Estimating the Longest Increasing Subsequence in Nearly Optimal Time

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Abstract

Longest Increasing Subsequence (LIS) is a fundamental statistic of a sequence, and has been studied for decades. While the LIS of a sequence of length $n$ can be computed exactly in time $O(n \log n)$, the complexity of estimating the (length of the) LIS in sublinear time, especially when $\text{LIS} \ll n$, is still open.

We show that for any $n \in \mathbb{N}$ and $\lambda = o(1)$, there exists a (randomized) non-adaptive algorithm that, given a sequence of length $n$ with $\text{LIS} \geq \lambda n$, approximates the LIS up to a factor of $1/\lambda^{o(1)}$ in $n^{o(1)}/\lambda$ time. Our algorithm improves upon prior work substantially in terms of both approximation and run-time: (i) we provide the first sub-polynomial approximation for LIS in sub-linear time; and (ii) our run-time complexity essentially matches the trivial sample complexity lower bound of $\Omega(1/\lambda)$, which is required to obtain any non-trivial approximation of the LIS.

As part of our solution, we develop two novel ideas which may be of independent interest: First, we define a new Genuine-LIS problem, where each sequence element may either be genuine or corrupted. In this model, the user receives unrestricted access to actual sequence, but does not know apriori which elements are genuine. The goal is to estimate the LIS using genuine elements only, with the minimal number of “genuiness tests”. The second idea, Precision Forest, enables accurate estimations for composition of general functions from “coarse” (sub-)estimates. Precision Forest essentially generalizes classical precision sampling, which works only for summations. As a central tool, the Precision Forest is initially pre-processed on a set of samples, which thereafter is repeatedly reused by multiple sub-parts of the algorithm, improving their amortized complexity.

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1 Introduction

Longest Increasing Subsequence (LIS) is a fundamental measure of a sequence, and has been studied for decades. Near linear-time algorithms have been known for a long time, for example, the Patience Sorting algorithm [Ham72, Mal73] finds the LIS of a sequence of length \( n \) in time \( O(n \log n) \). The celebrated Ulam’s problem asks for the length of LIS in a random permutation; see discussion and results in [AD99]. LIS is also an important special case of the problem of finding the Longest Common Subsequence (LCS) between two strings, as LIS is LCS when one of the strings is monotonically increasing. Moreover, LIS is often a subroutine in LCS algorithms: for example, when strings are only mildly repetitive [Gus97, Chapter 12], or more recently in approximation algorithms [HSSS19]. Longest increasing subsequences have multiple applications in areas such as random matrix theory, representation theory, and physics [AD99], and the related LCS problem also has multiple applications in bioinformatics, and is used for data comparisons such as in the diff command.

In the quest for faster algorithms, researchers started studying whether we can estimate the length of LIS (denoted LIS as well) in sublinear time. An early version of this question underpins one of the first sublinear-time algorithms: to test whether an array is sorted, or monotonically increasing [EKK+00] — i.e., whether the length of the LIS is \( n \) or is at most \((1-\epsilon)n\). [EKK+00] gave a \( O(\frac{1}{\epsilon} \log n) \) time algorithm, and this running time was later shown to be tight [ACCL07, Fis04]. Since then, there have been numerous influential results on testing monotonicity and other similar properties; see, e.g., [DGL+99, PRR06, Fis01, CS14, BCS20, BS19, PRW20, BCLW19, AN10, SW07] and the book [Gol17].

While monotonicity results focus on the case when LIS \( \approx n \), the case when \( \lambda \triangleq \frac{\text{LIS}}{n} \ll 1 \) has seen much less progress. The first result for the case when LIS \( \ll n \) [SS17] shows how to \((1+\epsilon)\) approximate the length when the LIS is still large: they distinguish the case when LIS \( \geq \lambda n \) from the case when LIS \( \leq (\lambda - \epsilon)n \) using \( (1/\epsilon)O(1/\epsilon) \log^{O(1)} n \) time, which only gives a \((1+\epsilon)\) factor approximation in truly sublinear time if \( \lambda = \Omega(\log(\log(1/\epsilon))) \). For arbitrary \( \lambda \), [RSSS19] gave an algorithm that achieves \( O(1/\lambda^2) \)-factor approximation of LIS in \( \tilde{O}(\sqrt{n}/\lambda^2) \) time. Very recently, [MS21] improved upon this result by presenting an algorithm with approximation \( O(1/\lambda) \) and runtime \( O(n^{1-O(1/\epsilon)}(\log n/\lambda)^{O(1/\epsilon)}) \). Independently, [NV21] obtained a non-adaptive \( O(1/\lambda) \)-approximation algorithm using \( \tilde{O}(\sqrt{r}/\lambda^2) \) queries, where \( r \) is the number of distinct values. The authors of [NV21] also proved a lower bound, showing that any non-adaptive algorithm that estimates the LIS to within an additive error of \( \epsilon n \) requires \( (\log n)^{O(1/\epsilon)} \) queries.\footnote{We insist that the algorithm must return an estimate \( \hat{\text{LIS}} \) such that \( \hat{\text{LIS}} \leq \text{LIS} \). Otherwise, the estimate \( \hat{\text{LIS}} = n \) is trivially \( 1/\lambda \)-approximate, but it corresponds to an infeasible solution unless the sequence is monotonically increasing.}

To put the above results into context, contrast LIS with the problem of estimating the weight of a binary vector of length \( n \): when the weight is \( \Omega(\lambda n) \), we can approximate the weight up to a factor of \( 1+\epsilon \) by sampling \( O(1/\lambda) \) positions, which is optimal. So far, one cannot rule out that a similar performance is achievable for estimating LIS too, at least for smaller values of \( \lambda < 1/\log n \). The aforementioned results not only have approximation factors that are polynomial in \( 1/\lambda \) but also time / sample complexities that are polynomial in \( n \) and \( 1/\lambda \). Hence the following guiding question remains open:

Can we estimate the length of LIS in essentially the time needed to estimate the weight of a binary vector?

In this paper we come close to answering this question in the affirmative, by obtaining an algorithm that runs in time near-linear in \( 1/\lambda \), and achieves sub-polynomial in \( 1/\lambda \) approximation. We show the following:

**Theorem 1.1** (Main theorem). For \( n \in \mathbb{N} \), and any \( \lambda = o(1) \), there exists a (randomized) non-adaptive algorithm that, given a sequence of length \( n \) with LIS \( \geq \lambda n \), approximates the LIS up to a \( 1/\lambda o(1) \) factor in \( O\left(\frac{1}{\lambda} \cdot n^{o(1)}\right) \) time with high probability.
We find this result quite surprising, as one may guess that in a random permutation of \([n]\), when \(\lambda \approx 1/\sqrt{n}\), one would need to read essentially the entire sequence (which would imply a \(\Omega(1/\lambda^2)\) lower bound on the number of queries needed). On the contrary, when \(\lambda \approx 1/\sqrt{n}\), our run-time (and hence sample complexity) nearly matches the streaming complexity from [GJKK07, EJ08] (albeit with worse approximation).

One of our main contributions is a new problem, which we call Genuine-LIS and which may be of independent interest. In the Genuine-LIS problem, we are given a sequence \(y \in \mathbb{N}^n\), with a caveat that only some of the elements of \(y\) are “genuine” and the others are “corrupted”, a property we can test a character for. The goal is to estimate the length of LIS among the genuine elements of \(y\), using as few tests as possible, while the values of \(y\) are known to the algorithm “for free”. We relate the Genuine-LIS problem to the LIS problem, by showing reductions in both directions, as components of our main algorithm.

While this paper makes progress in understanding the complexity of estimating the LIS, the following important question remains open:

**Open Question** ((1 + \(\epsilon\))-approximation). *Does there exist an algorithm that, given a sequence of length \(n\) with LIS \(\geq \lambda n\), estimates the LIS up to a 1 + \(\epsilon\) factor in \(O\left(\frac{1}{\lambda} \cdot n^{o(1)}\right)\) time with probability 2/3?*

We believe that finding an improved approximation algorithm for the Genuine-LIS problem above may lead to a (1 + \(\epsilon\)) LIS approximation algorithm.

We give a technical overview of our algorithm in Section 2, before setting up preliminaries in Section 3. Section 4 contains the proof of the main theorem, assuming results proved in subsequent sections. In Section 5, we develop the Precision-Trees data structure that is used to improve the sample and time complexity of the main algorithm. The guarantees of the two primary subroutines are proved in Section 6 and Section 7. In Section 8, we extend these algorithms in certain ways critical to our final application.

### 1.1 Related work

Computing the length of LIS has also been studied in the streaming model, where settling its complexity is a major open problem [P44]. In this setting, the main question is to determine the minimum space required to estimate the length of LIS by an algorithm reading the sequence left-to-right. [GJKK07, EJ08] gave deterministic one-pass algorithms to (1 + \(\epsilon\))-approximate the LIS using \(O(\sqrt{n})\) space, and matching lower bounds against deterministic algorithms were given by [EJ08, GG07]. These lower bounds are derived using deterministic communication complexity lower bounds and provably fail to extend to randomized algorithms [Cha12]. However, no randomized algorithm requiring \(o(\sqrt{n})\) space is known for this problem either. We note that our algorithm can be used in the streaming setting, yielding streaming complexity \(O(n^{1/2+\epsilon})\) for approximation \(n^{o(1)}\).

There has been much more success on the “complement” problem of estimating the distance to monotonicity, i.e., \(d_m := n - \text{LIS}\), in both the sublinear-time and the streaming settings. In the random-access setting, the problem was first studied in [ACCL07], and later in [SS17], who gave an algorithm that achieves (1 + \(\epsilon\))-approximation in time \(\text{poly}(1/d_m, \log n)\) for any constant \(\epsilon > 0\). These algorithms have been also used, indirectly, for faster algorithms for estimating Ulam distance [AIK09, NSS17] and smoothed edit distance [AK12].

In the streaming setting, several results were obtained which achieve \(O(1)\)-approximation of \(d_m\) using polylog\((n)\) space [GJKK07, EJ08]. This culminated in a randomized (1 + \(\epsilon\))-approximation algorithm using polylog\((n)\) space by Saks and Seshadhri [SS13], and a deterministic (1 + \(\epsilon\))-approximation algorithm using polylog\((n)\) space by Naumovitz and Saks [NS15]. [NS15] also showed space lower bounds against (1 + \(\epsilon\))-approximation streaming algorithms of \(\Omega(\log^2 n/\epsilon)\) (deterministic) and \(\tilde{\Omega}(\log^2 n/\epsilon)\) (randomized).

The LIS problem has also been studied recently in other settings, such as the MPC and the fully dynamic settings. [IMS17] gave a (1 + \(\epsilon\))-approximation, \(O(1/\epsilon^2)\)-round MPC algorithm for LIS whenever
the space per machine is \( n^{3/4+\Omega(1)} \). In the dynamic setting, a sequence of works [MS20, GJ21a] culminating in [KS21], gave the first exact dynamic LIS algorithm with sublinear update time, and also gave a deterministic algorithm with update time \( n^{o(1)} \) and approximation factor \( 1 - o(1) \). [GJ21b] showed conditional lower bounds on update time for certain variants of the dynamic LIS problem.

## 2 Technical overview

### 2.1 Obstacles from prior work

Our starting point is the algorithm of [RSSS19] which achieves \( O(1/\lambda^3) \) approximation in time \( \tilde{O}(\sqrt{n}/\lambda^7) \). Let \( \text{OPT} \) be an optimum solution (LIS) with length \(|\text{OPT}| \geq \lambda n \). The algorithm consists of 2 main steps:

1. The input sequence \( y \) is divided into \( \sqrt{n} \) contiguous blocks \( y^{(1)}, \ldots, y^{(\sqrt{n})} \) of length \( \sqrt{n} \) each. For each block \( y^{(i)} \), it "guesses" the interval \( [s_i, \ell_i] \), where \( s_i \) and \( \ell_i \) are the minimum and maximum values in \( y^{(i)} \cap \text{OPT} \) respectively. This is done by sampling \( O(1/\lambda) \) elements in \( y^{(i)} \), and considering the \( O(1/\lambda^2) \) intervals bounded by all possible pairs of sampled elements, called candidate intervals.

2. It generates a set of mutually disjoint pseudo-solutions, which are sequences of \( \Omega(\sqrt{n}/\lambda) \) candidate intervals that are monotone, i.e., all values in a candidate interval in block \( i \) are less than all values in a candidate interval in block \( j \) for all \( i < j \). It estimates the quality of each pseudo-solution (the sum of LIS of the candidate intervals in it) using sub-sampling, and outputs the largest quality.

The gist of why this generates the claimed approximation: for each block \( y^{(i)} \), some candidate interval is a good approximation of the interval \( [s_i, \ell_i] \) w.h.p., so the union of the LIS in all pseudo-solutions essentially covers \( \text{OPT} \). Moreover, there are \( O(1/\lambda^3) \) pseudo-solutions. Hence the output is a \( O(1/\lambda^3) \)-approximation of the LIS. The runtime of \( \tilde{O}(\sqrt{n}/\lambda^7) \) is dominated by the time required to evaluate the quality of pseudo-solutions to sufficient accuracy. Recursively applying this technique leads to better runtime but worse approximation.

We now sketch several improvements to the above result. In [MS21], this result is improved by giving an algorithm for LIS with approximation factor \( O(1/\epsilon) \) and runtime \( O(n^{1-\Omega(\epsilon)} \cdot (\log n/\lambda)^{1/\epsilon}) \) for any constant \( \epsilon > 0 \). This is obtained by first giving an algorithm with approximation factor \( O(1/\lambda^{3/4}) \) using a grid packing technique introduced in [MS20], and then recursing on this algorithm \( O(1/\epsilon) \) times, thus improving the approximation at the expense of increased runtime. In particular, let \( k = O(1/\epsilon) \) be the number of recursion levels, and \( \chi = n^{1/10k} \) be the grid length (which needs to be \( \text{poly}(n) \) in order to achieve truly sublinear overall runtime). The approximation factor is \( O(1/\lambda^{3/k}) = O(1/\epsilon) \), the polynomial dependence on \( 1/\lambda \) being essential since the algorithm in [RSSS19] is invoked at the final recursion step. The runtime turns out to be \( O((n/\chi) \cdot (\log n/\lambda)^{O(k)}) = O(n^{1-\Omega(\epsilon)} \cdot (\log n/\lambda)^{O(1/\epsilon)}) \).

In [NV21], a non-adaptive algorithm is presented with sample complexity \( \tilde{O}(\sqrt{r}/\lambda^2) \) and approximation \( O(1/\lambda) \), where \( r \) is the number of distinct values. The first step is to partition the indices and values into \( O(\sqrt{r}) \) parts each, thus giving rise to \( O(r) \) boxes. It then identifies monotone sequences of dense boxes which together capture a significant part of \( \text{OPT} \), and outputs the estimated length of the longest sequence. There are \( O(1/\lambda) \) such monotone sequences, which immediately leads to approximation factor of \( O(1/\lambda) \). Again, the polynomial dependence of the sample complexity on \( r \) and \( 1/\lambda \) seems intrinsic.

There are several fundamental obstacles in improving these bounds, and in particular, getting the runtime down all the way to \( 1/\lambda \) while improving the approximation. One such obstacle is the straight-forward approach (as in prior work) of independent recursions cannot obtain better than \( 1/\lambda^2 \) run-time in the worst-case. Consider the case when the LIS contains small slopes (i.e., sections where the distance between 2 LIS elements on \( y \)-axis is much smaller than on the \( x \)-axis), then even if one is able to efficiently isolate the ranges of all possible slopes (there are \( \approx 1/\lambda \) many slopes), one still needs
to sample coordinates without knowing a priori which coordinates contain the values they care about, and finding just a single LIS element would take $1/\lambda$ samples in each instance. We will get back to this obstacle later.

2.2 Our approach

Similarly to [RSSS19], we sample some anchor elements uniformly and generate candidate $y$-range intervals based on the location and values of the anchors and then look for alignments of intervals and estimate the local LIS of a small number of sampled candidate intervals.

However, from that point our algorithm is entirely different. The crux of our new algorithm lies in how we create and “handle” those candidate intervals. In particular, to handle such candidate intervals, we formulate a new problem, termed Genuine-LIS, which is of independent interest. We derive two complementary solutions for it: one “direct”, and the other by reducing the problem to standard sublinear-time LIS estimation. This allows us to improve upon the polynomial approximation achieved in previous works. In particular, we generalize the problem of aligning candidate intervals to the Genuine-LIS problem, and overall, reduce the LIS problem to the composition of 2 sub-problems: the Genuine-LIS problem takes care of the “global alignment” of candidate-intervals, while the Block-LIS problem recursively estimates the “local alignment” (these ideas and terms will be made precise later). In addition, we generate candidate intervals in a succinct manner: we manage to reduce the number of sampled anchor needed to $\approx 1/\lambda$, as well as limiting the number of candidate intervals to be near-linear in the number of sampled points. Last, we recurse over smaller instances in precision-sampling manner, thereby avoiding approximation blow up.

We now formulate the two LIS problems which are at the core of our construction: 1) Block-LIS, a slight generalization of the LIS problem, and 2) Genuine-LIS, the “genuine LIS” problem, as well as the interplay between them. Following that, we will provide an overview of the algorithms to solve these 2 problems. The high-level flow of our algorithm is provided in Figure 1.

The Block-LIS problem. The first problem, Block-LIS, is very similar to the standard LIS problem, with two extensions. First, the main input consists of $n$ blocks of (at most) $k$ elements each, and each block can contribute at most one of its elements to a subsequence. Second, we are also given a set of possible values $Y$, such that each element of a subsequence must be in $Y$.

Formally, we define the Block-LIS as follows: given a sequence $y \in \mathbb{N}^{n \times k}$ and a range of values $Y$, compute the length of a maximal sub-sequence $\text{OPT} \triangleq \{(w, y_w)\}_{w}$, where each $w = (w_1, w_2) \in [n] \times [k]$ and $\text{OPT}$ does not contain 2 elements from the same block (i.e., with the same value of $w_1$), such that $(w_1, y_w)$ is monotone. Sometimes, we also restrict the set of blocks to an interval $X \subseteq [n]$, and we define the quantity $\text{Block-LIS}(y, X, Y)$ as the maximal increasing subsequence of $y \in \mathbb{N}^{X \times [k]}$ using elements of $Y$.

