Two-Pion Decay Widths of Excited Charm Mesons

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Abstract

The widths for $\pi\pi$ decay of the $L = 1$ charm mesons are calculated by describing the pion coupling to light constituents quarks by the lowest order chiral interaction. The wavefunctions of the charm mesons are obtained as solutions to the covariant Blankenbecler-Sugar equation. These solutions correspond to an interaction Hamiltonian modeled as the sum of a linear scalar confining and a screened one-gluon exchange (OGE) interaction. This interaction induces a two-quark contribution to the amplitude for two-pion decay, which is found to interfere destructively with the single quark amplitude. For the currently known $L = 1$ $D$ mesons, the total $\pi\pi$ decay widths are found to be $\sim 1$ MeV for the $D_1(2420)$ and $\sim 3$ MeV for the $D_2^*(2460)$ if the axial coupling of the constituent quark is taken to be $g_A^q = 1$. The as yet undiscovered spin singlet $D_1^*$ state is predicted to have a larger width of 7 - 10 MeV for $\pi\pi$ decay.
1 Introduction

The pion decay widths of excited charm mesons, the $D$ mesons, which are formed of one light flavor quark (antiquark) and a charm antiquark (quark) provide the best observables for determining the coupling of pions to light constituent quarks, as the pions do not couple to the charm quarks. The coupling strength of the pion to the light flavor constituent quark is determined by the axial coupling constant $g_A^q$ of the quark. The theoretically indicated values for $g_A^q$ [1, 2, 3, 4], which fall in the range 0.75 to 1.0, have been shown to be consistent with extant empirical information on the single pion decay widths of the ground state and excited charm mesons [5, 6]. The uncertainty margin derives from the fact that only upper limits have been set experimentally [7] on the widths of the $D^{*}$ mesons, while the total decay widths of the orbitally excited $L = 1$ $D_1(2420)$ and $D_2^*(2460)$ charm meson states are known only within a very wide uncertainty range. As the latter lie well above the threshold, not only for single pion but also for two pion decay, it would be particularly instructive to obtain theoretical predictions for as well as empirical information on the branching ratios for the latter decay modes.

We here report a calculation of the two-pion decay widths of the excited $L = 1$ charm meson states, by extending a similar earlier calculation of the widths of their single pion decay modes [6]. The charm mesons consisting of a light quark (or antiquark) $q$ and a heavy antiquark (or quark) $\bar{Q}$ are here treated as relativistic two-particle systems described by wavefunctions which have been obtained as solutions to the covariant Blankenbecler-Sugar equation [8, 9] in ref. [10]. The interaction between the $q$ and the $\bar{Q}$ (or the $\bar{q}$ and the $Q$) is described as a combined linear confining and screened one-gluon exchange interaction, with parameters determined so as to obtain a satisfactory prediction of the empirically known part of the $D$ meson spectrum [10].

The model for emission of two pions from the light constituent quarks employed is the conventional chiral pion-quark pseudovector coupling model, which includes a Weinberg-Tomozawa type term for constituent quarks. The only parameter in this amplitude is the axial coupling of the light constituent quarks $g_A^q$, as the quark mass is fixed by the Hamiltonian model used to determine the $D$ meson spectrum and wavefunctions. In addition to the single quark amplitude for two-pion emission, we also consider the interaction current contribution to the vector current of the $q\bar{Q}$ system, which is associated with intermediate antiquark terms. This exchange current contribution to the Weinberg-Tomozawa interaction is found to interfere destructively with the single quark amplitude and to bring about a reduction of the calculated width for the two pion decay of the $L = 1$ $D$ mesons by about 25%.

The amplitude for two-pion decay is found to be mainly spin-independent, which is in accordance with the current empirical information on the $\pi\pi$ decays of the analogous strange $K$ mesons. The $\pi\pi$ decay width of the $D_2^*$ meson is predicted to be about 3 MeV. For the $D_1$ meson, the corresponding number is found to be 1 MeV. Estimates are also given for the $\pi\pi$ decay widths of the hitherto undiscovered $D$ meson states with $L = 1$, i.e. the spin triplet $D_3^*$ and the spin singlet $D_1^*$. The masses of these two states are predicted in ref. [10] to be 2340 and 2390 MeV respectively. The $\pi\pi$ decay width of the $D_3^*$ is here found to be small (of the order one tenth of an MeV, whereas that of the $D_1^*$ is predicted to be about 7 MeV. This variation is a natural consequence of the rather small phase space available for these decays, combined with the significant spin-spin and spin-orbit splittings between various $D$ meson states.

This paper is divided into 4 sections. In section 2 the operator for two-pion emission from a single quark is described along with a derivation of the corresponding decay widths. In section 3 the exchange current contribution to the two-pion decay width is calculated. Section 4 contains a concluding discussion.
2 The Decay Rate for $\pi\pi$ decay

2.1 Single Quark Amplitude for Two-Pion Decay

The emission of two pions from a $D$ meson may be described by the pseudovector Lagrangian, which constitutes the lowest order chiral coupling for pions to constituent quarks:

$$\mathcal{L} = -\frac{g^q}{2f_\pi} \bar{q} \gamma_\mu \partial_\mu \vec{\phi}_\pi \cdot \vec{\tau} \psi_q. \quad (1)$$

Here $g^q_A$ denotes the axial coupling constant of pions to light constituent quarks, and $f_\pi$ is the pion decay constant, the empirical value of which is 93 MeV. The axial coupling constant is conventionally taken to be equal to, or somewhat less than, unity [1, 2]. This coupling gives rise to Born term and crossed Born amplitudes of conventional form, Fig. 1 (a,b), for the emission of two pions from an interacting constituent quark.

The chiral model for the two-pion decay amplitude is completed by the Weinberg-Tomozawa (WT) interaction, which is described by the Lagrangian

$$\mathcal{L}_{WT} = -\frac{i}{4f_\pi^2} \bar{q} \gamma_\mu \vec{\tau} \cdot \vec{\phi}_\pi \times \partial_\mu \vec{\phi}_\pi \psi_q. \quad (2)$$

This interaction leads to the contact amplitude that is described by the diagram c) in Fig. 1. The Born amplitude (Fig. 1a) may be expressed as

$$T_B = -i \left( \frac{g^q_A}{2f_\pi} \right)^2 \bar{q} \gamma_\mu \gamma_5 \cdot k_b \frac{1}{\gamma \cdot p_a + i m_q} \gamma_5 \gamma \cdot k_a \psi_q \tau_b \tau_a. \quad (3)$$

Here $m_q$ denotes the mass of the light constituent quark, which in ref. [10] was obtained as 450 MeV, as this value led to an optimal description of the charm meson spectrum within the model considered. Similarly the crossed Born amplitude (Fig. 1b) takes the form

$$T_{CB} = -i \left( \frac{g^q_A}{2f_\pi} \right)^2 \bar{q} \gamma_\mu \gamma_5 \cdot k_a \frac{1}{\gamma \cdot p_b - i m_q} \gamma_5 \gamma \cdot k_b \psi_q \tau_a \tau_b. \quad (4)$$

The general isospin decomposition of the two-pion emission amplitude for constituent quarks is, in analogy with that for nucleons,

$$T = \delta_{ab} T^+ + \frac{1}{2} [\tau_b, \tau_a] T^- . \quad (5)$$

The general invariant expression for for the amplitudes $T^+$ and $T^-$ is in turn

$$T^\pm = \bar{u}(p') \left( A^\pm - i \gamma \cdot Q B^\pm \right) u(p). \quad (6)$$

