Strong and electromagnetic contributions to the $U_A(1)$ anomaly and the $P^0 \to \gamma \gamma$ decays

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Abstract

Using flavor basis we relate flavor axial anomalies to the mass matrix of pseudoscalar isoscalar fields in the context of a Linear Sigma Model which includes $U_A(1)$ symmetry breaking. We incorporate additional contributions to these anomalies due to external electromagnetic fields invoking ’t Hooft’s argument on anomaly matching and work out the predictions of this formalism for $\eta \to \gamma \gamma$ and $\eta' \to \gamma \gamma$ decays. We show that the only effect of the $U_A(1)$ anomaly in these processes is in the formation of the $\eta$ and $\eta'$ systems. From experimental data on these decays we extract the pseudoscalar mixing angle in flavor basis as $\phi_P \in [38.4^\circ, 41.0^\circ]$.

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I. INTRODUCTION.

The understanding of the mechanisms leading to the mixing of pseudoscalar mesons is an important task in hadronic physics and this topic has been actively investigated during the last years [1]. From the OZI rule perspective this mixing is unusually large, in contrast with e.g. vector mesons which are close to the ideal mixing composition dictated by the OZI rule. Mixing of pseudoscalar mesons has been traditionally described in the Gell-Mann basis for $SU(3)$. In this framework, octet axial currents are conserved in the massless quark limit

$$\partial^\mu A^a_\mu \equiv \partial^\mu \bar{q} \frac{\lambda^a}{2} \gamma_\mu \gamma_5 q = 0, \quad a = 1, ..., 8,$$

whereas the singlet axial current has a non-vanishing divergence due to the gluonic ABJ triangular anomaly

$$\partial^\mu A^0_\mu = \frac{1}{\sqrt{6}} \frac{n_f \alpha_s}{4\pi} C^a_{\alpha \beta} \tilde{G}^{a \alpha \beta}$$

where $\tilde{G}^{a \alpha \beta} \equiv \frac{1}{2} \varepsilon^{\alpha \beta \mu \nu} G^a_{\mu \nu}$. This has far-reaching consequences for hadronic systems, in particular to those sharing the quantum numbers of the operators on the left of Eq.(2). These relations must be modified in the presence of external vector fields to account for
additional contributions coming from the coupling of fermions to these fields. The typical example of modifications to these relations is the case of massless QCD coupled to external electromagnetic fields. In this case, e.g. the divergence of $A_\mu^3$ gets a contribution from the ABJ photon anomaly

$$\partial^\mu A_\mu^3 = \frac{\alpha}{4\pi} F_{\alpha\beta} \tilde{F}^{\alpha\beta}$$

where $F_{\alpha\beta}$ stands for the electromagnetic strength field tensor. This result, and the characterization of the matrix element of $A_\mu^3$ between the vacuum and two photons on the basis of Lorentz covariance, gauge invariance, parity etc. lead to the existence of a singularity at $q^2 = 0$ in this matrix element [2]. In the confining theory, is meaningless to speak about the quarks as physical degrees of freedom and the only explanation for this singularity invokes 't Hooft’s consistency condition [3], i.e. that singular contributions in $\langle 0 \mid A_\mu^3 \mid \gamma\gamma \rangle$ at the level of quarks and at the level of hadrons must match. This leads to a massless hadronic excitation (the pion) which couples to $A_\mu^3$ and to two photons. This mechanism successfully describe the $\pi^0 \rightarrow \gamma\gamma$ decay which in the absence of the anomaly would be forbidden (actually of a lower power in $m_\pi$ [4]) which is inconsistent with experimental results.

In principle, one could try a similar calculation for the $\eta \rightarrow \gamma\gamma$ and $\eta' \rightarrow \gamma\gamma$ decays. However, here we encounter the problem of how to quantify the effects of the strong contribution to the $U_A(1)$ anomaly in Eq.(2) at the hadron level and how these effects influence the mixing of pseudoscalar isoscalar fields.

In the conventional singlet-octet basis, the octet current gets no contributions from the gluon anomaly and a similar procedure to the case of the pion can be used to estimate the decay amplitude for $\eta_s \rightarrow \gamma\gamma$, although the extrapolation from $q^2 = 0 \rightarrow m_{\eta_s}^2$ is more severe in this case. Relating this amplitude to the $\eta \rightarrow \gamma\gamma$ decay in principle requires a careful analysis of the mechanisms for mixing of pseudoscalars which nevertheless, in this case, seems to be quantitatively not so relevant due to the experimental fact that the naive mixing angle is small in this basis. In the case of the singlet we run in trouble due to the strong contribution to the singlet axial anomaly. Relating the physical amplitudes to the singlet-octet ones is more problematic in this case and definitively requires to clarify the role of the axial anomaly in the mixing of pseudoscalars (or stated differently, to quantify the gluon content of pseudoscalar mesons).

