ON THE LIE ALGEBRA STRUCTURE OF $HH^1(A)$ OF A
FINITE-DIMENSIONAL ALGEBRA $A$

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Abstract. Let $A$ be a split finite-dimensional associative unital algebra over a field. The first main result of this note shows that if the Ext-quiver of $A$ is a simple directed graph, then $HH^1(A)$ is a solvable Lie algebra. The second main result shows that if the Ext-quiver of $A$ has no loops and at most two parallel arrows in any direction, and if $HH^1(A)$ is a simple Lie algebra, then $\text{char}(k) \neq 2$ and $HH^1(A) \cong \mathfrak{sl}_2(k)$. The third result investigates symmetric algebras with a quiver which has a vertex with a single loop.

1. Introduction

Let $k$ be a field. Our first result is a sufficient criterion for $HH^1(A)$ to be a solvable Lie algebra, where $A$ is a split finite-dimensional $k$-algebra (where the term ‘algebra’ without any further specifications means an associative and unital algebra).

**Theorem 1.1.** Let $A$ be a split finite-dimensional $k$-algebra. Suppose that the Ext-quiver of $A$ is a simple directed graph. Then the derived Lie subalgebra of $HH^1(A)$ is nilpotent; in particular the Lie algebra $HH^1(A)$ is solvable.

The recent papers [4] and [8] contain comprehensive results regarding the solvability of $HH^1(A)$ of tame algebras and blocks, and [8] also contains a proof of Theorem 1.1 with different methods. We will prove Theorem 1.1 in Section 3 as part of the more precise Theorem 3.1 bounding the derived length of the Lie algebra $HH^1(A)$ and the nilpotency class of the derived Lie subalgebra of $HH^1(A)$ in terms of the Loewy length $\ell(A)$ of $A$. The hypothesis on the quiver of $A$ is equivalent to requiring that $\text{Ext}_A^1(S,S) = \{0\}$ for any simple $A$-module $S$ and $\dim_k(\text{Ext}_A^1(S,T)) \leq 1$ for any two simple $A$-modules $S$, $T$. If in addition $A$ is monomial, then Theorem 1.1 follows from work of Strametz [10]. The hypotheses on $A$ are not necessary for the derived Lie subalgebra of $HH^1(A)$ to be nilpotent or for $HH^1(A)$ to be solvable; see [2, Theorem 1.1] or [8] for examples.

The Lie algebra structure of $HH^1(A)$ is invariant under derived equivalences, and for symmetric algebras, also invariant under stable equivalences of Morita type. Therefore, the conclusions of Theorem 1.1 remain true for any finite-dimensional $k$-algebra $B$ which is derived equivalent to an algebra $A$ satisfying the hypotheses of this theorem, or for a symmetric $k$-algebra $B$ which is stably equivalent of Morita type to a symmetric algebra $A$ satisfying the hypotheses of the theorem.

If we allow up to two parallel arrows in the same direction in the quiver of $A$ but no loops, then it is possible for $HH^1(A)$ to be simple as a Lie algebra. The only simple Lie algebra to arise in that case is $\mathfrak{sl}_2(k)$, with $\text{char}(k) \neq 2$.

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Theorem 1.2. Let $A$ be a split finite-dimensional $k$-algebra. Suppose that $\text{Ext}^1_A(S, S) = \{0\}$ for any simple $A$-module $S$ and that $\dim_k(\text{Ext}^1_A(S, T)) \leq 2$ for any two simple $A$-modules $S, T$. If $HH^1(A)$ is not solvable, then $\text{char}(k) \neq 2$ and $HH^1(A) / \text{rad}(HH^1(A))$ is a direct product of finitely many copies of $\mathfrak{sl}_2(k)$. In particular, the following hold.

(i) If $HH^1(A)$ is a simple Lie algebra, then $\text{char}(k) \neq 2$, and $HH^1(A) \cong \mathfrak{sl}_2(k)$.

(ii) If $\text{char}(k) = 2$, then $HH^1(A)$ is a solvable Lie algebra.

This will be proved in Section 3 for monomial algebras this follows as before from Strametz [10]. An example of an algebra $A$ satisfying the hypotheses of this theorem is the Kronecker algebra, a 4-dimensional $k$-algebra, with $\text{char}(k) \neq 2$, given by the directed quiver with two vertices $e_0, e_1$ and two parallel arrows $\alpha, \beta$ from $e_0$ to $e_1$. This example is a special case of more general results on monomial algebras; see in particular [10] Corollary 4.17. As in the case of the previous Theorem, the conclusions of Theorem 1.2 remain true for an algebra $B$ which is derived equivalent to an algebra $A$ satisfying the hypotheses of this theorem, or for a symmetric algebra $B$ which is stably equivalent of Morita type to a symmetric algebra $A$ satisfying the hypotheses of the theorem.

We have the following partial result for symmetric algebras whose quiver has a single loop at some vertex.

Theorem 1.3. Suppose that $k$ is algebraically closed. Let $A$ be a finite-dimensional symmetric $k$-algebra, and let $S$ be a simple $A$-module. Suppose that $\dim_k(\text{Ext}^1_A(S, S)) = 1$ and that for any primitive idempotent $i$ in $A$ satisfying $iS \neq \{0\}$ we have $J(iAi)^2 = iJ(A)^2i$. If $HH^1(A)$ is a simple Lie algebra, then $\text{char}(k) = p > 2$ and $HH^1(A)$ is isomorphic to either $\mathfrak{sl}_2(k)$ or the Witt Lie algebra $W = \text{Der}(k[x]/(x^p))$.

This will be proved in Section 4, along with some general observations regarding the compatibility of Schur functors and the Lie algebra structure of $HH^1(A)$. Section 5 contains some examples.

