Abstract. Given an \( m \)-periodic link \( L \subset S^3 \), we show that the Khovanov spectrum \( \mathcal{X}_L \) constructed by Lipshitz and Sarkar admits a homology group action. We relate the Borel cohomology of \( \mathcal{X}_L \) to the equivariant Khovanov homology of \( L \) constructed by the second author. The action of Steenrod algebra on the cohomology of \( \mathcal{X}_L \) gives an extra structure of the periodic link.

1. Introduction

Khovanov homology [14] assigns to a diagram \( D \) of a link \( L \subset S^3 \) a bigraded chain complex \( CKh^*(D; \mathbb{Z}) \), whose homology groups, \( Kh^*(D; \mathbb{Z}) \), are a link invariant. The Khovanov homotopy type was constructed by Lipschitz and Sarkar as a refinement of Khovanov homology.

Theorem 1.1. [17, Theorem 1.1] Let \( D \) be a diagram representing a link \( L \subset S^3 \). Then for any \( q \in \mathbb{Z} \) there exists a topological space \( \mathcal{X}_D^q \) such that the reduced singular cochain complex \( \tilde{C}^*(\mathcal{X}_D^q; \mathbb{Z}) \) is a copy of the Khovanov complex \( CKh^*(D; \mathbb{Z}) \). In particular \( \tilde{H}^i(\mathcal{X}_D^q; \mathbb{Z}) \) is equal to \( Kh^{i,q}(D; \mathbb{Z}) \). The stable homotopy type of \( \mathcal{X}_D^q \) is an invariant of the link \( L \).

Define \( \mathcal{X}_D = \bigvee_q \mathcal{X}_D^q \). We will often write \( \mathcal{X}_L \) instead of \( \mathcal{X}_D \) noting that \( \mathcal{X}_L \) is defined up to stable homotopy. The space \( \mathcal{X}_L \) is called a Khovanov homotopy type. There are various constructions of a Khovanov homotopy type, see [11,15,17], we refer to [20] for a survey.

A link \( L \) in \( S^3 \) is said to be \( m \)-periodic if it is invariant with respect to the action of \( \mathbb{Z}_m \) on \( \mathbb{R}^2 \) by rotation and \( D \) is disjoint from the center of the rotation. Any \( m \)-periodic link admits an \( m \)-periodic link diagram.

The first main result of this paper is the following.

Theorem 1.2. Let \( D \) be a \( m \)-periodic link diagram. Then, \( \mathcal{X}_D^q \) admits an action of \( \mathbb{Z}_m \). Moreover, the equivariant stable homotopy type of \( \mathcal{X}_D^q \) is an invariant of the associated \( m \)-periodic link.

As a consequence we show the following result.

Theorem 1.3. Let \( Sq^i : Kh^{*,*}(L; \mathbb{Z}_2) \to Kh^{*,*}(L; \mathbb{Z}_2) \) be the Steenrod operation on Khovanov homology as constructed in [19]. If \( L \) is \( p^n \)-periodic, \( p > 2 \), then \( Sq^i \) preserves the \( \mathbb{Z}_2[\mathbb{Z}_{p^n}] \) module structure of \( Kh^{*,*}(L; \mathbb{Z}_2) \). In particular, the Steenrod squares preserve the decomposition of \( Kh^{*,*}(L; \mathbb{Z}_2) \) into irreducible representations of \( \mathbb{Z}_2[\mathbb{Z}_{p^n}] \).

Theorem 1.3 gives more insight into the structure of the ordinary \( \mathbb{Z}_2 \) Khovanov homology of periodic knots, including torus knots. Another application is a refinement of the periodicity criterion of [2]. As the criterion is rather technical, we describe it in Section 7. Another application of Theorem 1.2 is an isomorphism between Borel cohomology of \( \mathcal{X}_D^q \) and the equivariant Khovanov homology constructed by the second author, see Corollary 5.6.

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The construction of the group action seems to be a straightforward generalization of the construction of Lawson, Lipshitz and Sarkar: a group action on a periodic link diagram induces a group action on a cube of resolutions, hence a group action on the Khovanov flow category. Then a Khovanov flow category admits an equivariant neat embedding and, consequently, an equivariant geometric realization. This simple sketch is in fact a rough idea behind our construction, some technical problems have to be overcome. In order to even spell the notion of an equivariant neat embedding we need to work in the category of equivariant manifolds and equivariant vector bundles. While the final statement (Theorem 1.2) has a rather simple form, the intermediate stages of the construction are significantly more complicated than those in the original article of Lipshitz and Sarkar. We point out that our construction is not an equivariant version of the construction of Cohen, Jones and Segal [5]. This is because we follow a simpler but more specific approach of Lawson, Lipshitz and Sarkar.

The structure of the paper is the following. In Section 2 we review the construction of Lawson, Lipschitz and Sarkar of the Khovanov homotopy type, we mostly follow [15]. In Section 3 we pass to study group actions on flow categories, we prove that a flow category admitting a group action admits an equivariant geometric realization, to this end some tools from equivariant differential and algebraic topology are used. All these ingredients allow us to give a proof of Theorem 1.2 which is given in Section 4. In Section 5 we review the construction of Politarczyk [24] of equivariant Khovanov homology and show the decomposition into irreducible representations that we mentioned in the introduction. In Section 6 we give a proof of Theorem 1.3. Section 7 strengthens the periodicity criterion of [2]. We show the case of the Hopf link in Section 8. Finally, the appendix contains some results on $\langle n \rangle$-manifolds and permutohedra, which would break the flow of the argument if kept in the body of the text.

There is an independent construction of the equivariant Khovanov homotopy type by [23], which uses different approach.

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where we use the notation \( \nu(C_{u,v}) = \nu(u,v) \).

A resolution configuration \( \mathcal{D} \) is a pair \( (Z(\mathcal{D}), A(\mathcal{D})) \) where \( Z(\mathcal{D}) \) is a set of pairwise disjoint embedded circles in \( S^2 \) and \( A(\mathcal{D}) \) is a totally-ordered set consisting of disjoint embedded arcs in \( S^2 \) such that the boundary of every arc lies in \( Z(\mathcal{D}) \). The index of the resolution configuration \( \mathcal{D} \), \( \text{ind}(\mathcal{D}) \), is the cardinality of \( A(\mathcal{D}) \). A labeled resolution configuration is a pair \( (\mathcal{D}, x) \) consisting of a resolution configuration \( \mathcal{D} \) and a map \( x \) assigning a label, \( x_+ \) or \( x_- \), to each element of \( Z(\mathcal{D}) \).

Given two resolution configurations \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \), we define the resolution configuration \( \mathcal{D}_1 \setminus \mathcal{D}_2 \) by declaring

\[
Z(\mathcal{D}_1 \setminus \mathcal{D}_2) = Z(\mathcal{D}_1) \setminus Z(\mathcal{D}_2), \quad A(\mathcal{D}_1 \setminus \mathcal{D}_2) = \{ A \in A(\mathcal{D}_1) : \forall z \in Z(\mathcal{D}_2) \partial A \cap z = \emptyset \}.
\]

For a resolution configuration \( \mathcal{D} \) we can choose a subset \( B \subset A(\mathcal{D}) \) and obtain a new resolution configuration \( s_B(\mathcal{D}) \), called surgery of \( \mathcal{D} \) along \( B \), by performing a surgery along the arcs in \( B \).

We can define a partial ordering on the set of labeled resolution configurations. Let \( (\mathcal{D}, x) \) and \( (\mathcal{E}, y) \) be two resolution configurations such that \( \text{ind}(\mathcal{D}) = \text{ind}(\mathcal{E}) \). We say that \( (\mathcal{D}, x) \prec (\mathcal{E}, y) \) if \( \mathcal{E} \) can be obtained from \( \mathcal{D} \) by surgery along a single arc in \( A \in A(\mathcal{D}) \) and one of the following conditions holds:

1. If \( \partial A \) is on a single circle \( Z \) which splits during the surgery into two circles \( Z_1 \) and \( Z_2 \), then
   * If \( x(Z) = x_+ \) then either \( y(Z_1) = x_+ \) and \( y(Z_2) = x_- \) or \( y(Z_1) = x_- \) and \( y(Z_2) = x_+ \),
   * If \( x(Z) = x_- \) then \( y(Z_1) = x_+ \) and \( y(Z_2) = x_- \).
2. If \( \partial A \) lies on two circles \( Z_1 \) and \( Z_2 \) which are merged during the surgery into a single circle \( Z \), then
   * If \( x(Z_1) = x(Z_2) = x_+ \), then \( y(Z) = x_+ \).
   * If \( x(Z_1) = x_+ \) and \( x(Z_2) = x_- \) or \( x(Z_1) = x_- \) and \( x(Z_2) = x_+ \), then \( y(Z) = x_- \).

For general labeled resolution configurations the partial order is defined as the transitive closure of the above relation.

A decorated resolution configuration is a triple \( (\mathcal{D}, x, y) \) where \( (\mathcal{D}, y) \) and \( (s_{A(\mathcal{D})}(\mathcal{D}), x) \) are labeled resolution configurations such that \( (\mathcal{D}, y) \prec (s_{A(\mathcal{D})}(\mathcal{D}), x) \).

Let \( D \) be an oriented link diagram with \( n = n_+ + n_- \) ordered crossings, where \( n_\pm \) denotes the number of positive, respectively negative crossings. For every \( v \in \{0,1\}^n \) we can define a resolution configuration \( D_D(v) = (Z(D_D(v)), A(D_D(v))) \) obtained by smoothing the \( i \)th coordinate of \( v \), as depicted in Figure 1. To avoid the cumbersome notation, we will drop the subscript \( D \) when it is clear from the context. Note that the arcs in \( A(D_D(v)) \) are in one to one correspondence with 0 coordinates in \( v \). Moreover, the vector space \( V(D_D(v)) \) generated by all possible labeled resolution configurations \( (D_D(v), x) \), can be identified with \( V \otimes [Z(D_D(v))] \).

Define the Khovanov complex of \( D \) in homological grading \( i = |v| - n_- \) as

\[
\text{CKh}^i(D; \mathbb{F}) = \bigoplus_{v \in \{0,1\}^n, |v| = i + n_-} V(D_D(v)).
\]
The space $\text{CKh}^i$ inherits the quantum grading from $V$. To be more precise, to an element $x = x_{\epsilon_1} \otimes \ldots \otimes x_{\epsilon_t} \in V(D(v))$, $\epsilon_i \in \{+, -\}$, we associate the grading $g(x) = \sum q(x_{\epsilon_i}) + n_+ - 2n_- + |v|$. The space $\text{CKh}^i$ splits as a direct sum of spaces $\text{CKh}^{i,q}$, where the second index denotes the quantum grading.

In order to make $\text{CKh}^*$ into a cochain complex, we need to choose a sign assignment $\nu$. This done, we define the differential of an element $(D(v), x)$ with homological grading $i$ as

$$\partial_i(D(v), x) = \sum_{|u| = |v| + 1} (-1)^{\nu(u,v)} (D(u), y).$$

The cohomology groups of the complex $\text{CKh}$, that is $\text{Kh}^i \text{Hom}_\text{C}(\text{CKh}, V)$, satisfy the following three conditions:

1. Ob$(\text{Cube}(n)) = \{0, 1\}^n$ with grading defined by $g^r(u) = |u| = \sum_i u_i$, where $u = (u_1, u_2, \ldots, u_n)$.

2. For two objects $u > v$ of the flow category with $g^r(u) - g^r(v) = d$ we define

$$\mathcal{M}_{\text{Cube}(n)}(u, v) = \Pi_{d-1} \subseteq \prod_{i: u_i > v_i} \mathbb{R},$$

where $\Pi_{d-1}$ is a $(d - 1)$-dimensional permutohedron as in Definition B.1.

3. Composition of morphisms

$$\mathcal{M}_{\text{Cube}(n)}(v, w) \times \mathcal{M}_{\text{Cube}(n)}(u, v) \to \mathcal{M}_{\text{Cube}(n)}(u, w)$$

is defined with the aid of identification from Lemma B.5. Namely let us choose a triple of objects $u > v > w$ such that $g^r(u) - g^r(v) = k$, $g^r(v) - g^r(w) = l$ and assume that

2.2. Flow categories. In this section we are using notion of an $\langle n \rangle$-manifold introduced in the Appendix (Section A). The necessary background on permutohedra is given in Section B.

Definition 2.2. A flow category is a topological category $\mathcal{C}$ such that the set of objects is discrete and is equipped with a grading function $\text{gr}_\mathcal{C}$: $\text{Ob}(\mathcal{C}) \to \mathbb{Z}$. Moreover, morphism spaces satisfy the following three conditions:

1. For any $x \in \text{Ob}(\mathcal{C})$, $\text{hom}_\mathcal{C}(x, x) = \{\text{id}\}$.

2. For any $x, y \in \text{Ob}(\mathcal{C})$ with $\text{gr}^\mathcal{C}(x) - \text{gr}^\mathcal{C}(y) = d$, $\text{hom}_\mathcal{C}(x, y)$ is a (possibly empty) $(d - 1)$-dimensional $(d - 1)$-manifold.

3. If $\text{gr}^\mathcal{C}(x) - \text{gr}^\mathcal{C}(y) = d$, then the composition maps induce diffeomorphisms of $(d - 2)$-manifolds

$$\bigcup_{z \in \text{Ob}(\mathcal{C}) \backslash \{x, y\}} \text{hom}_\mathcal{C}(z, y) \times \text{hom}_\mathcal{C}(x, z) \cong \partial_i \text{hom}_\mathcal{C}(x, y).$$

Moreover, for any $x, y \in \text{Ob}(\mathcal{C})$ we define the moduli space from $x$ to $y$ as

$$\mathcal{M}_\mathcal{C}(x, y) = \begin{cases} \text{hom}_\mathcal{C}(x, y), & \text{if } x \neq y, \\ \emptyset, & \text{otherwise.} \end{cases}$$

As we often go back and forth from one category to another, we will not drop the subindex $\mathcal{C}$.

If $k \in \mathbb{Z}$ we can define the $k$-th suspension of $\mathcal{C}$, $\Sigma^k(\mathcal{C})$, to be the flow category with the same objects and morphisms and associated grading function

$$\text{gr}^{\Sigma^k(\mathcal{C})}(x) = \text{gr}^\mathcal{C}(x) + k.$$
For each \(x,y\) and the morphism spaces are moduli spaces of non-parametrized gradient flow lines of \(\text{Morse-Smale function}\). Example 2.6. In [17, Definition 3.14] there is a method to assign a flow category \(\mathcal{C}_f\) to every Morse-Smale function \(f: M \to \mathbb{R}\), where \(M\) is a smooth compact manifold. Basically, objects of \(\mathcal{C}_f\) are critical points of \(f\), the grading of an object is the index of the associated critical point, and the morphism spaces are moduli spaces of non-parametrized gradient flow lines of \(f\).

Suppose \(C = [0,1]^n\), \(f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_1(x_n)\), where \(f_1(x) = 3x^2 - 2x^3\). Then the cube flow category is the flow category assigned to \(f\).

Definition 2.7 (see [15, Section 3.2]). A cubical flow category is a flow category \(\mathcal{C}\) equipped with a grading-preserving functor \(\mathfrak{f}: \Sigma^k \mathcal{C} \to \text{Cube}(n)\), for some \(k \in \mathbb{Z}\), \(n \in \mathbb{N}\), such that for any pair of objects \(x, y\) of \(\mathcal{C}\) the map \(\mathfrak{f}_{x,y}: \mathcal{M}_C(x,y) \to \mathcal{M}_{\text{Cube}(n)}(\mathfrak{f}(x), \mathfrak{f}(y))\) is a covering map.

To conclude this subsection we recall a definition of [17, Section 3.4.2].

Definition 2.8. Given a flow category \(\mathcal{C}\), we say that a subcategory \(\mathcal{C}'\) is downward closed (respectively upward closed) if for any \(x, y \in \text{Ob}(\mathcal{C})\) with \(\mathcal{M}_C(x,y) \neq 0\), \(x \in \text{Ob}(\mathcal{C}')\) implies that \(y \in \text{Ob}(\mathcal{C}')\) (respectively, \(y \in \text{Ob}(\mathcal{C}')\) implies that \(x \in \text{Ob}(\mathcal{C}')\)).

2.3. Neat embeddings. Recall that Lawson, Lipshitz and Sarkar described in [15, Section 3] a construction that turns a cubical flow category into a CW-complex. The construction is a simplification of the construction of Lipshitz and Sarkar in [17]. In Sections 2.3, 2.4 and 2.5 we give a brief review.

Let \((\mathcal{C}, \mathfrak{f})\) be a cubical flow category, and fix \(d_\bullet = (d_0, d_1, \ldots, d_{n-1}) \in \mathbb{N}^n\) and \(R > 0\). For any \(u > v\) in \(\text{Ob}(\text{Cube}(n))\) define

\[
E_{u,v} = \prod_{i=v}^{u-1} \left[ -R, R \right]_{[d_i]} \times \mathcal{M}_{\text{Cube}(n)}(u,v).
\]

For any triple of objects \(u > v > w\) there is a map \(E_{v,w} \times E_{u,v} \to E_{u,w}\) defined as the following composition

\[
E_{v,w} \times E_{u,v} \cong \prod_{i=w}^{v-1} \left[ -R, R \right]_{[d_i]} \times \mathcal{M}_{\text{Cube}(n)}(v,w) \times \prod_{j=v}^{u-1} \left[ -R, R \right]_{[d_j]} \times \mathcal{M}_{\text{Cube}(n)}(u,v)
\]

\[
\cong \prod_{i=w}^{v-1} \left[ -R, R \right]_{[d_i]} \times \mathcal{M}_{\text{Cube}(n)}(v,w) \times \mathcal{M}_{\text{Cube}(n)}(u,v)
\]

\[
\hookrightarrow \prod_{i=w}^{v-1} \left[ -R, R \right]_{[d_i]} \times \mathcal{M}_{\text{Cube}(n)}(u,w).
\]

A cubical neat embedding \(\iota\) of a cubical flow category \((\mathcal{C}, \mathfrak{f})\) relative to \(d_\bullet = (d_0, d_1, \ldots, d_{n-1}) \in \mathbb{N}^n\) is a collection of neat embeddings

\[
\iota_{x,y}: \mathcal{M}_C(x,y) \hookrightarrow \mathcal{E}_{\mathfrak{f}(x),\mathfrak{f}(y)}
\]

such that

(CNE-1) For each \(x, y \in \text{Ob}(\mathcal{C})\) the following diagram commutes
(CNE-2) For any \( u, v \in \text{Ob}(\text{Cube}(n)) \) the map
\[
\bigoplus_{f(x) = u, f(y) = v} \mathcal{M}_C(x, y) \xrightarrow{\ell_{x, y}} \mathcal{M}_C(x, y) \hookrightarrow E_{f(x), f(y)}
\]
is a neat embedding (see Definition A.4).

(CNE-3) For any triple \( x > y > z \in \text{Ob}(\mathcal{C}) \) the following diagram commutes
\[
\begin{array}{ccc}
\mathcal{M}_C(y, z) \times \mathcal{M}_C(x, y) & \longrightarrow & \mathcal{M}_C(x, z) \\
\downarrow & & \downarrow \\
E_{f(y), f(z)} \times E_{f(x), f(y)} & \longrightarrow & E_{f(x), f(z)}.
\end{array}
\]

2.4. **Framed cubical neat embeddings.** To perform the construction of Lawson, Lipshitz and Sarkar, we need to construct an extension of \( \iota \) to a framed cubical neat embedding \( \bar{\iota} \), i.e. a collection of embeddings
\[
\bar{\iota}_{x, y} : \prod_{i = [f(y)]}^{[f(x)]-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_C(x, y) \rightarrow E_{f(x), f(y)},
\]
for some \( \epsilon > 0 \), in such a way that the commutativity from (CNE-3) is preserved. In general \( \bar{\iota} \) can be constructed as follows:
\[
\bar{\iota}_{x, y} : \prod_{i = [f(y)]}^{[f(x)]-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_C(x, y) \rightarrow E_{f(x), f(y)} = \prod_{i = [f(y)]}^{[f(x)]-1} [-R, R]^{d_i} \times \mathcal{M}_{\text{Cube}(n)}(f(x), f(y))
\]
(2.9)
\[
(t, \gamma) \mapsto (t + \pi^R_{u,v, t_{x, y}}(\gamma), \pi^M_{u,v, t_{x, y}}(\gamma)),
\]
where
\[
\pi^R_{u,v} : \prod_{i = [v]}^{[u]-1} [-R, R]^{d_i} \times \mathcal{M}_{\text{Cube}(n)}(u, v) \rightarrow \prod_{i = [v]}^{[u]-1} [-R, R]^{d_i},
\]
(2.10)
\[
\pi^M_{u,v} : \prod_{i = [v]}^{[u]-1} [-R, R]^{d_i} \times \mathcal{M}_{\text{Cube}(n)}(u, v) \rightarrow \mathcal{M}_{\text{Cube}(n)}(u, v)
\]
(2.11)
are projections, with \( f(x) = u, f(y) = v \).

If \( \bar{\iota} \) is any framed cubical neat embedding of the cube flow category, then \( \bar{\iota} \) determines a sign assignment. Namely, for \( u, v \in \{0, 1\}^n \) such that \( gr(u) - gr(v) = 1 \), we set \( \nu(u, v) = 0 \) if \( t_{u,v}(\mathcal{M}(u, v)) \) is framed positively with respect to the standard framing of \([-R, R]^{d_{|v|}}\), and \( \nu(u, v) = 1 \) otherwise. In this case, we say that \( \bar{\iota} \) refines \( \nu \).

