WEAK COHERENCE OF GROUPS AND FINITE DECOMPOSITION COMPLEXITY

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Abstract. The weak regular coherence (WRC) property of a finitely generated group $\Gamma$ was introduced by G. Carlsson and this author to play the role of a weakening of Waldhausen’s regular coherence as part of computation of the integral $K$-theoretic assembly map. This property is a coarse invariant of the word metric. It was shown that groups with finite asymptotic dimension (FAD) have this property.

A new class of metric spaces (sFDC) was introduced recently by A. Dranishnikov and M. Zarichnyi. This class includes most notably the spaces with finite decomposition complexity (FDC) studied by E. Guentner, D. Ramras, R. Tessera, and G. Yu and so is larger than FAD. The main theorem of this paper shows that a group that belongs to sFDC is weakly regular coherent.

As a consequence, groups $\Gamma$ with sFDC have some remarkable algebraic and topological properties. One such property is vanishing of the Whitehead group $\text{Wh}(\Gamma)$ for any group that has finite $K(\Gamma,1)$ and FDC, which verifies a long-standing conjecture for this class of groups.

Introduction

Given a topological manifold, there are many consequences of the fact that the fundamental group has finite asymptotic dimension (FAD). It is known that the Novikov conjecture is true for such manifolds from the pioneering work of Yu [15]. The Novikov conjecture in integral $K$-theory was proved for groups with FAD by Bartels [2]. Further, the integral Borel conjecture for groups with finite $K(\pi,1)$ and FAD was verified by Carlsson and the author [3, 4, 5, 6]. The latter fact, in particular, has the consequence that the Whitehead group of such $\Gamma$ is trivial.

The ultimate goal of this paper is to extend these results further to groups that have finite decomposition complexity (FDC) as defined by Guentner–Tessera–Yu [10]. The work of Carlsson–Goldfarb shows that the Borel isomorphism conjecture follows from the combination of the coarse integral Novikov conjecture and the weak regular coherence property. The integral Novikov conjecture for these groups has already been verified by Ramras–Tessera–Yu [13].

The groups with FDC or sFDC are intermediate between the class FAD and the groups with Yu’s property A [8]. Some well-known examples of groups with infinite asymptotic dimension, such as the wreath product of two copies of the integers $\mathbb{Z}$, belong to both classes. Further examples of FDC groups established in [11] are all finitely generated subgroups of $GL_n(k)$, where $k$ is a field. For other interesting infinite dimensional groups such as Thompson’s group, Grave’s group, Gromov’s...
random groups the answer is unknown. It is also unknown where groups such as
Out($F_n$) belong in this hierarchy.

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paper [8] and other related communications.

1. Geometric preliminaries

The following is a summary of relationships between some coarse properties of
metric spaces.

The original definition of asymptotic dimension by M. Gromov is a coarse ana-
logue of the covering dimension of topological spaces.

**Definition 1.1.** The asymptotic dimension of a metric space $X$ is defined as the
smallest number $n$ such that for any $d > 0$ there is a uniformly bounded cover $U$ of
$X$ such that any metric ball of radius $d$ in $X$ meets no more than $n + 1$ elements
of the cover $U$. If such number exists for $X$, one says that $X$ has finite asymptotic
dimension.

A number of authors generalized FAD in several directions in recent years. One
such generalization is asymptotic property C defined in Dranishnikov [7].

Given a subset $S$ of a metric space $X$, we will use the notation $S[b]$ for the
$b$-enlargement of $S$, that is the subset \( \{ x \in X \mid d(x,s) \leq b \text{ for some } s \in S \} \). So,
in particular, the metric ball centered at $x$ with radius $r$ is denoted by $x[r]$. Also,
given a number $R > 0$, a collection of disjoint subsets $S_\alpha$ of $X$ is called $R$-disjoint
if $S_\alpha[R]$ is disjoint from the union $\bigcup_{\beta \neq \alpha} S_\beta$, for all $\alpha$.

**Definition 1.2** (APC). A metric space $X$ has the asymptotic property C if for
every sequence of positive numbers $R_1 \leq R_2 \leq \ldots$ there exists a natural number $n$
and uniformly bounded $R_i$-disjoint families $W_i$, $1 \leq i \leq n$, such that the union of
all $n$ families is a covering of $X$.

