On the convergences and applications of the inertial-like proximal point methods for null point problems

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Abstract
Motivated and inspired by the discretization of the nonsmooth system of a nonlinear oscillator with damping, we propose what we call the inertial-like proximal point algorithms for finding the null point of the sum of two maximal operators, which has many applied backgrounds, such as, convex optimization and variational inequality problems, compressed sensing etc. The common feature of the presented algorithms is using the new inertial-like proximal point method which does not involve the computation for the norm of the difference between two adjacent iterates $x_n$ and $x_{n-1}$ in advance, and avoids complex inertial parameters satisfying the traditional and difficult checking conditions. Numerical experiments are presented to illustrate the performances of the algorithms.

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1 Introduction

Let $A : H \to 2^H$ be a set-valued operator in real Hilbert space $H$.

(1) The graph of $A$ is given by
$$gphA = \{(x, y) \in H \times H : y \in Ax\}.$$

(2) The operator $A$ is said to be monotone if
$$\langle x - y, u - v \rangle \geq 0, u \in Ax, v \in Ay, \forall x, y \in H.$$

(3) The operator $A$ is said to be maximal monotone if $gphA$ is not properly contained in the graph of any other monotone operator.
Let $A$, $B$ be two maximal monotone operators in $H$. We are concern with the well-studied null point problem that is formulated as follows:

$$0 \in (A + B)x^*$$  \hspace{1cm} (1.1)

with solution set denoted by $\Omega$.

A special interesting case of (1.1) is a minimization of sum of two proper, lower semi-continuous and convex functions $f, g : H \to \mathbb{R}$, that is,

$$\min_{x \in H} \{f(x) + g(x)\}. \hspace{1cm} (1.2)$$

So this is equivalent to (1.1) with $A = \partial f$ and $B = \partial g$ being the subdifferentials of $f$ and $g$, respectively.

We recall the resolvent operator $J_r^A = (I + rA)^{-1}, r > 0$, which is called the backward operator and plays an significant role in the approximation theory for zero points of maximal monotone operators. Due to the work of Aoyama et al. [3], we have the following properties:

$$\langle J_r^A x - y, x - J_r^A x \rangle \geq 0, \quad y \in A^{-1}(0), \hspace{1cm} (1.3)$$

where $A^{-1}(0) = \{z \in H : 0 \in Az\}$. Moreover, the following key facts that will be needed in the sequel.

**Fact 1:** The resolvent is not only always single-valued, but also firmly monotone:

$$\|x - y\|^2 - \|(I - J_r^A)x - (I - J_r^A)y\|^2 \geq \|J_r^A x - J_r^A y\|^2. \hspace{1cm} (1.4)$$

**Fact 2:** Using the resolvent operator, we can write down inclusion problem (1.1) as a fixed point problem. It is known that

$$x^* = J_\lambda^A (I - \lambda B)x^*, \lambda > 0.$$

Douglas-Rachford’s splitting method (DRSM) (or forward-backward splitting method) was first introduced in [10] as an operator splitting technique to solve partial differential equations arising in heat conduction and soon later has been extended to find solutions for the sum of two maximal monotone operators by Lions and Mercier [13]. Douglas-Rachford’s splitting method(DRSM) is formulated as

$$x_{n+1} = J_\lambda^A (I - \lambda B)x_n, \hspace{1cm} (1.5)$$

where $\lambda > 0$, and $(I - \lambda B)$ is called the forward operator.

Basing on the method (1.5), many researchers improved and modified the algorithms for the inclusion problem (1.1) and obtained nice results, see e.g., Boikanyo [7], Dadashi and Postolache [8], Kazmi and Rizvi [12], Moudafi [20][19], Sitthithakerngkiet et al. [27].

On the other hand, one classical way of looking at the null point problem $0 \in Ax$ is to consider the forward discretization for $\frac{dx}{dt} \approx \frac{x_{n+1} - x_n}{h_n}, \forall h_n > 0$ in the evolution system:

$$\begin{cases}
\frac{dx}{dt} + Ax(t) \geq 0,
\end{cases} \hspace{1cm} (1.6)$$

$$x(0) = x_0,$$
and then the evolution system is discretized as
\[
\frac{x_{n+1} - x_n}{h_n} + Ax_{n+1} \ni 0 \iff (I + h_nF)x_{n+1} = x_n,
\]
which inspired Alvarez and Attouch [2] to introduce the inertial method for the following nonsmooth case of a nonlinear oscillator with damping
\[
\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + A(x(t)) = 0, \quad a.e. t \geq 0.
\]
By discretizing, Alvarez and Attouch [2] obtained the implicit iterative sequence
\[
x_{n+1} = x_n + \alpha_n(x_n - x_{n-1}) + \lambda_n A(x_{n+1}) \ni 0,
\]
where \(\alpha_n = 1 - \gamma h_n\) and \(\lambda_n = h_n^2\), which yielded the Inertial-Prox algorithm
\[
x_{n+1} = J^A_{\lambda_n}(x_n + \alpha_n(x_n - x_{n-1})),
\]
the extrapolation term \(\alpha_n(x_n - x_{n-1})\) is called inertial term.

Note that when \(\alpha_n \equiv 0\), the recursion (1.8) corresponds to the standard proximal iteration
\[
x_{n+1} - x_n + \lambda_n A(x_{n+1}) \ni 0,
\]
which has been well studied by Martinet [15] and Moreau [17] and other researchers, and the weak convergence of \(x_n\) to a solution of \(0 \in Ax\) has been well known since the classical work of Rockafellar [25].

For ensuring the convergence of the Inertial-Prox sequence, Alvarez and Attouch [2] pointed the following key assumption: there exists \(\alpha \in (0, 1)\) such that \(\alpha_n \in [0, \alpha]\) and
\[
\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\|^2 < \infty.
\]

Inertial method is shown to have nice convergence properties in the field of continuous optimization and is studied intensively in split inverse problem by many authors soon later (because which could be utilized in some situations to accelerate the convergence of the sequences). For some recent works applied to various fields, see Alvarez [1], Attouch et al. [4–6], Ochs et al. [22, 23].

Although Alvarez and Attouch [2] pointed that one can choose appropriate rule to enable the assumption (1.10) applicable, the parameter \(\alpha_n\) involving the iterates \(x_n\) and \(x_{n-1}\) should be computed in advance. Namely, to make the condition (1.10) hold, the researchers have to set constraints on the inertia coefficient \(\alpha_n\) and estimate the value of the \(\|x_n - x_{n-1}\|\) before choosing \(\alpha_n\). In recent works, Gibali et al. [11] improved the inertial control condition by constraining inertia coefficient \(\alpha_n\) such that \(0 < \alpha_n < \bar{\alpha}_n\), where \(\alpha \in (0, 1), \epsilon_n \in [0, \infty)\) and \(\sum_{n=0}^{\infty} \epsilon_n < \infty\),
\[
\bar{\alpha}_n = \begin{cases} 
\min \{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|^2}\}, & \text{if } x_n \neq x_{n-1}, \\
\alpha, & \text{otherwise}.
\end{cases}
\]
More works on inertial methods, one can refer Dang et al. [9], Moudafi et al. [18], Suantai et al. [26], Tang [28], and therein.

Theoretically, the condition (1.10) on the parameter $\alpha_n$ is too strict for the convergence of the inertial algorithm. Practically, estimating the value of the $\|x_n - x_{n-1}\|$ before choosing the inertial parameter $\alpha_n$ may need large amount of computation. The two drawbacks may make the Inertial-Prox method inconvenient in the practical test in the sense. So it is natural to think about the following question:

**Question 1.1** Can we delete the condition (1.10) in inertial method? Namely, can we construct a new inertial algorithm for solving (1.1) without any constraint on the the inertial parameter or the computation of norm of the difference between $x_n$ and $x_{n-1}$ before choosing the inertial parameter $\alpha_n$?

The purpose of this paper is to present an affirmative answer to the above question. In this paper, we study the convergence problem of a new inertial-like technique for the solution of the null point problem (1.1) without the assumption (1.10) and the prior computation of the $\|x_n - x_{n-1}\|$ before choosing the inertial parameter $\theta_n$.

The outline of the paper is as follows. In section 2, we collect definitions and results which are needed for our further analysis. In section 3, our novel approach for the null point problem is introduced and analyzed, the convergence theorems of the presented algorithms are obtained. Moreover, convex optimization and variational inequality problem are studied as the applications of the null point problem in section 4. Finally, in section 5, some numerical experiments, using the inertial-like method, are carried out in order to support our approach.

## 2 Preliminaries

Let $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ be the inner product and the induced norm in a Hilbert space $H$, respectively. For a sequence $\{x_n\}$ in $H$, denote $x_n \to x$ and $x_n \rightharpoonup x$ by the strong and weak convergence to $x$ of $\{x_n\}$, respectively. Moreover, the symbol $\omega_w(x_n)$ represents the $\omega$-weak limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) := \{ x \in H : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\} \}.$$  

The identity below is useful:

$$\| \alpha x + \beta y + \gamma z \|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2$$

$$-\alpha \beta \|x - y\|^2 - \beta \gamma \|y - z\|^2 - \gamma \alpha \|x - z\|^2$$  \quad (2.1)

for all $x, y, z \in R$ and $\alpha + \beta + \gamma = 1$.

**Definition 2.1.** Let $H$ be a real Hilbert space, $D \subset H$ and $T : D \to H$ some given operator.

1. The operator $T$ is said to be **Lipschitz continuous** with constant $\kappa > 0$ on $D$ if

$$\|T(x) - T(y)\| \leq \kappa \|x - y\|, \ \forall x, y \in D.$$
The operator $T$ is said to be $\gamma$-cocoercive if there exists $\gamma > 0$ such that
\[
\langle Tx - Ty, x - y \rangle \geq \gamma \|T(x) - T(y)\|^2, \quad \forall x, y \in D.
\]

**Remark 2.2.**
(1) If $T$ is $\gamma$-cocoercive, then it is $\frac{1}{\gamma}$-Lipschitz continuous.
(2) From **Fact 1**, we can conclude that $J_A^r$ is a non-expansive operator if $A$ is a maximal monotone mapping.

