Sub-Lorentzian distance and spheres on the Heisenberg group

Yuri L. Sachkov · Elena F. Sachkova

Received: 8 August 2022 / Revised: 6 February 2023 / Accepted: 17 April 2023 / Published online: 4 July 2023

Abstract
The left-invariant sub-Lorentzian problem on the Heisenberg group is considered. An optimal synthesis is constructed, the sub-Lorentzian distance and spheres are described.

Keywords Sub-Lorentzian geometry · Heisenberg group · Geometric control

Mathematics Subject Classification (2010) 53C22 · 49K15

1 Introduction

A sub-Riemannian structure on a smooth manifold $M$ is a vector distribution $\Delta \subset TM$ endowed with a Riemannian metric $g$ (a positive definite quadratic form). Sub-Riemannian geometry is a rich theory and an active domain of research during the last decades [1–8].

A sub-Lorentzian structure is a variation of a sub-Riemannian one for which the quadratic form $g$ in a distribution $\Delta$ is a Lorentzian metric (a nondegenerate quadratic form of index 1). Sub-Lorentzian geometry tries to develop a theory similar to the sub-Riemannian geometry, and it is still in its childhood. For example, the left-invariant sub-Riemannian structure on the Heisenberg group is a classic subject covered in almost every textbook or survey on sub-Riemannian geometry. On the other hand, the left-invariant sub-Lorentzian structure on the Heisenberg group is not studied in detail. This paper aims to fill this gap.

The paper has the following structure. In Section 2 we recall the basic notions of the sub-Lorentzian geometry. In Section 3 we state the left-invariant sub-Lorentzian structure on the Heisenberg group studied in this paper. Results obtained previously for this problem by M. Grochowski are recalled in Section 4. In Section 5 we apply the Pontryagin maximum principle and compute extremal trajectories; as a consequence, almost all extremal trajectories (timelike ones) are parametrized by the exponential mapping. In Section 5.1 we show that the exponential mapping is a diffeomorphism and find explicitly its inverse. On this basis in...
Section 5.2 we study optimality of extremal trajectories and construct an optimal synthesis. In Section 6 we describe explicitly the sub-Lorentzian distance, in Section 7 we find its symmetries, and in Section 7.1 we study in detail the sub-Lorentzian spheres of positive and zero radii. Finally, in Section 7.2 we discuss the results obtained and pose some questions for further research.

A short announcement of a part of results of this paper was published in [9].

2 Sub-Lorentzian geometry

A sub-Lorentzian structure on a smooth manifold $M$ is a pair $(\Delta, g)$ consisting of a vector distribution $\Delta \subset TM$ and a Lorentzian metric $g$ on $\Delta$, i.e., a nondegenerate quadratic form $g$ of index 1. Sub-Lorentzian geometry attempts to transfer the rich theory of sub-Riemannian geometry (in which the quadratic form $g$ is positive definite) to the case of Lorentzian metric $g$. Research in sub-Lorentzian geometry was started by M. Grochowski [10–15], see also [16–19].

Let us recall some basic definitions of sub-Lorentzian geometry. A vector $v \in T_q M$, $q \in M$, is called horizontal if $v \in \Delta_q$. A horizontal vector $v$ is called:

- timelike if $g(v) < 0$,
- spacelike if $g(v) > 0$ or $v = 0$,
- lightlike (or null) if $g(v) = 0$ and $v \neq 0$,
- nonspacelike if $g(v) \leq 0$.

A Lipschitzian curve in $M$ is called timelike if it has timelike velocity vector a.e.; spacelike, lightlike and nonspacelike curves are defined similarly.

A time orientation $X_0$ is an arbitrary timelike vector field in $M$. A nonspacelike vector $v \in \Delta_q$ is future directed if $g(v, X_0(q)) < 0$, and past directed if $g(v, X_0(q)) > 0$.

A future directed timelike curve $q(t), t \in [0, t_1]$, is called arclength parametrized if $g(\dot{q}(t), \dot{q}(t)) \equiv -1$. Any future directed timelike curve can be parametrized by arclength, similarly to the arclength parametrization of a horizontal curve in sub-Riemannian geometry.

The length of a nonspacelike curve $\gamma \in \text{Lip}([0, t_1], M)$ is

$$l(\gamma) = \int_0^{t_1} |g(\dot{\gamma}, \ddot{\gamma})|^{1/2} dt.$$ 

For points $q_0, q_1 \in M$ denote by $\Omega_{q_0 q_1}$ the set of all future directed nonspacelike curves in $M$ that connect $q_0$ to $q_1$. In the case $\Omega_{q_0 q_1} \neq \emptyset$ define the sub-Lorentzian distance from the point $q_0$ to the point $q_1$ as

$$d(q_0, q_1) = \sup \{l(\gamma) \mid \gamma \in \Omega_{q_0 q_1}\}.$$ (2.1)

Notice that in papers [14, 15] (and in Lorentzian geometry [26]) in the case $\Omega_{q_0 q_1} = \emptyset$ it is set $d(q_0, q_1) = 0$. We prefer not to consider $d(q_0, q_1)$ in this case.

A future directed nonspacelike curve $\gamma$ is called a sub-Lorentzian length maximizer if it realizes the supremum in (2.1) between its endpoints $\gamma(0) = q_0, \gamma(t_1) = q_1$.

The causal future of a point $q_0 \in M$ is the set $J^+(q_0)$ of points $q_1 \in M$ for which there exists a future directed nonspacelike curve $\gamma$ that connects $q_0$ and $q_1$. The chronological future $I^+(q_0)$ of a point $q_0 \in M$ is defined similarly via future directed timelike curves $\gamma$. 

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Let $q_0 \in M$, $q_1 \in J^+(q_0)$. The search for sub-Lorentzian length maximizers that connect $q_0$ with $q_1$ reduces to the search for future directed nonspacelike curves $\gamma$ that solve the problem
\[ l(\gamma) \rightarrow \max, \quad \gamma(0) = q_0, \quad \gamma(t_1) = q_1. \tag{2.2} \]

A set of vector fields $X_1, \ldots, X_k \in \text{Vec}(M)$ is an orthonormal frame for a sub-Lorentzian structure $(\Delta, g)$ if for all $q \in M$
\[
\Delta_q = \text{span}(X_1(q), \ldots, X_k(q)),
g_q(X_1, X_1) = -1, \quad g_q(X_i, X_i) = 1, \quad i = 2, \ldots, k,
g_q(X_i, X_j) = 0, \quad i \neq j.
\]

Assume that time orientation is defined by a timelike vector field $X \in \text{Vec}(M)$ for which $g(X, X) < 0$ (e.g., $X = X_1$). Then the sub-Lorentzian problem for the sub-Lorentzian structure with the orthonormal frame $X_1, \ldots, X_k$ is stated as the following optimal control problem:
\[
\dot{q} = \sum_{i=1}^{k} u_i X_i(q), \quad q \in M, \\
u \in U = \left\{ (u_1, \ldots, u_k) \in \mathbb{R}^k \mid u_1 \geq \sqrt{u_2^2 + \cdots + u_k^2} \right\}, \\
_q(0) = q_0, \quad q(t_1) = q_1, \\
l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2 - \cdots - u_k^2} \, dt \rightarrow \max.
\]

**Remark 1** The sub-Lorentzian length is preserved under monotone Lipschitzian time reparametrizations $t(s)$, $s \in [0, s_1]$. Thus if $q(t)$, $t \in [0, t_1]$, is a sub-Lorentzian length maximizer, then so is any its reparametrization $q(t(s))$, $s \in [0, s_1]$.

In this paper we choose primarily the following parametrization of trajectories: the arclength parametrization $(u_1^2 - u_2^2 - \cdots - u_k^2 \equiv 1)$ for timelike trajectories, and the parametrization with $u_1(t) \equiv 1$ for future directed lightlike trajectories. Another reasonable choice is to set $u_1(t) \equiv 1$ for all future directed nonspacelike trajectories.

**Remark 2** In Lorentzian geometry, only nonspacelike curves have a physical meaning since according to the Relativity Theory information cannot move with a speed greater than the speed of light [25–27]. By this reason, in sub-Lorentzian geometry typically only nonspacelike curves are studied [10–15].

Geometrically, spacelike curves may well be considered. For rank 2 sub-Lorentzian structures there is not much geometric difference between timelike and spacelike curves since the first ones are obtained from the second ones by a change of sub-Lorentzian form $g \leftrightarrow -g$, or, equivalently, by a change of controls $(u_1, u_2) \leftrightarrow (u_2, u_1)$. Although, for sub-Lorentzian structures of rank greater than 2 the spacelike cone is nonconvex, so the optimization problem of finding the longest spacelike curve is not well-defined (optimal trajectories do not exist).