The mixing of the pseudoscalar isoscalar fields can also be formulated in flavor basis $\{\eta_{ns}, \eta_s\}$. The most general form of the mass Lagrangian is

$$L_{mass} = -\frac{1}{2}(m_{\eta_{ns}}^2 \eta_{ns}^2 + m_{\eta_s}^2 \eta_s^2 + 2m_{s-ns}^2 \eta_s \eta_{ns}).$$

where the last term account for the OZI rule violating $\eta_{ns} - \eta_s$ transitions. The precise origin of this term is still unclear. The standard chiral expansion uses the octet and singlet fields, hence, whatever the mechanism for mixing of flavor fields be, it is considered from the very start in this formalism.

In the model-independent formulation of mixing in Eq.(4), we can relate the mixing angle to the non-diagonal term $m_{s-ns}^2$. The extraction of the pseudoscalar mixing angle following this procedure requires the precise quantification of all the mechanisms contributing to meson masses which is a difficult task. In [3,4] it was shown that the $U_A(1)$ anomaly gives a
sizeable contribution to $m_{\pi}^2 - m_{\rho}^2$ via its coupling to the spontaneous breaking of chiral symmetry. Assuming that this is the only mechanism for mixing of flavor fields, a pseudoscalar mixing angle is obtained consistent with Gell-Mann’s $SU(3)$ symmetry when pseudoscalar meson masses are used as input to fix the values of the free parameters of the model. Although interesting from the conceptual point of view, this mechanism does not account for the experimentally measured mixing angle which is close to, but definitely different from the $SU(3)$ value. The precise description of the mixing angle calls for considering further mechanisms among which Lipkin’s loops cancellation [6] looks appealing as pointed out in [6]. Another possibility for the extraction of this angle is to consider processes where this angle be involved but sensitivity to pseudoscalar meson masses be reduced.

Photonic decays of pseudoscalar mesons are appropriate to this end since, as we shall see below, in this case strong contributions to the $U_A(1)$ anomaly can play a role in the conformation of physical pseudoscalar mesons only and effects due meson masses on the corresponding widths are softened.

In this work we relate breaking of axial symmetry in the isoscalar channels to the pseudoscalar mass matrix in Eq. (4), within a Linear Sigma Model which incorporates $U_A(1)$ symmetry breaking. Contributions of external electromagnetic fields to the breaking of axial symmetry in the isoscalar channels are introduced using ’t Hooft’s anomaly matching condition. We calculate $P^0 \rightarrow \gamma \gamma$ decays in this framework and extract the pseudoscalar mixing angle from experimental data on these processes.

In order to state notation we briefly review the model in the next section and work out its predictions for the isoscalar weak decay constants and the anomalies in the isoscalar axial currents. In section III we introduce effects of external electromagnetic fields and work out the predictions of this formalism for the $P^0 \rightarrow \gamma \gamma$ decays. In section IV we give our conclusions.

II. AXIAL ANOMALIES AND THE MASS MATRIX OF ISOSCALAR PSEUDOSCALAR FIELDS.

A. The model.

The chirally symmetric $[U(3)_L \otimes U(3)_R]$ meson Lagrangian [5,8–10], describes a scalar and a pseudoscalar nonet, in turn denoted by $(\sigma_i)$ and $(P_i)$,

$$\mathcal{L} = \mathcal{L}_{sym} + \mathcal{L}_{U_A(1)} + \mathcal{L}_{SB} \, .$$

Here,

$$\mathcal{L}_{sym} = \frac{1}{2} \text{tr} \left[ (\partial_\mu M)(\partial^\mu M^\dagger) \right] - \frac{\mu^2}{2} X(\sigma, P) - \frac{\lambda}{2} Y(\sigma, P) - \frac{\lambda'}{4} X^2(\sigma, P) \, ,$$

$$M = \sigma + iP, \text{ and } X, Y \text{ stand in turn for the left-right symmetric traces}$$

$$X(\sigma, P) = \text{tr} \left[ MM^\dagger \right], \quad Y(\sigma, P) = \text{tr} \left[ (MM^\dagger)^2 \right]$$
The pseudoscalar and scalar matrix fields $P$ and $\sigma$ are written in terms of a specific basis spanned by seven of the standard Gell-Mann matrices, namely $\lambda_i$ ($i = 1, \ldots, 7$), and by two non-standard matrices $\lambda_{ns} = \text{diag}(1,1,0)$, and $\lambda_s = \sqrt{2} \text{ diag}(0,0,1)$, respectively. The decomposition obtained in this way reads $P \equiv \frac{1}{\sqrt{2}} \lambda_i P_i$ with $i = ns, s, 1, \ldots, 7$ and similarly for the scalar field. The instanton-induced interaction in (5) is

$$L_{UA(1)} = -\beta \left( \det(M) + \det(M^\dagger) \right).$$

(8)