**Definition 2.3.** Let $C$ be a nonempty closed convex subset of $H$. We use $P_C$ to denote the projection from $H$ onto $C$; namely,
\[
P_Cx = \arg \min \{\|x - y\| : y \in C\}, \quad x \in H.
\]

The following significant characterization of the projection $P_C$ should be recalled: Given $x \in H$ and $y \in C$,
\[
P_Cx = z \iff \langle x - z, y - z \rangle \leq 0, \quad y \in C. \tag{2.2}
\]

**Lemma 2.4.** (Xu [29]) Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that
\[
a_{n+1} \leq (1 - \beta_n)a_n + \beta_nb_n + c_n, \quad n \geq 0,
\]
where $\{\beta_n\}$ is a sequence in $(0,1)$ and $\{c_n\} \subset (0,\infty)$ and $\{b_n\} \subset \mathbb{R}$ such that
(1) $\sum_{n=1}^{\infty} \beta_n = \infty$;
(2) $\limsup_{n \to \infty} b_n \leq 0$ or $\sum_{n=1}^{\infty} |\beta_n|b_n < \infty$;
(3) $\sum_{n=1}^{\infty} c_n < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

**Lemma 2.5.** (see e.g., Opial [21]) Let $H$ be a real Hilbert space and $\{x_n\}$ be a bounded sequence in $H$. Assume there exists a nonempty subset $S \subset H$ satisfying the properties:
(i) $\lim_{n \to \infty} \|x_n - z\|$ exists for every $z \in S$,
(ii) $\omega_w(x_n) \subset S$.

Then, there exists $\bar{x} \in S$ such that $\{x_n\}$ converges weakly to $\bar{x}$.

**Lemma 2.6.** (Maingé [16]) Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at the infinity in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_{j+1}}$ for all $j \geq 0$. Also consider the sequence of integers $\{\sigma(n)\}_{n \geq n_0}$ defined by
\[
\sigma(n) = \max\{k \leq n : \Gamma_k \leq \Gamma_{k+1}\}.
\]

Then $\{\sigma(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n \to \infty} \sigma(n) = \infty$ and, for all $n \geq n_0$,
\[
\max\{\Gamma_{\sigma(n)}, \Gamma_n\} \leq \Gamma_{\sigma(n)+1}.
\]
3 Main Results

3.1 Motivation of Inertial-Like Proximal Technique

Inspired and motivated by the discretization (1.8), we consider the following iterative sequence

$$x_{n+1} - x_{n-1} - \theta_n (x_n - x_{n-1}) + \lambda_n A(x_{n+1}) \ni 0,$$

where $x_0, x_1$ are two arbitrary initial points, and $\{\lambda_n\}$ is a real nonnegative number sequence. This recursion can be rewritten as

$$\begin{align*}
    w_n &= x_{n-1} + \theta_n (x_n - x_{n-1}) \\
    x_{n+1} &= J_{\lambda_n}^A w_n. 
\end{align*}$$

(3.1)

The discretization sequence $\{x_n\}$ in (3.1) always exists because the sequence $\{x_n\}$ satisfying (1.8) always exists for any choice of the sequence $\{\alpha_n\}$ according to Alvarez and Attouch [2]. In addition, it can be deduced that if $\theta_n = 1 + \alpha_n$, the formula (3.1) can cover Alvarez and Attouch’s Inertial-Prox algorithm. More relevantly, the inertial coefficient $\theta_n$ can be considered 1 in our new inertial proximal point algorithms. We thus obtain what we call the Inertial-Like Proximal point algorithm.

3.2 Some Conventions and Inertial-like Proximal Algorithms

C1: Throughout the rest of this paper, we always assume that $H$ is a Hilbert space. We rephrase the null point problem as follows:

$$0 \in (A + B)x^*$$

(3.2)

where $A, B : H \to 2^H$ are two maximal monotone set-valued operators with $B \gamma$-cocoercive.

C2: Denote by $\Omega$ the solution set of the null point problem; namely,

$$\Omega = \{x^* \in H : 0 \in (A + B)x^*\}$$

and we always assume $\Omega \neq \emptyset$.

Now, combining the Fact 2 and the inertial-like technique (3.1), we introduce the following algorithms.

Algorithm 3.1

**Initialization:** Choose a positive sequence $\{\theta_n\} \subset [0, 1]$. Select arbitrary initial points $x_0, x_1$.

**Iterative Step:** After the $n$-iterate $x_n$ is constructed, compute

$$w_n = x_{n-1} + \theta_n (x_n - x_{n-1}), \quad n \geq 1,$$

(3.3)
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and define the \((n+1)\)th iterate by
\[
x_{n+1} = J_{\tau_n}^A(I - \tau_n B)w_n. \tag{3.4}
\]

**Remark 3.1** It is not hard to find that if \(w_n = x_{n+1}\), for some \(n \geq 0\), then \(x_{n+1}\) is a solution of the inclusion problem (3.2), and the iteration process is terminated in finite iterations. If \(\theta_n = 1\), Algorithm 3.1 reduces to the general *forward-backward* algorithm in Moudafi [19].

**Algorithm 3.2**

**Initialization:** Choose a sequence \(\{\theta_n\} \subset [0, 1]\) satisfying one of the three cases: (I.) \(\theta_n \in (0, 1)\) such that \(\lim_{n \to \infty} \theta_n(1 - \theta_n) > 0\); (II.) \(\theta_n \equiv 0\); (III.) \(\theta_n \equiv 1\). Choose \(\{\alpha_n\}\) and \(\{\beta_n\}\) in \((0, 1)\) such that
\[
\lim_{n \to \infty} \alpha_n > 0; \quad \lim_{n \to \infty} \alpha_n < 1; \quad \lim_{n \to \infty} \beta_n = 0, \quad \sum_{n=0}^{\infty} \beta_n = \infty.
\]

Select arbitrary initial points \(x_0, x_1\).

**Iterative Step:** After the \(n\)-iterate \(x_n\) is constructed, compute
\[
w_n = x_{n-1} + \theta_n(x_n - x_{n-1}),
\]
and define the \((n+1)\)th iterate by
\[
x_{n+1} = (1 - \alpha_n - \beta_n)w_n + \alpha_n J_{\tau_n}^A(I - \tau_n B)w_n. \tag{3.5}
\]

**Remark 3.2.** In the subsequent convergence analysis, we will always assume that the two algorithms generate an infinite sequence, namely, the algorithms are not terminated in finite iterations. In addition, in the simulation experiments, in order to be practical, we will give a stop criterion to end the iteration for practice. Otherwise, set \(n := n + 1\) and return to Iterative Step.

### 3.3 Convergence Analysis of Algorithms

**Theorem 3.1.** If the assumptions C1 – C2 are satisfied and \(\tau_n \in (\epsilon, 2\gamma - \epsilon)\) for some given \(\epsilon > 0\) small enough, we obtain the weak convergence result, namely the sequence \(\{x_n\}\) generated by Algorithm 3.1 converges weakly to a point \(\bar{x} \in \Omega\).

**Proof.** Without loss of generality, we take \(z \in \Omega\) and then we have get \(z = J_{\tau_n}^A(I - \tau_n B)z\) from Fact 2.

It turns out from (2.1) and (3.3) that
\[
\|w_n - z\|^2 = \|x_{n-1} + \theta_n(x_n - x_{n-1}) - z\|^2 = \theta_n\|x_n - z\|^2 + (1 - \theta_n)\|x_{n-1} - z\|^2 - \theta_n(1 - \theta_n)\|x_n - x_{n-1}\|^2. \tag{3.6}
\]
Since $J^A_\tau$ is firmly nonexpansive, it follows from (3.4) and Fact 1 that
\[
\|x_{n+1} - z\|^2 = \|J^A_\tau(I - \tau_nB)w_n - z\|^2 \\
\leq \|(I - \tau_nB)w_n - (I - \tau_nB)z\|^2 - \|(I - J^A_\tau)(I - \tau_nB)w_n - (I - J^A_\tau)(I - \tau_nB)z\|^2 \\
= \|(w_n - z) - \tau_n(Bw_n - Bz)\|^2 - \|w_n - x_{n+1} - \tau_n(Bw_n - Bz)\|^2 \\
= \|w_n - z\|^2 - 2\langle w_n - z, \tau_n(Bw_n - Bz) \rangle + \tau_n^2\|Bw_n - Bz\|^2 \\
- \|w_n - x_{n+1}\|^2 + 2\langle w_n - x_{n+1}, \tau_n(Bw_n - Bz) \rangle - \tau_n^2\|Bw_n - Bz\|^2 \\
= \|w_n - z\|^2 - \|w_n - x_{n+1}\|^2 - 2\langle w_n - z, \tau_n(Bw_n - Bz) \rangle \\
+ 2\tau_n\langle w_n - x_{n+1}, Bw_n - Bz \rangle. \tag{3.7}
\]
It follows from the fact $B$ is $\gamma$-cocoercive that
\[
\langle Bw_n - Bz, w_n - z \rangle \geq \gamma\|Bw_n - Bz\|^2,
\]
so we have from (3.7) that
\[
\|x_{n+1} - z\|^2 \leq \|w_n - z\|^2 - \|w_n - x_{n+1}\|^2 - 2\gamma\tau_n\|Bw_n - Bz\|^2 \\
+ 2\tau_n\langle w_n - x_{n+1}, Bw_n - Bz \rangle.
\]
On the other hand, we have
\[
2\gamma\tau_n\|Bw_n - Bz - \frac{w_n - x_{n+1}}{2\gamma}\|^2 \\
= 2\gamma\tau_n\left(\|Bw_n - Bz\|^2 + \left\|\frac{w_n - x_{n+1}}{2\gamma}\right\|^2 - 2\left\langle Bw_n - Bz, \frac{w_n - x_{n+1}}{2\gamma}\right\rangle\right) \\
= 2\gamma\tau_n\|Bw_n - Bz\|^2 + \frac{\tau_n}{2\gamma}\|w_n - x_{n+1}\|^2 - 2\tau_n\langle Bw_n - Bz, w_n - x_{n+1}\rangle
\]
and, furthermore,
\[
2\gamma\tau_n\|Bw_n - Bz\|^2 - 2\tau_n\langle Bw_n - Bz, w_n - x_{n+1}\rangle \\
= 2\gamma\tau_n\left\|Bw_n - Bz - \frac{w_n - x_{n+1}}{2\gamma}\right\|^2 - \frac{\tau_n}{2\gamma}\|w_n - x_{n+1}\|^2.
\]
Hence we obtain from (3.6), (3.7) that
\[
\|x_{n+1} - z\|^2 \leq \|w_n - z\|^2 - \|w_n - x_{n+1}\|^2 - 2\gamma\tau_n\|Bw_n - Bz\|^2 \\
+ 2\tau_n\langle w_n - x_{n+1}, Bw_n - Bz \rangle \\
= \|w_n - z\|^2 - 2\gamma\tau_n\left\|Bw_n - Bz - \frac{w_n - x_{n+1}}{2\gamma}\right\|^2 \\
+ \left(\frac{\tau_n}{2\gamma} - 1\right)\|w_n - x_{n+1}\|^2 \\
= \theta_n\|x_n - z\|^2 + (1 - \theta_n)\|x_{n-1} - z\|^2 - \theta_n(1 - \theta_n)\|x_n - x_{n-1}\|^2 \\
- 2\gamma\tau_n\left\|Bw_n - Bz - \frac{w_n - x_{n+1}}{2\gamma}\right\|^2 + \left(\frac{\tau_n}{2\gamma} - 1\right)\|w_n - x_{n+1}\|^2. \tag{3.8}
\]
Since $\tau_n \leq (\epsilon, 2\gamma - \epsilon)$, it follows from (3.3) that
\[
\|x_{n+1} - z\|^2 \leq \theta_n \|x_n - z\|^2 + (1 - \theta_n)\|x_{n-1} - z\|^2 \\
\leq \max\{\|x_n - z\|^2, \|x_{n-1} - z\|^2\} \\
\leq \ldots \\
\leq \max\{\|x_1 - z\|^2, \|x_0 - z\|^2\},
\]
which means that the sequence $\{\|x_n - z\|\}$ is bounded, and so in turn $\{w_n\}$ is.

