A GENERALIZATION OF THE PROBABILITY THAT THE COMMUTATOR OF TWO GROUP ELEMENTS IS EQUAL TO A GIVEN ELEMENT

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Abstract. The probability that the commutator of two group elements is equal to a given element has been introduced in literature few years ago. Several authors have investigated this notion with methods of the representation theory and with combinatorial techniques. Here we illustrate that a wider context may be considered and show some structural restrictions on the group.

1. Different formulations of the commutativity degree

Given two elements \(x\) and \(y\) of a group \(G\), several authors studied the probability that a randomly chosen commutator \([x, y]\) of \(G\) satisfies a prescribed property. P. Erdős and P. Turán [6] began to investigate the case \([x, y] = 1\), noting some structural restrictions on \(G\) from bounds of statistical nature. Their approach involved combinatorial techniques, which were developed successively in [2, 3, 4, 5, 7, 9, 10, 12, 13, 15, 17] and extended to the infinite case in [8, 13, 18]. On another hand, P. X. Gallagher [11] investigated the case \([x, y] = 1\), using character theory, and opened another line of research, illustrated in [3, 4, 12, 16, 19]. The literature shows that it is possible to variate the condition on \([x, y]\) involving arbitrary words, which could not be the commutator word \([x, y]\). From now, all the groups which we consider will be finite.

Given two subgroups \(H\) and \(K\) of \(G\) and two integers \(n, m \geq 1\), we define

\[
p_{y}^{(n,m)}(H, K) = \frac{|\{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in H^n \times K^m \mid [x_1, \ldots, x_n, y_1, \ldots, y_m] = y\}|}{|H|^n |K|^m},
\]

as the probability that a randomly chosen commutator of weight \(n + m\) of \(H \times K\) is equal to a given element of \(G\). Denoting

\[
\mathcal{A} = \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in H^n \times K^m \mid [x_1, \ldots, x_n, y_1, \ldots, y_m] = y\},
\]

\(|\mathcal{A}| = |H|^n |K|^m \cdot p_{y}^{(n,m)}(H, K)\). The case \(n = m = 1\) can be found in [4] and is called generalized commutativity degree of \(G\). For \(n = m = 1\) and \(H = K = G\),

\[
p_{y}^{(1,1)}(G, G) = p_{y}(G) = \frac{|\{(x, y) \in G^2 \mid [x, y] = y\}|}{|G|^2},
\]

is the probability that the commutator of two group elements of \(G\) is equal to a given element of \(G\) in [16].

It is well known (see for instance [11, Exercise 3, p. 183]) that the function \(\psi(g) = |\{(x, y) \in G \times G \mid [x, y] = g\}|\) is a character of \(G\) and we have

\[
\psi = \sum_{\chi \in \text{Irr}(G)} \frac{\chi(g)}{\chi(1)} \chi,
\]

where \(\text{Irr}(G)\) denotes the set of all irreducible complex characters of \(G\). However, the authors exploited this fact in [16, Theorem 2.1], writing (1.3) as

\[
p_{y}(G) = \frac{1}{|G|} \sum_{\chi \in \text{Irr}(G)} \chi(g) \chi(1),
\]

For terminology and notations in character theory we refer to [14].

Now for \(g = 1\),

\[
p_{1}^{(1,1)}(G, G) = p_{1}(G) = d(G) = \frac{|\{(x, y) \in G^2 \mid [x, y] = 1\}|}{|G|^2} = \frac{|\text{Irr}(G)|}{|G|}.
\]
Remark 2.1. If $S = \{[x_1, \ldots, x_n, y_1, \ldots, y_m] \mid x_1, \ldots, x_n \in H; y_1, \ldots, y_m \in K\}$, then $p_{(n,m)}(H, K) = 0$ if and only if $g \notin S$. On another hand, $p_{(n,m)}(H, K) = 1$ if and only if $[H, \ldots, H, K, \ldots, K] = [nH, mK] = 1$.

Remark 2.2. The equation (1.1) assigns by default the map
\[
p_g^{(n,m)} : (x_1, \ldots, x_n, y_1, \ldots, y_m) \in H^n \times K^m \mapsto p_g^{(n,m)}(H, K) \in [0, 1],
\]
which is a probability measure on $H^n \times K^m$, satisfying a series of standard properties such as being multiplicative, symmetric and monotone.

