On Hermitian Eisenstein Series of Degree 2

by

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Abstract. We consider the Hermitian Eisenstein series $E_{k}^{(K)}$ of degree 2 and weight $k$ associated with an imaginary-quadratic number field $K$ and determine the influence of $K$ on the arithmetic and the growth of its Fourier coefficients. We find that they satisfy the identity $E_{4}^{(K)}^{2} = E_{8}^{(K)}$, which is well-known for Siegel modular forms of degree 2, if and only if $K = \mathbb{Q}(\sqrt{-3})$. As an application, we show that the Eisenstein series $E_{k}^{(K)}$, $k = 4, 6, 8, 10, 12$ are algebraically independent whenever $K \neq \mathbb{Q}(\sqrt{-3})$. The difference between the Siegel and the restriction of the Hermitian to the Siegel half-space is a cusp form in the Maass space that does not vanish identically for sufficiently large weight; however, when the weight is fixed, we will see that it tends to 0 as the discriminant tends to $-\infty$. Finally, we show that these forms generate the space of cusp forms in the Maass Spezialschar as a module over the Hecke algebra as $K$ varies over imaginary-quadratic number fields.

Keywords: Hermitian Eisenstein series, Siegel Eisenstein series, Maass space

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1 Introduction

Eisenstein series are the most common examples of modular forms in several variables. In the case of Hermitian modular forms associated with an imaginary-quadratic number field $\mathbb{K}$, they were introduced by H. Braun [2]. In this paper we consider Hermitian Eisenstein series of degree 2. Its Fourier expansion is determined by the Maaß condition and has been worked out explicitly (cf. [16], [10]).

This knowledge leads to new insights on the influence of $\mathbb{K}$ on the arithmetic and the growth of the Fourier coefficients. We will demonstrate that the Eisenstein series $E^k_{\mathbb{K}}$ satisfy the identity $E^k_{\mathbb{K}} = E^8_{\mathbb{K}}$, whose analogue for Siegel modular forms of degree 2 is well known, if and only if $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$. This allows us to show that the Eisenstein series $E^k_{\mathbb{K}}$, $k = 4, 6, 8, 10, 12$, are algebraically independent whenever $\mathbb{K} \neq \mathbb{Q}(\sqrt{-3})$.

Finally we consider the difference between $E^k_{\mathbb{K}}$ restricted to the Siegel half-space and the Siegel Eisenstein series of weight $k$. This is a Siegel cusp form in the Maaß space. When the weight $k$ is fixed, its limit is 0 as the discriminant of $\mathbb{K}$ tends to $-\infty$. On the other hand, it does not vanish identically whenever the weight is sufficiently large. Moreover the vanishing of the above difference can be characterized by the vanishing of a Shimura lift. This allows us to show that the subspace of cusp forms in the Maaß space is generated by these restrictions as a module over the Hecke algebra, when $\mathbb{K}$ varies over all imaginary-quadratic number fields.

2 An identity in weight 8

The Hermitian half-space $\mathbb{H}_2$ and the Siegel half-space $S_2$ of degree 2 are given by

$$\mathbb{H}_2 := \left\{ Z \in \mathbb{C}^{2 \times 2}; \frac{1}{2I}(Z - Z^t) > 0 \right\} \supset S_2 := \left\{ Z \in \mathbb{H}_2; Z = Z^t \right\}.$$

Throughout the paper we let $\mathbb{K}$ be an imaginary-quadratic number field of discriminant $\Delta = \Delta_{\mathbb{K}}$ with ring of integers $\mathcal{O}_{\mathbb{K}}$ and inverse different $\mathcal{O}^\#_{\mathbb{K}} = \mathcal{O}_{\mathbb{K}}/\sqrt{\Delta_{\mathbb{K}}}$. If $D$ is a fundamental discriminant, let $\chi_D$ denote the associated Dirichlet character; in particular, $\chi_{\mathbb{K}} = \chi_{\Delta}$. The Hermitian modular group of degree 2 is

$$\Gamma_2^{(\mathbb{K})} := \left\{ M \in \mathcal{O}_{\mathbb{K}}^{4 \times 4}; M^tJM = J, \det M = 1 \right\}, \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\Gamma_2 := \Gamma_2^{(\mathbb{K})} \cap \mathbb{R}^{4 \times 4}$$

is the Siegel modular group of degree 2. The space $\mathcal{M}_k(\Gamma_2^{(\mathbb{K})})$ of Hermitian modular forms of weight $k$ consists of all holomorphic functions $F : \mathbb{H}_2 \to \mathbb{C}$ satisfying

$$F((AZ + B)(CZ + D)^{-1}) = \det(CZ + D)^kF(Z) \quad \text{for all} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2^{(\mathbb{K})}.$$
Any such $F$ has a Fourier expansion of the form

$$F(Z) = \sum_{T \in \Lambda_2, T \geq 0} \alpha_F(T) e^{2\pi i \text{trace}(TZ)},$$

where

$$\Lambda_2 = \left\{ T = \begin{pmatrix} n & t \\ m & \ast \end{pmatrix} : m, n \in \mathbb{N}_0, t \in \mathfrak{c}_K^2 \right\}.$$ 

