ON A SUM INVOLVING THE EULER FUNCTION

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Abstract. We obtain reasonably tight upper and lower bounds on the sum \( \sum_{n \leq x} \varphi(\lfloor x/n \rfloor) \), involving the Euler functions \( \varphi \) and the integer parts \( \lfloor x/n \rfloor \) of the reciprocals of integers.

1. Background and motivation

Let, as usual, for an integer \( n \geq 1 \), \( \varphi(n) \) denote the Euler function, that is, the number of units in the residue ring \( \mathbb{Z}/n\mathbb{Z} \).

By a classical result of Walfisz [22], we have the following asymptotic formula for the summary function of the Euler function

\[
\sum_{n \leq x} \varphi(n) = \frac{x^2}{2\zeta(2)} + O \left( x(\log x)^{2/3}(\log \log x)^{4/3} \right),
\]

see also [8, Theorem 6.44].

Furthermore, for any real number \( x \) we denote by \( \lfloor x \rfloor \) its integer part, that is, the greatest integer that does not exceed \( x \). The most straightforward sum of the floor function is related to the divisor summatory function since

\[
\sum_{n \leq x} \lfloor x/n \rfloor = \sum_{n \leq x} \sum_{k \leq x/n} 1 = \sum_{n \leq x} \tau(n),
\]

where \( \tau(n) \) is the number of divisors of \( n \). From [11, Theorem 2] we infer

\[
\sum_{n \leq x} \lfloor x/n \rfloor = x \log x + x(2\gamma - 1) + O \left( x^{517/1648+o(1)} \right),
\]

where \( \gamma \) is the Euler–Mascheroni constant, in particular \( \gamma \approx 0.57722 \).
Here we combine both functions and consider an apparently new type of sums, namely,
\[ S(x) = \sum_{n \leq x} \varphi \left( \lfloor x/n \rfloor \right). \]

The sum \( S(x) \) is also a mean value of a certain divisor function, as it may be seen by interchanging the summations. More precisely, if \( \tau_x \) is the divisor function defined by
\[
\tau_x(n) = \sum_{\gcd(d, \lfloor dx/n \rfloor) = 1} 1
\]
then
\[ S(x) = \sum_{n \leq x} \tau_x(n). \]

Note that, for each fixed real number \( x \geq 1 \), the arithmetic function \( \tau_x \) is not multiplicative, which explains why an asymptotic formula for \( S(x) \) is quite difficult to get. However, the aim of this work is to obtain reasonably tight upper and lower bounds for this sum.

We also consider more general sums of arithmetic functions with \( \lfloor x/n \rfloor \), and in the case of functions growing slower than the Euler function we obtain asymptotic formulas for such sums.

We remark our work is partially motivated by the extensive body of research on arithmetic functions with integer parts of real-valued functions, most commonly, with Beatty \( \lfloor \alpha n + \beta \rfloor \) sequences, see, for example, [1, 3, 6, 12, 13], and Piatetski–Shapiro \( \lfloor n^\gamma \rfloor \) sequences, see, for example, [2, 4, 5, 7, 14, 15], with real \( \alpha, \beta \) and \( \gamma \). In particular, we obtain an analogue of the result of Morgenbesser [15] on the sum of digits of \( \lfloor n^\epsilon \rfloor \) for the sequence \( \lfloor x/n \rfloor \), see Example 3.4 below.

2. Main Results

2.1. The Euler function. We start with upper and lower bounds on \( S(x) \).

**Theorem 2.1.** Uniformly, for all \( x \geq 3 \),
\[
\left( \frac{2629}{4009} \cdot \frac{1}{\zeta(2)} + \frac{1380}{4009} + o(1) \right) x \log x \leq S(x) \leq \left( \frac{2629}{4009} \cdot \frac{1}{\zeta(2)} + o(1) \right) x \log x,
\]
as \( x \to \infty \).
The proofs of both lower and upper bounds of Theorem 2.1 relying on the theory of exponent pairs, see [8, Chapter 6]. In particular, to obtain the numerically strongest result, we use the recently discovered exponent pair of Bourgain [10] combined with so called \(A\)- and \(B\)-processes, see [8, Sections 6.4.2 and 6.6.2]. We remark that in the lower of Theorem 2.1 the quantity in \(o(1)\) is negative.

We note it is natural to ask the following:

**Question 2.2.** Is it true that

\[
S(x) = \left(\frac{1}{\zeta(2)} + o(1)\right) x \log x
\]

as \(x \to \infty\)?

