Perturbative Expansion around the Gaussian Effective Action:
The Background Field Method

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Abstract

We develop a systematic method of the perturbative expansion around the Gaussian effective action based on the background field method. We show, by applying the method to the quantum mechanical anharmonic oscillator problem, that even the first non-trivial correction terms greatly improve the Gaussian approximation.

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I. INTRODUCTION

The variational Gaussian approximation method has provided a convenient and practical device in obtaining non-perturbative information from various quantum field theories, and has played an important role in enlarging our knowledge on the non-perturbative nature of the theories [1]. The main drawback of this method, however, is that it does not provide a systematic procedure to compute the correction terms to the approximation. One natural way to improve the Gaussian approximation is to use more general non-Gaussian trial wave functional for the variational approximation [2]. Although this approach can easily be applied to the case of quantum mechanics, it has not been very successful in the case of quantum field theory.

The efforts to establish the systematic approximation method based on the variational techniques have been made by authors of ref. [3] and the so-called optimized expansion method and variational perturbation theory have been formulated. It is based on the fact that the Gaussian approximation is to approximate an interacting theory by a free field theory with a new mass, that is determined by variational gap equation. One constructs a complete set of energy eigenstates for this free theory and use it as a basis for perturbative expansion of the interacting theory. Cea and Tedesco [3] used this variational perturbation theory to evaluate the second order correction terms to the Gaussian effective potential for the (1+1) and (2+1) dimensional $\phi^4$ theories. This perturbation theory is conceptually simple and provides a very convenient systematic procedure to evaluate the correction terms to the Gaussian effective potentials for time-independent systems. For time-dependent systems such as those treated by Eboli, Jackiw and Pi [1], however, one needs a method that can deal with more general effective action.

It is the purpose of this paper to develop a systematic method of computing correction terms to the Gaussian effective action based on the background field method (BFM) [4]. The background field method provides a very effective and simple way of computing effective action in the loop-expansion. The reason why the procedure of computing effective action
in BFM becomes simplified is that one has rearranged the summation of diagrams in such a way that the propagator used in the Feynman rule already contains the part of the 1-loop interaction effect. We proceed one step further so that the zeroth order term in the expansion would give the Gaussian approximation, and the propagator used in the expansion would be the one obtained in the Gaussian approximation. This provides a very convenient and simple method in computing order by order correction terms to the Gaussian approximation of the effective action.

In the next section we give a brief review of BFM and develop the perturbative expansion method around the Gaussian effective action by using scalar $\phi^4$ theory. In section II, we illustrate the effectiveness of the method by evaluating the first non-trivial correction terms to the Gaussian effective potential of the $(0+1)$-dimensional $\phi^4$ theory (anharmonic oscillator) and compare the result with those of various other methods. We conclude with some discussions in the last section.

II. PERTURBATIVE EXPANSION AROUND THE GAUSSIAN EFFECTIVE ACTION

For a system described by the classical action,

$$S[\phi] = \int d^4x \mathcal{L}[\phi(x), \partial_\mu \phi(x)],$$

the generating functional for Green’s functions is defined by the path integral,

$$< 0|0 >^J \equiv Z[J] \equiv e^{iW[J]} \equiv \int \mathcal{D}\phi e^{iS[\phi]+iJ\phi},$$

where $J(x)$ is the external source, $W[J]$ is the generating functional for connected Green’s functions, and the integral convention, $J\phi \equiv \int d^n x J(x) \phi(x)$ is used in the exponent. The Green’s functions can be obtained by

$$< 0_+ | T[\phi(x_1) \cdots \phi(x_n)] | 0_- >^J = (\frac{\delta}{i \delta J(x_1)}) \cdots (\frac{\delta}{i \delta J(x_n)}) e^{iW[J]}.$$

3
The vacuum expectation value of the field operator in the presence of external source is defined by

\[ \varphi \equiv \langle \phi(x) \rangle' = \frac{\delta}{\delta J(x)} W[J]. \] (4)

The effective action is then defined by the Legendre transformation;

\[ \Gamma[\varphi] \equiv W[J] - \varphi J. \] (5)

Taking the derivative of \( \Gamma[\varphi] \) with respect to the external source \( J \) gives

\[ \frac{\delta}{\delta \varphi} \Gamma[\varphi] = -J, \] (6)

which is of the same form as the classical equation of motion.

