A proof for the fundamental conjecture in RWRE

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Abstract: We prove the fundamental conjecture concerning multidimensional random walks in random environments (RWRE). This conjecture asserts that: "any RWRE which is directionally transient in an open set of directions, is also ballistic." Specifically, for i.i.d. random environments we prove the strong form of this conjecture: "any RWRE which is directionally transient in an open set of directions satisfies condition T."

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1. Introduction

The main objective of this article is to derive a proof for the fundamental conjecture of $d-$ dimensional random walks in random environments (RWRE, $d \geq 2$). That conjecture states the existence of a non-vanishing deterministic velocity under the assumption of directional transience in an open set of directions. To a great extent, that belief is based on the absence of counterexamples. Nevertheless, theoretical and more rational viewpoints indicated the non-existence of traps in the higher dimensional case $d \geq 2$, under conditions related to transience. These conditions are the so-called ballisticity assumptions (cf. [Sz00] for a pioneer study of traps and [BS02]-[Ze04] for general reviews). The coined term trap makes reference to finite but arbitrary large sets where the random walk process wastes a relatively large time with relatively high probability. Alongside, this article provides a further and definitive study concerning the non-existence of such traps, which is diagrammatically depicted as follows:

A transient RWRE confined to start from the origin and inside of large box of size $L$, chooses an exit path of length proportional to $L$ with full annealed probability as $L$ goes to infinity, in particular there are not loops made by trace of the walker. Large loops are negligible in the long time behaviour.

Let us introduce the standard setting in order to properly explain our work and the conjecture. We focus on the relevant multidimensional case, thus the underlying dimension $d$ of the random walk is an integer greater than 1. The environment prescribes at each site in $\mathbb{Z}^d$ the transitions governing the evolution

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of the random particle. Specifically we let $\kappa \in (0, 1/(2d)]$ and define the simplex:

$$\mathcal{P}_\kappa := \left\{ z \in \mathbb{R}^{2d} : \sum_{i=1}^{2d} z_i = 1, \, z_i \geq \kappa \forall i \in [1, 2d] \right\}. \quad (1.1)$$

We will denote norms $\ell^1$ and $\ell^2$, by $\cdot_1$ and $\cdot_2$, respectively. The set of environments is $\Omega := \mathcal{P}_\kappa^{2d}$ and we denote an element $\omega \in \Omega$ in the form $\omega := \omega(x, e) = \omega(x, \cdot)$, $x \in \mathbb{Z}^d$, $e \in \mathbb{Z}^d$, with $|e|_1 = 1$. We also use the notation $\omega_x := \omega(x, \cdot)$, for $x \in \mathbb{Z}^d$.

For the time being, assume a given ergodic probability measure $\mathbb{P}$ on the $\sigma-$algebra $\mathcal{F}_\Omega$, generated by cylinder sets in $\Omega$. Let $\omega \in \Omega$ and $x \in \mathbb{Z}^d$ and define the quenched law $P_{x, \omega}$ as the probability measure of the Markov chain $(X_n)_{n \geq 0}$ with state space in $\mathbb{Z}^d$ starting from $x$ and stationary transition probabilities to nearest neighbour sites, given by the environment, i.e.

$$P_{x, \omega}[X_0 = x] = 1$$

$$P_{x, \omega}[X_{n+1} = X_n + e|X_n] = \omega(X_n, e), \quad \text{for } e \in \mathbb{Z}^d \text{ with } |e| = 1.$$

We then define for $x \in \mathbb{Z}^d$ the annealed probability measure $P_x$ via the semidirect product $P_x := \mathbb{P} \times P_{x, \omega}$ on $\Omega \times (\mathbb{Z}^d)^n$ endowed with its canonical $\sigma-$algebra.

With a little abuse of notation, we will denote as well by $P_x$, the marginal law of the process $(X_n)_{n \geq 0}$ under $P_{x}$ itself. We use symbols $(\mathcal{F}_n)_{n \geq 0}$ and $\mathcal{F}$ to indicate the natural filtration and $\sigma-$algebra of the random walk process, respectively.

We study the fundamental conjecture in strong mixing random environments, following certain extension of X. Guo in [Gu14]. For a universal set $U$, and a subset $A \subset U$ we write $U \setminus A$ the complement of $A$, and we simply write this by $A^c$ whenever $U$ is clear from the context.

We use the notation $|\cdot|_1$ and $|\cdot|_2$ to denote the $\ell_1$ and $\ell_2$-distance on $\mathbb{R}^d$ respectively; and furthermore, for $A, B \subset \mathbb{Z}^d$, $i \in \{1, 2\}$, the notation $d_i(A, B)$ stands for the canonical $\ell_i$-distance between sets $A, B$, i.e. $d_i(A, B) := \inf\{|x - y|_i, \, x \in A, y \in B\}$.

We recall the definition of Markovian field.

**Definition 1.1.** Markovian field on $\mathbb{Z}^d$. For $r \geq 1$ and $V \subset \mathbb{Z}^d$, let $\partial^r V = \{ z \in V^c : d_1(z, V) \leq r \}$ be the $r-$boundary of the set $V$. To simplify notation we will also write $\partial^r V = \partial V$ for sets $V \subset \mathbb{Z}^d$. A random environment $(\mathbb{P}, \mathfrak{F}_\Omega)$ on $\mathbb{Z}^d$ is called $r$-Markovian if for any finite $V \subset \mathbb{Z}^d$, $\mathbb{P}-$a.s.

$$\mathbb{P}[(\omega_x)_{x \in V} \in |\mathfrak{F}_V^c|] = \mathbb{P}[(\omega_x)_{x \in V} \in |\mathfrak{F}_{\partial^r V}|],$$

where $\mathfrak{F}_\Lambda = \sigma(\omega_x, \, x \in \Lambda)$.

Thus, we introduce the type of randomness on the environment that we are interested.

**Definition 1.2.** Strong mixing environments. Let $C$ and $g$ be positive real numbers. We will say that an $r$-Markovian field $(\mathbb{P}, \mathfrak{F}_\Omega)$ satisfies the strong mixing
condition \((SM)_{C,g}\) if for all finite subsets \(\Delta \subset V \subset \mathbb{Z}^d\) with \(d_1(\Delta, V^c) \geq r\), and \(A \subset V^c\),
\[
\frac{d\mathbb{P}[(\omega_x)_{x \in \Delta} \in \cdot | \eta]}{d\mathbb{P}[(\omega_x)_{x \in \Delta} \in \cdot | \eta']} \leq \exp \left( C \sum_{x \in \partial^r \Delta, y \in \partial^r A} e^{-g|x-y|} \right) \tag{1.2}
\]
for \(\mathbb{P}\)–a.s. all pairs of configurations \(\eta, \eta' \in \mathcal{P}^\infty_{\mathbb{Z}^d}\) which agree over the set \(V^c \setminus A\).

Here we have used the notation
\[
\mathbb{P}[(\omega_x)_{x \in \Delta} \in \cdot | \eta] = \mathbb{P}[(\omega_x)_{x \in \Delta} \in \cdot | \eta| (\omega_x)_{x \in V^c} = \eta].
\]

We introduce the so-called ballisticity conditions, nevertheless we first need to establish some further terminology. We define the unit sphere \(S^{d-1}\) by
\[
S^{d-1} := \{ x \in \mathbb{R}^d : \sum_{i=1}^{d} x_i^2 = 1 \}.
\]
We then define for \(L \in \mathbb{R}\) and \(\ell \in S^{d-1}\), the following \((F_n)_{n \geq 0}\) stopping times:
\[
T^L_\ell := \inf\{ n \geq 0 : X_n \cdot \ell \geq L \} \quad \text{and} \quad \tilde{T}^L_\ell := \inf\{ n \geq 0 : X_n \cdot \ell \leq L \}. \tag{1.3}
\]

We define classic Sznitman \(T\)–types of ballisticity conditions.

**Definition 1.3.** Let \(\gamma \in (0,1)\) and \(\ell \in S^{d-1}\). We say that condition \((T^\gamma|\ell)\) holds, if for each \(b > 0\) there exists some neighbourhood \(U_\ell\) of \(\ell\) in \(S^{d-1}\) such that for each \(\ell' \in U_\ell\),
\[
\limsup_{L \to \infty} L^{-\gamma} \log P_0 \left[ \tilde{T}^{\ell'}_{-bL} < T^L_\ell \right] < 0 \tag{1.4}
\]
is fulfilled. We further define condition \((T)|\ell\) as simply \((T^1)|\ell\), and condition \((T^\gamma)|\ell\) as the requirement that \((T^\gamma)|\ell\) is fulfilled for each \(\gamma \in (0,1)\).

The priori weaker condition of transience along an open set of directions will be the standard assumption throughout this work.

**Definition 1.4.** For \(\ell \in S^{d-1}\), we say that the RWRE is transient along an open set of directions containing \(\ell\) or directional transience along an open is fulfilled, if there exists a neighbourhood \(U_\ell\) of \(\ell\) in \(S^{d-1}\) such that for any \(\ell' \in U_\ell\), we have
\[
P_0 \left[ \lim_{n \to \infty} X_n \cdot \ell' = \infty \right] = 1. \tag{1.5}
\]

We now introduce the definition of ballistic asymptotic behaviour:

**Definition 1.5.** Non-vanishing limiting velocity Let \(\ell \in S^{d-1}\). We say that the RWRE satisfies a ballistic strong law of large numbers along direction \(\ell\), if there exists a deterministic non-vanishing velocity \(v \in \mathbb{R}^d\) with \(v \cdot \ell > 0\) such that \(P_0\)–a.s.
\[
\lim_{n \to \infty} \frac{X_n}{n} = v. \tag{1.6}
\]
In these terms, we consider the particular case when $P = \mu^{\otimes \mathbb{Z}^d}$ for certain fixed probability $\mu$ on the canonical $\sigma-$ algebra for $\mathcal{P}_\kappa$. We call this environmental framework by an i.i.d. random environment. Then the fundamental conjecture can be settle as the following theorem asserts.

**Theorem 1.6.** Let $\ell \in \mathbb{S}^{d-1}$, then for any random walk in an i.i.d. uniform elliptic random environment the following assertions are equivalents:

(i) Directional transience along an open set $U_\ell \subset \mathbb{S}^{d-1}$ is fulfilled, with $\ell \in U_\ell$.

(ii) A ballistic strong law of large numbers along direction $\ell$ with velocity $v \in \mathbb{R}^d$ holds.

This was the crucial problem in the field of RWRE. For dimension $d = 1$ the characterization problem of the ballistic regime was solved by F. Solomon [So75] (cf. (1.16) Theorem on page 7), a student of F. Spitzer. Indeed, Solomon’s result says that the conjecture is not true for the one dimensional case.

A first work in the higher dimensional case connecting some large time behaviour with a ballistic type of condition was [Ka81]. Therein, S. Kalikow a student of H. Kesten, proved transience under a posteriori coined Kalikow’s condition. Kalikow’s condition was assumed in several investigations, for instance in [SZ99] by A-S. Sznitman and M. Zerner, where they proved a ballistic strong law of large numbers. It was also used as a standing assumption for environments where the randomness was harder, as the case of mixing ones, we refer for example to the article of F. Comets and O. Zeitouni [CZ04], as well as the work of F. Rassoul-Agha [RA03] among others. However, it is highly expected that the fundamental conjecture was already known, even in the times of Kalikow’s result. Thus, the condition was possibly assumed for that time with the later purpose of approving of disapproving its equivalence with transience along an open set.

Alongside, A-S. Sznitman under Kalikow’s condition performed certain renormalization procedures to get the so-called *atypical quenched estimate* (cf. Proposition 3.1 in [Sz00]), which essentially implies the strong law of large numbers. The method displayed him the stronger character of Kalikow’s condition. Indeed, Sznitman realized that the entire process could be replicated and even improved under a weaker assumption: the so-called $(T)$ condition of Sznitman (cf. [Sz01] Theorem 3.4). Moreover, Sznitman proved that Kalikow’s condition is strictly stronger than condition $(T)$ in dimension $d = 1$ (cf. [SZ99] Remark 2.5 and [Sz01] Proposition 2.6). From that point, Kalikow’s condition was ruled out of the so-called ballisticity assumptions.

In subsequent developments, a great deal of efforts were focused on to weaken the ballisticity assumptions. A common subject of study was to obtain the ballistic behaviour for the random walk precess from these assumptions. As was mentioned, in the seminal articles [Sz01]-[Sz02], Sznitman introduced $(T^\gamma)$ types assumptions for $\gamma \in (0,1]$, proved the ballistic behaviour and several large deviation estimates under $(T^\gamma)$. Sznitman also proved that $(T^\gamma)$ conditions are equivalents for any $\gamma \in (1/2,1)$. Later on, in the article [BDR14], the authors
besides some problems in the proof argument solved in [GVV19], proved the conjecture about the equivalence between all \((T^\gamma)\) conditions for any \(\gamma \in (0,1)\). Then, in [GR18] a proof concerning the equivalence of condition \((T)\) with any \((T^\gamma), \gamma \in (0,1)\) was obtained after a first try in [GR15], both articles in collaboration with A. F. Ramírez.

Overall, the new ideas involved in this article have come to light from the investigations in mixing random environments in [Gue19], and [GVV19] in collaboration with G. Valle and M. E. Vares.

We will prove the conjecture by proving in turn the following stronger result.

**Theorem 1.7.** Let \(\ell \in \mathbb{S}^{d-1}\), then for any random walk in a strong mixing uniform elliptic random environment the following assertions are equivalents:

(i) Directional transience along an open set \(U_\ell \subset \mathbb{S}^{d-1}\) is fulfilled, with \(\ell \in U_\ell\).

(ii) \((T^\prime)\)|\(\ell\) is fulfilled.

