Learning Weakly Convex Sets in Metric Spaces

Eike Stadtländer¹, Tamás Horváth¹,²,³, and Stefan Wrobel¹,²,³

¹ Dept. of Computer Science, University of Bonn, Bonn, Germany
² Fraunhofer IAIS, Schloss Birlinghoven, Sankt Augustin, Germany
³ Fraunhofer Center for Machine Learning, Sankt Augustin, Germany
{stadtländer,horvath,wrobel}@cs.uni-bonn.de

Abstract. We introduce the notion of weak convexity in metric spaces, a generalization of ordinary convexity commonly used in machine learning. It is shown that weakly convex sets can be characterized by a closure operator and have a unique decomposition into a set of pairwise disjoint connected blocks. We give two generic efficient algorithms, an extensional and an intensional one for learning weakly convex concepts and study their formal properties. Our experimental results concerning vertex classification clearly demonstrate the excellent predictive performance of the extensional algorithm. Two non-trivial applications of the intensional algorithm to polynomial PAC-learnability are presented. The first one deals with learning k-convex Boolean functions, which are already known to be efficiently PAC-learnable. It is shown how to derive this positive result in a fairly easy way by the generic intensional algorithm. The second one is concerned with the Euclidean space equipped with the Manhattan distance. For this metric space, weakly convex sets form a union of pairwise disjoint axis-aligned hyperrectangles. We show that a weakly convex set that is consistent with a set of examples and contains a minimum number of hyperrectangles can be found in polynomial time. In contrast, this problem is known to be NP-complete if the hyperrectangles may be overlapping.

Keywords: abstract convexity · concept learning · vertex classification

1 Introduction

Several results in the theory of machine learning are concerned with concept classes defined by various forms of convexity (e.g., polygons formed by the intersection of a bounded number of half-spaces [2], conjunctions [14], or geodesic convexity in graphs [11]). In a broad sense, convex sets constitute contiguous subsets of the domain. This property can, however, be a drawback when the target concept cannot be represented by a single convex set. To overcome this problem, we relax the notion of convexity by introducing that of weak convexity for metric spaces. More precisely, a subset A of a metric space is weakly convex if for all x, y ∈ A and for all points z in the ground set, z belongs to A whenever x and y are near to each other and the three points satisfy the triangle inequality with equality. This definition has been inspired by the following relaxation of
convexity in the Hamming metric space [6]: A Boolean function is \( k \)-convex for some positive integer \( k \) if for all true points \( x \) and \( y \) having a Hamming distance of at most \( k \), all points on all shortest paths between \( x \) and \( y \) are also true. Our definition of weak convexity generalizes this notion to arbitrary metric spaces.

We present some properties of weakly convex sets of a metric space. In particular, we show that they form a convexity space [15] and hence, a closure system. Furthermore, they give rise to a unique decomposition into a set of “connected” blocks that have a pairwise minimum distance. We also study two scenarios for learning weakly convex sets. The first one considers the case that the metric space is finite and weakly convex sets are given extensionally. For this setting we define a preclosure operator and show that weakly convex sets can be characterized by a closure operator defined by the fixed points of the iterative applications of this preclosure operator. This characterization gives rise to an efficient algorithm computing the weakly convex hull for any set of points. We then prove that a weakly convex set that is consistent with a set of examples and has the smallest number of blocks can be found in polynomial time. This result makes use of the unique decomposition of weakly convex sets. As a proof of concept, we experimentally demonstrate on graph vertex classification that a remarkable accuracy can be obtained already with a relatively small training data set.

The second scenario deals with the case that the metric spaces are not necessarily finite and that weakly convex sets are given intensionally, using some compact representation. We present a simple generic algorithm, which iteratively “merges” weakly convex connected blocks and give sufficient conditions for the efficiency of a more sophisticated version of this naive algorithm. Similarly to the extensional setting, we prove that a weakly convex set consistent with a set of examples and containing a minimum number of blocks can be found in polynomial time if certain conditions are fulfilled. We also present two non-trivial applications of this general result to polynomial PAC-learnability [14]. The first one deals with learning \( k \)-convex Boolean functions, for which there already exists a positive PAC result [6]. We still consider this problem because we show that the same result can be obtained in a very simple way by our intensional learning algorithm. Furthermore, our general purpose algorithm calculates the \( k \)-convex Boolean function for a set of examples in the same asymptotic time complexity as the domain specific one in [6]. The second application deals with the metric space defined by \( \mathbb{R}^d \) endowed with the Manhattan (or \( L_1 \)) distance. Weakly convex sets for this case are the union of a set of pairwise disjoint axis-aligned closed hyperrectangles. Using our general learning algorithm, we prove in a very simple way that the concept class formed by weakly convex sets containing at most \( k \) hyperrectangles is polynomially PAC-learnable. To underline the strength and utility of our approach, we note that the consistent hypothesis finding problem for the related problem that the hyperrectangles are not required to be pairwise disjoint is NP-complete, even for \( d = 2 \) (see, e.g., [1]).

**Related Work** To the best of our knowledge, our notion of weak convexity in metric spaces is new. As mentioned above, it has been inspired by the definition of \( k \)-convex Boolean functions introduced in [6]. In fact, our notion generalizes
that for $k$-convex Boolean functions to a broad class of metric spaces, including infinite ones as well. We also mention the somewhat related, but fundamentally distinct notion of $\alpha$-hulls (see, e.g., [5]). They are defined as the intersection of enclosing generalized disks, but only for finite subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$. Furthermore, it is known that the $\alpha$-hull operator is not idempotent [8]. In contrast, our notion results in an abstract convexity structure in the sense of [15] and has therefore a corresponding closure operator defined for arbitrary subsets of a broad class of metric spaces. Last, but not least, even though our definitions resemble those of the density-based clustering approach [7], DBSCAN clusters are generally not weakly convex, except for very specific parameter values.

Outline The rest of the paper is organized as follows. We collect the necessary notions and fix the notation in Section 2. In Section 3 we define weakly convex sets in metric spaces and prove some of their basic properties. Sections 4 and 5 are devoted to learning weakly convex sets in the extensional and intensional problem settings. Finally, we conclude in Section 6 and mention some problems for future work. Most of the proofs are omitted for space limitations. They can be found in [13].

2 Preliminaries

In this section we collect the necessary notions and fix the notation. For any $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$. The family of all finite subsets of a set $X$ is denoted by $[X]^{<\omega}$. A metric space is a pair $(X, D)$, where $X$ is a set and $D$ is a metric on $X$ (i.e., (i) $D(x, y) = 0$ iff $x = y$, (ii) $D(x, y) = D(y, x)$, and (iii) $D(x, z) + D(z, y) \leq D(x, y)$ for all $x, y, z \in X$).

