Time averaging of weak values—consequences for time-energy and coordinate-momentum uncertainty

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Abstract

Using the quantum transition path time probability distribution we show that time averaging of weak values leads to unexpected results. We prove a weak value time-energy uncertainty principle and time-energy commutation relation. We also find that time averaging allows one to predict in advance the momentum of a particle at a post selected point in space with accuracy greater than the limit of $\hbar/2$ as dictated by the uncertainty principle. This comes at a cost—it is not possible at the same time to predict when the particle will arrive at the post selected point. A specific example is provided for one dimensional scattering from a square barrier.

1. Introduction

The Heisenberg uncertainty principle [1, 2] is a cornerstone of quantum theory. It establishes a lower bound on the product of the variances of non-commuting observables. More specifically, let $\hat{q}$ and $\hat{p}$ denote the position and momentum operators and $|\Psi_0\rangle = \exp\left(-\frac{i\hat{H}t}{\hbar}\right)|\Psi_0\rangle$ a wavefunction evolved to time $t$ from its form $|\Psi_0\rangle$ at time 0 under the Hamiltonian operator $\hat{H}$. The mean values of the position and the momentum at any time $t$ are $\bar{q}_t = \langle \Psi_t | \hat{q} | \Psi_t \rangle$, $\bar{p}_t = \langle \Psi_t | \hat{p} | \Psi_t \rangle$. The Heisenberg uncertainty principle then assures us that at any time $t$ the product of the variances $\Delta q^2_t = \langle \Psi_t | \hat{q}^2 | \Psi_t \rangle - \langle \Psi_t | \hat{q} | \Psi_t \rangle^2$ and $\Delta p^2_t = \langle \Psi_t | \hat{p}^2 | \Psi_t \rangle - \langle \Psi_t | \hat{p} | \Psi_t \rangle^2$ is bounded from below - $\Delta q^2_t \Delta p^2_t \geq \hbar^2/4$. In these relations time is considered to be the external time as defined by Aharonov and Bohm [3] and by Busch [4]. In Busch’s words, it is ‘the time period between the preparation and the instant at which a measurement of, say, position is performed.’ The concept of time which we will consistently use in this paper is this external time.

The experimental implications of the uncertainty relation are well understood. Repeated independent measurements of the momentum and position of particles, lead to an uncertainty which is greater or equal to the Heisenberg lower bound [2]. More specifically, equal copies of the same particle are prepared at some initial time $t = 0$. These are described by the wavefunction $|\Psi_0\rangle$. The particles are allowed to evolve up to some time $t$ and at this time, either their position or their momentum is measured. Multiple repetition of such a measurement will lead to the observation of a mean momentum and position and standard deviations from the means whose product is bounded from below by $\hbar/2$. The uncertainty relation as presented above differs from the more general measurement-disturbance relation of Ozawa [5] which was subsequently derived using weak values [6] and verified experimentally by Steinberg and coworkers [7].

It should be emphasized that the constraint on the standard deviations holds only if measurements are carried out after the same time interval $t$ passed between preparation ($t = 0$) and measurement. If one measures the momentum at time $t_1$ and the position at time $t_2 = t_1$ then the uncertainty product relation for the standard deviations no longer holds in the simple form presented above. It is important to note that the Heisenberg principle does not prevent one from determining with high precision either the coordinate or the momentum at

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a specified time \( t \). It only disallows the precise simultaneous determination of the values of both the coordinate and the momentum at the same time \( t \).

A central result of this paper is an analog of these last statements in connection with a post selected value of the position \( x \), analogous to the precisely determined external time interval \( t \) and determination of the momentum. We will show that the Heisenberg relation does not prevent us from constructing a scenario in which we can predict with a certainty which is greater than the Heisenberg limitation, the momentum of a particle when it reaches a post selected position \( x \). The scenario we present precludes the ability to predict accurately the time at which the particle arrives at the point \( x \). Different particles may arrive at \( x \) at different times and therefore the product of their uncertainties in position and momentum are not limited by the Heisenberg relations. We will also show that the ability to pinpoint the momentum and position of the particle comes at the price of not being able to predict at what time the particle will reach the point \( x \), if at all. We do know that if it reaches the point \( x \) it will have with some certainty a known momentum \( p \).

