Classical mechanics of the Hydrogen atom perturbed by Van der Waals potential interacting with combined electric and magnetic fields

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Abstract. The hydrogen atom perturbed by Van der Waals potential interacting with combined electric and magnetic fields is a non-integrable system, except for some cases, one of which is separable in elliptical coordinates. A detailed study of the classical mechanics of the system is presented giving a complete description of the real phase space structure of the system, we also determine all the generic bifurcations of Liouville tori. This study is supported by numerical investigations via the Poincare surface of section and the phase space trajectories method. The dynamic character of this system depends on the van der Waals potential and the intensity of the magnetic field. Numerical calculations show that if the separability condition is verified, the classical dynamics is regular. However, with a small change in the condition, the dynamic property of the hydrogen atom begins to change. Since the condition is far from integrable behavior, almost all phase space trajectories are chaotic. On the other hand, the intensity of the electric field has no influence on the system.

1. Introduction

It is well known that most natural phenomena are chaotic and the others are quasi-periodic as well as these phenomena are modeled by dynamic systems which are described by ordinary or partial differential equations, linear or nonlinear. The discussion of differential equations prompts us to talk about Hamiltonian systems and their integrability. Among the most important methods and approaches to study the integrability of Hamiltonian systems are: the Liouville theorem [1], the Galois differential theory [2] the Ziglin criterion [3] [4], the Poincaré sections [5], and the Painlevé analysis [6] [7] [8] [9]. These methods make it possible to prove either complete integrability or integrability for exceptional cases, or non-integrability. The best example of fully integrable systems is the hydrogen atom, which is why we find several theoretical and experimental studies [10] - [11], but sometimes we find that the Hamiltonian system is integrable in exceptional cases as in the case of the hydrogen atom disturbed by the Van der Waals potential [12] [13], and non-integrable as in the case of the hydrogen atom perturbed by the quadratic effect of Zeeman [14].

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Several studies have been done concerning the hydrogen atom such as: the study of the classical dynamics of a hydrogen atom in a generalized van der Waals potential, in which the author has removed the singularity of the problem from the Levi-Civita regularization. Moreover, converts the problem into two coupled anharmonic sex oscillators. In addition, identified the integrable cases of the system by the analysis of Painlevé. Finally a numerical study in make the transition of the type chaos-order-chaos [10]. The study of the classical dynamics of a hydrogen atom of Rydberg in a generalized van der Waals potential plus a magnetic field by the method of phase space trajectories and the surface section of Poincaré [15]. The study of the classical dynamics of a hydrogen atom in the presence of uniform magnetic and quadrupole electric fields, in which the author to explore the structure and evolution of phase space by means of section surfaces of Poincaré [11].

In our work, we will examine the dynamic behavior of the hydrogen atom disturbed by the Van der Waals potential in interaction with combined electric and magnetic fields, this system is described by the following Hamiltonian:

\[
H = \frac{1}{2} p^2 + V(x^2 + y^2 + \lambda z^2) + \gamma (x^2 + y^2) + \beta z
\]  

and the corresponding equations of motion are

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p_x} = p_x \\
\dot{y} &= \frac{\partial H}{\partial p_y} = p_y \\
\dot{z} &= \frac{\partial H}{\partial p_z} = p_z
\end{align*}
\]  

\[
\begin{align*}
\dot{p}_x &= \frac{\partial H}{\partial x} = x \left( \frac{1}{r^{3/2}} + 2(\alpha + \gamma) \right) \\
\dot{p}_y &= \frac{\partial H}{\partial y} = y \left( \frac{1}{r^{3/2}} + 2(\alpha + \gamma) \right) \\
\dot{p}_z &= \frac{\partial H}{\partial z} = z \left( \frac{1}{r^{3/2}} + 2\alpha \lambda^2 \right) + \beta
\end{align*}
\]  

Where \((X) = (dX/dt), r^2 = x^2 + y^2 + z^2, P^2 = P_x^2 + P_y^2 + P_z^2, \) and \(\alpha, \lambda, \gamma \) and \(\beta\) are parameters.