It stands for the bosonization of 't Hooft's effective quark-quark interaction which has a determinant structure in flavor space \cite{11,12}. Finally, there is the standard quark mass term

$$L_{SB} = \text{tr} [c\sigma] = \text{tr} \left[ \frac{b_0}{\sqrt{2}} \mathcal{M}_q(M + M^\dagger) \right]$$

(9)

which breaks the left-right symmetry explicitly. The $c$ matrix is spanned by the same basis $c \equiv \frac{1}{\sqrt{2}} \lambda_i c_i$, where the nine expansion coefficients $c_i$ are independent constants. It is related to the quark mass matrix by $c = \sqrt{2} b_0 \mathcal{M}_q$ and has $\frac{c_{ns}}{\sqrt{2}} = \sqrt{2} m_{ns} b_0$ and $c_s = \sqrt{2} m_s b_0$ as the only non-vanishing entries. Here, $b_0$ is an unknown parameter with dimensions of squared mass. We work in the exact isospin limit, $\hat{m} = m_u = m_d$ in the following. The linear $\sigma$ term in Eq. (9) induces $\sigma$-vacuum transitions which supply the scalar fields with non-zero vacuum expectation values (v.e.v) (hereafter denoted by $\langle \cdots \rangle$). To simplify notations, let us re-denote $\langle \sigma \rangle$ by $V$ with $V = \text{diag} (a, a, b)$, where $a$ and $b$ in turn denote the vacuum expectation values of the strange and non-strange quarkonium, respectively,

$$a = \frac{1}{\sqrt{2}} \langle \sigma_{ns} \rangle, \quad b = \langle \sigma_s \rangle.$$  

(10)

We now shift, as usual, the old $\sigma$ field to a new scalar field $S = \sigma - V$ such that $\langle S \rangle = 0$. In this way, new mass terms, three-meson interactions, and a linear term are generated. All these terms are affected – via the 't Hooft determinant by the $U_A(1)$ anomaly which get coupled to the v.e.v’s of the scalar fields by the spontaneous breaking of chiral symmetry. The consequence of all these effects is the breaking of the original symmetry down to $SU(2)_I$ isospin. The masses of the seven unmixed pseudoscalar corresponding to the original Gell-Mann matrices $\lambda_i$ ($i = 1, \ldots, 7$), namely the isovector pseudoscalar ($\pi$) mesons as well as the two isodoublets of pseudoscalar ($K$) mesons, are obtained as

$$m_{\pi}^2 = \xi + 2\beta b + \lambda a^2, \quad m_{K}^2 = \xi + 2\beta a + \lambda(a^2 - ab + b^2);$$

(11)

where we used the convenient short-hand notation $\xi \equiv \mu^2 + \lambda'(2a^2 + b^2)$. The elimination of the linear terms imposes the following constraints on the explicit-symmetry-breaking terms $c_{ns}$, and $c_s$:

$$c_{ns} = \sqrt{2} a m_{\pi}^2, \quad c_s + \frac{c_{ns}}{\sqrt{2}} = (a + b)m_{K}^2.$$  

(12)

or in term of the quark masses

$$a m_{\pi}^2 = \sqrt{2} \hat{m} b_0, \quad (a + b)m_{K}^2 = \sqrt{2}(\hat{m} + m_{s}) b_0.$$  

(13)
In Ref. \cite{8} the PCAC relations for the pion and kaon field are discussed. These relations yield
\[ f_\pi = \sqrt{2} a, \quad f_K = \frac{1}{\sqrt{2}} (a + b), \] (14)
which when used in (13) yields
\[ f_\pi m_\pi^2 = 2 \hat{m} b_0, \quad f_K m_K^2 = (\hat{m} + m_s) b_0. \] (15)

