SHARP VERSION OF THE GOLDBERG-SACHS THEOREM

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Abstract. We reexamine from first principles the classical Goldberg-Sachs theorem from General Relativity. We cast it into the form valid for complex metrics, as well as real metrics of any signature. We obtain the sharpest conditions on the derivatives of the curvature that are sufficient for the implication \([\text{integrability of a field of alpha planes}] \Rightarrow \) (algebraic degeneracy of the Weyl tensor). With every integrable field of alpha planes we associate a natural connection, in terms of which these conditions have a very simple form.

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1. Introduction

The original Goldberg-Sachs theorem of General Relativity [4] is a statement about Ricci flat 4-dimensional Lorentzian manifolds. Nowadays it is often stated in the following, slightly stronger, form:

**Theorem 1.1.** Let \((M, g)\) be a 4-dimensional Lorentzian manifold which satisfies the Einstein equations \(\text{Ric}(g) = \Lambda g\). Then it locally admits a congruence of null and shearfree geodesics if and only if its Weyl tensor is algebraically special.

If \((M, g)\) is conformally flat, then such a spacetime admits infinitely many congruences of null and shearfree geodesics.

This theorem proved to be very useful in General Relativity, especially during the ‘golden era’ of General Relativity in the 1960s, when the important Einstein spacetimes, such as Kerr-Newman, were constructed.

Remarkably, years after the Lorentzian version was first stated, it was pointed out that the theorem has a Riemannian analog [21]. This gives a very powerful local result in 4-dimensional Riemannian geometry, which can be stated as follows [16, 15]:

**Theorem 1.2.** Let \((M, g)\) be a 4-dimensional Riemannian manifold which satisfies the Einstein equations \(\text{Ric}(g) = \Lambda g\). Then it is locally a hermitian manifold if and only if its Weyl tensor is algebraically special.

Note that the notion of a congruence of null and shearfree geodesics, in the Lorentzian case, is replaced by the notion of a complex surface with an orthogonal complex structure, in the Riemannian case. Also in this case, if \((M, g)\) is conformally flat, it admits infinitely many local hermitian structures.

Theorem 1.2 was in particular used by LeBrun [11] to obtain all compact complex surfaces, which admit an Einstein metric that is hermitean but not Kähler, (see also [3, 12]).

The only other signature which, in addition to the Lorentzian and Euclidean signatures, a four dimensional metric may have, is the ‘split signature’: \((+, +, -, -)\). It is again remarkable, that the Goldberg-Sachs theorem has also its split signature version. Here, however, the situation is more complicated and the theorem should be split into two statements:

**Theorem 1.3.** Let \((M, g)\) be a 4-dimensional manifold equipped with a split signature metric which satisfies the Einstein equations \(\text{Ric}(g) = \Lambda g\). If in addition \((M, g)\) is either locally a pseudohermitian manifold, or it is locally foliated by real 2-dimensional totally null submanifolds, then \((M, g)\) has an algebraically special Weyl tensor.

**Theorem 1.4.** Let \((M, g)\) be a 4-dimensional manifold equipped with a split signature metric which satisfies the Einstein equations \(\text{Ric}(g) = \Lambda g\) and which is conformally non flat. If in addition \((M, g)\) has an algebraically special Weyl tensor with a multiple principal totally null field of 2-planes having locally constant real index, then it is either locally a pseudohermitian manifold, or it is locally foliated by real 2-dimensional totally null submanifolds.

In these two theorems the term ‘pseudohermitian manifold’ means: ‘a complex manifold with a complex structure which is an orthogonal transformation for the split signature metric \(g\)’. The more complicated terms such as ‘multiple principal
Totally null field of 2-planes having locally constant real index' will be explained in Section 3.

All four theorems have in common the part concerned with the Einstein assumption and algebraic speciality of the Weyl tensor. But they look quite different on the other side of the equivalence. The similarity in the first part suggests that also the second part should have a unified description. This is indeed the case. As will be shown in the sequel, these theorems are consequences, or better said, appropriate interpretations, of the following complex theorem [19, 20]:

**Theorem 1.5.** Let \((M, g)\) be a 4-dimensional manifold equipped with a complex valued metric \(g\) which is Einstein. Then the following two conditions are equivalent:

1. \((M, g)\) admits a complex two-dimensional totally null distribution \(N \subset T^C M\), which is integrable in the sense that \([N, N] \subset N\).
2. The Weyl tensor of \((M, g)\) is algebraically special.

2. Convenient sharper versions

Our motivation for reexamining these theorems is as follows:

First, as remarked e.g. by Trautman [26], all the theorems have an aesthetic defect. This is due to the fact that both equivalence conditions, such as (i) and (ii) in Theorem 1.5, are conformal properties of \((M, g)\); the Einstein assumption does not share this symmetry. Of course, a way out is to replace the Einstein assumption by an assumption about \((M, g)\) being conformal to Einstein, see e.g. [3]. Thus, in the complex version of the theorem the assumption should be: \((M, g)\) is conformal to Einstein.

This leads to the question about the weakest conformal assumption involving (the derivatives of) the Ricci part of the curvature that is sufficient to ensure the thesis of the Goldberg-Sachs theorem. Several authors have proposed their assumptions here (see [9, 18, 22, 23, 24]). For example the authors of [9, 18, 24] use an assumption, which involves contractions of (the derivatives of) the Ricci tensor with the vectors spanning the totally null distribution \(N\).

Trautman in [26] has a different point of view. He proposes that there should be a conformally invariant assumption which does not refer to the thesis of the theorem. Trautman conjectures that a proper replacement for the assumption is: \((M, g)\) is Bach flat. This, in four dimension, is certainly conformal, does not refer to \(N\), and is necessary for \(g\) to be conformal to Einstein.

In this paper, among other things, we show that the approach of [9, 18, 24] is the proper one. In particular in Section 7.4 we show that, in the case of a Riemannian signature metric, Trautman’s conjecture is not true.

Our new analysis of the Goldberg-Sachs theorem starts with Theorem 5.10. Its proof shows that it is rather hard to find a single curvature condition, different than the conformally Einstein one, which would guarantee equivalence in the thesis of Goldberg and Sachs. This proof also clearly shows that it is the implication (algebraical speciality) \(\Rightarrow\) (integrability of totally null 2-planes) that causes the difficulties. Then in Section 5.2 we give various generalizations of the Goldberg-Sachs theorem to the conformal setting, starting with the conformal replacement of the assumption of Theorem 5.10 which implies (algebraical speciality) \(\Rightarrow\) (integrability of totally null 2-planes). This culminates in a slight improvement of the theorem of Penrose and Rindler [18], which we give in our Theorem 5.28, and in Theorems 5.31 and 5.32, which treat more special cases. These three theorems we
consider as the sharpest conformal improvement of the classical Goldberg-Sachs theorem, in a sense that they include both implications \((\text{algebraical speciality}) \Rightarrow (\text{integrability of totally null 2−planes})\) and \((\text{algebraical speciality}) \Leftrightarrow (\text{integrability of totally null 2−planes})\). In Section 7 the real versions of theorems from Section 5.2 are considered, the most striking of them being:

**Theorem 2.1.** Let \(\mathcal{M}\) be a 4-dimensional oriented manifold with a (real) metric \(g\) of Riemannian signature, whose selfdual part of the Weyl tensor is nonvanishing. Let \(J\) be a metric compatible almost complex structure on \(\mathcal{M}\) such that its holomorphic distribution \(\mathcal{N} = T^{(1,0)}\mathcal{M}\) is selfdual. Then any two of the following imply the third:

1. The Cotton tensor of \(g\) is degenerate on \(\mathcal{N}\), \(A_{\mathcal{N}} \equiv 0\).
2. \(J\) has vanishing Nijenhuis tensor on \(\mathcal{M}\), meaning that \((\mathcal{M}, g, J)\) is a hermitian manifold.
3. The selfdual part of the Weyl tensor is algebraically special on \(\mathcal{M}\) with \(\mathcal{N}\) as a field of multiple principal selfdual totally null 2-planes.

This theorem in its (more complicated) Lorentzian version is present in [9, 18, 24]. The Riemannian version is implicit there, once one understands the relation between fields of totally null 2-planes and almost hermitian structures, as for example, explained in [15, 16], (see also [1] where these developments are related to global issues on compact Riemannian manifolds.)

When one is only interested in the implication \((\text{algebraical speciality}) \Rightarrow (\text{integrability of totally null 2−planes})\), our proposal for the sharpest version of the Goldberg-Sachs theorem, is given in Theorem 5.21. This gets its final and very elegant (but equivalent) version in Theorem 6.5. This last theorem utilizes a new object which we introduce in this paper, namely a connection, which is naturally associated with each integrable field of totally null 2-planes \(\mathcal{N}\). We call this connection the characteristic connection of a field of totally null 2-planes.

If \(\mathcal{N}\) satisfies the integrability conditions \([\mathcal{N}, \mathcal{N}] \subset \mathcal{N}\), we prove in Theorem 6.1 the existence of a class of connections \(\nabla^w\), which are characterized by the following two conditions:

\[
\nabla^w X \mathcal{N} \subset \mathcal{N} \quad \text{for all } X \in T\mathcal{N}.
\]

\[
\nabla^w X g = -B(X)g
\]

These connections are not canonical - they define the 1-form \(B\) only partially. However, they naturally restrict to a unique (partial) connection \(\nabla\) on \(\mathcal{N}\). This by definition is the characteristic connection of \(\mathcal{N}\). In general this connection is complex. It is defined everywhere on \(\mathcal{M}\), but it only enables one to differentiate vectors from \(\mathcal{N}\) along vectors from \(\mathcal{N}\). Thus the connection \(\nabla\) is effectively 2-dimensional, and as such, its curvature \(\hat{R}^A_{\ BCD}\) has only one independent component. It follows that

\[
\hat{R}^A_{\ BCD} = 4\Psi_1\delta^A_B\epsilon_{CD},
\]

where \(\Psi_1\) is the Weyl tensor component whose nonvanishing is the obstruction to the algebraic speciality of the metric. The symbol \(\delta^A_B\) is the Kronecker delta (i.e. the identity) on \(\mathcal{N}\) and the \(\epsilon_{CD}\) is the 2-dimensional antisymmetric tensor. The Ricci tensor \(\hat{R}_{AB} = \hat{R}^C_{\ ABC}\) for \(\nabla\) is then \(\hat{R}_{AB} = 4\Psi_1\epsilon_{AB}\) and is antisymmetric.
Now the replacement for the Einstein condition in the Goldberg-Sachs theorem, in its \((\text{integrability of } \mathcal{N}) \Rightarrow (\text{algebraical speciality})\) part, is

\[
\nabla_A \nabla_B R_{CD} \equiv 0,
\]

as is explained in Theorem 6.5.

An interesting situation occurs in the Riemannian (and also in the split signature) case. There, the reality conditions imposed on the 1-form \(v\) and the metric \(g\) yields more information than the partial connection. Using this connection we get Theorem 7.16, which is a slightly more elegant (pseudo)hermitian version of the signature independent Theorem 6.5.

3. Totally null 2-planes in four dimensions

To discuss the geometrical meaning of the complex version of the Goldberg-Sachs theorem we recall the known \([7]\) properties of totally null 2-planes as we range over the possible signatures of 4-dimensional metrics.

Let \(V\) be a 4-dimensional real vector space equipped with a metric \(g\), of some signature. Given \(V\) and \(g\) we consider their complexifications. Thus we have \(V^C\) and the metric \(g\) which is extended to act on complexified vectors of the form \(v_1 + iv_2\), \(v_1, v_2 \in V\), via: \(g(v_1 + iv_2, v_1' + iv_2') = g(v_1, v_1') - g(v_2, v_2') + i(g(v_1, v_2') + g(v_2, v_1'))\).

Let \(\mathcal{N}\) be a 2-complex-dimensional vector subspace in \(V^C\), \(\mathcal{N} \subset V^C\), with the property that \(g\) identically vanishes on \(\mathcal{N}\), \(g_{\mathcal{N}} \equiv 0\). In other words: \(\mathcal{N}\) is a 2-complex-dimensional vector subspace of \(V^C\) such that for all \(n_1, n_2\) from \(V^C\) we have \(g(n_1, n_2) = 0\). This is the definition of \(\mathcal{N}\) being \textit{totally null}.

Such \(\mathcal{N}\)'s exist irrespectively of the signature of \(g\). In fact, let \((e_1, e_2, e_3, e_4)\) be an orthonormal basis for \(g\) in \(V\). Then, if the metric has signature \((+, +, +, +)\), an example of \(\mathcal{N}\) is given by

\[
\mathcal{N}_E = \text{Span}_C(e_1 + ie_2, e_3 + ie_4).
\]

If the metric has Lorentzian signature \((+, +, +, -)\) then we choose the basis so that \(g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1 = -g(e_4, e_4)\), and as an example of \(\mathcal{N}\) we take

\[
\mathcal{N}_L = \text{Span}_C(e_1 + ie_2, e_3 + e_4).
\]

In the case of split signature \((+, +, -,-)\) we have \(g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = g(e_4, e_4) = -1\), and we distinguish two different classes of 2-dimensional totally null \(\mathcal{N}\)'s. As an example of the first class we take

\[
\mathcal{N}_{S_1} = \text{Span}_C(e_1 + ie_2, e_3 + ie_4),
\]

and as an example of the second class we take

\[
\mathcal{N}_{S_2} = \text{Span}_C(e_1 + e_2, e_3 + e_4).
\]

If \(V\) is a \textit{complex} 4-dimensional vector space with a \textit{complex} metric \(g\), the notion of a totally null 2-dimensional vector subspace \(\mathcal{N}\) still makes sense: these are simply 2-dimensional complex vector subspaces \(\mathcal{N} \subset V\) for which \(g_{\mathcal{N}} \equiv 0\).

Irrespectively of the fact if the 2-dimensional totally null vector space \(\mathcal{N}\) is defined in terms of a complex vector space \(V\) with a complex metric, or in terms of \((V^C, g)\) in which \(V\) is real and \(g\) is the complexified real metric, choosing an orientation in \(V\), one can check that \(\mathcal{N}\) is always either \textit{selfdual} or \textit{antiselfdual} (see \(e.g.\) [17]).

By this we mean that we always have
• either: \( *(n_1 \wedge n_2) = n_1 \wedge n_2 \) for all \( n_1, n_2 \in \mathcal{N} \),
• or: \( *(n_1 \wedge n_2) = -n_1 \wedge n_2 \) for all \( n_1, n_2 \in \mathcal{N} \),

where \( * \) denotes the Hodge star operator. Thus the property of being selfdual or antselfdual (partially) characterizes totally null 2-planes.

In case of real \( V \), irrespective of the metric signature, totally null spaces in \( V^\mathbb{C} \) may be further characterized by their real index [7]. This is defined as follows:

Given a vector subspace \( \mathcal{N} \subset V^\mathbb{C} \) one considers its complex conjugate \( \bar{\mathcal{N}} = \{ w \in V^\mathbb{C} \mid \bar{w} \in \mathcal{N} \} \).

Then the intersection \( \mathcal{N} \cap \bar{\mathcal{N}} \) is the complexification of a real vector space, say \( \mathcal{K} \), and the real index of \( \mathcal{N} \) is by definition the real dimension of \( \mathcal{K} \), or the complex dimension of \( \mathcal{N} \cap \bar{\mathcal{N}} \), which is the same.

In our examples above, \( \mathcal{N}_E \) and \( \mathcal{N}_S \) have real index zero, \( \mathcal{N}_L \) has real index one and \( \mathcal{N}_S' \) has real index two. These are examples of a general fact, discussed in any dimension in [7], which when specialized to a four dimensional \( V \), reads:

- If \( g \) has Euclidean signature, \((+,+,+,+)\), then every 2-dimensional totally null space \( \mathcal{N} \) in the complexification \( V^\mathbb{C} \) has real index zero;
- If \( g \) has Lorentzian signature, \((+,+,+,-)\), then every 2-dimensional totally null space \( \mathcal{N} \) in the complexification \( V^\mathbb{C} \) has real index one;
- If \( g \) has split signature, \((+,+,+,+)\), then a 2-dimensional totally null space \( \mathcal{N} \) in the complexification \( V^\mathbb{C} \) has either real index zero or two;
- In either signature the spaces of all \( \mathcal{N} \)s with indices zero or one are generic - they form real 2-dimensional manifolds; In the split signature the spaces of all \( \mathcal{N} \)s with index two are special - they form a real manifold of dimension one.

If we have a 2-dimensional totally null \( \mathcal{N} \) with real index zero then \( V^\mathbb{C} = \mathcal{N} \oplus \bar{\mathcal{N}} \).

This enables us to equip the real vector space \( V \) with a complex structure \( J \), by declaring that the holomorphic vector space \( V^{(1,0)} \) of this complex structure is \( \mathcal{N} \). In other words, \( J \) is defined as a linear operator in \( V \) such that, after complexification, \( J(\mathcal{N}) = i \bar{\mathcal{N}} \). Due to the fact that \( \mathcal{N} \) is totally null, the so defined \( J \) is hermitian, \( g(Jv_1, Jv_2) = g(v_1, v_2) \) for all \( v_1, v_2 \in V \). Thus a totally null \( \mathcal{N} \) of real index zero in dimension four defines a hermitian structure \( J \) in the corresponding 4-dimensional real vector space \((V, g)\). Also the converse is true. For if we have \((V, g, J)\) in real dimension four, we define \( \mathcal{N} \) by \( \mathcal{N} = V^{(1,0)} \), i.e. we declare that \( \mathcal{N} \) is just the holomorphic vector space for \( J \). Due to the fact that \( J \) is hermitian, and because of the assumed Euclidean or split signature of the metric, \( \mathcal{N} \) is totally null and has real index zero. This proves the following

**Proposition 3.1.** There is a one to one correspondence between (pseudo)hermitian structures \( J \) in a four dimensional real vector space \((V, g)\), equipped with a metric of either Euclidean or split signature, and 2-dimensional totally null planes \( \mathcal{N} \subset V^\mathbb{C} \) with real index zero.

In the Lorentzian case, where all \( \mathcal{N} \)s have index one, every \( \mathcal{N} \) defines a 1-real-dimensional vector space \( \mathcal{K} \). This is spanned by a real vector, say \( k \), which is null, as it is a vector from \( \mathcal{N} \). The space \( \mathcal{K}^\perp \) orthogonal to \( \mathcal{K} \) includes \( \mathcal{K} \), \( \mathcal{K} \subset \mathcal{K}^\perp \).

Its complexification \((\mathcal{K}^\perp)^\mathbb{C} = \mathcal{N} + \bar{\mathcal{N}} \). The quotient space \( \mathcal{H} = \mathcal{K}^\perp / \mathcal{K} \) has real dimension two, and acquires a complex structure in a similar way as \( V \) did in the Euclidean/split case. Indeed, we define \( J \) in \( \mathcal{H} \) by declaring that its holomorphic
space $H^{(1,0)}$ coincides with the 2-dimensional complex vector space $(N + \bar{N})/(N \cap \bar{N})$. This shows that a 2-dimensional totally null $N$, in the complexification of a Lorentzian 4-dimensional $(V, g)$, defines a real null direction $k$ in $V$ together with a complex structure $J$ in the quotient space $K^\perp/K$, $K = \mathbb{R}k$. One can easily see that also the converse is true, and we have the following

Proposition 3.2. There is a one to one correspondence between 2-dimensional totally null planes $N$, in the complexification of a four dimensional oriented and time oriented Lorentzian vector space $(V, g)$, and null directions $K = \mathbb{R}k$ in $V$ together with their associated complex structures $J$ in $K^\perp/K$.

The last case, in which the signature of $g$ is split, $(+, +, -, -)$, and in which the $N$'s have real index 2, provides us with a real 2-dimensional totally null plane in $V$. Thus we have

Proposition 3.3. There is a one to one correspondence between 2-dimensional totally null planes $N$ with real index two, in the complexification of a four dimensional split signature vector space $(V, g)$, and real totally null 2-planes in $V$.

We now pass to the analogous considerations on 4-manifolds. Thus we consider a 4-dimensional manifold $M$, with a metric $g$, equipped in addition with a smooth distribution $N$ of complex totally null 2-planes $N_x, x \in M$, of a fixed index. Applying the above propositions we see that, depending on the index of $N$, such an $M$ is equipped either with an almost hermitian structure $(M, g, J)$ (in case of index 0), or with an almost optical structure $(M, g, K, J_{K^\perp/K})$ (in case of index 1), or with a real distribution of totally null 2-planes (in case of index 2). The interesting question about the integrability conditions for these three different real structures has a uniform answer in terms of the integrability of the complex distribution $N$. Actually, by inspection of the three cases determined by the real indices of $N$, one proves the following [16]

Proposition 3.4. Let $M$ be a 4-dimensional real manifold and $g$ be a real metric on it. Let $N$ be a complex 2-dimensional distribution on $M$ such that $g|_N \equiv 0$. Then the integrability condition,

$$[N, N] \subset N,$$

for the distribution $N$ is equivalent to

- the Newlander-Nirenberg integrability condition for the corresponding $J$, if $N$ has index zero;
- the geodesic and shear-free condition for the corresponding real null direction field $k$, if $N$ has index one. In this case the 3-dimensional space of integral curves of $k$ has (locally) the structure of 3-dimensional CR manifold.
- the classical Fröbenius integrability for the real distribution corresponding to $N$, if $N$ has index two. In this case we have a foliation of $M$ by 2-dimensional real manifolds corresponding to the leaves of $N$.

Returning to the complex Goldberg-Sachs theorem 1.5, we see that one part of its thesis, which is concerned with the integrability condition $[N, N] \subset N$, has a very nice geometric interpretation in each of the real signatures. In particular, in the real index zero case, the theorem gives if and only if conditions for the local existence of a hermitian structure on a 4-manifold [16, 15].
4. Signature independent Newman-Penrose formalism

The purpose of this section is to establish a version of the Newman-Penrose formalism [14] - a very convenient tool to study the properties of 4-dimensional manifolds equipped with a metric - in such a way that it will be usable in the following different settings. These are:

(a) \(\mathcal{M}\) is a complex 4-dimensional manifold, and \(g\) is a holomorphic metric on \(\mathcal{M}\),
(b) \(\mathcal{M}\) is a real 4-dimensional manifold, and \(g\) is a complex valued metric on \(\mathcal{M}\),
(c) \(\mathcal{M}\) is a real 4-dimensional manifold, and \(g\) is:
   (ci) real of Lorentzian signature,
   (cii) real of Euclidean signature,
   (ciii) real of split signature,
   (civ) a complexification of a real metric having one of the above signatures.