The fact that (2.1) is multiplicative is described by the next result.

Proposition 2.3. Let $E$ and $F$ be two groups such that $e \in E, f \in F, A, C \leq E$ and $B, D \leq F$. Then
\[
p_{(e,f)}^{(n,m)}(A \times C, B \times D) = p_{e}^{(n,m)}(A, B) \cdot p_{f}^{(n,m)}(C, D).
\]

Proof. It is enough to note that
\[
[(a_1, \ldots, a_n), (c_1, \ldots, c_n), ([b_1, \ldots, b_m], [d_1, \ldots, d_m])] = ([(a_1, \ldots, a_n), [b_1, \ldots, b_m]], [c_1, \ldots, c_n], [d_1, \ldots, d_m])].
\]

Proposition 2.4. With the notations of (1.1), $p_g^{(n,m)}(H, K) = p_{g^{-1}}^{(n,m)}(K, H)$. Moreover, if $H$, or $K$, is normal in $G$, then $p_g^{(n,m)}(H, K) = p_g^{(n,m)}(K, H) = p_{g^{-1}}^{(n,m)}(H, K)$.

Proof. The commutator rule $[x, y]^{-1} = [y, x]$ implies the first part of the result. Now let $H$ be normal in $G$, $n \leq m$ and $B = \{[y_1, \ldots, y_m, x_1, \ldots, x_n] \in K^m \times H^n \mid [y_1, \ldots, y_m, x_1, \ldots, x_n] = g\}$. The map $\varphi : (x_1, \ldots, x_n, y_1, \ldots, y_m) \in A \mapsto (y_1^{-1}, y_2^{-1}, \ldots, y_m^{-1}, b_1, \ldots, b_m, y_1 x_1 y_1^{-1}, y_2 x_2 y_2^{-1}, \ldots, y_n x_n y_n^{-1}) \in B$ is bijective and so the remaining equalities follow. A similar argument can be applied, when the assumption $H$ is normal in $G$ is replaced by $K$ is normal in $G$.

The fact that (2.1) is monotone is more delicate to prove, since this is a situation in which we may find upper bounds for (1.1). Details are given later on. Now we will get another expression for (1.1). With the notations of (1.1), $C_l(K([x_1, \ldots, x_n])$ denotes the $K$-conjugacy class of $[x_1, \ldots, x_n] \in H$. 

2. TECHNICAL PROPERTIES AND SOME COMPUTATIONS

We begin with two elementary observations on (1.1).

is the probability of commuting pairs of $G$ (or briefly the commutativity degree of $G$), largely studied in [2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 15, 16, 17, 19]. In particular, 

(\begin{equation} 
\label{eq:2.1} 
p_{(n,1)}^{(n,1)}(G, G) = \frac{|\{(x_1, \ldots, x_n, x_{n+1}) \in G^{n+1} \mid [x_1, \ldots, x_n, x_{n+1}] = 1\}|}{|G|^{n+1}} = d_h(n, G), 
\end{equation} 

is the $n$-th nilpotency degree of $G$ in [2, 3, 9, 17, 18] and that

(\begin{equation} 
\label{eq:2.2} 
p_{(1,1)}^{(n,1)}(H, G) = \frac{|\{(x_1, \ldots, x_n) \in H^{n} \times G \mid [x_1, \ldots, x_n] = 1\}|}{|H|^n |G|} = d_h(n, H, G), 
\end{equation} 

is the relative $n$-th nilpotency degree of $H$ in $G$, studied in [7, 9, 17, 18]. We may express (1.7) not necessarily with $g = 1$; assuming that $H$ is normal in $G$, Equation (4) and Theorem 4.2 imply

(\begin{equation} 
\label{eq:2.3} 
p_{g}^{(1,1)}(H, G) = \frac{|\{(x, y) \in H \times G \mid [x, y] = 1\}|}{|H| |G|} = \frac{1}{|H||G|} \sum_{\chi \in \text{Irr}(G)} |H| \langle \chi, \chi \rangle = \chi(g), 
\end{equation} 

where $\chi_H$ denotes the restriction of $\chi$ to $H$ and $\langle , \rangle$ the usual inner product. Our purpose is to study (1.1), extending the previous contributions in [2, 3, 7, 10, 17]. The main results of the present paper are in Section 3, in which the general considerations of Section 2 are applied.
Proposition 2.5. With the notations of (1.1),

\[(2.2)\quad p_g^{(n,m)}(H,K) = \frac{1}{|H|^n |K|^m} \sum_{x_1,\ldots,x_n \in H \atop g^{-1}[x_1,\ldots,x_n] \in \text{Cl}_K([x_1,\ldots,x_n])} |C_K([x_1,\ldots,x_n])|^m.\]

