Hyperinvariant subspaces for sets of polynomially compact operators

J. Bračič1 · M. Kandić2,3

Received: 23 May 2022 / Accepted: 10 September 2022 / Published online: 29 September 2022
© Tusi Mathematical Research Group (TMRG) 2022

Abstract
We prove the existence of a non-trivial hyperinvariant subspace for several sets of polynomially compact operators. The main results of the paper are: (i) a non-trivial norm closed algebra \( \mathcal{A} \subseteq \mathcal{B}(\mathcal{X}) \) which consists of polynomially compact quasinilpotent operators has a non-trivial hyperinvariant subspace; (ii) if there exists a non-zero compact operator in the norm closure of the algebra generated by an operator band \( \mathcal{S} \), then \( \mathcal{S} \) has a non-trivial hyperinvariant subspace.

Keywords Polynomially compact operator · Hyperinvariant subspace

Mathematics Subject Classification 47A15 · 47B07 · 47L10 · 47B10

1 Introduction

Let \( \mathcal{X} \) be a complex Banach space. Denote by \( \mathcal{B}(\mathcal{X}) \) the algebra of all bounded linear operators on \( \mathcal{X} \). A closed subspace \( \mathcal{M} \subseteq \mathcal{X} \) is said to be invariant for an operator \( T \in \mathcal{B}(\mathcal{X}) \) if \( T \mathcal{M} \subseteq \mathcal{M} \). Let \( \mathcal{S} \subseteq \mathcal{B}(\mathcal{X}) \) be a non-empty set of operators. Then, \( \mathcal{M} \) is an invariant subspace of \( \mathcal{S} \) if it is invariant for every operator in \( \mathcal{S} \). If \( \mathcal{M} \) is invariant for every operator in \( \mathcal{S} \) and for every operator in the commutant

Communicated by Mostafa Mbekhta.

\( \Box \) M. Kandić
marko.kandic@fmf.uni-lj.si

J. Bračič
janko.bracic@ntf.uni-lj.si

1 Faculty of Natural Sciences and Engineering, University of Ljubljana, Aškerčeva Cesta 12, 1000 Ljubljana, SI, Slovenia

2 Faculty of Mathematics and Physics, University of Ljubljana, Jadranska Ulica 19, 1000 Ljubljana, SI, Slovenia

3 Institute of Mathematics, Physics and Mechanics, Jadranska Ulica 19, 1000 Ljubljana, SI, Slovenia
Let \( \mathcal{S}' = \{ T \in \mathcal{B}(\mathcal{X}) ; TS = ST \text{ for every } S \in \mathcal{S} \} \), then it is a hyperinvariant subspace of \( \mathcal{S} \). Of course, the trivial subspaces \( \{0\} \) and \( \mathcal{X} \) are (hyper)invariant for any set of operators. We are interested in the existence of non-trivial invariant and hyperinvariant subspaces. The problem of existence of invariant and hyperinvariant subspaces for a given operator or a non-empty set of operators is an extensively studied topic in operator theory. The problem is solved in the finite-dimensional setting by Burnside’s theorem (see [14, Theorem 1.2.2]). In the context of infinite-dimensional Banach spaces, the problem is open for reflexive Banach spaces, in particular, for the infinite-dimensional separable Hilbert space. However, there are some Banach spaces for which we know either that every operator has a non-trivial invariant subspace or that there exist operators without it. For instance, Argyros and Haydon [1] have proved the existence of an infinite-dimensional Banach spaces \( \mathcal{X} \) such that every operator in \( \mathcal{B}(\mathcal{X}) \) is of the form \( \lambda I + K \), where \( I \) is the identity operator and \( K \) is compact. It follows, by the celebrated von Neumann-Aronszajn-Smith theorem [2] and Lomonosov’s theorem [12], that any operator in \( \mathcal{B}(\mathcal{X}) \) has a non-trivial invariant subspace and any non-scalar operator in \( \mathcal{B}(\mathcal{X}) \) has a non-trivial hyperinvariant subspace. On the other hand, several examples of Banach spaces (including \( c_0 \)) with operators without a non-trivial invariant subspace are known (see [3, 15] for the first examples and [6] for a general approach to Read’s type constructions of operators without non-trivial invariant closed subspaces).

With the von Neumann–Aronszajn–Smith theorem and Lomonosov’s theorem in mind, it is not a surprise that suitable compactness conditions imply existence of non-trivial invariant and hyperinvariant subspaces for different classes of operators and sets of operators. For instance, Shulman [16, Theorem 2] proved that an algebra of operators whose radical contains a non-zero compact operator has a non-trivial hyperinvariant subspace. Turovskii [17, Corollary 5] extended this result to semigroups of quasinilpotent operators. Another type of results are those related to triangularizability of a set of operators. Recall that a non-empty set \( \mathcal{S} \subseteq \mathcal{B}(\mathcal{X}) \) is triangularizable if there exists a chain \( \mathcal{C} \) which is maximal as a chain of subspaces of \( \mathcal{X} \) and every subspace in \( \mathcal{C} \) is invariant for all operators in \( \mathcal{S} \). Every commutative set of compact operators is triangularizable (see [14, Theorem 7.2.1]). Konvalinka [10, Corollary 2.6] has extended this result by showing that a commuting family of polynomially compact operators is triangularizable. Another result in this direction, obtained by the second author [9], says that for a norm closed subalgebra \( \mathcal{A} \subseteq \mathcal{B}(\mathcal{X}) \) of power compact operators the following assertions are equivalent: (a) \( \mathcal{A} \) is triangularizable; (b) the Jacobson radical \( \mathcal{R}(\mathcal{A}) \) consists precisely of quasinilpotent operators in \( \mathcal{A} \); (c) the quotient algebra \( \mathcal{A}/\mathcal{R}(\mathcal{A}) \) is commutative.

