Asymptotic analysis for the elasticity system with Tresca and maximal monotone graph conditions

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Abstract

In this paper, we consider the stationary problem in three dimensional thin domain $\Omega^\varepsilon$ with maximal monotone graph and Tresca conditions. In the first step, we present the problem statement and give the variational formulation. We then study the asymptotic behavior when one dimension of the domain tends to zero. In the latter case a specific Reynolds limit equation is obtained and the uniqueness of the displacement of the limit problem are proved.

Keywords: Asymptotic approach, maximal monotone graph, Tresca law, variational problem, weak solution.

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1. Introduction

The asymptotic behavior of linear elasticity has been studied by several authors. The study of the non linear problem in a stationary regime in a three dimensional thin domain with non linear friction of Tresca has been considered in [3]. In [13], Lions studied theoretically a problem governed by the Laplace equation with Dirichlet boundary conditions. He proved the existence of a solution based essentially on the method of compactness and the uniqueness of the solution by imposing conditions on the data. The asymptotic analysis of a contact problem in a three dimensional with friction between two elastic bodies was investigated in [12]. The authors in [4] studied the non linear boundary value problem governed by partial differential equations which describe the evolution of linear elastic materials. The asymptotic behavior of a Bingham fluid in a thin domain with boundary conditions non linear was studied in [10]. In [22] the authors worked on the asymptotic convergence of a dynamical problem of a non isothermal linear elasticity with friction of Tresca type. In the last few years, some research papers have been written dealing with the asymptotic analysis of an incompressible fluid in a three-dimensional thin domain, when one dimension of the fluid domain tends to zero (see, e.g., [1, 6, 7] and the references cited therein). More recently, the authors in [2] have studied the asymptotic analysis of a dynamical problem of isothermal elasticity with non linear friction of Tresca type but without the intervention of the nonlinear term. The

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The purpose of this paper is to study the asymptotic analysis of a stationary problem for the linear elasticity represented by a thin domain $\Omega^\varepsilon$ in $\mathbb{R}^3$ with Tresca and maximal monotone graph boundary conditions. We consider the Dirichlet boundary conditions on $\Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon$ where $\Gamma_L^\varepsilon$ is the lateral one, the maximal monotone graph condition at the top surface $\Gamma_1^\varepsilon$, finally, a non-linear Tresca interface condition at the bottom one $\omega$.

This work is organized as follows. In Section 2 we introduce some notations and give the problem statement and variational formulation. In Section 3, we use the change of variable $z = \frac{x_3}{\varepsilon}$, to transform the initial problem posed in the domain $\Omega^\varepsilon$ into a new problem posed in a fixed domain $\Omega$ independent of the parameter $\varepsilon$. We find some estimates on the displacement of the small parameter and prove the convergence theorem by using several inequalities. In Section 4, we investigate the convergence results of the weak problem and its uniqueness.

2. Problem statement and variational formulation

We consider a homogeneous, elastic and isotropic body domain defined by:

$$\Omega^\varepsilon = \{(x', x_3) \in \mathbb{R}^3 : x' = (x_1, x_2) \in \omega, 0 < x_3 < \varepsilon h(x')\},$$

where $\omega$ is a non-empty bounded domain of $\mathbb{R}^2$ with a Lipschitz continuous boundary, $h(.)$ is a Lipschitz continuous function defined on $\omega$ such that $0 < h_* \leq h(x') \leq h^*, \forall (x', 0) \in \omega$ and $\varepsilon$ is a small parameter that will tend to zero. We decompose the boundary of $\Omega^\varepsilon$ as $\Gamma^\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon \cup \omega$ with

$$\omega = \{(x', x_3) \in \bar{\Omega}^\varepsilon : x_3 = 0\},$$
$$\Gamma_1^\varepsilon = \{(x', x_3) \in \bar{\Omega}^\varepsilon : (x', 0) \in \omega, \ x_3 = \varepsilon h(x')\},$$
$$\Gamma_L^\varepsilon = \{(x', x_3) \in \bar{\Omega}^\varepsilon : x' \in \partial \omega, 0 < x_3 < \varepsilon h(x')\},$$

where $\omega$ is the bottom of the domain, $\Gamma_1^\varepsilon$ is the upper surface and $\Gamma_L^\varepsilon$ the lateral part of $\Gamma^\varepsilon$ (see Fig. 1). Let $u^\varepsilon(.) : \Omega^\varepsilon \rightarrow \mathbb{R}^3$ be the displacement and $S$ the set of all symmetric $3 \times 3$ matrices. Then $D(u^\varepsilon) \subset S$ denotes the symmetric gradient of $u^\varepsilon$ whose components are $\frac{1}{2} \left( \frac{\partial u^\varepsilon_i}{\partial x_j} + \frac{\partial u^\varepsilon_j}{\partial x_i} \right)$, $1 \leq i, j \leq 3$. Let $\sigma^\varepsilon = \left( \sigma_{ij}^\varepsilon \right)$, $i, j = 1, 2, 3$ denotes the stress tensor, with

$$\sigma_{ij}^\varepsilon (u^\varepsilon) = 2\mu d_{ij} (u^\varepsilon) + \lambda d_{kk} (u^\varepsilon) \delta_{ij},$$

where $\mu$ and $\lambda$ are elasticity coefficients with $\lambda > 0$ and $\lambda + \mu \geq 0$, $\delta_{ij}$ the symbol of Kronecker.

