Relative torsion and bordism classes of positive scalar curvature metrics on manifolds with boundary

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Abstract
We define a relative $L^2$-$\rho$-invariant for Dirac operators on odd-dimensional spin manifolds with boundary and show that they are invariants of the bordism classes of positive scalar curvature metrics which are collared near the boundary. As an application, we show that if a $4k + 3$-dimensional spin manifold with boundary admits such a metric and if, roughly speaking, there exists a torsion element in the difference of the fundamental groups of the manifold and its boundary, then there are infinitely many bordism classes of such psc metrics on the given manifold. This result implies that the moduli-space of psc metrics on such manifolds has infinitely many path components. We also indicate how to define delocalised $\eta$-invariants for odd-dimensional spin manifolds with boundary.

Keywords  Manifolds with boundary · Positive scalar curvature · Rho-invariants · Moduli space

Mathematics Subject Classification  19K56 · 53C27 · 53C22 · 58J28

1 Introduction
In recent years, there have been many contributions to the study of the existence and classification problem for positive scalar curvature metrics on manifolds. The index theory of (twisted) Dirac operators on spin manifolds has been extensively and successfully used in this study. Some results on the topology of the space of psc metrics can be found in [5, 6, 12, 17, 28] and [9]. Apart from [5] and [9], most of these results apply to closed manifolds and statements on the existence and classification problem for psc metrics on manifolds with boundary are more scarce in the literature. The recent works [5] and [9] give very precise information about the (nontriviality) of the homotopy groups of the space of psc metrics which are collared near the boundary and restrict to a fixed metric near the boundary. How-
ever, the invariants utilised by them cannot be used to gain information about the space of bordism classes of such metrics. The following paper is a contribution to the study of psc metrics on manifolds with boundary. Here, we develop invariants which allow us to study the bordism classes of psc metrics on a manifold with boundary which are collared near the boundary. Our observations lead to results concerning the path components of the (moduli) space of psc metrics which do not necessarily restrict to a fixed metric at the boundary.

In the following, we will not discuss the question whether a given spin manifold with boundary admits a metric with positive scalar curvature (with certain boundary conditions) and we will concern ourselves with the secondary question of classification of such metrics. More precisely, we will show that under certain conditions on the fundamental groups of the manifold and its boundary and the dimension of the manifold, the space of psc metrics which are collared near the boundary has infinitely many path components. The same statement is true for the moduli space of such metrics (modulo the action of the diffeomorphism group). Furthermore, we show that on manifolds satisfying our conditions, there exist infinitely many psc metrics (collared near the boundary) which are not even bordant (in an equivariant sense) to each other. Our results together with those of Bär–Hanke in [3] also give information about the space of psc metrics for which the boundary has nonnegative mean curvature and the induced metric on the boundary has psc. Our results can, for example, be applied to 4k + 3-dimensional manifolds obtained by removing a disk from closed spin manifolds which admit a psc metric and whose fundamental group has torsion.

In [6], Botvinnik and Gilkey showed that if a closed odd-dimensional spin manifold with finite fundamental group admits a positive scalar curvature metric, then it admits infinitely many such metrics which are not (equivariantly) bordant to each other. This implies, in particular, that these metrics do not lie in the same path component of the space of psc metrics. The invariants they used to differentiate psc metrics was the APS-\(\rho\)-invariants introduced in [2]. These are the \(\eta\)-invariants of the spin Dirac operator twisted with virtual representations of virtual dimension 0. In [17], Piazza and Schick used the approach of Botvinnik and Gilkey and obtained similar results for manifold with arbitrary fundamental groups containing torsion. The invariant used by them was the \(L^2-\rho\)-invariant of Cheeger and Gromov introduced in [8]. This is the difference of the \(L^2-\eta\)-invariant of the Dirac operator associated the universal cover and the \(\eta\)-invariant of the manifold. It can also be seen as the generalisation of the APS-\(\rho\)-invariant associated to a virtual representation arising from the regular and trivial representations of the fundamental group. To obtain results in dimensions \(4k + 1\), Piazza and Schick also made use of the delocalised \(\eta\)-invariant of John Lott.

The main novelty of this paper is the definition of the relative \(L^2-\rho\)-invariant for odd-dimensional compact spin manifolds with boundary. In most of the paper, we are concerned with establishing the fundamental properties of this invariant. The geometric applications stated above are then obtained by following the same path as the one taken in [17]. In a remark, we also point out how our technique can be used to define delocalised \(\eta\)-invariants associated to metrics on manifolds with boundaries. Using this, there does not seem to be any problem in obtaining the analogues of the more quantitative results in [17] for manifolds with boundary.

We quickly describe how our relative \(L^2-\rho\)-invariant is defined. Let \(M\) be a compact spin manifold with boundary \(N\). Suppose that the Riemannian metric is collared near the boundary. By the van Kampen theorem the double of \(M\) along \(N\) has fundamental group \(\pi_1(M) \ast_{\pi_1(N)} \pi_1(M)\). Denote by \(\text{Im}(\pi_1(N))^{\pi_1(M)}\) the normal closure of the image of \(\pi_1(N)\) in \(\pi_1(M)\) under the map of fundamental groups induced by the inclusion \(N \to M\). One can then twist the spin Dirac operator of the double with a bundle of Hilbert modules over the group von Neumann algebra of \(\pi_1(M)/\text{Im}(\pi_1(N))^{\pi_1(M)}\). This bundle is defined using a suitable representation of
\( \pi_1(M) \ast_{\pi_1(N)} \pi_1(M) \) on the group von Neumann algebra \( \mathcal{N}(\pi_1(M)/\text{Im}(\pi_1(N)))^{\pi_1(M)} \). One can define an \( \eta \)-invariant with values in the abelianisation of \( \mathcal{N}(\pi_1(M)/\text{Im}(\pi_1(N)))^{\pi_1(M)} \) and our \( L^2 - \rho \)-invariant is then obtained from the latter \( \eta \)-invariant using the canonical trace of \( \mathcal{N}(\pi_1(M)/\text{Im}(\pi_1(N)))^{\pi_1(M)} \). Using a pair of finite-dimensional representations of \( \pi_1(M) \) agreeing on \( \pi_1(N) \), we can also define a relative version of the APS \( \rho \)-invariants and in this case our invariants seem to be strongly related to the invariants defined by Bunke in [7], which has inspired our construction.

The fact that the relative \( L^2 - \rho \)-invariant is an invariant of the bordism class of psc metrics is then established by a modification of a standard argument using the (higher) APS index formula. We later prove the important fact that the relative \( L^2 - \rho \)-invariant can also be obtained as the \( L^2 - \eta \)-invariant of a suitable cover of the double of the manifold with boundary. We make use of this characterisation of our invariant as an \( L^2 - \eta \)-invariant to establish further fundamental properties of the relative \( \rho \)-invariant.

The paper is organised as follows. In Sect. 2, we define in detail the relative versions of the groups appearing in the Stolz positive scalar curvature sequence, which again fit in a long exact sequence. In Sect. 3, we recall some material from higher index theory, which will be used in the rest of the paper. In Sect. 4, we define the relative \( L^2 - \rho \)-invariant associated to Riemannian metrics which are collared near the boundary and prove that it is an invariant of the bordism class of a psc metric. This is followed by the characterisation of the relative \( L^2 - \rho \)-invariant as an \( L^2 - \eta \)-invariant. In Sect. 5, we use the latter characterisation to establish the fundamental properties of our invariant. In Sect. 6, we use the developed theory and the techniques of [17] to obtain some geometric applications.

## 2 Relative Stolz psc sequence

In this section we define the relative version of the positive scalar curvature exact sequence of Stolz. For the definitions in the absolute case we refer the reader to [18, Definition 1.26] and [20, Definition 5.4]. Let \( \alpha : Y \to X \) be a continuous map of the topological spaces \( Y \) and \( X \). First we recall the definition of the relative spin bordism group.

**Definition 2.1** \( \Omega_n^{\text{Spin}}(\alpha) \) will denote the usual relative spin bordism group. Cycles are given by tuples \((M, N, F : M \to X, f : N \to Y)\) such that \( M \) is an \( n \)-dimensional compact spin manifold with boundary \( N \) and \( F|_N = \alpha \circ f \). This cycle is null-bordant if there exists \((V, E, e)\) where

- \( V \) is an \( n + 1 \)-dimensional manifold with corner with \( \partial V = M \cup \partial_i V \) and corner of \( V \) is \( \partial V = M \cap \partial_i V = N \).
- \( E \) is a continuous map \( V \to X \) with \( E|_M = F \).
- \( e \) is a continuous map \( \partial_i V \to Y \) with \( E|_{\partial V} = \alpha \circ e \) and \( e|_N = f \).