In Section 7 we present some numerical data which makes us rather cautiously believe that the answer to Question 2.2 is positive.

### 2.2. Slowly growing arithmetic functions

One of the difficulties in investigating the sum \(S(x)\) is a large size of \(\varphi(n)\). In particular, some individual terms of the sum \(S(x)\) are only logarithmically smaller than the entire sum. However, for slowly growing arithmetic functions \(f(n)\) in similar sums,

\[
S_f(x) = \sum_{n \leq x} f \left(\lfloor x/n \rfloor\right),
\]

we are able to get an asymptotic formula.

Let \(\tau_k(n)\) denotes the generalised divisor function, which is defined as the number of ordered representations \(n = d_1 \ldots d_k\) with integer numbers \(d_1, \ldots, d_k \geq 1\). In particular \(\tau_1(n) = 1\).

We also define \(\varepsilon_1(x) = 0\) and

\[
(2.1) \quad \varepsilon_k(x) = \sqrt{\frac{k \log \log \log x}{\log \log x}} \left( k - 1 + \frac{30}{\log \log \log x} \right),
\]

for \(k \geq 2\). Now we have the obvious estimate \(\varepsilon_k(x) = o(1)\) as \(x \to \infty\).

We write \(O_k\) to indicate that in the relations \(U = O_k(V)\) the implied constant may depend on \(k\). We also write \(\mathbb{Z}_{\geq k}\) for the set

\[
\mathbb{Z}_{\geq k} = \mathbb{Z} \cap [k, \infty).
\]
Theorem 2.3. Let \( f \) be a complex-valued arithmetic function such that there exist \( A > 0 \) and \( k \in \mathbb{Z}_{\geq 1} \) such that \( |f| \leq A \tau_k \). Then
\[
\sum_{n \leq x} f \left( \lfloor x/n \rfloor \right) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O_k \left( Ax^{1/2}(\log x)^{\delta_k + \varepsilon_{k+1}(x)/2} \right),
\]
where \( \delta_1 = 0 \) if \( k = 1 \), \( \delta_k = k - 1/2 \) if \( k \geq 2 \), and where \( \varepsilon_{k+1}(x) \) is defined in (2.1).

In particular, applying Theorem 2.3 to \( f(n) = \varphi(n)/n \) (and using \( k = 1 \)) we obtain:

**Corollary 2.4.** We have
\[
\sum_{n \leq x} \frac{\varphi \left( \lfloor x/n \rfloor \right)}{\lfloor x/n \rfloor} = \kappa x + O \left( x^{1/2} \right),
\]
where
\[
\kappa = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2(n+1)} \approx 0.78838.
\]

Finally, the method of proof of Theorem 2.3 can be extended to more general and faster growing arithmetic functions at the cost of a weaker error term.

We use
\[
\phi = \frac{1 + \sqrt{5}}{2} \approx 1.61803
\]
to denote the Golden ratio.

**Theorem 2.5.** Let \( f \) be a complex-valued arithmetic function and assume that there exists \( A > 0 \) such that
\[
|f(n)| \ll n^{\phi - 1} (\log en)^{-A}.
\]
Then
\[
\sum_{n \leq x} f \left( \lfloor x/n \rfloor \right) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O \left( x(\log x)^{-A(\phi - 1)} \right).
\]

We also have a result which depends on the average behaviour of arithmetic functions, which is very useful for functions with irregular behaviour. We give several examples of such functions in Section 3.

**Theorem 2.6.** Let \( f \) be a complex-valued arithmetic function and assume that there exists \( 0 < \alpha < 2 \) such that
\[
\sum_{n \leq x} |f(n)|^2 \ll x^\alpha.
\]
Then
\[\sum_{n \leq x} f \left( \lfloor x/n \rfloor \right) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O \left( x^{(\alpha+1)/3} (\log x)^{(1+\alpha)(2+\varepsilon_2(x))/6} \right)\]

where \(\varepsilon_2(x)\) is given in (2.1).

In particular, if in Theorem 2.1 one replaces the sum \(S(x)\) with \(\varphi(n)\) with a similar sum with \(\varphi(n)^\beta\) for some \(\beta < 1/2\), then Theorem 2.6 immediately applies and implies an asymptotic formula.

3. Some applications

Here we give some examples of interesting arithmetic functions to which we can apply our results.