The classical Newman-Penrose formalism was devised for the case where \(\mathcal{M}\) is real, and \(g\) is Lorentzian. Although the generalization of the formalism, applicable to all the above settings, is implicit in the formulation given in the Penrose and Rindler monograph [18], one needs to have some experience to use it in the cases (cii) and (ciii). For this reason we decided to derive the formalism from first principles, emphasizing from the very beginning how to apply it to the above different situations. To achieve our goal of very easy applicability of this formalism to these different situations, we have introduced a convenient notation, in various instances quite different from the Newman-Penrose original. Since the Newman-Penrose formalism proved to be a great tool in the study of Lorentzian 4-manifolds, we believe that our formulation, explained here from the basics, will help the community of mathematicians working with 4-manifolds having metrics of Euclidean or split signature to appreciate this tool.

From now on \((\mathcal{M}, g)\) is a 4-dimensional real or complex manifold equipped with a complex valued metric. This means that the metric \(g\) is a nondegenerate symmetric bilinear form, \(g : T^\mathbb{C}\mathcal{M} \times T^\mathbb{C}\mathcal{M} \rightarrow \mathbb{C}\), with values in the complex numbers [17].

Given \(g\) we use a (local) null coframe \((\theta^1, \theta^2, \theta^3, \theta^4) = (M, P, N, K)\) on \(\mathcal{M}\) in which \(g\) is

\[ g = g_{ab} \theta^a \theta^b = 2(MP + NK). \]

Here, and in the following, formulae like \(\theta^a \theta^b\) denote the symmetrized tensor product of the complex valued 1-forms \(\theta^a\) and \(\theta^b\): \(\theta^a \theta^b = \frac{1}{2}(\theta^a \otimes \theta^b + \theta^b \otimes \theta^a)\).

Remark 4.1. Note that our setting, although in general complex, includes all the real cases. These cases correspond to metrics \(g\) such that \(g(X, Y)\) is real for all real vector fields \(X, Y \in T\mathcal{M}\). In other words, in such cases the metric \(g\) restricted to the tangent space \(T\mathcal{M}\) of \(\mathcal{M}\) is real. If \(\mathcal{M}\) is equipped with a metric \(g\) satisfying this condition, then we always locally have a null coframe \((\theta^1, \theta^2, \theta^3, \theta^4) = (M, P, N, K)\) in which

- \((E)\) \(P = \bar{M}\) and \(K = \bar{N}\) if the metric \(g|_{T\mathcal{M}}\) has Euclidean signature,
- \((S_c)\) \(P = \bar{M}\) and \(K = -\bar{N}\), if the metric \(g|_{T\mathcal{M}}\) has split signature,
- \((L)\) \(P = M\), \(N = \bar{N}\) and \(K = K\), if the metric \(g|_{T\mathcal{M}}\) has Lorentzian signature.

Remark 4.2. The main statement above about the cases \((E), (S_c)\) and \((L)\) can be rephrased as follows: In the complexification of the cotangent space of \(T^\mathbb{C}\mathcal{M}\), one
can introduce three different real structures by appropriate conjugation operators: ‘bar’. On the basis of the 1-forms \((\theta^1, \theta^2, \theta^3, \theta^4) = (M, N, P, K)\) these are defined according to:

\begin{align*}
(E) & \quad M = P, \quad P = M, \quad \bar{N} = K \quad \text{and} \quad \bar{K} = N. \quad \text{With this choice of the conjugation,} \quad g_{\mid T M} \text{is real and has Euclidean signature.} \\
(S_c) & \quad M = P, \quad \bar{P} = M, \quad \bar{N} = -K \quad \text{and} \quad \bar{K} = -N. \quad \text{With this choice of the conjugation,} \quad g_{\mid T M} \text{is real and has split signature.} \\
(L) & \quad M = P, \quad \bar{P} = M, \quad \bar{N} = N \quad \text{and} \quad \bar{K} = K. \quad \text{With this choice of the conjugation,} \quad g_{\mid T M} \text{is real and has Lorentzian signature.}
\end{align*}

Note also that the labels \(a = 1, 2, 3, 4\) of the null coframe components \(\theta^a\), behave in the following way under these conjugations:

\begin{align*}
(E) & \quad \bar{1} \to 2, \quad \bar{2} \to 1, \quad \bar{3} \to 4, \quad \bar{4} \to 3 \quad \text{in the Euclidean case,} \\
(S_c) & \quad \bar{1} \to 2, \quad \bar{2} \to 1, \quad \bar{3} \to -4, \quad \bar{4} \to -3 \quad \text{in the split case,} \\
(L) & \quad \bar{1} \to 2, \quad \bar{2} \to 1, \quad \bar{3} \to 3, \quad \bar{4} \to 4 \quad \text{in the Lorentzian case.}
\end{align*}

These transformations of indices under the respective complex conjugations will be important when we perform complex conjugations on multiindexed quantities, such as for example, \(R_{abcd}\). In particular, the above transformation of indices imply, for example, that in the \((S_c)\) case \(\bar{R}_{1323} = R_{2414}, \quad \bar{R}_{1321} = -R_{2412}\), and so on.

**Remark 4.3.** We denoted the split signature case by the letter \(S\) with a subscript \(c\) to distinguish this case from the case \(r\) in which the field of 2-planes annihilating the coframe 1-forms \(P\) and \(K\) in \((S_c)\) is totally real. It is well known [7], that if the metric \(g_{\mid T M}\) has split signature, one can choose a totally real null coframe on \(\mathcal{M}\), such that

\begin{align*}
(S_r) & \quad \bar{M} = M, \quad \bar{P} = P, \quad \bar{N} = N, \quad \bar{K} = K.
\end{align*}

This situation, although less generic [7] than \((S_c)\) is worthy of consideration, since in the integrable case of the Goldberg-Sachs theorem it leads to the foliation of \(\mathcal{M}\) by real 2-dimensional leaves, corresponding to the distribution of totally null 2-planes.

Given a null coframe \((\theta^a)\) we calculate the differentials of its components

\begin{equation}
\begin{aligned}
\text{d} \theta^a &= -\frac{1}{2} c^a_{\quad bc} \theta^b \wedge \theta^c. \\
\end{aligned}
\end{equation}

Following Newman and Penrose [14], and the tradition in General Relativity literature [8], we will assign Greek letter names to the coefficient functions \(c^a_{\quad bc}\). As is well known these coefficients naturally split onto two groups with 12 complex coefficients in each group. They correspond to two spin connections associated with the metric \(g\). The 12 coefficients from the first group will be denoted by \(\alpha, \beta, \gamma, \lambda, \mu, \nu, \rho, \sigma, \tau, \varepsilon, \kappa, \pi\). The 12 coefficients from the second group will be denoted by putting primes on the same Greek letters. The ‘primed’ and ‘unprimed’ quantities, as describing two different spinorial connections, will be treated as independent objects in the complex setting. Their relations to the complex conjugation in the real settings will be described in Remark 4.4. This said, we write the four
equations (2) as:

\[
\begin{align*}
\mathrm{d}\theta^1 &= (\alpha - \beta')\theta^1 \wedge \theta^2 + (\gamma - \gamma' - \mu)\theta^1 \wedge \theta^3 + (\varepsilon - \varepsilon' - \rho')\theta^1 \wedge \theta^4 - \\
&\quad \lambda \theta^2 \wedge \theta^3 - \sigma' \theta^2 \wedge \theta^4 + (\pi - \tau')\theta^3 \wedge \theta^4 \\
\mathrm{d}\theta^2 &= (\beta - \alpha')\theta^1 \wedge \theta^2 - \lambda' \theta^1 \wedge \theta^3 - \sigma \theta^1 \wedge \theta^4 + \\
&\quad (\gamma' - \gamma - \mu')\theta^2 \wedge \theta^3 + (\varepsilon' - \varepsilon - \rho)\theta^2 \wedge \theta^4 + (\pi' - \tau)\theta^3 \wedge \theta^4 \\
\mathrm{d}\theta^3 &= (\rho' - \rho)\theta^1 \wedge \theta^2 + (\alpha' + \beta - \tau)\theta^1 \wedge \theta^3 - \kappa \theta^1 \wedge \theta^4 + \\
&\quad (\alpha + \beta' - \tau')\theta^2 \wedge \theta^3 - \kappa' \theta^2 \wedge \theta^4 - (\varepsilon + \varepsilon')\theta^3 \wedge \theta^4 \\
\mathrm{d}\theta^4 &= (\mu - \mu')\theta^1 \wedge \theta^2 - \nu \theta^1 \wedge \theta^3 - (\alpha' + \beta + \pi')\theta^1 \wedge \theta^4 - \\
&\quad \nu' \theta^2 \wedge \theta^3 - (\alpha + \beta' + \pi)\theta^2 \wedge \theta^4 - (\gamma' + \gamma)\theta^3 \wedge \theta^4.
\end{align*}
\]

This notation for the coefficient functions \(c_{\mu\nu}^a\), although ugly at first sight, has many advantages. One of them is the already mentioned property of separating the two spin connections associated with the metric \(g\) by associating them with the respective ‘primed’ and ‘unprimed’ objects. More explicitly, defining the Levi-Civita connection 1-forms \(\Gamma^a_{\mu\nu}\) by

\[
\Gamma_{ab} = -\Gamma_{ba}, \quad \Gamma_{ab} = g_{ac}\Gamma^c_{\mu\nu},
\]

we get the following expressions for \(\Gamma_{ab}^c\):

\[
\begin{array}{c}
\frac{1}{2}(\Gamma_{12} + \Gamma_{34}) = \alpha' \theta^1 + \beta' \theta^2 + \gamma' \theta^3 + \varepsilon' \theta^4 \\
\Gamma_{13} = \lambda \theta^1 + \mu \theta^2 + \nu' \theta^3 + \pi' \theta^4 \\
\Gamma_{24} = \rho' \theta^1 + \sigma' \theta^2 + \tau' \theta^3 + \kappa' \theta^4.
\end{array}
\]

\[
\begin{array}{c}
\frac{1}{2}(-\Gamma_{12} + \Gamma_{34}) = \beta \theta^1 + \alpha \theta^2 + \gamma \theta^3 + \varepsilon \theta^4 \\
\Gamma_{23} = \mu \theta^1 + \lambda \theta^2 + \nu \theta^3 + \pi \theta^4 \\
\Gamma_{14} = \sigma \theta^1 + \rho \theta^2 + \tau \theta^3 + \kappa \theta^4.
\end{array}
\]

The two spin connections correspond to \(\chi' = (\Gamma_{24}, \frac{1}{2}(\Gamma_{12} + \Gamma_{34}), \Gamma_{13})\) and \(\chi = (\Gamma_{14}, \frac{1}{2}(-\Gamma_{12} + \Gamma_{34}), \Gamma_{23})\), respectively.

**Remark 4.4.** The above notation is an adaptation of the Lorentzian version of the Newman-Penrose formalism. This can be easily seen, taking into account the reality conditions discussed in Remarks 4.1, 4.2. In particular, in the Lorentzian case \((L)\), the complex conjugation defined in Remark 4.2, applied to the quantities \(\alpha, \beta, \gamma, \ldots\), yields:

\[
\begin{pmatrix}
\bar{\alpha} & \bar{\beta} & \bar{\gamma} & \bar{\varepsilon} \\
\bar{\lambda} & \bar{\mu} & \bar{\nu} & \bar{\pi} \\
\bar{\rho} & \bar{\sigma} & \bar{\tau} & \bar{\kappa}
\end{pmatrix}
= 
\begin{pmatrix}
\alpha' & \beta' & \gamma' & \varepsilon' \\
\lambda' & \mu' & \nu' & \pi' \\
\rho' & \sigma' & \tau' & \kappa'
\end{pmatrix}
\]

Thus in the Lorentzian case the complex conjugation changes ‘unprimed’ Greek letters into ‘primed’ ones and vice versa. Therefore in this signature the ‘primed’ Greek letter quantities are totally determined by the ‘unprimed’ ones. The situation is drastically different in the two other real signatures. There the ‘primed’ Greek letter quantities are independent of the ‘unprimed’ ones. On the other hand in these two cases, there are some relations between the quantities within each of the ‘primed’ and ‘unprimed’ family. In the Euclidean case they are given by
\[ (E) \quad \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} & \tilde{\varepsilon} \\ \lambda & \mu & \nu & \pi \\ \rho & \sigma & \tau & \kappa \end{pmatrix} = \begin{pmatrix} -\beta & -\alpha & -\varepsilon & -\gamma \\ \sigma & \rho & \kappa & \tau \\ \mu & \lambda & \pi & \nu \end{pmatrix}, \]

with the same relations after the replacement of all ‘unprimed’ quantities by their ‘primed’ counterparts on both sides.

In the split signature cases, we have

\[ (S_c) \quad \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} & \tilde{\varepsilon} \\ \lambda & \mu & \nu & \pi \\ \rho & \sigma & \tau & \kappa \end{pmatrix} = \begin{pmatrix} -\beta & -\alpha & \varepsilon & \gamma \\ -\sigma & -\rho & \kappa & \tau \\ -\mu & -\lambda & \pi & \nu \end{pmatrix}, \]

and

\[ (S_r) \quad \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} & \tilde{\varepsilon} \\ \lambda & \mu & \nu & \pi \\ \rho & \sigma & \tau & \kappa \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \varepsilon \\ \lambda & \mu & \nu & \pi \\ \rho & \sigma & \tau & \kappa \end{pmatrix}, \]

again with the identical relations for the ‘primed’ quantities.

Now we pass to the ‘prime’-‘unprime’ decomposition of the curvature. The Riemann tensor coefficients \( R^a_{bcd} \) are defined by Cartan’s second structure equations:

\[ (7) \quad \frac{1}{2} \frac{d}{dx}^a_b \Gamma^c_{ab} \wedge \Gamma^d_{cd} = \frac{1}{2} R^a_{bcd} = \frac{1}{2} R^a_{bcd} \theta^c \wedge \theta^d. \]

Due to our conventions, modulo symmetry, the only nonzero components of the metric are \( g_{12} = g_{34} = 1 \). The inverse of the metric, \( g^{ab} \), again modulo symmetry, has \( g^{12} = g^{34} = 1 \) as the only nonvanishing components. The Ricci tensor is defined as \( R_{ab} = R_{acb} \). Its scalar is: \( R = R_{ab} g^{ab} \), and its tracefree part is: \( \bar{R}_{ab} = R_{ab} - \frac{1}{2} \bar{R} g_{ab} \). Using the metric \( g_{ab} \) we also define \( R_{abcd} = g_{ae} R^e_{bcd} \). This is further used to define the covariant components of the Weyl tensor \( C^a_{bcd} \) via:

\[ C_{abcd} = R_{abcd} - \frac{1}{12} R(g_{ae} g_{db} - g_{ad} g_{eb}) + \frac{1}{2} (g_{ad} \bar{R}_{eb} - g_{ae} \bar{R}_{db} + g_{be} \bar{R}_{da} - g_{bd} \bar{R}_{ca}). \]

In the context of the present paper, in which the conformal properties matter, it is convenient to use the Schouten tensor \( P \), with help of which we can write the above displayed equality as

\[ (8) \quad C_{abcd} = R_{abcd} + g_{ae} P_{cb} - g_{ae} P_{db} + g_{be} P_{da} - g_{bd} P_{ca}. \]

The Schouten tensor \( P \) is a ‘trace-corrected’ Ricci tensor, with the explicit relation given by

\[ P_{ab} = \frac{1}{2} R_{ab} - \frac{1}{12} R g_{ab}. \]

In the Newman-Penrose formalism, the 10 components of the Weyl tensor are encoded in 10 complex quantities \( \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4 \) and \( \Psi'_0, \Psi'_1, \Psi'_2, \Psi'_3, \Psi'_4 \). Five of them have ‘primes’, to emphasize that they are associated with the ‘primed’ spin connection. Another way of understanding this notation is to say that the ‘unprimed’ \( \Psi \)s are five components of the self-dual part of the Weyl tensor, and the ‘primed’ \( \Psi \)s are the components of the anti-self-dual part of the Weyl.

The Ricci and Schouten tensors are mixed ‘prime’-‘unprime’ objects, and as such are not very nicely denoted in the ‘prime’ vs ‘unprime’ setting. For this reason, when referring to \( R_{ab}, \bar{R}_{ab} \) and \( P_{ab} \), we will not use the Newman-Penrose notation, and will express these objects using the standard four-dimensional indices \( a = 1, 2, 3, 4 \), as e.g. in \( 12[P_{12} + P_{34}] = 2(R_{12} + R_{34}) = R \).

Having said all of this we express Cartan’s second structure equations \((7)\), and in particular the curvature coefficients \( R^a_{bcd} \), in terms of \( \Psi, \Psi' \)s, \( P \) and the null
coframe \((\theta^i)\) as follows:

\[
\frac{1}{2} d(\Gamma_{12} + \Gamma_{34}) + \Gamma_{24} \wedge \Gamma_{13} = \\
- \Psi'_2 \theta^1 \wedge \theta^3 + \Psi'_2 \theta^2 \wedge \theta^4 + \frac{1}{2} (2 \Psi'_2 - P_{12} - P_{34}) (\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4) + \\
P_{23} \theta^2 \wedge \theta^3 - P_{14} \theta^1 \wedge \theta^4 - \frac{1}{2} (P_{12} - P_{34}) (\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4)
\]

\[
d\Gamma_{13} + (\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{13} = \\
\Psi'_2 \theta^1 \wedge \theta^3 + (\Psi'_2 + P_{12} + P_{34}) \theta^2 \wedge \theta^4 - \Psi'_2 (\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4) + \\
P_{22} \theta^2 \wedge \theta^3 + P_{44} \theta^1 \wedge \theta^4 + P_{24} (\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4),
\]

with analogous equations for the 'unprimed' objects:

\[
\frac{1}{2} d(-\Gamma_{12} + \Gamma_{34}) + \Gamma_{14} \wedge \Gamma_{23} = \\
- \Psi'_2 \theta^2 \wedge \theta^3 + \Psi'_2 \theta^1 \wedge \theta^4 - \frac{1}{2} (2 \Psi'_2 - P_{12} - P_{34}) (\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) + \\
P_{13} \theta^1 \wedge \theta^3 - P_{24} \theta^2 \wedge \theta^4 + \frac{1}{2} (P_{12} - P_{34}) (\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4)
\]

\[
d\Gamma_{23} + (-\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{23} = \\
\Psi'_2 \theta^2 \wedge \theta^3 + (\Psi'_2 + P_{12} + P_{34}) \theta^1 \wedge \theta^4 + \Psi'_2 (\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) + \\
P_{33} \theta^1 \wedge \theta^3 + P_{22} \theta^2 \wedge \theta^4 + P_{23} (\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4)
\]

\[
d\Gamma_{14} + \Gamma_{14} \wedge (-\Gamma_{12} + \Gamma_{34}) = \\
(\Psi'_2 + P_{12} + P_{34}) \theta^2 \wedge \theta^3 + \Psi'_2 \theta^1 \wedge \theta^4 - \Psi'_2 (\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) + \\
P_{11} \theta^1 \wedge \theta^3 + P_{44} \theta^2 \wedge \theta^4 - P_{14} (\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4).
\]

Note that in the first part (9) of the structure equations, the full traceless part of the Schouten tensor \(P\), represented by its nine components \(P_{11}, P_{13}, P_{14}, P_{22}, P_{23}, P_{24}, P_{33}, P_{44}\) and \(P_{12} - P_{34}\), stays with the basis of the selfdual 2-forms:

\[
\Sigma = (\theta^2 \wedge \theta^3, \theta^1 \wedge \theta^4, \theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4).
\]

In the second part (10) of the structure equations, the full traceless part of the Schouten tensor \(P\) appears again, but now at the basis of the antiselfdual 2-forms:

\[
\Sigma' = (\theta^1 \wedge \theta^3, \theta^2 \wedge \theta^4, \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4).
\]

On the other hand the selfdual and the antiselfdual parts of the Weyl tensor, corresponding to the respective \(\Psi\)'s and \(\Psi'\)'s, are separated: in equations (9) we only have \(\Psi\)'s, whereas in (10) we only have \(\Psi'\)'s. The trace of the Schouten tensor \(2(P_{12} + P_{34})\), proportional to the Ricci scalar \(R\), appears in both sets of equations, always together with the respective Weyl tensor components \(\Psi_2\) and \(\Psi'_2\). It is also worthwhile to mention that if one uses the following basis

\[
E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},
\]
of the Lie algebra \( \mathfrak{sl}(2) \), and if one defines
\[
\Gamma = \Gamma_{14}E_+ + \frac{1}{2}(\Gamma_{12} + \Gamma_{34})E_0 + \Gamma_{23}E_- ,
\]
\[
\Gamma' = \Gamma_{24}E_+ + \frac{1}{2}(\Gamma_{12} + \Gamma_{34})E_0 + \Gamma_{13}E_+ ,
\]
then the left hand sides of equations (9)-(10) appear in the formulae
\[
d\Gamma + \Gamma \wedge \Gamma = \left( \frac{1}{2}d(-\Gamma_{12} + \Gamma_{34}) + \Gamma_{14} \wedge \Gamma_{23} - d\Gamma_{23} - (-\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{23} \right) ,
\]
\[
d\Gamma' + \Gamma' \wedge \Gamma' = \left( \frac{1}{2}d(\Gamma_{12} + \Gamma_{34}) + \Gamma_{24} \wedge \Gamma_{13} - d\Gamma_{13} - (\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{13} \right) ,
\]
This explains the term 'spin connections' assigned to the previously defined quantities \( \chi \) and \( \chi' \). It also justifies the 'prime'-unprime' notation, which is rooted in the speciality of 4-dimensions, stating that for \( n \geq 3 \) the Lie algebra \( \mathfrak{so}(n,\mathbb{C}) \) is not simple only when \( n = 4 \), and in that case it has the symmetric split: \( \mathfrak{so}(4,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \). This enables us to split the \( \mathfrak{so}(4,\mathbb{C}) \)-valued Levi-Civita connection into the well defined \( \mathfrak{sl}(2,\mathbb{C}) \)-valued 'primed' and 'unprimed' parts, which are totally independent. In real signatures we have an analogous split for \( \mathfrak{so}(n-p,p) = \mathfrak{g} \oplus \mathfrak{g}' \), \( p = 0,1,2 \), where now \( \mathfrak{g} \) and \( \mathfrak{g}' \) are two copies of the appropriate real form of \( \mathfrak{sl}(2,\mathbb{C}) \). This again enables us to split the Levi-Civita connection into the 'primed' and 'unprimed' connections, with the appropriate reality conditions, as in \((E), (S_+), (S_-) \text{ or } (L)\).

Comparing equations (5)-(6) with (9)-(10), one finds relations between the curvature quantities \( \Psi \) and \( \Psi' \) and the first derivatives of the connection coefficients \( \alpha, \beta, \ldots, \alpha', \beta', \ldots \). These relations are called the Newman-Penrose equations [14]. We present them in the Appendix. In these equations, and in the rest of the paper, we denote the vector fields dual on \( \mathcal{M} \) to the null coframe \((M,P,N,K)\) by the respective symbols \((\delta, \partial, \Delta, D)\). Thus we have e.g. \( \delta \partial M = 1 \), and zero on all the other coframe components, \( D \partial N = 0 \), etc. Also, when applying these vector fields to functions on \( \mathcal{M} \) we omit parentheses. Thus, instead of writing \( D(\alpha) \) to denote the derivative of a connection coefficient \( \alpha \) in the direction of the basis vector field \( D \), we simply write \( D\alpha \).