Notice that by the main result in [GR18] and Theorem 3.6 of [Sz01], the conjecture stated in Theorem 1.6 and the result announced in the abstract are indeed implied by the last theorem. This stronger version of the conjecture was expected as well, for instance in page 227 of [Szn02] that conjecture was slightly mentioned. Somehow, the conjecture is based on to combine the multiplicative property given by the Markov property under the quenched law with the uniform elliptic assumption. Roughly speaking, the elliptic assumption ensures that the walk escapes from traps, and the multiplicative property makes any decay equivalent to an exponential one, under an appropriate unknown construction. In these terms, this article provides the right renormalization scheme combining both viewpoints.

We denote by \([\cdot]\) the integer part function. We gather results in [GR18], Theorem 1.1. in [RS09], Theorem 3.6 in [Sz01] and Theorem 1 in [GVV19], to get the next corollary.

**Corollary 1.8.** We let \(\ell \in \mathbb{S}^{d-1}\). Then, for any random walk in an i.i.d. uniform elliptic random environment the following assertions are equivalents:

(i) Directional transience along an open set \(U_\ell \subset \mathbb{S}^{d-1}\) is fulfilled, with \(\ell \in U_\ell\).

(ii) A ballistic strong law of large numbers along direction \(\ell\) with velocity \(v \in \mathbb{R}^d\) holds.

(iii) Condition \((T)\)|\(\ell\) holds.

(iv) Condition \((T^\prime)\)|\(\ell\) holds.

(v) For some \(\gamma \in (0,1)\), condition \((T^\gamma)\)|\(\ell\) holds.

(vi) In addition to (ii), considering the random element of the Skorohod space \(D(\mathbb{R}^+, \mathbb{R}^d)\):

\[
B_n^\gamma := \frac{X_{[n]} - [n]v}{\sqrt{n}}
\]

then \(B_n^\gamma\) converges in law under \(P_0\) to a non-degenerate \(d\)-dimensional Brownian motion.
(vii) In addition to (ii), considering $B^n$ as in (vi), we have $\mathbb{P}$-a.s. $B^n$ converges in law under $P_{0,\omega}$ to a non-degenerate $d-$dimensional Brownian motion.

We shall now outline the structure of this article which essentially compose the renormalization argument. In Section 2, we will prove a suitable seed estimate under the assumption 1.4. We next construct our scaling estimates, which provide to iterate the process starting from the seed estimate along successive scales running over integers $k \geq 0$. We indeed end Section 3 proving a stretched exponential decay similar as the one in Definition 1.4. Finally, we prove Theorems 1.6 and 1.7 in Section 4 by collecting the previous results.

2. Triggering condition for the renormalization procedure: Fixing the seed estimate

The main subject to be treated in the current section will be to construct an appropriate seed estimate to be used into a renormalization procedure. This so-called seed estimate will be provided as soon as transience along a neighbourhood of directions is fulfilled. On the other hand, the scaling procedure will be performed in the next section. Our proof ends in the last Section 4, which combines the seed estimate and the successive scaling estimates in the next section to prove condition $(T)$ and $(T')$ under transience, when we assume i.i.d. and strong mixing random environment, respectively.

Let us begin with a preliminary result concerning a suitable form to express transience in a given direction $\ell \in S^{d-1}$.

**Lemma 2.1.** Assume that directional transience along direction $\ell \in S^{d-1}$ holds. Then, for any constant $b > 0$ there exists a function $\vartheta : (3\sqrt{d}, \infty) \mapsto (0, \infty)$, with

$$\lim_{L \to \infty} \vartheta(L) = 0,$$

such that for all large $L$, we have

$$P_0 \left[ T^d_{-bL} < T^d_L \right] \leq \vartheta(L).$$

**Proof.** Notice that under the assumption of transience in direction $\ell$, we have that

$$P_0 \left[ \lim_{L \to \infty} X_n \cdot \ell = \infty \right] = 1$$

holds. Therefore, the complementary event to the one entering into the probability in (2.2) has null probability. As a result,

$$P_0 \left[ \bigcup_{L \geq 1} \cap_{n \geq 1} \cup_{k \geq n} \{ X_k \cdot \ell < L \} \right] = 0,$$

(2.3)
and notice that this says there are not infinite many \( n \in \mathbb{N} \) such that \( X_n \cdot \ell < L \) for any \( L \in \mathbb{N} \). In virtue of this implication, we assume that (2.1) is false and derive a contradiction. Observe that when
\[
\lim_{L \to \infty} P_0 \left[ \bar{T}^{\ell}_{-bL} < T^\ell_L \right] \to 0,
\]
by definition there exist constants \( b, \epsilon > 0 \) such that for any \( L \in \mathbb{N} \) there exists and integer \( L' > L \) with
\[
P_0 \left[ \bar{T}^{\ell}_{-bL'} < T^\ell_{L'} \right] > \epsilon. \tag{2.4}
\]
From the last assertion we do construct an strictly increasing sequence \((L_k)_{k \geq 0}\) of integers so that,
\[
P_0 \left[ \bar{T}^{\ell}_{-bL_k} < T^\ell_{L_k} \right] > \epsilon.
\]
On the other hand, since the walk is transience we have, \( P_0 - \text{ a.s. for any } k \in \mathbb{N} \)
\[
\{ \bar{T}^{\ell}_{-bL_k} < T^\ell_{L_k} \} \subset \{ \bar{T}^{\ell}_{-bL_k} < \infty \}.
\]
Thus, we use the facts that \( P_0 - \text{ a.s. one has} \)
\[
X_{\bar{T}^{\ell}_{-bL_k}} \cdot \ell < 0,
\]
along with (cf. 1.3)
\[
\bar{T}^{\ell}_{-bL_k} < \bar{T}^{\ell}_{-bL_{k+1}} \quad (\text{we use that } (L_k)_{k \geq 0} \text{ is strcitly increasing}),
\]
in order to derive a contradiction with the assumption (2.3) implied by tran-
sience along direction \( \ell \).

For a set \( A \subset \mathbb{Z}^d \), we introduce the \((\mathcal{F}_n)_{n \geq 0} - \text{ stopping time,} \)
\[
T_A := \inf \{ n \geq 0 : X_n \notin A \}. \tag{2.5}
\]
Starting from the result of the previous lemma, we can state and prove a propo-
sition which implies decay of an unlikely exit from a box. We introduce some further terminology concerning such solid blocks that we call by the term boxes. Let \( \ell \in \mathbb{Z}^{d-1} \) and \( R \) be a rotation of \( \mathbb{R}^d \) satisfying \( R(e_1) = \ell \). We introduce for real numbers \( L, \hat{L} > 0 \) and \( \bar{L} > 0 \), the box \( B(L, \hat{L}, \bar{L}, \ell) \), defined by
\[
B(L, \hat{L}, \bar{L}, \ell) := R \left( (-\hat{L}, L) \times (-\bar{L}, L)^{d-1} \right) \cap \mathbb{Z}^d. \tag{2.6}
\]
We stress that the exact form of rotation \( R \) is immaterial for our purposes. We also define for that box its boundary positive part \( \partial^+ B(L, \hat{L}, \bar{L}, \ell) \), as follows
\[
\partial^+ B(L, \hat{L}, \bar{L}, \ell) := \partial B(L, \hat{L}, \bar{L}, \ell) \cap \{ z \in \mathbb{Z}^d : z \cdot \ell \geq L \}. \tag{2.7}
\]
We provide the formal statement of the mentioned decay in the next proposition.
Proposition 2.2. Assume directional transience along an open set $U_\ell \subset S^{d-1}$ of directions, with $\ell \in U_\ell$. Then there exist some constant $\tilde{c} > 1$, together with some function $\phi : (3\sqrt{d}, \infty) \mapsto (0, \infty)$, satisfying
\[
\lim_{M \to \infty} \phi(M) = 0,
\]
such that for large $M$, we have
\[
P_0 \left[ X_{T_B(M(1+1/M), \ell, \ell_\ell)} \notin \partial^+ B \left( M, M \left( 1 + \frac{1}{11} \right), \tilde{c}M, \ell \right) \right] \leq \phi(M)
\]
holds.

Proof. Assume that the RWRE is transient in an open set of directions $U_\ell$, containing $\ell \in S^{d-1}$. We fix a rotation $R$ of $\mathbb{R}^d$, such that $R(e_1) = \ell$.

Using Lemma 2.1, under the assumption of directional transience along an open set $U_\ell \subset S^{d-1}$ of directions, we have that for any $b > 0$ and $\ell' \in U_\ell$, there exists a function $\phi_{\ell'} : (3\sqrt{d}, \infty) \mapsto [0, \infty)$ (depending also on $b$, but it is nonmaterial), such that
\[
\lim_{L \to \infty} \phi_{\ell'}(L) = 0,
\]
together with for all large $L$,
\[
P_0 \left[ T_{bL} < T_{L_{\ell'}} \right] \leq \phi_{\ell'}(L).
\]

(2.8)

Since the set $U_\ell$ is open in $S^{d-1}$, there exists a strictly positive number $\alpha$, such that the $2(d-1)$ vectors $\ell_i^\pm$, for integer $i \in [2, d]$ defined by
\[
\ell_i^+ := \frac{\ell + \alpha R(e_i)}{\|\ell + \alpha R(e_i)\|_2} \quad \text{and} \quad \ell_i^- := \frac{\ell - \alpha R(e_i)}{\|\ell - \alpha R(e_i)\|_2},
\]
are elements of set $U_\ell$. Notice that we can and do assume that $\alpha < 1/2$. Observe that setting $\ell' = \ell_i^\pm$ for $i \in [2, d]$, there exists $L_{0, \ell'} > 3\sqrt{d}$ such that (2.8) holds, whenever $L > L_{0, \ell'}$. Hence, we first define
\[
L_0 := \max_{\ell' \in \{\ell_i^\pm, i \in [2, d]\}} L_{0, \ell'},
\]
and then, define real numbers:
\[
b := \frac{11}{36} \sqrt{\frac{1 - \alpha^2}{1 + \alpha}}
\]
\[
M := b\sqrt{\alpha^2 + 1}L \quad \text{(for $L \geq L_0$ in (2.9), and)}
\]
\[
\tilde{c} := 2 \times \max \left\{ \frac{1}{\alpha} + \frac{1 + \sqrt{\alpha^2 + 1}}{b(\alpha^2 + 1)}, \frac{23}{11\alpha} \right\}.
\]

Thus under these terminology, for fixed $L > L_0$ we define the set $D$ by
\[
D := \{ z \in \mathbb{Z}^d : z \cdot \ell \in (-M, M(1 + 1/11)), \forall i \in [2, d] \ z \cdot \ell_i^\pm > -bL \}.
\]
along with its boundary frontal part \( \partial^+ D \),
\[
\partial^+ D := \partial D \cap \{ z \in \mathbb{Z}^d : z \cdot \ell \geq M(1 + 1/11) \}.
\]
We plainly have that for the box \( B := B(M, M, \bar{c}M, \ell) \),
\[
P_0 \left[ X_{T_B} \notin \partial^+ B \right] \leq P_0 \left[ X_{T_D} \notin \partial^+ D \right]
\leq \sum_{i=2}^d P_0 \left[ \tilde{T}_{-bL}^{\ell_i} < T_L^{\ell_i} \right] + P_0 \left[ \tilde{T}_{-bL}^{\ell_i} < T_L^{\ell_i} \right]
\leq \sum_{i=2}^d \phi_{\ell_i}(L) + \phi_{\ell_i}(L).
\]
The proof is finished by taking \( \phi(M) := \sum_{i=2}^d \left( \phi_{\ell_i} + \phi_{\ell_i} \right) \left( \frac{M}{\tilde{c}d\sqrt{d} + 1} \right) \).

We proceed with a further step to get of what we have called a seed estimate. That coined term makes reference to the procedure of inserting this triggering condition into a renormalization procedure, the formal construction will be given in the next section. We provide the statement and proof of the main result in this section. We let \( x \in \mathbb{R}^d, c > 0 \) and \( L > 3\sqrt{d} \), we first define blocks (\( R \) is a rotation as in the beginning of the section):
\[
\tilde{B}_1(x, c, L) := R \left( x + [0, L] \times [0, 3cL]^d \right) \cap \mathbb{Z}^d \tag{2.10}
\]
\[
B_2(x, c, L) := R \left( x + (-L, L(1 + 1/11)) \times (-cL, 4cL)^{d-1} \right) \cap \mathbb{Z}^d. \tag{2.11}
\]
We also define the frontal boundary part of \( B_2(x, c, L) \), denoted by \( \partial^+ B_2(x, c, L) \) and defined as follows:
\[
\partial^+ B_2(x, c, L) := \partial B_2(x, c, L) \cap \{ z \in \mathbb{Z}^d : (z - x) \cdot \ell \in L(1 + 1/11) \}.
\]
Under this notation, we have:

**Proposition 2.3.** Assume directional transience along an open set \( \mathcal{U}_\ell \subset \mathbb{S}^{d-1} \) of directions, with \( \ell \in \mathcal{U}_\ell \). Then, there exist a strictly increasing sequence of positive integers \( (L_k)_{k \geq 0} \), a constant \( \bar{c} > 1 \) and a function \( \phi : (3\sqrt{d}, \infty) \rightarrow [0, \infty) \) with
\[
\lim_{L \to \infty} \phi(L) = 0,
\]
such that for each \( L_k \geq L_0 \),
\[
\mathbb{E} \left[ \sup_{x \in \tilde{B}_1(0, \bar{c}, L_k)} P_{x, \omega} \left[ X_{T_{B_2(0, \bar{c}, L)}} \notin \partial^+ B_2(0, \bar{c}, L_k) \right] \right] \leq \phi(L_k). \tag{2.12}
\]

**Remark 2.4.** The conclusion of Proposition 2.3 is that there exists a subsequence of integers \( (L_k)_{k \geq 0} \) such that the random variable depending on \( L_k \) and entering at (2.12) converges to 0 in the norm of \( L^1(\mathbb{P}, \mathcal{F}_\Omega) \).
Proof. We shall apply the Proposition 2.2. For large $L$ and a positive number $\tilde{c}$ as in Proposition 2.2 (switching letter $M$ by $L$), we start by introducing the notation $B_{L,\tilde{c}}$ to denote box $B(L(1 + \frac{1}{T}), L, \tilde{c}L, \ell)$ (cf. (2.6)).

