Morrey smoothness spaces: A new approach

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Abstract In the recent years, the so-called Morrey smoothness spaces attracted a lot of interest. They can (also) be understood as generalisations of the classical spaces $A^s_{p,q}(\mathbb{R}^n)$ with $A \in \{B, F\}$ in $\mathbb{R}^n$, where the parameters satisfy $s \in \mathbb{R}$ (smoothness), $0 < p \leq \infty$ (integrability) and $0 < q \leq \infty$ (summability). In the case of Morrey smoothness spaces, additional parameters are involved. In our opinion, among the various approaches at least two scales enjoy special attention, also in view of applications: the scales $A^s_{u,p,q}(\mathbb{R}^n)$ with $A \in \{N, E\}$ and $u \geq p$, and $A^{s,r}_{p,q}(\mathbb{R}^n)$ with $A \in \{B, F\}$ and $r \geq 0$.

We reorganise these two prominent types of Morrey smoothness spaces by adding to $(s, p, q)$ the so-called slope parameter $\rho$, preferably (but not exclusively) with $-n \leq \rho < 0$. It comes out that $|\rho|$ replaces $n$, and $\min(|\rho|, 1)$ replaces 1 in slopes of (broken) lines in the $\left(\frac{1}{p}, s\right)$-diagram characterising distinguished properties of the spaces $A^s_{p,q}(\mathbb{R}^n)$ and their Morrey counterparts. Special attention will be paid to low-slope spaces with $-1 < \rho < 0$, where the corresponding properties are quite often independent of $n \in \mathbb{N}$.

Our aim is two-fold. On the one hand, we reformulate some assertions already available in the literature (many of which are quite recent). On the other hand, we establish on this basis new properties, a few of which become visible only in the context of the offered new approach, governed, now, by the four parameters $(s, p, q, \rho)$.

Keywords Morrey space, smoothness space of Morrey type, Besov-Morrey space, Triebel-Lizorkin-Morrey space, Besov-type space, Triebel-Lizorkin-type space

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1 Introduction

The nowadays classical spaces

$$A^s_{p,q}(\mathbb{R}^n) \quad \text{with } A \in \{B, F\}, \quad s \in \mathbb{R} \quad \text{and} \quad 0 < p, q \leq \infty$$

have been extended in several directions, most notably into two types of Morrey smoothness spaces,

$$N^s_{u,p,q}(\mathbb{R}^n) \quad \text{and} \quad E^s_{u,p,q}(\mathbb{R}^n) \quad \text{with } p \leq u < \infty,$$

on the one hand and

$$A^{s,r}_{p,q}(\mathbb{R}^n) \quad \text{with } A \in \{B, F\} \quad \text{and} \quad 0 \leq r < \infty,$$
on the other hand. With \( p = u \) in (1.2) and \( \tau = 0, \, p < \infty \) in (1.3), one obtains the corresponding spaces in (1.1).

These scales of the spaces (1.2) and (1.3) were investigated intensively in recent years, and we add some further historic remarks below. A strong motivation to study extensions from the scales of the function spaces (1.1) to this Morrey setting came from possible applications to PDEs (partial differential equations), as it was already the case for the ‘basic’ Morrey spaces extending \( L_p \). Some Besov-Morrey spaces were first introduced by Netrusov [51] by means of differences. One of the milestones in this direction is the famous paper by Kozono and Yamazaki [37], where they used spaces of type (1.2) to study Navier-Stokes equations; we also refer in this context to the papers by Mazzucato [46], Ferreira and Postigo [10], or by Yang et al. [91], and Lemarié-Rieusset [38,39], as well as to the monographs [60,61,94].

It is natural to ask for counterparts of distinguished properties of the spaces \( A^*_{p,q}(\mathbb{R}^n) \) in the context of these Morrey smoothness spaces. Typical examples are embeddings in \( L_\infty(\mathbb{R}^n) \) and in \( L_1^{\text{loc}}(\mathbb{R}^n) \) (which regular distributions) or traces on hyper-planes. One may also be interested in finding out those spaces to which the \( \delta \)-distribution or the characteristic function \( \chi_Q \) of the unit cube \( Q = (0,1)^n \) \((n \in \mathbb{N})\) belongs. Some final answers have been obtained in recent times. But the related conditions for the above questions are not very appealing, producing quite often curved lines in the well-known \((\frac{1}{p},s)\)-diagram, where any space of type \( A^*_{p,q} \) is indicated by its smoothness parameter \( s \) and integrability parameter \( p \), neglecting the fine index \( q \) for the moment; we refer to [24] for such examples. This suggests searching for a re-parametrisation of the spaces in (1.2) and (1.3) such that the outcome produces natural and transparent results, based on independent arguments, can thus not only be understood in a better way, detached from the (sometimes quite involved) technical requirements. But one might also observe more intrinsic reasons for common phenomena. This will also constitute some basis for unified results and, occasionally, lead to appropriate conjectures.

The classical parameters \( s \) (smoothness), \( p \) (integrability) and \( q \) (summability) in (1.1), and also in (1.2) and (1.3), are untouchable. But we replace \( u \) in (1.2) and \( \tau \) in (1.3) by the common parameter \( \rho \), typically (but not exclusively) with \( -n \leq \rho < 0 \). The corresponding spaces
\[
\rho A^*_{p,q}(\mathbb{R}^n) = \{ A^\rho A^*_{p,q}(\mathbb{R}^n), A^\rho A^*_{p,q}(\mathbb{R}^n): A \in \{ B,F \} \}
\]  
cover all the spaces in (1.2) and (1.3) with
\[
A^\rho A^*_{p,q}(\mathbb{R}^n) = A^\rho A^*_{p,q}(\mathbb{R}^n). \tag{1.5}
\]
We call \( \rho \) the slope parameter because \(|\rho|\) quite often takes over the role of the slope \( n \) in distinguished (broken) lines in the \((\frac{1}{p},s)\)-diagram. For example, the sharp embedding
\[
A^\rho_{p,q}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n), \quad \text{if } s > \frac{n}{p}, \quad 0 < p < \infty, \tag{1.6}
\]
as far as the breaking line is concerned, has now the sharp counterpart
\[
\rho A^\rho_{p,q}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n), \quad \text{if } s > \frac{\rho}{p}, \quad 0 < p < \infty. \tag{1.7}
\]
We refer to Theorems 5.3 and 5.18 and Figure 1 on page 1328 below.

This also applies to other distinguished properties, and many of them are discovered only recently. We touch on two of them. Let
\[
\sigma_p^t = t \left( \max \left( \frac{1}{p}, 1 \right) - 1 \right), \quad t \geq 0, \quad 0 < p \leq \infty. \tag{1.8}
\]
Then the sharp inclusion
\[
A^\rho_{p,q}(\mathbb{R}^n) \subset L_1^{\text{loc}}(\mathbb{R}^n), \quad \text{if } s > \sigma_p^t, \quad 0 < p < \infty, \tag{1.9}
\]

on the other hand. With \( p = u \) in (1.2) and \( \tau = 0, \, p < \infty \) in (1.3), one obtains the corresponding spaces in (1.1).

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\]
as far as the breaking line is concerned, has now the sharp counterpart
\[ \varphi A_{p,q}^s(\mathbb{R}^n) \subset L_1^{\text{loc}}(\mathbb{R}^n), \quad \text{if } s > \sigma \frac{|\varphi|}{p}, \quad 0 < p < \infty. \]  
(1.10)
This can be found in Theorem 5.7 and Figure 2 on page 1330 below.

For the characteristic function \( \chi_Q \) of the cube \( Q = (0,1)^n \), the sharp assertion
\[ \chi_Q \in \varphi A_{p,q}^s(\mathbb{R}^n), \quad \text{if } s < \frac{1}{p}, \quad 0 < p < \infty, \]  
(1.11)
as far the breaking line is concerned, has now the sharp counterpart
\[ \chi_Q \in \varphi A_{p,q}^s(\mathbb{R}^n), \quad \text{if } s < \frac{1}{p} \min(|\varphi|, 1), \quad 0 < p < \infty. \]  
(1.12)
The generalisation of the slope \( n \) in (1.6) and (1.9) by \( |\varphi| \) in (1.7) and (1.10) obeys the so-called 
Slope-\( n \)-Rule, whereas the replacement of 1 in (1.11) by \( \min(|\varphi|, 1) \) in (1.12) is a typical example of the 
Slope-1-Rule, as formulated in Subsection 2.2 below. It is one of the main aims of this paper to reformulate already existing assertions for the spaces in (1.2) and (1.3) in terms of these slope rules for the spaces in (1.4). This will be complemented by some new properties. Then detailed proofs will be given.

The rest of this paper is organised as follows. In Section 2, we introduce the above-mentioned spaces in (1.1)–(1.4) and discuss in detail their interrelations, including coincidences and diversities. It is central for our approach to structure the somewhat bewildering plethora of the Morrey smoothness spaces in common use according to (1.2) and (1.3) into \( \varphi \)-clans consisting, roughly speaking, of the spaces in (1.4) such that the above-mentioned slope rules can be applied. But the reorganisation of the above Morrey smoothness spaces into \( \varphi \)-clans is not only an efficient technical device. Any \( \varphi \)-clan is an organic entity on equal footing with the classical spaces in (1.1). Its families in (1.4) have different abilities complementing each other symbiotically. In Section 3, we collect some tools: wavelet characterisation, interpolations, lifts, characterisation by derivatives and the useful Fatou property. Section 4 deals with the so-called key properties (smooth pointwise multipliers, diffeomorphisms, extensions of the corresponding spaces in \( \mathbb{R}_+^n \) to their counterparts in \( \mathbb{R}^n \), and traces on hyper-planes). First we try to justify in Subsection 4.1 why the classical spaces in (1.1) and now their Morrey generalisations in (1.4) (as reformulations of (1.2) and (1.3)) deserve to be studied. It comes out that the first three of the above-mentioned key properties can be treated rather quickly based on what is already known. The situation is different as far as traces of suitable spaces in (1.4) on \( \mathbb{R}^{n-1} \) are concerned. This gives us the possibility to show how our approach can be used to obtain new substantial assertions about traces and related extensions. In Section 5, we discuss essential features like the precise versions of (1.7), (1.10) and (1.12). This will be complemented by some further topics such as truncations for spaces on \( \mathbb{R}^n \), expansions by Haar wavelets, and Faber expansions. Finally, in Section 6 we concentrate on embeddings again, beginning with the situation of embeddings of spaces on bounded domains, including related compactness and entropy number results. We comment on growth envelope functions which represent some tool to ‘measure’ unboundedness in function spaces. In particular, we always pay attention to these results formulated in the context of the \( \varphi \)-clan according to the slope rules. We end our paper by an outlook when—different from our approach so far—we consider different \( \varphi \)-clans, for example, embeddings between spaces belonging to the \( \varphi_1 \)- and \( \varphi_2 \)-clans with \( \varphi_1 \neq \varphi_2 \). In that sense we ‘cross borders’ between the \( \varphi \)-clan structures we propose here. In our opinion the results, also illustrated in some further diagrams there, are very much convincing to use the new \( \varphi \)-clan setting and follow this line of research further.

2 Definitions and basic properties

2.1 Definitions

We use the standard notations. Let \( \mathbb{N} \) be the collection of all the natural numbers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( \mathbb{R}^n \) be the Euclidean \( n \)-space, where \( n \in \mathbb{N} \). Furthermore, we set \( \mathbb{R} = \mathbb{R}^1 \), and \( \mathbb{C} \) is the complex plane.
Let $S(\mathbb{R}^n)$ be the usual Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, and $S'(\mathbb{R}^n)$ be the dual space of all the tempered distributions on $\mathbb{R}^n$. Let $L_p(\mathbb{R}^n)$ with $0 < p \leq \infty$ be the standard complex quasi-Banach space with respect to the Lebesgue measure in $\mathbb{R}^n$, quasi-normed by

$$
\|f \|_{L_p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}
$$

(2.1)

with the natural modification for $p = \infty$. Similarly, we define $L_p(M)$, where $M$ is a Lebesgue-measurable subset of $\mathbb{R}^n$. As usual $\mathbb{Z}$ is the collection of all integers, and $\mathbb{Z}^n$ where $n \in \mathbb{N}$ denotes the lattice of all the points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. If $\varphi \in S(\mathbb{R}^n)$, then

$$
\hat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi x} \varphi(x) dx, \quad \xi \in \mathbb{R}^n
$$

(2.2)

denotes the Fourier transform of $\varphi$. As usual, $F^{-1} \varphi$ and $\varphi^\vee$ stand for the inverse Fourier transform, which is given by the right-hand side of (2.2) with i instead of $-i$. Note that $x\xi = \sum_{j=1}^n x_j \xi_j$ stands for the scalar product in $\mathbb{R}^n$. Both $F$ and $F^{-1}$ are extended to $S'(\mathbb{R}^n)$ in the standard way.

Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$
\varphi_0(x) = 1, \quad \text{if} \ |x| \leq 1 \quad \text{and} \quad \varphi_0(x) = 0, \quad \text{if} \ |x| > 3/2,
$$

(2.3)

and let

$$
\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}.
$$

(2.4)

Since

$$
\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for} \ x \in \mathbb{R}^n,
$$

(2.5)

$\varphi_j$’s form a dyadic resolution of unity. The entire analytic functions $(\varphi_j \hat{f})^\vee(x)$ make sense pointwise in $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$. Let

$$
Q_{J,M} = 2^{-J}M + 2^{-J}(0,1)^n, \quad J \in \mathbb{Z}, \quad M \in \mathbb{Z}^n
$$

(2.6)

be the usual dyadic cube in $\mathbb{R}^n$ ($n \in \mathbb{N}$) with sides of length $2^{-J}$ parallel to the coordinate axes and with the lower left corner of $2^{-J}M$. If $Q$ is a cube in $\mathbb{R}^n$ and $d > 0$, then $dQ$ is the cube in $\mathbb{R}^n$ concentric to $Q$ whose side-length is $d$ times the side-length of $Q$. Let $|\Omega|$ be the Lebesgue measure of the Lebesgue measurable set $\Omega$ in $\mathbb{R}^n$. Let $a^+ = \max(a,0)$ for $a \in \mathbb{R}$. Furthermore,

$$
a_i \sim b_i \quad \text{for} \ i \in I \quad (\text{equivalence})
$$

(2.7)

for two sets of positive numbers $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ means that there are two positive numbers $c_1$ and $c_2$ such that

$$
c_1a_i \leq b_i \leq c_2a_i \quad \text{for all} \ i \in I.
$$

(2.8)

**Definition 2.1.** Let $n \in \mathbb{N}$ and $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the above dyadic resolution of unity.

(i) Let

$$
0 < p \leq \infty, \quad 0 < q \leq \infty, \quad s \in \mathbb{R}.
$$

(2.9)

Then $B^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$
\|f \|_{B^s_{p,q}(\mathbb{R}^n),\varphi} = \left( \sum_{j=0}^{\infty} 2^{js} ||(\varphi_j \hat{f})^\vee ||_{L_p(\mathbb{R}^n)}^q \right)^{1/q}
$$

(2.10)

is finite (with the usual modification if $q = \infty$).

(ii) Let

$$
0 < p < \infty, \quad 0 < q \leq \infty, \quad s \in \mathbb{R}.
$$

(2.11)
Then $F_{p,q}^a(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$
\|f \| F_{p,q}^a(\mathbb{R}^n) \varphi = \left( \sum_{j=0}^{\infty} 2^{jnq} |(\varphi_j \hat{f})^{\vee}(\cdot)|^q \right)^{1/q} \| L_p(\mathbb{R}^n) \|
$$

(2.12)
is finite (with the usual modification if $q = \infty$).

(iii) Let $0 < q \leq \infty$ and $s \in \mathbb{R}$. Then $F_{\infty,q}^a(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$
\|f \| F_{\infty,q}^a(\mathbb{R}^n) \varphi = \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{jn/q} \left( \int_{Q_{J,M}} \sum_{j > J} 2^{jsq} |(\varphi_j \hat{f})^{\vee}(x)|^q dx \right)^{1/q}
$$

(2.13)
is finite (with the usual modification if $q = \infty$ as explained below).

**Remark 2.2.** These are the classical spaces $A_{p,q}^a(\mathbb{R}^n)$ according to (1.1). The above definition coincides with [82, Definition 1.1, p. 2, including the part (iii)]. There one also finds some discussions and (historical) references, especially about the spaces $F_{\infty,q}^a(\mathbb{R}^n)$. Let us mention here, in particular, the series of monographs [72, 73, 75] and the long paper [12]. In particular, it is well known that the spaces in the above definition are independent of the chosen resolution of unity $\varphi$ (with equivalent quasi-norms). This justifies our omission of the subscript $\varphi$ in (2.10), (2.12) and (2.13) in the sequel. They are quasi-Banach spaces (and Banach spaces if $p \geq 1$ and $q \geq 1$). As remarked in [82, p. 3], one can replace $J \in \mathbb{Z}$ in (2.13) by $J \in \mathbb{N}_0$ (with equivalent quasi-norms), i.e.,

$$
\|f \| F_{\infty,q}^a(\mathbb{R}^n) \varphi = \sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} 2^{jn/q} \left( \int_{Q_{J,M}} \sum_{j > J} 2^{jsq} |(\varphi_j \hat{f})^{\vee}(x)|^q dx \right)^{1/q}.
$$

(2.14)

If $q = \infty$, then (2.13) and (2.14) must be understood as

$$
\|f \| F_{\infty,\infty}^a(\mathbb{R}^n) \varphi \sim \sup_{j \in \mathbb{N}_0, x \in \mathbb{R}^n} 2^{jn} |(\varphi_j \hat{f})^{\vee}(x)| = \|f \| B_{\infty,\infty}^a(\mathbb{R}^n) \|
$$

(2.15)

In other words, the Hölder-Zygmund spaces $C_s^s(\mathbb{R}^n) = B_{s,\infty}^a(\mathbb{R}^n)$ can be incorporated into the $F$-scale as

$$
F_{s,\infty}^a(\mathbb{R}^n) = B_{s,\infty}^a(\mathbb{R}^n) = C^s(\mathbb{R}^n), \quad s \in \mathbb{R}.
$$

(2.16)

The spaces in (1.2) generalise the above spaces $A_{p,q}^a(\mathbb{R}^n)$ with $p < \infty$ replacing there the Lebesgue spaces $L_p(\mathbb{R}^n)$ by the well-known Morrey spaces which we adapt to our later purposes as follows. As usual, $f \in L_p^{loc}(\mathbb{R}^n)$ means that the restriction of $f$ to any bounded Lebesgue-measurable set $M$ in $\mathbb{R}^n$ belongs to $L_p(M)$ ($0 < p \lesssim \infty$). Let again $Q_{J,M}$ be the cubes in (2.6).

**Definition 2.3.** Let $n \in \mathbb{N}$, $0 < p < \infty$ and $-n \leq g < 0$. Then $\Lambda_p^g(\mathbb{R}^n)$ collects all $f \in L_p^{loc}(\mathbb{R}^n)$ such that

$$
\|f \| \Lambda_p^g(\mathbb{R}^n) \varphi = \sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} 2^{jn/2} \|f \| L_p(Q_{J,M})
$$

(2.17)
is finite.

**Remark 2.4.** Obviously, $\Lambda_p^g(\mathbb{R}^n)$ are quasi-Banach spaces (and Banach spaces if $p \geq 1$). Note that by a Lebesgue point argument one observes immediately that $\Lambda_p^g(\mathbb{R}^n) = \{0\}$ when $g > 0$ or $g < -n$, and $\Lambda_p^0(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. This shows that the restriction $-n \leq g < 0$ in Definition 2.3 is natural. Furthermore,

$$
\Lambda_p^{-n}(\mathbb{R}^n) = L_p(\mathbb{R}^n), \quad 0 < p < \infty.
$$

(2.18)

The Morrey space $M_p^a(\mathbb{R}^n)$ with $0 < p \leq u < \infty$ in common use collects all $f \in L_p^{loc}(\mathbb{R}^n)$ such that

$$
\|f \| M_p^a(\mathbb{R}^n) \varphi = \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{jn(\frac{1}{u} - \frac{1}{p})} \|f \| L_p(Q_{J,M})
$$

(2.19)
is finite. Compared with (2.17) one obtains

$$
\Lambda_p^g(\mathbb{R}^n) = M_p^a(\mathbb{R}^n), \quad 0 < p < \infty, \quad ug + np = 0.
$$

(2.20)
Remark 2.5. The idea of Morrey spaces $\mathcal{M}^p_u(\mathbb{R}^n)$ ($0 < p \leq u < \infty$) goes back to Morrey [49], dealing with the regularity of solutions of some partial differential equations. They are part of a wider class of Morrey-Campanato spaces (see [52]), and can be seen as a complement to $L_p$ spaces, since $\mathcal{M}^p_u(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. In an analogous way, one can define the spaces $\mathcal{M}^p_u(\mathbb{R}^n)$ ($p \in (0, \infty)$), but using the Lebesgue differentiation theorem, one can easily prove that $\mathcal{M}^\infty_u(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. So we usually restrict ourselves to $u < \infty$.

We replace $L_p(\mathbb{R}^n)$ with $p < \infty$ in Definition 2.1 by the above spaces $\Lambda^s_p(\mathbb{R}^n)$ whereas all the other notations have the same meaning as there.

Definition 2.6. Let $n \in \mathbb{N}$, $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and $-n \leq \varrho < 0$. Then $\Lambda^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \mid \Lambda^s_{p,q}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jq} \|\varphi_j \hat{f}\|_q \mid \Lambda^s_p(\mathbb{R}^n)\|_q\right)^{1/q}$$

(2.21)

is finite (with the usual modification if $q = \infty$) and $\Lambda^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \mid \Lambda^s_{p,q}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jq} \|\varphi_j \hat{f}\|_q \mid \Lambda^s_p(\mathbb{R}^n)\|_q\right)^{1/q}$$

(2.22)

is finite (with the usual modification if $q = \infty$).

Remark 2.7. By (2.18), one has

$$\Lambda^{-n}_{p,q}(\mathbb{R}^n) = A^s_{p,q}(\mathbb{R}^n), \quad A \in \{B, F\}.$$ 

(2.23)

Using (2.20) and the standard definitions in the literature, we can see that it comes out that

$$\Lambda^s_{p,q}(\mathbb{R}^n) = \Lambda^s_{u,p,q}(\mathbb{R}^n), \quad u_p + np = 0$$

(2.24)

and

$$\Lambda^s_{p,q}(\mathbb{R}^n) = \mathcal{E}^s_{u,p,q}(\mathbb{R}^n), \quad u_p + np = 0.$$ 

(2.25)

These spaces attracted some attention in the last decades. In particular, they are quasi-Banach spaces which are independent of $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ (with equivalent quasi-norms).

Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^n)$ were introduced by Kozono and Yamazaki [37]. They studied semi-linear heat equations and Navier-Stokes equations with initial data belonging to Besov-Morrey spaces. The investigations were continued by Mazzucato [47], where one can find the atomic decomposition of some spaces. The Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^n)$ were later introduced by Tang and Xu [70]. The ideas were further developed by Sawano and Tanaka [56, 57, 62, 63]. The most systematic and general approach to the spaces of this type can be found in the monograph [94] or in the survey papers by Sickel [67, 68], which we also recommend for further up-to-date references on this subject. We refer to the recent monographs [60, 61] for applications.

It turns out that many of the results from the classical situation have their counterparts for the spaces $\mathcal{A}^s_{u,p,q}(\mathbb{R}^n)$, e.g., in view of elementary embeddings. However, there also exist some differences. Sawano [56] proved that for $s \in \mathbb{R}$ and $0 < p < u < \infty$,

$$\mathcal{N}^s_{u,p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^s_{u,p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{N}^s_{u,p,\infty}(\mathbb{R}^n),$$

(2.26)

where in the latter embedding $\infty$ cannot be improved—unlike in the classical case of $u = p$. On the other hand, Mazzucato [47, Proposition 4.1] has shown that

$$\mathcal{E}^0_{u,p,q}(\mathbb{R}^n) = \mathcal{M}^u_p(\mathbb{R}^n), \quad 1 < p \leq u < \infty.$$ 

Next, we introduce a second type of Morrey smoothness spaces. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ and $Q_{l,M}$ be as in (2.3)–(2.6).
Definition 2.8. Let \( n \in \mathbb{N}, s \in \mathbb{R} \) and \( 0 < p < \infty, 0 < q \leq \infty. \) Let \(-n \leq \varrho < \infty.\) Then \( \Lambda^e B^s_{p,q}(\mathbb{R}^n) \) is the collection of all \( f \in S'((\mathbb{R}^n)) \) such that

\[
\|f| \Lambda^e B^s_{p,q}(\mathbb{R}^n)\| = \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{\frac{s}{2}(n+\varrho)} \left( \sum_{j \geq J^+} 2^{sjq} \|(|f_j|^{1/q} \| L_p(Q,J,M) \|^q \right)^{1/q} \tag{2.27}
\]

is finite and \( \Lambda^e F^s_{p,q}(\mathbb{R}^n) \) is the collection of all \( f \in S'((\mathbb{R}^n)) \) such that

\[
\|f| \Lambda^e F^s_{p,q}(\mathbb{R}^n)\| = \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{\frac{s}{2}(n+\varrho)} \left( \sum_{j \geq J^+} 2^{sjq} \|(|f_j|^{1/q} \| L_p(Q,J,M) \|^q \right)^{1/q} \tag{2.28}
\]

is finite (with the usual modification if \( q = \infty \)).

Remark 2.9. By measure-theoretical arguments, one has

\[
\Lambda^{-n} A^s_{p,q}(\mathbb{R}^n) = A^s_{p,q}(\mathbb{R}^n), \quad A \in \{B,F\} \tag{2.29}
\]

for all admitted \( s, p, q. \) Otherwise these spaces are reformulations of the corresponding Morrey smoothness spaces \( A^s_{p,q}(\mathbb{R}^n) \) which attracted a lot of attention, i.e.,

\[
\Lambda^e A^s_{p,q}(\mathbb{R}^n) = A^s_{p,q}(\mathbb{R}^n), \quad s \in \mathbb{R}, \quad 0 < p < \infty, \quad 0 < q \leq \infty, \tag{2.30}
\]

where

\[
\tau = \frac{1}{p} \left( 1 + \frac{\varrho}{n} \right), \quad -n \leq \varrho < \infty. \tag{2.31}
\]

The so-called Besov-type spaces \( B^s_{p,q} \) and Triebel-Lizorkin-type spaces \( F^s_{p,q}, \) commonly denoted by \( A^s_{p,q}, \) nowadays with \( A \in \{B,F\}, \) were introduced in [94] and proved therein to be quasi-Banach spaces. In the Banach case, the scale of Besov type spaces had already been introduced and investigated in [1–3], and also by Yuan and Yang [87, 88]. We refer again to the monograph [94] and the fine survey papers [67, 68] for further background information and references. In [42] an even more general approach was studied.

These spaces coincide for all the admitted parameters with the hybrid spaces

\[
L^r A^s_{p,q}(\mathbb{R}^n) = \Lambda^e A^s_{p,q}(\mathbb{R}^n), \quad -n \leq \varrho < \infty, \quad 0 < p < \infty, \quad r = \frac{q}{p} \quad 0 < q \leq \infty, \tag{2.32}
\]

according to [82, Definition 1.6, p. 6] and the references given there to [80, Definition 3.36, pp. 68–69] and [79]. We refer to [95] for some discussion of the different approaches. There and in [79, 82] one also finds detailed (historical) references. In contrast to Definition 2.6, we admitted in Definition 2.8 also \( \varrho \geq 0 \) (in good agreement with \( \tau \geq 0 \) in (2.31)). But Proposition 2.12 below makes clear why we concentrate later on in both Definitions 2.6 and 2.8 on \( -n < \varrho < 0.\)

Remark 2.10. We briefly compare the two scales \( A^s_{u,p,q}(\mathbb{R}^n) \) and \( A^s_{p,q}(\mathbb{R}^n) \). It is known that

\[
\Lambda^e A^u_{u,p,q}(\mathbb{R}^n) \to B^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad \tau = 1/p - 1/u \tag{2.33}
\]

(see [94, Corollary 3.3]). Moreover, this embedding is proper if \( \tau > 0 \) and \( q < \infty. \) If \( \tau = 0 \) or \( q = \infty, \) then both spaces coincide with each other, i.e., \( \Lambda^e A^s_{u,p,q}(\mathbb{R}^n) = B^s_{p,\infty}(\mathbb{R}^n). \) As for the \( F \)-spaces, if \( 0 \leq \tau < 1/p, \) then

\[
F^s_{p,q}(\mathbb{R}^n) = E^s_{u,p,q}(\mathbb{R}^n) \quad \text{with} \quad \tau = 1/p - 1/u, \quad 0 < p \leq u < \infty \tag{2.34}
\]

(see [94, Corollary 3.3], [64, Theorem 1.1(ii), p. 74] and [59, Theorem 6.35, p. 794]). Moreover, if \( p \in (0, \infty) \) and \( q \in (0, \infty), \) then

\[
F^s_{p,q} (\mathbb{R}^n) = F^s_{\infty,q} (\mathbb{R}^n) = B^s_{q,\infty} (\mathbb{R}^n) \tag{2.35}
\]

(see [67, Propositions 3.4 and 3.5] and [68, Remark 10]).
Remark 2.11. Recall that the space $\text{bmo}(\mathbb{R}^n)$ is covered by the above scales: consider the local (non-homogeneous) space of functions of bounded mean oscillation, $\text{bmo}(\mathbb{R}^n)$, consisting of all the locally integrable functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfying that

$$
\|f\|_{\text{bmo}} := \sup_{|Q| \leq 1} \frac{1}{|Q|} \int_Q |f(x) - x_Q| dx + \sup_{|Q| > 1} \frac{1}{|Q|} \int_Q |f(x)| dx < \infty,
$$

where $Q$ appearing in the above definition runs over all the cubes in $\mathbb{R}^n$, and $x_Q$ denotes the mean value of $f$ with respect to $Q$, namely, $x_Q := \frac{1}{|Q|} \int_Q f(x) dx$ (see [72, 2.2.2(viii)]). The space $\text{bmo}(\mathbb{R}^n)$ coincides with $F^0_{\infty,2}(\mathbb{R}^n)$ (see [72, Theorem 2.5.8/2]). Hence, the above result (2.35) implies, in particular,

$$
\text{bmo}(\mathbb{R}^n) = F^0_{\infty,2}(\mathbb{R}^n) = F^{0,1/p}_{p,2}(\mathbb{R}^n) = B^{0,1/2}_{2,2}(\mathbb{R}^n), \quad 0 < p < \infty.
$$

Recall that $C^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n)$ ($\sigma \in \mathbb{R}$) are the above Hölder-Zygmund spaces in (2.16). A continuous embedding, indicated as usual by $\hookrightarrow$, is called strict if the two spaces involved do not coincide.