Block-LIS is a generalization of the standard LIS problem, where we have $k = 1$ and $Y = \mathbb{N}$, and in fact, we instantiate the original input sequence $y \in \mathbb{N}^n$ in our Block-LIS algorithm. The main advantage of this generalization appears when the sequence is sparse in $Y$. Then, we show the multiplicative approximation factor for Block-LIS is not only a function of the additive error $\lambda n$, but also of the total number of integers in range $Y$ (i.e., $|Y \cap y|$). In particular, we show that this approximation is a function of $\frac{\lambda n}{|Y \cap y|}$.

As for the use of blocks, while its use is not a core necessity of the algorithm (in fact, one can consider the values in each block in descending manner, yielding an equivalent problem with no blocks required), its main use comes from our need to instantiate several overlapping instances using the same Precision-Tree data-structure (to be discussed later).

The Genuine-LIS problem. The second (and more novel) problem, Genuine-LIS, is defined as follows.
The input is a sequence $g$ where each element is associated with an additional flag signifying whether it is genuine or not, i.e., $g \in (\mathbb{N} \times \{0, 1\})^{n \times k}$. One receives unrestricted access to the elements (i.e., the first coordinates) which is denoted by $g_1 \in \mathbb{N}^{n \times k}$. However, one must “pay” to test whether an element is genuine or not; this is determined by the second coordinates $g_2 \in \{0, 1\}^{n \times k}$, referred to as genuineness flags. The goal is to compute the length of the longest increasing subsequence of $g$ restricted to genuine characters, i.e., Block-LIS$(g_{(1)}^{-1}(1), [n], \mathbb{N}))$ using as few tests for genuineness as possible. Of course, the previous restriction that each block can contribute at most one element to the subsequence still applies.

### 2.3 Overview of the algorithm EstimateGenuineLIS for the Genuine-LIS problem

We will need the notion of an $(\alpha, \beta)$-approximation going forward. For $\alpha \geq 1$ and $\beta > 0$, an $(\alpha, \beta)$-approximation $\hat{q}$ of a quantity $q$ is an $\alpha$-multiplicative and $\beta$ additive estimation of $q$, i.e., $\hat{q} \in [q/\alpha - \beta, q]$.

A (sub-optimal) approximate solution to Genuine-LIS problem follows from the framework of [RSSS19]:

1. Greedily and iteratively look for the maximal increasing sequences containing $\gtrapprox \lambda n$ letters, generating solutions $P_1, \ldots, P_t$ for some $t \lesssim 1/\lambda$.

2. Sub-sample $\approx 1/\lambda$ coordinates from the union $P = \bigcup_{i\in t} P_i$ and check if each one is genuine or not. Let $\kappa$ be the number of Genuine letters.

3. Output $\lambda n \kappa$.

It is easy to see that this algorithm yields a $(1/\lambda, \lambda^2 n)$-approximation with approximately $1/\lambda$ tests for genuineness, since not all elements of $P$ may be genuine. We refer to the portion of genuine elements in $P$ as the genuine density $d \in [\lambda, 1]$. Observe that the approximation is in fact precisely $1/d$, and that the worst approximation happens when $P$ is “sparse”, i.e., when only $\approx \lambda n$ elements in $P$ are genuine. For such a sparse case, however, one can instantiate the Genuine-LIS algorithm as a Block-LIS one, restricted to the genuine elements. In fact, this “sparse” case is precisely where the approximation factor is minimal, i.e., $\frac{\lambda n}{|P| |g|} = 1$. Our improved approximation stems mainly from balancing between the dense and sparse cases as above.

**Speeding up LIS extraction using dynamic LIS.** The Genuine-LIS instances we generate can be of size $n \cdot k \lesssim 1/\lambda$, and our goal is to obtain overall run-time that is near-linear in $1/\lambda$ as well. The standard, dynamic-programming solution for finding and extracting the optimal LIS each time for item (1) above can potentially incur an overhead that is quadratic in the instance size (each LIS extraction would take linear time, and we need to greedily extract up to a near-linear number of solutions), and hence this simple procedure is prohibitive for our fine-grained applications. To ensure near-linear time, we implement a fast dynamic LIS data-structure from [GJ21a], and iteratively extract near-maximal pseudo-solutions.

**Extensions to the Genuine-LIS algorithm.** We need two extensions to the algorithm for the Genuine-LIS problem, described below:

- **Sparse (unbalanced) instances:** Some Genuine-LIS instances we generate are sparse, with many “null” elements\footnote{We remark “null” is different from “non-genuine”, since a null element is void, doesn’t need to be tested.}, meaning, some blocks have less than $k$ elements. When the instance $g$ consists of $m \ll nk$ non-null elements, we would like to improve the approximation and runtime bounds to be a function of $m/n$ instead of $k$. To obtain the improved bound, we partition the blocks based on an exponential discretization of the number of non-null elements, and output the maximum over all instances, where each instance consists only of blocks containing a similar amount of non-null elements.
• **Genuine-LIS over intervals:** The first coordinates of the Genuine-LIS instances we generate initially consist of intervals rather than integers. That is an issue for us, since the space of intervals $I$ admits a partial order relation only. To exemplify the issue, the dynamic LIS data-structures recently developed in [MS20, GJ21a, KS21] which are useful in speeding up our algorithm, cannot immediately handle partial order sequences. Our ideal solution involves a mapping $\varphi : I \rightarrow \mathbb{N}$ that approximately preserves the overall LIS over all subsequences, and therefore also preserves the overall LIS over the genuine elements. While we are unable to show a single mapping that works for all intervals, we partition the space of intervals into $\log(\frac{k}{\lambda})$ sets $I_\ell$ based on an exponential discretization of the interval lengths $\ell$ and provide a mapping $\varphi_\ell : I_\ell \rightarrow \mathbb{N}$ for each set. We eventually output the maximal integer Genuine-LIS result over all such $\varphi_\ell$ maps at merely another $\log(\frac{k}{\lambda})$-factor approximation, and a small additive error.

The formal statement for the algorithm to solve Genuine-LIS, named EstimateGenuineLIS, is presented in Section 4, and its description and analysis are in Section 6. The Genuine-LIS extensions are presented and analyzed in Section 8.

### 2.4 Overview of the algorithm EstimateBlockLIS for the Block-LIS problem

The algorithm EstimateBlockLIS for Block-LIS$(y, X, Y)$ starts by partitioning $X$ into consecutive intervals of equal length $X_1, \ldots, X_\tau$, where $\tau$ is a dynamic, carefully chosen branching factor, and is always (slightly) sub-polynomial in the instance size. Next, we simulate i.i.d subsampling of $\approx \frac{\tau}{\lambda}$ blocks $(w_j, y_{w_j})$ called anchors, generating sets of anchors $S_i$ using the sampled blocks in each corresponding $X_i$.

Using each set $S_i$, we construct candidate intervals $Y_i$. Here again the prior work is insufficient for us, since we require the number of candidate intervals to be near-linear in $|S_i|$, and hence we develop new techniques for constructing sparse covering sets where $|Y_i| \approx |S_i|$ while still “approximately covering all options”.

Next, we break down the overall Block-LIS problem into a “global” Genuine-LIS instance over $\tau$ blocks. The first coordinates of the Genuine-LIS instance are the candidate intervals $Y_i$ themselves, while the second coordinates (i.e., the genuineness flags) indicate whether the corresponding “local” Block-LIS$(y, X_i, Y')$ estimated values are above a certain threshold $\kappa$, for each $Y' \in Y_i$.

This threshold $\kappa$ itself depends on a parameter $\rho$, which characterizes the relation between the “global” Genuine-LIS instance and the “local” Block-LIS instances. Specifically, $1/\rho$ is the average fraction of intervals $X_i$ participating in an optimal solution. Consider 2 extreme cases, one where the LIS is uniformly distributed among all $X_i$, and one where the LIS is maximally concentrated among a small subset of the $x$-intervals. Intuitively, it is more difficult to certify an increasing subsequence when the LIS is sparse. So, we can exploit the trade-off in the complexity of certifying the global LIS (which is higher in the second case) against the complexity of certifying the local Block-LIS (which is higher in the first case). Since we do not know $\rho$ a priori, we simply iterate over all possible magnitudes (again, by exponential discretization) and output the maximum over all estimators.

Once we formulate the overall Block-LIS problem as a composition of a “global” Genuine-LIS over multiple “local” Block-LIS instances, we decompose the problem through a procedure called Precision-Tree decomposition which will be described next.

The formal statement for the Block-LIS algorithm is presented in Section 4, and the algorithm description and analysis are in Section 7.
2.5 Precision Forest

Note that a straight-forward instantiation of the above algorithm will yield the right approximation, but will require at least $\approx 1/\lambda^2$ input queries and run-time, and moreover, require the algorithm to be adaptive. This stems precisely because of the “small slopes” obstacle discussed earlier on. To improve the complexities, and to allow our sampling algorithm to be non-adaptive, we introduce the notion of Precision Trees (Section 5). The overall idea is to preprocess a tree data-structure for input $y$ with precision parameter $\lambda$ (denoted $T_\lambda(y)$), and perform all the sampling a priori in a Precision Sampling, tree-wise manner, such that any sub-tree under any node, is another precision-tree instantiation with a different (random) parameter. Our algorithms will use such property to decompose Precision-Trees into several smaller trees over different inputs / precision parameters, and reuse the randomness of the original tree to deduce statistics (and in particular, concentration bounds) over many different Genuine-LIS and Block-LIS instances without ever needing to access the original input again. Since the tree is recursively decomposed into multiple disjoined, smaller trees, we named this novel technique “Precision Forest”.

Defining a Precision-Tree Access. Precision sampling was introduced in [AKO11] to accurately estimate a summation function $a = \sum_{i=1}^n a_i$, for unknown $a_i \geq 0$, from “coarse” estimates for $a_i$. For our delicate applications, we need to generalize precision sampling from simple addition to general functions, allowing one to approximate $g \circ f$ where $g$ is a general function over $n$ coordinates and $f = (f_1, \ldots, f_n)$ consists of $n$ independent functions on different parts of the input, sharing the same co-domain.

To define precision tree access, we first define Precision-Tree as follows. Given a vector of elements $\mathcal{E} = \{e_1, \ldots, e_n\}$, and a parameter $\beta$, we can define a $\beta$-ary precision tree $T$. Each leaf is associated with an integer representing its location in the tree. Each internal node $v$ is associated with an interval representing the unions of its leaves (e.g., the root of $T$ is associated with the entire $[1, n]$). We denote $\mathcal{E}_v$ or $\mathcal{E}(v)$ the set of elements under $v$.

Given $\mathcal{E}$ and $\beta$, we can define a precision tree $T_\beta(\mathcal{E})$ as follows. First, conceptually, we create the $\beta$-ary decision tree as described above. Next, we will assign a precision score $P_v$ to each node $v$ be the following recursive procedure. We set $P_{\text{root}(T)} = \lambda$. Recursively we can define the precision score of a node $v$ by

$$P_v = Z_v \cdot P_{\text{parent}(v)} \text{ where } Z_v \sim \text{i.i.d. Uniform}\{1, 2, 3, \ldots, \beta/4\}.$$  \hspace{1cm} (1)

We construct Precision-Trees in two ways:

- **Precision-Tree preprocessing**: Initially, when we are given non-restricted access to the string, and only care about bounding the sample complexity, we build a precision tree with the string elements as leaves.

- **Precision-Tree decomposition**: Thereafter, when we define the overall Block-LIS problem as a composition of Genuine-LIS and Block-LIS functions, we decompose the tree and compute each function independently. Here, we cannot precompute a tree but instead we reuse the randomness of the original tree to generate multiple trees according to the above definition.

Preprocessing a Precision-Tree: Sample Complexity Bound. We initially preprocess a precision tree $T$ by the following recursive procedure. First we include the root. Then, for each node $v$ just included, if $P_v > 1$ we stop the recursion. Otherwise, if $v$ is a leaf, we store the input in the associated location of $v$ in $v$. If $v$ is internal node, we recurse on all children of $v$.

We call such a leaf-sampling mechanism as Precision-Tree $T_\lambda(e_1, \ldots e_n)$ access to the set of elements $e_1, \ldots e_n$. While we a priori generate such a tree only once for the actual string $y$, we will simulate Precision-Tree access for different trees over different inputs and parameters in our algorithms using a certain procedure which we call the Trim-Tree algorithm (see Figure 2).
Overall, our entire sampling mechanism is merely a simple, one-round non-adaptive precision-sampling tree over the coordinates of $y$, which we store in a data structure $T$ of size $O^*(1/\lambda)$ with convenient fast access.\footnote{We abuse notation and use $T$ to denote both the conceptual Precision-Tree and the data structure implementing it.} After such preprocessing, the rest of the algorithm has no access to $y$, but only to $T$.

The following lemma, which bounds the expected number of leaves sampled, shows that we do not sample too many leaves. It is proved in Section 5.

**Lemma 2.1 (Sample complexity bound).** The expected number of elements in $y$ that are sampled is $\log^{O(\log \beta)}(\beta)/\lambda$.

We note that we can obtain a high probability result using standard boosting techniques.

The Precision-Tree technique is a key enabler for several important properties we need. First, the fact any sub-tree is another precision-tree instantiation allows us to compose the Genuine-LIS and Block-LIS algorithms, by decomposing the Precision-Tree. Second, it allows us to “zoom into” different location of the string with the right precision. Last and most importantly, we get improved amortized complexity, by reusing the randomness of the precision-tree for different $y$-value ranges we care about.

![Figure 1: High level flow of the main algorithm (Theorem 1.1)](image)

## 3 Preliminaries

**Sequences and intervals.** Given a set $\mathcal{X}$ and $n \in \mathbb{N}$, a sequence $y = (y_1, y_2, \ldots, y_n) \in \mathcal{X}^n$ is an ordered collection of elements in $\mathcal{X}$, and can be identified with the set $P_y = \{(i, y_i) \mid i \in [n]\} \subset \mathbb{N} \times \mathcal{X}$. For $\ell \in [n]$, we say that $z = (z_1, \cdots, z_{\ell})$ is a subsequence of $x$ of length $\ell$, and denote it by $\{y_{i_j}\}_{j}$, if there exist integers $1 \leq i_1 < i_2 < \cdots < i_{\ell} \leq n$ such that $z_j = y_{i_j}$ for all $j \in [\ell]$. Again, we can identify $z$ with $P_z = \{(i_j, y_{i_j}) \mid j \in [\ell]\}$. We will use this correspondence extensively henceforth.

Also, define the interval space $I \trianglerighteq \{[a,b] \mid a, b \in \mathbb{N}, a \leq b\} \cup \{(a,b) \mid a \in \mathbb{N}, b \in [1,\infty), a < b\}$. For $I \in I$, we use $|I|$ to denote $|I \cap \mathbb{N}|$, i.e., the number of natural numbers contained in interval $I$. For a sequence $y \in \mathbb{N}^n$ and an interval $Y \in I$, we write $y \cap Y$ to denote the multi-set of elements in $y$ that are also in $Y$.

**Monotonicity.** We define monotone sets as follows:

\footnote{The notation $O^*(\cdot)$ hides a $n^{o(1)}$ factor.}
Definition 3.1 (Monotone sets). Fix a partially ordered set \((\mathcal{X},<)\). We say that a set \(P \subseteq \mathbb{N} \times \mathcal{X}\) is monotone if for all \(((i,k),(j,l)) \in P \times P\), we have \(i < j \leftrightarrow k < l\).

Note that this definition captures the notion of an increasing subsequence. In particular, for the standard notion of an increasing subsequence over natural numbers, we take \(\mathcal{X} = \mathbb{N}\) and \(<\) as the usual “less than” relation over \(\mathbb{N}\) (a total order). However, we will need this more general definition to consider increasing subsequences over other partially ordered sets, like the space of intervals \(I\).

For a finite set \(P \subseteq \mathbb{N} \times \mathcal{X}\), a longest increasing subsequence (LIS) of \(P\), is a monotone set \(Q \subseteq P\) of maximum cardinality, with the additional constraint that \(Q\) does not contain two distinct elements \((i,y_1),(i,y_2)\) for any \(i\). We often use \(\text{OPT}\) to refer to a particular LIS, and use \(|\text{OPT}|\) to denote its length.

Distributions. For \(p \in [0,1]\), we use \(\text{Ber}(p)\) to denote the Bernoulli distribution with success probability \(p\), and \(\text{Bin}(n,p)\) to denote the binomial distribution with parameters \(n\) and \(p\). By convention, we project \(p\) to the range \([0,1]\) whenever \(p > 1\) or \(p < 0\).

We use “i.i.d. random variables” to mean that a collection of random variables is independent and identically distributed, and use “sub-sampling with i.i.d. probability \(p\)” to mean that each element is sampled independently with equal probability \(p\).

Operations on Vectors, Sets, Functions. For a set \(A \subset \mathbb{R}\) and a number \(\alpha\), we define \(A + \alpha := \{a + \alpha : a \in A\}\) and \(\alpha A := \{\alpha a : a \in A\}\). We write \(\ln\) to denote natural logarithm and \(\log\) to denote logarithm to the base 2.

The notation \(\circ\) is used for function composition (i.e., \(g \circ f(x) = g(f(x))\)), and \(\oplus\) is used for direct sum. We use the notation \(*\) as argument of a function, by which we mean a vector of all possible entries. For example, \(f(*)\) is a vector of \(f(i)\) for \(i\) ranging over the domain of \(f\) (usually clear from the context).

Definition 3.2 ((\(\alpha,\beta\)-Approximation). For \(\alpha \geq 1\) and \(\beta > 0\), an \((\alpha,\beta)\)-approximation \(\hat{q}\) of a quantity \(q\) is an \(\alpha\)-multiplicative and \(\beta\) additive estimation of \(q\), i.e., \(\hat{q} \in [q/\alpha - \beta,q]\).

We use \(E_b(k)\) for \(b,k \in \mathbb{N} \cup \{\infty\}\) to denote the set of powers of \(b\) bounded by \(k\), i.e., \([b^i : i \in \mathbb{N}] \cap [1,k]\). We also define \(E_b = E_b(\infty)\). We use \(\mathbb{R}_+\) to denote the set of non-negative real numbers.

Other notation. Notation \(O(\cdot)\) hides polynlog\((n)\) factors, while \(O^*(\cdot)\) hides a factor of \(n^{o(1)}\).

