On the spectral radius of a class of non-odd-bipartite even uniform hypergraphs*

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Abstract: In order to investigate the non-odd-bipartiteness of even uniform hypergraphs, we introduce a class of $k$-uniform hypergraphs $G$, called $(k, \frac{k}{2})$-hypergraphs, which satisfy the property: $k$ is even, every edge $e$ of $G$ can be divided into two disjoint $\frac{k}{2}$-vertex sets say $e_1$ and $e_2$ and for any $e'$ incident to $e$, $e \cap e' = e_1$ or $e_2$. Such graph $G$ can be constructed from a simple graph, which is called the underlying graph of $G$. We show that $G$ is non-odd-bipartite if and only if the underlying graph of $G$ is non-bipartite. We obtain some results for the spectral radius of weakly irreducible nonnegative tensors, and use them to discuss the perturbation of the spectral radius of the adjacency tensor or signless Laplacian tensor of a $(k, \frac{k}{2})$-hypergraph after an edge is subdivided. Finally we show that among all $(k, \frac{k}{2})$-hypergraphs with $n$ half edges, the minimum spectral radius of the adjacency tensor (resp. signless Laplacian tensor) is achieved uniquely for $C_n$ when $n$ is odd and for $C_{n-1} + e$ when $n$ is even.

Keywords: Hypergraph; non-odd-bipartiteness; adjacency tensor; signless Laplacian tensor; spectral radius

1 Introduction

Hypergraphs are a generalization of simple graphs. They are really handy to show complex relationships found in the real world. A hypergraph $G = (V(G), E(G))$ is a set of vertices say $V(G) = \{v_1, v_2, \ldots, v_n\}$ and a set of edges, say $E(G) = \{e_1, e_2, \ldots, e_m\}$ where $e_j \subseteq V(G)$. If $|e_j| = k$ for each $j = 1, 2, \ldots, m$, then $G$ is called a $k$-uniform hypergraph. In particular, the 2-uniform hypergraphs are exactly the classical simple graphs. The degree $d_v$ of a vertex $v \in V(G)$ is defined as $d_v = |\{e_j : v \in e_j \in E(G)\}|$. A walk $W$ of length $l$ in a hypergraph is a sequences of alternate vertices and edges: $v_0, e_1, v_1, e_2, \ldots, e_l, v_l$, where $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 0, 1, \ldots, l-1$. If $v_0 = v_l$, then $W$ is called a circuit. A walk is called a path if no vertices or edges are repeated. A circuit is called a cycle if no vertices or edges are repeated. A hypergraph is said to be connected.

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if every two vertices are connected by a walk. Let $G$ and $H$ be two hypergraphs. We say $H$ is a sub-hypergraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and $H$ is a proper sub-hypergraph of $G$ if $V(H) \subsetneq V(G)$ or $E(H) \subsetneq E(G)$.

In recent years spectral hypergraph theory has emerged as an important field in algebraic graph theory. Let $G$ be a $k$-uniform hypergraph. The adjacency tensor $A = A(G) = (a_{i_1i_2...i_k})$ of $G$ is a $k$th order $n$-dimensional symmetric tensor, where

$$a_{i_1i_2...i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{v_{i_1}, v_{i_2}, ..., v_{i_k}\} \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Let $D = D(G)$ be a $k$th order $n$-dimensional diagonal tensor, whose diagonal element $d_{i...i}$ is $d_{vi}$ for all $i \in [n] := \{1, 2, ..., n\}$. Then $L = L(G) = D(G) - A(G)$ is the Laplacian tensor of the hypergraph $G$, and $Q = Q(G) = D(G) + A(G)$ is the signless Laplacian tensor of $G$.

Qi [9] showed that $\rho(L(G)) \leq \rho(Q(G))$, and posed a question of identifying the conditions under which the equality holds. Hu et. al [5] proved that if $G$ is connected, then the equality holds if and only if $k$ is even and $G$ is odd-bipartite. Here an even uniform hypergraph $G$ is called odd-bipartite if $V(G)$ has a bipartition $V(G) = V_1 \cup V_2$ such that each edge has an odd number of vertices in both $V_1$ and $V_2$. Such partition will be called an odd-bipartition of $G$. Shao et. al [11] proved a stronger result that the Laplacian $H$-spectrum (resp. Laplacian spectrum) and signless Laplacian $H$-spectrum (resp. Laplacian spectrum) of a connected $k$-uniform hypergraph $G$ are equal if and only if $k$ is even and $G$ is odd-bipartite. They also proved that the adjacency $H$-spectrum of $G$ (resp. adjacency spectrum) is symmetric with respect to the origin if and only if $k$ is even and $G$ is odd-bipartite.

So, the non-odd-bipartite even uniform hypergraphs are more interesting on distinguishing the Laplacian spectrum and signless Laplacian spectrum and studying the non-symmetric adjacency spectrum. In the past, Hu, Qi, Shao [6] introduced the cored hypergraphs and the power hypergraphs, where the cored hypergraph is one such that each edge contains at least one vertex of degree 1, and the $k$-th power of a simple graph is obtained by replacing each edge (a 2-subset) with a $k$-subset by adding $k - 2$ new vertices. These two kinds of hypergraphs are both odd-bipartite. Peng introduced $s$-path and $s$-cycle in [8]. It is stated in [6] that for $2 \leq s < \frac{k}{2}$, these kind of hypergraphs are cored hypergraphs and hence are odd-bipartite.

In this paper, we first investigate the odd-bipartiteness of $s$-paths and $s$-cycles when $\frac{k}{2} \leq s \leq k - 1$ in Section 2. An $s$-path is always odd-bipartite, but this does not hold for $s$-cycles. When $s = \frac{k}{2}$, we prove that an $s$-cycle is odd-bipartite if and only if its length is even. Motivated by the discussion of $s$-cycles, we introduce a class of $k$-uniform hypergraphs $G$, called $(k, \frac{k}{2})$-hypergraphs, which satisfy the property: $k$ is even, every edge $e$ of $G$ can be divided into two disjoint $\frac{k}{2}$-vertex sets say $e_1$ and $e_2$ and for any $e'$ incident to $e$, $e \cap e' = e_1$ or $e_2$. Such graph $G$ can be constructed from a simple graph, which is called the underlying graph of $G$. We show that $G$ is non-odd-bipartite if and only if the underlying graph of $G$ is non-bipartite.

