Gradient Descent Converges to Ridgelet Spectrum

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Abstract

Deep learning achieves a high generalization performance in practice, despite the non-convexity of the gradient descent learning problem. Recently, the inductive bias in deep learning has been studied through the characterization of local minima. In this study, we show that the distribution of parameters learned by gradient descent converges to a spectrum of the ridgelet transform based on a ridgelet analysis, which is a wavelet-like analysis developed for neural networks. This convergence is stronger than those shown in previous results, and guarantees the shape of the parameter distribution has been identified with the ridgelet spectrum. In numerical experiments with finite models, we visually confirm the resemblance between the distribution of learned parameters and the ridgelet spectrum. Our study provides a better understanding of the theoretical background of an inductive bias theory based on lazy regimes.

1 Introduction

Characterizing the local minima of the gradient descent (GD) learning is important for the theoretical study of neural networks. Due to the non-convexity of the learning problem, it is a hard and challenging problem. The over-parametrization, an assumption that the parameter number is sufficiently larger than the sample size, is considered to be an important factor to prove the better performance of deep learning (Arora et al., 2019b; Allen-Zhu et al., 2019; Nitanda and Suzuki, 2017; Rotskoff and Vanden-Eijnden, 2018; Mei et al., 2018; Sirignano and Spiliopoulos, 2020a; Chizat and Bach, 2018; Jacot et al., 2018; Lee et al., 2019; Frankle and Carbin, 2019). By regarding the weights of parameters in a neural network as a signed distribution, we analyze the over-parametrized regime by means of the integral representation (Barron, 1993; Murata, 1996; Sonoda and Murata, 2017): $S[\gamma] := \int \gamma(a, b) \sigma(a \cdot x - b) da db$. Here, $\gamma$ represents the weights, or the parameter distribution, and $\sigma(a \cdot x - b)$ represents a hidden unit with an activation function $\sigma$, input $x$ and hidden parameters $(a, b)$. This is a weighted integral of infinite hidden units, but we remark that by formally letting a singular measure $\sum_{i=1}^{p} c_i \delta(a_i, b_i)$ as $\gamma$, we can also represent a weighted sum of finite hidden units as $\sum_{i=1}^{p} c_i \sigma(a_i \cdot x - b_i)$.

Since all the hidden parameters $(a, b)$ are integrated out, we do not need to update hidden parameters during the training, and we only need to update the parameter distribution $\gamma$. This is a strong advantage because the learning problem regains the convexity in the function space. This convexification trick has been known and employed in the integral representation theory (Barron, 1993) a.k.a. ridgelet analysis (Murata, 1996; Candes, 1999), convex neural networks (Bengio et al., 2006) and random Fourier features (Rahimi and Recht, 2008). Recently, the mean-field regime (Mei et al., 2018; Rotskoff and Vanden-Eijnden, 2018; Sirignano and Spiliopoulos, 2020a), a.k.a. the Wasserstein gradient flow theory (Chizat and Bach, 2018; Nitanda and Suzuki, 2017) also adopted this formulation, and the integral representation has been recognized as a crucial tool to prove the global convergence of deep learning.

Thus far, we know less about the minimizers themselves, due to the non-trivial null space of the integral operator $S$. In fact, there are infinitely many different parameter distributions, say $\gamma_1$ and $\gamma_2$, that indicate the same function: $S[\gamma_1] = S[\gamma_2]$ (see Appendix C.3 for more details). In this study, we consider a regularized square loss minimization problem and provide a unique explicit representation of the global minimizer in terms of the ridgelet transform on the torus.
The ridgelet transform, which is a wavelet-like integral transform, is originally developed in (Murata, 1996; Candès, 1998; Sonoda and Murata, 2017), and has a remarkable application to analysis of neural networks. Whereas in the original studies of the ridgelet transform, they employed the Fourier analysis on the Euclidean space, we utilize the Fourier analysis on the torus, and develop a simple but flexible framework to study the neural networks with modern activation functions such as the rectified linear unit (ReLU). Although the Fourier analysis on the torus imposes the periodicity on the activation function, we theoretically show a periodic activation function still provides a sufficient and effective power to analyze the over-parametrized neural networks.

To be precise, our main theorem is described as follows:

**Main Theorem.** Let \( F := L^2(\mathbb{R}^m, P) \) be the space of data generating functions (\( P \): finite Borel measure with density \( q \)) and \( G := L^2([-A, A]^m \times [-T/2, T/2]; \text{d}a\text{d}b) \) the space of parameter distributions. Here \( A \) and \( T \) be a bounds of the hidden parameters. We assume the activation function \( \sigma \) is periodic with period \( T \). For any \( f \in F \) and \( \beta > 0 \), the unique solution of the following minimization problem

\[
\gamma_{A,\beta}^*[f] := \arg \min_{\gamma \in G} \left\| f - \int \gamma(a, b)\sigma(a \cdot x - b)\text{d}a\text{d}b \right\|_F^2 + \beta \|\gamma\|_G^2.
\]

is uniquely represented by the ridgelet transform:

\[
\gamma_{A,\beta}^*[f] = \int_{\mathbb{R}^m} f(x)q(x)\frac{1}{q(x) + \beta}\sigma(a \cdot x - b)\text{d}x + (\text{residual terms depending only on } P, A \text{ and } \beta).
\]

Here, the residual terms tend to 0 as \( A \to \infty \).

Numerical simulation confirms our main results, namely, the scatter plot of parameter distributions learned by GD shows a similar pattern to the ridgelet spectrum. As a consequence, we can also gain a better understanding of the theoretical background of lazy learning, a recent trend of inductive bias theory stating that the learned parameters are very close to the initial parameters, such as the neural tangent kernel (Jacot et al., 2018; Lee et al., 2019; Arora et al., 2019a) and the strong lottery ticket hypothesis (Frankle and Carbin, 2019).

The structure of this paper is as follows: In Section 2, we develop the theory of the ridgelet transform on the torus. It is a theoretical basis to provide an explicit representation of the over-parametrized neural network at the global minimum. Then we introduce a positive definite kernel to prove universality of neural networks with modern activation functions such as the rectified linear unit (ReLU). Although the Fourier analysis on the torus imposes the periodicity on the activation function, we theoretically show a periodic activation function still provides a sufficient and effective power to analyze the over-parametrized neural networks.

**Notation.** For a measurable space \( X \) with a positive measure \( \mu \), we denote by \( L^2(X, \mu) \) the square integrable functions on \( X \) with respect to \( \mu \).

## 2 Ridgelet Transforms on the Torus

In this section, we establish the ridgelet transform on the torus, which is a theoretical basis of this study. We fix \( T > 0 \), and denote by \( \mathbb{T} \) the torus \( \mathbb{R}/T\mathbb{Z} \), and often regard \( \mathbb{T} \) as the interval \([-T/2, T/2]\). We fix a bounded measurable function \( \sigma : \mathbb{T} \to \mathbb{R} \), or equivalently, a bounded measurable periodic function \( \sigma \) on \( \mathbb{R} \) with period \( T \) \((\sigma(x + T) = \sigma(x))\). For an integer \( n \), we write \( \hat{\sigma}(n) := (1/T) \int_{-T/2}^{T/2} \sigma(x)e^{2\piinx/T} \). Originally, the ridgelet transform has been defined on the Euclidean space (Murata, 1996; Candès, 1999). However, the original definition excludes non-integrable activation functions such as Tanh and ReLU. Sonoda and Murata (Sonoda and Murata, 2017) have extended the ridgelet transform to accept such non-integrable activation functions, by introducing an auxiliary dual activation function. However, their theory sacrifices the Plancherel formula, which we need in this study. Therefore, in order to cover the non-integrable activation functions, we come to suppose periodic activation functions.
2.1 Ridgelet transform

We introduce the ridgelet transform and its reconstruction formula.