The aim of this paper is to consider the problem of existence of a non-trivial hyperinvariant subspace for sets of polynomially compact operators. For instance, we prove (Theorem 3.3) that a non-trivial norm closed algebra \( \mathcal{A} \subseteq \mathcal{B}(\mathcal{X}) \) which consists of polynomially compact quasinilpotent operators has a non-trivial hyperinvariant subspace. Another result (Theorem 4.3) which we mention here is related to operator bands, that is, to semigroups of idempotent operators. It says that an operator band \( \mathcal{S} \) has a non-trivial hyperinvariant subspace if there exists a non-zero compact operator in the norm closure of the algebra generated by \( \mathcal{S} \).
2 Preliminaries

2.1 Notation

Let $\mathcal{X}$ be a non-trivial complex Banach space. Since the results proved in this paper are either trivial or well known when $\mathcal{X}$ is finite-dimensional, we always assume that $\dim(\mathcal{X}) = \infty$. Let $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of all bounded linear operators on $\mathcal{X}$ and let $\mathcal{K}(\mathcal{X}) \subseteq \mathcal{B}(\mathcal{X})$ be the ideal of compact operators. The identity operator is denoted by $I$ and an operator is said to be scalar if it is a scalar multiple of $I$. The norm closure of a set $S \subseteq \mathcal{B}(\mathcal{X})$ is denoted by $\overline{S}$.

For two operators $S_1, S_2 \in \mathcal{B}(\mathcal{X})$, we denote their commutator $S_1S_2 - S_2S_1$ by $[S_1, S_2]$. A non-empty set $S \subseteq \mathcal{B}(\mathcal{X})$ is commutative if any two operators from $S$ commute, that is, $[S_1, S_2] = 0$ for all $S_1, S_2 \in S$. Similarly, $S$ is essentially commutative if $[S_1, S_2] \in \mathcal{K}(\mathcal{X})$. For an arbitrary non-empty subset $S \subseteq \mathcal{B}(\mathcal{X})$ the commutant of $S$ is $S' = \{T \in \mathcal{B}(\mathcal{X}); [T, S] = 0, \forall S \in S\}$. It is clear that $S'$ is a closed subalgebra of $\mathcal{B}(\mathcal{X})$. If $S$ is commutative, then $S \subseteq S'$.

2.2 Invariant subspaces

A non-empty subset $\mathcal{M} \subseteq \mathcal{X}$ is a subspace if it is a closed linear manifold. It is said that a subspace $\mathcal{M}$ of $\mathcal{X}$ is invariant for the operator $T$ if $TM \subseteq \mathcal{M}$. An invariant subspace $\mathcal{M}$ is non-trivial if $\{0\} \neq \mathcal{M} \neq \mathcal{X}$. A non-empty set $S \subseteq \mathcal{B}(\mathcal{X})$ is reducible if there exists a non-trivial subspace $\mathcal{M} \subseteq \mathcal{X}$ which is invariant for every $T \in S$. If there exists a chain $\mathcal{C}$ which is maximal as a chain of subspaces of $\mathcal{X}$ and every subspace in $\mathcal{C}$ is invariant for all operators in $S$, then $S$ is said to be triangularizable.

If a subspace $\mathcal{M}$ is invariant for every operator $T$ in a set $S$ and in its commutant $S'$, then $\mathcal{M}$ is said to be a hyperinvariant subspace for $S$.

Next theorem (see Assertion in the end of [12]) is one of the deepest results in the theory of invariant subspaces.

**Theorem 2.1** (Lomonosov) Every non-scalar operator which commutes with a non-zero compact operator has a non-trivial hyperinvariant subspace.

The proof of Theorem 2.1 relies on the following useful lemma.

**Lemma 2.2** (Lomonosov’s Lemma) Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{X})$ be an algebra. If $\mathcal{A}$ is not reducible, then for every non-zero compact operator $K \in \mathcal{B}(\mathcal{X})$ there exist an operator $A \in \mathcal{A}$ and a non-zero vector $x \in \mathcal{X}$ such that $AKx = x$. 
2.3 Spectral radius

The spectrum of an operator \( T \in \mathcal{B} \) is denoted by \( \sigma(T) \) and the spectral radius of \( T \) is \( \rho(T) = \max \{|z|; z \in \sigma(T)\} \). By the spectral radius formula (Gelfand’s formula), \( \rho(T) = \lim_{n \to \infty} \|T^n\|^\frac{1}{n} \). An operator \( T \in \mathcal{B} \) is quasinilpotent if \( \rho(T) = 0 \). A compact quasinilpotent operator is called a Volterra operator.