In these expressions, the four-vector $Q$ denotes the combination $Q = (k_b - k_a)/2$, where $k_a$ and $k_b$ are the four-momenta of the emitted pions. The commutator of the SU(2) generators may be expressed as $[\tau_b, \tau_a] = 2i \epsilon_{abc} \tau_c$. In this notation the Born, crossed Born and Weinberg-Tomozawa amplitudes are, respectively

$$T_B = -i \left( \frac{g^q_A}{2f_\pi} \right)^2 \left( -\gamma \cdot Q + 2 i m_q + 4 m_q^2 \gamma \cdot Q Q^2 + m_q^2 \right) \left( \delta_{ba} + \frac{1}{2} [\tau_b, \tau_a] \right) , \quad (7)$$
\[ T_{\text{CB}} = -i \left( \frac{g_q}{2f_\pi} \right)^2 \left( \gamma \cdot Q + 2im_q^2 \frac{\gamma \cdot Q}{p_b^2 + m_q^2} \right) \left( \delta_{ba} - \frac{1}{2} [\tau_b, \tau_a] \right), \]
\[ T_{\text{WT}} = -i \gamma \cdot Q \frac{1}{2f_\pi^2} \frac{1}{2} [\tau_b, \tau_a]. \]

Figure 1: Feynman diagrams that correspond to the interaction Lagrangians of eqs. (1) and (2). The diagrams a) and b) describe the Born and crossed Born amplitudes and the diagram c) describes the contact vertex that is associated with the Weinberg-Tomozawa Lagrangian, eq. (2). In these diagrams \( p_a = p - k_a, p_b = p - k_b \) and \( k_a + k_b = p - p' \).

Comparison of these amplitudes with eq. (3) yields the desired expressions for the sub-amplitudes \( A^\pm \) and \( B^\pm \), which are

\[ A^+ = \left( \frac{g_q}{2f_\pi} \right)^2 4m_q, \]
\[ A^- = 0, \]
\[ B^+ = -\left( \frac{g_q}{2f_\pi} \right)^2 4m_q^2 \left[ \frac{1}{s - m_q^2} - \frac{1}{u - m_q^2} \right], \]
\[ B^- = -\left( \frac{g_q}{2f_\pi} \right)^2 \left( 2 + 4m_q^2 \left[ \frac{1}{s - m_q^2} + \frac{1}{u - m_q^2} \right] \right) + \frac{1}{2f_\pi^2}. \]
Here the identities $p_i^2 = -s$ and $p_i^2 = -u$, where $s$ and $u$ are the invariant Mandelstam variables, have been used. These results for the $A$ and $B$ amplitudes are formally equivalent to the corresponding results for the two-pion emission amplitude for nucleons $^{[3]}$. Note that in eq. (13), the contribution from the Weinberg-Tomozawa interaction tends to cancel the constant term in the $B^-$ amplitude that arises from the Born terms. If the axial coupling constant $g_A^q$ is taken to equal 1, then this cancellation is exact.

2.2 The Width for Two-Pion Decay

When excited $D$ mesons decay by emission of pions, it is only the light constituent quark component that couples to the pions. Although pion decay from a single non-interacting quark is kinematically impossible, the pion emission amplitude nevertheless involves the charm antiquark indirectly through the bound state wave function in the matrix element of the single quark pion emission operator. The role of two quark emission operators that involve the interaction between the light quark and charm antiquark, with pion emission from the former, will be considered below in section 3.

The general expression for the two pion decay width of an excited $D$ meson may be written in the form:

$$\Gamma = (2\pi)^4 \int \frac{d^3k_a}{(2\pi)^3} \frac{d^3k_b}{(2\pi)^3} \frac{d^3P_f}{(2\pi)^3} \frac{|T_{f1}|^2}{4\omega_a\omega_b} \delta(4)(P_f + k_a + k_b - P_i).$$  (14)

Here $k_a$ and $k_b$ denote the four-momenta of the two emitted pions, $P_i$ and $P_f$ denote those of the initial and final state $D$ mesons while $\omega_a$ and $\omega_b$ denote the energies of the emitted pions. Since the constituents of the $D$ mesons form bound states, their normalization factors are included in the spinors $\bar{u}(p')$ and $u(p)$ in eq. (3). In the laboratory frame $P_i^0 = M_i$. By introducing the variables $\bar{Q} = (\vec{k}_b - \vec{k}_a)/2$ and $\vec{q} = \vec{k}_b + \vec{k}_a$, the decay width expression may be rewritten as

$$\Gamma = \int \frac{d^3q d^3Q |T_{f1}|^2}{(2\pi)^5 4\omega_a\omega_b} \delta \left( \sqrt{q^2 + M_f^2} + \omega_a + \omega_b - M_i \right).$$  (15)

Here the energy factors are defined as $\omega_a = \sqrt{m_a^2 + (\vec{q}/2 - \bar{Q})^2}$ and $\omega_b = \sqrt{m_b^2 + (\vec{q}/2 + \bar{Q})^2}$ respectively. The remaining delta function may be used to fix the variable $Q$, so that finally the expression for the differential width becomes

$$\frac{d\Gamma}{d\Omega_q} = \frac{1}{4} \frac{1}{(2\pi)^4} \int_0^{\bar{Q}} dq q^2 \int_{-1}^{1} dz \frac{Q_f^2(q, z)}{\omega_a(q, z) (Q_f + \frac{q^2}{2}) + \omega_b(q, z) (Q_f - \frac{q^2}{2})} |T_{f1}|^2.$$  (16)

Here it is understood that in order to obtain the total widths for two-pion decay, eq. (16) is to be integrated over $\Omega_q$ yielding an additional factor $4\pi$. In eq. (16), the variable $z$ is defined by $Q \cdot \bar{q} = Qqz$. With this notation the pion energies $\omega_a$ and $\omega_b$ are given by the expressions $\omega_a = \sqrt{m_a^2 + Q_f^2 + q^2/4 - Q_f qz}$ and $\omega_b = \sqrt{m_b^2 + Q_f^2 + q^2/4 + Q_f qz}$, where the fixed variable $Q_f$ is given by

$$Q_f^2 = \frac{(E_f - M_i)^2 - (4m_{\pi}^2 + q^2)(E_f - M_i)^2}{4(E_f - M_i)^2 - 4q^2 z^2}.$$  (17)

Here $E_f$ denotes the energy of the final state $D$ meson and is given by $E_f = \sqrt{q^2 + M_f^2}$. In all these expressions, $m_{\pi}$ denotes the pion mass. Since different charge states are not considered in this paper,
an average pion mass of 137 MeV has been used throughout the calculations. The cutoff momentum \( q_f \) corresponds to the maximal momentum of any one of the final state particles, e.g. the final state \( D \) meson. Thus \( q_f \) corresponds to the \( q \)-value of a decay of the form \( D' \rightarrow DX \), where \( D' \) and \( D \) are the appropriate \( D \) meson states and \( X \) is a particle with mass \( M_X = 2m_\pi \). The appropriate values for \( q_f \) are listed along with the employed \( D \) meson masses in Table 1.

The decomposition in eq. (6) is convenient with respect to the summation over the isospins of the pions, leading to

\[
\sum_{\text{isospin}} |T_{fi}|^2 = 3T_+^+T^+ + 6T_+^-T^-.
\]  

For calculational purposes, it is useful to split eq. (6) into spin independent and spin-dependent parts as

\[
T^\pm = \alpha^\pm + i\vec{\sigma}_q \cdot \vec{\beta}^\pm.
\]  

Here, the amplitude \( \alpha \) will contain, in addition to the spin independent amplitude \( A \), the spin-independent contributions that arise from the \( \gamma \cdot Q \) term in eq. (3), while \( \vec{\beta} \) will contain all the spin-dependent terms contained in the expression eq. (6). Consider first the non-relativistic limit. In that limit, the expressions for \( \alpha^\pm \) and \( \vec{\beta}^\pm \) become

\[
\alpha^\pm = A^\pm + \left( Q_0 - \frac{\vec{P} \cdot \vec{Q}}{m_q} \right) B^\pm,
\]

\[
\vec{\beta}^\pm = \frac{\vec{q} \times \vec{Q}}{2m_q} B^\pm,
\]

where \( Q_0 \) is defined as \( (\omega_b - \omega_a)/2 \) and \( \vec{P} \) is defined as \( \vec{P} = (\vec{p}' + \vec{p}_q)/2 \). Because two pion decays of charm mesons with \( L = 1 \) to the ground states with \( L = 0 \) are considered here, the largest contribution is expected to arise from the \( \vec{P} \) dependent term in the spin independent amplitude \( \alpha^\pm \).