Figure 1: The domain $\Omega^\varepsilon$. We consider now the following mechanical problem.
**Problem 2.1.** Find a displacement field $u^\varepsilon: \Omega^\varepsilon \to \mathbb{R}^3$ such that
\begin{equation}
-\text{div} \sigma^\varepsilon + (\alpha^\varepsilon)^2 u^\varepsilon = f^\varepsilon \quad \text{in} \quad \Omega^\varepsilon, \\
u^\varepsilon = 0 \quad \text{on} \quad \Gamma^\varepsilon_L, \\
P(u^\varepsilon) - \sigma^\varepsilon(u^\varepsilon) \cdot n = \beta^\varepsilon(u^\varepsilon) \quad \text{on} \quad \Gamma^\varepsilon_F, \\
u^\varepsilon \cdot n = 0 \quad \text{on} \quad \Gamma^\varepsilon_F \cup \Gamma^\varepsilon_T,
\end{equation}
where $n = (n_1, n_2, n_3)$ is the unit outward normal to $\Gamma^\varepsilon$, and
\begin{align*}
u^\varepsilon_n &= u^\varepsilon \cdot n, \\
u^\varepsilon_T &= u^\varepsilon - (u^\varepsilon_n)n, \\
_1^\varepsilon &= (\sigma^\varepsilon \cdot n)n, \\
_2^\varepsilon &= \sigma^\varepsilon - (\sigma^\varepsilon \cdot n)n,
\end{align*}
are, respectively the normal and the tangential components of $u^\varepsilon$ on the boundary $\omega$ and the components of the normal and the tangential stress tensor on $\varepsilon$. We denote by $P$ the differential operator of the first order to coefficients lipschitzians and $\beta^\varepsilon$ defined by $\beta^\varepsilon = \varepsilon^{-1}(1 + J^\varepsilon_L)$ with $\varepsilon > 0$, the Yosida’s approach of $\beta$ where $J^\varepsilon_L = -(1 + \varepsilon J^\varepsilon)\beta^{-1}$ is the resolvante of $\beta$ which is the maximal monotone graph such that $0 \in \beta(0)$. Boundary condition (2.2) is the regularized condition of Yosida. Condition (2.3) represents a Tresca friction law on $\omega$ where $k^\varepsilon$ is the friction coefficient and $\ell^\varepsilon = \{f^\varepsilon_i\}_{1 \leq i \leq 3}$ is the body forces. Furthermore, the equation (2.1) represents the equilibrium equation, here $\alpha^\varepsilon > 0$.

To get a weak formulation, we introduce the closed convex
\begin{equation*}
V^\varepsilon = \left\{ v \in H^1(\Omega^\varepsilon)^3 : v = 0 \text{ on } \Gamma^\varepsilon_L \text{ and } v \cdot n = 0 \text{ on } \varepsilon \cup \Gamma^\varepsilon_T \right\}.
\end{equation*}
By standard calculations, the variational formulation of the problem 2.1 is given by following.

**Problem 2.2.** Find a displacement field $u^\varepsilon \in V^\varepsilon$ such that
\begin{equation}
an(u^\varepsilon, \varphi - u^\varepsilon) - \int_{\Gamma^\varepsilon_F} P(u^\varepsilon)(\varphi - u^\varepsilon) \, d\tau + \int_{\Gamma^\varepsilon_L} \beta^\varepsilon(u^\varepsilon)(\varphi - u^\varepsilon) \, d\tau \\
+ j^\varepsilon(\varphi) - j^\varepsilon(u^\varepsilon) \geq (f^\varepsilon, \varphi - u^\varepsilon) \quad \forall \varphi \in V^\varepsilon,
\end{equation}
where
\begin{align*}
an(u^\varepsilon, \varphi) &= 2\mu \int_{\Omega^\varepsilon} \sigma^\varepsilon(u^\varepsilon) \cdot \overline{\sigma^\varepsilon}(\varphi) \, dx + \lambda \int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla (\varphi) \, dx + (\alpha^\varepsilon)^2 \int_{\Omega^\varepsilon} u^\varepsilon \varphi \, dx, \\
j^\varepsilon(\varphi) &= \int_{\omega} k^\varepsilon |\varphi| |\varphi|' \, dx, \\
(f^\varepsilon, \varphi) &= \int_{\Omega^\varepsilon} f^\varepsilon \varphi \, dx.
\end{align*}

**Theorem 2.3.** Problems 2.1 and 2.2 are equivalent.

**Proof.** See [3].

### 3. Change of the domain and some estimates

For the asymptotical analysis of problem 2.1, we use the approach which consist in transporting the initially posed problem in the domain $\Omega^\varepsilon$ which depends on a small parameter $\varepsilon$ to an equivalent problem with a fixed domain $\Omega$ which is independent of $\varepsilon$. For that, we introduce the change of the variable $z = \frac{x_3}{\varepsilon}$, which changes $(x', x_3)$ in $\Omega^\varepsilon$ to $(x', z)$ in $\Omega$ where
\[
\Omega = \{(x', z) \in \mathbb{R}^3, (x', 0) \in \varepsilon \text{ and } 0 < z < h(x') \},
\]
and we denote by $\Gamma = \partial \Omega \cup \Gamma_1 \cup \Gamma_L$ its boundary, then we define the following functions in $\Omega$,
\[
\begin{cases}
\hat{u}_1^\varepsilon(x',z) = u_1^\varepsilon(x',x_3), & i = 1, 2, \\
\hat{u}_3^\varepsilon(x',z) = \varepsilon^{-1}u_3^\varepsilon(x',x_3).
\end{cases}
\]

Let us assume the following
\[
\hat{f}(x',z) = \varepsilon^2f^\varepsilon(x',x_3), \quad \hat{k} = \varepsilon k^\varepsilon \text{ and } \hat{\alpha} = \varepsilon \alpha^\varepsilon.
\]

Let
\[
V = \left\{ \phi \in H^1(\Omega)^3 : \phi = 0 \text{ on } \Gamma_L, \phi \cdot n = 0 \text{ on } \omega \cup \Gamma_1 \right\},
\]
\[
V_z = \left\{ v = (v_1, v_2) \in L^2(\Omega)^2 : \varepsilon \frac{\partial v_i}{\partial z} \in L^2(\Omega), i = 1, 2 \text{ and } v = 0 \text{ on } \Gamma_L \right\},
\]
\[
\Pi(V) = \left\{ \varphi = (\varphi_1, \varphi_2) \in H^1(\Omega)^2 : \varphi = 0 \text{ on } \Gamma_L \right\}.
\]

$V_z$ is the Banach space with norm
\[
\|v\|_{V_z} = \left( \sum_{i=1}^{2} \left( \|v_i\|_{L^2(\Omega)}^2 + \left\| \frac{\partial v_i}{\partial z} \right\|_{L^2(\Omega)}^2 \right) \right)^{\frac{1}{2}}.
\]

