Existence and Non-existence of Length Averages for Foliations

Yushi Nakano¹, Tomoo Yokoyama²

¹ Department of Mathematics, Tokai University, 4-1-1 Kitakaname, Hirituka, Kanagawa 259-1292, Japan.
E-mail: yushi.nakano@tsc.u-tokai.ac.jp

² Department of Mathematics, Kyoto University of Education/JST Presto, 1 Fujinomori, Fukakusa,
Fushimi-ku, Kyoto 612-8522, Japan. E-mail: tomoo@kyokyo-u.ac.jp

Received: 8 August 2018 / Accepted: 22 April 2019
Published online: 22 June 2019 – © Springer-Verlag GmbH Germany, part of Springer Nature 2019

Dedicated to Atsuro Sannami for his 60th birthday

Abstract: Since the pioneering work of Ghys et al., it has been known that several methods of dynamical systems theory can be adopted to study of foliations. Our aim in this paper is to investigate complexity of foliations, by generalising existence problem of time averages in dynamical systems theory to foliations: It has recently been realised that a positive Lebesgue measure set of points without time averages only appears for complicated dynamical systems, such as dynamical systems with heteroclinic connections or homoclinic tangencies. In this paper, we introduce the concept of length averages to singular foliations, and attempt to collect interesting examples with/without length averages. In particular, we demonstrate that length averages exist everywhere for any codimension one $C^1$ orientable singular foliation without degenerate singularities on a compact surface under a mild condition on quasi-minimal sets of the foliation, which is in strong contrast to time averages of surface flows.

1. Introduction

There are plenty of papers in which several methods of dynamical systems theory were successfully adopted to study of foliations. One of the pioneering work was done by Ghys et al. [11], in which entropies of foliations were first introduced (a close link between the entropy of foliations and the Godbillon-Vey class of foliations can be found in unpublished earlier works by Duminy, see a paper [7] and the preface of [40]). A standard reference for connection between the topology and dynamics of foliations is a monograph of Walczak [40].

Our aim in this paper is to investigate “complexity” of foliations, by generalising existence problem of time averages in dynamical systems theory to foliations. We first recall time averages for flows. A $C^r$ flow $F$ on a smooth manifold $M$ with $r \geq 1$ is given...
as a $\mathcal{C}^r$ mapping from $\mathbb{R} \times M$ to $M$ such that $f^t \equiv F(t, \cdot)$ is a $\mathcal{C}^r$ diffeomorphism for each $t \in \mathbb{R}$, and that $f^0 = \text{id}_M$ and $f^{s+t} = f^s \circ f^t$ for each $s, t \in \mathbb{R}$. For each point $x \in M$ and continuous function $\varphi : M \to \mathbb{R}$, we call

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(f^t(x))dt
$$

(1.1)

the time average of $\varphi$ at $x$. By Birkhoff’s ergodic theorem (cf. [17]), if $\mu$ is an invariant measure of $F$, then the time average of any continuous function exists $\mu$-almost everywhere. Hence, when the time average of some continuous function does not exist at a point, the point is called a non-typical point or an irregular point (see e.g. [3,36]; such a point is also called a point with historic behaviour in another context [20,22,28,34]).

There are a wide variety of examples $F$ for which time averages exist for any continuous function and Lebesgue almost every point. (We notice that it is rather special that time averages exist at every point; see e.g. [3,36].) The simplest example is a conservative dynamical system (i.e. a dynamical system for which a Lebesgue measure is invariant), due to Birkhoff’s ergodic theorem. Another famous example may be a $\mathcal{C}^r$ Axiom A flow on a compact smooth Riemannian manifold with no cycles for $r > 1$ ([4,27]). It is also known that time averages exist Lebesgue almost everywhere for large classes of non-uniformly hyperbolic dynamical systems [1,2]. Moreover, from classical theorems by Denjoy and Siegel ([35, §1]; see also [17, §11] and Sect. 2.1), we can immediately conclude that if $F$ is a $\mathcal{C}^r$ flow on a torus with no singular points for $r \geq 1$, then time averages exist at (not only Lebesgue almost every point but also) every point in the torus.

On the other hand, it also has been known that for several (non-hyperbolic) dynamical systems, the time average of some continuous function does not exist on a Lebesgue positive measure set. One of the most famous example is Bowen’s folklore example, which is a smooth surface flow with two heteroclinically connected singular points (refer to [33]; see also Remark 9 and Fig. 1). One can also find another interesting example without time averages constructed by Hofbauer and Keller [16] for some quadratic maps. Furthermore, recently Kiriki and Soma [20] showed that there is a locally dense class of $\mathcal{C}^r$ surface diffeomorphisms for which some time average does not exist on a positive Lebesgue measure set, by employing dynamical systems with homoclinic tangencies (for a $\mathcal{C}^r$ diffeomorphism $P$, the time average of a continuous function $\varphi$ at a point $x$ is given by $\lim_{n \to \infty} 1/n \sum_{j=0}^{n-1} \varphi(P^j(x))$; their result was further extended to three-dimensional flows in [22]).

From that background we here consider length averages for $\mathcal{C}^r$ singular foliations with $r \geq 1$. A standard reference of $\mathcal{C}^r$ (regular) foliations is [5,6,13,14]. We refer

![Fig. 1. Bowen’s flow]
to Stefan [31] and Sussmann [32] for the definition of singular foliations on a smooth manifold \( M \) without boundary. We here recall that it is shown in [21] that when \( r \geq 1 \), a \( \mathcal{C}^r \) singular foliation \( \mathcal{F} \) on a smooth manifold \( M \) without boundary is equivalent to a partition into immersed connected submanifolds such that for any point \( x \in M \), there is a fibred chart at \( x \) with respect to \( \mathcal{F} \) (see [21] for the definition of fibred charts; note that fibred charts are not well defined if \( M \) has boundary), so that each element \( L \) of \( \mathcal{F} \) (called a leaf) admits a dimension, denoted by \( \dim(L) \). Furthermore, we say that a partition \( \mathcal{F} \) is a \( \mathcal{C}^r \) singular foliation if the induced partition \( \hat{\mathcal{F}} \) of \( \mathcal{F} \) on the double of \( M \) (i.e. \( M \times \{0, 1\} \sim \), where \( (x, 0) \sim (x, 1) \) for \( x \in \partial M \); see Fig. 3) is a \( \mathcal{C}^r \) singular foliation (the pair \( (M, \mathcal{F}) \) is called a \( \mathcal{C}^r \)-foliated manifold), and define the dimension of each leaf \( L \in \mathcal{F} \) through a boundary point as the dimension of the lifted leaf of \( L \) in \( \hat{\mathcal{F}} \). The integer \( \dim(\mathcal{F}) := \max(\dim(L) \mid L \in \mathcal{F}) \) is called the dimension of \( \mathcal{F} \), and \( \dim(M) - \dim(\mathcal{F}) \) the codimension of \( \mathcal{F} \). A leaf \( L \) is said to be regular if \( \dim(L) = \dim(\mathcal{F}) \) and singular if \( \dim(L) < \dim(\mathcal{F}) \). The union of singular leaves is called the singular set and denoted by \( \text{Sing}(\mathcal{F}) \). With a usual abuse of notation, we simply say that \( \mathcal{F} \) is a regular foliation if \( \text{Sing}(\mathcal{F}) = \emptyset \).

When \( \dim(\mathcal{F}) = 1 \) (in particular, when \( \mathcal{F} \) is a codimension one singular foliation \( \mathcal{F} \) on a surface \( M \)), we have that \( \text{Sing}(\mathcal{F}) = \bigcup_{L \in \mathcal{S}} L \) with \( \mathcal{S} = \{ L \in \mathcal{F} \mid \#L = 1 \} \), so that \( \mathcal{F} - \mathcal{S} \) is a \( \mathcal{C}^1 \) regular foliation of \( M - \text{Sing}(\mathcal{F}) \). A point in \( \text{Sing}(\mathcal{F}) \) is called a singularity or a singular point. A singularity \( x \) of \( M \) is said to be (metrically) non-degenerate if there are a neighbourhood \( U \) of \( x \) and a \( \mathcal{C}^1 \) vector field \( A \) on \( M \) such that the set of orbits of the flow generated by \( A \) corresponds to \( \mathcal{F} \) on \( U \) and that \( x \) is a non-degenerate singularity of \( A \) (i.e. \( A(x) = 0 \) and each eigenvalue of \( DA \) at \( x \) is non-zero). Otherwise, a singularity is said to be (metrically) degenerate. Notice that, by definition, each boundary component of a foliation with no degenerate singular points on a surface is either a circle which is transverse to the foliation or a union of leaves. In particular, each center does not belong to the boundary. We also note that if there is no degenerate singularities, then there are at most finitely many non-degenerate singularities. In other words, any accumulation point of infinitely many non-degenerate singularities need to be degenerate.