It is known that a uniform hypergraph is connected if and only if the adjacency tensor of the graph is weakly irreducible. There are many results on the spectral theory of irreducible
Lemma 2.1

In the other part with blue. Note that we color the vertices in one part of the bipartition with red, and color the vertices in the other part with blue. Without loss of generality, we will use the notions (of length $s$ an $d$-path, say $v_1, v_2, \ldots, v_{s+d(k-s)}$, such that \{v_{1+j(k-s)}, v_{2+j(k-s)}, \ldots, v_{s+(j+1)(k-s)}\} is an edge of $P$ for $j = 0, \ldots, d - 1$. An $s$-cycle $C$ of length $d$ is a $k$-uniform hypergraph on $d(k-s)$ vertices, say $v_1, v_2, \ldots, v_{d(k-s)}$, such that \{v_{1+j(k-s)}, v_{2+j(k-s)}, \ldots, v_{s+(j+1)(k-s)}\} is an edge of $C$ for $j = 0, \ldots, d - 1$, where $v_{d(k-s)+j} = v_j$ for $j = 1, \ldots, s$. When $1 \leq s < \frac{k}{2}$, an $s$-path or $s$-cycle is a cored hypergraph and hence it is odd-bipartite. For more accuracy, we will use the notions $(k, s)$-path and $(k, s)$-cycle instead to indicate that they are $k$-uniform hypergraphs.

Lemma 2.1 A $(k, s)$-path is always odd-bipartite where $\frac{k}{2} \leq s \leq k - 1$.

Proof. Let $P$ be a $s$-path of length $d$. If $d = 1$, the assertion holds clearly. Assume the assertion holds for all $s$-paths of length $d < m$. We prove it by induction on number of edges. Consider an $s$-path $P$ of length $m$. Let $e_m$ be the last edge of $P$. Note that $P - e_m$ is an $s$-path, say $P'$ of length $m - 1$, together with $k - s$ isolated vertices. By induction $P'$ is odd-bipartite, which has an odd-bipartition $V(P') = V_1 \cup V_2$. Now, if $|V_1 \cap e_m|$ is odd, put all vertices of $e_m \backslash V(P')$ in $V_2$. Otherwise, take one vertex from $e_m \backslash V(P')$ and put it in $V_1$, and put the remaining in $V_2$. Then we get an odd-bipartition of $P$.

What about the odd-bipartiteness of $(k, s)$-cycles when $\frac{k}{2} \leq s \leq k - 1$? We first discuss the case of $s = \frac{k}{2}$.

Lemma 2.2 Let $C$ be a $(k, \frac{k}{2})$-cycle of length $m$. Then $C$ is odd-bipartite if and only if $m$ is even.

Proof. We have a partition of $V(C) = V_1 \cup V_2 \cup \cdots \cup V_m$ such that $e_i := V_i \cup V_{i+1}$ is an edge of $C$ for $i = 1, 2, \ldots, m$, where $V_{m+1} = V_1$. Suppose that $C$ is odd-bipartite, which has an odd-bipartition. We color the vertices in one part of the bipartition with red, and color the vertices in the other part with blue. Note that $e_1 = V_1 \cup V_2$ contains an odd number of red vertices. Without loss of generality, $V_1$ contains an odd number of red vertices. So $V_2$ contains an even number of red vertices, and then $V_3$ contains an odd number of red vertices by considering the
edge $e_2$. Repeat the discussion, we get $V_m$ contains an odd number of red vertices if $m$ is odd, and even number of red vertices otherwise. However, if $m$ is odd, then the edge $e_m = V_m \cup V_1$ contains an even number of red vertices, a contradiction. So $m$ is necessarily even. On the other hand, if $m$ is even, it is easy to give an odd-bipartition of $C$. \hfill \Box

For general case, it may be difficult to determine under which conditions a $(k, s)$-cycle is odd-bipartite when $\frac{k}{2} < s \leq k-1$. For example, let $C_1$ be a $(4, 3)$-cycle of length 8. By the definition of $s$-cycle, take $V(C_1) = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $E(C_1) = \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \ldots, \{8, 1, 2, 3\}\}$. Then $V_1 = \{1, 5\}$ and $V_2 = V \setminus V_1$ constitute an odd-bipartition of $C_1$, and hence $C_1$ is odd-bipartite. However the $(4, 3)$-cycle of length 6 is non-odd bipartite. So the length is not the only condition to determine whether a $(k, s)$-cycle is odd-bipartite or not.

Before proceeding further, we state that a lot of work has been done in spectral hypergraph theory when the graphs under discussion are odd-bipartite, while non-odd-bipartite graphs did not enjoy such an attention. The above discussion on $(k, s)$-cycles provides a ground for introducing a class of hypergraphs which can be easily classified into odd-bipartite and non odd-bipartite. Now let us move on to our definition.

**Definition 2.3** Let $G$ be a $k$-uniform hypergraph ($k$ even) on $n$ vertices such that every edge $e$ can be divided into two disjoint $\frac{k}{2}$-vertex sets say $e_1$ and $e_2$ and for any edge $e'$ incident to $e$, $e \cap e' = e_1$ or $e_2$. Such graph $G$ will be called a $(k, \frac{k}{2})$-hypergraph, and $e_1$ or $e_2$ will be called a half edge of $G$.