**Lemma 2.12.** Any sign assignment \( \nu \) determines a framed cubical neat embedding of the cube flow category which refines \( \nu \).

**Proof.** The lemma follow directly from [17 Proposition 4.12].
2.5. **Geometric realizations.** Let us fix a cubical flow category \((C, f)\), a neat embedding \(\iota\) of \(C\) relative to a tuple \(d_\bullet = (d_0, d_1, \ldots, d_{n-1})\) and fix \(\epsilon > 0\) in such a way that the map (2.9) is an embedding. We construct a CW-complex \(|C|\) in the following way:

(1) For any \(x \in \text{Ob}(C)\), if \(u = f(x)\), we define the cell associated to \(x\) as

\[
P(x) = \prod_{i=0}^{[u]-1} [-R, R]^{d_i} \times \prod_{i=[u]}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \tilde{M}_{\text{Cube}(n)}(u, 0),
\]

where \(\tilde{M}_{\text{Cube}(n)}(u, 0)\) is defined to be \(\{0\}\) if \(u = 0\) and \([0, 1] \times M_{\text{Cube}(n)}(u, 0)\) otherwise.

(2) The cells \(P(x)\) are glued together inductively. First we start with a disjoint union of cells \(P(y)\) for \(\{y: f(y) = 0 \in \{0, 1\}^n\}\). For arbitrary \(x \in \text{Ob}(C)\), the cell \(P(x)\) is glued to the union \(\bigcup_{y: f(x) > f(y)} P(y)\). The gluing map is described below.

(3) For any \(x, y \in \text{Ob}(C)\) with \(f(x) = u > v = f(y)\) the cubical embedding provides an embedding \(\theta_{y,x}: P(y) \times M_{C}(x, y) \to P(x)\) given by

\[
P(y) \times M_{C}(x, y) = \prod_{i=0}^{[u]-1} [-R, R]^{d_i} \times \prod_{i=[u]}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \tilde{M}_{\text{Cube}(n)}(v, 0) \times M_{C}(x, y)
\]

\[
\cong \prod_{i=[v]}^{[u]-1} [-R, R]^{d_i} \times \prod_{i=[u]}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \tilde{M}_{\text{Cube}(n)}(v, 0) \times \left(\prod_{i=[v]}^{[u]-1} [-\epsilon, \epsilon]^{d_i} \times M_{C}(x, y)\right)
\]

(4) The attaching map for \(P(x)\) sends \(P_y(x) \cong P(y) \times M_{C}(x, y)\) to \(P(y)\) via the projection onto the first factor. The complement of \(\bigcup_{y} P_y(x)\) in \(\partial P(x)\) is mapped to the base point.

**Remark 2.15.** It is proved in [15, Lemma 3.16] that the attaching map is well-defined. This boils down to showing that if \(x, y, z \in \text{Ob}(C)\) are such that \(f(x) > f(y) > f(z)\) then there exists a map \(\kappa_{x,y,z}\) that makes the following diagram commute.

\[
\begin{array}{ccc}
P_z(x) \cap P_y(x) & \xrightarrow{\kappa_{x,y,z}} & P_y(x) \\
P_z(x) & \xrightarrow{\partial P(y)} & P(y) \\
P(z) & \xrightarrow{\partial P(y)} & P_z(y)
\end{array}
\]

\[\text{1We remark that in many papers the cell was denoted by } C(x), \text{ but we think the letter } C \text{ is too overloaded in the present article.}\]
The cubical realization of \((C, f)\) is defined to be the formal desuspension
\[
\mathcal{X}(C) = \sum_{k=-d_0-d_1-\ldots-d_{n-1}}^k \|C\|.
\]

2.6. Chain complex associated with a cubical flow category. For completeness of the
exposition we recall how to compute the singular cohomology of the cubical realization. A
detailed account is given in [17, Section 3] and [15, Section 3.2].

Starting from a cubical flow category \((C, f)\), define a cochain complex \(C^*(C, f)\) in the following way:

* The group \(C^k(C)\) is freely generated over \(\mathbb{Z}\) by the objects of \(C\) whose grading is equal
to \(k\);
* If \(x \in \text{Ob}(C)\) has grading \(k\), then we define

\[
\partial(x) = \sum_{y \in \text{Ob}(C), \ gr(y) = k+1} n_{x,y} \langle y \rangle,
\]

where \(n_{x,y}\) is the signed count of points in \(M(x, y)\). In particular, if we choose a framed
cubical neat embedding which refines a sign assignment \(\nu\), then

\[
(2.18) \quad n_{x,y} = (-1)^{\nu(i(f(x), f(y)))} \# M(x, y).
\]

**Lemma 2.19.** \(C^*(C, f)\) is a cochain complex, that is, \(\partial^2 = 0\), and its associated cohomology is
equal to the cohomology of \(\|C\|\), the geometric realization of \((C, f)\).

2.7. Khovanov homotopy type. In this subsection we apply constructions described in previous
sections to a specific flow category, the Khovanov flow category, which is at the heart of
the Lipshitz-Sarkar construction. Let \(D\) be an oriented link diagram with \(n = n_+ + n_-\)
ordered crossings. The starting point of the construction is to assign to every decorated reso-
lution configuration \((D(v), x, y)\) the moduli space \(M_{Kh}(D(v), x, y)\), which is a disjoint union of
permutohedra \(\Pi_{m-1}\), with \(m = \text{ind}(D(v))\). If \(m = 1\), \(M_{Kh}(D(v), x, y)\) consists of a single point.
If \(m = 2\), the moduli space \(M_{Kh}(D(v), x, y)\) can be defined once we choose another piece of data
called the ladybug matching, for details refer to [17, Section 5.1]. For \(m > 2\) the moduli spaces
\(M_{Kh}(D(v), x, y)\) can be constructed inductively.

**Definition 2.20.** ([17]. The Khovanov flow category, \(C_{Kh}(D)\), is a cubical flow category such that:

* \(\text{Ob}(C_{Kh}(D))\) consists of all labeled resolution configurations \((D(v), x)\), where \(v \in \{0, 1\}^n\).
The grading of an object is equal to its homological grading \(i(D(v), x) = |v| - n_-\) (recall that
each object has an additional quantum grading, as explained in Section 2.1);
* The morphism space is defined in the following way

\[
\mathcal{M}_{Kh}((D(v), x), (D(u), y)) = \begin{cases} 
M_{Kh}(D(u) \setminus D(v), x', y'), & \text{if } (D(u), y) \prec (D(v), x), \\
\emptyset, & \text{otherwise},
\end{cases}
\]

where \(x'\) and \(y'\) are the restrictions of \(x\) and \(y\), respectively, to \(D(u) \setminus D(v)\) and \(s(D(u) \setminus D(v))\).
* The functor \(f: \Sigma^n C_{Kh}(D) \to \text{Cube}(n)\) maps a labeled resolution configuration \((D(v), x)\)
to \(v\).

**Remark 2.21.** It is worth to stress that while \(M_{Kh}\) denotes the moduli space associated with a
pair of configurations \((D(v), x, y)\), the morphism space for the Khovanov flow category \(C_{Kh}\) is
denoted by \(\mathcal{M}_{Kh}\).

In this setting, and after making some choices (such as a framing and a neat embedding of
\(C_{Kh}(D)\)), we obtain the geometric realization of Khovanov flow category, \(\|C_{Kh}(D)\|\). This CW-complex is called Khovanov space. By construction, the Khovanov homology of \(D\), as constructed
in Subsection 2.1, is canonically isomorphic with the reduced cohomology of \(\|C_{Kh}(D)\|\), up to
grading shift. Finally, the formal desuspension of the Khovanov space $\Sigma^\infty - \|C_{Kh}(D)\|$ is the Khovanov homotopy type $\chi_D$, constructed in [15,17].

3. Equivariant Flow Categories

In this section we adapt the construction from Section 2 to the equivariant setting. First, we will introduce some terminology from equivariant differential topology. General references include [8,22,26].

3.1. Terminology. Let $G$ be a finite group. An orthogonal representation of $G$ is a homomorphism $\rho: G \to O(V)$, where $O(V)$ denotes the group of orthogonal automorphisms of some inner product space $V$. In particular, $V$ is implicitly equipped with an inner product which is preserved by $G$.

Two representations $\rho_1$ and $\rho_2$ are equivalent if they differ by a base change, i.e. if there exists $A \in O(V)$ such that for any $g \in G$, $\rho_1(g) = A \cdot \rho_2(g) \cdot A^{-1}$. If it does not lead to confusion, we will refer to a representation $\rho: G \to O(V)$ simply as $V$. In particular, for a subgroup $H \subset G$, the notation $V|_H$ means the representation $\rho|_H: H \to O(V)$. For two representations $V,W$ we denote by $\text{hom}_G(V,W)$ the space of $G$-equivariant maps from $V$ to $W$.

If $W \subset V$ are two $G$-representations, then by $V-W$ we denote the orthogonal complement of $W$ in $V$. A representation is called irreducible if it does not contain any proper subrepresentation.

**Theorem 3.1** (Wedderburn decomposition). There are irreducible $G$-representations $W_1, W_2, \ldots, W_n$ such that, if $V$ is any $G$-representation, there exists an isomorphism of representations

\[ V \cong \bigoplus_{i=1}^{n} W_i^{\alpha_i} \quad \text{with } \alpha_i > 0 \text{ for } 1 \leq i \leq n. \]

Subrepresentations $W_i^{\alpha_i}$ are called isotypic components of $V$.

**Lemma 3.2** (Schur’s Lemma). Let $V$ and $W$ be two irreducible representations. If $V \cong W$, then $\text{hom}_G(V,W)$ is a division algebra (i.e. every nonzero element is invertible). On the other hand, if $V$ and $W$ are not equivalent, then $\text{hom}_G(V,W) = \{0\}$.

Representation ring $RO(G)$ is a ring whose elements are formal differences $V-W$ of orthogonal $G$-representations, where $V_1 - W_1 = V_2 - W_2$ in $RO(G)$ if $V_1 \oplus W_2$ is equivalent to $V_2 \oplus W_1$. Notice that if $W \subset V$, then $V-W$ is isomorphic in $RO(G)$ to the orthogonal complement of $W$ in $V$. Addition is induced by direct sums and multiplication is induced by tensor products over $\mathbb{R}$.

**Example 3.3.** In this paper we will be mainly interested in representations of the finite cyclic group of order $m > 1$

\[ \mathbb{Z}_m = \{ t \mid t^m = 1 \}. \]

The simplest representation is the trivial representation

\[ t \mapsto 1 \in O(\mathbb{R}) \]

denoted by $\mathbb{R}$. If $m$ is even there is also the sign representation

\[ t \mapsto -1 \in O(\mathbb{R}) \]

which we denote by $\mathbb{R}_-$. For any $1 \leq \alpha \leq m/2$ there is a representation $V_\alpha$

\[ t \mapsto \begin{pmatrix} \cos \left( \frac{2\pi \alpha}{m} \right) & -\sin \left( \frac{2\pi \alpha}{m} \right) \\ \sin \left( \frac{2\pi \alpha}{m} \right) & \cos \left( \frac{2\pi \alpha}{m} \right) \end{pmatrix} \in O(\mathbb{R}^2). \]

It is easy to check that $\mathbb{R}$, $\mathbb{R}_-$ and $V_\alpha$, for $1 \leq \alpha \leq m/2$, are irreducible. In fact, every irreducible representation of $\mathbb{Z}_m$ is equivalent to one of those representations.
**Example 3.4.** The regular representation of a finite group $G$ is
\[ \mathbb{R}[G] = \text{span}_{\mathbb{R}} \{ g \in G \}. \]
and the action is given by right multiplication by a group element. Notice that $\mathbb{R}[G]$ is an $\mathbb{R}$-algebra called the group algebra and every $G$-representation is a module over $\mathbb{R}[G]$ and vice versa.

For $G = \mathbb{Z}_m$ we have
\[ \mathbb{R}[G] = \begin{cases} \mathbb{R} \oplus \mathbb{R} \oplus \bigoplus_{\alpha=1}^{m/2} V_\alpha, & \text{if } m \text{ is even}, \\ \mathbb{R} \oplus \bigoplus_{\alpha=1}^{m/2} V_\alpha, & \text{if } m \text{ is odd}. \end{cases} \]

### 3.2. Group actions on manifolds.

We review now some basic results on group actions on manifolds. Our purpose is not only to gather results for further use, but the ideas and the language introduced in this section are used in the construction of equivariant flow categories later on. Several more technical results on group actions on manifolds are given in the Appendix. General references for group actions on manifolds include [8, 13, 26].

We say that $M$ is a $G$-manifold, if it is a manifold (possibly with boundary) equipped with a smooth action of a finite group $G$. Given a point $x \in M$, we denote by $G_x$ the isotropy group of $x$, i.e.
\[ G_x = \{ g \in G \mid g \cdot x = x \}. \]
The orbit type of $x$ is defined to be the conjugacy class of $G_x$ in $G$. Note that for any $g \in G$, $G_{gx} = gG_xg^{-1}$. Equivalently we can say that the orbit of $x$ is diffeomorphic to $G/G_x$. Observe also that $G_x$ acts on the tangent space $T_x M$. By abuse of notation we will denote by $T_x M$ the tangent representation of $G_x$. For any subgroup $H \subset G$ define
\[ M^H = \{ x \in M : \forall h \in H \; h \cdot x = x \} = \{ x \in M : H \subset G_x \}. \]
We say that $M$ is of dimension $V - W \in RO(G)$, if for any $x \in M$ there exists an isomorphism of $G_x$-representations $T_x M \oplus W|_{G_x} \cong V|_{G_x}$.

**Example 3.5.** Let $\rho_m$ denote a rotation of $\mathbb{R}^3$ (with coordinates $x, y, z$) around the z-axis by the angle $2\pi \alpha/m$. $S^2 = \mathbb{R}^3 \setminus \mathbb{R}^2$ is invariant under $\rho_m$, hence $\mathbb{Z}_m$ acts on $S^2$. It is easy to verify that if $1 \leq \alpha < m/2$, then $S^2$ is of dimension $V_\alpha$, and if $m/2 < \alpha \leq m$, then $S^2$ is of dimension $V_{m-\alpha}$.

Let $M$ be a compact $G$-manifold and let $p : E \to M$ be a vector bundle over $M$. We say that $E$ is a $G$-vector bundle if there exists an action of $G$ on $E$ by vector bundle morphisms such that $p$ commutes with the action of $G$ on $E$ and $M$. If $V$ is a $G$-representation, then a $V$-bundle is a $G$-vector bundle $p : E \to M$ such that for any $x \in M$ there exists an isomorphism of $G_x$ representations between $V$ and $p^{-1}(x)$. We denote by $\underline{V}_M$ the trivial $V$-bundle over $M$, i.e. $\underline{V}_M = V \times M$.

**Example 3.6.** Let $M$ be a $G$-manifold and let $V$ be a $G$-representation. It is easy to check that the tangent bundle $TM$ is a $V$-bundle if and only if $M$ is of dimension $V$.

**Example 3.7.** Let $G = \mathbb{Z}_m$, $H \subset G$ and consider the homogeneous space $G/H$ (i.e. the set of $H$-cosets). Any $G$-vector bundle over $G/H$ is determined by its fiber $V_eH$ over $eH$, the coset of the identity element, which is an $H$-representation. If $V$ is any $H$-representation, define
\[ G \times_H V = G \times V/\sim, \]
where the relation is given by $(g, v) \sim (g \cdot h, h^{-1} \cdot v)$, for $h \in H$. It is not hard to check that $G \times_H V$ is a $G$-vector bundle over $G/H$ with the projection map
\[ [g, v] \mapsto gH, \]
where $[g, v]$ denotes the equivalence class of $(g, v)$. 
A framing of a $V$-bundle is a choice of an isomorphism of $V$-bundles $\phi: E \to V_M$. A stable framing is a choice of an isomorphism of $V$-bundles $\phi: E \oplus W_M \to V_M \oplus W_M$ for some trivial bundle $W_M$. A framing of a $V$-bundle determines an orientation of the bundle.

**Definition 3.8.** With the previous notation, $M$ is said to be subordinate to $V$ if for each $x \in M$ there exists an invariant neighborhood $U_x$ of $x$, and an equivariant differentiable embedding of $U_x$ in $V^t \setminus \{0\}$ for some $t$. We write $G(V)$ for the category whose objects are $G$-manifolds subordinate to $V$ and whose maps are continuous equivariant maps.

It is always possible to choose a $G$-invariant metric $m$ on $M$ in such a way that $T_xM$ becomes an orthogonal representation of $G_x$ (the group action preserves the scalar product $m_x$ on $T_xM$).

In the non-equivariant setting the exponential map

$$\exp: T_xM \to M$$

maps diffeomorphically a neighborhood of $0 \in T_xM$ onto a neighborhood of $x \in M$. In the equivariant setting we have the following analogous result (see for instance [8, Theorem I.5.6]).

**Theorem 3.9** (Slice Theorem). The exponential map

$$\exp: G \times G_x T_xM \to M$$

yields an equivariant diffeomorphism

$$\exp: G \times G_x D_\epsilon(T_xM) \to U_x$$

onto an invariant neighborhood of the orbit $x \cdot G$, where $D_\epsilon(T_xM)$ denotes the disk of radius $\epsilon$ (with respect to $m_x$) centered at $0$ in $T_xM$.

**Corollary 3.10.** If $M$ is a compact $G$-manifold, then there exists a $G$-representation $V$ such that $M$ is subordinate to $V$.

**Proof.** Choose $x \in M$. The group algebra $\mathbb{R}[G_x]$ is a subalgebra of $\mathbb{R}[G]$. Moreover, $\mathbb{R}[G]$ is a free module of rank $|G/H|$ over $\mathbb{R}[G_x]$. Define a $G$-representation $W_x = \mathbb{R}[G] \otimes_{\mathbb{R}[G_x]} T_xM$. By Theorem 3.9, we can choose an invariant neighborhood $U_x$ of $x$ equivariantly diffeomorphic to $G \times G_x D_1(T_xM)$. It is easy to see that $U_x$ embeds in $W_x \setminus \{0\}$.

Compactness of $M$ implies that we can choose a finite family $\mathcal{F}$ of $G$-representations in such a way that every $W_x$ is isomorphic to a member of $\mathcal{F}$. Therefore we can take

$$V = \bigoplus_{U \in \mathcal{F}} U.$$ 

□

**Definition 3.11.** Let $H \subset G$ be a subgroup and let $V$ be an $H$-representation. A $G$-cell of type $(H,V)$, denoted by $E(H,V)$ is $G \times_H D_R(V)$, where $D_R(V)$ denotes the disk in $V$ of radius $R > 0$. Notice that if $V$ is a $G$-representation, then $E(H,V|H) \cong G/H \times V$. A $G$-cell complex is a topological space $X$ with a filtration

$$X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots$$

such that

* $X_0$ is a disjoint union of orbits,
* for any $n > 0$, $X_n = X_{n-1} \cup_I E(H_n, V_n)$, where

$$f: \partial E(H_n, V_n) \to X_{n-1}$$

is an equivariant map,
* $X = \text{colim}_n X_n$. 

□
3.3. Group actions on flow categories. We are now ready to introduce the definition of a group action on a flow category. The definition given below is quite involved. It might help the reader to keep in mind that the construction is modeled on the flow category associated to an equivariant Morse function.

**Definition 3.12.** Let $G$ be a finite group and let $\mathcal{C}$ be a flow category. We say that $\mathcal{C}$ is a $G$-equivariant flow category (as usual, we will omit $G$ when it is clear from the context) if it is equipped with the following data:

1. For any $g \in G$ there exists a grading preserving functor $F_g : \mathcal{C} \to \mathcal{C}$.

2. There is an equivariant grading function $gr_G : \text{Ob}(\mathcal{C}) \to \bigsqcup_{H \subseteq G} \text{RO}(H)$.

Moreover, these data must satisfy the following conditions:

**EFC-1** $F_e$ is the identity functor.

**EFC-2** For any $g_1, g_2 \in G$ we have $F_{g_1} \circ F_{g_2} = F_{g_1 \cdot g_2}$.

**EFC-3** $(F_{g})_{x,y} : M(x, y) \to M(F_{g}(x), F_{g}(y))$ is a diffeomorphism of $(\text{ind}(x) - \text{ind}(y) - 1)$-manifolds, which satisfies the following property

$$(F_{g})_{x,y} \mid_{M(z,y) \times M(z, x)} = (F_{g})_{z,y} \times (F_{g})_{z, x},$$

where $z$ is an object of $\mathcal{C}$ such that $\text{ind}(y) < \text{ind}(z) < \text{ind}(x)$.

**EFC-4** $gr_G(x) \in \text{RO}(G_x)$, where $G_x = \{g \in G : F_g(x) = x\}$.

**EFC-5** $\dim gr_G(x) = gr(x)$.

**EFC-6** If there exists $g \in G$ such that $F_g(x_1) = x_2$, for some $x_1, x_2 \in \text{Ob}(\mathcal{C})$, then $gr_G(x_2) = v_g(gr_G(x_1))$, where $v_g : \text{RO}(G_x) \to \text{RO}(G_{g \cdot x})$ is induced by a map $G_x \ni h \mapsto ghg^{-1} \in gG_x g^{-1} = G_{g \cdot x}$.