Proof. It is straightforward to check that
\[(2.3)\quad C_K^m([x_1,\ldots,x_n]) = \underbrace{C_K([x_1,\ldots,x_n]) \times \cdots \times C_K([x_1,\ldots,x_n])}_{\text{m-times}}.\]

In particular, \(|C_K^m([x_1,\ldots,x_n])| = \underbrace{|C_K([x_1,\ldots,x_n])|^m}_{\text{m-times}}.\]

\[A = \bigcup_{[x_1,\ldots,x_n] \in H} \{[x_1,\ldots,x_n] \times T_{[x_1,\ldots,x_n]}\}, \text{ where } T_{[x_1,\ldots,x_n]} = \{(y_1,\ldots,y_m) \in K^m \mid [x_1,\ldots,x_n,y_1,\ldots,y_m] = g\}.\]

Obviously, \(T_{[x_1,\ldots,x_n]} \neq \emptyset\) if and only if \(g^{-1}[x_1,\ldots,x_n] \in \text{Cl}_K([x_1,\ldots,x_n]).\) Let \(T_{[x_1,\ldots,x_n]} \neq \emptyset.\) Then \(|T_{[x_1,\ldots,x_n]}| = |C_K^m([x_1,\ldots,x_n])|,\) because the map \(\psi : [y_1,\ldots,y_m] \mapsto g[y_1,\ldots,y_m] \) is bijective, where \([y_1,\ldots,y_m]\) is a fixed element of \(T_{[x_1,\ldots,x_n]}.\) We deduce that
\[(2.4)\quad |A| = \sum_{[x_1,\ldots,x_n] \in H} |T_{[x_1,\ldots,x_n]}| = \sum_{x_1,\ldots,x_n \in H \atop g^{-1}[x_1,\ldots,x_n] \in \text{Cl}_K([x_1,\ldots,x_n])} |C_K^m([x_1,\ldots,x_n])|^m\]
and the result follows. \(\square\)

Special cases of Proposition 2.5 are listed below.

Corollary 2.6. In Proposition 2.5, if \(m = 1\) and \(G = K,\) then
\[(2.5)\quad p_g^{(n,1)}(H,G) = \frac{1}{|H|^n |G|} \sum_{x_1,\ldots,x_n \in H \atop g^{-1}[x_1,\ldots,x_n] \in \text{Cl}_G([x_1,\ldots,x_n])} |C_G([x_1,\ldots,x_n])|.\]

Corollary 2.7 (See [4], Theorem 2.3). In Proposition 2.5, if \(m = n = 1,\) then
\[(2.6)\quad p_g^{(1,1)}(H,K) = \frac{1}{|H| |K|} \sum_{x \in H \atop g^{-1}x \in \text{Cl}_K(x)} |C_K(x)|.\]

In particular, if \(G = K,\) then \(p_g^{(1,1)}(H,G) = \frac{1}{|H| |G|} \sum_{x \in H \atop g^{-1}x \in \text{Cl}_G(x)} |C_G(x)|.\)

Corollary 2.8 (See [7], Proof of Lemma 4.2). In Proposition 2.5, if \(m = 1\) and \(G = K,\) then
\[(2.7)\quad p_1^{(n,1)}(H,G) = d^{(n)}(H,G) = \frac{1}{|H|^n |G|} \sum_{x_1,\ldots,x_n \in H} |C_G([x_1,\ldots,x_n])|.\]

Corollary 2.9. In Proposition 2.5, if \(C_K([x_1,\ldots,x_n]) = 1,\) then
\[(2.8)\quad p_1^{(n,m)}(H,K) = \frac{1}{|H|^n} + \frac{1}{|K|^m} - \frac{1}{|H|^n |K|^m}.\]

