Complexity of operators generated by quantum mechanical Hamiltonians

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We propose how to compute the complexity of operators generated by Hamiltonians in quantum field theory (QFT) and quantum mechanics (QM). The Hamiltonians in QFT/QM and quantum circuit have a few essential differences, for which we introduce new principles and methods for complexity. We show that the complexity geometry corresponding to one-dimensional quadratic Hamiltonians is equivalent to AdS\textsubscript{3} spacetime. Here, the requirement that the Hamiltonian is bounded below corresponds to the speed of a particle is not superluminal. Our proposal proves the complexity of the operator generated by a free Hamiltonian is zero, as expected. We also show that the complexity can be used as an indicator of quantum phase transitions. By studying non-relativistic particles in compact Riemannian manifolds we find the complexity is given by the global geometric property of the space. In particular, we show that in low energy limit the critical spacetime dimension to insure nonnegative complexity is $3+1$ dimension.

\section{Introduction}

The recent developments of the quantum information theory and holographic duality show that some concepts from quantum information are useful in understanding foundations of gravity and quantum field theory (QFT). One of such concepts is the entanglement entropy, from which the spacetime geometry may emerge (e.g., see Refs. [1–6]). The “complexity” is another important concept since it may have a relation with physics from quantum information are useful in understanding foundations of gravity and quantum field theory (QFT). The continuous versions of quantum circuits of a given $U$ are identified with various curves connecting the identity and $U$ in the geometry. The complexity is defined by the minimal length of those curves.

This idea of complexity geometry of unitary operators is very attractive but there are two important shortcomings yet. First, the geometry is not determined by some physical principle, but given by hand. It is acceptable for quantum circuit problems since we can design the circuit as we want. However, in general QFT/QM, there must be constraints given by nature not by our hands. Second, it is only valid for SU($n$) operators with finite $n$. It is enough for quantum circuit problems, but to deal with the operators generated by general QFT/QM systems such as $H = p^2/2m + V(x,p)$ we need to develop the formalism for the Hamiltonian with infinite dimensional Hilbert space.

In this paper, we propose how to remedy these shortcomings by generalizing the complexity geometry for finite qubit systems to general QFT/QM. In particular, we take into account the fact that the generating functional in QFT/QM plays a crucial role contrary to quantum circuits. We also, for the first time, note the importance of the lower boundedness of the Hamiltonian in QFT/QM. As a result, we will uncover novel interesting results which cannot be simply inferred from finite qubit systems.

\section{Complexity for U($n$) Operators}

Let us first review basic idea of right-invariant complexity geometry for SU($n$) groups based on [26–28]. Consider the space of operators in SU($n$) group with finite $n$. Suppose that a curve ($c(s) \in \text{SU}(n)$) is generated by a generator $H(s)$ as follows.

\begin{equation}
    c(s) = e^{\int s \, iH(s)\, ds} \quad \text{or} \quad \dot{c}(s) = iH(s)c(s), \quad (1)
\end{equation}

\begin{equation}
    e^{\int s \, iH(s)\, ds} = \frac{1}{N} e^{\int s \, \text{tr} \, iH(s)\, ds},
\end{equation}

where every points corresponds to a unitary operator. The continuous versions of quantum circuits of a given $U$ are identified with various curves connecting the identity and $U$ in the geometry. The complexity is defined by the minimal length of those curves.
We assume that the line element of this curve is given by a certain function of a generator only:

\[ dl = \bar{F}(H(s)) ds := \sqrt{\bar{g}(H(s), H(s))} ds, \tag{2} \]

where \( \bar{g} (\cdot, \cdot) \) is a metric defined in Lie algebra \( \mathfrak{su}(n) \) and we call \( \bar{F}(H) \) the norm of \( H \). In bases \( \{ e_I \} \), \( H = \epsilon_I Y^I \) and the metric components can be expressed as

\[ \bar{g}_{IJ} = \frac{1}{2} \partial^2 \bar{F}(H)^2 / (\partial Y^I \partial Y^J). \tag{3} \]

To obtain the metric in the group manifold with the coordinate \( X^I \), the metric needs to be transformed by a coordinate transformation

\[ g_{IJ}(X) = \tilde{g}_{KL} M^K_I (X) M^L_J (X), \tag{4} \]

where the transformation matrix is defined as \( Y^I (s) ds = M^K_I (X) dX^K \). The complexity of an operator \( \bar{W}(s) := \bar{T} e^{i H s} \bar{g}(\bar{H}) d\bar{s} \), denoted by \( \mathcal{C} (\bar{W}(s)) \), is defined by the minimal length of all curves which connect \( W(s) \) to identity:

\[ \mathcal{C}(\tilde{W}(s)) = \min \int_0^s \tilde{F}(H(\bar{s})) d\bar{s}, \tag{5} \]

where \( H(\bar{s}) \) satisfy \( \bar{W}(s) = \bar{T} e^{i H(\bar{s})} d\bar{s} \).

Note that the geometry and complexity is right-invariant, because \( H \) itself is invariant under the right-translation \( c \rightarrow c \bar{x} \) for \( \forall \bar{x} \in \text{SU}(n) \).

However, there is another symmetry which must be required only for the complexity of QFT/QM but not for quantum circuits. As a generator \( H \) in (1) let us consider a Hamiltonian of QFT/QM. Recall that \( H \) and \( \bar{U} H \bar{U}^\dagger \) give the same generating functional so physically equivalent. Therefore, if the complexity in QFT/QM is an observable or a physical quantity yielding observables it is natural that the complexity of the unitary operators generated by \( H \) and \( \bar{U} H \bar{U}^\dagger \) are also same. In other words, there is a symmetry, which we will call a unitary invariance

\[ \bar{F}(\tilde{H}) = \bar{F}(\bar{U} H \bar{U}^\dagger). \tag{6} \]

For more details on this point, please see appendix A. Also, in appendix B, we show in a concrete example that Eq. (6) is a necessary condition so that the complexity should respect the fundamental symmetries in QFT/QM.

There are also other arguments to support Eq. (6), e.g., see Refs. [42, 43].

The right-invariance and unitary invariance (6) together imply the complexity in QFT/QM should be also left-invariant, so bi-invariant. This bi-invariance implies two important results. First, the theory of Lie algebra proves that the Riemannian metric \( \tilde{g}(\cdot, \cdot) \) is uniquely determined by the Killing form up to a constant factor \( \lambda > 0 \)

\[ \tilde{g}(H, H) = \bar{F}(H)^2 = \lambda^2 \text{Tr}(HH^\dagger) = \lambda^2 \text{Tr}(H^2). \tag{7} \]

Second, the geodesic in a bi-invariant metric is given by a constant generator, say \( \bar{H} \), [52] so

\[ \mathcal{C}(\exp(i \bar{H})) = \bar{F}(\bar{H}). \tag{8} \]

Let us now consider an operator generated by a physical Hamiltonian, denoted by \( \mathcal{H} \). In general \( \mathcal{H} \) has a nonzero trace and its Hilbert space may be infinite dimensional so Eq. (7) needs to be generalized accordingly.

