Research Article

Numerical Solution of Burgers’ Equation Based on Mixed Finite Volume Element Methods

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Received 22 July 2019; Accepted 16 January 2020; Published 18 March 2020

Academic Editor: Fabio Tramontana

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In this article, mixed finite volume element (MFVE) methods are proposed for solving the numerical solution of Burgers’ equation. By introducing a transfer operator, semidiscrete and fully discrete MFVE schemes are constructed. The existence, uniqueness, and stability analyses for semidiscrete and fully discrete MFVE schemes are given in detail. The optimal $a$ priori error estimates for the unknown and auxiliary variables in the $L^2(\Omega)$ norm are derived by using the stability results. Finally, numerical results are given to verify the feasibility and effectiveness.

1. Introduction

In this article, we consider the following one-dimensional Burgers’ equation:

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad (x, t) \in \Omega \times \Gamma,$$

with initial and boundary conditions

$$\begin{cases} u(a, t) = u(b, t) = 0, & t \in \Gamma; \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

where $\Omega = (a, b)$, $\Gamma = (0, T]$ with $0 < T < \infty$, $\alpha$ is a positive constant, $\nu > 0$ is the viscosity coefficient, and $u_0(x)$ is the given initial function.

Burgers’ equation is a famous nonlinear evolution equation which was first derived by Bateman [1] in 1915. Burgers [2] utilized this equation to model turbulence behavior. Since its appearance, this equation has been widely concerned by researchers because of its various practical applications, such as gas dynamics, shock theory, traffic flows, viscous flow, and turbulence. The exact solutions can be expressed as a Fourier series expansion by introducing a Hopf–Cole transformation [3, 4]. Benton and Platzman [5] gave the exact solutions of Burgers’ equation in one-dimensional spatial regions for some different initial functions. In the past several decades, many numerical techniques had been constructed and tested to solve Burgers’ equation [6–13], such as finite element methods, finite difference methods, least-squares finite element methods, and spectral methods.

In recent years, the mixed finite element (MFE) methods and finite volume (FV) methods have been used to solve Burgers’ equation by many researchers. Luo and Liu [14] proposed an MFE method to solve one-dimensional Burgers’ equation by introducing a flux function as an auxiliary variable and gave the existence, uniqueness, and error analyses for the discrete solutions. Chen and Jiang [15] constructed a characteristic MFE scheme to solve one-dimensional Burgers’ equation and obtained the optimal $a$ priori error estimates for the velocity and flux (gradient) in the $L^2$ norm. Pany et al. [16] applied an $H^1$-Galerkin MFE method to approximate the velocity and flux of one-dimensional Burgers’ equation and gave a priori error estimates and numerical experiments. Shi et al. [17] provided a low-order least-squares nonconforming characteristic MFE scheme to solve two-dimensional Burgers’ equation and obtained the optimal order error estimates in the $L^2$ norm. Hu et al. [18]
constructed a Crank–Nicolson time discretization MFE scheme to treat two-dimensional Burgers’ equation by using the $P_0^h - P_1$ pair and gave the optimal error analysis and numerical experiments. Nascimento et al. [19] applied a Fourier pseudospectral method and an FV method to solve one-dimensional Burgers’ equation and gave the numerical comparison. Guo et al. [20] proposed a fifth-order FV weighted compact scheme to solve one-dimensional Burgers’ equation and gave numerical experiments. Sheng and Zhang [21] proposed an FV method to solve two-dimensional Burgers’ equation and obtained the optimal error estimate in the $H^1$ norm.

The aim of this article is to develop mixed finite volume element (MFVE) methods to solve one-dimensional Burgers’ equation by combining the MFVE methods [22–25] with the finite volume element (FVE) methods [26–30]. The MFVE methods, also called mixed covolume methods, were first proposed by Russell [31] to solve the elliptic equation. Now, the methods have been applied to solve second-order elliptic equations [32–34], integrodifferential equations [35], parabolic equations [36, 37], time-fractional partial differential equations [38], and so on. In this article, we introduce a flux function as an auxiliary variable, rewrite (1) as the first-order system, and construct the semidiscrete and nonlinear backward Euler fully discrete MFVE scheme. We apply the Brouwer fixed-point theorem to prove the existence and use the Sobolev embedding theoretical results in the analysis of the fully discrete MFVE scheme.

The rest of this article is organized as follows: In Section 2, we use a flux function as an auxiliary variable and give the mixed variational formulation and the semidiscrete MFVE scheme. The existence, uniqueness, stability, and convergence analyses for semidiscrete and fully discrete schemes are given in Section 3 and 4, respectively. In Section 5, a numerical example is given to verify the theoretical results. In this article, the standard definitions and notations of the Sobolev spaces as in [39] are used. Furthermore, we use the symbol $C$ to represent a generic constant which is independent of the space and time mesh parameters $h$ and $\Delta t$.

## 2. Semidiscrete MFVE Scheme

We introduce a flux function $p(x, t) = (\alpha/2) f(u(x, t)) - \nu u_x(x, t)$ as an auxiliary variable, where $f(u) = u^2$. Then, we can rewrite equations (1) and (2) as the following first-order system:

$$\begin{align*}
\begin{cases}
  u_t + p_x = 0, & (x, t) \in \Omega \times \Gamma, \\
  p - \frac{\alpha}{2} f(u) + \nu u_x = 0, & (x, t) \in \Omega \times \Gamma, \\
  u(a, t) = u(b, t) = 0, & t \in \Gamma, \\
  u(x, 0) = u_0(x), & x \in \overline{\Omega}.
\end{cases}
\end{align*}$$  (3)

The mixed variational formulation of (3) is to find \{u(t), p(t)\} $\in H^1_0(\Omega) \times L^2(\Omega)$ such that

$$\begin{align*}
\begin{cases}
  (u_t, \nu v) - (v_x, p) = 0, & \forall v \in H^1_0(\Omega), \\
  (p, q) - \frac{\alpha}{2} (f(u), q) + (u_x, q) = 0, & \forall q \in L^2(\Omega), \\
  u(x, 0) = u_0(x), & \forall x \in \overline{\Omega}.
\end{cases}
\end{align*}$$  (4)

Now, we construct the primal mesh for the interval $\overline{\Omega} = [a, b]$ with the nodes $a = x_0 < x_1 < x_2 < \cdots < x_N = b$, where $N$ is some positive integer. Then, the primal mesh is denoted by $\mathcal{S}_h = \{A_i = [x_i, x_{i+1}]: i = 0, 1, \ldots, N - 1\}$, and the diameter of the primal mesh is defined by $h = \max_{0 \leq i \leq N - 1} h_i$, where $h_i = x_{i+1} - x_i$. We assume that the mesh satisfies the quasiuniform condition $h_i \geq c_0 h (0 \leq i \leq N - 1)$ for some constant $c_0 > 0$.

Next, the corresponding dual mesh is constructed by the nodes $a = x_0 < x_{1/2} < x_{3/2} < \cdots < x_{N-1/2} < x_N = b$, where $x_{i+1/2} = (x_i + x_{i+1})/2 (i = 0, 1, \ldots, N - 1)$. Denote $A_i^* = [x_{i-1/2}, x_{i+1/2}]$, $A_i^* = [x_{i-1/2}, x_{i+1/2}]$ \((i = 1, 2, \ldots, N - 1)\), and $A_N^* = [x_{N-1/2}, x_N]$, then the dual mesh is defined by $\mathcal{S}_h^* = \{A_i^* : i = 0, 1, \ldots, N\}$.

