ON HOMEOMORPHIC PRODUCT MEASURES ON
THE CANTOR SET

RANDALL DOUGHERTY AND R. DANIEL MAULDIN

ABSTRACT. Let $\mu(r)$ be the Bernoulli measure on the Cantor space given as the infinite product of two-point measures with weights $r$ and $1 - r$. It is a long-standing open problem to characterize those $r$ and $s$ such that $\mu(r)$ and $\mu(s)$ are topologically equivalent (i.e., there is a homeomorphism from the Cantor space to itself sending $\mu(r)$ to $\mu(s)$). The (possibly) weaker property of $\mu(r)$ and $\mu(s)$ being continuously reducible to each other is equivalent to a property of $r$ and $s$ called binomial equivalence. In this paper we define an algebraic property called “refinability” and show that, if $r$ and $s$ are refinable and binomially equivalent, then $\mu(r)$ and $\mu(s)$ are topologically equivalent. We then give a class of examples of refinable numbers; in particular, the positive numbers $r$ and $s$ such that $s = r^2$ and $r = 1 - s^2$ are refinable, so the corresponding measures are topologically equivalent.

Two measures $\mu$ and $\nu$ defined on the family of Borel subsets of a topological space $X$ are said to be homeomorphic or topologically equivalent provided there exists a homeomorphism $h$ of $X$ onto $X$ such that $\mu$ is the image measure of $\nu$ under $h$: $\mu = \nu h^{-1}$. This means $\mu(E) = \nu(h^{-1}(E))$ for each Borel subset $E$ of $X$.

One may be interested in the structure of these equivalence classes of measures or in a particular equivalence class. For example, a probability measure $\mu$ on $[0, 1]$ is topologically equivalent to Lebesgue measure if and only if $\mu$ gives every point measure 0 and every non-empty open set positive measure. (The distribution function of $\mu$ is a homeomorphism on $[0, 1]$ witnessing this equivalence.) This is a special case of a result of Oxtoby and Ulam [10], who characterized those probability measures $\mu$ on finite dimensional cubes $[0, 1]^n$ which are homeomorphic to Lebesgue measure. For this to be so, $\mu$ must give points measure 0, non-empty open sets positive measure, and the boundary of the cube

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measure 0. Later Oxtoby and Prasad \cite{9} extended this result to the Hilbert cube. These results have been extended and applied to various manifolds. The book of Alpern and Prasad \cite{2} is an excellent source for these developments. Oxtoby \cite{8} also characterized those measures on the space of irrational numbers in \([0, 1]\) which are homeomorphic to Lebesgue measure.

It is natural to ask what measures are homeomorphic to Lebesgue measure on \(C = \{0, 1\}^N\), the Cantor space, where by Lebesgue measure we mean Haar measure or infinite product measure \(\mu(1/2)\) resulting from fair coin tossing. The topology on \(C\) is the standard product topology; we will use as basic open (actually clopen) sets for this topology the sets \(\langle e \rangle\) for all finite sequences \(e\) from \(\{0, 1\}\), where \(\langle e \rangle\) is the set of infinite sequences in \(C\) which begin with the finite sequence \(e\). (These basic clopen sets are sometimes called cylinders.) We will say that the length of a basic clopen set \(\langle e \rangle\) is the length of the finite sequence \(e\).

It turns out that the Cantor space is more rigid than \([0, 1]^n\) for measure homeomorphisms – it is not true that a measure \(\nu\) on \(C\) which gives points measure 0 and non-empty open sets positive measure is equivalent to Lebesgue measure. In fact, even among the product measures the only one which is equivalent to Lebesgue measure is Lebesgue measure itself. To describe the situation let us use the following notation. For each number \(r\), \(0 < r < 1\), let \(\mu(r)\) be the infinite product measure determined by coin tossing with probability of success \(r\). Consider the equivalence relation on \([0, 1]\), \(r \sim_{\text{top}} s\) if and only if \(\mu(r)\) is topologically equivalent to \(\mu(s)\). (We will sometimes abuse terminology by saying that \(r\) is topologically equivalent to \(s\).) It turns out that this equivalence relation is closely related to an algebraic/combinatorial relation. To explain this, we make the following definition.

**Definition 1.** Let \(0 < r, s < 1\). The number \(s\) is said to be binomially reducible to \(r\) provided

\[
s = \sum_{i=0}^{n} a_i \ r^i (1 - r)^{n-i},
\]

where \(n\) is a non-negative integer and each \(a_i\) is an integer with \(0 \leq a_i \leq \binom{n}{i}\).

