TROPICAL ARITHMETIC AND TROPICAL MATRIX ALGEBRA

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Abstract. This paper introduces a new structure of commutative semiring, generalizing the tropical semiring, and having an arithmetic that modifies the standard tropical operations, i.e. summation and maximum. Although our framework is combinatorial, notions of regularity and invertibility arise naturally for matrices over this semiring; we show that a tropical matrix is invertible if and only if it is regular.

Introduction

Traditionally, researchers have been able to frame mathematical theories using formal structures provided by algebra; geometry is often a source for interesting phenomena in the core of these theories. The semiring structure introduced in this paper emerges from the combinatorics within max-plus algebra and its corresponding polyhedral geometry, called tropical geometry. Although our ground structure is a semiring, much of the theory of standard commutative algebra can be formulated on this semiring, leading to application in combinatorics, semigroup theory, polynomials algebra, and algebraic geometry.

Tropical mathematics takes place over the tropical semiring \( (\mathbb{R} \cup \{-\infty\}, \max, +) \), the real numbers equipped with the operations of maximum and summation, respectively, addition and multiplication \([4, 6, 12]\), and it interacts with a number of fields of study including algebraic geometry, polyhedral geometry, commutative algebra, and combinatorics. Polyhedral complexes, resembling algebraic varieties over a field with real non-archimedean valuation, are the main objects of the tropical geometry, where their geometric combinatorial structure is a maximal degeneration of a complex structure on a manifold.

Over the past few years, much effort has been invested in the attempt to characterize a tropical analogous to classical linear algebra, \([3, 7, 12]\), and to determine connections between the classical and the tropical worlds \([11, 13, 14]\). Despite the progress that has been achieved in these tropical studies, some fundamental issues have not been settled yet; the idempotency of addition in \((\mathbb{R} \cup \{-\infty\}, \max, +)\) is maybe one of the main reasons for that. Addressing this reason, and other algebro-geometric needs, our goals are:

(a) Introducing a new structure of a partial idempotent semiring having its own arithmetic that generalizes the max-plus arithmetic and also carries a tropical geometric meaning;

(b) Presenting a novel approach for a theory of matrix algebra over partial idempotent semirings that includes notions of regularity and semigroup invertibility, analogous as possible to that of matrices over fields.

The latter goal is central issue in the study of Green’s relations over semigroups and is essential toward developing a linear representations of semigroups. Our new approach answers these goals and paves a way to treat other needs like having a notions of linear dependency and rank.

Our new structure, which we call extended tropical semiring, is built on the disjoint union of two copies of \(\mathbb{R}\), denoted \(\mathbb{R}\) and \(\mathbb{R}^\nu\), together with the formal element \(-\infty\) that serves as the gluing point of \(\mathbb{R}\) and \(\mathbb{R}^\nu\). Thus,

\[ T := \mathbb{R} \cup \{-\infty\} \cup \mathbb{R}^\nu \]
is provided with an order, $\prec$, extending the usual order on $\mathbb{R}$, and endowed with the addition $\oplus$ and the multiplication $\odot$ that modify the familiar operations $\max$ and $+$. By this setting, $(\mathbb{T}, \oplus, \odot)$ has the structure of a commutative semiring, $\oplus$ is idempotent only on $\mathbb{R}^+$, and $(\mathbb{T}, \oplus, \odot)$ allows to define a homomorphic relation to a field with real non-Archimedean valuation. From the point of view of algebraic geometry, $\oplus$ encodes an additive multiplicity that enables to define tropical algebraic sets in a natural manner.

The second part of the paper focuses mainly on introducing a theory of matrix algebra over $(\mathbb{T}, \oplus, \odot)$, reassembling the classical theory of matrices over fields, that includes notions of regularity and invertibility in a natural way with the following relation:

**Theorem 3.7.** A tropical matrix is pseudo invertible if and only if it is tropically regular.

We provide also an explicit characterization of the pseudo inverse matrix $\mathcal{A}^\nabla$ of a regular matrix $\mathcal{A}$, which turns out to be similar to that of the classical theory. Concerning semigroup theory, we show that the monoid $M_n(\mathbb{T})$ of matrices over $(\mathbb{T}, \oplus, \odot)$ can be related to as an E-dense monoid in which our invertibility suits E-denseness, that is the products $\mathcal{A}\mathcal{A}^\nabla$ and $\mathcal{A}^\nabla\mathcal{A}$ are idempotent matrices [10].

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1. Extended Tropical Arithmetic – A New Approach

With two goals in minds, geometrically and algebraically derived, our objective is to introduce a new concept of idempotent semiring extensions, applied here to the classical tropical semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$, including also the relation to non-Archimedean fields with real valuations. Although related topics have been discussed earlier for $(\mathbb{R} \cup \{-\infty\}, \max, +)$, cf. [1, 2, 3, 15], in this paper we use a different approach implemented on a semiring structure having a modified arithmetic. We open by describing the standard tropical framework, then we present the basics of our new concept and the associated semiring structure.

1.1. The tropical semiring. Tropical mathematics is the mathematics over idempotent semirings, the tropical semiring is usually taken to be $(\mathbb{R} \cup \{-\infty\}, \max, +)$, the real numbers together with the formal element $-\infty$, and with the operations of tropical addition and tropical multiplication

$$a + b := \max\{a, b\}, \quad a \cdot b := a + b,$$

cf. [11] [12]. We write $\hat{\mathbb{R}}$ for $\mathbb{R} \cup \{-\infty\}$ and equip $\hat{\mathbb{R}}^n := \mathbb{R}$ with the Euclidean topology, assuming that $\mathbb{R}$ is homeomorphic to $[0, \infty)$. The tropical semiring contains the **max-plus algebra** [2] [12] and it emerges as a target of non-Archimedean fields with real valuation; it is an idempotent semiring, i.e. $a + a = a$, with the unit $1_{\hat{\mathbb{R}}} := 0$, and the zero element $0_{\hat{\mathbb{R}}} := -\infty$.

Elements of the semiring $\mathbb{R}[\lambda_1, \ldots, \lambda_n]$ are called tropical polynomials in $n$ variables over $\mathbb{R}$ and are of the form

$$f = \max_{i \in \Omega}\{\langle \Lambda, i \rangle + a_i\},$$

where $\langle \cdot, \cdot \rangle$ stands for the standard scalar product, $\Omega \subset \mathbb{Z}^n$ is a finite nonempty set of points $i = (i_1, \ldots, i_n)$ with nonnegative coordinates, $a_i \in \mathbb{R}$ for all $i \in \Omega$, and $\Lambda = (\lambda_1, \ldots, \lambda_n)$. The addition and multiplication of polynomials are defined according to the familiar law.

Any tropical polynomial $f \in \mathbb{R}[\lambda_1, \ldots, \lambda_n] \setminus \{-\infty\}$ determines a piecewise linear convex function $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}$. But, in the tropical case, the map $f \mapsto \tilde{f}$ is not injective, and one can reduce the polynomial semiring so as to have only those elements needed to describe functions.

A tropical hypersurface is defined to be the domain of non-differentiability, also called the corner locus, of $\tilde{f}$ for some $f \in \mathbb{R}[\lambda_1, \ldots, \lambda_n] \setminus \{-\infty\}$. Therefore, points of a tropical hypersurface can be specified as the points on which the value of $\tilde{f}$ is attained by at least two monomials of $f$. This property is crucial for understanding the purpose of incorporating additive multiplicities, it will be used later to distinguish the corner locus from the other points of a domain.
One of our goals is to establish a semiring structure that allows one to realize (algebraically) the points of a corner locus as a “zero” locus of a polynomial; namely, to have the ability to form algebraic sets. Therefore, we would like to have a structure that not only provides the operation of maximum, but also encodes an indication about its additive multiplicity. In other word, in some sense, to “resolve” the idempotency of \((\mathbb{R}, \max, +)\).

Remark 1.1. Indeed, to address this goal, one may suggest an alternative arithmetic that defines the addition of two equal elements to be \(-\infty\), which we write as \(a + a = -\infty\), and the addition of different elements to be their maximum. We denote this structure as \((\mathbb{R}, \text{“max”}, +)\). Unfortunately, this type of addition is not associative; for example, for \(b < a\) we have "\((b + (a + a)) = (b + (-\infty)) = b\) while "\((b + a) + a = (a) + a = -\infty\)."

Our next development addresses this algebro-geometric issue; later we show that it also servers a solid base for developing a theory of matrix algebra over semirings that have the notions of regularity and invertibility.