If $\varepsilon(T) := \max\{\ell \in \mathbb{N}; \frac{1}{2} T \in \Lambda_2\}$ for $T \neq 0$, we can define the Hermitian Eisenstein series of even weight $k \geq 4$ due to [16] and [10] as a Maaß lift via

$$E^{(K)}_k(Z) = 1 + \sum_{0 \neq T \in \Lambda_2, T \geq 0} \sum_{d|\varepsilon(T)} d^{k-1} \alpha_k^* (|\Delta| \det T/d^2) e^{2\pi i \text{trace}(TZ)}, \quad Z \in \mathbb{H}_2,$$

where $\alpha_k^* = \alpha_k^*|_{\Delta}$ is given by

$$\alpha_k^*(\ell) = \begin{cases} 0, & \text{if } \ell \neq 0, a_\Delta(\ell) = 0, \\
-2k/B_k, & \text{if } \ell = 0, \\
r_{k,\Delta} \sum_{t|\ell} \varepsilon_{t,\ell}(\ell/t) k^{-2}, & \text{if } \ell > 0, a_\Delta(\ell) \neq 0,
\end{cases}$$

where

$$r_{k,\Delta} = -4k(k-1)/B_k B_{k-1,\chi} > 0,$$

$$\varepsilon_{t,\ell}^{(\Delta)} = \frac{1}{a_\Delta(\ell)} \sum_{D_1 D_2 = \Delta \; D_i \text{ fund. discr.}} \chi_{D_1}(t) \chi_{D_2}(-\ell/t),$$

$$a_\Delta(\ell) = \sharp \{ u : \mathfrak{o}_K^2/\mathfrak{o}_K; \Delta u \equiv \ell \mod \Delta \} = \prod_{j=1}^{r} (1 + \chi_j(-\ell)),$$

if $\Delta = \Delta_1 \cdots \Delta_r$ is the decomposition into prime discriminants and $\chi_j = \chi_{\Delta_j}$. If $\ell \in \mathbb{N}$ and $a_\Delta(\ell) > 0$, then any $t | \ell$ satisfies

$$\varepsilon_{t,\ell}^{(\Delta)} = \prod_{j=1}^{r} \chi_j(t) + \chi_j(-\ell/t) / 1 + \chi_j(-\ell)$$

$$= \prod_{j: \gcd(t,\Delta_j) = 1} \chi_j(t) \cdot \prod_{j: \gcd(t,\Delta_j) > 1} \chi_j(-\ell/t).$$

If $k > 4$ is even, we have the absolutely convergent series

$$E^{(K)}_k(Z) = \sum_{(A B; C D) \in \Gamma_2^{(K)}} \det(CZ + D)^{-k}, \quad Z \in \mathbb{H}_2.$$
We derive a first result on the growth and the arithmetic of the Fourier coefficients depending on $K$.

**Theorem 1.** Let $\mathbb{K}$ be an imaginary-quadratic number field and let $k \geq 4$ be even. Then

a) $\varepsilon_{t,\ell}^{(\Delta)} = \chi_{D_t}(t)\chi_{D_t}(-\ell/t)$ holds for all $t, \ell \in \mathbb{N}$, $a_\Delta(\ell) > 0$, where $D_t, D'_t$ are fundamental discriminants satisfying $D_tD'_t = \Delta$, $|D_t| = \gcd(t^\infty, \Delta)$.

b) $0 < r_{k,\Delta}(2 - \zeta(k - 2))\ell^{k-2} \leq \alpha_{k,\Delta}(\ell) \leq r_{k,\Delta}\zeta(k - 2)\ell^{k-2}$ holds for all $\ell \in \mathbb{N}$ with $a_\Delta(\ell) > 0$.

c) One has

\[
0 < \frac{(2\pi)^{2k-1}}{\zeta(k-1)\zeta(k)(k-2)!(k-1)!} \cdot \frac{1}{|\Delta|^{k-3/2}} \leq r_{k,\Delta} \leq \frac{(2\pi)^{2k-1}}{(2 - \zeta(k - 1))\zeta(k)(k-2)!(k-1)!} \cdot \frac{1}{|\Delta|^{k-3/2}}.
\]

d) If $\ell_1, \ell_2 \in \mathbb{N}$ are coprime with $a_\Delta(\ell_j) > 0$ and $\gcd(\ell_1\ell_2, \Delta) = 1$, then

\[
\alpha_{k,\Delta}(\ell_1) \cdot \alpha_{k,\Delta}(\ell_2) = r_{k,\Delta} \cdot \alpha_{k,\Delta}(\ell_1\ell_2).
\]

