An elementary approach to component sizes in some critical random graphs

Umberto De Ambroggio

January 19, 2021

Abstract

In this article we show that part of the argument used in [10] to derive the correct upper bound for the probability of observing an unusually large maximal component in the near-critical Erdős-Rényi graph can be used to analyse other models of random graphs when considered at criticality. Specifically, we apply our method to a model of random intersection graph, a random graph obtained through $p$-bond percolation on a general $d$-regular graph, and a model of inhomogeneous random graph.

1 Introduction

Let us start by introducing the required notation. Let $G = (V,E)$ be any (undirected) graph. Given two vertices $v, u \in V$, we write $v \sim u$ if the edge $\{v, u\}$ is present in $G$ and say that vertices $u$ and $v$ are neighbours. We write $v \leftrightarrow u$ if there exists a path of occupied edges connecting vertices $v$ and $u$ and we adopt the convention that $v \leftrightarrow v$ for every $v \in V$. We denote by $C(v) := \{u \in V : v \leftrightarrow u\}$ the component (or cluster) of vertex $v \in V$ and its size by $|C(v)|$. Moreover, we define the largest component $C_{\text{max}}$ to be any cluster $C(v)$ for which $|C(v)|$ is maximal, so that $|C_{\text{max}}| = \max_{v \in V} |C(v)|$.

The Erdős-Rényi random graph on $[n] := \{1, \ldots, n\}$, denoted by $G(n,p)$, is the random graph obtained from the complete graph on $n$ vertices by independently retaining each edge with probability $p \in [0,1]$ and deleting it with probability $1 - p$. One of the most surprising aspects of this model is that when $p$ is of the form $p = p(n) = \gamma/n$, then the $G(n,p)$ random graph undergoes a phase transition as $\gamma$ passes 1. Specifically, if $\gamma < 1$, then $|C_{\text{max}}|$ is of order $\log(n)$; if $\gamma = 1$, then $|C_{\text{max}}|$ is of order $n^{2/3}$; and if $\gamma > 1$, then $|C_{\text{max}}|$ is of order $n$. See for instance the books [4],[14] or [29] for proofs of these statements and other interesting properties of this model. See also Krivelevich and Sudakov [18] for a simple proof of the phase transition in $G(n,p)$.

In [10] the authors introduced a ballot-type result (Lemma 5.1 below) to provide a new, purely probabilistic proof of the fact that in the $G(n,p)$ model considered in the so-called critical window, i.e. when $p$ is of the form $p = p(n) = n^{-1} + \lambda n^{-4/3}$, the probability of observing a maximal cluster of size larger than

---

*University of Bath, Department of Mathematical Sciences - umbidea@gmail.com
$An^{2/3}$ tends to zero (as $n \to \infty$) exponentially fast in $A$. More precisely, they proved that, for large enough $n$,

$$
\frac{c}{A^{3/2}}e^{-\frac{A^3}{2} + \frac{A^2}{2} - \frac{A^2}{8}} \leq P(|C_{max}| > An^{2/3}) \leq \frac{c'}{A^{3/2}}e^{-\frac{A^3}{2} + \frac{A^2}{2} - \frac{A^2}{8}},
$$

(1)

where $c, c'$ are two positive constants, thus showing that in the near-critical $G(n, p)$ model the number of vertices contained in the maximal component is unlikely to be much larger than $n^{2/3}$.

We remark that the correct asymptotic for $P(|C_{max}| > An^{2/3})$ in this critical model was obtained first by Pittel [25] (whose paper is partially based on an earlier article by Luczak, Pittel and Wierman [20]) and more recently by Roberts [26]. We also mention that Nachmias and Peres [22] used a general martingale argument to establish an exponential upper bound for the probability in (1), but their bound is not optimal.

The purpose of this work is to show that part of the argument used in [10] to prove the upper bound in (1) is quite general and can be used to obtain, in a surprisingly simple way, polynomial upper bounds for $P(|C_{max}| > k)$ in different models of random graphs when considered at criticality. Specifically, we apply our method to three different models, namely a model of random intersection graph, a random graph obtained through $p$-bond percolation on a general $d$-regular graph, and a model of inhomogeneous random graph, and we show that $|C_{max}|$ is unlikely to be much larger than $n^{2/3}$ in these models. In this sense, these random graphs exhibit a similar critical behaviour.

2 Results

In order to better understand the statement of our main result (Proposition 2.1 below), we first need to recall the definition of an exploration process, which is an algorithmic procedure used to reveal the components of a given graph; see e.g. [10], [22], [21], [26] and references therein. As we will see in a moment, when the graph under investigation is random such exploration process reduces the study of component sizes to the analysis of the trajectory of a random process, which looks like (but it is not quite) a random walk.

Let $G = ([n], E)$ be any (undirected) graph, and let $V_n$ be a vertex selected uniformly at random from $[n]$. During the exploration of $C(V_n)$, each vertex will be either active, explored or unseen, and its status will change during the course of the exploration. At each step $t \in \{0\} \cup [n]$ of the algorithm, the number of explored vertices will be $t$ whereas the number of active vertices will be denoted by $Y_t$. At time $t = 0$, if $V_0$ is an isolated vertex we stop the procedure; otherwise, there exists some vertex $u \in [n] \setminus V_0$ with $\{V_0, u\} \in E$. In this case, vertices $V_0$ and $u$ are declared active, whereas all other vertices are declared unseen (so that $Y_0 = 2$). At each step $t \in [n]$ of the algorithm, if $Y_{t-1} > 0$ we let $u_t$ be the active vertex with the smallest label; if $Y_{t-1} = 0$, we let $u_t$ be the unseen vertex with the smallest label. We reveal all unseen neighbours of $u_t$ in $G$ and change the status of these vertices to active. Then, we set $u_t$ itself explored. Denoting by $\eta_t$ the number of unseen vertices in $G$ which become active at step $t$ of the exploration process, we see that:

(i) if $Y_{t-1} > 0$ then $Y_t = Y_{t-1} + \eta_t - 1$;
(ii) if $Y_{t-1} = 0$ then $Y_t = \eta_t$.

**Remark 1.** We remark that our description of an exploration process is slightly different with respect to the one provided e.g. in [10], [22] and [20]. Indeed, in our setting the algorithm is actually run only in the case where $V_n$ is not an isolated vertex, and the exploration starts from two active vertices and not from one active vertex, as it usually happens. This small modification will be particularly useful in one of our applications.

