Random walks on uniform and non-uniform combs and brushes

Alex V Plyukhin\(^1\) and Dan Plyukhin\(^2\)

\(^1\) Department of Mathematics, Saint Anselm College, Manchester, NH, United States of America
\(^2\) Department of Computer Science, University of Toronto, Toronto, ON, Canada
E-mail: aplyukhin@anselm.edu and dplyukhin@cs.toronto.edu

Received 19 March 2017
Accepted for publication 14 June 2017
Published 18 July 2017

Online at stacks.iop.org/JSTAT/2017/073204
https://doi.org/10.1088/1742-5468/aa79b4

Abstract. We consider random walks on comb- and brush-like graphs consisting of a base (of fractal dimension \(D\)) decorated with attached side-groups. The graphs are also characterized by the fractal dimension \(D_a\) of a set of anchor points where side-groups are attached to the base. Two types of graphs are considered. Graphs of the first type are uniform in the sense that anchor points are distributed periodically over the base, and thus form a subset of the base with dimension \(D_a = D\). Graphs of the second type are decorated with side-groups in a regular yet non-uniform way: the set of anchor points has a fractal dimension smaller than that of the base, \(D_a < D\). For uniform graphs, a qualitative method for evaluating the sub-diffusion exponent suggested by Forte \textit{et al} for combs \((D = 1)\) is extended for brushes \((D > 1)\) and numerically tested for the Sierpinski brush (with the base and anchor set built on the same Sierpinski gasket). As an example of nonuniform graphs we consider the Cantor comb composed of a one-dimensional base and side-groups, the latter attached to the former at anchor points forming the Cantor set. A peculiar feature of this and other nonuniform systems is a long-lived regime of super-diffusive transport when side-groups are of a finite size.

Keywords: classical Monte Carlo simulations, communication, supply and information networks, diffusion, transport properties
1. Introduction

Comb- and brush-like graphs and networks are branched systems consisting of a base of fractal dimension $D$ and a collection of identical side-groups, each of fractal dimension $d$, attached to the base according to a certain protocol. Structures with a one-dimensional base $D = 1$ are often referred to as combs (see figure 1) and those with $D > 1$ as brushes (see figure 2); the unifying term ‘bundled structures’ was coined for both graph types in [1].

In this paper we will find it instructive to further classify bundled structures by the fractal dimension $D_a$ of the anchor set, i.e. the subset of the base consisting of the points (called anchor points) at which the side-groups are attached; see figure 1. We shall say that a bundled structure is uniform (or uniformly decorated) if side-groups are attached to the base in a periodic manner, so that $D_a = D$; see figures 1 and 2. Otherwise, if $D_a < D$, the structure will be called non-uniform. An example of a non-uniform bundled structure is the Cantor comb ($D = d = 1$, $D_a = \ln 2 / \ln 3$); see figure 6. We will show that uniform and non-uniform bundled structures may differ considerably in their transport properties, as well as in the methods for their analysis.

Random walks on uniform comb-like structures were first studied as a simplified model of transport on percolation clusters [2, 3], but in recent years the problem has also been addressed from different perspectives, and found to be relevant to many other natural and man-made systems and processes (e.g. transport in branched polymers, porous media, spiny dendrites of neuron cells, and artificial networks); see [1, 4–9] and references therein. Typically, transport characteristics (like the random walk dimension) of the base or a side-group in isolation are already known, and one is interested in finding those characteristics for the corresponding composition. Of particular interest is the exponent $\alpha$, which describes the mean-square displacement along the base $\langle r^2(t) \rangle \sim t^\alpha$. For bundled systems with infinitely extended side-groups, transport along the base usually exhibits sub-linear ($\alpha < 1$ or even slower, e.g. logarithmic) time dependence. For example, for a simple uniform comb composed of an unbounded one-dimensional base and side-groups ($D = D_a = d = 1$), transport along the base is sub-diffusive with $\alpha = 1/2$. Diffusion along the base of a uniform brush, composed of a two-dimensional base ($D = D_a = 2$)
Random walks on uniform and non-uniform combs and brushes

