Remarks Towards the Spectrum of the Heisenberg Spin Chain Type Models

Č. Burdík\textsuperscript{b}, J. Fuksa\textsuperscript{a,b}, A. P. Isaev\textsuperscript{a}, S. O. Krivonos\textsuperscript{a}, and O. Navrátil\textsuperscript{c}

\textsuperscript{a}Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Moscow region, 141980 Russia
\textsuperscript{b}Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering Czech Technical University in Prague
\textsuperscript{c}Department of Mathematics, Faculty of Transportation Sciences Czech Technical University in Prague

e-mail: fuksa@theor.jinr.ru, isaevap@theor.jinr.ru, krivonos@theor.jinr.ru, burdices@kmlinux.fjfi.cvut.cz, navratil@fd.cvut.cz

Abstract—The integrable close and open chain models can be formulated in terms of generators of the Hecke algebras. In this review paper, we describe in detail the Bethe ansatz for the XXX and the XXZ integrable close chain models. We find the Bethe vectors for two-component and inhomogeneous models. We also find the Bethe vectors for the fermionic realization of the integrable XXX and XXZ close chain models by means of the algebraic and coordinate Bethe ansatz. Special modification of the XXZ closed spin chain model (“small polaron model”) is considered. Finally, we discuss some questions relating to the general open Hecke chain models.

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1. INTRODUCTION

A braid group $B_L$ in the Artin presentation is generated by invertible elements $T_i$ ($i = 1, \ldots, L - 1$) subject to the relations:

$$
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad \text{for} \quad i \neq j \pm 1. \tag{1.1}
$$

An $A$-Type Hecke algebra $H_L(q)$ (see, e.g., [1]) is a quotient of the group algebra of $B_L$ by additional Hecke relations

$$
T_i^2 = (q - q^{-1}) T_i + 1, \quad (i = 1, \ldots, L - 1), \tag{1.2}
$$

where $q$ is a parameter (deformation parameter). Let $x$ be a parameter (spectral parameter) and we define elements

$$
T_k(x) := x^{-1/2} T_k - x^{1/2} T_k^{-1} \in H_L(q), \tag{1.3}
$$

which called baxterized elements. By using (1.1) and (1.2) one can check that the baxterized elements (1.3) satisfy the Yang–Baxter equation in the braid group form

$$
T_k(x) T_{k+1}(xy) T_k(y) = T_{k+1}(y) T_k(xy) T_{k+1}(x), \tag{1.4}
$$

and

$$
T_k(x) T_k(y) = (q - q^{-1}) T_k(xy) + (x^{-1/2} - x^{1/2})(y^{-1/2} - y^{1/2}). \tag{1.5}
$$

Equations (1.4) and (1.5) are baxterized analogs of the first relation in (1.1) and Hecke condition (1.2).

The Hamiltonian of the open Hecke chain model of the length $L$ is

$$
\mathbb{H}_L = \sum_{k=1}^{L-1} T_k \in H_L(q), \tag{1.6}
$$

(see, e.g., [2] and references therein). Any representation $\rho$ of the Hecke algebra gives an integrable open spin chain with the Hamiltonian $\rho(\mathbb{H}_L) = \sum_{k=1}^{L-1} \rho(T_k)$.

Define the closed Hecke algebra $\hat{H}_L(q)$ by adding additional generator $T_L$ to the set $\{T_1, \ldots, T_{L-1}\}$ such that $T_L$ satisfies the same relations (1.1) and (1.2), where we impose the periodic condition $T_{L+k} = T_k$. Then the closed Hecke chain of the length $L$ is described by the Hamiltonian

$$
\hat{H}_L = \sum_{k=1}^{L} T_k \in \hat{H}_L(q) \quad \text{and any representation} \quad \rho(\hat{H}_L) \quad \text{leads to the integrable closed spin chain with the Hamiltonian}
$$

$$
\rho(\hat{H}_L) = \sum_{k=1}^{L} \rho(T_k). \tag{1.7}
$$

In Sections 2–4, special representations $\rho = \rho_R$ of the algebra $H_L(q)$, called the $R$-matrix representations, are considered. In the case of $GL_q(2)$-type $R$-matrix representation $\rho_R$, the Hamiltonian (1.7) coincides with the XXZ spin chain Hamiltonian. It is clear that in the case of $q = 1$ we recover the XXX spin chain. The integrable structures for XXX spin chain...
are introduced in Subsection 2.1. We discuss some results of the algebraic Bethe ansatz for these models. In Section 3, we formulate the so-called two-component model (see [3, 4] and references therein). The two-component model was introduced to avoid problems with computation of correlation functions for local operators attached to some site $x$ of the chain. Using this approach we obtain in Section 4 the explicit formulas for the Bethe vectors, which show the equivalence of the algebraic and coordinate Bethe ansatzes.

In Section 5 we generalize the results of Sections 2—4 to the case of inhomogeneous XXX spin chain.

The realization of the XXX spin chains in terms of free fermions is considered in Sections 6—8. Here we explicitly construct Bethe vectors for XXX spin chains in the sectors of one, two and three magnons. In Section 9, we discuss another special representation $\rho$ of the Hecke algebra $H_{q}(q)$ which we call the fermionic representation. In this representation the Hamiltonian (1.7) describes the so-called “small polaron model” (see [5] and references therein). In Sections 10 and 11, we construct the Bethe vectors and obtain the Bethe ansatz equations for the “small polaron model” and for the XXZ closed spin chains by means of the coordinate Bethe ansatz. Then we compare these results with those obtained by means of the algebraic Bethe ansatz in Section 2. We show that the Hamiltonian of the “small polaron model” has the different spectrum comparing to the XXZ model in the sector of an even number of magnons.

Finally, in Section 12, we discuss the general open Hecke chain models which are formulated in terms of the elements of the Hecke algebra $H_{q}(q).$ We present the characteristic polynomials (in the case of the finite length of the chain) which define the spectrum of the Hamiltonian of this model in some special irreducible representations of $H_{q}(q).$ The method of construction of irreducible representations of the algebra $H_{q}(q)$ is formulated at the end of Section 12.

In Appendix, we give some details of our calculations.

### 2. Algebraic Bethe Ansatz

At the beginning, we describe some basic features of the algebraic Bethe ansatz. The method was formulated as a part of the quantum inverse scattering method proposed by Faddeev, Sklyanin and Takhtadjan [6, 7]. The main object of this method is the Yang—Baxter algebra generated by matrix elements of the monodromy matrix. The main rules for the Yang—Baxter algebra were elaborated in the very first papers [9—11]. Many quantum integrable systems were described in terms of this method, cf. [13—15]. We strongly recommend the review paper [8] for introductory reading and [12] for more detailed review.

### 2.1. L-Operator and Transfer Matrix for XXX Spin Chain

Suppose we have a chain of $L$ sites. The local Hilbert space $h_i$ corresponds to the $i$-th site. For our purposes, it is sufficient to suppose $h_i = \mathbb{C}^2.$ The total Hilbert space of the chain is

$$\mathcal{H} = \bigotimes_{i=1}^{L} h_i. \quad (2.1)$$

The basic tool of algebraic Bethe ansatz is the Lax operator. For its definition, we need an auxiliary vector space $V_A = \mathbb{C}^2.$ The Lax operator is a parameter depending object acting on the tensor product $V_A \otimes h_i$

$$L_{a,i} : V_A \otimes h_i \rightarrow V_A \otimes h_i \quad (2.2)$$

explicitly defined as

$$L_{a,i}(\lambda) = \left(\lambda \frac{1}{2} + \sum_{\alpha=1}^{3} \sigma_\alpha^a \sigma_\alpha^i \right), \quad (2.3)$$

where $\sigma_\alpha^i = 1/2 \sigma_\alpha$ is the spin operator on the $i$-th site, $\sigma_\alpha^a = (\sigma_x^a, \sigma_y^a, \sigma_z^a)$ — are Pauli sigma-matrices which act in the space $V_A$ ($\sigma_\alpha$ — are Pauli sigma-matrices which act in the space $h_i$) and $\mathbb{1}_{a,i}$ is the identity matrix in $V_A \otimes h_i.$ Operator $L_{a,i}(\lambda)$ can be expressed as a matrix in the auxiliary space

$$L_{a,i}(\lambda) = \begin{pmatrix} \lambda + \frac{1}{2} + S_i^z & S_i^- \\ S_i^+ & \lambda + \frac{1}{2} - S_i^z \end{pmatrix}. \quad (2.4)$$

Its matrix elements form an associative algebra of local operators in the quantum space $h_i.$

Introducing the permutation operator $P$

$$P = \frac{1}{2} \left( \mathbb{1} \otimes \mathbb{1} + \sum_{\alpha=1}^{3} \sigma_\alpha^a \otimes \sigma_\alpha^i \right) \quad (2.5)$$

(here $\mathbb{1}$ denotes a $2 \times 2$ unit matrix) we can rewrite the Lax operator as

$$L_{a,i}(\lambda) = \lambda \mathbb{1}_{a,i} + P_{a,i}. \quad (2.6)$$

Assume two Lax operators $L_{a,i}(\lambda)$ resp. $L_{b,i}(\mu)$ in the same quantum space $h_i$ but in different auxiliary spaces $V_A$ resp. $V_B.$ The product of $L_{a,i}(\lambda)$ and $L_{b,i}(\mu)$ makes sense in the tensor product $V_A \otimes V_B \otimes h_i.$ It turns out that there is an operator $R_{ab}(\lambda - \mu)$ acting nontrivially in $V_A \otimes V_B$ such that the following equality holds:

$$R_{ab}(\lambda - \mu) L_{a,i}(\lambda) L_{b,i}(\mu) = L_{b,i}(\mu) L_{a,i}(\lambda) R_{ab}(\lambda - \mu). \quad (2.7)$$
Relation (2.7) is called the fundamental commutation relation. The explicit expression for $R_{ab}(\lambda - \mu)$ is
\[
R_{ab}(\lambda - \mu) = (\lambda - \mu)1_{a,b} + P_{a,b},
\]
where $1_{a,b}$ resp. $P_{a,b}$ is identity resp. permutation operator in $V_a \otimes V_b$. In the matrix form we get for $R_{ab}(\lambda - \mu)$
\[
R_{ab}(\lambda - \mu) = \begin{pmatrix}
\lambda - \mu + 1 & 0 & 0 & 0 \\
0 & \lambda - \mu & 1 & 0 \\
0 & 0 & \lambda - \mu & 0 \\
0 & 0 & 0 & \lambda - \mu + 1
\end{pmatrix}.
\]

The operator $R_{ab}(\lambda - \mu)$ is called the $R$-matrix. It satisfies the Yang–Baxter equation
\[
R_{ab}(\lambda - \mu) R_{ac}(\lambda) R_{bc}(\mu) = R_{ac}(\mu) R_{bc}(\lambda) R_{ab}(\lambda - \mu)
\]
in $V_a \otimes V_b \otimes V_c$. Comparing (2.6) and (2.8) we see that the Lax operator and the $R$-matrix are the same.

We define a monodromy matrix
\[
T_a(\lambda) = L_{a,1}(\lambda)L_{a,2}(\lambda)\ldots L_{a,L}(\lambda)
\]
as a product of the Lax operators along the chain, i.e. over all quantum spaces $h_i$. As a matrix in the auxiliary space $V_a$, the monodromy matrix
\[
T_a(\lambda) = \begin{pmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{pmatrix}
\]
defines an algebra of global operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$ on the Hilbert space $\mathcal{H}$. It is called the Yang–Baxter algebra. The monodromy matrix $T_a(\lambda)$ is a step from local observables $S_i^a$ on $h_i$ to global observables on $\mathcal{H}$.

The trace
\[
\tau(\lambda) \equiv \text{Tr}_a T_a(\lambda) = A(\lambda) + D(\lambda)
\]
of $T_a(\lambda)$ in the auxiliary space $V_a$ is called the transfer matrix. It constitutes a generating function for commutative conserved charges. Assume the Lax operators
\[
L_a = L_{a,1}(\lambda), \quad L_b = L_{b,i}(\mu), \quad L'_a = L_{a,i+1}(\lambda), \quad L'_b = L_{b,i+1}(\mu)
\]
and $R_{ab} = R_{ab}(\lambda - \mu)$ in the tensor product $V_a \otimes V_b \otimes \mathcal{H}$, then
\[
R_{ab} L_a L'_a L_b L'_b = R_{ab} L_a L_b L'_a L'_b = L_b L_a R_{ab} L'_a L'_b
\]
\[
= L_b L_a L'_a L'_b R_{ab} = L_b L_a L'_a L'_b R_{ab}.
\]
Here we used (2.10) and the fact that operators acting nontrivially in different vector spaces commute. Hence, we can deduce commutation relations between the elements of the monodromy matrix
\[
R_{ab}(\lambda - \mu) T_a(\lambda) T_b(\mu) = T_b(\mu) T_a(\lambda) R_{ab}(\lambda - \mu).
\]
Equation (2.15) is a consequence of (2.7) for global observables $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$. We call it the global fundamental commutation relation. The commutativity of transfer matrices obviously follows from (2.15). After multiplying (2.15) by $R_{ab}^{-1}(\lambda - \mu)$ we get
\[
T_a(\lambda) T_b(\mu) = R_{ab}^{-1}(\lambda - \mu) T_b(\mu) T_a(\lambda) R_{ab}(\lambda - \mu).
\]
Taking the trace over auxiliary spaces $V_a$ and $V_b$ we obtain
\[
\tau(\lambda) \tau(\mu) = \tau(\mu) \tau(\lambda).
\]
Obviously, the monodromy matrix (2.11) is a polynomial of degree $L$ with respect to the parameter $\lambda$.
\[
T_a(\lambda) = \left(\lambda + \frac{1}{2}\right)^{L-1} \left(\lambda + \frac{1}{2}ight)^{L-1} \sum_{j=1}^{L-2} \sum_{k=1}^{L-2} \sigma_\alpha^a \otimes S_j^a + \ldots.
\]

Therefore, the transfer matrix $\tau(\lambda)$ is also a polynomial of degree $L$
\[
\tau(\lambda) = 2 \left(\lambda + \frac{1}{2}\right)^{L} + \sum_{k=0}^{L-2} \lambda^k Q_k.
\]

The term of order $\lambda^{L-1}$ vanishes because Pauli matrices are traceless. Due to commutativity (2.17) of transfer matrices also
\[
[Q_a, Q_\lambda] = 0.
\]
We see that the transfer matrix is a generating function for a set of commuting observables.

The Hamiltonian of the system appears naturally amongst the observables $Q_k$. From the definition of the Lax operator (2.3) we see that
\[
L_{a,1}(-1/2) = P_{a,1}.
\]
Hence,
\[
T_a(-1/2) = P_{a,1} P_{a,2} \cdots P_{a,L}
\]
\[
= P_{L-1,1} \cdots P_{2,1} P_{1,2} P_{a,1}.
\]

If we differentiate $T_a(\lambda)$ with respect to $\lambda$, we get
\[
\frac{dT_a(\lambda)}{d\lambda} = \sum_{k=1}^{L} P_{a,1} \cdots P_{a,k} \cdots P_{a,L}
\]

\[
= \sum_{k=1}^{L} P_{L-1,1} \cdots P_{k-1,1} \cdots P_{1,2} P_{a,1}.
\]
where we set
\[ R \]
with arbitrary spin in each site. Relations (2.7) with
\[ \text{su} \]
matrix
\[ GLq \]
and we take generators in any representation of
\[ \text{su} \]
valid for this generalized spin chain models as well.

The Hamiltonian of the system is
\[ H = \sum_{k=1}^{L} \sum_{\alpha=1}^{3} S_{k}^{a} S_{k+1}^{a} = \frac{1}{2} \sum_{k=1}^{L} P_{k}^{a} + \frac{L}{4}, \]
where we set \( S_{n+L} = S_{n} \) resp. \( P_{L+n+L} = P_{L+n} \). We can see that
\[ H = \frac{1}{2} \frac{d}{d\lambda} \ln \tau(\lambda) \bigg|_{\lambda=-1/2} - \frac{L}{4}. \]
This is the reason why we can say that the transfer matrix \( \tau(\lambda) \) is a generating function for commuting conserved charges.

**Remark.** Let \( S_{i}^{a} \) be generators of the Lie algebra \( su(2) \) in \( i \)-th site
\[ [S_{i}^{a}, S_{j}^{b}] = i \epsilon^{ab} S_{i}^{c} \delta_{ij}, \]
and we take generators \( S_{i}^{a} \) in any representation of
\[ su(2) \] which acts in the space \( h_{i} \). Then Eqs. (2.3) and (2.4) define \( L \)-operator for the integrable chain model with arbitrary spin in each site. Relations (2.7) with \( R \)-matrix (2.8) are equivalent to the defining relations (2.28). Formulas (2.11)–(2.13), (2.17)–(2.20) are valid for this generalized spin chain models as well.

### 2.2. Some Remarks on the XXZ Chain

The fundamental \( R \)-matrix for the quantum group
\[ GL_{q}(N) \] is [16, 17]
\[ \hat{R} = q^{N} \sum_{i=1}^{N} e_{ii} \otimes e_{ii} \]
\[ + \sum_{i \neq j} e_{ij} \otimes e_{ji} + (q-q^{-1}) \sum_{ij} e_{ii} \otimes e_{jj}, \]
where \( e_{ii} \) is the \( N \times N \) matrix unity. In a particular case of \( GL_{q}(2) \), the \( R \)-matrix (2.29) can be written in terms of Pauli matrices
\[ \hat{R} = \frac{1}{2} \left( \sigma^{x} \otimes \sigma^{x} + \sigma^{y} \otimes \sigma^{y} + \frac{q+q^{-1}}{2} \sigma^{z} \otimes \sigma^{z} \right) \]
\[ + \frac{q-q^{-1}}{4} \left( \sigma^{z} \otimes \mathbb{1}_{2} - \mathbb{1}_{2} \otimes \sigma^{z} \right) + \frac{3q-q^{-1}}{4} \mathbb{1}_{2} \otimes \mathbb{1}_{2}, \]
Here and below we use notation \( \mathbb{1}_{N} \) for the \( N \times N \) unit matrix. The fundamental \( R \)-matrix (2.29) satisfies the Hecke condition (1.2)
\[ \hat{R}^{2} = (q-q^{-1}) \hat{R} + \mathbb{1}_{N} \otimes \mathbb{1}_{N}. \]
If we define
\[ \hat{R}_{k+1}^{(q)} = \mathbb{1}_{N}^{\otimes(k-1)} \otimes \hat{R} \otimes \mathbb{1}_{N}^{\otimes(L-k-1)}, \]
we obtain the \( R \)-matrix representation \( \rho_{R} \) of the Hecke algebra (1.1), (1.2)
\[ \rho_{R} : T_{k} \rightarrow \hat{R}_{k+1}^{(q)}. \]
Then, the baxterized \( R \)-matrix is (see Eq. (1.3))
\[ \hat{R}_{k+1}^{(q)}(\mu) = \rho_{R}(\mu^{-1/2} T_{k} - \mu^{1/2} T_{k}^{1}) \]
\[ = \mu^{-1/2} \hat{R}_{k+1}^{(q)} - \mu^{1/2} \hat{R}_{k+1}^{(q)} \]
\[ = (\mu^{-1/2} - \mu^{1/2}) \hat{R}_{k+1}^{(q)} + \mu^{1/2} (q-q^{-1}). \]
This \( R \)-matrix is a solution of the Yang–Baxter equation in the braid group form
\[ R_{k+1}(\lambda) R_{k+1+k+2}(\lambda, \mu) R_{k+1}(\mu) \]
\[ = R_{k+1+k+2}(\mu) R_{k+1}(\lambda) R_{k+1+k+2}(\lambda). \]
Note that if there is a solution of Eq. (2.35), the solution of the equation
\[ R_{k+1}(\lambda) R_{k+1+k+2}(\lambda, \mu) R_{k+1+k+2}(\mu) \]
\[ = R_{k+1+k+2}(\mu) R_{k+1}(\lambda) R_{k+1+k+2}(\lambda) \]
can be easily found as
\[ R_{k+1+k+2}(\lambda) = \tilde{R}_{k+1}(\lambda) R_{k+1+k+2}(\lambda). \]
The \( R \)-matrix \( R_{k+1+k+2}(\lambda) \) has to be defined as
\( P_{k+1+k+2}(\lambda) \). The validity of (2.36) is very important for correct definition of the transfer matrix. We are able to define the Lax operator as the \( R \)-matrix
\[ L_{a,i}(\lambda) = R_{a,i}(\lambda) \]
and the monodromy matrix in the form (2.11). Commutativity of the transfer matrix is just a matter of proving
\[ R_{ab}(\mu) T_{a}(\lambda, \mu) T_{b}(\lambda) = T_{a}(\lambda) T_{b}(\lambda, \mu) R_{ab}(\mu). \]
The \( R \)-matrices (2.29), (2.34) for \( N = 2 \) are the basic building blocks for the XXZ spin chain. Let us write (2.37) for \( N = 2 \) as following
\[ R_{k+1}(\lambda) = \mathbb{1}_{2}^{\otimes(k-1)} \otimes R(\lambda) \otimes \mathbb{1}_{2}^{\otimes(L-k-1)}, \]
where \( R(\lambda) = (\lambda^{-1/2} \hat{R} - \lambda^{1/2} \hat{R}^{+}) \cdot P \). The matrix \( \hat{R} \) is given in (2.30) and \( R(\lambda) \) has the matrix form
which is important to write the commutation relations (2.39) in components. We see that the form (2.41) of the R-matrix is not symmetric to transposition, as usually appears in literature, cf. [4, 8] etc. We use the Drinfel’d—Reshetikhin twist to symmetrize it, cf. [18].