By injecting the new data and unknown factor in (2.4) and after multiplication by $\varepsilon$, we deduce
\[
\mu \varepsilon^2 \sum_{i,j=1}^{2} \int_{\Omega} \left( \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_3} \right) \frac{\partial}{\partial x_j} (\hat{\phi}_i - \hat{u}_i^\varepsilon) \, dx' \, dz
+ \mu \sum_{i,j=1}^{2} \int_{\Omega} \left( \frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_3} \right) \left[ \frac{\partial}{\partial z} (\hat{\phi}_i - \hat{u}_i^\varepsilon) + \varepsilon^2 \frac{\partial}{\partial x_3} (\hat{\phi}_3 - \hat{u}_3^\varepsilon) \right] \, dx' \, dz
+ 2\mu \varepsilon^2 \int_{\Omega} \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \frac{\partial}{\partial z} (\hat{\phi}_3 - \hat{u}_3^\varepsilon) \, dx' \, dz + \lambda \varepsilon^2 \int_{\Omega} \text{div}(\hat{u}^\varepsilon) \text{div}(\hat{\phi} - \hat{u}^\varepsilon) \, dx' \, dz
+ \lambda^2 \sum_{i,j=1}^{2} \int_{\Omega} \hat{u}_i^\varepsilon (\hat{\phi}_i - \hat{u}_i^\varepsilon) \, dx' \, dz + \varepsilon^2 \lambda^2 \int_{\Omega} \hat{u}_3^\varepsilon (\hat{\phi}_3 - \hat{u}_3^\varepsilon) \, dx' \, dz + \int_{\omega} \hat{k} (|\hat{\phi}| - |\hat{u}^\varepsilon|) \, dx'
- \varepsilon \sum_{i=1}^{2} \int_{\omega} P_1(\hat{u}_i^\varepsilon) (\hat{\phi}_i - \hat{u}_i^\varepsilon) \sqrt{1 + |\nabla h^\varepsilon(x')|^2} \, dx' - \varepsilon^3 \int_{\omega} P_1(\hat{u}_3^\varepsilon) (\hat{\phi}_3 - \hat{u}_3^\varepsilon) \sqrt{1 + |\nabla h^\varepsilon(x')|^2} \, dx'
+ 2\sum_{i=1}^{2} \int_{\omega} \hat{\beta}_E(\hat{u}_i^\varepsilon) (\hat{\phi}_i - \hat{u}_i^\varepsilon) \sqrt{1 + |\nabla h^\varepsilon(x')|^2} \, dx' + \varepsilon \int_{\omega} \hat{\beta}_E(\hat{u}_3^\varepsilon) (\hat{\phi}_3 - \hat{u}_3^\varepsilon) \sqrt{1 + |\nabla h^\varepsilon(x')|^2} \, dx'
\geq \sum_{i=1}^{2} \int_{\omega} \hat{f}_i (\hat{\phi}_i - \hat{u}_i^\varepsilon) \, dx' \, dz + \varepsilon \int_{\Omega} \hat{f}_3 (\hat{\phi}_3 - \hat{u}_3^\varepsilon) \, dx' \, dz, \quad \forall \hat{\phi} \in V,
\]
where
\[
P_1 = \sum_{i=1}^{2} a_i(x') \frac{\partial}{\partial x_i} + a_0(x'),
\]
P_1 is differential operator of the first order and
\[
\hat{\beta}_E(\hat{u}^\varepsilon) = \varepsilon \beta_E(u^\varepsilon).
\]

We introduce some results which will be used later.
Lemma 3.1 ([4, Poincaré inequality]). We have
\[ \int_{\Omega} |\epsilon u^\varepsilon|^2 \, dx \leq 2\varepsilon h^* \int_{\Omega} |\epsilon u^\varepsilon|^2 \, d\tau + 2(\varepsilon h^*)^2 \int_{\Omega} \left| \frac{\partial u^\varepsilon}{\partial x_3} \right|^2 \, dx. \]  
(3.1)

Lemma 3.2 ([4, Korn inequality]). We have
\[ \|\nabla u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq 2 \int_{\Omega_\varepsilon} |D(u^\varepsilon)|^2 \, dx + C(\Gamma_\varepsilon) \int_{\Gamma_\varepsilon} |u^\varepsilon|^2 \, d\tau, \]  
where
\[ C(\Gamma_\varepsilon) = \left\| \frac{\partial}{\partial x_2} \frac{h^\varepsilon}{h} \right\|_{C(\bar{\omega})} \left(1 + \left\| \frac{\partial}{\partial x_1} \frac{h^\varepsilon}{h} \right\|^2_{C(\bar{\omega})}\right). \]

Lemma 3.3 ([4]). We have
\[ \int_{\Gamma_\varepsilon} |\epsilon u^\varepsilon|^2 \, d\tau' \leq C(\Omega) \int_{\Omega} \left( |\epsilon u^\varepsilon|^2 + |\nabla \epsilon u^\varepsilon|^2 \right) \, dx' \, dz, \]  
where \( C(\Omega) \) is a constant independent of \( \epsilon \).

Lemma 3.4 ([12]). Let \( a_k(x') \in C^{0,1} \) for \( k = 0, 1, 2 \), we get
\[ \int_{\Omega} \omega_1 \epsilon(\epsilon u^\varepsilon) \, dx' \leq c^* \int_{\Omega} |\epsilon u^\varepsilon| \, dx', \]  
where \( c^* \) is a constant independent of \( \epsilon \).

Now we will obtain a priori estimates for the displacement field \( \epsilon u^\varepsilon \) in the domain \( \Omega \).

