Modified Goodness of Fit Tests for Rayleigh Distribution

Vanida Pongsakchat¹ and Pattaraporn Tonhaseng²

¹ Department of Mathematics, Faculty of Science, Burapha University
Chonburi, Thailand
Email: vanida [AT] buu.ac.th

² Department of Mathematics, Faculty of Science, Burapha University
Chonburi, Thailand
Email: pattarapornotonhaseng [AT] gmail.com

ABSTRACT— The modified goodness of fit tests for the Rayleigh distribution are studied. The critical values of modified Kolmogorov-Smirnlov, Cramer-von-Mises and Anderson-Darling tests are obtained by Monte Carlo simulation for different sample sizes and significant levels. The type I error rate and power of these tests are studied and compared. The results show that all of the three tests have type I error rate close to the significant levels. Under several alternative distributions, it is found that when the sample size is large, modified Anderson-Darling has the largest power in all cases. However, when the sample size is small, skewness of the distribution plays an important role. For the more skewed distribution, the modified Anderson-Darling test has more power than the others, while the modified Cramer-von-Mises has the largest power when the distribution is less skewed.

Keywords— Goodness-of-fit test, Rayleigh distribution, Monte Carlo simulation, Power of the test

1. INTRODUCTION

The Rayleigh distribution was introduced by Lord Rayleigh in 1880 [8]. It has been used in many fields such as in medical research, estimate the noise variance in an MRI image from background data, in physics, model processes such as wave heights, sound and light radiation, radio signals, wind power and ultrasound image modeling etc. The Rayleigh distribution is also used in the field of reliability theory and survival analysis [2][4].

If $X$ is a Rayleigh random variable, from Johnson, Kotz & Balakrishnan [5] the probability density function and the cumulative distribution function with one parameter ($\sigma$) are given by

$$f(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, \quad 0 < x, \sigma > 0$$

(1)

and

$$F(x) = 1 - e^{-x^2/2\sigma^2}, \quad 0 < x, \sigma > 0$$

(2)

Often, it is important to know whether the data come from the certain distribution. The statistical test called goodness of fit can be used for this purpose. The well-known goodness of fit tests are Kolmogorov-Smirnov (KS), Cramer-von-Mises (CvM) and Anderson-Darling (AD) tests. These tests are based on the empirical cumulative distribution. However, in practice, the parameters of the hypothesized distribution need to be estimated from the data. In this case, the standard critical values tables of these tests are no longer valid. The Kolmogorov-Smirnov, Cramer-von-Mises and Anderson-Darling tests are called the modified tests when the parameters of the hypothesized distribution must be estimated. Many researchers proposed the table of critical values of the modified tests for some distributions using Monte Carlo techniques.

Lilliefors [6,7] obtained the critical values tables for a modified Kolmogorov-Smirnov test for the normal and exponential distributions. Among the many authors that have constructed the tables of critical values for various modified goodness of fit tests for different types of distribution. Further details can be seen in [1,3,9,10,11].

In this paper, critical values for modified KS, modified CvM and modified AD using Monte Carlo techniques are obtained for Rayleigh distribution with unknown parameter. Tables of critical values for various sample and significant levels are provided. The type I error rate and power of these tests are compared and discussed.
2. MODIFIED GOODNESS-OF-FIT TESTS

A single random sample of size \( n \) is drawn from a population with unknown cumulative distribution function \( F_n(x) \) and we wish to test the hypotheses

\[
H_0 : F_n(x) = F_0(x), \quad \text{for all } x \\
H_a : F_n(x) \neq F_0(x), \quad \text{for some } x
\]

where \( F_0(x) \) is hypothesized cumulative distribution function.

In this section, we introduce the modified KS, CvM and AD tests for Rayleigh distribution when the parameter are estimated. The estimator are used in \( F_n(x) \) for modified goodness of fit tests.

