Explicit Cutoff Regularization in Coordinate Representation

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Abstract
In this paper, we study a special type of cutoff regularization in the coordinate representation. We show how this approach unites such concepts and properties as an explicit cut, a spectral representation, a homogenization, and a covariance. Besides that, we present new formulae to work with the regularization and give additional calculations of the infrared asymptotics for some regularized Green’s functions appearing in the pure four-dimensional Yang–Mills theory and in the standard two-dimensional Sigma-model.

Keywords: cutoff regularization, Green’s function, spectral representation, homogenization, covariance, infrared asymptotics

1. Introduction
Divergent integrals appear in physical and mathematical calculations quite frequently. For example, the perturbative quantum field theory [1, 2] is based on the study of Feynman diagrams, which contain divergent quantities of different types due to a ‘bad’ infrared behavior of Green’s functions (propagators). Fortunately, the renormalization theory [3, 4] was created, and this allows us to obtain physically meaningful results from the divergencies. However, such approach depends on a regularization scheme, see [5]. There are a lot of ways to introduce it, but we are interested in only one of them, the so-called cutoff regularization.

In this work we are going to discuss the cutoff regularization of a special type. We follow the papers [6–8] and introduce an explicit cut in the coordinate representation. However, we want the regularization to possess and unite some additional properties, such as an explicit cutoff procedure, a spectral representation, a homogenization, and a covariance. All these concepts and conditions can be satisfied. In the next section we will explain our point of view in detail.
Description of some other cutoff regularizations can be found in [9–14], but they do not include all the properties suggested here.

Of course, there are other types of regularization, which have been actively investigated in recent years. Among them we can note a regularization by higher covariant derivatives [15–17] and an implicit regularization [18, 19]. But they are not considered in the paper, because the first one leads to the asymptotics for an operator of order more than two, while the second one does not give an explicit infrared regularization of the Green’s function asymptotics.

The paper has the following structure. First of all, in section 2, we describe the main ideas of our approach and explain some conditions, which the regularization should satisfy. Then, in section 3.1, we take the first steps to realize the project. In the section a one-dimensional case on \( \mathbb{R} \) is studied. In section 3.2, we calculate necessary spectral functions to give the formulation for an arbitrary dimension value. So, this section is related to the case \( \mathbb{R}^d \) for \( d > 1 \). In section 4, we consider some generalizations and study cases, applicable to concrete models, such as the pure four-dimensional Yang–Mills theory and the standard two-dimensional Sigma-model, see theorems 1 and 2. Some additional remarks and discussion are presented in section 5.

2. Approach description

Let us consider a Laplace-type operator \( L \) on \( \mathbb{R}^d \), where \( d \in \mathbb{N} \), with smooth coefficients. Then, we introduce eigenvalues \( \lambda^2 \) and eigenfunctions \( \phi_\lambda \) for the operator, such that \( L(x)\phi_\lambda(x) = \lambda^2 \phi_\lambda(x) \) for all \( x \in \mathbb{R}^d \). Furthermore, we assume that all \( \lambda^2 > 0 \). Of course, we imply the presence of suitable boundary conditions, so that the problem is well posed. We do not discuss the spectral problem in detail here, because we want to explain the main idea of our approach. However, we assume that the Green’s function \( G \) exists and its kernel has the following representation
\[
G(x, y) = \int_{\mathbb{R}} d\mu(\lambda) \phi_\lambda(x) \frac{1}{\lambda^2} \phi_\lambda^*(y), \quad x, y \in \mathbb{R}^d,
\]  
where the integral is written using a measure \( d\mu(\lambda) \), which corresponds to the operator \( L \), see [20]. The star denotes the Hermitian conjugation. Let us note that if the last measure contains only a discrete spectrum, then we can replace the integral by a sum over all eigenvalues, taking into account their multiplicities. For example, in the case of the operator \( L(x) = -\partial_x^\mu \partial_x^\mu \) on \( \mathbb{R}^d \), we have a continuous spectrum, and the transition to the spectral (momentum) representation is performed by the standard Fourier transform.

As it was mentioned in the introduction, Green’s functions play an important role in the perturbative quantum field theory, because they are constructive blocks for the Feynman diagrams. So, they can appear under an integration in various nonlinear combinations. This fact leads to the appearance of divergences, which permeate modern theoretical physics. For example, on the four-dimensional space, the right hand side of formula (1) has the asymptotics \( (1 + o(1))/(4\pi |x - y|^2) \) in the range \( x \sim y \). Therefore, the Green’s function has the singularity at \( x = y \). Such situation arises in the Yang–Mills theory, for instance.

To avoid the singularities and, moreover, divergences, we need to introduce some type of regularization. There are a lot of ways to perform this, but we are interested only in a cutoff regularization in the coordinate representation. Let us describe several conditions we are going to take into account.

(a) Explicit cutoff. First of all, the regularization should be controlled by a clear procedure. In the introduction, we have cited some ways. However, now we suggest to use the one studied and successfully applied in the works [6–8]. In the next sections we give precise
definitions, whereas here we want to consider only one example. Returning to the four-
dimensional case mentioned above, the procedure means that the main term of the Green’s
function has the following deformation

\[ |x - y|_\Lambda = \begin{cases} |x - y|, & |x - y| > 1/\Lambda; \\ 1/\Lambda, & |x - y| \leq 1/\Lambda, \end{cases} \tag{2} \]

where the limit \( \Lambda \to +\infty \) removes the regularization. The non-leading terms also have
deformations, but they contain ambiguities that should be controlled by additional condi-
tions, described below. These ambiguities appear due to the ability to add to the deformed
Green’s function an additional component \( g_\Lambda(x, y) \), such that \( Lg_\Lambda \to 0 \) for \( \Lambda \to +\infty \) in the
sense of generalized functions [21, 22].

(b) **Spectral representation.** The procedure should have an explicit spectral meaning, because
we want to know how spectral functions of the operator \( L \) are deformed. The most expected
way is to represent the regularization by an operator of integration \( \hat{J} \), whose kernel \( \hat{J}(x, y) \)
is equal to the following integral

\[ \hat{J}_\Lambda(x, y) = \int_{\mathbb{R}^4} d\mu(\lambda) \phi_\Lambda(x) \rho(\lambda/\Lambda) \phi_\Lambda^*(y), \tag{3} \]

where the function \( \rho(\cdot) \) has properties, sufficient to work with the representation. It is clear
that we need to have the equality \( \rho(0) = 1 \), because the limit transition \( \Lambda \to +\infty \) removes
the regularization. After applying such operator to the Green’s function, we obtain the fol-
lowing transformation \( 1/\lambda^2 \to \rho(\lambda/\Lambda)/\lambda^2 \) in formula (1).

(c) **Homogenization.** Another question which arises during the construction of the cutoff reg-
ularization concerns the possibility of its representation by a ‘classical’ integration operator.
In addition, we expect this integration operator, if it exists, to be a homogenization
operator on a sphere centered at some point. For example, in the four-dimensional case
mentioned above, this question can be reformulated for the first order as follows: does there exist a smooth kernel \( \omega(y) \), such that the relations hold

\[ \frac{1}{4\pi|x|^3} = \frac{1}{S_3} \int_{S^3} d\hat{S}(y) \frac{\omega(y)}{4\pi|y/\Lambda + x|^2}, \quad \frac{1}{S_3} \int_{S^3} d\hat{S}(y) \omega(y) = 1, \tag{4} \]

where \( S^3 \) is the unit sphere in \( \mathbb{R}^4 \) at the origin, \( S_3 \) is its surface area, and \( d\hat{S}(y) \) is a measure
on the sphere? The answer is positive, as we will see it below.

(d) **Covariance.** This point means that the kernel \( \hat{J}_\Lambda(x, y) \) should change covariantly with
respect to gauge transformations, if, of course, such type of transformation is applicable
to the operator \( L \). Actually, the feature follows from decomposition (3) and, therefore, the
kernel \( \hat{J}_\Lambda(x, y) \) inherits properties of the operator \( L(x) \). Indeed, from the change

\[ L(x) \to U(x)L(x)U^*(x) \text{ we obtain } \hat{J}_\Lambda(x, y) \to U(x)\hat{J}_\Lambda(x, y)U^*(y) \tag{5} \]

due to the appropriate change of the eigenfunctions \( \phi_\Lambda(x) \to U(x)\phi_\Lambda(x) \), where \( U(x) \) is a
smooth element of a transformation group.