4 Main Algorithm for Estimating LIS

Our main algorithm is composed of two algorithms, for solving Block-LIS and Genuine-LIS. These algorithms recursively call each other, with access to a Precision-Tree data structure. This data structure queries the input sequence at the beginning (non-adaptively and non-uniformly), and our algorithms access the sequence only via this data structure. We describe its details in Section 5, and for now refer to it as a tree \(T_{\lambda}(x)\) for some parameter \(\lambda < 1\) and an input string \(y\).

We now state the main guarantees of the two algorithms; their proof will appear in later sections. We then show how these two algorithms yield our main algorithm for estimating LIS. Below, for an instance \(g \in (\mathbb{N} \times \{0,1\})^{n \times k}\), let \(g\) restricted to first/second coordinate be \(g_1 \in \mathbb{N}^{n \times k}\) and \(g_2 \in \{0,1\}^{n \times k}\) respectively.

Theorem 4.1 (EstimateGenuineLIS algorithm; Section 6). Fix integers \(n,k\) and \(\lambda \in (1/n,1)\). Fix an instance \(g \in (\mathbb{N} \times \{0,1\})^{n \times k}\). For some monotone functions \(a_s : \mathbb{R}_+^2 \to [1,\infty)\) and \(c_s : \mathbb{R}_+^2 \to [1,\infty)\). Suppose there exists a randomized algorithm \(A_{BL}\) that, given \(X \subseteq [n], Y \subseteq \mathbb{N},\) parameter \(\tau_s\), and a Precision-Tree \(T_{\frac{1}{c_s}(|X|,\tau_s,\lambda \frac{|X|}{|Y|}\frac{1}{|\mathbb{N}^k|})}\) (\(y\) for some \(y \in \mathbb{N}^{n \times k}\), can produce an \((\alpha_s,\lambda |X|)\)-approximation for Block-LIS\((y,X,Y)\) where \(\alpha_s = a_s\left(|X|,\tau_s,\frac{\lambda |X|}{|Y|\frac{1}{|\mathbb{N}^k|}}\right)\) w.h.p. in time \(t_s\).
Fix any parameter $\gamma, \tau \geq 1$, and let $c \geq \tilde{O}(1/\gamma) + c_s(n, \tau, \lambda, \gamma/\lambda/k)$. Then, there exists an algorithm $A$, that, given free access to $g_1$ and Precision-Tree access to $g_2, T_{1/c}(g_2)$, produces a $(\alpha_g, \lambda n)$-approximation for $\text{Genuine-LIS}(g)$ w.h.p., where $\alpha_g = \log(k/\lambda) \cdot \max\{\gamma, a_s(n, \tau, \gamma/\lambda/k)\}$. The algorithm $A$ runs in time $\tilde{O}(nk) + O(t_s)$.

**Theorem 4.2 (EstimateBlockLIS algorithm; Section 7).** Fix monotone functions $t_s, a_g : \mathbb{R}_+^2 \rightarrow [1, \infty), a_s, c_g : \mathbb{R}_+^3 \rightarrow [1, \infty)$, satisfying, for all $r, \tau \in E, m \in \mathbb{N}, \lambda < 1, \lambda_1, \lambda_2 \in [1, 1]$, and $k' \in [1, 1/\lambda]$ with $\lambda_1 \lambda_2 = \Omega(\lambda k')$:

- $a_s(r, \tau, \lambda_1/m) \geq \text{polylog}(\lambda_1) \cdot a_g(\tau, \lambda_2/m) \cdot a_s\left(\frac{\tau}{\lambda_2}, \tau, \lambda_1/m\right)$;
- $c_s(r, \tau, \lambda, \lambda_1/m) \geq \beta^{O(1)} \cdot \frac{1}{\lambda} + c_g(\tau, \lambda_2, k') \cdot c_s\left(\frac{\tau}{\lambda}, \Theta(\lambda_1/k'), \lambda_1/m\right)$; and,
- $t_s(r, \tau) \geq \log^{O(\log \beta)}(\beta) \cdot t_s\left(\frac{\tau}{\lambda}, \tau\right)$.

Suppose there exists and algorithm $A_{GL}$ with the following guarantee: given $\text{Genuine-LIS}$ instance $g \in (\mathbb{N} \times \{0, 1\})^{n_0 \times k_0}$ with $\beta$-ary Precision-Tree $T_{1/c_0}(g_0, g_2)(g_2)$ access, $A_{GL}$ outputs $(a_g(n_0, \lambda_0, \lambda_0), \lambda_0 n_0)$-approximation to $\text{Genuine-LIS}(g)$ in time $\tilde{O}(n_0 \cdot k_0)$ w.h.p.

Now fix input $y \in \mathbb{N}^{n \times k}$, a block interval $X \subseteq [n]$, value range interval $Y \subseteq \mathbb{N}$, parameters $\lambda \in (0, 1)$, $\beta \in \mathbb{N}$, and $\tau \in E$. Then, given a $\beta$-ary Precision-Tree $T_{1/c}(g)$, we can produce a $(\alpha, |X|)$-approximation for $\text{Block-LIS}(y, X, Y)$ w.h.p., as long as $\alpha \geq a_s\left(|X|, \tau, \frac{\lambda |X|}{|Y|(X,Y)}\right)$ and $c \geq c_s\left(|X|, \tau, \lambda, \frac{|X|}{|Y|(X,Y)}\right)$.

The algorithm’s expected run-time is at most $c \cdot t_s(|X|, \tau) \cdot \frac{|Y|(X,Y)}{|X|}$.

The proofs are deferred to Section 6 and Section 7. Combining the two algorithms from above, we obtain the following theorem.

**Theorem 4.3.** Fix $k = 1$, any $\lambda = o(1)$ and $\epsilon < 1$. There exists a randomized non-adaptive algorithm $A_{BL}$ that solves $\text{Block-LIS}$ up to $(\alpha, \lambda n)$-approximation, where $\alpha = (1/\lambda)^{\sqrt{\tau}} \cdot (\log 1/\lambda)^{O((\log^2 1/\epsilon^{1/\sqrt{\tau}}))}$ using $\frac{1}{\lambda} \cdot n^{O((\sqrt{\tau} \log 1/\epsilon))}$ time (and hence samples from the input).

The algorithm for Theorem 4.3 follows the outline from Figure 1. In particular, we first build a $\beta$-ary precision tree $T_{1/c}$ for $\beta = \Theta(\log n)$, and $c = c_s(n, n', \lambda, \lambda) = \frac{1}{\lambda} \cdot n^{O((\sqrt{\tau} \log 1/\epsilon))}$. We then apply the algorithms of Theorems 4.1 and 4.2 recursively with carefully chosen parameters. Most importantly $\tau$ and $\gamma$ are carefully chosen, as a function of the other parameters, to balance approximation and complexity. Informally, we pick $\tau \approx |X|^\epsilon$ in each Block-LIS instance, and $\gamma \approx (k/\lambda)^{\sqrt{\tau}}$ for a recursion of depth $\approx 1/\sqrt{\tau}$, then we stop and use the dense estimator only by setting $\gamma$ to be maximal. The proof follows from the above theorems but by rather tedious calculations, and is deferred to Appendix A. We now complete Theorem 1.1, by instantiating the LIS$_n$ problem with the parameters above set suitably.

Proof of Theorem 1.1. Let $y \in \mathbb{N}^n$ be an input to LIS$_n$. We solve Block-LIS using the algorithm of Theorem 4.3 with inputs $y, \lambda$, and $\epsilon = 1/\log \log 1/\lambda$, noting that $\alpha = \lambda^\theta(1)$ and runtime complexity is $\frac{1}{\lambda} \cdot n^{O(1/\sqrt{\log \log 1/\lambda})} = \frac{1}{\lambda} \cdot n^{o(1)}$.\qed

5 Precision Trees Data Structure

In this section we discuss the Precision-Trees data structure used to improve our bounds, and the properties of this data structure.
5.1 Tree Sampling Data Structure

We equip each precision-tree with a Sampling Oracle Data Structure (denoted SODS) that should be seen as data structure wrapper with access to Precision-Tree, allowing efficient sampling of leaves.

First, we show one can simulate uniform samples of elements given a Precision-Tree access.

**Lemma 5.1 (Simulating Random Samples).** Fix $\delta \leq 1$ and Precision-Tree $T = T_\delta(e_1, \ldots, e_n)$, as well as arbitrary $\delta' \geq \delta$, with $1/\delta' \in \mathbb{N}$. Using access to $T$ only, we can generate a set of elements $S \subseteq \{e_1, \ldots, e_n\}$ such that the distribution of $S$ is identical to the distribution where each $e_i$ is included i.i.d. with probability $1/\delta' n$. The runtime to generate $S$ is $O((\log \beta)^{\log_4 n}/\delta)$.

We remark that, while two subsequent invocations of the above lemma may give different sets $S, S'$, each with the above distribution, they are dependent between each other.

**Proof.** The main task here is to show independence, i.e., that we can choose a set of i.i.d. samples by reusing the randomness of the Precision-Tree. We prove by induction on the tree height, starting with $T_1$, the node is included in the Precision-Tree, and we can subsample it with any required probability as needed. Now consider a non-leaf node $v$; by induction the statement holds for its children.

If $\delta' \leq 4/\beta$, we are basically done since any child $u$ of $v$, has score $P_u \leq \frac{P_u \beta}{\delta'} \leq \frac{\delta' \beta}{\delta} \leq 1$, while the number of elements $n_u = \frac{n_v}{\delta'}$, so we can simply subsample leafs in $u$'s tree, each with probability $\frac{1}{\delta' n_v} = \frac{1}{\delta \beta n_u} \leq \frac{1}{P_u n_u}$ as needed (note that $\delta' n_v / n_u = \beta \delta' \geq P_u$ as required by the inductive hypothesis).

It remains to show this for $\delta' > 4/\beta$. For this case we claim we would like to use the tree-randomness to generate the samples.

For distributions $P, Q$, we say that $P$ stochastically dominates $Q$ if $F_P(t) \leq F_Q(t)$ for all $t$, with strict inequality for some $t$, where $F_P$ and $F_Q$ are the cumulative distribution functions (CDFs) of $P$ and $Q$ respectively. We first claim the following:

**Claim 5.2.** Fix $k \in \mathbb{N}$, $0 < p < 1$ and $F \geq 1$ such that $F \cdot p F_k < 1/4$. Let $X_1, \ldots, X_k \sim_{i.i.d.} \text{Ber}(p)$ (i.e., i.i.d. Bernoulli random variables with bias $p$) and let $X = \sum_i X_i$. Let $Y_1, \ldots, Y_k \sim_{i.i.d.} \text{Ber}(2p F_k)$ and let $Y = Y^* \cdot \sum_i Y_i$ where $Y^* \sim \text{Ber}(1/F)$, independent of the other variables. Then, $Y$ stochastically dominates $X$.

**Proof.** $X$ and $Y$ are both supported on $\{0, 1, \cdots, k\}$. So, it suffices to show that $\Pr[Y \geq i] > \Pr[X \geq i]$ for all $i \in [k]$. Note that $2p F \leq 2p F_k < 1/2$, so $2p F$ is a valid probability. For $i \in [k]$, let

$$f(i) := \frac{\Pr[Y = i]}{\Pr[X = i]} = \binom{k}{i} \cdot (2p F)^i \cdot (1 - 2p F)^{k-i} = \frac{F \cdot (k)}{i} \cdot p^i \cdot (1 - p)^{k-i} = 2^i \cdot F^{i-1} \cdot \left( \frac{1 - 2p F}{1 - p} \right)^{k-i}.$$

To show that $\Pr[Y \geq i] > \Pr[X \geq i]$ for all $i \in [k]$, it suffices to show that $f(i) > 1$ for all $i \in [k]$. Using the conditions that $2p F < 1/2$, $p F k < 1/4$ and $F \geq 1$, along with the fact that $1 - x \geq 2^{-2x}$ for all $x \in [0, 1/2]$, we obtain:

$$f(i) > 2(1 - 2p F)^k \geq 2 \cdot 2^{-4p F k} > 2 \cdot 2^{-1} = 1,$$

for all $i \in [k]$. This concludes the proof. \qed
We show one can simulate sub-sampling of each leaf of \( u \) with i.i.d. probability \( 1/\delta' n_u \) by the following process. If \( \mathcal{P}_u > 1 \), then output no elements. Otherwise, we can (by the inductive hypothesis) sub-sample each leaf of \( u \) with i.i.d. probability \( 1/n_u \).

Now, let
\[
F \triangleq \frac{1}{[1/\delta']^2} \geq \frac{1}{[1/P_u]} \geq \frac{1}{[1/\Pr[P_u \leq 1]]}.
\]
This process generates \( \mu \) i.i.d. leaves of \( u \) with distribution
\[
\mu \sim \Pr[T \mid \Pr[P_u \leq 1]] \cdot \sum_{l \text{ leaf of } u} \Pr[T \mid \Pr[P_i \leq 1]] \sim \Pr[T] \cdot \sum_{l \text{ leaf of } u} \Pr[T \mid \Pr[P_i \leq 1]].
\]

Noting that \( n_u = \frac{n}{\beta} \geq \frac{\delta' n_u}{2F} \) and \( \frac{nuF}{\delta'n_u} \leq \frac{1}{4} \), we invoke Claim 5.2 to obtain that the distribution \( \mu \) stochastically dominates the required sample-size distribution \( \sum_{l \text{ leaf of } u} \Pr[T \mid \Pr[P_i \leq 1]] \), and hence we can simulate the distribution we need over leaves of each child, and hence also for \( v \).

For run-time, we note that this process takes time (at most) proportional to the size of the tree. For the latter, by Lemma 2.1, the time is \( O((\log \beta)^{\log_{\beta} n} / \delta) \).

Next, we show that if each element of the tree is a block of integers, then one can construct a data structure that sub-samples with interval range restriction, in time proportional to the sample size.

**Corollary 5.3** (Conditional sub-sampling data structure). Fix Precision-Tree \( T = T_{\delta}(y) \) with \( y \in \mathbb{N}^{n \times k} \). There exists a data structure, that given any interval \( Y \in \mathcal{I} \) and sub-sampling probability \( \eta \leq 1/\delta n \) with \( 1/\eta \in E_2(n) \), sub-sample block-coordinates \( X' \subseteq [n] \) with i.i.d. probability \( \eta \) and outputs a set of all coordinates \( W \subseteq X' \times [k] \) such that \( y_W \in Y \), i.e. \( W \triangleq \bigcup_{i \in X'} \{(i, j) \mid y_{i,j} \in Y\} \). In time \( O(|W|) \). The preprocessing time is, in expectation, \( \tilde{O}\left(\frac{n \cdot ||m||}{\delta n}\right) \), where \( m \leq k \) is the number of non-null entries in \( y_i \).

**Proof.** To preprocess, we prepare for each possible \( \eta \leq 1/\delta n \) with \( 1/\eta \in E_2(n) \) by subsampling each block with i.i.d. probability \( \eta \) using Lemma 5.1. Then we compute a set of coordinates \( Z_{\eta} \subseteq [n] \) by combining all coordinates from all sampled blocks. Let \( U_{\eta} \triangleq \bigcup_{w \in Z_{\eta} \times [k]} \{y_w\} \). Compute \( U_{\eta} \) and store it as a sorted array with a pointer to \( y^{-1}(j) \cap (Z_{\eta} \times [k]) \) for each \( j \in U_{\eta} \). Now, for each \((\eta, Y)\) query, locate the range of entries in \( U_{\eta} \) containing precisely the elements in \( Y \) using binary search of \( \min(Y), \max(Y) \). This will be our output \( W \).

The correctness follows immediately from the construction. Preprocessing runtime is as claimed as, in expectation, we need to sort \( ||m||/1/\delta n \) elements, and there are \( O(\log n) \) different values of \( \eta \) to consider. Runtime is only \( O(\log n) \) plus the size of the output (as it is stored as contiguous block). \( \square \)

### 5.2 Decomposing Precision-Trees

We now show that if one can compute some “local” function over intervals of elements given some local Precision-Tree parameter access to each interval, then we can redefine the tree-access as a tree-access to the global problem assuming all local problems were computed successfully. For the analysis, we assume that the number of leaves is an integral power of \( \beta \). The results extend to general tree sizes (for example by padding the input).

The algorithm follows the following steps (the full algorithm is described in Figure 2):

1. Identify the “correct” level \( \ell \) in \( T \) where each node corresponds to the right number of elements.
2. Find all nodes \( v \) at level \( \ell \) with enough precision to compute the local function \( f_v \) and compute the local function on these nodes by “detaching” the sub-trees rooted at each \( v \).
3. Consider the global tree $T$, trimmed to levels $\leq \ell$ (ie, the top of the tree). Augment the precision parameters by dividing each score with the precision we need to use for the local computation. This creates another “simulated” precision tree where we have access to the value at all leaves with $\mathcal{P}_v \leq 1$.

4. Compute the global function $g$ with tree access to the locally computable values.

Overall, we constructively show the following:

**Lemma 5.4 (Tree Decomposition).** Fix $\beta, \tau, n \in \mathbb{N}$ such that $n, \tau$ are each a power of $\beta$. Fix a domain of elements $\mathcal{E} \equiv \{e_1, \ldots, e_n\}$ and let $X_1, \ldots, X_\tau$ be a partition of $\mathcal{E}$ into disjoint intervals of equal width. Fix $\delta, \eta \in (0, 1)$ and (possibly randomized) functions $g : T_\delta(\mathcal{F}^\tau) \rightarrow \mathcal{G}$ and $\{f_i : T_\eta(X_i) \rightarrow \mathcal{F}\}_{i \in [\tau]}$ for some spaces $\mathcal{G}$ and $\mathcal{F}$. Then, algorithm TrimTree (see Figure 2), given a Precision-Tree $T_{\delta\eta}(\mathcal{E})$, outputs $g(T_\delta(f_1(T_\eta(X_1)), \ldots, f_\tau(T_\eta(X_\tau))))$. The algorithm’s expected run-time is $\tilde{O}(\tau + t_g + \frac{1}{\beta \tau} \cdot \|f\|_1 \cdot \log^{O(\log_\beta(\tau)}(\beta))$, where $t_g$ is the time to compute $g$, and $(t_f)_i$ is the expected time to compute $f_i$.

