These lectures start with an elementary introduction to the subject of magnetic monopoles which should be accessible from any physics background. In the Weinberg-Salam model of electroweak interactions, magnetic monopoles appear at the ends of a type of non-topological vortices called 

\textit{electroweak strings}. These will also be discussed, as well as recent simulations of their formation during a phase transition which indicate that, in the (unphysical) range of parameters in which the strings are classically stable, they can form with a density comparable to topological vortices.

1. Introduction

Last year marked the 50th anniversary of one of P.A.M. Dirac’s most profound and famous papers - on magnetic monopoles. While his 1931 paper is usually considered the official birthday of magnetic monopoles, his 1948 paper is where he really showed that it was possible to have a consistent quantum theory of magnetic poles in conjunction with electric charges, and described the interaction between them [1].

A standard reference on magnetic monopoles is Preskill’s lectures in the 1985 Les Houches school [2]. Here I have tried to present magnetic monopoles and vortices in a way that makes them accessible to physicists who are not so familiar with the language of high energy physics, in particular with \textit{e.g.} non-abelian gauge theories. As a result, these lectures are much less technical.
The theme of this school is the use of topological defects as a tool to understand the dynamics of phase transitions out of equilibrium. It turns out that, since magnetic monopoles and electroweak vortices are non-topological in the Weinberg-Salam model, their study can be particularly interesting in order to understand the role of gauge fields in defect formation during phase transitions.

2. The elusive monopole

The problem with magnetic monopoles is well known in the context of Maxwell’s equations. If the electromagnetic field is described by the vector potential $\vec{A}$, then
\[ \vec{\nabla} \cdot \vec{B} = 0 \] (1)
so there can be no sources or sinks for the magnetic field.

It would be fair to say that the experimental evidence for the existence of magnetic monopoles is not good. The 1998 Review of Particle Properties by the Particle Data Group [3] shows the result of monopole searches in particle accelerators: not a monopole in sight. On the other hand, cosmic ray searches have essentially only one event for which there seems to be no alternative explanation, observed by Cabrera in Stanford in 1982 [4]. The Cabrera detector, like many others that failed to find anything before or after it, consisted of a superconducting ring where a persistent current was monitored for a long time; in [4] the loop had an area of 20 cm$^2$ and was monitored for a total of 151 days. During this time a single event was recorded which could be ascribed to a magnetically charged particle with one Dirac unit of magnetic charge $q_m = 2\pi \hbar c/e$. A magnetic monopole.

Under the circumstances, the experimental evidence is neatly summarized by the sentence (whose author is unfortunately unknown to me) “It is not clear that nobody has ever seen a magnetic monopole; what is clear is that nobody has ever seen two”. And yet since Dirac’s seminal work there have been over three thousand papers in the literature about magnetic monopoles! [5]

The reasons behind this fascination with monopoles have evolved with time, but they are basically three:
- the existence of monopoles would explain the quantisation of electric charge (for which there is no alternative explanation to this day). In his 1948 paper Dirac says: “If one supposes that a particle with a single magnetic pole can exist and that it interacts with charged particles, the laws of quantum mechanics lead to the requirement that the electric charges shall be quantized – all charges must be integral multiples of a unit charge e connected with the pole strength by the formula $eg = \frac{1}{2} \hbar c$. Since electric charges are known to be quantized and no reason for this has yet been proposed apart from
the existence of magnetic poles, we have here a reason for taking magnetic monopoles seriously”. He then goes on to say that the fact that they have not yet been observed may be ascribed to the large value of the quantum of the pole.

- a large class of theories that include electromagnetism as a subset predict magnetic monopoles as solitons, as was shown by ’t Hooft and Polyakov in 1974 [6], and

- if magnetic monopoles exist, Maxwell’s equations are symmetric under the exchange of electric and magnetic fields. This duality symmetry relates small electric charge to large magnetic charge and vice versa. A generalization of this symmetry to non-abelian theories would mean that the dual theory (of weakly coupled monopoles) could be used to understand strongly coupled non-abelian gauge theories and, in particular, confinement, for which there is no other analytic approach. While the idea is not new, some of the most important developments in this area are fairly recent; but they fall outside the scope of these lectures, and I refer the reader to an excellent review by Harvey [7].

In what follows I will take \( c = \hbar = 1 \). I will also depart from Dirac’s notation and use \( q_m \) to refer to the magnetic charge; \( g \) will be the \( SU(2) \) coupling constant.