As we know, Lagrangian densities for the standard field theories are local, because they
contain a finite number of derivatives. At the same time property (b) can lead to a loss of
the locality, while condition (c) allows us to keep a quasi-locality. Indeed, we can replace
the regularization in the form (4) from the operator to the field, and, hence, we obtain the local
homogenization.

3. Results for Euclidean space \( \mathbb{R}^d \)

3.1. One-dimensional case

In this section we investigate one simple example, which shows the main idea of the approach.
Let us introduce the standard Laplace operator \( A(s) = -\partial_s \partial_s \) on the real axis \( \mathbb{R} \), which has a
twofold continuous spectrum. This means that the equation \( A(s) \psi(s) = \lambda^2 \psi(s) \) has two oscil-
lating solutions \( \exp(\pm i \lambda s) \) for all \( \lambda \in \mathbb{R}_+ \), which form a kernel for the Fourier transform and,
therefore, lead to the momentum representation, see paragraph 9 in [23].

It is known that the equation \( A(s) g(s) = \delta(s) \) has a set of solutions, which are called funda-
mental solutions. Now we are interested in \( g(s) \) of a special type, namely \( G(|s|) = -|s|/2 \).
It is easy to verify that other fundamental solutions can be obtained from \( G(|s|) \) by adding the
so-called ‘zero modes’ in the form \( \exp(\pm i \lambda s) \) for all \( \lambda \in \mathbb{R}_+ \), which form a kernel for the Fourier transform and, and it is equal to the average value of all the

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\[
A(s)G(|s|_\Lambda) = \frac{\partial_s}{2} \begin{cases} \text{sign}(s), & |s| > 1/\Lambda; \\ 0, & |s| \leq 1/\Lambda, \end{cases}
\]

where \( \Lambda > 0 \) is an auxiliary regularization parameter, which is quite large.

In this case the solution \( G(|s|) \) is transformed into the regularized one \( G(|s|_\Lambda) \), which satis-
fies the following limit transition \( G(|s|_\Lambda) \to G(|s|) \) for \( \Lambda \to +\infty \) in the sense of generalized
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\]

where \( \Lambda > 0 \) is an auxiliary regularization parameter, which is quite large.
Hence, the homogenization of the function can be reformulated as the homogenization of the exponential. Indeed, we can write the following relation

$$\frac{1}{S_0} \int_{S^d} d\sigma(r) f(s - r/\Lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} dt e^{i\sigma t} \rho(t/\Lambda),$$

(10)

where

$$\rho(t/\Lambda) = \frac{1}{S_0} \int_{S^d} d\sigma(r) e^{i\sigma t/\Lambda} = \frac{1}{2} \sum_{\sigma = \pm 1} e^{i\sigma t/\Lambda} = \cos(t/\Lambda).$$

(11)

We see that the function $\rho(\cdot)$ gives additional oscillations, which lead to the regularization. Using the limit $\Lambda \to +\infty$, these oscillations can be excluded. In high dimensions, the function gets an additional decreasing at infinity.

On the other hand, we can use the Taylor expansion of the function $\cos(\cdot)$ and take it from the integration as derivatives. Then, we get

$$\frac{1}{S_0} \int_{S^d} d\sigma(r) f(s - r/\Lambda) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} \left(\Lambda^2 A(s)/\Lambda^2 \right)^n f(s) = \rho\left(\sqrt{\Lambda^2 A(s)}/\Lambda\right) f(s),$$

(12)

where we have introduced the operator-valued function, which is understood as the corresponding Taylor series.

To obtain a connection between functions (6) and (11) we need to take one more step. First of all, let us remember the Fourier transform of the generalized function $G(|s|)$

$$\int_{\mathbb{R}} ds e^{i\sigma t} G(|s|) = \frac{1}{P} \hat{G}(|\sigma|).$$

(13)

Then, using the representation $G(|s|_{\Lambda}) = G(|s|) + \left(G(|s|_{\Lambda}) - G(|s|)\right)$ and the fact that the difference $G(|s|_{\Lambda}) - G(|s|) = 0$ for all $|s| \geq 1/\Lambda$, we get the following chain of equalities

$$\int_{\mathbb{R}} ds e^{i\sigma t} G(|s|_{\Lambda}) = \frac{1}{P} + \int_{-1/\Lambda}^{1/\Lambda} ds e^{i\sigma t} \left(\frac{|s|}{2} - \frac{1}{2\Lambda}\right) = \frac{\rho(t/\Lambda)}{t^2} = \rho(t/\Lambda) \hat{G}(|\sigma|).$$

(14)

Finally, applying the inverse Fourier transform and formulae (9) and (10), we have

$$G(|s|_{\Lambda}) = \frac{1}{2\pi} \int_{\mathbb{R}} dt e^{i\sigma t} \hat{G}(|\sigma|) \rho(t/\Lambda) = \frac{1}{S_0} \int_{S^d} d\sigma(r) G(|s - r/\Lambda|).$$

(15)

After all the calculations, we have obtained two very interesting and remarkable relations. On the one hand, we have shown that our cutoff regularization (6) can be achieved by using the operator of homogenization in the form (10). And on the other hand, such type of the regularization has an explicit spectral meaning, because from the spectral decomposition of formula (13) we see that the eigenvalue $1/\lambda^2$ goes to $\rho(\lambda/\Lambda)/\lambda^2$. Additionally, let us note that the limit $\Lambda \to +\infty$ removes the regularization, because $\rho(0) = 1$.

In the next sections, we describe our approach for all dimensions and more intricate Laplace-type operators. Also, we make some additional calculations for several special operators.

3.2. Multidimensional case

To move on, let us introduce some additional convenient notations. First of all, we abandon the restriction on the dimension, so we study $d \in \mathbb{N}$. Then, for all $x \in \mathbb{R}^d$, we can define an auxiliary function $G(|x|)$ as
\begin{align}
G(|x|) &= \begin{cases} 
-|x|/2, & d = 1; \\
-\ln(|x|)/2\pi, & d = 2; \\
|x|^{2-d}/(d-2)S_{d-1}, & d \geq 3,
\end{cases}
\end{align}

where \(S_{d-1} = 2\pi^{d/2}/\Gamma(d/2)\) is the surface area of \(S^{d-1}\), the \((d-1)\)-dimensional unit sphere, centered at the origin. Also, we have used \(|x| = \sqrt{x_\mu x^\mu}\) with the corresponding summation over repeated indices. Of course, the last function \(G(\cdot)\) depends on the dimension value, which will be omitted as a rule, because this does not cause confusion. Let us note that function (16) satisfies the very well known equality \(-\partial_\mu \partial_\nu G(|x-y|) = \delta(x-y)\), where \(x, y \in \mathbb{R}^d\).

In the next calculations, we repeat all steps from the previous section. The generalization means that the dimension parameter takes positive integer values.

Following the main idea, we need to find the Fourier transformation for the regularized Green’s function \(G(|x|_\Lambda)\), see (2).

**Lemma 1.** Let all the conditions described above be valid. Also, let \(x, k \in \mathbb{R}^d\), \(s = |x|\), \(t = |k|\), and \(S_x\) is the corresponding deformation according to formula (6), where \(\Lambda > 0\). Then, we have the following relation

\begin{equation}
\int_{\mathbb{R}^d} d^d x e^{ik \cdot x} G(s_\Lambda) = \frac{1}{S_{d-1} t^2} \int_{\mathbb{R}^{d-1}} d\sigma(x) e^{ik \cdot x/\Lambda}.
\end{equation}

**Proof.** Let us proceed in stages. The situation \(d = 1\) was studied in the previous section. In the case \(d \geq 3\) we should add and subtract the function \(1/x^{d-2}\) to the integrand in the region \(s\Lambda < 1\). Then, using the Fourier transform for the standard Green’s function, we obtain

\begin{equation}
\frac{1}{(d-2)S_{d-1}} \int_{\mathbb{R}^d} \frac{d^d x e^{ik \cdot x}}{s_\Lambda^d} = \frac{1}{(d-2)S_{d-1}} \int_{\mathbb{B}_{1/\Lambda}^d} d^d x e^{ik \cdot x} \left(\Lambda^{d-2} - \frac{1}{s^{d-2}}\right),
\end{equation}

where \(\mathbb{B}^d_{1/\Lambda}\) is the closed \(d\)-dimensional ball with the radius \(1/\Lambda\), centered at the origin.