**Proof.** We observe that

\[
\text{sgn } (B_kB_{k-1,\chi}) = \chi_{\mathbb{K}}(-1) = -1,
\]

(5)

\[
|B_k| = \frac{2k!\zeta(k)}{(2\pi)^k},
\]

(6)

\[
\frac{2(k-1)!|\Delta|^{k-3/2}}{(2\pi)^{k-1}}(2 - \zeta(k - 1)) \leq |B_{k-1,\chi}| \leq \frac{2(k-1)!|\Delta|^{k-3/2}}{(2\pi)^{k-1}}\zeta(k - 1),
\]

\[
\varepsilon_{t,\ell}^{(\Delta)} = \chi_{\mathbb{K}}(t), \text{ if } \gcd(t, \Delta) = 1.
\]

Then the claim follows from (2), (3) and (4).
Inserting estimates for the Riemann zeta function we get

\[
\begin{align*}
\frac{8792}{\sqrt{|\Delta|}} & \leq \alpha_4^4(|\Delta|) \leq \frac{61362}{\sqrt{|\Delta|}}, \\
\frac{181995}{\sqrt{|\Delta|}} & \leq \alpha_6^6(|\Delta|) \leq \frac{231109}{\sqrt{|\Delta|}}, \\
\frac{251164}{\sqrt{|\Delta|}} & \leq \alpha_8^8(|\Delta|) \leq \frac{264410}{\sqrt{|\Delta|}}, \\
\frac{99324}{\sqrt{|\Delta|}} & \leq \alpha_{10}^{10}(|\Delta|) \leq \frac{100541}{\sqrt{|\Delta|}}, \\
\frac{15720}{\sqrt{|\Delta|}} & \leq \alpha_{12}^{12}(|\Delta|) \leq \frac{15768}{\sqrt{|\Delta|}}.
\end{align*}
\]

(7)

**Corollary 1.** Let \( \mathbb{K} \) be an imaginary-quadratic number field. Then

\[ E_4^{(\mathbb{K})}^2 = E_8^{(\mathbb{K})} \]

holds if and only if \( \mathbb{K} = \mathbb{Q}(\sqrt{-3}) \).

**Proof.** If \( \mathbb{K} = \mathbb{Q}(\sqrt{-3}) \), then the identity follows from [4], Theorem 6. Suppose \(|\Delta| \geq 4\). The Fourier coefficient of \( I \) in \( E_4^{(\mathbb{K})}^2 - E_8^{(\mathbb{K})} \) is

\[
2\alpha_4^4(|\Delta|) + 2\alpha_4^4(0)^2 - \alpha_8^8(|\Delta|) \geq \frac{17584}{\sqrt{|\Delta|}} + 115200 - \frac{264410}{\sqrt{|\Delta|}} > 0,
\]

according to (7), whenever \(|\Delta| \geq 5\). For \( \Delta = -4 \) a direct computation shows that this Fourier coefficient is nonzero. \( \square \)

Clearly the restriction of \( E_4^{(\mathbb{K})}^2 - E_8^{(\mathbb{K})} \) to \( S_2 \) vanishes because \( \dim \mathcal{M}_8(\Gamma_2) = 1 \). If \( \mathbb{K} = \mathbb{Q}(\sqrt{-1}) \) then [4], Theorem 10, yields

\[ E_4^{(\mathbb{K})}^2 - E_8^{(\mathbb{K})} = c \phi_4^2 \quad \text{for} \quad c = 230400/61. \]

(8)

**Corollary 2.** Let \( \mathbb{K} = \mathbb{Q}(\sqrt{-1}) \). Then the graded ring of symmetric Hermitian modular forms with respect to the maximal discrete extension of \( \Gamma_2^{(\mathbb{K})} \) is the polynomial ring in

\[ E_k^{(\mathbb{K})}, \quad k = 4, 6, 8, 10, 12. \]

**Proof.** [4], Corollary 9 and (8). \( \square \)

Let \( e_{k,m}^{(\mathbb{K})} : \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) denote the \( m \)-th Fourier-Jacobi coefficient of \( E_k^{(\mathbb{K})} \) belonging to \( J_{k,m}(\sigma_{\mathbb{K}}) \), the space of Hermitian Jacobi forms of weight \( k \) and index \( m \) (cf. [11]). Note that the first Fourier-Jacobi coefficient of \( E_4^{(\mathbb{K})}^2 - E_8^{(\mathbb{K})} \) vanishes on the submanifold
\{(\tau, z, z); \tau \in \mathbb{H}_1, z \in \mathbb{C}\}$. Let \(M_k(\Gamma_1)\) stand for the space of elliptic modular forms of weight \(k\). Then the result of Eichler-Zagier \([6]\), Theorem 3.5, yields

**Corollary 3.** Let \(K\) be an imaginary-quadratic number field, \(K \neq \mathbb{Q}(\sqrt{-3})\). If \(k \geq 4\) is even, the mapping

\[
M_{k-4}(\Gamma_1) \times M_{k-6}(\Gamma_1) \times M_{k-8}(\Gamma_1) \to J_{k,1}(O_K),
\]

\[
(f, g, h) \mapsto f e_{4,1}^{(K)} + g e_{6,1}^{(K)} + h e_{8,1}^{(K)}
\]

is an injective homomorphism of the vector spaces, which turns out to be an isomorphism for \(K = \mathbb{Q}(\sqrt{-1})\).

