Primary Non-QE Graphs on Six Vertices

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Abstract  A connected graph is called of non-QE class if it does not admit a quadratic embedding in a Euclidean space. A non-QE graph is called primary if it does not contain a non-QE graph as an isometrically embedded proper subgraph. The graphs on six vertices are completely classified into the classes of QE graphs, of non-QE graphs, and of primary non-QE graphs.

Key words  distance matrix, quadratic embedding, quadratic embedding constant, primary non-QE graph

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1 Introduction

A quadratic embedding of a (finite connected) graph $G = (V, E)$ in a Euclidean space $\mathbb{R}^N$ is a map $\varphi : V \to \mathbb{R}^N$ satisfying

$$
||\varphi(x) - \varphi(y)||^2 = d(x, y), \quad x, y \in V,
$$

where the left-hand side is the square of the Euclidean distance between two points $\varphi(x)$ and $\varphi(y)$, and the right-hand side is the graph distance. We say that a graph $G$ is of QE class or of non-QE class according as it admits a quadratic embedding or not. The concept of quadratic embedding (of a metric space in general) traces back to the early works of Schoenberg [25, 26] and has been discussed along with the so-called Euclidean distance geometry [1, 3, 11, 12, 14], for wider aspects see also [7, 8, 16] and references cited therein.

There are interesting questions both on graphs of QE class and on those of non-QE class. One of the important questions on non-QE graphs is to obtain a sufficiently rich list of non-QE graphs. If a graph $H$ is isometrically embedded
in a graph \( G \) and if \( G \) is of QE class, so is \( H \). In other words, if \( H \) is isometrically embedded in a graph \( G \) and if \( H \) is of non-QE class, so is \( G \). Thus, upon classifying graphs of non-QE class it is important to specify a primary non-QE graph, that is, a non-QE graph \( G \) which does not contain a non-QE graph \( H \) as an isometrically embedded proper subgraph.

In this paper we complete the classification of graphs on six vertices, which is seen as a milestone of our new approach of the QE constant explained below. As a result, among 112 graphs on six vertices there are 3 primary non-QE graphs, 24 non-primary non-QE graphs, and 85 QE graphs. Employing the list of small connected graphs due to McKay [17], which is reproduced in Appendix C, the results are stated as follows.

**Theorem 1.1.** Among 112 graphs on six vertices the primary non-QE graphs are \( G6-30 \), \( G6-60 \) and \( G6-84 \), see Figures 1 and 2.

**Theorem 1.2.** Among 112 graphs on six vertices the non-primary non-QE graphs are \( G6-22 \), \( G6-36 \), \( G6-40 \), \( G6-45 \), \( G6-49 \), \( G6-53 \), \( G6-54 \), \( G6-55 \), \( G6-64 \), \( G6-67 \), \( G6-71 \), \( G6-72 \), \( G6-73 \), \( G6-78 \), \( G6-85 \), \( G6-86 \), \( G6-88 \), \( G6-91 \), \( G6-94 \), \( G6-96 \), \( G6-101 \), \( G6-102 \), \( G6-103 \) and \( G6-107 \). Moreover, these graphs contain \( G5-10 \) or \( G5-17 \) as an isometrically embedded proper subgraph, see Figure 3.

**Theorem 1.3.** Among 112 graphs on six vertices the graphs not listed in Theorems 1.1 and 1.2 are of QE class.

![Figure 1: G6-30 (left) and G6-60 (right)](image1)

![Figure 2: G6-84](image2)
We briefly mention our quantitative approach. Let $G = (V, E)$ be a graph (always assumed to be finite and connected). Let $C(V)$ be the space of $\mathbb{R}$-valued functions on $V$ equipped with the canonical inner product denoted by $\langle \cdot, \cdot \rangle$. The distance matrix $D = [d(x, y)]_{x,y \in V}$ acts on $C(V)$ from the left as usual. It follows from Schoenberg’s theorem \cite{25,26} that $G$ is of QE class if and only if the distance matrix $D = [d(x, y)]_{x,y \in V}$ is conditionally negative definite, namely, $\langle f, D f \rangle \leq 0$ for all $f \in C(V)$ with $\langle 1, f \rangle = 0$, where $1 \in C(V)$ stands for the constant function taking value one. It is then natural to consider the conditional maximum defined by

$$\text{QEC}(G) = \max \{ \langle f, D f \rangle ; \langle f, f \rangle = 1, \langle 1, f \rangle = 0 \},$$

(1.1)

which is called the quadratic embedding constant (QE constant for short) of $G$. Apparently, the QE constant provides a criterion for a graph to be of QE class or not. In fact, a graph $G$ on at least two vertices is of QE class if and only if $\text{QEC}(G) \leq 0$.

In the recent paper \cite{23} we started a systematic study of the QE constant as a new quantitative invariant of a graph, where we obtained basic formulas as well as examples. Since then we have collected explicit values of the QE constants of particular graphs \cite{10,18,21,24} and formulas in relation to graph operations \cite{15,19}. We are also interested in classifying the finite connected graphs in terms of the QE constants \cite{4}, where we started an attempt to characterize graphs along with the increasing sequence of the QE constants of paths.

Moreover, the QE constant is interesting from the point of view of spectral analysis of distance matrices \cite{2,28}. In fact, $\text{QEC}(G)$ lies between the largest and the second largest eigenvalues of the distance matrix $D$ in such a way that $\lambda_2(D) \leq \text{QEC}(G) < \lambda_1(D)$. It is an interesting open question to characterize graphs $G$ with $\lambda_2(D) = \text{QEC}(G)$. Any transmission-regular graph, of which the distance matrix has a constant row sum by definition, has this property and also any path $P_{2n}$ on even vertices \cite{18}. While, the equality holds for any distance regular graph and all distance-regular graphs with $\text{QEC}(G) \leq 0$ are listed in \cite{13}.

As another interesting aspect of the QE constant we mention a relation to noncommutative harmonic analysis. The entry-wise exponential matrix of the distance matrix is called the $Q$-matrix and is defined by $Q = Q_q = [q^{d(x,y)}]_{x,y \in V}$, where $-1 \leq q \leq 1$. Let $\pi(G) \subset [-1, 1]$ be the region of $q$ such that $Q = Q_q$ is positive definite (allowing zero eigenvalue). By the well-known general theory on positive definite kernels we know that $D$ is conditionally negative definite if and only if $Q = Q_q$ is positive definite for all $0 \leq q \leq 1$, that is $[0, 1] \subset \pi(G)$. The last condition is essential for noncommutative harmonic analysis, first appeared in \cite{5} in relation to the length function of a free group, and also for $q$-deformed spectral analysis of growing graphs \cite{9}. Here an interesting question related to $\text{QEC}(G)$ arises to determine the positivity region $\pi(G)$. Some concrete examples are found.
in $[20]$ and the graphs $G$ with $\pi(G) = [-1, 1]$ are recently characterized in $[27]$. This paper is organized as follows. In Section 2 we prepare basic notions and notations, for more details see $[19, 23]$. In Section 3 we mention some criteria for a graph to be of QE or of non-QE class and sieve the graphs on six vertices. In particular, we achieve Theorem 1.2 by showing that a non-primary non-QE graph on six vertices contains $G_{5-10}$ or $G_{5-17}$ (these are the primary non-QE graphs on five vertices) as an isometrically embedded subgraph. In Section 4 we complete the classification by calculating the QE constants of the remainders which are not judged during the previous section. Appendices A and B contain formulas for $\text{QEC}(K_n \wedge K_m, 1)$ and $\text{QEC}(K_n \backslash P_4)$, respectively. Appendix C contains the list of graphs on six vertices, which is reproduced from McKay $[17]$.