The mass term of the Lagrangian involving the mixed isoscalar pseudoscalar fields, which correspond to the \( \lambda_{ns} \) and \( \lambda_s \) matrices, has the structure of the mass Lagrangian in Eq.(4) with the specific values
\[ m_{\eta_{ns}}^2 = \xi - 2 \beta b + \lambda a^2, \quad m_{\eta_s}^2 = \xi + \lambda b^2, \quad m_{s-ns}^2 = -2 \sqrt{2} \beta a \] (16)
Here, \( m_{\eta_s} \) and \( m_{\eta_{ns}} \) are the masses of the strange and non-strange pseudoscalar quarkonia respectively, while \( m_{s-ns}^2 \) denotes the transition mass-matrix elements of the strange–non-strange pseudoscalar quarkonia, which in this model is due to the interplay between ’t Hooft interaction and the spontaneous breakdown of chiral symmetry.

This mass Lagrangian can be diagonalized by rotating to the physical basis
\[ \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = R(\phi_P) \begin{pmatrix} \eta_{ns} \\ \eta_s \end{pmatrix} \quad \text{with} \quad R(\phi_P) = \begin{pmatrix} \cos \phi_P & -\sin \phi_P \\ \sin \phi_P & \cos \phi_P \end{pmatrix}, \] (17)
such that the diagonal physical mass matrix \( M_D \equiv \text{Diag} \left( m_{\eta_s}^2, m_{\eta_{ns}}^2 \right) \) is related to the same matrix in the flavor fields representation \( M_F \) as
\[ M_D = R(\phi_P) M_F R^\dagger(\phi_P) \quad \text{where} \quad M_F = \begin{pmatrix} m_{\eta_{ns}}^2, & m_{s-ns}^2 \\ m_{s-ns}^2, & m_{\eta_s}^2 \end{pmatrix}. \] (18)

**B. Flavor weak decay constants**

The flavor weak decay constants \( f_{ns}, f_s \) are defined by:
\[ \langle 0 | A^{ns}_\mu(0) | \eta^{ns}(q) \rangle = i f_{ns} q_\mu, \quad \langle 0 | A^{s}_\mu(0) | \eta^s(q) \rangle = i f_s q_\mu. \] (19)

In the literature we also find weak decay constants related to the following matrix elements \cite{13}
\[ \langle 0 | A^{ns}_\mu(0) | \eta(q) \rangle = i f_{\eta}^{ns} q_\mu, \quad \langle 0 | A^{ns}_\mu(0) | \eta'(q) \rangle = i f_{\eta'}^{ns} q_\mu, \]
\[ \langle 0 | A^{s}_\mu(0) | \eta(q) \rangle = i f_{\eta}^s q_\mu, \quad \langle 0 | A^{s}_\mu(0) | \eta'(q) \rangle = i f_{\eta'}^s q_\mu. \] (20)

Let us analyze the predictions of the model for the divergences of the isoscalar currents and their implications for the weak decay constants. Under the axial transformations
\[ \delta M = - \frac{i}{\sqrt{2}} \{ \varepsilon, M \}, \quad \delta M^\dagger = \frac{i}{\sqrt{2}} \{ \varepsilon, M^\dagger \}, \] (21)
the Lagrangian in Eq.(3) is no longer invariant due to the breaking terms. A calculation of the divergences of the strange and non-strange axial currents in the model yields:

$$\partial_{\mu} A_{\mu}^{ns} = c_{ns} \eta_{ns} + 2\beta W, \quad \partial_{\mu} A_{\mu} = \sqrt{2} c_{s} \eta_{s} + \sqrt{2} \beta W$$

where \(W\) stands for the contribution coming from \(t\)’Hooft interaction and contains trilinear, bilinear and linear terms in the fields. Explicitly

$$W = i(det(M) - detM^\dagger) = -2\sqrt{2}ab\eta_{ns} - 2a^2\eta_{s} + \text{bilinear} + \text{tril.}$$

The bilinear and trilinear terms in Eq.(23) give vanishing contributions at tree level to the quantities to be considered here, hence they will be dropped in the following. Inserting (22, 23) in Eqs.(19) we obtain

$$f_{ns} m_{ns}^2 \eta_{ns} = c_{ns} - 4\sqrt{2}\beta ab, \quad f_{s} m_{s}^2 = \sqrt{2} c_{s} - 2\sqrt{2}\beta a^2.$$  

