Research Article

Weighted Differentiation Composition Operators to Bloch-Type Spaces

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all functions holomorphic on $\mathbb{D}$, $dA(z) = (1/\pi) dx dy$ the normalized area measure on $\mathbb{D}$, and $H^\infty$ the space of all bounded holomorphic functions with the norm

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

Let $\alpha > 0$. The $\alpha$-Bloch space $B^\alpha$ on $\mathbb{D}$ is the space of all holomorphic functions $f$ on $\mathbb{D}$ such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

The little $\alpha$-Bloch space $B_0^\alpha$ consists of all $f \in B^\alpha$ such that

$$\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0.$$

Both spaces $B^\alpha$ and $B_0^\alpha$ are Banach spaces with the norm

$$\|f\|_{B^\alpha} = \|f(0)\| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|,$$

and $B_0^\alpha$ is a closed subspace of $B^\alpha$. If $\alpha = 1$, they become the classical Bloch space $B$ and little Bloch space $B_0$, respectively. For any $\alpha > 0$, the space $B_0^\infty$ consists of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{B_0^\infty} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty.$$

For information of such spaces, see, for example, [1–4].

Let $\varphi$ and $\psi$ be holomorphic maps on the open unit disk $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. For a nonnegative integer $n$, we define a linear operator $D^{\alpha}_n$ as follows:

$$D^{\alpha}_n f = \psi \cdot (f^{(n)} \circ \varphi), \quad f \in H(\mathbb{D}).$$

We call it weighted differentiation composition operators, which was defined in [6,7]. If $n = 0$ and $\psi \equiv 1$, $D^{\alpha}_n$ becomes $C_{\varphi}$ induced by $\varphi$, defined as $C_{\varphi} f = f \circ \varphi$, $f \in H(\mathbb{D})$. If $\psi = 1$ and $\varphi(z) = z$, then $D^\alpha$ is the differentiation operator defined as $D^\alpha f = f^{(n)}$. If $n = 0$, then we get the weighted...
composition operator $\psi C_\varphi$ defined as $\psi C_\varphi f = \psi \cdot (f \circ \varphi)$. If $n = 1$ and $\varphi(z) = \varphi'(z)$, then $D^n_{\varphi \psi}$ reduces to $DC_\varphi$. When $\varphi \equiv 1$, then $D^n_{\varphi \psi}$ reduces to differentiation composition operator $C_\varphi D^n$ (also named as product of differentiation and composition operator). If we put $\varphi(z) = z$, then $D^n_{\varphi \psi} = M_\varphi D^n$, the product of multiplication and differentiation operator.

The boundedness and compactness of differentiation composition operator between spaces of holomorphic functions have been studied extensively. For example, Hibschwei- ler; Portnoy and Ohno studied differentiation composition operator $C_\varphi D$ on Hardy and Bergman spaces in [8, 9]; Li; Stević and Ohno studied $C_\varphi D$ on Bloch type spaces in [10–12]; Wu and Wulan gave a new compactness criterion of $C_\varphi D^n$ on the Bloch space in [13]. Recently, the weighted differentiation composition operator between different function spaces has also been investigated by several authors (see, for example, [14–21]).

Boundedness, compactness, and essential norm of weighted composition operator $\psi C_\varphi$ between Bloch-type spaces have been studied in [22–24]. Recently, Manhas and Zhao [25] and Hyvärinen and Lindström [26] gave a new characterization of boundedness and compactness of $\psi C_\varphi$, in terms of the norm of $\varphi^n$ (for the compactness of composition operator, see [27, 28]).

Motivated by [13, 25, 26], we study the operator $D^n_{\varphi \psi}$ ($n \geq 1$) from BMOA and Bloch space to Bloch-type spaces.

Throughout this paper, constants are denoted by $C$; they are positive and not necessarily the same at each occurrence. The notation $A \leq B$ means that there is a positive constant $C$ such that $A \leq CB$. When $A \leq B$ and $B \leq A$, we write $A \approx B$.

### 2. Some Lemmas

It is well known that $H^\infty \subset \text{BMOA} \subset \mathcal{B}$. From the definition of the norm, we know

$$\|f\|_{\text{BMOA}} \leq \|f\|_\infty, \quad f \in H^\infty. \quad (9)$$

Indeed, Girela proved that

$$\|f\|_\mathcal{B} \leq \|f\|_{\text{BMOA}}, \quad (10)$$

in Corollary 5.2 of [5]. The following lemma is from Lemma 5 in [29] (see also Lemma 4.12 of [4]).

**Lemma 1.** If $f \in H(\mathbb{D})$, then

$$|f(0)|^2 \leq 2 \int_{\mathbb{D}} |f(z)|^2 \log \frac{1}{|z|} dA(z). \quad (11)$$

The following lemma may be known, but we fail to find its reference; so we give a proof for the completeness of the paper.