In particular, for any $g_1, g_2 \in G$, $v_{g_1} \circ v_{g_2} = v_{g_1 \cdot g_2}$.

**EFC-7** Let $x, y \in \text{Ob}(\mathcal{C})$ and define $G_{x,y} = \{g \in G : F_g(M_C(x, y)) \subseteq M_C(x, y)\}$. $M_C(x, y)$ is a compact $G_{x,y}$-manifold of dimension $gr_G(x) |_{G_{x,y}} - gr_G(y) |_{G_{x,y}} - \mathbb{R}$.

In the non-equivariant setting it is possible to define the suspension $\Sigma^k \mathcal{C}$ of a flow category $\mathcal{C}$ by shifting the grading function by $k$. In the equivariant setting we can define the suspension of a flow category $\mathcal{C}$ by any virtual representation $V - W \in \text{RO}(G)$. $\Sigma^V - W \mathcal{C}$ has the same objects and morphisms as $\mathcal{C}$ but different grading function

$$(gr_G)_{\Sigma^V - W \mathcal{C}}(x) = (gr_G)_{\mathcal{C}}(x) + (V - W)|_{G_x} \in \text{RO}(G_x).$$

**Example 3.13.** Let $f$ be the Morse-Smale function from Example 2.6. It is immediate to check that it is invariant under cyclic permutation of coordinates. Therefore, the cube flow category becomes a $\mathbb{Z}_n$-equivariant flow category with respect to this action.

**Definition 3.14.** Given two $G$-equivariant flow categories $\mathcal{C}_1$ and $\mathcal{C}_2$, a functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ is said to be a $G$-equivariant functor if

* $f$ commutes with group actions on $\mathcal{C}_1$ and $\mathcal{C}_2$,
* for any object $x$ in $\mathcal{C}_1$ there is a $G_x$-invariant map

$$(3.15) \quad f_{gr_G(x)} : gr_G(x) \to gr_G(f(x)),$$

such that for any $g \in G$, we have

$$v_g \circ f_{gr_G(x)} = f_{gr_G(F_g(x))} \circ v_g.$$
Definition 3.16. A $G$-equivariant functor $f: C_1 \to C_2$ is called a $G$-cover if for any $x, y \in \text{Ob}(C_1)$ the map $f_{x,y}: \mathcal{M}_{C_1}(x, y) \to \mathcal{M}_{C_2}(f(x), f(y))$ is a cover and for any $x$ the $G_x$-equivariant map $f_{g, x}(x)$ is an isomorphism of $G_x$-representations.

3.4. Equivariant cube flow category. Recall that objects of the cube flow category are elements of $\{0, 1\}^n$. If $\sigma \in S_n$ is a permutation such that $\sigma^m = id$, then $\sigma$ induces an action of $\mathbb{Z}_m$ on $\{0, 1\}^n$.

Proposition 3.17. Let $\sigma \in S_n$ satisfying $\sigma^m = id$. The cube flow category $\text{Cube}(n)$ can be equipped with the structure of a $\mathbb{Z}_m$-equivariant flow category such that the action on the set of objects is given by $\sigma$. Moreover, for any object $x$ we have $\text{gr}_G(x) = \mathbb{R}[G_x]_{\text{gr}(x)/|G_x|}$.

Proof. For $x \in \text{Ob}(\text{Cube}(n))$ and $1 \leq k \leq m$ we define $\mathcal{F}_\sigma(x) = \sigma(x)$. In order to define $\mathcal{F}_\sigma$ on morphisms, recall that we regard $\mathcal{M}_{\text{Cube}(n)}(x, y)$ as a subset of $\prod_i: x_i > y_i \in \mathbb{R} \subset \mathbb{R}^n$. Now, $\sigma$ yields a linear isomorphism

\[
\mathcal{M}_{\text{Cube}(n)}(x, y) \xrightarrow{\bar{\sigma}} \mathcal{M}_{\text{Cube}(n)}(\bar{\sigma}(x), \bar{\sigma}(y))
\]

\[
\prod_i: x_i > y_i \in \mathbb{R} \xrightarrow{\bar{\sigma}} \prod_i: \sigma(x_i) > \sigma(y_i) \in \mathbb{R}.
\]

Therefore, we can define $(\mathcal{F}_\sigma)_{x,y} = \bar{\sigma}|_{\mathcal{M}_{\text{Cube}(n)}(x, y)}$. Lemma B.25 implies that conditions [EFC-1], [EFC-2] and [EFC-3] are satisfied.

In order to define the grading function

\[
\text{gr}_G: \text{Ob}(\text{Cube}(n)) \to \bigcup_{H \subset \mathbb{Z}_m} RO(H)
\]

note first that if $\emptyset = (0, 0, \ldots, 0) \in \text{Ob}(\text{Cube}(n))$, then for any $x \in \text{Ob}(\text{Cube}(n))$, the space $\mathcal{M}_{\text{Cube}(n)}(x, \emptyset)$ is a $(\mathbb{R}[G_x]_{\text{gr}(x)/|G_x|} - \mathbb{R})$-dimensional manifold; see Lemma B.25. The only possible choice for $\text{gr}_G(x)$ is then $\text{gr}_G(x) = \mathbb{R}[G_x]_{\text{gr}(x)/|G_x|}$. Conditions [EFC-4], [EFC-5] and [EFC-7] are satisfied automatically. Condition [EFC-6] is satisfied, because we can identify $G_x$-representations $\text{gr}_G(x) \cong \mathbb{R}[G_x]_{\text{gr}(x)/|G_x|} \cong \prod_i: x_i > 0 \mathbb{R}$. If $g = \sigma^k \in G$ and $y = x \cdot g$, then the map

\[
\sigma^k: \prod_i: x_i > 0 \mathbb{R} \to \prod_i: y_i > 0 \mathbb{R}
\]

gives the required identification of a $G_x$-representation $\text{gr}_G(x)$ and a $(gG_xg^{-1})$-representation $\text{gr}_G(y)$. \hfill \Box

Notation 3.18. Given $\sigma \in S_n$ such that $\sigma^m = id$, we denote by $\text{Cube}_n(n)$ the $\mathbb{Z}_m$-equivariant cube flow category for which the action on objects is given by $\sigma$.

Definition 3.19. Let $\mathcal{C}$ be a $\mathbb{Z}_m$-equivariant flow category. We say that $\mathcal{C}$ is a $\mathbb{Z}_m$-equivariant cubical flow category if it is a cubical flow category and, for some $\mathbb{Z}_m$-module $V$ and some $\sigma \in S_n$ satisfying $\sigma^m = id$, the functor $f: \Sigma^\mathcal{C} \to \text{Cube}_n(n)$ is a $\mathbb{Z}_m$-equivariant cover.

3.5. Equivariant Khovanov category. Our goal is to construct a group action on the Khovanov flow category. Recall that $D$ is a diagram of an $m$-periodic link and $G = \mathbb{Z}_m$ acts on $\mathbb{R}^2$ by rotations, preserving the diagram $D$. The action of $G$ permutes crossings of $D$. Write $\sigma$ for the permutation corresponding to a generator of $G$. We have $\sigma^m = id$. The following proposition shows how to extend the action on the set of objects to the action on the Khovanov flow category.
Proposition 3.20. The action of $\mathbb{Z}_m$ on $D$ induces a group action on the Khovanov flow category $\mathcal{C}_{\text{Kh}}(D)$. The assignment $f: (D(v), x) \mapsto |v|$ can be extended to an equivariant cubical functor $f: \Sigma^\mathbb{Z}_m[\mathbb{R}]^n \to \mathcal{C}_{\text{Kh}}(D) \to \text{Cube}_\sigma(n)$; in particular $\mathcal{C}_{\text{Kh}}(D)$ is an equivariant cubical flow category.

Proof. The permutation $\sigma$ induces an action of $\mathbb{Z}_m$ on $\{0, 1\}^n$ (we will denote this action by $\sigma$ as well). In order to define the action of $\mathbb{Z}_m$ on the set of objects of the Khovanov flow category, we let $(D(v), x)$ for $v \in \{0, 1\}^n$, be a labeled resolution configuration. We define $F_\sigma(D(v), x) = (D(\sigma(v)), x \circ \sigma^{-1})$; see Figure 2. Clearly $F_\sigma$ induces an action of $\mathbb{Z}_m$ on the set of objects of $\mathcal{C}_{\text{Kh}}(D)$.

We need to describe the action on morphisms, that is, on the moduli spaces $M_{\text{Kh}}(D(v), x, y)$; see Section 2.7 for the definition of $M_{\text{Kh}}$ and $M_{\text{Ch}}$ and the relation between the two.

Recall that the action of $\mathbb{Z}_m$ on the cube flow category was linear when restricted to any moduli space $M_{\text{Cube}}(f(x), f(y))$. Therefore, the group action on morphisms in the cube category was completely determined by its restriction to the set of vertices of the respective permutohedra. We shall first take care of axioms (EFC-1)–(EFC-7) hold as well.

The construction of $F_\sigma$ is straightforward for index 1 decorated configurations. Namely, $M_{\text{Kh}}(D(v), x, y)$ is a single point by construction (see the construction of $M_{\text{Kh}}(D(v), x, y)$ in [17] Section 5). This means that if $x, y \in \text{Ob}(\mathcal{C}_{\text{Kh}})$ have $\text{ind}(y) = \text{ind}(x) - 1$ and $\mathcal{M}_{\text{Ch}}(x, y)$ is non-empty, the functor $f$ induces a diffeomorphism between the moduli spaces $M_{\text{Kh}}(x, y)$ and $M_{\text{Cube}}(f(x), f(y))$, because each of them consists of a single point. Therefore $F_\sigma$ on zero-dimensional moduli space is uniquely determined by the action of $F_\sigma$ on the cube flow category $\text{Cube}(n)$.

Likewise, once we have constructed the map $F_\sigma$ for moduli spaces corresponding to index $k$ decorated configurations and $k > 2$, then the extension for index $k + 1$ decorated configurations is straightforward. Namely, take the moduli space $M(x, y)$ with $x = (D(v), x), y = (D(u), y), x, y \in \text{Ob}(\mathcal{C}_{\text{Kh}})$. Assume it has dimension $k$. Consider the corresponding index $k + 1$ decorated configuration $(D(u) \setminus D(v), x, y)$. By the inductive assumption the map $F_\sigma$ is already defined on the boundary of each connected component of $M_{\text{Kh}}(x, y) = M_{\text{Kh}}(D(v) \setminus D(u), x, y)$ and takes it to the boundary of $M_{\text{Ch}}(F_\sigma x, F_\sigma y) = M_{\text{Kh}}(D(\sigma(v)) \setminus D(\sigma(u)), x \circ \sigma^{-1}, y \circ \sigma^{-1})$. Moreover, the following diagram is commutative

\[
\begin{array}{ccc}
\partial M_{\text{Kh}}(x, y) & \xrightarrow{F_\sigma} & \partial M_{\text{Kh}}(F_\sigma x, F_\sigma y) \\
\downarrow f_{x,y} & & \downarrow f_{F_\sigma x, F_\sigma y} \\
\partial M_{\text{Cube}}(f(x), f(y)) & \xrightarrow{F^\text{Cube}_\sigma} & \partial M_{\text{Cube}}(f(F_\sigma x), f(F_\sigma y))
\end{array}
\]
The extension of $\mathcal{F}_\sigma$ can be defined on $\mathcal{M}_{\text{Kh}}(D(v), x, y)$ as follows. Let
\[
\mathcal{M}_{\text{Kh}}(x, y) = C_1 \cup C_2 \cup \ldots \cup C_k
\]
\[
\mathcal{M}_{\text{Kh}}(\mathcal{F}_\sigma x, \mathcal{F}_\sigma y) = C'_1 \cup C'_2 \cup \ldots \cup C'_{k'}
\]
de note connected components. Without loss of generality we may and will assume that $\mathcal{F}_\sigma$ maps $\partial C_i$ onto $\partial C'_i$. For any $1 \leq i \leq k$ we define
\[
(3.21) \quad \mathcal{F}_\sigma|_{C_i} = (f_{\sigma(D(v), x, y)}|_{C'_i})^{-1} \circ \mathcal{F}_\sigma \circ f_{(D(v), x, y)}|_{C_i}.
\]
The axioms [EFC-1] and [EFC-2] are trivially satisfied and [EFC-3] is guaranteed by the fact that the construction is performed inductively.

It remains to construct $\mathcal{F}_\sigma$ for index 2 decorated configurations. Let $(D(v), x, y)$ be a decorated resolution configuration of index 2. Recall that $\mathcal{M}_{\text{Kh}}(D(v), x, y)$ consists of either a single interval or two copies of an interval depending on whether it is a ladybug configuration or not. If the configuration in question is not a ladybug configuration, $f$ can be extended over the whole $\mathcal{M}_{\text{Kh}}(D(v), x, y)$ and the extension of $\mathcal{F}_\sigma$ is given by (3.21).

If, on the other hand, we are dealing with a ladybug configuration, the action of $\mathbb{Z}^m$ preserves the ladybug matching by [17] Lemma 5.8. Therefore, we again obtain a well-defined extension of $f$ to the whole $\mathcal{M}_{\text{Kh}}(D(v), x, y)$ and the extension of $\mathcal{F}_\sigma$ is given again by (3.21). This completes the construction of the group action on the flow category $\mathcal{C}_{\text{Kh}}(D)$. Conditions [EFC-1]–[EFC-3] are trivially satisfied.

Next step is the definition of the grading. The grading is essentially inherited from the grading on the cube category up to a shift. Motivated by condition (3.15), for an element $y = (D(v), x) \in \text{Ob}(\mathcal{C}_{\text{Kh}}(D))$ we define
\[
(3.22) \quad gr_G(y) = gr_{G_{(v)}}(f(y))|_{G_y} - \mathbb{R}[G_y]^{n-\bar{m}} = \mathbb{R}[G_y]^{gr(y)|_{G_y}}|_{G_y} - \mathbb{R}[G_y]^{n-\bar{m}} = \mathbb{R}[G_y]^{gr(y)-\bar{m}}|_{G_y}.
\]
In the last equality in (3.22) we have used the fact that $\mathbb{R}[G]|_{H} = \mathbb{R}[H]^{[G]/|H|}$.

With this definition, the functor $f: \sum \mathbb{R}[\mathbb{Z}^m]^{n-\bar{m}} \mathcal{C}_{\text{Kh}}(D) \rightarrow \text{Cube}_{\sigma}(m)$ preserves the grading. Conditions [EFC-4] and [EFC-5] are immediately satisfied. Properties [EFC-6] and [EFC-7] are deduced from the analogous properties of the category $\text{Cube}_{\sigma}(n)$ and the fact that $f$ is equivariant and preserves the dimension of the moduli space.

Remark 3.23. We remark that shifting by $\mathbb{R} \mathbb{Z}^m|^{n-\bar{m}}$ is an overall shift corresponding to the grading shift by $\bar{m}$ in the classical setting.

3.6. Equivariant neat embedding. Let $G = \mathbb{Z}^m$ and let $(\mathcal{C}, f)$ be a $G$-equivariant cubical flow category. Fix a sequence $e_* = (e_0, e_1, \ldots, e_{n-1})$ of positive integers. For an orthogonal $G$-representation $V$ and any $u > v \in \text{Ob}(\text{Cube}(n))$ define
\[
E(V)_{u,v} = \prod_{i = |v|}^{u-1} D_R(V_{u,v})^{e_i} \times \mathcal{M}_{\text{Cube}(n)}(u, v)
\]
where $D_R(V)$ denotes the ball of radius $R$ in $V$ and $V_{u,v} = V|_{G_{u,v}}$. Recall that the symbol $V|_H$ denotes the restriction of the representation to the subgroup $H$, the underlying linear space is the same.

For any triple of objects $u > v > w$ there exists an equivariant map
\[
(E(V)_{v,w})|_{G_{u,v,w}} \times (E(V)_{u,v})|_{G_{u,v,w}} \rightarrow (E(V)_{u,w})|_{G_{u,w}}
\]
where $G_{u,v,w} = G_u \cap G_v \cap G_w$. 
An equivariant cubical neat embedding of a cubical flow category $(\mathcal{C}, \mathfrak{f})$ relative to $e_\bullet$ and relative to the representation $V$, is a collection of $G_{x,y}$-equivariant neat embeddings for $x, y \in \text{Ob}(\mathcal{C})$ (notice that $G_x \subset G_{f(x)}$ and $G_y \subset G_{f(y)}$, so $G_{x,y} \subset G_{(f(x),f(y))}$)

\[ \iota_{x,y}: \mathcal{M}_C(x, y) \to E(V)_{f(x),f(y)}, \]

such that analogues of conditions [CNE-1], [CNE-2] and [CNE-3] are satisfied. Namely:

(ECNE-1) For any $x, y \in \text{Ob}(\mathcal{C})$ the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{M}_C(x, y) & \xrightarrow{\iota_{x,y}} & E(V)_{f(x),f(y)} \\
\downarrow & & \downarrow \text{projection} \\
\mathcal{M}_\text{Cube(n)}(f(x), f(y)). & & \\
\end{array}
\]

(ECNE-2) For any $x, y \in \text{Ob}(\mathcal{C})$ and any $g \in G$ the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{M}_C(x, y) & \xrightarrow{\iota_{x,y}} & E(V)_{x,y} \\
(F_g)_{x,y} \downarrow & & \downarrow g \cdot (-) \\
\mathcal{M}_C(\sigma(x), \sigma(y)) & \xrightarrow{\iota_{\sigma(x),\sigma(y)}} & E(V)_{\sigma(x),\sigma(y)}. \\
\end{array}
\]

The right vertical arrow is labelled by $g \cdot (-)$, which should be read that the map is induced by the group action.

(ECNE-3) For any triple of objects $x, y, z \in \text{Ob}(\mathcal{C})$ such that $f(x) > f(y) > f(z)$ the following diagram of $G_{x,y,z}$-equivariant maps is commutative

\[
\begin{array}{ccc}
\mathcal{M}_C(y, z) \times \mathcal{M}_C(x, y) & \xrightarrow{\iota_{x,y}} & \mathcal{M}_C(x, z) \\
\downarrow & & \downarrow \\
E(V)_{f(y),f(z)} \times E(V)_{f(x),f(y)} & \xrightarrow{\iota_{x,y}} & E(V)_{f(x),f(z)}. \\
\end{array}
\]

**Proposition 3.24.** Any equivariant cubical flow category admits an equivariant cubical neat embedding.

**Proof.** Consider $x, y \in \text{Ob}(\mathcal{C})$. The space $\mathcal{M}_C(x, y)$ is, by (EFC-7) a compact $G_{x,y}$ manifold of dimension $\text{gr}_C(x)|G_{x,y} - \text{gr}_C(y)|G_{x,y} - \mathbb{R}$. In particular, by Mostow–Palais Theorem (see Theorem A.8) there exists a representation $W_{x,y}$ such that $\mathcal{M}_C(x, y)$ embeds in $W_{x,y}$ (we do not require $W_{x,y}$ to be irreducible). Define $V$ to be the direct sum of all $W_{x,y}$ over pairs $x, y \in \text{Ob}(\mathcal{C})$.

We want to construct embeddings $\iota_{x,y}: \mathcal{M}_C(x, y) \to E(V)_{f(x),f(y)}$. Recall we have $E(V)_{u,v} = \prod_{i=[u]}^{[v]-1} D_R(V_{u,v})^{e_i} \times \mathcal{M}_\text{Cube(n)}(u, v)$. The map $\iota_{x,y}$ will be a product $j_{x,y} \times \mathfrak{f}$, where

\[ j_{x,y}: \mathcal{M}_C(x, y) \to \prod_{i=[f(y)]}^{[f(x)]-1} D_R(V_{f(x),f(y)})^{e_i} \]

and $\mathfrak{f}: \mathcal{M}_C(x, y) \to \mathcal{M}_\text{Cube(n)}(f(x), f(y))$ is given by the definition of the cubical flow category (see Definition 2.7 above).

Our task is therefore to construct the map $j_{x,y}$ and we shall proceed by induction on $\delta = |f(x)| - |f(y)|$. For $\delta = 1$, the space $\mathcal{M}_C(x, y)$ is a finite set of points. The construction of $j_{x,y}$ in this case is obvious. Conditions (ECNE-1), (ECNE-2) are satisfied, while (ECNE-3) is empty.
Suppose the embedding has been constructed for all \( x, y \) with \( \delta < k \) and we aim to construct a map \( j_{x,y} \) for \( |f(x)| - |f(y)| = k \). By the induction assumption, the map \( j_{x,y} \) is defined already on the boundary of \( \mathcal{M}_C(x, y) \). We extend this map to a \( G \)-equivariant map on the whole of \( \mathcal{M}_C(x, y) \) by Lemma 5.7, maybe increasing the values of some of the \( e_i \). Condition (ECNE-1) for \( j_{x,y} \) is trivially satisfied and (ECNE-2) follows from the \( G \)-equivariance. Condition (ECNE-3) follows from the construction, because \( j_{x,z} \) on the interior of \( \mathcal{M}_C(x, z) \) is an extension of \( j_{x,z} \) on the boundary.