Evaluation of the expression (18) requires spin sums for both the spin independent and spin-dependent terms in eq. (13). The spin independent part of the amplitude requires evaluation of the spin sums

\[
\frac{1}{2J + 1} \sum_{M = -J}^{J} \langle LSJM| \alpha^* \left( \frac{1 - \vec{\sigma}_q \cdot \vec{\sigma}_Q}{4} \right) \alpha |LSJM\rangle,
\]

\[
\frac{1}{2J + 1} \sum_{M = -J}^{J} \langle LSJM| \alpha^* \left( \frac{3 + \vec{\sigma}_q \cdot \vec{\sigma}_Q}{4} \right) \alpha |LSJM\rangle,
\]

for decay to the spin singlet \( D \) and spin triplet \( D^* \) mesons respectively. For two-pion decay of the spin triplet \( D_1(2420) \) and \( D_2^*(2460) \) mesons the spin independent amplitude gives no contribution for decay to the ground state \( D \) meson (22). The contribution given by the expressions above is \( |\alpha|^2 \) to the decay rate to the spin triplet \( D^* \) meson (23).

The corresponding spin sums for the \( \vec{\sigma}_q \cdot \vec{\beta} \) dependent term in eq. (13) are somewhat more complicated. For initial states with \( L = 1, S = 1 \) they may be expressed as in ref. (13), giving
\[
\frac{1}{2J+1} \sum_{M=-J}^{J} \langle 11JM | \frac{\beta^2}{3} - \frac{S_{12}(\beta)}{6} | 11JM \rangle ,
\]
for decay to spin singlet \((D)\) final states, and
\[
\frac{1}{2J+1} \sum_{M=-J}^{J} \langle 11JM | \frac{2\beta^2}{3} + \frac{S_{12}(\beta)}{6} | 11JM \rangle ,
\]
for decay to spin triplet \((D^*)\) final states. In eqs. (24) and (25), \(S_{12}(\beta)\) is defined as the tensor operator
\[
S_{12}(\hat{\beta}) = 3 \hat{\sigma}_q \cdot \hat{\beta} \hat{\sigma}_Q \cdot \hat{\beta} - \hat{\sigma}_q \cdot \hat{\sigma}_Q .
\]
Given these spin sums, and the state vectors for the spin 1 \(P\)-state \(D\) mesons,
\[
| 1sJM \rangle = \sum_{ls} (11ls|JM) \frac{u_1(r)}{r} Y_{1l}(\hat{r}) |1s\rangle + \sum_{l} (10ls|JM) \frac{u_1(r)}{r} Y_{1l}(\hat{r}) |00\rangle ,
\]
where \(|1s\rangle\) denotes a spin triplet state with \(s_z = s\), the following spin summed squared amplitudes for the two-pion decays of the spin triplet \(D_{2}^*\), \(D_1\) and \(D_0^*\) mesons are obtained:
\[
|T|_{D_{2}^* \rightarrow D^*}^2 = 3 \left[ |\alpha^+|^2 + 2|\alpha^-|^2 + \frac{3}{5} \left( |\beta^+|^2 + 2|\beta^-|^2 \right) \right],
\]
\[
|T|_{D_1 \rightarrow D^*}^2 = 3 \left[ |\alpha^+|^2 + 2|\alpha^-|^2 + |\beta^+|^2 + 2|\beta^-|^2 \right],
\]
\[
|T|_{D_0^* \rightarrow D^*}^2 = 3 \left[ |\alpha^+|^2 + 2|\alpha^-|^2 \right],
\]
\[
|T|_{D_{2} \rightarrow D}^2 = 3 \left[ \frac{2}{5} \left( |\beta^+|^2 + 2|\beta^-|^2 \right) \right],
\]
\[
|T|_{D_1 \rightarrow D}^2 = 0,
\]
\[
|T|_{D_0^* \rightarrow D}^2 = 3 \left[ |\beta^+|^2 + 2|\beta^-|^2 \right].
\]

From the above results, it follows that for the \(\pi\pi\) decays of the spin triplet \(D_1\) meson, there is no contribution for two-pion decay to the \(D\) states. For the spin singlet \(D_{1}^*\) meson, similar expressions hold, with the modification that the spin-independent amplitudes now only contribute to decay to \(D\) states. For the spin-dependent terms, one has to consider the following additional spin sums
\[
\frac{1}{3} \sum_{M=-1}^{1} \langle 101M | - \frac{S_{12}(\beta)}{6} | 101M \rangle ,
\]
for decay to spin singlet \(D\) final states, and
\[
\frac{1}{3} \sum_{M=-1}^{1} \langle 101M | \beta^2 + \frac{S_{12}(\beta)}{6} | 101M \rangle ,
\]
for decay to spin triplet \(D^*\) final states. Application of these spin sums yields then the desired expressions for the spin summed squared amplitudes for two-pion decay of the spin singlet \(D_{1}^*\) meson:
\[ |T|_{D_1^* \rightarrow D^*}^2 = 3 \left[ |\beta^+|^2 + 2|\beta^-|^2 \right], \]  
\[ |T|_{D_1^* \rightarrow D}^2 = 3 \left[ |\alpha^+|^2 + 2|\alpha^-|^2 \right]. \]  

In the non-relativistic limit, the following two radial matrix elements are required for the numerical evaluation of the above amplitudes:

\[ \mathcal{M}_0 = \frac{1}{3m_q} \int_0^\infty dr \left[ u_0^2(r(1) - u_0(r)u_1(r) - 2\frac{u_0(r)u_1(r)}{r} \right] j_0 \left( \frac{qr}{2} \right), \]  
\[ \mathcal{M}_1 = \int_0^\infty dr \, u_0(r)u_1(r) j_1 \left( \frac{qr}{2} \right). \]

Here \( u_0 \) and \( u_1 \) denote the reduced radial wavefunctions that are obtained by numerical solution of the Blankenbecler-Sugar equation in ref. [10] for states with \( L = 0, 1 \) respectively.

In the following calculation, the non-local combinations of the kinematic variables, which appear in the denominators in the Born term amplitudes \( B^+ \) and \( B^- \), have been approximated by their corresponding expectation values. For calculational convenience, the approximation \( E_q \approx m_q \) has likewise been made.

