Hydrodynamics of Sakai-Sugimoto model in the quenched approximation

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Abstract

We study transport properties of the finite temperature Sakai-Sugimoto model. The model represents a holographic dual to $4 + 1$ dimensional supersymmetric $SU(N_c)$ gauge theory compactified on a circle with anti-periodic boundary conditions for fermions, coupled to $N_f$ left-handed quarks and $N_f$ right-handed quarks localized at different points on the compact circle. We analytically compute the speed of sound and the sound wave attenuation in the quenched approximation. Since confinement/deconfinement (and the chiral symmetry restoration) phase transitions are first order in this model, we do not see any signature of these phase transitions in the transport properties.

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1 Introduction

Recently Sakai and Sugimoto (SS) \[1, 2\] introduced a supergravity model realizing holographic dual \[3, 4\] to four-dimensional, large $N_c$ QCD with massless flavors. Specifically, they considered five-dimensional $SU(N_c)$ maximally supersymmetric Yang-Mills (SYM) theory compactified on a circle with anti-periodic boundary conditions for the fermions\(^1\), and coupled to $N_f$ left-handed quarks and $N_f$ right-handed quarks localized at different points on the compact circle. At weak coupling the model can be represented by intersecting $D4/D8/D8$ brane system in type IIA string theory compactified on a circle. Coincident $N_c$ $D4$ branes wrap the compactification circle, while two stacks (with $N_f$ branes in each) of $D8$ and $D\bar{8}$ branes are localized at different points on this compactification circle. At strong coupling, the wrapped $D4$ branes are replaced with an appropriate near horizon geometry, while the $D8$ and $D\bar{8}$ branes are treated in the probe approximation\(^2\). The probe brane approximation is valid in the low-energy limit, and as long as $N_f \ll N_c$. The probe approximation of the dual holographic description corresponds to a quenched approximation for the fundamental quarks on the gauge theory side.

One of the most interesting aspects of the SS model is that it provides a simple holographic realization of the nonabelian chiral symmetry breaking. In \[5, 6\] the authors studied finite temperature confinement/deconfinement and chiral symmetry restoration in SS model. Rather interestingly, these two phase transitions are not necessarily simultaneous: for small enough separation of quarks on the circle, it was found \[5\] that there is a phase of the hot SS gauge theory which is deconfined, but with a broken chiral symmetry. As the separation between quarks exceeds certain critical value, both the deconfinement and the chiral symmetry restoration occur at the same temperature. For all range of parameters, all of these phase transitions are of first order. Moreover, each phase, even being thermodynamically unfavorable, i.e., having a larger free energy, appears to exist at arbitrary temperature\(^3\).

In this paper we study transport properties of the hot SS gauge theory plasma

\(^{1}\)Such boundary conditions completely break the supersymmetry and give masses to adjoint fermions of the 5d SYM theory.

\(^{2}\)In other words, the eight-brane backreaction on the $D4$ brane bulk geometry is neglected.

\(^{3}\)This should be contrasted with a model with abelian chiral symmetry breaking \[7\], where the chirally symmetric phase is believed to exist only for sufficiently high temperature \[8, 9, 10, 11\] even though the chiral symmetry restoration phase transition is expected to be first order.
in the quenched approximation. Since the backreaction of the $D8$ and $\overline{D8}$ branes is neglected, effectively, we study the hydrodynamics of near-extremal $D4$ branes wrapped on a circle with anti-periodic boundary conditions for the fermions. The latter model was discussed in [12] as the first example of the confining theory constructed within gauge theory-string theory correspondence. Since the background geometry satisfies condition of [13, 14], the shear viscosity $\eta$ of the SS plasma saturates the universal viscosity bound proposed$^4$ in [15]

\[ \frac{\eta}{s} = \frac{1}{4\pi}, \]  
(1.1)

where $s$ is the entropy density. On the other hand, the speed of sound and the sound wave attenuation is gauge theory specific. Given the dispersion relation for the sound waves

\[ \omega(q) = v_s q - i \frac{2q^2}{3T} \frac{\eta}{s} \left(1 + \frac{3\zeta}{4\eta}\right), \]  
(1.2)

where $v_s, \zeta$ are the plasma sound speed, and bulk viscosity correspondingly, for the SS model (in the deconfined phase) we find

\[ v_s = \frac{1}{\sqrt{5}}, \quad \zeta = \frac{4}{15}. \]  
(1.3)