Replacing every half edge of a $(k, \frac{k}{2})$-hypergraph $G$ by a vertex and preserving the adjacency, we will get a simple graph, called the underlying graph of $G$ and is denote it by $u(G)$. So $G$ has $n = m \cdot \frac{k}{2}$ vertices, where $m$ is exactly the number of vertices of $u(G)$. On the other hand, the $(k, \frac{k}{2})$-hypergraph $G$ is obtained from the underlying graph $u(G)$ by replacing every vertex $v$ with a $\frac{k}{2}$-vertex set $\mathbf{v}$ such that $vu \in E(u(G))$ if and only if $\mathbf{v} \cup u \in E(G)$; see Fig. 2.1. The $(k, \frac{k}{2})$-hypergraphs may be considered as another version of power of simple graphs.

For convenience, a half edge of a $(k, \frac{k}{2})$-hypergraph will be written in black font, e.g. $\mathbf{v}$. If $\mathbf{v} \cup u \in E(G)$, we will simply write it as $\mathbf{v}u \in E(G)$. In addition, all vertices in a half edge say $\mathbf{v}$ have the same degree, so we simply write $d_{\mathbf{v}}$ instead of $d_v$ for all $v \in \mathbf{v}$.

![Fig. 2.1 Constructing a $(k, \frac{k}{2})$-hypergraph from a simple graph](image-url)

From the definition, a $(k, \frac{k}{2})$-cycle $C$ of length $m$ is a $(k, \frac{k}{2})$-hypergraph, and $u(C)$ is exactly the simple cycle of length $m$. By Lemma 2.2, $C$ is non-odd-bipartite if and only if $u(C)$ is non-bipartite. We generalize this fact as follows.
Theorem 2.4 Let $G$ be a connected $(k, \frac{k}{2})$-hypergraph. Then $G$ is non-odd-bipartite if and only if $u(G)$ is non-bipartite.

Proof. We prove an equivalent assertion: $G$ is odd-bipartite if and only if $u(G)$ is bipartite. Assume that $G$ is odd-bipartite. If $u(G)$ is a tree, surely $u(G)$ is bipartite. Otherwise, let $C$ be an arbitrary $(k, \frac{k}{2})$-cycle of $G$, then $C$ is also odd-bipartite and hence $u(C)$ is bipartite of even length by Lemma 2.2. So, $u(G)$ is bipartite.

On the contrary, assume that $u(G)$ is bipartite, with a bipartition $(V_1, V_2)$. Extend this bipartition to another bipartition $(V_1, V_2)$ of $G$, that is, $V_1$ (respectively, $V_2$) is obtained by replacing each vertex in $V_1$ (respectively $V_2$) by a half edge. Choose an arbitrary vertex from each half edge in $V_1$, form a new set $U_1$. Then $(U_1, V(G) \setminus U_1)$ is an odd-bipartition of $G$. □

3 Spectral radius and eigenvectors of hypergraphs

For integers $k \geq 3$ and $n \geq 2$, a real tensor (also called hypermatrix) $T = (t_{i_1\ldots i_k})$ of order $k$ and dimension $n$ refers to a multidimensional array with entries $t_{i_1\ldots i_k} \in \mathbb{R}$ for all $i_j \in [n]$ and $j \in [k]$. The tensor $T$ is called symmetric if its entries are invariant under any permutation of their indices. Given a vector $x \in \mathbb{R}^n$, $T x^k$ is a real number, and $T x^{k-1}$ is an $n$-dimensional vector, which are defined as follows:

$$T x^k = \sum_{i_1, i_2, \ldots, i_k \in [n]} t_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad (T x^{k-1})_i = \sum_{i_2, \ldots, i_k \in [n]} t_{i_1 i_2 \ldots i_k} x_{i_2} \cdots x_{i_k} \text{ for } i \in [n].$$

Let $I$ be the identity tensor of order $k$ and dimension $n$, that is, $i_{i_1 i_2 \ldots i_k} = 1$ if and only if $i_1 = i_2 = \cdots = i_k \in [n]$ and zero otherwise.

Definition 3.1 [10] Let $T$ be a $k$th order $n$-dimensional real tensor. For some $\lambda \in \mathbb{R}$, if the polynomial system $(\lambda I - T) x^{k-1} = 0$, or equivalently $T x^{k-1} = \lambda x^{k-1}$, has a solution $x \in \mathbb{R}^n \setminus \{0\}$, then $\lambda$ is called an $H$-eigenvalue of $T$ and $x$ is an $H$-eigenvector of $T$ associated with $\lambda$, where $x^{k-1} := (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1}) \in \mathbb{R}^n$.

If $\lambda \in \mathbb{R}$ is an eigenvalue of $T$ with multiplicity 1, we say that it is simple. If $x \in \mathbb{R}^n_+$ (the set of nonnegative vectors of dimension $n$), then $\lambda$ is called an $H^+$-eigenvalue of $T$. If $x \in \mathbb{R}^n_{++}$ (the set of positive vectors of dimension $n$), then $\lambda$ is said to be an $H^{++}$-eigenvalue of $T$. The spectral radius of $T$ is defined as

$$\rho(T) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } T\}.$$

To generalize the classical Perron-Frobenius Theorem from nonnegative matrices to nonnegative tensors, we need the definition of the irreducibility of tensor. Chang et. al [11] introduced the irreducibility of tensor. A tensor $T = (t_{i_1\ldots i_k})$ of order $k$ and dimension $n$ is called reducible if there exists a nonempty proper subset $I \subset [n]$ such that $t_{i_1 i_2 \ldots i_k} = 0$ for any $i_1 \in I$ and any $i_2, \ldots, i_k \notin I$. If $T$ is not reducible, then it is called irreducible.
Friedland et. al \cite{3} proposed a weak version of irreducible nonnegative tensors $\mathcal{T}$. The graph associated with $\mathcal{T}$, denoted by $G(\mathcal{T})$, is the directed graph with vertices $1, \ldots, n$ and an edge from $i$ to $j$ if and only if $t_{ii_1i_2\ldots i_k} > 0$ for some $i_l = j$, $l = 2, \ldots, m$. The tensor $T$ is called \textit{weakly irreducible} if $G(\mathcal{T})$ is strongly connected. Surely, an irreducible tensor is always weakly irreducible. Pearson and Zhang \cite{7} proved that the adjacency tensor of a uniform hypergraph $G$ is weakly irreducible if and only if $G$ is connected. Clearly, this shows that if $G$ is connected, then $\mathcal{A}(G)$, $\mathcal{L}(G)$ and $\mathcal{Q}(G)$ are all weakly irreducible.