We denote by \( \mathcal{F}(x) \) the leaf of a foliation \( \mathcal{F} \) through a point \( x \in M \). Let \( d \) be the distance on leaves of \( \mathcal{F} \) induced by a Riemannian metric of \( M \).

**Definition 1.** For a point \( x \in M \) and a continuous function \( \varphi \) on \( M \), we define the length average of \( \varphi \) at \( x \) by

\[
\lim_{r \to \infty} \frac{1}{|B_r^{\mathcal{F}}(x)|} \int_{B_r^{\mathcal{F}}(x)} \varphi(y)dy,
\]

(1.2)

where \( B_r^{\mathcal{F}}(x) = \{ y \in \mathcal{F}(x) \mid d(x, y) < r \} \) and \( |B_r^{\mathcal{F}}(x)| = \int_{B_r^{\mathcal{F}}(x)} dy \) is a \( p \)-dimensional volume of \( B_r^{\mathcal{F}}(x) \) with \( p = \dim(\mathcal{F}(x)) \). If the length average of \( \varphi \) at \( x \) exists for every continuous function \( \varphi \), then we simply say that length averages exist at \( x \).

---

1. Existence of length averages (given in Definition 1) is independent of the choice of Riemannian structures when the manifold is compact, because any two Riemannian metrics on a compact manifold are Lipschitz equivalent (for two metrics \( g_1 \) and \( g_2 \) on \( M \), a mapping \( v \mapsto g_2(v, v) / \sqrt{g_1(v, v)} \) on the unit tangent bundle of \( M \) is continuous and strictly positive, and by compactness, it is bounded above and below by positive constants). On the other hand, it can depend on the choice of Riemannian structures when the manifold is non-compact, as indicated in Examples 17 and 18.
We emphasise that the length average of $\phi$ for the foliation generated by a flow does not coincide with the time average of $\phi$ for the flow in general (see Remark 2 for details). On the other hand, it also should be noticed that these two averages coincide with each other in the regular case (note that we can reparametrise the flow into a flow with unit velocity, because the compactness of the surface implies that the velocities at any points of the flow are bounded and away from zero). Finally, we would like to say that we are strongly interested in any physical meaning and application of length averages.

1.1. Foliations without length averages. For foliations on open manifolds or foliations with degenerate singular points, there are abundant examples for which some length average does not exist. In Sect. 3, we will provide with the following examples without length averages:

- Trivial foliation on a Euclidean space (Example 16), and foliation by the (regularised) Koch curves on an open disk (Example 17).
- Foliation by the (regularised) Koch curves on a compact surface with one degenerate singularity (Example 18).

These examples will make clear that the compactness of $M$ and the non-degeneracy of singularities of $\mathcal{F}$ might be necessary for investigating interesting examples without length averages. Moreover, in higher codimensional cases, we can easily construct foliations without length averages. We will have the following example:

- Foliation generated by the suspension flow of any diffeomorphism $P$ without time averages (that is, $P$ admits a positive Lebesgue measure set consisting of points without time averages, such as the time-one map of the Bowen’s surface flow, see Remark 9 and Fig. 1). We refer to Example 20. Since time averages exist everywhere for any one-dimensional diffeomorphism (see [17, §11] e.g.), the dimension of the phase space of $P$ has to be $\geq 2$, so that this is a codimension two foliation of a compact manifold with no singularities.

1.2. Foliations with length averages. In contrast to the examples in the previous subsection, length averages exist at every point for any codimension one $C^1$ singular foliation on a compact surface without degenerate singularities under a mild condition on quasi-minimal sets (which is a significantly stronger conclusion than one can expect from a straightforward dynamical analogy, as indicated in Remark 2). We will also show that the conclusion holds for Lebesgue almost every point under another condition on quasi-minimal sets.

**Theorem A.** For any codimension one $C^1$ singular foliation without degenerate singularities on a compact surface, the following holds:

1. Length averages exist everywhere if the union of quasi-minimal sets of the foliation is empty or uniquely ergodic (see Definition 15).
2. Length averages exist Lebesgue almost everywhere if the union of quasi-minimal sets of the foliation is locally dense (see Definition 15).

We will see in Remark 11 that every quasi-minimal set is uniquely ergodic for “almost every” foliation on a compact surface, and for every foliation on a compact surface with orientable genus less than two or non-orientable genus less than four.

**Remark 2.** One can see an essential difference between time averages and length averages for the Bowen’s classical surface flow, which heuristically explains the strong contrast between Theorem A and time averages of surface flows. Let $\mathcal{F}$ be a singular foliation
given as the set of all orbits of a continuous flow $F$, i.e. each leaf $\mathcal{F}(x)$ of $F$ through $x$ is given as the orbit $O(x) := \{ f^t(x) \mid t \in \mathbb{R} \}$ of $x$. We call $\mathcal{F}$ the singular foliation generated by $F$ (such singular foliations will be intensively studied in Sect. 2). Let $F$ be the Bowen’s flow on a compact surface $S$, and $\mathcal{F}$ the singular foliation generated by $F$. Then, it follows from [33] that there exists a positive Lebesgue measure set $D \subset S$ such that for any $x \in D$, the time average $\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \varphi(f^t(x)) dt$ of some continuous function $\varphi : S \to \mathbb{R}$ along the orbit $O(x)$ of $x$ does not exist (by virtue of the non-existence of the time average (1.1) of some continuous function $\varphi$ along the forward orbit $O_+(x) := \{ f^t(x) \mid t \in \mathbb{R}_+ \}$ and by the convergence of $f^{-t}(x)$ to a repelling point as $t \to \infty$). On the other hand, since $\mathcal{F}$ is a codimension one singular foliation with no degenerate singularities on the compact surface $S$ and it does not have any quasi-minimal set, we can apply Theorem A: for any $x \in D$, the length average (1.2) of any continuous function $\varphi : M \to \mathbb{R}$ along the leaf $\mathcal{F}(x) = O(x)$ through $x$ does exist. See Remark 9 for detail. Compare also with Example 20.

As an application of the item (1) of Theorem A together with Remark 11, we can immediately obtain the following corollary for regular foliations:

**Corollary B.** Length averages exist everywhere for any codimension one $C^1$ regular foliation on a compact surface.

One may realise that Corollary B is rather a generalisation of classical theorems by Denjoy and Siegel for toral flows without singularities (this is contrastive to the non-regular case as in Remark 2). In fact, existence of time averages at every point for surface flows (in particular, toral flows) without singularities is a direct consequence of Corollary B by the remark after Definition 1 for regular foliations. This similarity between flows and foliations seems to be reminiscent of the previous works for connection between the geometric and dynamics of foliations (such as the geometric entropy of a foliation and the topological entropy of the associated holonomy group [11, Theorem 3.2]). Furthermore, existence of time averages for the group generated by the “Poincaré map” of foliations plays an important role in the proof of Theorem A (see Sect. 2.1), and thus, it is likely that analysis of holonomy group is also crucial for analysis of length averages. (Technically, moreover, the difference between time averages and length averages is closely related with the first return time to a cross-section, see Remark 9.)

According to the examples in the previous subsection without length averages and Theorem A, we may naturally ask the following problem:

**Problem 3.** Does length averages exist everywhere for any codimension one foliation on a compact smooth Riemannian manifold?

Theorem A is a satisfactory but not complete answer to Problem 3 for foliations on compact surfaces. One may ask whether length averages exist everywhere without the assumption on quasi-minimal sets in the item (1) of Theorem A. This can be reduced to existence problem of time averages for any interval exchange transformations, see Remark 11 and the proof of Proposition 13. In higher dimensions, statistical behaviours of foliations are more complicated due to the number of ends: in surfaces the numbers of ends of foliations are either zero or two (since 1-D paracompact manifolds are either the circle or the real line), while in higher dimensional manifolds they can be infinite. Therefore, investigating codimension one foliations with infinite ends (e.g. Hirsch foliations [15] or Sacksteder foliations [29]) would also be a first step to the answer of Problem 3. We again note that for analysis of length averages, it seems important to understand the dynamics of associated holonomy groups, so recent development in ergodic theory of
codimension-one foliations and group actions by circle diffeomorphisms (e.g. [8]) might be helpful, see also Example 20 and Remark 21.

