} C_{ijl}^l F_{lk1}
\]

where the structure constants are defined by the generating function

\[
C_{ij}^l(z_1, z_2) = -\frac{z_1^{-l} - z_2^{-l}}{l(p_1 - p_2)}.
\]

Since \(F\) obeys \((11)\), one may rewrite it in an equivalent form:

\[
C_{ij}^l(z_1, z_2) = -\frac{z_1^{-l} - z_2^{-l}}{l(z_1 - z_2)} e^{-D(z_1)D(z_2)F}.
\]

It is easy to see from \((9)\) that the infinite sum in \((23)\) is actually always finite: it truncates at \(l = i + j\).

Let us show that \(F\) obeys the WDVV \((3)\), with each index running over natural numbers. In terms of generating functions this means that \(X(z_1, z_2, z_3, z_4)\) given by \((11)\) is totally symmetric w.r.t. permutations of \(z_1\ldots z_4\). It is enough to prove the symmetry w.r.t. the permutation of \(z_2\) and \(z_3\), which is equivalent to the relation

\[
13z_1 e^{D_1 D_3 F} (D_1 - D_2) D_3 D_4 F = 12z_1 e^{D_1 D_3 F} (D_1 - D_3) D_2 D_4 F,
\]

where \(z_{ik} = z_i - z_k\). Using \((12)\) it is straightforward to bring \((26)\) into the form

\[
D_4 \left( 13z_1 e^{D_1 D_3 F} - 12z_1 e^{D_1 D_2 F} - 23z_1 e^{D_3 D_4 F} \right) = 0
\]

which is the \(D_4\)-derivative of \((12)\) and therefore the WDVV follows from the Hirota equation.

One can prove in a similar way WDVV for \(F\) choosing \(D(z_a)F_{ij}\) rather than \(F_{1ij}\) as a metric. Apply \(D(z_3)\) to the symmetric form of the Hirota equation \((12)\) written for the three points \(z_1, z_2, z_a\). One obtains:

\[
D_1 D_2 D_3 F = \frac{1}{p_{12}} (p_{1a} D_1 - p_{2a} D_2) D_3 D_a F,
\]

where \(p_{ik} \equiv (z_i - z_k) e^{D_i D_k F}\). Similarly to the previous case, which is reproduced in the limit \(z_a \to \infty\), this equality defines structure constants. The WDVV equations are then equivalent to

\[
p_{1a} (p_{13} D_1 D_3 D_4 F - p_{12} D_1 D_2 D_4 F) = (p_{13} p_{2a} - p_{12} p_{3a}) D_2 D_3 D_4 F
\]

Plugging the \(D_4\)-derivative of \((12)\), write the l.h.s. of \((29)\) as \(p_{1a} p_{23} D_2 D_3 D_4 F\). It is clear then that \((29)\) is equivalent to the identity \(p_{1a} p_{23} = p_{13} p_{2a} - p_{12} p_{3a}\) which is automatically satisfied by \(p_{ij} = p_i - p_j\), as is indeed the case due to \((12)\).
3.3 Realization of associative algebra and the residue formula

To give a realization of the associative algebra with the structure constants defined by (24), we introduce the polynomials

\[
\phi_k(p) = \frac{dH_k(p)}{dp}, \quad k \geq 1.
\]

Expanding both sides of the identity

\[
\frac{1}{(p-p_1)(p-p_2)} = \frac{1}{p_1-p_2} \left( \frac{1}{p-p_1} - \frac{1}{p-p_2} \right)
\]
in \(z_1^{-1}, z_2^{-1}\), using (21), and comparing the coefficients, we obtain the algebra

\[
\phi_i(p)\phi_j(p) = \sum_{l \geq 1} C_{ij}^l \phi_l(p)
\]

where the structure constants are exactly those defined by (24). This infinite-dimensional algebra is just the ring of polynomials of arbitrary degree.