A closure system over some ground set $X$ is a pair $(X, C)$ with $C \subseteq 2^X$ such that $C$ is closed under arbitrary intersection, where $2^X$ denotes the power set of $X$. We require that $X \in C$. The elements of $C$ are called closed sets. One of the elementary properties of closure systems is that they can be characterized in terms of closure operators (see, e.g., [3]). More precisely, a function $\rho : 2^X \to 2^X$ is a closure operator if it satisfies the following properties for all $A, B \subseteq X$: (i) $A \subseteq \rho(A)$ (extensivity), (ii) $\rho(A) \subseteq \rho(B)$ whenever $A \subseteq B$ (monotonicity), and (iii) $\rho(\rho(A)) = \rho(A)$ (idempotency). If $\rho$ is extensive and monotone, but not necessarily idempotent, then it is a preclosure operator. The fixed points of a closure operator are called closed sets and the set system $(X, C_\rho)$ with $C_\rho = \{ A \subseteq X : \rho(A) = A \}$ is always a closure system. Conversely, for any closure system $(X, C)$, the function $\rho : 2^X \to 2^X$ with $\rho : A \mapsto \bigcap\{ C \in C : A \subseteq C \}$ is a closure operator satisfying $C = \{ \rho(A) : A \subseteq X \}$. Finally, a convexity space [15] over a set $X$ is a closure system $(X, C)$ such that (i) $\emptyset, X \in C$ and (ii) $C$ is closed under nested unions (i.e., $\bigcup D \in C$ for any $D \subseteq C$ that is totally ordered w.r.t. set inclusion).

Our notion of weak convexity defined in the next section is inspired by that of $k$-convexity introduced in the seminal paper by Ekin, Hammer, and Kogan [6]. More precisely, consider the metric space $(\mathbb{H}_d, D_H)$, where $\mathbb{H}_d = \{0, 1\}^d$ is the
\(d\)-dimensional Hamming cube and \(D_H\) is the Hamming distance. A subset \(X\) of \(\mathbb{H}_d\) is \(k\)-convex for an integer \(k \geq 1\) if for all \(x, y \in X\) with \(D_H(x, y) \leq k\) and for all \(z \in \mathbb{H}_d\), \(z \in X\) whenever the triangle inequality holds with equality (i.e., \(D_H(x, y) = D_H(x, z) + D_H(z, y)\)).

An (undirected) graph is a pair \(G = (V, E)\), where \(V\) is a finite set of vertices and \(E \subseteq \{e \subseteq V : |e| = 2\}\) is a set of edges. An edge \(\{x, y\}\) will sometimes be denoted by \(xy\). A graph \(G' = (V', E')\) is a subgraph of \(G\) if \(V' \subseteq V\) and \(E' \subseteq E\). A path is a graph \(P = (V_P, E_P)\) with \(V_P = \{v_1, \ldots, v_n\}\) and \(E_P = \{v_i v_{i+1} : i \in [n-1]\}\). The length of a path \(P\) is the number of edges it contains. A graph is connected if all pairs of its vertices are connected by a path. If two vertices of a graph \(G\) are connected by a path, we define their geodesic distance by the length of a shortest path connecting them. Note that it is a metric on the set of vertices for connected graphs. A subset \(X \subseteq V\) is called (geodesically) convex in a graph \(G = (V, E)\) if for all \(u, v \in V\) and for all shortest paths \(P = (V_P, E_P)\) connecting \(u\) and \(v\), we have \(V_P \subseteq X\).

For the standard definitions of concepts, concept classes, VC-dimension, and polynomial PAC-learnability from computational learning theory, the reader is referred to some standard text book about learning theory (see, e.g., [9]). Let \(C\) be a concept class over some domain \(X\). The \(k\)-fold union of \(C\) for some \(k \geq 1\) integer is defined by \(C_k \bigcup = \{C_1 \cup \ldots \cup C_k : C_i \in C\} for all \(i \in [k]\}\). Note that the definition does not require the \(C_i\)'s to be pairwise different. The following problem is central to concept learning:

**Problem 1 (The Consistency Problem).** Given a concept class \(C \subseteq 2^X\) over some domain \(X\) and disjoint sets \(E^+, E^- \subseteq X\) of examples, return a concept \(C' \in C\) that is consistent with \(E^+\) and \(E^-\), i.e., \(E^+ \subseteq C\) and \(E^- \cap C = \emptyset\) if such a concept exists; o/w return “No”.

In order to prove polynomial PAC-learnability, we will use the following results from computational learning theory [2].

**Theorem 1.** Let \(C \subseteq 2^X\) be a concept class over some domain \(X\) with VC-dimension \(d > 0\).

(i) \(C\) is polynomially PAC-learnable if \(d\) is bounded by a polynomial of its parameters and Problem 1 can be solved in polynomial time in the parameters.

(ii) For all \(k \geq 1\), the VC-dimension of \(C_k\) is at most \(2dk \log(3k)\).

### 3 Weak Convexity in Metric Spaces

In this section we relax the notion of convexity defined for Euclidean spaces to weak convexity in metric spaces and discuss some basic formal properties of weakly convex sets. The main result of this section is formulated in Thm. 2. It states that weakly convex sets have a unique decomposition into a set of weakly convex “connected” blocks that have a pairwise minimum distance from each other. To define weak convexity, recall that a subset \(A \subseteq \mathbb{R}^d\) is convex if

\[
D_2(x, z) + D_2(z, y) = D_2(x, y) \text{ implies } z \in A
\] (1)
for all $x, y \in A$ and for all $z \in \mathbb{R}^d$, where $D_2$ is the Euclidean distance. Our notion of weak convexity in metric spaces incorporates a relaxation of (1), motivated by the fact that convex sets defined by (1) are always “contiguous” and cannot therefore capture well-separated regions of the domain. We address this problem by adapting the idea of $k$-convexity over Hamming metric spaces [6] to arbitrary ones. Analogously to [6], we do not require (1) to hold for all points $x$ and $y$, but only for such pairs which have a distance of at most a user-specified threshold. In other words, while ordinary convexity is based on a global condition resulting in a single “contiguous” region, our notion of weak convexity relies on a local one, resulting in potentially several isolated regions, where the spread of locality is controlled by the above mentioned user-specified threshold. This consideration yields the following formal definition of weakly convex sets in metric spaces:

**Definition 1.** Let $(X, D)$ be a metric space and $\theta \geq 0$. A set $A \subseteq X$ is $\theta$-convex (or simply, weakly convex) if for all $x, y \in A$ and $z \in X$ it holds that $z \in A$ whenever $D(x, y) \leq \theta$ and $z \in \triangle_=(x, y)$, where

$$\triangle_=(x, y) = \{ z \in X : D(x, z) + D(z, y) = D(x, y) \} .$$

(2)

Notice that (2) does not require $x \neq y$. In particular, $\triangle_=(x, x) = \{ x \}$ for all $x \in X$. The family of all weakly convex sets is denoted by $C_{\theta, D}$; we omit $D$ if it is clear from the context. It always holds that $C_{0, D} = 2^X$.