We choose the mixed finite element space $H_{0h} \times L_h$ as the trial function space, where

$$\begin{align*}
H_{0h} &= \{w_h \in H^1_0(\Omega): w_h|_A \in P_1(A), \forall A \in \mathcal{S}_h\}, \\
L_h &= \{v_h \in L^2(\Omega): v_h|_A \in P_0(A), \forall A \in \mathcal{S}_h^*\}. 
\end{align*}$$  (5)

Let $\{\phi_i: i = 1, 2, \ldots, N - 1\}$ and $\{\chi_A: i = 0, 1, \ldots, N - 1\}$ be the basis of the spaces $H_{0h}$ and $L_h$, respectively, where $\phi_i$ is the piecewise linear polynomial defined in [40] and $\chi_A$ is the characteristic function of the set $A$. The system (3) is integrated as follows:

$$\begin{align*}
\int_{A_i} u_t \, dx + \int_{A_i} p_x \, dx, & \quad t \in \Gamma, 1 \leq i \leq N - 1, \\
\int_{A_i} (p + \nu u_x) \, dx = \frac{\alpha}{2} \int_{A_i} f(u) \, dx, & \quad t \in \Gamma, 0 \leq i \leq N - 1.
\end{align*}$$  (6)

Now, we define the transfer operator $\gamma_h: H_{0h} \rightarrow L^2(\Omega)$ (see [40]) as follows:
Theoretical Analysis for the Semidiscrete MFVE Scheme

3.1. Some Lemmas. For theoretical analysis, we first give some properties of the transfer operator \( y_h \) and two projection operators.

Lemma 1 (see [40]). The transfer operator \( y_h \) is bounded, that is,
\[
\| y_h w_h \| \leq \sqrt{3} \| w_h \|, \quad \forall w_h \in H_{0h}.
\]
(12)

Lemma 2 (see [40]). The transfer operator \( y_h \) satisfies the following symmetry relation:
\[
(w_h, y_h v_h) = (y_h w_h, v_h), \quad \forall v_h, w_h \in H_{0h}.
\]
(13)

Lemma 3 (see [40]). The transfer operator \( y_h \) satisfies the following positivity:
\[
(y_h w_h, w_h) \geq \frac{3}{4} \| w_h \|^2, \quad \forall w_h \in H_{0h}.
\]
(14)

Lemma 4 (see [40]). Let \( I \) be an identity operator, then the transfer operator \( y_h \) satisfies the following properties:
\[
\| (I - y_h) w_h \| \leq \frac{\sqrt{12}}{12} h \| w_h \|, \quad \forall w_h \in H_{0h},
\]
\[
\| (y_h - I) w_h \| \leq \frac{\sqrt{12}}{12} h \| w_h \|, \quad \forall v_h, w_h \in H_{0h}.
\]
(15)

By a simple calculation, it is easy to have that \( b(y_h v_h, q_h) = -(v_h x, q_h), \quad \forall v_h \in H_{0h}, \forall q_h \in L_h \). Then, we get the semidiscrete MFVE scheme to find \( \{ u_h, p_h \} : [0, T] \rightarrow H_{0h} \times L_h \) such that
\[
\begin{align*}
(u_h, y_h v_h) - (v_h x, p_h) &= 0, \quad \forall v_h \in H_{0h}, \\
(p, q_h) + (u_hx, q_h) &= \frac{\alpha}{2} (f(u_h), q_h), \quad \forall q_h \in L_h,
\end{align*}
\]
(10)

where \( \{ u_h(0), p_h(0) \} \) satisfies
\[
\begin{align*}
(u_h(0), y_h v_h(0)) &= (0, z_h), \\
(p_h(0), q_h) &= \frac{\alpha}{2} (f(u_h(0), q_h), \quad \forall q_h \in L_h.
\end{align*}
\]
(11)

Lemma 5 (see [40]). There exists a constant \( C > 0 \) such that
\[
\| (y, I - y_h) w_h \| \leq Ch \| w_h \|, \quad \forall w_h \in H_{0h}.
\]
(16)

Now, the elliptic projection operator \( \Pi_h : H^1_0(\Omega) \rightarrow H_{0h} \) is introduced below, which satisfies
\[
(\omega - \Pi_h \omega, z_h) = 0, \quad \forall z_h \in L_h, \quad \forall \omega \in H^1_0(\Omega).
\]
(17)

At the same time, the \( L^2 \) orthogonal projection operator \( R_h : L^2(\Omega) \rightarrow L_h \) is introduced to satisfy
\[
(\varphi - R_h \varphi, q_h) = 0, \quad \forall q_h \in L_h, \quad \forall \varphi \in L^2(\Omega).
\]
(18)

Referring to References [24, 25], we can know that the projection operators \( \Pi_h \) and \( R_h \) satisfy the following estimate properties.

Lemma 6. There exists a constant \( C > 0 \) such that, for \( i = 0, 1, \)
\[
\begin{align*}
\| \frac{\partial^i \omega}{\partial t^i} - \frac{\partial^i (\Pi_h \omega)}{\partial t^i} \|_{1} &\leq Ch \| \frac{\partial^i \omega}{\partial t^i} \|_{1}, \quad \frac{\partial^i \omega}{\partial t^i} \in H^1_0(\Omega), \\
\| \frac{\partial^i \omega}{\partial t^i} - \frac{\partial^i (R_h \omega)}{\partial t^i} \|_{1} &\leq Ch \| \frac{\partial^i \omega}{\partial t^i} \|_{1}, \quad \frac{\partial^i \omega}{\partial t^i} \in H^1(\Omega) \\
\| \frac{\partial^i \varphi}{\partial t^i} - \frac{\partial^i (R_h \varphi)}{\partial t^i} \|_{1} &\leq Ch \| \frac{\partial^i \varphi}{\partial t^i} \|_{1}, \quad \frac{\partial^i \varphi}{\partial t^i} \in H^1(\Omega). \end{align*}
\]
(19)
3.2. Existence, Uniqueness, and Stability Analyses

**Theorem 1.** There exists a unique discrete solution for the semidiscrete MFVE scheme (10).

**Proof.** Obviously, there exists a unique solution \( \{u_h(0), p_h(0)\} \) for the scheme (11). Let \( \{\phi_i\}_{i=1}^{N-1} \) and \( \{X_{A_j}\}_{j=0}^{N-1} \) be the basis functions of the spaces \( H_{oh} \) and \( L_h \), respectively, then \( u_h \in H_{oh} \) and \( p_h \in L_h \) can be expressed as follows:

\[
 u_h(x, t) = \sum_{i=1}^{N-1} u_{h_i}(t)\phi_i(x), \quad p_h(x, t) = \sum_{j=0}^{N-1} p_{h_j}(t)X_{A_j}(x).
\]

(20)

Choosing \( v_h = \phi_i (1 \leq i \leq N - 1) \) and \( q_h = \chi_{A_j} (0 \leq j \leq N - 1) \) in (10), we rewrite the semidiscrete scheme (10) in the matrix form to find \( \{U_h(t), P_h(t)\} \) such that

\[
\begin{align*}
 DU_h(t) &- B^T P_h(t) = 0, \\
 AP_h(t) + \nu BU_h(t) & = \frac{\alpha}{2} F(U_h(t)),
\end{align*}
\]