Let \( \mathcal{F} \) be a non-empty set of operators in \( \mathcal{B} \). For each \( n \in \mathbb{N} \), let \( \mathcal{F}^{(n)} = \{T_1 \cdots T_n; T_1, \ldots, T_n \in \mathcal{F}\} \). By \( ||\mathcal{F}|| = \sup\{|T|; T \in \mathcal{F}\} \) we denote the joint norm of \( \mathcal{F} \) and by \( \rho(\mathcal{F}) \), we denote the joint spectral radius of \( \mathcal{F} \) defined as

\[
\rho(\mathcal{F}) = \lim_{n \to \infty} ||\mathcal{F}^{(n)}||^\frac{1}{n}.
\]

A subalgebra \( \mathcal{A} \subseteq \mathcal{B} \) is said to be finitely quasinilpotent if \( \rho(\mathcal{F}) = 0 \) for every finite subset \( \mathcal{F} \) of \( \mathcal{A} \). By [16, Theorem 1], every subalgebra \( \mathcal{A} \subseteq \mathcal{B} \) of Volterra operators is finitely quasinilpotent.

2.4 Polynomially compact operators

An operator \( T \in \mathcal{B} \) is polynomially compact if there exists a non-zero complex polynomial \( p \) such that \( p(T) \) is a compact operator. In particular, algebraic operators (nilpotents, idempotents, etc.) are polynomially compact. Hence, \( T \) is polynomially compact if and only if \( \pi(T) \), where \( \pi : \mathcal{B} \to \mathcal{B}/\mathcal{K} \) is the quotient projection, is an algebraic element in the Calkin algebra \( \mathcal{B}/\mathcal{K} \). If \( T^n \) is compact for some \( n \in \mathbb{N} \), then \( T \) is said to be power compact. For a polynomially compact operator \( T \), there exists a unique monic polynomial \( m_T \) of the smallest degree such that \( m_T(T) \) is a compact operator. The polynomial \( m_T \) is called the minimal polynomial of \( T \). The following is the structure theorem for polynomially compact operators proved by Gilfeather [4, Theorem 1].

**Theorem 2.3** Let \( T \in \mathcal{B} \) be a polynomially compact operator with minimal polynomial \( m_T(z) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k} \). Then there exist invariant subspaces \( \mathcal{X}_1, \ldots, \mathcal{X}_k \) for \( T \) such that \( \mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_k \) and \( T = T_1 \oplus \cdots \oplus T_k \), where \( T_i \) is the restriction of \( T \) to \( \mathcal{X}_i \). The operators \( (T_j - \lambda_j I_j)^{n_j} \) are all compact.

The spectrum of \( T \) consists of countably many points with \( \{\lambda_1, \ldots, \lambda_k\} \) as the only possible limit points and such that all but possibly \( \{\lambda_1, \ldots, \lambda_k\} \) are eigenvalues with finite-dimensional generalized eigenspaces. Each point \( \lambda_j (j = 1, \ldots, k) \) is either the limit of eigenvalues of \( T \) or else \( \mathcal{X}_j \) is infinite-dimensional and \( T_j - \lambda_j I_j \) is a quasinilpotent operator on \( \mathcal{X}_j \).

**Corollary 2.4** A quasinilpotent operator \( T \in \mathcal{B} \) is polynomially compact if and only if it is power compact.
Hyperinvariant subspaces for sets of polynomially compact operators

Proof It is clear that every power compact operator is polynomially compact. On the other hand, if $T$ is a polynomially compact and quasinilpotent, then $\sigma(T) = \{0\}$ and therefore $m_T(z) = z^n$ for a positive integer $n$, by Theorem 2.3, that is, $T$ is power compact.

\[\Box\]

2.5 Algebras and ideals

For a non-empty set of operators $S \subseteq B(\mathcal{X})$, let $A(S)$ be the subalgebra of $B(\mathcal{X})$ generated by $S$ and let $A_1(S)$ be the subalgebra of $B(\mathcal{X})$ generated by $S$ and $I$. By $\mathcal{H}(S)$ we denote the algebra which is generated by $S$ and its commutant $S'$. We will call it the hyperalgebra of $S$. If $S$ is a semigroup, then an operator $T \in B(\mathcal{X})$ is in $\mathcal{H}(S)$ if and only if there exist $n \in \mathbb{N}$ and operators $S_1, \ldots, S_n \in S$ and $T_0, T_1, \ldots, T_n \in S'$ such that

\[T = T_0 + S_1T_1 + \cdots + S_nT_n = T_0 + T_1S_1 + \cdots + T_nS_n.\]

Here, we used the fact that $I \in S'$. Since $S \subseteq \mathcal{H}(S)$ and $A(S)$ is the smallest algebra which contains $S$, we have $A(S) \subseteq \mathcal{H}(S)$. On the other hand, it is obvious that $S' = A(S)'$ and therefore $A(S)' \subseteq \mathcal{H}(S)$. We conclude that the hyperalgebra of $S$ is generated by $A(S)$ and $A(S)'$, that is, $\mathcal{H}(S) = \mathcal{H}(A(S))$.

If $\mathcal{M}$ and $\mathcal{N}$ are non-empty subsets of $B(\mathcal{X})$, then let $\mathcal{M} + \mathcal{N} = \{M + N; M \in \mathcal{M}, N \in \mathcal{N}\}$ and let $\mathcal{M}N$ be the set of all finite sums $M_1N_1 + \cdots + M_kN_k$, where $M_i \in \mathcal{M}$ and $N_i \in \mathcal{N}$ for each $i = 1, \ldots, k$. Hence, if $A$ is a subalgebra of $B(\mathcal{X})$, then its hyperalgebra is $\mathcal{H}(A) = A' + AA' = A' + A'A$.