Notice that transport coefficients (1.3) are not dissimilar from the transport properties of other examples of strongly coupled near-conformal gauge theory plasma [17, 18]

\[ \left(v_s^2 - \frac{1}{3}\right) \ll 1, \quad \frac{\zeta}{\eta} \simeq -\kappa \left(v_s^2 - \frac{1}{3}\right), \quad \kappa \sim 1, \]  
(1.4)

In fact, the precise value of $\kappa = 2$ from (1.3) is exactly the same as for the cascading gauge theory [18]. Also, there is no signature of the confinement/deconfinement phase transition in the sound wave dispersion relation. The latter is not unexpected, given that this phase transition is a first order.

The rest of this paper paper is the derivation of (1.3). In the next section we discuss effective five-dimensional action of the near-extremal wrapped $D4$ brane system. Interestingly, the resulting five-dimensional supergravity action is very similar to the one obtained in [11, 18]. In section 3 we study fluctuations of the corresponding black brane geometry dual to a sound wave mode of the quenched SS gauge theory plasma. We introduce gauge invariant fluctuations and obtain their equations of motion. These equations of motion are valid beyond the hydrodynamic approximation, and for arbitrary temperature. In section 4 we derive and solve fluctuation equations analytically

$^4$The universality of the shear viscosity in the supergravity approximation was proven in [13, 16, 14].
in the hydrodynamic limit. Imposing Dirichlet condition on the gauge invariant fluctuations at the boundary of the background black brane geometry determines the dispersion relation for the lowest quasinormal frequency. Finally, using the universality result, we can extract.

It would be very interesting to extend our computation beyond the probe approximation. We expect that the speed of sound waves and their attenuation would develop in this case dependence on the number of fundamental flavors \(N_f\), and the dependence on the radius of the compactification circle. Relevant discussion of the related supergravity background was presented in [20].

2 Consistent Kaluza-Klein Reduction to 5-dimensions

In this section we derive the 5-dimensional effective action from the type IIA 10-dimensional supergravity action by reduction on \(S^1 \times S^4\).

Consider the type IIA supergravity action in the Einstein frame:

\[
S_{IIA} = \frac{1}{2 \kappa_{10}^2} \int_{M_{10}} d^{10}x \left( -G^{(10)} \right)^{1/2} \left[ R^{(10)} - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - \frac{e^{\Phi/2}}{2} |F_4|^2 \right],
\]

where \(\Phi\) is the dilaton and \(F_4\) is the 4-form field strength, and the following metric ansatz:

\[
ds_{10}^2 = G^{(10)}_{MN} dx^M dx^N = e^{-\frac{4f}{3}} g_{\mu\nu} dx^\mu dx^\nu + e^{2f} \left[ e^{8w} (dS^1)^2 + e^{-2w} (dS^4)^2 \right],
\]

where the capital Latin indexes \((M, N, \ldots)\) run from 0 to 9 and the Greek indexes \((\mu, \nu, \ldots)\) run from 0 to 4. Moreover, the fields \(f\) and \(w\) as well as \(\Phi\) and \(F_4\) do not depend on the coordinates of \(S^1 \times S^4\). The 4-form field strength \(F_4\) is given by

\[
F_4 = \frac{1}{\sqrt{4!}} A \omega_{S^4},
\]

where \(A\) is a constant and \(\omega_{S^4}\) is the 4-sphere volume form. With such an ansatz, we obtain:

\[
\left( -G^{(10)} \right)^{1/2} = \left( -g \right)^{1/2} e^{-\frac{4f}{3}} (g_4)^{1/2},
\]

\[
\left( -G^{(10)} \right)^{1/2} |F_4|^2 = A^2 e^{-8(f-w)} \left( -g \right)^{1/2} e^{-\frac{4f}{3}} (g_4)^{1/2},
\]

\[
\left( -G^{(10)} \right)^{1/2} (\partial_\mu \Phi)(\partial^\mu \Phi) = \left( -g \right)^{1/2} (g_4)^{1/2} (\partial_\mu \Phi)(\partial^\mu \Phi),
\]