Also from Eqs(11,12,16) we obtain the following relations

$$m_{\eta_{ns}}^2 - m_{\eta_{s}}^2 = -4\beta b \quad bm_{\eta_{ns}}^2 + 2\beta a^2 = c_{s},$$

which when inserted in Eq.(24) predict [6]

$$f_{ns} = \sqrt{2}a = f_{\pi} \quad f_{s} = \sqrt{2}b = 2f_{K} - f_{\pi}.$$  

On the other hand, inserting Eqs.(23,24) in Eqs.(22) we obtain

$$\partial_{\mu} A_{\mu}^{ns} = f_{ns} m_{\eta_{ns}}^2 \eta_{ns} - 4\beta a^2 \eta_{s}, \quad \partial_{\mu} A_{\mu}^{s} = f_{s} m_{\eta_{s}}^2 \eta_{s} - 4\beta ab \eta_{ns}. $$

The last terms in the r.h.s of the previous two Eqs. are entirely due to the coupling of the \(U_A(1)\) anomaly to the v.e.v.’s of scalars. They are a manifestation, at the hadron level, of the gluon ABJ anomaly in QCD which we assume here as dominated by instantons which generate \(t\)’Hooft interaction. Eqs.(27) can be rewritten in a symmetric form

$$\frac{1}{f_{ns}} \partial_{\mu} A_{\mu}^{ns} = m_{\eta_{ns}}^2 \eta_{ns} + m_{\eta_{s}}^2 \eta_{s}, \quad \frac{1}{f_{s}} \partial_{\mu} A_{\mu}^{s} = m_{\eta_{s}}^2 \eta_{s} + m_{\eta_{ns}}^2 \eta_{ns}.$$  

which makes explicit the relation between the mass matrix of isoscalar pseudoscalars and the divergence of isosinglet axial currents. It is interesting to make also explicit terms driven by the quark masses and those induced by the \(U_A(1)\) anomaly which are not related to quark masses. Using relations (12,25), Eqs.(27) can also be rewritten as

$$\partial_{\mu} A_{\mu}^{ns} = (2b_0 \hat{m} - 2\sqrt{2}\beta f_{ns} f_{s}) \eta_{ns} - 2\beta f_{ns}^2 \eta_{s}, \quad \partial_{\mu} A_{\mu}^{s} = (2b_0 m_{s} - \sqrt{2}\beta f_{ns}^2) \eta_{s} - 2\beta f_{ns} f_{s} \eta_{ns}.$$  

which makes transparent that in the absence of the \(U_A(1)\) symmetry breaking \((\beta = 0)\) the divergences of flavor currents are driven by quark masses and flavor isoscalar pseudoscalar fields are pseudo-Goldstone bosons. Eq.(28) is the fundamental relation which we will exploit below in the description of the two photon decay of isoscalar pseudoscalar mesons. Before this, let us make two remarks on the consequences of Eq.(28). The first one concerns the
weak decay constants. Inserting (28) in the divergence of Eq.(20) and using Eq.(17) we obtain

\[ f_{\eta m}^2 = f_{\eta s} \left( m_{ns}^2 \cos \phi_P - m_{s-s}^2 \sin \phi_P \right), \]
\[ f_{\eta s}^2 = f_{\eta s} \left( m_{s-s}^2 \cos \phi_P - m_{s-s}^2 \sin \phi_P \right), \]
\[ f_{\eta m'}^2 = f_{\eta m} \left( m_{s-s}^2 \cos \phi_P + m_{s-s}^2 \sin \phi_P \right), \]
\[ f_{\eta s'}^2 = f_{\eta s} \left( m_{s-s}^2 \sin \phi_P + m_{s-s}^2 \cos \phi_P \right). \]

(30)

On the other hand, from Eqs. (18) we get

\[ m_{\eta m}^2 \cos \phi_P = m_{ns}^2 \cos \phi_P - m_{s-s}^2 \sin \phi_P, \]
\[ -m_{\eta s}^2 \sin \phi_P = m_{s-s}^2 \cos \phi_P - m_{s-s}^2 \sin \phi_P, \]
\[ m_{\eta s'}^2 \sin \phi_P = m_{s-s}^2 \sin \phi_P + m_{s-s}^2 \cos \phi_P, \]
\[ m_{\eta m'}^2 \cos \phi_P = m_{s-s}^2 \sin \phi_P + m_{s-s}^2 \cos \phi_P. \]

(31)

which when inserted in Eq.(30) yields

\[ \begin{pmatrix} f_{\eta m} & f_{\eta s} \\ f_{\eta m'} & f_{\eta s'} \end{pmatrix} = \begin{pmatrix} \cos \phi_P - \sin \phi_P \\ \sin \phi_P \cos \phi_P \end{pmatrix} \begin{pmatrix} f_{\eta s} & 0 \\ 0 & f_{\eta s} \end{pmatrix}. \]

(32)

This relation was postulated in [14] and taken as the basic assumption in the analysis of pseudoscalar mixing and weak decay constants.

