Vorticity and magnetic field production in relativistic ideal fluids

Jian-hua Gao,1,2, Bin Qi,1 and Shou-Yu Wang1

1Shandong Provincial Key Laboratory of Optical Astronomy and Solar-Terrestrial Environment, School of Space Science and Physics, Shandong University at Weihai, Weihai 264209, China
2Key Laboratory of Quark and Lepton Physics (MOE), Central China Normal University, Wuhan 430079, China

Abstract

In the framework of relativistic ideal hydrodynamics, we study the production mechanism for vorticity and magnetic field in relativistic ideal fluids. It is demonstrated that in the uncharged fluids the thermal vorticity will always satisfy the Kelvin’s theorem and the circulation must be conserved. However, in the charged fluids, the vorticity and magnetic field can be produced by the interaction between the entropy gradients and the fluid velocity gradients. Especially, in the multiple charged fluids, the vorticity and magnetic field can be produced by the interaction between the inhomogenous charge density ratio and the fluid velocity gradients even if the entropy distribution is homogeneous, which provides another mechanism for the production of vorticity and magnetic field in relativistic plasmas or in the early universe.

PACS numbers: 52.27.Ny, 95.30.Qd, 12.38.Mh

*Electronic address: gaojh@sdu.edu.cn
I. INTRODUCTION

It is well known that the Universe is filled with vorticities and magnetic fields on all scales. However the origin of these vorticities and magnetic fields is still one of the most challenging open problems in theoretical physics \[1–4\]. In the nonrelativistic ideal fluid, Kelvin’s circulation theorem in hydrodynamics or generalized version in magnetic hydrodynamics forbids the vorticity or magnetic fields to emerge from a zero initial value when the fluid is barotropic. In order to produce seed vorticity or magnetic field, we must resort to the baroclinic effects or go beyond the ideal fluids by including diffusive terms. The presence of large-scale magnetic fields in the Universe, especially the existence of an intergalactic magnetic field \[5–11\] indicates the very possibility that the magnetic fields should have been present in the early universe \[12–14\], in which the temperature of the Universe is very high and the velocities of the fluid and the particle components are both relativistic. Hence we need relativistic hydrodynamics to deal with these very hot fluid systems. Besides, relativistic hydrodynamics is also an very important theoretical tool in high energy heavy-ion physics. The ideal and dissipative hydrodynamics has succeeded greatly in describing the collective flow from the data of RHIC and LHC. The study of the vorticity and magnetic field production in relativistic ideal fluids is very relevant to the important chiral effects, called the chiral-magnetic effects \[16–18\], chiral-vorticity effects \[18, 19\], and local polarization effects \[20, 21\], which can be expected in the non-central heavy-ion collisions, because all these effects depend on the production of the vorticity and magnetic field in the quark-gluon plasma.

In the relativistic ideal fluids, there exist similar covariant version of the Kelvin’s circulation theorem \[22–24\]. It turns out that there are some subtleties when we deal with the relativistic case. In Refs. \[25, 26\], it is demonstrated that vorticity and magnetic field can be produced in relativistic purely ideal fluid from the interaction between the inhomogeneous flow fields and inhomogeneous entropy, which is caused totally by the special relativity. However all these relativistic investigations up to date on the vorticity and magnetic field, as far as we know, are only limited to the systems with single conserved charge and the particle components in the fluids are also specified with finite mass. In this paper, we will extend these investigation to more general cases by direct manipulation of the relativistic hydrodynamic equations. We will not assume in advance whether the particle components
are massive or not and the systems we will consider can have multiple conserving charge or have no conserving charge at all. We find that the thermal vorticity will always satisfy the Kelvin’s circulation theorem and be conserved in the uncharged fluids. However, in the charged fluids, especially in the multiple charged fluids, the vorticity and magnetic field can be produced not only by the interaction between inhomogeneous entropy and inhomogeneous fluid velocity magnitude but also by the interaction between inhomogeneous charge density ratio and inhomogenous fluid velocity magnitude. The latter provides another new mechanism for the production of vorticity and magnetic field in the early universe or in the quark gluon plasma produced in heavy-ion collision at RHIC or LHC.