Now let $G = ([n], E)$ be any (undirected) random graph, and imagine to run the above algorithm in order to reveal the components of $G$. It is clear that now the $\eta_i$ are random variables. Observe that, given any $k \in \mathbb{N} = \{1, 2, \ldots\}$, if $V_n$ is an isolated vertex then $|C(V_n)| = 1$ and hence, in particular, we can’t have $|C(V_n)| > k$ (as $k \geq 1$). On the other hand, if $V_n$ is not isolated then $|C(V_n)| > k$ implies that $Y_t = 2 + \sum_{i=1}^{t}(\eta_i - 1) > 0$ for all $t \in [k]$. Therefore we can write

$$P(|C(V_n)| > k) \leq P\left(2 + \sum_{i=1}^{t}(\eta_i - 1) > 0 \ \forall t \in [k]\right).$$

Note that the $\eta_i$ are not independent and, moreover, they have different distributions (one of the reasons is that the number of unseen vertices in the graph decreases during the course of the exploration). Therefore $Y_t$ does not define a random walk.

In order to bound from above the probability in (2), the idea is to produce a sequence of independent and identically distributed (i.i.d.) random variables $X_i$, bigger than the $\eta_i$, that allow us to replace the probability on the right hand side of (2) with the probability that a random walk (started at 2) stays positive up to time $k$. In some random graphs this is an immediate consequence of the model construction, while in other instances one needs more care in order to produce these $X_i$.

Here is our main result.

**Proposition 2.1.** Let $G = ([n], E)$ be any (undirected) random graph. Suppose that there exists a sequence of i.i.d. random variables $(X_i)_{i \geq 1}$ taking values in $\mathbb{N}_0$ such that, for all $k \in \mathbb{N}$, setting $S_t := \sum_{i=1}^{t}(X_i - 1)$ we have that $P(|C(V_n)| > k) \leq P\left(2 + S_t > 0 \ \forall t \in [k]\right)$ and

$$P\left(\sum_{i=1}^{k+1}X_i \geq k + 1 + \lambda\right) \leq \Phi e^{-\lambda/\Phi}, \ \lambda > 0,$$

where $\Phi = \Phi(k, n)$ does not depend on $\lambda$. If $P(X_1 = 3) > 0$ then

$$P(|C_{\text{max}}| > k) \leq \frac{1 + 2e^{-1} \Phi}{P(X_1 = 3)} \frac{n}{k^{3/2}} \left(1 + O(1/\sqrt{k})\right).$$

**Remark 2.** In all our applications the probability $P(X_1 = 3)$ is bounded away from zero and, moreover, $\Phi = O(1)$ as $n \to \infty$, so that if we take $k = \lceil An^{2/3} \rceil$ in (4) above we see that $P(|C_{\text{max}}| > An^{2/3})$ is indeed $O(A^{-3/2})$.

**Remark 3.** We remark that condition (4) might be stated in different (possibly more general) terms, but we decided to state it in this way because of its simplicity to be verified, as shown in our applications.
Our claim that the approach introduced in [10] is robust and that Proposition 2.1 leads to simple upper bounds for $\mathbb{P}(C_{\text{max}} > k)$ in several models of random graphs at criticality is justified in sub-sections 2.1, 2.2 and 2.3 below, where we apply Proposition 2.1 to obtain polynomial upper bounds for the above probability in three particular models of random graphs.

2.1 Critical random intersection graph

Our first application of Proposition 2.1 involves a model of random intersection graph; for an introduction to this class of models, we refer the reader to [12].

Here we are interested in the random graph described by Lagerås and Lindholm [19]. Such a random graph, denoted by $G(n, m, p)$, with a set of vertices $V = \{v_i : i \in [n]\}$ and a set of edges $E$, is constructed from a bipartite graph $B(n, m, p)$ with two sets of vertices: $A = \{a_j : j \in [m]\}$, which we call the set of auxiliary vertices, and $V$ (that is, the vertex set of $G(n, m, p)$). Edges in $B(n, m, p)$ between vertices and auxiliary vertices are present independently with probability $p \in [0, 1]$. Two distinct vertices $v_i$ and $v_j$ are neighbours in $G(n, m, p)$ (i.e. $\{v_i, v_j\} \in E$) if and only if there exists at least one $a_k \in A$ such that both edges $\{a_k, v_i\}$ and $\{a_k, v_j\}$ are present in the bipartite graph $B(n, m, p)$.

We are interested in the case where $p = p(n) := \gamma/n^{(1+\alpha)/2}$ and $m = m(n) := \lfloor \beta n \rfloor$, where $\alpha, \beta, \gamma > 0$ are fixed parameters of the model.

Stark [27] has shown that the vertex degree distribution (i.e. the distribution of the degree of a vertex selected uniformly at random) is highly dependent on the value of $\alpha$. However, as shown by Deijfen and Kets [11], the clustering is controllable only when $\alpha = 1$.

The component structure of the graph is studied for $\alpha \neq 1, \gamma > 0$ and $\beta = 1$ by Bherisch [3], whereas it is studied for $\alpha = 1$ and $\beta, \gamma > 0$ in [19]. Specifically, Lagerås and Lindholm [19] proved that the $G(n, m, p)$ model undergoes a phase transition as $\beta \gamma^2$ passes 1. Indeed, setting $\mu := \beta \gamma^2$, they proved that if $\mu < 1$ then with probability tending to one there is no component in $G(n, m, p)$ with more than $O(\log(n))$ vertices, while if $\mu > 1$ then, with probability tending to one, there exists a unique giant component of size $n \delta$ where $\delta \in (0, 1)$, and the size of the second largest component is at most of order $\log(n)$.

By means of Proposition 2.1 we show that, in the critical case $\mu = 1$, it is unlikely for the largest component to contain more than $n^{2/3}$ vertices.

Proposition 2.2. Let $G(n, m, p)$ be the random intersection graph described above. Let $m := \lfloor \beta n \rfloor$, $p := \gamma/n$ and $\mu := \beta \gamma^2$. If $\mu = 1$ then, given any $A > 1$, when $n$ is sufficiently large we have that

$$\mathbb{P}(\lfloor C_{\text{max}} \rfloor \leq \lfloor A n^{2/3} \rfloor) \geq 1 - \frac{c_1}{A^{4/2}},$$

where $c_1 = c_1(\gamma, \beta) > 1$ is a constant which depends on $\gamma$ and $\beta$.