In addition to the Newman-Penrose equations we will also need the commutators of the basis vector fields. These are given by the formulae dual to equations (3), and read:
\[
[\delta, \partial] = (\beta' - \alpha)\delta + (\alpha' - \beta)\partial + (\rho - \rho')\Delta + (\mu' - \mu)D
\]
\[
[\delta, \Delta] = (\mu + \gamma' - \gamma)\delta + \lambda'\partial + (\tau' - \alpha' - \beta)\Delta + \nu' D
\]
\[
[\partial, \Delta] = \lambda \delta + (\mu' + \gamma - \gamma')\partial + (\tau' - \alpha - \beta')\Delta + \nu D
\]
(13)
\[
[\delta, D] = (\rho' + \epsilon')\delta + \sigma\partial + \kappa\Delta + (\alpha' + \beta + \pi')D
\]
\[
[\partial, D] = \sigma'\delta + (\rho + \epsilon - \epsilon')\partial + \kappa'\Delta + (\alpha + \beta' + \pi)D
\]
\[
[\Delta, D] = (\tau' - \pi)\delta + (\tau - \pi')\partial + (\epsilon' + \epsilon)\Delta + (\gamma' + \gamma)D
\]

The Newman-Penrose equations are supplemented by the second Bianchi identities, which are crucial for the proof of the Goldberg-Sachs theorem. These are relations between the first derivatives of the curvature quantities \( \Psi \), \( \Psi' \) and \( P \) and the connection coefficients. These Bianchi identities are also presented in the Appendix.
5. Generalizations of the Goldberg-Sachs Theorem for Complex Metrics

The thesis of the Goldberg-Sachs theorem can be restated in the language of the Newman-Penrose formalism as follows:

To interpret the integrability condition $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$ on the totally null distribution $\mathcal{N}$, we align the Newman-Penrose coframe $(\theta^1, \theta^2, \theta^3, \theta^4) = (M, P, N, K)$ in such a way that the two null and mutually orthogonal frame vectors $e_1 = m = \delta$ and $e_4 = k = D \text{span } \mathcal{N}$, $\mathcal{N} = \text{Span}_\mathbb{C}(\delta, D)$. Such a coframe on $(M, g)$ will be called a coframe adapted to $\mathcal{N}$.

Then the integrability of $\mathcal{N}$ is totally determined by the commutator $[\delta, D]$ of these basis vectors. Looking at this commutator in (13), we see that the condition that $[\delta, D]$ is in the span of $\delta$ and $D$ is equivalent to $\kappa \equiv \sigma \equiv 0$. Thus we have

**Proposition 5.1.** Let $\mathcal{N}$ be a field of selfdual totally null 2-planes on a 4-dimensional manifold $M$ with the metric $g$. Let $(m, p, n, k)$ be a null frame in $\mathcal{U} \subset M$ adapted to $\mathcal{N}$. Then the field $\mathcal{N} = \text{Span}_\mathbb{C}(m, k)$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, in $\mathcal{U}$ if and only if the frame connection coefficients $\Gamma_{144} = \kappa$ and $\Gamma_{141} = \sigma$ vanish identically, $\kappa \equiv \sigma \equiv 0$, in $\mathcal{U}$.

To interpret the algebraic speciality of the selfdual part of the Weyl tensor, we focus on the condition

$$C(m, k, m, k) \equiv 0.$$  (14)

Here we consider the Weyl tensor $C_{abcd}$ as a linear map $C : \otimes^4 \mathfrak{T}^2 \mathcal{M} \to \mathbb{C}$. Note that, since the so understood Weyl tensor is *antisymmetric* in the first two arguments, as well as, independently, in the last two arguments, the vanishing in equation (14), although defined on a particular basis of $\mathcal{N}$, is basis independent. Actually, if we think of $C$ as a linear map $C : (\Lambda^2 \mathfrak{T}^2 \mathcal{M}) \otimes (\Lambda^2 \mathfrak{T}^2 \mathcal{M}) \to \mathbb{C}$, and identify a 2-dimensional totally null distribution $\mathcal{N}$ with the complex line bundle $\mathcal{N}_\lambda = \{w \in \Lambda^2 \mathfrak{T}^2 \mathcal{M} \mid w = v_1 \wedge v_2, v_1, v_2 \in \mathcal{N}\}$, then we say that $\mathcal{N}$ is a principal totally null distribution iff

$$C(\mathcal{N}_\lambda, \mathcal{N}_\lambda) \equiv 0.$$  (15)

**Remark 5.2.** The quantity $C(m, k, m, k)$ is a null counterpart of the sectional curvature from Riemannian geometry. In fact, given a 2-dimensional vector space $V = \text{Span}_\mathbb{R}(X, Y)$, the sectional curvature associated with $V$ is

$$K = K(X, Y) = \frac{g(R(X, Y)X, Y)}{|X \wedge Y|^2}.$$  

The appeareance of the denominator $|X \wedge Y|^2 = g(X, X)g(Y, Y) - g(X, Y)^2$ in this expression makes this quantity independent of the choice of $X, Y$ in $V$. The notion of sectional curvature loses its meaning for vector spaces $V$ which are totally null, since for them the metric $g$ when restricted to $V$ vanishes, making the denominator $|X \wedge Y|^2 \equiv 0$ for all $X, Y \in V$. To incorporate totally null vector spaces $V$, one needs to generalize the notion of sectional curvature, removing the denominator from its definition. This leads to the quantity

$$K_0 = K_0(X, Y) = g(R(X, Y)X, Y).$$
This, although basis dependent, transforms in a homogeneous fashion,

\[ K_0(X, Y) \to (ad - bc)^2 K_0(X, Y), \]

under the change of basis \( X \to aX + b Y, \ Y \to cX + dY \). Thus vanishing or not of \( K_0 \) is an invariant property of any 2-dimensional vector space \( V \subset T_x \mathcal{M} \). This property of having \( K_0 \) equal or not equal to zero, characterizes \( V \) and is well defined regardless of the fact if the metric is real or complex, including the cases when \( V \) is totally null.

Now, passing to the specific situation of 4-dimensional manifolds, we can choose \( V \) to be a field of selfdual totally null 2-planes \( \mathcal{N} \). More specifically, if \( \mathcal{N} = \text{Span}_\mathbb{C}(m, k) \), we easily check (see (10)) that \( K_0(m, k) = C(m, k, m, k) = \Psi_0 \). Thus \( K_0(m, k) \) is the \( \Psi_0 \) component of the selfdual part of the Weyl tensor. For an antisyelfdual totally null plan \( \mathcal{N}' = \text{Span}_\mathbb{C}(p, k) \) we have \( K_0(p, k) = C(p, k, p, k) = \Psi'_0 \), which is the corresponding component of the antisyelfdual part of the Weyl tensor. This shows that the principal selfdual totally null 2-planes are just those for which the quantity \( \Psi_0 \) vanishes. Thus, in a sense, the principal selfdual totally null 2-planes have vanishing sectional curvature. (We have also an analogous statement for the principal antiselfdual 2-planes; they are related to the antiselfdual part of the Weyl tensor, and are defined by the vanishing of the quantity \( \Psi'_0 \).)

Let us now choose a Newman-Penrose coframe \((M, P, N, K)\) which is not related to any particular choice of \( \mathcal{N} \). Thus we have \( g = 2(MP + NK) \). Then, at every point of \( \mathcal{M} \), we have two families \( \mathcal{N}_z \) and \( \mathcal{N}_{z'} \) of 2-dimensional totally null planes [17]. These two families are parametrized by a complex parameter \( z \) or \( z' \), respectively, and the 2-planes parametrized by \( z \) are selfdual, and those parametrized by \( z' \) are antiselfdual. In terms of the frame \((e_1, e_2, e_3, e_4) = (n, p, n, k) = (\delta, \partial, \triangle, D)\) dual to \((M, P, N, K)\), they are given by

\[ \mathcal{N}_z = \text{Span}_\mathbb{C}(m + zn, k - zp), \quad z \in \mathbb{C}, \tag{16} \]

and

\[ \mathcal{N}_{z'} = \text{Span}_\mathbb{C}(p + z'n, k - z'm), \quad z' \in \mathbb{C}. \tag{17} \]

Adding a totally null plane \( \mathcal{N}_\infty = \text{Span}_\mathbb{C}(n, p) \) to the first family, and \( \mathcal{N}_\infty = \text{Span}_\mathbb{C}(n, m) \) to the second family, we have two spheres of 2-dimensional totally null planes at each point of \( \mathcal{M} \). The first sphere consists of the selfdual 2-planes, the second of the antiselfdual 2-planes.

Now we find the principal 2-planes in each of these spheres. The principal 2-planes in the first sphere correspond to those \( z \) such that

\[ C(m + zn, k - zp, m + zn, k - zp) = 0. \tag{18} \]

The left hand side of this equation is a fourth order polynomial in the complex variable \( z \), thus (18) treated as an equation for \( z \), has four roots, some of which may be multiple roots. Moreover, equation (18) written explicitly in terms of the Newman-Penrose Weyl coefficients \( \Psi_s \) and \( \Psi'_s \), involves only the ‘unprimed’ quantities. Explicitly:

\[ C(m + zn, k - zp, m + zn, k - zp) = \Psi_4 z^4 - 4\Psi_3 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z + \Psi_0, \]

where we have used the conventions of the previous section, such as \( C(m, k, m, k) = \Psi_0 \), etc. Similar considerations for the second sphere lead to the following proposition:
Proposition 5.3. A selfdual totally null 2-plane $N_z = \text{Span}_C(m + zn, k - zp)$ is principal at $x \in M$ iff $z$ is a root of the equation

\begin{equation}
\Psi_4 z^4 - 4\Psi_3 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z + \Psi_0 = 0.
\end{equation}

An antiselfdual totally null 2-plane $N_{z'} = \text{Span}_C(m + z'k, n - z'p)$ is principal at $x \in M$ iff $z'$ is a root of the equation

\begin{equation}
\Psi'_4 z'^4 - 4\Psi'_3 z'^3 + 6\Psi'_2 z'^2 + 4\Psi'_1 z' + \Psi'_0 = 0.
\end{equation}

Thus at every point of $M$ we have at most four selfdual principal null 2-planes and at most four antiselfdual principal null 2-planes. If a principal null 2-plane corresponds to a multiple root of (19) or (20), then such a 2-plane is called a multiple principal null 2-plane. A selfdual or antiselfdual part of the Weyl tensor with multiple principal 2-planes at a point is called algebraically special at this point.

We also note that the number and the multiplicity of the roots in (19) or (20) is a conformal invariant of the metric at a point. Thus the algebraically special cases can be further stratified according to the number of the roots and their multiplicities.

The possibilities here for (19) are: a) three distinct roots, b) two distinct roots, with one of multiplicity three, c) two distinct roots, each with multiplicity two, d) one root of multiplicity four, e) selfdual part of the Weyl tensor is zero. We have also the corresponding possibilities a'), b') c'), d') and e') for (20).

Definition 5.4. The selfdual part of the Weyl tensor is of Petrov type II, III, D, N, or 0 at a point, if equation (19) has roots as in the respective cases a), b), c), d) and e) at this point. If the Petrov type of the selfdual part of the Weyl tensor varies in $M$, from point to point, but only between the types II and D, we say that it is of type $\Pi$. The analogous classification holds also for the antiselfdual part of the Weyl tensor.

Remark 5.5. Suppose that the selfdual part of the Weyl tensor of $(M, g)$ does not vanish at each point of a neighbourhood $U' \subset M$. Thus at every point of $U'$ we have at least one principal totally null 2-plane. We now take the principal null 2-plane which at $x \in U'$ has the smallest multiplicity $1 \leq q \leq 4$. There always exists a neighbourhood $U \subset U'$ of $x$ in which this principal totally null 2-plane extends to a field $N$ of principal totally null 2-planes of multiplicity not bigger than $q$. In $U$ we choose a null frame $(m, p, n, k)$ in such a way that $\text{Span}_C(m, k) = N$. In this frame the definition (16) shows that $N' = N_0$, i.e. that the corresponding $z = 0$ in $U$. Moreover since $N$, as a field of principal null 2-planes in $U$ satisfies (19), then $\Psi_0 \equiv 0$ everywhere in this frame.

This proves the following

Proposition 5.6. Around every point $x$ of a manifold $(M, g)$ with nowhere vanishing selfdual part of the Weyl tensor, there exists a neighbourhood $U$ and a null frame $(m, p, n, k)$ in $U$ in which $\Psi_0 \equiv 0$ everywhere.

Now if the selfdual part of the Weyl tensor is algebraically special of type II in $U$, with $N$ the corresponding principal multiple field of totally null 2-planes, then in $U$ we choose a null frame $(m, p, n, k)$ adapted to $N$. In this frame $N' = N_0 = \text{Span}(m, k)$, the value $z = 0$ is a double root of (19), and since this is true at every point of $U$, we have $\Psi_0 \equiv \Psi_1 \equiv 0$. Performing similar considerations for types III
and \( N \), and forcing \( z = 0 \) to be a root of the equation (19) with the respective locally constant multiplicity \( q = 1, 2, 3 \) and 4, we get the following.

**Proposition 5.7.** Let \( \mathcal{N} \) be a field of principal totally null 2-planes for the selfdual part of the Weyl tensor of a metric \( g \) on a 4-dimensional manifold \( M \). Assume that \( \mathcal{N} \) has a constant multiplicity \( q \) in a neighbourhood \( U \) in \( M \). Then one can choose a null frame \((m, p, n, k)\) in \( U \), with \( \mathcal{N} = \text{Span}(m, k) \) and \( q = 2(MP + NK) \), so that

- if \( q = 1 \) then in this frame \( \Psi_0 \equiv 0 \) and \( \Psi_1 \neq 0 \),
- if \( q = 2 \) then in this frame \( \Psi_0 \equiv \Psi_1 \equiv 0 \) and \( \Psi_2 \neq 0 \),
- if \( q = 3 \) then in this frame \( \Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv 0 \) and \( \Psi_3 \neq 0 \),
- if \( q = 4 \) then in this frame \( \Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv \Psi_3 \equiv 0 \) and \( \Psi_4 \neq 0 \).

Conversely, if we have a null frame in \( U \) in which

- \( \Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv \Psi_3 \equiv 0 \) and \( \Psi_4 \neq 0 \) then \( \mathcal{N} = \text{Span}(m, k) \) is a field of multiple principal 2-planes in \( U \) with multiplicity \( q = 4 \),
- \( \Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv 0 \) and \( \Psi_3 \neq 0 \) then \( \mathcal{N} = \text{Span}(m, k) \) is a field of multiple principal 2-planes in \( U \) with multiplicity \( q = 3 \),
- \( \Psi_0 \equiv \Psi_1 \equiv 0 \) and \( \Psi_2 \neq 0 \) then \( \mathcal{N} = \text{Span}(m, k) \) is a field of multiple principal 2-planes in \( U \) with multiplicity \( q = 2 \),
- \( \Psi_0 \equiv 0 \) and \( \Psi_1 \neq 0 \) then \( \mathcal{N} = \text{Span}(m, k) \) is a field of multiple principal 2-planes in \( U \) with multiplicity \( q = 1 \).

This immediately implies

**Corollary 5.8.** The selfdual part of the Weyl tensor of a metric \( g \) on a 4-dimensional manifold \( M \) is algebraically special in neighbourhood \( U \), with \( \mathcal{N} \) being a field of multiple principal 2-planes in \( U \) if and only if there exists a null frame \((m, p, n, k)\) in \( U \) in which \( \Psi_0 \equiv \Psi_1 \equiv 0 \) in \( U \). In this frame \( \mathcal{N} = \text{Span}_C(m, k) \).

5.1. **Generalizing the Przanowski-Plebański version.** The starting point for our generalizations of the Goldberg-Sachs theorem is to replace the Ricci flat condition from the classical version [4], by a condition on only that part of the Ricci tensor, which is 'visible' to the integrable totally null 2-plane \( \mathcal{N} \).

For this we consider the Ricci tensor of \((M, g)\) as a symmetric, possibly degenerate, bilinear form on \( M \). We denote it by \( \text{Ric} \) and extend it to the complexification \( T^C M \) by linearity. Now given a complex distribution \( Z \subset T^C M \) we say that the Ricci tensor is degenerate on \( Z \),

\[
\text{Ric}|_Z = 0, \quad \text{iff} \quad \text{Ric}(Z_1, Z_2) = 0, \quad \forall Z_1, Z_2 \in Z.
\]

Then we have the following theorem:

**Theorem 5.9.** Let \( \mathcal{N} \subset T^C M \) be a field of totally null 2-planes on a 4-dimensional manifold \((M, g)\) equipped with a real metric \( g \) of any signature. Assume that the Ricci tensor \( \text{Ric} \) of \((M, g)\), considered as a symmetric bilinear form on \( T^C M \), is degenerate on \( \mathcal{N} \),

\[
\text{Ric}|_{\mathcal{N}} = 0.
\]

If in addition the field \( \mathcal{N} \) is integrable, \([\mathcal{N}, \mathcal{N}] \subset \mathcal{N}\), everywhere on \( M \), then \((M, g)\) is algebraically special at every point, with a field of multiple principal totally null 2-planes tangent to \( \mathcal{N} \).

To prove it, we fix a null frame \((m, p, n, k)\) on \( M \) adapted to \( \mathcal{N} \). This means that \( \mathcal{N} = \text{Span}_C(m, k) \).
It is then very easy to see that the vanishing of the Ricci tensor on $\mathcal{N}$ is, due to our conventions, equivalent to the conditions

$$P_{11} \equiv P_{14} \equiv P_{44} \equiv 0.$$  

Instead of proving Theorem 5.9, we prove a theorem that implies it. This is the complex version of the Goldberg-Sachs theorem, which generalizes the Lorentzian version due to Przanowski and Plebanski [23]. When stated in the Newman-Penrose language, this reads as follows:

**Theorem 5.10.**

(1) Suppose that a 4-dimensional metric $g$ satisfies $P_{11} \equiv P_{14} \equiv P_{44} \equiv 0$ and $\kappa \equiv \sigma \equiv 0$. Then $\Psi_0 \equiv \Psi_1 \equiv 0$.

(2) If $g$ is Einstein, $\text{Ric}(g) = \Lambda g$, and has a nowhere vanishing selfdual part of the Weyl tensor, then $\Psi_0 \equiv \Psi_1 \equiv 0$ implies $\kappa = \sigma = 0$.

Before the proof we make the following remarks:

**Remark 5.11.** It is easy to see that part (1) of the above Theorem is equivalent to Theorem 5.9.

**Remark 5.12.** Note that $\text{Ric} = 0$ and more generally $\text{Ric} = \Lambda g$ are special cases of our condition $\text{Ric}|_{\mathcal{N}} = 0$.

**Proof.** (of Theorem 5.10). First we assume that $\kappa$ and $\sigma$ vanish everywhere on $\mathcal{M}$. To conclude that $\Psi_0 \equiv 0$ is very easy: Actually this conclusion is an immediate consequence of the Newman-Penrose equation (74). For if $\kappa$ and $\sigma$ are identically vanishing, then equation (74) gives $\Psi_0 \equiv 0$. Note that this conclusion holds even without any assumption about the components of the Schouten tensor $P$ (or the Ricci tensor).

Now we prove the following

**Lemma 5.13.** Suppose that a 4-dimensional metric $g$ satisfies $\kappa \equiv \sigma \equiv 0$ and

\begin{align*}
\delta \Psi_1 &= 2(\beta + 2\tau)\Psi_1, \\
D \Psi_1 &= 2(\varepsilon - 2\rho)\Psi_1.
\end{align*}

Then it also satisfies

$$\Psi_1 \equiv 0.$$  

**Proof.** We use the commutator (13), and the Newman-Penrose equations (75)-(77) to obtain the compatibility conditions for (21) and (22). This is a pure calculation.

We give its main steps below:

- applying $[\delta, D]$ to (21) and (22) we get:

  $$[\delta, D] \Psi_1 = 2\delta((\varepsilon - 2\rho)\Psi_1) - 2D((\beta + 2\tau)\Psi_1);$$

- next, using (13), and again (21) and (22), we transform this identity into:

  $$2(\rho' + \varepsilon' - \varepsilon)(\beta + 2\tau)\Psi_1 + 2(\alpha' + \beta + \pi')(\varepsilon - 2\rho)\Psi_1 \equiv 2\delta((\varepsilon - 2\rho)\Psi_1) - 2D((\beta + 2\tau)\Psi_1);$$

- now, the Leibniz rule, and a third use of (21) and (22), enables us to eliminate of the derivatives of $\Psi_1$ in (23);

- actually, simplifying (23), and using (21), (22) we get:

  $$2(\delta((\varepsilon - 2\rho)\Psi_1) - 2D((\beta + 2\tau)\Psi_1);$$

(24)  

$$\left(2\delta(\varepsilon - 2\rho) - 2D(\beta + 2\tau) + 2(\varepsilon - \varepsilon'(\beta + 2\tau) - 2(\alpha' + \beta + \pi')(\varepsilon - 2\rho)\right) \Psi_1 \equiv 0;$$
the last step in the proof of the lemma is to use the Newman-Penrose equations (75)-(77);
• these equations eliminate $\delta \varepsilon - D\beta$, (look at (75)), $\delta \rho$, (look at (76)), and $D\tau$, (look at (77)), from the identity (24);
• this makes the identity (24) derivative-free;
• actually it transforms (24) to a remarkable identity:

$$ (10\Psi_1)\Psi_1 \equiv 0; $$

This proves Lemma 5.13.

To conclude the proof of the part one of Theorem 5.10 we use our assumptions
$P_{11} \equiv P_{14} \equiv P_{44} \equiv 0, \kappa \equiv \sigma \equiv 0$, and their consequence $\Psi_0 \equiv 0$, and insert them
in the Bianchi identities (83) and (84). This trivially gives the relations (21) and (22), respectively. Then an obvious use of Lemma 5.13 finishes the proof of part one of Theorem 5.10.

We now pass to the proof of part two of Theorem 5.10.

When going from \( (\Psi_0 \equiv \Psi_1 \equiv 0) \) to \( (\kappa \equiv \sigma \equiv 0) \) we do as follows:

• Initially we only assume that $P_{11} \equiv P_{14} \equiv P_{44} \equiv 0$.
• Then the Bianchi identities (83) and (84) give:

$$ 2P_{13}\kappa + (3\Psi_2 + P_{12} - P_{34})\sigma \equiv 0 $$

and

$$ (3\Psi_2 - P_{12} + P_{34})\kappa + 2P_{24}\sigma \equiv 0, $$

respectively.