**Theorem 3.2 (The Perron-Frobenius Theorem for Nonnegative Tensors)**

1. (Yang and Yang 2010 \cite{12}) If $\mathcal{T}$ is a nonnegative tensor of order $k$ and dimension $n$, then $\rho(\mathcal{T})$ is an $H^+$-eigenvalue of $\mathcal{T}$.

2. (Frieland, Gaubert and Han 2011 \cite{3}) If furthermore $\mathcal{T}$ is weakly irreducible, then $\rho(\mathcal{T})$ is the unique $H^{++}$-eigenvalue of $\mathcal{T}$, with the unique eigenvector $x \in \mathbb{R}^n_+$, up to a positive scaling coefficient.

3. (Chang, Pearson and Zhang 2008 \cite{1}) If moreover $\mathcal{T}$ is irreducible, then $\rho(\mathcal{T})$ is the unique $H^+$-eigenvalue of $\mathcal{T}$, with the unique eigenvector $x \in \mathbb{R}^n_+$, up to a positive scaling coefficient.

**Theorem 3.3** \cite{12, 14} Let $\mathcal{B}, \mathcal{C}$ be order $k$ dimension $n$ tensors satisfying $|\mathcal{B}| \leq \mathcal{C}$, where $\mathcal{C}$ is weakly irreducible. Let $\beta$ be an eigenvalue of $\mathcal{B}$. Then

1. $|\beta| \leq \rho(\mathcal{C})$.

2. If $\beta = \rho(\mathcal{C})e^{i\varphi}$ and $y$ is corresponding eigenvector, then all entries of $y$ are nonzero, and $\mathcal{C} = e^{-i\varphi} \mathcal{B} \cdot D^{-(k-1)} \cdot D \cdots D$, where $D = \text{diag}(\frac{y_1}{|y_1|}, \frac{y_2}{|y_2|}, \ldots, \frac{y_n}{|y_n|})$.

**Corollary 3.4** Suppose $0 \leq \mathcal{B} \leq \mathcal{C}$, where $\mathcal{C}$ is weakly irreducible. Then $\rho(\mathcal{B}) < \rho(\mathcal{C})$.

**Proof.** By Theorem 3.2(1), $\rho(\mathcal{B})$ is an eigenvalue of $\mathcal{B}$, with a nonnegative eigenvector $y$. By Theorem 3.3, $\rho(\mathcal{B}) \leq \rho(\mathcal{C})$. If $\rho(\mathcal{B}) = \rho(\mathcal{C})$, then, also by Theorem 3.3, $y > 0$, and hence $\mathcal{B} = \mathcal{C}$; a contradiction. $\blacksquare$

**Corollary 3.5** Suppose $G$ is a connected $k$-uniform hypergraph and $H$ is a proper sub-hypergraph of $G$. Then $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(G))$ and $\rho(\mathcal{Q}(H)) < \rho(\mathcal{Q}(G))$.

**Proof.** We only consider the adjacency tensor. The other case can be discussed in a similar manner. Observe that $\mathcal{A}(G)$ is weakly irreducible. First assume $V(H) = V(G)$. Then $\mathcal{A}(H) \subseteq \mathcal{A}(G)$, which implies the result by Corollary 3.4. Secondly assume $V(H) \subsetneq V(G)$. Add the isolated vertices of $V(G) \setminus V(H)$ to $H$ such that the resulting hypergraph, say $H'$ has the same vertex set as $G$. Then $\rho(\mathcal{A}(H)) = \rho(\mathcal{A}(H'))$; or see \cite{15} Theroem 3.2. Noting that $\mathcal{A}(H') \subseteq \mathcal{A}(G)$, we also get the result. $\blacksquare$

Let $\mathcal{B} \geq 0$. Let $x \in \mathbb{R}^n_+$. Denote

$$r_i(\mathcal{B}) = \sum_{i_2, \ldots, i_k = 1}^{n} b_{i_1i_2\ldots i_k}, \quad s_i(\mathcal{B}, x) = \frac{(\mathcal{B} x^{k-1})_i}{x^{k-1}_i}, \quad \text{for } i = 1, 2, \ldots, n.$$
The following two results give bounds for the spectral radius \( \rho(B) \) of a general nonnegative tensor \( B \). Here we impose an additional condition on \( B \), that is, \( B \) is weakly irreducible, and characterize the equality cases.

**Lemma 3.6** [12, Lemma 5.2] *Let \( B \geq 0 \). Then*

\[
\min_{1 \leq i \leq n} r_i \leq \rho(B) \leq \max_{1 \leq i \leq n} r_i. \tag{3.1}
\]

**Lemma 3.7** [12, Lemma 5.3] *Let \( B \geq 0 \), and let \( x \in \mathbb{R}^n_{++} \). Then*

\[
\min_{1 \leq i \leq n} s_i(B, x) \leq \rho(B) \leq \max_{1 \leq i \leq n} s_i(B, x). \tag{3.2}
\]

**Lemma 3.8** *Let \( B, x \in \mathbb{R}^n_{++} \) as defined in Lemmas 3.6 and 3.7. Suppose that \( B \) is weakly irreducible. Then either equality in (3.1) holds if and only if \( r_1(B) = r_2(B) = \cdots = r_n(B) \); either equality in (3.2) holds if and only if \( B x^{k-1} = \rho(B) x^{k-1} \).*