2.2 Critical $p$-bond percolation on $d$-regular graph

In this section we consider a second application of Proposition 2.1. Here we analyse a random graph $G_p$ obtained through $p$-bond percolation on a general $d$-regular graph.
In [21] Nachmias and Peres adapted the martingale method developed by the same authors in [22] to prove that, for any $d \geq 3$, when $p \leq (d - 1)^{-1}$ then

$$\mathbb{P}(|C_{\text{max}}| > \lceil An^{2/3} \rceil) \leq \frac{8}{A^{3/2}},$$

(6)

see Proposition 1.2 in [21]. For a random regular graph $G(n, d, p)$ they were also able to sharpen the upper bound in (6) and to prove a corresponding lower bound. (The $(n, d, p)$ random graph is obtained by the following two-step procedure: first we draw uniformly at random a graph from the set of all simple and vertex-labelled $d$-regular graphs with vertex set $[n]$, and then we keep each edge with probability $p$ and delete it with probability $1 - p$.) Specifically, in Theorem 2 of [21] it is shown that, when $p$ is of the form

$$p = p(n, d) = (1 + \lambda n^{-1/3})/(d - 1) \ (\lambda \in \mathbb{R})$$

(7)

and $d \geq 3$ is fixed, then there are constants $C_1, C_2 \in (0, \infty)$ depending on $\lambda$ and $d$ such that, for every $A > 0$ and all $n$, $\mathbb{P}(|C_{\text{max}}| > A_n^{2/3}) \leq A^{-1}C_1 e^{-C_2 A}$. In [21] it is also shown that there exists a constant $C_3 \in (0, \infty)$ (also depending on $\lambda$ and $d$) such that, for small enough $\delta > 0$ and all $n$, then $\mathbb{P}(|C_{\text{max}}| < \lceil \delta n^{2/3} \rceil) \leq C_3 \delta^{1/2}$, thus proving that the size of $|C_{\text{max}}|$ is indeed of order $n^{2/3}$ in this model when considered at criticality.

We remark that in [21], the parameter $d$ is not allowed to depend on $n$. The problem of determining the size of $|C_{\text{max}}|$ in the critical $G(n, d, p)$ model when $d = d(n)$ depends on $n$ has been investigated by Joos and Perarnau [15], where the authors proved (among many other things) that for any $d \in \{3, \ldots, n - 1\}$ and when $p$ is of the form (7), then for all sufficiently large $n$ and $A = A(\lambda)$ we have that $\mathbb{P}(|C_{\text{max}}| \not\in [A^{-1} n^{2/3}, A n^{2/3}]) \leq 20/\sqrt{A}$.

Our goal here is to show that, by means of Proposition 2.1, we can recover (up to a multiplicative constant) the bound in (6), in a very simple way.

**Proposition 2.3.** Let $G$ be a $d$-regular graph, $d > 3$, and denote by $G_p$ the random graph obtained by bond percolation on $G$ with probability $p$. If $p \leq 1/(d - 1)$ then, given any $A > 1$, we have that

$$\mathbb{P}(|C_{\text{max}}| \leq \lceil An^{2/3} \rceil) \geq 1 - \frac{c_2}{A^{3/2}},$$

(8)

for some constant $c_2 > 1$.

## 2.3 Critical inhomogeneous random graph

In this section we discuss our final application of Proposition 2.1. In the random graph model that we investigate here, the $n$ vertices are endowed with weights, and edges between pair of vertices are placed independently with probabilities moderated by such weights.

Specifically, let $w = (w_i)_{i \in [n]} \subset (0, \infty)$ be a sequence of real numbers, which we call the sequence of vertex weights; we think of $w_i$ as the weight assigned to vertex $i \in [n]$. Define $l_n := \sum_{i \in [n]} w_i$, the sum of all weights.

We consider the so-called Norros-Reittu random graph [23] as described by Van der Hofstad [28]. This is an inhomogeneous random graph, that we denote
by $NR_n(w)$, in which the probability that the edge \(\{i, j\}\) is present in $NR_n(w)$ (for $1 \leq i < j \leq n$) is given by
\[
p_{ij}^{NR} := P(\{i, j\} \in E(NR_n(w))) := 1 - e^{-w_j / j / n},
\]
and edges are present independently.

Inhomogeneous random graphs were studied extensively by Bollobas, Janson and Riordan [5]. As explained by Janson [13] and further remarked by Van der Hofstad [28], the $NR_n(w)$ random graph is closely related to the models studied by Chung and Lu [6, 7, 8] and Norros and Reittu [23], so that the results proved for the $NR_n(w)$ random graph apply as well to these other models.

Other models of inhomogeneous random graphs have been studied more recently by Penrose [24] and by Kang, Pachon and R"odriguez [17].

It is clear that the topology of the $NR_n(w)$ model depends on the choice of the sequence $w$, which we now specify.

Let $F : \mathbb{R} \mapsto [0, 1]$ be a distribution function, and define
\[
[1 - F]^{-1}(u) := \inf \{s : [1 - F(s)] \leq u\}, \quad u \in (0, 1).
\]

By convention, we set $[1 - F]^{-1}(1) := 0$. We construct the weights as in [28], namely we set
\[
w_j := [1 - F]^{-1}(j/n), \quad j \in [n].
\]

In [28] (Theorem 3.13) it has been shown that in the $NR_n(w)$ random graph with vertex weights as in (10), the proportion of vertices having degree $k \geq 0$, denoted by $N_k$, converges in probability (as $n \to \infty$) to
\[
p_k := E \left( e^{-W \frac{W^k}{k!}} \right),
\]
where $W$ is a random variable taking values in $(0, \infty)$ with distribution function $F$. The limiting sequence $(p_k)_{k \geq 0}$ is a so-called mixed Poisson distribution with mixing distribution $F$.

In order to describe the phase transition for the size of the largest component, define
\[
\nu := E(W^2) / E(W).
\]

As explained by Van der Hofstad [28] (see also [9]), this (positive) real number corresponds to the asymptotic mean of the offspring distribution in a branching process approximation of the exploration of $C(V_n)$.