For completeness, we show how to derive the residue formula \([3, 16]\) for third order derivatives of \(\mathcal{F}\) directly from the Hirota equation. Substituting into (22) its particular case \(D_1D_2\partial t_1\mathcal{F} = -\frac{(D_1-D_2)\partial_{t_1}^2 \mathcal{F}}{p_1-p_2}\) (obtained in the limit \(z_3 \to \infty\)), one easily expresses \(D_1D_2D_3\mathcal{F}\) in terms of \(D_1\partial_{t_1}^2 \mathcal{F}\) only:

\[
D_1D_2D_3\mathcal{F} = \sum_{i=1}^3 \text{res}_{p_i} \left( \frac{D(z(p))\partial_{t_1}^2 \mathcal{F}}{(p-p_1)(p-p_2)(p-p_3)} \right) dp.
\]

In the numerator we have: \(D(z)\partial_{t_1}^2 \mathcal{F} = -\partial p(z)/\partial t_1\) which is equal to \(\partial z(p)/\partial t_1\) in terms of the independent variable \(p\) \((z'(p) \equiv dz(dp)\). Expanding both sides of the above formula in the series in \(z_{1,2,3}\) and using (21) we obtain:

\[
\mathcal{F}_{jkm} = \frac{1}{2\pi i} \oint_{C_\infty} \frac{\partial_{t_1} z(p)}{z'(p)} \phi_j(p)\phi_k(p)\phi_m(p) dp,
\]

where \(C_\infty\) is a small contour around infinity in the domain where \(p(z)\) is holomorphic and univalent. It suffices that \(z'(p)\) does not have zeros and singularities in the domain. One may think of (32) as coming from the scalar product

\[
\langle f \cdot g \rangle = \frac{1}{2\pi i} \oint_{C_\infty} \frac{\partial_{t_1} z(p)}{z'(p)} f(p)g(p) dp.
\]

defined for polynomials. Note that \(\eta_{jk} = \mathcal{F}_{j1k} = \langle \phi_j \cdot \phi_k \rangle\) (just because \(\phi_1 = 1\)). Therefore, the algebra (31) is in full agreement with (34):

\[
\mathcal{F}_{jkm} = \langle \phi_j \phi_k \cdot \phi_m \rangle = \sum_l C_{jk}^l \langle \phi_l \cdot \phi_m \rangle = \sum_l C_{jk}^l \mathcal{F}_{lm1}.
\]

Meanwhile, this gives another proof of WDVV for \(\mathcal{F}\): \(X_{jkmn} \equiv \langle \phi_j \phi_k \phi_m \phi_n \rangle\) (this representation is obviously equivalent to (31)) is symmetric due to apparent symmetry of the r.h.s. of (32).

If there exists a times-independent function \(\varphi(z)\) such that \(E(p) \equiv \varphi(z(p))\) is a meromorphic function of \(p\) with the number of poles being unchanged under variations of all \(t_j\), then the integral is equal to the sum of residues at zeros of \(E'(p)\) (cf. [16]):

\[
\mathcal{F}_{jkm} = \sum_{E'(p_a)=0} \text{res}_{p_a} \left( \frac{\partial_{t_1} E(p)}{E'(p)} \phi_j(p)\phi_k(p)\phi_m(p) dp \right).
\]
Indeed, $\partial_t z(p)/z'(p) = \partial_t E(p)/E'(p)$ and poles of $\partial_t E(p)$ do not contribute to the integral since they are canceled by those of $E'(p)$.

The existence of such a function $E$ means a finite-dimensional reduction of the hierarchy. In this case the WDVV algebra becomes finite-dimensional. This can be easily seen directly from the residue formula. Below we show this starting from the Hirota equation.

### 3.4 Finite-dimensional reductions

In terms of the Hirota equation the finite-dimensional reduction is a set of additional constraints for second order derivatives of tau-function:

$$\mathcal{F}_{1M} = Q_M(\mathcal{F}_{11}, \ldots, \mathcal{F}_{1N-1}), \quad M \geq N$$  \hspace{1cm} (36)

where functions $Q_M$ do not explicitly depend on times and are required to be consistent with the evolution equations. Any reduction of this kind leaves us with $N - 1$ independent primary fields which can be chosen to be $\mathcal{F}_{1j}$ with $1 \leq j \leq N - 1$. All other $\mathcal{F}_{1M}$ with $M \geq N$ (descendants) are expressed through the independent ones via formulas (36).

