SHARP ESTIMATES FOR SPREADING SPEED OF THE LOTKA-VOLTERRA DIFFUSION SYSTEM WITH STRONG COMPETITION

RUI PENG, CHANG-HONG WU, AND MAOLIN ZHOU

Abstract. This paper concerns the classical two-species Lotka-Volterra diffusion system with strong competition. The sharp dynamical behavior of the solution is established in two different situations: either one species is an invasive one and the other is a native one or both are invasive species. Our results seem to be the first that provide a precise spreading speed and profile for such a strong competition system. Among other things, our analysis relies on the construction of new types of supersolution and subsolution, which are optimal in certain sense.

1. Introduction

In this paper, we consider the classical two-species Lotka-Volterra competition-diffusion system:
\[
\begin{align*}
\begin{cases}
    u_t &= d u_{xx} + ru(1 - u - av), \\
    v_t &= v_{xx} + v(1 - v - bu),
\end{cases} 
\quad t > 0, \quad x \in \mathbb{R},
\end{align*}
\]
with the initial data
\[
\begin{align*}
    u(0, x) &= u_0(x), \\
    v(0, x) &= v_0(x), \quad x \in \mathbb{R},
\end{align*}
\]
where \( u(t, x) \) and \( v(t, x) \) represent the population density of two competing species at the position \( x \) and time \( t \); \( d \) stands for the diffusion rate of \( u \); \( r \) represents the intrinsic growth rate of \( u \); \( a \) and \( b \) represent the competition coefficient for two species, respectively. All parameters are assumed to be positive. Note that system (1.1) has been reduced into the dimensionless form using a standard scaling (see, e.g., [32]).

Since the pioneering works of Fisher [16] and Kolmogorov, Petrovsky and Piskunov [25], reaction-diffusion equations have been the subject of a large amount of research works aiming at the understanding of the spread dynamics of invasive species. More precisely, when an invasive species is introduced into a new environment, the mathematical approach of [16, 25] to describe the spreading of species is based on the study of the long time behavior of the solution of the following Fisher-KPP equation:
\[
\begin{align*}
\begin{cases}
    w_t &= d w_{xx} + r w(1 - w), \\
    w(0, t) &= w_0(x), \\
    w(0, x) &= w_0(x), \quad x \in \mathbb{R},
\end{cases} 
\end{align*}
\]
where \( w(t, x) \) stands for the population density for the invasive species at time \( t \) and position \( x \).

When \( w_0 \neq 0 \) is nonnegative with compact support in \( \mathbb{R} \), the classical result of Aronson and Weinberger [1, 2] shows that there exists a unique \( c^* = 2\sqrt{rd} \) such that the solution \( w \) to (1.3)
satisfies
\[
\lim_{t \to \infty} \max_{|x| \geq ct} w(x, t) = 0 \quad \text{for any } c > c^*; \\
\lim_{t \to \infty} \max_{|x| \leq ct} [1 - w(x, t)] = 0 \quad \text{for any } c \in (0, c^*).
\]

Such a spreading behavior describes the invading phenomenon of the unstable state 0 by the stable state 1, and the quantity \(c^*\) is often referred to as the (asymptotic) spreading speed of the species and has been then used to predict the spreading speed for various invasive species in reality [38]. Furthermore, \(c^*\) coincides with the minimal speed of the traveling wave solution of the form: \(w(x - ct)\) connecting 1 and 0; that is, if and only if \(c \geq c^*\), the following problem
\[
\begin{cases}
dw'' + cw' + rw(1 - w) = 0, & w > 0 \text{ in } \mathbb{R}, \\
w(-\infty) = 1, & w(\infty) = 0
\end{cases}
\]
(1.4)

admits a unique solution (up to translation).

In the absence of the species \(v\) (resp. \(u\)), system (1.1) is reduced to the Fisher-KPP equation (1.3), which admits a unique traveling wave solution (up to translation), denoted by \(U_{KPP}(x - ct)\) (resp. \(V_{KPP}(x - ct)\)) connecting 1 and 0 if and only if \(c \geq 2\sqrt{rd}\) (resp. \(c \geq 2\)). For sake of convenience, we denote in this paper
\[
c_u = 2\sqrt{rd}, \quad c_v = 2.
\]

Clearly, \(c_u\) (resp. \(c_v\)) is the spreading speed of the species \(u\) (resp. \(v\)) in the absence of the species \(v\) (resp. \(u\)) of (1.1).

Traveling wave solutions usually play a crucial role in understanding the spreading of invasive species. As far as one species is concerned, great progress have been made in recent decades to determine the spreading dynamics via the associated traveling wave solutions; one may refer to, for instance, [3, 6, 21, 26, 36, 37, 40] and references therein.

When multiple species interact, there is a wide literature on (asymptotic) spreading speeds for various evolutionary systems; see, e.g., [13, 27, 28, 29, 30, 41] and references therein. However, to the best of our knowledge, there have been only few papers devoted to the rigorous study of long-time dynamics of a multiple-species system. One of the mathematical difficulties lies in that in general different spreading speeds may occur in different species, which brings highly nontrivial challenges when one deals with the convergence of solutions. Indeed, even for the simplest yet most classical Lotka-Volterra system (1.1), its global dynamics is still poorly understood except for some cases which will be mentioned briefly below.

In the remarkable work [18], Girardin and Lam investigated system (1.1) in the strong-weak (also called as monostable) competition case (i.e., \(a < 1 < b\)) with the initial data being null or exponentially decaying in a right half-line. By constructing very technical pairs of supersolutions and subsolutions, they gained a rather complete understanding of the spreading properties of (1.1). Among other things, they found the acceleration phenomena during the period of invasion in some cases; see [18] for precise and more results. One may also refer to Lewis, Li and Weinberger [27, 28] for previous studies in the monostable case. On the other hand, the analogous problem
with free boundaries was addressed in \[11\], where the behavior of the slower species is determined by some semi-wave system studied in \[10\].

In the strong (also called as bistable) competition case (i.e., \(a, b > 1\)), Carrere \[5\] considered \(1.1\). It was proved that if the two species are initially absent from the right half-line \(x > 0\), and the slower one dominates the faster one on \(x < 0\), then the latter will invade the right space at its Fisher-KPP speed, and will be replaced by or will invade the former, depending on the parameters, at a slower speed. This shows that the system forms a propagating terrace, connecting the unstable state \((0, 0)\) to the two stable states \((1, 0)\) and \((0, 1)\). We also mention the work \[12\], therein the authors proved that prey-predator systems can develop different spreading speeds.

The current paper focuses on the strong competition case, and our primary goal is to derive the sharp dynamical behavior of the solution of \(1.1\). We are concerned with two typical situations: either one species is an invasive one and the other is a native one or both are invasive species. The obtained results substantially complement and improve those in \[5\]. To our knowledge, the main results of this paper seem to be the first that give the precise estimates for the spreading speed of system \(1.1\) with strong competition.

Since the competition model enjoys the comparison principle, our main results are established by the delicate construction of supersolutions and subsolutions. To this aim, we first derive some good decay estimates of the solution as \(t\) is sufficiently large. Based on such estimates, we then construct various types of supersolutions and subsolutions, which turn out to be very new and optimal in certain sense. It is worth mentioning that in \[18\], Girardin and Lam also adopted the approach of supersolution and subsolution to establish their main results. Nevertheless, the pairs of supersolutions and subsolutions constructed here are rather different from those used in \[18\], mainly due to the essential differences between the strong competition problem and strong-weak competition problem. On the other hand, to derive the convergence results including a Bramson correction (refer to Theorem 2 and Theorem 3 below), we reduce system \(1.1\) into a perturbed Fisher-KPP equation and then the argument used in \[21\] can be applied to obtain the Bramson correction. See also \[7\] for the Bramson correction in an SIS model.

Before presenting the main results of the paper, we need to state some assumptions and introduce some notations. From now on, we always assume that

\[(H1)\] the strong competition: \(a, b > 1\).

Under \((H1)\), let us recall the well-known results on traveling front solutions corresponding to system \(1.1\), which are vital in describing the global dynamics of \(1.1\). By a traveling front solution, we mean a solution of \(1.1\) with the form

\[(u(t, x), v(x, t)) = (U(x - ct), V(x - ct))\]

and the limits \((U, V)(\pm\infty)\) exists and unequal, where \(c\) is called the wave speed. From Gardner \[17\] and Kan-on \[22\], system \(1.1\) admits a unique (up to a translation) traveling front solution connecting steady states \((1, 0)\) and \((0, 1)\). More precisely, there exists a unique speed

\[c_{uv} \in (-2, 2\sqrt{rd})\]
such that when $c = c_{uv}$, the following problem

\[
\begin{aligned}
& \begin{cases}
  cU' + dU'' + rU(1 - U - aV) = 0, & \xi \in \mathbb{R}, \\
  cV' + V'' + V(1 - V - bU) = 0, & \xi \in \mathbb{R}, \\
  (U, V)(-\infty) = (1, 0), & (U, V)(\infty) = (0, 1), \\
  U' < 0, V' > 0, & \xi \in \mathbb{R}
\end{cases}
\end{aligned}
\]  

(1.5)

has a unique (up to a translation) solution $(U, V) \in [C^2(\mathbb{R})]^2$. By our notation, $c_{uv} < c_u$.

In this paper, we also assume that

\((H2)\) $c_{uv} > 0$.

The sufficient conditions to guarantee \((H2)\) will be mentioned later. It is noted that if \((H1)\) and $c_{uv} < 0$ are fulfilled, the global dynamics of (1.1) may depend on the initial repartition of $u$ and $v$; such a case shall not be studied in this paper.

Regarding the initial data $(u_0, v_0)$, we consider two different scenarios:

\(\text{(A1)}\) $u_0 \in C(\mathbb{R}) \setminus \{0\}$, $u_0 \geq 0$ with compact support; $v_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with a positive lower bound.

\(\text{(A2)}\) $u_0, v_0 \in C(\mathbb{R}) \setminus \{0\}$, $u_0, v_0 \geq 0$ with compact support.

Scenario \((\text{A1})\) means that species $u$ is the invasive species that initially occupies some bounded interval and species $v$ is the native species that has already occupied the whole space; while scenario \((\text{A2})\) means that both two species are invasive species that initially occupy only open bounded intervals.

For convenience, let us lump conditions \((H1)\) and \((H2)\) together as condition \((H)\). We are now in a position to present the main results obtained in this paper.

Our first main result concerns scenario \((\text{A1})\) and indicates the successful invasion of species $u$ if $v$ is the native species.

**Theorem 1.** Assume that \((H)\) and \((\text{A1})\) hold. Then there exists a constant $\hat{h}$ such that the solution $(u, v)$ of (1.1)-(1.2) satisfies

\[
\lim_{t \to \infty} \left[ \sup_{x \in [0, \infty)} \left| u(t, x) - U(x - c_{uv}t - \hat{h}) \right| + \sup_{x \in [0, \infty)} \left| v(t, x) - V(x - c_{uv}t - \hat{h}) \right| \right] = 0,
\]

(1.6)

where $(c_{uv}, U, V)$ is a solution of (1.5).

The result of Theorem 1 is related to the stability of traveling fronts; we refer to [14] for critical pulled fronts of (1.1) with $a < 1 < b$ and [39] for a buffered bistable system.

Our next two main results concern scenario \((\text{A2})\); that is, both species are invasive ones. It turns out that $c_u$ and $c_v$ play an important role to determine the dynamical behavior of solutions.

We first consider the case $c_u > c_v$.. In this case, the following result shows that $u$ spreads faster than $v$; $u$ will drive $v$ to extinction in the long-run while $u$ converges to a shifted traveling front with a Bramson correction [3, 21, 26, 40].
Theorem 2. Assume that (H) and (A2) hold. If $c_u > c_v$, then the solution $(u, v)$ of (1.1)-(1.2) satisfies
\[
\lim_{t \to \infty} \left[ \sup_{x \in [0, \infty)} \left| u(t, x) - U_{KPP} \left( x - c_u t + \frac{3}{c_u} \ln t + \omega(t) \right) \right| + \sup_{x \in [0, \infty)} \left| v(t, x) \right| \right] = 0,
\]
where $\omega$ is a bounded function defined on $[0, \infty)$.

Finally, we handle the case $c_u < c_v$. Then $c_{uv} < c_u < c_v$. In this case, the following result suggests that the species $u$ spreads at the slower speed $c_{uv}$ and the species $v$ spreads at the speed $c_v$ and thus a propagating terrace is formed. Though this phenomenon was proved in [5], our result gives the sharp estimates for the spreading speed of the solution.

Theorem 3. Assume that (H) and (A2) hold, and that $c_u < c_v$. Denote $c_0 = \frac{c_u + c_v}{2}$. Then the solution $(u, v)$ of (1.1)-(1.2) satisfies
\[
\lim_{t \to \infty} \left[ \sup_{x \in [c_0 t, \infty)} \left| v(t, x) - V_{KPP} \left( x - c_v t + \frac{3}{c_v} \ln t + \omega(t) \right) \right| + \sup_{x \in [c_0 t, \infty)} \left| u(t, x) \right| \right] = 0
\]
and
\[
\lim_{t \to \infty} \left[ \sup_{x \in [0, c_0 t]} \left| u(t, x) - U(x - c_{uv} t - h_1) \right| + \sup_{x \in [0, c_0 t]} \left| v(t, x) - V(x - c_{uv} t - h_1) \right| \right] = 0
\]
for some bounded function $\omega$ on $[0, \infty)$ and some $h_1 \in \mathbb{R}$, where $(c_{uv}, U, V)$ is a solution of (1.5).

Some comments on Theorem 1-3 are made in order as follows.

Remark 1.1. The sign of $c_{uv}$ has been investigated in the literature. Indeed, Kan-on [22] proved that $c_{uv}$ is decreasing in $a$ and is increasing in $b$. Guo and Lin [20] provided explicit conditions to determine the sign of $c_{uv}$; in particular, their results conclude that
(i) When $r = d$, then $c_{uv} > 0$ if $b > a > 1$, $c_{uv} = 0$ if $a = b > 1$ and $c_{uv} < 0$ if $a > b > 1$.
(ii) When $r > d$, then $c_{uv} > 0$ if $a > 1$ and $b \geq \left( \frac{b}{d} \right)^2 a$.
(iii) When $r < d$, then $c_{uv} < 0$ if $b > 1$ and $a \geq \left( \frac{d}{r} \right)^2 b$.
In addition, it can be shown that if $r, d > 0$ and $a > 1$ are fixed, $c_{uv} > 0$ for all large $b$. One may also see Girardin and Nadir [19], Rodrigo and Mimura [35] and Ma, Huang and Ou [31] for related discussion.

Remark 1.2. We would like to mention the following.

(i) Similar results of Theorem 1, Theorem 3 also hold for $x \in (-\infty, 0]$ since the arguments used on the right half-line work on its left half-line in the strong-competition system.
(ii) By slightly modifying the supersolutions and subsolutions constructed in this paper, we can see that Theorem 1 remains true if the initial datum $u_0$ in (A1) is assumed to satisfy $u_0 \to 0$ as $|x| \to \infty$; similarly, Theorem 2 and Theorem 2 hold if the initial data $u_0, v_0$ in (A2) are assumed to satisfy $u_0, v_0 \to 0$ as $|x| \to \infty$.
(iii) The techniques developed in this paper may be applicable to more general competition systems (1.1) as well as other parabolic systems including cooperative systems with arbitrary size.
The remainder of this paper is organized as follows. In section 2, we shall prepare some well-known results and provide important estimates of the solution of (1.1)-(1.2) that will be used in both \((A1)\) and \((A2)\). Section 3 is devoted to the proof of Theorem 1, and Theorem 2 and Theorem 3 are proved in Section 4.

2. Preliminaries

In this section, we prepare some preliminary results that will be used in both cases: \((A1)\) and \((A2)\). In the first subsection, we recall the exact exponential decays of traveling front solution of (1.5) connecting \((0,1)\) and \((1,0)\). In the second subsection, we recall the comparison principle for system (1.1)-(1.2). Some crucial estimates of solutions to system (1.1)-(1.2) are given in the third subsection.

2.1. The asymptotic behavior of bistable fronts. The asymptotic behavior of the traveling front solution for (1.1) with \(c = c_{uv} \neq 0\) as \(\xi \to \pm \infty\) is well known; we refer to [23] or [33, section 2]. Here we state the results that will be used in the rest of this paper.