The addition of two cycles is defined to be their disjoint union. The inverse of a cycle is obtained by reversing the spin structure. Two cycles are equivalent if their difference is null-bordant. The group \( \Omega_n^{\text{Spin}}(\alpha) \) is defined as the set of equivalence classes of the above cycles and the above addition induces the group operation. In the following, manifolds such as \( V \) providing a cobordism between other manifolds will be called bordisms and \( \partial_i V \) will be referred to as the internal boundary of the bordism \( V \).

1 [18, Definition 1.26.2] contains a typo. The group defined there is \( R_{n+1}^{\text{Spin}}(X) \).
Definition 2.2 The cycles for \( \text{Pos}_{n}^{\text{Spin}}(\alpha) \) have the form
\[
(M, N, F : M \to X, f : N \to Y, g),
\]
where \((M, N, F, f)\) defines a cycle for \( \Omega_{n}^{\text{Spin}}(\alpha) \) and \( g \) is a metric with positive scalar curvature on \( M \) which is collared near \( N \). A cycle is null-bordant if there exists \((V, E, e, G)\) where
- \((V, E, e)\) provides null-bordism for \((M, N, F, f)\) seen as a cycle in \( \Omega_{n}^{\text{Spin}}(\alpha) \).
- \( G \) is a positive scalar curvature metric on \( V \) which is collared near \( \partial V \) and restricts to \( g \) on \( M \). We assume that \( G \) has the form \( G|_{\partial V} + dt^2 + ds^2 \) in a bicollar neighbourhood of the corner \( \partial_c V \).

Addition and inversion of cycles are defined by taking the disjoint union and reversing the relation. Again, addition of cycles induces the group operation.

Definition 2.3 Cycles of \( R_{n+1}^{\text{Spin}}(\alpha) \) are given by tuples
\[
(Z, M, W, H : Z \to X, F : M \to X, h : W \to Y, g)
\]
where
- \( M \) and \( W \) are compact \( n \)-dimensional spin manifolds with \( N := \partial M = \partial W \) and \((M, N, F, f := h|_{N}, g)\) defines a cycle of \( \text{Pos}_{n}^{\text{Spin}}(\alpha) \).
- \((Z, H, h)\) provides an equivalence of \((M, N, F, f)\) with the empty set, seen as cycles of \( \Omega_{n}^{\text{Spin}}(\alpha) \), in particular we are assuming \( Z \) to be manifold with corner with \( \partial Z = W \cup_{\partial Z} M \) and corner \( \partial_{c}Z = N \).

This tuple is null-bordant if
- There exists \((V, E, e, G)\) providing an equivalence between \((M, N, F, f := h|_{N}, g)\) the empty tuple seen as cycles for \( \text{Pos}_{n}^{\text{Spin}}(\alpha) \). We note that \( \partial_{c}V \) provides an equivalence between \((N, f, g|_{N})\) and the empty tuple considered as cycles of \( \text{Pos}_{n-1}^{\text{Spin}}(Y) \).
- There exists \((U, L : U \to Y)\) where
  - the spin manifold with corner \( U \) is a null-bordism for \( W \),
  - \( \partial_{c}U = \partial_{c}V \) and
  - the map \( L \) extends the maps \( H \).

In particular, \((U, L, G|_{\partial_{c}V})\) provides an equivalence between \((W, h, g|_{N})\) and the empty tuple considered as cycles in \( R_{n}^{\text{Spin}}(Y) \).
- There exists an \( n+1 \)-dimensional spin manifold with corner \( V' \) with \( \partial V' = V \cup_{\partial_{c}V} U \) and a continuous map \( E' : V' \to X \) extending \( H \) satisfying \( E'|_{U} = \alpha \circ L \).

Addition and inversion of cycles are defined by taking the disjoint union and reversing the spin structure, respectively. Two cycles are called equivalent if their difference is null-bordant and the group \( R_{n+1}^{\text{Spin}}(\alpha) \) is defined as the quotient of the set of cycles by this equivalence relation. Again, addition of cycles induces the group operation.

Given maps \( \beta : C \to D, s : C \to Y \) and \( t : D \to X \) which make the diagram
\[
\begin{array}{ccc}
C & \xrightarrow{s} & Y \\
\downarrow{\beta} & & \downarrow{\alpha} \\
D & \xrightarrow{t} & X
\end{array}
\]
commutative, one obtains group homomorphisms

$$\Omega_n^{Spin} (\beta) \to \Omega_n^{Spin} (\alpha), \quad \text{Pos}_n^{Spin} (\beta) \to \text{Pos}_n^{Spin} (\alpha) \quad \text{and} \quad R_n^{Spin} (\beta) \to R_n^{Spin} (\alpha),$$

by postcomposing the maps in the definition of cycles for these groups with \( s \) and \( t \) suitably. It is then easy to see that the relative \( \Omega_n^{Spin}, \text{Pos}_n^{Spin} \) and \( R_n^{Spin} \) groups defined above form functors on the category whose objects are continuous maps of topological spaces and where morphisms are given by pairs \((t, s)\) as above.

First, observe that the relative groups associated to \( \alpha : \emptyset \to X \) coincide with the absolute groups associated to \( X \). Using this observation and the commutativity of the diagram

$$\begin{array}{ccc}
\emptyset & \to & \emptyset \\
& \alpha & \downarrow \\
Y & \to & X \\
\downarrow & & \downarrow \text{id}_X \\
X & \to & X
\end{array}$$

we obtain maps

$$\alpha_* : \gamma_n^{Spin}(Y) \to \gamma_n^{Spin}(X) \quad \text{and} \quad (\text{id}_X)_* : \gamma_n^{Spin}(X) \to \gamma_n^{Spin}(\alpha).$$

Moreover, the following maps are well defined:

- the boundary map \( R_{n+1}^{Spin}(\alpha) \to \text{Pos}_{n+1}^{Spin}(\alpha) \) which sends the tuple given by \((Z, M, W, H : Z \to X, F : M \to X, h : W \to Y, g)\) to \((M, F : M \to X, h \mid N : N \to Y, g)\);
- the forgetful map \( \text{Pos}_n^{Spin}(\alpha) \to \Omega_n^{Spin}(\alpha) \) which sends a cycle \((M, N, F : M \to X, f : N \to Y)\) to \((M, N, F : M \to X, f : N \to Y)\);
- the map \( \Omega_n^{Spin}(\alpha) \to R_n^{Spin}(\alpha) \) which sends \((M, N, F : M \to X, f : N \to Y)\) to the cycle \((M, \emptyset, N, F : M \to X, \emptyset, f : N \to Y, \emptyset)\).

The following two propositions follow immediately from the definitions.

**Proposition 2.4** There exist a long exact sequence

$$\cdots \to \Omega_{n+1}^{Spin}(Y) \xrightarrow{\alpha_*} \Omega_{n+1}^{Spin}(X) \xrightarrow{(\text{id}_X)_*} \Omega_{n+1}^{Spin}(\alpha) \xrightarrow{\partial} \Omega_n^{Spin}(Y) \xrightarrow{\delta} \cdots$$

where \( \partial \) is given at the level of cycles by sending \((M, N, F : M \to X, f : N \to Y)\) to \((f : N \to Y)\). Similar long exact sequences exist for the \( \text{Pos}_*^{Spin} \) and \( R_*^{Spin} \) functors. The long exact sequence for \( \text{Pos}_*^{Spin} \) terminates at \( \text{Pos}_2^{Spin}(\alpha) \) and the long exact sequence for \( R_*^{Spin} \) terminates at \( R_3^{Spin}(\alpha) \).

**Proposition 2.5** We have a relative Stolz long exact sequence

$$\cdots \to R_{n+1}^{Spin}(\alpha) \to \text{Pos}_n^{Spin}(\alpha) \to \Omega_n^{Spin}(\alpha) \to R_n^{Spin}(\alpha) \to \cdots \quad (1)$$

Furthermore, the arrows in this exact sequence are natural transformation of functors.

### 3 Index-theoretic preliminaries

In this section, we recall some background material from higher index theory on manifolds with boundary. For a more general, yet concise review of related material we refer the reader to [16, Sections 2 and 3].
Let $W$ be a compact spin manifold with boundary $X$. Let $\mathcal{A}$ be a $C^*$-algebra and $\mathcal{L}$ be a flat bundle of finitely generated projective Hilbert $C^*$-modules over $\mathcal{A}$. Denote by $\mathcal{D}_W$ the spin Dirac operator of $W$. This is a first order differential operator acting on sections of the spinor bundle $S_W$ on $W$. Twisting $\mathcal{D}_W$ by the bundle $\mathcal{L}$ we obtain an $\mathcal{A}$-linear differential operator $\mathcal{D}_{W,\mathcal{L}}$ acting on the smooth sections of $S \otimes \mathcal{L}$. In what follows, we will suppose that the Riemannian metric of $W$ is collared near the boundary and has positive scalar curvature there. Then one can define a higher index $\text{ind}(\mathcal{D}_{W,\mathcal{L}})$ for the operator $\mathcal{D}_{W,\mathcal{L}}$ in $K_{\dim W}(\mathcal{A})$.\(^2\)

**Remark 3.1** A priori, $\text{ind}(\mathcal{D}_{W,\mathcal{L}})$ depends on the positive scalar curvature metric on the boundary. If the metric has positive scalar curvature everywhere, then $\text{ind}(\mathcal{D}_{W,\mathcal{L}})$ vanishes. We will need this fact in the proof of the well-definedness of the relative $L^2$-$p$-invariant.