and one-dimensional side-groups ($d = 1$), is governed by the same transport exponent $\alpha = 1/2$. While these results for $\alpha$ can be obtained theoretically in a simple manner [2, 3, 6], its evaluation for more complicated structures may be quite involved [1, 8], particularly for those with non-trivial fractal dimensions $D, D_a, d$. On the other hand, for the special case of uniform comb-like structures ($D = D_a = 1$), Forte et al recently suggested an attractive qualitative method to determine the transport exponent $\alpha$ without detailed combinatorial calculations, based on a simple matching argument [7].
The first goal of this paper is to extend the qualitative approach by Forte et al to uniform brush-like structures with a fractal base dimension, $D = D_a > 1$; this is quite straightforward and will be discussed in section 2, with support in section 3 by numerical simulations on the uniform Sierpinski brush ($D = D_a = \ln 3/\ln 2$ and $d = 1$). Simulation of transport in bundled structures, characterized by non-integer dimensions, has some interesting peculiarities which we feel have not been fully addressed in the literature so far.

The second goal of the paper is to simulate random walks on non-uniform bundled structures, like the Cantor comb (see figure 6), for which $D_a < D$. In section 4 we confirm a theoretical result for the transport exponent $\alpha$ suggested for such systems in [8]. We also studied non-uniform combs with finite side-groups, observing for that case a very long transient regime of super-diffusive transport, with mean-square displacement increasing faster than linearly with time. We show that the long duration of this transient makes the heuristic method of Forte et al inapplicable to non-uniform bundled structures.

2. Uniform systems: matching argument

Let us recapitulate the matching argument of Forte et al [7]. Consider a generic uniform comb ($D = D_a = 1$) decorated with identical side-groups, each characterized by fractal dimension $d$, random walk dimension $d_w$, and spectral dimension $d_s = 2d/d_w \leq 2$; see figure 1. Assume that the base and side-groups are discrete and that random walks are characterized by a single hopping rate $w$ and a hopping distance of unit length. Diffusion on the base in isolation (ignoring side-groups) is normal, whereas diffusion on an isolated side-group (detached from the base) may be either normal or anomalous. The corresponding mean-square displacements as functions of time are

$$\langle x^2(t) \rangle_b \sim w t, \quad \langle r^2(t) \rangle_{sg} \sim (w t)^{2/d_w},$$

where subscripts $b$ and $sg$ refer to the isolated base and an isolated side-group, respectively. Given (1), one wishes to evaluate the exponent $\alpha$ for the mean-square displacement

$$\langle x^2(t) \rangle \sim (w t)^{\alpha}$$

along the base of the comb when decorated with infinitely extended side-groups.

To this end, consider first a comb composed of the infinite base decorated with finite side-groups of linear size $L$. For such a system, on sufficiently long time scales $t \geq t_c(L)$ the diffusion along the base is expected to be normal, but with an effective $L$-dependent hopping rate $W(L) < w$,

$$\langle x^2(t) \rangle \sim W(L) t, \quad t \geq t_c(L).$$

The characteristic time $t_c(L)$ can be estimated as the one needed for a random walker to diffuse along an isolated side-group to the distance of order $L$, $\langle r^2(t_c) \rangle_{sg} \sim (w t_c)^{2/d_w} \sim L^2$, which gives

$$t_c(L) \sim w^{-1} L^{d_w}.$$
As for the effective hopping rate $W(L)$, one expects it to be proportional to the occupation probability of the anchor point for random walks on an isolated side-group of size $L$. The assumed condition on the spectral dimension $d_s = 2d/d_w \leq 2$ implies that diffusion on an isolated side-group is compact [3], so one can evaluate the effective rate on a time scale $t > t_c(L)$ as

$$W(L) \sim w L^{-d}. \quad (5)$$

Equations (3)–(5) completely specify diffusion on the long time scale, $t > t_c(L)$.

