Delay-independent design for chaotic synchronization in delay-coupled Bernoulli map networks

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Abstract: The present paper considers chaotic synchronization in a network consisting of Bernoulli maps with a time-delay connection. We demonstrate that a connection strength and a map parameter at which chaotic synchronization occurs can be systematically designed, even for cases in which the connection delay, the number of maps, and the detailed topology information are unknown. The primary advantage of the proposed design is that the designed connection strength and map parameter are valid for any connection delay. This result is due to the fact that the stability of the synchronized state is the same as that of a time-invariant linear system having both an uncertain dimension and an uncertain parameter. For such a linear system, it is quite difficult to obtain the necessary and sufficient condition of the stability. However, a simple sufficient condition enables us to provide the design. The analytical results are confirmed through numerical examples.

Key Words: chaotic synchronization, delay-coupled map network, Bernoulli map, delay-independent design

1. Introduction

Synchronization is a typical nonlinear phenomenon in coupled oscillators \cite{1}. The local stability of a synchronized state is the same as that of a time-varying linear system \cite{2,3}. Since an analytical condition for such a time-varying linear system to be stable is not easy to obtain, numerical simulations are required for analyzing the stability of the synchronized state.

Connections in coupled oscillators are generally realized by information interaction between oscillators. The propagation speed of information, which is inversely proportional to the time delay in the connection, naturally affects the dynamics of the coupled oscillators. In recent years, synchronizations
in delay-coupled oscillators have attracted a great deal of attention in the field of nonlinear science [4, 5].

Local stability analysis of a synchronized state in delay-coupled oscillators is difficult. This is because a time-varying linear system with delay governing local stability demonstrates that (i) the dimension of the system becomes infinite due to its delay and (ii) the system has a time-varying parameter that depends on the synchronized state. On the other hand, delay-coupled map networks do not exhibit (i) because they have discrete-time dynamics. Thus, their stability analysis is easier than that of delay-coupled oscillators. In addition, for such map networks, it is possible to numerically calculate high-precision orbits quite rapidly due to the fact that numerical integration is not required. Therefore, map networks are suitable for investigation of the dynamics of large scale networks with delay.

A number of studies have investigated synchronizations in delay-coupled map networks [6]. However, few studies have examined the analytical conditions for local stability, because (ii) remains. Recently, in order to avoid (ii), Bernoulli maps are often used as network maps, because local stability of the synchronized state can be described by the stability of a time-invariant linear system. Then, we can easily derive an analytical condition for stability. As a result, several researchers have investigated synchronizations in delay-coupled Bernoulli map networks. Kestler et al. found a sublattice synchronization [7, 8]; Kinzel et al. provided an analytical result indicating that synchronization rarely occurs when the connection delay is large [9]. Englert et al. analyzed the dynamics with multiple delay times in connection [10]. Zigzag et al. found a ratio between the self-feedback delay and the connection delay [11, 12]. Kanter et al. investigated synchronizations in networks with a unidirectional connection [13, 14]. Zeeb et al. analyzed the attractor dimension in networks [15] and the response of the synchronized state in networks to external perturbation [16].

Previous studies on coupled oscillators attempted to clarify a mechanism for various nonlinear phenomena. Based on such studies, a systematic design procedure for the connection to induce a desired phenomenon in coupled oscillators has created considerable interest in recent years. As a result, a design problem for inducing the desired phase dynamics in coupled oscillators has been proposed [17–20], and design problems for inducing the amplitude death phenomenon have been solved using a robust control theory [21–24].

The present study proposes a systematic design procedure for a map parameter and a connection strength for inducing chaotic synchronization in delay-coupled Bernoulli map networks. The proposed procedure deals with a practical situation in which the number of maps, the time delay in connection, and the detailed topology information are unknown. We demonstrate that the local stability is the same as that of a time-invariant linear system having both an uncertain dimension and an uncertain parameter, where the dimension depends on both the number of maps and the time delay in connection and the parameter depends on both the number of maps and the network topology. Therefore, it is quite difficult to provide a necessary and sufficient condition for local stability because the linear system has such uncertain dimension and parameters. As such, we introduce a simple sufficient stability criterion for time-invariant linear systems, which is derived from the field of control theory [25]. This criterion is the basis of the proposed design procedure for inducing chaotic synchronization. Numerical examples are presented in order to confirm the validity of the proposed design procedure. The present paper is a substantially extended version of our conference paper [26].