We now introduce a new action \( S[\phi + B] \) which is obtained by shifting the field \( \phi \) by a background field \( B \). This new action gives a new generating functional for Green’s functions in the presence of the background field \( B \),

\[ \tilde{Z}[J, B] \equiv e^{i\tilde{W}[J, B]} \]
\[ = \int \mathcal{D}\phi e^{iS[\phi + B] + iJ\phi}, \] (7)

and the vacuum expectation value of field operator is now defined by

\[ \tilde{\varphi} \equiv \frac{\delta}{\delta J} \tilde{W}[J, B] = \frac{\delta}{i\delta J} \log(\tilde{Z}[J, B]). \] (8)

Since Eq. (7) can be written as

\[ \tilde{Z}[J, B] = \int \mathcal{D}\phi e^{iS[\phi] + iJ(\phi - B)} = Z[J]e^{-iJB}, \] (9)

the new generating functional \( \tilde{W}[J, B] \) for connected Green’s functions in the presence of the background field is related to the old one by,

\[ \tilde{W}[J, B] = W[J] - JB. \] (10)

By taking the derivative of Eq. (10) with respect to \( J \), one obtains the relation between \( \tilde{\varphi} \) and \( \varphi \);
The new effective action \( \tilde{\Gamma}[\tilde{\varphi}, B] \) in the presence of the background field \( B \) can then be written as

\[
\tilde{\Gamma}[\tilde{\varphi}, B] \equiv \tilde{W}[J, B] - J\tilde{\varphi} = (W[J] - JB) - J(\varphi - B) = \Gamma[\tilde{\varphi} + B].
\]

If we set \( \tilde{\varphi} = 0 \), then Eq. (12) becomes

\[
\Gamma[B] = \tilde{\Gamma}[\tilde{\varphi} = 0, B],
\]

which implies that the effective action \( \Gamma[B] \) consists of one \( \tilde{\varphi} \)-particle irreducible diagrams with no external \( \tilde{\varphi} \)-lines. Due to Eq. (8) and (12), the effective action can be considered to be the extremum value of \( \tilde{W}[J, B] \) with respect to the variation of the external source \( J \).

We now consider the n-dimensional scalar \( \phi^4 \) theory described by the Lagrangian density,

\[
\mathcal{L} = -\frac{1}{2} \phi(i\mathcal{D}^{-1})\phi - \frac{\lambda}{4!}\phi^4,
\]

where \( \mathcal{D}^{-1} \) is defined by

\[
i\mathcal{D}^{-1} \equiv (\partial^2 + \mu^2).
\]

The generating functional for Green’s functions in the presence of the background field \( B \) is given by

\[
\tilde{Z}[J, B] \equiv \int \mathcal{D}\phi e^{iS(\phi + B) + i\phi J}
\]

\[
= \text{det} \sqrt{-K} \exp\left[\frac{1}{2}BD^{-1}B - \frac{i\lambda}{4!}B^4\right]
\]

\[
\times \exp\left[(BD^{-1} - \frac{i\lambda}{3!}B^3)\frac{\delta}{i\delta J}\right] \exp\left[-\frac{i\lambda}{4!}(\frac{\delta}{i\delta J})^4 - \frac{i\lambda}{3!}B(\frac{\delta}{i\delta J})^3\right]e^{\frac{i}{2}JKJ},
\]

where \( K \) is defined by

\[
K_{xy}^{-1} = \mathcal{D}_{xy}^{-1} - \frac{i\lambda}{2}B_x^2\delta_{xy}.
\]
The effective action is given by

\[ e^{i\Gamma[B]} = \hat{Z}[J, B]|_{\hat{\varphi}=0}. \]  