Theorem 3.5. Assuming that \( f \in L^2(\Omega)^3 \), the friction coefficient \( F \in L^\infty(\omega) \) and under the assumptions
\[ \varepsilon C(\Gamma_\varepsilon) \leq \frac{1}{\mu}, \quad \frac{15}{16} \mu > \left(1 + c^* + \frac{\mu}{2}\right) C(\Omega, h) = \bar{c}, \quad \alpha^2 > \left(1 + c^* + \frac{\mu}{2}\right) C(\Omega, h) = \bar{c}, \]  
there exists a constant \( c_1 > 0, i = 1, 2, 3 \) independent of \( \epsilon \) such that
\[ \varepsilon^2 \sum_{i,j=1}^2 \left| \frac{\partial \epsilon u^\varepsilon}{\partial x_i} \right|^2_{L^2(\Omega)} + \sum_{i=1}^2 \left( \left| \frac{\partial \epsilon u^\varepsilon}{\partial x_i} \right|^2_{L^2(\Omega)} + \varepsilon^4 \left| \frac{\partial \epsilon u^\varepsilon}{\partial x_i} \right|^2_{L^2(\Omega)} \right) + \varepsilon^2 \left| \frac{\partial \epsilon u^\varepsilon}{\partial z} \right|^2_{L^2(\Omega)} \leq c_1, \]  
(3.4)
\[ \|\epsilon u^\varepsilon\|_{L^2(\Omega)} \leq c_2, \quad \text{for } i = 1, 2, \]  
(3.5)
\[ \|\epsilon u^\varepsilon\|_{L^2(\Omega)} \leq c_3. \]  
(3.6)

Proof. Let \( u^\varepsilon \) be a solution of the variational problem (2.4). Choosing \( \varphi = 0 \) as test function in (2.4), we get
\[ a(u^\varepsilon, u^\varepsilon) = \int_{\Omega_\varepsilon} \beta(\epsilon u^\varepsilon) u^\varepsilon \, d\tau + \int_{\Gamma_\varepsilon} \beta(\epsilon u^\varepsilon) u^\varepsilon \, d\tau + j(\epsilon u^\varepsilon) (f^\varepsilon, u^\varepsilon). \]

As \( j(\epsilon u^\varepsilon) \) is positive and \( \beta(\epsilon u^\varepsilon) \) is increasing lipschitzian with \( \beta(0) = 0 \), we have
\[ 2\mu \int_{\Omega} |D(\epsilon u^\varepsilon)|^2 \, dx + (\alpha^2) \int_{\Omega_\varepsilon} |\epsilon u^\varepsilon|^2 \, dx - \int_{\Gamma_\varepsilon} \beta(\epsilon u^\varepsilon) u^\varepsilon \, d\tau \leq (f^\varepsilon, u^\varepsilon). \]  
(3.7)

By (3.2) and (3.3), we have
\[ 2\mu \int_{\Omega_\varepsilon} |D(\epsilon u^\varepsilon)|^2 \, dx \geq \mu \|\nabla u^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} - \mu C(\Gamma_\varepsilon) \int_{\Gamma_\varepsilon} |\epsilon u^\varepsilon|^2 \, d\tau \]  
(3.8)
\[
\begin{align*}
&\geq \mu \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} - \mu C(\varepsilon) \int_{\omega} |u^\varepsilon|^2 \sqrt{1 + |\nabla h^\varepsilon(x')|^2} dx' \\
&\geq \mu \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} - \mu C(\varepsilon) \max_{x' \in \omega} \sqrt{1 + |\nabla h^\varepsilon(x')|^2} \int_{\omega} |u^\varepsilon|^2 dx' \\
&\geq \mu \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} - \mu C(\varepsilon) \max_{x' \in \omega} \sqrt{1 + |\nabla h^\varepsilon(x')|^2} \int_{\Gamma_1} |\hat{u}^\varepsilon|^2 d\tau \\
&\geq \mu \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} - \mu C(\varepsilon) C(\Omega, h) \int_{\Omega} \left(|\hat{u}^\varepsilon|^2 + |\nabla \hat{u}^\varepsilon|^2\right) dx' dz,
\end{align*}
\]
where
\[
C(\Omega, h) = C(\Omega) \max_{x' \in \omega} \sqrt{1 + |\nabla h(x')|^2},
\]

as
\[
(f^\varepsilon, u^\varepsilon) \leq \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \|u^\varepsilon\|_{L^2(\Omega^\varepsilon)},
\]
by (3.1) (Poincaré inequality), we obtain
\[
(f^\varepsilon, u^\varepsilon) \leq \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \left(\sqrt{2\varepsilon h^*} \left(\int_{\Gamma_1} |u^\varepsilon|^2 d\tau\right)^{\frac{1}{2}} + \sqrt{2\varepsilon h^*} \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}\right),
\]
we have
\[
\sqrt{2\varepsilon h^*} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)} = \sqrt{2\varepsilon h^*} \frac{\varepsilon}{\mu} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}.
\]
Using Young’s inequality, we get
\[
\sqrt{2\varepsilon h^*} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq \frac{8\varepsilon^2 h^2}{\mu} \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu}{16} \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}.
\]
In the same way, we have
\[
\sqrt{2\varepsilon h^*} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \left(\int_{\Gamma_1} |u^\varepsilon|^2 d\tau\right)^{\frac{1}{2}} = \sqrt{2\varepsilon h^*} \frac{\varepsilon}{\mu} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \left(\int_{\Gamma_1} |u^\varepsilon|^2 d\tau\right)^{\frac{1}{2}}
\]
\[
\leq \frac{\varepsilon^2 h^2}{\mu} \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu}{2\varepsilon} \int_{\Gamma_1} |u^\varepsilon|^2 d\tau
\]
\[
\leq \frac{\varepsilon^2 h^*}{\mu} \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu}{2\varepsilon} \max_{x' \in \omega} \sqrt{1 + |\nabla h^\varepsilon(x')|^2} \int_{\omega} |u^\varepsilon|^2 dx'
\]
\[
\leq \frac{\varepsilon^2 h^*}{\mu} \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu}{2\varepsilon} C(\Omega, h) \int_{\Omega} \left(|\hat{u}^\varepsilon|^2 + |\nabla \hat{u}^\varepsilon|^2\right) dx' dz.
\]
Thus
\[
(f^\varepsilon, u^\varepsilon) \leq \left[\frac{8\varepsilon^2 h^2}{\mu} + \frac{\varepsilon^2 h^*}{\mu}\right] \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu}{16} \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu}{2\varepsilon} C(\Omega, h) \int_{\Omega} \left(|\hat{u}^\varepsilon|^2 + |\nabla \hat{u}^\varepsilon|^2\right) dx' dz. \tag{3.9}
\]
On the other hand, by Lemma 3.4, we have
\[
\int_{\Gamma_1} P(u^\varepsilon) u^\varepsilon d\tau = \sum_{i=1}^{2} \int_{\omega} P_1(\hat{u}^\varepsilon_{i}) \hat{u}^\varepsilon_{i} \sqrt{1 + |\nabla h^\varepsilon(x')|^2} dx' + \varepsilon^2 \int_{\omega} P_1(\hat{u}^\varepsilon_{i}) \hat{u}^\varepsilon_{i} \sqrt{1 + |\nabla h^\varepsilon(x')|^2} dx'
\]
\[
\leq \max_{x' \in \omega} \sqrt{1 + |\nabla h^\varepsilon(x')|^2} \left(\sum_{i=1}^{2} \int_{\omega} P_1(\hat{u}^\varepsilon_{i}) \hat{u}^\varepsilon_{i} dx' + \varepsilon^2 \int_{\omega} P_1(\hat{u}^\varepsilon_{i}) \hat{u}^\varepsilon_{i} dx'\right)
\]
\[ \leq c^* \max_{x' \in \omega} \sqrt{1 + |\nabla h^\varepsilon(x')|^2} \left( \sum_{i=1}^{2} \left| \hat{\mu}^\varepsilon_i \right|^2 \, dx' + \varepsilon^2 \left| \hat{\mu}^\varepsilon_{2} \right|^2 \, dx' \right) \] (3.10)