1.2. The extended tropical semiring. Roughly speaking, the central idea of our new approach is a generalization of \((\mathbb{R}, \max, +)\) to a semiring structure having a partial idempotent addition that distinguishes between sums of similar elements and sums of different elements. Set theoretically, our semiring is composed from the disjoint union of two copies of \(\mathbb{R}\), denoted \(\mathbb{R}\) and \(\mathbb{R}^\nu\), which glued along the formal element \(-\infty\) to create the set

\[
\mathbb{T} := \mathbb{R} \cup \{-\infty\} \cup \mathbb{R}^\nu.
\]

In what follows we denote the unions \(\mathbb{R} \cup \{-\infty\}\) and \(\mathbb{R}^\nu \cup \{-\infty\}\) respectively by \(\mathbb{R}\) and \(\mathbb{R}^\nu\), write \(\mathbb{T}^\nu\) for \(\mathbb{T} \setminus \{-\infty\}\), and call the elements of \(\mathbb{R}\) reals.

We use the generic notation that \(a, b \in \mathbb{R}\) for reals, \(a^\nu, b^\nu \in \mathbb{R}^\nu\) where \(a, b \in \mathbb{R}\), and \(x, y \in \mathbb{T}\). Thus, \(\mathbb{T}\) is provided with the following order \(<\) extending the usual order on \(\mathbb{R}\):

**Axiom 1.2.** The order \(<\) on \(\mathbb{T}\) is defined as:

1. \(-\infty < x\), \(\forall x \in \mathbb{T}^\nu\);
2. for any real numbers \(a < b\), we have \(a < b^\nu\), \(a^\nu < b\), and \(a^\nu < b^\nu\);
3. \(a < a^\nu\) for all \(a \in \mathbb{R}\).

One can verify that the corresponding partial order, \(\preceq\), holds only in the cases where both elements are in \(\mathbb{R}\) or both are in \(\mathbb{R}^\nu\).

**Example 1.3.** Assume \(a < b < c\) are reals; then

\[-\infty < a < a^\nu < b < b^\nu < c < c^\nu\].

According to the rules of \(<\), cf. Axiom 1.2. \(\mathbb{T}\) is then endowed with the two operations \(\oplus\) and \(\odot\), addition and multiplication respectively, defined as below. (We use the notation \(\max_{\prec}\) to denote the maximum with respect to the order \(\prec\).)

**Axiom 1.4.** The laws of the extended tropical arithmetic are:

1. \(-\infty \oplus x = x \oplus -\infty = x\) for each \(x \in \mathbb{T}\);
2. \(x \oplus y = \max_{\prec}\{x, y\}\) unless \(x = y\);
3. \(a \oplus a = a^\nu \oplus a^\nu = a^\nu\);
4. \(-\infty \odot x = x \odot -\infty = -\infty\) for each \(x \in \mathbb{T}\);
5. \(a \odot b = a + b\) for all \(a, b \in \mathbb{R}\);
6. \(a^\nu \odot b = a \odot b^\nu = a^\nu \odot b^\nu = (a + b)^\nu\).

We call the triple \((\mathbb{T}, \oplus, \odot)\) the **extended tropical semiring**; later we show that \((\mathbb{T}, \oplus, \odot)\) indeed have the structure of commutative semiring with unit \(1_T := 0\) and \(0_T := -\infty\).

Recall that two preliminary essential demands have been required on \(\oplus\), validity of associativity and, simultaneously, differentiation between addition of similar reals and addition of different reals.
The first requirement is satisfied by Axiom 1.2 (3) and Axiom 1.4 (2); that is, for reals, we have the following:

\[
(1.2) \quad b \oplus a'' = b \oplus (a \oplus a) \quad \uparrow \quad (b \oplus a) \oplus a = \begin{cases} 
(a) \oplus a = a'', \quad a \succ b, \\
(a'') \oplus a = a'', \quad a = b, \\
(b) \oplus a = b, \quad b \succ a, 
\end{cases}
\]

the equality is then derived from Axiom 1.4 (2).

**Remark 1.5.**

1. The addition \( \oplus \) (in comparison to that of \((\mathbb{R}, \max, +)\)) is not idempotent, since \( a \oplus a = a'' \); this is one of the main aspects of our approach.

2. \( \mathbb{R}' \) is an ideal of \( \mathbb{T} \) where sometimes we want to think about as a set of pseudo zeros, namely consisting of those elements to be ignored. On the other hand, by Axiom 1.4 (3), \( \mathbb{R}' \) can be also realized as a “shadow” copy of \( \mathbb{R} \) whose elements carry additive multiplicities \( > 1 \), received as tropical sums of identical reals. This view is important for understanding the linkage between our arithmetic and the notion of tropicalization.

In the context of semigroups, both \((\mathbb{T}, \oplus)\) and \((\mathbb{T}, \odot)\) are monoids but not groups and thus, invertibility is invalid for both \( \oplus \) and \( \odot \). Yet, for \( \odot \), one can talk about partial invertibility which is well defined on reals only.

**Definition 1.6.** The division, denoted \( \odot' \), of \( x, y \in \mathbb{T} \), with \( y \neq -\infty \), is defined as \( x \odot' y = x \odot (-y) \), where \( -y = (-a)' \) when \( y = a'' \).

Note that \( \odot' \) is not well defined over all \( \mathbb{T} \), but suits our purpose. The cancellation law, \( x \odot y = x \odot z \Rightarrow y = z \), does not always hold; for example, the equality \( a'' \odot b = a'' \odot b'' \) does not satisfy cancellation.

**Remark 1.7.** The structure of \((\mathbb{T}, \oplus, \odot)\) has been formulated on two disjoint copies of \( \mathbb{R} \) with the modification of the operations \( \max \) and \( + \); the same construction can be performed for any idempotent semiring with the property \( a + b \in \{a, b\} \) and in particular for \((\mathbb{Z}, \max, +)\).

1.3. **Properties of the extended tropical arithmetic.** Having formulated extended tropical arithmetic, we address the its basic properties. We describe only the main cases in detail; therefore, the trivial cases involving \(-\infty\) are omitted. To clarify the exposition, sometimes, we treat the elements of \( \mathbb{R} \) and \( \mathbb{R}' \) separately.

**Commutativity:** Axiomatic (cf. Axiom 1.4).

**Associativity:** By definition \((a \oplus b)' = a'' \oplus b'' \) and \((a \odot b)' = a'' \odot b'' \). Thus, for different elements in \( \mathbb{T} \), the associativity of \( \oplus \) and \( \odot \) is clear by the associativity of \( \max \) and \( + \) which also provides the associativity of \( \odot \) for all \( \mathbb{T} \). The case in which identical reals are involved has already been examined in (1.2). For the case of two similar elements in \( \mathbb{R}' \), we have:

\[
(a \oplus b'') \oplus c'' = \begin{cases} 
b'' \oplus c'' = (b \oplus c)'', \quad b \succ a, \\
a \oplus c'' \downarrow \quad a \succ b, \\
a \succ a, b, 
\end{cases}
\]

and

\[
a \oplus (b'' \oplus c'') = a \oplus (b \oplus c)' = \begin{cases} 
c'', \quad c \succ a, b, \\
b'', \quad b \succ a, c, \\
a, \quad a \succ b, c, 
\end{cases}
\]

which have equal evaluations. (The other cases of compound expressions are obtained by the same way.)
**Distributivity:** To verify distributivity of \( \odot \) over \( \oplus \), for the case when all elements are reals, write
\[
\begin{aligned}
a \odot (b \oplus c) &= \begin{cases} 
a \odot b, & b \triangleright c, 
a \odot b^\nu, & b = c, 
a \odot c, & c \triangleright b, \end{cases} 
\end{aligned}
\]
and
\[
\begin{aligned}
(a \odot b) \oplus (a \odot c) &= \begin{cases} 
a \odot b, & b \triangleright c, 
(a \odot b)^\nu, & b = c, 
a \odot c, & c \triangleright b, \end{cases} 
\end{aligned}
\]
and compare the evaluations with respect to the different ordering of the involved arguments.

When elements of both \( \mathbb{R} \) and \( \mathbb{R}^\nu \) are involved, use the above specification together with Axiom 1.4 for example,
\[
a^\nu \odot (b \oplus c) = (a \odot (b \oplus c))^\nu = ((a \odot b) \oplus (a \odot c))^\nu = (a \odot b^\nu) \oplus (a \odot c^\nu). \]

**Zero:** By definition \( 0_\mathbb{T} := -\infty \) is the additive identity of \( \mathbb{T} \) (cf. Axiom 1.4 (1)), and it annihilates \( \mathbb{T} \) (cf. Axiom 1.4 (4)).