To justify this assertion, it is necessary to consider the time-energy uncertainty relation. One of the challenges is that the simple derivation of an uncertainty relation for the coordinate and momentum does not exist for energy and time. Busch argues [4] that ‘there is no unique universal relation that could stand on equal footing with the position–momentum uncertainty relation’. With respect to the external time he notes that ‘external time is sharply defined at all scales relevant to a given experiment. Hence there is no scope for an uncertainty interpretation with respect to external time.’ Hilgevoord [8] claimed that ‘there is no reason why a Heisenberg relation should hold ... between the time coordinate and the energy of the system.’

As originally discussed by Pauli [9] one of the difficulties in defining time-energy uncertainties comes from the fact that the energy spectrum of the Hamiltonian operator is typically bounded from below. These assertions notwithstanding, a second central result of this paper is, following the methodology of Robertson [10] for the ‘standard’ uncertainty principle, the derivation of a general energy—external time uncertainty principle based on time averaging of weak values. To prevent any misunderstanding, the time-energy uncertainty relation we derive does not involve a time operator. All that is needed is a time operator. It also does not relate to the measurement questions. It only expresses in rigorous terms a time averaging of weak values. To prevent any misunderstanding, the time-energy uncertainty relation stressing again, that there is no need to define a time operator. All that is needed are time of flight measurements.

2. Time averaging and a weak value time-energy uncertainty principle

Limiting the discussion to one dimension, we use the definition of the normalized transition path time distribution for finding a particle at the post selected point \( x \) as in [16]:

\[
P(t; x) = \frac{\langle \Psi_t | \delta (\hat{q} - x) | \Psi_t \rangle}{\int_0^\infty dt \langle \Psi_t | \delta (\hat{q} - x) | \Psi_t \rangle} = \frac{\langle x | \Psi_t \rangle^2}{N(x)}.
\]  

(1)

It gives the probability of finding the system at time \( t \) at the post selected point \( x \). Implicit in this definition is the assumption that the normalization time integral in the denominator converges.

We note here at the outset that the transition path time distribution is in principle measurable using a suitably defined time of flight experiment. One places a screen at the post selected point \( x \) and two synchronized clocks, one at the orifice of the emerging particles and the other at the screen. For example, in single atom time of flight experiments [17], particles are released from a trap [18] at time zero, and the arrival time at a detector screen is recorded.

The weak value of an operator \( \hat{O} \) at the post selected point \( x \) is defined as [15]

\[
O_w(x, t) = \frac{\langle x | \hat{O} | \Psi_t \rangle}{\langle x | \Psi_t \rangle}.
\]  

(2)

It is well known that the spatial average of the weak value is identical to the result of a strong measurement, that is:
\[
\int_{-\infty}^{\infty} \mathrm{d}x \, |\langle x|\Psi_t\rangle|^2 O_{m}(x, t) = \langle \Psi_t|\hat{O}|\Psi_t\rangle. \tag{3}
\]

This does not mean that one cannot measure the weak value precisely, indeed, repeated weak measurement experiments in which the weak value is measured after the same time interval \( t \) and at the same post selected position \( x \) will give the weak value as defined in equation (3). However, at different post selected values of the coordinate \( x \), one will find different weak values. Their spatial average as defined in equation (3) will lead to the strong value.