Remark 4. It is natural to ask whether or not further dynamical analogies hold for length averages of foliations, such as a version of Birkhoff’s ergodic theorem for length averages. Furthermore, it seems interesting to investigate a generalisation of Theorem A (and Corollary B) to \( C^0 \) singular foliations in an appropriate sense. We assumed singular foliations in Theorem A to be \( C^1 \), partly because \( C^1 \) makes it possible to define the length \( r \) in each leaf, which is needed to make sense Definition 1. It is not difficult to define length averages for piecewise \( C^1 \) singular foliations, including foliations by the classical Koch curves (see Example 18). Possibly, one can even define length averages for \( C^0 \) singular foliations if one considers, instead of a length \( r \) and the ball \( B_{\varepsilon} x \) in \( F(x) \), a number \( N \) and the set of plaques \( \tilde{\ell}_1, \ldots, \tilde{\ell}_N \) such that \( x \in \tilde{\ell}_1, \ell = \tilde{\ell}_N \) and \( \tilde{\ell}_j \cap \tilde{\ell}_{j+1} \neq \emptyset \) (and takes the infimum over all foliated atlases, if necessary). We think that this definition would help connecting length averages for foliations to time averages for their “Poincaré map”, as in Example 20. We also note that in the argument of the proof of Theorem A and Corollary B, the \( C^1 \) regularity is (essentially) necessary only in Lemma 7.

2. Proof of Theorem A and Corollary B

2.1. Singular foliations generated by flows. We shall deduce Theorem A from the following theorem for length averages of singular foliations generated by surface flows, which is our goal in this subsection.

Theorem 5. For any \( C^1 \) singular foliation generated by a \( C^1 \) flow without degenerate singular points on an orientable compact surface, the following holds:

1. Length averages exist everywhere if the union of quasi-minimal sets of the flow is empty or uniquely ergodic (see Definition 10).
2. Length averages exist Lebesgue almost everywhere if the union of quasi-minimal sets of the flow is locally dense (see Definition 10).

Let \( F \) be a \( C^1 \) flow without degenerate singularities on an orientable compact surface \( S \). An orbit of a point \( x \) is called recurrent if \( x \in \omega(x) \cup \alpha(x) \), where \( \omega(x) := \bigcap_{t \in \mathbb{R}} \{ f^t(x) \mid t > s \} \) and \( \alpha(x) := \bigcap_{t \in \mathbb{R}} \{ f^t(x) \mid t < s \} \) (called \( \omega \)-limit set and \( \alpha \)-limit set of \( x \), respectively). Recall that a quasi-minimal set is an orbit closure of a non-closed recurrent orbit. We say that a subset of \( S \) is a circuit if it is an image of a circle composed of a finite number of singularities together with homoclinic and heteroclinic orbits connecting these singularities. A circuit \( \gamma \) is said to be attracting if there is a continuous mapping \( q \) from \([0, 1] \times S^1\) to a neighbourhood of \( \gamma \) such that \( q(0, \cdot) \) is a continuous mapping from the unit circle \( S^1 \) to the circuit, and that \( q((0, 1] \times S^1) \) is an embedded annulus satisfying that \( \bigcup_{t \geq 0} f^t(q((0, 1] \times S^1)) \subset q((0, 1] \times S^1) \). The image of \( q \) is called a collar of attraction of \( \gamma \). We start from the classification of limit sets in (a generalisation of) Poincaré-Bendixson theorem.

Theorem 6 (Theorem 2.6.1 of [26]). Each \( \omega \)-limit set of a \( C^1 \) flow \( F \) with finitely many singular points on a compact surface is one and only one of the following four types: a singular point, a periodic orbit, an attracting circuit, or a quasi-minimal set.
It is obvious that an analogous statement for $\omega$-limit sets also holds due to Theorem 2.6.1 of [26]. We will show Lemma 5 according to the four cases of $\omega(x)$ (and $\alpha(x)$) in Theorem 6. For convenience, we say that the limit in (1.2) for a continuous function $\varphi : S \to \mathbb{R}$ with $B_{\gamma}^F(x)$ replaced by

$$B_{\gamma}^F(x) = \{ y \in O_{\gamma}(x) \mid d(x, y) < r \}$$

is the length average of $\varphi$ along the forward orbit of $x$, where $d$ is the distance on $O_{\gamma}(x)$.

As mentioned in Remark 2, we need to note that the time average of a continuous function along an orbit does not coincide with the length average of the function along the orbit in general.

We say that a subset $\Sigma$ of $S$ is a cross-section of $F$ if $\Sigma$ is either a (closed or open) segment or a circle such that $F$ is transverse to $\Sigma$ and the first return time $T_\Sigma$ of $x \in \Sigma$ to $\Sigma$ (i.e. the positive number $t$ such that $f^t(x) \in \Sigma$ and $f^s(x) \not\in \Sigma$ for all $0 < s < t$) is well defined and finite. (Notice that our definition is slightly different with the standard one, cf. [30].) Let $\gamma_x = \{ f^t(x) \mid 0 \leq t \leq T_\Sigma \}$ and $|\gamma_x|$ the length of $\gamma_x$ for $x \in \Sigma$. The following elementary lemma will be used repeatedly.

**Lemma 7.** Let $\Sigma$ be a cross-section of a $C^1$ flow $F$ on a compact surface $S$. Assume that $\tilde{x} \in \Sigma$ satisfies that $f^t(\tilde{x}) \not\in \partial \Sigma$ for all $0 < t \leq T_\tilde{x}$, where $\partial \Sigma$ is the boundary of $\Sigma$. Then, $x \mapsto |\gamma_x|$ is continuous at $\tilde{x}$.

**Proof.** Take a real number $\epsilon > 0$. Let $A$ be the vector field generating $F$, and $\ell_A(x) = |A(x)|_x$ for each $x \in \Sigma$, where $|v|_x$ is the length of $v \in T_x M$ with respect to the Riemannian metric of $S$ at $x$. Let $\epsilon_1$ be a positive number smaller than $\epsilon(2T_{\tilde{x}})^{-1}$ and $\|\ell_A\|_\infty = \sup_{x \in \Sigma} |\ell_A(x)|$. For each $\rho \in (0, T_{\tilde{x}})$ and each positive numbers $\delta$, we consider a flow box $B_{\rho, \delta}$ given by $B_{\rho, \delta} = \{ f^t(x) \mid t \in [0, \rho], x \in U_\delta \}$, where $U_\delta = \{ x \in \Sigma \mid d_\Sigma(\tilde{x}, x) < \delta \}$. Let $\rho$ and $\delta$ be sufficiently small positive numbers such that $\rho < \min\{ T_{\tilde{x}}, \epsilon(4\|\ell_A\|_\infty)^{-1} \}$, that $\frac{T_{\tilde{x}}}{\rho}$ is not an integer, and that $|\ell_A(y) - \ell_A(\tilde{y})| \leq \epsilon_1$ whenever $y$ and $\tilde{y}$ are in $f^{j\rho}(B_{\rho, \delta})$ with some $0 \leq j \leq N := \lceil \frac{T_{\tilde{x}}}{\rho} \rceil$ (note that $\ell_A$ is uniformly continuous).

Since $\frac{T_{\tilde{x}}}{\rho}$ is not an integer, we have $N\rho < T_{\tilde{x}} < (N + 1)\rho$. Furthermore, due to the hypothesis, we can take $\delta > 0$ such that each flow segment of $f^{N\rho}(B_{\rho, \delta})$ intersects $\Sigma$ (that is, $\{ f^t(x) \mid t \in [N\rho, (N + 1)\rho] \}$ intersects $\Sigma$ for each $x \in U_\delta$).