**Lemma 2.** Let $f \in H(\mathbb{D})$. Then,

$$\|f\|_{\mathcal{B}} \leq \|f\|_{\text{BMOA}}. \quad (12)$$

**Proof.** Applying Littlewood-Paley identity

$$\|f\|_{H^2}^2 = |f(0)|^2 + 2 \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z) \quad (13)$$

and Lemma 1, we have

$$\sup_{a \in \mathbb{D}} \|f \circ \sigma_n - f(a)\|_{H^2} \leq \sup_{a \in \mathbb{D}} \left(2 \int_{\mathbb{D}} \left|f'(\sigma_n(z))\right|^2 \log \frac{1}{|z|} dA(z) \right)^{1/2} \geq \sup_{a \in \mathbb{D}} \left(1 - |a|^2 \right) \left|f'(a)\right|. \quad (14)$$

It follows from the definitions of Bloch space and BMOA space that

$$\|f\|_\mathcal{B} \leq \|f\|_{\text{BMOA}}. \quad (15)$$

By Theorem 6.2 of [5] and the proof of Theorem 1 of [30], we have the following lemma.

**Lemma 3.** Let $n$ be a fixed positive integer and $f \in \mathcal{B}$ with $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. If

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^2 \left(1 - |z|^2\right)^{2n-2} \left(1 - |\sigma_n(z)|^2\right) dA(z) \leq 1,$$

then $\|f\|_{\text{BMOA}} \leq 1$.

**Lemma 4.** Suppose that $n$ is a fixed positive integer. Let $k \in \mathbb{N}^+$, $0 \leq x \leq 1$, and

$$H_k^n(x) = \begin{cases} k(k-1) \cdots (k-n+1)(1-x)^n x^{-n} & \text{if } k > n, \\ n!(1-x)^n & \text{if } k = n. \end{cases} \quad (17)$$

If $k \geq n$, then there are two positive constants $c_n$ and $C_n$, depending only on $n$, such that

$$c_n \leq H_k^n(x) \leq C_n, \quad \text{for } \frac{k-n}{k} \leq x \leq \frac{k-n+1}{k+1}. \quad (18)$$

**Proof.** The proof is similar to that of Lemma 2.2 of [13] and is so omitted. \qed

### 3. Boundedness of $D^n_{\varphi \psi}$

In this section, we characterize the boundedness of $D^n_{\varphi \psi}$ from BMOA and the Bloch space to Bloch-type spaces.

**Theorem 5.** Let $\alpha > 0$, $\psi \in H(\mathbb{D})$, $n \in \mathbb{N}^+$, and $\varphi$ a holomorphic self-map of $\mathbb{D}$. Then, the following statements are equivalent:

(a) $D^n_{\varphi \psi} : \text{BMOA} \to \mathcal{B}^\alpha$ is bounded.

(b) $D^n_{\varphi \psi} : \mathcal{B}^{\alpha_0} \to \mathcal{B}^{\alpha_0}$ and $D^{n+1}_{\varphi \psi} : \text{BMOA} \to \mathcal{B}^{\alpha_0}$ are bounded.

(c) $D^n_{\varphi \psi} : \mathcal{B}_0 \to \mathcal{B}^\alpha$ is bounded.
We obtain
\[
D_{\varphi,\psi}^{n} f_{\lambda} \geq D_{\varphi,\psi}^{n} f_{\lambda}^{(n)} (\varphi (\lambda)) + \psi (\lambda) \varphi' (\lambda) f_{\lambda}^{(n+1)} (\varphi (\lambda)) \geq (n+1)! \left( \frac{1 - |\varphi (\lambda)|^{2}}{1 - |\varphi (\lambda)|^{2}} \right)^{\frac{1}{n+1}} |\varphi (\lambda)| \varphi' (\lambda).
\]
(25)

Thus, for any \( r_{0} \in (0, 1) \), we have
\[
\sup_{|\lambda| \leq r_{0}} \left( \frac{1 - |\varphi (\lambda)|^{2}}{1 - |\varphi (\lambda)|^{2}} \right)^{\frac{1}{n+1}} |\varphi (\lambda)| \varphi' (\lambda) < \infty.
\]
(26)

Using (21) yields
\[
\sup_{|\lambda| \leq r_{0}} \left( \frac{1 - |\varphi (\lambda)|^{2}}{1 - |\varphi (\lambda)|^{2}} \right)^{\frac{1}{n+1}} |\varphi (\lambda)| \varphi' (\lambda) \leq \frac{1}{(1 - r_{0}^{2})^{n+1}} \sup_{\lambda \in \mathbb{D}} \left( \frac{1 - |\varphi (\lambda)|^{2}}{1 - |\varphi (\lambda)|^{2}} \right)^{\frac{1}{n+1}} |\varphi (\lambda)| \varphi' (\lambda) < \infty.
\]
(27)