Further, let us transform the second term in the last equality. For that we represent the exponential in the following form \(e^{ik \cdot x} = -\partial_\mu \partial_\nu \sigma \cdot x^{\mu}\nu / t^2\) and then integrate by parts twice

\begin{equation}
\int_{\mathbb{B}^d_{1/\Lambda}} d^d x \left(\partial_\mu \partial_\nu \sigma \cdot x^{\mu}\nu\right) \left(\Lambda^{d-2} - \frac{1}{s^{d-2}}\right) = (d-2)S_{d-1} - (d-2) \int_{\mathbb{R}^{d-1}} d\sigma(x) e^{ik \cdot x/\Lambda},
\end{equation}

where we have applied the equality \(\partial_\mu \partial_\nu \sigma \cdot x^{\mu}\nu s^{2-d} = -(d-2)S_{d-1} \delta(x)\).

Hence, substituting relation (19) in formula (18) and making the change of variables \(x^{\mu} \rightarrow x^{\nu}/\Lambda\), we obtain the statement of the lemma for \(d \geq 2\).

The equality for \(d = 2\) can be achieved in the same manner. We note that all the calculations are performed in the sense of generalized functions on the space of functions with a compact support.\hfill\Box

We see from the last lemma that the Fourier transformation for \(G(|x|_\Lambda)\) contains an additional factor, compared to the non-regularized function. This factor is formulated as an integral. Fortunately, it can be calculated explicitly.

**Lemma 2.** Let \(k \in \mathbb{R}^d\), \(t = |k|\), and \(\Lambda > 0\). Then, under the conditions described above, the relation holds

\begin{equation}
\rho(|t|/\Lambda) = \frac{1}{S_{d-1}} \int_{\mathbb{R}^{d-1}} d\sigma(y) e^{ik \cdot y/\Lambda} = \Gamma(d/2) \left(\frac{|t|}{2\Lambda}\right)^{1-d/2} J_{d/2-1}(|t|/\Lambda).
\end{equation}
To verify the equality we need to apply the inverse Fourier transform to both sides of (17). Hence, the left hand side gives (16), while for the right hand side we can write out the following chain of equalities

\[
\left. \frac{\partial_s \partial_{\mu}}{(4\pi)^d} \int_{\mathbb{R}^d} d^k k e^{ik\cdot x} e^{ik\cdot y} e^{ik\cdot \Lambda} \right|_{x=S} = \frac{1}{S_{d-1}} \int_{\mathbb{R}^{d-1}} d\sigma(y) G(|x-y/\Lambda|) \quad \text{(28)}
\]

Proof. To obtain the statement of the lemma, we need to derive one auxiliary relation. Let the hat denote that the corresponding vector has unit length, so \( \hat{y}^\mu = y^\mu/|y| \) for all \( y \in \mathbb{R}^d \). Let \( m = 2n \) with \( n \in \mathbb{N} \cup \{0\}, \ c = 2^{d+1/2} \Gamma(n + d/2) \), and \( A_{\mu_1 \cdots \mu_n} \) is a symmetric tensor. Then, analogously to derivation of (31) in [24], we have the following chain of equalities

\[
\int_{S^{d-1}} d\sigma(y) \hat{y}^{\mu_1} \cdots \hat{y}^{\mu_n} A_{\mu_1 \cdots \mu_n} = \frac{2}{c} \int_{\mathbb{R}^d} dy e^{-r^2/2} \rho^{m+d-1} \int_{S^{d-1}} d\sigma(y) \hat{y}^{\mu_1} \cdots \hat{y}^{\mu_n} A_{\mu_1 \cdots \mu_n} \quad \text{(21)}
\]

\[
= \frac{2}{c} \int_{\mathbb{R}^d} dy e^{-|\hat{y}|^2/2} \hat{y}^{\mu_1} \cdots \hat{y}^{\mu_n} A_{\mu_1 \cdots \mu_n} \quad \text{(22)}
\]

\[
= \frac{2(2\pi)^{d/2}}{c} \partial_{\mu_1} \cdots \partial_{\mu_n} A_{\mu_1 \cdots \mu_n} \left( \frac{|\hat{y}|^2}{2} \right)^{n/2} \quad \text{(23)}
\]

\[
= \frac{2(2\pi)^{d/2}}{c} \left( \frac{2n!}{2^n n!} \right) A_{\mu_1 \cdots \mu_n}. \quad \text{(24)}
\]

Hence, using the Taylor expansion for the exponential on the left hand side of (20), applying the last relation to each term, and then summing the resulting series with the usage of

\[
\sum_{n=0}^{+\infty} \left( \frac{-r^2/4}{n!} \right)^n \frac{1}{n \Gamma(n + d/2)} = \frac{1}{2} (r/2)^{-d/2} J_{d/2-1}(r),
\]

(25)

where \( r \geq 0 \), we obtain the statement of the lemma. \( \square \)

From the last lemma it follows that the factor \( \rho(\cdot) \) is presented by the oscillating function, which depends on the dimension parameter. We have omitted this parameter in definition (20), because this does not lead to confusion. For example, in the case \( d = 1 \), we get the known result from formula (11). Indeed, we get

\[
\Gamma(1/2) \left( \frac{|t|}{2\Lambda} \right)^{1/2} J_{-1/2}(|t|/\Lambda) = \cos(|t|/\Lambda).
\]

(26)

So, the obtained result is consistent with the previous one.

Finally, we can derive a general representation formula using an integration over \((d - 1)\)-dimensional sphere. Such formula gives a connection between the regularization and the homogenization, see section 2.

Lemma 3. Let \( x \in \mathbb{R}^d, s = |x|, \Lambda > 0, \) and \( s_{\Lambda} \) is the corresponding deformation from (6). Then, under the conditions described above, we have the following additional representation for (16)

\[
G(s_{\Lambda}) = \frac{1}{S_{d-1}} \int_{S^{d-1}} d\sigma(y) G(|x-y/\Lambda|).
\]

(27)

Proof. To verify the equality we need to apply the inverse Fourier transform to both sides of (17). Hence, the left hand side gives (16), while for the right hand side we can write out the following chain of equalities

\[
\int_{\mathbb{R}^d} d\sigma(y) e^{ik\cdot x} e^{ik\cdot \Lambda} = \frac{1}{S_{d-1}} \int_{S^{d-1}} d\sigma(y) \delta(x-y/\Lambda) \quad \text{(28)}
\]

\[
= \frac{\partial_s \partial_{\mu}}{S_{d-1}} \int_{S^{d-1}} d\sigma(y) G(|x-y/\Lambda|), \quad \text{(29)}
\]
from which the statement of the lemma follows. Note that on the last step we have excluded the operator $\partial_{\mu} \partial_{\nu}$ and then we have fixed the local zero modes with the use of the obvious value at the origin $G([0]/\Lambda) = G(1/\Lambda)$.

The last three lemmas lead to the main result discussed above. On the one hand, we have the connection between the explicit cutoff regularization of the Green’s function and the homogenization with the use of the $(d-1)$-dimensional sphere, see formulae (2) and (27). On the other hand, we have obtained the relation between the regularized Green’s function and the corresponding momentum (spectral) representation. These equations can be written for all $x \in \mathbb{R}^d$ in the form

$$G(|x|/\Lambda) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d^n y e^{i|y|} \tilde{G}(|y|) \rho(|y|/\Lambda) = \frac{1}{S_{d-1}} \int_{S^{d-1}} d\sigma(y) G(|x - y/\Lambda|),$$

(30)

which is actually the generalization of the one-dimensional case, see (15), to an arbitrary dimension. Additionally, we note that an analog of the first equality from (30) is known in the general theory of Fourier transforms, see theorem 4.15 in [25], but in the context of our paper they are new and give a new point of view on the process of the regularization.