Let $\gamma_{x,j} := \{ f^t(x) \mid t \in [j\rho, (j + 1)\rho] \}$ for each $x \in \Sigma$ and $0 \leq j \leq N$. Then, it is straightforward to see that

$$||\gamma_{x,j}|| - |\gamma_{x,j}| \leq \rho \epsilon_1$$

for any $x \in U_\delta$ and $0 \leq j \leq N$. Consequently, it holds that for each $x \in U_\delta$,

$$||\gamma_x|| - |\gamma_x| \leq N\rho \epsilon_1 + |\gamma_{x,N}| + |\gamma_{x,N}| - |\gamma_x| \leq T_{\tilde{x}} \epsilon_1 + 2\rho \|\ell_A\|_\infty < \epsilon.$$

Since $\epsilon$ is arbitrary, this completes the proof. □

When $\omega(x)$ is a singular point, it is straightforward to see that the length average of any continuous function along the forward orbit of $x$ exists. (Note that the length of each connected component of the intersection of any orbit and a small neighbourhood of any non-degenerate singularity is finite; compare with Example 17.) On the other hand, we need to work a little harder even in the periodic orbit case.
Proposition 8. Let \( x \) be a point in \( S \) whose \( \omega \)-limit set is a periodic orbit or an attracting circuit \( \gamma \). Then the length average of any continuous function \( \varphi : S \to \mathbb{R} \) along the forward orbit of \( x \) exists.

Proof. We assume that \( x \not\in \gamma \) because we can immediately get the conclusion when \( x \in \gamma \). Let \( \Sigma_{+} \) be a closed segment which is transverse to \( \gamma \) at \( s_{0} \) in the boundary of \( \Sigma_{+} \). Let \( t_{n} \) be the \( n \)-th hitting time of \( x \) to \( \Sigma_{+} \) with \( n \geq 1 \) (i.e. \( 0 \leq t_{n} < t_{n+1} \) and \( f^{t}(x) \in \Sigma_{+} \) if and only if \( t \in \{ t_{1}, t_{2}, \ldots \} \)), and \( x_{n} = f^{t_{n}}(x) \). Then, by the assumption \( \gamma = \omega(x) \) and the continuity of the vector field generating \( F \), one can find \( n_{0} \geq 1 \) such that

\[
d_{\Sigma_{+}}(x_{n+1}, s_{0}) < d_{\Sigma_{+}}(x_{n}, s_{0}) \quad \text{for each } n \geq n_{0}, \quad (2.1)
\]

Let \( \Sigma_{+}(a, b) \subset \Sigma_{+} \) be the open segment connecting \( a \) and \( b \). One can find \( n_{1} \geq n_{0} \) such that the \( \omega \)-limit set of each point in \( \Sigma_{+}(x_{n_{1}}, s_{0}) \) does not include singular points, because otherwise one can find infinitely many singular points, which leads to contradiction due to the non-existence of degenerate singularities (see comments above Definition 1). Similarly, one can find \( n_{2} \geq n_{1} \) such that the \( \omega \)-limit set of each point in \( \Sigma_{+}(x_{n_{2}}, s_{0}) \) includes no periodic orbits, because otherwise one can find infinitely many periodic orbits arbitrary close to \( \gamma \), which contradicts to the continuity of the vector field generating \( F \) and \( \omega(x) = \gamma \). Moreover, by the attracting property of \( \gamma \), one can find \( n_{3} \geq n_{2} \) such that the \( \omega \)-limit set of each point in \( \Sigma_{+}(x_{n_{3}}, s_{0}) \) is contained in a collar of attraction of \( \gamma \). Since no quasi-minimal sets are contained in annuli, by virtue of Theorem 6, the \( \omega \)-limit set of each point in \( \Sigma_{+}(x_{n_{3}}, s_{0}) \) is \( \gamma \) and so the first return time \( T_{\tilde{x}} \) of each point \( \tilde{x} \) in \( \Sigma := \Sigma_{+}(x_{n_{3}}, s_{0}) \) to \( \Sigma \) is well defined and finite. Furthermore, it follows from (2.1) that

\[
d_{\Sigma_{+}}(P(\tilde{x}), s_{0}) < d_{\Sigma_{+}}(\tilde{x}, s_{0}) \quad \text{for each } \tilde{x} \in \Sigma, \quad (2.2)
\]

where \( P : \Sigma \to \Sigma \) is the forward Poincaré map on \( \Sigma \).

By Lemma 7, \( \tilde{x} \mapsto |\gamma_{\tilde{x}}| \) is continuous on the cross-section \( \Sigma \), and thus, a function \( \tilde{\varphi} : \Sigma \to \mathbb{R} \) given by

\[
\tilde{\varphi}(\tilde{x}) = \int_{\gamma_{\tilde{x}}} \varphi(y) dy \quad (\tilde{x} \in \Sigma)
\]

(2.3)
is also continuous for any continuous function \( \varphi : S \to \mathbb{R} \), where \( \gamma_{\tilde{x}} = \{ f^{t}(\tilde{x}) \mid 0 \leq t \leq T_{\tilde{x}} \} \). By (2.2), for any continuous function \( \tilde{\varphi}_{1} : \Sigma_{+} \to \mathbb{R} \) and \( \tilde{x} \in \Sigma \), we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} \tilde{\varphi}_{1}(P^{j}(\tilde{x})) = \lim_{\tilde{y} \to s_{0}} \tilde{\varphi}_{1}(\tilde{y}), \quad (2.4)
\]
in particular, the time averages of \( \tilde{\varphi}_{1} \) at \( \tilde{x} \) exists. On the other hand, for all large \( r \), denoting by \( N_{r} \) the maximal integer such that \( P^{j}(x_{n_{3}}) \in B_{r}^{+}(x) \) for each \( 0 \leq j \leq N_{r} - 1 \), we can rewrite the integral of \( \varphi \) along the forward orbit of \( x \) by

\[
\int_{B_{r}^{+}(x)} \varphi(y) dy = \int_{\tilde{y}_{0}} \varphi(y) dy + \sum_{j=0}^{N_{r}-1} \tilde{\varphi}(P^{j}(x_{n_{3}})) + \int_{\tilde{y}_{1}} \varphi(y) dy, \quad (2.5)
\]

where \( \tilde{y}_{0} = \{ f^{t}(x) \mid 0 \leq t \leq t_{n_{3}} \} \) and \( \tilde{y}_{1} = B_{r}(x) - \tilde{y}_{0} - \bigcup_{j=0}^{N_{r}-1} \gamma_{P^{j}(x_{n_{3}})} \), whose lengths are bounded by a constant independently of \( r \).
Note that \( \lim_{\tilde{x} \to s_0} |\gamma_1| \in (0, \infty) \) because the vector field generating \( F \) is continuous and \( |\gamma| \in (0, \infty) \), and so we have \( \lim_{\tilde{x} \to s_0} |\tilde{\varphi}(\tilde{x})| < \infty \) due to the form of \( \tilde{\varphi} \) in (2.3). Applying (2.5) to \( \varphi \equiv 1 \) and (2.4) to \( \tilde{\varphi}_1(\tilde{x}) = |\gamma_1| \), we get that

\[
\lim_{r \to \infty} \frac{|B_r^+(x)|}{N_r} = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} |\gamma_j(x_{n_3})| = \lim_{\tilde{x} \to s_0} \frac{1}{\omega(\tilde{x})} \left( \lim_{\tilde{x} \to s_0} |\gamma_1| \right).
\]

Hence, it follows from (2.5) that

\[
\lim_{r \to \infty} \frac{1}{|B_r^+(x)|} \int_{B_r^+(x)} \varphi(y) dy = \left( \lim_{\tilde{x} \to s_0} |\gamma_1| \right)^{-1} \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} \tilde{\varphi}(P_j(x_{n_3})),
\]

if the limits exist. By using (2.4) again, we can see that the limit in the right-hand side exists and coincides with \( \lim_{\tilde{x} \to s_0} \tilde{\varphi}(\tilde{x})/|\gamma_1| \). This completes the proof. \( \square \)

**Remark 9.** The Bowen flow has an open set \( D \) surrounded by an attracting circuit consisting of two saddle singularities \( p_1 \) and \( p_2 \) and two heteroclinic orbits \( \gamma_1 \) and \( \gamma_2 \) connecting these singularities such that the time average of some continuous function does not exist at \( x \) and \( \omega(x) \) is the attracting circuit \( \gamma := \{ p_1 \} \cup \{ p_2 \} \cup \gamma_1 \cup \gamma_2 \) for every point \( x \) in \( D \) except the source \( \hat{p} \) of the flow ( [33]), see Fig. 1. On the other hand, length averages exist at every point in \( D \) for the singular foliation generated by the Bowen’s flow due to Proposition 8. The key is that the length of \( \gamma_1 \) in the proof of Proposition 8 is bounded by a constant independently of \( r \), while the time to pass \( \gamma_1 \), which is transverse to \( O \) such that the forward Poincaré map \( P : \Sigma \to \Sigma \) is well-defined and topologically semi-conjugate to a minimal interval exchange transformation \( E : \mathbb{S}^1 \to \mathbb{S}^1 \) on the circle \( \mathbb{S}^1 \) (recall that an interval exchange transformation is said to be minimal if every orbit is dense; refer to [38] for definition and basic properties of interval exchange transformations). Moreover, the semi-conjugacy \( h : \Sigma \to \mathbb{S}^1 \) is constructed as the quotient map induced by pairwise disjoint (possibly empty) closed intervals \( \{ I_n \}_{n \geq 1} \) of \( \Sigma \), that is, \( h(I_n) \) is a point set for each \( n \geq 1 \), the restriction of \( h \) on \( \Sigma - \bigcup_{n \geq 1} I_n \) is injective and \( h \circ P = E \circ h \). We call \( E \) the associated interval exchange transformation of the quasi-minimal set \( \tilde{O} \).