On time-scales shorter than $t_c(L)$, a random walker would not feel the boundary of the side-groups, and the mean-square displacement along the base is expected to have the same form as in the case of unbounded side-groups,

$$\langle x^2(t) \rangle \sim (w t) \alpha, \quad t < t_c(L). \quad (6)$$

Assuming that the transition from the short-time (6) to long-time (3) behaviours is sufficiently abrupt, one expects that the two expressions must approximately match at $t = t_c(L)$, which gives

$$W(L) t_c(L) = [w t_c(L)]^{\alpha}. \quad (7)$$

Substituting expression (4) for $t_c(L)$ and (5) for $W(L)$, one finds [7]

$$\alpha = 1 - \frac{d}{d_w} = 1 - \frac{d_s}{2}. \quad (8)$$

This result coincides with that of the rigorous combinatorial approach [1] and is also in accordance with analytical and numerical studies of specific comb-like structures reported in the literature [2–4, 6].

Our first goal is to extend the above reasoning for brush-like systems with base dimension $D > 1$. As before, we shall assume that the base is decorated with side-groups uniformly, i.e. $D_a = D$, see figure 2. For brush-like systems, diffusion may be anomalous not only on isolated side-groups but also on the isolated base,

$$\langle r^2(t) \rangle_b \sim (w t)^2/D_w, \quad \langle r^2(t) \rangle_{sg} \sim (w t)^{2/d_w}. \quad (9)$$

Here $D_w$ and $d_w$ are random walk dimensions for an isolated base and side-group, respectively. Consider first a brush with finite side-groups of linear size $L$. As for combs with finite side-groups, we expect that on a sufficiently long time scale $t > t_c(L)$ diffusion along the base is of the same type, i.e. has the same random walk dimension $D_w$ as for an isolated base, but with a re-normalized $L$-dependent hopping rate $W(L)$,

$$\langle r^2(t) \rangle \sim [W(L) t]^{2/D_w}, \quad t \geq t_c(L), \quad (10)$$

while for $t < t_c(L)$ the mean-square displacement follows the same law $\langle r^2(t) \rangle \sim (w t)^\alpha$ as for the infinite system. Assuming that the transition between the two diffusion regimes occurs sufficiently fast, one expects the corresponding expressions for the mean-square displacement to be approximately equal at the transition time $t = t_c(L)$,

$$[W(L) t_c(L)]^{2/D_w} = [w t_c(L)]^{\alpha}. \quad (11)$$

This matching relation is analogous to that for combs, equation (7).
The next assumption is that the effective rate $W(L)$ and the crossover time $t_c(L)$ depend on the structure of side-groups but not on that of the base, and therefore estimations (4) and (5) for those quantities obtained above for combs should be valid for brush-like structures also. Then substituting (4) and (5) into (11), one obtains

$$\alpha = \frac{2}{D_w} \left( 1 - \frac{d}{d_w} \right) = \frac{D_s}{D} \left( 1 - \frac{d_s}{2} \right), \tag{12}$$

where $D_s = 2D/D_w$ and $d_s = 2d/d_w$ are spectral dimensions of the isolated base and side-group, respectively. This result is in agreement with the combinatorial evaluation of [1], and in the next section we shall verify it with numerical simulations for a specific brush structure with one-dimensional side-groups ($d = 1$, $d_w = 2$). For the latter case, the expression (12) takes the simple form $\alpha = 1/D_w$.

As the structure of (12) suggests, that expression holds for $d_s < 2$, which is the condition of compactness of diffusion on isolated side-groups. Also, recall that both expressions (8) for combs and (12) for brushes hold for uniform systems only, assuming the condition $D_a = D$. We postpone the discussion of non-uniform structures with $D_a < D$ until section 4.