**Notation:** We write “$G_{m-n}$” for the graph on $m$ vertices with the item number $n$ in the list of small connected graphs due to McKay $[17]$. The list of graphs on six vertices is reproduced in Appendix C.

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## 2 Quadratic Embedding of Graphs

### 2.1 Distance Matrices

A graph $G = (V, E)$ is a pair of a non-empty set $V$ of vertices and a set $E$ of edges, where an edge is a set of two distinct vertices in $V$. For $x, y \in V$ we write $x \sim y$ if $\{x, y\} \in E$. A graph is called connected if any pair of vertices $x, y \in V$ there exists a finite sequence of vertices $x_0, x_1, \ldots, x_m \in V$ such that $x = x_0 \sim x_1 \sim \cdots \sim x_m = y$. In that case the sequence of vertices is called a walk from $x$ to $y$ of length $m$. Unless otherwise stated, by a graph we mean a finite connected graph throughout this paper.

Let $G = (V, E)$ be a graph (assumed to be finite and connected). For $x, y \in V$ with $x \neq y$ let $d(x, y) = d_G(x, y)$ denote the length of a shortest walk connecting $x$ and $y$. By definition we set $d(x, x) = 0$. Then $d(x, y)$ becomes a metric on $V$, which we call the graph distance. The distance matrix of $G$ is defined by

$$D = [d(x, y)]_{x,y \in V},$$

which is a matrix with index set $V \times V$.

Let $C(V)$ be the linear space of $\mathbb{R}$-valued functions $f$ on $V$. We always identify $f \in C(V)$ with a column vector $[f_x]_{x \in V}$ through $f_x = f(x)$. The canonical inner
product on \( C(V) \) is defined
\[
\langle f, g \rangle = \sum_{x \in V} f(x)g(x), \quad f, g \in C(V).
\]
As usual, the distance matrix \( D \) gives rise to a linear transformation on \( C(V) \) by matrix multiplication. Since \( D \) is symmetric, we have \( \langle f, Dg \rangle = \langle Df, g \rangle \).

### 2.2 Quadratic Embedding

A **quadratic embedding** of a graph \( G = (V, E) \) in a Euclidean space \( \mathbb{R}^N \) is a map \( \varphi : V \to \mathbb{R}^N \) satisfying
\[
\|\varphi(x) - \varphi(y)\|^2 = d(x, y), \quad x, y \in V,
\]
where the left-hand side is the square of the Euclidean distance between two points \( \varphi(x) \) and \( \varphi(y) \). A graph \( G \) is called of **QE class** or of **non-QE class** according as it admits a quadratic embedding or not.

In general, a graph \( H = (V', E') \) is called a **subgraph** of \( G = (V, E) \) if \( V' \subset V \) and \( E' \subset E \). If both \( G \) and \( H \) are connected, they have their own graph distances. If they coincide in such a way that
\[
d_H(x, y) = d_G(x, y), \quad x, y \in V',
\]
we say that \( H \) is **isometrically embedded** in \( G \). The next assertion is obvious by definition but useful.

**Lemma 2.1.** Let \( G = (V, E) \) and \( H = (V', E') \) be graphs and assume that \( H \) is isometrically embedded in \( G \).

1. If \( G \) is of QE class, so is \( H \).
2. If \( H \) is of non-QE class, so is \( G \).

The following definition is then adequate for classifying graphs of non-QE class.

**Definition 2.2.** A graph of non-QE class is called **primary** if it contains no isometrically embedded proper subgraphs of non-QE class.

Recall that the **diameter** of a graph \( G = (V, E) \) is defined by
\[
\text{diam}(G) = \max\{d(x, y) ; x, y \in V\}.
\]
The next result is useful, of which the proof is straightforward, see also [22].

**Lemma 2.3.** Let \( G = (V, E) \) be a graph and \( H = (V', E') \) a subgraph.

1. If \( H \) is isometrically embedded in \( G \), then \( H \) is an induced subgraph of \( G \).
2. If \( H \) is an induced subgraph of \( G \) and \( \text{diam}(H) \leq 2 \), then \( H \) is isometrically embedded in \( G \).
2.3 Quadratic Embedding Constants

Let \( G = (V, E) \) be a graph with \(|V| \geq 2\). The quadratic embedding constant (QE constant for short) of \( G \) is defined by

\[
\text{QEC}(G) = \max \{ \langle f, Df \rangle : f \in C(V), \langle f, f \rangle = 1, \langle 1, f \rangle = 0 \},
\]
where \( 1 \in C(V) \) is defined by \( 1(x) = 1 \) for all \( x \in V \).

It follows from Schoenberg [25, 26] that a graph \( G \) admits a quadratic embedding if and only if the distance matrix \( D = [d(x, y)] \) is conditionally negative definite, i.e., \( \langle f, Df \rangle \leq 0 \) for all \( f \in C(V) \) with \( \langle 1, f \rangle = 0 \). Hence a graph \( G \) is of QE class (resp. of non-QE class) if and only if \( \text{QEC}(G) \leq 0 \) (resp. \( \text{QEC}(G) > 0 \)).

In order to get the value of \( \text{QEC}(G) \) we have established a standard method on the basis of Lagrange’s multipliers.

**Proposition 2.4** ([23]). Let \( D \) be the distance matrix of a graph \( G \) on \( n \) vertices with \( n \geq 3 \). Let \( S \) be the set of all stationary points \( (f, \lambda, \mu) \) of

\[
\varphi(f, \lambda, \mu) = \langle f, Df \rangle - \lambda(\langle f, f \rangle - 1) - \mu(1, f),
\]
where \( f \in C(V) \cong \mathbb{R}^n \), \( \lambda \in \mathbb{R} \) and \( \mu \in \mathbb{R} \). Then we have

\[
\text{QEC}(G) = \max \{ \lambda : (f, \lambda, \mu) \in S \}.
\]

For the range of \( \text{QEC}(G) \) we have the following results.

**Proposition 2.5** ([4]). For any graph \( G \) we have \( \text{QEC}(G) \geq -1 \) and the equality holds if and only if \( G \) is a complete graph.

Thus, in order to determine \( \text{QEC}(G) \) of a graph \( G \) which is not a complete graph, it is sufficient to seek out the stationary points of \( \varphi(f, \lambda, \mu) \) with \( \lambda > -1 \) and then to specify the maximum of \( \lambda \) appearing therein.

A table of the QE constants of graphs on \( n \leq 5 \) vertices is available [23]. By direct observation we have the following

**Theorem 2.6** ([23]). All graphs on \( n \leq 4 \) vertices are of QE class. There are just two non-QE graphs on five vertices, that are \( G_5-10 \) and \( G_5-17 \), see Figure 3. Their QE constants are given by

\[
\text{QEC}(G_5-10) = \frac{2}{5}, \quad \text{QEC}(G_5-17) = \frac{4}{11 + \sqrt{161}} \approx 0.1689.
\]

Moreover, both are primary non-QE graphs.