In [3] (Theorem 3.1) it is shown that the graph undergoes a phase transition as $\nu$ passes 1. In particular, if $\nu > 1$, the largest component contains a positive proportion of the total number of vertices, whereas if $\nu \leq 1$ the largest component contains a vanishing proportion of vertices. When $\nu > 1$ the corresponding random graph is said to be super-critical, and when $\nu < 1$ it is called sub-critical. Finally, when $\nu = 1$ the random graph is said to be critical.

Van der Hofstad [28] provided a complete picture of the component structure in the critical $NR_n(w)$ model. Specifically, he proved that in the case where
\[
\lim_{x \to \infty} x^{-(\tau - 1)}(1 - F(x)) = c_F
\]
for some constant $c_F > 0$ and some $3 < \tau < 4$, then there is a constant $b > 0$ such that for all $\omega > 1$ and $n \geq 1$, the $NR_n(w)$ random graph satisfies
\[
P(\omega^{-1} n^{(\tau - 2)/(\tau - 1)}) \leq |C_{\text{max}}| \leq \omega n^{(\tau - 2)/(\tau - 1)} \geq 1 - b / \omega.
\]
On the other hand, when

\[ 1 - F(x) \leq c_F x^{-(\tau - 1)} \ (x \geq 0) \]  

(14)

for some \( c_F > 0 \) and some \( \tau > 4 \), then there is a constant \( b > 0 \) such that, for all \( n \geq 1 \) and all \( \omega > 1 \), the \( NR_n(w) \) random graph satisfies

\[ \mathbb{P}(\omega^{-1} n^{2/3} \leq |C_{max}| \leq \omega n^{2/3}) \geq 1 - b/\omega. \]  

(15)

(Actually Van der Hofstad \[28\] proved a more general result, namely that the lower bounds (13) and (15) remain valid also after a small perturbation of the vertex weights; see Theorems 1.1 and 1.2 in \[28\].)

For an heuristic explanation of the critical behaviour described by (13) and (15), we refer to section 1.3 in \[28\].

We also mention that in \[9\] the authors used the first part of the martingale argument introduced by Nachmias and Peres \[22\] to obtain simple upper bounds for the probability of observing unusually large maximal components in the (critical) \( NR_n(w) \) random graph for both regimes \( \tau \in (3, 4) \) and \( \tau > 4 \), even if in the former case (i.e. for \( \tau \in (3, 4) \)) the distribution function \( F \) is required to satisfy a stronger condition with respect to the one stated in (12).

Our goal here is to use Proposition 2.1 to provide a very simple proof of the fact that, in the critical \( NR_n(w) \) model with vertex weights as in (10) and distribution function \( F \) satisfying (14), the largest component is unlikely to contain more than \( n^{2/3} \). More precisely, we prove the following

**Proposition 2.4.** Consider the \( NR_n(w) \) random graph with weights defined as in (10) above. Suppose that there exists a constant \( c_F > 0 \) and a \( \tau > 4 \) such that

\[ 1 - F(x) \leq c_F x^{-(\tau - 1)} \ (x \geq 0) \]  

for all \( x \geq 0 \). Then, given any \( A > 1 \), when \( n \) is large enough we have that

\[ \mathbb{P}(|C_{max}| \leq \lceil An^{2/3} \rceil) \geq 1 - \frac{c_3}{A^{3/2}}, \]

where \( c_3 = c_3(c_F, \tau, F) > 1 \) is a constant which depends on \( c_F, \tau \) and \( F \).

### 3 Proofs

Here we are going to prove the results stated in Section 2. We start by proving Proposition 2.1 and subsequently we prove the remaining results, namely Propositions 2.2, 2.3, and 2.4.

#### 3.1 Preliminary results

The proof of Proposition 2.1 relies on the following ballot-type estimate, which is taken from \[10\]. For a general introduction to classical ballot theorems and their generalisations, see for instance \[1, 16\] and references therein.

**Lemma 3.1.** Fix \( n \in \mathbb{N} \) and let \( (W_i)_{i \geq 1} \) be a sequence of i.i.d. valued random variables taking values in \( Z \), whose distribution may depend on \( n \). Let \( r \in \mathbb{N} \), and suppose that \( \mathbb{P}(W_1 = r) > 0 \). Define \( S_t = \sum_{i=1}^{t} W_i \) for \( t \in \mathbb{N}_0 \). Then for any \( j \geq 1 \) we have

\[ \mathbb{P}(r + S_t > 0 \ \forall t \in [n], r + S_n = j) \leq \mathbb{P}(X_1 = r)^{-1} \frac{j}{n+1} \mathbb{P}(S_{n+1} = j). \]

7
We also make use of the following

**Lemma 3.2** (Theorem 1 in [2]). If $f$ is a positive and strictly decreasing function on $[1, \infty)$, then there is a positive constant $C(f) < f(1)$ and a sequence $(E_f(n))_n$ with $0 < E_f(n) < f(n)$ such that

$$\sum_{i=1}^n f(i) = \int_1^n f(x)dx + C(f) + E_f(n), \quad n \geq 2.$$ 

### 3.2 Proof of Proposition 2.1

Let $k = k(n) \in \mathbb{N}$. By hypothesis, there is a sequence of i.i.d. random variables $X_i$ taking values in $\mathbb{N}_0$ such that, setting $S_t := \sum_{i=1}^t (X_i - 1)$, then

$$P(|C(V_n)| > k) \leq P(2 + S_t > 0 \forall t \in [k]). \quad (16)$$

Using Lemma 3.1 with $W_i = X_i - 1$ and $r = 2$ we obtain

$$\sum_{h=1}^\infty P(2 + S_t > 0 \forall t \in [k], 2 + S_k = h) \leq c \sum_{h=1}^\infty \frac{h}{k+1} P(S_{k+1} = h), \quad (17)$$

where $c := 1/P(X_1 = 3)$. Now let $m$ be a non-negative integer to be specified later. By splitting the series in (17) at $h = m$ we can write

$$\sum_{h=1}^\infty \frac{h}{k+1} P(S_{k+1} = h). \quad (18)$$

Now the series in (18) equals

$$\frac{c}{k+1} \sum_{h=m+1}^\infty h P\left(\sum_{i=1}^{k+1} X_i = h + k + 1\right)$$

$$= \frac{c}{k+1} \sum_{z=m+k+2}^\infty z P\left(\sum_{i=1}^{k+1} X_i = z\right) - c P\left(\sum_{i=1}^{k+1} X_i \geq m + k + 2\right). \quad (19)$$