Combining (26) with (27), we get
\[
\sup_{|\lambda| \leq r_{0}} \left( \frac{1 - |\varphi (\lambda)|^{2}}{1 - |\varphi (\lambda)|^{2}} \right)^{\frac{1}{n+1}} |\varphi (\lambda)| \varphi' (\lambda) < \infty.
\]
(28)

We next consider the function
\[
g_{\lambda} (z) = (n+2) \frac{1 - |\varphi (\lambda)|^{2}}{1 - |\varphi (\lambda)| z} - \left( \frac{1 - |\varphi (\lambda)|^{2}}{1 - |\varphi (\lambda)| z} \right)^{n+1}, \quad \lambda \in \mathbb{D}.
\]
(29)

Similarly, we get \( g_{\lambda} \in \mathcal{B}_{0} \cap \text{BMOA} \) and
\[
\| g_{\lambda} \|_{\text{BMOA}} \leq \| g_{\lambda} \|_{\infty} \leq 1.
\]
(30)

Moreover,
\[
g_{\lambda}^{(n)} (z) = n! \left( \frac{1 - |\varphi (\lambda)|^{2}}{1 - |\varphi (\lambda)| z} \right) - \left( \frac{1 - |\varphi (\lambda)|^{2}}{1 - |\varphi (\lambda)| z} \right)^{n+1} \left( \frac{1 - |\varphi (\lambda)|^{2}}{1 - |\varphi (\lambda)| z} \right)^{n+2}.
\]
(31)

So
\[
g_{\lambda}^{(n)} (\varphi (\lambda)) = \frac{n! \left( \frac{1 - |\varphi (\lambda)|^{2}}{1 - |\varphi (\lambda)| z} \right)^{n}}{\left( 1 - |\varphi (\lambda)|^{2} \right)^{n+1}}.
\]
(32)
and $g(z) \in BMO$. We have, as above,
\[
\left\| D^n_{\phi,\psi} f \right\|_{BMO} \geq \left\| D^n_{\phi,\psi} g \right\|_{BMO}
\]
\[
\geq n \left( \frac{1-|\lambda|^2}{1-|\phi(\lambda)|^2} \right)^{\alpha} |\psi'(\lambda)|.
\]
(33)
Thus, for any $s_0 \in (0, 1),$
\[
\sup_{s \leq s_0 < \lambda} \left( \frac{1-|\lambda|^2}{1-|\phi(\lambda)|^2} \right) |\psi'(\lambda)| < \infty.
\]
(34)
Applying (20), we get
\[
\sup_{\lambda \in \mathbb{D}} \left( \frac{1-|\lambda|^2}{1-|\phi(\lambda)|^2} \right) |\psi'(\lambda)| < \infty.
\]
(35)
Combining (34) with (35) yields
\[
\sup_{\lambda \in \mathbb{D}} \left( \frac{1-|\lambda|^2}{1-|\phi(\lambda)|^2} \right) |\psi'(\lambda)| < \infty.
\]
(36)
If $\eta = \infty$, then for any positive integer $N$, we can find $b \in \mathbb{D}$ such that
\[
\frac{(1-|a|^2)^{\alpha}}{(1-|\phi(b)|^2)^{\alpha}} |\psi'(b)| > N.
\]
(37)
If $\phi(b) = 0$, then choose the test function $g(z) = z^n$. It is clear that $g \in BMO$. From Lemma 2, we have
\[
\|g\|_{BMO} \leq \|g\|_{BMO} \leq \|g\|_\infty = 1.
\]
(39)
So
\[
\left\| D^n_{\phi,\psi} f \right\|_{BMO} \geq \left\| D^n_{\phi,\psi} g \right\|_{BMO} \geq \left( 1-|b|^2 \right)^{\alpha} |\psi'(b)| > N.
\]
(40)
If $\phi(b) \neq 0$, consider the function
\[
g(z) = \frac{(1-|a|^2)^n}{(1-|a|^2)^n} \sum_{j=0}^{\infty} c_j z^j,
\]
(41)
where $a = \phi(b)$. Let $F(z) = \sum_{j=0}^{\infty} c_j z^j$. Then, $F(0) = F'(0) = \cdots = F^{(n-1)}(0) = 0$ and
\[
F^{(n)}(z) = \left( 1-|a|^2 \right)^n.
\]
(42)
It is easy to see that
\[
(1-|z|^2)^n |F^{(n)}(z)| = \left( 1-|a|^2 \right)^n \leq 1.
\]
(43)
So, by Theorems 5.4 and 5.13 of [4], we have $F \in \mathcal{B}_0$ and $\|F\|_{BMO} \leq 1$. By Lemma 1 of [31] and Lemma 3, we get
\[
\|F\|_{BMO} \leq 1.
\]
(44)
Since $N$ is arbitrary, we get $\|D^n_{\phi,\psi} f\|_{BMO} = \infty$. This contradicts the boundedness of $D^n_{\phi,\psi} : BMO \rightarrow A^\infty$. Now, suppose that $D^n_{\phi,\psi} : BMO \rightarrow A^\infty$ is bounded or $D^{n+1}_{\phi,\psi} : BMO \rightarrow A^\infty$ is bounded. Set
\[
\eta = \sup_{z \in \mathbb{D}} \left( 1-|\phi(z)|^2 \right)^{\alpha} |\psi(z)\psi'(z)|.
\]
(45)
If $\eta = \infty$, then for any positive integer $M$, exists $u \in \mathbb{D}$ such that
\[
\frac{(1-|u|^2)^{\alpha}}{(1-|\phi(u)|^2)^{\alpha}} |\psi(u)\psi'(u)| > M.
\]
(46)
If $\psi(u) = 0$, then set $g(z) = z^{n+1}$. The process as above gives
\[
\left\| D^{n+1}_{\phi,\psi} f \right\|_{A^\infty} \geq \left\| D^{n+1}_{\phi,\psi} g \right\|_{A^\infty} > M.
\]
(47)
If $\psi(u) \neq 0$, consider the function
\[
g(z) = \frac{(1-|a|^2)^{\frac{n+1}{2}}}{(1-|a|^2)^{\frac{n+1}{2}}} \sum_{j=0}^{\infty} c_j z^j,
\]
(48)
where $a = \psi(u)$. Let $F(z) = \sum_{j=0}^{\infty} c_j z^j$. Then, $F(0) = F'(0) = \cdots = F^{(n)}(0) = 0$ and
\[
F^{(n+1)}(z) = \left( 1-|a|^2 \right)^{n+1} |\psi(u)\psi'(u)| > M.
\]
(49)
Applying Theorems 5.4 and 5.13 of [4] again yields $F \in \mathcal{B}_0$ and $\|F\|_{BMO} \leq 1$. We get $\|F\|_{BMO} \leq 1$ and
\[
\left\| D^{n+1}_{\phi,\psi} f \right\|_{A^\infty} \geq \left\| D^{n+1}_{\phi,\psi} g \right\|_{A^\infty} > M.
\]
(50)
Since $M$ is arbitrary, we have $\|D^{n+1}_{\phi,\psi} f\|_{A^\infty} = \infty$. This contradicts the boundedness of $D^{n+1}_{\phi,\psi}$.\]
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(III): (g) ⇒ (e), (g) ⇒ (f). Note that
\[
\|D^n_{\varphi,\psi}f\|_{_{\mathcal{B}^a}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| \psi(z) \varphi'(z) f^{(n+1)}(z) \right| \\
+ \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| \psi'(z) \varphi(z) \right|
\]
\[
\leq \|f\|_{_{\mathcal{B}^a}} \left[ \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| \psi(z) \varphi'(z) \right| \\
+ \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| \psi'(z) \right| \right],
\]
\[
\|D^{n+1}_{\varphi,\psi}f\|_{_{\mathcal{B}^a}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| \psi(z) \varphi'(z) f^{(n+1)}(z) \right| \\
\leq \|f\|_{_{\mathcal{B}^a}} \left[ \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| \psi(z) \varphi'(z) \right| \\
+ \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| \psi'(z) \right| \right].
\]