Similarly, and this is a central new concept introduced in this paper, one may define a time averaged mean of the weak value

\[
\langle O(x) \rangle \equiv \int_0^\infty \mathrm{d}t P(t; x) O_{m}(x, t). \tag{4}
\]

Measuring the weak value at different times will give different results. Time averaging them will give the time averaged weak value as defined in equation (4). In different words, consider the time of flight experiment. The screen is located at the post selected point \( x \). The particle will arrive at the screen at different times. For each fixed time there will be a weak value which may be measured. It will though be different at different times. Its time average is defined in equation (4). In the following we will use the notation

\[
\langle O_1(x)O_2(x) \rangle \equiv \int_0^\infty \mathrm{d}t P(t; x) O_{1,m}(x, t)O_{2,m}(x, t) \tag{5}
\]

to denote the time average of a product of weak values. We will also use the bracket notation for moments of the time parameter:

\[
\langle t^n(x) \rangle = \int_0^\infty \mathrm{d}t t^n P(t; x). \tag{6}
\]

With these preliminaries, following the standard derivation of the uncertainty principle \([10, 19]\) we consider the inequality

\[
0 \leq \frac{1}{N(x)} \int_0^\infty \mathrm{d}t \langle x|\hat{I}|\Psi_t\rangle \langle \Psi_t|\hat{I}\rangle \langle \Psi_t|\hat{H}|x\rangle \langle x|\hat{H}\rangle \langle \Psi_t|\Psi_t\rangle + \lambda \int_0^\infty \mathrm{d}t \langle x|\hat{H}|\Psi_t\rangle \langle \Psi_t|\hat{H}\rangle |x\rangle \langle x|\hat{H}\rangle \langle \Psi_t|\Psi_t\rangle - \langle t^2(x) \rangle - \lambda \hbar + \lambda^2 \langle H(x)H^*(x) \rangle, \tag{7}
\]

where \( \lambda \) is an arbitrary real number, \( t \) is the scalar value of the time, \( \hat{I} \) is the identity operator and \( \hat{H} \) is the Hamiltonian operator. We stress that the time as used here is just a parameter, not an operator. It multiplies the identity operator which is of course hermitian. Therefore the product as defined on the rhs of equation (7) is necessarily positive. Noting that

\[
\hat{H}|\Psi_t\rangle = i\hbar \frac{\partial}{\partial t}|\Psi_t\rangle \tag{8}
\]

allows us to rewrite the inequality in 7 as:

\[
0 \leq \int_0^\infty \mathrm{d}t \langle x|\hat{I}|\Psi_t\rangle \langle \Psi_t|\hat{I}\rangle \langle \Psi_t|\hat{H}|x\rangle \langle x|\hat{H}\rangle \langle \Psi_t|\Psi_t\rangle + \lambda \hbar \int_0^\infty \mathrm{d}t \langle x|\hat{H}|\Psi_t\rangle \langle \Psi_t|\hat{H}\rangle |x\rangle \langle x|\hat{H}\rangle \langle \Psi_t|\Psi_t\rangle + \lambda^2 \int_0^\infty \mathrm{d}t \langle x|\hat{H}|\Psi_t\rangle \langle \Psi_t|\hat{H}\rangle |x\rangle \langle x|\hat{H}\rangle \langle \Psi_t|\Psi_t\rangle. \tag{9}
\]

Due to the introduction of time averaging and the assumption that the normalization integral \( N(x) < \infty \), one may integrate the middle term on the right hand side by parts to find:

\[
0 \leq \langle t^2(x) \rangle - \lambda \hbar + \lambda^2 \langle H(x)H^*(x) \rangle. \tag{10}
\]

Minimizing with respect to \( \lambda \) leads to the time averaged weak value second moment energy and time relation:

\[
\langle t^2(x) \rangle \langle H(x)H^*(x) \rangle \geq \frac{\hbar^2}{4}. \tag{11}
\]

Continuing in this vein, consider the relation between the standard deviations. Denoting

\[
\Delta t = t - \langle t(x) \rangle, \quad \Delta \hat{H} = \hat{H} - \langle H(x) \rangle \tag{12}
\]

and using as before the inequality

\[
0 \leq \frac{1}{N(x)} \int_0^\infty \mathrm{d}t \langle x|\Delta t\hat{I} - i\lambda \Delta \hat{H}|\Psi_t\rangle \langle \Psi_t|\Delta t\hat{I} + i\lambda \Delta \hat{H}|x\rangle \langle x|\hat{H}\rangle \langle \Psi_t|\Psi_t\rangle, \tag{13}
\]

the relationship as in (8) and integrating by parts one readily finds

\[
0 \leq \langle \Delta t^2(x) \rangle + \lambda^2 \langle \Delta \hat{H}(x)\Delta \hat{H}^*(x) \rangle - \lambda \hbar [1 - \langle t(x) \rangle P(0; x)]. \tag{14}
\]