Notice also that curves $q(\cdot)$ of variable causality ($\text{sign}(\dot{q}) \not\equiv \text{const}$) cannot be optimal: it is easy to show that the causal character of extremal trajectories is preserved.

**Remark 3** The sub-Lorentzian distance is defined by maximization (2.1), not by minimization as in sub-Riemannian geometry. In Lorentzian geometry, the distance means physically the...
space-time interval between events in a space-time \([25–27]\). On the other hand, the minimum
of sub-Lorentzian length is always zero (by virtue of lightlike trajectories), so the minimization
problem here is not very interesting.

Notice also that the sub-Lorentzian distance \(d\) is not a distance in the sense of metric
spaces since \(d\) is not symmetric and satisfies the reverse
triangle inequality.

3 Statement of the sub-Lorentzian problem on the Heisenberg group

The Heisenberg group is the space \(M \simeq \mathbb{R}^3_{x,y,z}\) with the product rule

\[(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + (x_1y_2 - x_2y_1)/2)\].

It is a three-dimensional nilpotent Lie group with a left-invariant frame

\[X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z},\]  

(3.1)

with the only nonzero Lie bracket \([X_1, X_2] = X_3\).

Consider the left-invariant sub-Lorentzian structure on the Heisenberg group \(M\) defined
by the orthonormal frame \((X_1, X_2)\), with the time orientation \(X_1\). Sub-Lorentzian length
maximizers for this sub-Lorentzian structure are solutions to the optimal control problem

\[\dot{q} = u_1 X_1 + u_2 X_2, \quad q \in M,\]  

(3.2)

\[u \in U = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1 \geq |u_2|\},\]  

(3.3)

\[q(0) = q_0 = \text{Id} = (0, 0, 0), \quad q(t_1) = q_1,\]  

(3.4)

\[l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2} \, dt \to \max.\]  

(3.5)

Along with this (full) sub-Lorentzian problem, we will also consider a reduced sub-Lo-
rentzian problem

\[\dot{q} = u_1 X_1 + u_2 X_2, \quad q \in M,\]  

(3.6)

\[u \in \text{int} \ U = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1 > |u_2|\},\]  

(3.7)

\[q(0) = q_0 = \text{Id} = (0, 0, 0), \quad q(t_1) = q_1,\]  

(3.8)

\[l(q(\cdot)) = \int_0^{t_1} \sqrt{u_1^2 - u_2^2} \, dt \to \max.\]  

(3.9)

In the full problem (3.2)–(3.5) admissible trajectories \(q(\cdot)\) are future directed nonspacelike
ones, while in the reduced problem (3.6)–(3.9) admissible trajectories \(q(\cdot)\) are only future
directed timelike ones. Passing to arclength-parametrized future directed timelike trajectories,
we obtain a time-maximal problem equivalent to the reduced sub-Lorentzian problem (3.6)–
(3.9):

\[\dot{q} = u_1 X_1 + u_2 X_2, \quad q \in M,\]  

(3.10)

\[u_1^2 - u_2^2 = 1, \quad u_1 > 0,\]  

(3.11)

\[q(0) = q_0 = \text{Id} = (0, 0, 0), \quad q(t_1) = q_1,\]  

(3.12)

\[t_1 \to \max.\]  

(3.13)
4 Previously obtained results

The sub-Lorentzian problem on the Heisenberg group (3.2)–(3.5) was studied by M. Grochowski [14, 15]. In this section we present results of these works related to our results.

(1) Sub-Lorentzian extremal trajectories were parametrized by hyperbolic and linear functions: were obtained formulas equivalent to our formulas (5.2), (5.3).

(2) It was proved that there exists a domain in $M$ containing $q_0 = \text{Id}$ in its boundary at which the sub-Lorentzian distance $d(q_0, q)$ is smooth.

(3) The attainable sets of the sub-Lorentzian structure from the point $q_0 = \text{Id}$ were computed: the chronological future of the point $q_0$

$$I^+(q_0) = \{(x, y, z) \in M \mid -x^2 + y^2 + 4|z| < 0, \ x > 0\},$$

and the causal future of the point $q_0$

$$J^+(q_0) = \{(x, y, z) \in M \mid -x^2 + y^2 + 4|z| \leq 0, \ x \geq 0\}. \quad (4.1)$$

In the standard language of control theory [4], $I^+(q_0)$ is the attainable set of the reduced system (3.6), (3.7) from the point $q_0$ for arbitrary positive time. Thus the attainable set of the reduced system (3.6), (3.7) from the point $q_0$ for arbitrary nonnegative time is

$$A = I^+(q_0) \cup \{q_0\}.$$

The attainable set of the full system (3.2), (3.3) from the point $q_0$ for arbitrary nonnegative time is

$$\text{cl}(A) = J^+(q_0).$$

Fig. 1 The Heisenberg beak $\partial A$
The attainable set $\mathcal{A}$ was also computed in paper [20], where its boundary was called the Heisenberg beak. See the set $\partial \mathcal{A}$ in Figs. 1, 20, and its views from the $y$- and $z$-axes in Figs. 2 and 3 respectively.
5 Pontryagin maximum principle

In this section we compute extremal trajectories of the full sub-Lorentzian problem (3.2)–(3.5). The majority of results of this section were obtained by M. Grochowski [14, 15] in another notation, we present these results here for further reference.

Notice that we prove optimality of all extremal trajectories in Section 5.2 without a priori theorem on existence of optimal trajectories. Such a theorem was recently proved [23], and it can shorten the proof of optimality in our work.

Denote points of the cotangent bundle $T^*M$ as $\lambda$. Introduce linear on fibers of $T^*M$ Hamiltonians $h_\nu(\lambda) = \langle \lambda, X_i \rangle$, $i = 1, 2, 3$. Define the Hamiltonian of the Pontryagin maximum principle (PMP) for the sub-Lorentzian problem (3.2)–(3.5)

$$h_\nu^v(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda) - v \sqrt{u_1^2 - u_2^2}, \quad \lambda \in T^*M, \quad u \in U, \quad v \in \mathbb{R}.$$  

It follows from PMP [4, 21] that if $u(t)$, $t \in [0, t_1]$, is an optimal control in problem (3.2)–(3.5), and $q(t)$, $t \in [0, t_1]$, is the corresponding optimal trajectory, then there exists a curve $\lambda_\nu \in \text{Lip}([0, t_1], T^*M)$, $\pi(\lambda_\nu) = q(t)^1$, and a number $v \in \{0, -1\}$ for which there hold the conditions for a.e. $t \in [0, t_1]$:

1. the Hamiltonian system $\dot{\lambda}_t = \vec{h}_u^v(\lambda_\nu)^2$,
2. the maximality condition $h_\nu^v(\lambda_\nu) = \max_{v \in U} h_\nu^v(\lambda_\nu) \equiv 0$,
3. the nontriviality condition $(v, \lambda_\nu) \neq (0, 0)$.

A curve $\lambda_\nu$ that satisfies PMP is called an extremal, and the corresponding control $u(\cdot)$ and trajectory $q(\cdot)$ are called extremal control and trajectory.

5.1 Abnormal case

Theorem 1 In the abnormal case $v = 0$ extremals $\lambda_\nu$ and controls $u(t)$ have the following form for some $\tau_1, \tau_2 \geq 0$:

1) $h_3(\lambda_\nu) \equiv \text{const} > 0$:

$$t \in (0, \tau_1) \Rightarrow \quad h_1(\lambda_\nu) = h_2(\lambda_\nu) < 0, \quad u_1(t) = -u_2(t),$$

$$t \in (\tau_1, \tau_1 + \tau_2) \Rightarrow \quad h_1(\lambda_\nu) = -h_2(\lambda_\nu) < 0, \quad u_1(t) = u_2(t).$$

1 where $\pi : T^*M \to M$ is the canonical projection, $\pi(\lambda) = q, \lambda \in T_q^*M$

2 where $\vec{h}(\lambda)$ is the Hamiltonian vector field on $T^*M$ with the Hamiltonian function $h(\lambda)$
(2) \(h_3(\lambda_t) \equiv \text{const} < 0:\)

\[
\begin{align*}
t \in (0, \tau_1) \Rightarrow & \quad h_1(\lambda_t) = -h_2(\lambda_t) < 0, \quad u_1(t) = u_2(t), \\
t \in (\tau_1, \tau_1 + \tau_2) \Rightarrow & \quad h_1(\lambda_t) = h_2(\lambda_t) < 0, \quad u_1(t) = -u_2(t).
\end{align*}
\]

(3) \(h_3(\lambda_t) \equiv 0:\)

\[
\begin{align*}
(h_1, h_2)(\lambda_t) & \equiv \text{const} \neq (0, 0), \quad h_1(\lambda_t) \equiv -|h_2(\lambda_t)|, \\
u(t) & \equiv \text{const}, \quad u_1(t) \equiv \pm u_2(t), \quad \pm = -\text{sgn}(h_1 h_2(\lambda_t)).
\end{align*}
\]