First, to deal with the Hamiltonian of nonzero trace, we define a “mean value” \( \bar{H} \) by

\[ \text{Tr}(\mathcal{H} - \bar{\mathcal{H}}) = 0, \quad \bar{\mathcal{H}} = \text{identity}, \tag{9} \]

such that \( (\mathcal{H} - \bar{\mathcal{H}}) \in \mathfrak{su}(n) \). Because \( U(1) \) is just a phase transformation, \( (\mathcal{H} - \bar{\mathcal{H}}) \) and \( \mathcal{H} \) generate equivalent transformations and they should give the same complexity. Thus, we may use Eq. (7) for the norm of \( \mathcal{H} \)

\[ \bar{F}(\mathcal{H}) = \lambda \sqrt{\text{Tr}[(\mathcal{H} - \bar{\mathcal{H}})^2]]. \tag{10} \]

However, if the Hilbert space of \( \mathcal{H} \) is infinitely dimensional trace in Eq. (9) and (10) is divergent so we need to renormalize it. We will show how to do it for two important cases: a quadratic Hamiltonian in one dimensional space and a non-relativistic particle in compact Riemannian manifolds.

III. ADS \(_3\) SPACE TIME AS A COMPLEXITY GEOMETRY

Let us consider a general quadratic Hamiltonian in one-dimension space,

\[ \mathcal{H} = \frac{Y^+}{2} \hat{x}^2 + \frac{Y^-}{2} \hat{p}^2 + \frac{Y^0}{4} (\hat{x} \hat{p} + \hat{p} \hat{x}). \tag{11} \]

Here, we want to emphasize that the \( \{ Y^I \} \) should satisfy the constraints

\[ 4Y^+ Y^- - (Y^0)^2 \geq 0 \quad \text{and} \quad Y^\pm \geq 0 \tag{12} \]

so that the Hamiltonian is bounded below. It can be seen from Eq. (14). This is another difference between

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2 See appendix C for a concrete example.

3 Under the left translation \( c(s) \rightarrow \bar{U} c(s) \), \( \bar{F}(H) \rightarrow \bar{F}(\bar{U} H \bar{U}^{-1}) = \bar{F}(H) \).

4 For an operator \( \exp(i \bar{H}) \) with a given \( \bar{H} \), there may be many \( H_k \) satisfying \( \exp(i H_k) = \exp(i \bar{H}) \). The complexity is given by the minimal \( \bar{F}(H_k) \) among all \( H_k \). Thus, the complexity of the operator \( \exp(i \bar{H}) \) may stop growing after large time \( t \) [43].

5 In a mathematical jargon, the Hamiltonian may not be in the trace class.
QFT/QM and quantum circuits. The Hermit operators in quantum circuits have finite number of eigenvalues and are always bounded below. However, the Hermit operators in QFT/QM may not be bounded below in general, so we should be careful in defining parameter ranges.

To compute the norm of $H$ we make a canonical transformation \((x, p) \to (x', p')\) as follows,

$$H = \frac{c_1}{2} \dot{x}^2 + \frac{c_2}{2} \dot{p}^2,$$

where $c_1$, $c_2$ and $Y^\pm, Y^0$ satisfy

$$c_1 + c_2 = Y^+ + Y^-, \quad c_1c_2 = Y^+Y^- - (Y^0)^2/4. \quad (14)$$

One can read the eigenvalues of $H - \bar{H}$ as the analytic continuation of the following sum \([53]\)

$$E_n = \omega(n + 1/2 - \bar{H}, \quad n = 0, 1, 2, \cdots \quad (15)\text{ with } \omega = \sqrt{Y^+Y^- - (Y^0)^2/4}. \text{ The norm of } H \text{ will be divergent and we need to consider a proper renormalization of Eq. (10).}$$

In general, we can define the $\zeta$-function for a positive definite Hermitian operator $O$ with eigenvalues \(\{E_n\}\) as analytic continuation of the following sum

$$\zeta_O(s) := \sum_{n=0}^{\infty} \frac{1}{E_n^s}. \quad (16)$$

In particular, for the operator $O = H - \bar{H}$ with $H$ in (13), we have

$$\zeta_{H - \bar{H}}(s) = \sum_{n=0}^{\infty} \frac{1}{(n + 1/2 - \bar{H}/\omega)^s} = \omega^{-s}\zeta(s, 1/2 - \bar{H}/\omega), \quad (17)$$

where $\zeta(s, q)$ is the Hurwitz-$\zeta$ function, which is defined as the analytic continuation of the following sum \([53]\)

$$\zeta(s, q) = \sum_{n=0}^{\infty} \frac{1}{(n + q)^s}. \quad (18)$$

We may define the renormalized traces as

$$\text{Tr}(H - \bar{H}) := \zeta_{H - \bar{H}}(-1), \quad (19)$$

and the renormalized norm square as

$$\tilde{F}_{re}(H)^2 = \text{Tr}[(H - \bar{H})^2] = \lambda^2 \zeta_{H - \bar{H}}(-2). \quad (20)$$

First, by Eq. (9), Eq. (19) determines $\bar{H}$:

$$\zeta(-1, -\bar{H}/\omega + 1/2) = 0 \Rightarrow \frac{\bar{H}^2}{\omega^2} - \frac{1}{12} = 0, \quad (21)$$

which gives two solutions $\bar{H} = \pm \omega/(2\sqrt{3})$. Thus, the Eq. (20) becomes

$$\tilde{F}_{re}(H)^2 = \lambda^2 \omega^2 \zeta \left(-2, \pm \frac{1}{2\sqrt{3}} + \frac{1}{2}\right) = \pm \frac{\sqrt{3}\lambda^2}{108} \omega^2. \quad (22)$$

Because $\tilde{F}_{re}(H)^2$ should be nonnegative when coefficients $Y'$ satisfy Eq. (12), we have to choose $\bar{H} = -\omega/(2\sqrt{3})$. Defining the parameter $\lambda^2 = 108\lambda^2/(2\sqrt{3})$ we obtain

$$\tilde{F}_{re}(H)^2 = \lambda^2 \omega^2 = \lambda_0^2 (4Y^+Y^- - (Y^0)^2), \quad (23)$$

Indeed, our result (23) is consistent with the theoretic consideration. The Hamiltonian (11) is written as $H = Y'\epsilon I$ with

$$e_+ = \frac{1}{2} \hat{x}^2, \quad e_- = \frac{1}{2} \hat{p}^2, \quad e_0 = \frac{1}{4} (\hat{x}\hat{p} + \hat{p}\hat{x}), \quad (24)$$

where \(\{i\epsilon I\}\) consists of the $su(1, 1)$ Lie algebra \(^6\)

$$[i\epsilon_0, i\epsilon_\pm] = \pm i\epsilon_\pm, \quad [i\epsilon_-, i\epsilon_+] = 2i\epsilon_0. \quad (25)$$

As the $su(1, 1)$ is simple, its bi-invariant metric $\tilde{g}(\cdot, \cdot)$ is proportional to the Killing form. In other words, $\tilde{g}$ can be computed simply by the structure constants of the Lie algebra. As a result

$$\tilde{g}_{IJ}Y'^IY'^J \sim 4Y^+Y^- - (Y^0)^2, \quad (26)$$

which serves as a nice consistent check for our method. In appendix B, we provide an alternative independent method to obtain metric (26) without assuming the symmetry (6).