For Rayleigh distribution, parameter \( \sigma \) can be estimated using maximum likelihood estimation. From [5] the maximum likelihood estimator of \( \sigma \) is

\[
\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}}
\]  
(3)

1. Modified Kolmogorov-Smirnov statistic

\[
D^+_n = \max \left\{ \frac{i}{n} - F_0(x_i; \hat{\sigma}) \right\};
\]

\[
D^-_n = \max \left\{ F_0(x_i; \hat{\sigma}) - \frac{i-1}{n} \right\};
\]

\[
D_n = \max \{ D^+_n, D^-_n \}.
\]  
(4)

2. Modified Cramer-von-Mises statistic

\[
W^2_n = \frac{1}{12n} + \sum_{i=1}^{n} \left( F_0(x_i; \hat{\sigma}) - \frac{2i-1}{2n} \right)^2.
\]  
(5)

3. Modified Anderson-Darling statistic

\[
A^2_n = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \{ \log \{ F_0(x_i; \hat{\sigma}) \} + \log \{ 1 - F_0(x_{n-i+1}; \hat{\sigma}) \} \}.
\]  
(6)

3. CRITICAL VALUES TABLES

For each of the three tests (KS, CvM, AD), each sample size \( n = 5(5)30(10)50 \) and 100, a random sample \( x_1, x_2, ..., x_n \) is generated from Rayleigh distribution. The random sample is used to estimate the MLE estimator \( \sigma \) and then used to determine \( F_n(x) \). The test statistics are calculated and recorded for a given values of \( n \). The process is repeated 10,000 times. The 10,000 number of statistics values are arranged in ascending order and the 80%, 85%, 90%, 95% and 99% quantiles are founded thus establishing the critical value for the particular test and sample size. Table 1-3 show critical values for modified KS, CvM and AD tests.

Based on Table 1-3, critical values of modified KS, CvM and AD test increase as the significant level (\( \alpha \)) decrease. For modified CvM and AD tests, the critical values increase as the sample size increases while the critical values for modified KS decrease as sample size increases. Moreover, for the specifc sample size the critical values for modified AD test is the biggest for all significant levels.
### Table 1: Critical values for modified KS test

| n   | $\alpha = 0.20$ | $\alpha = 0.15$ | $\alpha = 0.10$ | $\alpha = 0.05$ | $\alpha = 0.01$ |
|-----|----------------|----------------|----------------|----------------|----------------|
| 5   | 0.3604         | 0.3769         | 0.4045         | 0.4420         | 0.5122         |
| 10  | 0.2626         | 0.2768         | 0.2955         | 0.3247         | 0.3814         |
| 15  | 0.2171         | 0.2289         | 0.2444         | 0.2687         | 0.3172         |
| 20  | 0.1893         | 0.1996         | 0.2133         | 0.2346         | 0.2770         |
| 25  | 0.1701         | 0.1783         | 0.1917         | 0.2106         | 0.2492         |
| 30  | 0.1560         | 0.1645         | 0.1757         | 0.1932         | 0.2283         |
| 40  | 0.1357         | 0.1431         | 0.1528         | 0.1680         | 0.1989         |
| 50  | 0.1217         | 0.1284         | 0.1360         | 0.1508         | 0.1787         |
| 100 | 0.0869         | 0.0916         | 0.0978         | 0.1075         | 0.1274         |

### Table 2: Critical values for modified CvM test

| n   | $\alpha = 0.20$ | $\alpha = 0.15$ | $\alpha = 0.10$ | $\alpha = 0.05$ | $\alpha = 0.01$ |
|-----|----------------|----------------|----------------|----------------|----------------|
| 5   | 0.1293         | 0.1463         | 0.1704         | 0.2102         | 0.2987         |
| 10  | 0.1295         | 0.1473         | 0.1727         | 0.2164         | 0.3204         |
| 15  | 0.1297         | 0.1476         | 0.1734         | 0.2183         | 0.3263         |
| 20  | 0.1297         | 0.1478         | 0.1736         | 0.2193         | 0.3304         |
| 25  | 0.1297         | 0.1478         | 0.1737         | 0.2194         | 0.3297         |
| 30  | 0.1300         | 0.1479         | 0.1740         | 0.2201         | 0.3316         |
| 40  | 0.1296         | 0.1477         | 0.1739         | 0.2202         | 0.3333         |
| 50  | 0.1296         | 0.1477         | 0.1740         | 0.2207         | 0.3343         |
| 100 | 0.1300         | 0.1482         | 0.1746         | 0.2212         | 0.3363         |