At this stage the following remark is in order:

**Remark 5.14.** If we were able to conclude that the rank of the matrix

$$ m = \begin{pmatrix} 2P_{13} & 3\Psi_2 + P_{12} - P_{34} \\ 3\Psi_2 - P_{12} + P_{34} & 2P_{24} \end{pmatrix} $$

was **identically** equal to two, this would immediately yield $\kappa \equiv \sigma \equiv 0$, which would conclude the proof. On the other extreme, if we were sure that the matrix $m$ was **identically** equal to zero (i.e. if it had rank identically equal to zero), we would argue as follows: The identically zero rank of $m$ means that in addition to $P_{11} \equiv P_{14} \equiv P_{44} \equiv 0$ we have: $P_{13} \equiv P_{24} \equiv P_{12} - P_{34} \equiv \Psi_2 \equiv 0$. Then, combining the Bianchi identities (85) and (91), we get

$$ 2P_{33}\kappa + 2(P_{23} - 3\Psi_3)\sigma \equiv 0. $$

Similarly, using the Bianchi identities (86) and (92) we get:

$$ 2(P_{23} + 3\Psi_3)\kappa + 2P_{22}\sigma \equiv 0. $$

Thus, in such case, the situation is similar to the previously considered case with the matrix $m$: Now we have

$$ m_1 = \begin{pmatrix} P_{33} & -3\Psi_3 + P_{23} \\ 3\Psi_3 + P_{23} & P_{22} \end{pmatrix}, $$

and if $m_1$ has rank **identically** equal to two, we conclude that $\kappa \equiv \sigma \equiv 0$. If it has rank **identically** equal to zero, we in addition have $P_{33} \equiv P_{22} \equiv P_{23} \equiv \Psi_3 \equiv 0.$
This, due to the Bianchi identities, implies also that $P_{12} \equiv P_{34} \equiv \text{const}$. Comparing this with (87) and (88) leads to

$$\Psi_4 \sigma \equiv \Psi_4 \kappa \equiv 0,$$

which if we assume $\Psi_4 \neq 0$, yields $\kappa \equiv \sigma \equiv 0$.

This remark emphasizes that the local properties of the matrices $m$ and $m_1$ are crucial for the behaviour of $\kappa$ and $\sigma$. Since we have no guarantee that rank of e.g. $m_1$ is locally constant, returning to our proof, we must strengthen our assumptions on $g$ by requiring that it satisfies more curvature conditions than $P_{11} \equiv P_{14} \equiv P_{44} \equiv 0$.

- The additional conditions which enable us to get $\kappa \equiv \sigma \equiv 0$ are:

$$P_{13} \equiv P_{22} \equiv P_{23} \equiv P_{24} \equiv P_{33} \equiv P_{12} - P_{34} \equiv 0.$$

These, with the already assumed $P_{11} \equiv P_{14} \equiv P_{44} \equiv 0$, constitute the full set of Einstein conditions $\text{Ric}(g) = \Lambda g$, for the metric $g$.

- Under the Einstein assumption $\text{Ric}(g) = \Lambda g$ and the requirement that the selfdual part of the Weyl tensor is nonvanishing, we get $\kappa \equiv \sigma \equiv 0$ in a very easy way, by a successive inspection of the Bianchi identities (83), (84), (85),(86),(87),(88).

- Indeed, the assumed Einstein equations $P_{11} \equiv P_{14} \equiv P_{44} \equiv 0$, $P_{22} \equiv P_{23} \equiv P_{24} \equiv P_{33} \equiv P_{12} - P_{34} \equiv 0$, the algebraic speciality conditions $\Psi_0 \equiv \Psi_1 \equiv 0$, and the Bianchi identities (83), (84), give $\sigma \Psi_2 \equiv 0$ and $\kappa \Psi_2 \equiv 0$. This means that whenever $\Psi_2 \neq 0$ we have $\kappa \equiv \sigma \equiv 0$. By continuity the points in which $\kappa$ or $\sigma$ are nonzero form open sets in $M$. On these sets $\Psi_2 \equiv 0$ everywhere. Thus the discussed situation has only two possible outcomes: either $\kappa \equiv \sigma \equiv 0$ (which finishes the proof), or we have $\Psi_2 \equiv 0$ in an open set, in addition to the assumed $\Psi_0 \equiv \Psi_1 \equiv 0$.

- In this latter case we look at the Bianchi identities (85) and (86), obtaining: $\sigma \Psi_3 \equiv 0$ and $\kappa \Psi_3 \equiv 0$. This again leads to either $\sigma \equiv \kappa \equiv 0$ or to $\Psi_3 \equiv 0$ in addition to $\Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv 0$.

- If $\Psi_3 \equiv 0$ the Bianchi identities (87) and (88) give: $\sigma \Psi_4 \equiv 0$ and $\kappa \Psi_4 \equiv 0$.

Thus if we want to have nonvanishing selfdual part of the Weyl tensor, we are forced to have $\kappa \equiv \sigma \equiv 0$.

- This finishes the proof in this direction.

Thus in going from $\left( \Psi_0 \equiv \Psi_1 \equiv 0 \right)$ to $\left( \kappa \equiv \sigma \equiv 0 \right)$, we are only able to prove the theorem in the classical (although with a possibly nonzero cosmological constant) Goldberg-Sachs version, namely Theorem 5.10, (2).

Whether it is possible to weaken the Einstein assumption above to $\text{Ric}_{\mathcal{N}^2} \equiv 0$ is an open question.

5.2. **Generalizing the Kundt-Thompson and the Robinson-Schild version.**

As noted by Kundt and Thompson [9] and Robinson and Schild [24], to achieve the algebraic speciality of the metric, when $\kappa \equiv \sigma \equiv 0$ has been assumed, it is sufficient to use weaker conditions than $P_{11} \equiv P_{14} \equiv P_{44} \equiv 0$. There are various approaches to obtain these conditions in the General Relativity literature (see e.g. [18]). In this section we present our approach, which is signature independent.

We first assume that $P_{11} \equiv P_{14} \equiv P_{44} \equiv 0$ holds *only conformally*. Thus we merely assume that there exists a scale $\Upsilon : \mathcal{M} \to \mathbb{R}$ such that the rescaled metric
\[ \hat{g} = e^{2\Upsilon} g \] satisfies
\[ \hat{Ric}_{|\mathcal{N}} = 0, \]
where \( \mathcal{N} = \text{Span}_C(m, k) \). This means that choosing a null coframe \((M, P, N, K)\) for \( g \), and the corresponding rescaled null coframe \( \hat{M} = e^{\Upsilon} M, \hat{P} = e^{\Upsilon} P, \hat{N} = e^{\Upsilon} N \) and \( \hat{K} = e^{\Upsilon} K \) for \( \hat{g} \) we have
\[ \hat{P}_{11} \equiv \hat{P}_{14} \equiv \hat{P}_{44} = 0. \]

Note that for this to be satisfied we do not need to assume \( \hat{P}_{11} \equiv \hat{P}_{14} \equiv \hat{P}_{44} = 0 \).

Our aim now is to deduce what restrictions on \( g \) are imposed by equations (29).

As it is well known (see e.g. [5]) the rescaled Schouten tensor \( \hat{P} \) is related to \( P \) via:
\[ \hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} \Upsilon_c \Upsilon^c g_{ab}, \]
with \( \Upsilon_a = \nabla_a \Upsilon \). Now, applying the covariant derivative \( \nabla_c \) on both sides of this equation, antisymmetrizing over the indices \( \{ca\} \) and using again this equation to eliminate the covariant derivatives of \( \Upsilon_a \) we get
\[ \nabla_c [\hat{P}_{ab}] + \Upsilon_{[a} \hat{P}_{b]c} + \Upsilon^d \hat{P}_{d[a} g_{b]} \equiv \frac{1}{2} (A_{bca} + C_{acb} \Upsilon_d). \]

Here \( A_{abc} \) is the Cotton tensor
\[ A_{abc} = 2\nabla_{[a} P_{c]b}, \]
and \( C_{abcd} \) is the Weyl tensor. Note that in addition to
\[ A_{abc} = -A_{acb}, \]
as a consequence of the first and the second Bianchi identities, we also have:
\[ A_{abc} + A_{cab} + A_{bca} = 0, \]
and
\[ A_{abc} = \nabla_d C_{dabc}, \quad A_{ab}^d = 0, \]
respectively.

The obtained identity (30) is a generalization of the identity known in the theory of conformally Einstein spaces (see e.g. [5]). It is interesting on its own, but it is particularly useful in our situation of equations (29).

Let as assume that in addition to (29) the distribution of totally null planes \( \mathcal{N} \) is integrable. This means that in the frame \((m, p, n, k)\) we have
\[ \kappa \equiv \sigma = 0, \]
which is the same as assuming that the respective connection coefficients satisfy
\[ \Gamma_{414} \equiv \Gamma_{411} = 0. \]

As we proved in the previous section this implies that the Weyl tensor coefficient
\[ \Psi_0 = 0. \]

Now, using the frame \((m, p, n, k)\) and our assumptions (29) and (33) on the l.h.s of the identity (30), we directly check that the following proposition is true:

**Proposition 5.15.** Suppose that a distribution of totally null 2-planes \( \mathcal{N} \) on \((M, g)\) be integrable, \([\mathcal{N}, \mathcal{N}] \subset \mathcal{N} \), and that the Schouten tensor \( \hat{P} \) of the rescaled metric \( \hat{g} = e^{2\Upsilon} g \) is degenerate on \( \mathcal{N} \), \( \hat{P}_{|\mathcal{N}} \equiv 0 \). Then for every three vector fields \( X, Y, Z \in \mathcal{N} \) we have:
\[ X^a Y^b Z^c (\nabla_c [\hat{P}_{ab}] + \Upsilon_{[a} \hat{P}_{b]c} + \Upsilon^d \hat{P}_{d[a} g_{b]} \equiv 0. \]
Since, in addition, in the coframe \((m, p, n, k)\) the Weyl tensor coefficient \(\Psi_0 \equiv 0\), the r.h.s. of (30), after being contracted with vectors \(X, Y, Z\) from \(\mathcal{N}\), includes only the Weyl tensor coefficient \(\Psi_1\). Thus the considered identity, when restricted to \(\mathcal{N}\), reduces to two complex equations:
\[
(34) \quad A_{441} - \Psi_1 \delta \Upsilon \equiv 0,
\]
and
\[
(35) \quad A_{414} - \Psi_1 D \Upsilon \equiv 0.
\]
This relates the components \(\{141\}\) and \(\{441\}\) of the Cotton tensor \(\text{algebraically}\) to the Weyl tensor coefficient \(\Psi_1\), and proves the following

**Proposition 5.16.** A metric \(g\) with an integrable field of selfdual totally null 2-planes \(\mathcal{N}\) on a 4-dimensional manifold \(\mathcal{M}\) admits a conformal scale \(\Upsilon : \mathcal{M} \to \mathbb{R}\) such that the rescaled metric \(\hat{g}\) has Ricci tensor \(\hat{\text{Ric}}\) degenerate on \(\mathcal{N}\),
\[
\hat{\text{Ric}}|_{\mathcal{N}} \equiv 0,
\]
only if the Cotton tensor \(A\) of the original metric satisfies equations (34), (35) in a null coframe in which \(\kappa \equiv \sigma \equiv 0\).

It is interesting that the expressions (34) and (35) appear also in the following

**Proposition 5.17.** Suppose that a metric \(g\) admits an integrable maximal totally null field of 2-planes. Then the Cotton tensor components \(A_{141}\) and \(A_{441}\) in the null coframe \((M, P, N, K)\) in which \(\kappa \equiv \sigma \equiv 0\) are related to the Cotton tensor components \(\hat{A}_{141}\) and \(\hat{A}_{441}\) in the null coframe \((e^\Upsilon M, e^\Upsilon P, e^\Upsilon N, e^\Upsilon K)\) of the rescaled metric \(\hat{g} = e^{3\Upsilon} g\) via
\[
(36) \quad \hat{A}_{141} = e^{-3\Upsilon} (A_{141} - \Psi_1 \delta \Upsilon),
\]
\[
(37) \quad \hat{A}_{441} = e^{-3\Upsilon} (A_{441} - \Psi_1 D \Upsilon).
\]

The proof of this fact is straightforward. For example it can be checked in the Newman-Penrose formalism with \(\kappa = \sigma = 0\), in which the relevant components of the Cotton tensor read:
\[
(38) \quad A_{141} = DP_{11} - 2\delta P_{14} + (2\epsilon - 2\epsilon + \rho')P_{11} + (2\beta + 2\pi')P_{14} - \lambda P_{44},
\]
\[
(39) \quad A_{441} = DP_{14} - 2\delta P_{44} - \kappa P_{11} + (2\rho' - 2\epsilon)P_{14} + (2\alpha' + 2\beta + \pi')P_{44}.
\]

Now, treating the Cotton tensor \(A\) as a linear map \(T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}\), we recall that \(A\) is \text{degenerate} on a vector distribution \(Z, A_z = 0\), iff \(A(Z_1, Z_2, Z_3) = 0\) for all \(Z_1, Z_2, Z_3 \in \mathcal{Z}\). Then, if we take \(\mathcal{N} = \text{Span}_c(m, k)\), where \((m, p, n, k)\) is a null frame, we see that \(A_{441} = A_{141} = 0\) if and only if \(A|_{\mathcal{N}} = 0\). This together with Propositions 5.16 and 5.17 imply the following

**Corollary 5.18.** Suppose that a metric \(g\) admits an integrable maximal totally null field \(\mathcal{N}\) of 2-planes. If the metric can be conformally rescaled to \(\hat{g}\) so that the rescaled Ricci tensor \(\hat{\text{Ric}}\) is degenerate on \(\mathcal{N}\), \(\hat{\text{Ric}}|_{\mathcal{N}} \equiv 0\), then in this scale the rescaled Cotton tensor \(\hat{A}\) is degenerate on \(\mathcal{N}\), \(\hat{A}|_{\mathcal{N}} \equiv 0\).

**Remark 5.19.** We note that given an integrable totally null field of 2-planes \(\mathcal{N}\) the condition \(A|_{\mathcal{N}} \equiv 0\) is weaker than \(\text{Ric}|_{\mathcal{N}} \equiv 0\). We saw that \(\text{Ric}|_{\mathcal{N}} \equiv 0\) implies \(\hat{A}|_{\mathcal{N}} \equiv 0\), but the converse is not guaranteed.
Now we use the Bianchi identities (93) and (94), which we display here as the following

**Lemma 5.20.** On any 4-dimensional manifold with a metric $g$ as in (1) we have

\begin{align}
A_{141} & \equiv \triangle \Psi_0 + (\mu - 4\gamma)\Psi_0 - 3\sigma \Psi_2 \\
A_{441} & \equiv \partial \Psi_0 - (\pi + 4\alpha)\Psi_0 + D\Psi_1 + 2(2\rho - \varepsilon)\Psi_1 + 3\kappa \Psi_2.
\end{align}

**Proof.** This is proved in the Appendix, but we can also see this by observing that subtracting (40) from (38) and, respectively (41) from (39) we obtain the respective Bianchi identities (83) and (84). \qed

This Lemma is crucial for the rest of our arguments in this section. It has various consequences, the first being the following sharper version of part one of Theorem 5.10:

**Theorem 5.21.** Let $N \subset T^\mathbb{C}M$ be a field of totally null 2-planes on a 4-dimensional manifold $(M, g)$ equipped with metric $g$. Assume that the Cotton tensor $A$ of the metric $g$, considered as a threeform on $T^\mathbb{C}M$, is degenerate on $N$.

\[ A_{|N} \equiv 0. \]

If in addition the field $N$ is integrable, $[N, N] \subset N$, everywhere on $M$, then $(M, g)$ is algebraically special at every point, with a field of multiple principal totally null 2-planes tangent to $N$.

**Proof.** In an adapted null coframe $(M, P, N, K)$ our integrability assumption is $\kappa \equiv \sigma \equiv 0$, which as we know, implies $\Psi_0 \equiv 0$. The assumption about the degeneracy of the Cotton tensor means $A_{141} \equiv A_{441} \equiv 0$, which together with $\Psi_0 \equiv 0$ and Lemma 5.20 gives the identities: $\delta \Psi_1 = 2(\beta + 2\gamma)\Psi_1$ and $D\Psi_1 = 2(\varepsilon - 2\rho)\Psi_1$. This implies $\Psi_1 \equiv 0$ by Lemma 5.13. Thus the field of (principal) totally null 2-planes $N$ is multiple. \qed

**Remark 5.22.** Note that as a result of this theorem, the assumption $A_{|N} \equiv 0$ is conformal. Without knowing that $\kappa \equiv \sigma \equiv 0$, we would like to have an assumption about vanishing of still higher order derivatives of the curvature, that together with $\kappa \equiv \sigma \equiv 0$ would imply $\Psi_1 \equiv 0$, the assumption $A_{141} \equiv A_{441} \equiv 0$ seemed to be not conformal, because of the inhomogeneous terms in the transformations (36)-(37). But since under the assumptions $\kappa \equiv \sigma \equiv 0$ and $A_{141} \equiv A_{441} \equiv 0$ we were able to discover that actually $\Psi_1 \equiv 0$, then $A_{141}$ and $A_{441}$ transform homogeneously under the conformal rescaling. Thus in such case the condition $A_{|N} \equiv 0$ is conformal.

The second application of Lemma 5.20 is included in the following

**Remark 5.23.** Suppose that we would like to have a still sharper (than in Theorem 5.21) version of part one of Theorem 5.10. Thus instead of assuming $\text{Ric}_{|N} \equiv 0$, or the weaker condition $A_{|N} \equiv 0$, we would like to have an assumption about vanishing of still higher order derivatives of the curvature, that together with $\kappa \equiv \sigma \equiv 0$ would imply $\Psi_1 \equiv 0$. Then Lemma (5.20) assures that it is impossible, and the condition $A_{|N} \equiv 0$ can not be weakened. Indeed, denoting such hypothetical condition by $S \equiv 0$, we would have $(\kappa \equiv \sigma \equiv 0 \& S \equiv 0) \Rightarrow (\Psi_1 \equiv 0)$. But since $\kappa \equiv \sigma \equiv 0$, in addition, implies that $\Psi_0 \equiv 0$, then Lemma 5.20 implies $A_{141} \equiv A_{441} \equiv 0$. Thus the hypothetically weaker than $A_{|N} \equiv 0$ condition $S \equiv 0$, in turn, implies $A_{|N} \equiv 0$. Since this alone, according to Theorem 5.21, is already sufficient to imply $\Psi_1 \equiv 0$, we do not need condition $S \equiv 0$ to obtain the desired result. This proves the following
Theorem 5.24. The weakest curvature condition which together with the integrability condition \([N, N] \subset N\), implies that the field of totally null 2-planes \(N\) is principal and multiple is the degeneracy of the Cotton tensor on \(N\), \(A_{|N|} \equiv 0\).

Example 5.25. An example of a condition \(S \equiv 0\) which is a priori weaker than \(A_{|N|} \equiv 0\) may be obtained as follows. The procedure used in the proof of Lemma 5.13 may be equally applied to the situation in which the conditions (21)-(22) are replaced by the Bianchi identities (40) and (41). Then, under the assumption that \(\kappa \equiv \sigma \equiv 0\), and hence \(\Psi_0 \equiv 0\), we literally repeat all the steps from the proof of Lemma 5.13. Indeed, starting with the application of \(\delta\) on both sides of \(A_{141} \equiv -\delta \Psi_1 + 2(2\tau + \beta)\Psi_1\) and \(D\) on both sides of \(A_{441} \equiv D\Psi_1 + 2(2\rho - \varepsilon)\Psi_1\), after subtraction and use of the commutator (13), we obtain the following identity:

\[
DA_{141} - \delta A_{441} - (3\epsilon - \rho' - \epsilon' - 4\rho)A_{141} + (3\beta + \alpha' + \pi' + 4\tau)A_{441}.
\]

This, is satisfied always when \(\kappa \equiv \sigma \equiv 0\). Thus the vanishing of the r.h.s of (42) implies \(\Psi_1 \equiv 0\). Moreover, since when \(\kappa \equiv \sigma \equiv 0\) the vanishing of \(\Psi_1\) is a conformal property, then the vanishing of the r.h.s. of (42) is a conformal property. In fact a direct calculation shows that if in a null coframe \((M, P, N, K)\) we have \(\kappa \equiv \sigma \equiv 0\) and

\[
S = DA_{141} - \delta A_{441} - (3\epsilon - \rho' - \epsilon' - 4\rho)A_{141} + (3\beta + \alpha' + \pi' + 4\tau)A_{441}
\]

then in the conformally rescaled metric \(\hat{g} = e^{2\gamma}g\) and in the corresponding rescaled null coframe \((e^\gamma M, e^\gamma P, e^\gamma N, e^\gamma K)\) we have \(\hat{\kappa} \equiv \hat{\sigma} \equiv 0\) and

\[
\hat{S} = e^{-4\gamma}S.
\]

Now using the explicit formulae for the covariant derivatives of the Cotton tensor components \(A_{141}\) and \(A_{441}\):

\[
\nabla_4 A_{141} = DA_{141} - (3\epsilon - \epsilon')A_{141} + \pi' A_{441},
\]

\[
\nabla_1 A_{441} = \delta A_{441} - (3\beta + \alpha')A_{441} - \rho'A_{141},
\]

solving this for \(DA_{141}\) and \(\delta A_{441}\) and inserting in (43), we get

\[
S = \nabla_4 A_{141} - \nabla_1 A_{441} + 4\rho A_{141} + 4\tau A_{441}.
\]

We thus have a condition \(S \equiv 0\), which together with \(\kappa \equiv \sigma \equiv 0\) is conformal and implies that \(\Psi_1 \equiv 0\). It is always satisfied when \(A_{441} \equiv A_{141} \equiv 0\), i.e. we have \((A_{441} \equiv A_{141} \equiv 0) \Rightarrow S \equiv 0\), and at the first glance there is no reason for the implication \((S \equiv 0) \Rightarrow (A_{441} \equiv A_{141} \equiv 0)\). However, this implication is true, on the ground of the discussion in Remark 5.23. As a consequence we have

**Proposition 5.26.** Under the assumption that the distribution of selfdual totally null 2-planes \(N\) is integrable, \([N, N] \subset N\), the following two, conformally invariant, conditions are equivalent

- the Cotton tensor of the metric \(g\) is degenerate on \(N\), \(A_{|N|} \equiv 0\)
- the scalar \(S\) of the metric \(g\), as defined in (44), identically vanishes, \(S \equiv 0\).