If $\mathcal{J}$ is an ideal in an algebra $A \subseteq B(\mathcal{X})$, then we write $\mathcal{J} \triangleleft A$. We will denote by $\mathcal{J}_\mathcal{N}$ the ideal in $\mathcal{H}(A)$ generated by $\mathcal{J}$.

Lemma 2.5 Let $A \subseteq B(\mathcal{X})$ be an algebra. If $\mathcal{J} \triangleleft A$, then $\mathcal{J}_\mathcal{N} = A' \mathcal{J} = \mathcal{J}A'$.

Proof From equalities $\mathcal{J}_\mathcal{N} = \mathcal{J} + \mathcal{H}(A)\mathcal{J} + \mathcal{J}\mathcal{H}(A) + \mathcal{H}(A)\mathcal{J}\mathcal{H}(A)$ and $\mathcal{H}(A) = A' + A'A$, we conclude

\[
\mathcal{J}_\mathcal{N} = \mathcal{J} + \mathcal{H}(A)\mathcal{J} + \mathcal{H}(A)A + \mathcal{H}(A)\mathcal{J}\mathcal{H}(A) \\
= \mathcal{J} + \mathcal{J} =\mathcal{J} = \mathcal{J} = \mathcal{J} =\mathcal{J} = \mathcal{J} =\mathcal{J} = \mathcal{J} + A' \mathcal{J} \subseteq A' \mathcal{J}
\]

as $A'$ contains the identity operator. On the other hand, since we also have $A' \mathcal{J} \subseteq \mathcal{J}_\mathcal{N}$, we obtain the equality $\mathcal{J}_\mathcal{N} = A' \mathcal{J}$. $\Box$

An algebra $A$ over an arbitrary field is said to be a nil-algebra if every element of $A$ is nilpotent. A nil-algebra $A$ is of bounded nil-index if there exists a positive integer $n$ such that $x^n = 0$ for each $x \in A$. If there exists $n \in \mathbb{N}$ such that $a_1 \cdots a_n = 0$ for all $a_1, \ldots, a_n \in A$, then $A$ is said to be a nilpotent algebra. By the celebrated Nagata-Higman theorem (see [8, 13]), every nil-algebra of bounded nil-index is nilpotent.
Lemma 2.6 Let $A \subseteq B(\mathcal{X})$ be an algebra. An ideal $I \triangleleft A$ is nilpotent if and only if $I^* \triangleleft \mathcal{H}(A)$ is nilpotent. The nilpotency indices of $I$ and $I^*$ are equal.

**Proof** Since $A'$ and $I$ commute, an easy induction shows that for each $n \in \mathbb{N}$ we have $(A'I)^n = A'I^n$. Hence, if $I^n = \{0\}$, then $(A'I)^n = \{0\}$, as well. If $(A'I)^n = \{0\}$, then $I^n \subseteq A'I^n = (A'I)^n = \{0\}$ yields that $I^n = \{0\}$. $\square$

If $I$ is a nil-ideal of bounded nil-index, then $I$ is nilpotent by the Nagata-Higman theorem. This immediately implies that $I^*$ is nilpotent. In particular, if $A$ is a nilpotent algebra, then $AA'$ is a nilpotent ideal in the hyperalgebra $\mathcal{H}(A)$.

3 Hyperinvariant subspaces of algebras of polynomially compact operators

The simplest polynomially compact operators which are not necessary compact are algebraic operators, in particular nilpotent operators. Hadwin et al. [7, Corollary 4.2] proved that a norm closed algebra of nilpotent operators on the separable infinite-dimensional complex Hilbert space is triangularizable. The following proposition shows that a norm closed algebra of nilpotent operators on an arbitrary complex Banach space has a non-trivial hyperinvariant subspace.

**Proposition 3.1** If a subalgebra $\{0\} \neq A \subseteq B(\mathcal{X})$ consists of nilpotent operators, then its hyperalgebra $\mathcal{H}(A)$ is reducible in either of the following cases.

(a) The algebra $A$ is nilpotent.
(b) The algebra $A$ is norm closed.

**Proof** (a) There exists $n \in \mathbb{N}$ such that an arbitrary product of at least $n$ operators from $A$ is the zero operator. Let $n_0$ be the smallest positive integer with this property. Since $A \neq \{0\}$ we have $n_0 > 1$. There exist operators $A_1, \ldots, A_{n_0-1} \in A$ such that $A_0 := A_1, \ldots, A_{n_0-1} \neq 0$. Note that $A_0T = TA_0 = 0$ for every operator $T \in A$, that is, $A \subseteq (A_0)'$, where $(A_0)'$ is the commutant of $A_0$. It is clear that $A' \subseteq (A_0)'$. Hence, $\mathcal{H}(A) \subseteq (A_0)'$. Since $A_0 \neq 0$ the kernel $\ker(A_0)$ is a non-trivial subspace of $\mathcal{X}$ and it is hyperinvariant for $A_0$. It follows that $\ker(A_0)$ is a non-trivial hyperinvariant subspace for $A$, that is, $\mathcal{H}(A)$ is reducible.