3. Simulation: Sierpinski brush

Consider a uniform brush with one-dimensional side-groups and a Sierpinski gasket [3] as a base; see figure 2. For this bundled structure, which we call the Sierpinski brush, the set of relevant dimensions is

$$d = 1, \quad d_w = 2, \quad D = D_a = \ln 3/\ln 2, \quad D_w = \ln 5/\ln 2, \tag{13}$$

and according to result (12) of the previous section, the mean-square displacement along the base is expected to follow the sub-diffusive law $\langle r^2(t) \rangle \sim t^\alpha$ with $\alpha = 1/D_w = \ln 2/\ln 5 \approx 0.43$. Numerical simulation results, presented in figure 4, support this prediction. In addition, they also show that the convergence to the above asymptotic dependence occurs very slowly, taking about $10^3$ steps. Below we discuss some details of the simulation which, due to the fractal geometry of the base, are of interest on their own.

Numerical simulation of random walks on fractals typically involves the following two points. Firstly, in order to minimize finite size effects one needs to run random walks on a fractal set of such a large size that explicitly storing the coordinates of each site in computer memory (say, in a multidimensional array) is infeasible. Since a coordinate system is prerequisite to knowing how to navigate the fractal, one needs an effective (and efficient) algorithm for enumerating the hopping sites and determining the labels of their neighbors. Secondly, once a pertinent labeling algorithm for a fractal set is developed, it may still be a nontrivial problem to express with it the metric properties of the set. This second problem might be avoided if one is interested in such properties as the probability of return to the origin, but becomes inevitable when one wishes to directly evaluate the mean-square displacement, drift in an external field, etc. As shown below (see also [10]), for the Sierpinski brush and similar systems both difficulties can
be readily resolved by embedding the fractal in a 2-dimensional Euclidean lattice and writing the Cartesian coordinates of fractal sites as binary numbers.

Unlike the usual definition of the Sierpinski gasket, which is defined ‘from the outside’, in this paper we construct the base of the Sierpinski brush ‘from the inside, out’ as the limit of a sequence of inductively defined generations (see figure 3): the first generation $G_1$ is composed of three quarters of a square, and specifies the pattern for higher generations; each subsequent generation $G_{n+1}$ is defined according to this pattern as three copies of $G_n$. For the purposes of simulation it suffices to let the base be a generation of sufficient size, so that finite size-effects are negligible; the presented results use $G_{30}$, with approximately $10^{14}$ jumping sites. Note that we identify the jumping sites with the cells of the base (depicted as shaded squares in figures 2 and 3), rather than the base’s vertices. The brush construction is finalised by attaching one-dimensional discrete side-groups to each cell of the base.

As mentioned above, in order to identify a cell from the base in the embedding 2D lattice, it is convenient to enumerate the cells of the lattice by pairs of Cartesian coordinates $(x, y)$, each expressed in binary form. In $G_k$, the binary coordinates of each cell consist $k$ bits, i.e. digits, which are each either zero or one:

$$x = (a_1a_2 \cdots a_k), \quad y = (b_1b_2 \cdots b_k), \quad a_i, b_i \in \{0, 1\}. \quad \quad \quad (14)$$

As seen from figure 3, the cells belonging to the base (i.e. the Sierpinski gasket cells) are those, and only those, cells of the 2D lattice for which the sum of bits in every position of the binary address is either zero or one but not two:

$$a_i + b_i < 2, \quad \text{for} \quad i = 1, 2, \ldots, k. \quad \quad \quad (15)$$

For example, in the base of second generation $G_2$ (see figure 3) the cell with Cartesian coordinates $x = 3, y = 1$ has the binary address $x = (11), y = (01)$. This cell is not in
Consider first random walks on the isolated base, i.e. on the Sierpinski gasket without side-groups. An initial site is selected by randomly generating a binary address of the form (14) subject to condition (15). A particle is placed on the initial site and performs random walks according to the following two step protocol: first, the particle makes a virtual jump to a randomly selected nearest neighbour cell on the embedding 2D lattice. In our simulation the coordination number of the embedding lattice is $n_c = 8$, so that the particle chooses randomly between eight neighboring sites. Second, we check that the new site satisfies (15) to determine if the new site belongs to the base, i.e. if it is a cell of the Sierpinski gasket. If the test is passed, the virtual jump is accepted. Otherwise (the new site does not belong to the base), the virtual jump is rejected, and the particle remains on the original site.