The kinematical variables \( Q \) and \( q \) have, in addition to the energies of the emitted pions, been treated without approximation. The resulting expressions are then obtained as

\[ B^+ \approx -4m_\pi^2 \left( \frac{g^4}{2f_\pi^2} \right) \left\{ m_\pi^2 + \left( \langle \vec{P} \cdot \vec{q} \rangle - 2 \langle \vec{P} \cdot Q \rangle + \frac{q^2}{2} - Q \cdot \vec{q} - 2m_q\omega_a \right) \right\}^{-1} \]
\[ - \left\{ m_\pi^2 + \left( \langle \vec{P} \cdot \vec{q} \rangle + 2 \langle \vec{P} \cdot Q \rangle + \frac{q^2}{2} + Q \cdot \vec{q} - 2m_q\omega_b \right) \right\}^{-1}. \]  

\[ B^- \approx -4m_\pi^2 \left( \frac{g^4}{2f_\pi^2} \right) \left\{ m_\pi^2 + \left( \langle \vec{P} \cdot \vec{q} \rangle - 2 \langle \vec{P} \cdot Q \rangle + \frac{q^2}{2} - Q \cdot \vec{q} - 2m_q\omega_a \right) \right\}^{-1} \]
\[ + \left\{ m_\pi^2 + \left( \langle \vec{P} \cdot \vec{q} \rangle + 2 \langle \vec{P} \cdot Q \rangle + \frac{q^2}{2} + Q \cdot \vec{q} - 2m_q\omega_b \right) \right\}^{-1} + \frac{1 - \frac{g^2}{2f_\pi^2}}{2}. \]

The contribution to the decay rate from the spin independent \( \alpha^+ \) amplitude in (39) may then be expressed as

\[ |T|_{\alpha^+}^2 = 3|\alpha^+|^2 = 3 \left[ A^+ M_1 + B^+ \left( \frac{\omega_b - \omega_a}{2} M_1 - \frac{Q f z}{2} M_0 \right) \right]^2 \]  

for all decays to \( D^* \) final states. Note that the matrix elements here are \( q \)-dependent. Similarly, by using the \( B^- \) amplitude, the contribution to \( |T|^2 \) from the \( \alpha^- \) term becomes

\[ |T|_{\alpha^-}^2 = 6|\alpha^-|^2 = 6 \left( B^- \right)^2 \left[ \frac{\omega_b - \omega_a}{2} M_1 - \frac{Q f z}{2} M_0 \right]^2, \]  

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for all decays to $D^*$ final states. Here $\omega_a$ and $\omega_b$ are defined as for eq. (16). In addition, the $B^+$ and $B^-$ amplitudes also contribute through the spin dependent amplitudes $\beta^+$ and $\beta^-$ (19). In the case of the decay mode $D_2^* \rightarrow D^*\pi\pi'$, these contributions may be expressed as

$$|T|_{\beta^-}^2 = \frac{18}{5} |\beta^-|^2 = \frac{3}{2} \frac{q^2 Q^2}{m_q^2} (B^-)^2 \frac{3}{5} M_1^2,$$

and

$$|T|_{\beta^+}^2 = \frac{9}{5} |\beta^+|^2 = \frac{3}{4} \frac{q^2 Q^2 (1 - z^2)}{m_q^2} (B^+)^2 \frac{3}{5} M_1^2.$$

The corresponding expressions for the remaining decay modes of the spin triplet $L = 1$ $D$ mesons to which the spin-dependent amplitudes contribute may be inferred from eqs. (27-32): For the decay $D_2^* \rightarrow D^*\pi\pi'$, the factors $3/5$ on the r.h.s. of eqs. (43) and (44) are to be replaced by $2/5$, and for the decays $D_0^* \rightarrow D^*\pi\pi'$ and $D_1 \rightarrow D^*\pi\pi'$, those factors equal $1$. That is also the case for the decay of the spin singlet $D_1^*$ meson to $D^*\pi\pi'$ (36). The total decay widths are obtained by adding the contributions from eqs. (41,42,43) and (44), and inserting them into eq. (16). In the numerical evaluation of the expressions (39) and (40), the terms $\vec{P} \cdot \vec{q}$ and $\vec{P} \cdot \vec{Q}$ in the denominators have been approximated by their respective expectation values:

$$\left\langle \vec{P} \cdot \vec{q} \right\rangle = \frac{q m_q}{2} M_0,$$

$$2 \left\langle \vec{P} \cdot \vec{Q} \right\rangle = Q f z m_q M_0.$$ (45)

(46)

Note that despite the appearance of the matrix element $M_0$ in the above expressions, no non-relativistic approximation is implied.

As the velocities of the confined quarks and antiquarks in the $D$ mesons are close to that of light, the non-relativistic approximation for the amplitudes is not reliable. Therefore, eqs. (20) and (21) should be replaced by the analogous unapproximated forms

$$\alpha^\pm = \sqrt{\frac{E' + m_q}{2E'}} \sqrt{\frac{E + m_q}{2E}} \left[ A^\pm \left(1 - \frac{P^2 - q^2/4}{(E' + m_q)(E + m_q)}\right) + B^\pm \left(1 + \frac{P^2 - q^2/4}{(E' + m_q)(E + m_q)}\right) \right],$$

and

$$\beta^\pm = \sqrt{\frac{E' + m_q}{2E'}} \sqrt{\frac{E + m_q}{2E}} \left\{ \frac{A^\pm - B^\pm Q_0}{(E + m_q)(E' + m_q)} \vec{q} \times \vec{P} + B^\pm \left(\vec{P} \times \vec{Q} \left(\frac{1}{E + m_q} - \frac{1}{E' + m_q}\right)\right) \right\}.$$ (47)

(48)
Thus, relativistic counterparts to eqs. (41-44) are required. As a consequence, eqs. (41) and (42) should be replaced by the expressions

\[
|T|_{\alpha^+}^2 = 3|\alpha^+|^2 = 3 \left[ A^+ \mathcal{M}_{1+}^{\text{rel}} + B^+ \left( \frac{\omega_b - \omega_a}{2} \mathcal{M}_{1-}^{\text{rel}} - \frac{Q_I z}{2} \mathcal{M}_{0}^{\text{rel}} \right) \right]^2 ,
\]

and

\[
|T|_{\alpha^-}^2 = 6|\alpha^-|^2 = 6 \left( B^- \right)^2 \left[ \frac{\omega_b - \omega_a}{2} \mathcal{M}_{1-}^{\text{rel}} - \frac{Q_I z}{2} \mathcal{M}_{0}^{\text{rel}} \right]^2 .
\]

Here the relativistic matrix elements \( \mathcal{M}_{1\pm}^{\text{rel}} \) are defined as

\[
\mathcal{M}_{1\pm}^{\text{rel}} = \frac{1}{\pi} \int_0^\infty dv' r' u_0(r') \int_0^\infty dr r u_1(r) \int_0^\infty dP P^2 \int_{-1}^1 dv f_{\text{BS}}(P, v) \frac{q/4 + P v}{\sqrt{P^2 + q^2/16 + P q v/2}} \sqrt{E' + m_q} \sqrt{E + m_q} \left( 1 + \frac{P^2 - q^2/4}{(E' + m_q)(E + m_q)} \right) j_0 \left( r' \sqrt{P^2 + \frac{q^2}{16} - \frac{P q v}{2}} \right) j_1 \left( r \sqrt{P^2 + \frac{q^2}{16} + \frac{P q v}{2}} \right) ,
\]

where the last term in eq. (47) has been neglected because of its smallness. Similarly, the relativistic matrix element \( \mathcal{M}_0^{\text{rel}} \) is obtained as

\[
\mathcal{M}_0^{\text{rel}} = \frac{1}{3\pi} \int_0^\infty dv' r' \int_0^\infty dr r \int_0^\infty dP P^2 \int_{-1}^1 dv f_{\text{BS}}(P, v) \sqrt{E' + m_q} \sqrt{E + m_q} \left( \frac{1}{E + m_q} + \frac{1}{E' + m_q} \right) \left[ u'_0(r') u_1(r) - u_0(r') u'_1(r) - 2 \frac{u_0(r') u_1(r)}{r} \right] j_0 \left( r' \sqrt{P^2 + \frac{q^2}{16} - \frac{P q v}{2}} \right) j_0 \left( r \sqrt{P^2 + \frac{q^2}{16} + \frac{P q v}{2}} \right) .
\]