Minimizing with respect to \( \lambda \) gives the central result of this section, namely the uncertainty relation for the standard deviations:
\[ \sqrt{\langle \Delta t^2(x) \rangle \langle \Delta \hat{H}(x) \Delta \hat{H}^*(x) \rangle} \geq \frac{\hbar}{2} \left[ 1 - \langle \hat{t}(x); P(0; x) \rangle \right]. \]  

(15)

If the post selected coordinate \( x \) is sufficiently far away from the incident wavepacket then \( P(0; x) \approx 0 \) and we have regained a time-energy uncertainty relation for the time averaged weak values which is identical to Heisenberg’s result for coordinate-momentum uncertainty.

The term \( \langle \hat{t}(x); P(0; x) \rangle \) appears in (15) since we have imposed that the time measurement is in the interval \([0, \infty]\) and not \([-\infty, \infty]\). In a typical experimental setup, the particle cannot be initially found at the point \( x \) so that effectively we have regained the lower limit of \( \hbar/2 \). This uncertainty relation implies that if the standard deviation of the time averaged weak value of the energy is small, then the uncertainty in the time of arrival at the point \( x \) becomes very large.

The time averaged weak value of the time-energy uncertainty relation is intimately related to a time averaged weak value commutator of the energy and the time defined by taking into consideration that the weak value of the Hamiltonian operator is complex. One may define a weak time value

\[ t_w(x) = \frac{\langle x|\hat{t}|\Psi_0 \rangle}{\langle x|\Psi_0 \rangle} = t. \]

(16)

To prevent misunderstanding, here too, the time is not considered as an operator, as before, it is the external time and therefore this value of the time is just the time itself. One then readily finds, using equation (8) that

\[ \langle [H(x), t(x)] \rangle \equiv \langle [i\hat{t}H(x) - H(x)i\hat{t}] \rangle \]

\[ = \int_0^\infty dt P(t; x) \left[ \frac{\langle \Psi_0|\hat{H}|x \rangle}{\langle \Psi_0|x \rangle} - \frac{\langle x|\hat{H}|\Psi_0 \rangle}{\langle x|\Psi_0 \rangle} \right] = i\hbar. \]

(17)

### 3. Predicting the momentum of a particle at a point \( x \)

In this section we shall show that time averaging of weak values leads to the conclusion that it is possible to predict the momentum of a particle at a post selected value of the coordinate \( x \) with arbitrary accuracy. For this purpose we consider a scattering system, with a potential \( V(x) \) of finite range localized about \( x = 0 \). The Hamiltonian of the particle whose mass is \( M \) is:

\[ \hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{x}). \]

(18)

To simplify we impose the condition that the potential function \( V(x) \) goes to 0 as \( x \to \pm \infty \). Initially the system is prepared in a coherent state \( |\Psi_0 \rangle \) localized about the initial position \( x_i \) chosen to be sufficiently far to the left of the potential such that \( V(x_i) = 0 \), and incident mean momentum \( p_i > 0 \):

\[ \langle x|\Psi_0 \rangle = \left( \frac{\Gamma}{\pi} \right)^{1/4} \exp \left( -\frac{\Gamma(x - x_i)^2}{2} \right) + \frac{i}{\hbar} p_i (x - x_i). \]

(19)

We will also assume that the probability of the particle initially leaking into the interaction region \((x \sim 0)\) is negligible \((\Gamma x_i^2 \gg 1)\) where \( \Gamma \) is the width parameter of the coherent state. This implies that initially the particle is a free particle with positive mean momentum \( p_i \) in the \( x \) direction. This initial state obeys the Heisenberg position-momentum uncertainty relation. The transition path time distribution and the mean time of arrival at the post selected point are well defined since for this generic scattering system it has been shown by Muga [20] that at long time \( \langle x|\Psi_0 \rangle \sim t^{-3/2} \). In the following we will choose the position \( x > 0 \) far enough in the asymptotic region of the potential. Under such conditions, the normalization integral \( N(x) \) (see equation (1)) becomes independent of \( x \).

