On the high temperature limit of the Casimir energy

Y. Koohsarian\textsuperscript{1}, K. Javidan\textsuperscript{2}

Department of Physics, Ferdowsi University of Mashhad
P.O.Box 1436, Mashhad, Iran

Abstract

We introduce a useful approach to find asymptotically explicit expressions for the Casimir free energy at large temperature. The resulting expressions contain the classical terms as well as the few first terms of the corresponding heat-kernel expansion, as expected. This technique works well for many familiar configurations in Euclidean as well as non-Euclidean spaces. By utilizing this approach, we provide some new numerically considerable results for the Casimir pressure in some rectangular ideal-metal cavities. For instance, we show that at sufficiently large temperature, the Casimir pressure acting on the sidewalls of a rectangular tube can be up to twice that of the two parallel planes. We also apply this technique for calculating the Casimir free energy on a 3-torus as well as a 3-sphere. We show that a nonzero mass term for both scalar and spinor fields as well on the torus as on the sphere, violates the third law of thermodynamics. We obtain some negative values for the Casimir entropy on the 3-torus as well as on the 3-sphere. We speculate that these negative Casimir entropies can be interpreted thermodynamically as an instability of the vacuum state at finite temperatures.

Keywords
Casimir energy, Regularization, High temperature limit, Heat kernel expansion, Negative Casimir entropy, Unstable vacuum state, Rectangular cavities, Casimir energy in non-Euclidean spaces.

1 Introduction

It is well known that the total zero-point energy of a quantum field restricted by some physical boundaries, can have some macroscopic manifestations, such as the famous Casimir effect \cite{1}. Nowadays various aspects of the Casimir effect have been investigated extensively for various configurations, see \cite{2} \cite{7} as reviews. One of the important, still controversial \cite{8} \cite{12} aspects of the Casimir energy is its temperature dependence. In the framework of the quantum field theory, as we know, the vacuum energy at finite temperature is obtained generally by applying the known Matsubara imaginary-time formalism. This formalism is actually equivalent to the quantum thermodynamics approach, where the finite-temperature Casimir energy, known also as the Casimir free energy, is given by the total free energy of the quantized field modes, see e.g. \cite{2} \cite{5}.

\textsuperscript{1}yo.koohsarian@mail.um.ac.ir
\textsuperscript{2}Javidan@um.ac.ir
The representations obtained in the literature for the Casimir free energy usually have no asymptotically explicit expressions at high temperatures. Conventionally the high-temperature asymptotic expression of the Casimir energy is obtained by applying the known heat kernel expansion, which has some complications e.g. in calculating the classical terms [4]. In this paper, we introduce another useful approach to obtain exact representations for the Casimir energy having asymptotically explicit expression for the high-temperature limits. This is a useful technique specially for fully bounded configurations, such as the rectangular cavities, which are familiar configurations in the Casimir effect literature [20–27], also for some compact manifolds with nontrivial topology and curvature such as torus and sphere, which have significant role in some cosmological models [2,4]. In this approach all the high-temperature limiting terms including the classical terms as well as the main first terms of the heat-kernel expansion would be automatically obtained in exact forms. These terms would have explicit dependence on the mass parameter, providing asymptotically suitable expressions as well for small as for large masses. Such asymptotically explicit expressions might not be simply obtained through the conventional approaches, such as the heat-kernel expansion.

In section 2, we introduce a rather general approach to find representations asymptotically suitable at high temperatures, for the Casimir free energy. We use the mentioned approach to obtain new representations for the Casimir free energy, for the scalar as well as the electromagnetic field in configurations with flat boundary conditions specifically in rectangular cavities (section 3). We provide some new numerical results for the Casimir pressures in rectangular tube and rectangular box, and show that at sufficiently large temperature, the Casimir pressure in the tube and the box can be larger, by factors of 2 and 1.5 respectively, than that of the parallel planes. In section 4, we apply the mentioned approach for the scalar as well as the spinor field in the non-Euclidean space, specifically on a 3-torus and a 3-sphere. We show that a nonzero mass term for both scalar and spinor fields as well on the torus as on the sphere, violates the third law of thermodynamics. In some cases we obtain negative values for the Casimir entropies, and interpret these negative entropies as an instability in the vacuum state.

2 A useful approach

From the functional formalism of the quantum field theory, the vacuum energy of a scalar field restricted by some appropriate boundary conditions, can be written as [4]

\[ E_0 = -\frac{i}{2} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \sum J \ln \left[ -\xi^2 + k_J^2 + \mu^2 \right] \]  \hspace{1cm} (1)

where, “\(\omega\)” and “\(k\)” represent the energy and the momentum of off-mass-shell scalar modes, and the collective index \(J\) labels the mode numbers. At finite temperature, the mode frequencies, \(\omega_l\)’s, are replaced with the (imaginary) Matsubara frequencies

\[-i\omega_l = \frac{2\pi}{\beta} l \quad ; \quad l = 0, \pm 1, \pm 2, ... \]  \hspace{1cm} (2)
in which \( \beta \equiv 1/T \), with \( T \) as the temperature. As a result, the scalar free energy takes the form

\[
E_0(T) = \frac{1}{2\beta} \sum_{J} \sum_{l=-\infty}^{\infty} \ln \left( \left( \frac{2\pi}{\beta} \right)^2 l^2 + k_J^2 + \mu^2 \right).
\]

(3)

which after regularization and subtracting the contribution of the free vacuum, gives the Casimir free energy of the scalar field. For a Dirac 4-spinor, in topologically flat spaces, one should just multiply the above equation by \( \frac{-4}{-4} \).

Now, as is conventional, by introducing a regularization parameter, the scalar free energy takes the form

\[
E_0 = -\frac{1}{2\beta} \lim_{s \to 0} \frac{\partial}{\partial s} \sum_{l=-\infty}^{\infty} \left( l^2 + \lambda_J^2 + \lambda_\mu^2 \right)^{-s}
\]

(4)

where by discarding an irrelevant constant term, we have introduced the dimensionless parameters \( \lambda_J \equiv \beta k_J/2\pi \) and \( \lambda_\mu \equiv \beta \mu/2\pi \). A sum of the form can be written in terms of an inhomogeneous Epstein-like zeta function as

\[
Z_{\{b_1,b_2,\ldots,b_p\}}(a_1, a_2, \ldots, a_p, c; s) = \sum_{\{n_i\}=0}^{p} \left[ \sum_{i=1}^{p} a_i^2(n_i + b_i)^2 + c^2 \right]^{-s} ;
\]

\[ s < 0, \quad a_1 \leq a_2 \leq \ldots \leq a_p. \]  

(5)

Then using the familiar gamma function, as is conventional, we rewrite the above expression as a parametric integral;

\[
Z_{\{b_i\}}(a_i, c; s) = \sum_{\{n_i\}=0}^{\infty} \int_{0}^{\infty} \frac{dt}{t^{s}} \exp \left[-t \left( \sum_{i=1}^{p} a_i^2(n_i + b_i)^2 + c^2 \right) \right]
\]

(6)

