Volterra-composition operators on the weighted Bergman space with exponential type weights

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Abstract. The properties of Volterra-composition operators on the weighted Bergman space with exponential type weights are investigated in this paper. For a certain class of subharmonic function $\psi : \mathbb{D} \to \mathbb{R}$, we state some necessary and sufficient conditions that a Volterra-composition operator $V^g_\psi$ from the weighted Bergman space $AL^2_\psi(\mathbb{D})$ to Bloch type space $B_\psi(\mathbb{D})$ (or little Bloch type space $B_{\psi,0}(\mathbb{D})$) must satisfy for $V^g_\psi$ to be bounded or compact.

Keywords The weighted Bergman space; Bloch type space(little Bloch type space); Volterra-composition operator; bounded; compact.

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1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $dA$ be the normalized area measure on $\mathbb{D}$. Let $L^2(\mathbb{D}, dA)$ be the space of square integrable functions and let $L^\infty(\mathbb{D}, dA)$ be the space of essential bounded measurable functions. We will use abbreviations $L^2(\mathbb{D})$ for $L^2(\mathbb{D}, dA)$ and $L^\infty(\mathbb{D})$ for $L^\infty(\mathbb{D}, dA)$.

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For a subharmonic function $\psi : \mathbb{D} \to \mathbb{R}$, let $L^\infty_\psi(\mathbb{D})$ be the space of measurable functions $f$ on $\mathbb{D}$ such that $e^{-\psi}f \in L^\infty(\mathbb{D})$ and let $L^2_\psi(\mathbb{D})$ be the Hilbert space of measurable functions $f$ on $\mathbb{D}$ such that

$$\|f\|_{L^2_\psi} = \left( \int_{\mathbb{D}} |f|^2 e^{-2\psi} dA \right)^{\frac{1}{2}} < \infty$$

The inner product of $L^2_\psi(\mathbb{D})$ is given by

$$\langle f, g \rangle_{L^2_\psi} = \int_{\mathbb{D}} f \overline{g} e^{-2\psi} dA$$

Let $H^\infty_\psi(\mathbb{D})$ be the subspace of $L^\infty_\psi(\mathbb{D})$ consisting of analytic functions and $AL^2_\psi(\mathbb{D})$ be the closed subspace of $L^2_\psi(\mathbb{D})$ consisting of analytic functions. $AL^2_\psi(\mathbb{D})$ is called the weighted Bergman space with exponential type weights (see [3], [4]).

An analytic function $f$ is said to belong to the Bloch-type space $B_\psi(\mathbb{D})$ if

$$b_\psi(f) = \sup_{z \in \mathbb{D}} e^{-2\psi(z)} |f'(z)| < \infty.$$ 

Let $B_\psi,0(\mathbb{D})$ denote the subspace of $B_\psi(\mathbb{D})$ such that

$$\lim_{|z| \to 1} e^{-2\psi(z)} |f'(z)| = 0.$$ 

This space is called little Bloch-type space.

**Definition 1.1.** [1], [2] For real valued function $\psi \in C^2(\mathbb{D})$ with $\Delta \psi > 0$, where $\Delta$ is the Laplace operator. Let $\tau(z) = (\Delta \psi(z))^{-\frac{1}{2}}$. We say that $\psi \in \mathcal{D}$ if the following conditions are satisfied.

1. There exists a constant $C_1 > 0$ such that $|\tau(z) - \tau(\xi)| \leq C_1 |z - \xi|$ for $z, \xi \in \mathbb{D}$.
2. There exists a constant $C_2 > 0$ such that $\tau(z) \leq C_2 (1 - |z|)$ for $z \in \mathbb{D}$.
3. There exist constants $0 < C_3 < 1$ and $a > 0$ such that $\tau(z) \leq \tau(\xi) + C_3 (1 - |z|)$ for $|z - \xi| > a \tau(\xi)$.

Let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$ and suppose that $g : \mathbb{D} \to \mathbb{C}$ is an analytic function, $\varphi$ is an analytic self-map of the unit disk and $f \in H(\mathbb{D})$, the Riemann-Stieltjes operator is defined by

$$T_g f(z) = \int_0^1 f(tz) z g'(tz) dt = \int_0^z f(t) g'(t) dt, \quad z \in \mathbb{D}.$$ 

The composition operator $C_\varphi$ is defined as $C_\varphi f \triangleq f \circ \varphi$.

Let $V^g_\varphi$ be the Volterra-composition operator which is defined as

$$V^g_\varphi f(z) \triangleq (T_g \circ C_\varphi f)(z) = \int_0^z f(\varphi(t)) g'(t) dt, \quad \forall f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$
Note that when $\varphi$ is an analytic self-map of the unit disk and $f, g \in H(\mathbb{D})$, we have

$$(V_\varphi^g f)'(z) = f(\varphi(z)) g'(z).$$

In this paper, we will characterize the boundedness and compactness of the operator $V_\varphi^g$ from the weighted Bergman space with exponential type weights to the Bloch-type space (or little Bloch-type space). These properties of Hankel operators have been considered (see [1], [2]).