Definition 2.1. We define the ridgelet transform \( R : L^2(\mathbb{R}^m, dx) \to L^2(\mathbb{R}^m \times \mathbb{T}, dadb) \) by

\[
R[f](a, b) := \int_{\mathbb{R}^m} f(x) \sigma(a \cdot x - b) dx.
\]  

To be precise, we define \( R[f] \) for all \( f \in L^2(\mathbb{R}^m) \) via bounded extension, an essentially the same arguments in the definition of the \( L^2 \)-Fourier transform. Namely, We first define \( R[f] \) for \( f \in L^1(\mathbb{R}^m) \), which is absolutely convergent because \( \sigma \in L^\infty(\mathbb{T}) \). Then, we extend \( R[f] \) for \( f \in L^2(\mathbb{R}^m) \) as a common limit of \( R[f_i] \), where \( f_i \) is any sequence in \( L^1(\mathbb{R}^m, dx) \cap L^2(\mathbb{R}^m, dx) \) that converges to \( f \) in \( L^2 \). Let us introduce the admissible condition on \( \sigma \):

Assumption 2.2 (admissible condition). The function \( \sigma \in L^\infty(\mathbb{T}) \) satisfies the following two conditions: (1) \( \hat{\sigma}(0) = 0 \), and (2) \( \sum_{n \neq 0} |\hat{\sigma}(n)|^2 / |n|^m = T^{-m-1} \).

We need the admissibility condition in the proof of the reconstruction formula blow. It is not at all strong. In fact, the infinite sum of the second condition always converge because \( \sigma \) is square integrable, thus, we may replace \( \sigma \) with a function satisfying these condition via only multiplying and subtracting constants. In particular, restrictions of Tanh and ReLU to \( \mathbb{T} \) can satisfy this assumption with slight modifications on the constants. Under the admissible condition, the ridgelet transform meets the reconstruction formula and the Plancherel formula as follows:

Theorem 2.3. Impose Assumption 2.2 on \( \sigma \). Then for \( f, g \in L^2(\mathbb{R}^m, dx) \), we have

\[
\int_{\mathbb{R}^m \times \mathbb{T}} R[f](a, b) \sigma(a \cdot x - b) dadb = f(x), \tag{4}
\]

\[
\langle R[f], R[g] \rangle_{L^2(\mathbb{R}^m \times \mathbb{T}, dadb)} = \langle f, g \rangle_{L^2(\mathbb{R}^m, dx)} \tag{5}.
\]

By the Plancherel formula (5), the adjoint operator of \( R \) is calculated as \( R^*[\gamma](x) = \int \gamma(a, b) \sigma(a \cdot x - b) dadb \) for \( \gamma \in \text{Im}(R) \). This might be regarded as an integral representation of the neural network. However, this integral transform \( R^* \) is defined only on the image of \( R \), which is hard to specify (due to the non-triviality of the null space \( \ker S \)). Thus, we will introduce the modified version of it in Section 2.3. By discretizing the integral in (4), we have a well-known universality in \( L^2(\mathbb{R}^m, P) \) (\( P \) is a finite Borel measure) of 2-layer neural networks with the activation \( \sigma \) as a corollary of Theorem 2.3:

Corollary 2.4. For any finite Borel measure \( P \) on \( \mathbb{R}^m \), the linear space generated by \( \{ \sigma_{a,b} : (a, b) \in \mathbb{R}^m \times \mathbb{T} \} \) is dense in \( L^2(\mathbb{R}^m, P) \).

2.2 Reproducing kernel Hilbert space with inner product of features

In this section, we introduce an RKHS, which is an effective framework to analyze behaviors of the parameters and expressive power of neural networks, and we prove a stronger universality result (Corollary 2.7) for 2-layer neural networks with parameters restriction for later use.

We fix a positive number \( A > 0 \) and let \( I_A := [-A, A] \). We define a positive definite kernel on \( \mathbb{R}^m \) by

\[
k_A(x, y) := \int_{\mathbb{T}^2} \sigma(a \cdot x - b) \sigma(a \cdot y - b) dadb.
\]  

We denote by \( \mathcal{H} \) the RKHS associated with the kernel \( k_A \). We call \( \mathcal{H} \) the RKHS with inner product of features. We remark that \( k_A \) is a continuous and bounded kernel. Next we discuss the characteristic property and \( c_{01} \)-universality ((Sriperumbudur et al., 2010, p.2392)) of \( \mathcal{H} \), namely, density properties in function spaces. To deal with this problem, let us introduce the following mild assumption on \( \sigma \):

Assumption 2.5. The bounded measurable function \( \sigma \) on \( \mathbb{T} \) satisfies \( \# \{ n \in \mathbb{Z} : \hat{\sigma}(n) \neq 0 \} = \infty \).
In other words, the $\sigma$ cannot be a finite sum of trigonometric polynomials, and thus any discontinuous square integrable function on $\mathbb{T}$ satisfies this assumption. For example, $\text{ReLU}|_{[-T/2,T/2]}$ and $\tanh|_{[-T/2,T/2]}$ satisfy this. Under this assumption we have the following theorem:

**Theorem 2.6.** Under Assumption 2.5, the $k_A$ is characteristic. If we additionally impose Assumption 2.2 on $\sigma$, $k_A$ is $c_0$-universal.

By means of Theorem 2.6, we prove a stronger form of universality as follows:

**Corollary 2.7.** For any finite Borel measure $P$ on $\mathbb{R}^m$ and $A > 0$, the linear space generated by $\{\sigma_{a,b} : (a,b) \in \mathbb{I}_m^\ast \times \mathbb{T}\}$ is dense in $L^2(\mathbb{R}^m, P)$. Here we define $\sigma_{a,b}(x) := \sigma(a \cdot x - b)$.

**Proof.** We here denote by $\langle , \rangle$ the inner product in $L^2(\mathbb{R}^m, P)$. It suffices to show that for any $f \in L^2(\mathbb{R}^m, P)$, $f = 0$ if $\langle f, \sigma_{a,b} \rangle = 0$ for all $(a,b) \in \mathbb{I}_m^\ast \times \mathbb{T}$. Since $\sigma_{a,b}$ is a nonzero constant function for $a,b \in \mathbb{T}$, we see that $\langle f, 1 \rangle = 0$. Let $k_{A,y}(x) := k_A(x,y)$. Then we have $\langle f, k_{A,y} \rangle = \int_{\mathbb{I}_m^\ast \times \mathbb{T}} f(a,b) \sigma_y(a,b) \, da \, db = 0$ for all $y \in \mathbb{R}^m$ where $\sigma_y(a,b) := \sigma(a \cdot x - b)$. Since $\mathcal{H}$ is generated by $k_{A,x}$’s, thus we conclude that $f$ is contained in the orthogonal complement of $\mathbb{R} + \mathcal{H}$. Since $k_A$ is characteristic, the space $\mathbb{R} + \mathcal{H}$ is dense in $L^2(\mathbb{R}, P)$ (cf. (Keni Fujumizy and Jordan, 2009, Proposition 5), (Sriperumbudur et al., 2010, Section 3.2)). Hence we have $f = 0$.