**Proposition 2.12.** Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and $A \in \{B,F\}$.

(i) Then

$$
\Lambda^{-n} A^s_{p,q}(\mathbb{R}^n) = \Lambda^{-n} A^s_{p,q}(\mathbb{R}^n) = \Lambda^s_{p,q}(\mathbb{R}^n).
$$

(ii) Let $q > 0$. Then

$$
\Lambda^s A^s_{p,q}(\mathbb{R}^n) = C^{s+\frac{\tau}{p}}(\mathbb{R}^n).
$$

(iii) Let in addition $0 < q_1 < q_2 < \infty$. Then

$$
B^s_{\infty,q_1}(\mathbb{R}^n) \hookrightarrow \Lambda^s B^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow \Lambda^s B^s_{p,q_2}(\mathbb{R}^n) \hookrightarrow \Lambda^s B^s_{p,\infty}(\mathbb{R}^n) = C^s(\mathbb{R}^n).
$$

All the embeddings are strict. Furthermore,

$$
\Lambda^s F^s_{p,q}(\mathbb{R}^n) = F^s_{q,q}(\mathbb{R}^n)
$$

and

$$
\Lambda^s F^s_{p,p}(\mathbb{R}^n) = \Lambda^s B^s_{p,p}(\mathbb{R}^n) = F^s_{\infty,p}(\mathbb{R}^n).
$$

**Proof.** Part (i) summarises (2.23) and (2.29). Part (ii) is covered by [80, Proposition 3.54, p.92] and (2.32). The first strict embedding in (2.38), the last coincidence in (2.38) and also (2.39) follow from (2.30) with (2.31) together with (2.35), and may also be found in [82, Proposition 1.18, pp.12–13]. The related proofs rely on wavelet arguments. This can also be used to justify that the remaining (rather obvious) embeddings in (2.38) are also strict. Finally one obtains (2.40) from (2.27), (2.28) with $q = p$ and (2.13). \qed

Remark 2.13. Note that (2.37), using the coincidence (2.30) with (2.31), was already obtained in [89] as $A^\tau s = B^{s+n(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^n)$ whenever $\tau > \frac{1}{p}$ or $\tau = \frac{1}{p}$ and $q = \infty$. In general, for arbitrary $\tau \geq 0$, then $A^\tau s = B^{s+n(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^n)$ (see [94, Proposition 2.6]).

**Remark 2.14.** It makes sense to extend the definition of the spaces $\Lambda^s B^s_{p,q}(\mathbb{R}^n)$ to $p = \infty$. But it follows from (2.27) that the related spaces coincide with $B^s_{\infty,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$ and $0 < q \leq \infty$. The situation is somewhat different as far as the corresponding hybrid spaces

$$
L^r B^s_{\infty,q}(\mathbb{R}^n), \quad r \geq 0, \quad s \in \mathbb{R}, \quad 0 < q \leq \infty
$$

are concerned. For this purpose, one has first to replace the factor $2^{\frac{n}{2}(n+\theta)}$ in (2.27) by $2^{(\frac{n}{2}+\tau)}$ and to choose afterwards $p = \infty$. This gives the factor $2^{\frac{n}{2}r}$. If $r = 0$, then one has again $B^s_{\infty,q}(\mathbb{R}^n)$. If $r > 0$, then it follows from [80, Proposition 3.54, p.92] that the corresponding spaces coincide with $C^{s+r}(\mathbb{R}^n)$.\]
2.2 The $\varrho$-clans

Based on the above definitions and coincidences, it is obvious that the classical spaces

$$A^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad A \in \{B,F\}, \quad s \in \mathbb{R} \quad \text{and} \quad 0 < p, q \leq \infty \quad (2.42)$$

(including $F^s_{\infty,q}(\mathbb{R}^n)$) have to be complemented by

$$\Lambda^eA^s_{p,q}(\mathbb{R}^n) \quad \text{and} \quad A^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad A \in \{B,F\}, \quad s \in \mathbb{R}, \quad 0 < p < \infty, \quad 0 < q \leq \infty, \quad -n < \varrho < 0, \quad (2.43)$$
on the one hand and

$$\Lambda^0B^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad s \in \mathbb{R} \quad \text{and} \quad 0 < p, q < \infty, \quad (2.44)$$
on the other hand, as far as the most prominent scales of Morrey smoothness spaces are concerned. This can be seen from Definitions 2.6 and 2.8 together with (2.24) and (2.25) for the spaces of type $A^s_{p,q}(\mathbb{R}^n)$, from (2.30) with (2.31) for the spaces of type $A^e_{p,q}(\mathbb{R}^n)$, and from Remark 2.14 for the hybrid spaces. According to Proposition 2.12 we regain the classical spaces (2.42) from $\Lambda^eA^s_{p,q}(\mathbb{R}^n)$ and $A^s_{p,q}(\mathbb{R}^n)$ for $\varrho = -n$ and $\varrho = 0$, respectively. For the somewhat peculiar spaces in (2.44), one has (2.38) and (2.40) indicating that they are closely related to the spaces $A^e_{\infty,q}(\mathbb{R}^n)$. We return to them occasionally, but concentrate otherwise on the spaces in (2.43). One may ask how these spaces are related to each other. This attracted some attention in the literature and will be discussed later on in some details. Above all we rely on the coincidence

$$\Lambda^eF^s_{p,q}(\mathbb{R}^n) = \Lambda^eF^s_{p,q}(\mathbb{R}^n), \quad s \in \mathbb{R}, \quad 0 < p < \infty, \quad 0 < q \leq \infty, \quad -n \leq \varrho < 0, \quad (2.45)$$
extending (2.36) to the related $F$-spaces with $-n < \varrho < 0$. This follows from (2.25), (2.30) and (2.31), inserted in the well-known coincidence (2.34). In other words, one has only one $\Lambda^eF = \Lambda^eF$ scale in (2.43), but two $\Lambda^eB$ and $\Lambda^eB$ scales.

**Definition 2.15.** Let $n \in \mathbb{N}$.

(i) The $n$-clan consists of the spaces according to (2.42).

(ii) For $-n < \varrho < 0$, the $\varrho$-clan $\varrho\Lambda^e_{A^s_{p,q}}(\mathbb{R}^n)$ consists of the three families

$$\Lambda^eB^s_{p,q}(\mathbb{R}^n), \quad \Lambda^eB^s_{p,q}(\mathbb{R}^n) \quad \text{and} \quad \Lambda^eF^s_{p,q}(\mathbb{R}^n) = \Lambda^eF^s_{p,q}(\mathbb{R}^n) \quad (2.46)$$

with

$$s \in \mathbb{R}, \quad 0 < p < \infty, \quad 0 < q \leq \infty. \quad (2.47)$$

(iii) The 0-clan consists of the spaces according to (2.44).

For convenience, we have sketched the situation for the new spaces defined above in Remark 2.23 below.

**Remark 2.16.** As already said, we are mainly interested in the $\varrho$-clans in the part (ii) of the above definition. The theory of the classical spaces in (2.42), now called the $n$-clan, has been developed over decades. This is well reflected in many books, including [72, 73, 75, 82]. In particular, [82] may be considered as a collection of final answers about distinguished problems for these spaces, including $F^s_{\infty,q}(\mathbb{R}^n)$. This can be taken as a guide asking the same questions for the larger community covered by the above definition. It is the main aim of this paper to make clear that the above classification of the Morrey smoothness spaces in common use produces transparent answers having the same impressive beauty as their ancestors in the $n$-clan. The first examples had already been described in Section 1, (1.7) compared with (1.6), (1.10) compared with (1.9), and also (1.12) compared with (1.11). We formulate these expectations as follows.

**Slope Rules 2.17.** Let $n \in \mathbb{N}$ and $-n < \varrho < 0$.

(i) An extension of an adequate assertion for the spaces $A^s_{p,q}(\mathbb{R}^n)$ of the $n$-clan to the spaces (2.46) and (2.47) of the $\varrho$-clan $\varrho\Lambda^e_{A^s_{p,q}}(\mathbb{R}^n)$ is subject to the Slope-1-Rule if it depends on $\frac{1}{p} \min(|\varrho|, 1)$ (instead of $\frac{1}{p}$).
(ii) An extension of an adequate assertion for the spaces $A^s_{p,q}(\mathbb{R}^n)$ of the $n$-clan to the spaces (2.46) and (2.47) of the $\varrho$-clan $\varrho A^s_{p,q}(\mathbb{R}^n)$ is subject to the Slope-$n$-Rule if it depends on $|1/\varrho|$ (instead of $1/\varrho$).

**Remark 2.18.** These slope rules are illustrated by the examples in Section 1. We call $\varrho$ the slope parameter although not $\varrho$ itself but $|\varrho|$ is the slope of the corresponding lines in the $(1,1,n)$-diagram, as it can be seen exemplarily in Figure 1 on page 1328 below. This suggests replacing $-n < \varrho < 0$ by $0 < \varrho < n$. But we stick to the usual habit that larger values of a fixed parameter produce smaller spaces. This applies to the smoothness $s$, and also to the integrability $p$ for related spaces on bounded domains, but not to the fine index $q$. It is also in good agreement with the parameters $\tau$ in (2.30) and (2.31), $r$ in (2.32) and $\varrho > 0$ in (2.37).

### 2.3 Relations and coincidences

The already indicated sharp embeddings in (1.7) and (1.10) (as far as breaking lines in the $(1,1,n)$-diagrams in Figure 1 on page 1328 and Figure 2 on page 1330 are concerned) suggest that, in general, spaces belonging to different $\varrho$-clans do not coincide. At least it seems to be reasonable to discuss relations and coincidences within a fixed $\varrho$-clan. Again one can take the $n$-clan, consisting of the spaces $A^s_{p,q}(\mathbb{R}^n)$ in (2.42) as a guide. We repeat the final answer as it may be found in [72, Subsection 2.3.9, pp.61–62] (under the additional assumption $p_1,p_2 < \infty$ in the $F$-case), and in [82, Theorem 2.10, p.28] in the above complete form.

**Proposition 2.19.** Let $n \in \mathbb{N}$, $0 < p_1, p_2, q_1, q_2 \leq \infty$ and $s_1 \in \mathbb{R}, s_2 \in \mathbb{R}$. Then

$$B^s_{p_1,q_1}(\mathbb{R}^n) = B^s_{p_2,q_2}(\mathbb{R}^n) \text{ if and only if } s_1 = s_2, \quad p_1 = p_2, \quad q_1 = q_2,$$

and

$$F^s_{p_1,q_1}(\mathbb{R}^n) = F^s_{p_2,q_2}(\mathbb{R}^n) \text{ if and only if } s_1 = s_2, \quad p_1 = p_2, \quad q_1 = q_2.$$

**Remark 2.20.** By [82, Theorem 2.9, p.26], one has for $s \in \mathbb{R}$ and $0 < p,q,u,v \leq \infty$ that

$$B^s_{p,q}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,u}(\mathbb{R}^n)$$

if and only if

$$0 < u \leq \min(p,q) \quad \text{and} \quad \max(p,q) \leq v \leq \infty.$$

If $p < \infty$, then this is just the well-known result in [69, Theorem 3.1.1].

One may ask for counterparts within the fixed $\varrho$-clan $\varrho A^s_{p,q}(\mathbb{R}^n)$. Instead of the two families $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ for the $n$-clan, one has now to cope with the three families in (2.46). By (2.21) and (2.27), the corresponding $B$-spaces are quasi-normed by

$$\|f\|_{\Lambda_{\varrho}B^s_{p,q}(\mathbb{R}^n)} = \left( \sum_{j=0}^{\infty} 2^{jq} \sup_{Q \in \mathcal{Z}, M \in \mathcal{Z}^n} 2^{j(n+\varrho)q} \|\varphi_j f\|_{L_p(Q,M)}^q \right)^{1/q}$$

and

$$\|f\|_{\Lambda_{\varrho}E^s_{p,q}(\mathbb{R}^n)} = \sup_{Q \in \mathcal{Z}, M \in \mathcal{Z}^n} 2^{j(n+\varrho)q} \left( \sum_{j \in \mathcal{J}} 2^{jq} \|\varphi_j f\|_{L_p(Q,M)}^q \right)^{1/q}$$

(with the usual modification if $q = \infty$). These explicit versions will be of some use, now and later on. Recall that the continuous embedding $\hookrightarrow$ is called strict if the two spaces in question do not coincide.

**Theorem 2.21.** Let $n \in \mathbb{N}$, $-n < \varrho < 0$ and $s_1, s_2, s_3 \in \mathbb{R}$. Let $0 < p, p_1, p_2 < \infty$ and $0 < q, q_1, q_2 \leq \infty$.

(i) The spaces $\Lambda_{\varrho}B^s_{p_1,q_1}(\mathbb{R}^n)$ and $\Lambda_{\varrho}B^s_{p_2,q_2}(\mathbb{R}^n)$ coincide if and only if $p_1 = p_2, q_1 = q_2, s_1 = s_2$.

Similarly the spaces $\Lambda_{\varrho}F^s_{p_1,q_1}(\mathbb{R}^n)$ and $\Lambda_{\varrho}F^s_{p_2,q_2}(\mathbb{R}^n)$ coincide if and only if $p_1 = p_2, q_1 = q_2, s_1 = s_2$. 
(ii) The spaces \( \Lambda_\varepsilon B_{p,1}^{s_1}(\mathbb{R}^n) \) and \( \Lambda_\varepsilon F_{p_2,q_2}^{s_2}(\mathbb{R}^n) \) do not coincide. Furthermore,

\[
\Lambda_\varepsilon B_{p,\text{min}(p,q)}^{s}(\mathbb{R}^n) \hookrightarrow \Lambda_\varepsilon F_{p,q}^{s}(\mathbb{R}^n) \hookrightarrow \Lambda_\varepsilon B_{p,\infty}^{s}(\mathbb{R}^n). \tag{2.55}
\]

(iii) The spaces \( \Lambda_\varepsilon F_{p_1,q_1}^{s_1}(\mathbb{R}^n) \) and \( \Lambda_\varepsilon F_{p_2,q_2}^{s_2}(\mathbb{R}^n) \) coincide if and only if \( p_1 = p_2, q_1 = q_2, s_1 = s_2 \). Furthermore,

\[
\Lambda_\varepsilon B_{p,\text{min}(p,q)}^{s}(\mathbb{R}^n) \hookrightarrow \Lambda_\varepsilon F_{p,q}^{s}(\mathbb{R}^n) \hookrightarrow \Lambda_\varepsilon B_{p,\text{max}(p,q)}^{s}(\mathbb{R}^n) \tag{2.56}
\]

and

\[
\Lambda_\varepsilon B_{p,p}^{s}(\mathbb{R}^n) = \Lambda_\varepsilon F_{p,p}^{s}(\mathbb{R}^n). \tag{2.57}
\]

(iv) If \( q < \infty \), then

\[
\Lambda_\varepsilon B_{p,q}^{s}(\mathbb{R}^n) \hookrightarrow \Lambda_\varepsilon F_{p,q}^{s}(\mathbb{R}^n) \tag{2.58}
\]

is a strict embedding. Furthermore,

\[
\Lambda_\varepsilon B_{s}^{s}(\mathbb{R}^n) = \Lambda_\varepsilon B_{s,\infty}^{s}(\mathbb{R}^n) \tag{2.59}
\]

and

\[
\Lambda_\varepsilon F_{s}^{s}(\mathbb{R}^n) = \Lambda_\varepsilon F_{s,\infty}^{s}(\mathbb{R}^n). \tag{2.60}
\]

Proof. Step 1. The parts (i) and (ii) are covered by [56, Proposition 1.3 and Theorem 1.7, pp. 96 and 98]. The property (2.60) has already been mentioned in (2.45). There one also finds related references. Then it follows from the part (i) that the spaces \( \Lambda_\varepsilon F_{p_1,q_1}^{s_1}(\mathbb{R}^n) \) and \( \Lambda_\varepsilon F_{p_2,q_2}^{s_2}(\mathbb{R}^n) \) coincide only trivially. The well-known embeddings (2.56) and also (2.57) can be obtained by the same arguments as for the classical spaces \( \Lambda_{p,q}^{s}(\mathbb{R}^n) \), based on Definitions 2.1 and 2.8 and also (2.51) and (2.52). We refer to [94, Proposition 2.1].

Step 2. The embedding (2.58) follows from (2.53) and (2.54). According to [67, Proposition 3.1, p. 121], one can replace \( j \geq J^+ \) in (2.54) by \( j \in \mathbb{N}_0 \) (with equivalent quasi-norms). Then (2.53) and (2.54) with \( q = \infty \) coincide. This proves (2.59). It remains to prove that the embedding (2.58) is strict if \( q < \infty \). This follows from Theorem 5.18 below if, in addition, \( s = \frac{ml}{p} - n \), and can be extended to all \( s \in \mathbb{R} \) by lifting according to Theorem 3.8 below. □

Remark 2.22. The assertions of the above theorem are more or less known. But they are somewhat scattered in the literature and not related to the \( \rho \)-class. Furthermore, we want to show that Theorem 5.18 can be used to justify that the embedding (2.58) is strict. As far as the coincidences of the spaces \( \Lambda_\varepsilon A_{p,q}^{s}(\mathbb{R}^n) \) are concerned we refer the reader again to [56] for a more general version. In particular, there is no need to fix \( \rho \) from the very beginning. The homogeneous counterpart of the part (iv) may be found in [64, Theorem 1.1, p. 74]. The justification of (2.59) for the inhomogeneous version requires that one can replace the natural condition \( j \geq J^{-} \) in (2.54) by \( j \in \mathbb{N}_0 \). This has been used in Step 2 with a reference to [67]. We would like to point to the difference between (2.55) and (2.56) concerning the last embedding: while in the case of spaces \( \Lambda_\varepsilon A_{p,q}^{s}(\mathbb{R}^n) \), the well-known behaviour as described in Remark 2.20 is preserved in view of the fine index \( q \), this is no longer true for spaces \( \Lambda_\varepsilon A_{p,q}^{s}(\mathbb{R}^n) \) where \( q = \infty \) cannot be improved—unlike in the case of \( \rho = -n \) (see [56]).

Remark 2.23. In view of the coincidences and embeddings within the \( B \)- and \( F \)-scales according to Theorem 2.21, we may illustrate the situation for the new spaces \( \rho A_{p,q}^{s}(\mathbb{R}^n) \) with \( -n < \rho < 0 \) defined in Definition 2.15 as follows:

\[
\rho A_{p,q}^{s}(\mathbb{R}^n)
\]

\[
\Lambda_\varepsilon A_{p,q}^{s}(\mathbb{R}^n) \twoheadrightarrow \Lambda_\varepsilon B_{p,q}^{s}(\mathbb{R}^n) \hookrightarrow \Lambda_\varepsilon F_{p,q}^{s}(\mathbb{R}^n) \hookrightarrow \Lambda_\varepsilon A_{p,q}^{s}(\mathbb{R}^n).
\]
By Proposition 2.19, there are no other coincidences within the \( n \)-clan, the classical spaces in (2.42), than the obvious ones. This suggests asking for coincidences within a fixed \( \varrho \)-clan as introduced in Definition 2.15(ii) or within all the spaces covered by Definition 2.15 for fixed \( n \in \mathbb{N} \). We fix the expected ideal outcome that within a fixed \( \varrho \)-clan there are no further coincidences than the ones listed in Theorem 2.21.

**Conjecture 2.24.** Let \( n \in \mathbb{N} \), \( -n < \varrho < 0 \) and \( s, s_1, s_2 \in \mathbb{R} \). Let \( 0 < p, p_1, p_2 < \infty \) and \( 0 < q, q_1, q_2 \leq \infty \).

(i) Theorem 2.21(i) remains valid if one replaces \( \Lambda^{\varrho} \) by \( \Lambda^{\varrho} \).

(ii) There are no further coincidences for the spaces belonging to the same \( \varrho \)-clan than (2.57), (2.59) and (2.60).

**Remark 2.25.** We did not deal with coincidences, diversities and strict embeddings within a fixed \( \varrho \)-clan other than the ones stated in Theorem 2.21. The above conjecture may be considered as a request to pay more attention to the questions of this type. One could try to use wavelet expansions as described in Subsection 3.1 for all the spaces covered by Definition 2.8 to prove or disprove the above conjecture or parts of it. Then \( \varrho < 0 \) has to play a decisive role. If \( \varrho \geq 0 \), then Proposition 2.12 shows that there are many coincidences. For example, it follows from (2.37) that

\[
\Lambda^{\varrho} F^{s_1}_{p_1,q_1}(\mathbb{R}^n) = \Lambda^{\varrho} F^{s_2}_{p_2,q_2}(\mathbb{R}^n), \quad \text{if} \quad \varrho > 0 \quad \text{and} \quad s_1 + \frac{\varrho}{p_1} = s_2 + \frac{\varrho}{p_2}. \tag{2.61}
\]

One could also discuss the above conjecture in the limelight of distinguished properties of the above spaces. For example, if \(-1 < \varrho < 0\), then it follows from Theorem 4.12 below that spaces belonging to the same \( \varrho \)-clan cannot coincide if their differential dimensions \( s - \frac{\varrho}{p} \) are different. It is an even more challenging request to extend Proposition 2.19 from the \( n \)-clan to all the clans covered by Definition 2.15. Then Proposition 2.12(iii) shows that there are further coincidences.

### 3 Tools

#### 3.1 Wavelets

Some proofs below rely on wavelet arguments. We collect what we need restricting us to the bare minimum. We follow [82, Subsection 1.2, pp.7–12], where one finds further (and more comprehensive) assertions, explanations and references. Let us, in particular, refer to the standard monographs [7,43,48,85].

As usual, \( C^u(\mathbb{R}) \) with \( u \in \mathbb{N} \) collects all the bounded complex-valued continuous functions on \( \mathbb{R} \) having continuous bounded derivatives up to order \( u \) inclusively (see also (4.4) below for its \( n \)-dimensional version). Let

\[
\psi_F \in C^u(\mathbb{R}), \quad \psi_M \in C^u(\mathbb{R}), \quad u \in \mathbb{N} \tag{3.1}
\]

be real compactly supported Daubechies wavelets with

\[
\int_{\mathbb{R}} \psi_M(t) t^v dt = 0 \quad \text{for all} \quad v \in \mathbb{N}_0 \quad \text{with} \quad v < u. \tag{3.2}
\]

Recall that \( \psi_F \) is called the scaling function (father wavelet) and \( \psi_M \) is called the associated wavelet (mother wavelet). We extend these wavelets from \( \mathbb{R} \) to \( \mathbb{R}^n \) by the usual multiresolution procedure. Let \( n \in \mathbb{N} \), and

\[
G = (G_1, \ldots, G_n) \in G^0 = \{F,M\}^n, \tag{3.3}
\]

which means that \( G_r \) is either \( F \) or \( M \). Furthermore, let

\[
G = (G_1, \ldots, G_n) \in G^n = G^j = \{F,M\}^{n*}, \quad j \in \mathbb{N}, \tag{3.4}
\]
which means that $G_r$ is either $F$ or $M$, where $s$ indicates that at least one of the components of $G$ must be an $M$. Hence, $G^0$ has $2^n$ elements, whereas $G^j$ with $j \in \mathbb{N}$ and $G^*$ have $2^n - 1$ elements. Let

$$
\psi_{G,m}^j(x) = \prod_{i=1}^n \psi_{G_i}(2^j x_i - m_i), \quad G \in G^j, \quad m \in \mathbb{Z}^n, \quad x \in \mathbb{R}^n,
$$

where (now) $j \in \mathbb{N}_0$. We always assume that $\psi_F$ and $\psi_M$ in (3.1) have $L_2$-norm 1. Then

$$
\{2^{jn/2} \psi_{G,m}^j : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n \}
$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$ (for any $u \in \mathbb{N}$) and

$$
f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_{m,G}^j \psi_{G,m}^j\n$$

with

$$
\lambda_{m,G}^j = \lambda_{m,G}^j(f) = 2^{jn} \int_{\mathbb{R}^n} f(x) \psi_{G,m}^j(x) dx = 2^{jn} (f, \psi_{G,m}^j)
$$

is the corresponding expansion. Recall that $f \in S'(\mathbb{R}^n)$ is an element of $B^s_{p,q}(\mathbb{R}^n)$, where $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ if and only if it can be expanded by (3.7) and (3.8) (with sufficiently large $u$ in (3.1) and (3.2)) such that

$$
\|f \mid B^s_{p,q}(\mathbb{R}^n)\| \sim \left( \sum_{j=0}^{\infty} 2^{j(n - \frac{q}{p})} \sum_{G \in G^j} \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{m,G}^j(f)|^p \right)^{q/p} \right)^{1/q}
$$

(with the usual modification if $\max(p, q) = \infty$). Explanations and references may be found in [82, Subsection 1.2.1, pp.7–10] and [80, Subsection 3.2.3, pp.51–54]. The related counterparts for the spaces $F^s_{p,q}(\mathbb{R}^n)$ do not play a substantial role here as we can often restrict ourselves to the $B$-setting in this paper. This wavelet expansion can be extended to the $B$-spaces of the $\varrho$-clan in (2.46) as follows.

**Proposition 3.1.** Let $n \in \mathbb{N}$, $-n < \varrho < 0$, $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Then $f \in S'(\mathbb{R}^n)$ is an element of $\Lambda^\varrho B^s_{p,q}(\mathbb{R}^n)$ if and only if it can be expanded by (3.7) and (3.8) (with sufficiently large $u$ in (3.1) and (3.2)) such that

$$
\|f \mid \Lambda^\varrho B^s_{p,q}(\mathbb{R}^n)\| \sim \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{\frac{J(n + \varrho)}{p}} \left( \sum_{j \geq J+} \sum_{m \in \mathbb{Z}^n} |\lambda_{m,G}^j(f)|^p \right)^{q/p} \right)^{1/q}
$$

(with the usual modification if $q = \infty$) is finite (with equivalent quasi-norms). Similarly, $f \in S'(\mathbb{R}^n)$ is an element of $\Lambda^\varrho B^s_{p,q}(\mathbb{R}^n)$ if and only if it can be expanded by (3.7) and (3.8) (with sufficiently large $u$ in (3.1) and (3.2)) such that

$$
\|f \mid \Lambda^\varrho B^s_{p,q}(\mathbb{R}^n)\| \sim \left( \sum_{j=0}^{\infty} 2^{j(n - \frac{q}{p})} \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{\frac{J(n + \varrho)}{p}} \left( \sum_{m \in \mathbb{Z}^n, G \in G^j} |\lambda_{m,G}^j(f)|^p \right)^{q/p} \right)^{1/q}
$$

(with the usual modification if $q = \infty$) is finite (with equivalent quasi-norms).

**Remark 3.2.** The assertion about the spaces $\Lambda^\varrho B^s_{p,q}(\mathbb{R}^n)$ is covered by [82, Proposition 1.16, p.11] with a reference to [80, Theorem 3.26, p.64] and (2.32). It remains valid with $\varrho = 0$ for the spaces $\Lambda^0 B^s_{p,q}(\mathbb{R}^n)$ in (2.44). Its counterpart for the spaces $\Lambda^\varrho B^s_{p,q}(\mathbb{R}^n)$ goes back to [56, Theorem 1.5, p.97], appropriately reformulated (see also [54]). There are related $F$-counterparts. They play only a marginal role in this paper. We will give precise references if needed. The above formulations are sufficient for our purpose. But they are a little bit sloppy from a technical point of view. More precisely, one first justifies that the expansion (3.7) converges unconditionally in $S'(\mathbb{R}^n)$ if the coefficients $\{\lambda_{m,G}^j\}$ belong to
the sequence spaces in (3.9)–(3.11) with $\lambda^G_m$ in place of $\lambda^G_m(f)$. Afterwards one ensures that $f$ belongs to the related spaces with the uniquely determined coefficients $\lambda^G_m = \lambda^G_m(f)$ in (3.8). One may consult [80, Theorem 3.26, p.64] and the (technical) explanations given there. We return to this point later on in Subsection 5.7 in connection with Haar wavelets.