2. Delay-coupled Bernoulli map networks

Let us consider the following chaotic maps:

\[ x_i(n + 1) = f[x_i(n)] + \varepsilon u_i(n), \quad (i = 1, \ldots, N), \]  

(1)

where \( x_i(n) \in \mathbb{R} \) and \( u_i(n) \in \mathbb{R} \) are the state and connection signal of map \( i \) at time \( n \in \mathbb{Z} \), respectively. \( f : [0, 1) \rightarrow [0, 1) \) denotes the Bernoulli map\(^1\) \( f(x) := (ax) \mod 1 \), where \( a > 0 \) is the map parameter.

\(^1\)A Bernoulli map produces a sequence of independent identically distributed random variables. Thus, it has significant applications in digital communication systems [27].
Also, \( N \in \mathbb{Z}^+ \) is the number of maps, and \( \varepsilon \in [0,1] \) is the connection strength. The connection signal \( u_i(n) \) is described by

\[
u_i(n) = \frac{1}{d_i} \left\{ \sum_{j=1}^{N} c_{ij} f[x_j(n-\tau)] \right\} - f[x_i(n)],
\]

where \( x_j(n-\tau) \) is the past state of map \( j \), and \( \tau \in \mathbb{Z}^+ \) is the time delay in connection \([6]^{2}\). Here, \( c_{ij} \) governs the network topology. If map \( i \) is connected to map \( j \), then \( c_{ij} = c_{ji} = 1 \), otherwise \( c_{ij} = c_{ji} = 0 \). The self-delayed feedback is forbidden: \( c_{ii} = 0 \). The degree of map \( i \) is denoted by \( d_i := \sum_{j=1}^{N} c_{ij} \). In the present study, we assume that there is no isolated map (i.e., \( d_i > 0 \)). The network topology is described by a matrix:

\[
C := \begin{bmatrix}
0 & c_{12}/d_1 & \cdots & c_{1N}/d_1 \\
c_{21}/d_2 & 0 & \cdots & c_{2N}/d_2 \\
\vdots & \vdots & \ddots & \vdots \\
c_{N1}/d_N & c_{N2}/d_N & \cdots & 0
\end{bmatrix}.
\]

Note that the eigenvalues \( \rho_q \) \((q = 1, \ldots, N)\) of asymmetric matrix \( M := I_N - C \) are always within the range \( \rho_q \in [0,2] \) \([22, 23, 28, 29]\), i.e.,

\[
0 = \rho_1 \leq \rho_2 \leq \cdots \leq \rho_N \leq 2,
\]

for any network topology. Furthermore, there is an eigenvector \([1 \ \cdots \ 1]^T\) corresponding to \( \rho_1 = 0 \).

The synchronized manifold in networks is described by

\[
s(n) := x_1(n) = x_2(n) = \cdots = x_N(n).
\]

Substituting this state into map network (1) and (2), we obtain the dynamics of the synchronized state \( s(n) \):

\[
s(n+1) = (1 - \varepsilon)f[s(n)] + \varepsilon f[s(n-\tau)].
\]

Note that \( s(n) \) is governed only by a single delayed map (6), where \( \varepsilon \) is the weight of the current state \( f[s(n)] \) and the past state \( f[s(n-\tau)] \). Next, we attempt to solve the main problem. This problem involves the systematic design of the connection strength \( \varepsilon \) and the map parameter \( a \) for the case in which the connection delay \( \tau \), number of maps \( N \), and the connection topology \( c_{ij} \) are all unknown, where a part of the information about network topology (i.e., the lower limit \( \rho_{\min} \leq \rho_2 \) and upper limit \( \rho_{\max} \geq \rho_N \) of the eigenvalues) is obtained in advance. The main reason why we consider the above problem is that, in the real world, it is difficult to obtain the exact value of \( \rho_2 \) and \( \rho_N \) for large-scale networks.