A particular example is the familiar $N$-KdV reduction, for which $\mathcal{F}$ is independent of $t_N, t_{2N}, t_{3N}, \ldots$. In this case functions $Q_M$ are certain polynomials with rational coefficients such that $E(p) = z^N(p)$ is a polynomial. Other reductions, when $z^N$ is a rational function, are also known \[17, 14\]. In the sequel we do not refer to explicit form of (36).

Hereafter, we use capital letters (like $J, M, L, \ldots$) for the descendants, i.e. for indices larger than $N - 1$. Summation over small (primary) indices (from 1 to $N - 1$) is always indicated explicitly.

In the case of a reduction the metric $\mathcal{F}_{lk1}$ in (23) becomes a degenerate matrix of rank $N - 1$. Indeed, taking the derivative of (36) w.r.t. $t_k$, we find that $J$-th line of the matrix $\mathcal{F}_{jk1}$ is a linear combination of the first $N - 1$ lines. The finite WDVV equations are obtained by means of a projection on the nondegenerate $N - 1$-dimensional subspace.

Let us separate the sums over primary and descendant fields in (23) and use (36) to express $\mathcal{F}_{Lk1}$ through $\mathcal{F}_{lk1}$:

$$\mathcal{F}_{ijk} = \sum_{l=1}^{N-1} \left( \bar{C}_{ij}^l + \sum_{L} \bar{C}_{ij}^L \frac{\partial Q_L}{\partial \mathcal{F}_{1l}} \right) \mathcal{F}_{lk1}. \hspace{1cm} (37)$$

The object

$$\bar{C}_{ij}^l = C_{ij}^l + \sum_{L} C_{ij}^L \frac{\partial Q_L}{\partial \mathcal{F}_{1l}}$$

at $i, j < N$ defines the structure constants of a finite dimensional algebra formed by the primary fields. The system (23) becomes finite:

$$\mathcal{F}_{ijk} = \sum_{l=1}^{N-1} \bar{C}_{ij}^l \mathcal{F}_{lk1}, \hspace{1cm} (38)$$

where the metric $\mathcal{F}_{lk1}$ is non-degenerate on the small space $(l, k = 1, \ldots, N - 1)$. The structure constants of the primary fields obey the finite-dimensional WDVV equations. In other words, $\bar{X}_{ijkm}$ defined by $\bar{X}_{ijkm} = \sum_{l=1}^{N-1} \bar{C}_{ij}^l \mathcal{F}_{lkm}$ is symmetric with respect to the permutations of the (small) indices $i, j, k, m$. This follows from the fact that $\bar{X}_{ijkm} = X_{ijkm}$ for $i, j, k, m < N$.  

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The proof is elementary. We write
\[ X_{ijkm} = \sum_{l=1}^{N-1} C^l_{ij} F_{lkm} + \sum_{L} C^L_{ij} Y_{Lkm}, \] (39)
and substitute \( C^l_{ij} = \tilde{C}^l_{ij} - \sum_{L} C^L_{ij} \partial Q_L / \partial F_{l1} \) into eq. (39): \( X_{ijkm} = \tilde{X}_{ijkm} + \sum_{L} C^L_{ij} Y_{Lkm} \), where \( Y_{Lkm} = F_{Lkm} - \sum_{l<N} \partial Q_L / \partial F_{l1} F_{lkm} \) vanishes. To see this, express \( F_{lkm} \) inside the sum in terms of \( F_{l1} \) using (38) and interchange the order of summation. Using (38) once again, one obtains the result.