Let us consider the case, $Y^+ = Y^0 = 0$ and $Y^- > 0$, which corresponds to the free particle case. From (23) we find that

$$\tilde{F}_{re}(H) = 0, \quad (27)$$

which is the first explicit realization of the fact \([11]\) (the Sec. VIII B 4) that the free Hamiltonian cannot mix information across its degrees of freedom so the complexity of the operator generated by free Hamiltonian is zero.

By parameterizing the $SU(1, 1)$ group as

$$\hat{U}(y, z, u) = \exp(i\epsilon_0) \exp(i\epsilon_+) \exp(i\epsilon_-), \quad (28)$$

we obtain the metric in the group manifold (see Eq. (4) and appendix C for details)

$$d\ell^2 = \lambda_0^2 (-dy^2/4 + dz dy du + dz du). \quad (29)$$

As the signature of this metric is $(-, -, +)$, it is interesting to define a “spacetime interval” $ds^2 = -d\ell^2 < 0$

$$ds^2 = g^{(st)}_{IJ} dx^I dx^J := \lambda_0^2 (dy^2/4 - z dy du - dz du), \quad (30)$$

which yields the Riemann tensor $R^{(st)}_{IJKL}$

$$R^{(st)}_{IJKL} = -\frac{1}{\lambda_0^2} (9 g^{(st)}_{IK} g^{(st)}_{JL} - g^{(st)}_{IL} g^{(st)}_{JK}). \quad (31)$$

\(^6\) This is also the symplectic Lie algebra $sp(2, \mathbb{R})$, the special linear Lie algebra $sl(2, \mathbb{R})$ and Lorentz Lie algebra $so(2, 1)$.

\(^7\) The complexity related to $su(1, 1)$ algebra has been studied in Refs. [21, 29, 37, 47], in only right-invariant geometry and without the restriction (12).
This means that the AdS$_3$ spacetime with the AdS radius $\ell_{\text{AdS}} = \lambda_0$ emerges as our complexity geometry! We find some interesting correspondences between the complexity geometry of SU(1,1) group and AdS$_3$ spacetime. For example, the physical Hamiltonians with (12) correspond to the time-like tangent vectors in AdS$_3$ spacetime. In Tab. 1, we make more comparison between the complexity geometry of SU(1,1) generated by quadratic Hamiltonian and spacetime geometry of AdS$_3$.

IV. COMPLEXITY AND QUANTUM PHASE TRANSITIONS

We will show an interesting relation between the complexity and quantum phase transitions (QPTs). For example, let us consider the Lipkin-Meshkov-Glick (LMG) model [54]. The LMG model was introduced in nuclear physics. It describes a cluster of mutually interacting spins in a transverse magnetic field and has found applications in many other areas of physics like quantum spin systems [55], ion traps [56], Bose-Einstein condensates in double wells [57], and in circuit quantum electrodynamics experiments [58]. The Hamiltonian of LMG model reads [59]

$$\mathcal{H} = -\frac{1}{N} \sum_{i<j} (\alpha \sigma^x_i \sigma^x_j + \gamma \sigma^y_i \sigma^y_j) - h \sum_i \sigma^z_i,$$

where $N$ is the number of spins and $h$ stands for an external magnetic field along the $z$-direction. $\alpha$ and $\gamma$ are two real numbers. Without loss generality, we can set $\alpha > 0$ and $|\gamma / \alpha| \leq 1$. In the limit $N \to \infty$, it has been shown that there is a QPT at $h_c = \alpha$. For $h > h_c$ the system is in paramagnetic phase with zero magnetization along the $xy$-plane while for $h < h_c$ it is in the symmetry-broken phase with nonzero magnetization in $xy$-plane [55].

In the large $N$ limit, if $h \geq h_c$, the effective Hamiltonian reads [60]

$$\mathcal{H}_\gamma = -hN + (2h - \gamma - \alpha) \hat{a}^\dagger \hat{a} - \frac{\alpha - \gamma}{2} (\hat{a}^2 + \hat{a}^\dagger \hat{a})^2,$$

which may be written as

$$\mathcal{H}_\gamma = (h - \gamma) \hat{p}^2 + (h - \alpha) \hat{x}^2 - hN + (2h - \gamma - \alpha)/2,$$

by using $\hat{a} = \frac{1}{\sqrt{2}} (\hat{x} + i \hat{p})$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}} (\hat{x} - i \hat{p})$. Similarly, if $h < h_c$, the effective Hamiltonian reads [60]

$$\mathcal{H}_\gamma = (\alpha - \gamma) \hat{p}^2 + (\alpha^2 - h^2) \hat{x}^2 - (N(\alpha^2 + h^2) + (\gamma \alpha + 2h^2 - 3\alpha^2))/(2\alpha).$$

By identifying $Y'$ in (11) and using the complexity formula in (23) we obtain

$$C(\exp(i\mathcal{H}_\gamma)) = 2\lambda_0 \sqrt{(h - \gamma)(h - h_c),}$$

$$C(\exp(i\mathcal{H}_\gamma)) = 2\lambda_0 \sqrt{\alpha^{-1}(\alpha - \gamma)(\alpha^2 - h^2)}.$$

At the QPT point $h = h_c = \alpha$, the change of the complexity $\partial C / \partial h$ is not only discontinuous but also divergent. This example implies that the change of the complexity may be used as a detector of QPT.

V. COMPLEXITY IN A COMPACT RIEMANNIAN MANIFOLD

Let us consider a non-relativistic particle with mass $m$ and a bounded below potential $V(x)$ in a $n$-dimensional compact Riemannian manifold $M$ ($\partial M = 0$) with arbitrary positive definite metric $G_{\mu\nu}$. The Hamiltonian $H$ is given by

$$H = -\frac{\nabla^2}{2m} + V(x),$$

where $\nabla$ is the covariant derivative associated to metric $G_{\mu\nu}$.