**Proof.** Note that the dimensions on both sides are equal to \([\frac{k}{4}]\) due to \([4]\), Theorem 3, whenever \(K = \mathbb{Q}(\sqrt{-1})\).

We give a precise description of \(e_{k,1}^{(K)}\).

**Lemma 1.** Let \(K\) be an imaginary-quadratic number field and let \(k \geq 4\) be even. Then the first Fourier-Jacobi coefficient of \(E_k^{(K)}\) is given by

\[
e_{k,1}^{(K)}(\tau, z, w) = \frac{1}{2} \sum_{c,d \in \mathbb{Z}} \sum_{\lambda \in O_K} (c \tau + d)^{-k} \exp \left(2\pi i \left[ (a \tau + b)\lambda \bar{X} - czw + (z\lambda + w\bar{X}) \right] / (c \tau + d) \right).
\]

**Proof.** Proceed in the same way as Eichler/Zagier \([6]\) in § 6. One knows that \(E_k^{(K)}\) is an eigenform under all Hecke operators \(T_2(p)\) for all inert primes \(p\) from \([16]\). On the other hand the Jacobi-Eisenstein series is an eigenform under

\[
T_f(p) = \Gamma_f^{(K)} \diag(1, p, p^2, p) \Gamma_f^{(K)},
\]

\[
\Gamma_f^{(K)} = \left\{ \begin{pmatrix} \ast & \ast & \ast & \ast \\ 0 & 0 & 0 & 1 \\ \ast & \ast & \ast & \ast \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_2^{(K)} \right\}, \quad p \in \mathbb{P} \ \text{inert}.
\]

As in both cases the constant Fourier coefficient is 1, the claim follows.

**Remark 1.** a) If \(E_k\) denotes the normalized Eisenstein series of weight \(k\) for some group \(\Gamma\), then the identity \(E_4^2 = E_8\) is well-known for elliptic modular forms and Siegel modular forms of degree 2 (cf. \([19], [4]\)). But it also holds for modular forms of degree 2 with respect to the Hurwitz order (cf. \([15]\), p. 89) as well as the integral Cayley numbers (cf. \([5]\)), i.e. for \(O(2, 6)\) and \(O(2, 10)\). Hence this identity is a hint at the influence of the arithmetic of the attached number field on the modular forms.

b) Note that \(E_{k}^{(K)}\) is a modular form with respect to the maximal discrete extension of \(\Gamma_2^{(K)}\) (cf. \([17], [22]\)).

c) It follows from \([4]\) that the Fourier coefficients \(\varepsilon_{t,\ell}^{(\Delta)}\) are also multiplicative in \(\Delta\), i.e.

\[
\varepsilon_{t,\ell}^{(\Delta)} = \varepsilon_{t,\ell}^{(\Delta_1)} \cdot \cdots \cdot \varepsilon_{t,\ell}^{(\Delta_r)}.
\]
Due to Corollary 3 the dimension of the Maaß space in \( \mathcal{M}_k(\Gamma_2) \) is \( \geq \left\lfloor \frac{k}{4} \right\rfloor \) for \( \mathbb{K} \neq \mathbb{Q}(\sqrt{-3}) \) and equal to \( \left\lfloor \frac{k+2}{6} \right\rfloor \) for \( \mathbb{K} = \mathbb{Q}(\sqrt{-3}) \) (cf. [4]), if \( k \in \mathbb{N} \) is even.

e) If \( k > 4 \) is even we can improve the estimate from [10] slightly for all \( T \in \Lambda_2 \), \( T > 0 \):

\[
\frac{(2\pi)^{2k-1}}{(k-2)!(k-1)!} \cdot \frac{2 - \zeta(k-2)}{\zeta(k-1)\zeta(k)} \cdot \frac{1}{\sqrt{\Delta}} \cdot (\det T)^{k-2} \leq \alpha_k(T)
\]

\[
\leq \frac{(2\pi)^{2k-1}}{(k-2)!(k-1)!} \cdot \frac{\zeta(k-3)\zeta(k-2)}{(2 - \zeta(k-1))\zeta(k)} \cdot \frac{1}{\sqrt{\Delta}} \cdot (\det T)^{k-2}.
\]

3 Algebraic independence

It is well-known that there are exactly 5 algebraically independent Hermitian modular forms. In this section we explicitly determine algebraically independent Eisenstein series.