(21)

where

\[
\begin{align*}
 U_h(t) &= (u_{h_1}(t), u_{h_2}(t), \ldots, u_{h(N-1)}(t))^T, \\
 P_h(t) &= (p_{h_0}(t), p_{h_1}(t), \ldots, p_{h(N-1)}(t))^T, \\
 A &= \left(\left(\chi_{A_j}, X_{A_i}\right)\right)_{i=0,\ldots,N-1}^{j=0,\ldots,N-1}, \\
 B &= \left(\left(\phi_i, \chi_{A_j}\right)\right)_{i=1,\ldots,N-1; j=0,\ldots,N-1}^{j=0,\ldots,N-1}, \\
 D &= \left(\left(\phi_i, \chi_{A_j}\right)\right)_{i=1,\ldots,N-1; j=0,\ldots,N-1}^{j=0,\ldots,N-1}, \\
 F(U_h(t)) &= \left(\left(f(U_h(t), X_{A_i}\right)\right)_{j=0,\ldots,N-1}^{j=0,\ldots,N-1}.
\end{align*}
\]

(22)

It is easy to know that matrices \( A \) and \( D \) are symmetrically positive definite. Then, the system (21) can be rewritten as

\[
\begin{align*}
 DU_h(t) &+ \nu B^T A^{-1} BU_h(t) - \frac{\alpha}{2} B^T A^{-1} F(U_h(t)) = 0, \\
 P_h(t) & = \frac{\alpha}{2} A^{-1} F(U_h(t)) - \nu A^{-1} B U_h(t).
\end{align*}
\]

(23)

According to the theory of differential equation, we can see that the system (23) has a unique solution, which shows that there exists a unique discrete solution for the semidiscrete MFVE scheme (10).

**Theorem 2.** Let \( \{u_h, p_h\} \) be the discrete solution of the semidiscrete scheme (10), then there exists a constant \( C > 0 \) such that

\[
\begin{align*}
\|u_h(t)\|_{\infty} &\leq C \|u_h(0)\|_1 + \|u_0\|_1 e^{C \|u_0\|_1}, \\
\|p_h(t)\| &\leq C \|u_h(0)\|_1 + \|u_0\|_1 e^{C \|u_0\|_1}.
\end{align*}
\]

(24)

**Proof.** Choosing \( v_h = u_h \) and \( q_h = u_{hx} \) in (10), we have

\[
(u_{ult}, \gamma_1 u_t) + \nu (u_{hx}, u_{hx}) = \frac{\alpha}{2} (f(u_h), u_{hx}).
\]

(25)

Noting that \( (f(u_h), u_{hx}) = 0 \), we rewrite (25) as

\[
\frac{1}{2} \frac{d}{dt} (u_h, u_{hx}) + \nu \|u_{hx}\|^2 = 0.
\]

(26)

Integrating (26) from 0 to \( t \), we get

\[
(u_h(t), u_{hx}(t)) + 2\nu \int_0^t \|u_{hx}\|^2 \, dt = (u_h(0), u_{hx}(0)).
\]

(27)

Applying Lemma 1 and Lemma 3, we obtain

\[
\|u_h(t)\|^2 + \frac{8\nu}{3} \int_0^t \|u_{hx}\|^2 \, dt \leq \frac{4\nu^2}{3} \|u_h(0)\|^2.
\]

(28)

Calculating the derivative of (10) with respect to \( t \), we get

\[
(p_{ht}, q_h) = \frac{\alpha}{2} (2u_{ult}, q_h).
\]

(29)

Setting \( v_h = u_{ult} \) in (10) and \( q_h = p_h \) in (29), we have

\[
(u_{ult}, u_{hx}) + \frac{1}{\nu} (p_{ht}, p_h) = \frac{\alpha}{2} (u_{ult}, p_h).
\]

(30)

Applying the Sobolev embedding theorem to estimate the right-hand side of (30), we obtain

\[
\frac{\alpha}{2} \left(\|u_{ult}, p_h\|\right) \leq \frac{\alpha}{2} \|u_{ult}\| \cdot \|p_h\| \leq \frac{3}{8} \|u_{ult}\|^2 + \frac{\alpha^2}{3\nu^2} \|u_{ult}\|_2 \|p_h\|^2 + \frac{3\nu^2}{8} \|u_{ult}\|^2.
\]

(31)

Making use of (31) and Lemma 3 in (30), we have

\[
\frac{3}{4} \|u_{ult}\|^2 + \frac{1}{2\nu} \frac{d}{dt} \|p_h(t)\|^2 \leq \frac{3}{8} \|u_{ult}\|^2 + C \|u_{hx}\|^2 \|p_h\|^2.
\]

(32)

Integrating (32) from 0 to \( t \), we obtain

\[
\|p_h(t)\|^2 \leq \|p_h(0)\|^2 + C \int_0^t \|u_{hx}\|^2 \, dt \leq \|p_h(0)\|^2 e^{C \int_0^t \|u_{hx}\|^2 \, dt}.
\]

(33)

Apply the Gronwall lemma and (28) in (33), we obtain

\[
\|p_h(t)\|^2 \leq \|p_h(0)\|^2 e^{C \int_0^t \|u_{hx}\|^2 \, dt} \leq \|p_h(0)\|^2 e^{C \|u_{hx}(0)\|^2}.
\]

(34)

Next, choosing \( q_h = u_{hx} \) in (10), we get

\[
\|u_{hx}(t)\|_{\infty} \leq C \|u_{hx}(0)\|_1 + \|u_0\|_1 e^{C \|u_0\|_1}.
\]

(24)
\( (p_h, u_{h{x}}) + \|u_{h{x}}\|^2 = \frac{\alpha}{2} (f (u_{h}), u_{h{x}}). \)

Noting that \( (f (u_{h}), u_{h{x}}) = 0 \), we have
\[ \|u_{h{x}}(t)\|^2 \leq \|p_h(t)\|_\Omega u_{h{x}}(t). \]

Substituting (34) into the above inequality, we have
\[ \|u_{h{x}}(t)\| \leq \frac{1}{2} \|p_h(t)\| \leq \frac{1}{2} \|p_h(0)\| e^{\frac{\alpha}{2} \|u_{h}{0}\|^2}. \]

Thus, applying the Sobolev embedding theorem, we obtain
\[ \|u_h(t)\|_\infty \leq C \|u_{h{x}}(t)\|_1 \leq C \|u_{h}{0}\| \leq C \|p_h(0)\| e^{\frac{\alpha}{2} \|u_{h}{0}\|^2}. \]

Now, we estimate \( \|u_h(0)\|_1 \) and \( \|p_h(0)\| \). We choose \( z_h = u_{h{x}}(0) \) in (11) to obtain
\[ \|u_h(0)\|_\infty \leq C \|u_{h{x}}(0)\|_1 \leq C \|u_{h}{0}\| \leq C \|u_{h}{0}\|_1. \]

Choosing \( q_h = p_h(0) \) in (11), we have
\[ \begin{cases} 
\langle \eta_h, y_h v_h \rangle - (v_{h{x}}, \eta_h) = -\langle \xi, y_h \nu_h \rangle - (u_t, \langle I - \gamma_h \rangle \nu_h), & \forall \nu_h \in H(t), \\
(\epsilon, q_h) + v(\eta_h, q_h) = \frac{\alpha}{2} (f(u) - f(u_{h}), q_h), & \forall q_h \in L_2,
\end{cases} \]

where \( \eta(0) \) and \( \epsilon(0) \) satisfy
\[ \begin{cases} 
\eta(0) = 0, \\
(\epsilon(0), q_h) + v(\eta_h(0), q_h) = \frac{\alpha}{2} (f(u(0)) - f(u_{h}(0)), q_h), & \forall q_h \in L_2.
\end{cases} \]

**Theorem 3.** Let \( [u, p] \) and \( [u_{h}, p_h] \) be the solutions of the systems (4) and (10), respectively. Assume that the initial solution \( [u_{h}(0), p_h(0)] \) satisfies (11), then there exists a constant \( C > 0 \) such that
\[ \|u(t) - u_h(t)\| \leq C h \left( \|u(t)\|_1 + \left( \int_0^t \left( \|u(t)\|_1^2 + \|u_{h}(t)\|_1^2 \right) dt \right)^{1/2} \right), \]
\[ \|p(t) - p_h(t)\| \leq C_1 h \left( \|p(t)\|_1 + \|u_{h}(0)\|_1^2 + \left( \int_0^t \left( \|u(t)\|_1^2 + \|u_{h}(t)\|_1^2 \right) dt \right)^{1/2} \right), \]

where \( C_1 = C \left( \|u\|_{L^\infty((0,T];L^1)} \right) \) represents the function of \( \|u\|_{L^\infty((0,T];L^1)} \) and \( \|u\|_1 \).