Observe that using Proposition 2.2 and under the notation therein, the constant $\tilde{c} > 1$ is such that for all large $L$ we have

$$\mathbb{E} \left[ P_{0,\omega} \left[ X_{TB_{L,\tilde{c}}(0,0,L_k)} \notin \partial^+ B_{L,\tilde{c}} \right] \right] \xrightarrow{L \to \infty} 0.$$ 

Therefore, restricting $L$ to integer numbers we see that the random variable $P_{0,\omega} \left[ X_{TB_{L,\tilde{c}}(0,0,L_k)} \notin \partial^+ B_{L,\tilde{c}} \right] \to 0$ in $\mathbb{P}$- probability. Moreover, using an standard result of Analysis (cf. [HS65], Theorem 11.26) we can find a deterministic subsequence $(L_k)_{k \geq 0}$ (which is strictly increasing by definition of subsequence of integer numbers), so that for any $L_k \geq L_0$ we have that, $\mathbb{P}$- a.s.

$$P_{0,\omega} \left[ X_{TB_{L_k,\tilde{c}}(0,0,L_k)} \notin \partial^+ B_{L_k,\tilde{c}} \right] \xrightarrow{k \to \infty} 0.$$ 

(2.13)

We assert that the claim:

$$\mathbb{P} \left[ \sup_{x \in B(0,\tilde{c},L_k)} P_{x,\omega} \left[ X_{TB_2(0,\tilde{c},L_k)} \notin \partial^+ B_2(0,\tilde{c},L_k) \right] \to 0 \right] = 1$$

(2.14)

holds.

In order to prove (2.14), we let $k \geq 0$ be a subindex of the subsequence of integers $(L_k)_{k \geq 0}$ as above. Notice that by construction for each $x \in B_1(0,\tilde{c},L_k)$, we have $\mathbb{P}$- a.s.

$$P_{x,\omega} \left[ X_{TB_2(0,\tilde{c},L_k)} \notin \partial^+ B_2(0,\tilde{c},L_k) \right] \leq P_{x,\omega} \left[ X_{T_{x+B_{L_k,\tilde{c}}}} \notin x + \partial^+ B_{L_k,\tilde{c}} \right].$$

(2.15)

Therefore, we estimate the probability of the complementary event involved in (2.14) as follows:

$$\mathbb{P} \left[ \sup_{x \in B_1(0,\tilde{c},L_k)} P_{x,\omega} \left[ X_{TB_2(0,\tilde{c},L_k)} \notin \partial^+ B_2(0,\tilde{c},L_k) \right] \to 0 \right]$$

$$\leq \mathbb{P} \left[ \bigcup_{x \in B_1(0,\tilde{c},L_k)} \left\{ P_{x,\omega} \left[ X_{TB_2(0,\tilde{c},L_k)} \notin \partial^+ B_2(0,\tilde{c},L_k) \right] \to 0 \right\} \right]$$

$$\leq \mathbb{P} \left[ \bigcup_{x \in B_1(0,\tilde{c},L_k)} \left\{ P_{x,\omega} \left[ X_{T_{x+B_{L_k,\tilde{c}}}} \notin x + \partial^+ B_{L_k,\tilde{c}} \right] \to 0 \right\} \right]$$

$$\leq \sum_{x \in \mathbb{Z}^d} \mathbb{P} \left[ P_{0,\omega} \left[ X_{TB_{L_k,\tilde{c}}(0,0,L_k)} \notin \partial^+ B_{L_k,\tilde{c}} \right] \to 0 \right] = 0,$$

where we have used above: (2.15) to get the third line and translation invariance of $\mathbb{P}$ along with (2.13) to get the last line, hence we have proven claim (2.14).

The assertion of the proposition follows after an application of Lebesgue’s dominated convergence, with dominating function $\mathbb{I}_\Omega$ and the help of (2.14).
We close this section with a reinforcement of the previous proposition. In spite of the stronger character of the result to be displayed soon, Proposition 2.3 is actually enough for our purposes in the next section.

An examination of the proof argument in Proposition 2.3 shows that under the notation therein, we indeed have for any sequence of positive real numbers \((L_k)_{k \geq 0}\) tending to \(\infty\), as \(k \to \infty\), there exists a subsequence \((L_{k_n})_{n \geq 0}\) of \((L_k)_{k \geq 0}\) such that,

\[
E \left[ \sup_{x \in \tilde{B}_1(0, L_{k_n})} P_{x, \omega} \left[ X_{T_{B_2(0, \tilde{c}, L_{k_n})}} \notin \partial^+ B_2(0, \tilde{c}, L_{k_n}) \right] \right] \to 0 \quad (2.16)
\]

as \(n \to \infty\). Therefore, we assert that this implies that:

\[
\lim_{L \to \infty} E \left[ \sup_{x \in \tilde{B}_1(0, L)} P_{x, \omega} \left[ X_{T_{B_2(0, \tilde{c}, L)}} \notin \partial^+ B_2(0, \tilde{c}, L) \right] \right] = 0. \quad (2.17)
\]

So as to prove (2.17), we argue by reduction ad absurdum, i.e. we assume that (2.17) does not hold. Then, there exists a sequence \((L_k)_{k \geq 0}\) of positive real numbers tending to \(\infty\) as \(k \to \infty\) such that (2.17) does not hold when the limit \(L \to \infty\) is switched by \(k \to \infty\) and \(L\) is switched by \(L_k\) in the expression inside of its expectation. Thus, there exists an \(\epsilon > 0\), such that for all \(L_k\), there exists \(L_k' > L_k\) with

\[
E \left[ \sup_{x \in \tilde{B}_1(0, L_k')} P_{x, \omega} \left[ X_{T_{B_2(0, \tilde{c}, L_k')}} \notin \partial^+ B_2(0, \tilde{c}, L_k) \right] \right] > \epsilon. \quad (2.18)
\]

The last statement makes able the construction of a subsequence \((L_{k_n})_{n \geq 0}\) of \((L_k)_{k \geq 0}\), satisfying the inequality (2.18) when \(L'\) is switched by \(L_{k_n}\). However, by the preceding discussion above (2.16), \((L_{k_n})_{n \geq 0}\) has a further subsequence \((L_{k_{nm}})_{m \geq 0}\) such that

\[
E \left[ \sup_{x \in \tilde{B}_1(0, L_{k_{nm}})} P_{x, \omega} \left[ X_{T_{B_2(0, \tilde{c}, L_{k_{nm}})}} \notin \partial^+ B_2(0, \tilde{c}, L_{k_{nm}}) \right] \right] \to 0,
\]

which is impossible given (2.18) and hence, we obtain the required contradiction.

**Corollary 2.5.** Under the notation and assumptions of Proposition 2.3. We have that there exists \(\tilde{c} > 1\) such that,

\[
\lim_{L \to \infty} E \left[ \sup_{x \in \tilde{B}_1(0, L)} P_{x, \omega} \left[ X_{T_{B_2(0, \tilde{c}, L)}} \notin \partial^+ B_2(0, \tilde{c}, L) \right] \right] = 0.
\]
3. Constructing the renormalization scheme producing stretched exponential decay

We main aim in this section will be to construct a re-scaling method turning out stronger or sharper estimates starting from weaker ones. Commonly, these type of theoretical constructions are called renormalization procedures. In order to the entire process works, we need a so-called seed estimate, along with an inductive estimate to pass from scale $k$ to $k+1$, for any positive integer number $k$. The seed estimate was the target in the previous section, thus we focus on the inductive step here. The final stronger control is turned out by localize a large parameter between consecutive scales $k$ and $k+1$. Whenever the inductive estimate has been a fruitful one, the final decay will be improved.

Let us begin with the formal development to obtain appropriate inductive estimates.

Similarly as in Section 2, throughout this section we fix a direction $\ell \in S^{d-1}$ and a rotation $R$ of $\mathbb{R}^d$ such that $R(e_1) = \ell$.

We introduce the successive dimensions of the boxes involved in the corresponding scales.

Specifically, we consider parameters $(L_k)_{k \geq 0}$ and $(\bar{L}_k)_{k \geq 0}$:

\begin{align*}
3\sqrt{d} < L_0 < L_1, & \quad N_0 := \frac{L_1}{L_0} = 1100d^3 \in \mathbb{N}, \quad (3.1) \\
3\sqrt{d} < \bar{L}_0 = L_0 < \bar{L}_1, & \quad \bar{N}_0 := \frac{\bar{L}_1}{L_0} = 11d^3N_0^2 \in \mathbb{N}, \quad (3.2) \\
\text{and for } k \geq 1, \text{ we define: } L_{k+1} = N_0L_k, & \quad \bar{L}_{k+1} = \bar{N}_0\bar{L}_k. \quad (3.3)
\end{align*}

Notice that we have for $k \geq 1$,

$$L_k = N_0^k L_0, \quad \bar{L}_k = \bar{N}_0^k \bar{L}_0, \text{ and } \bar{L}_k < L_3^3.$$ 

Further restrictions on the scaling sequences $(L_k)_{k \geq 0}$ and $(\bar{L}_k)_{k \geq 0}$ will be prescribed later on. Essentially, we require that $L_0 \geq c_1$, where the constant $c_1 > 0$ might depend in turn, on model constants, for instance $\kappa, d, g$ and $C$. Keeping the record of the precise value for the model constant $c$ is possible, nevertheless it is immaterial as we will remark once we have introduced our triggering condition in Definition 3.2.

With the purpose of applying Proposition 2.3, we recall the constant $\bar{c}$ introduced therein. We denote $\mathcal{L}_k$ for integer $k \geq 0$, the set:

$$\mathcal{L}_k := L_k \mathbb{Z} \times 3\bar{c}\bar{L}_k \mathbb{Z}^{d-1}.$$ 

Moreover, for integers $k \geq 0$ and $x \in \mathcal{L}_k$, we consider boxes $\bar{B}_1(x, \bar{c}, L_k, \bar{L}_k)$,
One can see that, |

**Definition 3.2** |

"containing \( \ell \) in (3.5) will be called "quasi-cover property".

Notice that in virtue of the choice for the scales in (3.1)-(3.3) and definition above, we have

\[
B_2(x, \bar{c}, L_0, \bar{L}_0) = B_2(x, \bar{c}, L_0), \quad \text{and} \quad \bar{B}_1(x, \bar{c}, L_0, \bar{L}_0) = \bar{B}_1(x, \bar{c}, L_0).
\]

As well, we introduce a further block \( \bar{B}_1(x, \bar{c}, L_k, \bar{L}_k) \), defined by

\[
\bar{B}_1(x, \bar{c}, L_k, \bar{L}_k) := R \left( x + (0, L_k) \times (0, 3\bar{c}\bar{L}_k)^d \right) \cap \mathbb{Z}^d. \tag{3.4}
\]

It will be also useful to consider the set of boxes in scale \( k \geq 0 \), denoted by \( \mathfrak{B}_k \) and defined as follows

\[
\mathfrak{B}_k := \left\{ B_2(x, \bar{c}, L_k, \bar{L}_k), \ x \in \mathfrak{L}_k \right\}.
\]

**Remark 3.1.** Let \( k \geq 0 \) be an integer and constant \( \bar{c} \) be as in Proposition 2.3.

It will be useful to note that the choice of scales given (3.1)-3.3 and the boxes \( \bar{B}_1(x, \bar{c}, L_k, \bar{L}_k) \), \( B_2(x, \bar{c}, L_k, \bar{L}_k) \) along with \( \bar{B}_1(x, \bar{c}, L_k, \bar{L}_k) \) constructed with those scales, has an important property:

For \( k \geq 1 \) and \( x \in \mathfrak{L}_k \), consider for fixed \( B_2(x, \bar{c}, L_k, \bar{L}_k) \), the set:

\[
\mathfrak{B}_{2,k,x} := \left\{ \bar{B}_1(y, \bar{c}, L_{k-1}, \bar{L}_{k-1}), \ y \in \mathfrak{L}_{k-1}, \right.
\]

\[
\text{such that } \bar{B}_1(y, \bar{c}, L_{k-1}, \bar{L}_{k-1}) \subset B_2(x, \bar{c}, L_k, \bar{L}_k) \right\}.
\]

One can see that,

\[
B_2(x, \bar{c}, L_k, \bar{L}_k) \subset \bigcup_{y \in \mathfrak{L}_{k-1}, \bar{B}_1(y, \bar{c}, L_{k-1}, \bar{L}_{k-1}) \in \mathfrak{B}_{2,k,x}} \bar{B}_1(x, \bar{c}, L_{k-1}, \bar{L}_{k-1}). \tag{3.5}
\]

The property prescribed in (3.5) will be called "quasi-cover property".

Recall that we now that by Proposition 2.3 or Corollary 2.5, we have that the following definition is fulfilled under transience in a neighbourhood of direction containing \( \ell \in \mathbb{S}^d \). The expression "a number \( L_0 \) large but finite" stands for "\( L_0 \) is a finite number larger than any prescribed constant of the RWRE model.”