We assume that $H_\psi^\infty(\mathbb{D})$ is dense in $AL_\psi^2(\mathbb{D})$, the set of all polynomials is dense in $AL_\psi^2(\mathbb{D})$.

2 Preliminaries

In order to prove the main results, we need the following lemmas.

**Lemma 2.1.** Let

$$\|f\|_{B_\psi} = |f(0)| + \sup_{z \in \mathbb{D}} e^{-2\psi(z)}|f'(z)|,$$

then Bloch-type space $B_\psi(\mathbb{D})$ is a Banach space with the norm $\| \cdot \|_{B_\psi}$.

**Proof.** At first, we will show that $\| \cdot \|_{B_\psi}$ is a norm. For a function $f \in B_\psi(\mathbb{D})$, we define

$$b_\psi(f) = \sup_{z \in \mathbb{D}} e^{-2\psi(z)}|f'(z)|.$$

It is easy to see that

$$\|\alpha f\|_{B_\psi} = |\alpha f(0)| + b_\psi(\alpha f) = |\alpha||f(0)| + |\alpha|b_\psi(f) = |\alpha||f\|_{B_\psi}.$$  

$$\|f + g\|_{B_\psi} = |f(0) + g(0)| + b_\psi(f + g) \leq |f(0)| + b_\psi(f) + |g(0)| + b_\psi(g) = \|f\|_{B_\psi} + \|g\|_{B_\psi}.$$  

So $\| \cdot \|_{B_\psi}$ is a semi-norm.

Assume that $\|f\|_{B_\psi} = |f(0)| + b_\psi(f) = 0$, we have $f(0) = 0$ and $b_\psi(f) = 0$, which means that

$$\sup_{z \in \mathbb{D}} e^{-2\psi(z)}|f'(z)| = 0.$$  

Since $e^{-2\psi(z)} \neq 0$ when $z \in \mathbb{D}$, it must be $|f'(z)| \equiv 0$. It follows that $f(z) \equiv C$ for some constant $C$. Since $f(0) = 0$, we obtain $C = 0$. It implies that $f = 0$.

Now we are going to prove the completeness of $\| \cdot \|_{B_\psi}$. Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence of $B_\psi(\mathbb{D})$, i.e.

$$\|f_n - f_m\|_{B_\psi} = |f_n(0) - f_m(0)| + \sup_{z \in \mathbb{D}} e^{-2\psi(z)}|f_n'(z) - f_m'(z)| \to 0 \quad (n, m \to \infty).$$  

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Since
\[
|f_n(z) - f_m(z)| = \left| \int_0^z [f'_n(t) - f'_m(t)]dt + [f_n(0) - f_m(0)] \right| \\
\leq \int_0^z |f'_n(t) - f'_m(t)|dt + |f_n(0) - f_m(0)|,
\]
for each \(z \in \mathbb{D}\), there exists a \(f \in H(\mathbb{D})\) such that \(f_n\) uniformly converges to \(f\) on any compact subsets of \(\mathbb{D}\). By Cauchy integral formula, we have \(f'_n\) uniformly converges to \(f'\) on any compact subsets of \(\mathbb{D}\).

Since \(b_\psi(f_n - f_m) \to 0\), for any \(\varepsilon > 0\), there exists a positive integer \(N\), such that when \(n,m > N\), we have
\[
e^{-2\psi(z)}|f'_n(z) - f'_m(z)| < \varepsilon, \quad \text{for all } z \in \mathbb{D}.
\]
Letting \(m \to \infty\) in the above inequality, we can see that
\[
e^{-2\psi(z)}|f'_n(z) - f'(z)| \leq \varepsilon, \quad \text{for all } z \in \mathbb{D},
\]
this implies that \(b_\psi(f_n - f) \to 0\) \(\quad (n \to \infty)\). It follows that \(\|f_n - f\|_{B_\psi} \to 0\) \(\quad (n \to \infty)\). It is easy to see that there exists a positive integer \(n\) such that for all \(z \in \mathbb{D}\)
\[
e^{-2\psi(z)}|f'(z)| \leq 1 + e^{-2\psi(z)}|f'_n(z)| \leq 1 + b_\psi(f_n) < \infty.
\]
i.e. \(f \in B_\psi\), thus the completeness is proved.

