Boundedness of semistable principal bundles on a curve, with classical semisimple structure groups

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0 Introduction

Let $k$ be an algebraically closed field of characteristic $\neq 2$. Let $G$ be a semisimple, simply connected algebraic group over $k$ which is of classical type, that is, $G$ is special linear, special orthogonal, or symplectic. Let $X$ be a nonsingular irreducible projective curve over $k$. We consider étale locally trivial principal $G$-bundles on $X$ which are semistable. The main result in this paper is the following:

**Theorem 1** If $G$ is a semisimple, simply connected algebraic group of classical type then semistable principal $G$-bundles on $X$ form a bounded family.

In other words, we prove that there exists a family of principal $G$-bundles on $X$ parametrized by a scheme $T$ of finite type over $k$ such that every semistable $G$-bundle on $X$ is isomorphic to the restriction of this family to $\{t\} \times_k X$ for some closed point $t$ of $T$.

In characteristic zero, the theorem follows from the Narasimhan-Seshadri theorem, and $G$ can be any semisimple group, not necessarily of classical type (see Ramanathan [R]). The proof given below is characteristic free, but needs restriction to classical type.

The theorem is already known when $G$ is a special linear group $SL_n$, as in that case a semistable principal $G$-bundle on $X$ is the same as a semistable vector bundle on $X$ whose rank is $n$ and determinant is trivial. We therefore have to prove the theorem only when $G$ is special orthogonal group $SO_n$ or symplectic group $Sp_{2n}$. This is done as follows.

A principal $G$-bundle in these cases is the same as a pair $(V, b)$ where $V$ is a vector bundle on $X$ with trivial determinant and $b : V \otimes_{\mathcal{O}_X} V \to \mathcal{O}_X$ is a non-degenerate bilinear form, which is either symmetric or skew-symmetric. Let $F \subset V$ be a totally isotropic vector subbundle of the maximal possible rank $r$ (isotropic means the restriction $b : F \otimes_{\mathcal{O}_X} F \to \mathcal{O}_X$ is identically zero, and the maximal possible rank is the rank of $G$). The crucial step is to show that the maximal possible degree of such subbundles is bounded below, where the bound depends only on $X$ and $G$. This we do by a generalization of the Mukai-Sakai theorem (see [M-S]) for vector bundles to
the present situation. On the other hand, if the principal $G$-bundle corresponding to $(V, b)$ is semistable, then an elementary calculation shows that the degree of any isotropic subbundle is less than or equal to zero.

It follows that the set of all isotropic subbundles $F$ of maximal rank and maximal degrees of all possible $(V, b)$ corresponding to semistable $G$-bundles (where $G$ is fixed) constitutes a bounded family of vector bundles. From this we can deduce that the semistable $G$-bundles form a bounded family.

The arrangement of this paper is as follows. In section 1 we recall the basic definitions, and prove that a pair $(V, b)$ is semistable (if and only) if every isotropic subbundle of $(V, b)$ has degree $\leq 0$. In section 2, we prove an analogue of the Mukai-Sakai theorem, showing that the maximal possible degree of isotropic subbundles is not too small. In section 3 we complete the proof of theorem 1.

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1 Semistability and isotropic subbundles

Let $E$ be a principal $G$-bundle, and $P \subset G$ a maximal parabolic subgroup. Let $\sigma : X \to E/P$ be a section, which is the same as a reduction of the structure group to $P$. Let $T_\pi$ be the relative tangent bundle to the projection $\pi : E/P \to X$. Recall (see Ramanathan [R]) that the bundle $E$ is said to be semistable if for every maximal parabolic $P \subset G$ and every reduction $\sigma : X \to E/P$, the following inequality holds:

\begin{equation}
\text{deg}(\sigma^*(T_\pi)) \geq 0
\end{equation}

If $G = SL_n$, then the principal bundle $E$ corresponds to vector bundle $V$ with trivial determinant. A reduction to a maximal parabolic corresponds to giving a vector subbundle $F \subset V$, and the above inequality is then equivalent to the usual definition of semistability for vector bundles. Next we consider the case when $G$ is a special orthogonal or symplectic group. Let $W$ be the vector space $k^{2n}$ (or $k^{2n+1}$), with standard basis $(e_i)$. We put $u_i = e_i$ and $v_i = e_{i+n}$ for $1 \leq i \leq n$ (and $w = e_{2n+1}$). Consider the bilinear form $b$ defined in one of the following ways.

For the group $SO_{2n+1}$, we put

\begin{equation}
\begin{align*}
b(u_i, v_i) &= b(v_i, u_i) = b(w, w) = 1 \text{ for } 1 \leq i \leq n, \\
\end{align*}
\end{equation}

while all other pairings of basis vectors are zero.