\[ \leq c^* \max_{x' \in \omega} \sqrt{1 + |\nabla h^\varepsilon(x')|^2} \int_\omega |\hat{u}^\varepsilon|^2 \, dx' \]

\[ \leq c^* \max_{x' \in \omega} \sqrt{1 + |\nabla h^\varepsilon(x')|^2} \int_{\Gamma_1} |\hat{u}^\varepsilon|^2 \, d\tau' \]

\[ \leq c^* C(\Omega, h) \int_\Omega \left( |\hat{u}^\varepsilon|^2 + |\nabla \hat{u}^\varepsilon|^2 \right) \, dx' \, dz. \]

By injecting (3.8), (3.9), and (3.10) in (3.7) it becomes

\[ \frac{15}{16} \mu \left\| \nabla u^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} + (\alpha^\varepsilon)^2 \int_{\Omega^\varepsilon} |u^\varepsilon|^2 \, dx - \left( \mu C(\Gamma^\varepsilon) + c^* + \frac{\mu}{2\varepsilon} \right) C(\Omega, h) \int_\Omega \left( |\hat{u}^\varepsilon|^2 + |\nabla \hat{u}^\varepsilon|^2 \right) \, dx' \, dz \]

\[ \leq \varepsilon^2 \frac{h^*}{\mu} (8h^* + 1) \left\| \hat{f}^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)}. \] (3.11)

Multiplying inequality (3.11) by \( \varepsilon \) and using

\[ \varepsilon^2 \left\| \hat{f}^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1} \left\| \hat{f}^\varepsilon \right\|^2_{L^2(\Omega)}, \]

we obtain

\[ \left[ \frac{15}{16} \mu - \left( 1 + c^* + \frac{\mu}{2} \right) C(\Omega, h) \right] \left\| \nabla u^\varepsilon \right\|^2_{L^2(\Omega)} + \left[ \alpha^2 - \left( 1 + c^* + \frac{\mu}{2} \right) C(\Omega, h) \right] \left\| \hat{u}^\varepsilon \right\|^2_{L^2(\Omega)} \]

\[ \leq \frac{h^*}{\mu} (8h^* + 1) \left\| \hat{f}^\varepsilon \right\|^2_{L^2(\Omega)}. \]

Assume that

\[ \frac{15}{16} \mu > \left( 1 + c^* + \frac{\mu}{2} \right) C(\Omega, h) = \hat{c}, \quad \hat{\alpha}^2 > \left( 1 + c^* + \frac{\mu}{2} \right) C(\Omega, h) = \hat{c}, \]

so

\[ \left\| \nabla u^\varepsilon \right\|^2_{L^2(\Omega)} \leq c_1, \]

we deduce (3.4) with

\[ c_1 = \frac{h^*}{\mu \hat{c}} (8h^* + 1) \left\| \hat{f}^\varepsilon \right\|^2_{L^2(\Omega)}, \]

and

\[ \left\| \hat{u}^\varepsilon_i \right\|^2_{L^2(\Omega)} \leq c_2, \text{ for } i = 1, 2, \quad \left\| \varepsilon \hat{u}^\varepsilon_i \right\|^2_{L^2(\Omega)} \leq c_3. \]

\[
\square
\]

4. Convergence results and the limit problem

**Theorem 4.1.** Under the same assumptions as in Theorem 3.5, there exists \( u^* = (u^*_1, u^*_2) \in V_z \) such that

\[ \hat{u}^\varepsilon_i \rightarrow u^*_i, \quad i = 1, 2, \text{ weakly in } V_z, \] (4.1)

\[ \varepsilon \frac{\partial \hat{u}^\varepsilon_i}{\partial x_j} \rightarrow 0, \quad i, j = 1, 2, \text{ weakly in } L^2(\Omega), \] (4.2)

\[ \varepsilon \frac{\partial \hat{u}^\varepsilon_3}{\partial z} \rightarrow 0, \quad \text{weakly in } L^2(\Omega), \] (4.3)

\[ \varepsilon \hat{u}^\varepsilon_3 \rightarrow 0, \quad \text{weakly in } L^2(\Omega). \]
Proof. From (3.4) and (3.5), we deduce (4.1). Also (4.2) follows from (3.4) and (4.1). From (3.4) and (3.6) we obtain (4.3).