Denote by $[\mathcal{A}, \mathcal{A}]$ the closure of the linear subspace of $\mathcal{A}$ generated by elements of the form $ab - ba$ with $a, b \in \mathcal{A}$. The quotient $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$ is called the abelianisation of $\mathcal{A}$ and will be denoted by $\mathcal{A}_{ab}$. The canonical projection map $\mathcal{A} \to \mathcal{A}_{ab}$ then defines a trace on $\mathcal{A}$. We will refer to this map as the algebraic trace and we will denote it by $\text{tr}_{\text{alg}}$. Note that any trace on $\mathcal{A}$ factors uniquely through the algebraic trace. The algebraic trace then induces a map $K_0(\mathcal{A}) \to \mathcal{A}_{ab}$, which we will also denote by $\text{tr}_{\text{alg}}$.

**Definition 3.2** Suppose that the manifold $W$ is even-dimensional. The $\mathcal{A}_{ab}$-valued index of the operator $\mathcal{D}_{W,\mathcal{L}}$ is defined to be the image of $\text{ind}(\mathcal{D}_{W,\mathcal{L}}) \in K_{\dim W} \cong K_0(\mathcal{A})$ under the algebraic trace map and will be denoted by $\text{ind}_{[0]}(\mathcal{D}_{W,\mathcal{L}})$.

We will now recall a higher APS index formula for the $\mathcal{A}_{ab}$-valued index. Recall that, for a manifold $Y$, a $C^*$-algebra $\mathcal{A}$ and a bundle $E \to Y$ of finitely generated projective Hilbert $\mathcal{A}$-modules modules, the trace $\text{tr}_{\text{alg}}$ induces a homomorphism $C^\infty(Y, \text{End}_\mathcal{A}(E)) \to C^\infty(Y, \mathcal{A}_{ab})$.

Let us first define the $\mathcal{A}_{ab}$-valued $\eta$-invariant of the twisted Dirac operator on the boundary. Denote by $\mathcal{D}_X$ the spin Dirac operator on $X$ and by $\mathcal{D}_X,\mathcal{L}$ the Dirac operator twisted with the restriction of $\mathcal{L}$ to $X$. The operator $\mathcal{D}_X,\mathcal{L} e^{-t \mathcal{D}_X^2} \mathcal{L}$ is then a smoothing operator in the Mischenko–Fomenko calculus for each $t > 0$. Let $K_t$ denote the kernel of the latter operator. For each $x \in X$, $K_t(x, x)$ can then be seen as an $\mathcal{A}$-linear endomorphism of $(S_X \otimes \mathcal{L})_x$ and thus we can take its algebraic trace, which we will denote by $\text{tr}_{\text{alg}}(K_t(x, x))$. See [22, Proposition 2.26 and Lemma 2.29] for more details.

**Definition 3.3**

- Define $\text{TR}$ by the following integral

$$\text{TR}(\mathcal{D}_X,\mathcal{L} e^{-t \mathcal{D}_X^2} \mathcal{L}) := \int_X \text{tr}_{\text{alg}}(K_t(x, x)) \, dx \in \mathcal{A}_{ab}.$$  

- The $\mathcal{A}_{ab}$-valued $\eta$-invariant of $\mathcal{D}_X,\mathcal{L}$ is defined by the integral

$$\eta_{[0]}(\mathcal{D}_X,\mathcal{L}) := \frac{1}{\sqrt{\pi}} \int_0^\infty \text{TR}(\mathcal{D}_X,\mathcal{L} e^{-t \mathcal{D}_X^2} \mathcal{L}) \frac{dt}{\sqrt{t}} \in \mathcal{A}_{ab}.$$  

**Remark 3.4** The above definition makes sense even if we forget about $W$ and start with a flat bundle $\mathcal{L}$ of finitely generated projective $\mathcal{A}$-modules on a closed odd-dimensional spin manifold $X$ which has positive scalar curvature.

\(^2\)See Wu [27].
Secondly, for a compact manifold $Y$ and a bundle of finitely generated projective Hilbert $A$-modules $E \to Y$ with connection $\nabla^E$ and curvature $\Omega^E \in \Omega^2(Y, \text{End}_A(E))$, recall that the $A_{ab}$-valued Chern character of $E$ is defined as the even-degree $A_{ab}$-valued differential form

$$\text{ch}(E) := \text{tr}^{\text{alg}} \left( \exp \left( \Omega^E \right) \right) \in \Omega^2(Y, A_{ab}).$$

We are now ready to state the following APS index formula.

**Theorem 3.5** [16, Theorem 2.24] The $A_{ab}$-valued index $\text{ind}_{[0]}(D_W, \mathcal{L})$ is given by

$$\int_W \widehat{\Lambda}(W)(x) \wedge \text{ch}([W]) \, dx - \frac{1}{2} \eta_{[0]}(D_{X, \mathcal{L}}).$$

Here, $\widehat{\Lambda}(W)(x)$ denotes the value at the point $x$ of the differential form constructed using Chern–Weil theory representing the $\widehat{\Lambda}$-class of the manifold $W$.

### 4 The relative $L^2$-$\rho$-invariant

Let $\Lambda$ and $\Gamma$ be discrete groups and let $\varphi: \Lambda \to \Gamma$ be a group homomorphism. $\varphi$ induces a map $B\varphi: B\Lambda \to B\Gamma$, which can be assumed to be injective.\(^3\) We will construct a group homomorphism

$$\rho^{\varphi}_{(2)}: \text{PosSpin}_n(B\varphi) \to \mathbb{R}$$

for all odd $n$ using $\eta$-invariants of suitable Dirac operators.

Denote by $\text{Im}(\varphi)^\Gamma$ the normal subgroup generated by the image of $\Lambda$ in $\Gamma$ and by $\mathcal{N}(\Gamma/\text{Im}(\varphi)^\Gamma)$ the von Neumann algebra of the group $\Gamma/\text{Im}(\varphi)^\Gamma$. Denote by $\pi_r: \Gamma \to \Gamma/\text{Im}(\varphi)^\Gamma$ the canonical projection map and by $\pi_{\text{triv}}: \Gamma \to \Gamma/\text{Im}(\varphi)^\Gamma$ the trivial group homomorphism. The homomorphisms $\pi_r$ and $\pi_{\text{triv}}$ give rise to a homomorphism

$$\pi_r \ast \pi_{\text{triv}}: \Gamma \ast \Lambda \Gamma \to \Gamma/\text{Im}(\varphi)^\Gamma$$

by the universal property of the amalgamated product. Let $\Pi$ be the composition

$$\Gamma \ast \Lambda \Gamma \xrightarrow{\pi_r \ast \pi_{\text{triv}}} \Gamma/\text{Im}(\varphi)^\Gamma \hookrightarrow \mathcal{N}\left(\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}\right),$$

where the second arrow is the inclusion of $\Gamma/\text{Im}(\varphi)^\Gamma$ into its von Neumann algebra. There are two natural traces on $\mathcal{N}(\Gamma/\text{Im}(\varphi)^\Gamma)$: the algebraic trace

$$\text{tr}^{\text{alg}}: \mathcal{N}\left(\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}\right) \to \mathcal{N}\left(\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}\right)_{ab},$$

which is given by the quotient map, and the canonical trace

$$\tau_{\text{can}}: \mathcal{N}\left(\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}\right) \to \mathbb{C}$$

which is the continuous extension of the canonical trace $\mathbb{C}(\Gamma/\text{Im}(\varphi)^\Gamma) \to \mathbb{C}$ on the group ring, mapping an element to the coefficient of the neutral element. Note that the canonical

\(^3\) Replace $B\Gamma$ by the mapping cylinder of $B\varphi$. 
trace factorises through the universal trace: \( \tau_{\text{can}} = \tau_{\text{can}}' \circ \text{tr}^{\text{alg}} \) with \( \tau_{\text{can}}' \) a unique linear map \( \mathcal{N}(\Gamma/\text{Im}(\varphi)^\Gamma)_{ab} \rightarrow \mathbb{C} \).