**Proof.** Taking \( v_h = \eta \) and \( q_h = \epsilon \) in (43), we have
\[ \langle \eta, y_h \eta \rangle + \frac{1}{v} (\epsilon, \epsilon) = -\langle \xi, y_h \eta \rangle - (u_t, \langle I - \gamma_h \rangle \eta) \]
\[ - \frac{\alpha}{2v} (f(u) - f(u), \epsilon). \]
Applying Theorem 2, we have \( \|f(u) - f(u_h)\| \leq C (\|\xi\| + \|\eta\|) \). We apply the Cauchy–Schwarz inequality and the Young inequality in (46) to obtain
\[
\frac{1}{2} \frac{d}{dt} (\eta, \gamma_{\eta}\eta) + \frac{1}{2} \|\xi\|^2 \leq \frac{1}{2\nu} \|\xi\|^2 + C \left( \|\xi\|^2 + \|\eta\|^2 + \|\xi\|^2 + \|\eta\|^2 \right).
\]
(47)

Integrating (47) from 0 to \( t \), and applying Lemma 3, we obtain
\[
\|\eta\|^2 + \frac{8}{3\nu} \int_0^t \|\xi\|^2 \, dt \leq C \int_0^t \left( \|\eta\|^2 + \|\xi\|^2 + \|\xi\|^2 + \|\eta\|^2 \right) \, dt.
\]
(48)

Applying the Gronwall lemma in (48) yields
\[
\|\eta\|^2 + \frac{8}{3\nu} \int_0^t \|\xi\|^2 \, dt \leq C \int_0^t \left( \|\eta\|^2 + \|\xi\|^2 + \|\xi\|^2 + \|\eta\|^2 \right) \, dt.
\]
(49)

Next, calculating the derivative of (43) with respect to \( t \), we obtain
\[
\langle \epsilon_t, q_h \rangle + \nu (\eta_{tt}, q_h) = \alpha (u \cdot u_t - u_h \cdot u_{ht}, q_h).
\]
(50)

Choosing \( q_h = \epsilon \) in (50) and \( \nu_t = \eta_t \) in (43), we have
\[
(\eta_t, \gamma_{\eta_t}\eta_t) + \frac{1}{\nu} (\epsilon_t, \epsilon) = (-\langle \xi_t, \gamma_{\eta_t}\eta_t \rangle - (u_t, \xi_t - \gamma_{\eta_t}\eta_t) + \alpha (u \cdot u_t - u_h \cdot u_{ht}, \epsilon)
\]
(51)

Applying the stability results in Theorem 2, we have
\[
\alpha \frac{\nu}{\nu} (u \cdot u_t - u_h \cdot u_{ht}, \epsilon) = \alpha \left( (u \cdot u_t - u_h \cdot u_t + u_t u_t - u_h u_{ht}) \right)
\]
\[
= \alpha (\xi_t + \eta_t + \eta_{\eta_t} u_{ht}, \epsilon)
\]
\[
\leq \frac{\alpha}{\nu} \left( \|u_t\| + \|\xi_t\| + \|\eta_t\| + \|\eta_{\eta_t}\| + \|\xi_{\eta_t}\| \right) \|\epsilon\|
\]
\[
\leq \frac{\alpha}{\nu} \left( \|u_t\| + \|\xi_t\| + \|\eta_t\| + \|\eta_{\eta_t}\| + \|\xi_{\eta_t}\| \right) \|\epsilon\|
\]
\[
+ C \left( \|\xi_{\eta_t}\| + \|\eta_{\eta_t}\| \right) \|\epsilon\|
\]
\[
\leq \frac{3}{16} \|\eta_t\|^2 + C \left( \|\xi_t\|^2 + \|\eta_t\|^2 \right)
\]
\[
+ \|\xi_{\eta_t}\|^2 + \|\eta_{\eta_t}\|^2.
\]
(52)

where \( C_1 = C_1 (\|u_t\|_\infty, (u_t, u_{ht}), \|u_{ht}\|_\infty) \) represents the function of \( \|u_t\|_\infty, \|u_{ht}\|_\infty, \) and \( \|u_{ht}\|_\infty. \) Applying Lemmas 3–5 in (51), and making use of (52), we obtain
\[
\frac{3}{4} \|\eta\|^2 + \frac{1}{2\nu} \frac{d}{dt} \|\xi\|^2 \leq \frac{3}{8} \|\eta_t\|^2 + C_1 \left( \|\xi_t\|^2 + \|\eta_t\|^2 + \|\xi_{\eta_t}\|^2 + \|\eta_{\eta_t}\|^2 \right).
\]
(53)

Integrating (53) from 0 to \( t \), we get
\[
\int_0^t \frac{3}{4} \|\eta\|^2 + \|\xi(t)\|^2 \, dt \leq \int_0^t \|\eta(0)\|^2 + C_1 \int_0^t \left( \|\xi(t)\|^2 + \|\eta(t)\|^2 + \|\xi_{\eta_t}\|^2 + \|\eta_{\eta_t}\|^2 \right) \, dt.
\]
(54)

Applying the Gronwall lemma in (54), we obtain
\[
\|\xi(t)\|^2 \leq C_1 \left( \|\xi(0)\|^2 + \int_0^t \left( \|\xi(t)\|^2 + \|\eta(t)\|^2 + \|\xi_{\eta_t}\|^2 + \|\eta_{\eta_t}\|^2 \right) \, dt \right).
\]
(55)

We estimate \( \|\xi(0)\| \) by choosing \( q_h = \epsilon(0) \) in (44) and noting that \( \eta(0) = 0 \):
\[
\|\xi(0)\|^2 \leq \frac{\alpha}{2} \left( \langle u(0) + u_h(0), \xi(0) + \eta(0) \rangle, \epsilon(0) \right)
\]
\[
\leq C \|u_0\|_\infty \|\xi_0\|_\infty \|\epsilon(0)\|_\infty.
\]
(56)

Applying Lemma 6 in (56), we have
\[
\|\epsilon(0)\| \leq C \|u_0\|_\infty \|\xi_0\|_\infty \|\epsilon(0)\|_\infty \leq C \|u_0\|_\infty \|\xi_0\|_\infty.
\]
(57)

Finally, we apply Lemma 6 and the triangle inequality to obtain the desired conclusion.

\[\square\]

4. Fully Discrete MFVE Scheme and Its Theoretical Analysis

4.1. Fully Discrete MFVE Scheme. Let \( 0 = t_0 < t_1 < \cdots < t_M = T \) be an equidistant partition of the time interval \([0, T]\), where \( M \) is a positive integer. Denote \( \Delta t = T/M \) to represent the step length and \( t_n = n \Delta t, n = 0, 1, \ldots, M \). And denote \( \psi^n = \psi(t_n) \) and \( \partial_t \psi^n = (\psi^n - \psi^{n-1})/\Delta t \) for a function \( \psi \).