[4] Proposition 3.4 follows from Corollary 2.9 when \(m = n = 1.\)

Remark 2.10. Equation (1.7) makes equivalent the study of \(p_1^{(n,1)}(H,G)\) and that of \(d^{(n)}(H,G).\) This is illustrated in Corollary 2.8 and noted here for the first time. Therefore there are many information from [2] [7] [9] [17] and [4] [3] [16] which can be connected. It is relevant to point out that these concepts were treated independently and with different methods in the last years.
Let \( \chi \) be a character of \( G \) and \( \theta \) be a character of \( H \leq G \). The Frobenius Reciprocity Law \([14\, Lemma\, 5.2]\) gives a link between the restriction \( \chi_H \) of \( \chi \) to \( H \) and the induced character \( \theta^G \) of \( \theta \). Therefore \( \langle \chi, \theta^G \rangle_G = \langle \chi_H, \theta \rangle_H \). Write this number as \( e_{(\chi, \theta)} = \langle \chi, \theta^G \rangle_G = \langle \chi_H, \theta \rangle_H \). If \( e_{(\chi, \theta)} = 0 \), then \( \theta \) does not appear in \( \chi_H \) and so \( \chi \) does not appear in \( \theta^G \). Recall from \([14]\) that, if \( e_{(\chi, \theta)} \neq 0 \), then \( \chi \) covers \( \theta \) (or also \( \theta \) belongs to the constituents of \( \chi_H \)). In particular, if \( \theta = \chi_H \), then \( e_{(\chi, \chi_H)} = \langle \chi, \chi_H^G \rangle_G = \langle \chi_H, \chi_H \rangle_H \). From a classic relation (see \([14\, Lemma\, 2.29]\)), \( e_{(\chi, \chi_H)} = \langle \chi, \chi_H \rangle_G = \langle \chi_H, \chi_H \rangle_H \leq |G : H| \langle \chi, \chi \rangle_G = |G : H| e_{(\chi, \chi)} \) and the equality holds if and only if \( \chi(x) = 0 \) for all \( x \in G - H \). In particular, if \( \chi \in \text{Irr}(G) \), then \( \langle \chi_H, \chi_H \rangle_H = |G : H| \) if and only if \( \chi(x) = 0 \), for all \( x \in G - H \). Therefore the following result is straightforward.

**Corollary 2.11.** With the notations of \([14]\), \( p_g^{(1,1)}(H, G) \leq |G : H| \; p_1(g) \) and the equality holds if and only if all the characters vanish on \( G - H \).

At this point, \([4\, Theorem\, 4.2]\) becomes

(2.9) \quad \zeta(g) = |H| \sum_{\chi \in \text{Irr}(G)} \frac{e_{(\chi, \chi g)}}{\chi(1)} \chi(g) = |\{(x, y) \in H \times G \mid [x, y] = g\}| = \sum_{x \in H} \sum_{g^{-1}x \in C_G(x)} |C_G(x)|,

where \( \zeta(g) \) is the number of solutions \((x, y) \in H \times G\) of the equation \([x, y] = g\). Note that \((2.9)\) and \([1\, Exercise\, 3, p. 183]\) give a short argument to prove that \( \zeta(g) \) is a character of \( G \) with respect to the argument in \([4\, Corollary\, 4.3]\). The equation \((2.9)\) becomes

(2.10) \quad p_g^{(1,1)}(H, G) = \frac{\zeta(g)}{|H| |G|}.

For the general case that \( n > 1, m > 1 \) and \( G = K \),

(2.11) \quad p_g^{(n,m)}(H, G) = \frac{\zeta^{(n,m)}(g)}{|G|^m} \frac{1}{|K|^n} \sum_{x_1, \ldots, x_n \in H} \sum_{g^{-1}[x_1, \ldots, x_n] \in C_G([x_1, \ldots, x_n])} |C_G([x_1, \ldots, x_n])|^m,

where

(2.12) \quad \zeta^{(n,m)}(g) = \sum_{x_1, \ldots, x_n \in H} \sum_{g^{-1}[x_1, \ldots, x_n] \in C_G([x_1, \ldots, x_n])} |C_G([x_1, \ldots, x_n])|^m

is the number of solutions \((x_1, \ldots, x_n, y_1, \ldots, y_m) \in H^n \times G^m\) of \([x_1, \ldots, x_n, y_1, \ldots, y_m] = g\).