Now we claim that there exists the limit of the sequence $\{\|x_n - z\|\}$. For this purpose, two situations are discussed as follows:

**Case 1.** There exists an integer $N_0 \geq 0$ such that $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $n \geq N_0$. Then there exists the limit of the sequence $\{\|x_n - z\|\}$, denoted by $l = \lim_{n \to \infty} \|x_n - z\|$, and so
\[
\lim_{n \to \infty} (\|x_n - z\|^2 - \|x_{n+1} - z\|^2) = 0.
\]

In addition, we have
\[
\sum_{n=0}^{\infty} (\|x_{n+1} - z\|^2 - \|x_n - z\|^2) = \lim_{n \to \infty} (\|x_{n+1} - z\| - \|x_0 - z\|) < \infty
\]
and therefore from (3.8) we get
\[
\lim_{n \to \infty} \theta_n (1 - \theta_n)\|x_n - x_{n-1}\|^2 \leq (\|x_n - z\|^2 - \|x_{n+1} - z\|^2) + (1 - \theta_n)(\|x_{n-1} - z\|^2 - \|x_n - z\|^2)
\]
and so
\[
\lim_{n \to \infty} \theta_n (1 - \theta_n)\|x_n - x_{n-1}\|^2 = 0, \quad \sum_{n=0}^{\infty} \theta_n (1 - \theta_n)\|x_n - x_{n-1}\|^2 < \infty.
\]

Now it remains to show that
\[
\omega_w(x_n) \subset \Omega.
\]
Since the sequence $\{x_n\}$ is bounded, let $\bar{x} \in \omega_w(x_n)$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ weakly converging to $\bar{x}$. To this end, it remains to verify that $\bar{x} \in (A + B)^{-1}(0)$.

Notice again (3.8), we have
\[
2\gamma \tau_n \left\|Bw_n - Bz - \frac{w_n - x_{n+1}}{2\gamma}\right\|^2 + (1 - \frac{\tau_n}{2\gamma})\|w_n - x_{n+1}\|^2 + \theta_n (1 - \theta_n)\|x_n - x_{n-1}\|^2 \\
\leq \theta_n \|x_n - z\|^2 + (1 - \theta_n)(\|x_{n-1} - z\|^2 - \|x_{n+1} - z\|^2) \\
= (\|x_n - z\|^2 - \|x_{n+1} - z\|^2) + (1 - \theta_n)(\|x_{n-1} - z\|^2 - \|x_n - z\|^2),
\]
which means that
\[
\left\|Bw_n - Bz - \frac{w_n - x_{n+1}}{2\gamma}\right\|^2 \to 0;
\]
and
\[ \|w_n - x_{n+1}\|^2 \to 0. \]

At the same time, it follows from (3.4) that
\[ \frac{w_n - x_{n+1}}{\tau_n} - (Bw_n - Bx_{n+1}) \in (A + B)x_{n+1}, \tag{3.9} \]

Since \( B \) is \( \gamma \)-cocoercive, we have that \( B \) is \( \frac{1}{\gamma} \)-Lipschitz continuous. Passing to the limit on the subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) converging weakly to \( \bar{x} \) in (3.9) and taking account that \( A + B \) is maximal monotone and thus its graph is weakly-strongly closed, it follows that
\[ 0 \in (A + B)\bar{x}, \]

which means that \( \bar{x} \in \Omega \).

In view of the fact that the choice of \( \bar{x} \) in \( \omega_w(x_n) \) was arbitrary, namely \( \omega_w(x_n) \subset \Omega \), and we conclude from Lemma 2.5 that \( \{x_n\} \) converges weakly to \( \bar{x} \in \Omega \).

**Case 2.** If the sequence \( \{\|x_{n_k} - z\|\} \) does not decrease at infinity in the sense, then there exists a sub-sequence \( \{n_k\} \) of \( \{n\} \) such that \( \|x_{n_k} - z\| \leq \|x_{n_{k+1}} - z\| \) for all \( k \geq 0 \). Furthermore, by Lemma 2.6 there exists an integer, non-decreasing sequence \( \sigma(n) \) for \( n \geq N_1 \) (for some \( N_1 \) large enough) such that \( \sigma(n) \to \infty \) as \( n \to \infty \),
\[ \|x_{\sigma(n)} - z\| \leq \|x_{\sigma(n)+1} - z\|, \|x_n - z\| \leq \|x_{\sigma(n)+1} - z\| \]
for each \( n \geq 0 \).

Notice the boundedness of the sequence \( \{\|x_n - z\|\} \), which implies that there exists the limit of the sequence \( \{\|x_{\sigma(n)} - z\|\} \) and hence we conclude that
\[ \lim_{n \to \infty} (\|x_{\sigma(n)+1} - z\|^2 - \|x_{\sigma(n)} - z\|^2) = 0. \]

From (3.8) with \( n \) replaced by \( \sigma(n) \), we have
\[ \|x_{\sigma(n)+1} - z\|^2 - \|x_{\sigma(n)} - z\|^2 \leq (\theta_{\sigma(n)} - 1)(\|x_{\sigma(n)} - z\|^2 - \|x_{\sigma(n)-1} - z\|^2) - \theta_{\sigma(n)}(1 - \theta_{\sigma(n)})\|x_{\sigma(n)} - x_{\sigma(n)-1}\|^2 \]
\[ + 2\gamma \tau_{\sigma(n)} \left\| Bw_{\sigma(n)} - Bz - \frac{w_{\sigma(n)} - x_{\sigma(n)+1}}{2\gamma} \right\|^2 + \left( \frac{\tau_{\sigma(n)}}{2\gamma} - 1 \right)\|w_{\sigma(n)} - x_{\sigma(n)+1}\|^2. \]

By a similar argument to **Case 1**, we obtain
\[ \left\| Bw_{\sigma(n)} - Bz - \frac{w_{\sigma(n)} - x_{\sigma(n)+1}}{2\gamma} \right\|^2 \to 0; \]

and
\[ \|w_{\sigma(n)} - x_{\sigma(n)+1}\|^2 \to 0. \]
These together with (3.4) implies that
\[
\frac{w_{\sigma(n)} - x_{\sigma(n)+1}}{\tau_{\sigma(n)}} - (Bw_{\sigma(n)} - Bx_{\sigma(n)+1}) \in (A + B)x_{\sigma(n)+1}.
\]

Since the sequence \( \{x_n\} \) is bounded, let \( \bar{x} \in \omega_w(x_n) \) and \( \{x_{\sigma(n)}\} \) be a subsequence of \( \{x_n\} \) weakly converging to \( \bar{x} \). Passing to the limit on the subsequence \( \{x_{\sigma(n)}\} \) of \( \{x_n\} \) converging weakly to \( \bar{x} \) in the above inequality and taking account that \( A + B \) is maximal monotone and thus its graph is weakly-strongly closed, it follows that
\[
0 \in (A + B)\bar{x},
\]
which means that \( \bar{x} \in \Omega \). In view of the fact that the choice of \( \bar{x} \) in \( \omega_w(x_n) \) was arbitrary, namely \( \omega_w(x_n) \subset \Omega \), and we conclude from Lemma 2.5 that \( \{x_n\} \) converges weakly to \( \bar{x} \in \Omega \).

This completes the proof.

Next we prove the strong convergence of Algorithm 3.2.

**Theorem 3.2.** If the assumptions C1 – C2 are satisfied and \( \tau_n \in (\epsilon, 2\gamma - \epsilon) \) for some given \( \epsilon > 0 \) small enough, we obtain the strong convergence result, namely the sequence \( \{x_n\} \) generated by Algorithm 3.2 converges in norm to \( z = P_\Omega(0) \) (i.e., the minimum-norm element of the solution set \( \Omega \)).

**Proof.** To illustrate the result clearly, the following three situations should be discussed: (I). \( \theta_n \in (0, 1) \); (II). \( \theta_n \equiv 0 \) and (III). \( \theta_n \equiv 1 \), respectively.

(I). First we consider the strong convergence under the assumption \( \theta_n \in (0, 1) \).