In the following, let \( n \in \mathbb{N} \) be odd. A class in \( \text{Pos}_{n}^{\text{Spin}}(B\varphi) \) is represented by a tuple \((M, N, c, g)\) with \( M \) a compact \( n\)-dimensional spin manifold with boundary \( N \), endowed with a positive scalar curvature metric \( g \) which is collared near the boundary and a continuous map \( c : (M, N) \rightarrow (B\Gamma', \Lambda\Lambda) \).\(^4\) In the notation of Definition 2.2, this cycle will be denoted by \((M, N, c, c|_N, g)\). Denote by \( DM \) the double \( M \cup_N (-M) \) of \( M \), where \(-M\) denotes the manifold \( M \) with the reversed spin structure. Since the metric of \( M \) is collared near the boundary, we naturally obtain a Riemannian metric on the double with positive scalar curvature. The map \( c \) then induces a map

\[ Dc : DM \rightarrow B\Gamma \cup_{BA} B\Gamma \rightarrow B(\Gamma^{\star}A \Gamma). \]

Denote by \( \widetilde{DM} \) the cover associated to the map \( Dc \), and consider the bundle of von Neumann algebras

\[ \mathcal{L} := \widetilde{DM} \times_{\Gamma^{\star}A \Gamma} \mathcal{N} \left( \frac{\Gamma}{\text{Im}(\varphi)^\Gamma} \right), \]

where an element of \( \Gamma^{\star}A \Gamma \) acts on \( \mathcal{N}(\Gamma/\text{Im}(\varphi)^\Gamma) \) via left multiplication by its image under the map \( \Pi \) from (3). Let \( \mathcal{D}_{DM, L} \) be the spin Dirac operator of \( DM \) twisted by \( \mathcal{L} \). We are now ready to define the relative \( L^2-\rho\)-invariant.

**Definition 4.1** The relative \( L^2-\rho\)-invariant of a cycle \((M, N, c, g)\) representing a class in \( \text{Pos}_{n}^{\text{Spin}}(B\varphi) \) is

\[ \rho_{(2)}^\varphi (M, N, c, g) := \tau_{\text{can}}'(\eta|_0)(\mathcal{D}_{DM, L}). \]

**Remark 4.2** When the boundary of \( M \) is empty, the double \( DM \) is just the disjoint union of two copies of \( M \) with opposite orientations and \( \widetilde{DM} \) is the universal covering on one copy of \( M \) and the trivial \( \Gamma\)-covering on \( M \) on the other copy of \( M \). In this situation, we recover the classical Cheeger-Gromov \( \rho\)-invariant as the difference of the \( L^2-\eta\)-invariant and the classical \( \eta\)-invariant of \( M \).

**Proposition 4.3** The relative \( L^2-\rho\)-invariant descends to a group homomorphism

\[ \rho_{(2)}^\varphi : \text{Pos}_{n}^{\text{Spin}}(B\varphi) \rightarrow \mathbb{R}. \]

**Proof** We need to show that if \((M, N, c, g)\) and \((M', N', c', g')\) represent the same class in \( \text{Pos}_{n}^{\text{Spin}}(B\varphi) \), then

\[ \rho_{(2)}^\varphi (M, N, c, g) = \rho_{(2)}^\varphi (M', N', c', g'). \]

This is equivalent to showing that if \((M, N, c, g)\) is null-bordant, namely represents zero in \( \text{Pos}_{n}^{\text{Spin}}(B\varphi) \), then \( \rho_{(2)}^\varphi (M, N, c, g) = 0 \).

Let \((M, N, c, g)\) be null-bordant. This means that there exists a tuple \((V, b, b|_N, G)\) as in Definition 2.2 providing an equivalence between \((M, N, c|_N, G)\) and the empty set. Consider the manifold \( W := V \cup_{\partial a} V - V \), which is a compact spin manifold with boundary \( DM \). Since \( G \) is collared near the boundary and bicollared near the corner of \( V \), we obtain naturally a psc metric on \( W \) which is collared near the boundary. On the boundary this metric

\[^4\text{We are using the assumption that the map } B\varphi \text{ is injective.}\]
coincides with the psc metric on $DM$ defined using $g$. Furthermore, using $b$ we get a map $Db: W \rightarrow B\Gamma \cup_{BA} B\Gamma \rightarrow B(\Gamma \ast \Lambda \Gamma)$ making the diagram

$$
\begin{array}{ccc}
W & \xrightarrow{Db} & B(\Gamma \ast \Lambda \Gamma) \\
\downarrow & & \\
DM & \xrightarrow{Dc} & B(\Gamma \ast \Lambda \Gamma)
\end{array}
$$

commutative. Let $\tilde{W}$ be the cover associated to $Db$ and set

$$
L_W := \tilde{W} \times_{\Gamma \ast \Lambda \Gamma} N(\Gamma / \text{Im}(\phi))
$$

where the action of $\Gamma \ast \Lambda \Gamma$ on $N(\Gamma / \text{Im}(\phi))$ is, as above, induced by $\Pi$ from (3). Clearly, $L_W$ restricts to $L$ on $DM$. Since the metric on $DM$ has positive scalar curvature and $W$ is even-dimensional, one can define an index $\text{ind}(D_{W,L}) \in K_0(N(\Gamma / \text{Im}(\phi))^{\text{ab}})$. By Theorem 3.5,

$$
\text{tr}^{\text{alg}}(\text{ind}(D_{W,L})) = \text{ind}_0(D_{W,L}) = \int_W \hat{A}(W)(x) \wedge \text{ch}(L)(x)dx - \frac{1}{2} \eta_0(\mathcal{D}_{DM,L}).
$$

Since $L$ is a flat bundle with each fibre a free $N(\Gamma / \text{Im}(\phi))^{\text{ab}}$-module of rank 1, we have $\text{ch}(L)(x) = 1 \in N(\Gamma / \text{Im}(\phi))_\text{ab}$. Hence,

$$
\text{ind}_0(D_{W,L}) = \int_W \hat{A}(W)(x)dx \cdot 1 - \frac{1}{2} \eta_0(\mathcal{D}_{DM,L}).
$$

On the other hand,

$$
\int_W \hat{A}(W) = \int_V \hat{A}(W) + \int_{-V} \hat{A}(W) = 0.
$$

The metric on $W$ has positive scalar curvature; therefore, $\text{ind}(D_{W,L})$ and hence $\text{ind}_0(D_{W,L})$ vanish. This implies that

$$
\eta_0(\mathcal{D}_{DM,L}) = 0
$$

Applying $\tau'_\text{can}$ to both sides gives

$$
\rho_{(2)}^\phi(M, N, c, g) = 0.
$$

Thus we have shown that if $(M, N, c, g)$ represents the zero class in $\text{Pos}^{\text{Spin}}_n(B\phi)$ then $\rho_{(2)}^\phi(M, N, c, g) = 0$, which proves that $\rho_{(2)}^\phi$ descends to a map

$$
\text{Pos}^{\text{Spin}}_n(B\phi) \rightarrow \mathbb{R}.
$$

Since addition in $\text{Pos}^{\text{Spin}}_n(B\phi)$ is induced by taking disjoint unions, this map is clearly a group homomorphism.

\begin{remark}
An element $\gamma \in \Gamma / \text{Im}(\Lambda)^\Gamma$ whose conjugacy class has polynomial growth gives rise to a trace $K_0(N(\Gamma / \text{Im}(\Lambda)^\Gamma)) \rightarrow \mathbb{C}$. We can then define a relative version of John Lott’s delocalised $\eta$-invariant (see [13, 14]) by applying this trace to $\eta_0(\mathcal{D}_{DM,L})$. The techniques used below, combined with the approach of [17], should give an analogue of Theorem 6.5 for $4k + 1$-dimensional manifolds without much difficulty.
\end{remark}
Remark 4.5 A pair of finite dimensional representations of $\Gamma$ agreeing on $\Lambda$ gives rise to a representation of $\Gamma_\ast \Lambda \Gamma$, which can be used to define a finite dimensional flat bundle on $DM$ (see above for the notation). The $\eta$-invariant of the spin Dirac operator of $DM$ twisted with the latter bundle can be shown to be an invariant of the bordism class of the psc metric on $M$ (using an argument similar to the one used in the proof of the previous theorem). This can be seen as a generalisation of the APS $\rho$-invariant associated to a pair of representations for manifolds with boundary and seems to be strongly related to the invariants considered by Bunke in [7], which have inspired our construction.

Now we show that the relative $L^2$-$\rho$-invariant is equal to the $L^2$-$\eta$-invariant of a suitable $\Gamma/\text{Im}(\varphi)\Gamma$-cover of $DM$.

