On complexified analytic Hamiltonian flows and geodesics on the
space of Kähler metrics

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Abstract

In the case of a compact real analytic symplectic manifold \((M, \omega)\) we describe an approach
to the complexification of Hamiltonian flows \cite{Sc, Do, Th} and corresponding geodesics on
the space of Kähler metrics. In this approach, motivated by recent work on quantization, the
complexified Hamiltonian flows act, through the Gröbner theory of Lie series, on the sheaf of
complex valued real analytic functions, changing the sheaves of holomorphic functions. This
defines an action on the space of (equivalent) complex structures on \(M\) and also a direct action
on \(M\). This description is related to the approach of \cite{BLU} where one has an action on a com-
plexification \(M_C\) of \(M\) followed by projection to \(M\). Our approach allows for the study of some
Hamiltonian functions which are not real analytic. It also leads naturally to the consideration
of continuous degenerations of diffeomorphisms and of Kähler structures of \(M\). Hence, one
can link continuously (geometric quantization) real, and more general non-Kähler, polarizations
with Kähler polarizations. This corresponds to the extension of the geodesics to the boundary
of the space of Kähler metrics. Three illustrative examples are considered.

We find an explicit formula for the complex time evolution of the Kähler potential under
the flow. For integral symplectic forms, this formula corresponds to the complexification of the
prequantization of Hamiltonian symplectomorphisms. We verify that certain families of Kähler
structures, which have been studied in geometric quantization, are geodesic families.

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1 Introduction

The formal complexification $\mathcal{H}am_{\mathcal{C}}(M,\omega)$ of the group of Hamiltonian diffeomorphisms $\mathcal{H}am(M,\omega)$ of a symplectic manifold $(M,\omega)$ has been studied recently in two main contexts.

In Kähler geometry, the observation that the space, $\mathcal{K}_{\omega}$, of Kähler metrics with a fixed cohomology class has the structure of an infinite-dimensional symmetric space of the form $\mathcal{H}am_{\mathcal{C}}(M,\omega)/\mathcal{H}am(M,\omega)$, led Semmes [Se] and Donaldson [Do1, Do2] to the result that the Hamiltonian flows in imaginary time lead to geodesics in $\mathcal{K}_{\omega}$ with respect to the Donaldson-Semmes-Mabuchi metric [Ma]. These geodesics play an important role in the recent breakthrough in Kähler geometry, completing the proof of the link of $K$-stability of a Fano variety with the existence of a Kähler-Einstein metric [Ti, St, Be, CDS]. (See [PS] for a review.)

The second context in which $\mathcal{H}am_{\mathcal{C}}(M,\omega)$ has been studied has been in quantization. The applications of imaginary time Hamiltonian flows to quantization originated in the work of Thiemann [Th1, Th2] in the context on nonperturbative quantum gravity. There, a complex canonical transformation is used to take the gravitational spin $SU(2)$-connection to the complex Ashtekar $SL(2,\mathbb{C})$ connection.

In the context of geometric quantization, $\mathcal{H}am_{\mathcal{C}}(M,\omega)$ acts naturally in the space of (geometric quantization) polarizations by acting on the local functions defining the polarization. This idea has been adapted to the framework of geometric quantization by Hall and Kirwin in [HK1]. It has been further investigated, in the case of complexifications of Lie groups of compact type, in [KMN1, KMN2]. Previous work where these ideas were also present include [FMMN1, FMMN2, BFMN] for symplectic toric manifolds and [LS3, LS2]. In a series of papers [RZ1, RZ2, RZ3], Rubinstein and Zelditch have also related ideas of quantization to the study of geodesics on the space of Kähler metrics.

For geometric quantization the study of complex time Hamiltonian flows has the advantage of linking continuously the (mathematically easier to define) quantum Hilbert spaces for Kähler polarizations with the more difficult to define, especially in the case of singular polarizations, quantum Hilbert spaces for real and mixed polarizations. [HK1, KW1, KW2, KMN1, KMN2]. One motivation for the present work was the verification that the Kähler families considered in the
geometric quantization of cotangent bundles of Lie groups, abelian varieties and toric varieties correspond in fact to geodesic families.

As we show below in examples, the imaginary time Hamiltonian flows in geometric quantization extends the flows considered in Kähler geometry past the boundary of the space of Kähler polarizations. (For geodesics hitting the boundary of the space of Kähler metrics see, for example, [RZ2].) While the “metrics” beyond the boundary will be degenerate or become indefinite on subsets, the corresponding polarizations can still be interesting from a geometric quantization point of view.

In [BLU], the authors also study properties of complexified Hamiltonian flows in the case of a real analytic manifold \((M, \omega)\). They describe the complexified Hamiltonian “flows” in \((M, \omega)\) in terms of flows on a complexification \((M_C, \omega_C)\), which is an holomorphic symplectic manifold. The “flow” on \((M, \omega)\) is then obtained by projection along leaves of a time evolving holomorphic foliation of \((M_C, \omega_C)\). The projection of the flow does not, in general, lead to a flow on \((M, \omega)\).

In the present work, motivated by [HK1, KMN1], we give a complementary approach in the case of real analytic \((M, \omega)\), using the Gröbner formalism of Lie series [Gr1, Gr2]. Let \(C^{an}(M)\) be the space of complex valued real analytic functions on \(M\). This formalism defines the exponential \(e^{tX} h\), for the Hamiltonian flow of (real) \(h \in C^{an}(M)\) acting (locally) on \(C^{an}(M)\), as actual power series. This point of view in particular, allows for explicit formulas for the geodesics and for the complex time evolution of the Kähler potential. In this formalism, as we show in a few examples, by letting the complexified Hamiltonian flow go to the boundary of the space of Kähler metrics, one can study real and mixed polarizations (in the sense of geometric quantization). The close link to geometric quantization also clarifies the power series expression for the geodesics in the space of Kähler potentials, as explained in Section 5. When it hits this boundary, the complexified Hamiltonian flow can still continue to exist on the complexification \(M_C\) of \(M\), but the projection to \(M\) ceases to be holomorphic. Notice also that, in our approach, we can also in some cases weaken the requirement of analiticity of \(h\) and of \(M\) and consider Hamiltonian flows of non real analytic Hamiltonian functions \(h \in C^{\infty}(M)\). (See Section 8.3.)

Further applications to Kähler geometry, geometric tropicalization and quantization will be studied in [MN, KMN3].

2 Complexification of analytic flows and action on complex structures

We will use the formalism of Lie series introduced by Gröbner [Gr1, Gr2]. Let \(M\) be a compact real analytic manifold and \(X\) be a real analytic vector field on \(M\) with analytic flow \(\varphi_t\). Let \(C^{an}(M)\) denote the algebra of complex valued real analytic functions on \(M\). If \(f \in C^{an}(M)\) then \(f \circ \varphi_t \in C^{an}(M), \forall t \in \mathbb{R}\). Moreover, as shown in [Gr1, Gr2], there exists a \(T_f > 0\) such that

\[
e^{tX} f := \sum_{k=0}^{\infty} \frac{X^k(f)}{k!} t^k,
\]

is an absolutely convergent power series in \(t\), for \(|t| < T_f\). Note that, as a consequence,

\[
f \circ \varphi_t = e^{tX} f, \quad \text{for } |t| < T_f.
\]

Consequently,

**Lemma 2.1** Let \(M\) be a compact real analytic manifold and \(X\) a real analytic vector field on \(M\). For each \(f \in C^{an}(M)\) there exists \(T_f > 0\) such that for \(\tau \in D_0 = \{ \tau \in \mathbb{C} : |\tau| < T_f \}\) the power
Proof. Let us consider an open neighbourhood of complex structure $\mathcal{J}$ on $M \times D_0$.

We will use the following natural notation

$$e^{\tau X} \cdot f := \sum_{k=0}^{\infty} \frac{X^k(f)}{k!} \tau^k,$$

whenever the right hand side is an absolutely convergent power series. In this case, we will say that $e^{\tau X}$ can be applied to $f$.

Remark 2.2 The operator $e^{\tau X}$ (see Theorem 5 in [Gr2]) acts as a (local) automorphism of the algebra $C^\text{an}(M)$, that is, for $f, g \in C^\text{an}(M)$ and $|\tau|$ sufficiently small,

$$e^{\tau X} \cdot (fg) = (e^{\tau X} \cdot f)(e^{\tau X} \cdot g).$$

Moreover, as a direct consequence of Theorem 6 of [Gr2],

Theorem 2.3 Let $D \subset \mathbb{R}^n$ be an open disk centered at the origin, $X$ a real analytic vector field on $D$ and $f \in C^\text{an}(D)$ such that the power series

$$f(x^1, \ldots, x^n) = \sum_{I \in \mathbb{N}_0^n} a_I x^I, \quad I = (i^1, \ldots, i^n),$$

where $x^I = (x^1)^{i^1} \cdots (x^n)^{i^n}$, converges in $D$. Assume that there exists $T > 0$ such that, for $t \in \mathbb{R}$ with $|t| < T$, $e^{tX}$ can be applied to $f, x^1, \ldots, x^n$ in a neighbourhood of $0 \in D$. Then, one has, for $|t|$ sufficiently small,

$$e^{tX} \cdot f(x^1, \ldots, x^n) = \sum_{k=0}^{\infty} \frac{X^k(f)}{k!} t^k(x^1, \ldots, x^n) = f(e^{tX} x^1, \ldots, e^{tX} x^n) = f \circ \varphi_t(x^1, \ldots, x^n)$$

in a neighbourhood of $0 \in D$, where all the power series in $t$ are absolutely convergent.

Let now $\dim M = 2n$ and let $J_0$ be a complex structure on $M$. Let $\{z_i\}_{i=1, \ldots, n}$ be a local system of $J_0$-holomorphic coordinates in an open neighbourhood $U$ of $p \in M$.

Theorem 2.4 There exists $T > 0$ such that for every $\tau \in D_T = \{\tau \in \mathbb{C} : |\tau| < T\}$ the functions

$$z_i^\tau = e^{\tau X} \cdot z_i, \quad i = 1, \ldots, n,$$

form a system of complex coordinates on some open neighbourhood $V \subset U$ of $p$, defining a new complex structure $J_{\tau}$ on $V$ for which the coordinates $\{z_i^\tau\}_{i=1, \ldots, n}$ are holomorphic.

Proof. Let us consider an open neighbourhood of $p$, $V' \subset U$, with compact support $\overline{V'} \subset U$. The existence of $T$ such that the functions $z_i^\tau$ are well defined on $V'$ for $|\tau| < T$ follows from Lemma 2.1. By taking a smaller $T$ if necessary, by continuity in $\tau$ and the fact that $dz_1^\tau \wedge \cdots \wedge dz_n^\tau \neq 0$ at every point in $V'$, we get that $dz_1^\tau \wedge \cdots \wedge dz_i^\tau \wedge \cdots \wedge d\bar{z}_n^\tau \neq 0$ at every point in $V'$, for $|\tau| < T$. By the inverse function theorem, we obtain that $\{z_i^\tau\}_{i=1, \ldots, n}$ form a new system of complex coordinates (that is, the real and imaginary parts form a system of real analytic coordinates) on some, possibly smaller, open neighbourhood $V \subset V'$ of $p$. □
Theorem 2.5 There exists $T > 0$ such that for $\tau \in D_T = \{\tau \in \mathbb{C} : |\tau| < T\}$ there exist a global complex structure $J_\tau$ on $M$, extending the complex structure given in local $J_0$-holomorphic charts by Theorem 2.4 and a unique biholomorphism

$$\varphi_\tau : (M, J_\tau) \rightarrow (M, J_0),$$

which, on local $J_0$-holomorphic coordinates, acts as $e^{\tau \bar{X}}$.