To discuss the next application of Lemma 5.20 we introduce

**Definition 5.27.** A metric \(g\) on a 4-dimensional manifold \(M\) is called **II-generic** if and only if the points in which its selfdual part of the Weyl tensor degenerates to Petrov types III, N or 0 are rare, in the sense that they belong to closed sets without interior in \(M\).
In particular every metric with selfdual part of the Weyl tensor being at each point of \( M \) algebraically general, or of mixed type: algebraically general on some subsets and type II or type D on their complements, is II-generic; a metric which is e.g. of type III in an open set of \( M \) is not II-generic.

Now we are ready to discuss a slight generalization of the known conformal versions of the Goldberg-Sachs theorem. In the Lorentzian case such versions were given by Kundt and Thompson [9] and Robinson and Schild [24]. Penrose and Rindler [18] gave a complex (spinorial) version of the Kundt-Thompson/Robinson-Schild theorem. Here we quote our complex version, which is a slight generalization:

**Theorem 5.28.** Let \( M \) be a 4-dimensional manifold with a II-generic metric \( g \). Let \( N \) be a field of selfdual totally null 2-planes on \( M \). Then any two of the following imply the third:

1. The Cotton tensor of \( g \) is degenerate on \( N \), \( A_{1N} \equiv 0 \).
2. \( N \) is integrable, \( [N, N] \subset N \).
3. The selfdual part of the Weyl tensor is algebraically special on \( M \) with \( N \) being a multiple principal field of selfdual totally null 2-planes.

**Proof.** First we observe that the implication \( (0) \& (i) \Rightarrow (ii) \) is true, as a simple application of Theorem 5.21. Note that for this we do not need the genericity assumption about the Weyl tensor.

To prove the other two implications we choose a null coframe on \((M, g)\) so that \( N = \text{Span}_\mathbb{C}(m, k) \) and \( g = 2(MP + NK) \) as in (1). Then

- the condition (0) is: \( A_{141} \equiv A_{441} \equiv 0 \),
- the condition (i) is: \( \kappa \equiv \sigma \equiv 0 \),
- the condition (ii) is: \( \Psi_0 \equiv \Psi_1 \equiv 0 \).

Now, the proof of \( (i) \& (ii) \Rightarrow (0) \) is an immediate consequence of Lemma 5.20, since the assumptions (i) \& (ii) imply the identical vanishing of the r.h.s. of identities (40)-(41), which means that also their l.h.s. identically vanish, \( A_{141} \equiv A_{441} \equiv 0 \). Note that also in the proof of this statement the genericity assumption about the Weyl tensor was not needed.

This assumption is however needed to get the last implication \( (0) \& (ii) \Rightarrow (i) \).

Indeed assuming (i) \& (ii), the identities (40)-(41) from Lemma 5.20 reduce to the identities \(-3\sigma \Psi_2 \equiv 0\) and \(3\kappa \Psi_2 \equiv 0\). Now, similarly as in the proof of part two of the Theorem 5.10, to conclude that \( \kappa \equiv \sigma \equiv 0 \) in a neighbourhood \( U \subset M \), it is enough to assume that \( \Psi_2 \neq 0 \) on the complement of the closed sets without interior in \( U \). Since in our coframe in \( U \), according to (ii), we have \( \Psi_0 \equiv \Psi_1 \equiv 0 \), Proposition 5.7 assures that the coefficient \( \Psi_2 \) of the Weyl tensor is nonvanishing on the complement of the closed sets without interior in \( U \) if and only if the metric is II-generic in \( U \). Since this is the main assumption of Theorem 5.28 we see that \( 3\sigma \Psi_2 \equiv 0 \) and \( 3\kappa \Psi_2 \equiv 0 \) imply \( \kappa \equiv \sigma \equiv 0 \) in \( U \). This proves the part \( (0) \& (ii) \Rightarrow (i) \) of the theorem. \( \square \)

As a consequence of this proof we also have the following

**Corollary 5.29.** Let \( M \) be a 4-dimensional manifold with a metric \( g \) and let \( N \) be a field of selfdual totally null 2-planes on \( M \). Assume that \( N \) is integrable, \( [N, N] \subset N \), and that the selfdual part of the Weyl tensor is algebraically special
on \( M \), with \( N \) being a multiple principal field of selfdual totally 2-planes. Then the Cotton tensor of \( g \) is degenerate on \( N \). \( A|_N \equiv 0 \).

To discuss the sharpening of the Theorem 5.28 with respect to the implication \((0) \& (\text{ii}) \Rightarrow (\text{i})\) we introduce two more notions analogous to the II-genericticity.

**Definition 5.30.** A metric \( g \) on a 4-dimensional manifold \( M \) is called III-generic if and only if the points in which its selfdual part of the Weyl tensor degenerates to Petrov types \( N \) or 0 belong to closed sets without interior in \( M \). Similarly, a metric \( g \) on a 4-dimensional manifold \( M \) is called \( N \)-generic if and only if the points in which its selfdual part of the Weyl tensor vanishes belong to closed sets without interior in \( M \).

For the III-generic metrics we have the following

**Theorem 5.31.** Let \( M \) be a 4-dimensional manifold with a III-generic metric \( g \), whose selfdual part of the Weyl tensor is in addition algebraically special at all points of \( M \). Let \( N \) be the corresponding field of multiple principal totally null 2-planes on \( M \). If the Cotton tensor \( A \) of the metric \( g \) satisfies
\[
A(\cdot, Z_1, Z_2) \equiv 0, \quad \forall Z_1, Z_2 \in N,
\]
then the field \( N \) is integrable, \([N, N] \subset N\), on \( M \).

Similarly for the \( N \)-generic metrics we have

**Theorem 5.32.** Let \( M \) be a 4-dimensional manifold with an \( N \)-generic metric \( g \), whose selfdual part of the Weyl tensor is in addition algebraically special at all points of \( M \). Let \( N \) be the corresponding field of multiple principal totally null 2-planes on \( M \). Consider the 2-forms \( A_Z = A(Z, \cdot, \cdot) \), where \( A \) is the Cotton tensor of the metric \( g \) and \( Z \) is a complex-valued vector field on \( M \). If for every vector field \( Z \in N \) the two form \( A_Z \) is antiselfdual at each point of \( M \), then the field \( N \) is integrable, \([N, N] \subset N\), on \( M \).

We first prove Theorem 5.31.

**Proof.** Again we choose a null coframe on \((M, g)\) so that \( N = \text{Span}_{\mathbb{C}}(m, k) \) and \( g = 2(MP + NK) \) as in (1). Since \( N \) consists of multiple principal null 2-planes, according to Proposition 5.7, we have \( \Psi_0 \equiv \Psi_1 \equiv 0 \) in this coframe. Moreover, in this coframe the condition \( A(\cdot, Z_1, Z_2) \equiv 0 \ \forall Z_1, Z_2 \in N \) means that the coframe components \( A_{i41}, i = 1, 2, 3, 4 \) satisfy
\[
(45) \quad A_{141} \equiv A_{214} \equiv A_{341} \equiv A_{414} \equiv 0.
\]

Now we again use the Bianchi identities (93)-(94) which reduce to
\[
-3\sigma \Psi_2 \equiv 0, \quad \text{and} \quad 3\kappa \Psi_2 \equiv 0.
\]

Similarly as in the proof of the second part of the Theorem 5.10 this yields \( \kappa \equiv \sigma \equiv 0 \), with the exception when \( \Psi_2 \equiv 0 \). In such a case we have
\[
(46) \quad \Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv 0,
\]
and these two Bianchi identities are tautologies. Thus to conclude something about \( \kappa \) and \( \sigma \) we need to use another pair of Bianchi identities. These are given by (95)-(96) and refer to the respective components \( A_{341} \) and \( 214 \) of the Cotton tensor. Now, with the assumed (45) and (46) these identities reduce to
\[
2\sigma \Psi_3 \equiv 0, \quad \text{and} \quad 2\kappa \Psi_3 \equiv 0.
\]
This does not yield \( \kappa \equiv \sigma \equiv 0 \) only if \( \Psi_3 \equiv 0 \) in the neighbourhood. But this is forbidden by our assumption that the metric is III-generic in the considered neighbourhood.

Thus if the metric is III-generic in the neighbourhood we proved that \( \kappa \equiv \sigma \equiv 0 \) in a frame adapted to \( \mathcal{N} \), which according to Proposition 5.1, means that \( \mathcal{N} \) is integrable. \( \square \)

Proof of Theorem 5.32. Choosing the null frame as in the above proof we first interpret the condition about the Cotton tensor 2-forms \( A_Z \) being all antiselfdual. Since \( \mathcal{N} \) is spanned by \( m \) and \( k \) we only need to consider the 2-forms \( A_m = A(m, \cdot , \cdot ) \) and \( A_k = A(k, \cdot , \cdot ) \). We have:

\[
A_m = A_{123} \theta^2 \wedge \theta^3 + \frac{1}{2}(A_{112} - A_{134})(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) + A_{114} \theta^1 \wedge \theta^4 + A_{115} \theta^1 \wedge \theta^3 + \frac{1}{2}(A_{112} + A_{134})(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4) + A_{124} \theta^2 \wedge \theta^4
\]

and

\[
A_k = A_{423} \theta^2 \wedge \theta^3 + \frac{1}{2}(A_{412} - A_{434})(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) + A_{414} \theta^1 \wedge \theta^4 + A_{413} \theta^1 \wedge \theta^3 + \frac{1}{2}(A_{412} + A_{434})(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4) + A_{424} \theta^2 \wedge \theta^4.
\]

So looking at the bases (11) and (12) of the selfdual and antiselfdual 2-forms \( \Sigma \) and \( \Sigma' \), we conclude that these 2-forms are antiselfdual if the following six conditions for the coframe components of the Cotton tensor are satisfied:

\[
\begin{align*}
A_{114} &\equiv A_{414} \equiv 0 & \text{(47)} \\
A_{112} - A_{134} &\equiv 0 & \text{(48)} \\
A_{412} - A_{434} &\equiv 0 & \text{(49)} \\
A_{123} &\equiv A_{423} \equiv 0 & \text{(50)}
\end{align*}
\]

Now we use the symmetries of the Cotton tensor to give equivalent forms of the conditions (48)-(49). Using (32) we get 

\[
A_{112} \equiv A_{341} - A_{413}
\]

and using (31) we get

\[
A_{134} \equiv -A_{413} - A_{341}.
\]

Subtracting the latter from the former we get the identity

\[
A_{112} - A_{134} \equiv 2A_{341}.
\]

In the similar way we prove the identity

\[
A_{412} - A_{434} \equiv 2A_{214}.
\]

Comparing these two identities with (47)-(50) we conclude that the condition that \( A_Z \) is antiselfdual for all \( Z \in \mathcal{N} \), in our coframe, is equivalent to the six conditions

\[
\begin{align*}
A_{114} &\equiv A_{414} \equiv 0 & \text{and} \\
A_{112} - A_{134} &\equiv 0 & \text{and} \\
A_{412} - A_{434} &\equiv 0 & \text{and} \\
A_{123} &\equiv A_{423} \equiv 0.
\end{align*}
\]

Since the first four conditions are precisely \( A(\cdot , Z_1, Z_2) \equiv 0 \) for \( Z_1, Z_2 \in \mathcal{N} \), we now use Theorem 5.31 to conclude \( \kappa \equiv \sigma \equiv 0 \), provided that we are not in the situation when

\[
\Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv \Psi_3 \equiv 0
\]

in the neighbourhood. If this is the case, to show that we still have \( \kappa \equiv \sigma \equiv 0 \) we need the additional assumption (30). With this and (51) being assumed, using the Bianchi identities (97)-(98), we easily obtain

\[
-\sigma \Psi_4 \equiv 0 \quad \text{and} \quad \kappa \Psi_4 \equiv 0.
\]
This implies that \( \kappa \equiv \sigma \equiv 0 \) in the neighbourhood, on the ground of the \( N \)-genericity of the metric. This finishes the proof. \( \square \)

As a counterpart to Corollary 5.29 we have

**Corollary 5.33.** Let \( M \) be a 4-dimensional manifold with a metric \( g \) and let \( N \) be a field of selfdual totally null 2-planes on \( M \). Assume that \( N \) is integrable, \( [N, N] \subset N \), and that the selfdual part of the Weyl tensor is algebraically special on \( M \) with \( N \) being a multiple principal field of selfdual totally null 2-planes. Then if \( N \) has multiplicity equal to three the Cotton tensor of \( g \) satisfies \( A(\cdot, Z_1, Z_2) \equiv 0 \) for all \( Z_1, Z_2 \in N \). If \( N \) has multiplicity equal to four the 2-form \( A_Z \) of the Cotton tensor \( A \) of \( g \) is antiselfdual.

**Proof.** The proof is an immediate application of the Bianchi identities (93)-(98). \( \square \)

### 6. Interpretation in terms of a characteristic connection

The terms \( 4\rho A_{414} + 4\tau A_{441} \) that appear in formula (44) defining \( S \) in Example 5.25 suggests that to describe the geometry of manifolds with \( \kappa \equiv \sigma \equiv 0 \) it would be useful to have a vectorial object, say \( B_a \), with components \( B_a \) being roughly

\[
B_a = (B_1, B_2, B_3, B_4) = (4s^{-1}\tau, B_2, B_3, -4s^{-1}\rho),
\]

where \( s \) is a complex constant. If we were able to find a geometric way of distinguishing such \( B_a \), then the formula for \( S \) would be \( S = (\nabla_4 - sB_4)A_{441} - (\nabla_1 - sB_1)A_{441} \) and would have an explicit geometric meaning. Note that the values of components \( B_2 \) and \( B_3 \) are totally irrelevant here! In this section we show how to geometrically distinguish such (partially determined) \( B_a \).

**6.1. Characteristic connection of a totally null 2-plane.** Let us chose an arbitrary 1-form \( B = B_a \theta^a \) on \( (M, g = g_{ab}\theta^a\theta^b) \). Given a choice of \( B \) one defines a new connection \( \nabla^W \) on \( M \), which is related to the Levi-Civita connection as follows.

Let \( \Gamma_{ab} = \Gamma_{abc}\theta^c \), be the Levi-Civita connection 1-forms as given in (4). Define

\[
\Gamma^W_{abc} = \Gamma_{abc} + \frac{1}{2}(g_{ca}B_b - g_{cb}B_a + g_{ab}B_c).
\]

Then the new connection \( \nabla^W \) is defined on \( M \) by

\[
\nabla^W_X e_b = X^c \nabla^W_X e_b = X^c \nabla^W_{\theta^c} e_b, \quad \Gamma^W_{a\,bc} = g^{ad} \Gamma_{dbc},
\]

where \( (e_a) \) is a frame dual to the coframe \( (\theta^a) \), \( e_a \cdot \theta^b = \delta^b_a \).

The connection \( \nabla^W \) is called the **Weyl connection**. It is the unique **torsionless** connection satisfying

\[
\nabla^W g = -Bg.
\]

It has the nice property of being **conformal** in the sense that if the metric \( g \) undergoes a transformation \( g \to \hat{g} = e^{2\phi} g \), then equation (55) is preserved,

\[
\nabla^W \hat{g} = -\hat{B}\hat{g},
\]

with a mere change \( B \to \hat{B} = B - 2d\phi \).

The conformal properties of Weyl connections would be very interesting for our purpose of describing conformal conditions for the Goldberg-Sachs theorem, provided that, we were able to associate a unique Weyl form \( B \) with the main object
of this theorem namely a field of totally null 2-planes \( \mathcal{N} \). The following theorem shows that although such a natural way of choosing \( B \) is possible only partially, it nevertheless enables us to define a canonical connection on \( \mathcal{N} \), which encodes its conformal properties.

**Theorem 6.1.** Let \( \mathcal{N} \) be a field of totally null 2-planes on \((\mathcal{M}, g)\), where \( g \) is a 4-dimensional metric of any (including complex) signature. Let us assume that \( \mathcal{N} \) is integrable [\( \mathcal{N}, \mathcal{N} \) \( \subset \mathcal{N} \). Then there exists a unique connection \( \nabla \) on \( \mathcal{N} \), which encodes the conformal properties of this field of totally null 2-planes.

**Proof.** We define the connection \( \nabla \) in two steps.

**Step One.** We first look for a Weyl connection \( \nabla \) on \( \mathcal{M} \), as in (53)-(54), which has the property that it preserves \( \mathcal{N} \). This means that we ask if there exists a Weyl connection \( \nabla \) on \( \mathcal{N} \), such that

\[
(w_\nabla e_1) \wedge e_1 \wedge e_4 = 0, \quad (w_\nabla e_4) \wedge e_1 \wedge e_4 = 0, \quad \forall \ c = 1, 2, 3, 4,
\]

where we abbreviated \( w_\nabla e_c \) to \( w_\nabla e_c = w_\nabla c \). It is very easy to see that, since in the chosen frame the coefficients of the metric \( g_{ab} \) are all zero, except \( g_{12} = g_{21} = g_{34} = g_{43} = 1 \), then these conditions are equivalent to:

\[
(w_\nabla e_1) \wedge e_1 \wedge e_4 = \Gamma_{11c}e_2 \wedge e_1 \wedge e_4 + \Gamma_{41c}e_3 \wedge e_1 \wedge e_4 = 0 \quad \forall \ c = 1, 2, 3, 4
\]

\[
(w_\nabla e_4) \wedge e_1 \wedge e_4 = \Gamma_{14c}e_2 \wedge e_1 \wedge e_4 + \Gamma_{44c}e_3 \wedge e_1 \wedge e_4 = 0
\]

or, what is the same,

\[
\Gamma_{11c} = \Gamma_{41c} = \Gamma_{44c} = 0, \quad \forall \ c = 1, 2, 3, 4.
\]

Comparing these last equations with (53), we easily see that \( \Gamma_{11c} = \Gamma_{44c} = 0 \) is automatically satisfied for all \( c = 1, 2, 3, 4 \), and then, by considering the remaining conditions \( \Gamma_{14c} = \Gamma_{41c} = 0 \), we see that (56) is equivalent to:

(57) \[ \Gamma_{14c} + \frac{1}{2}(g_{c1}B_4 - g_{c4}B_1) = 0 \quad \forall \ c = 1, 2, 3, 4. \]

Now examining these equations for \( c = 1 \) and \( c = 4 \) we get the conditions that the Levi-Civita connection coefficients \( \Gamma_{141} \) and \( \Gamma_{144} \) must satisfy

(58) \[ \Gamma_{141} = \Gamma_{144} = 0. \]

Examining the equations (57) for \( c = 2 \) and \( c = 3 \), we get the relations between the components \( B_1 \) and \( B_4 \) of the 1-form \( B \) and the Levi-Civita connection coefficients \( \Gamma_{143} \) and \( \Gamma_{142} \). These are:

(59) \[ B_3 = 2\Gamma_{143}, \quad B_4 = -2\Gamma_{142}. \]

Thus, the requirement that there is a Weyl connection preserving \( \mathcal{N} \) is equivalent to the fact that in a coframe adapted to \( \mathcal{N} \), we have (58) and (59). Since \( \Gamma_{141} \) and \( \Gamma_{144} \), in the coframe adapted to \( \mathcal{N} \), are \( \Gamma_{141} = \sigma \) and \( \Gamma_{144} = \kappa \), then we see that the
The connection $\nabla^w$ exists only if the field of totally null 2-planes $\mathcal{N}$ is integrable. When $\mathcal{N}$ is integrable then, in the adapted coframe $(\theta^i)$, the two of the components of the Weyl 1-form $B$, namely $B_1$ and $B_4$, are totally determined. They are equal to

$$B_1 = 2\tau, \quad B_4 = -2\rho,$$

as desired in (52), with $s = 2$.

Concluding this part of the proof, we say that the condition (56) that the Weyl connection preserves $\mathcal{N}$ determines this connection only up to the terms $B_2$ and $B_3$ in the Weyl 1-form. In step two of the proof we restrict this connection to $\mathcal{N}$.

**Step two.** Since $\nabla^w$ preserves $\mathcal{N}$ in any direction then, in particular, it preserves it along $\mathcal{N}$. Thus $\nabla^w$, with any choice of $B_2$ and $B_3$, restricts naturally to $\mathcal{N}$. But *apriori* this restriction may depend on the choice of $B_2$ and $B_3$. That this is not the case follows from the following.

First observe that because of (60), we have

$$\begin{align*}
\Gamma^w_{211} &= \Gamma_{211} + B_1, \quad \Gamma^w_{111} = 0, \quad \Gamma^w_{411} = 0, \quad \Gamma^w_{311} = \Gamma_{311} \\
\Gamma^w_{214} &= \Gamma_{214} + \frac{1}{2}B_2, \quad \Gamma^w_{114} = 0, \quad \Gamma^w_{414} = 0, \quad \Gamma^w_{314} = \Gamma_{314} + \frac{1}{2}B_2 \\
\Gamma^w_{241} &= \Gamma_{241} + \frac{1}{2}B_3, \quad \Gamma^w_{141} = 0, \quad \Gamma^w_{441} = 0, \quad \Gamma^w_{341} = \Gamma_{341} + \frac{1}{2}B_3 \\
\Gamma^w_{244} &= \Gamma_{244}, \quad \Gamma^w_{144} = 0, \quad \Gamma^w_{444} = 0, \quad \Gamma^w_{344} = \Gamma_{344} + B_4.
\end{align*}$$

Thus the covariant derivatives

$$\begin{align*}
\nabla^w_{1}e_1 &= \Gamma^w_{11}e_1 = \frac{1}{2}\Gamma_{111}e_1 + \frac{1}{2}\Gamma_{111}e_2 + \frac{1}{2}\Gamma_{411}e_3 + \frac{1}{2}\Gamma_{311}e_4 \\
\nabla^w_{4}e_1 &= \Gamma^w_{14}e_1 = \frac{1}{2}\Gamma_{141}e_1 + \frac{1}{2}\Gamma_{141}e_2 + \frac{1}{2}\Gamma_{441}e_3 + \frac{1}{2}\Gamma_{341}e_4 \\
\nabla^w_{1}e_4 &= \Gamma^w_{41}e_4 = \frac{1}{2}\Gamma_{411}e_1 + \frac{1}{2}\Gamma_{411}e_2 + \frac{1}{2}\Gamma_{441}e_3 + \frac{1}{2}\Gamma_{341}e_4 \\
\nabla^w_{4}e_4 &= \Gamma^w_{44}e_4 = \frac{1}{2}\Gamma_{441}e_1 + \frac{1}{2}\Gamma_{441}e_2 + \frac{1}{2}\Gamma_{444}e_3 + \frac{1}{2}\Gamma_{344}e_4
\end{align*}$$

of vectors $(e_1, e_4)$ in the directions $e_1$ and $e_4$ spanning $\mathcal{N}$, are expressible purely in terms of the Levi-Civita connection coefficients $\Gamma^{abc}$ and the totally determined part of $B$. In these relations the unknown coefficients of $B$, namely $B_2$ and $B_3$, do not appear!