For the group $Sp_{2n}$, we put
\[(1.3) \quad b(u_i, v_i) = -b(v_i, u_i) = 1 \text{ for } 1 \leq i \leq n, \]

while all other pairings of basis vectors are zero.

For the group \(SO_{2n}\), we put

\[(1.4) \quad b(u_i, v_i) = b(v_i, u_i) = 1 \text{ for } 1 \leq i \leq n, \]

while all other pairings of basis vectors are zero.

Let \(G \subset SL(W)\) be the subgroup preserving \(b\). Let \(P_r \subset G\) be the subgroup which carries the \(r\)-dimensional linear subspace \(W_r = \langle u_1, \ldots, u_r \rangle\) into itself. Then \(P_r\) is a maximal parabolic subgroup of \(G\). Let \(T \subset G\) be the standard maximal torus, consisting of matrices of the form

\[\text{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1})\]

when \(W\) is even dimensional (that is, \(G\) is \(SO_{2n}\) or \(Sp_{2n}\)), and of the form

\[\text{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}, 1)\]

when \(W\) is odd dimensional (that is, \(G\) is \(SO_{2n+1}\)). Let for \(1 \leq i \leq n\), the multiplicative character

\[\lambda_i : T \to \mathbb{G}_m\]

be defined to take the value \(a_i\) on the above diagonal matrices.

Let \(g\) and \(p_r\) be the Lie algebras of \(G\) and \(P_r\) respectively. Then \(T\) acts by adjoint action on the quotient \(g/p_r\), and \(T\) acts on \(W_r\) by restriction of the defining representation of \(G\) on \(W\).

The following lemma can be proved by an elementary calculation, which we omit.

**Lemma 1.5**

(i) The torus \(T\) acts on \(\text{det}(W_r)\) by the character \(\alpha_r = \lambda_1 + \ldots + \lambda_r\).

(ii) The torus \(T\) acts on \(\text{det}(g/p_r)\) by the character \(\chi_r\), given case by case as follows.

\[\chi_r = \begin{cases} 
-(2n - r)\alpha_r & \text{if } G = SO_{2n+1} \\
-(2n - r + 1)\alpha_r & \text{if } G = Sp_{2n} \\
-(2n - r - 1)\alpha_r & \text{if } G = SO_{2n} 
\end{cases}\]

(iii) In particular, in each case \(\chi_r\) is a negative multiple of the character \(\alpha_r\) on \(\text{det}(W_r)\), which we write as

\[\chi_r = -C(n, r)\alpha_r\]

where \(C(n, r) > 0\) is an integer given as above.

Now let \(E\) be a principal \(G\)-bundle, where \(G\) is as above. Let \((V, b)\) be the associated vector bundle together with a bilinear form \(b\). Then for any isotropic subbundle \(F_r \subset V\) of rank \(r\), we get a reduction of structure group to (upto isomorphism) the parabolic subgroup \(P_r\). If the transition functions are contained in the torus \(T\), then it follows from the above lemma that the associated \(g/p_r\)-bundle \(\mathcal{F}_r\) has degree equal to the strictly negative multiple.
\[ \deg(F_r) = -C(n, r) \deg(F_r) \]

of the degree of \( F_r \). The above relation holds even if the structure group is not reduced to \( T \), as we can have a flat deformation which reduces the structure group to \( T \), and such a deformation does not affect the degrees.

Note that the bundle \( F_r \) is canonically isomorphic to the pullback \( \sigma^*(T_\pi) \) of the relative tangent bundle of \( \pi : E/P \to X \) by the parabolic reduction \( \sigma \). Hence if \( E \) is semistable, we must have \( \deg(F_r) \geq 0 \). Hence the equation \( \text{(1.6)} \) now completes the proof of the ‘only if’ part of the following proposition.

**Proposition 1.7** The principal bundle corresponding to a pair \((V, b)\) is semistable if and only if \( \deg(F) \leq 0 \) for any isotropic subbundle \( F \) of \((V, b)\).

For the ‘if’ part, just observe that any maximal parabolic subgroup of \( G \) is conjugate to one of the subgroups \( P_r \) that we have explicitly described.

**Remark 1.8** In particular, if \( E \) is semistable, then any subbundle of an isotropic subbundle \( F \) of \((V, b)\) has non-positive degree, so the Harder-Narasimhan type of \( F \) is bounded above.

## 2 Isotropic subbundles of maximal degrees

In this section, we generalize the result of Mukai-Sakai [M-S] to isotropic subbundles of \((V, b)\). The method is a straightforward generalization of [M-S]. In this section, we do not assume that \((V, b)\) is stable.

