I. INTRODUCTION

Nowadays non-Hermitian physics has emerged as a versatile platform for the exploring of functional devices that are absent or difficult to realize in Hermitian regime [1–14]. The passive and active \( PT \)-symmetric non-Hermitian systems with balanced gain and loss are investigated theoretically and experimentally in various setups in optics [15–22]. The non-Hermiticity induces nonunitary dynamic including the power oscillation [3, 5] and the unidirectional reflectionless [6, 7]. However, the \( PT \) symmetry protects the symmetry of transmission [23]–[26]. An ideal building block possessing asymmetric transmission is the asymmetric dimer with an unequal hopping strength [27]. Moreover, asymmetric transmission is possible under gain and loss associated with effective magnetic flux [28–31]. Nonreciprocal photonics in the non-Hermitian physics are revealed to be useful for the applications in optics [32–34].

Non-Hermitian system at exceptional point (EP) has a coalescence state [39, 40]. We propose a novel application of the \( PT \)-symmetric non-Hermitian Su-Schrieffer-Heeger (SSH) model by embedded it in a two-dimensional square lattice tube. The coalescence state at the exceptional point of non-Hermitian SSH model is chiral and selectively controls helical transport and amplification. Two typical helicity-dependent scattering dynamics are observed. If the incidence has an identical helicity with the embedded non-Hermitian SSH model, we observe a perfect transmission without reflection. However, if the incidence has an opposite helicity with the embedded non-Hermitian SSH model, except for a full transmission, we observe an amplified transmission with different helicity from the incidence; but the amplified reflection has identical helicity with the incidence. These intriguing features are completely unexpected in Hermitian system. Moreover, the helical amplification at high efficiency can be triggered by an arbitrary excitation. The different dynamics between incidences with opposite helicities are results of unidirectional tunneling, which is revealed to be capable of realizing without introducing magnetic field. Our findings open a direction in all-optical device and provide perspectives in non-Hermitian transport.

II. CHIRAL SCATTERING CENTER

We start with an infinite square lattice tube. The tube has infinite sites in the \( x \) direction, and the supported momentum in the \( x \) direction is continuous in the region \( k_x \in [−\pi, \pi] \). The tube is periodic along the \( y \) direction and has total \( 2M \) sites, and the supported momentum in the \( y \) direction is discrete \( k_y = n\pi/M \) (integer \( n = 1, 2, \ldots, 2M \)). The eigenstate of the tube is

\[
|\Psi\rangle = \frac{1}{\sqrt{2M}} \sum_{j=-\infty}^{\infty} \sum_{l=1}^{2M} e^{ik_y j} e^{ik_x l} |j, l\rangle ,
\]

where \( j \) (\( l \)) is the index of the lattice in the \( x \) (\( y \)) direction. We consider a non-Hermitian engineering of the tube, whose schematic is illustrated in Fig. [1]a. A non-Hermitian SSH ring is embedded in the center (\( j = 0 \)) of...
The eigen energies of the non-Hermitian SSH ring are significantly different from the eigen energies of the uniform ring. The energy levels of the non-Hermitian SSH ring and the uniform ring are compared in Fig. 2(a); correspondingly, the supported discrete momenta $k_y$ are mismatch. Consequently, the incidences in the tube can not pass the non-Hermitian SSH ring center and are blocked. However, if the non-Hermitian SSH ring $H_{SSH}$ is engineered at the EP, it has a coalescence eigenstate with zero energy. The coalesced zero mode is

$$|\psi_-\rangle = \frac{1}{\sqrt{2M}} \sum_{l=1}^{2M} e^{-il\pi/2} |0, l\rangle . \quad (2)$$

The intriguing feature of the coalescence eigenstate $|\psi_-\rangle$ is that it is identical to one of the two degenerate zero energy eigenstates of the uniform ring (at the momentum $k_y = \pm \pi/2$) as elaborated in Fig. 2(b). For the Hermitian conjugation SSH ring $H_{SSH}^\dagger$, it differs from $H_{SSH}$ only in the sense that the gain and loss are interchanged. $H_{SSH}^\dagger$ is also at the EP and supports a coalescence zero energy eigenstate in the form of

$$|\psi_+\rangle = \frac{1}{\sqrt{2M}} \sum_{l=1}^{2M} e^{+il\pi/2} |0, l\rangle . \quad (3)$$