\[\text{We use conventions of [21] and keep only relevant fields.}\]
where $g_4$ is the determinant of the metric of the 4-sphere, and the curvature scalar $R^{(10)}$ is given by:

$$R^{(10)} = e^{\frac{10}{3} f} \left[ R^{(5)} - 20 g^{\mu \nu} (\partial_\mu w)(\partial_\nu w) - \frac{40}{3} g^{\mu \nu} (\partial_\mu f)(\partial_\nu f) \right] + 12 e^{-2( f - w)} ,$$ \hspace{1cm} (2.5)

where $R^{(5)}$ is the curvature scalar with respect to the 5-dimensional metric $g_{\mu \nu}$. From (2.4) and (2.5), and integrating over $S^1 \times S^4$ in the action (2.1), the effective 5-dimensional action follows:

$$S_5 = \frac{2\pi V_4}{2\kappa_5^2} \int_{\mathcal{M}_5} d^5 x \left( -g \right)^{1/2} \left[ R^{(5)} - \frac{40}{3} (\partial f)^2 - 20 (\partial w)^2 - \frac{1}{2} (\partial \Phi)^2 - \mathcal{P} \right] ,$$ \hspace{1cm} (2.6)

where $V_4$ is the volume of the 4-sphere and

$$\mathcal{P} \equiv A^2 e^\frac{1}{2} f e^{-\frac{34}{3} f + 8w} - 12 e^{-\frac{16}{3} f + 2w} .$$ \hspace{1cm} (2.7)

Notice that (2.6) is very similar to the five dimensional effective action of the cascading gauge theory derived in [11, 18]. From (2.6) we obtain the following equations of motion:

$$\Box f - \frac{3}{80} \frac{\partial \mathcal{P}}{\partial f} = 0 ,$$ \hspace{1cm} (2.8)

$$\Box w - \frac{1}{40} \frac{\partial \mathcal{P}}{\partial w} = 0 ,$$ \hspace{1cm} (2.9)

$$\Box \Phi - \frac{\partial \mathcal{P}}{\partial \Phi} = 0 ,$$ \hspace{1cm} (2.10)

$$R^{(5)}_{\mu \nu} = \frac{40}{3} \partial_\mu f \partial_\nu f + 20 \partial_\mu w \partial_\nu w + \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{3} g_{\mu \nu} \mathcal{P} .$$ \hspace{1cm} (2.11)

Consider now the following ansatz for the 5-dimensional metric:

$$ds_5^2 = -c_1^2 dt^2 + c_2^2 d\vec{x}^2 + c_3^2 dr^2 ,$$ \hspace{1cm} (2.12)

the correspondent 10-dimensional line element therefore takes the form:

$$ds_{10}^2 = e^{-\frac{10}{3} f} \left[ -c_1^2 dt^2 + c_2^2 d\vec{x}^2 + c_3^2 dr^2 \right] + e^{2f+8w} (dS^1)^2 + e^{2(f-w)} (dS^4)^2 .$$ \hspace{1cm} (2.13)
A comparison with the finite temperature bulk geometry of the Sakai-Sugimoto model in the deconfined phase which is given by \([12, 5]\):

\[
ds^2_{ss} = g_s^{-\frac{1}{2}} \left\{ \left( \frac{r}{R_{D4}} \right)^{\frac{9}{8}} \left[ -\triangle(r) \ dt^2 + d\vec{x}^2 + (dS^4)^2 \right] \right. \\
+ \left. \left( \frac{R_{D4}}{r} \right)^{15/8} \left[ \frac{dr^2}{\triangle(r)} + r^2 (dS^4)^2 \right] \right\}, \quad F_4 = \frac{2\pi N_c \omega_{s^4}}{V_4},
\]

leads to the following identifications:

\[
w = \frac{1}{10} \ln r - \frac{3}{10} \ln R_{D4}, \\
f = -\frac{1}{4} \ln g_s + \frac{51}{80} \ln R_{D4} + \frac{13}{80} \ln r, \\
c_1 = g_s^{-\frac{2}{3}} R_{D4}^{\frac{1}{6}} r^\frac{5}{6} [\triangle(r)]^{\frac{1}{2}}, \\
c_2 = c_1 [\triangle(r)]^{-\frac{1}{2}}, \\
c_3 = g_s^{-\frac{2}{3}} R_{D4}^{\frac{2}{3}} r^{-\frac{2}{3}} [\triangle(r)]^{-\frac{1}{2}}.
\]