The next step in the construction of Lawson, Lipshitz and Sarkar is the construction of a framing for the cubical neat embedding. In the equivariant setting we construct the framing in Section 2.5. For each \( (x, y) \) with \( |C| \) given in Section 2.5 we define the associated \( G \)-cell

\[
(3.25) \quad \text{EP}(x) = \prod_{i=0}^{[u]-1} D_R(V)^{e_i} \times \prod_{i=[v]}^{n-1} D_\epsilon(V)^{e_i} \times \widetilde{M}_\text{Cube}(0, u, 0),
\]

where \( \widetilde{M}(u, 0) = [0, 1] \times M(u, 0) \) if \( u \neq 0 \) and \( \widetilde{M}(0, 0) = \{0\} \). The group action on the interval \([0, 1]\) is assumed to be trivial. Note that \( \text{EP}(x) \) is homeomorphic to the cell \( P(x) \) constructed in Section 2.5 (we need to set \( d_i = e_i \dim V \)), the point is that the present construction is equivariant.

For \( x, y \in \text{Ob}(\mathcal{C}) \) such that \( f(x) = u > v = f(y) \) we construct a map \( E\theta_{y,x} : \text{EP}(y) \times \mathcal{M}_C(x, y) \rightarrow \text{EP}(x) \) by an analogous formula as \( (2.14) \) in Section 2.5, namely

\[
(3.26) \quad \text{EP}(y) \times \mathcal{M}_C(x, y) = \prod_{i=0}^{[v]} D_R(V)^{e_i} \times \prod_{i=[u]}^{n-1} D_\epsilon(V)^{e_i} \times \text{EP}(x).
\]

In fact, with the choice of \( d_i = e_i \dim V \) and an identification \( \text{EP}(x) \cong P(x) \), \( E\theta \) is exactly the same map as \( \theta \). Again the key point is that \( E\theta_{y,x} \) is \( G_{x,y} \)-equivariant. Write \( \text{EP}_y(x) \subset \text{EP}(x) \) for the image of \( E\theta(y) \).

Analogously to the non-equivariant case, the complex \( ||C|| \) is constructed inductively by taking the cells \( \text{EP}(x) \) and the attaching map taking \( \text{EP}_y(x) \) to \( \text{EP}(y) \) via the projection \( \text{EP}_y(x) \cong \text{EP}(y) \times \mathcal{M}_C(x, y) \rightarrow \text{EP}(y) \). As in the non-equivariant case, the remaining part \( \partial \text{EP}(x) \setminus \bigcup_y \text{EP}_y(x) \) is mapped to the base point.
Remark 3.27. If \( x_1, x_2, \ldots, x_k \) is an orbit of \( x_1 \in \text{Ob}(C) \), then there exists an equivariant homeomorphism

\[
\text{EP}(x_1) \sqcup \text{EP}(x_2) \sqcup \ldots \sqcup \text{EP}(x_k) \cong G \times_{G_{x}} \left( \prod_{i=0}^{n-1} D_{R}(V)^{c_i} \times \prod_{i=|u|} D_{e}(V)^{c_i} \times gr_{G}(x_1) \right),
\]

i.e. a \( G \)-cell of type \((G_x, V^{c_1+\cdots+c_{n-1}} \oplus gr_{G}(x_1))\), compare Definition 3.11.

Remark 3.28. The map \( E\theta_{x,x} \) gives a well-defined attaching map, see item (4) in Section 2.5 and equation (2.16). This is because, as we mentioned above, \( E\theta \) is essentially the map \( \theta \) from Section 2.5. Another possibility is to observe that the map \( \kappa_{x,y,z} \) constructed in the proof of \([15\text{ Lemma 3.16}] \) is equivariant because of the axioms (EFC-1)–(EFC-3). We omit the details.

Definition 3.29. The **equivariant cubical realization of** \( C \) is defined to be the formal desuspension \( \Sigma^{-W-\nu_0+\cdots+\nu_{n-1}}||C|| \), where \( W \) denotes a representation of \( G \) such that \( f : \Sigma^W C \to \text{Cube}(n) \) is the cubical functor.

The following result is a direct consequence of the construction:

**Proposition 3.30.** Let \((\mathcal{C}, f : \Sigma^W C \to \text{Cube}(n))\) be a \( G \)-equivariant cubical flow category. Let \( \iota \) be an equivariant cubical neat embedding relative to \( \epsilon_\bullet = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}) \in \mathbb{N}^n \) and relative to an orthogonal \( G \)-representation \( V \). There exists a CW-complex \( ||C|| \) equipped with a cellular action of \( G \), such that every object \( x \in \text{Ob}(C) \) corresponds to a single cell of \( ||C|| \) of dimension \( gr_{G}(x) \). Moreover, the forgetful functor (i.e. the one which forgets the action of \( G \)) maps \( ||C|| \) to the stable homotopy type constructed by Lawson, Lipshitz and Sarkar in [17].

### 3.8. Fixed points of the geometric realization.

The purpose of this subsection is to study the fixed point sets (with respect to a subgroup \( H \)) of the group action on the geometric realization. The results are of interest on their own, but they are also used in the proof of the invariance of the equivariant Khovanov homotopy type under Reidemeister moves. Recall that \( X^H \) denotes the set of fixed points of \( H \), that is, \( X^H = \{ x \in X | x \cdot h = x, \forall h \in H \} \).

Let \( C \) be an equivariant cubical flow category. For any \( H \subset G \) define the **\( H \)-fixed subcategory** \( C^H \) in the following way:

- The objects of \( C^H \) are those objects of \( C \) that are fixed under the action of \( H \), that is, \( \text{Ob}(C^H) = \text{Ob}(C)^H \);
- The morphisms between objects are given by fixed point submanifolds, that is,

\[
\mathcal{M}_{C^H}(x, y) = \begin{cases} 
\mathcal{M}_C(x, y)^H & x \neq y \\
\{id\} & x = y; 
\end{cases}
\]

- The grading of \( x \in \text{Ob}(C)^H \) is \( \dim gr_{G}(x)^H \).

**Remark 3.31.** If \( H \) is a normal subgroup of \( G \) (in the paper we work with \( G \) cyclic, so this is always the case), then it is possible to endow \( C^H \) a structure of a \( G/H \)-category. We do not use this structure throughout this paper.

The following example yields the instance of an \( H \)-fixed subcategory, that is the most important in our approach.

**Example 3.32.** Consider \( \text{Cube}_{\sigma}(n) \) for \( \sigma \in S_n \) such that \( \sigma \) is a product of disjoint cycles of length \( m \) and has no fixed points. Without loss of generality, we can assume that

\[
\sigma = (1, 2, \ldots, m)(m + 1, m + 2, \ldots, 2m) \cdots (n - m + 1, n - m + 2, \ldots, n).
\]

Let \( \mathbb{Z}_k = H \subset G = \mathbb{Z}_m \), where \( k \) divides \( m \). \( \text{Cube}_{\sigma}(n)^H \) consists of objects and morphisms that are invariant under \( \sigma^{n/k} \). We will show that there exists a functor \( r_H : \text{Cube}_{\sigma}(n)^H \to \text{Cube}(n/k) \) that induces an isomorphism of categories.
First, if \((v_1, \ldots, v_n)\) is an object in \(\text{Ob}(\text{Cube}_\sigma(n)^H)\), then by definition it is an object in \(\text{Cube}_\sigma(n)\) fixed by the action of \(\mathbb{Z}_k\). This amounts to saying that, for \(l = 0, \ldots, n/m - 1\) and \(j = 1, \ldots, m/k\), we have

\[
v_{lm+j} = v_{lm+j+m/k} = v_{lm+j+2m/k} = v_{lm+j+(k-1)m/k}.
\]

The functor \(r_H\) on objects is defined as

\[
(3.33) \quad r_H: (v_1, \ldots, v_n) = (v_1, \ldots, v_{m/k}, v_{m+1}, \ldots, v_{m+m/k}, v_{2m+1}, \ldots, v_{(n/m-1)m+m/k}).
\]

We now define \(r_H\) on morphisms. Consider first \(0_1 = (0, \ldots, 0)\) and \(1_n = (1, \ldots, 1)\) in \(\text{Cube}_\sigma(n)\). They are clearly fixed under the action of any subgroup \(H \subset G\). The space \(\mathcal{M}_{\text{Cube}_\sigma(n)}(1_n, 0_n)\) is, by definition, the permutohedron \(\Pi_{n-1} \subset \mathbb{R}^n\). The set \(\mathcal{M}_{\text{Cube}_\sigma(n)}(1_n, 0_n)^H\) of fixed points under \(H\) is given by \(\Pi_{n-1} \cap L\), where \(L\) is a linear subspace of \(\mathbb{R}^n\) given by

\[
L = \bigcap_{i=0}^{n/m-1} \left\{ x_{lm+j} = x_{lm+j+m/k} = \cdots = x_{lm+j+(k-1)m/k} \right\}.
\]

Let \(s = (n/m)(m/k)(k-1)\) be the codimension of \(L\). By Theorem [3.21] there is an identification \(\psi\) of \(\Pi_{n-1} \cap L\) with \(\Pi_{n-1-s}\). Choose one such \(\psi\). The map \(\psi\) identifies \(\mathcal{M}_{\text{Cube}_\sigma(n)}(1_n, 0_n)^H\) with \(\mathcal{M}_{\text{Cube}(n/k)}(1_{n/k}, 0_{n/k})\).

Take now general \(u, v \in \text{Ob}(\text{Cube}_\sigma(n)^H)\) with \(u > v\) and consider the product

\[
\Pi_{u,v} = \mathcal{M}_{\text{Cube}_\sigma(n)}(1_n, u) \times \mathcal{M}_{\text{Cube}_\sigma(n)}(u, v) \times \mathcal{M}_{\text{Cube}_\sigma(n)}(v, 0_n).
\]

By the axioms of the cube category \(\Pi_{u,v}\) embeds as a codimension 2 facet in the moduli space \(\mathcal{M}_{\text{Cube}_\sigma(n)}(1_n, 0_n) = \Pi_{n-1}\). In fact, \(\Pi_{u,v}\) corresponds to a facet \(\Pi_{\mathfrak{p}}\) of \(\Pi_{n-1}\) for some partition \(\mathfrak{p}\) into three subsets \(\Pi_{1_n,u}, \Pi_{u,v}\) and \(\Pi_{v,0_n}\). The three sets are sets of indices at which the entries of the corresponding vectors are different. For example \(\Pi_{1_n,u}\) is the set of indices \(i\) such that \((1_n)_i \neq v_i\) (here \((1_n)_i\) is obviously 1), the sets \(\Pi_{u,v}\) and \(\Pi_{v,0_n}\) are defined analogously.

We define now the map \(r_H: \mathcal{M}_{\text{Cube}_\sigma(n)}(u, v)^H \to \mathcal{M}_{\text{Cube}(n/k)}(r_{HU}, r_{HV})\) as a composition:

\[
\mathcal{M}_{\text{Cube}_\sigma(n)}(u, v)^H \to \mathcal{M}_{\text{Cube}_\sigma(n)}(1_n, u)^H \times \mathcal{M}_{\text{Cube}_\sigma(n)}(u, v)^H \times \mathcal{M}_{\text{Cube}_\sigma(n)}(v, 0_n)^H \\
\psi \to \mathcal{M}_{\text{Cube}(n/k)}(1_{n/k}, r_{HU}) \times \mathcal{M}_{\text{Cube}(n/k)}(r_{HU}, r_{HV}) \times \mathcal{M}_{\text{Cube}(n/k)}(r_{HV}, 0_{n/k}) \\
\to \mathcal{M}_{\text{Cube}(n/k)}(r_{HU}, r_{HV}).
\]

The first map is an embedding to a fiber \(\{pt\} \times \mathcal{M}_{\text{Cube}_\sigma(n)}(u, v)^H \times \{pt\}\) for two chosen points in \(\mathcal{M}_{\text{Cube}_\sigma(n)}(1_n, u)^H\) and \(\mathcal{M}_{\text{Cube}_\sigma(n)}(v, 0_n)^H\), respectively, it is evident that \(r_H\) does not depend on the choice. The last map is the projection onto the second factor. The map \(\psi\) was defined above as a map from \(\Pi_{n-1} \cap L\) to \(\Pi_{n-1-s}\). However, the property of this map is that a facet \(\Pi_{\mathfrak{p}}\) is taken to the facet \(\Pi'_{\mathfrak{p}}\), where \(\Pi'_{\mathfrak{p}} = \Pi_{n-1-s}\) and \(\mathfrak{p}'\) is a reduction of \(\mathfrak{p}\) as in Theorem [3.21]. A straightforward calculation using \([3.33]\) and the precise construction of \(\mathfrak{p}'\) given in Theorem [3.21] reveals that \(\mathfrak{p}'\) is a partition into three subsets \(\Pi'_{1_{n/k},r_{HU}}, \Pi'_{r_{HU},r_{HV}}\) and \(\Pi'_{r_{HV},0_{n/k}}\), where \(\Pi'_{\mathfrak{p}}\) is again the subset of indices at which the corresponding vectors differ. This means that \(\Pi'_{\mathfrak{p}}\) is exactly \(\mathcal{M}_{\text{Cube}(n/k)}(1_{n/k}, r_{HU}) \times \mathcal{M}_{\text{Cube}(n/k)}(r_{HU}, r_{HV}) \times \mathcal{M}_{\text{Cube}(n/k)}(r_{HV}, 0_{n/k})\); we omit the details.

We briefly sketch that \(r_H\) behaves well with respect to the compositions. Suppose that \(u, v, w \in \text{Ob}(\text{Cube}_\sigma(n)^H)\) with \(u > w > v\), \(gr(u) - gr(w) = k\), \(gr(w) - gr(v) = l\). Write \(r_{HU}, r_{HW}, r_{HV}\) for the corresponding objects in \(\text{Cube}(n/k)\). We need to show that the following
diagram commutes.

\[
\begin{array}{ccc}
\mathcal{M}_{\text{Cube}_r(n)}(u,w) \times \mathcal{M}_{\text{Cube}_r(n)}(w,v) & \rightarrow & \mathcal{M}_{\text{Cube}_r(n)}(u,v) \\
\downarrow r_H & & \downarrow r_H \\
\mathcal{M}_{\text{Cube}(n/k)}(r_H u, r_H w) \times \mathcal{M}_{\text{Cube}(n/k)}(r_H w, r_H v) & \rightarrow & \mathcal{M}_{\text{Cube}(n/k)}(r_H u, r_H v).
\end{array}
\]

This commutativity is true if \( \psi \) takes

\[ A = \mathcal{M}_{\text{Cube}_r(n)}(1_n, u)^H \times \mathcal{M}_{\text{Cube}_r(n)}(u, w)^H \times \mathcal{M}_{\text{Cube}_r(n)}(w, v)^H \times \mathcal{M}_{\text{Cube}_r(n)}(v, 0_n)^H \]

to

\[ B = \mathcal{M}_{\text{Cube}(n/k)}(1_{n/k}, r_H u) \times \mathcal{M}_{\text{Cube}(n/k)}(r_H u, r_H w) \times \mathcal{M}_{\text{Cube}(n/k)}(r_H w, r_H v) \times \mathcal{M}_{\text{Cube}(n/k)}(r_H v, 0_{n/k}). \]

The first of the two spaces corresponds to a refinement of the partition \( p \) from the construction of \( r_H \) on morphisms. The second is a refinement of the partition \( p' \). The fact that \( \psi \) takes \( A \) to \( B \) follows from the fact that taking refinements commutes with reductions, see Lemma 3.12

We omit straightforward but tedious details.

Finally, the equivariant grading on \( \text{Cube}_r(n) \) described in Proposition 3.17 has the property that if \( x \in \text{Ob}(\text{Cube}(n))^H \), then \( gr_Q(x)^H \) is equal to the grading of \( r_H(x) \). This is a straightforward verification, we omit the details.

In the remaining part of this subsection we will use the notation \( Z_k = H \subset G = Z_m \), with \( k \) a divisor of \( m \).

**Lemma 3.35.** The pair \( (C^H, f)^H \), where \( f^H = r H \circ f|_{CH} \) and \( r \) is as in the previous example, is a cubical flow category.

**Proof.** In order to prove that \( C^H \) is a flow category, we need to verify the axioms (FC-1), (FC-2) and (FC-3). The axiom (FC-1) is obvious. The axiom (FC-2) follows from the axiom (EFC-7) and Proposition B.14. The axiom (FC-3) follows from the axiom (EFC-3). This shows that \( C^H \) is a flow category. It remains to prove that the functor \( f^H \) makes it a cubical flow category.

Since \( f \) commutes with the group action, it takes objects in \( C \) that are fixed under \( H \) to objects of \( \text{Cube}(n) \) that are fixed under \( H \). In particular \( f^H \) is well-defined on objects.

To show that it is well-defined on morphisms, observe that for any \( x, y \in \text{Ob}(C)^H \), the map

\[ f_{x,y} : \mathcal{M}_C(x, y)^H \rightarrow \mathcal{M}_{\text{Cube}(n)}(f(x), f(y))^H \]

is a diffeomorphism when restricted to any connected component of \( \mathcal{M}_C(x, y)^H \). In particular \( r_H \circ f_{x,y} \) is a covering map. This means that \( f^H \) makes \( C^H \) into a cubical flow category. \( \square \)

**Lemma 3.36.** Let \( C \) be a framed cube flow category and \( \iota \) a neat embedding of \( C \) relative to \( e_\bullet = (e_1, e_2, \ldots, e_{n-1}) \) and relative to a representation \( V \). Then, for any \( H \subset G \), \( \iota \) yields a neat embedding of \( C^H \), denoted by \( \iota^H \), relative to

\[ e^H_\bullet = (e_0 + e_1 + \cdots + e_{k-1}, e_k + e_{k+1} + \cdots + e_{2k-1}, \ldots, e_{n-k} + e_{n-k+1} + \cdots + e_{n-1}). \]

**Remark 3.37** (Remark B.3 continued). One can construct \( \iota^H \) in such a way that it is a \( G/H \)-equivariant neat embedding.

**Proof.** Following the notation introduced in Subsection 3.6 an equivariant neat embedding of \( C \) is given by a collection of maps \( \iota_{x,y} : \mathcal{M}_C(x, y) \rightarrow E(V)_{f(x), f(y)} \) satisfying axioms (ECNE-1), (ECNE-2) and (ECNE-3). An equivariant neat embedding

\[ \iota_{x,y} : \mathcal{M}_C(x, y) \rightarrow E(V)_{f(x), f(y)}, \]

where \( x, y \in \text{Ob}(C)^H \), yields an embedding

\[ \iota_{x,y}|_H : \mathcal{M}_C(x, y)^H \rightarrow E(V)^H_{f(x), f(y)}. \]
Observe that
\[ E(V)^H_{f(x),f(y)} = \prod_{i=|[f(x)]|}^{|[f(x)]-1} \left( D_R(V^H_{f(x),f(y)}) \right)^{e_i} \times M_{\text{Cube}(n)}(f(x), f(y))^H, \]
because \( H \subset G_{f(x),f(y)} \). Since \( |f(x)| = k \cdot |f^H(x)| \) and \( |f(y)| = k \cdot |f(y)|^H \), there exists an equivariant linear embedding
\[ \eta: \prod_{i=|[f(x)]|}^{|[f(x)]-1} D_R(V^H_{f(x),f(y)})^{e_i} \hookrightarrow \prod_{i=|[f(y)]|}^{|[f(y)]-1} D_R(V^H_{f^H(x),f^H(y)})^{e_{k,i}+e_{k,i+1}+\cdots+e_{k,(i+1)-1}}, \]
for some \( R' > R \). Using the map \( r_H \) from Example 3.32 we can construct a neat embedding
\[ \eta \times r_H: E(V)^H_{f(x),f(y)} \hookrightarrow E(V^H)^{f(x),f(y)}, \]
and therefore we can define
\[ \tilde{f}^H_{f(x),f(y)} = (\eta \times R) \circ (i_{x,y})^H. \]
Properties (CNE-1), (CNE-2) and (CNE-3) follow directly from axioms (ECNE-1), (ECNE-2) and (ECNE-3).

**Proposition 3.38.** Suppose \( ||C|| \) is an equivariant geometric realization of an equivariant cube flow category \( C \). Then the fixed point set \( ||C||^H \) is homeomorphic to a realization of the fixed point flow category \( C^H \).

**Proof.** We need to show essentially two facts: the equality of cells, and the equality of attaching maps. First, if \( x \in \text{Ob}(C)^H \), we can construct a cell \( P_H(x) \) using the construction of Section 2.5 taking \( C^H \) as the starting category. This corresponds to a cell used for constructing \( ||C^H|| \). Alternatively we can take \( EP(x)^H \) to be the set of \( H \)-fixed points of the cell \( EP(x) \) constructed in Section 3.7. We claim that \( P_H(x) \cong EP(x)^H \) once we have set \( d_i = e_i \dim V^H \).

To see this recall that by (3.25) we have
\[ EP(x)^H = \prod_{i=0}^{|[f(x)]-1} D_R(V^H_{x,x}^{e_i}) \times \prod_{i=|[f(x)]|}^{n-1} D_{e_i}(V^H_{x,x}^{e_i}) \times \tilde{M}_{\text{Cube}(n)}(f(x), 0)^H, \]
\[ P_H(x) = \prod_{i=0}^{|[f(x)]-1} D_R(V^H_{x,x}^{e_i}) \times \prod_{i=|[f(x)]|}^{n/k-1} D_{e_i}(V^H_{x,x}^{e_i}) \times \tilde{M}_{\text{Cube}(n/k)}(f(x), 0)^H, \]
where \( e_i = e_{k,i} + e_{k,i+1} + \cdots + e_{k,(i+1)-1} \). Discussion in Example 3.32 implies that \( EP(x)^H \cong P_H(x) \), for any \( x \in \text{Ob}(C^H) \).