Similar to the previous proof of weak convergence, we begin by showing the boundedness of the sequence \( \{x_n\} \). To see this, we denote \( z_n = J_{\theta_n}^\beta ((I - \tau_nB)w_n) \) and we use the projection \( z := P_\Omega(0) \) in a similar way to the proof of (3.7) and (3.8) of Theorem 3.1 to get that
\[
\|z_n - z\|^2 \leq \|w_n - z\|^2 - \|w_n - z_n\|^2 - 2\langle w_n - z, \tau_n(Bw_n - Bz) \rangle + 2\tau_n\langle w_n - z_n, Bw_n - Bz \rangle
\]
\[
\leq \|w_n - z\|^2 + \left( \frac{\tau_n}{2\gamma} - 1 \right) \|w_n - z_n\|^2 - 2\gamma\tau_n\|Bw_n - Bz - \frac{w_n - z_n}{2\gamma}\|^2
\]
(3.10)
hence one can see \( \|z_n - z\| \leq \|w_n - z\| \).

It turns out from (3.3) and (3.5) that
\[
\|x_{n+1} - z\| = \|(1 - \alpha_n - \beta_n)w_n + \alpha_n z_n - z\|
\]
\[
= \|(1 - \alpha_n - \beta_n)(w_n - z) + \alpha_n(z_n - z) + \beta_n(-z)\|
\]
\[
\leq (1 - \alpha_n - \beta_n)\|w_n - z\| + \alpha_n\|z_n - z\| + \beta_n\|z\|
\]
\[
\leq (1 - \beta_n)(\theta_n\|x_n - z\| + (1 - \theta_n)\|x_{n-1} - z\|) + \beta_n\|z\|
\]
\[
\leq (1 - \beta_n)(\max\{\|x_n - z\|, \|x_{n-1} - z\|\}) + \beta_n\|z\|
\]
\[
\leq \ldots
\]
\[
\leq \max\{\|x_0 - z\|, \|x_1 - z\|, \|z\|\},
\]
which implies that the sequence \( \{x_n\} \) is bounded, and so are the sequences \( \{w_n\}, \{z_n\} \).

Applying the identity (2.1) we deduce that
\[
\|x_{n+1} - z\|^2 = \|(1 - \alpha_n - \beta_n)w_n + \alpha_n z_n - z\|^2
\]
\[
= \|(1 - \alpha_n - \beta_n)(w_n - z) + \alpha_n (z_n - z) + \beta_n (-z)\|^2
\]
\[
\leq (1 - \alpha_n - \beta_n)\|w_n - z\|^2 + \alpha_n \|z_n - z\|^2 + \beta_n \|z\|^2
\]
\[
- (1 - \alpha_n - \beta_n)\alpha_n \|z_n - w_n\|^2.
\]
(3.11)

Substituting (3.10) into (3.11) and after some manipulations, we obtain
\[
\|x_{n+1} - z\|^2 \leq (1 - \alpha_n - \beta_n)\|w_n - z\|^2 + \alpha_n \|w_n - z\|^2 + (\frac{\tau_n}{2\gamma} - 1)\|w_n - z_n\|^2
\]
\[
- 2\gamma \tau_n \|Bw_n - Bz - \frac{w_n - z_n}{2\gamma}\|^2 + \beta_n \|z\|^2
\]
\[
- (1 - \alpha_n - \beta_n)\alpha_n \|z_n - w_n\|^2.
\]

Combining (3.6) we have
\[
\|x_{n+1} - z\|^2 \leq (1 - \beta_n)\|w_n - z\|^2 + \beta_n \|z\|^2 - (1 - \alpha_n - \beta_n)\alpha_n \|z_n - w_n\|^2
\]
\[
+ \alpha_n (\frac{\tau_n}{2\gamma} - 1)\|w_n - z_n\|^2 - 2\gamma \tau_n \|Bw_n - Bz - \frac{w_n - z_n}{2\gamma}\|^2
\]
\[
= (1 - \beta_n)[\theta_n \|x_n - z\|^2 + (1 - \theta_n)\|x_{n-1} - z\|^2 - \theta_n (1 - \theta_n)\|x_n - x_{n-1}\|^2]
\]
\[
+ \beta_n \|z\|^2 + \alpha_n (\frac{\tau_n}{2\gamma} - 2 + \alpha_n + \beta_n)\|w_n - z_n\|^2
\]
\[
- 2\gamma \tau_n \alpha_n \|Bw_n - Bz - \frac{w_n - z_n}{2\gamma}\|^2.
\]
(3.12)

We explain the strong convergence under two situations, respectively.

**Case 1.** There exists an integer \( N_0 \geq 0 \) such that \( \|x_{n+1} - z\| \leq \|x_n - z\| \) for all \( n \geq N_0 \). Then there exists the limit of the sequence \( \{\|x_n - z\|\} \), denoted by \( l = \lim_{n \to \infty} \|x_n - z\| \), and so
\[
\lim_{n \to \infty}(\|x_n - z\|^2 - \|x_{n+1} - z\|^2) = 0.
\]

In addition, we have
\[
\sum_{n=0}^{\infty}(\|x_{n+1} - z\|^2 - \|x_n - z\|^2) = \lim_{n \to \infty}(\|x_{n+1} - z\| - \|x_0 - z\|) < \infty
\]
and therefore from (3.12) we obtain
\[
\alpha_n (2 - \frac{\tau_n}{2\gamma} - \alpha_n - \beta_n)\|w_n - z_n\|^2 + 2\gamma \tau_n \alpha_n \|Bw_n - Bz - \frac{w_n - z_n}{2\gamma}\|^2
\]
\[
+ \theta_n (1 - \theta_n) (1 - \beta_n)\|x_n - x_{n-1}\|^2
\]
\[
\leq (1 - \beta_n)\theta_n \|x_n - z\|^2 + (1 - \theta_n)\|x_{n-1} - z\|^2 + \beta_n \|z\|^2 - \|x_{n+1} - z\|^2
\]
\[
\leq \theta_n \|x_n - z\|^2 + (1 - \theta_n)\|x_{n-1} - z\|^2 + \beta_n \|z\|^2 - \|x_{n+1} - z\|^2
\]
\[
= \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + (1 - \theta_n)(\|x_{n-1} - z\|^2 - \|x_n - z\|^2) + \beta_n \|z\|^2,
\]
and then from the condition $\beta_n \to 0$, we get

$$
\lim_{n \to \infty} \theta_n (1 - \theta_n) (1 - \beta_n) \| x_n - x_{n-1} \|^2 = 0;
\lim_{n \to \infty} \alpha_n (2 - \frac{\tau_n}{2\gamma} - \alpha_n - \beta_n) \| w_n - z_n \|^2 = 0;
\lim_{n \to \infty} \gamma \tau_n \alpha_n \left\| B w_n - B z - \frac{w_n - z_n}{2\gamma} \right\|^2 = 0.
$$

Notice $\alpha_n (2 - \frac{\tau_n}{2\gamma} - \alpha_n - \beta_n) = \alpha_n (1 - \alpha_n - \beta_n) + \alpha_n (1 - \frac{\tau_n}{2\gamma})$ and notice the assumptions on the parameters $\theta_n, \beta_n, \tau_n$ and $\alpha_n$, hence we have

$$
\lim_{n \to \infty} \| x_n - x_{n-1} \|^2 = 0; \quad \lim_{n \to \infty} \| w_n - z_n \|^2 = 0;
\lim_{n \to \infty} \left\| B w_n - B z - \frac{w_n - z_n}{2\gamma} \right\|^2 = 0,
$$

which imply that $\| w_n - x_n \| = |1 - \theta_n| \cdot \| x_n - x_{n-1} \| \to 0$ and $\| x_{n+1} - w_n \| \leq \alpha_n \| z_n - w_n \| + \beta_n \| w_n \| \to 0$ as $n \to \infty$, and then

$$
\| x_{n+1} - x_n \| \leq \| x_{n+1} - w_n \| + \| w_n - x_n \| \to 0, \quad n \to \infty
$$

This proves the asymptotic regularity of $\{x_n\}$.

By repeating the relevant part of the proof of Theorem 3.1, we get $\omega_w(x_n) \subset \Omega$.

It is now at the position to prove the sequence $\{x_n\}$ strongly converges to $z = P_\Omega(0)$.

To this end, we rewrite $x_{n+1}$ as $x_{n+1} = (1 - \beta_n) v_n + \beta_n \alpha_n (z_n - w_n)$, where $v_n = (1 - \alpha_n) w_n + \alpha_n z_n$, and making use of the inequality $\| u + v \|^2 \leq \| u \|^2 + 2 \langle v, u + v \rangle$ which holds for all $u, v \in H$, we get

$$
\| x_{n+1} - z \|^2 = \| (1 - \beta_n) (v_n - z) + \beta_n (\alpha_n (z_n - w_n) - z) \|^2 \\
\leq (1 - \beta_n)^2 \| v_n - z \|^2 + 2 \beta_n \left\langle \alpha_n (z_n - w_n) - z, x_{n+1} - z \right\rangle.
$$

It follows from (2.1) that

$$
\| v_n - z \|^2 = (1 - \alpha_n) \| w_n - z \|^2 + \alpha_n \| z_n - z \|^2 - \alpha_n (1 - \alpha_n) \| z_n - w_n \|^2,
$$

and then

$$
\| x_{n+1} - z \|^2 \leq (1 - \beta_n)^2 \| (1 - \alpha_n) \| w_n - z \|^2 + \alpha_n \| z_n - z \|^2 - \alpha_n (1 - \alpha_n) \| z_n - w_n \|^2 \\
+ 2 \beta_n \left\langle \alpha_n (z_n - w_n) - z, x_{n+1} - z \right\rangle.
$$