Proof. Since $M$ is compact, we can cover it with a finite atlas $\{(U_\alpha, z_\alpha)\}_{\alpha=1,\ldots,N}$ of $J_0$-holomorphic local charts. Let $T_{\alpha,p} > 0$, $V_{\alpha,p}$ be open sets, $p \in V_{\alpha,p} \subset U_\alpha$ such that the functions

$$z_{\alpha,p,\tau} = e^{\tau X} z_{\alpha,p} = (e^{\tau X} \cdot z_{\alpha,p}^1, \ldots, e^{\tau X} \cdot z_{\alpha,p}^n),$$

where $z_{\alpha,p} = (z_\alpha)|_{V_{\alpha,p}}$, are defined on $V_{\alpha,p}$, are holomorphic functions of $\tau$ and

$$z_{\alpha,p,\tau}(V_{\alpha,p}) \subset z_\alpha(U_\alpha),$$

for $\tau \in D_{T_{\alpha,p}}$. Let $\{V_{\alpha_j,p_j}\}_{j=1,\ldots,K}$ be a finite subcover of (the infinite open cover) $\{V_{\alpha,p}\}_{\alpha=1,\ldots,N,p \in M}$ and $\bar{T} = \min_j \{T_{\alpha_j,p_j}\}$. Let $\phi_{\alpha_j} \circ \tau_{\alpha_k}$ be the coordinate transition functions. From Theorem 2.3 it follows that there exists $T, 0 < T \leq \bar{T}$ such that

$$z_{\alpha_j,p_j,\tau} = e^{\tau X} \cdot z_{\alpha_j,p_j} = e^{\tau X} \cdot (\phi_{\alpha_j} \circ \tau_{\alpha_k} \circ z_{\alpha_k,p_k}) = \phi_{\alpha_j} \circ \tau_{\alpha_k} \circ z_{\alpha_k,p_k,\tau} \forall \tau \in D_T.
$$

We have therefore defined a new atlas

$$\{(V_{\alpha_j,p_j}, z_{\alpha_j,p_j,\tau})\}_{j=1,\ldots,K}$$

on $M$, with the same transition functions restricted to the smaller open sets $V_{\alpha_j,p_j} \cap V_{\alpha_k,p_k}$. Therefore the atlas $\{(V_{\alpha_j,p_j}, z_{\alpha_j,p_j,\tau})\}_{j=1,\ldots,K}$ defines a new complex structure $J_\tau$ on $M$, equivalent to $J_0$. For $\alpha = 1, \ldots, N$, denote by $\phi_{\alpha} : z_\alpha(U_\alpha) \subset \mathbb{C}^n \rightarrow U_\alpha$ the inverse of the coordinate function $z_\alpha$. For $\tau \in D_T$, define the maps

$$\varphi_{\tau,j} = \phi_{\alpha_j} \circ z_{\alpha_j,p_j,\tau} : V_{\alpha_j,p_j} \rightarrow U_\alpha.$$

We have, from (2.2), that the maps $\{\varphi_{\tau,j}\}_{j=1,\ldots,K}$ glue together to give a well defined global bijective map $\varphi_\tau : M \rightarrow M$. It is clear that $\varphi_\tau$ gives the unique biholomorphism from $(M, J_\tau)$ to $(M, J_0)$, such that $z_{\alpha_j,p_j,\tau} = z_{\alpha_j,p_j} \circ \varphi_\tau$, which proves the theorem. \qed

Remark 2.6 Note that $J_\tau = \varphi_{\tau}^{-1} J_0 := \varphi_{\tau}^{-1} \circ J_0 \circ \varphi_{\tau}$ is the push-forward of $J_0$ by $\varphi_{\tau}^{-1}$. \hfill \Diamond

Corollary 2.7 Under the conditions of Theorem 2.5 we have

$$\varphi_\tau^* \mathcal{O}_{(M, J_0)} = \mathcal{O}_{(M, J_\tau)},$$

where $\mathcal{O}_{(M, J)}$ denotes the structure sheaf of $M$ with respect to the complex structure $J$.

Corollary 2.8 Let $g \in C^\infty(M)$ be a, not necessarily $J_0$-holomorphic function and let $\{z_i\}_{i=1,\ldots,n}$ be local $J_0$-holomorphic coordinates on $M$. For $T$ as in Theorem 2.5 and $\tau \in D_T$, we have

$$\varphi_{\tau}^* g(z_1^1, \ldots, z_n^1, z_1^2, \ldots, z_n^n) = g(z_1^1, \ldots, z_n^1, z_\tau^1, \ldots, z_\tau^n),$$

where $\{z_i^\tau\}_{i=1,\ldots,n}$ are defined in Theorem 2.4 and

$$\bar{z}_\tau = \bar{z}_{\tau} = e^{\tau X} \cdot \bar{z},$$

with $\bar{z} = (z_1^1, \ldots, z_n^1), \bar{z}_\tau = (\bar{z}_1^1, \ldots, \bar{z}_n^1)$. 5
Remark 2.9 If $\tau = t \in \mathbb{R}$, then $\varphi_t$ in (2.1) is the usual time $t$ flow of $X$ and is therefore $J_0$-independent. For $\tau \notin \mathbb{R}$, $\varphi_\tau$ in general will depend on $J_\tau$. Moreover, if $\tau \notin \mathbb{R}$, $\varphi_\tau$ does not in general define a flow, in the sense that $\varphi_{\tau+\sigma} \neq \varphi_\tau \circ \varphi_\sigma$, even all if three of these diffeomorphisms are defined.

We now note that if $|\tau|$ becomes too large it may happen that the complex structure $J_\tau$ no longer exists even though the functions $\{z^i_{\alpha, \tau}\}_{i=1, \ldots, n}$ are well defined, and functionally independent, for all the open sets in the cover $\{U_\alpha\}$ of $M$. This happens, for instance, if for some $\alpha$ the $2n$-form $dz^1_{\alpha, \tau} \wedge \cdots \wedge dz^n_{\alpha, \tau} \wedge dz^1_{\alpha, \tau} \wedge \cdots \wedge dz^n_{\alpha, \tau}$ has zeros. Note that

Lemma 2.10 In the above notation, if $\tau \in \mathbb{C}$ is such that the functions $\{z^i_{\alpha, \tau}\}$ are well defined on an open set $U_\alpha$, then the $J_\tau$-holomorphic form $dz^1_{\alpha, \tau} \wedge \cdots \wedge dz^n_{\alpha, \tau}$ does not have zeros on $U_\alpha$.

Proof. The action of the operator $e^{\tau X}$ on holomorphic functions can be inverted, since

$$e^{-\tau X} z^j_{\alpha, \tau} = z^j_{\alpha}. \quad \blacksquare$$

On the other hand, the zeros of $dz^1_{\alpha, \tau} \wedge \cdots \wedge dz^n_{\alpha, \tau} \wedge dz^1_{\alpha, \tau} \wedge \cdots \wedge dz^n_{\alpha, \tau}$ mean that some linear combination of $dz^j_{\alpha, \tau}, j = 1, \ldots, n$, becomes real and the corresponding polarization becomes mixed. For these values of $\tau$, which cannot be treated by a flow on $M_C$ followed by projection to $M$, it is still interesting, for geometric quantization purposes, to study the associated mixed polarizations. In the next example, we illustrate this situation.

Example 2.11 Consider the complex manifold $(\mathbb{R}^2, J_0)$, where $J_0$ is the homogeneous complex structure defined by the global holomorphic coordinate $z = x + \tau_0 y$, with $\text{Im} \tau_0 \neq 0$. Even though this is a noncompact example, it serves to illustrate the results above. Let $X = y \frac{\partial}{\partial x}$, so that

$$z_\tau = e^{\tau X} \cdot z = x + (\tau_0 + \tau)y.$$ 

Letting $\tau_0 = r_0 + is_0, \tau = r + is, r_0, s_0, r, s \in \mathbb{R}$, we obtain that $\varphi_\tau : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map defined by the matrix

$$
\begin{pmatrix}
1 & r + r_0 - \frac{(s+s_0)r_0}{s_0} \\
0 & \frac{s+s_0}{s_0}
\end{pmatrix}
$$

in the canonical basis. This complexified "flow" of diffeomorphisms defines, for $s \neq -s_0$, a linear isomorphism of $\mathbb{R}^2$. For $s = -s_0$, $\varphi_{\tau-s_0}$ maps $\mathbb{R}^2$ to a real dimension one linear subspace. Consider on the complex one-parameter subgroup of the formal group $\text{Diff}(\mathbb{R}^2)_C$,

$$C = \{e^{\tau X}, \tau \in \mathbb{C}\},$$

the three subsets $C = C_+ \cup C_0 \cup C_-$ according to whether $\text{Im}(\tau_0 + \tau)$ is positive, equal to zero or negative, respectively. Then, the map $e^{\tau X} \mapsto \varphi_\tau$ breaks up into maps

$$
C_+ \to GL_+(\mathbb{R}^2) \subset \text{Diff}_+(\mathbb{R}^2) \\
C_0 \to M_2(\mathbb{R}) \subset C^\infty(\mathbb{R}^2, \mathbb{R}^2) \\
C_- \to GL_-(\mathbb{R}^2) \subset \text{Diff}_-(\mathbb{R}^2).
$$

\footnote{Recall that on a symplectic manifold $(M, \omega)$ a (geometric quantization) polarization, $\mathcal{P}$, is a Lagrangian distribution in the complexified tangent bundle $TM \otimes \mathbb{C}$. $\mathcal{P}$ is said to be real if, at each point of $M$, $\mathcal{P} = \mathcal{P}$. If $(M, \omega, J)$ is a Kähler manifold, then $T^{(1,0)}M$ defines a Kähler polarization $\mathcal{P}_J$, such that $\mathcal{P}_J \cap \mathcal{P}_J = 0$. (See, for example, [Wo].)
Note that when \( \tau \) is such that \( s = -s_0 \), we still have a well defined, albeit real, function
\[
z_{\tau} = x + (r_0 + r)y = \bar{z}_\tau.
\]

At this value of \( s \), the complex polarization associated to \( J_\tau \) becomes real, see Example [7.1].

Remark 2.12 Under the conditions of Theorem 2.5 note that for \( J_0 \)-holomorphic \((k,0)\)-forms and \(|\tau|\) sufficiently small we have
\[
\varphi_{\tau}^* f(z) d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_k} = e^{\tau X_h} f(z) d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_k},
\]
while for \( J_0 \)-anti-holomorphic forms of type \((0, k)\),
\[
\varphi_{\tau}^* f(z) d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_k} =: e^{\tau X_h} f(z) d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_k}.
\]

Note that these expressions are convergent power series in \( \tau \) and \( \bar{\tau} \) respectively.