Subsequently we use a generalized form of the Poisson summation formula,

\[
\sum_{n=0}^{\infty} \exp \left[ -\alpha^2(n + \theta)^2 \right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \alpha^{2n} \zeta(-2n, \theta) + \frac{\sqrt{\pi}}{2\alpha} \cos(2\pi \theta) \sum_{n=1}^{\infty} \exp \left[ -\frac{n^2 \pi^2}{\alpha^2} \right].
\]

(7)

in which,

\[
\zeta(s, r) = \sum_{n=0}^{\infty} (n + r)^{-s}
\]

(8)

is the known Hurwitz zeta function. Now we apply the formula just to the \( n_i \)-sum with the smallest \( a_i \). This condition, as we see in the next sections, is necessary for obtaining asymptotically suitable expressions for the Casimir energy as well for low as for high temperature limits. So here by applying Eq. to the \( n_1 \)-sum in Eq. one can find

\[
Z_{\{b_1, b_2, \ldots, b_p\}}(a_1, a_2, \ldots, a_p, c; s) = \ldots
\]
the recurrence (9) can be simplified as
\[
\sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} a_1^{2n_1} \zeta(-2n_1, b_1) \frac{\Gamma(s + n_1)}{\Gamma(s)} Z_{p-1,\{b_2,3,...\}} (a_2, ..., a_p, c; s)
+ \frac{\sqrt{\pi}}{2a_1 \Gamma(s)} \left( \Gamma(s - 1/2) Z_{p-1,\{b_2,3,...\}} (a_2, b_2, ..., a_p, b_p, c; s - 1/2) + G_{p,\{b_2,3,...\}} (a_1, a_2, ..., a_p, c; s - 1/2) \right)
\] (9)
in which
\[
G_{p,\{b_2,3,...\}} (a_1, a_2, ..., a_p, c; s) \equiv 4 \sum_{n_1=1}^{\infty} \sum_{n_2,..,n_p=0}^{\infty} \left( \sum_{i=2}^{p} a_i^2 (n_i + b_i)^2 + c^2 \right)^{-s/2}
\times K_{-s} \left[ 2\pi \left( \sum_{i=2}^{p} a_i^2 (n_i + b_i)^2 + c^2 \right) n_i^2/a_i^2 \right]^{1/2}
\] (10)
with
\[
K_{-s}(z) = \frac{(z/2)^{s/2} \Gamma(1/2)}{\Gamma(\nu + 1/2)} \int_1^{\infty} e^{-zt} (t^2 - 1)^{-(2\nu-1)/2} dt,
\] (11)
as a modified Bessel function of the second kind, and we have used the following integral relation
\[
\int_0^{\infty} \frac{dt}{t} t^{-r} \exp \left[ -x^2 t - y^2/t \right] dt = 2(x/y)^r K_r(2xy).
\] (12)
Now as is seen Eq. (9) has a recurrence form, see Eq. (4.19) of [15], hence we need just to find a regularized form for \( Z_1 \). But such a sum can be directly regularized by utilizing the known Abel-Plana formula [18], so by using the recurrence (9) one can find exact regularized expressions for other \( Z \) functions.

In this work we are interested specifically in the cases with \( b_i = 0, \frac{1}{2} \). In these cases by using
\[
\zeta(0, 0) = \frac{1}{2} \quad \zeta(-2n, 0) = 0 \quad n = 1, 2, ...
\]
\[
\zeta(-2n_1, 1/2) = 0 \quad n = 0, 1, ...
\]
the recurrence (9) can be simplified as
\[
Z_{p,0} (a_1, a_2, ..., a_p, c; s) = -\frac{1}{2} Z_{p-1,0} (a_2, a_3, ..., a_p, c, s)
+ \frac{\sqrt{\pi}}{2a_1 \Gamma(s)} \left( \Gamma(s - 1/2) Z_{p-1,0} (a_2, ..., a_p, c; s - 1/2) + G_{p,0} (a_1, a_2, ..., a_p, c; s - 1/2) \right)
\] (13)
\[
Z_{p,\frac{1}{2}} (a_1, a_2, ..., a_p, c; s) =
\frac{\sqrt{\pi}}{2a_1 \Gamma(s)} \left( \Gamma(s - 1/2) Z_{p-1,\frac{1}{2}} (a_2, ..., a_p, c; s - 1/2) - G_{p,\frac{1}{2}} (a_1, a_2, ..., a_p, c; s - 1/2) \right)
\] (14)
in which

\[ Z_{p,0}(a_1, a_2, ..., a_p, c; s) \equiv \sum_{\{n_i\}=1}^{p} \left[ \sum_{i=1}^{p} a_i^2 n_i^2 + c^2 \right]^{-s} \]  

(15)

\[ Z_{p,\frac{1}{2}}(a_1, a_2, ..., a_p, c; s) \equiv \sum_{\{n_i\}=0}^{p} \left[ \sum_{i=1}^{p} a_i^2 (n_i + 1/2)^2 + c^2 \right]^{-s} \]  

(16)

and \( G_{p,x} \equiv G_{p,\{b_2, b_3, ..., b_p-x\}} \), see Eq. (10). Now the recurrences (13) and (14) start with the two simplest cases

\[ Z_{1,0}(a_1, c; s) = \sum_{n_1=1}^{\infty} \left[ a_1^2 n_1^2 + c^2 \right]^{-s} \]  

(17)

\[ Z_{1,\frac{1}{2}}(a_1, c; s) = \sum_{n_1=0}^{\infty} \left[ a_1^2 (n_1 + 1/2)^2 + c^2 \right]^{-s} \]  

(18)

The above sums can be directly regularized by utilizing two specific generalized forms of the Abel-Plana formula [2]

\[ \sum_{n=1}^{\infty} (n^2 + q^2)^{-\nu} - \int_{0}^{\infty} du \left( u^2 + q^2 \right)^{-\nu} = -\frac{q^{-2s}}{2} - 2 \sin \pi \nu \int_{q}^{\infty} du \frac{(u^2 - q^2)^{-\nu}}{e^{2\pi u} - 1}, \]  

(19)

\[ \sum_{n=0}^{\infty} [(n + 1/2)^2 + q^2]^{-\nu} - \int_{0}^{\infty} du \left( u^2 + q^2 \right)^{-\nu} = 2 \sin \pi \nu \int_{q}^{\infty} du \frac{(u^2 - q^2)^{-\nu}}{e^{2\pi u} + 1}, \]  

(20)

where the right-hand sides of the above equations are the regularized forms of the sums. As a result one can directly obtain

\[ Z_{1,0}(a_1, c; s) = -\frac{e^{-2s}}{2} + 2 \sin(\pi s) \int_{c/a_1}^{\infty} du \frac{(a_1^2 u^2 - c^2)^{-s}}{e^{2\pi u} - 1} \]  

(21)

\[ Z_{1,\frac{1}{2}}(a_1, c; s) = -2 \sin(\pi s) \int_{c/a_1}^{\infty} du \frac{(a_1^2 u^2 - c^2)^{-s}}{e^{2\pi u} + 1} \]  

(22)