4 Associativity equations in dToda hierarchy

4.1 Hirota relations for dToda hierarchy and WDVV equations

The dToda hierarchy reveals the WDVV algebra in a similar manner. The arguments almost completely repeat those of the previous sections, the Hirota equations being somewhat different. The independent variables of the dToda hierarchy are \( t_k \), where \( k \) is any integer number: \( t_0, t_{\pm 1}, t_{\pm 2}, \ldots \). Let us introduce two functions:
\[ w^\pm(z) = z \exp \left( -\frac{1}{2} \partial_{t_0}^2 F - \partial_{t_0} D^\pm(z) F \right) \] (40)
where \( D^\pm(z) = \sum_{k=1}^{\infty} \frac{z^{-k}}{k} \partial / \partial t_{\pm k} \) and assume that they are regular and univalent in some domain around infinity. The Hirota equations for the dToda hierarchy read [13, 11]
\[ w^\pm(z_1) - w^\pm(z_2) = (z_1 - z_2) e^{-\frac{1}{2} \partial_{t_0}^2 F} e^{-D_1^\pm D_2^\pm F}, \] (41)
\[ 1 - \frac{1}{w^\pm(z_1) w^\pm(z_2)} = e^{-D_1^\pm D_2^\pm F}. \] (42)

Similarly to the arguments of Sec. 3.1 Hirota equations (41, 42) define the dToda hierarchy with commuting flows generated by
\[ H_{\pm j}(w) = \left( (z^\pm(w^\pm 1))^j \right)_\pm + \frac{1}{2} \left( (z^\pm(w^\pm 1))^j \right)_0, \quad j \geq 1, \quad H_0(w) = \log w. \] (43)
Here \( z^\pm(w) \) is the inverse function of \( w^\pm(z) \) and \( (...)_\pm, (...)_0 \) means strictly positive, negative, and constant part of the Laurent series, respectively. The Lax-Sato equations read
\[ \frac{\partial z^\pm(w^\pm 1)}{\partial t_j} = \text{sign} j \{ H_j(w), z^\pm(w^\pm 1) \}_{\text{Toda}}, \] (44)
where the Poisson bracket is defined as \( \{ f, g \}_{\text{Toda}} = w \frac{\partial f}{\partial w} \frac{\partial g}{\partial t_0} - w \frac{\partial f}{\partial t_0} \frac{\partial g}{\partial w} \). The dToda analog of the generating function (21) for derivatives of \( H_j \) is
\[ \frac{w^\pm}{w^\pm(z_1) - w^\pm} = \pm \sum_{k \geq 1} \frac{z_1^{-k}}{k} w \frac{d}{dw} H_{\pm k}(w). \] (45)
We define the metric to be \( \eta_{ij} = F_{0ij} \), where the indices take integer values, and the structure constants as
\[
F_{ijk} = \sum_{l=-\infty}^{\infty} C_{ij}^l F_{lk0}.
\] (46)

From the definition of the metric, we have \( C_{0ij}^l = \delta_j^l \) for all \( l, j \). To find other structure constants, apply \( \partial_{t_k} \) to the Hirota equations (41) and (42):
\[
D_1^+ D_2^+ \partial_{t_k} F = \frac{1}{w_1^+ - w_2^+} \left( w_2^+ D_2^+ F_{k0} - w_1^+ D_1^+ F_{k0} \right),
\] (47)
\[
D_1^+ D_2^+ \partial_{t_k} F = \frac{1}{w_1^+ w_2^- - 1} \left( F_{k00} + D_1^+ F_{k0} + D_2^- F_{k0} \right).
\] (48)

Here and below \( w_1^\pm = w^\pm (z_i) \). In complete analogy with eq. (3) generating functions for structure constants are read from the r.h.s. of these equations. We conclude from (47) that \( C_{ij}^l = 0 \) whenever \( i, j \) are both positive and \( l \leq 0 \) or both negative and \( l \geq 0 \). If all the indices are positive or all negative, we have:
\[
\sum_{i \geq 1} \sum_{j \leq -1} C_{ij}^l \frac{z_1^{-i} z_2^j}{i} \frac{z_2^{-i} z_2^j}{j} = - \frac{w_1^+ w_2^+ - w_2^+ w_2^-}{\pm l (w_1^+ - w_2^-)} , \quad \pm l \geq 1.
\] (49)

