A FANO COMPACTIFICATION OF THE $\text{SL}_2(\mathbb{C})$ FREE GROUP CHARACTER VARIETY

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Abstract. We show that a certain compactification $\mathcal{X}_g$ of the $\text{SL}_2(\mathbb{C})$ free group character variety $\mathcal{X}(F_g, \text{SL}_2(\mathbb{C}))$ is Fano. This compactification has been studied previously by the second author, and separately by Biswas, Lawton, and Ramras. Part of the proof of this result involves the construction of a large family of integral reflexive polytopes.

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1. Introduction

Let $\text{SL}_2(\mathbb{C})$ denote the group of $2 \times 2$ complex matrices with determinant 1, and let $F_g$ denote the free group on $g$ letters. The character variety $\mathcal{X}(F_g, \text{SL}_2(\mathbb{C}))$ is the moduli space of representations $\rho : F_g \rightarrow \text{SL}_2(\mathbb{C})$. Character varieties emerge naturally as moduli of local systems on a punctured Riemann surface and as generalizations of Teichmüller space [Gol88], [Sim94a], [Sim94b], [Vog89]. In this paper we study a compactification $\mathcal{X}_g$ of $\mathcal{X}(F_g, \text{SL}_2(\mathbb{C}))$, of a form constructed by the second author in [Man18b], and essentially by Biswas, Lawton, and Ramras [BLR19]. The following is our main theorem.

Theorem 1.1. The boundary $\mathcal{X}_g \setminus \mathcal{X}(F_g, \text{SL}_2(\mathbb{C}))$ is the union of $g$ irreducible divisors $D_i$, $1 \leq i \leq g$, and the anti-canonical class of $\mathcal{X}_g$ is $3(\sum_{i=1}^g D_i)$. The divisor $\sum_{i=1}^g D_i$ is very ample, in particular $\mathcal{X}_g$ is Fano.

The proof of Theorem 1.1 is by toric degeneration, in particular we show that $\mathcal{X}_g$ is Fano by finding a degeneration to a Fano toric variety. For each choice of a certain type of graph $\Gamma$ with edge set $E(\Gamma)$ and a spanning tree $T$ (see Section 3), we show that there is a toric flat family $\pi_\Gamma : \overline{X}_\Gamma \rightarrow \mathbb{C}^{E(\Gamma)}$ with general fiber $\mathcal{X}_g$ and special fiber a Fano toric variety $Y_\Gamma$. For an introduction to toric flat families see [KM]. The toric variety $Y_\Gamma$ is the projective toric variety associated to a polytope $P(\Gamma, T) \subset \mathbb{R}^{E(\Gamma)}$. The following is then a critical ingredient in the proof of Theorem 1.1.

Theorem 1.2. For each choice of $\Gamma$, $T$ as above, the polytope $P(\Gamma, T)$ is normal, and its third Minkowski sum $3P(\Gamma, T)$ is (an integral translate of) an integral reflexive polytope.
Figure 1. The polytopes $3P(\Gamma, T)$ for the two genus 2 graphs. Each is reflexive after translating by $(-2, -2, -2)$. The vertices are labelled with the corresponding labelled graphs.

A broader class of graphs $\Gamma$ than those used in Theorem 1.2 can be used to find toric degenerations of $\mathcal{X}_g$, however it is not always the case that the corresponding polytopes are normal or reflexive. In Section 3 we prove several combinatorial results along these lines.

We mention a conjecture formulated by Simpson [Sim16] that a relative character variety should have a log Calabi-Yau compactification. In [Wha20], Whang verifies this for relative $\SL_2(\mathbb{C})$ character varieties of punctured surfaces by constructing a compactification which, while different from $\mathcal{X}_g$, has a similar construction. Both compactifications can be viewed as solutions to certain inequalities on the values of the valuations on the coordinate ring of $\mathcal{X}(F_g, \SL_2(\mathbb{C}))$ constructed in [Man18b]. In principle, the techniques in Section 4 can also be used to construct a toric degeneration of Whang’s compactification. We expect that the elegant combinatorial constructions found by Whang in [Wha20, Section 6] have a polyhedral explanation, and will likewise have analogues in our setting due to the reflexivity of $3P(\Gamma, T)$.

The compactification $\mathcal{X}_g$ is from a family of such compactifications constructed by the second author in [Man18b]. A similar construction is explored for all simple groups $G$ of adjoint type by Biswas, Lawton, and Ramras in [BLR19], see also [Man18b, Sections 4 and 5]. Their construction relies on the use of the wonderful compactification $G \subset \overline{G}$ in the role of the compactification $\SL_2(\mathbb{C}) \subset X$ from Section 2. Roughly speaking, by extending $\SL_2(\mathbb{C})$ to $X$, we allow the values of the generators of $F_g$ to take values in the (equivalence classes of) singular matrices. Similarly, Biswas, Lawton, and Ramras allow the generators of $F_g$ to take values in the wonderful compactification of a more general group $G$. We expect that the techniques used in this paper can be generalized to show compactifications like those in [BLR19] are Fano.

Example 1.3. The polytopes in Figure 1 correspond to the anti-canonical embedding of the special fiber $Y_\Gamma$ for each genus 2 trivalent graph $\Gamma$. 
2. The compactification \( \mathcal{X}_g \)

In [Man18b] a divisorial compactification \( \mathcal{X}(F_g, \text{SL}_2(\mathbb{C})) \subset X_\Gamma \) is constructed for every ribbon graph \( \Gamma \) with first Betti number \( g \). It can be shown that \( X_\Gamma \) does not depend on the ribbon structure of \( \Gamma \). By [Man18b, Proposition 8.2], the boundary \( X_\Gamma \setminus \mathcal{X}(F_g, \text{SL}_2(\mathbb{C})) \) is of normal crossings type, with one irreducible divisor \( D_e \) for each edge \( e \in E(\Gamma) \). These divisors are sufficient to give the class group of \( X_\Gamma \).

**Proposition 2.1.** For any ribbon graph \( \Gamma \) with first Betti number \( g \), the class group is

\[
\text{Cl}(X_\Gamma) \cong \bigoplus_{e \in E(\Gamma)} \mathbb{Z} D_e.
\]

**Proof.** The divisors \( D_e \) are the irreducible components of the complement \( X_\Gamma \setminus \mathcal{X}(F_g, \text{SL}_2(\mathbb{C})) \), so there is an exact sequence of groups:

\[
\mathbb{C}[\mathcal{X}(F_g, \text{SL}_2(\mathbb{C}))]^* \to \bigoplus_{e \in E(\Gamma)} \mathbb{Z} D_e \to \text{Cl}(X_\Gamma) \to \text{Cl}(\mathcal{X}(F_g, \text{SL}_2(\mathbb{C}))) \to 0.
\]

The character variety \( \mathcal{X}(F_g, \text{SL}_2(\mathbb{C})) \) is factorial [LM16], and the only units in the coordinate ring \( \mathbb{C}[\mathcal{X}(F_g, \text{SL}_2(\mathbb{C}))] \) are constants. \( \square \)

If the graph \( \Gamma \) is trivalent, then it determines two cones \( C_\Gamma, P(\Gamma) \) which give \( \mathcal{X}(F_g, \text{SL}_2(\mathbb{C})) \) many of the features of an affine toric variety. Points \( \gamma \in C_\Gamma \cong \mathbb{R}_{>0}^{E(\Gamma)} \) each determine a real-valued valuation \( v_\gamma \) on \( \mathbb{C}[\mathcal{X}(F_g, \text{SL}_2(\mathbb{C})]) \), called a length function. This cone can be identified with a *prime cone* in the tropical variety of \( \mathcal{X}(F_g, \text{SL}_2(\mathbb{C})) \) associated to its embedding by a certain finite collection \( Y_g \) of regular functions, [Man18b, Proposition 6.3]. We will use a compactification of the toric flat family associated to the integral points of \( C_\Gamma \) for our main construction in Section 4.