3.2 Interpolation

If one wishes to study the interpolation of the Morrey spaces in (2.20), or the Morrey smoothness spaces in (2.24), (2.25) and (2.30), (2.31), then one finds interesting and deep results in the literature; we refer to [37,46,63] and recommend [68, Section 2] for a more detailed discussion. Such interpolation formulas fit well in the above concept with $\theta$ as the decisive parameter. But this is not our topic here in its full generality. We concentrate on very few assertions which will be of some use later on. Let $(A_0, A_1)_{\theta,q}$ with $0 < \theta < 1$ and $0 < q \leq \infty$ be the classical real interpolation for interpolation couples $(A_0, A_1)$ of quasi-Banach spaces; we refer to the standard literature [5, 71]. First, we deal with the real interpolation of the three families belonging to a fixed $\varrho$-clan according to Definition 2.15(ii).

Proposition 3.3. Let $n \in \mathbb{N}$ and $-n < \varrho < 0$. Let $0 < p < \infty$ and $0 < q, q_1, q_2 \leq \infty$. Let $0 < \theta < 1$ and

$$-\infty < s_1 < s < s_2 < \infty, \quad s = (1 - \theta)s_1 + \theta s_2.$$  

Then

$$\Lambda_{\varrho}B^{s}_{p,q}(\mathbb{R}^n) = (\Lambda_{\varrho}B^{s_1}_{p,q_1}(\mathbb{R}^n), \Lambda_{\varrho}B^{s_2}_{p,q_2}(\mathbb{R}^n))_{\varrho,q}.$$  

Remark 3.4. This assertion goes back to [68, Theorem 2.2, p.91]. As mentioned there one can replace $\Lambda_{\varrho}B^{s_1}_{p,q_1}(\mathbb{R}^n)$ and $\Lambda_{\varrho}B^{s_2}_{p,q_2}(\mathbb{R}^n)$ in (3.13) independently by $\Lambda_{\varrho}F^{s_1}_{p,q_1}(\mathbb{R}^n)$ and/or $\Lambda_{\varrho}F^{s_2}_{p,q_2}(\mathbb{R}^n)$. This follows immediately from (3.13) and (2.55). Using in addition (2.60), one obtains

$$\Lambda_{\varrho}B^{s}_{p,q}(\mathbb{R}^n) = (\Lambda^\varrho F^{s_1}_{p,q_1}(\mathbb{R}^n), \Lambda^\varrho F^{s_2}_{p,q_2}(\mathbb{R}^n))_{\varrho,q}.$$  

This relation will be of some service for us later on. It shows that properties already proved for the spaces $\Lambda^\varrho F^{s}_{p,q}(\mathbb{R}^n)$ can be transferred to the third family $\Lambda_{\varrho}B^{s}_{p,q}(\mathbb{R}^n)$ of the $\varrho$-clan (if interpolation can be applied).

Quite obviously, (3.13) and (3.14) extend the well-known real interpolation for the classical spaces in (2.42) from the $n$-clan to the $\varrho$-clan. One may ask what happens with other interpolation methods, in particular diverse types of complex interpolation. This attracted some attention in the literature, both for the Morrey spaces in (2.20) and also for their smooth generalisations in (2.24), (2.25) and (2.30), (2.31). Satisfactory results can only be obtained within a fixed $\varrho$-clan. As far as the complex interpolation of the Morrey spaces $\Lambda^\varrho N^{s}_{p,q}(\mathbb{R}^n)$ is concerned one may consult [40, 41] (see also [18,19,45] for some more recent contributions). The complex interpolation especially of $\Lambda_{\varrho}F^{s}_{p,q}(\mathbb{R}^n)$ and their restrictions $\Lambda_{\varrho}F^{s}_{p,q}(\Omega)$ to bounded Lipschitz domains $\Omega$ has been studied in [16,17,90] and, quite recently, in [98,99]. There one also finds further discussions and references. In any case, it seems to be a rather tricky undertaking. Fortunately enough there is a very efficient interpolation method in the context of the above spaces which will be of some service. We give a description.

The so-called $\pm$-method goes back to [15]. A description and further references may be found in [96, Subsection 2.2, p.1842]. It applies to arbitrary interpolation couples $(A_1, A_2)$ of quasi-Banach spaces and produces interpolation spaces $(A_1, A_2, \theta)$ ($0 < \theta < 1$), having the desired interpolation property. We concentrate on the $\varrho$-clan according to Definition 2.15(ii) consisting of the three families according to (2.46) and (2.47).

Theorem 3.5. Let $n \in \mathbb{N}$, $-n < \varrho < 0$, $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$. Let $0 < p_1, p_2 < \infty$ and $0 < q_1, q_2 \leq \infty$. Let $0 < \theta < 1$ and

$$s = (1 - \theta)s_1 + \theta s_2, \quad \frac{1}{p} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1 - \theta}{q_1} + \frac{\theta}{q_2}.$$  

Then

$$\langle \Lambda_{\varrho}B^{s_1}_{p_1,q_1}(\mathbb{R}^n), \Lambda_{\varrho}B^{s_2}_{p_2,q_2}(\mathbb{R}^n), \theta \rangle = \Lambda_{\varrho}B^{s}_{p,q}(\mathbb{R}^n),$$  

(3.16)
\begin{equation}
\langle \Lambda^\varphi B_{p_1,q_1}(\mathbb{R}^n), \Lambda^\varphi B_{p_2,q_2}(\mathbb{R}^n), \theta \rangle = \Lambda^\varphi B_{p,q}(\mathbb{R}^n)
\end{equation}

and
\begin{equation}
\langle \Lambda_\varphi F_{p_1,q_1}(\mathbb{R}^n), \Lambda_\varphi F_{p_2,q_2}(\mathbb{R}^n), \theta \rangle = \Lambda_\varphi F_{p,q}(\mathbb{R}^n).
\end{equation}

\textbf{Proof.} This remarkable assertion is covered by [96, Theorem 2.12, p. 1843] reformulated according to (2.24), (2.25) and (2.30), (2.31). \qed

\textbf{Remark 3.6.} According to [96], one can extend the above assertions to the \(n\)-clan in Definition 2.15(i) consisting of the classical spaces in (2.42) with the following outcome. Let \(n \in \mathbb{N}, s_1, s_2 \in \mathbb{R}\) and \(0 < p_1, p_2, q_1, q_2 \leq \infty\). Let \(0 < \theta < 1\) and \(s, p, q\) be as in (3.15). Then
\begin{equation}
\langle B^s_{p_1,q_1}(\mathbb{R}^n), B^s_{p_2,q_2}(\mathbb{R}^n), \theta \rangle = B^s_{p,q}(\mathbb{R}^n)
\end{equation}

and
\begin{equation}
\langle F^s_{p_1,q_1}(\mathbb{R}^n), F^s_{p_2,q_2}(\mathbb{R}^n), \theta \rangle = F^s_{p,q}(\mathbb{R}^n).
\end{equation}

The remarkable incorporation of \(p_1 = \infty\) and/or \(p_2 = \infty\) in (3.20) goes back to [12, Theorem 8.5, pp. 98 and 134]. Otherwise (3.16)–(3.18) compared with (3.19) and (3.20) fit in the Slope-\(n\)-Rule as formulated in Slope Rules 2.17(ii). On the other hand nothing seems to be known if one wishes to interpolate spaces belonging to different \(g\)-clans.

\textbf{Remark 3.7.} It has been shown in [96, Corollary 2.14, p. 1844] that also the basic spaces underlying the above smoothness spaces interpolate as expected. Let \(-n \leq \varrho < 0\),
\begin{equation}
0 < p_1 \leq p_2 < \infty, \quad 0 < \theta < 1, \quad \frac{1}{p} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_2}.
\end{equation}

Then
\begin{equation}
\langle \Lambda^\varphi_p(\mathbb{R}^n), \Lambda^\varphi_{p_2}(\mathbb{R}^n), \theta \rangle = \Lambda^\varphi_p(\mathbb{R}^n)
\end{equation}

extends
\begin{equation}
\langle L_{p_1}(\mathbb{R}^n), L_{p_2}(\mathbb{R}^n), \theta \rangle = L_p(\mathbb{R}^n)
\end{equation}

from the Lebesgue spaces \(L_p(\mathbb{R}^n)\) to the Morrey spaces \(\Lambda^\varphi_p(\mathbb{R}^n)\) as introduced in Definitions 2.3.

\section{3.3 Lifts}

It is well known that the classical lift operator
\begin{equation}
I_\sigma : f \mapsto (\langle \xi \rangle^{-\sigma} \hat{f})^\vee \quad \text{with} \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2}, \quad \xi \in \mathbb{R}^n, \quad \sigma \in \mathbb{R}
\end{equation}

maps the space \(A^\varphi_{p,q}(\mathbb{R}^n)\) isomorphically onto \(A^{\varphi+\sigma}_{p,q}(\mathbb{R}^n)\),
\begin{equation}
\| I_\sigma f \|_{A^{\varphi+\sigma}_{p,q}(\mathbb{R}^n)} \sim \| f \|_{A^\varphi_{p,q}(\mathbb{R}^n)}, \quad f \in A^\varphi_{p,q}(\mathbb{R}^n).
\end{equation}

This goes back to [72, Theorem 2.3.8, pp. 58–59] with \(p < \infty\) for the \(F\)-spaces and has been extended in [82, Theorem 1.22, p. 16] to all the members \(A^\varphi_{p,q}(\mathbb{R}^n)\) in (2.42) of the \(n\)-clan in Definition 2.15(i) (now including \(F^\infty_{\infty,q}(\mathbb{R}^n)\)). We fix the more or less obvious counterpart for spaces belonging to a \(g\)-clan according to Definition 2.15(ii).

\textbf{Theorem 3.8.} Let \(n \in \mathbb{N}, -n < \varrho < 0\) and
\begin{equation}
s \in \mathbb{R}, \quad 0 < p < \infty, \quad 0 < q \leq \infty.
\end{equation}

Let \(\sigma \in \mathbb{R}\). Then \(I_\sigma\) maps \(\Lambda^\varphi A^s_{p,q}(\mathbb{R}^n)\) isomorphically onto \(\Lambda^\varphi A^{s+\sigma}_{p,q}(\mathbb{R}^n)\), and \(\Lambda_\varphi A^s_{p,q}(\mathbb{R}^n)\) isomorphically onto \(\Lambda_\varphi A^{s+\sigma}_{p,q}(\mathbb{R}^n)\), \(A \in \{B, F\}\),
\begin{equation}
I_\sigma \Lambda^\varphi A^s_{p,q}(\mathbb{R}^n) = \Lambda^\varphi A^{s+\sigma}_{p,q}(\mathbb{R}^n), \quad A \in \{B, F\},
\end{equation}
\begin{equation}
I_\sigma \Lambda_\varphi A^s_{p,q}(\mathbb{R}^n) = \Lambda_\varphi A^{s+\sigma}_{p,q}(\mathbb{R}^n), \quad A \in \{B, F\}.
\end{equation}
Proof. The assertion (3.27) follows from [80, Theorem 3.72, p.102] and (2.32). The real interpolation (3.14) shows that this property can be extended to the spaces $\Lambda_{\varrho}B_{p,q}^s(\mathbb{R}^n)$ in (3.28), while the result for $\Lambda_{\varrho}F_{p,q}^s(\mathbb{R}^n)$ is a consequence of the coincidence (2.45) and (3.27).

Remark 3.9. In view of Definition 2.15, we can summarise (3.27) and (3.28) as

$$I_{\varrho}(pA_{p,q}^s(\mathbb{R}^n)) = pA_{p,q}^{s+\sigma}(\mathbb{R}^n). \quad A \in \{B,F\}. \tag{3.29}$$

The result (3.28) for spaces $\Lambda_{\varrho}B_{p,q}^s(\mathbb{R}^n)$ can be found in [70] already, while (3.27) is covered by [94, Proposition 5.1] (see also [67, Corollary 3.2]). Again, using the identity (2.45), we see that this finally covers the case of $\Lambda_{\varrho}F_{p,q}^s(\mathbb{R}^n)$ in (3.28).

The related mapping property

$$I_{\varrho}\Lambda^0B_{p,q}^s(\mathbb{R}^n) = \Lambda^0B_{p,q}^{s+\sigma}(\mathbb{R}^n) \tag{3.29}$$

of the members (2.44) of the 0-clan in Definition 2.15(iii) is also covered by [80, Theorem 3.72, p.102] and (2.32) with $r = \varrho = 0$.

3.4 Derivatives

Derivatives are defined as usual as $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\partial^\alpha_j = \partial x_j^{\alpha_j}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N}_0$, $|\alpha| = \sum_{j=1}^n \alpha_j$. The classical assertion

$$\|f\|_{A_p^s(\mathbb{R}^n)} \sim \sup_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_{A_p^{s-m}(\mathbb{R}^n)} \tag{3.30}$$

(with equivalent quasi-norms) for all the spaces of the $n$-clan in (2.42) (including $F_{p,q}^s(\mathbb{R}^n)$) may be found in [82, Theorem 1.24, p.17]. We are interested in an extension of this assertion to all the families of the $p$-clan in Definition 2.15(ii). We agree that (3.30) (and its counterparts in the following theorem) includes the assertion that $f \in A_{p,q}^s(\mathbb{R}^n)$ if and only if $D^\alpha f \in A_{p,q}^{s-m}(\mathbb{R}^n)$ for all $\alpha$ with $0 \leq |\alpha| \leq m$.

Theorem 3.10. Let $n \in \mathbb{N}$, $-n < \varrho < 0$ and

$$s \in \mathbb{R}, \quad 0 < p < \infty, \quad 0 < q \leq \infty. \quad (3.31)$$

Let $m \in \mathbb{N}$. Then

$$\|f\|_{\Lambda^\varrho A_{p,q}^s(\mathbb{R}^n)} \sim \sup_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_{\Lambda^\varrho A_{p,q}^{s-m}(\mathbb{R}^n)}, \quad A \in \{B,F\}. \tag{3.32}$$

(with equivalent quasi-norms) and

$$\|f\|_{\Lambda^\varrho A_{p,q}^s(\mathbb{R}^n)} \sim \sup_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_{\Lambda^\varrho A_{p,q}^{s-m}(\mathbb{R}^n)}, \quad A \in \{B,F\} \tag{3.33}$$

(with equivalent quasi-norms).

Proof. The equivalence (3.32), including the above agreement, follows from [80, Corollary 3.66, p.98] and (2.32) again. Then the real interpolation formula (3.14) also shows that

$$\sup_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_{\Lambda^\varrho B_{p,q}^{s-m}(\mathbb{R}^n)} \leq c \|f\|_{\Lambda^\varrho B_{p,q}^s(\mathbb{R}^n)}, \quad f \in \Lambda^\varrho B_{p,q}^s(\mathbb{R}^n), \quad m \in \mathbb{N}. \tag{3.34}$$

The reverse inequality can be based on the lifting (3.28), i.e.,

$$\|(1 - \Delta)^{\sigma/2} f\|_{\Lambda^\varrho B_{p,q}^{s-\sigma}(\mathbb{R}^n)} \sim \|f\|_{\Lambda^\varrho B_{p,q}^s(\mathbb{R}^n)}, \quad \sigma \in \mathbb{R}. \tag{3.35}$$

If $m \in \mathbb{N}$ is even, then the converse of (3.34) follows from (3.35) with $\sigma = m$. Let $m \in \mathbb{N}$ be odd and $f \in \Lambda^\varrho B_{p,q}^s(\mathbb{R}^n)$. We assume that for each $\varepsilon > 0$, there is an $f_\varepsilon \in \Lambda^\varrho B_{p,q}^s(\mathbb{R}^n)$ such that

$$\sup_{0 \leq |\alpha| \leq m} \|D^\alpha f_\varepsilon\|_{\Lambda^\varrho B_{p,q}^{s-m}(\mathbb{R}^n)} \leq \varepsilon \|f_\varepsilon\|_{\Lambda^\varrho B_{p,q}^s(\mathbb{R}^n)}. \tag{3.36}$$
Application of what one already knows shows that there is a corresponding inequality with \( m + 1 \) in place of \( m \). But this is a contradiction. This proves (3.33) for all \( m \in \mathbb{N} \) for spaces \( \Lambda_q B_{p,q}^s(\mathbb{R}^n) \), while the result for \( \Lambda_q F_{p,q}^s(\mathbb{R}^n) \) follows from (2.45) and (3.32) again.

**Remark 3.11.** We refer to [70]. For the first results of type (3.32) in the case of spaces \( F_{p,q}^s(\mathbb{R}^n) \), recall (2.30) and (2.31), and for the spaces of type \( N_{p,q}^s(\mathbb{R}^n) = \Lambda_q B_{p,q}^s(\mathbb{R}^n) \), recall (2.24) (see also [62]). Again we may summarise (3.32) and (3.33) by

\[
\|f|g_A^p(\mathbb{R}^n)\| \sim \sup_{0<|\alpha|<m} \| D^\alpha f |g_A^{p,-m}(\mathbb{R}^n)\|, \quad A \in \{ B, F \}
\]

in the sense of equivalent quasi-norms; recall Definition 2.15.

Based on [80, Corollary 3.66, p.98], one can extend (3.32) to the \( q \)-clan in Definition 2.15(iii), i.e.,

\[
\|f|\Lambda^0 B_{p,q}^s(\mathbb{R}^n)\| \sim \sup_{0<|\alpha|<m} \| D^\alpha f |\Lambda^0 B_{p,q}^{s-m}(\mathbb{R}^n)\|, \quad f \in \Lambda^0 B_{p,q}^s(\mathbb{R}^n). \tag{3.37}
\]

**Remark 3.12.** We need Theorem 3.10 as a tool in our later considerations. On the other hand, derivatives and differences played a decisive role in the theory of function spaces from the very beginning. One may expect that characterisations of some spaces \( A_{p,q}^s(\mathbb{R}^n) \) in (2.42), the \( n \)-clan, in terms of the differences

\[
\Delta_n^h f(x) = f(x+h)-f(x), \quad (\Delta_n^{h+1} f)(x) = \Delta_n^h (\Delta_n^h f)(x), \tag{3.38}
\]

\( x \in \mathbb{R}^n, \ h \in \mathbb{R}^n, \ k \in \mathbb{N} \), and their ball means

\[
d_n^h f(x) = \left( t^{m+1} \int_{|h| \leq t} |\Delta_n^h f(x)|^m dh \right)^{1/m}, \quad x \in \mathbb{R}^n, \quad t > 0, \tag{3.39}
\]

\( 0 < v \leq \infty \), have suitable counterparts for the spaces in (2.46) and (2.47) of the \( g \)-clan, \( -n < g < 0 \), subject to the (modified) Slope-\( n \)-Rule (see Slope Rules 2.17(ii)). The related theory for the spaces \( A_{p,q}^s(\mathbb{R}^n) \) with \( p < \infty \) for \( F \)-spaces in [73, Theorem 3.5.3, p.194] has been complemented quite recently by the corresponding assertions for \( F_{p,\infty}^s(\mathbb{R}^n) \). Discussions and detailed references may be found in [82, Subsection 4.3.3, pp.134–135]. The first substantial results about characterisations of some spaces in (2.46) and (2.47) in terms of (3.38) and (3.39) may be found in [32,34]. In [86] further equivalent characterisations of the spaces \( \Lambda^0 A_{p,q}^s(\mathbb{R}^n) \) via derivatives can be found.

### 3.5 Fatou property

Let \( A(\mathbb{R}^n) \) be a quasi-normed space in \( S'(\mathbb{R}^n) \) with \( A(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n) \) (a continuous embedding). Then \( A(\mathbb{R}^n) \) is said to have the Fatou property if there is a positive constant \( c \) such that from

\[
\sup_{j \in \mathbb{N}} \|g_j | A(\mathbb{R}^n)\| < \infty \quad \text{and} \quad g_j \rightarrow g \text{ in } S'(\mathbb{R}^n), \tag{3.40}
\]

it follows that \( g \in A(\mathbb{R}^n) \) and

\[
\|g | A(\mathbb{R}^n)\| \leq c \sup_{j \in \mathbb{N}} \|g_j | A(\mathbb{R}^n)\|. \tag{3.41}
\]

We took over the above formulation from [82, Subsection 1.3.4, pp.18–19]. But the Fatou property for this type of function spaces has some history which may be found there. The surprisingly simple argument ensuring that all the spaces in (2.42) of the \( n \)-clan have the Fatou property applies equally to all the spaces in Definition 2.15.

**Theorem 3.13.** Let \( -n < g < 0 \). Then all the spaces (2.46) and (2.47) of the \( g \)-clan according to Definition 2.15(ii) have the Fatou property.

**Proof.** Let \( \psi \in S(\mathbb{R}^n) \). Then

\[
(\psi \hat{f})^\vee(x) = c(f, \psi^\vee(x - \cdot)), \quad f \in S'(\mathbb{R}^n), \quad x \in \mathbb{R}^n. \tag{3.42}
\]

This reduces the Fatou property for the Fourier-analytically defined spaces in (2.21) and (2.22) based on (2.17), (2.27) and (2.28) to the classical measure-theoretical Fatou property for \( L_p \)-spaces. \( \square \)
4 Key problems

4.1 Motivations

The spaces $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$ with $0 < p, q \leq \infty$ (and $p < \infty$ for $F$-spaces) and $s \in \mathbb{R}$ according to the parts (i) and (ii) of Definition 2.1 were introduced from the mid 1960s to the mid 1970s. Related (historical) references may be found in [82, Subsection 4.3.4, pp. 135–136]. The incorporation of the spaces $F_{\infty,q}^s(\mathbb{R}^n)$ in Definition 2.1(iii) came somewhat later and had been discussed in detail in [82, pp. 3–5]. At the end of the 1970s, the question arose whether these spaces are worth to be studied, especially their extensions from $p \geq 1$ to $p < 1$. As a good criterion at this time (and maybe up to now), one may ask whether the restriction $A_{p,q}^s(\Omega)$ of $A_{p,q}^s(\mathbb{R}^n)$, $A \in \{B,F\}$ to bounded smooth domains $\Omega$ in $\mathbb{R}^n$ can be used to study elliptic boundary value problems in $\Omega$. This requires that these spaces have the following distinguished properties, called key theorems in [73] (but treated already in [72]):

1. pointwise (smooth) multipliers in $A_{p,q}^s(\mathbb{R}^n)$,
2. diffeomorphisms of $A_{p,q}^s(\mathbb{R}^n)$,
3. extensions of the corresponding spaces $A_{p,q}^s(\mathbb{R}^n)$ to $A_{p,q}^s(\mathbb{R}^n)$, and
4. traces of $A_{p,q}^s(\mathbb{R}^n)$ on $\mathbb{R}^{n-1}$ ($2 \leq n \in \mathbb{N}$).

Here,

$$\mathbb{R}^n_+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \}, \quad n \in \mathbb{N}. \quad (4.1)$$

Final assertions and historical references may be found in [82]. Beginning with the early 1990s, the classical spaces $A_{p,q}^s(\mathbb{R}^n)$ have been extended by the two types of Morrey smoothness spaces (1.2) and (1.3). They have been studied in detail in numerous papers. An examination of the obtained assertions (some of which are quite recent) suggests not looking at these two types of Morrey smoothness spaces separately, but uniting and reorganising them into the clans as described in Definition 2.15 based on (2.24), (2.25) and (2.30), (2.31). The emerging fourth parameter $\varrho$ is now on equal footing with the classical parameters $s$ (smoothness), $p$ (integrability) and $q$ (summability), where $|\varrho|$ is quite often the slope of distinguished (broken) lines in the $(\frac{1}{p}, s)$-diagram. In addition, the three families in (2.46) of a fixed $\varrho$-clan are sibyntically related to each other, where the real interpolation (3.14) is a good and useful example. In what follows we reformulate many already existing assertions in terms of the $\varrho$-clans and complement them by some new properties which illuminate our approach. It is quite natural to deal first with the above key problems. It comes out that the first three of them can be treated rather quickly using already existing properties. But related assertions about traces in the literature are not really satisfactory. We deal with them in detail in the context of our approach. A new phenomenon comes out (here and at other occasions). The behaviour of the low-slope spaces, which means $0 < |\varrho| < 1$, is more or less the same in all the dimensions $n$. In particular, there is no breaking point at $p = 1$ as usual for the spaces $A_{p,q}^s(\mathbb{R}^n)$ ($2 \leq n \in \mathbb{N}$), and also for high slope spaces with $1 < |\varrho| < n$. This is in good agreement with the Slope-1-Rule as formulated in Slope Rules 2.17(i).

4.2 Pointwise multipliers

Pointwise multipliers for the classical spaces $A_{p,q}^s(\mathbb{R}^n)$ in (2.42), the $n$-clan according to Definition 2.15, have been studied with great intensity over decades. Rather final results have been obtained (now including $F_{\infty,q}^s(\mathbb{R}^n)$). One may consult [82, Subsection 2.4], where one finds also detailed references. It might be a challenging task to develop a comparable theory for the spaces within the $\varrho$-clans as introduced...
in Definition 2.15. Our aim here is rather modest. We deal with smooth pointwise multipliers as requested in the first of the four key problems listed above in Subsection 4.1.

Let $A(\mathbb{R}^n)$ be a quasi-Banach space in $\mathbb{R}^n$ with the continuous embedding

$$S(\mathbb{R}^n) \hookrightarrow A(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n).$$

(4.2)

Then $g \in L_\infty(\mathbb{R}^n)$ is said to be a pointwise multiplier for $A(\mathbb{R}^n)$ if

$$f \mapsto gf$$

generates a bounded map in $A(\mathbb{R}^n)$. (4.3)

Of course one has to say what this multiplication means in this generality. References for related detailed natural number all the complex-valued continuous functions in $\mathbb{R}^n$ with classical continuous derivatives up to order $k$ inclusively with

$$\|g | C^k(\mathbb{R}^n)\| = \sup_{|n| \leq k,x \in \mathbb{R}^n} |D^n g(x)| < \infty.$$  

(4.4)

For our purpose, it is sufficient to know that for any classical space $A^s_{p,q}(\mathbb{R}^n)$ in (2.42) there exist a natural number $k \in \mathbb{N}$ and a constant $c > 0$ such that

$$\|gf | A^s_{p,q}(\mathbb{R}^n)\| \leq c \|g | C^k(\mathbb{R}^n)\| \cdot \|f | A^s_{p,q}(\mathbb{R}^n)\|$$

(4.5)

for all $g \in C^k(\mathbb{R}^n)$ and $f \in A^s_{p,q}(\mathbb{R}^n)$.

**Theorem 4.1.** For any space in (2.46) with $-n < s < 0$, $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$, there exist a natural number $k \in \mathbb{N}$ and a constant $c > 0$ such that

$$\|gf | A^{s}_{p,q}(\mathbb{R}^n)\| \leq c \|g | C^k(\mathbb{R}^n)\| \cdot \|f | A^{s}_{p,q}(\mathbb{R}^n)\|$$

(4.6)

for all $g \in C^k(\mathbb{R}^n)$ and $f \in A^{s}_{p,q}(\mathbb{R}^n)$, $A \in \{B,F\}$, and

$$\|gf | \Lambda^e A^{s}_{p,q}(\mathbb{R}^n)\| \leq c \|g | C^k(\mathbb{R}^n)\| \cdot \|f | \Lambda^e A^{s}_{p,q}(\mathbb{R}^n)\|$$

(4.7)

for all $g \in C^k(\mathbb{R}^n)$ and $f \in \Lambda^e A^{s}_{p,q}(\mathbb{R}^n)$, $A \in \{B,F\}$.

**Proof.** Let $Q_{J,M}$ be as in (2.6) and $2Q_{J,M}$ be as explained there. Let $\Lambda^e A^{s}_{p,q}(\mathbb{R}^n)$ be again the spaces as introduced in Definition 2.8 (now with $-n < s < 0$). Let $s < m \in \mathbb{N}_0$. Then it follows from [80, Corollary 3.66, p.98] and (2.32) that $f \in \Lambda^e A^{s}_{p,q}(\mathbb{R}^n)$ if and only if

$$\sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^j(2^{(n+q)} \|D^m f | A^{s-m}_{p,q}(2Q_{J,M})\|$$

(4.8)

is finite (with equivalent quasi-norms). Here, $A^{s}_{p,q}(\Omega)$ is the usual restriction of $A^{s}_{p,q}(\mathbb{R}^n)$ to the domain (which is an open set) $\Omega$ in $\mathbb{R}^n$. In particular, (4.5) remains valid with the same $c$ and $g$ if one replaces there $A^{s}_{p,q}(\mathbb{R}^n)$ by $A^{s}_{p,q}(\Omega)$ (independently of $\Omega$). Then (4.6) follows from (4.8) and (4.5) (with a different $k \in \mathbb{N}$). Now, in view of the coincidence (2.45), it is sufficient to prove (4.7) in the case of $A = B$. This can be obtained by the real interpolation (3.14).