When \( i \) and \( j \) have different signs we use eq. (48) to obtain:
\[
\sum_{i \geq 1} \sum_{j \leq -1} C_{ij}^l \frac{z_1^{-i} z_2^j}{i} = - \frac{1}{l (1 - w_1^+ w_2^-)} , \quad l \geq 1,
\] (50)
\[
\sum_{i \geq 1} \sum_{j \leq -1} C_{ij}^l \frac{z_1^{-i} z_2^j}{i} = - \frac{1}{l (1 - w_1^+ w_2^-)} , \quad l \leq -1,
\] (51)
\[
\sum_{i \geq 1} \sum_{j \leq -1} C_{ij}^l \frac{z_1^{-i} z_2^j}{i} = \frac{1}{1 - w_1^+ w_2^-}.
\] (52)

With this definition of the structure constants at hand, one can prove WDVV for any solution to the dToda hierarchy\footnote{In a very particular case, the infinite WDVV in dToda hierarchy were found in \cite{18}.}
\[
\sum_{l=-\infty}^{\infty} C_{ij}^l F_{ikm} = \sum_{l=-\infty}^{\infty} C_{ik}^l F_{ijm}
\] (53)
in the same way as for the dKP-case. The calculation is somewhat longer since the cases when all indices have different possible signs have to be considered separately. Details of the proof are given in the appendix.

The realization of the associative algebra defined by the structure constants (48)–(52) is obtained with the help of eq. (45) exactly in the same way as in the dKP-case. The generators
\[
\phi_i(w) = w \frac{dH_i}{dw}
\] (54)
for all integer \( i \) span the ring of Laurent polynomials of arbitrary degree. In this basis the structure constants of the algebra are given by (54)–(52).
The derivation of the residue formulas from Hirota relations is also parallel to the dKP-case. Consider for simplicity the case when all the indices are positive. We have:

\[ D_1^+ D_2^+ D_3^+ \mathcal{F} = \sum_{\alpha=1}^{3} \text{res}_{w_\alpha} \left( \frac{D^+(z^+(w))\partial_{t_1}^2 \mathcal{F}}{(w-w_1)(w-w_2)(w-w_3)} dw \right). \]

By virtue of (43), this is equivalent to

\[ \mathcal{F}_{jkm} = \frac{1}{2\pi i} \oint_{C_\infty} \frac{\partial_{t_0} z^+(w)}{z^+(w)^{j+1}} \phi_j(w)\phi_k(w)\phi_m(w) \frac{dw}{w^2}, \quad j, k, m \geq 1, \tag{55} \]

where again \( z^+(w)' = dz^+/dw \). Similar formulas can be written for non-positive indices.

### 4.2 The associative algebra related to conformal maps

As shown in [11], a particular solution \( \mathcal{F}^{(0)} \) to the dToda hierarchy describes evolution of conformal mapping of a complex domain with respect to deformations of the domain. This solution is specified by the reality conditions \( t_{-k} = \bar{t}_k \), \( w^-(z) = \bar{w}^+(z) \) consistent with the hierarchy. Under the reality conditions \( \mathcal{F}^{(0)} \) is a real-valued function of times. The relation to conformal maps is as follows. Let \( z(w) = rw + \sum_{k \geq 0} u_k z^{-k} \) be the univalent conformal map\(^3\) from the exterior of the unit circle \(|w| > 1\) to the exterior of a given analytic curve \( \gamma \), the normalization being fixed by the conditions that infinity is taken to infinity and \( r \) is real and positive. Then we set \( z^+(w) = z(w), \ z^-(w) = \bar{z}(w^{-1}) \). The function \( w(z) = w^+(z) \) is the inverse map. It has been shown [11] that evolution of the map is described by the dToda hierarchy with the generators of commuting flows given by (43). The reality conditions imply \( H_{-j}(w) = H_j(w^{-1}) \). The times are harmonic moments of the exterior domain:

\[ t_k = \frac{1}{2\pi i} \oint_{\gamma} z^{-k} \bar{z} dz \]

with the origin assumed to be outside the domain. The “initial conditions” for the solution are given by the dispersionless limit of the string equation:

\[ \{ z(w), \ \bar{z}(w^{-1}) \} + \mathcal{F}^{(0)} = 1. \tag{57} \]