**Proof** Apply the PMP for the case \(\nu = 0\). \(\Box\)

**Corollary 1** Along abnormal extremals \(H(\lambda_t) \equiv 0\), where \(H = \frac{1}{2}(h_2^2 - h_1^2)\).

### 5.2 Normal case

In the normal case \((\nu = -1)\) extremals exist only for \(h_1 \leq -|h_2|\).\(^3\) In the case \(h_1 = -|h_2|\) normal controls and extremal trajectories coincide with the abnormal ones. And in the domain \(\{\lambda \in T^*M \mid h_1 < -|h_2|\}\) extremals are reparametrizations of trajectories of the Hamiltonian vector field \(\tilde{H}\) with the Hamiltonian \(H = \frac{1}{2}(h_2^2 - h_1^2)\). In the arclength parametrization, the extremal controls are

\[
(u_1, u_2)(t) = (-h_1(\lambda_t), h_2(\lambda_t)),
\]

and the extremals satisfy the Hamiltonian ODE \(\dot{\lambda} = \tilde{H}(\lambda)\) and belong to the level surface \(\{H(\lambda) = \frac{1}{2}\}\), in coordinates:

\[
\begin{align*}
\dot{h}_1 &= -h_2 h_3, \quad \dot{h}_2 = -h_1 h_3, \quad \dot{h}_3 = 0, \\
\dot{q} &= \cosh \psi X_1 + \sinh \psi X_2, \\
\dot{h}_1 &= -\cosh \psi, \quad h_2 = \sinh \psi, \quad \psi \in \mathbb{R}.
\end{align*}
\]

We denote \(c = h_3\) and obtain a parametrization of normal trajectories \(q(t) = \pi \circ e^{t\tilde{H}}(\lambda_0),\)

\[\lambda_0 \in H^{-1}\left(\frac{1}{2}\right) \cap T^*_{Id}M.\]

If \(c = 0\), then

\[
x = t \cosh \psi, \quad y = t \sinh \psi, \quad z = 0.
\]

If \(c \neq 0\), then

\[
x = \frac{\sinh(\psi + ct) - \sinh \psi}{c}, \quad y = \frac{\cosh(\psi + ct) - \cosh \psi}{c}, \quad z = \frac{\sinh(ct) - ct}{2c^2},
\]

Summing up, we obtain the following characterization of normal trajectories in the full sub-Lorentzian problem (3.2)–(3.5).

**Theorem 2** Normal controls and trajectories either coincide with abnormal ones (in the case \(h_1(\lambda_t) = -|h_2(\lambda_t)|\), see Theorem 1), or can be arclength parametrized to get controls (5.1) and future directed timelike trajectories (5.2) if \(c = 0\), or (5.3) if \(c \neq 0\).

In particular, along each normal extremal \(H(\lambda_t) \equiv \text{const} \in \{0, \frac{1}{2}\}\).

Consequently, normal trajectories are either nonstrictly normal (i.e., simultaneously normal and abnormal) in the case \(H = 0\), or strictly normal (i.e., normal but not abnormal)\(^3\)

---

\(^3\) The set \(\{(h_1, h_2) \in (\mathbb{R}^2)^{\times} \mid h_1 \leq -|h_2|\}\) is the polar set to \(U\) in the sense of convex analysis.
in the case $H = \frac{1}{2}$. Strictly normal arclength-parametrized trajectories are described by the exponential mapping

$$\text{Exp} : N \to \tilde{A}, \quad (\lambda, t) \mapsto q(t) = \pi \circ e^{t\tilde{H}}(\lambda),$$

$$N = C \times \mathbb{R}_+, \quad \mathbb{R}_+ = (0, +\infty), \quad C = T^*_{\text{ld}} M \cap H^{-1}\left(\frac{1}{2}\right) \simeq \mathbb{R}^2_{\psi, c},$$

$$\tilde{A} = \text{int} A = I^+(q_0)$$

(5.4)

given explicitly by formulas (5.2), (5.3).

In papers [14, 15] were obtained formulas equivalent to (5.2), (5.3).

**Remark 4** Projections of strictly normal (future directed timelike) trajectories to the plane $(x, y)$ are:

- *either rays* $y = kx$, $x \geq 0$, $k \in (-1, 1)$ (for $c = 0$), see Fig. 4,
- *or arcs of hyperbolas with asymptotes* $x = \pm y > 0$ (for $c \neq 0$), see Fig. 5.

*Projections of nonstrictly normal (future directed lightlike) trajectories to the plane $(x, y)$ are broken lines with one or two edges parallel to the rays $x = \pm y > 0$, see Fig. 6.*

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Projections of all extremal trajectories (as well as of all admissible trajectories) to the plane \((x, y)\) are contained in the angle \(\{(x, y) \in \mathbb{R}^2 \mid x \geq |y|\}\), which is the projection of the attainable set \(J^+(q_0)\) to this plane.

Remark 5 The Hamiltonian \(H = \frac{1}{2}(h_2^2 - h_1^2)\) is preserved on each extremal. On the other hand, since the problem is left-invariant, the extremals respect the symplectic foliation\(^4\) on the dual of the Heisenberg Lie algebra \(T^*_{\text{Id}}M = \{(h_1, h_2, h_3)\}\) consisting of 2-dimensional symplectic leaves \(\{h_3 = \text{const} \neq 0\}\) and 0-dimensional leaves \(\{h_3 = 0, (h_1, h_2) = \text{const}\}\). Thus projections of extremals to \(T^*_{\text{Id}}M = \{(h_1, h_2, h_3)\}\) belong to intersections of the level surfaces \(\{H = \text{const} \in \{0, \frac{1}{2}\}\}\) with the symplectic leaves:

- branches of hyperbolas \(h_1^2 - h_2^2 = 1, h_1 < 0, h_3 \neq 0,\)
- points \((h_1, h_2) = \text{const}, H \in \{0, \frac{1}{2}\}, h_1 \leq -|h_2|, h_3 = 0,\)
- angles \(h_1 = -|h_2|, h_3 \neq 0.\)

See Figs. 7, 8.

\(^4\) The symplectic foliation is the decomposition of the dual of a Lie algebra into coadjoint orbits, see e.g. [2]
Remark 6 In the sense of work [14], strictly normal extremal trajectories \( q(t) = \pi \circ e^{t\tilde{H}}(\lambda) \), \( \lambda \in C \), are Hamiltonian since they are projections of trajectories of the Hamiltonian vector field \( \tilde{H} \).

On the other hand, nonstrictly normal extremal trajectories given by items (1), (2) of Theorem 1 are non-Hamiltonian, e.g., the broken curves

\[
\begin{align*}
& e^{t(X_1+X_2)}, & t \in [0, \tau_1], \\
& e^{(t-\tau_1)(X_1-X_2)} \cdot e^{\tau_1(X_1+X_2)}, & t \in [\tau_1, \tau_2],
\end{align*}
\]

(5.5)

and

\[
\begin{align*}
& e^{t(X_1-X_2)}, & t \in [0, \tau_1], \\
& e^{(t-\tau_1)(X_1+X_2)} \cdot e^{\tau_1(X_1-X_2)}, & t \in [\tau_1, \tau_2],
\end{align*}
\]

(5.6)

for \( 0 < \tau_1 < \tau_2 \). See item (5) in Section 4. Although, each smooth arc of the broken trajectories (5.5), (5.6) is a reparametrization of projection of a trajectory of the Hamiltonian vector field \( \tilde{H} \) contained in a face of the angle \((h_1, h_2, h_3) \in T^*_\text{id}M \mid h_1 = -|h_2| \)\), see Fig. 8.
Fig. 7 Strictly normal \((h_1(t), h_2(t), h_3(t))\)

Fig. 8 Nonstrictly normal \((h_1(t), h_2(t), h_3(t))\)
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Fig. 9 Plot of $\alpha(p)$

6 Inversion of the exponential mapping

**Theorem 3** The exponential mapping $\text{Exp} : N \to \tilde{A}$ is a real-analytic diffeomorphism. The inverse mapping $\text{Exp}^{-1} : \tilde{A} \to N, (x, y, z) \mapsto (\psi, c, t)$, is given by the following formulas:

\begin{align}
    z = 0 & \Rightarrow \psi = \text{artanh} \frac{y}{x}, \quad c = 0, \quad t = \sqrt{x^2 - y^2}, \quad (6.1) \\
    z \neq 0 & \Rightarrow \psi = \text{artanh} \frac{y}{x} - p, \quad c = (\text{sgn} z) \frac{\sinh 2p - 2p}{2z}, \quad t = \frac{2p}{c}, \quad (6.2)
\end{align}

where $p = \beta \left( \frac{z}{x^2 - y^2} \right)$, and $\beta : (-\frac{1}{4}, \frac{1}{4}) \to \mathbb{R}$ is the inverse function to the diffeomorphism $\alpha : \mathbb{R} \to \left( -\frac{1}{4}, \frac{1}{4} \right)$, $\alpha(p) = \frac{\sinh 2p - 2p}{8 \sinh^2 p}$.