**Remark 4.2.** We refer the reader to [80, Theorem 3.69 and Remarks 3.70 and 3.71, p.101]. There one finds (4.6) based on the same arguments as above, further explanations and references. In particular, (4.6) can be extended to the spaces in (2.44) of the $0$-clan in Definition 2.15(iii). The simple proof of (4.7) based on (4.6) underlines again the close relationship of the three families in (2.46) of a $p$-clan among each other.
4.3 Diffeomorphisms

A continuous one-to-one map of $\mathbb{R}^n$ ($n \in \mathbb{N}$), onto itself,

\[
y = \psi(x) = (\psi_1(x), \ldots, \psi_n(x)), \quad x \in \mathbb{R}^n,
\]

\[
x = \psi^{-1}(y) = (\psi_1^{-1}(y), \ldots, \psi_n^{-1}(y)), \quad y \in \mathbb{R}^n,
\]

is called a diffeomorphism if all the components $\psi_j(x)$ and $\psi_j^{-1}(y)$ are real $C^\infty$ functions on $\mathbb{R}^n$ for $j = 1, \ldots, n$.

\[
\sup_{x \in \mathbb{R}^n}(|D^\alpha \psi_j(x)| + |D^\alpha \psi_j^{-1}(x)|) < \infty \quad \text{for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| > 0.
\]

We used the standard notations again. In particular, $\mathbb{N}_0^n$ collects all $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N}_0$ and $|\alpha| = \sum_{j=1}^n \alpha_j$. Then $\varphi \mapsto \varphi \circ \psi$, given by $(\varphi \circ \psi)(x) = \varphi(\psi(x))$, is a one-to-one map of $S(\mathbb{R}^n)$ onto itself.

This can be extended by standard arguments to a one-to-one map of $S'(\mathbb{R}^n)$ onto itself. Some related details may be found in [82, Subsection 2.3, pp. 39–40].

**Proposition 4.3.** Let $\psi$ be the above diffeomorphic map of $\mathbb{R}^n$ onto itself. Then

\[
D_\psi : A^s_{p,q}(\mathbb{R}^n) \hookrightarrow A^s_{p,q}(\mathbb{R}^n), \quad D_\psi f = f \circ \psi
\]

is an isomorphic map for all the spaces in (4.22), the $n$-clan.

**Remark 4.4.** This diffeomorphism is a tricky problem and attracted some attention over decades. Related references may be found in [82, p. 39]. Finally, in [81, Subsection 1.3.8, pp. 66–67], a streamlined proof of less than one page could be given for all the spaces with $p < \infty$ for $F$-spaces. The incorporation of the spaces $F^s_{p,q}(\mathbb{R}^n)$ in [82, Theorem 2.25, p. 39] had been based on (2.39) and (4.8) with $\varrho = 0$ and $A^{s-m}_{p,q}(\mathbb{R}^n) = F^{s-m}_{p,q}(\mathbb{R}^n)$, $p < \infty$, $s < m \in \mathbb{N}_0$. But this reduction works for all the spaces $\Lambda^pA^s_{p,q}(\mathbb{R}^n)$ and had already been used in [80, Theorem 3.69, p. 101] in connection with the above problem. We formulate the outcome.

**Theorem 4.5.** Let $\psi$ be the above diffeomorphic map of $\mathbb{R}^n$ onto itself. For $-n < \varrho < 0$, let $\Lambda^pA^s_{p,q}(\mathbb{R}^n)$ and $A^s_{p,q}(\mathbb{R}^n)$, $A \in \{B, F\}$ be the three families of the $\varrho$-clan $\varrho A^s_{p,q}(\mathbb{R}^n)$ according to Definition 2.15(ii).

Then

\[
D_\psi : \Lambda^pA^s_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^pA^s_{p,q}(\mathbb{R}^n), \quad D_\psi f = f \circ \psi
\]

and

\[
D_\psi : A^s_{p,q}(\mathbb{R}^n) \hookrightarrow A^s_{p,q}(\mathbb{R}^n), \quad D_\psi f = f \circ \psi
\]

are isomorphic maps.

**Proof.** One has for some $d > 0$ that the controlled distortion

\[
\psi(Q_{J,M}) \subset dQ_{J,M'} \quad \text{for all } J \in \mathbb{Z}, \ M \in \mathbb{Z}^n \text{ and } M' = M'(J, M) \in \mathbb{Z}^n.
\]

Then the application of (4.5) and Proposition 4.3 to (4.8) with $f \circ \psi$ in place of $f$ shows that $D_\psi$ in (4.13) maps $\Lambda^pA^s_{p,q}(\mathbb{R}^n)$ into itself. This argument can also be applied to $\psi^{-1}$. This ensures that $D_\psi$ is a map onto. The incorporation of $D_\psi$ in (4.14) is again a matter of the real interpolation (3.14), where again we may restrict ourselves to the case of $A = B$ in view of (2.45).

**Remark 4.6.** It follows by the same argument that

\[
D_\psi : \Lambda^0B^s_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^0B^s_{p,q}(\mathbb{R}^n), \quad D_\psi f = f \circ \psi
\]

is an isomorphic map for the spaces in (2.44) of the 0-clan according to Definition 2.15(iii).
4.4 Extensions

Next, we deal with the third of the four key problems described in Subsection 4.1. Let again

\[ \mathbb{R}_n^+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \}, \quad n \in \mathbb{N}. \]  \hspace{1cm} (4.17)

As usual, \( D'(\mathbb{R}_n^+) \) denotes the set of all the distributions in \( \mathbb{R}_n^+ \). Furthermore, \( g \mid \mathbb{R}_n^+ \in D'(\mathbb{R}_n^+) \) stands for the restriction of \( g \in S'(\mathbb{R}) \) to \( \mathbb{R}_n^+ \).

**Definition 4.7.** Let \( A(\mathbb{R}^n) \) be a space covered by Definition 2.15. Then

\[ A(\mathbb{R}_n^+) = \{ f \in D'(\mathbb{R}_n^+) : f = g \mid \mathbb{R}_n^+ \text{ for some } g \in A(\mathbb{R}^n) \} , \]  \hspace{1cm} (4.18)

\[ \| f \mid A(\mathbb{R}_n^+) \| \leq \inf \| g \mid A(\mathbb{R}^n) \| , \]  \hspace{1cm} (4.19)

where the infimum is taken over all \( g \in A(\mathbb{R}^n) \) with \( g \mid \mathbb{R}_n^+ = f \).

**Remark 4.8.** It follows from standard arguments that \( A(\mathbb{R}_n^+) \) is a quasi-Banach space (and a Banach space for \( p \geq 1 \) and \( q \geq 1 \)), continuously embedded in \( D'(\mathbb{R}_n^+) \) or the restriction of \( S'(\mathbb{R}^n) \) to \( \mathbb{R}_n^+ \).

The restriction operator \( re \),

\[ re f = f \mid \mathbb{R}_n^+ : S'(\mathbb{R}^n) \hookrightarrow D'(\mathbb{R}_n^+) \]  \hspace{1cm} (4.20)

generates a linear and bounded map from \( A(\mathbb{R}^n) \) onto \( A(\mathbb{R}_n^+) \). One asks for a linear and bounded extension operator \( ext \),

\[ ext : A(\mathbb{R}_n^+) \hookrightarrow A(\mathbb{R}^n) \]  \hspace{1cm} (4.21)

such that

\[ re \circ ext = id, \quad \text{identity in } A(\mathbb{R}_n^+). \]  \hspace{1cm} (4.22)

Of interest are common extension operators for substantial parts of the spaces in Definition 2.15. In the case of the \( g \)-clan with \( -n < g < 0 \), there are for any \( 0 < \varepsilon < 1 \) common extension operators for all the related spaces restricted by

\[ \varepsilon < p < \infty, \quad \varepsilon < q \leq \infty, \quad |s| < \varepsilon^{-1}, \quad -n < g < -\varepsilon, \]  \hspace{1cm} (4.23)

where the \( n \)-clan, consisting of the classical spaces in \( (2.42) \), and the 0-clan can be later incorporated; we refer to Remark 4.11 below. This ensures that the interpolation can be applied. This will be of some use for us. The extension as described above for the classical spaces \( A^\varepsilon_{p,q}(\mathbb{R}^n) \) in \( (2.42) \) was a central topic in the theory of these function spaces from the very beginning. In [82, Subsection 2.5, pp. 57–62], one finds final assertions, including in particular \( F_{\infty,q}^s(\mathbb{R}^n) \), and related references which will not be repeated here. But the incorporation of \( F_{\infty,q}^s(\mathbb{R}^n) \) into the already existing theory for the spaces \( A^\varepsilon_{p,q}(\mathbb{R}^n) \) (with \( p < \infty \) for \( F \)-spaces) is based on \( (2.39) \),

\[ A^0 F_{p,q}^s(\mathbb{R}^n) = F_{\infty,q}^s(\mathbb{R}^n), \quad s \in \mathbb{R}, \quad 0 < p < \infty, \quad 0 < q \leq \infty, \]  \hspace{1cm} (4.24)

and rather peculiar lifting properties for the spaces \( A^\varepsilon_{p,q}(\mathbb{R}^n) \) and their restrictions to \( \mathbb{R}_n^+ \). A detailed description may be found in [82, Proposition 2.68, p. 59]. The method itself goes back to [71, Subsection 2.10.3, pp. 231–233]. It has been extended in [50, 58] to all the spaces \( A^\varepsilon_{p,q}(\mathbb{R}^n) \) covered by Definition 2.8. One may also consult [94, Subsection 6.4.1, pp. 168–172]. We fix the outcome.

**Theorem 4.9.** Let

\[ A_{p,q,\varepsilon}(\mathbb{R}_n^+), \quad \Lambda^p B_{p,q}(\mathbb{R}_n^+), \quad \text{and} \quad A^\varepsilon_{p,q}(\mathbb{R}_n^+) = \Lambda^\varepsilon F_{p,q}^s(\mathbb{R}_n^+) \]  \hspace{1cm} (4.25)

be the restrictions of the spaces \( (2.46) \) and \( (2.47) \) on \( \mathbb{R}_n^+ \) of the \( g \)-clan \( g A^\varepsilon_{p,q}(\mathbb{R}^n) \) with \( -n < g < 0 \). Then for any \( 0 < \varepsilon < 1 \) and \( p, q, s, g \) as in \( (4.23) \), there is a common extension operator from these spaces to their \( \mathbb{R}^n \)-counterparts.

**Proof.** This follows from [50, Theorems 3.7 and 3.15, pp. 327 and 336], reformulated according to \( (2.24), (2.25) \) and \( (2.30), (2.31) \). \( \square \)
Rem 4.10. In [75, Subsection 1.11.8, pp.69–72], it had been explained how restrictions and extensions can be used to shift interpolation formulas from $\mathbb{R}^n$ to bounded Lipschitz domains $\Omega$. The arguments work equally well for $\mathbb{R}^\circ_+$ in place of $\Omega$. In particular, Proposition 3.3, the useful interpolation formula (3.14) and Theorem 3.5 remain valid if one replaces there $\mathbb{R}^n$ by $\mathbb{R}^\circ_+$.

Rem 4.11. By the given references, in particular [50, Theorem 3.15], one can extend Theorem 4.9 to the spaces of the 0-clan in Definition 2.15(iii); recall (2.30), (2.31) and their restrictions to $\mathbb{R}^n_+$.

4.5 Traces

We come to the fourth, i.e., the last key problem as listed in Subsection 4.1. Let $n \in \mathbb{N}$ and $n \geq 2$. Let $x = (x', x_n), x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. The question arises for which spaces covered by Definition 2.15 the trace $\text{tr}$,

$$\text{tr} f = f(x', 0), \quad x' \in \mathbb{R}^{n-1}, \quad f \in \Lambda^\circ A_{p,q}^s (\mathbb{R}^n) \quad \text{or} \quad f \in \Lambda_\circ A_{p,q}^s (\mathbb{R}^n)$$

(4.26)
generates a linear and bounded operator into (or better onto) some related spaces on $\mathbb{R}^{n-1}$. For the classical cases in (2.42), the $n$-clan, one has rather final answers. One may consult [82, Subsection 2.2, pp.29–38] and the references therein, covering also some technical explanations which will not be repeated here. It was the main aim of [82, Theorem 2.13, p.32] to complement the already existing assertions for $\mathbb{R}^n_+$ with $p < \infty$ for the $F$-spaces by

$$\text{tr} : F_{\infty,q}^s (\mathbb{R}^n) \hookrightarrow C^s (\mathbb{R}^{n-1}), \quad 0 < q \leq \infty, \quad s > 0$$

(4.27)

and

$$\text{ext} : C^s (\mathbb{R}^{n-1}) \hookrightarrow F_{\infty,q}^s (\mathbb{R}^n), \quad 0 < q \leq \infty, \quad s > 0$$

(4.28)

with

$$\text{tr} \circ \text{ext} = \text{id}, \quad \text{identity in } C^s (\mathbb{R}^{n-1}).$$

(4.29)

It comes out that the underlying arguments based on (4.24) can also be used to study traces of the spaces $\Lambda^\circ A_{p,q}^s (\mathbb{R}^n)$. This will be done below. But first we discuss some assertions about traces available in the literature.

Let $n \geq 2, 0 < p < \infty, 0 < q \leq \infty, -n \leq \varrho < -1$ and

$$s - \frac{1}{p} > \sigma_{p,n}^{n-1} = (n-1) \left( \max \left( \frac{1}{p}, 1 \right) - 1 \right);$$

(4.30)

recall (1.8). Then $\text{tr}$ is a linear and bounded operator such that

$$\text{tr} \Lambda^e B_{p,q}^s (\mathbb{R}^n) = \Lambda^e B_{p,q}^{s-\frac{1}{p}} (\mathbb{R}^{n-1})$$

(4.31)

and

$$\text{tr} \Lambda^e F_{p,q}^s (\mathbb{R}^n) = \Lambda^e B_{p,q}^{s-\frac{1}{p}} (\mathbb{R}^{n-1})$$

(4.32)

These assertions are covered by [67, Theorem 3.10.8, p.144] based on [64, Theorem 1.3, p.75] and [94, Theorem 6.8, p.163] reformulated according to (2.30) and (2.31). The related counterpart

$$\text{tr} \Lambda_\circ B_{p,q}^s (\mathbb{R}^n) = \Lambda_\circ B_{p,q}^{s-\frac{1}{p}} (\mathbb{R}^{n-1}) \quad \text{and} \quad \text{tr} \Lambda_\circ F_{p,q}^s (\mathbb{R}^n) = \Lambda_\circ B_{p,q}^{s-\frac{1}{p}} (\mathbb{R}^{n-1})$$

(4.33)

under the same restrictions for the parameters $p, q$ and $s$ as above goes back to [50, Theorem 3.8, p.329]. Again the case, $A = F$ in (4.33) follows from (4.32) due to the coincidence (2.45), whereas Theorem 2.21(iv) shows that one cannot replace there $\Lambda^e$ by $\Lambda_{e+1}$ as one might expect. It follows from (2.36) that one recovers the classical assertions if one chooses $\varrho = -n$ in (4.31)–(4.33). The Slope-$n$-Rule (see Slope Rules 2.17(ii)) suggests that $s - \frac{\varrho}{p}$ is the so-called differential dimension for the spaces on the left-hand sides of (4.31)–(4.33), and similarly for the spaces on the related right-hand sides. It follows from

$$s - \frac{|\varrho|}{p} = s - \frac{1}{p} + \frac{\varrho + 1}{p} = s - \frac{1}{p} - \frac{|\varrho + 1|}{p}$$

(4.34)
that they are the same both for the original spaces in $\mathbb{R}^n$ and the related target spaces in $\mathbb{R}^{n-1}$. But the above assertions for the spaces $\rho A^s_{p,q}(\mathbb{R}^n)$ in (2.46) of the $\rho$-clan with $-n < \rho < 0$ are less complete than their classical counterparts in [82, Theorem 2.13, p.32]. There arise several questions. The condition (4.30) is natural for the classical spaces $A^s_{p,q}(\mathbb{R}^n)$, but not for the spaces in (2.46) subject to Slope Rules 2.17. The above assertions exclude spaces with $-1 \leq \rho < 0$, deserving special attention as will be clear later on. The (preferably atomic) arguments in the papers mentioned above do not produce linear and bounded extension operators in generalisation of (4.28) and (4.29). But this is crucial both for the theory itself and also for applications, for example to boundary value problems for elliptic differential operators in, say, smooth bounded domains in $\mathbb{R}^n$. We try to answer at least some of these questions and to raise the trace theory for the spaces in (2.46) to the same level as the other key problems treated above.

Let again $C^s(\mathbb{R}^n)$ ($s \in \mathbb{R}$) be the Hölder-Zygmund spaces as introduced in (2.16). Let $\Lambda^p A^s_{p,q}(\mathbb{R}^n)$ and $\Lambda^p A^s_{p,q}(\mathbb{R}^n)$, $A \in \{B, F\}$ be the spaces in (2.46) of the related $\rho$-clan with $-n < \rho < 0$. From [80, Proposition 3.54, p.92] and (2.32), it follows that

$$
\Lambda^p A^s_{p,q}(\mathbb{R}^n) \hookrightarrow C^{s-\frac{|\rho|}{p}}(\mathbb{R}^n). 
$$

(4.35)

If $s > \frac{|\rho|}{p}$, then $C^{s-\frac{|\rho|}{p}}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$, where $C(\mathbb{R}^n) = C^0(\mathbb{R}^n)$ is the usual space of all the bounded continuous functions in $\mathbb{R}^n$ normed according to (4.4). Then the trace on $\mathbb{R}^{n-1}$ makes sense pointwise. Combined with the (almost but not totally obvious) restriction

$$
\text{tr} C^s(\mathbb{R}^n) = C^s(\mathbb{R}^{n-1}), \quad \sigma > 0
$$

(4.36)

(see [82, Theorem 2.13, p.32]), one obtains

$$
\text{tr} : \Lambda^p A^s_{p,q}(\mathbb{R}^n) \hookrightarrow C^{s-\frac{|\rho|}{p}}(\mathbb{R}^{n-1}), \quad 0 < p < \infty, \quad 0 < q \leq \infty, \quad s > |\rho|/p.
$$

(4.37)

We wish to show that the space on the right-hand side is the related trace space and that there is a common linear and bounded extension operator which applies not only to a fixed space $\Lambda^0 A^s_{p_0,q_0}(\mathbb{R}^n)$ but also to spaces $\Lambda^s A^s_{p,q}(\mathbb{R}^n)$ with the parameters $(s, p, q, \rho)$ close to $(s_0, p_0, q_0, \rho_0)$. We will not stress this point, but it justifies interpolation ensuring below that the extension operator applies also to related spaces $\Lambda^s B^s_{p,q}(\mathbb{R}^n)$.

**Theorem 4.12.** Let $n \geq 2$, $-1 < \rho < 0$,

$$
0 < p < \infty, \quad 0 < q \leq \infty \quad \text{and} \quad s > |\rho|/p.
$$

(4.38)

Then

$$
\text{tr} : \Lambda^p A^s_{p,q}(\mathbb{R}^n) \hookrightarrow C^{s-\frac{|\rho|}{p}}(\mathbb{R}^{n-1}), \quad A \in \{B, F\},
$$

(4.39)

$$
\text{tr} : \Lambda^s B^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^{s-\frac{|\rho|}{p}}_{\infty,q}(\mathbb{R}^{n-1})
$$

(4.40)

and

$$
\text{tr} : \Lambda^s F^s_{p,q}(\mathbb{R}^n) \hookrightarrow C^{s-\frac{|\rho|}{p}}(\mathbb{R}^{n-1}).
$$

(4.41)

Furthermore, there are linear and bounded extension operators $\text{ext}$ with

$$
\text{tr} \circ \text{ext} = \text{id}, \quad \text{identity in } C^{s-\frac{|\rho|}{p}}(\mathbb{R}^{n-1}) \text{ and } B^{s-\frac{|\rho|}{p}}_{\infty,q}(\mathbb{R}^{n-1})
$$

(4.42)

such that

$$
\text{ext} : C^{s-\frac{|\rho|}{p}}(\mathbb{R}^{n-1}) \hookrightarrow \Lambda^p A^s_{p,q}(\mathbb{R}^n),
$$

(4.43)

$$
\text{ext} : B^{s-\frac{|\rho|}{p}}_{\infty,q}(\mathbb{R}^{n-1}) \hookrightarrow \Lambda^s B^s_{p,q}(\mathbb{R}^n)
$$

(4.44)

and

$$
\text{ext} : C^{s-\frac{|\rho|}{p}}(\mathbb{R}^{n-1}) \hookrightarrow \Lambda^s F^s_{p,q}(\mathbb{R}^n).
$$

(4.45)
Proof. **Step 1.** By (4.37), one has (4.39). Then (3.14) and its classical counterpart prove (4.40) by the real interpolation.

**Step 2.** We prove (4.43) with $A = B$. For this purpose, we rely on the same wavelet arguments as in the proof of [82, Theorem 2.13, pp.32–34]. We expand $f \in C^\sigma(\mathbb{R}^{n-1})$, $\sigma = s - \frac{|\ell|}{p} > 0$ in $\mathbb{R}^{n-1}$ similarly as in (3.7)–(3.9),

$$f = \sum_{j \in N_0, G \in G^j, m \in \mathbb{Z}^{n-1}} \lambda_{m}^{j,G} \psi_{G,m}(x')$$

where $x' = (x_1, \ldots, x_{n-1})$,

$$\lambda_{m}^{j,G} = \lambda_{m}^{j,G}(f) = 2^{j(n-1)}(f, \psi_{G,m}), \quad j \in N_0, \quad m \in \mathbb{Z}^{n-1}$$

with $G^0 = \{F, M\}^{n-1}$, $G^j = \{F, M\}^{n-1,*}$ for $j \in \mathbb{N}$ and

$$\|f \|_{C^\sigma(\mathbb{R}^{n-1})} \sim \sup_{j \in N_0, G \in G^j, m \in \mathbb{Z}^{n-1}} 2^{j\sigma} |\lambda_{m}^{j,G}(f)|.$$  

We use now a simplified version of the wavelet-friendly extension as constructed in [76, Subsection 5.1.3, pp.139–147]. Let

$$\chi \in D(\mathbb{R}) = C_0^\infty(\mathbb{R}), \quad \text{supp } \chi \subset (-1,1), \quad \chi(t) = 1, \quad \text{if } |t| \leq 1/2,$$

and $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. Then

$$g = \text{ext } f = \sum_{j \in N_0, G \in G^j, m \in \mathbb{Z}^{n-1}} \lambda_{m}^{j,G}(f) \psi_{G,m}(x') \chi(2^j x_n)$$

with

$$g(x', 0) = f(x'), \quad x' \in \mathbb{R}^{n-1}$$

is an atomic expansion for the spaces $\Lambda^p B^s_{p,q}(\mathbb{R}^n)$ as described in [80, Theorem 3.33, p.67] (including all the requested moment conditions, if required). In particular, according to (3.10) one has to estimate

$$\|g \|_{\Lambda^p B^s_{p,q}(\mathbb{R}^n)} \leq c \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{J(n+\theta)} \left( \sum_{j \in J^+, M} 2^{j(s-\frac{\theta}{p})} \sum_{m \in Q, G \in G^j} |\lambda_{m}^{j,G}(f)|^p \right)^{1/p}.$$  

We may assume $q < \infty$. By (4.46) and (4.48) with $\sigma = s - \frac{|\ell|}{p}$ only $2^{(n-1)(j-J)}$ terms contribute to the last sum. Then

$$\|g \|_{\Lambda^p B^s_{p,q}(\mathbb{R}^n)} \leq c \|f \|_{C^\sigma(\mathbb{R}^{n-1})} \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{J(n+\theta)} \sum_{j \in J^+, M} 2^{j(s-\frac{\theta}{p})} \sum_{m \in Q, G \in G^j} |\lambda_{m}^{j,G}(f)|^p \leq c' \|f \|_{C^\sigma(\mathbb{R}^{n-1})} \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{J(1+\theta)} \sum_{j \in J^+, M} 2^{-j\frac{\theta}{p}(1-|\ell|)}$$

where we used $0 < |\ell| < 1$. This proves (4.43) with $A = B$.

**Step 3.** The corresponding assertion for the spaces in (4.43) with $A = F$ follows from (2.56). The real interpolation (3.14) and its classical counterpart prove (4.44). Clearly, (4.41) and (4.45) are consequences of (4.39) and (4.43) together with the coincidence (2.45).
The above arguments can also be used to justify (4.31)–(4.33) and to show that ext as constructed in (4.50) is also an extension operator for the related trace spaces. As there it applies not only to a fixed space \( \Lambda^{\alpha}A_{p_0,q_0}^r(\mathbb{R}^n) \) or \( \Lambda^{\alpha}F_{p_0,q_0}^{r,s}(\mathbb{R}^n) \) but also to the corresponding spaces with neighbouring \((s,p,q,\varrho)\). We formulate the outcome and indicate the respective modifications. We concentrate on the three families in (2.46) of the related \( \varrho \)-clan again. Let \( \sigma_p^{n-1} \) be as in (1.8).

**Theorem 4.13.** Let \( n \geq 2, -n < \varrho < -1, \)

\[
0 < p < \infty, \quad 0 < q < \infty \quad \text{and} \quad s - \frac{1}{p} > \sigma_p^{n-1}. \tag{4.54}
\]

Then

\[
\text{tr} : \Lambda^{\varrho}B_{p,q}^s(\mathbb{R}^n) \hookrightarrow \Lambda^{\varrho+1}B_{p,q}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}), \tag{4.55}
\]

\[
\text{tr} : \Lambda^{\varrho}F_{p,q}^s(\mathbb{R}^n) \hookrightarrow \Lambda^{\varrho+1}B_{p,p}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}), \tag{4.56}
\]

\[
\text{tr} : \Lambda^{\varrho}F_{p,q}^s(\mathbb{R}^n) \hookrightarrow \Lambda^{\varrho+1}B_{p,q}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}), \tag{4.57}
\]

and

\[
\text{tr} : \Lambda^{\varrho}F_{p,q}^s(\mathbb{R}^n) \hookrightarrow \Lambda^{\varrho+1}B_{p,p}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}). \tag{4.58}
\]

Furthermore, there are linear and bounded extension operators \( \text{ext} \) with

\[
\text{tr} \circ \text{ext} = \text{id}, \quad \text{identity in} \quad \Lambda^{\varrho+1}B_{p,q}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}) \quad \text{and} \quad \Lambda^{\varrho+1}B_{p,p}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}). \tag{4.59}
\]

such that

\[
\text{ext} : \Lambda^{\varrho+1}B_{p,q}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}) \hookrightarrow \Lambda^{\varrho}B_{p,q}^s(\mathbb{R}^n), \tag{4.60}
\]

\[
\text{ext} : \Lambda^{\varrho+1}B_{p,p}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}) \hookrightarrow \Lambda^{\varrho}F_{p,q}^s(\mathbb{R}^n), \tag{4.61}
\]

\[
\text{ext} : \Lambda^{\varrho+1}B_{p,q}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}) \hookrightarrow \Lambda^{\varrho}B_{p,q}^s(\mathbb{R}^n) \tag{4.62}
\]

and

\[
\text{ext} : \Lambda^{\varrho+1}B_{p,p}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}) \hookrightarrow \Lambda^{\varrho}F_{p,q}^s(\mathbb{R}^n). \tag{4.63}
\]

**Proof. Step 1.** We prove (4.55) and expand \( f \in \Lambda^{\varrho}B_{p,q}^s(\mathbb{R}^n) \) as in (3.7) and (3.8) with the equivalent quasi-norm (3.10). Then \( f(x',0) \) looks like (4.46). But one must be aware that there are terms with \( G_l = F \) for all \( l = 1, \ldots, n-1 \). Then one has no moment conditions and the outcome must be considered as an expansion in \( \mathbb{R}^{n-1} \) by atoms without moment conditions. The prototypes of the remaining terms on the right-hand side of (3.10) are now \( M = (M',0), M' \in \mathbb{Z}^{n-1} \) and related counterparts of \( Q_{j,m} \subset Q_{l,M} \). Using

\[
n + q = n - 1 + (\varrho + 1) \quad \text{and} \quad s - \frac{n}{p} = s - \frac{1}{p} - \frac{n - 1}{p}, \tag{4.64}
\]

we see that \(-n + 1 \leq \varrho + 1 < 0 \) and \( s - \frac{1}{p} > \sigma_p^{n-1} \) ensure that \( f(x',0) \) is an expansion in \( \mathbb{R}^{n-1} \) by atoms without moment conditions. Now it follows from [80, Theorem 3.33, p.67] that \( f(x',0) \in \Lambda^{\varrho+1}B_{p,q}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}) \) and

\[
\|f(x',0)\|_{\Lambda^{\varrho+1}B_{p,q}^{s-\frac{1}{p}}(\mathbb{R}^{n-1})} \leq c \|f\|_{\Lambda^{\varrho}B_{p,q}^s(\mathbb{R}^n)}. \tag{4.65}
\]

This proves (4.55). As far as (4.56) is concerned we rely on the wavelet expansion for \( f \in \Lambda^{\varrho}F_{p,q}^s(\mathbb{R}^n) = L^rF_{p,q}^s(\mathbb{R}^n) \) with \( r = g/p \) in [80, Theorem 3.26, p.64]. Then the sequence space \( \Lambda^{\varrho}F_{p,q}^s(\mathbb{R}^n) \) on the right-hand side of (3.10) must be replaced by its counterpart \( \Lambda^{\varrho}F_{p,q}^s(\mathbb{R}^n) = L^{\varrho/p}F_{p,q}^s(\mathbb{R}^n) \) in [80, Definition 3.24, p.63]. But for the restriction of these wavelet expansions to terms contributing to \( f(x',0) \) \((x' \in \mathbb{R}^{n-1})\), one is in the same situation as in [76, p.145, (5.98)] and the references given there. For these terms the corresponding quasi-norm \( \Lambda^{\varrho}F_{p,q}^s(\mathbb{R}^n) \) is equivalent to \( \Lambda^{\varrho}B_{p,q}^s(\mathbb{R}^n) \), which means the right-hand side...
of (3.10) with \( q = p \). But then (4.56) follows from the above arguments applied to \( \Lambda^{\varepsilon}B_{p,p}(\mathbb{R}^n) \). The interpolation (3.14) extends the above assertion to (4.57).