Compared with Proposition 2.4, Corollary 2.7 provides a practically important conclusion, namely, it says even if the parameters in the hidden layer are bounded, 2-layer neural networks have sufficient expressive power under Assumption 2.5.

### 2.3 Integral representation of neural networks

In this section, we define an integral representation of a 2-layer neural network. It is also regarded as a truncated version of the adjoint operator $R^*$ of the ridgelet transform $R$. Although the theory of the ridgelet transform on $\mathbb{R}^m \times \mathbb{T}$ is very clear, it has a flaw to analyze the neural networks. In fact, because $L^2(\mathbb{R}^m, dx)$ does not contain $\sigma_{a,b}(x) := \sigma(a \cdot x - b)$, thus any finite neural networks, we cannot see the direct connection between finite neural networks and integral representations of neural networks. To circumvent this technical issue, we consider a $P$-weighted version (since $\sigma_{a,b} \in L^2(\mathbb{R}^m, P)$).

**Definition 2.8.** We define an integral representation of a neural network $S_A : L^2(\mathbb{I}_m^\ast \times \mathbb{T}, da \, db) \to L^2(\mathbb{R}^m, P)$ by

$$S_A[\gamma](x) := \int_{\mathbb{I}_m^\ast \times \mathbb{T}} \gamma(a,b) \sigma(a \cdot x - b) \, da \, db.$$  \hspace{1cm} (7)

The operator $S_A$ can be regarded as a limit of neural networks of the form $\sum_i c_i \sigma(a_i \cdot x - b_i)$ whose hidden parameters $(a_i, b_i)$ are contained in $\mathbb{I}_m^\ast \times \mathbb{T}$. By simple computation, we see that the adjoint operator $S_A^*$ is explicitly represented as $S_A^*[f](a,b) = \int f(x) \sigma(a \cdot x - b) \, dx \, dp(x)$. Thus $S_A$ is the adjoint operator of a weighted analogue of the ridgelet transform (c.f. Definition 2.1). Then we have the following proposition describing the expressive power of $S_A$:

**Proposition 2.9.** Assume $\sigma$ is continuous at a point $b_0$ and $\sigma(b_0) \neq 0$. The image of $S_A$ is dense in $L^2(\mathbb{R}^m, P)$.

**Proof.** Denote by $\langle , \rangle$ the inner product of $L^2(\mathbb{R}^m, P)$. Since the image of $S_A$ is the same as the orthogonal complement of the adjoint operator $S_A^*$, it suffices to show that $S_A^*[f] = 0$ implies $f = 0$. In fact, if $S_A^*[f] = 0$, then $S_A^*[f](a,b) = \langle f, \sigma_{a,b} \rangle = 0$ for almost every $(a,b)$. Since $S_A^*[f]$ is continuous on $\{a \neq 0\}$, we see that $\langle f, \sigma_{a,b} \rangle = 0$ for all $(a,b)$ with $a \neq 0$. In addition, $0 = \lim_{|a| \to 0} S_A^*[f](a,b_0) = \sigma(b_0)(f,1)$. Thus by Corollary 2.7, we have $f = 0$. \hfill $\square$
3 Main Results

In this section, we describe the formulation of our problem and main results. We impose Assumptions 2.2 and 2.5 on the bounded measurable map $\sigma$ on $T$. We fix an absolutely continuous finite Borel measure $P$ on $\mathbb{R}^m$. We assume $P$ has a bounded density function $q$.

3.1 Square loss minimization for the integral representation

Our main goal is to provide an explicit representation of the global minimizer of the learning problem:

$$\min_{\gamma} \|f - S_A(\gamma)\|_{L^2(\mathbb{R}^m, P)}^2. \quad (8)$$

We regard the function $\gamma \in L^2(I^n \times T, d\alpha db)$ as a distribution $\gamma(a, b)d\alpha db$. We will give an explicit representation to the distribution attaining the solution of (8) in terms of the ridgelet transform.

For $f \in L^2(\mathbb{R}^m, P)$, $A > 0$, and $\beta > 0$, we consider the $L^2$-regularized square loss of the integral representation $S_A$:

$$L_{\beta, A}(\gamma; f) := \|f - S_A(\gamma)\|_{L^2(\mathbb{R}^m, P)}^2 + \beta \|\gamma\|_{L^2(I^n \times T, d\alpha db)}^2. \quad (9)$$

We denote by $\gamma_{\beta, A}^*[f]$ the unique element that attains min$_\gamma L_{\beta, A}(\gamma; f)$, which always exists as long as $S_A$ is densely defined closed operator. See Appendix D for more details. Although $\|\gamma\|_{L^2(I^n \times T, d\alpha db)}$ can tend to infinity as $\beta \to 0$, by Proposition 2.9, we have the following proposition:

**Proposition 3.1.** Assume $\sigma$ is continuous at a point $b_0$ and $\sigma(b_0) \neq 0$. Then the square loss $\|f - S_A(\gamma_{\beta, A}^*[f])\|_{L^2(\mathbb{R}^m, P)}^2$ converges to 0 as $\beta \to 0$.

**Proof.** Let $\varepsilon > 0$ be an arbitrary positive number. By Proposition 2.9, there exists an element of $L^2(I^n \times T, d\alpha db)$ such that $\|f - S_A(\gamma)\|_{L^2(\mathbb{R}^m, P)}^2 < \varepsilon/2$. Take $\beta = \varepsilon/2 \|\gamma\|_{L^2(I^n \times T, d\alpha db)}$. Then $\gamma_{\beta, A}^*[f]$ satisfies $\varepsilon > L_{\beta, A}(\gamma_{\beta, A}^*[f]; f) > \|f - S_A(\gamma_{\beta, A}^*[f])\|_{L^2(\mathbb{R}^m, P)}^2$. Thus we have $\|f - S_A(\gamma_{\beta, A}^*[f])\|_{L^2(\mathbb{R}^m, P)}^2 \to 0$. \qed

3.2 An explicit representation of the global minimizer

Our first main result is the explicit representation of the minimizer of the regularized square loss minimization problem in terms of the ridgelet transform.

**Theorem 3.2.** Let $P$ be an absolutely continuous finite Borel measure on $\mathbb{R}^m$ with bounded density function $q$. Let $f \in L^2(\mathbb{R}^m, dx)$. Then, automatically $f \in L^2(\mathbb{R}^m, P)$ and we have

$$\gamma_{\beta, A}^*[f] = R \left[ \frac{qf}{\beta + q} \right] + \Delta_{\beta, A}[f], \quad (10)$$

where $\Delta_{\beta, A}[f]$ is an element of $L^2(I^n \times T, d\alpha db)$ such that

$$\lim_{A \to \infty} \|\Delta_{\beta, A}[f]\|_{L^2(I^n \times T, d\alpha db)} = 0. \quad (11)$$

By formally completing square in the Hilbert space, we can verify the unique existence of the minimizer $\gamma_{\beta, A}^*$. However, the concrete property of $\gamma_{\beta, A}^*$ is not clear. Theorem 3.2 provides the explicit representation of the minimizer via ridgelet transform.