**Example 3.1.** Let \(f(n) = \sqrt{3} \Omega(n)\). From [18, Chapter I.3, Exercise 58(f)], we have
\[\sum_{n \leq x} f(n)^2 \ll x^{\log 3/\log 2}\]

(indeed one verifies that the functions \(f(\vartheta)\) and \(g(\vartheta)\) involved in the asymptotic formulas of [18, Chapter I.3, Exercise 58(f)] are both monotonically decreasing). Hence, Theorem 2.6 gives
\[\sum_{n \leq x} \sqrt{3}^{\Omega(\lfloor x/n \rfloor)} = x \sum_{n=1}^{\infty} \sqrt{3}^{\Omega(n)} \frac{\sqrt{3}^{\Omega(n)}}{n(n+1)} + O \left( x^{\log 6/\log 8} (\log x)^{\log 6/\log 8+o(1)} \right)\].

Note that
\[\sum_{n=1}^{\infty} \sqrt{3}^{\Omega(n)} \approx 1.77694\]

Clearly in Example 3.1 one can take \(f(n) = \lambda^{\Omega(n)}\) with any \(\lambda \leq \sqrt{3}\) and still have an asymptotic formula. We now show that one can also take a slightly larger values of \(\lambda\).

**Example 3.2.** Let \(f(n) = \lambda^{\Omega(n)}\) with \(\sqrt{3} \leq \lambda < 2\). Combining the trivial bound
\[\Omega(n) \leq \frac{\log n}{\log 2}\]
with [18, Chapter I.3, Exercise 58(f)], we derive
\[
\sum_{n \leq x} f(n)^2 = \sum_{n \leq x} 3^{\Omega(n)} \left( \frac{\lambda^2}{3} \right)^{\Omega(n)} \\
\leq x^{\log(\lambda^2/3)/\log 2} \sum_{n \leq x} 3^{\Omega(n)} \ll x^{2 \log \lambda/\log 2}.
\]

Hence by Theorem 2.6, for any positive \( \lambda < 2 \) there exists some \( \kappa > 0 \) such that
\[
\sum_{n \leq x} \lambda^{\Omega([x/n])} = x \sum_{n=1}^{\infty} \lambda^{\Omega(n)} \frac{1}{n(n+1)} + O \left( x^{1-\kappa} \right).
\]

We now give an application of Theorem 2.6 to a very different function.

**Example 3.3.** Let \( k \in \mathbb{Z}_{\geq 2} \) and define \( M_k(n) \) to be the maximal \( k \)-full divisor of \( n \) (see [17]). Since
\[
L \left( s, M_k^2 \right) = \zeta(s) \prod_p \left( 1 + \frac{p^{2k} - 1}{p^{ks}} \right) \quad (\sigma > 2 + \frac{1}{k})
\]
we infer that
\[
\sum_{n \leq x} M_k(n)^2 \leq x^{2 + 1/k + o_k(1)},
\]
where \( o_k(1) \) denotes a quantity which for a fixed \( k \) tends to zero as \( x \to \infty \). Now let \( f_k(n) = n^{-1/k} M_k(n) \). By partial summation, we obtain from the above estimate
\[
\sum_{n \leq x} f_k(n)^2 \leq x^{2 - 1/k + o_k(1)}.
\]

Applying Theorem 2.6 we derive
\[
\sum_{n \leq x} [x/n]^{-1/k} M_k \left( \lfloor x/n \rfloor \right) = x \sum_{n=1}^{\infty} \frac{M_k(n)}{n^{1+1/k} (n+1)} + O \left( x^{1-1/(3k)+o_k(1)} \right).
\]

Furthermore, if for an integer \( q \geq 2 \) we use \( \sigma_q(n) \) to denote the sum of \( q \)-ary digits of \( n \), then we see that Theorem 2.1 immediately implies:

**Example 3.4.** For any integer \( q \geq 2 \), we have
\[
\sum_{n \leq x} \sigma_q \left( \lfloor x/n \rfloor \right) = x \sum_{n=1}^{\infty} \frac{\sigma_q(n)}{n(n+1)} + O \left( x^{2/3+o(1)} \right).
\]
4.1.1. Vaaler polynomials. For a real $z \in \mathbb{R}$ we denote

\[
\psi(z) = z - \lfloor z \rfloor - \frac{1}{2} \quad \text{and} \quad e(z) = \exp(2\pi iz).
\]

We need a result of Vaaler \[21\], approximating $\psi(z)$ via trigonometric polynomials which we present in the form given by \[8\], Theorem 6.1. For this, for any $0 < |t| < 1$ we put $\Phi(t) = \pi t (1 - |t|) \cot(\pi t) + |t|$. Note that $0 < \Phi(t) < 1$ for $0 < |t| < 1$.