2. ON DERIVATIONS AND THE RADICAL

We start with a brief review of some basic terminology. Let $k$ be a field. The nilpotency class of a nilpotent Lie algebra $\mathcal{L}$ is the smallest positive integer $m$ such that $\mathcal{L}^m = \{0\}$, where $\mathcal{L}^1 = \mathcal{L}'$ and $\mathcal{L}^{m+1} = [\mathcal{L}, \mathcal{L}^m]$ for $m \geq 1$. In addition, the derived length of a solvable Lie algebra is the smallest positive integer $n$ such that $\mathcal{L}^{(n)} = \{0\}$, where $\mathcal{L}^{(1)} = \mathcal{L}'$ and $\mathcal{L}^{(n+1)} = [\mathcal{L}^{(n)}, \mathcal{L}^{(n)}]$ for $n \geq 1$. A Lie algebra $\mathcal{L}$ is called strongly solvable if its derived subalgebra is nilpotent. A Lie algebra $\mathcal{L}$ of finite dimension $n$ is called completely solvable (also called supersolvable) if there exists a sequence of ideals $\mathcal{L}_1 = \mathcal{L} \supset \mathcal{L}_2 \supset \cdots \supset \mathcal{L}_n \supset \{0\}$ such that $\dim_k(\mathcal{L}_i) = n + 1 - i$ for $1 \leq i \leq n$.

Remark 2.1. If $k$ is algebraically closed of characteristic zero, then the classes of strongly and completely solvable Lie algebras coincide with the class of solvable Lie algebras as a consequence of Lie’s theorem. Lie’s theorem does not hold in positive characteristic. If $k$ is algebraically closed of prime characteristic $p$, then by [3] Theorem 3, a finite-dimensional Lie algebra $\mathcal{L}$ over $k$ is strongly solvable if and only if $\mathcal{L}$ is completely solvable.

Let $A$ be a finite-dimensional $k$-algebra. We denote by $\ell(A)$ the number of isomorphism classes of simple $A$-modules. The Loewy length $\ell(A)$ of $A$ is the smallest positive integer $m$ such that $J(A)^m = \{0\}$, where $J(A)$ denotes the Jacobson radical of $A$. We denote by $[A, A]$ the $k$-subspace of $A$ generated by the set of additive commutators $ab - ba$, where $a, b \in A$. A derivation on $A$ is a $k$-linear map $f : A \to A$ satisfying $f(ab) = f(a)b + af(b)$ for all $a, b \in A$. If $f, g$ are derivations
on $A$, then so is $[f, g] = f \circ g - g \circ f$, and the space $\text{Der}(A)$ of derivations on $A$ becomes a Lie algebra in this way. If $c \in A$, then the map $[c, -]$ defined by $[c, a] = ca - ac$ is a derivation; any derivation of this form is called an inner derivation. The space $\text{IDer}(A)$ of inner derivations is a Lie ideal in $\text{Der}(A)$, and we have a canonical isomorphism $HH^1(A) \cong \text{Der}(A)/\text{IDer}(A)$; see [12] Lemma 9.2.1. It is easy to see that any derivation on $A$ preserves the subspace $[A, A]$, and that any inner derivation of $A$ preserves any ideal in $A$. A finite-dimensional $k$-algebra $A$ is called split if $\text{End}_A(S) \cong k$ for every simple $A$-module $S$. If $A$ is split, then by the Wedderburn-Malcev Theorem, $A$ has a separable subalgebra $E$ such that $A = E \oplus J(A)$. Moreover, $E$ is unique up to conjugation by elements in the group $A^\times$ of invertible elements in $A$. A primitive decomposition $I$ of $1$ in $E$ remains a primitive decomposition of $1$ in $A$.

For convenience, we mention the following well-known descriptions of certain $\text{Ext}^1$-spaces.

**Lemma 2.2.** Let $A$ be a split finite-dimensional $k$-algebra, let $i$ be a primitive idempotent in $A$. Set $S = Ai/J(A)i$ and $S^\vee = iA/iJ(A)$. We have $k$-linear isomorphisms

$$HH^1(A; S \otimes_k S^\vee) \cong \text{Ext}^1_A(S, S) \cong \text{Hom}_A(J(A)i/J(A)^2i, S) \cong \text{Hom}_{A \otimes_k A \rightarrow}(J(A)/J(A)^2, S \otimes_k S^\vee).$$

**Lemma 2.3.** Let $A$ be a split finite-dimensional $k$-algebra. Let $i$ be a primitive idempotent in $A$, and set $S = Ai/J(A)i$. We have $\text{Ext}^1_A(S, S) = \{0\}$ if and only if $iJ(A)i \subseteq J(A)^2$.

**Proof.** By Lemma [2.2] we have $\text{Ext}^1_A(S, S) = \{0\}$ if and only if $J(A)/J(A)^2$ has no simple bimodule summand isomorphic to $S \otimes_k S^\vee$. This is equivalent to $i(J(A)/J(A)^2) = \{0\}$, hence to $iJ(A)i \subseteq J(A)^2$ as stated.

**Lemma 2.4.** Let $A$ be a split finite-dimensional $k$-algebra, and let $E$ be a separable subalgebra of $A$ such that $A = E \oplus J(A)$. Every class in $HH^1(A)$ has a representative $f \in \text{Der}(A)$ satisfying $E \subseteq \ker(f)$.

**Proof.** Let $f : A \rightarrow A$ be a derivation. Since $E$ is separable, it follows that for any $E$-$E$-bimodule $M$ we have $HH^1(E; M) = \{0\}$. In particular, the derivation $f|_E : E \rightarrow E$ is inner; that is, there is an element $c \in A$ such that $f(x) = [c, x]$ for all $x \in E$. Thus the derivation $f - [c, -]$ on $A$ vanishes on $E$ and represents the same class as $f$ in $HH^1(A)$.