As the eigenvalues of such Hamiltonian cannot be computed analytically in general, we cannot use $\zeta$-function method. In order to regularize the trace for this Hamiltonian, we define the regularized trace of any function of $H$, say $f(H)$ as

$$\text{Tr}_{\tau} [f(H)] := \frac{1}{K_H(\tau)} \text{Tr} [f(H)e^{-\tau H}],$$

where

$$K_H(\tau) := \text{Tr}(e^{-\tau H}) \quad \text{and} \quad \tau \to 0^+.$$

The operator $e^{-\tau H}$ is usually called the “heat kernel” [61, 62]. Here, we take $\tau > 0$ so that Eq. (39) is finite since the Hamiltonian is bounded from below.

First, we compute $\mathcal{H}$ by using Eq. (9) with the regularized trace (39)

$$\mathcal{H} = \text{Tr}_\tau(H)/\text{Tr}_\tau(\hat{1}) = \text{Tr}_\tau(H) = -\frac{d}{d\tau} \ln K_H(\tau).$$

Thus, we generalize Eq. (10) as

$$\tilde{F}(H)^2 = \lambda^2 \{ \text{Tr}_\tau(H^2) - [\text{Tr}_\tau(H)]^2 \}
= \lambda^2 \frac{d^2}{d\tau^2} \ln K_H(\tau).$$

For finite dimensional cases, Eq. (42) agrees with Eq. (10) after we take limit $\tau \to 0^+$. Without loss of generality, we may set $m = 1/2$ in following discussion. For a general $G_{\mu\nu}$, it is difficult to find even the ground state and the first eigenvalue of $H$. However, in the case of $\tau \to 0^+$, the trace of the heat kernel can be computed in terms of a serie of $\tau$ [62],

$$K_H(\tau) = \frac{1}{(4\pi \tau)^{n/2}} \sum_{n=0}^{\infty} a_n \tau^n,$$
Spacetime geometry

Future-toward light-cone

Points in AdS spacetime

Speed of particle cannot be superluminal

Trajectories with future-toward time-like tangent vectors

Future-toward light-cone

TABLE I. Comparison between the complexity geometry of SU(1,1) group and spacetime geometry of AdS.

| Complexity geometry | Spacetime geometry |
|---------------------|--------------------|
| Unitary operators generated by quadratic Hamiltonian should be bounded below | Points in AdS spacetime |
| Curves with positive complexity | Speed of particle cannot be superluminal |
| Set of physically realizable unitary operators | Trajectories with future-toward time-like tangent vectors |

where $a_0 = \int_M \sqrt{G}dx^n$, $a_1 = \int_M (R/6 + V) \sqrt{G}dx^n$ and $a_2 = b + c$ with

$$b := \frac{1}{2} \int_M \sqrt{G}(V + R/6)dx^n,$$

$$c := \frac{1}{180} \int_M \sqrt{G}(R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - R_{\mu\nu}R^{\mu\nu})dx^n. \quad (45)$$

Here $G$ is the determinant of $G_{\mu\nu}$, $R_{\mu\nu\sigma\rho}$ is the Riemannian tensor, $R_{\mu\nu}$ is the Ricci tensor and $R$ is the scalar curvature.

Plugging Eq. (43) into Eq. (42) and setting $\tau \rightarrow 0^+$, we obtain

$$\lambda^{-2} \tilde{F}(H)^2 = \frac{n}{2\tau^2} + \frac{2a_2 a_0 - a_1^2}{a_0^2} + O(\tau). \quad (46)$$

The divergent term does not contribute to the metric defined by Eq. (3), so it can be removed without changing the complexity geometry. Thus we obtain the renormalized norm square

$$\tilde{F}_{re}(H)^2 := \lim_{\tau \rightarrow 0^+} \tilde{F}(H)^2 - \frac{\lambda^2 n}{2\tau^2} = \lambda^2 \frac{2a_2 a_0 - a_1^2}{a_0^2}. \quad (47)$$

For a “free particle”, i.e., a flat metric $G_{\mu\nu} = \delta_{\mu\nu}$ with a constant potential $V(x) = V_0$, we obtain $\tilde{F}_{re}(H)^2 = 0$, which is consistent with (27). Eq. (47) also gives us a new insight for the relation between the complexity and the geometry of the space. According to the definitions of coefficients $a_0, a_1$ and $a_2$, it turns out that the complexity (47) is always well-defined (non-negative) only if the spacetime dimension is 3+1 or less, which is compatible to the spacetime of our world in low energy limit. (see appendix D for a proof.)

VI. CONCLUSIONS

For the complexity to be a useful tool for gravity and QFT/QM, we first have to fill up the conceptual gap between the quantum circuits and QFT/QM in defining complexity. By noting that the generating functional plays a central role in QFT/QM contrary to quantum circuits we propose an additional symmetry (6) for the complexity in QFT/QM. It leads to a simple and unique formula for the complexity of SU(n) operators. Even though the formula is unique, its result is still rich. In particular, when we study complexity of the operators generated by Hamiltonians in the infinite dimensional Hilbert space, it shows novel results which can not be obtained from finite qubit systems. Interestingly enough, the complexity geometry corresponding to a general quadratic Hamiltonian in one-dimension is equivalent to AdS$_3$ space. Here, we noted the lower boundedness of the Hamiltonian gives an constraint for the non-negativity of the complexity, which has not been appreciated before.

Our formula proves that the complexity of the operator generated by free Hamiltonians vanishes, which was intuitively plausible. We also showed that the complexity can be used as an indicator of a QPT. It will be interesting to analyze the critical behaviors of the complexity in detail for more QPT systems. We uncovered the connection between the complexity and the background geometry. In particular, the fact that the critical dimension to insure a nonnegative complexity in low energy limit is just 3+1 dimension is worthy of more investigations.

This paper focused on a few quantum mechanical systems in order to simplify technical details. However, we expect that our proposal may result in more novel properties in QFT, especially at strong interactions or in the curved spacetime.

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Appendix A: Unitary invariance of the complexity

The symmetry of the complexity of operators is implied by the symmetry of $\tilde{F}$ as shown in Eq. (5). If complexity is observable or a physical quantity yielding any observable, so is norm $\tilde{F}$, and vice verse. Thus, we discuss the symmetry of the norm $\tilde{F}$.

Let us consider two systems of which dynamics are given by time independent Hamiltonians $H$ and $UHU^\dagger$ respectively. Recall that the generating functional in
QFT is the same for both $H$ and $UHU^\dagger$ so two systems are indistinguishable as far as the observables derived from the generating function. Therefore, if the complexity is an observable, it is natural to have $\tilde{F}(H) = \tilde{F}(\hat{U}HU^\dagger)$. Note that a generating functional is not essential for the quantum circuit system so $\tilde{F}(H) = \tilde{F}(\hat{U}HU^\dagger)$ may not be necessary for this perspective.