**Lemma 4.1.** For any real number $x \geq 1$ and any positive integer $H$,

\[
\psi(z) = -\sum_{0 < |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{e(hz)}{2\pi i h} + R_H(z),
\]

where the error term $R_H(z)$ satisfies

\[
|R_H(z)| \leq \frac{1}{2H+2} \sum_{|h| \leq H} \left(1 - \frac{|h|}{H+1}\right) e(hz).
\]

4.1.2. Initial transformation. Let $x$ be sufficiently large and $J$ be any real number satisfying $x^{1/2} < J \leq x$. Clearly

\[
S(x) = \sum_{n \leq x/J} \varphi(\lfloor x/n \rfloor) + \sum_{x/J < n \leq x} \varphi(\lfloor x/n \rfloor)
\leq \sum_{n \leq x/J} \varphi(\lfloor x/n \rfloor) + \sum_{n \leq J} \varphi(n) \left(\frac{x}{n} - \frac{x}{n+1}\right)
\leq \sum_{n \leq x/J} \varphi(\lfloor x/n \rfloor) + x \sum_{n \leq J} \frac{\varphi(n)}{n(n+1)}
\]

\[
+ \sum_{n \leq J} \varphi(n) \left(\psi\left(\frac{x}{n+1}\right) - \psi\left(\frac{x}{n}\right)\right).
\]

where $\psi(z)$ is given by (4.1) Now, splitting the last sum into two ranges $n \leq x^{1/2}$ and $x^{1/2} < n \leq J$ we obtain

\[
S(x) = S_0(x) + S_1(x) + S_2(x) + S_3(x),
\]
where
\[
S_0(x) = \sum_{n \leq x/J} \varphi \left( \frac{\lfloor x/n \rfloor}{J} \right),
\]
\[
S_1(x) = x \sum_{n \leq J} \frac{\varphi(n)}{n(n+1)},
\]
\[
S_2(x) = \sum_{n \leq x/2} \varphi(n) \left( \psi \left( \frac{x}{n+1} \right) - \psi \left( \frac{x}{n} \right) \right),
\]
\[
S_3(x) = \sum_{x/2 < n \leq J} \varphi(n) \left( \psi \left( \frac{x}{n+1} \right) - \psi \left( \frac{x}{n} \right) \right).
\]

4.2. The lower bound.

4.2.1. Exponential sums twisted by the Euler totient. We refer to [8, Sections 6.6.3] for the definition and basic properties of exponent pairs.

Lemma 4.2. Let \( N, N_1 \in \mathbb{Z}_{\geq 1} \) and \( x > 0 \) such that \( N < N_1 \leq 2N \) and \( N \leq x \). If \((k, \ell)\) is an exponent pair, then
\[
\sum_{N < n \leq N_1} \varphi(n) e \left( \frac{x}{n} \right) \ll x^k N^{1+\ell-2k} \log N + N^3 x^{-1} + N.
\]
Furthermore, if \((k, \ell) \neq \left( \frac{1}{2}, \frac{1}{2} \right)\), then the factor \( \log N \) may be omitted.

Proof. For any arithmetic functions \( f \) and \( g \), \( f * g \) is the usual Dirichlet convolution product of \( f \) and \( g \), defined as
\[
(f * g)(n) = \sum_{d|n} f(d)g(n/d).
\]

Using \( \varphi = \mu \ast \text{Id} \), see [8, Equation (4.7)], we obtain
\[
\sum_{N < n \leq N_1} \varphi(n) e \left( \frac{x}{n} \right) = \sum_{N < n \leq N_1} (\mu \ast \text{Id})(n) e \left( \frac{x}{n} \right)
\]
\[
= \sum_{n \leq N_1} \mu(n) \sum_{N/n < m \leq N_1/n} m e \left( \frac{x}{mn} \right)
\]
\[
= \sum_{n \leq N} \mu(n) \sum_{N/n < m \leq N_1/n} m e \left( \frac{x}{mn} \right) + O(N).
\]
For all $M, M_1 \in \mathbb{Z}_{\geq 1}$ such that $M < M_1 \leq 2M$, if $(k, \ell)$ is an exponent pair, then by Abel summation

$$
\sum_{M < m \leq M_1} me\left(\frac{x}{mn}\right) \ll M \max_{M \leq n \leq M_1} \left| \sum_{M \leq m \leq n} e\left(\frac{x}{mn}\right) \right|
$$