(18)

Note that the first line of Eq. (16) gives rise to the 1-loop effective action and the last line, upon setting \( \hat{\varphi} = 0 \), represents the higher loop contributions to the effective action. This higher loop contributions can be evaluated by using the Feynman rule with the propagator given by \( K_{xy} \) of Eq. (17). Thus the effective action consists of one \( \hat{\varphi} \)-particle irreducible bubble diagrams with no external lines, and the procedure of evaluating the higher loop contributions is simplified compared to the conventional method. The reason why it is simplified is that the propagators used in the Feynman rule, Eq. (17), already contains the interaction effect through the background field \( B \), and the generating functional has been rearranged in such a way that the zeroth order term in the expansion, the first line of Eq. (16), is the one-loop effective action.

We now proceed one step further so that the propagator used for the perturbative expansion would be the one obtained in the Gaussian approximation. In other words, we want to rearrange \( e^{i\Gamma[B]} \) in such a way that the zeroth order term in the expansion would give the Gaussian approximation of the effective action. To do this, we consider the relations:

\[
\frac{\delta}{\delta J_x} e^{\frac{1}{2} \int J_y G_{yz} J_z} = G_{xy} J_y e^{\frac{1}{2} \int J_y G_{yz} J_z} \equiv (GJ)_x e^{\frac{1}{2} JGJ} 
\]

(19)

\[
\frac{\delta^2}{\delta J_x^2} e^{\frac{1}{2} \int J_y G_{yz} J_z} = ((GJ)_x^2 + G_{xx}) e^{\frac{1}{2} JGJ} 
\]

(20)

\[
\frac{\delta^3}{\delta J_x^3} e^{\frac{1}{2} \int J_y G_{yz} J_z} = ((GJ)_x^3 + 3G_{xx}(GJ)_x) e^{\frac{1}{2} JGJ} 
\]

(21)

\[
\frac{\delta^4}{\delta J_x^4} e^{\frac{1}{2} \int J_y G_{yz} J_z} = ((GJ)_x^4 + 6G_{xx}(GJ)_x^2 + 3G_{xx}^2) e^{\frac{1}{2} JGJ}, 
\]

(22)

which would appear in the expansion of the last line of Eq. (16). Note that Eq. (21) and (22) contain the diagrams where internal lines come out of a point and go back to the same point as shown in Fig. 1 and 2, and the Gaussian approximation of the effective action consists of such diagrams.

To extract such diagrams out of the last line of Eq. (16), we define new functional derivatives in such a way that
Comparing Eqs. (21) and (22) and Eqs. (23) and (24), we find that the primed derivatives must be defined by

\[ \left( \frac{\delta^3}{\delta J_x^3} \right)' \equiv \frac{\delta^3}{\delta J_x^3} - 3G_{xx} \frac{\delta}{\delta J_x} \]

\[ \left( \frac{\delta^4}{\delta J_x^4} \right)' \equiv \frac{\delta^4}{\delta J_x^4} - 6G_{xx} \frac{\delta^2}{\delta J_x^2} + 3G_{xx}^2. \]

We note that the primed derivatives operated on \( e^{\frac{1}{2} J G J} \) do not generate the diagrams that contribute to the Gaussian approximation, such as those shown in Fig. 1(b), Fig. 2(b) and (c).

This implies that by using the primed derivatives one may extract the diagrams that contribute to the Gaussian effective action, out of the last line of Eq. (16). To do this we need to find a Green’s function \( G \) which satisfies,

\[ \exp\left[ \lambda B \left( \frac{\delta}{\delta J} \right)^3 - \frac{i\lambda}{4!} \left( \frac{\delta}{\delta J} \right)^4 \right] e^{\frac{1}{2} J K J} = \]

\[ N \exp[A \frac{\delta}{\delta J}] \exp\left[ \frac{\lambda B}{3!} \left( \frac{\delta^3}{\delta J^3} \right)' - \frac{i\lambda}{4!} \left( \frac{\delta^4}{\delta J^4} \right)' \right] e^{\frac{1}{2} J G J}, \]