The weak value of the momentum at the post selected point \( x \) is by definition

\[ p_w(t; x) = \frac{\langle x|\hat{p}|\Psi_0 \rangle}{\langle x|\Psi_0 \rangle} = -i\hbar \frac{\partial \ln \langle x|\Psi_0 \rangle}{\partial x}. \]

(20)

In this formulation, the position is known precisely, its post selected value is \( x \). It is well known [15] that the spatial average of the weak value is the mean value of the momentum:

\[ \langle \Psi_0|\hat{p}|\Psi_0 \rangle = \int_{-\infty}^\infty dx \left( \langle x|\Psi(t) \rangle \right)^2 p_w(t; x). \]

(21)

As discussed already in the previous section, instead of considering the spatial average we will consider the time averaged weak value of the momentum \( \langle p(x) \rangle \) and its variance

\[ \langle \Delta p^2 \rangle = \langle (p(x))^2 \rangle - \langle p(x) \rangle^2 \]

(22)

using the transition path time distribution and as before the brackets denote time averages.
The weak value of the momentum is a complex quantity. Since we chose the post selected value of $x$ to be large enough such that the normalization integral $N(x)$ is independent of $x$ we find that the imaginary part of the weak value of the momentum is

$$\text{Im}[p_w(t; x)] = -\frac{\hbar}{2} \frac{\partial \ln P(t; x)}{\partial x}. \quad (23)$$

This means that the imaginary part of the time averaged value of the weak momentum vanishes ($\text{Im} \langle p(x) \rangle = 0$).

To obtain further insight into time averaged weak values, we consider expressly the time evolved wavefunction, expanding it in terms of the scattering eigenstates of the Hamiltonian:

$$\langle x|\Psi(t) \rangle = \int_{-\infty}^{\infty} dp \exp\left(-i \frac{p^2t}{2\hbar}\right) \langle x|p^+ \rangle \langle p^+|\Psi \rangle. \quad (24)$$

Asymptotically, the eigenfunctions $\langle x|p^+ \rangle$ have the form (with $p > 0$):

$$\langle x|p^+ \rangle = \begin{cases} \frac{1}{\sqrt{2\pi \hbar}} \left[ \exp\left(\frac{ipx}{\hbar}\right) + R(p) \exp\left(-\frac{ipx}{\hbar}\right) \right], & x \to -\infty \\ \frac{1}{\sqrt{2\pi \hbar}} T(p) \exp\left(\frac{ipx}{\hbar}\right), & x \to \infty \end{cases} \quad (25)$$

and $R(p)$, $T(p)$ are the reflection and transmission amplitudes, respectively. It is straightforward to evaluate the time dependent wavefunction in the asymptotic region. Since the wavefunction is initially localized outside of the range of the potential, the overlap $\langle p^+|\Psi \rangle = \langle p|\Psi \rangle + R^2(p) \langle -p|\Psi \rangle$ where $\langle p|\Psi \rangle$ is just the momentum representation of the initial wavefunction. Similarly, by choosing the post selected value of $x$ to be positive and large we have that $\langle x|p^+ \rangle = T(p) \langle x|p \rangle$. One is thus left with a quadrature to obtain the time dependent wavefunction $\langle x|\Psi(t) \rangle$ and its associated weak momentum value.