Remark 2.13 A geometric interpretation of \( \varphi_{\tau} \) can be given in terms of the complexification of \( M \), as in [BLU]. (See also Section 8.)

3 Complexification of Hamiltonian flows

Let \( U \subset \mathbb{R}^{2n} \) be an open set. Let \( J_0 \) be the standard complex structure on \( \mathbb{R}^{2n} \cong \mathbb{C}^n \) and consider now \((U, J_0)\) with symplectic form \( \omega \) such that \((U, \omega, J_0)\) is a Kähler manifold. For \( h \in C^\infty(U) \), let \( X_h \) be its (real analytic) Hamiltonian vector field with flow \( \varphi_t \). In this Section, we will consider Theorem 2.5 in the context of complexified Hamiltonian flows and will study some additional properties valid in this case. Letting \( z = (z^1, \ldots, z^n) \) be local \( J_0 \)-holomorphic coordinates on \( U \), recall that \( \omega \) is a type \((1,1)\)-form.

Theorem 3.1 Let \( V \subset U \) be an open set and let \( f \in C^\infty(V) \) such that \( e^{\tau X_h} \cdot f \) is well defined for \(|\tau| < T\) for some \( T > 0 \). Then, the real-analytic Hamiltonian vector field of \( e^{\tau X_h} \cdot f \) has a power series expansion
\[
X_{e^{\tau X_h} f} = \sum_{k=0}^{\infty} \frac{\tau^k P_h^k(X_f)}{k!} =: e^{\tau X_h} X_f, \quad |\tau| < T. \tag{3.1}
\]

Proof. Observe that, for \( k \in \mathbb{N}, g \in C^\infty(V), \)
\[
\left( \mathcal{L}^{P_h}_{X_h} X_f \right)(g) = X_{h}^{X_h(f)}(g).
\]
From real analyticity of \( \varphi_t \), for real time \( t \), we then have
\[
X_{e^{t X_h} f} = e^{t X_h} X_f. \tag{3.2}
\]
The equality (3.1) follows by taking the analytic continuation of (3.2) in \( t \). ■

Remark 3.2 For real time \( t \), as long as the flow of \( X_h \) is defined, (3.2) is the series expansion, for the symplectomorphism \( \varphi_t \), of the equality \((\varphi_t^{-1})_* X_f = X_{e^{t X_h} f} \). However, note that for \( \text{Im} \tau \neq 0 \), \( \varphi_\tau \) will not, in general, be a symplectomorphism. In this case, for \( f \) not \( J_0 \)-holomorphic, we have, in general, \( \varphi_\tau^* f \neq e^{\tau X_h} \cdot f \).

Remark 3.3 Note that \( z_\tau = (e^{\tau X_h} \cdot z) = e^{\tau X_h} \cdot \bar{z} \neq e^{\tau X_h} \cdot \bar{z} = \bar{z}_\tau \).
We have, for the Hamiltonian vector fields for the holomorphic coordinate functions,

**Lemma 3.4** Under the conditions of Theorem 3.1,

\[ e^{\tau \mathcal{L}_{X_{\bar{h}}}} X_{z^i} = X_{z^i}, \quad e^{\tau \mathcal{L}_{X_{\bar{h}}}} X_{\bar{z}^i} = X_{\bar{z}^i} \]

where the left hand sides are convergent power series (in \( \tau \) and \( \bar{\tau} \) respectively).

**Proof.** We have \( \mathcal{L}_{X_{\bar{h}}} \omega = 0 \) and

\[ dz^i = X_{z^i} \omega = i e^{\tau \mathcal{L}_{X_{\bar{h}}}} X_{z^i} \omega, \]

so that \( X_{z^i} = e^{\tau \mathcal{L}_{X_{\bar{h}}}} X_{z^i}, i = 1, \ldots, n \). Since \( z^i \) is given by a convergent power series in \( \tau \), all the expressions make sense as convergent power series.

4 **Action on Kähler structures**

**Theorem 4.1** Let \((M, \omega, J_0, \gamma_0)\) be a real analytic compact Kähler manifold with Kähler form \( \omega \), complex structure \( J_0 \) and Riemannian metric \( \gamma_0 \). Assume \( \omega, J_0, \gamma_0 \) are analytic forms. Let \( h \in C^{an}(M) \).

There exists \( T > 0 \) such that for each

\[ \tau \in D_T := \{ \tau \in \mathbb{C} : |\tau| < T \} \]

one has that:

(i) \((M, \omega, J_\tau)\) is a Kähler manifold, where \( J_\tau = \varphi_{\tau}^* J_0 \) and \( \varphi_{\tau} \) is the biholomorphism of Theorem 2.5.

(ii) Let \( \kappa_0 \) be a local analytic Kähler potential for \((M, \omega, J_0)\). A local Kähler potential for \((M, \omega, J_\tau)\) is then given by

\[ \kappa_\tau = -2\text{Im} \psi_\tau, \quad (4.1) \]

with

\[ \psi_\tau = -\frac{i}{2} e^{\tau X_{\bar{h}}} \cdot \kappa_0 + \tau h - \alpha_\tau, \quad (4.2) \]

where \( \alpha_\tau \) is the analytic continuation in \( t \) of

\[ \alpha_t = \int_0^t e^{t' X_{\bar{h}}} (\theta(X_{\bar{h}})) dt', \quad (4.3) \]

\( \theta \) is the real local potential for \(-\omega\) defined by \( \theta(0,1) = \bar{\partial}_0 \kappa_0 \), with \( \bar{\partial}_0 \) the \( \bar{\partial} \)-operator relative to the complex structure \( J_0 \).

**Remark 4.2** One has \( d\alpha_\tau = e^{\tau d\omega} X_{\bar{h}} \theta - \theta \) and also

\[ \alpha_\tau = \sum_{k=1}^{\infty} \frac{\tau^k}{k!} X_{\bar{h}}^{k-1}(\theta(X_{\bar{h}})), \]

for \( \alpha_\tau \) in (4.3).
Proof. To prove (i) it is enough to show that $\omega$ is of type $(1, 1)$ with respect to $J_\tau$. Positivity, for small enough $|\tau|$, follows from continuity at $\tau = 0$. From Lemma 3.3

$$X_{z_{i\alpha}} = e^{\tau L_h} X_{z_{i\alpha}}.$$

Note that the local vector fields $\{X_{z_{i\alpha}}\}_{i=1,\ldots,n}$ are linearly independent at every point of an open neighborhood. Since $L_{X_h} \omega = 0$ and $\omega$ is of type $(1, 1)$ with respect to $J_0$, we obtain

$$\{ z^i_{\alpha}, z^j_{\alpha} \} = 0, i, j = 1, \ldots, n.$$

Therefore,

$$dz^i_{\alpha}(X_{z_{i\alpha}}) = \omega(X_{z_{i\alpha}}, X_{z_{i\alpha}}) = 0,$$

so that $\{X_{z_{i\alpha}}\}_{i=1,\ldots,n}$ generate the $(0, 1)$ tangent space with respect to $J_\tau$. Therefore, $\omega$ is of type $(1, 1)$ with respect to $J_\tau$ which concludes the proof of (i).

To prove (ii), let $\theta = \theta^{(1,0)} + \theta^{(0,1)}$, be the decomposition of $\theta$ into $(1, 0)$ and $(0, 1)$ pieces with respect to $J_\tau$. Since $\omega$ is of type $(1, 1)$ with respect to $J_\tau$, $\partial_\tau \theta^{(0,1)} = 0$, where $\partial_\tau, \bar{\partial}_\tau$ are the $\partial, \bar{\partial}$-operators with respect to $J_\tau$. Therefore, by the $\bar{\partial}$-lemma, $\theta^{(0,1)} = \bar{\partial}_\tau \psi$, for some locally defined complex valued analytic function $\psi$. A Kähler potential for $\omega$ with respect to $J_\tau$ will then be given by $\kappa_\tau = -2\text{Im} \psi$. Recall that $\{X_{z_{i\alpha}}\}_{i=1,\ldots,n}$ is a basis of the $(0, 1)$ tangent space for $J_\tau$. Since $\omega = -d\theta$, we have, expanding in powers of $\tau$,

$$e^{\tau L_h} \theta = \theta - \tau dh + d\alpha_\tau.$$

Then, since $\theta^{(0,1)} = -\frac{i}{2} \bar{\partial}_0 \kappa_0$ and $X_{z_{i\alpha}} = e^{\tau L_h} X_{z_{i\alpha}}$, the validity of (ii) follows from the equality

$$\theta(X_{z_{i\alpha}}) = -\frac{i}{2} X_{z_{i\alpha}}(e^{\tau L_h} \kappa_0) + \tau dh(X_{z_{i\alpha}}) - d\alpha_\tau(X_{z_{i\alpha}}), i = 1, \ldots, n,$$

where we use $e^{\tau L_h} \theta(X_{z_{i\alpha}}) = e^{\tau L_h} \theta(X_{z_{i\alpha}}) = e^{\tau X_h} \theta(X_{z_{i\alpha}})$ due to the fact that, as a consequence of Remark 2.2, $e^{\tau L_h}$ acts as a (local) automorphism of the algebras of tensor fields on $M$.

Note that $\varphi_t$ defines a symplectomorphism of $(M, \omega)$ for all $t \in \mathbb{R}$. However, for complex $\tau$, $\varphi_\tau$ will not, in general, be a symplectomorphism, as already noted in Remark 3.2. Therefore, the Kähler structures $(M, J_0, \omega, \gamma_0)$ and $(M, J_\tau, \omega, \gamma_\tau)$ will, in general, be non-equivalent.

That the evolution in real time does not change the diffeomorphism equivalence class of the Kähler structure, can be checked explicitly by verifying that under the flow of the Hamiltonian symplectomorphisms $\varphi_t$, $t \in \mathbb{R}$ the Kähler potential just evolves by composition with the flow. In fact,

**Proposition 4.3** Under the conditions of Theorem 4.1 let $\tau_0 \in DT, s \in \mathbb{R}$ such that $\tau + s \in DT$. We then have,

$$\kappa_{\tau + s} = \varphi_s^* \kappa_\tau.$$

**Proof.** From formula (4.3) it is easy to check that, for real $t, s$, one has

$$\varphi_s^* \alpha_t = e^{s X_h} \cdot \alpha_t = \alpha_{t+s} - \alpha_s.$$

By analytic continuation in $t$, we obtain,

$$e^{s X_h} \alpha_\tau = \alpha_{t+s} - \alpha_s.$$
Since, for real $s$, we have $\alpha_s = \bar{\alpha}_s$, and using $X_h(h) = 0$, it is straightforward to verify that
\[
\kappa_{\tau + s} = e^{sX_h} \kappa_{\tau}.
\]

If $M$ is not compact or if $|\tau|$ becomes too large, in general the maps $e^{\tau X_h}$ may develop singularities, even if they stay as local biholomorphisms on large subsets of $M$. Also, on some subsets of $X$ one may lose the positivity of the metric $\gamma$. Nevertheless, it can still be very useful to consider these maps and their action on the set of polarizations of the symplectic manifold $(M, \omega)$. Note that the action of the diffeomorphisms $\varphi_{\tau}$ on polarizations is given by the following

**Theorem 4.4** Under the conditions of Theorem 4.1, the Kähler polarization (that is the $(0, 1)$-tangent space) for $(M, \omega, J_{\tau})$ is given by
\[
\mathcal{P}_{\tau} = (\varphi_{\tau}^{-1})_* \mathcal{P}_0 = e^{\tau L_{X_h}} \mathcal{P}_0,
\]
where this expression can be interpreted as a convergent power series in $\tau$ if $e^{\tau L_{X_h}}$ is applied to appropriate sections of $\mathcal{P}_0$ such as $X_{z_i}$, $i = 1, \ldots, n$, where $\{z^i\}_{i=1,\ldots,n}$ are local holomorphic coordinates on $(M, J_0)$.