The second cone \( P(\Gamma) \subset \mathbb{R}^{E(\Gamma)} \) is the set of \( a \in \mathbb{R}^{E(\Gamma)} \) for which

\[
a(e) \leq a(f) + a(g), \\
a(f) \leq a(e) + a(g), \\
a(g) \leq a(e) + a(f),
\]

for any three edges \( e, f, g \in E(\Gamma) \) which contain a common vertex. Accordingly, we say \( a \in P(\Gamma) \) satisfies the *triangle inequalities* at each vertex of \( \Gamma \). There is a natural pairing \( \langle -, - \rangle_\Gamma : C_\Gamma \times P(\Gamma) \to \mathbb{R} \) induced by the inner product on \( \mathbb{R}^{E(\Gamma)} \). We often consider the cone \( P(\Gamma) \) with respect to the sublattice \( M_\Gamma \subset \mathbb{Z}^{E(\Gamma)} \) defined by the condition \( a(e) + a(f) + a(g) \in 2\mathbb{Z} \) for every \( e, f, g \in E(\Gamma) \) which share a common vertex. We let \( N_\Gamma = \text{Hom}(M_\Gamma, \mathbb{Z}) \) denote the dual lattice.

The trivalent graph \( \Gamma \) also determines a basis \( \mathcal{B}_\Gamma \subset \mathbb{C}[\mathcal{X}(F_g, \text{SL}_2(\mathbb{C}))] \) of the regular functions on \( \mathcal{X}(F_g, \text{SL}_2(\mathbb{C})) \). In particular, there is an element \( \Phi_a \in \mathcal{B}_\Gamma \), called a spin diagram, for each lattice point \( a \in P(\Gamma) \cap M_\Gamma \). For \( a \in P(\Gamma) \cap M_\Gamma \) and \( \gamma \in C_\Gamma \) we have \( v_\gamma(\Phi_a) = \langle \gamma, a \rangle_\Gamma \) by [Man18b, Corollary 6.12]. Moreover, by [Man18b, Proposition 8.3], the section space \( H^0(X_\Gamma, \sum n_e D_e) \) is the subspace of \( \mathbb{C}[\mathcal{X}(F_g, \text{SL}_2(\mathbb{C}))] \) spanned by those spin diagrams \( \Phi_a \) with \( a(e) \leq n_e \). If \( \hat{\Gamma} \) is a graph with first Betti number \( g \), we can still obtain a nice basis of each section space of \( X_\Gamma \) by finding a trivalent graph \( \Gamma \) and a sufficiently nice surjection \( \pi : \Gamma \to \hat{\Gamma} \).

**Proposition 2.2.** Let \( \Gamma \) and \( \hat{\Gamma} \) be graphs with first Betti number equal to \( g \), and suppose that \( \Gamma \) is trivalent. Let \( \pi : \Gamma \to \hat{\Gamma} \) be a map of graphs which collapses edges \( f_1, \ldots, f_r \) while mapping the remaining edges bijectively onto the edges of \( \hat{\Gamma} \), then the section space \( H^0(X_\Gamma, \sum n_e D_e) \) has a basis given by the elements of \( \mathcal{B}_\Gamma \) with \( a(\pi(e')) \leq n_{\pi(e')} \).
Proof. This follows from [Man18b, Proposition 8.3].

In the sequel we consider the case where \( \hat{\Gamma} \) is a wedge of \( g \) loops at a vertex, and \( \Gamma \) is a trivalent graph with a choice of spanning tree \( \mathcal{T} \subset \Gamma \). When \( \mathcal{T} \) is collapsed to a single vertex, the result is \( \hat{\Gamma} \). We denote the compactification given by a wedge of \( g \) loops by \( \mathcal{X}_g \). Following [Man18b], \( \mathcal{X}_g \) is constructed as a GIT quotient by an action of \( \text{SL}_2(\mathbb{C}) \) on a product \( X^g \). Here \( X \) is an \( \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \) projective compactification of \( \text{SL}_2(\mathbb{C}) \), equivariantly embedded in \( \mathbb{P}^4 \) as the 0-locus of the quadratic polynomial \( AD - BC - t^2 \). We let \( i : X \to \mathbb{P}^4 \) be the inclusion map, and \( D \subset X \) denote the divisor obtained by setting \( t = 0 \). The sheaf of sections of \( D \) is then the pullback of \( \mathcal{O}(1) \) on \( \mathbb{P}^4 \), and can be identified with the sections of the pullback line bundle \( \mathcal{L} = i^*\mathcal{O}(1) \). The space \( \mathcal{X}_g \) is the GIT quotient:

\[
\mathcal{X}_g = \text{SL}_2(\mathbb{C})\backslash \mathcal{L}^{\alpha_2}X^g,
\]

where \( \text{SL}_2(\mathbb{C}) \) acts through the diagonal of \( \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \) on each component of \( X^g \). The divisor \( D \subset X \) can be regarded as a projectivization of the singular \( 2 \times 2 \) matrices. In particular, two singular matrices \( A_1, A_2 \) define the same point of \( D \) if \( A_1 = dA_2 \), where \( d \in \mathbb{C}^* \). In this way, \( \mathcal{X}_g \) is a space of \( \text{SL}_2(\mathbb{C}) \) representations of \( F_g \) where the values of the generators of \( F_g \) are allowed to degenerate to equivalence classes of singular matrices.

3. The Polytopes

In this section we introduce the polytope \( P(\Gamma, \mathcal{T}) \) where \( \Gamma \) is a trivalent graph with first Betti number \( g \), and \( \mathcal{T} \subset \Gamma \) is a spanning tree. In Section 4, we argue that \( 3P(\Gamma, \mathcal{T}) \) is the moment polytope for a toric degeneration of the compactification \( \mathcal{X}_g \). We let \( E(\Gamma) \) and \( V(\Gamma) \) denote the edge and vertex sets of \( \Gamma \), and we let \( \ell_1, \ldots, \ell_g \) denote the elements in the set \( F(\Gamma, \mathcal{T}) = E(\Gamma) \setminus E(\mathcal{T}) \).

**Definition 3.1.** Let \( P(\Gamma, \mathcal{T}) \subset P(\Gamma) \subset \mathbb{R}^{E(\Gamma)} \) be the set of points \( a \) such that \( a(\ell) \leq 1 \) for any \( \ell \in F(\Gamma, \mathcal{T}) \). We refer to these conditions as the boundary inequalities.

**Lemma 3.2.** \( P(\Gamma, \mathcal{T}) \) is a polytope of dimension \( 3g - 3 \).

**Proof.** We begin by showing \( P(\Gamma, \mathcal{T}) \) is bounded. By definition, if \( a \in P(\Gamma, \mathcal{T}) \), then \( 0 \leq a(e) \leq 1 \) for all \( e \in F(\Gamma, \mathcal{T}) \). Let \( \mathcal{T}_0 = \mathcal{T} \) and let \( L_0 \) be the set of leaves of \( \mathcal{T}_0 \). Each leaf in \( \mathcal{T}_0 \) must be either adjacent to a loop or two edges in \( F(\Gamma, \mathcal{T}) \); as these edges are bounded by 1, we have that \( 0 \leq a(e) \leq 2 \).

