Boundary Conditions in Stepwise Sine-Gordon Equation and Multi-Soliton Solutions

N. Riazi * and A. Sheykhi †

Physics Department and Biruni Observatory, Shiraz University, Shiraz 71454, Iran

We study the stepwise sine-Gordon equation, in which the system parameter is different for positive and negative values of the scalar field. By applying appropriate boundary conditions, we derive relations between the soliton velocities before and after collisions. We investigate the possibility of formation of heavy soliton pairs from light ones and vice versa. The concept of soliton gun is introduced for the first time; a light pair is produced moving with high velocity, after the annihilation of a bound, heavy pair. We also apply boundary conditions to static, periodic and quasi-periodic solutions.

Keywords: Soliton theory, sine-Gordon equation, nonlinear interactions.

I. INTRODUCTION

The Sine-Gordon equation (SGE) is a nonlinear equation which is encountered in various contexts [1]. As an example, is can be shown that the classical string on a two-sphere is more or less equivalent to the sine-Gordon model [2]. Mikhailov [3] has considered a system which can be understood as a T-dual to the classical string on a two-sphere. He has shown that there is a projection map from the phase space of this model to the phase space of the sine-Gordon model.

One traditional way to obtain SGE in 1+1 dimensions is in the context of differential geometry and the theory of surfaces. The fundamental forms of a surface are not independent of each other and must satisfy the Gauss-Codazzi equations. It can be easily shown that the Gauss-Codazzi equations lead to the Sine-Gordon equation for a surface of constant negative curvature. The spatial shapes of the surfaces corresponding to the one and two soliton solutions have been shown in [4].

The SGE can be written in the following form, using an appropriate choice of coordinates:

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{c^2 \partial t^2} = a \sin \varphi,$$

(1)

where \(a\) is a constant parameter. In this form, SGE is a relativistic equation, which can be obtained...
from the following Lagrangian density:
\[ \mathcal{L} = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - a(1 - \cos \varphi). \tag{2} \]

The energy momentum tensor can be obtained from the Noether’s theorem \[5\]:
\[ T^{\mu \nu} = \partial^\mu \varphi \partial^\nu \varphi - \eta^{\mu \nu} \mathcal{L}, \tag{3} \]
where \( \eta^{\mu \nu} \) is the Minkowski metric for the 1+1 dimensional spacetime. The SGE is an integrable equation. An integrable system has as many constants of motion as the number of the system’s degrees of freedom \[6\]. Since the SGE have infinite degrees of freedom (as in any other continuous medium), its integrability means that it has infinitely many constants of motion. Integrable equations have peculiar solutions named solitons (or kinks in the case of SGE). Solitons are localized waves which do not disperse as they move in the nonlinear medium. They also preserve their shape after the collision with other solitons or localized inhomogeneities in the medium. It is usually claimed that a soliton reobtains its initial velocity after collision with another soliton and only a phase shift results form the collision \[4\]. For the system we will investigate in this paper, we will show that this claim is indeed not correct.

The one soliton solution (kink) of (1) is
\[ \varphi(x, t) = 4 \arctan e^{\gamma a(x-vt)}. \tag{4} \]
This solution can be obtained by different means (e.g via Backlund transformations or by direct integration and applying a Lorentz boost). The effective width of a kink at rest is about \( a^{-1} \).

The total energy and linear momentum of a single soliton (kink) of the SGE can be obtained from the following relations:
\[ E = \int T^0_0 dx = \gamma M c^2 \quad P = \int T^1_0 dx = \gamma M v. \tag{5} \]
Here, \( \gamma \) is related to the kink velocity via the \( \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \), and the kink’s rest energy (mass) is given by \( M = 8 \sqrt{a} \frac{c}{c^2} \).

For one soliton solutions, it can be shown that:
\[ E^2 = P^2 c^2 + M^2 c^4. \tag{6} \]
In this respect and other respects to be discussed, the kink behaves like a classical relativistic particle, although it is an extended object and has wave nature inherent in it. Many applications
have been found for 1+1 dimensional SGE (see [1] for a review). Some generalizations of this equation can be found in [8] and [9].

The organization of this paper is as follows. In section II we will introduce the stepwise sine-Gordon equation in which the parameter $a$ is different for negative and positive values of the scalar field $\varphi$. By applying appropriate boundary conditions, we derive relations between the kink velocities before and after interactions. We investigate the possibility of formation of heavy soliton pairs from light ones and vice versa. In section III we briefly study the relation between the simplex structure and topological charges of different solutions. In section IV we apply boundary conditions to static, periodic and quasi-periodic solutions. Section V is devoted to summary and conclusions.