**Step 2.** Let \( \text{ext} \) be the same extension operator as in (4.49)–(4.51) (common for neighbouring spaces) with \( f \in \Lambda^{\varepsilon+1}B_{p,q}^{1/2}(\mathbb{R}^{n-1}) \). Then (4.60) follows from (4.64) applied to (3.10) with \( q \) in place of \( f \) and its counterpart on \( \mathbb{R}^{n-1} \) for \( f \). Recall that the atoms in (4.50) have all the moment conditions one needs.

Then (4.61) follows again from (3.10) with \( q = p < \infty \) and the comments at the end of the preceding Step 1. Finally, (4.62) is a matter of interpolation in the same way as above. Again (4.58) and (4.63) result from (4.56), (4.61) and the coincidence (2.45).

**Remark 4.14.** The proof of the atomic expansions for the spaces \( \Lambda^{\varepsilon}A_{p,q}^s(\mathbb{R}^n) = L^{\varepsilon/p}A_{p,q}^s(\mathbb{R}^n) \) in [80, Theorem 3.33, pp. 67–68] based on [79, Theorem 1.37, pp. 28–31] relies on the related technicalities for the classical spaces \( A_{p,q}^s(\mathbb{R}^n) \). This explains that one still requires \( s > \sigma_p^0 \) with \( t = n \) in (1.8), to ensure atomic expansions in \( \Lambda^{\varepsilon}B_{p,q}(\mathbb{R}^n) \) without moment conditions. One may ask whether this condition can be replaced by \( s > \sigma_p^{0|\varepsilon|} \) as suggested by the Slope-n-Rule (see Slope Rules 2.17(ii)). It would be desirable to confirm this replacement. If so, then \( s - \frac{1}{p} > \sigma_p^{-n-1} \) in (4.54) originating from the right-hand sides of (4.55) and (4.56), could be replaced by \( s - \frac{1}{p} > \sigma_p^{|\varepsilon|^{-1}} \) where \( -n < \varepsilon < -1 \). Combining Theorem 4.12, one obtains the reasonable restriction

\[
s > \frac{1}{p} \min(|\varepsilon|, 1) + \sigma_p^{\max(|\varepsilon|, 1)-1}, \quad 0 < p < \infty, \quad -n < \varepsilon < 0 \tag{4.66}
\]

for traces of the spaces in (2.46) of the \( \varepsilon \)-clan in \( \mathbb{R}^n \). If confirmed it would be an outstanding example of the Slope-1-Rule (see Slope Rules 2.17(ii)) in the interpretation that \( |\varepsilon| = 1 \) is a breaking point. Conditions of type (4.66) seem to be quite natural. They also appear in other occasions, for example in Theorem 5.29 below.

Theorems 4.12 and 4.13 apply to all the \( \varepsilon \)-clans, \( -n < \varepsilon < 0 \), according to Definition 2.15(ii) with the exception of \( \varepsilon = -1 \). It is quite clear that \( |\varepsilon| = 1 \) is a breaking point for trace spaces. Theorem 4.13 cannot be extended to \( \Lambda_{-1}B_{p,q}(\mathbb{R}^n) \) because possible trace spaces \( \Lambda_{-1}B_{p,q}^{s-1/2}(\mathbb{R}^{n-1}) \) are not covered by Definition 2.6. The situation is better for the spaces \( \Lambda^{\varepsilon}A_{p,q}^s(\mathbb{R}^n) \) as introduced in Definition 2.8 for all \( \varepsilon \) with \( -n < \varepsilon < \infty \). Furthermore, one can extend the arguments in the proof of Theorem 4.13 to the corresponding spaces with \( \varepsilon = -1 \). We formulate the outcome.

**Corollary 4.15.** Let \( n \geq 2 \),

\[
0 < p < \infty, \quad 0 < q \leq \infty \quad \text{and} \quad s - \frac{1}{p} > \sigma_p^{-n-1}. \tag{4.67}
\]

Then

\[
\text{tr} : \Lambda^{-1}B_{p,q}^s(\mathbb{R}^n) \hookrightarrow \Lambda^{0}B_{p,q}^{s-1/2}(\mathbb{R}^{n-1}) \tag{4.68}
\]

and

\[
\text{tr} : \Lambda^{-1}F_{p,q}^s(\mathbb{R}^n) \hookrightarrow \Lambda^{0}B_{p,q}^{s-1/2}(\mathbb{R}^{n-1}) \tag{4.69}
\]

Furthermore, there are linear and bounded extension operators \( \text{ext} \) with

\[
\text{tr} \circ \text{ext} = \text{id}, \quad \text{identity in } \Lambda^0B_{p,q}^{s-1/2}(\mathbb{R}^{n-1}) \tag{4.70}
\]

such that

\[
\text{ext} : \Lambda^0B_{p,q}^{s-1/2}(\mathbb{R}^{n-1}) \hookrightarrow \Lambda^{-1}B_{p,q}^s(\mathbb{R}^n) \tag{4.71}
\]

and

\[
\text{ext} : \Lambda^0B_{p,q}^{s-1/2}(\mathbb{R}^{n-1}) \hookrightarrow \Lambda^{-1}F_{p,q}^s(\mathbb{R}^n). \tag{4.72}
\]

**Proof.** As mentioned in Remark 3.2, Proposition 3.1 remains valid for the spaces \( \Lambda^0B_{p,q}^s(\mathbb{R}^n) \). Then the arguments in the proof of Theorem 4.13 also apply to the above spaces, including the references about atomic expansions and the indicated technicalities as far as the \( F \)-spaces are concerned. 

\[ \Box \]
Remark 4.16. Some consequences of the above corollary might be of interest. It follows from (2.38) and (2.40) that
\[
\text{tr} \Lambda^{-1}B^s_{p,\infty}(\mathbb{R}^n) = C^{s-\frac{1}{p}}(\mathbb{R}^{n-1}), \quad 0 < p < \infty, \quad s > \frac{1}{p}
\] (4.73)
and
\[
\text{tr} \Lambda^{-1}F^s_{p,q}(\mathbb{R}^n) = F^{s-\frac{1}{p}}_{\infty,q}(\mathbb{R}^{n-1}), \quad 0 < p < \infty, \quad 0 < q \leq \infty, \quad s - \frac{1}{p} > \sigma_p^{n-1}.
\] (4.74)
In (4.73), one does not need the sharper restriction \( s - \frac{1}{p} > \sigma_p^{n-1} \) because no moment conditions for atomic expansions in \( C^s(\mathbb{R}^{n-1}) \) with \( \sigma > 0 \) are requested. Furthermore, (4.53) appropriately modified can also be applied to \( |\varrho| = 1 \) and \( q = \infty \). In other words, (4.73) complements Theorem 4.12.

5 Essential features

5.1 Embeddings in \( L_\infty(\mathbb{R}^n) \) and \( C(\mathbb{R}^n) \)

First we deal with embeddings of the spaces covered by Definition 2.15 in \( L_\infty(\mathbb{R}^n) \) and \( C(\mathbb{R}^n) \) as target spaces. We always assume \( f(x) = 0 \) if \( x \in \mathbb{R}^n \) is not a Lebesgue point of the locally Lebesgue-integrable function \( f(x) \) in \( \mathbb{R}^n \). Then
\[
\| f \|_{L_\infty(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |f(x)|
\] (5.1)
is the norm in the usual space \( L_\infty(\mathbb{R}^n) \) of (essentially) bounded complex-valued functions in \( \mathbb{R}^n \). Let \( C(\mathbb{R}^n) \) be the space of all the continuous bounded complex-valued functions in \( \mathbb{R}^n \) normed by (5.1). Both \( L_\infty(\mathbb{R}^n) \) and \( C(\mathbb{R}^n) \) are considered as subspaces of \( S'(\mathbb{R}^n) \). As above \( \hookrightarrow \) indicates the continuous embedding. We concentrate on the \( \varrho \)-clan with \( -n < \varrho < 0 \) consisting of the three families (2.46) and (2.47). But it comes out that there are some peculiarities compared with the related embeddings of the classical spaces (2.42), the \( n \)-clan. This may justify that we first recall the corresponding assertion.

Proposition 5.1. Let \( A^n_{p,q}(\mathbb{R}^n) \) with \( A \in \{B, F\} \), \( s \in \mathbb{R} \), \( 0 < p \leq \infty \) and \( 0 < q \leq \infty \) be the spaces according to Definition 2.1, the \( n \)-clan (2.42). Then
\[
B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)
\] (5.2)
if and only if
\[
\left\{ \begin{array}{l}
either s > \frac{n}{p}, & 0 < q \leq \infty, \\
or s = \frac{n}{p}, & 0 < q \leq 1,
\end{array} \right.
\] (5.3)
and
\[
F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)
\] (5.4)
if and only if
\[
\left\{ \begin{array}{l}
either s > \frac{n}{p}, & 0 < q \leq \infty, \\
or s = \frac{n}{p}, & 0 < p \leq 1, \quad 0 < q \leq \infty.
\end{array} \right.
\] (5.5)
Furthermore, one can replace \( L_\infty(\mathbb{R}^n) \) in (5.2) and (5.4) by \( C(\mathbb{R}^n) \).

Remark 5.2. This coincides with [82, Theorem 2.3, pp. 22–23] where we complemented already known assertions for the spaces \( A^n_{p,q}(\mathbb{R}^n), A \in \{B, F\} \) with \( p < \infty \) for the \( F \)-spaces by \( F^\mu_{\infty,q}(\mathbb{R}^n) \). There one also finds related references, like [69, Theorem 3.3.1].

For the spaces in (2.46) and (2.47) of the \( \varrho \)-clan according to Definition 2.15(ii) with \( -n < \varrho < 0 \), one has the following counterpart.
Theorem 5.3. Let
\[ s \in \mathbb{R}, \quad 0 < p < \infty, \quad 0 < q \leq \infty \quad \text{and} \quad -n < \varrho < 0. \]  
(5.6)

Then
\[ \Lambda^\varrho A^p_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \quad \text{if and only if} \quad s > \frac{|\varrho|}{p}, \]  
(5.7)

\[ A \in \{B,F\}, \quad \text{and} \]
\[ \Lambda^\varrho B^p_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \quad \text{if and only if} \]
\[ \begin{cases} 
\text{either} & s > \frac{|\varrho|}{p}, \quad 0 < q \leq \infty, \\
\text{or} & s = \frac{|\varrho|}{p}, \quad 0 < q \leq 1,
\end{cases} \]  
(5.8)

\[ \Lambda^\varrho F^p_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \quad \text{if and only if} \quad s > \frac{|\varrho|}{p}. \]  
(5.9)

Furthermore, one can replace \( L_\infty(\mathbb{R}^n) \) in (5.7)–(5.9) by \( C(\mathbb{R}^n) \).

Remark 5.4. The embedding (5.7) is covered by [93, Proposition 5.4, p.334] whereas (5.8) goes back to [28, Proposition 5.5, p.140], and (5.9) to [29, Proposition 3.8], reformulated according to (2.24), (2.30) and (2.31). They may also be found in [24, Proposition 3.1, p.225]. These embeddings, compared with their classical counterparts in Proposition 5.1, are examples of the Slope-n-Rule (see Slope Rules 2.17(ii)). But there are also some peculiarities as far as limiting embeddings with

\[ s = \frac{n}{p} \quad \text{and} \quad s = \frac{|\varrho|}{p} \]

are concerned. Further related discussions especially about limiting embeddings may be found in [25, 29, 92]. One may extend (5.7) to \( \varrho = 0 \) in the following way. The 0-clan consists of the spaces \( \Lambda^0 B^p_{p,q}(\mathbb{R}^n) \) in (2.44), and also the spaces \( \Lambda^0 F^p_{p,q}(\mathbb{R}^n) \) given by (2.39), and they can be incorporated: using the coincidences (2.30) with (2.31), as well as (2.35) and (2.39), we see that [92, Proposition 4.1] yields

\[ \Lambda^0 A^p_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \quad \text{if and only if} \quad s > 0, \]

where again \( A \in \{B,F\} \) and \( L_\infty(\mathbb{R}^n) \) can be replaced by \( C(\mathbb{R}^n) \). The limiting embedding in (5.8) compared with (5.7) also illuminates the strict embedding (2.58).

\[ \begin{array}{c}
|s| = \frac{n}{p} - n \\
\hline
|\varrho|/p
\end{array} \]

Figure 1 (Color online) Embeddings in \( L_\infty \) and \( \delta \)-distribution
5.2 Embeddings in the space of locally integrable functions

Let again $A(\mathbb{R}^n)$ be a quasi-Banach space with

$$S(\mathbb{R}^n) \hookrightarrow A(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n).$$

(5.10)

Recall that $L^{1_{oc}}(\mathbb{R}^n)$ collects all the Lebesgue-measurable (complex-valued) functions in $\mathbb{R}^n$ which are integrable on any bounded domain in $\mathbb{R}^n$. Then

$$A(\mathbb{R}^n) \subset L^{1_{oc}}(\mathbb{R}^n)$$

(5.11)

means that any $f \in A(\mathbb{R}^n)$ can be represented as a locally integrable function such that

$$f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \quad \varphi \in D(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n).$$

(5.12)

First we recall the final conditions ensuring (5.11) for the $n$-clan, consisting of the classical spaces in (2.42). We rely on our notation (1.8) again.

**Proposition 5.5.** Let $A_{p,q}^s(\mathbb{R}^n)$ with $A \in \{B, F\}$, $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$ be the spaces according to Definition 2.1, the $n$-clan (2.42). Then

$$B_{p,q}^s(\mathbb{R}^n) \subset L^{1_{oc}}(\mathbb{R}^n)$$

(5.13)

if and only if

$$\begin{cases} 
\text{either} & 0 < p \leq \infty, \quad s > \sigma_p^s, \quad 0 < q \leq \infty, \\
\text{or} & 0 < p \leq 1, \quad s = \sigma_p^s, \quad 0 < q \leq 1, \\
\text{or} & 1 < p \leq \infty, \quad s = 0, \quad 0 < q \leq \min(p, 2), 
\end{cases}$$

(5.14)

and

$$F_{p,q}^s(\mathbb{R}^n) \subset L^{1_{oc}}(\mathbb{R}^n)$$

(5.15)

if and only if

$$\begin{cases} 
\text{either} & 0 < p < 1, \quad s \geq \sigma_p^s, \quad 0 < q \leq \infty, \\
\text{or} & 1 \leq p \leq \infty, \quad s > 0, \quad 0 < q \leq \infty, \\
\text{or} & 1 \leq p \leq \infty, \quad s = 0, \quad 0 < q \leq 2. 
\end{cases}$$

(5.16)

**Remark 5.6.** This coincides with [82, Theorem 2.4, pp.23–24], where already known assertions for the spaces $A_{p,q}^s(\mathbb{R}^n)$ with $p < \infty$ for $F$-spaces (see [69, Theorem 3.3.2]), had been complemented by $F_{\infty,q}^s(\mathbb{R}^n)$. There one finds further references.

The Slope-$n$-Rule (see Slope Rules 2.17(ii)) suggests that one has to replace the breaking line $s = \sigma_p^s$ in the $(\frac{1}{p}, s)$-diagram for the spaces $A_{p,q}^s(\mathbb{R}^n)$ by $s = \sigma_p^{|\varphi|}$ for the spaces (2.46) and (2.47) of the $\varphi$-clan, $-n < \varphi < 0$. This is the case. But there is so far no final assertion of what happens for the limiting spaces with $s = \sigma_p^{|\varphi|}$. This may justify that we concentrate on the non-limiting spaces with $s \neq \sigma_p^{|\varphi|}$.

**Theorem 5.7.** Let

$$s \in \mathbb{R}, \quad 0 < p < \infty, \quad 0 < q \leq \infty \quad \text{and} \quad -n < \varphi < 0.$$

(5.17)

Then

$$\varphi A_{p,q}^s(\mathbb{R}^n) \subset L^{1_{oc}}(\mathbb{R}^n) \quad \text{if} \quad s > \sigma_p^{|\varphi|}$$

(5.18)

and

$$\varphi A_{p,q}^s(\mathbb{R}^n) \not\subset L^{1_{oc}}(\mathbb{R}^n) \quad \text{if} \quad s < \sigma_p^{|\varphi|},$$

(5.19)

$A \in \{B, F\}$.
Remark 5.8. These assertions are covered by [24, Theorems 3.3 and 3.6, pp. 228 and 232] reformulated according to (2.24), (2.30) and (2.31). In [24, Theorems 3.4 and 3.8, pp. 228 and 233], there are detailed discussions of what happens if $s = \sigma_p|\rho|p$, at least partly comparable with $s = \sigma_n|\rho|p$ in Proposition 5.5 but less final. In [25, Theorems 3.2 and 3.4], the complete characterisation in the case of $\rho - A_s^{\rho,p,q} = \Lambda \rho A_s^{\rho,p,q}$ could be obtained for the limiting case of $s = \sigma_p|\rho|p$, which reads as

$$\Lambda \rho B_{p,q}^{\sigma|\rho|p}(\mathbb{R}^n) \subset L_1^{\text{loc}}(\mathbb{R}^n) \quad \text{if and only if} \quad 0 < q \leq \min(\max(p,1),2),$$

and

$$\Lambda \rho F_{p,q}^{\sigma|\rho|p}(\mathbb{R}^n) \subset L_1^{\text{loc}}(\mathbb{R}^n) \quad \text{if and only if} \quad \begin{cases} \text{either} & p \geq 1 \quad \text{and} \quad 0 < q \leq 2, \\ \text{or} & 0 < p < 1. \end{cases}$$

This is literally the extension of (5.13)–(5.16) when $s = \sigma_n|\rho|p$ in the classical case there. The coincidence (2.45) thus answers the question for $\Lambda \rho B_{p,q}^{\sigma|\rho|p}(\mathbb{R}^n)$ in the case of $s = \sigma_p|\rho|p$, too, i.e.,

$$\Lambda \rho F_{p,q}^{\sigma|\rho|p}(\mathbb{R}^n) \subset L_1^{\text{loc}}(\mathbb{R}^n) \quad \text{if and only if} \quad \begin{cases} \text{either} & p \geq 1 \quad \text{and} \quad 0 < q \leq 2, \\ \text{or} & 0 < p < 1. \end{cases}$$

The only gap for the moment concerns the characterisation of $\Lambda \rho B_{p,q}^{\sigma|\rho|p}(\mathbb{R}^n) \subset L_1^{\text{loc}}(\mathbb{R}^n)$, i.e., the counterpart of (5.20); we refer to [24, Theorem 3.8(i)] for the details.

5.3 Multiplication algebras

Recall that a quasi-Banach space $A(\mathbb{R}^n)$ on $\mathbb{R}^n$ with

$$S(\mathbb{R}^n) \hookrightarrow A(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n), \quad A(\mathbb{R}^n) \subset L_1^{\text{loc}}(\mathbb{R}^n)$$

is said to be a multiplication algebra if $f_1f_2 \in A(\mathbb{R}^n)$ whenever $f_1 \in A(\mathbb{R}^n)$, $f_2 \in A(\mathbb{R}^n)$ and if there is a constant $c > 0$ such that

$$\|f_1f_2\|_A(\mathbb{R}^n) \leq c\|f_1\|_A(\mathbb{R}^n)\cdot\|f_2\|_A(\mathbb{R}^n)$$

for all $f_1 \in A(\mathbb{R}^n)$ and $f_2 \in A(\mathbb{R}^n)$. Note that we assume that $A(\mathbb{R}^n)$ consists entirely of regular distributions. Then the product of two elements of these spaces makes sense at least pointwise a.e. (almost everywhere). One may ask for additional properties of these products. Of special interest is the question whether the product of two elements of $A(\mathbb{R}^n)$ is an element of this space again. This coincides with [82, Definition 2.37, p. 44]. There one finds related (historical) references, also about possible applications in the theory of distinguished non-linear PDEs, including the Navier-Stokes equations.

**Proposition 5.9.** Let $A(\mathbb{R}^n)$ be a space covered by Definition 2.15 satisfying in addition (5.23). If $A(\mathbb{R}^n)$ is a multiplication algebra, then

$$A(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n).$$
Remark 5.10. The short proof of this well-known assertion in [79, Proposition 2.41, p. 90] applies to all the spaces in question. In other words, the search for possible multiplication algebras is now shifted from Proposition 5.5 and Theorem 5.7 to Proposition 5.1 and Theorem 5.3. First we recall the final assertion for the classical spaces $A^s_{p,q}(\mathbb{R}^n)$.

Proposition 5.11. Let $A^s_{p,q}(\mathbb{R}^n)$ be the classical spaces in (2.42), the $n$-clan, satisfying in addition (5.23) with $A(\mathbb{R}^n) = A^s_{p,q}(\mathbb{R}^n)$. Then $B^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra if and only if

\[
\begin{aligned}
\text{either } s > n/p, & \text{ where } 0 < p, q \leq \infty, \\
or s = n/p, & \text{ where } 0 < p < \infty, 0 < q \leq 1,
\end{aligned}
\]  

(5.26)

and $F^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra if and only if

\[
\begin{aligned}
\text{either } s > n/p, & \text{ where } 0 < p, q \leq \infty, \\
or s = n/p, & \text{ where } 0 < p \leq 1, 0 < q \leq \infty.
\end{aligned}
\]  

(5.27)

Remark 5.12. This coincides with [82, Theorem 2.41, p. 45] where the related assertion for the spaces $A^s_{p,q}(\mathbb{R}^n)$ with $p < \infty$ for $F$-spaces has been extended to $F^s_{p,q}(\mathbb{R}^n)$. In [82, Remark 2.42, p. 46], one also finds detailed references about the substantial history of this problem, including [44] as one of the earliest contributions.

We ask for a counterpart of Proposition 5.11 for the spaces (2.46) of the $\varrho$-clan with $-n < \varrho < 0$. Let again

\[
\sigma^{|\varrho|}_p = |\varrho| \left( \max \left( \frac{1}{p}, 1 \right) - 1 \right), \quad 0 < p \leq \infty
\]  

(5.28)

be as in (1.8) with $t = |\varrho|$.

Theorem 5.13. Let $A(\mathbb{R}^n)$ be one of the spaces in (2.46) and (2.47) of the $\varrho$-clan with $-n < \varrho < 0$ satisfying in addition (5.23).

(i) Then $A(\mathbb{R}^n) = \Lambda^{\varrho} A^s_{p,q}(\mathbb{R}^n)$, $A \in \{B, F\}$ is a multiplication algebra if and only if $s > |\varrho|/p$.

(ii) If $A(\mathbb{R}^n) = \Lambda^{\varrho} B^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra, then either $s > |\varrho|/p$, $0 < q \leq \infty$, or $s = |\varrho|/p$, $0 < q \leq 1$. Conversely, if $s > |\varrho|/p$, $0 < q \leq \infty$, then $\Lambda^{\varrho} B^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra.

(iii) Then $A(\mathbb{R}^n) = \Lambda^{\varrho} F^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra if and only if $s > |\varrho|/p$.

Proof. Step 1. If $A(\mathbb{R}^n)$ is a multiplication algebra, then the indicated conditions for $s, p, q$ follow from (5.25) and Theorem 5.3.

Step 2. If $s > |\varrho|/p$, then it follows from (2.32) and [80, Theorem 3.60, p. 95] with a reference to [79, Theorem 2.43, p. 91] that the spaces $\Lambda^{\varrho} A^s_{p,q}(\mathbb{R}^n)$ are multiplication algebras. This proves the part (i). The corresponding proof (of the local version) of the spaces $\Lambda^{\varrho} B^s_{p,q}(\mathbb{R}^n)$ in [79, pp. 91–93] relies on (the local version of) the wavelet expansion according to Proposition 3.1 with the right-hand side of (3.10) as the related sequence space. But it also works for the right-hand side of (3.11) at least as long as $s > |\varrho|/p$. This proves the part (ii). Part (iii) is a consequence of (2.45) and (i).

Corollary 5.14. A space $\Lambda^{\varrho} A^s_{p,q}(\mathbb{R}^n)$, $A \in \{B, F\}$ of the $\varrho$-clan according to Definition 2.15(ii) with $-n < \varrho < 0$ is a multiplication algebra if and only if it is continuously embedded in $C(\mathbb{R}^n)$.

Proof. This follows immediately from Theorem 5.5 and the part (i) of the above theorem.

Remark 5.15. The above-mentioned somewhat sketchy proof in [79, pp. 91–93] relies decisively on $s > |\varrho|/p$. But it is not clear whether the related arguments can be extended to the spaces $\Lambda^{\varrho} B^s_{p,q}(\mathbb{R}^n)$ with $s = |\varrho|/p$ and $0 < q \leq 1$. If this is the case (as we expect), then Corollary 5.14 can be extended to all the spaces in (2.46) of the $\varrho$-clan with $-n < \varrho < 0$. But this is not obvious as the situation for the $n$-clan shows. The conditions (5.3) and (5.26) differ for $p = \infty$. This somewhat tricky point has its own history and was clarified eventually in [69, Remark 3.3.2, p. 114 and Corollary 4.3.2 and Remark 4.3.5, p. 120]. In any case, Theorem 5.13, compared with Proposition 5.11, is a further example of the Slope-$n$-Rule (see Slope Rules 2.17(ii)).
5.4 The δ-distribution

As a preparation of our later arguments we clarify to which spaces covered by Definition 2.15 the δ-distribution, \( \delta(\varphi) = \varphi(0), \varphi \in S(\mathbb{R}^n), \) belongs. First we deal with the \( n \)-clan, consisting of the spaces in (2.42), complementing related very classical assertions for these spaces with \( p < \infty \) for the \( F \)-spaces by \( F_{\infty,q}^s(\mathbb{R}^n) \).

Proposition 5.16. Let \( n \in \mathbb{N} \) and let \( A_{p,q}^s(\mathbb{R}^n) \) be the spaces according to (2.42). Then

\[
\delta \in B_{p,q}^s(\mathbb{R}^n) \quad \text{if and only if} \quad \begin{cases} \text{either} & 0 < p \leq \infty, \quad s < n\left(\frac{1}{p} - 1\right), \quad 0 < q \leq \infty, \\ \text{or} & 0 < p \leq \infty, \quad s = n\left(\frac{1}{p} - 1\right), \quad q = \infty \end{cases} \tag{5.29}
\]

and

\[
\delta \in F_{p,q}^s(\mathbb{R}^n) \quad \text{if and only if} \quad \begin{cases} \text{either} & 0 < p < \infty, \quad s < n\left(\frac{1}{p} - 1\right), \quad 0 < q \leq \infty, \\ \text{or} & p = \infty, \quad s \leq -n, \quad 0 < q \leq \infty. \end{cases} \tag{5.30}
\]

Proof. Let \( \{\varphi_j\}_{j=0}^\infty \) be the usual dyadic resolution of unity according to (2.3)–(2.5) and let \( \varphi_j(\xi) = \varphi(2^{-j}\xi), j \in \mathbb{N} \) and \( \xi \in \mathbb{R}^n \). Using \( \hat{\delta} = c \neq 0, \) we have from

\[
(\varphi_j\hat{\delta})^\vee(x) = c2^j n^\vee(2^j x), \quad j \in \mathbb{N}, \quad x \in \mathbb{R}^n \tag{5.31}
\]

that

\[
\|\varphi_j\hat{\delta}\|_{L_p(\mathbb{R}^n)} \sim 2^n(1-\frac{1}{p}), \quad j \in \mathbb{N}, \quad 0 < p \leq \infty, \tag{5.32}
\]

If we insert (5.32) into (2.10), we obtain

\[
\|\delta \|_{B_{p,q}^s(\mathbb{R}^n)} \sim \left( \sum_{j=0}^\infty 2^n 2^{jn+n(1-s)} \right)^{1/q}. \tag{5.33}
\]

This proves (5.29). Let \( 0 < p_0 < p < p_1 \leq \infty, \) \( 0 < q \leq \infty \) and

\[
s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}. \tag{5.34}
\]

Then (5.30) follows from (5.29) and the well-known sharp embeddings

\[
B_{p_0,\infty}^{s_0}(\mathbb{R}^n) \hookrightarrow B_{p,q}^s(\mathbb{R}^n) \hookrightarrow B_{p_1,\infty}^{s_1}(\mathbb{R}^n) \tag{5.35}
\]

with \( 0 < u \leq p \leq v \leq \infty, \) often called the Franke-Jawerth embedding nowadays to honour the contributions by Franke [11], concerning the left-hand embedding, and Jawerth [35], concerning the right-hand embedding. The sharp result can be found in [69]. Let us also refer to the more recent, elegant new proof of Vybíral [83]. We recommend [82, Theorem 2.5, p. 25] for further discussions. Let again \( \chi_Q \) be the characteristic function of the unit cube \( Q = (0,1)^n, \) One has according to [82, Proposition 2.43, p. 46] that

\[
\chi_Q \in F_{\infty,q}^0(\mathbb{R}^n) \quad \text{for all} \quad 0 < q \leq \infty. \tag{5.36}
\]

Let \( \partial^n = \partial_1 \cdots \partial_n. \) Then (5.30) with \( p = \infty \) follows from

\[
\partial^n \chi_Q = \delta + \cdots \in F_{\infty,q}^{-n}(\mathbb{R}^n), \tag{5.37}
\]

where \( + \cdots \) indicates \( \delta \)-distributions with the remaining corners of \( Q \) as off-points. Here, we used in addition the special case

\[
\|f \|_{F_{\infty,q}^0(\mathbb{R}^n)} \sim \sup_{0 \leq |a| \leq n} \|D^a f \|_{F_{\infty,q}^{-n}(\mathbb{R}^n)} \| \tag{5.38}
\]

of (3.30), \( F_{\infty,q}^0(\mathbb{R}^n) \hookrightarrow C^s(\mathbb{R}^n), \) [82, Theorem 2.9, p. 26] and (5.29). □
Remark 5.17. We incorporated $F_{\infty,q}^a(\mathbb{R}^n)$ into the otherwise classical assertion for the spaces $A^s_{p,q}(\mathbb{R}^n)$. But we have no reference at hand. We inserted the above proposition because it will again illuminate the difference between the spaces in (2.42) of the $n$-clan and the spaces in (2.46) and (2.47) of the $r$-clans, $-n < r < 0$, in limiting situations.