(b) Since $A$ is a closed subalgebra of $B(\mathcal{X})$, it is a Banach algebra. Therefore, $A$ is a Banach algebra which is also a nil-algebra, so that by Grabiner’s theorem [5], $A$ is a nilpotent algebra. Now, we apply (a). $\square$

Hadwin et al. have constructed a semi-simple algebra of nilpotent operators on a separable Hilbert space such that for each pair $(x, y)$ of vectors, where $x \neq 0$, there exists an operator $A \in A$ such that $Ax = y$ (see [7, Section 4]). Hence, Proposition 3.1(b) (and consequently Theorem 3.3 below) does not hold, in general,
for non-closed algebras. However, as the following example shows there are simple special cases when a not necessarily closed algebra generated by a set of nilpotents has a non-trivial hyperinvariant subspace.

**Example** The idea for this example is from [7, Theorem 4.3] where quadratic nilpotent operators are considered. Choose and fix \( \lambda \in \mathbb{C} \). Let \( S \subseteq \mathcal{B}(\mathcal{H}) \) be a non-empty set of operators such that \( A^2 = \lambda A \) for every \( A \in S \). If \( S \) is a linear manifold and contains a non-scalar operator, then \( \mathcal{A}(S) \) has a non-trivial hyperinvariant subspace. To see this, choose a non-scalar operator \( A \in S \). Then its kernel \( \ker(A) \) is a non-trivial hyperinvariant subspace for \( A \). Since \( \mathcal{A}(S)' \subseteq (A)' \), we see that \( \ker(A) \) is invariant for every operator from \( \mathcal{A}(S)' \). Let \( B \in S \) be arbitrary. It follows from \( (A + B)^2 = \lambda(A + B) \) that \( AB = -BA \). Hence, \( ABx = -BAx = 0 \) for every \( x \in \ker(A) \), that is, \( \ker(A) \) is invariant for every \( B \in S \). We conclude that \( \ker(A) \) is invariant for every operator from the hyperalgebra \( \mathcal{H}(S) \).

**Proposition 3.2** Let \( A \subseteq \mathcal{B}(\mathcal{H}) \) be a non-trivial subalgebra. If there exists a non-trivial nilpotent ideal \( J \triangleleft A \), then \( A \) has a non-trivial hyperinvariant subspace.

**Proof** Let \( J \) be a non-trivial nilpotent ideal in \( \mathcal{A}(S) \). Then, by Lemma 2.6, the ideal \( J_n \), which is generated by \( J \) in the hyperalgebra \( \mathcal{H}(A) \) is nilpotent, as well. By Proposition 3.1(a) the ideal \( J_n \) has a non-trivial hyperinvariant subspace, in particular, it is reducible. By [14, Lemma 7.4.6], \( \mathcal{H}(A) \) is reducible, as well. \( \square \)

The following two theorems show that an algebra \( A \) of polynomially compact operators has a non-trivial hyperinvariant subspace if \( A \) satisfies some additional condition. For instance, as we already mentioned, the key assumption in Theorem 3.3 is that the involved algebra is norm closed.

**Theorem 3.3** Let \( \{0\} \neq A \subseteq \mathcal{B}(\mathcal{H}) \) be a norm closed algebra. If every operator in \( A \) is quasinilpotent and polynomially compact, then \( A \) has a non-trivial hyperinvariant subspace.

**Proof** Note first that each operator in \( A \) is power compact, by Corollary 2.4. If each operator in \( A \) is nilpotent, then \( \mathcal{H}(A) \) is reducible, by Proposition 3.1(b). Assume therefore that there exists an operator \( T \in A \) which is not nilpotent. Since \( T \) is power compact there exists \( m \in \mathbb{N} \) such that \( K = T^m \neq 0 \) is compact. Let \( J \) be the two-sided ideal in \( A \), generated by \( K \). It is clear that \( K \in J \subseteq A \). Hence, \( J \) is an algebra of Volterra operators. By [16, Theorem 1], \( J \) is finitely quasinilpotent. Let \( A = B_0 + \sum_{i=1}^n A_i B_i \), where \( A_i \in A \) and \( B_i \in A' \), be an arbitrary operator in \( \mathcal{H}(A) \) and let \( M = \{ K, A_1 K, \ldots, A_n K \} \). Since \( M \) is a finite subset of \( J \) it is a quasinilpotent set, that is, \( \rho(M) = 0 \). Let \( N = \{ B_0, B_1, \ldots, B_n \} \subseteq A' \). Since \( B_0, B_1, \ldots, B_n \) commute with operators from \( M \) we have \( \rho(AK) \leq (n + 1)\rho(M)\rho(N) = 0 \), by [16, Lemma 1]. This shows that the operator \( AK \) is quasinilpotent for each \( A \in \mathcal{H}(A) \). It follows, by Lemma 2.2, that the hyperalgebra \( \mathcal{H}(A) \) is reducible.
Theorem 3.4 Let $S \subseteq \mathcal{B}(\mathcal{H})$ be an essentially commutative set of polynomially compact operators which contains at least one non-scalar operator. If $S$ is triangularizable, then $\mathcal{A}(S)$ has a non-trivial hyperinvariant subspace.