In this setting the residue formula (55) can be written in a more transparent form. Since \( z(w) \) maps the exterior of the unit circle in a conformal manner, for some region in the space of \( t_k \), neither zeros of \( z'(w) \) nor poles or other singularities of \( z(w) \) are in the domain \(|w| > 1\). Therefore, the function under the integral in (55) is regular everywhere outside the unit circle except infinity. So, the integration contour can be taken to be the unit circle \(|w| = 1\):

\[ \mathcal{F}^{(0)}_{jkm} = \frac{1}{2\pi i} \oint_{|w|=1} \frac{\partial_{t_0} z(w)}{z'(w)} \phi_j(w)\phi_k(w)\phi_m(w) \frac{dw}{w^2}, \quad j, k, m \geq 1. \tag{58} \]

The string equation (57) reads

\[ \frac{\partial_{t_0} z(w)}{z'(w)} = -w^2 \frac{\partial_{t_0} \bar{z}(w^{-1})}{\bar{z}'(w^{-1})} + \frac{w}{z'(w)\bar{z}'(w^{-1})} \]

\(^3\)Here and below bar means complex conjugation and for any series \( f(z) = \sum f_k z^k \) we set \( \bar{f}(z) = \sum \bar{f}_k z^k \).

\(^4\)To ensure conformality and univalentness, absolute values of the coefficients \( u_k \) should be restricted by some inequalities which we do not discuss here (see e.g. [11]).
where \( \bar{z}'(w^{-1}) \) is the derivative \( d\bar{z}/dw \) taken at the point \( w^{-1} \). Plugging this into (58) and taking into account that the function \( \partial_{t_0} \bar{z}(w^{-1})/\bar{z}'(w^{-1}) \) is regular inside the unit circle, we come to

\[
\mathcal{F}_{jkm}^{(0)} = \frac{1}{2\pi i} \oint_{|w|=1} \frac{\phi_j(w)\phi_k(w)\phi_m(w)}{z'(w)\bar{z}'(w^{-1})} \frac{dw}{w} = -\frac{1}{2\pi i} \oint_{\gamma} \frac{dH_jdH_kdH_m}{dzd\bar{z}}.
\]  

(59)

A more detailed analysis shows that this formula is valid, up to an overall sign, for all integer indices, not only for positive ones. This formula was first derived by I. Krichever \[20\] within a different approach.

One may again interpret (59) as the scalar product

\[
\langle f \cdot g \rangle = \frac{1}{2\pi i} \oint_{|w|=1} \frac{df}{wz'(w)\bar{z}'(w^{-1})} f(w)g(w)
\]

(60)

on the infinite-dimensional ring of Laurent polynomials. In the basis (54) the algebra reads \( \phi_i(w)\phi_j(w) = \sum_k C_{ij}^k \phi_k(w) \), where the structure constants are given by (49)-(52). The proof is the same as for (54).

Comparing with the dKP residue formula (55), one may say that the requirement of reality together with that of conformality and univalency effectively defines a reduction: in both cases these conditions ensure that the integral in (54) is saturated by singularities coming from the denominator. From this point of view, one may regard conformal maps as an infinite-dimensional reduction of the dToda hierarchy.