**One:** One can easily check that \( 1_\mathbb{T} := 0 \) is the multiplicative identity of \( \mathbb{T} \).

**Theorem 1.8.** The set \( \mathbb{T} \) equipped with the addition \( \oplus \) and the multiplication \( \odot \) is a (non-idempotent) commutative semiring, \( (\mathbb{R}^\nu, \oplus) \) is an additive semigroup, and \( (\mathbb{R}, \odot) \) and \( (\mathbb{R}^\nu, \odot) \) are multiplicative semigroups.

**Remark 1.9.** In the view of Axiom 1.4, \( \nu \) is realized as the onto order preserving projection
\[
\nu : (\mathbb{T}, \oplus, \odot) \rightarrow (\mathbb{R}^\nu, \oplus, \odot),
\]
where \( \nu : a \mapsto a^\nu, \nu : a^\nu \mapsto a^\nu, \) and \( \nu : -\infty \mapsto -\infty \). Then, \( \nu \) is a semiring homomorphism and we write \( x^\nu \) for the image of \( x \in \mathbb{T} \) in \( \mathbb{R}^\nu \), where \( \nu \) is is the identity for each \( x \in \mathbb{R}^\nu \). Accordingly, call \( a^\nu \) the \( \nu \)-value of \( a \). Given \( x, y \in \mathbb{T} \), we say that \( x \) is greater than \( y \), or maximal, up to \( \nu \) if \( x^\nu \triangleright y^\nu \), similarly, when \( x^\nu = y^\nu \) we say that \( x \) and \( y \) are equal up to \( \nu \).

Writing \( x^n \) for the tropical product \( x \odot x \odot \cdots \odot x \) of \( n \) factors we have:

**Lemma 1.10.** \( (x \odot y)^n = x^n \odot y^n, \ n \in \mathbb{N}, \) for any \( x, y \in \mathbb{T} \).

**Proof.** Assume \( n > 1 \), by induction:
\[
(x \odot y)^n = (x \odot y)(x \odot y)^{n-1} = (x \odot y)(x^{n-1} \odot y^{n-1}) = x^n \odot x^{n-1}y \odot xy^{n-1} \odot y^n.
\]
Suppose \( x \triangleright y \), then
\[
x^n \triangleright x^{n-1}y \oplus xy^{n-1} \oplus y^n \triangleright y^n
\]
and \( (x \odot y)^n = x^n \). Similarly, if \( y \triangleright x \), then \( (x \odot y)^n = y^n \). In the case of \( x = y \), we have \( x \odot y \in \mathbb{R}^\nu \), \( x^n \odot y^n \in \mathbb{R}^\nu \), and \( x^n \odot y^n = x^n = (x \odot y)^n \).

**Corollary 1.11.** \( (\bigoplus_{i=1}^s x_i)^n = \bigoplus_{i=1}^s x_i^n, \ n \in \mathbb{N}, \) for any \( x_1, \ldots, x_s \in \mathbb{T} \).

**Corollary 1.12.** The “Cauchy” inequality
\[
x_1 \odot x_2 \odot \cdots \odot x_n \preceq x_1^n \oplus x_2^n \oplus \cdots \oplus x_n^n
\]
holds for any \( x_1, \ldots, x_n \in \mathbb{T} \); equality occurs only if \( \nu(x_1) = \nu(x_2) = \cdots = \nu(x_n) \) and at least one \( x_i \) is in \( \mathbb{R}^\nu \).
1.4. Tropical arithmetics and tropicalization. The informal term tropicalization is used to describe a map, based on a real valuation, of objects defined over a non-Archimedean field $K$ with real valuation to objects defined over $(\mathbb{R}, \max, +)$; objects are either varieties or polynomials. The tropicalization of a variety $W \subset K^n$ is a polyhedral complex in $\mathbb{R}^n$, while a polynomial in $n$ variables in $K[\lambda_1, \ldots, \lambda_n]$ is mapped to a tropical polynomial in $n$ variables in $\mathbb{R}[\lambda_1, \ldots, \lambda_n]$, which we recall determines an affine piecewise linear function.

Let $K$ be an algebraically closed field with a real non-Archimedean valuation

\[ Val : (K, +, \cdot) \rightarrow (\mathbb{R}, \max, +); \]

for example, assume $K$ is the field of locally convergent complex Puiseux series, of the form

\[ f(t) = \sum_{a \in R} c_a t^a, \quad c_a \in \mathbb{C}, \]

where $R \subset \mathbb{Q}$ is bounded from below and the elements of $R$ have a bounded denominator. Then,

\[ Val(f) = \begin{cases} \min \{a \in R : c_a \neq 0\}, & f \in K[\lambda_1, \ldots, \lambda_n] \setminus \{0\} \\ -\infty, & f = 0 \end{cases}, \]

is a real valuation satisfying the rules of being non-Archimedean,

\[ Val(f \cdot g) = Val(f) + Val(g), \]

\[ Val(f + g) \leq \max\{Val(f), Val(g)\}. \]

(Note that $Val$ is not a homomorphism, since it does not preserve associativity.) Thus, in the sense of tropicalization, the arithmetic operations of $K$ are replaced with the correspondence: $\cdot \mapsto +$ and $+ \mapsto \max$.

**Remark 1.13.** Taking $f, g \in K$ with $Val(f) = Val(g) = a$, then $Val(f + g)$ can be any point of the ray $[-\infty, a]$. These cases provide the motivation for the use of $(\mathbb{T}, \oplus, \odot)$ as the target of $Val$ that allows to distinguish between the cases in which Formula (1.6)(ii) is interpreted as equality and the cases it is inequality.

In order to realize $(\mathbb{T}, \oplus, \odot)$ as the target of $Val$, to each point $a' \in \mathbb{R}^n$ we assign the ray $P_{a'} := [-\infty, a']$ and to each $x \in \mathbb{R}$ we assign the singleton $P_a := \{a\}$, in particular $P_{-\infty} := \{-\infty\}$; therefore $x \in P_x$ for each $x \in \mathbb{T}$. With this construction we obtain the inclusions:

\[ P_{-\infty}, P_a \subset P_{a'}, \quad P_{a'} \subset P_{b'} \iff a < b, \quad \forall a, b \in \mathbb{R}. \]

(Recall that two series in $K$ that are vanished in order 1 must vanished on order at least 1; the inclusions (1.7) address this property.)

Let $G(\mathbb{T}) := \{P_x : x \in \mathbb{T}\}$, then $Val(f) \in P_x$ for some $P_x \in G(\mathbb{T})$ which clearly needs not be unique. Accordingly, for each pair $f, g \in K$ and $x \in \mathbb{T}$ we define the relation

\[ Val(f) \in P_x \quad \text{or} \quad Val(f) \notin P_x \]

determined by the inclusion of $Val(f)$ in $P_x$.

**Theorem 1.14.** Formula (1.8) yields a homomorphism; that is, for any $f, g \in K$ with $Val(f) \in P_x$ and $Val(g) \in P_y$ we have $Val(f \cdot g) \in P_{x \odot y}$ and $Val(f + g) \in P_{x \oplus y}$.

**Proof.** Suppose $Val(f) = a$, $Val(g) = b$. Then, since $x \in P_x$ for each $x \in \mathbb{T}$,

\[ Val(f \cdot g) = Val(f) + Val(g) = a \odot b \in P_{a \odot b}, \]

and $Val(f \cdot 0) = Val(f) + Val(0) = a \odot (-\infty) \in P_{-\infty}$. For the additive relation, write

\[ Val(f + g) \leq \max\{Val(f), Val(g)\} = \]

\[ \max\{a, b\} = \begin{cases} a \in P_a = P_{a \odot b} & a > b, \\ a \in P_a \subset P_{a'} = P_{a \oplus a} & a = b, \\ b \in P_b = P_{a \odot b} & b > a, \end{cases} \]

and use the inclusion $P_a \subset P_{a'}$, cf. (1.7). The case of $Val(f + 0)$ is trivial. \qed
1.5. The relation to the max-plus arithmetic. The structure of \((\mathbb{T}, \oplus, \odot)\) provides a much richer structure, generalizing both the max-plus semiring and the one suggested in Remark 1.12 and achieves the best of both worlds.

**Lemma 1.15.** The map

\[ \pi : (\mathbb{T}, \oplus, \odot) \rightarrow (\mathbb{R}, \max, +) , \]

\[ \pi : a^\nu \mapsto a, \pi : a \mapsto a, \text{ and } \pi : -\infty \mapsto -\infty, \text{ is a semiring epimorphism.} \]

**Proof.** Clearly, \(\pi\) is onto. Assume \(\pi(x) = a\) and \(\pi(y) = b\), where \(x, y \in \mathbb{T}\), then \(\pi(x \odot y) = \max\{a, b\} = \max\{\pi(x), \pi(y)\}\) and \(\pi(x \odot y) = a + b = \pi(x) + \pi(y)\).