Depending on how one handles rejected jumps, the above protocol can be performed in two ways. In the so-called ‘blind ant scenario’, rejected jumps are counted as steps with zero displacement but have the same duration as accepted jumps. On the other hand, in the ‘myopic ant scenario’ rejected jumps are totally virtual and not counted at all, i.e. have zero duration. In other words, a myopic ant selects a new site to jump only among those nearest neighbour sites which belong to the gasket. For both scenarios we found the random walk dimension to be the same, and very close to $D_w = \ln 5/\ln 2 \approx 2.32$. This value is the same as for the more familiar triangular
Sierpinski gasket [3]. The equality of $D_w$ for the triangular and rectangular versions of the Sierpinski gasket, though intuitively expected, is perhaps not quite obvious considering that the two fractals are not completely equivalent: for a triangular Sierpinski gasket the coordination number for all cells (except the three apex cells) is 3, while for our rectangular gasket this number is either 3 or 4. In our simulation this difference was found to have no effect on the asymptotic parameters of random walks.

The extension of the above simulation scheme for the Sierpinski brush, i.e. when one-dimensional side-groups are attached to every cell of the Sierpinski gasket (see figure 2) is straightforward. The results presented in figure 4 confirm theoretical prediction (12) for brushes with unbounded side-groups, showing the asymptotic dependence $\langle r^2(t) \rangle \sim t^{\alpha}$ for the mean-square displacement along the base, with $\alpha = 1/D_w$. An important bonus result is the significant duration (more than $10^3$ steps) of the initial transient regime; see the inset of figure 4.

Recall that the matching argument leading to prediction (12) is based on the assumption of a relative abruptness of the crossover between the two asymptotic behaviors of the mean-square displacement in a system with finite side-groups. Our simulation confirms the validity of this assumption. Figure 5 shows $\langle r^2(t) \rangle$ for a Sierpinski brush with side-groups of length $2L$ for $L = 100$ (and also for $L = 50$ and $L = 200$ in the inset). One observes that for short times $t < t_c(L)$ the mean-square displacement follows the law $\langle r^2(t) \rangle \sim t^{\alpha}$ with $\alpha = 1/D_w \approx 0.43$ as for the brush with infinite side-groups, while for longer times $t > t_c(L)$ there occurs a transition to the dependence $\langle r^2(t) \rangle \sim t^{2/D_w} \approx t^{0.86}$ as for the isolated base. As can be seen in the inset, the characteristic transition time $t_c(L)$ increases with $L$ in a manner consistent with theoretical expectation (4).
So far we have focused on uniform bundled structures for which side-groups are anchored to each site of the base, or for which they decorate the base in a periodical manner. For such systems the fractal dimension $D_a$ of the set of anchor points is the same as that of the base, $D_a = D$. As an example of a non-uniform system with $D_a < D$, let us consider a comb-like structure composed of a one-dimensional base and side-groups, with side-groups attached to the base at anchor points forming a Cantor set; see figure 6. For this system, which we call the Cantor comb, the set of relevant dimensions is

\begin{align}
\begin{aligned}
d &= D = 1, \\
d_w &= D_w = 2, \\
D_a &= \ln 2 / \ln 3 \approx 0.63.
\end{aligned}
\end{align}

Compared to uniform systems, the presence of the additional dimension $D_a$ makes a theoretical analysis of non-uniform bundled structures more complicated. To the best of our knowledge, so far there are no general results in the literature on transport properties of non-uniform bundled structures with an arbitrary set of dimensions. A specific class of combs with a one-dimensional base and side-groups, $d = D = 1$, and arbitrary $D_a < 1$ (to which the Cantor comb belongs) was considered by Iomin [8]. Using an analytical yet somewhat ambiguous approach, he suggested two possible expressions for the exponent $\alpha$ governing the mean-square displacement along the base $\langle x^2(t) \rangle \sim t^\alpha$. We found one of those expressions, namely

\begin{align}
\alpha &= 1 - D_a/2,
\end{align}

Figure 6. A recursive composition and cell enumeration of the Cantor comb. The first three generations, $G_1$, $G_2$, $G_3$, are shown. Below each base cell is its address in decimal notation, and above is the corresponding ternary (base-3) representation. Anchor cells, where side-groups are attached to the base, form the Cantor set and have ternary addresses not containing the digit 1.