For \( i > 0 \), set \( \mathcal{T}_i = \mathcal{T}_{i-1} \setminus L_{i-1} \) and let \( L_i \) be the set of leaves in \( \mathcal{T}_i \). We claim that \( 0 \leq a(e) \leq 2^{i+1} \) for all \( e \in L_i \). By induction, we can assume that for every \( e \in L_{i-1} \), we have \( 0 \leq a(e) \leq 2^{i-1} \). Suppose that \( e \in L_i \). Then \( e \) is adjacent to two edges \( f_1, f_2 \in F(\Gamma, \mathcal{T}) \cup L_0 \cup \cdots \cup L_{i-1} \). The triangle inequalities imply that

\[
0 \leq a(e) \leq 2^{j+1} + 2^{k+1} \leq 2^{i+1}
\]

for some \( -1 \leq j, k < i \) finishing the inductive step. This process terminates after finitely many, say \( n \), steps; hence, we see that \( P(\Gamma, \mathcal{T}) \subset [0, 2^{n+1}]^{E(\Gamma)} \).

To see that \( \dim P(\Gamma, \mathcal{T}) = 3g - 3 \), we need to show it is full-dimensional as \( |E(\Gamma)| = 3g - 3 \). To this end, note that \( p = (2/3, \ldots, 2/3) \in \mathbb{R}^{E(\Gamma)} \) lies in the interior of \( P(\Gamma, \mathcal{T}) \) since it strictly satisfies all the boundary and triangle inequalities. Letting \( r \) be the minimum distance from \( p \) to any one of the supporting hyperplanes of \( P(\Gamma, \mathcal{T}) \), we see that \( P(\Gamma, \mathcal{T}) \) contains the ball of radius \( r \) centered at \( p \); hence, \( P(\Gamma, \mathcal{T}) \) is full-dimensional. \( \square \)

Suppose \( \Gamma \) is obtained by adding a loop at every leaf of \( \mathcal{T} \). In Section 3.1, we show that the third Minkowski dilate of \( P(\Gamma, \mathcal{T}) \) is an integral translate of a reflexive polytope, and in Section
3.2, we show that $P(\Gamma, \mathcal{T})$ is a normal lattice polytope. As we shall see, these results do not hold for general pairs $\Gamma, \mathcal{T}$. Before presenting the proofs of these two results, we make a final remark about the lattices $M_\Gamma$ and $N_\Gamma$ when $\Gamma, \mathcal{T}$ is of this prescribed form.

**Lemma 3.3.** Let $\mathcal{T}$ be a trivalent tree with $g$ leaves, and let $\Gamma$ be the graph obtained from $\mathcal{T}$ by adding a loop to each leaf. The lattice $M_\Gamma \subseteq \mathbb{Z}^{\mathcal{E}(\Gamma)}$ is equal to

$$\mathbb{Z}^{F(\Gamma, \mathcal{T})} \oplus (2\mathbb{Z})^{\mathcal{E}(\mathcal{T})}$$

and $N_\Gamma$ is

$$\mathbb{Z}^{F(\Gamma, \mathcal{T})} \oplus \left(\frac{1}{2}\mathbb{Z}\right)^{\mathcal{E}(\mathcal{T})}.$$ 

**Proof.** Let $a \in M_\Gamma$. Set $\mathcal{T}_0 = \mathcal{T}$ and consider the leaves of $\mathcal{T}_0$. Each leaf $e$ is connected to a loop $\ell$. Since $a \in M_\Gamma$, we have that $2a(\ell) + a(e) \in 2\mathbb{Z}$. It follows that $a(\ell) \in \mathbb{Z}$ and $a(e) \in 2\mathbb{Z}$.

For $i > 0$, let $\mathcal{T}_i$ be the subtree of $\mathcal{T}_{i-1}$ obtained by removing all the leaves of $\mathcal{T}_i$. Assume inductively that for each leaf $e$ of $\mathcal{T}_{i-1}$, we have that $a(e) \in 2\mathbb{Z}$. Consider a leaf $f$ of $\mathcal{T}_i$. Since $\mathcal{T}$ is trivalent, it is adjacent to edges $e_1, e_2 \in E(\mathcal{T}_{i-1})$. As $a \in M_\Gamma$, we have that $a(e_1) + a(e_2) + a(f) \in 2\mathbb{Z}$. Since $a(e_1)$ and $a(e_2)$ are even, we must have that $a(f)$ is even as well. Continuing in this way until $\mathcal{T}_j$ is empty, we see that $M_\Gamma$ is $\mathbb{Z}^{F(\Gamma, \mathcal{T})} \oplus (2\mathbb{Z})^{\mathcal{E}(\mathcal{T})}$.

The second statement about $N_\Gamma$ follows since it is the set of all $b \in \mathbb{R}^{\mathcal{E}(\Gamma)}$ for which $a \cdot b \in \mathbb{Z}$ for all $a \in M_\Gamma$. □

3.1. **The Gorenstein-Fano property.** As currently defined, $P(\Gamma, \mathcal{T})$ has the origin as a vertex; thus, no dilate will be reflexive. To amend this issue, we consider a new polytope $Q(\Gamma, \mathcal{T})$ defined by the following inequalities:

**Definition 3.4.** Let $Q(\Gamma, \mathcal{T}) \subseteq \mathbb{R}^{\mathcal{E}(\Gamma)}$ be the set of points satisfying the following conditions:
Figure 3. The two trivalent genus 2 graphs \( \Gamma_1 \) and \( \Gamma_2 \), respectively.

(1) For any three edges \( e, f, g \) which share a common vertex we must have:

\[
\frac{w(e) + w(f) - w(g)}{2} \geq -1, \\
\frac{w(e) - w(f) + w(g)}{2} \geq -1, \\
\frac{-w(e) + w(f) + w(g)}{2} \geq -1.
\]

(2) For any edge \( \ell \in F(\Gamma, T) \) we must have:

\[-w(\ell) \geq -1.\]

Remark 3.5. The \( \frac{1}{2} \)'s are introduced so that the inward pointing normal vectors of \( Q(\Gamma, T) \) are primitive with respect to the lattice \( N_\Gamma \). In other words, the polar dual \( Q(\Gamma, T)^0 \) is a lattice polytope with respect to \( N_\Gamma \).

Lemma 3.6. The translation of \( Q(\Gamma, T) \) by the vector consisting of all 2's is the third dilate of \( P(\Gamma, T) \).

Proof. Let \( 2 = (2, \ldots, 2)^T \in \mathbb{R}^{E(\Gamma)} \). Let \( w \in Q(\Gamma, T) \). Then we see that \( w + 2 \) satisfies the following inequalities:

(1) For any edges \( e, f, g \) which share a common vertex, we have

\[
\frac{(w(e) + 2) + (w(f) + 2) - (w(g) + 2)}{2} \geq 0, \\
\frac{(w(e) + 2) - (w(f) + 2) + (w(g) + 2)}{2} \geq 0, \\
\frac{-(w(e) + 2) + (w(f) + 2) + (w(g) + 2)}{2} \geq 0.
\]

(2) For any edge \( \ell \in F(\Gamma, T) \), we have

\[-(w(\ell) + 2) \geq -3.\]

It follows that \( Q(\Gamma, T) + 2 \subseteq 3P(\Gamma, T) \). Similarly, if \( v \in 3P(\Gamma, T) \), then \( v - 2 \in Q(\Gamma, T) \); hence, we have equality. \( \square \)

The main result of this section is that \( Q(\Gamma, T) \) is reflexive when \( F(\Gamma, T) \) is a collection of \( g \) loops. The proof is by induction on the first Betti number of the graph. Thus, we begin by showing the dumbbell graph in Figure 3a yields a reflexive polytope. For this, it is enough to argue that \( Q(\Gamma_1, T) \) is a lattice polytope with respect to \( M_{\Gamma_1} \).
We denote the two loops by $\ell_1$ and $\ell_2$ and the bridge by $e$ and order the standard basis vectors of $\mathbb{R}^{E(\Gamma_1)}$ in this way. Then $Q(\Gamma_1, T)$ is the set of all $x \in \mathbb{R}^3$ satisfying the following inequalities

$$
\begin{pmatrix}
1 & 0 & -\frac{1}{2} \\
0 & 1 & -\frac{1}{2} \\
-1 & 0 & 0 \\
0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\geq
\begin{pmatrix}
-1 \\
-1 \\
-1 \\
-1
\end{pmatrix}.
$$

The vertices of $Q(\Gamma_1, T)$ are the columns of the matrix below.

$$
\begin{pmatrix}
-2 & 1 & -2 & 1 & 1 \\
-2 & -2 & 1 & 1 & 1 \\
-2 & -2 & -2 & -2 & 4
\end{pmatrix}
$$

In particular, $Q(\Gamma_1, T)$ is a lattice polytope with respect $M_{\Gamma_1} = \mathbb{Z}\ell_1 \oplus \mathbb{Z}\ell_2 \oplus 2\mathbb{Z}e$.