In what follows we set \(g_s = 1\) and \(R_{D4} = 1\) (in this case \(A^2 = \frac{9}{2}\)), so that the relations (2.15) are:

\[
w = \frac{1}{10} \ln r, \\
f = \frac{13}{80} \ln r, \\
c_1 = r^\frac{5}{6} [\triangle(r)]^{\frac{1}{2}}, \\
c_2 = r^\frac{5}{6}, \\
c_3 = r^{-\frac{2}{3}} [\triangle(r)]^{-\frac{1}{2}}.
\]

We conclude this section with short comments on the thermodynamics of the black brane configuration (2.14). The Hawking temperature \(T\) is related to the nonextremality parameter \(r_\Lambda\) as

\[
\frac{1}{T} = \frac{4\pi}{3 r_\Lambda^{1/2}}.
\]

From (2.14) the entropy density \(s\) of the black branes is

\[
s \propto r_\Lambda^{5/2} \propto T^5.
\]
The first law of thermodynamics
\[-dP = dF = -s \, dT,\]
where \(P\) is the pressure and \(F\) is the free energy density, then implies that
\[-P = F = -\frac{1}{6} \, sT \Rightarrow \epsilon = \frac{5}{6} \, sT \Rightarrow P = \frac{1}{5} \, \epsilon, \quad (2.19)\]
where \(\epsilon\) is the energy density. From the equation of state (2.19) we find that
\[v_s^2 = \frac{\partial P}{\partial \epsilon} = \frac{1}{5}. \quad (2.20)\]
In the next section we reproduce (2.20) from the dispersion relation for the pole in stress-energy tensor two point correlation function in the sound wave channel, or equivalently [19], from the dispersion relation for the lowest sound channel quasinormal mode in the non-extremal geometry (2.14).

### 3 Fluctuations

Now we study fluctuations in the background geometry
\[g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu},\]
\[f \rightarrow f + \delta f,\]
\[w \rightarrow w + \delta w,\]
\[\Phi \rightarrow \Phi + \delta \Phi, \quad (3.1)\]
where \(\{g_{\mu\nu}, f, w, \Phi\}\) are the black brane background configuration (satisfying (2.16)), and \(\{h_{\mu\nu}, \delta f, \delta w, \delta \Phi\}\) are the fluctuations. We choose the gauge
\[h_{tr} = h_{x_i r} = h_{rr} = 0. \quad (3.2)\]
Additionally, we assume that all the fluctuations depend only on \((t, x_3, r)\), i.e., we have an \(O(2)\) rotational symmetry in the \(x_1 x_2\) plane.

At a linearized level we find that the following sets of fluctuations decouple from each other
\[
\{h_{x_1 x_2}\},
\{h_{x_1 x_1} - h_{x_2 x_2}\},
\{h_{tx_1}, h_{x_1 x_3}\},
\{h_{tx_2}, h_{x_2 x_3}\},
\{h_{tt}, h_{aa} \equiv h_{x_1 x_1} + h_{x_2 x_2}, h_{tx_3}, h_{x_3 x_3}, \delta f, \delta w, \delta \Phi\}. \quad (3.3)
\]
The last set of fluctuations is a holographic dual to the sound waves in quenched SS
gauge theory plasma which is of interest here. Introduce

\[ h_{tt} = c_1^2 \hat{h}_{tt} = e^{-i\omega t + iqx} c_1^2 H_{tt}, \]
\[ h_{tz} = c_2^2 \hat{h}_{tz} = e^{-i\omega t + iqx} c_2^2 H_{tz}, \]
\[ h_{aa} = c_2^2 \hat{h}_{aa} = e^{-i\omega t + iqx} c_2^2 H_{aa}, \]
\[ h_{zz} = c_2^2 \hat{h}_{zz} = e^{-i\omega t + iqx} c_2^2 H_{zz}, \]
\[ \delta f = e^{-i\omega t + iqx} F, \]
\[ \delta w = e^{-i\omega t + iqx} \Omega, \]
\[ \delta \Phi = e^{-i\omega t + iqx} p, \]
\[ \hat{h}_{ii} = \hat{h}_{aa} + \hat{h}_{zz}, \quad H_{ii} = H_{aa} + H_{zz}, \] (3.4)