Thus $\nabla^w$ restricts to a *unique* and totally determined connection on $\mathcal{N}$. We define

$$\nabla = \nabla^w_{|\mathcal{N}} \quad \text{on} \quad \mathcal{N}.$$ 

Since this connection is constructed with only conformal objects, it is manifestly conformal.

The formulae for this connection in the Newman-Penrose formalism are:

$$\begin{align*}
\nabla_m m &= (\beta - \alpha' + 2\tau)m - \lambda'k \\
\nabla_k m &= (\varepsilon - \varepsilon' - \rho)m + (\tau - \pi')k \\
\nabla_m k &= (\rho' - \rho)m + (\alpha' + \beta + \tau)k \\
\nabla_k k &= \kappa'm + (\varepsilon + \varepsilon' - 2\rho)k.
\end{align*}$$

□
The connection $\nabla$ defined in Theorem 6.1 is called the characteristic connection of an integrable totally null 2-plane $\mathcal{N}$ field.

Now, having any three (complex-valued) vector fields $X, Y, Z \in \mathcal{N}$, we define the torsion $\mathcal{T}$ and the curvature $\mathcal{R}$ of $\nabla$ via the usual:

$$
\mathcal{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],
$$

$$
\mathcal{R}(X,Y)Z = [\nabla_X,\nabla_Y]Z - \nabla_{[X,Y]}Z.
$$

By construction these are conformal tensors defined on $\mathcal{N}$. Since both $\mathcal{T}$ and $\mathcal{R}$ are antisymmetric in $X, Y$ they may have at most two, respectively four, independent components. Actually we have the following

**Theorem 6.2.** The characteristic connection $\nabla$ of an integrable $\mathcal{N}$ is torsionless, $\mathcal{T} \equiv 0$.

Its curvature, $\mathcal{R}$, is given by

$$
\mathcal{R}(m,k)m = 4\Psi_1 m,
$$

$$
\mathcal{R}(m,k)k = 4\Psi_1 k,
$$

where $\Psi_1$ is the Weyl tensor coefficient of the Levi-Civita connection as defined in (10).

**Proof.** The torsionless property of the connection and the formulae (63)-(64) for the curvature can be checked by a direct calculation. Indeed, for the torsionless we only have to show that $\mathcal{T}(m,k) = 0$. One checks that this is a direct consequence of the definitions (61), (60) and the commutation relation $[\delta, D]$ from (13). To check (63) one uses the definition (62), the commutator $[\delta, D]$ and the Newman-Penrose equations (75), (77), (78), (79) and (82). Similarly, to check (64) one uses (62), (13) and the Newman-Penrose equations (75), (76), (79), (80) and (81). In all of these expressions one has to put the integrability conditions $\kappa \equiv \sigma \equiv 0$. The rest of the proof is easy pure algebra. $\square$

Thus we see that the curvature of $\nabla$ has only one independent component, which is a constant multiple of $\Psi_1$. Moreover, the entire curvature, which may be identified with the curvature operator $\mathcal{R}(m,k) : \mathcal{N} \rightarrow \mathcal{N}$, satisfies

$$
\mathcal{R}(m,k) = (4\Psi_1)\text{Id}_\mathcal{N}.
$$

Recalling that $\Psi_1$ is that part of the selfdual part of the Weyl tensor, which if vanishes, makes it algebraically special, we have the following

**Corollary 6.3.** A 4-dimensional manifold $\mathcal{M}$ with a metric $g$ and an integrable field of totally null 2-planes $\mathcal{N}$ is algebraically special if and only if the characteristic connection $\nabla$ of $\mathcal{N}$ is flat, i.e. iff its curvature $\mathcal{R} \equiv 0$.

This proves the following Proposition.

**Proposition 6.4.** A 4-dimensional manifold $(\mathcal{M},g)$ is algebraically special iff it possesses an integrable field of totally null 2-planes whose characteristic connection is flat.
6.2. Characteristic connection and the sharpest Goldberg-Sachs theorem. Given an integrable field of totally null 2-planes $N$ we have the corresponding characteristic connection $\nabla$. Let $(f_A) = (f_1, f_2)$ be a frame in $N$. In the previous section we found that the curvature of $\nabla$ in the basis $(f_A) = (m, k)$ is

$$\tilde{R}^A_{BCD} = 4\Psi_1 \delta^A_B \epsilon_{CD},$$

where $A, B, C, D = 1, 2, (\delta^A_B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $(\epsilon_{CD}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus, in particular, the ‘Ricci tensor’ $\tilde{R}_{AB} = \tilde{R}^C_{ACB}$ of this connection is antisymmetric and equal to

$$\tilde{R}_{AB} = 4\Psi_1 \epsilon_{AB}.$$

Since the curvature has only one component, it is obvious that the other possible contraction, namely $\tilde{R}^C_{ACB}$, is proportional to $\tilde{R}_{AB}$: $\tilde{R}^C_{ACB} = 2\tilde{R}_{AB}$. Using this Ricci tensor we are able to formulate the following, quite elegant, strengthening of the generalization of the Goldberg-Sachs theorem given in Theorem 5.21.

**Theorem 6.5.** Let $N \subset T^2 \mathcal{M}$ be an integrable field of totally null 2-planes on a 4-dimensional manifold $(\mathcal{M}, g)$ equipped with metric $g$. Assume that the tensor $\nabla_{(C \nabla_D)} \tilde{R}_{AB}$ vanishes everywhere on $\mathcal{M}$,

$$\nabla_{(C \nabla_D)} \tilde{R}_{AB} \equiv 0. \tag{65}$$

Then $(\mathcal{M}, g)$ is algebraically special at every point of $\mathcal{M}$, with a multiple field of principal totally null 2-planes tangent to $N$.

**Proof.** For every connection $\nabla_A$, the action of the operator $\nabla_{(C \nabla_D)}$ on any tensor is a suitable linear action of the curvature of $\nabla_A$ on this tensor. Since for $\nabla_A$ the curvature has only one component, namely $\nabla_{(C \nabla_D)} \tilde{R}_{AB}$ only involves a constant coefficient sum of terms of the form $\Psi_1 \tilde{R}_{AB}$. Since $\tilde{R}_{AB}$ itself is proportional to $\Psi_1$, because of the symmetry, we conclude that

$$\nabla_{(C \nabla_D)} \tilde{R}_{AB} = c \Psi_1^2 \epsilon_{AB} \epsilon_{CD}, \quad c = \text{const.}$$

The constant $c$ may be calculated in a particular basis, e.g. in the basis $(f_A) = (m, k)$. Using this basis, the definitions (60) and the Newman-Penrose equations from the Appendix, it is a matter of algebra to check that $c = -16$.

Now, if $\nabla_{[C \nabla_D]} \tilde{R}_{AB} \equiv 0$, then also $\Psi_1 \equiv 0$, and hence $\Psi_1 \equiv 0$. Since $N$ is integrable, then we also have $\Psi_0 \equiv 0$, which means that $N$ is a multiple totally null 2-plane. This finishes the proof. \(\square\)

**Remark 6.6.** Since $S$ as in (44) is equal to $-10\Psi_1^2$, and this is turn is $8/5$ of the only component of the conformal tensor $\nabla_{[C \nabla_D]} \tilde{R}_{AB}$, it is now clear why an ‘ad hoc’ defined object $S$ in (43) is a weighted scalar.

**Remark 6.7.** According to the discussion in Example 5.25, the assumption about the conformal tensor $\nabla_{[C \nabla_D]} \tilde{R}_{AB} \equiv 0$, replacing the Ricci flatness condition from the original Goldberg-Sachs theorem, can not be weakened if one wants to get the implication ($\kappa \equiv \sigma \equiv 0 \Rightarrow (\Psi_0 \equiv \Psi_1 \equiv 0)$. Thus, although the connection $\nabla$ provides plenty of a priori “weaker” conditions, such as for example $\nabla_{[E \nabla_F]} \nabla_{[C \nabla_D]} \tilde{R}_{AB} \equiv 0$, or conditions with more iterations of the curvature operator $\nabla_{[C \nabla_D]}$, they all are equivalent to the simplest condition $\nabla_{[C \nabla_D]} \tilde{R}_{AB} \equiv 0$. 

7. Generalizations of the Goldberg-Sachs theorem for real metrics

Theorems 5.10, 5.21, 5.24, 5.28, 5.31 and 5.32 were proved assuming that the metric $g$ is complex. The proofs also work when $g$ is real. To see this it is enough to look at the proofs assuming one of the reality conditions $(L)$, $(E)$, $(S_c)$ or $(S_r)$ of Remarks 4.1, 4.2, 4.3 and 4.4. They impose relations between the components of the Weyl tensor $\Psi_\mu$ and $\Psi'_\nu$, between the Schouten tensor components $P_{\mu\nu}$ and between the Cotton tensor components $A_{abc}$. These relations are harmless for the arguments in the proofs. They however may be used to shorten the proofs and may cause that some assumptions appearing in the complex versions can be dropped off.

We first discuss the Euclidean case.

7.1. Euclidean case. In this case, in every null coframe $(M, P, N, K)$, as in (1), the reality conditions $(E)$ imply that in particular:

$$(66) \quad \Psi_4 = \Psi_0, \quad \Psi_3 = \Psi_1, \quad \Psi_2 = \overline{\Psi}_2, \quad \Psi_1' = \overline{\Psi}_0, \quad \Psi_0' = \overline{\Psi}_1', \quad \Psi_0'' = \overline{\Psi}_2''.$$

In the rest of this section we consider the selfdual part of the Weyl tensor and principal null 2-planes associated with it. The analysis of the antiselfdual case is analogous.

Relations (66), when compared with the equations (19) defining the principal 2-planes, imply the following:

**Proposition 7.1.** If $z = z_1$ is a solution of $\Psi_4 z^4 - 4\Psi_3 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z + \Psi_0 = 0$ then is so $z_2 = -\frac{1}{z_1}$.

**Proof.** Inserting (66) and $z = z_1$ in the equation defining the principal null 2-planes (19) we get

$$\overline{\Psi}_0 z_1^4 - 4\overline{\Psi}_1 z_1^3 + 6\overline{\Psi}_2 z_1^2 + 4\overline{\Psi}_1 z_1 + \Psi_0 = 0.$$

Now dividing this by $z_1^{-4}$ and taking the complex conjugation of the result, we get

$$\overline{\Psi}_0 z_2^4 - 4\overline{\Psi}_1 z_2^3 + 6\overline{\Psi}_2 z_2^2 + 4\overline{\Psi}_1 z_2 + \Psi_0 = 0,$$

which finishes the proof.

Comparing this with Proposition 3.1 we have

**Corollary 7.2.** Principal null 2-planes always appear in pairs corresponding to pairs of solutions $(z_1, z_2) = (z_1, -\frac{1}{z_1})$ of equation (19).

A pair of solutions $(z_1, z_2) = (z_1, -\frac{1}{z_1})$ of equation (19) at a point $x$ distinguishes a pair $(J(z_1), J(\frac{1}{z_1}))$ of principal hermitian structures $J(z_1)$ and $J(\frac{1}{z_1})$ at $x$, which are conjugate to each other, $J(\frac{1}{z_1}) = -J(z_1)$.

**Proof.** The only thing to be proven is $J(\frac{1}{z_1}) = -J(z_1)$. By definition of these two structures we have $J(z)(m + zn) = i(m + zn), \quad J(z)(k - zp) = i(k - zp)$ and $J(\frac{1}{z})(m - \frac{1}{z}n) = i(m - \frac{1}{z}n), \quad J(\frac{1}{z})(k + \frac{1}{z}p) = i(k + \frac{1}{z}p).$ The second set of equations is equivalent to $J(\frac{1}{z})(\bar{m}n - \bar{n}m) = i(\bar{m}n - \bar{n}m)$ and $J(\frac{1}{z})(\bar{k}p + \bar{p}k) = i(\bar{k}p + \bar{p}k)$, which after conjugation and the use of the reality conditions $(E)$ gives:

$$J(\frac{1}{z})(k - zp) = -i(k - zp) = -J(z)(k - zp),$$

$$J(\frac{1}{z})(m + zn) = -i(m + zn) = -J(z)(m + zn).$$

□
This corollary implies that at each point $x$ of $M$ the selfdual part of the Weyl tensor may be in one of the following Petrov types:

- **type G**: the generic type, in which the selfdual part of the Weyl tensor does not vanish at $x$, and in which we have two distinct pairs $(z_1, z_2) = (z_1, -\frac{1}{z_1})$ and $(z_3, z_4) = (z_3, -\frac{1}{z_3})$, $z_1 \neq z_3$, of solutions of equation (19). In such case the pairs $(z_1, z_2)$ and $(z_3, z_4)$ correspond to two pairs of different mutually conjugate principal hermitian structures $(J(z_1), J(z_2))$ and $(J(z_4), J(z_4))$ at $x$.

- **type D**: this is the degeneracy of type G. It occurs when $z_1$ is a double root of (19), i.e. when $z_3 = z_1$. In such case we have only one pair of double principal hermitian structures $(J(z_1), J(z_2))$ at $x$.

- **type 0**: this is the antiselfdual type in which the selfdual part of the Weyl tensor vanishes at $x$. In this case the sphere of selfdual 2-planes has no distinguished points.

Note that always we may choose a Newman-Penrose frame in which $\Psi_0 = 0$ at $x$. In types G or D it is achieved by choosing the Newman-Penrose vectors $m$ and $k$ such that they span the principal null 2-plane corresponding to $z_1$. Then, in such a frame, the algebraically special type D is characterized by $\Psi_1 = 0$ and $\Psi_2 \neq 0$ at $x$. If in such a frame $\Psi_1 \neq 0$, then the selfdual part of the Weyl tensor is algebraically general (of type G) at $x$.

This proves the following

**Theorem 7.3.** At every point of a 4-dimensional manifold $M$ equipped with a real euclidean-signature metric $g$ the selfdual part of the Weyl tensor may be of one of the types G, D, and 0, with the analogous types for the antiselfdual part of the Weyl tensor. Thus, at every point of a 4-manifold $M$ equipped with a euclidean signature metric $g$ we have $3 \times 3 = 9$ ‘Petrov’ types.

Thus the Euclidean reality conditions ($E$) imply that the number of possible Petrov types in the Euclidean case is much smaller than in the complex case. This implies that the complex theorems of the previous Section have much stronger Euclidean versions. In particular, the proof of Theorem 5.28, when the reality conditions ($E$) are assumed, goes through as in the complex version, with the only exception, that the II-genericticity property of $g$ may now be weakened to the assumption that the selfdual part of the Weyl tensor is nowhere vanishing (or even to a still weaker assumption that the points at which the selfdual part of the Weyl tensor vanishes form closed sets without interior). Indeed, in the Euclidean case, the assumption $\Psi_0 = \Psi_1 \equiv 0$ and $\Psi_2 \neq 0$, which is needed for the conclusion that $\kappa \equiv \sigma \equiv 0$, means only that the selfdual part of the Weyl tensor is nonvanishing, since now $\Psi_0 = \Psi_1 \equiv 0$ implies that $\Psi_4 \equiv \Psi_3 \equiv 0$. This proves the Riemannian version of the Goldberg-Sachs Theorem 2.1.

One of the corollaries from the complex Theorem 5.28 is also the following

**Corollary 7.4.** If the selfdual part of the Weyl tensor of a real metric $g$ of Riemannian signature does not vanish on a 4-dimensional manifold $M$, then modulo complex conjugation, such a metric admits at most two hermitean structures that agree with the orientation. If such hermitean structures exist their spaces of $(1,0)$ vectors coincide with the selfdual principal totally null 2-planes. In particular, in type D we may have only one hermitean structure, which exists if and only if the Cotton tensor for $g$ vanishes on its space of $(1,0)$ vectors.
The Euclidean version of Theorem 5.10 is also worth quoting. We have

**Corollary 7.5.** Assume that a 4-dimensional manifold $M$ equipped with a real metric of Riemannian signature $g$ has a nonvanishing selfdual part of the Weyl tensor $C^+$. Suppose that it admits a hermitean structure $J$ which agrees with the orientation, and that its Ricci tensor vanishes on the space $N$ of $(1,0)$ vectors of $J$. Then $C^+$ is of type $D$, with $N$ being the only principal selfdual null 2-plane.

### 7.2. Split signature case.

To spell out all the possible Petrov types and their interpretations in this case we first consider the Newman-Penrose coframe $(M, P, N, K)$ with the reality conditions $(S_e)$ from Remarks 4.1 and 4.2. In this coframe the sphere of selfdual totally null 2-planes $N_{z}$ is spanned by $m + zn$ and $k - zm$ as in (16). Now, having the reality conditions $S_{e}$, we ask which values of $z \in \mathbb{C}$ correspond to the nongeneric selfdual totally null 2-planes which have real index equal to two. We have the following

**Proposition 7.6.** A selfdual 2-plane $N_{z}$ has real index equal to two if and only if the complex parameter $z \in \mathbb{C}$ lies on the unit circle $z \bar{z} = 1$.

**Proof.** Due to the reality conditions $(S_e)$ a real nonvanishing vector $v = a(m + zn) + b(k - zm)$ from $N_{z}$ must satisfy

$$a(m + zn) + b(k - zm) = \bar{a}(p - \bar{z}k) + \bar{b}(-n - \bar{z}m).$$

Equating to zero the respective coefficients at $m, p, n, k$ we easily get that this is possible if and only if $z \bar{z} = 1$. Thus $N_{z}$ includes real nonzero vectors if and only if $z \bar{z} = 1$. We further observe that if $z \bar{z} = 1$ then $v$ is real if and only if $b = -\bar{a} \bar{z}$. Thus, when $z$ is fixed, we have a 1-complex-parameter-family $v = v(a)$ of real vectors in $N_{z}$. Choosing two different values of $a$ we get

$$v(a) \wedge v(a') = (a \bar{a}' - a' \bar{a})(m \wedge p - \bar{z}m \wedge k - \bar{z}p \wedge n - n \wedge k).$$

This shows that $N_{z}$ with $z \bar{z} = 1$ includes independent real vectors (take e.g. $a = 1$ and $a' = i$), thus it has real index two. This finishes the proof.

Let us now choose a Newman-Penrose coframe as in (1). Then the reality conditions $(S_e)$ imply that we have:

$$\Psi_4 = \Psi_0, \quad \Psi_3 = -\Psi_1, \quad \Psi_2 = \Psi_2, \quad \Psi'_4 = \Psi'_0, \quad \Psi'_3 = -\Psi'_1, \quad \Psi'_2 = \Psi'_2,$$

and the reality conditions $(S_{r})$ mean that all Weyl tensor coefficients $\Psi$ and $\Psi'$ are real:

$$\Psi_0 = \bar{\Psi}_0, \quad \Psi_1 = \bar{\Psi}_1, \quad \Psi_2 = \bar{\Psi}_2, \quad \Psi_3 = \bar{\Psi}_3, \quad \Psi_4 = \bar{\Psi}_4,$$

(we also have analogous relations for $\Psi'$).

We pass to the split signature version of the Petrov classification. We perform the analysis for the selfdual part of the Weyl tensor; the classification for the antiselfdual case is analogous.

Let us fix a point $x \in M$. Let $(M, P, N, K)$ be a Newman-Penrose coframe around $x$ satisfying the reality conditions $(S_r)$, and as a consequence (67). We have the following

**Proposition 7.7.** If $z = z_1$ is a solution of $\Psi_4 z^4 - 4 \Psi_3 z^3 + 6 \Psi_2 z^2 + 4 \Psi_1 z + \Psi_0 = 0$ then is so $z_2 = \frac{1}{z_1}$.
Proof. Inserting (67) and \( z = z_1 \) in the equation defining the principal null 2-planes (19) we get
\[
\bar{\Psi}_0 z_1^4 + 4\bar{\Psi}_1 z_1^3 + 6\bar{\Psi}_2 z_1^2 + 4\bar{\Psi}_1 z_1 + \Psi_0 = 0.
\]
Now dividing this by \( z_1^{-4} \) and taking the complex conjugation of the result, we get
\[
\bar{\Psi}_0 z_2^4 + 4\bar{\Psi}_1 z_2^3 + 6\bar{\Psi}_2 z_2^2 + 4\bar{\Psi}_1 z_2 + \Psi_0 = 0,
\]
which finishes the proof. \( \Box \)

Comparing this Proposition with Proposition 7.6 we get

**Corollary 7.8.** Selfdual principal null 2-planes always appear in pairs corresponding to pairs of solutions \((z_1, z_2) = (z_1, \frac{1}{z_1})\) of equation (19). The situation in which \( z_1 = z_2 \) happens only if the principal selfdual null 2-plane has real index two.

Using Proposition 3.1 we may also reinterpret this corollary as follows

**Corollary 7.9.** If equation (19) at a point \( x \) admits a principal selfdual null 2-plane of real index zero, then at this point we have two distinguished hermitian structures \( J(z_1) \) and \( J(\frac{1}{z_1}) \) associated with the solution \( z_1 \) of (19). Moreover these two structures are conjugate to each other.

Proof. The only thing to be proven is \( J(\frac{1}{z_1}) = -J(z_1) \). By definition of these two structures we have \( J(z)(m + zn) = i(m + zn) \), \( J(z)(k - zp) = i(k - zp) \) and \( J(\frac{1}{z})(m + \frac{1}{zn}) = i(m + \frac{1}{zn}) \), \( J(\frac{1}{z})(k - \frac{1}{zp}) = i(k - \frac{1}{zp}) \). The second set of equations is equivalent to \( J(\frac{1}{z})(\bar{z}m + n) = i(\bar{z}m + n) \) and \( J(\frac{1}{z})(\bar{z}k - p) = i(\bar{z}k - p) \), which after conjugation and the use of the reality conditions \((S_c)\) gives:
\[
\begin{align*}
\bar{J}(\frac{1}{z})(k - zp) &= -i(k - zp) = -J(z)(k - zp), \\
\bar{J}(\frac{1}{z})(m + zn) &= -i(m + zn) = -J(z)(m + zn).
\end{align*}
\]
\( \Box \)

Because of quite different reality conditions (67) and (68) at each point \( x \in \mathcal{M} \) we need to consider separately two different cases: the generic one a) in which the selfdual part of the Weyl tensor admits at least one principal totally null 2-plane of real index zero at \( x \), and the less generic one b) in which all principal null planes have real index two at \( x \).