See plots of the functions $\alpha(p)$ and $\beta(z)$ in Figs. 9 and 10 respectively.

**Proof** The exponential mapping is real-analytic since the strictly normal extremals are trajectories of the real-analytic Hamiltonian vector field $\tilde{H}$. We show that $\text{Exp}$ is bijective.

Formulas (6.1) follow immediately from (5.2).

Let $c \neq 0$. Then formulas (5.3) yield

\begin{align}
    x &= \frac{2}{c} \sinh p \cosh \tau, \quad y = \frac{2}{c} \sinh p \sinh \tau, \quad z = \frac{1}{2c^2} (\sinh 2p - 2p), \quad (6.3) \\
    p &= \frac{ct}{2}, \quad t = \psi + \frac{ct}{2}, \quad (6.4)
\end{align}

Thus

\begin{align}
    x^2 - y^2 &= \frac{4}{c^2} \sinh^2 p, \\
    \frac{z}{x^2 - y^2} &= \frac{\sinh 2p - 2p}{8 \sinh^2 p} = \alpha(p). \quad (6.5)
\end{align}
The function $\alpha(p)$ is a diffeomorphism from $\mathbb{R}$ to $(-\frac{1}{4}, \frac{1}{4})$, thus it has an inverse function, a diffeomorphism $\beta : (-\frac{1}{4}, \frac{1}{4}) \rightarrow \mathbb{R}$. So $p = \beta \left( \frac{z}{\sqrt{x^2 - y^2}} \right)$. Now formulas (6.2) follow from (6.3), (6.4).

So Exp is a smooth bijection with a smooth inverse, i.e., a diffeomorphism. \hfill \Box

7 Optimality of extremal trajectories

We prove optimality of extremal trajectories and in such a way show existence of optimal trajectories. The main tool is a sufficient optimality condition (Theorem 4) based on a field of extremals (see [4], Section 17.1).
7.1 Sufficient optimality condition

Let $M$ be a smooth manifold, then the cotangent bundle $T^*M$ bears the Liouville 1-form $s = pdq \in \Lambda^1(T^*M)$ and the symplectic 2-form $\sigma = ds = dp \wedge dq \in \Lambda^2(T^*M)$. A submanifold $\mathcal{L} \subset T^*M$ is called a Lagrangian manifold if $\dim \mathcal{L} = \dim M$ and $\sigma|_\mathcal{L} = 0$.

Consider an optimal control problem

$$
\dot{q} = f(q,u), \quad q \in M, \quad u \in U,
q(t_0) = q_0, \quad q(t_1) = q_1,
J[q(\cdot)] = \int_{t_0}^{t_1} \varphi(q,u) \, dt \to \min,
$$

$t_0$ is fixed, $t_1$ is free.

Let $g_u(\lambda) = \langle \lambda, f(q,u) \rangle - \varphi(q,u), \lambda \in T^*M, q = \pi(\lambda), u \in U$, be the normal Hamiltonian of PMP. Suppose that the maximized normal Hamiltonian $G(\lambda) = \max_{u \in U} g_u(\lambda)$ is smooth in an open domain $O \subset T^*M$, and let the Hamiltonian vector field $\tilde{G} \in \text{VEC}(O)$ be complete.

**Theorem 4** Let $\mathcal{L} \subset G^{-1}(0) \cap O$ be a Lagrangian submanifold such that the form $s|_\mathcal{L}$ is exact. Let the projection $\pi : \mathcal{L} \to \pi(\mathcal{L})$ be a diffeomorphism on a domain in $M$. Consider an extremal $\tilde{\lambda}_t = e^{t\tilde{G}}(\lambda_0), t \in [t_0,t_1]$, contained in $\mathcal{L}$, and the corresponding extremal trajectory $\tilde{q}(t) = \pi(\tilde{\lambda}_t)$. Consider also any other trajectory $q(t) \in \pi(\mathcal{L}), t \in [t_0,\tau]$, such that $q(t_0) = \tilde{q}(t_0), q(\tau) = \tilde{q}(t_1)$. Then $J[\tilde{q}(\cdot)] < J[q(\cdot)]$.

**Proof** Completely similarly to the proof of Theorem 17.2 [4].

7.2 Optimality in the reduced sub-Lorentzian problem on the Heisenberg group

We apply Theorem 4 to the time-maximal problem (3.10)–(3.13). For this problem the maximized Hamiltonian $G = 1 - \sqrt{h_1^2 - h_2^2}$ is smooth on the domain $O = \{ \lambda \in T^*M \mid h_1 < -|h_2| \}$, and the Hamiltonian vector field $\tilde{G} \in \text{VEC}(O)$ is complete. In the domain $O$ the Hamiltonian vector fields $\tilde{G}$ and $\tilde{H}$ have the same trajectories up to a monotone time reparametrization; moreover, on the level surface $\{ H = \frac{1}{2} \} = \{ G = 0 \}$ they just coincide between themselves.

Define the set

$$
\mathcal{L} = \left\{ e^{t\tilde{G}}(\lambda_0) \mid \lambda_0 \in C, \, t > 0 \right\}. \tag{7.1}
$$

**Lemma 1** $\mathcal{L} \subset T^*M$ is a Lagrangian manifold such that $s|_\mathcal{L}$ is exact.

**Proof** Consider a smooth mapping

$$
\Phi : (T_{id}^*M \cap G^{-1}(0)) \times \mathbb{R}_+ \to T^*M, \quad (\lambda_0, t) \mapsto e^{t\tilde{G}}(\lambda_0).
$$
Since
\[
\text{rank}\left(\frac{\partial \Phi}{\partial (t, \lambda_0)}\right) = \text{rank}\left(\tilde{G}(\lambda), e^*_s\tilde{G}\left(h_2 \frac{\partial}{\partial h_1} + h_1 \frac{\partial}{\partial h_2}\right), e^*_s\frac{\partial}{\partial h_3}\right)
\]
\[
= \text{rank}\left(\tilde{G}(\lambda_0), h_2 \frac{\partial}{\partial h_1} + h_1 \frac{\partial}{\partial h_2}, \frac{\partial}{\partial h_3}\right)
\]
\[
= \text{rank}\left(-h_1X_1 + h_2X_2, h_2 \frac{\partial}{\partial h_1} + h_1 \frac{\partial}{\partial h_2}, \frac{\partial}{\partial h_3}\right)
\]
\[
= 3,
\]
then \(L\) is a smooth 3-dimensional manifold.

Further, \(\pi(L) = \exp(N) = \tilde{A}\) by Theorem 3. Moreover, since \(\exp = \pi \circ \Phi\) and \(\exp : N \to \tilde{A}\) is a diffeomorphism by Theorem 3, then \(\pi : L \to \tilde{A}\) is a diffeomorphism as well.

Let us show that \(\sigma|_L = 0\). Take any \(\lambda = e^{t\tilde{G}(\lambda_0)} \in \mathcal{L}, (\lambda_0, t) \in N\), then \(T_{\lambda}L = \mathbb{R}\tilde{G}(\lambda) \oplus e^*_s(T_{\lambda_0}C)\). Take any two vectors \(T_{\lambda}L \ni v_i = r_i\tilde{G}(\lambda) + e^*_s\tilde{G}(\lambda_0), w_i \in T_{\lambda_0}C, i = 1, 2\). Then
\[
\sigma(v_1, v_2) = r_1\sigma(\tilde{G}(\lambda_0), w_2) + r_2\sigma(w_1, \tilde{G}(\lambda_0)) = 0
\]
since \(\sigma(w_1, \tilde{G}(\lambda_0)) = (dG, w_1) = 0\) by virtue of \(w_i \in T_{\lambda_0}C = \ker dG\), and since \(w_i \in T_{\lambda_0}C \subset T_{\lambda_0}(T_{\lambda_0}^*M)\), \(\sigma|_{T_{\lambda_0}(T_{\lambda_0}^*M)} = 0\).

So the 1-form \(s|_L\) is closed. But \(\tilde{A}\) is simply connected, thus \(L\) is simply connected as well. Consequently, \(s|_L\) is exact by the Poincaré lemma.

\(\square\)

**Theorem 5** For any point \(q_1 \in \text{int} \mathcal{A} = \mathcal{I}^+(q_0)\) the strictly normal trajectory \(q(t) = \exp(\lambda, t), t \in [0, t_1]\), is the unique optimal trajectory of the time-maximal problem (3.10)–(3.13) connecting \(q_0\) with \(q_1\), where \((\lambda, t_1) = \exp^{-1}(q_1) \in N\).