**Definition 10.** A quasi-minimal set of a \( C^1 \) flow \( F \) is said to be uniquely ergodic if the associated interval exchange transformation is uniquely ergodic (i.e. there is a unique invariant probability measure of the transformation). Furthermore, a quasi-minimal set of a \( C^1 \) flow \( F \) is said to be locally dense if \( h \) is injective. When every quasi-minimal set of \( F \) is uniquely ergodic or locally dense, we simply say that the union of quasi-minimal set of \( F \) is uniquely ergodic or locally dense, respectively.

**Remark 11.** There is an example of a minimal but non-uniquely ergodic interval exchange transformation ( [19]). On the other hand, it is rather typical for interval
exchange transformations to be uniquely ergodic: Recall that an interval exchange transformation is determined by the number of intervals $N \geq 2$, a length vector $(\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{R}^N$ and a permutation of $\{1, 2, \ldots, N\}$. It is a famous deep result ([24,37]) that for each $N$, $\pi$ and Lebesgue almost every $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, the interval exchange transformation given by $N$, $\lambda$ and $\pi$ is uniquely ergodic.

Furthermore, it is known that any interval exchange transformation with $N \leq 3$ is uniquely ergodic ([18]). We show that $N \leq 2$ always holds for the associated interval exchange transformations of a $C^1$ flow $F$ on a compact surface $S$ if the orientable genus of $S$ is less than two or the non-orientable genus of $S$ is less than four. It is known that the total number of quasi-minimal sets of $F$ cannot exceed $g$ if $S$ is an orientable surface of genus $g$ [23], and $h - 1$ if $S$ is a non-orientable surface of genus $h$ [25]. So, $S$ has no quasi-minimal set when the orientable genus of $S$ is zero or the non-orientable genus of $S$ is less than three. Moreover, since each boundary component of a foliated surface with no degenerate singularities is either transverse to the foliation or a union of leaves, we can assume that $S$ is a closed surface. Therefore, it is a direct consequence of Denjoy-Siegel Theorem that $N \leq 2$ when the orientable genus of $S$ is one.

We consider the number of intervals $N$ for a closed surface $S$ with non-orientable genus three. By classification theorem of closed surfaces, the Euler characteristic of $S$ is $2 - 3 = -1$. So, by cutting $S$ along a circle $\Sigma$ which is transverse to a quasi-minimal set of $F$, we obtain a projective plane with two punctures (note that its Euler characteristic is also $1 - 2 = -1$). The resulting surface is drawn on the left-side of Fig. 2, and two small circles (with $+$ and $-$) on the left-hand side correspond to $\Sigma$. Now we consider two (parts of) trajectories $a$ and $b$ which start from the circle with $+$ and land on the circle with $-$ (i.e. transversely return to $\Sigma$). When $N \geq 2$, we can take $a$ and $b$ as they are not parallel as depicted in Fig. 2. Here by parallel we mean that there is a trivial flow box whose transverse boundaries are $a$ and $b$ and whose tangential boundaries are the circles with $+$ and $-$. However, another trajectory $c$ has to be parallel to $a$ or $b$ (see the right-hand side of Fig. 2), and thus, we immediately get $N \leq 2$.

**Proposition 12.** There is a Lebesgue zero measure set $A$ such that if $x$ is outside of $A$ and the $\omega$-limit set of $x$ is a locally dense quasi-minimal set, then the length average of any continuous function along the forward orbit of $x$ exists.

**Proof.** Let $P$ be the Poincaré map on a circle $\Sigma$ which transversely intersects a quasi-minimal set $\overline{O}$. Since any interval exchange transformation preserves a Lebesgue measure, by the hypothesis, there is a zero Lebesgue measure set $A_0 \subset \Sigma$ such that the time average of $P$ for any continuous function on $\Sigma$ exists at every $x$ outside of $A_0$. Let $O_-(A_0) = \bigcup_{t \leq 0} f^t(A_0)$, then $O_-(A_0)$ be a Lebesgue zero measure set of $S$ (note that $O_-(A_0)$ can be decomposed into countably many zero measure sets
Let $y$ be a point outside of $O_-(A_0)$ such that the $\omega$-limit set of $y$ is $\overline{O}$. Then, one can find a real number $t_1 \geq 0$ such that $f^{t_1}(y) \in \Sigma$ because $\omega(y) = \overline{O}$, and it follows from $y \notin O_-(A_0)$ that $f^{t_1}(y) \notin A_0$. Therefore, the length average of any continuous function exists along the forward orbit of $y$ by Lemma 7 (see the proof of Proposition 8). Since there are at most finitely many quasi-minimal sets (cf. [12,26]), this completes the proof. □

**Proposition 13.** Let $x$ be a point whose $\omega$-limit set is a uniquely ergodic quasi-minimal set. Then, the length average of any continuous function along the forward orbit of $x$ exists.

**Proof.** The proof is basically same as the proof of Proposition 12 (recall that if a map is uniquely ergodic, then time averages exist everywhere for the map [17]). The only difference is that the semi-conjugacy $h$ between the associated Poincaré map $P : \Sigma \to \Sigma$ and the associated interval exchange transformation $E : S^1 \to S^1$ of the quasi-minimal set of $x$ may be not injective, that is, there is pairwise disjoint nonempty closed intervals $I_n$ of $\Sigma$ such that $h(I_n)$ is a point set for each $n \geq 1$. On the other hand, it can be managed due to the unique ergodicity of $E$: Let $\mu$ be an invariant measure of $P$, then the pushforward measure $h_*\mu$ of $\mu$ by $h$ (given by $h_*\mu(A) = \mu(h^{-1}A)$ for each Borel set $A \subset S^1$) is an invariant measure of $E$. Moreover, since $E$ is a uniquely ergodic interval exchange transformation, $h_*\mu$ is the Lebesgue measure of $S^1$, and thus we have $\mu(I_n) = h_*\mu(h(I_n)) = 0$ for each $n \geq 1$ because $h(I_n)$ is a point set and $I_n \subset h^{-1} \circ h(I_n)$. This immediately concludes that $P : \Sigma \to \Sigma$ is uniquely ergodic, and we complete the proof by repeating the argument in the proof of Proposition 12. □

Theorem 5 immediately follows from Theorem 6, Propositions 8, 12 and 13.

### 2.2. Proof of main theorems.

We will reduce existence of length averages for a $C^1$ singular foliation without degenerate singularities in Theorem A to existence of length averages for a singular foliation generated by a $C^1$ flow in Theorem 5. We say that a singular foliation $\mathcal{F}$ on a compact surface $S$ is orientable if the restriction of $\mathcal{F}$ on $S - \text{Sing}(\mathcal{F})$ is orientable. We need the following observation from [13, Remark 2.3.2].

**Lemma 14.** Any one-dimensional $C^1$ singular foliation on a closed manifold (i.e. a compact manifold without boundaries) is orientable if and only if it can be generated by a $C^1$ flow.

Fix a compact surface $S$ and a codimension one $C^1$ singular foliation $\mathcal{F}$ without degenerate singularities on $S$. To define unique ergodicity for $\mathcal{F}$ (in Definition 15), we need construct a codimension one $C^1$ orientable singular foliation $\mathcal{F}^*$ on an orientable compact surface $\hat{S}$ without boundary, together with a continuous surjection $p : \hat{S} \to S$ such that $\hat{\mathcal{F}}$ is the lifted singular foliation of $\mathcal{F}$ by $p$.