To proceed, we observe the following: if $X$ is a random variable taking values in $\mathbb{N}_0$, then for any $h \geq 1$, we have

$$\mathbb{E}[X \mathbb{1}_{\{X \geq h\}}] = \mathbb{E}\left[\sum_{i=1}^\infty \mathbb{1}_{\{i \leq X\}} \mathbb{1}_{\{X \geq h\}}\right] = h P(X \geq h) + \sum_{i=h+1}^\infty P(X \geq i).$$

Thus the series in (19) equals

$$\frac{c}{k+1} \sum_{i=m+k+3}^\infty P\left(\sum_{i=1}^{k+1} X_i \geq m + k + 2\right) + \frac{c}{k+1} \sum_{z=m+k+3}^\infty P\left(\sum_{i=1}^{k+1} X_i \geq z\right).$$
Substituting the series in (19) with these three terms we obtain

\[
\frac{c}{k + 1} \sum_{h=m+1}^{\infty} h P(S_{k+1} = h) = \frac{c}{k + 1} m P\left(\sum_{i=1}^{k+1} X_i \geq m + k + 2\right) + \frac{c}{k + 1} \sum_{h=m+1}^{\infty} P\left(\sum_{i=1}^{k+1} X_i \geq h\right).
\]  

(20)

Now observe that the series in (20) can be rewritten as follows:

\[
\frac{c}{k + 1} \sum_{z=m+k+2}^{\infty} P\left(\sum_{i=1}^{k+1} X_i \geq z\right) = \sum_{h=m+1}^{\infty} P\left(\sum_{i=1}^{k+1} X_i \geq h + k + 1\right).
\]

Summarizing, so far we have shown that

\[
P(|C(V_n)| > k) \leq \frac{c}{k + 1} m + \frac{c}{k + 1} \sum_{i=1}^{k+1} X_i \geq k + 1 + (m + 1)
\]  

(21)

By hypothesis, for all \(h \geq m + 1\) we have that

\[
P\left(\sum_{i=1}^{k+1} X_i \geq k + 1 + h\right) \leq \Phi e^{-\frac{h}{\sqrt{k+1}}},
\]

where \(\Phi = \Phi(k, n)\) does not depend on \(h\). Therefore we can bound from above the series in (21) as follows:

\[
\frac{c}{k + 1} \sum_{h=m+1}^{\infty} P\left(\sum_{i=1}^{k+1} X_i \geq h + k + 1\right) \leq \frac{c}{k + 1} \sum_{h=m+1}^{\infty} e^{-\frac{h}{\sqrt{k+1}}}.
\]  

(22)

Now by Lemma 3.2 we see that, for any \(L > m + 1\),

\[
\sum_{h=m+1}^{L} e^{-\frac{h}{\sqrt{k+1}}} \leq \sum_{i=1}^{L} e^{-\frac{h}{\sqrt{k+1}}} \leq \int_{1}^{L} e^{-\frac{x}{\sqrt{k+1}}} dx + e^{-\frac{m+1}{\sqrt{k+1}}} + e^{-\frac{k+1}{\sqrt{k+1}}} = \sqrt{k+1} e^{-\frac{m+1}{\sqrt{k+1}}} \left(1 + \frac{1}{\sqrt{k+1}}\right) - \sqrt{k+1} e^{-\frac{k+1}{\sqrt{k+1}}} \left(1 - \frac{1}{\sqrt{k+1}}\right).
\]

Therefore

\[
\sum_{h=m+1}^{\infty} e^{-\frac{h}{\sqrt{k+1}}} = \lim_{L \to \infty} \sum_{h=m+1}^{L} e^{-\frac{h}{\sqrt{k+1}}} \leq \sqrt{k+1} e^{-\frac{m+1}{\sqrt{k+1}}} \left(1 + O(1/\sqrt{k})\right)
\]

and hence we arrive at

\[
(22) \leq \frac{c}{\sqrt{k+1}} \Phi e^{-\frac{m+1}{\sqrt{k+1}}} \left(1 + O(1/\sqrt{k})\right).
\]
Thus we obtain
\[ P(\lvert C(v) \rvert > k) \leq c \left( \frac{m}{k+1} + \frac{m}{k+1} \Phi e^{-\frac{\Phi}{\sqrt{k+1}}} + e^{-\frac{\Phi}{\sqrt{k+1}}} \left( 1 + O(1/\sqrt{k}) \right) \right). \]

Taking \( m = \lceil \sqrt{k+1} \rceil \) we see that
\[ P(\lvert C(v) \rvert > k) \leq c \frac{\sqrt{k+1}}{k+1} + 2c e^{-\frac{\Phi}{\sqrt{k+1}}} \left( 1 + O(1/\sqrt{k}) \right). \]

Finally, denoting by \( N_k := \sum_{v \in [n]} 1 \{ \lvert C(v) \rvert > k \} \) the number of vertices that are contained in components of size at least \( k \) we obtain
\[ P(C_{\text{max}} > k) = P(N_k > k) \leq \frac{1}{k} E(N_k) = \frac{n}{k} P(C(V_n) > k) \]
\[ \leq c(1 + 2e^{-1} \Phi) \frac{n}{k^{3/2}} \left( 1 + O(1/\sqrt{k}) \right), \]
completing the proof of the proposition.

### 3.3 Proof of Proposition 2.2

Let \( H(n, m, p) \) be a random (multi-)graph constructed from the bipartite graph \( B(n, m, p) \) by letting the number of edges between \( v_i, v_j \in V \) equal the number of auxiliary vertices \( a_k \) that are adjacent to both \( v_i \) and \( v_j \). (Recall that \( V \) is the vertex set of the random intersection graph \( G(n, m, p) \) under investigation.) Notice that \( G(n, m, p) \) can be obtained from \( H(n, m, p) \) by coalescing multiple edges between vertices into one single edge. Hence, thanks to this construction, we see that the degree distribution in \( G(n, m, p) \) is dominated by the degree distribution in \( H(n, m, p) \). Notice that the latter is a compound binomial distribution with moment generating function
\[ E[e^{rX_1}] = \left( 1 - \frac{\gamma}{n} \right)^n + \frac{\gamma}{n} \left( 1 - \frac{\gamma}{n} + \frac{\gamma}{n} e^r \right)^{n-1} \right)^{\beta n}, \tag{23} \]
since (by construction) a vertex \( v \in H(n, m, p) \) is connected to a \( \text{Bin}(m, p) \) number of auxiliary vertices, and each one of them is connected to an independent \( \text{Bin}(n-1, p) \) number of vertices in \( V \setminus \{v\} \).