The second remark concerns the structure of the mass matrix and the contributions coming from quark mass terms. From Eqs.(29) it is possible to write the mass matrix in the flavor basis as

\[ M_F = \begin{pmatrix} m_{qq}^2 + 2\alpha^2 & \sqrt{2}\alpha^2 y \\ \sqrt{2}\alpha^2 y & m_{ss}^2 + \alpha^2 y^2 \end{pmatrix} \]

(33)

where

\[ m_{qq}^2 = \frac{2\tilde{m}}{f_{\eta s}} b_0, \quad m_{ss}^2 = \frac{2m_s}{f_{\eta s}} b_0, \quad \alpha^2 = -2\beta b, \quad y = \frac{f_{\eta s}}{f_{\eta s}} = \frac{a}{b}. \]

(34)

It was shown in [14] that this result can be derived directly from QCD if we assume Eq.(32). In terms of QCD quantities we identify

\[ b_0 = \langle 0 | \bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d | \eta_{ns} \rangle, \quad \alpha^2 = \frac{1}{\sqrt{2}f_{\eta s}} \langle 0 | \frac{\alpha_s}{4\pi} G G | \eta_{ns} \rangle. \]

(35)

III. ELECTROMAGNETIC CONTRIBUTIONS TO $U_A(1)$-SYMMETRY BREAKING AND $P^0 \to \gamma\gamma$ DECAYS.

In the presence of external electromagnetic fields, relations(28) must be modified to account for the ABJ terms due to the photons. This modification can be obtained using ’t Hooft’s argument on the matching of the anomalies [3] to translate exactly the same form of
the anomaly as calculated at the level of the fundamental theory (QCD) to the composite theory. For purposes of comparison below we include also the corresponding relation for pions

\[
\partial^\mu A^\mu_\lambda = f_\pi m_\pi^2 \alpha^0 + \frac{\alpha}{4\pi} N_c D^3 \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma},
\]

\[
\partial^\mu A^{\mu}_{qns} = f_{ns} \left( m_{ns}^2 \eta_{ns} + m_{s-ns}^2 \eta_{ns} \right) + \frac{\alpha}{4\pi} N_c D^{ns} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma},
\]

\[
\partial^\mu A^{\mu}_{s} = f_{s} \left( m_{s}^2 \eta_{s} + m_{s-ns}^2 \eta_{ns} \right) + \frac{\alpha}{4\pi} N_c D^{s} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}.
\]

Here, \( D^x = tr\{\{Q, Q\}_X^x\} \) with \( X = 3, ns, s \) and \( q \) stands the quark charge matrix. The last two relations can be written in terms of physical pseudoscalar fields and in compact matrix form read

\[
\left( \frac{1}{f_{P}} \partial A^{\nu}_{ns} \right) = R^\dagger(\phi_P) M_D \left( \eta_{\eta'} \right) + \left( \frac{1}{f_{s}} D^{ns} \right) \xi
\]

(37)

where \( \xi \equiv \frac{\alpha}{4\pi} N_c \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \), \( M_D \) denotes the diagonal mass matrix which is related to the same matrix in the flavor basis by Eq.(18).

The three-point function for an axial and two electromagnetic currents

\[
T^{\alpha}_{\mu\nu\lambda}(k_1, k_2, q) \equiv i \int d^4x d^4y d^4z e^{i(k_1 x + k_2 y - q z)} \langle 0 | T j_\mu(x) j_\nu(y) A^{\alpha}_\lambda(z) | 0 \rangle,
\]

(38)

satisfy

\[
q^{\lambda} T^{\alpha}_{\mu\nu\lambda}(k_1, k_2, q) = \int d^4x d^4y d^4z e^{i(k_1 x + k_2 y - q z)} \langle 0 | T j_\mu(x) j_\nu(y) \partial^\alpha_\lambda A^{\alpha}_\lambda(z) | 0 \rangle.
\]

(39)