**Proof** Take any \(\lambda_0 \in C, t_1 > t_0 > 0\). Then the Lagrangian manifold \(L(7.1)\) and the extremal \(\tilde{\lambda}_t = e^{t\tilde{G}(\lambda_0)}, t \in [t_0, t_1]\), satisfy hypotheses of Theorem 4. Thus the trajectory \(\tilde{q}(t) = \pi(\tilde{\lambda}_t), t \in [t_0, t_1]\), is a strict maximizer for the time-maximal problem (3.10)–(3.13).

Take any \(\lambda_1 \in C, t_2 > 0\), and consider the extremal trajectory \(\tilde{q}(t) = \exp(\lambda_1, t), t \in [0, t_2]\). Take any \(\tilde{q} \in \tilde{A}\). The set \(\mathcal{A}\) is an attainable set of a left-invariant control system on a Lie group, thus it is a semigroup. Consequently, \(\tilde{q} \cdot \tilde{q}(t)\) is an extremal trajectory contained in \(\tilde{A}\). By the previous paragraph, this trajectory is a strict maximizer for the time-maximal problem (3.10)–(3.13). By left invariance of this problem, the same holds for the trajectory \(\tilde{q}(t), t \in [0, t_2]\).

Denote the cost function for the equivalent reduced sub-Lorentzian problem (3.6)–(3.9) and time-maximal problem (3.10)–(3.13):
\[
\tilde{d}(q_1) = \sup\{l(q(\cdot)) \mid \text{traj. } q(\cdot) \text{ of (3.6)–(3.9), } q(0) = q_0, q(t_1) = q_1\}
\]
\[
= \sup\{t_1 > 0 \mid \exists \text{ traj. } q(\cdot) \text{ of (3.10)–(3.13) s.t. } q(0) = q_0, q(t_1) = q_1\},
\]
where \(q_1 \in \text{int} \mathcal{A} = \mathcal{I}^+(q_0)\). This function has the following description and regularity property.

**Theorem 6** Let \(q = (x, y, z) \in I^+(q_0)\). Then
\[
\tilde{d}(q) = \sqrt{x^2 - y^2} \cdot \frac{p}{\sinh p}, \quad p = \beta \left(\frac{z}{x^2 - y^2}\right).
\]

The function \(\tilde{d} : \mathcal{I}^+(q_0) \to \mathbb{R}_+\) is real-analytic.

\(\square\)
Proof Let \( q \in I^+(q_0) \), then the sub-Lorentzian length maximizer from \( q_0 \) to \( q \) for the time-maximal problem (3.10)–(3.13) is described in Theorem 5, and the expression for \( \tilde{d}(q) \) in (7.2) follows from the expression for \( t \) in (6.2).

The both functions \( \sqrt{x^2 - y^2} \) and \( \frac{p}{\sinh p} \) are real-analytic on \( I^+(q_0) \), thus \( \tilde{d} \) is real-analytic as well. \( \Box \)

7.3 Optimality in the full sub-Lorentzian problem on the Heisenberg group

In this subsection we consider the full sub-Lorentzian problem (3.2)–(3.5).

Theorem 7 Let \( q_1 \in I^+(q_0) \). Then the sub-Lorentzian length maximizers for the full problem (3.2)–(3.5) are reparametrizations of the corresponding optimal trajectory for the time-maximal problem (3.10)–(3.13) described in Theorem 5.

In particular, \( d|_{I^+(q_0)} = \tilde{d} \).

Proof Let \( q(t), t \in [0, t_1] \), be a trajectory of the full problem (3.2)–(3.5) such that \( q(0) = q_0 \), \( q(t_1) = q_1 \), and let \( q(\cdot) \) be not a trajectory of the reduced problem (3.6)–(3.9) (that is, there exist \( 0 \leq t_1 < t_2 \leq t_1 \) such that \( (u_1 - |u_2|)|_{[t_1, t_2]} = 0 \). Let \( \tilde{q}(t), t \in [0, \tilde{t}_1] \), be the optimal trajectory in the time-maximal problem (3.10)–(3.13) connecting \( q_0 \) with \( q_1 \). We show that \( l(q(\cdot)) < l(\tilde{q}(\cdot)) \). By contradiction, suppose that \( l(q(\cdot)) \geq l(\tilde{q}(\cdot)) \).

Let \( l(q(\cdot)) = l(\tilde{q}(\cdot)) \). It is easy to see that \( q(\cdot) \) is not optimal for the full problem (3.2)–(3.5). Indeed, if \( q(\cdot) \) is optimal, then PMP (see Section 5) implies that \( u_1(t) - |u_2(t)| = 0 \), \( t \in [0, t_1] \), thus \( q(\cdot) \) is lightlike. Then \( q_1 = q(t_1) \notin I^+(q_0) \), a contradiction. So the trajectory \( q(\cdot) \) is not optimal for the full problem (3.2)–(3.5). Thus there exists a trajectory \( \tilde{q}(\cdot) \) of this problem with the same endpoints and \( l(\tilde{q}(\cdot)) > l(q(\cdot)) \). The curve \( \tilde{q}(\cdot) \) cannot be a trajectory of the reduced system since its length is greater than the maximum \( l(\tilde{q}(\cdot)) \) in this problem. So we can denote \( \tilde{q}(\cdot) \) as \( q(\cdot) \) and assume that \( l(q(\cdot)) > l(\tilde{q}(\cdot)) \).

After time reparametrization we obtain that the control \( u(t) = (u_1(t), u_2(t)) \) corresponding to the trajectory \( q(t), t \in [0, t_1] \), satisfies \( u_1(t) \equiv 1 \), thus \( |u_2(t)| \leq 1 \).

For any \( \delta \in (0, 1) \) define a function

\[
\tilde{u}_2^\delta(t) = \begin{cases} u_2(t) & \text{for } |u_2(t)| \leq 1 - \delta, \\ 1 - \delta & \text{for } u_2(t) > 1 - \delta, \\ \delta - 1 & \text{for } u_2(t) < \delta - 1, \end{cases}
\]

so that

\[
|\tilde{u}_2^\delta(t)| \leq 1 - \delta, \quad |u_2^\delta(t) - u_2(t)| \leq \delta, \quad t \in [0, t_1]. \tag{7.3}
\]

Define an admissible control \( \tilde{u}_2(t) = (1, u_2^\delta(t)), t \in [0, t_1] \), and consider the corresponding trajectory \( \tilde{q}_2^\delta(t), t \in [0, t_1] \), of the reduced problem (3.6)–(3.9) with \( q_2^\delta(0) = q_0 \). Denote its endpoint \( \tilde{q}_2^\delta(t_1) = q_1^\delta \). By virtue of the second inequality in (7.3),

\[
\max_{t \in [0, t_1]} \|q_2^\delta(t) - q(t)\| \to 0
\]
as \( \delta \to +0 \). So for sufficiently small \( \delta > 0 \) we have

\[
l(q_2^\delta(\cdot)) > l(\tilde{q}(\cdot)) \quad \text{and} \quad \|q_1^\delta - q_1\| \text{ is small},
\]

where \( \| \cdot \| \) is any norm in \( M \cong \mathbb{R}^3 \). In particular, \( q_1^\delta \in I^+(q_0) \) for small \( \delta > 0 \).
Now let \( \hat{q}^\delta(t) \), \( t \in \left[0, \hat{t}^\delta_1\right] \), be the optimal trajectory in the time-maximal problem (3.10)–(3.13) with the boundary conditions \( \hat{q}^\delta(0) = q_0, \hat{q}^\delta(\hat{t}^\delta_1) = q_1^\delta \). Then for small \( \delta > 0 \)

\[
\begin{align*}
& l \left( \hat{q}^\delta(\cdot) \right) \geq l(q^\delta(\cdot)) > l(\hat{q}(\cdot)), \\
& \| q^\delta_1 - q_1 \| = \| \hat{q}^\delta(\hat{t}^\delta_1) - \hat{q}(t_1) \| \text{ is small}.
\end{align*}
\]

By virtue of Theorem 6, the sub-Lorentzian distance \( \hat{d}: I^+(q_0) \to \mathbb{R}_+ \) in the time-maximal problem (3.10)–(3.13) is continuous, thus for small \( \delta > 0 \)

\[
| l \left( \hat{q}^\delta(\cdot) \right) - l(\hat{q}(\cdot)) | = | \hat{d}(q^\delta_1) - \hat{d}(q_1) | \text{ is small}.
\]