Notably, the two coalescence eigenstates in $H_{SSH}$ and $H_{SSH}^\dagger$ are the two degenerate zero modes of the uniform ring, being orthogonal $\langle \psi_- | \psi_+ \rangle = 0$. The two coalescence eigenstates possess opposite chiralities classified by the type of EP \[39, 40, 47-49\]. The non-Hermitian SSH ring $H_{SSH}$ engineered at the EP only supports the resonant transmission for the incident wave with one of two momenta $k_y = \pm \pi/2$.

To comprehensively address the scattering inside the tube, we write down the tube Hamiltonian in the momentum space $H = \sum_{k_y} H_{k_y}$. $H_{k_y}$ describes a ladder system as schematically illustrated in Fig. 1(b). The
Hamiltonian in the momentum subspace $k_y$ reads

$$H_{k_y} = J_x \sum_{j=-\infty}^{\infty} \sum_{\lambda=1,0} |j - 1, k_y \rangle \langle j, k_y|$$

$$+ 2J_y \cos k_y \sum_{j\neq 0,j=-\infty}^{\infty} \langle j, k_y \rangle_{10} \langle j, k_y|$$

$$+ \left(we^{ik_y} + we^{-ik_y}\right) |0, k_y \rangle_{10} \langle 0, k_y| + \text{H.c.}$$

$$-i(w - v)(-1)^\lambda \sum_{\lambda=1,0} |0, k_y \rangle_{\lambda\lambda} \langle 0, k_y|.$$  \hspace{1cm} (4)

The Fourier transformations applied to the square lattice tube is $|j, k_y \rangle_{\lambda} = M^{-1/2} \sum_{l=1}^{M} e^{ik_y(2l-\lambda)} |j, 2l - \lambda\rangle$, where $\lambda = 1, 0$ and $k_y = n\pi/M$, ($n = 1, 2, ..., M$). $H_{k_y}$ with different momenta commute

$$[H_{k_y}, H_{k_y'}] = 0.$$  \hspace{1cm} (5)

This means that $H$ can be decomposed into $M$-fold independent sub-Hamiltonians. The ladder $H_{k_y}$ has a non-Hermitian dimer embedded in the center. Remarkably, any distortion $w \neq v$ generates two opposite effective magnetic fluxes in the two plaquettes with gain and loss.

An interesting situation is when the ladder has zero effective magnetic fluxes at $k_y = \pi/2$ although the presence of nonreciprocal Hermitian coupling $i(w - v)$ with a Peierls phase factor $i = e^{i\pi/2}$. This structure is equivalent to the ladder with asymmetric coupling in Fig. 1(c) without employing effective magnetic flux \cite{50, 51}. From the analysis of the $k_y = \pi/2$ subspace, we can obtain the previous conclusion discussed.

III. UNIDIRECTIONAL TUNNELING AND HELICAL RESONANT TRANSPORT

In the $k_y = \pi/2$ subspace, term with coupling $2J_y \cos k_y$ vanish, and the ladder Hamiltonian in Eq. (4) reduces to the form of

$$H_{\pi/2} = J_x \sum_{j=-\infty}^{\infty} \sum_{\lambda=1,0} |j - 1, \pi/2 \rangle_{\lambda\lambda} \langle j, \pi/2|$$

$$+ i(w - v) |0, \pi/2 \rangle_{10} \langle 0, \pi/2| + \text{H.c.}$$

$$-i(w - v)(-1)^\lambda \sum_{\lambda=1,0} |0, \pi/2 \rangle_{\lambda\lambda} \langle 0, \pi/2|.$$  \hspace{1cm} (6)