**Remark 3.7.** We also note that $\Gamma_2$ from Figure 3b yields a reflexive polytope. Label the edges $\ell_1, \ell_2,$ and $e$ arbitrarily and set $T = e$. Then $Q(\Gamma_2, T)$ is the set of all $x \in \mathbb{R}^3$ satisfying

$$
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-1 & 0 & 0 \\
0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\geq
\begin{pmatrix}
-1 \\
-1 \\
-1 \\
-1
\end{pmatrix}.
$$

The vertices of $Q(\Gamma_2, T)$ are the columns in the matrix below.

$$
\begin{pmatrix}
-2 & 1 & 1 & -2 & 1 \\
-2 & 1 & -2 & 1 & 1 \\
-2 & -2 & 1 & 1 & 4
\end{pmatrix}
$$

The column sums are all even; hence, each vertex is in $M_{\Gamma_2}$.

We need a lemma concerning certain faces of $Q(\Gamma, T)$.

**Lemma 3.8.** Let $\Gamma$ be a trivalent genus $g \geq 3$ graph with $g$ loops, and let $T$ be its unique spanning tree. For $e \in E(T)$, then define

$$
F_e = 3P(\Gamma, T) \cap \{w \in \mathbb{R}^{E(\Gamma)} \mid w(e) = 0\}.
$$

$F_e$ is a face of $3P(\Gamma, T)$, and $F_e$ is isomorphic to

$$
3P(\Gamma_1, T_1) \times 3P(\Gamma_2, T_2).
$$

$\Gamma_1$ and $\Gamma_2$ are the connected components of the graph obtained by deleting $e$ contracting the pairs of edges adjacent to $e$; and $T_1$ and $T_2$ are their spanning trees, respectively. If $e$ is a leaf of $T$ and $\Gamma_1$ is just a single loop, then we take $P(\Gamma_1, T_1) = [0, 1]$.

**Proof.** $F_e$ is a face since it is the intersection of $3P(\Gamma, T)$ with the two supporting hyperplanes \{w | w(e) + w(f) - w(h) = 0\} and \{w | w(e) - w(f) + w(h) = 0\} where $e, f,$ and $h$ are edges meeting at a vertex.

Suppose $e \in E(T)$ is a leaf adjacent to a loop $\ell$, and let $w \in F_e$. The triangle and boundary inequalities imply that $w(\ell) \in [0, 3]$. The other pair of edges $f, h$ adjacent to $e$ must lie in $E(T)$ and since $w(e) = 0$, we see that $w(f) = w(h)$. Thus, any point $w$ in $F_e$ gives a point in $w' \in [0, 3] \times 3P(\Gamma_2, T_2)$ by setting $w'(fh) = w(f) = w(h)$ and $w'(k) = w(k)$ for all other edges. On the other hand, a point $w' \in [0, 3] \times 3P(\Gamma_2, T_2)$ yields a point in $w \in 3P(\Gamma, T)$ by
setting \(w(\ell)\) to be whatever was in the first component, \(w(e) = 0, w(f) = w(h) = w'(fh)\), and \(w(k) = w'(k)\) for all other edges. Thus, the claim holds when \(e \in E(\mathcal{T})\) is a leaf.

Suppose \(e \in E(\mathcal{T})\) is not a leaf, and it is adjacent to two pairs of edges \(f, h\) and \(f', h'\) all in \(E(\mathcal{T})\). Again, we have that \(w(f) = w(h)\) and \(w(f') = w(h')\). The map which sends \(w \in F_e\) to \(w' \in 3P(\Gamma_1, T_1) \times 3P(\Gamma_2, T_2)\) defined by \(w'(fh) = w(f) = w(h)\) and \(w'(f'h') = w'(f') = w(h')\) and \(w'(k) = w(k)\) for all other edges \(k\) is well-defined and bijective, so the claim holds in this case as well. \(\square\)

**Theorem 3.9.** \(Q(\Gamma, \mathcal{T})\) is reflexive whenever the pair \(\Gamma, \mathcal{T}\) is such that \(F(\Gamma, \mathcal{T})\) is a collection of \(g\) loops.

**Proof.** Let \(\Gamma\) be a graph with first Betti number \(g \geq 3\) consisting of a spanning tree \(\mathcal{T}\) and \(g\) loops. It is enough to show that \(Q(\Gamma, \mathcal{T}) + 2 = 3P(\Gamma, \mathcal{T})\) is a lattice polytope with respect to \(M_\Gamma\). Recall that for any \(w \in 3P(\Gamma, \mathcal{T})\) we have the triangle inequalities for each \(e, f, h\) sharing a vertex:

\[
\begin{align*}
w(e) + w(f) - w(h) & \geq 0, \\
w(e) - w(f) + w(h) & \geq 0, \\
-w(e) + w(f) + w(h) & \geq 0,
\end{align*}
\]

and we have boundary inequalities for each \(\ell \in F(\Gamma, \mathcal{T})\):

\[w(\ell) \leq 3.\]

Let \(a\) be a vertex of \(3P(\Gamma, \mathcal{T})\). The dimension of \(3P(\Gamma, \mathcal{T})\) is \(3g - 3\); thus, \(a\) is the intersection of at least \(3g - 3\) supporting hyperplanes. First, suppose there are three edges \(e, f, h\) meeting at a vertex so that at least two of the three triangle inequalities are equalities on \(a\), i.e.

\[a(e) + a(f) = a(h)\text{ and }a(e) + a(h) = a(f)\]

This with the third triangle inequality imply that \(a(e) = 0\) and \(a(f) = a(h)\). We can assume that \(e \in E(\mathcal{T})\), because if say \(e = f\) were a loop and \(a(e) = 0\), then we would have that \(a(h) = 0\) with \(h \in E(\mathcal{T})\), and we could work with \(h\) instead. Moreover, if \(f'\) and \(h'\) are the (possibly not distinct) edges which are also adjacent to \(e\), then the facts that \(a(e) = 0\) and the triangle inequalities for \(e, f', h'\) imply that \(a(f') = a(h')\).

\[
\begin{array}{c}
\text{0} \\
f' \\
\text{h'} \\
f \\
h
\end{array}
\]

Now, consider the graph \(\Gamma'\) obtained by deleting \(e\) and concatenating the pairs of edges \(f, h\) and \(f', h'\). Note that \(\Gamma'\) has two connected components both of which are trees with loops attached at each leaf, and their first Betti numbers are strictly smaller than \(g\).

Let \(a'\) be the induced labelling on \(\Gamma'\) via the map from Lemma 3.8. By Lemma 3.8, we have that \(a'\) is a vertex of \(3P(\Gamma_1, T_1) \times 3P(\Gamma_2, T_2)\). By the induction hypothesis, this is a lattice polytope with respect to \(M_{\Gamma_1} \times M_{\Gamma_2}\) (if \(\Gamma_1\) is a loop then we take \(M_{\Gamma_1}\) to be \(\mathbb{Z}\)). Thus, in order to see that \(a \in M_{\Gamma}\), we only need to check that \(a(e) + a(f) + a(h) \in 2\mathbb{Z}\) and \(a(e) + a(f') + a(h') \in 2\mathbb{Z}\). This follows since \(a'(fh)\) and \(a'(f'h')\) are in \(\mathbb{Z}\) and since \(a'(fh) = a(f) = a(h)\) and \(a'(f'h') = a(f') = a(h')\).