**Proof** It is obvious that every subspace of $\mathcal{H}$ which is invariant for $S$ is invariant for $\mathcal{A}(S)$, as well. Hence, $\mathcal{A}(S)$ is triangularizable. Since $S$ consists of polynomially compact operators, the same holds for the algebra $\mathcal{A}(S)$, by [10, Theorem 1.5]. Denote by $\pi(\mathcal{A}(S))$ and $\pi(S)$ the image of $\mathcal{A}(S)$ and $S$, respectively, in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Since $\pi(\mathcal{A}(S))$ is the subalgebra of the Calkin algebra generated by the commutative set $\pi(S)$ we see that $\pi(\mathcal{A}(S))$ is commutative, as well. It follows that $\mathcal{A}(S)$ is an essentially commutative subalgebra of $\mathcal{B}(\mathcal{H})$. By [9, Theorem 3.5], for an algebra of essentially commuting polynomially compact operators triangularizability of $\mathcal{A}(S)$ is equivalent to the fact that each commutator $[S, T]$, where $S, T \in \mathcal{A}(S)$, is a quasinilpotent operator.

Suppose that there exist operators $S, T \in \mathcal{A}(S)$ such that the commutator $K := [S, T]$ is non-zero. Let $J$ be the ideal in $\mathcal{A}(S)$ generated by $K$. Of course, $J$ consists of compact operators and, by Ringrose’s Theorem (see [14, Theorem 7.2.3]), every operator in $J$ is quasinilpotent. Hence, $J$ is an algebra of Volterra operators. Let $J^1$ be the ideal in the hyperalgebra $\mathcal{H}(S)$ generated by $J$. It is clear that $J^1 \subseteq J$. We claim that $J^1$ consists of quasinilpotent operators, as well. Choose an arbitrary operator $A \in J^1$. By Lemma 2.5, there exist operators $I_1, \ldots, I_n \in J$ and operators $B_1, \ldots, B_n \in \mathcal{A}(S)'$ such that $A = \sum_{i=1}^n I_i B_i$. By [16, Lemma 1], $\rho(A) \leq \eta n \rho(\{I_1, \ldots, I_n\}) \rho(\{B_1, \ldots, B_n\})$. Since each finite subset of an algebra of Volterra operators is quasinilpotent, by [16, Theorem 1], we have $\rho(\{I_1, \ldots, I_n\}) = 0$ and consequently $\rho(A) = 0$. We have proved that $J^1$ is a non-trivial ideal of Volterra operators. By [16, Theorem 2], the hyperalgebra $\mathcal{H}(S)$ is reducible.

Suppose now that $\mathcal{A}(S)$ is commutative. Hence, $\mathcal{A}(S) \subseteq \mathcal{A}(S)'$. Let $T$ be any non-scalar polynomially compact operator in $\mathcal{A}(S)$ and let $m_T$ be its minimal polynomial. Hence, $m_T(T)$ is a compact operator. If $m(T)$ is a non-zero compact operator, then it has a non-trivial hyperinvariant subspace $\mathcal{Y}$, by Lomonosov’s Theorem. Since $\mathcal{A}(S) \subseteq \mathcal{A}(S)' \subseteq (m_T(T))'$ subspace $\mathcal{Y}$ is invariant for every operator in $\mathcal{A}(S)$ and $\mathcal{A}(S)'$. Thus, $\mathcal{H}(S)$ is reducible. On the other hand, if $m_T(T) = 0$, then $T$ is a non-scalar algebraic operator. Hence, for every $\lambda \in \sigma(T)$, the kernel $\ker(T - \lambda I)$ is a non-trivial hyperinvariant subspace for $T$, and consequently, for $\mathcal{A}(S)$ as $\mathcal{A}(S) \subseteq \mathcal{A}(S)' \subseteq (T)'$. \qed

4 Hyperinvariant subspaces for operator bands

An operator band on a Banach space $\mathcal{H}$ is a (multiplicative) semigroup $S \subseteq \mathcal{B}(\mathcal{H})$ of idempotents, that is, $S^2 = S$ for each $S \in S$. The linear span of an operator band $S$ is the algebra $\mathcal{A}(S)$ called a band algebra. We will denote by $\mathcal{N}(S)$, the set of all nilpotent operators in the band algebra $\mathcal{A}(S)$. By [11, Theorem 5.2], $\mathcal{N}(S)$ is the linear
span of \([S, S]\), that is, \(N(S) = \{ \sum_{i=1}^{n} [A_i, B_i]; n \in \mathbb{N}, A_i, B_i \in \mathcal{A}(S) \}\). By \([11, \text{Corollary 5.5}]\), the set \(N(S)\) of nilpotent operators in \(\mathcal{A}(S)\) coincides with the Jacobson radical \(\mathcal{R}(\mathcal{A}(S))\) of \(\mathcal{A}(S)\).

**Proposition 4.1** Let \(\{0\} \neq S \subseteq \mathcal{B}(\mathcal{X})\) be an operator band. If \(N \subseteq N(S)\) is a non-empty set, then the ideal \(J_M\) in the hyperalgebra \(\mathcal{H}(S)\) generated by \(N\) is a nil-ideal.

**Proof** Denote by \(J\) the ideal in \(\mathcal{A}(S)\) generated by \(N\). We have already mentioned that \(\mathcal{R}(\mathcal{A}(S)) = N(S)\). Since the radical is an ideal, we have that \(J \subseteq \mathcal{R}(\mathcal{A}(S))\). Hence, \(J\) is a nil-ideal in \(\mathcal{A}(S)\).

