ON BILINEAR MAXIMAL BOCHNER-RIESZ OPERATORS

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Abstract. We prove that the bilinear maximal Bochner-Riesz operator $T_\lambda^*$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for appropriate $(p_1, p_2, p)$ when $\lambda > (4n + 3)/5$.

1. Introduction

Let $\mathbb{T}^n$ be the torus $[0,1]_n$. For a function $f$ in $L^1(\mathbb{T}^n)$, its Fourier coefficient $a_k$ for $k \in \mathbb{Z}^n$ is defined by

$$a_k = \int_{\mathbb{T}^n} f(x) e^{-2\pi ik \cdot x} \, dx.$$ (1)

The series $\sum_{k \in \mathbb{Z}^n} a_k e^{2\pi ik \cdot x}$ is called the Fourier series related to $f$.

The pointwise convergence of the Fourier series (1) has been a central problem of Fourier analysis. Carleson’s celebrated work [4], answering Lusin’s conjecture, shows that the pointwise convergence is valid a.e. for $f$ in $L^2(\mathbb{T})$. This was generalized later by Hunt [13] for any $f$ in $L^p(\mathbb{T})$ with $1 < p \leq \infty$. These pointwise convergence results were reproved by Fefferman [8] and Lacey and Thiele [14].

For higher dimensions the Fourier series could naturally be interpreted as the limit of the spherical partial sum $\sum_{|k| \leq N} a_k e^{2\pi ik \cdot x}$. Unfortunately Fefferman’s counterexample [7] indicates that the pointwise convergence fails for such sums when $n \geq 2$ and $p \neq 2$. For this reason, it is natural to consider smoother versions of spherical sums, known as the Bochner-Riesz means, for which several analogous pointwise convergence results have been obtained; see, for instance Carbery [2], Christ [5], Carbery, Rubio de Francia and Vega [3], and Tao [18].

We can also consider analogous bilinear questions. In the bilinear theory, developed in the past decades, we study the restriction of the output of linear operators on the diagonal, when the input is of tensor
product form. In our case, we ask what type of convergence can we obtain for the operator

$$A_\lambda^t(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)(1 - (|t\xi|^2 + |t\eta|^2))^{\lambda}_+ e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta,$$

which can be regarded as $B_{1/t}^\lambda(f \otimes g)(x,x)$ with $B_{1/t}^\lambda$ the Bochner-Riesz operator on $\mathbb{R}^{2n}$ and $x \in \mathbb{R}^n$. For test functions we should have $A_\lambda^t(f,g) \to fg$ as $t \to 0$ in the $L^p$ or pointwise sense. Grafakos and Li [12] and Bernicot Grafakos, Song and Yan [1] have proved some partial positive results for $\lambda = 0$ and $\lambda > 0$ respectively concerning the $L^p$ convergence.

In this paper, we are concerned with the pointwise convergence of the means (2), in particular with the boundedness of the bilinear maximal Bochner-Riesz operator, which of course implies the boundedness of the bilinear Bochner-Riesz operators. The bilinear maximal Bochner-Riesz operator for $\lambda > 0$ is defined as

$$T_\lambda^*(f,g)(x) = \sup_{t > 0} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m(t\xi, t\eta) \hat{f}(\xi)\hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \right|,$$

where $m(\xi, \eta) = m^\lambda(\xi, \eta) = (1 - (|\xi|^2 + |\eta|^2))^{\lambda}_+$, which is equal to $(1 - (|\xi|^2 + |\eta|^2))^{\lambda}$ when $|(\xi, \eta)| \leq 1$ and 0 when $|(\xi, \eta)| > 1$. For simplicity we occasionally denote $T_\lambda^*$ by $T_*$ when there is no confusion. $T_*$ is a natural generalization of the (linear) maximal Bochner-Riesz operator from which it naturally inherits its name.

Our main theorem is as follows.

**Theorem 1.1.** When $\lambda > (4n + 3)/5$, $T_\lambda^*$ is bounded from $L^{p_1} \times L^{p_2}$ to $L^p$ when $p > (2n-11)/(10\lambda-6n-17)$, $p_1$, $p_2 > (4n-22)/(10\lambda-6n-17)$ and $1/p = 1/p_1 + 1/p_2$.

To prove this theorem, we need to interpolate between the positive result for $\lambda$ above the critical index (Proposition 2.1) and the $L^2 \times L^2 \to L^1$ boundedness (Theorem 1.3) for $\lambda$ down to $(4n + 3)/5$. The former of these results is standard but the latter is novel and constitutes the main contribution of this paper. Assuming these two results, we prove Theorem 1.1 via the standard interpolation technique known from the linear case; we outline some ideas of this interpolation technique below and omit the details, which can be found in [15] and [16].

**Proof of Theorem 1.1.** First we (bi)linearize this maximal operator $T_\lambda^*$. Let $\mathcal{A}$ be the class of nonnegative measurable functions on $\mathbb{R}^n$ with finitely many distinct values. For each function $R \in \mathcal{A}$, we can define
the bilinear operator $T_{R(x)}(f, g) = T_{R(x)}^\lambda(f, g)$ by the integral

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 - R(x)^2(|\xi|^2 + |\eta|^2))^{\lambda/2} \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot \xi} d\xi d\eta.
$$

It is not hard to verify that for Schwartz functions $f$ and $g$ we have

$$
\sup_{R(x) \in \mathcal{A}} \|T_{R(x)}(f, g)\|_p = \|T_*(f, g)\|_p.
$$

Hence we know that $T_R$ is bounded for all indices we have proved for $T_*$. And our claim will be established if we can prove the corresponding results for all $T_R$ with $R \in \mathcal{A}$ such that the constants involved are independent of $R$.

The advantage of $T_{R(x)}^\lambda$ is that it is bilinear so that we are allowed to apply interpolation results. The specific one we use here is Theorem 7.2.9 of [10]. We can verify that $T_{R(x)}^\lambda(f, g)$, as can be defined for all complex numbers $\lambda$, is analytic in $\lambda$ and actually admissible when $f$ and $g$ are simple functions. We know that $T_{R(x)}^{\lambda_0}$ is bounded from $L^2 \times L^2$ to $L^1$ for $\lambda_0$ whose real part is strictly great than $(4n + 3)/5$, and $T_{R(x)}^{\lambda_1}$ is bounded from $L^{q_1} \times L^{q_2}$ to $L^q$ with $\text{Re}\lambda_1 > n - 1/2$, $q_1, q_2 > 1$ and $1/q = 1/q_1 + 1/q_2$. The bounds for these two cases are of admissible growth in the imaginary parts of $\lambda_j$ for $j = 0, 1$, and they both are independent of $R \in \mathcal{A}$. Consequently we can interpolate between these two points using Theorem 7.2.9 of [10] and obtain that $\|T_{R(x)}^{\lambda}(f, g)\|_{L^p} \leq C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$ for $p_1, p_2 > (4n - 22)/(10\lambda - 6n - 17)$ and $1/p = 1/p_1 + 1/p_2$, which implies the claimed conclusion.

\[ \square \]

**Remark 1.1.** We have all strong boundedness points for $T_*$ when $\lambda$ is larger than the critical index $n - 1/2$ by Proposition 2.1, which is greater than $(4n + 3)/5$ when $n \geq 6$.