The document is organized as follows: in section 2, in terms of elliptic coordinates, we eliminate the singularity of the problem as well as the latter make separable and consequently the equations of motion are regularized. Then in Section 3, we present a detailed study of the real phase space topology for non-critical values of the motion constants \(h\) and \(f\). As for the critical values, all the generic bifurcations of Liouville tori will be described using Fomenko surgery. In section 4, using Poincaré section surfaces we explore the structure and evolution of the phase space when control parameters vary.

2. Equations of motion

In the Hamiltonian (1), there is a singularity at \(r = 0\). It is, therefore, necessary to introduce a suitable coordinate transformation to remove it. To do this, we can use two canonical transformations, the first is

\[
\begin{align*}
\chi &= \rho \cos \theta \\
y &= \rho \sin \theta \\
z &= z
\end{align*}
\]

The Hamiltonian function takes the form

\[
H = \frac{1}{2} \left( \frac{p^2}{\rho^2} + \frac{p^2_\theta}{\rho^2} + P_z^2 \right) - \frac{1}{\sqrt{\rho^2 + \lambda^2 z^2}} + \alpha (\rho^2 + \lambda^2 z^2) + \gamma \rho^2 + \beta z
\]  

Where \(P_\rho, P_\theta\) and \(P_z\) are the canonical momenta conjugate to the coordinates \(\rho, \theta\) and \(z\) respectively.

In this equation, \(\theta\) is a cyclic variable, and the corresponding canonically conjugate momenta \(P_\theta\) is conserved, \(P_\theta = n\) with \(n\) a constant. Then, equation (4) can be rewritten as

\[
H = \frac{1}{2} \left( p^2_\rho + p^2_z \right) - \frac{1}{\sqrt{\rho^2 + z^2}} + \frac{n^2}{2 p^2} + \alpha (\rho^2 + \lambda^2 z^2) + \gamma \rho^2 + \beta z
\]  

(5)
after ignoring the cyclic integral associated to the cyclic coordinate \( \theta \), \( P_\theta = n = 0 \), the equation (5) takes the form

\[
H = \frac{1}{2} (p_\theta^2 + p_\varphi^2) - \frac{1}{\sqrt{\rho^2 + z^2}} + \alpha (\rho^2 + \lambda^2 z^2) + \gamma \rho^2 + \beta z
\]  

(6)

The second canonical transformation is:

\[
\begin{align*}
\rho &= -\frac{\beta}{2(\alpha + \gamma)} \sinh \mu \sin \nu \\
z &= -\frac{\beta}{2(\alpha + \gamma)} (1 + \cosh \mu \cos \nu)
\end{align*}
\]

\[
P_\rho = -\frac{2(\alpha + \gamma)}{\beta (\cosh(\mu))^2 - \cos(\nu)^2)} (\cosh \mu \sin \nu P_\theta + \sinh \mu \cos \nu P_\varphi)
\]

\[
P_z = -\frac{2(\alpha + \gamma)}{\beta (\cosh(\mu))^2 - \cos(\nu)^2)} (\sinh \mu \cos \nu P_\theta - \cosh \mu \sin \nu P_\varphi)
\]  

(7)

The Hamiltonian (6) has a collision singularity at \( r^2 = \rho^2 + z^2 = 0 \), which can be removed with the following change in the independent variable

\[
\tau = \frac{4(\alpha + \gamma)^2}{\beta^2} \int dt \frac{1}{\cosh(\mu(t))^2 - \cos(\nu(t))^2}
\]

After this change, the equation (6) takes the form

\[
\widetilde{H} = 0 = \frac{1}{2} (p_\mu^2 + p_\nu^2) - \frac{\beta}{2(\alpha + \gamma)} (\cosh \mu - \cos(\nu)) + \frac{\beta^4}{16(\alpha + \gamma)^3} (\cosh(\mu)^2 - \cos(\nu)^2) \times
\]

\[
\left[ \sinh(\mu)^2 \sin(\nu)^2 + \frac{\alpha \lambda^2}{\alpha + \gamma} (1 + \cosh \mu \cos(\nu)^2 - 2(1 + \cosh(\mu) \cos(\nu)) - \frac{4h(\alpha + \gamma)}{\beta^2} \right]
\]  

(8)

