Stability and Stabilization of Nash Equilibrium for Uncertain Noncooperative Dynamical Systems With Zero-Sum Tax/Subsidy Approach

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Abstract—A zero-sum tax/subsidy approach for stabilizing unstable Nash equilibria in pseudo-gradient-based noncooperative dynamical systems is proposed without the information of agents’ personal sensitivity parameters. Specifically, we first present several sufficient conditions for guaranteeing stability of an unstable Nash equilibrium in the face of uncertainty. Furthermore, we develop a framework where a system manager constructs a zero-sum tax/subsidy incentive structure by collecting taxes from some agents and giving the same amount of subsidy in total to other agents so that the agents’ payoff structure is properly modified. Finally, we present several numerical examples to illustrate the utility of the zero-sum tax/subsidy approach.

Index Terms—Compensation by incentives, gradient play, Nash equilibrium, noncooperative system, stabilization, tax/subsidy approach (TSA).

I. INTRODUCTION

GAME THEORY has widely contributed in last decades for investigating noncooperative multiagent systems where many applications are found in both engineering and economics, for example, wireless sensor networks [1], communication channel allocation [2], signal interference avoidance [3], data security in intelligent transportation systems [4], and electricity market [5], to name but a few. Agents in the noncooperative systems mutually affect the selfish decision making of the other agents through the interconnected relations of their utilities or payoffs.

It is common knowledge that in noncooperative systems, the agents’ selfish decision making may degrade the social welfare [6], [7]. For example, the tragedy of the commons describes a social trap involving the conflict between the individual interests and the public interest in the allocation of resources [8]. In such a situation, without a person who is entitled to control the entire noncooperative system, every agent expands its demand independently according to his own self-interest, and the limited resources are destined to be over-exploited by the unrestricted demands, which eventually harms the common good of all agents in the common resource systems.

For the aggregation of such self-interested agents, it has turned out that the imposition of external policies or explicit incentive mechanisms changes agents’ decision-making tendencies and hence, results in the endogenously cooperative behaviors in the noncooperative systems [9]–[11]. As a coercion policy, which agents cannot escape once in place, a tax/subsidy approach (TSA) was proposed by [12] to reward or penalize the deviations from the average contribution of the other competitors to the public goods. In contrast to the coercion policy, Varian [13] investigated a compensation mechanism where agents are allowed to voluntarily subsidize the other agents in the prestige when the other agents’ decisions are not made yet. The compensation mechanisms are understood as a liberal solution as agents have freedom to escape the mechanism. In usual, the liberal solution works as a weak external rule to the noncooperative system and is expected to be less efficient than the coercion solution.

In order to describe the state change of noncooperative systems, several models are proposed in the literature. Specifically, agents’ dynamic decision behaviors are typically characterized by the best-response dynamics (or called dynamic fictitious play) [14], [15] and myopic pseudogradient dynamics (or called better response dynamics, or dynamic gradient play) [16]–[18] for discrete-time and continuous-time systems, respectively. In the pseudogradient dynamics setup, the agents continuously change their state according to the pseudogradient projection onto their own local state space without having foresight. For example, Singh et al. [19] analyzed agents’ behaviors in a noncooperative system with two agents and quadratic payoff functions. Bowling and Veloso [20] investigated the agents’ behaviors with a variable learning rate for the case where an agent wins (possesses higher utility than the opponent) in the two-agent noncooperative system. The article [21] proposed a congestion control framework for data traffic with the pseudogradient dynamics for the users on the Internet. The article [22] explored the stability change of a noncooperative system with loss-averse agents while [23] discussed the relationship between the positively invariant set and the set of positive externalities for a
pseudogradient-based noncooperative system with two agents and quadratic payoffs.

To improve the social utility level, it is preferable to develop a compensation mechanism that collects taxes from some agents and gives subsidies to some other agents. Specifically, Alpcan et al. [24] modified agents’ original payoff functions in order to reach the highest social welfare by adding a pricing term among the agents. For stabilizing minimum latency flows in the Braess graphs, [25] considered the capitation tax and subsidy. Morimoto et al. [26] imposed a subsidy mechanism to achieve stabilization for heterogeneous replicator dynamics. It is necessary to emphasize that in the above works the existence of a system manager is assumed and he/she is characterized as a resource owner or distributor who is able to give additional subsidies. However, the system manger in many economic applications serves merely as a mediator and does not have productivity to pay the additional profits to the agents. In such a case, every subsidy has to be financed by taxes taken from the others [27] and hence, the tax/subsidy mechanism ought to be designed in a zero-sum fashion, for example, [28].

Ideally, the system manager has all the knowledge about the noncooperative system, including the payoff functions and the decision dynamics of the agents. In reality, it is often difficult to observe perfect information about the activities of the noncooperative agents. This hidden information is called private information in economics [29] and this uncertainty can be obstructive for designing the incentive mechanisms. Even though in the existing gradient-based Nash equilibrium seeking problems [24], [30]–[32], the seeking speed is predetermined, the rational agents in a noncooperative dynamical system, in general, change their states according to their own inherent sensitivities which may not be observed by the system manager. The work in [33] provided an explicit mechanism by side payments with the idea of transferring the utility in a two agent system, which induces cooperation and drives the noncooperative system to the socially maximum welfare state, but unfortunately, the case with more agents and the sensitivity parameters is not considered. Indeed, even though for a two-agent noncooperative system, the sensitivity parameters do not change the stability property of Nash equilibria [34], they may change the stability property in the system with more than two agents and bring agents’ state to a worse utility state.

In this article, we develop a utility-transfer framework for noncooperative systems to remodel agents’ dynamical decision making in the face of agents’ private information. Specifically, we assume that that the sensitivity parameters in the pseudogradient dynamics are uncertain to the system manager. Under this uncertainty, the system manager is expected to construct a zero-sum tax/subsidy mechanism to (globally) stabilize a Nash equilibrium. To deal with the uncertainty, we first characterize the stability of the Nash equilibrium for arbitrary values of sensitivity and then investigate the zero-sum tax/subsidy framework without knowing the sensitivity parameters. In the proposed TSA, the system manager defines the utility-transfer structure dividing the agents into subgroups so that the utility transfers are completed within the subgroups in a zero-sum and distributed manner. The amounts of tax (negative incentive) and subsidy (positive incentive) for each agent are determined by quadratic incentive functions with well-chosen control parameters. It turns out from the numerical examples that the proposed framework can guarantee global asymptotic stabilizability for some noncooperative systems with nonquadratic payoff functions.

The article is organized as follows. In Section II, we characterize the pseudogradient-based noncooperative dynamical systems. In Section III, we discuss the stability of a Nash equilibrium for multiagent noncooperative systems without knowing agents’ sensitivity parameter. In Section IV, we first introduce our zero-sum tax/subsidy mechanism for two-agent noncooperative systems, and then extend it to more general multiagent systems. Furthermore, in Section V, we present a couple of illustrative numerical examples. Finally, in Section VI, the conclusion is given.

Notations: We use the following notations in this article. We write \( \mathbb{Z} \) for the set of positive integers, \( \mathbb{R} \) for the set of real numbers, \( \mathbb{R}^+ \) for the set of positive real numbers, \( \mathbb{R}^{m \times n} \) for the set of \( m \times n \) real matrices, and \( \mathbb{R}^n \) for the set of \( n \times 1 \) real column vectors. Furthermore, we write \( \text{det}(\cdot) \) for determinant, \( (\cdot)^\top \) for transpose, and \( \text{diag}(\cdot) \) for diagonal matrices. Finally, we write the identity matrix and the ones vector of dimension \( n \) by \( I_n \) and \( 1_n \), respectively.

II. PRELIMINARY, MOTIVATIONS, AND PROBLEM STATEMENT

A. System Description

Consider the noncooperative system with payoff functions \( J_i : \mathbb{R}^N \rightarrow \mathbb{R} \) for agent \( i \in \mathcal{N} \), where \( \mathcal{N} \triangleq \{1, \ldots, N\} \) denotes the set of agents. Each agent \( i \in \mathcal{N} \) controls its state (strategy) \( x_i \in \mathbb{R} \), \( i \in \mathcal{N} \). Let \( x = (x_i, x_{-i}) \in \mathbb{R}^n \) denote all agents’ state (strategy) profile, where \( x_{-i} \in \mathbb{R}^{N-1} \) denotes the agents’ state profile except agent \( i \). In this article, we suppose that each agent \( i \) aims to increase its own payoff \( J_i(x_i, x_{-i}) \), where \( J_i \) may depend on all the agents’ state. We denote the noncooperative system by \( \mathcal{G}(J) \) with \( J \triangleq \{J_i| i \in \mathcal{N}\} \).