The numbers \(r^a(1 - r)^b\) for integers \(a, b \geq 0\) will be referred to as cylinder sizes for \(r\); they are the measures of the basic clopen sets under \(\mu(r)\). So the right side of (1) is the general form for the measure of a clopen set under \(\mu(r)\).

Let us note some basic facts about reducibility to be used later. (See \cite{5} for this and further background information.) Note that if \(s\) is

binomially reducible to \( r \), so is \( 1 - s \) (change \( a_i \) to \( \binom{n}{i} - a_i \)). If \( s_1 \) and \( s_2 \) are reducible to \( r \), so is \( s_1s_2 \). (If \( s_1 = \sum_{i=0}^{n} a_ir^i(1-r)^{n-i} \) and \( s_2 = \sum_{j=0}^{m} b_jr^j(1-r)^{m-j} \), then \( s_1s_2 = \sum_{i=0}^{n} \sum_{j=0}^{m} a_ib_jr^{i+j}(1-r)^{n+m-i-j} = \sum_{k=0}^{n+m} \sum_{i+j=k} a_ib_j r^k(1-r)^{n+m-k} \) and \( \sum_{i+j=k} a_ib_j \leq \sum_{i+j=k} \binom{n}{i} \binom{m}{j} = \binom{n+m}{k} \). Hence, if \( s \) is binomially reducible to \( r \), so is \( s^n(1-s)^b \) for any \( a, b \geq 0 \). Also, it is known that \( \mu(s) \) is continuously reducible to or is a continuous image of the measure \( \mu(r) \) (i.e., \( \mu(s) = \mu(r) \circ g^{-1} \) for some continuous \( g : \mathbb{C} \to \mathbb{C} \)) if and only if \( s \) is binomially reducible to \( r \) [5]. Thus, we have another natural equivalence relation on \([0,1]\).

**Definition 2.** Let \( 0 < r, s < 1 \). Then \( r \) is binomially equivalent to \( s \), denoted \( r \approx s \), provided \( r \) is binomially reducible to \( s \) and \( s \) is binomially reducible to \( r \), or, equivalently, each of the measures \( \mu(r) \) and \( \mu(s) \) is a continuous image of the other.

Among several still unsolved problems concerning these relations is the following.

**Problem ([6] Problem 1065).** Is it true that the product measures \( \mu(r) \) and \( \mu(s) \) are homeomorphic if and only if each is a continuous image of the other, or, equivalently, each of the numbers \( r \) and \( s \) is binomially reducible to the other?

(Note: After this paper was first circulated, the above problem was solved in the negative by Austin [3].)

One can think of this problem in the following way. Suppose we have \( \mu(s) = \mu(r) \circ g^{-1} \) and \( \mu(r) = \mu(s) \circ h^{-1} \), where the maps \( g \) and \( h \) are continuous. Is there some sort of Cantor-Bernstein or back-and-forth argument for the Cantor set which, given \( g \) and \( h \), produces not just a one-to-one onto map, but a homeomorphism taking \( \mu(r) \) to \( \mu(s) \)?

Many cases of this problem have already been settled. Let us note that, for a given \( n \), the functions \( r^i(1-r)^{n-i} \) for \( 0 \leq i \leq n \) are linearly independent polynomials, since their trailing terms (i.e., nonzero terms of least degree) have distinct degrees. Therefore, \( \sum_{i=0}^{n} a_ir^i(1-r)^{n-i} \) as in Definition 1 is a polynomial of degree \( > 1 \) unless it is 0 (when \( a_i = 0 \) for all \( i \)), 1 (when \( a_i = \binom{n}{i} \)), 1 - \( r \) (\( a_i = \binom{n-1}{i} \)), or \( r \) (\( a_i = \binom{n-1}{i-1} \)). Therefore, if \( r \) and \( s \) are binomially reducible to each other, \( r \neq s \), and \( r \neq 1 - s \), then \( s = P(r) \) and \( r = Q(s) \) where \( \deg(P), \deg(Q) > 1 \), so \( r = Q \circ P(r) \) and \( \deg(Q \circ P) > 1 \). Thus, \( r \) is algebraic. Also, in this case, \( r \) and \( s \) have the same algebraic degree. Moreover, \( r \) is an algebraic integer if and only if \( s \) is. Huang [4] showed that if \( r \) is an algebraic integer of degree 2, and \( r \approx s \), then \( r = s \) or \( r = 1 - s \). In fact, Navarro-Bermudez [6] showed that if \( r \) is rational or transcendental and
$r \approx s$, then $r = s$ or $r = 1 - s$. We gather these facts in the following theorem.