The equations of motion associated with \( \widetilde{H}(\xi, \mu) \) in the new time \( \tau \) are:

\[
\begin{align*}
P_\mu &= \frac{\beta \sinh \mu}{2(\alpha + \gamma)} (1 + \frac{\beta h}{\alpha + \gamma} \cosh \mu) - \frac{\alpha \lambda^2 \beta^4}{8(\alpha + \gamma)^4} \sinh \mu (1 + \cosh \mu \cos(\nu))(2 \cosh(\mu)^2 \cos(\nu) + \cosh(\mu) - \cos(\nu)^3) \\
&\quad - \frac{\beta^4}{8(\alpha + \gamma)^4} [\sinh \mu \cosh(\mu)^2 \sin(\nu)^2 - 2(1 + \cosh(\mu) \cos(\nu))]
\end{align*}
\]

\[
P_\nu = \frac{\beta \sin \nu}{2(\alpha + \gamma)} (1 + \frac{\beta h}{\alpha + \gamma} \cos(\nu)) - \frac{\alpha \lambda^2 \beta^4}{8(\alpha + \gamma)^4} \sin \nu (1 + \cosh \mu \cos(\nu))(2 \cosh \mu \cos(\nu)^2 + \cos(\nu) - \cosh(\mu)^3)
\]

\[
- \frac{\beta^4}{8(\alpha + \gamma)^4} [\cos \nu \sin(\nu)^2 \sinh(\mu)^2 \sin(\nu)^2 - 2(1 + \cosh(\mu) \cos(\nu))]
\]  

(9)

3. Topological analysis

The two-dimensional Hamiltonian (8) is separable if the relations (10) are verified

\[
[\alpha \lambda^2 / (\alpha + \gamma)] = 1 \iff \lambda = \pm (1 + \gamma / \alpha)^{1/2}
\]  

(10)

this case the second integral of motion reads

\[
F = f = -\frac{p_\mu^2}{2} + \frac{\beta^2 h}{4(\alpha + \gamma)^2} \cosh(\mu)^2 - \frac{\beta^4}{16(\alpha + \gamma)^3} (\cosh(\mu)^4 - 2 \cosh(\mu)^2)
\]

\[
+ \frac{\beta \cosh(\mu)}{2(\alpha + \gamma)}
\]  

(11)

\[
F = f = \frac{p_\nu^2}{2} + \frac{\beta^2 h}{4(\alpha + \gamma)^2} \cos(\nu)^2 - \frac{\beta^4}{16(\alpha + \gamma)^3} (\cos(\nu)^4 - 2 \cos(\nu)^2) + \frac{\beta \cos \nu}{2(\alpha + \gamma)}
\]  

(12)

It is easy to verify that Equation (7) becomes
\[
\begin{align*}
\rho &= -\frac{\beta}{2(\alpha + \gamma)} \sinh \mu \sin \nu \\
\chi &= -\frac{\beta}{2(\alpha + \gamma)} (1 + \cosh \mu \cos \nu)
\end{align*}
\]
\[P_\mu = \frac{\rho}{\beta [\cosh(\mu)^2 - \cos(\nu)^2]} \left( \cosh \mu \sin \nu \sqrt{Q_\mu} + \sinh \mu \cos \nu \sqrt{Q_2(\nu)} \right) \]
\[P_\nu = \frac{\rho}{\beta [\cosh(\mu)^2 - \cos(\nu)^2]} \left( \sinh \mu \cos \nu \sqrt{Q_\mu} - \cosh \mu \sin \nu \sqrt{Q_2(\nu)} \right) \]

where
\[
\begin{align*}
Q_1(\mu) &= P_\mu^2 = 2 \left( -f + \frac{\beta \cosh \mu}{2(\alpha + \gamma)} + \frac{\beta^2 h \cosh(\mu)^2}{4(\alpha + \gamma)^2} \right) \\
&\quad - \frac{\beta^4}{16(\alpha + \gamma)^3} (\cosh(\mu)^4 - 2 \cosh(\mu)^2)
\end{align*}
\]
\[
Q_2(\nu) = P_\nu^2 = 2 \left( f - \frac{\beta \cos \nu}{2(\alpha + \gamma)} - \frac{\beta^2 h}{4(\alpha + \gamma)^2} \cos(\nu)^2 + \frac{\beta^4}{16(\alpha + \gamma)^3} (\cos(\nu)^4 - 2 \cos(\nu)^2) \right)
\]

in these conditions, the equations of motion are
\[
\begin{align*}
\dot{\chi} &= \sqrt{Q_1(\mu)} \\
\dot{\eta} &= \sqrt{Q_2(\nu)} \\
\dot{P_\mu} &= \sinh \mu V(\cosh \mu) \\
\dot{P_\nu} &= \sin \nu V(\cos \nu)
\end{align*}
\]