Firstly, if $S$ is orientable, simply set $\hat{S}_1 = S$ and $\hat{\mathcal{F}}_1 = \mathcal{F}$, together with the identity map $p_0 : \hat{S}_1 \to S$. In the case when $S$ is not orientable, we consider the usual orientation double covering $\hat{S}_1$ of $S$ (cf. [9]) and its induced foliation $\hat{\mathcal{F}}_1$ by the canonical projection $p_0 : \hat{S}_1 \to S$. (That is, we construct $\hat{S}_1$ as the set of pairs $(y, o)$, where $y$ is a point in $S$ and $o \in \{+,-\}$ is the orientation of $S$, and $p_0$ is given by $p_0(y, o) = y$.)

Secondly, when $\hat{S}_1$ has no boundaries, simply set $\hat{S}_2 = \hat{S}_1$ and $\hat{\mathcal{F}}_2 = \hat{\mathcal{F}}_1$, together with the identity map $p_1 : \hat{S}_2 \to \hat{S}_1$. When $\hat{S}_1$ has a boundary, let $\hat{S}_2$ be the double of $\hat{S}_1$ (i.e., $\hat{S}_2 = \hat{S}_1 \times [0,1]/\sim$, where $(x, 0) \sim (x, 1)$ for all $x \in \partial \hat{S}_1$). Note that $\hat{S}_2$ has no boundaries. See Fig. 3. We let $p_1 : \hat{S}_2 \to \hat{S}_1$ be the canonical projection, given
by $p_1(y, j) = y$ for $y \in \hat{S}_1 - \partial \hat{S}_1$ and $j = 0, 1$, and $p_1([y, 0]) = y$ for $y \in \partial \hat{S}_1$. Furthermore, we let $\hat{F}_2$ be the lifted foliation of $\hat{F}_1$ on $\hat{S}_2$ (denoted by $\hat{F}_1 \sqcup_{\partial} -\hat{F}_1$ in Fig. 3).

Thirdly, when $\hat{F}_2$ is orientable, simply set $\hat{S} = \hat{S}_2$ and $\hat{F} = \hat{F}_2$, together with the identity map $p_2 : \hat{S} \rightarrow \hat{S}_2$. In the case when $\hat{F}_2$ is not orientable, we consider the tangent orientation double covering ($\hat{S}', \hat{F}')$ of $(\hat{S}_2 - \text{Sing}(\hat{F}_2), \hat{F}_2 - \hat{S}_2)$ (cf. 2.3.5, p.16 of [13]) with the canonical projection $p'_2 : \hat{S}' \rightarrow \hat{S}_2$ (i.e. $p'_2(y, o) = y$ for $y \in \hat{S}_2 - \text{Sing}(\hat{F}_2)$ and the orientation $o \in \{+, -\}$ of $\hat{F}_2$ at $y$), where $\hat{S}_2$ is the set of singular leaves of $\hat{F}_2$. Let $\hat{S} = \hat{S}' \sqcup (\text{Sing}(\hat{F}_2) \times \{0, 1\})$, and define $p_2 : \hat{S} \rightarrow \hat{S}_2$ by $p_2(x) = p'_2(x)$ for $x \in \hat{S}'$ and $p_2(s, j) = s$ for $s \in \text{Sing}(\hat{F}_2)$ and $j \in \{0, 1\}$. We inductively define a metric $d_{\hat{S}}$ on $\hat{S}$ as follows: Let $d_{\hat{S}}(x, y) = d_{\hat{S}'}(x, y)$ for each $x, y \in \hat{S}'$, where $d_{\hat{S}'}$ is the induced metric of $d_{\hat{S}_2}$ by the double covering $p'_2$. Since $\text{Sing}(\hat{F}_2) \times \{0, 1\}$ is a finite set (see comments above Definition 1), we can denote it by $\{s_1, s_2, \ldots, s_N\}$. Assume that $d_{\hat{S}}$ is defined on $\hat{S}' \sqcup \{s_1, s_2, \ldots, s_k\}$ with some $0 \leq k < N$ (we let $\{s_1, \ldots, s_k\} = \emptyset$ if $k = 0$, for convenience). Then, it is straightforward to see that the restriction of $\hat{F}_2$ on a small ball $U$ centred at $p_2(s_{k+1})$ is orientable (because each point in $\text{Sing}(\hat{F}_2)$ is a sink, a source, a center or a saddle of a closed manifold $\hat{S}_2$), and thus, $p_2^{-1}(U - \{p_2(s_{k+1})\})$ consists of two connected components, (at least) one of which has a positive distance from $\{s_1, \ldots, s_k\}$. Denote the connected component by $A_{k+1}$, so that we have $d_{\hat{S}}(A_{k+1}, \{s_1, \ldots, s_k\}) > 0$. For each $x \in \hat{S}' \sqcup \{s_1, s_2, \ldots, s_k\}$, let $d_{\hat{S}}(s_{k+1}, x) := \lim_{n \to \infty} d_{\hat{S}}(y_n, x)$ with some sequence $\{y_n\}_{n \geq 1} \subset A_{k+1}$ such that $\lim_{n \to \infty} d_{\hat{S}_2}(p_2(s_{k+1}), p_2(y_n)) = 0$. By construction, it is not difficult to see that $p_2 : \hat{S} \rightarrow \hat{S}_2$ is a double covering, and that the smooth structure of $\hat{S}_2$ can induce a smooth structure of $\hat{S}$ by $p_2$. Since the restriction of $\hat{F}$ is orientable both on $p_2^{-1}(\hat{S}_2 - \text{Sing}(\hat{F}_2))$ (due to the orientability of $\hat{F}'$) and on a small neighbourhood of $p_2^{-1}(\text{Sing}(\hat{F}_2))$, $\hat{F}$ is an orientable foliation on an orientable closed surface $\hat{S}$.

We call the resulting foliation $\hat{F}$ the associated foliation of $F$. Note that $\hat{F}$ can be generated by a $C^1$ flow due to Lemma 14. We need the following definition.

**Definition 15.** We say that the union of quasi-minimal sets of a codimension one $C^1$ singular foliation $F$ on a compact surface $S$ is **uniquely ergodic** or **locally dense** if the union of quasi-minimal sets of the flow generating the associated foliation $\hat{F}$ is uniquely ergodic or locally dense, respectively (see Definition 10).
We are now ready to prove Theorem A and Corollary B.

Proof of Theorem A. Fix a codimension one $C^1$ singular foliation $\mathcal{F}$ on a compact surface $S$. Let $(\hat{S}, \hat{\mathcal{F}})$ be the associated foliated manifold of $(S, \mathcal{F})$ given above. We will show that if length averages exist at $x \in \hat{S}$ for $\hat{\mathcal{F}}$, then length averages also exist at $p(x)$ for $\mathcal{F}$, which immediately completes the proof by Theorem 5 and Lemma 14.

First we will show that if length averages exist at $x \in \hat{S}_1$ for $\hat{\mathcal{F}}_1$, then length averages also exist at $p_0(x) \in S$ for $\mathcal{F}$. It is trivial in the case $(\hat{S}_1, \hat{\mathcal{F}}_1) = (S, \mathcal{F})$, so we consider the other case. Assume that length averages exist at $x \in \hat{S}_1$ for $\hat{\mathcal{F}}_1$. We also assume that $\hat{\mathcal{F}}_1(p_0(x))$ is not compact because otherwise the claim obviously holds. Let $\varphi : S \to \mathbb{R}$ be a continuous function. Then, $\hat{\varphi} := \varphi \circ p_0$ is continuous, and thus the length average of $\hat{\varphi}$ along the leaf of $\hat{\mathcal{F}}_1$ through $x$, given by

$$\lim_{r \to \infty} \frac{1}{|B_r^{\hat{\mathcal{F}}_1}(x)|} \int_{B_r^{\hat{\mathcal{F}}_1}(x)} \hat{\varphi}(\check{x}) d\check{x},$$

exists because of the assumption for length averages on $x$. On the other hand, it is straightforward to see that $|\det Dp_0| \equiv 1$ and $p_0(B_r^{\hat{\mathcal{F}}_1}(x)) = B_r^\mathcal{F}(p_0(x))$. (Note that, although $p_0$ is not injective, $p_0|_L$ is injective for all non-compact $L \in \hat{\mathcal{F}}_1$ since $p_0(L)$ is a leaf.) Therefore, by changing the variables formula, we have that

$$\frac{1}{|B_r^\mathcal{F}(p_0(x))|} \int_{B_r^\mathcal{F}(p_0(x))} \varphi(\check{y}) d\check{y} = \frac{1}{|B_r^{\hat{\mathcal{F}}_1}(x)|} \int_{B_r^{\hat{\mathcal{F}}_1}(x)} \hat{\varphi}(\check{x}) d\check{x}. \ (2.7)$$

Since $\varphi$ is arbitrary, it follows from (2.6) and (2.7) that length averages exist at $p_0(x)$ for $\mathcal{F}$.