**Lemma 2.5.** Let $A$ be a split finite-dimensional $k$-algebra, and let $E$ be a separable subalgebra of $A$ such that $A = E \oplus J(A)$. Let $f : A \rightarrow A$ be a derivation such that $E \subseteq \ker(f)$. For any two idempotents $i, j$ in $E$ we have $f(iAj) \subseteq iAj$ and $f(AiAj) \subseteq AiAj$.

**Proof.** Let $i, j$ be idempotents in $E$, and let $a, b \in A$. We have $f(iaj) = f(i^2aj) = if(iaj) + f(i)iaj = if(iaj)$, since $i \in E \subseteq \ker(f)$. Thus $f(iaj) \in iA$. A similar argument shows that $f(iaj) \in Aj$, and hence $f(iaj) \in iAj$. This shows the first statement. The second statement follows from this and the equality $f(biaj) = f(b)iaj + bf(iaj)$.

**Lemma 2.6.** Let $A$ be a split finite-dimensional $k$-algebra such that $\text{Ext}^1_A(S, S) = \{0\}$ for all simple $A$-modules $S$. Then for any derivation $f : A \rightarrow A$ we have $f(J(A)) \subseteq J(A)$.

**Proof.** Let $E$ be a separable subalgebra of $A$ such that $A = E \oplus J(A)$. Let $I$ be a primitive decomposition of $1$ in $E$ (hence also in $A$). Note that if $i, j \in I$ are not conjugate in $A^\times$, then $iAj \subseteq J(A)$. The hypotheses on $A$ imply that $J(A)i/J(A)^2i$ has no summand isomorphic to $Ai/J(A)i$, and hence that $iJ(A)i \subseteq J(A)^2$ for any $i \in I$. Then $iJ(A)i \subseteq J(A)^2$ for any two $i, j \in I$ which are conjugate in $A^\times$. Let now $f : A \rightarrow A$ be a derivation. As noted above,
any inner derivation preserves \( J(A) \). Thus, by Lemma 2.4 we may assume that \( f|_E = 0 \). Since \( J(A) = \oplus_{i \in I} J(A)_i \), it suffices to show that \( f(J(A)) \subseteq J(A)_i \), where \( i \in I \). If \( j \) is conjugate to \( i \), then \( AjJ(A)_i \subseteq J(A)_i \). Since \( J(A)_i = \sum_{j \in I} AjJ(A)_i \), it follows from Nakayama’s Lemma that \( J(A)_i = \sum_j AjAi \), where \( j \) runs over the subset \( I' \) of all \( j \) in \( I \) which are not conjugate to \( i \). Now \( f \) preserves the submodules \( AjAi \) in this sum, thanks to Lemma 2.3. The result follows. \( \square \)

The following observations are variations of the statements in [6, Proposition 3.5].

**Proposition 2.7.** Let \( A \) be a split finite-dimensional \( k \)-algebra, and let \( E \) be a separable subalgebra of \( A \) such that \( A = E \oplus J(A) \). For \( m \geq 1 \), denote by \( D_m \) the subspace of \( \text{Der}(A) \) consisting of all derivations \( f : A \to A \) such that \( E \subseteq \ker(f) \) and such that \( f(J(A)) \subseteq J(A)^m \). The following hold.

(i) For any positive integers \( m, n \) we have \( [D_m, D_n] \subseteq D_{m+n-1} \).

(ii) The space \( D_1 \) is a Lie subalgebra of \( \text{Der}(A) \), and for any positive integer \( m \), the space \( D_m \)

is a Lie ideal in \( D_1 \).

(iii) The space \( D_2 \) is a nilpotent ideal in \( D_1 \). More precisely, if \( \theta(A) \leq 2 \), then \( D_2 = \{0\} \), and if \( \theta(A) > 2 \), then the nilpotency class of \( D_2 \) is at most \( \theta(A) - 2 \).

**Proof.** The space of derivations on \( A \) which vanish on \( E \) is easily seen to be closed under the Lie bracket on \( \text{Der}(A) \). Thus statement (i) follows from [6, Lemma 3.4]. Statement (ii) is an immediate consequence of (i). If \( m \geq \theta(A) \), then \( J(A)^m = \{0\} \), and hence \( D_m = \{0\} \). Together with (i), this implies (iii). \( \square \)

**Proposition 2.8.** Let \( A \) be a split finite-dimensional \( k \)-algebra, and let \( E \) be a separable subalgebra of \( A \) such that \( A = E \oplus J(A) \). For \( m \geq 1 \), denote by \( D_m \) the subspace of \( \text{Der}(A) \) consisting of all derivations \( f : A \to A \) such that \( E \subseteq \ker(f) \) and such that \( f(J(A)) \subseteq J(A)^m \). Suppose that every derivation \( f \) on \( A \) satisfies \( f(J(A)) \subseteq J(A) \). Then the canonical algebra homomorphism \( A \to A/J(A)^2 \) induces a Lie algebra homomorphism \( \Phi : \text{HH}^1(A) \to \text{HH}^1(A/J(A)^2) \), and the following hold.

(i) The canonical surjection \( \text{Der}(A) \to \text{HH}^1(A) \) maps \( D_1 \) onto \( \text{HH}^1(A) \).

(ii) The canonical surjection \( \text{Der}(A) \to \text{HH}^1(A) \) maps \( D_2 \) onto \( \ker(\Phi) \); in particular, \( \ker(\Phi) \)

is a nilpotent ideal in the Lie algebra \( \text{HH}^1(A) \).

(iii) The Lie algebra \( \text{HH}^1(A) \) is solvable if and only if \( \text{HH}^1(A)/\ker(\Phi) \) is solvable.

(iv) If the derived Lie algebra of \( \text{HH}^1(A) \) is contained in \( \ker(\Phi) \), then \( \text{HH}^1(A) \) is nilpotent.

(v) If the Lie algebra \( \text{HH}^1(A) \) is simple, then \( \Phi \) is injective.