Taking the unitary transformation

$$|j, \pi/2 \rangle_\alpha = \frac{1}{\sqrt{2}} \left(|j, \pi/2 \rangle_1 - |j, \pi/2 \rangle_0\right),$$

$$|j, \pi/2 \rangle_\beta = -\frac{i}{\sqrt{2}} \left(|j, \pi/2 \rangle_1 + |j, \pi/2 \rangle_0\right).$$  \hspace{1cm} (7)

with $j = 0, \pm 1, \pm 2, ...$, the Hamiltonian can be rewritten in the form of

$$H_{\pi/2} = J_x \sum_{j=-\infty}^{\infty} \langle j - 1, \pi/2 |_{\alpha\alpha} \langle j, \pi/2|$$

$$+ |j - 1, \pi/2 \rangle_{\beta\beta} \langle j, \pi/2| + \text{H.c.}$$

$$+ 2(w - v) \langle 0, \pi/2 |_{\alpha\beta} \langle 0, \pi/2|,$$  \hspace{1cm} (8)

which is schematically illustrated in Fig. 1(c). We note that the non-Hermiticity of $H_{\pi/2}$ only arises from the unidirectional hopping term $2(w - v) |0, \pi/2 \rangle_{\alpha\beta}$ and $H_{\pi/2}$ still has parity symmetry. The composed asymmetric coupling takes the advantages of the two leads structure, which differs from other cases in the literatures \cite{25, 54}.

A tight-binding network is constructed topologically by the sites and various connections between them. There are three types of basic non-Hermitian clusters leading to the non-Hermiticity of a discrete non-Hermitian system: i) complex on-site potential denoted as $(V + i\gamma) |l\rangle \langle l|; ii)$ non-Hermitian dimer denoted as $e^{i\varphi} |l\rangle \langle l| + |j\rangle \langle j|$, and iii) asymmetric hopping amplitude dimer denoted as $\mu |l\rangle \langle j| + \nu |j\rangle \langle l|$ ($\mu \neq \nu$), where $\nu, \gamma, \varphi, \mu, \nu$ are real numbers. The asymmetric hopping induces imaginary magnetic flux and has been used in modeling a delocalization phenomenon \cite{52}. The unidirectional hopping is defined as $\mu \nu = 0$, which is an extreme non-Hermitian term, only allows the particle tunneling from A to B.

The unidirectional hopping we encounter here is a basic non-Hermitian element in the context of tight-binding network, which leads to unidirectional tunneling between

FIG. 3. Schematic of the helical transport. The helical filter is the SSH ring at EP. (a) Perfect transmission with zero transverse bandwidth for incident wave possessing identical helicity with the SSH ring helical filter. (b) Full transmission associating with out going wave for incidence possessing opposite helicity with the SSH ring helical filter.
two Hermitian subsystems A and B [Fig. 1(d)]: On one hand, when an initial state is set in subsystem B, the particle is always confined in B and thus the Dirac probability is conservative. On the other hand, when an initial state is set in subsystem A, the particle can tunnel to B and thus the Dirac probability is not conservative. Subsystem can be regarded as a conditional invariant subspace, which is an exclusive feature of a non-Hermitian system. These features can be seen by considering the scattering problem of Hamiltonian Eq. (8).

We take the momentum \( k_y = \pm \pi/2 \) as an inner degree of freedom; \( k_y = -\pi/2 \) indicates the circling toward the positive direction of \( y \), thus, an angular momentum associated with \( k_y = -\pi/2 \) is toward the negative direction of \( x \); in contrast, an angular momentum associated with \( k_y = \pi/2 \) is toward the positive direction of \( x \). For incidence with \( k_y = -\pi/2 \) (\( k_y = \pi/2 \)) moving toward the positive direction of \( x \), the direction of angular momentum associated with \( k_y \) is opposite (identical) to its propagation direction along \( x \), thus, we refer to the helicity as the left-handed (right-handed).