It is clear that \(N\) and ideal \(J\) generate the same ideal \(J_M\) in the hyperalgebra \(\mathcal{H}(S)\). Let \(A \in J_M\) be arbitrary. By Lemma 2.5, there exist \(n \in \mathbb{N}\), \(J_i \in J\) and \(B_i \in \mathcal{A}(S)\) such that \(A = \sum_{i=1}^{n} J_iB_i\). Let \(F\) be a finite subset of \(S\) such that \(J_1, \ldots, J_n\) are contained in the algebra \(\mathcal{A}(F)\) generated by \(F\). Note that such finite sets exist, because each \(J_i\) is of the form \(p_i(S_{i_1}^{(i)}, \ldots, S_{i_k}^{(i)})\), where \(p_i\) is a polynomial of \(k_i\) non-commuting variables and \(S_{i_1}^{(i)}, \ldots, S_{i_k}^{(i)}\) are idempotents from \(S\). Denote by \(S_0\) the operator band generated by \(F\). Thus, \(\mathcal{A}(F) \subseteq \mathcal{A}(S_0)\). Since \(F\) is finite, the operator band \(S_0\) is finite, as well, by the Green-Rees theorem (see \([14, \text{Theorem 9.3.11}]\)). Now, we apply \([14, \text{Theorem 9.3.15}]\) which says that there exists a finite chain \(0 = \mathcal{X}_0 \subseteq \mathcal{X}_1 \subseteq \cdots \subseteq \mathcal{X}_m = \mathcal{X}\) of invariant subspaces for \(S_0\) such that for each \(E \in S_0\), the operator induced by \(E\) on \(\mathcal{X}_i / \mathcal{X}_{i-1}\) is either zero or the identity operator. This implies that operator induced by \(J_i\) on \(\mathcal{X}_i / \mathcal{X}_{i-1}\) is a scalar multiple of the identity operator. Since each \(J_i\) is nilpotent, every operator \(J_i\) induces the zero operator on \(\mathcal{X}_i / \mathcal{X}_{i-1}\). Hence, for each \(i\) and each \(j\), we have \(J_i(\mathcal{X}_j) \subseteq \mathcal{X}_{j-1}\). This implies that an arbitrary product of length at least \(m\) with letters from \(\{J_1, \ldots, J_n\}\) is zero. Now it is obvious that \(A^m = (\sum_{i=1}^{n} J_iB_i)^m = 0\) as \(J_i\) and \(B_i\) commute. \(\square\)

**Proposition 4.2** Let \(\{0\} \neq S \subseteq \mathcal{B}(\mathcal{X})\) be an operator band. If \(K \in \overline{\mathcal{A}(S)}\) is a compact operator, then for each operator \(A \in \overline{\mathcal{H}(S)}\) the commutator \([K, A]\) is in the Jacobson radical of \(\mathcal{H}(S)\), that is, \([K, \mathcal{H}(S)] \subseteq \mathcal{R}(\mathcal{H}(S))\).

**Proof** Let \(K \in \overline{\mathcal{A}(S)}\) be a compact operator and let \(A \in \overline{\mathcal{H}(S)}\) be an arbitrary operator. To prove that the commutator \([K, A]\) is in \(\mathcal{R}(\overline{\mathcal{H}(S)})\), we need to show that \([K, A]C\) is quasinilpotent for each \(C \in \mathcal{H}(S)\). Since \([K, A]C\) is compact, the continuity of the spectral radius at compact operators yields that it suffices to prove that \([K, A]C\) is quasinilpotent for all \(A, C \in \mathcal{H}(S)\).

Let \(A, C \in \mathcal{H}(S)\) be arbitrary. Since \(K\) belongs to the closure of \(\mathcal{A}(S)\), there is a sequence \((K_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}(S)\) which converges to \(K\). We claim that for each \(n \in \mathbb{N}\) the commutator \([K_n, A]\) is nilpotent and belongs to the ideal \(N(S)\) in \(\mathcal{H}(S)\) generated by the Jacobson radical \(N(S)\) of the band algebra \(\mathcal{A}(S)\). Since \(A \in \mathcal{H}(S)\), there exist operators \(A_1, \ldots, A_k \in \mathcal{A}(S)\) and \(B_0, \ldots, B_k \in \mathcal{A}(S)'\) such that \(A = B_0 + \sum_{i=1}^{k} A_iB_i\). It follows that \([K_n, A] = \sum_{i=1}^{k} [K_n, A_i]B_i\). Since every commutator \([K_n, A_i]\) is in \(N(S)\)
and $\mathcal{N}(S) \subseteq \mathcal{N}(S)'$, it follows $[K_n, A_i] \in \mathcal{N}(S)'$. Since $\mathcal{N}(S)'$ is an ideal in the hyper-algebra $\mathcal{H}(S)$ we have $[K_n, A]C \in \mathcal{N}(S)'$. By Proposition 4.1, the operator $[K_n, A]C$ is nilpotent. To finish the proof, we apply the fact that the spectral radius is continuous at compact operators and that the compact operator $[K, A]C$ is the limit of the sequence $([K_n, A]C)_{n \in \mathbb{N}}$ of nilpotent operators.

\end{proof}

**Theorem 4.3** Let $S \subseteq \mathcal{B}(\mathcal{H})$ be an operator band. If there exists a non-zero compact operator in $\overline{\mathcal{A}(S)}$, then $S$ has a non-trivial hyperinvariant subspace.