**Proof of Lemma 5.4.** Consider the full Precision-Tree $T_{\delta\eta}(\mathcal{E})$. Take the top log$_\beta$ $\tau$ levels of the tree to get to the level where each node $v$ has $n/\tau$ leaves in its subtree. This top portion (trimmed tree) cannot be seen as $T_\tau$ as follows: the precision $\mathcal{P}_v$ in $T_\delta$ is simply defined to be $\mathcal{P}_v/\eta$. Hence for leaf $v$ in $T_\delta$, we have that $\mathcal{P}_v' \leq 1$ iff $\mathcal{P}_v \leq \eta$. Hence for each such $v$, if $\mathcal{P}_v' \leq 1$, we have another $T_\eta$ rooted at $v$ (in the full Precision-Tree), allowing us to compute the corresponding $f_i(T_\eta(X_i))$. Hence we will be able to also compute $g(T_\delta(f_1(T_\eta(X_1)), \ldots, f_\tau(T_\eta(X_\tau))))$ as required.

For runtime, we note that by Lemma 2.1, we have $\frac{1}{\beta} \cdot \log^{O(\log_\beta(\tau)}(\beta)$ leaves $v$ with $\mathcal{P}_v' \leq 1$. We only preprocess SODS and compute the function $f_i$ on those leaves, and all $f_i$ have the same probability to be computed. Also, we note from the independence of $Z_i$’s, once we fix precision parameter $\eta_i$, the time to compute each $f_i(T_\eta(X_i))$ is independent from the random choice of $i$ where $f_i$ is computed, hence we have the expected runtime is the product of expectations. Finally, computing the function $g$ takes $t_g$ time.

**Proof of Lemma 2.1.** Let $s_v = 1/\mathcal{P}_v$. Since we only sample leaves with $s_v \geq 1$, then the total sampled leaves is at most $\sum_{v \in V_{\log_\beta(n)}} s_v$.

Now, since $Z_v$ are chosen i.i.d. uniformly in $\{1, \ldots, \Omega(\beta)\}$, then we deduce the recursive relation:

$$E[s_v] \leq E\left[\frac{1}{\mathcal{P}_v}\right] = \frac{O(\log(\beta)}{\beta} \cdot E\left[\frac{1}{\mathcal{P}_{\text{parent}(v)}}\right] \quad (2)$$

Let $\tau_\ell \equiv \sum_{v \in V_\ell} s_v$. Then $\tau_0 = 1/\lambda$, and for level $\ell$, summing and using Equation (2), we obtain

$$E[\tau_\ell] = E\left[\sum_{v \in V_\ell} s_v\right] \leq \beta \cdot E\left[\sum_{v \in V_{\ell-1}} O(\log(\beta)}{\beta} \cdot s_v\right] \leq O(\log(\beta) E\left[\sum_{v \in V_{\ell-1}} s_v\right] = O(\log(\beta)\tau_{\ell-1}.$$
Algorithm 1: TrimTree

Input: Disjoint intervals of elements \( \{X_i\}_{i \in [r]} \) of length \( r \) each, global function \( g : T_δ(\mathcal{F}^+) \to \mathcal{G} \), functions \( \{f_i : T_\eta(X_i) \to \mathcal{F}\}_{i \in [r]} \), Precision-Tree \( T \), local precision parameter \( \eta \).

Output: \( g([f_i(T_\eta(X_i))]_{i \in [r]}) \)

1. Let \( \ell = \log_2 \tau \) be the level in \( T \) such that for \( v \in V_\ell \), \( |X_v| = r \).
2. \( T_g \leftarrow \) the induced sub-tree of \( V_{\leq \ell} \).
3. for \( v \in V_\ell \) do
   4. if \( P_v \leq \eta \) then
      5. Preprocess SODS for the precision tree rooted at \( v \) using Corollary 5.3.
      6. Compute \( f_i(T_\eta(X_i)) \) using the sub-tree rooted at \( v \) and store the result in \( v \).
   7. \( P_v \leftarrow P_v/\eta \) for all \( v \in T_g \).
8. Preprocess SODS for \( T_g \) using Corollary 5.3.
9. Compute \( q \leftarrow g([f_i(T_\eta(X_i))]_{i \in [r]}) \) using \( T_g \).
10. return \( q \).

Figure 2: Algorithm TrimTree for decomposing Precision-Trees (Lemma 5.4).

Then \( \tau_\ell \leq O(\log(\beta)^{\ell}/\lambda) \), and we conclude \( \mathbb{E}\left[\sum_{v \in V_{\log(\beta)n \eta}} s_v\right] = O(\log(\beta))^{\log_2 n}/\lambda \) as needed.

6 Algorithm for Genuine-LIS: Proof of Theorem 4.1

In this section we describe and analyze the algorithm for the Genuine-LIS problem, which is called EstimateGenuineLIS.

First, we introduce an extension to the Block-LIS algorithm. In particular, we allow for a few “heavy blocks” which will not affect the approximation or complexity guarantees of Theorem 4.1. We use the following definition:

Definition 6.1. Fix input \( y \in \mathbb{N}^{X \times [k]} \), parameter \( \delta < 1 \) and value range interval \( Y \subseteq \mathcal{I} \). Let \( X^\delta \) be the \( \delta|X| \) blocks in \( X \) for which the quantity \( |y(X^\delta) \cap Y| \) is maximized. The \( \delta \)-heavy trimmed input \( y_{-\delta,Y} \in \mathbb{N}^{X \times [k]} \) is \( y \) after setting \( y_{-\delta,Y}(X^\delta) = \emptyset \) (i.e., removing all integers from \( y(X^\delta) \)).

For a range of \( y \)-values \( Y' \), we also define \( y_{-\delta}(X', Y') \equiv y(X'_{-\delta,Y'}) \cap Y' \) (i.e., the multiset of values in \( y(X'_{-\delta,Y'}) \) restricted to \( Y' \)).

The stronger version of Theorem 4.1 is as follows:

Lemma 6.2 (EstimateGenuineLIS algorithm, extended). Fix integers \( n,k \) and \( \lambda \in (1/n, 1) \). Fix an instance \( g \in \mathbb{N} \times \{0,1\} \times [k] \). For some monotone functions \( a_s : \mathbb{R}_+^2 \to [1, \infty) \) and \( c_s : \mathbb{R}_+^4 \to [1, \infty) \), suppose there exists a randomized algorithm \( A_{BL} \) that, given \( X \subseteq [n] \), \( Y \subseteq \mathbb{N} \), parameter \( \tau_s \), and a Precision-Tree \( T_{1/c_s}[X|\tau_s;X,Y] \gamma_{\lambda/2}(X,Y) \) for \( \text{Block-LIS}(y,X,Y) \) where \( a_s = a_s[X|\tau_s,\gamma_{\lambda/2}(X,Y)] \) w.h.p. in time \( t_s \).

Fix any parameter \( \gamma, \tau \geq 1 \), and let \( c \geq \tilde{O}(1/\lambda) + c_s(n, \tau, \lambda, \gamma \lambda/k) \). Then, there exists an algorithm \( A \) that, given free access to \( g_1 \) and Precision-Tree access to \( g_2, T_{1/c}(g_2) \), produces a \((\alpha, \lambda n)\)-approximation
for Genuine-LIS\((g)\) w.h.p., where \(\alpha_g = \log(k/\lambda) \cdot \max\{\gamma, a_s(n, \tau, \gamma \lambda/k)\}\). The algorithm \(A\) runs in time \(\tilde{O}(nk) + O(t_s)\).

To prove Lemma 6.2, we use the following reduction, as described above:

**Lemma 6.3.** Suppose there exists an algorithm \(A\) that \((f\left(\frac{|X|}{|y|/|Y|}\right), \lambda n)\)-approximates Block-LIS w.h.p given tree access \(T_{1/c}(g)\), in time \(t = t\left(\frac{|X|}{|y|/|Y|}\right)\), where \(c = c\left(\frac{|X|}{|y|/|Y|}\right)\) and \(f, c, t\) are some functions. Then there exists an algorithm \(A'\) that \((O\left(f\left(\frac{|X|}{|y|/2(|X,Y|)}\right)\right), \lambda n)\)-approximates Block-LIS with tree access \(T_{1/c}\left(\frac{|X|}{|y|/2(|X,Y|)}\right)\) in time \(O(t) + \tilde{O}(1/\lambda)\).

The proof of Lemma 6.3 is deferred to Section 8.

**Proof of Theorem 4.1 using Lemmas 6.2 and 6.3.** The proof is immediate since the extended algorithm assumed in Lemma 6.2 reduce to the standard versions using Lemma 6.3.

The rest of this section is devoted to prove Lemma 6.2.

### 6.1 Algorithm Description

The algorithm \textsc{EstimateGenuineLIS} is presented in Figure 3. It consists of the following mechanisms.

**Extracting pseudo-solutions.** We first perform some preprocessing. For this step, we treat all elements the same, whether they are genuine or not, and greedily compute and extract disjoint increasing subsequences, each of length \(\gtrapprox \lambda n\), denoted \(\{P_i\}\) (which are called pseudo-solutions), until there is no longer an increasing subsequence of length \(\lambda n\) remaining. For reasons that will become clear later, we also need to guarantee that each time we extract a sequence, such a sequence is approximately proportional to the longest one at that time, i.e., \textit{after removing all previous sequences}, and therefore we use a greedy algorithm.

We note that we cannot afford to use the standard dynamic-programming solution for finding an optimal LIS. Although it is simple and provides an exact greedy solution, its runtime is prohibitive for our delicate bounds. Instead, we use a fast data-structure for approximate greedy extraction, based on the fully dynamic data-structure in [GJ21a].

**Theorem 6.4.** [GJ21a] Given an input \(x \in \mathbb{N}^n\), there exists a dynamic data-structure \(D\) with three operations: 1) inserting/deleting of an element, in time \((\log n/\epsilon)\tilde{O}(1)\), and 2) finding an approximate longest increasing sub-sequence of length within \((1 + \epsilon)\)-factor of the optimum, \(\text{OPT}\), in time \(O(|\text{OPT}|)\).

This theorem leads to the following corollary for Block-LIS, with \(\epsilon = 1\).

**Corollary 6.5** (Extracting Pseudo-solution, approximate). There exists a data-structure that given a sequence of integer blocks \(y \in \mathbb{N}^{n \times k}\), extracts (i.e., finds and removes) an increasing sequence of length at least \(|\text{OPT}|/2\), where \(\text{OPT}\) is the current LIS. The pre-processing time is \(\tilde{O}(n \cdot k)\).

**Proof.** To preprocess the data-structure, we insert all elements to \(D\) one by one, where elements of each block are inserted in non-increasing order. This takes time \(\tilde{O}(nk)\). Finally, we answer the extraction queries by LIS-querying \(D\), and remove all elements in the 2-approximate LIS solution one by one from \(D\). 

\(\square\)
Pseudo-Solution Partitioning / Discretization. Next, we partition \( \{ P_i \}_i \) into \( \log(1/\lambda) \) disjoint buckets denoted \( \{ P_{\ell} \}_\ell \), where each \( P \in P_{\ell} \) is an increasing sub-sequence of length \( \approx \ell \lambda n \) (not necessarily containing only genuine elements). Our algorithm generates an estimator for Genuine-LIS for each scale \( \ell \), based purely on the elements of the subsequences \( \{ P \mid P \in P_{\ell} \} \), and outputs the maximal estimator. Clearly, one of the buckets must contain a significant fraction of the Genuine-LIS; we focus on that bucket henceforth.

Estimator Computation. We next describe our estimator for each bucket as above, which by itself is generated by taking the maximum of 2 estimators, denoted Dense Estimator and Sparse Estimator. The Dense Estimator is a fairly straightforward one. Here, we subsample from the coordinates of all pseudo-solutions \( U_{\ell} \triangleq \cup_{P \in P_{\ell}} P \), compute the number of genuine elements in the sampled set, and return this count after re-scaling. The main idea is that, if \( U_{\ell} \) has many genuine elements, then the aforementioned sampling procedure gives a high-fidelity estimate.

The opposite situation is when the genuine elements are sparse within the union of subsequences of length approximately \( \ell \). Then, if the Genuine-LIS itself is quite large, it must be the case that a high proportion of the genuine elements participate in the LIS. Here, we invoke the algorithm for Block-LIS, restricted to genuine elements only. Using the fact that Block-LIS approximation is a function of the proportion of LIS elements from the total “relevant” elements (in this case, genuine elements), and the sparse case makes progress by reducing the number of relevant elements, resulting in improved approximation. The overall estimator for length \( \ell \) is the maximum of the 2 estimators.

\[
\text{Algorithm 2: EstimateGenuineLIS} \\
\text{Input: } g \in (\mathbb{N} \times \{0,1\})^{n \times k}, \lambda \in [1/n,1], \gamma \geq 1, \text{Precision-Tree } T = T_{1/c}(g) \text{ for the parameter } c \\
\text{ defined in Lemma 6.2.} \\
\text{Output: } \text{A number } \hat{G} \in [0,n]. \\
1 \text{ Greedily and iteratively extract 2-approximate maximal increasing sequences from } g_1, \text{ as long as } \\
\text{ they contain at least } \lambda n/4 \text{ elements using Corollary 6.5, generating pseudo-solutions } P_1, \ldots, P_t \\
\text{ for some } t = O(k/\lambda), \text{ where each } P_i \subseteq [n] \times [k] \text{ is a set of coordinates.} \\
2 \text{ Using Corollary 5.3 and } T, \text{ sub-sample blocks } S \subseteq [n] \text{ with i.i.d. probability } 1/\lambda \cdot \frac{10\zeta}{n} \text{ each, for } \\
\zeta = O(\log n), \text{ and let } W \leftarrow \{(i,j) \in S \times [k] \mid g(i,j)_2 = 1\} \text{ be the set of genuine coordinates.} \\
3 \text{ Compute } B \subseteq S \leftarrow \text{the } \zeta \text{ blocks in } S \text{ for which } |B \cap W| \text{ is maximized.} \\
4 \text{ for } \ell \in E_2(4/\lambda)/4 \text{ do } \\
5 \quad \text{Let } P_\ell \triangleq \{ P_i \mid |P_i| \in (\lambda n \ell/2, \lambda n \ell) \}, \text{ and let } U_\ell \triangleq \cup_{P \in P_\ell} P. \\
6 \quad \kappa_\ell \leftarrow |W \cap (S \setminus B) \times [k] \cap U_\ell|. \\
7 \text{return } \\
\hat{G} \triangleq \max \max_{\ell} \left\{ \frac{\lambda^2 \ell n \kappa_\ell}{4k \zeta} - \frac{\lambda n}{4}, \text{ESTIMATEBLOCKLIS} \left(g(*)_1 \mid_{U_\ell \cap g_2^{-1}(1)}, n, \lambda, T \right) \right\}.
\]

Figure 3: Description of the algorithm EstimateGenuineLIS. \( g(*)_1 \mid_{U_\ell \cap g_2^{-1}(1)} \) denote the sequence \( g(*)_1 \) restricted to positions in \( U_\ell \) for which \( g(.)_2 = 1 \). From preliminaries, \( E_b(k) \triangleq \{ b^i \mid i \in \{0, \ldots, \lfloor \log_b k \rfloor \} \} \).
6.2 Analysis

We now analyze the algorithm, proving Lemma 6.2.

Proof of Lemma 6.2. Let \( \text{OPT} \) be the coordinates of an optimal solution of length \( \text{Genuine-LIS}(g) \). Let \( U \triangleq \cup \ell U_\ell \), and \( \overline{U} \triangleq [n] \times [k] \setminus U \). Define \( \tilde{\text{OPT}} \triangleq \text{OPT} \cap U \) (i.e., restricted to elements in \( U \)). Since \( \overline{U} \) does not contain an increasing sequence of length \( \frac{1}{2} \lambda n \), we have that \( |\tilde{\text{OPT}}| \geq |\text{OPT}| - |\text{OPT} \cap U| \geq |\text{OPT}| - \frac{1}{2} \lambda n \).

For \( \ell \in E_2(4/\lambda) \), let us also define \( \tilde{\text{OPT}}_\ell \triangleq \text{OPT} \cap U_\ell \).

To bound our estimators, we first bound the quantity \( \kappa_\ell \). For this task, we introduce more notations. Let \( M \subset [n] \) be the \( \lambda n/2 \) blocks containing the highest number of genuine elements. We also define the following:

\[
\begin{align*}
\tilde{\kappa}_\ell &\leftarrow |W \cap U_\ell| , \\
q^{(\ell)} &\triangleq \frac{1}{nk} \cdot |U_\ell \cap g_2^{-1}(1)| , \\
q^{(\ell)}_{\lambda/2} &\triangleq \frac{1}{nk} \cdot |U_\ell \cap g_2^{-1}(1) \cap ([n] \setminus M) \times [k]| , \\
m^{(\ell)}_{\lambda/2} &\triangleq \frac{1}{k} \cdot \min_{i \in M} \{ |U_\ell \cap g_2^{-1}(1) \cap (i, \ast)| \} , \\
G_{\ell,1} &\triangleq \frac{\lambda^2 \ell n \kappa_\ell}{40-k \zeta} - \frac{\lambda n}{4} , \\
G_{\ell,2} &\triangleq \text{the output of EstimateBlockLIS} \left( g^{(\ast)}_1 \mid_{U_\ell \cap g_2^{-1}(1)} , n , \lambda , T \right) ,
\end{align*}
\]

We have,

\[
\mathbb{E} [\tilde{\kappa}_\ell] = q^{(\ell)} \cdot k \cdot \frac{1}{\chi} \cdot 10 \zeta = q^{(\ell)} \cdot \frac{10k \zeta}{\chi} .
\]

We need to have a good estimator w.h.p. and therefore use the following statistical fact:

Fact 6.6. Fix \( \zeta \in \mathbb{N} \). Let \( X_1 , \ldots , X_m \) be non-negative independent random variables and let \( X = \sum_i X_i \). Let \( Y \) be the sum of the \( \zeta \) largest (empirical) \( X_i \). Then, \( X - Y \leq 2 \mathbb{E}[X] \) with probability \( 1 - \exp(-\Omega(\zeta)) \).