**Definition 3.2 (Condition (3)) \( L_0, y, \bar{c}, \phi \).** We say that condition (3) \( L_0, y, \bar{c}, \phi \) is satisfied if there exists a large but finite number \( L_0 \) such that for \( y \in \mathfrak{L}_0 \) there exist a constant \( \bar{c} > 0 \) and a function \( \phi : (3\sqrt{d}, \infty) \to [0, \infty) \) with

\[
\lim_{L \to \infty} \phi(L) = 0,
\]
such that
\[
\mathbb{E} \left[ \sup_{x \in B_1(y, \tilde{c}, L_0)} P_{x, \omega} \left[ X_{T_{B_2(y, \tilde{c}, L_0)}} \notin \partial^+ B_2(y, \tilde{c}, L_0) \right] \right] \leq \phi(L_0).
\]

Notice that by Proposition 2.3, condition $(\mathfrak{T})_{L_0, y, \tilde{c}, \phi}$ is fulfilled under directional transience in a neighbourhood of $\ell \in \mathbb{S}^{d-1}$. However, it is straightforward to see that $y = 0 \in \mathbb{Z}^d$ is the worst case (in general $y \notin \mathbb{Z}^d$ when $y \in \mathcal{L}_0$). Thus under directional transience in a open set containing $\ell$, we do assume that there exists an $L_0$ large but finite such that for certain parameters $\tilde{c}$ and $\phi$ involved in Definition 3.2, condition $\mathfrak{T}_{L_0, y, \tilde{c}, \phi}$ is fulfilled for any $y \in \mathcal{L}_0$.

Throughout the rest of the section, we will assume condition $(\mathfrak{T})_{L_0, y, \tilde{c}, \phi}$ for any $y \in \mathcal{L}_0$ and we fix its parameters $L_0$, $\tilde{c}$ and $\phi$. We consider the sequences of scales $(L_k)_{k \geq 0}$ satisfying (3.1)-(3.3), where $L_0$ is as the previous number. Furthermore, for easy in the writing, we will use the following notation for $k \geq 0$,

\begin{align}
\tilde{B}_{1,k}(x) &:= \tilde{B}_1(x, \tilde{c}, L_k, \tilde{L}_k), \quad B_{2,k}(x) := B_2(x, \tilde{c}, L_k, \tilde{L}_k) \quad (3.6) \\
\check{B}_{1,k}(x) &:= \check{B}_1(x, \tilde{c}, L_k, \check{L}_k), \quad \partial^+ B_{2,k}(x) := \partial^+ B_2(x, \tilde{c}, L_k, \check{L}_k). \quad (3.7)
\end{align}

In the next definition we introduce the term of Good box in scale $k \geq 0$.

**Definition 3.3 (Good Box).** For $x \in \mathcal{L}_0$, we say that box $B_{2,0}(x)$ is $L_0$–Good if

\[
\sup_{x \in \tilde{B}_{1,0}(x)} P_{x, \omega} \left[ X_{T_{B_{2,0}(x)}} \notin \partial^+ B_{2,0}(x) \right] < \phi(\hat{L}_0).
\]

Otherwise, we say that the box $B_{2,0}(x)$ is $L_0$–Bad.

Recursively, we say that for $k \geq 1$ and $x \in \mathcal{L}_k$, the box $B_{2,k}(x)$ is $L_k$–Good if: There exists a box $B_{2,k-1}(y) \in \mathcal{B}_{k-1}$, $y \in \mathcal{L}_{k-1}$, with $\check{B}_{1,k-1}(y) \subset B_{2,k}(x)$, such that for any other box $B_{2,k-1}(z) \in \mathcal{B}_{k-1}$, with $z \in \mathcal{L}_{k-1}$, $\check{B}_{1,k-1}(y) \subset B_{2,k}(x)$ and $B_{2,k-1}(y) \cap B_{2,k-1}(z) = \emptyset$, we have that $B_{2,k-1}(z)$ is $L_{k-1}$–Good. Otherwise, we say that $B_{2,k}(x)$ is $L_k$–Bad.

As was mentioned in Remark 3 of [GVV19] and we shall establish in a remark below, the previous definition makes able to apply mixing assumptions. We also stress that, roughly speaking, for $k \geq 0$ and $x \in \mathcal{L}_k$, the box $B_{2,k}(x)$ is $L_k$–Good whenever there is at most one box $B_{2,k-1}(y)$, $y \in \mathcal{L}_{k-1}$ which is $L_{k-1}$–Bad and contained in $B_{2,k}(x)$. Nevertheless, neither $B_{2,k-1} \subset B_{2,k}$ nor there is at most one bad box contained in $B_{2,k}(x)$.

The next remark will be useful in several parts of the remaining section.

\[\text{imsart-generic ver. 2014/10/16 file: transientconjectureGuerra.tex date: June 2, 2020}\]
Remark 3.4. Notice that for integer \( k \geq 0 \) and \( x \in \mathcal{L}_k \), the environment event "the box \( B_{2,k}(x) \) is \( L_k - \text{Good} \)" depends at most on transitions in the set:

\[
B_{k,x} := R \left( x + \left( -A_k, L_k + \frac{A_k}{11} \right) \times \left( -\tilde{c}A_k, 3\tilde{c}L_k + c\tilde{A}_k \right)^{d-1} \right) \cap \mathbb{Z}^d, \quad (3.8)
\]

where \( A_k := \sum_{i=0}^{k} L_i \) and \( \tilde{A}_k := \sum_{i=0}^{k} \tilde{L}_i \).

Moreover, we observe that for a box \( B_{2,k}(x) \) as above, the number of boxes in \( \mathcal{B}_k \) intersecting it along a straight line along direction \( \ell = R(e_i) \) is five: two at each direction \( \pm \ell \) point out, besides itself. The remaining of the boxes \( B_{2,k}(y) \), with centre \( y \in \mathcal{L}_k \) out of the slab:

\[
\mathcal{H}_{x,k,1} := \{ z \in \mathbb{R}^d : |(z-x) \cdot \ell| \leq (5/2)L_k \},
\]

are at least separated an \( \ell^1 \)– distance of \((10/11)L_k\). Thus, this makes able to apply mixing assumptions on the environment (cf. Definition 1.2).

Analogously, for straight line through direction \( R(e_i) \), where the integer \( i \in [2,d] \) there exist at most three boxes in \( \mathcal{B}_k \) intersecting \( B_{2,k}(x) \). The remaining boxes with centres out of the slab:

\[
\mathcal{H}_{x,k,i} := \{ z \in \mathbb{R}^d : (-1/2) < (z-x) \cdot R(e_i) < (7/2)L_k \}
\]

are at least separated an \( \ell^1 \)– distance of \( \tilde{L}_k \).

On the other hand, we plainly have that for any integer \( k \geq 1 \),

\[
A_{k-1} \leq (1/11)L_k, \quad \tilde{A}_{k-1} \leq (1/11)\tilde{L}_k.
\]

As a result of the precedent discussions, for \( k \geq 1 \) any disjoint boxes \( B_{2,k-1}(y_1) \), \( B_{2,k-1}(y_2) \) where the points \( y_1, y_2 \in \mathcal{L}_{k-1} \) in the quasi-cover of \( B_{2,k} \) (cf. Remark 3.1), its respective set of site transitions:

\[
B_{k-1,y_1} \text{ and } B_{k-1,y_2},
\]

are at least separated in \( \ell^1 \)– a distance of \((9/11)L_k\). This remark will be extremely important when we need to apply mixing condition (1.2).

Recall that we are assuming condition \((\Sigma)\mathcal{L}_0, y, \tilde{c}, \phi\) and this tacitly implies that \( L_0 \) is greater than any prescribed positive constant of the model. A starting point will be provided by the next proposition.

Proposition 3.5. Let \( k \) be a non-negative integer and \( x \in \mathcal{L}_k \). For \( k=0 \), and any \( x \in \mathcal{L}_k \), we have that

\[
P \left[ B_{2,k}(x) \text{ is } L_k - \text{Bad} \right] \leq \phi^3(L_0). \quad (3.9)
\]

Further, for \( k \geq 1 \) we have that there exists a constant \( \eta_1 := \eta_1(d, L_0) > 0 \) such that for any \( x \in \mathcal{L}_k \),

\[
P \left[ B_{2,k}(x) \text{ is } L_k - \text{Bad} \right] \leq e^{-\eta_1 \cdot 2^k}. \quad (3.10)
\]
that for the choice of $L_\nu$ afterwards, we shall prove that there exist integer $k$ for this end, it will be convenient to prove by induction that we have for any integer $k \geq 0$ and $x \in \mathcal{L}_k$, the inequality:

$$
P [B_{2,k}(x) \text{ is } L_k - \text{Bad}] \leq e^{-ck2^k},$$

where the sequence $(c_k)_{k \geq 0}$ is defined as follows (recall constants $C, g$ and $r$ in Definition 1.2):

$$c_0 := \ln \left( \frac{1}{\phi^2(L_0)} \right), \quad \text{and for } k \geq 0$$

$$c_{k+1} := c_k - \frac{\ln(\lambda(d))}{2^{k+1}} - \frac{\exp (-g(9/11)L_k)9r^{2d}L_k^2(6\tilde{c}\tilde{L}_k)^{2(d-1)}C}{2^{k+1}}.$$

Afterwards, we shall prove that there exist $\nu_1 > 0$ and $\nu_2 := \nu_2(\nu_1) > 0$, such that for the choice of $L_0 > \nu_1$ turns out

$$\inf_{k \geq 0} c_k > \nu_2,$$

and this will end our proof. Notice that the case $k = 0$ was already proven, thus we have to prove the inductive step.

Therefore, we assume that (3.11) holds for $k \geq 0$ and we will see that (3.11) is satisfied when $k$ is replaced by $k + 1$. Note that it is enough to prove it under the assumption of $x = 0 \in \mathbb{Z}^d$.

Observe now that using Definition 3.3, the environment event "$B_{2,k+1}(0)$ is $L_{k+1} - \text{Bad}$" is contained in the following event:

$$\mathfrak{M}_k := \{ \exists B_{2,k}(y_1), B_{2,k}(y_2) \in \mathfrak{B}_k : \hat{B}_{1,k}(y_1), \hat{B}_{1,k} \subset B_{2,k+1}(0),$$

$$B_{2,k}(y_1) \cap B_{2,k}(y_2) = \emptyset, B_{2,k}(y_1), B_{2,k}(y_2) \text{ are } L_k - \text{Bad} \}. \quad (3.13)$$

We apply Remark 3.4, together with Definition 1.2 to find that, $P[\mathfrak{M}_k]$ is bounded from above by:

$$\sum_{(y_1, y_2) \in \mathcal{N}_{2,k}} \mathcal{F}_M(y_1, y_2) P[B_{2,k}(y_1) \text{ is } L_k - \text{Bad}] P[B_{2,k}(y_2) \text{ is } L_k - \text{Bad}],$$

provided we define the set $\mathcal{N}_{2,k}$, as follows:

$$\mathcal{N}_{2,k} := \left\{ (z_1, z_2) \in \mathbb{Z}^d \times \mathbb{Z}^d : \hat{B}_{1,k}(z_1), \hat{B}_{1,k}(z_2) \subset B_{2,k+1}(0),$$

$$B_{2,k}(z_1) \cap B_{2,k}(z_2) = \emptyset \right\},$$

along with for $(y_1, y_2) \in \mathcal{N}_{2,k}$, we define the mixing correction $\mathcal{F}_M(y_1, y_2)$ by (cf. Definitions 1.1 and 1.2 for notation),

$$\mathcal{F}_M(y_1, y_2) := \exp \left( \sum_{z_1 \in \partial' B_{2,k}(y_1)} \sum_{z_2 \in \partial' B_{2,k}(y_2)} C e^{-g|y_1 - y_2|_1} \right).$$
Where in turn, we have assumed $L_0 > 10r$ (cf. Definition 1.1), in order to apply the mixing assumption of Definition 1.2.

Let us remark that for the induction hypothesis (3.11) applied twice, we get the estimate:

$$\mathbb{P} \left[ B_{2,k}(y_1) \text{ is } L_k - \text{Bad} \right] \mathbb{P} \left[ B_{2,k}(y_2) \text{ is } L_k - \text{Bad} \right] \leq e^{-c_k 2^{k+1}},$$

(3.15)

for each $(y_1, y_2) \in \mathcal{N}_{2,k}$. On the other hand, we use a rough counting argument to obtain the following upper bounds,

$$|\mathcal{N}_{2,k}| \leq \left( \frac{5}{3} \tilde{N}_0 \right)^{d-1} \times \left( \frac{23}{11} N_0 \right) =: \lambda(d)$$

(3.16)

$$\mathcal{F}_M \leq \exp \left( g(9/11) L_k \right) 9 r^{2d} L_k^2 (6 \tilde{c} \tilde{L}_k)^{2(d-1)C},$$

where for a set $A \subset \mathbb{Z}^d$, we denote by $|A|$ its cardinality. Let us stress that the last bound is uniform on $(y_1, y_2) \in \mathcal{N}_{2,k}$.