### Table 3: Critical values for modified AD test

| n   | $\alpha = 0.20$ | $\alpha = 0.15$ | $\alpha = 0.10$ | $\alpha = 0.05$ | $\alpha = 0.01$ |
|-----|----------------|----------------|----------------|----------------|----------------|
| 5   | 0.7622         | 0.8533         | 0.9809         | 1.1751         | 1.6975         |
| 10  | 0.7884         | 0.8848         | 1.0239         | 1.2683         | 1.8990         |
| 15  | 0.7971         | 0.8956         | 1.0376         | 1.2863         | 1.9156         |
| 20  | 0.8013         | 0.9002         | 1.0431         | 1.2965         | 1.9319         |
| 25  | 0.8040         | 0.9036         | 1.0468         | 1.2996         | 1.9312         |
| 30  | 0.8063         | 0.9064         | 1.0501         | 1.3057         | 1.9378         |
| 40  | 0.8071         | 0.9078         | 1.0521         | 1.3084         | 1.9445         |
| 50  | 0.8076         | 0.9081         | 1.0542         | 1.3101         | 1.9506         |
| 100 | 0.8128         | 0.9133         | 1.0583         | 1.3163         | 1.9538         |

4. **TYPE I ERROR RATE**

Type I error rate of the modified KS, CvM and AD test are obtained for sample size $n = 10(10)50$ and 100 selected from Rayleigh distribution with $\sigma = 0.1, 1.2, 2.8$ and $5.5$. For each selected test, sample size, and hypothesize distribution, 10,000 random samples are generated and the tests are conducted using critical values at significant level $\alpha = 0.05$ and 0.01 in this paper. The proportion of rejections are recorded as the type I error rate and reported in Table 4-5.

From Table 4 and 5, it is founded that type I error rate of the three modified tests are close to the significant levels for all sample sizes and $\sigma$. Hence, these tests can control probability of type I error.
### Table 4 Type I error rate for modified KS, CvM and AD test (α =0.05)

| n  | Test | \( \sigma = 0.1 \) | \( \sigma = 1.2 \) | \( \sigma = 2.8 \) | \( \sigma = 5.5 \) |
|----|------|--------------------|--------------------|--------------------|--------------------|
| 10 | KS   | 0.0502             | 0.0481             | 0.0520             | 0.0466             |
|    | CvM  | 0.0542             | 0.0486             | 0.0527             | 0.0464             |
|    | AD   | 0.0540             | 0.0486             | 0.0513             | 0.0482             |
| 20 | KS   | 0.0523             | 0.0479             | 0.0471             | 0.0487             |
|    | CvM  | 0.0488             | 0.0488             | 0.0480             | 0.0487             |
|    | AD   | 0.0482             | 0.0494             | 0.0478             | 0.0463             |
| 30 | KS   | 0.0478             | 0.0500             | 0.0516             | 0.0463             |
|    | CvM  | 0.0462             | 0.0504             | 0.0494             | 0.0466             |
|    | AD   | 0.0486             | 0.0491             | 0.0507             | 0.0469             |
| 40 | KS   | 0.0501             | 0.0528             | 0.0511             | 0.0478             |
|    | CvM  | 0.0487             | 0.0522             | 0.0492             | 0.0503             |
|    | AD   | 0.0488             | 0.0492             | 0.0491             | 0.0484             |
| 50 | KS   | 0.0535             | 0.0515             | 0.0480             | 0.0491             |
|    | CvM  | 0.0507             | 0.052             | 0.0475             | 0.0488             |
|    | AD   | 0.0513             | 0.0514             | 0.0481             | 0.0502             |
| 100| KS   | 0.0526             | 0.0516             | 0.0506             | 0.0488             |
|    | CvM  | 0.0494             | 0.0478             | 0.0511             | 0.0492             |
|    | AD   | 0.0480             | 0.0503             | 0.0492             | 0.0477             |