Notice that $\| z_n - z \|^2 \leq \| w_n - z \|^2$ from (3.10), and notice $(1 - \beta_n)^2 < 1 - \beta_n$, hence we obtain

$$
\| x_{n+1} - z \|^2 \leq (1 - \beta_n) \| w_n - z \|^2 - \alpha_n (1 - \alpha_n) (1 - \beta_n) \| z_n - w_n \|^2 \\
+ 2 \beta_n \left\langle \alpha_n (z_n - w_n) - z, x_{n+1} - z \right\rangle.
$$
Submitting (3.6) into the above inequality, we have
\[ \|x_{n+1} - z\|^2 \leq (1 - \beta_n)\|x_n - z\|^2 + (1 - \theta_n)\|x_{n-1} - z\|^2 - \theta_n(1 - \theta_n)\|x_n - x_{n-1}\|^2 \]
\[ - \alpha_n(1 - \alpha_n)(1 - \beta_n)^2\|z_n - w\|^2 + 2\beta_n\alpha_n(z_n - w_n) - z, x_{n+1} - z \], (3.13)
and then noticing the \( \|x_{n+1} - z\| \leq \|x_n - z\| \) for all \( n \geq N_0 + 1 \), we have
\[ \|x_{n+1} - z\|^2 \leq (1 - \beta_n)\|x_n - z\|^2 - \theta_n(1 - \theta_n)(1 - \beta_n)\|x_n - x_{n-1}\|^2 \]
\[ - \alpha_n(1 - \alpha_n)(1 - \beta_n)^2\|z_n - w\|^2 + 2\beta_n\alpha_n(z_n - w_n) - z, x_{n+1} - z \]
\[ \leq (1 - \beta_n)\|x_n - z\|^2 + 2\beta_n\alpha_n(z_n - w_n) - z, x_{n+1} - z \]
\[ = (1 - \beta_n)\|x_n - z\|^2 - (1 - \beta_n)\|x_n - z\|^2 + (1 - \beta_n)\|x_{n-1} - z\|^2 \]
\[ + 2\beta_n\alpha_n(z_n - w_n) - z, x_{n+1} - z \]
\[ = (1 - \beta_n)\|x_n - z\|^2 + (1 - \beta_n)(\|x_{n-1} - z\|^2 - \|x_n - z\|^2) \]
\[ + 2\beta_n\alpha_n(z_n - w_n) - z, x_{n+1} - z \]
\[ = (1 - \beta_n)a_n + \beta_nb_n + c_n, \] (3.14)
where \( a_n = \|x_n - z\|^2, \ b_n = 2[\alpha_n(z_n - w_n, x_{n+1} - z) + \langle -z, x_{n+1} - z \rangle] \) and \( c_n = (1 - \beta_n)(\|x_{n-1} - z\|^2 - \|x_n - z\|^2) \).

Since \( \omega_n(x_n) \subset \Omega \) and \( z = P_{\Omega}(0) \), which implies from (2.2) that \( \langle -z, q - z \rangle \leq 0 \) for all \( q \in \Omega \), we deduce that
\[ \limsup_{n \to \infty} \langle -z, x_{n+1} - z \rangle = \max_{q \in \omega_n(x_n)} \langle -z, q - z \rangle \leq 0. \] (3.15)
Since \( \|w_n - z_n\| \to 0 \), \( \lim_{n \to \infty} a_n = 0 \) and \( \|x_{n+1} - z\| \) is bounded, combining (3.15) implies that
\[ \limsup_{n \to \infty} b_n = \limsup_{n \to \infty} 2\alpha_n(z_n - w_n, x_{n+1} - z) + \langle -z, x_{n+1} - z \rangle \]
\[ = \limsup_{n \to \infty} 2\langle -z, x_{n+1} - z \rangle \leq 0. \]

In addition, we have
\[ \sum_{n=1}^{\infty} c_n \leq \sum_{n=1}^{\infty} (\|x_{n-1} - z\|^2 - \|x_n - z\|^2) < \infty. \]

These enable us to apply Lemma 2.4 to (3.14) to obtain that \( a_n \to 0 \). Namely, \( x_n \to z \) in norm and the proof of Case 1 is complete.

Case 2. If the sequence \( \{\|x_n - z\|\} \) does not decrease at infinity in the sense that there exists a sub-sequence \( \{n_k\} \) of \( \{n\} \) such that \( \|x_{n_k} - z\| \leq \|x_{n_{k+1}} - z\| \) for all \( k \geq 0 \). Furthermore, by Lemma 2.6, there exists an integer, non-decreasing sequence \( \sigma(n) \) for \( n \geq N_1 \) (for some \( N_1 \) large enough) such that \( \sigma(n) \to \infty \) as \( n \to \infty \),
\[ \|x_{\sigma(n)} - z\| \leq \|x_{\sigma(n)+1} - z\|, \|x_n - z\| \leq \|x_{\sigma(n)+1} - z\| \]
for each \( n \geq 0 \).

Notice the boundedness of the sequence \( \{ \| x_n - z \| \} \), which implies that there exists the limit of the sequence \( \{ \| x_{\sigma(n)} - z \| \} \) and hence we conclude that
\[
\lim_{n \to \infty} (\| x_{\sigma(n)+1} - z \|^2 - \| x_{\sigma(n)} - z \|^2) = 0.
\]
As a matter of fact, observe that \((3.12)\) holds for each \( n \), so from \((3.12)\) with \( n \) replaced by \( \sigma(n) \) and using the relation \( \| x_{\sigma(n)} - z \|^2 \leq \| x_{\sigma(n)+1} - z \|^2 \), we have
\[
\theta_{\sigma(n)}(1 - \theta_{\sigma(n)}) (1 - \beta_{\sigma(n)}) \| x_{\sigma(n)} - x_{\sigma(n)-1} \|^2 + \alpha_{\sigma(n)} (2 - \frac{\tau_{\sigma(n)}}{2\gamma} - \alpha_{\sigma(n)} - \beta_{\sigma(n)}) \| w_{\sigma(n)} - z_{\sigma(n)} \|^2 + 2\gamma \tau_{\sigma(n)} \alpha_{\sigma(n)} \| B w_{\sigma(n)} - B z - \frac{w_{\sigma(n)} - z_{\sigma(n)}}{2\gamma} \|^2 \\
\leq (1 - \beta_{\sigma(n)}) (\theta_{\sigma(n)} \| x_{\sigma(n)} - z \|^2 + (1 - \theta_{\sigma(n)}) \| x_{\sigma(n)-1} - z \|^2) - \| x_{\sigma(n)+1} - z \|^2 + \beta_{\sigma(n)} \| z \|^2 \\
\leq (1 - \beta_{\sigma(n)}) \| x_{\sigma(n)} - z \|^2 - \| x_{\sigma(n)+1} - z \|^2 + \beta_{\sigma(n)} \| z \|^2 \\
= \| x_{\sigma(n)} - z \|^2 - \| x_{\sigma(n)+1} - z \|^2 - \beta_{\sigma(n)} (\| x_{\sigma(n)} - z \|^2 - \| z \|^2)
\]

Notice the assumptions on the parameters \( \theta_{\sigma(n)}, \beta_{\sigma(n)}, \tau_{\sigma(n)} \) and \( \alpha_{\sigma(n)} \), Taking the limit by letting \( n \to \infty \) yields
\[
\lim_{n \to \infty} \| w_{\sigma(n)} - z_{\sigma(n)} \| = 0; \quad (3.16)
\]
\[
\lim_{n \to \infty} \| x_{\sigma(n)} - x_{\sigma(n)-1} \|^2 = 0; \quad (3.17)
\]
\[
\lim_{n \to \infty} \| B w_{\sigma(n)} - B z - \frac{w_{\sigma(n)} - z_{\sigma(n)}}{2\gamma} \|^2 = 0. \quad (3.18)
\]

Note that we still have \( \| x_{\sigma(n)+1} - x_{\sigma(n)} \| \to 0 \) and then that the relations \((3.16)-(3.18)\) are sufficient to guarantee that \( \omega_w(x_{\sigma(n)}) \subset \Omega \).

Next we prove \( x_{\sigma(n)} \to z \).

Observe that \((3.13)\) holds for each \( n \). So replacing \( n \) with \( \sigma(n) \) in \((3.13)\) and using the relation \( \| x_{\sigma(n)} - z \|^2 \leq \| x_{\sigma(n)+1} - z \|^2 \) again for \( n \geq N_1 \), we obtain
\[
\| x_{\sigma(n)+1} - z \|^2 = (1 - \beta_{\sigma(n)}) (\theta_{\sigma(n)} \| x_{\sigma(n)} - z \|^2 + (1 - \theta_{\sigma(n)}) \| x_{\sigma(n)-1} - z \|^2 + \gamma \tau_{\sigma(n)} \alpha_{\sigma(n)} (\| x_{\sigma(n)} - z \|^2 - \| x_{\sigma(n)-1} - z \|^2) + \alpha_{\sigma(n)} (1 - \alpha_{\sigma(n)}) (1 - \beta_{\sigma(n)}) \| z_{\sigma(n)} - w_{\sigma(n)} \|^2 + 2\beta_{\sigma(n)} \| \alpha_{\sigma(n)} (z_{\sigma(n)} - w_{\sigma(n)}) - z, x_{\sigma(n)+1} - z \| \).
\]
and then we obtain
\[
\beta_{\sigma(n)} \| x_{\sigma(n)} - z \|^2 \leq \| x_{\sigma(n)} - z \|^2 - \| x_{\sigma(n)+1} - z \|^2 + 2\beta_{\sigma(n)} \| \alpha_{\sigma(n)} (z_{\sigma(n)} - w_{\sigma(n)}) - z, x_{\sigma(n)+1} - z \|,
\]
and then we obtain
\[
\beta_{\sigma(n)} \| x_{\sigma(n)} - z \|^2 \leq 2\beta_{\sigma(n)} \| \alpha_{\sigma(n)} (z_{\sigma(n)} - w_{\sigma(n)}) - z, x_{\sigma(n)+1} - z \|,
\]
Two possible cases will be shown as follows.

which means that there exists a constant $M$ such that $M \geq 2\|x_n - z\|$ for all $n$ and

$$
\|x_{\sigma(n)} - z\|^2 \leq 2 \left( \alpha_{\sigma(n)}(z_{\sigma(n)} - w_{\sigma(n)}) - z, x_{\sigma(n)+1} - z \right) \\
\leq M \|z_{\sigma(n)} - w_{\sigma(n)}\| + 2 \left( -z, x_{\sigma(n)+1} - z \right). 
$$

(3.19)

Now since $\|x_{\sigma(n)+1} - x_{\sigma(n)}\| \to 0$, we have

$$
\limsup_{n \to \infty} \langle -z, x_{\sigma(n)+1} - z \rangle = \limsup_{n \to \infty} \langle -z, x_{\sigma(n)} - z \rangle = \max_{q \in \omega(x_{\sigma(n)})} \langle -z, q - z \rangle \leq 0
$$

by virtue of the facts $z = P_\Omega(0)$ and $\omega(x_{\sigma(n)}) \subset \Omega$.