where \( \{H_{tt}, H_{tz}, H_{aa}, H_{zz}, F, \Omega, p\} \) are functions of a radial coordinate only. Expanding
at a linearized level Eqs. (2.8)-(2.11) with Eq. (3.1) and Eq. (3.4) we find the following
coupled system of ODE’s

\[ 0 = H_{tt}'' + H_{tt}' \left[ \ln \frac{c_1^2 c_2^3}{c_3} \right]' - H_{ii}' \left[ \ln c_1 \right]' - \frac{c_2^3}{c_1^2} \left( q^2 \frac{c_1^2}{c_2^2} H_{tt} + \omega^2 H_{ii} + 2q \omega H_{tz} \right) \]
\[ - \frac{2}{3} \frac{c_2^2}{c_3} \left( \frac{\partial P}{\partial f} F + \frac{\partial P}{\partial w} \Omega + \frac{\partial P}{\partial \Phi} p \right), \] (3.5)

\[ 0 = H_{tz}'' + H_{tz}' \left[ \ln \frac{c_5}{c_1 c_3} \right]' + \frac{c_2^2}{c_2} \omega q H_{aa}, \] (3.6)

\[ 0 = H_{aa}'' + H_{aa}' \left[ \ln \frac{c_1^2 c_5}{c_3} \right]' + (H_{zz}' - H_{tt}') \left[ \ln c_2^2 \right]' + \frac{c_3^3}{c_1^2} \left( \omega^2 - q^2 \frac{c_1^2}{c_2^2} \right) H_{aa} \]
\[ + \frac{4}{3} \frac{c_2^3}{c_3} \left( \frac{\partial P}{\partial f} F + \frac{\partial P}{\partial w} \Omega + \frac{\partial P}{\partial \Phi} p \right), \] (3.7)

\[ 0 = H_{zz}'' + H_{zz}' \left[ \ln \frac{c_1^2 c_5}{c_3} \right]' + (H_{aa}' - H_{tt}') \left[ \ln c_2^2 \right]' \]
\[ + \frac{c_2^3}{c_1^2} \left( \omega^2 H_{zz} + 2q \omega H_{tz} + q^2 \frac{c_1^2}{c_2^2} (H_{tt} - H_{aa}) \right) \]
\[ + \frac{2}{3} \frac{c_2^3}{c_3} \left( \frac{\partial P}{\partial f} F + \frac{\partial P}{\partial w} \Omega + \frac{\partial P}{\partial \Phi} p \right), \] (3.8)
\[ 0 = \mathcal{F}'' + \mathcal{F}' \left[ \ln \left( \frac{c_1 c_3^2}{c_2} \right)' + \frac{1}{2} f' [H_{ii} - H_{tt}]' + \frac{c_3}{c_1^2} \left( \omega^2 - q^2 c_1^2 \right) \right] \mathcal{F} \]
\[ - \frac{3}{80} c_3^2 \left( \frac{\partial^2 \mathcal{P}}{\partial f^2} \mathcal{F} + \frac{\partial^2 \mathcal{P}}{\partial f \partial w} \Omega + \frac{\partial^2 \mathcal{P}}{\partial f \partial \Phi} p \right), \tag{3.9} \]

\[ 0 = \Omega'' + \omega' \left[ \ln \left( \frac{c_1 c_3^2}{c_2} \right)' + \frac{1}{2} w' [H_{ii} - H_{tt}]' + \frac{c_3}{c_1^2} \left( \omega^2 - q^2 c_1^2 \right) \right] \Omega \]
\[ - \frac{1}{40} c_3^2 \left( \frac{\partial^2 \mathcal{P}}{\partial w \partial f} \mathcal{F} + \frac{\partial^2 \mathcal{P}}{\partial w \partial \omega} \Omega + \frac{\partial^2 \mathcal{P}}{\partial w \partial \Phi} p \right), \tag{3.10} \]