Summing up, for small \( \delta > 0 \) the difference

\[
| l(q(\cdot)) - l(\hat{q}(\cdot)) | < | l(q(\cdot)) - l(q^\delta(\cdot)) | + | l(q^\delta(\cdot)) - l(\hat{q}(\cdot)) |
\]

becomes arbitrarily small, a contradiction. Thus \( \hat{q}(\cdot) \) is optimal and \( q(\cdot) \) is not optimal in the full sub-Lorentzian problem (3.2)–(3.5).

\[ \square \]

**Theorem 8** Let \( q_1 = (x_1, y_1, z_1) \in \partial A = J^+(q_0) \setminus I^+(q_0), q_1 \neq q_0 \). Then an optimal trajectory in the full sub-Lorentzian problem (3.2)–(3.5) is a future directed lightlike piecewise smooth trajectory with one or two subarcs generated by the vector fields \( X_1 \pm X_2 \). In detail, up to a reparametrization:

1. If \( z_1 = 0 \), then
   \[
u(t) \equiv \text{const} = (1, \pm 1), \quad q(t) = e^{t(X_1 \pm X_2)} = (t, \pm t, 0), \quad t \in [0, t_1], \quad t_1 = x_1.
\]
2. If \( z_1 > 0 \), then
   \[
t \in [0, t_1] \Rightarrow u(t) \equiv (1, -1), \quad q(t) = e^{t(X_1 - X_2)} = (t, -t, 0), \\
t \in [t_1, t_1 + t_2] \Rightarrow u(t) \equiv (1, 1), \\
q(t) = e^{(t-t_1)(X_1+X_2)}e^{t_1(X_1-X_2)} = (t, t-2t_1, t_1(t-t_1)), \\
t_1 = \frac{x_1 - y_1}{2}, \quad t_2 = \frac{x_1 + y_1}{2}.
\]
3. If \( z_1 < 0 \), then
   \[
t \in [0, t_1] \Rightarrow u(t) \equiv (1, 1), \quad q(t) = e^{t(X_1+X_2)} = (t, t, 0), \\
t \in [t_1, t_1 + t_2] \Rightarrow u(t) \equiv (1, -1), \\
q(t) = e^{(t-t_1)(X_1-X_2)}e^{t_1(X_1+X_2)} = (t, 2t_1 - t, -t_1(t-t_1)), \\
t_1 = \frac{x_1 + y_1}{2}, \quad t_2 = \frac{x_1 - y_1}{2}.
\]

The broken lightlike trajectories with two arcs described in items (1), (2) of Theorem 8 are shown in Fig. 21.

**Proof** Let \( q(t), t \in [0, t_1] \), be a future directed nonspacelike trajectory connecting \( q_0 \) and \( q_1 \). If \( q(\cdot) \) is not lightlike, then there exists a future directed timelike arc \( q(t), t \in [s_1, s_2], 0 \leq s_1 < s_2 \leq t_1 \), thus \( q(t_1) \in \text{int} \, A \), a contradiction. Thus \( q(\cdot) \) is lightlike, and the statement follows by direct computation of trajectories of the lightlike vector fields \( X_1 \pm X_2 \).

\[ \square \]

**Corollary 2** For any \( q_1 \in J^+(q_0), q_1 \neq q_0 \), there is a unique, up to reparametrization, sub-Lorentzian length minimizer in the full problem (3.2)–(3.5) that connects \( q_0 \) and \( q_1 \):

\[ \square \]
• if $q_1 \in \text{int} \ A = I^+(q_0)$, then $q(\cdot)$ is a future directed timelike strictly normal trajectory described in Theorems 5, 7.
• if $q_1 \in \partial A = J^+(q) \setminus I^+(q_0)$, then $q(\cdot)$ is a future directed lightlike nonstrictly normal trajectory described in Theorem 8.

Corollary 3 Any extremal trajectory of problem (3.2)–(3.5) is optimal, thus it contains neither conjugate points nor cut points.

Corollary 4 Any sub-Lorentzian length maximizer of problem (3.2)–(3.5) of positive length is timelike and strictly normal.

Remark 7 The broken trajectories described in items (2), (3) of Theorem 8 are optimal in the sub-Lorentzian problem, while in sub-Riemannian problems trajectories with angle points cannot be optimal, see [22]. Moreover, these broken trajectories are normal and nonsmooth, which is also impossible in sub-Riemannian geometry.

8 Sub-Lorentzian distance

Denote $d(q) := d(q_0, q), q \in J^+(q_0)$.

Theorem 9 Let $q = (x, y, z) \in J^+(q_0)$. Then

$$d(q) = \sqrt{x^2 - y^2} \cdot \frac{p}{\sinh p}, \quad p = \beta \left( \frac{z}{x^2 - y^2} \right).$$

(8.1)
In particular:

(1) $z = 0 \iff d(q) = \sqrt{x^2 - y^2},$
(2) $q \in J^+(q_0) \setminus I^+(q_0) \iff d(q) = 0.$

Remark 8 In the right-hand side of the first equality in (8.1), we assume by continuity that $\frac{p}{\sinh p} = 1$ for $p = 0$ and $\frac{p}{\sinh p} = 0$ for $p = \infty$. See the plot of the function $\frac{p}{\sinh p}$ in Fig. 11.

Proof Let $q \in I^+(q_0)$, then the sub-Lorentzian length maximizers from $q_0$ to $q$ are described in Theorem 7 and the expression for $d[\tilde{x}] = \tilde{d}$ was obtained in Theorem 6. In particular, if $z = 0$, then $p = 0$ and $d(q) = \sqrt{x^2 - y^2}$, and vice versa.

Let $q \in J^+(q_0) \setminus I^+(q_0)$, then the sub-Lorentzian length maximizers from $q_0$ to $q$ are described in Theorem 8. Thus $d(q) = 0$, which agrees with (8.1) since in this case $\frac{|z|}{x^2 - y^2} = \frac{1}{4}$, so $p = \infty$. \hfill \Box

Fig. 11 Plot of $\frac{p}{\sinh p}$
We plot restrictions of the sub-Lorentzian distance to several planar domains:

- $d|_{z=0} = \sqrt{x^2 - y^2}$ to the domain $J^+(q_0) \cap \{z = 0\} = \{x \geq |y|, z = 0\}$, see Fig. 12,
- $d|_{y=0}$ to the domain $J^+(q_0) \cap \{y = 0\} = \{-x^2/4 \leq z \leq x^2/4, y = 0\}$, see Fig. 13,
- $d|_{x=1}$ to the domain $J^+(q_0) \cap \{x = 1\} = \{y^2 + 4|z| \leq 1, x = 1\}$, see Fig. 14.

The sub-Lorentzian distance has the following regularity properties.

**Theorem 10** (1) The function $d(\cdot)$ is continuous on $J^+(q_0)$ and real-analytic on $I^+(q_0)$.
(2) The function $d(\cdot)$ is not Lipschitz near points $q = (x, y, z)$ with $x = |y| > 0, z = 0$.

**Proof** (1) follows from representation (8.1).
(2) follows from item (1) of Theorem 9 since the function $d|_{z=0} = \sqrt{x^2 - y^2}$ is not Lipschitz near points with $x = |y| > 0$. \(\square\)

**Remark 9** Item (1) of Theorem 10 improves item (2) of Section 4.

**Fig. 12** Plot of $d|_{z=0}$

**Fig. 13** Plot of $d|_{y=0}$
Fig. 14 Plot of \( d|_{x=1} \)

**Remark 10**  
Item (2) of Theorem 10 is visualized in Fig. 12 since the cone given by the plot of \( d|_{z=0} = \sqrt{x^2 - y^2} \) has vertical tangent planes at points \( x = |y| > 0 \).

Moreover, item (2) of Theorem 10 can be essentially detailed by a precise description of the asymptotics of the sub-Lorentzian distance \( d(q) \) as \( q \to \partial A \), this will be done in a forthcoming paper [24].

**Remark 11**  
The sub-Lorentzian distance \( d : J^+(q_0) \to [0, +\infty) \) is not uniformly continuous since the same holds for its restriction \( d|_{z=0} = \sqrt{x^2 - y^2} \) on the angle \( \{ x \geq |y| \} \).

As was shown in [15], the sub-Lorentzian distance \( d(q) \) admits a lower bound by the function \( \sqrt{x^2 - y^2 - 4|z|} \) and does not admit an upper bound by this function multiplied by any constant (see item (4) in Section 4). Here we precise this statement and prove another upper bound.

**Theorem 11**  
(1) The ratio \( \sqrt{x^2 - y^2 - 4|z|} / d(q) \) takes any values in the segment \([0, 1]\) for \( q = (x, y, z) \in J^+(q_0) \).

(2) For any \( q = (x, y, z) \in J^+(q_0) \) there holds the bound \( d(q) \leq \sqrt{x^2 - y^2} \), moreover, the ratio \( d(q) / \sqrt{x^2 - y^2} \) takes any values in the segment \([0, 1]\).