On the other hand, the following is one of the equivalent definitions of the finite
decomposition complexity from Guentner–Tessera–Yu.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two families of metric spaces. Let $R > 0$. The family $\mathcal{X}$ is called
$R$-decomposable over $\mathcal{Y}$ if for any space $X$ in $\mathcal{X}$ there are collections of subsets
$\{U_{1,\alpha}\}$ and $\{U_{2,\beta}\}$ such that

$$X = \bigcup_{i=1,2} \bigcup_{\gamma=\alpha,\beta} U_{i,\gamma},$$

each $U_{i,\gamma}$ is a member of the family $\mathcal{Y}$, and each of the collections $\{U_{1,\alpha}\}$ and $\{U_{2,\beta}\}$
is $R$-disjoint. A family of metric spaces is called bounded if there is a uniform bound
on the diameters of the spaces in the family.

One of the equivalent definitions of finite decomposition complexity of the metric
space $X$ is in terms of a winning strategy for the following game between two players.
In round number 1 the first player selects a number $R_1 > 0$, the second player has
to select a family of metric spaces $\mathcal{Y}_1$ and an $R_1$-decomposition of $\{X\}$ over $\mathcal{Y}_1$. In
each succeeding round number $i$ the first player selects a number $R_i > 0$, the
second player has to select a family of metric spaces $\mathcal{Y}_i$ and an $R_i$-decomposition
of $\mathcal{Y}_{i-1}$ over $\mathcal{Y}_i$. The second player wins the game if for some finite value of $i$ the
family $\mathcal{Y}_i$ is bounded.
Definition 1.3 (FDC). A metric space $X$ has \textit{finite decomposition complexity} if the second player possesses a winning strategy in every game played over $X$.

The following property was defined by A. Dranishnikov and M. Zarichnyi in [8].

Definition 1.4 (sFDC). A metric space $X$ has \textit{straight finite decomposition complexity} if, for any sequence $R_1 \leq R_2 \leq \ldots$ of positive numbers, there exists a finite sequence of metric families $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ such that $X$ is $R_1$-decomposable over $\mathcal{V}_1, \mathcal{V}_1$ is $R_2$-decomposable over $\mathcal{V}_2$, etc., and the family $\mathcal{V}_n$ is bounded.

The relationship between the four geometric conditions on a metric space is established in [8] and is expressed by the diagram

\[
\begin{array}{ccc}
\text{FAD} & \leftrightarrow & \text{FDC} \\
\downarrow & & \downarrow \\
\text{APC} & \leftrightarrow & \text{sFDC}
\end{array}
\]

All four classes satisfy Yu’s property A.

In this paper we generalize [4] to a larger class of $R[\Gamma]$-modules by weakening the geometric condition on the group, promoting the main theorem from $\Gamma$ with FAD to $\Gamma$ with sFDC.

2. Filtered modules over a metric space

We start by recasting some of the controlled algebra from [5].

Let $R$ be a left noetherian ring with unit and let $\text{Mod}(R)$ be the category of left $R$-modules. For a general set $X$, the power set $\mathcal{P}(X)$ partially ordered by inclusion is a category with subsets of $X$ as objects and unique morphisms $(S,T)$ when $S \subset T$.

Definition 2.1. An $X$-filtered $R$-module is a functor $F: \mathcal{P}(X) \to \text{Mod}(R)$ where all structure maps $F(S,T)$ are monomorphisms and $F(\emptyset) = 0$.

We will abuse notation by referring to $F(S,T)$ as $F$. For any covariant functor $F: \mathcal{P}(X) \to \text{Mod}(R)$ with $F(\emptyset) = 0$ there exists an associated $X$-filtered $R$-module $F_X$ given by $F_X(S) = \text{im} F(S,X)$.

Given a filtered module $F$ and an arbitrary submodule $F'$, the submodule has the \textit{standard $X$-filtration} given by $F'(S) = F(S) \cap F'$. For example, for each subset $T \subset X$, the submodule $F(T)$ of $F$ has the canonical filtration and gives a filtered module $F_T$ with $F_T(S) = F(S) \cap F(T)$.

We will assume that $X$ is a proper metric space in the sense that all closed metric balls in $X$ are compact.

Definition 2.2. Given a number $b \geq 0$, an $R$-homomorphism $\varphi: F_1 \to F_2$ is called \textit{$b$-controlled} or simply \textit{controlled} if the image $\varphi(F_1(S))$ is contained in the submodule $F_2(S[b])$ for all subsets $S \subset X$.