Where
\[
V(\nu) = \frac{\beta}{2(\alpha + \gamma)} + \frac{\beta^2 h}{2(\alpha + \gamma)^2} q - \frac{\beta^4}{4(\alpha + \gamma)^3} q(q^2 - 1)
\]

In order to give a complete description of the topology of the phase space \( \mathcal{M}_R \), we take this transformation
\[
\begin{align*}
\zeta &= \cosh \mu \\
\eta &= \cos \nu \\
P_\zeta &= (\zeta^2 - 1) P_\zeta \\
P_\eta &= (1 - \eta^2) P_\eta
\end{align*}
\]

From equations (14)-(15) and (17), we find
\[
P_\zeta = \pm \frac{\sqrt{Q_1(\zeta)}}{(\zeta^2 - 1)} \text{ and } P_\eta = \pm \frac{\sqrt{Q_2(\eta)}}{(1 - \eta^2)}
\]

Where
\[
\begin{align*}
Q_1(\zeta) &= 2(\zeta^2 - 1) \left( -f + \frac{\beta}{2(\alpha + \gamma)} \zeta + \frac{\beta^2 h}{4(\alpha + \gamma)^2} \zeta^2 - \frac{\beta^4}{16(\alpha + \gamma)^3} (\zeta^4 - 2 \zeta^2) \right) \\
Q_2(\eta) &= 2(1 - \eta^2) \left( f - \frac{\beta}{2(\alpha + \gamma)} \eta - \frac{\beta^2 h}{4(\alpha + \gamma)^2} \eta^2 + \frac{\beta^4}{16(\alpha + \gamma)^3} (\eta^4 - 2 \eta^2) \right)
\end{align*}
\]

\( H=h \) and \( F=f \) are the first integrals of motion, functions of \( (\mu, \nu, P_\mu, P_\nu) \) which are constant along the solutions of Equation (16).

For the following calculations, we used \( G(\eta) \) instead of \( G(\zeta) \) and \( G(\mu) \) because having the same form.

### 3.1. Topology of Regular Level Set

In this section, we will give the admissible regions on the bifurcation diagrams as well as a detailed description of the topology of regular-level sets, that is, the topology of the real phase space:

\[
\mathcal{M}_R = \{ (\zeta, \eta, P_\zeta, P_\eta) \in \mathbb{R}^4 : H = h, F = f \} \subset \mathbb{R}^4
\]

the set of the critical values of the energy-momentum mapping

\( (\zeta, \eta, P_\zeta, P_\eta) \to (H, F) \)

**Definition.** The bifurcation diagram of an integrable system is defined to be the region of possible motion depicted on the plane of first integrals \( (h, f) \) [16]
It turns out (like in [17]-[19]), that the bifurcation diagram $B$ is exactly the discriminant locus of the polynomial $Q(u)$ whose coefficients are functions in $h, f$ and the parameters of the system $(\alpha, \beta, \gamma)$.

$$B = \{(h, f, \alpha, \beta, \gamma, \lambda) \in \mathbb{R}^6 : \text{discr}(Q(u)) = 0\}$$

all parameters are positive except $\alpha$ which can be positive or negative, that’s why we consider two cases $\alpha > 0$ and $\alpha < 0$.

For the case $\alpha > 0$

In order to plot the bifurcation diagram, the polynomial $Q(u)$ must have real roots, so it is necessary that constants $\alpha, \beta, \gamma$ and $\lambda$ verified the equation (10)

$$\left(h, f, \alpha > 0, \gamma, \beta, \frac{1}{\alpha}, \sqrt{1 + \frac{\beta}{\alpha}} \right) \in \mathbb{R}^4 : \text{discr}(Q(u)) = 0$$

The set $\mathcal{M} \setminus B \cap \{\alpha > 0\}$ consists of 8 connected components (as it is shown in Figure 1). Thus, in each connected component of the set $\mathcal{M} \setminus B$ the level set $\mathcal{M}$ has the same topological type, and this latter may be changed only if $(h, f)$ passes through $B \cap \{\alpha > 0\}$.