The two-sided bound
\[
\sqrt{x^2 - y^2 - 4|z|} \leq d(q) \leq \sqrt{x^2 - y^2}, \quad q \in J^+(q_0),
\]  
is visualized in Fig. 15, which shows plots of the surfaces (from below to top):
\[
\sqrt{x^2 - y^2} = 1, \quad d(q) = 1, \quad \sqrt{x^2 - y^2 - 4|z|} = 1, \quad q \in J^+(q_0).
\]

**Proof**  
(1) It follows from (8.1) that
\[
\frac{x^2 - y^2 - 4|z|}{d^2(q)} = \frac{\sinh^2 p - \sinh p \cosh p + p}{p^2},
\]
and the function in the right-hand side takes all values in the segment $[0, 1]$ for $q \in J^+(q_0)$.

(2) It follows from (8.1) that $\frac{d(q)}{\sqrt{x^2 - y^2}} = \frac{p}{\sinh p}$. When $q \in J^+(q_0)$, the ratio $\frac{p}{\sinh p}$ takes all values in the segment $[0, 1]$, see Remark 8 after Theorem 9.

\[ \Box \]

9 Symmetries

**Theorem 12**

(1) The hyperbolic rotations generated by the flow of the vector field $X^0 = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ and reflections $\varepsilon^1 : (x, y, z) \mapsto (x, -y, z), \varepsilon^2 : (x, y, z) \mapsto (x, y, -z)$ preserve $d(\cdot)$.

(2) The dilations generated by the flow of the vector field $Y = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2z \frac{\partial}{\partial z}$ stretch $d(\cdot)$:

\[ d(e^sY(q)) = e^s d(q), \quad s \in \mathbb{R}, \quad q \in J^+(q_0). \]

**Proof**

(1) The flow of the hyperbolic rotations

\[ e^{x^0} : (x, y, z) \mapsto (x \cosh s + y \sinh s, x \sinh s + y \cosh s, z), \quad s \in \mathbb{R}, \quad (x, y, z) \in M, \]

preserves the exponential mapping:

\[ e^{x^0} \circ \text{Exp}(\psi, c, t) = \text{Exp}(\psi + s, c, t), \quad (\psi, c, t) \in N, \quad s \in \mathbb{R}, \]

thus $d(e^{x^0}(q)) = d(q)$ for $q \in I^+(q_0)$. Moreover, the flow $e^{x^0}$ preserves the boundary $\partial A = J^+(q_0) \setminus I^+(q_0)$, thus $d(e^{x^0}(q)) = d(q) = 0$ for $q \in J^+(q_0) \setminus I^+(q_0)$.

Further, it is obvious from (8.1) that the reflections $\varepsilon^1, \varepsilon^2$ preserve $d(\cdot)$.

(2) The flow of the dilations

\[ e^{sY} : (x, y, z) \mapsto (xe^s, ye^s, ze^{2s}), \quad s \in \mathbb{R}, \quad (x, y, z) \in M, \]

acts on the exponential mapping as follows:

\[ e^{sY} \circ \text{Exp}(\psi, c, t) = \text{Exp}(\psi, ce^{-2s}, te^s), \quad (\psi, c, t) \in N, \quad s \in \mathbb{R}, \]
thus $d(e^{sY}(q)) = e^s d(q)$ for $q \in I^+(q_0)$. The equality $d(e^{sY}(q)) = e^s d(q) = 0$ for $q \in J^+(q_0) \setminus I^+(q_0)$ follows since the flow $e^{sY}$ preserves the boundary $\partial A = J^+(q_0) \setminus I^+(q_0)$. \hfill \Box

## 10 Sub-Lorentzian spheres

### 10.1 Spheres of positive radius

Sub-Lorentzian spheres

$$S(R) = \{q \in M \mid d(q) = R\}, \quad R > 0,$$

are transformed one into another by dilations:

$$S(e^s R) = e^{sY}(S(R)), \quad s \in \mathbb{R},$$

thus we describe the unit sphere

$$S = S(1) = \{\exp(\lambda, 1) \mid \lambda \in C\}. \tag{10.1}$$

**Theorem 13** (1) The unit sub-Lorentzian sphere $S$ is a regular real-analytic manifold diffeomorphic to $\mathbb{R}^2$.

(2) Let $q = \exp(\psi, c, 1) \in S$, $(\psi, c) \in C$, then the tangent space

$$T_q S = \left\{ v = \sum_{i=1}^{3} v_i X_i(q) \mid -v_1 \cosh(\psi + c) + v_2 \sinh(\psi + c) + v_3 c = 0 \right\}. \tag{10.2}$$

(3) $S$ is the graph of the function $x = \sqrt{y^2 + f(z)}$, where $f(z) = e \circ k(z)$, $e(w) = \frac{\sinh^2 w}{w^2}$, $k(z) = b(z)/2$, $b = a^{-1}$, $a(c) = \frac{\sinh c - c}{2c^2}$.

(4) The function $f(z)$ is real-analytic, even, strictly convex, unboundedly and strictly increasing for $z \geq 0$. This function has a Taylor decomposition $f(z) = 1 + 12z^2 + O(z^4)$ as $z \to 0$ and an asymptote $4|z|$ as $z \to \infty$:

$$\lim_{z \to \infty} (f(z) - 4|z|) = 0. \tag{10.3}$$

(5) The function $f(z)$ satisfies the bounds

$$4|z| < f(z) < 4|z| + 1, \quad z \neq 0. \tag{10.4}$$

(6) A section of the sphere $S$ by a plane $\{z = \text{const}\}$ is a branch of the hyperbola $x^2 - y^2 = f(z)$, $x > 0$. A section of the sphere $S$ by a plane $\{x = \text{const} > 1\}$ is a strictly convex curve $y^2 + f(z) = x^2$ diffeomorphic to $S^1$.

(7) The sub-Lorentzian distance from the point $q_0$ to a point $q = (x, y, z) \in \tilde{A} \cap \mathbb{R}$ may be expressed as $d(q) = R$, where $x^2 - y^2 = R^2 f(z/R^2)$.

(8) The sub-Lorentzian ball $B = \{q \in M \mid d(q) \leq 1\}$ has infinite volume in the coordinates $x, y, z$.

See in Fig. 16 a plot of the sphere $S$ (above in red) and the Heisenberg beak $\partial A$ (at the bottom in blue). Different sub-Lorentzian length maximizers connecting $q_0$ and $S$ are shown in Fig. 17. A plot of the function $f(z)$ illustrating bound (10.4) is shown in Fig. 18. Sections of the sphere $S$ by the planes $\{x = 1, 2, 3\}$ are shown in Fig. 19.
Proof (1) Since $\text{Exp} : C \times \mathbb{R}_+ \to \tilde{A}$ is a diffeomorphism, the parametrization (10.1) of the sphere $S$ implies that it is a smooth 2-dimensional manifold diffeomorphic to $\mathbb{R}^2$. Moreover, the exponential mapping is real-analytic, thus $S$ is real-analytic as well.

(2) Let $q = \text{Exp}(\lambda, 1) \in S$, $\lambda = (\psi, c, q_0) \in C$, and let $t = e^{\tilde{H}(\lambda_0)}$. Then

$$T_qS = \lambda_1 = \{v \in T_qM \mid \langle \lambda_1, v \rangle = 0\}. \quad (10.5)$$

Since $h_1(\lambda_1) = -\cosh(\psi + c)$, $h_2(\lambda_1) = \sinh(\psi + c)$, $h_3(\lambda_1) = c$, representation (10.2) follows from (10.5).

(3) It follows from (10.2) that the 2-dimensional manifold $S$ projects regularly to the coordinate plane $(y, z)$, thus it is a graph of a real-analytic function $x = F(y, z)$. Since $e^{tX^0}(S) = S, t \in \mathbb{R}$, then

$$0 = X^0(F(y, z) - x)|_S = F(y, z) \frac{\partial F}{\partial y}(y, z) - y.$$ 

Integrating this differential equation, we get $F(y, z) = \sqrt{y^2 + f(z)}$ for a real-analytic function $f(z)$.

Since $S \cap \{z = 0\} = \{x = \sqrt{y^2 + 1}, z = 0\}$, then $f(0) = 1$.