Next, we deal with the $r$-clans according to Definition 2.15(ii) consisting of the three families (2.46) with $-n < r < 0$, $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$.

Theorem 5.18. Let $n \in \mathbb{N}$, $-n < r < 0$ and $0 < p < \infty$. Let $0 < q \leq \infty$ and $s \in \mathbb{R}$. Then

$$
\delta \in \Lambda_\varepsilon A^s_{p,q}(\mathbb{R}^n) \quad \text{if and only if} \quad s \leq \frac{|\rho|}{p} - n, \quad 0 < q \leq \infty,
$$

(5.39)

where $A \in \{B, F\}$,

$$
\delta \in \Lambda_\varepsilon B^s_{p,q}(\mathbb{R}^n) \quad \text{if and only if} \quad \begin{cases} 
\text{either} & s < \frac{|\rho|}{p} - n, \quad 0 < q \leq \infty, \\
\text{or} & s = \frac{|\rho|}{p} - n, \quad q = \infty
\end{cases},
$$

(5.40)

and

$$
\delta \in \Lambda_\varepsilon F^s_{p,q}(\mathbb{R}^n) \quad \text{if and only if} \quad s \leq \frac{|\rho|}{p} - n, \quad 0 < q \leq \infty.
$$

(5.41)

Proof. Step 1. We prove (5.40). This means by (2.53) that we have to check whether

$$
\|\delta \mid \Lambda_\varepsilon B^s_{p,q}(\mathbb{R}^n)\| = \left( \sum_{j=0}^{\infty} 2^{jsq} \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{j/2} |(\varphi_j \delta)^2 \parallel L_p(Q_{J,M})|^q \right)^{1/q}
$$

(5.42)

is finite or not (with the usual modification if $q = \infty$). By (5.31), one has

$$
\sup_{M \in \mathbb{Z}^n} \|((\varphi_j \delta)^2 \parallel L_p(Q_{J,M})\| \sim \begin{cases} 
2^{jn(1-\frac{1}{p})}, & \text{if } j \geq J, \\
2^n 2^{-\frac{s}{p}}, & \text{if } j < J.
\end{cases}
$$

(5.43)

Using $-n < r < 0$, we obtain that

$$
\|\delta \mid \Lambda_\varepsilon B^s_{p,q}(\mathbb{R}^n)\| \sim \left( \sum_{j=0}^{\infty} 2^{q(s+n+\frac{r}{p})} \right)^{1/q}
$$

(5.44)

(again with the usual modification if $q = \infty$). This proves (5.40).

Step 2. We prove (5.39) for $A = B$. This means by (2.54) that we have to check whether

$$
\|\delta \mid \Lambda_\varepsilon B^s_{p,q}(\mathbb{R}^n)\| = \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{j/2} |(\varphi_j \delta)^2 \parallel L_p(Q_{J,M})|^q \left( \sum_{j \geq J} 2^{jsq} |(\varphi_j \delta)^2 \parallel L_p(Q_{J,M})|^q \right)^{1/q}
$$

(5.45)

is finite or not (with the usual modification if $q = \infty$). By (5.43) (with $M = 0$), one has

$$
\|\delta \mid \Lambda_\varepsilon B^s_{p,q}(\mathbb{R}^n)\| \sim \sup_{J \in \mathbb{Z}} 2^{j/2} |(\varphi_j \delta)^2 \parallel L_p(Q_{J,M})|^q \left( \sum_{j \geq J} 2^{jsq} |(\varphi_j \delta)^2 \parallel L_p(Q_{J,M})|^q \right)^{1/q}
$$

(5.46)

Since $|\rho| < n$, it is sufficient to justify (5.39) for $s < \frac{n}{p} - n$. Using in addition $n + \rho > 0$, we have

$$
\|\delta \mid \Lambda_\varepsilon B^s_{p,q}(\mathbb{R}^n)\| \sim \sup_{J \in \mathbb{Z}} 2^{j/2} |(\varphi_j \delta)^2 \parallel L_p(Q_{J,M})|^q 2^{j/2} |(\varphi_j \delta)^2 \parallel L_p(Q_{J,M})|^q.
$$

(5.47)

This proves (5.39) for $A = B$. One obtains the corresponding assertion for the spaces $\Lambda_\varepsilon F^s_{p,q}(\mathbb{R}^n)$ from (2.56). This finally covers (5.41) in view of the coincidence (2.45).
Remark 5.19. The arguments of Step 2 of the above proof also apply to the 0-clan according to Definition 2.15(iii) with the outcome that
\[ \delta \in \Lambda^0 B_{p,q}^s(\mathbb{R}^n) \quad \text{with} \quad 0 < p, q < \infty \quad \text{if and only if} \quad s \leq -n. \] (5.48)

The parallel result holds for \( \Lambda^0 F_{p,q}^s(\mathbb{R}^n) \) due to (2.35) and (2.39), as then [24, Example 3.2] provides
\[ \delta \in \Lambda^0 F_{p,q}^s(\mathbb{R}^n) \quad \text{with} \quad 0 < p, q < \infty \quad \text{if and only if} \quad s \leq -n. \]

Remark 5.20. The above theorem and also the comments in Remark 5.19 are already covered by [24, Example 3.2, pp. 225–227], appropriately reformulated. This also applies to the proof, relying as above, on the Fourier-analytical definition of these spaces. However, comparing the corresponding diagrams in [24] with the above Figure 1 on page 1328 which indicate the parameters \( (\frac{1}{p}, s) \) such that \( \delta \in \varrho A_{p,q}^s(\mathbb{R}^n) \), we see that the advantage of the new setting is obvious.

Remark 5.21. Theorem 5.18 compared with Proposition 5.16 is again an example of the Slope-Rule (see Slope Rules 2.17(ii)), where \( \frac{1}{p} \) is replaced by \( \frac{|\varrho|}{p} \). But there are also some differences in limiting situations, which means \( s = \frac{n}{p} - n \) in Proposition 5.16 and \( s = \frac{|\varrho|}{p} - n \), \( 0 < |\varrho| < n \) in Theorem 5.18. This also applies to (5.48) compared with \( B_{\infty,n}^s(\mathbb{R}^n) \). Combined with the lifts as described in (3.25) and (3.29), it shows again that the first embedding in (2.38) is strict. Furthermore if \( q < \infty \), then it follows from the above theorem combined with the lifts as described in Theorem 3.8 that the spaces \( \Lambda^0 B_{p,q}^s(\mathbb{R}^n) \) and \( \Lambda^0 s_{p,q}^n(\mathbb{R}^n) \), \( n < q \leq 0 \), do not coincide. This had already been used in Step 2 of the proof of Theorem 2.21 to justify that the embedding (2.58) is strict.

5.5 The characteristic function \( \chi_Q \)

Expansions in terms of Haar wavelets in suitable spaces \( A_{p,q}^s(\mathbb{R}^n) \) attracted a lot of attention up to our time. A description including impressive recent results may be found in [82, Subsection 3.5, pp. 98–103]. It is quite natural to ask for the corresponding expansions in appropriate spaces according to (2.46) and (2.47), the \( q \)-clan, \( -n < q \leq 0 \). These spaces are not separable: the corresponding proof for the Morrey spaces \( A_{p,q}^s(\mathbb{R}^n) \) with \( -n < q \leq 0 \) according to Definition 2.3 in [80, Subsection 2.3.4, pp. 23–24] can be extended to these spaces. Then the related Haar wavelets cannot be a basis. However, of course, expansions within the dual pairing \( (S(\mathbb{R}^n), S'(\mathbb{R}^n)) \) make sense, quite similar to the wavelet expansions in Subsection 3.1. First steps in this direction have been done quite recently in [97], discussing the question to which spaces the characteristic function \( \chi_Q \) of the unit cube \( Q = (0,1)^n \) belongs and in which spaces \( \chi_Q \) generates a linear and bounded functional. We deal with these topics in the context of our preceding considerations.

First we recall to which classical spaces \( A_{p,q}^s(\mathbb{R}^n) \) the characteristic function \( \chi_Q \) of \( Q = (0,1)^n \), \( n \in \mathbb{N} \), belongs.

**Proposition 5.22.** Let \( A_{p,q}^s(\mathbb{R}^n) \) be the spaces according to (2.42), the \( n \)-clan. Then
\[ \chi_Q \in B_{p,q}^{1/p}(\mathbb{R}^n) \quad \text{if and only if} \quad q = \infty \] (5.49)
and
\[ \chi_Q \in F_{p,q}^{1/p}(\mathbb{R}^n) \quad \text{if and only if} \quad p = \infty. \] (5.50)
Furthermore, \( \chi_Q \in A_{p,q}^s(\mathbb{R}^n) \) if and only if either \( s < 1/p \) or as in (5.49) and (5.50).

**Remark 5.23.** This coincides with [82, Proposition 2.50, p. 50]. The proof consists of two steps. First one deals with \( n = 1 \) and secondly one reduces higher dimensions to one dimension. This method also works for the other spaces covered by Definition 2.15.

**Theorem 5.24.** Let \( \Lambda^e A_{p,q}^s(\mathbb{R}^n) \) and \( \Lambda^e s_{p,q}^n(\mathbb{R}^n) \) with \( A \in \{B,F\} \) be the spaces according to (2.46) and (2.47) of the \( q \)-clan in Definition 2.15(ii) with \( -n < q \leq 0 \). Then
\[ \chi_Q \in \Lambda^e A_{p,q}^s(\mathbb{R}^n) \] (5.51)
if and only if
\[-\infty < s \leq \frac{1}{p} \min(|\varrho|, 1) \quad \text{with} \quad q = \infty, \quad \text{if} \quad A = B \quad \text{and} \quad s = \frac{1}{p}. \quad (5.52)\]

Furthermore,
\[\chi_Q \in \Lambda_\varrho^* B^p_{p,q}(\mathbb{R}^n) \quad (5.53)\]
if and only if
\[-\infty < s \leq \frac{1}{p} \min(|\varrho|, 1) \quad \text{with} \quad q = \infty, \quad \text{if} \quad s = \frac{\min(|\varrho|, 1)}{p}, \quad (5.54)\]
and
\[\chi_Q \in \Lambda_\varrho^* F^p_{p,q}(\mathbb{R}^n) \quad (5.55)\]
if and only if
\[-\infty < s \leq \frac{1}{p} \min(|\varrho|, 1). \quad (5.56)\]

One can sketch the parameter areas according to (5.52) and (5.54) as shown in Figure 3 below.

**Proof of Theorem 5.24.** **Step 1.** Let \( n = 1 \) and \(-1 < \varrho < 0\). Let \( \chi_I \) be the characteristic function of the unit interval \( I = (0, 1) \). Then \( \chi'_I = c(\delta - \delta_1) \) and \( c \neq 0 \), where \( \delta_1 \) is the shifted \( \delta \)-distribution with the off-point 1. Now Theorem 3.10 shows that
\[\chi_I \in \Lambda_\varrho^* A^p_{p,q}(\mathbb{R}) \quad \text{if and only if} \quad \delta \in \Lambda_\varrho^* A^{p-1}_p(\mathbb{R}) \quad (5.57)\]
and
\[\chi_I \in \Lambda_\varrho^* B^p_{p,q}(\mathbb{R}) \quad \text{if and only if} \quad \delta \in \Lambda_\varrho^* B^{p-1}_p(\mathbb{R}). \quad (5.58)\]

Application of Theorem 5.18 proves the one-dimensional case of the theorem.

**Step 2.** Let \( 2 \leq n \in \mathbb{N} \). We prove (5.51) and (5.52) for \( A = B \). For this purpose, we expand \( f = \chi_Q \) according to (3.7) and (3.8). Then we are in the same position as in [82, p.51], especially
\[\lambda^G_{m,n}(\chi_Q) = \lambda^{j,M}_{m,n}(\chi_I), \quad m = (m_1, \ldots, m_n), \quad G = (F, \ldots, F, M) \quad (5.59)\]
is the prototype to be considered, relying on the same reasoning as there. In (3.10) and (3.11) with \( f = \chi_Q \) we may assume \( J \in \mathbb{N}_0 \) instead of \( J \in \mathbb{Z} \). This follows from \( \varrho + n > 0 \). For fixed \( m_n \) in (5.59), one has now \( \sim 2^{(j-J)(n-1)} \) terms in (5.59) with \( m = (m', m_n) \) and \( Q_{j,m} \subset Q_{J,M} \). Then
\[2^{\frac{n}{p}(n+\varrho)+j(s-\frac{1}{p})} \left( \sum_{m', Q_{j,m} \subset Q_{J,M}} |\lambda^{j,M}_{m,n}(\chi_Q)|^p \right)^{1/p} \sim 2^{\frac{n}{p}(1+\varrho)+j(s-\frac{1}{p})} |\lambda^{j,M}_{m,n}(\chi_I)|. \quad (5.60)\]
If $-n < q \leq -1$, then $J = 0$ is the largest term in (5.60). This reduces (3.10) with $f = \chi_Q$ to the one-dimensional case of (3.9) with $f = \chi_I$. In other words,

$$\|\chi_Q \mid \Lambda^\theta B^\theta_{p,q}(\mathbb{R}^n)\| \sim \|\chi_I \mid B^\theta_{p,q}(\mathbb{R})\|, \quad -n < q \leq -1.$$  \hspace{1cm} (5.61)

Then (5.51) with $A = B$ and (5.52) with $|\varrho| > 1$ follow from Proposition 5.22. If $-1 < q < 0$, then one has by (5.60) and (3.10) that

$$\|\chi_Q \mid \Lambda^\theta B^\theta_{p,q}(\mathbb{R}^n)\| \sim \|\chi_I \mid \Lambda^\theta B^\theta_{p,q}(\mathbb{R})\|, \quad -1 < q < 0.$$  \hspace{1cm} (5.62)

This reduces (5.51) and (5.52) to Step 1.

**Step 3.** The embedding (2.56) and Step 2 prove (5.51) and (5.52) with $A = F$ for all the relevant spaces $\Lambda^\theta F^\theta_{p,q}(\mathbb{R}^n)$ with the exception of $|\varrho| \geq 1$, $s = \frac{1}{p}$ and $q = \infty$. In this case, one relies on the wavelet expansion (3.7) with (3.8) for $\Lambda^\theta F^1_{p,\infty}(\mathbb{R}^n)$ with a related counterpart of the right-hand side of (3.10) as described in [82, Proposition 1.16, pp. 11–12] based on a reference to [80, Theorem 3.26, p. 64]. Using again (5.59), one obtains

$$\|\chi_Q \mid \Lambda^\theta F^1_{p,\infty}(\mathbb{R}^n)\| \sim \|\chi_I \mid F^1_{p,\infty}(\mathbb{R})\|, \quad 0 < p < \infty, \quad -n < q \leq -1,$$  \hspace{1cm} (5.63)

as the counterpart of (5.61). The one-dimensional case of (5.50) shows that $\chi_Q$ does not belong to $\Lambda^\theta F^1_{p,\infty}(\mathbb{R}^n)$.

**Step 4.** The proof of (5.53) and (5.54) follows the same scheme as in Step 2. One relies now on (3.11) and (5.60). Instead of (5.61) and (5.62), one has now

$$\|\chi_Q \mid \Lambda^\theta B^\theta_{p,q}(\mathbb{R}^n)\| \sim \|\chi_I \mid B^\theta_{p,q}(\mathbb{R})\|, \quad -n < q \leq -1$$  \hspace{1cm} (5.64)

and

$$\|\chi_Q \mid \Lambda^\theta B^\theta_{p,q}(\mathbb{R}^n)\| \sim \|\chi_I \mid \Lambda^\theta B^\theta_{p,q}(\mathbb{R})\|, \quad -1 < q < 0.$$  \hspace{1cm} (5.65)

Then (5.53) and (5.54) follow from (5.49) and Step 1. Again (5.55) with (5.56) is a consequence of (5.51) and (5.52) (with $A = F$) based on the coincidence (2.45).

**Remark 5.25.** Proposition 5.22 shows that the above theorem is a typical example of the Slope-1-Rule (see Slope Rules 2.17(i)). Note that the latter part of (5.52) means, in particular, that for $|\varrho| < 1$ no further assumption appears in the limiting case, also for $A = B$, in contrast to the situation for $|\varrho| \geq 1$, when $q = \infty$ is needed in the limiting case for $A = B$.

**Remark 5.26.** The above arguments can also be applied to the spaces in (2.44) of the 0-clan in Definition 2.15(iii) with the outcome that

$$\chi_Q \in \Lambda^\theta B^\theta_{p,q}(\mathbb{R}^n) \quad \text{with} \quad 0 < p, q < \infty \quad \text{if and only if} \quad s \leq 0.$$  \hspace{1cm} (5.66)

This follows from the above Step 1 now using (5.48) and (5.60) with $\varrho = 0$, based on Proposition 3.1 extended to $\varrho = 0$ according to a related comment in Remark 3.2. If we compare (5.49) with the assertion that

$$\chi_Q \in B^0_{\infty,q}(\mathbb{R}^n) \quad \text{if and only if} \quad q = \infty,$$  \hspace{1cm} (5.67)

it again illuminates the first strict embedding in (2.38).

The assertions (5.51) and (5.52) with $A = B$ and also (5.66) go back to [97, Theorem 2.1] reformulated according to (2.30) and (2.31). The second main topic of this paper addresses the problem in which spaces $\chi_Q$ generates a linear and bounded functional. This means

$$\|\langle f, \chi_Q \rangle \| \leq c\|f \mid A(\mathbb{R}^n)\|, \quad f \in A(\mathbb{R}^n),$$  \hspace{1cm} (5.68)

where $A(\mathbb{R}^n)$ is a space covered by Definition 2.15. We deal first with the one-dimensional case.
Proposition 5.27. Let $\Lambda^eA^s_{p,q}(\mathbb{R})$ and $\Lambda^eA^s_{p,q}((0,1))$ be the spaces according to (2.46) and (2.47) of the $\rho$-clan $\varrho A^s_{p,q}(\mathbb{R})$ in Definition 2.15(ii) with $n = 1$ and $-1 < \rho < 0$. Then the characteristic function $\chi_I$ of the unit interval $I = (0,1)$ generates a linear and bounded functional in $\Lambda^eA^s_{p,q}(\mathbb{R})$ if and only if $s > \frac{1}{p} - 1$, (5.69)

in

$$
\Lambda^eB^s_{p,q}(\mathbb{R}) \quad \text{if and only if} \quad \begin{cases} 
\text{either} & s > \frac{1}{p} - 1, \quad 0 < q \leq \infty, \\
\text{or} & s = \frac{1}{p} - 1, \quad 0 < q \leq 1,
\end{cases}
$$

and in

$$
\Lambda^eF^s_{p,q}(\mathbb{R}) \quad \text{if and only if} \quad s > \frac{1}{p} - 1.
$$

Proof. Recall that $\chi'_I = c(\delta - \delta_1)$ and $c \neq 0$, where $\delta_1$ is the shifted $\delta$-distribution with the off-point 1. Let $\varphi$ be a smooth function on $\mathbb{R}$ with compact support near the origin 0. Then

$$(\chi_I, \varphi') = -(\chi'_I, \varphi) = -c \varphi(0) = -c(\delta, \varphi).$$

This can be extended by completion to arbitrary continuous functions with compact support near the origin. Let $I_2 = (-2, 2)$. By Theorem 3.10, one has

$$
\|f| \Lambda^eA^{s+1}_{p,q}(\mathbb{R})\| \sim \|f| \Lambda^eA^s_{p,q}(\mathbb{R})\| + \|f'| \Lambda^eA^s_{p,q}(\mathbb{R})\|
$$

$$
\sim \|f'| \Lambda^eA^{s+1}_{p,q}(\mathbb{R})\|, \quad f \in \Lambda^eA^{s+1}_{p,q}(\mathbb{R}), \quad \text{supp } f \subset I_2,
$$

where the second equivalence can be obtained in the usual way based on the compact embedding of $\Lambda^eA^{s+1}_{p,q}(I_2)$ into $\Lambda^eA^s_{p,q}(I_2)$ being a special case of Theorem 6.5 below. Similarly,

$$
\|f| \Lambda^eB^{s+1}_{p,q}(\mathbb{R})\| \sim \|f'| \Lambda^eB^s_{p,q}(\mathbb{R})\|, \quad f \in \Lambda^eB^{s+1}_{p,q}(\mathbb{R}), \quad \text{supp } f \subset I_2.
$$

Let $f \in \Lambda^eA^{s+1}_{p,q}(\mathbb{R})$ and $\text{supp } f \subset I_2$ with $s + 1 > \frac{1}{p}$. Then it follows from (5.72), (5.73) and Theorem 5.3 that

$$
\|f| C(\mathbb{R})\| \leq c\|f| \Lambda^eA^{s+1}_{p,q}(\mathbb{R})\| \leq c\|f'| \Lambda^eA^{s+1}_{p,q}(\mathbb{R})\| \leq c\|f'| \Lambda^eA^s_{p,q}(\mathbb{R})\|.
$$

(5.75)

If $g$ is a smooth function with $\text{supp } g \subset I_2$, then one has $g = f'$ in $I_2$ for a suitable smooth function with, say, $\text{supp } f \subset [-4, 4]$. Hence, one can replace $f'$ in (5.75) by $g$. This can be extended by a Fatou argument to all $g \in \Lambda^eA^s_{p,q}(\mathbb{R})$ with $\text{supp } g \subset I_2$. Then it follows that $\chi_I$ generates a linear and bounded functional in $\Lambda^eA^s_{p,q}(\mathbb{R})$ with $s > \frac{1}{p} - 1$, similarly for the spaces in (5.70) based on Theorem 5.3 again. Conversely, if $\chi_I$ generates a linear and bounded functional in some space $\Lambda^eA^s_{p,q}(\mathbb{R})$, then it follows from (5.72) that

$$
|f(0)| \leq c|(\chi_I, f')| \leq c\|f| \Lambda^eA^s_{p,q}(\mathbb{R})\| \leq c\|f| \Lambda^eA^{s+1}_{p,q}(\mathbb{R})\|
$$

(5.76)

for compactly supported smooth functions. This requires

$$
\Lambda^eA^{s+1}_{p,q}(\mathbb{R}) \hookrightarrow C(\mathbb{R}),
$$

similarly for $\Lambda^eB^{s+1}_{p,q}(\mathbb{R})$. Then the only-if parts in (5.69) and (5.70) follow from Theorem 5.3 again. The assertion (5.71) is covered by (5.69) with the coincidence (2.45).}

**Remark 5.28.** The counterpart of the above proposition for the classical spaces in (2.42) with $n = 1$ can be obtained if one uses Proposition 5.1 with $n = 1$ instead of Theorem 5.3 with $n = 1$. We do not formulate the more or less known outcome. But it is quite clear that the above proposition is an example of the Slope-1-Rule again (see Slope Rules 2.17(ii)).
It is not clear how the above arguments can be extended from one dimension to higher dimensions. But the question in which spaces $A(\mathbb{R}^n)$ the characteristic function $\chi_Q$ of the unit cube $Q = (0, 1)^n$ generates a linear and bounded functional according to (5.68) has been treated in detail in [97] with the following remarkable outcome. Recall the notation (1.8).

**Theorem 5.29.** Let $n \in \mathbb{N}$. Let $\Lambda^\epsilon A^s_{p,q}(\mathbb{R}^n)$ and $\Lambda_\epsilon A^s_{p,q}(\mathbb{R}^n)$ with $A \in \{B, F\}$ be the spaces according to (2.46) and (2.47) of the $\epsilon$-clan $\Lambda^\epsilon A^s_{p,q}(\mathbb{R}^n)$ in Definition 2.15(ii) with $-n < \epsilon < 0$. Then $\chi_Q$ generates a linear and bounded functional in these spaces if, in addition,

$$s > \frac{1}{p} \min(|\epsilon|, 1) - 1 + \sigma_p^{\max(|\epsilon|, 1)} - 1.$$  \hspace{1cm} (5.78)

It does not generate a linear and bounded functional in these spaces if, in addition,

$$s < \frac{1}{p} \min(|\epsilon|, 1) - 1 + \sigma_p^{\max(|\epsilon|, 1)} - 1.$$  \hspace{1cm} (5.79)

**Proof.** Both assertions are covered by [97, Theorems 2.3 and 2.5] for the spaces $B^s_\tau(\mathbb{R}^n) = \Lambda^\epsilon B^s_{p,q}(\mathbb{R}^n)$ according to (2.30) and (2.31). The remaining spaces (2.46) and (2.47) can be incorporated by an elementary embedding based on the monotonicity of these spaces with respect to $s$, Theorem 2.21 and the equivalence (2.45).

We indicate the parameter area given by (5.78) according to the different cases of $\epsilon$ in Figure 4 below.

**Remark 5.30.** In [97], one also finds a detailed and (almost) final discussion of what happens on the breaking line

$$s = \frac{1}{p} \min(|\epsilon|, 1) - 1 + \sigma_p^{\max(|\epsilon|, 1)} - 1$$ \hspace{1cm} (5.80)

for the spaces $\Lambda^\epsilon B^s_{p,q}(\mathbb{R}^n)$. The arguments can also be applied to the spaces in (2.42) of the $n$-clan in Definition 2.15. Replacing $|\epsilon|$ in (5.80) by $n \in \mathbb{N}$, one obtains the well-known breaking line

$$s = \frac{1}{p} - 1 + \sigma_p^{n-1} = \max \left( \frac{1}{p} - 1, n \left( \frac{1}{p} - 1 \right) \right)$$ \hspace{1cm} (5.81)

for the classical spaces $A^s_{p,q}(\mathbb{R}^n)$. For $-1 \leq \epsilon < 0$ (as it is always in one dimension), one has the breaking line $s = \frac{|\epsilon|}{p} - 1$ for these low-slope spaces, independently of $n \in \mathbb{N}$. In any case, (5.80) is an example of the Slope-1-Rule (see Slope Rules 2.17(i)), at least on the understanding that $|\epsilon| = 1$ is a breaking point.

![Figure 4](Color online) $\chi_Q$ generates a bounded linear functional
5.6 Truncation

A (complex-valued) quasi-Banach space $A(\mathbb{R}^n)$ on $\mathbb{R}^n$ with

$$S(\mathbb{R}^n) \hookrightarrow A(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n), \quad A(\mathbb{R}^n) \subset L_{t}^{\text{loc}}(\mathbb{R}^n)$$

is said to have the truncation property if $|f| \in A(\mathbb{R}^n)$ for any real $f \in A(\mathbb{R}^n)$ and if there is a constant $c > 0$ such that

$$\|\|f\| | A(\mathbb{R}^n) \| \leq c\|f | A(\mathbb{R}^n) \|, \quad f \in A(\mathbb{R}^n) \text{ real.}$$

This innocent-looking but nevertheless rather tricky problem attracted a lot of attention for the classical spaces in (2.42), the $n$-clan according to Definition 2.15. At least for the spaces $B^s_{p,q}(\mathbb{R}^n)$, one has the following satisfactory assertion. Let again $\sigma^*_p$ be as in (1.8).

**Proposition 5.31.** The spaces

$$B^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad 0 < p, q \leq \infty \quad \text{and} \quad s \in \left( \sigma_p^*, 1 + \frac{1}{p} \right)$$

have the truncation property and the spaces

$$B^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad 0 < p, q \leq \infty \quad \text{and} \quad s \notin \left[ \sigma_p^*, 1 + \frac{1}{p} \right]$$

do not have the truncation property.

**Remark 5.32.** If $\sigma_p^* \geq 1 + \frac{1}{p}$, then (5.84) is empty and does not apply to any $s$. If $\sigma_p^* > 1 + \frac{1}{p}$, then the condition in (5.85) means $s \in \mathbb{R}$. The above result is covered by [74, Theorem 25.8, p.364] where one finds a similar assertion for the spaces $F^p_{q,s}(\mathbb{R}^n)$, being less final if $q < 1$.