One final comment. I think everyone attending this school is aware of the language problems between high energy and condensed matter physicists. A relatively common source of confusion is the use of the word gauge symmetry, which can mean different things to the two communities. We all agree that electromagnetism has a gauge symmetry, it is the symmetry that allows local (that is, position-dependent) changes in the phase of the wave function and a compensating change of gauge in the vector potential

\[
\psi(t, \vec{x}) \to e^{ie\chi(t, \vec{x})} \psi(t, \vec{x}) \\
A_\mu(t, \vec{x}) \to A_\mu(t, \vec{x}) + \nabla_\mu \chi(t, \vec{x}) \quad \mu = 0, 1, 2, 3.
\tag{2}
\]

The electromagnetic tensor \( F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu \), also known as the field strength, is unchanged by this transformation, while the covariant derivative of \( \psi \), \( \nabla_\mu \psi - ieA_\mu \psi \), transforms in the same way as \( \psi \) itself (thus the name covariant).

High energy physicists use the term “gauge symmetry” to indicate any symmetry which is local, whether or not it corresponds to a \( U(1) \) transformation. Most condensed matter physicists, on the other hand, will talk about gauge symmetries to indicate a change in the phase of the wave function (a \( U(1) \) transformation), whether or not there are vector potentials around. Thus, the transformation

\[
\Psi \to e^{i\alpha} \Psi \quad \alpha = \text{const}
\tag{3}
\]
is a gauge transformation in the condensed matter literature, but not in
the high energy literature (where it would be called a *global U(1) trans-
morphism*). On the other hand a high energy physicist will talk about
e.g. an SU(2) or SO(3) gauge transformation, meaning what is best described as
a “position-dependent rotation” in internal space. Since there are three de-
grees of freedom associated with rotations, we need three vector potentials

\[ W_\mu = (W^1_\mu, W^2_\mu, W^3_\mu) \]

\[ W_\mu \equiv W_\mu \cdot \tau = \frac{1}{2} \begin{pmatrix} W^3_\mu & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & -W^3_\mu \end{pmatrix}, \]

where \( \tau = (\tau^1, \tau^2, \tau^3) \) are the Pauli spin matrices. The transformation law
for the gauge potentials is

\[ W_\mu(t, \vec{x}) \rightarrow M^{-1}(t, \vec{x}) W_\mu(t, \vec{x}) M(t, \vec{x}) + \frac{1}{g} M^{-1}(t, \vec{x}) \nabla_\mu M(t, \vec{x}) \]

where \( M(t, \vec{x}) \) is a SU(2) matrix at each point in spacetime; note that if
\( M \) were independent of position, as in the case of spin, there would be no
need for gauge potentials and the symmetry would be called a *global SU(2)*
symmetry.

The transformation is non-abelian and moreover the field strength,
which has to be generalized from

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g [W_\mu W_\nu - W_\nu W_\mu] \]

or

\[ G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + g W_\mu \times W_\nu \]

(with the cross product taken in internal space) is no longer invariant but
changes like

\[ G_{\mu\nu} \cdot \tau \rightarrow M^{-1}(t, \vec{x}) G_{\mu\nu} \cdot \tau M(t, \vec{x}) \]

The transformation law of the scalars depends on the group representa-
tion to which they belong and we list here two that will be relevant later.
The fundamental representation of SU(2) is a doublet of complex fields
\( \Phi = (\phi_1, \phi_2) \) whose transformation law and covariant derivative are

\[ \Phi \rightarrow M^{-1}(t, \vec{x}) \Phi \quad \text{and} \quad D_\mu \Phi \equiv \nabla_\mu \Phi + g W_\mu \Phi \]

respectively. (9)

The adjoint representation is a triplet of real scalars \( \phi = (\phi^1, \phi^2, \phi^3) \) which,
like the gauge potentials, can be assembled into a matrix \( \Phi \equiv \phi \cdot \tau \) with
transformation law

\[ \Phi(t, \vec{x}) \rightarrow M^{-1}(t, \vec{x}) \Phi(t, \vec{x}) M(t, \vec{x}) \]

and covariant derivative

\[ D_\mu \Phi \equiv \nabla_\mu \Phi + g [W_\mu \Phi - \Phi W_\mu] \]
or, equivalently,

\[ D_\mu \Phi \equiv \nabla_\mu \Phi + g W_\mu \times \Phi. \]  

(12)

In these expressions, \( \mu = 0, 1, 2, 3 \). In what follows, we will “work in temporal gauge”, setting the time component of the gauge fields to zero. Thus, the vector potential (or gauge potential) will be a three vector and we will use the notation \( \vec{\nabla} \times \vec{W} + g \vec{W} \times \vec{W} \) for the field strength; the expressions above should serve to clarify whether the cross product is taken in internal space, in real space or in both.