On the other hand one can also define:

**Lemma 1.16.** The map

\[ \theta : (\mathbb{R}, \max, +) \rightarrow (\mathbb{R}_\nu, \odot, \circ) , \]

\[ \theta : a \mapsto a^\nu \text{ and } \theta : -\infty \mapsto -\infty, \text{ is a semiring isomorphism that embeds } (\mathbb{R}, \max, +) \text{ in } (\mathbb{T}, \oplus, \odot) . \]

**Proof.** Take \(a, b \in \mathbb{R}\), then \(\theta(\max\{a, b\}) = (\max\{a, b\})^\nu = a^\nu \odot b^\nu = \theta(a) \odot \theta(b)\), and \(\theta(a + b) = (a + b)^\nu = a^\nu \circ b^\nu = \theta(a) \circ \theta(b)\). \(\mathbb{R}_\nu \subset \mathbb{T}\), so \(\theta\) embeds \((\mathbb{R}, \max, +)\) in \((\mathbb{T}, \oplus, \odot)\).

**Corollary 1.17.** Categorically, by Remark 1.14. Lemma 1.15 and Lemma 1.16, the diagram

\[ (\mathbb{T}, \oplus, \odot) \xrightarrow{\pi} (\mathbb{R}, \max, +) \]

\[ \nu \quad \theta \]

\[ (\mathbb{R}_\nu, \odot, \circ) \]

commutes.

Corollary 1.17 displays \((\mathbb{T}, \oplus, \odot)\) as a generalization of \((\mathbb{R}, \max, +)\) which is endowed with a richer structure in the sense that it encodes an indication about the additive multiplicity of elements in \(\mathbb{R}\). Namely, since \(a \oplus a = a^\nu\) and \(a \oplus a^\nu = a^\nu\), \(a^\nu\) can be realized as a point with additive multiplicity \(\geq 1\). (Clearly, computations for \((\mathbb{R}, \max, +)\) can be performed on \((\mathbb{T}, \oplus, \odot)\) and then to be sent back to \((\mathbb{R}, \max, +)\).)

As for the arithmetic suggested in Remark 1.11 (i.e. defined with “\(a + a = -\infty\)”), one may suggest the map

\[ \phi : (\mathbb{T}, \oplus, \odot) \rightarrow (\mathbb{R}, \max^\nu, +) , \]

\[ \phi : a \mapsto a, \phi : a^\nu \mapsto -\infty, \text{ and } \phi : -\infty \mapsto -\infty; \text{ but, since } (\mathbb{R}, \max^\nu, +) \text{ is not associative, } \phi \text{ is not a homomorphism.} \]

1.6. Geometric view. Let us remind that one of our goals was to obtain a semiring structure that enables us to treat algebracially the points of a corner locus of tropical functions, namely, to define tropical algebraic set. To present only the frame of this idea, given a tropical polynomial \(f \in \mathbb{T}[\lambda_1, \ldots, \lambda_n]\) we define the tropical algebraic set of the corresponding function \(\tilde{f}\) to be

\[ Z(\tilde{f}) = \{(x_1, \ldots, x_n) \in \mathbb{T}^n : \tilde{f}(x_1, \ldots, x_n) \in \mathbb{R}_\nu\}. \]

Therefore, the corner locus of \(f \in \mathbb{R}[\lambda_1, \ldots, \lambda_n]\) over \((\mathbb{R}, \max, +)\) is just the restriction of \(Z(\tilde{f})\), considered as a polynomial over \((\mathbb{T}, \oplus, \odot)\), to the real points, i.e. \(Z(\tilde{f}) \cap \mathbb{R}^n\).

**Example 1.18.** Consider the similar linear functions \(f(x) = x \oplus a\) over \((\mathbb{T}, \oplus, \odot)\) and \(f(x) = \max\{x, a\}\) over \((\mathbb{R}, \max, +)\), see Figure 1. Restricting the domain to \(\mathbb{R}\) only, over \((\mathbb{T}, \oplus, \odot)\), the image of the corner locus, which contains the single point \(a\), is distinguished and is now mapped to \(\mathbb{R}_\nu\).

The study of polynomial algebras and tropical algebraic sets over \((\mathbb{T}, \oplus, \odot)\) will be treated in a forthcoming paper.
2. Matrix Algebra

Our forthcoming study is dedicated to introducing the fundamentals of the matrix algebra over \((\mathbb{T}, \oplus, \odot)\) whose operations of are typically combinatorial. Yet, developing an algebraic theory, analogous to classical theory of matrix algebra over fields, with a view to combinatorics, is our main goal. This goal is supported by the connections to graph theory \([9]\), the theory of automata \([10]\), and semiring theory \([5]\).

Notations: For the rest of the paper, assuming the nuances of the different arithmetics are already familiar, we write \(xy\) for the product \(x \odot y\), \(\frac{x}{y}\) for the division \(x \odot y\), and \(x^n\) for \(x \odot \cdots \odot x\) repeated \(n\) times.

2.1. Tropical matrices. It is standard that if \(R\) is a semiring then we have the semiring \(M_n(R)\) of \(n \times n\) matrices with entries in \(R\), where addition and multiplication are induced from \(R\) as in the familiar matrix construction. Accordingly, we define the semiring of tropical matrices \(M_n(\mathbb{T})\) over \((\mathbb{T}, \oplus, \odot)\), whose unit is the matrix

\[
I = \begin{pmatrix}
0 & \ldots & -\infty \\
\vdots & \ddots & \vdots \\
-\infty & \ldots & 0
\end{pmatrix}
\]

and whose zero matrix is \(Z = (-\infty)I\); therefore, \(M_n(\mathbb{T})\) is also a multiplicative monoid. We write \(A = (a_{ij})\) for a tropical matrices \(A \in M_n(\mathbb{T})\) and denote the entries of \(A\) as \(a_{ij}\). Since \(\mathbb{T}\) is a commutative semiring, \(xA = Ax\) for any \(x \in \mathbb{T}\) and \(A \in M_n(\mathbb{T})\).

As in the familiar way, we define the transpose of \(A = (a_{ij})\) to be \(A^t = (a_{ji})\), and have the relation

\[
(AB)^t = B^tA^t.
\]

(The proof is standard by the commutativity and the associativity of \(\oplus\) and \(\odot\) over \(\mathbb{T}\).)

The minor \(A_{ij}\) is obtained by deleting the \(i\) row and \(j\) column of \(A\). We define the tropical determinant to be

\[
|A| = \bigoplus_{\sigma \in S_n} (a_{1\sigma(1)} \cdots a_{n\sigma(n)}),
\]

where \(S_n\) is the set of all the permutations on \(\{1, \ldots, n\}\). Equivalently, \(|A|\) can be written in terms of minors as

\[
|A| = \bigoplus_{j} a_{1\sigma j} |A_{1\sigma j}|,
\]
for some fixed index $i_o$. Indeed, in the classical sense, since parity of indices’ sums are not involved in Formula (2.2), the tropical determinant is a permanent, which makes the tropical determinant a pure combinatorial function. The adjoint matrix $\text{Adj}(A)$ of $A = (a_{ij})$ is defined as the matrix $(a'_{ij})^\text{t}$ where $a'_{ij} = |A|_{ij}$.

**Observation 2.2.** The tropical determinant has the following properties:

1. Transposition and reordering of rows or columns leave the determinant unchanged;
2. The determinant is linear with respect to scalar multiplication of any given row or column.

### 2.2. Regularity of matrices.

Using the special structure of $(\mathbb{T}, \oplus, \ominus)$, the algebraic formulation of combinatorial properties becomes possible.

**Definition 2.3.** A matrix $A \in M_n(\mathbb{T})$ is said to be **tropically singular**, or short, whenever $|A| \in \mathbb{R}^\nu$, otherwise $A$ is called **tropically regular**, or regular, for short.

In particular, when two or more different permutations, $\sigma \in S_n$, achieve the $\nu$-value of $|A|$ simultaneously, or the permutation that reaches the $\nu$-value of $|A|$ involves an entry in $\mathbb{R}^\nu$, then $A$ is singular.