II. VORTICITY IN RELATIVISTIC IDEAL UNCHARGED FLUIDS

In this section, we consider relativistic fluids without any conserving current, in which the hydrodynamical equations are just the energy-momentum conservation

\[ \partial_\nu T^{\mu\nu} = 0. \]  

(1)

where \( T^{\mu\nu} \) is the energy-momentum tensor. In the ideal hydrodynamics, \( T^{\mu\nu} \) can be decomposed into the following form

\[ T^{\mu\nu} = (\varepsilon + P) u^\mu u^\nu - P g^{\mu\nu} \]  

(2)

where \( \varepsilon \) is the energy density in the local frame, \( P \) is the pressure of the fluid, and the fluid 4-velocity \( u^\mu = (\gamma, \gamma v) \) with the relativistic kinematic factor \( \gamma = 1/\sqrt{1 - v^2} \) and the normalization \( u^2 = 1 \). Substituting Eq.(2) into Eq.(1) and contracting both sides with fluid velocity \( u^\mu \), we can have

\[ u^\nu \partial_\nu \varepsilon + (\varepsilon + P) \partial_\nu u^\nu = 0. \]  

(3)

With the general equations from thermodynamics

\[ T ds = d\varepsilon, \]  

(4)

\[ Ts = \varepsilon + P, \]  

(5)

it is easy to verify that Eq. (3) is just the entropy current conservation

\[ \partial_\mu (su^\mu) = 0. \]  

(6)
Using Eq. (5), we can rewrite the energy-momentum tensor as
\[ T^{\mu \nu} = T_{su}^{\mu} u^\nu - Pg^{\mu \nu} \]

(7)

With the entropy conservation (6), the energy-momentum conservation can be rewritten by
\[ su^\nu \partial_\nu (T u^\mu) - \partial^\mu P = 0. \]

(8)

Using Gibbs relation \( dp = sdT \), we can have the following identity
\[ u^\nu \partial_\nu (T u^\mu) - \partial^\mu T = 0. \]

(9)

It is convenient to define the antisymmetric thermal vorticity tensor \( \Xi^{\mu \nu} \) by
\[ \Xi^{\mu \nu} = \partial_\nu (T u^\mu) - \partial^\mu (T u^\nu) \]

(10)

which is very similar to the EM strength tensor \( F^{\mu \nu} \) with the EM potential \( A^\mu \) replaced by the temperature current \( T u^\mu \). With such definition, we can rewrite Eq.(9) as
\[ \Xi^{\mu \nu} u_\nu = 0 \]

(11)

The circulation of the 4-vector temperature current \( T u^\mu \) along the covariant loop \( L(s) \) where \( s \) denotes the proper time is given by
\[ \frac{d}{ds} \oint_{L(s)} T u^\mu dx_\mu = \oint_{L(s)} \Xi^{\mu \nu} u_\nu dx_\mu = 0 \]

(12)

which is just the relativistic Kelvin circulation theorem. For a specific observer, the vorticity or magnetic field is always defined in a fixed frame, hence we need to consider the vorticity circulation of the synchronic loop \( L(t) \). We specify the components of thermal vorticity tensor \( \Xi^{\mu \nu} \) in the 3-dimension space as
\[ \Xi^{\mu \nu} = \begin{pmatrix}
0 & -\mathcal{E}^1 & -\mathcal{E}^2 & -\mathcal{E}^3 \\
\mathcal{E}^1 & 0 & -\mathcal{B}^3 & \mathcal{B}^2 \\
\mathcal{E}^2 & \mathcal{B}^3 & 0 & -\mathcal{B}^1 \\
\mathcal{E}^3 & -\mathcal{B}^2 & \mathcal{B}^1 & 0
\end{pmatrix} \]

(13)

with the 3-vector definition
\[ \mathcal{E} = (\mathcal{E}^1, \mathcal{E}^2, \mathcal{E}^3) = [\nabla (T \gamma) - \partial_t (T \gamma v)], \]
\[ \mathcal{B} = (\mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3) = \nabla \times (T \gamma v) \]

(14)
With the above definition, we can express the space components of Eq. (11) as

\[ \mathcal{E} + \mathbf{v} \times \mathcal{B} = 0 \]  

or,

\[ \partial_t \left( \frac{T \mathbf{v}}{\sqrt{1 - \mathbf{v}^2}} \right) + \left( \nabla \times \frac{T \mathbf{v}}{\sqrt{1 - \mathbf{v}^2}} \right) \times \mathbf{v} = -\nabla \frac{T}{\sqrt{1 - \mathbf{v}^2}} \]  