Definition 1 [35]: For the noncooperative system \( \mathcal{G}(J) \), the state profile \( x^* \in \mathbb{R}^N \) is called a Nash equilibrium of \( \mathcal{G}(J) \) if

\[
J_i(x^*_i, x^*_{-i}) \geq J_i(x_i, x^*_{-i}), \quad x_i \in \mathbb{R}, \quad i \in \mathcal{N}.
\]

At a Nash equilibrium, no agent has any incentive to deviate unilaterally from the equilibrium state if the other agents’ state does not change.

Assumption 1: The payoff functions \( J_i(x), i \in \mathcal{N} \), are twice continuously differentiable.

Note that the noncooperative system \( \mathcal{G}(J) \) may not possess any Nash equilibrium. Some sufficient conditions for existence of a Nash equilibrium with the closed convex domain can be found in [16] and [36, Ch. 2]. However, in general, guaranteeing the existence of a Nash equilibrium for an unbounded state space is a complicated problem. In this article, we suppose that there exists at least one Nash equilibrium. In this case, under Assumption 1, since the Nash equilibrium \( x^* \) satisfies \( x^*_i = \arg \max_{x_i \in \mathbb{R}} J_i(x_i, x^*_{-i}) \) for all \( i \in \mathcal{N} \), it follows that:

\[
\frac{\partial J_i(x^*)}{\partial x_i} = 0, \quad i \in \mathcal{N}.
\]
Moreover, it is important to note that the Nash equilibrium is characterized independent of the underlying dynamics.

B. Myopic Pseudogradient Dynamics

In this article, we suppose that each agent continuously changes its state (strategy) of the noncooperative system $G(J)$ in the unbounded state space $\mathbb{R}^N$ in order to increase its own payoff. Specifically, we assume that the state profile $x(\cdot)$ is available for all the agents and each agent follows the pseudogradient dynamics given by

$$\dot{x}_i(t) = \alpha_i \frac{\partial J_i(x(t))}{\partial x_i}, \quad i \in N \tag{3}$$

where $\alpha_i$, $i \in N$, are agent-dependent positive constant parameters representing sensitivity to the increasing/decreasing payoff per unit state change [16]. In this case, agents selfishly concern their own payoffs and myopaically change their states (strategies) according to the current information without any foresight on the future state of the other agents. The pseudogradient dynamics are widely used as the dynamics for rational foresight on the future state of the other agents. The pseudo-strategies according to the current information without any concern their own payoffs and myopaically change their states moving rates given by (3) are characterized to be proportional to the projection of the gradient of $J_i(x)$ onto $x_i$-axis, which is called the pseudo-gradient, but the sensitivity parameters $\alpha_i$, $i \in N$, which decide how fast the agents move, are in many cases private so that they are not observed. It is important to note that at the Nash equilibrium $x^*$, $\dot{x}(t) = 0$ since (2) holds.

C. Motivations and Problem Statement

1) Motivation: Some of the Nash equilibria may be unstable in the noncooperative system $G(J)$, since agents’ payoff functions are generally different from each other. For instance, Fig. 1 shows the payoff functions of each agent in a two-agent noncooperative system with an unstable Nash equilibrium. Assume there is a system manager, for example, the governor of the markets, who controls the amount of tax and subsidy to converge to it. Assuming all the information of the payoff parameters representing sensitivity to the increasing/decreasing payoff per unit state change [16]. In this case, agents selfishly concern their own payoffs and myopaically change their states (strategies) according to the current information without any foresight on the future state of the other agents. The pseudogradient dynamics are widely used as the dynamics for rational foresight on the future state of the other agents. The pseudo-strategies according to the current information without any concern their own payoffs and myopaically change their states moving rates given by (3) are characterized to be proportional to the projection of the gradient of $J_i(x)$ onto $x_i$-axis, which is called the pseudo-gradient, but the sensitivity parameters $\alpha_i$, $i \in N$, which decide how fast the agents move, are in many cases private so that they are not observed. It is important to note that at the Nash equilibrium $x^*$, $\dot{x}(t) = 0$ since (2) holds.

2) Problem: Consider the the target Nash equilibrium $x^*$ with uncertain sensitivity parameters $\alpha_i$, $i \in N$, for the system manager. Our main objectives are two folds: 1) find the condition for determining the stability property of the Nash equilibrium $x^*$ with arbitrary $\alpha_i$, $i \in N$ and 2) design an explicit incentive mechanism to stabilize the possibly unstable Nash equilibrium $x^*$ with the unknown sensitivity parameters $\alpha_i$, $i \in N$.

III. Stability Analysis of Nash Equilibrium with Unknown Sensitivity Parameters

In this section, we characterize stability properties of the Nash equilibrium of the noncooperative system $G(J)$. Specifically, we first present the results for the general $N$-agent case, and then specialize the results to 3-agent and 2-agent cases. For the statement of the following results, let $\alpha \triangleq (\alpha_1, \ldots, \alpha_N)$ and define:

$$A(J, \alpha, x) \triangleq \begin{bmatrix} \alpha_1 \frac{\partial^2 J_1(x)}{\partial x_1^2} & \cdots & \alpha_1 \frac{\partial^2 J_1(x)}{\partial x_1 \partial x_N} \\ \vdots & \ddots & \vdots \\ \alpha_N \frac{\partial^2 J_N(x)}{\partial x_N \partial x_1} & \cdots & \alpha_N \frac{\partial^2 J_N(x)}{\partial x_N^2} \end{bmatrix} \tag{4}$$

Note that under Assumption 1, since the functions $J_i(x)$, $i \in N$, are twice continuously differentiable, the matrix (4) is a continuous function with respect to $x$. Moreover, under Assumption 2, the diagonal terms $\alpha_i[\partial^2 J_i(x^*)/\partial x_i^2]$, $i \in N$, in $A(J, \alpha, x^*)$ are all negative. This fact is used in the analysis of the following results.

A. Stability Analysis for $N$-Agent Noncooperative Systems

The sensitivity parameters $\alpha_i$, $i \in N$, are inherent to each of the agents and are not exactly observed. Without knowing
the value of $\alpha$ for the $N$-agent noncooperative system, the following results provide several ways to determine stability of the Nash equilibrium.

**Corollary 1:** Consider the Nash equilibrium $x^* \in \mathbb{R}^N$ for the $N$-agent noncooperative system $G(J)$ with myopic pseudogradient dynamics (3). If the payoff functions $J_i(x)$, $i \in N$, satisfy

$$(-1)^N \det A(J, 1_N, x^*) < 0$$  \hspace{1cm} (5)

then the Nash equilibrium $x^*$ is unstable for any positive constants $\alpha_i$, $i \in N$.

**Proof:** First, let $\tilde{x} \triangleq x - x^*$. Note that linearizing the system dynamics (3) around $x^*$ yields

$$\dot{\tilde{x}}(t) = A(J, \alpha, x^*) \tilde{x}(t).$$  \hspace{1cm} (6)

The result is a direct consequence of Lyapunov’s indirect method. Specifically, consider the characteristic equation $\det (sI - A(J, \alpha, x^*)) = s^N + a_{N-1}s^{N-1} + \cdots + a_1s + a_0 = 0$ of $A(J, \alpha, x^*)$, where $a_0, \ldots, a_{N-1}$ are appropriate constants. In particular, $a_0 = (-1)^N \det A(J, \alpha, x^*) = (-1)^N \det A(J, 1_N, x^*) \times \prod_{i \in N} \alpha_i$. Now, since $a_i > 0$, $i \in N$, it follows from (5) that $a_0 < 0$. Hence, it follows from the Routh or Hurwitz criterion that the Nash equilibrium $x^*$ is unstable.

The fictitious sensitivity $1_N$ in (5) can be replaced by any $\hat{\alpha} \in \mathbb{R}_+^N$ to determine instability because it does not change the sign of the determinant of $A(J, \cdot, x^*)$.

Relation of payoff dependency between the agents can be characterized by defining a graph. For specific graph structures, we can specialize condition (5) as shown in the following examples.