\[ 0 = \Phi'' + \phi' \left[ \ln \left( \frac{c_1 c_3^2}{c_2} \right)' + \frac{1}{2} \Phi' [H_{ii} - H_{tt}]' + \frac{c_3}{c_1^2} \left( \omega^2 - q^2 c_1^2 \right) \right] \Phi 
\[ - c_3^2 \left( \frac{\partial^2 \mathcal{P}}{\partial \Phi \partial f} \mathcal{F} + \frac{\partial^2 \mathcal{P}}{\partial \Phi \partial w} \Omega + \frac{\partial^2 \mathcal{P}}{\partial \Phi \partial \Phi} p \right), \tag{3.11} \]

where all derivatives \( \partial \mathcal{P} \) are evaluated on the background geometry. Additionally, there are three first order constraints associated with the (partially) fixed diffeomorphism invariance

\[ 0 = \omega \left( H_{ii}' + \left[ \ln \left( \frac{c_2}{c_1} \right)' \right] H_{ii} \right) + \omega \left( H_{tt}' + \left[ \ln \left( \frac{c_2}{c_1} \right)' \right] H_{tt} \right) \]
\[ + \omega \left( \frac{80}{3} f' \mathcal{F} + 40 w' \Omega + \Phi' p \right), \tag{3.12} \]

\[ 0 = q \left( H_{tt}' - \left[ \ln \left( \frac{c_2}{c_1} \right)' \right] H_{tt} \right) + \frac{c_2}{c_1} \omega \left( H_{tt}' - q H_{aa}' - q \left( \frac{80}{3} f' \mathcal{F} + 40 w' \Omega + \Phi' p \right) \right), \tag{3.13} \]

\[ 0 = \left[ \ln \left( c_1 c_2^2 \right)' \right] H_{ii}' - \left[ \ln c_2^2 \right]' H_{ii}' + \frac{c_3}{c_1^2} \left( \omega^2 H_{ii} + 2 \omega q H_{tz} + q^2 \frac{c_1^2}{c_2^2} (H_{tt} - H_{aa}) \right) \]
\[ + c_3^2 \left( \frac{\partial \mathcal{P}}{\partial f} \mathcal{F} + \frac{\partial \mathcal{P}}{\partial w} \Omega + \frac{\partial \mathcal{P}}{\partial \Phi} p \right) - \left( \frac{80}{3} f' \mathcal{F}' + 40 w' \Omega' + \Phi' p' \right). \tag{3.14} \]

We explicitly verified that Eqs. (3.5)-(3.11) are consistent with constraints (3.12)-(3.14).
Introducing the gauge invariant fluctuations

\[ Z_H = 4 \frac{q}{\omega} H_{tz} + 2 H_{zz} - H_{aa} \left( 1 - \frac{q^2 c_1 c_2}{\omega^2 c_1' c_2} \right) + 2 \frac{q^2 c_1^2}{\omega^2 c_2} H_{tt} , \]

\[ Z_f = F - \frac{f'}{[\ln c_2']} H_{aa} , \]

\[ Z_w = \Omega - \frac{\omega'}{[\ln c_2']} H_{aa} , \]

\[ Z_\Phi = p - \frac{\Phi'}{[\ln c_2']} H_{aa} , \]

and a new radial coordinate

\[ x \equiv \frac{c_1}{c_2} , \]

we find from Eqs. (3.15), (3.16), (3.17), (3.18), decoupled set of equations of motion for \( Z \)'s :

\[ 0 = \frac{d^2 Z_H}{dx^2} + \frac{3 q^2 (2 x^2 - 1) + 5 \omega^2}{x (5 \omega^2 - q^2 (3 + 2 x^2))} \frac{dZ_H}{dx} + \frac{4}{9} \frac{(-\omega^2 + q^2 x^2) (9 q^2 (3 + 2 x^2) - 5 \omega^2) - 18 q^2 r_\Lambda x^2 (1 - x^2)^{5/3}}{(5 \omega^2 - q^2 (3 + 2 x^2)) (1 - x^2)^{5/3} x^2 r_\Lambda} Z_H \]

\[ + \frac{4}{15} \frac{q^2 (-3 q^2 + 5 \omega^2)}{\omega^2 (5 \omega^2 - q^2 (3 + 2 x^2))} (48 Z_w + 9 Z_\Phi + 52 Z_f) , \]

