Dimensional crossover with a continuum of critical exponents for NLS on doubly periodic metric graphs

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Abstract

We investigate the existence of ground states for the focusing nonlinear Schrödinger equation on a prototypical doubly periodic metric graph. When the nonlinearity power is below 4, ground states exist for every value of the mass, while, for every nonlinearity power between 4 (included) and 6 (excluded), a mark of $L^2$-criticality arises, as ground states exist if and only if the mass exceeds a threshold value that depends on the power. This phenomenon can be interpreted as a continuous transition from a two-dimensional regime, for which the only critical power is 4, to a one-dimensional behavior, in which criticality corresponds to the power 6. We show that such a dimensional crossover is rooted in the coexistence of one-dimensional and two-dimensional Sobolev inequalities, leading to a new family of Gagliardo-Nirenberg inequalities that account for this continuum of critical exponents.

1 Introduction

Since the first appearance of branched structures in the modelization of organic molecules [22], through the development of the mathematical theory of quantum graphs [9, 21], networks (or metric graphs) have provided a general and flexible tool to describe dynamics in complex structures like systems of quantum wires, Josephson junctions, propagation of signals through waveguides, and some related technologies. Pioneering studies about nonlinear systems on metric graphs appeared in [7–8], but more recently the research on such topics has grown rapidly, and several results have been achieved on propagation of solitary waves [11, 13, 24] and on stationary states (23, 12, 18, 19, 20, 16). In a series of recent works (2, 3, 4) we investigated the problem of existence of ground states for the energy functional associated to the focusing, $L^2$-subcritical and critical nonlinear Schrödinger (NLS) equation

\[ i\partial_t u(t) = -u''(t) - |u(t)|^{p-2}u(t) \]

on finite non-compact metric graphs, i.e. branched structures with a finite number of vertices and edges, and at least one infinite edge (i.e. a half-line).
Specifically, by ground state on a metric graph $G$ we mean every global minimizer of the energy functional

$$E_p(u) = \frac{1}{2} \int_G |u'|^2 \, dx - \frac{1}{p} \int_G |u|^p \, dx,$$

in the class of $H^1(G)$ functions with fixed $L^2$-norm (or mass) $\mu > 0$. The constraint is dynamically meaningful as the mass, as well as the energy, is conserved by the NLS flow, and the problem of the existence of ground states is particularly relevant in the physics of Bose-Einstein condensates (see e.g. Section 1 in [6] and [2, 3, 4]).

In this paper we extend the analysis of the existence of ground states to a prototypical doubly periodic metric graph $G$, particularly relevant in the applications, for which the techniques developed in previous works (where non-compactness was due to one or more unbounded edges) do not apply: a two-dimensional infinite grid isometrically embedded in $\mathbb{R}^2$, with vertices on the lattice $\mathbb{Z}^2$ and edges of unit length (see Figure 1).

It appears, roughly speaking, that macroscopically the grid $G$ has dimension two, while microscopically it is of dimension one. This peculiarity is absent in graphs with a finite number of half-lines, where the two-dimensional scale is lacking, as well as in other two-dimensional structures as $\mathbb{Z}^2$, where edges are missing and there is of course no microscopic one-dimensional structure [25]. The presence of two scales in $G$ results in a transition from a one-dimensional to a two-dimensional behavior, that emerges in functional inequalities and influences the existence of ground states. We shall refer to this phenomenon as dimensional crossover.

Before commenting further on this point, it is convenient to state our main results in a precise form. We define, for $\mu > 0$, the mass-constrained set

$$H^1_\mu(G) = \left\{ u \in H^1(G) : \int_G |u|^2 \, dx = \mu \right\},$$

and the corresponding “ground-state energy level”

$$E_p(\mu) = \inf_{u \in H^1_\mu(G)} E_p(u),$$
considered as a function $\mathcal{E}_p : (0, +\infty) \rightarrow \mathbb{R} \cup \{-\infty\}$ of the mass $\mu$. By a “ground states of mass $\mu$” we mean a function $u \in H^1_\mu(G)$ such that

$$E_p(u) = \mathcal{E}_p(\mu).$$

When $p \in (2, 4)$, ground states exist for every prescribed mass.

**Theorem 1.1** (Subcritical case). Assume $2 < p < 4$. Then for every $\mu > 0$ there exists a ground state of mass $\mu$, and $\mathcal{E}_p(\mu) < 0$.

The picture changes as the exponent of the nonlinearity increases.

**Theorem 1.2** (Dimensional crossover). For every $p \in [4, 6]$ there exists a critical mass $\mu_p > 0$ such that

(i) If $p \in (4, 6)$ then ground states of mass $\mu$ exist if and only if $\mu \geq \mu_p$, and

$$\mathcal{E}_p(\mu) \begin{cases} = 0 & \text{if } \mu \leq \mu_p \\ < 0 & \text{if } \mu > \mu_p. \end{cases}$$

(ii) If $p = 4$ then ground states of mass $\mu$ exist if $\mu > \mu_4$, whereas they do not exist if $\mu < \mu_4$. Moreover (5) is valid also when $p = 4$.

(iii) If $p = 6$ then there are no ground states, regardless of the value of $\mu$, and

$$\mathcal{E}_6(\mu) \begin{cases} = 0 & \text{if } \mu \leq \mu_6 \\ = -\infty & \text{if } \mu > \mu_6. \end{cases}$$

We point out that, when $p = 4$, the existence of ground states of mass $\mu = \mu_4$ is still an open problem. For the sake of completeness, we also mention that when $p > 2$ one has $\mathcal{E}_p(\mu) \equiv -\infty$ for every $\mu$, as one can easily see by a scaling argument.

In order to interpret Theorems 1.1 and 1.2 let us recall that in $\mathbb{R}^d$, for the minimization of the NLS energy under a mass constraint, there exists a critical exponent $p^*_d$ such that

1. if $p < p^*_d$, for every mass $\mu > 0$ the ground-state energy level is finite and negative, and is attained by a ground state;

2. if $p > p^*_d$, for every mass $\mu > 0$ the ground-state energy level equals $-\infty$.

It is well-known (1.4) that $p^*_d = \frac{4}{d} + 2$ for the NLS in $\mathbb{R}^d$, yielding $p^*_1 = 6$ for $\mathbb{R}$ and $p^*_2 = 4$ for $\mathbb{R}^2$. Furthermore, it has been proved in [2, 3] that for finite non-compact graphs (i.e. graphs with finitely many edges, at least one of them being unbounded) the critical exponent is 6, exactly as for $\mathbb{R}$. Thus the exponents considered in Theorem 1.1 are subcritical both in dimension one and two, which reflects into the typical subcritical flavor of the result.