(4.4)

$$
\ll M \left\{ \left(\frac{x}{Mn}\right)^k M^{\ell-k} + \frac{M^2 n}{x} \right\}
$$

$$
\ll \left(\frac{x}{n}\right)^k M^{1+\ell-2k} + \frac{M^3 n}{x}.
$$

Inserting (4.4) with $M = \frac{N}{n}$ and $M_1 = \frac{N_1}{n}$ in (4.3), we obtain

$$
\sum_{N < n \leq N_1} \phi(n) e\left(\frac{x}{n}\right) \ll x^k N^{1+\ell-2k} \sum_{n \leq N} \frac{1}{n^{1+\ell-k}} + \frac{N^3}{x} \sum_{n \leq N} \frac{1}{n^2} + N
$$

$$
\ll x^k N^{1+\ell-2k} (\log N)^\alpha + \frac{N^3}{x} + N,
$$

where $\alpha = 1$ if $(k, \ell) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\alpha = 0$ otherwise, giving the asserted result. \(\square\)

4.2.2. Exponent pairs and a lower bound on $S(x)$. The desired lower bound on $S(x)$ is a particular case of the following more general result, which may have its own interest.

**Lemma 4.3.** Let $x \geq e$ be sufficiently large and let $J$ be any real number satisfying $x^{1/2} < J \leq x$ and $(k, \ell)$ be an exponent pair. Then

$$
S(x) \geq \frac{x \log J}{\zeta(2)} + O(\mathfrak{J}(\log J)^2 + x),
$$

where

$$
\mathfrak{J} = (J^{\ell+1}x^{k+1})^{1/(k+2)} + (J^{2(\ell+1)}x^k)^{1/(k+2)}
$$

$$
\quad + (J^{3k-\ell+5}x^{-k-1})^{1/(k+2)} + J^3 / x.
$$

**Proof.** Recalling (4.2) and using that $S_0(x) \geq 0$ we write

$$
S(x) \geq S_1(x) + S_2(x) + S_3(x).
$$

Now

$$
S_1(x) = x \sum_{n \leq J} \frac{\phi(n)}{n^2} + O(x) = \frac{x \log J}{\zeta(2)} + O(x)
$$
and obviously

\[ S_2(x) \ll x. \]

It remains to estimate \( S_3 \). Covering the interval \([x^{1/2}, J]\) by \( L \ll \log J \)
dyadic intervals of the form \([N, 2N]\), we have

\[
S_3(x) \leq \left| \sum_{x^{1/2} < n \leq J} \varphi(n) \psi \left( \frac{x}{n + 1} \right) \right| + \left| \sum_{x^{1/2} < n \leq J} \varphi(n) \psi \left( \frac{x}{n} \right) \right| \\
\ll L \max_{\vartheta \in \{0, 1\}} \max_{x^{1/2} < N \leq J} \left| \sum_{N < n \leq 2N} \varphi(n) \psi \left( \frac{x}{n + \vartheta} \right) \right|.
\]

Now, by Lemma 4.1, for any integer \( H \geq 1 \),

\[
S_3(x) \ll L \max_{\vartheta \in \{0, 1\}} \max_{x^{1/2} < N \leq J} \left( \frac{N^2}{H} + \sum_{h \leq H} \frac{1}{h} \left| \sum_{N < n \leq 2N} \varphi(n) e \left( \frac{hx}{n + \vartheta} \right) \right| \right).
\]

Note that the function

\[ n \mapsto \frac{hx}{n(n + 1)} \]

is non-increasing and bounded by \( HxN^{-2} \) so that, by partial summation,

\[
\sum_{N < n \leq 2N} \varphi(n) e \left( \frac{hx}{n(n + 1)} \right) = \sum_{N < n \leq 2N} \varphi(n) e \left( \frac{hx}{n} \right) e \left( -\frac{hx}{n(n + 1)} \right) \\
\ll \left( 1 + \frac{Hx}{N^2} \right) \left| \sum_{N < n \leq 2N} \varphi(n) e \left( \frac{hx}{n} \right) \right|
\]

and therefore

\[
(4.5) \quad S_3(x) \ll L \max_{x^{1/2} < N \leq J} \left( \frac{N^2}{H} + \left( 1 + \frac{Hx}{N^2} \right) W \right),
\]

where

\[
W = \sum_{h \leq H} \frac{1}{h} \left| \sum_{N < n \leq 2N} \varphi(n) e \left( \frac{hx}{n} \right) \right|.
\]