Proof. Define \( p_i \triangleq \Pr[X_i > 2 \mathbb{E}[X]/\zeta] \), and notice that \( i \sum_i p_i < \zeta/2 \) using Markov’s inequality. Define also \( Z_i \triangleq \max\{0, X_i - 2 \mathbb{E}[X]/\zeta\} \), and \( Z = \sum_i Z_i \). Then \( X - Z \) is a sum of non-negative independent random variables bounded by \( 2 \mathbb{E}[Z]/\zeta \), and hence with probability \( \exp(-\Omega(\zeta)) \), \( X - Z > 2 \mathbb{E}[X] \). Also, notice that \( \Pr[Z_i > 0] = p_i \), and hence, with probability \( \exp(-\Omega(\zeta)) \), \( \sum_i 1[Z_i > 0] > \sum_i p_i + \zeta/2 = \zeta \), implying \( Z > Y \) with such probability as well. By the union bound, we conclude that \( X - Y \leq 2 \mathbb{E}[X] \) with probability \( 1 - \exp(-\Omega(\zeta)) \) as needed. \( \square \)

Now, notice that \( \tilde{\kappa}_\ell \) is a sum of independent random variables and hence from Fact 6.6, w.h.p. we have

\[
\kappa_\ell \leq 2 \mathbb{E}[\tilde{\kappa}_\ell] = 20 q^{(\ell)} \cdot k \zeta \cdot \frac{10k \zeta}{\chi} .
\]

For the lower bound, we derive a lower bound for \( \kappa_\ell \) using \( q^{(\ell)}_{\lambda/2} \). Notice that each \( i \in M \) gets sampled i.i.d. with probability \( \frac{10k \zeta}{M} \). Hence \( |S \cap M| > 2 \zeta \) w.h.p, which implies that \( |(S \setminus B) \cap M| > \zeta \). This means that the contribution of blocks in \( M \) to \( \kappa_\ell \) is at least \( \zeta \cdot k \cdot m^{(\ell)}_{\lambda/2} \). Next, consider the contribution \( \psi_\ell \) of blocks in \([n] \setminus M \) to \( \kappa_\ell \). Then \( \psi_\ell \) is a sum of independent random variables in \( k \cdot [0 , m^{(\ell)}_{\lambda/2}] \) and hence, using the Chernoff bound, we have

\[
\psi_\ell \geq \mathbb{E}[\psi_\ell] - \zeta \cdot k \cdot m^{(\ell)}_{\lambda/2} = \frac{10k \zeta}{\chi} \cdot q^{(\ell)}_{\lambda/2} - \zeta \cdot k \cdot m^{(\ell)}_{\lambda/2} .
\]

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Therefore \( \kappa_\ell \geq \psi_\ell + \zeta \cdot k \cdot m^{(\ell)}_{\lambda/2} \geq \frac{10C}{\lambda} \cdot q^{(\ell)}_{\lambda/2} \) w.h.p. as well. Hence we can finally bound \( G_{\ell,1} \), w.h.p., as follows:

\[
G_{\ell,1} = \frac{\lambda^2 \ell n \kappa_\ell}{40k^2} - \frac{\lambda n}{4} \leq \left[ \frac{\lambda n}{4} \left( \ell q^{(\ell)}_{\lambda/2} - 1 \right) \right] + \frac{1}{2} \lambda n \cdot q^{(\ell)}.
\]

For the second estimator, we have that \( G_{\ell,2} \) is a \((a_{\kappa_\ell}^{(\ell)}(n, \tau, \lambda), \lambda n)\)-approximation of \( \text{Genuine-LIS}(g(U_\ell)) \).

We now bound the maximum over these quantities.

**Upper Bound:** W.h.p., for all \( \ell \), we have \( G_{\ell,1} \leq \frac{1}{4} \cdot \lambda n \cdot q^{(\ell)} \). Now, by construction, we have that all pseudo-solutions \( P \in \mathcal{P}_\ell \) are of length \( |P| \geq \lambda n \ell/2 \), and hence \( |\mathcal{P}_\ell| \leq \frac{2k}{\lambda^2} \). Therefore, there must exist some increasing sub-sequence \( P^* \in \mathcal{P}_\ell \) such that \( |P^* \cap g^{-1}_{\ell}(1)| \geq \frac{1}{2} \cdot \lambda n \cdot q^{(\ell)} \), i.e., that contains at least that many genuine elements. This implies \( |\text{OPT}_{\ell}| \geq G_{\ell,1} \).

Similarly, for \( G_{\ell,2} \), the Block-LIS approximation algorithm outputs a lower bound on the LIS, and hence \( G_{\ell,2} \leq |\text{OPT}_{\ell}| \).

We conclude that \( \max_{\ell} \max \{G_{\ell,1}, G_{\ell,2}\} \leq |\text{OPT}_{\ell}| \leq |\text{OPT}| \).

**Lower Bound:** Since \( \text{OPT}_{\ell} = \bigcup_{\ell} \text{OPT}_{\ell} \), then there exists \( \ell^* \in E_2(4/\lambda) \) such that \( |\text{OPT}_{\ell^*}| \geq \frac{|\text{OPT}|}{\log(1/\lambda)} \).

Fix \( U^* \triangleq U_{\ell^*} \setminus (M \times [k]) \), \( q^* = q^{(\ell^*)}_{\lambda/2} \), \( \text{OPT} \triangleq \text{OPT}_{\ell^*} \), and \( G^* = G_{\ell^*, \ell^*} \).

Consider the following two cases:

**Dense Case:** First, suppose \( q^* \geq \frac{1}{7} \). Observe that from Corollary 6.5, we may only extract a solution of length at most \( \lambda n \ell \) once the remaining LIS is at most \( 2\lambda n \ell \) (i.e., after extracting longer subsequences). Hence \( |\text{OPT}_{\ell}| \leq 2\lambda n \ell \). Therefore,

\[
G^*_1 \geq \frac{\lambda^2 n}{47} - \frac{1}{4} \lambda n \geq \frac{|\text{OPT}_{\ell}|}{47} - \frac{1}{4} \lambda n \geq \frac{|\text{OPT}_{\ell}|}{47 \log(1/\lambda)} - \frac{1}{2} \lambda n \geq \frac{|\text{OPT}|}{47 \log(1/\lambda)} - \lambda n.
\]

**Sparse Case:** Now, suppose \( q^* < \frac{1}{7} \). Then \( |U^* \cap g_{\ell}^{-1}(1)| < \frac{n k}{7} \) and hence for any \( \tau = \tau_\ell \) such that \( T = T_{1/\ell}(g) \), we have

\[
G^*_2 \geq |\text{OPT}_{\ell}| - \lambda n \geq \frac{|\text{OPT}|}{\log(1/\lambda)} - \lambda n,
\]

where

\[
\alpha = a_n\left(n, \tau, \frac{\lambda n}{|U^* \cap g_{\ell}^{-1}(1)|}\right) \leq a_n\left(n, \tau, \frac{\lambda}{\kappa}\right),
\]

\[
c = c_n\left(n, \tau, \lambda, \frac{\lambda n}{|U^* \cap g_{\ell}^{-1}(1)|}\right) \leq c_n\left(n, \tau, \lambda, \frac{\lambda}{\kappa}\right).
\]

Since we are either in the dense or sparse case as described above, we obtain that

\[
\max\{G^*_1, G^*_2\} \geq \frac{|\text{OPT}|}{A \log(1/\lambda)} - \lambda n,
\]

where \( A = \max\{\gamma, a_n(n, \tau, \frac{\lambda}{\kappa})\} \).

We conclude that \( \tilde{G} = \max\{G_\cdot\} \) is a \((\log(1/\lambda)A, \lambda n)\)-approximation for \( |\text{OPT}| \), as needed.

**Runtime Complexity:** The algorithm’s runtime is dominated by the first step, i.e., generating increasing subsequences, which requires \( \tilde{O}(nk) \) time using Corollary 6.5. In addition, we require \( t_s \) time for computing the second estimator. Thus, the overall algorithm for \text{ESTIMATEGENUINELIS} runs in time \( \tilde{O}(nk) + t_s \).
7 Algorithm for Block-LIS: Proof of Theorem 4.2

Here we present our construction for the Block-LIS problem, and prove Theorem 4.2. Recall that we are given an interval \( X \subseteq [n] \), a sequence \( y \in \mathbb{N}^{|X|} \times [k] \) and a range of values \( Y \subseteq \mathbb{N} \), and the goal is to approximate Block-LIS\((y, X, Y)\), i.e., the length of a maximal sub-sequence OPT, specified by a set of indices \( w_1, w_2, \ldots, w_\ell \subseteq X \times [k] \) such that the set of (first coordinate, value) pairs \( \{(w_i)_1, y_{w_i}\}\) are given an interval can be found “for free”.

Algorithm for Block-LIS (Theorem 4.2, extended)

Lemma 7.1

Consider the pair \( (X_i, Y_i) \), where \( i \in [\ell] \), which together forms a block \( X_i \subseteq X \times Y \) of\( \beta \)-approximate\( X_i \) and any \( \beta \)-approximate to Block-LIS\((y, X_i, Y_i')\) is above a certain threshold. For each \( x \)-interval \( X_i \), we compute a set of intervals \( Y_i \) (called candidate intervals), which together forms a block in the Genuine-LIS instance. Intuitively, one can think of the Genuine-LIS instance as solving the global LIS, while each Block-LIS instance is solving some local LIS over (sub-)interval \( X_i \), over some range of \( y \)-values. Overall, we will show that such a formulation is equivalent to a composition of functions, and use the Tree Decomposition Lemma to obtain our correctness and complexity guarantees.

7.1 Extending the Genuine-LIS problem

Before describing our construction for proving Theorem 4.2, we first introduce a slightly stronger version of it, which will be easier for us to work with. In particular, we introduce 2 extensions of the requirements of the Genuine-LIS algorithm. First, we allow the inputs to be blocks of intervals rather than integers, and second, we require the approximation to improve if there are many “null” entries, i.e., if most blocks have much less than \( k \) entries.\(^5\)

Overall, the stronger version is the following:

Lemma 7.1 (Theorem 4.2, extended). Fix monotone functions \( a_g : \mathbb{R}_+^2 \to [1, \infty) \), \( a_s, c_g : \mathbb{R}_+^3 \to [1, \infty) \) and \( c_s : \mathbb{R}_+ \to [1, \infty) \), satisfying, for all \( r, \tau \in E_\beta \), \( m \in \mathbb{N} \), \( \lambda < 1 \), \( \lambda_1, \lambda_2 \in [\lambda, 1] \), and \( k' \in [1, 1/\lambda] \) with \( \lambda_1 \lambda_2 = \Omega(\lambda^k) \):

- \( a_s(r, \tau, \lambda m) \geq \text{polylog}\left(\frac{k}{\lambda}\right) \cdot a_g\left(\tau, \frac{\lambda}{k}\right) \cdot a_s\left(r, \tau, \frac{\lambda}{m}\right) \); and
- \( c_s(r, \tau, \lambda, \lambda m) \geq \beta^O(1) \cdot \frac{r}{k'} + c_g(\tau, \lambda, k') \cdot c_s\left(\frac{r}{k'}, \tau, \Theta(\lambda_1 / k') \cdot \frac{\lambda}{k}\right) \).

Fix input \( y \in \mathbb{N}^{n \times k} \), a block interval \( X \subseteq [n] \), value range interval \( Y \subseteq \mathbb{N} \), parameters \( \lambda \in (0, 1), \beta \in \mathbb{N}, \tau \in E_\beta \). Suppose there exist the following algorithms:

\( A_{GL} \): Given Genuine-LIS instance \( g \in (\mathbb{N} \times \{0, 1\})^{n \times k} \) with \( \beta \)-ary Precision-Tree \( T_{1/c_g(n_g, \lambda_g, m_g, n_g)}(g_2) \) access, \( A_{GL} \) outputs \( a_g\left(n_g, \frac{\lambda_g n_g}{m_g}\right) \)-approximation to Genuine-LIS\((g)\) in time \( O(n_g + m_g) \) w.h.p.

\( A_{BL} \): Fix \( t > 1 \). For any interval \( X' \subseteq X \), value range \( Y' \subseteq Y \), and any \( \lambda_s < 1 \), given a \( \beta \)-ary Precision-Tree \( T_{1/c_s(\lambda_s X', \tau, \lambda_s, \frac{\lambda_s |X'|}{y_{-\lambda_s / \tau} |y(X')|})} (y(X')) \), \( A_{BL} \) outputs \( a_s, \lambda_s |X'| \)-approximation to

\(^5\)We highlight that “null” is different from “not genuine”; while “not genuine” entries are expensive to find, “null” entries can be found “for free”.

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Block-LIS\((y, X', Y')\) \(\text{w.h.p.}\), where \(a_s \triangleq a_s\left(|X'|, \tau, \frac{\lambda_s |X'|}{|y - \lambda/2(X', Y')|}\right)\). The expected run-time is at most \(t_{BL} = t \cdot c_s\left(|X'|, \tau, \lambda_s, \frac{\lambda_s |X'|}{|y - \lambda/2(X', Y')|}\right)\).

Then, given a \(\beta\)-ary Precision-Tree \(T_1/c(y)\), we can produce a \((\alpha, \lambda |X'|)\)-approximation for Block-LIS\((y, X, Y)\) \(\text{w.h.p.}\), as long as \(\alpha \geq a_s\left(|X|, \tau, \frac{\lambda |X|}{|y - \lambda/2(X, Y)|}\right)\) and \(c \geq c_s\left(|X|, \tau, \lambda, \frac{\lambda |X|}{|y (X, Y)|}\right)\).

The algorithm’s expected run-time is at most \(c \cdot t \cdot \log^{O((\log \beta) / (\beta))\cdot \frac{|y (X, Y)|}{|X|}}\).

To prove Theorem 4.2, we use the reduction from Lemma 6.3, along with the following reduction:

**Lemma 7.2.** Suppose the algorithm EstimateGenuineLIS \((f(\lambda/k), \lambda n)\)-approximates Genuine-LIS for some function \(f\) in time \(t(n-k)\) over integers, then there exists an algorithm \(A'\) that \((O(f(\lambda n/m) \log(1/\lambda)), \lambda n)\)-approximates Genuine-LIS over intervals with \(m\) non-null entries. The run-time is \(t(m) + O(n + m)\).

The proof of Lemma 7.2 is deferred to Section 8.

**Proof of Theorem 4.2 using Lemmas 6.3, 7.1 and 7.2.** The proof follows by induction. We assume Theorem 4.2 holds for \(X' \subset X\), setting \(t = t_s(|X'|, \tau)\) (noting that for the base case, we have \(a_s(1, \cdot, \cdot) = 1\) and \(c_s(1, \cdot, \cdot, \cdot) = 1\), simply by outputting 1 iff the block is not empty). Also note the extended algorithms assumed in Lemma 7.1 reduce to the standard version using Lemma 7.2, with \(O(\log 1/\lambda_9)\) additional approximation (which can be absorbed into the polylog\((k/\lambda)\)-approximation of parameter \(\alpha\)). Finally, one can observe that the additive time \(\tilde{O}(n_g + m_g)\) does not change the asymptotic time complexity.

The rest of this section is devoted to prove Lemma 7.1.

### 7.2 Algorithm

The algorithm for the Block-LIS problem, named EstimateBlockLIS, is described in Figure 6.

#### 7.2.1 Global LIS vs. Local LIS

We first provide some intuition underlying the algorithm, and in particular, for balancing the “global LIS” and “local LIS”, a necessary ingredient of the Block-LIS algorithm. Suppose that \(k = 1\), so that the problem reduces to a standard LIS problem. In this special case, it might be useful to visualize the instance as a set of points on a standard two-dimensional grid, with \(X\) being an interval on the \(x\)-axis, and the element \((i, y_i)\) being represented by the point with \(x\)-coordinate \(i\) and \(y\)-coordinate \(y_i\). In this equivalent formulation, the objective is to determine the maximum length of a subsequence that is increasing with respect to both axes.

Assume further that a longest increasing subsequence \(\text{OPT}\) is of length approximately \(\lambda |X|\). Note that \(\text{OPT}\) can be distributed arbitrarily with respect to \(X\). To approximate \(\text{OPT}\), we fix some length \(r\), and use the standard, in-order partitioning of \(X\) into \(\tau = |X|/r\) mutually disjoint and covering intervals \(X_1, \cdots, X_r\) of length \(r\) each. There are two potential extreme scenarios for the distribution of \(\text{OPT}\):

1. All elements in \(\text{OPT}\) belong to approximately \(\lambda |X|/r\) intervals \(X_i\), meaning that all integers in those intervals are part of the \(\text{OPT}\), and the other intervals do not contribute at all.
2. \(\text{OPT}\) is uniformly distributed across all intervals, i.e., each interval \(X_i\) contributes approximately \(\lambda r\) integers to \(\text{OPT}\).
Intuitively speaking, one can consider the problem of certifying an alignment to be more difficult if fewer integers participate in an alignment. Therefore, in the first case above, one has to “work harder” on the global alignment to find the participating intervals; however, little effort is required to verify the local LIS in each interval of interest, i.e., to get a lower bound on the LIS within each interval. In the second case, on the other hand, it should be easier to determine the global alignment, but it is more difficult to approximate the local LIS (within each interval).

Of course, an optimal solution can lie anywhere between these extreme scenarios. However, one can show that there must exist some $\rho \in [1, 1/\lambda]$, such that there are approximately $1/\rho$ fraction of intervals, each having approximately $\lambda \rho r$ “local” contribution to $OPT$.

7.2.2 No repetitions

The algorithm assumes no element repeats itself in the sequence. To remove such assumption, one can reconfigure the element values setting $y_{i,j} \leftarrow n \cdot y_{i,j} - i$.

7.2.3 Algorithm Overview

Now we consider the general Block-LIS problem. Before describing the main algorithm, we describe its main subroutines and mechanism, setting up some concepts and notation along the way. The notation is summarized in Table 1.

**Sampling.** The algorithm starts by simulating uniformly random blocks $(w_j, y_{w_j})$ and then accumulating all $y_{w,j}$-samples in $Y$ within the sampled blocks. The main idea is to generate a set $S_i$ of sampled values, such that the “distance in $S_i$” between any 2 values in $S_i$ is approximately proportional to the number of integers in range over the entire $y(X_i)$. To obtain such a guarantee, we simply use members of $S_i$ whose rank is a multiple of $O(\log n)$. We also want to make sure there are not too many total samples in all blocks, i.e., that $\sum_j |S_i|$ is close to its expectation. To control the tail bounds of such a quantity, we “throw away” the $\Theta(\log n)$ blocks with the largest number of samples in $Y$. The sampling algorithm is presented in Figure 4, and the guarantees are formally stated in Claim 7.7, as part of the analysis.

**Algorithm 3: SampleAndPartition**

**Input:** A Precision-Tree $T = T_{1/c}(y)$, where $y \in \mathbb{N}^{X \times [k]}$ consists of $|X|$ blocks with at most $k$ integers each, value range $Y \in \mathcal{I}$, branching parameter $\tau$, and sample size $s$.