**Proof.** For completeness we restate the proof of (3.1) and (3.2) as in [12]. We first consider the equality cases of (3.1). Let \( \alpha = \min_{1 \leq i \leq n} r_i \). If \( \alpha = 0 \), surely \( \rho(B) \geq \alpha = 0 \), and \( \rho(B) = 0 \) if and only if \( B = 0 \) (see [15, Theorem 3.1]). So we assume that \( \alpha > 0 \). Let \( C \) be a tensor with the same order and dimension as \( B \) whose entries are defined as \( c_{i_1 i_2 \ldots i_k} = \frac{\alpha}{r_1(B)} b_{i_1 i_2 \ldots i_k} \). Then \( 0 \leq C \leq B \), and by Theorem 3.3(1), \( \rho(C) \geq \rho(B) \). In addition, \( r_i(C) = \alpha \) for each \( i = 1, 2, \ldots, n \), which implies \( \rho(C) = \alpha \) (see [12, Lemma 5.1]). So we get \( \rho(B) \geq \rho(C) = \alpha \). If \( \rho(B) = \alpha \), then \( \rho(B) = \rho(C) \), and then \( B = C \) by Corollary 3.3. This implies that \( r_i(B) = \alpha \) for each \( i = 1, 2, \ldots, n \), and the necessity holds. The sufficiency is easily verified by [12, Lemma 5.1]. For the right equality of (3.1), the proof is same.

Next we consider the equality cases of (3.2). Let \( D = \text{diag}(x_1, x_2, \ldots, x_n) \), and let \( E = B \cdot D^{-(k-1)} \cdot D \cdots D \). Then \( E \) and \( B \) have the same eigenvalues ([14, Theorem 2.7]), which yields \( \rho(B) = \rho(E) \). In addition, \( E \) is also weakly irreducible. Noting that \( r_i(E) = s_i(B, x) \) for \( i = 1, 2, \ldots, n \). By (3.1),

\[
\min_{1 \leq i \leq n} s_i(B, x) = \min_{1 \leq i \leq n} r_i(E) \leq \rho(E) \leq \max_{1 \leq i \leq n} r_i(E) = \max_{1 \leq i \leq n} s_i(B, x).
\]

If the right equality holds, then all \( r_i(E) \), and hence all \( s_i(B, x) \), have the same value, i.e. \( B x^{k-1} = \rho(B) x^{k-1} \). The proof for the right equality is similar.

**Corollary 3.9** *Suppose that \( B \) is a weakly irreducible nonnegative tensor. If there exists a vector \( y \geq 0 \) such that \( By^{k-1} \leq \mu y^{k-1} \) (resp. \( By^{k-1} \geq \mu y^{k-1} \)), then \( \rho(B) < \mu \) (resp. \( \rho(B) > \mu \)).

**Proof.** Assume that \( B y^{k-1} \leq \mu y^{k-1} \). By a similar discussion as in the proof of Theorem 1.4 (1) of [1], we get \( y > 0 \). From the inequality and by Lemma 3.7

\[
\rho(B) \leq \max_{1 \leq i \leq n} s_i(B, y) \leq \mu.
\]
If \( \rho(B) = \mu \), then \( \rho(B) = \max_{1 \leq i \leq n} s_i(B, y) \), which implies that \( B y^{k-1} = \rho(B) y^{k-1} \), a contradiction to the assumption.

Next we assume that \( B y^{k-1} \geq \mu y^{k-1} \). By Theorem 5.3 of [12],

\[
\rho(B) = \max_{x \geq 0, \|x\|_1 > 0} \frac{\lambda}{\mu} \geq \mu.
\]

If \( \rho(B) = \mu \), then \( B y^{k-1} \geq \rho(B) y^{k-1} \), which implies that \( B y^{k-1} = \rho(B) y^{k-1} \) by Lemma 3.5 of [14] as \( B \) is weakly irreducible; a contradiction. ■

For the adjacency tensor of a \( k \)-uniform hypergraph \( G \), denote

\[
N(u) = \{ \{u_2 u_3 \ldots u_k\} : \{u_2 u_3 \ldots u_k\} \in E(G) \}.
\]

The eigenvector equation \( A(G)x^{k-1} = \lambda x^{k-1} \) could be interpreted as

\[
\lambda x^{k-1}_u = \sum_{\{u_2 u_3 \ldots u_k\} \in N(u)} x_{u_2} x_{u_3} \ldots x_{u_k}, \text{ for each } u \in V(G). \tag{3.3}
\]

The eigenvector equation \( Q(G)x^{k-1} = \lambda x^{k-1} \) could be interpreted as

\[
[\lambda - d(u)]x^{k-1}_u = \sum_{\{u_2 u_3 \ldots u_k\} \in N(u)} x_{u_2} x_{u_3} \ldots x_{u_k}, \text{ for each } u \in V(G). \tag{3.4}
\]

A hypergraph \( G \) is isomorphic to a hypergraph \( H \), written as \( G \cong H \), if there exists a bijection \( \sigma : V(G) \rightarrow V(H) \) such that \( \{v_1, v_2, \ldots, v_k\} \in E(G) \) if and only if \( \{\sigma(v_1), \sigma(v_2), \ldots, \sigma(v_k)\} \in E(H) \). The bijection \( \sigma \) is called an isomorphism of \( G \) and \( H \). If \( G = H \), then \( \sigma \) is called an automorphism of \( G \). Let \( x \) be a vector defined on \( V(G) \). Denote \( x_\sigma \) be the vector such that \( (x_\sigma)_u = x_{\sigma(u)} \) for each \( u \in V(G) \).

**Lemma 3.10** Let \( G \) be a \( k \)-uniform hypergraph and \( \sigma \) be an automorphism of \( G \). Let \( X \) be an eigenvector of \( A(G) \) (resp. \( L(G) \), \( Q(G) \)) associated with an eigenvalue \( \lambda \). Then \( X_\sigma \) is also an eigenvector of \( A(G) \) (resp. \( L(G) \), \( Q(G) \)) associated with \( \lambda \), where \( X_\sigma \) is defined such that \( (X_\sigma)_u = x_{\sigma(u)} \) for each \( u \in V(G) \).