**Theorem 1** (various authors). For $r$ rational, transcendental, or an algebraic integer of degree 2, the $\sim_{\text{top}}$ equivalence class containing $r$ and the $\approx$ equivalence class containing $r$ are both equal to $\{r, 1 - r\}$.

On the other hand, it is known that for every $n \geq 3$, there are algebraic integers $r$ of degree $n$ such that the $\approx$ equivalence class containing $r$ has at least 4 elements [4]. (In fact, Pinch [11] showed that, if $n = 2^k + 1$, then there is an algebraic integer $r$ of degree $n$ with at least $2k$ distinct numbers binomially equivalent to it.) The simplest of these is the solution of

$$r^3 + r^2 - 1 = 0$$

lying in the open interval $(0, 1)$. For this value of $r$, it turns out that $s = r^2 \approx r$, and Navarro-Bermudez and Oxtoby [7] proved that $r \sim_{\text{top}} s$ via a simple homeomorphism. Until now this has been the only nontrivial example of topologically equivalent product measures.

The purpose of this paper is to present a new condition under which binomially equivalent numbers are topologically equivalent. First, we define a condition called “refinable” on numbers in $[0, 1]$. Next, we show that if $r$ and $s$ are binomially equivalent and both $r$ and $s$ are refinable, then the measures $\mu(r)$ and $\mu(s)$ are homeomorphic. Finally, we apply our condition to the root $r$ of

$$r^4 + r - 1 = 0,$$

with $r$ between 0 and 1, and to $s = r^2$. We show both $r$ and $s$ are refinable and $r$ and $s$ are binomially equivalent. Thus, $\mu(r)$ and $\mu(s)$ are topologically equivalent via a very non-trivial homeomorphism.

Any cylinder size $r^a(1 - r)^b$ can be split into two cylinder sizes $r^{a+1}(1 - r)^b$ and $r^a(1 - r)^{b+1}$. Either or both of these can be split in the same way, and so on. After finitely many steps, one has partitioned the original cylinder size into finitely many cylinder sizes. We will call a partition obtained in this way a tree partition of $r^a(1 - r)^b$ (named after the representation of the Cantor space as the set of paths through a complete infinite binary tree). A tree partition corresponds to a partition of a basic clopen set in $\mathcal{C}$ into basic clopen subsets. Note that any tree partition can be split further by steps as above to yield a new tree partition in which all the final cylinder sizes have the same length, say $a + b + n$; in this final partition the cylinder size $r^{a+i}(1 - r)^{b+n-i}$ will occur $\binom{n}{i}$ times.

On the other hand, one may be able to partition a cylinder size for $r$ into a finite collection of smaller cylinder sizes (whose sum is the
original cylinder size; repetitions are allowed) in a way which is not a tree partition. For instance, one can partition the cylinder size 1 into \( \{ r^3, (1 - r)^3, r(1 - r), r(1 - r) \} \) (or, written more briefly, \( \{ r^3, (1 - r)^3, 3r(1 - r) \} \); we will treat a positive integer coefficient of a cylinder size as a multiplicity). For specific values of \( r \), there may be many more such partitions.

Recall the definition of refinement: given partitions \( P \) and \( P' \) of the same set, we say that \( P' \) is a refinement of \( P \) (or \( P \) has been refined to \( P' \)) if every member of \( P \) is a union of members of \( P' \). The corresponding definition for partitions of a number (e.g., a cylinder size) is: \( P' \) is a refinement of \( P \) if one can write \( P' \) as the union (respecting multiplicities) of collections \( S_t \) for \( t \in P \) such that, for each \( t \) in \( P \), the sum of \( S_t \) is \( t \).

**Definition 3.** A number \( r \) is refinable provided every partition of a cylinder size for \( r \) into smaller cylinder sizes can be refined to a tree partition.