Therefore, by revealing the components of \( G(n, m, p) \) using the exploration process described at the beginning of Section 2, we can write
\[ P(\lvert C(V_n) \rvert > k) \leq P \left( 2 + \sum_{i=1}^t (X_i - 1) > 0 \ \forall t \in [k] \right), \]
where the \( X_i \) are i.i.d. compound binomial random variables with moment generating function given in (23).

Using the probability generating function of \( X_1 \) (which coincides with (23) after substituting \( e^r \) with \( r \)), it is not difficult to show that (for large enough \( n \)) the probability \( P(X_1 = 3) \) is bounded from below by a positive constant which depends solely on \( \gamma \) and \( \beta \).
Next, in order to apply Proposition 2.1, we simply need to prove an exponential upper bound for
\[ P \left( \sum_{i=1}^{k+1} X_i \geq k + 1 + \lambda \right) \quad (\lambda > 0). \] (24)
Setting \( Z := \exp \{-r\lambda - r(k + 1)\} \) and recalling that the \( X_i \) are i.i.d., an application of Markov’s inequality yields
\[ (24) \leq Z \mathbb{E} \left( \prod_{i=1}^{k+1} \exp\{rX_i\} \right) = Z \mathbb{E} (\exp\{rX_1\})^{k+1}. \] (25)
Recalling the expression of the moment generating function of \( X_1 \) given in (23), we obtain
\[ (25) = Z \exp \left\{ \beta n (k + 1) \log \left( 1 - \gamma + \gamma n \left[ 1 + \frac{2}{n} (e^r - 1) \right]^{n-1} \right) \right\}. \] (26)
Taking \( r \in (0, 1) \) we have that \( e^r - 1 \leq r + r^2 \). Then, since \( 1 + x \leq e^x \) for all \( x \geq 0 \), we obtain
\[ (26) \leq Z \exp \left\{ \beta n (k + 1) \log \left( 1 + \frac{2}{n} \left( \exp \left\{ \gamma (r + r^2) \right\} - 1 \right) \right) \right\}. \] (27)
Taking \( r \) so small that \( 0 < \gamma (r + r^2) < 1 \) we obtain
\[ \exp \left\{ \gamma (r + r^2) \right\} - 1 \leq \gamma (r + r^2) + \gamma^2 (r + r^2)^2 \]
and hence, using the fact that \( \log(1 + x) \leq x \) for all \( x > -1 \) we can write
\[ (27) \leq Z \exp \left\{ \beta n (k + 1) \left( \frac{\gamma^2}{n} (r + r^2) + \frac{\gamma^3}{n} (r + r^2)^2 \right) \right\}. \] (28)
Recalling the definition of \( Z \) and taking \( r = 1/\sqrt{k+1} \) (where \( k \) is taken large enough so that the previous conditions on \( r \) are satisfied), we see that
\[ (28) \leq \Phi e^{-\lambda/\sqrt{k+1}}, \]
where we set \( \Phi := e^{1+2\gamma} \). Hence, taking \( k = \lceil An^{2/3} \rceil \) and applying Proposition 2.1 we obtain that
\[ \mathbb{P}(\lvert C_{\text{max}} \rvert > \lceil An^{2/3} \rceil) \leq \frac{1 + 2e^{2\gamma}}{\mathbb{P}(X_1 = 3)} \frac{1}{A^{3/2}} \left( 1 + O \left( \frac{1}{\sqrt{An^{1/3}}} \right) \right). \]
Recalling that \( \mathbb{P}(X_1 = 3) \) is bounded from below (for large enough \( n \)) by a constant depending solely on \( \gamma \) and \( \beta \), we obtain the desired result.

3.4 Proof of Proposition 2.3
Since \( G \) is \( d \)-regular, we can use the exploration process described at the beginning of Section 2 to conclude that
\[ \mathbb{P}(\lvert C(V_n) \rvert > k) \leq \mathbb{P} \left( 2 + \sum_{i=1}^{k} (X_i - 1) > 0 \quad \forall t \in [k] \right), \]
where the $X_i$ are i.i.d. random variables with $X_i \sim \text{Bin}(d - 1, p)$, so that $\sum_{i=1}^{k+1} X_i \sim \text{Bin}((k + 1)(d - 1), p)$. (Notice that, if we would have started the exploration process with only one active vertex, now we would have $\eta_1 \sim \text{Bin}(d, p)$ and hence in particular it would be impossible to dominate $\eta_1$ with a $\text{Bin}(d - 1, p)$ random variable.) Using a monotonicity argument we can focus on the (critical) case $p = 1/(d - 1)$. Note that, since $d > 3$,

$$P(X_1 = 3) = \frac{1}{6} \frac{(d - 1)(d - 3)}{(d - 2)^2} \left(1 - \frac{1}{d - 1}\right)^{d-1} \geq e^{-4/3}/24.$$  

Next, by Markov’s inequality it is easy to see that, for any $\lambda > 0$ and $r \in (0, 1)$,

$$P\left(\sum_{i=1}^{k+1} X_i \geq k + 1 + \lambda\right) \leq e^{-r\lambda + r^2(k+1)}. \quad (29)$$

Taking $r = 1/\sqrt{k+1}$ we obtain

$$\text{(29)} \leq \Phi e^{-\sqrt{k+1}/r},$$

where we set $\Phi := e^{-1}$. Hence, using Proposition 2.1 we arrive at

$$P(|C_{\max}| > k) \leq \frac{3}{P(X_1 = 3)} \left(1 + O(1/\sqrt{k})\right).$$

Taking $k = \lceil A n^{2/3} \rceil$ we obtain the desired bound.

### 3.5 Proof of Proposition 2.4

Before starting with the actual proof, we need to recall the definition of size-biased distribution of a non-negative random variable and to introduce a few facts.