Now we consider the scattering problem for a unidirectional scattering center, which is exactly solvable and is of significant not only for the non-Hermitian physics but also for applications in optics. A direct motivation is that the equivalent Hamiltonian of the present Hamiltonian in an invariant subspace is a concrete example with unidirectional tunneling. Two solutions of the scattering wave function in the subspace \( H_{x/2} \) can be obtained by the Bethe ansatz method (see Appendix A). For incidence with momentum \( k_x \) in the \( x \) direction, the resonant transmission solution is

\[
|\psi^L_{k_x}\rangle = \frac{1}{\sqrt{2M}} \sum_{j=-\infty}^{\infty} \sum_{l=1}^{M} e^{ik_x j} (-1)^j (|j, 2l - 1\rangle - i |j, 2l\rangle),
\]

(9)
and the interfered transmission solution is
\[
|\psi_{k_x}^R\rangle = \frac{1}{\sqrt{2M}} \sum_{j=-\infty}^{\infty} \sum_{l=1}^{M} e^{ik_xj} (-1)^l (|j, 2l - 1\rangle + i |j, 2l\rangle) + A_{k_x} e^{ik_x|j|} (-1)^l (|j, 2l - 1\rangle - i |j, 2l\rangle),
\]
with \(A_{k_x} = (w - v) / (J_x \sin k_x)\) the \(k_x\)-dependent amplified amplitude.

Two solutions have the following implications when we consider incidences with opposite helicities. (i) \(|\psi_{k_x}^L\rangle\) indicates a perfect resonant transmission for the \(|j, 2l - 1\rangle - i |j, 2l\rangle\)-type (left-handed) incident wave, where \(-i\) indicates the momentum \(k_y = -\pi/2\). (ii) \(|\psi_{k_x}^R\rangle\) indicates a combination of perfect resonant transmission and an equal amplitude \(|j, 2l - 1\rangle - i |j, 2l\rangle\)-type out going waves with \(k_x\)-dependent amplitude for the \(|j, 2l - 1\rangle + i |j, 2l\rangle\)-type (right-handed) incident wave, where \(+i\) indicates the momentum \(k_y = +\pi/2\). The amplitude is inversely proportional to the group velocity \(2J_x \sin k_x\) of the incidence, and diverges only for zero group velocity. In the region \(j > 0\), we have
\[
|\langle j, 2l - \lambda |\psi_{k_x}^R\rangle|^2 = \frac{1}{2M} \left[ (-1)^\lambda \frac{w - v}{J_x \sin k_x} \right]^2,
\]
which indicates an interference pattern along the transverse direction \((y\) direction). In particular, for \(\sin k_x = \pm (w - v) / J_x\), it exhibits transverse standing-wave mode, which indicates that the transmitted waves only appear in the tube leads embedded the blue sites or the yellow sites.

So far we have given a complete analysis of the scattering problem for the incidence with transverse wave vector \(k_y = \pm \pi/2\). For other incidences with \(k_y \neq \pm \pi/2\), the coalescing state of the non-Hermitian SSH ring is a forbidden channel due to the mismatch of the transverse momentum \(k_y\). Other channel in the scattering center can also be eliminated adiabatically at large staggered coupling \(w, v \gg J_x, J_y\), resulting in near perfect reflection. As a temporary summary, we can conclude that the present system allows the coexistence of two types of helical resonant transport: (i) zero transverse bandwidth perfect transmission, (ii) zero transverse bandwidth amplified transmission and reflection. These phenomena are schematically simulated in Fig. 3.

To verify and demonstrate the performance of the proposed scheme, we simulate the time evolutions for both two typical cases. We consider two kinds of Gaussian wavepacket with opposite helicities as the initial excitation
\[
|\Psi_G^L(0)\rangle = \Omega^{-\frac{1}{2}} \sum_{j=-\infty}^{\infty} \sum_{l=1}^{2M} e^{-\frac{\alpha_w^2 (j - N_c)^2}{2}} e^{ik_xj} e^{+i\frac{\pi}{2} l} |j, l\rangle,
\]
where \(\Omega = 2M \sqrt{\pi / \alpha_w}\) is the normalization constant. The parameter \(\alpha_w\) determines the width of the Gaussian wavepacket and \(N_c\) is the initial position of its center. We numerically calculate the evolved wave function

![FIG. 5. Snapshots of the intensity for single-site excitation in the y direction. The parameters of the initial excitation are \(\alpha_w = 0.3, k_x = -\pi/2\) and \(N_c = -35\), and of the system are \(J_x = J_y = J = 0.25, w = 1.5, v = 0.5\). The time unit is \(J^{-1}\).](image)

\[
|\Psi_G^R(t)\rangle = e^{-iHt} |\Psi_G^L(0)\rangle
\]
for the system with \(M = 2\) [see Fig. 3(a)]. The non-Hermitian SSH ring as the filter is located at the position \(x = 0\). We present the numerical results, including the profiles of the evolved excitations and the probability distribution in \(y\) direction for the evolved excitations in Figs. 4(b)-(e). We also present the numerical results for the system with \(M = 4\) in Fig. B1 (see Appendix B). Both cases verify our predictions.