We define the Siegel Eisenstein series \( S_k \) of degree 2 for even \( k \geq 4 \)

\[
S_k(Z) = \sum_{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2 : (CZ + D)^{-k}, \ Z \in \mathbb{S}_2,} \det(CZ + D)^{-k}
\]

and denote its Fourier coefficients by \( \gamma_k(R) \). Clearly \( E_k(\mathbb{K})|_{\mathbb{S}_2} = S_k \) holds for \( k = 4, 6, 8 \).

The following Fourier coefficients of \( S_k \) were computed by Igusa [12] and are given by

| \( f \) | \( \gamma_f \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) | \( \gamma_f \left( \begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix} \right) \) | \( \gamma_f \left( \begin{smallmatrix} 1/2 & 1/2 \\ 0 & 1 \end{smallmatrix} \right) \) |
|---|---|---|---|
| \( S_4 \) | 240 | 30240 | 13440 |
| \( S_6 \) | -504 | 166320 | 44352 |
| \( S_4S_6 \) | -264 | -45360 | 57792 |
| \( X_{12} \) | 65520 | 402585120 | 39957120 |

\( X_{12} := 441S_4^3 + 250S_6^2 \).

Lemma 2. Let \( \mathbb{K} \) be an imaginary quadratic number field. Then

\[
F_{10}^{(\mathbb{K})} := E_{10}^{(\mathbb{K})} - E_4^{(\mathbb{K})}E_6^{(\mathbb{K})}, \quad F_{12}^{(\mathbb{K})} := E_{12}^{(\mathbb{K})} - \frac{441}{691}E_4^{(\mathbb{K})^3} - \frac{250}{691}E_6^{(\mathbb{K})^2}
\]

are Hermitian cusp forms of weight 10 resp. 12, whose restrictions to \( \mathbb{S}_2 \) do not vanish identically.

Proof. If \( F = F_{10}^{(\mathbb{K})} \), \( F_{12}^{(\mathbb{K})} \), then [11] - [3] show that \( \alpha_F(T) = 0 \) for all \( T \in \Lambda_2 \), \( \det T = 0 \).
Hence $F$ is a cusp form. The Fourier coefficients $\beta_F(R)$ of $F|_{S_2}$ are given by

$$
\beta_F(R) = \sum_{T \in \Lambda_2, T \geq 0 \atop T+T=2R} \alpha_f(T).
$$

Note that $E_k^{(K)}|_{S_2} = S_k$ for $k = 4, 6, 8$. If $k = 10$, then Theorem I and the above table yield $\beta_F(I) > 0$.

If $k = 12$, then for $\Delta \neq -4$

$$
\beta_F(I) = \alpha_{12}^*(\Delta) + 2 \sum_{1 \leq j < \sqrt{|\Delta|}} \alpha_{12}^*(|\Delta| - j^2) - \frac{402\,585\,120}{691}
$$

(9)

$$
\leq 15\,768 \left( \frac{1}{\sqrt{|\Delta|}} + 2 \right) - \frac{402\,585\,120}{691} < 0
$$

by means of Theorem I. If $\Delta = -4$ then

$$
\beta_F(I) = -\frac{20\,026\,621\,440\,000}{34\,910\,011} < 0. \quad \square
$$

A simple consequence is

**Theorem 2.** Let $K$ be an imaginary-quadratic number field.

a) The graded ring of Siegel modular forms of even weight is generated by

$$
E_k^{(K)}|_{S_2}, \quad k = 4, 6, 10, 12.
$$

b) If $K \neq \mathbb{Q}(\sqrt{-3})$ the Eisenstein series

$$
E_k^{(K)} \quad k = 4, 6, 8, 10, 12
$$

are algebraically independent.

**Proof.** a) Use Lemma 2

b) Apply a) as well as Corollary II. \(\square\)

If $K = \mathbb{Q}(\sqrt{-3})$, we already know the graded ring of Hermitian modular forms (cf. IV, Theorem 6).