The R-matrix $R_{ab}(\lambda)$ acts in the tensor product of the auxiliary spaces $V_a \otimes V_b$. The monodromy matrix $T_a(\lambda)$ acts in $V_a \otimes \mathcal{H}$. Let $U$ be a diagonal matrix. It can be easily seen that $[U \otimes I_b + I_a \otimes U, R_{ab}(\lambda)] = 0$. We introduce the twisted R-matrix resp. the monodromy matrix

$$R_{ab}(\lambda) = (\lambda^V \otimes I_b) R_{ab}(\lambda) (\lambda^{-V} \otimes I_b),$$

$$T_a(\lambda) = (\lambda^U \otimes I_{\mathcal{H}}) T_a(\lambda) (\lambda^{-U} \otimes I_{\mathcal{H}}).$$

If

$$R_{ab}(\lambda) T_a(\mu) T_b(\mu) = T_b(\mu) T_a(\lambda) R_{ab}(\lambda)$$

(2.44)

is satisfied, then also

$$\tilde{R}(\lambda) = \begin{pmatrix}
\lambda^{1/2} q - \lambda^{-1/2} q^{-1} & 0 & 0 & 0 \\
0 & \lambda^{1/2} - \lambda^{-1/2} q^{-1} & 0 & 0 \\
0 & 0 & \lambda^{1/2} q - \lambda^{-1/2} q & 0 \\
0 & 0 & 0 & \lambda^{-1/2} q - \lambda^{-1/2} q^{-1}
\end{pmatrix}$$

(2.46)

which corresponds to the R-matrix appearing in [4, 8]. Moreover, it is easy to see that

$$\tilde{T}_a(\lambda) = R_{a,1}(\lambda) \tilde{R}_{a,2}(\lambda) \cdots \tilde{R}_{a,t}(\lambda).$$

(2.47)

It can also be seen that

$$\tilde{T}_a(\lambda) = \left( \begin{array}{c c c c}
\tilde{A}(\lambda) & \tilde{B}(\lambda) \\
\tilde{C}(\lambda) & \tilde{D}(\lambda)
\end{array} \right)$$

(2.48)

$$\tilde{A}(\lambda) = \frac{1}{2} \left( \begin{array}{c c c c}
\lambda^{1/2} - \lambda^{-1/2} & 0 & 0 & 0 \\
0 & \lambda^{1/2} - \lambda^{-1/2} q^{-1} & 0 & 0 \\
0 & 0 & \lambda^{1/2} q - \lambda^{-1/2} q & 0 \\
0 & 0 & 0 & \lambda^{-1/2} q - \lambda^{-1/2} q^{-1}
\end{array} \right)$$

(2.49)

$$\tilde{B}(\lambda) = (q - q^{-1}) \sigma^-, \quad \tilde{C}(\lambda) = (q - q^{-1}) \sigma^+, \quad \tilde{D}(\lambda) = \frac{1}{2} \left( \begin{array}{c c c c}
\lambda^{1/2} - \lambda^{-1/2} & 0 & 0 & 0 \\
0 & \lambda^{1/2} - \lambda^{-1/2} q^{-1} & 0 & 0 \\
0 & 0 & \lambda^{1/2} q - \lambda^{-1/2} q & 0 \\
0 & 0 & 0 & \lambda^{-1/2} q - \lambda^{-1/2} q^{-1}
\end{array} \right)$$

(2.50)

where $\sigma^\pm = 1/2(\sigma^x \pm i \sigma^y)$. The twisted R-matrix $R_{ab}(\lambda)$ resp. the twisted monodromy matrix $\tilde{T}_a(\lambda)$ will be used throughout the text.
2.3. Global Fundamental Commutation Relations

Global commutation relations are determined by Eq. (2.15) resp. (2.44) for XXX resp. XXZ in the tensor product $V_a \otimes V_b \otimes \mathcal{H}$. They are explicitly expressed by multiplication of matrices in the tensor product of the auxiliary spaces $V_a \otimes V_b$. After simple factorization, the $R$-matrices (2.9) resp. (2.46) can be written uniformly in the following way:

$$R_{ab}(\lambda) = \begin{pmatrix} f(\lambda) & 0 & 0 & 0 \\ 0 & 1 & g(\lambda) & 0 \\ 0 & g(\lambda) & 1 & 0 \\ 0 & 0 & 0 & f(\lambda) \end{pmatrix},$$ (2.53)

where for the XXX chain we have

$$f(\lambda) = \frac{\lambda + 1}{\lambda}, \quad g(\lambda) = \frac{1}{\lambda},$$ (2.54)

and for XXZ

$$f(\lambda) = \frac{\lambda - 1/2 - q^{1/2} q^{-1}}{\lambda - 1/2}, \quad g(\lambda) = \frac{q - q^{-1}}{\lambda - 1/2}.$$ (2.55)

We take the monodromy matrix (2.43) resp. (2.47) for the XXZ chain. For more comfort, we omit the tilde over the corresponding operators. The matrices $T_a(\lambda)$ resp. $T_b(\mu)$ take the form

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$ (2.56)

resp.

$$T_a(\mu) = \begin{pmatrix} A(\mu) & B(\mu) \\ A(\mu) & B(\mu) \\ C(\mu) & D(\mu) \\ C(\mu) & D(\mu) \end{pmatrix}.$$ (2.57)

Multiplying and comparing the left- and right-hand side of (2.15) resp. (2.45), we obtain the set of commutation relations. Comparing the matrix elements on the positions (1, 1), (1, 4), (4, 1), (4, 4) we obtain

$$[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = [D(\lambda), D(\mu)] = 0.$$ (2.58)

Comparing (2, 3) resp. (2, 2)

$$[B(\lambda), C(\mu)] = g(\lambda, \mu)(D(\mu)A(\lambda) - D(\lambda)A(\mu)),$$ (2.59)

$$[A(\lambda), D(\mu)] = g(\lambda, \mu)(C(\mu)B(\lambda) - C(\lambda)B(\mu)).$$ (2.60)

From a comparison of the matrix elements (1, 3), (3, 4), (2, 1) resp. (4, 3) we obtain

$$A(\mu)B(\lambda) = f(\lambda, \mu)B(\lambda)A(\mu) + g(\mu, \lambda)B(\mu)A(\lambda),$$
$$D(\mu)B(\lambda) = f(\mu, \lambda)B(\lambda)D(\mu) + g(\mu, \lambda)B(\mu)D(\lambda),$$
$$A(\mu)C(\lambda) = f(\mu, \lambda)C(\lambda)A(\mu) + g(\mu, \lambda)C(\mu)A(\lambda),$$
$$D(\mu)C(\lambda) = f(\mu, \lambda)C(\lambda)D(\mu) + g(\mu, \lambda)C(\mu)D(\lambda),$$

where for the XXX chain we have

$$f(\lambda, \mu) = f(\lambda - \mu) = \frac{\lambda - \mu + 1}{\lambda - \mu},$$ (2.61)
$$g(\lambda, \mu) = g(\lambda - \mu) = \frac{1}{\lambda - \mu},$$ (2.62)

and for XXZ

$$f(\lambda, \mu) = f(\lambda / \mu) = \frac{\mu - \sqrt{\mu - 1}}{\mu - 1},$$ (2.63)
$$g(\lambda, \mu) = g(\lambda / \mu) = \frac{\sqrt{\lambda - 1}}{\mu - 1}.$$ (2.64)

We see that $g(\lambda, \mu) = -g(\lambda, \mu)$.

2.4. Eigenstates of the Transfer Matrix

To uncover the spectrum of the transfer matrix $\tau(\lambda) = A(\lambda) + D(\lambda)$ is now the natural next step. In Section 2.3, we get four operators $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ under commutation relations (2.58)–(2.65). They generate an associative algebra. Relations (2.58)–(2.65) together with an assumption that the Hilbert space $\mathcal{H}$ has the structure of the Fock space are sufficient to find the spectrum $\tau(\lambda)$. From the beginning, we work on the Hilbert space $\mathcal{H} = (C^2)^{\otimes k}$, i.e. we choose a specific representation. But the content of this chapter is valid in general, i.e. also for other representations.

To uncover the Fock space structure in $\mathcal{H}$, let us find a pseudovacuum vector $|0\rangle \in \mathcal{H}$ such that $C(\lambda)|0\rangle = 0$ which is an eigenvector of the operators $A(\lambda)$ and $D(\lambda)$

$$A(\lambda)|0\rangle = \alpha(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = \delta(\lambda)|0\rangle.$$ (2.66)

Let us remind that there is a state $|0\rangle_k$ in each $h_k$ such that the corresponding Lax operator is of the upper triangular form

$$L_{a, k}(\lambda)|0\rangle_k = \begin{pmatrix} a(\lambda) & \text{something} \\ 0 & d(\lambda) \end{pmatrix}|0\rangle_k,$$ (2.67)

where $a(\lambda), d(\lambda)$ are the functions of the parameter $\lambda$.

We see that for XXX

$$a(\lambda) = \lambda + 1, \quad d(\lambda) = \lambda.$$ (2.68)
and for XXZ
\[ a(\lambda) = \lambda^{-1/2} q - \lambda^{-1/2} q^{-1}, \]
\[ d(\lambda) = \lambda^{-1/2} - \lambda^{1/2}. \]
(2.70)

The vector \(|0\rangle \in \mathcal{H}\) is of the form
\[ |0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_L. \]
(2.71)

In our particular representation, \( h_k = \mathbb{C}^2 \), we have
\[ |0\rangle_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]
It can be easily seen that
\[ T_a(\lambda)|0\rangle = \begin{pmatrix} a(\lambda) \text{ something} \\ 0 \\ d(\lambda) \end{pmatrix}|0\rangle. \]
(2.72)

We have found that the state \(|0\rangle \in \mathcal{H}\) satisfies
\[ C(\lambda)|0\rangle = 0, \quad A(\lambda)|0\rangle = \alpha(\lambda)|0\rangle, \]
\[ D(\lambda)|0\rangle = \delta(\lambda)|0\rangle, \]
where
\[ \alpha(\lambda) = \alpha(\lambda)^L, \quad \delta(\lambda) = d(\lambda)^L, \]
i.e., \(|0\rangle \in \mathcal{H}\) is an eigenstate of the transfer matrix \( \tau(\lambda) = A(\lambda) + D(\lambda) \).

Other eigenstates of the transfer matrix (2.13) are of the form
\[ |\{\lambda\}\rangle = |\lambda_1, \ldots, \lambda_M\rangle \]
\[ \equiv B(\lambda_1)B(\lambda_2) \cdots B(\lambda_M)|0\rangle \equiv |M\rangle. \]
(2.75)

They are called the Bethe vectors. For \( M \in \mathbb{N} \) we will call the Bethe vector \(|\lambda_1, \ldots, \lambda_M\rangle\) the \( M \)-magnon state. It turns out that there have to be some restrictions on the parameters \( \{\lambda\} = \{\lambda_1, \ldots, \lambda_M\} \) to get the eigenstates of the transfer matrix. First, we note that in view of commutativity of the operators \( B \) (2.58) we have
\[ |\lambda_1, \ldots, \lambda_M\rangle = |\sigma(\lambda_1, \ldots, \lambda_M)\rangle, \]
(2.76)

for any permutation \( \sigma \in S_M \) of \( \{\lambda_1, \ldots, \lambda_M\} \). Then, using (2.61), (2.73) and (2.76), we deduce
\[ A(\lambda_1, \ldots, \lambda_M) = A(\lambda_1)B(\lambda_1) \cdots B(\lambda_M)|0\rangle \]
\[ = (f(\lambda_1, \lambda_1)B(\lambda_1)A(\lambda_1) + g(\lambda_1, \lambda_1)B(\lambda_1)A(\lambda_1)) \]
\[ \times B(\lambda_2) \cdots B(\lambda_M)|0\rangle = (f(\lambda_1, \lambda) + g(\lambda_1, \lambda_1)P_{\lambda_1 \lambda}) \]
\[ \times B(\lambda_1)A(\lambda_1)B(\lambda_2) \cdots B(\lambda_M)|0\rangle \]
\[ = \ldots = \prod_{k=1}^{M} (f(\lambda_k, \lambda) + g(\lambda_k, \lambda_1)P_{\lambda_k \lambda}) |\lambda_1, \ldots, \lambda_M\rangle \]
\[ = \alpha(\lambda) \prod_{k=1}^{M} (f(\lambda_k, \lambda)|\lambda_1, \ldots, \lambda_M\rangle \]
\[ + \sum_{i=1}^{M} \Phi_i(\lambda_1, \lambda_1, \ldots, \lambda_M)|\lambda_1, \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots, \lambda_M\rangle, \]
where \( P_{\lambda \mu} \) is a permutation operator of the parameters \( \lambda \) and \( \lambda_k \) and it is clear that
\[ \Phi_i(\lambda, \lambda_1, \ldots, \lambda_M) \]
\[ = \alpha(\lambda_i)g(\lambda_i, \lambda) \prod_{k=1}^{M} f(\lambda_k, \lambda_i). \]
(2.78)

Since the left-hand side of (2.77) is symmetric under all permutations of \( \{\lambda_1, \ldots, \lambda_M\} \), we obtain
\[ \Phi(\lambda_1, \lambda_1, \ldots, \lambda_M) \]
\[ = \alpha(\lambda_i)g(\lambda_i, \lambda) \prod_{k=1}^{M} f(\lambda_k, \lambda_i), \quad \forall i = 1, \ldots, M. \]
(2.79)

In the same way by using (2.62), (2.73) and (2.76) we deduce
\[ D(\lambda)|\lambda_1, \ldots, \lambda_M\rangle = D(\lambda)B(\lambda_1) \cdots B(\lambda_M)|0\rangle \]
\[ = \prod_{k=1}^{M} (f(\lambda, \lambda_k) + g(\lambda, \lambda_k)P_{\lambda \lambda_k}) \delta(\lambda)|\lambda_1, \ldots, \lambda_M\rangle \]
\[ = \delta(\lambda) \prod_{k=1}^{M} f(\lambda, \lambda_k)|\lambda_1, \ldots, \lambda_M\rangle \]
\[ + \sum_{i=1}^{M} \Psi_i(\lambda, \lambda_1, \ldots, \lambda_M)|\lambda_1, \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots, \lambda_M\rangle, \]
(2.80)

where
\[ \Psi_i(\lambda, \lambda_1, \ldots, \lambda_M) = \delta(\lambda_i)g(\lambda_i, \lambda) \prod_{k=1}^{M} f(\lambda_i, \lambda_k), \]
\[ \forall i = 1, \ldots, M. \]

The combination of (2.77) and (2.80) gives that \( |\lambda_1, \ldots, \lambda_M\rangle \) is the eigenvector of the transfer matrix (2.13) \( \tau(\lambda) = A(\lambda) + D(\lambda) \)
\[ (A(\lambda) + D(\lambda))|\lambda\rangle = \Lambda(\lambda, \{\lambda\})|\lambda\rangle, \]
\[ \Lambda(\lambda, \{\lambda\}) = \alpha(\lambda) \prod_{i=1}^{M} f(\lambda_i, \lambda) + \delta(\lambda) \prod_{i=1}^{M} f(\lambda_i, \lambda), \]
\[ \forall i = 1, \ldots, M. \]
(2.82)

if the set of parameters \( \{\lambda_1, \ldots, \lambda_M\} \) satisfies the so-called Bethe equations:
\[ \Phi(\lambda_1, \lambda_1, \ldots, \lambda_M) + \Psi_i(\lambda_1, \lambda_1, \ldots, \lambda_M) = 0 \]
\[ \Rightarrow \alpha(\lambda_i)g(\lambda_i, \lambda) \prod_{k=1}^{M} f(\lambda_k, \lambda_i) \]
\[ + \delta(\lambda_i)g(\lambda_i, \lambda) \prod_{k=1}^{M} f(\lambda_k, \lambda) = 0. \]
3. GENERALIZATION OF THE TWO-COMPONENT MODEL

In the literature, cf. [3, 4] etc., there appears a so-called two-component model. The two-component model was introduced to avoid problems with computation of correlation functions for local operators attached to some site $x$ of the chain in the algebra of global operators (2.12) $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ defined on the chain as a whole.