However, it is possible that the complexity itself is not an observable but a physical quantity yielding observables. For example, a wavefunction of the Schroedinger equation and the gauge potential in electromagnetism are such quantities. We will show also in this case the condition $\tilde{F}(H) = \tilde{F}(\hat{U}HU^\dagger)$ is required.

Let us first consider the case that the complexity itself leads to an observable. This means that the norm $\tilde{F}$ can lead to an observable, i.e., an observable $O(\hat{F})$ as a function of $\tilde{F}$. Indeed, this one variable case is trivial. In order for $O$ to be an observable $\tilde{F}$ must be an observable since $\tilde{F}$ is a scalar. Notwithstanding, we describe it for completeness. We assume that $O(x)$ is a non-constant and smooth function, which means

$$\forall x, \exists k \in \mathbb{N}^+, \text{ such that } \frac{d^kO(x)}{dx^k} \neq 0. \quad (A1)$$

Because $O$ is an observable, for arbitrary $H$ and $\hat{U}(s)$,

$$O(\tilde{F}(H)) = O(\tilde{F}(\hat{U}(s)\hat{H}U^\dagger(s))) \Leftrightarrow \frac{dO}{ds} = 0, \quad (A2)$$

where $s$ is a continuous parameter for $\hat{U}(s)$. If $dO(x)/dx \neq 0$ Eq. (A2) implies

$$\frac{d}{ds} \tilde{F}(\hat{U}(s)\hat{H}U^\dagger(s)) = 0. \quad (A3)$$

because

$$\frac{dO}{ds} = \frac{dO(x)}{dx} \frac{d}{ds} \tilde{F}(\hat{U}(s)\hat{H}U^\dagger(s)) = 0. \quad (A4)$$

If $dO(x)/dx = 0$, we can find a $k \geq 2$ such that $d^kO(x)/dx^k \neq 0$ (Eq. (A1)) and obtain the same result Eq. (A3). This shows $\tilde{F}(H) = \tilde{F}(\hat{U}HU^\dagger)$ for arbitrary $\hat{U}$.

In more general cases, one complexity itself is not an observable but we can obtain an observable by several complexities. For example, one may think that the complexity is not a direct observable but the difference between two complexities is observable, which implies that neither $\tilde{F}(H_1)$ nor $\tilde{F}(H_2)$ is observable but $\tilde{F}(H_1) - \tilde{F}(H_2)$ is an observable. Thus we have $\tilde{F}(H_1) - \tilde{F}(H_2) = \tilde{F}(\hat{U}H_1U^\dagger) - \tilde{F}(\hat{U}H_2U^\dagger)$. Then we can choose $O := O(x_1, x_2) = x_1 - x_2$ and so $O(\tilde{F}(H_1), \tilde{F}(H_2)) = O(\tilde{F}(\hat{U}H_1U^\dagger), \tilde{F}(\hat{U}H_2U^\dagger))$. In general an observable $O$ may have more than two arguments.

If $O(x_1, x_2, \cdots, x_n)$ is a non-constant smooth function, we can find that $\forall x_1, x_2, \cdots, x_n, \exists k \in \mathbb{N}^+$ and $\exists l$ such that $\partial^kO/\partial x_1^k \neq 0$. Without loss generality, we take $l = 1$ and obtain

$$\exists k \in \mathbb{N}^+, \partial^kO/\partial x_1^k \neq 0, \forall x_1, x_2, \cdots, x_n. \quad (A5)$$

Next, as $O(\tilde{F}(H_1), \tilde{F}(H_2), \cdots, \tilde{F}(H_n))$ is an observable, we require:

$$O(\tilde{F}(H_1), \tilde{F}(H_2), \cdots, \tilde{F}(H_n)) = O(\tilde{F}(\hat{U}H_1U^\dagger), \tilde{F}(\hat{U}H_2U^\dagger), \cdots, \tilde{F}(\hat{U}H_nU^\dagger)), \quad (A6)$$

for arbitrary $H_j$ and $\hat{U}$. i.e. $H_1$ and $\hat{U}H_2U^\dagger$ gives the same observables. The basic idea of the proof is to reduce Eq. (A6) to the trivial one variable case.

$$O(\tilde{F}(H), \tilde{F}(H'), \cdots, \tilde{F}(H')) = O(\tilde{F}(\hat{U}(s)\hat{H}(s)\hat{H}(s)^\dagger), \tilde{F}(H'), \cdots, \tilde{F}(H')). \quad (A7)$$

It can be done by choosing $H_1 = H, H_2 = H_3 = \cdots H_n = H'$ and $U(s) = \exp(H'(s))$ with $\hat{U}(s)\hat{H}(s)(\hat{H}(s)^\dagger) \neq H$. By the essentially same procedure for the one variable case, we conclude $\tilde{F}(H) = \tilde{F}(\hat{U}HU^\dagger)$.

### Appendix B: Complexity metric for quadratic Hamiltonians from symmetries

In this appendix, we provide another independent method to obtain the complexity geometry for quadratic Hamiltonians. Here, we do not assume the complexity geometry is bi-invariant but only assume right-invariant. Just by using $U(1)$ gauge symmetry and canonical transformation symmetry, we can determine the unique complexity geometry and show that the complexity geometry of general quadratic Hamiltonians must be bi-invariant as a result.

Let us consider a Hamiltonian of a non-relativistic particle with potential $V(\vec{x}) = \frac{k}{2}\vec{x}^2$ and a magnetic vector potential $A(\vec{x})$,

$$\mathcal{H} = \frac{1}{2m} [\vec{p} + \vec{A}(\vec{x})]^2 + \frac{k}{2}\vec{x}^2. \quad (B1)$$

Without loss generality, we set charge $q = 1$. If we take a pure gauge, $\vec{A}(\vec{x}) = \nabla \phi(\vec{x})$ for an arbitrary scalar function $\phi(\vec{x})$, the Hamiltonian

$$\mathcal{H}_\phi = \frac{1}{2m} [\vec{p} + \nabla \phi(\vec{x})]^2 + \frac{k}{2}\vec{x}^2. \quad (B2)$$

and the harmonic oscillator Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2m} \vec{p}^2 + \frac{k}{2}\vec{x}^2, \quad (B3)$$

describe the same physical system. Thus, they should give the same complexity and

$$\tilde{F}(\mathcal{H}_0) = \tilde{F}(\mathcal{H}_\phi), \quad (B4)$$
for all $\phi(\tilde{x})$. For the Hamiltonian (B3), we can take a canonical transformation $(\tilde{x}, \tilde{p}) \to (\gamma \tilde{x}, \gamma^{-1} \tilde{p})$, which leads to

$$\mathcal{H}_0(\gamma) = \frac{1}{2m\gamma^2} \tilde{p}^2 + \frac{k\gamma^2}{2} \tilde{x}^2.$$  \hfill (B5)

Because this transformation is canonical, $\mathcal{H}_0(\gamma)$ and $\mathcal{H}_0$ describe the same physical system so

$$\tilde{F}(\mathcal{H}_0) = \tilde{F}(\mathcal{H}_0(\gamma)), \hfill (B6)$$

for all $\gamma \neq 0$.