Let $z \neq 0$. Then $z = \frac{\sinh c - c}{2c^2} = a(c)$ by virtue of (5.3). The function $a : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, denote the inverse function $b = a^{-1}$. By virtue of (6.5), we have $f(z) = x^2 - y^2 = \frac{4}{c^2} \sinh^2 p$, whence $f(a(c)) = \frac{4}{c^2} \sinh^2 p, \text{thus } f(a) = e^{\tilde{H}(\lambda_0)}$, where $e(x) = \frac{\sinh x}{x^2}$. Item (3) follows.
Fig. 17  Maximizers connecting $q_0$ and $S$

Fig. 18  Plot of $f(z)$ and bound (10.4)
(4) We have already proved that $f(z)$ is real-analytic. Since $\varepsilon^1(S) = S$, then $f$ is even. Immediate computation shows that $k'(z) > 0$, $z > 0$, and $e'(x) > 0$, $x > 0$, whence $f''(z) > 0$, $z > 0$. Similarly it follows that $f''(z) > 0$ for $z > 0$. By virtue of the expansions

$$k(z) = 6z + O(z^2), \quad e(x) = 1 + \frac{x^2}{3} + O(x^4),$$

we get $f(z) = 1 + 12z^2 + O(z^4)$, $z \to 0$. Finally, it easily follows from the definition of the function $f(z)$ that

$$\lim_{z \to \infty} (f(z) - 4|z|) = 0.$$  

(5) follows from (4).

(6) It is straightforward that $S \cap \{z = \text{const}\} = \{x^2 - y^2 = f(z), \quad x > 0, \quad z = \text{const}\}$ is a branch of a hyperbola.

The section $S \cap \{x = \text{const} > 1\} = \{y^2 + f(z) = x^2, \quad x = \text{const} > 1\}$ is a smooth compact curve, thus diffeomorphic to $S^1$. If $y \geq 0$, then this curve is given by the equation $y = \sqrt{x^2 - f(z)}$, which is a strictly concave function (this follows by twice differentiation).

(7) Take any point $q = (x, y, z) \in \tilde{A}$, then there exists $s \in \mathbb{R}$ such that $e^{-sY}(q) \in S$, i.e.,

$$d(q) = e^s,$$

see item (2) of Theorem 12. Denoting $R = e^s$, we get $\frac{R}{x} = \sqrt{\frac{y^2}{R^2} + f\left(\frac{z}{R^2}\right)}$, and item (7) of this theorem follows.

(8) The unit ball is given explicitly by

$$B = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \sqrt{y^2 + 4|z|} \leq x \leq \sqrt{y^2 + f(z)} \right\},$$

thus its volume is evaluated by the integral

$$V(B) = \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \left( \sqrt{y^2 + f(z)} - \sqrt{y^2 + 4|z|} \right) = +\infty.$$  

\begin{remark}
Thanks to bound (10.4) of the function $f(z)$, the sphere $S = \left\{ x = \sqrt{y^2 + f(z)} \right\}$ is contained in the domain

$$\left\{ q = (x, y, z) \in M \mid \sqrt{y^2 + 4|z|} < x \leq \sqrt{y^2 + 4|z| + 1} \right\}.$$
\end{remark}
The flows of the vector fields $Y$ and $X^0$ provide an approximation of the function $\sqrt{y^2 + f(z)}$ defining $S$ up to the accuracy
\[
\sqrt{y^2 + 4|z| + 1 - \sqrt{y^2 + 4|z|}} = \frac{1}{\sqrt{y^2 + 4|z| + 1 + \sqrt{y^2 + 4|z|}}} \leq \min \left( 1, \frac{2}{|y|}, \frac{1}{\sqrt{|z|}} \right).
\]

10.2 Sphere of zero radius

Now consider the zero radius sphere
\[ S(0) = \{q \in M \mid d(q) = 0\}. \]

Theorem 14

1. $S(0) = J^+(q_0) \setminus I^+(q_0) = \partial J^+(q_0) = \partial I^+(q_0) = \partial A$.
2. $S(0)$ is the graph of a continuous function $x = \Phi(y, z) := \sqrt{y^2 + 4|z|}$, thus a 2-dimensional topological manifold.
3. The function $\Phi(y, z)$ is even in $y$ and $z$, real-analytic for $z \neq 0$, Lipschitz near $z = 0$, $y \neq 0$, and Hölder with constant $\frac{1}{2}$, non-Lipschitz near $(y, z) = (0, 0)$.
4. $S(0)$ is filled by broken lightlike trajectories with one or two edges described in Theorem 8, and is parametrized by them as follows:
\[
S(0) = \left\{ e^{r_2(X_1-X_2)}e^{r_1(X_1+X_2)} = (\tau_1 + \tau_2, \tau_1 - \tau_2, -\tau_1\tau_2) \mid \tau_i \geq 0 \right\}
\]
\[
\cup \left\{ e^{r_2(X_1+X_2)}e^{r_1(X_1-X_2)} = (\tau_1 + \tau_2, \tau_1 - \tau_2, \tau_1\tau_2) \mid \tau_i \geq 0 \right\}.
\]
5. The flows of the vector fields $Y, X^0$ preserve $S(0)$. Moreover, the symmetries $Y, X^0$ provide a regular parametrization of
\[
S(0) \cap \{\text{sgn} z = \pm 1\} = \left\{ e^{sY} \circ e^{sX^0}(q_{\pm}) \mid r, s > 0 \right\}, \tag{10.6}
\]
where $q_{\pm} = (x_{\pm}, y_{\pm}, z_{\pm})$ is any point in $S(0) \cap \{\text{sgn} z = \pm 1\}$.
6. The sphere $S(0) = \{16z^2 = (x^2 - y^2)^2, \ x^2 - y^2 \geq 0, \ x \geq 0\}$ is a semi-algebraic set.
7. The zero-radius sphere is a Whitney stratified set with the stratification
\[
S(0) = (S(0) \cap \{z > 0\}) \cup (S(0) \cap \{z < 0\})
\]
\[
\cup (S(0) \cap \{z = 0, \ y > 0\}) \cup (S(0) \cap \{z = 0, \ y < 0\}) \cup \{q_0\}.
\]
8. Intersection of the sphere $S(0)$ with a plane $\{z = \text{const} \neq 0\}$ is a branch of a hyperbola $\{x^2 - y^2 = 4|z|, \ x > 0, \ z = \text{const}\}$, intersection with a plane $\{z = 0\}$ is an angle $\{x = |y|, \ z = 0\}$, intersection with a plane $\{y = kx\}$, $k \in (-1, 1)$, is a union of two half-parabolas $\{4z = \pm(1 - k^2)x^2, \ x \geq 0, \ y = kx\}$, and intersection with a plane $\{y = \pm x\}$ is a ray $\{y = \pm x, \ z = 0\}$.

The Heisenberg beak $S(0) = \partial A$ is plotted in Figs. 1, 2 and 3 as a graph of the function $x = \sqrt{y^2 + 4|z|}$ by virtue of (4.1), and in Fig. 20 as a parametrized surface by virtue of (10.6) with $q_{\pm} = (2, 0, \pm 1)$.

Proof

(1), (2) follow from item (2) of Theorem 9 and item (3) of Section 4.
(3) and (6)–(8) are obvious.
(4) follows from Theorem 8.
(5) follows from Theorem 12.

Lightlike maximizers filling $S(0)$ are shown in Fig. 21. Sub-Lorentzian spheres of radii 0, 1, 2, 3 are shown in Fig. 22.
Remark 13  The spheres  

\[
S(1) = \left\{ (x, y, z) \in M \mid x = \sqrt{y^2 + f(z)}, \ y, z \in \mathbb{R} \right\},
\]

\[
S(0) = \left\{ (x, y, z) \in M \mid x = \sqrt{y^2 + 4|z|}, \ y, z \in \mathbb{R} \right\},
\]

tend one to another as \( z \to \infty \) since for any \( y \in \mathbb{R} \)

\[
\lim_{z \to \infty} \left( \sqrt{y^2 + f(z)} - \sqrt{y^2 + 4|z|} \right) = 0
\]

by virtue of (10.3). The same holds for any spheres \( S(R_1), S(R_2), R_i \in [0, +\infty) \).

11 Conclusion

The results obtained in this paper for the sub-Lorentzian problem on the Heisenberg group differ drastically from the known results for the sub-Riemannian problem on the same group:

1. The sub-Lorentzian problem is not completely controllable.
2. Filippov’s existence theorem for optimal controls cannot be immediately applied to the sub-Lorentzian problem.
3. In the sub-Lorentzian problem all extremal trajectories are infinitely optimal, thus the cut locus and the conjugate locus for them are empty.
4. The sub-Lorentzian length maximizers coming to the zero-radius sphere are nonsmooth (concatenations of two smooth arcs forming a corner, nonstrictly normal extremal trajectories).
5. Sub-Lorentzian spheres and sub-Lorentzian distance are real-analytic if \( d > 0 \).

On the other hand, all these properties hold both in the sub-Lorentzian problem on the Heisenberg group and in the Lorentzian problem on the flat Minkowski space.
It would be interesting to understand which of these properties persist for more general sub-Lorentzian problems (e.g., for left-invariant problems on Carnot groups).

The authors thank A.A. Agrachev, L.V. Lokutsievskiy, and M. Grochowski for valuable discussions of the problem considered.

The authors are also grateful to the reviewers whose comments helped to improve presentation of the paper.

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