**Proof.** The hypotheses on \( \text{Der}(A) \) together with Lemma 2.3 imply that \( \text{HH}^1(A) \) is equal to the image of the space \( D_1 \) in \( \text{HH}^1(A) \), whence (i). The canonical surjection \( \text{Der}(A) \to \text{HH}^1(A) \) clearly maps \( D_2 \) to \( \ker(\Phi) \); we need to show the surjectivity of the induced map \( D_2 \to \ker(\Phi) \). Note first that any inner derivation in \( D_1 \) is of the form \([c, -] \) for some \( c \) which centralises \( E \). Note further that the centraliser \( C_A(E) \) of \( E \) in \( A \) is canonically isomorphic to \( \text{Hom}_{E \otimes_k E \text{-op}}(E, A) \) (via the map sending an \( E-E \)-bimodule homomorphism \( \alpha : E \to A \) to \( \alpha(1) \)). Since \( E \) is separable, hence projective as an \( E-E \)-bimodule, it follows that the functor \( \text{Hom}_{E \otimes_k E \text{-op}}(E, -) \) is exact. In particular, the surjection \( A \to A/J(A)^2 \) induces a surjection \( C_A(E) \to C_{A/J(A)^2}(E) \), where we identify \( E \) with its image in \( A/J(A)^2 \). Let \( f \in D_1 \) such that the class of \( f \) is in \( \ker(\Phi) \), or equivalently, such that the induced derivation, denoted \( \bar{f} \), on \( A/J(A)^2 \) is inner. Then there is \( c \in A \) such that \( \bar{f} = [\bar{c}, -] \), where \( \bar{c} = c + J(A)^2 \) centralises the image of \( E \) in \( A/J(A)^2 \). By the above, we may choose \( c \) such that \( c \) centralises \( E \) in \( A \). Then the derivation \( f - [c, -] \) represents the same
class as \( f \), still belongs to \( D_1 \), and induces the zero map on \( A/J(A)^2 \). Thus \( f - [c, -] \) belongs in fact to \( D_2 \), proving (ii). The remaining statements are immediate consequences of (ii).

The next result includes the special case of Theorem 1.1 where \( \delta(A) \leq 2 \).

**Proposition 2.9.** Let \( A \) be a split finite-dimensional \( k \)-algebra such that \( J(A)^2 = \{0\} \). Suppose that for every simple \( A \)-module \( S \) we have \( \text{Ext}^1_A(S, S) = \{0\} \) and that for any two simple \( A \)-modules \( S, T \) we have \( \dim_k(\text{Ext}^1_A(S, T)) \leq 1 \). Let \( E \) be a separable subalgebra of \( A \) such that \( A = E \oplus J(A) \). The following hold:

(i) If \( A \) is basic and if \( f, g \) are derivations on \( A \) which vanish on \( E \), then \([f, g] = 0\).

(ii) The Lie algebra \( HH^1(A) \) is abelian.

(iii) Let \( e(A) \) be the number of edges in the quiver of \( A \). We have

\[
\dim_k(HH^1(A)) = e(A) - \ell(A) + 1 \leq (\ell(A) - 1)^2.
\]

**Proof.** In order to prove (i), suppose that \( A \) is basic. Let \( I \) be a primitive decomposition of \( 1 \) in \( A \) such that \( E = \prod_{i \in I} k_i \). Let \( f \) and \( g \) be derivations on \( A \) which vanish on \( E \). Then \( f, g \) are determined by their restrictions to \( J(A) \). By Lemma 2.6, the derivations \( f, g \) preserve \( J(A) \).

By the assumptions, each summand \( iAj \) in the vector space decomposition \( A = \bigoplus_{i,j \in I} iAj \) has dimension at most one. By Lemma 2.3, any derivation on \( A \) which vanishes on \( E \) preserves this decomposition. Therefore, if \( X \) is a basis of \( J(A) \) consisting of elements of the subspaces \( iAj \), \( i, j \in I \), which are nonzero, then \( f|_{J(A)} : J(A) \to J(A) \) is represented by a diagonal matrix. Similarly for \( g \). But then the restrictions of \( f \) and \( g \) to \( J(A) \) commute. Since both \( f, g \) vanish on \( E \), this implies that \([f, g] = 0\), whence (i). If \( A \) is basic, then clearly (i) and Lemma 2.3 together imply (ii).

Since the hypotheses of the Lemma as well as the Lie algebra \( HH^1(A) \) are invariant under Morita equivalences, statement (ii) follows for general \( A \). In order to prove (iii), assume again that \( A \) is basic. By the assumptions, \( e(A) = \dim_k(J(A)) = |X| \). One verifies that the extension to \( A \) by zero on \( I \) of any linear map on \( J(A) \) which preserves the summands \( iAj \) (with \( i \neq j \) or equivalently, which preserves the one-dimensional spaces \( kx \), where \( x \in X \), is in fact a derivation. By Lemma 2.3, any class in \( HH^1(A) \) is represented by such a derivation. Thus the space of derivations on \( A \) which vanish on \( I \) is equal to \( \dim_k(J(A)) = e(A) \). Each \( i \in I \) contributes an inner derivation. The only \( k \)-linear combination of elements in \( I \) which belongs to \( Z(A) \) are the multiples of \( I = \sum_{i \in I} i \). Thus the space of inner derivations which annihilate \( I \) has dimension \( \ell(A) - 1 \), whence the first equality. Since there are at most \( \ell(A) - 1 \) arrows starting at any given vertex, it follows that \( e(A) \leq (\ell(A) - 1)\ell(A) \), whence the inequality as stated.

The above Proposition can also be proved as a consequence of more general work of Strametz [10], calculating the Lie algebra \( HH^1(A) \) for \( A \) a split finite-dimensional monomial algebra.

3. Proofs of Theorems 1.1 and 1.2

Theorem 1.1 is a part of the following slightly more precise result. Let \( k \) be a field.