We divide the chain $[1, \ldots, L]$ into two components $[1, \ldots, x]$ and $[x + 1, \ldots, L]$. Then we have the Hilbert space splitted into two parts $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{H}_1 = h_1 \otimes \ldots \otimes h_1$ and $\mathcal{H}_2 = h_{x+1} \otimes \ldots \otimes h_L$. We see that pseudovacuum $|0\rangle \in \mathcal{H}$ is of the form $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$ where $|0\rangle_1 \otimes \mathcal{H}_1$ and $|0\rangle_2 \in \mathcal{H}_2$. We define on $V_\alpha \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$ the monodromy matrix for each component

$$T_1(\lambda) = L_{a,1}(\lambda) \cdots L_{a,x}(\lambda) = \left( A_1(\lambda) \ B_1(\lambda) \ \ C_1(\lambda) \ D_1(\lambda) \right),$$

resp.

$$T_2(\lambda) = L_{a,x+1}(\lambda) \cdots L_{a,L}(\lambda) = \left( A_2(\lambda) \ B_2(\lambda) \ \ C_2(\lambda) \ D_2(\lambda) \right).$$

Each of these monodromy matrices satisfies exactly the same commutation relations (2.15) as the original undivided monodromy matrix (2.11). Moreover, we have

$$A_1(\lambda)|0\rangle_j = \alpha_1(\lambda)|0\rangle_j, \quad D_1(\lambda)|0\rangle_j = \delta_1(\lambda)|0\rangle_j,$$

$$C_1(\lambda)|0\rangle_j = 0.$$  

Operators corresponding to different components mutually commute. From construction, it is easy to see that

$$\alpha(\lambda) = \alpha_1(\lambda)\alpha_2(\lambda), \quad \delta(\lambda) = \delta_1(\lambda)\delta_2(\lambda).$$

For the whole chain $[1, \ldots, L]$ the full monodromy matrix $T$ is

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = T_1(\lambda)T_2(\lambda) = \begin{pmatrix} A_1(\lambda)A_2(\lambda) + B_1(\lambda)C_2(\lambda) & A_1(\lambda)B_2(\lambda) + B_1(\lambda)D_2(\lambda) \\ C_1(\lambda)A_2(\lambda) + D_1(\lambda)C_2(\lambda) & C_1(\lambda)B_2(\lambda) + D_1(\lambda)D_2(\lambda) \end{pmatrix},$$

and the $M$-magnon state is represented in the form

$$|\lambda_1, \ldots, \lambda_M\rangle = \prod_{k=1}^M B(\lambda_k)|0\rangle = \prod_{k=1}^M (A_1(\lambda_k)B_2(\lambda_k) + B_1(\lambda_k)C_2(\lambda_k))|0\rangle_1 \otimes |0\rangle_2.  \quad (3.6)$$

The beautiful result of Izergin and Korepin [3] states that the Bethe vectors of the full model can be expressed in terms of the Bethe vectors of its components. To obtain this expression, we should commute in (3.6) all operators $A_1(\lambda_k)$ and $D_1(\lambda_k)$ to the right with the help of (2.61) and (2.62) and then use (3.3). Finally, we obtain the following result [3].
Proposition 1. An arbitrary Bethe vector corresponding to the full system can be expressed in terms of the Bethe vectors of the first and second component. Let $I = \{\lambda_1, \ldots, \lambda_M\}$ be a finite set of spectral parameters. To concise notation below, we will consider the set $I$ as a finite set of indices $I = \{1, \ldots, M\}$, then

$$\prod_{k \in I} B(\lambda_k) |0\rangle = \sum_{I_1 \cup I_2} \prod_{k_1 \in I_1} (\delta_2(\lambda_{k_1}) B_1(\lambda_{k_1})) \prod_{k_2 \in I_2} (\alpha_1(\lambda_{k_2}) B_2(\lambda_{k_2})) |0\rangle_1 \otimes |0\rangle_2 \prod_{k_1, k_2 \in I_2} f(\lambda_{k_1}, \lambda_{k_2}),$$

(3.7)

where $f(\lambda_{k_1}, \lambda_{k_2})$ is defined in (2.65) resp. (2.66) and the summation is performed over all divisions of the index set $I$ into two disjoint subsets $I_1$ and $I_2$ where $I = I_1 \cup I_2$.

Proof. The proof is just a matter of commutation relations (2.15) resp. (2.61)–(2.65). We use induction on the number of elements $M$ of the index set $I$. We see that

$$B(\lambda) |0\rangle = (A_1(\lambda) B_2(\lambda) + B_1(\lambda) D_2(\lambda))$$

(3.8)

which is exactly the formula (3.7) for $M = 1$. Let us suppose that (3.7) is valid for the index set $I = \{1, \ldots, M-1\}$. Then we have

$$B(\lambda) \prod_{k \in I} B(\lambda_k) |0\rangle = (A_1(\lambda) B_2(\lambda) + B_1(\lambda) D_2(\lambda))$$

$$\times \sum_{I_1 \cup I_2} \prod_{k_1 \in I_1} (\delta_2(\lambda_{k_1}) B_1(\lambda_{k_1})) \prod_{k_2 \in I_2} (\alpha_1(\lambda_{k_2}) B_2(\lambda_{k_2})) |0\rangle_1 \otimes |0\rangle_2 \prod_{k_1, k_2 \in I_2} f(\lambda_{k_1}, \lambda_{k_2})$$

(3.9)

\[= \sum_{I_1 \cup I_2} \left( A_1(\lambda) \prod_{k \in I_1} \delta_2(\lambda_k) B_1(\lambda_k) \right) \left( B_2(\lambda) \prod_{k_2 \in I_2} \alpha_1(\lambda_{k_2}) B_2(\lambda_{k_2}) \right) |0\rangle_1 \otimes |0\rangle_2 \prod_{k_1, k_2 \in I_2} f(\lambda_{k_1}, \lambda_{k_2}) + \sum_{I_1 \cup I_2} \left( B_1(\lambda) \prod_{k \in I_1} \delta_2(\lambda_k) B_1(\lambda_k) \right) \left( D_2(\lambda) \prod_{k_2 \in I_2} \alpha_1(\lambda_{k_2}) B_2(\lambda_{k_2}) \right) |0\rangle_1 \otimes |0\rangle_2 \prod_{k_1, k_2 \in I_2} f(\lambda_{k_1}, \lambda_{k_2}). \]

In the first sum we use (2.77) to commute $A_1(\lambda)$ with $\prod_{k \in I_1} B_1(\lambda_k)$ resp. (2.80) to commute $D_2(\lambda)$ with $\prod_{k \in I_2} B_2(\lambda_k)$ in the second sum. Using just the second term in (2.77) we get for the first sum:

$$\sum_{I_1 \cup I_2} \sum_{k \in I_1} g(\lambda, \lambda_k) \alpha_1(\lambda_k) \delta_2(\lambda_k) B_1(\lambda_k) B_2(\lambda) \prod_{j \in I_2} \delta_2(\lambda_j) B_1(\lambda_j) \prod_{i \in I_2} \alpha_1(\lambda_i) B_2(\lambda_i) |0\rangle_1 \otimes |0\rangle_2 \prod_{k_1, k_2 \in I_2} f(\lambda_{k_1}, \lambda_{k_2}) \prod_{i \in I_1 \setminus k} f(\lambda_i, \lambda_k).$$

(3.10)

Similarly, using just the second term in (2.80) we get for the second sum:

$$\sum_{I_1 \cup I_2} \sum_{k \in I_2} g(\lambda_k, \lambda) \alpha_1(\lambda_k) \delta_2(\lambda_k) B_1(\lambda_k) B_2(\lambda) \prod_{j \in I_1} \delta_2(\lambda_j) B_1(\lambda_j) \prod_{i \in I_1} \alpha_1(\lambda_i) B_2(\lambda_i) |0\rangle_1 \otimes |0\rangle_2 \prod_{k_1, k_2 \in I_2} f(\lambda_{k_1}, \lambda_{k_2}) \prod_{i \in I_1 \setminus k} f(\lambda_i, \lambda_k)$$

(3.11)

$$\times \left( \prod_{i \in I_1 \setminus k} \delta_2(\lambda_i) B_1(\lambda_i) \right) \left( \prod_{i \in I_2} \alpha_1(\lambda_i) B_2(\lambda_i) \right) |0\rangle_1 \otimes |0\rangle_2 \prod_{k_1, k_2 \in I_2} f(\lambda_{k_1}, \lambda_{k_2}) \prod_{i \in I_2 \setminus k} f(\lambda_i, \lambda_k).$$
where we introduced new partition $I_1' = I_1 \cup \{k\}$ and $I_2' = I_2 \setminus \{k\}$. We see that (3.11) is almost the same as (3.10) with only one difference. In (3.10) there appears a factor $g(\lambda, \lambda_k)$ and in (3.11) there appears $g(\lambda_k, \lambda)$. Using the fact that $g(\lambda, \lambda_k) = -g(\lambda_k, \lambda)$, cf. (2.65), we see that these two sums cancel each other. Therefore, only the first parts of (2.77) and (2.80) contribute to (3.9). We get

$$B(\lambda) \prod_{k \in I} B(\lambda_k) |0\rangle = \sum_{I_1, I_2} \left( \alpha_1(\lambda) \prod_{k_1 \in I_1} f(\lambda_{k_1}, \lambda) \delta_2(\lambda_{k_2}) B_1(\lambda_{k_2}) \right) \times \left( B_2(\lambda) \prod_{k_2 \in I_2} \alpha_1(\lambda_{k_2}) B_2(\lambda_{k_2}) |0\rangle_1 \otimes |0\rangle_2 \right) \prod_{k_1 \in I_1} f(\lambda_{k_1}, \lambda_k) \delta_2(\lambda_{k_2}) B_1(\lambda_{k_2})$$

which proves the induction.

This result can be straightforwardly generalized to an arbitrary number of components $N \leq L$.

$$\prod_{k \in I} B(\lambda_k) |0\rangle = \sum_{I_1, I_2, \ldots, I_N} \prod_{k_1 \in I_1} \prod_{k_2 \in I_2} \cdots \prod_{k_N \in I_N} \left( \alpha_1(\lambda_{k_1}) \delta_2(\lambda_{k_2}) f(\lambda_{k_2}, \lambda) \right) B_1(\lambda_{k_1}) B_2(\lambda_{k_2}) |0\rangle_1 \otimes \cdots \otimes B_N(\lambda_{k_N}) |0\rangle_N,$$

where summation is performed over all divisions of the set $I$ into its $N$ mutually disjoint subsets $I_1, I_2, \ldots, I_N$.

**Proof.** The proof is simply performed by induction on the number of components $N$ and by using (3.7). For $N=2$ is (3.13) just (3.7). Let us suppose that (3.13) is valid for some $N < L$ and make induction step to $N+1$. The chain $[1, \ldots, L]$ is divided into $N$ subchains $[1, \ldots, x_1], [x_1+1, \ldots, x_2], \ldots, [x_N-1+1, \ldots, L]$. Let us divide the last interval, if possible, into two subchains $[x_N-1+1, \ldots, x_N]$ and $[x_N+1, \ldots, L]$ and apply (3.7) to set of $B$ operators $\prod_{k \in I_N} B_N(\lambda_{k_N}) |0\rangle_N$. We get

$$\prod_{k \in I_N} B_N(\lambda_{k_N}) |0\rangle_N = \sum_{I_1' \cup I_2' \cup \cdots \cup I_{N+1}} \prod_{k \in I_N} \delta_{N+1}(\lambda_{k_1}) \alpha_N^*(\lambda_{k_N}, \lambda_{k_{N+1}}) f(\lambda_{k_N}, \lambda_{k_{N+1}})$$

where the sum goes over all divisions of $I_N$ into its two disjoint subsets $I_N$ and $I_{N+1}$ such that $I_N = I_N' \cup I_{N+1}$ and operators $B_N'(\lambda)$ and $B_N'(\lambda)$ act on the new subchains $[x_N-1+1, \ldots, x_N]$ resp. $[x_N+1, \ldots, L]$; the same for $\alpha_N'(\lambda)$ resp. $\delta_N'(\lambda)$ and the pseudovacuum vectors $|0\rangle_N'$ resp. $|0\rangle_{N+1}'$. Let us remind that

$$\prod_{k \in I_N} = \prod_{k \in I_N} \prod_{k \in I_N'} \prod_{k \in I_{N+1}}.$$

Inserting (3.14) into induction assumption (3.13) we get

$$\prod_{k \in I} B(\lambda_k) |0\rangle = \sum_{I_1, I_2, \ldots, I_N} \prod_{k_1 \in I_1} \prod_{k_2 \in I_2} \cdots \prod_{k_N \in I_N} \left( \alpha_1(\lambda_{k_1}) \delta_2(\lambda_{k_2}) f(\lambda_{k_2}, \lambda) \right) \times B_1(\lambda_{k_1}) |0\rangle_1 \otimes B_2(\lambda_{k_2}) |0\rangle_2 \otimes \cdots \otimes B_N'(\lambda_{k_N}) |0\rangle_N' \otimes B_N'(\lambda_{k_{N+1}}) |0\rangle_{N+1}'$$

which proves the induction.
4. BETHE VECTORS

In this section, we will see that computation of the Bethe vectors in the algebraic Bethe ansatz is just a matter of using Proposition 2. By assumption we have a chain of length \( L \). Let us divide it into \( L \) components, i.e. into \( L \) subchains of length one (1-chains). Using proposition 2 we get for the \( M \)-magnon (Bethe vector) with \( M \leq L \):

\[
\prod_{k=1}^{M} B(\lambda_k)[0] = \sum_{1 \leq n_1 < \ldots < n_M \leq L} \left| \sum_{\sigma_{ij}} \prod_{k=1}^{M} \prod_{1 \leq i,j \leq L} (\alpha(\lambda_k) \delta(\lambda_k f(\lambda_k, \lambda_k))) \right|^{M} 
\times B_{n_1}(\lambda_1) B_{n_2}(\lambda_2) \cdots B_{n_M}(\lambda_M) [0_1 \otimes 0_2 \otimes \cdots \otimes 0_L].
\]

(4.1)

It can be easily seen that for 1-chain, i.e. for a chain with Hilbert space \( \mathbb{C}^2 \),

\[
B(\lambda)B(\mu)[0] = 0.
\]

(4.2)

Therefore, the sum over all divisions of \( \{1, \ldots, M\} \) into \( L \) subsets contains just divisions into subsets containing at most one element, i.e. \( |f| = 0 \), \( \lambda \). Moreover, only \( M \) of them is nonempty, let us denote them \( I_{n_1}, I_{n_2}, \ldots, I_{n_M} \). We have to sum over all possible combinations of such sets, i.e. over all \( M \)-tuples \( n_1 < n_2 < \ldots < n_M \). Next, we have to sum over all distributions of the parameters \( \lambda_1, \lambda_2, \ldots, \lambda_M \) into the sets \( I_{n_1}, \ldots, I_{n_M} \). We can simplify our life assuming that \( \lambda_j \in I_{n_j} \). Then, by summing over all permutations \( \sigma \in S_M \) of \( \{\lambda_1, \ldots, \lambda_M\} \), we get exactly all the other distributions.

Let us study what happens to the coefficient

\[
\prod_{k=1}^{M} B(\lambda_k)[0] = \sum_{1 \leq n_1 < \ldots < n_M \leq L} \left| \sum_{\sigma_{ij}} \prod_{k=1}^{M} \prod_{1 \leq i,j \leq L} (\alpha(\lambda_k) \delta(\lambda_k f(\lambda_k, \lambda_k))) \right|^{M} 
\times B_{n_1}(\lambda_1) B_{n_2}(\lambda_2) \cdots B_{n_M}(\lambda_M) [0_1 \otimes 0_2 \otimes \cdots \otimes 0_L].
\]

(4.6)

For 1-chain, it holds that \( B(\lambda) = B \) is parameter independent. Moreover, eigenvalues \( \alpha(\lambda) = a(\lambda), \delta(\lambda) = d(\lambda) \) are still the same for all components \( i = 1, \ldots, L \), where \( a(\lambda) \) and \( d(\lambda) \) are defined in (2.69) resp. (2.70). We get

\[
\prod_{k=1}^{M} B(\lambda_k)[0] = \sum_{1 \leq n_1 < \ldots < n_M \leq L} \left| \sum_{\sigma_{ij}} \prod_{k=1}^{M} \prod_{1 \leq i,j \leq L} (\alpha(\lambda_k) \delta(\lambda_k f(\lambda_k, \lambda_k))) \right|^{M} 
\times B_{n_1}(\lambda_1) B_{n_2}(\lambda_2) \cdots B_{n_M}(\lambda_M) [0_1 \otimes 0_2 \otimes \cdots \otimes 0_L].
\]

(4.7)
5. INHOMOGENEOUS BETHE ANSatz

We start with the inhomogeneous monodromy matrix

\[ T^\delta_a(\lambda) \]

\[ = L_{a,1}(\lambda + \xi_1) \cdots L_{a,x}(\lambda + \xi_x) \cdots L_{a,L}(\lambda + \xi_L), \]

where \( L_{a,j}(\lambda) \) are the Lax operators defined in (2.3) resp. (2.46) depending on whether we consider XXX or XXZ spin chain. Let us remark that for the XXZ chain the monodromy matrix is of the form

\[ \begin{pmatrix} \mathcal{A}^\delta(\lambda) & \mathcal{B}^\delta(\lambda) \\ \mathcal{C}^\delta(\lambda) & \mathcal{D}^\delta(\lambda) \end{pmatrix} \]

where, again, the operators \( \mathcal{A}^\delta(\lambda), \mathcal{B}^\delta(\lambda), \mathcal{C}^\delta(\lambda) \) and \( \mathcal{D}^\delta(\lambda) \) act in \( \mathcal{H} = h_1 \otimes \cdots \otimes h_L \). Acting on the pseudovacuum vector \( \langle 0 \rangle \in \mathcal{H} \) we get

\[ \mathcal{A}^\delta(\lambda)\langle 0 \rangle = \alpha^\delta(\lambda)\langle 0 \rangle, \]

\[ \mathcal{D}^\delta(\lambda)\langle 0 \rangle = \delta^\delta(\lambda)\langle 0 \rangle, \]

\[ \mathcal{C}^\delta(\lambda)\langle 0 \rangle = 0, \]

where

\[ \alpha^\delta(\lambda) = \alpha(\lambda + \xi_1)\alpha(\lambda + \xi_2)\cdots \alpha(\lambda + \xi_L), \]

\[ \delta^\delta(\lambda) = d(\lambda + \xi_1)d(\lambda + \xi_2)\cdots d(\lambda + \xi_L). \]

In what follows, we will use the notation connected with the XXX chain but we can do for the XXZ chain the same as well.

Expressing \( T^\delta_a(\lambda) \) in the auxiliary space \( V_a \) we get

\[ T^\delta_a(\lambda) = \mathcal{A}^\delta(\lambda) \mathcal{B}^\delta(\lambda) \mathcal{C}^\delta(\lambda) \mathcal{D}^\delta(\lambda) \]

We have

\[ \mathcal{A}^\delta(\lambda)\langle 0 \rangle = \alpha^\delta(\lambda)\langle 0 \rangle, \]

\[ \mathcal{D}^\delta(\lambda)\langle 0 \rangle = \delta^\delta(\lambda)\langle 0 \rangle. \]

A very important property of the inhomogeneous chain is that its operators \( \mathcal{A}^\delta(\lambda), \mathcal{B}^\delta(\lambda), \mathcal{C}^\delta(\lambda) \) and \( \mathcal{D}^\delta(\lambda) \) satisfy the same fundamental commutation relations as the homogeneous chain (2.58)–(2.65), i.e.

\[ \text{commutation relations are independent of the inhomogeneity parameters } \xi. \]

Therefore, an analogy of propositions 1 and 2 can be easily formulated.

**Proposition 3.** Let \( N \leq L \). An arbitrary Bethe vector of the full system can be expressed in terms of the Bethe vectors of its \( N \) components

\[ \prod_{k = 1}^{N} \mathcal{B}^\delta(\lambda_k)\langle 0 \rangle = \sum_{I_1 \cup \cdots \cup I_N = \{1, \ldots, L\}} \prod_{k \in I_1} \cdots \prod_{k \in I_N} (\alpha^\delta(\lambda_k)\delta^\delta(\lambda_k))(\lambda_k; \lambda_k) \]

\[ \times \mathcal{B}^\delta_1(\lambda_{k_1})\mathcal{B}^\delta_2(\lambda_{k_2})\cdots \mathcal{B}^\delta_N(\lambda_{k_N})\langle 0 \rangle. \]
\[
\prod_{k=1}^{M} B^\pm(\lambda_k)|0\rangle = \sum_{1 \leq n_1 < \ldots < n_M \leq L} \sum_{\Sigma} \sigma_{\lambda} \prod_{j=1}^{n_1-1} \prod_{i=1}^{\lambda_j} \prod_{j=n_1+1}^{M} \prod_{i=1}^{\lambda_j} \prod_{i=1}^{j-1} f(\lambda_\mu, \lambda_j)
\times B_{n_1}^\pm(\lambda_1) \cdots B_{n_M}^\pm(\lambda_M)|0\rangle = \sum_{1 \leq n_1 < \ldots < n_M \leq L} \sum_{\Sigma} \sigma_{\lambda} \prod_{j=1}^{n_1-1} \prod_{i=1}^{\lambda_j} \prod_{j=n_1+1}^{M} \prod_{i=1}^{\lambda_j} \prod_{i=1}^{j-1} f(\lambda_\mu, \lambda_j) B_{n_1} \cdots B_{n_M}|0\rangle
\]

where again the \(B\)-operators \(B_{n_j}(\lambda) = B_{n_j}\) are parameter independent for 1-chains.

