EXTENSIONS OF $c_0$

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Abstract. If $X$ is a closed subspace of a Banach space $L$ which embeds into a Banach lattice not containing $\ell_\infty^n$'s uniformly and $L/X$ contains $\ell_\infty^n$'s uniformly, then $X$ cannot have local unconditional structure in the sense of Gordon-Lewis (GL-l.u.st.).

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0. Introduction.

Fifteen years ago, Bourgain [Bou] gave the first example of an uncomplemented subspace of an $L_1$ space which is itself isomorphic to an $L_1$ space. He asked whether there was a “natural” example of this phenomenon. In particular, if one takes the kernel $X$ of the quotient mapping from $L_1$ onto $c_0$ given by $f \mapsto \{ \int f \cdot r_n \}_{n=1}^{\infty}$, where $\{r_n\}_{n=1}^{\infty}$ are the Rademacher functions, Bourgain asked whether $X$ is isomorphic to $L_1$ or whether at least $X$ is a $L_1$ space. Of course, this space $X$ is not complemented in $L_1$ because the quotient space $L_1/X \equiv c_0$ does not embed into $L_1$. Actually, Bourgain attributes these questions to Pisier; at any rate, both of them as well as e.g. Kisliakov, Zippin, Schechtman, and I thought about them around that time. Recently Kalton and Pełczyński [KP] solved these problems in the negative. In fact, they showed that if $X$ is a subspace of $L_1$ and $c_0$ embeds into $L_1/X$, then $X$ is uncomplemented in its bidual (so that $X$ is not isomorphic to an $L_1$ space) and there is an operator from $X$ into a Hilbert space which is not absolutely summing (so that $X$ is not a $L_1$ space [LP]). While lecturing on their results in 1995 and 1996, the authors of [KP] asked whether such an $X$ could have local unconditional structure (l.u.st.), [DPR]. In this note we give a negative answer to this question and go on to show in Corollary 2.2 that such an $X$ cannot even have GL-l.u.st. [GL].

The algebraic point of view, important for [KP], is critical for this paper. In fact, once one draws the diagram (2.3) and completes it to (2.4), one realizes that the answer to the Kalton-Pełczyński question is already contained in their paper [KP]! While the proof of the stronger result that $X$ fails GL-l.u.st. if $L_1/X$ contains a copy of $c_0$ does use some new analytical lemmas which are generalizations of lemmas in [KP], no doubt the authors of [KP] would have discovered and proved them had they looked at (2.4).

For the most part we use standard Banach space theory terminology, as can be found in [LT1], [LT2]. However, since the algebraic point of view is so important for us, in section 1 we introduce some standard algebraic terminology and rephrase in the language of homological algebra some known and essentially known results from Banach space theory.

I thank Mariusz Wodzicki for reminding me that it is OK to “think algebraically”, and Alvaro Arias for reading and correcting a preliminary version of this paper.
1. Algebraic preliminaries.

In this section we review some facts about Banach spaces used in the sequel, but phrase them in the language of homological algebra. The analytical facts, except for Proposition 1.7, which is from [KP], have either been known for twenty years or are small generalizations of such facts. The algebraic point of view provides a good framework for organizing these analytical results and makes it much easier to see how to approach the problem of Bourgain–Pisier and the related one of Kalton–Pełczyński mentioned in the introduction. While the algebraic point of view is important in the work of Kalton and Pelczyński [KP], the language used in [KP] is more standard for Banach space theory. Some of what we describe appears in Domański’s paper [Dom1] and dissertation [Dom2] and the draft of the book of Castillo and González [CG]. Also, Kalton himself [Kal] exposed some of what we treat in the process of developing a $\mathcal{L}_{p}$-space theory for $0 < p < 1$.

We have included rather more material in this section than is needed for solving the problem of Kalton and Pelczyński mentioned in the Introduction in the expectation that the algebraic point of view will be useful for attacking other problems in Banach space theory.

The category we work in is usually denoted by $\text{Ban}$; the objects are Banach spaces and the morphisms are bounded linear operators. A sequence $\cdots \rightarrow X_{j} \rightarrow X_{j+1} \rightarrow X_{j+2} \rightarrow \cdots$ of morphisms in $\text{Ban}$ is called exact provided it is exact in the larger Abelian category $\text{Vect}$ of vector spaces with linear maps as morphisms. This just means the range of each of the bounded linear operators $X_{j} \rightarrow X_{j+1}$ is the kernel of the succeeding one $X_{j+1} \rightarrow X_{j+2}$. So the diagram $0 \rightarrow X \overset{J}{\rightarrow} L \overset{Q}{\rightarrow} Y \rightarrow 0$ is a short exact sequence exactly when $J$ is an isomorphic embedding and $Q$ is surjective with kernel $JX$; that is, up to the usual identifications (here we avoid discussing “natural isomorphisms”), $X$ is a subspace of $L$ and $Y$ is the quotient space $L/X$. The short exact sequence $0 \rightarrow X \overset{J}{\rightarrow} L \overset{Q}{\rightarrow} Y \rightarrow 0$ is said to be an extension of $Y$ and a coextension of $X$. We abuse language by calling both the sequence itself and the space $L$ an extension of $Y$ by $X$ and a coextension of $X$ by $Y$. In Banach space theory it is more common to call $L$ a twisted sum of $X$ with $Y$, but here we shall use the categorical language.

Given a diagram

$$
\begin{array}{ccc}
L & \overset{Q}{\rightarrow} & Y \\
\uparrow & & \uparrow u \\
Z & & \\
\end{array}
$$

we say $u$ factors through $Q$ or, when $Q$ is understood, lifts to $L$ provided there is an operator $\bar{u} : Z \rightarrow L$ making the following diagram commute:

$$
\begin{array}{ccc}
L & \overset{Q}{\rightarrow} & Y \\
\downarrow & \bar{u} & \uparrow u \\
Z & & \\
\end{array}
$$

Dually, given the diagram

$$
\begin{array}{ccc}
X & \overset{J}{\rightarrow} & L \\
\downarrow u & & \\
Z & & \\
\end{array}
$$

we say $u$ factors through $J$ or, when $J$ is understood, lifts to $X$ provided there is an operator $\bar{u} : Z \rightarrow X$ making the following diagram commute:

$$
\begin{array}{ccc}
X & \overset{J}{\rightarrow} & L \\
\downarrow & \bar{u} & \uparrow u \\
Z & & \\
\end{array}
$$
we say that \( u \) factors through \( J \) or, when \( J \) is understood, extends to \( L \), provided there is an operator \( \hat{u} : L \to Z \) making the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{J} & L \\
u \downarrow & & \downarrow \hat{u} \\
Z
\end{array}
\]

Usually when these concepts are used, \( Q \) is surjective and \( J \) is an isomorphic embedding.