![Figure 1. Bifurcation diagram $B \cap \{\alpha = \text{const}\}$ for $\alpha > 0$](image.png)

**Theorem 1.** The set $\mathcal{M} \setminus B \cap \{\alpha > 0\}$ consists of 8 connected and nonintersecting with each other domains. The sections of these components with the plane $\{\alpha = \text{const}\}$ are shown on figure 1. The topological type of $\mathcal{M} \setminus B$ is a disjoint union of two-dimensional two-tori $2T^2$, two-dimensional tori $T^2$ and the empty set $\emptyset$ as it is shown in table 1.

**Table 1.** Real roots of the polynomial $Q(u),$ admissible ovals for $(h, f) \in \mathbb{R}^2 \setminus B$ and topological type of $\mathcal{M}.$

| Domain | Roots of $Q(u)$ | Projection of the admissible ovals on $z$-plane | Topological Type $\mathcal{M}$ |
|--------|----------------|-----------------------------------------------|-------------------------------|
| 1      | $-1 < 1$       | $[-1, 1]$                                      | $\emptyset$                   |
| 2      | $-1 < u_1 < u_2 < 1$ | $[-1, u_1] \cup [u_2, 1]$                     | $\emptyset$                   |
| 3      | $-1 < u_1 < 1 < u_2$ | $[u_1, 1]$                                     | $2T^2$                        |
| 4      | $u_1 < -1 < 1 < u_2$ | $[u_1, u_2]$                                   | $\emptyset$                   |
| 5      | $u_1 < -1 < u_2 < u_3 < 1 < u_4$ | $[u_1, u_2] \cup [1, u_4]$                   | $2T^2$                        |
| 6      | $u_1 < u_2 < -1 < u_3 < 1 < u_4$ | $[-1, u_1]$                                   | $2T^2$                        |
| 7      | $u_1 < u_2 < -1 < u_3 < u_4$ | $[u_1, 1] \cup [u_3, u_4]$                   | $2T^2$                        |
| 8      | $-1 < 1 < u_1 < u_2$ | $[-1, 1]$                                      | $T^2$                         |

**Proof.** Consider the complexified system

...
\[ \mathcal{M}_c = (\zeta, \eta, P_1, P_0) \in \mathbb{C}^4; H = h = cte, F = f = cte \]  

Consider also the hyperelliptic curves

\[ \Omega_1: \{\omega_1^2 = Q_1(\zeta)\} \quad \text{and} \quad \Omega_2: \{\omega_2^2 = Q_2(\eta)\} \]

and the corresponding Riemann surfaces \( R_1 \) and \( R_2 \) of the same genus \( j_1 = j_2 = 2 \). We obtain the explicit solutions of the initial problem (16) by solving the Jacobi inversion problem [20].

Thus, \( \rho, z, P, Pz \) can be expressed in terms of hyperelliptic functions living in the Jacobi variety \( \Omega = \Omega_1 \otimes \Omega_2 \) (where \( \otimes \) is the symmetric product). These functions however are not single valued as can be seen from formulae (13) and (16).

Indeed, to each point on the symmetric product \( \Omega_1 \otimes \Omega_2 \) there correspond two values of \( (\rho, z, P, Pz) \). Thus, we define the natural projection

\[ \mathcal{M}_c \rightarrow \Omega_1 \otimes \Omega_2 \]  

Corresponding to the involution

\[ i: (\rho, z, P, Pz) \rightarrow (\rho, z, -P, -Pz) \]

the real level sets \( \mathcal{M}_R = \Re(\mathcal{M}_c) \) is the set of fixed points of the complex conjugation on \( \mathcal{M}_c \):

\[ \chi: (\rho, z, P, Pz) \rightarrow (\rho, \bar{z}, \bar{P}, \bar{P}z) \]

Consider also the natural projection \( \psi \) on the Riemann surfaces \( R = R_1 \otimes R_2 \) given in \( \zeta, \eta \) coordinates by:

\[ \psi: (\zeta, \eta) \rightarrow (\bar{\zeta}, \bar{\eta}) \]