We combine the discussion leading to define (3.13), the estimates in (3.16) and the induction hypothesis in inequality (3.15) to get that $\mathbb{P} \left[ B_{2,k+1}(0) \text{ is } L_{k+1} - \text{Bad} \right]$ is bounded by above by:

$$\exp \left( -2^{k+1} \left( c_k - \frac{\ln(\lambda(d))}{2^{k+1}} - \frac{\exp \left( -g(9/11)L_k \right) 9 r^{2d} L_k^2 (6 \tilde{c} \tilde{L}_k)^{2(d-1)C} \right)}{2^{k+1}} \right).$$

On the other hand, by the very definition of the constants $c_k$, $k \geq 0$ in (3.12), we have finished the proof of (3.11). As was mentioned, it is convenient at this point to find positive constants $\nu_1$ and $\nu_2$ such that:

$$\inf_{k \geq 0} c_k > \nu_2,$$

(3.17)

whenever $L_0 \geq \nu_1$. Nevertheless, note that whenever $L_0$ is chosen so that (recall $L_0 = \tilde{L}_0$, cf. (3.2)):

$$\exp \left( -g(9/11)L_0 \right) 9 \tilde{L}_0^2 (6 \tilde{c} \tilde{L}_0)^{2(d-1)C} < e^{-g(1/30)L_0}$$

one has the following estimate for the series entering at the definition of sequence $(c_k)_{k \geq 0}$ in (3.12),

$$\inf_{k \geq 0} c_k \geq c_0 - \left( \sum_{k=1}^{\infty} \ln(\lambda(d)) + e^{-g(1/30)L_0} \right)$$

$$\ln \left( \frac{1}{\phi^2(L_0)} \right) - \left( \ln(\lambda(d)) + e^{-g(1/30)L_0} \right).$$

Thus, we plainly have that there exists certain choice of constants $\nu_1, \nu_2 > 0$, providing the inequality (3.17) whenever $L_0 > \nu_1$. This ends the proof of all the required claims in the proposition.
The next step into the renormalization construction will be to obtain a quenched estimate for the random walk exit from a given Good box in $\mathfrak{B}_k$, for each integer $k \geq 0$. This is the harder and more extensive part of our proof. As the proof shall depict, a more involved argument will be needed, when it is compared to the one given in [GVV19], Proposition and Section 5. Roughly speaking, in order to bound from above the unlikely exit by the boundary side where $-\ell$ point out, we avoid here the use of uniform ellipticity prescribed in (1.1), instead we shall successively apply the strong Markov property.

**Proposition 3.6.** Let $k$ be a non-negative integer and $x \in \mathfrak{L}_k$. Assume that the box $B_{2,k}(x)$ is $L_k-$ Good, then there exists a constant $\eta_2 := \eta_2(d,L_0) > 0$ such that

$$
\sup_{y \in \tilde{B}_{1,k}(x)} P_{y,\omega} \left[ X_{T_{B_{2,k}(x)}} \notin \partial^+ B_{2,k}(x) \right] \leq e^{-\eta_2 \nu_k},
$$

where $\nu_k := \frac{N_2}{4}$

**Proof.** Let us prove by using induction the following claim:

Let $(c_k)_{k \geq 0}$ be a sequence defined by:

$$
c_k := \frac{1}{4^k L_0} \ln \left( \frac{1}{\phi^+(L_0)} \right), \quad k \geq 0.
$$

Then, for any $k \geq 0$ and $x \in \mathfrak{L}_k$ we have that,

$$
\sup_{y \in \tilde{B}_{1,k}(x)} P_{y,\omega} \left[ X_{T_{B_{2,k}(x)}} \notin \partial^+ B_{2,k}(x) \right] \leq e^{-c_k L_k}.
$$

We then see that the assertion of the proposition is implied by this claim, with constant $\eta_2 := L_0 c_0$.

We prove the required claim given in (3.20) by induction on $k$. The case $k = 0$ and $x \in \mathfrak{L}_0$ is straightforward using Definition 3.3, we indeed have the estimate,

$$
\sup_{y \in \tilde{B}_{1,k}(x)} P_{y,\omega} \left[ X_{T_{B_{2,k}(x)}} \notin \partial^+ B_{2,k}(x) \right] < e^{-\ln \left( \frac{1}{\phi^+(L_0)} \right)} = e^{-c_0 u^0}.
$$

As a result, it suffices that we assume that (3.20), and prove the analogous estimate (3.20) when $k$ is switched by $k + 1$.

Furthermore, we notice that by stationarity of the probability measure $\mathbb{P}$, the worst case to estimate (3.18) is $x = 0$, other points in $\mathfrak{L}_k$ does not belong to $\mathbb{Z}^d$ necessarily. Thus we can a do assume $x = 0$ and consider the left expression in (3.20) when $k$ is replaced by $k + 1$, we assume that the box $B_{2,k+1}(0)$ is $L_{k+1} -$ Good as well. We introduce the $(\mathcal{F}_n)_{n \geq 0} -$ stopping times $\sigma^+_u$ and $\sigma^-_u$ for $u \in \mathbb{R}$ and integer $i \in [2,d]$

$$
\sigma^+_u := \inf \{ n \geq 0 : (X_n - X_0) \cdot R(e_i) \geq u \}, \quad \sigma^-_u := \inf \{ n \geq 0 : (X_n - X_0) \cdot R(e_i) \leq u \}.
$$

(3.21)
It will be convenient to introduce the path space event of lateral exit from the box $B_{2,k+1}(0)$, denoted by $I_k$ and defined by (recall notation (1.3) and (2.5))

$$I_k := \left\{ \exists i \in [2,d]: \sigma^+_{i\tilde{c}_{L_{k+1}}} < T_{B_{2,k+1}(0)}, \text{ or } \sigma^-_{-\tilde{c}_{L_{k+1}}} < T_{B_{2,k+1}(0)} \right\}.$$ 

Observe that the following decomposition for any $y \in \tilde{B}_{1,k+1}(0)$,

$$P_{y,\omega}[X_{T_{B_{2,k+1}(0)}} \notin \partial^+ B_{2,k+1}(0)] \leq P_{y,\omega}[I_k] + P_{y,\omega}[I^c_k \cap \{X_{T_{B_{2,k+1}(0)}} \cdot \ell \leq -L_{k+1}\}],$$

holds. At this point of our argument, we use the induction hypothesis to split the proof into getting suitable upper bounds for the expressions:

$$P_{y,\omega}[I_k], \text{ along with, (3.23)}$$

$$P_{y,\omega}[I^c_k \cap \{X_{T_{B_{2,k+1}(0)}} \cdot \ell \leq -L_{k+1}\}].$$

We begin with an estimate for the probability in (3.23). Notice first that for arbitrary $y \in \tilde{B}_{1,k+1}(0)$, we can further decompose that probability as follows:

$$P_{y,\omega}[I_k] \leq \sum_{i=2}^{d} \left( P_{y,\omega}[\sigma^+_{i\tilde{c}_{L_{k+1}}} < T_{B_{2,k}(0)}] + P_{y,\omega}[\sigma^-_{i\tilde{c}_{L_{k+1}}} < T_{B_{2,k}(0)}] \right).$$

Following a close analysis as the argument to prove Proposition 5 in [GVV19], we will obtain an upper bound for the probability:

$$P_{y,\omega}[\sigma^+_{2\tilde{c}_{L_{k+1}}} < T_{B_{2,k}(0)}].$$

The other terms inside the sum in (3.25) could be similarly bounded. Indeed, our method will display that all of them have the same upper bound. In order to bound the probability in (3.26), it will be useful to set

$$n_k := \frac{23}{11}N_0 + 1$$

Let us indicate that $n_k$ is the amount of successive boxes $B_{2,k}(z)$, $z \in \mathcal{L}_k$ along a straight line along direction $\ell$, such that $B_{1,k}(z) \subset B_{2,k+1}(0)$. We introduce as well, integer parameters $J_k$ and $T_k$ defined by:

$$J_k := \left\lceil \frac{N_0}{4(n_k + 1)} \right\rceil,$$ 

and

$$T_k := 4\tilde{c}(n_k + 1)\tilde{L}_k.$$ 

Since $J_kT_k \leq \tilde{c}\tilde{L}_{k+1}$, for any $y \in \tilde{B}_{1,k+1}(0)$ one sees that $P_{y,\omega}$-- a.s.

$$\{\sigma^+_{J_k\tilde{c}_{L_{k+1}}} < T_{B_{2,k+1}(0)}\} \subset \{\sigma^+_{2\tilde{c}_{L_{k+1}}} < T_{B_{2,k+1}(0)}\}. $$

(3.29)
For integer $j > 0$, we introduce the set $c_\perp(j, k)$ defined by
\[ c_\perp(j, k) := \{ z \in \mathbb{Z}^d : z \cdot R(e_2) \in \mathcal{L}_k[j, j+1], z \in B_{2,k+1}(0) \}. \tag{3.30} \]

Let us denote by $| \cdot |_\perp$ the semi-norm on $\mathbb{R}^d$ given by
\[ |w|_\perp := \sup_{i \in [2,d]} |R(e_i)|, \text{ for } w \in \mathbb{R}^d. \]

It will be useful to introduce as well, for $j \in \mathbb{Z}$ the cylinder set,
\[ \mathcal{L}(j, k) := \{ z \in \mathbb{Z}^d : \inf_{w \in c_\perp(j, k)} |z - w|_\perp \leq cn_k \mathcal{L}_k, z \in B_{2,k+1}(0) \}. \tag{3.31} \]

Throughout the remaining of this part in the proof, and for easy of notation when $u \in \mathbb{R}$, we will denote by $\sigma_u$, the stopping time $\sigma_u^+$. Observe that applying the strong Markov property, for an arbitrary $y \in B_{1,k+1}(0)$ we get (recall notation introduced in (3.21)),
\[ P_{y,\omega} \left[ \sigma_{\mathcal{L}_k+1} < T_{B_{2,k+1}(0)} \right] \leq P_{y,\omega} \left[ \sigma_{J_k \mathcal{L}_k} < T_{B_{2,k+1}(0)} \right] \leq P_{y,\omega} \left[ \sigma_{(J_k-2) \mathcal{L}_k} < T_{B_{2,k+1}(0)}, \ P_{X_{\sigma_{(J_k-2) \mathcal{L}_k}},\omega} \left[ \sigma_{2 \mathcal{L}_k} < T_{B_{2,k+1}(0)} \right] \right]. \tag{3.32} \]

We will need some further terminology so as to introduce a useful remark, providing a suitable strategy to get an upper bound recursively on the expression (3.32). Let us first recall that under definitions (3.1)-(3.3), together with the boxes introduced in (3.6), we have that for any given integer $k \geq 0$, $\mathcal{B}_k = \mathbb{Z}^d$.

We let $\theta$ be the canonical time shift function on the random walk process $(X_n)_{n \geq 0}$, i.e. for natural numbers $n$, we define $\theta_n : (\mathbb{Z}^d)^N \to (\mathbb{Z}^d)^N$, such that $\theta_n((X_m)_{m \geq 0}) = (X_{m+n})_{m \geq 0}$.

We introduce a sequence of stopping times $(H^i)_{i \geq 0}$, along with sequences $(Z^i)_{i \geq 0}$ and $(Y^i)_{i \geq 0}$ denoting successive random positions of the random walk, as follows
\begin{align*}
H^0 &= 0, \ Z_0 = X_0, \ Y_0 = \text{ an arbitrary point in } \{ z \in \mathcal{L}_k : Z_0 \in \mathcal{L}_k(z) \}, \\
H^1 &= T_{B_{2,k+1}(0)} \cap T_{B_{2,k}(Y_0)}, \ Z_1 = X_{H^1}, \ Y_1 = \text{ an arbitrary point in } \{ z \in \mathcal{L}_k : Z_1 \in \mathcal{L}_k(z) \}. \\
\text{Moreover, we recursively define for integer } i > 1, \\
H^i &= H^{i-1} + H^1 \circ \theta_{H^{i-1}}, \ Z_i = X_{H^i}, \ Y_i = \text{ an arbitrary point in } \{ z \in \mathcal{L}_k : Z_i \in \mathcal{L}_k(z) \}. \tag{3.33}\end{align*}

We introduce as well, the $(\mathcal{F}_n)_{n \geq 0}$—stopping time $S$ defined as
\[ S = \inf \{ n \geq 0 : X_n \in \partial B_{2,k}(Y_0) \setminus \partial^+ B_{2,k}(Y_0) \}. \]

We can thus establish the following crucial remark for our proof’s method:
Remark 3.7. For arbitrary $\omega \in \Omega$, $j$ an integer and $k$ fixed as above, we let $z \in c_\perp(j, k)$ and introduce the path space event

$$\mathcal{L}_{k, j, z} := \bigcap_{i=0}^{n_k-1} \{ X_0 = z, \theta_{H_i}^i \{ H^1 < S \} \},$$

since the height along direction $R(e_2)$ after successive exit from boxes of scale $k$ composing a quasi-cover of $B_{2, k+1}(0)$ (cf. Remark 3.1) is at most $4\tilde{c}_k n_k < 4\tilde{c}_k (n_k + 1)$ (see definitions (3.30 and (3.31))), for arbitrary $z_1 \in c_\perp(j - 2, k)$ we plainly see that the inequality,

$$P_{z_1, \omega} \left[ \sigma_{2T_{k}} \geq T_{B_{2, k+1}(0)} \right] \geq P_{z_1, \omega} \left[ \mathcal{L}_{k, j - 2, z_1} \right] \quad (3.34)$$

holds.