The estimate of Lemma 4.2 yields

\[
W \leq (Hx)^k N^{1+\ell-2k} \log N + \frac{N^3}{x} + N \log H \\
\leq \left( (Hx)^k N^{1+\ell-2k} + \frac{N^3}{x} \right) \log H,
\]

where we have used the fact that \( N > x^{1/2} \) implies that \( N^3x^{-1} \geq N \).
Inserting this estimate in (4.5), we derive
\[ S_3(x) \ll L^2 \max_{x^{1/2} < N \leq J} \left( \frac{N^2}{H} + \left( 1 + \frac{Hx}{N^2} \right) \left( (Hx)^k N^{1+\ell-2k} + \frac{N^3}{x} \right) \right). \]

Now choose
\[ H = \left\lceil \left( N^{2k-\ell+3} x^{-k-1} \right)^{1/(k+2)} \right\rceil. \]

Note that the condition \( N > x^{1/2} \) ensures that \( H \geq 1 \). We eventually obtain
\[ S_3(x) \ll L^2 \mathfrak{J}, \]
concluding the proof.

4.2.3. **Concluding the proof of the lower bound.** The lower bound of Theorem 2.1 follows from Lemma 4.3 at once by using the exponent pair of Bourgain [10, Theorem 6], coupled with several applications of van der Corput’s \( A \)- and \( B \)-processes, see [8, Sections 6.4.2 and 6.6.2]:
\[ (k, \ell) = BA^3 \left( BA^2 \right)^2 \left( \frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon \right) = \left( \frac{3071}{7887} + \varepsilon, \frac{1380}{2629} + \varepsilon \right) \]
and choosing \( J = x^{2629/4009 - \varepsilon} \).

4.3. **The upper bound.**

4.3.1. **Some explicit estimates.** The following estimate is a well-known result. For a proof, see [16, Lemma 2.1].

**Lemma 4.4.** For all \( x \geq 1 \)
\[ \left| \sum_{n \leq x} \frac{1}{n} - \log x - \gamma \right| \leq \frac{6}{11x}. \]

We also need some bounds on sums involving the Euler function, which follows from [20, Lemma 2.2] and partial summation.

**Lemma 4.5.** For all \( x \geq 1 \) we have
\[ \sum_{n \leq x} \frac{\varphi(n)}{n^2} \leq \frac{\log x}{\zeta(2)} + 2 + \frac{1}{\zeta(2)}. \]
4.3.2. Concluding the proof. We recall the representation of $S(x)$ as (4.2).

First, using Lemma 4.4

\[ S_0(x) \leq x \sum_{n \leq x/J} \frac{1}{n} \leq x \log(x/J) + O(x). \]  

Next, using Lemma 4.5 we derive

\[ S_1(x) \leq x \sum_{n \leq J} \frac{\varphi(n)}{n(n+1)} \leq x \sum_{n \leq J} \frac{\varphi(n)}{n^2} = x \frac{\log J}{\zeta(2)} + O(x). \]

Furthermore, we obviously have

\[ S_2(x) \leq \sum_{n \leq x^{1/2}} \varphi(n) \leq \sum_{n \leq x^{1/2}} n \ll x. \]

Now, choosing $J = x^{2629/4009 - \varepsilon}$ for a sufficiently small $\varepsilon > 0$ as in Section 4.2.3, we see that

\[ S_3(x) = o(x). \]

Now substituting the bound (4.6), (4.7), (4.8) and (4.9) in (4.2) implies the asserted upper bound Theorem 2.1.

5. Proof of Theorem 2.3

5.1. Initial transformation. Let $T \in [2\sqrt{x}, x]$ be a parameter at our disposal. Then

\[
S_f(x) = \sum_{n \leq x} f(n) \left( \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \\
= \sum_{n \leq T} f(n) \left( \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \\
+ \sum_{T < n \leq x} f(n) \left( \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \\
= \sum_{n \leq T} f(n) \left( \frac{x}{n(n+1)} + O(1) \right) \\
+ O \left( \sum_{T < n \leq x} |f(n)| \left( \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \right).
\]
Hence

\[ S_f(x) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O \left( R_1(x) + R_2(x) + R_3(x) \right), \tag{5.1} \]

where

\[ R_1(x) = x \sum_{n>T}^{\infty} \frac{|f(n)|}{n(n+1)}, \]
\[ R_2(x) = \sum_{n \leq T} |f(n)|, \]
\[ R_3(x) = \sum_{T < n \leq x} |f(n)| \left( \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right). \]

5.2. Bounding error terms. Using that

\[ \sum_{n \leq T} |f(n)| \leq A \sum_{n \leq T} \tau_k(n) \ll_k AT (\log T)^{k-1}, \]

see [8, Section 4.8, Exercise 13], and by partial summation we obtain

\[ R_1(x) \leq Ax \sum_{n>T}^{\infty} \frac{\tau_k(n)}{n^2} \ll_k AT^{-1}x(\log T)^{k-1}, \tag{5.2} \]

where the implied constant in \( U \ll_k V \) (which is equivalent to \( U = O_k(V) \)) may depend on \( k \).