It induces an involution on the Jacobi variety and hence on \( \mathcal{M}_c \) by the natural projection \( \sigma \). Formulae (11) and (12) imply that this involution \( \psi \) coincides with the complex conjugation (22) on \( \mathcal{M}_c \) the upshot is that in order to describe \( \mathcal{M}_R \) it is enough to study the projection \( \sigma \) and the pair \( (R, \psi) \)

\[ \sigma: \mathcal{M}_c \rightarrow \text{Jac}(R) = \Omega_1 \otimes \Omega_2 \]

Remark. The pair \( (R, \psi) \) where \( R \) is a Riemann surface and \( \sigma \) is an involution on \( R \) is called Klein surface [21].

Definition. A connected component of the set of fixed points of \( \chi \) on the curve \( \Omega_1 \) and \( \Omega_2 \) is called an oval.

To determine the ovals of \( \Omega_1 \) and \( \Omega_2 \) it suffices to study the real roots of the polynomial \( Q(u) \) for different values of \( h, f \) and \( \alpha \). These roots are shown on Table 1. Using the formulae (7), (13), (18) and the condition \( (\rho, z, P, Pz) \in \mathbb{R}^4 \), we find exactly two admissible ovals whose projections on the \( \zeta \)-plane and \( \eta \)-plane are given by \( \Delta_1 \) and \( \Delta_2 \) (see Table 1). The product of the admissible ovals in \( \Omega_1 \otimes \Omega_2 \) and the projection \( \sigma \) of \( \mathcal{M}_R \) such as, \( \mathcal{M}_R = \sigma^{-1}(\Omega_1 \otimes \Omega_2) = \Delta_1 \times \Delta_2 \), gives:

- \( \mathcal{M}_R \) is a two-dimensional two-tori \( 2T^2 \) in domain 5, 6 and 7.
- \( \mathcal{M}_R \) is a two-dimensional tori \( T^2 \) in domain 3 and 8.
- \( \mathcal{M}_R \) is the empty set \( \Phi \) in domain 1, 2 and 4.

3.1.2. For the case \( \alpha < 0 \)

In this case, the bifurcation diagram is defined by (29)

\[ B' = \left\{ (h, f, \alpha < 0, |\alpha| > \gamma, \beta, \sqrt{1 + \frac{2}{\alpha}} \in \mathbb{R}^4; \text{discr}(Q(u)) = 0 \right\} \]

Theorem 2. The set \( \mathcal{M}_R |_{B'} \cap \{ \alpha < 0 \} \) consists of 9 connected and nonintersecting with each other domains. The sections of these components with the plane \( \{ \alpha = \text{const} \} \) are shown on Figure 2. The topological type of \( \mathcal{M}_R \) is a two-dimensional tori \( T^2 \), to a disjoint union of cylinders \( C \), or it is the empty set \( \Phi \) as it is shown in Table 2.

The product of the admissible ovals in \( \Omega_1 \otimes \Omega_2 \) and the projection \( \sigma \) of \( \mathcal{M}_R \) such as, \( \mathcal{M}_R = \sigma^{-1}(\Omega_1 \otimes \Omega_2) = \Delta_1 \times \Delta_2 \), gives:

- \( \mathcal{M}_R \) is a two-cylinders \( 2C \) in domain 2, 3 and 4.
- \( \mathcal{M}_R \) is a four-cylinders \( 4C \) in domain 5 and 9.
• $\mathcal{M}_\mathbb{R}$ is a two-dimensional tori + two cylinders $T^2 + 2C$ in domain 6.

$\mathcal{M}_\mathbb{R}$ is the empty set $\Phi$ in domain 1, 7 and 8.