Here $S[b]$ stands for the metric $b$-enlargement of $S$ in $X$ defined as the subset $\{x \in X \mid d(x,S) \leq b\}$.

We say $\varphi$ is \textit{$b$-bicontrolled} or \textit{bicontrolled} if in addition to containments $\varphi(F_1(S)) \subset F_2(S[b])$
as above, there are containments

\[ \varphi(F_1) \cap F_2(S) \subset \varphi F_1(S[b]) \]

for all subsets \( S \) of \( X \).

The \( X \)-filtered objects \( F \) may be subject to the following constraints.

- \( F \) is \textit{locally finite} if \( F(V) \) is a finitely generated submodule of \( F \) whenever \( V \) is a bounded subset of \( X \).
- \( F \) is \textit{\( D \)-lean} or simply \textit{lean} if there is a number \( D \geq 0 \) such that we have

\[
F(U) \subset \sum_{x \in U} F(x[D])
\]

for any subset \( U \) of \( X \).
- \( F \) is \textit{\( \delta \)-split} or \textit{split} if there is a number \( \delta \geq 0 \) such that we have

\[
F(U_1 \cup U_2) \subset F(U_1[\delta]) + F(U_2[\delta])
\]

for any pair of subsets \( U_1, U_2 \) of \( X \).

- \( F \) is \textit{\( d \)-insular} or \textit{insular} if there is a number \( d \geq 0 \) such that

\[
F(U_1) \cap F(U_2) \subset F(U_1[d] \cap U_2[d])
\]

for any pair of subsets \( U_1, U_2 \) of \( X \).

It is clear that a \( D \)-lean module is \( D \)-split.

If \( F \) is \( X \)-filtered, the kernel \( K \) of an arbitrary homomorphism \( f: F \to G \) inherits the standard filtration \( K(S) = K \cap F(S) \).

**Proposition 2.3.** Given a bicontrolled epimorphism \( f: F \to G \), suppose \( F \) is split and \( G \) is insular. Then the kernel \( K \) with the standard filtration is split.

**Proof.** Suppose \( F \) is \( \delta \)-split and \( G \) is \( d \)-insular. Let \( i: K \to F \) be the inclusion of the kernel. Given \( z \in K(T \cup U) \), we have \( i(z) \in F(T \cup U) \subset F(T[\delta]) + F(U[\delta]) \), so we can write accordingly \( i(z) = y_1 + y_2 \) where \( f(y_1) = -f(y_2) \). Since \( G \) is \( d \)-insular,

\[
f(y_1) = -f(y_2) \in G(T[\delta + b]) \cap G(U[\delta + b]) \subset G(T[\delta + b + d] \cap U[\delta + b + d]),
\]

so we are able to find

\[
y \in F(T[\delta + 2b + d] \cap U[\delta + 2b + d])
\]

such that \( f(y) = f(y_1) = -f(y_2) \), because generally \( (S \cap P)[b] \subset S[b] \cap P[b] \). Thus

\[
i(z) = y_1 + y_2 = (y_1 - y) + (y_2 + y)
\]

and

\[
y_1 - y \in F(T[\delta + 2b + d]), \quad y_2 + y \in F(U[\delta + 2b + d]).
\]

Let \( z_1 = i^{-1}(y_1 - y) \) and \( z_2 = i^{-1}(y_2 + y) \), and we have \( z = z_1 + z_2 \) such that

\[
z_1 \in K(T[\delta + 2b + d]), \quad z_2 \in K(U[\delta + 2b + d]),
\]

so \( K \) is \((\delta + 2b + d)\)-split. \( \square \)

**Proposition 2.4.** Given a b-controlled homomorphism \( f: F \to G \), suppose \( F \) is \( D \)-lean and \( G \) is \( d \)-insular. Let \( K \) be the kernel of \( f \) and let \( U \) be the union of a \((2D + 2b + 2d)\)-disjoint collection \( \{U_\alpha\} \) of subsets of \( X \). Then \( K(U) \subset \sum K(U_\alpha[D]) \).
Proof. Since $F$ is $D$-lean, an arbitrary element $k \in K(U)$ can be written as a sum $k = \sum k_\alpha$ where $k_\alpha \in F(U_\alpha[D])$. To check that $k_\alpha \in K$, consider $k'_\alpha = \sum_{\beta \neq \alpha} k_\beta$ and $U'_\alpha = \bigcup_{\beta \neq \alpha} U_\beta$. Then $f(k_\alpha) \in G(U_\alpha[D + b])$ and $f(k'_\alpha) \in G(U'_\alpha[D + b])$. So $f(k_\alpha) - f(k'_\alpha) \in G(U_\alpha[D + b + d] \cap U'_\alpha[D + b + d]) = G(0)$.