3.3 Relation to the 2-layer finite neural networks

In this section, we prove that the over-parametrized finite neural networks converge to the minimizer in the integral representation (10) as the parameter number $p$ tends to infinity. The over-parametrization may let us
suppose that the parameter distribution \( \lambda_p := p^{-1} \sum_{i=1}^{p} \delta_{(a_i, b_i)} \) of a neural network \( g_p(x) := \sum_{i=1}^{p} c_i \sigma(a_i \cdot x - b_i) \) weakly converges to the uniform distribution \( U \) on \( \mathbb{R}^n \times \mathbb{T} \), and the problem reduces to the optimization of \( \{c_i\}_{i=1}^{p} \). Here, the weak convergence assumption is satisfied, for example, when the parameters \( \{a_i, b_i\}_{i=1}^{p} \) are i.i.d. samples drawn from \( U \). However, the randomness is not necessary in the proof. Let us consider the supervised learning problem as follows: Given a sequence \( \{\lambda_p\}_{p=1}^{\infty} \) of Borel measures that weakly converges to the Lebesgue measure \( \lambda \) on \( \mathbb{R}^n \times \mathbb{T} \), define \( S_p : L^2(\mathbb{R}^n \times \mathbb{T}, \lambda_p) \rightarrow L^2(\mathbb{R}^m, \mathbb{R}) \) by allocating \( f(a, b) \sigma(a \cdot x - b) d\lambda_p(a, b) = \sum_{i=1}^{p} \gamma(a_i, b_i) \sigma(a_i \cdot x - b_i) \) to \( \gamma \in L^2(\mathbb{R}^n \times \mathbb{T}, \lambda_p) \). Then for any bounded continuous function \( f \) on \( \mathbb{R}^n \times \mathbb{T} \), we approximately compute the ridgelet spectrum \( \sigma_p \). Theorem 3.3 implies the weak convergence of parameter distributions, which is a stronger convergence of over-parametrized neural networks to the global minimum than previous results:

**Theorem 3.3.** Assume \( \{\lambda_p\}_{p=1}^{\infty} \) weakly converges to \( U \) as \( p \rightarrow \infty \). Then the minimizer \( \gamma^p_{\beta, A} \) of (12) converges to the minimizer \( \gamma^*_{\beta, A} \) of (9) in the sense \( \|\gamma^p_{\beta, A} - \gamma^*_{\beta, A}\|_{L^2(\mathbb{R}^n \times \mathbb{T}, \lambda_p)} \rightarrow 0 \) as \( p \rightarrow \infty \).

Although we cannot catch the shape of the distribution of the optimal solution \( \gamma^p_{\beta, A} \) when the parameter number \( p \) is small, the over-parametrized neural networks accumulate the ridgelet spectrums. Combining Theorem 3.3 with Theorem 3.2, we obtain an explicit representation of the global minimizer via the ridgelet transform. Theorem 3.3 implies the weak convergence of parameter distributions, which is a stronger convergence of over-parametrized neural networks to the global minimum than previous results:

**Corollary 3.4.** The distribution \( \gamma^p_{\beta, A}(\lambda_p) \) weakly converges to an absolutely continuous distribution \( \gamma^*_{\beta, A}(\lambda) \) as \( p \rightarrow \infty \), namely, for any bounded continuous function \( f \) on \( \mathbb{R}^n \times \mathbb{T} \), we have \( \int f \gamma^p_{\beta, A}(\lambda_p) \rightarrow \int f \gamma^*_{\beta, A}(\lambda) \) as \( p \rightarrow \infty \).

In Section 4 below, we consider a learning problem a 2-layer neural network via gradient descent. We see the parameters of the over-parametrized neural networks accumulate the ridgelet spectrums.

## 4 Numerical Simulation

In order to verify the main results, we conducted numerical simulation with artificial datasets. Here, we only display the results of Experiment 1. The readers are also encouraged to refer supplementary materials for further experimental results.

### 4.1 Scatter plots of GD trained parameters.

Given a dataset \( D_n = \{(x_i, y_i)\}_{i=1}^{n} \), we repeatedly trained \( s = 1,000 \) neural networks \( g(x; \theta^{(t)}) = \sum_{j=1}^{n} c_j^{(t)} \sigma(a_{ij}^{(t)} x - b_{ij}^{(t)}) \), \( t \in [s] \) with activation function \( \sigma = \) Gaussian, Tanh and ReLU. The training is conducted by minimizing the square loss: \( L(\theta) = \frac{1}{n} \sum_{i=1}^{n} |y_i - g(x_i; \theta)|^2 \) using stochastic gradient descent with weight decay. Note that weight decay has an equivalent effect to \( L^2 \) regularization, which we assumed in the main theory. After the training, we obtained \( sp \) sets of parameters \( \{(a_{ij}^{(t)}, b_{ij}^{(t)}, c_j^{(t)})\}_{t \in [s], j \in [p]} \), and plotted them in the \((a, b, c)\)-space. (\( c \) is visualized in color.) See supplementary materials for more details on the settings.

### 4.2 Ridgelet spectrum

Given a dataset \( D_n = \{(x_i, y_i)\}_{i=1}^{n} \), we approximately compute the ridgelet spectrum \( R[f](a, b) \) of \( f \) at every sample points \((a, b)\) by numerical integration:

\[
R[f](a, b) \approx \frac{1}{n} \sum_{i=1}^{n} y_i \sigma(a x_i - b) \Delta x,
\]

where \( \Delta x \) is a normalizing constant, which is a constant because we assume that \( x_i \) be uniformly distributed. We remark that more sophisticated methods for the numerical computation of the ridgelet transform has been developed. See (Do and Vetterli, 2003) and (Sonoda and Murata, 2014) for example.
Figure 1: Parameter distributions $\gamma(a, b)$ trained by SGD (top) and ridgelet spectra $R[f](a, b)$ obtained by numerical integration (bottom) for the common data generating function $f(x) = \sin 2\pi x$, $x \in [-1, 1]$.

### 4.3 Results

In Figure 1, we have compared the scatter plot of gradient descent (GD) trained parameters and the ridgelet spectra. All six figures are obtained from the common data generating function $f(x) = \sin 2\pi x$ on $[-1, 1]$. Despite the fact that the scatter plot and ridgelet spectrum are obtained from different procedures: numerical optimization and numerical integration, both figures share characteristics in common. For example, red and blue parameters in the scatter plots (a-c) concentrate in the area where the ridgelet spectra (d-f) indicate the same colors. Due to the periodic assumption, the ridgelet spectrum spreads infinitely in $b$ with period $T = 1$. On the other hand, due to the weight decay assumption and initialized locations of parameters, the GD trained parameters gathers around the origin. Here, we used the uniform distribution $U([-1, 1])$ for the initialization. We can understand that these differences between the scatter plot and the spectrum as the residual term $\Delta_{A, B}$ in the main theorem. Another remarkable fact is that the GD trained parameters essentially did not change their positions in $(a, b)$ from the initialized value. This is possible because the support of initial parameters overlap the ridgelet spectrum from the beginning. We can understand this phenomenon as the so-called lazy regime.