**Theorem 3.1.** Let \( A \) be a split finite-dimensional \( k \)-algebra. Suppose that for every simple \( A \)-module \( S \) we have \( \text{Ext}^1_A(S, S) = \{0\} \) and that for any two simple \( A \)-modules \( S, T \) we have \( \dim_k(\text{Ext}^1_A(S, T)) \leq 1 \). Set \( L = HH^1(A) \), regarded as a Lie algebra.

(i) If \( \delta(A) \leq 2 \) then \( L \) is abelian.

(ii) If \( \theta(A) > 2 \), then the derived Lie algebra \( L' = [L, L] \) is nilpotent of nilpotency class at most \( \theta(A) - 2 \). The derived length of \( L \) is at most \( \log_2(\theta(A) - 1) + 1 \).

In particular, \( L \) is solvable, and if \( k \) is algebraically closed, then \( L \) is completely solvable.

Proof. If \( \theta(A) \leq 2 \), then \( J(A)^2 = \{0\} \), and hence (i) follows from Proposition 2.9 Suppose that \( \theta(A) > 2 \). We may assume that \( A \) is basic. Note that \( A \) and \( A/J(A)^2 \) have the same Ext-quiver, and hence we may apply Proposition 2.9 to the algebra \( A/J(A)^2 \); in particular, \( HH^1(A/J(A)^2) \) is abelian. Thus the kernel of the canonical Lie algebra homomorphism \( L = HH^1(A) \to HH^1(A/J(A)^2) \) contains \( L' \). Proposition 2.8 implies that \( L' \) is contained in the image of \( D_2 \), hence nilpotent of nilpotency class at most \( \theta(A) - 2 \) by Proposition 2.7. From the same proposition we have that if \( f \in L^{(n)} \), then \( f(J(A)) \subseteq J(A)^{2n-1+1} \) for \( n \geq 1 \). Therefore the derived length is at most \( \log_2(\theta(A) - 1) + 1 \). Since \( L' \) is nilpotent, it follows that if \( k \) is algebraically closed, then \( L \) is completely solvable. \( \square \)

Proof of Theorem 1.3. By Lemma 2.6 every derivation \( f : A \to A \) preserves \( J(A) \), and hence sends \( J(A)^2 \) to \( J(A)^2 \). Thus the canonical map \( A \to A/J(A)^2 \) induces a Lie algebra homomorphism \( \varphi : \text{Der}(A) \to \text{Der}(A/J(A)^2) \) which in turn induces a Lie algebra homomorphism \( \Phi : HH^1(A) \to HH^1(A/J(A)^2) \). By Proposition 2.8 \( \ker(\Phi) \) is a nilpotent ideal. If \( \text{char}(k) = 2 \), then \( HH^1(A/J(A)^2) \) is solvable by [10, Corollary 4.12], and hence \( HH^1(A) \) is solvable. Suppose now that \( HH^1(A) \) is not solvable. Then, by the above, we have \( \text{char}(k) \neq 2 \). Then, by [10, Corollary 4.11, Remark 4.16], the Lie algebra \( HH^1(A/J(A)^2) \) is a finite direct product of copies of \( \mathfrak{sl}_2(k) \). Thus \( HH^1(A)/\ker(\Phi) \) is a subalgebra of a finite direct product of copies of \( \mathfrak{sl}_2(k) \), and hence \( HH^1(A)/\text{rad}(HH^1(A)) \) is a subquotient of a finite direct product of copies of \( \mathfrak{sl}_2(k) \). Since any proper Lie subalgebra of \( \mathfrak{sl}_2(k) \) is solvable, it follows easily that the semisimple Lie algebra \( HH^1(A)/\text{rad}(HH^1(A)) \) is a finite direct product of copies of \( \mathfrak{sl}_2(k) \). \( \square \)

4. Schur functors and proof of Theorem 1.3

The hypothesis \( J(iAi)^2 = iJ(A)^2i \) in the statement of Theorem 1.3 means that for any primitive idempotent \( j \) not conjugate to \( i \) in \( A \) we have \( iA_jAi \subseteq J(iAi)^2 \); that is, the image in \( iAi \) of any path parallel to the loop at \( i \) which is different from that loop is contained in \( J(iAi)^2 \) We start by collecting some elementary observations which will be used in the proof of Theorem 1.3. Let \( k \) be a field.

Lemma 4.1. Let \( A \) be a \( k \)-algebra and \( e \) an idempotent in \( A \). Let \( f : A \to A \) be a derivation. The following hold.

(i) We have \( f(AeA) \subseteq AeA \).
(ii) We have \( ef(e)e = 0 \).
(iii) We have \( (1 - e)f(e)(1 - e) = 0 \).
(iv) We have \( f(e) \in eA(1 - e) \oplus (1 - e)Ae \).
(v) We have \( f(e) = [[f(e), e], e] \); equivalently, the derivation \( f - [[f(e), e], -] \) vanishes at \( e \).
(vi) If \( f(e) = 0 \), then for any \( a \in A \) we have \( f(eae) = ef(a)e \); in particular, \( f(eAe) \subseteq eAe \) and \( f \) induces a derivation on \( eAe \).
(vii) If \( f(e) = 0 \) and if \( f \) is an inner derivation on \( A \), then \( f \) restricts to an inner derivation on \( eAe \).
Proof. Let \( a, b \in A \). Then \( aeb = aeeb \), hence \( f(aeb) = acef(e) + f(ac)eb \in AeA \), implying the first statement. We have \( f(e) = f(e^2) = f(e)e + ef(e) \). Right multiplication of this equation by \( e \) yields \( f(e)e = f(e)e + ef(e)e \), whence the second statement. Right and left multiplication of the same equation by \( 1 - e \) yields the third statement. Statement (iv) follows from combining the statements (ii) and (iii). We have \( [[f(e), e], e] = [f(e)e - ef(e), e] \). Using that \( ef(e)e = 0 \) this is equal to \( f(e)e + ef(e) = f(e) \), since \( f \) is a derivation. This shows (v). Suppose that \( f(e) = 0 \). Let \( a \in A \). Then \( f(ea) = f(e)ae + ef(a)e + eaf(e) = ef(a)e \), whence (vi). If in addition \( f = [c, -] \) for some \( c \in A \), then the hypothesis \( f(e) = 0 \) implies that \( ec = ce \), and hence (vi) implies that the restriction of \( f \) to \( eAe \) is equal to the inner derivation \([ce, -]\). This completes the proof of the Lemma.