Next, we consider the case when \(a\) is the intersection of at least \(3g - 3\) supporting hyperplanes but for each triple of edges \(e, f, h\) sharing a common vertex at most one of the triangle inequalities
is an equality on $a$. In other words, there is no $e$ so that $a(e) = 0$. Since $\Gamma$ has $2g - 2$ vertices, there are two possibilities:

1. there is one triangle equality for every triple $e, f, h$ meeting at a vertex, and $a(\ell) = 3$ for all but possibly one \( \ell \in F(\Gamma, T) \), or
2. $a(\ell) = 3$ for all $\ell \in F(\Gamma, T)$, and there is a single triangle equality for all but exactly one triple $e, f, h$ meeting at a vertex,

In case (1), we can see that for any edge $a(e)$ is either the sum or difference of $a(f)$ and $a(h)$ where $e, f, h$ meet at a vertex. Start at a leaf of $T$. This leaf is adjacent to a loop in $F(\Gamma, T)$. Since the genus is at least 3, we can assume that this edge satisfies $a(\ell) = 3$. Using the triangle equality at this node, we see the value on the leaf is 6 which is even. Continuing in this fashion, we can solve for values on all edges in $\Gamma$, and we see that $a \in \mathbb{Z}^{E(\Gamma)}$, and in fact, we can see that $a(e) \in 2\mathbb{Z}$ for all edges in $T$. To see $a \in M_\Gamma$, consider a triple of edges sharing a vertex $e, f, h$. Without loss of generality, we can assume $a(e) = a(f) + a(h)$, so

$$a(e) + a(f) + a(h) = 2a(e) \in 2\mathbb{Z}.$$  

It follows that $a \in M_\Gamma$.

In case (2), the same argument works to show that $a \in \mathbb{Z}^{E(\Gamma)}$ and that $a(e) + a(f) + a(h) \in 2\mathbb{Z}$ for every triple meeting at a vertex except one. Call this triple $e', f', h'$. We must show that $a(e') + a(f') + a(h') \in 2\mathbb{Z}$. This follows since these are either all tree edges in which case, they are all even, or $e' = f'$ is a loop and $h'$ is a leaf in $T$ in which case $6 + a(h') \in 2\mathbb{Z}$ as $a(h')$ is even.

We end this section with a discussion about the hypotheses in Theorem 3.9. As shown earlier, the graph from Figure 3b does not satisfy the conditions of the theorem; however, $Q(\Gamma_2, T)$ is still reflexive. As we explain in the following example, $Q(K_4, T)$ is reflexive if and only if $T$ is trivalent.

**Example 3.10.** Consider the pair $K_4, T_1$ from Figure 4a, so $T_1$ is the set of solid edges in 4a. We label the edges in $F(K_4, T_1)$ $\ell_1, \ell_2, \ell_3$ clockwise starting at the top edge, and we label the tree edges $e_1, e_5, e_6$ starting at the edge adjacent to $\ell_1$ and $\ell_2$ and proceeding clockwise. Then $Q(K_4, T_1)$ is the defined by the four sets of triangle inequalities as well as the boundary inequalities $w(\ell_i) \leq 1$ for $i = 1, 2, 3$. The vertices of $Q(K_4, T_1)$ can be computed and are the columns in the matrix below. The rows (from top to bottom) correspond to the edges $\ell_1, \ell_2, \ell_3, e_4, e_5$, and $e_6$, respectively.

$$\begin{pmatrix}
-2 & 1 & -2 & 1 & 1 & 1 & -2 & 1 & -2 & 1 & 1 & 1 \\
-2 & 1 & 1 & -2 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & 1 \\
-2 & -2 & 1 & 1 & 4 & 1 & 1 & -2 & 2 & 4 & 1 & 1 & -2 & 4 \\
-2 & -2 & 1 & 1 & 4 & -2 & -2 & 1 & 1 & 4 & -2 & 1 & 4 & 4 \\
-2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 4 & 4
\end{pmatrix}$$

One can check that each vertex lies in $M_{K_4}$ by multiplying the matrix above by the matrix $M$ below and checking that all entries are indeed even.

$$M := \begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}$$

Now, consider $Q(K_4, T_2)$ where $T_2$ comprises of the edges $\ell_2, e_4$, and $e_6$. The polytope is defined in a similar way: it has the same twelve triangle inequalities, but the boundary inequalities are $w(\ell_1) \leq 1$, $w(\ell_3) \leq 1$, and $w(e_5) \leq 1$. The vertices of $Q(K_4, T_2)$ are the columns in the
Figure 4. Up to isomorphism, the pair \(K_4, \mathcal{T}\) is one of the two graphs above. In each the dashed edges are the edges in \(F(K_4, \mathcal{T})\) and the solid edges are \(\mathcal{T}\).

matrix below.

\[
\begin{pmatrix}
-2 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & 1 & 1 \\
-2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & 4 & 1 & -2 & 1 \\
-2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & 1 & 1 & 1 & 1 \\
-2 & 2 & 1 & 1 & 1 & 1 & -2 & 2 & 1 & 1 & 1 & 1 & 1 \\
-2 & -2 & 1 & 1 & -2 & -2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-2 & -2 & 1 & 1 & -2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-2 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & 1 & 1 & 1 & 1 \\
-2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

Using the matrix \(M\), we can check that all the vertices except the one colored in red is in \(M_{K_4}\).

Indeed, the following computation shows this vertex colored in red is not in \(M_{K_4}\).

\[
M \begin{pmatrix}
\begin{array}{c}
1 \\
1 \\
\frac{1}{1} \\
\frac{1}{1} \\
\end{array}
\end{pmatrix} = \begin{pmatrix}
0 \\
3 \\
3 \\
0 \\
\end{pmatrix}
\]

As Example 3.10 suggests, deciding whether \(Q(\Gamma, \mathcal{T})\) is reflexive or not is very sensitive with respect to both \(\Gamma\) and choice of spanning tree \(\mathcal{T}\). After some reflection, one can see that the inductive step in the proof of Lemma 3.8 fails for general pairs \(\Gamma, \mathcal{T}\). Indeed, when contracting edges, you may contract an edge \(\ell\) that lies in \(F(\Gamma, \mathcal{T})\) with a tree edge \(e\). Then the boundary condition on \(\ell\) imposes an extra condition, so one is left not with \(Q(\Gamma_1, \mathcal{T}_1) \times Q(\Gamma_2, \mathcal{T}_2)\) but rather this polytope with an extra boundary condition on an edge in the tree. This does not happen for the class of graphs considered in Lemma 3.8 since we are only every contracting tree edges. As of yet, we are unsure what pairs \(\Gamma, \mathcal{T}\) produce a reflexive polytope \(Q(\Gamma, \mathcal{T})\), and so we ask the following question.

**Question 3.11.** For what pairs \(\Gamma, \mathcal{T}\) is \(Q(\Gamma, \mathcal{T})\) reflexive?

We can answer Question 3.11 when the genus is 2 or 3, and we have an obstruction on \(\Gamma, \mathcal{T}\) for \(Q(\Gamma, \mathcal{T})\) being reflexive when \(g \geq 4\).

**Proposition 3.12.** Both trivalent graphs with first Betti number 2 yield reflexive polytopes. If \(\Gamma\) has first Betti number 3, then \(Q(\Gamma, \mathcal{T})\) is reflexive if and only if \(\Gamma, \mathcal{T}\) is one of the graphs in Figure 5.