**Remark 2.4.** Despite some classical properties hold for the tropical determinant, cf. Observation 2.2, the familiar relation $|AB| = |A||B|$ does not hold true on our setting; for example, take the matrix

$$(2.4) \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \end{pmatrix} \quad \text{with} \quad A^2 = AA = \begin{pmatrix} 3 & 4 & 4 \\ 5 & 6 & 6 \end{pmatrix},$$

then, $|A| = 4$ and $|A||A| = 8$, while $|A^2| = 9^\nu$. In the view of tropicalization, which ignores signs, the determinant of a matrix over $\mathbb{K}$ is assigned to the permanent of a matrix in $M_n(\mathbb{T})$; this explains the tropical situation in which the product of two regular matrices might be singular.

**Theorem 2.5.** A matrix with two identical rows or columns is singular.

**Proof.** Proof by induction on $n \geq 2$. The case of $n = 2$ is clear by direct computation. Assume the two first columns of $A$ are identical, and expand $|A|$ in terms of minors along the first row, that is $|A| = \bigoplus a_{i1} |A_{i1}|$. Since $a_{11} = a_{12}$ and $A_{11} = A_{12}$, then $a_{11} |A_{i1}| = a_{12} |A_{i2}|$, and so $a_{11} |A_{i1}| + a_{12} |A_{i2}| \mathbb{R}^\nu$. By induction hypothesis, for any $i > 2$, $A_{1i}$ is a matrix with identical columns, and is singular, that is $|A_{1i}| \mathbb{R}^\nu$ for all $i > 2$. When adding all together, $a_{i1} |A_{i1}| \mathbb{R}^\nu$ for all $i = 1, \ldots, n$, and thus $|A| \mathbb{R}^\nu$. \hfill \Box

**Theorem 2.6.** If $A$ and $B$ are regular matrices and their product $AB$ is also regular, then $|AB| = |A||B|$. When either $A$ or $B$ is singular, then $AB$ is also singular.

**Proof.** Let $S_n$ be the set of all the permutations on $N = \{1, \ldots, n\}$, and let $F_n = \{N \rightarrow N\}$ be the set of all maps from $N$ to itself, in particular $S_n \subset F_n$. Denoting the entries of $AB$ by $(ab)_{ij}$, we write the determinant $|AB|$ in the form of Formula (2.2) as:

$$|AB| = \bigoplus_{\sigma \in S_n} \bigoplus_{i} (ab)_{i\sigma(i)} = \bigoplus_{\sigma \in S_n} \bigoplus_{i} \left( \bigoplus_{k} (a_{ik} b_{k\sigma(i)}) \right) =$$

$$\bigoplus_{\sigma \in S_n} \left( (a_{11} b_{1\sigma(1)} + \cdots + a_{1n} b_{n\sigma(1)}) \cdots (a_{n1} b_{1\sigma(n)} + \cdots + a_{nn} b_{n\sigma(n)}) \right) =$$

$$(*) \quad \bigoplus_{\sigma \in S_n} \bigoplus_{\mu \in F_n} \left( \bigoplus_{i} (a_{i\mu(i)} b_{\mu(i)\sigma(i)}) \right) = \bigoplus_{\sigma \in S_n} \bigoplus_{\mu \in F_n} \left( \bigoplus_{i} a_{i\mu(i)} \bigoplus_{i} b_{\mu(i)\sigma(i)} \right).$$

By the structure of the left hand side of (*), we can see that the value of $|AB|$ is obtained when both $\bigoplus_{i} a_{i\mu(i)}$ and $\bigoplus_{i} b_{\mu(i)\sigma(i)}$ attain their maximal evaluation at the same time. We show that this is possible. Namely both reach their maximal evaluation on the same $\mu$, which we denote by $\mu_o$, the corresponding $\sigma \in S_n$ is then denoted by $\sigma_o$. Note that when $|AB| \mathbb{R}$ there must be exactly one pair, $\mu_o$ and $\sigma_o$; otherwise, by definition, $AB$ would not be regular.
Case I: Suppose \( \mu_0 \in S_n \) is a permutation which maximizes \( \bigcirc_i a_{i\mu_0(i)} \). We show that there is also a permutation \( \sigma_0 \in S_n \) that maximizes \( \bigcirc_i b_{\mu_0(i)\sigma_0(i)} \) for \( \mu_0 \). Assume \( |AB| \in \mathbb{R} \), with \( \sigma_t \in S_n \) maximizes \( \bigcirc_j b_{j\sigma_t(j)} \). Generally speaking, for any given \( \mu \in S_n \) and \( \sigma_t \in S_n \), there exits \( \sigma \in S_n \) which makes the diagram

\[
\begin{array}{c}
N_{[j]} \xrightarrow{\mu} N_{[i]} \\
\sigma_t \downarrow funds \downarrow \sigma \\
N
\end{array}
\]

commutative, where we use the notation \([ \cdot ]\) to indicate the appropriate indices. Accordingly, choosing \( \sigma_0 \in S_n \) for which \( \sigma_0 \circ \mu_0 = \sigma_t \), we obtain \( \bigcirc_i b_{\mu_0(i)\sigma_0(i)} = \bigcirc_j b_{j\sigma_t(j)} \); in this case the two components of (*) reach their maximum simultaneously and we can write:

\[
(*) = \left( \bigcirc_i a_{i\mu_0(i)} \right) \left( \bigcirc_j b_{j\mu_0(i)} \right) = \left( \bigcirc_i a_{i\mu_0(i)} \right) \left( \bigcirc_j b_{j\mu_0(i)} \right) = |A||B|.
\]

When \( A \) is singular, there are at least two different \( \mu_1, \mu_2 \in S_n \) that attain the \( \nu \)-value of \( |A| \) or a single \( \mu \in S_n \) that involves a non-real entry. The latter case is obvious, since (*) has a non-real multiplier and thus \( |AB| \in \mathbb{R}^\nu \). Suppose \( \sigma_1, \sigma_2 \in S_n \) are two permutations satisfying \( \nu_0 = \sigma_t \circ \mu_1, \mu_2; \) then

\[
(*) = \bigcirc_i a_{i\mu_1(i)} \bigcirc_i b_{i\mu_1(i)\sigma_1(i)} = \bigcirc_i a_{i\mu_2(i)} \bigcirc_i b_{i\mu_2(i)\sigma_2(i)},
\]

and hence \( |AB| \in \mathbb{R}^\nu \).

Case II: Suppose \( \mu_0 \in F_n \setminus S_n, |AB| \in \mathbb{R} \), and let \( \sigma_o \in S_n \) be the corresponding permutation which maximizes the product

\[
(**) \quad \bigcirc_i (a_{i\mu_0(i)} b_{i\mu_0(i)\sigma_0(i)}) = \bigcirc_i a_{i\mu_0(i)} \bigcirc_i b_{i\mu_0(i)\sigma_0(i)}.
\]

In particular, there is only one such pair, \( \mu_0 \) and \( \sigma_o \), for otherwise \( AB \) would not be regular. Since \( \mu_0 \notin S_n \), there are at least two indices \( i_1 \neq i_2 \) with \( \mu_0(i_1) = \mu_0(i_2) = k_o \). Let \( h_1 := \sigma_0(i_1) \) and \( h_2 := \sigma_0(i_2) \); then \( h_1 \neq h_2 \), since \( \sigma_o \in S_n \). Subject to \( \mu_0 \) and \( \sigma_o \), (***) can be rewritten as,

\[
(***) \quad \bigcirc_i a_{i,\mu_0(i)} \left( b_{k_o h_1, h_2} \bigcirc_{i \neq i_1, i_2} b_{i\mu_0(i)\sigma_0(i)} \right) = \bigcirc_i a_{i,\mu_0(i)} \left( b_{k_o h_1, h_2} \bigcirc_{i \neq i_1, i_2} b_{i\mu_0(i)\sigma_0(i)} \right).
\]

Denote by \( \bar{\sigma}_o \in S_n \) the permutation obtained by switching between the images of \( i_1 \) and \( i_2 \) in \( \sigma_o \), while all other correspondences remain as they are; explicitly, \( \bar{\sigma}_o(i_1) = h_2, \bar{\sigma}_o(i_2) = h_1, \) and \( \bar{\sigma}_o(i) = \sigma_o(i), \) for all \( i \neq i_1, i_2 \). With respect to \( \sigma_o \), we have the following combinatorial situation:

| \( h_1 \) | \( h_2 \) |
|---|---|
| \( \sigma_o(i_1) = h_1 \) | \( \sigma_o(i_2) = h_2 \) |
| \( \bar{\sigma}_o(i_1) \) | \( \bar{\sigma}_o(i_2) \) |
| \( k_o \) | \( * \) | \( * \) |
(The diagram helps us to understand the modification of $\mu_o$.) Using $\tilde{\sigma}$ we expand $(**) \text{ further,}$

$$(***) = \left( \bigodot_i a_{i\mu_o(i)} \right) \left( b_{\mu_o(i_2)\tilde{\sigma}_{i_2}(i_2)} b_{\mu_o(i_1)\tilde{\sigma}_{i_1}(i_1)} \right) \bigodot_{i \neq i_1 i_2} b_{\mu_o(i)\tilde{\sigma}_{i}(i)} = \left( \bigodot_i a_{i\mu_o(i)} \right) \left( \bigodot_i b_{\mu_o(i)\tilde{\sigma}_{i}(i)} \right),$$

which means that $\tilde{\sigma} \neq \sigma_o$ also attain the $\nu$-value of $|AB|$ and thus, $|AB| \in \tilde{R}^\nu$. This contradicts the assumption that $\mu_o \notin S_n$, so $\mu_o \in S_n$, and this case has already been discussed before. \hfill \Box