We first recall one definition of the $L^2$-$\eta$-invariant. Let $X$ be a closed odd-dimensional spin manifold and $\tilde{X} \to X$ be a $G$-cover of $X$ for some discrete group $G$. Denote by $\tilde{\mathcal{D}}$ the Dirac operator on $\tilde{X}$, constructed using the pullback of the metric on $X$, and by $k_t$ the Schwartz kernel of the operator $\tilde{\mathcal{D}}e^{-t(\tilde{\mathcal{D}})^2}$. Then the $L^2$-$\eta$-invariant of the spin Dirac operator associated to the cover $\tilde{X} \to X$ is given by

$$
\eta(\tilde{\mathcal{D}}):= \frac{1}{\sqrt{\pi}} \int_0^\infty \left( \int_{\mathcal{F}} k_t(x, x) \, dx \right) \frac{dt}{\sqrt{t}}
$$

where $\mathcal{F}$ is any fundamental domain of the action of $G$ on $\tilde{X}$. The homomorphism $\pi_r \ast \pi_{\text{triv}}$ defined in (2) gives rise to a map $B(\pi_r \ast \pi_{\text{triv}}): B(\Gamma_\ast \Lambda \Gamma) \to B\left(\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}\right)$ of classifying spaces. Denote by $(\text{Dec})'$ the composition

$$
DM \xrightarrow{\text{Dec}} B(\Gamma \ast \Lambda \Gamma) \xrightarrow{B(\pi_r \ast \pi_{\text{triv}})} B\left(\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}\right).
$$

Proposition 4.6 The relative $L^2$-$\rho$-invariant $\rho(2)(M, N, g, c)$ is given by the $L^2$-$\eta$-invariant of the cover of $DM$ associated to the classifying map $(\text{Dec})'$.

Proof Denote by $(\text{DM})'$ the cover associated to $(\text{Dec})'$. Thus, we have $(\text{DM})' = \tilde{DM}/\text{Ker}(\pi_r \ast \pi_{\text{triv}})$. The map

$$
\tilde{DM} \times N\left(\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}\right) \to (\text{Dec})' \times N\left(\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}\right)
$$

which is the natural projection on the first component and the identity on the second component induces an isomorphism of the bundles $\mathcal{L} = \tilde{DM} \times_{\Gamma_\ast \Lambda \Gamma} N\left(\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}\right)$ and $(\text{DM})' \times_{\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}} N\left(\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}\right)$. The $\eta$-invariant of the Dirac operator on $DM$ twisted with the bundle $(\text{DM})' \times_{\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}} N\left(\frac{\Gamma}{\text{Im}(\varphi)^\Gamma}\right)$ is the $L^2$-$\eta$-invariant of the cover $(\text{DM})'$ [16, Proposition E.11].

5 Basic properties of the relative $L^2$-$\rho$-invariant

In this section, we relate the relative $L^2$-$\rho$-invariant to the classical $L^2$-$\rho$-invariant and establish some of its fundamental properties.

The classical $L^2$-$\rho$-invariant gives rise to a group homomorphism

$$
\rho^G(2): \text{Pos}_{n_{\mathbb{R}}}^\text{Spin}(BG) \to \mathbb{R}
$$
for odd $n$. A cycle for $\text{Pos}_{n}^{\text{Spin}}(BG)$ is a tuple $(M, c, g)$ with $M$ a closed $n$-dimensional spin manifold, $g$ a positive scalar curvature metric on $M$ and $c$ a continuous map $M \rightarrow \Omega BG$. Denote by $\tilde{M}$ the $G$-cover associated to $c$ and by $\tilde{\mathcal{D}}$ and $\mathcal{D}$ the spin Dirac operator on $\tilde{M}$ and $M$ respectively. The classical $L^2$-$\rho$-invariant of $[(M, c, g)]$ is then defined to be

$$\rho_{(2)}^G([(M, c, g)] := \eta_2(\tilde{\mathcal{D}}) - \eta(\mathcal{D}),$$

where $\eta(\mathcal{D})$ is the usual $\eta$-invariant of $\mathcal{D}$ introduced in [1]. Let $H$ be a further discrete group and $f : H \rightarrow G$ a group homomorphism. Then there is a natural map

$$\text{Pos}_{n}^{\text{Spin}}(BG) \rightarrow \text{Pos}_{n}^{\text{Spin}}(Bf)$$

induced by sending a cycle $(M, c, g)$ for $\text{Pos}_{n}^{\text{Spin}}(BG)$ to the cycle $(M, \emptyset, c, \emptyset \rightarrow BH, g)$.

**Remark 5.1** We use the notation from the above discussion. There is a natural map $\Omega_{n}^{\text{Spin}}(BG) \rightarrow \Omega_{n}^{\text{Spin}}(Bf)$ induced by sending a cycle $(M, c)$ for $\Omega_{n}^{\text{Spin}}(BG)$ to the cycle $(M, \emptyset, c, \emptyset \rightarrow BH)$. The diagram

$$\begin{array}{ccc}
\text{Pos}_{n}^{\text{Spin}}(BG) & \longrightarrow & \Omega_{n}^{\text{Spin}}(BG) \\
\downarrow & & \downarrow \\
\text{Pos}_{n}^{\text{Spin}}(Bf) & \longrightarrow & \Omega_{n}^{\text{Spin}}(Bf)
\end{array}$$

is commutative.

**Proposition 5.2** Let $G$ be a discrete group and let $f : 1 \rightarrow G$ be the trivial homomorphism from the trivial group into $G$. Then the diagram

$$\begin{array}{ccc}
\text{Pos}_{n}^{\text{Spin}}(BG) & \longrightarrow & \text{Pos}_{n}^{\text{Spin}}(Bf) \\
\rho_{(2)}^G & \longrightarrow & \rho_{(2)}^G
\end{array}$$

is commutative.

**Proof** Let $(M, c, g)$ be a cycle for $\text{Pos}_{n}^{\text{Spin}}(BG)$. We use the notation from the previous section. $DM$ is then just the disjoint union $M \sqcup -M$. Since $B1$ can be taken to be a point, we have $BG \cup_{B1} BG = BG \vee BG$, the wedge sum. Analogously to (5), the map

$$BG \vee BG \rightarrow B(G \ast G) \rightarrow BG$$

induced by mapping the first component of $G \ast G$ via the identity and the second component trivially to $G$ is the identity on the first copy of $BG$ and crushes the second copy to the base point $B1$. Thus the map $(Dc)' : DM = M \sqcup -M \rightarrow BG$ restricted to $M$ is $c$ and restricted to $-M$ is the map to a point. Hence, the cover of $DM$ associated to $(Dc)'$ is the disjoint union of the $G$-cover of $M$ associated to $c$ and the trivial $G$-cover of $-M$. The $L^2$-$\eta$-invariant associated to this cover is thus the difference of the $L^2$-$\eta$-invariant associated to the cover classified by $c$ and the classical $\eta$-invariant of $M$. By Proposition 4.6, this is the same as $\rho_{(2)}^f([(M, \emptyset, c, \emptyset \rightarrow BG, g)]$. On the other hand, by (6), the latter difference is simply the definition of $\rho_{(2)}^G([(M, c, g)]$. The claim follows. \hfill $\square$
As above, let \( G \) and \( H \) be a pair of discrete groups and let \( f : H \to G \) be a group homomorphism. Furthermore, suppose there exist group homomorphisms \( s : H \to \Lambda \) and \( t : G \to \Gamma \) making the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{f} & \Lambda \\
\downarrow{f} & & \downarrow{\phi} \\
G & \xrightarrow{t} & \Gamma 
\end{array}
\] (7)

commutative. The pair \((t, s)\) then gives rise to a map of pairs \( B(t, s) : (BG, BH) \to (B\Gamma, B\Lambda) \) and the map

\[
B(t, s)_* : \text{PosSpin}^n(Bf) \to \text{PosSpin}^n(B\phi)
\]
is obtained by postcomposing the classifying maps for a cycle of \( \text{PosSpin}^n(Bf) \) with \( B(t, s) \). Moreover, the commutativity of the diagram implies that \( t \) induces a homomorphism \( p : \frac{G}{\text{Im}(f)G} \to \frac{\Gamma}{\text{Im}(\Lambda)\Gamma} \).

**Lemma 5.3** Let \( f : H \to G \) be an homomorphism of discrete finitely presented groups. Then:

- we can realize \( BH \) and \( BG \) so that the induced maps \( Bf \) is a continuous inclusions of topological space;
- there exists a model for the classifying space of \( G *_H G \) which contains as a subspace the push-out \( BG \sqcup_{BH} BG \).