In order to complete the proof of Proposition 3.38 we need to show that the attaching maps coincide. This holds, provided that \( \theta_H(y,x) = E\theta(y,x)^H \), where \( \theta_H(y,x) \) is the map \( \theta \) of Subsection 2.5 constructed for \( C^H \), and \( E\theta(y,x)^H \) is the restriction of \( E\theta \) to the set of fixed points. Choose \( x, y \in \text{Ob}(C^H) \). Going through the construction of \( \theta \) and \( E\theta \) (given in (2.14) and (2.26)) we see that the equality \( E\theta(y,x)^H = \theta_H \) follows from the commutativity of the diagram

\[ \begin{array}{ccc}
M_{\text{Cube}(n)}(f(y), 0)^H \times M_{\text{Cube}(n)}(f(x), f(y))^H & \xrightarrow{\phi^H} & M_{\text{Cube}(n)}(f(x), 0)^H \\
\downarrow r_H \times r_H & & \downarrow r_H \\
M_{\text{Cube}(n)}(f^H(y), 0) \times M_{\text{Cube}(n)}(f^H(x), f^H(y))^H & \xrightarrow{\phi^H} & M_{\text{Cube}(n)}(f^H(x), 0)^H,
\end{array} \]

where \( r_H \) is the map from Example 3.32 that is a map from the fixed point set of a permutohedron to a permutohedron of lower dimensions, as described in detail in Example 3.32.
Commutativity of the diagram follows from the construction of this map (see Proposition B.14 and Corollary B.23).

3.9. Equivariant chain complexes. In Section 2.6 we constructed a cochain complex $C^*(C, f)$, whose cohomology was equal to the cohomology of the geometric realization $|C|$. When trying to conduct this construction in the equivariant setting, that is, when trying to construct a group action on $C^*(C)$ we hit the following technical problem.

The differential of the chain complex (2.17) depends on the sign assignment $\nu$ on the cubical flow category, see (2.18). The symmetry group $\sigma$ acts on sign assignments via

$$\sigma(\nu)(x, y) = \nu(\sigma(x), \sigma(y)).$$

However, in general, we do not have $\sigma(\nu) = \nu$.

To remedy this, we recall that the sign assignments form a 1-chain in $[0, 1]^n$ with values in $\mathbb{F}_2$ (see Section 2.1). It is not necessarily trivial, but the difference of any two sign assignments satisfies a cocycle condition. Therefore, there exists a 0-cochain $c \in C^0([0, 1]^n; \mathbb{F}_2)$ such that $\sigma(\nu) - \nu = \partial c$. That is,

$$\nu(\sigma(f(x)), \sigma(f(y))) - \nu(f(x), f(y)) = c(f(x)) - c(f(y)). \tag{3.39}$$

**Lemma 3.40.** The map $F_\sigma : C^*(C) \to C^*(C)$ given by $x \mapsto (-1)^{c(f(x))}x$ commutes with the differential and therefore it generates the $\mathbb{Z}_m$ action on the chain complexes.

**Proof.** We need to check that the coefficient in $\partial F(y)$ at $F(x)$ is equal to the coefficient in $\partial y$ at $x$. The latter is equal to

$$(-1)^{c(f(x), f(y))}M(x, y), \tag{3.41}$$

compare to (2.18). We want to compute now the first one. Write $y' = \sigma(y), x' = \sigma(x)$. By (2.18) the coefficient in $\partial y'$ at $x'$ is equal to

$$(-1)^{c(f(x'), f(y'))}M(x', y') = (-1)^{c(f(x'), f(y'))}M(x, y), \tag{3.42}$$

Given the definition of $F_\sigma$, we have $F_\sigma(x) = (-1)^{c(f(x))}x'$ and $F_\sigma(y) = (-1)^{c(f(y))}y'$. Thus, in light of (3.42), the coefficient in $\partial F(y)$ at $F(x)$ is given by

$$(-1)^{c(f(x)) + c(f(y)) + \nu(f(\sigma(x)), f(\sigma(y)))}M(x, y). \tag{3.43}$$

Finally, to show the equality of (3.41) and (3.43) we need to guarantee that

$$c(f(x)) + c(f(y)) + \nu(f(\sigma(x)), f(\sigma(y))) = \nu(f(x), f(y)) \mod 2,$$

but this follows immediately from (3.39). \qed

**Remark 3.44.** This sign problem is not uncommon, it appears in the construction of the equivariant Khovanov homology [24, Section 2]. The approach in [24] is essentially the same as the one we use here.

3.10. Equivariant subcategories. Suppose that $C'$ is an equivariant downward closed subcategory of $C$. A full subcategory $C''$ of $C$ whose objects are objects not in $C'$ is clearly upward closed. We call it the complementary upward closed category of $C'$. As $C'$ is invariant under the group action, the subcategory $C''$ is also an invariant subcategory.

The following result is a direct generalization of [17, Lemma 3.32].

**Proposition 3.45.** If $C, C'$ and $C''$ are as above, then there exist three equivariant maps, an inclusion $i : C' \to C$, a collapse $\kappa : C \to C''$ and the Puppe map $\rho : C'' \to \Sigma C'$, that induce the following cohomology long exact sequence

$$\cdots \to \widetilde{H}^i(|C|) \xrightarrow{i_*} \widetilde{H}^i(|C'|) \xrightarrow{\kappa_*} \widetilde{H}^i(|C''|) \xrightarrow{\rho_*} \widetilde{H}^{i+1}(|C'|) \xrightarrow{\nu_*} \cdots \tag{3.46}$$
Suppose \( C \) is an equivariant cubical flow category, \( C' \) is a downward closed subcategory and \( C'' \) is the complementary upward closed category, and let the maps \( \iota, \kappa \) and \( \rho \) be as in Proposition 3.45. We ask under which conditions one of these maps is an equivariant homotopy equivalence. This holds under some extra assumptions that we spell in Lemma 3.47. Although these assumptions are harder to verify, the methods developed in Subsection 3.8 simplify the process.

Lemma 3.47. Let \( \iota, \kappa \) and \( \rho \) be as described in Proposition 3.45.

(a) If for any subgroup \( H \subset G \) the reduced homology \( \tilde{H}^*(||C''||^H) \) is trivial, then the map \( \iota \) is an equivariant stable homotopy equivalence.

(b) If for any subgroup \( H \subset G \) the reduced homology \( \tilde{H}^*(||C'||^H) \) is trivial, then the map \( \kappa \) is an equivariant stable homotopy equivalence.

(c) If for any subgroup \( H \subset G \) the reduced homology \( \tilde{H}^*(||C||^H) \) is trivial, then the map \( \rho \) is an equivariant stable homotopy equivalence.

Proof. We prove only part (a), since the proofs for the other two statements are analogous. Our assumptions imply that for any \( H \subset G \),

\[
\iota^H: ||C'||^H \to ||C||^H
\]

is a stable homotopy equivalence. Since \( ||C'|| \) and \( ||C|| \) are equivariantly homotopy equivalent to \( G\)-CW-complexes, the equivariant version of the Whitehead Theorem (see e.g. [22, Section VI.3]) implies that \( \iota \) is a stable equivariant homotopy equivalence.

□

4. Proof of Theorem 1.2

Coming back to link theory, suppose that \( D \) is an \( m \)-periodic diagram representing the \( m \)-periodic link \( L \). By Proposition 3.20 the Khovanov flow category \( \mathcal{C}_{Kh}(D) \) admits a group action. After a suitable desuspension, \( \mathcal{C}_{Kh}(D) \) becomes an equivariant cubical flow category. Proposition 3.30 shows that the geometric realization \( \mathcal{X}_D \) admits a \( \mathbb{Z}_m \)-action. In particular, it ensures the existence of a group action on the Khovanov homotopy type.

To conclude the proof of Theorem 1.2 we need to show that the stable equivariant homotopy type of \( \mathcal{X}_D \) does not depend on the choices made.

* Independence of \( R \) and \( \epsilon \). Arguing as in [17, Lemma 3.25], we see that different choice of parameters \( R \) and \( \epsilon \) yields an equivariantly homeomorphic space.

* Independence of \( e_* \). Any cubical neat embedding \( \iota \) of \( C \) relative to \( e_* = (e_1, \ldots, e_{n-1}) \) induces a cubical neat embedding \( \iota' \) relative to \( e'_* = (e_1, \ldots, e_i + 1, \ldots, e_{n-1}) \). Arguing as in the proof of [17, Lemma 3.26] we conclude that

\[
\Sigma^V||C||_{e_*} \simeq ||C||_{e'_*}.
\]

* Independence of \( V \). Let us introduce the following notation. Suppose \( \iota_V \) is a cubical neat embedding relative to \( e_* \), and to a representation \( V \). Let \( V \leftrightarrow W \) be a neat embedding. We write \( \iota_V^W \) for the neat embedding relative to \( e_* \) and \( W \), which is obtained by composing the neat embedding relative to \( V \) with the embedding \( V \leftrightarrow W \). We observe that if \( W = V \oplus V' \), then by construction

\[
\Sigma^V||C||_{\iota_V^W} \cong ||C||_{\iota_V^W}.
\]

Suppose \( \iota_V \) and \( \iota_W \) are two cubical neat embeddings relative to \( e_*V \) and \( V \), respectively to \( e_*W \) and \( W \). By increasing the entries of \( e_*V \) and \( e_*W \) and using independence on \( e_* \) discussed above we may and will assume that \( e_*V = e_*W = e_* \). We will also assume that the entries of \( e_* \) are sufficiently large.

Under the latter assumption, the two embeddings \( \iota_V^W \oplus W \) and \( \iota_W^V \oplus V \) are equivariantly isotopic by the Mostow-Palais Theorem (Theorem A.8). By this we mean, that for any \( x, y \in \text{Ob}(C) \) there exists an equivariant isotopy \( \iota_{t,x,y}^L \) (\( t \in [0, 1] \)) such that \( \iota_{t=x,y}^L = (\iota_{t=x,y}^V)^{W} \) and \( \iota_{t=x,y}^1 \) (\( t_{t=x,y}^W \)) satisfying compatibility relations for all \( t \in [0, 1] \). Such isotopy is constructed...
by defining $j_{x,y}^t$, once $j_{x,y}^0$ and $j_{x,y}^1$ have been defined (see proof of Proposition 3.24). The construction of $j_{x,y}^t$ is inductive as in Proposition 3.24, using Mostow-Palais Theorem at each stage. We omit straightforward details.

Given the isotopy, we obtain that $\|C\|_{V \oplus W}$ and $\|C\|_{W \oplus V}$ are equivariantly homotopy equivalent, hence $\|C\|_{V}$ and $\|C\|_{W}$ are stably equivariantly homotopy equivalent, as desired.

The independence on the choice of the diagram and on the ladybug matching is more complicated, we prove these results in Sections 4.1 and 4.2, respectively.

4.1. Independence of equivariant Reidemeister moves. Let $D_1$ be an $m$-periodic diagram representing an $m$-periodic link $L$. The link $L$ can be modified by an isotopy which commutes with the group action. This transformation produces another $m$-periodic diagram $D_2$ representing $L$. Such an equivariant isotopy can be realized as a sequence of orbits of Reidemeister moves from $D_1$ to $D_2$, and periodicity of the diagram is preserved at each step. An orbit of a single move is called an equivariant Reidemeister move (see Figure 3 for two such examples). Any two $m$-periodic diagrams representing the same link are related by a sequence of equivariant Reidemeister moves.

Proposition 4.1. The stable equivariant homotopy type of $X_D$ is invariant under the equivariant Reidemeister moves.

Proof. We prove the invariance under the equivariant R2-move, following the same idea as in [17, Proof of Proposition 6.2]. The proof of the invariance under other equivariant Reidemeister moves follows the same lines as in [17]; the necessary adjustments to make these proofs work in the equivariant case are the same as the adjustments for the proof of the invariance under equivariant R2-move, which we give in detail. The main difficulty in the proof is to verify the assumptions of Lemma 3.47.

Let $D$ be a diagram with crossings $c_1, \ldots, c_n$. Let $D'$ be the diagram obtained after performing an equivariant R2-move on $D$, and let $(c_{n+1}, c_{n+2}), (c_{n+3}, c_{n+4}), \ldots, (c_{n+2m-1}, c_{n+2m})$ be the $m$ pairs of new crossings created during the process; see Figure 4(a).

For each $v \in \{0,1\}^{n+2m}$, write $D'(v) = D_{D'}(v)$, and recall that $v_i$, the $i^{th}$ coordinate of $v$, corresponds to the type of resolution of the crossing $c_i$ of $D'$. 

**Figure 3.** Two examples of equivariant Reidemeister moves over the Hopf link and the trefoil knot.

**Figure 4.** The equivariant R2-move and the resolution configurations described in proof of Theorem 1.2 are shown.
Consider now the subcategory of $\mathcal{C}_{Kh}(D')$, denoted $\mathcal{C}_1$, consisting of those labeled resolution configurations $(D'(v), x)$ such that:

* either there exists a value of $k$ with $v_{n+2k-1} = 0$ and $v_{n+2k} = 1$, and $x$ assigns a label $x_+$ to the extra circle created (see Figure 4(b));
* or there exists a value of $k$ satisfying $v_{n+2k-1} = v_{n+2k} = 1$ (see Figure 4(c)).

It is clear that $\mathcal{C}_1$ is an upward closed category. In fact, it corresponds to the upward closed category $\mathcal{C}'_1$ in [17, Proof of Proposition 6.3].

**Lemma 4.2.** The subcategory $\mathcal{C}_1$ is $G$-invariant, and for each subgroup $H \subset G$ the corresponding complex $C^*(\mathcal{C}_1^H)$ is acyclic.

We defer the proof of Lemma 4.2 past the proof of Proposition 4.1.

Let $\mathcal{C}_2$ be the complementary downward closed subcategory of $\mathcal{C}_1$, consisting of those labeled resolutions which do not satisfy any of the two previous conditions. Next, we consider a subcategory $\mathcal{C}_3$ of $\mathcal{C}_2$. The objects of $\mathcal{C}_3$ are the labeled resolution configurations $(D'(v), x)$ such that there exists a value of $k$ satisfying:

* either $v_{n+2k-1} = 0$ and $v_{n+2k} = 1$, and the extra circle is labeled by $x_-$;
* or $v_{n+2k-1} = 0$ and $v_{n+2k} = 0$;

see Figure 4(d) and (e). We observe that $\mathcal{C}_3$ is an upward closed category.

**Lemma 4.3.** The subcategory $\mathcal{C}_3$ is $G$-invariant, and for any subgroup $H \subset G$ the complex $C^*(\mathcal{C}_3^H)$ is acyclic.

We omit the proof of Lemma 4.3, since it is analogous to the proof of Lemma 4.2.

The complementary category of $\mathcal{C}_3$ in $\mathcal{C}_2$ is $\mathcal{C}_4$: that is to say, $\mathcal{C}_4$ is the category such that there exists a value of $k$ satisfying $v_{n+2k-1} = 1$ and $v_{n+2k} = 0$ (see Figure 4(f)). Moreover, observe that $\mathcal{C}_4$ is isomorphic to the category $\mathcal{C}_{Kh}(D)$ corresponding to the original diagram $D$.

In this setting, we apply Lemma 3.47 twice to get the desired result. Namely, we first state that $||\mathcal{C}_{Kh}(D')||$ is equivariantly stably homotopy equivalent to $||\mathcal{C}_2||$ and then that $||\mathcal{C}_2||$ is equivariantly stably homotopy equivalent to $||\mathcal{C}_4|| = ||\mathcal{C}_{Kh}(D)||$. This concludes the proof of Proposition 4.1. \[\square\]

**Proof of Lemma 4.2.** Let, for any $a \in \{0, 1\}^m$, $C^*(a)$ denote the cochain complex generated by objects of $\mathcal{C}_1$ of the form $v_{n+2k-1} = a_i$ and $v_{n+2k} = 1$. Notice that $C^*(\{1, 1, \ldots, 1\})$ is a subcomplex of $C^*(\mathcal{C}_1)$. Moreover, the differential yields an isomorphism of chain complexes

$$C^*(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_m) \cong C^*(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_m);$$

see Figure 5. Therefore, there exists an isomorphism of chain complexes

$$f: C^*(\mathcal{C}_1) \cong C^*(1, \ldots, 1) \otimes C^*(\text{Cube}(m)).$$

Since $C^*(\text{Cube}(m))$ is acyclic, K"unneth Theorem implies that $C^*(\mathcal{C}_1)$ is also acyclic.
An analogous argument works for the fixed point sets categories. Let $H \subset G$ be a subgroup of order $k$ and consider $C^H_D$. Let $\sigma$ denote the permutation of crossings of $D'$. Notice that the subset of crossings $\{c_{n+1}, c_{n+2}, \ldots, c_{n+2m}\}$ consists of two orbits

$$c_{n+1}, c_{n+3}, c_{n+5}, \ldots, c_{n+2m-1} \quad \text{and} \quad c_{n+2}, c_{n+4}, \ldots, c_{n+2m},$$

and therefore we obtain an isomorphism of chain complexes

$$\psi: C^*(C^H_D) \cong C^*(1, 1, \ldots, 1)^H \otimes C^*(\text{Cube}(m/k)).$$

We can apply once more K"unneth Theorem to conclude that $C^*(C_1)^H$ is acyclic. \hfill \Box

4.2. Independence of the ladybug matching. Before we give the proof of independence of the ladybug matching, we introduce some notation: Given a link diagram $D$ and let $C$ be the reflection of the diagram $D$ along the $y$-axis after switching all crossings (that is, $D'$ is the result of rotating $D$ by 180 degrees). The diagrams $D$ and $D'$ represent the same link, but the rotation of the group action is inverted. In other words, $D$ and $D'$ represent the same equivariant link. Therefore these diagrams can be related by a sequence of equivariant Reidemeister moves. By Proposition 4.1 we have a stable equivariant homotopy equivalence between $||C_{Kh}(D)||$ and $||C_{Kh}(D')||$.

There is also a stable equivariant homotopy equivalence between $||C^K_{Kh}(D)||$ and $||C_{Kh}(D')||$. This is shown using the same argument as in the proof of [17] Proposition 6.5: the isomorphism of framed flow categories $C^K_{Kh}(D)$ and $C_{Kh}(D)$ is equivariant, if we revert the group action on one side.

The composition of the two stable equivariant homotopy equivalences yields the desired stable equivariant homotopy equivalence. \hfill \Box

5. Equivariant Khovanov homology

5.1. Review of the construction. We begin with a brief review of the construction of the equivariant Khovanov homology [24], later on we merge this construction with the construction of the equivariant Khovanov homotopy type that we introduced in previous sections.

Let $D$ be an $m$-periodic diagram representing an $m$-periodic link $L$. The symmetry of $L$ can be realized by a cobordism in $S^3 \times I$ in the following way. Suppose the rotation center is at $0 \in \mathbb{R}^2$. Consider $D \times I \subset \mathbb{R}^2 \times I$ and twist it by the diffeomorphism $\theta: \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2 \times [0, 1]$.
Let Proposition 5.4. X agree. In the rest of this section we denote by (5.3). Another way uses the construction of the Khovanov homotopy type. Namely we two ways of calculating equivariant homology from a Khovanov theory. One way is to use the equivariant Khovanov homology as Borel cohomology.

Khovanov homology is an invariant of periodic links [24]. Since Ext (5.3) \( E_{kh}^{k,q} \) Khovanov homology of \( L \) can be regarded as a \( \Lambda \)-module, where \( \Lambda = R[Z_m] \). For an \( R \)-module \( M \), define the equivariant Khovanov homology of \( L \) in gradings \( k \) and \( q \) as

\[
(5.3) \quad \text{EKh}^{k,q}(L; M) = \text{Ext}^k_{\Lambda}(M; \text{CKh}^{*,q}(D; R)).
\]

Since \( \text{Ext}^k_{\Lambda}(M; \text{CKh}^{*,q}(D; R)) \) does not depend on the chosen diagram \( D \), equivariant Khovanov homology is an invariant of periodic links [24].

5.2. Equivariant Khovanov homology as Borel cohomology. We have now essentially two ways of calculating equivariant homology from a Khovanov theory. One way is to use the definition (5.3). Another way uses the construction of the Khovanov homotopy type. Namely we can consider the Borel cohomology of the space \( \mathcal{X}_D \). We will now show that the two constructions agree. In the rest of this section we denote by \( \tilde{C}^*(\mathcal{X}_D; R) \) the reduced cellular cochain complex of \( \mathcal{X}_D \) associated to the CW-structure described in Section 3.7.

First we state a preparatory result.

Proposition 5.4. Let \( D \) be an \( m \)-periodic diagram of a link. There exists an identification of \( R[Z_m] \) cochain complexes

\[
(5.5) \quad \tilde{C}^*(\mathcal{X}_D; R) \cong \text{CKh}^*(D; R).
\]

Here it should be understood that the structure of the \( R[Z_m] \) cochain complex is given by the \( Z_m \)-action on \( \tilde{C}^*(\mathcal{X}_D; R) \) and on \( \text{CKh}(D) \).