**Output:** Partition of $X$ with sampled $y$-values of each part.

1. $\zeta \leftarrow O(\log n)$.
2. $X_1, \ldots, X_\tau \leftarrow$ a partition of $X$ into $\tau$ consecutive intervals, each consisting of $r = |X|/\tau$ blocks.
3. Using Corollary 5.3 and $T$, sub-sample blocks $S \subseteq X$ with i.i.d. probability $\frac{\zeta}{|X|}$, and let $W \leftarrow \{(i,j) \in S \times [k] \mid y_{i,j} \in Y\}$.
4. Compute $B \subseteq S \leftarrow$ the $\zeta$ blocks in $S$ for which $|B \cap W|$ is maximized.
5. for $i \in [\tau]$ do
6. $\{p^i_1, p^i_2, \ldots\} \leftarrow y(W \cap ((X_i \setminus B) \times [k]))$ where $p^i_m \leq p^i_{m+1}$ for all $m$.
7. $S_i \leftarrow \{p^i_m\}_{m \in [\rho] \cap \mathbb{N}}$.
8. return $(X_*, S_*)$.

Figure 4: Description of the sampling and partitioning subroutine.
Generation of Candidate Intervals. For each set \( S_i \), we construct \( O(\log(k/\lambda)) \) candidate interval sets \( \{Y_i, \Delta\} \). This process will be explained in detail in the next subsection.

Discretization. The algorithm performs exponential discretization over the following parameters:

- \( \rho \): inverse of the fraction of intervals participating in OPT. This parameter characterizes the relation between the “global” and “local” LIS. In particular, for the 2 extreme scenarios outlined above, the parameter \( \rho \) would be \( 1/\lambda \) and \( 1 \) respectively.

- \( \Delta \): a quantity proportional to the “candidate interval size” of the local-LIS (i.e., to \( |y-\lambda/2(y, X_i, Y')| \)) and inversely proportional to the “block size” of the global-LIS problem (i.e., to \( m_g/n_g \)).

Decomposition and Recursion. The next step is to determine, for each choice of parameters \( (\rho, \Delta) \), the largest set of monotone candidate intervals \( \{\{(i, Y') : Y' \in Y_i, \Delta\}\} \) such that each candidate interval has local LIS at least \( \kappa_{\rho} \equiv r/\rho \). To do this, the algorithm recursively defines the Block-LIS algorithm on each interval \( X_i \), marking the candidate \( y \)-intervals with a sufficiently long local LIS as “genuine”, and then solving a “global” Genuine-LIS problem over all genuine \( (X_i, Y') \) pairs. Thus, the problem is formulated as a composition of a Genuine-LIS problem over many smaller Block-LIS problems, and the Precision-Tree decomposition algorithm (Lemma 5.4) is used to answer this question.

Optimization. Finally, we output the solution with the maximum value among all combinations of parameters.

Algorithm 4: DECOMPOSE

Input: A Precision-Tree \( T = T_{1/\epsilon}(y) \), where \( y \in \mathbb{N}^{X \times [k]} \) consists of \( |X| \) blocks with at most \( k \) integers each, value range \( Y \subseteq \mathbb{N} \), \( x \)-partition \( \{X_i\}_i \) of \( X \), \( y \)-partition sets \( \{Y_i\}_i \) of \( Y \).

Output: An integer \( \hat{L} \in [0, \text{Block-LIS}(y, X, Y)] \).

1. Let \( \mathcal{A}_{GL}, \mathcal{A}_{BL} \) be the aforementioned algorithms from Lemma 7.1.
2. Let \( \kappa_{\rho} \equiv r/\rho \).
3. For each \( i \), let \( \Lambda_{\rho}(i, *) \equiv \left[ \left\{ Y' : 1 \left[ \mathcal{A}_{BL}\left(y(X_i), Y', \frac{1}{\rho}, \tau\right) > \kappa_{\rho}\right]\right\} \right]_{Y' \in Y_i} \).
4. Let \( G_{\rho} \equiv \mathcal{A}_{GL}(\Lambda_{\rho}, \lambda_{\rho}) \cdot 0.5\kappa_{\rho} \).
5. for \( \rho \in E_2(1/\lambda) \) do
6. Compute \( L_{\rho} \leftarrow G_{\rho} \circ \Lambda_{\rho} \) using Lemma 5.4 and \( T \).
7. return \( \hat{L} \leftarrow \max_{\rho} L_{\rho} \).

Figure 5: Description of the decomposition subroutine. Note that lines 2 – 4 are merely definitions for \( \rho \in E_2(1/\lambda) \), and not computational steps.

7.2.4 Candidate Intervals

Now we describe our construction of the candidate intervals \( Y_{i, \Delta} \) using the sampled \( y \)-values \( S_i \), for some fixed \( i, \Delta \). The high-level goal, as in [RSSS19], is to cover an optimal LIS solution using a small number of monotone interval sequences denoted as pseudo-solutions, so that the largest LIS within all pseudo-solutions would be a good approximation to the optimum. However, we need to choose such candidates more carefully, since, in addition to “capturing the local LIS”, we would like to ensure the following efficiency guarantees, for our delicate bounds:
1. There are not too many candidate intervals, i.e., \(|\mathcal{Y}_i| \lesssim \frac{|y(X_i) \cap Y|}{\lambda |X_i|}\), and more importantly,

2. The total number of integers in range over all candidate intervals do not cause significant overhead. In particular: \(\sum_{Y' \in \mathcal{Y}_i} |y(X_i) \cap Y'| \lesssim |y(X_i) \cap Y|\).

These guarantees are formally stated in Lemma 7.6, as part of the analysis.

We now describe the algorithm for generating such candidate intervals. First, we show how to generate a small “covering” set of candidate intervals over some fixed set \([m]\).

**Claim 7.3.** For any \(m \in \mathbb{N}\), there exists a clustering of \([m]\) into intervals \(\{J_{\Delta,1}, J_{\Delta,2}, \ldots, J_{\Delta,m/\Delta}\}_{\Delta \in E_2(m)}\) where each \(J_{\Delta,j} \subseteq I \cap 2^{|m|}\) and further:

1. \(|J_{\Delta,j}| = \Delta\) for all \(\Delta,j\).

2. For any interval \(I \in [m] \cap I\), there exists a set of intervals \(\mathcal{E}_I \subseteq \{J_{\Delta,j}\}_{\Delta,j}\) of size \(|\mathcal{E}_I| = O(\log \log m)\), such that \(I\) is precisely covered by \(\mathcal{E}_I\); i.e., \(I = \bigcup_{\mathcal{E}_I \in \mathcal{E}_I} J\).

**Proof.** Consider a binary tree \(T_m\) with leaves indexed by integers in \([m]\) and internal nodes indexed by the union of its children. The set of clusters corresponding to the indexing of all nodes in each level of the tree provides the above guarantees. For (1), we have a node at each height \(h \in [\log_2 m]\) of length \(\Delta = 2^h\) hence that bound is immediate.

For (2), we reconstruct \(I\) as follows: let \(a \triangleq \min\{I\}\) and \(b \triangleq \max\{I\}\). We start with \(v_a\) (the leaf with index \(a\)) and add it to \(\mathcal{E}_I\). Now we go up the tree level by level, each time adding the sibling of the current node if and only if it is contained in \(I\). Similarly, we do the same process with \(v_b\). We note that this process generates \(\mathcal{E}_I \subseteq \{J_{\Delta,j}\}_{\Delta,j}\) of size \(O(\log m)\). It is left to show that \(I = \bigcup_{\mathcal{E}_I \in \mathcal{E}_I} J\).

For this task, consider any \(c \in [m]\). One one hand, if \(c \notin I\), no ancestor of \(c\) will be added to \(\mathcal{E}_I\) by construction. On the other hand, for \(c \in I\), let \(v\) be the “lowest” node such that either \(a,c\) or \(b,c\) are common descendants of \(v\). Assume w.l.o.g. that \(a,c\) are the common descendants (otherwise consider the symmetric argument). Now, if \(c = a\), then we add \(v_a = v_c\) to \(\mathcal{E}_I\). Otherwise, \(c \in (a,b)\), then \(c \in \text{right}(v) \subseteq I\), and hence \(\text{right}(v)\) will be added to \(\mathcal{E}_I\). We conclude that \(c \in I\) if and only if \(c \in \bigcup_{\mathcal{E}_I \in \mathcal{E}_I} J\) as needed.
An immediate corollary is that, given a value set $S$ and a sub-additive set function $\psi$, we can generate a small set of candidate intervals which approximates, up to a logarithmic factor, all candidate intervals. Here we note that the set function $\psi_{X_i,y}(Y') \triangleq \text{Block-LIS}(X_i,Y',y)$ is sub-additive in $Y'$.

**Corollary 7.4.** Fix a finite set of values $S \subseteq \mathbb{N}$ of size $m$. There exist sets of intervals $\{\mathcal{C}_{\Delta}^S\}_{\Delta \in \mathcal{E}_m} \subseteq \mathcal{I}$ such that:

1. For all $\Delta$ and all $I \in \mathcal{C}^S_{\Delta}$: $|I \cap S| = \Delta$.
2. For all $\Delta$: $|\mathcal{C}^S_{\Delta}| = |S|/\Delta$.
3. Fix a sub-additive set function $\psi: \mathcal{I} \rightarrow \mathbb{R}$. For any $I \in \{(a,b) \in \mathcal{I} \mid a,b \in S\}$, there exists some $J \in \bigcup_{\Delta} \mathcal{C}_{\Delta}^S$ with $J \subset I$ such that $\psi(I) = O(\log |S|) \cdot \psi(J)$.

**Proof.** Index all values by rank such that $S_1 \leq S_2 \leq \ldots \leq S_m$. We now construct a clustering of $[m]$ as per Claim 7.3, and output $\mathcal{C}^S_{\Delta} \triangleq \{[S_a,S_b] \mid [a,b] \in \{S_{\Delta_{i}}\}_{i}\}$. Then all the above guarantees are immediate (the last guarantee is obtained by sub-additivity).

---

**Algorithm 5: EstimateBlockLIS**

- **Input:** A Precision-Tree $T = T_{1/c}(y)$, where $y \in \mathbb{N}^{X \times [k]}$ consists of $|X|$ blocks with at most $k$ integers each, value range $Y \subseteq \mathbb{N}$, error parameter $\lambda \in (1/n,1)$, branching parameter $\tau$.
- **Output:** An integer $\hat{L} \in [0, \text{Block-LIS}(y, X, Y)]$.

1. if $|X| < c$ then
   2. return $\hat{L} \leftarrow \text{Block-LIS}(y, X, Y)$ through exact computation.
3. $(X_*, S_*) \leftarrow \text{SampleAndPartition}(T, \tau, \frac{\lambda}{c}).$
4. $Y_{i,*} \leftarrow$ sets of intervals by invoking Corollary 7.4 using each $S_i \in S_*$.
5. return $\hat{L} \leftarrow \max_{\Delta} \{\text{Decompose}(T, X_*, Y_{*,\Delta})\}$.

---

Figure 6: Description of the algorithm EstimateBlockLIS.

### 7.3 Analysis

Before analyzing the estimator $\hat{L}$, we show several important properties. First, we claim that partitioning $X$ into intervals of smaller size and matching each one with a value range $Y$ monotonically cannot over-estimate the overall LIS:

**Lemma 7.5** (Upper Bound). Fix an interval $Y \subset \mathbb{N}$. Let $X_1, \ldots, X_{\tau}$ be a partition of $X$ into consecutive, disjoint, in-order block intervals (i.e., $X_1 < X_2 < \ldots < X_{\tau}$) and let $P \subset [\tau] \times \mathcal{I}$ be an arbitrary monotone set. Then, we have $\sum_{(i,Y_i) \in P} \text{Block-LIS}(y, X_i, Y_i \cap Y) \leq \text{Block-LIS}(y, X, Y)$.

**Proof.** For each $(i,Y_i) \in P$, let $\text{OPT}_i \subseteq (X_i,*) \cap y^{-1}(Y_i \cap Y)$ be a set of coordinates of an optimal increasing subsequence for $\text{Block-LIS}(y, X_i, Y_i \cap Y)$. Then $\sum_{(i,Y_i) \in P} |\text{OPT}_i| = \sum_{(i,Y_i) \in P} \text{Block-LIS}(y, X_i, Y_i \cap Y)$. On the other hand, since $P$ is a monotone set, we have $y(p) < y(q)$ for any $p \in \text{OPT}_i, q \in \text{OPT}_j$ with $i < j$. We conclude that the sequence $U_i y(\text{OPT}_i) \subseteq Y$ forms an increasing subsequence of $y(X)$ of length $\sum_{(i,Y_i) \in P} \text{Block-LIS}(y, X_i, Y_i \cap Y)$ and the claim follows. ☐
Next, we argue three essential invariants when constructing candidate intervals. These guarantees will be used later to lower bound our estimator:

**Lemma 7.6** (Candidate intervals guarantees). Fix input $X, Y, y$. With high probability, all the following invariants hold for the EstimateBlockLIS algorithm:

1. For all $i, \Delta$ and all $Y' \in \mathcal{Y}_{i, \Delta}$,
   \[ |y_{-\lambda/2}(X_i, Y')| = O(\lambda \Delta r) \, . \]
2. For all $\Delta$,
   \[ \mathbb{E}_{i \in [r]} ||\mathcal{Y}_{i, \Delta}|| = O\left(\frac{|y \cap Y|}{\lambda \Delta |X|}\right) \, , \]
3. There exist candidate intervals $Y' \in \bigoplus_{i \in [r]} \bigcup_{\Delta \in E_2(poly(k/\lambda))} \mathcal{Y}_{i, \Delta}$ such that $\{(i, Y'_i)\}_i$ is a monotone set; and
   \[ \sum_i \text{Block-LIS}(y, X_i, Y'_i) = \Omega\left(\frac{\text{Block-LIS}(y, X, Y) - \lambda |X|}{\log(k/\lambda)}\right) \, . \]

We analyze the correctness and complexity of our algorithm, proving Lemma 7.1 assuming Lemma 7.6. We prove Lemma 7.6 in Section 7.4.

Proof of Lemma 7.1 using Lemma 7.6. For all $i$ and all $Y' \in \mathcal{Y}_{i, \epsilon}$, let $\ell_{i,Y'} \triangleq \text{Block-LIS}(y, X_i, Y')$. By Lemma 7.6 (2) and (3), we have that for some $\Delta^*$, there exists a monotone set of pairs $P^* \in \bigoplus_i \{(i) \times \mathcal{Y}_{i, \Delta^*}\}$ such that:

\[ \sum_{(i, Y') \in P^*} \ell_{i,Y'} \geq \Omega\left(\frac{\text{Block-LIS}(y, X, Y) - \lambda |X|}{\log(k/\lambda)}\right) > \frac{\Omega(\text{Block-LIS}(y, X, Y) - \lambda |X|)}{\log^3(k/\lambda)} \, . \]

Therefore, we have for any $\alpha_s > 1$, some $\rho^* \in E_2(1/\lambda)$ for which\(^6\)

\[ \sum_{(i, Y') \in P^*} 1\left[\ell_{i,Y'} \geq 2\alpha_s \cdot \kappa_{\rho^*}\right] \geq \frac{1}{2\alpha_s \cdot \kappa_{\rho^*}} \cdot \frac{\Omega(\text{Block-LIS}(y, X, Y) - \lambda |X|)}{\log^3(k/\lambda)} \, . \]  

(3)

Let us first argue correctness. Define $\kappa^* \triangleq \kappa_{\rho^*} = \frac{r}{\rho^*}$ and $\mathcal{Y}_{i, \Delta^*} \triangleq \mathcal{Y}_{i, \Delta^*}$. Recall the following guarantees of $\mathcal{A}_{\text{BL}}$ and $\mathcal{A}_{\text{GL}}$ which hold for each pair $(i, Y') \in P^*$:

1. Let $\tilde{\ell}_{i,Y'}$ be the output of $\mathcal{A}_{\text{BL}}(y(X_i), Y', \frac{1}{\rho^*}, \tau)$. Then $\tilde{\ell}_{i,Y'}$ is a $(\alpha_s, \frac{r}{\rho^*})$-approximation of $\ell_{i,Y'}$, where $\alpha_s \triangleq a_s\left(\tau, \tau, \frac{r}{\rho^*} |y(X_i)\cap Y'|\right)$.
2. Let $\Lambda^* \triangleq \Lambda_{\Delta^*, \rho^*}$. The output of $\mathcal{A}_{\text{GL}}(\Lambda^*, \lambda \rho^*)$ is a $(\alpha_g, \lambda \rho^* \tau)$-approximation of $\sum_{(i, Y') \in P^*} 1 \left[\tilde{\ell}_{i,Y'} \geq \kappa^*\right]$, where $\alpha_g \triangleq a_g\left(\tau, \lambda \rho^*, \frac{Y'|Y_i|}{\mathbb{E}_i[|Y_i|]}\right)$.

We use the approximation guarantees from above to obtain a lower bound on $L^* \triangleq L_{\Delta^*, \rho^*}$. Assuming $c \geq c_\alpha(\cdot) \cdot c_\rho(\cdot)$, we invoke Lemma 5.4 to get that the quantity $L^*$ is a $(\alpha_g, \lambda \rho^* \tau)$-approximation for a Genuine-LIS instance, where each block $i$ is defined by all possible candidate intervals $\mathcal{Y}_{i}^*$, and each candidate interval $Y' \in \mathcal{Y}_{i}^*$ is genuine iff a $(\alpha_s, \frac{r}{\rho^*})$-local approximation $\tilde{\ell}_{i,Y'}$ passes the $\kappa_{\rho^*}$ threshold.