In the non-relativistic limit, these matrix elements reduce to the forms of eqs. (38) and (37) respectively. In the above equations, the factor \( f_{\text{BS}}(P, v) \) arises in the reduction of the amplitude from the form appropriate to the Bethe-Salpeter equation to that of the Blankenbecler-Sugar equation, and is defined as

\[
f_{\text{BS}}(P, v) = \frac{M_Q + m_q}{\sqrt{(E + E')(E' + E')}} ,
\]

where \( M_Q \) denotes the mass of the heavy (anti)quark. The energy factors \( E, E' \) and \( E_c, E'_c \) are defined as

\[
E = \sqrt{m_q^2 + P^2 + P q v + q^2/4} , \quad E' = \sqrt{m_q^2 + P^2 - P q v + q^2/4} ,
\]

for the light quark \( q \), and
for the heavy (anti)quark $Q$. In addition, eqs. (43) and (44) should also be replaced by analogous relativistic forms. These may be obtained as

$$E_c = \sqrt{M_Q^2 + P^2 + Pqv + q^2/4}, \quad E'_c = \sqrt{M_Q^2 + P^2 - Pqv + q^2/4},$$

for the heavy (anti)quark $Q$. In addition, eqs. (43) and (44) should also be replaced by analogous relativistic forms. These may be obtained as

$$|T|_{\beta^-}^2 = 6 (B^-)^2 \frac{3}{5} \left[ \frac{q Q f \sqrt{1 - z^2}}{2} \mathcal{M}_{\beta^1}^{\text{rel}} - \frac{\omega_b - \omega_a}{2} q \mathcal{M}_{\beta^2}^{\text{rel}} \right]^2,$$

for the contribution from $\bar{\beta}^-$, and

$$|T|_{\beta^+}^2 = 3 \frac{3}{5} \left\{ A^+ q \mathcal{M}_{\beta^2}^{\text{rel}} + B^+ \left[ \frac{q Q f \sqrt{1 - z^2}}{2} \mathcal{M}_{\beta^1}^{\text{rel}} - \frac{\omega_b - \omega_a}{2} q \mathcal{M}_{\beta^2}^{\text{rel}} \right] \right\}^2,$$

for the corresponding one from $\beta^+$. Here the matrix elements are defined as

$$\mathcal{M}_{\beta^1}^{\text{rel}} = \frac{1}{\pi} \int_0^\infty dr' r' u_0(r') \int_0^\infty dr \ r \ u_1(r) \int_0^\infty dP \ P^2 \int_{-1}^1 dv \ f_{\text{BS}}(P, v) \frac{q/4 + P v}{\sqrt{P^2 + q^2/16 + Pqv/2}} \sqrt{E' + m_q} \sqrt{E + m_q} \left( \frac{1}{E' + m_q} + \frac{1}{E + m_q} \right)$$

$$j_0 \left( r' \sqrt{P^2 + q^2/16 - Pqv/2} \right) j_1 \left( r \sqrt{P^2 + q^2/16 + Pqv/2} \right),$$

and

$$\mathcal{M}_{\beta^2}^{\text{rel}} = \frac{1}{\pi} \int_0^\infty dr' r' u_0(r') \int_0^\infty dr \ r \ u_1(r) \int_0^\infty dP \ P^3 \int_{-1}^1 dv \sqrt{1 - v^2} f_{\text{BS}}(P, v) \frac{q/4 + P v}{\sqrt{P^2 + q^2/16 + Pqv/2}} \sqrt{E' + m_q} \sqrt{E + m_q} \left( \frac{1}{E' + m_q} + \frac{1}{E + m_q} \right)$$

$$j_0 \left( r' \sqrt{P^2 + q^2/16 - Pqv/2} \right) j_1 \left( r \sqrt{P^2 + q^2/16 + Pqv/2} \right).$$

With exception of a factor $m_q^{-1}$, the matrix element (58) reduces to the form (59) in the non-relativistic limit. In these expressions, the $\vec{P} \times \vec{Q}$ term in eq. (48) has been dropped because of its smallness. Furthermore, it turns out that eq. (59) is numerically much smaller than eq. (58). Overall, the contributions from the amplitudes $\alpha^+$ and $\alpha^-$ are dominant, whereas those from $\beta^+$ and $\beta^-$ represent only small corrections. This feature is in accordance with the current experimental status for the analogous decays of the excited strange mesons, for which the decay mode $K_S^* \to K \pi \pi$ has not yet been seen, although the decays of the $K_S^*$ meson are otherwise well studied. In eqs. (56) and (57), it is again understood that the factor $3/5$ is to be replaced as for eqs. (43) and (44) when different decay modes are considered. The constituent quark and $D$ meson masses used are listed in Table 1. The numerical results obtained by using the relativistic expressions are given in Table 2. For comparison the corresponding non-relativistic results are presented in Table 3. The non-relativistic treatment leads to overpredictions by a factor $\sim 3$. 

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| Decay     | Initial state mass ($M_i$) | Final state mass ($M_f$) | $q_f$ |
|-----------|-----------------------------|--------------------------|--------|
| $D_2^* \rightarrow D^* \pi\pi$ | 2460                        | 2007                     | 327    |
| $D_2^* \rightarrow D\pi\pi$   | 2460                        | 1867                     | 462    |
| $D_1 \rightarrow D^* \pi\pi$  | 2420                        | 2007                     | 296    |
| $D_1 \rightarrow D\pi\pi$    | 2390                        | 2007                     | 244    |
| $D_0^* \rightarrow D^* \pi\pi$| 2341                        | 2007                     | 177    |
| $D_0^* \rightarrow D\pi\pi$   | 2389                        | 2007                     | 177    |

Table 1: Initial and final state $D$ meson masses and resulting maximal kinematically allowed momenta $q_f$ in MeV used in the calculations. The masses listed are to be viewed as averages over the different charge states. In the calculations, the values $m_q = 450$ MeV and $M_Q = 1580$ MeV, obtained by fits to the spectra in ref. [10], have been used for the constituent quark masses.

| Decay     | $|T|_{\alpha+}^2$ | $|T|_{\beta-}^2$ | $|T|_{\beta+}^2$ | Total ($g_A^{\pi\pi} = 1$) | Total ($g_A^{\pi\pi} = 0.87$) |
|-----------|------------------|------------------|------------------|-----------------------------|-----------------------------|
| $D_2^* \rightarrow D^* \pi\pi$ | 0.896            | 2.864            | $5.06 \cdot 10^{-3}$ | $1.17 \cdot 10^{-3}$ | 3.77 MeV                    | 2.39 MeV                    |
| $D_2^* \rightarrow D\pi\pi$   | 0.367            | 1.291            | $6.45 \cdot 10^{-2}$ | $7.51 \cdot 10^{-3}$ | 0.07 MeV                    | 0.05 MeV                    |
| $D_1 \rightarrow D^* \pi\pi$  | 2.749            | 7.915            | $6.54 \cdot 10^{-4}$ | $3.30 \cdot 10^{-4}$ | $\simeq 0$ MeV              | $\simeq 0$ MeV              |
| $D_0^* \rightarrow D\pi\pi$   | 0.020            | 0.098            | $1.43 \cdot 10^{-2}$ | $2.86 \cdot 10^{-3}$ | 0.12 MeV                    | 0.07 MeV                    |

Table 2: Numerical results using the single-quark amplitudes and the relativistic matrix elements for the two-pion decay widths of the spin triplet $D_2^*$, $D_1^*$, $D_0^*$ and the spin singlet $D_1^*$ mesons. The individual results from the spin independent ($\alpha^\pm$) and spin dependent amplitudes ($\beta^\pm$) are shown for $g_A^{\pi\pi} = 1$, and the total $\pi\pi$ decay widths for both $g_A^{\pi\pi} = 1$ and $g_A^{\pi\pi} = 0.87$.