**Lemma 2.7.** Let \( G = (V, E) \) and \( H = (V', E') \) be two graphs with \(|V| \geq 2\) and \(|V'| \geq 2\). If \( H \) is isometrically embedded in \( G \), we have \( \text{QEC}(H) \leq \text{QEC}(G) \).

The proof is obvious as the distance matrix of \( H \) is a principal submatrix of the one of \( G \), see [23]. Note that Lemma 2.7 follows by comparison of the QE constants.
3 Connected Graphs on Six Vertices

We will classify the graphs on six vertices into the classes of QE graphs, of non-QE graphs, and of primary non-QE graphs along the following steps:

Step 1 Sieve out all graphs which are the star products or Cartesian products of two graphs of QE class. Those graphs are of QE class.

Step 2 Sieve out all non-primary non-QE graphs.

Step 3 Use some special series of graphs of which the QE constants are known.

Step 4 Sieve out all graphs which are graph joins of two regular graphs.

Step 5 Construct explicitly quadratic embeddings from known ones of smaller graphs.

Step 6 Remaining graphs are judged by explicit calculation of the QE constant.

3.1 Star and Cartesian Products

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs. The Cartesian product of \( G_1 \) and \( G_2 \) is a graph on the vertex set \( V = V_1 \times V_2 \) with the adjacency relation

\[
(x_1, y_1) \sim (x_2, y_2) \iff x_1 = x_2, \ y_1 \sim y_2 \text{ or } x_1 \sim x_2, \ y_1 = y_2.
\]

The Cartesian product of \( G_1 \) and \( G_2 \) is denoted by \( G_1 \times G_2 \). Assume that \( V_1 \) and \( V_2 \) are disjoint, and choose \( o_1 \in V_1 \) and \( o_2 \in V_2 \) as distinguished vertices (or roots) of \( G_1 \) and \( G_2 \), respectively. Then the star product of \( G_1 \) and \( G_2 \) with respect to \( o_1 \) and \( o_2 \) is a graph obtained by putting two graphs together at the distinguished vertices \( o_1 \) and \( o_2 \), which are then identified. The star product depends on the choice of distinguished vertices and is denoted by \( G_1 \star (o_1,o_2) \) \( G_2 \). For simplicity we write \( G_1 \star G_2 \) whenever there is no danger of confusion.

Proposition 3.1 ([23]). If two graphs \( G_1 \) and \( G_2 \) are of QE class, so is their Cartesian product \( G_1 \times G_2 \).
Proposition 3.2 ([23]). If two graphs $G_1$ and $G_2$ are of QE class, so is their star product $G_1 \star G_2$.

Remark 3.3. By definition any graph $G$ admits a Cartesian product structure as $G = G \times K_1$, where $K_1$ is the graph consisting of a single vertex, and similarly a star product structure $G = G \star K_1$. Those trivial cases are not excluded from Propositions 3.1 and 3.2. In fact, a quadratic embedding of $K_1$ in a Euclidean space is trivial. However, for the QE constant we need to avoid $K_1$ since $\text{QEC}(K_1)$ is not defined.

Remark 3.4. It is shown [23] that $\text{QEC}(G_1 \times G_2) = 0$ for any non-trivial Cartesian product of two QE graphs $G_1$ and $G_2$. On the other hand, it is a highly non-trivial problem to describe $\text{QEC}(G_1 \star G_2)$ in terms of $\text{QEC}(G_1)$ and $\text{QEC}(G_2)$. Some useful estimates are known, see [4, 19].

In view of the unique non-trivial factorization $6 = 2 \times 3$ we can easily specify all graphs on six vertices which admit non-trivial Cartesian product structures. It is also a simple task to check whether a graph on six vertices admits a non-trivial star product structure or not. If $G = G_1 \times G_2$ or $G = G_1 \star G_2$ is such a non-trivial product, the numbers of vertices of $G_1$ and $G_2$ are less than or equal to five. On the other hand, we know that any graph on less than or equal to five vertices are of QE class except two graphs G5-10 and G5-17. We thus collect all graphs on six vertices which are non-trivial Cartesian or star products of two QE graphs.

Remark 3.5. If $G = G_1 \star G_2$ and $G_1$ is one of the two graphs G5-10 and G5-17, then $G_2$ is necessarily $K_2 = P_2$ and $G_1$ is isometrically embedded in $G$. It then follows from Lemma 2.1 that $G$ is of non-QE class. This case will be discussed in a more general context in Subsection 3.2.

The results are summarized in the following table. Among 112 graphs on six vertices there are 2 graphs which are non-trivial Cartesian products of two QE graphs, and 51 graphs which are non-trivial star products of two QE graphs. These two classes are mutually exclusive. Note that the six graphs G6-1\~G6-6 are trees.

| graphs                        | No.                                | QE/Non-QE |
|-------------------------------|------------------------------------|-----------|
| Cartesian products            | G6-35 ($K_2 \times P_3$), G6-80 ($K_2 \times K_3$) | QE        |
| star products of QE graphs    | G6-1\~18, G6-20, G6-21, G6-23\~29, G6-31, G6-33, G6-34, G6-37\~39, G6-41\~44, G6-47, G6-48, G6-51, G6-56, G6-57, G6-62, G6-68\~70, G6-74, G6-77, G6-83, G6-89, G6-98 | QE        |
3.2 Non-Primary Non-QE Graphs

Let $G$ be a non-QE graph on six vertices. If $G$ is not primary, it contains a non-QE graph $H$ as an isometrically embedded proper subgraph. Since $H$ has five or less vertices, it is necessarily $G_{5-10}$ or $G_{5-17}$. Thus, any non-primary non-QE graph on six vertices is obtained by adding a vertex $a$ to $G_{5-10}$ or $G_{5-17}$. Since $\text{diam}(G_{5-10}) = \text{diam}(G_{5-17}) = 2$, it follows from Lemma 2.3 that $G_{5-10}$ or $G_{5-17}$ is isometrically embedded in $G$ whatever vertices are connected to $a$. In this way the non-primary non-QE graphs on six vertices are specified.

**Proposition 3.6.** There are 11 graphs on six vertices containing $G_{5-10}$ as an isometrically embedded subgraph, which are classified by the degree of additional vertex $a$.

- $\text{deg}(a) = 1$: $G_6-22, G_6-36$,
- $\text{deg}(a) = 2$: $G_6-40, G_6-49, G_6-55$,
- $\text{deg}(a) = 3$: $G_6-64, G_6-72, G_6-73$,
- $\text{deg}(a) = 4$: $G_6-85, G_6-86$,
- $\text{deg}(a) = 5$: $G_6-96$.

**Proposition 3.7.** There are 17 graphs on six vertices containing $G_{5-17}$ as an isometrically embedded subgraph, which are classified by the degree of additional vertex $a$.

- $\text{deg}(a) = 1$: $G_6-45, G_6-53, G_6-54$,
- $\text{deg}(a) = 2$: $G_6-64, G_6-67, G_6-71, G_6-72, G_6-78$,
- $\text{deg}(a) = 3$: $G_6-85, G_6-86, G_6-88, G_6-91, G_6-94$,
- $\text{deg}(a) = 4$: $G_6-101, G_6-102, G_6-103$,
- $\text{deg}(a) = 5$: $G_6-107$.