II. STEPWISE SGE IN 1+1 DIMENSIONS AND BOUNDARY CONDITIONS

By the stepwise SGE, we mean imposing the following condition on equation (1):

$$a = a_1, \text{ for } \varphi < 0$$
$$a = a_2, \text{ for } \varphi > 0.$$  

For each of the regions $\varphi > 0$ and $\varphi < 0$, the scalar field $\varphi$ satisfies (11) with the corresponding value of the parameter $a$. The region needing particular attention is the boundary region separating the $\varphi < 0$ and $\varphi > 0$ regions. If we have $\varphi < 0$ for $t < 0$ and $\varphi > 0$ for $t > 0$ (or vice versa), we can show that the following boundary condition holds:

$$\forall x; \quad \left. \frac{\partial \varphi}{\partial t} \right|_0^+ = \left. \frac{\partial \varphi}{\partial t} \right|_0^- .$$  

In order to arrive at this boundary condition, we can integrate the dynamical equation from $t = -\epsilon$ to $t = +\epsilon$ and let $\epsilon \to 0$. In a similar way, if the two regions $\varphi > 0$ and $\varphi < 0$ connect to each other at $x = 0$, we obtain the following junction condition:

$$\forall t; \quad \left. \frac{\partial \varphi}{\partial x} \right|_0^+ = \left. \frac{\partial \varphi}{\partial x} \right|_0^- .$$  

This boundary condition is obtained by integrating the dynamical equation over $x$ from $x = -\epsilon$ to $+\epsilon$ and letting $\epsilon \to 0$. These conditions provide the key relations for studying the dynamics of two soliton interactions in the stepwise SGE.

Like the conventional SG equation, the stepwise SGE is a relativistically covariant equation. One expects the boundary conditions mentioned above to be expressible in a manifestly covariant...
form. This is indeed the case. In deriving equations (9) and (10), we have assumed that the boundary between $\varphi < 0$ and $\varphi > 0$ regions occur at $t = 0$ or $x = 0$ hyper-surfaces. In the examples which will follow, this assumption is actually fulfilled. However, in an arbitrary frame of reference, where the boundary between these two regions occurs at $x_b(t)$, the junction conditions should be written in the covariant form

$$\left( \frac{\partial \varphi}{\partial x^\mu} \right)_- = \left( \frac{\partial \varphi}{\partial x^\mu} \right)_+,$$

applied at $x = x_b(t)$ (the - and + signs correspond to either sides of the boundary).

A. Free Kink-Antikink Collision

The free kink-anti-kink solution of the SGE for $m_i^2 > 1$ is [10]:

$$\varphi(x, t) = -4 \arctan \left( \frac{m_i}{\sqrt{m_i^2 - 1}} \frac{\sinh(\sqrt{(m_i^2 - 1)a_i ct})}{\cosh(m_i \sqrt{a_i} x)} \right)$$

(12)

This solution holds for $t > 0$ (with $i=1$) and $t < 0$ (with $i=2$). If we write the suitable parameter for each interval, namely

$$a = a_1 \text{ and } m = m_1 \text{ for } t > 0$$

$$a = a_2 \text{ and } m = m_2 \text{ for } t < 0$$

(13) (14)

in the SGE in each region, then by use the first boundary condition (9) we easily obtain

$$m_1 \sqrt{a_1} = m_2 \sqrt{a_2}$$

(15)

It can be shown that $m_i$ is related to the soliton’s velocity far from the interaction region via

$$m_i = \gamma_i = \frac{1}{\sqrt{1 - \frac{v_i^2}{c^2}}}$$

(16)

And therefore using the relation $M_i c^2 = 8\sqrt{a_i}$, we can rewrite equation (15) as

$$\gamma_1 M_1 c^2 = \gamma_2 M_2 c^2$$

(17)

This equation shows the energy conservation in annihilation of a free kink-anti-kink pair and the creation of another one with different masses. This process has been shown in Figure [11].
B. Bound State Kink-Anti-Kink Pair (Breather Solution)

The kink-anti-kink solution for $m_i^2 < 1$ is

$$\varphi(x, t) = \pm 4 \arctan \left( \frac{m_i \sinh(\sqrt{(1 - m_i^2)} a_i ct)}{\sqrt{1 - m_i^2} \cosh(m_i \sqrt{a_i} x)} \right).$$  \hfill (18)