This quadrature needs to be carried out numerically, and is dependent on the specifics of the potential which determines the momentum dependence of the reflection and transmission amplitudes. However, one may readily obtain analytic expressions using a steepest descent estimate of the integrals. One finds that the important contribution to the time dependent wavefunction, when $x \gg 0$ is

$$\langle x|\Psi(t) \rangle \approx \left(\frac{\Gamma M^2}{\pi(M+i\hbar\Gamma)^2}\right)^{1/2} T\left(\frac{M \xi_i - i\hbar \Gamma(x_i - x)}{M + i\hbar\Gamma}\right) \cdot \exp\left(-\frac{p_i^2}{2\hbar^2 \Gamma} + \frac{M \Gamma}{2} \left(\frac{\xi(i - x) - \frac{\hbar}{\Gamma}}{M + i\hbar\Gamma}\right)^2\right). \quad (26)$$

so that

$$|\langle x|\Psi(t) \rangle|^2 \approx \frac{M \sqrt{\Gamma \pi}}{\pi [M^2 + i\hbar^2 \Gamma^2]} \left| T\left(\frac{p_i - i\hbar \Gamma(x_i - x)}{1 + \left(\frac{\Gamma M}{\frac{i\hbar \Gamma}{\Gamma}}\right)^2}\right) \right|^2 \exp\left(-\frac{\Gamma (x_i - x + \frac{\hbar I}{\Gamma})^2}{1 + \left(\frac{\Gamma M}{\frac{i\hbar \Gamma}{\Gamma}}\right)^2}\right). \quad (27)$$

The denominator of the transition path time distribution is estimated as:

$$\int_0^\infty dt |\langle x|\Psi(t) \rangle|^2 \approx M |T(p_i)|^2 / p_i \quad (28)$$

and as noted, is independent of $x$. Within this steepest descent evaluation the weak value of the momentum is:

$$p_w(t; x) \approx \frac{M [NP_i - \hbar \Gamma^2 (x_i - x)]}{[M^2 + (i\hbar \Gamma)^2]} + \frac{i\hbar \Gamma M (x - x_i) - p_i]}{[M^2 + (i\hbar \Gamma)^2]} \quad (29)$$

Time averaging this expression using the steepest descent estimate for the transition path time distribution gives the result:

$$\langle p(x) \rangle \approx p_i \quad (30)$$

or in other words, the time average of the weak value of the momentum equals the initial averaged incident momentum. It remains to consider the second moment of the weak value, and within the steepest descent approximation one finds

$$\langle \Delta p^2(x) \rangle = \int_0^\infty dt P(t; x) |\langle p_w(t; x) \rangle|^2 - \langle p(x) \rangle^2 \approx \frac{\hbar^2 \Gamma}{2} \quad (31)$$

which is precisely the momentum variance of the initial wavepacket. By reducing the width parameter $\Gamma$ this variance can become arbitrarily small.
We have thus demonstrated, using a steepest descent approximation that the time average of the weak value of the momentum and its variance at the post selected value of the coordinate $x$ are the same as the initial mean values $\bar{p}_i$ and $D\bar{p}^2_i$. However, the coordinate is post selected, it is known precisely. This means that the Heisenberg relation does not limit the precision with which the time averaged post selected weak value of the momentum and its variance may be determined. Moreover, when $\Gamma$ is sufficiently small, the weak value of the momentum will be very close to the incident mean value of the momentum. In other words, even for a single particle, we can predict in advance its momentum when reaching the post selected point $x$. This is a central result of this paper.

The steepest descent approximation for the transition path time distribution as given in equation (27) goes at long times as $t^{-1}$ so that strictly speaking the time integrals would diverge. As already mentioned, the correct long time dependence of the transition path time distribution goes as $-t^3$ so that there is no problem in reality.

The steepest descent approximation is correct for finite time, the long time tail is very small, it goes as $\exp \left(-\gamma t^3/h^3\right)$. To prevent any doubt, we have also undertaken a numerically exact study of a model system to demonstrate that indeed one may predict the momentum at $x$ with arbitrary accuracy.