**Theorem 4.2.** With the same assumptions of Theorem 3.5, the solution \( u^* \) satisfies the following relations

\[
-\mu \frac{\partial^2 u^*_i}{\partial z^2} + \hat{\alpha}^2 u^*_i = \hat{f}_i, \quad \text{for } i = 1, 2, \text{ in } L^2(\Omega),
\]

(4.4)

\[
\mu \sum_{i=1}^2 \int_{\Omega} \frac{\partial u^*_i}{\partial z} \frac{\partial}{\partial z} (\phi_i - u^*_i) \, dx \, dz + \hat{\alpha}^2 \sum_{i=1}^2 \int_{\Omega} u^*_i (\phi_i - u^*_i) \, dx \, dz + \int_{\Omega} \hat{k} (|\phi| - |u^*|) \, dx
\]

\[
\geq \sum_{i=1}^2 \int_{\Omega} \hat{f}_i (\phi_i - u^*_i) \, dx \, dz, \quad \forall \phi \in \Pi(V).
\]

(4.5)

Proof. The variational inequality (3.2) can be written as

\[
\sum_{i=1}^4 \int_{\Omega} \omega \frac{\partial u^*_i}{\partial z} \frac{\partial}{\partial z} (\phi_i - u^*_i) \, dx \, dz + \int_{\Omega} \hat{k} |\phi| \, dx
\]

\[
+ \hat{\alpha}^2 \sum_{i=1}^2 \int_{\Omega} \hat{\omega}_i (\phi_i - \hat{u}_i) \, dx \, dz + \epsilon^2 \hat{\alpha}^2 \int_{\Omega} \hat{\omega}_3 (\phi_3 - \hat{u}_3) \, dx \, dz
\]

\[
- \epsilon \sum_{i=1}^2 \int_{\Omega} \hat{P}_1 (\hat{u}_i) (\phi_i - \hat{u}_i) \sqrt{1 + \lambda h^2} \, dx \, dx' - \epsilon^3 \int_{\Omega} \hat{P}_1 (\hat{u}_3) (\phi_3 - \hat{u}_3) \sqrt{1 + \lambda h^2} \, dx' \, dx
\]

\[
+ \sum_{i=1}^2 \int_{\Omega} \hat{P}_3 (\hat{u}_i) (\phi_i - \hat{u}_i) \sqrt{1 + \lambda h^2} \, dx' \, dx + \epsilon \int_{\Omega} \hat{P}_3 (\hat{u}_3) (\phi_3 - \hat{u}_3) \sqrt{1 + \lambda h^2} \, dx' \, dx
\]

\[
- \int_{\Omega} \hat{k} |\hat{u}_i| \, dx \, dx' \geq \sum_{i=1}^2 \int_{\Omega} \hat{f}_i (\phi_i - \hat{u}_i) \, dx \, dz + \epsilon \int_{\Omega} \hat{f}_3 (\phi_3 - \hat{u}_3) \, dx \, dz,
\]

where

\[
I_1 = \mu \epsilon^2 \sum_{i,j=1}^2 \int_{\Omega} \left( \frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} (\phi_i - \hat{u}_i) \, dx \, dz,
\]

\[
I_2 = \mu \sum_{i=1}^2 \int_{\Omega} \left( \frac{\partial \hat{u}_i}{\partial z} + \epsilon^2 \frac{\partial \hat{u}_3}{\partial x_i} \right) \frac{\partial}{\partial z} (\phi_i - \hat{u}_i) \, dx \, dz,
\]

\[
I_3 = \mu \sum_{i=1}^2 \int_{\Omega} \epsilon^2 \left( \frac{\partial \hat{u}_i}{\partial z} + \epsilon^2 \frac{\partial \hat{u}_3}{\partial z} \right) \frac{\partial}{\partial x_i} (\phi_i - \hat{u}_i) \, dx \, dz,
\]

\[
I_4 = 2\mu \int_{\Omega} \epsilon^2 \frac{\partial \hat{u}_3}{\partial z} \frac{\partial}{\partial z} (\phi_3 - \hat{u}_3) \, dx \, dz.
\]

By the Theorem 4.1, we have

\[
\lim_{\epsilon \to 0} \sum_{i=1}^4 I_1 (\epsilon) = \mu \sum_{i=1}^2 \int_{\Omega} \frac{\partial u_i}{\partial z} \frac{\partial}{\partial z} (\phi_i - u_i) \, dx \, dz,
\]

\[
\lim_{\epsilon \to 0} \epsilon \int_{\Omega} \hat{f}_3 \phi_3 \, dx \, dz = 0.
\]
Thus we now choose in the variational inequality (4.6), and as \( \hat{\beta}_\varepsilon(\hat{u}^\varepsilon) \) \( \hat{\phi}_i - \hat{u}_i^\varepsilon \) \( \sqrt{1 + |\nabla h(x')|^2} \) \( \varepsilon = 0, \quad i = 1, 2, \)

\[
\lim_{\varepsilon \to 0} \int_\omega P_1(\hat{u}_i^\varepsilon) (\hat{\phi}_i - \hat{u}_i^\varepsilon) \sqrt{1 + |\nabla h(x')|^2} \, dx' = 0,
\]

\[
\lim_{\varepsilon \to 0} \int_\omega P_1(\hat{u}_j^\varepsilon) (\hat{\phi}_j - \hat{u}_j^\varepsilon) \sqrt{1 + |\nabla h(x')|^2} \, dx' = 0,
\]

\[
\lim_{\varepsilon \to 0} \sum_{i=1}^2 \int_\omega \beta_\varepsilon(\hat{u}_i^\varepsilon) (\hat{\phi}_i - \hat{u}_i^\varepsilon) \sqrt{1 + |\nabla h(x')|^2} \, dx' = 0, \quad i = 1, 2,
\]

\[
\lim_{\varepsilon \to 0} \int_\omega \beta_\varepsilon(\hat{u}_i^\varepsilon) (\hat{\phi}_i - \hat{u}_i^\varepsilon) \sqrt{1 + |\nabla h(x')|^2} \, dx' = 0,
\]

because

\[
\left\| \beta_\varepsilon(\hat{u}_i^\varepsilon) \right\|^2_{L^2(\omega)} = \left\| \beta_\varepsilon(u_i^\varepsilon) \right\|^2_{L^2(\omega)} \leq \frac{\varepsilon^2}{\varepsilon^2} \left\| u_i^\varepsilon \right\|^2_{L^2(\omega)} = \frac{\varepsilon^2}{\varepsilon^2} \int_\omega \left[ \sum_{i=1}^2 \hat{u}_i^\varepsilon \right]^2 + \varepsilon^2 \hat{u}_i^\varepsilon \right\] \, dx' \to 0.
\]