![Figure 2. Bifurcation diagram $B' \cap \{\alpha = \text{const}\}$ for $\alpha < 0$](image)

### Table 2. Real roots of the polynomial $Q(u)$, admissible ovals for $(h, f) \in \mathbb{R}^2 \setminus B'$ and topological type of $\mathcal{M}_\mathbb{R}$

| Domain | Roots of $Q(u)$ | Projection of the admissible ovals on $z$-plane | Topological Type $\mathcal{M}_\mathbb{R}$ |
|--------|-----------------|-----------------------------------------------|----------------------------------------|
| 1      | $-1 < 1$        | $\emptyset$                                   | $\emptyset$                            |
| 2      | $-1 < u_1 < u_2 < 1$ | $[u_3, u_4]$                  | $\emptyset$                            |
| 3      | $-1 < u_1 < 1 < u_2$ | $[u_3, 1]$                     | $\emptyset$                            |
| 4      | $u_1 < -1 < 1 < u_2$ | $[-1, 1]$                       | $\emptyset$                            |
| 5      | $u_1 < -1 < u_2 < u_3 < 1 < u_4$ | $[-1, u_3] \cup [u_3, u_4]$ | $\emptyset$                            |
| 6      | $u_1 < u_2 < -1 < u_3 < 1 < u_4$ | $[u_3, 1]$                     | $\emptyset$                            |
| 7      | $u_1 < u_2 < -1 < 1 < u_3 < u_4$ | $\emptyset$                       | $\emptyset$                            |
| 8      | $-1 < u_1 < u_2$ | $\emptyset$                       | $\emptyset$                            |
| 9      | $-1 < u_1 < u_2 < u_3 < 1 < u_4$ | $[u_1, u_2] \cup [u_3, 1]$ | $\emptyset$                            |

3.2. Topology of Singular Level Sets

Suppose now that the constants $h, f$ are changed in such a way that $(h, f)$ passes through the bifurcation diagram $B$. Then the topological type of $\mathcal{M}_\mathbb{R}$ may change and the bifurcation of Liouville tori takes place.

In this section, we will give the description of all generic bifurcations of the topological types of $\mathcal{M}_\mathbb{R}$ using the Fomenko bifurcation surgery on Liouville Tori [22].

In the first case $\alpha > 0$, we can have three types of bifurcation of the level set $\mathcal{M}_\mathbb{R}$ passing from a domain $i$ to domain $j$ (see table 3). To prove that, it suffices to look at the bifurcations of the roots of the polynomial and bifurcations of invariant Liouville tori is shown in figure 3.

### Table 3. Generic bifurcation of the level set $\mathcal{M}_\mathbb{R}$ passing from domain $i$ to domain $j$

| $i$  | $j$  | $T^2 \rightarrow \Phi$ | $2T^2 \rightarrow \Phi$ | $2T^2 \rightarrow T$ |
|------|------|------------------------|--------------------------|----------------------|
| 3    | 1    | 5 → 3                  | 6 → 3                    | 6 → 8                |
| 3    | 2    | 5 → 4                  | 7 → 3                    | 7 → 8                |
| 3    | 4    | 5 → 4                  | 7 → 8                    | 6 → 8                |
| 8    | 1    | 5 → 4                  | 6 → 8                    |                      |

We have three types of bifurcation:
Bifurcation $2T^2 \rightarrow S \times (S \land S) \rightarrow T^2$: The two-dimensional two-tori $2T^2$ merge into two dimensional tori $T^2$ by passing through the complex $S \times (S \land S)$ where $(S \land S)$ is a union of two circles having exactly one common point.

Bifurcation $2T^2 \rightarrow 2S \rightarrow \Phi$: The two-dimensional two-tori $2T^2$ are contracted to two circles corresponding to two periodic solutions, and then vanishes.

Bifurcation $T^2 \rightarrow S \rightarrow \Phi$: The two-dimensional tori $T^2$ shrinks to a circle corresponding to the periodic solution, and vanishes.

Figure 3. Correspondence between bifurcation of roots of the polynomial $Q(u)$ for $(h, f) \in B$ and bifurcations of invariant Liouville tori, where $(S \land S)$ is a union of two circles having exactly one common point.

For the second case $\alpha < 0$, the Fomenko surgery of bifurcation on Liouville tori [22] cannot be applied as its invariant level sets contain a non-compact component (cylinder $C$). we can have the type of bifurcation of the level set $\mathcal{M}_R$ passing from domain $i$ to domain $j$ (see table 4.). To prove that, it suffices to look at the bifurcations of roots of the polynomial $Q(u)$ as shown in figure 4.