**Proof.** Let \( u \in V(G) \) be an arbitrary but fixed vertex. By Eq. (3.3), we have

\[
(A(G)X^{k-1}_\sigma)_u = \sum_{\{u_2 u_3 \ldots u_k\} \in E(G)} (X_\sigma)_{u_2} (X_\sigma)_{u_3} \ldots (X_\sigma)_{u_k}
\]

\[
= \sum_{\{u_2 u_3 \ldots u_k\} \in E(G)} X_{\sigma(u_2)} X_{\sigma(u_3)} \ldots X_{\sigma(u_k)}
\]

\[
= \sum_{\{\sigma(u), \sigma(u_2), \ldots, \sigma(u_k)\} \in E(G)} X_{\sigma(u_2)} X_{\sigma(u_3)} \ldots X_{\sigma(u_k)}
\]

\[
= \lambda X^{k-1}_{\sigma(u)}
\]

\[
= \lambda X^{k-1}_{\sigma(u)}.
\]

where the fourth equality is obtained from the eigenvector equation \( A(G)x^{k-1} = \lambda x^{k-1} \). Hence \( X_\sigma \) is also an eigenvector of \( A(G) \) associated with the eigenvalue \( \lambda \). The proof for \( L(G) \) and \( Q(G) \) is similar by the fact \( d_u = d_{\sigma(u)} \) for each \( u \in V(G) \). ■
Lemma 3.11 Let $G$ be a connected $(k, \frac{k}{2})$-hypergraph, and let $x > 0$ be an eigenvector of $\mathcal{A}(G)$ (resp. $\mathcal{Q}(G)$). If $u$ and $v$ are the vertices in the same half edge of $G$, then $x_u = x_v$.

**Proof.** Let $\sigma$ be a permutation of $V(G)$ such that it interchanges $u$ and $v$ and fix all other vertices. It easily seen $\sigma$ is an automorphism of $G$. Then by Lemma 3.10, $x_{\sigma}$ is also an eigenvector of $\mathcal{A}(G)$ (resp. $\mathcal{Q}(G)$). By Theorem 3.2(2), $\mathcal{A}(G)$ (resp. $\mathcal{Q}(G)$) has a unique $H^+$-eigenvector up to a multiple, so $x_{\sigma} = x$, which implies the result. □

Let $G, x$ be defined as in Lemma 3.11. We will use $x_v$ to denote the common value of the vertices in the half edge $v$. At the end of this section, we determine the spectral radius of the adjacency tensor of a special $(k, \frac{k}{2})$-hypertree $T_n$, whose underlying simple graph $u(T_n)$ is obtained from a path $P_{n-4}$ ($n \geq 6$) by attaching two pendant edges at each of its end vertex; see Fig. 3.1.

![Fig. 3.1 A special $(k, \frac{k}{2})$-hypergraph $T_n$](image)

Let $x \in \mathbb{R}^n$ which assigns the values of the vertices of $T_n$ as follows: $x_u = 1$ for $i = 1, 2, 3, 4$, $x_v = 4^{\frac{k}{2}}$ for $i = 1, 2, \ldots, n - 4$. By Eq. (3.1), it is easy to check $x$ is an eigenvector of $\mathcal{A}(T_n)$ corresponding to the eigenvalue 2. By Theorem 3.2(2), $\rho(\mathcal{A}(T_n)) = 2$.

4 Adjacency spectral radius of $(k, \frac{k}{2})$-hypergraphs

Let $G$ be a $(k, \frac{k}{2})$-hypergraph and let $uw$ be an edge of $G$. Denote by $G_{uw}$ the hypergraph obtained from $G$ by subdividing the edge $uw$, that is by inserting a half edge say $v$ and forming two new edges $uv$ and $vw$ instead of the original edge $uw$; see Fig. 4.1.

![Fig. 4.1 Left: an edge $uw$; right: $uw$ is subdivided by inserting a half edge $v$.](image)

By the discussion at the end of last section, If $G = T_n$ or $G = C_n$ for some $n$, $\rho(G)$ remains the same, where $C_n$ is denoted to be a $(k, \frac{k}{2})$-cycle of length $n$. In the below we will discuss under what condition $\rho(G)$ will increase or decrease after subdividing an edge. An internal path of $G$ is a sequence of half edges $u_1, u_2, \ldots, u_l$, such that all $u_i$ are distinct (except possibly $u_1 = u_l$), $u_i u_{i+1}$ is an edge of $G$ for $i = 1, 2, \ldots, l - 1$, $d(u_1) \geq 3$, $d(u_2) = \cdots = d(u_{l-1}) = 2$ (unless $l = 2$), and $d(u_l) \geq 3$. 

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Lemma 4.1 Let $G$ be a connected $(k, \frac{k}{2})$-hypergraph, and let $uw$ be an edge of $G$ not on any internal path. If $G$ is not a $(k, \frac{k}{2})$-cycle, then $\rho(A(G)) < \rho(A(G_{u,w}))$.

Proof. The result follows by Corollary 3.5 as $G$ is a proper subgraph of $G_{u,w}$.

Lemma 4.2 Let $G$ be a connected $(k, \frac{k}{2})$-hypergraph with $n$ half edges, and let $uw$ be an edge of $G$ on an internal path. If $G \neq T_n$, then $\rho(A(G)) > \rho(A(G_{u,w}))$.

Proof. As $G \neq T_n$ and $G$ has an internal path, $G$ contains a proper sub-hypergraph $T_m$ ($m < n$). So $\rho(A(G)) > \rho(A(T_m)) = 2$ by Corollary 3.5. Let $u_0, u_1, \ldots, u_{l+1}$ be a sequence of half edges of $G$ forming an internal path, where $u = u_i, w = u_{i+1}$ for some $i = 0, 1, \ldots, l$. Note that $G_{u,w} \cong G_{u_0,u_1} \cong \cdots \cong G_{u_{l},u_{l+1}}$. So we may assume that $u$ be any one of $u_0, u_1, \ldots, u_l$ as convenient. In what follows, $A(G)x^{k-1} = \rho(A(G))x^{[k-1]}$, where $x > 0$.