The desired results follow.

(IV): (f) ⇔ (h). Suppose that (f) is true. It follows from Proposition 5.1 of [4] that \(\|z^k\|_{_{\mathcal{D}}} \leq \|z^k\|_{_{\mathcal{B}}} < \infty\). So,
\[
\sup_{k \in \mathbb{N}} \|D^n_{\varphi,\psi}(z^k)\|_{_{\mathcal{B}^a}} < \infty,
\]
\[
\sup_{k \in \mathbb{N}} \|D^{n+1}_{\varphi,\psi}(z^k)\|_{_{\mathcal{B}^a}} < \infty.
\]

Conversely, assume that (h) is true. It is easy to see that
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| \psi'(z) \right| \leq \|D^n_{\varphi,\psi}(z^n)\|_{_{\mathcal{B}^a}} < \infty,
\]
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| \psi(z) \varphi'(z) \right| \leq \|D^{n+1}_{\varphi,\psi}(z^{n+1})\|_{_{\mathcal{B}^a}} < \infty.
\]

If \(\|\varphi\|_{_{\infty}} < 1\), then
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| \psi'(z) \right| \\
< \frac{1}{(1 - |\varphi(z)|^2)^{\alpha}} \sup_{z \in \mathbb{D}} (1 - |\varphi(z)|^2)^{\alpha} \left| \psi'(z) \right| < \infty,
\]
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| \psi(z) \varphi'(z) \right| \\
< \frac{1}{(1 - |\varphi(z)|^2)^{\alpha+1}} \sup_{z \in \mathbb{D}} (1 - |\varphi(z)|^2)^{\alpha+1} \left| \psi(z) \varphi'(z) \right| < \infty.
\]

(54)

Hence, (g) is true. From (g) ⇒ (f), we obtain that (f) is also true.