Let us first consider what we can obtain from the symmetry (B4). For $\phi(\tilde{x}) = a m \tilde{x}^2$ with an arbitrary real number $a$ we obtain

$$\mathcal{H}_\phi = \mathcal{H}(a, k, m) = \sum_{i=1}^{3} \left[ \frac{1}{m} \tilde{p}_i^2 + 4a \tilde{x}_i \tilde{p}_i + \frac{\tilde{p}_i \tilde{x}_i}{4} + (4a^2 m + k) \tilde{x}_i^2 \right].$$  \hfill (B7)

Comparing with the bases defined in Eq. (24), we find that this Hamiltonian contains the triple copies of $su(1,1)$ Lie algebra. The symmetry (B4) implies

$$\tilde{F}(\mathcal{H}(a, k, m)) = \tilde{F}(\mathcal{H}(0, k, m)),$$  \hfill (B8)

for arbitrary $a, k$ and $m$. In general, $\tilde{F}(\mathcal{H})^2 = \tilde{g}_{IJ} Y^I Y^J$, where $\tilde{g}_{IJ}$ is a general metric for $su(1,1)$ Lie algebra. In our case,

$$\tilde{F}(\mathcal{H}(a, k, m))^2 = 3 \left[ \tilde{g}_{-} - \frac{1}{m^2} + \tilde{g}_{00} 16a^2 + \tilde{g}_{++}(4a^2 m + k)^2 \right.\left. + \tilde{g}^{-} 2 4a^2 m + k \right] + \tilde{g}_{++} 8a (4a^2 m + k) + \frac{\tilde{g}_0}{m} 8a \right].$$ \hfill (B9)

The overall coefficient 3 comes from the fact that $\mathcal{H}(a, k, m)$ contains triple copies of $su(1,1)$ Lie algebra. If $k = 0$ Eq. (B8) implies

$$\tilde{g}_{--} \frac{1}{m^2} = \left[ \tilde{g}_{-} - \frac{1}{m^2} + (2\tilde{g}_{00} + \tilde{g}_{-}) 8a^2 \right. \left. + \tilde{g}_{++} 16a^2 m + \tilde{g}_{00} 32a^2 m^3 + \frac{\tilde{g}_0}{m} 8a \right],$$ \hfill (B10)

for all $a$. This means that $\tilde{g}_{++} = \tilde{g}_{-} = \tilde{g}_{00} = 0$ and

$$\tilde{g}_{+-} = -2\tilde{g}_{00}. \hfill (B11)$$

Thus, Eq. (B9) boils down to

$$\tilde{F}(\mathcal{H}(a, k, m))^2 = 3 \left[ \tilde{g}_{--} \frac{1}{m^2} + 2\tilde{g}_{+-} \frac{k}{m} \right].$$ \hfill (B12)

We see that the gauge symmetry gives us very strong restriction on the metric for quadratic Hamiltonians.

Now let us consider what we can obtain from the symmetry (B6), which means

$$\tilde{F}(\mathcal{H}(0, k, m))^2 = \tilde{F}(\mathcal{H}(0, k^2, m^2))^2.$$ \hfill (B13)

By using Eq. (B12), we have

$$\tilde{g}_{--} \frac{1}{m^2} + 2\tilde{g}_{+-} \frac{k}{m} = \frac{\tilde{g}_{--} (m^2)^2}{2} + 2\tilde{g}_{+-} \frac{k}{m},$$ \hfill (B14)

for all $\gamma \neq 0$. This implies $\tilde{g}_{--} = 0$.

Thus, respecting the symmetries (B4) and (B6) we find that the nonzero components of metric $\tilde{g}_{IJ}$ are only $\{\tilde{g}_{++}, \tilde{g}_{00}\}$ and they satisfy the Eq. (B11). As a result, for a quadratic Hamiltonian $\mathcal{H} = Y^I e_I$ with $\{e^I\}$ defined in Eq. (24), we have

$$\tilde{F}(\mathcal{H})^2 = -g_{00} [4Y^+ Y^- - (Y^0)^2],$$ \hfill (B15)

which is the same as Eq. (23).

It shows that the complexity metric of quadratic Hamiltonians must be bi-invariant! By this alternative approach we may conclude that the bi-invariance (and so the symmetry (6)) is a necessary condition for the complexity to respect the fundamental symmetries of QFT/QM.

### Appendix C: Metric for SU(1,1) group

We will explain how to obtain Eq. (C8). Let us start with Eq. (1),

$$d\hat{U}(X^I)\hat{U}^\dagger(X^I) = i e_I Y^I ds,$$ \hfill (C1)

where $\hat{U}(X^I)$ is an element of the group $G$ parameterized by the coordinate $X^I$ and the Lie algebra $\mathfrak{g}$ is spanned by the bases $\{ie_I\}$.

For any representation of $G$, i.e. a map $\pi : G \mapsto GL(n, K)$ with $K = \mathbb{R}$ or $\mathbb{C}$, and its induced representation in the Lie algebra, i.e. $\pi_* : \mathfrak{g} \mapsto GL(n, K)$ Eq. (C1) reads

$$d[\pi(\hat{U})][\pi(\hat{U})]^{-1} = \pi_*(ie_I) Y^I ds.$$ \hfill (C2)

Because the coefficient $Y^I ds$ is independent of choice on faithful representations we may choose any faithful representation for our convenience.