The above results are summarized in the following table, where $G_6-64, G_6-72, G_6-85$ and $G_6-86$ occur in both classes. As a result, the non-primary non-QE graphs on six vertices are completely specified and the proof of Theorem 1.2 is completed. It is also noteworthy that any non-QE graph on six vertices not listed here is a primary non-QE graph.

| Graphs containing $G_{5-10}$ as an isometrically embedded subgraph | No. | QE/Non-QE |
|---------------------------------------------------------------|-----|-----------|
| $G_6-22, G_6-36, G_6-40, G_6-49, G_6-55, G_6-64, G_6-72, G_6-73, G_6-85, G_6-86, G_6-96$ | | Non-QE |

| Graphs containing $G_{5-17}$ as an isometrically embedded subgraph | No. | QE/Non-QE |
|---------------------------------------------------------------|-----|-----------|
| $G_6-45, G_6-53, G_6-54, G_6-64, G_6-67, G_6-71, G_6-72, G_6-78, G_6-85, G_6-86, G_6-88, G_6-91, G_6-94, G_6-101, G_6-102, G_6-103, G_6-107$ | | Non-QE |
3.3 Special Series of Graphs with Known QE Constants

**Proposition 3.8** ([23]). For the complete graph $K_n$ with $n \geq 2$ we have

$$QEC(K_n) = -1.$$  

**Proposition 3.9** ([18]). For the paths $P_n$ with $n \geq 2$ we have

$$QEC(P_n) = -\left(1 + \cos \frac{n\pi}{n}\right)^{-1}.$$  

**Proposition 3.10** ([23]). For the cycles on odd number of vertices we have

$$QEC(C_{2n+1}) = -\left(4 \cos^2 \frac{n\pi}{2n+1}\right)^{-1}, \quad n \geq 1,$$

and for those on even number of vertices we have

$$QEC(C_{2n}) = 0, \quad n \geq 2.$$  

**Proposition 3.11** ([22]). For the complete multipartite graph $K_{m_1, m_2, \ldots, m_k}$ with $k \geq 2$ and $m_1 \geq m_2 \geq \cdots \geq m_k \geq 1$, we have

$$QEC(K_{m_1, m_2, \ldots, m_k}) = \begin{cases} 
-2 + m_1, & \text{if } m_1 = m_2, \\
-2 - \alpha^*, & \text{if } m_1 > m_2,
\end{cases}$$

where $\alpha^*$ is the minimal solution to

$$\sum_{i=1}^{k} \frac{m_i}{\alpha + m_i} = 0.$$  

Note that $-m_1 < \alpha^* < -m_2$.

Finally, we also use the following new result, of which the proof and related discussion are deferred in the Appendix A.

**Proposition 3.12.** For $1 \leq m \leq n$ let $K_n \wedge K_{m,1}$ be the graph obtained by putting $m$ vertices of the complete graph $K_n$ and the $m$ end-vertices of the star $K_{m,1}$ together. Then we have

$$QEC(K_n \wedge K_{m,1}) = \frac{-2n + m - 1 + \sqrt{n(n-m)(m+1)}}{n+1}.$$  

10
It is an easy task to collect all complete multipartite graphs on six vertices together with their QE constants by Proposition 3.11. Among them there are three non-QE graphs which are not primary. On the other hand, it follows from Proposition 3.12 that

$$\text{QEC}(K_5 \wedge K_{m,1}) = -\frac{11 + m + \sqrt{5(5 - m)(m + 1)}}{6} < 0, \quad 1 \leq m \leq 5.$$  

Hence $K_5 \wedge K_{m,1}$ are of QE class for $1 \leq m \leq 5$. Note that $K_5 \wedge K_{1,1} = K_5 \star K_2$ and $K_5 \wedge K_{5,1} = K_6$. The results in this subsection are summarized in the following table.

| graphs                  | No.       | QE/Non-QE |
|-------------------------|-----------|-----------|
| complete graphs         | G6-112 ($K_6$) | QE       |
| paths                   | G6-6 ($P_6$)    | QE       |
| cycles                  | G6-19 ($C_6$)    | QE       |
| complete multipartite graphs | G6-1 ($K_{5,1}$), G6-61 ($K_{4,1,1}$) | QE       |
|                         | G6-104 ($K_{3,1,1,1}$), G6-108 ($K_{2,2,2}$) |           |
|                         | G6-110 ($K_{2,2,1,1}$), G6-111 ($K_{2,1,1,1,1}$) | Non-QE   |
|                         | G6-40 ($K_{4,2}$), G6-73 ($K_{3,3}$), G6-96 ($K_{3,2,1}$), |           |
| $K_5 \wedge K_{m,1}$   | G6-98 ($K_5 \wedge K_{1,1}$), G6-105 ($K_5 \wedge K_{2,1}$), G6-109 ($K_5 \wedge K_{3,1}$), G6-111 ($K_5 \wedge K_{4,1}$) G6-112 ($K_5 \wedge K_{5,1}$) | QE       |

### 3.4 Graph Joins of Regular Graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets. The graph join of $G_1$ and $G_2$ is a graph $G = (V, E)$, where

$$V = V_1 \cup V_2, \quad E = E_1 \cup E_2 \cup \{\{x, y\}; \ x \in V_1, \ y \in V_2\},$$

and is denoted by $G = G_1 + G_2$. Note that $G$ becomes a connected graph even if $G_1$ or $G_2$ is not connected. Quite a few graphs admit graph join structures but we have no general formula for $\text{QEC}(G_1 + G_2)$ except the case where both $G_1$ and $G_2$ are regular graphs.

**Proposition 3.13** ([15]). For $i = 1, 2$ let $G_i = (V_i, E_i)$ be a $r_i$-regular graph on $n_i = |V_i|$ vertices with $V_1 \cap V_2 = \emptyset$, where $r_i \geq 0$ and $n_i \geq 1$. Then we have

$$\text{QEC}(G_1 + G_2) = -2 + \max \left\{-\lambda_{\min}(G_1), -\lambda_{\min}(G_2), \frac{2n_1n_2 - r_1n_2 - r_2n_1}{n_1 + n_2}\right\},$$
where \( \lambda_{\text{min}}(G_i) \) is the minimal eigenvalue of the adjacency matrix of \( G_i \).

According to the partition \( 6 = 1 + 5 = 2 + 4 = 3 + 3 \) we can easily list the graphs on six vertices which are graph joins of two regular graphs. Their QE constants are obtained easily by the formula in Proposition 3.13 and thereby we can classify them into the classes of QE or non-QE. The results are summarized in the following table without mentioning the QE constants.

| graphs          | No.                               | QE/Non-QE |
|-----------------|-----------------------------------|-----------|
| graph joins     | G6-112 \((K_1 + K_5), G6-92 (K_1 + C_5)\) | QE        |
| (1 + 5)         | G6-1 \((K_1 + K_5)\)               |           |
| graph joins     | G6-112 \((K_2 + K_4), G6-110 (K_2 + C_4)\) | QE        |
| (2 + 4)         | G6-97 \((K_2 + (K_2 \cup K_2)), G6-61 (K_2 + \bar{K}_4), G6-111 (\bar{K}_2 + K_4), G108 (\bar{K}_2 + C_4)\) |           |
| graph joins     | G6-88 \((\bar{K}_2 + (K_2 \cup K_2)), G6-40 (K_2 + \bar{K}_4)\) | Non-QE    |
| (3 + 3)         | G6-112 \((K_3 + K_3), G6-104 (K_3 + \bar{K}_3)\) | QE        |
|                 | G6-73 \((\bar{K}_3 + \bar{K}_3)\) | Non-QE    |

Note that the three non-QE graphs in the above table appear already in the list of non-primary non-QE graphs, see Subsection 3.2.