From the sign of $\varphi$, we can distinguish two regions:

$$a = a_1 \quad m = m_1 \quad \text{for} \quad -\tau_1 < t < 0$$  \hfill (19)

$$a = a_2 \quad m = m_2 \quad \text{for} \quad 0 < t < \tau_2$$  \hfill (20)

where

$$\tau_1 = \frac{\pi}{\sqrt{(1 - m_1^2)} a_1 c} \quad \tau_2 = \frac{\pi}{\sqrt{(1 - m_2^2)} a_1 c}.$$  \hfill (21)

Imposing the boundary condition (9) we get:

$$m_1 \sqrt{a_1} = m_2 \sqrt{a_2}$$  \hfill (22)

Which leads to equation (17). From equation (21), we can write

$$\frac{\tau_2}{\tau_1} = \frac{a_1}{a_2} \sqrt{\frac{1 - m_1^2}{1 - \left(\frac{a_1}{a_2}\right)^2 m_1^2}}.$$  \hfill (23)
FIG. 2: The breather solution in the stepwise SGE.

FIG. 3: Annihilation of a heavy breather and creation of a light free pair.

It is clear that we have no breather in the case $m_2 > 1$ and $t > 0$. In this case, the breather will decay to free light pairs. This case is illustrated in Figures 2 and 3. So, we distinguish two cases for $a_2 < a_1$:

- $m_1 < \frac{a_2}{a_1}$. In this case one breather turns into another breather with a different oscillation period.
• $m_1 > \frac{a_2}{a_1}$. In this case one breather decays into a free one with lighter rest mass. Let us coin this phenomenon, which is the most important result of this work, as the soliton gun (See Fig. 3). This phenomenon is apparently similar to the annihilation of $\mu^+\mu^-$ bound pair and the creation of a free $e^+e^-$ pair.

By suitable choice of initial conditions, it is possible to create a heavy breather via light kink-anti-kink collision. This breather then decays back to the initial free pair. This is similar to the resonance phenomenon in elementary particle physics.

C. Kink-Kink Collision

This case corresponds to $m_1^2 > 1$. Using the kink-kink solution

$$\varphi(x,t) = 4 \arctan \left( \frac{\sqrt{m_1^2 - 1}}{m_i} \frac{\sinh(m_i \sqrt{a_i} x)}{\cosh(\sqrt{(m_i^2 - 1)a_i c t})} \right),$$

and applying the boundary condition (10), after a little calculation we obtain

$$\sqrt{(m_1^2 - 1)a_1} = \sqrt{(m_2^2 - 1)a_2}$$

or

$$\gamma_1 M_1 v_1 = \pm \gamma_2 M_2 v_2$$

After the collision, the velocity of each kink will change (see Figure 4). This is in contradiction with the old idea that solitons will reobtain their initial velocities after collision and only a time delay results. Since our system is relativistic, we can consider a reference frame observer in which one of the solitons is at rest before the collision and acquires a velocity after the collision. This is similar to the collision of a billiard ball to another ball initially at rest.

III. SIMPLEX STRUCTURE AND TOPOLOGICAL CHARGE FOR THE STEPWISE SGE

Simplexes are r-dimensional geometrical structures, which are shown by the symbol $\sigma_r = \langle p_0p_1...p_r \rangle$. For example, a 0-simplex $\langle p_0 \rangle$ is a point or a vertex, and a 1-simplex $\langle p_0p_1 \rangle$ is a line or an edge. A 2-simplex $\langle p_0p_1p_2 \rangle$ is defined to be a triangle with its interior included.
FIG. 4: Collision between two kinks with different rest masses in the stepwise SGE. The RHS kink is 5 times as massive as the LHS one.

and etc. The set of finite number of simplexes in $\mathbb{R}^m$ which are properly fitted together is called simplical complex and denoted by $K$. By 'properly', we mean that:

i) An arbitrary face of a simplex of $K$ belongs to $K$, that is, if $\sigma \in K$ and $\sigma' \leq \sigma$ then $\sigma' \in K$.

ii) If $\sigma$ and $\sigma'$ are two simplexes of $K$, the intersection $\sigma \cap \sigma'$ is either empty or a face of $\sigma$ and $\sigma'$; that is, if $\sigma, \sigma' \in K$ then either $\sigma' \cap \sigma = \emptyset$ or $\sigma \cap \sigma' \leq \sigma$ and $\sigma \cap \sigma' \leq \sigma'$.

If we assign orientation to an r-simplex for $r \geq 1$, it is called $r$-chain and is shown with $\sigma_r = (p_0p_1...p_r)(r > 0)$.