Subsequent cases are given by the recurrences (13) and (14). Some cases needed for the later calculations are given as follow:

\[ Z_{2,0}(a_1, a_2, c; s) = \frac{e^{-2s}}{4} - \frac{\sqrt{\pi} e^{1-2s}}{4a_1} \frac{\Gamma \left( s - \frac{1}{2} \right)}{\Gamma(s)} - \sin(\pi s) \int_{c/a_2}^{\infty} du \frac{(a_2^2 u^2 - c^2)^{-s}}{e^{2\pi u} - 1} \]  

\[ -\frac{\sqrt{\pi} \cos(\pi s)}{a_1} \frac{\Gamma \left( s - \frac{1}{2} \right)}{\Gamma(s)} \int_{c/a_2}^{\infty} du \frac{(a_2^2 u^2 - c^2)^{s-\frac{1}{2}}}{e^{2\pi u} - 1} + \frac{\sqrt{\pi}}{2a_1} \frac{G_{2,0}(a_1, a_2, c; s - \frac{1}{2})}{\Gamma(s)} \]  

(23)

\[ Z_{3,0}(a_1, a_2, a_3, c; s) = -\frac{e^{-2s}}{8} + \frac{\sqrt{\pi} e^{1-2s}}{8} \left( \frac{1}{a_1} + \frac{1}{a_2} \right) \frac{\Gamma \left( s - \frac{1}{2} \right)}{\Gamma(s)} - \frac{\pi e^{-2s}}{8a_1a_2} \frac{\Gamma \left( s - \frac{1}{2} \right)}{\Gamma(s)} \]
\[
\frac{\sqrt{\pi}}{2} \left( \frac{1}{a_1} + \frac{1}{a_2} \right) \cos(\pi s) \Gamma \left( s - \frac{1}{2} \right) \int_{c/a_3}^\infty du \frac{(a_3^2 u^2 - c^2)^{1/2 - s}}{e^{2\pi u} - 1} + \frac{\sin(\pi s)}{2} \int_{c/a_3}^\infty du \frac{(a_3^2 u^2 - c^2)^{-s}}{e^{2\pi u} - 1} - \pi \frac{\sin(\pi s)\Gamma(s - 1)}{\Gamma(s)} \int_{c/a_3}^\infty du \frac{(a_3^2 u^2 - c^2)^{1-s}}{e^{2\pi u} - 1}
\]

\[
\frac{\sqrt{\pi} G_{3.0} \left( a_1, a_2, a_3, c; s - \frac{1}{2} \right)}{2a_1} - \frac{\sqrt{\pi} G_{2.0} \left( a_2, a_3, c; s - \frac{1}{2} \right)}{4a_2} + \frac{\pi}{4a_1 a_2} G_{2.0} \left( a_2, a_3, c; s - \frac{1}{2} \right) \Gamma(s) \int_{c/a_4}^\infty du \frac{(a_2^2 u^2 - c^2)^{1-s}}{e^{2\pi u} - 1} \left( \frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \frac{1}{a_2 a_3} \right) \sin(\pi s) \Gamma(s - 1) \int_{c/a_4}^\infty du \frac{(a_2^2 u^2 - c^2)^{-s}}{e^{2\pi u} - 1} - \pi \sin(\pi s) \Gamma(s - 1) \int_{c/a_4}^\infty du \frac{(a_2^2 u^2 - c^2)^{1-s}}{e^{2\pi u} - 1} \left( \frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \frac{1}{a_2 a_3} \right) \cos(\pi s) \Gamma(s - 1) \int_{c/a_4}^\infty du \frac{(a_2^2 u^2 - c^2)^{-s}}{e^{2\pi u} - 1} \left( \frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \frac{1}{a_2 a_3} \right) \cos(\pi s) \Gamma(s - 1) \int_{c/a_4}^\infty du \frac{(a_2^2 u^2 - c^2)^{-s}}{e^{2\pi u} - 1} + \pi \frac{\sin(\pi s)\Gamma(s - 1)}{\Gamma(s)} \int_{c/a_4}^\infty du \frac{(a_2^2 u^2 - c^2)^{1-s}}{e^{2\pi u} - 1}
\]

Note that some infinite sums of the form \([5]\) has been studied in \([15]\) in a rather different manner based on the Riemann and/or the Hurwitz zeta regularization. However, the resulting expressions of the mentioned work are well-defined for \(c \neq 0\), such that the \(c = 0\) case has been considered separately. However, it is important for our work to find an exact expression for a finite “c”, resulting in an appropriate regularized form for \(c \to 0\), as here “c” has the role of the mass parameter.

Note that the summations over the entire integer numbers can also be written in terms of the above zeta functions. For instance,

\[
\sum_{n_1, n_2, n_3, n_4 = -\infty}^\infty \left[ a_1^2 n_1^2 + \ldots + a_4^2 n_4^2 + c^2 \right]^{-s} = e^{-2s} + 2 \left( Z_{1.0}(a_1, c; s) + Z_{1.0}(a_2, c; s) + Z_{1.0}(a_2, c; s) \right) + 4 \left( Z_{2.0}(a_1, a_2, c; s) + Z_{2.0}(a_1, a_3, c; s) + Z_{2.0}(a_2, a_3, c; s) + Z_{2.0}(a_2, a_4, c; s) + Z_{2.0}(a_3, a_4, c; s) \right)
\]
\[ +8 \left( Z_{3,0}(a_1, a_2, a_3, c; s) + Z_{3,0}(a_1, a_2, a_4, c; s) + Z_{3,0}(a_2, a_3, a_4, c; s) \right) \\
+ 16Z_{4,0}(a_1, a_2, a_3, a_4, c; s), \quad (26) \]

while
\[
\sum_{n_1, n_2 = \infty}^\infty \left[ a_1^2(n_1 + 1/2)^2 + \cdots + a_p^2(n_p + 1/2)^2 + c^2 \right]^{-s} = 2^s Z_{p+2}(a_1, \ldots, a_p, c; s). \quad (27) \]

Some other cases needed in this works can be obtained in a similar way:
\[
\sum_{n_1 = 1}^\infty \sum_{n_2 = 0}^\infty \left[ a_1^2(n_1 + 1/2)^2 + \cdots + a_p^2(n_p + 1/2)^2 + c^2 \right]^{-s} = \sin(\pi s) \int_{c/2}^\infty du \frac{(a_2^2u^2 - c^2)^{-s}}{e^{2\pi u} + 1} + \frac{\sqrt{\pi} \cos(\pi s) \Gamma(s - \frac{1}{2})}{\Gamma(s)} \int_{c/2}^\infty du \frac{(a_2^2u^2 - c^2)^{\frac{1}{2} - s}}{e^{2\pi u} + 1} + \frac{\sqrt{\pi} G_{2,1} \left( a_1, a_2, c; s - \frac{1}{2} \right)}{2a_1} \quad (28) \]
\[
- \frac{\sqrt{\pi} \cos(\pi s) \Gamma(s - \frac{1}{2})}{\Gamma(s)} \int_{c/2}^\infty du \frac{(a_2^2u^2 - c^2)^{\frac{1}{2} - s}}{e^{2\pi u} - 1} - \frac{\sqrt{\pi} G_{2,0} \left( a_1, a_2, c; s - \frac{1}{2} \right)}{2a_1} \quad (29) \]

Note that for \( a_1 \ll a_2 \ll a_3 \ll a_4 \) all the \( G \) function terms in the above zeta expressions can be neglected with a good degree of approximation, see Eq. [10]. As a result, as we see in the next sections, by using the above zeta functions one can appropriately regularize the vacuum energy obtaining two different representation asymptotically suitable for low/high temperatures. Finally to obtain the Casimir energy one must also subtract the contribution terms of the free vacuum, which are given by the first few terms of the heat-kernel expansion \[2,19,27\]. As we see in the following sections, through the above approach, these contribution terms would be automatically appeared in the resulting asymptotic expressions for Casimir energy at high temperature.