4. Nonuniform systems: Cantor comb

Before outlining the simulation details, let us stress that in contrast to the uniform systems result, (17) presupposes (tacitly in [8]) a special type of initial conditions. Namely, initial sites of random walks are assumed to be chosen randomly among the anchor points, or within a finite distance from an anchor point. In this case, after time $t$, when the walker moved to the average distance $x(t)$ from an initial site, the average number of anchor points the walker encountered is estimated as $N_a(t) \sim x(t)^{D_a}$. 

https://doi.org/10.1088/1742-5468/aa79b4
For non-uniform systems, this estimation, assumed by the theory \[ \text{[8]} \], does not generally hold for other choices for initial positions. For example, if positions of initial sites are chosen to be not correlated to that of anchor points then \( N_a(t) \) would depend on the system size \( R \) and vanish in the limit \( R \to \infty \) (since the density of the anchor points decreases with the system size \( R \) as \( 1/R^{D_a} \)). In that case, in the limit \( R \to \infty \) side-groups do not affect diffusion at all, and instead of (17) one would trivially expect to find \( \alpha = 2/D_w \) (\( \alpha = 1 \) for the Cantor comb), i.e. the transport exponent for the isolated base. Our simulation of random walks on the Cantor comb with random initial positions, uncorrelated to positions of anchor points, indeed shows that sort of behavior.

Similar to the Sierpinski brush, it is convenient for simulation purposes to design the Cantor comb inductively from smaller to larger scales and to enumerate cells of the base with both decimal and ternary (base-3) integers \[ \text{[10]} \]. As seen from figure 6, this allows a simple test for an anchor cell: a cell is an anchor cell (i.e. belongs to the Cantor set) if and only if its ternary address has no 1-digits. The simulation results described below were obtained for the Cantor comb of generation \( G_{30} \) with a base consisting of \( 3^{30} \sim 10^{14} \) cells. Further increasing or slightly decreasing the generation order was found not to affect the results, which suggests that finite size effects were negligible in our simulation.

For the Cantor comb with infinitely extended side-groups, the simulation results are shown in figure 7. The long-time asymptotic behavior of the mean-square displacement along the base is well described by the sub-diffusion law \( \langle x^2(t) \rangle \sim t^\alpha \) with the exponent given by the theoretical prediction (17), \( \alpha = 1 - \ln 2/(2 \ln 3) \approx 0.68 \). Similar to the uniform Sierpinsky brush, this behavior is preceded by a long (about \( 10^3 \) steps) transient period of faster diffusion; see the inset of figure 7.

Now we consider random walks on the Cantor comb with side-groups of finite length \( 2L \). Not only is this a question of a practical interest, but also along this line...
one may hope to extend for non-uniform systems the matching argument discussed in section 2. As for uniform systems, for short times, while the diffusing particle does not reach the end points of side-groups, the mean-square displacement along the base of the Cantor comb is expected to follow the same sub-diffusion law as the comb with infinite side-groups, i.e. $\langle x^2(t) \rangle \sim t^\alpha$ with $\alpha \approx 0.68$ (dashed line) as for the system with infinite side-groups. For longer time scales there occurs a transition to the super-diffusive dependence $t^\beta$ with $\beta \approx 1.18$ (dashed line). For even longer time scales the exponent $\beta$ decreases and tends towards one. The transition to normal diffusion, $\beta \to 1$, is shown in the inset for a system with shorter side-groups, $L = 5$, for which the transition is noticeable within the simulation time range.