Let \( \{G_n\}_{n=1}^\infty \) be a sequence of finite dimensional spaces which is dense, in the sense of the Banach-Mazur distance, in the collection of all finite dimensional spaces, and let \( C_p \) be the \( \ell_p \)-sum of \( \{G_n\}_{n=1}^\infty \) when \( 1 \leq p \leq \infty \) and let \( C_0 \) be the \( c_0 \)-sum of \( \{G_n\}_{n=1}^\infty \) (this notation differs slightly from what we used [J1] when we introduced these spaces). It is also convenient to use nonseparable versions of these spaces, so given an infinite cardinal \( \kappa \), let \( C_p(\kappa) \) be the \( \ell_p \) sum of \( \kappa \) copies of \( C_p \) (\( c_0 \) sum when \( p = 0 \)). Actually, separability plays no role in our use of \( C_p \) and from a categorical perspective it would be more natural to use everywhere the \( \ell_p \) sum of all finite dimensional subspaces of \( \ell_\infty \), each repeated \( \kappa \)-times, where \( \kappa \) is suitably large, but...

We now come to the definitions of colocal extension and local lifting which are perhaps not so well known but play an important role in our investigation (and, implicitly, in that of [KP]). While seemingly particular to the category \( \text{Ban} \), Wodzicki has pointed out that there are analogues of these concepts in some categories studied by algebraists. Referring again to the lifting diagram (1.1), we say that \( u \) locally factors through \( Q \) or, when \( Q \) is understood, locally lifts to \( L \), provided that for every operator \( w : C_1 \to Z \), the composition \( uw \) factors through \( Q \). Notice that this is just an economical way of saying that for every finite dimensional subspace \( E \) of \( Z \), there is a factorization \( \tilde{u}_E \) through \( Q \) of the restriction \( u_E \) of \( u \) to \( E \) so that \( \sup_E ||\tilde{u}_E|| < \infty \).

If, in (1.1), \( Q \) is quotient mapping, then every operator from a \( L_1 \) space into \( Y \) locally lifts to \( L \).

Dually, referring to the diagram (1.2), we say that \( u \) colocally factors through \( J \) or, when \( J \) is understood, colocally extends to \( L \), provided that for every operator \( w : Z \to C_\infty \), the composition \( wu \) factors through \( J \). Notice that this is just an economical way of saying that for every finite dimensional quotient space \( E \) of \( Z \), there is a factorization \( \tilde{q}_E \tilde{u} \) through \( J \) of the composition of \( u \) with the quotient mapping \( q_E \) of \( Z \) onto \( E \) so that \( \sup_E ||\tilde{q}_E\tilde{u}|| < \infty \).

If, in (1.2), \( J \) is an isomorphic embedding, then every operator from \( X \) into a \( L_\infty \) space colocally extends to \( L \). While not obvious from the definition, this follows from Proposition 1.1 and the fact that the second dual of a \( L_\infty \) space is injective. In Proposition 1.1 we make use of the fact [FJT] that the identity on any Banach space colocally factors through the embedding of the space into its bidual.

Although we do not really need the spaces \( C_1 \) and \( C_\infty \) in the later sections, it is interesting to note that they allow the concepts of local lifting and colocal extensions to be expressed in the language of \( \text{Ban} \). Moreover, Wodzicki has pointed out that these spaces are useful in the study of deeper algebraic properties of \( \text{Ban} \).

From a categorical perspective, the definition of colocal extension is the “right” definition since it is evident that it is dual (in the sense of category theory) to the definition of local lifting. However, from the perspective of the local theory of Banach spaces, probably the most natural definition is item (2) in Proposition 1.1;
Proposition 1.1. Consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{J} & L \\
\downarrow u & & \downarrow \\
Z & & \\
\end{array}
\]

(1.2)

The following are equivalent:

(1) \(u\) coloca\-ly extends to \(L\).

(2) For every closed subspace \(W\) of \(L\) containing \(JX\) as a finite codimensional sub-

space, there is an operator \(u_W : W \to Z\) so that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{J} & W \\
\downarrow u & \nearrow u_W & \\
Z & & \\
\end{array}
\]

commutes and \(\sup_W \|u_W\| < \infty\).

(3) \(u^*\) factors through \(J^*\).

(4) \(u^{**}\) factors through \(J^{**}\).

(5) The operator \(X^{\downarrow u} \overset{i}{\to} Z^{**}\) factors through \(J\), where \(Z^{i\to}Z^{**}\) is the canonical em-

bedding.

(6) For every operator \(Z \overset{w}{\to} M\) with \(M\) a dual space, \(wu\) factors through \(J\).

(7) For each cardinal number \(\aleph\) and every operator \(Z \overset{w}{\to} C_\infty(\aleph)\), \(wu\) factors through \(J\).

Proof. (3) implies (4) by taking adjoints, while the reverse implication follows from

taking adjoints and using the fact that every dual space is norm one complemented

in its bidual.

(4) \(\implies\) (6) follows from the following commutative diagram (each vertical arrow

is the canonical embedding of the space into its bidual) and the complementation

of \(M\) in \(M^{**}\):

\[
\begin{array}{ccc}
X & \overset{u}{\to} & Z \overset{w}{\to} M \\
\downarrow & & \downarrow & & \downarrow \\
X^{**} & \overset{u^{**}}{\to} & Z^{**} \overset{w^{**}}{\to} M^{**} \\
J^{**} \downarrow & & \uparrow \\
L^{**} = & & L^{**} \\
\end{array}
\]

(6) implies (7) because \(C_\infty(\aleph) = C_1(\aleph)^\ast\), while (6) \(\implies\) (5) is formal.

(7) \(\implies\) (1) is formal and (1) \(\implies\) (7) is essentially obvious.

The implication (2) \(\implies\) (3) uses the “Lindenstrauss compactness method”

and involves only a small variation of an argument in [J2], so we just outline the

proof. Extend each of the operators \(u_W\) to (nonlinear, discontinuous) mappings \(v_W\)

from \(L\) to \(Z\) by defining \(v_W(y)\) to be 0 when \(y\) is not in \(W\). The \(W\)'s are directed

by inclusion and thereby generate a net of functions from \(Z^\ast\) into \(R^L\) defined by

\(v_W^\#(z^\ast)(y) = z^\ast(v_Wy)\). It is easy to verify that the net \(\{v_W^\#\}\) has a cluster point

\(v : Z^\ast \to R^L\) in the product space \((R^L)^{Z^\ast}\) and that \(v\) is in fact a bounded linear

operator from \(Z^\ast\) into \(L^\ast\), and that \(u^\ast = J^*v\).

The aforementioned Lindenstrauss compactness method was used in the same

way in [J2] to prove that if \(M\) is a separable Banach space, then \(M^\ast\) is isometrically

isomorphic to a norm one complemented subspace of \(C_\infty\). (Actually, that is why
we outlined the proof of (2) \(\Rightarrow\) (3) rather than the slightly simpler direct proof of (2) \(\Rightarrow\) (4).) A similar argument yields that if the density character of \(M\) is \(\aleph\), then \(M^*\) is isometrically isomorphic to a norm one complemented subspace of \(C_\infty(\aleph)\). This gives (7) \(\Rightarrow\) (6).