\(^6\)Recall that $\kappa_{\rho^*} = r/\rho^*$.
Now, to count those quantities together, we use Lemma 7.6 (1) and (2), to obtain for each $Y^i \in \mathcal{Y}_t^*$, for all $i$:

$$|y(X_i) \cap Y^i| = O\left(\frac{\lambda\kappa^*}{\tau} \cdot |X| \right);$$

and also, that:

$$\frac{m_g}{n_g} = \mathbb{E}_i [||\mathcal{Y}_t^*||] = O\left(\frac{|y(Y)|}{\lambda^* \Delta |X|} \right).$$

Where $n_g = \tau$ and $m_g = \sum_i [||\mathcal{Y}_t^*||]$, are, respectively, the number of blocks and non-null entries in the Genuine-LIS instance as above. So we have:

$$\alpha_s = a_s \left(\frac{r, \tau, r}{\rho^* |y(X_i) \cap Y^i|} \right) = a_s \left(\frac{r, \tau, 1}{\lambda^* \Delta \rho^*} \right).$$

$$\alpha_g = a_g \left(\frac{\lambda^* \rho^*}{\mathbb{E}_i [||\mathcal{Y}_t^*||]} \right) = a_g \left(\frac{\lambda^* \rho^*}{|y(Y)|} \right).$$

Therefore, letting $\lambda_s = \frac{|y(\mathcal{Y})|}{\lambda^* \Delta \rho^* |X|}$ and $\lambda_g = \lambda \rho^*$, we obtain that $\lambda = O\left(\frac{\lambda_s \rho^* r}{m_g}\right)$, and also that:

$$\alpha_s \cdot \alpha_g = a_s \left(\frac{r, \tau, \lambda^* |X| |y(Y)|}{|y(\mathcal{Y})|} \right) \cdot a_g \left(\frac{\lambda^* \rho^* |y(Y)|}{m_g}\right).$$

We can finally derive the following overall bound on $L^*$, and in particular, show it is a $(\alpha, \lambda |X|)$-approximation, for some $\alpha = O\left(\alpha_s \cdot \alpha_g \cdot \text{polylog}(k/\lambda)\right)$. For the lower bound, we have that:

$$L^* \geq \left(\frac{1}{\alpha_s} \sum_{(i,Y^i) \in P^*} \mathbb{1}_i \left[\ell_{i,Y^i} \geq \kappa^* + \lambda \rho^* \tau \right] - \lambda \rho^* \tau \right) \cdot 0.5 \kappa^*$$

$$\geq \sum_{(i,Y^i) \in P^*} \left(\frac{1}{\alpha_g} \mathbb{1}_i \left[\ell_{i,Y^i} \geq \alpha_s \cdot \left(\kappa^* + \frac{r}{\rho^*}\right) \right] - \lambda \rho^* \tau \right) \cdot 0.5 \kappa^*$$

$$\geq \frac{0.5 \kappa^*}{\alpha_g} \cdot \sum_{(i,Y^i) \in P^*} \left(\mathbb{1}_i \left[\ell_{i,Y^i} \geq \alpha_s \cdot \left(\kappa^* + \frac{r}{\rho^*}\right) \right] \right) - 0.5 \kappa^* \cdot \lambda \rho^* \tau$$

$$\geq \frac{0.5 \kappa^*}{\alpha_g} \cdot \sum_{(i,Y^i) \in P^*} \left(\mathbb{1}_i \left[\ell_{i,Y^i} \geq \alpha_s \cdot \left(\kappa^* + \kappa^*\right) \right] \right) - 0.5 \kappa^* \cdot \lambda \rho^* \tau$$

$$\geq \frac{0.5 \kappa^*}{\alpha_g} \cdot \frac{1}{2 \alpha_s \rho^*} \cdot \frac{\Omega(\text{Block-LIS}(y,X,Y)) - |\lambda |X|}{\log^4(k/\lambda)} - 0.5 \lambda |X|$$

$$\geq \Omega\left(\frac{\text{Block-LIS}(y,X,Y)}{\alpha_s \alpha_g \log^4(k/\lambda)}\right) - \lambda |X|.$$
Second, **computation of candidate intervals**, takes time \( \tilde{O}(|S|) = \tilde{O}(\frac{|Y| \cdot |Y'|}{X'}) \).

Last, and most important, are the recursive calls invoked by the **decomposition procedure**. Here, for each set of parameters \( \Delta, t \), the expected runtime by Lemma 5.4 is

\[
\tilde{O}\left( \tau + t_{GL} + \frac{1}{\delta \tau} \cdot \|t_f\|_1 \cdot \log^{O(\log_{\beta}(\tau))}(\beta) \right),
\]

where \( \delta = \frac{1}{c_g} \). The Lemma assumption provides the bound

\[
t_{GL} = \tilde{O}(m_g + n_g) = \tilde{O}(\tau + \sum_i |Y_i, \Delta|) = \tilde{O}(\tau + |S|) = \tilde{O}(\frac{|Y| \cdot |Y'|}{X'}),
\]

and hence the main quantity to analyze is \( \frac{1}{\delta} \cdot \|t_f\|_1 \cdot \log^{O(\log_{\beta}(\tau))}(\beta) \).

Notice that computing each \( f_i \) requires \( |Y_i, \Delta| \) invocations of \( A_{BL} \), where for each \( Y' \in Y_i, \Delta \), we pay (expected) time \( t \cdot c_s(\cdot) \cdot \frac{|y(X_i) \cap Y'|}{|X_i|} \). Furthermore, note that candidate intervals in \( Y_i, \Delta \) are disjoint and contained in \( Y \), and hence \( \sum_{Y' \in Y_i, \Delta} |y(X_i) \cap Y'| \leq |y(X_i) \cap Y| \). Therefore:

\[
\mathbb{E}_T \left[ \|t_f\|_1 \right] \leq \sum_i \sum_{Y' \in Y_i, \Delta} t \cdot c_s \cdot \frac{|y(X_i) \cap Y'|}{|X_i|} = \sum_i t \cdot c_s \cdot \frac{|y(X_i) \cap Y'|}{|X'|} = t \cdot c_s \cdot \frac{|y(Y) |}{|X'|}.
\]

Now, since \( c \geq c_s \cdot c_g \), then,

\[
\mathbb{E}_T \left[ \frac{1}{\delta} \cdot \|t_f\|_1 \right] = \frac{c_g}{\tau} \cdot \mathbb{E}[\|t_f\|_1] \leq t \cdot c_s \cdot c_g \cdot \frac{|y(Y) |}{|X'|} \leq t \cdot c \cdot \frac{|y(Y) |}{|X'|}.
\]

Counting the time taken by all procedures above, the overall expected runtime is:

\[
\tilde{O}\left( \tau + t \cdot c \cdot \log^{O(\log_{\beta}(\tau))}(\beta) \right) \cdot \frac{|y(Y) |}{|X'|} = t \cdot c \cdot \log^{O(\log_{\beta}(\tau))}(\beta) \cdot \frac{|y(Y) |}{|X'|},
\]

as needed.

\[
\square
\]

### 7.4 Analysis of Block-LIS invariants - proof of Lemma 7.6

To prove Lemma 7.6, we state the following claim:

**Claim 7.7.** The algorithm *SampleAndPartition* generates, with high probability, \( \{(X_i, S_i)\} \), pairs which satisfy the following for all \( i \in [\tau] \) and \( Y' \in \mathcal{I} \):

1. \( \sum_{w \in X_i} 1[w \cap Y' \neq \emptyset] \leq \lambda r \cdot (|Y' \cap S_i| + 3) \)
2. \( |y_{-\lambda/2}(X_i, Y')| \leq 2\lambda r \cdot (|Y' \cap S_i| + 3) \).

We now prove Lemma 7.6 using Claim 7.7.

**Proof of Lemma 7.6.** Fix \( i, \Delta \). By Corollary 7.4 (1), we have for all \( Y' \in Y_i, \Delta \) that \( |Y' \cap S_i| = \Delta + 1 \), and we use Claim 7.7 (2), to obtain the first invariant.

For the second invariant, show a bound on \( \mathbb{E}_{i \in [\tau]} \|Y_i, \Delta\| \) using the random sets \( W, B, S_s \) computed in the *SampleAndPartition* algorithm. Let \( S = \cup_i S_i \), we would like to show high probability bound on \( |S| \). Note that \( |S| \leq |W| + (B \times |k|) \). Now, each block is subsampled independently with probability \( \left( \frac{\zeta}{X'} \right) \). Then, \( \mathbb{E}[|W|] = \zeta \frac{|y(Y) |}{X'} \). For our task, we invoke Fact 6.6, which implies \( |S| = O\left( \frac{|y(Y) |}{X'} \right) \) w.h.p.

We also invoke Corollary 7.4 (2) to obtain \( |Y_i^*| = \frac{|S_i|}{\Delta} \), which implies

\[
\mathbb{E}_i [\|Y_i^*\|] = \mathbb{E}_i \left[ \frac{|S_i|}{\Delta} \right] = \frac{|S|}{\Delta \tau} = O\left( \frac{|y(Y) |}{X \Delta X'} \right),
\]

27
as needed.

For the third invariant, let OPT be (the indices of) an optimal increasing subsequence of Block-LIS\((y, X, Y)\). Let \(M_i \triangleq \max\{y(X_i \cap \text{OPT})\}\) be the maximal value of \(y(\text{OPT})\) in \(X_i\) (by convention, \(M_i = \perp\) if \(\text{OPT} \cap X_i = \emptyset\)). Let \(U_i \triangleq (M_j, M_i]\) for the largest \(j < i\) such that \(M_j\) is defined (similarly, let \(U_i = \perp\) if \(\text{OPT} \cap X_i = \emptyset\)). Last, define
\[
R_i \triangleq \operatorname{argmax}_{[j,k] \subseteq U_i \setminus \{j,k\} \in S_i} (k-j),
\]
i.e., the largest interval in \(U_i\) using the values of \(S_i\) only. We have the following bound on \(\text{Block-LIS}(y, X, Y)\):
\[
\text{Block-LIS}(y, X, Y) = \sum_i \text{Block-LIS}(y, X_i, U_i) \leq \sum_i \left(\text{Block-LIS}(y, X_i, R_i) + \text{Block-LIS}(y, X_i, U_i \setminus R_i)\right).
\]
Now, notice that by definition of \(R_i\), \(U_i \setminus R_i \cap S_i = \emptyset\). Notice also that \(U_i \setminus R_i\) consists of at most 2 intervals. We invoke Claim 7.7 (1) on each one of those intervals to obtain that \(\sum_{w \in X_i} 1[w \cap U_i \setminus R_i] \leq 4\lambda r\), implying that \(\text{Block-LIS}(y, X_i, U_i \setminus R_i) \leq 4\lambda r\) as well. We therefore have:
\[
\sum_i \text{Block-LIS}(y, X_i, R_i) \geq \text{Block-LIS}(y, X, Y) - 4\lambda r \cdot \tau = \text{Block-LIS}(y, X, Y) - 4\lambda|X|
\]
It remains to bound \(\sum_i \text{Block-LIS}(y, X_i, R_i)\). Here, we use Corollary 7.4 (3), with \(\psi(j) = \text{Block-LIS}(y, X_i, J)\) and \(I = R_i\), to obtain \(\text{Block-LIS}(y, X_i, R_i) \leq O(\log |S_i|) \cdot \text{Block-LIS}(y, X_i, Y_i)\), for some \(R_i \geq Y_i \in \mathcal{Y}_{i,\Delta}\).

Last, to avoid “too large \(\Delta\)”, we simply do not count intervals with many sampled pivots (for the rest \(\mathcal{Y}_{i,\Delta}\) is empty), absorbing them into the additive error. Overall:
\[
\sum_{i:|S_i| \leq k/\lambda^2} \text{Block-LIS}(y, X_i, Y_i) \geq \sum_{i:|S_i| \leq k/\lambda^2} \frac{\text{Block-LIS}(y, X_i, R_i)}{O(\log |S_i|)} \\
\geq \sum_{i \in [\tau]} \frac{\text{Block-LIS}(y, X_i, R_i) - \lambda \cdot \tau \cdot \tau}{O(\log(k/\lambda))} \\
= \Omega\left(\frac{\text{Block-LIS}(y, X, Y) - \lambda|X|}{\log(k/\lambda)}\right).
\]
We also note that since \(\{(i, R_i)\}_i\) are monotone, so are \(Y_i\). This concludes the proof.

It remains to prove Claim 7.7.

**Proof of Claim 7.7.** Fix \(Y' \in \mathcal{I}\). Assume w.l.o.g \(Y' \in \{(j,k) \mid \{j,k\} \in y(X_i) \cup \{-\infty, \infty\}\}\) (otherwise, consider the smallest interval in this set, which contains \(Y'\)). Notice that since there are at most \(\operatorname{poly}(n)\) such intervals, it suffices to prove this holds w.h.p. for a single \(Y'\) and the claim follows by the union bound.

To prove the claim, we also define \(\gamma \triangleq |Y' \cap S_i| + 3\) and use \(\gamma\) to get our bounds.

**Concentration Analysis for \(\gamma\).** First, we show concentration for \(\gamma\) using the quantities we care to bound. We observe that \(\gamma - 2 \in \frac{|p' \cap Y'| + |[0,2]|}{\zeta}\), and hence \(\gamma \in |p' \cap Y'| + [2, 4]\). Now, the quantity \(|p' \cap Y'\) can be lower bounded by \(\text{Bin}\left(\sum_{w \in X_i} 1[w \cap Y' \neq \emptyset], \frac{\tau\zeta}{|X|}\right) - \zeta\), which w.h.p. is at least:
\[
\tau\zeta \cdot \frac{\sum_{w \in X_i} 1[w \cap Y' \neq \emptyset]}{|X|} - 2\zeta = \zeta \cdot \left(\frac{\sum_{w \in X_i} 1[w \cap Y' \neq \emptyset]}{\lambda r} - 2\right).
\]
Hence \(\gamma \geq \sum_{w \in X_i} 1[w \cap Y' \neq \emptyset] / (\lambda r)\), and since \(|Y' \cap S_i| + 2 = \gamma\), we obtain part (1) of the claim.

To prove (2), we show that our sampling procedure guarantees that most blocks \(w \in X_i\) will have bounded number of values in \(Y'\).
All blocks in $y(X_i)_{-\lambda/2,Y'}$ contain $O(\gamma)$ elements in $Y'$. Let $T \triangleq \max_{w \in y(X_i)_{-\lambda/2,Y'}} |w \cap Y'|$. We argue $T = O(\gamma)$. By construction, we have $\frac{\gamma}{T^2}$ blocks in $y(X_i)$ with at least $T$ integers in $Y'$. So, the quantity $|p_i^* \cap Y'|$ is lower bounded by $\text{Bin} \left( \frac{\gamma}{T^2}, \frac{T}{\lambda |X|} \right)$. Hence, by Chernoff bound, we have

$$
\zeta \gamma \geq |p_i^* \cap Y'| \geq T \cdot \left( \frac{\gamma}{2} - O(\log n) \right) > \frac{\zeta T}{4}
$$

w.h.p. This implies that $T = O(\gamma)$.

Finally, observe that the quantity $|p_i^* \cap Y'|$ can also be lower bounded by a sum of $\zeta \tau/\lambda$ independent random variables with expectation $|y(X_i,Y')|/|X|$ and upper bounded by $O(\gamma)$, which w.h.p. is at least:

$$
|p_i^* \cap Y'| \geq \frac{\zeta}{\lambda} \cdot |y(X_i,Y')| - \frac{1}{2} \zeta \gamma.
$$

This implies that $\gamma/2 \geq |y(X_i,Y')|/(\lambda r)$, which further implies part (2) as well.

8 Genuine-LIS and Block-LIS Extensions Reducions, Proof of Lemmas 6.3 and 7.2

In this section we prove the reductions regarding the Genuine-LIS and Block-LIS extensions, namely Lemmas 6.3 and 7.2.

8.1 Block-LIS extension reduction

For the Block-LIS problem, all we need to show is that one can “ignore” not too many exceptionally large blocks. The algorithm is the natural one:

1. Using Corollary 5.3 and $T$, sub-sample blocks $S \subseteq X$ with i.i.d. probability $\frac{\zeta}{\lambda |X|}$, and let $W \leftarrow \{(i,j) \in S \times [k] \mid y_{i,j} \in Y\}$.

2. Compute $\Upsilon \leftarrow$ the $(1-0.6\lambda)$-quantile of $|\{(i,j) \in W\}|_{i \in S}$.

3. Run $A$ with $\lambda' = 0.3\lambda$. Whenever $A$ tries to access a block $i$ using $T$ we check first if $|y_i,^* \cap Y| > \Upsilon$, and if so, we treat the block as “empty” (say, by override).

Proof of Lemma 6.3. First, we argue that each block in $y(X)_{-\lambda/2,Y}$ has at least $\Upsilon$ elements in $Y$. Indeed, that holds w.h.p. by the Chernoff bound. Therefore, the instance we approximate is contained in $y(X,Y)$. It remains to show the additive error bounds. Here again, by Chernoff bound, we “omit” at most $0.7\lambda n$ blocks, and hence the total additive error is at most $0.3\lambda n + 0.7\lambda n = \lambda n$. This concludes the proof.

8.2 Genuine-LIS extension reduction

For the Genuine-LIS problem, we need to show two extensions:

Sparse instances. When the instance $g$ consists of $m \ll nk$ non-null entries\(^7\), we would like to improve the approximation and runtime bounds to be a function of $m/n$ instead of $k$.

To obtain the improved bound, we first exponentially discretize the blocks based on the number of non-null entries (denoted the block size $m_i$) and output the maximum over all instances. For each scale of block size $b \in E_2(k)$, we use the following extended version of Genuine-LIS:

\(^7\)Note that unlike “sparse genuine” instances where we have few genuine elements, here we have few elements overall.
Lemma 8.1 (EstimateGenuineLIS algorithm, sparse version). Suppose the algorithm EstimateGenuineLIS \((f(\lambda/k), \lambda n)\)-approximates Genuine-LIS for some function \(f\) in time \(t(n \cdot k)\), then there exists an algorithm \(A\) that \((O(f(\lambda n/dk)), \lambda n)\)-approximates Genuine-LIS with \(d\) non-empty blocks. The run-time is \(t(dk) + O(n)\).

We adapt algorithm EstimateGenuineLIS as follows. Let \(D\) be the set of non-empty blocks. We proceed by following the same algorithmic steps, using parameters \(n' \triangleq d\) and \(\lambda' \triangleq \lambda n/d\).

Proof. It suffices to show that the algorithm output guarantees do not deteriorate from the case where the input is \(g \in \mathbb{N}^{d \times k}\) with parameter \(\lambda'\) as above and does not include the empty blocks. Note that the additive error is \(\lambda n = \lambda' d\).

Indeed, for the dense estimator, we only count \(i \in D\) which are subsampled, and each one is independently subsampled with the same probability as above.

For the sparse estimator, we note that the guarantees of Theorem 4.3 are with respect to \(|Y \cap y|/|X|\), and increasing \(n\) can only help, since it gives us greater flexibility for choosing \(\tau \in E_\beta\).