Proof. This is a consequence of the construction of the cochain complex of \( \tilde{C}^*(\mathcal{X}_D) \). The cellular cochain complex \( C^*(\mathcal{X}_D; R) \) was constructed in Section 2.6. The construction is that the generators of \( \tilde{C}^*(\mathcal{X}_D; R) \) correspond to the generators of \( \text{CKh}^*(D; R) \). differential on \( \tilde{C}^*(\mathcal{X}_D; R) \) is the same as in \( \text{CKh}(D; R) \). In Section 3.9 it was shown that the induced group actions on \( \tilde{C}^*(\mathcal{X}_D; R) \) and \( \text{CKh}(D; R) \) coincide. \( \square \)

In order to state and prove next corollary, we will adopt the following convention. If \( C^* \) is a chain complex, we will associate to it a cochain complex \( C^*_r \) defined by \( C^*_r = C_r \) with the differential \( d^*_r \) defined by \( d_r = (-1)^n d_n \), where \( d_n : C_n \to C_{n-1} \) is the differential in \( C^* \). In particular, if \( C^* \) is a cochain complex, then a projective resolution \( P^* \) of \( C^* \) is a bounded above cochain complex (i.e. \( P^n = 0 \) for \( n \) big enough) such that there exists a quasi-isomorphism \( P^* \to C^* \). Hence if \( C^* \) and \( D^* \) are chain complexes, and \( P^*, Q^* \) are projective resolutions of \( C^*_r \) and \( D^*_r \), respectively, then

\[
\text{Tor}_p(C^*_r ; D^*_r) = H^{-p}(P^* \otimes Q^*) \cong H^{-p}(P^* \otimes D^*_r) \cong H^{-p}(C^*_r \otimes Q^*).
\]

For two cochain complexes \( C^*, D^* \) we define the Hom cochain complex

\[
\text{Hom}^n(C^*, D^*) = \prod_{p \in \mathbb{Z}} \text{Hom}(C^p , D^{p+n}).
\]
with the differential $d^n_{\text{Hom}}(f) = d^n_{g} \circ f - (-1)^n f \circ d^n_{r}$. If $P^*$ is a projective resolution of a cochain complex $C^*$ and $I^*$ is an injective resolution of $D^*$, then

$$\text{Ext}^n(C^*, D^*) = H^n(\text{Hom}^*(P^*, I^*)) \cong H^n(\text{Hom}^*(P^*, D^*)) \cong H^n(\text{Hom}^*(C^*, I^*)) .$$

Recall that to any topological group $G$ we can associate a contractible space $EG$ equipped with the free action of $G$. By $BG = EG/G$ we denote the classifying space of $G$. For a $G$-space $X$ and any $G$-module $M$ we define the Borel equivariant cohomology of $X$

$$H^*_G(X; M) = H^*(EG \times_G X; M) \cong \text{Ext}^*_{R[G]}(C^*_r(X; R); M) ,$$

where $C^*_r(X)$ denotes the cochain complex associated to the cellular cochain complex $C_*(X)$ of $X$ using the convention described above. In particular, we have $C^*_r(X; R) = \text{Hom}^*_R(C^*_r(X); R)$.

There is a natural map $EG \times X \to EG$ which, after taking quotient of both sides, yields a map

$$EG \times_G X \xrightarrow{p} BG .$$

Hence there is a map

$$H^*(BG; M) \xrightarrow{p^*} H^*_G(X; M) .$$

We define reduced Borel cohomology of $X$, $\widehat{H}^*_G(X; M)$, to be the cokernel of $p^*$. It is easy to check that

$$\widehat{H}^*_G(X; M) \cong \text{Ext}^*_{R[G]}(\tilde{C}^*_r(X; R); M) .$$

**Corollary 5.6.** Let $L$ be an $m$-periodic link and suppose that $R$ is a field. For any $R[G]$-module $M$, the equivariant Khovanov homology $EKh^{l,q}(L; M)$ is isomorphic to the reduced Borel cohomology of $\mathcal{X}_L$

$$EKh^{l,q}(L; M) \cong H^*_G(\mathcal{X}_L, \text{Hom}_R(M, R)) ,$$

where $g$ acts on $\text{Hom}_R(M, R)$

$$(g \cdot f)(x) = f(g^{-1} \cdot x) .$$

**Proof.** If $L$ is an $m$-periodic link and $D$ is an $m$-periodic diagram of $L$, then

$$EKh^{l,q}(L; M) = \text{Ext}^*_{R[G]}(M, CKh^{l,q}(D; R)) = \text{Ext}^*_{R[G]}(M, \tilde{C}^*(\mathcal{X}^q_D; R)) .$$

Let $P^*_M$ be some projective resolution of $M$. We have

$$\text{Ext}^*_{R[G]}(M, \tilde{C}^*(\mathcal{X}^q_D; R)) \cong \text{Ext}^*_{R[G]}(M, \text{Hom}_R(\tilde{C}^*_r(\mathcal{X}^q_D; R), R)) \cong \cong H^*(\text{Hom}^*_R(P_M^*, \text{Hom}_R(\tilde{C}^*_r(\mathcal{X}^q_D; R), R))) \cong H^*(\text{Hom}^*_R(P_M^* \otimes_{R[G]} \tilde{C}^*_r(\mathcal{X}^q_D; R), R)) \cong \cong H^*(\text{Hom}^*_R(\tilde{C}^*_r(\mathcal{X}^q_D; R), \text{Hom}_R(P_M^*, R))) ,$$

where the first isomorphism comes from the definition of the cochain complex, the second is the definition of the Ext functor and the last two isomorphisms come from [3, Exercise I.0.6]. Since $R$ is a field, any projective $R[G]$-module is also injective by [6, Exercise 1.10.24]. Moreover, the functor

$$M \mapsto \text{Hom}_R(M, R)$$

defined on the category of left $R[G]$-modules is exact by [6, Exercise 1.10.22] and maps projective modules to projective modules by [6, Corollary 1.10.29]. Therefore, $\text{Hom}^*(P^*_M, R)$ becomes an injective resolution of the $R[G]$-module $\text{Hom}_R(M, R)$. Consequently

$$\text{Ext}^*_{R[G]}(M, \tilde{C}^*(\mathcal{X}^q_D; R)) \cong \text{Ext}^*_{R[G]}(\tilde{C}^*_r(\mathcal{X}^q_D), \text{Hom}_R(M, R))$$

and the proof is finished. □
5.3. **Decomposition of** $\text{EKh}(L)$. In general, $\text{EKh}(L)$ can be quite intricate and much larger than Khovanov homology; in fact, the ordinary Khovanov homology can be recovered as a special case. The following result is stated in [24, proof of Theorem 3.7]; see also [2, Proposition 2.8].

**Proposition 5.7.** If $L$ is a periodic link, then

$$\text{EKh}(L; \Lambda) = \text{Kh}(L; R).$$

**Proof.** As it might be confusing, whether Proposition 5.7 holds under the assumption that $m$ is invertible in $R$ or in general, we present a quick argument that works for any $m$. We have $\text{Ext}^i_\Lambda(\Lambda; \text{CKh}_j^k(L; R)) = 0$ for $i > 0$, so the Cartan-Eilenberg spectral sequence degenerates yielding the desired statement. Clearly, vanishing of higher Ext groups does not require semisimplicity of $\Lambda$. \[\square\]

In this paper we are particularly interested in the case when $R = F$ is a field and $m$ is invertible in $F$. In this case the algebraic structure of the group algebra allows many simplifications.

Recall that if $m$ is invertible in $F$, the group algebra $\Lambda = F[Z_m]$ can be decomposed into a direct sum of simple modules

$$\Lambda = \bigoplus_{i=1}^N M_i.
(5.8)$$

Moreover, any finitely generated $\Lambda$-module decomposes into a direct sum and every summand is isomorphic to some $M_i$, for $1 \leq i \leq N$. The decomposition above implies that

$$\text{EKh}^*\ast(L; \Lambda) = \bigoplus_{i=1}^N \text{EKh}^*\ast(L; M_i).
(5.9)$$

These facts together with Proposition 5.7 lead to a decomposition of Khovanov homology into generalized eigenspaces of the action of $Z_m$ (see [2, Section 3]):

$$\text{Kh}^*\ast(L; F) = \bigoplus_{i=1}^N \text{EKh}^*\ast(L; M_i).
(5.10)$$

Let $F = F_r$ be a finite field of prime order $r$ and let $m = p^n$ be a prime power, with $p \neq r$. Assume that the order of $r$ in the multiplicative group of invertible elements of $Z_{p^n}$ is maximal, i.e.,

$$r^{(p-1)p^{n-1}} \equiv 1 \pmod{p^n} \text{ and } r^s \not\equiv 1 \pmod{p^n}, \text{ for } 1 \leq s < (p-1)p^{n-1}.
(5.11)$$

Now, for any $0 \leq s \leq n$, write $\Phi_{p^s}(X) \in \mathbb{Z}[X]$ for the $p^s$-th cyclotomic polynomial, given by

$$\Phi_1 = X - 1, \quad \Phi_{p^s}(X) = X^{p^{s-1}} + X^{p^{s-2}} + \ldots + X + 1, \quad \Phi_{p^s}(X) = \Phi_p(X^{p^{s-1}}), \text{ for } s > 1.$$  

It is easy to verify that

$$X^{p^n} - 1 = \prod_{s=0}^n \Phi_{p^s}(X),$$

and therefore $F_r(\xi_{p^s}) = F_r[X]/(\Phi_{p^s}(X))$ is a module over $\Lambda$, where the action of $t^k \in Z_{p^n}$ is given by multiplication by $X^{k}$.

Moreover, the Chinese Remainder Theorem gives the following decomposition of $\Lambda$ into simple modules

$$\Lambda = F_r[Z_{p^n}] = F_r[X]/(X^{p^n} - 1) \cong \bigoplus_{s=0}^n F_r(\xi_{p^s}).
(5.12)$$

**Remark 5.13.** It is worth pointing out that condition (5.11) guarantees that $\Phi_{p^s}(X)$ is indecomposable over $F_r$, for $s = 1, \ldots, n$. 

Proposition 5.14. [2] Let \( p \) and \( r \) be two different prime numbers satisfying condition \((5.11)\). Then there exists a decomposition

\[
(5.15) \quad \text{Kh}(L; \mathbb{F}_p) \cong \text{EKh}(L; \mathbb{F}_r[\mathbb{Z}_{p^s}]) = \bigoplus_{s=0}^{n} \text{EKh}(L; \mathbb{F}_r(\xi_{p^s})).
\]

6. Cohomology operations and equivariant homotopy type

6.1. Stable cohomology operations. Given two generalized cohomology theories \( X(\cdot) \) and \( Y(\cdot) \), a stable cohomology operation of degree \( k \) is a family of natural transformations between functors \( X^i(\cdot) \) and \( Y^{k+i}(\cdot) \) commuting with suspension. We focus on stable cohomology operations in singular homology over a finite field. These operations form a Steenrod algebra. Standard references include [9, Section 4.L] and [7, Section 10.4].

The Steenrod algebra \( A_2 \) over \( \mathbb{Z}_2 \) is generated by the Steenrod squares \( Sq^1: H^*(\cdot, \mathbb{Z}_2) \to H^{*+1}(\cdot, \mathbb{Z}_2) \), with \( Sq^1 \) being the Bockstein homomorphism corresponding to the short exact sequence \( 0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0 \).

For a prime \( p > 2 \), the Steenrod algebra \( A_p \) is generated by the Bockstein homomorphism \( \beta \), and operations \( P^k: H^*(\cdot, \mathbb{Z}_p) \to H^{*+2k(p-1)}(\cdot, \mathbb{Z}_p) \). The homomorphism \( \beta \) is of degree 1 and it is the connecting homomorphism of the long exact sequence of cohomology induced by the short exact sequence of groups \( 0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0 \).

Coming back to Khovanov homology we make the following observation, see [18][19].

Proposition 6.1. Let \( \alpha \) be a stable cohomology operation of degree \( k \) over \( \mathbb{Z}_p \). Then, given a link \( L \) and \( q \in \mathbb{Z} \), the map \( \alpha \) induces a well defined map

\[
\alpha_q: \text{Kh}^{*+q}(L; \mathbb{Z}_p) \to \text{Kh}^{*+k+q}(L; \mathbb{Z}_p).
\]

There appeared several algorithms for computing Steenrod squares in Khovanov homology, so the invariants coming from Steenrod squares can be effectively computed (see [19][21]). The knotkit package [25] implements the algorithm of [19]. We remark that the maps \( Sq^1 \) and \( \beta \) are determined from the integral Khovanov homology, see [19, Section 2.5].

6.2. Steenrod squares and group actions. The next statement shows that Steenrod operations commute with the group action.

Theorem 6.2. Let \( L \) be an \( m \)-periodic link and \( \mathbb{F} \) a field. Any stable cohomology operation

\[
\alpha: H^*(-; \mathbb{F}) \to H^{*+k}(-; \mathbb{F})
\]

commutes with the action of \( \mathbb{Z}_m \) on \( \text{Kh}(L; \mathbb{F}) \).

Proof. Cohomology operations are natural, so they commute with the group action on cohomology of \( \mathcal{X}_L \). On the other hand, Proposition 5.4 shows that the \( \mathbb{Z}_m \)-action on cohomology of \( \mathcal{X}_L \) commutes with the group action on the Khovanov homology of \( L \).

□

Corollary 6.3. Any stable cohomology operation

\[
\alpha: H^*(-; \mathbb{F}) \to H^{*+k}(-; \mathbb{F})
\]

preserves decomposition \((5.10)\).

Denote by \( \mathcal{A} \), the Steenrod algebra over \( \mathbb{F}_p \). By Corollary 6.3 \( \text{Kh}(L; \mathbb{F}_r) \) is a module over \( \mathcal{A}_r \). Notice that the action of \( \mathcal{A}_r \) on \( \text{Kh}(L; \mathbb{F}_r) \) factors through some finite-dimensional quotient \( \mathcal{B} = \mathcal{A}_r/I \). Therefore by the Krull-Schmidt-Azumaya Theorem [6, Theorem 6.12] there exists a decomposition

\[
(6.4) \quad \text{Kh}(L; \mathbb{F}_r) = \bigoplus_{i=0}^{N} S_i,
\]
where each $S_i$ is an indecomposable $A_r$-module ($1 \leq i \leq N$), that is, $S_i$ cannot be written as a direct sum of its nontrivial submodules. Moreover, this decomposition is unique up to isomorphism and reordering of the summands.

**Theorem 6.5.** Let $L$ be a $p^n$-periodic link. For any $0 \leq j \leq N$, its equivariant Khovanov homology $\text{EKh}(L; M_j)$, admits the following decomposition when treated as a $(\Lambda, A_r)$-bimodule

$$\text{EKh}(L; M_j) = \bigoplus_{i \in \mathbb{Z}} M_j \otimes_{\mathbb{F}_r} S_i,$$

where $S_i$ is an indecomposable $A_r$-module of the decomposition (6.4).

**Proof.** Notice that [2, Definition A.8] implies that every $M_j$ is a finite field extension of $\mathbb{F}_r$. Consequently $\text{EKh}(L; M_j)$ is a $M_j$-vector space and Theorem 6.2 implies that the action of $A_r$ extends naturally to the action of the algebra $M_j \otimes_{\mathbb{F}_r} A_r$, hence the statement follows. $\square$

7. Periodicity criterion

7.1. Periodicity criterion of [2]. The main result of [2] is obtained by studying carefully the equivariant analogues of the Lee and Bar-Natan spectral sequences together with Propositions 5.7 and 5.14. In particular, the presentation of the Khovanov polynomials $\text{KhP}$ in (7.2) as a sum of polynomials $\Delta P_j$ corresponds to the decomposition of the equivariant Khovanov homology in (5.15).

**Theorem 7.1.** [2, Theorem 1.1] Let $K$ be a $p^n$-periodic knot for some odd prime $p$. Let $r \neq p$ be a prime and suppose that condition (5.11) is satisfied. Write $s(K, \mathbb{F}_r)$ for the $s$-invariant of $K$ derived from the Lee or Bar-Natan theory (see [1, 16]), and set $c = 1$ if $\mathbb{F} = \mathbb{F}_2$ and $c = 2$ otherwise. Then the Khovanov polynomial $\text{KhP}(K; \mathbb{F}_r)$ decomposes as a sum

$$\text{KhP}(K; \mathbb{F}_r) = \Delta P_0 + \sum_{j=1}^n (p^j - p^{j-1}) \Delta P_j,$$

where

$$\Delta P_0, \Delta P_1, \ldots, \Delta P_n \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$$

are Laurent polynomials such that

1. $\Delta P_0 = q^{s(K, \mathbb{F}_r)}(q + q^{-1}) + \sum_{j=1}^\infty (1 + tq^{2cj}) \Delta S_{0j}$, with $\Delta S_{0j} \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ Laurent polynomials with non-negative coefficients.
2. $\Delta P_k = \sum_{j=1}^\infty (1 + tq^{2kj}) \Delta S_{kj}$, with $\Delta S_{kj} \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ Laurent polynomials with non-negative coefficients, for $1 \leq k \leq n$.
3. $\Delta P_k(-1, q) - \Delta P_{k+1}(-1, q) \equiv \Delta P_k(-1, q^{-1}) - \Delta P_{k+1}(-1, q^{-1}) \pmod{q^{p^n-k} - q^{-p^n-k}}$.
4. If the width of $\text{Kh}(K; \mathbb{F}_r)$ is equal to $w$, then $S_{kj} = 0$ for $j > \frac{cw}{2}$.

**Remark 7.3.** Theorem 7.1 can be generalized to the case of links, once we know the equivariant Lee (or Bar-Natan) homology of this theory. This generalization, with a collection of examples, is given in [2].

7.2. Periodicity criterion and Steenrod squares. In this subsection we present a new periodicity criterion, strengthening the one stated in [2].

Given an $m$-periodic link $L$, consider the decomposition of its Khovanov homology shown in (6.4), and write $PS_i$ for the Khovanov polynomial associated to the summand $S_i$. Therefore, we can decompose the Khovanov polynomial of $L$ as

$$(7.4) \qquad \text{KhP}(L; \mathbb{F}_r) = \bigoplus_{i=0}^{N} PS_i,$$

leading to a new condition on the polynomials $\Delta P_i$ in Theorem 7.1.
Proposition 7.5. Suppose $\Delta P_0, \ldots,$ are as in Theorem 7.1. Then each of the $\Delta P_j$ is a sum of Poincaré polynomials $P_{S_{i_1}, \ldots, S_{i_k}}$.

Proof. From the proof of Theorem 7.1 we know that the polynomials $\Delta P_i$ are the Poincaré polynomials of $E Kh(L; \mathbb{F}_r(\xi_{j, \epsilon}))$. The statement follows immediately from Theorem 6.5. □

We do not give examples where Theorem 7.1 fails to provide an obstruction for periodicity, but Proposition 7.5 obstructs periodicity. In many cases we checked, if a knot passed the criterion of Theorem 7.1 the number of different presentations of $KhP$ as a sum of $\Delta P_i$ that satisfy the conditions of Theorem 7.1 was enormous, seven digit numbers were not uncommon. On the other hand, condition (7.4) usually restricts the number of possible presentation to a smaller number, not to zero. In short, we believe that the applicability of our criterion is rather limited.

8. Equivariant Khovanov homotopy type of the Hopf link

Let $H$ be the Hopf link diagram shown in Figure 6. We write a superscript to refer to the quantum grading. By [17, Section 9.2], the stable homotopy type of $H$ is

$$\chi^0_{Kh}(H) \simeq \chi^2_{Kh}(H) \simeq S^0, \quad \chi^1_{Kh}(H) \simeq \chi^6_{Kh}(H) \simeq S^2.$$

We will study the equivariant Khovanov homotopy type of $H$ with respect to the $\mathbb{Z}_2$-symmetry of the Hopf link obtained by rotating the Hopf link by 180 degrees.

Generators of the Khovanov flow category are listed in Table 1. To give a flavor of the calculations we show how to determine $gr_{\mathbb{Z}_2}(d)$. We have $gr(d) = 2$ and and the isotropy group of $d$, $G_d$, is trivial. We also have $n_z = 0$. Hence (3.22) gives us $gr_{\mathbb{Z}_2}(d) = \mathbb{Z}/2^1 = \mathbb{Z}^2$.

In quantum grading 0 and 6 there is only one cell, therefore, we have

$$\chi^0_{Kh}(H) \simeq S^0, \quad \chi^6_{Kh}(H) \simeq S_1(\mathbb{R} \oplus \mathbb{R}[\mathbb{Z}_2]) = S_{\mathbb{R}[\mathbb{Z}_2]},$$

where $S_1(\mathbb{R} \oplus \mathbb{R}[\mathbb{Z}_2])$ denotes the unit sphere contained in the representation $\mathbb{R} \oplus \mathbb{R}[\mathbb{Z}_2]$, which is the same as the one-point compactification of $\mathbb{R}[\mathbb{Z}_2]$, i.e. $S^2$ with $\mathbb{Z}_2$ acting by a reflection.