And as \( j \) is convex and lower semi-continuous \( \left( \lim_{\varepsilon \to 0} \left( \inf \int_\omega \hat{k} |\hat{u}_i^\varepsilon| \, dx' \right) \geq \int_\omega \hat{k} |u_i^\varepsilon| \, dx' \right) \), we obtain

\[
\mu \sum_{i=1}^2 \int_\Omega \frac{\partial u_1^*}{\partial z} (\hat{\phi}_i - u_i^*) \, dx' \, dz + \alpha^2 \sum_{i=1}^2 \int_\Omega u_i^* (\hat{\phi}_i - u_i^*) \, dx' \, dz + \int_\Omega \hat{k} (|\hat{\phi}_i| - |u_i^*|) \, dx' \geq \sum_{i=1}^2 \int_\Omega \hat{f}_i (\hat{\phi}_i - u_i^*) \, dx' \, dz. \tag{4.6}
\]

We now choose in the variational inequality (4.6),

\[
\hat{\phi}_i = u_i^* \pm \psi_i, \quad \psi_i \in H_0^1(\Omega), \quad i = 1, 2,
\]

we get

\[
\mu \sum_{i=1}^2 \int_\Omega \frac{\partial u_1^*}{\partial z} \frac{\partial \psi_i}{\partial z} \, dx' \, dz + \alpha^2 \sum_{i=1}^2 \int_\Omega u_i^* \psi_i \, dx' \, dz = \sum_{i=1}^2 \int_\Omega \hat{f}_i \psi_i \, dx' \, dz.
\]

Using now the Green formula and choosing \( \psi_1 = 0 \) and \( \psi_2 \in H_0^1(\Omega) \), then \( \psi_2 = 0 \) and \( \psi_1 \in H_0^1(\Omega) \), we obtain

\[
- \int_\Omega \mu \frac{\partial}{\partial z} \left( \frac{\partial u_i^*}{\partial z} \right) \psi_i \, dx' \, dz + \alpha^2 \int_\Omega u_i^* \psi_i \, dx' \, dz = \int_\Omega \hat{f}_i \psi_i \, dx' \, dz.
\]

Thus

\[
- \mu \frac{\partial^2 u_i^*}{\partial z^2} + \alpha^2 u_i^* = \hat{f}_i, \quad \text{for } i = 1, 2, \text{ in } H^{-1}(\Omega), \tag{4.7}
\]

and as \( \hat{f} \in L^2(\Omega) \), then (4.7) is valid in \( L^2(\Omega) \).

**Theorem 4.3.** Under the same assumptions as in Theorem 4.2, we have the following inequality

\[
\int_\omega \hat{k} (|\psi + s^*| - |s^*|) \, dx' - \int_\omega \mu \tau^\varepsilon \psi \, dx' \geq 0, \quad \forall \psi \in L^2(\omega)^2, \tag{4.8}
\]

\[
\begin{cases} 
\mu |\tau^*| < \hat{k} & \Rightarrow s^* = 0, \\
\mu |\tau^*| = \hat{k} & \Rightarrow \exists \gamma > 0 \text{ such that } s^* = \gamma \tau^*,
\end{cases} \tag{4.9}
\]

where

\[
\tau^* = \frac{\partial u_i^*}{\partial z} (x', 0), \quad s^* (x') = u^* (x', 0), \quad \text{and } s^*_h (x') = u^* (x', h(x')).
\]
Also \( u^* \) and \( s^* \) satisfy the following equality of Reynolds
\[
\int_\omega \left( \int_0^h u^*(x', z) \, dz - \frac{h}{2} s^*(x') - \frac{h}{2} \tilde{s}^*(x') + \tilde{F}(x') + \tilde{U}^*(x') \right) \nabla \psi = 0, \quad \forall \psi \in H^1(\omega),
\]
(4.10)
where
\[
\tilde{F}(x') = \frac{1}{\mu} \int_0^h F(x', z) \, dz - \frac{h}{2\mu} \tilde{F}(x', h), \quad F(x', z) = \int_{\zeta}^z \tilde{f}(x', \eta) \, d\eta d\zeta,
\]
and
\[
\tilde{U}^*(x') = \frac{h\alpha^2}{2\mu} u^*(x', h) - \tilde{\alpha}^2 \int_0^h u^*(x', z) \, dz, \quad U^*(x', z) = \int_{\zeta}^z u^*(x', \eta) \, d\eta d\zeta.
\]

Proof. Passing to the limit in (3.2) and using the Green formula then choosing \( \hat{\phi}_i = u_i^* + \psi_i \), \( \psi_i \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega) \), \( i = 1, 2 \), where
\[
H^1_{\Gamma_1 \cup \Gamma_L}(\Omega) = \{ \psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_1 \cup \Gamma_L \},
\]
we obtain
\[
- \sum_{i=1}^2 \int_\Omega \mu \frac{\partial^2 u_i^*}{\partial z^2} \psi_i dx' \, dz + \sum_{i=1}^2 \int_{\Gamma} \mu \frac{\partial u_i^*}{\partial z} \nabla \psi_i d\sigma + \sum_{i=1}^2 \int_{\Gamma} \frac{\partial}{\partial z} \sum_{i=1}^{\tilde{\alpha}^2} \frac{\partial^2 u_i^*}{\partial z^2} \psi_i dx' \, dz + \int_\Omega \hat{k} \left| u^* + \psi \right| dx' - \int_\Omega \hat{k} \left| u^* \right| dx'
\geq \sum_{i=1}^2 \int_\Omega \hat{f}_i \psi_i dx' \, dz.
\]
But
\[
\int_{\Gamma} \mu \frac{\partial u_i^*}{\partial z} \nabla \psi_i d\sigma = - \int_\Omega \mu \frac{\partial u_i^*}{\partial z}(x', 0) \psi_i dx',
\]
then
\[
- \sum_{i=1}^2 \int_\Omega \mu \frac{\partial^2 u_i^*}{\partial z^2} \psi_i dx' \, dz + \sum_{i=1}^2 \int_{\Gamma} \frac{\partial}{\partial z} \sum_{i=1}^{\tilde{\alpha}^2} \frac{\partial^2 u_i^*}{\partial z^2} \psi_i dx' \, dz + \int_\Omega \hat{k} \left| \psi + s^* \right| dx' - \int_\Omega \mu \tau^* \psi_i dx'
\geq \sum_{i=1}^2 \int_\Omega \hat{f}_i \psi_i dx' \, dz.
\]
By (4.4), we deduce
\[
\int_\Omega \hat{k} \left| \psi + s^* \right| dx' - \int_\Omega \mu \tau^* \psi dx' \geq 0.
\]
This inequality remain valid for any \( \psi \in (D(\omega))^2 \) and by density of \( D(\omega) \) in \( L^2(\omega) \), we deduce (4.8). We also obtain (4.9) as in [1].