**Proof** Let $K$ be a non-zero compact operator in $\overline{\mathcal{A}(S)}$. Then, by Proposition 4.2, $[K, A]$ is a compact operator contained in the radical $\mathcal{R}(\mathcal{H}(S))$ for each $A \in \mathcal{H}(S)$. If $[K, A] = 0$ for every $A \in \mathcal{H}(S)$, then $\mathcal{H}(S) \subseteq (K)'$ and, therefore, every non-trivial hyperinvariant subspace of $K$ is a non-trivial hyperinvariant subspace for $\mathcal{A}(S)$. Hence, we may assume that for some $A \in \mathcal{H}(S)$ the commutator $[K, A]$ is a non-zero operator. Let $\mathcal{J}$ be the ideal in $\mathcal{H}(S)$ generated by $[K, A]$. Of course, each operator in $\mathcal{J}$ is compact. Since, by Proposition 4.2, $[K, A]$ is in the Jacobson ideal of $\mathcal{R}(\mathcal{H}(S))$ we have $\mathcal{J} = \mathcal{H}(S)[K, A] \subseteq \mathcal{H}(S)\mathcal{R}(\mathcal{H}(S)) \mathcal{H}(S) \subseteq \mathcal{R}(\mathcal{H}(S))$. It follows that operators in $\mathcal{J}$ are quasinilpotent. Thus, $\mathcal{J}$ is a Volterra ideal in $\overline{\mathcal{H}(S)}$. By [16, Theorem 2], $\mathcal{J}$ is reducible. Now we apply [14, Lemma 7.4.6] and conclude that $\mathcal{H}(S)$ is reducible.

\end{proof}

**Corollary 4.4** Every essentially commuting non-scalar operator band $S \subseteq \mathcal{B}(\mathcal{H})$ has a non-trivial hyperinvariant subspace.

**Proof** If $S$ is commutative, then $S \subseteq S'$. In this case the kernel of any non-scalar operator from $S$ is invariant for each $S \in S'$. If $S$ is not commutative, then there exist idempotents $E, F \in S$ with a non-zero compact commutator $EF - FE \in \mathcal{A}(S)$. The assertion follows, by Theorem 4.3.

\end{proof}

Since every essentially commuting band of operators has an invariant subspace, an application of the Triangularization lemma (see [14, Lemma 7.1.11]) immediately implies the following result.

**Corollary 4.5** Every essentially commuting band of operators on a Banach space is triangularizable.

**Acknowledgements** The authors would like to thank the anonymous referee for carefully reading the manuscript. The paper is a part of the project *Distinguished subspaces of a linear operator* and the work of the first author was partially supported by the Slovenian Research Agency through the research program P2-0268. The second author acknowledges financial support from the Slovenian Research Agency, Grants Nos. P1-0222, J1-2453 and J1-2454.
References

1. Argyros, S.A., Haydon, R.G.: A hereditarily indecomposable $L_\infty$-space that solves the scalar-plus-compact problem. Acta Math. 54, 1–54 (2011)
2. Aronszajn, N., Smith, K.T.: Invariant subspaces of completely continuous operators. Ann. Math. 60, 345–350 (1954)
3. Enflo, P.: On the invariant subspace problem for Banach spaces. Acta Math. 158, 213–313 (1987)
4. Gilfeather, F.: The structure and asymptotic behavior of polynomially compact operators. Proc. Am. Math. Soc. 25, 127–134 (1970)
5. Grabiner, S.: The nilpotency of Banach nil algebras. Proc. Am. Math. Soc. 21, 510 (1969)
6. Grivaux, S., Roginskaya, M.: A general approach to Read’s type constructions of operators without non-trivial invariant closed subspaces. Proc. Lond. Math. Soc. 109, 596–652 (2014)
7. Hadwin, D., Nordgren, E., Radjabaliour, M., Radjavi, H., Rosenthal, P.: On simultaneous triangularization of collections of operators. Houston J. Math. 17, 581–602 (1991)
8. Higman, G.: On a conjecture of Nagata. Math. Proc. Camb. Philos. Soc. 52, 1–4 (1956)
9. Kandić, M.: On algebras of polynomially compact operators. Linear Multilinear Algebra 64(6), 1185–1196 (2016)
10. Konvalinka, M.: Triangularizability of polynomially compact operators. Integr. Equ. Oper. Theory 52, 271–284 (2005)
11. Livshits, L., MacDonald, G., Mathes, B., Radjavi, H.: On band algebras. J. Oper. Theory 46, 545–560 (2001)
12. Lomonosov, V.I.: Invariant subspaces for the family of operators which commute with a completely continuous operator. Funct. Anal. Appl. 7, 213–214 (1973)
13. Nagata, M.: On the nilpotency of nil-algebras. J. Math. Soc. Jpn. 4, 296–301 (1952)
14. Radjavi, H., Rosenthal, P.: Simultaneous Triangularization. Springer, New York (2000)
15. Read, C.J.: A solution to the invariant subspace problem. Bull. Lond. Math. Soc. 16, 337–401 (1984)
16. Shulman, V.S.: On invariant subspaces of volterra operators. Funk. Anal. i Prilozhen. 18, 84–86 (1984). (in Russian)
17. Turovskii, Y.V.: Volterra semigroups have invariant subspaces. J. Funct. Anal. 162, 313–322 (1999)