Thus, diffusion along the base of the Cantor comb with finite side-groups shows not two, as for uniform systems, but three distinct regimes: initial sub-diffusion is followed up by super-diffusion, which very slowly evolves into normal diffusion. We also found a similar behavior for a non-uniform version of the Sierpinski brush (see figure 2) where the anchor points form a Sierpinski gasket, but the base spans the
entire two-dimensional lattice, \( d = 1, D_a = \ln 3/\ln 2, D = 2 \) (in contrast to the uniform Sierpinski brush, discussed in section 3, for which \( D_a = D = \ln 3/\ln 2 \)). We believe such a three-stage behavior is generic for non-uniform bundled systems with \( D_a < D \). Clearly, the matching argument of section 2 is not applicable here.

It is, of course, quite evident that a continuous transient from sub-diffusion to normal diffusion cannot be anything but super-diffusive. However, the very presence and significant duration of such a transient regime may not be obvious. One can intuitively interpret the origin of the super-diffusive regime in non-uniform bundled structures as follows: on time scales much longer than \( t(L) \sim L^{d_w} \) the role of side-groups of size \( L \) is to reduce the effective jumping rate of random walks along the base. As time progresses, a diffusing particle explores regions of larger spatial scale \( x \) of the base, and ‘sees’ the density of side-group sites decreasing as \( 1/x^{D-D_a} \). As a result, the effective jumping rate and diffusion coefficient increase with \( x \) which, as is well known [11], may result in super-diffusive transport. On the other hand, in the long time limit, when the diffusing particle explores a vast region for which the fraction of side-groups is negligible compared to the number of sites of the base, diffusion is expected to follow asymptotically the law \( \langle x^2(t) \rangle \sim t^{2/D_w} \) (that is, \( \langle x^2(t) \rangle \sim t \) for combs) as for the isolated base.

5. Conclusion

We have studied random walks on two groups of comb- and brush-like graphs. The first group is comprised of systems uniformly decorated with side-groups, for which the fractal dimensions of the base and the set of anchor points are equal, \( D = D_a \). The second group is that of non-uniform graphs for which anchor points form a proper subset of the base with dimension smaller than that of the base, \( D_a < D \).

For uniform graphs we extended the qualitative argument by Forte et al, originally developed for comb-like systems \( (D = 1) \), to brushes with arbitrary base dimension \( D \). The result is expression (12) for the exponent \( \alpha \), describing sub-diffusive transport along the base of the system with unbounded side-groups. We verified this expression in numerical simulations for the uniform Sierpinski brush with \( D = D_a = \ln 3/\ln 2 \). The simulation also shows that the regime of stationary sub-diffusion with \( \langle r^2(t) \rangle \sim t^{\alpha} \) is preceded by a transient regime of significant duration (about \( 10^3 \) steps) and therefore the former may not be observable in systems with finite and relatively short side-groups.

For uniform graphs with finite but sufficiently long side-groups, simulation shows that the mean-square displacement along the base experiences a transition from the time dependence \( t^\alpha \) (as in a system with infinite side-groups) at shorter time scales to \( t^{2/D_w} \) (as for the isolated base) at longer time scales. Simulations confirm the abruptness of this transition, which is tacitly implied in the argument by Forte et al.

In contrast, for nonuniform graphs like the Cantor comb \( (D = 1, D_a = \ln 2/\ln 3) \) with side-groups of finite length, the transition to the time dependence \( \langle x^2(t) \rangle \sim t^{2/D_w} \), characteristic for an isolated base, occurs only after a very long regime of super-diffusive transport. While the origin of the super-diffusive regime is intuitively clear, its theoretical description remains an open problem.