3 Ideal-metal planes

3.1 Two parallel planes

For two parallel planes with separation distance “\( a \)”, at temperature “\( T \)”, the total zero-point energy of electromagnetic field is well-known:
\[
E_0 = \frac{1}{\beta^2} \frac{A}{2\pi} \int_0^\infty k_\perp dk_\perp \sum_{n=1}^\infty \sum_{l=-\infty}^\infty \ln \left[ l^2 + \left( \frac{\beta}{2\pi} \right)^2 \left( k_\perp^2 + \left( \frac{n\pi}{a} \right)^2 \right) \right], \quad (30) \]

in which, “\( A \)” is the area of each plane, and \( k_\perp \) is the total momentum of a mode propagating parallel to the planes. Rewriting the above equation as a parametric integral, similarly as the previous section, one obtains
\[
E_0 = -\frac{\pi A}{\beta^2} \lim_{s \to 0} \frac{\partial}{\partial s} \frac{\Gamma(s-1)}{\Gamma(s)} \left( \sum_{n=1}^\infty (\lambda^2 n^2)^{1-s} + 2 \sum_{l,n=1}^\infty (l^2 + \lambda^2 n^2)^{1-s} \right) \quad (31) \]
where we have made use of a Gaussian integral for \( k_\perp \), and separated the \( l = 0 \) term. Now both the sum terms can be directly regularized in terms of \( Z_{1,0} \) and \( Z_{2,0} \). But according to the previous section, if in Eq. (23) we choose \( a_{1,2} = 1, \lambda_a \), the resulting expression would be asymptotically suitable for small temperatures (\( \lambda_a \geq 1 \)), while choosing \( a_{1,2} = \lambda_a, 1 \) would be appropriate for large temperatures (\( \lambda_a \leq 1 \));

\[
E_0 = -\frac{A}{\beta^3} \lim_{s \to 0} \frac{\partial}{\partial s} \Gamma(s-1) \left( Z_{1,0}(\lambda_a, 0; s-1) + 2Z_{2,0}(\lambda_a, 1, 0; s-1) \right). \tag{32}
\]

So using Eqs. (21) and (23) in the above equation, and after some calculations one can find the regularized vacuum energy as

\[
E_0(a, T) = -\frac{A}{\beta^3} \left[ \frac{8\pi^2}{3\lambda_a} \int_0^\infty du \frac{u^3}{e^{2\pi u} - 1} + 2\pi^2 \lambda_a^2 \int_0^\infty du \frac{u^2}{e^{2\pi u} - 1} \right. \\
+ \frac{4\pi}{\lambda_a} \sum_{n,l=1}^\infty \left( \frac{l}{n/\lambda_a} \right)^{3/2} K_{3/2} \left( 2\ln\pi \div \lambda_a \right) \left] \right. \\
(33)
\]

Now one can see that the first term of the above equation is just equal to the main term of the corresponding heat-kernel expansion \[27\], i.e. the free energy contribution of the black body radiation. Note that unlike the conventional regularization approaches, here the contribution of the free vacuum has been appeared automatically in the regularized zero-point energy. By subtracting this term one finds another exact expression for the Casimir energy, which, in contrast to the familiar expressions of the literature, has an asymptotically explicit form for sufficiently large temperatures;

\[
E_C(a, T) \approx -\frac{A}{8\pi a^2} \zeta(3) T; \quad T \gg 1/2a. \tag{34}
\]

This is the familiar high-temperature (classical) limiting expression of the Casimir energy of the parallel planes.

### 3.2 Rectangular box

The electromagnetic free energy in a rectangular ideal-metal box, can be written as

\[
E_0 = \frac{1}{2\beta} \sum_{l=\infty}^\infty \left[ 2 \sum_{n,m,r=1}^\infty \ln \left[ l^2 + \lambda_a^2 n^2 + \lambda_b^2 m^2 + \lambda_c^2 r^2 \right] + \sum_{m,r=1}^\infty \ln \left[ l^2 + \lambda_b^2 m^2 + \lambda_c^2 r^2 \right] \\
+ \sum_{n,r=1}^\infty \ln \left[ l^2 + \lambda_c^2 n^2 + \lambda_b^2 m^2 \right] + \sum_{n,m=1}^\infty \ln \left[ l^2 + \lambda_a^2 n^2 + \lambda_c^2 m^2 \right] \right), \tag{35}
\]

in which \( \lambda_a \equiv \beta/2a, \lambda_b \equiv \beta/2b, \lambda_c \equiv \beta/2c \) where \( a, b, c \) are the sidelengths of the box \( a \leq b \leq c \). Now according to the previous section, to find an asymptotically suitable expression for large temperatures (\( \lambda_c \leq \lambda_b \leq \lambda_a \leq 1 \)), the above equation can be written as

\[
E_0 = -\frac{1}{2\beta} \lim_{s \to 0} \frac{\partial}{\partial s} \left[ 2Z_{3,0}(\lambda_c, \lambda_b, \lambda_a, 0; s) + 4Z_{4,0}(\lambda_c, \lambda_b, \lambda_a, 1, 0; s) \right]
\]
The above equation can be directly regularized by using Eqs. (23), (24) and (25). As a result one can see that the first three terms of the heat kernel expansion, i.e. the terms proportional to the volume, total side-area and total side-length of the box, would be automatically appeared in the resulting regularized expression, see the forth term of Eq. (23), the forth and sixth terms of Eq. (24), etc. Then for sufficiently large temperatures we find

\[
E_{\text{em}C}(a, b, c, T) \approx -\frac{1}{\beta} \left( \frac{\zeta(3)}{8\pi} \frac{bc}{a^2} + \frac{\pi}{24b} \ln\left(\frac{\beta}{2a}\right) - \frac{\ln\left(\frac{\beta}{2b}\right)}{4} + \sum_{n,m} \frac{1}{m^2 + \frac{l^2}{b^2}} K_1 \left(2\pi n \frac{b}{a} \sqrt{m^2 + \frac{l^2}{b^2}}\right)\right) + 2c \sum_{n,m,l} \frac{1}{n^2 + m^2 + l^2(c/b)^2} K_1 \left(2\pi n \frac{b}{a} \sqrt{m^2 + \frac{l^2}{b^2}}\right) \]  

(38)

From the above equation, one can see that the Casimir forces, acting on the opposite sides of the box, can be attractive or repulsive, depending on the values of \(a, b\) and \(c\), which is generally in agreement with the known results \[2, 4, 20–22, 26\]. For a cube \((a=b=c)\) the above equation results in a limiting Casimir pressure as

\[
P_C(T \gg 1/a) \approx \frac{0.14}{a^3} T,
\]

(39)

Actually by using (37) it can be numerically shown that the above equation gives the largest pressure of the box. Note that at room temperature, the above equation is valid for \(a \gtrsim 6 \mu m\). Comparing the above equation with that of the parallel planes, one can see that the strength of Casimir pressure of a cube can be larger, by a factor of 1.5, than that of the two parallel planes.