\[\square\]

Proposition 4.2. Let \( A \) be a \( k \)-algebra, and let \( e \) be an idempotent in \( A \). For any derivation \( f \) on \( A \) satisfying \( f(e) = 0 \) denote by \( \varphi(f) \) the derivation on \( eAe \) sending \( eae \) to \( ef(a)e \), for all \( a \in A \). The correspondence \( f \mapsto \varphi(f) \) induces a Lie algebra homomorphism \( HH^1(A) \to HH^1(eAe) \). If \( A \) is an algebra over a field of prime characteristic \( p \), then this map is a homomorphism of \( p \)-restricted Lie algebras.

Proof. Let \( f \) be an arbitrary derivation on \( A \). By Lemma 4.1 (v), the derivation \( f - [[f(e), e], -] \) vanishes at \( e \). Thus every class in \( HH^1(A) \) has a representative in \( \text{Der}(A) \) which vanishes at \( e \). By Lemma 4.1 (vi), any derivation on \( A \) which vanishes at \( e \) restricts to a derivation on \( eAe \), and by Lemma 4.1 (vii), this restriction sends inner derivations on \( A \) to inner derivations on \( eAe \), hence induces a map \( HH^1(A) \to HH^1(eAe) \). A trivial verification shows that if \( f, g \) are two derivations on \( A \) which vanish at \( e \), the so does \([f, g]\), and an easy calculation shows that therefore the above map \( HH^1(A) \to HH^1(eAe) \) is a Lie algebra homomorphism. If \( A \) is an algebra over a field of characteristic \( p > 0 \), and if \( f \) is a derivation on \( A \) which vanishes at \( e \), then the derivation \( f^p \) vanishes on \( e \) and the restriction to \( eAe \) commutes with taking \( p \)-th powers by Lemma 4.1 (vi). This shows the last statement.

\[\square\]

We call the Lie algebra homomorphism \( HH^1(A) \to HH^1(eAe) \) in Proposition 12 the canonical Lie algebra homomorphism induced by the Schur functor given by multiplication with the idempotent \( e \).

For \( A \) a finite-dimensional \( k \)-algebra and \( n \) a positive integer, denote by \( HH^1_{(n)}(A) \) the subspace of \( HH^1(A) \) of classes which have a a representative \( f \in \text{Der}(A) \) satisfying \( f(J(A)) \subseteq J(A)^n \).

Proposition 4.3. Let \( A \) be a split finite-dimensional \( k \)-algebra. Let \( i \) be a primitive idempotent in \( A \). Set \( S = Ai/J(A)i \). Suppose that \( \text{Ext}^1_A(S, S) = \{0\} \). Then image of the canonical map \( HH^1(A) \to HH^1(iAi) \) is contained in \( HH^1_{(i)}(iAi) \).

Proof. By Lemma 2.3 we have \( i(A)i = i(J(A)^2)i \). By Lemma 4.1 (v), any class in \( HH^1(A) \) is represented by a derivation \( f \) satisfying \( f(i) = 0 \). Thus if \( a \in J(A) \), then \( iai = ibci \) for some \( b, c \in J(A) \), and hence \( f(iai) = if(b)ci + ibf(c)i \in iJ(A)i \).

\[\square\]

Proposition 4.4. Let \( A \) be a split symmetric \( k \)-algebra. Let \( i \) be a primitive idempotent in \( A \). Set \( S = Ai/J(A)i \). Suppose that \( \text{Ext}^1_A(S, S) \neq \{0\} \). Then the canonical Lie algebra homomorphism \( HH^1(A) \to HH^1(iAi) \) is nonzero.

Proof. Set \( S' = iA/iJ(A) \). Choose a maximal semisimple subalgebra \( E \) of \( A \). Since \( \text{Ext}^1_A(S, S) \) is nonzero, it follows from Lemma 2.2 that \( J(A)/J(A)^2 \) has a direct summand isomorphic to
$S \otimes_k S^\vee$ as an $A$-$A$-bimodule. Since $A$ is symmetric, we have $\text{soc}(A) \cong A/J(A)$, and hence $\text{soc}(A)$ has a bimodule summand isomorphic to $S \otimes_k S^\vee$. Thus there is a bimodule homomorphism $J(A)/J(A)^2 \to \text{soc}(A)$ with image isomorphic to $S \otimes_k S^\vee$. Composing with the canonical map $J(A) \to J(A)/J(A)^2$ yields a bimodule homomorphism $f : J(A) \to \text{soc}(A)$ with kernel containing $J(A)^2$ and with image isomorphic to $S \otimes_k S^\vee$. Extending $f$ by zero on $E$ yields a derivation $\hat{f}$ on $A$, by Lemma \[2,4\] Restricting $\hat{f}$ to $iJ(A)i$ sends $iJ(A)i$ to a nonzero subspace of $\text{soc}(A)$ isomorphic to $iS \otimes_k S^\vee i$, hence onto $\text{soc}(iAi)$. Thus the image of $\hat{f}$ under the canonical map $\text{Der}(A) \to \text{Der}(iAi)$ from Proposition \[1,2\] is a nonzero derivation with kernel containing $ki + J(iAi)^2$ and image in $\text{soc}(iAi)$. By \[2, Corollary 3.2\], the class in $\text{HH}^1(iAi)$ of this derivation is nonzero, whence the result.

\[\Box\]

**Proposition 4.5.** Let $p$ be an odd prime and suppose that $k$ is algebraically closed of characteristic $p$. Set $W = \text{Der}(k[x]/(x^p))$. For $−1 \leq i \leq p−2$ let $f_i$ be the derivation of $k[x]/(x^p)$ sending $x$ to $x^{i+1}$, where we identify $x$ with its image in $k[x]/(x^p)$. Let $L$ be a simple Lie subalgebra of $W$. Then either $L = W$, or $L \cong \mathfrak{s}_2(k)$.