To prove (4.10), we integrate twice (4.4) between 0 and \( z \), we obtain
\[
u^*(x', z) = s^*(x') + z\tau^* - \frac{1}{\mu} F(x', z) + \tilde{\alpha}^2 \mu U^*(x', z),
\]
(4.11)
where
\[
U^*(x', z) = \int_{\zeta}^z \int_0^\zeta u^*(x', \eta) \, d\eta d\zeta,
\]
replacing \( z \) by \( h \), we obtain
\[
u^*(x', h(x')) = s^*(x') + h\tau^* - \frac{1}{\mu} F(x', h(x')) + \frac{\tilde{\alpha}^2}{\mu} U^*(x', h(x')),
\]
whence
\[ h\tau^* = s_h^*(x') - s_h^*(x') + \frac{1}{\mu} F(x', h(x')) - \frac{\alpha^2}{\mu} U^*(x', h(x')) , \] (4.12)
integrating (4.11) with respect to \( z \) in the interval \((0, h(x'))\), we obtain
\[ \int_0^h u^*(x', z) \, dz = hs^*(x') + \frac{h^2}{2} \tau^* - \frac{1}{\mu} \int_0^h F(x', z) \, dz + \frac{\alpha^2}{\mu} \int_0^h U^*(x', z) \, dz. \]
From (4.11) and (4.12), we deduce that
\[ \int_0^h u^*(x', z) \, dz - \frac{h}{2} s^*(x') - \frac{h}{2} s_h^*(x') + \tilde{F}(x') + \tilde{U}^*(x') = 0, \]
with
\[ \tilde{F}(x') = \frac{h}{2\mu} \int_0^h F(x', z) \, dz - \frac{h}{2\mu} F(x', h), \quad F(x', z) = \int_0^z \tilde{f}(x', \eta) \, d\zeta d\eta, \]
and
\[ \tilde{U}^*(x') = \frac{h^2\alpha^2}{2\mu} U^*(x', h) - \frac{\alpha^2}{\mu} \int_0^h U^*(x', z) \, dz, \quad U^*(x', z) = \int_0^z \tilde{u}^*(x', \eta) \, d\zeta d\eta. \]
Then
\[ \int_\Omega \left( \int_0^h u^*(x', z) \, dz - \frac{h}{2} s^*(x') - \frac{h}{2} s_h^*(x') + \tilde{F}(x') + \tilde{U}^*(x') \right) \nabla \psi = 0. \]
\[ \square \]

**Theorem 4.4.** The solution \( u^* \) of our limit problem is unique.

**Proof.** Let \( u^1 \) and \( u^2 \) be two solutions of (4.5), then
\[ \mu \sum_{i=1}^2 \int_\Omega \frac{\partial u_i}{\partial z} \frac{\partial}{\partial z} (\phi_i - u_i^1) \, dx' \, dz + \alpha^2 \sum_{i=1}^2 \int_\Omega u_i^1 (\phi_i - u_i^1) \, dx' \, dz + j(\phi_i) - j(u_i) \geq \sum_{i=1}^2 (\tilde{f}_i, \phi_i - u_i^1), \] (4.13)
and
\[ \mu \sum_{i=1}^2 \int_\Omega \frac{\partial u_i^2}{\partial z} \frac{\partial}{\partial z} (\phi_i - u_i^2) \, dx' \, dz + \alpha^2 \sum_{i=1}^2 \int_\Omega u_i^2 (\phi_i - u_i^2) \, dx' \, dz + j(\phi_i) - j(u_i) \geq \sum_{i=1}^2 (\tilde{f}_i, \phi_i - u_i^2), \] (4.14)
where
\[ j(\phi) = \int_\omega \tilde{k} |\phi| \, dx'. \]
Taking \( \phi = u^2 \) in (4.13) and \( \phi = u^1 \) in (4.14), we get
\[ \mu \sum_{i=1}^2 \int_\Omega \frac{\partial u_i^1}{\partial z} \frac{\partial}{\partial z} (u_i^2 - u_i^1) \, dx' \, dz + \alpha^2 \sum_{i=1}^2 \int_\Omega u_i^1 (u_i^2 - u_i^1) \, dx' \, dz + j(u_i^2) - j(u_i^1) \geq \sum_{i=1}^2 (\tilde{f}_i, u_i^2 - u_i^1), \] (4.15)
\[ \mu \sum_{i=1}^2 \int_\Omega \frac{\partial u_i^2}{\partial z} \frac{\partial}{\partial z} (u_i^1 - u_i^2) \, dx' \, dz + \alpha^2 \sum_{i=1}^2 \int_\Omega u_i^2 (u_i^1 - u_i^2) \, dx' \, dz + j(u_i^1) - j(u_i^2) \geq \sum_{i=1}^2 (\tilde{f}_i, u_i^1 - u_i^2). \] (4.16)
By adding the two inequalities (4.15) and (4.16), we obtain
\[ \mu \sum_{i=1}^2 \int_\Omega \frac{\partial}{\partial z} (u_i^1 - u_i^2) \frac{\partial}{\partial z} (u_i^1 - u_i^2) \, dx' \, dz + \alpha^2 \sum_{i=1}^2 \int_\Omega (u_i^1 - u_i^2) (u_i^1 - u_i^2) \, dx' \, dz \leq 0, \]
this implies
\[ \mu \left\| \frac{\partial}{\partial z} (u^1_i - u^2_i) \right\|^2_{L^2(\Omega)} + \hat{\alpha}^2 \left\| u^1_i - u^2_i \right\|^2_{L^2(\Omega)} = 0. \]

By Poincaré inequality, we get
\[ \left\| u^1_i - u^2_i \right\|_{V_z} = 0, \]
so
\[ u^1 = u^2. \]

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