An equivalent definition in symbols: \( r \) is refinable iff, for every true equation of the form

\[
r^a (1 - r)^b = r^{c_1} (1 - r)^{d_1} + r^{c_2} (1 - r)^{d_2} + \ldots + r^{c_m} (1 - r)^{d_m},
\]

there exist \( n \geq 0 \) and nonnegative integers \( p_{ij} \) for \( 0 \leq i \leq n, 1 \leq j \leq m \) such that

\[
\binom{n}{i} = p_{i1} + p_{i2} + \ldots + p_{im}
\]

for \( 0 \leq i \leq n \) and

\[
r^{c_j} (1 - r)^{d_j} = \sum_{i=0}^{n} p_{ij} r^{a+i} (1 - r)^{b+n-i}
\]

for \( 1 \leq j \leq m \). We note that in this definition \( a, b, c_j, \) and \( d_j \) are assumed to be nonnegative integers, but the definition would be equivalent if we allowed them to be arbitrary integers.

We briefly compare this notion to that of a “good” measure as introduced by Akin [1]. A probability measure \( \mu \) on the Cantor space is good if, whenever \( U, V \) are clopen sets with \( \mu(U) < \mu(V) \), there exists a clopen subset \( W \) of \( V \) such that \( \mu(W) = \mu(U) \). We state a few facts without proof here. If a product measure \( \mu(r) \) is good, then \( r \) is refinable. If \( r \) is transcendental, then \( r \) is refinable, but \( \mu(r) \) is not good. If \( r \) is rational and \( r \neq 1/2 \), then \( r \) is not refinable and hence \( \mu(r) \) is not good.

Refinability is useful because of the following result.
Theorem 2. If $0 < r, s < 1$, $r$ and $s$ are binomially equivalent, and each of $r$ and $s$ is refinable, then the measures $\mu(r)$ and $\mu(s)$ are homeomorphic.

Proof. We construct partitions $P_n$ and $Q_n$ of $C$ into clopen sets for $n = 0, 1, 2, \ldots$ and bijections $\pi_n : P_n \mapsto Q_n$ satisfying the following properties:

1. $P_{n+1}$ is a refinement of $P_n$ and $Q_{n+1}$ is a refinement of $Q_n$,
2. each member of $P_{2n-1}$ and each member of $Q_{2n}$ is a basic clopen set of length $\geq n$,
3. for any $X \in P_n$ we have $\mu(s)(\pi_n(X)) = \mu(r)(X)$, and
4. if $X \in P_{n+1}$ and $X \subseteq X' \in P_n$, then $\pi_{n+1}(X) \subseteq \pi_n(X')$.

Given the above sequence, define $f : C \mapsto C$ by: for each $\alpha \in C$, let $X_n$ be the unique member of $P_n$ containing $\alpha$ and let $f(\alpha)$ be the unique element of $\bigcap_n \pi_n(X_n)$. It is straightforward to verify that $f$ is a well-defined homeomorphism of $C$ ($f^{-1}$ is defined by an analogous method from $Q_n$ to $P_n$), and $f(X) = \pi_n(X)$ for all $X \in P_n$, so that $\mu(s)(f(X)) = \mu(r)(X)$ for $X \in \bigcup_n P_n$. Since every clopen set is a finite disjoint union of sets each in $\bigcup_n P_n$, $f$ maps $\mu(r)$ to $\mu(s)$.

We build $P_n$, $Q_n$, and $\pi_n$ by a back-and-forth recursive construction. Let $P_0 = Q_0 = \{C\}$ with $\pi_0(C) = C$. Given $P_{2n}$, $Q_{2n}$, $\pi_{2n}$, let $P_{2n+1}$ be a refinement of $P_{2n}$ into basic clopen sets of length $\geq n+1$. Fix $Y \in Q_{2n}$, a basic clopen set, say of $\mu(s)$-measure $s^a(1-s)^b$. Now, $\pi_{2n}^{-1}(Y) \in P_{2n}$ is a union of basic clopen sets $X_1, \ldots, X_k \in P_{2n+1}$, each having $\mu(r)$-measure $r^p(1-r)^q$ for some integers $p, q$, and these measures add up to $s^a(1-s)^b$. Since $r$ is binomially reducible to $s$, so is each $r^p(1-r)^q$. Thus, each $\mu(r)(X_j)$ can be expressed as a finite sum of numbers $s^c(1-s)^d$. Putting these together for all such $X_j$’s, we get a list of numbers $s^{c_1}(1-s)^{d_1}, s^{c_2}(1-s)^{d_2}, \ldots, s^{c_m}(1-s)^{d_m}$ with sum $s^a(1-s)^b$.