Indeed, the last chain of equalities plays an important role in infrared decompositions of Green’s functions, because it allows us to combine several types of the representation of the standard fundamental solution. As we will see this in the following sections, such trick simplifies a number of calculations. We are going to demonstrate this on two types of Laplace operator, which appear in the pure four-dimensional Yang–Mills theory and the standard two-dimensional Sigma-model.

4. Applications

Now we want to study the application of the regularization to some special operators. First of all, we need to remember some basic concepts of the infrared, near the diagonal, expansion of the Green’s function using the corresponding Seeley–DeWitt (sometimes, they are named after Hadamard, Minakshisundaram [26], and Gilkey [27]) coefficients, see [28, 29].

Let $L$ be a Laplace-type operator on $\mathbb{R}^d$ with smooth coefficients. Moreover, we concretize its form using the following local formula

$$L(x) = -D_{\mu} D_{\nu} - v(x) \text{ for all } x \in \mathbb{R}^d,$$

(31)

where $D_{\mu} = \partial_{\mu} + B_{\mu}(x)$ is a covariant derivative, $B_{\mu}(x)$ are components of a connection 1-form, and $v(x)$ is an auxiliary potential. We note that the last objects can be non-commutative between each other.

In this case the Green’s function $G(x,y)$, which solves the equation $L(x)G(x,y) = \delta(x - y)$ and satisfies appropriate boundary conditions (decreasing at infinity, for example), can be expanded near the diagonal, where $|x - y|$ is bounded by some small fixed parameter, in the following form, see formulae (A14) and (A15) in [30],

$$G(x,y) = \sum_{n=0}^{+\infty} R_n(x - y) a_n(x,y) + PS(x,y).$$

(32)

Here, $a_n(x,y)$ for $n \in \mathbb{N} \cup \{0\}$ are the Seeley–DeWitt coefficients, which are constructed using the heat kernel method [31, 32] and can be found in the works [27, 33–37]. They satisfy the following recurrence relations.
(x − y)\mu D_{\nu\alpha} a_0(x, y) = 0, a_0(y, y) = 1, (n + (x − y)\mu D_{\nu\alpha})a_n(x, y) = −L(x)a_{n−1}(x, y), n > 0,
(33)

and can be defined by them. Then, the functions \( R_n(x − y) \) do not depend on the connection components and the potential. Their smoothness gets better with increasing of their order number. The first one has the worst smoothness, because it leads to the \( \delta \)-function according to the formula \( −\partial_\mu, \partial_\nu R_0(x) = \delta(x) \). Further, the part \( PS(x, y) \) is smooth and includes the global information about the operator and the boundary conditions, while the Seeley–DeWitt coefficients are local by construction. Additionally, we note that local zero modes, recently mentioned in [7, 38] and described in Corollary 1 of [39], are included into \( PS(x, y) \) part as well.

Hence, the non-smooth behavior of the Green’s function is included into the \( R_n \)-functions. We do not write appropriate recurrence relations for them, because their definitions can be varied. Indeed, we can subtract a smooth part from the sum and add it to \( PS(x, y) \). Anyway, in every example below we give explicit formulae.

In the next calculations, we want to demonstrate how to apply the process of regularization in some popular situations, such as the pure four-dimensional Yang–Mills theory and the two-dimensional Sigma-model.

Additionally, we note that the next operators will have an auxiliary small parameter \( s \in \mathbb{R}_+ \), such that the connection components and the potential would be small, and, moreover, they can be excluded with the usage of the transition \( s \to +0 \).

### 4.1. Pure 4D Yang–Mills theory

Let \( G \) be a compact semisimple Lie group, and \( \mathfrak{g} \) is its Lie algebra. Then, let \( \mathfrak{r}' \) be the generators of the algebra \( \mathfrak{g} \), where \( a = 1, \ldots, \dim \mathfrak{g} \), such that the relations hold

\[
[r^a, r^b] = f^{abc} r^c, \quad \text{tr}(r^a r^b) = −2\delta^{ab},
(34)
\]

where \( f^{abc} \) are antisymmetric structure constants for \( \mathfrak{g} \), and ‘tr’ is the Killing form. In this case the connection component \( B_{\mu}(x) \) mentioned above can be expanded as \( B_{\mu}^a(x) r^a \), where, as usual, we mean the standard summation over the repeating indices. Then, introducing the components of the field strength tensor for \( x \in \mathbb{R}^4 \) in the form

\[
F_{\mu\nu}(x) = F_{\mu\nu}^a(x)r^a, \quad F_{\mu\nu}^a(x) = \partial_\mu B_{\nu}^a(x) − \partial_\nu B_{\mu}^a(x) + f^{abc} B_{\mu}^b(x) B_{\nu}^c(x),
\]

we write out two Laplace-type operators

\[
M_{0}^{ab}(x) = −D_{\alpha}^{μ} D_{\alpha}^{μ}, \quad M_{1}^{ab}(x) = M_{0}^{ab}(x) + 2f^{abc} F_{\mu\nu}(x),
(35)
\]

where the matrix components \( D_{\alpha}^{μ} \) of the covariant derivative \( D_{\alpha}^{μ} \) have the form

\[
D_{\alpha}^{μ} = \partial_\alpha \delta^{ab} + f^{acb} B_{\alpha}^c(x).
(36)
\]

These operators play a crucial role in the perturbative expansions and in the Feynman diagram technique, see [7, 8, 15, 40–43], because they define the main quadratic forms, for gauge and ghost fields, and propagators, or Green’s functions, which are actually constructed blocks of the technique.

To obtain the results for both operators simultaneously, we introduce the operator of more general type

\[
M_{2}^{ab}(x; \alpha) = M_{0}^{ab}(x) + 2\alpha f^{abc} F_{\mu\nu}(x),
(37)
\]

where \( \alpha \in \mathbb{R} \). Then, we have two additional relations

\[
M_{0}^{ab}(x) = \frac{1}{4}M_{2}^{ab}(x; 0), \quad M_{1}^{ab}(x) = M_{2}^{ab}(x; 1).
(38)
\]
Green’s functions, corresponding to the operators from (35) and (37), are notated as

\[ G_0(x,y), \ G_{1\mu\nu}(x,y), \ \text{and} \ G_{2\mu\nu}(x,y;\alpha), \ \text{respectively.} \quad (39) \]

One of the ways to investigate the quantum Yang–Mills theory relates to the renormalization theory, namely, to studying its divergences. Fortunately, the main results of papers [7, 44] and formula (27) in [8], the form of the divergent part has a special view, which allows us to consider a simplified version of the connection components. They are equal to

\[ B^a_\mu(x) = \frac{s}{2} \delta^\nu_\mu \xi^a_\mu, \ \text{where} \]

\[ (\xi^a_\mu)_\mu = \frac{1}{\sqrt{8\dim G}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \quad \text{for all} \ a \in \{1, \ldots, \dim G\}, \quad (40) \]

and \( s \) is a small auxiliary positive parameter mentioned above. The corresponding components \( F^{a\mu}_\nu \) are equal to \( s \xi^a_\mu \xi^a_\nu \). It is quite convenient that we have commutativity and the following two normalization properties

\[ \sum_{a=1}^{\dim G} \sum_{\mu,\nu=1}^{4} \xi^a_\mu \xi^a_\nu = 1, \ \text{and} \ \sum_{\sigma=1}^{4} \xi^a_\sigma \xi^b_\sigma = -\frac{2\delta^a_\mu \delta^b_\nu}{8\dim G} \quad \text{for all} \ a, b, \mu, \nu. \quad (41) \]

Actually, the parameter \( s \) can be used in the asymptotic expansion instead of \( x - y \), if the last difference is from a finite neighborhood of the diagonal \( x = y \). In both situations every term of the asymptotics consists of a finite number of components, and the transition from one asymptotics to another can be achieved by an explicit procedure.