### 3.5 Explicit Construction of Quadratic Embeddings

#### 3.5.1 Graphs with pendant edges

A graph \( G = (V, E) \) is called with a pendant edge if there exist four distinct vertices \( a, b, a', b' \in V \) satisfying \( a \sim a' \sim b' \sim b \sim a, \quad \deg(a') = \deg(b') = 2 \).

The edge \( \{a', b'\} \) is called a pendant edge. We first note that the induced subgraph spanned by \( a, b, a', b' \) forms a cycle \( C_4 \). Since this cycle is isometrically embedded in \( G \), we have \( \text{QEC}(G) \geq \text{QEC}(C_4) = 0 \). On the other hand, the induced subgraph spanned by \( V \setminus \{a', b'\} \), denoted by \( H \), is also isometrically embedded in \( G \). Hence \( \text{QEC}(G) \geq \text{QEC}(H) \) and we have

\[
\max\{0, \text{QEC}(H)\} \leq \text{QEC}(G) \tag{3.1}
\]

**Proposition 3.14** (Graph with pendant edge). Let \( G = (V, E) \) be a graph with a pendant edge \( a' \sim b' \). Let \( H \) be the induced subgraph spanned by \( V \setminus \{a', b'\} \). If \( H \) is of QE class, so is \( G \) and \( \text{QEC}(G) = 0 \).
Proof. Let \( d(x, y) \) denote the graph distance of \( G \). As is mentioned above, \( d(x, y) \) coincides with the graph distance of \( H \). By assumption we take a quadratic embedding of \( H \), say, \( \varphi : V \setminus \{a', b'\} \to \mathbb{R}^N \). Then we have

\[
\|\varphi(x) - \varphi(y)\|^2 = d(x, y), \quad x, y \in V \setminus \{a', b'\}.
\]

In view of the natural inclusion \( \mathbb{R}^N \subset \mathbb{R}^{N+1} \), we define a map \( \tilde{\varphi} : V \to \mathbb{R}^{N+1} \) by

\[
\tilde{\varphi}(x) = \begin{cases} 
(\varphi(x), 0), & x \neq a', b', \\
(\varphi(a), 1), & x = a', \\
(\varphi(b), 1), & x = b'.
\end{cases}
\] (3.2)

We will prove that \( \tilde{\varphi} \) gives rise to a quadratic embedding of \( G \) by showing that

\[
\|\tilde{\varphi}(x) - \tilde{\varphi}(y)\|^2 = d(x, y), \quad x, y \in V.
\] (3.3)

In fact, for \( x, y \in V \setminus \{a', b'\} \) we see from (3.2) that

\[
d(x, y) = \|\varphi(x) - \varphi(y)\|^2 = \|\tilde{\varphi}(x) - \tilde{\varphi}(y)\|^2.
\]

For \( x \in V \) and \( y = a' \) we have

\[
d(x, a') = d(x, a) + 1
\] (3.4)

and

\[
\|\tilde{\varphi}(x) - \tilde{\varphi}(a')\|^2 = \|\varphi(x) - \varphi(a)\|^2 + 2.
\] (3.5)

Since \( x, a \in V \setminus \{a', b'\} \) we have \( d(x, a) = \|\varphi(x) - \varphi(a)\|^2 \), combining (3.4) and (3.5) we come to

\[
d(x, a') = d(x, a) + 1 = \|\varphi(x) - \varphi(a)\|^2 + 2 = \|\tilde{\varphi}(x) - \tilde{\varphi}(a')\|^2.
\]

Similarly, (3.3) is shown to be valid for \( x \in V, y = b' \) and \( x = a', y = b' \). Thus, \( \tilde{\varphi} \) is a quadratic embedding of \( G \) and \( \text{QEC}(G) \leq 0 \). Finally, applying (3.1), we conclude that \( \text{QEC}(G) = 0 \). \( \square \)

Among the graphs on six vertices the graphs with pendant edges are easily specified. They are G6-15, G6-18, G6-26, G6-34, G6-35, G6-46, G6-52 and G6-75. Those graphs are of QE class since all graphs on four vertices are of QE class.
3.5.2 Extension of a quadratic embedding of a smaller graph

Let $G$ be a graph on six vertices and $H$ an induced subgraph of $G$ which is isometrically embedded in $G$. Assume that $H$ is of QE class and given explicitly a quadratic embedding in a Euclidean space. Then we may seek a quadratic embedding of $G$ by extending the given one. This strategy works well particularly when $H$ contains a cycle $C_4$ as a (not necessarily induced) subgraph. Taking $G_6-79$ as an example, we will illustrate the procedure.

Let $G$ be the graph $G_6-79$ realized on $V = \{1, 2, \ldots, 6\}$ as in Figure 4. The induced subgraph $H$ spanned by $\{1, 2, 3, 4\}$ is $C_4$ and isometrically embedded in $G$. Let $e_1, e_2, \ldots$ be mutually orthogonal unit vectors in a Euclidean space of sufficiently high dimension. We note first that a quadratic embedding of $H$ is given explicitly by

$$\varphi(1) = 0, \quad \varphi(2) = e_1, \quad \varphi(3) = e_1 + e_2, \quad \varphi(4) = e_2.$$

For a quadratic embedding of $G$ it is sufficient to find two vectors in the Euclidean space corresponding to the vertices 5 and 6, which are denoted by $x$ and $y$, respectively. Then the conditions on $x$ and $y$ are written down easily.

$$
\|x - \varphi(1)\|^2 = \|x - \varphi(2)\|^2 = 1, \quad \|x - \varphi(3)\|^2 = \|x - \varphi(4)\|^2 = 2, \\
\|y - \varphi(1)\|^2 = \|y - \varphi(2)\|^2 = 1, \quad \|y - \varphi(3)\|^2 = \|y - \varphi(4)\|^2 = 2, \\
\|x - y\|^2 = 1.
$$

It follows easily from (3.6) that

$$\langle x, e_1 \rangle = \frac{1}{2}, \quad \langle x, e_2 \rangle = 0.$$

Then we set

$$x = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 + x' = \frac{1}{2} e_1 + x',$$

where

$$\langle x', e_1 \rangle = \langle x', e_2 \rangle = 0, \quad \frac{1}{4} + \|x'\|^2 = 1.$$
Similarly, we have
\[ y = \frac{1}{2}e_2 + y', \quad (3.11) \]
and
\[ \langle y', e_1 \rangle = \langle y', e_2 \rangle = 0, \quad \frac{1}{4} + \|y'\|^2 = 1. \quad (3.12) \]
Inserting (3.9) and (3.11) into (3.8), we obtain
\[ \langle x', y' \rangle = \frac{1}{2}. \quad (3.13) \]
We then see that two vectors \( x' \) and \( y' \) satisfying (3.10), (3.12) and (3.13) certainly exist since
\[ -1 \leq \frac{\langle x', y' \rangle}{\|x'\| \cdot \|y'\|} = \frac{2}{3} \leq 1. \]
With such \( x' \) and \( y' \) we define \( x \) and \( y \) by (3.9) and (3.11), respectively. Then the quadratic embedding \( \varphi \) of \( H \) is extended to \( G \) by setting \( \varphi(5) = x \) and \( \varphi(6) = y \).