We consider scattering through a square barrier (atomic units are used throughout) with a particle of mass 1, a barrier height of unity, barrier width 2 and an incident momentum of $1/4$. The incident wavepacket is chosen such that the initial variance of its momentum is small, the width parameter of the coherent state is chosen to be $\Gamma = 0.001$. We then plot in figure 1 the transition path time probability distribution (1) for $x = -x_i = 100$ and compare it with its steepest descent approximation

$$P_{\text{SD}}(t; x) = \frac{\sqrt{\Gamma} p_i}{\sqrt{\pi[M^2 + t^2\hbar^2]^2}} \exp \left(-\frac{\Gamma(x_i - x + \frac{\bar{p}_i^2}{M})^3}{1 + \left(\frac{\hbar}{M}\right)^2} \right).$$

(32)

As is evident from the figure, the agreement is quantitative. The accurate transition path time distribution has its maximum at a time which is a bit shorter than the steepest descent approximation. Similarly, the value of the time averaged weak momentum for these parameters is 0.2522, slightly higher than the steepest descent estimate ($1/4$) which does not take into consideration the filtering effect of the transmission probability. Due to the tunneling, the transmission favors higher momenta as discussed in [16]. Decreasing the width parameter by a factor of 4, reduces the value of the time averaged weak momentum to 0.2505.

The deviation of the real and imaginary parts of the weak value of the momentum from their mean $\delta p_\rho (t; x) = p_\rho (t; x) - \langle p(x) \rangle$ are plotted as a function of time in figure 2 and compared with the steepest descent estimates of equation (29). The early times lead to positive values of the momentum differences, the later times to negative values, as might have been expected from a classical mechanics perspective. The agreement between the steepest descent estimates and the numerically exact estimates of the real and imaginary parts of the weak values is quantitative.

Finally, the standard deviation $\langle \sqrt{\Delta p^2(x)} \rangle$ of the time averaged weak value of the momentum with $\Gamma = 0.001$ is found to be 0.02228. From (31) one finds the value 0.02236. The standard deviation is an order of
magnitude less than that of the mean value of the momentum itself. Reducing the width parameter by a factor of 4 gives a time averaged mean of the weak momentum of 0.2505 and a standard deviation of 0.01117. As predicted in (31) one may arbitrarily reduce the standard deviation by reducing the width parameter of the initial coherent state. In other words, in principle, using time averaged weak values, one may accurately predict both the location and the momentum of a single particle.

Does this imply that we may construct a quantum trajectory analogous to the classical trajectory where the momentum and coordinate are known as functions of the time? Of course not. Strictly speaking, the variance of the mean time diverges logarithmically. Within the steepest descent estimate one finds

\[ \delta \left( \frac{\partial t}{x} \right) = \delta t - \delta \left( \frac{\partial t}{x} \right) \]

(33)

and this grows indefinitely as the precision with which the momentum is predetermined increases, that is, as \( \Gamma \to 0 \). In other words, the price paid for the precise determination of the momentum is imprecision in the knowledge of when the particle will actually reach the post selected point \( x \). Within the steepest descent approximation one finds that the time averaged weak value of the energy is

\[ \langle H(x) \rangle \approx \frac{p_t^2}{2M} + \frac{\hbar^2 \Gamma}{4M} \]

(34)

while the variance is given by:

\[ \langle |H(x)|^2 \rangle - \langle H(x) \rangle^2 \approx \hbar \Gamma \frac{p_t^2}{2M^2} + \frac{\hbar^4 \Gamma^2}{8M^2} \]

(35)

and as expected one regains the weak value time-energy uncertainty relation

\[ \langle \Delta t^2(x) \rangle \langle \Delta H(x) \Delta H^*(x) \rangle \approx \frac{\hbar^2}{4} + \frac{\hbar^4 \Gamma^2}{16p_t^2} \]

(36)

The picture that emerges is thus that determining the momentum of a particle at a point in space is analogous to determining the momentum of a particle at a point in time. In the latter case, if one knows precisely the momentum, then the position becomes fully indeterminate. In our case, pinpointing the momentum is the same as pinpointing the energy and it is the time which becomes indeterminate.