In fact, the main novelty of the paper emerges in Theorem 1.2 and lies in the “splitting” of the critical exponent $p^*_G$ induced by the twofold nature (one/two dimensional) of the grid. Indeed, on the grid $G$:

1. $p = 4$ is the supremum of those exponents $p$ such that $\mathcal{E}_p(\mu)$ is finite and negative (and attained by a ground state) for every $\mu > 0$;
2. \( p = 6 \) is the infimum of those exponents \( p \) such that \( E_p(\mu) = -\infty \) for every \( \mu > 0 \).

Besides, let us stress another remarkable aspect of the dimensional crossover. In \( \mathbb{R}^d \), as well as on non-compact finite graphs, the critical exponent is characterized by the existence of a critical mass in the following sense: for smaller masses every function has positive energy, while for larger masses there are functions with negative energy (as already mentioned, on a non-compact finite graph such a critical mass arises only when \( p = 6 \)).

On the contrary, on the grid \( G \) a similar notion of critical mass (the number \( \mu_p \) in Theorem 1.2) arises for every \( p \in [4, 6] \), so that, in this respect, every exponent within this range is, in fact, critical (see Remark 2.5). Beyond this critical mass, however, the energy is still bounded from below and a ground state exists, as if the problem had kept trace of the subcriticality of the exponent \( p < 6 \) at the microscopic scale.

From the point of view of functional analysis, the dimensional crossover is due to the simultaneous validity, for every function \( u \in W^{1,1}(G) \), of the two inequalities

\[
\|u\|_{L^\infty(G)} \leq \|u'\|_{L^1(G)}, \quad \|u\|_{L^2(G)} \leq \|u'\|_{L^1(G)}.
\]

Of these, the former is typical of dimension one, on the model of the well known inequality

\[
\|v\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2} \|v'\|_{L^1(\mathbb{R})}, \quad \forall v \in W^{1,1}(\mathbb{R}),
\]

while the latter is the formal analogue of the Sobolev inequality in \( \mathbb{R}^2 \)

\[
\|v\|_{L^2(\mathbb{R}^2)} \leq C \|\nabla v\|_{L^1(\mathbb{R}^2)}, \quad \forall v \in W^{1,1}(\mathbb{R}^2),
\]

and is typical of dimension two. As discussed in Section 2, either inequality in (7) entails a particular version of the Gagliardo-Nirenberg inequality in \( H^1(G) \) (12 and 13 respectively). By interpolation, one obtains the critical Gagliardo-Nirenberg inequalities

\[
\int_G |u|^p \, dx \leq K_p \left( \int_G |u|^2 \, dx \right)^{\frac{p-2}{2}} \int_G |u'|^2 \, dx, \quad \forall u \in H^1(G)
\]

which, being valid for every exponent \( p \in [4, 6] \), give rise to a continuum of critical exponents (see also Remark 2.5). Indeed, using (9), the NLS energy in (2) can be estimated from below as

\[
E_p(u) \geq \frac{1}{2} \left( 1 - \frac{2K_p \mu_p}{p} \right) \int_G |u'|^2 \, dx
\]

which shows that \( E_p(u) \geq 0 \) for every \( u \in H^1_p(G) \), as soon as

\[
\mu \leq \left( \frac{p}{2K_p} \right)^\frac{1}{p-2}.
\]

The number in the right-hand–side of this inequality is the critical mass \( \mu_p \) of Theorem 1.2.
Finally we would like to point out that we have chosen the grid $G$ to illustrate our results because it is the simplest doubly periodic metric graph, on which computations and proofs are particularly transparent. It should be clear however that many other doubly periodic graphs can be treated with the methods developed in the present work. Among these, we explicitly mention the hexagonal grid, a model for graphene.

At the core of the results stands the double periodicity of the graph, that is responsible for the occurrence of phenomena such as the dimensional crossover. To exploit the double periodicity on a concrete given graph one might of course have to alter some parts of the proofs presented in this paper to adapt them to the particular features of the graph under study. These modifications are by no means substantial, being of a technical nature. We plan to illustrate this with the detailed study of some other particular graphs, significatively relevant for the applications, in forthcoming papers.

2 Inequalities

In this section we establish some fundamental inequalities for functions on the grid.

For notational purposes, it is convenient to describe the grid $G$ as isometrically embedded in $\mathbb{R}^2$, with the lattice $\mathbb{Z}^2$ as set of vertices, and an edge of length one joining every pair of adjacent vertices. In this way, it is natural to interpret $G$ as the union of horizontal lines $\{H_j\}$ and vertical lines $\{V_k\}$, which cross at every vertex $(k, j) \in \mathbb{Z}^2$.

As on any metric graph, to deal with the energy functional (2), the natural functional framework is given by the standard spaces $L^p(G)$ and $H^1(G)$. With the notation for $G$ introduced above, for the $L^p$ norms we have

$$\|u\|_{L^p(G)}^p = \sum_{j \in \mathbb{Z}} \|u\|_{L^p(H_j)}^p + \sum_{k \in \mathbb{Z}} \|u\|_{L^p(V_k)}^p$$

$$= \sum_{j \in \mathbb{Z}} \int_{H_j} |u(x)|^p \, dx + \sum_{k \in \mathbb{Z}} \int_{V_k} |u(x)|^p \, dx < \infty$$

and

$$\|u\|_{L^\infty(G)} = \sup_{j,k} \left\{ \|u\|_{L^\infty(H_j)}, \|u\|_{L^\infty(V_k)} \right\},$$

while

$$\|u\|^2_{H^1(G)} = \|u\|^2_{L^2(G)} + \|u'\|^2_{L^2(G)}.$$ 

Here, as usual, $H^1(G)$ denotes the space of functions on $G$ whose restriction to every horizontal and vertical line belongs to $H^1(\mathbb{R})$, and that, in addition, are continuous at every vertex of $G$. In Theorem 2.2 we shall also need the space $W^{1,1}(G)$, similarly defined as the space of functions on $G$ whose restriction to every horizontal and vertical line belongs to $W^{1,1}(\mathbb{R})$ and that, in addition, are continuous at every vertex.

Remark. In the following, symbols like $\|u\|_p$ stand for $\|u\|_{L^p(G)}$. When the domain of integration is different from $G$, it will always be indicated in the norm.
First we recall the standard Gagliardo-Nirenberg inequality, which (up to a multiplicative constant $C > 1$ on the right-hand side) is valid on any noncompact metric graph (a proof in the general framework can be found in [4]). Here, for the sake of completeness, we shall give a short proof tailored to the grid $G$ which, by the way, yields a slightly sharper estimate.