3.3 Rectangular tube

The electromagnetic free energy in an ideal-metal rectangular tube can be written as

\[
E_0 = \frac{1}{2\beta^2} \frac{L}{2\pi} \int_{-\infty}^{\infty} du \sum_{l=-\infty}^{\infty} \left( 2 \sum_{n,m=1}^{\infty} \ln \left[ l^2 + \lambda_a^2 n^2 + \lambda_b^2 m^2 + (\beta/2\pi)^2 u^2 \right] + \sum_{n=1}^{\infty} \ln \left[ l^2 + \lambda_a^2 n^2 + (\beta/2\pi)^2 u^2 \right] + \sum_{m=1}^{\infty} \ln \left[ l^2 + \lambda_b^2 m^2 + (\beta/2\pi)^2 u^2 \right] \right) \]

(39)

in which \(\lambda_a = \beta/2a, \lambda_b = \beta/2b\), where \(a, b, L\) are the sidelengths of the tube \((a \leq b \ll L)\). Then similarly as before, a suitable expression for large temperatures \((\lambda_b \leq \lambda_a \leq 1)\) can
be obtained by regularizing the above equation as

$$E_0 = -\frac{L\sqrt{\pi}}{2\beta^2} \lim_{s \to 0} \frac{\partial}{\partial s} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \left[ Z_{1,0}(\lambda_a, 0; s - 1/2) + Z_{1,0}(\lambda_b, 0; s - 1/2) 
+ 2Z_{2,0}(\lambda_a, 1, 0; s - 1/2) + 2Z_{2,0}(\lambda_b, 1, 0; s - 1/2) 
+ 2Z_{2,0}(\lambda_b, \lambda_a, 0; s - 1/2) + 4Z_{3,0}(\lambda_b, \lambda_a, 1, 0; s - 1/2) \right] \quad (40)$$

As a result, after some calculations one can find

$$E_C(a, b, T) \approx -\frac{L^2}{2\beta^2} \left( \frac{\pi}{12b} + \frac{\zeta(3)}{4\pi} \frac{b}{a^2} + \frac{2}{a} \sum_{n,m=1}^\infty \frac{mK_1(2mn\pi b/a)}{nx} \right); \quad T \gg 1/2a. \quad (41)$$

One can see that the Casimir pressure for the rectangular tube can be attractive or repulsive depending on the values of the sidelengths “a” and “b”. The limiting Casimir pressure on two parallel sidewalls with the smallest separation distance, would be obtained as

$$P_{C,a} \approx -\frac{V(b/a)}{a^3} T; \quad T \gg 1/2a \quad (42)$$

in which,

$$V(x) \equiv \frac{\pi}{24x} + \frac{\zeta(3)}{4\pi} - \sum_{n,m=1}^\infty \left[ \frac{\pi m^2 K_0(2mn\pi x)}{nx} + \frac{\pi m^2 K_2(2mn\pi x)}{nx} - \frac{mK_1(2mn\pi x)}{nx} \right] \quad (43)$$

At room temperature, Eq. (42) is valid for $a, b \gtrsim 6\mu m$. By numerical computations one can find $V(x) \leq V(1) \approx 0.22$. So for sufficiently large temperatures, the pressure of the tube can be up to twice that of the parallel planes.

Note that the high-temperature limiting expressions (37) and (41) for the Casimir free energies of the box and tube, respectively, have not been given in other works. The familiar equations for the Casimir energy of the mentioned configurations, are asymptotically suitable for small temperatures, see e.g. [2, 3, 27]. These equations can also be obtained consistently through the approach of the section 2.

4 Non-Euclidean spaces

In this section we investigate the Casimir energy in spaces with nontrivial topology and curvature. We restrict ourselves on two familiar cases i.e. 3-torus and 3-sphere, which are of specific importance in some cosmological models.

4.1 3-torus

As we know, on a compact manifold with the topology of a torus, a scalar/spinor field satisfies a periodic identification condition in all coordinates, that is, for a field $\phi$ on a $d$-torus one can write

$$\phi(x_i + a_i) = \phi(x_i), \quad \partial_i \phi|_{x_i+\alpha_i} = \partial_i \phi|_{x_i}; \quad i = 1, 2, \ldots, d$$

$$\text{10}$$
in which \(x_i\)'s are the space coordinates, \(a_i\)'s are the identification parameters of the torus, and \(\partial_i \equiv \frac{\partial}{\partial x_i}\). As a result the free energy of a scalar field on a 3-torus can be written as

\[
E_0 = \frac{1}{2\beta} \sum_{l,m,n,r=-\infty}^{\infty} \ln \left[ t^2 + \lambda_a^2 m^2 + \lambda_b^2 n^2 + \lambda_c^2 r^2 + \lambda_d^2 \right],
\]

in which \(\lambda_{a,b,c}\) are given as before, and \(\lambda_\mu = \beta \mu / 2\pi\). For a 4-spinor the above equation is multiplied by “-4”. Then similarly as before, after some calculations using Eq. (26), for sufficiently small temperatures one obtains

\[
E_C(a, b, c, T) \approx -\frac{1}{2\beta} \left( \frac{4\pi^2 \lambda_\mu^3}{3\lambda_a \lambda_b \lambda_c} - \frac{\pi \lambda_\mu^2}{\lambda_a \lambda_c} (1 - 2 \ln (\lambda_\mu)) \right) - \frac{2\pi \lambda_\mu}{\lambda_c}
\]

\[
- \frac{16\pi^2}{3\lambda_a \lambda_b \lambda_c} \int_{\lambda_\mu}^{\infty} du \frac{u^2 - \lambda_\mu^2}{\exp(2\pi u) - 1} + \frac{16\pi^2}{3\lambda_b \lambda_c} \int_{\lambda_\mu/\lambda_a}^{\infty} du \frac{u^2 - (\lambda_\mu/\lambda_a)^2}{\exp(2\pi u) - 1}
\]

\[
+ \frac{4\pi^2 \lambda_\mu^2}{\lambda_c} \int_{\lambda_\mu/\lambda_b}^{\infty} du \frac{u^2 - (\lambda_\mu/\lambda_b)^2}{\exp(2\pi u) - 1} + 8\pi \lambda_\mu \int_{\lambda_\mu/\lambda_c}^{\infty} du \frac{u^2 - (\lambda_\mu/\lambda_c)^2}{\exp(2\pi u) - 1}
\]

\[
+ \frac{2\pi}{\lambda_c} \left( G_{2,0}(\lambda_c, \lambda_a, \lambda_\mu; -1) + G_{2,0}(\lambda_c, \lambda_b, \lambda_\mu; -1) + 2G_{3,0}(\lambda_c, \lambda_b, \lambda_a, \lambda_\mu; -1) \right)
\]