The results for the two-pion decay width of the positive parity charm mesons with $L = 1$ shown in Table 2 reveal a strong sensitivity to the strength of the pion coupling to the light constituent quarks as measured by the axial coupling constant of the constituent quarks. This is of course as expected, as the calculated decay widths are proportional to the 4th power of $g_A^{\pi\pi}$. Also, because of the large (140 MeV) splitting between the $D$ and $D^*$ states and the significant spin-orbit splittings of the $L = 1$ charm mesons, it turns out that the values of the maximal kinematically allowed momentum transfer $q_f$ as given in Table 1 show marked variation. Consequently, some decays are kinematically favored, while others, in particular $D_0^* \rightarrow D^* \pi\pi$, are strongly inhibited by the small phase space available. Hence the two-pion decay widths of the $L = 1$ $D$ mesons are very sensitive to the spin-orbit structure of the quark-antiquark interaction. The same conclusion was also reached in ref. [6] concerning the single pion decays of the $L = 1$ $D$ mesons. In the absence of empirical information and definite QCD lattice...
The energies of the hitherto undiscovered $D_0^*$ and $D_1^*$ mesons have here been taken to equal those found in the calculation of ref. [6].

The predicted total decay widths of the $L = 1$ $D$ mesons in the single quark approximation may be obtained by adding the calculated two-pion decay widths in Table 2 to those for single pion decay obtained in ref. [6]. With $g_A^q = 1$ the total calculated $\pi\pi$ decay width of the $D_2^*(2460)$ is 3.8 MeV and that of the $D_1(2420)$ is 1.7 MeV. If these values are added to the corresponding calculated values for single pion decay obtained in ref. [6], the total calculated width of the $D_2^*(2460)$ comes to 19.5 MeV, which is well within the uncertainty margin of the empirical value $25^{+8}_{-7}$ MeV for the total decay width [7].

In the case of the $D_1(2420)$ meson the total calculated width for single and two-pion decay comes to 15.3 MeV, which is close to the empirical uncertainty margin of the total decay width $18^{+9}_{-4.5}$ MeV [7]. Reduction of the value for $g_A^q$ to 0.87 brings the calculated values for the total width for $\pi$ and $\pi\pi$ decay a bit below the empirical values for the total widths. Likewise, employment of the two-quark contribution considered in the next section has the effect of reducing the calculated $\pi\pi$ widths. The final results for the $\pi\pi$ widths are listed in Table 3, along with predictions for the total widths of the $L = 1$ $D$ mesons.

| Decay        | $|T|_{\alpha}^2$ | $|T|_{\beta}^2$ | $|T|_{\alpha}^2$ | $|T|_{\beta}^2$ | Total ($g_A^q = 1$) | Total ($g_A^q = 0.87$) |
|--------------|-----------------|-----------------|-----------------|-----------------|-------------------|-------------------|
| $D_2^* \rightarrow D^*\pi\pi$ | 2.638 | 9.934 | 1.36 $\cdot$ 10^{-2} | 3.11 $\cdot$ 10^{-4} | 12.6 MeV | 8.01 MeV |
| $D_2^* \rightarrow D\pi\pi$ | – | – | 0.178 | 5.77 $\cdot$ 10^{-3} | 0.18 MeV | 0.12 MeV |
| $D_1 \rightarrow D^*\pi\pi$ | 1.083 | 4.539 | 6.21 $\cdot$ 10^{-3} | 1.16 $\cdot$ 10^{-4} | 5.63 MeV | 3.56 MeV |
| $D_1^* \rightarrow D^*\pi\pi$ | – | – | 1.74 $\cdot$ 10^{-3} | 2.60 $\cdot$ 10^{-5} | $\simeq$ 0 MeV | $\simeq$ 0 MeV |
| $D_1 \rightarrow D\pi\pi$ | 8.048 | 26.97 | – | – | 35.0 MeV | 22.4 MeV |
| $D_0^* \rightarrow D^*\pi\pi$ | 0.059 | 0.357 | – | – | 0.42 MeV | 0.26 MeV |
| $D_0 \rightarrow D\pi\pi$ | – | – | 3.84 $\cdot$ 10^{-2} | 9.51 $\cdot$ 10^{-4} | 0.04 MeV | 0.03 MeV |

Table 3: Numerical results for the two-pion decay widths of the spin triplet $D_2^*$, $D_1$, $D_0^*$ and the spin singlet $D_1^*$ mesons using the non-relativistic approximation to the single-quark amplitudes. The individual results from $\alpha^\pm$ and $\beta^\pm$ are shown for $g_A^q = 1$, and the total decay widths for both $g_A^q = 1$ and $g_A^q = 0.87$.

This apparent slight underprediction of the total decay widths may actually be a desirable situation in view of the fact that it is kinematically possible for other decay modes, mainly $\pi\pi\pi$ and $\eta$-meson decay, to contribute to the total widths of these mesons, even though there is very little phase space available for such decays. However, in view of the large systematical uncertainties involved in the experimental determination of the total widths of the $D_2^*$ and $D_1$ mesons, the uncertainty margins quoted by ref. [7] may be on the narrow side.
3 Two-Quark Contribution to the Decay Width

It is instructive, for the purpose of determining the two-quark contribution to the decay width, to rewrite the Weinberg-Tomozawa Lagrangian, eq. (2) in the form of a current-current coupling:

\[ \mathcal{L}_{WT} = -\frac{1}{4f^2_\pi} \vec{V}_\mu \cdot \vec{\phi}_\pi \times \partial_\mu \vec{\phi}_\pi. \] (60)

Here \( \vec{V}_\mu = i \bar{\psi}_q \gamma_\mu \psi_q \vec{\tau} \) is the isovector current of the light constituent quark and \( \vec{\phi}_\pi \times \partial_\mu \vec{\phi}_\pi \) is the current of the two-pion system. Given this expression it becomes very natural to describe the irreducible two-quark contribution to the two-pion production operator by means of two-quark interaction current contributions to the isovector current \( \vec{V}_\mu \).

The most significant interaction current contribution to the isovector current of the charm mesons is the "pair" current that is associated with the scalar confining interaction between the constituent quarks. This current, which is illustrated by the Feynman diagrams in Fig. 2, represents a relativistic correction term, which arises in the elimination of the small components of the quark spinors. In the non-relativistic approximation it can be described as a renormalization of the isovector current of the light constituent quark by the confining interaction, either in terms of interaction correction to the constituent quark mass or as an explicit additional term [4].

In the non-relativistic limit, the isovector current \( \vec{V}_\mu = (i\vec{V}_0, \vec{V}) \) of the light constituent quark takes the form

\[ \vec{V} = \frac{1}{2m_q} (\vec{p}_q' + \vec{p}_q - i\vec{\sigma}_q \times \vec{q}) \vec{\tau}, \] (61)
where \( m_q \) is the light constituent quark mass and \( \vec{p}_q \) and \( \vec{p}'_q \) are the initial and final quark momenta respectively. The corresponding expression for the "pair" current that is associated with the confining interaction is then obtained as [12]

\[
\vec{V}_c = -\frac{V_c(k)}{m_q} \vec{V}.
\]

(62)

Here \( V_c(k) \) is the (formal) Fourier transform of the confining interaction, and the momentum transfer \( \vec{k} \) is defined as the difference between the final and initial momenta of the charm (anti)quark: \( \vec{k} = \vec{p}'_Q - \vec{p}_Q \). Within the non-relativistic approximation it now becomes a straightforward task to take this two-quark contribution to the two-pion production operator [13] into account. The most significant contribution may be included by modifying that part of the amplitude \( B^- \) in the expression for \( \alpha^- \) (eq. (20)), which arises from the Weinberg-Tomozawa Lagrangian, according to the following prescription:

\[
\frac{1}{2f^2} \rightarrow \frac{1}{2f^2} \left( 1 - \frac{V_c(r)}{m_q} \right).
\]

(63)

Here \( V_c(r) \) is the scalar confining interaction, which in the wavefunction model of ref. [6] has the form \( V_c(r) = cr - b \), with \( c = 1120 \text{ MeV/fm} \) and \( b = 320 \text{ MeV} \). This replacement should only be made for terms that are associated with the spatial current coupling, i.e. for those that contain the momentum transfer \( \vec{Q} \). A similar modification should also be made in the expression for \( \beta^- \) (eq. (21)), but as that contribution is already very small compared to those from the spin-independent \( \alpha^- \) terms, see Table 2 it will not be considered in this paper.