Let \((c, U, V)\) be a solution of system (1.5). To describe the asymptotic behavior of \((U, V)\) near \(\xi = +\infty\), we need the following characteristic equations:

\[
\begin{align*}
\lambda_1 &= -c - \sqrt{c^2 + 4rd(a - 1)} / 2d, \\
\lambda_2 &= -c - \sqrt{c^2 + 4r} / 2.
\end{align*}
\]

Lemma 2.1 ([23, 33]). There exist two positive constants \(\ell_1\) and \(\ell_2\) such that

\[
\lim_{\xi \to +\infty} U(\xi) e^{\lambda_1 \xi} = \ell_1, \quad \lim_{\xi \to +\infty} \frac{1 - V(\xi)}{|\xi|^{\gamma+} e^{\Lambda_+ \xi}} = \ell_2,
\]

where \(\Lambda_+ := \max\{\lambda_1, \lambda_2\} < 0\) and

\[
\gamma_+ = \begin{cases} 
0, & \text{if } \lambda_1 \neq \lambda_2, \\
1, & \text{if } \lambda_1 = \lambda_2.
\end{cases}
\]

For the asymptotic behavior of \((U, V)\) near \(\xi = -\infty\), we need the following characteristic equations:

\[
\begin{align*}
\lambda_3 &= -c + \sqrt{c^2 + 4r} / 2d, \\
\lambda_4 &= -c + \sqrt{c^2 + 4(b - 1)} / 2.
\end{align*}
\]

Lemma 2.2 ([23, 33]). There exist two positive constants \(\ell_3\) and \(\ell_4\) such that

\[
\lim_{\xi \to -\infty} \frac{1 - U(\xi)}{|\xi|^{\gamma+} e^{\Lambda_- \xi}} = \ell_3, \quad \lim_{\xi \to -\infty} \frac{V(\xi)}{e^{\Lambda_+ \xi}} = \ell_4.
\]
where $\Lambda := \min\{\lambda_3, \lambda_4\} > 0$ and

$$
\gamma := \begin{cases} 
0, & \text{if } \lambda_3 \neq \lambda_4, \\
1, & \text{if } \lambda_3 = \lambda_4.
\end{cases}
$$

### 2.2. Comparison principle.

It is well known that system (1.1)-(1.2) can be reduced to a monotone system, which has the comparison principle (see, e.g., [4]). For reader’s convenience, we recall the notion of super and subsolutions and the comparison principle.

Define the differential operators

$$
N_1[u,v](t,x) := u_t - du_{xx} - ru(1 - u - av), \quad N_2[u,v](t,x) := v_t - v_{xx} - v(1 - v - bu).
$$

We say that $(\bar{u}, \bar{v})$ with $(\bar{u}, \bar{v}) \in [C(D) \cap C^2(D)]^2$ is a pair of supersolution of (1.1) in

$$
D := (\tau, T) \times (\zeta_1, \zeta_2), \quad 0 \leq \tau < T \leq \infty,
$$

if $(\bar{u}, \bar{v})$ satifies

$$
\begin{cases} 
N_1[\bar{u}, \bar{v}] \geq 0, & N_2[\bar{u}, \bar{v}] \leq 0 \quad \text{in } D, \\
\bar{u}(\tau, \cdot) \geq u_0(\cdot), & \bar{v}(\tau, \cdot) \leq v_0(\cdot) \quad \text{in } (\zeta_1, \zeta_2), \\
\bar{u}(t, \zeta_i) \geq u_0(\zeta_i), & \bar{v}(t, \zeta_i) \leq v_0(\zeta_i) \quad \text{for } t \in (\tau, T) \text{ and } i = 1, 2.
\end{cases}
$$

When $\zeta_1 = -\infty$ or $\zeta_2 = \infty$, the corresponding boundary condition (the third condition) in (2.5) is omitted. A pair of subsolution $(u, \bar{v})$ of (1.1)-(1.2) in $D$ can be defined analogously by reversing all inequalities.

The following is the standard comparison principle (see, e.g., [34]).

**Lemma 2.3 (Comparison Principle).** Suppose that $(\bar{u}, \bar{v})$ is a supersolution of (1.1) in $D := (\tau, T) \times (\zeta_1, \zeta_2)$, and $(u, \bar{v})$ is a subsolution of (1.1) in $D$. Then $\bar{u} \geq u$ and $\bar{v} \geq v$ in $D$.

**Remark 2.1.** The definition of super and subsolutions can be weakened slightly. For example, when both $(\bar{u}_1, \bar{v})$ and $(u_2, \bar{v})$ are subsolution in $D$, then $(\max\{\bar{u}_1, u_2\}, \bar{v})$ can be referred to as a subsolution in $D$ such that the comparison principle remains true. We refer to [18] for more discussion.

### 2.3. Some crucial estimates.

In this subsection, we present several lemmas to provide crucial estimates of the solution $(u, v)$ to problem (1.1)-(1.2), which play an important role in our analysis.

**Lemma 2.4.** There exist $M > 0$ and $T \gg 1$ such that

$$
egin{align*}
(2.6) & \quad u(t, x) \leq 1 + Me^{-rt}, \quad \forall t \geq T, \ x \in \mathbb{R}, \\
(2.7) & \quad v(t, x) \leq 1 + Me^{-t}, \quad \forall t \geq T, \ x \in \mathbb{R}.
\end{align*}
$$

**Proof.** Consider the ODE problem

$$
\begin{align*}
 w'(t) &= rw(1-w), \quad w(0) = \|u_0\|_{L^\infty} := w_0.
\end{align*}
$$

By an elementary calculation, we have

$$
 w(t) = \frac{w_0}{w_0 + (1 - w_0)e^{-rt}}, \quad t \geq 0.
$$
Clearly, there exist positive constants $T$ and $M$ such that $w(t) \leq 1 + Me^{-rt}$ for $t \geq T$. Then (2.6) follows by comparing $u(t, x)$ and $w(t)$. Similarly, (2.7) holds true.

**Lemma 2.5.** If $c > c_\epsilon := 2\sqrt{rd}$, then for any small $\varepsilon > 0$, there exist $M, \mu > 0$ and $T \gg 1$ such that

$$u(t, x) \leq Me^{-\mu[c-2\sqrt{rd(1+\varepsilon)}]t}, \quad \forall t \geq T, \ x > ct.$$  

**Proof.** Let $U_\varepsilon$ be the solution of

$$\begin{cases}
c\varepsilon U' + dU'' + r(1+\varepsilon - U)U = 0, \quad \xi \in \mathbb{R}, \\
U(-\infty) = 1 + \varepsilon, \quad U(+\infty) = 0, \quad U(0) = 1/2,
\end{cases}$$

where $c_\xi = 2\sqrt{rd(1+\varepsilon)}$ and $\varepsilon > 0$ is small enough such that $c_\epsilon < c_\xi < c$. By Lemma 2.4, there exists $T > 0$ such that

$$u(t, x) < 1 + \varepsilon/2 \quad \text{for all} \ t \geq T \text{ and } x \in \mathbb{R}.$$  

(2.8)

Recall from [25] that there exists $C > 0$ such that

$$U_\varepsilon(\xi) \sim C\xi e^{-[c_\xi/(2d)]\xi}, \quad \text{as} \ \xi \to \infty.$$

(2.9)

On the other hand, consider the Cauchy problem

$$\begin{cases}
w_t = dw_{xx} + rw, \ x \in \mathbb{R}, \ t > 0, \\
w(0, x) = u_0(x).
\end{cases}$$

Then, it is easily seen that

$$w(t, x) \leq K(t)e^{-x^2/(4dt)}, \quad \forall x \in \mathbb{R}, \ t > 0$$

for some $K > 0$ depending on $t$. By comparing $u$ and $w$, we thus obtain

$$u(T, x) = O(e^{-x^2/(4dT)}), \quad \text{as} \ x \to \infty.$$  

(2.10)

Therefore, by (2.8), (2.9) and (2.10), one can find $\hat{x} \gg 1$ such that $u(T, x) \leq U_\varepsilon(x - \hat{x})$ for all $x \in \mathbb{R}$.

Define

$$\overline{u}(t, x) := U_\varepsilon(x - \hat{x} - c_\xi(t - T)), \quad \underline{u}(t, x) = 0.$$  

It is easy to check that

$$N_1[\overline{u}, \underline{u}](t, x) = r\varepsilon U_\varepsilon \geq 0, \quad N_2[\overline{u}, \underline{u}](t, x) = 0 \quad \text{in} \ (T, \infty) \times \mathbb{R}.$$  

Thus, by comparison, we have $\overline{u}(t, x) \geq u(t, x)$ in $[T, \infty) \times \mathbb{R}$, and in turn that for all $t \geq T$ and $x > ct$,

$$u(t, x) \leq U_\varepsilon\left((c - c_\xi)t + c_\xi T - \hat{x}\right),$$

which together with (2.9), completes the proof. \qed

Next, we establish an exponential decay rate of $v$ when the spreading of $u$ occurs, which is important in order to construct a suitable subsolution.
Lemma 2.6. For any given \( c \in (0, c_{uv}) \) and small \( \epsilon > 0 \) such that
\[
\gamma_\epsilon := b(1-\epsilon) - 1 > 0,
\]
there exist positive constants \( T \) and \( M \) such that
\[
v(t, x) \leq Me^{-\gamma_\epsilon t}, \quad \forall t \geq T, \ x \in [-ct, ct].
\]

Proof. As in the proof of (9) in [5] we have
\[
\lim_{t \to \infty} \left[ \max_{x \in [-ct, ct]} |u(t,x) - 1| + \max_{x \in [-ct, ct]} v(t,x) \right] = 0. \tag{2.11}
\]
By (2.11), for any given small \( \epsilon > 0 \), there exists \( T \gg 1 \) such that
\[
0 \leq v(t,x) \leq \epsilon \quad \text{for all } t \geq T \text{ and } x \in [-ct, ct], \tag{2.12}
\]
\[
u(t,x) \geq 1 - \epsilon \quad \text{for all } t \geq T \text{ and } x \in [-ct, ct]. \tag{2.13}
\]
By the definition of \( \gamma_\epsilon \) and (2.13), we see from \( v \) equation in (1.1) that
\[
v_t \leq v_{xx} - \gamma_\epsilon v \quad \text{for all } t \geq T \text{ and } x \in [-ct, ct]. \tag{2.14}
\]
Given \( L > 0 \), consider the following fixed boundary problem
\[
\begin{cases}
\psi_t = \psi_{xx} - \gamma_\epsilon \psi, & t > 0, \ -L < x < L, \\
\psi(t, \pm L) = \epsilon, & t > 0, \\
\psi(0, x) = \epsilon - \chi(x) \leq \epsilon, & -L < x < L.
\end{cases}
\]
Note that the above problem admits the unique positive steady state
\[
\chi(x) := \frac{e^{\sqrt{\gamma_\epsilon x}} + e^{-\sqrt{\gamma_\epsilon x}}}{e^{\sqrt{\gamma_\epsilon L}} + e^{-\sqrt{\gamma_\epsilon L}}} \epsilon, \quad -L \leq x \leq L.
\]
Moreover, we have
\[
\lim_{t \to \infty} \psi(t,x) = \chi(x) \leq \epsilon \quad \text{uniformly for } x \in [-L,L].
\]
Denote
\[
\Psi(t, x) = \psi(t, x) - \chi(x).
\]
After some simple calculations, \( \Psi \) solves
\[
\begin{cases}
\Psi_t = \Psi_{xx} - \gamma_\epsilon \Psi, & t > 0, \ -L < x < L, \\
\Psi(t, \pm L) = 0, & t > 0, \\
\Psi(0, x) = \epsilon - \chi(x) \leq \epsilon, & -L < x < L.
\end{cases}
\]
By a simple comparison (with an obvious ODE problem), we have
\[
0 \leq \Psi(t,x) \leq \epsilon e^{-\gamma_\epsilon t} \quad \text{for } t > 0 \text{ and } -L \leq x \leq L,
\]
which gives
\[
\psi(t,x) \leq \epsilon \left( e^{-\gamma_\epsilon t} + \frac{e^{\sqrt{\gamma_\epsilon x}} + e^{-\sqrt{\gamma_\epsilon x}}}{e^{\sqrt{\gamma_\epsilon L}} + e^{-\sqrt{\gamma_\epsilon L}}} \right) \quad \text{for } t > 0 \text{ and } -L \leq x \leq L.
\]
In particular, taking any \( \sigma \in (0, 1/\sqrt{\gamma_\epsilon}) \), we deduce
\[
\psi(t,x) \leq \epsilon \left( e^{-\gamma_\epsilon t} + \frac{2e^{\sqrt{\gamma_\epsilon |x|}}}{e^{\sqrt{\gamma_\epsilon L}}} \right) \leq \epsilon \left( e^{-\gamma_\epsilon t} + 2e^{-\gamma_\epsilon \sigma L} \right) \leq \epsilon \left( e^{-\gamma_\epsilon t} + 2e^{-\gamma_\epsilon \sigma L} \right)
\]
for all \( t > 0 \) and \( x \in [-(1 - \sqrt{\gamma_\epsilon \sigma})L, (1 - \sqrt{\gamma_\epsilon \sigma})L]. \)
Let $L = t/\alpha$ with
\begin{equation}
0 < \alpha < \frac{1 - \sqrt{\gamma_\epsilon \sigma}}{c}.
\end{equation}
Then
\[ \psi(t, x) \leq \epsilon(e^{-\gamma_\epsilon t} + 2e^{-\gamma_\epsilon \sigma t/\alpha}) \]
for all $t > 0$ and $x \in [-(1 - \sqrt{\gamma_\epsilon \sigma})t/\alpha, (1 - \sqrt{\gamma_\epsilon \sigma})t/\alpha]$. Because of (2.15), we see that
\[ [-ct, ct] \subset \left[ -(1 - \sqrt{\gamma_\epsilon \sigma})t/\alpha, (1 - \sqrt{\gamma_\epsilon \sigma})t/\alpha \right], \]
which together with (2.12) and (2.14) enables one to apply comparison principle to assert that
\[ v(t + T, x) \leq \psi(t, x) \leq \epsilon(e^{-\gamma_\epsilon t} + 2e^{-\gamma_\epsilon \sigma t/\alpha}) \leq \epsilon \left\{ e^{-\gamma_\epsilon t} + 2e^{\frac{-c(\gamma_\epsilon \sigma t)}{1 - \sqrt{\gamma_\epsilon \sigma}}t} \right\} \]
for $t > 0$ and $x \in [-ct, ct]$. Taking $\sigma$ close to $1/\sqrt{\gamma_\epsilon}$, we thus complete the proof. \hfill \Box

When $c_u > c_v$, as in (12) of [5, p.2137], one has
\begin{equation}
\lim_{t \to \infty} \left[ \max_{x \in [-ct, ct]} |u(t, x) - 1| + \max_{x \in [-ct, ct]} v(t, x) \right] = 0
\end{equation}
for any $c \in (0, c_u)$. Then, replacing (2.11) by (2.16) and following the same line of the proof of Lemma 2.6 one can obtain

**Corollary 2.7.** Assume that $c_u > c_v$. For any given $c \in (0, c_u)$ and small $\epsilon > 0$, there exist positive constants $T$ and $M$ such that
\[ v(t, x) \leq M e^{-\gamma_\epsilon t}, \quad \forall t \geq T, \; x \in [-ct, ct], \]
where $\gamma_\epsilon$ is defined in Lemma 2.6.

**Lemma 2.8.** For any given $c \in (0, c_{uv})$, there exist positive constants $\delta, T$ and $M$ such that
\[ u(t, x) \geq 1 - M e^{-\delta t}, \quad \forall t \geq T, \; x \in [-ct, ct]. \]

**Proof.** Thanks to Lemma 2.6 there exist positive constants $T_1, M_1 > 0$ and $\delta_1 > 0$ such that
\[ v(t, x) \leq M_1 e^{-\delta_1 t}, \quad \forall t \geq T_1, \; x \in [-ct, ct]. \]
By (2.11), one can take $\eta > 0$ close to 1 and $\hat{T} \geq T_1$ such that
\[ u(t, x) \geq \eta, \quad \forall t \geq \hat{T}, \; x \in [-ct, ct], \]
which also yields that $u(1 - u) \geq \eta(1 - u)$ for all $u \in [\eta, 1]$.

To construct a subsolution of $u$-equation, we consider
\[ \begin{cases} 
\phi_t = d\phi_{xx} + \tau [\eta(1 - \phi) - aM_1 e^{-\delta_1(t+\hat{T})} \phi], & t > 0, \; -c\hat{T} < x < c\hat{T}, \\
\phi(t, \pm c\hat{T}) = \eta, & t \geq 0, \\
\phi(0, x) = \eta, & -c\hat{T} \leq x \leq c\hat{T}.
\end{cases} \]
Clearly, $\eta \leq \phi \leq 1$. It can be seen that $\phi$ is a subsolution for the equation solved by $u(t + \hat{T}, x)$ for $t > 0$ and $-c\hat{T} \leq x \leq c\hat{T}$.