In order to illustrate the notion of weak convexity, consider the finite set of points $A \subseteq \mathbb{R}^2$ depicted by filled dots in Fig. 1b. The strongly (i.e., ordinary) convex hull of $A$ is indicated by the gray area. In contrast, the $\subseteq$-smallest $\theta$-convex set containing $A$ for some suitable $\theta \geq 0$ is drawn in red. The most obvious difference is that there are three separated regions $A_1, A_2,$ and $A_3$, instead of a single contiguous area. In other words, weakly convex sets need not be connected despite that strongly convex sets in $\mathbb{R}^2$ do. This is a consequence of considering only such pairs for membership witnesses that have a distance of at most $\theta$ (see, also, Fig. 1a and 1c). For example, the points $p$ and $q$ in Fig. 1b have a distance strictly greater than $\theta$, implying that they do not witness the membership of the point $v$. Notice that in the same way as strongly convex sets, (parts of) weakly convex sets are not necessarily connected.
convex sets may degenerate. While $A_2$ and $A_3$ are regions with strictly positive area, $A_1$ is just a segment. We may even have isolated points as shown in Fig. 1a.

Despite this unconventional behavior of weakly convex sets, $(X, C_\theta)$ forms a convexity space. To see this, note that $\emptyset, X \in C_\theta$. Furthermore, $C_\theta$ is stable for arbitrary intersections and nested unions. Indeed, if $F \subseteq C_\theta$ is a family of $\theta$-convex sets, $x, y \in \bigcap F$ with $D(x, y) \leq \theta$ then $\triangle = (x, y) \subseteq F$ for all $F \in F$ implying that $\bigcap F$ is $\theta$-convex. If, in addition, $F$ is totally ordered by inclusion and $x, y \in \bigcup F$ with $D(x, y) \leq \theta$ then there are $F_x, F_y \in F$, say $F_x \subseteq F_y$, such that $x \in F_x$ and $y \in F_y$. Then, according to (2), $\triangle = (x, y) \subseteq F_y$ implying that $\bigcup F$ is $\theta$-convex. Hence, $C_\theta$ is a convexity space as claimed.

Since $C_\theta$ is stable for arbitrary intersections, it has an associated closure operator $\rho_\theta : 2^X \to 2^X$ with $A \mapsto \bigcap \{C \in C_\theta : A \subseteq C\}$ for all $A \subseteq X$. That is, $\rho_\theta$ maps a set $A$ to the $\subseteq$-smallest $\theta$-convex set containing $A$. It is called the weakly convex hull operator and its fixed points (i.e., the $\rho_\theta$-closed sets) form exactly $C_\theta$. Moreover, $\rho_\theta$ is domain finite [15], i.e., $\rho_\theta(A) = \bigcup\{\rho_\theta(F) : F \subseteq [A]^{<\omega}\}$.

### 3.1 Some Basic Properties of Weakly Convex Sets

We now present some basic properties of weakly convex sets that make this kind of closed sets interesting for machine learning from a practical as well as from a theoretical viewpoint. As already mentioned, weakly convex sets need not be contiguous (cf. Fig. 1), in contrast to, for instance, ordinary convex sets in the Euclidean space. Instead, one can observe regions that are separated from each other. This is again due to the fact that the notion of weak convexity utilizes a distance threshold $\theta$. As a consequence, separate regions may arise with a pairwise distance of at least $\theta$. In Thm. 2 below, which is one of our main technical results for this work, we formally state this property of weakly convex sets. We note that this result generalizes that stated in [6, Proposition 3.2] for the Hamming metric space to arbitrary metric spaces.

We first introduce some necessary notions. Let $M = (X, D)$ be a metric space, $\theta \geq 0$, and $A \subseteq X$. Two points $a, b \in A$ are $\theta$-connected w.r.t. $A$, denoted $a \sim_{\theta, A} b$, if there is a finite sequence $a = p_1, p_2, \ldots, p_r = b \in A$ such that $D(p_i, p_{i+1}) \leq \theta$ for all $i \in [r - 1]$. $A$ is $\theta$-connected if $a \sim_{\theta, A} b$ for all $a, b \in A$. Note that $\sim_{\theta, A}$ is an equivalence relation on $A$; its equivalence classes, denoted by $[a]_{\sim_{\theta, A}} = \{b \in A : a \sim_{\theta, A} b\}$ for all $a \in A$, will be referred to as $\theta$-connected components.

**Theorem 2.** Let $(X, D)$ be a metric space and $\theta \geq 0$. Then $A \subseteq X$ is $\theta$-convex iff there is a uniquely defined family of non-empty sets $(A_i \subseteq A)_{i \in I}$ for some index set $I$ satisfying the following conditions:

(i) $A = \bigcup_{i \in I} A_i$,
(ii) $A_i$ is $\theta$-convex for all $i \in I$,
(iii) $A_i$ is $\theta$-connected for all $i \in I$,
(iv) for all $i, j \in I$ with $i \neq j$, $D(a, b) > \theta$ for all $a \in A_i, b \in A_j$. 


In what follows, the family \((A_i)_{i \in I}\) satisfying conditions (i)-(iv) in Thm. 2 will be referred to as the \(\theta\)-decomposition of the \(\theta\)-convex set \(A\). Furthermore, the sets \(A_i\) in the \(\theta\)-decomposition of \(A\) will be called \(\theta\)-blocks or simply, blocks. The theorem above tells us that weakly convex sets can be partitioned uniquely into a family of non-empty blocks in a way that the distance between each pair of such weakly convex components is at least \(\theta\). The uniqueness of the \(\theta\)-decomposition in Thm. 2 gives rise to a naïve algorithm for computing the weakly convex hull of a finite set intensionally (cf. Alg. 2 in Section 5). The idea is to start with the singletons and enforce conditions (i)-(iv) by repeatedly merging invalid pairs of blocks. However, that requires the strict inequality in condition (iv) not only to hold for pairs of points, but also between blocks. Cor. 1 below is concerned with metric spaces in which this property holds.