### 3. Stability analysis

The errors around the synchronized manifold (5),

\[
\delta x_i(n) := x_i(n) - s(n), \ (i = 1, \ldots, N),
\]

are substituted into map network (1) and (2) and provide the error dynamics:

\[
\delta x(n+1) = (1 - \varepsilon)a\delta x(n) + \varepsilon aC\delta x(n-\tau),
\]

where \( \delta x(n) := \begin{bmatrix} \delta x_1(n) & \delta x_2(n) & \cdots & \delta x_N(n) \end{bmatrix}^T \). This is a time-invariant linear system, because \( \{df(x)/dx\}_{x=s_{(n)}} = a \) always holds. As a result, the stability of the synchronized state (5) can be described as that of linear system (8). Accordingly, we analyze the stability of linear system (8) below.

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\(^{2}\)Most connection delays in real-world networks are randomly distributed. However, in order to simplify the stability analysis, the present study deals only with networks in which all of the connection delays are identical.
Note that matrix $M = I_N - C$ can be diagonalized as $T^{-1}MT = \text{diag}(\rho_1, \ldots, \rho_N)$ with a diagonal transformation matrix $T$ [22, 23, 28, 29]. The transformation $\delta x(n) = T\xi(n)$ allows us to obtain

$$\xi(n + 1) = (1 - \varepsilon)a\xi(n) + \varepsilon a T^{-1}(I_N - M)T\xi(n - \tau).$$

Note also that the stability of synchronized manifold (5) can be reduced to that of dynamics (10) for all $q$ modes $q = 2, \ldots, N$. The second term on the right-hand side of Eq. (9) can be diagonalized. Thus, the dynamics of system (9) can be divided into $N$ modes $q$ ($q = 1, \ldots, N$), described by

$$\xi_q(n + 1) = (1 - \varepsilon)a\xi_q(n) + \varepsilon a(1 - \rho_q)\xi_q(n - \tau).$$

The characteristic polynomial of mode $q$ is given by

$$g(z, \rho_q) := z^{q+1} - (1 - \varepsilon)a z^q + \varepsilon a(\rho_q - 1).$$

The mode $q = 1$ indicates the time development of synchronized manifold (5), which is the dynamics of single delayed map (6). This map must behave chaotically in order to realize chaotic synchronization. The following lemma provides a condition for map (6) to be chaotic.

**Lemma 1.** If the map parameter satisfies $a > 1$, then the single delayed map (6) behaves chaotically for any $\varepsilon \in [0, 1]$ and any $\tau \in \{1, \ldots\}$.

**Proof.** The dynamics of the single delayed map (6) is governed by $q = 1$ mode dynamics:

$$\xi_1(n + 1) = (1 - \varepsilon)a\xi_1(n) + \varepsilon a\xi_1(n - \tau).$$

Its characteristic equation is written as

$$g(z, 0) = z^{q+1} - (1 - \varepsilon)a z^q - \varepsilon a.$$  

For $z = 1$ we have $g(1, 0) = 1 - a$. On the other hand, for real positive $z$, we see $\lim_{z \to +\infty} g(z, 0) = +\infty$. Thus, based on the continuity of $g(z, 0)$ over $z \in [1, +\infty)$, there exists at least one real root of $g(z, 0) = 0$ greater than 1, irrespective of both $\varepsilon$ and $\tau$, provided that $g(1, 0) < 0 \Leftrightarrow a > 1$ holds. Therefore, if $a > 1$, then the single delayed map (6) is unstable for any $\varepsilon \in [0, 1]$ and any $\tau \in \{1, \ldots\}$. Furthermore, in the right-hand side of Eq. (6), the first and second terms are respectively bounded in the intervals $[0, 1 - \varepsilon]$ and $[0, \varepsilon]$ due to $f : [0, 1) \to [0, 1)$. Thus, the single delayed map (6) is bounded in the interval $[0, 1)$. As a result, if $a > 1$, the single delayed map (6) behaves chaotically for any $\varepsilon \in [0, 1]$ and any $\tau \in \{1, \ldots\}$. \hfill $\square$

The extended version of Lemma 1, which is for the multiple-delay model, is shown in the previous study [10].