The argument in this subsection is applied to the graphs mentioned in the following table, of which the details are omitted.

| graphs                        | No.                  | QE/Non-QE |
|-------------------------------|----------------------|-----------|
| with pendant edges            | G6-15, G6-18, G6-26, G6-34, G6-35, G6-46, G6-52, G6-75 | QE        |
| with an explicit quadratic embedding | G6-50, G6-58, G6-63, G6-65, G6-70, G6-79, G6-81, G6-82, G6-87, G6-90, G6-93, G6-95, G6-99, G6-100 | QE        |

4 Completion of Classification

4.1 Calculating QE Constants

During the last section we have examined 105 graphs on six vertices and seven graphs remain for checking, which are G6-30, G6-32, G6-59, G6-60, G6-66, G6-84 and G6-106, see Figures 5–8.

The QE constants are calculated in a standard manner (Proposition 2.4). For G6-30, G6-59 and G6-66 the calculation is reduced to a quadratic equation and
we obtain

\[ QEC(G6-30) = -4 + \sqrt{19} \approx 0.1196..., \]
\[ QEC(G6-59) = -5 + \sqrt{19} \approx -0.2137..., \]
\[ QEC(G6-66) = -3 + \sqrt{5} \approx -0.3819.... \]

For G6-32, G6-60 and G6-84 the QE constants are solutions to algebraic equations of higher degrees. We have

\[ QEC(G6-32) = \lambda_1^* \approx -0.3121..., \]
\[ QEC(G6-60) = \lambda_2^* \approx 0.2034..., \]
\[ QEC(G6-84) = \lambda_3^* \approx 0.1313..., \]
where \( \lambda_1^*, \lambda_2^* \) and \( \lambda_3^* \) are the largest roots of the equations

\[
3\lambda^3 + 15\lambda^2 + 14\lambda + 3 = 0, \\
5\lambda^3 + 26\lambda^2 + 24\lambda - 6 = 0, \\
3\lambda^4 + 14\lambda^3 + 18\lambda^2 + 5\lambda - 1 = 0,
\]
respectively. Finally, we see that G6-106 is obtained from \( K_6 \) by deleting three edges which form a path \( P_4 \), that is, \( G-106 = K_6 \setminus P_4 \). Applying the general formula in Appendix B, we obtain

\[
\text{QEC}(G6-106) = \frac{-3 + \sqrt{5}}{2} \approx -0.3819....
\]

Remark 4.1. Apparently, graphs G6-30, G6-32, G6-59 and G6-60 share a common structure, namely, they are obtained by joining the cycle \( C_5 \) with the end-vertices of a star \( K_{m,1} \). Although not determined uniquely, such a graph is denoted by \( C_5 \land K_{m,1} \) for simplicity. Up to now no useful formula is known for the QE constants of graphs of this type.

The results of this subsection is summarized in the following table, where all graphs \( C_5 \land K_{m,1} \) with \( 1 \leq m \leq 5 \) are listed for completeness.

| graphs         | No.                                      | QE/Non-QE |
|----------------|------------------------------------------|-----------|
| \( C_5 \land K_{m,1} \) | G6-30 (\( C_5 \land K_{2,1} \)), G6-60 (\( C_5 \land K_{3,1} \)) | Non-QE    |
|                | G6-16 (\( C_5 \land K_{1,1} \)), G6-32 (\( C_5 \land K_{2,1} \)), G6-59 (\( C_5 \land K_{3,1} \)), G6-79 (\( C_5 \land K_{4,1} \)), G6-92 (\( C_5 \land K_{5,1} \)) | QE        |
| Remainders     | G6-84                                    | Non-QE    |
|                | G6-66, G6-106 (\( K_6 \setminus P_3 \)) | QE        |
4.2 Conclusion

During the last section all non-primary non-QE graphs on six vertices are found (Subsection 3.2) and no primary non-QE graphs are found. Hence the three graphs with positive QE constants found in the last subsection exhaust the primary non-QE graphs on six vertices, as is mentioned in Theorem 1.1. The proof of Theorem 1.2 is already completed in Subsection 3.2. Then the assertion of Theorem 1.3 is obvious.

Together with the two primary non-QE graphs on five vertices (Theorem 2.6) we have thus determined five primary non-QE graphs among the graphs on \( n \leq 6 \) vertices. It is a natural desire to explore new primary non-QE graphs on seven or more vertices. However, this question seems to be very difficult because of lack of constructive approach. In fact, our main argument to reach a primary non-QE graph on six vertices looks like the Eratosthenes’ sieve for prime numbers.

In the end of this paper we mention three examples of primary non-QE graphs on seven vertices. They are found as a byproduct of the formula for the QE constant of the complete multipartite graphs (Proposition 3.11).

Proposition 4.2 (22). Among the complete multipartite graphs \( K_{m_1,m_2,...,m_k} \) with \( k \geq 2 \) and \( m_1 \geq m_2 \geq ... \geq m_k \geq 1 \) there are exactly four primary non-QE graphs, that are \( K_{3,2}, K_{5,1,1}, K_{4,1,1,1} \), and \( K_{3,1,1,1,1} \). Moreover, any complete multipartite graph of non-QE class contains at least one of the above four primary ones as an isometrically embedded subgraph.

In other words, \( K_{5,1,1}, K_{4,1,1,1} \) and \( K_{3,1,1,1,1} \) are primary non-QE graphs on seven vertices. For more details see [22].

Appendix A: Calculating QEC\( (K_n \land K_{m,1}) \)

For \( 1 \leq m \leq n \) the graph \( K_n \land K_{m,1} \) is obtained by putting \( m \) vertices of the complete graph \( K_n \) with the \( m \) end-vertices of the star \( K_{m,1} \) together. We will prove the following formula.

Proposition A.1. For \( 1 \leq m \leq n \) we have

\[
\text{QEC}(K_n \land K_{m,1}) = -2n + m - 1 + \frac{\sqrt{n(n - m)(m + 1)}}{n + 1}.
\]

We adopt a realization of \( K_n \land K_{m,1} \) on the vertex set \( \{0, 1, 2, \ldots, n\} \). Consider the complete graph \( K_n = (V, E) \), where \( V = \{1, 2, \ldots, n\} \) and \( E = \{\{x, y\}; x, y \in V, x \neq y\} \). Then \( K_n \land K_{m,1} = (\bar{V}, \bar{E}) \) is given by

\[
\bar{V} = V \cup \{0\} = \{0, 1, 2, \ldots, n\}, \quad \bar{E} = E \cup \{\{0, x\}; x = 1, 2, \ldots, m\}.
\]
The vertex set of $K_n \land K_{m,1}$ being divided into three parts $\{0\}, \{1, 2, \ldots, m\}$ and $\{m + 1, \ldots, n\}$, the distance matrix of $K_n \land K_{m,1}$ admits a natural block matrix form:

$$D = \begin{bmatrix} 0 & 1 & 21 \\ 1 & J - I & J \\ 21 & J & J - I \end{bmatrix},$$

where $1$ is a column or row vector with all the entries are one, $J$ the matrix with all entries being one, and $I$ the identity matrix, and their sizes are understood in the context.