### 5 Related Works

In the past, many authors have investigated the local minima of deep learning. However, these results have often posed strong assumptions such as that (A1) the activation function is limited to linear or ReLUs (Kawaguchi, 2016; Soudry and Carmon, 2016; Nguyen and Hein, 2017; Hardt and Ma, 2017; Lu and Kawaguchi, 2017; Yun et al., 2018); (A2) the parameters are random (Choromanska et al., 2015; Poole et al., 2016; Pennington et al., 2018; Jacot et al., 2018; Lee et al., 2019; Frankie and Carbin, 2019); (A3) the input is subject to normal distribution (Brutzkus and Globerson, 2017); or (A4) the target functions are low-degree polynomials or another sparse neural network (Yehudai and Shamir, 2019; Ghorbani et al., 2019). Due to these simplifying assumptions, we know very little about the minimizers themselves. In this study, from the perspective of functional analysis, we present a stronger characterization of the distribution of parameters in the over-parametrized setting. As a result, our theory (A1’) accepts a wide range of activation functions, (A2’) need not assume the randomness of parameter distributions, (A3’) need not specify the data distribution, and (A4’) preserves the universal approximation property of neural networks such as the density in $L^2$.

The mean-field regime theory (Rotskoff and Vanden-Eijnden, 2018; Mei et al., 2018; Sirignano and Spiliopoulos,
We believe this section is not applicable to this paper because of the theoretical nature of this study.

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A Proofs

A.1 Theorem 2.3

These formula follow from the computations described in "Reconstruction formula" in Appendix C.2.

A.2 Theorem 2.6

Theorem A.1. Under Assumption 2.5, the $k_A$ is characteristic. If we additionally impose Assumption 2.2 on $\sigma$, $k_A$ is $c_0$-universal.

Proof. By direct computation, we have

\[
k_A(x, y) = \int_{T^2 \times T} \sigma(a \cdot x - b)\sigma(a \cdot y - b) \, da \, db
\]

\[
= \int_{T^2} \sum_{n \in \mathbb{Z}} |\tilde{\sigma}(n)|^2 e^{2\pi i n \cdot (x - y)/T} \, da
\]

\[
= |\tilde{\sigma}(0)|^2 (2A)^m + \int_{T^2} \sum_{n \in \mathbb{Z} \setminus \{0\}} |\tilde{\sigma}(n)|^2 \frac{|n/T|^m}{|n/T|^m} 1_{[-nA/T,nA/T]}(a) e^{2\pi i a \cdot (x - y)} \, da.
\]

Put $C := |\tilde{\sigma}(0)|^2 (2A)^m$. Since we assume $\#\{n \mid \tilde{\sigma}(n) \neq 0\} = \infty$, the support of the function $\lambda_A$ is $\mathbb{R}^m$. Therefore, we see that $k_A - C$ is universal (see Section 3.2. of (Sriperumbudur et al., 2010)), thus $k_A$ is characteristic. Under Assumption 2.2, we have $C = 0$, and it implies $k_A$ itself is $c_0$-universal.  

A.3 Theorem 3.2

Here, we define

\[
K((a, b), (a', b')) := \int \sigma(a \cdot x - b)\sigma(a' \cdot x - b') \, dP(x).
\]

and for $(a, b) \in \mathbb{R}^m \times T$, we define

\[
T_A[\gamma](a, b) := \int_{T^2 \times T} \gamma(a', b') \cdot K((a', b'), (a, b)) \, da' \, db'.
\]

We define a bounded absolutely integrable function $\lambda_A$ by

\[
\lambda_A(x) := \sum_{n \neq 0} \frac{|\tilde{\sigma}(n)|^2}{|n/T|^m} 1_{[-nA/T,nA/T]}(x).
\]

Lemma A.2. The correspondence $x \mapsto \sigma_x$ is bounded and continuous mapping from $\mathbb{R}^m$ to $L^2(I_A \times T, dadb)$.

Proof. We may assume $\sigma$ is continuous function, thus we immediately see the continuity. The boundedness is obvious.  

Corollary A.3. Let $f \in L^1(\mathbb{R}^m, dx)$, and let $B$ be a bounded linear operator on $L^2(I_A \times T, dadb)$. Then for any $A' > 0$, $\int f(x)B[\sigma_x] \, dx$ is a well-defined element in $L^2(I_{A'} \times T, dadb)$ and satisfy for any $h \in L^2(I_{A'} \times T, dadb)$,

\[
\left\langle \gamma, \int f(x)B[\sigma_x] \, dx \right\rangle_{L^2(I_{A'} \times T, dadb)} = \int \langle \gamma, f(x)B[\sigma_x] \rangle_{L^2(I_{A'} \times T, dadb)} \, dx.
\]

Lemma A.4. For $g \in L^1(\mathbb{R}^m, dx)$, we have

\[
\left\| \int g(x)T_A[\sigma_x] \, dx \right\|_{L^2(I_A \times T, dadb)} \leq \left\| [g \lambda_A] \right\|_{L^2(\mathbb{R}^m, dx)}.
\]
Proof. Put \( \phi_A(x) := \lambda^2(x) \). Since
\[
T_A[\sigma_x](a, b) = \int \sigma_{a,b}(y) \lambda^2_A(x-y) dy,
\]
for \( B > 0 \), by direct computation, we have
\[
\left\| \int g(x) T_A[\sigma_x] dx \right\|_{L^2(\mathbb{T}^m, d\alpha dB)}^2
= \int g(x) g(y) q(z) \lambda^2_B(z-w) \lambda^2_A(y-w) dx dy dw dz
= \int [g * \lambda^2_A] g(z) \lambda^2_B(w-z) dw dz
= \int \left[ (g * \lambda^2_A)^2 \right](x) \lambda_B(x) dx.
\]
By taking \( B \) to \( \infty \), we have the formula. \( \square \)

Corollary A.5. For any \( g \in L^2(\mathbb{R}, dx) \), the integral \( \int g(x) T_A[\sigma_x] dx \) is well-defined in the similar manner with the Fourier transform. Moreover, we have
\[
\lim_{A \to \infty} \left\| \int g(x) T_A[\sigma_x] dx - g \right\|_{L^2(\mathbb{R}^m, d\alpha dB)} = 0.
\]

Theorem A.6. Let \( P \) be an absolutely continuous finite Borel measure on \( \mathbb{R}^m \) with density function \( q \). Let \( f \in L^2(\mathbb{R}^d, P) \). Assume \( q \) is bounded and \( f \in L^2(\mathbb{R}^d, dx) \). Then we have
\[
\gamma^*_\beta,A[f] = R\left[ \frac{qf}{\beta + q} \right] + \Delta_{\beta,A}[f],
\]
where \( \Delta_{\beta,A}[f] \) is an element of \( L^2(\mathbb{T}^m, d\alpha dB) \) such that
\[
\lim_{A \to \infty} \| \Delta_{\beta,A}[f] \|_{L^2(\mathbb{T}^m, d\alpha dB)} = 0.
\]