From now on, we assume that \(\|\varphi\|_{_{\infty}} = 1\). For any integer \(k \geq n\), let
\[
\Delta^*_k = \{z \in \mathbb{D} : \frac{k-n}{k+1} \leq |\varphi(z)| \leq \frac{k-n+1}{k+1}\}.
\]

(55)

Let \(m \geq n\) be the smallest positive integer such that \(\Delta^*_m \neq \emptyset\). Since \(\Delta^*_k\) is not empty for every integer \(k \geq m\) and \(D = \cup_{k=m}^{\infty} \Delta^*_k\). By Lemma 4, for \(f \in \mathcal{B}\),
\[
\|D^n_{\varphi,\psi}f\|_{_{\mathcal{B}^a}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left| f^{(n)}(z) \right| \left| \psi(z) \right| \\
= \sup_{k \geq m} \sup_{z \in \Delta^*_k} \frac{1}{(1 - |\varphi(z)|^2)^{\alpha}} \left| f^{(n)}(z) \right| \left| \psi(z) \right| \\
\leq \frac{1}{c_n} \|f\|_{_{\mathcal{B}}} \sup_{k \in \mathbb{N}} \|D^n_{\varphi,\psi}(z^k)\|_{_{\mathcal{B}^a}}.
\]

(56)

So, \(D^n_{\varphi,\psi} : \mathcal{B} \rightarrow \mathcal{B}^a\) is bounded. Similar argument implies
\[
\|D^{n+1}_{\varphi,\psi}f\|_{_{\mathcal{B}^a}} = \sup_{k \geq m+1} \sup_{z \in \Delta^*_k} \left(1 - |z|^2\right)^{\alpha} \left| f^{(n+1)}(z) \right| \\
\leq \frac{1}{c_{n+1}} \|f\|_{_{\mathcal{B}}} \sup_{k \in \mathbb{N}} \|D^{n+1}_{\varphi,\psi}(z^k)\|_{_{\mathcal{B}^a}}.
\]

Thus, \(D^{n+1}_{\varphi,\psi} : \mathcal{B} \rightarrow \mathcal{B}^a\) is bounded. Theorem 5 is proved. □
4. Compactness of $D^n_{\psi,\varphi}$

The following criterion for the compactness is a useful tool and it follows from standard arguments, for example, Proposition 3.11 of [32] or Lemma 2.10 of [33].

Lemma 6. Let $\alpha > 0$, $n \in \mathbb{N}^+$, and $X = B_0, B$, or BMOA. Suppose that $\psi$ and $\varphi$ are in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then, $D^n_{\psi,\varphi} : X \to \mathcal{A}^\alpha$ is compact if and only if for any sequence $\{f_m\}$ in $X$ with $\sup_{z \in \mathbb{D}} \|f_m\|_X < \infty$, which converges to zero locally uniformly on $\mathbb{D}$; we have $\lim_{m \to \infty} \|D^n_{\psi,\varphi} f_m\|_{\mathcal{A}^\alpha} = 0$.

We now give the compactness of $D^n_{\psi,\varphi}$ from BMOA and the Bloch space to Bloch-type spaces.

Theorem 7. Let $\alpha > 0$, $\psi \in H(\mathbb{D})$, $n \in \mathbb{N}^+$, and $\varphi$ a holomorphic self-map of $\mathbb{D}$. Then, the following statements are equivalent:

(a) $D^n_{\psi,\varphi} : \text{BMOA} \to \mathcal{A}^\alpha$ is compact.

(b) $D^n_{\psi,\varphi'} : \text{BMOA} \to \mathcal{A}^{n+1} \text{co}$ is compact and $D^{n+1}_{\psi,\varphi'} : \text{BMOA} \to \mathcal{A}^\alpha \text{co}$ is compact.

(c) $D^n_{\psi,\varphi} : B_0 \to \mathcal{A}^\alpha$ is compact.

(d) $D^n_{\psi,\varphi'} : B_0 \to \mathcal{A}^\alpha \text{co}$ is compact and $D^{n+1}_{\psi,\varphi'} : B_0 \to \mathcal{A}^\alpha \text{co}$ is compact.

(e) $D^n_{\psi,\varphi} : B \to \mathcal{A}^\alpha$ is compact.

(f) $D^n_{\psi,\varphi'} : B \to \mathcal{A}^\alpha \text{co}$ is compact and $D^{n+1}_{\psi,\varphi'} : B \to \mathcal{A}^\alpha \text{co}$ is compact.

(g) $\psi \in \mathcal{A}^\alpha$, $\psi' \in \mathcal{A}^\alpha \text{co}$.

lim_{|\psi(z)| \to 1} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{n+1}} |\psi'(z)| = 0,

lim_{|\psi(z)| \to 1} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{n+1}} \psi(z) \varphi'(z) = 0.