We now investigate the long-time behavior of $\phi$. For convenience, let us define
\[ q(t) := 1 + \frac{aM_1}{\eta} e^{-\delta_1(t+\hat{T})}. \]
Then, we can rewrite
\[ r[\eta(1 - \phi) - aM_1e^{-\delta_1(t+T)}\phi] = r\eta - r\eta q(t)\phi. \]
Let us further define
\[ \Phi(t, x) := e^{Q(t)}[\phi(t, x) - \eta], \quad Q(t) := (r\eta)t - \frac{raM_1}{\delta_1}e^{-\delta_1(t+T)} \]
such that \( Q'(t) = r\eta q(t) \). A straightforward computation brings \( \phi \)-equation into \( \Phi \)-equation:
\[
\begin{cases}
\Phi_t = d\Phi_{xx} + r\eta e^{Q(t)}[1 - \eta q(t)], & t > 0, \quad -c^\hat{T} < x < c^\hat{T}, \\
\Phi(t, \pm c^\hat{T}) = 0, & t \geq 0, \\
\Phi(0, x) = 0, & -c^\hat{T} \leq x \leq c^\hat{T}.
\end{cases}
\]
Using the Green function of heat equation, we have
\[
\Phi(t, x) = r\eta \int_0^t e^{Q(\tau)}[1 - \eta q(\tau)] \int_{c^\hat{T}}^{c^\hat{T}} \tilde{G}(t, x; \tau, \xi)d\xi d\tau, \quad t > 0, \quad -c^\hat{T} < x < c^\hat{T},
\]
where \( \tilde{G}(t, x; \tau, \xi) \) is the green function defined by
\[
\tilde{G}(t, x; \tau, \xi) = \sum_{n \in \mathbb{Z}} (-1)^n G(t - \tau, x - \xi - 2nc^\hat{T}),
\]
with the heat kernel \( G \) given by
\[
G(t, x; \tau, \xi) = \frac{1}{\sqrt{4\pi d(t - \tau)}} e^{-\frac{(x-\xi)^2}{2d(t-\tau)}}.
\]

In what follows, we will use an estimate given in [8, Lemma 6.5] (note that although \( d = 1 \) therein, the same argument in [8] can yield the estimate for general \( d \)): for any \( \epsilon \in (0, 1) \), there exists \( T^* \gg 1 \) such that for all \( \hat{T} \geq T^* \),
\[
\int_{c^\hat{T}}^{c^\hat{T}} \tilde{G}(t, x; \tau, \xi)d\xi d\tau \geq 1 - \frac{4}{\sqrt{\pi}} e^{-\frac{\epsilon^2}{2\sqrt{d}}} \quad \text{for all } (x, t) \in \hat{D}_\epsilon,
\]
where \( \hat{D}_\epsilon \) is defined by
\[
\hat{D}_\epsilon := \left\{ (t, x) : 0 < t \leq \frac{\epsilon e^2c^\hat{T}}{4\sqrt{d}}, \quad |x| \leq (1 - \epsilon)c^\hat{T} \right\}.
\]
In light of this estimate, we obtain
\[
\Phi(t, x) \geq r\eta \left( 1 - \frac{4}{\sqrt{\pi}} e^{-\frac{\epsilon^2}{2\sqrt{d}}} \right) \int_0^t e^{Q(\tau)}[1 - \eta q(\tau)]d\tau \\
\geq r\eta \left( 1 - \frac{4}{\sqrt{\pi}} e^{-\frac{\epsilon^2}{2\sqrt{d}}} \right) \left( 1 - \eta - aM_1e^{-\delta_1\hat{T}} \right) \int_0^t e^{Q(\tau)}d\tau
\]
for all \( (t, x) \in \hat{D}_\epsilon \).
Recalling the definition of \( \Phi \), we have
\[
\phi(t, x) = e^{-Q(t)}\Phi(t, x) + \eta.
\]
Then
\[
(2.17) \quad \phi(t, x) \geq \left( 1 - \frac{4}{\sqrt{\pi}} e^{-\frac{\epsilon^2}{2\sqrt{d}}} \right) \left( 1 - \eta - aM_1e^{-\delta_1\hat{T}} \right) r\eta e^{-Q(t)} \int_0^t e^{Q(\tau)}d\tau + \eta
\]
for all \((t, x) \in \hat{D}_\epsilon\). By some simple calculations, we see that
\[
\begin{align*}
    r_\eta e^{-Q(t)} \int_0^t e^{Q(\tau)} d\tau &= (r_\eta) e^{-r_\eta t + \gamma e^{-\delta_1 (t + \hat{T})}} \left[ \int_0^t e^{r_\eta \tau - \gamma e^{-\delta_1 (\tau + \hat{T})}} d\tau \right] \\
    &\geq (r_\eta) e^{-r_\eta t + \gamma e^{-\delta_1 (\tau + \hat{T})}} \frac{1}{r_\eta} e^{r_\eta t} \left| t \right|_0 \\
    &= e^{\gamma e^{-\delta_1 (\tau + \hat{T})}} (1 - e^{-r_\eta t}),
\end{align*}
\]
where \(K := raM_1/\delta_1\). Plugging this estimate into (2.17), we have
\[
    \phi(t, x) \geq e^{K[e^{-\delta_1 \hat{T}} (1 - e^{-\delta_1 t})]} (1 - e^{-r_\eta t}) \left[ 1 - \frac{4}{\sqrt{\pi}} e^{-\frac{\phi}{2\sqrt{n}}} \right] \left[ 1 - M_1 e^{-\delta_1 \hat{T}} \right]
\]
for all \((t, x) \in \hat{D}_\epsilon\). By the fact that \(e^x \geq 1 - x\) for all \(x\), we then obtain
\[
    \phi(t, x) \geq [1 - K e^{-\delta_1 \hat{T}} (1 - e^{-\delta_1 t})] (1 - e^{-r_\eta t}) \left[ 1 - \frac{4}{\sqrt{\pi}} e^{-\frac{\phi}{2\sqrt{n}}} \right] \left[ 1 - M_1 e^{-\delta_1 \hat{T}} \right]
\]
for all \((t, x) \in \hat{D}_\epsilon\) by taking \(\hat{T}\) larger if necessary.

Set \(t = e^2 c^2 \hat{T}/(4\sqrt{a})\) and \(\epsilon > 0\) small enough such that
\[
    \frac{r_\eta e^2 c^2}{4\sqrt{a}} < \delta_1,
\]
we obtain
\[
    \phi\left( \frac{e^2 c^2 \hat{T}}{4\sqrt{a}}, x \right) \geq 1 - K e^{-\delta_1 \hat{T}} - e^{-r_\eta e^2 c^2 \hat{T}/(4\sqrt{a})} \\
    \geq 1 - (K + 1) e^{-r_\eta e^2 c^2 \hat{T}/(4\sqrt{a})},
\]
The parabolic comparison principle gives \(u(t + \hat{T}, x) \geq \phi(t, x)\), which together with (2.18) implies
\[
    u\left( \frac{e^2 c^2 \hat{T}}{4\sqrt{a}} + \hat{T}, x \right) \geq 1 - (K + 1) e^{-r_\eta e^2 c^2 \hat{T}/(4\sqrt{a})}
\]
for all \(|x| \leq (1 - \epsilon)e\hat{T}\). Note that
\[
    t = \frac{e^2 c^2 \hat{T}}{4\sqrt{a}} + \hat{T} \iff \hat{T} = \left(1 + \frac{e^2 c^2}{4\sqrt{a}}\right)^{-1} t.
\]
This yields that
\[
    u(t, x) \geq 1 - M e^{\delta_2 t} \quad \text{for } |x| \leq c\left(1 + \frac{e^2 c^2}{4\sqrt{a}}\right)^{-1} t, \quad t \geq T^{**},
\]
where
\[
    M = K + 1, \quad \delta_2 := r_\eta \left( \frac{e^2 c^2}{4\sqrt{a}} \right) \left(1 + \frac{e^2 c^2}{4\sqrt{a}}\right)^{-1} > 0, \quad T^{**} := T^* + \frac{e^2 c^2}{4\sqrt{a}} T^*.
\]
Since \(c\) can be arbitrarily close to \(c_{uv}\) and \(\epsilon > 0\) can be arbitrarily small, we thus completes the proof. \(\Square\)
3. Proof of Theorem \[1\] scenario (A1)

This section is devoted to the proof of Theorem \[1\]. To this aim, we shall construct suitable pairs of supersolutions and subsolutions. In this section, we always assume that \((u_0, v_0)\) satisfies (A1).

3.1. The construction of super-subsolutions.

3.1.1. The construction of a subsolution. Denote a subsolution \((u, v)\) by

\[
\begin{align*}
    u(t,x) &:= \max\{U(x - c_u t + \eta(t)) - p(t), 0\}, \\
    v(t,x) &:= (1 + q(t))V(x - c_v t + \eta(t)),
\end{align*}
\]

where

\[
\begin{align*}
    p(t) &= p_0 e^{-\alpha t}, \\
    q(t) &= q_0 e^{-\alpha t}, \\
    \eta(t) &= \eta_0 - \eta_1 e^{-\alpha t/2}
\end{align*}
\]

for some constants \(p_0 > 0, q_0 > 0, \alpha > 0\) and \(\eta_i \in \mathbb{R} (i = 0, 1)\) that will be determined later.

Lemma 3.1. For any \(p_0, q_0, \alpha, \eta_1 > 0\) satisfying

\[
\alpha < \min\{r, 1, (a - 1)r\}, \quad p_0 < \min\left\{1 - \frac{\alpha}{r}, \frac{q_0}{b}\right\},
\]

there exists \(T^* \geq 0\) such that

\[
N_1[u, v] \leq 0, \quad N_2[u, v] \geq 0 \quad \text{in} \ [T^*, \infty) \times (-\infty, \infty)
\]

for all \(\eta_0 \in \mathbb{R}\), where \(u\) and \(v\) are defined in (3.1).

Proof. Given any small \(\epsilon > 0\) satisfying

\[
\begin{align*}
    \epsilon &< \frac{(r - \alpha - rp_0)p_0}{r(2p_0 + aq_0)}, \\
    \epsilon &< \frac{(a - 1)r - \alpha}{2ra(1 + q_0p_0)}, \\
    \epsilon &< \frac{1 - \alpha}{4}.
\end{align*}
\]

Since \((U,V)(-\infty) = (1, 0)\) and \((U,V)(\infty) = (0, 1)\), there exists a sufficiently large constant \(M\) such that

\[
\begin{align*}
1 > U(\xi) > 1 - \epsilon, \quad V(\xi) < \epsilon \quad \text{for all} \ \xi \leq -M, \\
U(\xi) < \epsilon, \quad 1 > V(\xi) > 1 - \epsilon \quad \text{for all} \ \xi \geq M.
\end{align*}
\]

For simplicity, we set \(\xi = x - c_u t + \eta(t)\) and write \(U = U(\xi)\) (resp., \(V = V(\xi)\)). Also, we assume \(u > 0\) first, i.e., \(u(t,x) = U(\xi) - p(t) > 0\).

Then, by direct computations, we get from the first equation of (1.5) that

\[
\begin{align*}
N_1[u, v](t,x) &\quad = \eta'U' - c_u U' - p' - dU'' - r(U - p)[1 - U + p - a(1 + q)V] \\
&\quad = \eta'U' + r U(1 - U - aV) - p' - r(U - p)[1 - U + p - a(1 + q)V] \\
&\quad = \eta'U' - p' - rU(p - aV) + rp[1 - U + p - a(1 + q)V].
\end{align*}
\]
Also, by the second equation of (1.3), we have

\[(3.11) \quad N_2[u, v](t, x) = q'V + (1 + q)(-c_{uv} + \eta'V' - (1 + q)V'' - (1 + q)V[1 - (1 + q)V - b(U - p)])
\]

\[= q'V + (1 + q)[V(1 - V - bU) + \eta'V'] - (1 + q)V[(1 - (1 + q)V - bU + bp]
\]

\[= q'V + (1 + q)\eta'V' - (1 + q)V(bp - qV).
\]

Notice that if \( u = 0 \), then clearly \( N_1[u, v] = 0 \); while from (3.11) we see that \( u = 0 \) does not affect the equality in (3.11). Hence we can only consider that \( u(t, x) = U(\xi) - p(t) \).

We now divide our discussion into three cases:

(i) \( \xi < -M \); (ii) \( |\xi| \leq M \); (iii) \( \xi > M \).

**Case (i).** By the fact that \( \eta' > 0 \) (since \( \alpha, \eta_1 > 0 \)) and \( U' < 0 \), we have \( \eta'U' < 0 \). Combined with (3.8) and (3.10) we deduce

\[N_1[u, v](t, x) \leq -p' - rU(p - aqV) + rp[1 - U + p]
\]

\[\leq -p' - r(1 - \epsilon)p + rae + rp(\epsilon + p)
\]

\[= -p' - rp + rp^2 + 2re + reaq
\]

\[= [(\alpha - r + rp_0e^{\alpha t})p + r(2p_0 + aq_0)]e^{\alpha t}.
\]

Thanks to (3.5), we see that \( N_1[u, v] \leq 0 \) for all \( \xi < -M \).

On the other hand, since \( V'(\cdot)/V(\cdot) \geq \kappa_0 \) in \((-\infty, -M) \) for some \( \kappa_0 > 0 \) (due to Lemma 2.2), from (3.11) we have

\[N_2[u, v](t, x) \geq [\frac{q'}{1 + q} + \kappa_0\eta' - bp) (1 + q)V
\]

\[\geq e^{-(\alpha/2)t}[\kappa_0q_0e^{-(\alpha/2)t} + \kappa_0\kappa_1e^{-\alpha t} - bp_0e^{-\alpha t}(1 + q)V.
\]

Thus, one can find \( T_1 \gg 1 \) such that \( N_2[u, v](t, x) \geq 0 \) for all \( (x, t) \) satisfying \( \xi < -M \) and \( t \geq T_1 \).

**Case (ii).** Since \( U' < 0 \) in \( \mathbb{R} \), we have \( \max_{\xi \in [-M, M]} U'(\xi) = -\kappa_1 < 0 \). Also, by virtue of \( V' \leq 1 \), it is easily seen that

\[N_1[u, v](t, x) \leq -\eta\kappa_1 - p' - rU(p - aq) + rp(1 + p)
\]

\[= \frac{\alpha}{2}e^{-(\alpha/2)t} + O(1)e^{\alpha t}.
\]

Therefore, there exists \( T_2 \gg 1 \) such that \( N_1[u, v](t, x) \leq 0 \) for all \( (x, t) \) satisfying \( |\xi| \leq -M \) and \( t \geq T_2 \).

Since \( V' > 0 \) in \( \mathbb{R} \), we have \( \min_{\xi \in [-M, M]} V'(\xi) = \kappa_2 > 0 \). Then, it holds

\[N_2[u, v](t, x) \geq q'V + \kappa_2\eta' - (1 + q)Vbp = \frac{\alpha}{2}e^{-(\alpha/2)t} - O(1)e^{-\alpha t}.
\]

Hence, there exists \( T_3 \gg 1 \) such that \( N_2[u, v](t, x) \geq 0 \) for all \( (x, t) \) satisfying \( |\xi| \leq -M \) and \( t \geq T_3 \).
Lemma 3.2. Since by Lemma 3.1, there exists \( T \) such that (3.3) holds. By (3.7), there exists \( T \) such that (3.8) holds. In view of (3.7), we deduce that (3.4) holds for all \( x, t \) such that \( \xi > M \). Hence, there exists \( T \) such that (3.5) holds for all \( (x,t) \) such that \( \xi > M \) and \( t \geq T_4 \).

On the other hand, by means of \( \eta'V' > 0 \) and (3.9), we obtain

\[
N_1[u, \bar{v}](t, x) \leq -p' + raqUV + rp + rp^2 - rpaV \\
\leq -p' + raq\epsilon + rp + rp^2 - rpa(1 - \epsilon) \quad \text{thanks to (3.9)} \\
= -p' - (a - 1)rp + rp^2 + ra(p + q)\epsilon \\
= \left[ \alpha - (a - 1)r + rp_0e^{-at} + ra\left(1 + \frac{q_0}{p_0}\right)\epsilon \right]p_0e^{-at} \\
\leq \left[ \frac{\alpha - (a - 1)r}{2} + rp_0e^{-at} \right]p_0e^{-at} \quad \text{(using (3.6)).}
\]

By (3.3), there exists \( T_4 \gg 1 \) such that \( N_1[u, \bar{v}](t, x) \leq 0 \) for all \( (x, t) \) satisfying \( \xi > M \) and \( t \geq T_4 \).

Combining the discussions in cases (i)-(iii) and taking \( T_* := \max\{T_1, T_2, T_3, T_4, T_5\} \geq 0 \), we have proved (3.4) for all \( x \in \mathbb{R} \) and \( t \geq T_* \). This completes the proof. \( \square \)

Next, we shall show that the parameters in \((u, \bar{v})\) can be chosen suitably such that it can compare with the solution \((u, v)\) of (1.1)-(1.2) from a large time.

Lemma 3.2. Let \((u, \bar{v})\) be defined in (3.1) and satisfy (3.3). Then there exist small \( \alpha^* > 0 \) and large \( T^* > 0 \) and \( \eta_0 > 0 \) such that

\[
u \geq u, \quad \bar{v} \geq v \quad \text{in } \mathbb{R} \times [T^*, \infty) \times [0, \infty),
\]

provided that \( \alpha \in (0, \alpha^*) \) and \( \eta_0 \geq \eta^* \).