Using relations (37) we obtain

\[
q^{\lambda} T^{ns}_{\mu\nu\lambda}(k_1, k_2, q) = f_{ns} \left( \frac{m_{s}^2 \cos \phi_P}{q^2 + m_{s}^2} \Gamma^\eta_{\mu\nu}(k_1, k_2) + \frac{m_{s}^2 \sin \phi_P}{q^2 + m_{s}^2} \Gamma^\eta_{\mu\nu}(k_1, k_2) \right) - D^{ns} \xi_{\mu\nu}(k_1, k_2).
\]

\[
q^{\lambda} T^{s}_{\mu\nu\lambda}(k_1, k_2, q) = f_{s} \left( \frac{-m_{s}^2 \sin \phi_P}{q^2 + m_{s}^2} \Gamma^\eta_{\mu\nu}(k_1, k_2) + \frac{m_{s}^2 \cos \phi_P}{q^2 + m_{s}^2} \Gamma^\eta_{\mu\nu}(k_1, k_2) \right) - D^{s} \xi_{\mu\nu}(k_1, k_2).
\]

(40)

where

\[
\Gamma^P_{\mu\nu}(k_1, k_2) \equiv i \int d^4x d^4y d^4z e^{i(k_1 x + k_2 y - q z)} \langle 0 | T j_\mu(x) j_\nu(y) P(z) | 0 \rangle,
\]

\[
\xi_{\mu\nu}(k_1, k_2) \equiv i \int d^4x d^4y d^4z e^{i(k_1 x + k_2 y - q z)} \langle 0 | T j_\mu(x) j_\nu(y) \xi(z) | 0 \rangle.
\]

(41)

In the limit \( q \to 0 \) the l.h.s in Eqs.(39) vanishes yielding the following relations

\[
\begin{pmatrix}
    f_{ns} \cos \phi_P & f_{ns} \sin \phi_P \\
    -f_{s} \sin \phi_P & f_{s} \cos \phi_P
\end{pmatrix}
\begin{pmatrix}
    \Gamma^\eta_{\mu\nu}(k_1, k_2) \\
    \Gamma^\eta_{\mu\nu}(k_1, k_2)
\end{pmatrix}
= \begin{pmatrix}
    D^{ns} \xi_{\mu\nu}(k_1, k_2) \\
    D^{s} \xi_{\mu\nu}(k_1, k_2)
\end{pmatrix}
\]

(42)

Using these results we obtain the invariant matrix element for neutral pseudoscalars decaying into two photons
$$M(P^0(q, \eta) \rightarrow \gamma(k_1, \epsilon_1) \gamma(k_2, \epsilon_2)) = M(P^0 \rightarrow \gamma \gamma)\varepsilon(\epsilon_1, k_1, \epsilon_2, k_2).$$

(43)

where

$$M(\pi^0 \rightarrow \gamma \gamma) = \frac{\alpha}{\pi f_\pi},$$

$$M(\eta \rightarrow \gamma \gamma) = \frac{\alpha}{3\pi} \left( \frac{5}{f_{\text{ms}}} \cos \phi_P - \sqrt{2} \frac{y_f}{f_s} \sin \phi_P \right),$$

$$M(\eta' \rightarrow \gamma \gamma) = \frac{\alpha}{3\pi} \left( \frac{5}{f_{\text{ms}}} \sin \phi_P + \sqrt{2} \frac{y_f}{f_s} \cos \phi_P \right).$$

(44)

It is worth remarking that the only effect of the $U_A(1)$ anomaly in these processes concerns the formation of the physical states $\eta$ and $\eta'$ from the flavor states $\eta^{\text{as}}, \eta^{\text{s}}$. This is reflected in the $\sin \phi_P$ and $\cos \phi_P$ factors appearing in Eqs. (44). Additional effects of 't Hooft interaction can be expected in the formation of final states. These effects can be important in the case of hadronic final states. For photonic final states the $\bar{q}q - \gamma \gamma$ instanton induced interaction is highly suppressed. We must also be clear that the results in Eqs. (44) are valid in the soft limit $q \rightarrow 0$. Extrapolation to the physical $q^2 = m_\eta^2$ and $q^2 = m_{\eta'}^2$ are necessary to compare with existing experimental data. Finally, notice that in the soft limit all the information on pseudoscalar meson masses cancels out. Thus the extraction of the mixing angle using photonic decays of pseudoscalar mesons is free of the uncertainties attributed to mechanisms for generation of meson masses at this point.