(h) $\lim_{|\psi(z)| \to 1} \sup_k \|D^n_{\psi,\varphi}(z^k)\|_{\mathcal{A}^\alpha} = 0$

and $\lim_{|\psi(z)| \to 1} \sup_k \|D^{n+1}_{\psi,\varphi}(z^k)\|_{\mathcal{A}^\alpha} = 0$.

Proof. The proof is a modification of that of Theorem 5; so we give a sketch of the proof. We will prove the theorem according to the following steps. (I): (a) $\Rightarrow$ (g), (c) $\Rightarrow$ (g).

(II): (b) $\Rightarrow$ (g), (d) $\Rightarrow$ (g). (III): (g) $\Rightarrow$ (e), (g) $\Rightarrow$ (f).

(IV): (f) $\Rightarrow$ (h).

(I): (a) $\Rightarrow$ (g), (c) $\Rightarrow$ (g). Suppose that (a) or (c) holds. Then by Theorem 5, we have

\[ \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^{\alpha} \left|\psi'(z)\right| \leq \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^{\alpha} \left|\psi'(z)\right| \leq \sup_{z \in \mathbb{D}} \left(1 - |\varphi(z)|^2\right)^{n} < \infty, \]

\[ \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^{\alpha} \left|\psi(z) \varphi'(z)\right| \leq \sup_{z \in \mathbb{D}} \left(1 - |\varphi(z)|^2\right)^{n+1} \psi(z) \varphi'(z) \leq \sup_{z \in \mathbb{D}} \left(1 - |\varphi(z)|^2\right)^{n+1} \psi(z) \varphi'(z) \leq \infty. \]

That is, $\psi \in \mathcal{A}^\alpha$, $\psi' \in \mathcal{A}^\alpha \text{co}$.

Let $\{z_j\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Now, we consider the function

\[ f_j(z) = (n+2) \left(1 - \frac{\varphi(z_j)}{1 - \varphi(z_j) z}\right)^n \left(1 - \frac{\varphi(z_j)}{1 - \varphi(z_j) z}\right)^{n+1} \varphi(z_j) \varphi'(z_j). \]

Simple computation shows that $f_j \in B_0 \cap \text{BMOA}$ and

\[ \|f_j\|_{\text{BMOA}} \leq \|f_j\|_{\infty} \leq 1. \]

It is also easy to check that $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Moreover,

\[ f_j^{(n)}(z) = (n+1) \left(\frac{\varphi(z_j)}{1 - \varphi(z_j) z}\right)^n \times \left[\frac{1 - |\varphi(z_j)|^2}{1 - \varphi(z_j) z} - \frac{1 - |\varphi(z_j)|^2}{1 - \varphi(z_j) z}\right]^{n+1}. \]

We have

\[ \|D^n_{\psi,\varphi} f_j\|_{\mathcal{A}^\alpha} \geq (n+1)! \left(\frac{1 - |z_j|^2}{1 - |\varphi(z_j)|^2}\right)^{n+1} \psi(z_j) \varphi'(z_j). \]

By Lemma 6, we get

\[ \lim_{|\psi(z)| \to 1} \sup_k \|D^n_{\psi,\varphi}(z^k)\|_{\mathcal{A}^\alpha} = 0 \]

and $\lim_{|\psi(z)| \to 1} \sup_k \|D^{n+1}_{\psi,\varphi}(z^k)\|_{\mathcal{A}^\alpha} = 0$.

We next consider the function

\[ g_j(z) = (n+2) \left(1 - \frac{\varphi(z_j)}{1 - \varphi(z_j) z}\right)^n \left(1 - \frac{\varphi(z_j)}{1 - \varphi(z_j) z}\right)^{n+1} \varphi(z_j) \varphi'(z_j). \]

Similarly, we get $g_j \in B_0 \cap \text{BMOA}$ and

\[ \|g_j\|_{\text{BMOA}} \leq \|g_j\|_{\infty} \leq 1. \]
It is easy to see that \(g_j\) converges to zero uniformly on compact subsets of \(D\) as \(j \to \infty\) and
\[
g_j^{(n)}(z) = n! \left( \varphi\left( z_j \right) \right)^n \\
\times \left[ (n + 2) \frac{1 - |\varphi(z_j)|^2}{n} \left( 1 - \frac{|\varphi(z_j)|}{|z|} \right)^m \right] \right] \\
- (n + 1) \left( \frac{1 - |\varphi(z_j)|^2}{n} \right)^2 \\
\left( 1 - \frac{|\varphi(z_j)|}{|z|} \right)^{m+2}. \tag{67}
\]
Thus,
\[
\left\| D_{\varphi,\psi}^n g_j \right\|_{B^\alpha} \geq n! \left( \frac{1 - |z_j|^2}{n} \right)^\alpha \left| \varphi(z_j) \right|^n \left| \psi'(z_j) \right|. \tag{68}
\]
Applying Lemma 6 again, we have
\[
\lim_{|\varphi(z_j)| \to 1} \left[ (n + 2) \frac{1 - |\varphi(z_j)|^2}{n} \left( 1 - \frac{|\varphi(z_j)|}{|z|} \right)^m \right] \right] \\
- (n + 1) \left( \frac{1 - |\varphi(z_j)|^2}{n} \right)^2 \\
\left( 1 - \frac{|\varphi(z_j)|}{|z|} \right)^{m+2} = 0. \tag{69}
\]
Since \(z_j \in D\) is arbitrary, we proved that (g) is true.