3. Do-it-yourself magnetic monopoles

This section is “adapted” (i.e. taken) from Coleman’s 1974 Erice lectures [8]. He refers to this as “the monopole hoax”, a joke to be played (or at least attempted) by a cunning theorist on a gullible experimenter. In view of the number of cunning experimenters in this audience I will refrain from comments and just describe here how to build your own magnetic monopole.

1) Take a solenoid. It has to be very long and very thin so as to be invisible; as Coleman says, it helps if the solenoid is many miles long and considerably thinner than a fermi (this is very much a gedanken hoax).

2) Put one end at the experimenter’s laboratory,

3) hide the other end, and

4) turn on the current.

For a gullible theorist this may pass as a magnetic monopole, but of course there is a way in which the solenoid could be detected: through Aharonov-Bohm scattering.

The interference pattern in a double-slit experiment is shifted when a solenoid is placed between the slits and the screen. Even if the particle trajectories remain well outside the solenoid, their wave functions \( \psi_1, \psi_2 \) acquire a phase \( \exp \left[ i e \int \vec{A} \cdot d\vec{l} \right] \) (with \( e \) the electric charge of the particle and the integral taken along the particle’s path); if the paths are on either side of the solenoid, the interference pattern changes because the probability amplitude \( \left| \psi_1 + \psi_2 \right|^2 \) becomes

\[ \left| e^{ie \int_1 \vec{A} \cdot d\vec{l}} \psi_1 + e^{ie \int_2 \vec{A} \cdot d\vec{l}} \psi_2 \right|^2 = \left| \psi_1 + e^{ie \oint \vec{A} \cdot d\vec{l}} \psi_2 \right|^2, \]  

(13)

where \( \oint \vec{A} \cdot d\vec{l} \), taken around the solenoid, measures its magnetic flux.

Notice that, if the flux is an integer multiple of \( 2\pi/e \), the solenoid becomes undetectable even quantum mechanically in our gedanken experiment – it is called a Dirac string.

The vector potential of a monopole whose Dirac string is along the negative z-axis is given (in spherical coordinates \( (r, \theta, \varphi) \) centred on the
monopole) by
\[ \vec{A}_N \cdot d\vec{x} = \frac{q_m}{4\pi} (1 - \cos \theta) d\varphi, \quad \theta \neq \pi \quad (14) \]
giving rise to a radial magnetic flux\(^1\)
\[ \vec{B} \cdot d\vec{S} = \frac{q_m}{4\pi} \sin \theta d\theta d\varphi \quad \text{or} \quad \vec{B} = \frac{q_m}{4\pi} \frac{1}{r^2} \hat{r} \quad (15) \]
where \(q_m\) is the magnetic charge. The vector potential is singular on the Dirac string, which is located at \(\theta = \pi\), but regular everywhere else.

Since the magnetic flux of the monopole is supplied by the Dirac string, the condition that the flux through the string should be a multiple of \(2\pi/e\) gives rise to the famous Dirac quantization condition for the magnetic charge \(q_m\) of the monopole:
\[ q_m = \frac{2\pi N}{e} \quad \text{or, reintroducing } h \text{ and } c, \quad \frac{eq_m}{4\pi} = \frac{N}{2} (h\hbar) \quad (16) \]
Thus, the existence of one monopole would be enough to force electric charge to be quantized!

Note that we can use gauge invariance to change the position of the Dirac string; an equivalent description of this monopole is given by the vector potential
\[ \vec{A}_S \cdot d\vec{x} = -\frac{q_m}{4\pi} (1 + \cos \theta) d\varphi, \quad \theta \neq 0 \quad (17) \]
which is singular only at \(\theta = 0\) (the gauge transformation between \(\vec{A}_N\) and \(\vec{A}_S\) is singular in the position of both the old and new strings, of course).

You should not worry about these singularities – in the next section we will eliminate the Dirac string altogether.

4. The Wu-Yang construction of Dirac monopoles

Electromagnetism is not a theory of gauge potentials \textit{per se}, but rather of equivalence classes of gauge potentials. This was exploited by Wu and Yang [9] to give a non-singular description of magnetic monopoles by “patching up” vector potentials that are regular in different regions, provided they are equivalent on the overlaps.