Proof. We shall show that \((u, \bar{v})\) is a subsolution for \( 0 \leq x < \infty \) and \( t \geq T_* \) for some \( T_* \gg 1 \).

First, by Lemma 3.1, there exists \( T_1 \gg 1 \) such that

\[
N_1[u, \bar{v}](t, x) \leq 0, \quad N_2[u, \bar{v}](t, x) \geq 0 \quad \text{in } \mathbb{R} \times [0, \infty).
\]

By Lemma 2.8 one can find positive constants \( M, \delta \) and \( T_2 \) such that

\[
u(t, 0) \geq 1 - Me^{-\delta t} \quad \text{for all } t \geq T_2.
\]

Since

\[
u(t, 0) \leq U(-c_{uv}t + \eta_0 + \eta_1e^{-(\alpha/2)t}) - p_0e^{-at} \leq 1 - p_0e^{-at},
\]

by taking \( \alpha < \delta \), there exists \( T_3 > T_2 \) such that

\[
u(t, 0) \leq u(t, 0) \quad \text{for } t \geq T_3.
\]
Next, we show that there exists $T_4 > 0$ such that
\[
(3.14) \quad \overline{v}(t, 0) \geq v(t, 0) \quad \text{for } t \geq T_4.
\]
In view of Lemma 2.6, there exists a positive constant $K_1$ such that
\[
(3.15) \quad v(t, 0) \leq K_1 e^{-\gamma_\epsilon t} \quad \text{for all large } t.
\]
By the definition of $\overline{v}(t, 0)$ and Lemma 2.2, we have
\[
(3.16) \quad \overline{v}(t, 0) \geq V(x - c_{uv} t + \eta(t)) \sim C e^{-c_{uv} \lambda_4 t} \quad \text{for all large } t \text{ and some } C > 0,
\]
where
\[
\lambda_4 = \frac{-c_{uv} + \sqrt{c_{uv}^2 + 4(b - 1)}}{2} = \frac{2(b - 1)}{c_{uv} + \sqrt{c_{uv}^2 + 4(b - 1)}} > 0.
\]
Now, choosing $\epsilon > 0$ sufficiently small such that
\[
c_{uv} \lambda_4 = \frac{2(b - 1)}{1 + \sqrt{1 + \frac{4(b - 1)}{c_{uv}^2}}} < b(1 - \epsilon) - 1 =: \gamma_\epsilon.
\]
Then (3.14) follows from (3.15) and (3.16).

Take $T^* > \max\{T_1, T_2, T_3, T_4\}$. Then we see that
\[
\overline{v}(T^*, x) = (1 + q_0 e^{-\alpha T^*}) V(x - c_{uv} T^* + \eta_0 - \eta_1 e^{-(\alpha/2) T^*}).
\]
Thanks to (2.7) and the definition of $u$, one can choose $\alpha^* > 0$ sufficiently small and $\eta_0^* > 0$ sufficiently large such that
\[
(3.17) \quad u(T^*, \cdot) \leq u(T^*, \cdot), \quad \overline{v}(T^*, \cdot) \geq v(T^*, \cdot) \quad \text{in } [0, \infty)
\]
for all $\alpha \in (0, \alpha^*)$ and $\eta_0 \geq \eta_0^*$.

Combining (3.12), (3.13), (3.14) and (3.17), the desired result follows from the comparison principle. \hfill \Box

### 3.1.2. The construction of a supersolution.

To seek a pair of supersolution, we define
\[
(3.18) \quad \begin{cases}
\overline{u}(t, x) = (1 + q(t)) U(x - c_{uv} t + \eta(t)), \\
\underline{u}(t, x) = \max\{V(x - c_{uv} t + \eta(t)) - p(t), 0\},
\end{cases}
\]
where $p, q$ and $\eta$ have the same form as in (3.2).

The following lemma is parallel to Lemma 3.1; we only give some sketch of the proof.

**Lemma 3.3.** For any $p_0, q_0 > 0, \alpha \in (0, 1)$ and $\eta_1 < 0$ satisfying
\[
(3.19) \quad \alpha < \min\{r, 1, b - 1\}, \quad p_0 < \min\left\{1 - \alpha, \frac{q_0 \eta_1}{\alpha \left(1 - \frac{\alpha}{2}\right)}\right\},
\]
there exists $T^{**} \geq 0$ such that
\[
(3.20) \quad N_1[\overline{u}, \underline{v}] \geq 0, \quad N_2[\overline{u}, \underline{v}] \leq 0 \quad \text{in } [T^{**}, \infty) \times (-\infty, \infty)
\]
for all $\eta_0 \in \mathbb{R}$, where $\overline{u}$ and $\underline{v}$ are defined in (3.18).
Proof. As in the proof of Lemma 3.1, for any sufficiently small $\epsilon > 0$, there exists a sufficiently large constant $M$ such that (3.8) and (3.9) hold. Denote $x - c_u t + \eta(t)$ by $\xi$ and write $U = U(\xi)$ (resp., $V = V(\xi)$). By direct computations, we have

$$N_1[\bar{\mu}, \bar{\eta}](t, x) = q' U + (1 + q)(-c_u + \eta')U' - (1 + q)U'' - r(1 + q)U[1 - (1 + q)U - aV + ap]$$

and

$$N_2[\bar{\mu}, \bar{\eta}](t, x) = \eta' V' - c_u V' - dV'' - \eta' - (V - p)[1 - V + p - b(1 + q)U]$$

Similar to the proof of Lemma 3.1 we divide our discussion into three cases:

(i) $\xi < -M$; (ii) $|\xi| \leq M$; (iii) $\xi > M$.

Case (i): this part can be done similarly as in Case (iii) of the proof of Lemma 3.1. By (3.8) and the fact that $\eta' U' > 0$ (since $\eta_1 < 0$ and $U' < 0$), from (3.21) it follows

$$N_1[\bar{\mu}, \bar{\eta}](t, x) \geq q' U + r(1 + q)U(qU - ap) \geq U[-\alpha q_0 + q_0(1 - \epsilon)^2 - r(1 + q_0 e^{-\alpha t})ap_0 e^{-\alpha t}]e^{-\alpha t}.$$ 

Due to (3.19) and the fact that $\epsilon$ can be chosen smaller than $(1 - \alpha)/4$, we further have

$$q_0((1 - \epsilon)^2 - \alpha) - ap_0 \geq q_0(1 - 2\epsilon - \alpha) - ap_0 \geq q_0\left(\frac{1 - \alpha}{2}\right) - ap_0 > 0.$$ 

Then there exists $T_1 \gg 1$ such that $N_1[\bar{\mu}, \bar{\eta}](t, x) \geq 0$ for all $(x, t)$ fulfilling $\xi < -M$ and $t \geq T_1$.

On the other hand, in view of $\eta' U' > 0$ and the behavior of $U$ and $V$ near $-\infty$, one also knows that

$$N_2[\bar{\mu}, \bar{\eta}](t, x) \leq -p' + b\epsilon + p + p^2 - bp(1 - \epsilon) = [\alpha - (b - 1) + p_0 e^{-\alpha t} + O(1)\epsilon]p_0 e^{-\alpha t}.$$ 

Hence, thanks to (3.19) and the fact that $\epsilon$ can be chosen smaller if necessary, there exists $T_2 \gg 1$ such that $N_2[\bar{\mu}, \bar{\eta}](t, x) \geq 0$ for all $(x, t)$ satisfying $\xi < -M$ and $t \geq T_2$.

Case (ii) and Case (iii) can be handled by the similar process as in Case (ii) and Case (i) of the proof of Lemma 3.1 respectively; we omit the details here.

According to the above analysis, we see that there exists $T^{**} \geq 0$ such that (3.20) holds, which completes the proof. \qed

Lemma 3.4. For each $c > c_u := 2\sqrt{rd}$, $v(x, t)$ converges to 1 uniformly for $x \in [ct, \infty)$ as $t \to \infty$.

Proof. The argument is similar to that of [5, Lemma 2] with minor modifications; we omit the details here. \qed
Lemma 3.5. Let \((u,v)\) be defined in (3.18) and satisfy (3.19). Then there exist small \(\alpha^{**} > 0\) and large \(\tau_0, T_0 > 0\) and \(\eta^{**}_0 < 0\) such that

\[
u(t + \tau_0) \leq \hat{u}(t,x), \quad \nu(t + \tau_0, x) \geq \hat{v}(t, x) \quad \text{in} \quad [T_0, \infty) \times [0, \infty),
\]

provided that \(\alpha \in (0, \alpha^{**}]\) and \(\eta_0 \leq \eta^{**}_0\).

Proof. By Lemma 3.3, there exists \(T_1 > 0\) such that

\[
u(t, x) \geq \hat{v}(t, x) \quad \text{for all} \quad t < T_1.
\]

To see this, similar to (2.10), we have (3.26)

\[
u(t, 0) \geq u(t, \tau, 0), \quad \nu(t, 0) \leq v(t, \tau, 0) \quad \text{for all} \quad t \geq T_2 \quad \text{and} \quad \tau \geq 0.
\]

To do so, we first notice that \(U' < 0, U(-\infty) = 1 \quad \text{and} \quad \alpha < r \quad \text{(from condition (3.19))}\). Then, one can choose \(T_2\) larger such that for any \(t \geq T_2\),

\[
u(t, 0) \geq (1 + q_0 e^{-\alpha t}) U(-c_{uv} t + \eta_0 - \eta_1 e^{-(\alpha/2)t}) \geq 1 + M e^{-rt}
\]

for all \(\eta_0 \leq 0\), where \(M\) is defined in Lemma 2.4. It follows from (2.6) and (3.25) that, for any \(\eta_0 \leq 0\) and \(\tau \geq 0, \nu(t, 0) \geq u(t + \tau, 0)\) for \(t \geq T_2\). On the other hand, if necessary one can take \(\alpha\) smaller and \(T_2\) larger such that for any \(\eta_0 \leq 0\) and \(\tau \geq 0, v(t, 0) = 0 < v(t + \tau, 0)\) for \(t \geq T_2\).

Hence, (3.24) holds for all \(\eta_0 \leq 0\), provided \(\alpha\) is small enough.

Define \(T_0 = \max\{T_1, T_2\}\). We show that for any large negative \(\eta_0\),

\[
u(T_0, x) \geq u(T_0 + \tau_0, x), \quad \nu(T_0, x) \leq v(T_0 + \tau_0, x) \quad \text{for} \quad 0 \leq x < \infty.
\]

To see this, similar to (2.10), we have \(u(T_0 + \tau_0, x) = O(e^{-x^2/[4d(T_0 + \tau_0)]})\) as \(x \to \infty\). Thanks to Lemma 2.4 we obtain

\[
u(T_0 + \tau_0, x) \leq (1 + q_0 e^{-\alpha T_0}) U(x - cT_0 - \eta_1 e^{-(\alpha/2)T_0}) \quad \text{for all large} \quad x.
\]

Moreover, since \(U(-\infty) = 1 \quad \text{and} \quad U' < 0\), it follows from (2.6) and (3.27) that there exists \(\tilde{\eta} = \tilde{\eta}(T_0)\) such that

\[
u(T_0 + \tau_0, x) \leq (1 + q_0 e^{-\alpha T_0}) U(x - cT_0 - \eta_0 - \eta_1 e^{-(\alpha/2)T_0}) \quad \text{for all} \quad x \in [0, \infty),
\]

provided \(\eta_0 \leq \tilde{\eta}\). This implies that \(\nu(T_0, x) \geq u(T_0 + \tau_0, x)\) for all \(x \in [0, \infty)\) as long as \(\eta_0 \leq \tilde{\eta}\).

On the other hand, by Lemma 3.3 and the definition of \(\nu(T_0, \cdot)\), there exists \(\tau_0 \gg 1\) such that for \(\eta_0 = 0, v(T_0 + \tau_0, x) \geq \nu(T_0, x)\) for all sufficiently large \(x\). Then, there exists \(\tilde{\eta}\) such that for all \(\eta_0 \leq \tilde{\eta}\), \(v(T_0 + \tau_0, x) \geq \nu(T_0, x)\) for all \(x \geq 0\). Therefore, (3.26) holds provided \(\eta_0 \leq \eta^{**}_0 := \min\{\tilde{\eta}, \tilde{\eta}\}\).

The above discussion shows that there exists \(\alpha^{**} > 0\) sufficiently small such that the comparison principle can be applied to obtain (3.22), provided \(\alpha \in (0, \alpha^{**}]\) and \(\eta_0 \leq \eta^{**}_0\). This completes the proof. \(\square\)

Let us consider the long time behavior of the solution of (1.1) for \(x \geq 0\). Set

\[\xi = x - c_{uv} t \quad \text{with} \quad x \geq 0.\]

Then one can define the solution of (1.1)-(1.2) defined for \(x \geq 0\) and \(t > 0\) as

\[(\hat{u}, \hat{v})(t, \xi) = (u, v)(t, x) = (u, v)(t, \xi + c_{uv} t), \quad t > 0, \quad \xi \geq -c_{uv} t.\]
Then $(\hat{u}, \hat{v})$ satisfies

\[
\begin{align*}
\dot{u}_t &= d\hat{u}_\xi + c_{uv}\hat{u}_\xi + r\hat{u}(1 - \hat{u} - a\hat{v}), \\
\dot{v}_t &= \hat{v}_\xi + c_{uv}\hat{v}_\xi + \hat{v}(1 - \hat{v} - b\hat{u}), \quad t > 0, \quad \xi \geq -c_{uv}t.
\end{align*}
\]

Thanks to Lemma 3.2 and Lemma 3.5 we can obtain the following result immediately.

**Lemma 3.6.** Let $(c_{uv}, U, V)$ be a solution of (1.5). Then there exist constants $p_0, q_0, \alpha > 0$ and $\eta^*_t, \eta^{**}_t \in \mathbb{R}$, $i = 0, 1$, and $T > 0$ such that

\[
\begin{align*}
U(\xi + \eta^{**}_t - \eta^{*}_1 e^{-(\alpha/2)t}) - p_0e^{-\alpha t} &\leq \hat{u}(t, \xi) \leq (1 + q_0 e^{-\alpha t})U(\xi + \eta^*_1 - \eta^{*}_1 e^{-(\alpha/2)t}), \\
V(\xi + \eta^{**}_t - \eta^{*}_1 e^{-(\alpha/2)t}) - p_0e^{-\alpha t} &\leq \hat{v}(t, \xi) \leq (1 + q_0 e^{-\alpha t})V(\xi + \eta^*_1 - \eta^{*}_1 e^{-(\alpha/2)t})
\end{align*}
\]

for all $t \geq T$ and $\xi \geq -c_{uv}t$.

By Lemma 3.6 and the comparison principle, we have the following result.