**Example 1:** Consider the noncooperative system with the payoff dependency given by the center-sponsored star network illustrated in Fig. 2(a), where agent 1 is the center of the network. In this case, note that since

$$A(J, 1_N, x^*) = A(J, 1_N, x^*) =
\begin{bmatrix}
\frac{\partial^2 J_1(x^*)}{\partial x_1^2} & \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_N} \\
\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 J_2(x^*)}{\partial x_2^2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 J_N(x^*)}{\partial x_N \partial x_1} & 0 & \cdots & \frac{\partial^2 J_N(x^*)}{\partial x_N^2}
\end{bmatrix}$$

the left-hand side of (5) is given by

$$(-1)^N \left( \frac{\partial^2 J_1(x^*)}{\partial x_1^2} - \sum_{i=2}^N \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1} \right) \prod_{i=2}^N \frac{\partial^2 J_i(x^*)}{\partial x_i^2} = 0
$$

Noting that Assumption 2 implies $(-1)^N \prod_{i=2}^N \frac{\partial^2 J_i(x^*)}{\partial x_i^2}$ is negative, it follows from Corollary 1 that if the payoff functions $J_i(x)$, $i \in N$, satisfy:

$$\frac{\partial^2 J_1(x^*)}{\partial x_1^2} - \sum_{i=2}^N \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1} > 0$$

then the Nash equilibrium $x^*$ is unstable for any positive constants $\alpha_i$, $i \in N$.

**Example 2:** Consider the noncooperative system with the payoff dependency given by the directed ring network illustrated in Fig. 2(b). In this case, note that since

$$\dot{x}(t) = A(J, 1_N, x^*) \hat{\alpha} \tilde{x}(t) \Rightarrow (\hat{\alpha} A(J, 1_N, x^*) \tilde{x}(t))$$

the left-hand side of (5) is $\prod_{i=1}^N (-\frac{\partial^2 J_i(x^*)}{\partial x_i^2}) - \prod_{i=1}^N (\prod_{i=1}^N (-\frac{\partial^2 J_i(x^*)}{\partial x_i^2})) > 0$, where $x_{N+1}$ is understood as $x_1$. Thus, it follows from Corollary 1 that if the payoff functions $J_i(x)$, $i \in N$, satisfy:

$$\prod_{i=1}^N \left( -\frac{\partial^2 J_i(x^*)}{\partial x_i^2} \right) < \prod_{i=1}^N \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_{i+1}}$$

then the Nash equilibrium $x^*$ is unstable for any positive constants $\alpha_i$, $i \in N$.

Now, a sufficient condition is provided to guarantee stability without knowing $\alpha_i$, $i \in N$, in the following theorem.

**Theorem 1:** Consider the Nash equilibrium $x^* \in \mathbb{R}^N$ for the $N$-agent noncooperative system $G(J)$ with pseudogradient dynamics (3). If there exists $\hat{\alpha} \in \mathbb{R}_+^N$ such that

$$A^T(J, \hat{\alpha}, x^*) + \hat{\alpha} A(J, \hat{\alpha}, x^*) < 0$$

then the Nash equilibrium $x^*$ is locally asymptotically stable for any positive constants $\alpha_i$, $i \in N$.

**Proof:** Letting $\tilde{x} = x - x^*$, consider the Lyapunov function candidate $V(\tilde{x}) = \tilde{x}^T \tilde{x}$ with the positive-definite matrix $P = \text{diag}(\frac{\partial^2 J_1(x^*)}{\partial x_1^2}, \ldots, \frac{\partial^2 J_N(x^*)}{\partial x_N^2})$. Since

$$A^T(J, \alpha, x^*)P + PA(J, \alpha, x^*) = A^T(J, \hat{\alpha}, x^*) + \hat{\alpha} A(J, \hat{\alpha}, x^*) < 0$$

is satisfied, it follows using the linearized dynamics (6) that:

$$\dot{V}(\tilde{x}(t)) = \dot{x}^T(t)(A(J, \hat{\alpha}, x^*) + A(J, \hat{\alpha}, x^*))\tilde{x}(t) < 0$$

around $x^*$ and hence, the Nash equilibrium $x^*$ is asymptotically stable for all positive sensitivity parameters $\alpha_i$, $i \in N$. ■

**Remark 1:** The result in Theorem 1 appears to be similar to [16, Ths. 8 and 9] but it is certainly different in that
Theorem 1 guarantees asymptotic stability for arbitrary \( \alpha \) by evaluating the sign definiteness of \( A^T (J, \hat{\alpha}, x^*) + A(J, \hat{\alpha}, x^*) \) for a particular \( \hat{\alpha} \). To determine whether such \( \hat{\alpha} \) exists, we can address the linear matrix inequality (LMI) feasibility problem given by

\[
\text{diag}[\hat{\alpha}] A(J, 1_N, x^*) + A^T (J, 1_N, x^*) \text{diag}[\hat{\alpha}] < 0 \quad (12)
\]

assuming that all the information of \( J \) is known.

**Remark 2:** Because of the continuity of \( A(J, \hat{\alpha}, x) \) with respect to \( x \), (10) implies that there exists a connected set

\[
D_1^\delta \triangleq \{ x \in \mathbb{R}^N : A^T (J, \hat{\alpha}, x) + A(J, \hat{\alpha}, x) < 0 \} \quad (13)
\]

containing \( x^* \). Let \( f(x) \triangleq [\alpha_1(\partial J_1(x)/\partial x_1), \ldots, \alpha_N(\partial J_N(x)/\partial x_N)]^T \) denote the vector field of the pseudogradient dynamics and let \( V(x) \triangleq f^T(x)Pf(x) \). It is important to note that a subset of the region of attraction can be characterized by

\[
D_2^\delta \triangleq \{ x \in \mathbb{R}^N : V(x) < \delta \} \quad (14)
\]

with the maximum attainable \( \delta \in \mathbb{R}_+ \) such that \( D_2^\delta \subseteq D_1^\delta \) and \( D_2^\delta \) is connected in the neighborhood of \( x^* \) for all \( \delta < \delta \). This is because \( V(x) \) is understood as a Lyapunov function and it satisfies \( V(x(t)) = f^T(x(t))(A^T (J, \hat{\alpha}, x(t)) + A(J, \hat{\alpha}, x(t)) f(x(t)) < 0 \) for all \( x(t) \in D_2^\delta \setminus \{ x^* \} \). It is important to note that the estimated region of attraction \( D_2^\delta \) depends on the choice of \( \hat{\alpha} \) in \( A(J, \hat{\alpha}, x^*) \) and can be substantially smaller than the actual region of attraction. But for the special case where \( A^T (J, \hat{\alpha}, x) + A(J, \hat{\alpha}, x) < 0 \) holds for all \( x \in \mathbb{R}^N \), since it can be shown that \( f(x) = 0 \) only when \( x = x^* \) in \( \mathbb{R}^N \), it follows that the Nash equilibrium \( x^* \) is globally asymptotically stable for arbitrary \( \alpha \). For instance, if the payoff functions are quadratic, then (10) guarantees global asymptotic stability as (4) is a constant matrix.

**Remark 3:** For the noncooperative system with quadratic payoff functions where \( (\partial^2 J_i(x^*)/\partial x_i \partial x_j) \geq 0 \), \( i, j \in \mathcal{N}, i \neq j \), it follows from the properties of the Metzler matrix that (10) in Theorem 1 is also a necessary condition for the Nash equilibrium \( x^* \) to be asymptotically stable for arbitrary \( \alpha \).