**Theorem 2.1** (One-dimensional Gagliardo-Nirenberg inequality). For every $p \in [2, \infty)$ one has

\begin{equation}
\|u\|_p \leq \|u\|_2^{\frac{1}{2}} \|u'\|_2^{\frac{1}{2}} \quad \forall u \in H^1(G)
\end{equation}

and, moreover,

\begin{equation}
\|u\|_\infty \leq \|u\|_2^{\frac{1}{2}} \|u'\|_2^{\frac{1}{2}} \quad \forall u \in H^1(G).
\end{equation}

**Proof.** Since $\|u\|_p \leq \|u\|_\infty^{\frac{2}{p}} \|u\|_2^{\frac{2}{p}}$, it suffices to prove (13). On the other hand, given $u \in H^1(G)$, we have $u^2 \in W^{1,1}(H_j)$ for every horizontal line $H_j$ of $G$. Then, applying (8) with $v = u^2$ on $H_j$ yields

$$
\|u\|_{L^\infty(H_j)} \leq \int_{H_j} |u(x)u'(x)| \, dx \leq \|u\|_{L^2(H_j)} \|u'\|_{L^2(H_j)} \leq \|u\|_{L^2(G)} \|u'\|_{L^2(G)}.
$$

Since clearly this inequality remains true if we replace $H_j$ with any vertical line $V_k$, (13) follows immediately from (11).

As already mentioned, inequalities like (12) and (13) hold for every noncompact graph. On the contrary the next inequality, and its consequences below, rely on the two-dimensional web structure of the grid $G$.

**Theorem 2.2** (Two-dimensional Sobolev inequality). For every $u \in W^{1,1}(G)$,

\begin{equation}
\|u\|_2 \leq \frac{1}{2} \|u'\|_1.
\end{equation}

**Proof.** Given $u \in W^{1,1}(G)$, we have

\begin{equation}
\|u\|_2^2 = \sum_{j \in \mathbb{Z}} \int_{H_j} |u(x)|^2 \, dx + \sum_{k \in \mathbb{Z}} \int_{V_k} |u(y)|^2 \, dy.
\end{equation}

First observe that, for each $k$, using (8) we obtain

\begin{equation}
\int_{V_k} |u(y)|^2 \, dy \leq \|u\|_{L^\infty(V_k)} \int_{V_k} |u(y)| \, dy \leq \frac{1}{2} \|u'\|_{L^1(V_k)} \int_{V_k} |u(y)| \, dy.
\end{equation}

Then, for each $j \in \mathbb{Z}$, consider the horizontal lines $H_j$ and $H_{j+1}$, and denote by $P_j$ the path in $G$ obtained by joining together the halfline of $H_j$ to the left of $V_k$, the vertical segment of $V_k$ between $H_j$ and $H_{j+1}$ (which we denote by $I_j$), and the halfline of $H_{j+1}$ to the right of $V_k$ (see Figure 2).

Since in particular $u \in W^{1,1}(P_j)$, and the metric graph $P_j$ is isometric to $\mathbb{R}$, we find from (8)

$$
|u(y)| \leq \frac{1}{2} \int_{P_j} |u'(x)| \, dx \quad \forall y \in I_j.
$$
Figure 2: The path $P_j$ (thick in the picture).

and, since $I_j$ has length one, integrating this inequality over $I_j$ yields

$$\int_{I_j} |u(y)| \, dy \leq \frac{1}{2} \int_{P_j} |u'(x)| \, dx, \quad \forall j \in \mathbb{Z}. \quad (17)$$

Now observe that

$$V_k = \bigcup_{j \in \mathbb{Z}} I_j, \quad \bigcup_{j \in \mathbb{Z}} P_j = V_k \cup \bigcup_{j \in \mathbb{Z}} H_j,$$

and moreover, up to a negligible set, the paths $\{P_j\}$ $(j \in \mathbb{Z})$ are mutually disjoint: therefore, summing (17) over $j \in \mathbb{Z}$ yields

$$\int_{V_k} |u(y)| \, dy \leq \frac{1}{2} \left( \int_{V_k} |u'(y)| \, dy + \sum_j \int_{H_j} |u'(x)| \, dx \right) = \frac{1}{2} \left( v_k + \sum_j h_j \right)$$

having set, for brevity, $v_k = \int_{V_k} |u'(y)| \, dy$ and $h_j = \int_{H_j} |u'(x)| \, dx$. Combining with (16), and summing over $k$, one obtains

$$\sum_k \int_{V_k} |u(y)|^2 \, dy \leq \frac{1}{4} \sum_k v_k \left( v_k + \sum_j h_j \right).$$

Of course, by the symmetry of $\mathcal{G}$, we also have

$$\sum_j \int_{H_j} |u(x)|^2 \, dx \leq \frac{1}{4} \sum_j h_j \left( h_j + \sum_k v_k \right),$$

and summing the last two inequalities we find

$$\|u\|^2_{L^2(\mathcal{G})} \leq \frac{1}{4} \left( \sum_k (h_k^2 + v_k^2) + 2 \sum_{j,k} h_j v_k \right) \leq \frac{1}{4} \left( \sum_k h_k + v_k \right)^2 = \frac{1}{4} \|u'\|^2_{L^1(\mathcal{G})}.$$
Theorem 2.3 (Two-dimensional Gagliardo-Nirenberg inequality). For every $p \in [2, \infty)$ one has

$$\|u\|_p \leq C \|u\|_2^{\frac{p}{2}} \cdot \|u'\|_2^{1-\frac{p}{2}} \quad \forall u \in H^1(\mathcal{G}),$$

where $C$ is an absolute constant.

Proof. Given $p \in [2, \infty)$, we have

$$\|u\|_p \leq \|u\|_2^{1-\theta} \cdot \|u\|_{p+2}^\theta$$

where

$$\frac{1}{2} - \frac{\theta}{p+2} + \frac{\theta}{p + 2} = \frac{1}{p}, \quad \text{i.e.} \quad \theta = 1 - \frac{4}{p^2}.$$

Now observe that $u \in L^\infty(\mathcal{G})$ by (13), and hence $u^{1+p/2}$ belongs to $W^{1,1}(\mathcal{G})$ since $p \geq 2$. Therefore, we can replace $u$ with $u^{1+p/2}$ in (14), thus obtaining

$$\|u\|_{p+2}^{1+\frac{\theta}{p}} \leq \frac{p + 2}{4} \int_\mathcal{G} |u(x)|^{\frac{p}{2}} |u'(x)| \, dx \leq \frac{p + 2}{4} \|u\|_p^{\frac{p}{2}} \cdot \|u'\|_2.$$

Raising to the power $2/(p+2)$ we find

$$\|u\|_{p+2} \leq C \|u\|_2^{\frac{p+2}{4}} \cdot \|u'\|_2^{\frac{p+2}{4}}, \quad C = \sup_{p \geq 2} \left( \frac{p + 2}{4} \right)^{\frac{p}{p+2}}$$

(one may take e.g. $C = 3/2$). Plugging this inequality into (19) gives

$$\|u\|_p \leq \|u\|_2^{1-\theta} \cdot \|u\|_{p+2}^{\theta} \cdot \|u'\|_2^{\theta},$$

and (18) follows using (20), after elementary computations.