One may ask for counterparts for the other spaces covered by Definition 2.15. According to Theorem 5.7, the natural restriction $s > \sigma^*_p|\cdot|_p$ ensures (5.82). Furthermore, $1 + \frac{1}{p}$ in (5.84) and (5.85) is a typical candidate for the Slope-1-Rule in Slope Rules 2.17(i). This suggests asking for the following counterpart of the above proposition for the $B$-spaces in (2.46). Let $\sigma^*_p$ be as in (1.8).

**Conjecture 5.33.** The spaces

$$\Lambda^s_{p}B^s_{p,q}(\mathbb{R}^n) \quad \text{and} \quad \Lambda^eB^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad 0 < p < \infty, \quad 0 < q \leq \infty \quad \text{and} \quad -n < p < 0$$

have the truncation property if

$$s \in \left( \sigma^*_p|\cdot|_p, 1 + \frac{1}{p} \min(|\cdot|, 1) \right)$$

and they do not have the truncation property if

$$s \notin \left[ \sigma^*_p|\cdot|_p, 1 + \frac{1}{p} \min(|\cdot|, 1) \right].$$

**Remark 5.34.** As above (5.87) is empty if $\sigma^*_p|\cdot|_p = a \geq b = 1 + \frac{1}{p} \min(|\cdot|, 1)$ and (5.88) means $s \in \mathbb{R}$ if $a > b$. If one replaces $|\cdot|_p$ by $n$, then the above conditions coincide with the related ones in Proposition 5.31. If $0 < |\cdot| \leq 1$, then (5.87) and (5.88) are a strip in the $(\frac{1}{p}, s)$-diagram similarly as in Proposition 5.31 with $n = 1$. The first results about these problems for the spaces (2.24) with $1 \leq p < \infty$ and $1 \leq q \leq \infty$ have been obtained recently in [33, Theorem 2]. There is a technical assumption for the spaces with $s \geq 1$ caused by the method but surely not by the topic. Neglecting this additional condition, one obtains by reformulation according to (2.24) that the spaces $\Lambda^s_{p}B^s_{p,q}(\mathbb{R}^n)$ with $1 \leq p < \infty, \quad 1 \leq q \leq \infty$ and

$$-n < p < 0$$

have the truncation property if and only if

$$0 < s < 1 + \frac{1}{p} \min(|\cdot|, 1).$$

This fits in with the scheme of Conjecture 5.33 (under the indicated additional technical assumption) (see Figure 5 below).
Remark 5.35. As mentioned above there are less final counterparts of (5.84) and (5.85) for the spaces $F_{p,q}^s(\mathbb{R}^n)$, especially if $p < 1$ or $q < 1$. But at least under the restriction $p \geq 1$, $q \geq 1$ one would expect an $F$-version of the above Conjecture 5.33. This has been confirmed in [33, Theorem 3] (again under some additional restrictions) for the spaces on the right-hand side of (2.25) and their reformulations in terms of $\Lambda_\varphi B_{p,q}^s(\mathbb{R}^n)$.

5.7 Haar wavelets

We recalled in Proposition 3.1 the characterisation of the spaces $\Lambda_\varphi B_{p,q}^s(\mathbb{R}^n)$ and $\Lambda_\varphi B_{p,q}^s(\mathbb{R}^n)$ with $n \in \mathbb{N}$, $-n < \varphi < 0$, $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$ in terms of compactly supported Daubechies wavelets. We used these assertions as tools. But they are also of self-contained interest, including related $F$-spaces and other types of wavelets, above all the Haar wavelets, which attracted a lot of attention.

In [82, Subsection 3.5, pp.98–103], one finds what is known nowadays about Haar expansions for the classical spaces $A_{p,q}^s(\mathbb{R}^n)$, including detailed related references. One has now natural restrictions for the parameters $s, p, q$ under which Haar expansions in $A_{p,q}^s(\mathbb{R}^n)$ can be expected. This had been used in [80, Theorem 3.41, p.74] to prove Haar expansions for the spaces $\Lambda_\varphi A_{p,q}^s(\mathbb{R}^n)$ with the same restrictions for $s, p, q$ as for the spaces $A_{p,q}^s(\mathbb{R}^n)$ complemented by $s < |\varphi|/p$. But for fixed $\varphi$ with $-n < \varphi < 0$, one cannot expect that the same conditions for $s, p, q$ as for the spaces $A_{p,q}^s(\mathbb{R}^n)$ are still natural for $\Lambda_\varphi A_{p,q}^s(\mathbb{R}^n)$. It is just the main aim of [97], underlying the above Theorems 5.24 and 5.29, to collect basic ingredients indicating how suitable natural substitutes for the spaces $\Lambda_\varphi B_{p,q}^s(\mathbb{R}^n)$ may look. But we are not aware of resulting satisfactory Haar expansions for $\Lambda_\varphi B_{p,q}^s(\mathbb{R}^n)$ in terms of natural restrictions for $s, p, q$. Clipping together the above ingredients, we formulate the expected outcome as a conjecture. But first we collect some basic notation and fix what is known for the spaces $B_{p,q}^s(\mathbb{R}^n)$. We follow [82, Subsection 3.5, pp.98–103] which in turn is based on the related references mentioned there.

Let for $y \in \mathbb{R}$,

$$h_M(y) = \begin{cases} 1, & \text{if } 0 < y < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \leq y < 1, \\ 0, & \text{if } y \notin (0,1). \end{cases}$$

(5.90)

Let $h_F(y) = |h_M(y)| = \chi_I(y)$ be the characteristic function of the unit interval $I = (0,1)$. This is the counterpart of (3.1) and (3.2). We now use the same construction as in (3.3)–(3.8). Let again $n \in \mathbb{N}$, and let

$$G = (G_1, \ldots, G_n) \in G^0 = \{F, M\}^n,$$

(5.91)

which means that $G_r$ is either $F$ or $M$. Let

$$G = (G_1, \ldots, G_n) \in G^* = G^j \in \{F, M\}^n, \quad j \in \mathbb{N},$$

(5.92)
which means that $G_r$ is either $F$ or $M$, where $*$ indicates that at least one of the components of $G$ must be an $M$. Let
\[
h^j_{G,m}(x) = \prod_{i=1}^{n} h_{G_i}(2^j x_i - m_i), \quad G \in G^j, \quad m \in \mathbb{Z}^n, \quad x \in \mathbb{R}^n,
\]
where (now) $j \in \mathbb{N}_0$. It is well known that
\[
\{2^{jn/2}h^j_{G,m} : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n\}
\]
is an orthonormal basis in $L_2(\mathbb{R}^n)$ and
\[
f = \sum_{j \in \mathbb{N}_0} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^j_m h^j_{G,m}
\]
with
\[
\lambda^j_m = \lambda^j_m(f) = 2^{jn} \int_{\mathbb{R}^n} f(x)h^j_{G,m}(x)dx = 2^{jn}(f, h^j_{G,m})
\]
is the corresponding expansion. In contrast to (3.7)–(3.9) and Proposition 3.1, we are interested now in expansions by Haar wavelets not as a tool but as a self-contained topic. This suggests offering a more careful formulation as already indicated in Remark 3.2. In particular, the quasi-Banach space $b^s_{p,q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ collects all the sequences
\[
\lambda = \{\lambda^j_m \in \mathbb{C} : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n\}
\]
such that
\[
\|\lambda \|_{b^s_{p,q}(\mathbb{R}^n)} = \left( \sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p})q} \left( \sum_{m \in \mathbb{Z}^n, G \in G^j} |\lambda^j_m|^p \right)^{q/p} \right)^{1/q}
\]
is finite (with the usual modification if $\max(p, q) = \infty$).

**Proposition 5.36.** Let $n \in \mathbb{N}$,
\[
0 < p, q \leq \infty \quad \text{and} \quad \max\left( n\left(\frac{1}{p} - 1\right), \frac{1}{p} - 1 \right) < s \leq \min\left( \frac{1}{p}, 1 \right)
\]
as indicated in Figure 6 on page 1342 below. Let $f \in S'(\mathbb{R}^n)$. Then $f \in B^s_{p,q}(\mathbb{R}^n)$ if and only if it can be represented by
\[
f = \sum_{j \in \mathbb{N}_0, G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^j_m h^j_{G,m}, \quad \lambda \in b^s_{p,q}(\mathbb{R}^n),
\]
the unconditional convergence being in $S'(\mathbb{R}^n)$. The representation (5.100) is unique,
\[
\lambda^j_m = \lambda^j_m(f) = 2^{jn}(f, h^j_{G,m})
\]
and
\[
I : f \mapsto \{\lambda^j_m(f)\}
\]
is an isomorphic map of $B^s_{p,q}(\mathbb{R}^n)$ onto $b^s_{p,q}(\mathbb{R}^n)$,
\[
\|f \|_{B^s_{p,q}(\mathbb{R}^n)} \sim \|\lambda(f)\|_{b^s_{p,q}(\mathbb{R}^n)}.
\]

**Remark 5.37.** This is the $B$-part of [82, Theorem 3.18, p.99]. There one also finds explanations and detailed references. The restriction (5.99) is natural: if $(\frac{1}{p}, s)$ does not belong to the set
\[
\left\{(\frac{1}{p}, s) : 0 < p \leq \infty \quad \text{and} \quad \max\left( n\left(\frac{1}{p} - 1\right), \frac{1}{p} - 1 \right) \leq s \leq \min\left( \frac{1}{p}, 1 \right) \right\}
\]
in the $\{\frac{1}{p}, s) : 0 \leq \frac{1}{p} < \infty, s \in \mathbb{R}\}$ half-plane in Figure 6, then the above proposition is no longer valid. In recent times, it has been studied in detail what can be said about Haar expansions in $B^s_{p,q}(\mathbb{R}^n)$ (and also in $F^s_{p,q}(\mathbb{R}^n)$) if $(\frac{1}{p}, s)$ belongs to the difference of the two sets in (5.104) and (5.99) (the related boundary of the set (5.104)).
One may ask for a counterpart of the above proposition for the $B$-spaces $\mathcal{B}_{p,q}^s(\mathbb{R}^n)$ in (2.46) of the $\mathcal{g}$-clan in $\mathbb{R}^n$ according to Definition 2.15. Let $n \in \mathbb{N}$, $-n < \epsilon < 0$, $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Let $\lambda$ be the same sequence as in (5.97). Then the quasi-Banach space $\Lambda^s_{p,q}(\mathbb{R}^n)$ collects all the sequences $\lambda$ such that

$$\|\lambda \cap \Lambda^s_{p,q}(\mathbb{R}^n))\| = \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{\frac{1}{p}(n+\epsilon)} \left( \sum_{j \geq J^*} 2^{j(n+\epsilon)} \left( \sum_{m:Q_m \subset Q_{j,M}} |\lambda^{j,G}_m|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{q}}$$

is finite and the quasi-Banach space $\Lambda_{p,q}^s(\mathbb{R}^n)$ collects all the sequences $\lambda$ such that

$$\|f \cap \Lambda_{p,q}^s(\mathbb{R}^n))\| = \left( \sum_{j=0}^{\infty} 2^{j(n+\epsilon)} \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{\frac{1}{p}(n+\epsilon)} \left( \sum_{m:Q_m \subset Q_{j,M}} |\lambda^{j,G}_m|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{q}}$$

is finite. Here, all the notations have the same meaning as in Proposition 3.1 and as already indicated in Remark 3.2. According to [80, Theorem 3.41, p. 74], one has the following counterpart of Proposition 5.36 for the spaces $\Lambda^s_{p,q}(\mathbb{R}^n)$. Let $n \in \mathbb{N}$, $-n < \epsilon < 0$,

$$0 < p < \infty, \quad 0 < q \leq \infty \quad \text{and} \quad \max \left( n \left( \frac{1}{p} - 1 \right), \frac{1}{p} - 1 \right) < s < \min \left( \frac{1}{p}, \frac{|\epsilon|}{p}, 1 \right).$$

(5.107)

Let $f \in S'(\mathbb{R}^n)$. Then $f \in \Lambda^s_{p,q}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{j \in \mathbb{N}, G \in G^s} \lambda^{j,G}_{m} h^j_{G,m}, \quad \lambda \in \Lambda^s_{p,q}(\mathbb{R}^n),$$

(5.108)

the unconditional convergence being in $S'(\mathbb{R}^n)$. The representation (5.108) is unique with $\lambda^{j,G}_{m}$ as in (5.101), and $I$ in (5.102) is an isomorphic map of $\Lambda^s_{p,q}(\mathbb{R}^n)$ onto $\Lambda^s_{p,q}(\mathbb{R}^n)$. Theorem 5.24 shows that the restriction $s < \frac{1}{p} \min(|\epsilon|, 1)$ in (5.107), compared with $s < \frac{1}{p}$ in (5.99), is natural. The restriction $s < 1$ in (5.99) comes from the use of local means with the Haar functions $h^j_{G,m}$ as kernels and their restricted cancellation properties. One may consult the proof of [77, Proposition 2.8, p. 79] based on [77, Theorem 1.15, p. 7]. Although we have no reference, one can expect that this argument also applies to the spaces $\Lambda^s_{p,q}(\mathbb{R}^n)$. In other words, the right-hand sides of (5.99) and (5.107) are natural. If one asks for Haar expansions in $\mathcal{B}_{p,q}^s(\mathbb{R}^n)$, then the related conditions for $\chi_Q$ as described in Proposition 5.22 and Remark 5.30, especially (5.81), are necessary. According to Proposition 5.36, they are also sufficient for the spaces $\mathcal{B}_{p,q}^s(\mathbb{R}^n)$ at least as far as the breaking lines are concerned, in sharp contrast to Haar expansions for $F_{p,q}^s(\mathbb{R}^n)$ where $q$ comes from [82, Theorem 3.18, p. 99]. One can take this observation as a guide for
B-spaces. Then it is clear that the left-hand side of (5.107) as a natural restriction for Haar expansions in $\Lambda^\varphi B^s_{p,q}(\mathbb{R}^n)$ ($-n < \varphi < 0$), is questionable. Based on Theorems 5.24 and 5.29, the following expectation is at least reasonable. Let

$$R_\sigma = \left\{ \left( \frac{1}{p}, s \right) : 0 < p < \infty, \frac{\sigma}{p} - 1 < s < \min \left( 1, \frac{\sigma}{p} \right) \right\}, \quad 0 < \sigma \leq 1,$$

(5.109)

$$R_\sigma = \left\{ \left( \frac{1}{p}, s \right) : 0 < p < \infty, \max \left( \frac{1}{p} - 1, \sigma \left( \frac{1}{p} - 1 \right) \right) < s < \min \left( 1, \frac{1}{p} \right) \right\}, \quad \sigma > 1$$

(5.110)

and

$$\overline{R}_\sigma = \left\{ \left( \frac{1}{p}, s \right) : 0 < p < \infty, \frac{\sigma}{p} - 1 \leq s \leq \min \left( 1, \frac{\sigma}{p} \right) \right\}, \quad 0 < \sigma \leq 1,$$

(5.111)

$$\overline{R}_\sigma = \left\{ \left( \frac{1}{p}, s \right) : 0 < p < \infty, \max \left( \frac{1}{p} - 1, \sigma \left( \frac{1}{p} - 1 \right) \right) \leq s \leq \min \left( 1, \frac{1}{p} \right) \right\}, \quad \sigma > 1$$

(5.112)

in the $\{ \left( \frac{1}{p}, s \right) : 0 \leq \frac{1}{p} < \infty, s \in \mathbb{R} \}$ half-plane (see Figure 7 above).

**Conjecture 5.38.** Let $n \in \mathbb{N}$, $-n < \varphi < 0$, $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Let $g B^s_{p,q}(\mathbb{R}^n)$ be the $B$-spaces $\Lambda^\varphi B^s_{p,q}(\mathbb{R}^n)$ or $\Lambda^\varphi B^s_{p,q}(\mathbb{R}^n)$ of the $\varphi$-clan according to Definition 2.15(ii) and $g B^s_{p,q}(\mathbb{R}^n)$ be the respective sequence spaces $\Lambda^\varphi b^s_{p,q}(\mathbb{R}^n)$ or $\Lambda^\varphi b^s_{p,q}(\mathbb{R}^n)$ as introduced above. Let $f \in S'(\mathbb{R}^n)$ and $(\frac{1}{p}, s) \in R_{|\varphi|}$. Then $f \in g B^s_{p,q}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{j \in \mathbb{N}_0, G \subseteq G'} \lambda_{m}^{i,G} h_{G,m}^i, \quad \lambda \in g b^s_{p,q}(\mathbb{R}^n),$$

(5.113)

the unconditional convergence being in $S'(\mathbb{R}^n)$. The representation in (5.113) is unique with $\lambda_{m}^{i,G}$ as in (5.101), and $I$ in (5.102) is an isomorphic map of $g B^s_{p,q}(\mathbb{R}^n)$ onto $g b^s_{p,q}(\mathbb{R}^n)$. The spaces $g B^s_{p,q}(\mathbb{R}^n)$ with $(\frac{1}{p}, s) \not\in R_{|\varphi|}$ do not admit characterisations in terms of Haar wavelets.

**Remark 5.39.** This is the $\varphi$-counterpart of the corresponding assertion for the spaces $B^s_{p,q}(\mathbb{R}^n)$ as described in Proposition 5.36 and Remark 5.37.

Take it for granted that $s \leq 1$ is always necessary for Haar expansions, and then the above-quoted assertion, based on (5.107), and the preceding considerations confirm Conjecture 5.38 for the spaces

$$\Lambda^\varphi B^s_{p,q}(\mathbb{R}^n), \quad n \geq 2, \quad -n < \varphi \leq -1, \quad 0 < q \leq \infty, \quad 1 \leq p < \infty,$$

(5.114)

whereas $R_n$ and $R_{|\varphi|}$ differ if $p < 1$ (see Figure 8 below).

![Figure 7](image-url)  
*Figure 7* (Color online) Parameter area for Haar expansions
The situation for the low-slope spaces $\Lambda^\varepsilon B_{p,q}^s(\mathbb{R}^n)$, $n \in \mathbb{N}$, $-1 < q < 0$ is different and related Haar expansions are ensured so far only in the region

$$0 < p < \infty, \quad 0 < q \leq \infty \quad \text{and} \quad \max \left(n \left(\frac{1}{p} - 1\right), \frac{1}{p} - 1\right) < s < \min \left(\frac{|\varrho|}{p}, 1\right) \quad (5.15)$$

instead of $R_{|\varrho|}$ as conjectured (depending on $n$ and $|\varrho|$ and not only on $|\varrho|$ exclusively).

**Remark 5.40.** It is sufficient to confirm the above conjecture for the spaces $\Lambda^\varepsilon B_{p,q}^s(\mathbb{R}^n)$. Using (2.57) and (3.14), we have from $\Lambda^\varepsilon B_{p,q}^s(\mathbb{R}^n) = (\Lambda^\varepsilon B_{p,p}^{s_1}(\mathbb{R}^n), \Lambda^\varepsilon B_{p,p}^{s_2}(\mathbb{R}^n))_{q,q}$

and its sequence counterpart based on the indicated wavelet isomorphisms, that any affirmative assertion about Haar expansions for the spaces $\Lambda^\varepsilon B_{p,q}^s(\mathbb{R}^n)$ can be transferred to the spaces $\Lambda^\varepsilon B_{p,q}^s(\mathbb{R}^n)$. The sharpness of the breaking lines in the above conjecture for the spaces $\Lambda^\varepsilon B_{p,q}^s(\mathbb{R}^n)$ follows afterwards from (3.13) with $q = \infty$ and (2.59).

### 5.8 A proposal: Faber expansions

Expansions by Haar systems is a central topic in the theory of function spaces. It is of interest for its own sake. But they may also serve as a starting point for numerous other observations. In one dimension one can step from Haar systems to related Faber systems on the unit interval $I = (0, 1)$ or on $\mathbb{R}$, roughly speaking, by integration. This includes Faber expansions in suitable function spaces on $\mathbb{R}$, sampling and so-called numerical integration. We refer to [77] and, in particular, to [78, Subsection 3.2]. One may ask what happens if one replaces $B_{p,q}^s(\mathbb{R})$ with

$$0 < p, q \leq \infty, \quad \frac{1}{p} < s < 1 + \min \left(\frac{1}{p}, 1\right) \quad (5.16)$$

by, say, $\Lambda^\varepsilon B_{p,q}^s(\mathbb{R})$ ($-1 < q < 0$). Take it for granted that one has related affirmative answers for the above Conjecture 5.38, and then

$$0 < p < \infty, \quad 0 < q \leq \infty, \quad \frac{|\varrho|}{p} < s < 1 + \min \left(1, \frac{|\varrho|}{p}\right) \quad (5.17)$$

is the counterpart of (5.117), as indicated in Figure 9 below.

Faber systems can be extended from one dimension to higher dimensions by tensor products in the context of spaces with dominating mixed smoothness

$$S_{p,q}^r B(\mathbb{R}^n) \quad \text{with} \quad 0 < p, q \leq \infty \quad \text{and} \quad r \in \mathbb{R}. \quad (5.18)$$
This, in turn, can be used for numerical integration and discrepancy. As for basic definitions and related properties, one may consult \cite{65,77,78,81} and the references within. One may ask for Morrey versions $\mathcal{S}_{p,q}^{\varepsilon}(\mathbb{R}^n)$, $-1 < \varepsilon < 0$, $0 < p < \infty$, $0 < q \leq \infty$ and $r \in \mathbb{R}$ of these spaces and whether they are of some use in connection with sampling, discrepancy and numerical integration. We are not aware of any efforts in these directions so far.

6 Embeddings—revisited

In Subsections 5.1 and 5.2, we already studied embeddings of spaces $\mathcal{S}_{p,q}^{\varepsilon}(\mathbb{R}^n)$ into certain target spaces, in addition to the more basic embedding result which can be found in Section 2, in particular in Proposition 2.12 and Theorem 2.21. It is our intention now to return to the topic and modify it appropriately. At first we pose the question of continuous embeddings within the same $\varepsilon$-clan to spaces, defined on bounded domains. As is well known, then we may also ask for compact embeddings—unlike in the case of spaces on $\mathbb{R}^n$. This will be our first goal. Secondly we shed some light on the so-called growth envelope functions, some tool to ‘measure’ unboundedness of tempered distributions by very classical tools. It turns out again that Slope Rules 2.17 can be observed. Finally, in the last part, we leave the realm of embeddings within one and the same $\varepsilon$-clan and briefly discuss some phenomena when ‘crossing borders’ (between $\varepsilon$-clans).

6.1 Spaces on domains

Similarly as in Definition 4.7, one introduces spaces on domains $\Omega$ in $\mathbb{R}^n$ by restriction. As usual, $D'(\Omega)$ denotes the set of all the distributions on $\Omega$. Furthermore, $g \mid \Omega \in D'(\Omega)$ stands for the restriction of $g \in S'(\mathbb{R}^n)$ to $\Omega$.

**Definition 6.1.** Let $\Omega$ be an (arbitrary) domain (which is an open set) on $\mathbb{R}^n$ ($n \in \mathbb{N}$). Let $A(\mathbb{R}^n)$ be a space covered by Definition 2.15. Then

\[
A(\Omega) = \{f \in D'(\Omega) : f = g \mid \Omega \text{ for some } g \in A(\mathbb{R}^n)\},
\]

\[
\|f \mid A(\Omega)\| = \inf \|g \mid A(\mathbb{R}^n)\|,
\]

where the infimum is taken over all $g \in A(\mathbb{R}^n)$ with $g \mid \Omega = f$.

**Remark 6.2.** It follows from standard arguments that $A(\Omega)$ is a quasi-Banach space (and a Banach space if $p \geq 1$ and $q \geq 1$), continuously embedded in $D'(\Omega)$ and in the restriction of $S'(\mathbb{R}^n)$ to $\Omega$. Let

\[
g-\mathcal{S}_{p,q}^{\varepsilon}(\mathbb{R}^n), \quad -1 < \varepsilon < 0, \quad 0 < p < \infty, \quad 0 < q \leq \infty \quad \text{and} \quad r \in \mathbb{R}
\]
us mention, for completeness, that in the recent papers [14, 98, 99] also the question of bounded uniform extension operators has been studied for Lipschitz domains, based on the Rychkov approach [55].

The classical spaces \( A^s_{p,q}(\Omega) \) are the restrictions of the spaces in (2.42), the \( n \)-clan, now with respect to \( \Omega \). They have been studied in numerous papers and books, including [82, Subsection 2.6, pp. 62–74] where one finds some recent results (covering also \( F^s_{\infty,q}(\Omega) \)) and (historical) references. One may ask for counterparts of related assertions for the spaces

\[
A_s B^s_{p,q}(\Omega), \quad A^s B^s_{p,q}(\Omega) \quad \text{and} \quad A_s F^s_{p,q}(\Omega) = \Lambda^s F^s_{p,q}(\Omega)
\]  

(6.3)

with

\[
s \in \mathbb{R}, \quad 0 < p < \infty, \quad 0 < q \leq \infty \quad \text{and} \quad -n < p < 0,
\]

(6.4)

the \( \varrho \)-clan according to Definition 2.15(ii) now with respect to \( \Omega \). But this is not our topic. We are interested here exclusively in the description of compact embeddings between these spaces measured in terms of entropy numbers.

Let \( A \) and \( B \) be two quasi-Banach spaces and let \( U_A = \{ a \in A : \| a \| \leq 1 \} \) and \( U_B = \{ b \in B : \| b \| \leq 1 \} \) be the related unit balls. Let \( T : A \rightarrow B \) be a linear and compact map of \( A \) into \( B \). Then the \( k \)-th entropy number \( e_k(T) \) \((k \in \mathbb{N})\) is the infimum of all \( \varepsilon > 0 \) such that

\[
T(U_A) \subset \bigcup_{j=1}^{2^k-1} (b_j + \varepsilon U_B) \quad \text{for} \quad \{ b_j \}_{j=1}^{2^k-1} \subset B.
\]

(6.5)

Basic properties and references may be found in the monographs [6, 8, 36, 53] (restricted to the case of Banach spaces), and [9] for some extensions to quasi-Banach spaces (see also [75, Subsection 1.10, pp. 55–58]).

**Proposition 6.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) \((n \in \mathbb{N})\). Let \( 0 < p_1, p_2, q \leq \infty \) and \( s_1, s_2 \in \mathbb{R} \).

(i) Then there is a continuous embedding

\[
\text{id}_\Omega : B^s_{p_1,q}(\Omega) \hookrightarrow B^s_{p_2,q}(\Omega)
\]

(6.6)

if and only if

\[
s_1 - s_2 \geq n \cdot \max \left( \frac{1}{p_1} - \frac{1}{p_2}, 0 \right).
\]

(6.7)

(ii) The embedding \( \text{id}_\Omega \) is compact if and only if

\[
s_1 - s_2 \geq n \cdot \max \left( \frac{1}{p_1} - \frac{1}{p_2}, 0 \right).
\]

(6.8)

Furthermore,

\[
e_k(\text{id}_\Omega) \sim k^{\frac{-p_1 - 2}{p_2}}, \quad k \in \mathbb{N}.
\]

(6.9)

**Remark 6.4.** Detailed discussions and references about embeddings of the spaces \( A^s_{p,q}(\Omega) \) may be found in [82, Subsection 2.6.5, pp. 69–72]. The equivalence (6.9) for bounded \( C^\infty \) domains \( \Omega \) goes back to [9, Subsection 3.3, pp. 105–118] and the references given there. Its extension to arbitrary bounded domains \( \Omega \) in \( \mathbb{R}^n \) may be found in [75, Theorem 1.97 and Remark 1.98, p. 61] and related references. It is based on the observation that the quality of the bounded domain \( \Omega \) does not play any role. Everything can be reduced to the embedding

\[
\{ f \in B^s_{p_1,q}(\mathbb{R}^n) : \text{supp} f \subset [0,1]^n \} \hookrightarrow B^s_{p_2,q}(\mathbb{R}^n).
\]

(6.10)

This also applies to the counterparts of the above assertions for the spaces in (6.3) and (6.4) as discussed below. Nowadays everything can be reduced to wavelet representations for these spaces and mapping properties between related sequence spaces. In this way, one can also justify immediately the only-if parts of the above proposition. They are more or less obvious but not explicitly mentioned in the quoted literature.
The above proposition can be extended from $B^s_{p,q}(\Omega)$ to the spaces $A^s_{p,q}(\Omega)$ and $\Lambda^s_{p,q}(\Omega)$ with $-n < \varrho < 0$ in (6.3) and (6.4) as follows.