**Example 3:** Consider the N-agent noncooperative system with the payoff dependency given by the center-sponsored star network illustrated in Fig. 2(a). To investigate the conditions for the payoff functions \( J_i(x), i \in \mathcal{N} \), such that \( \hat{\alpha} \in \mathbb{R}^N \) exists to satisfy (10), note that the \( k \)-th order leading principal minor of \( A^T (J, \hat{\alpha}, x^*) + A(J, \hat{\alpha}, x^*) \) with \( \hat{\alpha}_1 = 1 \) is given by

\[
L_k \triangleq \left( \begin{array}{c}
\frac{\partial^2 J_1(x^*)}{\partial x_1^2} - \sum_{i=2}^{k} \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1} + \hat{\alpha}_1 \frac{\partial^2 J_i(x^*)}{\partial x_i^2} \\
\end{array} \right)
\times \prod_{i=2}^{k} \left( 2 \hat{\alpha}_i \frac{\partial^2 J_i(x^*)}{\partial x_i^2} \right)
\]

for \( k = 2, \ldots, N \). Since, by Assumption 2, \( (\partial^2 J_1(x^*)/\partial x_1^2) < 0 \), \( i = 2, \ldots, N \), and hence, \( (-1)^k \prod_{i=2}^{k} (2 \hat{\alpha}_i (\partial^2 J_i(x^*)/\partial x_i^2)) < 0 \), \( k = 2, \ldots, N \), the inequality \((-1)^k L_k > 0\) for guaranteeing (10) is equivalent to

\[
\frac{\partial^2 J_1(x^*)}{\partial x_1^2} < \frac{1}{2} \sum_{i=2}^{k} \left( \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1} + \hat{\alpha}_1 \frac{\partial^2 J_i(x^*)}{\partial x_i^2} \right)^2
\]

for \( k = 2, \ldots, N \). Therefore, since all the terms in the right-hand side are negative, the existence problem of \( \hat{\alpha} \) in satisfying (10) is equivalent to finding a solution \( \hat{\alpha} = (1, \hat{\alpha}_2, \ldots, \hat{\alpha}_N) \) for (15) with \( k = N \). Now, such \( \hat{\alpha} \) exists if and only if the simple condition

\[
\frac{\partial^2 J_1(x^*)}{\partial x_1^2} < \frac{1}{2} \sum_{i=2}^{N} \max_{x_1 \in \mathbb{R}_+} \left( \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1} + \hat{\alpha}_1 \frac{\partial^2 J_i(x^*)}{\partial x_i^2} \right)^2
\]

is satisfied for \( \mathcal{N}_0 \triangleq \{ i \in \mathcal{N} : (\partial^2 J_1(x^*)/\partial x_1 \partial x_1), (\partial^2 J_i(x^*)/\partial x_i \partial x_1) > 0 \} \), where in (16) we used max_{x_1 \in \mathbb{R}_+} (A + \alpha B)^2/2\alpha C = (2AB/C) for \( AB > 0 \) and \( C < 0 \).

**Remark 4:** Note that the local stability of the Nash equilibrium \( x^* \) under the dynamics (3) can also be directly derived if the matrix \( A(J, \alpha, x^*) \) (or, equivalently, \( A(J, 1_N, x^*) \)) is strictly diagonally dominant (i.e., \( (\partial^2 J_i(x^*)/\partial x_i^2) < \sum_{j \neq i} |(\partial^2 J_i(x^*)/\partial x_i \partial x_j)| \) for all \( i \in \mathcal{N} \)). The proof is based on Gershgorin’s circle theorem [42].

**B. Stability Analysis for 3-Agent Noncooperative Systems**

Recall that based on Lyapunov’s stability method, Theorem 1 requires us to look for \( \hat{\alpha} \) to make the symmetric part of \( A(J, \hat{\alpha}, x^*) \) negative definite to guarantee stability. For the case of \( N = 3 \), it is possible to characterize a different set of stability conditions on the payoff functions based on the Hurwitz criterion.

**Proposition 1:** Consider the Nash equilibrium \( x^* \in \mathbb{R}^3 \) for the 3-agent noncooperative system \( G(J_1, J_2, J_3) \) with pseudogradient dynamics (3). If the payoff functions \( J_i(x), i \in \{1, 2, 3\} \), satisfy

\[
\begin{align*}
\det A(J, 1_N, x^*) &< 0, \\
\frac{\partial^2 J_i(x^*)}{\partial x_i^2} &< 0, \\
\frac{\partial^2 J_1(x^*)}{\partial x_1^2} - \frac{\partial^2 J_1(x^*)}{\partial x_2^2} &< 0, \\
\frac{\partial^2 J_1(x^*)}{\partial x_1^2} - \frac{\partial^2 J_1(x^*)}{\partial x_3^2} &< 0, \\
\frac{\partial^2 J_2(x^*)}{\partial x_1^2} - \frac{\partial^2 J_2(x^*)}{\partial x_2^2} &< 0, \\
\frac{\partial^2 J_2(x^*)}{\partial x_1^2} - \frac{\partial^2 J_2(x^*)}{\partial x_3^2} &< 0, \\
\frac{\partial^2 J_3(x^*)}{\partial x_1^2} - \frac{\partial^2 J_3(x^*)}{\partial x_2^2} &< 0, \\
\frac{\partial^2 J_3(x^*)}{\partial x_1^2} - \frac{\partial^2 J_3(x^*)}{\partial x_3^2} &< 0,
\end{align*}
\]

for \( i, j \in \{1, 2, 3\}, i \neq j \).
then the Nash equilibrium $x^*$ is asymptotically stable for any positive constants $\alpha_1$, $\alpha_2$, and $\alpha_3$.

Proof: Consider the characteristic polynomial $s^3 + a_2 s^2 + a_1 s + a_0$ of $A(J, \alpha, x^*)$, where

\[
a_2 = -\sum_{i \notin N} \alpha_i \frac{\partial^2 J_i(x^*)}{\partial x_i^2}, \quad (20)
\]

\[
a_1 = \sum_{i \neq j} \alpha_i \alpha_j \left( \frac{\partial^2 J_i(x^*)}{\partial x_i^2} \frac{\partial^2 J_j(x^*)}{\partial x_j^2} - \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i} \right), \quad (21)
\]

\[
a_0 = -\det A(J, \alpha, x^*) = -\alpha_1 \alpha_2 \alpha_3 \det A(J, 1, x^*). \quad (22)
\]

Note that Assumption 2 implies $a_2 > 0$, and (17) and (18) imply $a_0 > 0$ and $a_1 > 0$, respectively. Furthermore, it follows from (18) and (19) that:

\[
a_2 a_1 - a_0 = \sum_{i \neq j} \left( \alpha_i \alpha_j \frac{\partial^2 J_i(x^*)}{\partial x_i^2} \frac{\partial^2 J_j(x^*)}{\partial x_j^2} \right) \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i}
\]

\[
\quad - \alpha_1 \alpha_2 \alpha_3 \left( 2 \frac{\partial^2 J_i(x^*)}{\partial x_i^2} \frac{\partial^2 J_j(x^*)}{\partial x_j^2} \frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i} - \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i} \right) > 0.
\]

Hence, it follows from the Hurwitz criterion that the Nash equilibrium $x^*$ is stable for any positive constants $\alpha_1$, $\alpha_2$, and $\alpha_3$.

Remark 5: The conditions in Proposition 1 provide different sufficient conditions from the one in Theorem 1.

For example, $A(J, 1, x^*) = \begin{bmatrix} -1 & 0 & 50 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix}$ satisfies (17)–(19), but there does not exist $\hat{\alpha} \in \mathbb{R}^3_+$ such that $A^T(J, \hat{\alpha}, x^*) + A(J, \hat{\alpha}, x^*) < 0$. On the contrary, $A^T(J, 1, x^*) + A(J, 1, x^*) < 0$ for $A(J, 1, x^*) = \begin{bmatrix} -6 & -5 & 1 \\ -2 & -2 & -5 \\ -5 & 3 & -1 \end{bmatrix}$, but in this case, the condition in (19) is false.

For a special case of the payoff dependency, it is interesting to observe that the conditions in Proposition 1 are equivalent to (10) in Theorem 1. In such a case, (17)–(19) guarantee the existence of $\hat{\alpha}$ for $A^T(J, \hat{\alpha}, x^*) + A(J, \hat{\alpha}, x^*) < 0$ as shown in the following remark.

Remark 6: Consider the 3-agent cooperative system with the payoff dependency given by the undirected serial graph, which is a special case of the center-sponsored star network discussed in Example 3. Note that $\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_3} = \frac{\partial^2 J_3(x^*)}{\partial x_3 \partial x_2} = 0$ because $J_2(x)$ and $J_3(x)$ are not the functions of $x_3$ and $x_2$, respectively. In this case, inequality (19) is automatically satisfied.