Corollary 2.4 (Interdimensional Gagliardo-Nirenberg inequality). There exists a universal constant $C > 0$ such that, for every $p \in [2, \infty)$,

$$\|u\|_p \leq C \|u\|_2^{1-\alpha} \|u'\|_2^{\alpha} \quad \forall \alpha \in \left[ \frac{p - 2}{2p}, \frac{p - 2}{p} \right], \quad \forall u \in H^1(\mathcal{G}).$$

In particular, for every $p \in [4, 6]$ there exists a constant $K_p$, depending only on $p$, such that

$$\|u\|_p \leq K_p \|u\|_2^{\frac{p}{2}} \|u'\|_2^{\frac{p}{2}} \quad \forall u \in H^1(\mathcal{G}).$$

Proof. Observe that (21) reduces to (12) when $\alpha = \frac{2 - p}{2p}$, while it reduces to (18) when $\alpha = \frac{p - 2}{p}$. Then (21) is established also for every intermediate value of $\alpha$, since the right-hand side is a convex function of $\alpha$.

Finally, when $p \in [4, 6]$, (22) is obtained letting $\alpha = 2/p$ in (21) (the condition $p \in [4, 6]$ guarantees that this choice of $\alpha$ is admissible).
Remark 2.5. In \( \mathbb{R}^d \), when dealing with the NLS energy
\[
\frac{1}{2} \| \nabla u \|^2_{L^2(\mathbb{R}^d)} - \frac{1}{p} \| u \|^p_{L^p(\mathbb{R}^d)}
\]
in presence of an \( L^2 \) mass constraint, the relevant version of the Gagliardo-Nirenberg (G-N) inequality is
\[
(23) \quad \| u \|_{L^p(\mathbb{R}^d)} \leq c \| u \|^{1-\alpha}_{L^2(\mathbb{R}^d)} \| \nabla u \|^\alpha_{L^2(\mathbb{R}^d)}, \quad \alpha = \frac{d(p-2)}{2p},
\]
valid as soon as \( \alpha \in [0,1) \) (see [17]). When \( p = 2 + 4/d \), this inequality becomes \( \text{critical} \) for the NLS energy because \( \alpha = 2/p \) (i.e. the exponents in the inequality become as in (22)), and a critical mass \( \mu_p \) comes into play. Now, while in (23) this critical exponent \( p = 2 + 4/d \) is uniquely determined by the ambient space \( \mathbb{R}^d \), on the grid \( G \) every \( p \in [4,6] \) is critical for the NLS energy, since one can let \( \alpha = 2/p \) in (21) (and obtain (22)) not just for one particular \( p \), but for every \( p \in [4,6] \).

Formally, solving for \( d \) in (23), for fixed \( \alpha \) we can interpret (21) as a G-N inequality in dimension \( d = \frac{2\alpha p}{p-2} \): we call (21) \( \text{interdimensional} \) since \( d \) ranges over \( [1,2] \), as \( \alpha \) varies (this is in contrast with (23), where the exponent \( \alpha \) is uniquely determined by \( p \) and the space dimension \( d \)). With this interpretation (22) (which is just (21) with \( \alpha = 2/p \)) can be seen as a critical G-N inequality in dimension \( d = 4/(p-2) \) so that, formally, every \( p \in [4,6] \) can be seen as the critical exponent \( p = 2 + 4/d \), in a fractal scaling dimension \( d \in [1,2] \).

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Remark 3.1. Note that, for every \( \mu > 0 \) and \( p < 6 \), the one-dimensional Gagliardo-Nirenberg (12) ensures that \( E_p(\mu) \) is finite and \( E_p \) is coercive on \( H^1(\mathbb{R}) \) ([11]).

Recalling (3) and (4), we first prove a dichotomy lemma for minimizing sequences, useful to prove the existence of ground states.

Lemma 3.2 (Dichotomy). Given \( \mu > 0 \) and \( p \in (2,6) \), let \( \{u_n\} \subset H^1(\mathbb{R}) \) be a minimizing sequence for \( E_p \), i.e.
\[
\lim_{n \to \infty} E_p(u_n) = E_p(\mu),
\]
and assume that \( u_n \rightharpoonup u \) weakly in \( H^1(\mathbb{R}) \) and pointwise a.e. on \( \mathcal{G} \). If
\[
(24) \quad m := \mu - \| u \|^2_2 \in [0,\mu]
\]
denotes the loss of mass in the limit, then either \( m = 0 \) or \( m = \mu \).

Proof. We assume that \( 0 < m < \mu \) and seek a contradiction. According to the Brezis–Lieb Lemma (11), we can write
\[
E_p(u_n) = E_p(u_n - u) + E_p(u) + o(1) \quad \text{as} \ n \to \infty,
\]
and, since $u_n \to u$ in $L^2(\mathcal{G})$, 
\begin{equation}
\|u_n - u\|_2^2 = \|u_n\|_2^2 + \|u\|_2^2 - 2\langle u_n, u \rangle_2 \to \mu - \|u\|_2^2 = m
\end{equation}
as $n \to \infty$. Now, for $n$ large enough,
\[
E_p(\mu) \leq E_p\left(\frac{\sqrt{\mu}}{\|u_n - u\|_2} (u_n - u)\right)
= \frac{1}{2} \frac{\mu}{\|u_n - u\|_2^2} \|u'_n - u'\|_2^2 - \frac{1}{p} \frac{\mu^{p/2}}{\|u_n - u\|_2^p} \|u_n - u\|_p^p
< \frac{\mu}{\|u_n - u\|_2^2} E_p(u_n - u),
\]
since $\|u_n - u\|_p^p \neq 0$ and $\|u_n - u\|_2^2 < \mu$. Thus,
\[
E_p(u_n - u) > \frac{\|u_n - u\|_2^2}{\mu} E_p(\mu),
\]
and by (26)
\[
\liminf_n E_p(u_n - u) \geq \frac{m}{\mu} E_p(\mu).
\]
Thus, taking the liminf in (25) we find
\begin{equation}
E_p(\mu) \geq \frac{m}{\mu} E_p(\mu) + E_p(u).
\end{equation}
Similarly, since $u \neq 0$ we also have
\begin{equation}
E_p(\mu) \leq \frac{1}{2} \frac{\mu}{\mu - m} \|u'_n\|_2^2 - \frac{1}{p} \left(\frac{\mu}{\mu - m}\right)^{\frac{2}{p}} \|u\|_p^p < \frac{\mu}{\mu - m} E_p(u)
\end{equation}
and, as $E_p(\mu) > -\infty$ by Remark 3.1 from (27) we finally obtain
\[
E_p(\mu) > \frac{m}{\mu} E_p(\mu) + \frac{\mu - m}{\mu} E_p(\mu) = E_p(\mu),
\]
a contradiction. \hfill \Box