**Proposition 3.13.** Let \(\Gamma\) be a trivalent graph of with first Betti number \(g\) with \(k\) loops where \(2 \leq k < g\). Let \(\mathcal{T}\) be a spanning tree of \(\Gamma\), and consider a non-loop edge \(f \in F(\Gamma, \mathcal{T})\) which is not adjacent to any other edges in \(F(\Gamma, \mathcal{T})\). Under these conditions, the polytope \(Q(\Gamma, \mathcal{T})\) is not reflexive.

**Proof.** We will show that \(Q(\Gamma, \mathcal{T}) + 2\) is not a lattice polytope. Fix a spanning tree \(\mathcal{T}\), a non-loop edge \(f \in F(\Gamma, \mathcal{T})\), and two loops \(\ell_1, \ell_2\). There is a unique path \(P\) starting at the vertex of \(\ell_1\) and ending at the vertex of \(\ell_2\) passing through \(f\) where all other edges are in \(\mathcal{T}\). Since \(\ell_1\) and \(\ell_2\) are adjacent to bridges in \(\Gamma\), we know that the path is of length at least 5. Suppose the edges in this path are \(\{f_1, f_2, \ldots, f_i\}\), so some \(f_j = f\) and every other \(f_k \in \mathcal{T}\). We also know that \(f_1\)
Up to isomorphism, these are the three pairs $\Gamma, \mathcal{T}$ of genus 3 graphs where $Q(\Gamma, \mathcal{T})$ is reflexive. The edges in each spanning tree are marked with solid edges, while the edges in $F(\Gamma, \mathcal{T})$ are dashed.

Figure 5. Up to isomorphism, these are the three pairs $\Gamma, \mathcal{T}$ of genus 3 graphs where $Q(\Gamma, \mathcal{T})$ is reflexive. The edges in each spanning tree are marked with solid edges, while the edges in $F(\Gamma, \mathcal{T})$ are dashed.

is the bridge connected to $\ell_1$ and $f_i$ is the bridge connected to $\ell_2$. Let $v \in \mathbb{R}^{E(\Gamma)}$ be the point below

$$v(e) = \begin{cases} 
3 & \text{if } e = f_k \text{ for } 1 \leq k \leq i \\
\frac{3}{2} & \text{if } e = \ell_1, \ell_2 \\
0 & \text{otherwise}
\end{cases}$$

To see that $v$ is a vertex, note that $v$ is the unique solution to the following set of equations and inequalities.

1. For each triple meeting at a vertex $f_j, f_{j+1}, e$ where $0 \leq j \leq i - 1$, we have
   $$-x_{f_j} + x_{f_{j+1}} + x_e = 0,$$
   $$x_{f_j} - x_{f_{j+1}} + x_e = 0,$$
   $$x_{f_j} + x_{f_{j+1}} - x_e \geq 0,$$
   and we have that
   $$2x_\ell - x_{f_i} = 0.$$

2. For every other triple $e, f, h$ where $e, f, h \notin P$, we have
   $$x_e + x_f - x_h = 0,$$
   $$x_e - x_f + x_h = 0,$$
   $$-x_e + x_f + x_h = 0.$$

3. For every $e \in F(\Gamma, \mathcal{T})$, we have
   $$x_e \leq 3,$$
   and for $f_j$, we have
   $$x_{f_j} = 3.$$

The equalities in (1) and (3) ensure that $v$ is as prescribed on $P$, and the equations in (2) and the inequalities in (1) ensure that $v(e) = 0$ away from $P$. It follows that $v$ is a vertex of $Q(\Gamma, \mathcal{T}) + 2$; thus, $Q(\Gamma, \mathcal{T})$ is not a lattice polytope.

**Remark 3.14.** This is not the only obstruction on $\Gamma, \mathcal{T}$ to $Q(\Gamma, \mathcal{T})$ being reflexive. For example, we have checked $Q(\Gamma, \mathcal{T})$ is not reflexive for the complete bipartite graph $K_{3,3}$ and for the Petersen graph with various choices of spanning tree in each, yet these graphs are simple.
3.2. Normality. Now we show that $P(\Gamma, T)$ is a normal lattice polytope with respect to the lattice $M_\Gamma$ when $\Gamma$ is constructed by adding $g$ loops at the leaves of $T$.

**Definition 3.15.** Let $CP(\Gamma, T) \subset \mathbb{R}^{E(\Gamma)} \times \mathbb{R}$ be the cone over the polytope $P(\Gamma, T) \times \{1\}$. Let $S(\Gamma, T)$ be the affine semigroup $CP(\Gamma, T) \cap (M_\Gamma \times \mathbb{Z})$.

**Theorem 3.16.** Let $\Gamma$ be obtained from a trivalent tree $T$ by adding loops at the leaves of $T$, then the affine semigroup $S(\Gamma, T)$ is generated by the lattice points $(P(\Gamma, T) \times \{1\}) \cap (M_\Gamma \times \mathbb{Z})$. In particular, $P(\Gamma, T)$ is a normal lattice polytope.

The proof of Theorem 3.16 relies on the normality of a closely-related polytope $\Delta(T)$.

**Definition 3.17.** Let $T$ be a trivalent tree, and let $\Delta(T) \subset \mathbb{R}^{E(T)}$ be the set of points $w$ satisfying the following conditions:

1. For any three edges $e, f, g \in E(T)$ which share a common vertex we must have:
   
   \[ w(e) \leq w(f) + w(g), \]
   
   \[ w(f) \leq w(e) + w(g), \]
   
   \[ w(g) \leq w(e) + w(f). \]

2. For any edge $e$ containing a leaf of $T$ we must have:

\[ w(e) \leq 2. \]

Let $M_T \subset \mathbb{Z}^{E(T)} \subset \mathbb{R}^{E(T)}$ denote the sublattice of even integer tuples.

It may be more efficient to define $\Delta(T)$ with respect to the usual integer lattice, and make the second inequality of the definition $w(e) \leq 1$. We choose to present $\Delta(T)$ in the manner above to make a connection with a class of polyhedra which play a prominent role in the study of the moduli of weighted configurations on the projective line [HMSV09], and the moduli of rank 2 parabolic vector bundles on a pointed projective line [Man18a, Man10].

**Lemma 3.18.** The polytope $\Delta(T)$ is a normal lattice polytope with respect to the lattice $M_T$.

**Proof.** Let $T'$ be the trivalent tree obtained by adding two new edges at every leaf of $T$. The graded affine semigroup consisting of the lattice points of the Minkowski sums $k\Delta(T)$, $k \geq 1$ can be seen to be the semigroup $S_{T'}(1, \ldots, 1)$ from [Man10, Definition 1.2]. The lemma is a consequence of [Man10, Theorem 1.8]. \qed

To relate $P(\Gamma, T)$ to $\Delta(T)$ we imagine splitting each loop in $\Gamma$ into two edges. The result is the tree $T'$ from the proof of Lemma 3.18. This construction does not result in an isomorphism between $P(\Gamma, T)$ and $\Delta(T)$, however it does allow us to use the normality of $\Delta(T)$ to decompose a degree $L$ element of $S(\Gamma, T)$ into $L$ degree 1 elements.

**Proposition 3.19.** Let $w \in (L \cdot P(\Gamma, T) \times \{1\}) \cap (M_\Gamma \times \mathbb{Z})$ be a degree $L$ element of $S(\Gamma, T)$, then there are $L$ lattice points $w_1, \ldots, w_L \in (P(\Gamma, T) \times \{1\}) \cap (M_\Gamma \times \mathbb{Z})$ such that $w_1 + \cdots + w_L = w$.