In the case a) we chose a Newman-Penrose coframe \((M, P, N, K)\) around \( x \) such that it satisfies the reality conditions \((S_c)\) and that the principal totally null 2-plane of real index zero corresponds to the solution \( z = 0 \) of (19). Then in such a coframe \( \Psi_0 = 0 \), and the equation defining the principal null 2-planes becomes
\[
2\bar{\Psi}_1 z^2 + 3\bar{\Psi}_2 z + 2\bar{\Psi}_1 = 0.
\]
Thus in this coframe we have two solutions \((z_1, z_2) = (0, \infty)\) corresponding to the mutually conjugate principal (almost) hermitian structures associated with two fields of principal 2-planes of index zero, and the rest of the principal 2-planes has to be determined as solutions to the quadratic equation (69). The roots of this equations are obviously
\[
z_{3,4} = -3\Psi_2 \pm \sqrt{9\Psi_2^2 - 16\Psi_1\Psi_4} \over 4\Psi_1.
\]
The interpretation depends on the sign of \(9\Psi_2^2 - 16\Psi_1\bar{\Psi}_1\) and on whether \(\Psi_1\) vanishes or not. It follows that at each point \(x \in \mathcal{M}\) we have now four cases:

**type G**: the generic case in which \(z_3 \neq z_4 = \frac{1}{z_3}\), \(z_3\bar{z}_3 \neq 1\), \(z_3 \neq 0\) and \(z_3 \neq \infty\). In such case we have two pairs of different mutually conjugate principal hermitian structures at \(x\) corresponding to \((J(0), J(\infty))\) and \((J(z_3), J(\frac{1}{z_3}))\). This case happens when \(9\Psi_2^2 > 16\Psi_1\bar{\Psi}_1\) and \(\Psi_1 \neq 0\) at \(x\).

**type SG**: in this case \(z_3 \neq z_4 = \frac{1}{z_3}\), \(z_3\bar{z}_3 = 1\). Here, in addition to the pair of mutually conjugate principal hermitian structures \((J(0), J(\infty))\) at \(x\), we have two different principal totally null 2-planes of real index two at \(x\). These real 2-planes are associated with the solutions \(z_3\) and \(z_4\), which lie on the circle \(z\bar{z} = 1\). This case happens when \(9\Psi_2^2 < 16\Psi_1\bar{\Psi}_1\) at \(x\).

**type II**: this is the degenerate case of the type SG. It happens when \(9\Psi_2^2 = 16\Psi_1\bar{\Psi}_1\) and \(\Psi_1 \neq 0\) at \(x\), and the equation (69) has double root \(z_3 = z_4 = 0\) at \(x\). We necessarily have \(z_3\bar{z}_3 = 1\) in this case, and thus, in addition to the pair of mutually conjugate principal hermitian structures \((J(0), J(\infty))\) we have also one double principal null 2-plane of real index two at \(x\).

**type D**: this is another degeneration of the type G. Now \(\Psi_1 = 0\) at \(x\) and we have \(z_3 = 0\) and \(z_4 = \infty\) as solutions of (69). Thus in this case the points \(z = 0\) and \(z = \infty\) have multiplicity two, and we have only one pair of double principal hermitian structures \((J(0), J(\infty))\) at \(x\).

We now pass to the cases in which we do not have a single principal null 2-plane which has a real index zero at \(x\). The analysis here could still be performed in the Newman-Penrose coframe satisfying the reality conditions \(S_r\), but since now all the solution of equation (19) would have to satisfy \(z\bar{z} = 1\), we would not be able to choose the frame in such a way that \(\Psi_0\) would be zero at \(x\). This would lead to the analysis of the roots of the quartic equation (19), and it is why it is now much easier to reason in the coframe that satisfies the reality conditions \((S_r)\). So now, we choose a Newman-Penrose coframe \((M, P, N, K)\) around \(x\), which satisfies the reality conditions \((S_r)\) and, since now we have at least one principal null 2-plane of real index two at \(x\), we may assume that we have \(\Psi_0 = 0\) at \(x\). In this coframe our principal totally null 2-plane of real index two corresponds to \(z_1 = 0\) and the other principal 2-planes are determined by

\[
\Psi_4 z^4 - 4\Psi_3 z^2 + 6\Psi_2 z^2 + 4\Psi_1 = 0.
\]

Here all the \(\Psi_1, \Psi_2, \Psi_3\) and \(\Psi_4\) are real and we admit only real solutions for \(z\). (If the solution is complex, it corresponds to a 2-plane with real index zero, and corresponds to one of the cases G, SG, II, or D, considered earlier.)

Now, a XVth century substitution \(z \rightarrow z - \frac{16\Psi_1}{3\Psi_3}\), brings this equation into the form

\[
z^3 + pz + q = 0,
\]

which has three real roots for \(z\) iff \(27p^2 + 4q^3 \geq 0\). This inequality gives the restriction on the Weyl tensor, which determines the situation we are talking about here. If the selfdual part of the Weyl tensor satisfies this restriction, the equation (19) has four real roots. This, in addition to G, SG, II and D, defines the five new Petrov types:

**type G_r**: equation (19), written in the coframe with reality conditions \((S_r)\), has four different real roots, meaning that we have four different principal null 2-planes of real index two at \(x\).

**type II_r**: equation (19), written in the coframe with reality conditions \((S_r)\), has one double and two different real roots, meaning that we have three different
principal null 2-planes of real index two at \( x \), one of them with multiplicity two,

type \( \Pi I_r \): equation (19), written in the coframe with reality conditions \( (S_r) \), has one triple and one distinct real roots, meaning that we have two different principal null 2-planes of real index two at \( x \), one of them with multiplicity three,

type \( N_r \): equation (19), written in the coframe with reality conditions \( (S_r) \), has one quadruple root, meaning that we have a single quadruple principal null 2-plane of real index two at \( x \),

type \( D_r \): equation (19), written in the coframe with reality conditions \( (S_r) \), has two distinct double real roots, meaning that we have two different principal null 2-planes of real index two at \( x \), each of them having multiplicity two.

Finally we have the Petrov type corresponding to the situation when the selfdual part of the Weyl tensor vanishes at \( x \) (the metric is antiselfdual at \( x \)).

This proves the following

\textbf{Theorem 7.10.} At every point of a 4-dimensional manifold \( M \) equipped with a real split-signature metric \( g \) the selfdual part of the Weyl tensor may be of one of the types \( G, S, I I, D, I I_r, I I I, N, D_r, 0 \), with the analogous types for the antiselfdual part of the Weyl tensor. Thus, at every point of a 4-manifold \( M \) equipped with a split signature metric \( g \) we have \( 10 \times 10 = 100 \) 'Petrov' types.

The above analysis also suggest the following terminology: the name algebraically special for the selfdual part of the Weyl tensor in the split signature case is reserved to the types \( I I, D, I I_r, I I I, N_r, D_r \) and 0 only. Although the types \( S \) and \( G_r \) are algebraically (and geometrically!) distinguished from the most general case \( G \), we also call them algebraically general. With this terminology, Theorems 1.3 and 1.4 follow from our Theorem 5.10.

Because of the huge number of the algebraically special cases to be considered, we skip the discussion of the split signature versions of further theorems from Section 5 here. Such a discussion deserves a separate paper. This should also answer several interesting questions, such as for example, the following: 'are there split-signature Einstein metrics of type \( I I \)?', 'is it possible to have a split signature Einstein 4-manifold on which an integrable totally null 2-plane can change its real index from 0 to 2', etc.

We close this section by mentioning the recent paper [10]. It is entirely devoted to the Newman-Penrose formalism adapted to the split signature situation, and it provides a version of the split-signature Goldberg-Sachs theorem.

\textbf{7.3. Lorentzian case.} Here the Petrov types are precisely the same as in the complex case described by the Definition 5.4, i.e. we have types \( G, I I, D, I I I, N \) and 0 here. The Lorentzian reality conditions \( (L) \) do not make any restriction on the Weyl tensor coefficients \( \Psi_{\mu} \). What they do is, they give a simple relation between the self-dual part of the Weyl tensor and the antiselfdual one. We have \( \Psi'_{\mu} = \Psi_{\mu} \), so here the antiselfdual part of the Weyl tensor is totally determined by the selfdual one. Since in the proofs in Section 5 the coefficients \( \Psi_{\mu} \) never appear, and only \( \Psi'_{\mu} \)s matter, all the proofs, and the theorems presented in Section 5 restrict naturally to the Lorentzian case without any alteration.

However, since in the Lorentzian signature the fields of totally null 2-planes have always real index one, it is customary to formulate the Lorentzian theorems in
terms of the real vector field $k$ such that $\text{Span}_C(k) = \mathcal{N} \cap \overline{\mathcal{N}}$. In particular, such a null real vector field is said to be \textit{geodesic and shear-free} [25] if it satisfies

$$\mathcal{L}_k g = ag + g(k)\omega,$$

with a function $a$ and a 1-form $\omega$ on $\mathcal{M}$. Here $g(k)$ is a 1-form on $\mathcal{M}$ such that $X \perp g(k) = g(k, X)$ for any vector field $X \in T\mathcal{M}$. When written in terms of the field $\mathcal{N}$ of the associated totally null 2-planes, condition (70) is equivalent to

$$[\mathcal{N}, \mathcal{N}] \subset \mathcal{N},$$

i.e. to the formal integrability condition for $\mathcal{N}$.

Suppose now the Weyl tensor $C_{abcd}$ of $(\mathcal{M}, g)$ is \textit{nonvanishing}. It is well known [2] that the algebraic equation

$$k_{[e} C_{a][bc]d} k^b k^c = 0,$$

for a null vector $k$ has at most \textit{four} solutions at every point $x \in \mathcal{M}$. The solutions $k$ of equation (71) at $x \in \mathcal{M}$ are called the \textit{principal null directions} (PNDs) at $x$. If equation (71) admits \textit{exactly} four PNDs at $x \in \mathcal{M}$ then $(\mathcal{M}, g)$ is said to be \textit{algebraically general} at $x$. If the number $q$ of solutions to (71) at $x \in \mathcal{M}$ is $1 \leq q \leq 3$ then $(\mathcal{M}, g)$ is called \textit{algebraically special} at $x$. In such case the quartic equation (71) has at least one \textit{multiple root}, and the solution $k$ corresponding to it is called a \textit{multiple PND}. This notion of the algebraical speciality coincides with the one in terms of the principal null 2-planes, since on a Lorentzian oriented and time oriented 4-manifold $\mathcal{M}$, there is one to one correspondence between fields of totally null 2-planes in the complexification and real null vector fields, defined by the intersection of the 2-planes with their complex conjugations.

Having said this, we present the Lorentzian version of our complex Theorem 5.9.

\textbf{Theorem 7.11.} Let $\mathcal{N} \subset \mathcal{T}^C \mathcal{M}$ be a field of totally null 2-planes on a Lorentzian 4-dimensional manifold $(\mathcal{M}, g)$. Assume that the Ricci tensor $\text{Ric}$ of $(\mathcal{M}, g)$, considered as a symmetric bilinear form on $\mathcal{T}^C \mathcal{M}$, is degenerate on $\mathcal{N}$,

$$\text{Ric}|_{\mathcal{N}} = 0.$$

If in addition the field $\mathcal{N}$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, everywhere on $\mathcal{M}$, or what is the same, if $k$ such that $\text{Span}_C(k) = \mathcal{N} \cap \overline{\mathcal{N}}$, is geodesic and shear-free, then $(\mathcal{M}, g)$ is algebraically special at every point, with a multiple PND tangent to $k$.

\textbf{Remark 7.12.} In [6] we used Theorem 7.11 without proof, since it would have made an already long paper even longer. Actually some statements equivalent to Theorem 7.11 are known to a few general relativists, see e.g. Lemma 2.2 on p. 577 of [23]. Since this equivalence is not easy to decipher, we decided to present this theorem here, as a corollary from the complex Theorem 5.9.

7.4. \textbf{Counterexample to Trautman’s conjecture.} Trautman in [26] asked if there exists an example of a 4-dimensional Bach flat metric with nonvanishing selfdual part of the Weyl tensor $C^+$, for which an integrable field of selfdual totally null 2-planes would not be principal for $C^+$. He \textit{conjectured} that the answer to this question is ‘no’. Although the question was formulated in the Lorentzian setting, it makes sense in any signature. It is also very closely related to the Goldberg-Sachs theorem.

Our analysis of this theorem from Section 5.2, especially the discussion in Example 5.25, suggests that the examples Trautman asks about, should be possible.
This is because, the conditions needed for 'if and only if' between conditions (i) and (ii) in Theorem 5.28 are related to those derivatives of the Cotton tensor that are not present in the Bach tensor. This is clear from Example 5.25: the integrability conditions for \( A_N = 0 \), give \( S = 0 \), where \( S \) is given by (44). And although the Bach tensor components may be obtained by differentiating some components of the Cotton tensor, the derivatives of the Cotton tensor appearing in \( S \) are not (at least algebraically) expressible in terms of the components of the Bach tensor.

Below in this Section we present a simple example of a metric with Euclidean signature which is Bach-flat, admits an integrable hermitean structure which agrees with the orientation, and whose selfdual part of the Weyl tensor is of general type G.

On \( \mathbb{R}^4 \), with local coordinates \((x^1, x^2, x^3, x^4)\), consider \( z = x^1 + ix^2 \) and \( w = x^3 + ix^4 \), and a complex-valued function \( f = f(w, z) \) holomorphic in both arguments \( w \) and \( z \). Given \( f \) define a Riemannian metric

\[
g = 2 \left( dw d\bar{w} + \exp(f(w, z) + \bar{f}(\bar{w}, \bar{z})) dz d\bar{z} \right).
\]

Now introduce the Newman-Penrose coframe by setting

\[
M = dw, \quad P = dw, \quad N = e^f dz, \quad K = e^f d\bar{z}.
\]

They obviously satisfy the Euclidean reality conditions \((E)\). A short calculation shows, that modulo the complex conjugation, the only nonvanishing Newman-Penrose coefficients are:

\[
\alpha = -\frac{i}{2} \pi = \beta' = -\frac{1}{2} \tau' = \frac{1}{4} f_w.
\]

In particular \( \kappa = \sigma = 0 \), which is obvious since the field of selfdual totally null 2-planes \( N \) spanned by \( m = \partial_w \) and \( k = e^{-f} \partial_{\bar{z}} \) is integrable. Now our main point is that the only nonvanishing components of the Weyl tensor are:

\[
\Psi_3 = \bar{\Psi}_4 = \frac{1}{4} e^{-f} f_{wz}.
\]

This in particular means that the field \( N \) is principal (since \( \Psi_0 = 0 \)), but when \( f_{wz} \neq 0 \) it is not multiple \( (\Psi_3 \neq 0 \neq \Psi_1) \). Moreover, since \( \Psi'_0 = \Psi'_1 = \Psi'_2 = \Psi'_3 = \Psi'_4 \equiv 0 \), i.e. the full antiselfdual part of the Weyl tensor identically vanishes, the metric is Bach flat. This answers in positive the question of Trautman we mentioned at the beginning of this Section. Moreover, if \( f_{wz} \neq 0 \), due to the Corollary 7.5, this selfdual metric can not have Ricci tensor vanishing on \( N \), and as such is never conformal to an Einstein metric.

7.5. Characteristic connection in real signatures. We now reexamine the arguments from Section 6 from the point of view of the reality conditions.

From Step one of the proof of Theorem 6.1 we know that the Weyl form \( B \) of the Weyl connection which preserves an integrable \( N \), in an adapted to \( N \) coframe is given by \( B = 2\tau M + B_2 P + B_3 N - 2\mu K \). Thus in the complex case (or in the real cases in which we do not insist on \( B \) to be real) the Weyl 1-form is not totally determined by \( N \).

The situation is quite different in the Riemannian \((E)\) and the split signature \((S_c)\). In these two cases, the requirements that \( B \) is real determines it completely! Indeed, it is easy to see that the reality conditions \((E)\) or \((S_c)\) together with the requirement that \( B \) be real implies that \( B \) is equal to

\[
B = 2\tau M + 2\pi P - 2\mu N - 2\rho K.
\]
or, what is the same,
\[
\frac{1}{2}B = \Gamma_{143}^1 \theta^1 + \Gamma_{234}^2 \theta^2 + \Gamma_{321}^3 \theta^3 + \Gamma_{412}^4 \theta^4.
\]

This proves the following theorem

**Theorem 7.13.** Let \( N \) be a field of totally null 2-planes on \((M, g)\), where \( g \) is a 4-dimensional metric of Riemannian or split signature. Let us assume that \( N \) is integrable \([N, N] \subset N\) and that it has a real index 0 everywhere on \( M \). Then there exists a canonical Weyl connection \( \nabla^w \) on \( M \), which encodes the conformal properties of the structure \((M, g, N)\).

The connection \( \nabla^w \) is uniquely determined by the requirements that

- it is real,
- it is torsionless,
- it satisfies: \( \nabla^w g = -Bg \),
- it satisfies: \( \nabla^w X \mathcal{N} \subset \mathcal{N} \) for all \( X \in \mathcal{T}M \).

In terms of a coframe \((\theta^a)\) adapted to \( \mathcal{N} \) and the connection 1-forms \( \Gamma^w_{ab} = g^{ad} \Gamma^w_{db} \theta^c \), the connection \( \nabla^w \) is given by
\[
\Gamma^w_{abc} = \Gamma_{abc} + \frac{1}{2} (g_{ca} B_b - g_{cb} B_a + g_{ab} B_c)
\]
with
\[
\frac{1}{2}B = \Gamma_{143}^1 \theta^1 + \Gamma_{234}^2 \theta^2 + \Gamma_{321}^3 \theta^3 + \Gamma_{412}^4 \theta^4.
\]

Here \( \Gamma_{abc} \) are the Levi-Civita connection coefficients in the adapted coframe.

**Definition 7.14.** Let \( J \) be a hermitean (or pseudohermitean) structure on an \( 2n \)-dimensional manifold \((M, g)\) with a metric of Riemannian (or split) signature. A torsionless connection \( \nabla^w \) on \((M, g, J)\) is called (pseudo)hermitean-Weyl iff

- \( \nabla^w J = 0 \),
- and \( \nabla^w g = -Bg \) for some real 1-form \( B \) on \( M \).

According to our discussion in Section 3, integrable totally null 2-planes of real index 0 on a 4-dimensional manifold \((M, g)\) are in one-to-one correspondence with (pseudo)hermitean structures \( J \) on \((M, g)\), thus Theorem 7.13 can be reformulated as:

**Theorem 7.15.** Every 4-dimensional (pseudo)hermitean manifold \((M, g, J)\) defines a canonical (pseudo)hermitean-Weyl connection \( \nabla^w \). This connection encodes the conformal properties of the structure \((M, g, J)\). It is given by \( \nabla^w = \nabla^w \), where \( \nabla^w \) is as in Theorem 7.13.

Thus in the (pseudo)hermitean case there is a better connection, namely \( \nabla^w \), than the characteristic connection \( \nabla \). It is better, since it enables to differentiate any vector from the tangent space of \( M \) along any other vector from \( \mathcal{T}M \). The connection \( \nabla^w \) enables for the differentiation along \( \mathcal{N} = T^{(1,0)}M \) only. And, \( \nabla^w \) is better, because it contains much more information than \( \nabla \). In particular, \( \nabla^w \) is simply the restriction of \( \nabla^w \) to \( \mathcal{N} \).
We now pass to the (pseudo)hermitean part of our elegant Goldberg-Sachs Theorem 6.5.

We need some preparations:

Given the (pseudo)hermitean-Weyl connection $\nabla$, as in Theorem 7.15, we use the formula (53) to pass to the connection 1-forms $\Gamma_{ab} = \Gamma_{abc} \theta^c$. Here $(\theta^c)$ is a coframe adapted to $J$. The word ‘adapted’ (in accordance with the discussion in Section 3) means that the considered coframe is adapted to $N = T^{(1,0)}M$ as in the definition of this notion at the beginning of Section 5. Now, there is a sequence of definitions, which closely mimics the situation in Riemannian geometry:

Having the connection 1-forms $\Gamma_{ab}$, the metric $g$ and its inverse, represented by $g^{ab}$, we also have the 1-forms $\Gamma^a_b = g^{ac} \Gamma_{cb}$. Using them, we define the curvature of the connection $\nabla$. We do it, in terms of the curvature 2-forms $\Omega^a_b$, analogous to those given in the formula (7), by:

$$\frac{1}{2} R^a_{bcd} \theta^c \wedge \theta^d = d \Gamma^a_b + \Gamma^a_c \wedge \Gamma^c_b.$$  

Here $R^a_{bcd}$ are the curvature coefficients in the coframe $(\theta^a)$. Then we define the Ricci tensor

$$R^a_{ab} = \Omega^a_{acb},$$  

and its scalar

$$R = g^{ab} R^a_{ab}.$$  

The next step is to define the Schouten tensor

$$P^a_{ab} = \frac{1}{2} R^a_{ab} - \frac{1}{12} R g^a_{ab},$$  

and the Cotton tensor

$$A^a_{abc} = 2 \nabla_b P^a_{ca}.$$  

This defines a linear map

$$A : TM \times TM \times TM \to \mathbb{R}$$  

given by

$$A = \frac{1}{2} A^a_{abc} \theta^a \otimes (\theta^b \wedge \theta^c).$$  

Then the (pseudo)hermitean part of Theorem 6.5 is:

**Theorem 7.16.** Let $(M, g, J)$ be a 4-dimensional (pseudo)hermitean manifold and let $\nabla$ be its canonical (pseudo)hermitean-Weyl connection $\nabla$. Assume that

$$\nabla_X A(Y, X, Y) \equiv \nabla_Y A(X, X, Y)$$  

for all vectors $X, Y \in N = T^{(1,0)}M$.

Then the selfdual part of the Weyl tensor for $(M, g)$ is algebraically special at every point of $M$, with $J$ being the multiple principal hermitean structure on $N$.

**Proof.** The proof of this Theorem consists of straightforward calculations using the above definitions. The key point in these calculations is that $\nabla_X A(Y, X, Y) - \nabla_Y A(X, X, Y)$, when $X, Y$ run through all the vectors from $N$, is always proportional to $\nabla_{A_{141}} - \nabla_{A_{141}}$. Here the indices 1 an 4 are the components from the coframe adapted to $J$, in which $e_1 = m$ and $e_4 = k$. By a direct calculation one can...
to be meaningful in case of relation to be true, we need to take without the (pseudo)hermitean reality conditions (E) or (S). For this crucial relation to be true, we need to take $B$ as in (72) and to assume the integrability of $\mathcal{N}$, i.e. to assume $\kappa \equiv \sigma \equiv 0$. If these two assumptions are satisfied then $\nabla_4 A_{441} - \nabla_1 A_{441} = 16\Psi_1^2$ irrespective of the signature of the metric. It is even true when the metric is complex! Thus the Weyl connection $\nabla$ with $B$ as in (72) seems to be meaningful in case of $g$ being complex, or having any signature. The only trouble with such a connection is that in the Lorentzian case it is complex. If one can live with this, one can replace the condition (65) in Theorem 6.5 by (73) and Theorem 6.5 will be true for complex metrics, as well for metrics of all the other real signatures.