**Proposition 3.3.** Assume that $p < 6$ and that $E_p(\mu)$ is strictly negative. Then there exists $u \in H^1_\mu(\mathcal{G})$ such that
\[
E_p(u) = E_p(\mu).
\]

**Proof.** Let $\{u_n\} \subset H^1_\mu(\mathcal{G})$ be a minimizing sequence for $E_p$. Since $p < 6$, Remark 3.1 yields that $E_p(\mu) > -\infty$ and $u_n$ is bounded in $H^1(\mathcal{G})$, and by translating each $u_n$ (exploiting the periodicity of $\mathcal{G}$) we can also assume that $u_n$ attains its $L^\infty$-norm on a compact set $\mathcal{K} \subset \mathcal{G}$ independent of $n$. Therefore, up to subsequences, $u_n$ converges weakly in $H^1(\mathcal{G})$, and strongly in $L^\infty(\mathcal{G})$, to some function $u \in H^1(\mathcal{G})$. Setting $m := \mu - \|u\|_2^2$, from Lemma 3.2 one sees that either $m = 0$ or $m = \mu$. If $m = \mu$ then $u \equiv 0$, but in this case $u_n \to 0$ in $L^\infty(\mathcal{G})$, since in particular, $u_n \to u \equiv 0$ uniformly on $\mathcal{K}$. Therefore we would have
\[
E_p(u_n) \geq -\frac{1}{p} \|u_n\|_\infty^{p-2} \int_\mathcal{G} |u_n|^2 \, dx = -\frac{\mu}{p} \|u_n\|_\infty^{p-2} \to 0,
\]

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contradicting the fact that $\mathcal{E}_p(\mu) < 0$.

Thus it must be $m = 0$, so that $u_n \to u$ strongly in $L^2(G)$ and therefore $u \in H^1_p(G)$. Moreover, since $u_n$ is bounded in $L^\infty(G)$, $u_n \to u$ strongly also in $L^p(G)$. Then

$$E_p(u) \leq \liminf_n E_p(u_n) = \mathcal{E}_p(\mu)$$

by weak lower semicontinuity, and the proof is complete.

**Remark 3.4.** It is interesting to compare Proposition 3.3 with Theorem 3.3 in [4]. According to that result, in a finite non-compact graph the energy threshold under which the existence of a ground state of a given mass is guaranteed equals the energy of the soliton on $\mathbb{R}$ with the same mass. On the contrary, on the grid $G$ the absence of half-lines and the periodicity pushes the energy threshold up to zero. This makes some proofs easier, since finding a function with negative energy is far easier than finding a function whose energy lies below a particular negative number. In fact, this task is immediately accomplished when $p < 4$, as we now show.

**Proof of Theorem 1.1.** In view of Proposition 3.3, it suffices to construct a function in $H^1_p(G)$ with negative energy. Given $\mu > 0$, for $\varepsilon > 0$ let

$$\kappa_\varepsilon = \left(\frac{\varepsilon \mu}{2} \frac{1 - e^{-2\varepsilon}}{1 + e^{-2\varepsilon}}\right)^{1/2}$$

and consider the function of two variables

$$\varphi(x, y) = \kappa_\varepsilon e^{-\varepsilon(|x| + |y|)}, \quad (x, y) \in \mathbb{R}^2.$$

Now, as described in Section 2, we can consider $G$ isometrically embedded in $\mathbb{R}^2$, with its vertices on the lattice $\mathbb{Z}^2$, and we can define $u : G \to \mathbb{R}$ as the restriction of $\varphi$ to the grid $G$. Observe that, on every horizontal line $H_j$ of $G$, $u$ takes the form $\kappa_\varepsilon e^{-\varepsilon(|x| + |j|)}$, and a similar expression holds on vertical lines. Since for every $\lambda > 0$

$$\int_\mathbb{R} e^{-\lambda |x|} \, dx = \frac{2}{\lambda \varepsilon} \quad \text{and} \quad \sum_{j \in \mathbb{Z}} e^{-\lambda |j|} = \frac{1 + e^{-\lambda \varepsilon}}{1 - e^{-\lambda \varepsilon}},$$

recalling (20) we obtain

$$\int_G |u_\varepsilon|^2 \, dx = 2 \sum_{j \in \mathbb{Z}} \int_{H_j} |u_\varepsilon|^2 \, dx = 2 \kappa_\varepsilon^2 \sum_{j \in \mathbb{Z}} e^{-2\varepsilon |j|} \int_\mathbb{R} e^{-2\varepsilon |x|} \, dx = \mu$$

and, since $|u'_\varepsilon(x)| = \varepsilon |u_\varepsilon(x)|$,

$$\int_G |u'_\varepsilon|^2 \, dx = \varepsilon^2 \mu.$$

This shows in particular that $u_\varepsilon \in H^1_p(G)$. Similarly, observing that $\kappa_\varepsilon \sim \varepsilon \sqrt{\mu/2}$ as $\varepsilon \to 0$, we obtain the expansion

$$\int_G |u_\varepsilon|^p \, dx = 2 \sum_{j \in \mathbb{Z}} \int_{H_j} |u_\varepsilon|^p \, dx = 2 \kappa_\varepsilon^p \frac{2}{\varepsilon^p} \frac{1 + e^{-\varepsilon p}}{1 - e^{-\varepsilon p}} \sim C \mu^{p/2} \varepsilon^{p-2}$$

as $\varepsilon \to 0.$
where $C$ depends only on $p$. Therefore, as $\varepsilon \to 0$,

$$E_p(u_\varepsilon) \sim \frac{1}{2}\varepsilon^2 \mu - \frac{1}{p} C \mu^{p/2} \varepsilon^{p-2},$$

so that $E_p(u_\varepsilon) < 0$ (for $\varepsilon$ small enough) when $p < 4$. This proves that, when $p < 4$, $E_p(\mu) < 0$ for every $\mu > 0$. Moreover, since in particular $p < 6$, Remark 3.1 guarantees that $E_p(\mu)$ is finite. The result then follows from Proposition 3.3.