\[ 0 = \frac{d^2 Z_f}{dx^2} + \frac{1}{x} \frac{dZ_f}{dx} - \frac{4}{225} \frac{25 (1 - x^2) (-\omega^2 + q^2 x^2) + 243 r_\Lambda x^2 (1 - x^2)^{2/3}}{r_\Lambda x^2 (1 - x^2)^{8/3}} Z_f \]

\[ + \frac{9}{25 (1 - x^2)^2} (Z_\Phi + 12 Z_w) , \]

\[ 0 = \frac{d^2 Z_w}{dx^2} + \frac{1}{x} \frac{dZ_w}{dx} - \frac{4}{225} \frac{25 (1 - x^2) (-\omega^2 + q^2 x^2) + 162 r_\Lambda x^2 (1 - x^2)^{2/3}}{r_\Lambda x^2 (1 - x^2)^{8/3}} Z_w \]

\[ + \frac{6}{25 (1 - x^2)^2} (12 Z_f - Z_\Phi) , \]

\[ 0 = \frac{d^2 Z_\Phi}{dx^2} + \frac{1}{x} \frac{dZ_\Phi}{dx} - \frac{4}{45} \frac{5 (1 - x^2) (-\omega^2 + q^2 x^2) + 9 r_\Lambda x^2 (1 - x^2)^{2/3}}{r_\Lambda x^2 (1 - x^2)^{8/3}} Z_\Phi \]

\[ - \frac{48}{5 (1 - x^2)^2} (Z_w - Z_f) . \]
Further decoupling occurs if we introduce
\[ \kappa \equiv 48 Z_w + 9 Z_\Phi + 52 Z_f. \] (3.21)

In this case we find\(^6\):
\[
0 = \frac{d^2 Z_H}{dx^2} + \frac{3 q^2 (2 x^2 - 1) + 5 \omega^2}{x (5 \omega^2 - q^2 (3 + 2 x^2))} \frac{dZ_H}{dx} \\
+ \frac{4}{9} \left( (\omega^2 + q^2 x^2) (q^2 (3 + 2 x^2) - 5 \omega^2) - 18 q^2 r_\Lambda x^2 (1 - x^2)^{5/3} \right) Z_H \\
+ \frac{4}{15} \frac{q^2 (-3 q^2 + 5 \omega^2)}{\omega^2 (5 \omega^2 - q^2 (3 + 2 x^2))} \kappa, \tag{3.22}
\]

\[
0 = \frac{d^2 \kappa}{dx^2} + \frac{1}{x} \frac{d\kappa}{dx} + \frac{4 (\omega^2 - q^2 x^2)}{9 r_\Lambda x^2 (1 - x^2)^{5/3}} \kappa. \tag{3.23}
\]

## 4 Hydrodynamic limit

We study now physical fluctuation equations (3.22), (3.23) in the hydrodynamics approximation, \( \omega \to 0, q \to 0 \) with \( \frac{\omega}{q} \) kept constant. Similar to the computations in [17, 18], we would need only leading and next-to-leading (in \( q \)) solution of (3.22) and (3.23). We find that at the horizon, \( x \to 0_+ \), \( Z_H \propto x^{\pm i\omega/(2\pi T)} \), and similarly for \( \kappa \).

Incoming boundary conditions on all physical modes implies that
\[
Z_H(x) = x^{-i\omega} z_H(x), \quad \kappa(x) = x^{-i\omega} \kappa(x), \tag{4.1}
\]
where \( \{z_H, \kappa\} \) are regular at the horizon; we further introduced
\[
\omega \equiv \frac{\omega}{2\pi T}, \quad q \equiv \frac{q}{2\pi T}. \tag{4.2}
\]

There is a single integration constant for these physical modes, namely, the overall scale. Without the loss of generality the latter can be fixed as
\[
z_H(x) \bigg|_{x \to 0_+} = 1. \tag{4.3}
\]

\(^6\)This set of gauge invariant fluctuations will be sufficient to determine the sound wave dispersion relation.
In this case, the pole dispersion relation is simply determined as

\[ z_H(x) \bigg|_{x \to 1^-} = 0. \] (4.4)

The other boundary condition (besides regularity at the horizon and (4.4)) is

\[ \mathcal{K}(x) \bigg|_{x \to 1^-} = 0. \] (4.5)