Next, we consider the stabilization of synchronized state (5), which is defined by

$$\lim_{n \to +\infty} |x_i(n) - x_j(n)| = 0, \forall i, j \in \{1, \ldots, N\}. \quad (12)$$

Note that the stability of synchronized manifold (5) can be reduced to that of dynamics (10) for all modes $q = 2, \ldots, N$ except mode $q = 1$. The necessary and sufficient condition for the stability of synchronized manifold (5) is governed by

$$G(z) := \prod_{q=2}^N g(z, \rho_q). \quad (13)$$

As mentioned in the preceding section, it is supposed that we can obtain only the upper ($\rho_{\text{max}}$) and lower ($\rho_{\text{min}}$) limits of the eigenvalues $\rho_q$, ($q = 2, \ldots, N$), and that the number of maps $N$, the connection delay $\tau$, and detailed information on topology are unknown. Since $g(z, \rho_q)$ depends on these pieces of unknown information, it is quite difficult to guarantee the stability of $G(z)$. The following section, however, shows that a simple sufficient stability criterion proposed in the field of control theory allows us to provide a simple systematic procedure for designing $\varepsilon$ and $a$ such that $G(z)$ is stable.
4. Systematic design procedure

The Jury stability criterion, a popular criterion [30] in control theory, gives us the necessary and sufficient condition for the stability of each mode dynamics \( g(z, \rho_q) \). Obviously, this condition depends strongly on the connection parameters \((\tau, \varepsilon)\), the number of maps \( N \), network topology \( \rho_q \), and system parameter \( a \). The design problem with unknown \( \tau, N, a \) and \( \rho_q \) of the present study cannot be solved by this criterion. The present study applies a simple sufficient stability criterion proposed by Mori and Kokame [25] to the problem of the present study.

**Lemma 2** ([25]). The \( m \)-dimensional real polynomial,

\[
h(z) = z^m + \alpha_1 z^{m-1} + \cdots + \alpha_{m-1} z + \alpha_m,
\]

is stable if

\[
1 - \alpha_m^2 > |\alpha_1 - \alpha_m \alpha_{m-1}| + |\alpha_2 - \alpha_m \alpha_{m-2}| + \cdots + |\alpha_{m-1} - \alpha_m \alpha_1|,
\]

is satisfied.

This criterion enables us to derive a sufficient condition for the stability of synchronized manifold (5), as follows:

**Theorem 1.** Consider delay-coupled Bernoulli map networks consisting of maps (1) and connection (2). Here, \( s(n) \) is chaotic, and its synchronized manifold (5) is stable for any \( \tau \in \{1, \ldots \} \) and any topology if all inequalities,

\[
1 < a, \quad -\frac{1-a}{\varepsilon a} < \rho_{\min}, \quad \rho_{\max} < \frac{1-a}{\varepsilon a} + 2,
\]

are satisfied.

**Proof.** The term \( g(z, \rho) \) in Eq. (11) fits the form (14) if we take

\[
\alpha_1 := -(1 - \varepsilon) a, \quad \alpha_2 := \alpha_3 = \cdots = \alpha_{m-1} = 0, \quad \alpha_m := \varepsilon a (\rho - 1), \quad m := \tau + 1.
\]

Hence, Eq. (17) simplifies condition (15):

\[
1 - \alpha_m^2 > |\alpha_1| + |\alpha_m \alpha_1|.
\]

As \( \varepsilon \in [0,1] \), we have \( \alpha_1 \leq 0 \). Then, condition (18) can be further simplified to

\[
1 + \alpha_1 > |\alpha_m|.
\]

Substituting Eq. (17) into this condition results in

\[
-\frac{1-a}{\varepsilon a} < \rho < \frac{1-a}{\varepsilon a} + 2.
\]

Consequently, note that if all eigenvalues \( \rho_q \) \((q = 2, \ldots, N)\) satisfy condition (20), i.e., condition (16) holds, then synchronized manifold (5) is stable. Furthermore, Lemma 1 guarantees that \( s(n) \) is chaotic due to \( 1 < a \).