**Lemma 3.7.** Let $(c_{uv}, U, V)$ be a solution of (1.5). Then there exists a function $\nu(\epsilon)$ defined for small $\epsilon$ with $\nu(\epsilon) \to 0$ as $\epsilon \downarrow 0$ satisfying the following property: if $|\hat{u}(t_0, \xi) - U(\xi - \xi_0)| + |\hat{v}(t_0, \xi) - V(\xi - \xi_0)| < \epsilon$ for some $t_0, \xi_0 \in \mathbb{R}$, then

\[
|\hat{u}(t, \xi) - U(\xi - \xi_0)| + |\hat{v}(t, \xi) - V(\xi - \xi_0)| < \nu(\epsilon) \quad \text{for all } t \geq t_0 \text{ and } \xi \geq -c_{uv}t.
\]

### 3.2. The proof of Theorem 1

Let $(\hat{u}, \hat{v})$ be defined in (3.28) and $(c_{uv}, U, V)$ be a solution of (1.5). In addition, let $\{t_n\}$ be an arbitrary sequence such that $t_n > T$ ($T$ is defined in Lemma 3.6) for each $n$ and $t_n \to \infty$ as $n \to \infty$. Set

\[
\begin{align*}
\dot{u}_n(t, \xi) &= \hat{u}(t + t_n, \xi), \quad \dot{v}_n(t, \xi) = \hat{v}(t + t_n, \xi), \quad n \in \mathbb{N}.
\end{align*}
\]

By the standard parabolic regularity theory and passing to a subsequence, we may assume that

\[
(\dot{u}_n, \dot{v}_n) \to (u^\infty, v^\infty) \quad \text{in } C^{(1+\beta)/2, 1+\beta}(\mathbb{R} \times \mathbb{R}), \quad \text{as } n \to \infty,
\]

where $\beta \in (0, 1)$ and $(u^\infty, v^\infty)$ satisfies

\[
\begin{align*}
\dot{u}_\xi &= d\dot{u}_\xi + c_{uv}\dot{u}_\xi + rw^\infty(1 - u^\infty - av^\infty), \\
\dot{v}_\xi &= \dot{v}_\xi + c_{uv}\dot{v}_\xi + v^\infty(1 - v^\infty - bu^\infty), \quad t \in \mathbb{R}, \quad \xi \in \mathbb{R}.
\end{align*}
\]

In addition, let us replace $t$ by $t + t_n$ in the inequalities of Lemma 3.6 and take $n \to \infty$. Then we have

\[
\begin{align*}
\begin{cases}
U(\xi + \eta^*_1) \leq u^\infty(t, \xi) \leq U(\xi + \eta^{**}_1), & \forall t, \xi \in \mathbb{R}, \\
V(\xi + \eta^*_1) \leq v^\infty(t, \xi) \leq V(\xi + \eta^{**}_1), & \forall t, \xi \in \mathbb{R}.
\end{cases}
\end{align*}
\]

Define

\[
\begin{align*}
h_1 &= \inf\{h \in \mathbb{R} : u^\infty(t, \xi) \leq U(\xi - h) \text{ and } v^\infty(t, \xi) \geq V(\xi - h), \forall t, \xi \in \mathbb{R}\}, \\
h_2 &= \sup\{h \in \mathbb{R} : u^\infty(t, \xi) \geq U(\xi - h) \text{ and } v^\infty(t, \xi) \leq V(\xi - h), \forall t, \xi \in \mathbb{R}\}.
\end{align*}
\]

Notice that $h_1$ and $h_2$ are finite because of (3.30). Also, by the continuity,

\[
\begin{align*}
u^\infty(t, \xi) &\leq U(\xi - h_1) \text{ and } v^\infty(t, \xi) \geq V(\xi - h_1), \quad \forall t, \xi \in \mathbb{R}, \\
u^\infty(t, \xi) &\geq U(\xi - h_2) \text{ and } v^\infty(t, \xi) \leq V(\xi - h_2), \quad \forall t, \xi \in \mathbb{R}.
\end{align*}
\]
Clearly, \( h_1 \geq h_2 \). Below we are going to assert \( h_1 = h_2 \). Since the proof is rather long, we prove this assertion in the following lemma.

**Lemma 3.8.** Let \( h_1, h_2 \) be defined as above. Then \( h_1 = h_2 \).

**Proof.** For contradiction we assume that \( h_1 > h_2 \). First of all, we claim the following

\[ v^\infty(t, \xi) < U(\xi - h_1) \text{ and } v^\infty(t, \xi) > V(\xi - h_1), \quad \forall t, \xi \in \mathbb{R}. \]

If \((3.33)\) is false, then there exists \( t_0 \in \mathbb{R} \) and \( \xi_0 \in \mathbb{R} \) such that \( u^\infty(t_0, \xi_0) = U(\xi_0 - h_1) \) or \( v^\infty(t_0, \xi_0) = V(\xi_0 - h_1) \). Observe that \((U(\xi - h_1), V(\xi - h_1))\) also satisfies \((3.29)\). Using \((3.31)\) and the strong maximum principle, we obtain

\[ u^\infty(t, \xi) = U(\xi - h_1), \quad v^\infty(t, \xi) = V(\xi - h_1) \]

for all \( t \leq t_0 \) and \( \xi \in \mathbb{R} \). By the uniqueness of solutions to the corresponding Cauchy problem of \((3.29)\), we then conclude that \((3.34)\) is valid for all \( t \in \mathbb{R} \) and \( \xi \in \mathbb{R} \), contradicting the definition of \( h_2 \) due to \( h_2 < h_1 \). Therefore, \((3.33)\) holds.

Define

\[ \omega_1(\xi) := \inf_{t \in \mathbb{R}}[U(\xi - h_1) - u^\infty(t, \xi)], \quad \omega_2(\xi) := \inf_{t \in \mathbb{R}}[v^\infty(t, \xi) - V(\xi - h_1)], \quad \xi \in \mathbb{R}. \]

By \((3.33)\), we see that \( \omega_i(\xi) \geq 0 \) for all \( \xi \in \mathbb{R} \) and \( i = 1, 2 \).

In what follows, we divide our discussion into two cases:

- **Case 1:** there exists \( z_0 \in \mathbb{R} \) such that \( \omega_1(z_0) = 0 \) or \( \omega_2(z_0) = 0 \).
- **Case 2:** it holds that \( \omega_i(\xi) > 0 \) for all \( \xi \in \mathbb{R} \) and \( i = 1, 2 \).

We first consider **Case 1.** Without loss of generality, we may assume that \( \omega_1(z_0) = 0 \). Then, there exists \( \{\tau_n\} \) such that \( |\tau_n| \to \infty \) and \( \lim_{n \to \infty} u^\infty(\tau_n, z_0) = U(z_0 - h_1) \). Denote

\[ (\tilde{U}_n, \tilde{V}_n)(t, \xi) := (u^\infty, v^\infty)(t + \tau_n, \xi). \]

By standard parabolic regularity theory and passing to a subsequence we may assume that, for some \( \beta \in (0, 1) \),

\[ (\tilde{U}_n, \tilde{V}_n) \to (\tilde{U}^\infty, \tilde{V}^\infty) \quad \text{in} \quad C_{\text{loc}}^{\frac{1 + \beta}{2}, \frac{1 + \beta}{2}}(\mathbb{R} \times \mathbb{R}), \quad \text{as} \quad n \to \infty, \]

where \((\tilde{U}^\infty, \tilde{V}^\infty)\) satisfies \( \tilde{U}^\infty(0, z_0) = U(z_0 - h_1) \) and

\[ \begin{aligned}
\tilde{U}^\infty_t &= d\tilde{U}^\infty_{\xi} + c_{uv}\tilde{U}^\infty_{\xi} + r\tilde{U}^\infty(1 - \tilde{U}^\infty - a\tilde{V}^\infty), \quad t, \xi \in \mathbb{R}, \\
\tilde{V}^\infty_t &= \tilde{V}^\infty_{\xi} + c_{uv}\tilde{V}^\infty_{\xi} + \tilde{V}^\infty(1 - \tilde{V}^\infty - b\tilde{U}^\infty), \quad t, \xi \in \mathbb{R}.
\end{aligned} \]

Furthermore, from \((3.32)\) we see that

\[ \tilde{U}^\infty(t, \xi) \leq U(\xi - h_1) \text{ and } \tilde{V}^\infty(t, \xi) \geq V(\xi - h_1). \]

Notice that \((U(\xi - h_1), V(\xi - h_1))\) satisfies \((3.33)\) and \( \tilde{U}^\infty(0, z_0) = U(z_0 - h_1) \). Thus, the strong maximum principle and the uniqueness of solutions of the corresponding Cauchy problem yield that

\[ \tilde{U}^\infty(t, \xi) \equiv U(\xi - h_1) \text{ and } \tilde{V}^\infty(t, \xi) \equiv V(\xi - h_1) \quad \text{for all} \quad t, \xi \in \mathbb{R}, \]

which implies that

\[ (\tilde{U}_n, \tilde{V}_n)(0, \xi) \to (U, V)(\xi - h_1) \quad \text{as} \quad n \to \infty \text{ locally uniformly for} \quad \xi \in \mathbb{R}. \]
In fact, the convergence of (3.37) is uniform for $\xi \in \mathbb{R}$. Indeed, from (3.39) and (3.32) and the fact that $(U, V)(-\infty) = (1, 0)$ and $(U, V)(\infty) = (0, 1)$, we see that for each $\epsilon > 0$, there exists $N > 0$ and $M > 0$ such that when $n \geq N$,

$$
\|(\hat{U}_n, \hat{V}_n)(0, \cdot) - (U, V)(\cdot - h_1)\|_{L^\infty(\mathbb{R}\setminus[-M, M])} < \epsilon.
$$

Together with (3.37), it follows that $(\hat{U}_n, \hat{V}_n)(0, \xi) \to (U, V)(\xi - h_1)$ as $n \to \infty$ uniformly for $\xi \in \mathbb{R}$, or equivalently,

$$
(3.38) \quad (u^\infty, v^\infty) (\tau_n, \xi) \to (U, V)(\xi - h_1), \quad \text{as } n \to \infty \text{ uniformly for } \xi \in \mathbb{R}.
$$

Recall that the time sequence $\{\tau_n\}$ satisfies $|\tau_n| \to \infty$. Without loss of generality we may assume that $\tau_n \to -\infty$ or $\tau_n \to +\infty$ (if necessary we can take a subsequence). Suppose that $\tau_n \to -\infty$. Then, from (3.38) and Lemma 3.4 we see that

$$
(3.39) \quad \lim_{t \to \infty} \|(u^\infty, v^\infty)(t, \cdot) - (U, V)(\cdot - h_1)\|_{L^\infty(\mathbb{R})} = 0.
$$

We now define

$$
\sigma_1(\xi) := \inf_{t \in \mathbb{R}} [u^\infty(t, \xi) - U(\xi - h_2)], \quad \sigma_2(\xi) := \inf_{t \in \mathbb{R}} [V(\xi - h_2) - v^\infty(t, \xi)], \quad \xi \in \mathbb{R}.
$$

By (3.32), we see that $\sigma_i(\xi) \geq 0$ for all $\xi \in \mathbb{R}$ and $i = 1, 2$. Then, we have

**Claim 1:** It holds

$$
\sigma_i(\xi) > 0 \quad \text{for all } \xi \in \mathbb{R} \text{ and } i = 1, 2.
$$

If Claim 1 is not true, there exists $\zeta_0 \in \mathbb{R}$ such that $\sigma_1(\zeta_0) = 0$ or $\sigma_2(\zeta_0) = 0$. Without loss of generality, we may assume that $\sigma_1(\zeta_0) = 0$. By (3.39) we see that there exists $\{\tilde{\tau}_n\}$ such that $\tilde{\tau}_n \to -\infty$ and $\lim_{n \to \infty} u^\infty(\tilde{\tau}_n, \zeta_0) = U(\zeta_0 - h_2)$.

Denote

$$
(\hat{U}_n, \hat{V}_n)(t, \xi) := (u^\infty, v^\infty)(t + \tilde{\tau}_n, \xi).
$$

By standard parabolic regularity theory and passing to a subsequence we may assume that, for some $\beta \in (0, 1)$,

$$
(\hat{U}_n, \hat{V}_n) \to (\hat{U}^\infty, \hat{V}^\infty) \quad \text{in } C^{(1+\beta)/2, 1+\beta}_{loc}(\mathbb{R} \times \mathbb{R}), \quad \text{as } n \to \infty,
$$

where $(\hat{U}^\infty, \hat{V}^\infty)$ satisfies $\hat{U}^\infty(0, \zeta_0) = U(\zeta_0 - h_2)$ and

$$
(3.40) \begin{cases}
\hat{U}^\infty_t = d\hat{U}^\infty_{\xi\xi} + cu\hat{U}^\infty_{\xi} + r\hat{U}^\infty(1 - \hat{U}^\infty - a\hat{V}^\infty), \quad &\forall t, \xi \in \mathbb{R}, \\
\hat{V}^\infty_t = \hat{V}^\infty_{\xi\xi} + cu\hat{V}^\infty_{\xi} + \hat{V}^\infty(1 - \hat{V}^\infty - b\hat{U}^\infty), \quad &\forall t, \xi \in \mathbb{R}.
\end{cases}
$$

Then, similar to (3.36), we have

$$
\hat{U}^\infty(t, \xi) \equiv U(\xi - h_2) \quad \text{and} \quad \hat{V}^\infty(t, \xi) \equiv V(\xi - h_2), \quad \forall t, \xi \in \mathbb{R}.
$$

The same process as in deriving (3.38) gives

$$
(3.41) \quad (u^\infty, v^\infty) (\tilde{\tau}_n, \xi) \to (U, V)(\xi - h_2), \quad \text{as } n \to \infty \text{ uniformly for } \xi \in \mathbb{R}.
$$
Since $\tau_n \to -\infty$, it follows from (3.11) and Lemma 3.7 that

$$(u^\infty, v^\infty)(t, \xi) = (U, V)(\xi - h_2) \quad \text{for all } t, \xi \in \mathbb{R}.$$ 

which contradicts (3.33) and we thus obtain Claim 1.

Due to Claim 1, one can use the sliding method to further assert that

Claim 2: There exists $\epsilon > 0$ sufficiently small such that

$$u^\infty(t, \xi) \geq U(\xi - (h_2 + \epsilon)), \quad v^\infty(t, \xi) \leq V(\xi - (h_2 + \epsilon)), \quad \forall t, \xi \in \mathbb{R}.$$ 

Once Claim 2 is proved, we will obtain a contradiction with the definition of $h_2$.

We now verify Claim 2. Inspired by [9], we consider the following auxiliary system:

$$
\begin{cases}
Pt = dP_{\xi\xi} + c_u P_{\xi} + f(P, Q), & t > 0, \xi \geq \xi_0, \\
Qt = Q_{\xi\xi} + c_u Q_{\xi} + g(P, Q), & t > 0, \xi \geq \xi_0, \\
P(t, \xi_0) = U(\xi_0 - (h_2 + \epsilon)), & t > 0, \\
P(0, \xi) = 0, & \xi \geq \xi_0,
\end{cases}
$$

where

$$f(P, Q) := rP(1 - P - aQ), \quad g(P, Q) := Q(1 - Q - bP),$$

and $\xi_0 \gg 1$ and $0 < \epsilon \ll 1$ will be determined later.

Note that the initial function $(0, 1)$ forms a pair of subsolution of the corresponding stationary problem of (3.42). Hence, $P(t, \cdot)$ is increasing in $t$ and $Q(t, \cdot)$ is decreasing in $t$. Also, because $(U(\xi - (h_2 + \epsilon)), V(\xi - (h_2 + \epsilon)))$ satisfies the first two equations and the boundary condition of (3.42), one can apply the comparison principle to deduce that

$$0 \leq P(t, \xi) \leq U(\xi - (h_2 + \epsilon)), \quad V(\xi - (h_2 + \epsilon)) \leq Q(t, \xi) \leq 1$$

for all $t > 0$ and $\xi \geq \xi_0$.

Define the limit functions

$$P^*(\xi) := \lim_{t \to \infty} P(t, \xi), \quad Q^*(\xi) := \lim_{t \to \infty} Q(t, \xi), \quad \xi > \xi_0.$$ 

Then, one has

$$P^*(\xi) \leq U(\xi - (h_2 + \epsilon)), \quad V(\xi - (h_2 + \epsilon)) \leq Q^*(\xi), \quad \xi > \xi_0.$$ 

Furthermore, $(P^*, Q^*)$ satisfies

$$
\begin{cases}
0 = dP^*_{\xi\xi} + c_u P^*_{\xi} + f(P^*, Q^*), & \xi \geq \xi_0, \\
0 = Q^*_{\xi\xi} + c_u Q^*_{\xi} + g(P^*, Q^*), & \xi \geq \xi_0, \\
P^*(\xi_0) = U(\xi_0 - (h_2 + \epsilon)), & Q^*(\xi_0) = V(\xi_0 - (h_2 + \epsilon)), \\
P^*(\infty) = 0, & Q^*(\infty) = 1.
\end{cases}
$$

In the sequel, we are going to conclude

Claim 3: It holds

$$P^*(\xi) = U(\xi - (h_2 + \epsilon)), \quad Q^*(\xi) = V(\xi - (h_2 + \epsilon)), \quad \xi \geq \xi_0.$$ 

To verify Claim 3, we introduce

$$Z_1(\xi) := U(\xi - (h_2 + \epsilon)) - P^*(\xi), \quad Z_2(\xi) := Q^*(\xi) - V(\xi - (h_2 + \epsilon)).$$
From (3.43) it follows that

\[(3.44)\]

\[Z_i(\xi_0) = 0, \quad Z_i(\xi) \geq 0 \quad \text{for all } \xi \geq \xi_0 \text{ and } i = 1, 2.\]

For convenience, we write \(U_\varepsilon(\xi) = U(\xi - (h_2 + \varepsilon))\) and \(V_\varepsilon(\xi) = V(\xi - (h_2 + \varepsilon))\). By direct computations, we have

\[
dZ_1'' + c_{uv} \xi_1' = -rU_\varepsilon(1 - U_\varepsilon - aV_\varepsilon) + rP^*(1 - P^* - aQ^*)
\]

\[= r[\{aV_\varepsilon + P^* + U_\varepsilon\}Z_1 - aP^*Z_2], \quad \xi \geq \xi_0,\]

\[
Z_2'' + c_{uv} Z_2' = -Q^*(1 - Q^* - bP^*) + V_\varepsilon(1 - V_\varepsilon - bU_\varepsilon)
\]

\[= (bP^* + Q^* + V_\varepsilon - 1)Z_2 - bV_\varepsilon Z_1
\]

\[\geq (2V_\varepsilon - 1)Z_2 - bV_\varepsilon Z_1, \quad \xi \geq \xi_0 \quad (\text{due to (3.43)}).
\]

Because \(U_\varepsilon(+\infty) = 0\) and \(V_\varepsilon(+\infty) = 1\), for any \(\delta \in (0, a - 1)\), one can pick \(\xi_0 \gg 1\) such that

\[(3.45)\]

\[U_\varepsilon(\xi) < \delta, \quad V_\varepsilon(\xi) > 1 - \frac{\delta}{a} \quad \text{for all } \xi \geq \xi_0.
\]

Since \(Z_i(\xi_0) = 0 \leq Z_i(\xi)\) for \(\xi \geq \xi_0\) and \(Z_i(+\infty) = 0\), one can define

\[Z_i(\xi_i) = \max_{\xi \in [\xi_0, \infty)} Z_i(\xi) \geq 0, \quad i = 1, 2.
\]

Then, Claim 3 is equivalent to

\[(3.46)\]

\[Z_i(\xi_i) = 0 \quad \text{for } i = 1, 2.
\]

Suppose that \(Z_1(\xi_1) > 0\).