Example 2.7. Take the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}, \quad \text{then} \quad AB = \begin{pmatrix} 4 & 4 \\ 5 & 5 \end{pmatrix}. $$

$A$ is singular with $|A| = 4^\nu$, $B$ is regular with $|B| = 5$, and $AB$ is singular with $|AB| = 9^\nu$. On the other hand, $B^2 = \begin{pmatrix} 6 & 4 \\ 3 & 4 \end{pmatrix}$ is regular with $|B^2| = 10$, so $|B||B| = |B^2|$.

3. Invertibility of Matrices

We introduce a new notion of semigroup invertibility, and present it for the matrix monoid $M_n(\mathbb{T})$: this type of invertibility can be adopted to any abstract semigroup having a distinguished subset. Although our framework is typically combinatorial, we show how classical results are carried naturally on our setting.

3.1. Basic definitions. We open with an abstract definition.

Definition 3.1. Let $S$ be semigroup, and let $U \subset S$ be a proper subset with the property that for any $u \in U$ there exists some $v \in U$ for which $vu \in U$ and $vw \in U$. We call $U$ a distinguished subset of $S$.

An element $x \in S$ is said to be pseudo invariable if there is $y \in S$ for which $xy \in U$ and $yx \in U$, in particular all the members of $U$ are pseudo invertible. When $U$ consists of all idempotents elements of $S$, the pseudo invertibility is then called $E$-denseness \[10\]. A monoid is called $E$-dense if all of its elements are $E$-denseness.

To emphasize, for the purpose of pseudo invertibility, $U$ needs not be closed under the law of $S$. The notion of $E$-denseness is already known in literature, while the weaker version of pseudo invertibility is new.

To apply the notion of pseudo invariability to $M_n(\mathbb{T})$, viewed as monoid, we define a pseudo unit matrix to be a regular matrix of the form

$$\tilde{I} = \begin{pmatrix} 0 & \ldots & \ell_{ij}^{\nu} \\ \vdots & \ddots & \vdots \\ \ell_{ij}^{\nu} & \ldots & 0 \end{pmatrix},$$

that is $\ell_{ij} \in \tilde{R}^\nu$ for all $i \neq j$, and $\ell_{ii} = 0$, for each $i = 1, \ldots, n$. Since $\tilde{I}$ is regular we necessarily have $|\tilde{I}| = 0$, and in particular the unit matrix $I$, cf. \[(2.1)\], is also a pseudo unit. We define the distinguished subset $U_n(\mathbb{T}) \subset M_n(\mathbb{T})$ to be

$$\bar{U}_n(\mathbb{T}) = \left\{ \tilde{I} : \tilde{I} \text{ is a pseudo unit matrix} \right\} ,$$

Therefore, $\tilde{I} \in U_n(\mathbb{T})$ and hence $\tilde{I} \tilde{I} = \tilde{I} \tilde{I} = \tilde{I}$, for each $\tilde{I} \in U_n(\mathbb{T})$, which makes $U_n(\mathbb{T})$ a distinguished subset satisfies the condition of Definition \[(3.1)\].

Correspondingly, we define the distinguished subset $U_n^{idem}(\mathbb{T}) \subset M_n(\mathbb{T})$ to be

$$U_n^{idem}(\mathbb{T}) = \left\{ \tilde{I} : \tilde{I} \text{ is an idempotent pseudo unit matrix} \right\} .$$
Remark 3.2. It easy to show that any \( \bar{I} \in U_2(\mathbb{T}) \) is idempotent. For \( n > 2 \), not all of the pseudo units are idempotents; for example, take the triangular matrix
\[
\bar{I} = \begin{pmatrix}
0 & a^r & b^r \\
-\infty & 0 & c^r \\
-\infty & -\infty & 0
\end{pmatrix},
\]
with \( a^r c^r > b^r \).

Using \( U_n(\mathbb{T}) \) we explicitly define pseudo invertibility on \( M_n(\mathbb{T}) \):

Definition 3.3. A matrix \( A \in M_n(\mathbb{T}) \) is said to be pseudo invertible if there exists a matrix \( B \in M_n(\mathbb{T}) \) such that \( AB \in U_n(\mathbb{T}) \) and \( BA \in U_n(\mathbb{T}) \). If \( A \) is pseudo invertible, then we call \( B \) a pseudo inverse matrix of \( A \) and denote it as \( A^\triangledown \).

We use the notation of \( A^\triangledown \) since the pseudo matrix needs not be unique; moreover, in our setting \( AA^\triangledown \) is not necessarily equal to \( A^\triangledown A \), and thus might be evaluated for different pseudo units.

Example 3.4. Consider the following matrices:
\[
A = \begin{pmatrix}
0 & -2 & -1 \\
-2 & 0 & (-3)^r \\
-1 & (-3)^r & 0
\end{pmatrix}
\quad \text{and} \quad
A' = \begin{pmatrix}
0 & -2 & -1 \\
-2 & 0 & -3 \\
-1 & -3 & 0
\end{pmatrix}.
\]
For these matrices we have, \( AA' \in U_n(\mathbb{T}) \), \( A'A \in U_n(\mathbb{T}) \), and also \( AA \in U_n(\mathbb{T}) \). Namely, \( A \) has at least two pseudo inverses.

Remark 3.5. For the case of \( M_2(\mathbb{T}) \), our notion of pseudo invertibility coincides with the notion of general invertibility in a semigroup in the sense of Von-Neumann regularity \( \mathbb{R} \), but not for \( M_n(\mathbb{T}) \) with \( n > 2 \).

Remark 3.6. When one intends to use the other semirings structures, either \((\mathbb{R}, \max, +)\) or \((\mathbb{R}, \text{“max”}, +)\), in order to define an inverse matrix, or pseudo inverse, it appears to be very restricted or even impossible. Over \((\mathbb{R}, \max, +)\), unless \(-\infty\) is involved, the zero element \(-\infty\) is unreachable by tropical sums and products of entries. Thus, obtaining the unit matrix \( I \) as products of matrices is very restricted. On the other hand, when using \((\mathbb{R}, \text{“max”}, +)\), as suggested in Remark \[7\], multiplication is not associative, which makes implementation very difficult.

3.2. Theorem on tropical pseudo inverse matrix.

Theorem 3.7. A matrix \( A \in M_n(\mathbb{T}) \) is pseudo invertible if and only if is tropically regular. In case \( A \) is regular, \( A^\triangledown \) can be defined as
\[
A^\triangledown = \frac{\text{Adj}(A)}{|A|}.
\]

Before proving the theorem, we recall some definitions and present new notation:

(a) Division in \( \mathbb{T} \) is denoted by \( \div \) and interpreted as the substraction \( a - b \) in the classical sense. We write \( a^{-1} \) for \( \frac{a}{a} \) and \( a^m \) for the tropical product of \( a \) repeated \( m \) times (which is just \( m \cdot a \) in the usual sense).

(b) We use the notation \( A_{ih,jk} \) for \( (A_{ij})_{hk} \), that is \((h,k)\)-minor of the minor \( A_{ij} \), where \( h \neq i \) and \( k \neq j \) with respect to the initial indices of \( A \). Accordingly, \( |A_{ij}| \) is written in terms of minors as \( |A_{ij}| = \bigoplus_{k \neq j} a_{hk} |A_{ih,jk}| \), where \( h \neq i \).

Proof. We prove only multiplication on right, \( AA^\triangledown \in U_n(\mathbb{T}) \); the multiplication on left is proved in the same way.

\((\Leftarrow)\) Assume \( |A| \in \mathbb{R}^\nu \) and at the same time there exists a pseudo inverse \( A^\triangledown \). Then, by Theorem 2.6 \( |AA^\triangledown| \in \mathbb{R}^\nu \) and their product is singular. Recalling that \( |\bar{I}| = 0 \) for all \( \bar{I} \in U_n(\mathbb{T}) \), we have \( AA^\triangledown \notin U_n(\mathbb{T}) \).