### IV. PURIFICATION AND AMPLIFICATION BY THE HELICAL FILTER

For the incidence possessing an opposite helicity with the non-Hermitian SSH ring filter, the filter generates a \((w - v) / (J_x \sin k_x)\) times amplified out going wave possessing both identical and opposite helicities with the filter toward different directions. If \((w - v) \gg J_x\) or \(k_x\) approaches to 0 and \(\pi\), the reflection and transmission beams dominate after the incidence is scattered at the filter.

We first consider an initial single-site excitation in the \(y\) direction in comparison with a plane wave excitation in Eq. (12). The filter has the left-handed helicity and the initial excitation is Gaussian in the \(x\) direction in the form of
\[
|\Psi_G(0)\rangle = \Omega^{-\frac{1}{2}} \sum_{j=-\infty}^{\infty} e^{-\frac{\alpha_w^2 (j - N_c)^2}{2}} e^{+i\frac{\pi}{2} j} |j, 2\rangle,
\]
where \(\Omega = \sqrt{\pi / \alpha_w}\) is the normalization constant. From the simulation in Fig. 5 we notice that the evolved excitation is amplified after being scattered at the filter. In the \(x > 0\) region, the light and dark areas as a typical result of interference is observed. This indicates that the output in this region consists of two components with opposite helicities. The single-site excitation in the \(y\) direction consists all the possible discrete momentum \(k_y\) determined by the size \(2M\). Consequently, we notice the spreading of single-site excitation at the moment \(t = 10\) before reaching the filter. After scattering, all the components with momentum \(k_y \neq \pm \pi/2\) are reflected back to the \(x < 0\) region because their energies mismatch the
energies of the filter. The component with left-handed helicity ($k_y = -\pi/2$) or right-handed helicity ($k_y = \pi/2$) is resonant with the coalesced zero mode of the filter. The component with left-handed helicity ($k_y = -\pi/2$) is perfectly transmitted. However, the component with right-handed helicity ($k_y = \pi/2$), except for a perfect transmission, can induce the right-handed reflected and left-handed transmitted wave due to the gain in the filter; and the right-handed reflected and left-handed transmitted wave are the dominant components after scattering.

Then, we consider a single-site excitation in the tube. The tube has a hard boundary on its left side (Fig. 7), which perfectly reflects all the waves. The left boundary together with the filter form a chamber. For a single-site excitation, the excitation includes the momentum components in the full region of $k_x$ and $k_y$. However, only wave with right-handed left-handed helicity ($k_y = \pm \pi/2$) can pass the filter and are amplified with one dominant helicity identical to the filter in the output. The waves with different momentum $k_x$ have different amplified ratio and moving velocity in the $x$ direction. Figure 7 schematically explains the dynamics of a single-site excitation. In Fig. 7(a) we consider an initial single-site excitation in the chamber. The profiles of the evolved wave functions are simulated. The output wave well demonstrates our description. In Fig. 7(a), the fringes indicate the purified passing waves with left-handed helicity (being amplified and then being dominant) similar as the incidence in Figs. 4(b) and 4(c); the intensity spreading in the $x$ direction indicates the output with different velocities. In Fig. 7(b), the intensities of different helicities as functions of time are depicted. The output after scattered by the helical filter has a dominant left helicity, which possesses identical helicity with the filter.