In quantum grading 4 we have 5 generators $a, b, c, d, e$; see Figure 7. The action of $\mathbb{Z}_2$ on objects is given by

$$\sigma(a) = a, \quad \sigma(b) = c, \quad \sigma(d) = e.$$

| Name | Generator | $gr_{\mathbb{Z}_2}$ | $gr$ | $q$ |
|------|-----------|---------------------|------|-----|
| a    | $(D_H(00), x_+ x_+)$ | $\mathbb{R}^0$ | 0    | 4   |
| v    | $(D_H(00), x_+ x_-)$ | $\mathbb{R}^0$ | 0    | 2   |
| w    | $(D_H(00), x_- x_+)$ | $\mathbb{R}^0$ | 0    | 2   |
| m    | $(D_H(00), x_- x_-)$ | $\mathbb{R}^0$ | 0    | 0   |
| b    | $(D_H(10), x_+)$     | $\mathbb{R}^1$ | 1    | 4   |
| x    | $(D_H(10), x_-)$     | $\mathbb{R}^1$ | 1    | 2   |
| c    | $(D_H(01), x_+)$     | $\mathbb{R}^1$ | 1    | 4   |
| y    | $(D_H(01), x_-)$     | $\mathbb{R}^1$ | 1    | 2   |
| e    | $(D_H(11), x_+ x_+)$ | $\mathbb{R}^2$ | 2    | 4   |
| z    | $(D_H(11), x_- x_-)$ | $\mathbb{R}[\mathbb{Z}_2]$ | 2 | 2 |

Table 1. Generators of the Khovanov flow category

*Figure 6. 2-periodic diagram of the Hopf link*
Figure 7. Generators $a, \ldots, e$. The arrows denote gluing of corresponding cells, the differentials in the Khovanov chain complex have the arrows reversed.

Therefore, the action on 0-dimensional moduli spaces is as follows:

- $\sigma : \mathcal{M}(b, a) \rightarrow \mathcal{M}(c, a)$,
- $\sigma : \mathcal{M}(d, b) \rightarrow \mathcal{M}(e, c)$,
- $\sigma : \mathcal{M}(d, c) \rightarrow \mathcal{M}(e, b)$.

Analogously on 1-dimensional moduli spaces

- $\sigma : \mathcal{M}(d, a) \rightarrow \mathcal{M}(e, a)$,

is orientation reversing (with respect to the non-equivariant orientations). Indeed, since $f(d) = f(e) = 11$ and $gr_G(11) = \mathbb{R} \oplus \mathbb{R}$, then the map

$$\nu_\sigma : gr_G(d) \rightarrow gr_G(e)$$

from (EFC-6) is given by $(x, y) \mapsto (x, -y)$. This follows from the commutativity of the following diagram

$$
\begin{array}{ccc}
gr_G(d) & \xrightarrow{\nu_\sigma} & gr_G(e) \\
\downarrow{f_{gr(G)}} & & \downarrow{f_{gr(G)}} \\
gr_G(11) & \xrightarrow{\nu_\sigma} & gr_G(11)
\end{array}
$$

Here the bottom $\nu_\sigma$ is calculated in the category $\text{Cube}_\sigma(2)$, it is the group action on $\mathbb{R} \oplus \mathbb{R}$, so indeed we obtain $(x, y) \mapsto (x, -y)$. The vertical maps are as in Definition 3.14. Let $\iota$ be a cubical neat embedding of $\mathcal{C}_{\text{Kh}}(H)$ with respect to $d_\bullet = (1, 1)$ and representation $V = \mathbb{R}$.

We have three $G$-cells:

- $EP_2 = EP(a) \cong D_\epsilon(\mathbb{R}) \times D_\epsilon(\mathbb{R}) \times \{0\}$,
- $EP_3 = EP(b) \sqcup EP(c) \cong \mathbb{Z}_2 \times (D_R(\mathbb{R}) \times D_\epsilon(\mathbb{R}) \times [0, 1])$,
- $EP_4 = EP(d) \sqcup EP(e) \cong \mathbb{Z}_2 \times (D_R(\mathbb{R}) \times D_R(\mathbb{R}) \times [0, 1] \times \mathcal{M}(d, a))$.

The group action on these cells is given by maps.

- $\sigma : EP(b) \rightarrow EP(c)$, $(x, y, z) \mapsto (-x, -y, z)$,
- $\sigma : EP(d) \rightarrow EP(e)$, $(x, y, z, t) \mapsto (-x, -y, -z, -t)$.

Arguing as in [17, Section 9.2] we see that after collapsing the boundary of $EP_2$ to a point we obtain the two-dimensional skeleton

$$\Sigma^{\mathbb{R}^2} \mathcal{X}_{\text{Kh}}^4(H)^{(2)} = S^{\mathbb{R}^2} = S_1(\mathbb{R}^2 \oplus \mathbb{R})$$

After attaching $EP_4$ we obtain the three-dimensional skeleton.

$$\Sigma^{\mathbb{R}^2} \mathcal{X}_{\text{Kh}}^4(H)^{(3)} = S^{\mathbb{R}^3} = S_1(\mathbb{R}^3 \oplus \mathbb{R})$$
The additional trivial direction comes from the [0,1] factor in EP₄. Arguing as in the previous case, we can see that attaching of EP₆ to $\Sigma^{R^2} A^4_{Kh}(H)^{(3)}$ yields

$$\Sigma^{R^2} A^4_{Kh}(H) = S^{R^2} = S_1(R^4_+ \oplus R).$$

Therefore,

$$A^4_{Kh}(H) \simeq S^{R^2}.$$  

In quantum grading 2, we have five generators $v, w, x, y, z$ such that

$$\sigma(v) = w, \quad \sigma(x) = y, \quad \sigma(z) = z.$$  

The action on moduli spaces is the following

$$\sigma : M(x, v) \rightarrow M(y, w), \quad \sigma : M(x, w) \rightarrow M(y, v), \quad \sigma : M(z, v) \rightarrow M(z, w).$$

Once more we take the equivariant cubical neat embedding with respect to $d_* = (1, 1)$ and $V = R_-$. Arguing as in [15, Section 9.2] we verify that $A^2_{Kh}(H) \simeq S^0$.

Summarizing we obtain

$$A^2_{Kh}(H) = S^0 \vee S^0 \vee S^{R^2} \vee S^{R[Z]}.$$  

An application of the Mayer-Vietoris sequence implies that reduced Borel cohomology of $S^0$, $S^{R^2}$ and $S^{R[Z]}$ are free $H^*_c(pt)$-modules of rank one generated in homological degree 0, 2 and 2, respectively. This is consistent with computations done in [24, Section 5].

**Appendix A. (n)-manifolds**

A.1. Manifolds with corners. We say that $M$ is a $k$-manifold with corners if it is locally modelled on open subsets of $R^k_+$, where $R^k_+ = [0, \infty)$. In other words, $M$ is equipped with an atlas $A = \{(U, \psi_U)\}$ such that every $U$ is an open subset in $R^k_+$ and the transition maps $\psi_V^{-1} \circ \psi_U$, for any $U$ and $V$ such that $U \cap V \neq \emptyset$, are $C^\infty$ diffeomorphisms. Compare [12, Definition 2.1].

For every point $x \in M$ we can define its codimension, denoted by $c(x)$, which records the number of coordinates of $\psi_U$ which are zero for any chart $(U, \psi_U)$ for which $x \in U$. It is easy to verify that $c(x)$ is well-defined. Moreover we define codimension-i boundary to be the set $\{x \in M : c(x) = i\}$. A connected face of $M$ is the closure of a connected component of the codimension-1 boundary of $M$. A face is a (possibly empty) disjoint union of connected faces. A k-dimensional manifold with faces is a k-manifold with corners such that every point $x \in M$ belongs to exactly $c(x)$ connected faces of $M$. Moreover, we say that $M$ is an k-dimensional (n)-manifold if $M$ is an k-dimensional manifold and there exists a decomposition $\partial M = \partial_1 M \cup \partial_2 M \cup \ldots \cup \partial_n M$ such that

- $\partial_i M$ is a face of $M$, for every $1 \leq i \leq n$,
- $\partial_i M \cap \partial_j M$ is a face of both $\partial_i M$ and $\partial_j M$, for every $1 \leq i < j \leq n$.

We refer to [12, Section 2] for discussion of the notion of an (n)-manifold.

**Example A.1** (see [17, Definition 3.9]). For an (n+1)-tuple $d_* = (d_0, d_1, \ldots, d_n)$ of non-negative integers define

$$E_n^{d_*} = R^{d_0} \times R^{d_1} \times \ldots \times R^{d_n}.$$  

We make $E_n^{d_*}$ an (n)-manifold by declaring that

$$\partial_i E_n^{d_*} = R^{d_0} \times R^{d_1} \times \ldots \times R^{d_i-1} \times \{0\} \times R^{d_{i+1}} \times \ldots \times R^{d_n}.$$  

**Example A.2.** An n-dimensional permutohedron has a structure of an (n)-manifold, see Section [3.1] below.
In some cases it is more convenient to view \( (n) \)-manifolds as certain functors to the category of topological spaces. Let \( 2^{n} \) denote the category consisting of two objects 0 and 1 with a single non-identity morphism \( 0 \to 1 \). For an integer \( n > 1 \) let

\[
2^{n} = \underbrace{2^{1} \times 2^{1} \times \ldots \times 2^{1}}_{n}.
\]

For two objects \( a, b \in 2^{n} \), we set \( b \leq a \) if \( b_{i} \leq a_{i} \), for any \( 1 \leq i \leq n \), where \( a = (a_{1}, a_{2}, \ldots, a_{n}) \) and \( b = (b_{1}, b_{2}, \ldots, b_{n}) \). An \( n \)-\textit{diagram}, for \( n \geq 1 \), is a functor from the category \( 2^{n} \) to the category of topological spaces.

We can associate an \( n \)-\textit{diagram} to a given \( (n) \)-manifold \( X \) by declaring, for every \( a = (a_{1}, a_{2}, \ldots, a_{n}) \in 2^{n} \),

\[
X(a) = \begin{cases} 
X, & \text{if } a = (1, 1, \ldots, 1), \\
\cap_{i:a_{i}=0} \partial X, & \text{otherwise}.
\end{cases}
\]

Moreover, for any \( b \leq a \) in \( 2^{n} \) the map \( X(b) \to X(a) \) is the inclusion. We point out that \( X(a) \) is an \( \langle |a| \rangle \)-\textit{manifold} with the corresponding \( |a| \)-\textit{diagram} obtained by restricting \( X \) to the full subcategory of objects \( b \) such that \( b \leq a \) (recall that \( |a| = \sum a_{i} \)).

Let \( M \) be a manifold with corners. Choose a Riemannian metric on \( M \). We have the following generalization of the classical result.

**Proposition A.3.** There is an open tubular neighborhood \( U \) of \( \partial M \), homeomorphic to \( M \times [0, 1] \) and a subset \( V \subset TM|_{\partial M} \) such that the exp map yields a diffeomorphism between \( U \) and \( V \).

**Proof.** The proof is analogous to the proof of the collar neighborhood theorem, see for instance \([10] \) Section 4.6.

We recall now the concept of a neat embedding, which roughly means an embedding with no pathological behavior near the boundary. Various similar notions are discussed in detail in \([12] \) Section 3.

**Definition A.4.** Let \( X \) and \( Y \) be two \( (n) \)-manifolds. A neat embedding is an embedding \( \iota: X \to Y \) such that:

1. \( \iota \) is an \( n \)-map, i.e. \( \iota^{-1}(\partial Y) = \partial X \), for any \( 1 \leq i \leq n \),
2. the intersection of \( X(a) \) and \( Y(b) \) is perpendicular with respect to some Riemannian metric on \( Y \), for \( b < a \) in \( 2^{n} \).

**Theorem A.5 \(([17]) \).** Every compact \( (n) \)-manifold admits a neat embedding

\[
\iota: X \hookrightarrow E_{n}^{d_{\bullet}},
\]

for some \( d_{\bullet} \in \mathbb{N}^{n+1} \).

**Proof.** This is a direct consequence of \([17] \) Lemma 3.10.

**A.2. Group actions on \( (n) \)-manifolds.** Let \( \text{Diff}_{n}(X) \) denote the group of diffeomorphisms of the \( (n) \)-manifold \( X \) that are also \( n \)-maps. If \( G \) is a finite group, then a smooth action of \( G \) on \( X \) is a homomorphism

\[
\gamma: G \to \text{Diff}_{n}(X).
\]

An action of \( G \) is effective if \( \gamma \) is injective. Throughout this paper we assume that group actions are effective. Moreover, we will often identify \( g \in G \) with its image \( \gamma(g) \in \text{Diff}_{n}(X) \).

Let \( V - W \in RO(G) \). We say that \( X \) is of dimension \( V - W \), and denote it by \( \dim X = V - W \), if for any interior point \( x \) there exists an isomorphism of representations

\[
T_{x}X \oplus W|_{G_{x}} \cong V|_{G_{x}}.
\]

We have the following equivariant analogue of Proposition A.3.
Proposition A.6. Let $M$ be an \langle n \rangle-manifold with an action of a finite group $G$. Choose a $G$-invariant Riemannian metric on $M$. Then $\partial M$ admits a $G$-equivariant tubular neighborhood $U$ such that there exists a $G$-invariant subset $V \subset TM|_{\partial M}$ such that the exp-map takes $V$ diffeomorphically and $G$-equivariantly to $U$.

Proof. The proof for standard manifolds with boundary is given in [13, Section 3]. The case of manifolds with corners is analogous. □

A.3. Equivariant embeddings. The aim of this section is to prove Lemma [A.7], which is an equivariant version of [17, Lemma 3.10]. It is needed in the proof of Proposition 3.24.

Lemma A.7. Let $M$ be a compact \langle n \rangle-manifold with a group action. Suppose $M$ is subordinate to a representation $V$. If $\iota_0: \partial M \to V^d$ is a $G$-equivariant embedding, then there exists $d' \geq d$ and a $G$-equivariant embedding $\iota: M \to V^{d'}$ such that $|\iota|_{\partial M} = \iota_0$.

A key ingredient in the proof is the equivariant version of Whitney embedding theorem, due to Mostov and Palais.

Theorem A.8 (Mostow-Palais Theorem [26, Corollary 1.10]). If $M$ is in $\mathcal{G}(V)$ and the (real) dimension of $M$ is $n$, then any $G$-equivariant map
\[ f: M \to V^t \]
can be uniformly $C^k$-approximated by an equivariant immersion if $t \geq 2n$, and by an equivariant 1-1 immersion if $t \geq 2n + 1$. Moreover, if $A$ is a closed subset of $M$ and $f|_A$ is a 1-1 immersion, then the approximation may be chosen in such a way that it agrees with $f$ on $A$.

Proof of Lemma A.7. Let $U$ be a $G$-equivariant collar neighborhood of $\partial M$. Using Proposition A.6 we identify $U$ with a subset $Z \subset T_{\partial M}M$ via a $G$-equivariant diffeomorphism $\psi: U \to Z$ that takes $\partial M$ to the zero section of $T_{\partial M}M$. The $G$-bundle $T_{\partial M}M$ is a subbundle of a trivial bundle $V^{d_0}$ for some $d_0 > 0$. Then $\iota_0$ can be extended to $U$ via the composition
\[ Z \xrightarrow{\psi} T_{\partial M}M \hookrightarrow V^{d_0} \times \partial M \xrightarrow{\text{id} \times \iota_0} V^{d_0} \times V^d. \]
Suppose $d_0 + d > 2 \dim M$, otherwise increase $d_0$. Extend the embedding of $Z$ to a smooth $G$-equivariant map $\tilde{\iota}: M \to V^{d_0} \times V^d$. Then, $\tilde{\iota}$ can be perturbed to an equivariant embedding by the Mostow–Palais Theorem (Theorem A.8). By the second part of this theorem, we can keep $\tilde{\iota}$ equal to $\iota$ in a neighborhood of $\partial M$. □

Appendix B. Permutohedra

B.1. The construction. We refer to [4, Chapter 1] for general properties of permutohedra.

Definition B.1. Choose an increasing sequence $S = \{s_1, \ldots, s_r\}$ of positive integers. The permutohedron $\Pi_S$ is the convex hull in $\mathbb{R}^r$ of the set of points
\[ (s_{\sigma(s_1)}, \ldots, s_{\sigma(s_r)}), \]
where $\sigma$ runs through all permutations of the set $S$.

We write $\Pi_{r-1}$ for the permutohedron in the special case when $S = \{1, \ldots, r\}$. The subscript is $r-1$ and not $r$, because $\dim \Pi_{r-1} = r-1$.

Example B.2. A permutohedron for $S = \{1, 2, 3\}$ is depicted in Figure 8.

To describe the permutohedron in a combinatorial way, set
\[ \tau_i = \sum_{j=1}^i s_j. \]
Lemma B.4 (see [4, Theorem 1.5.7]). The permutohedron $\Pi_S$ is given by the equation $\sum x_i = \tau_r$ and the set of inequalities

$$\sum_{i \in P} x_i \geq \tau_{|P|},$$

where $P$ runs through non-empty proper subsets of $S$.

From Lemma B.4 we deduce the following fact [15, Section 2].

Lemma B.5. For any subset $P$ of $\{1, \ldots, r\}$ such that $0 < |P| < r$, the intersection of $\Pi_S$ with the hyperplane $\sum_{i \in P} x_i = \tau_{|P|}$ defines a face of $\Pi_S$, which is diffeomorphic to $\Pi_{S_1} \times \Pi_{S_2}$, where $S_1 = \{s_1, \ldots, s_{|P|}\}$ and $S_2 = S \setminus S_{|P|} = \{s_{|P|+1}, \ldots, s_r\}$.

Example B.6 (Example B.2 continued). For $S = \{1, 2, 3\}$ and $P = \{1, 2\}$, the face is given by $x_1 + x_2 = 3$. This is a product of a one-dimensional permutohedron $\Pi_1$ spanned by $(1, 2)$ and $(2, 1)$ in the $(x_1, x_2)$-coordinates, and a zero-dimensional permutohedron $\Pi_{\{3\}}$ given by $\{x_3 = 3\} \subset \mathbb{R}$. For $P = \{3\}$ we obtain the opposite face of the hexagon. It is given by $\{x_3 = 1\}$, $\{x_1 + x_2 = 5\}$. See Figure 9.

Sketch of proof of Lemma B.5. If $P = \{a_1, \ldots, a_{|P|}\}$ and $\{1, \ldots, r\} \setminus P = \{b_1, \ldots, b_{r-|P|}\}$, the diffeomorphism takes the element $(x_1, \ldots, x_r)$ to an element $(x_{a_1}, \ldots, x_{a_{|P|}}) \times (x_{b_1}, \ldots, x_{b_{r-|P|}}) \in \mathbb{R}^{|P|} \times \mathbb{R}^{r-|P|}$. It is routine to verify, using Lemma B.4, that the image is indeed equal to $\Pi_{S_1} \times \Pi_{S_2}$. □

Definition B.7. The face corresponding to $P$ is denoted by $\Pi_{P,S\setminus P}$ (often simply $\Pi_P$) and it is called the face associated with subset $P$.

In the following corollary we use the notation of the proof of Lemma B.5.
Corollary B.8. For a subset $\mathcal{P} \subset \{1, \ldots, r\}$, $0 < |\mathcal{P}| < r$, the face $\Pi_{\mathcal{P}, S \setminus \mathcal{P}}$ is contained in the set

\[
\{x_{a_1}, \ldots, x_{a_{|\mathcal{P}|}}\} \leq s|\mathcal{P}|,
\]
\[
\{x_{b_1}, \ldots, x_{r-|\mathcal{P}|}\} \geq s|\mathcal{P}| + 1.
\]

From Lemma B.5, we can obtain an inductive description of codimension $k$ faces of $\Pi_S$. They correspond to partitions $p = (\mathcal{P}_1, \ldots, \mathcal{P}_{k+1})$ of $\{1, \ldots, r\}$ into $k + 1$ non-empty subsets $\mathcal{P}_1, \ldots, \mathcal{P}_{k+1}$. Each such face, denoted by $\Pi(\mathcal{P}_1, \ldots, \mathcal{P}_{k+1})$, or, in short $\Pi_p$, is a product of $(k + 1)$ permutohedra $\Pi_{S_1} \times \Pi_{S_2} \times \cdots \times \Pi_{S_{k+1}}$, where

\[
S_j = \{s_1: |\mathcal{P}_1| + \cdots + |\mathcal{P}_{j-1}| < t \leq |\mathcal{P}_1| + \cdots + |\mathcal{P}_j| \}.
\]

In particular if we set

\[
\partial_i \Pi_S = \bigsqcup_{P \subset S, |P| = i} \Pi_{(P, S \setminus P)},
\]

for $1 \leq i \leq r - 1$, $\Pi_S$ becomes an $(r - 1)$-dimensional $(r - 1)$-manifold.

Remark B.9. For consistency of the notation, we observe that the interior of $\Pi_S$ corresponds to the trivial partition $\{1, \ldots, r\}$ into a single subset.

B.2. Intersecting a permutohedron with a hyperplane. We describe the intersection of a permutohedron with hyperplanes given by sets of equations $\{x_{a_i} = x_{b_i}\}$. It turns out that this intersection is a lower dimensional permutohedron. The key statement in this section is Proposition B.14, which identifies the intersection of a permutohedron $\Pi_S \cap L$ with $\Pi_{n-2}$. The identification of Proposition B.14 is such that the combinatorial structure of the boundary is preserved. In order to spell correctly this control over the combinatorial structure, we need to introduce a few simple notions.

Definition B.10. Let $p = (\mathcal{P}_1, \ldots, \mathcal{P}_{k+1})$ be a partition of $\{1, \ldots, r\}$. A refinement of $p$ is a partition of $\{1, \ldots, r\}$ obtained from $p$ by dividing a member of $p$ into two sets.