**Corollary 1.** Let \(M = (X, D)\) be a metric space, \(\theta \geq 0\), \(A \subseteq X\), and \((A_i)_{i \in I}\) the \(\theta\)-decomposition of \(\rho_\theta(A)\).

(i) We have \(|I| \leq |A|\). In particular, if \(A\) is finite, then \(I\) is finite.

(ii) If \(M\) is complete and \(A_i\) and \(A_j\) are (topologically) closed for some \(i \neq j\) then \(D(A_i, A_j) = \inf_{a \in A_i, b \in A_j} D(a, b) > \theta\).

Accordingly, Cor. 1 motivates the following definition of well-behaved metric spaces. A metric space \(M = (X, D)\) is compatible with the convexity space \((X, C_\theta)\) if \(\rho_\theta(A)\) is (topologically) closed for all \(A \in [X]^{<\omega}\) [15]. If, in addition, \(M\) is complete, we call it well-behaved.

Finally, we claim that the weakly convex hull operator is monotone w.r.t. \(\theta\). This property will be utilized by our consistent hypothesis finding algorithms.

**Proposition 1.** Let \((X, D)\) be a metric space and \(0 \leq \theta \leq \theta'\). Then for all \(A \subseteq X\), (i) \(\rho_\theta(A) \subseteq \rho_{\theta'}(A)\) and (ii) for all \(x, y \in \rho_\theta(A)\), \(x, y\) are in the same \(\theta'\)-block of the \(\theta'\)-decomposition of \(\rho_{\theta'}(A)\) if they are in the same \(\theta\)-block of the \(\theta\)-decomposition of \(\rho_\theta(A)\).

4 Learning in the Extensional Problem Setting

In this section we consider the case that the underlying metric space is finite and weakly convex sets are represented extensionally, e.g. because they have no (natural) compact representation. Examples of this scenario include, among others, the case that the metric space is given by the set of vertices of a graph together with some distance on vertices. To formulate some basic properties of \(\rho_\theta\) introduced in Section 3, we define a preclosure operator \(\hat{\rho}_\theta\) over \(X\). More precisely, let \(M = (X, D)\) be a finite metric space and \(\theta \geq 0\). For all \(x, y \in X\), let \(W_\theta(x, y) = \Delta_\Delta(x, y)\) if \(D(x, y) \leq \theta\); o/w \(W_\theta(x, y) = \emptyset\). Finally, define the function \(\hat{\rho}_\theta : 2^X \rightarrow 2^X\) by \(\hat{\rho}_\theta(A) = \bigcup_{x, y \in A} W_\theta(x, y)\) for all \(A \subseteq X\).

**Lemma 1.** The function \(\hat{\rho}_\theta\) over \(M\) is a preclosure operator.
Let $\hat{\rho}_0^0(A) = A$ and $\hat{\rho}_0^{i+1}(A) = \hat{\rho}_0(\hat{\rho}_0^i(A))$ for all $i \in \mathbb{N}$ and $A \subseteq X$. Since $\hat{\rho}_0$ is monotone by Lemma 1 and $X$ is finite, for all $A \subseteq X$ there exists a positive integer $\gamma(A)$ such that $\hat{\rho}_0^{\gamma(A)}(A) = \hat{\rho}_0^{\gamma(A)+1}(A)$, implying
\[ \hat{\rho}_0^{\gamma(A)}(A) = \hat{\rho}_0^{\gamma(A)+1}(A) \]  
for all $l \geq 0$. Furthermore, $\Gamma = \max\{\gamma(A) : A \subseteq X\} < \infty$. In the theorem below we claim that $\hat{\rho}_0^\Gamma$ yields exactly $\rho_0$.

**Theorem 3.** Let $(X, D)$ be a finite metric space and $\theta \geq 0$. Then

(i) $\rho : 2^X \rightarrow 2^X$ with $\rho(A) = \hat{\rho}_0^\Gamma(A)$ for all $A \subseteq X$ is a closure operator and

(ii) for all $A \subseteq X$, $\rho_0(A) = A$ iff $\rho(A) = A$.

We now consider the problem of computing weakly convex sets for the case that the metric space is finite and weakly convex sets are represented extensionally. More precisely, we are interested in the following problem setting:

**Problem 2.** *(The Extensional Weakly Convex Hull Problem).* Given a finite metric space $\mathcal{M} = (X, D)$ with $|X| = n$, a set $A \subseteq X$, and a threshold $\theta \geq 0$, compute the $\theta$-decomposition $A_1, \ldots, A_\ell$ of $\rho_0(A)$, where the $A_i$s are given extensionally.

The algorithm solving Problem 2 is given in Alg. 1. Its input consists of a set $A \subseteq X$ for some finite metric space $(X, D)$ and a non-negative real number $\theta$. The algorithm assumes that the pairwise distances for $(X, D)$ are given explicitly and that each element $x \in X$ is associated with a sorted sequence $S_x$ of pairs $(x', D(x, x'))$, for all $x' \in X \setminus \{x\}$. We assume that these sequences are calculated and stored once in a preprocessing step. The reason behind this assumption is that in order to solve a related consistency problem defined below, Alg. 1 will
be called with different values of \( \theta \). For any \( \delta \geq 0 \), these sequences allow the \( \delta \)-neighborhood \( N_\delta(x) = \{ y \in X : D(x,y) \leq \delta \} \) of a point \( x \in X \) to be calculated in time \( O(|N_\delta(x)|) \) for all \( \delta \geq 0 \) (cf. lines 5 and 7 of Alg. 1).

Alg. 1 maintains three variables. In particular, one can show in the proof of the theorem below, the set \( \rho_0(A) \) is calculated in \( C \). All elements of \( C \) are added first to the queue \( Q \), which is initialized with \( A \) (cf. line 1). The elements of \( Q \) are processed one by one (cf. lines 4–9). In particular, for the element \( x \) of \( Q \) considered in line 4, we move \( x \) from \( Q \) to \( C \) (line 4) and take all elements \( y \) in the \( \theta \)-neighborhood of \( x \) that have already been added to \( C \subseteq \rho_0(A) \) (line 5).

In variable \( E \) we maintain the set of edges of the \( \theta \)-neighborhood graph over \( C \), i.e., two elements of \( C \) are connected by an edge iff their distance is at most \( \theta \). As \( x \) is a new element in \( C \), in line 6 we connect it with all \( y \) considered in line 5.