The hyperfine interaction that is modeled as screened one-gluon exchange (OGE) between the quark and the antiquark also contains a central component which gives a contribution to the isovector exchange current. This contribution may be obtained as [10]:

\[
\vec{V}_g = -V_g(k) \left[ \frac{m_q}{M_Q} (\vec{p}_Q' + \vec{p}_Q + i\vec{\sigma}_Q \times \vec{k}) + i\vec{\sigma}_q \times \vec{k} \right].
\]

(64)

In the above expression for the OGE exchange current, the \( M_Q^{-1} \) dependent term arises from the spatial current coupling \( \vec{r} \cdot \vec{r}_Q \) term, and the second term from the charge coupling \( \gamma_4^q \gamma_4^\bar{Q} \) term in the OGE interaction. \( V_g(k) \) denotes the form of the OGE interaction in momentum space, which is conveniently expressed as

\[
V_g(k) = -\frac{16\pi}{3} \frac{\alpha_S(k^2)}{k^2},
\]

(65)

where a color factor of 4/3 has been included. In the interaction model used in ref. [11], the running coupling constant was taken to have the following screened form:

\[
\alpha_S(k^2) = \frac{12\pi}{27} \frac{1}{\ln((k^2 + 4m_g^2)/\Lambda_0^2)},
\]

(66)

where the parameters \( m_g \) and \( \Lambda_0 \) were obtained as \( m_g = 240 \text{ MeV} \) and \( \Lambda_0 = 280 \text{ MeV} \) respectively. In case of the \( \pi\pi \) decay of the \( D \) meson states with \( L = 1 \), the term with the sum of charm quark momenta in eq. (64), gives the largest contribution. As this term is inversely proportional to the mass of the charm quark, it is evident that the gluon exchange contribution will be smaller than the corresponding contribution from the exchange current (62) that is associated with the confining interaction. This main
term of the isovector exchange current that is associated with the OGE interaction may be taken into account in the non-relativistic approximation by modifying eq. (63) to read

\[
\frac{1}{2f_\pi^2} \rightarrow \frac{1}{2f_\pi^2} \left( 1 - \frac{V_g(r)}{m_q} + \frac{V_g(r)}{M_Q} \right),
\]

where \(V_g(r)\) is the central component of the one-gluon exchange potential. For a bare OGE interaction one would have \(V_g(r) = -4\alpha_S/3r\). This simple expression is however significantly modified by the running coupling as given by eq. (66). This effect may be included as in ref. [10], by expressing \(V_g(r)\) as

\[
V_g(r) = -\frac{4}{3\pi^2} \int_0^\infty dk j_0(kr) \alpha_S(k^2).
\]

This form reduces to the static OGE potential given above, if the running coupling \(\alpha_S\) is taken to be constant. Note that the relativistic corrections arising from the quark and antiquark spinors have not been included in the above expression, since it is more natural to place them in the expression for the matrix element instead. Moreover, comparison of these expressions with those obtained for the confining interaction shows that the net effect of the exchange current that is associated with OGE will have the same sign as that, which is associated with the confining interaction.

In the expressions (62) and (64), one factor of \(1/m_q\) represents the static limit of the propagator of the intermediate negative energy light constituent quark. In view of the results obtained in the previous section, this static limit is expected to give rise to a considerable overestimate of the two-quark amplitude contribution. A more realistic treatment may be obtained if the static propagator \(1/m_q\) is replaced by the symmetrized form \(4/(2m_q + E + E')\) as in ref. [1]. Thus in the relativistic case, the appropriate replacement of the Weinberg-Tomozawa term would be the following extension of the replacement (67):

\[
\frac{1}{2f_\pi^2} \rightarrow \frac{1}{2f_\pi^2} \left[ 1 - \sqrt{\frac{(E_c + M_Q)(E_c' + M_Q)}{4E_cE_c'}} \right] \left\{ V_c \left( \frac{|\vec{r} + \vec{r}'|}{2} \right) \left( 2m_q + E + E' \right) \left( 1 - \frac{P^2 - q^2/4}{(E_c' + M_Q)(E_c + M_Q)} \right) \right\}
\]

where the factors containing \(E_c'\) and \(E_c\) arise from the spinors of the heavy antiquark line in Fig. 2. To take the relativistic effects into account demands employment of the relativistic version of the matrix element \(\mathcal{M}_0^{\text{rel}}\) also in the case of the exchange current contribution. First of all, the contribution to the decay rate from the \(\alpha^-\) term, eq. (50), should be modified to take into account the two-quark contributions to the Weinberg-Tomozawa interaction. This can be accomplished by replacing eq. (50) with

\[
|T|^2_{\alpha^-} = 6 \left[ B^{-} \left( \frac{\omega_b - \omega_a}{2} - \frac{Q l_z}{2} \mathcal{M}_1^{\text{rel}} - \frac{Q l_z}{2} \mathcal{M}_0^{\text{rel}} \right) + \frac{1}{2f_\pi^2} \frac{Q l_z}{2} \left( \mathcal{M}_0^{\text{Conf}} - \mathcal{M}_0^{\text{OGE}} \right) \right]^2.
\]

The appropriate forms for the relativistic two-quark matrix elements in the above equation can be obtained by multiplying the expression for \(\mathcal{M}_0^{\text{rel}}\) with the relativistic factors included in eq. (68) above. In addition, the argument \(|\vec{r} + \vec{r}'|/2\) of the potentials will be approximated by the expression \((r^2 + r'^2)/2\). The resulting forms for the two-quark matrix elements in eq. (70) are then obtained as
\[
\mathcal{M}_0^{\text{Conf}} = \frac{1}{3\pi} \int_0^\infty dr' r' \int_0^\infty dr \left[ u'_0(r') u_1(r) - u_0(r') u'_1(r) - 2 \frac{u_0(r') u_1(r)}{r} \right] \int_0^\infty dP P^2 \int_{-1}^1 dv f_{\text{HS}}(P, v) \\
V_c \left( \sqrt{\frac{r'^2 + r^2}{2}} \right) \frac{4}{2m_q + E + E'} \sqrt{\frac{(E + m_q)(E' + m_q)}{4EE'}} \left( \frac{1}{E + m_q} + \frac{1}{E' + m_q} \right) \left( \frac{(E + M_Q)(E' + M_Q)}{4E_c E'_c} \right) \left( \frac{1}{E_c + M_Q} + \frac{1}{E'_c + M_Q} \right) \int_0^\infty dP P^2 \int_{-1}^1 dv f_{\text{HS}}(P, v) \\
j_0 \left( r \sqrt{P^2 + \frac{q^2}{16} + \frac{P_{qv}}{2}} \right),
\]