Next we will show that if length averages exist at $x \in \hat{S}_2$ for $\hat{\mathcal{F}}_2$, then length averages also exist at $p_1(x) \in \hat{S}_1$ for $\hat{\mathcal{F}}_1$. Only the case when $\hat{S}_2$ is the double of $\hat{S}_1$ is considered because the other case is trivial. Let $x \in \hat{S}_2$ be a point at which length averages exist. If $\hat{\mathcal{F}}_1(p_1(x))$ does not have an intersection with $\partial \hat{S}_1$, then it is straightforward to see that for any continuous function $\varphi$ on $\hat{S}_2$, the length average of $\varphi$ along $\hat{\mathcal{F}}_2(x)$ coincides with the length average of a continuous function $\hat{\varphi} = \varphi \circ p_1$ along $\hat{\mathcal{F}}_1(p_1(x))$ (see the argument in the previous paragraph), so length averages at $p_1(x)$ exist. Moreover, if $\hat{\mathcal{F}}_1(p_1(x))$ is included in $\partial \hat{S}_1$, then the length of $\mathcal{F}(p_1(x))$ is finite and length averages at $p_1(x)$ exist. Therefore, we consider the case when $\hat{\mathcal{F}}_1(p_1(x)) \cap \partial \hat{S}_1 \neq \emptyset$ and $\hat{\mathcal{F}}_1(p_1(x)) \nsubseteq \partial \hat{S}_1$. If $\hat{\mathcal{F}}_1(p_1(x))$ is transverse to $\partial \hat{S}_1$ with respect to both direction, then $\hat{\mathcal{F}}_1(p_1(x))$ is (diffeomorphic to) a closed interval, and so length averages at $p_1(x)$ exist. If $\hat{\mathcal{F}}_1(p_1(x))$ is transverse to $\partial \hat{S}_1$ at exactly one point $p_1(y)$ with $y \in \partial \hat{S}_2$, then by repeating the argument in the previous paragraph, one can check that for any continuous function $\varphi$ on $\hat{S}_2$, the length average of $\varphi$ along $\hat{\mathcal{F}}_2(x) = \hat{\mathcal{F}}_2(y)$ coincides with double of the length average of a continuous function $\hat{\varphi} = \varphi \circ p_1$ along $\hat{\mathcal{F}}_1(p_1(x)) = \hat{\mathcal{F}}_1(p_1(y))$, so length averages exist at $p_1(x)$.

Finally we need to show that if length averages exist at $x \in \hat{S}$ for $\hat{\mathcal{F}}$, then length averages also exist at $p_2(x) \in \hat{S}_2$ for $\hat{\mathcal{F}}_2$. However, it is completely analogous to the previous argument for the double covering $p_0 : \hat{S}_1 \to S$ because $p_2 : \hat{S} \to \hat{S}_2$ is also a double covering. This completes the proof. □
Proof of Corollary B. Let $S$ be a compact surface with a regular foliation $\mathcal{F}$. Since the Euler characteristic of any manifold with a regular foliation is 0 by the Poincaré-Hopf theorem, $S$ is a torus, an annulus, a Möbius band or a Klein bottle. On the other hand, the total number of quasi-minimal sets of $\mathcal{F}$ cannot exceed $g$ if $S$ is an orientable surface of genus $g$ [23], and $\frac{g+1}{2}$ if $S$ is a non-orientable surface of genus $h$ [25]. So, when $S$ is not a torus, there is no quasi-minimal set of $\mathcal{F}$. When $S$ is a torus, the union of quasi-minimal sets of $\mathcal{F}$ is uniquely ergodic by Denjoy-Siegel theorem. Therefore, we complete the proof of Corollary B by Theorem A. □

3. Examples

Example 16. There is a smooth codimension one regular foliation on an unbounded manifold without length averages. Let $M = \mathbb{R}^k \times \mathbb{R}^{m-k}$ and $\mathcal{F} = \{ \mathbb{R}^k \times \{ t \} \mid t \in \mathbb{R}^{m-k} \}$. Let $\varphi : M \to [-1, 1]$ be a continuous function such that $\varphi(x, y) = 1$ if $|x| \in \bigcup_{n \in \mathbb{N}}(10^{2n-1} + 1, 10^{2n} - 1)$ and $\varphi(x, y) = -1$ if $|x| \in \bigcup_{n \in \mathbb{N}}(10^{2n} + 1, 10^{2n+1} - 1)$ with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{m-k}$. Then, it follows from a straightforward calculation that the length average of $\varphi$ does not exist. Moreover, if the above foliated manifold can be isometrically embedded into a foliated $m$-dimensional manifold with $k$-dimensional leaves, then there is a nonempty open subset which consists of leaves without length averages.

Example 17. There is a smooth codimension one regular foliation on an incomplete manifold without length averages. We are indebted to Masayuki Asaoka for the construction. Let $M$ be an open disk in $\mathbb{R}^2$. Let $D_n \subset M$ ($n \geq 1$) be a square whose side has a length $(\frac{3}{4})^n$ such that $D_n$ borders on $D_{n+1}$ in the right side and that $D_n$ accumulates to the boundary of $M$. Also, define a piecewise linear leaf $L$ of $M$ as $L \cap D_n$ is the Koch curve of level $n$ scaled by $(\frac{3}{4})^n$ (refer to [10,39] for the definition of the Koch curve). See Figure 4. Since the length of the Koch curve of level $n$ is $(\frac{3}{4})^n$, the length of $L \cap D_n$ is 1. Finally, let $U$ be a small neighbourhood of $\cup_{n \in \mathbb{N}}D_n$ such that $L \cap (U - \cup_{n \in \mathbb{N}}D_n)$ has finite length. Let $\varphi : M \to \mathbb{R}$ be a uniformly bounded continuous function such that $\varphi(x) = 1$ if $x \in D_n (10^{2m-1} + 1 \leq n \leq 10^{2m} - 1, m \geq 1)$, $\varphi(x) = -1$ if $x \in D_n (10^{2m} + 1 \leq n \leq 10^{2m+1} - 1, m \geq 1)$ and $\varphi(x) = 0$ if $x \notin U$. Then, it is easy to check that on $L$, the length average of $\varphi$ (in the sense of piecewise $C^1$) does not exist. Furthermore, one can find a smooth curve along which the length average of $\varphi$ does not exist by applying a standard argument for mollifiers to the piecewise linear curve $L$.

Example 18. One can find points without length averages for a smooth codimension one foliation with degenerate singularities on a compact surface. The idea of the construction is basically same as the previous example. Let $M$ be a compact surface and $L$ be a leaf constructed in the same manner as in the previous example, except that $D_n$ accumulates

\[ L \cap (\cup_{n \in \mathbb{N}}D_n) = \cup_{n \in \mathbb{N}}(\ell_j(n - 1, n)) \] and $L \cap (U - \cup_{n \in \mathbb{N}}D_n)$ has finite length, the length average of $\varphi$ also does not exist."
to a point in $M$. Then, on the leaf $L$, some length average does not exist by virtue of the argument in the previous example (so that the accumulation point must be a degenerate singular point by Theorem A).

**Remark 19.** Examples 17 and 18 may show that (non-)existence of length averages for foliations is not preserved by some homeomorphisms of foliations (but preserved by any $C^1$ diffeomorphisms of foliations), while (non-)existence of time averages for flows is preserved by topological conjugacy.