Let \( u_h^n, p_h^n \) be the fully discrete solutions of \( u \) and \( p \) at \( t = t_n \), respectively. We can obtain the following nonlinear backward Euler MFVE scheme to find \( (u_h^n, p_h^n) \in H_{0h} \times L_h \) (\( n = 1, 2, \ldots, M \)),

\[
\begin{cases}
(\partial_t u_h^n, v_h), (v_h, p_h^n) = 0, \quad \forall v_h \in H_{0h}, \\
(p_h^n, q_h) + \nu (u_h^n, q_h) = \frac{\alpha}{2} \langle f(u_h^n), q_h \rangle, \quad \forall q_h \in L_h.
\end{cases}
\]
(58)

where the initial value \( \{u_h^0, p_h^0\} \) satisfies the following equations:

\[
\begin{cases}
(u_h^0, z_h) = (u_{0h}, z_h), \quad \forall z_h \in L_h, \\
(p_h^0, q_h) + \nu (u_h^0, q_h) = \frac{\alpha}{2} \langle f(u_h^0), q_h \rangle, \quad \forall q_h \in L_h.
\end{cases}
\]
(59)
Remark 1. The fully discrete MFVE scheme (58) is implicit in time. In the actual calculation of \( \{u^n_h, p^n_h\} \) \( 1 \leq n \leq M \), we need to make predictions first by using the linear backward Euler MFVE scheme defined as follows:

\[
\begin{align*}
(\partial_t u^n_h, y_h v_h) - (v_{nx}, p^n_h) &= 0, \\
(p^n_h, q_h) + \gamma(u_{nx}^n, q_h) &= \frac{\alpha}{2} (f(u_{n-1}^n), q_h),
\end{align*}
\]
\[
\forall v_h \in H_{0h}, n \geq 1,
\]
\[
\forall q_h \in L_{0h}, n \geq 1.
\]

Similar to the proof process of Theorem 4 in Reference [40], we can also obtain that there exists a unique discrete solution for the linear MFVE scheme (60).

4.2. Existence, Uniqueness, and Stability Analyses. For the existence analysis, we first give the Brouwer fixed-point theorem [41, 42].

Lemma 7. Let \( (H, (\cdot, \cdot)_H) \) be a finite-dimensional inner product space with a norm \( \| \cdot \|_H \). Let \( g : H \rightarrow H \) be a continuous operator. Assume that there exists \( \beta > 0 \) such that \( (g(z), z) \geq 0 \) for \( z \in H \) with \( z_{||_H} = \beta \). Then, there exists \( z^* \in H \) such that \( g(z^*) = 0 \) and \( ||z^*|| \leq \beta \).

Theorem 4. Suppose that \( \{u^n_h, p^n_h\}_{n=0}^{n-1} \) has been given, then there exists a unique fully discrete solution \( \{u^n_h, p^n_h\} \) for the nonlinear backward Euler MFVE scheme (58).

Proof. Choosing \( q_h = v_{hx} \) in (58), we have

\[
(\partial_t u^n_h, y_h v_h) + \gamma(u_{nx}^n, v_{hx}) = \frac{\alpha}{2} (f(u^n_h), v_{hx}).
\]  \hspace{1cm} (61)

The operator \( g : H_{0h} \rightarrow H_{0h} \) is defined as follows:

\[
(g(V), v_{hx})_{H^2} = (V, y_h v_{hx}) + \Delta t \gamma(V_{x}, v_{hx}) - (u_{nx}^{n-1}, y_h v_{hx})
\]

\[
- \frac{\alpha}{2} (f(V), v_{hx}).
\]  \hspace{1cm} (62)

It is easy to know that the operator \( g \) is continuous. Setting \( v_h = V \) in (62), we have

\[
(g(V), V)_{H^2} = (V, y_h V) + \Delta t \gamma(V_{x}, V_{x}) - (u_{nx}^{n-1}, y_h V)
\]

\[
- \frac{\alpha}{2} (f(V), V_x)
\]

\[
\geq \frac{3}{4} \|V\|^2 + \Delta t \|V_x\|^2 - \sqrt{3} \|u_{nx}^{n-1}\| \|V\|
\]

\[
\geq \frac{3}{4} \|V\|^2 + \frac{4 \sqrt{3}}{3} \|u_{nx}^{n-1}\| + \Delta t \|V_x\|^2.
\]  \hspace{1cm} (63)

Selecting \( V \in H_{0h} \) to satisfy \( \|V\| = 1 + (4 \sqrt{3} / 3) \|u_{nx}^{n-1}\| \), we have \( (g(V), V)_{H^2} > 0 \). Making use of Lemma 7, we know that there exists \( V^* \in H_{0h} \) such that \( g(V^*) = 0 \). Thus, we choose \( u^n_h = V^* \) to satisfy (61). Furthermore, selecting \( u^n_h = V^* \) in (58), we get

\[
(p^n_h, q_h) = -\gamma(V_{x}, q_{hx}) + \frac{\alpha}{2} (f(V^*), q_{hx}).
\]  \hspace{1cm} (64)

Thus, there exists a solution \( p^n_h \in L_{0h} \) which satisfies (64). Then, it is obviously known that \( \{u^n_h, p^n_h\} \) satisfies the scheme (58), which proves the existence of the fully discrete solutions.

Next, we give the uniqueness for the fully discrete scheme (58). Let \( \{V^n, Q^n\} \in H_{0h} \times L_{0h} \) be another solution of the scheme (58) with the initial value \( V^0 = u^0_h \), then we have

\[
(\partial_t V^n, y_h v_{hx}) + \gamma(V^n_{nx}, v_{hx}) = \frac{\alpha}{2} (f(V^n), v_{hx}).
\]  \hspace{1cm} (65)

Let \( E^n = V^n - V^0 \), then we have

\[
(\partial_t E^n, y_h v_{hx}) + \gamma(E^n_{nx}, v_{hx}) = \frac{\alpha}{2} (u^n_h \cdot u^n_h - u^0_h \cdot V^n - V^n. v_{hx})
\]

\[
\frac{\alpha}{2} (u^n_h \cdot u^n_h - u^0_h \cdot V^n + u^n_h V^n)
\]

\[
- V^n. V^n, v_{hx}
\]

\[
= \frac{\alpha}{2} (u^n_h E^n + V^n E^n, v_{hx}).
\]  \hspace{1cm} (66)

The following proof is based on the mathematical induction. First, \( E^0 = u^0_h - V^0 = 0 \); next assume \( E^{n-1} = 0 \) and then choose \( v_h = E^n \) in (66) to obtain

\[
(\partial_t E^n, y_h E^n) + \gamma \|E^n\|^2 = \frac{\alpha}{2} (u^n_h E^n + V^n E^n, E^n).
\]  \hspace{1cm} (67)

Applying the stability results in Theorem 5 (because the existence of the discrete solutions has been proved), we have

\[
\frac{1}{\Delta t} (E^n, y_h E^n) + \gamma \|E^n\|^2 \leq \frac{\gamma}{2} \|E^n\|^2 + C(\|u^n_h\|_{C^0} + \|V^n\|_{C^0}) \|E^n\|^2
\]

\[
\leq \frac{\gamma}{2} \|E^n\|^2 + C \|p^n_h\|^2 e^{C \|u^n_h\|_{C^0}} \|E^n\|^2.
\]  \hspace{1cm} (68)

Multiplying (68) by \( \Delta t \), and making use of Lemma 3, we obtain

\[
\frac{3}{4} \|E^n\|^2 + \frac{\gamma}{2} \Delta t \|E^n\|^2 \leq C \Delta t \|p^n_h\|^2 e^{C \|u^n_h\|_{C^0}} \|E^n\|^2.
\]  \hspace{1cm} (69)
Selecting $\Delta t$ to satisfy $\frac{\alpha}{2} \leq \frac{\Delta t}{2}$, we have $\frac{\Delta t}{2} \leq 1/2$, which indicates that $V^n = u^n_\alpha$. Then $Q^n = p^n_\alpha$ is known from (58). Thus, we complete the proof of Theorem 4.