6. FREE FERMIONS

In this Section we recall the well-known construction [19] of \(L\)-dimensional free fermion algebra in terms of the Pauli matrices. First, in \(\mathbb{C}^2\) one can easily define 1-dimensional fermions using the properties of the Pauli matrices. Let

\[
\psi = \sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y), \quad \overline{\psi} = \sigma^- = \frac{1}{2}(\sigma^x - i\sigma^y).
\]

(6.1)

Thus defined \(\psi, \overline{\psi}\) satisfy the fermionic relations

\[
[\psi, \overline{\psi}]_+ = \mathbb{1}, \quad \psi^2 = 0, \quad \overline{\psi}^2 = 0.
\]

(6.2)

For a tensor product of \(L\) copies of \(\mathbb{C}^2\) we can define fermions as

\[
\psi_k \equiv \prod_{j=1}^{k-1} \sigma_j^+ \psi_k, \quad \overline{\psi}_k \equiv \prod_{j=1}^{k-1} \sigma_j^- \overline{\psi}_k, \quad (k = 1, \ldots, L),
\]

(6.3)

where \(\sigma_j^\alpha\) denotes the sigma matrix attached to the \(j\)-th vector space, i.e.,

\[
\sigma_j^\alpha = \mathbb{1}^\otimes(j-1) \otimes \sigma^\alpha \otimes \mathbb{1}^\otimes(L-j).
\]

(6.4)

This concise notation is used throughout the whole text. Commutation relations for the fermions (6.3) are of the form

\[
[\overline{\psi}_k, \psi_j]_\pm = \delta_{kj} \mathbb{1}, \quad [\overline{\psi}_k, \overline{\psi}_j]_\pm = 0, \quad [\psi_k, \psi_j]_\pm = 0.
\]

(6.5)

It is a straightforward task to check the following identities:

\[
\overline{\psi}_{k+1} \psi_k + \psi_k \overline{\psi}_{k+1} + \psi_k \psi_{k+1} + \psi_{k+1} \psi_k = \sigma_k^\tau \sigma_{k+1}^\tau,
\]

(6.6)

\[
\psi_{k+1} \overline{\psi}_k + \overline{\psi}_k \psi_{k+1} + \psi_{k+1} \overline{\psi}_k + \overline{\psi}_k \psi_{k+1} = \sigma_k^\tau \sigma_{k+1}^\tau.
\]

(6.7)

\[
\sigma_k^\tau \sigma_{k+1}^\tau = \sigma_k^\tau \sigma_{k+1}^\tau.
\]

(6.8)

\[
(1 - 2\psi_k \psi_k)(1 - 2\overline{\psi}_{k+1} \psi_{k+1}) = \sigma_k^\tau \sigma_{k+1}^\tau.
\]

(6.9)

7. FERMIONIC REALIZATION OF XXX

We have seen that our definition (2.3) of the Lax operator \(L_\alpha(\lambda)\) led to expression (2.4) which is in fact identical to the definition of the \(R\)-matrix (2.8). Let us remind that the identity operator \(\mathbb{1}\) is a member of the algebra of fermions because of commutation relation (6.5). Therefore, from expression (2.4) for \(L_\alpha(\lambda)\) we see that it remains to know a fermionic realization only for the permutation operator \(P_{\alpha}.\)

Let us start with the permutation operator \(P_{k, k+1}\) which permutes the neighboring vector spaces \(h_k, h_{k+1}\). Due to identities (6.6)–(6.9) and definition of permutation operator (2.5) it is straightforward to check that

\[
P_{k, k+1} = \mathbb{1} + \overline{\psi}_{k+1} \psi_k - \overline{\psi}_k \psi_{k+1} - \psi_k \overline{\psi}_{k+1} + \psi_{k+1} \overline{\psi}_k.
\]

(7.1)

Problems appear when we try to find a fermionic realization of the permutation operator \(P_{j, k}\) in non-neighboring vector spaces \(h_j, h_k\) where \(j < k - 1\). It turns out that \(P_{j, k}\) becomes non-local in terms of fermions. Using properties of the Pauli matrices, \(P_{j, k}\) could be rewritten as

\[
P_{j, k} = \frac{1}{2}((\mathbb{1} - \sigma_j^\tau \sigma_k^\tau) + (\sigma_j^\tau \sigma_k^\tau + \sigma_j^\tau \sigma_k^\tau)).
\]

(7.2)

The first part is local even in terms of fermions

\[
\frac{1}{2}((\mathbb{1} - \sigma_j^\tau \sigma_k^\tau) = \overline{\psi}_j \psi_k - \overline{\psi}_k \psi_j + \overline{\psi}_j \psi_k \overline{\psi}_j) = \overline{\psi}_j \psi_k - \overline{\psi}_k \psi_j + 2 \overline{\psi}_j \psi_k \overline{\psi}_j.
\]

(7.3)
Moreover, we know, due to Eqs. (7.1) and (7.7), how to express the monodromy matrix (2.11) in terms of such nonlocal operators. We need to avoid the nonlocality.

The natural next step is to express the monodromy matrix (2.11) in terms of fermions. For this purpose we rewrite (7.6) as

\[
\hat{R}_{a,b}(\lambda) = \hat{R}_{a,b}(\nu) P_{a,b} \lambda \]

which satisfies

\[
\lambda(\psi_{k+1} \psi_{j+1} + \psi_{k} \psi_{j+1} - \psi_{k} \psi_{j}) + 2 \psi_{k} \psi_{j+1} \psi_{k+1} + 2 \psi_{k} \psi_{j+1} \psi_{k+1}.
\]

We substitute \( L_{a,i}(\lambda) = \hat{R}_{a,b}(\nu) P_{a,i} \), in the monodromy matrix (2.11) and obtain a very convenient expression

\[
T_{a}(\lambda) = L_{a,1}(\lambda)L_{a,2}(\lambda)\ldots L_{a,L}(\lambda)
\]

acts nontrivially only in the quantum spaces \( \mathcal{H} = h_{1} \otimes \cdots \otimes h_{L} \) and is a scalar in the auxiliary space \( V_{a} \).

Moreover, we know, due to Eqs. (7.1) and (7.7), how to express \( X(\lambda) \) in terms of fermions.

What remains is to express \( \hat{R}_{a,1} \) and \( P_{a,1} \) as the \( 2 \times 2 \) matrix in the auxiliary space \( V_{a} \). The permutation matrix (2.5) can be rewritten as

\[
P_{a,1} = \frac{1}{2}(\otimes 0 \sigma^x + \sigma^x \otimes \sigma^x + \sigma^x \otimes \sigma^x + \sigma^x \otimes \sigma^x)
\]

and using (6.3) and (6.8) we get

\[
T_{a}(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \]

where \( N_{1} = \psi_{1} \psi_{1} \). For \( \hat{R}_{a,1}(\lambda) \), we get

\[
\hat{R}_{a,1}(\lambda) = \begin{pmatrix} \lambda - \lambda N_{1} & \lambda \psi_{1} \\ \lambda & \lambda N_{1} + \lambda \end{pmatrix}
\]

Using (7.11) and (7.12), the monodromy matrix (7.8) can be written in the following form:

\[
T_{a}(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \]

where

\[
B(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \]

and

\[
C(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.
\]
BETHE VECTORS OF XXX

The goal of our text is to find expression for the Bethe vectors (2.75)

\[ |\hat{\lambda}_1, \ldots, \hat{\lambda}_M> = B(\lambda_1) \ldots B(\lambda_M)|0> \] (8.1)

For this purpose, the fermionic realization (7.15) of the creation operator \( B(\lambda) \) is very convenient. The operator \( X(\lambda) = \hat{R}_{12}(\lambda) \ldots \hat{R}_{L-1,L}(\lambda) P_{L-1,L} \ldots P_{12} \) can be written in terms of fermions due to Eqs. (7.11) and (7.12). From Eq. (2.71), our special representation, where \( |0> \) is defined by (7.2), and the definition of free fermions (6.3) we can see that

\[ \psi_k|0> = 0, \] (8.2)

for all \( k = 1, \ldots, L \).

If we were to write \( B(\lambda) \) in the normal form, our work would be simple. Unfortunately, it seems a rather difficult task. Instead, we have to use the "weak approach", i.e. to apply \( B(\lambda) \) to the pseudovacuum \( |0> \)

\[ B(v)B(\mu)B(\lambda)|0> = n(v)n(\mu)n(\lambda) \]

From the results (8.3), (8.5) and (8.6) we can conjecture that the general \( M \)-magnon state is of the form

\[ |\hat{\lambda}_1, \ldots, \hat{\lambda}_M> = B(\lambda_1) \ldots B(\lambda_M)|0> \]

\[ \prod_{i=1}^{M} n(\lambda_i) \sum_{1 \leq k_1 < \ldots < k_M \leq L} \sum_{\sigma} \sigma_\lambda \times \prod_{i=1}^{M} \left( \begin{array}{c} \lambda_i - \lambda_j + 1 \\ \lambda_i - \lambda_j \\ \end{array} \right) \psi_{k_1} \cdots \psi_{k_M}|0> \] (8.7)

\[ \equiv \prod_{i=1}^{M} n(\lambda_i) \prod_{i<j} \frac{1}{\lambda_i - \lambda_j} \sum_{1 \leq k_1 < \ldots < k_M \leq L} \sum_{\sigma} (-1)^{\sigma(\lambda_i)} \sigma_\lambda \prod_{i<j} \left( \begin{array}{c} \lambda_i - \lambda_j + 1 \\ \lambda_i - \lambda_j \\ \end{array} \right) \psi_{k_1} \cdots \psi_{k_M}|0> , \]

where \( \sigma_\lambda \) is a permutation of the parameters \( \lambda_1, \ldots, \lambda_M \), \( p(\sigma_\lambda) = 0, 1 \text{ (mod 2)} \) is the parity of the permutation \( \sigma_\lambda \) and \( \sum_{\sigma_\lambda \in S_M} \) is the sum over all such permutations. However, in the light of previous results this is no more a conjecture but a special representation of (4.7).


did try to commute the fermions \( \psi_k \) to the left and see what happens.

The details of this section are postponed to Appendix A. Here, we only write down the results.

We get the 1-magnon simply by application of (7.15) to pseudovacuum (2.71)

\[ B(\mu)|0> = n(\mu) \sum_{k=1}^{L} [\mu]_{k}^\psi_{k}|0>, \] (8.3)

where we use the concise notation

\[ [\mu] = \frac{\mu^{L} + 1}{\mu^{L}}, \] (8.4)

The 2-magnon state is of the form

\[ B(\mu)B(\lambda)|0> = n(\mu)n(\lambda) \sum_{1 \leq k \leq L} \sum_{\lambda} \left( \begin{array}{c} \lambda - \mu + 1 \\ \lambda - \mu \\ \end{array} \right) \psi_{k}\psi_{k+1}|0> \] (8.5)

and the 3-magnon state is

\[ \prod_{i<j} \left( \begin{array}{c} \lambda_i - \lambda_j + 1 \\ \lambda_i - \lambda_j \\ \end{array} \right) \psi_{k_i} \psi_{k_j}|0> \] (8.6)

FERMIONIC REALIZATION OF XXZ

Substituting (6.6)–(6.9) into (2.30) gives a fermionic representation for the generators (2.32) of the Hecke algebra

\[ \hat{R}_{k,k+1}^{(q)} = \psi_{k+1}\psi_{k} + \psi_{k}\psi_{k+1} - q\psi_{k}\psi_{k} - q^{-1}\psi_{k+1}\psi_{k+1} + (q + q^{-1})\psi_{k}\psi_{k+1}\psi_{k+1} \] (9.1)

In the following, we will use the baxterized \( R \)-matrix (2.34) multiplied by \( \mu^{1/2} \) for a simpler formula, which is of the form

\[ \hat{R}_{k,k+1}(\mu) = (1 - \mu) \]

\[ \times \left( \begin{array}{c} \lambda_i - \lambda_j + 1 \\ \lambda_i - \lambda_j \\ \end{array} \right) \psi_{k_i} \cdots \psi_{k_M}|0> \] (9.2)

\[ \equiv \hat{R}_{\lambda_1 \psi_{k_1}} \hat{R}_{\lambda_2 \psi_{k_2}} \cdots \hat{R}_{\lambda_M \psi_{k_M}} \] (9.3)

\[ T_\sigma(\mu) \]

\[ = \hat{R}_{\lambda_1 \psi_{k_1}} \hat{R}_{\lambda_2 \psi_{k_2}} \cdots \hat{R}_{\lambda_M \psi_{k_M}} \psi_{k_M} \cdots \psi_{k_1} \] (9.3)
The fermionic representation of $X(\mu)$ is obtained by (7.11) and (9.2).

As we have seen, we need to express the monodromy matrix (7.8) as a matrix in the auxiliary space $V_a$. The generator of the Hecke algebra (2.30) is of the form

$$\hat{R}_{a,1}(\mu) = (1 - \mu^q)\hat{R}_{a,1}^{(q)} + \mu(q^{-1})\mu = \begin{pmatrix} (q - \mu q^{-1})\mu - (1 - \mu)q^{-1}N_1 & (1 - \mu)\bar{\psi}_1 \\ (1 - \mu)\psi_1 & (1 - \mu)qN_1 + \mu(q^{-1}) \end{pmatrix}. \quad (9.5)$$

Then (2.34) is

$$n_q(\mu) = \frac{(q - \mu^{-1})(1 - \mu)L}{q - \mu^q}. \quad (10.4)$$

The 2-magnon state is obtained in the following form:

$$|\lambda, \mu\rangle = B(\lambda, \mu) = n_q(\lambda, \mu) n_q(\mu) \times \sum_{1 \leq r < s} \frac{\lambda q^{-1} - \mu q^{-1} \lambda \rho^{-1} \sigma^{-1} \sigma_{1..L}}{\lambda - \mu} \bar{\psi}_r \bar{\psi}_s |0\rangle. \quad (10.5)$$

We can see that the situation is very similar to that in Section 8. Again, we propose that the general $M$-magnon state possess the form

$$|\lambda_1, \lambda_M\rangle = \prod_{l=1}^M n_q(\lambda_l) \sum_{1 \leq k_1 < \cdots < k_M = S_M} \sum_{\sigma_{1..M} = S_M} \prod_{i<j} \frac{\lambda_i q^{-1} - \lambda_j q^{-1}}{\lambda_i - \lambda_j} \prod_{i=1}^M \bar{\psi}_{k_i} \cdots \bar{\psi}_{k_M} |0\rangle, \quad (10.6)$$

where $S_M$ is the symmetric group of order $M$ and $\sigma \in S_M$ permutes the parameters $\{\lambda_1, \ldots, \lambda_M\}$. Again, this is just a special representation of (4.7). In the next section we prove formula (10.6) by using the coordinate Bethe ansatz.

11. FERMIONIC MODELS AND COORDINATE BETHE ANSATZ

In this Section we will use the coordinate Bethe ansatz method to construct Bethe vectors for the periodic chain models which are formulated in terms of free fermions. The coordinate Bethe ansatz method is named after the seminal work by Hans Bethe [20]. Bethe found eigenfunctions and spectrum of the one-dimensional spin-$1/2$ isotropic magnet (which we called above as XXX Heisenberg closed spin chain model). The review of the applications of the coordin-
nate Bethe ansatz method can be found in the book [21] (see also [22] and references therein).

11.1. R-Matrix, Hamiltonian and a Vacuum State

Recall that the fermionic representation of the Hecke algebra (2.33) is based on the realization of the R-matrix in the form

$$R_{k,k+1} = \bar{\psi}_{k+1} \psi_{k} + \bar{\psi}_{k} \psi_{k+1} - q \bar{\psi}_{k+1} \psi_{k} - q^{-1} \bar{\psi}_{k} \psi_{k+1} + q.$$ 

Consider the Hamiltonian for the periodic fermionic chain model (“small polaron model”, see [5] and references therein)

$$H = \sum_{k=1}^{L-1} R_{k,k+1} + R_{L,1} - q L$$

$$= \sum_{k=1}^{L-1} (\bar{\psi}_{k+1} \psi_{k} + \bar{\psi}_{k} \psi_{k+1} - q \bar{\psi}_{k+1} \psi_{k} - q^{-1} \bar{\psi}_{k} \psi_{k+1} + q)$$

$$+ (q + q^{-1}) (\bar{\psi}_{k} \psi_{k+1} + \bar{\psi}_{k+1} \psi_{k}) + \bar{\psi}_{1} \psi_{L} + \bar{\psi}_{L} \psi_{1}$$

$$- q \bar{\psi}_{L} \psi_{1} - q^{-1} \bar{\psi}_{1} \psi_{L} + (q + q^{-1}) \bar{\psi}_{L} \psi_{1} \psi_{1} \psi_{L}$$

$$= \sum_{k=1}^{L-1} (\bar{\psi}_{k+1} \psi_{k} + \bar{\psi}_{k} \psi_{k+1} + (q + q^{-1}) \bar{\psi}_{k} \psi_{k+1} + q)$$

$$+ \bar{\psi}_{1} \psi_{L} + \bar{\psi}_{L} \psi_{1} + (q + q^{-1}) \bar{\psi}_{L} \psi_{1} \psi_{1} \psi_{L}$$

$$- (q + q^{-1}) \sum_{k=1}^{L} \bar{\psi}_{k} \psi_{k}.$$ 

This model is not coincident with the XXZ spin chain in view of the representation of the matrix $R_{L,1}$ given in (2.30) in terms of fermions (6.3). In the XXZ case the fermionic representation of $R_{L,1}$ is nonlocal.

The vacuum state $|0\rangle$ of the Hamiltonian is defined by the equations $\psi_{n}|0\rangle = 0$ for $k = 1, 2, \ldots, L$.

11.2. The 1-Magnon States

We look for the 1-magnon solution in the form

$$|1\rangle = \sum_{n=1}^{L} c_n \psi_n|0\rangle.$$ 

Substitution of (11.1) and (11.2) in the eigenvalue problem $H|1\rangle = E|1\rangle$ gives the following equation for the coefficients $c_n$ (the 4-fermionic term in (11.1) does not contribute to the equations):

$$c_{n-1} + c_{n+1} = (E + (q + q^{-1}))c_n,$$

$$1 \leq n \leq L,$$

where $c_{n+L} = c_n$, i.e., $c_0 = c_L$ and $c_{L+1} = c_1$. Since Eq. (11.3) is the discrete version of the ordinary differential equation of the second order with constant coefficients, one can solve (11.3) if we insert $c_n = X^n$. As a result, we obtain the condition

$$E + (q + q^{-1}) = X + X^{-1},$$

which is symmetric under the exchange $X \leftrightarrow X^{-1}$. Thus, the general solution of (11.3) is

$$c_n = A_1 X^n + A_2 X^{-n},$$

where arbitrary constants $A_1, A_2$ are independent of $n$. The boundary conditions $c_k = c_{L+k}$ lead to the equation for $X$:

$$X^L = 1.$$ 

However, in this case, we have $X^{-n} = X^{-L-n}$, and linearly independent solutions are

$$c_n = X^n,$$ 

where $X^L = 1$.