We now describe a second approach which also leads to determination of the momentum at a given spatial point. Given the transition path time distribution we have that the mean time it takes the particle to reach the point \( x \) is
\[ \langle t(x) \rangle = \int_0^\infty \! dt P(t; x). \quad (37) \]

The mean is well defined, since as noted, the long time tail of the transition path time distribution goes as \( t^{-3} \). We then consider the mean time difference for the particle to reach two points in the scattering direction \((x)\) which are chosen to be close to each other:

\[ \langle \delta t(x, \delta x) \rangle = \int_0^\infty \! dt [P(t; x + \delta x/2) - P(t; x - \delta x/2)]. \quad (38) \]

The momentum in the scattering direction is then by definition

\[ \vec{p}(x) = \lim_{\delta x \to 0} \frac{M \delta x}{\delta t(x, \delta x)} = M \left( \frac{\partial \langle t(x) \rangle}{\partial x} \right)^{-1}. \quad (39) \]

With this protocol, which does not invoke a weak value, we determined a mean momentum of the particle in the \(x\) direction at the precise location \((x)\). Within the steepest descent approximation one readily finds that

\[ \vec{p}(x)^{-1} \approx \frac{2}{M} \int_0^\infty \! dt \rho_{SD}(t; x) \left[ \frac{\Gamma (x_i - x + \frac{p_f}{M})}{\left[ 1 + \left( \frac{m t}{M} \right)^2 \right]^3} \right] \approx \frac{1}{P_t}. \quad (40) \]

Using the square barrier model as above and performing all integrations numerically exactly, we find that for \( \Gamma = 0.001 \), the momentum \( \vec{p}(x) = 0.2502 \). We thus find that also with this approach the mean momentum at the post selected pointed \( x \) is to a good approximation equal to the mean incident momentum.

4. Discussion

The introduction of time averaging of weak values, using a transition path time probability distribution leads to unexpected important results. We showed that through time averaging it becomes possible to derive a rigorous uncertainty relation for the product of the time averaged weak value of the energy and the \( (x) \). Within this formalism, the energy and the time are analogous to the coordinate and momentum operators in quantum mechanics. The two pairs obey the same uncertainty and commutation relations. This result indicates that when considering time in quantum mechanics, one need not construct a time operator. It is sufficient to consider time as a parameter in the time dependent Schrödinger equation, or in the terminology of Busch as an external time. Equivalently it may be considered to be its weak value as associated with the time dependent wavefunction at the post selected point \( x \).

The coordinate-momentum uncertainty principle is derived for a measurement of the two at the same value of the time interval, which is post selected. The time-energy uncertainty relationship is derived for a fixed value of the coordinate, which is post selected. At a given time it is impossible to determine accurately both the momentum and the coordinate of a particle. Similarly, at a given point in space, it is impossible to determine accurately both the energy of the particle and the time at which it will pass through the given point. On the other hand, just as it is possible to determine accurately the position, or alternatively the momentum of a particle at a fixed time, so it is possible to determine accurately the momentum of a particle at a fixed position. We have demonstrated these general relationships by considering explicitly the scattering of a particle through a square well potential.

Finally, the localization of the position and the momentum is measurable since the transition path time distribution is in principle measurable as already noted in section II. One places a screen at the post selected point \( x \) and two synchronized clocks, one at the orifice of the emerging particles and the other at the screen. In fact, single atom time of flight experiments [17] have been implemented, particles are released from a trap [18] at time zero, and the arrival time at a detector screen is recorded. Similarly, a weak measurement of the momentum at a post selected point has been demonstrated experimentally [21, 22]. This means that if we prepare a source of particles such that their mean momentum and spatial width \( (1/\sqrt{\Gamma}) \) are known, then we can predict with some certainty the momentum of one of these particles when it arrives at the post selected (screen) point \( x \). We cannot however, predict the arrival time of the single particle with certainty.

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