**Proof.** The $(0, 1)$-tangent space with respect to $J_{\tau}$ has a basis $\{X_{z_i}\}_{i=1,\ldots,n}$, where $X_{z_i} = e^{\tau L_{X_h}} X_{z_i}$ and $\{X_{z_i}\}_{i=1,\ldots,n}$ is a basis of the $(0, 1)$-tangent space with respect to $J_0$.

With a view to the application to half-form quantization, note also that we have the following result.

**Proposition 4.5** Under the conditions of Theorem 4.1, the fiber of canonical bundle $K_{\tau}$ of $(M, J_{\tau})$ is generated at each point by
\[
\Omega_{\tau} = e^{\tau L_{X_h}} \Omega_0,
\]
where $\Omega_0$ generates the fiber of the canonical bundle $K_0$ of $(M, J_0)$ and where the expression makes sense as a convergent power series in $\tau$ if $e^{\tau L_{X_h}}$ is applied to appropriate local sections of $K_0$ such as $\Omega_0 = dz^1 \wedge \cdots \wedge dz^n$.

**Proof.** We only need to note that
\[
dz_{\alpha_{\tau}} = de^{\tau L_{X_h}} \cdot z_{\alpha} = e^{\tau L_{X_h}} dz_{\alpha},
\]
where the equality is valid for real $\tau$ and follows for complex $\tau$, with small enough $|\tau|$, by analytic continuation.

## 5 Geometric quantization interpretation

For $[\omega] \in H^2(M, \mathbb{Z})$ integral, let $L \to M$ be a prequantum line bundle with connection $\nabla$ such that $F_\nabla = -i\omega$. Let $\hat{f}$ denote the prequantization of $f \in C^{\text{an}}(M)$ acting on sections of $L$,
\[
\hat{f} = i \nabla_{X_f} + f = iX_f + f - \theta(X_f).
\]

In this case, there is a clearer interpretation of the action on the Kähler potentials as $e^{i\psi_{\tau}}$ can be considered as a section of $L$ (written in the unitary frame corresponding to $\theta$) such that $e^{i\psi_{\tau}}$ is a section of $L \otimes L^{-1}$. Then,
**Theorem 5.1** The complex time evolution of the Kähler potential $\kappa_0$, corresponds to the analytic continuation to complex time of the prequantization of $e^{iX_h}$ acting on $e^{i\psi_0}$

$$e^{i\psi t} = e^{-i\tau \hat{h}} (e^{i\psi_0}) = e^{i(e^{tX_h}\psi_0 + \tau h - \alpha t)}.$$  \hspace{1cm} (5.1)

**Proof.** The formula is valid for real time $\tau = t \in \mathbb{R}$. To see this, we only need to show that

$$\frac{d}{dt} e^{i\psi t} = i \dot{\psi} e^{i\psi t} = -i\hat{h} e^{i\psi t}.$$ 

This follows by direct differentiation and by using $\dot{\alpha}_t = e^{tX_h} \theta(X_h)$ and $X_h \cdot \alpha_t = e^{tX_h} \theta(X_h) - \theta(X_h)$.

The theorem then follows by analytic continuation in $t$.  

Note that this is the analytic continuation to complex time of the usual formula for the evolution of wave functions in geometric quantization, under the Hamiltonian flow of $X_h$. (See, for example, Chapters 8 and 9 in [W0]. For other examples of the validity (in complex time) of the same formula, see [KMN1].) For a related description see also [Do3].

**Remark 5.2** The reason to have $e^{i\psi_0}$ evolve with $e^{-i\tau \hat{h}}$ rather than with (a lifting to $L$ of) $\varphi_\tau$ is due to the fact that a $\mathcal{P}_0$-polarized section of $L$ evolves under the flow to a $\mathcal{P}_\tau$-polarized section. Then, the product of a $J_0$-holomorphic section $F_1$ with a $J_0$-antiholomorphic $F_2$ evolves with $F_1^\tau F_2^\tau = e^{-i\tau \hat{h}} (F_1) e^{i\bar{\tau} \hat{h}} (F_2)$, thus corresponding the evolution of $e^{-\kappa_0}$ above.  

\hfill \Box

### 6 Symplectic and complex pictures

Let $(M, \omega, J_0)$ be a compact Kähler manifold. In [Do1] (see also [Do2, Do3]), Donaldson has considered a formal “complexification” of the infinite-dimensional group of Hamiltonian symplectomorphisms of $(M, \omega)$ and related it with geodesics in the space of Kähler metrics with fixed cohomology class and Mabuchi metric [Ma]. In the present section, we will recall some aspects of this description and will explain in which sense it can be extended in an interesting way for geometric quantization. As the section is mainly illustrative, and intended to give just a general qualitative perspective, we will not consider any technicalities arising from infinite-dimensions.

#### 6.1 Complex picture

Let $G$ denote the group of Hamiltonian symplectomorphisms of $(M, \omega)$ and $G_C$ its formal complexification. Following Donaldson [Do1], in the present paper we are studying two types of “orbits” of $G_C$. The orbit of first type for the pair $(\omega, J_0)$ coincides with the space of Kähler metrics on $(M, J_0)$ with fixed cohomology class

$$\mathcal{H}(\omega, J_0) = \{ \varphi^* \omega_0, \ \varphi \in \text{Diff}(M), [\varphi^* \omega] = [\omega], \text{the pair } (\varphi^* \omega, J_0) \text{ is Kähler} \} \cong \{ \eta \in C^\infty(M) : \omega + i\partial_0 \overline{\partial} \eta > 0 \} / \mathbb{R}.$$ 

where $\partial_0$ is the $\partial$-operator relative to the complex structure $J_0$. Morally

$$\mathcal{H}(\omega, J_0) = (G_C^* \cdot \omega, J_0)$$ 

so that, for $(\omega, J_0)$, $G_C$ corresponds to the following subset (not subgroup) of $\text{Diff}(M)$,

$$G_C^{(\omega, J_0)} = \{ \varphi \in \text{Diff}(M) : \varphi^* \omega \in \mathcal{H}(\omega, J_0) \}$$

11
and

$$\mathcal{H}(\omega, J_0) \cong G_{C}(\omega, J_0)/G$$

has the structure of an infinite-dimensional symmetric space with constant negative curvature \(\text{Se, DoI}\) for the Mabuchi metric \(\text{Ma}\). The transitivity of \(G_C\) in (6.1) is a consequence of Moser’s theorem. Choosing a path, \(t \mapsto \omega_t, t \in [0, 1]\), between two symplectic structures in \(\mathcal{H}(\omega, J_0)\), there are diffeomorphisms \(\Phi_t : M \rightarrow M\), obtained by integrating a \(t\)-dependent vector field on \(M\), such that \(\Phi_t^* \omega_t = \omega_0\). Notice that the family of Moser maps is unique up to composition with a family of symplectomorphisms of \((M, \omega)\). We will call the study of orbits \(\mathcal{H}(\omega, J_0)\) the “complex picture” as the complex structure is kept constant.

The Mabuchi metric on the space of Kähler potentials reads

$$||\delta \phi||^2 = \int_M |\delta \phi|^2 \frac{\omega^n}{n!},$$

where \(\dim M = 2n\) and \(\delta \phi \in C^\infty(M)\) is a tangent vector at the metric \(\gamma_\phi = \omega(\cdot, J_0 \cdot)\). This metric induces a metric on \(\mathcal{H}(\omega, J_0)\),

$$\mathcal{H}(\omega, J_0) \cong \left\{ \eta \in C^\infty(M) : \omega + i\partial_0 \overline{\partial_0} \eta > 0, \int_M \eta \omega^n_\phi = 0 \right\}.$$

Consider a path of Kähler potentials \(\phi_t\), with \(\phi_0 = 0\), corresponding to the Kähler forms

$$\omega_t = \omega_0 + i\partial_0 \overline{\partial_0} \phi_t.$$

The condition for this path to be a geodesic in the above metric reads

$$\ddot{\phi} = \frac{1}{2} ||\nabla \dot{\phi}||^2_{\phi},$$

where \(||\nabla \dot{\phi}||^2_{\phi}\) is the squared norm of the gradient of \(\dot{\phi}\) with respect to the Kähler metric determined by \(\phi\).

6.2 Symplectic picture

The second type of “\(G_C\) orbits” corresponds to acting on the complex structure with fixed symplectic structure

$$G(\omega, J_0) = (\omega, G_{C*} \cdot J_0) = \left\{ (\omega, \varphi * J_0), \varphi \in G_{C}(\omega, J_0) \right\} \quad (6.2)$$

If \(\text{Aut}(M, J_0)\) is discrete, then the “action” of \(G_C\) is free and

$$G(\omega, J_0) \cong G_C.$$

In that case we get a natural map

$$\pi : G(\omega, J_0) \longrightarrow \mathcal{H}(\omega, J_0)$$

\((\omega, \varphi * J_0) \mapsto \varphi^*(\omega, \varphi * J_0) = (\varphi^* \omega, J_0)\),

(6.3)

giving \(G(\omega, J_0)\) the structure of a principal \(G\)-bundle over \(\mathcal{H}(\omega, J_0)\).

If \(\text{Aut}(M, J_0)\) has continuous subgroups, then

$$G(\omega, J_0) \cong G_C/\text{Aut}(M, J_0),$$
where $Aut_0(M,J_0)$ is the connected component at the identity of $Aut(M,J_0)$. In that case, instead of (6.3) one gets the diagram

$$
\begin{array}{ccc}
\mathcal{G}(\omega, J_0) & \xrightarrow{\pi_1} & \mathcal{H}(\omega, J_0) \\
\pi_2 \downarrow & & \downarrow \\
G_C & & 
\end{array}
$$

(6.4)

We see that a map from orbits of imaginary time one-parameter Hamiltonian subgroups in $\mathcal{G}(\omega, J_0)$ to $\mathcal{H}(\omega, J_0)$ is defined if and only if these orbits do not correspond to subgroups of $Aut(M,J_0)$.

The study of $\mathcal{G}(\omega, J_0)$ is natural in geometric quantization, where it is natural to fix a symplectic structure and study the dependence of quantization on the choice of complex structure (even for complex structures related by a symplectomorphism). We call the study of $\mathcal{G}(\omega, J_0)$ the “symplectic picture”.