The implication (5) \(\Rightarrow\) (2) follows from Lemma 2.9 in [FJT] (which is, in turn, a simple consequence of the Principle of Local Reflexivity [LR] in the form given in [JRZ]), which says that condition (2) is true in the special case when \(L\) is the space \(X^{**}\), \(J\) is the canonical embedding, and \(u\) is the identity operator on \(X\). Indeed, let \(\alpha : L \to Z^{**}\) satisfy the factorization identity \(\alpha J = iu\), and for \(W\) as in item (2), let \(Z_W\) be the linear span in \(Z^{**}\) of \((iZ) \cup (\alpha W)\), and notice that \(iZ\) has finite codimension in \(Z_W\) because \(\overline{JX}\) has finite codimension in \(W\). Given \(\epsilon > 0\), Lemma 2.9 in [FJT] says that there is an operator \(P_W : Z_W \to Z\) so that \((P_W) i = I_Z\) and \(\|P_W\| < 3 + \epsilon\). Setting \(u_W = (P_W)\alpha|W\), we see that \(u = u_W J\) and \(\sup_W \|u_W\| < (3 + \epsilon)\|\alpha\|\).

In order to characterize when the operator \(u\) in (1.1) locally lifts to \(L\), it is convenient to introduce a weaker concept of factorization. In (1.1), say that \(u\) \(\text{approximately factors through} Q\) or, when \(Q\) is understood, \(\text{approximately locally lifts to} L\), provided that for each \(\epsilon > 0\) there is an operator \(u_\epsilon : Z \to L\) so that \(\|u - Q u_\epsilon\| < \epsilon\) and \(\sup_\epsilon \|u_\epsilon\| < \infty\). Similarly, say that \(u\) \(\text{approximately locally factors through} Q\) or, when \(Q\) is understood, \(\text{approximately locally lifts to} L\), provided that for every operator \(w : C_I \to Z\), the composition \(u w\) \(\text{approximately factors through} Q\). This is equivalent to saying that for every finite dimensional subspace \(E\) and \(\epsilon > 0\), there is an operator \(u_{E,\epsilon} : Z \to L\) so that \(\|u - Q u_{E,\epsilon}\| < \epsilon\) and \(\sup_{E,\epsilon} \|u_{E,\epsilon}\| < \infty\).

For a typical example of an operator which approximately factors but does not factor, set in (1.1) \(L = \ell_1\), \(Y = \ell_2\), \(Z = R\), let \(Q\) be the linear extension of the the operator which takes the \(n\)-th unit basis vector \(e_n\) in \(\ell_1\) to \(e_1 + \frac{1}{n} e_{n+1}\) in \(\ell_2\), and define \(u(t) = te_1\).

**Proposition 1.2.** Consider the diagram

\[
\begin{array}{ccc}
L & \xrightarrow{Q} & Y \\
\uparrow u & & \downarrow \\
Z & & \\
\end{array}
\]

The following are equivalent:

1. \(u\) \(\text{approximately locally lifts to} L\).
2. For every quotient \(L \xrightarrow{Q_w} W\) of \(L\) by a finite codimensional subspace of \(\ker Q\) and every \(\epsilon > 0\), there is an operator \(u_{W,\epsilon} : Z \to W\) so that \(\|Q_W u_{W,\epsilon} - u\| < \epsilon\) and \(\sup_{W,\epsilon} \|u_{W,\epsilon}\| < \infty\), where \(W \xrightarrow{Q_w} Y\) is the mapping induced by \(Q\).
3. \(u^*\) \(\text{factors through} Q^*\).
4. \(u^{**}\) \(\text{factors through} Q^{**}\).
5. The operator \(u u\) \(\text{factors through} Q^{**}\), where \(Y \xrightarrow{i} Y^{**}\) is the canonical embedding.

**Proof.** As in the proof of Proposition 1.1, (3) and (4) are easily seen to be equivalent, and (1) \(\Rightarrow\) (3) (or (1) \(\Rightarrow\) (4)) follows from a simple compactness argument. (4) \(\Rightarrow\) (5) is formal.

For (5) \(\Rightarrow\) (1), get \(Z \xrightarrow{w} L^{**}\) so that \(Q^{**} w = iu\) and fix a finite dimensional subspace \(E\) of \(Z\). By the principle of local reflexivity, there is a net \(\{u_n\}\) of operators...
from $wE$ into $L$ so that $\lim_\delta ||v_\delta|| = 1$ and $\{v_\delta w\}$ weak* converges to $wz$ for each $z$ in $E$. Since $Q^{**}$ is weak* continuous and extends $Q$, $\{iQv_\delta w\}$ weak* converges in $Y^{**}$ to $Q^{**}wz$ for each $z$ in $E$. But for $z$ in $E$, $Q^{**}wz = iuz$, so in fact $\{Qv_\delta wz\}$ converges weakly in $Y$ to $uz$. Therefore we can get a net of far out convex combinations of $\{v_\delta\}$, which we continue to denote by $\{v_\delta\}$, so that for each $z$ in $E$, $\lim_\delta ||Qv_\delta wz - uz|| = 0$, and hence even $\lim_\delta ||Qv_\delta wz - u|| = 0$.

We included (2) mostly because it is the “approximate” dual condition to item (2) in Proposition 1.1 and so omit the proof that it is equivalent to the other conditions in Proposition 1.2.

Proposition 1.2 combines with the Proposition 1.3 to give a characterization of when in (1.1) $u$ locally factors through $Q$ when $Q$ has closed range (the case of interest to us in the next section).

**Proposition 1.3.** Consider the diagram:

$$
\begin{array}{cccc}
L & \longrightarrow & Y & \\
\downarrow & & \uparrow u & \\
& Z & \\
\end{array}
$$

If $Q$ has closed range and $u$ approximately locally factors through $Q$, then $u$ locally factors through $Q$.

**Proof.** It is clear from the definition that if an operator $w$ approximately locally factors through an operator $v$, then the range of $w$ is contained in the closure of the range of $v$. Consequently, since $Q$ has closed range, we can assume that $Q$ is surjective.

Note that there is a constant $C_n$ so that for every $n$-dimensional subspace $F$ of $Y$, there is an operator $w_F : F \rightarrow L$ so that $||w_F|| \leq C_n$ and $Qw_F = I_F$. Indeed, since $Q$ is surjective, $Q\text{Ball}(L) \supset \delta \text{Ball}(Y)$ for some $\delta > 0$. Take in $F$ an Auerbach basis $\{y_j, y_j^*\}_{j=1}^n$; that is, $y_j^*(y_i) = \delta_{i,j}$ and $||y_j^*|| = 1 = ||y_j||$; and choose $x_j$ in $L$ with $||x_j|| \leq \frac{1}{\delta}$ and $Qx_j = y_j$. Set $w_Fy_j = x_j$ and extend linearly to $F$. Then $||w_F|| \leq \frac{n}{\delta}$ and $Qw_F = I_F$.

Choose $C$ so that for every finite dimensional subspace $E$ of $Z$ and $\epsilon > 0$, there is an operator $u_{E,\epsilon} : E \rightarrow L$ so that $||Qu_{E,\epsilon} - u_E|| < \epsilon$ and $||u_{E,\epsilon}|| \leq C$. Fix a finite dimensional subspace $E$ of $Z$ and set $n = \dim E$. Given $\epsilon > 0$, set $F_\epsilon = (Qu_{E,\epsilon} - u)E$. Then $v = u_{E,\epsilon} - w_{F_\epsilon}(Qu_{E,\epsilon} - u) : E \rightarrow L$ satisfies $||v|| \leq C + \epsilon C_n$ and $Qv = u_{E,\epsilon}$.