It will be convenient to introduce some definitions so as to get a suitable lower bound for the right hand side of inequality (3.34). We fix $k \geq 0$ as above, and thus for integer $i \in [0, n_k - 1]$ and $j \in \mathbb{Z}$, we define the set $\Theta_{i, j} \subset \mathbb{Z}^d$,

$$\Theta_{i, j} := \{ z \in \mathbb{Z}^d : \exists w \in \mathcal{L}_k, z \in \tilde{B}_{1, k}(w) \} \cdot \ell = -L_{k+1} + iL_k, B_{2, k}(w) \subset c_\perp(j, k) \}.$$  

We then introduce the random variables

$$\phi_{i, j} := \inf_{x \in \tilde{B}_{1, k}(w)} P_{x, \omega}[S > H^1],$$

$$\zeta_j := \inf_{x \in c_\perp(j, k)} P_{x, \omega}[\mathcal{L}_{k, j, x}]$$

and notice that a successive application of the strong Markov property on (3.34), for any arbitrary point $z_1 \in c_\perp(j - 2, k)$, provides us with the following estimate

$$P_{z_1, \omega} \left[ \sigma_{2T_{k}} \geq T_{B_{2, k+1}(0)} \right] \geq \left( \prod_{i=0}^{n_k-1} \phi_{i, j - 2} \right) \lor (\zeta_{j - 2}).$$

Hence, we obtain as a result

$$P_{z_1, \omega} \left[ \sigma_{2T_{k}} < T_{B_{2, k+1}(0)} \right] \leq \left( 1 - \prod_{i=0}^{n_k-1} \phi_{i, j - 2} \right) \land (1 - \zeta_{j - 2}) =: \varphi(j - 2).$$

On the other hand, it is a matter of fact that for any $y \in \tilde{B}_{1, k+1}(0)$, we have $P_{0, \omega} - a.s.$

$$X_{\sigma_{(j - 2)\tilde{T}_k}} \in c_\perp(j - 2, k).$$

We combine in turn the estimate (3.35) with the inequality established in (3.32) and the fact above, to get that

$$P_{y, \omega}[\sigma_{(j - 2)\tilde{T}_k} < T_{B_{2, k+1}(0)}]$$

$$\leq P_{y, \omega}[\sigma_{j_k T_k} < T_{B_{2, k+1}(0)}]$$

$$\leq P_{y, \omega}[\sigma_{(j - 2)\tilde{T}_k} < T_{B_{2, k+1}(0)}] \varphi(j - 2),$$

$$\quad (3.36)$$
for any $y \in \tilde{B}_{1,k+1}(0)$.

Furthermore, we observe that the strategy prescribed by Remark 3.7, when is applied at $z_1 \in c_\bot (J_k - 5, k)$ instead of $z_1 \in c_\bot (J_k - 2, k)$, in virtue of the precedent inequality (3.36), we obtain

$$P_{y,\omega}[\sigma_{J_k T_k} < T_{B_{2,k+1}(0)}] \leq P_{y,\omega}[\sigma_{(J_k - 5) T_k} < T_{B_{2,k+1}(0)}] \varphi(J_k - 2) \varphi(J_k - 5).$$

Therefore, by an induction argument we find that (recall that $[\cdot] : \mathbb{R} \mapsto \mathbb{Z}$ denotes the integer part function)

$$P_{y,\omega}[\sigma_{\tilde{c}_{L_{k+1}}} < T_{B_{2,k+1}(0)}] \leq \prod_{i=0}^{[(J_k-2)/3]-1} \varphi(J_k - 2 - 3i), \quad (3.37)$$

where the factors inside the product are random variables on the environment, which in turn depends on disjoint transition sites. The last remark will be important to apply the induction hypothesis, since we are assuming that box $B_{2,k+1}(0)$ is $L_{k+1} - \text{Good}$ as in the hypothesis of the Proposition.

We indeed apply the induction hypothesis of each factor into the product on the right hand side of inequality (3.37), besides the at most two possible factors containing the at most three $L_k - \text{Bad}$ boxes along direction $R(e_2)$ (see Remark 3.4), which will bound by one. Explicitly, we have

$$\varphi(J_k - 2 - 3i) \leq 1 - \left(1 - e^{-c_k L_k}\right)^{nk},$$

for those integers $i \in [0, [(J_k - 2)/3] - 1]$ such that all of the boxes $B_{2,k}(w), w \in \mathcal{L}_k$ contained $\sigma(J_k - 2 - 3i, k)$ are $L_k - \text{Good}$. Otherwise, i.e. for those possibly two integers $i$ such that some of them contain the at most three $L_k - \text{Bad}$ boxes, we use the bound,

$$\varphi(J_k - 2 - 3i) \leq \sup_{x \in c_\bot (J_k - 2 - 3i, k)} P_{x,\omega}[\mathcal{L}_{k,J_k - 2 - 3i,x}] \leq 1.$$

Furthermore, using the inequality $1 - q^n \leq n(1 - q)$ for $q \in (0, 1)$, the aforementioned procedure leads us to find that,

$$P_{y,\omega}[\sigma_{\tilde{c}_{L_{k+1}}} < T_{B_{2,k+1}(0)}] \leq \prod_{i=0}^{[(J_k-2)/3]-3} (n_k e^{-c_k L_k}) \leq \exp\left(-(J_k/8)(c_k L_k - \ln(n_k))\right)$$

for an arbitrary point $y \in \tilde{B}_{1,k+1}(0)$.

As was mentioned at the beginning of the estimate for the term in direction $R(e_2)$ of the sum in (3.25), the previous upper bound is also satisfied for the
other directions in the set \( \{ \pm R(e_i), i \in [2, d] \} \) entering at inequality (3.25). Thus, our final estimate for the probability in (3.23) is the following,

\[
P_{y,\omega}[I_k] \leq \exp \left( -(J_k/8)(c_kL_k - \ln(2(d-1)n_k))\right),
\]

(3.38)

for arbitrary \( y \in \tilde{B}_{1,k+1}(0) \).

We now turn to estimate the probability displayed in (3.24). The main strategy will be the introduction of Markov chain techniques to avoid the use of uniform elliptic assumption (1.1). The method will improve the analogous estimate in [GVV19], Proposition 5 of Section 5. By Definition 3.3, one can pick a box \( B_{2,k}(y) \), \( y \in \mathcal{Y}_k \) composing the quasi-cover of \( B_{2,k+1}(0) \) (cf. Remark 3.1), such that any other box composing the quasi-cover of box \( B_{2,k+1}(0) \) and not intersecting box \( B_{2,k}(y) \), is \( L_k - \text{Good} \). Thus, let us start by introducing some terminology which localizes the box prescribed by Definition 3.3 in a certain subset of \( B_{2,k+1}(0) \).

We define \( \mathcal{B}_{k,i} \) for integer \( i \in [1, N_0] \) and \( k \) fixed as above, the set of boxes in \( \mathcal{B}_k \) at position \( i \) towards direction \( -\ell \) points out, as follows

\[
\mathcal{B}_{k,i} := \{ B_{2,k}(w), w \in \mathcal{Y}_k, \ w \cdot \ell = -iL_k, \ \tilde{B}_{1,k}(w) \subset B_{2,k+1}(0) \}.
\]

By hypothesis \( B_{2,k+1}(0) \) is \( L_k - \text{Good} \), thus Remark 3.4 says that there exist at most five consecutive integers \( i \in [1, N_0] \), such that the sets \( \mathcal{B}_{k,i} \) contain \( L_k - \text{Bad} \) boxes, and all another box composing a quasi-cover as in Remark 3.1 is \( L_k - \text{Good} \). Therefore, in the worst case of Definition 3.3, we can choose an index \( i \in [1, N_0] \) so that the sets \( \mathcal{B}_{k,i} \), with \( i \in [\tilde{i}, \tilde{i} + 4] \) contain the at most five bad boxes along direction \( \ell \).

Note that there exists a further case, i.e. when the bad boxes along direction \( \ell \) are located toward \( +\ell \) points out, nevertheless our argument will show that in this case the estimates are sharper (cf. (3.45), comments below (3.47) and Remark 3.8).

We split the argument into three cases:

(i) Case \( \tilde{i} \in [N_0 - 9, N_0] \).

In this case, we use the following inequality

\[
P_{y,\omega}[I_k, \ X_{T_{B_{2,k+1}(0)}}, \ell \leq -L_{k+1} \} \leq P_{y,\omega}[I_k, \ X_{T_{B_{2,k+1}(0)}}, \ell \leq -(N_0 - 9)L_k \}]
\]

(3.39)

which is fulfilled for any \( y \in \tilde{B}_{1,k+1}(0) \). We need to introduce some further definitions in order to provide an upper bound based on one-dimensional exact formulae. Recall that we have a given box \( B_{2,k+1}(0) \) which is \( L_{k+1} - \text{Good} \), an arbitrary point \( y \in \tilde{B}_{1,k+1}(0) \), we are assuming the induction hypothesis (3.20) and thus the integer \( k \) is fixed. For \( i \in \mathbb{Z} \), we define the strip \( \mathcal{H}_i \) by

\[
\mathcal{H}_i := \{ x \in \mathbb{Z}^d : \exists z \in \mathbb{Z}^d \ |x - z|_1 = 1, \ (z - iL_k)(x - iL_k) \leq 0 \}.
\]
Furthermore, we introduce the truncated strip \( \tilde{\mathcal{H}}_i \), defined by \((y \in \tilde{B}_{1,k+1}(0)\) is fixed as above

\[
\tilde{\mathcal{H}}_i := \left\{ x \in \mathbb{Z}^d : \forall i \in [2,d] |(x-y) \cdot R(e_i)| < \tilde{c}L_{k+1} \right\}
\]  

(3.40)

We also define a function \( I : \mathbb{Z}^d \to \mathbb{Z} \) such that \( I(z) = i \) on \( \{ x \in \mathbb{Z}^d : x \cdot \ell \in [iL_k-(L_k/2),iL_k+(L_k/2)) \} \). Notice that under our choice of \( L_0 \) in (3.1), we have \( I(z) = i \) for \( z \in \mathcal{H}_i \). It will be useful as well to introduce a sequence \((V_n)_{n \geq 0}\) of \((\mathcal{F}_n)_{n \geq 0}\) stopping times, recording the successive visits to different strips \( \mathcal{H}_i \), \( i \in \mathbb{Z} \). We define recursively,

\[
V_0 = 0, \ V_1 = \inf \{ n \geq 0 : X_n \in \mathcal{H}_{I(X_0)+1} \cup \mathcal{H}_{I(X_0)-1} \}, \text{ and for } j > 1 \ V_j = V_{j-1} + V_1 \circ \theta_{V_{j-1}}.
\]

We define random variables \( P_z \) and \( Q_z \),

\[
P_z(\omega) := P_{z,\omega}[X_{V_1} \in \mathcal{H}_{I(X_0)+1}] \text{ and } Q_z(\omega) := P_{z,\omega}[X_{V_1} \in \mathcal{H}_{I(X_0)-1}].
\]

for \( z \in \mathbb{Z}^d \) (notice that \( P_z(\omega) + Q_z(\omega) = 1 \)). For integer \( i \) we further define the random variable \( \rho_i \), via

\[
\rho_i(\omega) := \sup\left\{ \frac{Q_z(\omega)}{P_z(\omega)} : z \in \tilde{\mathcal{H}}_i \right\}.
\]

(3.41)

For fixed \( \omega \in \Omega \) and \( w_0 := N_0(1+(1/11)) \), let us now introduce a function \( f_\omega : \mathbb{Z} \to (0, \infty) \) such that

\[
f_\omega(j) = 0, \text{ for } j \geq w_0 + 1.
\]

\[
f_\omega(j) = \sum_{j \leq n \leq w_0 \atop n < m \leq w_0} \rho_{m-1}^{-1}(\omega) \text{ otherwise.}
\]

(3.42)

Since the environment \( \omega \) will remain fixed along the proof, with a little abuse of notation, we denote by \( P_z \), \( Q_z \) and \( \rho_i \) the values of the same functions in \( \omega \). We also drop \( \omega \) from the environmental function \( f_\omega \). In these terms, we claim that

\[
P_{z,\omega}[T_{\infty}, \{ XT_{B_{2,k+1}(0)} : \ell \leq -(N_0-9)L_k \}] \leq \frac{f(0)}{f(-(N_0-9))}.
\]

(3.43)

for an arbitrary point \( z_1 \in \tilde{\mathcal{H}}_0 \) (recall that \( \tilde{\mathcal{H}}_0 \) depends on \( y \in \tilde{B}_{1,k+1}(0) \), see (3.40)).

In order to prove claim (3.43), we introduce the \((\mathcal{F}_{V_n})_{n \geq 0}\) stopping time

\[
\tau := \inf\{ n \geq 0 : X_{V_n} \in \mathcal{H}_{-(N_0-9)} \cup \mathcal{H}_{N_0(1+(1/11))} \},
\]

along with the \((\mathcal{F}_n)_{n \geq 0}\) stopping time \( \tilde{T}_y \),

\[
\tilde{T}_y = \inf \{ n \geq 0 : |(X_n-y) \cdot R(e_j)| \geq \tilde{c}L_{k+1} \text{ for some } j \in [2,d] \}.
\]

(3.44)
Observe that for each point \( \vartheta \) we have
\[
\vartheta\left(I(X_{m\wedge\tau})\right), \ V_{m\wedge\tau}\leq \tilde{T}_y ,
\]
and we assert it is decreasing in \( m \in \mathbb{N} \). Indeed, we decompose the expression defining \( \vartheta(m+1) \) as follows,
\[
\vartheta(m+1)\leq E_{z_{1},\omega}\left[f(I(X_{m\wedge\tau}))\right], \ V_{m\wedge\tau}\leq \tilde{T}_y , \ \tau \leq m
\]
\[+ E_{z_{1},\omega}\left[f(I(X_{m+1})), \ V_{m}\leq \tilde{T}_y , \ \tau > m\right].
\]
Nevertheless, we note that by the strong Markov property we obtain
\[
E_{z_{1},\omega}\left[f(I(X_{m+1})), \ V_{m}\leq \tilde{T}_y , \ \tau > m\right]
\leq E_{z_{1},\omega}\left[\tau > m, \ V_{m}\leq \tilde{T}_y , \ E_{X_{m}}[f(I(X_{1})] \right].
\]
In turn, a routine computation shows that on the set \( \{ \tau > m, \ V_{m}\leq \tilde{T}_y \} \), we have \( P_{z_{1},\omega}\) a.s.
\[
E_{X_{m}}[f(I(X_{1})] = P_{X_{m}} (f(I(X_{m}) + 1)) + Q_{X_{m}} (f(I(X_{m}) - 1))\]
\[= f(I(X_{m})) + \prod_{I(X_{m})<j\leq w_{0}} \rho_{-1}^{-1} \left[-P_{X_{m}} + Q_{X_{m}} \rho_{-1}^{-1} \right].\]
Since \( P_{z_{1},\omega}\) a.s. one has that \( X_{m} \in \mathcal{H}_{I(X_{m})} \) for \( m \geq 0 \) and the very definition of the random variable \( \rho_{i} \) for \( i \in \mathbb{Z} \) (cf. (3.41)), the expression inside of parentheses above is negative. Thus, the decreasing property of function \( \vartheta \) follows. Moreover, Fatou’s lemma together with the fact that \( P_{z_{1},\omega}\) a.s. the random variable \( \tau \) is finite, imply the required inequality of claim (3.43).