Consequently, following from $\|z_{\sigma(n)} - w_{\sigma(n)}\| \to 0$, the relation (3.19) assures that $x_{\sigma(n)} \to z$, which further implies from Lemma 2.6 that

$$
\|x_n - z\| \leq \|x_{\sigma(n)+1} - z\| \leq \|x_{\sigma(n)+1} - x_{\sigma(n)}\| + \|x_{\sigma(n)} - z\| \to 0.
$$

Namely, $x_n \to z$ in norm, and the proof of Case 2 is complete.

(II). Now, it is time to show the strong convergence when $\theta_n \equiv 0$. Clearly, if $\theta_n = 0$, then $u_n = x_{n-1}$ and $x_{n+1} = (1 - \alpha_n - \beta_n)x_{n-1} + \alpha_n z_{n-1}$, where $z_{n-1} = J_{\tau_n}^\infty(I - \tau_n B)x_{n-1}$.

Repeating the steps from (3.10)-(3.12), we have the following similar inequality:

$$
\begin{align*}
\|x_{n+1} - z\|^2 &\leq (1 - \beta_n)\|x_{n-1} - z\|^2 + \beta_n\|z\|^2 + \alpha_n \left( \frac{\tau_n}{2\gamma} - 2 + \alpha_n + \beta_n \right) \|x_{n-1} - z_{n-1}\|^2 \\
&\quad - 2\gamma\tau_n\alpha_n \left\| Bx_{n-1} - Bz - \frac{x_{n-1} - z_{n-1}}{2\gamma} \right\|^2.
\end{align*}
$$

(3.20)

Two possible cases will be shown as follows.

Case 1. There exists an integer $N_0 \geq 0$ such that $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $n \geq N_0$. Then there exists the limit of the sequence $\{\|x_n - z\|\}$, denoted by $l = \lim_{n \to \infty} \|x_n - z\|^2$, and so

$$
\lim_{n \to \infty} (\|x_n - z\|^2 - \|x_{n+1} - z\|^2) = 0.
$$

In addition, we have

$$
\sum_{n=0}^{\infty} (\|x_{n+1} - z\|^2 - \|x_n - z\|^2) = \lim_{n \to \infty} (\|x_{n+1} - z\|^2 - \|x_0 - z\|^2) < \infty
$$

and therefore from (3.20) we obtain

$$
\begin{align*}
\alpha_n \left( 2 - \frac{\tau_n}{2\gamma} + \alpha_n + \beta_n \right) \|x_{n-1} - z_{n-1}\|^2 &\leq (1 - \beta_n)\|x_{n-1} - z\|^2 + \beta_n\|z\|^2 - \|x_{n+1} - z\|^2 \\
&= (\|x_{n-1} - z\|^2 - \|x_n - z\|^2) + (\|x_n - z\|^2 - \|x_{n+1} - z\|^2) + \beta_n(\|z\|^2 - \|x_{n-1} - z\|^2).
\end{align*}
$$

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Notice the assumptions on the parameters $\alpha_n, \beta_n$ and $\tau_n$, we have
\[
\lim_{n \to \infty} \|x_{n-1} - z_{n-1}\|^2 = 0; \quad \lim_{n \to \infty} \|Bx_{n-1} - Bz - \frac{x_{n-1} - z_{n-1}}{2\gamma}\|^2 = 0,
\]
which imply that $\|x_{n+1} - x_{n-1}\| \leq \alpha_n \|z_{n-1} - x_{n-1}\| + \beta_n \|x_{n-1}\| \to 0$ as $n \to \infty$.

Next we show the asymptotic regularity of $\{x_n\}$. Indeed, it follows from the relation between the norm and inner product that
\[
\|x_{n+1} - x_n\|^2 = \|x_{n+1} - z + z - x_n\|^2
= \|x_{n+1} - z\|^2 + \|z - x_n\|^2 + 2\langle x_{n+1} - z, z - x_n \rangle
\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\|x_{n+1} - z\| \cdot \|x_{n+1} - x_n\|
\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + 2(M + m) \cdot \|x_{n+1} - x_n\|,
\]
where $M$ is a constant such that $M \geq \|x_n - z\|$ for all $n$ and $m > 0$ is a given constant, which means that
\[
\|x_{n+1} - x_n\|^2 - 2(M + m) \cdot \|x_{n+1} - x_n\| \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2,
\]
and then
\[
\sum_{n=0}^{\infty} (\|x_{n+1} - x_n\| - 2(M + m)) \cdot \|x_{n+1} - x_n\| \leq \sum_{n=0}^{\infty} (\|x_n - z\|^2 - \|x_{n+1} - z\|^2) < \infty.
\]
Therefore we obtain $\|x_{n+1} - x_n\| - 2(M + m) \to 0$ as $n \to \infty$.

Since $\|x_{n+1} - x_n\| \leq \|x_{n+1} - z\| + \|x_n - z\| \leq 2M$, we have $\|x_{n+1} - x_n\| \to 0$. This proves the asymptotic regularity of $\{x_n\}$.

By repeating the relevant part of the proof of Case 1 in (I), we get $\omega_w(x_n) \subset \Omega$ and
\[
\|x_{n+1} - z\|^2 \leq (1 - \beta_n)\|x_{n-1} - z\|^2 - \alpha_n(1 - \alpha_n)(1 - \beta_n)^2\|z_{n-1} - x_{n-1}\|^2
+ 2\beta_n \langle \alpha_n(z_{n-1} - x_{n-1}) - z, x_{n+1} - z \rangle
\leq (1 - \beta_n)\|x_n - z\|^2 + (1 - \beta_n)(\|x_{n-1} - z\|^2 - \|x_n - z\|^2)
+ 2\beta_n \langle \alpha_n(z_{n-1} - x_{n-1}) - z, x_{n+1} - z \rangle
= (1 - \beta_n)a_n + \beta_nb_n + c_n;
\]
where $a_n = \|x_n - z\|^2$, $b_n = 2[\alpha_n(z_{n-1} - x_{n-1}, x_{n+1} - z) + (-z, x_{n+1} - z)]$ and $c_n = (1 - \beta_n)(\|z_{n-1} - z\|^2 - \|x_n - z\|^2)$.

The rest of proof is consistent with Case 1 in (I). So we get $x_n \to z$ in norm.

Case 2. If the sequence $\{\|x_n - z\|\}$ does not decrease at infinity in the sense that there exists a sub-sequence $\{n_k\}$ of $\{n\}$ such that $\|x_{n_k} - z\| \leq \|x_{n_k+1} - z\|$ for all $k \geq 0$. Furthermore, by
Lemma 2.6 there exists an integer, non-decreasing sequence \( \sigma(n) \) for \( n \geq N_1 \) (for some \( N_1 \) large enough) such that \( \sigma(n) \to \infty \) as \( n \to \infty \),

\[
\|x_{\sigma(n)} - z\| \leq \|x_{\sigma(n)+1} - z\|, \quad \|x_n - z\| \leq \|x_{\sigma(n)+1} - z\|
\]

for each \( n \geq 0 \).

Note that we still have \( \|x_{\sigma(n)+1} - x_{\sigma(n)}\| \to 0 \).

By repeating the relevant part of the proof of Case 2 in (I), for \( n \geq N_1 \), we have

\[
\alpha_{\sigma(n)}(2 - \frac{\tau_{\sigma(n)}}{2\gamma} - \alpha_{\sigma(n)} - \beta_{\sigma(n)})\|x_{\sigma(n)-1} - z_{\sigma(n)-1}\|^2
\]

\[
+ 2\gamma \tau_{\sigma(n)} \alpha_{\sigma(n)} \left| Bx_{\sigma(n)-1} - Bz - \frac{x_{\sigma(n)-1} - z_{\sigma(n)-1}}{2\gamma} \right|^2
\]

\[
\leq \left( 1 - \beta_{\sigma(n)} \right) \|x_{\sigma(n)-1} - z\|^2 - \|x_{\sigma(n)+1} - z\|^2 + \beta_{\sigma(n)}\|z\|^2
\]

\[
\leq \left( 1 - \beta_{\sigma(n)} \right) \|x_n - z\|^2 - \|x_{\sigma(n)+1} - z\|^2 + \beta_{\sigma(n)}\|z\|^2
\]

\[
= \|x_n - z\|^2 - \|x_{\sigma(n)+1} - z\|^2 - \beta_{\sigma(n)}(\|x_{\sigma(n)} - z\|^2 - \|z\|^2)
\]

Notice the assumptions on the parameters \( \alpha_{\sigma(n)}, \beta_{\sigma(n)} \) and \( \tau_{\sigma(n)} \), taking the limit by letting \( n \to \infty \) yields

\[
\lim_{n \to \infty} \|x_{\sigma(n)-1} - z_{\sigma(n)-1}\| = 0;
\]

\[
\lim_{n \to \infty} \left| Bx_{\sigma(n)-1} - Bz - \frac{x_{\sigma(n)-1} - z_{\sigma(n)-1}}{2\gamma} \right|^2 = 0,
\]

and that these relations are sufficient to guarantee that \( \omega_w(x_{\sigma(n)}) \subset \Omega \).

Observe that (3.21) holds for each \( n \). So replacing \( n \) with \( \sigma(n) \) in (3.21) and using the relation

\[
\|x_{\sigma(n)} - z\|^2 < \|x_{\sigma(n)+1} - z\|^2
\]

again for \( n \geq N_1 \), we obtain

\[
\|x_{\sigma(n)+1} - z\|^2 \leq \left( 1 - \beta_{\sigma(n)} \right) \|x_n - z\|^2 - \alpha_{\sigma(n)}(1 - \alpha_{\sigma(n)})(1 - \beta_{\sigma(n)})^2\|z_{\sigma(n)-1} - x_{\sigma(n)-1}\|^2
\]

\[
+ 2\beta_{\sigma(n)}\alpha_{\sigma(n)}(z_{\sigma(n)-1} - w_{\sigma(n)-1}) - z, x_{\sigma(n)+1} - z
\]

\[
\leq \left( 1 - \beta_{\sigma(n)} \right) \|x_n - z\|^2 + 2\beta_{\sigma(n)}\alpha_{\sigma(n)}(z_{\sigma(n)-1} - x_{\sigma(n)-1}) - z, x_{\sigma(n)+1} - z,
\]

The rest of proof is consistent with Case 2 in (I). So we get \( x_n \to z \) in norm.