We then have to distinguish two cases:

(i) \((a - 1 - \delta)Z_1(\xi_1) > a\delta Z_2(\xi_2)\); (ii) \((a - 1 - \delta)Z_1(\xi_1) \leq a\delta Z_2(\xi_2)\).

When case (i) happens, one can use the equation of \(Z_1\), \((3.45)\) and the fact that \(P^* \leq U_\varepsilon\) to deduce

\[
0 \geq dZ_1''(\xi_1) + c_{uv} Z_1'(\xi_1) > r\{a\delta - \delta\}Z_1(\xi_1) - a\delta Z_2(\xi_2)
\]

\[\geq r\{a\delta - \delta\}Z_1(\xi_1) - a\delta Z_2(\xi_2) > 0,
\]

which reaches a contradiction and (i) thus cannot occur.

On the other hand, if case (ii) happens, one can use the equation of \(Z_2\) and \((3.45)\) to deduce

\[
0 \geq Z_2''(\xi_2) + c_{uv} Z_2'(\xi_2) > \left[2 \left(1 - \frac{\delta}{a}\right) - 1\right]Z_2(\xi_2) - bZ_1(\xi_2)
\]

\[\geq \frac{a - 1 - \delta}{a\delta} \left[1 - \frac{2\delta}{a}\right]Z_1(\xi_1) - bZ_1(\xi_1) > 0,
\]

where the last inequality holds as long as we choose a sufficiently small \(\delta\). Again, we arrive at a contradiction. Therefore, \(Z_1(\xi_1) = 0\), or equivalently, \(Z_1(\xi) = 0\) for all \(\xi \geq \xi_0\). Together with \((3.44)\) and the equation of \(Z_2\), we have

\[Z_2'' + c_{uv} Z_2' - (bP^* + Q^* + V_\varepsilon - 1)Z_2 = 0 \quad \text{for } \xi \geq \xi_0; \quad Z_2(\xi_0) = 0 \leq Z_2(\xi) \quad \text{for } \xi \geq \xi_0.
\]

As \(Z_2(+\infty) = 0\), the strong maximum principle implies that \(Z_2(\xi) = 0\) for all \(\xi \geq \xi_0\). Thus, we have proved \((3.46)\) and then Claim 3 holds.
We now complete the proof of Claim 2. Because of Claim 1, one can fix \( \epsilon > 0 \) sufficiently small such that
\[
u^\infty(t, \xi_0) \geq U(\xi_0 - (h_2 + \epsilon)), \quad v^\infty(t, \xi_0) \leq V(\xi_0 - (h_2 + \epsilon)) \quad \text{for all } t \in \mathbb{R}.
\]
Also, notice that \( u^\infty(t, \xi_0) \geq 0 = P(0, \xi) \) and \( v^\infty(t, \xi_0) \leq 1 = Q(0, \xi) \) for all \( \xi \geq \xi_0 \). Using the comparison principle, we obtain
\[
u^\infty(s + t, \xi) \geq P(t, \xi), \quad v^\infty(s + t, \xi) \leq Q(t, \xi) \quad \text{for all } t > 0, s \in \mathbb{R} \text{ and } \xi \geq \xi_0,
\]
which is equivalent to
\[
u^\infty(t, \xi) \geq P(t - s, \xi), \quad v^\infty(t, \xi) \leq Q(t - s, \xi) \quad \text{for all } t > s, s \in \mathbb{R} \text{ and } \xi \geq \xi_0.
\]
By taking \( s \to -\infty \) and using Claim 3, we have
\[
u^\infty(t, \xi) \geq P^*(\xi) = U(\xi - (h_2 + \epsilon)), \quad v^\infty(t, \xi) \leq Q^*(\xi) = V(\xi - (h_2 + \epsilon))
\]
for all \( t \in \mathbb{R} \) and \( \xi \geq \xi_0 \).

By a similar process used as above, we can conclude that there exists \( \xi_1 \gg 1 \) such that
\[
u^\infty(t, \xi) \geq U(\xi - (h_2 + \epsilon)), \quad v^\infty(t, \xi) \leq V(\xi - (h_2 + \epsilon))
\]
for all \( t \in \mathbb{R} \) and \( \xi \leq -\xi_1 \) by taking \( \epsilon > 0 \) smaller if necessary.

Notice that by the continuity, \( (3.47) \) still holds for all \( t \in \mathbb{R} \) and \( \xi \in [-\xi_1, \xi_0] \) by choosing \( \epsilon > 0 \) further smaller if necessary. Therefore, we have proved Claim 2. However, this contradicts the definition of \( h_2 \). Hence, we must have \( h_1 = h_2 \) when Case 1 occurs.

We now treat Case 2. In this case, one can apply the sliding method used above to show that
\[
u^\infty(t, \xi) \leq U(\xi - (h_1 - \epsilon)), \quad v^\infty(t, \xi) \geq V(\xi - (h_1 - \epsilon)), \quad \forall t, \xi \in \mathbb{R}
\]
for some small \( \epsilon > 0 \). This contradicts the definition of \( h_1 \), which means that \( h_1 > h_2 \) is impossible. Hence, it is necessary that \( h_1 = h_2 \) when Case 2 occurs. The proof is thus complete. \( \square \)

With the aid of Lemma 3.8, we are now ready to present

Proof of Theorem 4. Lemma 3.8 tells us that
\[
u^\infty(t, \xi) = U(\xi - \hat{h}), \quad v^\infty(t, \xi) = V(\xi - \hat{h}) \quad \text{for all } t, \xi \in \mathbb{R}
\]
with \( \hat{h} = h_1 = h_2 \). It then follows that
\[
limit_{n \to \infty} (\hat{u}, \hat{v})(t + t_n, \xi) = (U, V)(\xi - \hat{h}) \quad \text{in } C^{(1+\beta)/2, 1+\beta}_loc(\mathbb{R} \times \mathbb{R}).
\]
Since the time sequence \( \{t_n\} \) can be chosen arbitrarily, we have
\[
limit_{t \to \infty} (\hat{u}, \hat{v})(t, \xi) = (U, V)(\xi - \hat{h}) \quad \text{uniformly for } \xi \text{ in any compact subset of } \mathbb{R}.
\]
By (3.28), we thus obtain
\[
(3.48) \lim_{t \to \infty} (u, v)(t, x) = (U, V)(x - c_{uv}t + \hat{h}) \quad \text{locally uniformly in } x - c_{uv}t \text{ with } x \geq 0.
\]
Moreover, from (3.31) and (3.32) and the fact that \( (U, V)(-\infty) = (1, 0) \) and \( (U, V)(\infty) = (0, 1) \), it is clear to see that for each \( \epsilon > 0 \), there exists \( N' > 0 \) and \( M' > 0 \) such that \( t \geq N' \) implies that
\[
|u(t, x) - (U, V)(x - c_{uv}t - \hat{h})| < \epsilon \quad \text{for } 0 \leq x \leq c_{uv}t - M' \text{ and } x \geq c_{uv}t + M',
\]
and
which, combined with (3.48), yields (1.6). This completes the proof.

\[ \square \]

4. Proof of Theorems 2 and 3 scenario (A2)

In this section, we prove Theorems 2 and 3 unless otherwise specified, it is assumed that \((u_0, v_0)\) satisfies (A2) throughout this section.

4.1. The proof of Theorem 2. Let \((u, v)\) be the solution of (1.1)-(1.2). Given \(m \in (0, 1)\), we define \(E_m(t)\) as the set of points in \((0, \infty)\) such that \(u(t, \cdot) = m\). Namely,

\[ E_m(t) = \{ x > 0 : u(t, x) = m \}. \]

**Lemma 4.1.** For any \(m \in (0, 1)\), there exist \(M > 0\) and \(T > 0\) such that

\[ \max_{E_m(t)} \leq c u_t - \frac{3}{c_u} \ln t + M, \quad \forall t \geq T. \]

**Proof.** Let \(\bar{u}\) be the solution of the problem

\[ \bar{u}_t = d\bar{u}_{xx} + r(1 - \bar{u})\bar{u} \quad \text{in} \quad (0, \infty) \times \mathbb{R}; \quad \bar{u}(0, \cdot) = u_0. \]

From [3] or [21, Theorem 1.1], we see that there exist \(M > 0\) and \(T > 0\) such that

\[ \bar{E}_m(t) \subset \left[ c_u t - \frac{3}{c_u} \ln t - M, \quad c_u t - \frac{3}{c_u} \ln t + M \right] \quad \text{for all} \quad t \geq T, \]

where \(\bar{E}_m(t) = \{ x > 0 : \bar{u}(t, x) = m \}\).

Since \(\bar{u}_t \geq d\bar{u}_{xx} + r(1 - \bar{u} - av)\bar{u}\) in \((0, \infty) \times \mathbb{R}\), one can apply the comparison principle to deduce \(\bar{u} \geq u\), which implies that \(\max_{E_m(t)} \leq \max_{\bar{E}_m(t)} \) for \(t \geq T\). Using (4.1), we thus complete the proof. \(\square\)

**Lemma 4.2.** Assume that \(c_u > c_v\). Then there exist \(C, \mu, T > 0\) such that

\[ \sup_{x \in \mathbb{R}^+} v(x, t) \leq Ce^{-\mu t}, \quad \forall t \geq T. \]

**Proof.** Since \(v_0\) is of compact support, by the proof of Lemma 2.5 (just exchanging the role of \(u\) and \(v\)), we have the following result: if \(c > c_v := 2\), then for any small \(\varepsilon > 0\), there exist \(M, \mu > 0\) and \(T \gg 1\) such that

\[ v(t, x) \leq Me^{-\mu(c - 2\sqrt{1+\varepsilon})t} \quad \text{for all} \quad t \geq T \quad \text{and} \quad x > ct. \]

Together with Corollary 2.7, we thus complete the proof. \(\square\)

We next derive a lower estimate of \(\min_{E_m(t)}\). For our purpose, consider

\[ w_t = d w_{xx} + w(r - ru - C_0e^{-\mu t}) \quad \text{in} \quad (0, \infty) \times \mathbb{R}, \]

where \(C_0 := raC\), where \(\mu, C > 0\) is defined in Lemma 4.2. We shall apply the method developed by Hamel, Nolen, Roquejoffre, and Ryzhik [21] to estimate \(\min_{E_m(t)}\). To do so, we consider the linearized equation of (1.2) with the Dirichlet boundary condition along a suitable curve \(x = X(t)\). Namely,

\[ w_t = d w_{xx} + w(r - C_0e^{-\mu t}) \quad \text{in} \quad (0, \infty) \times (X(t), \infty) \quad \text{with} \quad w(t, X(t)) = 0, \]

where \(w(0, \cdot) = w_0 \geq (\neq 0) \) in \((0, \infty)\) and is of compact support.
Motivated by [21], we define
\[ X(t) := cu t - \frac{3d}{cu} \ln(t + t_0), \quad z(t, x') = w(t, x), \quad x' = x - X(t), \]
where \( t_0 > 0 \) will be determined later. After some simple calculations and dropping the prime sign, (4.3) becomes
\[ z_t = \frac{dz}{dx} + \left[ c_u - 3 \cdot \frac{d}{cu(t + t_0)} \right] z_x + (r - C_0 e^{-\mu t}) z \quad \text{in} \ (0, \infty) \times (0, \infty), \]
where \( z(t, 0) = 0 \) and \( z(0, \cdot) = z_0 \geq (\neq 0) \) in \( (0, \infty) \) and is of compact support.

We shall prove that \( z(t, x) \) has both positive upper and lower bounds over \([1, \infty) \times [a, b]\) for any given \( 0 < a < b < \infty \) using the argument of [21, Lemma 2.1]. To the end, we need the following lemma given in [21].

**Lemma 4.3.** ([21, Lemma 2.2]) Suppose that \( p(t, y) \) satisfies
\[ p_t + \mathcal{L}p = -\varepsilon e^{-\tau/2} p_y, \quad \tau > 0, \quad y > 0; \quad p(\tau, 0) = 0, \]
where
\[ \mathcal{L}p := -p_{yy} - yp_y/2 - p. \]
Then there exists \( \varepsilon_0 > 0 \) such that for any compact set \( K \) of \( \mathbb{R}_+ \), there exists \( C_K > 0 \) such that for \( 0 < \varepsilon < \varepsilon_0 \),
\[ p(\tau, y) = \frac{e^{-y^2/4}}{2\sqrt{\pi}} \left( \int_0^\infty \xi p(0, \xi) d\xi + O(\varepsilon) \right) + e^{-\tau/2} \tilde{p}(\tau, y), \]
where \( |\tilde{p}(\tau, y)| \leq C_K e^{-y^2/8} \) for all \( \tau > 0 \) and \( y \in K \); and \( O(\varepsilon) \) denote a function of \( (\tau, y) \) for \( \tau > 0 \) and \( y \in K \).

Due to Lemma 4.3, we have the following estimate for \( z \).

**Lemma 4.4.** Let \( z \) satisfy (4.4). Then there exists \( t_0 > 0 \) depending on \( z_0 \) such that for any \( 0 < a < b < \infty \), it holds
\[ 0 < \inf_{t \geq 1, a \leq x \leq b} z(t, x) \leq \sup_{t \geq 1, a \leq x \leq b} z(t, x) < \infty. \]

**Proof.** Our proof is based on [21, Lemma 2.1]. Define
\[ q(t, x) = e^{\frac{3t}{2a^2}} z(t, x). \]
Then, \( q \) satisfies
\[ q_t = dq_{xx} - \frac{3d}{cu(t + t_0)} q_x - \frac{3}{2(t + t_0)} q - C_0 e^{-\mu t} q \quad \text{in} \ (0, \infty) \times (0, \infty) \]
with \( q(t, 0) = 0 \). Using the self-similar variables
\[ \tau = \ln(t + t_0) - \ln t_0, \quad y = \frac{x}{[d(t + t_0)]^{1/2}}, \]
and setting \( Q(\tau, y) := q(t, x) \), direct computations yield that
\[ Q_\tau - \mathcal{L}Q = -\varepsilon Q_y + \left[ \frac{1}{2} - C_0 t_0 e^{\tau - \mu t_0 (e^\tau - 1)} \right] Q \quad \text{in} \ (0, \infty) \times (0, \infty) \]
with $Q(\tau, 0) = 0$, where $L$ is defined in Lemma 4.3 and 

$$\varepsilon := \frac{3\sqrt{7}e^{-\tau/2}}{c_u\sqrt{t_0}}.$$ 

Define $J(\tau) := C_0\theta e^{\tau-\mu t_0}(e^\tau-1)$ and 

$$I(\tau) := \exp \left[ \int_0^\tau \left( \frac{1}{2} - J(s) \right) ds \right].$$

Then, by Lemma 4.3, we have 

$$Q(\tau, y) = e^{\int_0^\tau J(s)ds} y \left[ \frac{e^{-y^2/4}}{2\sqrt{\pi}} \left( \int_0^\infty \xi Q(0, \xi) d\xi + O(\varepsilon) \right) + e^{-\tau/2} \tilde{Q}(\tau, y) \right]$$

$$= e^{\tau/2} e^{-\int_0^\tau J(s)ds} y \left[ \frac{e^{-y^2/4}}{2\sqrt{\pi}} \left( \int_0^\infty \xi Q(0, \xi) d\xi + O(\varepsilon) \right) + e^{-\tau/2} \tilde{Q}(\tau, y) \right],$$

where $|\tilde{Q}(\tau, y)| \leq C_K e^{-y^2/8}$ for all $\tau > 0$ and $y \in K$ for any compact set $K$. It follows that 

$$z(t, x) = \frac{x e^{\int_0^t J(s)ds}}{\sqrt{dt_0}} e^{-\int_0^{[t+t_0]/t_0} J(s)ds} \left[ C e^{-x^2/[4d(t+t_0)]} + \tilde{z}(t, x) \right],$$

where for any $0 < a < b < \infty$, 

$$\limsup_{t \to \infty} |\tilde{z}(t, x)| < \frac{C}{2}.$$ 

Furthermore, it is easily checked that there exist two positive constants $C_1$ and $C_2$ such that 

$$C_1 \leq e^{-\int_0^{[t+t_0]/t_0} J(s)ds} \leq C_2 \quad \text{for all } t \geq 0.$$ 

It follows that for any given $0 < a < b < \infty$, $z(t, x)$ has a positive lower bound and a positive upper bound for $x \in [a, b]$ and $t \geq t_0$, provided $t_0$ is large enough. For $1 \leq t \leq t_0$, one can use the strong maximum principle to assert that $z(t, x)$ has a positive lower bound and a positive upper bound for $x \in [a, b]$ and $t \in [1, t_0]$. The proof is thus complete. \(\square\)

Based on Lemma 4.4, one can apply the argument in [21] to derive a lower estimate of $\min E_m(t)$ under the condition $c_u > c_v$.