Using the last simplifications, we can write out the first three Seeley–DeWitt coefficients for the Green’s function \( G_{2\mu\nu}(x,y;\alpha) \) according to decomposition (32) as

\[ a_{0\mu\nu}(x,y) = \Phi(x,y)\delta_{\mu\nu}, \ a_{1\mu\nu}(x,y) = \Phi(x,y) \left( 2\alpha x \xi_{\mu\nu} + \frac{s^2}{12} \delta_{\mu\nu} (x-y)^{\sigma\rho} \xi_{\sigma\rho} \xi_{\rho\beta} \right), \quad (42) \]

\[ a_{2\mu\nu}(x,y) = s^2 \Phi(x,y) \left( \frac{\delta_{\mu\nu}}{12} \xi_{\rho\beta} \xi_{\rho\beta} + 2\alpha^2 \xi_{\mu\sigma} \xi_{\sigma\nu} \right) + o(s^3), \quad (43) \]

where, in the last equality, we have used that \( s \rightarrow +0, \xi_{\mu\nu} \) denotes the \((\dim g \times \dim g)\) matrix with the elements \( f^{abc} g^{\mu\nu} \), and

\[ \Phi(x,y) = \exp \left( x_{\mu\nu} y^{\mu\nu}/2 \right). \quad (44) \]

Then, for convenience, we make a transformation to the Fock–Schwinger gauge, see the first theorem in [37] or article [44]. So, we define the following auxiliary operator

\[ \hat{M}_{2\mu\nu}(x-y;\alpha) = \Phi(y,x)M_{2\mu\nu}(x;\alpha)\Phi(x,y). \quad (45) \]

The explicit calculation gives

\[ \hat{M}_{2\mu\nu}(x-y;\alpha) = -\delta_{\mu\nu} \left( \partial_{\alpha} \partial_{\nu} + s(x-y)^{\sigma\rho} \xi_{\sigma\rho} \partial_{\nu} + \frac{s^2}{4} (x-y)^{\sigma\rho} \xi_{\sigma\rho} \xi_{\rho\beta} \right) - 2\alpha x \xi_{\mu\nu}. \quad (46) \]

In the case, when \( s \rightarrow +0 \) and \( z = x - y \) is bounded, the Green’s function for the gauge transformed operator can be written as, see [28, 30, 37, 39],
\[ \hat{G}_{2\mu
u}(z;\alpha) = R_0(z)\delta_{\mu\nu} + 2\alpha s\xi_{\mu\nu}R_1(z) - \frac{\alpha^2 s^2\delta_{\mu\nu}\xi_{\sigma\beta}\xi_{\sigma\beta}}{2}\hat{R}_2(z) \]
\[ - \frac{s^2\delta_{\mu\nu}\xi_{\sigma\beta}\xi_{\sigma\beta}|z|^4(1 - 2\alpha^2)}{2^9\pi^2} + o(s^2), \]

where we have used the relation \( \xi_{\mu\nu}\xi_{\sigma\beta} = -\delta_{\mu\nu}\xi_{\rho\beta}\xi_{\rho\beta}/4, \) and

\[ R_0(x) = \frac{1}{4\pi^2|x|^2}, \quad R_1(x) = -\frac{\ln(|x|^2\mu^2)}{16\pi^2}, \quad R_2(x) = \frac{|x|^2(\ln(|x|^2\mu^2) - 1)}{64\pi^2}, \]
which satisfy the following equations

\[ -\partial_{\mu\nu}\partial_{\mu\nu}R_1(x) = R_0(x), \quad -\partial_{\mu\nu}\partial_{\mu\nu}R_2(x) = 2R_1(x) - \frac{1}{16\pi^2}. \]

Now we are ready to formulate the main task of the subsection. We want to find an asymptotic expansion, when \( s \to 0 \) and \( x \) is from a finite neighborhood of \( x = 0 \), of the operator

\[ \rho\left( \sqrt{M_2(x;\alpha)/\Lambda} \right) \hat{G}_2(x;\alpha), \] where \( \rho(r) = \frac{2J_1(r)}{r} \),

up to the \( s^2 \). We have written \( x \) instead of \( z = x - y \), see formulae (46) and (47), because the operators depend on the difference. Hence, for simplicity, we can study the case \( y = 0 \). In the last formula, we understand all the operators as the matrix-valued ones of size \( (4\dim g) \times (4\dim g) \). Our idea means that we need to find the Taylor expansion for the \( \rho \)-operator up to \( s^2 \) and, then, apply this to the Green’s function expansion (47).

Let us write out some useful lemmas and their proofs.

**Lemma 4.** Let \( \Lambda > 0 \) be quite large, \( L = \ln(\Lambda/\mu), A(x) = -\partial_{\mu\nu}\partial_{\mu\nu}, \) and \( \rho(\cdot) \) is the function defined in (50). Then, under the conditions described above, we have the following equalities

\[ \rho\left( \sqrt{A(x)/\Lambda} \right) (|x|^2 + a) = |x|^2 + \Lambda^{-2} + a, \]
and

\[ \rho\left( \sqrt{A(x)/\Lambda} \right) R_i(x) = \bar{R}_i(x), \]

where \( a \in \mathbb{R}, i = 0, 1, 2, \) and \( \bar{R}_i(x) \) are defined as

\[ \bar{R}_0(x) = \frac{1}{4\pi^2} \left\{ \begin{array}{ll}
|x|^2, & |x| > 1/\Lambda; \\
\Lambda^2, & |x| \leq 1/\Lambda,
\end{array} \right. \]

\[ \bar{R}_1(x) = \frac{1}{4\pi^2} \left\{ \begin{array}{ll}
-\frac{1}{2}\ln(|x|^2\mu^2) - \frac{1}{8}|x|^{-2}\Lambda^{-2}, & |x| > 1/\Lambda; \\
\frac{1}{2}L - \frac{1}{8}|x|^2\Lambda^2, & |x| \leq 1/\Lambda,
\end{array} \right. \]

\[ \bar{R}_2(x) = \frac{1}{4\pi^2} \left\{ \begin{array}{ll}
\frac{1}{16}|x|^2(\ln(|x|^2\mu^2) - 1) + \frac{1}{16}\Lambda^{-2}\ln(|x|^2\mu^2) + \frac{1}{32}|x|^{-2}\Lambda^{-4} + \frac{1}{16}\Lambda^{-2}, & |x| > 1/\Lambda; \\
-\frac{1}{8}\Lambda^{-2}L - \frac{1}{8}|x|^2L + \frac{1}{32}|x|^4\Lambda^2 + \frac{1}{16}|x|^2 - \frac{1}{16}\Lambda^{-2}, & |x| \leq 1/\Lambda.
\end{array} \right. \]

**Proof.** The first relation can be obtained quite easily using the series representation for the function \( \rho(\cdot) \) and the fact that we can analyze only the first two terms, because \( |x|^2 + a \) is from the kernel of \( A^4(x) \) for \( k > 1 \). So, we move on to relation (52).
Actually, we need to verify the relation only for \( i = 1, 2 \), because the case \( i = 0 \) was studied in the previous section. Indeed, the function \( R_0(x) \) coincides with the function \( G(|x|) \) from (16), when \( d = 4 \). Hence, we can use all the formulae from section 3.2, and, in particular, formula (30), from which we obtain equality (53), because \( R_0(x) = G(|x|) \).

Then, to simplify calculations, we need to note some useful points. Firstly, the operator \( A(x) \) commutes with \( \rho \left( \sqrt{A(x)}/\Lambda \right) \). Hence, we can expand relations (49) on the functions \( \tilde{R}_i(x) \).

Secondly, the function \( \tilde{R}_0(x) \) is continuous. Therefore, the functions \( \tilde{R}_1(x) \) and \( \tilde{R}_2(x) \) should have the following, additional to (49), properties

\[
\tilde{R}_i(x) \big|_{x=1/\Lambda-0} = \tilde{R}_i(x) \big|_{x=1/\Lambda+0}, \quad \partial_{x_i} \tilde{R}_i(x) \big|_{x=1/\Lambda-0} = \partial_{x_i} \tilde{R}_i(x) \big|_{x=1/\Lambda+0}, \tag{56}
\]

where \( i = 1, 2 \). It is quite easy to verify that the suggested functions satisfy all the conditions mentioned above.

However, the last conditions do not fix one arbitrariness, shift by a constant. To fix this, it is convenient to use the integral representation for \( \rho \)-operator. Additional relations are

\[
\rho \left( \sqrt{A(x)}/\Lambda \right) R_i(x) \big|_{x=0} = R_i(x) \big|_{x=1/\Lambda}, \tag{57}
\]

where \( i = 1, 2, 3 \). Substituting the functions \( \tilde{R}_i(x) \) into the right hand side, we verify the validity of the last relations. Hence, we get the main statement of the lemma.