Therefore, \(B_\psi(\mathbb{D})\) is a Banach space with the norm \(\| \cdot \|_{B_\psi}\). \(\quad \square\)

**Notation:** we write \(f(z, w) \sim g(z, w)\) if there exist positive constants \(C\) and \(C'\) such that
\[
Cg(z, w) \leq f(z, w) \leq C'g(z, w).
\]

**Lemma 2.2.** \([2]\) Let \(\psi \in \mathcal{D}\), then we have
\[
K(z, z)e^{-2\psi(z)} \sim (\tau(z))^{-2} = \Delta_\psi(z), \quad z \in \mathbb{D}.
\]
where \(K(z, w)\) is the reproducing kernel of \(AL^2_\psi(\mathbb{D})\).

**Lemma 2.3.** Let \(f \in AL^2_\psi(\mathbb{D})\) and \(\psi \in \mathcal{D}\), then
\[
|f(z)| \leq \sqrt{K(z, z)}\|f\|_{L^2_\psi}.
\]
**Proof.** \(|f(z)| = |\langle f, K_z \rangle| \leq \|f\|_{L^2_\psi}\|K_z\|_{L^2_\psi} = \sqrt{K(z, z)}\|f\|_{L^2_\psi}.\) \(\quad \square\)
Lemma 2.4. Let $\varphi$ be an analytic self-map of the unit disk $\mathbb{D}$ and $g \in H(\mathbb{D})$. Assume that $V_{\varphi}^g : AL_{\psi}^2(\mathbb{D}) \to B_{\psi}(\mathbb{D})$ is bounded. Then $V_{\varphi}^g$ is compact if and only if $V_{\varphi}^g f_k$ converges to zero as $k \to \infty$, for any bounded sequence $\{f_k\}_{k \in \mathbb{N}}$ in $AL_{\psi}^2(\mathbb{D})$ which uniformly converges to zero on compact subsets of $\mathbb{D}$.

Proof. First, assume that $V_{\varphi}^g$ is compact. Let $\{f_k\}_{k \in \mathbb{N}}$ be any bounded sequence in $AL_{\psi}^2(\mathbb{D})$ and uniformly converges to zero on compact subsets of $\mathbb{D}$. Since $V_{\varphi}^g$ is a compact operator, there exists a subsequence of $\{f_k\}_{k \in \mathbb{N}}$ (without loss of generality, we assume it is $\{f_k\}_{k \in \mathbb{N}}$) and a function $h \in B_{\psi}(\mathbb{D})$ such that

$$\|V_{\varphi}^g f_k - h\|_{B_{\psi}} \to 0 (k \to \infty).$$

That is,

$$|V_{\varphi}^g f_k(0) - h(0)| + \sup_{z \in \mathbb{D}} e^{-2\psi(z)} |f_k(\varphi(z))g'(z) - h'(z)| \to 0 (k \to \infty).$$

By the definition of $V_{\varphi}^g$, it is obvious that $V_{\varphi}^g f_k(0) = 0$. It follows that $h(0) = 0$. It is not difficult to see that $f_k(\varphi(z))g'(z)$ uniformly converges to $h'(z)$ on any compact subsets of $\mathbb{D}$. Besides, $\{f_k\}_{k \in \mathbb{N}}$ uniformly converges to zero on any compact subsets of $\mathbb{D}$ and $g \in H(\mathbb{D})$, it follows that $h'(z) \equiv 0$. Also, as $h(0) = 0$, we have $h(z) \equiv 0$, that is

$$\|V_{\varphi}^g f_k\|_{B_{\psi}} \to 0 (k \to \infty).$$

Conversely, let $\{f_k\}_{k \in \mathbb{N}}$ be any bounded sequence in $AL_{\psi}^2(\mathbb{D})$. By Lemma 2.3, we have $|f_k(z)| \leq \sqrt{K}(z, z) \|f_k\|_{L_{\psi}^2}$. Therefore, $\{f_k\}_{k \in \mathbb{N}}$ is uniformly bounded on any compact subset $K$ of $\mathbb{D}$.

By Montel Theorem, there exists a subsequence $\{f_{k_n}\}_{k_n \in \mathbb{N}}$ of $\{f_k\}_{k \in \mathbb{N}}$ and an analytic function $f$, such that $\{f_{k_n}\}$ converges to $f$ uniformly on any compact subsets $K$ of the unit disk $\mathbb{D}$.

According to the Fatou Lemma, $f \in AL_{\psi}^2(\mathbb{D})$. Together with the assumption, we have $\|V_{\varphi}^g (f_{k_n} - f)\|_{B_{\psi}} \to 0 (n \to \infty)$. Therefore, $\{V_{\varphi}^g f_k\}_{k \in \mathbb{N}}$ has subsequence $\{V_{\varphi}^g f_{k_n}\}_{n \in \mathbb{N}}$ which converges to $V_{\varphi}^g f \in AL_{\psi}^2(\mathbb{D})$. From the characterization of compact operators, we see that $V_{\varphi}^g : AL_{\psi}^2(\mathbb{D}) \to B_{\psi}(\mathbb{D})$ is compact. \qed

Lemma 2.5. Let $\psi \in \mathcal{D}$, then the bounded closed set $K$ of $B_{\psi,0}(\mathbb{D})$ is compact if and only if $\limsup_{|z| \to 1, f \in K} e^{-2\psi(z)} |f'(z)| = 0$.