**Theorem 5.** Let $\{u^n_\alpha, p^n_\alpha\}_{n=1}^M$ and $\{u^n_\alpha, p^n_\alpha\}_{n=1}^M$ be the discrete solutions of the systems (58) and (59), respectively. Then, there exists a constant $C > 0$ such that

$$\|P^n_\alpha\| \leq C\|P^n_0\|e^{C\|u^n_\alpha\|},$$

$$\|u^n_{\alpha,\infty}\| \leq C\|u^n_\alpha\| \leq C\|P^n_0\|e^{C\|u^n_\alpha\|}.$$  

Furthermore, the initial solution $\{u^n_\alpha, p^n_\alpha\}_{n=1}^M$ satisfies

$$\|P^n_\alpha\| \leq C\left(\|u^n_\alpha\| + \|u^n_{\infty,\alpha}\|\right),$$

$$\|u^n_{\alpha,\infty}\| \leq C\|u^n_\alpha\| \leq C\|u^n_\alpha\|.$$  

**Proof.** Similarly to the proof process of Theorem 2, we can obtain the estimate of the initial solution $\{u^n_\alpha, p^n_\alpha\}_{n=1}^M$. Choose $v_h = u^n_\alpha$ and $q_h = u^n_{\infty,\alpha}$ in (58) to obtain

$$\left(\begin{array}{c} v_h \cr q_h \end{array}\right) + \nu \left(\begin{array}{c} \delta_t u^n_\alpha \cr \delta_t u^n_{\infty,\alpha} \end{array}\right) = \frac{\alpha}{2} \left( f(u^n_\alpha), u^n_{\infty,\alpha} \right).$$

Taking note of $(f(u^n_\alpha), u^n_\alpha) = 0$ and $(\delta_t u^n_\alpha, \delta_t u^n_{\infty,\alpha}) = (1/2\Delta t)\left(\begin{array}{c} u^n_\alpha, u^n_{\infty,\alpha} \
 u^n_{\alpha,\infty}, u^n_{\alpha,\infty} \end{array}\right)$, we get

$$\left(\begin{array}{c} v_h \cr q_h \end{array}\right) = \frac{\alpha}{2} \left( f(u^n_\alpha), u^n_{\infty,\alpha} \right).$$

Multiplying (73) by $\Delta t$, summing from 1 to $n$, and applying Lemma 1 and Lemma 3, we have

$$\frac{3}{4}\|u^n_\alpha\|^2 + 2\Delta t \sum_{k=1}^n \|u^n_{\infty,\alpha}\|^2 \leq \sqrt{3}\|u^n_\alpha\|^2.$$  

Next, we make use of (58) and (59) to obtain

$$\left(\begin{array}{c} \delta_t u^n_\alpha \cr \delta_t u^n_{\infty,\alpha} \end{array}\right) = \frac{\alpha}{2\nu} \left( u^n_\alpha - u^n_{\alpha,\infty} - u^n_{\alpha,\infty} - u^n_{\alpha,\infty} \right).$$

Choosing $v_h = \delta_t u^n_\alpha$ in (58) and $q_h = p^n_\alpha$ in (75), we get

$$\left(\begin{array}{c} \delta_t u^n_\alpha \cr \delta_t u^n_{\infty,\alpha} \end{array}\right) = \frac{\alpha}{2\nu} \left( u^n_\alpha - u^n_{\alpha,\infty} - u^n_{\alpha,\infty} - u^n_{\alpha,\infty} \right).$$

Noting that

$$\frac{\alpha}{2\nu} \left( u^n_\alpha - u^n_{\alpha,\infty} - u^n_{\alpha,\infty} - u^n_{\alpha,\infty} \right) = \frac{\alpha}{2\nu} \left( u^n_\alpha - u^n_{\alpha,\infty} - u^n_{\alpha,\infty} - u^n_{\alpha,\infty} \right).$$

Choosing $\delta_t u^n_\alpha$ in (74) and $\delta_t u^n_{\infty,\alpha}$ in (77), we have

$$\|u^n_{\alpha,\infty}\|^2 \leq C\|u^n_\alpha\|^2 e^{C\|u^n_\alpha\|^2}.$$  

Choosing $u^n_\alpha$ in (74) and applying the Sobolev embedding theorem, we obtain

$$\|u^n_\alpha\|^2 \leq C\|u^n_\alpha\|^2 e^{C\|u^n_\alpha\|^2}.$$  

Thus, we complete the proof of Theorem 5.  

**4.3. A Priori Error Estimates.** First, the errors are expressed as

$$u(t_n) - u^n_\alpha = u(t_n) - u^n_\alpha = \Pi u(t_n) - u^n_\alpha = \xi^n + \eta^n,$$

$$p(t_n) - p^n_\alpha = p(t_n) - p^n_\alpha = \rho(t_n) - R_h p(t_n) - p^n_\alpha = \rho^n + \varepsilon^n.$$  

$$u(t_n) - u^n_\alpha = u(t_n) - u^n_\alpha = \Pi u(t_n) - u^n_\alpha = \xi^n + \eta^n,$$

$$p(t_n) - p^n_\alpha = p(t_n) - p^n_\alpha = \rho(t_n) - R_h p(t_n) - p^n_\alpha = \rho^n + \varepsilon^n.$$  

$$u(t_n) - u^n_\alpha = u(t_n) - u^n_\alpha = \Pi u(t_n) - u^n_\alpha = \xi^n + \eta^n,$$

$$p(t_n) - p^n_\alpha = p(t_n) - p^n_\alpha = \rho(t_n) - R_h p(t_n) - p^n_\alpha = \rho^n + \varepsilon^n.$$  

$$u(t_n) - u^n_\alpha = u(t_n) - u^n_\alpha = \Pi u(t_n) - u^n_\alpha = \xi^n + \eta^n,$$

$$p(t_n) - p^n_\alpha = p(t_n) - p^n_\alpha = \rho(t_n) - R_h p(t_n) - p^n_\alpha = \rho^n + \varepsilon^n.$$  

$$u(t_n) - u^n_\alpha = u(t_n) - u^n_\alpha = \Pi u(t_n) - u^n_\alpha = \xi^n + \eta^n,$$

$$p(t_n) - p^n_\alpha = p(t_n) - p^n_\alpha = \rho(t_n) - R_h p(t_n) - p^n_\alpha = \rho^n + \varepsilon^n.$$  

$$u(t_n) - u^n_\alpha = u(t_n) - u^n_\alpha = \Pi u(t_n) - u^n_\alpha = \xi^n + \eta^n,$$

$$p(t_n) - p^n_\alpha = p(t_n) - p^n_\alpha = \rho(t_n) - R_h p(t_n) - p^n_\alpha = \rho^n + \varepsilon^n.$$
By subtracting (58) from (4), we can obtain the error equations about \( \eta^m \) and \( \varepsilon^m \) as follows:

\[
\left \{ \begin{array}{l}
(\partial_t \eta^m, \gamma_h v_h) - (v_{hx}, \varepsilon^m) = -(u^m, (I - \gamma_h) v_h) - (\theta^m, \gamma_h v_h) - (\partial_t \xi^m, \gamma_h v_h), \forall v_h \in H_0h, \\
(\varepsilon^m, q_h) + \gamma(\eta^m, q_h) = \frac{\alpha}{2} (f(u^m) - f(u_h^m)) q_h, \forall q_h \in L_h,
\end{array} \right.
\]

where \( \theta^m = u^m - \partial_t u^m \) and \( \eta^m \) and \( \varepsilon^m \) satisfy

\[
\begin{cases}
\eta^0 = 0, \\
(\varepsilon^m, q_h) + \gamma(\eta^m, q_h) = \frac{\alpha}{2} (f(u^m) - f(u_h^m)) q_h, \forall q_h \in L_h.
\end{cases}
\]