As a corollary of Theorem 1.1, we obtain the pointwise convergence, as $t \to 0$, of the operator $A_{\lambda(t)}^*(f, g)(x)$, which we denote by $A_t(f, g)(x)$ as well.

**Proposition 1.2.** Suppose $\lambda > \min\{(4n + 3)/5, n - 1/2\}$, then for $f \in L^{p_1}$ and $g \in L^{p_2}$ with $p_1, p_2$ as in Theorem 1.1 we have

$$
\lim_{t \to 0} A_t(f, g)(x) \to f(x)g(x) \quad a.e.
$$

The proof of this proposition is similar to the linear case, but we sketch it here for completeness.

**Proof.** It is easy to establish (4) when both $f$ and $g$ are Schwartz functions. To prove (4) for $f \in L^{p_1}$ and $g \in L^{p_2}$ it suffices to show that for
any given $\delta > 0$ the set $E_{f,g}(\delta) = \{ y \in \mathbb{R}^n : O_{f,g}(y) > \delta \}$ has measure 0, where

$$O_{f,g}(y) = \limsup_{\theta \to 0} \limsup_{\epsilon \to 0} |A_\theta(f,g)(y) - A_\epsilon(f,g)(y)|.$$  

For any positive number $\eta$ smaller than $\|f\|_{L^p_1}, \|g\|_{L^p_2}$, there exist Schwartz functions $f_1 = f - a$ and $g_1 = g - b$ such that both $\|a\|_{L^p_1}$, and $\|b\|_{L^p_2}$ are bounded by $\eta$. We observe that

$$|E_{f,g}(\delta)| \leq |E_{f_1,g_1}(\delta/4)| + |E_{a,g_1}(\delta/4)| + |E_{f_1,b}(\delta/4)| + |E_{a,b}(\delta/4)|.$$  

Notice that $|E_{f_1,g_1}(\delta/4)| = 0$ since (4) is valid for $f_1, g_1$. To control the remaining three terms, we observe that, for instance,

$$|E_{a,g_1}(\delta/4)| \leq \{ y : 2T_\lambda(a, g_1)(y) > \delta/4 \} \leq C \left( \frac{\|a\|_{L^p_1} \|g_1\|_{L^p_2}}{\delta} \right)^p \leq C \left( \frac{\eta \|g\|_{L^p_2}}{\delta} \right)^p,$$

where the last term goes to 0 as $\eta \to 0$ since $g$ and $\delta$ are fixed. □

The boundedness of $T_\lambda^*$ when $\lambda > n - \frac{1}{2}$ is straightforward from its kernel, whose proof we postpone to next section, so what remains is that $T_\lambda^*$ is bounded from $L^2 \times L^2$ to $L^1$ for the claimed range of $\lambda$, which will be the main focus of the rest of this paper.

**Theorem 1.3.** When $\lambda > (4n + 3)/5$, for $T_*$ in (3) we have that

$$\|T_*(f,g)\|_{L^1} \leq C \|f\|_{L^2} \|g\|_{L^2}.$$  

Throughout this paper, we use the notation

$$\|T\|_{L^p_1 \times L^p_1 \to L^p} = \sup_{\|f\|_{L^p_1} \leq 1} \sup_{\|g\|_{L^p_2} \leq 1} \|T(f,g)\|_{L^p}.$$  

By a bilinear operator $T$ related to the multiplier $\sigma$ we mean $T$ is defined by

$$T(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta$$

for all Schwartz functions $f$ and $g$.

**2. The Decompositions**

Let us fix a nonnegative nonincreasing smooth function $\varphi(s)$ on $\mathbb{R}$ such that $\varphi(s) = 1$ for $s \leq 1/2$ and $\varphi(s) = 0$ for $s \geq 1$. Define $\varphi_j(s) = \varphi(2^{j+1}(s + 2^{-j} - 1))$ for $j \geq 1$. Denote by $\psi_j(s)$ the function $\varphi_1(s)$ when $j = 0$ and $\varphi_{j+1}(s) - \varphi_j(s)$ when $j \geq 1$. Notice that $\psi_0$ is
supported in $(-\infty, 3/4]$ and $\psi_j$ is supported in $[1 - 2^{-j}, 1 - 2^{-j-2}]$ for $j \geq 1$. Moreover $\sum_{j=0}^{\infty} \psi_j(s) = \chi_{(-\infty,1)}(s)$.

We decompose the multiplier $m$ smoothly so that $m = \sum_{j \geq 0} m_j$, where $m_j(\xi, \eta) = m(\xi, \eta)\psi_j(|(\xi, \eta)|)$ is supported in an annulus of the form

$$\{(\xi, \eta) \in \mathbb{R}^{2n} : 1 - 2^{-j} \leq |(\xi, \eta)| \leq 1 - 2^{-j-2}\}$$

for $j \geq 1$ and $m_0$ is supported in a ball of radius $3/4$ centered at the origin.

Let $T_j(f, g)(x) = \sup_{t>0} |\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)m_j(t\xi, t\eta)e^{2\pi i x \cdot (\xi + \eta)}d\xi d\eta|$, then $T_*(f, g)(x) \leq \sum_{j=0}^{\infty} T_j(f, g)(x)$. We first present some trivial results on $T_*$ and $T_j$.

**Proposition 2.1.** Assume $1 < p_1, p_2 \leq \infty$, and $1/p = 1/p_1 + 1/p_2$. Then for $\lambda > n - 1/2$, there exists a constant $C = C(p_1, p_2)$ such that $\|T_\ast\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C$. For any fixed $j$, there exists a constant $C_j$ such that $\|T_j\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C_j$.

**Proof.** Let us consider the kernel $K(y, z) = M^\vee(y, z)$ of $A_1$ defined in (2), which satisfies that $|K(y, z)| \leq C(1 + |y| + |z|)^{-\alpha - (n+\lambda+1)/2}$ (see, for example, [9]), hence for $\lambda > n - 1/2$, we have

$$|A_t(f, g)(x)| = \left| \int_{\mathbb{R}^{2n}} \sum_{j=0}^{\infty} t^{-2n}K\left(\frac{x-y}{t}, \frac{x-z}{t}\right)f(y)g(z)dydz \right|$$

$$\leq C(\varphi_t \ast f)(x)(\varphi_t \ast g)(x)$$

$$\leq CM(f)(x)M(g)(x),$$

where $M$ is the Hardy-Littlewood maximal function, and $\varphi_t(y) = t^{-n}\varphi(y/t)$ with $\varphi(y) = (1 + |y|)^{-\alpha - (n+\lambda+1)/2}$, which is integrable when $\lambda > n - 1/2$. Then $T_*(f, g)(x) \leq CM(f)(x)M(g)(x)$, which implies that $\|T_\ast(f, g)\|_{L^p} \leq C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$ for $1 < p_1, p_2 \leq \infty$ with $1/p = 1/p_1 + 1/p_2$, in view of the boundedness of the Hardy-Littlewood maximal function.