**Remark 2.12.** There are many evidences from the computations that \( \zeta^{(n,m)}(g) \) is a character of \( G \).

Now we may prove upper bounds for \((2.1)\) and find that \((2.3)\) is monotone.

**Proposition 2.13.** With the notations of \([14]\), if \( H \leq K \), then \( p_g^{(n,m)}(H, G) \geq p_g^{(n,m)}(K, G) \). The equality holds if and only if \( \text{Cl}_H(x) = \text{Cl}_K(x) \) for all \( x \in G \).

**Proof.** We note that \( \frac{1}{|K|} \leq \frac{1}{|H|} \) and then \( \frac{1}{|K|^n} \leq \frac{1}{|H|^n} \). By Proposition \(2.5\)

\[
|G|^m \cdot p_g^{(n,m)}(K, G) = \frac{1}{|K|^n} \sum_{x_1, \ldots, x_n \in K} |C_G([x_1, \ldots, x_n])|
\]

(2.13) \quad \leq \frac{1}{|H|^n} \sum_{x_1, \ldots, x_n \in K} |C_G([x_1, \ldots, x_n])|

in particular the last relation is true for \( x_1, \ldots, x_n \in H \leq K \) and continuing

(2.14) \quad \frac{1}{|H|^n} \sum_{x_1, \ldots, x_n \in H} |C_G([x_1, \ldots, x_n])| = |G|^m \cdot p_g^{(n,m)}(H, G).

The rest of the proof is clear. (\qed)

The next result shows an upper bound, which generalizes \([7\, Theorem\, 4.6]\).
Proposition 2.14. With the notations of (1.1), if $N$ is a normal subgroup of $G$ such that $H \leq N$, then $p_g^{(n,m)}(H, G) \leq p_g^{(n,m)}(N, H)$. Moreover, if $N \cap [nH, mG] = 1$, then the equality holds.

Proof. We have
\[
|H|^n \cdot |G|^m \cdot p_g^{(n,m)}(H, G) = |A|
\]
\[
= |\{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in H^n \times G^m \mid [x_1, \ldots, x_n, y_1, \ldots, y_m] \cdot g^{-1} = 1\}|
\]
\[
= |\{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in H^n \times G^m \mid [x_1, \ldots, x_n, y_1, \ldots, y_m, g^{-1}] = 1\}|
\]
\[
= \sum_{x_1 \in H} \cdots \sum_{x_n \in H} \sum_{y_1 \in G} \cdots \sum_{y_m \in G} |C_G([x_1, \ldots, x_n, y_1, \ldots, y_m])|
\]
\[
= \sum_{x_1 \in H} \cdots \sum_{x_n \in H} \sum_{y_1 \in G} \cdots \sum_{y_m \in G} \left|C_G([x_1, \ldots, x_n, y_1, \ldots, y_m])\right| |N|^{n+m}
\]
\[
\leq \sum_{x_1 \in H} \cdots \sum_{x_n \in H} \sum_{y_1 \in G} \cdots \sum_{y_m \in G} |C_G(N)\left([x_1, \ldots, x_n, y_1, \ldots, y_m]\right)|
\]
\[
= \sum_{s_1 \in H/N} \sum_{x_1 \in s_1} \cdots \sum_{s_n \in H/N} \sum_{x_1 \in s_1} \cdots \sum_{x_n \in s_n} \sum_{y_1 \in G/N} \cdots \sum_{y_m \in G/N} |C_G(N)\left([s_1, \ldots, s_n, T_1, \ldots, T_m]\right)|
\]
\[
= N^{n+m} \sum_{s_1 \in H/N} \sum_{x_1 \in s_1} \cdots \sum_{x_n \in s_n} \sum_{y_1 \in G/N} \cdots \sum_{y_m \in G/N} |C_G(N)\left([s_1, \ldots, s_n, T_1, \ldots, T_m]\right)|
\]
\[
= \left( \frac{H}{N} \right)^n \cdot \left( \frac{G}{N} \right)^m \cdot p_g^{(n,m)}(N, H) \cdot |N|^{n+m} = \left|H^n \cdot |G|^m \cdot p_g^{(n,m)}(N, H) \right|
\]
The condition of equality in the above relations is satisfied exactly when $N \cap [nH, mG] = 1$. The result follows. □

Corollary 2.15. A special case of Proposition 2.14 is $p_g(N) \leq p_g(G/N)$.