A short exact sequence $0 \rightarrow X \rightarrow L \rightarrow Y \rightarrow 0$ is said to split provided the identity on $Y$ lifts to $L$. (Sometimes we abuse language by saying that the extension, $L$, of $Y$ by $X$ splits.) This is equivalent to saying that the identity on $X$ extends to $L$ which is just to say that $JX$ is a complemented subspace of $L$. Say that $0 \rightarrow X \rightarrow L \rightarrow Y \rightarrow 0$ locally splits provided that the identity on $Y$ locally lifts to $L$. The following Corollary, which is an immediate consequence of Proposition 1.1, Proposition 1.2, and Proposition 1.3, gives several equivalents to the concept of local splitting. Whatever novelty there may be in Propositions 1.2–1.3, most of Corollary 1.4 is in the literature. In particular, (1) $\implies$ (3) is Proposition 1 in [J2]. That (4) implies the version of (2) in Proposition 1.1(2) is (as noted in the proof of Proposition 1.1) essentially Lemma 2.9 in [FJT]; moreover, the equivalence of (4) with 1.1(2) is part of Theorem 3.5 in [Ko].
Corollary 1.4. The following are equivalent for the short exact sequence
$$0 \to X \overset{J}{\to} L \overset{Q}{\to} Y \to 0:$$
1. The sequence locally splits.
2. The identity on $X$ colocally extends to $L$.
3. The short exact sequence $0 \leftarrow X^* \overset{J^*}{\leftarrow} L^* \overset{Q^*}{\leftarrow} Y^* \leftarrow 0$ splits.
4. The short exact sequence $0 \to X^{**} \overset{J^{**}}{\to} L^{**} \overset{Q^{**}}{\to} Y^{**} \to 0$ splits.

The last categorical concepts we mention are those of pushouts and pullbacks. Given the diagram (1.1), a pullback of it is a commutative diagram
\[
\begin{array}{ccc}
L & \overset{Q}{\to} & Y \\
\alpha \uparrow & & \uparrow u \\
W & \overset{\beta}{\to} & Z
\end{array}
\]
which satisfies the minimality condition that if
\[
\begin{array}{ccc}
L & \overset{Q}{\to} & Y \\
\alpha_1 \uparrow & & \uparrow u \\
W_1 & \overset{\beta_1}{\to} & Z
\end{array}
\]
is another commutative diagram, then there is a unique morphism $W_1 \overset{w}{\to} W$ so that $\alpha_1 = \alpha w$ and $\beta_1 = \beta w$. In any category pullbacks are unique in an obvious sense whenever they exist. Pullbacks of course do exist in $\text{Ban}$: Given (1.1), $W$ in (1.3) is the subspace of $L \oplus_{\infty} Z$ of all pairs $(x, z)$ for which $Qx = uz$. The operator $\alpha$ (respectively, $\beta$) is the restriction to $W$ of the coordinate projection from $L \oplus_{\infty} Z$ onto $L$ (respectively, $Z$). We call this the canonical pullback construction. Forgetting norms, this is same construction that is used to build pullbacks in the Abelian category $\text{Vect}$, so general categorical principles apply. For example, it is clear from the construction that if $Q$ is surjective; respectively, injective, then so is $\beta$, but this follows also from general categorical principles: the epimorphisms in $\text{Vect}$ are the surjective linear maps and in both $\text{Ban}$ and $\text{Vect}$, the monomorphisms are the injective morphisms. On the other hand, the epimorphisms in $\text{Ban}$ are not the surjective operators but rather the operators with dense range and so need not be epimorphisms in $\text{Vect}$; consequently, one would not expect $\beta$ to be an epimorphism in $\text{Ban}$ whenever $Q$ is (for an example take $Q$ with dense proper range and $u$ so that $(uZ) \cap (ZL) = \{0\}$—this forces $W$ to be $\{0\}$).

From the canonical pullback construction it is also clear that if $Q$ has closed range, so does $\beta$. Thus if $Q$ is an isomorphic embedding, so is $\beta$. The reason for taking the $\ell_{\infty}$ sum of $L$ and $Z$ is that if $Q$ is an isometric embedding and $\|u\| \leq 1$, then $\beta$ is an isometric embedding, and if $Q$ is an isometric quotient mapping and $\|u\| \leq 1$, then $\beta$ is an isometric quotient mapping.

Notice also that in (1.3) the kernels of $Q$ and $\beta$ are isometrically isomorphic, and in fact (1.3) can be extended to a commutative diagram
\[
\begin{array}{ccc}
0 & \to & X & \overset{\alpha}{\to} & L & \overset{Q}{\to} & Y \\
\| & & \uparrow & & \alpha \uparrow & & \uparrow u \\
0 & \to & V & \overset{\beta}{\to} & W & \to & Z \\
\uparrow & & \uparrow & & W_0 & = & Z_0 \\
0 & \to & 0 & \to & 0
\end{array}
\]
with exact rows and columns, which of course cannot necessarily be completed to short exact sequences. However, if the top row of (1.4) can be extended to a short exact sequence, so can the second row—this is another way of saying that $\beta$ is surjective when $Q$ is surjective.

Sometimes one can easily determine whether a commutative diagram (1.3) is a pullback of (1.1). For example, if

$$\begin{array}{ccc}
0 & \rightarrow & X & \rightarrow & L & \overset{Q}{\rightarrow} & Y & \rightarrow & 0 \\
\| & \alpha & \uparrow & \uparrow u \\
0 & \rightarrow & V & \rightarrow & W & \overset{\beta}{\rightarrow} & Z & \rightarrow & 0
\end{array}$$

(1.5)

is a commutative exact diagram in $\text{Ban}$, then (1.3) is a pullback of (1.1). Also, if (1.4) is commutative and exact, and $QL \cap uZ = Q\alpha W$—which is automatic when $Q$ and $\beta$ are surjective—then (1.3) is a pullback of (1.1). Notice that it is enough to check these assertions in the nice category $\text{Vect}$, for then the unique bounded linear operator from $W$ to the corner of the canonical pullback of (1.1) which makes the relevant diagram commute must be a surjective vector space isomorphism, hence a surjective isomorphism in $\text{Ban}$ by the open mapping theorem.

Proposition 1.5, the first part of which is important for [KP], says that the pullback construction provides an alternate way of looking at the problem of when an operator factors or locally factors through a quotient mapping:

**Proposition 1.5.** Consider the exact commutative diagram

$$\begin{array}{ccc}
0 & \rightarrow & X & \rightarrow & L & \overset{Q}{\rightarrow} & Y & \rightarrow & 0 \\
\| & \alpha & \uparrow & \uparrow u \\
0 & \rightarrow & V & \rightarrow & W & \overset{\beta}{\rightarrow} & Z & \rightarrow & 0
\end{array}$$

(1.6)

(1) $u$ lifts to $L$ if and only if the second row splits.
(2) $u$ locally lifts to $L$ if and only if the second row locally splits.

**Proof.** The “if” direction is obvious both in (1) and (2). Assume now that $u$ lifts to $L$, say $u = Q\gamma$, where $\gamma : Z \rightarrow L$. From the discussion prior to the statement of Proposition 1.5, we can assume without loss of generality that (1.3), the right square of (1.6), is the canonical pullback of (1.1), the upper right triangle of (1.6). This makes it easy to define a lifting, $\tau$, of $I_L$ to $W$; namely, set $\tau z = (\gamma z, z)$. Since $\beta$ is the projection onto the second component, this gives (1).