(II) (b) \(\Rightarrow\) (e), (d) \(\Rightarrow\) (g). Suppose that (b) or (d) holds. A similar argument to (I) shows that \(\varphi \in B^\alpha, \varphi \psi' \in B^\infty\). Now, suppose that the equations in (g) are not true. Then, there exists a sequence \(\{z_j\}\) in \(D\) and \(\delta > 0\) such that \(|\varphi(z_j)| \to 1\) as \(j \to \infty\) and
\[
\left( \frac{1 - |z_j|^2}{n} \right)^\alpha \left| \psi'(z_j) \right| \geq \delta, \tag{70}
\]
\[
\left( \frac{1 - |z_j|^2}{n} \right)^\alpha \left| \psi'(z_j) \phi'(z) \right| > \delta.
\]
Choose a subsequence of \(\{z_j\}\) if necessary and suppose that \(\inf_j |\varphi(z_j)| > 1/2\). Let
\[
f_j(z) = \frac{1 - |\varphi(z_j)|^2}{1 - \varphi(z_j)z}, \quad z \in D. \tag{71}
\]
Then, it is easy to check that \(f_j \in B^\alpha \cap \text{BMO}\), \(f_j \to 0\), uniformly on compact subsets of \(D\) and
\[
f_j^{(n)}(z) = n! \left( \frac{1 - |\varphi(z_j)|^2}{1 - \varphi(z_j)z} \right)^n. \tag{72}
\]
Thus,
\[
\left\| D_{\varphi,\psi}^n f_j \right\|_{B^\alpha} \geq n! \left( \frac{1 - |z_j|^2}{n} \right)^\alpha \left| \varphi(z_j) \right|^n \left| \psi'(z_j) \right| > \frac{n!\delta}{2n},
\]
\[
\left\| D_{\varphi,\psi}^{n+1} f_j \right\|_{B^\alpha} \geq (n + 1)! \left( \frac{1 - |z_j|^2}{n} \right)^\alpha \left| \varphi(z_j) \right|^n \left| \psi'(z_j) \right| > \frac{(n + 1)!\delta}{2n+1}. \tag{73}
\]
Those contradict the compactness of \(D_{\varphi,\psi}^n\) and \(D_{\varphi,\psi}^{n+1}\).

(III) (g) \(\Rightarrow\) (e), (g) \(\Rightarrow\) (f). Let \(\{f_m\}\) be a norm bounded sequence in \(B\) that converges to zero uniformly on compact subsets of \(D\). Let \(M = \sup_{|z| < r_0} |f_m(z)| < \infty\). For \(\epsilon > 0\), there then exists \(r_0 \in (0, 1)\) such that for \(|\varphi(z)| \to r_0\), we have
\[
\left( \frac{1 - |z|^2}{n} \right)^\alpha \left| \varphi'(z) \right| \leq \epsilon,
\]
\[
\left( \frac{1 - |z|^2}{n} \right)^\alpha \left| \varphi'(z) \phi'(z) \right| < \epsilon. \tag{74}
\]
Thus, for \(z \in D\), we have
\[
\left\| D_{\varphi,\psi}^n f_m \right\|_{B^\alpha} \leq \left| \varphi(0) f_m^{(n)}(\varphi(0)) \right| \\
+ \sup_{|\varphi(z)| \to r_0} \left( \frac{1 - |z|^2}{n} \right)^\alpha |f_m^{(n)}(\varphi(z))| \left| \psi'(z) \right| \\
+ \sup_{|\varphi(z)| > r_0} \left( \frac{1 - |z|^2}{n} \right)^\alpha \left| \psi'(z) \right| \left\| f_m \right\|_{B^\alpha} \\
+ \sup_{|\varphi(z)| \to r_0} \left( \frac{1 - |z|^2}{n} \right)^\alpha \left| \varphi(\varphi(z)) \right| \left\| f_m \right\|_{B^\alpha} \\
+ \sup_{|\varphi(z)| > r_0} \left( \frac{1 - |z|^2}{n} \right)^\alpha \left| \varphi(\varphi(z)) \right| \left\| f_m \right\|_{B^\alpha} \\
\leq \left| \varphi(0) f_m^{(n)}(\varphi(0)) \right| + K_1 \sup_{|z| < r_0} |f_m^{(n)}(z)| \\
+ K_2 \sup_{|z| > r_0} |f_m^{(n-1)}(z)| + 2\epsilon M, \tag{75}
\]
where \(K_1 = \sup_{z \in D}(1 - |z|^2)^\alpha|\psi'(z)|\) and \(K_2 = \sup_{z \in D}(1 - |z|^2)^\alpha|\psi(\varphi(z))|\). Since \(f_m^{(n)} \to 0\) uniformly on compact subsets of \(D\) as \(m \to \infty\), we have \(\left\| D_{\varphi,\psi}^n f_m \right\|_{B^\alpha} \to 0\) as \(m \to \infty\). It follows from Lemma 6 that \(D_{\varphi,\psi}^n : B \to B^\alpha\) is compact.
From $f^{(n)} \to 0$ uniformly on compact subsets of $\mathbb{D}$, we have
\[
\|D^n_{\psi,\psi'} f_m\|_{\mathcal{A}^\infty} \to 0 \quad \text{and} \quad \|D^{n+1}_{\psi,\psi'} f_m\|_{\mathcal{A}^\infty} \to 0 \quad \text{as} \quad m \to \infty.
\]
So, $D^n_{\psi,\psi'}$, $D^{n+1}_{\psi,\psi'} : \mathcal{B} \to \mathcal{A}^\infty$ are compact.