Proof. Assume that the bounded closed set $K$ of $B_{\psi,0}(\mathbb{D})$ is compact. Let $\varepsilon > 0$, we have $\bigcup_{f \in K} B(f, \frac{\varepsilon}{2}) \supseteq K$, where $B(f, \frac{\varepsilon}{2})$ is a ball with center $f$ and radius $\frac{\varepsilon}{2}$. Because $K$ is compact, there exist $f_1, f_2, \ldots, f_n \in K$, such that for any $f \in K$, we have

$$\|f - f_j\|_{B_{\psi}} < \frac{\varepsilon}{2} \quad (1 \leq j \leq n, j \in \mathbb{N}^*).$$
Hence, for any \( z \in \mathbb{D} \), we have, for \( 1 \leq j \leq n \),

\[
e^{-2\psi(z)}|f'(z)| \leq e^{-2\psi(z)}|f_j'(z)| + \frac{\varepsilon}{2}.
\]

Since \( f_j \in B_{\psi,0}(\mathbb{D}) \), for each positive integer \( j \), there exists a positive real number \( r_j \in (0, 1) \) such that \( e^{-2\psi(z)}|f_j'(z)| < \frac{\varepsilon}{2} \) when \( r_j < |z| < 1 \). Let \( r = \max\{r_1, r_2, \ldots, r_n\} \), then we have, when \( r < |z| < 1 \),

\[
e^{-2\psi(z)}|f'(z)| \leq \varepsilon.
\]

Namely,

\[
\lim_{|z|\to 1} \sup_{f \in K} e^{-2\psi(z)}|f'(z)| = 0.
\]

Conversely, suppose that the sequence \( \{f_n\}_{n \in \mathbb{N}} \subseteq K \). Since \( K \) is bounded, there is a positive constant \( M \) such that for any \( n \in \mathbb{Z}^+ \), \( \|f_n\|_{B_\psi} \leq M \). By Lemma 2.3, it is easy to see \( \{f_n\}_{n \in \mathbb{N}} \) is uniformly bounded on any compact subsets of \( \mathbb{D} \). By Montel Theorem, there exists subsequence of \( \{f_n\}_{n \in \mathbb{N}} \) ( without loss of generality, we suppose it to be \( \{f_n\}_{n \in \mathbb{N}} \) ) and an analytic function \( f \), such that \( \{f_n\}_{n \in \mathbb{N}} \) converges to \( f \) uniformly on any compact subsets of \( \mathbb{D} \).

Since

\[
\lim_{|z|\to 1} \sup_{f \in K} e^{-2\psi(z)}|f'(z)| = 0
\]

and \( \{f_n\}_{n \in \mathbb{N}} \subseteq K \), for any \( \varepsilon > 0 \), there exists an \( r \in (0, 1) \), such that \( e^{-2\psi(z)}|f_n'(z)| < \frac{\varepsilon}{2} \) for all \( n \), when \( r < |z| < 1 \). Furthermore, \( e^{-2\psi(z)}|f'(z)| < \frac{\varepsilon}{2} \) when \( r < |z| < 1 \), therefore we have

\[
e^{-2\psi(z)}|f_n'(z) - f'(z)| < \varepsilon
\]

when \( r < |z| < 1 \).

Combines with Cauchy Integral Theorem, \( f_n' \) converges to \( f' \) uniformly on any compact subset of \( \mathbb{D} \). Note that \( e^{-2\psi(z)} \) is continuous on \( r\overline{\mathbb{D}} \), hence for any \( z \in r\overline{\mathbb{D}} \), we have \( |e^{-2\psi(z)}| \leq C \) for some constant \( C \). As \( f_n' \) converges to \( f' \) uniformly on \( r\overline{\mathbb{D}} \), there exists an integer \( G \), when \( n \geq G \), \( |f_n'(z) - f'(z)| < \frac{\varepsilon}{2} \) holds for any \( z \in r\overline{\mathbb{D}} \).