\[
+ \frac{2\pi^{3/2}}{\lambda_b \lambda_c} G_{2,0}(\lambda_b, \lambda_a, \lambda_\mu; -3/2)\right) ; \quad \lambda_{a,b,c} \gg 1.
\]

while for large temperatures we find

\[
E_C(a, b, c, T) \approx -\frac{1}{2\beta} \left( \frac{4\pi^2 \lambda_\mu^2}{\lambda_b \lambda_c} \int_{\lambda_\mu}^{\infty} du \frac{u^2 - (\lambda_\mu/\lambda_b)^2}{\exp(2\pi u) - 1}
\]

\[
+ \frac{8\pi \lambda_b}{\lambda_c} \int_{\lambda_\mu/\lambda_b}^{\infty} du \frac{u^2 - (\lambda_\mu/\lambda_b)^2}{\exp(2\pi u) - 1} + 4\pi \left( \int_{\lambda_\mu/\lambda_c}^{\infty} du \frac{u^2 - (\lambda_\mu/\lambda_c)^2}{\exp(2\pi u) - 1}\right)
\]

\[
+ \frac{2\pi^{1/2}}{\lambda_c} \left( G_{2,0}(\lambda_c, \lambda_a, \lambda_\mu; -1/2) + G_{2,0}(\lambda_c, \lambda_b, \lambda_\mu; -1/2) + 2G_{3,0}(\lambda_c, \lambda_b, \lambda_a, \lambda_\mu; -1/2) \right)
\]

\[
+ \frac{2\pi}{\lambda_b \lambda_c} G_{2,0}(\lambda_b, \lambda_a, \lambda_\mu; -1)\right) ; \quad \lambda_{a,b,c} \ll 1.
\]

Here we have subtracted the contribution of the free vacuum, that is, the vacuum of a 3-torus with \(a, b, c \to \infty\),

\[
E_0(a, b, c, T) = -\frac{1}{2\beta} \left( -\frac{4\pi^2 \lambda_\mu^2}{3\lambda_a \lambda_b \lambda_c} + \frac{\pi \lambda_\mu^2}{\lambda_a \lambda_c} (1 - 2 \ln (\lambda_\mu)) \right) + \frac{2\pi \lambda_\mu}{\lambda_c}
\]

\[
+ \frac{16\pi^2}{3\lambda_a \lambda_b \lambda_c} \int_{\lambda_\mu}^{\infty} du \frac{u^2 - \lambda_\mu^2}{\exp(2\pi u) - 1}\right).\]

In the case with sufficiently large values of \(b/a, c/a\) and \(c/b\), all the remaining G-function terms can also be neglected. As a result the limiting Casimir energies (43) and
for small masses can be approximated as

$$E_C(a, b, c, T) \approx -\frac{\pi^2 bc}{180a^3} - \frac{c\zeta(3)}{4\pi b^2} - \frac{\pi}{12c} + c\mu T + \frac{4\pi^2 abc}{45}T^4; \quad \mu \ll 1/c \gg T \quad (48)$$

while for large masses we find

$$E_C(a, b, c, T) \approx T \left[ -\frac{bc\zeta(3)}{2\pi a^2} - \frac{\pi c}{6b} + \ln(cT) + \left( \frac{2}{\pi} - 1 \right) c\mu \right]; \quad \mu \ll 1/a \ll T, \quad (49)$$

Eq. (48) is in agreement with the results of [28] for the zero-temperature massless case. From Eqs. (48) and (50) the limiting scalar Casimir entropies, $S_C \equiv -\left(\partial/\partial T\right)E_C$, on the 3-torus are obtained as

$$S_C \approx -\frac{c\mu}{16\pi^2 abcT^3}; \quad \mu \ll 1/c \gg T,$$

$$S_C \approx -\ln(cT); \quad \mu \ll 1/a \ll T. \quad (52)$$

As is seen, for $\mu \neq 0$ the scalar Casimir entropy of the torus takes nonzero values at zero temperature, violating the Nernst heat theorem. One can see that the violating term, i.e. the first term of the first line of the above equation, actually comes from the thermal contribution of the free vacuum (47). Also the entropy takes negative values as well for low as for high temperatures. Such a negative Casimir entropy appears also in some familiar configuration (in the Euclidean 3-space), such as the two parallel planes at sufficiently small temperatures, see e.g. [35–39]. In Ref. [39] the negative Casimir entropy for the parallel planes case has been justified by assigning a positive contribution to the conductor walls, such that the total entropy would be positive. However for a compact manifold such as the 3-torus the Casimir effect is not induced by any physical boundary walls. But the negative entropy of the torus arises actually as a result of subtracting the large positive contribution of the free vacuum, from the total scalar entropy of the 3-torus. That is, the scalar entropy of the torus is lower than that of the free vacuum. As a result, we speculate that here the negative entropy can be interpreted as an instability of the scalar vacuum state on the torus. Note that for a spinor field, the above Casimir entropies should be multiplied just by “-4”, in agreement with the results of [29] for the zero-temperature massless case. Hence the spinor vacuum entropy of the 3-torus is positive for sufficiently small as well as large temperatures.
4.2 3-sphere

The 00-component of the energy-momentum tensor for the vacuum state of the scalar
and spinor field on a 3-sphere are given as \[30, 31\]

\[
\langle 0 \left| T_{00} \right| 0 \rangle = \frac{1}{4\pi^2a^3} \sum_{n=1}^{\infty} n^2\omega_n; \quad \omega_n^2 = \mu^2 + \frac{n^2}{a^2}
\]

\[
\langle 0 \left| T_{00} \right| 0 \rangle = -\frac{1}{\pi^2a^3} \sum_{n=1}^{\infty} \left( \left( n + \frac{1}{2} \right)^2 - \frac{1}{4} \right) \omega_n; \quad \omega_n^2 = \mu^2 + \frac{(n + 1/2)^2}{a^2}
\]

respectively, in which \(\omega\) is the sphere radius, and \(\mu\) is the mass parameter. Therefore
the free energies of the scalar and spinor field can be given by

\[
E_0 = \frac{1}{2\beta} \sum_{l=-\infty}^{\infty} \sum_{n=1}^{\infty} n^2 \ln \left[ l^2 + \lambda_n^2 n^2 + \lambda_{\mu}^2 \right]
\]

\[
E_0 = -\frac{2}{\beta} \sum_{l=-\infty}^{\infty} \sum_{n=1}^{\infty} \left( \left( n + \frac{1}{2} \right)^2 - \frac{1}{4} \right) \ln \left[ l^2 + \lambda_n^2 \left( n + \frac{1}{2} \right)^2 + \lambda_{\mu}^2 \right]
\]

respectively, in which \(\lambda_n \equiv \beta/2\pi a\), and \(\lambda_{\mu} \equiv \beta\mu/2\pi\) as before. Then, similarly as carried
out in the section 3, by rewriting the above equations as parametric integrals and using
some relations such as