Thus, to each solution $X = X_k$ of Eq. (11.6)

$$X_k = \exp\left(\frac{2\pi i k}{L}\right)$$

we have two one-magnon states (orthogonal to each other)

$$|1\rangle_k = \sum_{n=1}^{L} X_k^n \psi_n|0\rangle, \quad |1\rangle_k' = \sum_{n=1}^{L} X_k^{-n} \psi_n|0\rangle$$

with the same energy

$$E = (q + q^{-1}) + (X_k + X_k') .$$

On the other hand, we have $X_k^{-1} = X_{L-k}$ and the set of vectors $|1\rangle_k$ coincides with the set of vectors $|1\rangle_k'$. All these solutions correspond to the spectrum of free fermions.

11.3. The 2-Magnon States

We write

$$|n_1, n_2\rangle = \psi_{n_1} \psi_{n_2}|0\rangle, \quad 1 \leq n_1 < n_2 \leq L.$$ 

It is easy to find that the action of the Hamiltonian on the vector $|2\rangle = \sum_{1 \leq n_1 < n_2 \leq L} c_{n_1 n_2} |n_1, n_2\rangle$ is

$$H|2\rangle = \sum_{1 \leq n_1 < n_2 \leq L} \left((1 - \delta_{n_1 - 1, n_2}) c_{n_1 - 1, n_2} \right)$$

$$+ \left(1 - \delta_{n_1 + 1, n_2} \right) (c_{n_1, n_2 - 1} + c_{n_1 + 1, n_2})$$

$$+ \left(1 - \delta_{n_2 - 1, n_2} \right) c_{n_1, n_2 + 1} - \delta_{n_1, n_2} (1 + \delta_{n_2 L}) c_{n_2 L}$$

$$- (1 - \delta_{n_1 - 1, n_2}) \delta_{n_2, L} c_{1, n_2} + (q + q^{-1})$$

$$\times \left(\delta_{n_1 + 1, n_2} + \delta_{n_1, n_2 + 1} - \delta_{n_1, n_2 L} - 2\right) c_{n_1, n_2} |n_1, n_2\rangle.$$
Equation $\mathcal{H}[2] = \mathcal{E}[2]$ is then equivalent to the system of equations

\[
(1 - \delta_{n,1})c_{n-1,n} + (1 - \delta_{n+1,n})
\]

\[
\times (c_{n,n-1} + c_{n,1,n}) + (1 - \delta_{n,1})c_{n-1,n+1}
\] 

\[
- \delta_{n,1} (1 - \delta_{n,1})c_{n-L} - (1 - \delta_{n,1})\delta_{n,L}c_{1,n}
\] 

\[
= (\mathcal{E} + 2(q + q^{-1}) - (q + q^{-1})
\]

\[
\times (\delta_{n+1,n} + \delta_{n,1}\delta_{n,L})c_{n,n}
\]

for any $1 \leq n_1 < n_2 \leq L$.

The coordinate Bethe ansatz is based on the idea to write

\[
\mathcal{E} + 2(q + q^{-1}) = X_1 + X_1^{-1} + X_2 + X_2^{-1}
\]

and to find solution of the system in the form

\[
c_{n,n} = A_{12}X_1^{n_2}X_2^{n_1} + A_{21}X_2^{n_2}X_1^{n_1},
\]

where $A_{12}$ and $A_{21}$ are independent of $n_1$ and $n_2$, but they can depend on $X_1$ and $X_2$.

Substituting this assumption into the equation we obtain

\[
\delta_{n_1+1,n_2}(A_{12}(X_1X_2 - (q + q^{-1})X_2 + 1)
\]

\[
+ A_{21}(X_1X_2 - (q + q^{-1})(X_1 + 1))(X_2)^{n_2}_1
\]

\[
+ (A_{12} + X_1^{n_1}A_{21})(\delta_{n_1,n_2}X_1^{n_2} + \delta_{n_2,L}X_2^{n_1})
\]

\[
+ (A_{21} + X_2^{n_2}A_{12})(\delta_{n_1,n_2}X_1^{n_2} + \delta_{n_2,L}X_2^{n_1})
\]

\[
= \delta_{n_1,n_2}L((X_1X_2)^L + X_4X_2)(A_{12} + A_{21})
\]

\[
+ (q + q^{-1})(A_{12}X_1X_2 + A_{21}X_1X_2).
\]

To fulfill these equations we put

\[
A_{12}(X_1X_2 - (q + q^{-1})X_2 + 1)
\]

\[
+ A_{21}(X_1X_2 - (q + q^{-1})(X_1 + 1) = 0,
\]

\[
A_{12} + X_1^{n_1}A_{21} = 0, \ A_{21} + X_2^{n_2}A_{12} = 0,
\]

or equivalently

\[
A_{21} = \frac{X_1X_2 - (q + q^{-1})X_2 + 1}{X_1X_2 - (q + q^{-1})X_1 + 1},
\]

\[
X_1^{L} = \frac{X_1X_2 - (q + q^{-1})X_1 + 1}{X_1X_2 - (q + q^{-1})X_1 + 1},
\]

\[
X_2^{L} = \frac{X_1X_2 - (q + q^{-1})X_2 + 1}{X_1X_2 - (q + q^{-1})X_1 + 1}.
\]

**11.4. The 3-Magnon States**

For $1 \leq n_1 < n_2 < n_3 \leq L$ we put $|n_1, n_2, n_3\rangle = |\psi_{n_1}\psi_{n_2}\psi_{n_3}\rangle$. The action of the Hamiltonian $\mathcal{H}$ on a vector $|3\rangle$ is

\[
\mathcal{H}[3] = \sum_{1 \leq n_1 < n_2 < n_3 \leq L} \left( (1 - \delta_{n_1,1})c_{n_1-1,n_2,n_3} 
\right.
\]

\[
+ (1 - \delta_{n_1,1})c_{n_1,n_2-1,n_3} + c_{n_1+1,n_2,n_3} 
\]

\[
+ (1 - \delta_{n_2,1})c_{n_1,n_2,n_3-1} + c_{n_1,n_2+1,n_3} 
\]

\[
+ (1 - \delta_{n_3,L})c_{n_1,n_2,n_3+1} + \delta_{n_1,1}(1 - \delta_{n_3,L})
\]

\[
\times c_{n_2,n_2,L} + \delta_{n_2,L}(1 - \delta_{n_1,1})c_{n_1,n_2,n_3} + (q + q^{-1})
\]

\[
\times (\delta_{n_1+1,n_2} + \delta_{n_2+1,n_3} + \delta_{n_3,L}\delta_{n_2,L} - 3) c_{n_1,n_2,n_3}|n_1,n_2,n_3\rangle.
\]

Equation $\mathcal{H}[3] = \mathcal{E}[3]$ is equivalent to the system of equation

\[
(1 - \delta_{n_1,1})c_{n_1-1,n_2,n_3} + (1 - \delta_{n_1,1})
\]

\[
\times (c_{n_1,n_2-1} + c_{n_1+1,n_2,n_3}) + (1 - \delta_{n_2,1})
\]

\[
\times (c_{n_1,n_2,n_3-1} + c_{n_1,n_2+1,n_3}) + (1 - \delta_{n_3,L})c_{n_1,n_2,n_3+1}
\]

\[
+ \delta_{n_1,1}(1 - \delta_{n_3,L})c_{n_1,n_2,n_3} + \delta_{n_2,L}(1 - \delta_{n_1,1})c_{n_1,n_2,n_3}
\]

\[
= (\mathcal{E} + 3(q + q^{-1}) - (q + q^{-1})
\]

\[
\times (\delta_{n_1+1,n_2} + \delta_{n_2+1,n_3} + \delta_{n_3,L}\delta_{n_2,L} - 3) c_{n_1,n_2,n_3}|n_1,n_2,n_3\rangle,
\]

where $1 \leq n_1 < n_2 < n_3 \leq L$. When we put

\[
\mathcal{E} + 3(q + q^{-1}) = X_1 + X_1^{-1} + X_2 + X_2^{-1} + X_3 + X_3^{-1}
\]

and look for solution of $c_{n_1,n_2,n_3}$ in the form

\[
c_{n_1,n_2,n_3} = \sum_{\sigma \in S_3} A_{\sigma}X_{\sigma(1)}^{n_1}X_{\sigma(2)}^{n_2}X_{\sigma(3)}^{n_3},
\]

we obtain the following system of the equations:

\[
\delta_{n_1+1,n_2} \sum_{\sigma \in S_3} A_{\sigma}(X_{\sigma(1)}X_{\sigma(2)})^{n_1}X_{\sigma(3)}^{n_2} - (q + q^{-1})X_{\sigma(2)} + 1)
\]

\[
\times (X_{\sigma(1)}X_{\sigma(2)})^{n_1}X_{\sigma(3)}^{n_2} + \delta_{n_1+1,n_2} \sum_{\sigma \in S_3} A_{\sigma}(X_{\sigma(2)}X_{\sigma(3)})^{n_1}
\]

\[
- (q + q^{-1})X_{\sigma(3)} + 1)X_{\sigma(1)}(X_{\sigma(2)}X_{\sigma(3)})^{n_2}
\]

\[
+ \delta_{n_1,L} \sum_{\sigma \in S_3} A_{\sigma}(X_{\sigma(2)}X_{\sigma(3)})^{n_1}X_{\sigma(1)}^{n_2}X_{\sigma(3)}^{n_2}
\]

\[
+ \delta_{n_1,L} \sum_{\sigma \in S_3} A_{\sigma}(X_{\sigma(1)}X_{\sigma(2)}X_{\sigma(3)})^{n_2}X_{\sigma(1)}^{n_1}X_{\sigma(3)}^{n_2}
\]

\[
+ \delta_{n_1,L} \sum_{\sigma \in S_3} A_{\sigma}(X_{\sigma(1)}X_{\sigma(2)}X_{\sigma(3)})^{n_2}X_{\sigma(1)}^{n_1}X_{\sigma(3)}^{n_2}
\]

\[
- (q + q^{-1})X_{\sigma(1)}X_{\sigma(2)}X_{\sigma(3)} = 0.
\]
Let \( \pi_i \) be the transposition \( 1 \leftrightarrow 2 \) and \( \pi_3 \) the transposition \( 2 \leftrightarrow 3 \). To cancel the terms at \( \delta_{\pi_i + 1, \pi_i} \), and \( \delta_{\pi_i + 1, \pi_i} \), it is sufficient for any \( \sigma \in S_3 \) to put

\[
A_\sigma(X_{\sigma(1)}X_{\sigma(2)} - (q + q^{-1})X_{\sigma(2)} + 1)
+ A_{\sigma \cdot \pi_i}(X_{\sigma(1)}X_{\sigma(2)} - (q + q^{-1})X_{\sigma(2)} + 1) = 0,
\]

\[
A_\sigma(X_{\sigma(2)}X_{\sigma(1)} - (q + q^{-1})X_{\sigma(3)} + 1)
+ A_{\sigma \cdot \pi_i}(X_{\sigma(2)}X_{\sigma(1)} - (q + q^{-1})X_{\sigma(3)} + 1) = 0.
\]

If we consider the element \( \xi \in S_3 \) defined by the relations \( \xi(1) = 3 \), \( \xi(2) = 1 \), \( \xi(3) = 2 \), we obtain

\[
\sum_{\sigma \in S_3} A_\sigma(X_{\sigma(2)}X_{\sigma(1)} - X_{\sigma(1)}X_{\sigma(2)}X_{\sigma(3)})
\]

\[
= \sum_{\sigma \in S_3} A_\sigma(X_{\sigma(1)}X_{\sigma(2)} - A_{\sigma \cdot \xi}(X_{\sigma(2)}X_{\sigma(1)}) A_{\sigma \cdot \xi}X_{\sigma(3)}).
\]

So we put for any \( \sigma \in S_3 \)

\[
A_\sigma - A_{\sigma \cdot \xi}X_{\sigma(1)} = 0, \text{ i.e. } A_{\sigma \cdot \xi} = A_\sigma X_{\sigma(3)}.
\]

It is easy to show that these three assumptions solve the whole system for \( c_{n_i, n_2, \pi_i} \).

We obtained for the \( A_\sigma \) conditions

\[
A_{\sigma \cdot \pi_i} = \frac{X_{\sigma(1)}X_{\sigma(2)} - (q + q^{-1})X_{\sigma(2)} + 1}{X_{\sigma(1)}X_{\sigma(2)} - (q + q^{-1})X_{\sigma(2)} + 1} A_\sigma,
\]

\[
A_{\sigma \cdot \pi_2} = \frac{X_{\sigma(2)}X_{\sigma(3)} - (q + q^{-1})X_{\sigma(3)} + 1}{X_{\sigma(2)}X_{\sigma(3)} - (q + q^{-1})X_{\sigma(3)} + 1} A_\sigma.
\]

It follows from these relations that for any \( \sigma \in S_3 \)

\[
A_{(\sigma \cdot \pi_i) \cdot \pi_2} = A_{\sigma \cdot \xi}.
\]

Moreover, we have

\[
A_{(\sigma \cdot \pi_i) \cdot \pi_2} = \frac{X_{\sigma(2)}X_{\sigma(3)} - (q + q^{-1})X_{\sigma(3)} + 1}{X_{\sigma(2)}X_{\sigma(3)} - (q + q^{-1})X_{\sigma(3)} + 1} A_{\sigma \cdot \pi_i}.
\]

So for any \( \sigma \in S_3 \) the relation \( A_{(\sigma \cdot \pi_i) \cdot \pi_2} = A_{\sigma} \) holds. Therefore, \( A_{\sigma} \) is really a function of the symmetric group \( S_3 \).

Since \( \xi = \pi^2 \cdot \pi_i \), the equality \( A_{\sigma \cdot \xi} = X_{\sigma(3)}A_{\sigma} \) leads to the relation

\[
A_{\sigma \cdot \xi} = A_{(\sigma \cdot \pi_i) \cdot \pi_2} = A_{\sigma} X_{\sigma(3)}A_{\sigma}.
\]

So for any \( \sigma \in S_3 \) the relation \( A_{(\sigma \cdot \pi_i) \cdot \pi_2} = A_{\sigma} \) holds. Therefore, \( A_{\sigma} \) is really a function of the symmetric group \( S_3 \).

Since \( \xi = \pi^2 \cdot \pi_i \), the equality \( A_{\sigma \cdot \xi} = X_{\sigma(3)}A_{\sigma} \) leads to the relation

\[
A_{\sigma \cdot \xi} = A_{\sigma} X_{\sigma(3)}A_{\sigma}.
\]
has to hold. It is possible to rewrite these relations in the form

\[ X_i^L = \prod_{k \neq i} \frac{X_k - (q + q^{-1})X_k + 1}{X_k - (q + q^{-1})X_k + 1}. \]  \hspace{7cm} (11.12)

\section*{11.5. The M-Magnon States}

For \( 1 \leq n_1 < n_2 < \ldots < n_{M-1} < n_M \leq L \) we denote

\[ |\tilde{\mathcal{H}}\rangle = |n_1, n_2, \ldots, n_M\rangle = \Psi_{n_1} \Psi_{n_2} \cdots \Psi_{n_M}|0\rangle \]

and take the vector

\[ |M\rangle = \sum_\mathcal{H} c_\mathcal{H} |\tilde{\mathcal{H}}\rangle \]

and substitute these assumptions into the system, we obtain

\[ (\mathcal{H} \pm \tilde{\mathcal{H}})|M\rangle = \sum_\mathcal{H} \left( (1 - \delta_{n_1,1})c_{\mathcal{H} + \tilde{\mathcal{H}}} + (1 - \delta_{n_{k+1},1})c_{\mathcal{H} - \tilde{\mathcal{H}}} \right) \]

\[ \times \left( c_{\mathcal{H} + \tilde{\mathcal{H}}} + c_{\mathcal{H} - \tilde{\mathcal{H}}} \right) + (1 - \delta_{n_1,1})c_{\mathcal{H} + \tilde{\mathcal{H}}} \]

\[ + (-1)^{M-1} \delta_{n_1,1} (1 - \delta_{n_1,1})c_{\mathcal{H} + \tilde{\mathcal{H}}} \]

\[ + (-1)^{M-1} \delta_{n_k,1} (1 - \delta_{n_1,1})c_{\mathcal{H} + \tilde{\mathcal{H}}} \]

\[ + \sum_{k=1}^{M-1} \delta_{n_{k+1},1} c_{\mathcal{H} + \tilde{\mathcal{H}}} + (q + q^{-1}) \delta_{n_1,1} \delta_{n_1,1} c_{\mathcal{H} + \tilde{\mathcal{H}}} \]

\[ - M(q + q^{-1})c_{\mathcal{H} + \tilde{\mathcal{H}}}, \]

Equation \( \mathcal{H}|M\rangle = \mathcal{E}|M\rangle \) is then equivalent to the system

\[ (1 - \delta_{n_1,1})c_{\mathcal{H} + \tilde{\mathcal{H}}} + (1 - \delta_{n_1,1})c_{\mathcal{H} - \tilde{\mathcal{H}}} \]

\[ + (1 - \delta_{n_1,1})c_{\mathcal{H} + \tilde{\mathcal{H}}} \]

\[ + (1 - \delta_{n_1,1})c_{\mathcal{H} - \tilde{\mathcal{H}}} \]

\[ = \left( \mathcal{E} + M(q + q^{-1}) - (q + q^{-1}) \sum_{k=1}^{M-1} \delta_{n_{k+1},1} \right) c_{\mathcal{H} + \tilde{\mathcal{H}}} \]

\[ - (q + q^{-1}) \delta_{n_1,1} \delta_{n_1,1} c_{\mathcal{H} + \tilde{\mathcal{H}}}, \]

When we write the eigenvalue of the Hamiltonian as

\[ \mathcal{E} = \sum_{k=1}^{M} (X_k + X_k^{-1}) - M(q + q^{-1}) \]

look for the solution in the form

\[ c_{\mathcal{H}} = \sum_{\sigma \in S_M} A_{\sigma} X_{\sigma(1)}^{n_1} X_{\sigma(2)}^{n_2} \cdots X_{\sigma(M)}^{n_M}, \]

\[ \text{and substitute these assumptions into the system, we obtain} \]

\[ \sum_{k=1}^{M-1} \delta_{n_{k+1},1} n_{k+1} \sum_{\sigma \in S_M} A_{\sigma} \left( 1 + X_{\sigma(k)} X_{\sigma(k+1)} - (q + q^{-1}) X_{\sigma(k+1)} \right) \]

\[ \times X_{\sigma(1)}^{n_1} \cdots X_{\sigma(k)}^{n_k} \cdots X_{\sigma(M)}^{n_M} \]

\[ + \delta_{n_1,1} \sum_{\sigma \in S_M} A_{\sigma} X_{\sigma(2)}^{n_2} \cdots X_{\sigma(M)}^{n_M} \]

\[ + (-1)^{M-1} X_{\sigma(1)}^{n_1} \cdots X_{\sigma(M)}^{n_M} \]

\[ - (1)^{M-1} X_{\sigma(1)}^{n_1} \cdots X_{\sigma(M)}^{n_M} \]

\[ + (-1)^{M-1} X_{\sigma(1)}^{n_1} \cdots X_{\sigma(M)}^{n_M} \]

\[ = 0. \]

Let \( \pi_k, k = 1, \ldots, M - 1, \) be transpositions \( k \leftrightarrow k + 1. \)

When the relation

\[ \left( X_{\pi(k)} X_{\pi(k+1)} - (q + q^{-1}) (X_{\pi(k+1)} + 1) A_{\pi} \right) \]

\[ + \left( X_{\pi(k)} X_{\pi(k+1)} - (q + q^{-1}) X_{\pi(k+1)} + 1 \right) A_{\pi} = 0, \]

is true for any \( \sigma \in S_M \) and \( k = 1, \ldots, M - 1, \) the terms at \( \delta_{n_{k+1},1} n_{k+1} \) vanish.