**Theorem 6.5.** Let $\Omega$ be a bounded domain (which is an open set) in $\mathbb{R}^n$ ($n \in \mathbb{N}$). Let $-n < \varrho < 0$, $0 < q \leq \infty$, $0 < p_1, p_2 < \infty$ and $s_1, s_2 \in \mathbb{R}$. Then there is a continuous embedding

$$\text{id}_\Omega : A^s_{p_1,q}(\Omega) \hookrightarrow A^s_{p_2,q}(\Omega)$$

if and only if

$$s_1 - s_2 > |\varrho| \cdot \max\left(\frac{1}{p_1} - \frac{1}{p_2}, 0\right).$$

The same is true for the embedding

$$\text{id}_\Omega : \Lambda^s_{p_1,q}(\Omega) \hookrightarrow \Lambda^s_{p_2,q}(\Omega).$$

(iii) The embedding

$$\text{id}_\Omega : \varrho A^s_{p_1,q}(\Omega) \hookrightarrow \varrho A^s_{p_2,q}(\Omega)$$

is compact if and only if

$$s_1 - s_2 > |\varrho| \cdot \max\left(\frac{1}{p_1} - \frac{1}{p_2}, 0\right).$$

Furthermore,

$$e_k(\text{id}_\Omega) \sim k^{-\frac{(\varrho - s_1 - s_2)}{n}}, \quad k \in \mathbb{N}.$$  

**Proof.** The assertions about the continuity and the compactness of the embedding $\text{id}_\Omega$ in (6.12) in the case of $\varrho A^s_{p_i,q}(\Omega) = A^s_{p_i,q}(\Omega)$ are covered by [28, Theorems 3.1 and 4.1, pp.126 and 135] (when $A = B$) and [29, Theorem 5.2] (when $A = F$), always reformulated according to (2.24) and (2.25). The asymptotic (6.17) for spaces $\varrho A^s_{p,q}(\Omega) = \varrho A^s_{p,q}(\Omega)$ can be found in [31, Case (a), Theorem 4.2, pp.3 and 19–20]. The compactness and entropy number result for $\varrho A^s_{p,q}(\Omega) = \Lambda^s_{p,q}(\Omega)$ can be found in [13, Theorems 3.2 and 4.1]. Finally, the characterisation for the continuity of $\text{id}_\Omega$ in the case of $\Lambda^s_{p,q}(\Omega)$ is a special case of [14, Corollary 4.14].

**Remark 6.6.** As far as the $B$-counterpart of (6.14) is concerned, i.e.,

$$\text{id}_\Omega : \Lambda^s_{p_i,q}(\Omega) \hookrightarrow \Lambda^s_{p_2,q}(\Omega),$$

we have no final answer yet. More precisely, there is a gap concerning the limiting situation of (6.13) when $p_1 < p_2$, i.e.,

$$s_1 - s_2 = |\varrho| \left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0.$$  

Then dealing with different parameters $q_1$ and $q_2$, we could prove in [14, Theorem 4.9] that

$$\text{id}_\Omega : \Lambda^s_{p_1,q_1}(\Omega) \hookrightarrow \Lambda^s_{p_2,q_2}(\Omega)$$

is continuous in that case, if $q_1 < \frac{p_1}{p_2} q_2$ which excludes $q_1 = q_2$. Conversely, the continuity of $\text{id}_\Omega$ given by (6.20) in the limiting situation of (6.19) implies $q_1 \leq q_2$, as desired. So apart from this small gap in this particular limiting case of (6.19) for spaces of type $\Lambda^s_{p,q}(\Omega)$, there is good reason to assume that in all the settings according to Definition 2.15,

$$\text{id}_\Omega : \varrho A^s_{p_1,q}(\Omega) \hookrightarrow \varrho A^s_{p_2,q}(\Omega)$$

is continuous if and only if (6.13) is satisfied (see Figure 10 below).
The above theorem (together with our preceding remarks) compared with Proposition 6.3 shows that continuity and compactness of $\text{id}_\Omega$ within a $\rho$-clan obey the Slope-$n$-Rule; recall Slope Rules 2.17(ii). But it is very remarkable that (6.17) = (6.9) is the same for all $\rho$-clans, $-n < \rho < 0$ and also the $n$-clan. The assertions in the papers [13,14,28,31] are proved in the context of the spaces on the right-hand side of (2.24), covering also continuous and compact embeddings between spaces belonging to different $\rho$-clans. There are also some estimates of the entropy numbers of the corresponding compact embeddings, as well as their approximation numbers (see [13,30]). They are in some sense remarkable, but less final and more technical than in the above cases within a fixed $\rho$-clan. Crossing the border between different clans is also in the theory of function spaces a brave undertaking, we return to this topic in Subsection 6.3 below.

6.2 Growth envelope functions

As mentioned in the beginning, we shall prove some specific unboundedness feature of function spaces of Morrey type. This will be done in the context of growth envelope functions. First we briefly recall this concept, before we come to our main results.

If $f$ is an extended complex-valued measurable function on $\mathbb{R}^n$ which is finite a.e., then the decreasing rearrangement of $f$ is the function defined on $(0, \infty)$ by

$$f^*(t) := \inf\{\sigma > 0 : \mu(f, \sigma) \leq t\}, \quad t \geq 0$$

(6.21)

with $\mu(f, \sigma)$ being the distribution function given by

$$\mu(f, \sigma) := \{|x \in \mathbb{R}^n : |f(x)| > \sigma|$$

$\sigma \geq 0$.

As usual, the convention $\inf \emptyset = \infty$ is assumed.

**Definition 6.7.** Let $X = X(\mathbb{R}^n)$ be some quasi-normed function space on $\mathbb{R}^n$. The growth envelope function $\mathcal{E}^X_G : (0, \infty) \to [0, \infty]$ of $X$ is defined by

$$\mathcal{E}^X_G(t) := \sup_{\|f\|_X \leq 1} f^*(t), \quad t \in (0, \infty).$$

This concept was first introduced and studied in [74, Chapter 2] and [20] (see also [21]). With a slight abuse of the notation, we shall not distinguish between representative and equivalence classes (of growth envelope functions) and stick to the notation introduced above. We refer to the above references for further details and only recall that $X(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$ holds if and only if $\mathcal{E}^X_G$ is bounded. This is the case where we are not further interested in. Let us also mention that in rearrangement-invariant
Banach function spaces $X(\mathbb{R}^n)$ there exists the notion of the fundamental function $\varphi_X$, i.e., for $t > 0$, and $A_t \subset \mathbb{R}^n$ with $|A_t| = t$, then $\varphi_X(t) = \|\chi_{A_t} | X\|$. In [21, Subsection 2.3], it is proved that in this case

$$E^X_G(t) \sim \frac{1}{\varphi_X(t)}, \quad t > 0.$$

**Example 6.8.** Basic examples for spaces $X$ are the well-known Lorentz spaces $L_{p,q}(\mathbb{R}^n)$; for definitions and further details, we refer to [4, Chapter 4, Definition 4.1], for example. This scale represents a natural refinement of the scale of Lebesgue spaces. The following result is proved in [21, Subsection 2.2]. Let $0 < p < \infty$ and $0 < q < \infty$. Then

$$E^L_{p,q}(\mathbb{R}^n)(t) \sim t^{\frac{1}{q}}, \quad t > 0.$$  
(6.22)

In general, if we consider spaces of distributions, the concept of growth envelopes makes sense only for spaces $X(\mathbb{R}^n) \subset L^{1,0}_{\text{loc}}(\mathbb{R}^n)$, i.e., when we deal with locally integrable functions. For such spaces we shall thus assume $X(\mathbb{R}^n) \subset L^{1,0}_{\text{loc}}(\mathbb{R}^n)$ and, moreover, $X(\mathbb{R}^n) \not\hookrightarrow L_{\infty}(\mathbb{R}^n)$, as we are interested in questions of unboundedness.

**Example 6.9.** Let $X = A^s_{p,q}(\mathbb{R}^n)$ and recall Propositions 5.1 and 5.5. Then the results for growth envelope functions in spaces $A^s_{p,q}(\mathbb{R}^n)$ for $0 < q \leq \infty$, $0 < p < \infty$ and $s > \sigma^p_n$ read as follows:

(i) Assume $s < \frac{n}{p}$. Then

$$E^G_{s,p,q}(\mathbb{R}^n)(t) \sim t^{-\frac{1}{q} + \frac{1}{p}}, \quad t \to 0.$$  
(6.23)

(ii) Let $1 < q \leq \infty$ and $\frac{1}{q} + \frac{1}{p'} = 1$, as usual. Then

$$E^G_{p,q}(\mathbb{R}^n)(t) \sim |\log t|^\frac{1}{q'}, \quad t \to 0.$$  
(6.24)

(iii) Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, as usual. Then

$$E^G_{p,q}(\mathbb{R}^n)(t) \sim |\log t|^\frac{1}{p'}, \quad t \to 0.$$  
(6.25)

For the proofs and further discussions, we refer to [21, Theorems 8.1 and 8.16] and [74, Subsections 13 and 15]. Results for the cases $s = \sigma^p_n$ can be found in [21, Propositions 8.12, 8.14 and 8.24] and [66, 84].

In contrast to Examples 6.8 and 6.9, it turns out that we have

$$E^G_{s}(\mathbb{R}^n)(t) = \infty, \quad t > 0, \quad -n < q < 0$$  
(6.26)

for the Morrey spaces $A^s(\mathbb{R}^n)$ according to Definition 2.3 with $0 < p < \infty$ and $-n < q < 0$ (recall the notation (2.20) and see [22, Theorem 3.7]) in sharp contrast to

$$E^G_{s}(\mathbb{R}^n)(t) \sim t^{-\frac{1}{q}}, \quad t > 0, \quad q = -n,$$

by using (2.18). Similarly, whenever $\varphi A^s_{p,q}(\mathbb{R}^n) \subset L^{1,0}_{\text{loc}}(\mathbb{R}^n)$ and $\varphi A^s_{p,q}(\mathbb{R}^n) \not\hookrightarrow L_{\infty}(\mathbb{R}^n)$, we proved in [24, Theorems 4.9 and 4.11 and Corollary 4.14] (with some forerunner in [22]) that

$$E^G_{s}(\mathbb{R}^n)(t) = \infty, \quad t > 0, \quad -n < q < 0.$$  
(6.27)

This is in a good agreement with the earlier result in [74, Subsection 13.7] that

$$E^\text{bmo}(\mathbb{R}^n)(t) = \infty, \quad t > 0;$$

recall the coincidence $\text{bmo}(\mathbb{R}^n) = A^0_{p,q}(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n)$ in (2.39) and Remark 2.11. However, for a bounded domain $\Omega$, one has

$$E^\text{bmo}(\Omega)(t) \sim |\log t|, \quad t \to 0$$

(see [74, Subsection 13.7]). So these assertions are connected with the underlying $\mathbb{R}^n$. 
In [23, 26], we studied the growth envelope function for spaces on bounded domains, e.g.,

\[ E_G^{\Lambda^p_{\rho}}(\Omega) \sim t^{-\frac{1}{p}} \quad t \to 0, \quad -n \leq \rho < 0 \]  

(6.28) (see [26]) (where some care is needed to define the spaces \( \Lambda^p_{\rho}(\Omega) \), also in dependence on the quality of \( \Omega \)).

Recall that we have parallel results to Theorem 5.3 and 5.7 now. So we are left to consider the strip

\[ \left\{ \left( \frac{1}{p}, s \right) : 0 < p < \infty, \sigma_p^{|\rho|} \leq s < \frac{|\rho|}{p} \right\} \]

as shown in Figure 11 below; recall Figure 1 on page 1328 and Figure 2 on page 1330.

For convenience, we restrict ourselves to the so-called sub-critical case of spaces \( \rho - A^s_{p,q}(\Omega) \) omitting the limiting situations of \( s = \sigma_p^{|\rho|} \) and \( s = \frac{|\rho|}{p} \) here. Then we could prove in [23] the following.

**Theorem 6.10.** Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^n \) \( (n \in \mathbb{N}) \). Let \( -n < \rho < 0, 0 < p < \infty, 0 < q \leq \infty \) and \( \sigma_p^{|\rho|} < s < \frac{|\rho|}{p} \). Then

\[ E_G^{\rho - A^s_{p,q}(\Omega)}(t) \sim t^{-\frac{1}{p} + \frac{s}{|\rho|}}, \quad t \to 0. \]  

(6.29)

We refer to [23, Proposition 4.1 and Corollary 4.2]. There one can also find further results on the borderline cases of \( s = \sigma_p^{|\rho|} \) and \( s = \frac{|\rho|}{p} \).

Let us denote the number \( r \in (1, \infty) \) given by \( \frac{1}{r} = \frac{1}{p} - \frac{s}{|\rho|} \), when \( \left( \frac{1}{p}, s \right) \) belongs to the above strip. Then our results (6.28) and (6.29) yield that

\[ E_G^{\Lambda^p_{\rho}}(\Omega) \sim E_G^{\rho - A^s_{p,q}(\Omega)}(t) \sim t^{-\frac{1}{p} + \frac{s}{|\rho|}} = t^{-\frac{1}{p}}, \quad t \to 0, \quad -n < \rho < 0, \]

(6.30)

i.e., all the function spaces \( \rho - A^s_{p,q}(\Omega) \) depicted on the line \( s = \frac{|\rho|}{p} \) possess the same growth envelope function—and in that sense the same ‘quality’ of unboundedness—like the space \( \Lambda^p_{\rho}(\Omega) \). This reflects exactly the outcome in the case of spaces \( A^s_{p,q}(\mathbb{R}^n) \) and \( L_p(\mathbb{R}^n) \), i.e., when \( \rho = -n \), as described in (6.23) which remains valid if one replaces \( \mathbb{R}^n \) by a bounded domain \( \Omega \).

In other words, this result represents another fine incidence of the Slope-n-Rule as formulated in Slope Rules 2.17(ii).

**Remark 6.11.** In the monographs [21, 74] as well as in the above-mentioned papers [23, 26] we studied, in addition, the so-called growth envelope \( E_G(X) = (E_G^X(t), u_G^X) \), consisting of the growth envelope functions \( E_G^X(t) \), together with some fine index \( u_G^X \in (0, \infty) \), refining the characterisation. In the case of \( L_{p,q}(\mathbb{R}^n) \) the result reads, for example, as \( E_G(L_{p,q}(\mathbb{R}^n)) = (t^{-\frac{1}{p} + \frac{1}{q}}, q) \) (recall Example 6.8), and in the case of spaces \( A^s_{p,q}(\mathbb{R}^n) \), we have usually \( u_{G_{p,q}^{A^s}}(\mathbb{R}^n) = q \) and \( u_{G_{p,q}^{A^s}}(\mathbb{R}^n) = p \); we refer to the above-mentioned literature for further details.
6.3 Crossing borders

As already announced, our idea is to leave the situation of (only) one $\varrho$-clan and ask for new phenomena then. But first we would like to complement the sharp embeddings in Subsections 5.1 and 5.2 as follows.

**Theorem 6.12.** Let $n \in \mathbb{N}$ and $-n < \varrho < 0$. Let $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$ and $s_1, s_2 \in \mathbb{R}$.

(i) Then there is a continuous embedding

$$
id_{\mathbb{R}^n} : g-B_{p_1,q_1}^{s_1}(\mathbb{R}^n) \hookrightarrow g-B_{p_2,q_2}^{s_2}(\mathbb{R}^n)$$

if and only if

$$p_2 \geq p_1 \quad \text{and} \quad \left\{ \begin{array}{ll}
\text{either} & s_1 = \frac{|\varrho|}{p_1} > s_2 = \frac{|\varrho|}{p_2}, \quad 0 < q_1, q_2 \leq \infty, \\
\text{or} & s_1 = \frac{|\varrho|}{p_1} = s_2 = \frac{|\varrho|}{p_2}, \quad 0 < q_1 \leq q_2.
\end{array} \right. \tag{6.32}$$

(ii) Then there is a continuous embedding

$$
id_{\mathbb{R}^n} : g-F_{p_1,q_1}^{s_1}(\mathbb{R}^n) \hookrightarrow g-F_{p_2,q_2}^{s_2}(\mathbb{R}^n)$$

if and only if

$$p_2 \geq p_1 \quad \text{and} \quad \left\{ \begin{array}{ll}
\text{either} & s_1 = \frac{|\varrho|}{p_1} > s_2 = \frac{|\varrho|}{p_2}, \quad 0 < q_1, q_2 \leq \infty, \\
\text{or} & s_1 = \frac{|\varrho|}{p_1} = s_2 = \frac{|\varrho|}{p_2}, \quad 0 < q_1 \leq q_2.
\end{array} \right. \tag{6.34}$$

There are no compact embeddings $g-A_{p_1,q_1}^{s_1}(\mathbb{R}^n) \hookrightarrow g-A_{p_2,q_2}^{s_2}(\mathbb{R}^n)$.

**Proof.** Part (i) in the case of $g-B_{p,q}^{s}(\mathbb{R}^n) = \Lambda_{p,q} B_{p,q}^{s}(\mathbb{R}^n)$ is a special case of [27, Theorem 3.3], whereas the case of $g-B_{p,q}^{s}(\mathbb{R}^n) = \Lambda^{p} B_{p,q}^{s}(\mathbb{R}^n)$ is covered by [93, Theorem 2.5] and [14, Corollary 4.21]. Part (ii) is covered by [29, Theorem 3.1] when $g-F_{p,q}^{s}(\mathbb{R}^n) = \Lambda_{p,q} F_{p,q}^{s}(\mathbb{R}^n)$, and by [93, Corollary 5.9] when $g-F_{p,q}^{s}(\mathbb{R}^n) = \Lambda^{p} F_{p,q}^{s}(\mathbb{R}^n)$ and also by (2.45).

**Remark 6.13.** We refer to the papers [14, 27, 29, 92, 93] for further embedding results of $g-A_{p,q}^{s}$ spaces on $\mathbb{R}^n$, in particular when $p_1 \neq p_2$ (as is always assumed in our setting here). The difference between Theorem 6.12 for spaces on $\mathbb{R}^n$ and Theorem 6.5 for spaces on a bounded domain $\Omega$ concerns (apart from the obviously missing compactness assertion) essentially the extension to parameters $p_1 > p_2$ which is a matter of Hölder’s inequality. This is well known from the classical spaces already and also reflected in the comparison of the corresponding diagrams Figure 10 on page 1348 and Figure 12 below.

**Remark 6.14.** Let us briefly mention some phenomenon which essentially distinguishes the $g$-clan $g-A_{p,q}^{s}(\mathbb{R}^n)$ with $\varrho > -n$ from its classical counterpart $A_{p,q}^{s}(\mathbb{R}^n)$, corresponding to $\varrho = -n$; recall Proposition 2.12(i). Whenever $\varrho > -n$, then for the embedding into a Lebesgue space $L_r(\mathbb{R}^n)$ ($0 < r \leq \infty$), we get

$$g-A_{p,q}^{s}(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n) \quad \text{if and only if} \quad r = \infty$$

for appropriately chosen $s$, $p$ and $q$. This somewhat surprising result has been proved in [29, Corollary 3.6] for spaces of type $\Lambda_{p,q} A_{p,q}^{s}(\mathbb{R}^n)$, and in [93, Proposition 5.7] for spaces of type $\Lambda^{p} A_{p,q}^{s}(\mathbb{R}^n)$.

In contrast to this situation, we have embeddings $g-A_{p,q}^{s}(\mathbb{R}^n) \hookrightarrow bmo(\mathbb{R}^n)$; recall Remark 2.11 for the definition of $bmo(\mathbb{R}^n)$. Then for $-n \leq \varrho < 0$,

$$\Lambda_{p,q} A_{p,q}^{s}(\mathbb{R}^n) \hookrightarrow bmo(\mathbb{R}^n) \quad \text{if and only if} \quad s \geq \frac{|\varrho|}{p}$$

(see [25, Corollary 5.1]). The parallel result for the case of $g-A_{p,q}^{s}(\mathbb{R}^n) = \Lambda^{p} A_{p,q}^{s}(\mathbb{R}^n)$ can be found in [93, Propositions 5.12 and 5.13].
So far we dealt with properties of the spaces $\varrho^{\cdot p,q}_{n}(\mathbb{R}^n)$ according to Definition 2.15 within a fixed $\varrho$-clan. One may ask how spaces belonging to different $\varrho$-clans are related to each other. Here, we summarise our results extending Theorem 6.12 for possibly different parameters $\varrho_1$ and $\varrho_2$.

Theorem 6.15. Let $n \in \mathbb{N}$ and $-n < \varrho_1, \varrho_2 < 0$. Let $0 < p_1, p_2 < \infty$ and $0 < q_1, q_2 \leq \infty$. Let $s_1, s_2 \in \mathbb{R}$.

(i) Then there is a continuous embedding

\[
\text{id}_{\varrho}: \Lambda_{\varrho_1}B_{s_1}^{p_1,q_1}(\mathbb{R}^n) \hookrightarrow \Lambda_{\varrho_2}B_{s_2}^{p_2,q_2}(\mathbb{R}^n) \tag{6.35}
\]

if and only if

\[
|\varrho_2| \leq |\varrho_1|, \quad \frac{|\varrho_1|}{p_1} \geq \frac{|\varrho_2|}{p_2} \tag{6.36}
\]

and

\[
\begin{cases}
\text{either } s_1 - \frac{|\varrho_1|}{p_1} > s_2 - \frac{|\varrho_2|}{p_2}, & 0 < q_1, q_2 \leq \infty, \\
\text{or } s_1 - \frac{|\varrho_1|}{p_1} = s_2 - \frac{|\varrho_2|}{p_2}, & 0 < q_1 \leq q_2. \tag{6.37}
\end{cases}
\]

(ii) Then there is a continuous embedding

\[
\Lambda_{\varrho_1}F_{s_1}^{p_1,q_1}(\mathbb{R}^n) \hookrightarrow \Lambda_{\varrho_2}F_{s_2}^{p_2,q_2}(\mathbb{R}^n) \tag{6.38}
\]

if and only if (6.36) is satisfied and

\[
\begin{cases}
\text{either } s_1 - \frac{|\varrho_1|}{p_1} > s_2 - \frac{|\varrho_2|}{p_2}, & 0 < q_1, q_2 \leq \infty, \\
\text{or } s_1 - \frac{|\varrho_1|}{p_1} = s_2 - \frac{|\varrho_2|}{p_2}, & p_1 < p_2, \quad 0 < q_1, q_2 \leq \infty, \tag{6.39}
\end{cases}
\]

or

\[
\begin{cases}
s_1 = s_2, \quad p_1 = p_2, & 0 < q_1 \leq q_2. \tag{6.40}
\end{cases}
\]

(iii) Then there is a continuous embedding

\[
\Lambda^{\varrho_1}B_{s_1}^{p_1,q_1}(\mathbb{R}^n) \hookrightarrow \Lambda^{\varrho_2}B_{s_2}^{p_2,q_2}(\mathbb{R}^n) \tag{6.40}
\]

if (6.36) is satisfied and

\[
s_1 - \frac{|\varrho_1|}{p_1} > s_2 - \frac{|\varrho_2|}{p_2}. \tag{6.41}
\]
Conversely, if (6.40) is continuous, then (6.36) is satisfied and \( s_1 \geq s_2, s_1 - \frac{|\varrho_1|}{p_1} \geq s_2 - \frac{|\varrho_2|}{p_2} \) and \( q_1 \leq q_2 \) if \( s_1 = s_2 \).

(iv) Then there is a continuous embedding
\[
A^{\varrho_1}F^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow A^{\varrho_2}F^{s_2}_{p_2,q_2}(\mathbb{R}^n)
\]
if and only if (6.36) and (6.39) are satisfied.

Remark 6.16. The above result (i) coincides with [27, Theorem 3.3], and (ii) with [29, Theorem 3.1]. In particular, there are no continuous embeddings if \( |\varrho_2| > |\varrho_1| \). Part (iii) is covered by [93, Theorem 2.5]. In fact, there are further sufficient conditions for the continuity of (6.40) in the limiting case which we omitted here for simplicity, and we also refer to some extension of this result in [14, Corollary 4.21]. Finally, the part (iv) coincides with [93, Corollary 5.9], now also covered by the part (ii) and (2.45). Note that Theorem 6.12 is just a special case of Theorem 6.15 for the case of \( \varrho = \varrho_1 = \varrho_2 \) where, in addition, we have a complete result also for the part (iii) above. So apart from the small gap in the limiting case of (iii), it is known that
\[
\text{id}_{\mathbb{R}^n} : A^{\varrho_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow A^{\varrho_2}_{p_2,q_2}(\mathbb{R}^n)
\]
if and only if (6.36) is satisfied and \( s_1 - \frac{|\varrho_1|}{p_1} \geq s_2 - \frac{|\varrho_2|}{p_2} \) with some additional assumptions in the limiting case. We illustrate the situation in Figure 12 on page 1352 in the case of \( \varrho_1 = \varrho_2 = \varrho \).

Remark 6.17. We excluded so far \( A^{\varrho}_{p_1,q}(\mathbb{R}^n) \) according to (2.42), i.e., the \( n \)-clan in terms of Definition 2.15(i), as target spaces. However, in view of the first condition in (6.36), one may have doubts whether there is a continuous embedding of type \( \varrho^{\varrho_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow A^{\varrho_2}_{p_2,q_2}(\mathbb{R}^n) \) when \(-n < \varrho < 0\). In fact, this turns out to be wrong, or more precisely,
\[
\varrho^{\varrho_1}_{p_1,q_1}(\mathbb{R}^n) \not\subseteq A^{\varrho_2}_{p_2,q_2}(\mathbb{R}^n)
\]
if \(-n < \varrho < 0, 0 < p_1, p_2 < \infty, 0 < q_1, q_2 \leq \infty \) and \( s_1, s_2 \in \mathbb{R} \). This—at first glance surprising—result is covered by the references given in Remark 6.16.

Remark 6.18. The case of \( F^{\varrho}_{\infty,q}(\mathbb{R}^n) \) can be incorporated by embeddings, but is also partly covered by (2.35) together with [93, Corollary 5.8] and [14, Corollary 4.23].

We return to Subsection 6.1, in particular to Theorem 6.5, and ask for continuous and compact embeddings between different \( \varrho \)-clans. For convenience we restrict ourselves to the question of compactness only.
Theorem 6.19. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}$). Let $-n < \varrho_1, \varrho_2 < 0$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$ and $s_1, s_2 \in \mathbb{R}$.

(i) Assume $|\varrho_1| \leq |\varrho_2|$. Then

$$\text{id}_\Omega : g_{1}A_{p_1,q_1}^{s_1}(\Omega) \hookrightarrow g_{2}A_{p_2,q_2}^{s_2}(\Omega)$$

(6.44)

is compact if and only if

$$s_1 - s_2 > |\varrho_1| \cdot \max \left( \frac{1}{p_1} - \frac{1}{p_2}, 0 \right).$$

(6.45)

(ii) Assume $|\varrho_1| \geq |\varrho_2|$. Then

$$\text{id}_\Omega : g_{1}A_{p_1,q_1}^{s_1}(\Omega) \hookrightarrow g_{2}A_{p_2,q_2}^{s_2}(\Omega)$$

(6.46)

is compact if and only if

$$s_1 - s_2 > \max \left( \frac{|\varrho_1|}{p_1}, \frac{|\varrho_2|}{p_2}, 0 \right).$$

(6.47)

Remark 6.20. The above results are covered by [13, 14, 28, 29, 31], including also further limiting cases for continuous embeddings. This follows by straightforward calculation and reformulation of the conditions according to the coincidences (2.24), (2.25) and (2.30) with (2.31). The extension to parameters $p_2 < p_1$ is due to Hölder’s inequality and the boundedness of $\Omega$.

In the case of $|\varrho_1| \leq |\varrho_2|$, the picture remains essentially the same; compare Figure 10 on page 1348 and Figure 14. However, when $|\varrho_1| \geq |\varrho_2|$, then we meet again Figure 13 on page 1353, extended to the right by Hölder’s inequality again (see Figure 15 below).

In contrast to Remark 6.17, we may now replace the target spaces $g_{2}A_{p_2,q_2}^{s_2}(\Omega)$ by their classical counterparts $A_{p_2,q_2}^{s_2}(\Omega)$. This corresponds to Theorem 6.19(i) and is covered by the references listed in Remark 6.20.

Corollary 6.21. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}$). Let $-n < \varrho < 0$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$ and $s_1, s_2 \in \mathbb{R}$. Then

$$\text{id}_\Omega : gA_{p_1,q_1}^{s_1}(\Omega) \hookrightarrow A_{p_2,q_2}^{s_2}(\Omega)$$

(6.48)

is compact if and only if

$$s_1 - s_2 > |\varrho| \cdot \max \left( \frac{1}{p_1} - \frac{1}{p_2}, 0 \right).$$

(6.49)
Figure 15 (Color online) Embeddings in the case of different $\rho$-clans of spaces on bounded domains, the case $|\rho_1| \geq |\rho_2|$. 

Remark 6.22. One can further derive entropy number results parallel to (6.17) from [13,31]. For example, for $\text{id}_\Omega: \rho_1-A^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \rho_2-A^{s_2}_{p_2,q_2}(\Omega)$ we have

$$e_k(\text{id}_\Omega) \sim k^{-\frac{1}{|\rho_1|-|\rho_2|}}$$

in all the cases except

$$|\rho_1| < |\rho_2| \quad \text{and} \quad 0 < |\rho_1| \left(\frac{1}{p_1} - \frac{1}{p_2}\right) < s_1 - s_2 \leq n \left(\frac{1}{p_1} - \frac{1}{p_2}\right).$$

In the latter case, we obtained in [13,31] that there exist some $c > 0$ and for any $\varepsilon > 0$ a number $c_\varepsilon > 0$ such that

$$ek^{-\alpha} \leq e_k(\text{id}_\Omega) \leq c_\varepsilon k^{-\alpha+\varepsilon}, \quad k \in \mathbb{N}$$

with

$$\alpha = \frac{1}{n - |\rho_1|} \left(s_1 - s_2 - |\rho_1| \left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right) > 0.$$

Note that this case cannot appear in the classical setting when $\rho_1 = -n$; recall Proposition 6.3. Further results in the context of Theorem 6.19 can be found in [13,31].

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