\[
n^2 \sum_n e^{-t\lambda_n^2 n^2} = -\frac{1}{2t\lambda_n} \frac{\partial}{\partial \lambda_n} \sum_n e^{-t\lambda_n^2 n^2}
\]

one can rewrite Eqs. \[53\] and \[54\] as

\[
E_0(a, T) = \frac{1}{4\lambda_n \beta} \frac{\partial}{\partial \lambda_n} \lim_{s \to 0} \frac{\Gamma(s - 1)}{\Gamma(s)} \sum_{l=-\infty}^{\infty} \sum_{n=1}^{\infty} \left[ l^2 + \lambda_n^2 n^2 + \lambda_{\mu}^2 \right]^{-(s-1)}
\]

and

\[
E_0(a, T) = -\frac{2}{\beta} \lim_{s \to 0} \frac{\partial}{\partial s} \left( \frac{1}{2\lambda_n} \frac{\partial}{\partial \lambda_n} \Gamma(s - 1) \sum_{l=-\infty}^{\infty} \sum_{n=1}^{\infty} \left[ l^2 + \lambda_n^2 (n + 1/2)^2 + \lambda_{\mu}^2 \right]^{-(s-1)} + \frac{1}{4} \sum_{l=-\infty}^{\infty} \sum_{n=1}^{\infty} \left[ l^2 + \lambda_n^2 (n + 1/2)^2 + \lambda_{\mu}^2 \right]^{-s} \right)
\]

respectively. Now after separating the \(l = 0\) term, the above equations can be regularized
in terms of \(Z_{2.0}, Z_{1.0}\) and \(Z_{1.4}\) giving two different representations as well for the scalar
as for the spinor Casimir energy.

For the scalar case by using Eqs. \[21\] and \[23\] for Eq. \[57\] after some calculations,
the limiting Casimir free energies are obtained as

\[
E_C(a, T) \approx \frac{\pi}{2\beta \lambda_n} \frac{\partial}{\partial \lambda_n} \left( \lambda_{\mu}^3 - \frac{4}{3\lambda_n} \int_{\lambda_{\mu}}^{\infty} \frac{[u^2 - \lambda_{\mu}^2]^{3/2}}{\exp(2\pi u) - 1} \right)
\]
Here the energy contribution of the free scalar vacuum i.e. the vacuum energy contribution of a 3-sphere with infinitely large radius $a \rightarrow \infty$, 

$$E_0(a, T) = -\frac{1}{4\beta\lambda_a} \frac{\partial}{\partial \lambda_a} \left( \frac{2\pi\lambda^3}{3\lambda_a} - 8\pi \int_{\lambda_a}^{\infty} \frac{[u^2 - (\lambda_{\mu}/\lambda_a)^2]^{3/2}}{\exp(2\pi u) - 1} \right)$$

has been subtracted. Then for sufficiently small masses the above limiting Casimir energies take the forms

$$E_C(a, T) \approx \frac{1}{240a} + \frac{\pi^4a^3}{45} T^4 - \left( \frac{1}{9} + \frac{\pi}{6} \right) a^3 \mu^3 T; \quad \mu \ll 1/a \gg T \quad (62)$$

while for sufficiently large masses we find

$$E_C(a, T) \approx \frac{\zeta(3)}{4\pi^2} T; \quad \mu \ll 1/a \ll T, \quad (63)$$

For the spinor field, from Eq. (58) and after some similar calculation using Eqs (28) and (29), we find

$$E_C(a, T) \approx -\frac{2}{\beta} \left( \frac{1}{2\lambda_a} \frac{\partial}{\partial \lambda_a} \left[ \frac{2\pi\lambda^3}{3\lambda_a} - 8\pi \int_{\lambda_a}^{\infty} \frac{[u^2 - (\lambda_{\mu}/\lambda_a)^2]^{3/2}}{\exp(2\pi u) + 1} \right] \right)$$

$$- \frac{8\pi\lambda^3}{3} \int_{\lambda_a/\lambda_a}^{\infty} du \frac{[u^2 - (\lambda_{\mu}/\lambda_a)^2]^{3/2}}{\exp(2\pi u) + 1} - \pi\lambda_a \int_{\lambda_a/\lambda_a}^{\infty} du \frac{[u^2 - (\lambda_{\mu}/\lambda_a)^2]^{1/2}}{\exp(2\pi u) + 1} \right) \quad \lambda_a \gg 1, \quad (66)$$

$$E_C(a, T) \approx \frac{2}{\beta} \left( \frac{1}{2\lambda_a} \frac{\partial}{\partial \lambda_a} \left[ 2\pi\lambda^3 \int_{\lambda_a/\lambda_a}^{\infty} du \frac{u^2 - (\lambda_{\mu}/\lambda_a)^2}{\exp(2\pi u) + 1} \right] \right)$$
\[ E_0^s(a, T) = -2 \left( \frac{1}{2} \frac{\partial}{\partial \lambda_a} - \frac{2 \pi}{3} \lambda_a^2 + \frac{8 \pi}{3} \int_{\lambda_a}^{\infty} \frac{du}{\exp(2 \pi u) + 1} \right); \quad \lambda_a \ll 1. \] (67)

Here the contribution of the free spinor vacuum energy of the 3-sphere is given as

\[ E_0^s(a, T) = -2 \left( \frac{1}{2} \frac{\partial}{\partial \lambda_a} + \frac{2 \pi}{3} \lambda_a^2 + \frac{8 \pi}{3} \int_{\lambda_a}^{\infty} \frac{du}{\exp(2 \pi u) + 1} \right) + \frac{\pi \lambda_a}{4} \int_{\lambda_a}^{\infty} \frac{du}{\exp(2 \pi u) + 1}. \] (68)

Then for sufficiently small masses, the spinor Casimir energies (66) and (67) take the forms

\[ E_C(a, T) \approx \frac{17}{480} a + \frac{\pi a^2 T^2}{12} + \frac{\pi a \mu}{2} T; \quad \mu \ll 1/a \gg T. \] (69)

\[ E_C(a, T) \approx \frac{3 \zeta(3) + \pi^2 \ln(4)}{4 \pi^2} T; \quad \mu \ll 1/a \ll T, \] (70)

while for sufficiently large masses we find

\[ E_C(a, T) \approx \frac{a^{3/2} \mu^{1/2}}{\pi} e^{-2 \pi a \mu} + \frac{2}{3} \pi^3 \mu^3 T; \quad \mu \gg 1/a \gg T. \] (71)

\[ E_C(a, T) \approx 2 a^2 \mu^2 e^{-2 \pi a \mu} T; \quad \mu \gg 1/a \gg T, \] (72)

Eqs. (62), (64), (69) and (71) are in agreement with the zero-temperature results of [30–34]. However the temperature corrections obtained here for the Casimir energy of the sphere as well as torus, have not been given in other works.