Since $s$ is refinable, $Y$ can be partitioned into clopen sets $\hat{Y}_1, \ldots, \hat{Y}_m$ with $\mu(\hat{Y}_i) = s^{c_i}(1-s)^{d_i}$ for each $i$. But the list above was obtained by joining the lists for the individual $X_j$’s together; hence, we can combine the $\hat{Y}_i$’s to get clopen sets $Y_1, Y_2, \ldots, Y_k$, still forming a partition of $Y$, such that $\mu(s)(Y_j) = \mu(r)(X_j)$, for each $j$. Put these sets $Y_j$ into $Q_{2n+1}$, letting $\pi_{2n+1}(X_j) = Y_j$. Once this is done for all $Y \in Q_{2n}$, we will have the desired partition $Q_{2n+1}$ and map $\pi_{2n+1}$.

We have finished refining the partition on the $P$ side; it is now the partition on the $Q$ side that needs to be refined next. So let $Q_{2n+2}$ be a refinement of $Q_{2n+1}$ into basic clopen sets of length $\geq n+1$, and apply the above procedure with $r$ and $s$ interchanged to get $P_{2n+2}$ and $\pi_{2n+2}$ (the map from $Q_{2n+2}$ to $P_{2n+2}$ will be $\pi_{2n+2}^{-1}$). This will complete the back-and-forth recursive step. \qed
To prove that a number \( r \) is refinable, it suffices to show the following. Given any two finite multisets (sets whose elements can have multiplicity greater than 1) \( A \) and \( B \) of cylinder sizes for \( r \) such that \( \sum A = \sum B \), one can transform \( A \) and \( B \) to a common multiset \( C \), where for \( B \) the allowed transform steps are arbitrary splits (meaning replace a cylinder size \( x \) with any collection of cylinder sizes that add to \( x \)), while for \( A \) the only allowed steps are tree splits (meaning replace \( x \) with \( xr \) and \( x(1-r) \)). If \( A \) is a singleton, \( A = \{r^a(1-r)^b\} \), and \( B \) and \( C \) are as just described, then \( C \) will be a tree partition of \( r^a(1-r)^b \) which is a refinement of the partition \( B \). [Note that we only need the case where \( A \) is a singleton, but proofs that one can transform \( A \) and \( B \) to \( C \) as above will usually work even when \( A \) is an arbitrary multiset.]

Equivalently, one can transform \( A \) and \( B \) to a common \( C' \) where one gets from \( B \) to \( C' \) by arbitrary splits and one get from \( A \) to \( C' \) by a sequence of tree splits followed by a sequence of merges (meaning replace a subcollection of the current multiset by its sum, which we may or may not require to be a cylinder size). This is because, if \( C \) is the multiset obtained from \( A \) by the tree splits alone, then \( C \) is obtained from \( C' \) by arbitrary splits (a split is the inverse of a merge) and hence from \( B \) by arbitrary splits.

In fact, it will suffice to transform \( A \) to \( C' \) by any sequence of merges and tree splits in any order, because a merge followed by a tree split is equivalent to one or more tree splits followed by a merge (if \( \sum_i x_i = x \), then \( \sum_i x_i r = xr \) and \( \sum_i x_i (1-r) = x(1-r) \)). Define a “tree move” to be a merge or a tree split. This is the method we will use in the refinability proofs to follow: given \( A \) and \( B \), transform \( A \) to \( A' \) by tree moves and \( B \) to \( B' \) by splits, and show that \( A' = B' \) (this is the common multiset \( C' \)).

We will describe such transformations as built up out of simple steps. For instance, suppose we have cylinder sizes \( a_1, a_2, b_1, b_2, \) and \( b_3 \), and we demonstrate how to transform \( \{a_1, a_2\} \) into \( \{b_1, b_2, b_3\} \) using, say, tree moves. (We may write this more briefly as “one can get from \( a_1, a_2 \) to \( b_1, b_2, b_3 \) by tree moves.”) Then, for any multiset \( A \), we can also transform \( A \cup \{a_1, a_2\} \) into \( A \cup \{b_1, b_2, b_3\} \) using tree moves. (Here \( \cup \) is multiset union, where multiplicities are added.) Such a transformation can then be used as part of further transformations. Also, if one can get from \( a_1, a_2 \) to \( b_1, b_2, b_3 \) by tree moves, then for any \( c \) one can get from \( r^c \) \( (1-r)^d \) \( a_1 \), \( r^c (1-r)^d a_2 \) to \( r^c \) \( (1-r)^d b_1 \), \( r^c (1-r)^d b_2 \), \( r^c (1-r)^d b_3 \) by tree moves (just multiply every cylinder size involved by \( r^c (1-r)^d \)). All of our transformations of multisets will be sum-preserving.