Theorem 2.5. Suppose $X$ is a metric space with sFDC. Given a bicontrolled epimorphism $f : F \to G$, suppose $F$ is lean and $G$ is insular. Then the kernel $K$ with the standard filtration is lean.

Proof. Since $F$ is $D$-split if it is $D$-lean, Proposition 2.3 can be applied showing that $K$ is $(D + 2b + d)$-split.

We start with the sequence of numbers $R_1 \leq R_2 \leq \ldots \leq R_k \leq \ldots$ where $R_k = 2(k + 1)D + 2b + 2d$. By the assumption, there is a sequence of metric families $\mathcal{Y}_1, \mathcal{Y}_2, \ldots$ with the corresponding $R_k$-decompositions so that for some $n$ the family $\mathcal{Y}_n$ is bounded.

To start the inductive argument, assume that $X = U_1 \cup V_1$ so that $U_1$ is the union of an $R_1 = 4D + 2b + 2d$-disjoint family $\{U_1,\alpha\}$ and $V_1$ is the union of an $R_1$-disjoint family $\{V_1,\beta\}$. Then each element $k$ in $K$ is the sum $k_1U + k_1V$ for some $k_1U \in K(U_1[D])$ and $k_1V \in K(V_1[D])$. Since both $\{U_1,\alpha\}$ and $\{V_1,\beta\}$ are $(2D + 2b + 2d)$-disjoint, we have $k_1U = \sum k_{1U,\alpha}$, $k_1V = \sum k_{1V,\beta}$ where $k_{1U,\alpha} \in K(U_1[D]), k_{1V,\beta} \in K(V_1[D])$. By the assumption, each $\{U_1,\alpha\}$ and $\{V_1,\beta\}$ is $R_2 = (6D + 2b + 2d)$-decomposable over $\gamma_2$. Therefore each $\{U_1,\alpha[D]\}$ and $\{V_1,\beta[D]\}$ is $(4D + 2b + 2d)$-decomposable over $\gamma_2$. In particular, if $U_1,\alpha = U_2,\alpha \cup V_2,\alpha$ so that $U_2,\alpha$ is the union of an $R_2$-disjoint family $\{U_2,\alpha,\gamma\}$, then $U_2,\alpha[D]$ is the union of a $R_2 - 2D = (4D + 2b + 2d)$-disjoint family $\{U_2,\alpha,\gamma[D]\}$. The decompositions can be continued inductively for the total of $n$ steps giving $k$ as the sum of kernel elements in $F(W[2nD])$, where $W \in \mathcal{Y}_n$. So $k$ is generated by the kernel elements with uniformly bounded support: if the sets in $\mathcal{Y}_n$ have mesh bounded by $M$ then $K$ is $(M + 2nD)$-lean.

We will view a finitely generated group $\Gamma$ as a metric space with the word metric generated from some choice of a finite generating set. Then $\Gamma$ acts by left translations on itself, and each left translation is an isometry.

Definition 2.6. A $\Gamma$-filtered $R[\Gamma]$-module is $\Gamma$-equivariant or simply a $\Gamma$-module if $F(\gamma S) = \gamma F(S)$ for all choices of $\gamma \in \Gamma$ and $S \subset \Gamma$.

Corollary 2.7. Suppose $\Gamma$ is a finitely generated group with sFDC. Given a bicontrolled $R[\Gamma]$-epimorphism $f : F \to G$ of $\Gamma$-modules, suppose $F$ is lean and $G$ is insular. Then the kernel $K$ is finitely generated.

Proof. The kernel is lean by Theorem 2.5. If $D$ is a leanness constant then $K$ is generated by $K \cap F(e[D])$, where $e$ is the identity element, as an $R[\Gamma]$-module. Since $R$ is noetherian, the $R$-module $K \cap F(e[D])$ is finitely generated, so $K$ is finitely generated as an $R[\Gamma]$-module.

3. Applications

3.1. Weak regular coherence of sFDC groups. Let $A$ be a ring with a unit.

Definition 3.1. A left $A$-module is said to be of type FP$_\infty$ if it has a resolution by finitely generated projective $A$-modules. The ring $A$ is called coherent if every
finitely presented \( A \)-module is of type \( \text{FP}_\infty \). It is called *regular coherent* if each resolution can be chosen to be finite.

