Dirac-like Affine Fields in 3D

I. Mišković and Dj. Šijački
Institute of Physics, P.O.Box 57, 11001 Belgrade, Yugoslavia

Abstract. A generalization of the Dirac field equation in three-dimensional Minkowski space-time to the case of the $\mathcal{SL}(3,R) \subset \mathcal{SA}(3,R)$ symmetry is considered. Constraints that ensure a correct physical interpretation of the corresponding particle states are presented. Dirac-like equations based on both multiplicity-free and generic infinite-component $\mathcal{SL}(3,R)$ representations are outlined.

Introduction. The Dirac equation and the corresponding fields, that describe relativistic point-like quantum objects of spin $\frac{1}{2}$, played a crucial role in various important Particle Physics developments. There is a long list starting with successful description of the electron field and its electromagnetic interactions that goes over many aspects of the gauge theories and the Standard Model of electroweak and strong interactions to super-symmetry and non-commutative geometry. The aim of this paper is to consider a generalization of the Dirac equation and the corresponding fields in 3-dimensional Minkowski space-time to the case of the affine symmetry $\mathcal{SA}(3,R)$. In this case the Lorentz subgroup $SO(2,1)$ of the Poincaré group in 3-dimensions is enlarged to the group of all (special) linear transformations $SL(3,R)$. The relevance of such a Dirac-like Affine Symmetry equation are for an effective description of the IR (confining) region of the Hadronic Physics as well as in the field of gravitational interactions of spinorial matter.

Hadronic Matter in the IR Region. There are two recent results that focus on the relevance of the group of special linear transformations for an effective description of the hadronic matter in QCD. (i) It has been shown, that a Yang-Mills theory based on the $SU(2)$ group in 3D can be recast in the form of the General Relativity theory [1]. In particular one finds [2] a spatial $SL(3,R)$ symmetry group. (ii) It has been demonstrated that the QCD theory in the IR region can be described by an effective gravity-like theory (Chromogravity) [3]. Here, instead of the local $SU(3)$ color symmetry
one has an induced $\text{Diff}(4, R)$ group of General Coordinate Transformations ($\text{GCT}$). The infinite-component $\overline{\text{SL}}(4, R) \subset \text{Diff}(4, R)$ representations describe the hadronic matter fields [4].

**General Covariance and Spinorial Matter.** In the standard approach to General Relativity one starts with the group of General Coordinate Transformations and the theory is set upon the principle of general covariance. A unified description of both tensors and spinors would require the existence of respectively tensorial and (double valued) spinorial representations of the $\text{GCT}$ group. It is well known that the finite-dimensional representations of $\text{GCT}$ are characterized by the corresponding ones of the $\overline{\text{SL}}(4, R)$ group, and $\overline{\text{SL}}(4, R)$ does not have finite spinorial representations. However there are infinite-dimensional spinors of $\overline{\text{SL}}(4, R)$ which are the true "world" (holonomic) spinors [5]. The anholonomic $\overline{\text{SL}}(4, R)$ spinors describe the fermionic matter fields of the Metric-Affine Theory of Gravity [6].

**Physical Requirements.** The affine group $\overline{\text{SA}}(3, R) = T_3 \ltimes \overline{\text{SL}}(3, R)$, is a semidirect product of translations and $\overline{\text{SL}}(3, R)$ generated by $Q_{\mu \nu}$ ($\mu, \nu = 0, 1, 2$). The antisymmetric operators $M_{\mu \nu} = \frac{1}{2}(Q_{\mu \nu} - Q_{\nu \mu})$ generate the Lorentz subgroup $\overline{\text{SO}}(2, 1)$, the symmetric traceless operators (shears) $T_{\mu \nu} = \frac{1}{2}(Q_{\mu \nu} + Q_{\nu \mu}) - \frac{1}{3} \eta_{\mu \nu} Q_{\sigma \rho}$ generate the proper 3-volume-preserving deformations.

**Unitarity.** As in the Poincaré case, the $\overline{\text{SA}}(3, R)$ unirreps are induced from the unirreps of the corresponding little group $T'_2 \ltimes \overline{\text{SL}}(2, R)$. In the physically most interesting case $T'_2$ is represented trivially. The corresponding particle states have to be described by the unitary $\overline{\text{SL}}(2, R)$ representations, which are infinite-dimensional owing to the $\overline{\text{SL}}(2, R)$ noncompactness. Therefore, the corresponding $\overline{\text{SL}}(3, R)$ matter fields $\Psi(x)$ are necessarily infinite-dimensional and when reduced with respect to the $\overline{\text{SL}}(2, R)$ subgroup should transform with respect to its unirreps.

**Particle Properties.** Had the whole $\overline{\text{SL}}(3, R)$ been represented unitarily, the Lorentz boost generators would have a hermitian intrinsic part; as a result, when boosting a particle, one would obtain a particle with a different spin, i.e. another particle - contrary to experience. There exists however a remarkable inner deunitarizing automorphism $\mathcal{A}$ [4], which leaves the $R_+ \otimes \overline{\text{SL}}(2, R)$ subgroup intact, and which maps the $T_{0k}, M_{0k}$
generators into \(iM_{0k}, iT_{0k}\) respectively \((k = 1, 2)\). The deunitarizing automorphism allows us to start with the unitary representations of the \(\overline{SL}(3, R)\) group, and upon its application, to identify the finite (unitary) representations of the abstract \(\overline{SO}(3)\) compact subgroup with nonunitary representations of the physical Lorentz group. In this way, we avoid a disease common to most of infinite-component wave equations, in particular those based on groups containing the \(\overline{SL}(4, R)\) group \([7]\).