Figure 1 is a sketch of the stability region (i.e., the shaded region) on the parameter space \((\varepsilon - \rho)\), where inequalities (20) are satisfied. Let us define three points: the intersection of the upper and lower curves (A), the intersection of the upper curve and line \( \varepsilon = 1 \) (B), and the intersection of the lower curve and line \( \varepsilon = 1 \) (C). We easily obtain these points as follows:

\[
A : \left( 1 - \frac{1}{a} , 1 \right), \quad B : \left( 1 , 1 + \frac{1}{a} \right), \quad C : \left( 1 , 1 - \frac{1}{a} \right).
\]
Next, we shall discuss the above analytical results. Based on Fig. 1, if both \( \rho_{\text{min}} \) and \( \rho_{\text{max}} \) are less than B and greater than C, there exists a connection strength \( \varepsilon \) for stabilization. For example, as shown in Fig. 1, stabilization can be guaranteed at \( \varepsilon = \varepsilon_2 \) but not at \( \varepsilon = \varepsilon_1 \). Furthermore, these three points move to point (1, 1), and the region shrinks with the increase in \( a \). Note that for \( a > 1 \), points B and C never touch lines \( \rho = 2 \) and \( \rho = 0 \), respectively. Thus, we can conclude that for \( \rho_{\text{min}} > 0 \) and \( \rho_{\text{max}} < 2 \), there exist a connection strength \( \varepsilon \) and a map parameter \( a \) for inducing stabilization.

The above discussion is summarized in the following corollaries.

**Corollary 1.** There exist a connection strength \( \varepsilon \) and a map parameter \( a \) satisfying Theorem 1 if \( 0 < \rho_{\text{min}} \) and \( \rho_{\text{max}} < 2 \) hold.

In contrast, if \( \rho_{\text{min}} = 0 \) or \( \rho_{\text{max}} = 2 \) hold, then we cannot use Theorem 1. A systematic design procedure based on Theorem 1 and Corollary 1 is described below.

**Corollary 2.** Assume \( 0 < \rho_{\text{min}} \) and \( \rho_{\text{max}} < 2 \) as mentioned in Corollary 1. If the map parameter \( a \) is chosen from

\[
 a \in (1, \pi], \quad \pi := \min \left\{ \frac{1}{|\rho_{\text{min}} - 1|}, \frac{1}{\rho_{\text{max}} - 1} \right\},
\]

and the connection strength \( \varepsilon \) is chosen from

\[
 \varepsilon \in (\xi, 1], \quad \xi := \max \left\{ -\frac{1 - a}{a\rho_{\text{min}}}, \frac{1 - a}{a(\rho_{\text{max}} - 2)} \right\},
\]

then Theorem 1 holds.

The flow chart in Fig. 2 presents a systematic design procedure for the map parameter \( a \) and the connection strength \( \varepsilon \) based on Corollaries 1 and 2.

Points B and C in Fig. 1 approach one another with increasing \( a \). Thus, we see that the stabilizable topologies are limited for large \( a \). Therefore, when considering various topologies, \( a > 1 \) should be as small as possible. Furthermore, for a weak connection, \( a > 1 \) should be small, and the topology should be changed such that \( \rho_{\text{min}} \) and \( \rho_{\text{max}} \) approach 1. In addition, the smallest connection-strength \( \varepsilon^* := 1 - 1/a \) can be used for complete network (i.e., all-to-all) topology with sufficiently large maps (i.e., \( N \gg 1 \)) because \( \rho_2 = \rho_N \approx 1 \) holds for such a topology. In other words, for \( \varepsilon < \xi^* \), stabilization with any topology cannot be guaranteed.

**5. Numerical examples**

Let us design \( a \) and \( \varepsilon \) according to the flow chart shown in Fig. 2. Suppose that \( \rho_{\text{min}} = 0.26 \) and \( \rho_{\text{max}} = 1.75 \) are given in advance and that they satisfy Corollary 1. Next, from Corollary 2, \( a = 1.2 \).
Fig. 2. Flow chart for designing system parameter $a$ and connection strength $\varepsilon$.