(III) Finally we consider the situation \( \theta_n \equiv 1 \). It is obvious that if \( \theta_n = 1 \), then \( w_n = x_n \) and \( x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n z_n \), where \( z_n = J_{\tau_n}^A(I - \tau_n B)x_n \).

Repeating the steps from (3.10)-(3.12), we have the following similar inequality

\[
\|x_{n+1} - z\|^2 \leq \left( 1 - \beta_n \right) \|x_n - z\|^2 + \beta_n\|z\|^2 + \alpha_n \left( \frac{\tau_n}{2\gamma} - 2 + \alpha_n + \beta_n \right)\|x_n - z_n\|^2
\]

\[
-2\gamma \tau_n \alpha_n \left| Bx_n - Bz - \frac{x_n - z_n}{2\gamma} \right|^2
\]

\[
\leq \left( 1 - \beta_n \right) \|x_n - z\|^2 + \beta_n\|z\|^2
\]

\[
\leq \max\{\|x_n - z\|^2, \|z\|^2\},
\]
which means that \( \|x_n - z\|^2 \) is bounded. Similar to the proof of the above situations, two possible cases will be shown as follows.

Two possible cases will be shown as follows.

**Case 1.** There exists an integer \( N_0 \geq 0 \) such that \( \|x_{n+1} - z\| \leq \|x_n - z\| \) for all \( n \geq N_0 \). Then there exists the limit of the sequence \( \{\|x_n - z\|\} \), denoted by \( l = \lim_{n \to \infty} \|x_n - z\|^2 \), and so

\[
\lim_{n \to \infty} (\|x_n - z\|^2 - \|x_{n+1} - z\|^2) = 0.
\]

In addition, we have

\[
\sum_{n=0}^{\infty} (\|x_{n+1} - z\|^2 - \|x_n - z\|^2) = \lim_{n \to \infty} (\|x_{n+1} - z\| - \|x_0 - z\|) < \infty.
\]

By repeating the relevant part of the proof of **Case 1** in (I), we have

\[
\lim_{n \to \infty} \|x_n - z_n\|^2 = 0; \quad \lim_{n \to \infty} \left\| Bx_n - Bz - \frac{x_n - z_n}{2\gamma} \right\|^2 = 0,
\]

which imply that \( \|x_{n+1} - x_n\| \leq \alpha_n \|z_n - x_n\| + \beta_n \|x_n\| \to 0 \) as \( n \to \infty \).

The rest of proof is consistent with **Case 1** in (I). So we get \( x_n \to z \) in norm.

**Case 2.** If the sequence \( \{\|x_n - z\|\} \) does not decrease at infinity in the sense that there exists a sub-sequence \( \{n_k\} \) of \( \{n\} \) such that \( \|x_{n_k} - z\| \leq \|x_{n_k+1} - z\| \) for all \( k \geq 0 \). Furthermore, by Lemma 2.6 there exists an integer, non-decreasing sequence \( \sigma(n) \) for \( n \geq N_1 \) (for some \( N_1 \) large enough) such that \( \sigma(n) \to \infty \) as \( n \to \infty \),

\[
\|x_{\sigma(n)} - z\| \leq \|x_{\sigma(n)+1} - z\|, \quad \|x_n - z\| \leq \|x_{\sigma(n)+1} - z\|
\]

for each \( n \geq 0 \).

Note that we still have \( \|x_{\sigma(n)+1} - x_{\sigma(n)}\| \to 0 \). The rest of proof is consistent with **Case 1** in (I). So we get \( x_n \to z \) in norm.

This completes the proof. \( \Box \)

**Remark 3.3.** It is easy to see that the convergence of our algorithms still holds even without the following condition:

\[
\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty.
\]

The above assumption is not necessary at all in our cases. To some extents, our inertial-like algorithms seem to have two merits:

1. Compared with the general inertial proximal algorithms, we do not need to calculate the values of \( \|x_n - x_{n-1}\| \) before choosing the parameters \( \theta_n \) in numerical simulations, which make the algorithms convenient and use-friendly.

2. Compared with the general inertial algorithms, the inertial factors \( \theta_n \) are chosen in \([0, 1]\) with \( \theta_n = 1 \) possible, which are new, natural and interesting algorithms in some ways. In particular, under more mild assumptions, our proofs are simpler and different from the others.
4 Applications

4.1 Convex Optimization

Let $C$ be a nonempty closed and convex subset of $H$ and $f, g$ be two proper, convex and lower semi-continuous functions. Moreover, assume that $g$ is differentiable with a $1/\gamma$–Lipschitz gradient. With this data, consider the following convex minimizing problem:

$$\min_{x \in C} \{ f(x) + g(x) \}. \quad (4.1)$$

Denoted by $\Omega = \{ x : \min_{x \in C} \{ f(x) + g(x) \} \}$. Recall that the subdifferential of $f$ at $x$, denote by $\partial f$:

$$\partial f(x) := \{ x^* \in C : f(y) - f(x) \geq \langle y - x, x^* \rangle, \forall y \in H \}.$$

So, by taking $A = \partial f$ and $B = \nabla g$ (the gradient of $g$) in (3.2), where $A$ and $B$ are maximal monotone operators and $B$ is $\gamma$–cocoercive, we can apply Theorem 3.1 and Theorem 3.2 and obtain the following results:

**Theorem 4.1.** Let $f$ and $g$ be two proper, convex and lower semi-continuous functions such that $\nabla g$ and $\partial f$ are maximal monotone operators and $\nabla g$ also $\gamma$–cocoercive. Assume that $\Omega \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be any two sequences generated by the following scheme (see, e.g.):

$$\begin{align*}
  w_n &= x_{n-1} + \theta_n (x_n - x_{n-1}), \\
  x_{n+1} &= J_{\tau_n}^{\partial f} (w_n - \tau_n \nabla g w_n),
\end{align*}$$

where $\theta_n$ is satisfying the selection criteria in Algorithm 3.2. If $\tau_n \in (\epsilon, 2\gamma - \epsilon)$ for some given $\epsilon > 0$ small enough, and $\theta_n \in [0, 1]$, then the sequences $\{x_n\}$ converges weakly to a point of $\Omega$.

**Theorem 4.2.** Let $f$ and $g$ be two proper, convex and lower semi-continuous functions such that $\nabla g$ and $\partial f$ are maximal monotone operators and $\nabla g$ also $\gamma$–cocoercive. Assume that $\Omega \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be any two sequences generated by the following scheme:

$$\begin{align*}
  w_n &= x_{n-1} + \theta_n (x_n - x_{n-1}), \\
  x_{n+1} &= (1 - \alpha_n - \beta_n)w_n + \alpha_n J_{\tau_n}^{\partial f} (w_n - \tau_n \nabla g w_n),
\end{align*}$$

where $\alpha_n, \beta_n, \theta_n$ are satisfying the selection criteria in Algorithm 3.2. If $\tau_n \in (\epsilon, 2\gamma - \epsilon)$ for some given $\epsilon > 0$ small enough, then the sequences $\{x_n\}$ converges strongly to $z = P_\Omega(0)$.

4.2 Variational Inequality Problem

Let $C$ be a nonempty closed and convex subset of $H$ and $B : H \to H$ a maximal and $\gamma$–coercive operator.
Consider the classical variational inequality (VI) problem of finding a point $x^* \in C$ such that
\[ \langle Bx^*, z - x^* \rangle \geq 0, \quad \forall z \in C. \quad (4.2) \]

Denote by $\Omega$ the solution set of the problem (VI) (4.2).

So, by taking $Ax := \{w \in H : \langle w, z - x \rangle \leq 0, \forall z \in C\}$ (the normal cone of the set $C$), the problem (VI) (4.2) is equivalent to finding zeroes of $A + B$ (see e.g., Peypouquet [24]).

We apply our algorithms to the variational inequality (VI) problem and have the following theorems.

**Theorem 4.3.** Let $C$ be a nonempty closed and convex subset of $H$, $B : H \to H$ a maximal and $\gamma$–coercive operator. Choose $\tau_n \in (\epsilon, 2\gamma - \epsilon)$ for some given $\epsilon > 0$ small enough and assume that $\Omega \neq \emptyset$. Construct the sequences $\{x_n\}$ and $\{w_n\}$ as follows:
\[
\begin{align*}
    w_n &= x_{n-1} + \theta_n(x_n - x_{n-1}), \\
    x_{n+1} &= J_{\tau_n}^A(I - \tau_n B)w_n.
\end{align*}
\]

If $\theta_n \in [0, 1]$, then the sequences $\{x_n\}$ converges weakly to a point of $\Omega$.

**Theorem 4.4.** Let $C$ be a nonempty closed and convex subset of $H$, $B : H \to H$ be a maximal and $\gamma$–cocoercive operator. Choose $\tau_n \in (\epsilon, 2\gamma - \epsilon)$ for some given $\epsilon > 0$ small enough and assume that $\Omega \neq \emptyset$. Construct the sequences $\{x_n\}$, $\{w_n\}$ as follows.
\[
\begin{align*}
    w_n &= x_{n-1} + \theta_n(x_n - x_{n-1}), \\
    x_{n+1} &= (1 - \alpha_n - \beta_n)w_n + \alpha_n J_{\tau_n}^A(I - \tau_n B)w_n,
\end{align*}
\]

where $\alpha_n, \beta_n, \theta_n$ are satisfying the selection criteria in Algorithm 3.2. Then the sequence $\{x_n\}$ converges strongly to $z = P_\Omega(0)$.

## 5 Numerical Examples

In this section, we first present two numerical examples in infinite and finite-dimensional Hilbert spaces to illustrate the applicability, efficiency and stability of Algorithm 3.1 and Algorithm 3.2, and then consider sparse signal recovery from real-world life in finite dimensional spaces. In addition, the comparison results with other algorithms are also described. All the codes are written in Matlab R2016b and are preformed on an LG dual core personal computer.