Further, let us write out the Taylor expansion for the \( \rho \)-operator up to \( s^2 \). For that let us note that three parts of the operator \( M_{2\mu\nu}(x; \alpha) \) have the dependence on the parameter \( s \)

\[
-8\delta_{\mu\nu}x^\alpha \xi_{\rho\sigma} \partial_{x_\sigma} - \delta_{\mu\nu} \frac{s^2}{4} x^\alpha x^\beta \xi_{\rho\beta} - 2s\alpha \xi_{\mu\nu}. \tag{58}
\]

Actually, we are interested only in the second and the third ones, because the first one gives zero after applying to a spherically-symmetric function. For example, to the function \( R_i(x) \).

Hence, we can represent the expansion in the form

\[
\rho \left( \sqrt{M_2(x; \alpha)}/\Lambda \right) \rho_{\mu\nu} = \delta_{\mu\nu} \rho \left( \sqrt{A(x)}/\Lambda \right) + s\rho_{1\mu\nu}(x) + s^2\rho_{2\mu\nu}(x) + (s\rho_{3\mu\nu}(x) + s^2\rho_{4\mu\nu}(x)) x_\sigma \xi_{\rho\beta} \partial_{x_\alpha} + o(s^3), \tag{59}
\]

where we have two functions \( \rho_{1\mu\nu}(x) \) and \( \rho_{2\mu\nu}(x) \), in which we are interested in, and two additional ones \( \rho_{3\mu\nu}(x) \) and \( \rho_{4\mu\nu}(x) \), which do not contribute in our case and, therefore, are not important.

The first function can be obtained in a very simple way using the first derivative with respect to the parameter \( s \). For that let us consider the following chain of equalities

\[
\rho_{1\mu\nu}(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left( -A(x)/4A^2 \right)^{k-1} \left( 2\omega \xi_{\mu\nu}/4A^2 \right)^{n-k} \frac{n!}{(n+2)!} \left( -A(x)/4A^2 \right)^{n-k} \tag{60}
\]

\[
\rho_{1\mu\nu}(x) = \frac{\alpha \xi_{\mu\nu}}{2A^2} \sum_{n=1}^{\infty} \left( -A(x)/4A^2 \right)^{n-1} \frac{(n-1)!}{(n+2)!} \left( -A(x)/A^2 \right)^{n-1} \left( J_2(r)/r^2 \right) \bigg|_{r^2 = -A(x)/A^2}. \tag{61}
\]

Indeed, we can reformulate the last relation using the derivative with respect to the parameter \( \Lambda \) as follows

\[
\rho_{1\mu\nu}(x) = 2\alpha \xi_{\mu\nu} \Lambda^2 A^{-1} \frac{\partial}{\partial \Lambda} \rho \left( \sqrt{A(x)}/\Lambda \right). \tag{62}
\]
Actually, we are interested in both representations, because it is convenient to derive the general answer with the use of the second one, while the first one helps us to fix an integration parameter.

**Lemma 5.** Under the conditions described above, we have

\[ \rho_{\mu
u}(x)\bar{R}_i(x) = 2\alpha\xi_{\mu\nu}\tilde{R}_i(x), \]

where \( i = 0, 1, \) and

\[ \tilde{R}_0(x) = \frac{1}{4\pi^2} \left\{ \begin{array}{ll} \frac{1}{2} |x|^{-2} \Lambda^{-2}, & |x| > 1/\Lambda; \\ \frac{1}{4} - \frac{1}{8}|x|^2\Lambda^2, & |x| \leq 1/\Lambda, \end{array} \right. \]

\[ \tilde{R}_1(x) = \frac{1}{4\pi^2} \frac{1}{16\Lambda^2} \left\{ -\frac{4}{3} \ln(|x|^2\mu^2) - \frac{1}{2} |x|^{-2} \Lambda^{-2}, \right. \]

\[ \left. |x| > 1/\Lambda; \right\} \]

\[ |x| \leq 1/\Lambda. \]

**Proof.** Let us start from formula (62) and apply it to \( R_0(x) \). We will proceed in stages, and firstly we use formulae (52) and (53) from Lemma 4. We have the following chain (without \( \alpha\xi_{\mu\nu} \))

\[ 2\Lambda^2A^{-1}(x)\frac{d}{d\Lambda^2} \left( \sqrt{A(x)/\Lambda} \right) R_0(x) = 2\Lambda^2 \frac{d}{d\Lambda^2} A^{-1}(x)\tilde{R}_0(x) = g(\Lambda) + 2\Lambda^2 \frac{d}{d\Lambda^2} \tilde{R}_1(x) \]

\[ = g(\Lambda) + \frac{1}{4\pi^2} \left\{ \begin{array}{ll} \frac{1}{2} |x|^{-2} \Lambda^{-2}, & |x| > 1/\Lambda; \\ \frac{1}{2} - \frac{1}{4}|x|^2\Lambda^2, & |x| \leq 1/\Lambda, \end{array} \right. \]

where we have introduced an auxiliary function \( g(\Lambda) \). Hence, we need to verify that the function \( g(\Lambda) \) is equal to zero. For that we use representation (61) and the Fourier transform for \( R_0(x) \). Using the transition to the spherical coordinates, we have (without \( \alpha\xi_{\mu\nu} \))

\[ \frac{2}{\Lambda^2(2\pi)^2} \int_{\mathbb{R}^4} d^4ye^{i\nu\alpha\sigma}J_2(|y|/\Lambda)|y|^{-4}\Lambda^2 = \frac{1}{4\pi^2} \int_{R^+} \frac{dr}{r} J_2(r) = \frac{1}{4\pi^2} \frac{1}{2}, \]

from which the equality \( g(\Lambda) = 0 \) follows.

The second relation of the Lemma can be obtained from the previous one with the use of the operator \( A^{-1}(x) \), because, due to the commutativity, we have the relation \( A(x)\bar{R}_i(x) = R_0(x) \), see (49). Then, satisfying additional conditions of smoothness, as it was performed in lemma 4, see (56), we obtain the result. Let us additionally note that the constant of integration can be found by comparison of asymptotics in the region \( |x| \gg 1/\Lambda \), where the function is smooth and \( A^k(x)\bar{R}_i(x) = 0 \) for all \( k > 1 \).

Now we are ready to find the last function under the consideration, see \( \rho_{2\mu\nu} \) in (59). In the case we need to investigate the second derivative with respect to the parameter \( \tilde{s}^2 \). Using the series representation, we can write out the following chain of equations

\[ \rho_{2\mu\nu}(x) = \sum_{n=0}^{\infty} \sum_{k=1}^{n} \left( -\frac{A(x)/4\Lambda^2}{n!\Gamma(n+2)} \right)^{k-1} (|x|^2\delta_{\mu\nu}\xi_{\sigma\beta}/4^3\Lambda^2) \left( -\frac{A(x)/4\Lambda^2}{n!\Gamma(n+2)} \right)^{n-k} \]

\[ - \frac{\alpha^2\delta_{\mu\nu}\xi_{\sigma\beta}/4^3\Lambda^2}{(4\Lambda^2)^2} \sum_{n=2}^{\infty} \left( -\frac{A(x)/4\Lambda^2}{n!\Gamma(n+2)} \right)^{n-2} \sum_{k=1}^{n} (k-1) \]

\[ \frac{1}{n!\Gamma(n+2)} \sum_{k=1}^{n} (k-1) \]
Under the conditions described above, we have

\[
\frac{|x|^2 \delta_{\mu \nu} \xi_{\sigma \beta} \xi_{\sigma \beta}}{2^6 \Lambda^2} \sum_{n=1}^{+\infty} \frac{(-A(x)/4\Lambda^2)^{n-1}}{(n-1)!\Gamma(n+2)} + \frac{\delta_{\mu \nu} \xi_{\sigma \beta} \xi_{\sigma \beta} \chi^{\nu} \partial_{\nu}}{2^8 \Lambda^4} \sum_{n=2}^{+\infty} \frac{(-A(x)/4\Lambda^2)^{n-2}}{(n-2)!\Gamma(n+2)} 
\]