Furthermore, note that

\[
\det A(J, 1, x^*) = -\frac{\partial^2 J_3(x^*)}{\partial x_3^2} \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2} - \frac{\partial^2 J_2(x^*)}{\partial x_2^2} \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_3} + \frac{\partial^2 J_1(x^*)}{\partial x_1^2} \frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_3}.
\]

Hence, the conditions (17)–(19) are satisfied if and only if

\[
\frac{\partial^2 J_1(x^*)}{\partial x_1^2} < \min \left\{ \frac{\partial^2 J_2(x^*)}{\partial x_2^2}, \frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_3} \right\}, \quad \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}
\]

where the right-hand side is same as (16). Therefore, for this special case of the payoff dependency, Proposition 1 provides exactly the same sufficient conditions as the one given in Theorem 1.

C. Stability Analysis for 2-Agent Noncooperative Systems

Now, we assume $N = 2$ for the noncooperative system $G([J_1, J_2])$. The following results are investigated in [34] and fundamental in constructing the incentive function that we develop in Section IV. First, we note that stability can be determined by the sign of the determinant of $A$.

Proposition 2 [34]: Consider the Nash equilibrium $x^* \in \mathbb{R}^2$ for the 2-agent noncooperative system $G([J_1, J_2])$ with pseudodifferential dynamics (3). If the payoff functions $J_1(x)$ and $J_2(x)$ satisfy

\[
det A([J_1, J_2], [1, 1], x^*) > 0 \quad (24)
\]

then the Nash equilibrium $x^*$ is asymptotically stable for any positive constants $\alpha_1, \alpha_2 > 0$.

Remark 7: The undirected graph topology of the payoff dependency for the 2-agent system is a special case of the center-sponsored star network discussed in Example 3. Note that (24) is equivalent to (16) by letting $N = 2$, and hence, (24) represents the necessary and sufficient condition for the existence of $\hat{\alpha}$ in Theorem 1.

It follows from Corollary 1 (for $N = 2$) and Proposition 2 that if $\det A([J_1, J_2], [1, 1], x^*) > 0$ (resp., < 0), then the Nash equilibrium $x^*$ is asymptotically stable (resp., unstable). This fact implies that the existence of $\hat{\alpha}$ for (10) is in fact the necessary and sufficient condition for stability of $x^*$ assuming that there is no eigenvalue of $A([J_1, J_2], [\alpha_1, \alpha_2], x^*)$ on the imaginary axis. In the case where $\det A([J_1, J_2], [1, 1], x^*) = 0$ implying that at least one of the eigenvalues of $A([J_1, J_2], [\alpha_1, \alpha_2], x^*)$ is zero, the Nash equilibrium $x^*$ of (3) may be stable or unstable depending on the payoff functions that the agents are associated with. For an example of addressing the center manifold to determine stability, see [34].
negative). (b) \( \det A \) is nonsingular, then there may exist infinitely many Nash equilibria. Alternatively, if \( \det A \) is nonsingular, then the Nash equilibrium is unique. Furthermore, if \( \det A \) is singular, then there may exist infinitely many Nash equilibria.

**Example 4:** Consider the 2-agent noncooperative system \( G((J_1, J_2)) \) with the quadratic payoff functions given by

\[
J_i(x) = -x^T A_i x + b_i^T x + c_i, \quad i = 1, 2 \tag{25}
\]

where \( A_i = \begin{bmatrix} a_{i1} & a_{i2} \\ a_{i2} & a_{i1} \end{bmatrix} \in \mathbb{R}^{2 \times 2} \) is symmetric with \( a_{ii} > 0, b_i \in \mathbb{R}^2, \text{ and } c_i \in \mathbb{R}, \ i = 1, 2 \). Note that different from the noncooperative system with quadratic payoff functions, if the Jacobian matrix \( A((J_1, J_2), \{a_1, a_2\}, x^*) \) is nonsingular, then the Nash equilibrium is unique. If \( A((J_1, J_2), \{a_1, a_2\}, x^*) \) is singular, then there may exist infinitely many Nash equilibria.

**Proposition 3 (34):** Consider the 2-agent noncooperative system \( G((J_1, J_2)) \) with the pseudogradient dynamics (3) then, it is a saddle point.

The next result shows the fact that the eigenvalues of the 2 \times 2 Jacobian matrix of an unstable Nash equilibrium does not possess complex conjugate eigenvalues.

**Proposition 3 (34):** Consider the 2-agent noncooperative system \( G((J_1, J_2)) \) with the pseudogradient dynamics (3) then, it is a saddle point.

**IV. STABILIZATION OF EXISTING NASH EQUILIBRIUM WITH ZERO-SUM TAX/SUBSIDY APPROACH**

In this section, we characterize the stabilization method, which is called a TSA around the target Nash equilibrium \( x^* \) for the noncooperative system without the knowledge of the sensitivity parameters \( \alpha_i, i \in N \). In this framework, the system manager imposes an incentive mechanism so that the possibly unstable Nash equilibrium state \( x^* \) is stabilized by transferring the utility between the agents in a zero-sum fashion, that is, the payoff functions of agents are changed to \( \tilde{J}(x) = \{\tilde{J}_i\}_{i \in N} \) such that

\[
\sum_{i \in N} \tilde{J}_i(x) = \sum_{i \in N} J_i(x). \tag{26}
\]

In this case, the pseudogradient dynamics (3) are consequently changed to

\[
\dot{x}(t) = \begin{bmatrix} \alpha_1 \frac{\partial \tilde{J}_1(x(t))}{\partial x_1} \\ \vdots \\ \alpha_N \frac{\partial \tilde{J}_N(x(t))}{\partial x_N} \end{bmatrix} = \begin{bmatrix} \alpha_1 \frac{\partial J_1(x(t))}{\partial x_1} + p^k(x) \\ \vdots \\ \alpha_N \frac{\partial J_N(x(t))}{\partial x_N} - p^k(x) \end{bmatrix} \tag{27}
\]

and the corresponding Jacobian matrix (4) at the Nash equilibrium is given by \( A((J, \alpha, x^*)) \). Here, we suppose that the amount of tax/subsidy affects the agents’ utility in the additive way. We begin by characterizing the TSA for the simple 2-agent noncooperative systems, then extend the approach to more general \( N \)-agent systems.

**A. Tax/Subsidy Approach for 2-Agent Case**

In this section, we discuss the TSA for the 2-agent noncooperative system \( G((J_1, J_2)) \). Specifically, consider the noncooperative system \( G((J_1, J_2)) \) with the adjusted payoff functions \( \tilde{J}_1(x), \tilde{J}_2(x) \) given by

\[
\tilde{J}_1(x) = J_1(x) + p^k(x) \\
\tilde{J}_2(x) = J_2(x) - p^k(x) \tag{28}
\]

where \( p^k : \mathbb{R}^2 \to \mathbb{R} \) denotes an incentive function, which is twice continuously differentiable, \( k \) is a scalar parameter, and \( J_1(x) \) and \( J_2(x) \) are the original payoff functions satisfying Assumption 2.

The incentive function \( p^k(x) \) can be considered to be a feedback that is designed by the system manager. Note that \( p^k(x) \) should be determined in such a way that \( x^* \) remains the Nash equilibrium of \( G((\tilde{J}_1, \tilde{J}_2)) \) and \( \tilde{J}_1(x), \tilde{J}_2(x) \) should be still partially strictly concave at the desired Nash equilibrium \( x^* \), that is

\[
\frac{\partial^2 \tilde{J}_i(x^*)}{\partial x_i^2} < 0, \quad i = 1, 2. \tag{29}
\]

Furthermore, \( p^k(x) \) should satisfy

\[
p^k(x^*) = 0, \quad \frac{\partial p^k(x^*)}{\partial x_i} = 0, \quad i = 1, 2 \tag{30}
\]

for all \( k \in \mathbb{R} \), which guarantee \( \tilde{J}_i(x^*) = J_i(x^*) \) and \( \partial \tilde{J}_i(x^*)/\partial x_i = 0, i = 1, 2 \). This framework indicates that the system manager collects tax \( p^k(x) \) from one agent and gives the same amount to the other agent as subsidy, so that the.
respective payoff functions are accordingly changed to stabilize the possibly desirable Nash equilibrium. Note that \((30)\) implies that there is no compensation once the agents reach the target Nash equilibrium.

**Corollary 2:** Consider the 2-agent noncooperative system \(G((J_1, J_2))\) with TSA \((28)\) and the pseudogradient dynamics \((27)\). If \(p_k(x)\) in \((28)\) satisfies
\[
det A(\{\tilde{J}_1, \tilde{J}_2\}, \{1, 1\}, x^*) > 0
\] (31) then the Nash equilibrium \(x^*\) is stabilized for any positive constants \(a_1\) and \(a_2\).