V. SUMMARY

In summary, we propose the helical transport controlled by a $\mathcal{PT}$-symmetric non-Hermitian SSH ring at the EP as a helical filter. The filter has a chiral coalescence state due to the chirality of EP; therefore, the filter only ensures the resonant transmission for the incidence possessing an identical helicity with it. For the incidence possessing an opposite helicity with the filter, except for a full transmission; additional amplified transmission possessing identical helicity and reflections possessing opposite helicity with the filter are stimulated. The transmissions with opposite helicities interfere and create an interference pattern, which depends on the velocity of the incidence. We first propose a helical incidence dependent scattering and discuss the multiple-channel scattering problem in a two-dimensional non-Hermitian square lattice. The non-Hermitian SSH ring as a helical filter purifies the excitation: the amplified component in the transmission possesses identical helicity with the filter and can be dominates, associated with a reflection of opposite helicity. Our findings provide a novel application of the non-Hermitian SSH system and are valuable for the design of optical device using non-Hermitian metamaterial.

FIG. 6. Schematic of purified amplification triggered by a single-site excitation. Both helical filter and passing wave have the left-handed helicity.

FIG. 7. (a) Snapshots of the intensity of evolved wave for a single-site excitation. The dynamics indicates an amplified single helicity output. The system parameters are $M = 2$, $J_x = J_y = J = 0.25$, $w = 1.5$, $v = 0.5$ and the time unit is $J^{-1}$. (b) Total transmitted intensity for (a). The blue and gold lines represent the transmitted intensity ($x > 0$) with the left-handed and right-handed helicities, respectively. The black dash line represents the summation of intensities for all the other momentum components.
APPENDIX

A. Bethe ansatz solution

In this Appendix, we present the derivations of the scattering solutions in Eqs. (9) and (10), which is the heart of this work. We first focus on the scattering solutions of the Hamiltonian in Eq. (8). The Bethe ansatz wave function of a scattering state $|\psi_{k_x}\rangle$ has the form

$$|\psi_{k_x}\rangle = \sum_{j=-\infty}^{\infty} \left( f_j^\alpha |j, \pi/2\rangle_\alpha + f_j^\beta |j, \pi/2\rangle_\beta \right).$$  \hspace{1cm} (A1)

The Schrödinger equation $H_{x/2} |\psi_{k_x}\rangle = E_{k_x} |\psi_{k_x}\rangle$ gives

$$J_x \left( f_{j+1}^\alpha + f_{j-1}^\alpha \right) = E_{k_x} f_j^\alpha, \quad j \neq 0,$$

$$J_x \left( f_j^\beta + f_{j+1}^\beta \right) = E_{k_x} f_j^\beta, \quad j \neq 0,$$

$$J_x f_j^\alpha + J_x f_{j-1}^\alpha + 2 (w-v) f_0^\alpha = E_{k_x} f_0^\alpha,$$

$$J_x \left( f_j^\beta + f_{j-1}^\beta \right) = E_{k_x} f_j^\beta. \quad (A2)$$

Considering an incident plane wave with momentum $k_x$ incoming from one side of lead $\alpha$, the ansatz wave function $f_j^{\alpha/\beta}$ has the form

$$f_j^\alpha = \begin{cases} e^{ik_{j}x} + r_{k_x}^{\alpha} e^{-ik_{j}x}, & j \leq -1 \\ t_{k_x}^{\alpha} e^{ik_{j}x}, & j \geq 0 \end{cases},$$

$$f_j^\beta = \begin{cases} r_{k_x}^{\beta} e^{-ik_{j}x}, & j \leq -1 \\ t_{k_x}^{\beta} e^{ik_{j}x}, & j \geq 0 \end{cases}. \quad (A3)$$

Here $r_{k_x}^{\alpha/\beta}$ and $t_{k_x}^{\alpha/\beta}$ are the reflection and transmission amplitudes of the incident wave. Substituting Eq. (A3) into Eq. (A2), we obtain the energy

$$E_{k_x} = 2J_x \cos k_x, \quad (A4)$$

and the reflection and transmission amplitudes

$$t_{k_x}^{\alpha} = 1, \quad r_{k_x}^{\alpha} = 0,$$

$$t_{k_x}^{\beta} = r_{k_x}^{\beta} = 0. \quad (A5)$$

Then the scattering wave function is

$$|\psi_{k_x}^L\rangle = \sum_{j=-\infty}^{\infty} e^{ik_{j}x} |j, \pi/2\rangle_\alpha. \quad (A6)$$