**Proof** Let \( \langle S_H, R_H \rangle \) and \( \langle S_G, R_G \rangle \) be two finite representations respectively of \( H \) and \( G \) such that \( S_G \) contains \( f(S_H) \). Construct \( BH \) in the following standard way. Start with one 0-cell and consider a bouquet of circles, with one circle for each generator in \( S_H \). Attach suitably a 2-cell for each relation in \( R_H \), then attach all the \( n \)-cells for \( n \geq 3 \) necessary to make all the higher homotopy groups trivial.

To construct \( BG \), let us start with the CW-complex \( BH \) as above. Attach a circle to the 0-cell for each generator in \( S_G \setminus f(S_H) \) and a 2-cell for each relation in \( R_G \) and to each circle in \( BH \) corresponding to an element in \( S_H \cap \ker f \). Finally, attach all the \( n \)-cells for \( n \geq 3 \) necessary to make all the higher homotopy groups trivial. This proves the first statement.

For the second statement, consider the pushout \( BG \sqcup_{BH} BG \). By Van Kampen’s Theorem, its fundamental group is \( G *_H G \). If its higher homotopy groups are non-trivial, attach all the necessary \( n \)-cells for \( n \geq 3 \) in order to obtain a \( K(G *_H G, 1) \) space, which we denote by \( BG \sqcup_{BH} BG \). This concludes the proof of the lemma. \( \square \)

**Remark 5.4** Observe that if we have a commutative square as (7), then the obviously defined “double” map

\[
DB(t, s) : BG \sqcup_{BH} BG \to B\Gamma \sqcup_{B\Lambda} B\Gamma
\] (8)
is well-defined and, by the construction of \( B\Gamma \sqcup_{B\Lambda} B\Gamma \), naturally extends to a map

\[
DB(t, s) : BG \sqcup_{BH} BG \to B\Gamma \sqcup_{B\Lambda} B\Gamma
\] (9)

which induces a group homomorphism \( t *_s t : G *_H G \to \Gamma *_{\Lambda} \Gamma \).
Finally, for any model of $B(t \ast s, t): B(G \ast H G) \rightarrow B(\Gamma \ast \Lambda \Gamma)$, we have a commutative diagram

$$
\begin{array}{ccc}
BG \sqcup_{BH} BG & \longrightarrow & B(G \ast_H G) \\
\downarrow \text{DB}(t, s) & & \downarrow B(t \ast s, t) \\
B \Gamma \sqcup_{B \Lambda} B \Gamma & \longrightarrow & B(\Gamma \ast \Lambda \Gamma)
\end{array}
$$

where the horizontal arrows are homotopy equivalences.

**Proposition 5.5** Let $G$ and $H$ be as above and suppose that the homomorphism $p: \frac{G}{\text{Im}(H)^G} \rightarrow \frac{\Gamma}{\text{Im}(\Lambda)^\Gamma}$ is injective. Then the diagram

$$
\begin{array}{ cc }
\text{Pos}_{\text{Spin}}(Bf) & \xrightarrow{B(t, s)_*} & \text{Pos}_{\text{Spin}}(B\varphi) \\
\rho_f^* & \downarrow \rho_{triv}^{(2)} & \rho_{triv} \downarrow \\
\mathbb{R} & \xrightarrow{\mathbb{R}} & \mathbb{R}
\end{array}
$$

is commutative.

**Proof** Denote by $\pi_f^G$ and $\pi_{triv}^\Gamma$ the canonical projection maps $G \rightarrow G/\text{Im}(H)^G$ and $\Gamma \rightarrow \Gamma/\text{Im}(\Lambda)^\Gamma$, respectively. Let $\pi_{triv}^G: G \rightarrow G/\text{Im}(H)^G$ and $\pi_{triv}^\Gamma: \Gamma \rightarrow \Gamma/\text{Im}(\Lambda)^\Gamma$ be the trivial group homomorphisms. Consider the diagram

$$
\begin{array}{ cc }
BG \sqcup_{BH} BG & \longrightarrow & B(G \ast_H G) \\
\downarrow \text{DB}(t, s) & & \downarrow B(t \ast s, t) \\
B \Gamma \sqcup_{B \Lambda} B \Gamma & \longrightarrow & B(\Gamma \ast \Lambda \Gamma)
\end{array}
$$

where the maps $Dc$ and $D(B(t, s) \circ c)$ denote the “doubles” of $c$ and $B(t, s) \circ c$ and where the maps of the classifying spaces are induced by the maps of the corresponding groups. Observe that by construction we have the equality $\text{DB}(t, s) \circ Dc = \text{DB}(t, s)$ and then the commutativity of the left-hand side triangle. The middle square is commutative by Remark 5.4. The commutativity of the last square is given by functoriality.

Let $[(M, N, g, c)]$ be an element of $\text{Pos}_{\text{Spin}}^n(Bf)$. We need to show the equality $\rho_{triv}^{(2)}(M, N, g, c) = \rho_{triv}^{(2)}(M, N, g, B(t, s) \circ c)$. From Proposition 4.6, we know that $\rho_{triv}^{(2)}(M, N, g, c)$ is the $L^2-\eta$-invariant of the cover of $DM$ associated to $B(\pi_f^G \ast \pi_{triv}^G) \circ Dc$ and that $\rho_{triv}^{(2)}(M, N, g, B(t, s) \circ c)$ is the $L^2-\eta$-invariant of the cover of $DM$ associated to $B(\pi_f^G \ast \pi_{triv}^G) \circ D(B(t, s) \circ c)$. The (homotopy) commutativity of the above diagram implies that the cover of $DM$ associated to $B(\pi_f^G \ast \pi_{triv}^G) \circ D(B(t, s) \circ c)$ is the same as the cover associated to $Bp \circ B(\pi_f^G \ast \pi_{triv}^G) \circ Dc$. Since $p$ is injective, the latter cover is a disjoint union of copies of the cover of $DM$ associated to the map $B(\pi_f^G \ast \pi_{triv}^G) \circ Dc$. The claim then follows from the explicit formula (4) for the $L^2-\eta$-invariant.

In the rest of this section, $M$ will denote a compact odd-dimensional spin manifold with boundary $N$. Denote by $\mathcal{R}_{sc}^{>0}(M, N)$ the space of positive scalar curvature metrics on $M$. 

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which are collared near the boundary. Denote by \( \iota_* \) the map \( \pi_1(N) \to \pi_1(M) \) induced by the inclusion \( \iota: N \to M \). Let \( c: M \to B\Gamma \) and \( c_N: N \to B\Lambda \) denote the classifying maps of the universal covers of \( M \) and \( N \), respectively. Consider \( B_{\iota_*}: B\Lambda \to B\Gamma \), which we will assume to be injective. Thanks to Lemma 5.3, the latter maps can be chosen to make the diagram

\[
\begin{array}{ccc}
N & \xrightarrow{c_N} & B\Lambda \\
\downarrow & & \downarrow B_{\iota_*} \\
M & \xrightarrow{c} & B\Gamma
\end{array}
\]

commutative. Thus, \( c \) can be seen as a map of pairs \((M, N) \to (B\Gamma, B\Lambda)\). There is a canonical map

\[
\text{cl}: \mathcal{R}^{\text{sc}>0}(M, N) \to \text{Pos}_{\dim M}^{\text{Spin}}(B_{\iota_*})
\]

sending a psc metric \( g \) to the cycle represented by \([ (M, N, c, g) ] \).

**Definition 5.6** Two metrics \( g_1, g_2 \in \mathcal{R}^{\text{sc}>0}(M, N) \) are said to be \((\pi_1(M), \pi_1(N))\)-bordant if \( \text{cl}(g_1) = \text{cl}(g_2) \). Moreover, \((\pi_1(M), \pi_1(N))\)-bordism defines an equivalence relation on \( \mathcal{R}^{\text{sc}>0}(M, N) \). We will denote the equivalence class by \( \text{Bord}^{\text{sc}>0}(M, N) \).

**Remark 5.7** Note that \( \text{cl}: \mathcal{R}^{\text{sc}>0}(M, N) \to \text{Pos}_{\dim M}^{\text{Spin}}(B_{\iota_*}) \) induces an inclusion of \( \text{Bord}^{\text{sc}>0}(M, N) \) into \( \text{Pos}_{\dim M}^{\text{Spin}}(B_{\iota_*}) \).

Denote by \( \text{Diff}^{\text{Spin}}(M) \) the group of spin structure preserving diffeomorphisms of the manifold \( M \). This is a group of diffeomorphisms \( \Psi: M \to M \) which restrict to a product diffeomorphism \( \Psi|_N \times \text{Id} \) near the boundary. This group acts on \( \mathcal{R}^{\text{sc}>0}(M, N) \) via pull-backs. Proposition 5.8 is an analogue of [17, Proposition 2.10] and in fact its proof can be almost reduced to the proof of [17, Proposition 2.10] by Piazza and Schick.