As a result, for any \( z_{1} \in \tilde{H}_{0} \)
\[
P_{z_{1},\omega}[\tilde{T}_{k} \in \{ X_{T_{B_{k+1}(0)}}, \cdot \leq - (N_{0} - 9) k \}]
\leq \frac{\sum_{0\leq n\leq w_{0}} \prod_{n<j\leq w_{0}} \rho_{-1}^{-1} \left[-(N_{0} - 9) j\right]}{\prod_{0\leq n\leq w_{0}} \rho_{-1}^{-1} \left[-(N_{0} - 9) j\right]} \prod_{0\leq n\leq w_{0}} \rho_{j}. \tag{3.45}
\]
Observe that for each point \( z \in \tilde{H}_{i}, \ i \in [-N_{0}, N_{0}(1 + (1/11)], \) there exists a point \( u := u(z) \in \tilde{B}_{i,k}(v) \) for some \( v \in \mathcal{S}_{k} \) (a box composing the quasi-cover of box \( B_{2,k+1}(0) \), cf. Remark 3.1), such that \( |z - u| \) together with \( u \cdot \ell \geq i N_{k} \). Therefore, in virtue of the precedent discussion and uniform ellipticity (1.1), we have
\[
\rho_{i} \leq \sup_{x \in \tilde{B}_{i,k}} \frac{1}{\bar{P}_{x} \left[ X_{T_{B_{k+1}(0)}} \notin \partial \bar{B}_{2,k}(v) \right]} \prod_{0\leq n\leq w_{0}} \rho_{-1}^{-1} \left[-(N_{0} - 9) j\right] \prod_{0\leq n\leq w_{0}} \rho_{j}. \tag{3.46}
\]
where for \( i \in [-N_{0}, N_{0}(1 + (1/11)], \) we have denoted by \( \tilde{B}_{i,k} \) the set \( \{ x \in \tilde{B}_{i,k}(v), \ v \in \mathcal{S}_{k}, \ \tilde{B}_{i,k}(v) \subset B_{2,k+1}(0), \ v \cdot \ell = i N_{k} \} \). Combining the induction
hypothesis \((3.20), (3.46)\) and \((3.45)\), we find that for arbitrary \(z_1 \in \hat{H}_0\)
\[
P_{z_1, \omega}[I_{c_k, \{X_{T_{B_2,k+1}(0)} \cdot \ell \leq -(N_0 - 9)L_k\}}] \leq 2e^{-c_kL_k(N_0 - 9)^{-L_k}(N_0 - 9)}, \tag{3.47}
\]
provided that \(L_0 \geq \nu_1\) for some dimensional constant \(\nu_1 > 0\).
It is now straightforward to see that the case of bad boxes located toward \(+\ell\)
points out is more handling.

Furthermore, we note that for any point \(y \in \tilde{B}_{1,k+1}(0)\), we can a do define \(\hat{H}_0\) as in \((3.40)\) and set
\[
T_0 := \inf\{n \geq 0 : X_n \in \hat{H}_0\}.
\]
Observe that on the set \(\{I_{c_k, \{X_{T_{B_2,k+1}(0)} \cdot \ell \leq -(N_0 - 9)L_k\}}\), \(P_{y, \omega} - \text{a.s.}\) we
have \(T_0 < T_{B_2,k+1}(0)\) (cf. \((2.5)\) for notation) and \(T_0 < \tilde{T}_y\) (cf. \((3.44)\)), as a result
of the strong Markov property and recalling the inequality given in \((3.39)\), for
an arbitrary \(y \in \tilde{B}_{1,k+1}(0)\) we have that
\[
P_{y, \omega}[I_{c_k, \{X_{T_{B_2,k+1}(0)} \cdot \ell \leq -(N_0 - 9)L_k\}}] \leq \sum_{z_1 \in \hat{H}_0} P_{z_1, \omega}[I_{c_k, \{X_{T_{B_2,k+1}(0)} \cdot \ell \leq -(N_0 - 9)L_k\}}] \tag{3.48}
\]
\[
\leq \left(2e^{-c_kL_k}\right)^{N_0 - 9}.
\]
This last estimate will be used once we have bounded all of the remaining two
cases.

(ii) Case \(\tilde{i} \in [1, 4]\).

In this case, we push the walk up to the last time it gets to truncated strip \(\hat{H}_{-9}\)
and then, we will perform a similar analysis as in case (i). We fix \(y \in \tilde{B}_{1,k+1}(0)\)
and define for integer \(u \in [-N_0, N_0(1 + (1/11))]\), the random time
\[
T_u := \sup\{n \geq 0 : X_n \in \hat{H}_u\}.
\]
Notice that on the event \(\{I_{c_k, \{X_{T_{B_2,k+1}(0)} \cdot \ell \leq -L_{k+1}\}}\), \(P_{y, \omega} - \text{a.s.}\) we have
\(T_{-9} < T_{B_2,k+1}(0)\) and \(T_{-9} < \tilde{T}_y\). Thus, in particular on \(\{I_{c_k, \{X_{T_{B_2,k+1}(0)} \cdot \ell \leq -L_{k+1}\}}\), the random time \(T_{-9}\) is \(P_{y, \omega} - \text{a.s.}\) finite and moreover, using the
Markov property we find that
\begin{equation}
P_{y,\omega}[\mathcal{T}^c_k, X_{T_{B_{2,k+1}(0)}} \cdot \ell \leq -L_{k+1}]
\end{equation}
\begin{align*}
&= \sum_{n \geq 0, \ z_1 \in \overline{\mathcal{H}}_{-9}} P_{y,\omega}[\mathcal{T} = n < T_{B_{2,k+1}(0)} \land \mathcal{T} = z_1] \\
&\quad \times P_{z_1,\omega}[\mathcal{T}^c_k, X_{T_{B_{2,k+1}(0)}} \cdot \ell \leq -L_{k+1}, \overline{\mathcal{H}}_{-9} = \infty] \\
&\leq \sup_{z_1 \in \overline{\mathcal{H}}_{-9}} P_{z_1,\omega}[\mathcal{T}^c_k, X_{T_{B_{2,k+1}(0)}} \cdot \ell \leq -L_{k+1}, \overline{\mathcal{H}}_{-9} = \infty],
\end{align*}
provided that for a set \( A \subset \mathbb{Z}^d \), we define the stopping time \( \overline{\mathcal{H}}_A := \inf\{ n \geq 1 : X_n \in A \} \). Moreover, we observe that for any \( z_1 \in \overline{\mathcal{H}}_{-9} \), by the Markov property we have that
\begin{equation}
E_{z_1,\omega}[\overline{\mathcal{H}}_{-9} < T_{B_{2,k+1}(0)}, X_{\overline{\mathcal{H}}_{-11}} = z] \leq \sup_{z_2 \in \overline{\mathcal{H}}_{-11}} P_{z_2,\omega}[\mathcal{T}^c_k, \overline{\mathcal{H}}_{-N_0} < \overline{\mathcal{H}}_{-10}],
\end{equation}
Focusing on the last inequality of (3.50), we note the crucial point in case (i) was the estimate of
\begin{equation}
P_{z_1,\omega}[\mathcal{T}^c_k, X_{T_{B_{2,k+1}(0)}} \cdot \ell \leq -(N_0 - 9)L_k]
\end{equation}
\begin{align*}
&\leq P_{z_1,\omega}[\mathcal{T}^c_k, \overline{\mathcal{H}}_{-N_0} < \overline{\mathcal{H}}_{N_0(1+(1/11))}],
\end{align*}
for any \( z_1 \in \overline{\mathcal{H}}_{1,k+1}(0) \), using a one-dimensional computation to bound from above the right hand side in (3.51). For reference purposes, we introduce a one-dimensional coupling in the next remark, which provides similar bounds as in case i.

**Remark 3.8.** For fixed \( \omega \in \Omega \), one can and do consider the one-dimensional random walk \( (M_n)_{n \geq 0} \) with absorbing barriers in integers \( l_i - 1 := -N_0 - 1 \) and \( l_j + 1 := N_0(1 + (1/11)) + 1 \), and law \( \tilde{P}_m \) for \( k \in [l_i - 1, l_j + 1] \), such that

For \( i \in [l_i, l_j] \), and \( n \geq 0 \), we define transitions:
\begin{align*}
\tilde{P}_m[M_{n+1} = i + 1|M_n = i] &= 1 - \tilde{P}_m[M_{n+1} = i - 1|M_n = i] := \frac{1}{1 + \rho_i}.
\end{align*}

For \( n \geq 0 \), the starting point \( m \) and the absorbing barriers are given by:
\begin{align*}
\tilde{P}_m[M_0 = l_k] &= 1, \\
\tilde{P}_m[M_{n+1} = l_i - 1|M_n = l_i - 1] &= \tilde{P}_m[M_{n+1} = l_j + 1|M_n = l_j + 1] = 1.
\end{align*}
We consider the coupling between the actual random walk \((X_n)_{n \geq 0}\) and the one-dimensional \((M_n)_{n \geq 0}\).

Roughly speaking, for fixed \(y \in \tilde{B}_{1,k+1}\) the one-dimensional random walk \((M_n)_{n \geq 0}\) has the worst choice for the stationary transition \(\hat{P}_k[M_{n+1} = i + 1 | M_n = i] =: \alpha_i, \ i \in [-N_0, N_0(1 + (1/11))]\) (cf. 3.41), when we consider the movement of \((X_n)_{n \geq 0}\) along the event \(\{T_k^c, \tilde{H}_{\tilde{R}_i} < \tilde{H}_{\tilde{R}_j}\}, \ i < j\). It is now straightforward from this viewpoint that, for any point \(x \in \tilde{H}_k\), where \(i \leq m \leq j\) we have

\[
P_{x,\omega}[T_k^c, \tilde{H}_{\tilde{R}_i} < \tilde{H}_{\tilde{R}_j}] \leq \hat{P}_m[(M_n)_{n \geq 0} \text{ hits } i \text{ before } j].\quad (3.52)
\]

The Poisson equation obtained by varying the starting point in the one dimensional setting, is:

\[
\Omega_m := P_m[(M_n)_{n \geq 0} \text{ hits } i \text{ before } j] = \alpha_i \Omega_{m+1} + (1 - \alpha_i) \Omega_{m-1}, \ m \in (i,j)
\]

\[
\Omega_i = 1, \text{ and } \Omega_j = 0.
\]

The system above has unique solution, and the solution is given by (cf. [Ch60] pp. 67-71)

\[
\Omega_m = \frac{\sum_{i \leq m \leq j} \prod_{n < l \leq j} \rho_l^{-1}}{\sum_{i \leq m \leq j} \prod_{n < l \leq j} \rho_l^1}.
\quad (3.53)
\]

Therefore, in view of (3.52) we get

\[
\sup_{x \in \tilde{H}_k} P_{x,\omega}[T_k^c, \tilde{H}_{\tilde{R}_i} < \tilde{H}_{\tilde{R}_j}] \leq \Omega_m,
\quad (3.54)
\]

where \(\Omega_m\) has the expression in display (3.53). We stress that a formal proof of inequality (3.54) might be obtained using the procedure performed in case (i), nevertheless we would rather the approach given in this remark.

Hence, as a result of applying the estimate (3.54) of Remark 3.8 in the inequality 3.50, in view of inequality (3.49) and the induction hypothesis (3.20) we see that

\[
P_{y,\omega}[T_k^c, X_{T_{B_{2,k+1}(0)}} \cdot \ell \leq -L_{k+1}] \leq \sup_{z_2 \in \tilde{H}_{-11}} P_{z_2,\omega}[T_k^c, \tilde{H}_{\tilde{R}_{-N_0}} < \tilde{H}_{\tilde{R}_{-10}}] \leq \frac{\sum_{-11 \leq n \leq -10} \prod_{n < j \leq -10} \rho_j^{-1}}{\sum_{-N_0 \leq n \leq -10} \prod_{n < j \leq -10} \rho_j^1} \leq \left(\frac{2e^{-\nu_2 L_k}}{\kappa}\right)^{N_0-11}
\quad (3.55)
\]

for any \(y \in \tilde{B}_{1,k+1}(0)\), provided that \(L_0 \geq \nu_2\) for certain dimensional constant \(\nu_2 > 0\). We note that we have used the same type of resource as in (3.46) in order to obtain the last line in inequality (3.55). We keep in mind this last estimate and we continue with the third case.

(iii) Case \(\tilde{i} \in (4, N_0 - 9)\).
In this case, we have an in-between hole of three possible bad boxes. For an arbitrary but fixed point \( y \in \tilde{B}_{1,k+1}(0) \), we define the sets \( \tilde{H}_i \), where \( i \in [-N_0,N_0(1+(1/11))] \), as in case (i). An analogous argument using the Markov property as the one given in cases (ii) and (i), shows that for an arbitrary \( y \in \tilde{B}_{1,k+1}(0) \)

\[
P_{y,\omega}[I_{c,k},X_{T_{B_{2,k+1}(0)}} \cdot \ell \leq -L_{k+1}]
\]

\[
\leq \sup_{z_1 \in \tilde{H}_0} P_{z_1,\omega}[\tilde{T}_k, \tilde{H}_{-i} < \tilde{H}_{N_0(1+(1/11))}, \tilde{H} < \tilde{H}_{-i+(5)}].
\]

We apply Remark 3.8 on the first term to the right side of inequality (3.56), and we get the estimate

\[
\sup_{z_1 \in \tilde{H}_0} P_{z_1,\omega}[\tilde{T}_k, \tilde{H}_{-i} < \tilde{H}_{N_0(1+(1/11))}] \leq \left( \frac{2}{\kappa} e^{-c_k L_k} \right)^{N_0-i-7},
\]

provided that \( L_0 \geq \nu_3 \), where \( \nu_3 > 0 \) is certain positive constant.