**Proof:** The result is a direct consequence of Proposition 2. ■

As a typical form of the TSA, we consider the case with a simple quadratic incentive function given by
\[
p_k(x) \triangleq k(x_1 - x_1^*)^2 (x_2 - x_2^*)^2 \tag{32}
\]
which satisfies \((29)\) and \((30)\) for all \(k \in \mathbb{R}\). In this case, since \(J_i(x_1, x_2^*) = J_i(x_1, x_1^*)\) implies \(\arg\max_{x \in \mathbb{R}} J_i(x_1, x^*_2) = \arg\max_{x \in \mathbb{R}} J_i(x_1, x^*_2) = x_2^*\) for each \(i = 1, 2\), the state profile \(x^*\) remains the Nash equilibrium of \(G(\{\tilde{J}_1, \tilde{J}_2\})\). Moreover, since \(A(\{\tilde{J}_1, \tilde{J}_2\}, \{1, 1\}, x^*) = A(\{\tilde{J}_1, \tilde{J}_2\}, \{1, 1\}, x^*) + k\left[\begin{array}{c} 0 \\ -1 \\ 0 \end{array}\right]\), the condition \((31)\) for \(k\) to stabilize the Nash equilibrium is given by
\[
k \in (-\infty, \gamma_1) \cup (\gamma_2, \infty)
\] (33)
where
\[
\gamma_1 = -\frac{1}{2} \frac{\partial^2 J_1(x^*)}{\partial x_1^2} \frac{\partial^2 J_2(x^*)}{\partial x_2^2} - 4 \frac{\partial^2 J_1(x^*)}{\partial x_1^2} \frac{\partial^2 J_2(x^*)}{\partial x_2^2} + \frac{1}{2} \frac{\partial^2 J_2(x^*)}{\partial x_2^2} \frac{\partial^2 J_1(x^*)}{\partial x_1^2} < 0
\] (34)
\[
\gamma_2 = -\frac{1}{2} \frac{\partial^2 J_2(x^*)}{\partial x_2^2} \frac{\partial^2 J_1(x^*)}{\partial x_1^2} - 4 \frac{\partial^2 J_1(x^*)}{\partial x_1^2} \frac{\partial^2 J_2(x^*)}{\partial x_2^2} + \frac{1}{2} \frac{\partial^2 J_1(x^*)}{\partial x_1^2} \frac{\partial^2 J_2(x^*)}{\partial x_2^2} > 0.
\] (35)

Similarly, consider the case with a simple quadratic incentive function
\[
p_k(x) \triangleq \frac{1}{2} k \left[ (x_1 - x_1^*)^2 + (x_2 - x_2^*)^2 \right], \quad k \leq 0 \tag{36}
\]
which satisfies \((29)\), \((30)\) for all \(k \leq 0\). In this case, since \((36)\) implies \(J_1(x_1, x_2^*) = J_1(x_1, x_1^*) + (1/2)k(x_1 - x_1^*)^2\) and \(J_2(x_2, x_1^*) = J_2(x_2, x_1^*) + (1/2)k(x_2 - x_2^*)^2\), it follows from \(\arg\max_{x \in \mathbb{R}} J_i(x_1, x_2^*) = x_1^*, \quad i = 1, 2\), that \(\arg\max_{x \in \mathbb{R}} J_i(x_1, x_2^*) = x_1^*, \quad i = 1, 2\), and hence, the state profile \(x^*\) remains the Nash equilibrium of \(G(\tilde{J})\). Moreover, since \(A(\{\tilde{J}_1, \tilde{J}_2\}, \{1, 1\}, x^*) = A(\{\tilde{J}_1, \tilde{J}_2\}, \{1, 1\}, x^*) + k_2\), the condition \((31)\) for \(k\) to stabilize the Nash equilibrium is given by
\[
k < \gamma = -\frac{1}{2} \frac{\partial^2 J_1(x^*)}{\partial x_1^2} \frac{\partial^2 J_2(x^*)}{\partial x_2^2} - 4 \det(\Psi)
\]
\[
- \frac{1}{2} \frac{\partial^2 J_2(x^*)}{\partial x_1^2} + \frac{\partial^2 J_2(x^*)}{\partial x_2^2}
\] (37)
where \(\Psi = A(\{\tilde{J}_1, \tilde{J}_2\}, \{1, 1\}, x^*)\).

For the case where the original payoff functions are quadratic as given in \((25)\), the stabilizing condition of \(k\) for the incentive function \((32)\) [resp., \((36)\)] is given by \((33)\) with \(\gamma_1 = a_{12}^2 + a_{12}^2 - \sqrt{(a_{12}^2 + a_{12}^2)^2 - 4a_{11}^2 a_{22}^2}\) and \(\gamma_2 = a_{12}^2 + a_{12}^2 + \sqrt{(a_{12}^2 + a_{12}^2)^2 - 4a_{11}^2 a_{22}^2}\) (resp., \(k < a_{11}^2 + a_{22}^2 - \sqrt{(a_{11}^2 - a_{22}^2)^2 + 4a_{11}^2 a_{22}^2}\)).

**B. Distributed Tax/Subsidy Approach for N-Agent Case**

In the following, we extend the TSA characterized in the previous section to a higher dimensional system \(G(J)\) with \(\mathcal{N} = \{1, \ldots, N\}\). In particular, we suppose that the system manager decomposes the agents into several subgroups and installs distributed controllers (computers) for each of the subgroups. Each of the distributed controllers defines a utility transfer structure represented by a graph within the subgroup, which we call the tax/subsidy adjustment graph, such that the graph is weakly connected. Even though the controllers work in a distributed manner, the system manager needs to know, \textit{a priori}, the information of the payoff functions of all the agents before the operation.

We suppose that the number of subgroups is \(c\) and the tax/subsidy adjustment graphs \(G_1, \ldots, G_c\) are chosen as undirected graphs in such a way that there is no isolated agent that is free from the compensation mechanism. It is important to note that each distributed controller \(\eta \in \{1, \ldots, c\}\) transfers the utilities between the agents consisting of \(G_\eta\) with the information from the same set of agents, that is, \(x_i, \; i \in \mathcal{V}_\eta\), where \(\mathcal{V}_\eta\) denotes the set of nodes constituting the tax/subsidy adjustment graph \(G_\eta\). Henceforth, let \(\mathcal{N}_i\) be the set of neighbor agents for agent \(i\).

Now, consider the adjusted payoff functions given by
\[
\tilde{J}_i(x) \triangleq J_i(x) + p^K_i(x), \quad i \in \mathcal{N} \tag{38}
\]
with the quadratic incentive functions
\[
p^K_i(x) \triangleq \frac{1}{2} k_{ii} (x_i - x_i^*)^2 - \frac{1}{2} \sum_{j \in \mathcal{N}_i} k_{ij} (x_j - x_j^*)^2 / N_j
\]
\[
+ \sum_{j \notin \mathcal{N}_i} k_j (x_i - x_i^*) (x_j - x_j^*), \quad i \in \mathcal{V}_\eta \tag{39}
\]
where \(K = \{k_{ij}, i, j \in \mathcal{N} \triangleq \{K \in \mathbb{R}^{N \times N} : k_{ii} \leq 0, i \in \mathcal{N}, k_{ij} = -k_{ji}, i, j \in \mathcal{N}, i \neq j, k_{ij} = 0, j \notin \mathcal{N}_i, i \in \mathcal{N}\} \) and \(N_i\) is the number of the agents in \(\mathcal{N}_i\). Note that \(p^K_i(x)\) depends only on part of the agents’ state \(x_i, i \in \mathcal{V}_\eta\), in the subgraph \(G_\eta\). Furthermore, if there are multiple subgroups, then \(K\) can be transformed to a block-diagonal matrix by reordering the labels of the agents. Notice that the incentive functions given by \((39)\) are a generalization of the combined functions of \((32)\).
and (36). Furthermore, (39) implies

$$\sum_{i \in N} p_i^K(x) = 0, \quad x \in \mathbb{R}^N, \quad K \in \mathcal{K}$$

(40)

$$p_i^K(x^*) = 0, \quad \frac{\partial^2 p_i^K(x^*)}{\partial x_i} = 0, \quad \frac{\partial^2 J_i(x^*)}{\partial x_i^2} < 0$$

(41)

for all $i \in N$. In this case, since (39) implies

$$\tilde{J}_i(x_i, x_{-i}^*) = J_i(x_i, x_{-i}^*) + (1/2)k_{ij}(x_i - x_{ij}^*)^2, \quad i \in N,$$

it follows from arg max$_{x_i \in \mathbb{R}} J_i(x_i, x_{-i}^*) = x_i^*, \quad i \in N$, that arg max$_{x_i \in \mathbb{R}} \tilde{J}_i(x_i, x_{-i}^*) = x_i^*, \quad i \in N$, and hence, the state profile $x^*$ remains the Nash equilibrium of $G(J)$. Consequently, the Jacobian matrix of the adjusted pseudogradient dynamics

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is written as $A(\tilde{\gamma}, x^*)$. Hence, the two results are direct consequences of Gershgorin’s circle theorem and Theorem 1, respectively.