**Lemma 4.5.** Assume that $c_u > c_v$. For any $m \in (0, 1)$, there exist $M > 0$ and $T > 0$ such that 

$$\min E_m(t) \geq c_u t - \frac{3}{c_u} \ln t - M, \quad \forall t \geq T.$$ 

**Proof.** Thanks to Lemma 4.4 we can follow the same line as that in [21] Proposition 3.1 and Corollary 3.2] to deduce that there exists $M' > 0$ and $T_0 > 0$ such that 

$$\min E_m(t) \geq c_u t - \frac{3}{c_u} \ln t - M', \quad \forall t \geq T_0,$$

where $E_m(t) = \{ x > 0 | u(t, x) = m \}$ and $u$ solves (4.2) with $u(0, \cdot) \geq (\not\equiv) 0$ and is of compact support. Using Lemma 4.2 and taking $u(t, \cdot) \leq u(T, \cdot)$ ($T$ is defined in Lemma 4.2), one can apply the comparison principle to deduce that $u(t + T, \cdot) \geq u(t, \cdot)$ for all $t \geq 0$, which in turn implies that 

$$\min E_m(t + T) \geq \min E_m(t), \quad \forall t \geq T_0.$$

By (4.5), we thus complete the proof. \(\square\)
We are ready to prove Theorem 2.

**Proof of Theorem 2.** By Lemma 4.2, we have \( \lim_{t \to \infty} \sup_{x \in [0, \infty)} |v(t, x)| = 0 \). Also, in view of Lemma 4.1 and Lemma 4.5, we can safely follow the same analysis as that in [21, Section 4] to conclude that there exist \( C > 0 \) and a bounded function \( \omega : [0, \infty) \to \mathbb{R} \) such that

\[
\lim_{t \to \infty} \sup_{x \in [0, \infty)} \left| u(t, x) - U_{KPP} \left( x - c_u t + \frac{3}{c_u} \ln t + \omega(t) \right) \right| = 0.
\]

Thus, the proof is complete. \( \square \)

4.2. **The proof of Theorem 3.** In this subsection, combining some arguments used in the proof of Theorem 1 and Theorem 2, we shall construct a new type of supersolution to establish Theorem 3.

**Lemma 4.6.** Assume that \( c_v > c_u \). Then for any \( c > c_{uw} \), there exist positive constants \( C, \mu, T \) such that

\[
\sup_{x \in [ct, \infty)} u(t, x) \leq Ce^{-\mu t}, \quad \forall t \geq T.
\]

**Proof.** First, we will show that for each \( c_{uw} < c^- < c^+ < c_v \), there exist \( C_1, \mu_1, T_1 > 0 \)

\[
\sup_{x \in [c^-t, c^+t]} u(t, x) \leq C_1 e^{-\mu_1 t}, \quad \forall t \geq T_1.
\]

Since \( c_v > c_u \), thanks to [5, Theorem 1] (following the proof there with slight modifications), we have

\[
\lim_{t \to \infty} \sup_{c_1 t \leq x \leq c_2 t} \left( |u(t, x)| + |v(t, x) - 1| \right) = 0 \quad \text{for all} \quad c_{uw} < c_1 < c_2 < c_v.
\]

Therefore, one can choose small \( \varepsilon > 0 \) and \( T_0 \gg 1 \) such that \( 0 < u < \varepsilon \) and \( v \geq 1 - \varepsilon \) in \([T_0, \infty) \times [c_1 t, c_2 t] \).

For notational convenience, let us denote

\[
\rho := -r[1 - a(1 - \varepsilon)].
\]

Here we may assume that \( \rho > 0 \) since \( a > 1 \) and \( 0 < \varepsilon \ll 1 \). This implies that

\[
u_t \leq du_{xx} - \rho u \quad \text{in} \quad [T_0, \infty) \times [c_1 t, c_2 t]; \quad u(t, c_i t) \in (0, \varepsilon), \quad \forall t \geq T_0, \; i = 1, 2.
\]

Set

\[
c^* := (c_1 + c_2)/2, \quad \hat{c} := (c_2 - c_1)/2,
\]

\[
y := x - c^* t, \quad (\hat{u}, \hat{v})(t, y) := (u, v)(t, y + c^* t).
\]

By (4.8), we have

\[
\begin{cases}
\hat{u}_t \leq d\hat{u}_{yy} + c^* \hat{u}_y - \rho \hat{u} \quad \text{in} \quad [T_0, \infty) \times [-\hat{c}t, \hat{c}t],
\hat{u}(t, \pm \hat{c}t) \in (0, \varepsilon) \quad \text{for} \quad t \geq T_0.
\end{cases}
\]

Fix \( T > T_0 \) and consider

\[
\begin{cases}
\phi_t = d\phi_{yy} + c^* \phi_y - \rho \phi, \quad t > 0, \quad -\hat{c}T < y < \hat{c}T, \\
\phi(t, \pm \hat{c}T) = \varepsilon, \quad t > 0, \\
\phi(0, x) = \varepsilon, \quad -\hat{c}T \leq y \leq \hat{c}T.
\end{cases}
\]
Then, by comparison, we have
\[ \phi(t, x) \geq \hat{u}(t + T, x) \quad \text{for } t \geq 0 \text{ and } -\hat{c}T \leq y \leq \hat{c}T. \]

Let \( \Phi(t, y) = e^{\rho t}(\varepsilon - \phi) \). Then system (4.9) is reduced to
\[
\begin{cases}
\Phi_t = d\Phi_{yy} + c^*\Phi_y + \varepsilon e^{\rho t}, & t > 0, \\
\Phi(t, \pm \hat{c}T) = 0, & t > 0, \\
\Phi(0, x) = 0, & \hat{c}T \leq y \leq \hat{c}T.
\end{cases}
\]

From the proof of [24, Proposition 3.2], we have: for any small \( \sigma > 0 \), there exists \( T^* \gg 1 \) and \( \nu(\sigma) > 0 \) such that for \( T \geq T^* \),
\[ \Phi(t, y) \geq \rho(e^{\rho t} - 1)(1 - C_1 e^{-\nu(\sigma)\hat{c}T}), \quad (t, x) \in D(\sigma), \]
where \( C_1 \) is a positive constant, \( \nu(\sigma) \) has a positive lower bound for all small \( \sigma \) and
\[ D(\sigma) := \left\{ (t, y) \left| 0 < t < \frac{(\sigma \hat{c})^2 T}{4\sqrt{\rho}}, \right| y \right\} \leq (1 - \sigma)\hat{c}T \right\}. \]

It follows that
\[ \phi(t, y) \leq \varepsilon - \varepsilon(1 - e^{-\rho t})(1 - C_1 e^{-\nu(\sigma)\hat{c}T}) \leq \varepsilon(C_1 e^{-\nu(\sigma)\hat{c}T} + e^{-\rho t}), \quad (t, x) \in D(\sigma). \]

Taking \( t = (\sigma \hat{c})^2 T/(4\sqrt{\rho}) \) and \( \sigma \) small enough such that \( \nu \hat{c} > \rho(\sigma \hat{c})^2/(4\sqrt{\rho}) \), we obtain
\[ (4.10) \quad \phi(t, y) \leq \varepsilon(C_1 + 1)e^{-\rho(\sigma \hat{c})^2 T/(4\sqrt{\rho})}, \quad |y| \leq (1 - \sigma)\hat{c}T. \]

Then, by comparison, \( \hat{u}(t + T, y) \leq \phi(t, y) \), which together with (4.10) gives
\[ \hat{u}\left(\frac{(\sigma \hat{c})^2 T}{4\sqrt{\rho}} + T, y\right) \leq \varepsilon(C_1 + 1)e^{-\rho(\sigma \hat{c})^2 T/(4\sqrt{\rho})}, \quad |y| \leq (1 - \sigma)\hat{c}T. \]

Note that
\[ t = \frac{(\sigma \hat{c})^2 T}{4\sqrt{\rho}} + T \iff T = \left(1 + \frac{\sigma^2 \hat{c}^2}{4\sqrt{\rho}}\right)^{-1} t. \]

Thus, we infer that
\[ \hat{u}(t, y) \leq \varepsilon(C_1 + 1)e^{-\delta t} \quad \text{for } t \geq T^{**} \text{ and } |y| \leq (1 - \sigma)\hat{c}t, \]
where
\[ \delta := \rho\left(\frac{\sigma^2 \hat{c}^2}{4\sqrt{\rho}}\right)\left(1 + \frac{\sigma^2 \hat{c}^2}{4\sqrt{\rho}}\right)^{-1} > 0, \quad T^{**} := T^* + \frac{\sigma^2 \hat{c}^2}{4\sqrt{\rho}} T^*. \]

Hence, it follows that
\[ u(t, x) \leq \varepsilon(C_1 + 1)e^{-\delta t} \quad \text{for } t \geq T^{**} \text{ and } [c^* - (1 - \sigma)\hat{c}]t \leq x \leq [c^* + (1 - \sigma)\hat{c}]t. \]

Since \( \sigma > 0 \) can be arbitrarily small and \( c_1 \) (resp., \( c_2 \)) can be arbitrarily close to \( c_{av} \) (resp., \( c_v \)) such that \( (1 - \sigma)c_1 < c^- < c^+ < (1 - \sigma)c_2 \), we see that (4.6) holds. Finally, due to the assumption \( c_u < c_v \), Lemma 4.4 follows from (4.6) and Lemma 2.5.

Thanks to Lemma 4.6 and (4.7), one can follow the same lines as in Lemma 2.8 (with minor modifications) to obtain the following result.
Lemma 4.7. Assume that $c_v > c_u$. Then for any $c_{uv} < c_1 < c_2 < c_v$, there exist positive constants $C', \nu$ and $T'$ such that

$$\inf_{x \in [c_1, c_2]} v(t, x) \geq 1 - C'e^{-\nu t}, \quad \forall t \geq T'.$$

Remark 4.1. We remark that the parallel proof of Lemma 4.6 and Lemma 4.7 also shows that if $c_v > c_u$, then there exist positive constants $C, \mu, \nu$ and $T$ such that

$$\sup_{x \in (-\infty, -ct]} u(t, x) \leq Ce^{-\mu t}, \quad \forall t \geq T \text{ if } c > c_{uv},$$

$$\inf_{x \in [-ct, -c_1 t]} v(t, x) \geq 1 - Ce^{-\nu t}, \quad \forall t \geq T \text{ if } c_{uv} < c_1 < c_2 < c_v.$$

A pair of supersolution $(\overline{u}, \overline{v})$ given in Section 3 cannot be used as a comparing function to obtain the asymptotic behavior of $(u, v)$. A new pair of supersolution is constructed in the following lemma.

Lemma 4.8. Assume that $c_v > c_u$. Then for any $c \in (c_{uv}, c_v)$, there exists $h_1 \in \mathbb{R}$ such that the solution of $(1.1), (1.2)$ satisfies

$$\lim_{t \to \infty} \left[ \sup_{x \in [0, ct]} \left| u(t, x) - U(x - c_{uv} t - h_1) \right| + \sup_{x \in [0, ct]} \left| v(t, x) - V(x - c_{uv} t - h_1) \right| \right] = 0.$$

Proof. Let $(\hat{u}, \hat{v})$ be the solution of $(1.1)$ with the initial data $(\hat{u}_0, \hat{v}_0)$ satisfying

$$\hat{u}_0(x) = u_0(x), \quad \hat{v}_0(x) > v_0(x) \quad \text{in } \mathbb{R},$$

and $\hat{v}_0(\cdot) \geq \rho$ in $\mathbb{R}$ for some $\rho > 0$. Thanks to (4.11), we can compare $(\hat{u}, \hat{v})$ with $(u, v)$ such that

$$\hat{u}(t, x) \leq u(t, x), \quad \hat{v}(t, x) \geq v(t, x) \quad \text{for } t \geq 0 \text{ and } x \in \mathbb{R}.$$

Denote $(\overline{u}, \overline{v})$ by $(\hat{u}, \hat{v})$ such that Lemma 4.1 holds. Since $(\hat{u}_0, \hat{v}_0)$ satisfies (A1), one can apply Lemma 3.2 (with suitable choice of parameters) to insure that $\overline{u} \leq \hat{u}$ and $\overline{v} \geq \hat{v}$ over $[T_0, \infty) \times [0, \infty)$ for some $T_0 \gg 1$. Together with (4.12), we have

$$\overline{u}(t, x) \leq u(t, x), \quad \overline{v}(t, x) \geq v(t, x) \quad \text{in } [T_0, \infty) \times [0, \infty).$$

Next, we need to find a suitable pair of supersolution $(\overline{u}, \overline{v})$. Denote

$$\overline{u}(t, x) = U(x - c_{uv} t + \zeta(t)) + U(-x - c_{uv} t + \zeta(t)) - 1 + \hat{p}(t), \quad t \geq 0, \quad x \in \mathbb{R},$$

$$\overline{v}(t, x) = (1 - \hat{q}(t)) \left[ V(x - c_{uv} t + \zeta(t)) + V(-x - c_{uv} t + \zeta(t)) \right], \quad t \geq 0, \quad x \in \mathbb{R},$$

where

$$\hat{p}(t) = \hat{p}_0 e^{-\beta t}, \quad \hat{q}(t) = \hat{q}_0 e^{-\beta t}, \quad \zeta(t) = \zeta_0 - \zeta_1 e^{-(\beta / 2)t}$$

for some $\hat{p}_0, \hat{q}_0, \beta > 0$ and $\zeta_i \in \mathbb{R}$ ($i = 0, 1$) that will be determined later. The form of $\overline{v}$ here is inspired by [15].

For notational convenience, we also denote

$$\xi_{\pm} = \pm x - c_{uv} t + \zeta(t), \quad (U_{\pm}, V_{\pm}) = (U(\xi_{\pm}), V(\xi_{\pm})).$$

Then after some direct computation, we obtain

$$N_1[\overline{u}, \overline{v}] = \left( -c_{uv} + \zeta'(t) \right) \left( U'_+ + U'_- \right) + \hat{p}'(t)$$

$$- d(U''_+ + U''_-) - f(U_+ + U_- - 1 + \hat{p}, (1 - \hat{q})(V_+ + V_-)).$$
where \( f(u,v) := ru(1 - u - av) \). Since \( -c_{av}U'_+ - dU''_+ = f(U_\pm, V_\pm) \), we thus have

\[
(4.14) \quad N_1[\pi, \nu] = \zeta'(t)(U'_+ + U'_-) + \hat{p}' + f(U_+, V_+) + f(U_-, V_-) - f(U_+ + U_- - 1 + \hat{p}, (1 - \hat{q})(V_+ + V_-)) = \zeta'(t)(U'_+ + U'_-) + \hat{p}' + f(U_+, V_+) + f(U_-, V_-) - f(U_+ + U_- - 1 + \hat{p}, V_+) - f(U_+ + U_- - 1 + \hat{p}, (1 - \hat{q})(V_+ + V_-)).
\]

Similarly, making use of \( -c_{av}V'_+ - V''_+ = g(U_\pm, V_\pm) \) we obtain

\[
(4.15) N_2[\pi, \nu] = -\hat{q}'(t)(V'_+ + V'_-) + (1 - \hat{q})\zeta'(t)(V'_+ + V'_-) + (1 - \hat{q})[g(U_+, V_+) + g(U_-, V_-)] - g(U_+ + U_- - 1 + \hat{p}, (1 - \hat{q})(V_+ + V_-)) = -\hat{q}'(t)(V'_+ + V'_-) + (1 - \hat{q})\zeta'(t)(V'_+ + V'_-) - (1 - \hat{q})V_+[-V_- + \hat{q}(V_+ + V_-) - b(U_- - 1 + \hat{p})] - (1 - \hat{q})V_-[-V_+ + \hat{q}(V_+ + V_-) - b(U_+ - 1 + \hat{p})],
\]

where \( g(u, v) = v(1 - v - bu). \)

We shall show that \( N_1[\pi, \nu] \geq 0 \) and \( N_2[\pi, \nu] \leq 0 \) for \( x \in \mathbb{R} \) and sufficiently large \( t \). Here we only consider the range \( x \geq 0 \) since a similar process can be used for the case \( x < 0 \). First, we take \( \zeta_1 < 0 \) such that \( \zeta' < 0 \). Since \( x \geq 0, U' < 0 \) and \( \zeta' < 0 \), we have

\[
1 - U_- = 1 - U(-x - c_{av}t + \zeta(t)) \leq 1 - U(-c_{av}t + \zeta(0)).
\]

We also require \( \zeta_0 < 0 \). Then, by Lemma 2.2, there exist two constants \( \lambda_u > 0 \) and \( K_1 > 0 \) independent of \( \Lambda := (\hat{p}_0, \hat{q}_0, \beta, \zeta_0, \zeta_1) \), such that

\[
(4.16) \quad 1 - U_- \leq Ke^{-\lambda_u(c_{av}t - \zeta(0))} \quad \text{for all} \quad x \geq 0 \quad \text{and} \quad t \geq 0.
\]

Without loss of generality, we may assume that \( U_- - 1 + \hat{p} > 0 \) since we may choose \( \beta < -\lambda_u c_{av} \) and \( -\zeta_0 \) sufficiently large.