**Example 5.1.** In this example, we take $H = \mathbb{R}^3$ with Euclidean norm. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $Ax = 3x$ and let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $Bx = \frac{x}{3} + (-1, 2, 0), \forall x \in \mathbb{R}^3$. We can see that $A$ and $B$ are maximal monotone mappings and $B$ with $\gamma$-cocoercive($0 < \gamma \leq 3$), respectively. Indeed, let $x, y \in \mathbb{R}^3$, then
\[
\langle Bx - By, x - y \rangle = \left(\frac{x}{3} - \frac{y}{3}, x - y\right) = \frac{1}{3}\|x - y\|^2,
\]
while

$$\|Bx - By\|^2 = \frac{1}{9}\|x - y\|^2,$$

which means $B$ is $\frac{1}{3}$-cocoercive. It is not hard to check that $(A + B)^{-1}(0) = (3/10, -6/10, 0)$.

In the simulation process, we choose $x_0 = [0.1; -0.2; 0.1], x_1 = [0.2; 0.1; -0.3]$ as two arbitrary initial points. In order to investigate the change and tendency of $\{x_n\}$ more clearly, we denote by $z = (3/10, -6/10, 0)$ and let $\|x_{n+1} - z\| \leq 10^{-5}$ be the stopping criterion. The experimental results are shown in Figs. 1-2, where $z$-axis represents the logarithm of the third coordinate value for each point. Indeed, if $\theta_n \equiv 1$, then Algorithm 3.1 coincides with the result of Moudafi [19]. In addition, from Figs. 1-2 we can see that when $\theta_n = 0$, two families of points alternate, and finally infinitely close to the exact null point $z = (3/10, -6/10, 0)$.

**Example 5.2.** In this example, we show the behaviors of our algorithms in $H = L^2([0, 1])$. In addition, the compared results with Dadashi and Postolach [8] are considered. In the simulation, we define two mappings $A$ and $B$ by $Bx(t) = \frac{x(t)}{2}$ and $Ax(t) = \frac{3x(t)}{4}$ for all $x(t) \in L^2([0, 1])$. Then it can be shown that $B$ is $\frac{1}{2}$-cocoercive. In the numerical experiment, the parameters are chosen as $\alpha_n = 0.5 - 1/(10n + 2), \beta_n = \frac{1}{n+1}$ in Algorithm 3.2 for all $n \geq 1$. In addition, $\|x_{n+1} - x_n\| < 10^{-4}$ is used as stopping criterion and the following three different choices of initial functions $x_0(t), x_1(t)$ are chosen:

**Case 1:** $x_0(t) = \frac{\sin(-3t) + \cos(-5t)}{2}$ and $x_1(t) = 2\sin(5t)$;

**Case 2:** $x_0(t) = \frac{2t \sin(3t)e^{-5t}}{200}$ and $x_1(t) = t^2 - e^{-2t}$.
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Figure 3: Behavior of Algorithm 3.1 and Algorithm 3.2 for Case 1 in Exp.5.2.

Figure 4: Behavior of Algorithm 3.1 and Algorithm 3.2 for Case 2 in Exp.5.2.

**Case 3:** \(x_0(t) = 2t^3e^{5t}\) and \(x_1(t) = \frac{e^{t\sin(3t)}}{100}\).

Figs [3-5] represent the numerical results for \(\theta_n\) neither 0 nor 1, Fig [6] shows the numerical results for \(\theta_n = 0\) and \(\theta_n = 1\), respectively. Fig [7] shows the comparing results with Dadashi and Postolach [8] for the initial points \(x_0(t) = \frac{e^{t\sin(3t)}}{100}\) and \(x_0(t) = t^2 - e^{-2t}\), respectively. Table [1] shows the comparisons with Dadashi and Postolach [8] for the initial point \(x_0(t) = t^2 - e^{-2t}\).

Figure 5: Behavior of Algorithm 3.1 and Algorithm 3.2 for Case 3 in Exp.5.2.
It is clear that our algorithms, especially Algorithm 3.2, is faster, more efficient, more stable.

**Example 5.3.** (Compressed Sensing) In this example, to show the effectiveness and applicability of our algorithms in the real world, we consider the recovery of a sparse and noisy signal from a limited number of sampling which is a problem from the field of compressed sensing. The sampling matrix $T \in \mathbb{R}^{m\times n}$, $m < n$, is stimulated by standard Gaussian distribution and vector $b = Tx + \epsilon$, where $\epsilon$ is additive noise. When $\epsilon = 0$, it means that there is no noise to the observed data. For further explanations the reader can consult Nguyen and Shin[14].

Let $x_0 \in \mathbb{R}^n$ be the $K$-sparse signal, where $K << n$. Our task is to recover the signal $x_0$ from

| Error  | Iter. | CPU in second |
|--------|-------|---------------|
| Alg.1  | Alg.2 | Alg. in Dadashi et al.[31] | Alg.1 | Alg.2 | Alg. in Dadashi et al.[31] |
| $10^{-3}$ | 41   | 21   | 26   | 6.2813 | 3.4688 | 3.8438 |
| $10^{-4}$ | 49   | 33   | 52   | 10.0445 | 7.6542 | 7.8461 |
| $10^{-5}$ | 57   | 45   | 110  | 10.2969 | 8.6768 | 24.3212 |
Table 2: The comparisons of Algorithm 3.1, Algorithm 3.2, Sitthithakerngkiet et al. [27], Kazmi et al. [12], Tang [28]

| Method | Iter. | Sec.  | Iter. | Sec.  |
|--------|-------|-------|-------|-------|
| Algorithm 3.1 | 43 | 0.0992 | 83 | 0.4431 |
| Algorithm 3.2 | 63 | 0.1635 | 79 | 0.6368 |
| Sitthithakerngkiet et al. [27] | 91 | 0.12 | 122 | 0.6942 |
| Kazmi et al. [12] | 54 | 0.0771 | 39 | 0.1981 |
| Algo.3.2-Tang [28] | 54 | 2.2632 | 67 | 3.2541 |

the data $b$. To this end, we transform it into finding the solution of LASSO problem:

$$\min_{x \in \mathbb{R}^n} \|Tx - b\|^2$$

$$\text{s.t. } \|x\|_1 \leq t,$$

where $t$ is a given positive constant. If we define

$$B_1(x) = \begin{cases} \{u : \sup_{\|x\|_1 \leq t} \langle x - y, u \rangle \leq 0\}, & \text{if } y \in \mathbb{R}^n, \\ \emptyset, & \text{otherwise}, \end{cases}$$

$$B_2(x) = \begin{cases} \mathbb{R}^m, & \text{if } x = b, \\ \emptyset, & \text{otherwise}, \end{cases}$$

then one can see that the LASSO problem coincides with the problem of finding $x^* \in \mathbb{R}^n$ such that

$$0 \in B_1(x^*) \text{ and } 0 \in B_2(Tx^*),$$

which associates with the the problem for finding the solution of the problem:

$$0 \in (A + B)x^*,$$

where $A = B_1$ and $B = T^*(I - J_\lambda B_2)T$ with $\gamma$--cocoercive, $\gamma = \frac{1}{\|I\cdot I\cdot I\|}$.

In addition to showing the behavior of our algorithms, the results of Sitthithakerngkiet et al. [27], Kazimi and Riviz [12] without inertial process and Tang [28] with general inertial method are compared. For the experiment setting, we choose the following parameters: $T \in \mathbb{R}^{m \times n}$ is generated randomly with $m = 2^6, 2^7$, $n = 2^8, 2^9$, $x_0 \in \mathbb{R}^n$ is $K$-spikes ($K = 40, 60$) with amplitude $\pm 1$ distributed in whole domain randomly.

For simplicity, we define the nonexpansive mappings $S_n : \mathbb{R}^3 \to \mathbb{R}^3$ as $S_n = I$ for Algorithm 3.1 in Sitthithakerngkiet et al. [27], and the strongly positive bounded linear operator $D = I$, the constant $\xi = 0.5$ and we give fixed point $u = 0.1$. Moreover, we take $\alpha_n = 0.5 - 1/(10 \ast n + 2)$, $\beta_n = 10^{-3}/(n + 1)$ in all compared algorithms. Let $t = K - 0.001$ and $\|x_{n+1} - x_n\| \leq 10^{-4}$ be the stopping criterion. All the numerical results are presented in Table 2 and Fig. 8.
Figure 8: Numerical results for $m = 2^6$, $m = 2^8$ and $K = 40$
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6 Conclusion

We have provided two new Inertial-Like Proximal iterative algorithms (Algorithms 3.1 and 3.2) for the null point problem. We have proved that, under some mild conditions, Algorithms 3.1 converges weakly and Algorithms 3.2 strongly to a solution of the null point problem. Thanks to the novel structure of the inertial-like technique, our Algorithms 3.1 and 3.2 seem to have the following merits:

(i) Theoretically, they do not need verify the traditional and difficult checking condition \( \sum_{n=1}^{\infty} \theta_n \| x_n - x_{n-1} \|^2 < \infty \), namely, our convergence theorems still hold even without this condition.

(ii) Practically, they do not involve the computations of the norm of the difference between \( x_n \) and \( x_{n-1} \) before choosing the inertial parameter \( \theta_n \), hence less computation cost, as opposed to almost all previous inertial algorithms in the existing literature, that is, the constraints on inertial parameter \( \theta_n \) are looser and more natural, so they are extremely attractive and friendly for users.

(iii) Different from the general inertial algorithms, the inertial factors \( \theta_n \) in our Inertial-Like Proximal algorithms are chosen in \([0, 1]\) with \( \theta_n = 1 \) possible, which are new, natural and interesting algorithms in some ways. In particular, under more mild assumptions, our proofs are simpler and different from the others.

We have included several numerical examples which show the efficiency and reliability of Algorithm 3.1 and Algorithm 3.2. We have also made comparisons of Algorithm 3.1 and Algorithm 3.2 with other four algorithms in Sitthithakerngkiet et al. [27], Kazimi and Riviz [12], Tang [28], Dadashi and Postolach [8] confirming some advantages of our novel inertial algorithms.

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