**Corollary 4.** If $K = \mathbb{Q}(\sqrt{-3})$ the graded ring of symmetric Hermitian modular forms of even weight with respect to $\Gamma_2^{(K)}$ is the polynomial ring in

$$
E_k^{(K)} \quad k = 4, 6, 10, 12, 18.
$$
Proof. Use \textsuperscript{11}, Theorem 6, and show that
\[ E_{18}^{(K)}, E_{12}^{(K)}, E_6^{(K)}, E_{10}^{(K)}, E_4(E_{18}^{(K)})^2, E_6(E_{12}^{(K)})^3, E_6(E_6^{(K)})^3, E_6(E_4(E_6^{(K)})^3) \]
are linearly independent by calculating a few Fourier coefficients using (1) - (4). \hfill \square

Remark 2. a) It follows from the results of \textsuperscript{11} that there is a non-trivial cusp form \( f_4^{(K)} \) of weight 4 for all discriminants except \( \Delta_K = -3, -4, -7, -8, -11, -15, -20, -23 \). As its restriction to the Siegel half-space vanishes identically, one may replace \( E_8^{(K)} \) by \( f_4^{(K)} \) in these cases in Theorem 2 b).

b) Using Theorem 2 resp. Corollary \textsuperscript{11} resp. part a) we can construct a non-trivial skew-symmetric Hermitian modular form of weight 44 resp. 54 resp. 40 by an application of a suitable differential operator (cf. \textsuperscript{11}).

4 A Siegel cusp form

We consider
\[ G_k^{(K)}(Z) : = E_k^{(K)}(Z) - S_k(Z), \quad Z \in \mathbb{S}_2; \]
(10)
where \( (c_1, c_2, c_3, c_4)^T = (c_4 - c_2, -c_3, c_1) \). For \( K = \mathbb{Q}(\sqrt{-1}) \) this modular form was introduced by Nagaoka and Nakamura \textsuperscript{20}.

Theorem 3. Let \( K \) be an imaginary-quadratic number field. If \( k \geq 10 \) is even, then \( G_k^{(K)} \) is a Siegel cusp form of degree 2 and weight \( k \) in the Maass space.

a) One has
(11)
\[ \lim_{|\Delta_K| \to \infty} G_k^{(K)}(Z) = 0 \text{ for all } Z \in \mathbb{S}_2. \]

b) \( G_k^{(K)} \not\equiv 0 \) holds whenever \( k \geq \frac{10}{3} |\Delta_K| + 1 \).

Proof. \( G_k^{(K)} \) is a cusp form, as all its Fourier-coefficients \( \beta_k(R) \) with \( \det R = 0 \) vanish due to \textsuperscript{11} - \textsuperscript{11}. It belongs to the Maass space by virtue of \textsuperscript{16} and \textsuperscript{10}.

a) Because \( \dim M_k(\Gamma_2) \leq 1 \) for \( k < 10 \) (cf. \textsuperscript{17}), we have \( G_k^{(K)} \equiv 0 \) for \( k = 4, 6, 8 \). Let \( k \geq 10 \) and
\[ S_1 = \begin{pmatrix} 0 & 0 & P \\ 0 & -S & 0 \\ P & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 2 & 0 \\ 0 & |\Delta|/2 \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} 2 & 1 \\ 1 & (|\Delta| + 1)/2 \end{pmatrix}, \]
if $\Delta$ is even resp. odd. Then the explicit isomorphism in \cite{17} yields

$$G_k^{(K)}(Z) = \sum_{h \in \mathbb{Z}^k, h_{4} \geq 1 \atop h^s S_1 h = 0, \gcd(S_1 h) = 1} \left( -h_1 \det Z + \text{trace} \left( \begin{pmatrix} h_2 & g \\ \overline{g} & h_5 \end{pmatrix} \cdot Z \right) \right) + h_6, \quad \sum_{h \in \mathbb{Z}^6, h_{4} \geq 1 \atop h^s S_1 h = 0, \gcd(S_1 h) = 1}$$

where $g = h_3 + h_4 \sqrt{\Delta}/2$ resp. $g = h_3 + h_4(1 + \sqrt{\Delta})/2$. By virtue of \cite{15}, V.2.5, it suffices to show that the series in \eqref{12} over the absolute values at $Z = i I$ tends to 0, as $|\Delta| \to \infty$. Hence we consider

$$I_{\Delta} := \sum_{h} |h_1 + h_6 + i(h_2 + h_5)|^{k}$$

$$= \sum_{h} \left( h_1^2 + h_6^2 + h_2^2 + h_3^2 + (h_3 h_4) S \left( \begin{array}{c} h_3 \\ h_4 \end{array} \right) \right)^{-k/2}$$

in view of $h^s S_1 h = 0$. As $h_4 \geq 1$ we get

$$I_{\Delta} \leq \sum_{h} \left( \frac{|\Delta| - 3}{2} + \frac{1}{2} h^s h \right)^{-k/2} \leq (|\Delta| - 3)^{-k/4} \cdot \sum_{h} (h^s h)^{-k/4}.$$
for $\Delta \neq -4$, if we use (5) and (6). Then Stirling’s formula leads to

$$\beta_k(I) \geq r_k, \Delta|\Delta|^{k-3/2} \left( \frac{2 - \zeta(k-2)}{\sqrt{|\Delta|}} - \frac{e^{1/6(k-1)}\zeta(k-1)^2}{\zeta(2k-2)} \frac{\pi}{k-1} \right),$$

as $k \geq 10$. The expression in the bracket is positive because

$$k - 1 \geq \frac{1}{10} |\Delta| > \pi \left( \frac{e^{1/54}\zeta(9)^2}{2 - \zeta(8)} \right)^2 |\Delta|.$$

If $\Delta = -4$ we compute $\beta_k(R)$, $R = \left( \frac{1}{1/2}, 1/2 \right)$, and proceed in the same way (cf. [20]).