The runtime is immediate, as the only overhead is skipping empty blocks.

Genuine-LIS over interval space. We also need to overcome the obstacle that the space \(\mathcal{I}\) admits a partial order relation only, as opposed to \(\mathbb{N}\), which admits a total order relation. Our ideal solution involves a mapping \(\varphi : \mathcal{I} \to \mathbb{N}\) that approximately preserves the overall LIS over all subsequences (with probability 1), and therefore also preserves the overall LIS over the genuine elements. We first note that such a mapping is applied a priori on the first coordinates of the input, and is not affected by the genuineness flags, nor does it require access to any genuineness samples.

While we are unable to exhibit a single mapping for the entire space, we show that one can discretize an arbitrary Genuine-LIS instance over intervals into \(\log(k/\lambda)\) “nicer” instances \(G_1, \ldots, G_{\log(k/\lambda)}\), which are mutually disjoint and whose union covers most of the original Genuine-LIS intervals (the unused intervals are discarded, incurring some additive error). Each such “nicer” instance is then mapped to a Genuine-LIS instance over integers, incurring a constant factor approximation over all its subsequences. The final output is the maximum Genuine-LIS over all integral instances \(\{\varphi_i(G_{i,1}), G_{i,2}\}_i\). This costs us another \(\log(k/\lambda)\)-factor approximation.

In particular, we use the following claim:

Claim 8.2. For \(\ell \in \mathbb{N}\), define \(\mathcal{I}_\ell\) to be the collection of all intervals in \(\mathcal{I}\) whose length is in \([\ell, 2\ell]\). Then, there exists a mapping \(\varphi_\ell : \mathcal{I}_\ell \to \mathbb{N}\), which can be computed in constant time, such that:

1. For \(I, J \in \mathcal{I}_\ell\): \(I < J \Rightarrow \varphi_\ell(I) < \varphi_\ell(J)\); and,

2. For all increasing sequences \(y \in \mathbb{N}^m\) and all \(q \in \mathcal{I}_\ell^m\) such that \(\varphi_\ell(q) = y\), \(q\) must have an increasing subsequence of length \(m/3\).

Proof. We use the mapping \(\varphi_\ell(I) \triangleq \lfloor \min(I)/\ell \rfloor\). First we show (1). We have

\[
I < J \Rightarrow \min(I) \leq \max(I) - \ell + 1 \leq \min(J) - \ell,
\]

which implies that \(\varphi_\ell(I) < \varphi_\ell(J)\).

For (2), fix \(y\) as above, and for \(i \in \{0, 1, 2\}\), let \(y^i\) be the sequence \(y\) restricted to integer values \(y_w\) satisfying \(y_w \pmod{3} = i\). Let \(q^i\) be the corresponding subsequences of \(q\). Then each subsequence \(q^i\) is increasing. To see this, consider two intervals \(q^i_j\) and \(q^i_{j+1}\). Then \(\max(q^i_j) < \min(q^i_j) + 2\ell\). On the other hand, \(\min(q^i_{j+1}) > \min(q^i_j) + 2\ell\), as \(y^i_{j+1} \geq y^i_j + 3\). So, \(\max(q^i_j) < \min(q^i_{j+1})\).

Since \(|q^0| + |q^1| + |q^2| = m\), at least one of them satisfies the length promise as well.
Proof of Lemma 7.2. Let $A$ be such that $A(b^{(\ell)})$ is a $(\alpha, \lambda n/2)$-approximation for $\text{Genuine-LIS}(b^{(\ell)})$ in time $t$. We need to show the reduction algorithm provides a $(O(\log(1/\alpha), \lambda n))$-approximation for $g$.

For the upper bound, we use Claim 8.2, item (1), to map each interval to an integer preserving its strict order relation with all other intervals. This implies that we always output an upper bound of the original interval LIS, as we only use subsets of elements of the original sequence, and do not add elements.

For the lower bound, first consider intervals of length at least $2k/\lambda$, which do not participate in any $b^{(\ell)}$. Any increasing subsequence of intervals contained in $[1, 2nk]$ can have at most $\lambda n/2$ intervals of such magnitude, hence by ignoring such intervals we “omit” at most $\lambda n/2$ LIS elements (additive error). Now, consider all intervals of length at most $2k/\lambda$. Each such interval participates in some instance $b^{(\ell)}$, and hence there exists an instance such that the intervals participating in it $(\log(k/\lambda), \frac{\lambda}{2}n)$-approximates the original instance. Now, using Claim 8.2, item (2), this instance is mapped to an integer instance which $(O(\log(k/\lambda)), \frac{\lambda}{2}n)$-approximates the original instance. Therefore, the lower bound holds for at least one $b^{(\ell)}$.

Next, we argue that partitioning $b^{(\ell)}$ into $\log(k)$ instances of similar block size guarantees that at least one instance has $1/\log(k)$ fraction of the LIS. Finally, we use the guarantees of Lemma 8.1, together with the above bounds on $b^{(\ell)}_B$, to conclude that $A(b^{(\ell)}_B)$ outputs a $(\alpha \cdot O(\log(1/\lambda)), \lambda n)$-approximation of the original LIS. This concludes the proof.

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A Proof of Theorem 4.3

The proof is by recursive application of Theorems 4.1 and 4.2 for solving Block-LIS, up to approximation \( a \left( \frac{|X|}{|Y|}, r, \frac{\lambda|X|}{|Y|} \right), \lambda|X| \), with access to a Precision-Tree \( T_{1/c} \), for \( 
abla \|X\| \), \( \lambda \|X\| \), \( \gamma \|X\| \), \( \frac{|X|}{|Y|} \).

We will establish the recurrences for quantities \( a(\cdot, \cdot, \cdot) \) and \( c(\cdot, \cdot, \cdot, \cdot) \).

Claim A.1. Fix \( \epsilon < 1 \) and \( \xi, \theta, n \in \mathbb{N} \). Consider the following recursive formulas:

\[
\begin{align*}
    a(m, \tau, 1/L) & \le \xi \tau \max_{t \in [1, L/\tau]} \max \left\{ \gamma, a(\tau, \tau^\epsilon, \gamma) \right\} \cdot a(m/\tau, \tau, \Omega(t/L)); \\
    c(m, \tau, 1/D, 1/L) & \le \theta \tau \cdot D + \max_{r \in [1, D]} \left( \tilde{O}(r) + c(\tau, \tau^\epsilon, 1/r, \gamma/t) \right) \cdot c(m/\tau, \tau, \Omega(r/D), \Omega(t/L)).
\end{align*}
\]

With the following base cases:

\[
\begin{align*}
    a(\cdot, \cdot, 1) = 1 & \quad a(m, \cdot, 1/m) = 1 & \quad a(1, \cdot, \cdot) = 1 \\
    c(\cdot, \cdot, \cdot, 1) = 1 & \quad c(1, \cdot, \cdot, \cdot) = 1 & \quad c(m, \cdot, \cdot, 1/m) = m
\end{align*}
\]
Then, there exists a function $\gamma(t, n, m, \epsilon)$, such that substituting $\gamma = \gamma(t, n, m, \epsilon)$ in the recursive formulas above yield the following upper bounds, for $H \triangleq \epsilon^{\Theta(1/\sqrt{\log 1/\epsilon})}$:

$$
a(n, n^\epsilon, 1/L) = L^{\sqrt{t}} \cdot \xi^{O(H^2)}$$
$$
c(n, n^\epsilon, 1/D, 1/L) = D \cdot m^{O(c \log(1/\epsilon)(H))} \cdot \theta^{O(H)}.
$$

The proof of Claim A.1 is deferred to Appendix A.1.

To prove Theorem 4.3, we deduce the recurrences on approximation and Precision-Tree complexity showing they match the ones from Claim A.1, from which we directly obtain the final bounds on approximation. Then we analyze the runtime, where we use the bound established on $c(\cdot, \cdot, \cdot, \cdot)$ as well.

**Proof of Theorem 4.3.** To deduce the recurrences, we first note that for base cases, we have $a(\cdot, \cdot, 1) = 1, c(\cdot, \cdot, 1) = 1$ as in this case $\lambda = 1$ and additive approximation $|X|$ is trivial. Similarly, for $|X| = 1$ (ie 1 letter string), we have $a(1, \cdot, \cdot) = c(1, \cdot, \cdot, \cdot) = 1$, as we can compute if LIS is 1 or 0 with access to a 1 bit string. Similarly, we need full access for exact approximation, and hence: $a(m, \cdot, 1/m) = 1, c(m, \cdot, \cdot, 1/m) = m$. For the other cases, we show the following:

**Approximation.** For convenience, let $m \triangleq |X|, L \triangleq \frac{\log(|X|Y)}{\lambda |X|} \leq k/\lambda$. We apply Theorem 4.2. Define $t \triangleq k_g/\lambda_g$, in which case $\frac{\lambda_g |X|}{\lambda_g |X|} = \Omega(t/L)$. Note that if $t > \Omega(L)$, then, we get (base case) $a(|X|, \cdot, 1) = 1$, and hence $a(m, \tau, 1/L) \leq (\log k/\lambda)^{O(1)} \cdot \max_{t \leq 1/\lambda} a_g(\tau, 1/t)$. Thus, including when $t$ is smaller, we have that:

$$
a(m, \tau, 1/L) \leq (\log k/\lambda)^{O(1)} \cdot \max_{t \in [1, O(L)]} a_g(\tau, 1/t) \cdot a(m/\tau, \tau, \Omega(t/L)).
$$

Plugging in $a_g$ from Theorem 4.1, (with $\tau_s = n^\epsilon$), we get, for our choice of $\gamma$:

$$
a(m, \tau, 1/L) \leq (\log k/\lambda)^{O(1)} \cdot \max_{t \in [1, O(L)]} \{ \gamma, a(\tau, \tau^\epsilon, \cdot) \} \cdot a(m/\tau, \tau, \Omega(t/L)).
$$

**Precision-Tree Complexity.** We now deduce the complexity bound $c$. Note that the recursion for $c$ will be using same parameters as above. We need to keep track of new parameters, in particular let $D = 1/\lambda$. We have, where $\lambda_s, \lambda_g, k_g$ are as in approximation bound above, with $\lambda_s \lambda_g = \Omega(k_g \lambda)$:

$$
c(m, \tau, 1/D, 1/L) \leq \text{polylog}(n) \cdot \tau D + (c_g(\tau, \lambda_g, k_g) \cdot c(m/\tau, \tau, \Omega(\lambda_s/k_g), \Omega(t/L))
$$

Here we set $r = 1/\lambda_g \in [1, D]$ and hence:

$$
c(m, \tau, 1/D, 1/L) \leq \text{polylog}(n) \cdot \tau L + c_g(\tau, 1/r, k_g) \cdot c(m/\tau, \tau, \Omega(r/D), \Omega(t/L)).
$$

Plugging in $c_g$ from Theorem 4.1, (again, with $\tau_s = n^\epsilon$), we get:

$$
c(m, \tau, 1/D, 1/L) \leq \text{polylog}(n) \cdot \tau D + \max_{r \in [1, D]} \left( \tilde{O}(r) + c(\tau, \tau^\epsilon, 1/r, \gamma/t) \right) \cdot c(m/\tau, \tau, \Omega(r/D), \Omega(t/L)),
$$

We now apply Claim A.1, with $\theta = \text{polylog}(n)$ and $\xi = \text{polylog}(k/\lambda)$ to obtain the bounds we need.

**Runtime complexity.** The recurrence relation $t_s(n, n^\epsilon) = k_s n^\epsilon$, where $k = \tilde{O} \left( \log^{O(\log^2(\tau))}(\beta) \right)$, with base case $t_s(n^\epsilon, n^\epsilon) = k$, can be expanded to obtain $T(n, n^\epsilon) = k^{1/\epsilon}$.

Therefore, the total expected runtime is $\kappa^{1/\epsilon} \cdot c(m, \tau, 1/D, 1/L) \cdot \frac{L}{D} = L \cdot m^{O(\sqrt{\log 1/\epsilon} \cdot \log n)^{2(1/\sqrt{\log^2 1/\epsilon})}} \cdot \tilde{O} \left( \log^{O(\log^2(\tau))}(\beta) \right)^{1/\epsilon} = \log^{O(\log^2(\tau))}(\beta) \cdot \log(n)^{O(1/\epsilon)}$. The preprocessing of the Precision-Tree takes time $c \cdot \tilde{O} \left( \log^{O(\log^2(\tau))}(\beta) \right)$, which is a lower order term.

\[ \square \]
A.1 Recursion Analysis

We will prove each recursion separately. Let \( b = n^{1/H} \). Our choice of \( \gamma = t^{\sqrt{r}} \), except for \( \tau = b \), then we set \( \gamma = t \) and get the base cases \( a(m, b, 1/L) \leq \xi \cdot \max_{t \in [1, O(L)]} \tau \cdot a(m/b, b, \Omega(t/L)) \) and \( c(m, b, 1/D, 1/L) \leq \theta \cdot b \cdot D + \max_{r \in [1, D]} \left( \tilde{O}(r) + c(m/b, b, \Omega(r/D), \Omega(t/L)) \right) \).

Proof of Claim A.1, function \( a \). Setting \( \tau = m^\epsilon \), the recursion formula can be written as:

\[
a(m, m^\epsilon, 1/L) \leq \xi \cdot \max_{t_1, ..., t_{1/\epsilon} \geq 1 \atop t_1 \cdot t_2 \cdot ... \cdot t_{1/\epsilon} \leq 2^{O(T)T}} \prod_i \max \{ \gamma_i, a(m^\epsilon, m^{\epsilon^2}, \gamma_i/t_i) \},
\]

where \( T = 1/\epsilon \) and \( \gamma_i = \gamma(t_i) \). Let \( f = T \cdot \log(\xi) \). Setting \( q_h(l) = \log_2 a(m^{h}, m^{h+1}, 1/2^l) \), we obtain from above, and substituting \( \gamma_i \) with \( \log_2 \gamma_i = \sqrt{t_i} \):

\[
q_h(l) \leq f + \max_{t_1 + t_2 + ... + t_T \leq l + O(T)} \sum_i \max \{ \gamma_i, q_{h+1}(t_i - \gamma_i) \}.
\]

Now let \( H' = \log_T(H) \) and hence \( b = n^{1/H} = n^{H'}. \) Then \( q_{H'}(l) \leq f + l + O(\tilde{T}) \) by the base case. Now for \( h \leq H' \) we prove, by induction, that

\[
q_h(l) \leq (f + O(\tilde{T})) \cdot T^{2^{(H'-h)}} + (1 - \sqrt{\epsilon})^{H'-h} \cdot l.
\]

Indeed, we first check the base case for \( h = H' \): \( q_{H'}(l) \leq f + l + T \). Now note that \( q_h(t_i - \gamma(t_i)) \leq f \cdot T^{H'-h} + (1 - \sqrt{\epsilon})^{H'-h+1} \cdot t_i \). Hence, we can verify the inductive step for \( h < H' \):

\[
q_h(l) \leq f + \max_{t_1 + t_2 + ... + t_T \leq l + O(T)} \sum_i \max \{ \gamma(t_i), q_{h+1}(t_i - \gamma(t_i)) \}.
\]

\[
\leq f + \max_{t_1 + t_2 + ... + t_T \leq l + O(T)} \sum_i (f + O(T)) \cdot T^{2^{(H'-h)-1}} + (1 - \sqrt{\epsilon})^{H'-h} \cdot t_i
\]

\[
\leq f + (f + O(T)) \cdot T^{2^{(H'-h)-1}} + (1 - \sqrt{\epsilon})^{H'-h} \cdot l + O(T)
\]

\[
\leq (f + O(T)) \cdot T^{2^{(H'-h)}} + (1 - \sqrt{\epsilon})^{H'-h} \cdot l
\]

where we omitted \( \gamma(t_i) \) from the max in the 2nd inequality since \( \sqrt{t_i} \) is smaller than the inductive hypothesis on \( q_{h+1}(t_i) \) (using the particular choice on the value of \( H \)).

In particular, this bound implies \( q_0(l) = O(f) \cdot T^{2^{H'} + \sqrt{\epsilon}l} = f \cdot O(H^2 \cdot T) + \sqrt{\epsilon}l \).

Substituting back, we obtain that \( a(m, m^\epsilon, 1/L) \leq 2^{O(HT \cdot f + \sqrt{T} \log(L))} \leq L^{\sqrt{T} \cdot \xi O(H^2)}. \)

Proof of Claim A.1, (function \( c \)). Similar to the previous proof, setting \( \tau = m^\epsilon \) with \( t_i, \tilde{T} \) from before too:

\[
c(m, m^\epsilon, 1/D, 1/L) \leq (\log(n))^{O(\tilde{T})} \cdot \left( \frac{\sum_{i=1}^{\tilde{T}} m^{\epsilon} D/r_1, r_1=1 \prod_{j<i} c(\tau, \tau^e, 1/r_j, \gamma(t_j)/t_j) \right),
\]

where the \( r_i \)'s satisfy \( r_1, ..., r_{\tilde{T}} \geq 1 \) and \( r_1r_2 ..., r_{\tilde{T}} = D \). We further can simplify by dividing by \( D \):

\[
\frac{1}{D} c(m, m^\epsilon, 1/D, 1/L) \leq (\log(n))^{O(\tilde{T})} \cdot \frac{m^\epsilon}{\log^2 n} \cdot \left( \sum_{i=1}^{\tilde{T}} \prod_{j<i} \frac{1}{r_j} c(\tau, \tau^e, 1/r_j, \gamma(t_j)/t_j) \right).
\]

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As before, let $c_h(D) = \frac{1}{D} c(m^h, m^{h+1}, 1/D, 1/L)$, for which we get:

$$c_h(D) = (\log n)^{O(\tau)} \cdot m^{h+1} \cdot \max_{j<i} \prod_{j<i} c_{h+1}(r_j),$$

with the base case $c_{H'}(D) = (\log n)^{O(1)}$ (when we are taking the maximal $\gamma$). Hence we obtain that

$$c_0(D) \leq (\log n)^{\tau^0(h')} \cdot m^{c \cdot H'}. \tag{37}$$

Plugging back into $c$ we obtain that:

$$c(m, m^\epsilon, 1/D, 1/L) \leq D \cdot (\log n)^{\tau^0(h')} \cdot m^{c \cdot H'}. \tag{38}$$

\qed