Part (2) follows from (1) by taking second adjoints in (1.6) and applying Corollary 1.4(4) and Proposition 1.2(4). Alternatively, let $w : C_1 \rightarrow Z$ be any operator and extend (1.6) via the pullback construction to a commutative diagram with exact rows:

$$\begin{array}{ccc}
0 & \rightarrow & X & \rightarrow & L & \overset{Q}{\rightarrow} & Y & \rightarrow & 0 \\
\| & \alpha & \uparrow & \uparrow u \\
0 & \rightarrow & V & \rightarrow & W & \overset{\beta}{\rightarrow} & Z & \rightarrow & 0 \\
\| & \uparrow & \uparrow w \\
0 & \rightarrow & V & \rightarrow & W & \rightarrow & C_1 & \rightarrow & 0
\end{array}$$

(1.7)
By hypothesis, the operator \( uw \) factors through \( Q \), so by part (1) of Proposition 1.5 the bottom row of (1.7) splits, hence \( w \) factors through \( \beta \). This gives (2).

It is of considerable interest to determine when a locally splitting short exact sequence must split. The sequence \( 0 \to X \to X^{**} \to X^{**}/X \to 0 \), where \( X \to X^{**} \) is the canonical injection, must locally split—this immediate consequence of Proposition 1.4 has long been known—and splits if and only if \( X \) is complemented in some dual space. This can be used to prove the following fact, which is a version of what is called in [KP] Lindenstrauss’ lifting criterion.

**Lemma 1.6.** Consider the diagram

\[
\begin{array}{c}
0 \to X \to L \xrightarrow{Q} Y \to 0 \\
\uparrow u \\
Z
\end{array}
\]

where the top row is exact. If \( u \) locally lifts to \( L \) and \( X \) is complemented in \( X^{**} \), then \( u \) lifts to \( L \).

**Proof.** Lindenstrauss’ argument [Lin] provides a simple enough proof, but it is even easier to use Proposition 1.5. Extend (1.8) to (1.6). The second row of (1.6) locally splits by Proposition 1.5. Now look at the commutative diagram

\[
\begin{array}{c}
0 \to V^{**} \to W^{**} \xrightarrow{\beta^{**}} Z^{**} \to 0 \\
\uparrow \ \\
0 \to V \to W \xrightarrow{\beta} Z \to 0
\end{array}
\]

where the vertical arrows are the canonical embeddings and the rows are exact. The top row of (1.9) splits by Corollary 1.4. The space \( V \), being isomorphic to \( X \), is complemented in some dual and hence in \( V^{**} \), so the bottom row of (1.9) splits. So \( u \) factors through \( Q \) by the trivial direction of Proposition 1.5 (2).

Unfortunately, Lemma 1.6 is not of much use in determining when a coextension of a \( C(K) \) space must split; this is a problem closely connected to the investigation of the so-called “extension property” considered in [JZ1], [JZ2].

It was shown in [KP] that pullbacks provide a quick proof (which, however, relies on deep results from Banach space theory) of the following result:

**Proposition 1.7.** If \( Z^* \) has cotype two and \( \ell_2 \) is a quotient of \( Z \); in particular, if \( Z = C[0,1] \); then there is an extension of \( Z \) by \( \ell_2 \) which does not split.

**Proof.** It is known [ELP], [KPec] that there is an extension, \( L \), of \( \ell_2 \) by \( \ell_2 \) which does not split. Using the pullback construct, we get a commuting diagram

\[
\begin{array}{c}
0 \to \ell_2 \to L \xrightarrow{Q} \ell_2 \to 0 \\
\| \ \\
0 \to \ell_2 \to W \xrightarrow{\beta} Z \to 0
\end{array}
\]

with exact rows, \( u \) a surjection, and the top row not splitting. The mapping \( \alpha \) is a surjection since \( u \) is, hence \( \alpha^* \) is an isomorphic embedding of \( L^* \) into \( W^* \). If the
bottom row splits, then $W^*$ is isomorphic to the direct sum of $\ell_2$ and $Z^*$, hence $W^*$, whence also $L^*$, has cotype two. But for the known constructions of such $L$’s, $L^*$ does not have cotype two. Actually, for any such $L$, $L^*$ cannot have cotype two since that would force $L$ to have type two (by Pisier’s theorem [Pis1] and the Maurey–Pisier duality theory [MP] for type–cotype in $K$-convex spaces), in which case $0 \to \ell_2 \to L \to \ell_2 \to 0$ would split by Maurey’s factorization theorem [Mau1].

Since every Banach space has quotients uniformly isomorphic to $\ell^n_2$ for all $n$, local Banach space theory considerations show that the hypothesis in Proposition 1.7 that $\ell_2$ be a quotient of $Z$ is not needed; it is that version which appears in [KP]. In [KP] it is also noted that if $Z$ contains subspaces uniformly isomorphic to $\ell^n_\infty$, then there is an extension of $Z$ by $\ell_2$ which does not split. This is also a consequence of Proposition 1.7 and local theory techniques.

We turn to the notion of pushout, which is dual (in the sense of category theory) to that of pullback. A commutative diagram

$$
\begin{array}{ccc}
X & \rightarrow & L \\
\downarrow \downarrow & & \downarrow \beta \\
Z & \rightarrow & W \\
\end{array}
$$

is called a pushout of (1.2) provided that for every commutative diagram

$$
\begin{array}{ccc}
X & \rightarrow & L \\
\downarrow \downarrow & & \downarrow \beta_1 \\
Z & \rightarrow & W_1 \\
\end{array}
$$

there is a unique morphism $W \overset{w}{\rightarrow} W_1$ so that $\alpha_1 = w\alpha$ and $\beta_1 = w\beta$. Pushouts are unique in an obvious sense whenever they exist. Pushouts exist in $Ban$. Having thought through pullbacks, one can build pushouts by first taking the adjoint of (1.2) and constructing the pullback of it:

$$
\begin{array}{ccc}
X^* & \overset{J^*}{\leftarrow} & \overset{\wedge}{L}^* \\
\uparrow \uparrow \alpha & & \uparrow \uparrow \beta \\
Z^* & \overset{\wedge}{\leftarrow} & \wedge \tilde{W} \\
\end{array}
$$

The subspace $\tilde{W} = \{(z^*, x^*) : u^*z^* = J^*x^*\}$ is weak* closed in $Z^* \oplus_\infty L^* = (Z \oplus_1 L)^*$ and the coordinate projections $\tilde{\alpha}$ and $\tilde{\beta}$ are weak* continuous. Moreover, if one takes the adjoint of the commutative diagram (1.12) and writes down the unique operator $W_1^* \rightarrow \tilde{W}$ which makes the relevant diagram commute, one sees that $W_1^* \rightarrow \tilde{W}$ is weak* continuous. Thus the preadjoint of (1.13) is a indeed a pushout of (1.2), which we call the canonical pushout; it is defined directly by setting $W = (Z \oplus_1 L) / \overline{K}$, where $K = \{(ux, -Jx) : x \in X\}$, with $\alpha$ and $\beta$ the compositions of the natural mappings of $Z$ and $L$ into $Z \oplus_1 L$ with the quotient map from $Z \oplus_1 L$ onto $W$. This is the construction of the pushout of (1.2) in $Vect$ only when $K$ is closed. A natural condition to guarantee that $K$ be closed is that $J$ have closed range. Actually, the case where $J$ is even an isomorphic embedding may be the only one considered in the Banach space literature; at any rate, it is
this case which has played an important role in Banach space theory. The first deep application I am aware of was due to Kisliakov [Kis]. The construction was also critical for Pisier’s fundamental paper [Pis2]. Of course, canonical pushouts play a major role in the Kalton–Pełczyński paper [KP]. However, the categorical aspects of the canonical pushout seem not to have been explicitly noted.