Consider any sphere with non-zero radius surrounding the monopole. In the northern hemisphere \(\theta \in [0, \pi/2 + \epsilon]\) take \(\vec{A} = \vec{A}_N\) and in the southern hemisphere \(\theta \in [\pi/2 - \epsilon, \pi]\) take \(\vec{A} = \vec{A}_S\). In the overlap region,

\(^1\)Note that \(\vec{B}\) does not include the singular contribution from the Dirac string.
\[ \pi/2 - \epsilon < \theta < \pi/2 + \epsilon, \] the two descriptions are related by a regular gauge transformation,

\[ (\vec{A}_N - \vec{A}_S) \cdot dx = \vec{\nabla}_\chi \cdot dx \quad \text{where} \quad \chi(\varphi) = \frac{q_m}{2\pi} \varphi. \] (18)

This object has magnetic charge \( q_m \), as can be seen by computing the magnetic flux through the two hemispheres (using Stokes’ theorem). Note that, even though \( \chi \) is not singlevalued, \( \vec{\nabla}_\chi \) is, and therefore the gauge transformation is well defined on the gauge potentials. Moreover, single-valuedness of the gauge transformation on the wave functions, \( e^{i\chi} \), in the overlap equatorial region implies the Dirac quantisation condition! \( q_m \equiv \chi(2\pi) - \chi(0) = 2\pi N/e \).

We have eliminated the need for a Dirac string – note that the only singularity in this description is at the origin, \( r = 0 \). But this must be a real singularity because the energy of the monopole diverges as \( r \to 0 \) due to the \( 1/r^2 \) behaviour of the magnetic field,

\[ E = \int d^3x \left( \frac{1}{2} (|\vec{E}|^2 + |\vec{B}|^2) \sim \int_0^\infty \frac{dr}{r^2} \right) \] (19)

Electromagnetism is not the only force in nature. There are the weak and strong nuclear forces, and also gravity. We think that forces may become unified into one kind of interaction (a Unified Theory) at high energies\(^2\). When theorists started to investigate possible unified theories they found a surprise...

5. ’t Hooft–Polyakov monopoles

One of the first attempts to unify the electromagnetic and weak interactions was the \( O(3) \) Georgi-Glashow model \([10]\) in which the fundamental fields are a triplet of (real) scalars and a triplet of gauge potentials.

\[ \vec{\phi} = (\phi^1, \phi^2, \phi^3) \]
\[ \vec{W} = (\vec{W}^1, \vec{W}^2, \vec{W}^3) \] (20)

The most important aspect from our point of view is the energy, since we are seeking to remove the divergence at \( r = 0 \). I am making many simplifying assumptions here (no time dependence, no electric fields), and only writing the terms in the energy that are relevant for the argument:

\[ E = \int d^3x \left[ (\vec{\nabla} \cdot \vec{\phi})^2 + (\vec{\nabla} \times \vec{\phi} + g \vec{W} \times \vec{\phi})^2 + \lambda (\vec{\phi} \cdot \vec{\phi} - \eta^2)^2 \right] \] (21)

\(^2\)A special class are so-called Grand Unified Theories, or GUTs, where the electromagnetic, weak and strong interactions are described by a single simple group.
First of all, the Georgi–Glashow model includes electromagnetism as a subset: the configuration

\[ \phi^1 = \phi^2 = 0 \,, \quad \phi^3 = \eta \,, \quad \vec{W}^1 = \vec{W}^2 = 0 \, , \quad \vec{W}^3 = \vec{A} \,, \]

where \( \vec{A} \) is any solution to Maxwell’s equations, is also a solution of the full non-abelian field equations of the Georgi–Glashow model. In particular, the Wu-Yang (or Dirac) monopole is a solution. However, it is not a stable solution!

Indeed, the \( 1/r^2 \) divergence in the monopole energy is now coming from \( \vec{\nabla} \times \vec{W}^3 \) in the \( f((\vec{\nabla} \times \vec{W}^3 + g\vec{W}^1 \times \vec{W}^2)^2 \) term and could be controlled if \( \vec{W}^1 \) and \( \vec{W}^2 \) acquired non-zero values \( \sim 1/r \). This is consistent with the fact that \( \vec{W}^\pm \equiv (\vec{W}^1 \mp i\vec{W}^2)/\sqrt{2} \) are charged fields (the \( W \)-bosons) with charge \( \pm g \) respectively and a magnetic moment \( ig\vec{W}^3 \times \vec{W}^+ \) which couples to the \( \vec{W}^3 \) magnetic field, so their presence can reduce the magnetic energy. On the other hand, such “\( \vec{W} \)-condensation” has two immediate effects: one is an increase in energy coming from the new, non-zero \( (\vec{\nabla} \times \vec{W}^{1,2} \pm g\vec{W}^3 \times \vec{W}^{2,1})^2 \) terms; the other is that the scalar gradients \( D_\mu \phi^{1,2} \sim g\vec{W}^{2,1} \times \vec{\phi} \) now diverge as \( 1/r \). However this problem is eliminated if \( \phi^3 \sim r \) as \( r \to 0 \). The condition \( \phi(r = 0) = 0 \) imposes a penalty in energy from the \( f(\vec{\phi} \cdot \vec{\phi} - \eta^2)^2 \) term, but this is finite – thus, the result is always energetically favourable to the singular abelian monopole that we started with.