For non-uniform fractal combs with infinite side-groups, our simulation supported that the asymptotic mean-square displacement along the base \( \langle x^2(t) \rangle \sim t^{\alpha} \) is
Random walks on uniform and non-uniform combs and brushes

classified by an exponent of the form $\alpha = 1 - D_a/2$, as suggested by Iomin [8]. It
still remains to be seen how to generalize this result for non-uniform bundled structures
with $d, D > 1$. As a reference point for a future theory we evaluated $\alpha$ for a non-uniform
brush (briefly mentioned in the previous section) with one-dimensional side-groups, a
two-dimensional base, and the anchor set given by the Sierpinski gasket,

$$
\begin{align*}
    d &= 1, \\
    d_w &= 2, \\
    D &= D_w = 2, \\
    D_a &= \ln 3/\ln 2.
\end{align*}
\tag{18}
$$

The structure is illustrated by the same figure 2 as for the Sierpinski brush discussed
in section 2, but now a walker is allowed to step on every cell of the 2D lattice. For
this non-uniformly decorated brush ($D_a < D$) we observed transport properties quali-
tatively similar to that of the Cantor comb, and for the mean-square exponent we
obtained the approximate value $\alpha \approx 0.72$.

The simulation schemes described in the paper can be readily implemented for
biased random walks to evaluate a directional drift, i.e. the mean displacement along
the base $\langle r(t) \rangle_f$ induced by a weak external constant field $f$. The latter is modeled
by replacing the unbiased jumping probability $p$ to $p \pm f$ for jumps in the direction
along/against the field. In previous works it was found that for uniform combs the
field induced drift $\langle r(t) \rangle_f$ and the mean-square displacement for unbiased
($f = 0$) random walks $\langle r^2(t) \rangle$ satisfy the (generalized) Einstein relation, both increasing with time
according to the same law [2, 3, 7, 12]

$$
\langle r(t) \rangle_f \sim \langle r^2(t) \rangle \sim t^\alpha.
\tag{19}
$$

Our simulations showed the validity of this relation also for uniform brushes like the
Sierpinski brush and for non-uniform structures like the Cantor comb. We found relation (19)
valid for systems with finite and infinite side-groups, and not only for asymptotic but for transient
(in particular, super-diffusive) regimes as well. However, and remarkably, for the Cantor comb we found the deviation from the Einstein relation (19)
to be noticeable already for very weak fields $f > 10^{-3}$.

References

[1] Cassi D and Regina S 1996 Random walks on bundled structures Phys. Rev. Lett. 76 2914
[2] Weiss G H and Havlin S 1985 Some properties of a random walk on a comb structure Physica A 134 474
[3] ben-Avraham D and Havlin S 2000 Diffusion and Reactions in Fractals and Disordered Systems
(Cambridge: Cambridge University Press)
[4] Mendez V, Iomin A, Campos D and Horsthemke W 2015 Mesoscopic description of random walks on combs
Phys. Rev. E 92 062112
[5] Mendez V and Iomin A 2013 Reaction-subdiffusion front propagation in a comblike model of spiny dendrites
Phys. Rev. E 88 012706
[6] Arkhincheev V E 2000 Anomalous diffusion and charge relaxation on comb model: exact solutions Physica A
280 304
[7] Forte G, Burioni R, Cecconi F and Vulpiani A 2013 Anomalous diffusion and response in branched systems:
a simple analysis J. Phys.: Condens. Matter 25 465106
[8] Iomin A 2011 Subdiffusion on a fractal comb Phys. Rev. E 83 052106
[9] Agliari E, Cassi D, Cattivelli L and Sartori F 2016 Two-particle problem in comblike structures Phys. Rev.
E 93 052111
[10] Plyukhin D and Plyukhin A V 2016 Random walks with fractally correlated traps: stretched exponential and
power law survival kinetics Phys. Rev. E 94 042132
[11] Risken H 1989 The Fokker-Planck Equation (Berlin: Springer)
[12] Gradenigo G, Sarracino A, Villamaina D and Vulpiani A 2013 Einstein relation in systems with anomalous
diffusion Acta. Phys. Pol. B 44 899

https://doi.org/10.1088/1742-5468/aa79b4 14