4 Proof of Theorem 1.2

In the following we assume that the constants $K_p$ in the Gagliardo-Nirenberg inequality (22) are the smallest possible. In other words, for $p \in [4, 6]$ we let

$$K_p = \sup_{u \in H^1(G) \setminus \{0\}} \frac{\|u\|_{L^p(G)}^p}{\|u\|_2^2 \cdot \|u'\|_2^2},$$

The critical masses $\mu_p$ mentioned in Theorem 1.2 are defined in terms of the constants $K_p$, as follows.

Definition 4.1. For every $p \in [4, 6]$ we define the critical mass $\mu_p$ as the positive solution of

$$\mu_p = \left( \frac{p}{2K_p} \right)^{\frac{p}{p-2}}.$$

This definition is natural due to the identity

$$E_p(u) = \frac{1}{2}\|u'\|_2^2 \left( 1 - \frac{2}{p} \frac{\|u\|_{L^p(G)}^p}{\|u\|_2^2 \cdot \|u'\|_2^2} \right) \quad \forall u \in H^1_p(G)$$

which, using $Q_p(u) \leq K_p$ and (32), leads to the lower bound

$$E_p(u) \geq \frac{1}{2}\|u'\|_2^2 \left( 1 - \left( \frac{\mu}{\mu_p} \right)^{\frac{p}{p-2}} \right), \quad \forall u \in H^1_p(G),$$

that will be widely used in the sequel.

Remark 4.2. On the real line $\mathbb{R}$, when $p = 6$ the ground-state level

$$E_6^\mathbb{R}(\mu) = \inf \left\{ \frac{1}{2}\|w'\|_{L^2(\mathbb{R})}^2 - \frac{1}{6}\|w\|_{L^6(\mathbb{R})}^6 \mid w \in H^1_p(\mathbb{R}) \right\}, \quad \mu > 0$$

is attained by a ground state if and only if $\mu = \mu_6^\mathbb{R}$, where the number

$$\mu_6^\mathbb{R} = \frac{\frac{\pi}{\sqrt{3}}}{2}$$

is the critical mass of the real line (see [3]). Up to sign and translations, the ground states (of mass $\mu_6^\mathbb{R}$) are the soliton $\varphi(x) = \text{sech}(2x/\sqrt{3})^{1/2}$ together with all its mass-preserving rescalings $\varphi_{\lambda}(x) = \sqrt{\lambda} \varphi(\lambda x) \ (\lambda > 0)$. There holds

$$E_6^\mathbb{R}(\mu) \begin{cases} = 0 & \text{if } \mu \leq \mu_6^\mathbb{R} \\ = -\infty & \text{if } \mu < \mu_6^\mathbb{R} \end{cases}$$
Therefore, 
\[ K_6 = \sup_{u \in L^2(\mathbb{R})} \frac{\|u\|_{L^6(\mathbb{R})}}{\|u\|_{L^2(\mathbb{R})}} = \frac{4}{\pi^2} \]

(note that \( \mu_6 = (3/K_6^6)^{1/2} \), which is formally consistent with \( \mu_6 \) when \( p = 6 \)).

The following proposition gives a complete picture of the problem on the grid \( \mathcal{G} \) when \( p = 6 \) and, moreover, provides the exact values of \( \mu_6 \) and \( K_6 \).

**Proposition 4.3.** There hold \( \mu_6 = \mu_6^0 = \pi \sqrt{3}/2 \) and \( K_6 = K_6^6 = 4/\pi^2 \). Moreover, there holds \( \mathcal{E}_0(\mu) = \mathcal{E}_0^6(\mu) \) for every \( \mu > 0 \), but the infimum

\[ \mathcal{E}_0(\mu) = \inf \left\{ \frac{1}{2} \|u\|_{L^6(\mathcal{G})}^2 - \frac{1}{6} \|u\|_{L^6(\mathcal{G})}^6 \mid u \in H^1_\mu(\mathcal{G}) \right\}, \quad \mu > 0 \]

is never attained.

**Proof.** By a density argument, the infimum in (35) can be restricted to functions \( w \in H^1_\mu(\mathbb{R}) \) having compact support. In fact, by a mass-preserving transformation \( w(x) \mapsto w(x/\varepsilon)/\varepsilon \), one can restrict to functions supported in the interval \( I = [-\frac{1}{2}, \frac{1}{2}] \). Then, by interpreting this interval as one of the edges of the grid \( \mathcal{G} \), any function \( w \in H^1_\mu(\mathbb{R}) \) supported in \( I \) can be embedded in \( H^1_\mu(\mathcal{G}) \) by setting \( w \equiv 0 \) on \( \mathcal{G} \setminus I \), thus providing an admissible function in (39). This proves that \( \mathcal{E}_0(\mu) \leq \mathcal{E}_0^6(\mu) \) for every \( \mu > 0 \). Similarly, starting from the supremum in (35), by the same argument one proves that \( K_6 \geq K_6^6 \).

To prove the opposite inequalities we argue as follows. Given a nonnegative function \( u \in H^1(\mathcal{G}) \) \((u \not \equiv 0)\), let \( x_0 \in \mathcal{G} \) be a point where \( u \) achieves its absolute maximum \( \|u\|_{\infty} \), and let \( P \) be any path in \( \mathcal{G} \) such that \( x_0 \in P \) and \( P \) is isometric to the real line \( \mathbb{R} \) (a natural choice for \( P \) is the horizontal/vertical line of \( \mathcal{G} \) that contains \( x_0 \)). Since \( u(x_0) = \|u\|_{\infty} \) and \( u(x) \to 0 \) as \( x \to \pm \infty \) along \( P \) (in both directions away from \( x_0 \), the continuity of \( u \) guarantees that \( N(t) \geq 2 \) for every \( t \in (0, \|u\|_{\infty}) \), where

\[ N(t) = \# \{ x \in \mathcal{G} \mid u(x) = t \} \]

counts the number of preimages in \( \mathcal{G} \). Then, if \( \tilde{u} \in H^1(\mathbb{R}) \) denotes the symmetric rearrangement of \( u \) on \( \mathbb{R} \), applying Proposition 3.1 of [3] we obtain

\[ \|u\|_{L^6(\mathcal{G})} = \|\tilde{u}\|_{L^6(\mathcal{G})} \leq \|u\|_{L^6(\mathbb{R})} \leq \|u\|_{L^2(\mathbb{R})} = \|\tilde{u}\|_{L^2(\mathbb{R})} \]

so that, by the definition of \( K_6^6 \) in (38), we can estimate

\[ \|u\|_{L^6(\mathcal{G})} \leq \|\tilde{u}\|_{L^6(\mathcal{G})} \leq K_6 \|\tilde{u}\|_{L^6(\mathbb{R})} \leq K_6 \|u\|_{L^2(\mathbb{R})} \] \( \forall \mathcal{G} \)