In the case when magnetic charge is two Dirac units, \( q_m = 4\pi/g \) (see below for an explanation of this condition), this instability leads to the ’t Hooft–Polyakov monopole [6], a spherically symmetric configuration describing a non-singular magnetic monopole of finite mass [11],

\[ \phi^1 = \phi^2 = 0 \,, \quad \phi^3 = \eta r^2 \phi(r) \,, \quad \vec{W}^1 = \vec{W}^2 = 0 \, , \quad \vec{W}^3 = \frac{1}{g} \frac{f(r)}{r} \left( \frac{\phi}{\sin \theta} - i \hat{\phi} \right), \]

with \( f(r) \to 1 \), \( f'(r) \sim -r \) and \( \rho(r) \sim r \) as \( r \to 0 \) and \( f(r) \to 0 \), \( \rho(r) \to 1 \) as \( r \to \infty \).

Note that only the small \( r \) behaviour of the fields has changed; in particular, the magnetic charge of the monopole remains the same. After a (singular) gauge transformation it reduces to the more familiar form [6]

\[ \vec{\phi} = \rho(r) \hat{\phi} \]

\[ \vec{W} = \frac{1}{g} (f(r) - 1) \hat{\phi} \times \vec{\nabla} \hat{\phi} \,, \]

\[ \phi^3 = \eta r^2 \phi(r) \]
which shows that the ’t Hooft–Polyakov monopole is a topological defect (usually called a hedgehog because of the way the scalar field points radially outwards). The zero value of the scalar field at the origin $r = 0$ is forced by the non-trivial winding of the scalar field.

It remains to explain why the restriction to two units of magnetic charge. In the Wu-Yang construction we started with a sphere of non-zero radius, say $R$, divided into two hemispheres overlapping at the equator. If the monopole has $N$ units of magnetic charge the gauge transformation in the overlap region is a phase rotation by $2\pi N$. W–condensation replaces the singularity at the origin by an everywhere regular core. Since nothing changes outside the core, the patching condition for the ’t Hooft–Polyakov monopole remains a $2\pi N$ rotation for all $R > r_{\text{core}}$. But this cannot be true inside the core: since there is no singularity, the gauge transformation must also change continuously so that it becomes the identity when we reach $r = 0$. If the gauge group is $U(1)$, this is simply not possible, and all monopoles are singular. But in $SU(2)$, a $2\pi$ rotation is not continuously connected to the identity whereas a $4\pi$ rotation is! Thus, only monopoles with even $N$ can be non-singular. Of all these, it turns out that only $N = 2$ remains spherically symmetric after W–condensation.

The existence of magnetic monopoles is a very generic prediction for a large class of theories containing electromagnetism. Moreover, they should be produced in large numbers in the early Universe [12]. The fact that we do not observe those monopoles is a serious challenge to cosmologists, and has become known as the monopole problem.

6. Magnetic monopoles in the Weinberg–Salam model; electroweak strings and dumbells

The standard model of electroweak interactions has a $SU(2) \times U(1)$ gauge symmetry, corresponding to weak isospin and hypercharge respectively. Its bosonic sector comprises a neutral scalar field $\phi^0$, a charged scalar field $\phi^+$, and the vector potentials corresponding to the massless photon $\vec{A}$ and three massive vector bosons: the charged W–bosons ($\vec{W}^\pm$) and the neutral $\vec{Z}$. The fermionic sector consists of the three families of quarks and leptons

$$
\begin{pmatrix}
\nu_e \\
e \\
u_\mu \\
\mu \\
u_\tau \\
\tau \\
t \\
b
\end{pmatrix}
$$

(25)

In the Weinberg–Salam model, the electromagnetic and Z-fields are combinations of the $SU(2)$ and $U(1)$ gauge potentials ($\vec{W}$ and $\vec{Y}$ respectively):

$$
\vec{Z} \equiv \cos \theta_w \vec{W}^3 - \sin \theta_w \vec{Y}, \quad \vec{A} \equiv \sin \theta_w \vec{W}^3 + \cos \theta_w \vec{Y},
$$

(26)
where $\theta_w$ is called the weak mixing, or Weinberg, angle. Its measured value is $\sin^2 \theta_w \approx 0.23$.