We continue to use the same notation as the previous section. Let \((V, b)\) correspond to a principal \( G \)-bundle as above. Let \( F_r \subset V \) be an isotropic subbundle of rank \( r \) such that \( F_r \) has the maximum possible degree (say \( d \)) amongst isotropic subbundles of \( V \) having the same rank \( r \). Note that by Riemann-Roch theorem for the curve \( X \), the Hilbert polynomial of \( V/F_r \) is

\[ h(t) = -d + (n - r)(1 - g_X) + (n - r)t \]

where \( g_X \) denotes the genus of \( X \). Let \( Q \) be the quot scheme parametrizing all equivalence classes of quotients

\[ q : V \to \mathcal{E} \]

where \( \mathcal{E} \) is a coherent sheaf with Hilbert polynomial \( h(t) \). Then on \( X \times_k Q \) we have a universal quotient

\[ q : p_1^* V \to \mathcal{G} \]

Then \( Q \) is projective over \( k \). By pulling back \( b \) under \( X \times_k Q \to X \), we get a bilinear form \( p_1^*(b) \) on \( p_1^* V \). Let \( Q^{iso} \subset Q \) be the closed subscheme where the kernel of \( q : p_1^* V \to \mathcal{E} \) is isotropic. Let \( Q_0^{iso} \) be an irreducible component of \( Q^{iso} \) which contains the quotient \( V \to V/F_r \) that we started with.
Lemma 2.1 If $F_r$ has maximal degree amongst all rank $r$ isotropic subbundles in $V$, then the restriction

$$G|X \times_k Q_0^{iso}$$

of the universal quotient $G$ to any irreducible component $Q_0^{iso}$ containing $V \rightarrow V/F_r$ is a locally free sheaf on $X \times_k Q_0^{iso}$.

Proof Suppose not, then there exists a closed point $t \in Q_0^{iso}$ such that the restriction $G_t = G|X \times_k \{t\}$ is not locally free. Let $F$ be the kernel of $V \rightarrow G_t$. Let $F'$ be the $O_X$-saturation of $F$. Then $F'$ is generically equal to $F$, so $F'$ is isotropic and of the same rank $r$. But as $\deg(F') > \deg(F)$, this contradicts the maximality of the degree of $F_r$ as $\deg(F) = \deg(F_r)$.

The following proposition generalizes the theorem of Mukai-Sakai to isotropic subbundles.

Proposition 2.2 Let $F_r$ be an isotropic subbundle of rank $r$ in $(V,b)$ such that the degree of $F_r$ is maximal amongst such subbundles. Then the following inequality holds

$$\deg(F_r) \geq -g_X \cdot \dim(G/P_r) \cdot \frac{C(n,r)}{C(n,r)}$$

where $C(n,r)$ is the positive integer given by lemma 1.3 and $G/P_r$ is the quotient of $G$ by the maximal parabolic $P_r$, and $g_X$ is the genus of $X$.

Proof As $G$ acts on the left on $G/P_r$, the principal $G$-bundle $E$ has an associated bundle $Y_r \rightarrow X$ with fiber $G/P_r$. Note that $Y_r$ is the same as the closed subscheme, defined by the condition of isotropy, of the Grassman bundle of rank $r$ subspaces of the fibers of $V$.

Let $Q_0^{iso}$ be an irreducible component of $Q^{iso}$ containing $V \rightarrow V/F_r$. Then by the above lemma, we have a short exact sequence of vector bundles

$$0 \rightarrow F \rightarrow p_1^*V \rightarrow G \rightarrow 0$$

on $X \times_k Q_0^{iso}$. Hence we get a morphism

$$\varphi : X \times_k Q_0^{iso} \rightarrow Y_r$$

over $X$, which is the classifying morphism for the isotropic subbundle $F \subset p_1^*V$.

Note that as both sides are projective, the morphism $\varphi$ is proper. We claim that in fact $\varphi$ is finite. For, as in the corresponding argument of Mukai-Sakai, otherwise there will exist a complete curve $B \subset Q_0^{iso}$ and a closed point $x \in X$ such that $\{x\} \times_k B$ lies in a fiber of $\varphi$. Then by rigidity, the restricted morphism

$$\varphi : X \times_k B \rightarrow Y_r$$

will factor through the projection $X \times_k B \rightarrow B$, giving a contradiction, as $\varphi$ is over $X$. This shows that $\varphi$ is finite, hence
Note that each $S$-valued point of $Q^0_{iso}$, where $S$ is a $k$-scheme, can be regarded as a section $\sigma : X \times_k S \to Y \times_k S$. For any $k$-scheme $S$ and any section $s$ of $Y \times_k S \to X \times_k S$, we get an isotropic subbundle of the pullback of $V$ to $X \times_k S$. This gives a morphism $U \to Q^0_{iso}$ where $U$ is the scheme of sections of $Y \to X$, which is clearly injective on $k$-valued points. It follows that the dimension of $Q^0_{iso}$ is greater than or equal to the maximum of the dimensions of irreducible components $U_0$ of $U$ which contain the section $\sigma_r : X \to Y$, which corresponds to the subbundle $F_r \subset V$.