From Eqs. (62) and (63) the limiting scalar Casimir entropies on the 3-sphere are given as

\[ S_C \approx \left( \frac{1}{6} + \frac{\pi}{6} \right) a^3 \mu^3 - \frac{4 \pi^4 a^3}{45} T; \quad \mu \ll 1/a \gg T \] (73)

while for the spinor case using Eqs. (69) and (70) we find

\[ S_C \approx -\frac{\zeta(3)}{4 \pi^2} T; \quad \mu \ll 1/a \ll T. \] (74)

Again, as is seen, a nonzero mass term violates the third law of thermodynamics, and yet the scalar as well as the spinor entropy of the 3-sphere is negative, which as before, results from subtracting the contribution of the free vacuum. As before, we interpret this negative entropy as an instability of the finite-temperature vacuum state of scalar as well as spinor fields on the 3-sphere. Note that again the thermal contribution of the free vacuum is responsible for the terms violating the basic laws of thermodynamics.
5 Concluding remarks

In this work we have introduced a new useful approach to find asymptotically suitable expressions for the Casimir free energy for sufficiently large temperatures. This approach is based on a rather different regularization technique for an inhomogeneous Epstein-like zeta function. This approach works well for many familiar configurations in Euclidean as well as non-Euclidean spaces. The resulting expressions for the Casimir energy contain the classical terms as well as the first few terms of the corresponding heat-kernel expansion, as expected. The resulting asymptotic expressions, specially for the classical part, might not be simply obtained through other conventional approaches. By utilizing this technique, we have provided some new numerical results specifically for the Casimir pressure inside rectangular ideal-metal cavities. We have shown that at sufficiently large temperature, the Casimir pressure in a rectangular tube and a rectangular box can be larger by factors of 2 and 1.5, respectively, than that of the two parallel ideal-metal planes. Note that for sufficiently small temperatures the Casimir pressures of the box and tube are close to that of the parallel planes. We have also applied this technique for calculating the Casimir free energy as well on a 3-torus as on a 3-sphere. We have shown that a nonzero mass term for both scalar and spinor fields as well on the torus as on the sphere, violates the third law of thermodynamics. We have obtained negative values for the scalar Casimir entropy on the torus, and for both scalar and spinor Casimir entropies on the torus as well as on the sphere. Actually, these negative entropies arise when one subtracts the contribution of the free vacuum. This means that the vacuum entropy of the system is lower than that of the free vacuum. Thermodynamically we speculate that this negative Casimir entropy can be interpreted as an instability of the finite-temperature vacuum state.

Acknowledgment

We thank G. Jafari for his valuable comments, and S. Qolibikloo for his helps during the numerical computations

References

[1] Casimir, H. B. G., Proc. K. Ned. Akad. Wet. B 51, 793-5 (1948)
[2] M. Bordag, U. Mohideen and V. M. Mostepanenko, Phys. Rep. 353, 1205 (2001).
[3] K. A. Milton, The Casimir effect - Physical manifestation of zero-point energy , World Scientific, Singapore, 2001.
[4] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Advances in the Casimir Effect, Oxford Science Publications, 2009.
[5] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Rev. Mod. Phys. 81, 1827 (2009).
[6] S. K. Lamoreaux, Reports on Progress in Physics, 68,1 (2004)
[7] V. M. Mostepanenko and N. N. Trunov, *The Casimir effect and its applications*, Clarendon Press, Oxford; New York, 1997.
[8] Decca R S, Lopez D, Fischbach E, Klimchitskaya G L, Krause D E and Mostepanenko V M 2005 Ann. Phys. (NY) 318 37; quant-ph/0503105v1
[9] Brevik I, Aarseth J B, Hye J S and Milton K A 2005 Phys. Rev. E 71 056101; quant-ph/0410231
[10] Bezerra V B, Decca R S, Fischbach E, Geyer B, Klimchitskaya G L, Krause D E, Lopez D, Mostepanenko V M and Romero C, 2006 Phys. Rev. E 73 028101, quant-ph/0503134.
[11] Hye J S, Brevik I, Aarseth J B and Milton K A 2006 J. Phys. A: Math. Gen. 39 6031; quant-ph/0506025 v4
[12] Iver Brevik, Simen A Ellingsen and Kimball A Milton, New Journal of Physics 8 (2006) 236
[13] Peskin, M. E. and Schroeder, D. V., *An Introduction to Quantum Field Theory* Addison-Wesley, 1995.
[14] E. Elizalde, J. Math. Phys. 31, 1, (1990)
[15] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*, Springer Heidelberg (2012).
[16] E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford University Press, Oxford (1948)
[17] Gradshteyn, I. S. and Ryzhik, I. M. (1994). *Table of Integrals, Series, and Products*. Academic Press, New York.
[18] Erdelyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. (1981). *Higher Transcendental Functions*, Vol. 1. Kriger, New York.
[19] J.S. Dowker, G. Kennedy, J. Phys. A: Math. Gen. 11, 895 (1978)
[20] W.Lukosz, Physica 56 (1971) 109.
[21] J.AmbjHrn, S.Wolfram, Ann.Phys.(N.Y.) 147 (1983) 1.
[22] G.J.Maclay, Phys.Rev.A 61 (2000) 052110
[23] S.Hacyan, R.J. auregui, C.Villarreal, Phys.Rev.A 47 (1993) 4204.
[24] F.Caruso, R.De Paola, N.F.Svaiter, Int.J.Mod.Phys.A 14 (1999) 2077.
[25] X.-Z.Li, H.-B.Cheng, J.-M.Li, X.-H.Zhai, Phys.Rev.D 56 (1997) 2155.

17
[26] V. Hushwater, Am. J. Phys. 65 (1997) 381.

[27] B. Geyer, G. L. Klimchitskaya, V. M. Mostepanenko, Eur. Phys. J. C (2008) 57: 823834

[28] Mamayev, S. G. and Trunov, N. N., (1979), Teor. Matem. Fiz. 38, 34554

[29] Mamayev, S. G. and Trunov, N. N., Izv. Vuzov, Fizika N7, 913 (Russ. Phys. J. 23, 5514) (1980)

[30] Grib, A. A., Mamayev, S. G., and Mostepanenko, V. M. (1994). Vacuum Quantum Effects in Strong Fields, Friedmann Laboratory Publishing, St. Petersburg.

[31] Grib, A. A., Mamayev, S. G., and Mostepanenko, V. M. (1980), Fort-schr. Phys. 28, 17399.

[32] Ford, L. H., Phys. Rev. D 11, 33707 (1975).

[33] Ford, L. H., Phys. Rev. D 14, 330413 (1976).

[34] Mamayev, S. G., Mostepanenko, V. M., and Starobinsky, A. A., Zh. Eksp. Teor. Fiz. 70, 157791 (Sov. Phys. JETP 43, 82330) (1976)

[35] V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko, C. Romero, Phys. Rev. A, 69, 022119 (2004)

[36] Revzen, M., Opher, R., Opher, M. and Mann, A., Journal of Physics A: Mathematical and General, 30(22), p.7783 (1997).

[37] Bordag, M. and Pirozhenko, I.G., Physical Review D, 82(12), p.125016 (2010).

[38] V. B. Bezerra, G. L. Klimchitskaya, and V. M. Mostepanenko, Phys. Rev. A 66, 062112 (2002).

[39] I. Brevik, J. Aarseth, J. S. Hye, and K. A. Milton, arXiv: quant-ph/0410231

18