In lines 7–9 we take all \( z \in W_\theta(x,y) \) that have not yet been considered, mark \( z \), and add it to the queue \( Q \). Regarding line 7, note that the triangle inequality implies that if \( z \in W_\theta(x,y) \) then \( D(x,z), D(y,z) \leq D(x,y) \). Finally, after we have processed all elements that have been added to \( Q \), in line 10 we calculate the connected components of the \( \theta \)-neighborhood graph \( G_\theta = (C,E) \) and return the family formed by the sets of vertices of the connected components.

**Theorem 4.** Alg. 1 is correct and solves Problem 2 in \( O(nd^2) \) time, where \( d \) is the degree of the \( \theta \)-neighborhood graph \( G_\theta = (C,E) \).

In Section 4.1 we will be concerned with an application scenario of the following consistent hypothesis finding (CHF) problem:

**Problem 3 (The CHF Problem for Extensional Weakly Convex Hulls).** Given a finite metric space \( \mathcal{M} = (X,D) \) with \( |X| = n \), disjoint sets \( E^+, E^- \subseteq X \) of positive and negative examples, and an integer \( k > 0 \), return “Yes” and the \( \theta \)-decomposition of a \( \theta \)-convex set consistent with \( E^+ \) and \( E^- \) that consists of at most \( k \) blocks, if it exists for some \( \theta \); o/w the answer “No”.

**Remark 1.** Note that if Problem 3 can be solved in polynomial time then, as \( k \) cannot be greater than \( |E^+| \) by (i) of Corollary 1, a consistent hypothesis with the smallest number of blocks can be found in polynomial time. It always exists, as \( \rho_0(E^+) = E^+ \) and \( E^+ \cap E^- = \emptyset \) by assumption.

**Theorem 5.** Problem 3 can be solved in \( O(T_P(\mathcal{M}) + n^3 \log n) \) time, where \( T_P(\mathcal{M}) \) is the time complexity of computing all pairwise distances for \( X \).

### 4.1 Application Scenario: Vertex Classification

As a proof of concept, in this section we empirically demonstrate the learnability of weakly convex concepts over graphs. More precisely, we consider the metric space \( \mathcal{M} = (V,D_\theta) \) for some undirected graph \( G = (V,E) \), where \( D_\theta \) is the geodesic distance on \( V \). In the learning setting, \( V \) is partitioned into \( V^+ \) and \( V^- \), such that \( V^+ \) is \( \theta \)-convex for some \( \theta \geq 0 \). The target concept \( V^+ \) as well as \( \theta \) are unknown to the learning algorithm. The problem we investigate empirically
is to approximate $V^+$ given some small labeled set $E = E^+ \cup E^-$ of positive and negative examples.

We solve this learning task by computing the hypothesis $C = \rho_{\theta'}(E^+)$ for the greatest $\theta' \leq \max_{u,v \in V} D(u, v)$ that is consistent with $E$. Such a $\theta'$ always exists (cf. Remark 1). Furthermore, $C$ is computed by performing a binary search for $\theta'$ (see [13] for the details). To measure the predictive performance, we use accuracy (i.e., number of correctly classified vertices in $V \setminus E$ over $|V \setminus E|$) and compare it to the baseline $\max\{|V^+|/|V|, |V^-|/|V|\}$ defined by majority vote. We stress that the only purpose of these experiments is to empirically demonstrate that weakly convex concepts can be learned with a remarkable accuracy, without utilizing any domain specific properties and with using only a few training examples.

An adaptation of our approach to the domain specific problem of learning on graphs and a rigorous empirical comparison of its predictive performance with state-of-the-art problem specific algorithms goes far beyond the scope of this paper.

We generated 50 random graphs for $|V| = 100, 250, 1000, \text{ and } 2500$ for the experiments as follows: According to Prop. 1, the diameter of a graph is an upper bound on the parameter $\theta$. In order to provide a diverse set of target concepts and possible values for $\theta$, we generated random graphs based on Delaunay triangulations [4] as follows: After choosing the respective number of nodes $V \subset [0,1]^2$ uniformly at random, we have computed the Delaunay triangulation. We then connected two nodes in $V$ by an (undirected) edge iff they co-occur in at least one simplex of the triangulation. We considered the two cases that the edges are unweighted or they are weighted with the Euclidean distance between their endpoints. However, the resulting graph often contains a small number of very long edges (in terms of the Euclidean distance), especially near the “outline” of the chosen point set. Since such edges reduce the graph’s diameter substantially, we removed the longest 5% of the edges, i.e., those that are not contained in the 95th percentile w.r.t. the Euclidean distance of their endpoints.
For each graph $G = (V, E)$ in the resulting dataset, we have generated random partitionings $(V^+, V^-)$ of $V$ in a way that $V^+$ and $V^-$ are balanced (i.e., $|V^+| \approx |V^-|$) and $V^+$ is $\theta$-convex. We note that for all random partitionings obtained, $V^+$ was not strongly (i.e., ordinary) convex. The training examples $E^+$ and $E^-$ have been sampled uniformly at random from $V^+$ and $V^-$, respectively, such that $|E^+| \approx |E^-|$. The number of training examples (i.e., $|E^+ \cup E^-|$) was varied over 20, 40, 60, 80, 100. This overall procedure generates 5,000 learning tasks (50 graphs \times 20 random target concepts \times 5 training set sizes), for each graph size $|V| = 100, 250, 1000, 2500$. In Fig. 2 we give an example graph with $|V| = 250$, together with the node prediction using 40 training examples. The training examples are marked with black outline and the predictions are encoded by colors. In particular, dark red corresponds to true positive, dark blue to true negative, and light red to false negative nodes. In the example we have no false positive node, which was the case for most graphs.

Fig. 3 shows the accuracy ($y$-axes) of the baseline (blue box plots) and our learner (orange box plots for unweighted and red ones for weighted edges) grouped by the number of provided examples ($x$-axes) and the graph sizes $|V|$. In all cases, our learner outperforms the baseline significantly by noting that for $|V| = 100$, the high accuracy results obtained from 60 training examples are less interesting. For $|V| > 100$ it is remarkable that the learner does not require much more examples with increasing graph size. For example, for graphs with 2,500 vertices, already 80 examples are sufficient to achieve an average accuracy of 0.94 for unweighted graphs. Notice that the baseline is in all cases very close to 0.6. This is due to our construction of the target concepts: We chose $\theta$ maximal such that $2|V^+| < |V|$. Therefore, in almost all cases there are about 10% less positive nodes than negative. We have tested the generated weakly convex sets for strong convexity: almost all of them were not strongly convex. In summary, our experimental results clearly show that a remarkable predictive accuracy can be obtained already with relatively small training sets with our generic approach, without utilizing any domain specific knowledge.
Algorithm 2 **Intensional Weakly Convex Hull (Naïve)**