We observe that each $m_j$ is smooth and compactly supported, hence for each fixed $j$ a similar argument gives that $\|T_j\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C_j$. \hfill \Box

With the aid of the preceding decomposition and the boundedness of $T_j$, the study of the boundedness of $T_*$ is reduced to locating the decay of $C_j$ in $j$; for the case $(p_1, p_2) = (2, 2)$, this is contained in the following proposition.

**Proposition 2.2.** $T_j$ satisfies that

$$\|T_j\|_{L^2 \times L^2 \to L^1} \leq C_n j 2^{-j(\lambda - 4n/3)}.$$

Let us prove Theorem 1.3 using Proposition 2.2.
Proof of Theorem 1.3. Since $T_*(f,g)(x) \leq \sum_{j=0}^{\infty} T_j(f,g)(x)$, and the bound of $T_j$ has an exponential decay in $j$ when $\lambda > (4n + 3)/5$ by Proposition 2.2, $\|T_*\|_{L^2 \times L^2 \to L^1}$ is finite. □

It suffices to consider the cases when $j$ is large. We will use the wavelet decomposition of the multipliers as in [11]. So we need to introduce this decomposition due to [6], and the exact form we use here can be found in [19].

Lemma 2.3 ([19]). For any fixed $k \in \mathbb{N}$ there exist real compactly supported functions $\psi_F, \psi_M \in \mathcal{C}^k(\mathbb{R})$, which satisfy that $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$ and $\int_{\mathbb{R}} x^\alpha \psi_M(x) \, dx = 0$ for $0 \leq \alpha \leq k$, such that, if $\Psi^G$ is defined by

$$\Psi^G(\vec{x}) = \psi_{G_1}(x_1) \cdots \psi_{G_{2n}}(x_{2n})$$

for $G = (G_1, \ldots, G_{2n})$ in the set

$$\mathcal{I} := \{(G_1, \ldots, G_{2n}) : G_i \in \{F, M\}\},$$

then the family of functions

$$\bigcup_{\vec{\mu} \in \mathbb{Z}^{2n}} \left\{ \Psi^{(F, \ldots, F)}(\vec{x} - \vec{\mu}) \bigcup_{\lambda=0}^{\infty} \left\{2^{\lambda n} \Psi^G(2^\lambda \vec{x} - \vec{\mu}) : G \in \mathcal{I} \setminus \{(F, \ldots, F)\} \right\} \right\}$$

forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$, where $\vec{x} = (x_1, \ldots, x_{2n})$.

A lemma concerning the decay of the coefficients related to the orthonormal basis in Lemma 2.3 is given below.

Lemma 2.4 ([11]). Suppose $\sigma(\xi, \eta)$ defined on $\mathbb{R}^{2n}$ satisfies that there exists a constant $C_M$ such that $\|\partial^n(\sigma(\xi, \eta))\|_{L^\infty} \leq C_M$ for each multiindex $|\alpha| \leq M$, where $M$ is the number of vanishing moments of $\psi_M$. Then for any nonnegative integer $\gamma \in \mathbb{N}_0 = \{n \in \mathbb{Z} : n \geq 0\}$ we have

$$|\langle \Psi^G_{\vec{\mu}}, \sigma \rangle| \leq C C_M 2^{-(M+n)\gamma}.$$ 

This lemma can be proved by applying Appendix B.2 in [10], and we delete the details which can be found in [11].

We now go back to the multipliers and consider their wavelet decompositions. For this purpose, and to apply Lemma 2.3 and Lemma 2.4 we should study kinds of norms of $m_j$.

Lemma 2.5. There exists a constant $C$ such that

$$\|m_j\|_{L^2} \leq C 2^{-j(\lambda+1/2)},$$

and for any multiindex $\alpha$,

$$\|\partial^\alpha m_j\|_{L^\infty} \leq C 2^{-j(\lambda-|\alpha|)}.$$
Proof. A change of variables using polar coordinates implies that
\[
\|m_j\|_{L^2} = \left( \int_{\mathbb{R}^{2n}} |m_j(\xi, \eta)|^2 d\xi d\eta \right)^{1/2}
\leq C \left( \int_{1-2^{-j}}^{1-2^{-j-2}} (1 - r^2)^{2\lambda} r^{2n-1} dr \right)^{1/2}
\leq C (2^{-2\lambda} 2^{-j})^{1/2}
= C 2^{-j(\lambda+1/2)}.
\]

To estimate the \(\alpha\)-th derivatives, we use the Leibniz’s rule to write
\[
\partial^\alpha m_j(\xi, \eta) = \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1} \partial^{\alpha_1} m(\xi, \eta) \partial^{\alpha_2} \psi_j(\| (\xi, \eta) \|).
\]
Noticing that \( |\partial^{\alpha_1} m(\xi, \eta)| \leq C 2^{-j(\lambda - |\alpha_1|)} \) and \( \partial^{\alpha_2} \psi_j(\| (\xi, \eta) \|) \leq C 2^{j|\alpha_2|} \), we derive the bound \( \partial^\alpha m_j(\xi, \eta) \) by \( 2^{j(\lambda - |\alpha|)} \).

Unfortunately we need a large number of derivatives in our analysis, and the estimate we have on \( \|\partial^\alpha m_j\| \) is not suitable. However, we observe that the support of \( m_j \) is very thin, which is an advantage we should make use of. To realize this, we may dilate it to get a fixed width so that we are at a good position to apply the wavelet decomposition implying a good estimate, since, as we will see later, a uniform bound for derivatives plays an important role in our theory.

Let us define \( M_j(\xi, \eta) = m_j(2^{-j} \xi, 2^{-j} \eta) \), which is supported in the annulus \( \{ (\xi, \eta) \in \mathbb{R}^{2n} : 2^j - 1 \leq \| (\xi, \eta) \| \leq 2^j - 1/4 \} \), whose width is 3/4. Based on Lemma 2.5 we have the following corollary.