From Eqs. (44) we obtain the following fractions

$$\frac{M(\eta \rightarrow \gamma \gamma)}{M(\pi^0 \rightarrow \gamma \gamma)} = \frac{1}{3} \left( 5 \cos \phi_P - \sqrt{2} \frac{y_f}{f_s} \sin \phi_P \right),$$

$$\frac{M(\eta' \rightarrow \gamma \gamma)}{M(\pi^0 \rightarrow \gamma \gamma)} = \frac{1}{3} \left( 5 \sin \phi_P + \sqrt{2} \frac{y_f}{f_s} \cos \phi_P \right).$$

(45)

These results are usually written in terms of the mixing angle in the singlet-octet basis $\theta_P = \phi_P - \phi_{\text{id}}$ where $\phi_{\text{id}} = 54.7^\circ$ stands for the ideal mixing angle: $\cos \phi_{\text{id}} = \sqrt{1/3}$, $\sin \phi_{\text{id}} = \sqrt{2/3}$. In terms of this angle our results read

$$\frac{M(\eta \rightarrow \gamma \gamma)}{M(\pi^0 \rightarrow \gamma \gamma)} = \frac{1}{\sqrt{3}} \left( \frac{5 - 2y}{3} \cos \theta_P - \frac{5 + y \sqrt{2}}{3} \sin \theta_P \right),$$

$$\frac{M(\eta' \rightarrow \gamma \gamma)}{M(\pi^0 \rightarrow \gamma \gamma)} = \frac{1}{3} \sqrt{2} \left( \frac{5 + y}{3} \left( \cos \theta_P + \frac{5 - 2y}{\sqrt{2}(5 + y)} \sin \theta_P \right) \right).$$

(46)

Notice that in the case $f_{\text{ms}} = f_s$ we recover results from $SU(3)$ symmetry or quark model considerations [13].

The reported data [13] for these fractions is

$$\frac{M(\eta \rightarrow \gamma \gamma)}{M(\pi^0 \rightarrow \gamma \gamma)} = \frac{1.73 \pm 0.18}{\sqrt{3}},$$

$$\frac{M(\eta' \rightarrow \gamma \gamma)}{M(\pi^0 \rightarrow \gamma \gamma)} = 2 \sqrt{\frac{2}{3}} (0.78 \pm 0.04).$$

(47)

Experimental data for $\eta \rightarrow \gamma \gamma$ decay constrains the mixing angle to the range $\phi_P \in [38.4^\circ, 47.2^\circ]$ whereas data on $\eta' \rightarrow \gamma \gamma$ restrains the same angle to the interval $[34.3^\circ, 41.0^\circ]$. In the whole, both decays yield
This result is consistent with the averaged value obtained in [13-14]. Indeed, our analytical results in Eq.(44) agree with the corresponding amplitudes in Eqs.(3.13) of [14]. Although both approaches share the virtue of using the flavor basis instead of the widely used singlet-octet basis, there are important differences in the way this result has been derived. In [14] relations (32) and the generalized PCAC relations (see [10], Eq.(4.2))

$$\partial^{\mu} A_{\mu}^{\text{ns}} = f^{\text{ns}} \eta^2 \eta + f^{\text{ns}} \eta^2 \eta'$$
$$\partial^{\mu} A_{\mu}^{\text{s}} = f^{\text{s}} \eta^2 \eta + f^{\text{s}} \eta^2 \eta'$$

have been assumed and used to calculate the two photon decay widths of neutral pseudoscalars. In the present work, we start with a chiral Lagrangian which incorporates $U_A(1)$ symmetry breaking in a way inspired by instanton calculations. As a result we obtain a dynamical mechanism for the (OZI rule violating) mixing of pseudoscalar strange and non-strange quarkonia, namely, the coupling of the $U_A(1)$ symmetry breaking to the v.e.v’s of scalars due to the spontaneous breakdown of chiral symmetry. In this framework we are able to calculate modifications to the naive PCAC relations due to the $U_A(1)$ anomaly. In general, these modifications involve linear, bilinear and trilinear combinations of meson fields. However, only linear terms contribute to two photon decay of pseudoscalars at tree level and for this particular case we derive relations (32) and the modified PCAC relations (50) (first term in the r.h.s. of Eq.(37)) which are the starting point in [14].

IV. CONCLUSIONS

We use a $U(3) \times U(3)$ effective chiral model incorporating $U_A(1)$ symmetry breaking to study the strong and electromagnetic contributions to the non-conservation of the isosinglet axial currents in flavor basis. Strong contributions are explicitly calculated in the model and related to the mass matrix of isoscalar pseudoscalar fields and the corresponding weak decay constants. Electromagnetic contributions are introduced using 't Hooft’s argument on anomaly matching. We calculate the $\rho^0 \to \gamma \gamma$ decays in this framework and show that the only effect of the strong anomaly in these processes is in the formation of the $\eta$ and $\eta'$ systems. We use these results to estimate the pseudoscalar mixing angle from experimental data on $\eta \to \gamma \gamma$ and $\eta' \to \gamma \gamma$ obtaining $\phi_P \in [38.4^\circ, 41.0^\circ]$
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