Use the description of $\gamma_k(R)$ by means of Cohen’s function in [1], p. 80. A comparison of the Fourier coefficients and the Hecke bound for the Fourier coefficients of cusp forms yield the following asymptotic.

**Corollary 5.** If $k \geq 4$ is even and $N \in \mathbb{N}$, $N \equiv 0, 3 \bmod 4$ one has

$$H(k-1, N) \sim \sum_{\substack{|j| \leq |\Delta|N \mod 2 \\
j \equiv \Delta N \bmod 2}} \alpha_k^* \Delta \left( (|\Delta|N - j^2)/4 \right)$$

as $N \to \infty$ for any imaginary-quadratic number field $\mathbb{K}$.

**Remark 3.** a) We know that $G_k^{(\mathbb{K})} \equiv 0$ for $k = 4, 6, 8$. Hence we get equality for $k = 4, 6, 8$ in Corollary [5]. We conjecture that $G_k^{(\mathbb{K})} \neq 0$ for any even $k \geq 10$ and any imaginary-quadratic number field $\mathbb{K}$. This has been verified for $|\Delta_{\mathbb{K}}| \leq 500$ by the authors. The Fourier coefficients are not always positive as in the proof of Theorem [4] (cf. [9]).

b) The paramodular group of level $t$ can be embedded into $\Gamma_{2}^{(\mathbb{K})}$, whenever $t$ is the norm of an element in $\mathcal{O}_\mathbb{K}$ (cf. [13]). Hence one can construct paramodular cusp forms in the Maass space in the same way.
5 The Maaß Spezialschar

At first we characterize the vanishing of $G_{k}^{(K)}$. Recall (e.g. [21], Theorem 3.14) that for even $k \in \mathbb{N}$ the Shimura lifts are maps

$$\text{Sh}_{k,t-1/2} : \mathcal{S}_{k-1/2}(\Gamma_{0}(4)) \to \mathcal{S}_{2k-2}(SL_{2}(\mathbb{Z}))$$

$$f(\tau) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n \tau} \mapsto \sum_{n=1}^{\infty} b_{t}(n)e^{2\pi i n \tau},$$

for squarefree $t \in \mathbb{N}$, where the coefficients $b_{t}(n)$ are given by

$$\sum_{n=1}^{\infty} b_{t}(n)n^{-s} = \sum_{n=1}^{\infty} \left( \frac{t}{n} \right) n^{k-s-3/2} \cdot \sum_{n=1}^{\infty} a(n^2)n^{-s}.$$ 

Let $\mathcal{S}_{k-N_{2}}^{+}(\Gamma_{0}(4))$ be Kohnen’s plus space, i.e.

$$a(n) = 0 \text{ for } n \equiv 1, 2 \text{ mod } 4.$$ 

Lemma 3. Let $k \geq 4$ be even and $K = \mathbb{Q}(\sqrt{-t})$, $t \in \mathbb{N}$ squarefree. Then the following are equivalent:

(i) The restriction of the Hermitian Eisenstein series $E_{k}^{(K)}|_{\mathcal{S}_{2}}$ is equal to the Siegel Eisenstein series $S_{k}$.

(ii) The $t$-Shimura lift $\text{Sh}_{t,k-1/2} : \mathcal{S}_{k-1/2}(\Gamma_{0}(4)) \to \mathcal{S}_{2k-2}(SL_{2}(\mathbb{Z}))$ is identically zero.

Proof. As $E_{k}^{(K)}|_{\mathcal{S}_{2}}$ is the Maaß lift of $e_{k}^{(K)}(\tau, z, z)$, condition (i) holds if and only if $e_{k}^{(K)}(\tau, z, z)$ is equal to the classical Jacobi-Eisenstein series in [6]. Due to Lemma 1 one has

$$e_{k}^{(K)}(\tau, z, z) = \sum_{\lambda \in \mathcal{O}_{K}} \sum_{M: \Gamma_{\infty} \setminus \Gamma_{1}} e^{2\pi i (\tau \lambda \bar{\lambda} + z(\lambda + \bar{\lambda}))}|_{k,1} M,$$

if we use the definition of the slash operator from [6]. For each fixed $\lambda$ the series

$$\sum_{M: \Gamma_{\infty} \setminus \Gamma_{1}} e^{2\pi i (\tau \lambda \bar{\lambda} + z(\lambda + \bar{\lambda}))}|_{k,1} M = P_{k,\lambda \bar{\lambda}, \lambda + \bar{\lambda}}(\tau, z)$$

is the Jacobi-Poincaré series (cf. [23]). Its Petersson inner product with a cusp form $\phi$ is, up to a factor depending on $k$ as well as $(\lambda - \bar{\lambda})^{6-4k}$, equal to the Fourier coefficient of the term $e^{2\pi i (\tau \lambda \bar{\lambda} + z(\lambda + \bar{\lambda}))}$. Thus (i) holds if and only if

$$\sum_{\lambda \in \mathcal{O}_{K}} (\lambda - \bar{\lambda})^{6-4k} c(\lambda \bar{\lambda}, \lambda + \bar{\lambda}) = 0$$