A boundary $\partial_r \sigma_r$ of $\sigma_r$ is an $(r - 1)$-chain defined by:

$$\partial_r(p_0p_1...p_r) \equiv \Sigma(-1)^i(p_0...\hat{p}_i...p_r)$$

(27)

where the point $p_i$ under 'hat' is omitted. The $\partial_r$ is called the boundary operator. The boundary of an $r$-chain is a set of $(r-1)$-chains. We can also define $r$-cycle, which is an $r$-dimensional orientated complexes which has no boundary namely if $c$ is a $r$-cycle, then $\partial_r c = 0$ [11].

We can consider each soliton of the stepwise sine-Gordon equation as a 1-chain in $\varphi$ space. The classical vacuum of the potential of the system is $\varphi_n = 2n\pi, \ n \in \mathbb{Z}$, which are discrete points. The kink solution is an oriented simplex or chain,($\varphi_n\varphi_{n+1}$) and the anti-kink can be shown as a chain in the form $(\varphi_{n+1}\varphi_n) = -(\varphi_n\varphi_{n+1})$. If we act the boundary operator on the one soliton solution (kink), we obtain

$$\partial(\varphi_n\varphi_{n+1}) = \varphi_{n+1} - \varphi_n = 2\pi$$

(28)
$J^\mu = \frac{1}{2\pi} \epsilon^\mu\nu \partial_\nu \varphi$ is the topological current which is identically conserved ($\partial^\mu J_\mu = 0$). The topological charge in the sine-Gordon system can be written as:

$$Q = \int J^\rho dx = \frac{1}{2\pi} [\varphi(+\infty) - \varphi(-\infty)] = \frac{1}{2\pi} \partial_1 (\varphi(+\infty), \varphi(-\infty)).$$

(29)

It is clear that topological charge can be obtained directly by applying the boundary operator on the soliton-chain. The topological charge for the different solutions discussed in previous sections are as follows. For kink ($Q = +1$), anti-kink ($Q = -1$), kink-kink ($Q = +2$), breather and free kink-anti-kink ($Q = 0$). The topological charge is quantized and is independent of the $a_i$ parameters in the stepwise SGE.

In a similar way, we can consider multi-soliton solutions as complexes in the scalar field $\varphi$. If we define the sine-Gordon equation on $S^1$ instead of $R$, the single valuedness condition of $\varphi$ implies that there could be no single soliton solutions, and the multi soliton solutions are as $1$-cycles. It is natural that the total charge of the multi soliton solutions on a compact manifold is zero ($Q = \frac{1}{2\pi} \partial_1 e = 0$). In other words, the total topological charge of the sine-Gordon equation (ordinary or stepwise), on the compact space $S^1$ is always zero.
FIG. 6: Static solution corresponding to a kink, smoothly joining to a part of the kink chain and then to an anti-kink in the stepwise SGE.

IV. STATIC SOLUTIONS

It is well-known that the SG system has also static, periodic and quasi-periodic solutions which can be interpreted as a chain of kinks and anti-kinks or a chain of kinks (anti-kinks). These are given by Jacobi elliptic functions. Consider the first integral of the static SG equation:

\[
\frac{1}{2} \left( \frac{d\varphi}{dx} \right)^2 + a \cos(\varphi) = C, \tag{30}
\]

where \( C \) is an integration constant. For single soliton solutions, it is easily seen that \( C = a \) due to boundary conditions at \( x \to \pm \infty \). For \( C < a \), the static solutions can be expressed in terms of the Jacobi SN functions [12]:

\[
\varphi(x) = (2n + 1)\pi \pm 2 \text{arcsin}(k\text{SN}(\sqrt{a}(x - x_1), k)), \tag{31}
\]

where \( k \) and \( x_1 \) are integration constants, and \( n \) is an integer. This solution oscillates around \( \varphi = \pi \), which corresponds to a lattice of alternating kinks and anti-kinks. As \( k \to 1 \), the distance between neighboring kinks and anti-kinks increases, and in the limit \( k = 1 \), we will have an isolated kink (or anti-kink) solution.

For \( C > a \), we will have the following static solution:

\[
\varphi(x) = (2n + 1)\pi \pm x \pm 2 \text{arcsin}(k\text{SN}(\sqrt{a}(x - x_0), k)). \tag{32}
\]
FIG. 7: Static solution corresponding to a kink chain, smoothly joining to an anti-kink chain in the stepwise SGE.