Accordingly, the quadratic form associated to $D$ is written as

$$\varphi(f, g, h, \lambda, \mu) = \left\langle \begin{bmatrix} f \\ g \\ h \end{bmatrix}, D \begin{bmatrix} f \\ g \\ h \end{bmatrix} \right\rangle - \lambda(f^2 + \langle g, g \rangle + \langle h, h \rangle - 1) - \mu(f + \langle 1, g \rangle + \langle 1, h \rangle).$$

After simple calculus we see that any stationary point of $\varphi(f, g, h, \lambda, \mu)$ is obtained as a solution to the following system of equations:

$$\frac{\partial \varphi}{\partial f} = -2\lambda f + 2\langle 1, g \rangle + 4\langle 1, h \rangle - \mu = 0,$$  \hspace{1cm} (A.1)

$$\frac{\partial \varphi}{\partial g_i} = 2f + 2\langle 1, g \rangle + 2\langle 1, h \rangle - 2(\lambda + 1)g_i - \mu = 0,$$  \hspace{1cm} (A.2)

$$\frac{\partial \varphi}{\partial h_j} = 4f + 2\langle 1, g \rangle + 2\langle 1, h \rangle - 2(\lambda + 1)h_j - \mu = 0,$$  \hspace{1cm} (A.3)

under the two constraints:

$$\frac{\partial \varphi}{\partial \lambda} = f^2 + \langle g, g \rangle + \langle h, h \rangle - 1 = 0,$$  \hspace{1cm} (A.4)

$$\frac{\partial \varphi}{\partial \mu} = f + \langle 1, g \rangle + \langle 1, h \rangle = 0.$$  \hspace{1cm} (A.5)

Let $S$ be the set of all solutions to (A.1)-(A.5).

Hereafter we assume that $1 \leq m < n$. Since QEC$(K_n \land K_{m,1}) > -1$ by Proposition 2.5, it is sufficient to seek out all solutions $(f, g, h, \lambda, \mu) \in S$ with $\lambda > -1$. It is obvious from (A.2) and (A.3) that $g_i$ and $h_j$ are constant. We set

$$g_i = \gamma, \quad 1 \leq i \leq m; \quad h_j = \delta, \quad 1 \leq j \leq n - m.$$
After simple algebra the equations (A.1)–(A.5) are reduced to the following

\[-\lambda f + m\gamma + 2(n-m)\delta = \frac{\mu}{2}, \quad (A.6)\]
\[f + (m-1-\lambda)\gamma + (n-m)\delta = \frac{\mu}{2}, \quad (A.7)\]
\[2f + m\gamma + (n-m-1-\lambda)\delta = \frac{\mu}{2}, \quad (A.8)\]
\[f^2 + m\gamma^2 + (n-m)\delta^2 = 1 \quad (A.9)\]
\[f + m\gamma + (n-m)\delta = 0. \quad (A.10)\]

From (A.10) we obtain
\[f = -m\gamma - (n-m)\delta. \quad (A.11)\]

Then (A.6)–(A.8) become

\[m(\lambda + 1)\gamma + (n-m)(\lambda + 2)\delta = \frac{\mu}{2}, \quad (A.12)\]
\[-(\lambda + 1)\gamma = \frac{\mu}{2}, \quad (A.13)\]
\[-m\gamma + (-n + m - 1 - \lambda)\delta = \frac{\mu}{2}, \quad (A.14)\]

respectively. Set \(\mu = 0\) in (A.12)–(A.14). From our condition \(1 \leq m < n\) and \(\lambda > -1\) we see immediately that \(\gamma = \delta = 0\), and hence \(f = 0\) by (A.11). But \(f = \gamma = \delta = 0\) does not fulfill (A.9). Thus, \(\mu \neq 0\). Coming back to (A.13), we have
\[\gamma = \frac{-1}{\lambda + 1} \frac{\mu}{2}. \quad (A.15)\]

Then (A.12) and (A.14) become

\[(n-m)(\lambda + 2)\delta = (m+1)\frac{\mu}{2}, \quad (A.16)\]

respectively. Hence

\[(\lambda + n - m + 1)(m+1)\frac{\mu}{2} - (n-m)(\lambda + 2)(\frac{m}{\lambda + 1} - 1)\frac{\mu}{2} = 0.\]

After simple algebra with \(\mu \neq 0\) we obtain

\[(n+1)\lambda^2 + 2(2n-m+1)\lambda + m^2 - mn + 3n - 2m + 1 = 0.\]

The above quadratic equation possesses two real roots given by

\[\lambda_{\pm} = \frac{-(2n-m+1) \pm \sqrt{n(n-m)(m+1)}}{n+1}. \quad (A.17)\]
For $\lambda = \lambda_s$ we may determine $f, \gamma$ and $\delta$ in such a way that (A.12)–(A.14) and (A.9) are satisfied. Thus, we conclude that $\lambda_s$ is the maximum of $\lambda$ appearing in $S$. Namely,

$$QEC(K_n \land K_{m,1}) = \frac{-(2n - m + 1) + \sqrt{n(n - m)(m + 1)}}{n + 1},$$

as desired.

For $m = n$ we have $QEC(K_n \land K_{m,1}) = QEC(K_{n+1}) = -1$. On the other hand, the value of the right-hand side of (A.17) for $m = n$ is $-1$. Consequently, the formula (A.17) is valid for all $1 \leq m \leq n$.

**Appendix B: Calculating $QEC(K_n \backslash P_4)$**

For $n \geq 5$ let $K_n \backslash P_4$ be the graph obtained by deleting three edges of $K_n$ chosen in such a way that the subgraph generated by them is isomorphic to the path $P_4$.

**Proposition B.1.** We have

$$QEC(K_n \backslash P_4) = \frac{-3 + \sqrt{5}}{2}, \quad n = 5, 6,$$

and

$$QEC(K_n \backslash P_4) = \frac{-(n + 6) + \sqrt{(n + 6)^2 + 4n(n - 10)}}{2n}, \quad n \geq 7.$$

Upon explicit calculation we consider the complete graph $K_n = (V, \tilde{E})$ with vertex set $V = \{1, 2, \ldots, n\}$ and realize $K_n \backslash P_4$ as $G = (V, E)$, where

$$E = \tilde{E} \backslash \{[1, 2], [2, 3], [3, 4]\}.$$

The distance matrix of $G$ is written in a block matrix form:

$$D = \begin{bmatrix} A & J \\ J & J - I \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

Then the quadratic form associated to $D$ is written as

$$\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, D \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle = \langle f, Af \rangle + 2\langle 1, f \rangle\langle 1, g \rangle + \langle 1, g \rangle^2 - \langle g, g \rangle,$$
where

\[
\mathbf{f} = \begin{bmatrix} f_1 \\ \\ \\ f_4 \end{bmatrix} \in \mathbb{R}^4, \quad \mathbf{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_{n-4} \end{bmatrix} \in \mathbb{R}^{n-4}.
\]

For \( \lambda, \mu \in \mathbb{R} \) we set

\[
\varphi(f, g, \lambda, \mu) = \langle f, A f \rangle + 2 \langle 1, f \rangle \langle 1, g \rangle + \langle 1, g \rangle^2 - \langle g, g \rangle
- \lambda(\langle f, f \rangle + \langle g, g \rangle - 1) - \mu(\langle 1, f \rangle + \langle 1, g \rangle).
\]