Thus, it is easy to see, when \( n \geq G \), that \( e^{-2\psi(z)}|f_n'(z) - f'(z)| < \varepsilon \) holds for any \( z \in \mathbb{D} \). This implies \( \|f_n - f\|_{B_\psi} \to 0 \) ( \( n \to \infty \)). Since \( K \) is closed, we have \( f \in K \), that means \( K \) is compact. \( \square \)

### 3 Main Results

**Theorem 3.1.** Let \( \varphi \) be an analytic self-map of the unit disk \( \mathbb{D} \) and \( g \in H(\mathbb{D}) \), \( \psi \in \mathcal{D} \). Then \( V_\varphi^g : AL^2_\psi(\mathbb{D}) \to B_\psi(\mathbb{D}) \) is bounded if and only if

\[
\sup_{z \in \mathbb{D}} e^{\psi(\varphi(z)) - 2\psi(z)} \sqrt{\Delta \psi(\varphi(z))}|g'(z)| < \infty.
\]
Proof. By Lemma 2.2,
\[ e^{\psi(\varphi(z))} \Delta \psi(\varphi(z)) \sim \sqrt{K(\varphi(z), \varphi(z))}, \]
therefore, it suffices for us to prove that \( V^g_\varphi \) is bounded if and only if
\[ \sup_{z \in D} \sqrt{K(\varphi(z), \varphi(z))} e^{-2\psi(z)} |g'(z)| < \infty. \]

First, assume that \( V^g_\varphi : AL^2_\psi(D) \to B_\psi(D) \) is bounded. For every \( w \in D \), let \( f_w(z) = k_w(z) = \frac{K(z,w)}{\sqrt{K(w,w)}} \). It is easy to check that \( f_w \in AL^2_\psi \) and \( \|f_w\|_{L^2_\psi} = 1 \) for any \( w \in D \).

Hence, for a fixed \( w \in D \),
\[
|f_w(\varphi(z))| |g'(z)| e^{-2\psi(z)} = |(V^g_\varphi f_w)'(z)| e^{-2\psi(z)} \\
\leq \|V^g_\varphi f_w\|_{B_\psi} \\
\leq \|V^g_\varphi\| \|f_w\|_{L^2_\psi} \\
= \|V^g_\varphi\| < \infty
\]
for any \( z \in D \). Noticing that
\[ f_w(\varphi(z)) = \frac{K(\varphi(z), w)}{\sqrt{K(w,w)}}, \quad \text{for all } z \in D, \]
by setting \( w = \varphi(z) \), we have
\[ f_{\varphi(z)}(\varphi(z)) = \frac{K(\varphi(z), \varphi(z))}{\sqrt{K(\varphi(z), \varphi(z))}} = \sqrt{K(\varphi(z), \varphi(z))}. \]
It follows that
\[ \sup_{z \in D} \sqrt{K(\varphi(z), \varphi(z))} |g'(z)| e^{-2\psi(z)} < \infty. \]

Conversely, assume that
\[ \sup_{z \in D} \sqrt{K(\varphi(z), \varphi(z))} |g'(z)| e^{-2\psi(z)} = M < \infty. \]
For any \( f \in AL^2_\psi(D) \). By Lemma 2.3, we have
\[
\|V^g_\varphi f\|_{B_\psi} = |V^g_\varphi f(0)| + \sup_{z \in D} |(V^g_\varphi f)'(z)| e^{-2\psi(z)} \\
\leq \sup_{z \in D} |f(\varphi(z))g'(z)| e^{-2\psi(z)} \\
\leq \sup_{z \in D} \|f\|_{L^2_\psi} \sqrt{K(\varphi(z), \varphi(z))} e^{-2\psi(z)} |g'(z)| \\
= \|f\|_{L^2_\psi} M.
\]
Therefore, \( V^g_\varphi \) is bounded. \( \square \)
Theorem 3.2. Let $\varphi$ be an analytic self-map of the unit disk $\mathbb{D}$ and $g \in H(\mathbb{D})$, $\psi \in \mathcal{D}$. Then $V^g_{\varphi} : AL^2_{\psi}(\mathbb{D}) \to B_{\psi,0}(\mathbb{D})$ is bounded if and only if $V^g_{\varphi} : AL^2_{\psi}(\mathbb{D}) \to B_{\psi}(\mathbb{D})$ is bounded and

$$\lim_{|z| \to 1} e^{-2\psi(z)}|g'(z)| = 0.$$  

Proof. Assume that $V^g_{\varphi} : AL^2_{\psi}(\mathbb{D}) \to B_{\psi,0}(\mathbb{D})$ is bounded. It is clear that $V^g_{\varphi} : AL^2_{\psi}(\mathbb{D}) \to B_{\psi}(\mathbb{D})$ is bounded. Taking $f(z) = 1 \in AL^2_{\psi}(\mathbb{D})$ and $V^g_{\varphi}f \in B_{\psi,0}(\mathbb{D})$, then

$$0 = \lim_{|z| \to 1} |(V^g_{\varphi}f)'(z)|e^{-2\psi(z)} = \lim_{|z| \to 1} |f(\varphi(z))||g'(z)|e^{-2\psi(z)}$$

$$= \lim_{|z| \to 1} |g'(z)|e^{-2\psi(z)}$$

Conversely, suppose that $V^g_{\varphi} : AL^2_{\psi}(\mathbb{D}) \to B_{\psi}(\mathbb{D})$ is bounded and $\lim_{|z| \to 1} e^{-2\psi(z)}|g'(z)| = 0$. For each polynomial $p(z)$, the following inequality holds