8. Appendix

The 36 signature independent Newman-Penrose equations, which include 16 first Bianchi identities, are:

(74) \[ \delta \kappa = D\sigma + \alpha' \kappa + 3\beta \kappa + \kappa' - 3\varepsilon \sigma + \varepsilon' \sigma + \rho \sigma + \rho' \sigma + \kappa \tau + \Psi_0 \]

(75) \[ \delta \kappa' = D\sigma' + \alpha' \kappa' + 3\beta' \kappa' + \kappa' \pi - 3\varepsilon' \sigma' + \varepsilon' \sigma' + \rho' \sigma' + \rho \sigma' + \kappa' \tau' + \Psi_0' \]

(76) \[ \delta \rho = \partial \kappa + \kappa' \mu - \kappa' \mu' + \alpha' \rho + \beta \rho - 3\alpha \sigma + \beta' \sigma - \rho' \tau + \rho \tau - \Psi_1 - P_{14} \]

(77) \[ \nabla \tau = \Delta \kappa - \kappa' \kappa' + \kappa' \kappa + 3\rho \sigma - \sigma' \tau - \varepsilon \tau + \varepsilon \tau - \rho \tau + \rho' \tau - \Psi_1 + P_{14} \]

(78) \[ \nabla \tau' = \Delta \kappa' - \kappa' \kappa' + 3\rho' \sigma' - \sigma' \tau - \varepsilon' \tau + \varepsilon' \tau - \rho' \tau' + \Psi_1 + P_{14} \]

\[ \Delta \rho = \partial \tau - \kappa \nu + \gamma \rho + \gamma' \rho' - \mu \rho - \lambda \sigma - \alpha \tau + \beta \tau - \tau \tau' - \Psi_2 - P_{12} - P_{34} \]

\[ \Delta \rho' = \partial \tau' - \kappa' \nu' + \gamma' \rho + \gamma \rho' - \mu' \rho' - \lambda' \sigma' - \alpha' \tau' + \beta \tau' - \tau \tau' - \Psi_2 - P_{12} - P_{34} \]

\[ \Delta \alpha = \partial \gamma + \beta' \gamma' + \alpha \gamma' - \beta \lambda - \alpha \mu' - \varepsilon \nu + \nu \rho - \lambda \tau - \gamma' \tau + \Psi_3 \]

\[ \Delta \alpha' = \delta \gamma + \beta' \gamma' + \alpha' \gamma - \beta' \lambda' - \alpha' \mu - \varepsilon' \nu' + \nu' \rho' - \lambda' \tau - \gamma' \tau + \Psi_3' \]

\[ \Delta \lambda = \partial \nu - 3\gamma \lambda + \gamma' \lambda - \lambda \mu - \lambda \mu' + 3\alpha \nu + \beta' \nu - \nu \pi - \nu \tau' - \Psi_4 \]

\[ \Delta \lambda' = \delta \nu - 3\gamma \nu + \gamma' \nu' + \lambda' \mu - \lambda' \mu - 3\alpha' \nu + \beta' \nu' + \nu' \pi' - \nu' \tau' - \Psi_4' \]

\[ D\lambda = \partial \pi - 3\gamma \lambda + \varepsilon \lambda - \kappa' \nu + \alpha \pi + \beta' \pi - \pi^2 - \lambda \rho - \mu \sigma' - P_{22} \]

\[ D\lambda' = \delta \pi - 3\varepsilon' \lambda + \kappa \nu + \alpha' \pi + \beta' \pi' - \pi^2 - \lambda' \rho' - \mu' \sigma - P_{11} \]

\[ D\mu = \delta \pi - \varepsilon \mu - \varepsilon' \mu - \kappa \nu - \alpha' \pi + \beta \pi - \pi \pi' - \mu \rho - \lambda \sigma - \Psi_2 - P_{12} - P_{34} \]
\[ D\mu' = \partial\pi' - \epsilon' \mu' - \epsilon' \rho' - \kappa' \nu' - \alpha' \pi' + \beta' \pi' - \pi' \rho' - \mu' \rho - \lambda' \sigma' - \Psi'_2 - P_{12} - P_{34} \]
\[ D\alpha = \partial\varepsilon + \alpha' \varepsilon - 2\alpha \varepsilon - \beta \varepsilon - \gamma \kappa - \kappa \lambda - \varepsilon \pi - \alpha \rho + \pi \rho + \beta \sigma' + P_{24} \]

(79) \[ D\alpha' = \delta' \varepsilon + \alpha' \varepsilon - 2\alpha' \varepsilon - \beta' \varepsilon - \gamma' \kappa - \kappa' \lambda - \varepsilon' \pi' - \alpha' \rho' + \pi' \rho' - \beta' \sigma + P_{14} \]
\[ \Delta \beta = \delta \gamma + \alpha' \gamma + 2\beta \gamma - \beta' \gamma - \alpha \lambda - \beta \mu - \varepsilon \nu' + \nu \sigma - \gamma \tau - \mu \tau - P_{13} \]
\[ \Delta \beta' = \delta' \gamma' + \alpha' \gamma' + 2\beta' \gamma' - \beta' \gamma' - \alpha' \lambda - \beta' \mu' - \varepsilon' \nu' + \nu' \sigma' - \gamma' \tau' - \mu' \tau' - P_{23} \]

(80) \[ D\rho = \partial \kappa - 3\alpha \kappa - \beta' \kappa + \varepsilon \rho + \varepsilon' \rho - \rho^2 - \sigma \sigma' - \kappa' \tau - P_{44} \]

(81) \[ D\rho' = \delta \kappa' - 3\alpha' \kappa' - \beta' \kappa' - \kappa' \pi' + \varepsilon' \rho' + \varepsilon' \rho - \rho^2 - \sigma \sigma' - \kappa' \tau - P_{44} \]
\[ \Delta \mu = \delta \nu + \lambda \lambda' - \gamma \mu - \gamma' \mu - \mu^2 + \alpha' \nu + 3\beta \nu - \nu' \pi - \nu \tau - P_{33} \]
\[ \Delta \mu' = \delta \nu' + \lambda \lambda' - \gamma' \mu' - \gamma' \mu - \mu'^2 + \alpha' \nu' + 3\beta' \nu' - \nu' \pi' - \nu' \tau' - P_{33} \]
\[ D\nu = \partial \pi + \varepsilon' \nu + 3\varepsilon' \nu + \lambda \pi' - \gamma' \pi + \gamma \pi + \mu \pi - \mu \pi' - \lambda \tau + \Psi_3 - P_{23} \]
\[ D\nu' = \partial \pi' + \varepsilon' \nu' + 3\varepsilon' \nu' + \lambda \pi' - \gamma' \pi' + \gamma' \pi + \mu' \pi' - \mu' \pi - \lambda' \tau' + \Psi_3' - P_{13} \]
\[ D\gamma = \partial \varepsilon - 2\varepsilon' \gamma - \varepsilon' \gamma - \varepsilon' \gamma - \beta' \pi' + \alpha' \pi - \alpha' \tau' + \pi' \tau' - \beta' \tau - \Psi_2 + P_{34} \]
\[ D\gamma' = \partial \varepsilon' - 2\varepsilon' \gamma' - \varepsilon' \gamma - \varepsilon' \gamma - \beta' \pi' + \alpha' \pi - \alpha' \tau' + \pi' \tau' - \beta' \tau - \Psi_2' + P_{34} \]

(82) \[ \delta \partial = \delta \sigma + \kappa' \sigma - \lambda' \rho - 3\gamma \sigma + \gamma' \sigma + \mu \sigma - \alpha' \tau + \beta \tau + \tau^2 + P_{11} \]
\[ \partial' \tau = \Delta \sigma + \kappa' \nu + \lambda' \rho - 3\gamma' \sigma + \gamma' \sigma + \mu' \sigma - \alpha' \tau + \beta' \tau + \tau^2 + P_{22} \]
\[ \delta \alpha = \delta \beta + \alpha' \beta + \beta' \beta' - \varepsilon \mu + \varepsilon' \mu + \gamma \rho + \mu \rho - \gamma \rho - \lambda \sigma - \Psi_2 + P_{12} \]
\[ \delta \alpha' = \delta \beta' + \alpha' \beta' + \beta' \beta' - \varepsilon' \mu + \varepsilon' \mu + \gamma' \rho + \mu' \rho - \gamma' \rho - \lambda' \sigma' - \Psi_2' + P_{12} \]

The 20 second Bianchi identities are:
\[ \delta \Psi_1 = \Delta \Psi_0 - DP_{11} + DP_{14} - 4\gamma \Psi_0 + \mu \Psi_0 + 2\beta \Psi_0 - 3\kappa \Psi_2 - 4\tau \Psi_1 - \]
\[ 2\alpha P_{13} + 2\varepsilon P_{11} - 2\varepsilon' P_{14} - 2\beta P_{14} - 2\pi P_{14} - \lambda' P_{44} - \rho P_{11} - \sigma P_{12} + \sigma' P_{34} \]
\[ \delta \Psi_1' = \Delta \Psi'_0 - DP_{22} + DP_{24} - 4\gamma' \Psi'_0 + \mu' \Psi'_0 + 2\beta' \Psi'_0 - 3\kappa' \Psi'_2 + 4\tau' \Psi'_1 - \]
\[ 2\kappa' P_{23} + 2\varepsilon' P_{22} - 2\varepsilon P_{22} - 2\beta' P_{24} - 2\pi P_{24} - \lambda P_{44} - \rho P_{22} - \sigma P_{12} + \sigma' P_{34} \]
\[ D\Psi_1 = -\Delta \Psi_0 - DP_{14} + DP_{44} + 4\alpha \Psi_0 + \pi \Psi_0 + 2\varepsilon \Psi_1 - 3\kappa \Psi_2 - 4\Psi_1 \rho + \kappa' P_{11} + \]
\[ \kappa P_{12} + 2\varepsilon P_{14} - \kappa' P_{34} - 2\alpha' P_{44} - 2\beta P_{44} - \pi' P_{44} - 2\rho P_{14} - 2\pi P_{24} \]
\[ D\Psi_1' = -\Delta \Psi'_0 - DP_{24} + DP_{44} + 4\alpha' \Psi'_0 + \pi' \Psi'_0 + 2\varepsilon' \Psi'_1 - 3\kappa' \Psi'_2 - 4\Psi'_1 \rho' + \kappa P_{22} + \]
\[ \kappa' P_{12} + 2\varepsilon' P_{24} - \kappa' P_{34} - 2\alpha P_{44} - 2\beta P_{44} - \pi' P_{44} - 2\rho P_{24} - 2\pi' P_{14} \]
\[ \Delta \Psi_1 = \Delta \Psi_0 + DP_{13} - \delta P_{34} + \nu_0 + 2\gamma \Psi_1 - 2\mu \Psi_1 - 2\sigma \Psi_3 - 3\tau \Psi_2 - \]
\[ \pi P_{11} - \pi' P_{12} + 2\varepsilon' P_{13} + \mu P_{14} + \lambda' P_{24} + \kappa P_{33} + \pi' P_{34} + \rho P_{13} + \sigma P_{23} \]
\[ \Delta \Psi_1' = \partial \Psi_1' + DP_{23} - \partial P_{34} + \nu' \Psi_0' + 2 \gamma' \Psi_1' - 2 \mu' \Psi_1' - 2 \sigma' \Psi_0' - \pi' P_{22} - \pi P_{12} + 2 \psi P_{23} + \mu' P_{24} + \lambda P_{14} + \kappa' P_{33} + \pi P_{34} + \rho P_{23} + \sigma' P_{13} \]

\[ \partial \Psi_1 = -DP_2 + DP_{12} - \delta P_{24} + \lambda P_0 + 2 \alpha \Psi_1 + 2 \pi \Psi_1 + 2 \kappa \Psi_3 - 3 \rho \Psi_2 + \]

\[ \delta \Psi_1 = -DP_2 + DP_{12} - \delta P_{14} + \lambda P_0 + 2 \alpha \Psi_1 + 2 \pi \Psi_1 + 2 \kappa \Psi_3 - 3 \rho \Psi_2 + \]

\[ \kappa' P_{13} + \kappa P_{23} + 2 \alpha P_{24} + \pi P_{24} - \mu P_{14} + \rho P_{12} - \rho P_{34} + \sigma P_{22} \]

\[ \Delta \Psi_2 = -\delta \Psi_3 + \Delta P_{12} - \partial P_{13} + 2 \nu \Psi_1 - 3 \mu \Psi_2 - 3 \beta \Psi_3 + 2 \tau \Psi_3 + \sigma P_4 + \]

\[ \lambda P_{11} + \mu P_{12} - 2 \beta P_{13} + \nu P_{14} + \nu P_{24} - \rho P_{33} - \mu P_{34} + \tau P_{23} + \tau' P_{13} \]

\[ \Delta \Psi_2 = -\delta \Psi_3 + \Delta P_{12} - \partial P_{13} + 2 \nu \Psi_1 - 3 \mu \Psi_2 - 3 \beta \Psi_3 + 2 \tau \Psi_3 + \sigma P_4 + \]

\[ \lambda P_{11} + \mu P_{12} - 2 \beta P_{13} + \nu P_{14} + \nu P_{24} - \rho P_{33} - \mu P_{34} + \tau P_{23} + \tau' P_{13} \]

\[ \Delta \Psi_3 = -\delta \Psi_4 - \Delta P_{23} + \partial P_{33} - 3 \nu P_2 - 2 \gamma P_3 - 4 \beta P_4 + \gamma P_4 + \nu P_12 - \]

\[ 2 \gamma P_{22} + \gamma P_{22} - \mu P_{22} + \lambda P_{12} + 2 \alpha P_{23} - 2 \beta P_{24} + \lambda P_{34} + \sigma P_{33} + 2 \tau P_{23} \]

\[ \delta \Psi_4 = \Delta \Psi_4 - \Delta P_{11} + \partial P_{13} - 3 \lambda \Psi_0' - 2 \alpha \Psi_3 + 4 \pi P_3 + 4 P_4 + \rho \Psi_4 + 2 \gamma P_{11} - 2 \gamma P_{11} - \mu P_{11} - \lambda P_{12} + 2 \alpha P_{13} - 2 \gamma P_{14} + \lambda P_{34} + \sigma P_{33} + 2 \tau P_{13} \]

\[ \delta \Psi_4 = \Delta \Psi_4 - \Delta P_{11} + \partial P_{13} - 3 \lambda \Psi_0' - 2 \alpha \Psi_3 + 4 \pi P_3 + 4 P_4 + \rho \Psi_4 + 2 \gamma P_{11} - 2 \gamma P_{11} - \mu P_{11} - \lambda P_{12} + 2 \alpha P_{13} - 2 \gamma P_{14} + \lambda P_{34} + \sigma P_{33} + 2 \tau P_{13} \]

\[ \partial P_{12} = \quad \Delta P_{23} + \Delta P_{24} + \partial P_{23} - 2 \delta P_{34} - 2 \alpha P_{11} + 2 \beta P_{11} - \pi P_{12} + 2 \tau P_{13} + \]

\[ 2 \alpha P_{24} - \pi P_{24} + \mu P_{24} + 2 \mu P_{24} + \lambda P_{14} + \kappa P_{33} + \pi P_{34} + \nu P_{44} + \rho P_{13} + \sigma P_{13} + \tau P_{12} + \tau P_{34} + \tau P_{11} \]

\[ \partial P_{12} = \quad \Delta P_{23} + \Delta P_{24} + \partial P_{23} - 2 \delta P_{34} - 2 \alpha P_{11} + 2 \beta P_{11} - \pi P_{12} + 2 \tau P_{13} + \]

\[ 2 \alpha P_{24} - \pi P_{24} + \mu P_{24} + 2 \mu P_{24} + \lambda P_{14} + \kappa P_{33} + \pi P_{34} + \nu P_{44} + \rho P_{13} + \sigma P_{13} + \tau P_{12} + \tau P_{34} + \tau P_{11} \]

\[ \Delta P_{34} = -2 \partial P_{12} + \Delta P_{44} + \delta P_{14} + \delta P_{24} - \rho P_{12} - \kappa P_{13} - 2 \alpha P_{14} - \pi P_{14} - \kappa P_{23} - \]

\[ 2 \alpha P_{24} - \pi P_{24} + \rho P_{24} - 2 \gamma P_{24} + \mu P_{24} + \mu P_{44} + \mu P_{44} - \rho P_{12} + \rho P_{34} - \sigma P_{22} + \sigma P_{11} - 2 \partial P_{24} + 2 \tau P_{14} \]

\[ \Delta P_{34} = -2 \partial P_{12} + \Delta P_{44} + \delta P_{14} + \delta P_{24} - \rho P_{12} - \kappa P_{13} - 2 \alpha P_{14} - \pi P_{14} - \kappa P_{23} - \]

\[ 2 \alpha P_{24} - \pi P_{24} + \rho P_{24} - 2 \gamma P_{24} + \mu P_{24} + \mu P_{44} + \mu P_{44} - \rho P_{12} + \rho P_{34} - \sigma P_{22} + \sigma P_{11} - 2 \partial P_{24} + 2 \partial P_{14} \]
Using relations (31)-(32) we can reexpress identities (83)-(90) in terms of the components of the Cotton tensor. After this the Cotton tensor components ‘hide’ the terms with the Schouten tensor components $P_{ij}$, and the respective identities assume a more compact form as follows:

\begin{align}
A_{141} &= \Delta \Psi_0 + (\mu - 4\gamma)\Psi_0 - \delta \Psi_1 + 2(2\tau + \beta)\Psi_1 - 3\sigma \Psi_2 \\
A_{144} &= \partial \Psi_0 - (\pi + 4\alpha)\Psi_0 + D \Psi_1 + 2(2\rho - \varepsilon)\Psi_1 + 3\varepsilon \Psi_2 \\
A_{441} &= \Delta \Psi_1 + 2(\mu - \gamma)\Psi_1 - \delta \Psi_2 + 3\tau \Psi_2 - \nu \Psi_0 + 2\sigma \Psi_3 \\
A_{214} &= \partial \Psi_1 - 2(\alpha + \pi)\Psi_1 + D \Psi_2 + 3\rho \Psi_2 - \lambda \Psi_0 - 2\kappa \Psi_3 \\
A_{132} &= \Delta \Psi_2 + 3\mu \Psi_2 + \delta \Psi_3 - 2(\beta - \tau) \Psi_3 - 2\nu \Psi_1 - \sigma \Psi_4 \\
A_{423} &= \partial \Psi_2 - 3\pi \Psi_2 - D \Psi_3 - 2(\varepsilon + \rho) \Psi_3 - 2\lambda \Psi_1 + \kappa \Psi_4 \\
A_{323} &= \Delta \Psi_3 + 2(\gamma + 2\mu) \Psi_3 + \delta \Psi_4 + (4\beta - \tau) \Psi_4 + 3\nu \Psi_2 \\
A_{223} &= \partial \Psi_3 + 2(\alpha - 2\pi) \Psi_3 - D \Psi_4 - (\rho + 4\varepsilon) \Psi_4 + 3\lambda \Psi_2,
\end{align}

with the analogous identities for the primed quantities.

**References**

[1] Apostolov V, Gauduchon P, [1997] The Riemannian Goldberg-Sachs Theorem, *Int. J. of Mathematics*, 8, 421-439.

[2] Cartan E, [1922] Sur les espaces conformes generalises et l’universe optique, *Compt. Rendus Acad. Sci. Paris* 174, 857-859.

[3] Chen X, LeBrun C, Weber B [2008] On conformally Kähler, Einstein manifolds, *J. Amer. Math. Soc.* 21 no. 4, 1137-1168.

[4] Goldberg J N, Sachs R K [1962] A theorem on Petrov types, *Acta Phys. Polon. Suppl.* 22, 13.

[5] Gover A R, Nurowski P [2006] Obstructions to conformally Einstein metrics in n dimensions, *J. Geom. Phys.* 56, 450-484.

[6] Hill C D, Lewandowski L, Nurowski P [2008] Einstein equations and the embedding of 3-dimensional CR manifolds, *Indiana Univ. Math. J.* 57, 3131-3176.

[7] Kopczyński W, Trautman A [1992] Simple spinors and real structures, *J. Math. Phys.* 33, 550-559.

[8] Kramer D, Stephani H, MacCallum, Herlt E [1980] *Exact solutions of Einstein’s field equations* VEB Deutscher Verlag der Wissenschaften, Berlin.

[9] Kundt W, Thompson A, [1962] Le tenseur de Weyl et une congruence associee de geodesiques isotropes sans distorsion, *C. R. Acad. Sci. (Paris)* 254, 4257.

[10] Law P R [2009] Spin Coefficients for Four-Dimensional Neutral Metrics, and Null Geometry, *J. Geom. Phys.* 59, 1087-1126, arXiv:0902.1761.

[11] LeBrun C [1997] Einstein Metrics on Complex Surfaces, in *Geometry and Physics*, Anderson et al. eds. pp. 167-176, Marcel Dekker, arXiv:dg-ga/9506012.

[12] LeBrun C [2009] Einstein metrics, complex surfaces, and symplectic 4-manifolds, *Math. Proc. Cambridge Philos. Soc.* 147, no. 1, 1-8, arXiv:0803.3743.

[13] Lewandowski J, Nurowski P, Tafel J [1990] Einstein’s equations and realizability of CR manifolds, *Class. Q. Grav.* 7, L241-246.

[14] Newman E T, Penrose R [1962] An approach to gravitational radiation by a method of spin coefficients, *Journ. Math. Phys.* 3, 866-902.

[15] Nurowski P [1993] *Einstein equations and Cauchy-Riemann geometry*, PhD Thesis, SISSA.

[16] Nurowski P [1996] Optical geometries and related structures, *Journ. Geom. Phys.* 18, 335-348.

[17] Nurowski P [1997] Twistor bundles, Einstein equations and real structures, *Class. Q. Grav.* 14, A261-A290.
[18] Penrose R, Rindler W, (1986) Spinors and space-time v.2, Cambridge University Press

[19] Plebanski J F, Hacyan S (1975) Null geodesic surfaces and Goldberg-Sachs theorem in complex Riemannian spaces, J. Math. Phys. 16 2403-2407

[20] Plebanski J F, Rozga K (1984) The optics of null strings, J. Math. Phys. 25 1930-1940

[21] Przanowski M, Broda B (1983) Locally Kähler gravitational instantons, Acta Phys. Polon. B 637-661

[22] Przanowski M, Plebanski J F (1979) Generalized Goldberg-Sachs theorems in complex and real space-times. I Acta Phys. Polon. B10 485-514

[23] Przanowski M, Plebanski J F (1979) Generalized Goldberg-Sachs theorems in complex and real space-times. II Acta Phys. Polon. B10 573-598

[24] Robinson I, Schild A (1963) Generalization of a theorem by Goldberg and Sachs, Journ. Math. Phys. 4 484-489

[25] Robinson I, Trautman A (1989) Optical geometry, in New theories in physics ed Ajduk Z at al., World Scientific, Singapore

[26] Trautman A A conjectured form of the Goldberg-Sachs theorem Twistor Newsletter

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