(71)

for the confining interaction, and

\[
\mathcal{M}_0^{\text{OGE}} = \frac{1}{3\pi} \int_0^\infty dr' r' \int_0^\infty dr \left[ u'_0(r') u_1(r) - u_0(r') u'_1(r) - 2 \frac{u_0(r') u_1(r)}{r} \right] \int_0^\infty dP P^2 \int_{-1}^1 dv f_{\text{HS}}(P, v) \\
V_g \left( \sqrt{\frac{r'^2 + r^2}{2}} \right) \frac{4m_q}{2m_q + E + E'} \sqrt{\frac{(E + m_q)(E' + m_q)}{4EE'}} \left( \frac{1}{E + m_q} + \frac{1}{E' + m_q} \right) \left( \frac{(E + M_Q)(E' + M_Q)}{4E_c E'_c} \right) \left( \frac{1}{E_c + M_Q} + \frac{1}{E'_c + M_Q} \right) \int_0^\infty dP P^2 \int_{-1}^1 dv f_{\text{HS}}(P, v) \\
j_0 \left( r \sqrt{P^2 + \frac{q^2}{16} + \frac{P_{qv}}{2}} \right),
\]

(72)

for the OGE interaction. The numerical results that follow when eq. (70) is employed are displayed for the most important decay modes in Table 4.

| Decay   | Rel | +OGE | Total $g_A^q = 1$ | Total $g_A^q = 0.87$ |
|---------|-----|------|------------------|------------------|
| $D_2 \rightarrow D^* \pi \pi$ | 2.864 | 2.377 | 2.144 | 3.05 MeV | 1.82 MeV |
| $D_1 \rightarrow D^* \pi \pi$ | 1.291 | 1.076 | 0.974 | 1.34 MeV | 0.80 MeV |
| $D_1^* \rightarrow D \pi \pi$ | 7.915 | 6.535 | 5.872 | 8.62 MeV | 5.17 MeV |

Table 4: Numerical results for the most important $\pi \pi$ decay modes obtained upon employment of the two-quark contributions to the Weinberg-Tomozawa interaction. The modifications to the $|T|^2_{\alpha -}$ amplitude with $g_A^q = 1$ are shown as follows: The column ”Rel” gives the one-quark result for each decay rate (cf. Table 3), in the column ”+Conf” the contribution from $\mathcal{M}_0^{\text{Conf}}$ has been added, and in ”+OGE”, the results that follow when both $\mathcal{M}_0^{\text{Conf}}$ and $\mathcal{M}_0^{\text{OGE}}$ are employed are given. The resulting total decay widths for $\pi \pi$ decay are also shown, for both $g_A^q = 1$ and $g_A^q = 0.87$. 

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4 Discussion

In the absence at the present time of experimental data on the two-pion decay widths of the D mesons, it is instructive to compare the calculated results with the empirical knowledge of the analogous pionic decay modes of the positive parity strange mesons. The \( K^*_2(1430) \) state, which is the strange analog of the \( D^*_2(2460) \) has a width of \( \sim 13 \text{ MeV} \) for \( \pi\pi \) decay \footnote{7}. Given the smaller phase space and much narrower radial wavefunctions, the present calculated \( \pi\pi \) decay width value of 3.1 MeV for the \( D^*_2(2460) \) hence appears to be reasonable. Furthermore, in ref. \footnote{7}, the branching ratio \( K^*_2 \to K^*\pi \) is given as \( (24.7 \pm 1.5\%) \) while that for \( K^*_2 \to K^*\pi\pi \) is reported to be \( (13.4 \pm 2.2\%) \). Thus, the available experimental data indicates that for this particular decay mode, the width for \( \pi\pi \) decay ought to be \( \sim 55\% \) of the width for \( \pi \) decay. If the current calculation is compared to that in ref. \footnote{6}, the ratio of \( D^*_2 \to D^*\pi\pi \) to \( D^*_2 \to D^*\pi \) is obtained as \( \sim 60\% \) for \( g_A^q = 1 \). One may thus conclude that a width for \( D^*_2 \to D^*\pi\pi \) of 3 MeV, as obtained in the current calculation, is what would be expected by comparison with the corresponding decays of the strange K mesons. The present results also appear realistic in view of the fact that the empirical nonobservation of the decay mode \( K^*_2 \to K\pi\pi \) indicates that the amplitude for two-pion decay is mainly of spin-independent character.

The strange analog of the \( D_1(2420) \) state is most likely a mixture of the \( K_1(1270) \) and the \( K_1(1400) \) states. The two-pion decay modes of these two states are dominated by \( \rho \) meson decay. As \( \rho \)-meson decay is kinematically impossible for the \( D_1(2420) \) as well as the other \( L = 1 \) D mesons, these decay modes have little bearing on the decays of the latter. Addition of the calculated decay widths for single- and two-pion decay gives predictions for the total widths of the \( L = 1 \) D meson states. These results are shown in Table \footnote{5}.

| \( D \) meson state | \( \pi \) width | \( \pi\pi \) width | Total \( g_A^q = 1 \) | Total \( g_A^q = 0.87 \) | Experiment |
|----------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( D^*_2 \)          | 15.7 MeV        | 3.05 MeV        | 18.8 MeV        | 13.7 MeV        | 25\text{+8}_-5 \) MeV |
| \( D_1 \)            | 13.6 MeV        | 1.34 MeV        | 14.9 MeV        | 11.1 MeV        | 18.9\text{+4.6}_-3.5 \) MeV |
| \( D^*_0 \)          | 27.7 MeV        | \( \sim 0.1 \) MeV | 28.8 MeV        | 21.0 MeV        | –               |
| \( D^*_1 \)          | 13.2 MeV        | 8.62 MeV        | 21.8 MeV        | 15.2 MeV        | –               |

Table 5: Total decay widths for strong decay of the \( L = 1 \) D mesons that follow when the results of the present calculation of the \( \pi\pi \) decay widths are added to those for single pion decay obtained in ref. \footnote{6}. The individual results for \( \pi \) and \( \pi\pi \) decay are shown for \( g_A^q = 1 \). The experimental results, when available, have been taken from ref. \footnote{7}. Note that the \( D_1 \) is a spin triplet state with \( J = 1 \) and \( D^*_1 \) is the corresponding spin singlet state.

As indicated by Table \footnote{5}, the present calculation of the two pion decay widths of the \( L = 1 \) charm mesons completes the calculation of the corresponding single pion decay widths of ref. \footnote{6}. The calculations show that, if the charm mesons are described as two-particle systems formed of one light flavor constituent quark and one charm antiquark, where only the light quark couples to pions, a fair description of the presently known decay widths of these mesons may be obtained. Given the chiral form for
the coupling between the pions and the light constituent quarks, a better description of the total pionic decay widths obtains with the value $g_A^q = 1$ than with smaller values, even though the empirical data are still very crude at this time. If other decay modes, such as $\pi\pi\pi$ and $\eta$ decay should turn out to give appreciable contributions to the total decay widths of the positive parity charm mesons, a somewhat smaller value for $g_A^q$ may be favored. However, as $\pi\pi\pi$ decay has so far not been detected for the strange $K_2^*$ meson [7] and since the $\eta$ decay width of that same state is empirically found to be very small, that possibility is apparently not supported by experiment at this time. From the calculations of ref. [4], one may conclude that the width of the decay mode $D_2^* \to D\eta$ is probably less than 0.25 MeV.

Furthermore there is the possibility of two-pion decay of the $D_2^*$ and $D_1$ mesons through an intermediate $D_1^*$ which is close to its mass shell, an effect which has been investigated by ref. [5]. There, this mechanism was found to contribute significantly to the two-pion decay widths, although the effect is very sensitive to the widths and spin-orbit splittings of the $L = 1$ $D$ mesons and is thus very difficult to estimate.

Finally it should be noted that the results are quite sensitive to the exact form of the $B^-$ amplitude, as given by eq. (40). Therefore, crude approximations have to be avoided in the case of this amplitude. If the $B^-$ amplitude is treated non-relativistically, which implies dropping the nonlocal and $\vec{Q}$ dependent terms altogether, large overestimates of that contribution to the $\pi\pi$ decay widths will result.

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