$$|(V^g_{\varphi}p)'(z)|e^{-2\psi(z)} = |p(\varphi(z))||g'(z)|e^{-2\psi(z)} \leq M_p|g'(z)|e^{-2\psi(z)}.$$  

where $M_p = \sup_{z \in \mathbb{D}}|p(z)|$. Since $M_p < \infty$ and $\lim_{|z| \to 1} |g'(z)|e^{-2\psi(z)} = 0$, then

$$\lim_{|z| \to 1} |(V^g_{\varphi}p)'(z)|e^{-2\psi(z)} = 0.$$  

That means for each polynomial $p$, $V^g_{\varphi}p(z) \in B_{\psi,0}(\mathbb{D})$. Since the set consisting of polynomials is dense in $AL^2_{\psi}(\mathbb{D})$, for every $f \in AL^2_{\psi}(\mathbb{D})$, there is a sequence of polynomials $\{p_k\}_{k \in \mathbb{N}}$ such that

$$\|f - p_k\|_{L^2_\psi} \to 0 \quad (k \to \infty).$$

Hence,

$$\|V^g_{\varphi}f - V^g_{\varphi}p_k\|_{B_\psi} \leq \|V^g_{\varphi}\|\|f - p_k\|_{L^2_\psi} \to 0 (k \to \infty).$$

Since $V^g_{\varphi}p_k \in B_{\psi,0}(\mathbb{D})$ and $B_{\psi,0}(\mathbb{D})$ is the the closed subset of $B_{\psi}(\mathbb{D})$, we have $V^g_{\varphi}(AL^2_{\psi}(\mathbb{D})) \subset B_{\psi,0}(\mathbb{D})$.

Since $V^g_{\varphi} : AL^2_{\psi}(\mathbb{D}) \to B_{\psi}(\mathbb{D})$ is bounded, we see that $V^g_{\varphi} : AL^2_{\psi}(\mathbb{D}) \to B_{\psi,0}(\mathbb{D})$ is bounded.  

Theorem 3.3. Let $\varphi$ be an analytic self-map of the unit disk $\mathbb{D}$ and $g \in H(\mathbb{D})$, $\psi \in \mathcal{D}$. Then $V^g_{\varphi} : AL^2_{\psi}(\mathbb{D}) \to B_{\psi}(\mathbb{D})$ is compact if and only if

$$\lim_{|\varphi(z)| \to 1} e^{\psi(\varphi(z)) - 2\psi(z)} \sqrt[2]{\Delta \psi(\varphi(z))}|g'(z)| = 0.$$  

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Proof. By Lemma 2.2,
\[ e^{\psi(\varphi(z))}\sqrt{\Delta \psi(\varphi(z))} \sim \sqrt{K(\varphi(z),\varphi(z))}, \]
therefore, we should only show that \( V_\varphi^g \) is compact if and only if
\[ \lim_{|\varphi(z)| \to 1} \sqrt{K(\varphi(z),\varphi(z))} e^{-2\psi(z)}|g'(z)| = 0. \]

First, assume that \( \lim_{|\varphi(z)| \to 1} \sqrt{K(\varphi(z),\varphi(z))} e^{-2\psi(z)}|g'(z)| = 0 \), then for any \( \varepsilon > 0 \), there is a positive real number \( r_0 \in (0,1) \) such that
\[ \sqrt{K(\varphi(z),\varphi(z))} e^{-2\psi(z)}|g'(z)| < \varepsilon \]
when \( r_0 < |\varphi(z)| < 1 \). Besides, \( \sqrt{K(\varphi(z),\varphi(z))} e^{-2\psi(z)}|g'(z)| \) is bounded when \( |\varphi(z)| \leq r_0 \), it is easy to see that
\[ \sup_{z \in \mathbb{D}} \sqrt{K(\varphi(z),\varphi(z))} e^{-2\psi(z)}|g'(z)| < \infty. \]

By Theorem 3.1, \( V_\varphi^g : AL^2_\psi(\mathbb{D}) \to B_\psi(\mathbb{D}) \) is bounded.