(71)

\[
\frac{\delta_{\mu \nu} \xi_{\sigma \beta} \xi_{\sigma \beta}}{2^6 3^{n-2} \Lambda^4} \sum_{n=2}^{+\infty} \frac{(-A(x)/4\Lambda^2)^{n-2}}{(n-2)!\Gamma(n+1)} - \frac{\alpha^2 \delta_{\mu \nu} \xi_{\sigma \beta} \xi_{\sigma \beta}}{2^{5} \Lambda^4} \sum_{n=2}^{+\infty} \frac{(-A(x)/4\Lambda^2)^{n-2}}{(n-2)!\Gamma(n+2)},
\]

(72)

where we have used the following commutation relation

\[
( - A(x) )^{k-1} |x|^2 = |x|^2 \left( - A(x) \right)^{k-1} + 4(k-1)x^\nu \partial_{\nu} \left( - A(x) \right)^{k-2}
\]

\[
+ 4k(k-1) \left( - A(x) \right)^{k-2},
\]

(73)

and two auxiliary equalities

\[
\sum_{k=1}^{n} (k-1) = \frac{n(n-1)}{2}, \quad \sum_{k=1}^{n} k(k-1) = \frac{(n+1)n(n-1)}{3}.
\]

(74)

Then, we need to apply the operator sums from (71) and (72) to the function \( R_0(x) \). But, firstly, we rewrite the terms in the operator form, preserving their order, as

\[
\rho_{2\mu \nu}(x) = \frac{\delta_{\mu \nu} \xi_{\sigma \beta} \xi_{\sigma \beta}}{2^5} \left( \frac{|x|^2}{2\Lambda^2} + \left( x^\nu \partial_{\nu} - 2\Lambda^4 \frac{\partial}{\partial \Lambda^2} \Lambda^{-2} - 4\alpha^2 \right) A^{-1}(x) \frac{\partial}{\partial \Lambda^2} \right)
\]

\[
\times \sum_{n=1}^{+\infty} \frac{(-A(x)/4\Lambda^2)^{n-1}}{(n-1)!\Gamma(n+2)}.
\]

(75)

**Lemma 6.** Under the conditions described above, we have

\[
\rho_{2\mu \nu}(x) R_0(x) = \frac{\delta_{\mu \nu} \xi_{\sigma \beta} \xi_{\sigma \beta}}{2^5} \left( 2|x|^2 \hat{R}_0(x) + \frac{1}{\Lambda^2} \hat{R}_0(x) - \frac{\alpha^2}{\Lambda^2} \left( \frac{4}{3} \hat{R}_0(x) + \hat{R}_0(x) \right) \right),
\]

(76)

where

\[
\hat{R}_0(x) = \frac{1}{4\pi^2} \left\{ \begin{array}{ll}
0, & |x| > 1/\Lambda; \\
\frac{1}{2} - \frac{1}{4} |x|^2 \Lambda^2 + \frac{1}{2} |x|^4 \Lambda^4, & |x| \leq 1/\Lambda.
\end{array} \right.
\]

(77)

**Proof.** We are going to use some useful formulae derived above. Firstly, using relations (61) and (63)–(65), we get

\[
\sum_{n=1}^{+\infty} \frac{(-A(x)/4\Lambda^2)^{n-1}}{(n-1)!\Gamma(n+2)} R_0(x) = 4\Lambda^2 \hat{R}_0(x),
\]

(78)

\[
A^{-1}(x) \frac{\partial}{\partial \Lambda^2} \sum_{n=1}^{+\infty} \frac{(-A(x)/4\Lambda^2)^{n-1}}{(n-1)!\Gamma(n+2)} R_0(x) = g(\Lambda) + \frac{\partial}{\partial \Lambda^2} 4\Lambda^2 \hat{R}_1(x),
\]

(79)

\[
g(\Lambda) = \frac{1}{4\pi^2} \left\{ \begin{array}{ll}
\frac{1}{2} |x|^2 \Lambda^{-4}, & |x| > 1/\Lambda; \\
\frac{1}{2} \Lambda^{-2} - \frac{1}{2} |x|^2 + \frac{1}{2} |x|^4 \Lambda^2, & |x| \leq 1/\Lambda.
\end{array} \right.
\]

(80)

where \( g(\Lambda) \) is a constant of integration, which is actually equal to zero. This is easy to verify using the Fourier transform of the function at zero. Indeed, the left hand side of (79) at \( x = 0 \) equals
The statements of the theorem follow from acting by operator $\hat{g}$ under the conditions described above, we have

$$\left. \frac{1}{4\Lambda^2} \sum_{n=2}^{\infty} \frac{(-A(x)/4\Lambda)^{n-2}}{(n-2)!\Gamma(n+2)} R_0(x) \right|_{x=0} = \frac{1}{4\Lambda^2} \int_{\mathbb{R}^4} d^4y \frac{8J_3(|y|/\Lambda)}{|y|^3} \Lambda^3$$

and

$$= \frac{1}{4\pi^2\Lambda^2} \int_{\mathbb{R}^4} dr \frac{J_3(r)}{r^3} = \frac{1}{4\pi^2} \frac{1}{8\Lambda^2},$$

where we have used the relation

$$\sum_{n=2}^{\infty} \frac{(-r^2/4)^{n-2}}{(n-2)!\Gamma(n+2)} = \frac{8J_3(r)}{r^3},$$

and transition to the spherical coordinates. Then, applying the differential operator from (75) to (80), we get the final result.

Now we are ready to formulate the final result of this subsection for the Green’s functions mentioned in (39), (47), and (50).

**Theorem 1.** Let $x, y \in \mathbb{R}^4$, $z = x - y$, $\Lambda \gg 0$, and $s$ is a quite small positive number. Then, under the conditions described above, we have

$$\rho \left( \sqrt{M_2(x; \alpha)/\Lambda} \right) G_{2\sigma\nu}(x, y; \alpha) = \Phi(x, y) \hat{G}^\Lambda_{2\mu\nu}(z; \alpha),$$

where

$$\hat{G}^\Lambda_{2\mu\nu}(z; \alpha) = \hat{R}_0(z)\delta_{\mu\nu} + 2\alpha s\xi_{\mu\nu} \left( \hat{R}_0(z) + \hat{R}_1(z) \right)$$

and

$$+ s^2 \delta_{\mu\nu} \xi_{\sigma\beta} \xi_{\sigma\beta} \left( \frac{3|z|^2\Lambda^2 - 2\alpha^2}{2^23\Lambda^2} \hat{R}_0(z) + \frac{1 - \alpha^2}{25\Lambda^4} \hat{R}_0(z) - \alpha^2 \hat{R}_1(z) \right) - \alpha^2 \frac{2}{2^2} \hat{R}_2(z) - \frac{|z|^2 + \Lambda^{-2}}{2^9\pi^2} (1 - 2\alpha^2) + O(s^3).$$

Moreover, as a particular case of the last formula we have results for both initial operators (38) and their Green’s functions (39)

$$\rho \left( \sqrt{M_0(x)/\Lambda} \right) G_0(x, y) = \frac{1}{4} \Phi(x, y) \hat{G}^\Lambda_{2\mu\nu}(z; 0),$$

and

$$\rho \left( \sqrt{M_1(x)/\Lambda} \right) G_{1\sigma\nu}(x, y) = \Phi(x, y) \hat{G}^\Lambda_{2\mu\nu}(z; 1).$$

**Proof.** The statements of the theorem follow from acting by operator (59) on decomposition (47) and using the corresponding lemmas proved above.

4.2. 2D Sigma-model

Now we move on to the next example, a two-dimensional Sigma-model. Moreover, we consider the simplest case, the classical action of which can be represented in the form, see problem statements in [45–47],

$$S[g] = \frac{1}{\gamma^2} \int_{\mathbb{R}^2} d^2x \text{tr} \left( \partial_{\mu} g(x) \left( \partial_{\mu} g^{-1}(x) \right) \right),$$

where $g(x) \in SU(N)$ for $N \in \mathbb{N}$, $\mu \in \{1, 2\}$, and $\gamma$ is a coupling constant.
Investigating the path integral with the classical action and using the background field method, we obtain the standard perturbative expansion. A propagator for such decomposition can be constructed with the use of the following Laplace-type operator

$$M^{ab}(x) = -D^{ab}_{\nu} \nabla_\nu,$$

(91)

where the covariant derivative is defined by the same formula (36), but for the two-dimensional case. The connection components take their values in the Lie algebra $\mathfrak{su}(N)$. Of course, we assume that the main relations from (34) for the generators hold as well.