Our next theorem shows that there are non-trivial examples of refinable numbers.
Theorem 3. If \( r \) is the positive root of \( x^n + x - 1 = 0 \), where \( n > 1 \) and \( n \not\equiv 5 \mod 6 \), then \( r \) is refinable.

Proof. By a theorem of Selmer [12], the trinomial \( x^n + x - 1 \) is irreducible when \( n \not\equiv 5 \mod 6 \). Hence, \( r \) is algebraic of degree \( n \). Let \( A \) and \( B \) be multisets of cylinder sizes for \( r \) with the same sum. Since \( 1 - r = r^n \), we have

\[ r^a(1 - r)^{b+1} = r^{a+n}(1 - r)^b. \]

The replacement of \( r^a(1 - r)^{b+1} \) by \( r^{a+n}(1 - r)^b \) can be thought of as both a trivial merge and a trivial split. Therefore, it can be applied repeatedly to both \( A \) and \( B \) to produce new multisets \( A' \) and \( B' \) containing only powers \( r^i, i \geq 0 \).

Next note that the replacement in a multiset

\[ r^a \rightarrow r^{a+1}, r^{a+n} \quad (2) \]

is a split which is obtainable by tree moves (split \( r^a \) to \( r^{a+1} \) and \( r^a(1-r) \) to \( r^{a+n} \), so it can be applied on both sides. Let \( k \) be the largest exponent such that \( r^k \) occurs in \( A' \) or \( B' \). By repeatedly applying the replacement (2) to each \( r^a \) with \( a \leq k - n \), we can get from \( A'' \) and \( B'' \) to \( A' \) and \( B' \) consisting entirely of powers \( r^a \) with \( k - n + 1 \leq a \leq k \). These \( n \) powers of \( r \) are linearly independent over the rationals because \( 1, r, \ldots, r^{n-1} \) are (since \( r \) is algebraic of degree \( n \)). So the only way for \( A' \) and \( B' \) to have the same sum is to have \( A' = B' \). This completes the proof. \( \square \)

If \( r \) is the positive root of \( x^n + x - 1 = 0 \), and \( s = r^d \) where \( d \) is a divisor of \( n \), then \( r \approx s \), because \( r = 1 - s^{n/d} \) (recall that, if \( t \) is binomially reducible to \( r \), then so is \( 1 - t \)). For most \( n \), the preceding theorem shows that \( r \) is refinable. Hence, to show that \( r \sim_{top} s \), it will suffice to show that \( s \) is refinable.

In our next theorem we prove this for the special case when \( n = 4 \).

Theorem 4. If \( r, s \in (0, 1) \), \( s = r^2 \), and \( r = 1 - s^2 \), then \( r \) and \( s \) are refinable and the measures \( \mu(r) \) and \( \mu(s) \) are topologically equivalent.

Proof. We have \( r = 1 - r^4 \), so \( r \approx s \) (as noted above), \( r \) is refinable (by Theorem 3), and \( r \) and \( s \) are algebraic of degree 4 (by Selmer’s theorem mentioned previously and the fact that binomial equivalence preserves degree). Also, we have

\[ s = (1 - s^2)^2 = (1 - s)^2(1 + 2s + s^2). \]  

(3)

Now, from the cylinder size 1, one can get to

\[ s^2, (1 - s)^2, s^2(1 - s), 2s(1 - s)^2, s^3(1 - s), s^2(1 - s)^2 \]  

(4)
by tree splits (this corresponds to partitioning \( C \) into \( \langle 1,1 \rangle, \langle 0,0 \rangle, \langle 1,0,1 \rangle, \langle 0,1,0 \rangle, \langle 0,1,1 \rangle, \langle 0,1,1,1 \rangle, \) and \( \langle 0,1,1,0 \rangle \)), and then to
\[
s, s^2(1-s), s^3(1-s) \tag{5}
\]
by a merge of the second, fourth, and sixth terms in \( \langle 4 \rangle \), using \( \langle 3 \rangle \); of course, one can also get from 1 to \( \langle 4 \rangle \) by an arbitrary split. As noted earlier, this implies that one can get from \( s^a(1-s)^b \) to
\[
s^{a+1}(1-s)^b, s^{a+2}(1-s)^b, s^{a+2}(1-s)^{b+1}, s^{a+3}(1-s)^{b+1}
\]
by an arbitrary split or by tree moves.