### 6.3 Connecting the pictures and analytic Cauchy problem

Let us, for simplicity, suppose in this section that $Aut(M,J_0)$ is discrete. To go from the symplectic picture to the complex picture we need a section $\pi$ in (6.3). To go from the complex picture to the symplectic picture we need a section of (6.3). If $G$ was a compact Lie group, a natural global section would be given by lifting the action of $\exp(i\text{Lie}(G))$ on $\omega$ to $J_0$. It turns out, however, that the Cauchy problem for the geodesics in $H(\omega, J_0)$ does not always have solution, which means that there are (non-analytic) $h \in C^\infty(M)$ for which $e^{it\text{Lie}(h)\omega}$, is not well defined for any $t \neq 0$ [Do1]. On the other hand, if we restrict the $G$-bundle in (6.3) to the real-analytic case,

$$
\pi^\text{an} : \mathcal{G}(\omega, J_0)^\text{an} \longrightarrow \mathcal{H}(\omega, J_0)^\text{an}
$$

where $\varphi \in \text{Diff}(M)^\text{an} \cap G_C$, then, for each vector in the tangent space at $(\omega, J_0)$ in $\mathcal{H}(\omega, J_0)^\text{an}$, the Cauchy problem has a solution given by acting on $\omega$ with imaginary time analytic Hamiltonian symplectomorphisms. This is a consequence of Proposition 9.1. More explicitly, there are sections $\sigma$ of (6.5) defined on neighborhoods of $0$ of every direction in $T_{(\omega, J_0)}\mathcal{H}(\omega, J_0)^\text{an}$ given by

$$
\sigma(\varphi_i^s \omega, J_0) = (\omega, (\varphi_i s)_* J_0), \forall h \in C^\text{an}(M),
$$

where $|s| < T_h$ and $T_h \in \mathbb{R}$ is an $h$-dependent positive real number. Notice, however, that the compactness of $M$ is crucial for this result to hold. (See Example 7.2 where we consider a complete real analytic Hamiltonian vector field on the plane for which the Cauchy problem has no solution.)

Consider now a path $(\omega_t, J_0)$ in $\mathcal{H}(\omega, J_0)$ starting at $(\omega_0, J_0) = (\omega, J_0)$. Fixing a family of Moser maps $\Phi_t \in \text{Diff}(M)$, with $\Phi_t^* \omega_t = \omega$, corresponds to fixing a lift $(\omega_t, J_t)$ of the path to $\mathcal{G}(\omega, J_0)$, with $J_t = \Phi_t^* J_0$. Of course, in this way we obtain equivalent Kähler structures

$$
(M, \omega_t, J_t, \gamma_t) = \Phi_t^* (M, \omega_t, J_0, \gamma_t).
$$

If we denote by $\partial_t$ the $\partial$-operator relative to the complex structure $J_t$ we then have, for a (local) Kähler potential $k_t$ for $\omega$ with respect to $J_t$, $\omega = i\partial_t \bar{\partial} k_t = \Phi_t^* \omega_t = \varphi_t^*(\omega + i\partial_0 \bar{\partial}_0 \phi_t)$.

(6.6)

It follows that

$$
\phi_t = k_t \circ \Phi_t^{-1} - k_0.
$$

(6.7)

where $k_0$ is a (local) Kähler potential for $\omega$ with respect to $J_0$. 

7 Examples of extension beyond the space of Kähler structures

**Example 7.1** Consider again Example 2.11 where now \( \mathbb{R}^2 \) is equipped with its standard symplectic structure \( \omega = dx \wedge dy \), so that \( (\mathbb{R}^2, J_0, \omega) \) is a Kähler manifold. The diffeomorphisms \( \varphi_r \) (for \( s \neq -s_0 \)) in this case were generated by \( X = y \frac{\partial}{\partial x} = X_h \), where \( h(x, y) = \frac{y^2}{2} \).

We have the family of Kähler polarizations of \( (\mathbb{R}^2, \omega) \),

\[
P_\tau = \left\{ \frac{\partial}{\partial \bar{z}_\tau} \right\}, s > -s_0,
\]

where we recall that \( \tau_0 = r_0 + is_0, \tau = r + is, r_0, s_0, r, s \in \mathbb{R} \). When \( s \to -s_0 \), we see that \( P_\tau \to P_{r-is_0} \) where

\[
P_{r-is_0} = \left\{ -(r_0 + r) \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right\}_C
\]

is a real polarization.

It is straightforward to obtain the metric \( \gamma_\tau \) defined by \( (\omega, J_\tau) \),

\[
\gamma_\tau = \frac{1}{(s_0 + s)} dx^2 + 2 \left( \frac{r_0 + r}{s_0 + s} \right) dx dy + \left( \frac{(s_0 + s)^2 + (r_0 + r)^2}{s_0 + s} \right) dy^2,
\]

where \( \tau_0 = r_0 + is_0, \tau = r + is \). Note that as \( s \to -s_0 \) the map \( \varphi_\tau \) in (2.5) becomes singular. In this limit, after a rescaling by \( (s_0 + s) \), the metric becomes

\[
\lim_{s \to -s_0} (s_0 + s) \gamma_\tau = d(x + (r_0 + r)y)^2,
\]

so that there is a metric collapse of \( \mathbb{R}^2 \) to \( \mathbb{R} \), with a degeneration along the kernel of the linear map \( \varphi_{-is_0} \).

**Example 7.2** Consider, again, the symplectic plane with standard symplectic structure and with initial Kähler structure given by the standard holomorphic coordinate \( z = x + iy \). Consider the Hamiltonian function

\[
h(x, p) = \frac{1}{2} (xy)^2,
\]

with Hamiltonian vector field

\[
X_h = xy \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right).
\]

Let \( \varphi_{it} \) be the map defined in Theorem 2.5 where \( t \in \mathbb{R} \).

**Lemma 7.3**

\[
z_{it} := e^{itX_h} z_0 = e^{itxy} x + ie^{-itxy} y.
\]

**Proof.** This follows easily from direct computation from \( X_h^2(z_0) = (xy)^2 z_0 \) and \( X_h^{2k+1}(z_0) = (xy)^{2k+1} z_0 \), for \( k = 0, 1, 2, \ldots \). ■

**Remark 7.4** As we will see below, in this (non-compact) case, the map \( \varphi_{it} \) for \( t \neq 0 \) never defines a global diffeomorphism of the plane. However, in the neighborhood of each point, for \( |t| \) small enough, \( \varphi_{it} \) will be a local diffeomorphism. □
Lemma 7.5 The family of Kähler potentials and Kähler forms are given by

$$
\kappa_{it} = \frac{1}{2} \cos(2txy)(x^2 + y^2) + tx^2y^2 
$$

and

$$
\omega = ig_{it}(x,y)dz_{it} \wedge d\bar{z}_{it}, 
$$

where

$$
g_{it}^{-1} = 2t (x^2 + y^2 - 2xy \sin(2txy)) + 2 \cos(2txy).
$$

Proof. The formula for \(k_{it}\) follows from (4.1) and (4.2) by noticing that, with \(\theta = -xdy\), one has \(X_h(\theta(X_h)) = 0\) so that \(\alpha_{it} = 2ith\). The formula for \(g_{it}\) follows by computing of \(dz_\tau\) and \(d\bar{z}_\tau\) from Lemma 7.3 and comparing with \(\omega = dx \wedge dy\). We see that for any value of \(t\) there is a sufficiently small neighborhood of the origin \(x = y = 0\) where \(g_{it}\) is positive and the metric is Kähler. On the other hand, for any \(t \neq 0\) there are regions where \(g_{it}\) becomes negative. Let us consider the case \(t > 0\). (The \(t < 0\) case is clear.) We then have

$$
g_{it}^{-1} \geq 2t \min\{(x-y)^2, (x+y)^2\} + 2 \cos(2txy).
$$

We see that negative values of \(g_{it}\) can arise only in the region

$$
\{(x,y) \in \mathbb{R}^2 : |x-y| < \frac{1}{\sqrt{t}}\} \cup \{(x,y) \in \mathbb{R}^2 : |x+y| < \frac{1}{\sqrt{t}}\}.
$$

As \(t\) increases, these two strips become more and more concentrated around \(x = y\) and \(x = -y\). To find regions where \(g_{it}\) becomes negative, let us consider \(x = y\) and \(2txy = \frac{\pi}{2} + 2k\pi, k \in \mathbb{N}\). At those points we have \(g_{it}^{-1} = 0\), implying that \(\varphi_{it}\) is not a global diffeomorphism. By Taylor expanding around one of those points, we see that for \(2txy\) bigger but sufficiently close to \(\frac{\pi}{2} + 2k_0\pi, k_0 \in \mathbb{N}\), the sign of \(g_{it}\) is negative. Therefore, for any value of \(t > 0\), there is a countable set of open regions where the polarization \(\mathcal{P}_{it}\) is pseudo-Kähler. This shows that the Cauchy problem for the geodesic starting at \((\omega, J_0)\) and in the direction of \(h\) does not have a solution.

Remark 7.6 Note that the polarizations \(\mathcal{P}_{it}\) on the boundary of the Kähler regions become real, since \(\left(\frac{\partial (z_{it}, \bar{z}_{it})}{\partial (x, y)}\right)\), whose determinant equals \(g_{it}^{-1}\), has rank one at those points.

8 Relation with the Burns-Lupercio-Uribe approach

In [BLU], the authors also study the complexification \(\mathcal{H}am_{\mathbb{C}}(M, \omega) = G_{\mathbb{C}}\) when \((M, \omega)\) is real analytic. In this work, the Hamiltonian flow in complex time is described via an actual flow on a complexification \((M_{\mathbb{C}}, \omega_{\mathbb{C}})\) of \(M\). (For the complexification of real analytic Riemannian manifolds see [GS1, GS2, LS1, Sz].) While the formalism of [BLU] is more geometric than the one we consider, it requires the introduction of \(M_{\mathbb{C}}\) which may not be convenient when considering some real non-analytic examples, as in subsection 8.3.

8.1 The case of diffeomorphisms

Let us briefly review the [BLU] construction and show that in the real analytic case and when \(\varphi_\tau\) is a diffeomorphism it is equivalent to ours.

Let \((M, \omega, J_0)\) be a Kähler manifold with real analytic \((\omega, J_0)\). The complexification \(M_{\mathbb{C}}\) is a holomorphic symplectic manifold, with holomorphic symplectic form \(\omega_{\mathbb{C}}\) given by the analytic continuation of \(\omega\), as described below. On \(M_{\mathbb{C}}\) there exists an anti-holomorphic involution \(\sigma :
where the holomorphic function \( g \) isomorphic projection \( \pi : M_C \to M \), such that if \( \iota : M \to M_C \) is the inclusion, \( \pi_0 \circ \iota = id_M \). Let us describe this construction locally.