In this case, the field that satisfies Maxwell's equations at low energy is $\vec{A}$ and, being massless, it is the only vector potential that can give rise to long-range electric and magnetic fields and thus magnetic monopoles. Its configuration far from the monopole will be exactly like what was discussed in sections 3 and 4. Very close to the monopole, though, we would expect other fields to condense due to the intense magnetic field, changing the core structure.

But there is a problem. Note that, since the electromagnetic field has a hypercharge component and hypercharge is an abelian field, isolated magnetic monopoles are always singular at the origin. We are back to square one!

In order to have monopoles with regular cores one has to embed the $SU(2) \times U(1)$ symmetry of the Weinberg–Salam model into larger symmetry groups. We already mentioned Grand Unified Theories (GUTs, for short), where one simple group not only contains the electroweak interaction, but also the strong interaction. These monopoles are very heavy, because the unification of these forces occurs at very high energies ($\sim 10^{16}$ GeV) and the fields that condense at the core are very massive. Far too heavy to be produced in a particle accelerator.

But there are also lighter magnetic monopoles in the Weinberg–Salam model: they occur as monopole–antimonopole pairs connected by a vortex (the vortex carries magnetic flux of the $Z$-boson, and it is usually called a $Z$-string or an electroweak string). Such configurations were called dumbells by Nambu, who first considered them in 1977 [13]. The dumbell is rotating to avoid longitudinal collapse, and its mass is estimated at a few TeV.

Their internal structure is rather interesting. The $SU(2)$ fields are those of a ’t Hooft–Polyakov monopole–antimonopole pair, while the hypercharge $U(1)$ field configuration resembles that of a solenoid joining the monopole and antimonopole. As a result, the combination inside the solenoid is precisely the magnetic part of the $Z$ field, whereas the magnetic field that emanates from the solenoid ends is the massless electromagnetic field, and there are no singularities anywhere.

In some respects, the structure of the $Z$–string is similar to that of a magnetic vortex in an Abrikosov lattice that appears in a type II superconductor subjected to an external magnetic field. In the Weinberg–Salam model, the role of the vector potential is taken by the $Z$–field, and the order parameter is the neutral Higgs field $\phi^0$. But there is a very important difference: electroweak strings are non–topological, and therefore not necessarily stable.
A review of electroweak strings where the stability issue is discussed in some detail can be found in [14]. Stability depends on the value of the Weinberg angle and the masses of the various fields. In the physical range of these parameters, it is found that infinitely long, straight, bare strings are classically unstable (they are only stable for $\theta_w \sim \pi/2$ and $m_{\text{Higgs}} < m_Z$). On the other hand, stability improves for short segments and in the presence of magnetic fields. Finally, the strings are superconducting and fermion modes on the string might also stabilize them, although this is still under discussion. The stability of dumbbells remains an open question.

Even in the region of parameter space where strings are (classically) stable, the fact that stability is not topological immediately raises the question of whether a network of strings would form in a phase transition via the Kibble mechanism. We now turn to this question, and we will focus on the (unphysical) limit of the Weinberg–Salam model in which the $SU(2)$ symmetry becomes global, while keeping the hypercharge $U(1)$ symmetry local. In this limit, known as the semilocal model, electroweak strings are classically stable.

7. Semilocal strings

The semilocal model is obtained when the complex scalar field $\phi$ in the Abelian Higgs (or Landau–Ginzburg) model is replaced by an $SU(2)$ doublet of complex fields $(\phi, \psi)$.

It is also (the bosonic sector of) the Weinberg–Salam model in the limit in which the $SU(2)$ gauge coupling is set to zero; in this limit, the $W$-bosons and the photon decouple and the symmetry is $SU(2)_{\text{global}} \times U(1)_{\text{local}}$. The only gauge field is the neutral $Z$ boson, which coincides with the hypercharge field. The model has vortices (semilocal strings) whose properties are intermediate between electroweak strings and Abrikosov–Nielsen–Olesen vortices (see [14]).

The energy per unit length of cylindrically symmetric configurations is

$$E = \int d^2x [||\vec{\nabla} - iq\vec{Y}\phi||^2 + ||\vec{\nabla} - iq\vec{Y}\psi||^2 + \frac{1}{2}((\vec{\nabla} \times \vec{Y})^2 + \lambda(\phi^2 + \psi^2 - \frac{\eta^2}{2})^2)]$$

(27)

and it turns out that, even though the vacuum manifold is a three-sphere,

$$\phi^2 + \psi^2 = \frac{\eta^2}{2}$$

(28)

which is simply connected, $\pi_1(S^3) = 1$, there are stable strings if the scalar mass is smaller than the vector mass. These strings are not only stable

$^3$Similar systems have been considered in the condensed matter literature [15, 16].
classically, they are also stable to semiclassical tunnelling and to breaking by monopole pairs. If \( m_{\text{scalar}} > m_{\text{vector}} \) the strings are classically unstable, and if the masses are equal there is a two-parameter family of configurations with the same energy where the quantised magnetic flux \( 2\pi/q \) spreads over an arbitrarily large core width.