In the rest of this section we discuss a further reduction leading to new solutions of finite-dimensional WDVV equations. Consider a class of conformal maps represented by Laurent polynomials of the form

\[
z(w) = rw + \sum_{l=0}^{N-1} u_l w^{-l}.
\]

(61)

As proved in \[11\], this class of functions represents conformal maps to domains with a finite number of non-zero moments, namely, with \( t_k = \tilde{t}_k = 0 \) for \( k > N \). The residue formula (54) with \( j, k, m \geq 0 \), specific to this case reads

\[
\mathcal{F}_{jkm}^{(0)} = \sum_{a=1}^{N} \frac{\phi_j(w_a)\phi_k(w_a)\phi_m(w_a)w_a^{N-1}}{P_N(w_a)Q_N(w_a)}
\]

(62)

where \( P_N(w) = w^N z'(w) \), \( Q_N = \bar{z}'(w^{-1}) \) are polynomials of \( N \)-th degree:

\[
P_N(w) = rw^N - \sum_{k=1}^{N-1} k u_k w^{N-k-1}, \quad Q_N(w) = r - \sum_{k=1}^{N-1} k \tilde{u}_k w^{k+1}
\]

and \( w_a \) are zeros of \( P_N(w) \). They are inside the unit circle whereas zeros of \( Q_N \) are outside. Recall that \( \phi_j(w) \) for \( j \geq 0 \) are polynomials in \( w \) (of \( j \)-th degree). In particular, \( \phi_0(w) = 1 \).

The algebra

\[
\phi_j(w)\phi_k(w) = \sum_{l=0}^{N-1} \tilde{C}_{jk}^l \phi_l(w) \quad (\text{mod } P_N(w)), \quad j, k = 0, 1, \ldots, N - 1
\]

(63)

is an \( N \)-dimensional associative algebra isomorphic to the ring of all polynomials factorized over the ideal generated by \( P_N(w) \). It is easy to see from the residue formula (52) that
the structure constants obey $\mathcal{F}^{(0)}_{ijk} = \sum_{l=0}^{N-1} \tilde{C}^{li}_{ij} \mathcal{F}^{(0)}_{lk0}$. To do that, one should apply (62) to the $N$-dimensional set of flows $t_0, t_1, \ldots, t_{N-1}$. Therefore, we conclude that (logarithm of) the tau-function for curves [10, 11], being restricted to the space where all the times $t_k, \bar{t}_k$ with $k > N$ are zero (and $t_N \neq 0$ plays a role of a parameter), provides a solution to the finite WDVV equations. More precisely,

$$f(t_0, t_1, \ldots, t_{N-1}) = \mathcal{F}^{(0)}(t_0; t_1, \ldots, t_{N-1}, 0, 0, \ldots; \bar{t}_1, \ldots, \bar{t}_N, 0, 0, \ldots) \bigg|_{t_N \neq 0, \bar{t}_j \text{ fixed}}$$

(64)
solves the WDVV system (5) with the matrices $(F_{i})_{jk} = \frac{\partial^3 f}{\partial t_i \partial t_j \partial t_k}$, $0 \leq i, j, k \leq N - 1$.

(We stress that the “antiholomorphic” times $\bar{t}_k$ and the highest non-zero time $t_N$ are kept constant under the differentiation.)

In contrast to solutions to the finite WDVV system discussed in Sec. 3.4, the solutions constructed here do not allow one to switch on the higher flows (in other words there is no large phase space) since they do not preserve the form (61).

For $N = 2$ (the curve is an ellipse), the WDVV are empty since the number of independent variables is less than three. The first non-trivial example is the case $N = 3$, where the tau-function is not available in an explicit form. This makes it difficult to confirm the result by a direct verification. One might find $\mathcal{F}^{(0)}_{ijk}$ directly from the residue formula using one or another parameterization of the conformal map (with enough number of independent parameters). In this way, we checked that the solutions obtained do obey WDVV for $N = 3$ and $N = 4$ using MAPLE.

We expect that a more general class of finite-dimensional solutions to WDVV is generated by the tau-function of curves in the case when $z'(w)$ is a rational function. We hope to discuss this elsewhere.