Using the above identity, we can immediately obtain the conservation of the thermal current circulation in synchronic space

\[ \frac{d}{dt} \oint_{L(t)} T \gamma \mathbf{v} \cdot d\mathbf{x} = \oint_{L(t)} \left[ \partial_t \left( \frac{T \mathbf{v}}{\sqrt{1 - \mathbf{v}^2}} \right) + \left( \nabla \times \frac{T \mathbf{v}}{\sqrt{1 - \mathbf{v}^2}} \right) \times \mathbf{v} \right] \cdot d\mathbf{x} = -\oint_{L(t)} \left[ \nabla \frac{T}{\sqrt{1 - \mathbf{v}^2}} \right] \cdot d\mathbf{x} = 0 \]  

which implies that the thermal vorticity cannot emerge from a zero initial value. It should be noted that, although the circulation of the thermal current \( T \gamma \mathbf{v} \) is conserved, that of the kinetic current \( \gamma \mathbf{v} \) can be not conserved generally when the temperature is inhomogeneous.

By applying the Stokes theorem, the conservation of circulation can be transformed into the conservation of the flux \( \mathcal{B} \) through the surface which moves along with the fluid

\[ \frac{d}{dt} \int_{S(t)} \mathcal{B} \cdot d\mathbf{S} = \frac{d}{dt} \oint_{L(t)} T \gamma \mathbf{v} \cdot d\mathbf{x} = 0 \]  

which means that the vorticity field flux is conserved or the vorticity lines are frozen-in.

III. VORTICITY AND MAGNETIC FIELDS IN RELATIVISTIC IDEAL MAGNETOHYDRODYNAMICS WITH MULTIPLE CURRENTS

In this section, we are devoted to discussing the relativistic fluids with multiple currents. We assign one of the currents to the electric current \( J^\mu \) from the local gauge symmetry which can interact with the fluids by the magnetohydrodynamic equations and the other \( m \) currents \( J_i^\mu (i = 1, 2, \ldots, m) \) are from the global symmetry, such as baryonic current, leptonic current and so on. Then the magnetohydrodynamic equations for such system are given by

\[ \partial_\nu T^{\mu\nu} = F^{\mu\nu} J_\nu, \]  

\[ \partial_\mu J^\mu = 0, \]  

\[ \partial_\mu J_i^\mu = 0, \quad (i = 1, 2, \ldots, m) \]
The constitutive equations for the ideal magnetohydrodynamics reads

\[ T^\mu{}^\nu = (\varepsilon + P)u^\mu u^\nu - Pg^\mu{}^\nu \] (22)

\[ J^\mu = nu^\mu \] (23)

\[ J_i^\mu = n_i u^\mu \] (24)

where \( n \) is the electric charge density and \( n_i \) is charge density corresponding to other global symmetry. The energy-momentum conservation \([19]\) and current conservation \([20]\) can yield

\[ nu^\nu \partial_\nu \left( \frac{\varepsilon + P}{n} u^\mu \right) - \partial^\mu P = n F^\mu{}^\nu u_\nu \] (25)

We can define the generalized thermal vorticity tensor \( \Xi^\mu{}^\nu \) by

\[ \Xi^\mu{}^\nu = F^\mu{}^\nu + \partial^\nu (fu^\mu) - \partial^\mu (fu^\nu) \] (26)

with \( f = (\varepsilon + P)/n \). We can rewrite Eq.(25) as

\[ n\partial^\mu \left( \frac{\varepsilon + P}{n} \right) - \partial^\mu P = n\Xi^\mu{}^\nu u_\nu \] (27)

Using the thermal equation

\[ Tds = d\varepsilon - \mu dn - \sum_i \mu_i dn_i, \] \hspace{1cm} (28)

\[ Ts = \varepsilon + P - \mu n - \sum_i \mu_i n_i \] \hspace{1cm} (29)

we can have the Gibbs relation corresponding to the multiple charge components

\[ Td \left( \frac{s_n}{n} \right) = d \left( \frac{\varepsilon}{n} \right) + Pd \left( \frac{1}{n} \right) - \sum_i \mu_i d \left( \frac{n_i}{n} \right) \] \hspace{1cm} (30)

where \( \mu \) and \( \mu_i \) denote the chemical potentials with respect to different conserving charge. Now we can rewrite Eq.(27) as

\[ \Xi^\mu{}^\nu u_\nu = T\partial^\mu \left( \frac{s_n}{n} \right) + \sum_i \mu_i \partial^\mu \left( \frac{n_i}{n} \right) \] \hspace{1cm} (31)