**Proof.** Note that the subalgebra $S$ of $W$ spanned by the $f_i$ with $0 \leq i \leq p−2$ is solvable. Thus $L$ is not contained in $S$. Note further that $\dim_k(L) \geq 3$. Therefore there exist derivations

\[
\begin{align*}
f &= \sum_{i=-1}^{p-1} \lambda_i f_i \\
g &= \sum_{i=t}^{p-2} \mu_i f_i
\end{align*}
\]

belonging to $L$ with $\lambda_{-1} = 1$, and $\mu_t = 1$, where $t$ is an integer such that $0 \leq t \leq p−2$. Choose $g$ such that $t$ is minimal with this property. But then $[f, g]$ belongs to $L$. Since $[f_{-1}, f_t] = (t+1)f_{t-1}$, the minimality of $t \geq 0$ forces $t = 0$; that is we have

\[
g = \sum_{i=0}^{p-2} \mu_i f_i
\]

and $\mu_0 = 1$. Since $\dim_k(L) \geq 3$, it follows that there is a third element $h$ in $L$ not in the span of $f$, $g$, and hence, after modifying $h$ by a linear combination of $f$ and $g$, we can choose $h$ such that

\[
h = \sum_{i=s}^{p-2} \nu_i f_i
\]

for some $s$ such that $1 \leq s \leq p−2$ and $\nu_s = 1$. Choose $h$ such that $s$ is minimal with this property. Again by considering $[f, h]$, one sees that the minimality of $s$ forces $s = 1$. If $L$ is 3-dimensional, then $L \cong \mathfrak{s}_2(k)$, where we use that $k$ is algebraically closed. If $\dim_k(L) \geq 4$, then $L$ contains an element of the form

\[
u = \sum_{i=r}^{p-2} \tau_i f_i
\]

with $2 \leq r \leq p−2$ and $\tau_r = 1$. But then applying $[f, −]$ and $[h, −]$ repeatedly to $u$ shows that $L$ contains a basis of $W$, hence $L = W$. \[\Box\]
Remark 4.6. Note that if char$(k) = p > 2$, then the Witt Lie algebra $W$ contains indeed a subalgebra isomorphic to $\mathfrak{s}_2(k)$. Let $f, e, h$ be elements of the basis of $\mathfrak{s}_2(k)$ such that $[e, f] = h$, $[h, f] = -2f$, and $[h, e] = 2e$. Then we have a Lie algebra isomorphism $\mathfrak{s}_2(k) \cong \langle f_1, f_0, f_1 \rangle$ sending $f$ to $f_1$, $h$ to $2f_0$, and $e$ to $-f_1$.

Proof of Theorem 4.3. We use the notation and hypotheses of the notation in Theorem 4.3 and we assume that the Lie algebra $HH^1(A)$ is simple. We show that this forces $HH^1(A)$ to be a Lie subalgebra of the Witt Lie algebra $W$ with char$(k) = p > 2$, and then the result follows from Proposition 4.4.

Since $HH^1(A)$ is simple and since $\text{Ext}_A^1(S, S)$ is nonzero, it follows from Proposition 4.4 that the canonical Lie algebra homomorphism $\Phi : HH^1(A) \to HH^1(\mathfrak{i}A\mathfrak{i})$ from Proposition 4.4 is injective. By the assumptions, $\mathfrak{i}A\mathfrak{i}$ is a local algebra whose quiver has only one loop. Therefore $A \cong k[x]/(v)$ for some polynomial $v \in k[x]$ of degree at least 1. Since $k$ is algebraically closed, $v$ is a product of powers of linear polynomials, say $\prod_i(x - \beta_i)^{n_i}$, with pairwise distinct $\beta_i$ and positive integers $n_i$. Therefore $HH^1(\mathfrak{i}A\mathfrak{i})$ is a direct product of the Lie algebras corresponding to these factors. It follows that $HH^1(A)$ is isomorphic to a Lie subalgebra of $HH^1(k[x]/((x - \beta_i)^n))$ for some positive integer $n$. After applying the automorphism $x \mapsto x + \beta$ of $k[x]$ we have that $HH^1(A)$ is isomorphic to a Lie subalgebra of $HH^1(k[x]/(x^n))$ for some positive integer $n$. If char$(k) = p$ does not divide $n$ or if char$(k) = 0$, then the linear map sending $x$ to 1 is not a derivation on $k[x]/(x^n)$, and therefore $HH^1(k[x]/(x^n))$ is solvable in that case. Since Lie subalgebras of solvable Lie algebras are solvable, this contradicts the fact that $HH^1(A)$ is simple. Thus we have char$(k) = p > 0$ and $n = pm$ for some positive integer $m$. Since char$(k) = p$, it follows that the canonical surjection $k[x]/(x^n) \to k[x]/(x^p)$ induces a Lie algebra homomorphism $HH^1(k[x]/(x^m)) \to W = HH^1(k[x]/(x^p))$ with a nilpotent kernel. Thus $HH^1(A)$ is not contained in that kernel, and hence $HH^1(A)$ is isomorphic to a Lie subalgebra of $W$. The result follows.

To conclude this section we note that although it is not clear which simple Lie algebras might occur as $HH^1(A)$ when $\text{Ext}_A^1(S, S) = \{0\}$ for all simple $A$-modules $S$, it easy to show that $HH^1(A)$ is not a simple graded Lie algebra (with respect to the Gerstenhaber bracket).