Let \( \{f_k\}_{k \in \mathbb{N}} \) be a bounded sequence in \( AL^2_\psi(\mathbb{D}) \) which uniformly converges to zero on any compact subset of \( \mathbb{D} \) as \( k \to \infty \). Assume that for any \( k \in \mathbb{N}, \|f_k\| \leq C \), for some positive constant \( C \). Note that
\[ \sup_{|\varphi(z)| \leq r_0} |g'(z)| e^{-2\psi(z)} \leq M \]
for some constant \( M \). It follows that
\[ |(V_\varphi^g f_k)'(z)| e^{-2\psi(z)} = |f_k(\varphi(z))| |g'(z)| e^{-2\psi(z)} \leq \sup_{|\varphi(z)| \leq r_0} |f_k(\varphi(z))||g'(z)| e^{-2\psi(z)} + \sup_{|\varphi(z)| > r_0} |f_k(\varphi(z))||g'(z)| e^{-2\psi(z)} \leq M \sup_{|\varphi(z)| \leq r_0} |f_k(\varphi(z))| + \|f_k\|_{L^2_\psi} \sup_{|\varphi(z)| > r_0} \sqrt{K(\varphi(z),\varphi(z))}|g'(z)| e^{-2\psi(z)} \leq M \sup_{|\varphi(z)| \leq r_0} |f_k(\varphi(z))| + \varepsilon \|f_k\|_{L^2_\psi} \leq M \sup_{|\varphi(z)| \leq r_0} |f_k(\varphi(z))| + \varepsilon C. \]

Since \( \{f_k\}_{k \in \mathbb{N}} \) uniformly converges to zero on any compact subset of \( \mathbb{D} \), there exists a \( K \in \mathbb{Z}^+ \) such that if \( k > K \), we have \( \sup_{|\varphi(z)| \leq r_0} |f_k(\varphi(z))| < \varepsilon \). Therefore,
∥V^g f_k∥_{B_ϕ} \to 0 \text{ as } k \to \infty. \text{ By Lemma 2.4, we see that } V^g_ϕ : AL^2_ϕ(D) \to B_ϕ(D) \text{ is compact.}

Conversely, suppose that V^g_ϕ : AL^2_ϕ(D) \to B_ϕ(D) \text{ is compact, then it is clear that } V^g_ϕ : AL^2_ϕ(D) \to B_ϕ(D) \text{ is bounded. Let } \{z_k\}_{k \in \mathbb{N}} \text{ be sequence in } D \text{ such that } \lim_{k \to \infty} |ϕ(z_k)| = 1. \text{ Let }

f_k(z) = \frac{K(z, φ(z_k))}{\sqrt{K(φ(z_k), φ(z_k))}},

then, \( f_k \in AL^2_ϕ(D) \) and ∥f_k∥_{L^2_ϕ} = 1.

Since the set consisting of polynomials is dense in AL^2_ϕ(D), for any ε > 0 and \( f \in AL^2_ϕ(D) \), there exists a polynomial \( P_{f,ε}(z) \in AL^2_ϕ(D) \) such that

∥P_{f,ε} − f∥_{L^2_ϕ} < \frac{ε}{2}.

As

|⟨f_k, f⟩| \leq |⟨f_k, f − P_{f,ε}⟩| + |⟨P_{f,ε}, f_k⟩| ≤ ∥f_k∥_{L^2_ϕ}∥f − P_{f,ε}∥_{L^2_ϕ} + |⟨P_{f,ε}, f_k⟩|

by Lemma 2.2 and Definition 1.1, we have

\sqrt{K(z, z)} \geq C_1 \tau(z)^{-1}e^{ϕ(z)} \geq \frac{C_2}{1 − |z|}

for some positive constants \( C_1 \) and \( C_2 \). Notice that \lim_{k \to \infty} |φ(z_k)| = 0, we have

⟨P_{f,ε}, f_k⟩ = \frac{1}{∥K_{φ(z_k)}∥_{L^2_ϕ}^2}P_{f,ε}(φ(z_k)) → 0 \quad (k \to \infty)

That means that \( f_k \) weakly converges to zero as \( k \to \infty \).

Because \( V^g_ϕ : AL^2_ϕ(D) \to B_ϕ(D) \) is compact, we see that \( ∥V^g f_k∥_{B_ϕ} \to 0 \quad (k \to \infty) \). From the following fact

∥V^g f_k∥_{B_ϕ} ≥ \sup_{z \in D} |f_k(φ(z))||g′(z)|e^{-2ϕ(z)} ≥ |f_k(φ(z_k))||g′(z_k)|e^{-2ϕ(z_k)} = \sqrt{K(φ(z_k), φ(z_k))}g′(z_k)|e^{-2ϕ(z_k)}

we immediately obtain that \lim_{\|ϕ(z)\|→1} \sqrt{K(φ(z), φ(z))}|g′(z)|e^{-2ϕ(z)} = 0. \text{ The proof is completed.} ∎
Theorem 3.4. Let \( \varphi \) be an analytic self-map of the unit disk \( \mathbb{D} \) and \( g \in H(\mathbb{D}) \), \( \psi \in \mathcal{D} \). Then the operator \( V_\varphi^g : AL^2_\psi(\mathbb{D}) \to B_{\psi,0}(\mathbb{D}) \) is compact if and only if \( V_\varphi^g : AL^2_\psi(\mathbb{D}) \to B_{\psi,0}(\mathbb{D}) \) is bounded and

\[
\lim_{|z| \to 1} e^{\psi(\varphi(z))} \sqrt{\Delta \psi(\varphi(z))} |g'(z)| = 0.
\]