**Proposition 5.8** The composition

\[
\mathcal{R}^{\text{sc}>0}(M, N) \to \text{Bord}^{\text{sc}>0}(M, N) \leftrightarrow \text{Pos}_{\dim M}^{\text{Spin}}(B_{\iota_*}) \to \mathbb{R}
\]

is constant on the orbits of the action of \( \text{Diff}^{\text{Spin}}(M) \).

**Proof** We only sketch the proof. Let \( \Psi \in \text{Diff}^{\text{Spin}}(M) \). By “doubling” \( \Psi \), we obtain a spin structure preserving diffeomorphism of the double \( D M \) of \( M \). Let \( g \in \mathcal{R}^{\text{sc}>0}(M, N) \) and denote by \( Dg \) the metric on \( D M \) obtained by doubling \( g \).

By [17, Proposition 2.10] the spin Dirac operators on the universal cover \( \widetilde{D M} \) of \( D M \) constructed using the pullbacks of \( Dg \in \mathcal{R}^{\text{sc}>0}(M, N) \) and \( (D\Psi)^* (Dg) \) are unitarily equivalent. We thus have a unitary equivalence of the spin Dirac operators on the quotient of \( D M \) by the kernel of the homomorphism \( \Gamma \to \Gamma \) defined in the beginning of Sect. 4.

Hence, the \( L^2-\eta \)-invariant of the latter operators are equal. On the other hand, by Proposition 4.6, these \( L^2-\eta \)-invariants coincide with the relative \( L^2-\rho \)-invariants associated to the psc metrics \( g \) and \( \Psi^* g \). The claim follows. \( \square \)

**Corollary 5.9** The map \( \mathcal{R}^{\text{sc}>0}(M, N) \to \mathbb{R} \) of Proposition 5.8 descends to a map \( \pi_0(\mathcal{R}^{\text{sc}>0}(M, N)/\text{Diff}^{\text{Spin}}(M)) \to \mathbb{R} \).
6 Geometric application

We now discuss how our relative $L^2$-$\rho$-invariant can be used to make statements about the richness of the (moduli) space of positive scalar curvature metrics on manifolds with boundary which are collared near the boundary.

Lemma 6.1 Let $f : 1 \to \mathbb{Z}_m$ be the inclusion of the trivial group into $\mathbb{Z}_m$ for $m \in \mathbb{N}$. Let $n = 4k + 3 > 4$. Denote by $K^f$ the kernel of the map $\text{Pos}_n^{\text{Spin}}(Bf) \to \Omega_n^{\text{Spin}}(Bf)$. Then $\rho^{f(2)}$ restricted to $K^f$ is nontrivial. In particular, $K^f$ is infinite.

Proof Denote by $K$ the kernel of the map $\text{Pos}_n^{\text{Spin}}(B\mathbb{Z}_m) \to \Omega_n^{\text{Spin}}(B\mathbb{Z}_m)$. From [6, Lemma 2.3] and [17, Example 2.13] it is known that the map $\rho_{\mathbb{Z}_m}^{(2)}$ is nontrivial. The fact that $\Omega_n^{\text{Spin}}(B\mathbb{Z}_m)$ is finite, then implies that $\rho_{\mathbb{Z}_m}^{(2)}|_K$ is nontrivial (see the beginning of the proof of [17, Theorem 2.25]). The commutativity of the diagram

\[
\begin{array}{c}
\text{Pos}_n^{\text{Spin}}(B\mathbb{Z}_m) \\
\downarrow
\end{array} \quad \begin{array}{c}
\Omega_n^{\text{Spin}}(B\mathbb{Z}_m) \\
\downarrow
\end{array} 
\begin{array}{c}
\text{Pos}_n^{\text{Spin}}(Bf) \\
\downarrow
\end{array} \quad \begin{array}{c}
\Omega_n^{\text{Spin}}(Bf) \\
\downarrow
\end{array}
\]

implies that the image of $K$ under the map $\text{Pos}_n^{\text{Spin}}(B\mathbb{Z}_m) \to \text{Pos}_n^{\text{Spin}}(Bf)$ lies in $K^f$. The claim follows from this observation and Proposition 5.2. $\square$

The following proposition follows immediately from the proof of Lemma 6.1.

Proposition 6.2 We use the notation of Lemma 6.1. There exists an infinite subset of $K^f$ with the following properties:

- its elements have pairwise different relative $L^2$-$\rho$-invariants and
- are represented by a cycle $(M, N, c, c|_N, g)$ with $\partial M = N = \emptyset$.

Definition 6.3 Let $\varphi : \Lambda \to \Gamma$ be a group homomorphism. An element $\gamma \in \Gamma$ is called a relative torsion element of order $m$ of the homomorphism $\varphi$ if it is a torsion element of order $m$ and if the subgroup generated by $\gamma$ intersects $\text{Im}(\Lambda)^\Gamma$ only at the neutral element. If such an element exists, we say that $\varphi$ has relative torsion.

Proposition 6.4 Let $\varphi : \Lambda \to \Gamma$ be a group homomorphism. Suppose that $\varphi$ has relative torsion. Let $n = 4k + 3 > 4$. Denote by $K^\varphi$ the kernel of the map $\text{Pos}_n^{\text{Spin}}(B\varphi) \to \Omega_n^{\text{Spin}}(B\varphi)$. Then there exists an infinite subset of $K^\varphi$ with the following properties:

- its elements have pairwise different relative $L^2$-$\rho$-invariants and
- are represented by a cycle $(M, N, c, c|_N, g)$ with $\partial M = N = \emptyset$.

Proof Let $\gamma \in \Gamma$ be a relative torsion element of order $m$ of $\varphi$. Sending the generator of $\mathbb{Z}_m$ to $\gamma$ gives an injective homomorphism $t : \mathbb{Z}_m \to \Gamma$. Denote by $s : 1 \to \Lambda$ the trivial homomorphism of the trivial group into $\Lambda$. The pair $(t, s)$ gives rise to map $B(t, s)_n : \text{Pos}_n^{\text{Spin}}(Bf) \to \text{Pos}_n^{\text{Spin}}(B\varphi)$, where $f : 1 \to \mathbb{Z}_m$ is the trivial homomorphism.
Postcomposition of the map \( t \) with the canonical projection \( \Gamma \to \Gamma/\text{Im}(\Lambda) \Gamma \) gives rise to an injective map. Thus, by Proposition 5.5, the diagram

\[
\begin{array}{ccc}
\text{Pos}_n^{\text{Spin}}(Bf) & \xrightarrow{B(t,s)_*} & \text{Pos}_n^{\text{Spin}}(B\varphi) \\
\rho_f^{(2)} & \downarrow & \rho_\varphi^{(2)} \\
\mathbb{R} & \xrightarrow{\iota} & \mathbb{R}
\end{array}
\]

is commutative and the subset of \( K^\varphi \) of the claim can be obtained by pushing the subset of \( K^f \) provided by Proposition 6.2 to \( \text{Pos}_n^{\text{Spin}}(B\varphi) \) using \( B(t,s)_* \). The fact that these elements lie in \( K^\varphi \) follows from the naturality of the relative Stolz positive scalar curvature sequence.

\[ \square \]

As in the previous section, let \( M \) be a compact spin odd-dimensional manifold with boundary \( N \) and let \( \text{Diff}^{\text{Spin}}(M) \) denote the group of spin structure preserving diffeomorphisms of \( M \). Furthermore, denote by \( \iota_* \) the map \( \pi_1(N) \to \pi_1(M) \) induced by the inclusion and by \( c \) the map of pairs \( (M,N) \to (\text{B}\pi_1(M), \text{B}\pi_1(N)) \) classifying the universal covers of \( M \) and \( N \).

**Theorem 6.5** Suppose that \( n:=\dim M > 4 \) is of the form \( 4k + 3 \) and that \( \iota_* \) has relative torsion. Furthermore, suppose that \( R^{\text{sc}>0}(M, N) \neq \emptyset \). Then there are infinitely many orbits of the action of \( \text{Diff}^{\text{Spin}}(M) \) on \( R^{\text{sc}>0}(M, N) \) such that metrics belonging to different orbits are not \( (\pi_1(M), \pi_1(N))-\text{bordant} \).

**Proof** The statement of the theorem follows from Proposition 5.8 and the following claim. \( \square \)

**Claim A** There are infinitely many classes in \( R^{\text{sc}>0}_{\text{bord}}(M, N) \) with different relative \( L^2\)-\( \rho \)-invariants.

