Model of the method of parallel approximation on the surface

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Abstract. The article considers the kinematic model of pursuit, transferred from plane to surface, by the method of parallel approach. Simulation of iterative pursuit processes is a topical continuation of autonomous unmanned vehicles. The aim of the article is to develop a model in which the trajectory of the pursuer is the result of following the predicted routes at each discrete moment in time. The research was carried out in the computer mathematics system MathCAD on a surface defined by a point basis. Situations with different initial states were simulated. Detailed results in the form of animated images based on materials, program codes can be found on the authors’ website and channel.

1. Introduction
Earlier, the method of parallel approach on a plane was considered in the works of R. Isaacs, L. S. Pontryagin, L.O. Petrosyan in which it was shown that this method is optimal for the pursuer to achieve a maneuvering target.

\begin{align*}
\vec{r}_{i} &= \frac{\text{Target}_{i-1} - \text{Pers}_{i-1}}{|\text{Target}_{i-1} - \text{Pers}_{i-1}|} \\
L(\mu) &= \text{Target}_{i} + \mu \cdot \vec{r} \\
( L(\mu) - \text{Pers}_{i-1} )^2 &= (V_{p} \cdot \Delta T)^2
\end{align*}

\textbf{Figure 1.} Iterative scheme of parallel approach on a plane

The problem of pursuit on a plane by the parallel approach method can be interpreted as shown in Figure 1:
If the target takes a step: \( \text{Target}_i = \text{Target}_{i-1} + \hat{V}_T \cdot \Delta T \), then the line \( L(u) \) is drawn from the point \( \text{Target}_i \), and we are looking for its intersection point with a circle of radius \( V_P \cdot \Delta T \) centered at point \( \text{Pers}_{i-1} \).

This article proposes a model of a similar iteration scheme, but in surface space.

2. Problem statement
This article discusses an iterative model of sequential calculation of points of the pursuer's trajectory by a method similar to the method of parallel approach in projection onto a horizontal plane.

![Figure 2. Calculation of the next step of the iterative process](image_url)

To calculate the next iteration step at the point where the pursuer is located \( \text{Pers}_{i-1} \) on the surface \( z = f(x, y) \) we make a sphere \( S_{i-1} \) with radius \( V_P \cdot \Delta T \) (Figure 2). Then, the intersection point \( \text{Pers}_i \) of the sphere \( S_{i-1} \) with the line \( L_i \), which is the desired point of the next step. At the point of finding the target \( \text{Target}_i \) a projection plane \( \Sigma_i \) is constructed, which is parallel to the line connecting the horizontal projections of the starting points of the points \( \text{Pers}_0 \) and \( \text{Target}_0 \), of the pursuer and the target. The line \( L_i \) it is the product of the surface crossing \( z = f(x, y) \) and the surface \( \Sigma_i \).

3. Theory
1. Trajectory of the target
We will assume that the surface of movement of the pursuer and the target is given explicitly \( z = f(x, y) \). In general, the proposed geometric model will work when the pursuer and the target will mutually influence each other's behavior. In the program written on the basis of the materials of the article, the trajectory of the target is predetermined and is set by the projection onto the plane \( (XY) \) as functions \( x_t(t) \) and \( y_t(t) \). Where \( t \) – is a formal parameter. After processing the projection, we get the equation of the target trajectory:
\[
\text{Target}(s) = \begin{bmatrix}
  x_t(t(s)) \\
  y_t(t(s)) \\
  f(x_t(t(s)), y_t(t(s)))
\end{bmatrix}, \text{where } s \text{ is an arc length parameter.}
\]

The total arc length differential will be: \(ds^2 = dx_t^2 + dy_t^2 + \left(\frac{\partial f}{\partial x_t} \cdot dx_t + \frac{\partial f}{\partial y_t} \cdot dy_t\right)^2\). From this we come to the differential equation:

\[
\frac{dt}{ds} = \frac{1}{\sqrt{\frac{dx_t^2}{dt^2} + \frac{dy_t^2}{dt^2} + \left(\frac{\partial f}{\partial x_t} \cdot \frac{dx_t}{dt} + \frac{\partial f}{\partial y_t} \cdot \frac{dy_t}{dt}\right)^2}}.
\]

We solve this equation in the test program by the Runge-Kutta method of the 4th order with the initial conditions \(t(0) = 0\). As a result of the solution, we get a functional dependency \(s = V_T \cdot t\). If the speed of the target on the surface is constant and equal in magnitude \(V_T\), then you can go to the time parameter: \(s = V_T \cdot t\).