An \( A \)-module has *finite projective dimension* if it has a finite resolution by finitely generated \( A \)-modules. The ring \( A \) has *finite (global) dimension* if every finitely generated \( A \)-module has a resolution by finitely generated projective \( A \)-modules of some fixed length.

Waldhausen [14] discovered a remarkable collection of discrete groups \( \Gamma \) such that all finitely presented modules over the group ring \( \mathbb{Z}[\Gamma] \) are regular coherent. It includes free groups, free abelian groups, torsion-free one relator groups, fundamental groups of submanifolds of the three-dimensional sphere, and their various amalgamated products and HNN extensions and so, in particular, the fundamental groups of submanifolds of the three-dimensional sphere. Waldhausen called this property of the group *regular coherence* and used it to compute the algebraic \( K \)-theory of these groups. He asked if a weaker property of the group ring would suffice in his argument, see for example the paragraph after the proof of Theorem 11.2 in [14]. The regular coherence property for groups seems to be very special—simply constructing individual finite dimensional modules over group rings is hard. The *weak regular coherence* property defined in Carlsson–Goldfarb [3] turned out to play that role in a different argument [6] with the same purpose of computing the integral assembly map in \( K \)-theory.

Notation 3.2. In the rest of this section, \( R \) will be a noetherian ring with unit.

The following are elementary facts and consequences of Corollary 2.7.

**Proposition 3.3.** (0) Whenever two choices of finite generating sets \( \Sigma_1 \) and \( \Sigma_2 \) give two filtrations \( F_1 \) and \( F_2 \) of the same \( R[\Gamma] \)-module \( F \), the identity map \( F_1 \to F_2 \) is bicontrolled.

(1) Every \( R[\Gamma] \)-homomorphism \( \phi: F \to G \) between equivariant \( \Gamma \)-filtered modules, where \( F \) is lean, is controlled.

(2) Every controlled epimorphism of lean filtered modules is bicontrolled. Therefore every \( R[\Gamma] \)-epimorphism of lean \( \Gamma \)-modules is bicontrolled.

(3) Let \( \Gamma \) be a finitely generated group with \( sFDC \) then the kernel of an \( R[\Gamma] \)-epimorphism of lean and insular \( \Gamma \)-modules is lean and insular. In particular, it is finitely generated.

(4) The cokernel and the image of a bicontrolled \( R[\Gamma] \)-homomorphism of lean and insular \( \Gamma \)-modules is lean and insular.

One has the following family of important examples.

**Example 3.4.** A controlled idempotent homomorphism of an equivariant filtered module is always bicontrolled. Indeed, if \( \phi: F \to F \) is an idempotent so that \( \phi^2 = \phi \) then \( \phi|I = \text{id} \), so \( \phi f(X) \cap f(S) \subset \phi f(S) \). As a consequence, the cokernels
and images of idempotents of finitely generated free $R[\Gamma]$-modules are lean insular $\Gamma$-modules.

**Definition 3.5.** An $R[\Gamma]$-module is **finitely presented** if it is the cokernel of a homomorphism, called *presentation*, between finitely generated free $R[\Gamma]$-modules. If the homomorphism is bi-controlled, we call the presentation **admissible**.

**Proposition 3.6.** Let $\Gamma$ be a finitely generated group with sFDC. Every lean insular $\Gamma$-module has an admissible presentation.

**Proof.** Suppose $F$ is $D$-lean. Since the $R$-submodule $F(e[D])$ is finitely generated, there is finitely generated free $R$-module $F_e$ and a surjective $R$-homomorphism $\phi_e : F_e \to F(e[D])$. One similarly has epimorphisms $\phi_\gamma : F_\gamma \to F(\gamma[D])$ using isomorphic copies $F_\gamma$ of $F_e$. We define a new locally finite $\Gamma$-filtered module $F_0$ by assigning $F_0(S) = \bigoplus_{\gamma \in S} F_\gamma$. Then clearly $F_0$ is a lean and insular $\Gamma$-module. There is an $R[\Gamma]$-homomorphism $\phi_0 : F_0 \to F$ sending $F_0(\gamma)$ onto $F(\gamma[D])$ via $\phi_\gamma$. Viewed as an $R$-homomorphism, this $\phi_0$ is $D$-bi-controlled, so from Theorem 2.5 the kernel $K$ of $\phi_0$ is a lean insular $\Gamma$-module. One similarly constructs a filtered module $F_1$ and a bi-controlled $R[\Gamma]$-homomorphism $\phi_1 : F_1 \to K$. The composition of $\phi_1$ and the inclusion of $K$ in $F_0$ gives a bi-controlled $\psi_1 : F_1 \to F_0$ with the cokernel $F$, as required. □

**Corollary 3.7.** Let $\Gamma$ be a finitely generated group with sFDC. Every lean insular $\Gamma$-module is of type $FP_\infty$.