In the following, we will prove Claim A. Denote by \( K^{\iota_*} \) the kernel of

\[
\text{Pos}_n^{\text{Spin}}(B\iota_*) \to \Omega_n^{\text{Spin}}(B\iota_*)
\]

and let \( S \) be an infinite subset of \( K^{\iota_*} \) provided by Proposition 6.4. Furthermore, let \( g_0 \in R^{\text{sc}>0}(M, N) \) be some psc metric on \( M \). For any \( s = [(M', \emptyset, c', \emptyset \to B\pi_1(N), g')] \in S \) we have

\[
\rho_{\iota_*}((s + [(M, N, c, g_0)]) = \rho_{\iota_*}((s) + \rho_{\iota_*}([(M, N, c, g_0)]).
\]

Thus \( s + [(M, N, c, g_0)] \neq s' + [(M, N, c, g_0)] \) if \( s \neq s' \). Claim A then follows from

**Claim B** For each \( s = [(M', \emptyset, c', \emptyset \to B\pi_1(N), g')] \in S \) there exists \( g_s \in R^{\text{sc}>0}(M, N) \) such that \( s + [(M, N, c, g_s)] \in \text{Pos}_n^{\text{Spin}}(B\iota_*) \).

In the following, we will prove Claim B. Claim B is a relative version of [6, Lemma 3.1] (see also [17, Proposition 2.4]). Botvinnik and Gilkey cite the results of [15] in the proof of [6, Lemma 3.1]. Since \( s \) is represented by a closed manifold, we can use essentially the same arguments. We sketch the arguments for the convenience of the reader and refer the reader to [15] for further details.

Note that since \( s \in K^{\iota_*} \) there exists \( (V', E') \) with \( \partial V' = M' \) and \( E': V' \to B\Gamma \) a continuous map extending \( c' \). Thus \( [(M', \emptyset, c', \emptyset \to B\pi_1(N))] + [(M, N, c)] = [(M, N, c)] \in \Omega_n^{\text{Spin}}(B\iota_*) \). An explicit equivalence (bordism) between the two cycles \((M', \emptyset, c', \emptyset \to B\pi_1(N)) + (M, N, c)\) and \((M, N, c)\) is given by \( V := V' \cup M \times [0,1], E) \),

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where \( E : (V, N \times [0, 1]) \to (B\pi_1(M), B\pi_1(N)) \) is the map of pairs which restricts to \( E' \) on \( V' \) and to \( c \circ \pi_M \) on \( M \times [0, 1] \). Here, \( \pi_M : M \times [0, 1] \to M \) denotes the projection on \( M \). Consider the connected sum of \( V' \) and \( M \times [0, 1] \) obtained by removing a disk from \( V' \) away from \( M' \) and a disk on \( M \times [0, 1] \) away from \( \partial(M \times [0, 1]) \) and then attaching the rest along the boundary of the removed disks. We denote the resulting space again by \( V \). Note that, since any two maps on a disk are homotopic we can define a map, also denoted by \( E \), out of the new \( V \) to \( B\pi_1(M) \) which coincides with the previous map on the complement of the removed disks. To summarise, we have a connected spin bordism \( V \) between \( M \times [0, 1] \sqcup M' \) and \( M \times [1] \) with internal boundary \( N \times [0, 1] \) and a map \( E : (V, N \times [0, 1]) \to (B\pi_1(M), B\pi_1(N)) \). Thanks to [24, Lemma 5.2] (see also [21, Proposition 3.2]) and the fact that the surgeries involved in its proof can be done away from \( \partial V \), we can assume that

- \( E : V \to B\pi_1(M) \) induces an isomorphism at the level of fundamental groups;
- the inclusion \( M \times [1] \hookrightarrow V \) is 2-connected (i.e. it induces an epimorphism between \( \pi_2 \)-groups).

From [25] \(^5\) (see also [15, Theorem 1.1]), it follows that there exists a self-indexing Morse function on the bordism \( V \) (see [15] for the definition) which takes the value \(-1/2 \) on \( M \times [1] \) and the value \( n + 1 + 1/2 \) on \( M' \sqcup M \times [0] \), has no critical points of index less than 3 and has no critical points near \( \partial V \). Furthermore, we can assume that in a collar neighbourhood of \( N \times [0, 1] \) it is a linear interpolation between \(-1/2 \) and \( n + 1 + 1/2 \). Using \( -f \), we can decompose \( V \) into traces of surgeries on embedded spheres of codimension bigger than 2. Note that, since \( -f \) has no critical points near \( N \times [0, 1] \) the result of each surgery has boundary \( N \). The improved surgery theorem (see [10] and [26, Theorem 0.2]) then implies that \( V \) can be endowed with a psc metric \( G \), which is collared near \( (M \times [0] \sqcup M' \times [1]) \) and extends the metric \( g' \sqcup g_0 \). Moreover, since the surgeries on the level sets are taking place away from \( N \), following the construction in [26] gives rise to \( G \) which is collared near \( N \times [0, 1] \) and even restricts to \( g_0|_N + dr^2 \) on \( N \times [0, 1] \). Setting \( g_s := G|_{M \times [1]} \) then finishes the proof of Claim A and the theorem.

Theorem 6.5 has the following immediate

**Corollary 6.6** Suppose that \( M \) satisfies the conditions of Theorem 6.5 then

- \( \#\text{Bord}^{sc>0}(M, N) = \infty \)
- \( \#\pi_0(\mathcal{R}^{sc>0}(M, N)) = \infty \)
- \( \#\pi_0(\mathcal{R}^{sc>0}(M, N)/\text{Diff}^{spin}(M)) = \infty \)

Let \( g \in \mathcal{R}^{sc>0}(M, N) \). Set \( h := g|_N \) and denote by \( \mathcal{R}^{sc>0}(M, N)_h \) the subspace of \( \mathcal{R}^{sc>0}(M, N) \) consisting of metrics which restrict to \( h \) on the boundary. Recall that we have a fibration

\[
\mathcal{R}^{sc>0}(M, N)_h \xrightarrow{j} \mathcal{R}^{sc>0}(M, N) \xrightarrow{\partial} \mathcal{R}^{sc>0}(N) \tag{11}
\]

which induces the following sequence of maps

\[
\pi_0(\mathcal{R}^{sc>0}(M, N)_h) \xrightarrow{j_*} \pi_0(\mathcal{R}^{sc>0}(M, N)) \xrightarrow{\partial_*} \pi_0(\mathcal{R}^{sc>0}(N)) \tag{12}
\]

Then from the proof of Theorem 6.5 we deduce the following.

\(^5\) Bordisms of manifolds with boundary such as \( V \) are dealt with in [25]. There, they are called cobordisms relative to the boundary.
Corollary 6.7 Suppose that $M$ satisfies the conditions of Theorem 6.5 and that $h$ is in the image of $\partial$. Then

- $\pi_0(\mathcal{R}^{sc>0}(M, N)_h)$ has infinitely many connected components and sits injectively into $\pi_0(\mathcal{R}^{sc>0}(M, N))$.
- Let $\text{Diff}^{\text{Spin}}_\partial(M)$ the subgroup of $\text{Diff}^{\text{Spin}}(M)$ consisting of diffeomorphisms which restrict to the identity map of $N$, then the quotient of

$$\mathcal{R}^{sc>0}(M, N)_h/\text{Diff}^{\text{Spin}}_\partial(M)$$

also has infinitely many connected components.

- Let $\mathcal{R}_\partial := \text{Im}(\partial_\ast)/\text{Diff}^{\text{Spin}}(M)$, then

$$\pi_0(\mathcal{R}^{sc>0}(M, N)/\text{Diff}^{\text{Spin}}(M)) = \bigsqcup_{[h] \in \mathcal{R}_\partial} \pi_0(\mathcal{R}^{sc>0}(M, N)_h)/\text{Diff}^{\text{Spin}}_\partial(M)$$

Remark 6.8 Let $N$ be a closed $4k+3$-dimensional spin manifold which admits a psc metric and whose fundamental group has torsion. Let $M$ be a manifold obtained from $N$ by removing a disk. It is known that $M$ will then admit a psc metric which is collared near its boundary. Theorem 6.5 and its corollary can then be applied to $M$.

Remark 6.9 Bär and Hanke have recently shown in [3] that the space of psc metrics on a manifold with boundary which are collared near the boundary is weakly homotopy equivalent to the space of psc metrics for which the mean curvature of the boundary is nonnegative and which restrict to a psc metric on the boundary. Corollary 6.6 can then be used to gain information about the path components of the space of psc metrics with the latter boundary condition.

Remark 6.10 In [11] and [4] the higher $\rho$-invariants associated to psc metrics on closed manifolds have been related to the classical APS and $L^2$-$\rho$-invariants and this relationship has been used to prove certain rigidity results for these numerical invariants. Recently, $K$-theoretic higher $\rho$-invariants associated to psc metrics on manifolds with boundary have been defined in [19] and [23] using different techniques. We plan to address the relationship between the latter $K$-theoretic invariants and the invariants defined in this paper in the future.

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