Next, one can get from \( s \) to
\[
s^3, s(1-s)^2, 2s^2(1-s)^2, 2s^4(1-s), s^3(1-s)^2, s^3(1-s)^3, s^5(1-s)^2, s^4(1-s)^3 \tag{6}
\]
by tree splits — multiply the steps from 1 to \( \langle 4 \rangle \) by \( s \), and then use three successive tree splits to replace the cylinder size \( s^3(1-s) \) with
\[
s^4(1-s), s^3(1-s)^3, s^5(1-s)^2, s^4(1-s)^3.
\]
One can then get from \( \langle 6 \rangle \) to
\[
s^3, 2s(1-s)^2, s^2(1-s)^2, 2s^4(1-s), s^5(1-s)^2 \tag{7}
\]
by a merge from \( s(1-s)^2 \) times \( \langle 5 \rangle \) to \( s(1-s)^2 \). But we have
\[
s = r^2 = (1-s)^2 = (1-s)^2 + 2s(1-s)^2 + s^2(1-s)^2,
\]
and \( s \) is also equal to the sum of the terms in \( \langle 7 \rangle \) since all of our moves are sum-preserving, so we must have
\[
(1-s)^2 = s^3 + 2s^4(1-s) + s^5(1-s)^2
\]
and hence we can get from \( \langle 7 \rangle \) to
\[
(1-s)^2, 2s(1-s)^2, s^2(1-s)^2, s^3(1-s)^2 \tag{8}
\]
by a merge. So, we can get from \( s \) to \( \langle 8 \rangle \) by tree moves as well as by a split.

Next, we can get from 1 to \( s, (1-s) \) by a tree split and then to
\[
s, s(1-s), s^2(1-s), s^2(1-s)^2, s^3(1-s)^2 \tag{9}
\]
by a \( \langle 5 \rangle \) move. (Here “by a \( \langle 5 \rangle \) move” is short for “by replacing a cylinder size \( t \) with \( t \) times \( \langle 5 \rangle \), which is a split and can also be accomplished by tree moves.”) In this case \( t = 1-s \). Finally, we can get from 1 to \( s, (1-s) \) by a tree split, then to
\[
(1-s), (1-s)^2, 2s(1-s)^2, s^2(1-s)^2 \tag{10}
\]
by a \( \langle 5 \rangle \) move, then to
\[
(1-s), 3s(1-s)^2, 2s^2(1-s)^2, s^2(1-s)^3, s^3(1-s)^3
\]
by a $\text{(3)}$ move.

Now let $A$ and $B$ be multisets of cylinder sizes with the same sum. First use $\text{(8)}$ moves repeatedly on both multisets to get rid of all cylinder sizes $s^a(1-s)^b$ with $a > b + 1$. Then use $\text{(5)}$ moves to get rid of all $s^a(1-s)^b$ with $a < b$. So only numbers $s^a(1-s)^b$ with $a = b$ or $a = b + 1$ occur in the new multisets. Let $k$ be the largest exponent $a$ which occurs. If either multiset contains a cylinder size $s^a(1-s)^b$ such that $b < k - 1$, then we can use a $\text{(9)}$ move to replace it with cylinder sizes with larger exponents; similarly, we can use a $\text{(10)}$ move to replace a cylinder size $s^{b+1}(1-s)^b$ such that $b < k - 2$. (Both of these steps yield new cylinder sizes $s^{a'}(1-s)^{b'}$ with $a' = b'$ or $a' = b' + 1$.) By performing these steps as many times as possible, one can change all cylinder sizes to one of the following five cylinder sizes:

$$s^{k-1}(1-s)^{k-2}, s^{k-1}(1-s)^{k-1}, s^k(1-s)^{k-1},$$
$$s^k(1-s)^k, \text{ or } s^{k+1}(1-s)^k.$$  