Similarly, thanks to Lemma 2.2, we may find two constants \( \lambda_v > 0 \) and \( K_2 > 0 \) (independent of \( \Lambda \)) such that

\[
(4.17) \quad V_- \leq Ke^{-\lambda_v(c_{av}t - \zeta(0))} \quad \text{for all} \quad x \geq 0 \quad \text{and} \quad t \geq 0.
\]

To derive the differential inequalities, we divide the discussion into three cases.

**Case 1:** \( 0 \leq U_+ \leq \delta \) and \( 1 - \delta \leq V_+ \leq 1 \) for some small \( \delta > 0 \). Since \( \delta \) is sufficiently small, over the range \( 0 \leq u \leq \delta \) and \( 1 - \delta \leq v \leq 1 \), there exists \( m_1 > 0 \) such that \( (\partial f / \partial u)(u, v) = r(1 - 2u - av) < -m_1 \) \( (a > 1 \) is also used). Thus, the mean value theorem gives

\[
(4.18) \quad f(U_+, V_+) - f(U_+ + U_- - 1 + \hat{p}, V_+) \geq m_1(U_- - 1 + \hat{p})
\]

for \( 0 \leq U_+ \leq \delta \) and \( 1 - \delta \leq V_+ \leq 1 \). Also, by some simple computations,

\[
(4.19) \quad f(U_+ + U_- - 1 + \hat{p}, V_+) - f(U_+ + U_- - 1 + \hat{p}, (1 - \hat{q})(V_+ + V_-)) = ra(U_+ + U_- - 1 + \hat{p})(V_+ + V_-) \geq -ra\hat{q}(U_+ + U_- - 1 + \hat{p})(V_+ + V_-).
\]
Due to the range of $U_+$ and $V_+$ in Case 1, we deduce from (4.19) that
\[(4.20) f(U_+ + U_- - 1 + \hat{p}, V_+) - f(U_+ + U_- - 1 + \hat{p}, (1 - \hat{q})(V_+ + V_-)) \geq -2\hat{r}q(\delta + \hat{p}).\]

Obviously, it holds
\[(4.21) f(U_-, V_-) \geq -r_a U_- V_- \geq -r_a V_-.
\]

As a consequence, by (4.18), (4.20), (4.21) and the fact $U'_\pm \zeta' > 0$, we see from (4.14) that
\[
N_1[\pi, \nu] \geq \hat{p}' + m_1(U_- - 1 + \hat{p}) - 2\hat{r}q(\delta + \hat{p}) - r_a V_-\]
for $0 \leq U_+ \leq \delta$ and $1 - \delta \leq V_+ \leq 1$. In view of (4.10) and (4.11), we obtain
\[
N_1[\nu, \nu] \geq (-\beta \hat{p}_0 + m_1 \hat{p}_0 - 2\hat{r}q_0(\delta + \hat{p}_0 e^{-\beta t}))e^{-\beta t} - m_1 K_1 e^{-\lambda_u(c_{uw}-\zeta(0))} - r_a K_2 e^{-\lambda_u(c_{uw}-\zeta(0))}.
\]
Hence, there exists $T_1 \gg 1$ such that $N_1[\nu, \nu] \geq 0$ for $x \geq 0$ and $t \geq T_1$ within the range in Case 1, provided $\beta > 0$ and $\delta > 0$ are sufficiently small.

We next consider the inequality of $N_2[\pi, \nu]$. Since $\zeta' V'_\pm < 0$, from (4.15) it follows that
\[
N_2[\pi, \nu] \leq -(1 - \hat{q})V_- - (1 - \hat{q})\hat{q}(V_+ + V_-) - b(U_- - 1 + \hat{p})\]
\[
\leq -(1 - \hat{q})V_- - (1 - \hat{q})\hat{q}(V_+ + V_-) + b(1 - \hat{q})\hat{p}(V_+ + V_-)\]
\[
\leq -2\hat{q}' + 2(1 - \hat{q})V_- - (1 - \hat{q})\hat{q}(1 - \delta)^2 + 2b(1 - \hat{q})\hat{p},
\]
where we have used $1 - \delta \leq V_+ \leq 1$ and $0 \leq V_- \leq 1$. This, together with (4.17), yields
\[
N_2[\pi, \nu] \leq 2\beta \hat{q}_0 e^{-\beta t} - 2(1 - \hat{q})K_2 e^{-\lambda_u(c_{uw}-\zeta(0))} - (1 - \hat{q})\hat{q}(1 - \delta)^2 - 2b\hat{p}_0).
\]

Then one can find $T_2 \gg 1$ such that $N_2[\pi, \nu] \leq 0$ for $x \geq 0$ and $t \geq T_2$ within the range in Case 1, provided $\beta > 0$ small enough and $\hat{q}_0(1 - \delta)^2 > 2b\hat{p}_0$.

**Case 2:** $1 - \delta \leq U_+ \leq 1$ and $0 \leq V_+ \leq \delta$ for some small $\delta > 0$. In this case, there exists $m_2 > 0$ such that $(\partial f/\partial u)(u, v) = r(1 - 2u - av) < -m_2$ for $1 - \delta \leq u \leq 1$ and $0 \leq v \leq \delta$. This allows us to apply the same argument in Case 1 to deduce that for some large $T_3 > 0$, $N_1[\pi, \nu] \geq 0$ for $t \geq T_3$. The details are omitted here.

To verify $N_2[\pi, \nu] \leq 0$, we first observe that $V_- \leq V_+ \leq \delta$ when $x \geq 0$. Thus, one can find $\kappa > 0$ such that $V'_\pm \geq \kappa V_\pm$. Recall that $\zeta' < 0$. Then we have
\[
(1 - \hat{q})\zeta'(V'_+, V'_-) \leq \kappa(1 - \hat{q})\zeta'(V_+ + V_-).
\]

From (4.15) (also see the computation of $N_2[\pi, \nu]$ in Case 1) we have
\[
N_2[\pi, \nu] \leq -(1 - \hat{q})\hat{q}(V_+ + V_-) + 2(1 - \hat{q})V_- \hat{q}(V_+ + V_-)^2 + b(1 - \hat{q})\hat{p}(V_+ + V_-)\]
\[
\leq (V_+ + V_-)(1 - \hat{q} + \kappa(1 - \hat{q})\zeta' + 2(1 - \hat{q})\frac{V_+ V_-}{V_+ + V_-} + b(1 - \hat{q})\hat{p})\]
\[
\leq (V_+ + V_-)[\beta \hat{q}_0 e^{-\beta t} - \kappa(1 - \hat{q})\hat{q} e^{-\beta t} \left(\frac{\beta t}{2} e^{-(\beta/2) t}\right) + 2(1 - \hat{q})e^{-\beta t} K_2 e^{-\lambda_u(c_{uw}-\zeta(0))} + b(1 - \hat{q})\hat{p} e^{-\beta t}]\].
Therefore, it is easily seen that, for some large $T_3 > 0$, $N_2[\overline{\mu}, \overline{v}] \leq 0$ for $t \geq T_3$ for all sufficiently small $\beta > 0$.

Case 3: $\delta \leq U_+ + V_+ \leq 1 - \delta$ for the small $\delta$ used in Cases 1 and 2. In this range, there exists $\kappa_1 > 0$ such that $U'_+ \leq -\kappa_1$, which together with $U' < 0$ and $\zeta' < 0$ implies that $\zeta'(U'_+ + U'_-) \geq -\zeta'\kappa_1$. For convenience, we use $C$ as a positive constant independent of $\Lambda$ and $\delta$, which may vary from inequality to inequality. By the Lipschitz continuity of $f$, there exists $C > 0$

$$|f(U_+, V_+) - f(U_+ + U_- - 1 + \hat{p}, V_+)| \leq C(\hat{p} - 1 + U_-).$$

Moreover, as seen in the calculations of (4.19) and (4.21),

$$f(U_+ + U_- - 1 + \hat{p}, V_+) - f(U_+ + U_- - 1 + \hat{p}, (1 - \hat{q})(V_+ + V_-)) \geq -C(1 + \hat{p}_0)\hat{q},$$

$$f(U_-, V_-) \geq -CV_-.$$ Therefore, using (4.17), from (4.14) we get

$$N_1[\overline{\mu}, \overline{v}] \geq -\kappa_1\zeta' + \hat{p}' - C[\hat{p} - 1 + U_- + (1 + \hat{p}_0)\hat{q} + V_-] \geq \kappa_1\beta \cdot \frac{2}{(1 + \hat{q})} - \beta\hat{p}_0e^{-\beta t} - C[\hat{p}_0e^{-\beta t} + (1 + \hat{p}_0)\hat{q}_0e^{-\beta t} + e^{-\lambda_0(c_{uv}t - \zeta(0))}].$$

Then there exists $T_4 > 1$ such that $N_1[\overline{\mu}, \overline{v}] \geq 0$ for all $t \geq T_4$, provided $\beta > 0$ is sufficiently small.

On the other hand, in this range there exists $\kappa_2 > 0$ such that $V'_+ \geq \kappa_2$, which together with $V'_+ > 0$ and $\zeta' < 0$ imply that $\zeta'(V'_+ + V'_-) \leq \zeta'\kappa_2$. Thanks to (4.17), we see from (4.15) that

$$N_2[\overline{\mu}, \overline{v}] \leq -\hat{q}'(V_+ + V_-) + \kappa_2(1 - \hat{q})\zeta' + 2(1 - \hat{q})V_+V_- + b(1 - \hat{q})\hat{p}(V_+ + V_-) \leq 2\hat{q}_0e^{-\beta t} + \kappa_2(1 - \hat{q}_0e^{-\beta t}) + 2\hat{q}_0e^{-\beta t} + 2|\zeta'|(V_+ + V_-) \leq 2\hat{q}_0e^{-\beta t} + \kappa_2(1 - \hat{q}_0e^{-\beta t}) + 2\hat{q}_0e^{-\beta t} + 2|\zeta'|.$$

Then there exists $T_5 > 1$ such that $N_2[\overline{\mu}, \overline{v}] \leq 0$ for all $t \geq T_5$, provided $\beta > 0$ is sufficiently small.

Combining the discussion in Cases 1-3 and taking $T^* = \max\{T_1, T_2, T_3, T_4, T_5\}$, indeed we have shown that there exists some small $\beta^* > 0$ such that

$$N_1[\overline{\mu}, \overline{v}] \geq 0, \ N_2[\overline{\mu}, \overline{v}] \leq 0 \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad t \geq T^*$$

provided $\beta \in (0, \beta^*)$ and $\hat{q}_0(2(1 - \delta)^2) > 2b\hat{q}_0$.

Next, for any given $c \in (c_{uv}, c_v)$, we shall show that for some large $T^{**} \geq T^*$ and small $\beta^{**}$,

$$3$$

It follows from (4.15) and Lemma 4.6 that for some $T_6 > 0$,

$$\overline{\mu}(t, \pm ct) = u(t, \pm ct) \quad \text{for all} \quad t \geq T_6,$$

where $C$, $\mu$ are given in Lemma 4.6. Therefore, taking $\beta^{**} < \min\{\lambda_0(c + c_{uv}), \mu\}$ and $T_6$ larger if necessary, we see that $\overline{\mu}(t, ct) = u(t, ct)$ for all $t \geq T_6$, provided $\beta \in (0, \beta^{**})$. Thanks to (4.17) and Lemma 4.7 there exists $T_7 > 0$ such that

$$v(t, ct) - \overline{v}(t, ct) \geq 1 - C' e^{-\mu t} - (1 - \hat{q}(t))[1 + V(-ct - c_{uv}t + \zeta(t))], \quad t \geq T_7,$$
where $C'$ and $\nu$ are given in Lemma 4.7. Taking $\beta^{**}$ smaller such that

$$\beta^{**} < \min\{\lambda_u(c + c_{uv}), \mu, \nu, \lambda_v(c + c_{uv})\}$$

and $T_7$ larger if necessary, we obtain that $v(t, ct) \geq \underline{v}(t, ct)$ for all $t \geq T_7$, provided $\beta \in (0, \beta^{**})$.

Since $\underline{u}(\cdot, t)$ and $\underline{v}(\cdot, t)$ are even, the similar process used in the above (see also Remark 4.1) can be applied to assert $\underline{u}(t, -ct) \geq u(t, -ct)$ and $\underline{v}(t, -ct) \geq \underline{v}(t, -ct)$ for $t \geq T_8$, provided $\beta \in (0, \beta^{**})$ ($\beta^{**}$ may become smaller), where $T_8$ is some large constant. Therefore, (4.23) follows with $T^{**} := \max\{T^*, T_6, T_7, T_8\}$.

To use $(\underline{u}, \underline{v})$ as a comparison function over $[T^{**}, \infty) \times [-cT^{**}, cT^{**}]$, we fix $\beta < \min\{\beta^*, \beta^{**}\}$ and $\hat{q}_0(1 - \delta)^2 > 2b\hat{p}_0$. Then, taking $\zeta_0$ close to $-\infty$ (this does not affect the choice of $\beta^*$ and $\beta^{**}$), from the definition of $(\underline{u}, \underline{v})$ we can easily see

$$\underline{u}(T^{**}, x) \geq u(T^{**}, x), \quad \underline{v}(T^{**}, x) \geq \underline{v}(T^{**}, x) \quad \text{for} \quad x \in [-cT^{**}, cT^{**}].$$

As a result, by (4.22), a simple comparison analysis yields

$$\underline{u}(t, x) \geq u(t, x), \quad \underline{v}(t, x) \geq \underline{v}(t, x) \quad \text{in} \quad [T^{**}, \infty) \times [-cT^{**}, cT^{**}].$$

Now, combining (4.13) and (4.24), we obtain that for all large time and $x \geq 0$,

$$\max\{U(x - c_{uv}t + \eta(t)) - p(t), 0\} \leq u(t, x) \leq U(x - c_{uv}t + \zeta(t)) + U(-x - c_{uv}t + \zeta(t)) - 1 + \hat{p}(t),$$

$$\leq v(t, x) \leq (1 + \hat{q}(t))V(x - c_{uv}t + \zeta(t)) + V(-x - c_{uv}t + \zeta(t)) \leq (1 + q(t))V(x - c_{uv}t + \eta(t)).$$

Then following the same line as in the proof Theorem 1, we can finish the proof of Lemma 4.8 and may safely omit the details. This completes the proof. □

We are in a position to verify Theorem 3.

Proof of Theorem 3. We first show that, for any $c > c_{uv}$,

$$\lim_{t \to \infty} \left[ \sup_{x \in [ct, \infty)} \left| v(t, x) - V_{KPP}(x - c_v t + \frac{3}{c_v} \ln t + \omega(t)) \right| + \sup_{x \in [ct, \infty)} \left| u(t, x) \right| \right] = 0,$$

where $\omega$ is a bounded function defined on $[0, \infty)$. Indeed, by Lemma 4.6 $u$ decays to zero exponentially for $x \in [ct, \infty)$, which allows us to estimate $v$ along the process in Section 4.1 by exchanging the role of $u$ and $v$ therein. Then we can deduce that there exists a bounded function $\omega : [0, \infty) \to \mathbb{R}$ such that

$$\lim_{t \to \infty} \sup_{x \in [ct, \infty)} \left| v(t, x) - V_{KPP}(x - c_v t + \frac{3}{c_v} \ln t + \omega(t)) \right| = 0.$$ 

Hence, (4.25) holds.

In view of $c_{uv} < c_u < c_v$ and $c_0 = \frac{c_{uv} + c_u}{2}$, Theorem 3 follows immediately from Lemma 4.8 and (4.25). The proof is thus complete. □

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School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, 221116, Jiangsu Province, People’s Republic of China

E-mail address: pengrui@163.com

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan, Republic of China

E-mail address: changhong@math.nctu.edu.tw

School of Science and Technology, University of New England, Armidale, NSW 2351, Australia

E-mail address: zhouutokyo@gmail.com