where a normalization constant \( N \) and a function \( A(x) \) are to be determined. By using the definitions Eq. (23) and (26), one easily find \( G, N \) and \( A \) that satisfy Eq. (27):

\[ N = \frac{\text{det} \sqrt{-G}}{\text{det} \sqrt{-\left( G^{-1} - \frac{i\lambda}{2} G_{xx} \right)^{-1}}} \exp\left[ \frac{i\lambda}{8} G_{xx}^2 \right], \]

\[ A = \frac{\lambda}{2} B G_{xx}, \]

\[ G_{xy}^{-1} = K_{xy}^{-1} + \frac{i\lambda}{2} G_{xx} \delta_{xy} \]

\[ = D_{xy} - \frac{i\lambda}{2} B_{xy}^2 \delta_{xy} + \frac{i\lambda}{2} G_{xx} \delta_{xy}. \]

Thus we can rewrite the generating functional for Green’s functions, Eq. (16) as

\[ \tilde{Z}[J, B] = \text{det} \sqrt{-G} \exp\left[ \frac{1}{2} B D^{-1} B - \frac{i\lambda}{4!} B^4 + \frac{i\lambda}{8} G_{xx}^2 \right] \]

\[ \times \exp\left[ \left( -i B D^{-1} - \frac{\lambda}{4!} B^3 + \frac{\lambda}{2} B G_{xx} \right) \frac{\delta}{\delta J} \right] \exp\left[ \frac{\lambda B}{3!} \left( \frac{\delta^3}{\delta J^3} \right)' - \frac{i\lambda}{4!} \left( \frac{\delta^4}{\delta J^4} \right)' \right] e^{\frac{1}{2} J G J}. \]
where the functional derivatives in the last line do not generate the cactus type diagrams such as those shown in Fig.2(c).

The effective action $\Gamma[B]$ is given by

$$
e^{i\Gamma[B]} = \tilde{Z}[J, B]|_{\tilde{\varphi}=0}
= \det \sqrt{-G} \exp\left[\frac{1}{2}B^2D^{-1}B - \frac{i\lambda}{4!}B^4 + \frac{i\lambda}{8}G_{xx}\right] I[B],
$$

where

$$I[B] = \exp\left[\left(-iBD^{-1}\right)\frac{\lambda}{3!}B^3 + \frac{\lambda}{2}BG_{xx}\right] \frac{\delta}{\delta J}
\times \exp\left[\frac{\lambda B}{3!}\left(\frac{\delta^3}{\delta J^2}\right)' - \frac{i\lambda}{4!}\left(\frac{\delta^4}{\delta J^4}\right)\right]\left|e^{\frac{i}{2}JGJ}|_{\tilde{\varphi}=0}\right..$$

Note that Eq. (31), which determines the Green’s function $G_{xy}$, is exactly the variational gap equation written in Minkowski space notation [1], and therefore the coefficient of $I[B]$ in Eq. (32) gives rise to the Gaussian effective action as we have alluded to above. Since $I[B]$ of Eq. (33) can be expanded as a power series in $\lambda$ as is done in the conventional BFM, we have the perturbative expansion of the effective action around the Gaussian approximation.

We now consider the contributions coming from $I[B]$. The linear term in the exponent of Eq. (33) generate tadpole diagrams. This term does not contribute to the effective action because the effective action $\Gamma[B]$ is the extremum value of $\tilde{W}[J, B]$ as explained earlier, and the tadpole term does not change the extremum value. We therefore see that $I[B]$ has the same structure as the last line of Eq. (16) except that the propagator is replaced by $G_{xy}$, and the functional derivatives are replaced by the primed derivatives. Thus $I[B]$ consists of one $\varphi$-particle irreducible bubble diagrams with no external $\varphi$-lines, and without the cactus type diagrams (with internal lines coming out of a point and going back to the same point).