We also have

\[ R_2(x) \leq \sum_{n \leq T} |f(n)| \leq A \sum_{n \leq T} \tau_k(n) \ll_k AT (\log T)^{k-1}. \tag{5.3} \]

If \( k = 1 \), then

\[ R_3(x) \leq A \sum_{T < n \leq x} \left( \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \]
\[ = A \left( \left\lfloor \frac{x}{[T]} \right\rfloor - \left\lfloor \frac{x}{[x]+1} \right\rfloor \right) \ll A T^{-1}. \tag{5.4} \]

Choosing \( T = 3\sqrt{x} \) and substituting (5.2), (5.3) and (5.4) in (5.1) we obtain the desired result for \( k = 1 \).

Now we assume that \( k \geq 2 \) and estimate \( R_3(x) \) in this case. We have

\[ R_3(x) \ll A \max_{T < N \leq x} \sum_{N} \log(x/T), \tag{5.5} \]
where
\[
\Sigma_N(x) = \sum_{N < d \leq 2N} \tau_k(d) \left( \left\lfloor \frac{x}{d} \right\rfloor - \left\lfloor \frac{x}{d+1} \right\rfloor \right).
\]

Intervening the summations we obtain
\[
\Sigma_N(x) = \sum_{N < d \leq 2N} \tau_k(d) \left( \left\lfloor \frac{x}{d} \right\rfloor - \left\lfloor \frac{x}{d} - \frac{x}{d(d+1)} \right\rfloor \right)
\leq \sum_{N < d \leq 2N} \tau_k(d) \left( \left\lfloor \frac{x}{d} \right\rfloor - \left\lfloor \frac{x}{d} - \frac{x-xN^{-1}}{d} \right\rfloor \right)
= \sum_{x-xN^{-1} < n \leq x} \sum_{\substack{d | n \\ \text{N < d \leq 2N}}} \tau_k(d).
\]  
(5.6)

Now [9, Lemma 5.2] yields
\[
\Sigma_N(x) \ll_k (\log N)^{k-1} \sum_{x-xN^{-1} < n \leq x} \Delta_{k+1}(n),
\]
where \(\Delta_r\) is the \(r\)th Hooley’s divisor function (see [9] and the references therein), and using [9, Lemma 5.1], for an arbitrary fixed \(\varepsilon > 0\), we obtain
\[
\Sigma_N(x) \ll_{k,\varepsilon} (xN^{-1}(\log x)^{\varepsilon_{k+1}(x)} + x^\varepsilon)(\log N)^{k-1},
\]
where \(\varepsilon_{k+1}(x)\) is defined in (2.1) (and the implied constant is now allowed to depend on \(\varepsilon\) as well).

5.3. **Concluding the proof.** Collecting the previous estimates finally gives
\[
\sum_{n \leq x} f \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = x \sum_{n=1}^\infty \frac{f(n)}{n(n+1)}
+ O_{k,\varepsilon} \left\{ A_T(\log x)^{k-1} + \frac{Ax}{T}(\log x)^{k+\varepsilon_k(x)} + Ax^\varepsilon \right\},
\]
and choosing \(T = 3\sqrt{x(\log x)^{1+\varepsilon_{k+1}(x)}}\), and \(\varepsilon = 1/2\), we see that the last term never dominates, which completes the proof.
6. Proofs of Theorems 2.5 and 2.6

6.1. Proof of Theorem 2.5. The proof follows closely that of Theorem 2.3 in the case $k = 1$ above. It is only sufficient to note that, since $|f(n)| \ll n^{\phi-1}(\log en)^A$, the bounds (5.2), (5.3) and (5.4) become here

$$R_1(x) \ll xT^{\phi-2}, \quad R_2(x) \ll T^\phi, \quad R_3(x) \ll x^{\phi}T^{-1}(\log T)^{-A},$$

respectively. Choosing $T = x^{\phi/(\phi+1)}(\log x)^{-A/(\phi+1)}$ and replacing $\log T$ and with $\log x$ in the bound on $R_3(x)$ yields the asserted result.