Therefore, \( K_6 \leq K_6^6 \) by (31). Similarly, for the NLS energy we have

\[ \frac{1}{2} \|u\|_{L^6(\mathcal{G})}^2 - \frac{1}{6} \|u\|_{L^6(\mathcal{G})}^6 \leq \frac{1}{2} \|u\|_{L^2(\mathbb{R})}^2 - \frac{1}{6} \|u\|_{L^6(\mathcal{G})}^6 \]

and, since \( \tilde{u} \in H^1(\mathbb{R}) \) whenever \( u \in H^1(\mathcal{G}) \), this proves that \( \mathcal{E}_0^6(\mu) \leq \mathcal{E}_0(\mu) \) for every \( \mu > 0 \). Now assume that, for some \( \mu \), a function \( u \in H^1(\mathcal{G}) \) achieves the
infinum $E_\theta(\mu)$ in (39). Then, since $E_\theta^6(\mu) = E_\theta(\mu)$, (42) shows that, necessarily:

(i) $\tilde{u}$ achieves the infimum $E_\theta^6(\mu)$ in (35), i.e. in (41). Now, condition (i) entails that $\tilde{u}$ is a soliton on $\mathbb{R}$ (necessarily of mass $\mu^6_0$), while (ii) implies (see Proposition 3.1 of [3]) that $N(t) = 2$ in (40), i.e. that $u^{-1}(t)$ has exactly 2 elements for almost every $t \in (0, \|u\|_\infty)$: then, since every vertex of $G$ has degree 4, $u$ must vanish at every vertex and is necessarily supported in a single edge of $G$. So $\tilde{u}$ has compact support too, which is incompatible with $\tilde{u}$ being a soliton. This contradiction shows the infimum in (39) is not achieved.

Finally, (32) with $p = 6$ yields $\mu_6 = \sqrt{3/K_6} = \pi \sqrt{3}/2$, hence $\mu_6 = \mu^6_0$ by (40).

Proof of Theorem 1.2. The case where $p = 6$ has already been proved through Proposition 4.3. The rest of the proof is divided into three parts.

Computation of $E_\theta(\mu)$ when $p \in [4,6)$. First observe that, in the proof of Theorem 1.1, no restriction on $p$ was used to construct $u_\varepsilon$ and obtain (40), which is therefore valid also when $p \geq 4$. As a consequence, in this case, letting $\varepsilon \to 0$ in (40) we obtain

\begin{equation}
E_\theta(\mu) \leq 0 \quad \forall p \geq 4, \quad \forall \mu > 0.
\end{equation}

Moreover, (43) shows that $E_\theta(\mu) \geq 0$ when $\mu \leq \mu_p$. This, combined with (43), proves the first part of (45), also when $p = 4$.

Now fix a mass $\mu > \mu_p$ and a number $\varepsilon > 0$. Since the quotient $Q_p(u)$ in (43) is unaltered if $u$ is replaced with $\lambda u$, there exists $u \in H^1_p(G)$ such that

\begin{equation}
Q_p(u) = \frac{\|u\|_p^p}{\mu^{\frac{p-2}{2}} \|u'\|_2^2} \geq K_p - \varepsilon.
\end{equation}

Plugging this into (43), and then using (42), we can estimate

\begin{equation}
E_\theta(u) \leq \frac{1}{2} \|u'\|_2^2 \left( 1 - \frac{2}{p} (K_p - \varepsilon) \mu^{\frac{p-2}{2}} \right) = \frac{1}{2} \|u'\|_2^2 \left( 1 - \left( \frac{\mu}{\mu_p} \right)^{\frac{p-2}{2}} + \frac{2\varepsilon}{p} \mu^{\frac{p-2}{2}} \right).
\end{equation}

Since $\mu > \mu_p$, this quantity is strictly negative if $\varepsilon$ is small enough. Thus, for $\mu > \mu_p$, $E_\theta(\mu) < 0$. Moreover, when $p < 6$, $E_\theta(\mu) > -\infty$ by Remark 6.1. This proves the second part of (45), also when $p = 4$.

Ground states when $p \in [4,6)$ and $\mu \neq \mu_p$. When $\mu > \mu_p$, (45) (valid also when $p = 4$) shows that $E_\theta(\mu)$ is finite and negative, hence a ground state exists by Proposition 3.3. When $\mu < \mu_p$, $E_\theta(\mu) = 0$ by (45), but (44) reveals that $E_\theta(u) > 0$ for every $u \in H^1_p(G)$. Therefore, no ground state exists in this case.

Ground states when $p \in (4,6)$ and $\mu = \mu_p$. Since by (45) $E_\theta(\mu) = 0$, we can no longer rely on Proposition 3.3 and another argument is needed to show that $E_\theta(\mu)$ is in fact achieved.

Arguing as for (45), let $u_\varepsilon \in H^1_{\mu_p}(G)$ be a sequence of functions such that

\begin{equation}
\lim_n Q_p(u_n) = \lim_n \frac{\|u_n\|_p^p}{\mu_p^{\frac{p-2}{2}} \|u'_n\|_2^2} = K_p.
\end{equation}
We shall bound \( Q_p(u_n) \) in two different ways. First, from the the Gagliardo-Nirenberg inequality \((12)\) we obtain

\[
Q_p(u_n) \leq \frac{\|u_n\|_2^{\frac{p+1}{p}} \|u'_n\|_2^{\frac{p-1}{p}}}{\mu_p^{\frac{p-2}{p}} \|u'_n\|_2^2} = \frac{\|u_n\|_2^p}{\mu_p^{\frac{p-2}{p}} \|u'_n\|_2^2}.
\]

Secondly, interpolating and then using \((22)\) with \( p = 4 \), we obtain

\[
Q_p(u_n) \leq \frac{\|u_n\|_\infty^2 \|u_n\|_2^4}{\mu_p^{\frac{p-2}{p}} \|u'_n\|_2^2} \leq \frac{\|u_n\|_\infty^2 \|u_n\|_2^2}{\mu_p^{\frac{p-2}{p}} \|u'_n\|_2^2} \leq \frac{\|u_n\|_\infty^2 K_4^2 \|u_n\|_2^2 \|u'_n\|_2^2}{\mu_p^{\frac{p-2}{p}} \|u'_n\|_2^2} = \frac{\|u_n\|_\infty^p K_4^2}{\mu_p^{2}}.
\]