Proof. By the theory of the Tikhonov regularization, \( \gamma^*_\beta,A \) is explicitly described as follows:
\[
\gamma^*_\beta,A[f] = (\beta + T_A)^{-1} R_A[f],
\]
where we write \( T_A := R_A S_A \). We denote \( \frac{qf}{\beta + q} \) by \( g \). By direct computation, we have
\[
(\beta + R_A S_A)^{-1} R_A[f] = R[g] + (\beta + T_A)^{-1} \Lambda_A
\]
where we define \( \Lambda_A \) by
\[
\Lambda_A := \int g(\sigma_x - T_A[\sigma_x]) dx = R[qg] - \int T_A[\sigma_x] dx
\]
By Lemma A.4, we have \( \Lambda_A \in L^2(\mathbb{R}^m \times \mathbb{T}, d\alpha dB) \). By Corollary A.5, we see that \( \lambda_A \to 0 \) in \( L^2(\mathbb{R}^m \times \mathbb{T}, d\alpha dB) \). Therefore, we define
\[
\Delta_{\beta,A}[f] := (\beta + T_A)^{-1} \Lambda_A
\]
and the limit of \( \Delta_{\beta,A}[f] \) is zero as \( A \to \infty \). \( \square \)

A.4 Theorem 3.3

Here we prove the following statement:
Theorem A.7 (Theorem 3.3). Let \( f \in L^2(\mathbb{R}^m, P) \). For every \( p \in \mathbb{N} \), let \( \lambda_p := \frac{(2A)^m}{p} \sum_{i=1}^p \delta_{(a_i, b_i)} \) with \((a_i, b_i) \in I_A^m \times T\). Assume that \( \lambda_p \) weakly converges to the Lebesgue measure \( da db \) on \( I_A^m \times T \). Here, the weak convergence is in the sense that \( \int_{I_A^m \times T} h d\lambda_p(a, b) \to \int_{I_A^m \times T} h da db \) for any bounded continuous function \( h \) on \( I_A^m \times T \). Then the minimizer \( \gamma_{\alpha, A}^p[f] \) of
\[
\min \left\{ \| f - S_p[\gamma]\|_{L^2(\mathbb{R}^m, P)} + \beta \| \gamma \|_{L^2(I_A^m \times T, \lambda_p)} \right\}
\]
converges to the minimizer \( \gamma_{\alpha, A}[f] \) of \( \min, L_{\alpha}(\gamma; f) \) in the sense \( \| \gamma_{\alpha, A}^p[f] - \gamma_{\alpha, A}[f]\|_{L^2(I_A^m \times T, \lambda_p)} \to 0 \) as \( p \to \infty \).

Proof. We denote by \( L^2(\lambda_p) \) the square integrable space \( L^2(I_A \times T, \lambda_p) \), and by \( z_i \) the point \((a_i, b_i)\). Let \( T_A := R_A S_A \), and define a linear operator \( T_p := S_p R_A \) on \( L^2(I_A \times T, \lambda_p) \). We denote by \( G_p \) (resp. \( G \)) the minimizer \( \gamma_{\alpha, A}^p[f] = (\beta + T_p)^{-1} S_p[f] \) (resp. \( \gamma_{\alpha, A}[f] = (\beta + T_A)^{-1} R_A[f] \)). Since for any \( \gamma \in L^2(I_A \times T, da db) \), \( T_A[\gamma] \) is bounded and continuous at any \((a, b) \neq 0, G := \beta^{-1}(\gamma_0 - T_A[G]) \) is also bounded and continuous at any \((a, b) \neq 0 \). By direct computation, we have
\[
\| G - G_p \|_{L^2(\lambda_p)}^2 \leq \beta^2 \| (\beta + T_p)[G] - S_p[f] \|_{L^2(\lambda_p)}^2
\]
\[
= - \beta^2 \| (T_A - T_p)G + R_Af - S_p[f] \|_{L^2(\lambda_p)}^2
\]
\[
= - \beta^2 \| T_p[G] - T_A[G] \|_{L^2(\lambda_p)}^2
\]
\[
= \frac{(2A)^m}{\beta^2} p \sum_{i=1}^p \left( \sum_{j=1}^p G(z_j)(z_i, (a, b)) - \int_{I_A \times T} G(a, b)K(z_i, (a, b))da db \right)^2,
\]
where we denote \( K = (\sigma(\alpha \cdot x - b)\sigma(\alpha' \cdot x - b'))dP(x) \). For the first inequality, we use \( \| (\beta + T_p)^{-1} \| \leq \beta^{-1} \). For the third equality, we note that \( (a, b) \in \{ z_i \}_{i=1}^p \) \( S_p[f](a, b) = \int f(x) \sigma(\alpha \cdot x - b) \) \( dP(x) = R_A[f](a, b) \). We estimate the last term \( (16) \). Let \( I_A \times T = \bigcup_{k=1}^{N-1} Q_k \bigcup Q_0 \) be a decomposition of \( I_A \times T \) where \( \{ Q_k \}_{k=1}^{N-1} \) are pairwise disjoint cubics of lengths \( A/N \), and \( Q_0 := [-A/N, A/N)^m \). For \( k \geq 0 \), we denote \( n_{k, p} := \# \{ z_i \in Q_k : i = 1, \ldots, p \} \). Then \( (16) \) is
\[
\frac{(2A)^m}{\beta^2} \sum_{k=1}^N \frac{1}{n_{k, p}} \sum_{z \in Q_k} G(a, b)K(z, (a, b)) - \frac{(2A)^m}{p} G(z, (a, b))da db \leq (2A)^m \max(1 + C, 3C)\epsilon,
\]
where \( \{ Q_k \} := (A/N)^m \). For \( w \in I_A^m \times T \) and \( z \in Q_k \), we denote by \( I_{k, p}(z, w) \) the sum \( \sum_{k=1}^N \frac{1}{n_{k, p}} \sum_{z \in Q_k} G(a, b)K(z, (a, b)) - \frac{(2A)^m}{p} G(z, (a, b))da db \). Let \( C := \sup_{w \in I_A \times T} |G(z, (a, b))| \). Let \( \epsilon \) be an arbitrary positive number. We take a large \( N \) and \( p \) such that \( |Q_0| < \epsilon \) and \( (2A)^m n_{k, p} |Q_k| < \epsilon \). Unless \( k = 0 \), by the continuity of the integrand, we have \( I_{k, p}(z, w) = \frac{|Q_k|(1 + C)\epsilon}{p} \). For \( k = 0 \), we have \( I_{0, p} \leq 3C\epsilon \). Thus we have
\[
(16) \leq (2A)^m \max(1 + C, 3C)\epsilon,
\]
\[
\left\| G - G_p \right\|_{L^2(\lambda_p)}^2 \leq (2A)^m \max(1 + C, 3C)\epsilon.
\]
\[
\square
\]
A.5 Corollary 3.4

Here, we prove the following statement:

Corollary A.8. Let \( f \in L^2(\mathbb{R}^m, P) \). Then for any bounded continuous function \( h \) on \( I_A^m \times T \), we have \( \int h d\gamma_{\alpha, A}[f] \to \int h d\gamma_{\alpha, A}[f] \) as \( p \to \infty \).