This solution corresponds to a lattice of kinks, with the distance between the kinks increasing as $k \to 1$.

Based on the fore-mentioned solutions, we now turn to the stepwise SG equation with

$$a = a_1 \quad \text{if} \quad \varphi < \pi,$$

and

$$a = a_2 \quad \text{if} \quad \varphi > \pi.$$  \hspace{1cm} (33) \hspace{1cm} (34)

We must now apply the boundary conditions at $x_j$, where $x_j$'s are the solutions of $\varphi(x_j) = \pi$. We can distinguish several categories of static solutions for the stepwise SG equation. Among different possibilities, let us introduce the following four cases:

- Kink lattice, attached to a single kink. The boundary condition leads to:

$$1 + 2k_1 \sqrt{a_1} = 2\sqrt{a_2}, \quad (k_2 = 1).$$ \hspace{1cm} (35)

A sample solution is shown in Figure 5.
• Kink, smoothly attached to a part of the kink-antikink chain and then to an anti-kink (Figure 6). The boundary condition leads to

$$\sqrt{a_1} = k_2 \sqrt{a_2}, \quad (k_1 = 1).$$

(36)

This is an interesting solution, since it is a non-topological, static soliton with zero topological charge. It is well-known that the conventional SG system does NOT have a static, zero charge (non-topological) solution.

• Kink chain, smoothly attached to a part of the kink-antikink chain and then to an anti-kink chain (Figure 7). The boundary condition leads to

$$1 + 2k_1 \sqrt{a_1} = 2k_2 \sqrt{a_2}.$$  

(37)

• Kink chain attached smoothly to another kink chain (see Figure 8):

$$k_1 \sqrt{a_1} = k_2 \sqrt{a_2}.$$  

(38)
V. SUMMARY AND CONCLUSION

In this paper, we studied the stepwise sine-Gordon system in which the mass parameter \(a\) is different for positive and negative values of the scalar field. By applying appropriate boundary conditions at the boundary between the two regions, which are derived from the field equation, we extracted the relation between the soliton velocities before and after the collision.

Similar to the boundary conditions in electrodynamics or in the Schrödinger equation, the boundary conditions of the stepwise SGE are obtained by integrating the field equation across the boundary between the \(\varphi < 0\) and \(\varphi > 0\) regions. These relations are consistent with energy and momentum conservation relations for the system under consideration. We considered different cases and showed that in such a system, it is possible to transform a heavy pair of solitons to a light pair and vice versa. The concept of soliton gun was introduced for the first time. In this process, a heavy bound pair annihilates into a free light mass pair, moving at high velocities. Since the kinks belonging to different \((\varphi < 0\) and \(\varphi > 0\)) sectors have different rest energies (masses), they cannot retrieve their initial velocities after the collisions. The stepwise SGE is therefore an interesting system for studying the collisions between solitons of different rest energies.

The similarity of the kinks and classical particles is further demonstrated in the framework of the stepwise SGE. We emphasize that in this system, the initial soliton velocities are not necessarily retrieved after the collisions.

The application of boundary conditions to static, periodic and quasi-periodic solutions was also worked out, leading to new results, not present in the conventional SG system.

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[1] N. Riazi and A. R. Gharaati, *Int. J. Theor. Phys.*, 37, 1081 (1998).
[2] K. Pohlmeyer, *Comm. Math. Phys.*, 46, 207 (1976).
[3] A. Mikhailov, [hep-th/0511069](https://arxiv.org/abs/hep-th/0511069).
[4] C. L. Terng, K. Uhlenbeck, *Notices of Am. Math. Soc.*, Jan. issue, 47, 17 (2000).
[5] M. Guidry, *Gauge Field Theories*, (John Wiley and Sons, New York, 1991).
[6] A. Das, *Integrable Models*, (World Scientific, Singapore, 1989).
[7] R. Rajaraman, *Solitons and Instantons*, (Elsevier, Amsterdam 1982).

[8] N. Riazi, A. Azizi and S. M. Zebarjad, *Phys. Rev.* **D66**, 065003 (2002).

[9] N. Riazi and K. Mansouri, *Int. J. Theor. Phys.*, **44**, 309 (2005).

[10] G. L. Lamb Jr., *Elements of Soliton Theory*, (John Wiley and Sons, New York 1980).

[11] M. Nakahara, *Geometry, Topology and Physics*, (IOP Publishing Ltd., Bristol, 1990).

[12] I. Bakas and C. Sourdis, *Fortschr. Phys.* **50**, 815 (2002).