### Table 5 Type I error rate for modified KS, CvM and AD test (α =0.01)

| n  | Test | \( \sigma = 0.1 \) | \( \sigma = 1.2 \) | \( \sigma = 2.8 \) | \( \sigma = 5.5 \) |
|----|------|--------------------|--------------------|--------------------|--------------------|
| 10 | KS   | 0.0123             | 0.0105             | 0.0100             | 0.0101             |
|    | CvM  | 0.0115             | 0.0101             | 0.0102             | 0.0092             |
|    | AD   | 0.0103             | 0.0097             | 0.0109             | 0.0106             |
| 20 | KS   | 0.0105             | 0.0104             | 0.0103             | 0.0075             |
|    | CvM  | 0.0106             | 0.0098             | 0.0088             | 0.0077             |
|    | AD   | 0.0100             | 0.0102             | 0.0093             | 0.0078             |
| 30 | KS   | 0.0115             | 0.0114             | 0.0098             | 0.0102             |
|    | CvM  | 0.0111             | 0.0110             | 0.0102             | 0.0111             |
|    | AD   | 0.0105             | 0.0103             | 0.0106             | 0.0105             |
| 40 | KS   | 0.0097             | 0.0107             | 0.0099             | 0.0103             |
|    | CvM  | 0.0089             | 0.0103             | 0.0106             | 0.0110             |
|    | AD   | 0.0090             | 0.0107             | 0.0098             | 0.0118             |
| 50 | KS   | 0.0087             | 0.0098             | 0.0097             | 0.0106             |
|    | CvM  | 0.0106             | 0.0085             | 0.0096             | 0.0095             |
|    | AD   | 0.0102             | 0.0091             | 0.0091             | 0.0090             |
| 100| KS   | 0.0096             | 0.0094             | 0.0111             | 0.0095             |
|    | CvM  | 0.0092             | 0.0100             | 0.0111             | 0.0107             |
|    | AD   | 0.0095             | 0.0104             | 0.0112             | 0.0104             |
5. POWER STUDY

Power of the modified KS, CvM and AD tests are calculated and compared for sample size $n = 10(10)50$ and 100 for selected alternative distributions. The selected distributions are Weibull distribution with shape parameter $\gamma = 1,3$, scale parameter $\beta = 1$ and log-normal distribution with shape parameter $\sigma = 0.5,0.8,1.0$ and scale parameter $\mu = 0$. The null hypothesis of Rayleigh distribution with unspecified parameter is tested at significant value 0.05 and 0.01. For each selected test, sample size and the alternative distribution, 10,000 random samples are generated from the alternative distribution and the tests are conducted using the critical values in this paper. The proportions of rejections are recorded as the power for that situation and reported in Table 6-7.

From Table 6 and Table 7, it can be seen that power of the three tests increase as sample size increase. However, when the sample size is small ($n = 10$) they perform quite poor. Overall, the modified AD test has more power than the others when the alternative distribution is more skewed (Weibull distribution with $\gamma = 1$ and log-normal distribution with $\sigma = 0.8,1.0$). When the alternative distribution is less skewed, the modified CvM has the biggest power. In addition, for the sample size equals to 100, the modified AD is the most powerful test for all cases.