It follows that the circulation of the 4-vector current \( fu^\mu + A^\mu \) along the covariant loop \( L(s) \) is given by

\[ \frac{d}{ds} \oint_{L(s)} (fu^\mu + A^\mu) dx_\mu = \oint_{L(s)} \Xi^\mu{}^\nu u_\nu dx_\mu \]

\[ = \oint_{L(s)} \left[ T\partial^\mu \left( \frac{s_n}{n} \right) + \sum_i \mu_i \partial^\mu \left( \frac{n_i}{n} \right) \right] dx_\mu \] \hspace{1cm} (32)
It is obvious that the circulation of this 4-vector current is conserved when $T$ and $\mu_i$ are constant. Just like we did in the last section, we need to consider the vorticity circulation of the synchronic loop $L(t)$. Let us define the 3-vector

$$\mathcal{E} = E + \nabla (f\gamma) - \partial_t (f\gamma v),$$
$$B = B + \nabla \times (f\gamma v)$$

(33)

Then the space components of Eq. (32) can be written as

$$\gamma (E + v \times B) = T \nabla \left( \frac{s}{n} \right) + \sum_i \mu_i \nabla \left( \frac{n_i}{n} \right)$$

(34)

It follows that

$$\frac{d}{dt} \oint_L (f\gamma v + A) \cdot dx = \frac{d}{dt} \int_S B \cdot dS$$

$$= - \int_S \left[ \nabla \left( \frac{T}{\gamma} \right) \times \nabla \left( \frac{s}{n} \right) \right] \cdot dS - \sum_i \int_S \left[ \nabla \left( \frac{\mu_i}{\gamma} \right) \times \nabla \left( \frac{n_i}{n} \right) \right] \cdot dS$$

(35)

where the right side of the equation is the source term which can lead to the vorticity or magnetic fields from the zero initial value. If we set $\mu_i = n_i = 0$, we will recover the results obtained in Ref. [25],

$$\frac{d}{dt} \oint_L (f\gamma v + A) \cdot dx = - \int_S \left[ \nabla \left( \frac{T}{\gamma} \right) \times \nabla \left( \frac{s}{n} \right) \right] \cdot dS$$

(36)

As pointed out in Ref. [25], the source term can be decomposed into the usual baroclinic term

$$S_b \equiv - \int_S \left[ \frac{1}{\gamma} \nabla T \times \nabla \left( \frac{s}{n} \right) \right] \cdot dS$$

(37)

and the pure relativistic term

$$S_r \equiv - \int_S \left[ T \nabla \left( \frac{1}{\gamma} \right) \times \nabla \left( \frac{s}{n} \right) \right] \cdot dS$$

(38)

which is absent in the nonrelativistic limit. Since the baroclinic term is rather difficult to create, the relativistic term will be the only vorticity or magnetic fields production mechanism in majority of physical situations. Now when the multiple currents are involved, we notice that an extra new term

$$S_n \equiv - \sum_i \int_S \left[ \nabla \left( \frac{\mu_i}{\gamma} \right) \times \nabla \left( \frac{n_i}{n} \right) \right] \cdot dS$$

(39)
arises. This is the principal result of this paper. It is very interesting that this term will generate the vorticity or magnetic field even when the entropy is homogeneous where the first term in the second line of Eq.(35) will vanish. This new term can be broken into two terms too, one is

$$S_{n\mu} \equiv -\sum_i \int_S \left[ \frac{1}{\gamma} \nabla \mu_i \times \nabla \left( \frac{n_i}{n} \right) \right] \cdot dS$$

(40)

and the other is

$$S_{nr} \equiv -\sum_i \int_S \left[ \mu_i \nabla \left( \frac{1}{\gamma} \right) \times \nabla \left( \frac{n_i}{n} \right) \right] \cdot dS$$

(41)

It can be expected that $S_{n\mu}$ will tend to vanish because $\nabla \mu_i$ tend to be parallel to $\nabla \left( \frac{n_i}{n} \right)$ as time goes on. Therefore the dominated contribution will be from $S_{nr}$ term. As long as the fluid velocity magnitude and the ratio of the different charge densities are inhomogeneous, the vorticity or magnetic fields will always be generated.

IV. DISCUSSION AND CONCLUSION

First, we emphasize that our result Eq.(17) for the ideal fluid without conserving current can not be derived from the result Eq.(35) with conserving currents by naively taking the limit of $n \to 0$ and $n_i \to 0$ because there exists $1/n$ term. That is why we must consider the ideal fluid without conserving currents separately. Although the result Eq.(36) with single current can be found in the literature everywhere, we failed to find the result Eq.(17) in the literature. Hence we have given the derivation of the result Eq.(17) in our paper in Sec.II. The result reveals that the thermal vorticity always satisfy the Kelvin’s circulation theorem and cannot emerge from a zero initial value.