Proof. At first, we note that

\[
\lim_{|z| \to 1} e^{\psi(\varphi(z))} \sqrt{\Delta \psi(\varphi(z))} |g'(z)| = 0
\]
equals to

\[
\lim_{|z| \to 1} \sqrt{K(\varphi(z), \varphi(z))} e^{-2\psi(z)} |g'(z)| = 0.
\]

Firstly, we prove the sufficiency. Let \( K = \{ f : f \in AL^2_\psi(\mathbb{D}), \|f\|_{L^2_\psi} \leq 1 \} \). As \( V_\varphi^g : AL^2_\psi(\mathbb{D}) \to B_{\psi,0}(\mathbb{D}) \) is bounded, \( \{ V_\varphi^g f : f \in K \} \) is the bounded closed set of \( B_{\psi,0}(\mathbb{D}) \). It suffices to show that \( \{ V_\varphi^g f : f \in K \} \) is compact in \( B_{\psi,0}(\mathbb{D}) \). By Lemma 2.5, it is only to prove

\[
\lim_{|z| \to 1} \sup_{\|f\|_{L^2_\psi} \leq 1} e^{-2\psi(z)} |(V_\varphi^g f)'(z)| = 0.
\]

By lemma 2.3, for any \( f \in K \), we have

\[
e^{-2\psi(z)} |(V_\varphi^g f)'(z)| = e^{-2\psi(z)}|f(\varphi(z))||g'(z)|
\]

\[
\leq \|f\|_{L^2_\psi} \sqrt{K(\varphi(z), \varphi(z))} e^{-2\psi(z)} |g'(z)|
\]

\[
\leq \sqrt{K(\varphi(z), \varphi(z))} e^{-2\psi(z)} |g'(z)|.
\]

Note that the condition \( \lim_{|z| \to 1} \sqrt{K(\varphi(z), \varphi(z))} e^{-2\psi(z)} |g'(z)| = 0 \), we have

\[
\lim_{|z| \to 1} \sup_{\|f\|_{L^2_\psi} \leq 1} e^{-2\psi(z)} |(V_\varphi^g f)'(z)| = 0.
\]

Therefore, the operator \( V_\varphi^g : AL^2_\psi(\mathbb{D}) \to B_{\psi,0}(\mathbb{D}) \) is compact.

Secondly, we will prove the necessity. Suppose \( V_\varphi^g : AL^2_\psi(\mathbb{D}) \to B_{\psi,0}(\mathbb{D}) \) is compact, it is obvious that \( V_\varphi^g : AL^2_\psi(\mathbb{D}) \to B_{\psi}(\mathbb{D}) \) is compact. By Theorem 3.3, we have

\[
\lim_{|\varphi(z)| \to 1} \sqrt{K(\varphi(z), \varphi(z))} e^{-2\psi(z)} |g'(z)| = 0.
\]

That is, for any \( \varepsilon > 0 \), there exists an \( r \in (0, 1) \), such that

\[
\sqrt{K(\varphi(z), \varphi(z))} e^{-2\psi(z)} |g'(z)| < \varepsilon. \quad (*)
\]
when $r < \varphi(z) < 1$.

Since $V_{\psi}^g : AL_{\psi}^2(\mathbb{D}) \rightarrow B_{\psi,0}(\mathbb{D})$ is bounded, by Theorem 3.2, we have

$$\lim_{|z| \rightarrow 1} e^{-2\psi(z)}|g'(z)| = 0.$$ 

Let $\varepsilon' = \frac{\varepsilon}{C_r}$, there exists a positive real number $\sigma > 0$, such that

$$e^{-2\psi(z)}|g'(z)| < \frac{\varepsilon}{C_r}$$

when $\sigma < |z| < 1$, where $C_r$ is the upper bound of $\sqrt{K(\varphi(z),\varphi(z))}$ when $|\varphi(z)| \leq r$. Therefore, we have

$$\sqrt{K(\varphi(z),\varphi(z))}e^{-2\psi(z)}|g'(z)| < C_r \frac{\varepsilon}{C_r} = \varepsilon.$$

when $\sigma < |z| < 1$ and $|\varphi(z)| \leq r$.

Combines with $(\ast)$, we see that $\sqrt{K(\varphi(z),\varphi(z))}e^{-2\psi(z)}|g'(z)| < \varepsilon$ when $\sigma < |z| < 1$. Therefore,

$$\lim_{|z| \rightarrow 1} \sqrt{K(\varphi(z),\varphi(z))}e^{-2\psi(z)}|g'(z)| = 0.$$

The proof is completed. \qed

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