Corollary 2.16 (See [7], Theorem 4.6). In Proposition 2.14, if $m = 1$ and $g = 1$, then $d^{(n)}(H, G) \leq d^{(n)}(H/N, G/N)$.

3. SOME UPPER AND LOWER BOUNDS

A relation among (1.1)–(1.5) is described below.

Theorem 3.1. With the notations of (1.1), $p_g^{(n, m)}(G, G) \leq p_g^{(n, m)}(H, K) \leq p_1^{(n, m)}(H, K) \leq p_1^{(n, m)}(H, G) \leq p_1^{(n, m)}(H, H)$.

Proof. From Proposition 2.13 $p_g^{(n, m)}(G, G) \leq p_g^{(n, m)}(G, H)$. From Proposition 2.5
\[
p_g^{(n, m)}(H, K) = \frac{1}{|H|^n \cdot |K|^m} \sum_{x_1, \ldots, x_n \in H} |C_K([x_1, \ldots, x_n])|^m
\]
and for $g = 1$ we get
\[
\leq \frac{1}{|H|^n \cdot |K|^m} \sum_{x_1, \ldots, x_n \in H} |C_K([x_1, \ldots, x_n])|^m = p_1^{(n, m)}(H, K),
\]
where in the last passage still Proposition 2.5 is used. From $C_K([x_1, \ldots, x_n]) \subseteq C_G([x_1, \ldots, x_n])$, we deduce
\[
\leq \sum_{x_1, \ldots, x_n \in H} |C_G([x_1, \ldots, x_n])|^m = p_1^{(n, m)}(H, G).
\]
Applying Proposition 3.3, \( p_1^{(n,m)}(H, G) = p_1^{(n,m)}(G, H) \) and so \( p_1^{(n,m)}(G, H) \leq p_1^{(n,m)}(H, H) \) by Proposition 2.13.

**Corollary 3.2.** With the notations of (11), if \( Z(G) = 1 \), then \( p_2^{(n,1)}(H, K) \leq 2^{n-1} \).

**Proof.** It follows from Theorem 6.1 and [7, Theorem 5.3].

Another significant restriction is the following.

**Theorem 3.3.** With the notations of (11), let \( p \) be the smallest prime divisor of \(|G|\). Then

(i) \( p_2^{(n,m)}(H, K) \leq \frac{2^{n} + p - 2}{p^{m+1}} \),

(ii) \( p_2^{(n,m)}(H, K) \geq \frac{\{1 - p\}Y_{H}n + \{p\}H^n}{|H|^n |K|^m} - \frac{\{(K+p)C_H(K)\}^n}{|H|^n |K|^m} \);

where \( Y_{H^n} = \{[x_1, \ldots, x_n] \in H^n | \ C_K([x_1, \ldots, x_n]) = 1\} \).

**Proof.** If \([n_{H,m}K] = 1\), then \( C_{H^n}(K^n) = H^n \) and \( Y_{H^n} \) equals \( H^n \) or an empty set according as \( K^n \) is trivial or nontrivial. Assume that \([n_{H,m}K] \neq 1\). Then \( Y_{H^n} \cap C_{H^n}(K^n) = Y_{H^n} \cap (C_{H}(K) \times C_{H}(K) \times \ldots \times C_{H}(K)) = Y_{H^n} \cap (C_{H}(K) \times C_{H}(K) \times \ldots \times C_{H}(K)) \neq \emptyset \) and

\[
\begin{align*}
\sum_{x_1, \ldots, x_n \in H} |C_{K^n}([x_1, \ldots, x_n])| = & \sum_{x_1, \ldots, x_n \in H} |C_{K}([x_1, \ldots, x_n])|^m \\
= & \sum_{x_1, \ldots, x_n \in Y_{H^n}} |C_{K}([x_1, \ldots, x_n])|^m + \sum_{x_1, \ldots, x_n \in C_{H^n}(K)} |C_{K}([x_1, \ldots, x_n])|^m \\
& + \sum_{x_1, \ldots, x_n \in H^n - (Y_{H^n} \cup C_{H^n}(K))} |C_{K}([x_1, \ldots, x_n])|^m.
\end{align*}
\]