6. CONCLUSION

In this paper, the critical values tables for modified Kolmogorov-Smirnov, Cramer-von-Mises and Anderson-Darling tests for Rayleigh distribution are obtained at various significant levels and sample sizes. For the power study of these tests by using critical values in the obtained critical values tables, in general, the modified Kolmogorov-Smirnov test has the smallest power values in all cases, while the modified Anderson-Darling test has the largest power when sample size is 100. However, when the sample size is smaller than 100, skewness of the alternative distribution plays an important role. For the more skewed distribution, the modified Anderson-Darling test has more power than the others, while the modified Cramer-von-Mises has the largest power when the distribution is less skewed.

| Table 6: Power of modified tests for Rayleigh distribution when the alternative distribution is Weibull distribution |
| --- |
| $n$ | Tests | $\alpha = 0.05$ | $\gamma = 1$ | $\gamma = 3$ | $\alpha = 0.01$ | $\gamma = 1$ | $\gamma = 3$ |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 10 | KS | 0.5693 | 0.2151 | 0.3843 | 0.0626 |
|  | CVM | 0.6156 | 0.2544 | 0.4497 | 0.0729 |
|  | AD | 0.7724 | 0.1885 | 0.6397 | 0.0296 |
| 20 | KS | 0.8651 | 0.3966 | 0.7220 | 0.1592 |
|  | CVM | 0.8989 | 0.4807 | 0.7934 | 0.2103 |
|  | AD | 0.9551 | 0.4367 | 0.9035 | 0.1533 |
| 30 | KS | 0.9642 | 0.5785 | 0.9065 | 0.2762 |
|  | CVM | 0.9809 | 0.6797 | 0.9417 | 0.3903 |
|  | AD | 0.9942 | 0.6552 | 0.9795 | 0.3381 |
| 40 | KS | 0.9910 | 0.7008 | 0.9666 | 0.4236 |
|  | CVM | 0.9960 | 0.8083 | 0.9848 | 0.5564 |
|  | AD | 0.9992 | 0.8011 | 0.9962 | 0.5306 |
| 50 | KS | 0.9988 | 0.8102 | 0.9915 | 0.5446 |
|  | CVM | 0.9995 | 0.9000 | 0.9955 | 0.7033 |
|  | AD | 1.0000 | 0.9028 | 0.9989 | 0.6885 |
| 100 | KS | 1.0000 | 0.9866 | 0.9999 | 0.9242 |
|  | CVM | 1.0000 | 0.9980 | 1.0000 | 0.9802 |
|  | AD | 1.0000 | 0.9980 | 1.0000 | 0.9830 |
Table 7: Power of modified tests for Rayleigh distribution when the alternative distribution is log-normal distribution

| n   | Tests | $\alpha = 0.05$ | $\alpha = 0.01$ |
|-----|-------|----------------|----------------|
|     |       | $\sigma = 0.5$ | $\sigma = 0.8$ | $\sigma = 1$ | $\sigma = 0.5$ | $\sigma = 0.8$ | $\sigma = 1$ |
| 10  | KS    | 0.0994 | 0.2836 | 0.6299 | 0.0307 | 0.2458 | 0.4684 |
|     | CvM   | 0.1086 | 0.4155 | 0.6669 | 0.0324 | 0.2900 | 0.5245 |
|     | AD    | 0.0856 | 0.4515 | 0.7222 | 0.0231 | 0.3175 | 0.5934 |
| 20  | KS    | 0.1332 | 0.6549 | 0.8979 | 0.0498 | 0.5102 | 0.7995 |
|     | CvM   | 0.1475 | 0.7008 | 0.9205 | 0.0571 | 0.5767 | 0.8499 |
|     | AD    | 0.1334 | 0.7151 | 0.9375 | 0.0443 | 0.5935 | 0.8837 |
| 30  | KS    | 0.1760 | 0.8275 | 0.9739 | 0.0651 | 0.7006 | 0.9407 |
|     | CvM   | 0.2005 | 0.8619 | 0.9841 | 0.0766 | 0.7656 | 0.9638 |
|     | AD    | 0.2003 | 0.8707 | 0.9884 | 0.0651 | 0.7764 | 0.9736 |
| 40  | KS    | 0.2133 | 0.9087 | 0.9949 | 0.0818 | 0.8268 | 0.9837 |
|     | CvM   | 0.2411 | 0.9334 | 0.9973 | 0.0985 | 0.8761 | 0.9923 |
|     | AD    | 0.2635 | 0.9358 | 0.9980 | 0.0942 | 0.8797 | 0.9949 |
| 50  | KS    | 0.2562 | 0.9614 | 0.9984 | 0.0981 | 0.8998 | 0.9947 |
|     | CvM   | 0.3027 | 0.9747 | 0.9993 | 0.1193 | 0.9358 | 0.9978 |
|     | AD    | 0.3434 | 0.9762 | 0.9997 | 0.1166 | 0.9370 | 0.9983 |
| 100 | KS    | 0.4587 | 0.9994 | 1.0000 | 0.1995 | 0.9970 | 1.0000 |
|     | CvM   | 0.5641 | 0.9997 | 1.0000 | 0.2646 | 0.9986 | 1.0000 |
|     | AD    | 0.7054 | 0.9997 | 1.0000 | 0.3441 | 0.9988 | 1.0000 |