**Dirac-like Infinite-component Equation.** Let us consider a Dirac-like equation,

\[
(X^\mu p_\mu - M)\Psi(x) = 0
\]

for the field \(\Psi(x)\) that transforms as follows,

\[
\Psi(x) \mapsto \Psi'(x') = D(\bar{A})\Psi(A(x - a)), \quad (a, \bar{A}) \in T_3 \ltimes \overline{SL}(3, R)^A.
\]

The \(X_\mu\) matrices generalize the Dirac \(\gamma_\mu\) ones, act in the space of infinite-component spinorial fields \(\Psi(x)\) and ensure the \(\overline{SL}(3, R)^A\) covariance \((X_\mu \mapsto D(\bar{A})X_\mu D^{-1}(\bar{A}))\).

All \(\overline{SL}(3, R)\) unirreps are known \([8]\), and explicitly given in terms of the representation labels \((\sigma, \delta)\), and the \(\overline{SO}(3)\) subgroup representations \(D^{(j)}\). An arbitrary 3-vector operator \((j = 1)\) is given by

\[
X_\alpha = aD^{A(1)}_{0\alpha}(g) + b[D^{A(1)}_{1\alpha}(g) + D^{A(1)}_{-1\alpha}(g)], \quad g \in \overline{SO}(3),
\]

where, in the spherical basis, \(\alpha = 0, \pm 1\). There are two distinct cases corresponding to the multiplicity-free and generic \(\overline{SL}(3, R)\) representations.

**Multiplicity-free Representations Case.** The \(\overline{SO}(3)\) representations content \(\{j\}\) of the multiplicity-free \(\overline{SL}(3, R)\) representation is characterized by the \(\Delta j = 2\) condition, and thus for the representation starting with \(j = \frac{1}{2}\) one has \(\{j\} = \{\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots\}\). The matrix elements of the \(X_\alpha\) vector operator read

\[
\langle \sigma' j' m' | X_\alpha | \sigma j m \rangle = a^{(\sigma' \sigma)}_{j' j}(-)^{j' - m'}\sqrt{(2j' + 1)(2j + 1)}\left( \begin{array}{cc} j' & 1 \\ -m' & \alpha \end{array} \right)^A.
\]

**Generic Representation Case.** In the nontrivial-multiplicity case we obtain the following expression for the \(X_\mu\) matrix elements:

\[
\langle \sigma' \delta' j' k' m' | X_\alpha | \sigma \delta j k m \rangle = (-)^{j' - k'}(-)^{j' - m'}\sqrt{(2j' + 1)(2j + 1)}\left( \begin{array}{cc} j' & 1 \\ -m' & \alpha \end{array} \right)^A \times
\]
× \left\{ a_{j'j}^{(\sigma'\delta' \sigma \delta)} \begin{pmatrix} \hat{j'} & 1 & \hat{j} \\ -k' & 0 & k \end{pmatrix} + b_{j'j}^{(\sigma' \delta' \sigma \delta)} \left[ \begin{pmatrix} \hat{j'} & 1 & \hat{j} \\ -k' & 0 & k \end{pmatrix} + \begin{pmatrix} \hat{j'} & 1 & \hat{j} \\ -k' & -1 & k \end{pmatrix} \right] \right\}.

The reduced matrix elements \( a_{j'j}^{(\sigma'\delta' \sigma \delta)} \) and \( b_{j'j}^{(\sigma' \delta' \sigma \delta)} \) are determined by first embedding the \( \overline{SL}(3, R) \) group into the \( \overline{SL}(4, R) \) one, then by identifying \( X_\mu \) as \( Q_\mu A \) generators, and finally by reducing the \( \overline{SL}(4, R) \) representations down to the \( \overline{SL}(3, R) \) ones.

In conclusion, we have shown explicitly the existence of a non-trivial Dirac-like \( \overline{SL}(3, R) \) covariant field equation fulfilling all relevant physical requirements.

References

[1] F.A. Lunev, Phys. Lett. B 295 92 (1992); D.Z. Freedman, P.E. Haagensen, K. Johnson and J.I. Latorre, [hep-th/9309045]; E.W. Mielke, Y.N. Obukhov and F.W Hehl, Phys. Lett A 192 153 (1994); V. Radovanović and Dj. Šijački, Class. Quant. Grav. 12 1791 (1995).

[2] M. Bauer, D.Z. Freedman and P.E. Haagensen, Nucl. Phys. B 428 147 (1994).

[3] Dj. Šijački and Y. Ne’eman, Phys. Lett. B 247 571 (1990); Y. Ne’eman and Dj. Šijački, Phys. Lett. B 276 173 (1992); *ibid.* Int. J. Mod. Phys. A 10 4399 (1995); *ibid.* Mod. Phys. Lett. A 11 217 (1996).

[4] Y. Ne’eman and Dj. Šijački, Phys. Rev. D 37 3267 (1988); *ibid.* D 47 4133 (1993).

[5] Y. Ne’eman and Dj. Šijački, Phys. Lett. B 157 275 (1985); *ibid.* Found. Phys. 27 1105 (1997); Dj. Šijački, Acta Phys. Pol. B 29 1089 (1998).

[6] F.W. Hehl, G.D. Kerlick and P. von der Heyde, Phys. Lett. B 63 446 (1976); Y. Ne’eman and Dj. Šijački, Ann. Phys. (N.Y.) 120 292 (1979); F.W. Hehl, J.D. McCrea, E.W. Mielke and Y. Ne’eman, Phys. Rep. 258 1 (1995).

[7] J. Mickelsson, Commun. Math. Phys. 88 551 (1983); A. Cant and Y. Ne’eman, J. Math. Phys. 26 3180 (1985).

[8] Dj. Šijački, J. Math. Phys. 16 298 (1975); Dj. Šijački, J. Math. Phys. 31 (1990) 1872.