**Example 20.** There is a codimension two regular foliation on a compact manifold without length averages. Let $P$ be a diffeomorphism on a compact manifold $N$ such that there exists a positive Lebesgue measure set $D$ consisting of points without time averages. As we mentioned in Sect. 1, there are several examples of diffeomorphisms $P$ satisfying the condition for time averages. Furthermore, for simplicity, we assume that the backward orbit of each $x \in D$ along $P$ accumulates to a source $\hat{p}$, such as the time-one map of the Bowen flow (pictured in Fig. 1, see Remark 9). Let $\mathcal{F}$ be the trivial suspension of $P$. That is, $M = N \times [0, 1] / \sim$, where $(\tilde{x}, 1) \sim (P(\tilde{x}), 0)$ for $\tilde{x} \in N$, and each leaf $L$ of $\mathcal{F}$ is of the form $\bigcup_{n \in \mathbb{Z}} \{ P^n(\tilde{x}) \} \times [0, 1] / \sim$. Fix $\tilde{x} \in D$ and a continuous function $\tilde{\phi} : N \to \mathbb{R}$ such that the time average of $\tilde{\phi}$ along the forward orbit of $\tilde{x}$ does not exist. Let $\phi$ be a continuous function on $M$ given by $\phi(y, s) = \tilde{\phi}(y)$ for each $(y, s) \in N \times (0, 1)$, so that $\tilde{\phi}(y) = \int_{B_{\frac{1}{\sqrt{2}}}((y, \frac{1}{2}))} \phi(z) dz$. Then, it is not difficult to check that the length average of $\phi$ along the leaf $\mathcal{F}(\tilde{x}, \frac{1}{2})$, given in (1.2), equals to $\left( \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \tilde{\phi}(P^j(\tilde{x})) + \tilde{\phi}(\hat{p}) \right) / 2$, which does not exist due to the choice of $\tilde{x}$ and $\tilde{\phi}$.

**Remark 21.** Example 20 implies a canonical correspondence between time averages for a suspension flow and length averages for the foliation generated by the flow. This observation may be helpful when one considers existence problem of length averages for higher dimensional foliations (in particular, codimension-one foliations), such as Sacksteder foliations ([29]).
Acknowledgements. We are deeply grateful to anonymous reviewers for many suggestions, all of which substantially improved the paper.

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

1. Alves, J.F., Bonatti, C., Viana, M.: SRB measures for partially hyperbolic systems whose central direction is mostly expanding. In: The Theory of Chaotic Attractors, pp. 443–490 (2000)
2. Alves, J.F.: SRB measures for non-hyperbolic systems with multidimensional expansion. Ann. Sci. Ecol. Norm. Supér. 33, 1–32 (2000)
3. Barreira, L., Schmeling, J.: Sets of “non-typical” points have full topological entropy and full Hausdorff dimension. Isr. J. Math. 116(1), 29–70 (2000)
4. Bowen, R., Ruelle, D.: The ergodic theory of Axiom A flows. In: The Theory of Chaotic Attractors, pp. 55–76 (1975)
5. Candel, A., Conlon, L.: Foliations I. Graduate Studies in Mathematics, vol. 23. American Mathematical Society, Providence (2000)
6. Candel, A., Conlon, L.: Foliations II. Graduate Studies in Mathematics, vol. 60. American Mathematical Society, Providence (2003)
7. Cantwell, J., Conlon, L.: Endsets of exceptional leaves: a theorem of G. Duminy. In: Foliations: Geometry and Dynamics, pp. 225–261 (2002)
8. Deroin, B., Kleptsyn, V., Navas, A.: On the ergodic theory of free group actions by real-analytic circle diffeomorphisms. Invent. Math. 212(3), 731–779 (2018)
9. Dold, A.: Lectures on Algebraic Topology, Classics in Mathematics, Springer, Berlin (Reprint of the 1972 edition) (1995)
10. Falconer, K.: Fractal Geometry: Mathematical Foundations and Applications. Wiley, New York (2004)
11. Ghys, E., Langevin, R., Walczak, P.: Entropie géométrique des feuilletages. Acta Math. 160(1), 105–142 (1988)
12. Gutierrez, C.: Smoothing continuous flows on two-manifolds and recurrences. Ergod. Theory Dyn. Syst. 6(1), 17–44 (1986)
13. Hector, G., Hirsch, U.: Introduction to the geometry of foliations. Part A. Foliations on compact surfaces, fundamentals for arbitrary codimension, and holonomy. In: Aspects of Mathematics, vol. 1, Friedr. Vieweg & Sohn, Braunschweig (1986)
14. Hector, G., Hirsch, U.: Introduction to the geometry of foliations. Part B. Foliations of codimension one. In: Aspects of Mathematics, vol. 3, Friedr. Vieweg & Sohn, Braunschweig (1987)
15. Hirsch, M.: A stable analytic foliation with only exceptional minimal sets. In: Manning, A. (ed.) Lecture Notes in Mathematics, vol. 468. Springer, Berlin (1975)
16. Hofbauer, F., Keller, G.: Quadratic maps without asymptotic measure. Commun. Math. Phys. 127(2), 319–337 (1990)
17. Katok, A., Hasselblatt, B.: Introduction to the Modern Theory of Dynamical Systems. Cambridge University Press, Cambridge (1995)
18. Keane, M.: Interval exchange transformations. Math. Z. 141(1), 25–31 (1975)
19. Keynes, H.B., Newton, D.: A “minimal”, non-uniquely ergodic interval exchange transformation. Math. Z. 148(2), 101–105 (1976)
20. King, S., O’Meara, K.: Takens’ last problem and existence of non-trivial wandering domains. Adv. Math. 306, 524–588 (2017)
21. Kubarski, J.: About Stefan’s definition of a foliation with singularities: a reduction of the axioms. Bull. Soc. Math. Fr. 118(4), 391–394 (1990)
22. Labouriau, I.S., Rodrigues, A.A.: On Takens’ last problem: tangencies and time averages near heteroclinic networks. Nonlinearity 30(5), 1876 (2017)
23. Markley, N.G.: On the number of recurrent orbit closures. In: Proceedings of the American Mathematical Society, pp. 413–416 (1970)
24. Masur, H.: Interval exchange transformations and measured foliations. Ann. Math. 115(1), 169–200 (1982)
25. Mayer, A.: Trajectories on the closed orientable surfaces. Rec. Math. 12(54), 71–84 (1943)
26. Nikolaev, I., Zhuzhoma, E.: Flows on 2-Dimensional Manifolds. Lecture Notes in Mathematics, vol. 1705. Springer, Berlin (1999)
27. Ruelle, D.: Resonances for Axiom A Flows. Institut des Hautes Etudes Scientifiques (1986)
28. Ruelle, D.: Historical behaviour in smooth dynamical systems. In: Global Analysis of Dynamical Systems, pp. 63–66 (2001)
29. Sacksteder, R.: On the existence of exceptional leaves in foliations of co-dimension one. In: Annales de l’institut Fourier, pp. 221–225 (1964)
30. Smale, S.: Differentiable dynamical systems. Bull. Am. Math. Soc. 73(6), 747–817 (1967)
31. Stefan, P.: Accessibility and foliations with singularities. Bull. Am. Math. Soc. 80(6), 1142–1145 (1974)
32. Sussmann, H.J.: Orbits of families of vector fields and integrability of distributions. Trans. Am. Math. Soc. 180, 171–188 (1973)
33. Takens, F.: Heteroclinic attractors: time averages and moduli of topological conjugacy. Bol. Soc. Bras. Mat. Bull. Braz. Math. Soc. 25(1), 107–120 (1994)
34. Takens, F.: Orbits with historic behaviour, or non-existence of averages. Nonlinearity 21(3), T33–T36 (2008)
35. Tamura, I.: Topology of foliations: an introduction. In: Translations of Mathematical Monographs, vol. 97, American Mathematical Society, Providence (Translated from the 1976 Japanese edition and with an afterword by Kiki Hudson, with a foreword by Takashi Tsuboi) (1992)
36. Thompson, D.: The irregular set for maps with the specification property has full topological pressure. Dyn. Syst. 25(1), 25–51 (2010)
37. Veech, W.A.: Gauss measures for transformations on the space of interval exchange maps. Ann. Math. 1, 201–242 (1982)
38. Viana, M.: Complutense. Rev. Mat. Complut. 19(1), 7–100 (2006)
39. von Koch, H.: Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire. Ark. Mat. Astron. Fys. 1, 681–704 (1904)
40. Walczak, P.: Dynamics of Foliations, Groups and Pseudogroups, vol. 64. Birkhäuser, Boston (2012)

Communicated by C. Liverani