As it follows from explicit calculations of divergences, the non-zero contribution gives only the quadratic density $B^a_{\mu}(x)B^a_{\nu}(x)$. This means that we can introduce some convenient simplifications. Firstly, we exclude the variable $x$ from the connection components. Hence, we obtain the constant fields, such that $\partial_\nu B_\nu(x) = 0$ for all $\mu$ and $\nu$. Secondly, we add the commutativity, such that the matrices $B_{\mu}$, whose components are $f^{\alpha\beta}_{\mu}B^\alpha_\nu$, commute with each other $[B_\mu, B_\nu] = 0$ for all $\mu$ and $\nu$. Thirdly, we want the connection components to be small. To satisfy the last condition, we multiply them by an auxiliary small parameter $s > 0$.

It is easy to verify that the first two conditions can be satisfied easily. For example, we can take the connection components in the form $B^a_\mu = 1$ for all $a$ and $\mu$.

Let us define two auxiliary matrix-valued operators

$$\Phi_0(x-y) = \exp \left(-s(x-y)^\mu B_\mu/2\right), \ m = -\frac{1}{2} B_\mu B_\mu. \quad (92)$$

Hence, after performing the transition to the Fock–Schwinger gauge condition, as it was made in (45), we obtain the following new operator under the study

$$\hat{M}(x) = \Phi_0(y-x)M(x)\Phi_0(x-y) = -\partial_\mu \partial_\nu - s^2 m. \quad (93)$$

Green’s functions, corresponding to the last operators, have the following expansions, see [30, 37, 39],

$$\hat{G}(x-y) = R_0(x-y) + s^2 m R_1(x-y) + o(s^2), \ G(x-y) = \Phi_0(x-y)\hat{G}(x-y), \quad (94)$$

where we have used the two-dimensional versions for the functions

$$R_0(x) = -\frac{1}{4\pi} \ln(|x|^2\mu^2), \ R_1(x) = \frac{|x|^2 (\ln(|x|^2\mu^2) - 2)}{16\pi}, \quad (95)$$

which satisfy the following equations

$$A(x)R_0(x) = \delta(x), \ A(x)R_1(x) = R_0(x), \ A(x) = -\partial_\mu \partial_\nu. \quad (96)$$

Now we are ready to formulate and prove the main result of this subsection.

**Theorem 2.** Let $x,y \in \mathbb{R}^2$, $\Lambda \gg 0$, $s \to +0$, and $\rho(r) = J_0(|r|)$. Then, under the conditions described above, we have

$$\rho \left( \sqrt{M(x)/\Lambda} \right) G(x-y) = R_0(x-y) + s^2 m \left( \hat{R}_1(x-y) + \hat{R}_0(x-y) \right) + o(s^2), \quad (97)$$

and

$$\rho \left( \sqrt{M(x)/\Lambda} \right) G(x-y) = \Phi_0(x-y)\rho \left( \sqrt{M(x)/\Lambda} \right) \hat{G}(x-y), \quad (98)$$

where

$$\hat{R}_0(x) = \rho \left( \sqrt{A(x)/\Lambda} \right) R_0(x) = \frac{1}{4\pi} \left\{ \begin{array}{ll} -\ln(|x|^2\mu^2), & |x| > 1/\Lambda; \\ 2L, & |x| < 1/\Lambda, \end{array} \right.$$

(99)
\[ \hat{R}_1(x) = \rho \left( \sqrt{x} / \Lambda \right) R_1(x) = \frac{1}{4\pi} \left\{ \begin{array}{ll} \frac{1}{2} |x|^2 \left( \ln(|x|^2 \mu^2) - 2 \right) + \frac{1}{2} \Lambda^{-2} \ln(|x|^2 \mu^2), & |x| > 1/\Lambda; \\ -\frac{1}{2}(L+1)\Lambda^{-2} - \frac{1}{2}L|x|^2, & |x| \leq 1/\Lambda, \end{array} \right. \]  
\[ \hat{R}_0(x) = m^{-1} \frac{\partial}{\partial x^2} R_0(x) = \frac{1}{4\pi} \left\{ \begin{array}{ll} -\frac{1}{2} \Lambda^{-2} \ln(|x|^2 \mu^2), & |x| > 1/\Lambda; \\ \frac{1}{2}L\Lambda^{-2} + \frac{1}{4}(\Lambda^{-2} - |x|^2), & |x| \leq 1/\Lambda. \end{array} \right. \]  

**Proof.** The statement from (98) follows from formula (97) with the use of the second equality from (94). Hence, we need to verify only (97), which, actually, can be obtained by the same steps that were performed in the previous section. Let us proceed in stages.

Firstly, formula (99) is the consequence of lemmas 1, 2, and 3, because in the two-dimensional space \( R_0(x) \) coincides with \( G(|x|) \) from (16), and \( \hat{R}_0(x) \) coincides with \( G(|x|/\Lambda) \).

Secondly, formula (100) can be obtained using the second equality from (96) and an explicit integration with the use of the following smoothness conditions

\[ \hat{R}_1(x) \big|_{|x|=1/\Lambda-0} = \hat{R}_1(x) \big|_{|x|=1/\Lambda+0}; \partial_{x^2} \hat{R}_1(x) \big|_{|x|=1/\Lambda-0} = \partial_{x^2} \hat{R}_1(x) \big|_{|x|=1/\Lambda+0}. \]  

Then, fixing the integration constant by comparing the asymptotics in the region \( |x| \gg 1/\Lambda \), we get the result formulated above. Finally, repeating all the last calculations with the use of (62), where we have substituted \( \hat{s}^2 m \) instead of \( 2\alpha \xi_{\mu\nu} \), we get the last statement of the theorem.

5. Conclusion

In this paper, we have studied the explicit cutoff regularization in the coordinate representation, see (2). We constructed it in such a way that three additional conditions would be satisfied: the spectral representation, the homogenization (special type of an integral representation), and the covariance, if it exists before the regularization introduced. All these restrictions were achieved on the example of the standard Laplace operator \( -\partial_{x^\mu} \partial^\mu \) for an arbitrary dimension value, see formula (30), and on the example of two auxiliary operators, appearing in the four-dimensional Yang–Mills theory and in the two-dimensional Sigma-model, see section 4 and theorems 1 and 2.

Of course, we need to give some comments about a behavior of (85)–(87) and (97) with respect to the auxiliary parameter \( s \). Indeed, we have not proven any result about their convergence, so we supposed that they are asymptotic. Finding an operator norm, with respect to which the convergence would be valid, is a separate task and this is out of the scope of our paper.

In our opinion, these results can be useful in multi-loop calculations in the case of the pure four-dimensional Yang–Mills theory \([35, 42, 43, 48]\) and the two-dimensional Sigma-model. The answers described above contain all the necessary terms of the infrared expansion, needed to study the divergent parts.

Let us note that at the very beginning of the paper, we have fixed the view of the regularization under study, see (6). This restriction is very strong, because it controls the form of the spectral function \( \rho(r) \), see formulae (3) and (20). Actually, we can abandon this restriction and define the spectral function in an arbitrary convenient way, not for the explicit cutoff regularization. For example, we can take \( \rho(r) = \exp(-r^2) \) or the case with the Heaviside step function (explicit cutoff regularization in the spectral representation \([49, 50]\)), which does not depend on the dimension value. Anyway, each particular case should be studied separately, and this problem is the subject of further research.
Additionally, it would be useful to expand the result on the case of potentials \( v(x) \), depending on the variable \( x \), see (31). Such generalizations are useful in study the multidimensional cases to make them more convenient and simple. Fortunately, the main standard models can be investigated using the methods described above. For example, we can study a \( \phi^3 \)-model in the six-dimensional space or a \( \phi^4 \)-model in the four-dimensional space, which lead to the Laplace operator \(-\partial_\mu \partial^\mu - v(x)\) with a smooth potential, depending on \( x \). In such situations, we can consider only several terms from the Taylor expansion of \( v(x) \).

Data availability statement

No new data were created or analysed in this study.

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