**Definition 3.1.** For a non-negative random variable $X$ with $\mathbb{E}(X) > 0$, the size-biased distribution of $X$, denote by $X^*$, is the random variable defined by

$$P(X^* \leq x) = \frac{\mathbb{E}[X \mathbb{1}_{\{X \leq x\}}]}{\mathbb{E}(X)}.$$  

For proofs of the assertions that appear in the statement of next result, see Lemma 4.1 and Proposition 4.1 in [9].

**Lemma 3.3.** Suppose that $1 - F(x) \leq c_F x^{-(\tau - 1)}$ for all $x \geq 0$, for some $c_F > 0$ and $\tau > 4$. Let $w_i$ be as in (10). Then $\max\{w_i : i \in [n]\} \leq (c_F n)^{1/(\tau - 1)}$. Moreover, defining

$$F_n(x) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{w_i \leq x\}}$$

and letting $W_n$ being a random variable with distribution function $F_n$ and size-biased distribution $W_n^*$, then $\mathbb{E}\left[(W_n^*)^2\right] \leq C_1$ and $|1 - \mathbb{E}[W_n^*]| \leq C_2 n^{-\frac{\tau}{\tau - 1}}$ for large enough $n$, where $C_1$ and $C_2$ are two positive constants which depend on $c_F$, $\tau$ and $F$.  

12
As explained in Van der Hofstad [28] (see also [9]), the cluster exploration of $V_n$ in the $\mathcal{N}_R(w)$ random graph can be dominated by the total progeny of a (marked mixed-Poisson) branching process. Specifically, following Van der Hofstad [28] we can write

$$P(|C(V_n)| > k) \leq P \left( 2 + \sum_{i=1}^{t} (X_i - 1) > 0 \ \forall t \in [k] \right),$$

(31)

where the $X_i$ are independent mixed Poisson random variables with $X_i \sim Poi(w_{M_i})$ and the $M_i$ are i.i.d. random variables, all distributed as a random variable $M$ with distribution given by

$$P(M = m) = \frac{w_m}{\lambda n}, \ m \in [n].$$

As remarked in [28], a $Poi(w_M)$ random variable converges in distribution to a mixed Poisson random variable with random parameter $W^*$, where $W^*$ is the size-biased distribution of $W$, the latter being a positive random variable with distribution function $F$. Therefore, $P(X_1 = 3)$ converges to $P(Z = 3)$, where $Z \sim Poi(W^*)$. It follows that $P(X_1 = 3) \geq P(Z = 3)/2$ for all large enough $n$, and hence we obtain

$$P(X_1 = 3) \geq \frac{1}{2} \mathbb{E} [P(Z = 3|W^*)] = \mathbb{E} \left[ e^{-W^*} (W^*)^3 \right] / 12 > 0.$$  

(32)

Now, by Markov’s inequality we have that

$$P \left( \sum_{i=1}^{k+1} X_i \geq k + 1 + \lambda \right) \leq e^{-r(k+1)-r\lambda} \mathbb{E} \left[ e^{rX_1} \right]^{k+1}$$

(33)

for all $r > 0$. Taking $r \in (0,1)$, we can write

$$\mathbb{E} \left[ e^{rX_1} \right] = \sum_{h=0}^{\infty} e^{rh} \sum_{i \in [n]} \frac{w_i}{\lambda n} e^{-w_i} \frac{w_i^h}{h!} = \sum_{i \in [n]} \frac{w_i}{\lambda n} e^{w_i(e^r - 1)} \leq \sum_{i \in [n]} \frac{w_i}{\lambda n} e^{w_i(r+r^2)}.$$

Therefore going back to (33) we obtain

$$\leq \exp \left\{ -r(k+1) - r\lambda + (k+1) \log \left( \sum_{i \in [n]} \frac{w_i}{\lambda n} e^{w_i(r+r^2)} \right) \right\}. \ (34)$$

Now let $r \leq (\zeta \max\{w_i : i \in [n]\})^{-1}$, where $\zeta$ is a (large) positive constant chosen so that $w_i(r + r^2) < 1$ for all $i \in [n]$. Then we can write

$$\log \left( \sum_{i \in [n]} \frac{w_i}{\lambda n} e^{w_i(r+r^2)} \right) \leq \log \left( \sum_{i \in [n]} \frac{w_i}{\lambda n} \left[ 1 + w_i(r + r^2) + w_i^2(r + r^2)^2 \right] \right).$$

As explained in Van der Hofstad [28] (as well as in [8]), if $W_n$ is a random variable with distribution function $F_n$ given in (30) and $W_n^*$ its size-biased distribution, then

$$\sum_{i \in [n]} \frac{w_i^2}{\lambda n} = \mathbb{E}[W_n^*] \quad \text{and} \quad \sum_{i \in [n]} \frac{w_i^3}{\lambda n} = \mathbb{E}\left[(W_n^*)^2\right].$$
Therefore we can write
\[
\log \left( \sum_{i \in [n]} \frac{w_i}{I_w} \left[ 1 + w_i(r + r^2) + w_i^2(r + r^2)^2 \right] \right)
\]
\[
= \log \left( 1 + (r + r^2)E[W_n^*] + (r + r^2)^2E[(W_n^*)^2] \right)
\]
\[
\leq (r + r^2)E[W_n^*] + (r + r^2)^2E[(W_n^*)^2].
\]

Set \( r = 1/\sqrt{k + 1} \). By Lemma 3.3 we know that \( \max\{w_i : i \in [n]\} = O(n^{1/(\tau - 1)}) \) and hence, if we take \( k = k(n) \) such that \( n^{1/(\tau - 1)} = o(k^{1/2}) \), then the condition \( r \leq (\max\{w_i : i \in [n]\})^{-1} \) is indeed satisfied. Thus we arrive at

\[
\leq \Phi \exp \left\{ \lambda / \sqrt{k + 1} \right\},
\]

with
\[
\Phi := \exp \left\{ -\sqrt{k + 1} (1 - \mathbb{E}[W_n^*]) + \mathbb{E}[W_n^*] + 4\mathbb{E}[(W_n^*)^2] \right\}.
\]

By Proposition 2.1 we conclude that
\[
\mathbb{P}(|C_{\max}| > k) \leq \frac{1 + 2e^{-1}\Phi}{\mathbb{P}(X_1 = 3)} \frac{n}{k^{3/2}} \left( 1 + O(1/\sqrt{k}) \right).
\]