Require: well-behaved metric space \( M = (X, D) \) and representation scheme \( \mu \) for \( M \)
Input: \( A \in [X]^{<\omega} \) and \( \theta \geq 0 \)
Output: \( \mu(\theta, A_1), \ldots, \mu(\theta, A_L) \), where \( A_1, \ldots, A_L \) is the \( \theta \)-decomposition of \( \rho_\theta(A) \)

1: \( D \leftarrow \{ \mu(\theta, \{ x \}) : x \in A \} \)
2: while \( \exists B_i, B_j \in D \) such that \( B_i \neq B_j \) and \( D(B_i, B_j) \leq \theta \) do
3: \( D \leftarrow (D \setminus \{ B_i, B_j \}) \cup \{ \text{Merge}(\theta, A_i, B_i, B_j) \} \)
4: return \( D \)

5 **The Intensional Problem Setting**

In this section we consider the *intensional* problem setting, that is, the scenario that weakly convex sets have some compact representation. In contrast to the extensional case, the metric spaces in this section are allowed to be *infinite*. They are, however, required to be *well-behaved* (see Sect. 3 for the definition).

To formulate the problem setting considered in this section, we introduce the following notion for a metric space \( M = (X, D) \): A *representation scheme* for \( M \) is a function \( \mu : \mathbb{R}_{\geq 0} \times [X]^{<\omega} \to \{0, 1\}^* \) satisfying \( \mu(\theta, A) = \mu(\theta, B) \) iff \( \rho_\theta(A) = \rho_\theta(B) \) for all \( A, B \in [X]^{<\omega} \) and \( \theta \geq 0 \). In other words, \( \mu \) returns some representation of \( \rho_\theta(A) \) for all finite subsets \( A \subseteq X \). Note that \( \rho_\theta(A) \) can be infinite. Analogously to Problem 2, we are interested in the following computational problem:

**Problem 4 (The Intensional Weakly Convex Hull Problem).** Given a well-behaved metric space \( M = (X, D) \), a representation scheme \( \mu \) for \( M \), a set \( A \subseteq [X]^{<\omega} \) with \(|A| = m\), and \( \theta \geq 0 \), compute \( \mu(\theta, A) \).

For page limitation, we give a very simple naïve algorithm for Problem 4 (see Alg. 2), by noting that it is not optimal. It assumes a well-behaved metric space \( M = (X, D) \) and some representation scheme \( \mu \) for \( M \). The input to the algorithm consists of a finite subset \( A \subseteq X \) and a distance threshold \( \theta \geq 0 \). Its output is the set \( \{ \mu(\theta, A_1), \ldots, \mu(\theta, A_L) \} \) of binary strings representing the blocks \( A_1, \ldots, A_L \) in the \( \theta \)-decomposition of \( \rho_\theta(A) \). The algorithm first initializes the variable \( D \) with the set of the representations of \( \rho_\theta(\{ x \}) = \{ x \} \) for all \( x \in A \) (cf. line 1). It then iteratively selects two different blocks \( B_i, B_j \in D \) such that \( D(B_i, B_j) = \min_{x \in B_i, y \in B_j} D(x, y) \geq \theta \). If there are no such \( B_i \) and \( B_j \), then it terminates by returning \( D \); o/w it updates \( D \) by removing \( B_i, B_j \) and adding their merge defined by \( \text{Merge}(\theta, A_i, B_i, B_j) = \mu(\theta, (\text{ext}(B_i) \cup \text{ext}(B_j)) \cap A) \) if \( D(B_i, B_j) \leq \theta \); o/w \( \text{Merge}(\theta, A_i, B_i, B_j) = \bot \), where \( \text{ext}(B_i), \text{ext}(B_j) \) denote the extensions of \( B_i, B_j \), respectively. The proof of the proposition below follows by induction on \(|D|\) from Thm. 2 and Corollary 1.

**Proposition 2.** Alg. 2 is correct.

Let \( T_S \), \( T_D \), and \( T_M \) denote the time complexity of computing \( \mu(\theta, \{ x \}) \), the distance between \( B_i \) and \( B_j \), and the merge of \( B_i \) and \( B_j \), respectively, for any
Learning Weakly Convex Sets in Metric Spaces

\( x \in X \) and \( \theta \)-blocks \( B_i \) and \( B_j \). One can easily check that the time complexity of Alg. 2 is \( O(mT_S + m^3T_D + mT_M) \). Using a more sophisticated version of Alg. 2 not presented in this paper for space limitations, we have the following improved complexity result.

**Theorem 6.** Problem 4 can be solved in time \( O(mT_S + m^3T_D + mT_M) \).

We consider the consistency problem also for the intensional scenario.

**Problem 5 (The Consistency Problem for Intensional Weakly Convex Hulls).** Given a well-behaved metric space \( \mathcal{M} = (X, D) \), representation scheme \( \mu \) for \( \mathcal{M} \), disjoint finite sets \( E^+, E^- \subseteq X \) of labeled examples with \( |E^+ \cup E^-| = m \), and \( k > 0 \), return "Yes" and the representations of the blocks in the \( \theta \)-decomposition of a \( \theta \)-convex set that is consistent with \( E^+ \) and \( E^- \) and has at most \( k \) blocks, if such a decomposition exists for some \( \theta > 0 \); o/w the answer "No".

Note that Remark 1 applies also to the problem above. Using the same idea as for the solution of Problem 3 (i.e., to decide whether a desired \( \theta \) exists, we perform a binary search on the sorted set of pairwise distances between the elements in \( A \)), we have the following result on the above problem:

**Theorem 7.** Problem 5 can be solved in \( O((mT_S + m^2T_D + mT_M) \log m) \) time.

In Sections 5.1 and 5.2 below we present two non-trivial applications of Thm. 7 to polynomial PAC-learnability.

### 5.1 Learning Weakly Convex Boolean Functions

As a first application of Thm. 7, we show that the concept class formed by weakly convex Boolean functions is efficiently PAC-learnable. This result is not new, it has been obtained with a domain specific algorithm in [6]. Still, we present it as an application because, as we show below, we can derive it in a very simple way by using Thm. 7. Furthermore, our general purpose algorithm solving Problem 4 has the same asymptotic complexity on this problem as the domain specific one in [6].