Semilocal strings, like electroweak strings, can have open ends; but the monopoles at the ends are global monopoles and have a long-range interaction.

The stability of semilocal strings is well understood in terms of the competition between gradient energy and potential energy. The gauge field can compensate gradients in the (complex) phase of either of the two scalar fields, but it can only compensate both gradients simultaneously if there is a correlation between the phases of the two scalars; this correlation is present in the string solution, which optimizes gradient energy, but at the expense of a large concentration of potential energy at the core. If \( \lambda/q^2 \) is very large, it becomes energetically favourable to break the phase correlation in order to reduce potential energy, and the string is destroyed. If \( \lambda \) is small, the cost in gradient energy to dissolve the string becomes too high and the string is stable.

Since the strings are non-topological, the question immediately arises as to whether they would form at all in a phase transition. In particular, stability depends on the mass ratio \( m_{\text{scalar}}/m_{\text{vector}} \) and we would expect the formation rates to reflect this dependence.

We now turn to a first attempt to answer these questions through numerical simulations. Note that setting \( \psi = 0 \) in the semilocal model gives the Abelian Higgs (or Landau–Ginzburg) model, thus making it possible to compare the rates of formation of semilocal strings to those of topological strings in a similar environment. The conclusion seems to be that semilocal strings do form, and in some cases with number densities comparable to those of their topological counterparts.

8. Numerical simulations of semilocal string formation

For details of the simulations we refer the reader to [17, 18]. Here we will just point out the main features and results, summarized in figure 5.

Space is discretized into a lattice with periodic boundary conditions. The equations of motion are solved numerically using a standard staggered leapfrog method, and a dissipation term \( (\eta \dot{\phi}, \eta \dot{\psi} \text{ or } \eta \dot{Y_i}) \) is added to each equation to reduce the relaxation time. A range of strengths of dissipation was tested, and it did not significantly affect the number densities obtained; the simulations displayed in the figures all have have \( \eta = 0.5 \). Note that in an expanding Universe the expansion rate would act as a sort of viscosity,
Figure 1. The flux tube structure in a two-dimensional semilocal string simulation with $\beta = 0.05$. The upper panel ($t = 0$) shows the initial condition after the process described in the text. The lower panel shows the configuration resolved into five flux tubes by a short period of dynamical evolution ($t = 100$). These flux tubes are semilocal vortices. Note the different numbers of upward and downward pointing flux tubes, despite the zero net flux boundary condition. The missing flux resides in the smaller 'nodules', made long-lived by the numerical viscosity; the expansion of the universe could have a similar effect and preserve these 'skyrmonic' configurations [20].

though $\eta$ would typically not be constant. However, this is not meant to be a cosmological simulation since spacetime is flat.

We work in temporal gauge. Then Gauss’ law becomes a constraint
derived from the gauge choice $Y_0 = 0$, and is used to test the stability of the code.

The inverse vector mass is taken as the unit of length and time, and $\eta^2$ as the unit of energy. In these units the dynamics is governed by a single parameter,

$$\beta = \frac{m_{\text{scalar}}^2}{m_{\text{vector}}^2} = \frac{2\lambda}{q^2}$$

which also determines the stability of the straight string solutions: they are stable if $\beta < 1$ and unstable if $\beta > 1$. 4

The number density of defects is estimated by a multi-step process. The initial conditions are obtained with a generalization to non-topological strings of what is known to cosmologists as the Vachaspati–Vilenkin algorithm [19], followed by a short period of dynamical evolution.

- First, we generate a random initial configuration for the scalar fields drawn from the vacuum manifold, which is not discretised. If space is a grid of dimension $N^3$, the correlation length is chosen to be some number $p$ of grid points ($p = 16$ in [17, 18]; the size of the lattice is either $N = 64$ or $N = 256$). To obtain a reasonably smooth configuration for the scalar fields, we assign random vacuum values on a $(N/p)^3$ subgrid and interpolate the scalar field smoothly onto the full grid.

- We then find the gauge field configuration that minimizes the energy in this fixed scalar background.

Two dimensional test simulations have shown that the energy minimization is redundant, since the early stages of dynamical evolution carry out this role anyway; for simplicity, we used in practice a gauge field configuration which was close, but not equal, to the real minimum.