\((\Rightarrow)\) We write \( A^\circ \) for the adjoint matrix \( \text{Adj}(A) \), for short, and denote the product \( AA^\circ \) by \( \mathcal{B} = (b_{ij}) \). Assuming \( A \) is regular, we need to prove that \( AA^\circ \in U_n(\mathbb{T}) \), or equivalently, that \( AA^\circ = |A|\bar{I} \) for some \( \bar{I} \in U_n(\mathbb{T}) \). To prove this, we need to verify the following conditions:
(1) $b_{ii} = |A|$ for each $i$; 
(2) $b_{ij} \in \mathbb{R}^\nu$, for any $i \neq j$;
(3) $|\frac{d}{dt}| = 0$.

**Diagonal entries:** When $i = j$,

$$b_{ii} = \bigoplus_k a_{ik}a_{ki} = \bigoplus_k a_{ik}|A_{ik}| = |A|,$$

since this is just the expansion of $|A|$ along row $i$ (cf. Equation 2.3).

**Non-diagonal entries:** For $i \neq j$,

$$b_{ij} = \bigoplus_k a_{ik}a_{kj} = \bigoplus_k a_{ik}|A_{jk}| \in \mathbb{R}^\nu,$$

since this is the expansion of the determinant of the matrix obtained from $A$ by replacing row $j$ with a copy of row $i$, and which therefore has two identical rows and is singular (Theorem 2.5).

**Regularity of product:** To prove $|\frac{d}{dt}| = 0$, we show equivalently that $|B| = |A|^n$. Let $S_n$ be the set of all permutations on $N = \{1, \ldots, n\}$ and let $F_n = \{N \rightarrow N\}$ be the set of all maps from $N$ to itself, i.e. $S_n \subset F_n$, and write the expansion of $|B|$ explicitly,

$$|B| = \bigoplus_{\sigma \in S_n} \left( \bigotimes_i b_{i\sigma(i)} \right) = \bigoplus_{\sigma \in S_n} \left( \bigotimes_i \left( \bigoplus_k a_{ik}|A_{\sigma(i)k}| \right) \right) =$$

$$\bigoplus_{\sigma \in S_n} \left( (a_{11}|A_{\sigma(1)1}| \oplus \cdots \oplus a_{1n}|A_{\sigma(1)n}|) \cdots (a_{n1}|A_{\sigma(n)1}| \oplus \cdots \oplus a_{nn}|A_{\sigma(n)n}|) \right) =$$

$$\bigoplus_{\sigma \in S_n} \bigoplus_{\mu \in F_n} \left( a_{\mu(i)}|A_{\sigma(i)\mu(i)}| \right).$$

Assume $\sigma_0 \in S_n$ and $\mu_0 \in M_n$ achieve the $\nu$-value of $|B|$. In case $\sigma_0$ is the identity, by Equation 3.6, $b_{ii} = |A|$, for each $i$, and thus,

$$\bigotimes_i b_{i\sigma_0(i)} = \bigotimes_i b_{ii} = |A|^n.$$

Otherwise, when $\sigma_0$ is not the identity, we write

$$c := \bigotimes_i \left( a_{i\mu_0(i)}|A_{\sigma_0(i)\mu_0(i)}| \right),$$

for the product that reaches the $\nu$-value of 3.6 and prove it always $\prec |A|^n$.

**Case I:** Assume $\mu_0 \in S_n$ is a permutation, then Formula 3.8 can be reordered to the form

(*)

$$a_{1\mu_0(1)}|A_{1\mu_0(1)}| \cdots a_{n\mu_0(n)}|A_{n\mu_0(n)}|.$$

If (*$\succ |A|^n$, then it must have at least one component $a_{j\mu_0(n)}|A_{j\mu_0(n)}| \succ |A|$, but this contradicts the maximality of $|A|$. On the other hand, if all $a_{j\mu_0(n)}|A_{j\mu_0(n)}| = |A|$ we get a contradiction to the regularity of $|A|$. Therefore, (*$\prec |A|^n$.

**Case II:** Assume $\mu_0 \in F_n \setminus S_n$, then there exist at least two indices $i_1 \neq i_2$ for which $\mu_0(i_1) = \mu_0(i_2) = j_0$. We show the existence of a permutation $\mu_1 \in S_n$ that reaches the same $\nu$-value for Formula 3.8 as $\mu_0$ reaches, the proof is then completed by Case I, applied to $\mu_1$. 

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For the two components $a_{ij} \sigma_{\sigma_0(i_1)j_0}$ and $a_{ij} \sigma_{\sigma_0(i_2)j_0}$ of (3.8), indexed by $\mu_0(i_1) = \mu_0(i_2) = j_o$, we have the following combinatorial layouts:

The diagrams are useful to understand the modification of $\mu_0$.

Since $\mu \in M_n \setminus S_n$, there exists at least one index $j_h \neq j_o$ in $N \setminus \text{Im}(\mu_0)$. Therefore, the corresponding component, $a_{ij_h} \sigma_{\sigma_0(i)j_h}$, is absent in (3.8). Without loss of generality, we take $a_{ij_0} \sigma_{\sigma_0(i)j_0}$ and modify it. Clearly, $\sigma_{\sigma_0(i_2)j_0}$ involves an entry $a_{ij_h}$, let $i_h$ be the index for which $\sigma(i_h) = j_h$. Then $A_{\sigma_0(i_2)j_0} = a_{i_h j_h} \sigma_{\sigma_0(i_2)j_0}$, and hence, by the maximality of $\mu_0$,

$$a_{ij_h} \sigma_{\sigma_0(i_2)j_0} = a_{ij_h} a_{i_h j_h} \sigma_{\sigma_0(i_2)j_0} = a_{i_h j_h} \sigma_{\sigma_0(i_2)j_0}.$$ 

Namely, we have specified another map $\mu_1 \in M_n$ with $\mu_1(i_1) = j_o$, $\mu_1(i_2) = j_h$, and $\mu_1(i) = \mu_0(i)$ for all $i \neq i_1, i_2$. Therefore, we reduced the number of indices sharing a same image in $\mu_0$ to have $\text{Im}(\mu_0) \subset \text{Im}(\mu_1) \subset N$. Proceeding inductively we get a chain

$$\text{Im}(\mu_o) \subset \text{Im}(\mu_1) \subset \ldots \subset \text{Im}(\mu_l) = N,$$

the left equality is due to the finiteness of $F_n$. Thus $\mu_l \in S_n$; the proof of Case II is then completed by Case I.

So, we have showed that the identity $\sigma_o$ is the single permutation that maximizes (3.6), and for which we have $B = A A^\top = |A| \mathbb{I}$. Since $|A| \in \mathbb{R}$ and $\mathbb{I}$ is regular, so is $B$. This completes the proof of Theorem 5.7 on pseudo invertibility of matrices over $(\mathbb{I}, \oplus, \circ)$. 

We push the result of Theorem 5.7 further:

**Theorem 3.8.** For each regular matrix $A \in M_n(\mathbb{I})$, the products $A A^\top$ and $A^\top A$ are idempotents.

**Proof.** Writing $\bar{I} = A A^\top$, with $\bar{I} = (\ell_{ij})$, we prove that $\bar{I} = \bar{I}^2$. Recall that $a_{ij}^\top = \frac{|A|}{|A|}$, then

$$t_{ij} = \bigoplus_{k} a_{ik} a_{kj}^\top = \bigoplus_{k} a_{ik} \frac{|A_{jk}|}{|A|} = a_{ik} \frac{|A_{jk}|}{|A|},$$

for some fixed $k_e$. Suppose $(\bar{I})^2 = (t_{ij}^{(2)})$, then

$$t_{ij}^{(2)} = \bigoplus_h t_{ih} t_{hj} = \bigoplus_h \left( \bigoplus_{k} a_{ik} \frac{|A_{hk}|}{|A|} \right) \left( \bigoplus_{l} a_{hl} \frac{|A_{lj}|}{|A|} \right) =$$

$$\bigoplus_h \bigoplus_{k} \bigoplus_{l} a_{ik} \frac{|A_{hk}|}{|A|} a_{hl} \frac{|A_{lj}|}{|A|},$$

and we need to prove the equality

$$|A| \bigoplus_{k} a_{ik} |A_{jk}| = \bigoplus_h \bigoplus_{k} \bigoplus_{l} a_{ik} |A_{hk}| a_{hl} |A_{lj}|.$$