**Corollary 2.6.** \( M_j \) defined as above satisfies that
\[
\| M_j \|_{L^2} \leq C 2^{jn} 2^{-j(\lambda+1/2)},
\]
\[
\| \partial^\alpha M_j \|_{L^\infty} \leq C 2^{-j\lambda} \quad \text{for all multiindex } \alpha,
\]
and
\[
\nabla m_j(\xi, \eta) = 2^j (\nabla M_j)(2^j \xi, 2^j \eta).
\]

**Proof.** A simple change of variables implies that
\[
\| M_j \|_{L^2} = \left( \int_{\mathbb{R}^{2n}} |m_j(2^{-j} \xi, 2^{-j} \eta)|^2 d\xi d\eta \right)^{1/2}
= 2^{jn} \| m_j \|_{L^2}
\leq C 2^{jn} 2^{-j(\lambda+1/2)}.
\]

We control \( \partial^\alpha M_j \) by \( |2^{-j|\alpha|}(\partial^\alpha m_j)(2^{-j} \xi, 2^{-j} \eta)| \leq C 2^{-j|\alpha|-j(\lambda-|\alpha|)} = C 2^{-j\lambda} \).

The verification of the last identity is straightforward. \( \square \)
Since the new multiplier $M_j$ is still in $L^2$, we have a wavelet decomposition using Lemma 2.3, i.e.

$$M_j = \sum a_\omega \omega,$$

where the summation is over all $\omega = \Psi^{\gamma,G}_\mu$ in the orthonormal basis described in Lemma 2.3, the order of cancellations of $\psi_M$ is $M = 4n + 6$, and $a_\omega = <M_j, \omega>$. Concerning the size of $a_\omega$, we have the following estimate, which is a direct implication of Lemma 2.4 and Corollary 2.6.

**Corollary 2.7.** The coefficient $a_\omega$ related to $\omega$ with dilation $\gamma$ is bounded by $C \lambda^{-2} \lambda^{-2(M+n)\gamma}$.

Before coming to the proof of Proposition 2.2, we make a remark. The functions $\psi_F$ and $\psi_M$ have compact supports, and all elements in a fixed level, which means that they have the same dilation factor $\gamma$, in the basis come from translations of finitely many products, so we can classify the elements in the basis into finitely many classes so that all elements in the same level in each class have disjoint supports. From now on, we can always assume that the supports of $\omega$’s related to a given dilation factor $\gamma$ are disjoint.

### 3. The Proof of Proposition 2.2

With the wavelet decompositions in hand, we are able to prove Proposition 2.2. The proof is inspired by the square function technique (see [16] and [2]) and [11]. We control $T_j$ by two integrals with the diagonal part and the off-diagonal parts. For the diagonal part we have just one term, which can be handled using product wavelets. For the off-diagonal parts we introduce two square operators with each one bounded by the Hardy-Littlewood maximal function and a bounded linear operator.

We need to decompose $M_j$ further. Take $N$ to be a fixed large enough number so that $N/10$ is greater than $d$, the diameters of all $\omega$ with dilation factor $\gamma = 0$. We write $\omega(\xi, \eta) = \omega_\mu(\xi, \eta) = \omega_{1,k}(\xi)\omega_{2,l}(\eta)$, where $\mu = (k,l)$ with $k,l \in \mathbb{Z}^n$, and denote the corresponding coefficient $<\omega_{k,l}, M_j>$ by $a_{k,l}$. We define

$$M_j^1 = \sum_{\gamma} \sum_{|k| \geq N} \sum_{|l| \geq N} a_{k,l} \omega_{1,k} \omega_{2,l}$$

(7)

and

$$M_j^2 = \sum_{\gamma} \sum_k \sum_{|l| \leq N} a_{k,l} \omega_{1,k} \omega_{2,l}$$

(8)
Here $M^i_j$ is the diagonal part whose support is away from both $\xi$ and $\eta$ axes, $M^2_j$ is the off-diagonal part with the support near the $\eta$ axis, and the support of $M^3_j$ is near the $\xi$ axis. Corresponding to $M^i_j$, we define 

$$m^i_j(\xi, \eta) = M^i_j(2^j\xi, 2^j\eta)$$

for $i = 1, 2, 3$. Denote

$$A^i_{j,t}(f, g)(x) = \int_{\mathbb{R}^{2n}} m^i_j(t\xi, t\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)}d\xi d\eta,$$

which equals $\sum_{i=1}^3 A^i_{j,t}(f, g)(x)$ with $A^i_{j,t}(f, g)(x)$ associated with $m^i_j$.

We need the following lemma to handle $A^1_{j,t}(f, g)(x)$.

**Lemma 3.1.** The gradients $\nabla M^j_i$ and $\nabla m^j_i$ are defined pointwisely for $i = 1, 2, 3$. Moreover we have the expression

$$\nabla M^j_i(\xi, \eta) = \sum_{\gamma} \sum_{k,l} a_{k,l} \nabla_{(\xi, \eta)}(\omega_{1,k} \otimes \omega_{2,l})(\xi, \eta),$$

with the explanation that the second summation is over allowed pairs $(k, l)$ related to $M^j_i$.

**Proof.** It suffices to verify that $\partial_{\xi_1} M^1_j(\xi, \eta)$ exists for any $(\xi, \eta)$. We claim that

$$\lim_{h \to 0} \frac{M^1_j(\xi + he_1, \eta) - M^1_j(\xi, \eta)}{h} = \sum_{\gamma} \sum_{k,l} a_{k,l} \partial_{\xi_1} \omega_{1,k}(\xi)\omega_{2,l}(\eta),$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. We call on the Lebesgue dominated convergence theorem, so what we need to show actually is that for $h$ small enough, there exists a constant $C$ independent of $h$ such that

$$\frac{|M^1_j(\xi + he_1, \eta) - M^1_j(\xi, \eta)|}{h} \leq C.$$

We split the levels depending on whether $h$ is much smaller than $C2^{-\gamma}$, the diameters of the supports of $\omega$’s. If $|h| \leq C2^{-\gamma}$, then the difference $\omega_{1,k}(\xi + he_1, \eta) - \omega_{1,k}(\xi, \eta)$ with $|k - k'| \leq 1$ is controlled by $C|h|(|\partial_{\xi_1} \omega_{1,k}(\xi, \eta)| + |\partial_{\xi_1} \omega_{1,k'}(\xi, \eta)|)$, which is dominated by $C|h|2^\gamma 2^{\gamma/2}$. Recall the disjointness of the supports of $\omega$’s, so in each level, there
exists at most one \( \omega \) such that \( \omega(\xi, \eta) \neq 0 \). Consequently,
\[
|M_j^1(\xi + he_1, \eta) - M_j^1(\xi, \eta)|
= \left| \sum_{\substack{C2^{-\gamma} \geq |h|}} a_{k,l}[\omega_{1,k}(\xi + he_1) - \omega_{1,k}(\xi)]\omega_{2,l}(\eta) + \sum_{\substack{C2^{-\gamma} \leq |h|}} [a_{k_1,l_1}\omega_{1,k_1}(\xi + he_1)\omega_{2,l_1}(\eta) - a_{k_2,l_2}\omega_{1,k_2}(\xi)\omega_{2,l_2}(\eta)] \right|
\leq \sum_{\substack{C2^{-\gamma} \geq |h|}} C2^{-j2^{(M+n)\gamma}}2^{2\gamma |h|} + \sum_{\substack{C2^{-\gamma} \leq |h|}} C2^{-j2^{-(M+n)\gamma}}2^{2\gamma n}
\leq C2^{-j\lambda}|h| + |h|^{M}
\leq C2^{-j\lambda}|h|.
\]
This concludes the proof of Lemma 3.1. \( \square \)