Fig. 3. Time series data of deviation from the first map state, $x_i(n) - x_1(n)$ for $i = 2, \ldots, 6$, in a network in which the connection delay and topology are varied, and diagram of the topologies of each time interval.

is chosen from $a \in (1, 1/0.75)$. Finally, from Corollary 2, $\varepsilon = 0.7$ is chosen from $\varepsilon \in (0.2/0.3, 1]$. This design procedure suggests that, if we use $a = 1.2$ and $\varepsilon = 0.7$, then chaotic synchronization occurs for any $\tau \in \{1, \ldots\}$ and for any topologies for which the lower and upper eigenvalues satisfy $\rho_2 > \rho_{\min} = 0.26$ and $\rho_N < \rho_{\max} = 1.75$.

The designed values $a = 1.2$ and $\varepsilon = 0.7$ are numerically applied to a network in which the connection delay and topology are changed, as shown in Fig. 3. The time series data$^3$ of deviation from the first map state, $x_i(n) - x_1(n)$ for $i = 2, \ldots, 6$, are shown in Fig. 3. For $n \in [0, 500)$ (time interval (I)), six isolated maps without connection run freely. For $n \in [500, 1000)$(time interval (II)), six maps ($N = 6$) are coupled with connection delay $\tau = 4$ on a ring topology. Since this topology, with $\rho_2 = 0.5000$ and $\rho_6 = 2.0000$, does not satisfy Corollary 1, the proposed procedure cannot guarantee stability. As indicated by time interval (II) in Fig. 3, some maps are synchronized with the first map, although others are not. For $n \in [1000, 1500)$ (time interval (III)), six maps ($N = 6$) are coupled with connection delay $\tau = 3$. Since this network has two shortcuts on the ring topology, we

$^3$The small uniformly distributed random signals within $[-1 \times 10^{-4}, +1 \times 10^{-4}]$ are added to the right-hand side of map dynamics (1) in order to confirm the local stability against a small external disturbance in our numerical examples.
Averaged transient time $T_{\text{ave}}$ for the convergence of chaotic synchronization against (a) the connection delay $\tau$ with $N = 9$ and (b) the number of maps $N$ with $\tau = 5$. The following two network topologies are used: the ring network, in which each map is connected to four neighbors, and the complete network. One hundred initial states, which are uniformly and randomly chosen from $(0, 1)$, are used for averaging.

have $\rho_2 = 0.5497$ and $\rho_6 = 1.7287$. Note that the designed $a = 1.2$ and $\varepsilon = 0.7$ are valid for stabilizing the synchronized state because we see that all maps are synchronized after some transient period.

For $n \in [1500, 2000)$ (time interval (IV)), five maps ($N = 5$) are coupled with connection delay $\tau = 2$, and one map is isolated. Since we have $\rho_2 = 0.7257$ and $\rho_5 = 1.6076$, the stability for synchronization is analytically guaranteed. The five maps are synchronized and the isolated map (i.e., map 5) runs freely. For $n \in [2000, 2500)$ (time interval (V)), four maps ($N = 4$) are coupled with connection delay $\tau = 4$ and two maps are isolated. The stability for synchronization is analytically guaranteed due to the fact that $\rho_2 = 1.0000$ and $\rho_4 = 1.6667$ and the four maps are synchronized, whereas the other isolated maps (i.e., maps 2 and 5) run freely.