Proof. It suffices to show that \( \langle h, \gamma_{\alpha, A}^p[f] - \gamma_{\alpha, A}[f] \rangle_{L^2(I_A^m \times T, \lambda_p)} \) goes to \( 0 \) as \( p \to 0 \). By using Schwartz inequality, we have
\[
\langle h, \gamma_{\alpha, A}^p[f] - \gamma_{\alpha, A}[f] \rangle_{L^2(I_A^m \times T, \lambda_p)} \leq \left\| \gamma_{\alpha, A}^p[f] - \gamma_{\alpha, A}[f] \right\|_{L^2(I_A^m \times T, \lambda_p)} \cdot (2A)^m \sup_{\mathbb{T} \times \mathbb{T}} |h|.
\]
The right hand side converges \( 0 \) as \( p \to \infty \) by Theorem A.7.
\[
\square
\]
Details on Numerical Simulation

For the sake of visualization, all the datasets are 1-in-1-out, so that the scatter plot will be displayed in a three-dimensional manner: \((a, b) \in \mathbb{R}^2\) in position and \(c \in \mathbb{R}\) in color. However, we remark that our main results are valid for any dimension. We always consider the uniform distribution \(x_i \sim U(-1, 1)\) for the input vectors, and generate \(n = 1,000\) samples for training, except for the case of Topologist’s Sine Curve (TSC) \(y_i = \sin \frac{2\pi}{x_i}\). For the TSC, we generate \(n = 10,000\) because the frequency tends to infinity as \(x\) tends to 0. We employ the stochastic gradient descent with learning rate \(\eta > 0\) and weight decay \(\beta > 0\) for the gradient descent training. We remark that the weight decay is equivalent to the \(L^2\)-regularization. The initial parameters are drawn from the uniform distribution \(U(-1, 1)\).
B.1 Results

**Experiment 1.** In order to see the differences among activation functions, we conduct the experiment on a common dataset: 
\[ y_i = \sin 2\pi x_i \]
with three activation functions: Gaussian \( \sigma(t) = \exp(-|kt|^2) \) with scale \( k = 6 \), Tanh \( \sigma(t) = \tanh(kt) \) with scale \( k = 6 \), and ReLU \( \sigma(t) = \max\{0, t\} \). In order to cover a characteristic part in the period \( T = [-1/2, 1/2) \), we introduced the scale parameter \( k \) for Gaussian and Tanh. As a result, all the three \( \sigma \)s have period \( T = 1 \). If an activation function is periodic with period \( T \), then the spectrum is periodic in \( b \) with period \( T \) because
\[
R[f; \sigma](a, b) = R[f; \sigma(\cdot - T)](a, b) = R[f; \sigma](a, b + T). \tag{17}
\]

We can verify that our theory accepts a variety of activation functions. For all the three settings, we trained \( s = 1000 \) networks, each single network has \( p = 100 \) hidden units, and the weight decay rate and learning rate were set to \( \beta = 0.001 \) and \( \eta = 0.01 \) respectively.

Figure 2: \( f(x) = \sin 2\pi x, \sigma(z) = \exp(-(kz)^2/2), \tanh(kz), \text{relu}(z) \) (from top to bottom)
**Experiment 2.** In order to focus on a structure as a ridgelet spectrum, we prepared translated datasets $y_i = \exp(-|x_i - \mu|^2/2)$ with $\mu = -0.5, 0, 0.5$. We employ the periodic ReLU on $\mathbb{T} = [-1/2, 1/2]$ for the activation function. According to the ridgelet transform, it satisfies the *translation (time-shifting) property*:

$$R[f(\cdot - y)](a, b) = R[f](a, b - a \cdot y).$$  \hspace{1cm} (18)

We can clearly observe this relation in the scatter plots. For all the three settings, we trained $s = 1000$ networks, each single network has $p = 100$ hidden units, and the weight decay rate and learning rate were set to $\beta = 0.001$ and $\eta = 0.01$ respectively.

![Figure 3: $f(x) = \exp(-|x - \mu|^2/2), (\mu = -0.5, 0.0, +0.5) \text{ (from top to bottom)}, \sigma(z) = \text{relu}(z)$](image_url)
Experiment 3. In order to see the effect of the discontinuity, we conduct the experiment on the square wave \( y_i = \text{sign} \circ \sin 2\pi x_i \), with ReLU on \( T = [-1/2, 1/2] \). According to the ridgelet transform, if the function has a point singularity, then the spectrum has a line singularity:

\[
R[\delta_{x_0}](a, b) = \int_{\mathbb{R}} \delta_{x_0}(x) \rho(a \cdot x - b) \, dx = \rho(a \cdot x_0 - b).
\] (19)

We can clearly observe a few lines in the scatter plot. We trained \( s = 1000 \) networks, each single network has \( p = 100 \) hidden units, and the weight decay rate and learning rate were set to \( \beta = 0.001 \) and \( \eta = 0.01 \) respectively.

![Figure 4: \( f(x) = \text{sign} \circ \sin(2\pi x) \), \( \sigma(z) = \text{relu}(z) \), \( \beta = 0.01 \)](image)

Experiment 4. In order to see the dependence in the high-frequency, we conduct the experiment on topologist’s sine curve: \( y_i = \text{sign} \circ \sin 2\pi x_i \), which contains an infinitely wide range of frequencies, with ReLU on \( T = [-1/2, 1/2] \). We used \( n = 10,000 \) datapoints and \( p = 100 \) hidden units. As we have seen in Experiments 2 and 3, any local changes in the real domain causes a line singularity in the spectrum. We can see dense lines in the scatter plot. We trained \( s = 1000 \) networks, each single network has \( p = 100 \) hidden units, and the weight decay rate and learning rate were set to \( \beta = 0.001 \) and \( \eta = 0.01 \) respectively.

![Figure 5: \( f(x) = \sin(2\pi/x) \), \( \sigma(z) = \text{relu}(z) \), \( \beta = 0.01 \)](image)
C Cheat Sheet for Ridgelet Transform on $T = \mathbb{R}/\mathbb{T}$

We identify $T$ as $[-T/2, T/2)$ some $T > 0$. We write $\omega_n := 2\pi n/T$ for every $n \in \mathbb{Z}$.

C.1 Fourier transforms and Fourier expansions

**Fourier transform on $T$, or Fourier series expansion.** Let $T > 0$. For any $f \in L^2([-T/2, T/2])$,

$$\hat{f}(n) := \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-i\omega_n t} dt,$$

$$f(t) = \lim_{N \to \infty} \sum_{n=-N}^{N} \hat{f}(n) e^{i\omega_n t}. \quad (21)$$

In particular, the convolution theorem holds:

$$\hat{f} \ast \hat{g}(n) = \frac{1}{T} \hat{f}(n) \hat{g}(n). \quad (22)$$

**Fourier transform on $\mathbb{R}^m$.** In order to avoid the potential confusion, we write $\hat{f}$ and $\check{f}$ for the Fourier transform on $\mathbb{R}^m$:

$$f^\sharp(\xi) := \int_{\mathbb{R}^m} f(x) e^{-ix \cdot \xi} dx, \quad x \in \mathbb{R}^m \quad (23)$$

$$f^\flat(\xi) := \int_{\mathbb{R}^m} f(x) e^{ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^m \quad (24)$$

$$f(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} f^\sharp(\xi) e^{ix \cdot \xi} d\xi, \quad x \in \mathbb{R}^m. \quad (25)$$

C.2 Ridgelet transforms

Here we introduce a general form of ridgelet transforms (26) in terms of another bounded periodic function $\rho$.

In the main body, we use this theory in the case of $\rho = \sigma$. Assumption 2.2 corresponds to (28) and (29).