Let \( \tilde{\mathcal{E}} \in S_M \) be defined by the relations \( \tilde{\mathcal{E}}(k) = k - 1 \) for \( k = 1, \ldots, M \) and \( \tilde{\mathcal{E}}(1) = M. \) If we require

\[ A_{\sigma} + (-1)^{M-1} X_{\tilde{\mathcal{E}(M)}} A_{\sigma} = 0, \]

\[ A_{\sigma} + (-1)^{M-1} X_{\tilde{\mathcal{E}(M)}} A_{\sigma} = 0, \]

for any \( \sigma \in S_M, \) the terms at \( \delta_{n_{k+1},1} \) and \( \delta_{n_{k+1},1} \) are annulled.
Combining (11.14) and (11.15) we get

\[ \sum_{\sigma \in \mathcal{S}_M} A_\sigma (X_{(1)}^{\sigma_1} X_{(2)}^{\sigma_2} \cdots X_{(M-1)}^{\sigma_{M-1}} X_{(M)}^{\sigma_M} ) + \sum_{\sigma \in \mathcal{S}_M} A_\sigma X_{(1)}^{\sigma_1} X_{(2)}^{\sigma_2} \cdots X_{(M-1)}^{\sigma_{M-1}} X_{(M-1)}^{\sigma_{M-1}} X_{(M)}^{\sigma_M} \] 

\[ + (-1)^M(q + q^{-1}) X_{(1)}^{\sigma_1} X_{(2)}^{\sigma_2} \cdots X_{(M-1)}^{\sigma_{M-1}} X_{(M)}^{\sigma_M} = 0. \]

Therefore, the assumptions (11.14) and (11.15) solve the system for \( c_\sigma \).

We rewrite relation (11.14) as

\[ A_{\sigma + \pi_k} = \frac{X_{\sigma(k)} X_{\sigma(k+1)} - (q + q^{-1}) X_{\sigma(k+1)} + 1}{X_{\sigma(k)} X_{\sigma(k)} - (q + q^{-1}) X_{\sigma(k)} + 1} A_{\sigma + \pi_{M-1} \cdots \pi_1} \]

From this relation it is easy to show that for any \( \sigma \in \mathcal{S}_M \) and \( k = 1, \ldots, M-1 \) the relations

\[ A_{(\sigma + \pi_k) \ast (\sigma + \pi_{k+1})} = A_{(\sigma \ast \pi_k) \ast \pi_{k+1}} \]

are valid. Therefore, \( A_\sigma \) is really a function on symmetry group \( \mathcal{S}_M \).

If we write \( \xi = \pi_{M-1} \ast \pi_{M-2} \ast \cdots \ast \pi_2 \ast \pi_1 \) and use (11.16), it is possible to rewrite (11.15) as

\[ A_{\sigma + \xi} = \frac{X_{\sigma(1)} X_{\sigma(M)} - (q + q^{-1}) X_{\sigma(M)} + 1}{X_{\sigma(1)} X_{\sigma(1)} - (q + q^{-1}) X_{\sigma(1)} + 1} A_{\sigma + \pi_{M-1} \cdots \pi_1} \]

\[ = (-1)^{M-1} \prod_{k=1}^{M-1} X_{\sigma(k)} X_{\sigma(M)} - (q + q^{-1}) X_{\sigma(M)} + 1 A_{\sigma + \pi_{M-1} \cdots \pi_1} \]

\[ \times X_{\sigma(2)} X_{\sigma(M)} - (q + q^{-1}) X_{\sigma(M)} + 1 A_{\sigma + \pi_{M-1} \cdots \pi_1} \]

\[ = (-1)^{M-1} \prod_{k=1}^{M-1} X_{\sigma(k)} X_{\sigma(M)} - (q + q^{-1}) X_{\sigma(M)} + 1 A_{\sigma + \pi_{M-1} \cdots \pi_1} \]

This implies that for any \( i = 1, 2, \ldots, M \) the relation

\[ X_i^L = \prod_{k=1}^M X_k X_k - (q + q^{-1}) X_k + 1 \]  

(11.17)

has to be true.

11.6. Comparison with the Standard XXZ Model

In the standard XXZ model the eigenvalues of the hamiltonian are also given by relation (11.13). More-
Remark. The representation theory of the Hecke algebras $H_n(q)$ is well known. For the details of this representation theory see, e.g., [24–27, 30–33, 36, 41] and references therein. Each irreducible representation (irrep) of the Hecke algebra $H_n(q)$ ($q$ is a generic parameter) corresponds to the Young diagram $\Lambda$ with $n$ nodes. The dimension of the irrep $\Lambda$ is given by the hook formula (see, e.g., [28] and [32])

$$\dim(\Lambda) = \frac{n!}{\prod_{\alpha \in \Lambda} h_\alpha},$$  \hspace{3cm} (12.3)

where $h_\alpha$ is a hook length of the nod $\alpha \in \Lambda$. Recall, that the Young diagram $\Lambda$ with $m$ rows of the lengths ($\lambda_1, \lambda_2, \ldots, \lambda_m$)

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m, \quad \sum_{k=1}^{m} \lambda_k = n,$$

is called dual to the diagram $\Lambda'$ if ($\lambda_1, \lambda_2, \ldots, \lambda_m$) are the lengths of the columns of $\Lambda'$. It is clear that $\dim(\Lambda) = \dim(\Lambda')$.

The quantum integrable systems with the Hamiltonians (12.1) were considered in [2, 23]. In the next subsection, we list the characteristic identities for the Hamiltonians $H_n(q)$ for the cases $n = 2, \ldots, 6$. These identities define the whole energy spectrum of the Hecke chains of the length $n = 2, \ldots, 6$.

12.1. Characteristic Identities for $H_n(q)$

Here, we use the notation

$$\lambda = q - q^{-1}, \quad \bar{q} = q + q^{-1}, \quad \nu = \frac{1}{2}(q + q^{-1}).$$

12.1.1. The case $n = 2$. The characteristic identity for the Hamiltonian $H_2(q) = T_1 - \lambda/2$ is

$$\left( H_2 - \frac{1}{2} \bar{q} \right) \left( H_2 + \frac{1}{2} \bar{q} \right) = 0.$$

Two eigenvalues $1/2\bar{q}$ and $-1/2\bar{q}$ correspond to the 1-dimensional irreps $T_1 = q$ and $T_1 = -q^{-1}$ labeled, respectively, by the Young diagrams (2) and (12).

12.1.2. The case $n = 3$. Here, we have the set of commuting elements [2]

$$j_1 = T_1 + T_2, \quad j_2 = T_1 T_2 + T_2 T_1, \quad j_3 = T_2 T_1 T_2 + T_1 + T_2.$$

(12.4)

Note that the elements $j_2, j_3$ are expressed in terms of $j_1$:

$$j_2 = j_1^2 - \lambda j_1 - 2,$n

$$j_3 = \frac{1}{2}(j_1^3 - 2\lambda j_1^2 + (\lambda^2 - 1)j_1 + 2\lambda).$$

The element $H_3 = j_1 - \lambda$ is the Hamiltonian (12.1) for the open Hecke chain and $j_1$ is a central element in $H_3$. The characteristic identity for the Hamiltonian $H_3$ is:

$$\left( H_3 + \bar{q} \right) \left( H_3 - \bar{q} \right) \left( H_3 - 1 \right) \left( H_3 + 1 \right) = 0.$$  \hspace{3cm} (12.5)

This means that $\text{Spec}(H_3) = \{ \pm \bar{q}, \pm 1 \}$. The first two eigenvalues $\pm \bar{q}$ correspond to the one dimensional representations $T_i = \pm q^{\pm 1}$ ($i = 1, 2$) of $H_2(q)$, which are related to the Young diagrams (3), (13). The eigenvalues $(\pm 1)$ correspond to the 2-dimensional irrep (2, 1) of $H_3(q)$.

12.1.3. The case $n = 4$. In this case we have the following set of commuting elements

$$j_1 = \sum_{i=1}^{3} T_i, \quad j_2 = \{ T_1, T_2 \} + \{ T_2, T_3 \} + 2T_3 T_1,$n

$$j_3 = \{ T_1 T_3 T_2, T_1 \} + \{ T_1 + T_3 T_2 \} T_1 + T_3 T_1$$

$$+ \lambda T_1 T_3 T_2 + 2 \sum_{i=1}^{3} T_i,$n

$$j_4 = \{ T_2 T_3 T_2, T_1 \} + \{ T_2 T_1 T_2, T_3 \} + \{ T_2 T_3, T_1 \} + \{ T_2, T_1 \}.$n

The element $j_3$ is a central element in $H_4(q)$. Therefore, the longest entl $e in $H_4(q)$ is $j_3 = \lambda_1 - \lambda_2$ and $j_4$ commutes with the Hamiltonian $H_3(q) = j_1 - 3/2\bar{q}$. (12.1). This Hamiltonian satisfies the characteristic identity

$$\left( H_4 + \frac{3}{2} \bar{q} \right) \left( H_4 - \frac{3}{2} \bar{q} \right) \left( H_4 + \frac{1}{2} \bar{q} \right) \left( H_4 + \frac{1}{2} \bar{q} \right)^2 = 0.$$  \hspace{3cm} (12.6)

Thus, the spectrum of $H_4(q)$ consists of the eigenvalues:

$$(3/2\bar{q}, -3/2\bar{q})$$

for the two dual 1-dim. irreps (4), (14) of the Hecke algebra $H_4(q)$;

$$(1/2\bar{q}, 1/2\bar{q} \pm \sqrt{2})$$

and $(-1/2\bar{q}, -1/2\bar{q} \pm \sqrt{2})$ for the two dual 3-dim. irreps (3, 1), (2, 13);

$$(\pm 1/2, \sqrt{2}^2)$$

for the 2-dim. irrep (2, 1) of $H_4(q)$.
an odd function of order 25, and the characteristic identity is

\[(1^5 \cdot 5) \cdot (H_5 + 2\bar{q})(H_5 - 2\bar{q}).\]

\[(3, 1^3) : \ H_6, (\mathbb{H}_3^5 - 1)(\mathbb{H}_6^2 - 5).\]

\[(2, 1^4) : \ \left( (H_6 + \bar{q})^2 - \left(\sqrt{5} + 1\right)^2 \right),\]

\[\times \left( (H_6 + \bar{q})^2 - \left(\sqrt{5} + 1\right)^2 \right).\]  \hspace{1cm} (12.7)

\[(4, 1) : \ \left( (H_6 - \bar{q})^2 - \left(\sqrt{5} + 1\right)^2 \right),\]

\[\times \left( (H_6 - \bar{q})^2 - \left(\sqrt{5} + 1\right)^2 \right).\]

\[(2^2, 1^2) : (H_6^2 + \bar{q}H_6 - 1)(H_6^3 + \bar{q}H_6^2 - 5H_6 - 2\bar{q}).\]

\[(3, 2) : (H_6^2 - \bar{q}H_6 - 1)(H_6^3 - \bar{q}H_6^2 - 5H_6 + 2\bar{q}) = 0.\]

The last two lines give the eigenvalues of \(H_6\), which correspond to the 5 dimensional representations labeled by two dual Young diagrams \((2^2, 1^2), (3, 2)\). The sum of the dimensions of the irreps for \(H_6(q)\) is equal to 26. We obtain the 25-th order of the characteristic identity since the eigenvalue 0 has multiplicity 2. This eigenvalue appears in the self-dual irrep \((3, 1^3)\).

12.1.5. The case \(n = 6\). For the Hamiltonian \(H_6 = \sum_{i=1}^{6} T_i - \frac{5}{2}\lambda\), the characteristic polynomial is an even function of \(H_6\) of order 72. The characteristic polynomial is much more complicated:

\[(1^6) \cdot (6) : \ (H_6 + \frac{5}{2}\bar{q})(H_6 - \frac{5}{2}\bar{q}).\]

\[(2, 1^4) : \ (H_6 + \frac{3}{2}\bar{q}) \left( (H_6 + \frac{3}{2}\bar{q})^2 - 1 \right) \left( (H_6 + \frac{3}{2}\bar{q})^2 - 3 \right).\]

\[(5, 1) : \ (H_6 - \frac{3}{2}\bar{q}) \left( (H_6 - \frac{3}{2}\bar{q})^2 - 1 \right) \left( (H_6 - \frac{3}{2}\bar{q})^2 - 3 \right).\]

\[(3, 1^3) : \ (H_6 + \frac{1}{2}\bar{q}) \left( (H_6 + \frac{1}{2}\bar{q})^2 - 1 \right) \left( (H_6 + \frac{1}{2}\bar{q})^2 - 3 \right).\]

\[(12.8) \\left( (H_6 + \frac{1}{2}\bar{q})^2 - (\sqrt{3} + i)^3 \right) \left( (H_6 + \frac{1}{2}\bar{q})^2 - (\sqrt{3} - i)^3 \right).\]

\[(4, 1^2) : \ (H_6 - \frac{1}{2}\bar{q}) \left( (H_6 - \frac{1}{2}\bar{q})^2 - 1 \right) \left( (H_6 - \frac{1}{2}\bar{q})^2 - 3 \right).\]

\[\left( (H_6 - \frac{1}{2}\bar{q})^2 - (\sqrt{3} + 1)^3 \right) \left( (H_6 - \frac{1}{2}\bar{q})^2 - (\sqrt{3} - 1)^3 \right).\]

\[(4, 2) : \ [(3\bar{q}^3 - 20\bar{q} + (24 - 14\bar{q}^2)H_6 + 20\bar{q}H_6^2 - 8H_6^3).\]

\[{9\bar{q}^6 - 228\bar{q}^4 + 512\bar{q}^2 - 256}\]

\[+ (768 - 2528\bar{q}^2 + 3164\bar{q}^4)H_6^2 + (2048\bar{q} - 608\bar{q}^3)H_6^3\]

\[+ (624\bar{q}^2 - 576)H_6^4 - 320\bar{q}H_6^5 + 64H_6^6 \].

\[(2^3) : \ (3\bar{q}^3 + 16\bar{q}^2 - 64 + (8\bar{q}^2 + 160\bar{q})H_6\]

\[+ (3\bar{q}^3 - 16\bar{q}^2 - 64)H_6^2 - 32\bar{q}H_6^3 - 16H_6^4).\]  \hspace{1cm} (12.9)

\[(3, 2, 1) : \ (\bar{q}^6 + 16\bar{q}^4 - 256\bar{q}^2 - 256\]

\[+ (2560 + 1024\bar{q}^2 + 64\bar{q}^4)H_6^3\]

\[+ (5120 + 1152\bar{q}^2 + 192\bar{q}^4 + 16\bar{q}^6)H_6^5\]

\[+ (2048 + 512\bar{q}^2)H_6^6 + (8448 + 1536\bar{q}^2 + 96\bar{q}^4)H_6^8\]

\[+ 1024\bar{q}^8 + 3072 + 256\bar{q}^2 + 256\bar{q}^6 + 256\bar{q}^8) \}\{ \mathbb{H}_6 \rightarrow -\mathbb{H}_6 \},\]

where in the left-hand-side we indicate the corresponding representations which have dimensions

\[\dim(6) = \dim(1^6) = 1, \dim(3, 1^3)\]

\[\dim(4, 1^2) = 10, \dim(5, 1) = \dim(2, 1^5) = 5.\]

The factors in (12.9) correspond to the representation \((2^3)\) with \(\dim = 5\), the representation \((3, 2, 1)\) with \(\dim = 16\), the representation \((4, 2)\) with \(\dim = 9\) and their dual irreps which can be obtained from the previous ones by substitution \(\mathbb{H}_6 \rightarrow -\mathbb{H}_6\). The sum of the dimensions (12.3) for all these representations of \(H_6\) is equal to 76. Since the order of the characteristic polynomial is equal to 72, we conclude that some of these eigenvalues are degenerated. Two of such eigenvalues appear in the dual hook-type irreps \((3, 1^3), (4, 1^2)\) and other two appear in the dual nontrivial irreps \((3, 2), (2^2)\) (see below). It is clear that the degenerated eigenvalues are \(\pm 1/2\bar{q}\) (with the multiplicities 3). The important problem is to find an additional operator \(j_k\) which commutes with the Hamiltonian \(H_6\) and removes this degeneracy.

**Remark.** The factor which corresponds to the self-dual representation \((3, 2, 1)\) with \(\dim = 8\) presented in (12.9), can also be written in the concise form (we remove the common factor \(2^3\))

\[\{ v^9 + v^4 - 4v^2 - 1 + 10x + 16x\bar{v}^2 + 4x\bar{v}^2 - 20x^2\]

\[- 18x^2v^2 - 12x^2v^4 - 4x^2v^6 - 8x^3v^2 + 33x^4 + 24x^4v^2\]

\[+ 6x^4v^4 + 4x^5 - 12x^6 - 4x^2v^2 + x^8\} \]

\[= Z^4Y^4 - Z^3Y^2(6x + 1)(2x - 1)\]

\[+ [4ZY + (4x^2 + 6x - 1)](2x - 1)^2,\]

where \(x = H_6, v = \frac{\bar{q}}{2}, Z = x + v, Y = x - v.\)
12.2. Characteristic Polynomials for $\mathbb{H}_n$ in the Representations $(n - 2, 2)$

Now we impose additional relations on the generators $T_k$ of the Hecke algebra (1.1), (1.2):

$$T_k(q^3) T_{k-1}(q^3) T_k(q^3) = 0, \quad T_k(q^3) T_{k+1}(q^3) T_k(q^3) = 0,$$

(12.11)

where $T_k(x)$ are the Baxtered elements (1.3). The factor of the Hecke algebra over the relations (12.11) is called the Temperley-Lieb algebra $TL_n$. It is known that all irreps of the algebra $TL_n$ coincide with irreps $\rho_{n-k,k}$ (here $n \geq 2k$) of the Hecke algebra numerated by the Young diagrams $(n-k,k)$ with only two rows. The spectrum of all Hamiltonians $\rho_{n-k,k}(\mathbb{H}_n)$ is characterized into two factors of the 12-th and 8-th orders. The spectrum of the Hamiltonians $\rho_{n-k,k}(\mathbb{H}_n)$ is characterized into two factors of the 8-th and 12-th orders.

12.2.1. The case $n = 4$ and representation $(2, 2)$ with dim = 2. The Hamiltonian $\mathbb{H}_4 = x$ (see (12.1)) has the characteristic polynomial (see (12.6))

$$(x^2 - v^2 - 2) = YZ - 2,$$

(12.12)

where $Z = x + v, Y = x - v$.

12.2.2. The case $n = 5$ and representation $(3, 2)$ with dim = 5. The Hamiltonian $\mathbb{H}_5 = x$ has the characteristic polynomial as a product of two factors of orders 2 and 3 (see (12.7))

$$(x^2 - qx - 1)(x^3 - qx^2 - 5x + 2q) = (YZ - 1) \{YZ^2 - (2Y + 3Z)\},$$

(12.13)

where $Z = x, Y = x - 2v$.