(3.4)

Since \( p^m \leq |C_K([x_1, \ldots, x_n])|^m \leq \frac{|K^m|}{p^m}, \ Y_{H^n} \leq |H^n| \) and \( p^n \leq |C_{H}(K)|^n \leq \frac{|H^n|}{p^n} \),

(3.5)

\[
\begin{align*}
\leq & \ Y_{H^n} + |K| |C_{H}(K)|^n + (|H^n| - (|Y_{H^n}| + |C_{H}(K)|^n)) \frac{|K^m|}{p^m} \\
& + \sum_{x_1, \ldots, x_n \in H^n - (Y_{H^n} \cup C_{H^n}(K))} |C_{K}([x_1, \ldots, x_n])|^m.
\end{align*}
\]

and then

(3.6)

\[
\begin{align*}
p_2^{(n,m)}(H, K) \leq \frac{\{1 - p\}Y_{H}n}{|H|^n |K|^m} + \frac{|K| |C_H(K)|^n}{|H|^n |K|^m} + \frac{1}{p^m} - \frac{|Y_{H^n}|}{p^m - |H^n|} - \frac{|C_{H}(K)|^n}{p^m - |H^n|} \\
& \leq \frac{1}{p^m} + \frac{1}{p^m - |H^n|} + \frac{1}{p^m} - \frac{1}{p^m - 1} = \frac{2^{n} + p - 2}{p^{m+1}}.
\end{align*}
\]

Hence (i) follows. On another hand, we may continue in the other direction

\[
\begin{align*}
& \geq |Y_{H^n}| + |K| |C_{H}(K)|^n + p (|H^n| - (|Y_{H^n}| + |C_{H}(K)|^n)) \\
& \geq |Y_{H^n}| + |K| |C_{H}(K)|^n + p \left( \frac{1 - p}{|H^n|} \right) \frac{|Y_{H^n}|}{|H^n| |K|^m} + \frac{1}{p^m} - \frac{|K| + p |C_{H}(K)|^n}{|H^n| |K|^m}.
\end{align*}
\]

Then (ii) follows.

The bound in Theorem 3.3 (i) is a little bit different from the bound in [4, Corollary 3.9], where it is proved that \( p_2^{(1,1)}(H, K) \leq \frac{2^{p-1}}{p} \) and in particular \( p_2^{(1,1)}(H, K) \leq \frac{3}{4} \). We conclude the following structural restriction.

**Corollary 3.4.** In Theorem 3.3, if \( p_2^{(n,m)}(H, K) = \frac{2^{p^n + p - 2}}{p^{m+1}} \), then

(3.9)

\[
|H : C_H(K)| \leq \left( \frac{p^{n+1} - p^3 - \frac{p^2}{2p} + p}{2p^2 + p - 2} \right)^\frac{1}{n}.
\]

**Proof.** Looking at (3.8) and the proof of Theorem 3.3 (i), we deduce

(3.10)

\[
\frac{2^{p^n + p - 2}}{p^{m+1}} \leq \frac{|Y_{H^n}|}{|H^n| |K|^m} + \frac{|K| |C_H(K)|^n}{|H^n| |K|^m} + \frac{1}{p^m} \leq \frac{1}{p^m} + \frac{1}{p^m - 1} \left( \frac{C_H(K)}{|H^n|} \right)^n + \frac{1}{p^m}
\]

\[
= \frac{1}{p^m} \left( \frac{2}{p} + \frac{|C_H(K)|}{|H^n|} \right)^n.
\]
and then $\frac{2p^n + p - 2}{p^n + 2} \leq 2 + \frac{\left| CH(K) \right|}{p}$. We conclude that $\frac{p^{n+1}}{2p^n + p - 2} \geq \frac{p}{2} + \frac{H}{\left| CH(K) \right|}$ and so

$$p^{n+1} \left( \frac{1}{2p^n + p - 2} - \frac{p}{2} \right) = \frac{p^{n+1} - p^3 - p^2 + p}{2p^n + p - 2} \geq \frac{H}{\left| CH(K) \right|} \cdot$$

The result follows, once we extract the $n$-th root. □

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