Let’s introduce

\[ z_H = z_{H,0} + i \, q \, z_{H,1}, \quad \mathcal{K} = \mathcal{K}_0 + i \, q \, \mathcal{K}_1, \] (4.6)

where the index refers to either the leading, \( \propto q^0 \), or to the next-to-leading, \( \propto q^1 \), order in the hydrodynamic approximation. Additionally, as we are interested in the hydrodynamic pole dispersion relation in the stress-energy correlation functions, we find it convenient to parameterize

\[ w = v_s \, q - i \, q^2 \, \Gamma, \] (4.7)

where the speed of sound \( v_s \) and the sound wave attenuation \( \Gamma \) are to be determined from the pole dispersion relation (4.4)

\[ z_{H,0} \bigg|_{x \to 1^-} = 0, \quad z_{H,1} \bigg|_{x \to 1^-} = 0. \] (4.8)

Using parameterizations (4.6), (4.7), we obtain from (3.22) and (3.23) the following ODE’s

\[ 0 = x \, \mathcal{K}_0'' + \mathcal{K}_0', \] (4.9)

\[ 0 = z_{H,0}'' - \frac{(6 \, x^2 - 3 + 5 \, v_s^2)}{x \, (2 \, x^2 - 5 \, v_s^2 + 3)} \, z_{H,0}' + \frac{8}{2 \, x^2 - 5 \, v_s^2 + 3} \, z_{H,0} - \frac{4 \, (-3 + 5 \, v_s^2) \, \mathcal{K}_0}{15 \, (2 \, x^2 - 5 \, v_s^2 + 3) \, v_s^2}, \] (4.10)

describing leading (\( \propto q^0 \)), and

\[ 0 = x \, \mathcal{K}_1'' + \mathcal{K}_1' - 2v_s \, \mathcal{K}_0', \] (4.11)
\[
0 = z_{H,1}'' - \frac{(6 x^2 - 3 + 5 v_s^2)}{x \left(2 x^2 - 5 v_s^2 + 3\right)} z_{H,1}' + \frac{8}{2 x^2 - 5 v_s^2 + 3} z_{H,1} \\
+ \frac{2 v_s \left(40 x^2 \Gamma + 20 x^2 v_s^2 - 25 v_s^4 + 30 v_s^2 - 4 x^4 - 12 x^2 - 9\right)}{x \left(2 x^2 - 5 v_s^2 + 3\right)^2} z_{H,0} \\
- \frac{8 v_s \left(-2 x^2 + 5 v_s^2 - 3 + 10 \Gamma\right)}{(2 x^2 - 5 v_s^2 + 3)^2} z_{H,0} + \frac{8}{15} \left(6 x^2 + 9 + 25 v_s^4 - 30 v_s^2\right) \mathcal{K}_0 \\
- \frac{4}{15} \left(-3 + 5 v_s^2\right) \mathcal{K}_1 \left(2 x^2 - 5 v_s^2 + 3\right)^2 v_s^2,
\]

(4.12)

describing next-to-leading (\(\propto q^1\)) order in the hydrodynamic approximation. Solving (4.9) and (4.11) subject to regularity at the horizon and the boundary condition (4.5) we find

\[
\mathcal{K}_0 = 0, \quad \mathcal{K}_1 = 0.
\]

(4.13)

Given (4.13), solution to (4.9) subject to regularity at the horizon and the boundary condition (4.8) is

\[
z_{H,0} = \frac{5 v_s^2 + 2 x^2 - 3}{5 v_s^2 - 3},
\]

(4.14)

which from (4.8) determines (in agreement with (2.20))

\[
v_s = \frac{1}{\sqrt{5}}.
\]

(4.15)

With (4.13)-(4.15), solution to (4.12) subject to regularity at the horizon is

\[
z_{H,1} = \mathcal{C} \left(1 - x^2\right) + \frac{1}{\sqrt{5}} \left(5 \Gamma - 2\right),
\]

(4.16)

where \(\mathcal{C}\) is an arbitrary integration constant. From (4.8) we conclude\(^7\)

\[
\Gamma = \frac{2}{5}.
\]

(4.17)

Finally, comparing (1.2) and (4.7), and using (1.1), we obtain from (4.17) result quoted in (1.3).

\(^7\)Boundary condition (4.3) fixes \(\mathcal{C} = 0\).
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