We have thus established the systematic correction method to the Gaussian approximation, where one can compute order by order correction terms to the Gaussian effective action. One can easily show that the $\lambda^2$ contribution of $I[B]$ consists of the diagrams shown in Fig.3.
If we consider the space-time independent background field in the above formulation, we obtain the effective potential defined by
\[ V_{\text{eff}}[B] \equiv -\frac{\Gamma[B]}{\int d^4x}. \] (34)
Since $B$ is a constant, $G_{xy}$ is a function of $(x - y)$. Defining the Fourier transformation of $G_{xy}$ as
\[ G_{xy} \equiv \int g(p)e^{ip(x-y)} \] (35)
and the effective mass as
\[ m^2 \equiv \mu^2 + \frac{\lambda}{2}B^2 - \frac{\lambda}{2} \int g(p), \] (36)
we can write Eq. (30) as
\[ g(p) = \frac{1}{i(p^2 - m^2)}, \] (37)
where
\[ \int_p \equiv \int \frac{d^n p}{(2\pi)^n}, \] (38)
and $n$ is the dimension of space-time. We note that Eqs. (36) and (37) are the well known mass gap equation for the Gaussian effective potential [1]. Thus the zeroth order contribution of Eq. (32) is the Gaussian effective potential and the first non-trivial correction terms to the Gaussian approximation are of the order $\lambda^2$ as shown in Fig.3.

Thus up to the $\lambda^2$ order contribution of $I[B]$ the effective potential is given by
\[ V_{\text{eff}} = V_G + V_P, \] (39)
where
\[ V_G \equiv \frac{\mu^2}{2}B^2 + \frac{\lambda}{4!}B^4 - \frac{1}{2\lambda}(m^2 - \mu^2 - \frac{\lambda}{2}B^2)^2 + i\text{Tr} \log \sqrt{-G}, \] (40)
\[ V_P \equiv \frac{i\lambda^2}{12}B^2G^3 - \frac{i\lambda^2}{48}G^4, \] (41)
and $G$ is given by Eqs. (15) $\sim$ (17). $V_G$ is the Gaussian effective potential, and $V_P$ is the first non-trivial contribution from $I[B]$ which is shown in Fig.3. This result is the same as that of Cea and Tedesco [3] computed by using the variational perturbation theory.
III. EFFECTIVE POTENTIAL FOR THE (0+1)-DIMENSIONAL $\phi^4$ THEORY

To illustrate how much our method improve the Gaussian approximation we now consider the (0+1)-dimensional $\phi^4$ theory, which is the quantum mechanical anharmonic oscillator, and compare the results with those of various other methods. By applying the Wick rotation, Eqs. (40) and (41) can be written in this case as

$$V_{\text{eff}}[B] = V_G + V_P,$$

(42)

where

$$V_G = \frac{\mu^2}{2} B^2 + \frac{\lambda}{24} B^4 - \frac{1}{2\lambda}(m^2 - \mu^2 - \frac{\lambda}{2} B^2)^2 + \hbar \sqrt{m^2},$$

(43)

$$V_P = -\frac{\lambda^2 \hbar^2}{144m^4} B^2 - \frac{\lambda^2 \hbar^3}{1536(\sqrt{m^2})^5}.$$  (44)

The effective mass, $m$, in Eqs. (43) and (44) is given by

$$m^2 = \mu^2 + \frac{\lambda}{2} B^2 + \frac{\lambda \hbar}{4 \sqrt{m^2}},$$

(45)

where we have put the Plank constant explicitly.

The improved Gaussian effective potential contains all diagrams up to 2-loop. Therefore, the 2-loop effective potential, which can be obtained by taking the terms up to $\hbar^2$ order from Eq. (12), is

$$V_{2-\text{loop}} = \frac{\mu^2}{2} B^2 + \frac{\lambda}{24} B^4 + \frac{\hbar}{2} \sqrt{\mu^2 + \frac{\lambda}{2} B^2}$$

$$+ \hbar^2 \left( \frac{\lambda}{32}(\mu^2 + \frac{\lambda}{2} B^2)^{-1} - \frac{\lambda^2}{144} B^2(\mu^2 + \frac{\lambda}{2} B^2)^{-2} \right).$$  (46)

Ignoring the $\hbar^2$ term, we obtain the 1-loop effective potential, which agrees with that of Ref. 3. Using these results, we evaluate the ground state energies of various approximation methods and compare their results.