**Ridgelet transform.**

$$R[f](a, b) := \int f(x) \rho(a \cdot x - b) dx, \quad (a, b) \in \mathbb{R}^m \times T \quad (26)$$

**Adjoint Operator.** For $\gamma \in \text{Im}(R)$,

$$R^*[\gamma](x) := \int_{\mathbb{R}^m \times T} \gamma(a, b) \sigma(a \cdot x - b) dadb, \quad x \in \mathbb{R}^m \quad (27)$$

**Reconstruction formula.** Let $\rho, \sigma \in L^2([-T/2, T/2])$ satisfies the admissibility conditions

$$\sum_{n \neq 0} \frac{\overline{\rho(n)} \overline{\sigma(n)}}{|n|^m} = 1, \quad (28)$$

$$\overline{\rho(0)} \overline{\sigma(0)} = 0 \iff \int_T \rho(0-t) \sigma(t) dt = \langle \sigma, \rho \rangle_{L^2(T)} = 0. \quad (29)$$

Then, for any $f \in L^1(\mathbb{R}^m) \cap L^2(\mathbb{R}^m)$,

$$R^*[R[f]](x) = f(x). \quad (30)$$
Proof.

\[
R^*[R[f]](x) = \int_{\mathbb{R}^m \times \mathbb{T} \times \mathbb{R}^m} f(x)\rho(a \cdot y - b)\sigma(a \cdot x - b) dy db
\]

\[
= \int_{\mathbb{R}^m \times \mathbb{R}^m} f(x)\rho\sigma(a \cdot (x - y)) dy da
\]

\[
= \int_{\mathbb{R}^m \times \mathbb{R}^m} f(x) \left[ T \sum_{n \neq 0} \hat{\rho}(n)\tilde{\sigma}(n) \exp \left\{ \frac{2\pi i n a \cdot (x - y)}{T} \right\} \right] dy da
\]

\[
= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m \times \mathbb{R}^m} f(x) \left[ T^{m+1} \sum_{n \neq 0} \hat{\rho}(n)\tilde{\sigma}(n) \right] e^{ia \cdot (x - y)} dy da
\]

\[
= f(x).
\]

\[
\square
\]

**Fourier slice theorem.** In particular, \( R \) has a Fourier expression:

\[
R[f](a,b) = \sum_{n=-\infty}^{\infty} f^2(\omega_n,a)\hat{\rho}(n)e^{i\omega_n b} \quad (31)
\]

Proof. Since \( \frac{1}{T} \int_{-T/2}^{T/2} \rho(a \cdot x - b)e^{-i\omega_n a \cdot x} db = \hat{\rho}(n)e^{-i\omega_n a \cdot x} \), we have

\[
R[f](a,b) = \sum_{n=-\infty}^{\infty} \left[ \int_{\mathbb{R}^m} f(x)\hat{\rho}(n)e^{-i\omega_n a \cdot x} dx \right] e^{i\omega_n b}
\]

\[
= \sum_{n=-\infty}^{\infty} f^2(\omega_n,a)\hat{\rho}(n)e^{i\omega_n b}
\]

\[
\square
\]

C.3 Non-injectivity and null space of \( S \)

The admissibility condition is not a strong requirement because it requires that \( \sigma \) and \( \rho \) are not orthogonal to each other in the \( |n|^{-m} \)-weighted \( \ell^2 \)-space. Since \( \ell^2 \)-space is an infinite-dimensional Hilbert space, there are infinitely many different solutions to the equation \( S[\gamma] = f \). Namely, suppose that two different functions \( \rho_1 \) and \( \rho_2 \) satisfy the admissibility condition, and let \( \gamma_1 := R[f;\rho_1] \) and \( \gamma_2 := R[f;\rho_2] \). Then, \( \gamma_1 \neq \gamma_2 \) but \( S[\gamma_1] = f \) and \( S[\gamma_2] = f \) by the reconstruction formula. This clearly implies the non-triviality of the null space \( \ker S \). In general, a complete specification of \( \ker S \) is very difficult.

One major conclusion of this study is that if the solutions are restricted by \( L^2 \)-regularization, then we have a unique ridgelet function \( \rho = \sigma \). In general, the \( L^2 \)-regularization provides the minimum norm solution. Therefore, we can understand that among infinitely many different solutions \( \gamma = R[f;\rho] \), the \( R[f;\sigma] \) achieves the minimum norm solution.

D Regularized Square Loss Minimization in Hilbert spaces

Let \( G, F \) be Hilbert spaces endowed with the inner products \( \langle \cdot, \cdot \rangle_G \) and \( \langle \cdot, \cdot \rangle_F \), respectively, and \( S : G \rightarrow F \) be a densely defined closed linear operator.

For a given \( f \in F \), we find \( \gamma \in G \) satisfying

\[
S[\gamma] = f. \quad (32)
\]

For this problem, we have the following.
Proposition D.1. Let \( f \in F \). Then for every \( \beta > 0 \), we have
\[
\arg \min_{\gamma \in G} (\|S[\gamma] - f\|_F^2 + \beta \|\gamma\|_G^2) = (\beta + S^*S)^{-1}S^*[f],
\] (33)
where \( S^* : F \to G \) denotes the adjoint operator of \( S \).

Proof. A direct computation gives
\[
\|S[\gamma] - f\|_F^2 + \beta \|\gamma\|_G^2
= \langle S[\gamma], S[\gamma] \rangle_F - 2\Re \langle S[\gamma], f \rangle_F + \langle f, f \rangle_F + \beta \langle \gamma, \gamma \rangle_G
= \langle \sqrt{\beta + S^*S}[\gamma], \sqrt{\beta + S^*S}[\gamma] \rangle_G - 2\Re \langle \sqrt{\beta + S^*S}[\gamma], \sqrt{\beta + S^*S}^{-1}S^*[f] \rangle_G + \langle f, f \rangle_F
= \|\sqrt{\beta + S^*S}[\gamma] - \sqrt{\beta + S^*S}^{-1}S^*[f]\|_G^2 + \text{(nonnegative)}.
\
\]
Therefore, the objective functional attains the minimum at \( \gamma^* = (\beta + S^*S)^{-1}S^*[f] \).
\(\Box\)

Proposition D.2. Suppose that \( \gamma_0 \in G \) satisfies \( f = S[\gamma_0] \). Then,
\[
\lim_{\beta \to 0} (\beta + S^*S)^{-1}S^*[f] = \text{Proj}_{G \to (\ker S)^\perp}[\gamma_0].
\] (34)

Proof. Using the right continuous resolution of the identity \( \{E_\mu\}_{\mu \in \mathbb{R}} \) for \( S^*S \),
\[
(\beta + S^*S)^{-1}S^*[f] = \int_{\mathbb{R}} \frac{\mu}{\beta + \mu} dE_\mu \gamma_0
\to \int_{\mathbb{R}} \chi_{\mathbb{R}\setminus\{0\}}(0) dE_\mu \gamma, \quad \text{as } \beta \to 0
= (E_G - E_{0-}) \gamma_0
= \text{Proj}_{G \to (\ker S)^\perp}[\gamma_0].
\]
\(\Box\)

Here, \( (E_G - E_{0-}) \gamma_0 \) follows from the projection nature of \( dE_\mu \gamma_0 \).