Proof: Note that since $k_{ij} = 0, \quad j \not\in V_\eta$, $N$ number of inequalities characterized by 1) make $A(\tilde{\gamma}, 1_N, x^*)$ strictly diagonally dominant (i.e., $|\partial^2 J_i(x^*)/\partial x_i^2| < -\sum_{j \in V_\eta} |\partial^2 J_i(x^*)/\partial x_i \partial x_j|$, for all $i \in N$ and the inequality in 2) makes $A^T(\tilde{\gamma}, \tilde{x}^*), A(\tilde{\gamma}, \tilde{x}) = \text{diag}(\tilde{\gamma})K + K^T\text{diag}(\tilde{\gamma}) + A^T(\tilde{\gamma}, \tilde{x}) = \text{diag}(\tilde{\gamma})K + \text{diag}(\tilde{\gamma}) + A^T(\tilde{J}, \tilde{x})$ negative definite. Hence, the two results are direct consequences of Gershgorin’s circle theorem and Theorem 1, respectively.

Remark 8: Corollary 3 indicates that with the information of agents’ original payoff functions $J_1, \ldots, J_N$, the system manager can command the distributed controllers to process the tax/subsidy framework (38), (39) by transmitting the information of corresponding elements of a well-chosen matrix $K$ to the distributed controllers. As such, the system manager can stabilize the target Nash equilibrium $x^*$ for arbitrary $\alpha_i, \quad i \in N$, even though the sensitivity parameters $\alpha_i, \quad i \in N$, are unknown to him/her.

It can be easily found that $N$ number of inequalities characterized by 1) are always solvable for $K \in \mathcal{K}$ such that $A(\tilde{\gamma}, 1_N, x^*)$ is strictly diagonally dominant, because $k_{ii}, \quad i \in N$, can be taken to be sufficiently small so that each agent’s own utility is dominant compared to the effect of other agents. Moreover, even though the inequality characterized in 2) is a special linear matrix inequality with the constraint $K \in \mathcal{K}$, it is possible to make 2) [i.e., $A(\tilde{\gamma}, \tilde{x}^*), A^T(\tilde{\gamma}, \tilde{x}^*)$] strictly diagonally dominant to make sure that it is negative definite, that is

$$\sum_{j \neq i} \left| \frac{\partial^2 J_i(x)}{\partial x_i \partial x_j} \right| + \frac{\partial^2 J_i(x)}{\partial x_i^2} + (\hat{\alpha}_i - \hat{\alpha}_j)k_{ij} < 0$$

$$< -2\hat{\alpha}_i \left( \frac{\partial^2 J_i(x)}{\partial x_i^2} + k_{ii} \right), \quad i \in N$$

(42)

for $x = x^*$. It is interesting to see that (42) can determine $[k_{ij}]_{i \in N, j \in (i + 1, \ldots, N)}$ with a $\hat{\alpha}$ satisfying $\hat{\alpha}_i - \hat{\alpha}_j \neq 0, \quad i \in N, \quad j \not\in N$, and $\hat{k}_{ii} \leq 0, \quad i \in N$. Furthermore, when (42) is satisfied for all $x \in \mathbb{R}^N$, it can be shown that the possibly unstable Nash equilibrium $x^*$ is globally asymptotically stabilized.

Remark 9: The condition 1) in Corollary 3 also indicates that each distributed controller $\eta \in \{1, \ldots, c\}$ can independently choose parameters $[k_{ij}]_{i \in N, \eta \in \{1, \ldots, c\}}$, if the information of $[\partial^2 J_i(x^*)/\partial x_i \partial x_j], \quad j \not\in N, \quad i \in V_\eta$, is given. In other words, each distributed controller $\eta$ can work in a decentralized way even for the case where the number of the agents is large.

Remark 10: In the case where the number $N$ of the agents is so large that the calculation of the target Nash equilibrium $x^*$ is infeasible, our proposed framework can be similarly implemented without calculating the Nash equilibria for $G(J)$. Specifically, by setting $\tilde{x}^*$ as the target state, the incentive functions for the subgroup $\eta, \quad \eta \in \{1, \ldots, c\}$, are given by

$$p_i^K(x) \triangleq \frac{1}{2} k_{ii}(x_i - \tilde{x}_i^*)^2 - \frac{1}{2} \sum_{j \in N_\eta} k_{ij}(x_i - \tilde{x}_j^*)^2/N_j$$

$$\quad + \sum_{j \not\in N_\eta} k_{ij}(x_i - \tilde{x}_j^*)^2/N_j - \sum_{j \not\in N_\eta} \beta_i(x_i - \tilde{x}_j^*)^2/N_j, \quad i \in V_\eta$$

(43)

with $\beta_i \in \mathbb{R}, \quad i \in V_\eta$, satisfying

$$\text{arg max}_{x_i \in \mathbb{R}} \tilde{J}_i(x_i, x_{-i}^*) = \tilde{x}_i^*, \quad i \in V_\eta$$

(44)

and $[k_{ij}]_{i \in N, j \not\in N}$ satisfying the condition 2) in Corollary 3 with $x^*$ replaced by $\tilde{x}^*$. Note that when the target state $\tilde{x}^*$ is not the original Nash equilibrium $x^*$ in $G(J)$, the linear terms $\beta_i(x_i - \tilde{x}_j^*)^2 - \sum_{j \not\in N_\eta} \beta_i(x_i - \tilde{x}_j^*)^2/N_j$ of the incentive functions (43) with $\beta_i \in \mathbb{R}, \quad i \in V_\eta$, satisfying (44), contribute to make the target state $\tilde{x}^*$ a Nash equilibrium in $G(J)$. In such a case, it is understood that the original Nash equilibrium $x^*$ in $G(J)$ is moved to the target state $\tilde{x}^*$ in $G(J)$ under the proposed TSA. Alternatively, when the target state $\tilde{x}^*$ happens to be the same as the original Nash equilibrium $x^*$ in $G(J)$, the condition (44), which is met by the distributed controller, requires $\beta_i = 0$ in order for (43) to reduce to (39). It is worth noting that the establishment of (43) does not force the system manager to collect global information of the payoff functions $J_i(x), \quad i \in N$, since the target state $\tilde{x}^*$ is not necessary to be the original Nash equilibrium $x^*$ in $G(J)$.

V. ILLUSTRATIVE NUMERICAL EXAMPLES

In this section, a couple of numerical examples are presented for illustrating the results and the conditions concerning the
The trajectories of agents’ states diverge without the TSA but converge to the target Nash equilibrium $x^*$ with the proposed TSA for the same set of sensitivity parameters.

The example exhibits diverging trajectory whereas the trajectory of the second example converges to one of the Nash equilibria, which is not the target one without the tax/subsidy mechanism.