Table 4. Generic bifurcation of the level set $\mathcal{M}_R$ passing from domain $i$ to domain $j$

| $i$ | $j$ | $i$ | $j$ | $i$ | $j$ | $i$ | $j$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 6   | 7   | 6   | 5   | 6   | 3   | 9   | 3   |
| 6   | 8   | 6   | 5   | 6   | 3   | 9   | 4   |
| $T^2 + 2C \rightarrow \Phi$ | $T^2 + 2C \rightarrow 4C$ | $T^2 + 2C \rightarrow 2C$ | $4C \rightarrow 2C$ | $2C \rightarrow \Phi$ | $2 \rightarrow 1$ | $3 \rightarrow 1$ | $5 \rightarrow 4$ | $5 \rightarrow 3$ | $4 \rightarrow 2C$ | $2C \rightarrow \Phi$ |
4. Numerical illustration

By using a set of software routines, implemented in Maple, for plotting 2D and 3D projections of Poincare surfaces of section map, this map is constructed using a clever method introduced by Poincare and extended by Henon [5]. We give a numerical analysis of the topological analysis studied in section 3 and show the integrable behavior of the system. For fixed values of constants $h, f, \beta, \lambda$ and $\alpha, \gamma$ vary, the Liouville tori contained in the level set $\{H=h, F=f\}$ change their topological type. However, with the change of one of the parameters of the system (control parameter), we observe the random dispersion of the points in the sections which show that the system has changed their behavior from the regularity to quasi-regularity and to chaotic behavior. For $\lambda:\alpha<0$, Figures 5 (a, b, c, red) respectively correspond to domain 5, 6, 7 show the existence of two tori, as well as in the integrable case $\lambda=(1+\gamma/\alpha)^{1/2}$ these figures show that the structure of the PSS is very regular. If we vary a little the condition where the system is integrable that is to say for $\lambda=(1+\gamma/1.5\alpha)^{1/2}$, an irregular movement appears, as is clearly visible in Figures (a, b, c, blue). However, with the condition $\lambda=(1+\gamma/3\alpha)^{1/2}$ that is just far from the integrable behavior, a randomly distributed set of points occupying a large part of the volume of the phase space, it indicates that classical movement is dominated by the chaotic behavior of the PSS is shown in Figures 5 (a, b, c, green). Regarding Figures 5 (d, e), the variety corresponds to a torus, we also observe the order-chaos transition when the integrability condition was changed $\lambda=(1+\gamma/\alpha)^{1/2}$, $\lambda=(1+\gamma/1.5\alpha)^{1/2}$, $\lambda=(1+\gamma/3\alpha)^{1/2}$.

In Figure 6, which corresponds to $\alpha>0$, we confirm that the variety in domain 6 corresponds to a torus, and we observe that the PSS is regular for $\lambda=(1+\gamma/\alpha)^{1/2}$ and for $\lambda=(1+\gamma/1.5\alpha)^{1/2}$ the chaotic regions increase in size (Figure 6 (f-blue)). Finally, for $\lambda=(1+\gamma/3\alpha)^{1/2}$ no regular structure is visible on the PSS (Figure 6 (f-green)).
5. Conclusion

In this study, we have dealt with the problem of the hydrogen atom disturbed by the Van der Waals potential in interaction with combined electric and magnetic fields (the Stark-Zeeman effect). We began by establishing canonical transformations, thus allowing the simplification of the Hamiltonian and the separation of the Jacobian equation, this equation is reduced to the so-called Jacobi form, from which we have based the algebraic structure of the system.

The most important studies, we have done are: topological analysis of the real invariant varieties $\mathcal{M}_\mathbb{R} = \{ (h, f) \neq \mathbb{R}^2 \setminus B, \alpha > 0 \}$ domain $5 \mathcal{M}_\mathbb{R} \approx 2T^2$, (b) domain $6 \mathcal{M}_\mathbb{R} \approx 2T^2$, (c) domain $7 \mathcal{M}_\mathbb{R} \approx 2T^2$, (d) domain $8 \mathcal{M}_\mathbb{R} \approx T^2$, (e) domain $3 \mathcal{M}_\mathbb{R} \approx T^2$; for $\lambda = (1 + \gamma / \alpha)^{1/2}$ (rouge), $\lambda = (1 + \gamma / 1.5 \alpha)^{1/2}$ (bleu), $\lambda = (1 + \gamma / 3 \alpha)^{1/2}$ (vert).

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