Case 1: $u_0 = u_{l+1}$. By Lemma 3.10, $x_{u_i} = x_{u_{i+1-i}}$ for $i = 1, 2, \ldots, l$, as there is an automorphism of $G$ interchanging $u_i$ and $u_{l+1-i}$ for $i = 1, 2, \ldots, l$, and fixing all other half edges. If $l$ is even, set $u = u_{l/2}, w = u_{l/2+1}$. Define a vector $y$ on $G_{u,w}$ such that $y_z = x_z$ for each half edge $z$ of $G$, and $y_v = x_u = x_w$ for the new half edge $v$ inserted into the edge $uw$. By Eq. (3.1), for each half edge $z$ of $G_{u,w}$ other than $v$,

$$[A(G_{u,w})y^{k-1}]_z = [A(G)x^{k-1}]_z = \rho(A(G))x^{k-1}_z = \rho(A(G))y^{k-1}_z.$$ 

Noting that $y_v = x_u = x_w$,

$$[A(G_{u,w})y^{k-1}]_v = 2x^{k-1}_v < \rho(A(G))x^{k-1}_v = \rho(A(G))y^{k-1}_v.$$ 

So $A(G_{u,w})y^{k-1} \leq \rho(A(G))y^{[k-1]}$, and then $\rho(A(G_{u,w})) < \rho(A(G))$ by Corollary 3.9.

If $l$ is odd, we note that $x_{u_{(l-1)/2}} = x_{u_{(l+3)/2}} > x_{u_{(l+1)/2}}$. This is followed by the fact $\rho(A(G)) > 2$ and the eigenvector equation (3.3) on the half edge $u_{(l+1)/2}$, that is,

$$\rho(A(G))x^{k-1}_{u_{(l+1)/2}} = 2x^{k/2}_{u_{(l-1)/2}}x^{k/2-1}_{u_{(l+1)/2}}.$$ 

Now set $u = u_{(l-1)/2}, w = u_{(l+1)/2},$ and construct a vector $y$ such that $y_z = x_z$ for each half edge $z$ of $G$, $y_v = x_w$ for the inserted half edge $v$. By Eq. (3.1), for each half edge $z$ of $G_{u,w}$ other than $v$ and $w$,

$$[A(G_{u,w})y^{k-1}]_z = \rho(A(G))y^{k-1}_z.$$ 

For the half edge $w$,

$$[A(G_{u,w})y^{k-1}]_w = x^{k-1}_w + k/2 x^{k/2-1}_{u_{(l+3)/2}}x^{k/2-1}_{u_{(l+1)/2}} + k/2 x^{k/2-1}_{u_{(l+1)/2}}x^{k/2-1}_{u_{(l+3)/2}} = \rho(A(G))y^{k-1}_w.$$
For the half edge \( v \),

\[
[A(G_{u,w})y^{k-1}]_v = x^{k/2-1}_w x^{k/2}_{u(t-1)/2} + x^{k/2-1}_w x^{k/2}_{u(t+3)/2} < x^{k/2-1}_w x^{k/2}_{u(t-1)/2} + x^{k/2-1}_w x^{k/2}_{u(t+3)/2} = \rho(A(G))x^{k-1}_w = \rho(A(G))y^{k-1}_v.
\]

So \( A(G_{u,w})y^{k-1} \leq \rho(A(G))y^{[k-1]} \), and the result follows.

**Case 2:** \( u_0 \neq u_{t+1} \). Let \( t = \min\{i : X_u = \min\{X_{u_0}, ..., X_{u_{t+1}}\}\} \). Without loss of generality assume that \( X_0 \leq X_{t+1} \) so that \( t < l + 1 \). We have two cases here. If \( t > 0 \), set \( u = u_t, w = u_{t+1} \), and define \( Y_v = X_u \). By a similar discussion as in the Case 1 for odd \( l \), we will get \( A(G_{u,w})y^{k-1} \leq \rho(A(G))y^{[k-1]} \). For the case of \( t = 0 \), denote \( s = \sum_{z \in S} X_z^{k/2} \), where \( S = \{z : zu_0 \in E(G), z \neq u_1\} \). We have two subcases:

**Subcase 2.1:** \( s \geq X_{u_0}^{k/2} \). Set \( u = u_0, w = u_1 \), and define \( Y_v = X_{u_0} \). Note that \( \max\{s, X_{u_1}^{k/2}\} > X_{u_0}^{k/2} \); otherwise by considering Eq. (3.1) on the vertices of \( u_0 \), we get \( \rho(A(G)) = 2 \). By a similar discussion, we have \( A(G_{u,w})y^{k-1} \leq \rho(A(G))y^{[k-1]} \).

**Subcase 2.2:** \( s < X_{u_0}^{k/2} \). Set \( u = u_0, w = u_1 \), and define \( Y_u = Y_{u_0} = s^{2/k}, Y_v = X_{u_0} \). We can show \( A(G_{u,w})y^{k-1} \leq \rho(A(G))y^{[k-1]} \) if we can prove

\[
s + X_{u_0}^{k/2} < \rho(A(G)) s.
\]

By Eq. (3.1) on each half edge \( z \in S \), \( \lambda(G)X_z^{k-1} \geq X_z^{k/2-1}X_{u_0}^{k/2} \). So \( \lambda(G)X_z^{k/2} \geq X_{u_0}^{k/2} \). Summing the above inequalities over all \( z \in S \),

\[
\rho(A(G))s \geq |S|X_{u_0}^{k/2} \geq 2X_{u_0}^{k/2} > X_{u_0}^{k/2} + s.
\]

The above two results are generalization of the result from classical spectral graph theory due to Hoffman and Smith, see Lemma 2.3 and Proposition 2.4 of [1].

**Theorem 4.3** Among all connected non-odd-bipartite \( (k, \frac{k}{2}) \)-hypergraphs with \( n \) half edges, where \( n \) is odd, the minimum spectral radius of the adjacency tensor is achieved for \( C_n \).