### 5 Conclusion

We have shown that given a solution to the dKP or dToda integrable hierarchies the Hamiltonians of commuting flows form an associative (in general infinite dimensional) algebra, isomorphic either to polynomial ring or to the ring of certain rational functions (Laurent polynomials). In the basis provided by integrable hierarchy the structure constants of the algebra are expressed through the third order derivatives of the logarithm of the tau-function. The algebra becomes finite dimensional in the case of special solutions known as finite dimensional or algebraic reductions of the hierarchy. Such algebras were previously known for the $N$-KdV reductions of the dKP hierarchy. We showed that the Hirota’s relations in the dispersionless limit determine the structure constants of the algebra and the associativity conditions or WDVV equations can be obtained as a simple consequence of Hirota’s equations.

This provides a motivation to investigate whether any solution to WDVV equations obeys some Hirota-type relations for second order derivatives of the $F$. The theory of the Whitham hierarchy on Riemann surfaces of arbitrary genus developed in [3] suggests a proper generalization of the dispersionless Hirota equations and associative algebras to higher genus.

Starting from the tau-function for conformal maps, we have constructed a family of solutions to WDVV equations with finite number of variables. They correspond to some finite-dimensional associative algebras of a special form.

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Appendix

Here we derive the WDVV equations for the dToda hierarchy starting from Hirota’s relations (41, 42).

Let all free indices in (53) be strictly positive. Passing to the generating functions, we rewrite the equation as

$$\frac{1}{w_{12}}(w_1 D_1 - w_2 D_2) D_3 D_4 F = \frac{1}{w_{13}}(w_1 D_1 - w_3 D_3) D_2 D_4 F,$$

where we set $w_i^+ = w_i$, $D_i^+ = D_i$ for simplicity of notation. Eq. (63) is equivalent to

$$w_1(w_{13} D_1 D_3 D_4 F - w_{12} D_1 D_2 D_4 F - w_{23} D_2 D_3 D_4 F) = 0,$$

which is true due to the identity

$$w_{12} D_1 D_2 D_4 F + w_{23} D_2 D_3 D_4 F + w_{31} D_1 D_3 D_4 F = 0.$$

Identity (67) can be easily obtained from (17) by passing to the generating functions (multiplying it by $z_4^{-k}$, summing over $k$) and then adding two more contributions with cyclic permutations of $1, 2, 3$. The same arguments go through without modifications if the index $m$ in (53) is non-positive.

Let $j$ be negative, and all others be positive. Below, we employ the simplified notation $w_i^- = \bar{w}_i$, $D_i^- = \bar{D}_i$. In terms of the generating functions the WDVV now reads

$$w_{13} D_1 D_3 D_4 F = -\frac{(D_1 + \bar{D}_2 + \partial_{t_0}) D_3 D_4 F}{w_1 \bar{w}_2 - 1}.$$

We express $D_2 D_1 D_4 F$ and $D_2 D_3 D_4 F$ from (18) and substitute them back into (68). It gives

$$w_{12} D_1 D_2 D_4 F + w_1 D_1 D_4 F_0 - w_3 D_3 D_4 F_0 = 0,$$

in which one readily recognizes eq. (17), which concludes the proof for this case. At last, let $j, k$ be negative, and $i, m$ be positive. In this case we have the following WDVV to prove:

$$\frac{(D_3 D_4 F_0 + \bar{D}_3 D_1 D_4 F + \bar{D}_3 D_2 D_4 F)}{(w_1 \bar{w}_2 - 1)} = \frac{(D_2 D_4 F_0 + \bar{D}_2 D_1 D_4 F + \bar{D}_2 D_3 D_4 F)}{(w_1 \bar{w}_3 - 1)}.$$

As in a previous case use analog of eq. (18) to substitute for $\bar{D}_3 D_1 D_4$ and $\bar{D}_2 D_1 D_4$. Also note that $\bar{D}_2 \bar{D}_3 D_4$ together with $\bar{D}_3 D_4 \partial_{t_0} F$ and $\bar{D}_2 D_4 \partial_{t_0} F$ form analog of (69). All these facts combined together show that (70) holds, which concludes the proof of WDVV for the Toda case.
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