12.2.3. The case $n = 6$ and representation $(4, 2)$ with dim = 9. The Hamiltonian $\mathbb{H}_6 = x$ has the characteristic polynomial (see (12.9)) which is factorized into two factors of the 3rd and 6th orders

$$\{-3v^3 + 5v + (7v^2 - 3)x - 5vx^2 + x^3\},$$

$$\{9v^6 - 57v^4 + 32v^2 - 4 + (-44v + 160v^3 - 42v^5)x + (12 - 158v^2 + 79v^4)x^2 + (64v - 76v^3)x^3 + (39v^2 - 9)v^4 - 10vx^5 + x^6\}.$$

(12.14)

In terms of the new variables $Z = x - v, Y = x - 3v$ the factors in (12.14) are simplified to be

$$\{YZ^2 - (5Y + Z)\} \{Y^2Z^2 - (5Y + 4Z)YZ^2 + 2(5Y + Z)Z - 4\}.$$

12.2.4. The case $n = 7$ and the representation $(5, 2)$ with dim = 14. For the Hamiltonian $\mathbb{H}_7 = x$ the characteristic polynomial in this representation is factorized into two factors of the 6th and 8th orders. In terms of the new variables $Z = x - 2v, Y = x - 4v$ it reads

$$\{Z^4Y^2 - 3Z^2(Y + Z) + 2Z + 4v\} - (5Z + 2Y)^2.$$

12.2.5. The case $n = 8$ and the representation $(6, 2)$ with dim = 20. For the Hamiltonian $\mathbb{H}_8 = x$ the characteristic polynomial in this representation is factorized into two factors of the 8th and 12th orders

$$\{Z^6Y^2 - (9Y + 5Z)Z^4Y + Z^2(Z + 6Y)(5Z + 2Y) - (5Z + 2Y)^2\}.$$

12.2.6. The case $n = 9$ and the representation $(7, 2)$ with dim = 27. For the Hamiltonian $\mathbb{H}_9 = x$ the characteristic polynomial in this representation is factorized into two factors of the 12th and 15th orders

$$\{Y^2Z^6 - 5Y^2Z^2(2Y + Z) + 3YZ^4(8Y^2 + 13ZY + 2Z^2) - Z^3(16Y^2 + 64ZY^2 + 38Z^2Y + Z^3) + 3Z^3(8Y^2 + 13ZY + 2Z^2) - 5Z(2Y + Z) + 1\}$$

$$\times \{Y^4Z^6 - 6Y^2Z^2(20Y + 7Z) + 30Y^2(216Y^2 + 121ZY + 14Z^2) - Z^6(304Y^3 + 620ZY^2 + 212Z^2Y + 7Z^3) + 2Z(2Y + 7Z)(126Y^2 + 121ZY + 14Z^2) - Z^2(2Y + 7Z)(20Y + 7Z) + (2Y + 7Z)^3\}.$$

where $Z = x - 4v, Y = x - 6v$.

12.2.7. The case $n = 10$ and the representation $(8, 2)$ with dim = 35. The characteristic polynomial in this representation is factorized into two factors of the 15th and 20th orders

$$P_{(8, 2)} = \{Z^{12}Y^3 - 3Z^{10}Y^2(5Y + 2Z) + Z^8Y(69Y^2 + 76ZY + 10Z^2) - Z^6(119Y^3 + 278ZY^2 + 109Z^2Y + 4Z^3) + Z^4(4Y + 4Z)(69Y^2 + 76ZY + 10Z^2) - Z^3(5Y + 2Z)(Y + 4Z)^2 + (Y + 4Z)^3\}$$

$$\times \{Z^6Y^4 - Z^{14}Y^3(27Y + 8Z) + Z^{12}Y^2(261Y^2 + 194ZY + 20Z^2) - Z^{10}Y(1143Y^3 + 1632ZY^2 + 4392^2Y + 16Z^3)(12.15) + Z^8(2349Y^4 + 5982ZY^3 + 3216Z^2Y^2 + 326Z^3Y + 2Z^4) - Z^6(2187Y^4 + 9720ZY^3$$
+ 9812Z^2Y^2 + 2124Z^3Y + 40Z^4
+ 3Z^4(243Y^4 + 2214Z^3Y + 4098Z^2Y^2
+ 1816Z^3Y + 84Z^4) - 2Z^5(729Y^3 + 3033Z^2Y^2
+ 2584Z^2Y + 304Z^3) + 36Z^2(27Y^3 + 54Z^2Y + 14Z^3
- 80Z(3Y + 2Z) + 16)

where \( x = H_{10}^1, Z = x - 5v, Y = x - 7v \).

12.2.8. The case \( n = 11 \) and the representation (9, 2) with dim = 44. The characteristic polynomial in this representation is factorized into two factors of the 20-th and 24-th order

\[
P_{(9,2)} = \{Z^{16}Y^4 - 7Z^{14}Y^3(3Y + 2Z) + 5Z^{12}Y^2(31Y^2 + 26Z^2Y + 3Z^3) - Z^{10}(510Y^3 + 822Z^2Y^2 + 249Z^2Y + 10Z^3) + Z^8(775Y^4 + 228Z^2Y + 315Z^2Y^2 + 153Z^2Y + 7Z^3) - Z^6(525Y^4 + 2635Z^2Y^3 + 3002Z^2Y^2 + 730Z^3Y + 15Z^4 + 2Z^4(125Y^4 + 1290Z^2Y^3 + 2697Z^2Y^2 + 1346Z^3Y + 69Z^4) - Z^3(200Y^2 + 941Z^2 + 190Z^2Y + 119Z^3) + 3Z^2(35Y^2 + 79Z^2Y + 23Z^2) - 5Z(4Y + 3Z + 1) \} \]

(12.16)

\times \{Z^{20}Y^4 - 18Z^8Y^3(3Y + 2Z) - 2Z^{16}Y^2(729Y^3 + 3033Z^2Y^2 + 2584Z^2Y + 304Z^3) + 36Z^2(27Y^3 + 54Z^2Y + 14Z^3 - 80Z(3Y + 2Z) + 16) \}

where \( x = H_{12}, Z = x - 6v, Y = x - 8v \).

12.2.9. The case \( n = 12 \) and the representation (10, 2) with dim = 54. The characteristic polynomial in this representation is factorized into two factors of the 24-th and 30-th order

\[
P_{(10,2)} = \{Z^{20}Y^4 - 4Z^{18}Y^3(7Y + 2Z) + 3Z^{16}Y^2(100Y^2 + 68Z^2Y + 7Z^3) - Z^{14}(1591Y^3 + 1954Z^2Y^2 + 491Z^2Y + 20Z^3) + Z^{12}(4508Y^4 + 9064Z^3Y + 421Z^3Y^2 + 436Z^3Y + 5Z^4) - Z^{10}(6907Y^4 + 21850Z^3Y + 17112Z^2Y^2 + 3406Z^3Y + 105Z^4 + Z^8(5527Y^3 + 27480Z^2Y^3 + 34909Z^2Y^2 + 12200Z^2Y + 775Z^4) - 2Z^6(Y + 5Z) \}

\times (1082Y^3 + 3150Z^2Y + 2061Z^2Y + 255Z^3) + Z^4(Y + 5Z^2) \}

(411Y^2 + 634ZY + 155Z^3) - 7Z^3(5Y + 32)(Y + 5Z^2) + (Y + 5Z)^4 \}

\times (Z^{15} - Z^{23}Y^4(22Y + 5Z) + Z^{21}Y^2(729Y^2 + 41Z^2Y + 35Z^2) - Z^{17}Y^2(7623Y^3 + 68662Z^2Y + 13552Z^2Y^2 + 50Z^3) + Z^{17}Y(43076Y^4 + 60390Z^3Y + 20954Z^2Y^2 + 1834Z^3Y + 25Z^4) - Z^{15}(147983Y^5 + 307010Z^2Y^4 + 168558Z^2Y^3 + 26910Z^2Y^2 + 883Z^2Y + 2Z^6) \}

(12.17)

\[ + Z_{13}(310123Y^5 + 930974Z^2Y^3 + 770091Z^2Y^3 + 196172Z^3Y^2 + 12139Z^3Y + 70Z^5) \]

- 2Z^4(194326Y^5 + 840587Z^2Y^4 + 1026993Z^2Y^3 + 404211Z^2Y^2 + 42041Z^2Y + 475Z^3) + Z^3(278179Y^5 + 1759824Z^4 + 3173478Z^3Y^2 + 1898244Z^4Y^2 + 317599Z^2Y + 6460Z^3)

- Z^2(102487Y^4 + 1011560Z^4Y + 2742586Z^3Y^2 + 2500438Z^2Y^3 + 665275Z^4Y + 23750Z^5)

+ Z^3(14641Y^4 + 284834Z^2Y^4 + 1251866Z^2Y^3 + 1769240Z^3Y^2 + 753437Z^4Y + 47766Z^5)

- 2Z^2(14641Y^4 + 135520Z^2Y^4 + 320474Z^2Y^2 + 219128Z^2Y + 25365Z^4 + 8Z^3(2662Y^3 + 13673Z^2Y^2 + 16106Z^2Y + 3325Z^3) - 8Z^2(847Y^2 + 2222Z^2Y + 855Z^3) + 80Z(11Y + 10Z - 32)

, where \( x = H_{12}, Z = x - 6v, Y = x - 8v \).

12.2.10. The case \( n = 13 \) and the representation (11, 2) with dim = 65. The characteristic polynomial in this representation is factorized into two factors of the 30-th and 35-th order

\[ Y^2Z^{25} - 9Y^4(4Y + Z)Z^{13} + 7Y^3(75Y^2 + 43Y^2 + 4Z^2) \]

\[ \times Z^{25} - Y^2(4056Y^3 + 4031Z^2Y^2 + 874Z^2Y + 35Z^3)Z^{19} + Y(18231Y^4 + 28222Z^2Y^4 + 10781Z^2Y^2 + 1030Z^3Y + 15Z^2)Z^{17} - (49380Y^5 + 113163Z^4Y^4 + 68496Z^2Y^3 + 11805Z^2Y^2 + 4243Z^2Y + 4X)Z^{15} + (80891Y^6 + 268257Z^4Y^4 + 244835Z^2Y^3 + 68531Z^2Y^2 + 4605Z^2Y + 28Z^2)Z^{13} - (78576Y^5 + 375429Z^4Y + 506270Z^3Y^2 + 219334Z^2Y^2 + 24905Z^2Y + 300Z^2)Z^{11} + (43200Y^5 + 301984Z^4Y + 601609Z^3Y + 396150Z^2Y^2 + 72643Z^2Y + 1591Z^3)Z^9 - 2(6048Y^5 + 66096Z^4Y + 197920Z^3Y^2 + 198881Z^2Y^2 + 58122Z^2Y + 2254Z^2)Z^7 + (1296Y^5 + 28080Z^4Y + 136554Z^3Y + 21284Z^2Y^2 + 99644Z^2Y + 6907Z^3)Z^5} \]
\[-(2160 Y^4 + 22176 Z Y^3 + 57906 Z^2 Y^2 + 43570 Z^3 Y + 5527 Z^4) + (1296 Y^3 + 7360 Z Y^2 + 9551 Z^2 Y + 2164 Z^3) Y^3 - 3 (112 Y^2 + 324 Z Y + 137 Z^2) Z^2 + 35 (Y + Z) Z - 15 Z^{20} - Y^{54} Z + 11 Z^{28} + Y^3 (1239 Y^2 + 563 Z Y + 44 Z^2) Z^{26} \] 
\[Y^2 (15894 Y^3 + 12153 Z Y^2 + 2140 Z^2 Y + 77 Z^3) Z^{24} + Y (126278 Y^4 + 145446 Z Y^3 + 43545 Z^2 Y^2 + 3578 Z^3 Y + 55 Z^4) Z^{22} - (650946 Y^5 + 1067749 Z Y^4 + 486798 Z^2 Y^3 + 689672 Z^3 Y^2 + 2466 Z^4 Y + 11 Z^5) Z^{20} + (2219569 Y^5 + 5028863 Z Y^4 + 3303181 Z^2 Y^3 + 723221 Z^3 Y^2 + 45485 Z^4 Y + 484 Z^5) Z^{18} - (5017266 Y^5 + 15459111 Z Y^4 + 1420313 Z^2 Y^3 + 4550870 Z^3 Y^2 + 451913 Z^4 Y + 8712 Z^5) Z^{16} + (7433784 Y^5 + 3099993 Z Y^4 + 3927627 Z^2 Y^3 + 1790164 Z^3 Y^2 + 2662000 Z^4 Y + 83853 Z^5) Z^{14} - 2 (3523048 Y^5 + 1996798 Z Y^4 + 3479055 Z^2 Y^3 + 2227509 Z^3 Y^2 + 4830078 Z^4 Y + 236918 Z^5) Z^{12} + (4121784 Y^5 + 3205756 Z Y^4 + 7739969 Z^2 Y^3 + 6959592 Z^3 Y^2 + 2177927 Z^4 Y + 1627813 Z^5) Z^{10} - (2 Y + 11 Z) (715128 Y^4 + 3714176 Z Y^3 + 5556166 Z^2 Y^2 + 2682086 Z^3 Y + 310123 Z^4) Z^8 + (2 Y + 11 Z) (71532 Y^3 + 233700 Z Y^2 + 1912792 Z^2 Y + 35332 Z^3) Z^6 - (2 Y + 11 Z) (3924 Z Y^2 + 7128 Z Y^4 + 2299 Z^2) Z^4 + 7 (2 Y + 11 Z)^2 (15 Y + 11 Z) Z^2 - (2 Y + 11 Z) Z^2 \]

where \(x = \mathbb{H}_{13}, Z = x - 8 v \) and \(Y = x - 10 v \).

In all examples considered above, the characteristic polynomials for \(p_{n-2,2} (\mathbb{H}_n) \) are factorized into two factors with integer coefficients. So we formulate the following Conjecture.

**Conjecture.** Let \(n \geq 4 \). For irrep of the Hecke algebra \(H_n(q) \) with Young diagram \((n-2, 2) \) and dimension \(n(n-3)/2 = p_{n-1} + p_n \), the characteristic polynomial for the Hamiltonian \(p_{n-2,2} (\mathbb{H}_n) = x \) is represented as the product of two polynomial factors: the short factor \(p_{n-1} \) of the order \(p_{n-1} \) and the long factor \(p_n \) of the order \(p_n \) with integer coefficients, where

\[
p_n = \frac{1}{8} (((-1)^n - 1) 3 - 4 n^2) = \begin{cases} 
\frac{1}{4} n(n - 2), & \text{for even } n \\
\frac{1}{4} (n + 1)(n - 3)n, & \text{for odd } n
\end{cases}
\]

I.e.,

\[
p_3 = 0, p_4 = 2, p_5 = 3, p_6 = 6, p_7 = 8, p_8 = 12, p_9 = 15, p_{10} = 20, p_{11} = 24, p_{12} = 30, p_{13} = 35, p_{14} = 42, \ldots
\]

These polynomial factors are \(p_n = k_n + \bar{k}_n\)

\[
\text{short}_n = Z^{5^{k_{n-1}} - 1} \{ (1 - (n - 4) Z^{1 \cdot Y})^{-1} - (n - 4)(n - 5)/2 \} Z^{-2} \]

\[
+ (n - 6)(n^2 - 7 n + 8)/2 Z^{-3} Y^{-1} \]

\[
+ (n - 6)(n - 7)(n^2 - 5 n - 4)/8 Z^{-4} \]

\[
- (n - 6)(n - 7)(n - 8)/6 Z^{-3} Y^{-3} \]

(12.19)

\[
- n^4 - 20 n^3 + 137 n^2 - 338 n + 116)/4 Z^{-4} Y^{-2} \]

\[
- (\ldots) Z^{3} Y^{-1} \}
\]

... \[
+ (-1)^{[n-2/4]}(1 + (-1)^n)\left(\frac{n}{2} - 1\right) X Y \right)^{k_{n-1}}
\]

- \[
+ (-1)^{[n-1/4]}(1 - (-1)^n),
\]

\[
\text{long}_n \]

\[
Z^{5^{k_{n-1}} - 1} \{ (1 - (n - 2) Z^{1 \cdot Y})^{-1} - (n - 1)(n - 4)/2 \} Z^{-2} \]

\[
+ (n - 6)(n - 1)(n^2 - 3 n - 12)/8 Z^{-4} \]

\[
- (n - 2)(n - 6)(n - 7)/6 Z^{-3} Y^{-3} \]

(12.20)

\[
- (n^4 - 12 n^3 + 37 n^2 + 18 n - 124)/4 Z^{-4} Y^{-2} \]

\[
- (n^5 - 12 n^4 + 23 n^3 + 128 n^2 - 252 n - 224)/8 Z^{-5} Y^{-4} \]

+ \ldots \}

\[
+ (-1)^{[n-1/4]}(1 - (-1)^n)\left(\frac{n}{2} - 1\right) Z + 2 Y \right)^{k_{n-1}}
\]

\[
+ (-1)^{[n-1/4]}(1 + (-1)^n)\left(\frac{n}{2} - 1\right) \}
\]

where \([x]—integer part of x (e.g., \lfloor n/4 \rfloor —integer part of \lfloor n/4 \rfloor \).
For $n$ — odd, one can write (12.19) and (12.20) as the series

$$
\text{long}_n \sim \text{short}_{n+1} \sim Z^{x} Y^{h} \left(1 - Z^{-2} (C_{2,1} Z + C_{2,0})\right) + Z^{4} \left(C_{4,2} Z^{2} + C_{4,1} Z + C_{4,0}\right) - Z^{6} \left(C_{6,3} Z^{3} + C_{6,2} Z^{2} + C_{6,1} Z + C_{6,0}\right) + Z^{8} \sum_{j=0}^{4} C_{8,j} \left(Z_{j}^{2}/Y\right) - \ldots
$$

$$= Z^{x} Y^{h} \left(\sum_{m=0}^{\frac{n-2}{3}} (-Z)^{m} \sum_{j=0}^{m} \sum_{k} C_{2m,2j} \left(Z_{j}^{2}/Y\right)\right).$$

For $n$ — even, Eqs. (12.19) and (12.20) also can be written as the series over $Z/Y$. But we do not present it explicitly here.

**Remark 1.** For the hook-type representations $(n - 1, 1)$ we have the following spectrum for the Hamiltonian (see [35, 36])

$$\text{Spec}(\mathbb{H}_n) = \text{Spec} \left(\sum_{k=1}^{n-1} (T_k - q)\right)$$

$$= 2 \cos \left(\frac{\pi m}{n}\right) - (q + q^{-1}) \quad m = 1, \ldots, n - 1.$$

If, as usual (cf. (11.4)), we substitute $2 \cos(\pi m/n) = X^{m/2} + X^{-m/2}$, then for $X$ we will have the characteristic identity $X^{n} - 1 = 0, X \neq 1$. We see that the spectrum of the open XXZ spin chain (for even $m$) contains the spectrum of one-magnon states (except for the case $X = 1$) for closed XXZ spin chain (see (11.10)).

**Remark 2.** We have calculated the characteristic polynomials $P_{n-3,3}$ for the Hamiltonian (12.1) in the irreps $(n - 3, 3)$ $(n = 6, 7, 8, 9)$ and observed the same factorization of $P_{n-3,3}$ into two factors, which are the polynomials with the integer coefficients.

**Remark 3.** The quantum inverse scattering (R-matrix) method and the algebraic Bethe ansatz method for the open XXZ spin chain were elaborated by Sklyanin in [37] (about analytical Bethe ansatz approach see [38, 39] and references therein). The quantum group symmetry in the open XXZ spin chain was discovered in [40].

### 12.3. Method of Calculation

In this subsection, we explain the method of construction of the explicit matrix irreps for the Hecke algebra $\hat{H}_{M+1}(q)$, related to the fixed Young diagram $\Lambda$. A similar method was also considered in [36, 41].