Consider the metric space \( \mathcal{M}_H = (\mathbb{H}_n, D_H) \) for some \( n \in \mathbb{N} \) (see Section 2). Clearly, the finiteness of \( \mathbb{H}_n \) implies that \( \mathcal{M}_H \) is well-behaved for all \( \theta \geq 0 \). A Boolean function \( f : \{0,1\}^n \to \{0,1\} \) (\( n \in \mathbb{N} \)) is \( \theta \)-convex for some \( \theta \geq 0 \) if for all \( x, y, z \in \mathbb{H}_n \), \( f(z) = 1 \) whenever \( f(x) = f(y) = 1 \), \( D_H(x, y) \leq \theta \), and \( z \in \Delta_{\infty}(x, y) \). Note that for \( \mathcal{M}_H \) it suffices to consider the values in \( [n] \) for \( \theta \).

Throughout this section we will use the following notation: \( L_n \) denotes the set \( \{x_1, \neg x_1, \ldots, x_n, \neg x_n\} \) of Boolean literals. A term \( T \) is a conjunction of literals from \( L_n \); \( T \) is sometimes regarded as the set of literals it contains. A conflict between two terms \( T_i \) and \( T_j \) over \( L_n \) is an integer \( p \in [n] \) such that \( x_p \in T_i \) and \( \neg x_p \in T_j \) or vice versa. Finally, for a Boolean function \( f \), \( \text{ext}(f) \) denotes the extension of \( f \) (i.e., \( \text{ext}(f) = \{x \in \mathbb{H}_n : f(x) = 1\} \)). We will use the following auxiliary result:
Lemma 2. Let $A \subseteq \mathbb{H}_n$ be $\theta$-convex and $\theta$-connected for some $\theta \geq 2$. Then $A$ is convex (i.e., $n$-convex) and can be represented by a term $T$ over $L_n$.

For any $n > 0$, the concept class $B_{n,k} \subseteq 2^{\mathbb{H}_n}$ is defined as follows: For all $A \subseteq \mathbb{H}_n$, $A \in B_{n,k}$ iff $A$ is $\theta$-convex for some $\theta \geq 0$ and its $\theta$-decomposition has at most $k$ blocks. Thm. 2 and Lemma 2 together imply that any such weakly convex set $A$ can be represented uniquely by a $k$-term DNF $F$ such that the extensions of the terms in $F$ represent precisely the blocks in the $\theta$-decomposition of $A$. Since the blocks are non-empty, no term contains a variable and its negation. This gives rise to the following definition of the representation scheme $\mu$ for $\mathcal{M}_H$:

For all $S \subseteq \mathbb{H}_n$, define $\mu_H : \mathbb{R}_{\geq 0} \times \mathbb{H}_n \rightarrow \{0, 1\}^*$ by $\mu_H(\theta, S) = F$, where $F$ is the unique DNF representation of $\rho_\theta(S)$, if $\rho_\theta(S)$ is $\theta$-connected; o/w by $\perp$.

Lemma 3. Problem 5 can be solved in $O(nm^2 \log m)$ time for $\mathcal{M}_H$.

Proof. Let $\mu$ in Problem 5 be defined by $\mu_H$. We show that $T_S, T_D, T_M$ in Thm. 7 are all in $O(n)$. For $T_S$, the claim follows from $\mu_H(\theta, \{x\}) = \bigwedge_{i=1}^k l_i$, where $l_i = x_i$ if $x_i = 1$; o/w $l_i = \neg x_i$. Let $T_i$ and $T_j$ be terms over $L_n$. Their distance $D_H(T_i, T_j)$ is equal to the number of conflicts between $T_i$ and $T_j$, implying $T_D \in O(n)$. Finally, if $D_H(T_i, T_j) \leq \theta$ then $\text{MERGE}(T_i, T_j)$ is the term $T$ with literals $T_i \cap T_j$. Thus, $T_{\text{MERGE}} = O(n)$. The statement then follows by Thm. 7. \qed

Theorem 8. For all $d, k \geq 0$, $B_{n,k}$ is polynomially PAC-learnable.

Proof. Since $B_{n,k} \subseteq (B_{n,1})_k^\cup$, $\text{VC-dim}(B_{n,k}) \leq \text{VC-dim}((B_{n,1})_k^\cup) \leq 4nk \log(3k)$ by $\text{VC-dim}(B_{n,1}) \leq 2n$ and by (ii) of Thm. 1. Hence, the VC-dimension of $B_{n,k}$ is polynomial in $n$ and $k$. Furthermore, by Lemma 3, the consistency problem for $B_{n,k}$ can be solved in time polynomial in $n$, $k$, and $m = |E^+ \cup E^-|$. The theorem then follows by (i) of Thm. 1. \qed

Note that if the extensions of the terms in the DNF are not required to be pairwise disjoint then, in contrast to the positive result in Thm. 8, $k$-term DNF formulas are not polynomially PAC-learnable for any $k \geq 2$ if $P \neq \text{RP}$ [10]. In [6] it is shown that the class of $\theta$-convex Boolean functions is not polynomially PAC-learnable for $\theta > n/2 − 1$. The reason is that the number of terms having a pairwise distance greater than $n/2 − 1$ can be exponential in $n$. Notice that the number of terms in $B_{n,k}$ is bounded by the parameter $k$ in our problem setting. Finally we note that the time complexity of the domain specific algorithm in [6] that solves Problem 4 for $\mathbb{H}_n$ is $O(m^2n)$, which is the same as that of the sophisticated version of our general purpose Algorithm 2 in [13].

5.2 Learning Weakly Convex Axis-Aligned Hyperrectangles

Our second application of Thm. 7 is concerned with polynomial PAC-learnability of weakly convex sets in $\mathcal{M}_R = (\mathbb{R}^d, D_1)$, where $D_1$ is the Manhattan (or $L_1$) distance, i.e., $D_1(x, y) = \sum_{i=1}^d |x_i − y_i|$ for all $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$. Note that $\mathcal{M}_R$ can be regarded as a generalization of $\mathcal{M}_H$ considered in the
Lemma 4. Problem 5 can be solved in \( T \) that contains \( x \) and \( y \). This implies that all axis-aligned closed hyperrectangles are \( \theta \)-convex for all \( \theta > 0 \) and \( \rho_0(A) \) is closed for all finite subsets \( A \subset R^d \).

All concepts in the concept class \( R_{d,k} \) considered in this section are defined by the union of at most \( k \) pairwise disjoint axis-aligned closed hyperrectangles in \( R^d \), for some \( d, k > 0 \). More precisely, for all \( R \subseteq R^d \), \( R \in R_{d,k} \) iff \( R \) is \( \theta \)-convex for some \( \theta > 0 \) with respect to \( M_R \) and the \( \theta \)-decomposition of \( R \) consists of at most \( k \) blocks (i.e., axis-aligned closed hyperrectangles). For all finite sets \( S \in [R^d]^{<\omega} \), define \( \mu_R : R^d_0 \times R^d_0 \times [R^d]^{<\omega} \rightarrow \{0,1\}^* \) by \( \mu_R(\theta,S) = (S_{\min},S_{\max}) \) if \( \rho_0(S) \) is \( \theta \)-connected; o/w by \( 1, 4 \) where \( S_{\min} \) (resp. \( S_{\max} \)) denotes the componentwise minimum (resp. maximum) of the points in \( S \).