To see that $t_{ij}^{(2)} \geq t_{ij}$, take $h = j$ to have $(II) = \bigoplus_h a_{hl} |A_{lj}| = \bigoplus_l a_{jl} |A_{lj}| = |A|$. By the way of contradiction, assume $t_{ij}^{(2)} > t_{ij}$, and suppose $k_o, h_o, l_o$ are the indices reaching the $\nu$-value of $t_{ij}^{(2)}$ in Formula (3.10), then

$$a_{ik_0} |A_{h_o k_o} a_{h_o l_o} |A_{lj}| > a_{ik_0} |A_{j k_o} |A|.$$
Clearly \( a_{ik} |A_{jk}| \geq a_{ik} |A_{jk}| \), since otherwise we would have a contradiction to the maximality of (3.9). Thus,

\[
a_{ik} |A_{jh}| |A_{lj}| \geq a_{ik} |A_{jk}||A| ,
\]

and hence

(3.12) \[ |A_{jh}| |A_{lj}| \geq |A_{jk}||A| . \]

Due to the maximality of \(|A|\), we also have \(|A| \geq a_{hk} |A_{hk}|\), and by (3.12) we get

\[
|A_{jh}| |A_{lj}| \geq |A_{jk}| |A| \geq |A_{hk}| |A_{hk}| .
\]

Namely,

(3.13) \[ a_{hk} |A_{lj}| \geq a_{hk} |A_{hk}| . \]

This contradicts the specification of \( k_o \) as the index that reaches the maximum for (3.11). This completes the proof that \((AA)^2 = AA^\triangledown\); the case of multiplication on left is proved in the same way.

\[ \Box \]

**Corollary 3.9.** A matrix \( A \) is \( E \)-dense in \( M_n(\mathbb{T}) \), with respect to \( U_n^{idm}(\mathbb{T}) \), if and only if is tropically regular.

**Example 3.10.** Take the regular matrix

\[
A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}, \quad \text{then} \quad A^\triangledown = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} (-3),
\]

where \(|A| = 3\). (Recall that, in tropical sense, multiplying by \((-3)\) means dividing by 3.) The product \( AA^\triangledown \) is then

\[
\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} (-3) = \begin{pmatrix} 3 & 0^\nu \\ 4^\nu & 3 \end{pmatrix} (-3) = \begin{pmatrix} 0 & (-3)^\nu \\ 1^\nu & 0 \end{pmatrix} \in U_n^{idm}(\mathbb{T}).
\]

On the other hand, if we take the singular matrix

\[
A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}, \quad \text{then} \quad A^\triangledown = \begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix} (-3)^\nu,
\]

where here \(|A| = 3^\nu\). Computing the product \( AA^\triangledown \) we get

\[
\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix} (-3)^\nu = \begin{pmatrix} 3^\nu & 0^\nu \\ 6^\nu & 3^\nu \end{pmatrix} (-3)^\nu \notin U_n(\mathbb{T}),
\]

which is not a regular matrix, and therefore \( AA^\triangledown \notin U_n(\mathbb{T})\).

A few immediate conclusions are derived from our last results:

**Corollary 3.11.** Assume \( A \) is a regular matrix, then

1. \( \text{Adj}(A) \) is also regular;
2. \(|A| = (|A^\triangledown|)^{-1}\), and if \( A = A^\triangledown \) then \(|A| = |A^\triangledown| = 0\).

**Proof.** The first assertion is obvious. \( A, A^\triangledown \), and \( AA^\triangledown \) are all regular, then by Theorem 2.6 \(|A||A^\triangledown| = |I| = 0\) and hence \(|A| = |A^\triangledown|^{-1}\).

The converse assertion of (2) is not true; for example, take the matrix

\[
A = \begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix}, \quad \text{then} \quad A^\triangledown = \begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix}.
\]

Although \(|A| = |A^\triangledown| = 0\), we have \( A \neq A^\triangledown\).
Remark 3.12. Contrary to the classical theory of matrices over fields, tropically, the relation \((AB)\vee = B\vee A\vee\) does not hold true; for example, take the regular matrix as in (2.4), then
\[
A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad A\vee = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} (-4), \quad \text{and} \quad A\vee A\vee = \begin{pmatrix} 6 & 4 \\ 5 & 3 \end{pmatrix} (-8).
\]
On the other hand, \(A^2\) is not regular, cf. Remark 2.4, and the computation of \(\text{Adj}(A^2)/|A^2|\) yields
\[
(AA)\vee = \begin{pmatrix} 6 & 4 \\ 5 & 3 \end{pmatrix} (-9)^e, \quad \text{where} \quad A^2 = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix};
\]
this shows that \((A\vee)^2 \neq (AA)\vee\).

3.3. Matrices with real entries. Denoting by \(M_n(\mathbb{R})\) the semiring of matrices over \((\mathbb{R}, \max, +)\), the epimorphism \(\pi : (\mathbb{T}, \oplus, \odot) \to (\mathbb{R}, \max, +)\), cf. (1.9), induces in the standard way the epimorphism
\[
\pi_* : M_n(\mathbb{T}) \to M_n(\mathbb{R})
\]
of matrix semirings. We write \(\pi_*(A)\) for the image of \(A \in M_n(\mathbb{T})\) in \(M_n(\mathbb{R})\). Conversely, set-theoretic, \(M_n(\mathbb{R}) \subset M_n(\mathbb{T})\).

Proposition 3.13. Suppose \(A \in M_n(\mathbb{T})\) is regular, where both \(A\) and \(A\vee\) have only real entries, \(AA\vee = \bar{T}'\), and \(A\vee A = \bar{T}''\). Then
\[
\pi_*(\bar{T}'A) = A, \quad \pi_*(A\vee \bar{T}') = A\vee, \quad \pi_*(\bar{T}'' A\vee) = A\vee, \quad \text{and} \quad \pi_*(\bar{T}' A\vee) = A.
\]

Proof. We prove the relation \(\pi_*(\bar{T}'A) = A\). Letting \(\bar{T}' = (\epsilon_{ij})\), we show that
\[
\pi(\bigoplus_k \epsilon_{ik} a_{kj}) = a_{ij},
\]
for all \(i, j\). Recall that \(\epsilon_{ik} = (\bigoplus_h a_{ih} |A_{jh}|) |A|^{-1}\), cf. Formula (3.5), and \(\epsilon_{ik} \in \mathbb{R}^e\) whenever \(i \neq k\). Composing together, we get
\[
(*) \quad \bigoplus_k \left( \bigoplus_h a_{ih} |A_{jh}| |A|^{-1} \right) a_{kj} = \left( \bigoplus_k a_{ih} |A_{jh}| a_{kj} \right) |A|^{-1}.
\]
Using Formulas (3.7) and (3.8), we see that the maximal value of \(|A_{jh}| a_{kj}\) is attained when \(k = h = j\) and it is \(|A|\). Thus,
\[
\pi ((*) ) = \pi(a_{ij} |A_{jj}| |A|^{-1}) = a_{ij}.
\]
The other relations are proved in the same way. \(\square\)

Remark 3.14. In the sense of Proposition 3.15, the matrices \(\bar{T}'\) and \(\bar{T}''\) are pseudo right/left identities of \(A\) and \(A\vee\) respectively.

Pushing the results of Proposition 3.15 forward, we conclude:

Corollary 3.15. Suppose \(A \in M_n(\mathbb{T})\) is regular. Let \(AA\vee = \bar{T}'\) and \(A\vee A = \bar{T}''\); then
\[
\pi_*(\bar{T}' A) = \pi_*(A), \quad \pi_*(A\vee \bar{T}') = \pi_*(A\vee), \quad \pi_*(\bar{T}'' A\vee) = \pi_*(A\vee), \quad \text{and} \quad \pi_*(\bar{T}' A\vee) = \pi_*(A).
\]

Example 3.16. Let
\[
A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \quad \text{then} \quad A\vee = \begin{pmatrix} -1 & -3 \\ -2 & -3 \end{pmatrix} \quad \text{and} \quad AA\vee = \bar{T}' = \begin{pmatrix} 0 & (-2)^e \\ 1^e & 0 \end{pmatrix}.
\]
Computing the products we have
\[
\bar{T}' A = \begin{pmatrix} 0 & (-2)^e \\ 1^e & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2^e & 3 \end{pmatrix},
\]
\[
A\vee \bar{T}' = \begin{pmatrix} -1 & -3 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 0 & (-2)^e \\ 1^e & 0 \end{pmatrix} = \begin{pmatrix} -1 & (-3)^e \\ (-2)^e & -3 \end{pmatrix},
\]
and it easily verify that \(\pi_*(\bar{T}' A) = A\) and \(\pi_*(A\vee \bar{T}') = A\vee\).
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