For example, we may parameterize SU(1,1) group as

$$\hat{U}(X^I) = \hat{U}(y, z, u) = \exp(iye_0) \exp(ize_+) \exp(ie_-),$$ \hfill (C3)

and choose a $2 \times 2$ matrix representation for $e_0$ and $e_\pm$ as follows. $\pi_*(ie_I) = K_I$ with

$$K_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K_+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad K_- : = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
In this representation the group element Eq. (C3) is expressed as
\[ \pi(\hat{U}(y, z, u)) = \left( e^{y/2(1 - zu)}, -e^{y/2z} e^{-y/2u}, e^{-y/2u} \right), \quad (C4) \]
which yields
\[ d[\pi(\hat{U})]\pi(\hat{U})^{-1} = (dy - 2zdud)K_0 + e^y(dz + z^2du)K_+ + e^{-y}duK_. \quad (C5) \]
Thus, by Eq. (C2) we find
\[ Y^I ds = (dy - 2zdud, e^y(dz + z^2du), e^{-y}du) \quad (C6) \]
Finally, the line element reads
\[ dl^2 = \tilde{g}_{IJ}Y^I Y^J ds^2 = \lambda_0^2(Y^+Y^- - (Y^0)^2/4)ds^2 \quad (C7) \]
where $\tilde{g}_{IJ}$ is given by Eq. (23) and (3). By Eq. (C6) we obtain the metric in the group manifold
\[ dl^2 = \lambda_0^2(-dy^2/4 + zdud + dzdu) \quad (C8) \]

**Appendix D: Non-negativity of the complexity and spacetime dimension**

First, Eq. (47) can be rewritten as
\[ 2a_2a_0 - a_1^2 = (2ba_0 - a_1^2) + 2ca_0 \quad (D1) \]
Thanks to the Cauchy-Schwarz inequality
\[ 2ba_0 = \left( \int_M (V + R/6)^2Gdx^n \right) \int_M 1^2Gdx^n \geq \left( \int_M (V + R/6) \times 1Gdx^n \right)^2 = a_1^2 \quad (D2) \]

The term in the parenthesis in Eq. (D1) is nonnegative so we focus on the last term, $2ca_0$.

For a flat space, $c = 0$ so Eq. (D1) is always non-negative in arbitrary dimension. It can be zero only for-free particles, i.e., $V(x)$ is constant. It is consistent with Eq. (27).

For a curved space, as $a_0$ is positive, we only need to consider the sign of $c$. For $n = 1$, $R_{\mu\nu\sigma\rho} = 0$ so $c = 0$. For $n = 2$, $R_{\mu\nu\sigma\rho}$ has only one independent term. Under a local orthonormalized frame $\{e_i\}$ the metric components read $G_{ij} = \delta_{ij}$ and the nonzero curvature component is $K := R_{1212}$. Thus, $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - R_{\mu\nu}R^{\mu\nu} = 2K^2 \geq 0$ and $c \geq 0$. For $n \geq 3$, we rewrite $c$ as
\[ c = c_{GB} + \frac{1}{180} \int_M \sqrt{G}(3R_{\mu\nu}R^{\mu\nu} - R^2)d\Omega^3, \quad (D3) \]
where the Gauss-Bonnet term is introduced:
\[ c_{GB} := \frac{1}{180} \int_M \sqrt{G}(R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2)d\Omega^n. \quad (D4) \]

For $n = 3$, $c_{GB} = 0$. In the local orthonormalized frame $\{e_i\}$, we can diagonalize the Ricci tensor and obtain three eigenvalues $\{k_1, k_2, k_3\}$. Thus, $3R_{\mu\nu}R^{\mu\nu} - R^2 = 3(k_1^2 + k_2^2 + k_3^2) - (k_1 + k_2 + k_3)^2 \geq 0$ and $c \geq 0$. However, for $n \geq 4$, $3R_{\mu\nu}R^{\mu\nu} - R^2$ and $c_{GB}$ can be negative for some geometries, so $c$ and Eq. (D1) can be negative.

The Hamiltonian given by Eq. (38) can only be used in low energy limit of quantum field theory. If the spacetime dimension in low energy limit is $3 + 1$ or less, then Eq. (47) is always nonnegative.

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[1] Shinsei Ryu and Tadashi Takayanagi, “Holographic derivation of entanglement entropy from the anti-de sitter space/conformal field theory correspondence,” Phys. Rev. Lett. 96, 181602 (2006).

[2] Tatsuma Nishioka, Shinsei Ryu, and Tadashi Takayanagi, “Holographic Entanglement Entropy: An Overview,” J. Phys. A42, 504008 (2009), arXiv:0905.0932 [hep-th].

[3] Mark Van Raamsdonk, “Building up spacetime with quantum entanglement,” Gen. Rel. Grav. 42, 2323-2329 (2010), [Int. J. Mod. Phys.D19,2429(2010)], arXiv:1005.3035 [hep-th].

[4] Masahiro Nozaki, Shinsei Ryu, and Tadashi Takayanagi, “Holographic Geometry of Entanglement Renormalization in Quantum Field Theories,” JHEP 10, 193 (2012), arXiv:1208.3469 [hep-th].

[5] Jennifer Lin, Matilde Marcolli, Hiroshi Ooguri, and Bogdan Stoica, “Locality of Gravitational Systems from Entanglement of Conformal Field Theories,” Phys. Rev. Lett. 114, 221601 (2015), arXiv:1412.1879 [hep-th].

[6] Patrick Hayden, Sephr Nezami, Xiao-Liang Qi, Nathaniel Thomas, Michael Walter, and Zhao Yang, “Holographic duality from random tensor networks,” JHEP 11, 009 (2016), arXiv:1601.01694 [hep-th].

[7] Daniel Harlow and Patrick Hayden, “Quantum Computation vs. Firewalls,” JHEP 06, 085 (2013), arXiv:1301.4504 [hep-th].

[8] Douglas Stanford and Leonard Susskind, “Complexity and Shock Wave Geometries,” Phys. Rev. D90, 126007 (2014), arXiv:1406.2678 [hep-th].

[9] Leonard Susskind, “Computational Complexity and Black Hole Horizons,” Fortsch. Phys. 64, 44-48 (2016), [Fortsch. Phys.64,24(2016)], arXiv:1403.5695 [hep-th].