For the case of an incident plane wave with momentum $k_x$ incoming from one side of lead $\beta$, we set $f_j^{\alpha/\beta}$ as the form

$$f_j^\alpha = \begin{cases} r_{k_x}^{\alpha} e^{-ik_{j}x}, & j \leq -1 \\ t_{k_x}^{\alpha} e^{ik_{j}x}, & j \geq 0 \end{cases},$$

$$f_j^\beta = \begin{cases} e^{ik_{j}x} + r_{k_x}^{\beta} e^{-ik_{j}x}, & j \leq -1 \\ t_{k_x}^{\beta} e^{ik_{j}x}, & j \geq 0 \end{cases}. \quad (A7)$$

Substituting it into Eq. (A2), we obtain the energy $E_{k_x} = 2J_x \cos k_x$ and the reflection and transmission amplitudes

$$r_{k_x}^{\alpha} = t_{k_x}^{\alpha} = \frac{(w-v)i}{J_x \sin k_x},$$

$$r_{k_x}^{\beta} = 0, \quad t_{k_x}^{\beta} = 1. \quad (A8)$$

Then the scattering wave function is

$$|\psi_{k_x}^R\rangle = \sum_{j=-\infty}^{\infty} \left[ \frac{(w-v)i}{J_x \sin k_x} e^{ik_{j}x} |j, \pi/2\rangle_\alpha + e^{ik_{j}x} |j, \pi/2\rangle_\beta \right]. \quad (A9)$$

After applies the inverse transformation of Eq. (11), the above two scattering solutions Eqs. (A6) and (A9) can be written as

$$|\psi_{k_x}^L\rangle = \frac{1}{\sqrt{2}} \sum_{j=-\infty}^{\infty} e^{ik_{j}x} \left( |j, \pi/2\rangle_1 - |j, \pi/2\rangle_0 \right), \quad (A10)$$

and

$$|\psi_{k_x}^R\rangle = \frac{-i}{\sqrt{2}} \sum_{j=-\infty}^{\infty} \left[ e^{ik_{j}x} \left( |j, \pi/2\rangle_1 + |j, \pi/2\rangle_0 \right) \right] - \frac{w-v}{J_x \sin k_x} e^{ik_{j}x} \left( |j, \pi/2\rangle_1 - |j, \pi/2\rangle_0 \right) + \text{A11}$$

FIG. B1. (a) Non-Hermitian lattice tube with $M = 4$. Snapshots of the intensity at various time moments for three typical initial excitations: (b) Perfect transmission of Gaussian profile $|\Psi_0^+(0)\rangle$ with identical helicity of the filter; (c) interfered transmission of Gaussian profile $|\Psi_0^+(0)\rangle$ with opposite helicity of the filter; (d) purification and amplification of the single-site excitation. The interference pattern in (c) is in accord with the result in Eq. (11), containing four light and four dark spots. Other parameters of the initial excitation and system are identical with the $M = 2$ case in Figs. 4 and 7.
in the space of Hamiltonian Eq. (6). Applying the inverse Fourier transformations with \( k_y = \pi/2 \), we obtain the solutions in real space

\[
|\psi_{k_x}^L\rangle = \frac{-i}{\sqrt{2M}} \sum_{j=-\infty}^{\infty} \sum_{l=1}^{M} (-1)^l e^{ikx_2l} (|j, 2l - 1\rangle - i |j, 2l\rangle),
\]

and

\[
|\psi_{k_x}^R\rangle = \frac{-1}{\sqrt{2M}} \sum_{j=-\infty}^{\infty} \sum_{l=1}^{M} (-1)^l e^{ikx_2l} (|j, 2l - 1\rangle + i |j, 2l\rangle) - \frac{i}{J_x \sin k_x} e^{ikx_2l} (|j, 2l - 1\rangle - i |j, 2l\rangle).
\]

\[\text{(A13)}\]

**B. Numerical results for } M = 4

In Fig. B1 we present the numerical simulations for the system with size \( M = 4 \). The dynamics of perfect transmission, interfered transmission, and purification and amplification for single-site excitation are shown.

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