Let \( \{x^j, y^j\}_{j=1,...,n=\dim M} \) be a local system of real analytic coordinates on a sufficiently small open set \( U \subset M \). Their analytic continuation, \( \{\bar{x}^j, \bar{y}^j\}_{j=1,...,n} \), then form a local system of holomorphic coordinates on the complexification \( U_C \subset M_C \) of \( U \). Let us assume that the real analytic coordinates are such that \( \{(x^j, y^j) = (\text{Re } z^j, \text{Im } z^j)\}_{j=1,...,n} \) where \( \{z^j\}_{j=1,...,n} \) is a local system of \( J_0 \)-holomorphic coordinates on \( U \), so that \( \omega \) is given by the type \( (1, 1) \) form

\[
\omega = \sum_{j,k=1}^{n} i g_{jk}(z, \bar{z}) dz^j \wedge d\bar{z}^k,
\]

where the functions \( g_{jk} = \overline{g_{kj}}, j, k = 1, \ldots, n \), are real analytic and \( z = (z^1, \ldots, z^n) \). Then, one has holomorphic coordinates \( (z_C, w_C) \) on \( U_C \) which are the analytic continuation of \( (z, \bar{z}) \) respectively. Concretely, one has \( z^j_C = x^j_C + iy^j_C, w^j_C = x^j_C - iy^j_C, j = 1, \ldots, n \). The holomorphic symplectic form on \( U_C \) reads

\[
\omega_C = \sum_{j,k=1}^{n} i g_{jk}^C (z_C, w_C) dz^j_C \wedge dw^k_C,
\]

where the holomorphic function \( g_{jk}^C \) is the analytic continuation of \( g_{jk} \).

The anti-holomorphic involution \( \sigma : U_C \to U_C \) is given by \( \sigma(z_C, w_C) = (\bar{w}_C, \bar{z}_C) \) and the holomorphic projection \( \pi_0 : U_C \to U \) is given by \( \pi_0(z_C, w_C) = z_C \). Note that both \( \sigma \) and \( \pi_0 \) are well defined globally, since if one has an holomorphic change of local coordinates on \( M, \bar{z} = F(z), F \) holomorphic, then also one has \( \bar{z}_C = F(z_C) \) and \( \bar{w}_C = \bar{F}(w_C) \). The subset \( U \subset U_C \) is described by the equations

\[
w_C = \bar{z}_C.
\]

Let now \( h \in C^{\text{an}}(M) \) be a real Hamiltonian function, let \( \tau \in \mathbb{C} \) and consider \( \tilde{h} = \tau h \).\(^2\) Let \( \tilde{h}_C \) be its analytic continuation to \( U_C \) (if necessary one takes a smaller neighborhood), so that

\[
\tilde{h}_C(z_C, w_C)|_{z_C = w_C} = \tau h(z, \bar{z}), z = z_C.
\]

The Hamiltonian vector field of \( \text{Re } \tilde{h}_C \) with respect to the real symplectic form \( \frac{1}{2}(\omega_C + \bar{\omega}_C) \), is given by

\[
X_{\frac{1}{2}(\omega_C + \bar{\omega}_C)}^{\text{Re } h_C} = X^\omega_{\tilde{h}_C} + X^{\bar{\omega}^\tau_{\tilde{h}_C}} = \tau X^\omega_{\tilde{h}_C} + \bar{\tau} X^\bar{\omega}_{\tilde{h}_C}.
\]

Let \( I \subset \mathbb{R} \) be a sufficiently small open neighborhood of zero, and let \( \eta_t, t \in I \) be the time-\( t \) flow of the Hamiltonian vector field \( -X_{\frac{1}{2}(\omega_C + \bar{\omega}_C)}^{\text{Re } h_C} \). Let \( \mathcal{F}_t \) be the foliation of \( U_C \) defined by the projection \( \pi_0 \). Denote by \( \mathcal{F}_t \) its push-forward by \( \eta_t \), and let \( \pi_t \), for \( |t| \) small enough, be the corresponding projection to \( U \). In \([\text{BLU}]\), the authors consider the map (see their equation \((1.1)\))

\[
\pi_t \circ \eta_t \circ \iota : U \to M.
\]

If \( M \) is compact as we are assuming, as shown in \([\text{BLU}]\), this globalizes to a well defined diffeomorphism of \( M \), for small enough \( |t| \).

We then have,

\(^2\)While in the present work we consider the complex time Hamiltonian flow of real Hamiltonian functions \( h \), in \([\text{BLU}]\) the authors prefer to take more general complex valued Hamiltonians \( \tilde{h} \) and to consider their flows in real time. Our formalism can be easily adapted to the case of general complex valued real analytic \( h \). In particular, as it will be seen below in Section \([\text{BLU}]\) to describe geodesics on \( \mathcal{H}(\omega, J_0) \) it suffices to consider the imaginary time flow real Hamiltonians or, equivalently, real time flow of (pure) imaginary Hamiltonians.
Proposition 8.1 Under the conditions of Theorems 2.5 and 4.1, the diffeomorphisms of $M$ defined in (2.1) and (8.1) are related by
\[
\varphi_{t\tau}^{-1} = \pi_t \circ \eta_t,
\]
or, equivalently, the following diagram is commutative
\[
\begin{array}{ccc}
M_C & \xrightarrow{\eta_t} & M_C \\
\downarrow \varphi_{t\tau} & & \downarrow \pi_t \\
(M, J_0) & \xleftarrow{\varphi_{t\tau}} & (M, J_0).
\end{array}
\]

Proof. Consider local coordinates $(z_C, w_C)$ on $M_C$ as above, and assume we are under the conditions and notations of Theorems 2.5 and 4.1. Let $(z, \bar{z}) \in M$. Then,
\[
F_{t|\eta_{t}(z, \bar{z})} = \eta_t(F_{0|z, \bar{z}}).
\]
The leaf $F_{0|z, \bar{z}}$ is determined by the equations $z_C = z$, so that the leaf $F_{t|\eta_{t}(z, \bar{z})}$ is determined by the equations
\[
((z_C - z) \circ \eta_{-t})(z_C, w_C) = 0 \iff (e^{t\tau X_{\Re h_C}} \cdot z_C)(z_C, w_C) = z,
\]
whose set of solutions intersects $M$ at the point $\varphi_{t\tau}^{-1}(z, \bar{z})$, as we wanted.

Remark 8.2 Note that $\eta_t$ is the flow of $-X_{\Re h_C}^\omega + \omega_C$ due to a relative minus sign between our conventions and the conventions of [BLU].

8.2 Real and mixed polarizations

Let us consider a generalization of the geometric setting of [BLU] motivated by our approach. Assume that for some $t_0 \neq 0$, the complex structure $J_{t_0}$ of Theorem 1.2 of [BLU] degenerates, and the corresponding polarization of $(M, \omega)$ becomes mixed. To study this situation, let us consider diagram (8.2) with complex structures replaced by polarizations (which are equivalent to complex structures in the Kähler case),
\[
\begin{array}{ccc}
M_C & \xrightarrow{\eta_t} & M_C \\
\downarrow \varphi_{t\tau} & & \downarrow \pi_t \\
(M, P_0) & \xleftarrow{\varphi_{t\tau}} & (M, P_{t\tau}).
\end{array}
\]

Note that, in the case when $\varphi_{t\tau}$ is a diffeomorphism, one has $\varphi_{t\tau*}P_{t\tau} = P_0$. The polarization $P_0$ can be described locally by the $P_0$-polarized (i.e., $J_0$-holomorphic) functions $\{z_{ja}\}_{j=1,\ldots,n}$. Then, $P_{t\tau}$ will be described locally by the functions
\[
\{\varphi_{t\tau*}z_{ja}\}_{j=1,\ldots,n}.
\]

In the case when $\varphi_{t\tau}$ is well defined but no longer a diffeomorphism, (8.4) still defines a polarization on $M$. To illustrate this situation, let us consider an example.

Example 8.3 Consider again example 7.1. As seen in example 2.11, if $\tau = -i\tau_0$ the linear map $\varphi_{-i\tau_0}$ is not invertible, mapping $\mathbb{R}^2$ to the dimension one subspace $\{y = 0\}$. The polarization $P_0$ is defined by the polarized (i.e., $J_0$-holomorphic) function $z_0 = x + \tau_0 y$. The polarization $P_{-i\tau_0}$ is defined by the polarized function $z_0 \circ \varphi_{-i\tau_0} = x + r_0 y$, as in example 7.1.
Note also that on the complexification $M_C = \mathbb{C}^2$, for some $\tau \in \mathbb{C}$, the leaves of the foliation $\mathcal{F}_t$ are given by the conditions $(z_{t\tau})_C = \tau x + (\tau_0 + \tau y)_C = \text{const.}$ When $\tau = -i\tau_0$ (or more generally $\text{Im}(\tau) = -s_0$), these leaves either do not intersect $M = \mathbb{R}^2 \subset \mathbb{C}^2$ or are contained in $\mathbb{R}^2$ so that the projection $\pi_t$ is not defined.

We see that, for a mixed polarization $P_{t\tau}$ in $\mathbb{R}^3$ the projection $\pi_t$ is in general not defined. In fact, it will always be the case that if $P_{t\tau}$ is real then the leaves of the associated foliation of $M_C$ (defined by the level sets of the analytic continuation of the $P_{t\tau}$-polarized functions) either do not intersect $M$ or are contained in $M$. On the other hand, since the projection $\pi_0$ is well defined, by assumption, by inverting the vertical and top horizontal arrows in diagram (8.3) Proposition 8.1 can be extended in the following sense.

**Proposition 8.4** Assume that $\eta_0$ is defined for some $t \in \mathbb{R}$. The composition $\pi_0 \circ \eta_{-t} \circ \iota$ defines the map $\varphi_{t\tau} : M \to M$, as in the diagram

$$
\begin{array}{c}
M_C \xleftarrow{\eta_{-t}} M_C \\
\downarrow \pi_0 \\
(M, \mathcal{P}_0) \xleftarrow{\varphi_{t\tau}} (M, \mathcal{P}_{t\tau}).
\end{array}
$$

Let $\{(U_{\alpha}, z_{\alpha})\}$ be a set of local holomorphic charts for $(M, J_0)$, covering $\pi_0 \circ \eta_{-t} \circ \iota(M)$. Then, the local functions $\{\varphi_{t\tau}^\ast(z_{\alpha})\}_{j=1, \ldots, n}$ define a polarization $P_{t\tau}$ on $M$.

**Proof.** The functions $\{\varphi_{t\tau}^\ast(z_{\alpha}^j)\}_{j=1, \ldots, n}$ are defined locally on $M$ and for each $p \in M$ there is at least one index $\alpha$ for which the functions $\varphi_{t\tau}^\ast(z_{\alpha}^j)$, $j = 1, \ldots, n$ are well defined in a open neighborhood of $p$. The analytic continuation of $z_{\alpha}^j$ from $U_{\alpha}$ to $\pi_0^{-1}(U_{\alpha})$ is $(z_{\alpha}^j)_C$. The local functions $\{(z_{\alpha}^j)_C\}_{j=1, \ldots, n}$ on $M_C$ are functionally independent and define a local Lagrangian (with respect to $\omega_C$) foliation of $M_C$, whose leaves coincide with the leaves of $\mathcal{F}_0$. Under pull-back by the Hamiltonian flow $\eta_{-t}$, which by assumption is well defined and is therefore a diffeomorphism, these functions remain functionally independent and continue to define a local Lagrangian distribution. Since $M$ is a totally real Lagrangian submanifold of $M_C$, the restriction to $M$ by $\iota$ still gives a set of functionally independent functions defining, locally, a polarization on $M$. The fact that these locally defined polarizations glue consistently to give a globally defined polarization $P_{t\tau}$ follows from the fact that if, locally, $(z_{\alpha})_C = \phi_{\alpha\beta}((z_{\beta})_C)$, with $\phi_{\alpha\beta}$ a local biholomorphism, then $\eta_{-t}^\ast(dz_{\alpha})_C$ is a linear combination of $\{\eta_{-t}^\ast(dz_{\beta})_C\}_{j=1, \ldots, n}$.