With respect to the result for the multiple currents in Eq.(35), the contribution from the terms of $S_n$ or $S_{n\mu}$ and $S_{nr}$ is new. These terms, especially $S_{nr}$ term, are very relevant to the production of vorticity or magnetic fields in the early universe, where the particles can carry different charges, such as leptonic charge, electric charge, baryonic charge and so on. Besides, it is very relevant to the quark gluon plasma produced in heavy-ion collision. If there is any inhomogeneous local distribution for some different charges, the vorticity or magnetic fields will be induced through the mechanism in Eq.(41). Then the chiral-magnetic effects, chiral-vorticity effects, and local polarization effects will follow in the non-central heavy-ion collisions.
Acknowledgments

Jian-Hua Gao was supported in part by the Major State Basic Research Development Program in China (Grant No. 2014CB845406), the National Natural Science Foundation of China under the Grant No. 11105137 and CCNU-QLPL Innovation Fund (QLPL2014P01). Bin Qi was supported in part by the National Natural Science Foundation of China under the Grant No. 11005069 and Shou-Yu Wang was supported in part by the National Natural Science Foundation of China under the Grant No. 11175108.

[1] B. J. T. Jones, Rev. Mod. Phys. 48, 107 (1976).
[2] L. M. Widrow, Rev. Mod. Phys. 74, 775 (2002).
[3] R. M. Kulsrud and E. G. Zweibel, Rept. Prog. Phys. 71, 0046091 (2008).
[4] A. J. Christopherson, K. A. Malik and D. R. Matravers, Phys. Rev. D 79, 123523 (2009).
[5] A. Kosowsky and A. Loeb, Astrophys. J. 469, 1 (1996).
[6] D. D. Harari, J. D. Hayward, and M. Zaldarriaga, Phys.Rev. D55, 1841 (1997).
[7] A. Kosowsky, T. Kahniashvili, G. Lavrelashvili, and B. Ratra Phys. Rev. D71, 043006 (2005).
[8] T. Kahniashvili, Y. Maravin, and A. Kosowsky, Phys.Rev. D80, 023009 (2009).
[9] A. Neronov and I. Vovk, Science 328, 73 (2010).
[10] W. Essey, S. ’i. Ando and A. Kusenko, Astropart. Phys. 35, 135 (2011).
[11] L. Pogosian, A. P. S. Yadav, Y. -F. Ng and T. Vachaspati, Phys. Rev. D 84, 043530 (2011) [Erratum-ibid. D 84, 089903 (2011)].
[12] M. Giovannini, Int. J. Mod. Phys. D 13, 391 (2004).
[13] M. Giovannini, Lect. Notes Phys. 737, 863 (2008).
[14] A. Kandus, K. E. Kunze and C. G. Tsagas, Phys. Rept. 505, 1 (2011).
[15] A. Vilenkin, Phys. Rev. D 22, 3080 (1980).
[16] D. E. Kharzeev, L. D. McLerran and H. J. Warringa, Nucl. Phys. A 803, 227 (2008).
[17] K. Fukushima, D. E. Kharzeev and H. J. Warringa, Phys. Rev. D 78, 074033 (2008).
[18] A. Vilenkin, Phys. Rev. D 20, 1807 (1979).
[19] D. E. Kharzeev and D. T. Son, Phys. Rev. Lett. 106, 062301 (2011).
[20] K. Landsteiner, E. Megias and F. Pena-Benitez, Phys. Rev. Lett. 107, 021601 (2011).
[21] J. -H. Gao, Z. -T. Liang, S. Pu, Q. Wang and X. -N. Wang, Phys. Rev. Lett. 109, 232301 (2012)

[22] J. D. Bekenstein and A. Oron, Phys. Rev. D 18, 1809 (1978)

[23] J. D. Bekenstein and A. Oron, Phys. Rev. E 62, 5594 (2000)

[24] K. Elsasser, Phys. Rev. D 62, 044007 (2000).

[25] S. M. Mahajan and Z. Yoshida, Phys. Rev. Lett.,105,095005 (2010).

[26] S. M. Mahajan and Z. Yoshida, Phys. Plasmas,18,055701 (2011).