A quite similar argument as the given above, with the help of Remark 3.8, the induction hypothesis (3.20) and the inequality (3.46) provides the estimate,

\[
\sup_{z_2 \in \tilde{H}_{-i+(6)}} P_{z_2,\omega}[\tilde{T}_k, \tilde{H}_{-N_0} < \tilde{H}_{-i+(5)}] \leq \left( \frac{2}{\kappa} e^{-c_k L_k} \right)^{N_0-i-8}
\]

provided that \( L_0 \geq \nu_4 \), where \( \nu_4 > 0 \) is certain positive constant.

Thus, combining both upper bounds (3.58)-(3.59), in virtue of the inequality (3.56), for any point \( y \in \tilde{B}_{1,k+1}(0) \) we obtain

\[
P_{y,\omega}[I_{c,k},X_{T_{B_{2,k+1}(0)}} \cdot \ell \leq -L_{k+1}] \leq \left( \frac{2}{\kappa} e^{-c_k L_k} \right)^{N_0-i-8}
\]

This finishes the analysis of case (iii) and close our required estimates for the probability in (3.24).

We now combine the estimates given in cases (i)-(iii) along with the lateral estimate (3.38). Specifically, in view of inequality (3.38), we use the inequalities

\[
\sum_{0 \leq n \leq N_0(1+(1/11))} \prod_{n \leq j \leq N_0(1+(1/11))} \rho_{j-1} \cdot \prod_{0 \leq n \leq N_0(1+(1/11))} \rho_{j-1}
\]
displayed in (3.48)- (3.55)- (3.60), in order to see that
\[
\sup_{y \in \tilde{B}_{1,k+1}(0)} P_{y,\omega}[X_{T_{B_{2,k+1}}(0)} \notin \partial^+ B_{2,k+1}(0)] \leq 2 \left( \frac{2}{\kappa} e^{-c_k L_k} \right)^{N_0 - 9} \leq e^{-\frac{c_k L_k + 1}{4}}
\]
provided that $L_0 > \nu_1$, for certain constant of the model $\nu_5 > 0$. We have used our scaling choice (3.1)-(3.3), which implies in particular that $N_0 - 9 > N_0/2$. Furthermore, we have chosen $L_0$ large enough so that
\[
2 \left( \frac{2}{\kappa} \phi^2(L_0) \right)^{N_0} \leq e^{-c_1 L_1} = \phi(N_0) (L_0).
\]
This ends the induction and proves (3.20) by using the expression of constant $(c_k)_{k \geq 0}$ in (3.19).

The requirement of the proposition follows, since the claim 3.20 plainly implies (3.18) as was mentioned at the beginning.

We now proceed to combine Proposition 3.5 and Proposition 3.6 to localize a generic box of scale $L$, for a large number $L$ between two consecutive boxes of scales $L_k$ and $L_{k+1}$. The estimates given in the propositions shall provide a stretched exponential decay for the annealed probability of a random walk process’s unlikely exit. We first introduce the stretched exponential condition to be proven.

**Definition 3.9.** Let $\gamma \in (0, 1]$, $\ell \in S^{d-1}$ and $R$ be a rotation of $\mathbb{R}^d$, such that $R(e_1) = \ell$.

We say that condition $(T_\gamma)|\ell$ holds, if
\[
\limsup_{L \to \infty} L^{-\gamma} \ln \left( P_0[X_{T_{B_{0,L}}} \notin \partial^+ B_{0,L}] \right) < 0, \quad (3.61)
\]
where $B_{0,L} := B(L, L, 2L^3, \ell)$ (cf. (2.6) for notation).

Let us mention that condition $(T_\gamma)|\ell$ is a priori weaker than condition $(T^\gamma)|\ell$ in Definition 1.4. The detail can be found in Lemma 2.2 of [Gue19] for the case $\gamma = 1$ and Appendix of [GVV19] for $\gamma \in (0, 1)$. Condition $(T^\gamma)|\ell$ is equivalent to a requirement as in (3.61), whenever the underlying box is $B(L, L, cL, \ell)$, for some $c > 0$. However, we shall prove that both formulations are equivalent as a result of Theorem 1.7.

Noting that we are assuming condition $(T)_L_{0,0,\infty,\infty}$ and we have fixed at the beginning of the section, a direction $\ell \in S^{d-1}$ along with a rotation $R$ of $\mathbb{R}^d$ satisfying $R(e_1) = \ell$. Observe in particular that we have dropped the tacit dependence of $(T)_L_{0,0,\infty,\infty}$ on $R$ and $\ell$. Under these assumptions, the formal statement of the last result in this section is as follows:

**Theorem 3.10.** There exists $\gamma > 0$, such that condition $(T_\gamma)|\ell$ holds.
Proof. Since $(\exists)L_{\theta_0,0,c,\varphi}$ holds, for $L_0$ large but finite we consider scales (3.1)-(3.3) and the renormalization construction provided by the successive blocks in $\mathcal{B}_k$ with centres at points in the set $\mathcal{L}_k$, with $k \geq 0$. The main argument will be apply Propositions 3.5 and 3.6, with the help of a similar strategy as the one prescribed in Remark 3.7. We then start by setting $\gamma := \ln(2)/(2 \ln(N_0)) \in (0,1)$.

Furthermore, for large $L$ we consider the first integer $k > 0$ such that $L_k \leq L$, consequently we have $L_k \leq L < L_{k+1}$. We recall the box $B_{0,L} := B(L, L, 2L^3, \ell)$ introduced in Definition 3.9 and we further introduce the environment event $\mathcal{E}_k$ of good boxes of scale $k$ intersecting $B_{0,L}$, defined by

$$\Phi_k := \{\forall B_{2,k}(w), w \in \mathcal{L}_k, \hat{B}_{1,k}(w) \subset B_{2,k+1}(0) \Rightarrow B_{2,k}(w) \text{ is } L_k - \text{Good}\} \tag{3.62}$$

We then split the required expectation into two terms, as follows

$$P_0[X_{T_{B_{0,L}}} \notin \partial^+ B_{0,L}] = \mathbb{E} \left[P_{0,\omega}[X_{T_{B_{0,L}}} \notin \partial^+ B_{0,L}] \right] \leq \mathbb{E}[\mathbb{1}_{\mathcal{E}_k}] + \mathbb{E}[P_{0,\omega}[X_{T_{B_{0,L}}} \notin \partial^+ B_{0,L}]\mathbb{1}_{\mathcal{E}_k}] \tag{3.63}$$

Observe that using the Proposition 3.5, the first expectation on the right hand side of (3.63) after a rough counting argument, can be bounded from above by

$$\mathbb{E}[\mathbb{1}_{\mathcal{E}_k}] \leq (N_0(2 + (1/11)) + 2)(5\text{c}\bar{N}_0 + 2)^{d-1}e^{-\eta L^2(k)}. \tag{3.64}$$

On the other hand, we introduce quite analogously as in Remark 3.7, for the second term in the decomposition (3.63), the strategy encoded by the stopping times $(\mathcal{H}_t)_{t \geq 0}$ and the random position $(Z_t)_{t \geq 0}$ together with $(Y_t)_{t \geq 0}$ defined in (3.33). We also let the stopping time $S$ be defined as below (3.33) and notice that

$$\mathbb{E} \left[P_{0,\omega}[X_{T_{B_{0,L}}} \notin \partial^+ B_{0,L}]\mathbb{1}_{\Phi_k} \right] \leq 1 - \mathbb{E} \left[P_{0,\omega} \left[\bigcap_{0 \leq i < N_0} \theta_{i}^{-1}\{H^1 < S\}\bigg| \mathbb{1}_{\Phi_k}\right]\right], \tag{3.65}$$

since for large $L$ and large but finite $L_0$, one has that $3\text{cL} \bar{L}(N_0 - 1) + 4\text{cL} \bar{L} < 2L^3$ (cf. (3.1)-(3.3)), which in turn implies $\mathbb{P}$-a.s.

$$P_{0,\omega}[X_{T_{B_{0,L}}} \in \partial^+ B_{0,L}] \geq P_{0,\omega} \left[\bigcap_{0 \leq i < N_0} \theta_{i}^{-1}\{H^1 < S\}\right].$$

We thus apply Proposition 3.6 on inequality (3.65) to see that

$$\mathbb{E} \left[P_{0,\omega}[X_{T_{B_{0,L}}} \notin \partial^+ B_{0,L}]\mathbb{1}_{\Phi_k} \right] \leq 1 - (1 - e^{-\eta L^2(k)})^{N_0} \leq N_0e^{-\eta L^2(k)}. \tag{3.66}$$
As a result, in view of using (3.64) and (3.66) into (3.63), we see that
\[
P_0[X_{T_{0, L}} \notin \partial^+ B_{0, L}] \leq 2(N_0(2 + (1/11)) + 2)(5\tilde{c}_0 + 2)d^{-1}e^{-\eta_1^2}
\]
\[
\leq 2(N_0(2 + (1/11)) + 2)(5\tilde{c}_0 + 2)d^{-1}\exp\left(-\eta_1 \frac{L}{L_0} \frac{\ln(2)}{2\ln(N_0)}\right) = e^{-\eta_3 L},
\]
for certain constant \(\eta_3 := \eta(L_0, d) > 0\). The last inequality proves the claim in the theorem.

We have concluded this section with Theorem 3.10, some minor details are missing in order to complete our proof and they will be seen in the next section.

4. End of the proof and final remarks

In this last section we gather results in Sections 2 and 3 to finish the proof of Theorem 1.6 and Theorem 1.7. Besides, we will review some further results which are implied by these strong results.

*Proof of Theorem 1.7.* Defining We use Theorem 3.10 to conclude that directional transience in a neighbourhood of direction \(\ell \in S^{d-1}\) implies condition \((\Sigma')|\ell\), with \(\gamma := \ln(2)/2\ln(N)\) (cf. (3.1) for notation). Moreover, in virtue of Definition 3.9 we plainly see that condition \((\tilde{T}'\Gamma)|\ell\) of Definition 9 in Section 5 [GVV19] holds. Therefore, we use Theorem 1 in [GVV19] to see that the stronger condition \((T')|\ell\) is fulfilled. This finishes the proof of Theorem 1.7.

Notice that under the assumption in (1.1) and \(g > 2\ln(2/\kappa)\) in Definition 1.2, Theorem 2 of [GVV19] shows us the existence of a deterministic velocity \(v \in \mathbb{R}^d \setminus \{0\}\) for the random walk process. It is an open question whether one can get rid the further assumption \(g > 2\ln(2/\kappa)\).

*Proof of Theorem 1.6.* Theorem 1.6 is derived for instance, from Theorem 3.6 in [Sz01], Theorem 1.7 and the main result of [GR18].

We recall as well that the use Theorem 1.6 in conjunction with the main result of article [GR18] in the case of i.i.d. random environments, provides a proof for the result mentioned in the abstract:

**Theorem 4.1** (Under (1.1), \(d \geq 2\)). Assume that a random walk in an i.i.d. random environment is directional transient in a open neighbourhood of \(\ell \in S^{d-1}\). Then, condition \((T)|\ell\) holds.
It is an open problem the following question: Is still holding true the previous theorem for strong mixing environments?

There is plenty of equivalences between quite different concepts when we use Theorem 1.6, perhaps the best result displaying that fact is Corollary 1.8. For instance, it is not direct to see that a ballistic strong law of large numbers implies diffusive scaling limit. Moreover, one cannot directly see that any type of annealed decay for the unlikely exit is fulfilled when one is starting from ballistic regime, however Corollary 1.8 proves that such decay is exponential.

We finally stress that we have found a characterization of the ballistic behaviour, thus we can consider a characterization given in Corollary 1.8 as the multidimensional analogous criterion of the one-dimensional of Solomon [So75].

On the other hand, Sznitman constructed important examples in [Sz03] using the so-called effective criterion in dimensions $d \geq 3$, which proved that Kalikow’s condition is strictly stronger than $(T')$. The method used therein was to prove that certain perturbations of simple symmetric random walks satisfy an effective condition, nicknamed the effective criterion. The modifier effective stands for the finite character of the condition, i.e. the effective criterion needs knowing the behaviour of the transitions on a finite subset of sites in $\mathbb{Z}^d$, as opposed to the knowing of the entire set of sites in $\mathbb{Z}^d$ which are needed to check condition $T'$s, see Definition 1.4. It was conjectured by Sznitman the equivalence between any ballisticity assumption and this effective criterion, which was proved as a result of articles [BDR14], [GVV19] and [GR18].

Nevertheless, our results allow to state a stronger fact: ”any of the so-called ballisticity conditions is indeed equivalent to ballistic regime for the random walk process”. Specifically, we have the next result (whose proof is a straightforward consequence from Theorem 1.6, see also Corollary 1.8):

**Theorem 4.2** (Under (1.1), $d \geq 2$). For any random walk in an i.i.d. random environment, a ballistic strong law of large numbers with velocity $v \in \mathbb{R}^d \setminus \{0\}$ is equivalent to condition $(T)|\ell$. As a result, it is also equivalent to the effective criterion in direction $\ell$ introduced in [Sz02].

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