### 8.3 Non real analytic cases

The approach described in Theorems 2.5 and 4.1 can still be useful in situations where the Hamiltonian $h$ is not real analytic. In this case, one cannot use a flow on $M_C$ to describe the evolution of Kähler structures on $M$ but the local complex time Hamiltonian action on local coordinates can sometimes still be considered.

The prototypal non real analytic example is given by $(M, \omega, J_0) = (\mathbb{R}^{2n}, \sum_{j=1}^n dx^j \wedge dy^j, J_0)$, where $J_0$ is the standard complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, when one considers a smooth (non real analytic) function Hamiltonian $h$ depending only on the variables $(y^1, \ldots, y^n)$. One has,

$$X_h = \sum_{j=1}^n \frac{\partial h}{\partial y^j} \frac{\partial}{\partial x^j}.$$

The flow of $X_h$ is given by $\gamma_t(x, y) = (x + t\frac{\partial h}{\partial y}, y)$, where $\frac{\partial h}{\partial y} = \left(\frac{\partial h}{\partial y^1}, \ldots, \frac{\partial h}{\partial y^n}\right)$. The Lie series corresponding to the global $J_0$-holomorphic coordinates, $z = x + iy$, has only the first two terms
different from zero and thus defines, for any \( \tau = r + is \in \mathbb{C}, r, s \in \mathbb{R} \), new global functions defining the new polarization, \( \mathcal{P}_\tau \),

\[
z_\tau = e^{\tau X_h}(z) = x + r \frac{\partial h}{\partial y} + i(y + s \frac{\partial h}{\partial y}).
\]

The map \( \varphi_\tau : (\mathbb{R}^{2n}, \mathcal{P}_\tau) \rightarrow (\mathbb{R}^{2n}, \mathcal{P}_0) \) ie such that

\[
z_\tau = (\varphi_\tau)^* z,
\]

is, gobally in \( \mathbb{R}^{2n} \times \mathbb{C} \), given by the following map, which is analytic in \( x \) but not in \( y \),

\[
\varphi_\tau(x, y) = (x + r \frac{\partial h}{\partial y}, y + s \frac{\partial h}{\partial y}).
\]

If \( h \) has a nonnegative Hessian then we see that \( \varphi_\tau \) are diffeomorphisms for \( s \geq 0 \). Therefore, in this situation, Theorem 2.5 is globally valid.

A more elaborate example where one can treat non real analytic Hamiltonian flows is given by the cotangent bundle of a compact Lie group, \( T^*K \), where \( h \) is a smooth \( K \)-bi-invariant function. This case was treated in [KMN1] and it is further described in Section 10.

### 8.4 Completing flows

The geometric setting of [BLU] makes it possible, in some cases, to extend our map \( \varphi_\tau \) in Theorem 2.5. Important examples are given by real algebraic completely integrable systems (aci) [Va], \((M, \omega, \mu)\), where \( \mu : M \rightarrow \mathbb{R}^n \), is the moment map of a \( \mathbb{R}^n \) action such that the complexification \((M_C, \omega_C, \mu_C)\) is a nonsingular affine variety and the regular level sets of \( \mu_C \) are affine parts of abelian varieties where the flows of \( X_\mu \) linearize. Therefore, there is an extension \((\widehat{M}, \widehat{\omega}, \widehat{\mu})\) of the original system, with \( M \) dense in \( \widehat{M} \), in which \( \varphi_\tau \) can be geometrically extended to \( \tau \in \mathbb{C} \). We will study aci systems in the present context in more detail in [MN].

### 9 Geodesics on the space of Kähler metrics

Let us consider the families of “complexified” Hamiltonian flows of the previous section. As remarked in [Do1], geodesics in the space of Kähler metrics, which is viewed as a symmetric space for the group of symplectomorphisms, can be thought of as analytic continuations of Hamiltonian flows which are the one-parameter subgroups of the group of symplectomorphisms. This can be made explicit as follows.

**Proposition 9.1** Under the conditions of Theorem 4.1, let \( \tau = it, t \in \mathbb{R} \). Then, the path of Kähler metrics \( \gamma_\tau \), in (i) in the Theorem, is a geodesic path.

**Proof.** Let \( \varphi_\tau \) be the diffeomorphism defined in Theorem 2.5 for \( \tau = it, t \in \mathbb{R} \), so that, in local complex coordinates with respect to \( J_0 \),

\[
\varphi_\tau(z, \bar{z}) = (e^{itX_h}z, e^{-itX_h}\bar{z}).
\]

Letting \( \omega_\tau = (\varphi_\tau^{-1})^* \omega \), note that, from the proof of Theorem 4.1 equations (5.6), (5.7) and from \( J_\tau = \varphi_\tau^*J_0 \), it follows that \( \omega_\tau \) is of type (1,1) with respect to \( J_0 \). In this way, we have the Moser maps

\[
\Phi_t = \varphi_\tau : (M, J_\tau, \omega) \rightarrow (M, J_0, \omega_\tau), \tau = it, t \in \mathbb{R}.
\]
Recall that, \( \kappa_\tau = -2\text{Im} \, \psi_\tau \), where

\[
\psi_\tau = -i \frac{1}{2} e^{\tau X_h} \cdot \kappa_0 + \tau h - \alpha_\tau,
\]

and \( d\alpha_\tau = e^{\tau d\phi_i X_h} \theta - \theta \), with \( \omega = -d\theta \). One has,

\[
\dot{\kappa}_\tau := \frac{d\kappa_\tau}{dt} = -X_h(\psi_\tau + \bar{\psi}_\tau) - 2h + 2\theta(X_h),
\]

where we used

\[
\dot{\alpha}_\tau = iX_h(\alpha_\tau) + i\theta(X_h).
\]

Using (6.7), let \( \phi_t = \kappa_\tau \circ \Phi_t^{-1} - \kappa_0 \), so that

\[
\phi_t(z, \bar{z}) = \kappa_\tau(e^{-\tau X_h} z, e^{\tau X_h} \bar{z}), \quad \tau = it, t \in \mathbb{R}.
\]

Then,

\[
\dot{\phi}_t = \kappa_\tau + \partial_t \kappa_\tau(-iX_h) + \bar{\partial}_t \kappa_\tau(iX_h).
\]

Using \( d\psi_t(X_h) = X_h(\psi_t) = \partial_t \psi_t(X_h) + \bar{\partial}_t \psi_t(X_h) \) and \( \theta^{(0,1)}_t = \bar{\partial}_t \psi_t \), we obtain finally

\[
\dot{\phi}_t = -2h \circ \Phi_t^{-1}.
\]

(9.1)

It follows that, for \( \tau = it, t \in \mathbb{R} \),

\[
\dot{\phi}_t \circ \Phi_t = 2i\partial_t h(X_h) - 2i\bar{\partial}_t h(X_h) = 2dh(J_t X_h) = 2\omega(X_h, J_t X_h) = 2||X_h||^2_{\gamma_t}.
\]

Since we are on a Kähler manifold we have

\[
||X_h||^2_{\gamma_t} = ||dh||^2_{\gamma_t}.
\]

On the other hand, since \( \Phi_t \circ \gamma_\tau = \gamma_t \), for \( \tau = it, t \in \mathbb{R} \), we have

\[
\nabla \dot{\phi}_t = -2\nabla h \circ \Phi_t^{-1},
\]

where on the left hand side we have the gradient of \( \dot{\phi}_t \) with respect to the metric \( \gamma_\tau \), while on the right hand side we have the gradient of \( -2h \) with respect to the metric \( \gamma_t \). Therefore, using \( ||\nabla h||^2_{\gamma_t} = ||dh||^2_{\gamma_t} \), we obtain the geodesic equation

\[
\ddot{\phi}_t = \frac{1}{2} ||\nabla \dot{\phi}_t||^2_{\gamma_t}, \quad \tau = it, t \in \mathbb{R},
\]

as we wanted. \( \Box \)

Remark 9.2 The minus sign in the expression for the velocity \( \dot{\phi}_t \) in (9.1) can be traced back to Theorem 2.5 where we see that the Moser map is \( \varphi_\tau \) and not \( \varphi_\tau^{-1} \). \( \Diamond \)

Remark 9.3 Note that in Proposition 9.1 the simplification of taking \( \tau \) to be pure imaginary gives no essential loss of generality. For real time \( \tau = s \in \mathbb{R} \), the diffeomorphisms \( \varphi_s \) form an actual Hamiltonian flow, with \( \omega_s = \omega \). In this case, as a consequence of Proposition 4.3 \( \kappa_s = \kappa_0 \circ \varphi_s \) and \( \phi_s = 0 \) so that we always obtain diffeomorphic Kähler structures \( (M, \omega, J_0, \gamma_0) \cong (M, \omega, J_s, \gamma_s) \), where \( J_s = \varphi_s^* J_0, \gamma_s = \varphi_s^* \gamma_0 \). So, to actually move in the “moduli space of Kähler structures modulo diffeomorphism” we need a non-zero imaginary part of \( \tau \). \( \Diamond \)

Remark 9.4 The geodesic imaginary time families of Kähler structures considered in Theorem 4.1 and in Proposition 9.1 have been studied in the case of symplectic toric manifolds [BFMN, KMN1] and in the case of cotangent bundles of Lie groups of compact type [KMN2]. It is straightforward to check that the Kähler potentials presented in those works are instances of formula (4.2). The case of cotangent bundles of compact Lie groups is described in Section 10. In these cases, the diffeomorphisms \( \varphi_\tau \) exist for large values of \( \tau \). Further investigation of the toric case will appear in [KMN3]. \( \Diamond \)
10 Cotangent bundle of a compact Lie group

In this section, for completeness, we recall some of the results of [KMN1] and show that the families of bi-invariant Kähler structures on cotangent bundles of compact Lie groups described there are geodesic families. (The case of the quadratic symplectic potential has also been studied in [HK2].) This is an example where one can analytically continue to complex time Hamiltonian flows of not necessarily real analytic Hamiltonian functions. (See Section 8.3)

Let $K$ be a compact Lie group. Infinite families of $K \times K$-invariant Kähler structures on $TK \cong K \times \text{Lie}(K) \cong K_C$, where $K_C$ is the complexification of $K$, were studied in [KMN1].