For simplicity, we consider the case of positive $\mu^2 = 1$, and compare the ground state energies of various approximation methods in the unit $\hbar = 1$. Then the classical potential has minimum $V_{cl} = 0$ at $B = 0$. The values of the coupling constant will be chosen to be that of 3. ($\lambda$ of 3 corresponds to our $\frac{\lambda}{4\hbar}$.)
Table I shows the ground state energy values of various approximations and their errors compared to the numerical results, for several coupling constants. This shows that the first non-trivial correction to the Gaussian approximation greatly improves the result for the small coupling region. For strong coupling, Gaussian and improved Gaussian approximation display much better results than the loop expansion, which shows the non-perturbative nature of the approximation methods. Though the GEP is accurate to within about 2% in the strong coupling region, the error of the improved GEP is smaller than about 0.8%. This shows that the perturbative expansion method greatly improves the Gaussian approximation even at the first non-trivial correction level.

IV. DISCUSSION

We have developed the perturbative expansion method around the Gaussian effective action by using the background field method. The method is based on the observation that the Gaussian effective action consists of cactus type diagrams, which were extracted out of the functional derivative part of the effective action, i.e., the last line of Eq. (14), by introducing the primed derivatives defined in Eqs. (25) and (26). This procedure effectively rearranged the diagrams in such a way that the propagator used in the perturbative expansion becomes that of Gaussian approximation, and the zeroth order term of the effective action consists of the Gaussian one.

In the last section we have considered the quantum mechanical anharmonic oscillator, and have shown that the perturbative correction greatly improves the Gaussian approximation even at the first non-trivial ($\lambda^2$ contribution) correction level.

One can easily compute the $\lambda^2$ contributions to the effective potentials for the (1+1) and (2+1) dimensional $\phi^4$ theories, and show that they agree with those of Cea and Tedesco [3]. Our method appears to be simpler than the variational perturbation theory in practical applications and can also be used in the cases of time-dependent systems. Our method can easily be generalized to the cases of fermionic and gauge field theories.
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Figure 1. Feynman diagrams for Eq. (21).

Figure 2. Feynman diagrams for Eq. (22).

Figure 3. Diagrams contributing to the $\lambda^2$-order corrections to the Gaussian effective action.

Table 1. Ground state energy for anharmonic oscillator.
FIG. 1. Feynman diagrams for Eq. (21)

FIG. 2. Feynman diagrams for Eq. (22)

FIG. 3. Diagrams contributing to the $\lambda^2$-order corrections to the Gaussian effective action
| $\lambda$ | 1-loop | 2-loop | Gaussian | improved Gaussian |
|-----------|---------|---------|----------|-------------------|
|           | $E_0$   | error(%)| $E_0$    | error(%)          | $E_0$    | error(%)|
| 0         | 0.5     | 0       | 0.5      | 0                 | 0.5      | 0       |
| 0.24      | 0.5     | -1.43   | 0.508    | 0.048             | 0.507    | 0.006   |
| 2.4       | 0.5     | -10.6   | 0.575    | 2.84              | 0.560    | 0.21    |
| 24        | 0.5     | -37.8   | 1.25     | 55.5              | 0.813    | 1.09    |
| 240       | 0.5     | -66.8   | 8.       | 432               | 1.53     | 1.75    |
| 2400      | 0.5     | -84.0   | 75.5     | 2311              | 3.19     | 1.95    |
| $\infty$  | $\cdot$ | $\cdot$ | $\cdot$  | $\cdot$           | 0.236 $\lambda$ | 2.01    |
|           |         |         |          |                   | 0.230 $\lambda$ | -0.82   |