**Lemma 4.** Problem 5 can be solved in \( O(dm^2 \log m) \) time for \( M = (R^d,D_1) \).

**Proof.** Let \( \mu \) in Problem 5 be defined by \( \mu_R \). We prove the claim by showing that \( T_S,T_D,T_{\text{merge}} \) in Thm. 7 are all in \( O(d) \). In particular, \( T_S \in O(d) \) follows from \( \mu_R(\theta,\{x\}) = (x,x) \). Let \( B_i \) (resp. \( B_j \)) be an axis-aligned closed hyperrectangle, \( u = \min B_i \), and \( v = \max B_i \). Then, \( T_D \in O(d) \) by the fact that \( D_1(B_i,B_j) = \sum_{i=1}^d D_1([u_i,v_i],[x_i,y_i]) \), where \( D_1([u_i,v_i],[x_i,y_i]) = \min\{|x_i - v_i|,|u_i - y_i|\} \) if \( [u_i,v_i] \cap [x_i,y_i] \neq \emptyset \); o/w \( D_1([u_i,v_i],[x_i,y_i]) = 0 \). Finally, if \( \overline{D}(B_i,B_j) \leq \theta \) then \( \text{merge}(B_i,B_j) \) is the smallest axis-aligned closed hyperrectangle containing \( \min\{u,x\} \) and \( \max\{v,y\} \), implying \( T_{\text{merge}} = O(d) \). The claim then follows by Thm. 7. \( \square \)

**Theorem 9.** For all \( d,k \geq 0 \), \( R_{d,k} \) is polynomially PAC-learnable.

**Proof.** Since \( R_{d,k} \subseteq (R_{d,1})^k \), \( VC-dim(R_{d,k}) \leq VC-dim((R_{d,1})^k) \leq 4dk \log(3k) \) by \( VC-dim(R_{d,1}) = 2d \) and by (ii) of Thm. 1. Hence, the VC-dimension of \( R_{d,k} \) is polynomial in \( d \) and \( k \). Furthermore, by Lemma 4, the consistency problem for \( R_{d,k} \) can be solved in time polynomial in \( d, k \), and \( |E^+ \cup E^-| \). Thus, the theorem follows by (i) of Theorem 1. \( \square \)

While Lemma 4 implies that a consistent hypothesis that has the smallest number of pairwise disjoint axis-aligned \( d \)-dimensional closed hyperrectangles can be found in polynomial time for all \( d \geq 1 \), this problem becomes NP-complete even for \( d = 2 \), if disjointness is not required (see, e.g., [1]).

## 6 Concluding Remarks

The theoretical and experimental results of this paper demonstrate the usefulness of weakly convex sets for machine learning. While our focus in this paper was solely on applications to machine learning, weakly convex sets seem to be useful for data mining applications (e.g., itemset mining, subgroup discovery) as well.

---

4 We assume that real numbers are represented in \( O(1) \) space up to a certain precision.
The notion of weak convexity can be uninteresting for certain metric spaces. For example, for finite subspaces of \((\mathbb{R}^d, D_2)\), \(\triangle_e(x, y) = \{x, y\}\) holds almost surely for all points \(x\) and \(y\). To overcome this problem, in the long version of this paper we allow the triangle inequality to hold up to some tolerance \(\varepsilon\) instead of equality. All results of Sect. 3 can be generalized to this relaxed definition.

There are several interesting questions for further research. Note, for example, that Alg. 2 is very similar to single linkage clustering, raising the following question: Can the time complexity in Thm. 7 be further improved by using techniques (e.g., in [12]) designed for single linkage clustering algorithms?

Acknowledgments

This work was partly supported by the Ministry of Education and Research of Germany (BMBF) under project ML2R (grant number 01/S18038C) and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy - EXC 2070 - 390732324.

References

1. Bereg, S., Cabello, S., Díaz-Báñez, J., Pérez-Lantero, P., Seara, C., Ventura, I.: The class cover problem with boxes. Computational Geometry 45(7), 294–304 (2012)
2. Blumer, A., Ehrenfeucht, A., Haussler, D., Warmuth, M.K.: Learnability and the vapnik-chervonenkis dimension. J. ACM 36(4), 929–965 (1989)
3. Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order. Cambridge University Press, 2nd edn. (2002)
4. Delaunay, B.N.: Sur la sphère vide. Bull. Acad. Sci. URSS (6), 793–800 (1934)
5. Edelsbrunner, H., Mücke, E.P.: Three-dimensional alpha shapes. ACM Trans. Graph. 13(1), 43–72 (1994)
6. Ekin, O., Hammer, P.L., Kogan, A.: Convexity and logical analysis of data. Theoretical Computer Science 244(1), 95 – 116 (2000)
7. Ester, M., Kriegel, H.P., Sander, J., Xu, X., et al.: A density-based algorithm for discovering clusters in large spatial databases with noise. In: Proceedings of the Second International Conference on Knowledge Discovery and Data Mining (KDD). vol. 96, pp. 226–231. AAAI Press (1996)
8. Hemmer, M., Portaneri, C., Alliez, P.: Alpha Hulls. Research report (Nov 2020)
9. Kearns, M.J., Vazirani, U.V.: An Introduction to Computational Learning Theory. MIT Press (1994)
10. Pitt, L., Valiant, L.G.: Computational limitations on learning from examples. J. ACM 35(4), 965–984 (1988)
11. Seiffarth, F., Horváth, T., Wrobel, S.: Maximal closed set and half-space separations in finite closure systems. In: Proc. of ECML PKDD 2019. Lecture Notes in Computer Science, vol. 11906, pp. 21–37. Springer (2019)
12. Sibson, R.: SLINK: an optimally efficient algorithm for the single-link cluster method. Comput. J. 16(1), 30–34 (1973)
13. Stadtländer, E., Horváth, T., Wrobel, S.: Learning weakly convex sets in metric spaces. CoRR abs/2105.06251 (2021), https://arxiv.org/abs/2105.06251
14. Valiant, L.G.: A theory of the learnable. Commun. ACM 27(11), 1134–1142 (1984)
15. van de Vel, M.: Theory of Convex Structures, North-Holland Mathematical Library, vol. 50. Elsevier (1993)