Example 5: Consider a wireless communication system being composed of $N$ senders who compete with each other on quality of service characterized by signal-to-interference-plus-noise-ratio for a unique receiver [43]. Each sender (agent) adjusts its transmission power $x_i$ in $\mathbb{R}_+$ to maximize its profit given by $J_i(x_i) = \rho_0 \log_{10}(1 + [g_i x_i L_i]) - \beta_j x_i$, $i \in N$, where $\rho_0 \in \mathbb{R}_+$ denotes the earning rate for service quality, $\sigma \in \mathbb{R}_+$ denotes the additive white noise, $L \in \mathbb{R}_+$ denotes the spreading gain, $g_i \in \mathbb{R}_+$, $i \in N$, denote the channel gain, and $\beta_i \in \mathbb{R}_+$, $i \in N$, denote the price per unit power. Suppose $N = 2$, $\rho_0 = 1$, $\sigma = 0.1$, $L = 0.5$, $g_1 = 1$, $g_2 = 2$, $\beta_1 = 0.1$, and $\beta_2 = 0.2$, so that there exists a unique Nash equilibrium $x^* = [1.3810, 0.6905]^T$. It follows from Corollary 1 that $x^*$ is unstable under the pseudogradient dynamics (3) for any $\alpha \in \mathbb{R}^5_+$ since det $A(J, 1, 2, x^*) = -0.0064 < 0$.

Now, it follows from Corollary 2 that the TSA (28) along with the incentive function (36) with $\kappa = -0.3 < \gamma = -0.0408$ satisfying (37) guarantees that the target Nash equilibrium $x^*$ is asymptotically stabilized for any $\alpha \in \mathbb{R}^5_+$. (In fact, the choice of $k = -0.3$ also satisfies (42) for all $x \in \mathbb{R}^5_+$ with $k_{11} = k_{22} = k$, $k_{12} = -k_{12} = 0$, and $\hat{\alpha}_1 = \hat{\alpha}_2 = 1$ so that global asymptotic stabilization is guaranteed.) The initial state is set to $x(0) = [1, 0, 1]^T$ in the simulation. Fig. 4 shows the trajectories of agents’ states under the pseudogradient dynamics (3) with ten different values of $\alpha$ satisfying $\alpha_1 \in [20, 50]$ and $\alpha_2 \in [30, 85]$. It can be seen from the figure that the agents’ state converges to $x^*$ with the TSA for all those various sensitivity parameters.

Example 6: Consider the noncooperative system being composed of five agents with nonquadratic payoff functions given by $J_1(x_1) = -(x_1 + \sin x_2 - 0.5 \sin x_3)^2 + e^{-(x_1^2 - x_2^2 - x_3^2)}$, $J_2(x) = -(1/2)(2x_2 - \sin x_1 + 2 \sin x_3 x_2)^2 + e^{-(x_2^2 - x_2^2 - x_3^2)}$, $J_3(x) = -(1/3)(3x_3 + 3 \sin x_1 - \sin x_2 + \sin x_4 + \sin x_5)^2 + e^{-(x_1^2 - x_2^2 - x_3^2)}$, $J_4(x) = -(x_4 - 2 \sin x_3 + \sin x_5)^2 + e^{-(x_1^2 - x_2^2 - x_3^2)}$, $J_5(x) = -(x_5 + 3 \sin x_3 + 2 \sin x_4)^2 + e^{-(x_1^2 - x_2^2 - x_3^2)}$, where the payoff dependency network topology is shown in Fig. 5. Note that the noncooperative system possesses multiple Nash equilibria and $x^* = [0, 0, 0, 0, 0]^T$ is one of the Nash equilibria which maximizes every agent’s payoff. In this example, since det $A(J, 1, 3)$ = 482.67 > 0, it follows from Corollary 1 that the Nash equilibrium is unstable under the pseudogradient dynamics (3) for any $\alpha \in \mathbb{R}^5_+$.

To achieve stabilization of the Nash equilibrium $x^*$ by employing (38) and (39), we decompose the agents into two subgroups and install distributed controllers for each of the subgroups. We let the distributed controllers’ tax/subsidy adjustment graphs $G_1$ and $G_2$ be given by Fig. 5 so that agents’ payoffs are transferred between agents 1 and 2 in $V_1 = \{1, 2\}$ and between agents 3 and 5, as well as between agents 4 and 5 in $V_2 = \{3, 4, 5\}$. In this case, only the parameters $\{k_{11}, k_{22}, k_{12}\}$ and $\{k_{33}, k_{44}, k_{55}, k_{45}, k_{54}\}$ should be designed because $K = \{k_{ij} | i, j \in \{1, \ldots, 5\}\}$ should belong to the class $K$. Specifically, suppose that the system manager provides the information of $[\partial^2 J(x^*)]/[\partial x_i \partial x_j]$, $j \in \{1, \ldots, 5\}$, $i \in V_1$ (resp., $i \in V_2$) to the distributed controller for $G_1$ (resp., $G_2$). Then, it follows from conditions 1) of Corollary 3 and Remark 9 that the TSA (38) along with the incentive functions (39) with the choice $k_{11} = -2$, $k_{22} = 0$, $k_{12} = 1$ for $G_1$ and $k_{33} = -4.2$, $k_{44} = -2$, $k_{55} = -10$, and $k_{35} = k_{45} = 1$ for $G_2$ guarantees that the target Nash equilibrium $x^*$ is asymptotically stabilized for arbitrary $\alpha \in \mathbb{R}^5_+$. Furthermore, since these parameters in $K$ happen to satisfy (42) for all $x \in \mathbb{R}^5$ with $\hat{\alpha}_1 = 0.6$, $\hat{\alpha}_2 = 0.2$, $\hat{\alpha}_3 = 0.3$, $\hat{\alpha}_4 = 0.2$, $\hat{\alpha}_5 = 0.05$, we can further guarantee (with the global knowledge of the payoff functions) that the target Nash equilibrium $x^*$ is globally asymptotically stabilized for arbitrary $\alpha \in \mathbb{R}^5_+$. In this case, the incentive functions (39) are given by $p_1^k(x) = -p_2^k(x) = -x_1^2 + x_1 x_2$ for $G_1$ and $p_3^k(x) = -2.1 x_3^2 + 3 x_3 x_5 + 2.5 x_3^2$, $p_4^k(x) = -x_3^2 + x_4 x_5 + 2.5 x_3^2$, $p_5^k(x) = -5 x_5^2 - x_3 x_5 - 2.1 x_3^2 + 2.1 x_3^2$ for $G_2$. The initial state is set to $x(0) = [2, 1, 0, -1, 2]^T$ in the simulation. Fig. 6 shows the trajectories of agents’ states under the pseudogradient dynamics (3) with eight different values of $\alpha$ satisfying $\alpha_1 \in \{1, 4\}$, $\alpha_2 \in \{2, 4\}$, $\alpha_3 \in \{1, 4\}$, $\alpha_4 \in \{2, 3\}$, and $\alpha_5 \in \{2, 3\}$. It can be seen from the figure that without TSA, the agents’ state converges to another Nash equilibrium $\bar{x}^* = [-0.1356, 0.1146, -0.2884, -1.075, 2.611]^T$ instead of the target Nash equilibrium $x^*$ at the origin for all those various sensitivity parameters. However, the agents’ state converges to $x^*$ when we apply the TSA for the same set of sensitivity parameters.
different sets of sensitivity parameters. Fig. 6. Trajectories of the states with and without the TSA under eight
Nash equilibrium parameters. Another Nash equilibrium
problems with malicious attackers, to name but a few.
time-varying communication links with unknown number of agents,
behavior in pseudogradient dynamics, stabilization under time-
dynamical systems; for example, the state jumps in the com-
stabilization of agent’s selfish behaviors in the noncooperative
two-agent and five-agent noncooperative systems.

VI. CONCLUSION

We investigated the Nash equilibrium stabilization problem
for noncooperative dynamical systems through a TSA. In
the proposed TSA, a system manager collects some taxes from
some of the agents and gives the same amount in total as sub-
сидies to the neighbor agents in the tax/subsidy adjustment
To deal with the uncertainty in terms of the pri-
ivate information, we explored the stability conditions of Nash
equilibria without knowing the private information, and also
obtained the conditions under which the state trajectory con-
verges to the originally unstable Nash equilibrium using incentive
functions. Finally, we provided the numerical examples for
demonstrating stabilization of unstable Nash equilibrium for
two-agent and five-agent noncooperative systems.

There still remain several open problems on the analysis and
stabilization of agent's selfish behaviors in the noncooperative
dynamical systems; for example, the state jumps in the com-
bined dynamics with myopic pseudo-gradient dynamics and
best-response dynamics, cognitive hierarchy and risk-averse
behavior in pseudogradient dynamics, stabilization under time-
varying communication links with unknown number of agents,
hierarchical structure of the payoff dependencies, and security
problems with malicious attackers, to name but a few.

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