Suppose that (1.11) is a pushout of (1.2). Either directly from the canonical construction or by taking adjoints and using the pullback theory, one checks basic facts: If \( J \) is surjective or an isomorphic embedding or has closed range, then \( \alpha \) has the same property. If \( ||u|| \leq 1 \) and \( J \) is an isometric quotient map (respectively, an isometric embedding), then \( \alpha \) is an isometric quotient map (respectively, an isometric embedding). If \( J \) is an epimorphism in \( \text{Ban} \) (that is, has dense range), so is \( \alpha \). The map \( \alpha \) need not be injective when \( J \) is (take \( J \) injective with dense proper range and let \( u \) be a linear functional in \( X^* \) which is not in \( J^*L^* \)).

The dual to the extension of the pullback diagram (1.4) (after relabeling to agree with our pullback notation) is:

\[
\begin{array}{cccc}
X & J & L & \longrightarrow & Y & \rightarrow & 0 \\
\downarrow & \alpha & \downarrow & \beta & \| & \\
Z & \longrightarrow & W & \rightarrow & Y_1 & \rightarrow & 0 \\
\end{array}
\]  

(1.14)

In order for the first row of (1.14) to be exact, \( J \) must have closed range, in which case if (1.11) is a pullback diagram \( \alpha \) also has closed range and the quotients \( Y \equiv L/JX \) and \( Y_1 \equiv W/\alpha Z \) are naturally isomorphic.

If (1.4) is a pullback diagram and both \( J \) and \( u \) have closed range, then the pushout construction produces the dual commutative diagram to (1.5)

\[
\begin{array}{cccc}
X & J & L & \longrightarrow & Y & \rightarrow & 0 \\
\downarrow & \alpha & \downarrow & \beta & \| & \\
Z & \longrightarrow & W & \rightarrow & Y_1 & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & = & V_1 & \rightarrow & Y_1 & \rightarrow & 0 \\
\end{array}
\]  

(1.15)

where the columns and rows are exact. If \( J \) is an isomorphic embedding, the rows in (1.14) can be extended to short exact sequences. When both \( J \) and \( u \) are isomorphic embeddings, we get the commuting diagram

\[
\begin{array}{cccc}
0 & 0 & \\
\downarrow & \downarrow & \\
0 & \rightarrow & X & J & L & \longrightarrow & Y & \rightarrow & 0 \\
\downarrow & \alpha & \downarrow & \beta & \| & \\
0 & \rightarrow & Z & \longrightarrow & W & \rightarrow & Y_1 & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & = & V_1 & \rightarrow & Y_1 & \rightarrow & 0 \\
\end{array}
\]  

(1.16)

where the rows and columns are exact; this is used in section 2.

Also used in section 2 is part (2) of Proposition 1.8 (part (1) is important for [Kis], [Pis2], and [KP]), which is dual to Proposition 1.5.
**Proposition 1.8.** Consider the commutative exact diagram

\[
\begin{array}{ccc}
0 & \rightarrow & X \\
\downarrow u & & \downarrow \beta \\
0 & \rightarrow & Z \\
\end{array}
\quad
\begin{array}{ccc}
& J & \\
\rightarrow & L & \rightarrow Y \rightarrow 0 \\
\rightarrow & W & \rightarrow Y_1 \rightarrow 0 \\
\end{array}
\]

(1.17)

(1) \(u\) extends to \(L\) if and only if the second row splits.

(2) \(u\) colocaoly extends to \(L\) if and only if the second row locally splits.

**Proof of (2).** If \(u\) colocaoly extends to \(L\), then by Proposition 1.1 \(u^*\) factors through \(J^*\), so that by Proposition 1.5 (take the adjoint diagram of (1.16)),

\[
0 \leftarrow Z^* \alpha^* W^* \leftarrow Y_1^* \leftarrow 0
\]

splits. Hence also \(0 \rightarrow Z^{**} \alpha^{**} W^{**} \rightarrow Y_1^{**} \rightarrow 0\) splits, whence again by Corollary 1.4, \(0 \rightarrow Z \alpha W \rightarrow Y_1 \rightarrow 0\) locally splits.

Conversely, if \(0 \rightarrow Z \alpha W \rightarrow Y_1 \rightarrow 0\) locally splits, then the sequence

\[
0 \leftarrow Z^* \alpha^* W^* \leftarrow Y_1^* \leftarrow 0
\]

splits by Corollary 1.4. Hence by Proposition 1.5, \(u^*\) factors through \(J^*\), whence \(u\) colocaoly extends to \(L\) by Proposition 1.1. ■
2. Extensions of \( c_0 \).

In this section we prove:

**Theorem 2.1.** Suppose that

\[
0 \rightarrow X \rightarrow L \rightarrow Y \rightarrow 0
\]

is exact with \( X \) separable, \( c_0 \) is isomorphic to a subspace of \( Y \), and \( L \) embeds into a Banach lattice which does not contain \( \ell_\infty^n \)'s uniformly. Suppose

\[
0 \rightarrow X \rightarrow Z \rightarrow V \rightarrow 0
\]

is a locally splitting short exact sequence with \( Z \) separable and \( Z \) embeds into a Banach lattice which does not contain \( \ell_\infty^n \)'s uniformly. Then \( Z \) is not complemented in its bidual.

This theorem has a corollary which can be stated in the language of Banach space theory as:

**Corollary 2.2.** Suppose that \( L \) embeds into a Banach lattice which does not contain \( \ell_\infty^n \)'s uniformly and \( Q \) is an operator from \( L \) onto some Banach space \( Y \). If \( \ker Q \) has GL-l.u.st., then \( Y \) does not contain \( \ell_\infty^n \)'s uniformly.

If the conclusion in Corollary 2.2 is weakened to “\( c_0 \) does not embed into \( Y \)”, the resulting statement (at least when \( X \) is separable) is immediate from Theorem 2.1 and known results. In the appendix we show how to deduce Corollary 2.2 from Theorem 2.1, preferring in this section to concentrate on the proof of Theorem 2.1 itself. The most important case is \( L = L_1 \), but this case is not easier than the general one. However, the case \( L = L_1 \) does lend an easier proof that \( X \) does not have l.u.st. in its original sense, and we mention in the proof of Theorem 2.1 how to streamline the proof to obtain just this.

Note that even when \( Y = c_0 \), the hypotheses on \( L \) in Theorem 2.1 cannot be replaced by the conditions that that \( c_0 \) does not embed into \( L \) and \( L \) is itself a lattice. Indeed, it is clear that the identity on \( c_0 \) locally factors through the natural quotient map from \((\sum_{n=1}^\infty \ell_\infty^n)_1\) onto \( c_0 \).