(IV): $(f) \Leftrightarrow (h)$. Suppose that $(f)$ is true. Note that $\|z^k\|_{\mathcal{A}} \leq \|z^k\|_{\mathcal{A}^\infty} = 1$ and $z^k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$; by Lemma 6, we have
\[
\lim_{k \to \infty} \|D^n_{\psi,\psi'} (z^k)\|_{\mathcal{A}^\infty} = 0,
\]
\[
\lim_{k \to -\infty} \|D^{n+1}_{\psi,\psi'} (z^k)\|_{\mathcal{A}^\infty} = 0.
\]
Conversely, assume that $(h)$ is true. It is easy to see that
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^a \|\psi' (z)\| \leq \|D^n_{\psi,\psi'} (z^n)\|_{\mathcal{A}^\infty}
\]
\[
\leq \sup_{k \in \mathbb{N}} \|D^n_{\psi,\psi'} (z^k)\|_{\mathcal{A}^\infty} < \infty,
\]
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^a \|\psi (z) \psi' (z)\| \leq \|D^{n+1}_{\psi,\psi'} (z^{n+1})\|_{\mathcal{A}^\infty}
\]
\[
\leq \sup_{k \in \mathbb{N}} \|D^{n+1}_{\psi,\psi'} (z^k)\|_{\mathcal{A}^\infty} < \infty.
\]

If $\|\psi\|_{\mathcal{A}^\infty} < 1$, from (g) $\Rightarrow$ (f), we get that $(f)$ is true. If $\|\psi\|_{\mathcal{A}^\infty} = 1$, as in the proof of Theorem 5, let
\[
\Delta^n_k = \left\{ z \in \mathbb{D} : \frac{k-n}{k} \leq |\psi (z)| \leq \frac{k-n+1}{k+1} \right\}.
\]
And let $m$ with $m \geq n$ be the smallest positive integer such that $\Delta^n_k \neq \emptyset$. For given $\varepsilon > 0$, there exists a large enough integer $M_1$ with $M_1 > m$ such that
\[
\|D^n_{\psi,\psi'} (z^n)\|_{\mathcal{A}^\infty} < \varepsilon,
\]
\[
\|\psi^n f_j\|_{\mathcal{A}^\infty} < \varepsilon,
\]
whenever $k > M_1$. Let $\{f_j\}$ be a norm bounded sequence in $\mathcal{B}$ that converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Denote $M = \sup_m \|f_m\|_{\mathcal{A}} < \infty$. We get
\[
\|D^n_{\psi,\psi'} f_j\|_{\mathcal{A}^\infty}
\]
\[
= \sup_{k \geq M} \sup_{z \in \Delta^n_k} (1 - |z|^2)^a \|f_j^{(n)} (\psi (z))\| \|\psi' (z)\|
\]
\[
= \left( \sup_{m \leq M} \sup_{k \geq M} \sup_{z \in \Delta^n_k} (1 - |z|^2)^a \|f_j^{(n)} (\psi (z))\| \|\psi' (z)\| \right)
\]
\[
\leq \|f_j\|_{\mathcal{A}} \sup_{m \leq M} \|D^n_{\psi,\psi'} (z^n)\|_{\mathcal{A}^\infty}
\]
\[
\leq \frac{1}{\varepsilon} \|f_j\|_{\mathcal{A}}.
\]

Since $f_j^{(n)} \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|D^n_{\psi,\psi'} f_j\|_{\mathcal{A}^\infty} \to 0$ as $j \to \infty$. Thus, by Lemma 6, $D^n_{\psi,\psi'} : \mathcal{B} \to \mathcal{A}^\infty$ is compact. Similar as above, we can prove that $D^{n+1}_{\psi,\psi'} : \mathcal{B} \to \mathcal{A}^\infty$ is compact. The proof is complete. \hfill \Box

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