Let $h \in C^\infty(\text{Lie}(K))$ be an Ad-invariant strictly convex function with $||H||$ bounded away from zero, where $H$ is the Hessian of $h$. Let $\{y^i\}_{i=1,\ldots,n}$ be cartesian coordinates on $\text{Lie}(K)$ associated to a choice of orthonormal basis with respect to the Killing metric. Let

$$u^i = \frac{\partial h}{\partial y^i}, \quad i = 1, \ldots, n.$$  \tag{10.1}

Then, the map

$$T^*K \cong K \times \text{Lie}(K) \xrightarrow{\psi_0} K_C \quad \quad (x, Y) \mapsto x e^{iu(Y)},$$

is a diffeomorphism and defines on $T^*K$ (and on $K_C$) a Kähler structure $(T^*K, \omega, J_0)$ [KMN1]. To study the action of the Lie series $e^{\tau X_h}$ on the polarization $P_0$ corresponding to $J_0$ it is sufficient to study its action on $J_0$-holomorphic functions of the form, $f_{\pi, E}(x, Y) = \text{tr}(E_\pi \pi(x e^{iu(Y)})$, where $\pi$ denotes a finite dimensional representation of $K_C$ and $E_\pi$ an endomorphism of the space of the representation.

**Proposition 10.1**

$$e^{\tau X_h}(f_{\pi, E})(x, Y) = \text{tr}(E_\pi \pi(x e^{i(\tau + i)u(Y)}))$$  \tag{10.2}

**Proof.** This is a consequence of Theorem 3.7 in [KMN1]. There, it is shown that the operator $e^{\tau X_h}$ applied to $\text{tr}(E_\pi \pi(x)) \in C^\infty(K)$ gives

$$e^{\tau X_h} \cdot \text{tr}(E_\pi \pi(x)) = \text{tr}(E_\pi \pi(x e^{\tau u(Y)})).$$

On the other hand, for $\tau' = \tau + i$,

$$e^{(\tau + i)X_h} \cdot \text{tr}(E_\pi \pi(x)) = e^{\tau X_h} \cdot \text{tr}(E_\pi \pi(x e^{iu(Y)})),$$

which proves the proposition. For a more explicit proof, note that, from [KMN1],

$$X_h = \sum_{i=1}^n u^i(Y) X_i,$$  \tag{10.3}

where $\{X_i\}_{i=1,\ldots,n}$ is a frame of left-invariant vector fields on $K \times \text{Lie}(K)$, with zero component along $\text{Lie}(K)$. Therefore, we have

$$X_h \cdot \pi(x e^{iu(Y)}) = \pi(x) \pi(u(Y)) \pi(e^{iu(Y)}),$$

where we also denote by $\pi$ the representation induced by $\pi$ on $\text{Lie}(K)$. Therefore,

$$e^{\tau X_h} \cdot \text{tr}(E_\pi \pi(x e^{iu(Y)})) = \text{tr}(E_\pi \pi(x \pi(e^{\tau u(Y)}) \pi(e^{iu(Y)}))) = \text{tr}(E_\pi \pi(x e^{(\tau + i)u(Y)})).$$
Remark 10.2 Notice that Proposition 10.1 does not require the smooth function \( h \) to be real analytic.

The functions (10.2) define a polarization \( P_\tau \), which is Kähler for \( s = \text{Im}(\tau) > -1 \) (see Theorem 10.3), real for \( s = -1 \) and pseudo-Kähler for \( s < -1 \). The map \( \phi_\tau : T^*K \to T^*K \), mapping \( J_0 \) holomorphic functions to \( P_\tau \) polarized functions, is easily seen to be given by

\[
\varphi_\tau = \psi_0^{-1} \circ \psi_\tau,
\]

where

\[
T^*K \cong K \times \text{Lie}(K) \quad \xymatrix{ \psi \ar[r] & K_C \\ (x,Y) \ar[r]^\psi & x e^{(i+\tau)u(Y)} }
\]

**Theorem 10.3** ([KMN1]) For \( s = \text{Im}(\tau) > -1 \), \((T^*K,\omega,\phi_\tau^*J_0)\) is a Kähler manifold, with \( K \times K \) invariant Kähler structure and Kähler potential obtained by a Legendre transform of the \( K \times K \)-invariant Hamiltonian \( h \),

\[
\kappa_\tau(u(Y)) = 2(s + 1)(Y \cdot u(Y) - h(Y)).
\]

It is straightforward to check that the Kähler potentials \( \kappa_\tau \) correspond to a geodesic in the space of Kähler metrics generated by the imaginary time “flow” \( e^{\tau X_h} \). In fact,

**Proposition 10.4** The Kähler potential \( \kappa_\tau \) in Theorem 10.3 satisfies equation (4.2) in Theorem 4.1.

**Proof.** A symplectic potential is \( \theta = \sum_{i=1}^n y_i^* w_i \), where \( \{w_i\}_{i=1,\ldots,n} \) is the frame of left-invariant 1-forms on \( K \) dual to \( \{X_i\}_{i=1,\ldots,n} \) and pulled-back to \( T^*K \) by the canonical projection. One then has, from (10.3), \( \theta(X_h) = u(Y) \cdot Y \) and \( \iota_{X_h} d(\theta(X_h)) = 0 \), so that \( \alpha_\tau \) in Theorem 4.1 is

\[
\alpha_\tau = \tau u(Y) \cdot Y.
\]

Then, according to Theorem 4.1

\[
\kappa_\tau = -2\text{Im} \left( -\frac{i}{2} e^{\tau X_h} \kappa_0 + \tau h - \tau u(Y) \cdot Y \right).
\]

Since (see [KMN1]) \( \kappa_0 = 2(Y \cdot u(Y) - h(Y)) \) in this case, from (10.3) we have \( X_h(\kappa_0) = 0 \) and \( \kappa_\tau = 2(\text{Im} \tau + 1)(u(Y) \cdot Y - h) \), as required.

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**References**

[Be] R. Berman, *K-polystability of Q-Fano varieties admitting Kähler-Einstein metrics*, arXiv:1205.6214.

[BLU] D. Burns, E. Lupercio and A. Uribe, *The exponential map of the complexification of Ham in the real-analytic case*, arXiv:1307.0493.
[BFMN] T. Baier, C. Florentino, J. M. Mourão, and J. P. Nunes, *Toric Kähler metrics seen from infinity, quantization and compact tropical amoébas*, J. Diff. Geom. **89** (2011), 411-454.

[CDS] X-X. Chen, S. Donaldson and S. Sun, *Kähler-Einstein metrics and stability*, arXiv:1210.7494.

[Do1] S. K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, Amer. Math. Soc. Transl. (2) **196** (1999), 13–33.

[Do2] S. K. Donaldson, *Moment maps and diffeomorphisms*, Asian, J. Math., **3** (1999), 1–16.

[Do3] S. Donaldson, *Scalar curvature and projective embeddings. I*, J. Differential Geom. **59** (2001), 479–522.

[FMMN1] C. Florentino, P. Matias, J. M. Mourão, and J. P. Nunes, *Geometric quantization, complex structures and the coherent state transform*, J. Funct. Anal. **221** (2005), no. 2, 303–322.

[FMMN2] C. Florentino, P. Matias, J. M. Mourão, and J. P. Nunes, *On the BKS pairing for Kähler quantizations of the cotangent bundle of a Lie group*, J. Funct. Anal. **234** (2006), no. 1, 180–198.

[Gr1] W. Grobner, *Die Lie-Reihen und ihre Anwendungen*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1960.

[Gr2] W. Gröbner, *General theory of the Lie series*, in *Contributions to the method of Lie series*, W. Gröbner and H. Knapp Eds., Bibliographisches Institut Manheim, 1967.

[GS1] V. Guillemin and M. Stenzel, *Grauert tubes and the homogeneous Monge-Ampère equation*, J. Diff. Geom. **34** (1991), no. 2, 561–570.

[GS2] V. Guillemin and M. Stenzel, *Grauert tubes and the homogeneous Monge-Ampère equation II*, J. Diff. Geom. **35** (1992), no. 3, 627–641.

[HK1] B. Hall and W. D. Kirwin, *Adapted complex structures and the geodesic flow*, Math. Ann. **350** (2011), 455–474.

[HK2] B. Hall and W. D. Kirwin, *Complex structures adapted to magnetic flows*, arXiv:1201.2142.

[KMN1] W. D. Kirwin, J. M. Mourão, and J. P. Nunes, *Complex time evolution in geometric quantization and generalized coherent state transforms*, Journ. Funct. Anal. **265** (2013) 1460-1493.

[KMN2] W. D. Kirwin, J. M. Mourão, and J. P. Nunes, *Complex time evolution and the Mackey-­Stone-­Von Neumann theorem*, arXiv:1211.2145.

[KMN3] W. D. Kirwin, J. M. Mourão, and J. P. Nunes, *Decomplexification of toric varieties, geodesics in the space of toric Kähler metrics and quantization*, in preparation.
W. D. Kirwin and S. Wu, *Geometric quantization, parallel transport and the Fourier transform*, Comm. Math. Phys. **266** (2006), 577–594.

W. D. Kirwin and S. Wu, *Momentum space for compact Lie groups and the Peter-Weyl theorem*, in preparation.

L. Lempert and R. Szőke, *Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of riemannian manifolds*, Math. Annalen **319** (1991), no. 4, 689–712.

L. Lempert and R. Szőke, *Uniqueness in geometric quantization*, arXiv:1004.4863

L. Lempert and R. Szőke, *A new look at adapted complex structures*, Bull. Lond. Math. Soc. **44** (2012), 367–374.

T. Mabuchi, *Some symplectic geometry on compact Kähler manifolds. I.*, Osaka J. Math. **24** (1987), 227–252.

J. Mourão and J. P. Nunes, *Decomplexification of integrable systems, metric collapse and quantization*, in preparation.

D.H. Phong and J. Sturm, *Lectures on stability and constant scalar curvature*, arXiv:0801.4179

Y. Rubinstein and S. Zelditch, *The Cauchy problem for the homogeneous Monge-Ampère equation, I. Toeplitz quantization*, J. Differential Geom. **90** (2012), 303–327.

Y. Rubinstein and S. Zelditch, *The Cauchy problem for the homogeneous Monge-Ampère equation, II. Legendre transform*, Adv. Math. **228** (2011), 2989-3025.

Y. Rubinstein and S. Zelditch, *The Cauchy problem for the homogeneous Monge-Ampère equation, III. Lifespan*, arXiv:arXiv:1205.4793

S. Semmes, *Complex Monge–Ampère and symplectic manifolds*, Amer.J. Math. **114** (1992), 495–550.

J. Stoppa, *K-stability of constant scalar curvature Kähler manifolds*, Adv. Math. **221** (2009), 13971408.

R. Szőke, *Complex structures on tangent bundles of riemannian manifolds*, Math. Annalen **291** (1991), no. 3, 409–428.

T. Thiemann, *Reality conditions inducing transforms for quantum gauge field theory and quantum gravity*, Class.Quant.Grav. **13** (1996), 1383–1404.

T. Thiemann, *Gauge field theory coherent states (GCS). I. General properties*, Classical Quantum Gravity **18** (2001), no. 11, 2025–2064.

G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), 137.
[Va] P. Vanhaecke, *Algebraic integrability: a survey.*, Phil. Transl. R. Soc. A 366 (2007), 1203–1224.

[Wo] N. Woodhouse, *Geometric quantization*, Oxford University Press, Oxford, 1991.