**Proof of Theorem 2.1.** We can assume that \( X \) is a subspace of \( L \) and that \( Y = L/X \). It is easy to see that there is a separable superspace \( L' \) of \( X \) in \( L \) so that \( L'/X \) is isomorphic to \( c_0 \). So by replacing \( L \) by \( L' \), we can assume that \( L \), and a fortiori also \( Y \), are separable. We are thus considering the exact diagram:

\[
0 \rightarrow X \rightarrow^J L \rightarrow^u Y \rightarrow 0
\]
where $L$ and $Z$ both embed into separable Banach lattices which do not contain $\ell_1^n$'s uniformly, $Y$ contains a copy of $c_0$ (which, since $Y$ is separable, is necessarily complemented because $c_0$ has the separable extension property ([Sob], [LT1, Th. 2.f.5]), and the column locally splits. In some sense, the main point is one that is obvious to any self-respecting algebraist: to study (2.3), one should complete it to a full diagram. But since $J$ and $u$ are both isomorphic embeddings, the pushout construction extends (2.3) to the exact commutative diagram (1.16), which we repeat as (2.4):

$$
\begin{array}{ccccccccc}
0 & \rightarrow & X & \overset{J}{\rightarrow} & L & \overset{q}{\rightarrow} & Y & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Z & \overset{\alpha}{\rightarrow} & W & \overset{q_1}{\rightarrow} & Y_1 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
V & = & V_1 & & & & & &
\end{array}
$$

(2.4)

Before proceeding further, let us see that when $L = L_1$, the space $X$ cannot have l.u.s.t.; this answers the question Kalton and Pełczyński posed in their lectures on [KP]. If $X$ has l.u.s.t., then it is known that there is a locally splitting short exact sequence (2.2) with $Z$ a separable Banach lattice which isomorphically embeds into $L_1$ (see the appendix). In view of Corollary 1.4, this implies that $V^{**}$ embeds into $Z^{**}$ which embeds into the abstract $L_1$-space $L_1^{**}$, and hence by [LP] $V$ embeds into $L_1$. Since the first column in (2.4) locally splits, so does the second. But then $W^{**}$ is isomorphic to $L_1^{**} \oplus V^{**}$ and hence $W$ also embeds into $L_1$. Now we need a key analytical lemma proved, but not stated, in [KP]. (See the proof of Proposition 2.2 in [KP]. Actually, in [KP] $P$ is constructed so that $PQ$ is even representable; that is, factors through $\ell_1.$) Later we prove a generalization of Lemma 2.3 in order to prove Theorem 2.1.

**Lemma 2.3.** If $W$ is a subspace of $L_1$ and $q_1$ is an operator from $W$ into a space $Y_1$ which contains a complemented copy of $c_0$, then there is a projection $P$ on $Y_1$ with $PY_1$ isomorphic to $c_0$ such that $Pq_1$ is completely continuous; that is, carries weakly convergent sequences into norm convergent sequences.

Let $P$ be given from Lemma 2.3 and let $v$ be an operator from $L_1$ into the isomorph $PY_1$ of $c_0$ which is not completely continuous; there are a wealth of such operators. $v$ cannot factor through the completely continuous operator $Pq_1$, which is to say that $v$ cannot factor through $q_1$. But $v$ locally factors through $q_1$ since $q_1$ is surjective. By Lemma 1.6, $Z$ is not complemented in its bidual. But every Banach lattice which does not contain an isomorph of $c_0$ is complemented in its bidual via a band projection [LT2].

This digression from the proof of Theorem 2.1 in fact motivates the proof, to which we now return. Since the properties of being a Banach lattice and of not containing $\ell_\infty^n$'s uniformly are both preserved under passage to biduals, just as in the digression we conclude that $W$ embeds into a Banach lattice which does not contain $\ell_\infty^n$'s uniformly. The further argument in the digression shows that in order to complete the proof of Theorem 2.1, it is enough to verify the following generalization of Lemma 2.3:
Lemma 2.4. If $W$ is a separable subspace of a Banach lattice which does not contain $\ell^n_\infty$’s uniformly and $q_1$ is an operator from $W$ into a space $Y_1$ which contains a complemented copy of $c_0$, then there is a projection $P$ on $Y_1$ with $PY_1$ isomorphic to $c_0$ such that $Pq_1w$ is completely continuous for every operator $w$ from $L_1$ into $W$.

For the proof of Lemma 2.4 we need one sublemma and a couple of known facts.

Sublemma 2.5. Let $W$ be a separable Banach lattice which does not contain $\ell^n_\infty$’s uniformly. Suppose that $\{f_n\}_{n=1}^\infty$ is a weak* null sequence in $W^*$. Then there exist $g_n$ in the convex hull of $\{f_k\}_{k=n}^\infty$ so that $\{|g_n|\}_{n=1}^\infty$ is weak* null in $W^*$.

Proof. It is known that there exists $q < \infty$, a measure $\mu$, and an operator $u$ from $L_q(\mu)$ into $W$ with dense range which is an interval preserving lattice homomorphism. Since standard texts do not include this fact, here is a sketch of the proof (unexplained terminology as well as the quoted theorems about lattices can be found in [LT2]): Let $x$ be a weak order unit for $W$ and let $X$ be the linear span of the order interval $[-x, x]$ with $[-x, x]$ as its unit ball. $X$ is then an abstract $M$-space and so can be identified, as a Banach lattice, with $C(K)$ for some compact Hausdorff space $K$ by Kakutani’s representation theorem. Since $W$ does not contain $\ell^n_\infty$’s uniformly, the injection $j$ from $X$ into $W$ is $q$-summing for some $q$ by a theorem of Maurey’s [DJT, p. 223]. Choose a Pietsch measure [DJT, p. 45] $\mu$ for $u$; then $j$ factors through the natural injection $i$ from $X = C(K)$ into $L_q(\mu)$; say, $j = ui$ where $L_q(\mu) \xrightarrow{u} W$. The operator $u$ is of course uniquely defined because $C(K)$ is dense in $L_q(\mu)$; that $u$ has the stated properties can be deduced from the fact that $j$ has those properties.

Taking adjoints, we see that $W^* \xrightarrow{u^*} L_p(\mu) (\frac{1}{p} + \frac{1}{q} = 1)$ is an injective (since $u$ has dense range) lattice homomorphism (since $u$ is interval preserving; see [AB, p. 92]). By the weak* continuity of $u^*$, $u^*f_n \rightarrow 0$ weakly in $L_p(\mu)$, and hence there exist $g_n$ in the convex hull of $\{f_k\}_{k=n}^\infty$ so that $||u^*g_n||_p \rightarrow 0$ and hence $||u^*g_n||_q \rightarrow 0$. But $|u^*g_n| = u^*|g_n|$ because $u^*$ is a lattice homomorphism. But then the only possible weak* cluster point in $W^*$ of $\{|g_n|\}_{n=1}^\infty$ is 0, so that in fact $\{|g_n|\}_{n=1}^\infty$ must converge weak* in $W^*$ to 0.

Fact 2.6. If $W$ is a Banach lattice which does not contain a subspace isomorphic to $c_0$ and $w$ is an operator from $L_1$ into $W$, then $w\text{Ball}(L_\infty)$ is order bounded.