[10] Adam R. Brown, Daniel A. Roberts, Leonard Susskind, Brian Swingle, and Ying Zhao, “Holographic Complexity Equals Bulk Action?” Phys. Rev. Lett. 116, 191301
Shira Chapman, Michal P. Heller, Hugo Marrochio, and Adam R. Brown and Leonard Susskind, “Second law of
Adam R. Brown, Leonard Susskind, and Ying Zhao, “Switchbacks and the
Leonard Susskind and Ying Zhao, “Switchbacks and the
Brian Swingle and Yixu Wang, “Holographic Complexity as geometry,” Science 311, 1133–1135 (2006),
http://science.sciencemag.org/content/311/5764/1133.full.pdf.
Michael A. Nielsen, “A geometric approach to quantum circuit lower bounds,” Quantum Info. Comput. 6, 213–262 (2006).
Mark R. Dowling and Michael A. Nielsen, “The geometry of quantum computation,” Quantum Info. Comput. 8, 861–899 (2008).
Shira Chapman, Michal P. Heller, Hugo Marrochio, and Fernando Pastawski, “Toward a Definition of Complexity
for Quantum Field Theory States,” Phys. Rev. Lett. 120, 121602 (2018), arXiv:1707.08582 [hep-th].
Pawel Caputa, Nilay Kundu, Masamichi Miyaji, Tadashi Takayanagi, and Kento Watanabe, “Anti-de Sitter Space from Optimization of Path Integrals in Conformal Field Theories,” Phys. Rev. Lett. 119, 071602 (2017), arXiv:1703.00456 [hep-th].
Pawel Caputa, Nilay Kundu, Masamichi Miyaji, Tadashi Takayanagi, and Kento Watanabe, “Liouville Action as Path-Integral Complexity: From Continuous Tensor Networks to AdS/CFT,” JHEP 11, 097 (2017), arXiv:1706.07056 [hep-th].
Arpan Bhattacharyya, Pawel Caputa, Sumit R. Das, Nilay Kundu, Masamichi Miyaji, and Tadashi Takayanagi, “Path-Integral Complexity for Perturbed CFTs,” JHEP 07, 086 (2018), arXiv:1804.01999 [hep-th].
Tadashi Takayanagi, “Holographic Spacetimes as Quantum Circuits of Path-Integrations,” (2018), arXiv:1808.09072 [hep-th].
Koji Hashimoto, Norihiro Iizuka, and Sotaro Sugishita, “Time evolution of complexity in Abelian gauge theories,” Phys. Rev. D96, 126001 (2017), arXiv:1707.03840 [hep-th].
Koji Hashimoto, Norihiro Iizuka, and Sotaro Sugishita, “Thoughts on Holographic Complexity and its Basis-dependence,” Phys. Rev. D98, 046002 (2018), arXiv:1805.04226 [hep-th].
Mario Flory and Nina Miekley, “Complexity change under conformal transformations in AdS_3/CFT_2,” (2018), arXiv:1806.08376 [hep-th].
Robert A. Jefferson and Robert C. Myers, “Circuit complexity in quantum field theory,” JHEP 10, 107 (2017), arXiv:1707.08570 [hep-th].
Run-Qiu Yang, “Complexity for quantum field theory states and applications to thermofield double states,” Phys. Rev. D97, 066004 (2018), arXiv:1709.00921 [hep-th].
Alan P. Reynolds and Simon F. Ross, “Complexity of the AdS Soliton,” Class. Quant. Grav. 35, 095006 (2018), arXiv:1712.03732 [hep-th].
Rifath Khan, Chethan Krishnan, and Sanchita Sharma, “Circuit Complexity in Fermionic Field Theory,” (2018), arXiv:1801.07620 [hep-th].
Lucas Hackl and Robert C. Myers, “Circuit complexity for free fermions,” JHEP 07, 139 (2018), arXiv:1803.10638 [hep-th].
Run-Qiu Yang, Yu-Sen An, Chao Niu, Cheng-Yong Zhang, and Keun-Young Kim, “Principles and symmetries of complexity in quantum field theory,” (2018), arXiv:1803.01797 [hep-th].
Run-Qiu Yang, Yu-Sen An, Chao Niu, Cheng-Yong Zhang, and Keun-Young Kim, “More on complexity of operators in quantum field theory,” (2018), arXiv:1809.06678 [hep-th].
Daniel W. F. Alves and Giancarlo Camilo, “Evolution of complexity following a quantum quench in free field theory,” JHEP 06, 029 (2018), arXiv:1804.00107 [hep-th].
Javier M. Magn, “Black holes, complexity and quantum chaos,” JHEP 09, 043 (2018), arXiv:1805.05839 [hep-th].
Pawel Caputa and Javier M. Magnan, “Quantum Computation as Gravity,” (2018), arXiv:1807.04422 [hep-th].
Hugo A. Camargo, Pawel Caputa, Diptarka Das, Michal P. Heller, and Ro Jefferson, “Complexity as a
novel probe of quantum quenches: universal scalings and purifications,” (2018), arXiv:1807.07075 [hep-th].

[48] Minyong Guo, Juan Hernandez, Robert C. Myers, and Shan-Ming Ruan, “Circuit Complexity for Coherent States,” JHEP 10, 011 (2018), arXiv:1807.07677 [hep-th].

[49] Arpan Bhattacharyya, Arvind Shekar, and Aninda Sinha, “Circuit complexity in interacting QFTs and RG flows,” (2018), arXiv:1808.03105 [hep-th].

[50] Jie Jiang, Jieru Shan, and Jianzhi Yang, “Circuit complexity for free Fermion with a mass quench,” (2018), arXiv:1810.00537 [hep-th].

[51] Shira Chapman, Jens Eisert, Lucas Hackl, Michal P. Heller, Ro Jefferson, Hugo Marrochio, and Robert C. Myers, “Complexity and entanglement for thermofield double states,” (2018), arXiv:1810.05151 [hep-th].

[52] Marcos M. Alexandrino and Renato G. Bettiol, “Lie Groups with Bi-invariant Metrics,” in Lie Groups and Geometric Aspects of Isometric Actions (Springer International Publishing, Cham, 2015) pp. 27–47.

[53] E Elizalde, Ten physical applications of spectral zeta functions (Springer, Heidelberg New York, 2012).

[54] H.J. Lipkin, N. Meshkov, and A.J. Glick, “Validity of many-body approximation methods for a solvable model: (i). exact solutions and perturbation theory,” Nuclear Physics 62, 188 – 198 (1965).

[55] R. Botet, R. Julien, and P. Pfeuty, “Size scaling for infinitely coordinated systems,” Phys. Rev. Lett. 49, 478–481 (1982).

[56] R. G. Unanyan and M. Fleischhauer, “Decoherence-free generation of many-particle entanglement by adiabatic ground-state transitions,” Phys. Rev. Lett. 90, 133601 (2003).

[57] Huan-Qiang Zhou, Jon Links, Ross H McKenzie, and Xi-Wen Guan, “Exact results for a tunnel-coupled pair of trapped bose-einstein condensates,” Journal of Physics A: Mathematical and General 36, L113 (2003).

[58] J. Larson, “Circuit QED scheme for the realization of the lipkin-meshkov-glick model,” EPL (Europhysics Letters) 90, 54001 (2010).

[59] Ho-Man Kwok, Wen-Qiang Ning, Shi-Jian Gu, and Hai-Qing Lin, “Quantum criticality of the Lipkin-Meshkov-Glick model in terms of fidelity susceptibility,” Phys. Rev. E 78, 032103 (2008).

[60] Sébastien Dusuel and Julien Vidal, “Continuous unitary transformations and finite-size scaling exponents in the Lipkin-Meshkov-Glick model,” Phys. Rev. B 71, 224420 (2005).

[61] Ivan G. Avramidi, “Heat kernel approach in quantum field theory,” Quantum gravity and spectral geometry. Proceedings, Nucl. Phys. Proc. Suppl. 104, 3–32 (2002), [3(2001)], arXiv:math-ph/0107018 [math-ph].

[62] D. V. Vassilevich, “Heat kernel expansion: User’s manual,” Phys. Rept. 388, 279–360 (2003), arXiv:hep-th/0306138 [hep-th].