(86)

**Theorem 6.** Let \( \{u_h^m, p_h^m\} \) be the discrete solution of the fully discrete scheme (58), and the exact solution \( \{u, p\} \) of the mixed variational formulation (4) satisfies the following regularity properties:

\[
\begin{align*}
&u, u_t \in L^\infty(\Omega), \\
&u_{tt} \in L^2(\Omega), \\
&p \in L^{\infty}(H^1(\Omega)).
\end{align*}
\]

(87)

Then, there exists a constant \( C > 0 \) such that

\[
\max_{1 \leq m \leq M} \left( \| u(t_m) - u_h^m \| + \| p(t_m) - p_h^m \| \right) \leq C (h + \Delta t).
\]

(88)

**Proof.** Choosing \( v_h = \eta^m \) and \( q_h = \eta^m \) in (85), we have

\[
(\partial_t \eta^m, \gamma_h \eta^m) + \gamma \| \eta^m \|^2 = -(u^m, (I - \gamma_h) \eta^m) - (\theta^m, \gamma_h \eta^m) - (\partial_t \xi^m, \gamma_h \eta^m) - \frac{\alpha}{2} (f(u^m) - f(u_h^m), \gamma^m).
\]

(89)

Applying Theorem 5, we get

\[
\| f(u^m) - f(u_h^m) \| \leq C (\| \xi^m \|^2 + \| \eta^m \|^2).
\]

(90)

Noting that

\[
(\partial_t \eta^m, \gamma_h \eta^m) \geq \frac{1}{2\Delta t} \left( (\eta^m, \gamma_h \eta^m) - (\eta^{m-1}, \gamma_h \eta^{m-1}) \right),
\]

we easily get that

\[
\frac{1}{2\Delta t} \left( (\eta^m, \gamma_h \eta^m) - (\eta^{m-1}, \gamma_h \eta^{m-1}) \right) + \gamma \| \eta^m \|^2 \\
\leq C \left( \| u^m \|^2 + \| \theta^m \|^2 + \| \xi^m \|^2 + \| \partial_t \xi^m \|^2 \right) + C \| \xi^m \|^2 + \gamma \| \eta^m \|^2.
\]

(91)

Multiplying (91) by \( 2\Delta t \), and summing from 1 to \( m \), we have

\[
(\partial_t \eta^m, q_h) + \gamma(\eta^m, q_h) = \frac{\alpha}{2} \left( \| u^m \|^2 - u_h^m q_h^m - u_h^{m-1} u^m - u_h^{m-1} u_h^{m-1} \right) q_h.
\]

(92)

Applying Lemma 3 in (92), we obtain

\[
\frac{3}{4} \| \eta^m \|^2 + \gamma \| \xi^m \|^2 + \gamma \| \partial_t \xi^m \|^2 + C \Delta t \sum_{n=1}^{m} \left( \| \xi^m \|^2 + h^2 \| u_h^m \|^2 + \| \theta^m \|^2 + \| \partial_t \xi^m \|^2 \right).
\]

(93)

Selecting \( \Delta t \) to satisfy \( C \Delta t < 3/8 \), and applying the discrete Gronwall lemma, we obtain

\[
\| \eta^m \|^2 + \frac{8\gamma}{3} \Delta t \sum_{n=1}^{m} \| \eta^m \|^2 \leq C \Delta t \sum_{n=1}^{m} \left( \| \xi^m \|^2 + h^2 \| u_h^m \|^2 + \| \theta^m \|^2 + \| \partial_t \xi^m \|^2 \right).
\]

(94)

According to the integral remainder formula of Taylor formula, we get

\[
\frac{\partial_t \xi^m}{\Delta t} \| \xi^m \|^2 \leq C \int_{t_{n-1}}^{t_n} \| \xi \|^2 \, dt, \quad \| \theta^m \|^2 \leq C \int_{t_{n-1}}^{t_n} \| u \|^2 \, dt, \quad n \geq 1.
\]

(95)

Applying the above inequalities and Lemma 6, we obtain

\[
\| \eta^m \|^2 \leq C \left( \| u \|_{L^\infty} + \| u_h^m \|_{L^\infty(\Omega)} + \| u_h^m \|_{L^2(\Omega)} \right) + C \| u_h^m \|_{L^2(\Omega)} \Delta t.
\]

(96)

Next, we estimate \( \| p(t_m) - p_h^m \| \). Making use of (85) and (86), we have

\[
(\partial_t u^m, q_h) + \gamma(\eta^m, q_h) = \frac{\alpha}{2} \left( \| u^m \|^2 - u_h^m q_h^m - u_h^{m-1} u^m - u_h^{m-1} u_h^{m-1} \right) q_h.
\]

(97)
Applying (98) and Lemmas 3–5, we obtain

\[
\begin{aligned}
&\left( \frac{u^n u^n - u_h^n u_h^n - u^{n-1} u^{n-1} + u_h^{n-1} u_h^{n-1}}{\Delta t}, q_h \right) \\
&= \left( \left( u^n + u_h^n - u^{n-1} + u_h^{n-1} \right) (\partial_t \xi^n + \partial_t \eta^n) + 2 \left( u^{n-1} - u_h^{n-1} \right) \partial_t u^n, q_h \right) \\
&\leq \left( \left\| u^n \right\|_{\infty} + \left\| u_h^n \right\|_{\infty} + \left\| u^{n-1} \right\|_{\infty} + \left\| u_h^{n-1} \right\|_{\infty} \right) \left( \left\| \partial_t \xi^n \right\| + \left\| \partial_t \eta^n \right\| \right) \left\| q_h \right\| + 2 \left\| u_t \right\|_{L^\infty(\Omega)} \left( \left\| \xi^{n-1} \right\| + \left\| \eta^{n-1} \right\| \right) \left\| q_h \right\| \\
&\leq K_1 \left( \left\| \partial_t \xi^n \right\| + \left\| \partial_t \eta^n \right\| \right) \left\| q_h \right\| + K_2 \left( \left\| \xi^{n-1} \right\| + \left\| \eta^{n-1} \right\| \right) \left\| q_h \right\|,
\end{aligned}
\]

where \( K_1 = C \left( \left\| u^n \right\|_{L^\infty(\Omega)} + \left( \left\| u_h^n \right\|_{L^\infty(\Omega)} \right) e^{\Delta t} \right) \) and \( K_2 = 2 \left\| u_t \right\|_{L^\infty(\Omega)} \). For the convenience of description, we use \( C_2 = C_2 \left( K_1, K_2 \right) \) to denote the function of \( K_1 \) and \( K_2 \).