6.2. Proof of Theorem 2.6. Again, the proof follows closely that of Theorem 2.3 in the case $k \geq 2$. Firstly, note that, using the assumption (2.2) and the Cauchy–Schwarz inequality, we obtain

$$\sum_{n \leq x} |f(n)| \leq x^{1/2} \left( \sum_{n \leq x} |f(n)|^2 \right)^{1/2} \ll x^{(\alpha+1)/2}.$$ 

Hence, for any $T \in (2x^{1/2}, x]$, the bounds for $R_1(x)$ and $R_2(x)$ become

$$R_1(x) \ll xT^{(\alpha-3)/2} \quad \text{and} \quad R_2(x) \ll T^{(\alpha+1)/2},$$

respectively.

To estimate $R_3(x)$, assume that $T \ll x^{1-\varepsilon}$ for some fixed $\varepsilon > 0$. Now similarly to our treatment in (5.5) and (5.6), we obtain

$$R_3(x) = \sum_{T < n \leq x} |f(n)| \left( \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right)$$

$$\ll \max_{T < N \leq x} \left( \sum_{N < n \leq 2N} |f(n)| \left( \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n} - \frac{x-1}{n} \right\rfloor \right) \log x \right)$$

$$\ll \max_{T < N \leq x} \left( \sum_{\frac{x}{N} < n \leq x} \sum_{d|n} |f(d)| \right) \log x.$$ 

Now using the Cauchy–Schwarz inequality and (2.2), we derive

$$\sum_{d|n, N < d \leq 2N} |f(d)| \leq \left( \sum_{n \leq 2N} |f(n)|^2 \right)^{1/2} \Delta(n)^{1/2} \ll N^{\alpha/2} \Delta(n)^{1/2}.$$
so that, if $N \gg x^{1-\varepsilon}$ and $0 < \alpha < 2$

$$R_3(x) \ll \max_{T < N \leq x} \left( N^{\alpha/2} x N^{-1} \right) \left( \log x \right)^{1+\varepsilon_2(x)/2} \ll xT^{-1+\alpha/2} \left( \log x \right)^{1+\varepsilon_1(x)/2}$$

and choosing $T = x^{2/3} \left( \log x \right)^{(2+\varepsilon_1(x))/3}$ gives the asserted result.

7. Numerical Results

As we have mentioned, it is not clear that a limit for

$$\rho(x) = \frac{S(x)}{x \log x}$$

exists and if it exists whether it coincides with

$$\frac{1}{\zeta(2)} \approx 0.60793.$$

Using Maple we can calculate approximate values of $\rho(x)$ for various values of $x$ as shown in the following table.

| $x$   | $\rho(x)$ |
|-------|-----------|
| $10^6$ | 0.5844    |
| $10^7$ | 0.5849    |
| $10^8$ | 0.5896    |
| $10^9$ | 0.5909    |
| $10^{10}$ | 0.5940   |

Unlike the normal totient summation, increasing $x$ by 1 changes all the previous summands. So the ratio $\rho(x)$ can meaningfully change for a small change in $x$ as shown here.

| $x$   | $\rho(x)$ |
|-------|-----------|
| $10^6$ | 0.5844    |
| $10^6 + 1$ | 0.6274  |
| $10^6 + 2$ | 0.5965  |
| $10^6 + 3$ | 0.6447  |
| $10^6 + 4$ | 0.6108  |

The meaningful change in the ratio $\rho(x)$ is also evident for small changes in $x$ above $10^{10}$. 
8. Comments

In the proof of the lower bound of Theorem 2.1, one can take the slightly weaker exponent pair

$$(k, \ell) = BA^3 (BA^2)^2 B (0, 1) = \left( \frac{97}{251}, \frac{132}{251} \right)$$

and choose $J = x^{251/383}(\log x)^{-1198/383}$ to obtain

$$\sum_{n \leq x} \phi \left( \left\lfloor x/n \right\rfloor \right) \geq \left( \frac{251}{383} - \frac{1198 \log \log x}{383 \log x} \right) \frac{x \log x}{\zeta(2)} + O(x)$$

and similarly for the upper bound. Note that

$$\frac{251}{383 \zeta(2)} \approx 0.39841 \quad \text{and} \quad \frac{2629}{4009 \zeta(2)} \approx 0.39866$$

while in the upper bound we have

$$\frac{251}{383} \cdot \frac{1}{\zeta(2)} + \frac{132}{383} \approx 0.74305 \quad \text{and} \quad \frac{2629}{4009} \cdot \frac{1}{\zeta(2)} + \frac{1380}{4009} \approx 0.74289.$$
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