Then one can use tree splits on the first two of these five and $\text{(8)}$ moves on the resulting occurrences of $s^k(1-s)^{k-2}$ to reduce everything to the cylinder sizes

$$s^k(1-s)^{k-1}, s^{k-1}(1-s)^k, s^k(1-s)^k, \text{ or } s^{k+1}(1-s)^k.$$  

Let $A'$ and $B'$ be the final multisets using these cylinder sizes only. These four sizes are linearly independent over the rationals. (One can verify this directly by noting that from $s$, $(1-s)$, $s(1-s)$, $s^2(1-s)$ one can get 1, $s$, $s^2$, $s^3$ as linear combinations, or one can just notice that our argument shows that the multisets $\{1\}, \{s\}, \{s^2\}, \{s^3\}$ can all be reduced to these four forms using the same $k$. We are using here that $s$ has algebraic degree 4.) Therefore, since $A'$ and $B'$ are both multisets of these numbers and $\sum A' = \sum A = \sum B = \sum B'$, we must have $A' = B'$. So $s$ is refinable, as desired. Finally, using Theorem 2 we see that $\mu(r)$ and $\mu(s)$ are homeomorphic. □

**Corollary 5.** If $0 < r < 1$ and $r = 1 - r^4$, then there are at least 4 product measures topologically equivalent to $\mu(r)$.

Let us mention some other results which can be proven using the techniques of this paper. It is not hard to show that any transcendental number is refinable, although this is not useful for proving topological equivalence. One can show that, if $r$ is the root of $r^3 + r^2 - 1 = 0$ in $(0,1)$ and $s = r^2$, then $r$ and $s$ are both refinable. This gives another proof of the theorem of Navarro-Bermudez and Oxtoby. However, it is simpler to produce the homeomorphism as they did. We have verified that, if $r$ is the positive root of $r^6 + r = 1$, then $r$, $r^2$, and $r^3$ are all
refinable. Thus, there are at least six numbers in $(0,1)$ topologically equivalent to $r$. We have also verified that the positive numbers $r$ and $s$ given by $s = r^4$ and $r = 1 - s^2$ are refinable.

A number of problems remain open. One of these is the problem stated after Definition 2, which can now be restated as follows.

**Problem (5 Problem 1065).** Are $\sim_{top}$ and $\approx$ the same equivalence relation on $[0,1]$?

As noted earlier, the above problem has been solved in the negative by Austin [3].

In connection with this problem, we note that there are relatively simple examples of two probability measures on the Cantor space each of which is a continuous image of the other but there is no homeomorphism taking one to the other. One such example is $\mu(1/2)$ and $\nu$, where $\nu$ is obtained from $\mu(1/2)$ by multiplying the measure of any subset of the left half of the Cantor space by $3/2$ and multiplying the measure of any subset of the right half of the Cantor space by $1/2$. (Equivalently, $\nu$ can be described as the disjoint sum of $(3/4)\mu(1/2)$ and $(1/4)\mu(1/2)$.) The reason for this is that the $3/4$ half has no clopen subset of measure $1/2^n$, since all the numerators are divisible by 3, whereas every clopen subset with positive $\mu(1/2)$ measure has a clopen subset with measure $1/2^n$ for large $n$.

**Problem (5 Problem 1067).** Is there an infinite $\sim_{top}$ equivalence class? Is there an infinite $\approx$ equivalence class?

**Problem.** Is every number in a non-trivial $\sim_{top}$ equivalence class refinable? (Are the corresponding measures good in Akin’s sense?)

The corresponding question about nontrivial $\approx$ equivalence classes has a negative answer, by Theorem 2 combined with Austin’s result.

Regarding this problem, we note that the number $1/3$ is not refinable. This is because the partition $1/3 + 1/3 + 1/3$ of 1 cannot be refined to a tree partition of 1 since in any such tree partition only one number will have odd numerator. In fact, one can show that no rational $r$ in $(0,1)$ other than $1/2$ is refinable.

A particular case of interest in the preceding problem is the remaining “Selmer-like” reals, and the numbers binomially equivalent to Selmer or Selmer-like reals, where the Selmer reals and the Selmer-like reals are the positive numbers $r$ satisfying an equation $x^n + x - 1 = 0$ where $n \not\equiv 5 \pmod{6}$ or $n \equiv 5 \pmod{6}$, respectively. We already saw that the Selmer reals are refinable; It turns out that one can show that the Selmer-like reals are refinable as well, and the corresponding measures $\mu(r)$ are good in both cases. But it remains open whether
the numbers binomially equivalent to them are refinable (and hence
topologically equivalent to them).

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IDA Center for Communications Research, 4320 Westerra Ct., San
Diego, CA 92121
E-mail address: rdough@ccrwest.org

Department of Mathematics, PO Box 311430, University of North
Texas, Denton, TX 76203
E-mail address: mauldin@unt.edu