Proposition 4.7. Let $A$ be a finite dimensional $k$-algebra, and assume that for every simple $A$-module $S$ we have $\text{Ext}_A^1(S, S) = \{0\}$. Then $HH^1(A)$ is not a perfect graded Lie algebra. In particular, $HH^*$ is not simple.

Proof. If $f \in C^1(A, A) := \text{Hom}_k(A, A)$ and if $g \in C^0(A, A) := \text{Hom}_k(k, A)$, then the Gerstenhaber bracket is given by $[f, g] = f(g)$, i.e. simply evaluating $f$ in $g$. Note that $1 \in Z(A) = HH^0(A)$. By Lemma 2.4 and Lemma 2.6, $f$ preserves $J(A)$ and we may assume $E \subseteq \ker(f)$. Therefore the derived Lie subalgebra of $HH^* (A)$ does not contain $1_A$.

Remark 4.8. Lemma 4.4 and Proposition 4.2 hold for algebras over an arbitrary commutative ring instead of $k$.

5. Examples

Theorem 4.4 applies to certain blocks of symmetric groups.

Proposition 5.1. Let $k$ be a field of prime characteristic $p$. Let $A$ be a defect 2 block of a symmetric group algebra $kS_n$ or the principal block of $kS_{2p}$. Then $HH^1(A)$ is a solvable Lie algebra.
Proof. From [9] Theorem 1 and from [7] Theorem 5.1 we have that the simple modules do not self-extend and the Ext^1-space between two simple modules is at most one-dimensional. The statement follows from Theorem 1.1.

Remark 5.2. A conjecture by Kleshchev and Martin predicts that simple kSn-modules in odd characteristic do not admit self-extensions.

Proposition 5.3. Let A be a tame symmetric algebra over a field k with 3 isomorphism classes of simple modules of type 3A or 3K. Then \( HH^1(A) \) is a solvable Lie algebra.

Proof. From the list at the end of Erdmann’s book [5] we have that the simple modules in these cases do not self-extend and that the Ext^1-space between two simple modules is at most one-dimensional. The statement follows from Theorem 1.1.

As mentioned in the introduction, the above Proposition is part of more general results on tame algebras in [4] and [8]. We note some other examples of algebras whose simple modules do not have nontrivial self-extensions.

Theorem 5.4 ([11] Theorem 3.4). Let G be a connected semisimple algebraic group defined and split over the field \( \mathbb{F}_p \) with \( p \) elements, and \( k \) be an algebraic closure of \( \mathbb{F}_p \). Assume G is almost simple and simply connected and let \( G(\mathbb{F}_q) \) be the finite Chevalley group consisting of \( \mathbb{F}_q \)-rational points of G where \( q = p^r \) for a non-negative integer \( r \). Let \( h \) be the Coxeter number of G. For \( r \geq 2 \) and \( p \geq 3(h-1) \), we have Ext^1_{kG(\mathbb{F}_q)}(S, S) = \{0\} for every simple \( kG(\mathbb{F}_q) \)-module \( S \).

Remark 5.5. Let G be a simple algebraic group over a field of characteristic \( p > 3 \), not of type A_1, G_2 and F_4. Proposition 1.4 in [11] implies that not having self-extensions does not allow to lift to characteristic zero certain simple modular representations. Therefore, for these cases the Lie structure of \( HH^1 \) plays a central role.

In the context of blocks with abelian defect groups one expects (by Broué’s abelian defect conjecture) every block of a finite group algebra with an abelian defect group \( P \) to be derived equivalent to a twisted group algebra of the form \( k_{\alpha}(P \times E) \), where \( E \) is the inertial quotient of the block and where \( \alpha \) is a class in \( H^2(E; k^*) \), inflated to \( P \times E \) via the canonical surjection \( P \times E \to E \). Thus the following observation is relevant in cases where Broué’s abelian defect conjecture is known to hold (this includes blocks with cyclic and Klein four defect).

Proposition 5.6. Let \( k \) be a field of prime characteristic \( p \). Let \( P \) be a finite \( p \)-group and \( E \) an abelian \( p^r \)-subgroup of \( \text{Aut}(P) \) such that \([P, E] = P\). Set \( A = k(P \times E) \). Suppose that \( k \) is large enough for \( E \), or equivalently, that \( A \) is split. For any simple \( A \)-module \( S \) we have \( \text{Ext}^1_A(S, S) = \{0\} \).

Proof. Since \( E \) is abelian, it follows that \( \dim_k(S) = 1 \), and hence that \( S \otimes_k - \) is a Morita equivalence. This Morita equivalence sends the trivial \( A \)-module \( k \) to \( S \), hence induces an isomorphism \( \text{Ext}^1_A(k, k) \cong \text{Ext}^1_A(S, S) \). It suffices therefore to show the statement for \( k \) instead of \( S \). That is, we need to show that \( H^1(P \times E; k) = \{0\} \), or equivalently, that there is no nonzero group homomorphism from \( P \times E \) to the additive group \( k \). Since \([P, E] = P\), it follows that every abelian quotient of \( P \times E \) is isomorphic to a quotient of \( E \), hence has order prime to \( p \). The result follows.

Example 5.7. If \( B \) is a block of a finite group algebra over an algebraically closed field \( k \) of characteristic \( p > 0 \) with a nontrivial cyclic defect group \( P \) and nontrivial inertial quotient \( E \),
then $HH^1(B)$ is a solvable Lie algebra, isomorphic to $HH^1(kP)^E$, where $E$ acts on $HH^1(kP)$ via the group action of $E$ on $P$. Indeed, $B$ is derived equivalent to the Nakayama algebra $k(P \rtimes E)$, which satisfies the hypotheses of Theorem [I.1] (thanks to the assumption $E \neq 1$, which implies $[P, E] = P$). Note that $kP$ is isomorphic to the truncated polynomial algebra $k[x]/(x^{p^d})$, where $p^d = |P|$.

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