**Proof.** First we observe that the point $w \in L \cdot P(\Gamma, T)$ gives a point $u \in L \cdot \Delta(T)$ by restricting $w : E(\Gamma) \to \mathbb{Z}$ to $E(T) \subset E(\Gamma)$. If $e \in E(T)$ shares a vertex with a loop $\ell \in F(\Gamma, T)$, then $w(e) \leq 2w(\ell) \leq 2L$. Moreover, the defining parity condition $w(e) + w(f) + w(g) \in 2\mathbb{Z}$ of $M_\Gamma$ implies that $w(e) \in 2\mathbb{Z}$. As a consequence, $w$ weights all edges of $T$ with even integers.

Next we use Lemma 3.18 to find $u_1, \ldots, u_L \in \Delta(T) \cap M_T$ such that $u_1 + \cdots + u_L = u$. We must show that each $u_i$ extends to a $w_i \in P(\Gamma, T) \cap M_\Gamma$ in such a way that $w_1 + \cdots + w_L = w$. It suffices to show that this extension can be carried out at each loop of $\Gamma$. As above, let $e \in E(T)$ be an
edge which shares a vertex with a loop \( \ell \in F(\Gamma, \mathcal{T}) \). For each \( i \), \( u_i(e) \) is either 0 or 2, and there are precisely \( N = \frac{1}{2}w(e) \leq w(\ell) \) weightings \( u_1, \ldots, u_N \) where \( u_j(e) = 2 \). We extend each of these weightings to \( \Gamma \) by letting \( w_j(\ell) = 1 \). To ensure that \( w(\ell) = w(\ell_1) + \cdots + w(\ell_L) \) it remains to set \( w_k(\ell) = 1 \) for some set of size \( w(\ell) - \frac{1}{2}w(e) \) such that \( w_k(e) = 0 \). As \( N = \frac{1}{2}w(e) \leq w(\ell) \), this is always possible. After making such a choice at each leaf \( \ell \) we obtain \( w_1, \ldots, w_L \in P(\Gamma, L) \cap M_\Gamma \) which sum to \( w \).

Theorem 3.16 is a consequence of Proposition 3.19.

4. The degeneration of \( X_g \)

The purpose of this section is to construct a flat degeneration of \( X_g \) to a Fano projective toric variety. Let \( \Gamma \) be a trivalent graph of genus \( g \), and fix a spanning tree \( \mathcal{T} \) of \( \Gamma \). Recall that \( E(\Gamma) \) denotes the set of edges of \( \Gamma \), and \( F(\Gamma, \mathcal{T}) \) is the set of edges of \( \Gamma \) not in \( \mathcal{T} \).

4.1. A piecewise-linear valuation on \( \mathcal{X}(F_g, \text{SL}_2(\mathbb{C})) \). Let \( A_g \) be the coordinate ring of the free group \( \text{SL}_2(\mathbb{C}) \)-character variety, and recall the basis \( \mathcal{B}_\Gamma \) of spin diagrams \( \Phi_a \) associated to the integral points \( a \in P(\Gamma) \cap M_\Gamma \). In the terminology of [KM19] and [KM], \( \mathcal{B}_\Gamma \) is a linear adapted basis for the valuations \( v_\gamma, \gamma \in C_\Gamma \). The following summarizes this property, for a proof see [Man18b, ].

**Proposition 4.1.** For any trivalent graph \( \Gamma \) and \( \gamma \in C_\Gamma \) we have:

1. \( v_\gamma(\Phi_a) = \langle \gamma, a \rangle \),
2. \( v_\gamma(\sum c_a \Phi_a) = \max \{v_\gamma(\Phi_a) \mid c_a \neq 0\} \).

There is a distinguished point of \( C_\Gamma \) for each edge \( e \in E(\Gamma) \), namely the function which assigns 0 to every element of \( E(\Gamma) \setminus \{e\} \) and assigns 1 to \( e \). In the sequel we abuse notation by referring to this point as \( e \).

The data of the valuations \( v_\gamma \) and basis \( \mathcal{B}_\Gamma \) can be viewed as a valuation \( v_\Gamma \) on \( A_g \) which takes values in the semifield \( \mathcal{O}_{C_\Gamma} \) of piecewise-linear functions on the cone \( C_\Gamma \), as described in [KM]. The basis \( \mathcal{B}_\Gamma \subset A_g \) is then a so-called linear adapted basis of \( v_\Gamma : A_g \to \mathcal{O}_{C_\Gamma} \). The following is a consequence of [KM19, Proposition 6.1] and [KM, Theorem 1.2]. Let \( Y_\Gamma \) denote the normal affine toric variety associated to the cone \( P(\Gamma) \) and the lattice \( M_\Gamma \).

**Proposition 4.2.** Let \( \Gamma \) be a trivalent graph with first Betti number \( g \). There is a toric flat family

\[
\pi_\Gamma : X_\Gamma \to \mathbb{C}^{E(\Gamma)}.
\]

Any fiber of this family over a point in \((\mathbb{C}^*)^{E(\Gamma)}\) is isomorphic to \( \mathcal{X}(F_g, \text{SL}_2(\mathbb{C})) \). The fiber over the origin of \( \mathbb{C}^{E(\Gamma)} \) is isomorphic to \( Y_\Gamma \).

We construct a degeneration of \( X_g \) for every graph \( \Gamma \) with first Betti number \( g \) along with choice of spanning tree \( \mathcal{T} \) by fiber-wise compactifying the family \( X_\Gamma \). This yields a projective flat family of the form

\[
\overline{\pi}_\Gamma : \overline{X}_\Gamma \to \mathbb{C}^{E(\Gamma)}
\]

where each fiber is a projective variety containing the original fiber as a dense open subset. Our construction is more or less a relative version of the compactifications described in [KM19, §6].
4.2. Compactifying the family. Let $\Gamma, \mathcal{T}$ be as in the previous section. We construct a new ring $R_g$ which is a graded subring of $A_g[t_e \mid e \in F(\Gamma, \mathcal{T})]$ by letting

$$(R_g)_b = \{f \mid \varpi_{\Gamma}(f)[e] \leq b(e)\} \subset A_g^b.$$ 

**Proposition 4.3.** For any graph $\Gamma$ with first Betti number $g$ and spanning tree $\mathcal{T}$, $R_g$ is the Cox ring of $X_g$:

$$R_g \cong \text{Cox}(X_g).$$

**Proof.** We use Proposition 2.2 on the graph $\Gamma$ and the image graph $\tilde{\Gamma}$ obtained by contracting $\mathcal{T}$ to a single vertex. The graded component $(R_g)_b \subset A_g$ is then seen to be the section space $H^0(X_g, \sum_{e \in E(\tilde{\Gamma})} b(e)D_e)$. By Proposition 2.1, the summands of $R_g$ run over all elements of $\text{Cl}(X_g)$. \hfill \Box

We consider an affine toric flat family $\hat{\Gamma}_T : \hat{X}_T \to \mathbb{C}^E(\Gamma)$. A general fiber of this family is isomorphic to $\text{Spec}(R_g)$, and the special fiber is a toric degeneration of $\text{Spec}(R_g)$. To make this construction we require a piecewise-linear valuation $\varpi_{\Gamma} : R_g \to \mathcal{O}_{C_\Gamma}$. We obtain $\varpi_{\Gamma}$ by extending $\varpi_T$ to the Laurent polynomial ring $A_g[t_e \mid e \in F(\Gamma, \mathcal{T})]$, and then considering the corresponding induced valuation on $R_g$. The next proposition summarizes consequences of this construction.