Concerning the term \( \Phi \), by lemma 3.3 we know that \( \mathbb{E}[(W_n^*)^2] = O(1) \) and, moreover,
\[
|\mathbb{E}[W_n^*] - 1| = O \left( n^{-(\tau - 3)/(\tau - 1)} \right).
\]

Taking \( k = \lceil An^{2/3} \rceil \) and recalling that \( \tau > 4 \) we obtain
\[
-\sqrt{\lceil An^{2/3} \rceil} + 1(1 - \mathbb{E}[W_n^*]) \leq |1 - \mathbb{E}[W_n^*]| \sqrt{\lceil An^{2/3} \rceil} + 1
\]
\[
= O \left( n^{-(\tau - 3)/(\tau - 1)} \sqrt{An^{1/3}} \right)
\]
\[
= O \left( A^{1/2}n^{-2(\tau - 3)/(3\tau - 1)} \right) = o(1)
\]
as \( n \to \infty \). (Notice that the value of \( k \) we have chosen satisfies the earlier condition \( n^{1/(\tau - 1)} = o(k^{1/2}) \).) Therefore we conclude that the term \( \Phi \) given in (33) satisfies \( \Phi = O(1) \). Hence, using (32) we obtain (for \( k = \lceil An^{2/3} \rceil \))

\[
\leq \frac{c_3}{A^{n^2}}
\]

for some constant \( c_3 > 1 \) which depends on \( c_F \), \( \tau \) and the distribution function \( F \).
Acknowledgements. The author would like to thank the Royal Society for his PhD scholarship and Matthew Roberts for useful suggestions that helped improving the presentation of the paper. Moreover, the author thanks Guillem Perarnau and Angelica Pachon for interesting discussions about random intersection graphs in occasion of the workshop Graphs and Randomness in Turin (January 2019).

References

[1] L. Addario-Berry and B.A. Reed. Ballot theorems, old and new. In Horizons of Combinatorics, pages 9–35. Springer, 2008.

[2] Tom M. Apostol. An elementary view of Euler’s summation formula. The American Mathematical Monthly, 106(5):409–418, 1999.

[3] Michael Behrisch. Component evolution in random intersection graphs. The Electronic Journal of Combinatorics, 14, 2007.

[4] Béla Bollobás. Random graphs, volume 73 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, second edition, 2001.

[5] Béla Bollobás, Svante Janson, and Oliver Riordan. The phase transition in inhomogeneous random graphs. Random Structures & Algorithms, 31(1):3–122, 2007.

[6] Fan Chung and Linyuan Lu. Connected components in random graphs with given expected degree sequences. Annals of combinatorics, 6(2):125–145, 2002.

[7] Fan Chung and Linyuan Lu. The average distance in a random graph with given expected degrees. Internet Mathematics, 1(1):91–113, 2004.

[8] Fan Chung and Linyuan Lu. The volume of the giant component of a random graph with given expected degrees. SIAM Journal on Discrete Mathematics, 20(2):395–411, 2006.

[9] Umberto De Ambroggio and Angelica Pachon. Simple upper bounds for the largest components in critical inhomogeneous random graphs. arXiv preprint arXiv:2012.09001, 2020.

[10] Umberto De Ambroggio and Matthew I. Roberts. Unusually large components in near-critical Erdős-Rényi graphs via ballot theorems. arXiv preprint arXiv:2101.05358, 2021.

[11] Maria Deijfen and Willemien Kets. Random intersection graphs with tunable degree distribution and clustering. Probability in the Engineering and Informational Sciences, 23(4):661–674, 2009.

[12] Alan Frieze and Michal Karonski. Introduction to Random Graphs. Cambridge University Press, 2015.
[13] Svante Janson. Asymptotic equivalence and contiguity of some random graphs. *Random Structures & Algorithms*, 36(1):26–45, 2010.

[14] Svante Janson, Tomasz Łuczak, and Andrzej Ruciński. *Random graphs*, volume 45. John Wiley & Sons, 2011.

[15] Felix Joos and Guillem Perarnau. Critical percolation on random regular graphs. *Proceedings of the American Mathematical Society*, pages 3321–3332, 2018.

[16] Wouter Kager. The hitting time theorem revisited. *The American Mathematical Monthly*, 118(8):735–737, 2011.

[17] Mihyun Kang, Angelica Pachon, and Pablo M Rodríguez. Evolution of a modified binomial random graph by agglomeration. *Journal of statistical physics*, 170(3):509–535, 2018.

[18] Michael Krivelevich and Benny Sudakov. The phase transition in random graphs: A simple proof. *Random Structures & Algorithms*, 43(2):131–138, 2013.

[19] Andreas N. Lagerås and Mathias Lindholm. A note on the component structure in random intersection graphs with tunable clustering. *The Electronic Journal of Combinatorics*, 15, 2008.

[20] Tomasz Łuczak, Boris Pittel, and John C. Wierman. The structure of a random graph at the point of the phase transition. *Transactions of the American Mathematical Society*, 341(2):721–748, 1994.

[21] Asaf Nachmias and Yuval Peres. Critical percolation on random regular graphs. *Random Structures & Algorithms*, 36(2):111–148, 2010.

[22] Asaf Nachmias and Yuval Peres. The critical random graph, with martingales. *Israel J. Math.*, 176:29–41, 2010.

[23] Ilkka Norros and Hannu Reittu. On a conditionally poissonian graph process. *Advances in Applied Probability*, 38(1):59–75, 2006.

[24] Mathew D. Penrose. Inhomogeneous random graphs, isolated vertices, and poisson approximation. *Journal of Applied Probability*, 55(1):1127–136, 2018.

[25] Boris Pittel. On the largest component of the random graph at a nearcritical stage. *J. Combin. Theory Ser. B*, 82(2):237–269, 2001.

[26] Matthew I. Roberts. The probability of unusually large components in the near-critical Erdős-Rényi graph. *Advances in Applied Probability*, 50(1):245–271, 2017.

[27] Dudley Stark. The vertex degree distribution of random intersection graphs. *Random Structures & Algorithms*, 24(3):249–258, 2004.

[28] Remco van der Hofstad. Critical behavior in inhomogeneous random graphs. *Random Structures & Algorithms*, 42(4):480–508, 2013.

[29] Remco Van Der Hofstad. *Random graphs and complex networks*, volume 1. Cambridge university press, 2016.