For \( f, g \in \mathcal{S}(\mathbb{R}^n) \), using Corollary 2.6, we can rewrite
\[
A_{j,t}(f, g)(x)
= \int \int m_j^1(t\xi, t\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)}d\xi d\eta
= \int \int (s\xi, s\eta) \cdot \nabla m_j^1(2^j s\xi, 2^j s\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)}d\xi d\eta\frac{ds}{s}
= \int \int (2^j s\xi, 2^j s\eta) \cdot \nabla M_j^1(2^j s\xi, 2^j s\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)}d\xi d\eta\frac{ds}{s},
\]
where the existence of \( \nabla M_j^1 \) and \( \nabla m_j^1 \) are ensured by Lemma 3.1.

Define the operator \( \tilde{B}_{j,s}^1(f, g)(x) \) related to \( (s\xi, s\eta) \cdot \nabla M_j^1(s\xi, s\eta) \)
as \( \int (s\xi, s\eta) \cdot \nabla M_j^1(s\xi, s\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)}d\xi d\eta \). Then we have the pointwise estimate
\[
(11) \quad T_j^1(f, g)(x) = \sup_{t > 0} |A_{j,t}^1(f, g)(x)| \leq \int_0^{\infty} |\tilde{B}_{j,t}^1(f, g)(x)|\frac{ds}{s}
\]

We now turn to the study of the boundedness of \( \tilde{B}_{j,s}^1 \). We set
\[
\tilde{B}_{j,s,\gamma}(f, g)(x) = \int (s\xi, s\eta) \cdot \nabla M_{j,\gamma}(s\xi, s\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)}d\xi d\eta,
\]
where \( \nabla M_{j,\gamma}(\xi, \eta) = \sum_k \sum_{l} a_{k,l} \nabla(\omega_{1,k} \otimes \omega_{2,l})(\xi, \eta) \). For the standard case \( s = 1 \), we have the following estimate.

**Proposition 3.2.** For \( \tilde{B}_{j,1,\gamma}^1 \) we have the estimate
\[
\|\tilde{B}_{j,1,\gamma}^1(f, g)\|_{L^1} \leq CC(j, \gamma)\|\hat{f}x\epsilon\|_{L^2}\|\hat{g}x\epsilon\|_{L^2},
\]
with \( C(j, \gamma) = C2^{-j((5\lambda-4n-3)/5)2^{-\gamma(M-4n-5)/5}}, \) where the set \( E \) is defined as \( \{\xi : C2^{-\gamma} \leq |\xi| \leq 2^j\} \).
In the estimate of the diagonal part in \([11]\) we use mainly (i) the size of \(a_{k,l}\), (ii) the disjointness of the supports of \(\omega\)'s, and (iii) that \(\omega\) is a tensor product. So it is easy to obtain Proposition 3.2 by examining the proof in \([11]\) carefully. We sketch the proof here for the sake of completeness.

**Proof.** It suffices to consider, for example, \(\xi_1 \partial_{\xi_1} M_{j,\gamma}(\xi, \eta)\), and by Lemma 3.1 this is \(\sum_k \sum_l a_{k,l} \partial_{\xi_1} \omega_{1,k}(\xi) \xi_1 \omega_{2,l}(\eta)\) for allowed \(k, l\) in \(M_j^1\). We rewrite this as \(\sum_k \sum_l a_{k,l} v_k(\xi) \omega_{2,l}(\eta)\), where \(v_k(\xi) = \partial_{\xi_1} \omega_{1,k}(\xi) \xi_1\), which has the same support as that of \(\omega_{1,k}\). We denote by \(\|a\|_\infty\) the \(\ell^\infty\) norm of \(\{a_{k,l}\}_{k,l}\) when \(\omega_{k,l}\) has the dilation factor \(\gamma\).

Define for \(r \geq 0\) the set

\[
U_r = \{(k, l) \in \mathbb{Z}^{2n} : 2^{-r-1} \|a\|_\infty < |a_{k,l}| \leq 2^{-r} \|a\|_\infty\},
\]

which has the cardinality at most \(C \|a\|_2^2 \|a\|_\infty^2 2^{-2r}\), where \(\|a\|_2^2\) is the \(\ell^2\) norm of \(\{a_{k,l}\}\) bounded by \(\|M_j\|_{L^2}\). Let \(N_1 = (2^r \|a\|_2^2 / \|a\|_\infty^2)^{2/5}\), and define

\[
U_r^1 = \{(k, l) \in U_r : \text{card}\{s : (k, s) \in U_r\} \geq N_1\},
\]

\[
U_r^2 = \{(k, l) \in U_r \setminus U_r^1 : \text{card}\{s : (s, l) \in U_r \setminus U_r^1\} \geq N_1\},
\]

and \(U_r^3\) as the remaining set in \(U_r\).

For \(U_r^1\) we define a related set \(E = \{k : \exists l \text{ s.t.} (k, l) \in U_r^1\}\), and its cardinality \(N_2 = |E|\) is bounded by \(C(N_1)^4\). We denote by \(S_1\) the bilinear operator related to the multipliers \(\sum_{(k,l) \in U_r^1} a_{k,l} v_k \omega_{2,l}\). By an argument similar to \([11]\) using the three facts we mentioned before this proof we see that \(\|S_1(f, g)\|_{L^1} \leq C N_1^2 2^{\gamma(n+1)} 2^{j/2} 2^{-r} \|a\|_\infty \|f\|_{L^2} \|g\|_{L^2}\). We can similarly define \(S_2\) and get the same estimate. We can classify \(U_r^3\) into \(N_3^2\) classes so that for \((k, l), (k', l')\) in the same class with \((k, l) \neq (k', l')\), we must have \(k \neq k'\) and \(l \neq l'\). This observation enables us to bound the norm of \(S_3\), the operator related to \(U_r^3\), by \(C N_1^2 2^{\gamma(n+1)} 2^{j/2} 2^{-r} \|a\|_\infty\).