Let $a, b \in \{1, \ldots, r\}$, $a < b$. Suppose $p = (\mathcal{P}_1, \ldots, \mathcal{P}_{k+1})$ is a partition of $\{1, \ldots, r\}$. A reduction of $p$ (with respect to $b$) is a partition $p'$ of $\{1, \ldots, r-1\}$ consisting of $\mathcal{P}_1', \ldots, \mathcal{P}_{k+1}'$ such that

* if $x \in \mathcal{P}_i$ and $x < b$, then $x \in \mathcal{P}_i'$;
* if $x \in \mathcal{P}_i$ and $x > b$, then $x - 1 \in \mathcal{P}_i'$.

An inverse operation is called an $(a, b)$-extension for $1 \leq a < b \leq r - 1$. It takes a partition $p' = (\mathcal{P}_1', \ldots, \mathcal{P}_{k+1}')$ of $\{1, \ldots, r-1\}$ and associates with it a partition $p = (\mathcal{P}_1, \ldots, \mathcal{P}_{k+1})$ of $\{1, \ldots, r\}$ such that

* if $x \in \mathcal{P}_i'$ and $x < b$, then $x \in \mathcal{P}_i$;
* if $x \in \mathcal{P}_i'$ and $x \geq b$, then $x + 1 \in \mathcal{P}_i$;
* $a$ and $b$ belong to the same subset $\mathcal{P}_i$.

Example B.11. Suppose $r = 7$ and $p = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ with $\mathcal{P}_1 = \{1, 3, 5\}$, $\mathcal{P}_2 = \{2, 6\}$ and $\mathcal{P}_3 = \{4, 7\}$. Let $b = 5$. A reduction of $p$ is the partition $p'$ given by $\mathcal{P}_1' = \{1, 3\}$, $\mathcal{P}_2' = \{2, 5\}$, $\mathcal{P}_3' = \{4, 6\}$. A $(3,5)$-extension of $p'$ is again $p$.

We also need the following consistency property of refinements and reductions.

Lemma B.12. Let $p$ be a partition such that $a$ and $b$ belong to the same subset of the partition. Let $p'$ be a $b$-reduction of $p$. Let $\bar{p}$ be a refinement of $p$ such that $a, b$ belong to the same subset of $\bar{p}$. Then a $b$-reduction of $\bar{p}$ is a refinement of $p'$, and all the refinements of $p'$ arise in this way.

The proof of Lemma B.12 in general case is left as an exercise, we provide an example.
Example B.13. Let \( p = (\{1, 2, 3\}, \{4, 5\}) \) and \( a = 2, b = 3 \). The refinements of \( p \) with 3 subsets are \((\{1\}, \{2, 3\}, \{4, 5\})\), \((\{2, 3\}, \{1\}, \{4, 5\})\), \((\{1, 2, 3\}, \{4\}, \{5\})\) and \((\{1, 2, 3\}, \{5\}, \{4\})\). The corresponding reductions of refinements are \((\{1\}, \{2\}, \{3, 4\})\), \((\{2\}, \{1\}, \{3, 4\})\), \((\{1, 2\}, \{3\}, \{4\})\) and \((\{1, 2\}, \{4\}, \{3\})\), which are precisely 3 subsets refinements of the reduction \( p' = (\{1, 2\}, \{3, 4\}) \).

Now we are ready to state the result on the intersections of permutohedra.

**Proposition B.14.** Let \( L \) be a hyperplane in \( \mathbb{R}^r \) given by \( \{x_a = x_b\} \) for some \( a \neq b \). Let \( \Pi_S \) be a permutohedron in \( \mathbb{R}^r \) for some increasing sequence \( S = (s_1, \ldots, s_r) \). Consider

\[
\Pi_L = \Pi_S \cap L.
\]

Then there exists a diffeomorphism \( \psi \) between \( \Pi_L \) and \( \Pi_{(s_1, \ldots, s_{r-1})} \). Moreover, if \( p \) is a partition of \( \{1, \ldots, r\} \) then

* if \( a, b \) do not belong to the same subset of the partition, then \( \Pi_p \cap L \) is empty;
* if \( a, b \) belong to the same subset of the partition, then \( \Pi_p \cap L \) is mapped by \( \psi \) diffeomorphically to \( \Pi_{p'} \subset \Pi_{(s_1, \ldots, s_{r-1})} \), where \( p' \) is the reduction of \( p \) with respect to \( b \).

**Example B.16.** Let \( r = 3 \) and \( L = \{x_1 = x_3\} \). The intersection of \( L \) with \( \Pi_S \) is an interval whose endpoints are \( (\frac{1 + s_2}{2}, s_2, \frac{1 + s_2}{2}) \) and \( (\frac{s_2 + s_3}{2}, s_1, \frac{2 + s_3}{2}) \). For \( S = \{1, 2, 3\} \) this yields the segment connecting \((1.5, 3, 1.5)\) with \((2.5, 1, 2.5)\). See Figure 10.

**Proof of Proposition B.14.** We will construct the isomorphism by induction, starting with the lowest codimension boundary. We need the following fact.

**Lemma B.17.** Suppose \( p = (\mathcal{P}_1, \ldots, \mathcal{P}_{k+1}) \) is a partition of \( \{1, \ldots, r\} \). Then \( \Pi_p \cap L \) is empty, unless there exists an index \( i \) such that \( a, b \in \mathcal{P}_i \).

**Proof.** We argue by contradiction. Suppose, \( a \in \mathcal{P}_i, b \in \mathcal{P}_j \) with \( i \neq j \). We have an inclusion \( \Pi_p \subset \Pi_{\mathcal{P}_i \setminus \mathcal{P}_j} \). By Corollary B.8 if \( x = (x_1, \ldots, x_r) \in \Pi_{\mathcal{P}_i \setminus \mathcal{P}_j} \), then \( x_a \leq s_{\mathcal{P}_i} \) and \( x_b \geq s_{\mathcal{P}_j} + 1 \). Since \( s_k \) is a strictly increasing sequence, we conclude that \( x_a < x_b \), hence \( \Pi_{\mathcal{P}_i \setminus \mathcal{P}_j} \cap L = \emptyset \). In particular \( \Pi_p \cap L = \emptyset \). \( \square \)

**Continuation of the proof of Proposition B.14.** Take a partition \( p \) of \( \{1, \ldots, r\} \) and consider the associated face \( \Pi_p \subset \Pi_S \). Assume that \( a, b \) belong to the same subset of the partition \( p \). We will construct a diffeomorphism \( \psi \) that takes \( \Pi_p \cap L \) to \( \Pi_{p'} \subset \Pi_{(s_1, \ldots, s_{r-1})} \). This is done inductively with respect to the length of the partition \( p \), starting from partitions of highest length.
If \( p \) has length \( r \), then necessarily all the subset entering the partition have one element, so \( a \) and \( b \) cannot belong to the same subset of the partition, so \( \Pi_p \cap L \) is empty.

Next suppose \( p \) is a length \( r - 1 \) partition such that \( a, b \) belong to the same subset \( \mathcal{P}_j \), for some \( 1 \leq j \leq k + 1 \). Then, all the other subsets of the partition \( p \) have a single element, say \( \mathcal{P}_i = \{ p_i \} \) for \( i \neq j \), where \( p_1, \ldots, p_{r-2} \) is a permutation of the set \( \{1, \ldots, r\} \setminus \{a, b\} \). By inductive application of Corollary \textit{B.8} we get a precise description of \( \Pi_p \). Namely, \( \Pi_p \) is given by

\[
\begin{align*}
  x_{p_i} &= i, \text{ if } i < j, \\
  x_a + x_b &= j + (j + 1), \\
  x_a, x_b &\in [j, j + 1], \\
  x_{p_i} &= i + 1, \text{ if } i > j.
\end{align*}
\]

The intersection of \( \Pi_p \) with \( x_a = x_b \) is clearly given by

\[
\begin{align*}
  x_{p_i} &= i, \text{ if } i < j, \\
  x_a &= x_b = j + \frac{1}{2}, \\
  x_{p_i} &= i + 1, \text{ if } i > j.
\end{align*}
\]

Now \( p' \) is a partition such that all its subsets have one element, namely

\[
\begin{align*}
  \mathcal{P}'_i &= \{ p_i \}, \text{ if } i \neq j, p_i < b, \\
  \mathcal{P}'_i &= \{ p_i - 1 \}, \text{ if } i \neq j, p_i > b, \\
  \mathcal{P}'_j &= \{ a \}.
\end{align*}
\]

Therefore, \( \Pi_{p'} \) is a single point, given by \( \{ x_{p_i} = s_i \} \) if \( p_i < b \), \( \{ x_{p_i-1} = s_i \} \) if \( p_i > b \), and \( x_a = s_j \). The map \( \psi \) takes this point to \( \Pi_{p'} \), because \( p' \) is a partition into one element subsets.

We pass now to the induction step. Suppose \( \psi \) has been constructed for all boundary components of codimension at least \( k + 1 \) and consider a partition \( p = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{k+1} \). Then, \( \Pi_p \) is a convex polytope whose boundary is a union of polytopes \( \Pi_p \) for all refinements \( \tilde{p} \) of \( p \). Therefore, the intersection \( \Pi_p \cap \{ x_a = x_b \} \) is a convex polytope whose boundary is the union of \( \Pi_p \cap \{ x_a = x_b \} \).

By Lemma \textit{B.17} for each such refinement \( \tilde{p} \), either \( \Pi_p \cap L \) is empty, or \( \tilde{p} \) admits an \((a, b)\)-reduction \( \tilde{p}' \). By the induction assumption, in the latter case the restriction of \( \psi \)

\[
\psi|_{\Pi_p \cap L} : \Pi_p \cap L \to \Pi_{p'}
\]

has already been constructed. This means that \( \psi \) restricted to the boundary of \( \Pi_p \cap L \) is an isomorphism onto the boundary of \( \Pi_{p'} \). Therefore, we can extend \( \psi \) to a diffeomorphism

\[
\psi : \Pi_p \cap L \to \Pi_{p'}.
\]

\( \square \)

\textit{Remark} B.18. The isomorphism \( \psi \) constructed in the proof of Proposition \textit{B.14} does not need to be affine. Two convex polytopes with the same combinatorics are not necessarily affine equivalent.

\textbf{Example B.19.} Consider the permutohedron \( \Pi_S \subset \mathbb{R}^4 \) for \( S = \{s_1, s_2, s_3, s_4\} \), and intersect it with the hyperplane \( \{ x_2 = x_3 \} \). There are six partitions of \( \{1, 2, 3, 4\} \) into 3 subsets, such that the intersection of the corresponding face \( \Pi_p \) with \( \{ x_2 = x_3 \} \) is non-empty. These are

\[
\begin{align*}
  &\{2, 3\}, \{1\}, \{4\}, \\
  &\{2, 3\}, \{4\}, \{1\}, \\
  &\{4\}, \{2, 3\}, \{1\}, \\
  &\{1\}, \{2, 3\}, \{4\}, \\
  &\{4\}, \{1\}, \{2, 3\} \text{ and} \\
  &\{1\}, \{4\}, \{2, 3\}.
\end{align*}
\]

For every such partition \( p \), the intersections of \( \Pi_p \) with \( x_2 = x_3 \) are single points with respective coordinates

\[
\begin{align*}
  &\left( s_3, \frac{1}{2}(s_1 + s_2), \frac{1}{2}(s_1 + s_2), s_3 \right), \\
  &\left( s_4, \frac{1}{2}(s_1 + s_2), s_3, s_4, \frac{1}{2}(s_2 + s_3), s_1 \right), \\
  &\left( s_1, \frac{1}{2}(s_2 + s_3), s_4 \right), \\
  &\left( s_2, \frac{1}{2}(s_3 + s_4), s_1 \right) \text{ and} \\
  &\left( s_1, \frac{1}{2}(s_3 + s_4), s_2 \right).
\end{align*}
\]

The \( 3 \)-reductions of these six partitions are

\[
\begin{align*}
  &\{2\}, \{1\}, \{3\}, \\
  &\{2\}, \{3\}, \{1\}, \\
  &\{3\}, \{2\}, \{1\}, \\
  &\{1\}, \{2\}, \{3\} \text{ and} \\
  &\{3\}, \{1\}, \{2\}.
\end{align*}
\]
Then $\Pi(p)(x)$ is non-empty. Consider one of these partitions, say $\Pi_p(\ldots)$. Next, we extend the map to the one-dimensional faces. There are again six partitions of the whole segment $(s_1, s_2, s_3)$. Namely

$\psi((s_3, \frac{1}{2}(s_1 + s_2), s_4)) = (s_2, s_1, s_3)$, 
$\psi((s_4, \frac{1}{2}(s_2 + s_3), s_1)) = (s_3, s_2, s_1)$, 
$\psi((s_2, \frac{1}{2}(s_3 + s_4), s_1)) = (s_2, s_3, s_1)$, 
$\psi((s_4, \frac{1}{2}(s_1 + s_2), s_3)) = (s_3, s_1, s_2)$, 
$\psi((s_1, \frac{1}{2}(s_2 + s_3), s_4)) = (s_1, s_2, s_3)$, 
$\psi((s_1, \frac{1}{2}(s_3 + s_4), s_2)) = (s_1, s_3, s_2)$.

Next, we extend the map to the one-dimensional faces. There are again six partitions of $\{1, 2, 3, 4\}$ into 2 subsets such that the intersection of the corresponding face $\Pi_p$ with $\{x_2 = x_3\}$ is non-empty. Consider one of these partitions, say $p = (\{1, 2, 3\}, \{4\})$, other cases are analogous. Then $\Pi_p$ is a permutohedron $\Pi_{(s_1, s_2, s_3)}$ in coordinates $(x_1, x_2, x_3)$ and the fourth coordinate is $x_4 = 4$. The intersection of $\Pi_p$ with $\{x_2 = x_3\}$ is given by the segment $\ell$ connecting $(s_3, \frac{1}{2}(s_1 + s_2), s_4)$ with $(s_1, \frac{1}{2}(s_2 + s_3), s_4)$; compare Example B.16. The map $\psi$ already takes $(s_3, \frac{1}{2}(s_1 + s_2), s_4)$ to $(s_2, s_1, s_3)$ and $(s_1, \frac{1}{2}(s_3 + s_4), s_4)$ to $(s_1, s_2, s_3)$, so it can be extended to the whole segment $\ell$ as an affine map. See Figure 11.

Finally, the two-dimensional face of $\Pi_S \cap \{x_2 = x_3\}$ corresponds to the reduction of the trivial partition $p = (\{1, 2, 3, 4\})$, that is to the trivial partition $p' = (\{1, 2, 3\})$.

Remark B.20. The reason why we use general $S$ in Example B.19 and not $S = (1, 2, 3, 4)$ is to avoid confusion. The partition $\{\{i\}, \{j\}, \{k\}\}$ for $(i, j, k)$ being a permutation of $(1, 2, 3)$ does not necessarily correspond to the point $(i, j, k)$. Leaving $S = (1, 2, 3, 4)$ would give only an illusion of transparency.

Now we state a result on intersections of $\Pi_S$ with more than one hyperplane.
Lemma B.25. Let trivial representation $R$ shows that the action of $\bar{\Pi}$ yields a representation isomorphic to $n/m$ a product of $\bar{\Pi} = \Pi_{[a_1, \ldots, a_{n-K}]}$, where $K = \sum(k_i - 1) = \text{codim} H$. The diffeomorphism takes the face $\Pi_0 \cap H$ to the face $\Pi'_{p'}$, where $\Pi'$ is the partition obtained by the subsequent reduction of $p$ with respect to $a_{12}, \ldots, a_{1k_1}, a_{2}, \ldots, a_{wk_w}$ and $p$ is a partition such that for any $j$ the elements $a_{j1}, \ldots, a_{jk_i}$ belong to the same subset of the partition.

Remark B.22. A little care is needed for defining a subsequent reduction of a partition $p$ with respect to a subset, because the result might depend on the order in which the reductions are performed. In Theorem B.21 the subsequent reduction with respect to a subset should be understood as a reduction with respect to the largest element of the subset, then with respect to the second largest element and so on.

Proof. The proof of Theorem B.21 follows by an inductive application of Proposition B.14 and it is left as an exercise. $\square$

Corollary B.23. Suppose $\sigma$ is a permutation of some of the coordinates of $\mathbb{R}^n$. Consider the permutohedron $\Pi_{n-1} \subset \mathbb{R}^n$. The fixed point set $\Pi_{n-1}'$ is again a permutohedron.

Proof. Suppose $\sigma$ is a product of cycles $(c_1, \ldots, c_{1k_1})(c_{21}, \ldots, c_{2k_2}) \cdots (c_{\ell1}, \ldots, c_{\ell k_\ell})$ with $c_{ij}$ pairwise different. Then the fixed point set of $\sigma$ on $\mathbb{R}^n$ is given by $x_{ci} = x_{c_{i,j+1}}$, for $i = 1, \ldots, \ell$, $j = 1, \ldots, k_i - 1$. The statement follows from Theorem B.21. $\square$

B.3. Permutohedra as $G$-manifolds. Consider the $(n-1)$-dimensional permutohedron $\Pi_{n-1} \subset \mathbb{R}^n$ and choose a permutation $\sigma \in S_n$ such that $\sigma^m = \text{id}$. We can define an action of a cyclic group $\mathbb{Z}_m$ on $\mathbb{R}^n$ in terms of $\sigma$:

$$\bar{\sigma}(x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

Since $\Pi_{n-1}$ is an invariant subset, $\mathbb{Z}_m$ acts on $\Pi_{n-1}$. From now on, we will assume that $\sigma$ is a product of $n/m$ disjoint cycles of length $m$. This implies that the representation obtained in this way is isomorphic to $\mathbb{R}[\mathbb{Z}_m]^{n/m}$.

Permutohedron $\Pi_{n-1}$ is contained in the affine hyperplane

$$L = \left\{ (x_1, x_2, \ldots, x_n): \sum_{i=1}^{n} x_i = \frac{n(n + 1)}{2} \right\},$$

which is invariant under $\mathbb{Z}_m$. Translating $L$ to the hyperplane

$$L_0 = \left\{ (x_1, x_2, \ldots, x_n): \sum_{i=1}^{n} x_i = 0 \right\} \cong \mathbb{R}[\mathbb{Z}_m]^{n/m} - \mathbb{R}$$

shows that the action of $\bar{\sigma}$ restricted to $L$ yields a representation isomorphic to $\mathbb{R}[\mathbb{Z}_m]^{n/m} - \mathbb{R}$.

Remark B.24. Recall that $\mathbb{R}[\mathbb{Z}_m]^{n/m} - \mathbb{R}$ denotes the orthogonal complements of a one-dimensional trivial representation $\mathbb{R}$ inside $\mathbb{R}[\mathbb{Z}_m]^{n/m}$.

Lemma B.25. Let $\bar{\sigma}$ be as above.

1. At every interior point $x$ of $\Pi_{n-1}$ the tangent representation $T_x \Pi_{n-1}$ is isomorphic to $(\mathbb{R}[\mathbb{Z}_m]^{n/m} - \mathbb{R})|_{(\mathbb{Z}_m)_x}$, where $(\mathbb{Z}_m)_x$ denotes the isotropy group at $x \in \Pi_{n-1}$.
2. If $P \subset \{1, 2, \ldots, n\}$ and $\Pi_P$ denotes the face of $\Pi_{n-1}$ defined in Lemma B.3 then $\bar{\sigma}(\Pi_P) = \Pi_{\sigma(P)}$.
3. $\bar{\sigma}$ restricted to $\Pi_{n-1}$ is an $n$-map, i.e., $\bar{\sigma}(\partial_i \Pi_{n-1}) = \partial_i \Pi_{n-1}$. 

Proof. First statement follows readily because $\Pi$ since $\overline{\Pi}$

If we identify $\Pi_\mathcal{P}$ with $\Pi_{\mathcal{P}_{-1}} \times \Pi_{n-|\mathcal{P}|-1}$ as in Lemma 3B.5, then

$$\overline{\sigma}|_{\Pi_\mathcal{P}} = (\tau_1 \circ \sigma|_{\mathcal{P}} \circ \tau_1^{-1}) \times (\tau_2 \circ \sigma|_{\{1, 2, \ldots, n\} \setminus \mathcal{P} \circ \tau_2^{-1})].$$

Proof. First statement follows readily because $\Pi_{n-1} \subset L \cong \mathbb{R}[\mathbb{Z}_m]^n/m$ has nonempty interior.

In order to prove the second statement recall that $\Pi_{\mathcal{P}_{\setminus \{1, 2, \ldots, n\}}} = \Pi_{n-1} \cap \partial L_\mathcal{P}$, where

$$L_\mathcal{P} = \left\{ (x_1, \ldots, x_n) : \sum_{i \in |\mathcal{P}|} x_i \geq \frac{|\mathcal{P}|(|\mathcal{P}| + 1)}{2} \right\}.$$ 

Since $\overline{\sigma}(L_\mathcal{P}) = L_{\sigma(\mathcal{P})}$, the statement (2) follows.

Next, recall that by the definition of $\partial_i$

$$\partial_i \Pi_{n-1} = \bigcup_{\mathcal{P} \subset \{1, 2, \ldots, n\} \atop |\mathcal{P}| = i} \Pi_{\mathcal{P}_{\setminus \{1, 2, \ldots, n\} \setminus \mathcal{P}}}.$$ 

The fact that $|\sigma(\mathcal{P})| = |\mathcal{P}|$ completes the proof of the third statement.

In order to prove the last assertion, notice that if $\mathcal{P}$ is invariant under $\sigma$, then its complement is also invariant. \qed

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