First of all, we define the affine extension $\hat{H}_{M+1}(q)$ of the Hecke algebra $H_{M+1}(q)$. The affine Hecke algebra $\hat{H}_{M+1}(q)$ (see, e.g., Chapter 12.3 in [1]) is an extension of the Hecke algebra $H_{M+1}(q)$ by the additional affine elements $y_k (k = 1, \ldots, M + 1)$ subjected to the relations:

$$y_{k+1} = T_{k} y_{k} T_{k}, \quad y_{k} y_{j} = y_{j} y_{k},$$

$$y_{i} y_{j} = y_{j} y_{i}, \quad (j \neq i, i + 1).$$

The elements $\{y_k\}$ form a commutative subalgebra in $\hat{H}_{M+1}$, while the symmetric functions in $y_k$ form the center in $\hat{H}_{M+1}$. Let us introduce the intertwining elements [29] (presented in another form in [27])

$$U_{n+1} = \frac{1}{f(y_{m}, y_{n+1})} \left(\sigma_{n} y_{n} - y_{n} \sigma_{n}\right) \in \hat{H}_{M+1}(q)$$

(1 $\leq n \leq M$),

where $f(y_{m}, y_{n+1})$ is an arbitrary function of the two variables $y_{m}, y_{n+1}$. The elements $U_i$ satisfy the relations

$$U_{n} U_{n+1} U_{n} = U_{n+1} U_{n} U_{n+1},$$

$$U_{n+1}^{2} = \frac{(q y_{n} - q^{-1} y_{n+1})(q y_{n+1} - q^{-1} y_{n})}{f(y_{m}, y_{n+1}) f(y_{n}, y_{n+1})},$$

$$U_{n+1} y_{n} = y_{n+1} U_{n+1}, \quad U_{n+1} y_{n+1} = y_{n} U_{n+1},$$

$$[U_{n+1}, y_{k}] = 0 \quad (k \neq n, n + 1).$$

As it is seen from (12.25), the operators $U_{n+1}$ “permute” the elements $y_{n}$ and $y_{n+1}$, and this confirms the statement that the center of the Hecke algebra $H_{M+1}(q)$ is generated by the symmetric functions in $\{y_i\}$ ($i = 2, \ldots, M + 1$).
One may check that the Hamiltonian (12.1) satisfies
\[
[H_{M+1}, y_k] = U_{k+1} f_{k+1} - U_k f_k, \\
[H_{M+1}, y_k^\dagger] = \overline{U}_{k+1} f_{k+1} - \overline{U}_k f_k,
\]
(12.26)
where \( f_{k+1} = f(y_{k+1}, y_{k+1}) \), \( \overline{U}_{k+1} = \overline{U}_{k+1}(y_{k+1}y_{k+1})^{-1} \) and \( U_1 = U_M + 2 = 0 \). From (12.26) follows that
\[
\left[ H_{M+1}, \sum_{i=1}^k y_k \right] = U_{k+1} f_{k+1}, \\
\left[ H_{M+1}, \sum_{i=1}^k y_i^\dagger \right] = \overline{U}_{k+1} f_{k+1}.
\]
(12.27)
Further, it is convenient to fix
\[f(y_k, y_{k+1}) = y_k - y_{k+1}.
\]
Now we have
\[
U_{n+1}^2 = \frac{(q y_{n+1} - q^{-1} y_n)(q^{-1} y_{n+1} - q y_n)}{(y_{n+1} - y_n)^2}, \tag{12.28}
\]
\[
U_{n+1} = \sigma_n + \frac{\lambda y_{n+1}}{(y_{n+1} - y_n)} + \left( \frac{\lambda}{2} \right) \frac{(y_{n+1} + y_n)}{(y_{n+1} - y_n)}, \tag{12.29}
\]
and, therefore,
\[
s_n = U_{n+1} + v_{n+1}, \tag{12.30}
\]
where
\[
v_{n+1} = \frac{\lambda (y_{n+1} + y_n)}{2 (y_{n+1} - y_n)}. \tag{12.31}
\]
Due to the relations \( s_n^2 = \tilde{q}^2 / 4 \), we conclude that
\[
U_{n+1}^2 + v_{n+1}^2 = \frac{(q + q^{-1})^2}{4},
\]
\[
U_{n+1} v_{n+1} + v_{n+1} U_{n+1} = 0.
\]
Finally, for the Hamiltonian (12.1) we obtain
\[
H_{M+1} = \sum_{n=1}^M s_n = \sum_{n=1}^M (U_{n+1} + v_{n+1})
\]
\[= \sum_{n=1}^M \left[ U_{n+1} + \frac{\lambda (y_{n+1} + y_n)}{2 (y_{n+1} - y_n)} \right], \tag{12.32}
\]
where the operators \( \tilde{U}_{n+1} \) permute indices in the Young diagram \( \Lambda \), or put \( \Lambda \) equal to zero.

**Example 1.** Consider the basis for the representation \((2, 1)\), which is related to the Young diagram:
\[
\begin{array}{c}
\lambda \\
\lambda + 1
\end{array}
\]
(12.33)
We have 2 standard tableaux
\[
\psi_0 = \begin{pmatrix} 1 & 2 \\ 3 \end{pmatrix}, \quad \psi_1 = U_1 \psi_0 = \begin{pmatrix} 1 & 3 \\ 2 \end{pmatrix}
\]
(12.34)
Using (12.29), we obtain the action of \( s_1 \) on the vectors \( \psi_0, \psi_1 \) (12.33)
\[
s_1 \psi_0 = \frac{1 - \tilde{q}}{2} \psi_0, \quad s_1 \psi_1 = -\frac{1}{2} \tilde{q} \psi_1,
\]
\[
s_2 \psi_0 = \psi_1 + \frac{\lambda (q^2 + q^{-2})}{2(q^2 - q^{-2})} \psi_0,
\]
\[
s_2 \psi_1 = U_1^2 \psi_1 + \frac{\lambda (q^2 + q^{-2})}{2(q^2 - q^{-2})} \psi_1
\]
where \( U_1^2 \psi_0 = (q^3 - q^{-3})(q - q^{-1})/(q^2 - q^{-2})^2 \psi_0 \).

**Example 2.** Consider the basis for the representation \((3^2)\) which is related to the Young diagram:
\[
\begin{array}{c}
\lambda \\
\lambda + 1 \\
\lambda + 2
\end{array}
\]
(12.35)
We have 5 standard tableaux (the operators \( \tilde{U}_{k+1} \) permute numbers \( k \) and \( k + 1 \) in the standard tableaux)
\[
\psi_0 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad \psi_1 = U_1 \psi_0 = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{pmatrix}
\]
(12.36)
\[
\psi_2 = U_2 U_1 \psi_0 = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}
\]
\[
\psi_3 = U_3 U_4 \psi_0 = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{pmatrix}
\]
\[
\psi_4 = U_3 U_4 \psi_0 = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}
\]
(12.37)
Using (12.29) we find the action of the operators \( s_n \) to the basis vectors \( \psi_i \) (12.35)
\[
s_1 \psi_0 = \frac{1 - \tilde{q}}{2} \psi_0, \quad s_1 \psi_1 = \frac{1}{2} \tilde{q} \psi_1, \quad s_1 \psi_2 = \frac{1}{2} \tilde{q} \psi_2,
\]
\[
s_1 \psi_3 = -\frac{1}{2} \tilde{q} \psi_3, \quad s_1 \psi_4 = -\frac{1}{2} \tilde{q} \psi_4,
\]
\[
s_2 \psi_0 = \frac{1}{2} \tilde{q} \psi_0, \quad s_2 \psi_1 = \psi_0 - \frac{\lambda (q^2 + q^{-2})}{2(q^2 - q^{-2})} \psi_0,
\]
\[
s_2 \psi_2 = \psi_1 - \frac{\lambda (q^2 + q^{-2})}{2(q^2 - q^{-2})} \psi_1,
\]
\[
s_2 \psi_3 = U_3^2 \psi_1 - \frac{\lambda (q^2 + q^{-2})}{2(q^2 - q^{-2})} \psi_3
\]
\[ s_2 \psi_4 = U_2^3 \psi_2 - \frac{\lambda(q^2 + q^2)}{2(q^2 - q^4)} \psi_4, \]
\[ s_3 \psi_0 = \psi_1 - \frac{\lambda(q^4 + q^2)}{2(q^4 - q^2)} \psi_0, \]
\[ s_3 \psi_1 = U_2^4 \psi_0 - \frac{\lambda(q^2 + q^4)}{2(q^2 - q^4)} \psi_1, \quad s_3 \psi_2 = \frac{1}{2} \bar{q} \psi_2, \]
\[ s_3 \psi_3 = \bar{q} \psi_3, \quad s_3 \psi_4 = -\frac{1}{2} \bar{q} \psi_4, \]
\[ s_4 \psi_0 = \frac{1}{2} \bar{q} \psi_0, \quad s_4 \psi_1 = \psi_2 - \frac{\lambda(q^4 + 1)}{2(q^4 - 1)} \psi_1, \]
\[ s_4 \psi_2 = U_2^3 \psi_1 - \frac{\lambda(1+q^4)}{2(1-q^4)} \psi_2, \]
\[ s_4 \psi_3 = \psi_4 - \frac{\lambda(q^4 + 1)}{2(q^4 - 1)} \psi_3, \]
\[ s_4 \psi_4 = \frac{1}{2} \bar{q} \psi_3, \quad s_4 \psi_4 = -\frac{1}{2} \bar{q} \psi_4, \]
\[ \text{where} \]
\[ U_2^3 \psi_i = \frac{(q^3 - q^3)(q^3 - q^4)}{(q^2 - q^2)^2} \psi_i, \quad (i = 1, 2), \]
\[ U_2^4 \psi_0 = \frac{(q^5 - q^3)(q^3 - q^4)}{(q^4 - q^2)^2} \psi_0, \]
\[ U_2^5 \psi_i = \frac{(q^5 - q^4)(q^3 - q^4)}{(q^4 - q^2)^2} \psi_i, \quad (i = 1, 3), \]

Then the equation for eigenvalues \( \nu \) of \( \mathbb{H}_6 \) and eigenvectors in the space of the irreducible representation \((12.34)\) is given as follows:

\[ \sum_{i=0}^{5} s_i - \frac{\nu}{2} \left( \psi_0 + a_1 \psi_1 + a_2 \psi_2 + a_3 \psi_3 + a_4 \psi_4 \right) = 0, \]

which is equivalent to the characteristic identity

\[ \nu = \mathbb{H}_6: \]

\[ \left( \nu - \frac{\bar{q}}{2} \right) \left( 3q^4 + 16q^2 - 64 - (8q^3 + 160q) \nu \right) \]
\[ + (-8q^2 + 128) \nu^2 + 32q \nu^3 - 16q^4 \nu = 0. \]

Note that eigenvalue \( \mathbb{H}_6 = 1/2 \bar{q} \) has multiplicity 2 since it has already been presented in \((4, 12)\) \((12.8)\).

The second factor in the characteristic identity \((12.36)\) is related to the Young diagram \((3^7)\) and dual to the factor presented in \((12.9)\).

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**APPENDIX A**

**DETAILS IN XXX**

Equations \((7.11)\) and \((7.12)\) result in the following useful identities

\[ P_{k,k+1} \psi_k = \psi_{k+1} - \psi_k N_k + \psi_k N_{k+1}, \quad (A.1) \]
\[ \hat{R}_{k,k+1}(\lambda) \psi_k = \psi_{k+1} + \lambda \psi_k + \lambda \psi_{k+1}, \quad (A.2) \]

where \( N_k = \psi_k \psi_k \). Again, we can see that

\[ P_{k,k+1} \psi_k(0) = \psi_{k+1}(0), \quad (A.3) \]
\[ \hat{R}_{k,k+1}(\mu) \psi_k(0) = \psi_{k+1}(0) + \mu \psi_k(0), \quad (A.4) \]

and

\[ \hat{R}_{k,k+1}(\mu)(0) = (\mu + 1)(0), \quad (A.5) \]
\[ P_{k,k+1}(0) = (0). \quad (A.6) \]

For higher magnons we will also need

\[ \hat{R}_{k,k+1}(\lambda) \psi_{k+1} = (\lambda + 1) \psi_k \psi_{k+1} \quad (A.7) \]

and

\[ \hat{R}_{k,k+1}(\lambda) \psi_{k+1} = \lambda \psi_{k+1} + \sum_{j=1}^{k-1-1} \left( \frac{\lambda + 1}{\lambda} \right)^j \psi_{k+1}(0), \quad (A.8) \]

\[ \hat{R}_{k,k+1}(\lambda) \psi_{k+1} = \lambda \psi_{k+1} + \sum_{j=1}^{k-1} \left( \frac{\lambda + 1}{\lambda} \right)^j \psi_{k+1}(0), \quad (A.9) \]

It is obvious that the second term in \((7.15)\) annihilates vacuum state. Then, using \((A.8)\), we get for the 1-magnon state
if we use the notation (8.4).

Using (A.8) and (A.9) we obtain

$$B(\lambda)\psi_k|0\rangle = (\lambda + 1)\lambda^{-L - k} \sum_{m=0}^{k-2} [\lambda]^{m+2} \psi_{m+1}\psi_k|0\rangle$$

We get for the 2-magnon state using (A.11)

$$|\lambda, \mu\rangle = B(\lambda)B(\mu)|0\rangle = n(\mu) \sum_{k=1}^{L} [\lambda]^{k} B(\lambda)\psi_k|0\rangle$$

The finite sum in (A.12) can be calculated explicitly by means of geometric progression

$$\frac{1}{\lambda(\lambda + 1)} \sum_{k=r+1}^{s-1} [\lambda]^{k+r-k} = \frac{\lambda^r}{\lambda(\lambda - \mu)} [\lambda]^{r+1} [\lambda]^{-1}$$

Substitution of (A.13) into (A.12) gives

$$B(\lambda)B(\mu)|0\rangle = n(\mu)n(\lambda)$$

For the 3-magnon we need at first

$$B(\nu)\psi_s|0\rangle = (\nu + 1 - \nu N_s)X_{1sL}(\nu)\psi_1\psi_s$$

The finite sum in (A.15) can be calculated explicitly...
The 3-magnon state is obtained from the 2-magnon state (A.14)
\[
|\nu, \mu, \lambda, \rangle = B(\nu)B(\mu)B(\lambda)|0\rangle
\]
where we denote for more comfort
\[
\langle \nu, \mu, \lambda, | = [\mu]\langle \nu, \mu, \lambda, | = [\mu] [\nu, \mu, \lambda, ]^{\mu - \lambda + 1}_{\mu - \lambda}.
\]
Using (A.15) we get
\[
|\nu, \mu, \lambda, \rangle = n(\nu)n(\mu)n(\lambda) \sum_{1 \leq q < r < s \leq L} \left[ |\nu, \mu, \lambda, | q + 2 \right] K_2(s, r)
\]
\[
+ \frac{1}{\nu} \sum_{l = 1}^{r - q - 1} \left[ |\nu, \mu, \lambda, | q + 2 \right] K_2(s, r - l)
\]
\[
+ \frac{1}{\nu} \sum_{j = 1}^{s - r - 1} \left[ |\nu, \mu, \lambda, | q + 2 \right] K_2(s - j, r)
\]
\[
+ \frac{1}{\nu^2 (\nu + 1)^2} \sum_{r = 1}^{s - q - 1} \sum_{l = 1}^{s - r - 1} \left[ |\nu, \mu, \lambda, | q + 2 \right] K_2(s - j, r - l)
\]
\[
+ \frac{1}{(\nu + 1)^3} \sum_{l = q + 1}^{s - 1} \left[ |\nu, \mu, \lambda, | q + 2 \right] K_2(s, r, l)
\]
\[
+ |\nu, \mu, \lambda, | q + 2 \right] K_2(s, q) + \frac{1}{(\nu + 1)^3} \sum_{l = r + 1}^{s - 1} \left[ |\nu, \mu, \lambda, | q + 2 \right] K_2(l, q)
\]
\[
\times \psi_q \psi_r \psi_s |0\rangle = n(\nu)n(\mu)n(\lambda)
\]
\[
\times \sum_{1 \leq q < r < s \leq L} \sum_{T \in T_1} T \left[ |\nu, \mu, \lambda, | q + 2 \right] [\nu, \mu, \lambda, ]^{\nu - \mu + 1}_{\nu - \mu}
\]
\[
\times \frac{\nu - \lambda + 1_{\nu - \lambda}}{\nu - \lambda + 1 - \mu} \psi_q \psi_r \psi_s |0\rangle.
\]
\[\text{APPENDIX B}\]
\[\text{DETAILS IN XXZ}\]
It is convenient to introduce the following notation:
\[d(\lambda) = 1 - \lambda, \quad b(\lambda) = \lambda - q - \lambda, \quad a(\lambda) = \lambda - q - \lambda.\]
We see that coefficient (10.3) resp. normalization (10.4) can be written as
\[\lambda_q = a(\lambda) d(\lambda), \quad q(\lambda) = d(\lambda) b(\lambda) a(\lambda).\]
The $R$-matrix $\hat{R}_{k, k+1}(\lambda)$ is of the form (9.2) and the operator $B(\lambda)$ is of the form (9.8). For computing of Bethe vectors, the following set of identities seems to be very useful:
\[\hat{R}_{k, k+1}(\lambda) \psi_k |0\rangle = d(\lambda) \psi_k |0\rangle + b(\lambda) \psi_{k+1} |0\rangle, \quad (B.3)\]
\[\hat{R}_{k, k+1}(\lambda) \psi_{k+1} |0\rangle = \lambda b(\lambda) \psi_k |0\rangle + d(\lambda) \psi_{k+1} |0\rangle, \quad (B.4)\]
\[\hat{R}_{k, k+1}(\lambda) \psi_j = a(\lambda) \psi_j \quad \text{for} \quad j \notin \{k, k+1\}, \quad (B.5)\]
and
\[\hat{R}_{l, i+1} \ldots \hat{R}_{k, k+1}(\lambda) \psi_k |0\rangle = d(\lambda)^{k-i} \psi_l |0\rangle, \quad (B.6)\]
\[\hat{R}_{k, k+1}(\lambda) \psi_k |0\rangle = \lambda b(\lambda) \psi_{k+1} |0\rangle + d(\lambda) \psi_k |0\rangle, \quad (B.7)\]
Using (B.6), we can straightforwardly calculate the $q$-deformed 1-magnon state for $B(\mu)$ defined in (9.8)
\[|\mu, \rangle = B(\mu)|0\rangle = \frac{d(\mu)^L b(\mu)}{a(\mu)} \times \sum_{k = 1}^{L} \left( a(\mu) \right)_k \psi_k |0\rangle, \quad (B.8)\]
recalling (10.3) and (10.4).
Using the formulas mentioned (B.3)–(B.7), we can calculate the $q$-deformed 2-magnon state. First of all we need $B(\lambda) \psi_k |0\rangle$
\[B(\lambda) \psi_k |0\rangle = b(\lambda) a(\lambda) d(\lambda)^{L-k} \sum_{i = 0}^{k-2} [\lambda_q^q \psi_{k-i+1}] \psi_k |0\rangle
\]
\[+ b(\lambda) a(\lambda) d(\lambda)^{L-k} \sum_{j = 1}^{k-1} [\lambda_q^q \psi_{k+j}] \psi_k |0\rangle, \quad (B.9)\]
Hence, we get the $q$-deformed 2-magnon state
\[|\lambda, \mu, \rangle = B(\lambda) B(\mu)|0\rangle
\]
\[= n_q(\mu) \sum_{k = 1}^{L} \left[ \lambda_q^q B(\mu) \psi_k |0\rangle \right] = n_q(\mu) n_q(\lambda), \quad (B.10)\]
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