The target step in our iterative model will be:

\[
\text{Target}_i = \text{Target}_{i-1} + \frac{d\text{Target}(s)}{ds} \cdot V_T \cdot \Delta T.
\]

2. Iterative process of calculating points of the pursuer trajectory

The course of solving this iterative process is determined by the initial position of the points of the pursuer and the target. Consider the initial position of the points of the pursuer and the target, \(\text{Pers}_0\) and \(\text{Target}_0\) (Figure 3). Their projections onto the plane \((XY)\) will be \(\text{Pers}_{XY0}\) and \(\text{Target}_{XY0}\). Let’s form a vector \(\tau = \text{Pers}_{XY0} - \text{Target}_{XY0}\) on the plane \((XY)\). Where \(\text{Pers}_{XY0}\) and \(\text{Target}_{XY0}\) are horizontal projections of the points \(\text{Pers}_0\) and \(\text{Target}_0\) of starting position. If the coordinates of the point \(\text{Target}_i\) are known, then its horizontal projection \(\text{Target}_{XY_i}\) is also known. From each projection \(\text{Target}_{XY_i}\) let’s set aside the vector \(\tau\) to get the point \(\text{Pers}_{XY_i}\). Further, on the plane \((XY)\) a parametric straight line \(L_{XY_i}(y) = (1-y) \cdot \text{Target}_{XY_i} + y \cdot \text{Pers}_{XY_i}\) * is formed. The line \(L_{XY_i}(y)\)

on the plane \((XY)\) can be decomposed by coordinates \(L_{XY_i}(y) = \begin{bmatrix} L_{X_i}(y) \\ L_{Y_i}(y) \end{bmatrix}\). From there we get the

parametric equation of the line \(L_i(y) = \begin{bmatrix} L_{X_i}(y) \\ L_{Y_i}(y) \\ f(L_{X_i}(y), L_{Y_i}(y)) \end{bmatrix}\) on the surface \(z = f(x, y)\).

To get the coordinates of the next iteration step \(\text{Pers}_i\), it is necessary to build a sphere with a radius \(V_p \cdot \Delta T\) at the previously calculated point \(\text{Pers}_{i-1}\). \(\Delta T\) - is time sampling period.

The built-in means of computer mathematics systems are used to solve the equation \(|L_i(y) - \text{Pers}_{i-1}| = V_p \cdot \Delta T\) regarding the parameter \(y\). In our test program, we used the Mueller method for the solution. Let’s substitute the equation of the line \(L_i(y)\), \(\text{Pers}_i = L_i(y)\) the found value \(y\). This value will be the desired point for the next step of our iterative process.
Based on the materials of the article, a test program was written that sequentially calculates the points of the trajectory of the pursuer (Figure 4). The full text of the program with comments is located on the resource [7].

Figure 4 shows many points of the target's trajectory. \{Target_i\}, the set of points \{Pers_i^*\} and one of the lines connecting the points \{Target_i\} and \{Pers_i^*\}. Figure 4 is supplemented with a link to an animated image [8].
Each of the points of the pursuer's trajectory is a product of the intersection of a sphere of radius $V_P \cdot \Delta T$ centered at the previous position of the pursuer. Figure 5 shows one sphere. It can be seen from the figure that the sphere passes through the point of the next step of the trajectory. Figure 5 is supplemented with a link to an animated image [9].

5. The discussion of the results
The parallel approach method in pursuit problems on a plane and in space has many practical implementations. I would like to implement an analog of the pursuit method on surfaces. Note that the target and the pursuer can be on different surfaces. The pursuer's surface can be equidistant to the target surface or close to it. With such pursuit, questions arise about the line of sight of the target, about its ability to hide in the "folds of the terrain." Our target was to develop a model that would be useful for developers of autonomous robotic systems with elements of artificial intelligence.

6. Conclusion
We have developed an algorithm and written a test program for constructing the trajectory of the pursuer. The pursuer follows a parallel approach strategy. This algorithm can be implemented in the problem of simultaneous achievement of the goal by several pursuers. When writing the article, the theoretical results presented in the works of the founders of game theory [1-6] were used. The program code can be viewed on the resource [7]. Animated images made on the basis of the article can be viewed in the sources [8], [9].

7. References
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[9] Video, visualization of spheres when modeling the method of parallel approach, https://www.youtube.com/watch?v=xszwIyTHUec

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