**Proof.** Let \( G \) be a non-odd-bipartite \( (k, \frac{k}{2}) \)-hypergraph with \( n \) half edges, and \( G \neq C_n \). It suffices to prove \( \rho(A(G)) > \rho(A(C_n)) = 2 \). Since \( G \) is non-odd-bipartite, by Theorem 2.4 it properly contains a \( (k, \frac{k}{2}) \)-cycle of odd length say \( C_l \). So, \( \rho(A(G)) > \rho(A(C_l)) = \rho(A(C_n)) \) by Corollary 3.9.

**Theorem 4.4** Among all \( (k, \frac{k}{2}) \)-hypergraphs with \( n \) half edges, where \( n \) is even, the minimum spectral radius of the adjacency tensor is achieved for \( C_{n-1} + e \), where \( C_{n-1} + e \) is obtained from \( C_{n-1} \) by adding a half edge together with an edge joining this half edge and an arbitrary half edge of the cycle.
Lemma 5.1 Let \( G \) be a connected \((k, \frac{k}{2})\)-hypergraph, and let \( uw \) be an edge of \( G \) not on any internal path. If \( G \) is not a \((k, \frac{k}{2})\)-cycle, then \( \rho(G) < \rho(G_{uw}) \).

Proof. The proof is similar to Lemma 4.1.

Lemma 5.2 Let \( G \) be a connected \((k, \frac{k}{2})\)-hypergraph, and let \( uw \) be an edge of \( G \) on an internal path. Then \( \rho(G) > \rho(G_{uw}) \).

Proof. Let \( u_0, u_1, \ldots, u_{l+1} \) be a sequence of half edges of \( G \) forming an internal path, where \( u = u_i, w = u_{i+1} \) for some \( i = 0, 1, \ldots, l \). In what follows, \( Q(G)x^{k-1} = \rho(Q(G))x^{k-1} \), where \( x > 0 \). Note that \( \rho(Q(G)) > 4 \) by the discussion at the beginning of this section.

Case 1: \( u_0 = u_{l+1} \). By Lemma 3.10 \( x_{u_i} = x_{u_{i+1}} \) for \( i = 1, 2, \ldots, l \). If \( l \) is even, set \( u = u_{l/2}, w = u_{l/2+1} \). Define a vector \( y \) on \( G_{uw} \) such that \( y_z = x_z \) for each half edge \( z \) of \( G \), and \( y_v = x_u = x_w \) for the new half edge \( v \) inserted into the edge \( uw \). By Eq. (3.2) and the fact \( \rho(Q(G)) > 4 \), \( Q(G_{uw})y^{k-1} < \rho(Q(G))y^{k-1} \), and then \( \rho(Q(G_{uw})) < \rho(Q(G)) \) by Corollary 3.9.

If \( l \) is odd, by (3.2) and the fact \( \rho(Q(G)) > 4 \), \( x_{u_{(l-1)/2}} = x_{u_{(l+3)/2}} > x_{u_{(l+1)/2}} \). Now set \( u = u_{(l-1)/2}, w = u_{(l+1)/2} \) and construct a vector \( y \) such that \( y_z = x_z \) for each half edge \( z \) of \( G \), \( y_v = x_w \) for the inserted half edge \( w \). By Eq. (3.2), \( Q(G_{uw})y^{k-1} < \rho(G)y^{k-1} \).

Case 2: \( u_0 \neq u_{l+1} \). Let \( t = \min\{i : X_u = \min\{X_{u_0}, \ldots, X_{u_{l+1}}\}\} \). Without loss of generality assume that \( X_u \leq X_{u+1} \) so that \( t < l+1 \). We have two cases here. If \( t > 0 \), set \( u = u_t, w = u_{t+1} \), and define \( Y_v = X_u \). By (3.2), we will get \( Q(G_{uw})y^{k-1} < \rho(G)y^{k-1} \).
For the case of $t = 0$, denote $s = \sum_{z \in S} X_z^{k/2}$, where $S = \{z : zu \in E(G), z \neq u_1\}$. Set $u = u_0, w = u_1$, and define $Y_{u} = X_{u_0}$. For the half edge $u_0$,

$$[Q(G_u,w)y_{u_0}^{k-1}]_{u_0} = d(u_0)X_{u_0}^{k-1} + sX_{u_0}^{k/2-1}X_{u_0}^{k/2}$$

$$\leq d(u_0)X_{u_0}^{k-1} + sX_{u_0}^{k/2-1}X_{u_1}^{k/2}$$

$$= \rho(G)x_{u_0}^{k-1} = \rho(G)y_{u_0}^{k-1}.$$

For the half edge $v$,

$$[Q(G_u,w)y_{u_0}^{k-1}]_{v} = 2X_{u_0}^{k-1} + X_{u_0}^{k/2-1}X_{u_1}^{k/2}$$

$$< d(u_0)X_{u_0}^{k-1} + sX_{u_0}^{k/2-1}X_{u_0}^{k/2}$$

$$= \rho(G)x_{u_0}^{k-1} = \rho(G)y_{v}^{k-1}.$$

So we also have $Q(G_u,w)y_{u_0}^{k-1} \leq \rho(G)y_{u_0}^{k-1}$. □

The above result is a generalization of the result from classical spectral graph theory due to Feng et. al [2].

**Theorem 5.3** Among all connected non-odd-bipartite $(k, \frac{k}{2})$-hypergraphs with $n$ half edges, where $n$ is odd, the minimum spectral radius of the signless Laplacian tensor is achieved for $C_n$.

**Proof.** The proof is similar to Theorem 4.3. □

**Theorem 5.4** Among all $(k, \frac{k}{2})$-hypergraphs with $n$ half edges, where $n$ is even, the minimum spectral radius of the adjacency tensor is achieved for $C_{n-1} + e$.

**Proof.** The proof is similar to Theorem 4.4. □

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