Using the expression of \(N_1\), we obtain that

\[
\|S_1 + S_2 + S_3\|_{L^1} \leq C (\|a\|_2^4/\|a\|_\infty^{1/5}) 2^{\gamma(n+1)} 2^{j/2} 2^{-r/5} \|f\|_{L^2} \|g\|_{L^2}.
\]

Summing over \(r\) and the observation that in the support of \(M_j^1\) we have \(\xi \in E_1 = \{\xi : 2^{-\gamma} \leq |\xi| \leq 2^\gamma\}\) imply that

\[
\|\tilde{B}_{j,1,1}^1(f, g)\|_{L^1} \leq C (\|M_j\|_{L^2}^{4/5} \|a\|_\infty^{1/5} 2^{\gamma(n+1)} 2^{j/2} \|\tilde{f}\chi_E\|_{L^2} \|\tilde{g}\chi_E\|_{L^2},
\]

where the bound is controlled by

\[
C(j, \gamma) = C 2^{-j(5\lambda - 4n - 3)/5} 2^{-\gamma(M - 4n - 5)/5}
\]

since \(\|M_j\|_{L^2} \leq C 2^{j/2} 2^{-j(\lambda + 1/2)}\) by Corollary 2.6.

\[\square\]
Notice that we will have enough decay in $j$ and $\gamma$ if $\lambda > (4n + 3)/5$ and $M > 4n + 5$, which are satisfied by our assumptions on $\lambda$ and choices of wavelets.

**Corollary 3.3.** For the diagonal part we have

$$\|T^1_j(f, g)\|_{L^1} \leq CC(j)\|f\|_{L^2}\|g\|_{L^2},$$

where $C(j) = \sum_\gamma C(j, \gamma)(j + \gamma)$.

**Proof.** From (11) we know that

$$\|T^1_j(f, g)\|_{L^1} \leq \int_0^\infty \|\tilde{B}^1_{j,s}(f, g)\|_{L^1} \frac{ds}{s} \leq \int_0^\infty \sum_\gamma \|\tilde{B}^1_{j,s,\gamma}(f_s, g_s)\|_{L^1} \frac{ds}{s},$$

where $\hat{f}_s(\xi) = s^{-n/2}\hat{f}(\xi/s)$. Applying Lemma 3.2 the last integral is dominated by

$$C \int_0^\infty \sum_\gamma C(j, \gamma)\|\hat{f}_s\|_{L^2}\|\hat{g}_s\|_{L^2} \frac{ds}{s} \leq C \sum_\gamma C(j, \gamma)(\int_{\mathbb{R}^n} \int_0^\infty |\hat{f}_s\chi_{E_1}|^2 \frac{ds}{s} d\xi)^{1/2} (\int_{\mathbb{R}^n} \int_0^\infty |\hat{g}_s\chi_{E_1}|^2 \frac{ds}{s} d\xi)^{1/2}.$$ 

The double integral involving $f_s$ is bounded by $\int |\hat{f}(\xi)|^2 \int_{C_2^{-\gamma}/|\xi|} \frac{ds}{s} d\xi$, which is less than $C(j + \gamma)\|f\|_{L^2}^2$. Hence the last sum over $\gamma$ is controlled by $C \sum_\gamma C(j, \gamma)(j + \gamma)\|f\|_{L^2}\|g\|_{L^2}$. \qed

This concludes the argument of the diagonal part. We next deal with the off-diagonal parts. More specifically, we consider $A^2_{j,t}$ since the analysis of $A^1_{j,t}$ is similar. Recall that

$$A^2_{j,t} = \int_{\mathbb{R}^{2n}} m^2_j(t \xi, t \eta) \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$ 

We denote $(\xi, \eta) \cdot (\nabla m^2_j)(\xi, \eta)$ by $\tilde{m}^2_j(\xi, \eta)$. Then similar to $A^2_{j,t}(f, g)(x)$ we define $\tilde{A}^2_{j,t}(f, g)(x)$ as $\int \int \tilde{m}^2_j(t \xi, t \eta) \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta$. Like before we can define $B^2_{j,s}$ and $\tilde{B}^2_{j,s}$ similar to $A^2_{j,s}$ and $\tilde{A}^2_{j,s}$ with the appearances of $m$ replace by $M$. A simple calculation shows that $A^2_{j,s} = B^2_{j,2s}$.
and \( \tilde{A}^2_{j,s} = \tilde{B}^2_{j,2j,s} \). With all these notations, we have

\[
(A^2_{j,t}(f,g)(x))^2 = 2 \int_0^t A^2_{j,s}(f,g)(x) s \frac{dA^2_{j,s}(f,g)(x)}{ds} ds \\
= 2 \int_0^t B^2_{j,2j,s}(f,g)(x) \tilde{B}^2_{j,2j,s}(f,g)(x) ds \\
\leq 2 \int_0^\infty |B^2_{j,s}(f,g)(x)| |\tilde{B}^2_{j,s}(f,g)(x)| ds \\
\leq 2G_{j,s}(f,g)(x)\tilde{G}_{j,s}(f,g)(x),
\]

where we set

\[
G_j(f,g)(x) = \left( \int_0^\infty |B^2_{j,s}(f,g)(x)|^2 ds \right)^{1/2} \\
\tilde{G}_j(f,g)(x) = \left( \int_0^\infty |\tilde{B}^2_{j,s}(f,g)(x)|^2 ds \right)^{1/2}.
\]

These \( g \)-functions are bounded from \( L^2 \times L^2 \) to \( L^1 \).

**Lemma 3.4.** There exists a constant \( C \) independent of \( j \) such that for all \( f, g \in S(\mathbb{R}^n) \),

\[
\|G_j(f,g)\|_{L^1} \leq C 2^{-j(\lambda+1/2)} \|f\|_{L^2} \|g\|_{L^2}
\]

and

\[
\|\tilde{G}_j(f,g)\|_{L^1} \leq C 2^{-j(\lambda-1/2)} \|f\|_{L^2} \|g\|_{L^2}.
\]

As in Lemma 3.2, we will delete some details which can be found in [11].

**Proof.** We will focus on \( \tilde{G}_j \) first. For \( G_j \) we need to consider two typical cases, the derivative falling on \( \xi \) and the derivative falling on \( \eta \).