Now, choosing \( v_h = \partial_t \eta^n \) in (85) and \( q_h = \xi^n \) in (97), we have

\[
\left( \partial_t \eta^n, \gamma_h \partial_t \eta^n \right) + \frac{1}{2v} \left( \partial_t \xi^n, \xi^n \right) = \frac{\alpha}{2} \left( \frac{u^n u^n - u_h^n u_h^n - u^{n-1} u^{n-1} + u_h^{n-1} u_h^{n-1}}{\Delta t}, \xi^n \right) - \left( u_t^n, \left( 1 - \gamma_h \right) \partial_t \eta^n \right) - \left( \theta^n, \gamma_h \partial_t \eta^n \right) - \left( \partial_t \xi^n, \gamma_h \partial_t \eta^n \right).
\]

Applying (98) and Lemmas 3–5, we obtain

\[
\begin{aligned}
&\frac{3}{4} \left\| \partial_t \eta^n \right\|^2 + \frac{1}{2v \Delta t} \left( \left\| \xi^n \right\|^2 - \left\| \xi^{n-1} \right\|^2 \right) \\
&\leq \frac{3}{8} \left\| \partial_t \eta^n \right\|^2 + CC_2 \left( \frac{1}{2} \left\| u^n \right\|_{\infty} \left\| \xi^n \right\|_{\infty} + \left\| \partial_t \xi^n \right\|^2 + \left\| \theta^n \right\|^2 + \left\| \eta^{n-1} \right\|^2 \\
&+ \left\| \eta^{n-1} \right\| \left\| \xi^n \right\| \right),
\end{aligned}
\]

Multiplying (100) by \( 2v \Delta t \), and summing from 1 to \( m \), we get

\[
\begin{aligned}
&\frac{3v}{4} \Delta t \sum_{n=1}^{m} \left\| \partial_t \eta^n \right\|^2 + \left\| \xi^n \right\|^2 \leq \left\| \xi^0 \right\|^2 + CC_2 \Delta t \sum_{n=1}^{m} \left\| \xi^n \right\|^2 \\
&+ C \Delta t \left\| \xi^n \right\|^2 + CC_2 \Delta t \sum_{n=1}^{m} \left( \frac{1}{2} \left\| u^n \right\|_{\infty} \left\| \xi^n \right\|_{\infty} + \left\| \partial_t \xi^n \right\|^2 + \left\| \theta^n \right\|^2 \\
&+ \left\| \eta^{n-1} \right\|^2 + \left\| \eta^{n-1} \right\| \right).\end{aligned}
\]

Selecting \( \Delta t \) to satisfy \( C_2 \Delta t < 1/2 \), and applying the discrete Gronwall lemma, we have

\[
\left\| \xi^n \right\|^2 \leq C \left( \left\| \xi^0 \right\|^2 + C_2 \Delta t \sum_{n=1}^{m} \left( \frac{1}{2} \left\| u^n \right\|_{\infty} \left\| \xi^n \right\|_{\infty} + \left\| \partial_t \xi^n \right\|^2 + \left\| \theta^n \right\|^2 \\
&+ \left\| \xi^{n-1} \right\|^2 + \left\| \eta^{n-1} \right\|^2 \right) \right).
\]

We estimate \( \left\| \xi^n \right\|^2 \) by choosing \( q_h = \xi^0 \) in (86):

\[
\left\| \xi^n \right\|^2 = \frac{\alpha}{2} \left( f(u^0) - f(u_h^0), \xi^0 \right) = \frac{\alpha}{2} \left( u^0 + u_h^0 \right) \left( \xi^0 + \eta^0 \right), \xi^0 \leq C \left\| u_0 \right\|_{\xi} \left\| \xi^0 \right\| \left\| \xi^0 \right\|.
\]

Applying Lemma 6, we have

\[
\left\| \xi^0 \right\| \leq Ch \left\| u_0 \right\|_{\xi}^2.
\]

Taking note of \( \eta^0 = 0 \) in (102), and applying (95), (96), and Lemma 6, we obtain

\[
\left\| \xi^n \right\| \leq CC_2 \Delta t h \left( \left\| u_0 \right\|_{H^1(\Omega)} + \left\| u_t \right\|_{L^\infty(\Pi^t)} + \left\| u_{tt} \right\|_{L^\infty(\Pi^t)} \right) \\
+ \left\| U_t \right\|_{L^2(\Pi^t)} + CC_2 \left\| u_t \right\|_{L^2(\Pi^t)} \Delta t.
\]

Finally, we apply the triangular inequality and Lemma 6 to complete the proof of Theorem 6.

\[ \square \]

### 5. Numerical Example

Now, we give a numerical example to verify the feasibility and effectiveness of the MFVE scheme. We consider Burgers’ equation as follows:

\[
\begin{align*}
\partial_t u + u \partial_x u - \nu \partial_{xx} u &= 0, & (x, t) &\in (0, 1) \times (0, 2], \\
u (0, t) &= u(1, t) = 0, & t &\in (0, 2], \\
u (x, 0) &= \sin (\pi x), & x &\in (0, 1),
\end{align*}
\]
where $v = 1/48$. Following Reference [4], we can obtain the exact solution (Fourier series solution) as follows:

$$u(x, t) = 2\pi \frac{1}{a_0 + \sum_{n=1}^{\infty} a_n \exp\left(-n^2 \pi^2 \omega t\right) \cos(n\pi x)},$$

where

$$a_0 = \int_{0}^{1} \exp\left[-(2\pi \omega)^{-1} \left[1 - \cos(n\pi x)\right]\right] dx,$$

$$a_n = 2 \int_{0}^{1} \exp\left[-(2\pi \omega)^{-1} \left[1 - \cos(n\pi x)\right]\right] \cos(n\pi x) dx,$$

$$\omega = \frac{\pi h}{2(2\omega)^{1/2}}.$$

The auxiliary variable $p(x, t) = (1/2)u^2(x, t) - \nu u_x(x, t)$ is determined by the above data.

In practical calculation, we use the equidistant grids of spatial regions and truncate the series of the exact solutions with $\sum_{n=1}^{N}$ when calculating error results in the $L^\infty(L^2(\Omega))$ norm of $u$ and $p$. We give the error results and convergence orders in Table 1 with mesh sizes $h = \Delta t = 1/20, 1/40, 1/80, 1/160$. We can see that the convergence orders are approximate to 1, and these results are consistent with Theorem 6. The space-time graphs of the exact solutions of $u$ and $p$ at $t = 2$ with $h = 1/40$ are shown in Figures 1 and 2, respectively. The space-time graphs of the nonlinear backward Euler fully discrete solutions $u_h$ and $p_h$ at $t = 2$ with $h = \Delta t = 1/20$ are shown in Figures 3 and 4. The figures show that the numerical solutions of $u$ and $p$ have the same numerical behavior as their exact solutions. The numerical results and figures verify that the MFVE method for Burgers’ equation is feasible and efficient.
6. Conclusions

In this article, our main aim is to construct the MFVE scheme for solving Burgers’ equation by introducing the auxiliary variable \( p(x,t) \) and the transfer operator \( \gamma_t \). We give the detailed theoretical results on existence, uniqueness, and stability for the semidiscrete and fully discrete schemes and obtain the optimal \( a \) priori error estimates in the \( L^2(\Omega) \) norm for the velocity \( u \) and flux \( p \) by using the stability results in Theorems 2 and 5. Moreover, we give a numerical example to verify that the proposed scheme for Burgers’ equation is feasible and efficient. In the future, we will apply the MFVE methods to solve some other nonlinear partial differential equations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally to the draft of the manuscript and read and approved the final manuscript.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11761053 and 11701299), the Natural Science Foundation of Inner Mongolia Autonomous Region (2016BS0105 and 2017MS0107), the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region (NJYT-17-A07), and the Prairie Talent Project of Inner Mongolia Autonomous Region.

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