Practically speaking, it is important to obtain information on the transient period for stabilization. Next, we investigate the relationship between the connection delay $\tau$ and the transient period from the numerical simulations. We consider a ring network ($N = 9$), in which each map is connected to four neighbors, and a complete network ($N = 9$). The parameters are set as $a = 1.2$ and $\varepsilon = 0.7$. The transient period is estimated by conducting the following procedure: (a) set $\tau = 1$; (b) choose initial states of all maps and their delayed states uniformly and randomly from the range $(0, 1)$; (c) run the network numerically and estimate the transient period $T$; (d) repeat steps (b) and (c) 100 times, and estimate the average period $T_{\text{ave}}$; (e) change $\tau$ and go back to step (b). The transient period $T$ in step (c) is numerically estimated by judging $\sum_{j=2}^{9} (x_1(n) - x_j(n))^2 < 0.001$ for any $n \in [T, T + 50]$. The upper limit of time development is set to $n = 5 \times 10^6$. The average period $T_{\text{ave}}$ estimated using the above procedure is plotted with respect to the connection delay $\tau$ in Fig. 4(a). The period increases exponentially with increasing delay for the two networks. This suggests that significant time is required to achieve chaotic synchronization with large connection delays. An explanation of the increase in the transient period is given below. Let us restate the following four facts: (1) since the synchronized manifold is locally stable, all maps with initial states close to the manifold converge on the manifold for a short time; (2) since the dimension of error dynamics (9) from the manifold is $N(\tau + 1)$, the volume of phase space for this dynamics increases exponentially with increasing $\tau$; (3) this manifold corresponds to a point in the $N(\tau + 1)$-dimensional phase space; and (4) all maps wander in the phase space during the transient period. Based on these facts, the period might be...
considered to be approximately proportional to the volume of the phase space.

In order to verify the above facts, we numerically estimate the relation between the transient period and the number of maps $N$. Again, we consider the ring network and the complete network. The parameters $a$ and $\varepsilon$ are the same as those in Fig. 4(a), and $\tau$ is fixed at $\tau = 5$. The estimated transient period for $N \in \{8, \ldots, 13\}$ is shown in Fig. 4(b). For the ring network, the transient period increases with increasing $N$. This result supports the fact that the period is approximately proportional to the volume of the phase space. However, for the complete network, the period decreases with increasing $N$, which suggests that the relation among $T$, $\tau$, $N$, and $C$ should still be considered as an open problem.

6. Discussion
The proposed design procedure is meant to be applied to the Bernoulli map, the slope $a$ of which does not depend on the map state $x_i(n)$. This implies that the proposed procedure cannot be used for general one-dimensional maps, the slopes of which are varied in accordance with the map state. We have analytically confirmed that Corollary 2 is valid if we use the maximum value of the time-varying slope instead of the map parameter $a$. Although this result guarantees the stability of the synchronized state, we do not always observe the chaotic synchronization; that is, periodic synchronization and fixed-point synchronization (i.e., amplitude death) are sometimes observed. The details of the above argument will be reported elsewhere.

Let us next clarify the relation between the present paper and previous studies on delay-coupled systems in the long delay limit [31, 32], the results of which are strongly related to those of the present study. Englert et al. investigated chaotic synchronization in delay-coupled Bernoulli maps, which are a general model of the maps given by Eqs. (1) (2), with the self-delayed feedback and multiple delays [10]. They analytically provided the necessary and sufficient condition for chaotic synchronization in the long delay limit. However, for not long delay, the stability of chaotic synchronization was numerically analyzed using the Schur-Cohn theorem. On the other hand, the present paper analytically guarantees the stability of chaotic synchronization for an arbitrary delay (i.e., from the short delay limit to the long delay limit). Note that sufficient condition (20) of the present paper is equivalent to their necessary and sufficient condition in the long delay limit. This means that the stability region of chaotic synchronization in the parameter space for the long delay limit is included in the parameter space for an arbitrarily delay. Moreover, in the long delay limit, chaos in delay-coupled systems is classified into two categories: strong chaos, in which two nearby orbits separate quickly, and weak chaos, in which two nearby orbits separate slowly. Synchronization does not occur in networks with strong chaos in the long delay limit [32, 33]. We clarified that the smallest connection-strength $\varepsilon^*$ for an arbitrary delay corresponds to the border between strong chaos ($\varepsilon < \varepsilon^*$) and weak chaos ($\varepsilon > \varepsilon^*$), which means that the necessary condition for chaotic synchronization in the long delay limit (i.e., weak chaos) can be considered as that for an arbitrary delay.

7. Conclusion
In the present study, a systematic design procedure for the connection strength and map parameter in delay-coupled Bernoulli map networks for chaotic synchronization is proposed based on the simple sufficient stability criterion for time-invariant linear systems found in control theory. The primary advantage of this procedure is that it can be applied to the severe situation in which the number of maps, the time delay in connection, and the detailed topology information are unknown. In order to confirm the validity of the proposed design procedure, numerical examples using computer simulations were presented.

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