- An example of the initial conditions generated with this algorithm can be seen in Figure 1 in the case of a two-dimensional toy model with translational invariance in one dimension, say $z$. The plot shows magnetic field on the $x$–$y$ plane, which has a complicated flux structure with extrema of different values (top panel of Fig. 1), and it is far from clear which of these, if any, might evolve to form semilocal vortices; in order to resolve this ambiguity, the initial configurations are evolved forward in time with zero initial velocities for the fields. After a short transient, in the unstable regime $\beta > 1$ the flux quickly dissipates leaving no strings. By contrast, in the stable regime $\beta < 1$ stringlike features emerge when configurations in the “basin of attraction” of the semilocal string relax unambiguously into vortices (bottom panel of Fig. 1). We only count vortices after this relaxation process.

4$\beta$ is also the parameter that distinguishes type I ($\beta < 1$) from type II ($\beta > 1$) superconductors.
Figure 2. Loop formation from semilocal string segments. The figure shows two snapshots, at \( t = 70 \) and \( t = 80 \), of a \( 64^3 \) numerical simulation of a network of semilocal strings with \( \beta = 0.05 \), where the ends of an open segment of string join up to form a closed loop. Subsequently the loops seem to behave like those of topological cosmic string, contracting and disappearing.

- One important point is that strings are always identified with the location of magnetic flux tubes, rather than by the zeroes of the scalar field. Figure 3 shows two snapshots of a simulation on a \( 256^3 \) lattice. In order to make the figure, and also to compute the number density, we need to set a magnetic flux threshold – the strings that can be seen in that figure are made of those points in which the magnetic field exceeds half the maximum value in an Abrikosov-Nielsen-Olesen vortex. After the short
transient, this threshold can be modified without significantly affecting the results, which are ratios of semilocal to topological string number densities (shown in figure 5). The error bars include, among other things, this threshold dependence as well as the dispersion between runs.

Since the initial conditions are somewhat artificial, the results were checked against various other choices of initial conditions, in particular different initial conditions for the gauge field and also initial conditions for the scalar field closer to a thermal environment (see Fig. 4). However, it cannot be sufficiently stressed that it is not a realistic thermal simulation of the phase transition. There is no thermal noise and all the initial conditions in [17, 18] had zero initial velocities for the fields – initial conditions with non-zero field momenta have not yet been investigated –. What the simulations show is that the rate of formation of these non-topological vortices is not only non-zero but in certain cases it could even be comparable to that of their topological counterparts. Obviously the details of the transition can and should now be investigated in full.

9. Discussion and outlook

The physical mechanism behind the observed string formation by accretion of magnetic field in Figure 1 or by growth of string segments in Figures 2 and 3 is not very different from the formation of Abrikosov lattices in a type II superconductor, even though in a system such as the early Universe it makes no sense to talk about external magnetic fields. Because of the \( Y^2 \Phi^2 \) term in the energy, the magnetic field does not like to coexist with the superconducting (\( \phi \neq 0 \)) phase of the scalars.

In the topological strings case one usually argues that the non-zero winding in the phase of the scalar field forces a zero of the Higgs, and magnetic flux gathers there in a vortex. Conversely, what we are observing in the semilocal model is that if there is a sufficiently large concentration of magnetic flux in a small region, for instance near the end of a string segment or maybe due to a fluctuation, a line of zeroes of the Higgs can develop there. Small segments of string, on the other hand, will tend to contract and disappear.

It seems clear that these results should extend to electroweak strings with \( g \neq 0 \) as long as we remain in the region of parameters where the strings are classically stable \(^5\). In the region of stability, we expect a non-zero density of electroweak vortices to form in a phase transition. Preliminary

\(^5\)Note that the monopoles at the ends of semilocal strings are global, whereas those at \( g \neq 0 \) will have finite size cores. However, the core size grows as \( 1/g \) and, for sufficiently small \( g \), it may be larger than the average distance between string segments, causing the strings to grow
Figure 3. The growth of string segments to form longer strings. The figure shows two snapshots, at time $t = 60$ and $t = 70$ of a large $256^3$ numerical simulation of a network of semilocal strings with $\beta = 0.05$. Note several joinings of string segments, e.g. two separate joinings on the long central string, and the disappearance of some loops. The different apparent thickness of strings is entirely an effect of perspective. The simulation was performed on the Cray T3E at the National Energy Research Scientific Computing Center (NERSC) in Berkeley.

results seem to confirm this picture [21], so obviously this problem deserves attention.

We may be a long way from understanding the formation of magnetic monopoles in a phase transition, but it is possible that particle accelerators will show signatures of monopole-antimonopole pairs in the not too distant
future. If Nambu’s prediction is correct, dumbells could be the first soliton-like objects in the standard model of particle physics to be observed.

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