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holds for every Jacobi cusp form

\[
\phi(\tau, z) = \sum_{n,r} c(n, r) e^{2\pi i (n\tau + rz)} \in J_{k,1}.
\]

We want to rephrase this in terms of half-integral weight modular forms for \(\Gamma_0(4)\) as in \(\text{[6], § 5}\). Given (13) we attach the modular form

\[
F(\tau) = \sum_{N \equiv 0, 3 \mod 4} c(N) e^{2\pi i N\tau} \in M_{k-1/2}(\Gamma_0(4)),
\]

satisfying Kohnen’s plus condition. This bijection respects the Petersson inner products up to a trivial factor. The half-integral weight modular form attached to \(P_{k,\lambda,\lambda+\bar{\lambda}}\) is therefore the projection to the Kohnen plus space of the Poincaré series of weight \(k - 1/2\) and index \(- (\lambda - \bar{\lambda})^2\). As \(\lambda\) runs through \(\sigma_\mathbb{K}\) these values run through \(|\Delta| n^2, n \in \mathbb{N}_0\).

Hence (i) holds if and only if

\[
\sum_{n=1}^{\infty} c(|\Delta| n^2) n^{3-2k} = 0
\]

for every cusp form \(F(\tau) = \sum_{N \equiv 0, 3 \mod 4} c(N) e^{2\pi i N\tau} \in S_{k-1/2}(\Gamma_0(4))\). If such an \(F\) is a Hecke eigenform, then its Shimura lift \(Sh_{t,k-1/2}(F)\) is either a Hecke eigenform with the same eigenvalues or identically 0. In the first case the Euler product of its \(L\)-function implies that

\[
\sum_{n=1}^{\infty} b_t(n) n^{3-2k} \neq 0
\]

and thus

\[
\sum_{n=1}^{\infty} a(t n^2) n^{3-2k} \neq 0.
\]

Therefore (i) holds if and only if all the Hecke eigenforms in \(S_{k-1/2}^+(\Gamma_0(4))\) map to 0 under \(Sh_{t,k-1/2}\).

This leads to our final result

**Corollary 6.** Let \(k \geq 10\) be even. Then the modular form

\[
G_k^{(\mathbb{K})} = E_k^{(\mathbb{K})} \mid_{S_2} - S_k
\]

generate the subspace of cusp forms in the Maaß Spezialschar as a module over the Hecke algebra, as \(\mathbb{K}\) varies over all imaginary-quadratic number fields.

**Proof.** Let \(A_{k-1/2}(\Gamma_0(4)) \subseteq S_{k-1/2}^+(\Gamma_0(4))\) be the subspace generated by the preimages of \(E_k^{\mathbb{Q}(\sqrt{t})} \mid_{S_2} - S_k\) under the Maaß lift as \(t\) runs through all squarefree numbers in \(\mathbb{N}\). In the proof of Lemma 3 it was shown that a Kohnen plus Hecke eigenform \(F \in S_{k-1/2}^+(\Gamma_0(4))\)
is orthogonal to $\mathcal{A}_{k-1/2}(\Gamma_0(4))$ if and only if its $t$-Shimura lifts $Sh_{t,k-1/2}(F)$ are 0 for all squarefree $t \in \mathbb{N}$. This cannot happen by a theorem of Kohnen [14], which guarantees that some linear combination of Shimura lifts yields a Hecke-equivariant isomorphism

$$S^+_{k-1/2}(\Gamma_0(4)) \sim S_{2k-2}(\Gamma_1).$$

It follows that $\mathcal{A}_{k-1/2}(\Gamma_0(4))$ generates $S^+_{k-1/2}(\Gamma_0(4))$ as a module over the Hecke algebra. Since Maaß lifts respect Hecke operators, we obtain the claim.

**Remark 4.** a) Lemma [3] is trivial for $k = 4, 6, 8$ because

$$S_k(\Gamma_2) = S_{2k-2}(\Gamma_1) = \{0\}.$$

b) It is an open question whether the modular forms $G_k^{(K)}$ span the space of cusp forms in the Maaß space of even weight $k \geq 10$, when $K$ runs over all imaginary-quadratic number fields.

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