Let us consider the operator with the multiplier \( \xi \partial_\xi M_j^2 \), which equals \( \sum_\gamma \sum_{k,l} a_{k,l} v_k(\xi) \omega_{2,l}(\eta) \) with \( v_k(\xi) = \xi_1 \partial_\xi \omega_{1,k}(\xi, \eta) \). The corresponding \( g \)-function will be denoted by \( G_j^1(f, g) \). Let us fix \( \gamma \) and for each \( \gamma \) at most \( N \) number of \( \omega_{2,l} \) are involved, so we can consider for a single fixed \( l \). Observe that

\[
\int_{\mathbb{R}^n} \sum_k a_{k,l} v_k(\xi) \omega_{2,l}(\eta) \tilde{f}(\xi) \tilde{g}(\eta) e^{2\pi i\langle \xi, \eta \rangle} d\xi d\eta \\
= \|a\|_\infty 2^{\gamma(n+2)/2j} \int_{\mathbb{R}^2} \omega_{2,l} \tilde{g}(\eta) e^{2\pi i\eta} d\eta \int_{\mathbb{R}^n} \sum_k a_{k,l} v_k(\xi) \tilde{f}(\xi) e^{2\pi i\xi} d\xi.
\]

By \( |v_k| \leq 2^{\gamma(n+2)/2j} |a_{k,l}| \leq \|a\|_\infty \) and the disjointness of the supports of \( v_k \), we know that \( (\sum_k a_{k,l} v_k(\xi))/\|a\|_\infty 2^{\gamma(n+2)/2j} \) is a compactly
supported bounded function. Hence we bound the operator related to the multiplier \( \sum_k a_{k,l} v_k(\xi) \omega_{2,l}(\eta) \) by \( C \|a\|_\infty 2^j 2^{(n+1)\gamma} M(g)(x) T_\sigma(x) \), where \( T_\sigma(f) \) satisfies that \( \|T_\sigma(f)\|_{L^2} \leq C \|f \chi_F\|_{L^2} \) with \( F = \{ \xi \in \mathbb{R}^n : 2^j - 1 \leq |\xi| \leq 2^j - 1/4 \} \).

The operator \( \tilde{G}_j^1 \) is then bounded. Indeed we can estimate it by a standard dilation argument as follows.

\[
\int_{\mathbb{R}^n} \tilde{G}_j^1(f, g)(x)dx \\
= \int_{\mathbb{R}^n} \left( \int_0^\infty \int_{\mathbb{R}^{2n}} \sum_k a_{k,l} v_k(s\xi) \omega_{2,l}(s\eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi, \eta)} d\xi d\eta \right)^{1/2} ds \\
\leq C \sum_\gamma \|a\|_\infty 2^{j2^{(n+1)\gamma}} \|M(g)\|_{L^2} \int_{\mathbb{R}^n} \left( \int_0^\infty s^{-\gamma} \hat{f}(\xi/s)^2 \hat{g}(\xi)d\xi \right)^{1/2} ds \\
\leq C \sum_\gamma \|a\|_\infty 2^{j2^{(n+1)\gamma}} \|g\|_{L^2} \left( \int_{\mathbb{R}^n} \hat{f}(\xi)^2 \int_{(2^j - 1)/|\xi|}^{(2^j - 1/4)/|\xi|} \frac{ds}{s} d\xi \right)^{1/2} \\
\]

The integral with respect to \( s \) is \( \log \frac{2^j - 1/4}{2^j - 1} \leq C 2^{-j} \). This, combined with the bound of \( \|a\|_\infty \), shows that the last summation is smaller than

\[
(12) \quad C \sum_\gamma \|a\|_\infty 2^{j2^{(n+1)\gamma}2^{-j/2}} \|g\|_{L^2} \|f\|_{L^2} \leq C 2^{-j(\lambda - 1/2)} \|g\|_{L^2} \|f\|_{L^2}. 
\]

For the case the derivative falls on \( \eta \), for example \( \eta_1 \), we have a similar representation

\[
\int_{\mathbb{R}^{2n}} \sum_k a_{k,l} \omega_{1,k}(\xi) v_l(\eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi, \eta)} d\xi d\eta \\
= \|a\|_\infty 2^{\gamma n/2} \int_{\mathbb{R}^n} v_l(\eta) \hat{g}(\eta) e^{2\pi i x \cdot \eta} d\eta \int_{\mathbb{R}^n} \sum_k \frac{a_{k,l} \omega_{1,k}(\xi)}{\|a\|_\infty 2^{\gamma n/2}} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi. 
\]

The first integral in the last line is dominated by \( 2^{\gamma n/2} M(g)(x) \) because for the \( \omega_{2,l} \) with \( \gamma = 0 \), we have both \( \partial_1(\omega_{2,l})' \) and \( (\omega_{2,l})'' \) are Schwartz functions, and the number of the second type of functions is finite because \( |l| \leq N \). The operator related to the multiplier \( \sum_k a_{k,l} \omega_{1,k}(\xi) v_l(\eta) \) is therefore bounded by the quantity \( C \|a\|_\infty 2^{(n+1)\gamma} M(g)(x) T_{\sigma'}(f)(x) \), where \( T_{\sigma'} \) satisfies the same property \( T_{\sigma'} \) has. For \( L^1 \) norm of the \( g \)-function \( \tilde{G}_j^2 \) related the multiplier \( \sum_k a_{k,l} \omega_{1,k}(\xi) v_l(\eta) \) we apply a similar argument used in estimating
∥\tilde{G}_j^1∥_{L^1} to control it by
\begin{equation}
C \sum_{\gamma} 2^{-j(\lambda+1/2)} 2^{-(M-1)\gamma} ∥g∥_{L^2} ∥f∥_{L^2} ≤ C 2^{-j(\lambda+1/2)} ∥f∥_{L^2} ∥g∥_{L^2}
\end{equation}

This estimate and (12) show that
\begin{equation}
∥\tilde{G}_j(f, g)∥_{L^1} ≤ C 2^{-j(\lambda-1/2)} ∥f∥_{L^2} ∥g∥_{L^2}.
\end{equation}

For \(G_j(f, g)\), a similar and simpler argument applying to the standard representation \(\sum a_{k,l} \omega_{1,k}(\xi) \omega_{2,l}(\eta)\) gives that
\begin{equation}
∥G_j(f, g)∥_{L^1} ≤ C 2^{-j(\lambda+1/2)} ∥f∥_{L^2} ∥g∥_{L^2}.
\end{equation}

Here the difference \(2^j\) comes from the fact that in the multiplier of \(B^2_{j,s}\), we miss the term \((\xi, \eta)\), which is just controlled by \(2^j\).

\textbf{Corollary 3.5.} For the off-diagonal part we have
\begin{equation}
∥T_j^2(f, g)∥_{L^1} ≤ C 2^{-j\lambda} ∥f∥_{L^2} ∥g∥_{L^2}.
\end{equation}

\textbf{Proof.} By the calculation before Lemma 3.4 we have the pointwise control \(T_j^2(f, g)(x) ≤ \sqrt{2}(G_j(f, g)(x)\tilde{G}_j(f, g)(x))^{1/2}\), which and Lemma 3.4 imply that
\begin{align*}
∥T_j^2(f, g)∥_{L^1} &≤ \sqrt{2}(∥G_j(f, g)(x)∥_{L^1} ∥\tilde{G}_j(f, g)(x)∥_{L^1})^{1/2} \\
&≤ C(∥G_j(f, g)(x)∥_{L^1} ∥\tilde{G}_j(f, g)(x)∥_{L^1})^{1/2} \\
&≤ C(2^{-j(\lambda+1/2)} 2^{-j(\lambda-1/2)} ∥f∥_{L^2}^2 ∥g∥_{L^2}^2)^{1/2} \\
&= C 2^{-j\lambda} ∥f∥_{L^2} ∥g∥_{L^2}
\end{align*}

In this case we have nice decay for \(T_j^2\) since \(\lambda > (4n+3)/5 > 0\).