In fact, the stated hypothesis implies that every operator from $L_1$ into $W$ has a modulus [AB, p. 249].

Fact 2.7. An operator $w$ from $L_1$ is completely continuous if and only if $w\text{Ball}(L_\infty)$ is relatively compact.

Fact 2.7 can be found in [Uhl]. Actually, we do not really need it because it is evident that there are operators from $L_1$ into $c_0$ which take the ball of $L_\infty$ into a non-relatively compact set.

Proof of Lemma 2.4. It is clear that there is no loss of generality in assuming that $Y_1 = c_0$. By replacing $W$ by the closure of the (necessarily separable) sublattice it generates in some containing Banach lattice which does not contain $\ell^n_\infty$’s uniformly, we can assume, since $c_0$ has the separable extension property, that $W$ itself is a Banach lattice.
Set $f_n = q^*_1 e^*_n$, where $\{e^*_n\}_{n=1}^{\infty}$ is the unit vector basis of $\ell_1 = c_0^*$. Clearly $\{f_n\}_{n=1}^{\infty}$ tends weak* in $W^*$ to 0, so from Sublemma 2.5 we get $n_1 < n_2 < n_3 \ldots$ and $d^*_n \in \text{co} \{e^*_i\}_{i=k_n+1}^{k_{n+1}}$ so that, setting $g_n = \sum_{i=k_n+1}^{k_{n+1}} e^*_i$, $\{|g_n|\}_{n=1}^{\infty}$ converges weak* in $W^*$ to 0. Define $x_n = \sum_{i=k_n+1}^{k_{n+1}} e^*_i$ in $c_0$ and let $P$ be the contractive projection on $c_0$ defined by $P = \sum d^*_n \otimes x_n$.

Let $w$ be any operator from $L_1$ into $W$. By Fact 2.7, in order to check that $P q_1 w$ is completely continuous, it is enough to show that $P q_1 w \text{Ball}(L_\infty)$ is relatively compact. This amounts to checking that

$$\sup_{g \in w \text{Ball}(L_\infty)} |\langle g, g_n \rangle| \to 0 \text{ as } n \to \infty. \tag{2.5}$$

But by Fact 2.6, there exists $h \geq 0$ in $W$ so that $w \text{Ball}(L_\infty)$ is contained in the order interval $[-h, h]$. We thus have for each $n$:

$$\sup_{g \in w \text{Ball}(L_\infty)} |\langle g, g_n \rangle| \leq \langle |h|, |g_n| \rangle. \tag{2.5}$$

Since $\{|g_n|\}_{n=1}^{\infty}$ is weak* null in $W^*$, (2.5) follows. This completes the proof of Lemma 2.4 and hence also the proof of Theorem 2.1. ■

Remark 2.8 If the Banach space $X$ fails to have GL-l.u.st. and $X \subset Y \subset X^{**}$, then the identity on $X$ colocally extends to $Y$ by Corollary 1.4 (2) and hence $Y$ also fails GL-l.u.st.. If $S$ is a Sidon subset of the compact Abelian group $G$, then $L_1(G)/L_\infty^G(G)$ is isomorphic to $c_0$ (see [KP] for background). Pełczyński pointed out that if we apply Remark 2.8 and Corollary 2.2 to this kind of example, we obtain the following information about the classical object $M_T(G)$, the set of finite measures on $G$ whose Fourier transforms are supported on $T$:

**Corollary 2.9.** If $S$ is a Sidon subset of the compact Abelian group $G$, then $M_T(G)$ does not have GL-l.u.st..
3. Appendix.

The main background needed for deriving Corollary 2.2 from Theorem 2.1 is the following theorem, which is a restatement of results from [FJT]:

**Theorem 3.1.** A Banach space $X$ has GL-l.u.st. if and only if there is a locally splitting short exact sequence $0 \to X \to Z \to V \to 0$ with $Z$ a Banach lattice. Moreover, if in addition to having GL-l.u.st. $X$ does not contain $\ell^n_\infty$'s uniformly, then $Z$ may be chosen not to contain $\ell^n_\infty$'s uniformly. Also, if $X$ has l.u.st., $Z$ may be chosen to be finitely crudely representable in $X$ (that is, the finite dimensional subspaces of $Z$ embed into $X$ with uniformly bounded isomorphism constants).

In [FJT] and also [Mau2] it was remarked that a space $X$ has GL-l.u.st. if and only if $X^{**}$ is complemented in a Banach lattice (see [DJT, p. 348] for a proof, but keep in mind that in [DJT] GL-l.u.st. is called l.u.st. while l.u.st. is called DPR-l.u.st.). This gives the first statement in Theorem 3.1. The “also” statement is a consequence of Corollary 2.2 in [FJT], while the “moreover” follows from Proposition 2.6(i) and Remark 2.8 of [FJT]. Of course, for us the definition of GL-l.u.st. is irrelevant, since in section 2 we use only the characterization given by Theorem 3.1.

Notice that the “also” statement yields that if $X$ is a subspace of an $L_1$ space and $X$ has l.u.st., then the Banach lattice $Z$ from Theorem 3.1 can be taken finitely crudely representable in $L_1$, hence $Z$ embeds into an $L_1$ space by [LP]. This was used in the “digression” part of the proof of Theorem 2.1.

**Proof of Corollary 2.2.** The reader who is not familiar with ultrapowers of Banach spaces will find enough in chapter 8 of [DJT] to make the verification of claims we make about ultrapowers easy. The statements we make about GL-l.u.st. are probably more obvious from the definition than from the equivalent form given by the first statement in Theorem 3.1; again, [DJT] is sufficient reference.

The property of $L$ in Corollary 2.2; namely, that $L$ embeds into a Banach lattice which does not contain $\ell^n_\infty$'s uniformly; is stable under the taking of ultrapowers, as is the property of having GL-l.u.st.. On the other hand, any ultrapower of a space which contains $\ell^n_\infty$'s uniformly must contain $c_0$. Consequently, to prove Corollary 2.2 it is enough to check that $c_0$ does not embed into $Y$. Assume, for contradiction, that $c_0$ does embed into $Y$. Choose a separable subspace $L'$ of $L$ so that $QL'$ is closed and contains a copy of $c_0$. Since $\ker Q$ has GL-l.u.st., there is a separable subspace $X$ of $\ker Q$ containing the intersection of $\ker Q$ with $L'$ and which has GL-l.u.st.. By replacing $L'$ with the closed span of $L' \cup X$, we can assume that $X = L' \cap \ker Q$. Thus we have a short exact sequence $0 \to X \to L' \to Y_0 \to 0$ with $L'$ a separable subspace of some Banach lattice which does not contain $\ell^n_\infty$'s uniformly, $X$ has GL-l.u.st., and $Y_0$ contains a copy of $c_0$. But by Theorem 3.1, there is a locally splitting short exact sequence $0 \to X \to Z \to V \to 0$ with $Z$ a Banach lattice which does not contain $\ell^n_\infty$'s uniformly. Moreover, by replacing $Z$ with the closed sublattice generated by $X$, we can assume that $Z$ is separable. The lattice $Z$ is complemented in its bidual because it does not contain a copy of $c_0$ [LT2]. This contradicts Theorem 2.1 and completes the proof of Corollary 2.2. ■
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