A stationary point of \( \varphi(f, g, \lambda, \mu) \) is characterized as a solution to the equations:

\[
\frac{\partial \varphi}{\partial f_i} = \frac{\partial \varphi}{\partial g_j} = 0, \quad 1 \leq i \leq 4, \quad 1 \leq j \leq n - 4, \tag{B.2}
\]

under the two constraints:

\[
\langle f, f \rangle + \langle g, g \rangle = 1, \tag{B.3}
\]
\[
\langle 1, f \rangle + \langle 1, g \rangle = 0. \tag{B.4}
\]

After direct computation of derivatives and application of (B.4) the equations (B.2) become

\[
-(\lambda + 1)f_1 + f_2 = \frac{\mu}{2}, \tag{B.5}
\]
\[
f_1 - (\lambda + 1)f_2 + f_3 = \frac{\mu}{2}, \tag{B.6}
\]
\[
f_2 - (\lambda + 1)f_3 + f_4 = \frac{\mu}{2}, \tag{B.7}
\]
\[
f_3 - (\lambda + 1)f_4 = \frac{\mu}{2}, \tag{B.8}
\]
\[-(\lambda + 1)g_j = \frac{\mu}{2}, \quad 1 \leq j \leq n - 4. \tag{B.9}
\]

Let \( S \) be the set of all solutions \((f, g, \lambda, \mu)\) to the equations (B.5)–(B.9) under the two constraints (B.3) and (B.4). By Propositions 2.4 and 2.5 it is sufficient to seek a solution \((f, g, \lambda, \mu)\) with \( \lambda > -1 \).

We first see from (B.9) that \( g_j = \gamma \) is constant independently of \( j \) and

\[
g_j = \gamma = \frac{-1}{\lambda + 1} \frac{\mu}{2}. \tag{B.10}
\]

With the help of (B.3) and (B.8) the variables \( f_2 \) and \( f_3 \) being eliminated, (B.6) and (B.7) become

\[
-(\lambda^2 - 2\lambda)f_1 + (\lambda + 1)f_4 = (\lambda + 1) \frac{\mu}{2}, \tag{B.11}
\]
\[
(\lambda + 1)f_1 + (-\lambda^2 - 2\lambda)f_4 = (\lambda + 1) \frac{\mu}{2}. \tag{B.12}
\]
respectively. The determinant of the coefficients of the left-hand side is given by
\[ \Delta(\lambda) = (-\lambda^2 - 2\lambda)^2 - (\lambda + 1)^2 = (\lambda^2 + 3\lambda + 1)(\lambda^2 + \lambda - 1). \]

(Case 1) \( \Delta(\lambda) \neq 0 \). The solution to (B.11) and (B.12) is unique and we obtain
\[ f_1 = f_4 = \frac{- (\lambda + 1) \mu}{\lambda^2 + \lambda - 1}, \quad f_2 = f_3 = \frac{- (\lambda + 2) \mu}{\lambda^2 + \lambda - 1}. \]  
(B.13)

Then, using (B.10) and (B.13), the constraint (B.4) becomes
\[ \left( \frac{4\lambda + 6}{\lambda^2 + \lambda - 1} + \frac{n - 4}{\lambda + 1} \right) \frac{\mu}{2} = 0. \]  
(B.14)

If \( \mu = 0 \), it follows from (B.10) and (B.13) that \( f_i = g_j = 0 \), which does not fulfill the constraint (B.3). Hence \( \mu \neq 0 \) and from (B.14) we obtain
\[ n\lambda^2 + (n + 6)\lambda - (n - 10) = 0. \]
The roots are real and given by
\[ \lambda_\pm = -\left(\frac{n + 6}{\lambda^2 + \lambda - 1}\right) \pm \sqrt{(n + 6)^2 + 4n(n - 10)} \]  
2n
(B.15)

For \( \lambda = \lambda_\pm \) we have \( \Delta(\lambda_\pm) \neq 0 \) and we see that \( f_i \) and \( g_j \) are non-zero multiples of \( \mu \). Hence we may determine \( \mu \) in such a way that (B.3) is satisfied. Consequently, \( \lambda_\pm \) appear in \( S \).

(Case 2) \( \lambda^2 + 3\lambda + 1 = 0 \). By \( \lambda^2 + 2\lambda = -\lambda - 1 \) and \( \lambda \neq -1 \) it is easy to see that (B.11) and (B.12) are reduced to
\[ f_1 + f_4 = \frac{\mu}{2}. \]  
(B.16)

From (B.5) and (B.8) we have
\[ f_2 = (\lambda + 1)f_1 + \frac{\mu}{2}, \quad f_3 = (\lambda + 1)f_4 + \frac{\mu}{2} \]  
(B.17)

so that
\[ f_2 + f_3 = (\lambda + 3) \frac{\mu}{2}. \]  
(B.18)

From (B.16), (B.18) and (B.10) we see that
\[ \langle 1, f \rangle = (\lambda + 1) \frac{\mu}{2}, \quad \langle 1, g \rangle = \frac{-(n - 4)}{\lambda + 1} \frac{\mu}{2}. \]

Then the constraint (B.4) becomes
\[ \frac{1}{\lambda + 1}(\lambda^2 + 5\lambda - (n - 8)) \frac{\mu}{2} = 0. \]
Since there is no common root of $\lambda^2 + 3\lambda + 1 = 0$ and $\lambda^2 + 5\lambda - (n - 8) = 0$, we obtain $\mu = 0$. Then by (B.16) and (B.17) we obtain

$$f_2 = (\lambda + 1)f_1, \quad f_3 = (\lambda + 1)f_4 = -(\lambda + 1)f_1, \quad f_4 = -f_1.$$ 

The second constraint (B.3) becomes

$$f_1^2 + (\lambda + 1)^2 f_2^2 + (\lambda + 1)^2 f_1 + f_1^2 = 1$$

and $f_1$ is determined. Thus, the root of $\lambda^2 + 3\lambda + 1 = 0$, that is,

$$\lambda = \frac{-3 \pm \sqrt{5}}{2}$$

appears in $S$.

(Case 3) $\lambda^2 + \lambda - 1 = 0$. In a similar manner as in Case 2, we obtain

$$\mu = 0, \quad g_j = \gamma = 0, \quad f_1 = f_2 = (\lambda + 1)f_3, \quad f_4 = f_3.$$ 

For the constraint (B.4) we have necessarily $f_i = g_j = 0$, which however do not fulfill (B.3). Namely, any root of $\lambda^2 + \lambda - 1 = 0$ does not appear in $S$.

From (Case 1)–(Case 3) we conclude that

$$\text{QEC}(G) = \max \left\{ \lambda_+, \frac{-3 + \sqrt{5}}{2} \right\},$$

where $\lambda_+$ is given as in (B.15). Finally, comparison of two numbers in the right-hand side is easy by simple algebra and we obtain the desired formula.

**Appendix C: List of All Graphs on Six Vertices**

For the self-contained reference we reproduce the list of all connected graphs on six vertices following McKay [17]. Another list by Cvetković–Petrić [6] is also useful, where the eigenvalues of the adjacency matrices and related quantities are mentioned.
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