Recalling \((45)\), from these two bounds we infer that \( \|u'_n\|_2 \leq C \) (compactness) and \( \|u_n\|_\infty \geq C^{-1} \) (non-degeneracy), for some constant \( C > 0 \) independent of \( n \). Thus \( \{u_n\} \) is bounded in \( H^1(G) \) and, up to translations, we can also assume that each \( u_n \) achieves its \( L^\infty \) norm on some compact set \( K \subset G \) independent of \( n \). Then, up to subsequences, \( u_n \rightharpoonup u \) in \( H^1(G) \) for some \( u \in H^1(G) \), and \( u_n \rightharpoonup u \) in \( L^\infty_m(G) \): in particular, \( u_n \rightharpoonup u \) uniformly on \( K \) and, since \( \|u_n\|_{L^\infty(K)} > C^{-1} \), \( u \) is not identically zero.

Finally, writing \((33)\) with \( u = u_n \) and \( \mu = \mu_p \), since \( \|u'_n\|_2 \leq C \) we find

\[
|E_p(u_n)| \leq \frac{C^2}{2} \left| 1 - \frac{2}{p} Q_p(u_n) \mu_p^{\frac{2}{p}} \right| = \frac{C^2}{2} \left| 1 - \frac{Q_p(u_n)}{K_p} \right|
\]

having used \((32)\). Therefore, \( E_p(u_n) \to 0 \) by \((43)\) and, since \( E_p(\mu_p) = 0 \), \( u_n \) is a minimizing sequence for \( E_p \), so that Lemma \((3.2)\) applies: since we already know that \( u \) is not identically zero, we obtain that \( \|u\|_2^2 = \mu_p \), i.e. \( u \in H^1_{\mu_p}(G) \). But then \( u \) is the required minimizer: indeed, \( u_n \rightharpoonup u \) strongly in \( L^2(G) \) hence also in \( L^p(G) \), and by weak lower semicontinuity we obtain

\[
E_p(u) \leq \liminf_{n} E_p(u_n) = E_p(\mu_p).
\]

\( \square \)

References

[1] R. Adami, C. Cacciapuoti, D. Finco, D. Noja, Fast solitons on star graphs, Rev. Math. Phys. 23(4), 409-451 (2011).

[2] R. Adami, E. Serra, P. Tilli, Lack of ground state for NLSE on bridge-type graph, Springer Proceedings in Mathematics and Statistics, 128 (2015).

[3] R. Adami, E. Serra, P. Tilli, NLS ground states on graphs, Calc. Var. and PDEs 54(1) (2015), 743–761.

[4] R. Adami, E. Serra, P. Tilli, Threshold phenomena and existence results for NLS ground states on graphs, J. Funct. An. 271(1), 201–223 (2016).

[5] R. Adami, E. Serra, P. Tilli, Negative energy states for the \( L^2 \)-critical NLSE on metric graphs, Commun. Math. Phys. 352(1), 387–406 (2017).

[6] R. Adami, E. Serra, P. Tilli, Nonlinear dynamics on branched structures and networks, Riv. Mat. Univ. Parma 8(1), 109–159 (2017).
[7] F. Ali Mehmeti, *Nonlinear waves in networks*, Akademie Verlag, Berlin (1994).

[8] F. Ali Mehmeti, J. von Below, S. Nicaise, *Partial differential equations on multistructures*, Lecture notes in pure and applied mathematics, vol. 219, Marcel Dekker, Inc., New York, Basel, 2001.

[9] G. Berkolaiko, P. Kuchment, *Introduction to Quantum Graphs*, Mathematical Survey and Monographs, Applied Mathematics, American Mathematical Society (2013).

[10] J. L. Bona, R. C. Cascaval, *Nonlinear dispersive waves on trees*, Can. Appl. Math. Q. 16 (2008), no. 1, 1–18.

[11] H. Brezis, E.H. Lieb, *A relation between pointwise convergence of functions and convergence of functional*, Proc. Amer. Math. Soc. 88 (3) (1983) 486–490.

[12] C. Cacciapuoti, D. Finco, D. Noja, *Topology-induced bifurcations for the nonlinear Schrödinger equation on the tadpole graph*, Phys. Rev. E 91 (1) (2015), 013206.

[13] V. Caudrelier, *On the inverse scattering method for integrable PDEs in a star graph*, Commun. Math. Phys. 338 (2) (2015), 893–917.

[14] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes 10. American Mathematical Society, Providence, RI (2003)

[15] T. Cazenave, P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, 85 (4) (1982), 549-561.

[16] S. Gnutzmann, D. Waltner, *Stationary waves on nonlinear quantum graphs: General framework and canonical perturbation theory*, Phys. Rev. E 93 (3) (2015), 032204.

[17] G. Leoni, *A first course in Sobolev Spaces*, Graduate Studies in Mathematics, AMS (2009).

[18] D. Noja, *Nonlinear Schrödinger equation on graphs: recent results and open problems*, Phil. Trans. Roy. Soc. A 372 (2013), 20130002.

[19] D. Noja, D. Pelinovsky, G. Shaikhova, *Bifurcation and stability of standing waves in the nonlinear Schrödinger equation on the tadpole graph*, Nonlinearity 28 (7) (2015), 2343–2378.

[20] D. Pelinovsky, G. Schneider, *Bifurcations of Standing Localized Waves on Periodic Graphs*, Ann. H. Poincaré 18 (4) (2017), 1185–1211.

[21] O. Post, *Spectral analysis on graph-like spaces*, Lecture Notes in Mathematics, 2039, Springer, Heidelberg (2012).

[22] K. Ruedenberg, C. W. Scherr, *Free-Electron Network Model for Conjugated Systems. I. Theory*, J. Chem. Phys. 21 (1953), 1565–1581.

[23] K. Sabirov, Z. Sobirov, D. Babajanov, D. Matrasulov, *Stationary non-linear Schrödinger on simplest graphs*, Phys. Lett. A, 377 (12) (2013), 860–865.
[24] Z. Sobirov, D. Matrasulov, K. Sabirov, S. Sawada, K. Nakamura, *Integrable nonlinear Schrödinger equation on simple networks: Connection formula at vertices*, Phys. Rev. E, **81** (6) (2010) 066602.

[25] M.I. Weinstein, *Excitation thresholds for nonlinear localized modes on lattices*, Nonlinearity **12**, 673–691 (1999).