We now can prove Proposition 2.2, the key result of our theory.

\textbf{Proof of Proposition 2.2.} When \(\lambda > (4n+3)/5\), the simple observation \(T_j ≤ T_j^1 + T_j^2 + T_j^3\), the Corollary 3.3 and Corollary 3.5 complete the proof of Proposition 2.2.

\textbf{4. A Final Remark}

Tao \cite{17} proves that a necessary condition so that the linear maximal Bochner-Riesz operator \(B^\lambda\) is bounded on \(L^p(\mathbb{R}^n)\) is that \(\lambda ≥ \frac{2n-1}{2p} - \frac{3}{2}\). We modify his argument in this section to show that a similar requirement is also needed in the bilinear setting.

\textbf{Proposition 4.1.} A necessary condition such that the bilinear maximal Bochner-Riesz operator \(T^\lambda\) is bounded from \(L^{p_1}(\mathbb{R}^n) × L^{p_2}(\mathbb{R}^n)\) to weak \(L^p(\mathbb{R}^n)\) is that \(\lambda ≥ \frac{2n-1}{2p_1} - \frac{2n-1}{2}\).
Remark 4.1. This result is meaningful only if $p < 1$. We observe that the kernel requires another necessary condition $(\lambda + (2n + 1)/2)p \geq n$, i.e., $\lambda \geq \frac{n}{p} - \frac{2n + 1}{2}$, which is less restrictive than $\lambda \geq \frac{2n - 1}{2p} - \frac{2n - 1}{2}$.

Proof. We prove this theorem by constructing a counterexample. Let $M$ be a large number and $\epsilon$ a small number. Define a smooth function

$$\varphi(x) = \varphi_{\epsilon,M}(x) = \psi(\epsilon^{-1}|x|)\psi(\epsilon^{-1}M^{-1/2}x_n),$$

where $x' = (x_1, \ldots, x_{n-1})$ and $\psi$ is a smooth bump function supported in the interval $[-1, 1]$. Define $f_M(y) = e^{2\pi i u_n}\varphi(y)$ and $S_M = \{x : M \leq |x'| \leq 2M, M \leq x_n \leq 2M\}$. Obviously we have $\|f_M\|_p \sim (\epsilon^n M^{1/2})^{1/p}$ and $|S_M| \sim M^n$.

We will show that $T^\lambda_\epsilon(f_M, f_M)(x)$ is bounded from below for $x \in S_M$. Let us take $R = R_x = \sqrt{2|x'|/x_n}$, which is comparable to 1. Recall that the kernel $K^\lambda$ has the asymptotic representation for $X \in \mathbb{R}^{2n}$, as $|X| \to \infty$,

$$K^\lambda(X) = \frac{\Gamma(\lambda + 1) J_{\lambda+n}(2\pi |X|)}{|X|^{\lambda+n}}$$

$$= Ce^{2\pi i |X|} |X|^{-(\lambda + \frac{2n + 1}{2})} + Ce^{-2\pi i |X|} |X|^{-(\lambda + \frac{2n + 3}{2})} + O(|X|^{-(\lambda + \frac{2n + 3}{2})}).$$

From this and $R$ is comparable to 1 we can control $A^{\lambda}_{i/R}(f_M, f_M)(x)$ from below by

$$C_1 \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{2\pi i R(x,x)} |(x - y, x - z)|^{-(\lambda + \frac{2n + 1}{2})} \varphi(y) \varphi(z) dydz \right|$$

$$- C_2 \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( e^{2\pi i (R((x - y, x - z)|+y_n+z_n))} - e^{2\pi i R(x)} \right) |(x - y, x - z)|^{-(\lambda + \frac{n + 1}{2})} \varphi(y) \varphi(z) dydz \right|$$

$$- C_3 \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{2\pi i (-R((x - y, x - z)|+y_n+z_n))} |(x - y, x - z)|^{-(\lambda + \frac{n + 1}{2})} \varphi(y) \varphi(z) dydz \right|$$

$$- C_4 \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(x - y, x - z)|^{-(\lambda + \frac{n + 3}{2})} \varphi(y) \varphi(z) dydz \right|$$

$$= C_1 I - C_2 II - C_3 III - C_4 IV.$$

The first term is the main term, for which we have $I \sim M^{-(\lambda + \frac{2n + 1}{2})} e^{2n} M$. For the second term, we use that $|R((x - y, x - z)|+y_n+z_n-R((x, x))| \leq C \epsilon$ to control it by $M^{-(\lambda + \frac{2n + 1}{2})} e^{2n+1} M$. To handle the third term, we apply integration by parts to the variable $y_n$ and can show that it is bounded by $M^{-(\lambda + \frac{2n + 1}{2})} e^{2n} M(M^{-1} + \epsilon^{-1} M^{-1/2})$. It is not hard to see that $IV \leq M^{-(\lambda + \frac{2n + 3}{2})} e^{2n+1} M$. So we see that $|T^\lambda_\epsilon(f_M, f_M)(x)| \geq C_1 M^{-(\lambda + \frac{2n - 1}{2})} e^{2n} [1 - C' \epsilon + M^{-1} + \epsilon^{-1} M^{-1/2}]$. So if we choose $\epsilon$ to
be a fixed small number and let $M \to \infty$, the last quantity is comparable to $M^{-(\lambda + \frac{2n-1}{2})}$. Set this number to be $\alpha$ and if $T^\lambda_*$ is bounded from $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ to weak $L^p(\mathbb{R}^n)$, we have $M^n \sim |S_M| \leq |\{x : |T^\lambda_*(f_M, f_M)(x)| \geq \alpha\}| \leq \|f_M\|_p^p \|f_M\|_{p_2}^p / \alpha^p \leq M^{1/2 + (\lambda + \frac{2n-1}{2})}p$, which gives the necessary condition $\lambda \geq \frac{2n-1}{2} - \frac{2n-1}{2}$.

What we have proved in Theorem 1.1 and Theorem 1.3 are far from the restriction given by Proposition 4.1, so is $T^\lambda_*$ actually bounded from $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ to weak $L^p(\mathbb{R}^n)$ for $p > \max(\frac{1}{2}, \frac{2n-1}{2\lambda + 2n-1})$? Or do we at least have that $T^\lambda_*$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when $\lambda > 0$?

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