7. REFERENCES

[1] Afify, W. M., Ramzy, A., “Modified Goodness-of-Fit Tests for the Inverse Flexible Weibull Distribution”, Advances and Applications in Statistics, vol. 48 no. 4, pp.257-272, 2016. DOI: 10.17654/AS048040257

[2] Aslam M., Tahir M., Hussain Z., Al-Zahrani B., “A 3-Component Mixture of Rayleigh Distributions: Properties and Estimation in Bayesian Framework”, PLoS ONE 10(5): e0126183, 2015. DOI: 10.1371/journal.pone.0126183

[3] Badr, M. M., “Goodness-of-Fit Tests for the Compound Rayleigh Distribution with Application to Real Data”, Heliyon, vol. 5, e0225, 2019.

[4] Jahanshahi, S. M. A., Habibirad, A., Fakoor, V., “Some New Goodness-of-fit Tests for Rayleigh Distribution”, Pakistan Journal of Statistics and Operation Research, vol. 16, no. 2, pp.305-315, 2020.

[5] Johnson, N. L., Kotz, S., Balakrishnan, N., Continuous Univariate Distributions Volume 1, 2nd edition, John Wiley & Sons, New York, 1994.

[6] Lilliefors, H. W., “On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown”, Journal of the American Statistical Association, vol. 62, no. 318, pp.399–402, 1967. DOI: 10.2307/2283970

[7] Lilliefors, H. W., “On the Kolmogorov-Smirnov Test for the Exponential Distribution with Mean Unknown”, Journal of the American Statistical Association, vol. 64, no. 325, pp.387-389, 1969. DOI: 10.1080/01621459.1969.10500983

[8] Lord Rayleigh F.R.S., “XII. On the resultant of a large number of vibrations of the same pitch and of arbitrary phase”, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, vol.10 no. 60, pp.73-78, 1880. DOI: 10.1080/14786448008626893

[9] Shawky, A. I., Bahoban, R. A., “Modified Goodness-of-Fit Tests for Exponentiated Gamma Distribution with Unknown Shape Parameter”, InterStat, vol. 3, Jan, pp.1-17, 2009.

[10] Woodruff, B. W., Viviano, P. J., Moore, A. H., Dunne, E.J., “Modified Goodness- of-Fit Tests for Gamma Distribution with Unknown Location and Scale Parameters”, IEEE Transactions on Reliability, vol. R-33, no. 3, 1984.

[11] Yen, V. C., Moore, A. H. “Modified Goodness-of-Fit Test for the Laplace Distribution”, Communications in Statistics - Simulation and Computation, vol. 17, no. 1, pp.275-281, 1988. DOI: 10.1080/03610918808812661