We now quote the following proposition, due to Mori.

**Proposition 2.4** Let $\pi : Y \to X$ be a projective morphism, and let $U$ be the scheme of all sections of $Y \to X$. Let $T_\pi$ be the relative tangent bundle to $\pi : Y \to X$, and let $\sigma : X \to Y$ be a closed point of $U$. Let $U_0$ be an irreducible component of $U$ of maximal dimension which contains $\sigma$. Then the following inequality holds.

$$\dim(U_0) \geq \dim H^0(X, \sigma^*(T_\pi)) - \dim H^1(X, \sigma^*(T_\pi))$$

**Proof** This follows from proposition 3 of Mori [M], by taking $Z$ to be empty in the notation of [M], and observing that $U$ is an open subscheme of the scheme $\text{Hom}(X, Y)$.

We now apply this to the present case, by taking $X$ to be our curve $X$, $Y$ to be $Y_r$, and $\sigma$ to be the section $\sigma_r$ corresponding to the maximal degree isotropic subbundle $F_r$. Then $\sigma^*(T_\pi)$ equals the associated $g/p_r$ bundle $F_r$ in the notation of section 1. By equation 1.6, we have $\deg(F_r) = -C(n, r) \cdot d$. Hence by Riemann-Roch for the curve $X$ we get

$$\dim H^0(X, \sigma^*(T_\pi)) - \dim H^1(X, \sigma^*(T_\pi)) = -C(n, r) \cdot d + \dim(G/P_r)(1 - g_X)$$

Combining the above equation with proposition 2.4, we get

$$-C(n, r) \cdot d + \dim(G/P_r)(1 - g_X) \leq \dim(U_0)$$

Now, $U_0$ embeds in $Q^0_{iso}$, hence combining the above with equation 2.3, we finally have

$$-C(n, r) \cdot d + \dim(G/P_r)(1 - g_X) \leq \dim(G/P_r)$$

Solving this for $d$ completes the proof of the proposition 2.2.

**3 Proof of theorem 1**

We first recall the following standard fact.
Remark 3.1  Let $\mathcal{A}$ be a set of isomorphism classes of vector bundles on $X$. We say that $\mathcal{A}$ is **bounded** if there exists a finite type scheme $T$ over $k$ and vector bundles $\mathcal{E}$ on $X \times_k T$ such that given any element $a \in \mathcal{A}$, there exists a closed point $t \in T$ such that the restriction $\mathcal{E}|(X \times_k \{t\})$ represents $a$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two sets of isomorphism classes of vector bundles on $X$. Let $\mathcal{C}$ be the set of all isomorphism classes of vector bundles $\mathcal{V}$ which fit in a short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{V} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $[\mathcal{E}] \in \mathcal{A}$ and $[\mathcal{F}] \in \mathcal{B}$. Then it is known that if $\mathcal{A}$ and $\mathcal{B}$ are bounded sets, then $\mathcal{C}$ is also a bounded set.

With this, we are ready to prove theorem 1.

**Proof**  The above remark implies that the set of all isomorphism classes of rank $n$ vector bundles $F$ on $X$ which occur as a maximal degree isotropic vector subbundle of $(V, b)$ where $(V, b)$ corresponds to a semistable principal $G$-bundle on $X$ (where $G$ is one of $SO_{2n+1}$, $Sp_{2n}$, $SO_{2n}$) is bounded, as the degrees of such $F$ are bounded below by proposition 2.2, and their Harder-Narasimhan types are bounded above by proposition 1.7. Note that as $F$ is isotropic of the maximal possible rank $n$, the bilinear form $b$ induces an isomorphism $F^* \cong V/F$. The remark 3.1 implies that the set of isomorphism classes of extensions of the type

$$0 \longrightarrow F \longrightarrow \mathcal{E} \longrightarrow F^* \longrightarrow 0$$

form a bounded set. This shows that the isomorphism classes of the underlying vector bundles $V$ of semistable $(V, b)$ form a bounded set.

Let $S$ be a finite type scheme over $k$ and $\mathcal{V}$ a vector bundle on $X \times S$ in which all such $[V]$ occur. Hence there is a linear scheme $T$ over $S$ which parametrizes the pairs $(V, b)$ where $b$ is a bilinear form on $V$. Then $T$ has a locally closed subscheme where $b$ is non-degenerate, and symmetric (or skew symmetric). This completes the proof that the isomorphism classes of semistable $(V, b)$ form a bounded set.

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