**Proposition 4.4.** There is a piecewise-linear valuation $\varpi_{\Gamma} : R_g \to \mathcal{O}_{C_\Gamma}$ with adapted basis $\{\Phi_a t^b \mid a \in P(\Gamma) \cap M_\Gamma, b \in \mathbb{Z}^{F(\Gamma, \mathcal{T})}, a(e) \leq b(e) \forall e \in F(\Gamma, \mathcal{T})\}$. The special fiber of the corresponding toric flat family $\hat{\pi}_T : \hat{X}_T \to \mathbb{C}^E(\Gamma)$ is the normal affine toric variety associated to the cone $\hat{P}(\Gamma) = \{(a, b) \mid a(e) \leq b(e) \forall e \in F(\Gamma, \mathcal{T})\} \subset P(\Gamma) \times \mathbb{R}^g$. The action of $(\mathbb{C}^*)^{F(\Gamma, \mathcal{T})}$ on $\text{Spec}(R_g)$ corresponding to the class group grading extends to a fiberwise action on the whole family $\hat{X}_T$.

Observe that Proposition 4.4 implies that the Cox ring $R_g$ is both finitely generated and Cohen-Macaulay.

Now consider the character $\chi : (\mathbb{C}^*)^{F(\Gamma, \mathcal{T})} \to \mathbb{C}^*$ given by $\chi((t_\ell)_{\ell \in F(\Gamma, \mathcal{T})}) = \prod_{\ell \in F(\Gamma, \mathcal{T})} t_\ell$. Our desired compactification is then given by the GIT quotient of this family with respect to $\chi$.

$$\pi_T : \overline{X}_T = \hat{X}_T \sslash \chi (\mathbb{C}^*)^{F(\Gamma, \mathcal{T})} \to \mathbb{C}^E(\Gamma).$$

**Proposition 4.5.** $\pi_T$ is a projective morphism.

**Proof.** We let $\mathbb{C}[s_f \mid f \in E(\Gamma)]$ denote the coordinate ring of $\mathbb{C}^E(\Gamma)$. The space $\hat{X}_T$ is the Proj of a graded algebra $S_{\chi} = \bigoplus_{N \geq 0} F_N$, where $F_N = \{\Phi_a t^b s^c \mid a(e) \leq N, e \in F(\Gamma, T), a(f) \leq c(f)f \in E(\Gamma)\}$. This is a summand of the Rees algebra of $\varpi_{\Gamma}$, and so defines an affine toric flat family over $\mathbb{C}^E(\Gamma)$. In particular, $S_{\chi}$ is a flat $\mathbb{C}[s_f \mid f \in E(\Gamma)]$ algebra. Specializing to the origin of $\mathbb{C}^E(\Gamma)$ yields the graded affine semigroup algebra $\mathbb{C}[S(\Gamma, \mathcal{T})]$, which is generated by its degree 1 component. The conclusion of [KM, Proposition 4.6] now implies that $F_1$ generates $S_{\chi}$ as a $\mathbb{C}[s_f \mid f \in E(\Gamma)]$ algebra. Accordingly, we can define an embedding of the family $\overline{X}_T$ into a projective space over $\mathbb{C}^E(\Gamma)$.

$\square$

**Proposition 4.6.** $\pi_T : \overline{X}_T \to \mathbb{C}^E(\Gamma)$ is a flat family and $\overline{X}_T$ is Cohen-Macaulay.

**Proof.** First, we show that $\overline{X}_T$ is Cohen-Macaulay. The special fiber of the family $\hat{X}_T$ is a saturated affine semigroup algebra, so it is Cohen-Macaulay. It follows by the Hochster-Roberts theorem that $\text{Proj}(\bigoplus_{N \geq 0} F_N)$ is Cohen-Macaulay as well. We also know that the fibers of $\pi_T$...
are all of dimension $3g - 3 = |E(\Gamma)|$ by [KM, Lemma 6.8], and we know the base $C^{E(\Gamma)}$ is smooth. By Hironaka’s criterion, we see that the family is flat.

Remark 4.7. The family $\overline{X}_g$, the compactification $X_g$, and the polytope $P_\Gamma$ are constructed by considering the spin diagrams $\Phi_a$ with graph $\Gamma$ which satisfy $\max_{e \in F(\Gamma, T)} a(e) \leq 1$. The compactification considered in [Wha20] corresponds to the condition $\sum_{e \in F(\Gamma, T)} a(e) \leq 1$. Accordingly, the toric degeneration techniques used in this section can be used on the compactification studied in [Wha20] as well. It would be interesting to compute the corresponding polytope in that case.

4.3. The Special Fiber. In this section, we study the special fiber of $\pi_{\Gamma} : \overline{X}_\Gamma \rightarrow C^{E(\Gamma)}$. In particular, we show that the anti-canonical divisor of $\overline{Y}_\Gamma$ corresponds to the polytope $Q(\Gamma, T)$; thus, $\overline{Y}_\Gamma$ is Gorenstein-Fano when $\Gamma$ is a tree $T$ with loops attached to the edges. Let $M_\Gamma$ and $P(\Gamma)$ be as in the previous section, and let $P^\vee(\Gamma)$ be the dual cone of $P(\Gamma)$.

Lemma 4.8. $P^\vee(\Gamma)$ is a full-dimensional strongly convex rational polyhedral cone in $N_\Gamma \otimes \mathbb{R}$.

The valuation $w_\Gamma$ has associated graded

$$\text{gr}_{w_\Gamma}(R_g) = \bigoplus_{(a,b) \in \hat{S}_\Gamma} C^{\ell(a,b)}$$

where $\hat{S}_\Gamma = \{ (a,b) \in (P(\Gamma) \cap M_\Gamma) \times \mathbb{Z}^{F(\Gamma, T)} \mid a(e) \leq b(e) \text{ for all } e \in F(\Gamma, T) \} = \hat{P}_\Gamma \cap M_\Gamma \times \mathbb{Z}^{F(\Gamma, T)}$. We can describe the rays of the dual cone $\hat{P}_\Gamma^\vee \subset (N_\Gamma \times \mathbb{Z}^{F(\Gamma, T)})_{\mathbb{R}}$. Let $d_1, \ldots, d_g$ be the standard basis vectors of $\mathbb{R}^{F(\Gamma, T)}$ and let $(b_f)_{f \in E(\Gamma)}$ be the standard basis vectors of $\mathbb{R}^{E(\Gamma)}$. Then, we have that

$$\hat{P}_\Gamma^\vee(1) = \{ \mathbb{R}_{\geq 0}(d_i - b_i) \mid \ell \in L \} \cup \{ \mathbb{R}_{\geq 0}(0, v) \mid v \in P^\vee_\Gamma(1) \}.$$
4.4. **Transferring the Fano Property.** Our last result pertains to transferring the Fano property from the special fiber \( Y_Γ \) to \( X_Γ \). Recall from Propositions 4.6 and 4.5 that the morphism \( π_Γ : X_Γ \rightarrow C^{E(Γ)} \) is flat and projective, and \( X_Γ \) is Cohen-Macaulay. The fact that the family is Cohen-Macaulay ensures that the canonical sheaves of the fibers are specializations of the relative canonical sheaf \( ω_{X_Γ/C^{E(Γ)}} \) by [Con00, Theorem 3.6.1]. Flatness provides us with the fact that \( X_Γ \) is Gorenstein, as the special fiber is Gorenstein. Finally, as the family is projective (hence proper) and the anti-canonical line bundle on the special fiber is ample, there is an open neighborhood \( U \) of \( 0 \in C^{E(Γ)} \) where \( ω_{X_Γ}^{-1} \) is ample for all \( t \in U \) by [Laz04, Theorem 1.2.17]. In particular, we can take \( t \in (C^*)^{E(Γ)} \) and see that \( X_Γ \) is Fano. This proves our main result, Theorem 1.1, from the introduction.

**Theorem 4.10.** For \( t \in (C^*)^{E(Γ)} \), \( π_Γ^{-1}(t) \simeq X_Γ \) is a Gorenstein Fano projective variety.

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