**Proof.** Applying the construction from the proof of Proposition 3.6, one obtains a projective $R[\Gamma]$-resolution by finitely generated $R[\Gamma]$-modules

$$
\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to F \to 0
$$

In this resolution, each homomorphism $\psi_n : F_n \to F_{n-1}$ factors through an epimorphism $\phi_n$ onto a lean insular $\Gamma$-submodule $K_{n-1}$ of $F_{n-1}$. □

The following are appropriate weakenings of the coherence conditions for finitely generated groups.

**Definition 3.8.** Let $R$ be an arbitrary noetherian ring with unit. The following definitions are relative to this choice.

We say that a finitely generated group $\Gamma$ is **weakly coherent** if every $R[\Gamma]$-module with an admissible presentation is of type $FP_\infty$. We define $\Gamma$ to be **weakly regular coherent** if every $R[\Gamma]$-module with an admissible presentation has finite projective dimension.

When these terms are used without explicit specification of the choice of $R$, it is assumed that $R$ is the ring of integers \( \mathbb{Z} \).

The following is the main result of the paper.

**Theorem 3.9.** Suppose $\Gamma$ is a finitely generated group with the sFDC property. Then $\Gamma$ is weakly coherent. If, in addition, $\Gamma$ belongs to the class $\text{LHF}$ and $R$ is coherent and finite dimensional then $\Gamma$ is weakly regular coherent.

**Proof.** The first statement follows directly from Corollary 3.7. For the second statement we examine the resulting projective resolution of $F$ with an admissible presentation. By Theorem A of Kropholler [12], there is $d \geq 0$ such that the $d$-th kernel $K_d$ is isomorphic to a direct summand of a polyelementary module. When $\Gamma$
is torsion-free, the elementary modules of the form $U \otimes_R R[\Gamma]$, where $U$ is projective over $R$, are themselves projective over $R[\Gamma]$. So the polyelementary modules are also projective. This makes $K_d$ a projective $R[\Gamma]$-module. □

It is worth noting that the integer $d$ in the proof may be strictly greater than the projective dimension of the ring $R$. There is still a possibility that for the lean insular $\Gamma$-modules, $R$-flatness implies $R[\Gamma]$-flatness, and therefore $R[\Gamma]$-projectivity since all lean insular $\Gamma$-modules are finitely presented. This is certainly not true in general due to phenomena related to Moore’s conjecture, cf. [1].

**Corollary 3.10.** Let $F$ be a lean insular $\Gamma$-filtered abelian group viewed as a $\mathbb{Z}$-module. Suppose $\Gamma$ has sFDC, and there is a finite $K(\Gamma,1)$ complex. Then $F$ has finite projective dimension over $\mathbb{Z}[\Gamma]$.

### 3.2. The Whitehead group of FDC groups.

The motivating goal of this paper is to extend the techniques used in [3, 4, 5, 6] to larger geometric contexts. The results of this paper in conjunction with the proof of the Novikov conjecture for the FDC groups in [13] prove the vanishing of the Whitehead groups for FDC groups with finite classifying spaces.

The lean insular $\Gamma$-modules form an exact category which contains the finitely generated projective $\mathbb{Z}[\Gamma]$-modules as an exact subcategory. For details we refer to [5]. It follows from Quillen’s Resolution Theorem and Corollary 3.10 that the inclusion induces the Cartan isomorphisms on the level of algebraic $K$-groups whenever $\Gamma$ has sFDC and a finite $K(\Gamma,1)$.

It is known that the Whitehead group $\text{Wh}(\Gamma)$ contains obstructions to some crucial constructions in the classification theory of manifolds with the fundamental group $\Gamma$. Vanishing of the entire Whitehead group is a very much desired fact. This group fits in a long exact sequence of groups where the pairs of adjacent groups are related by assembly maps, namely $A_0: H_0(\Gamma, K(\mathbb{Z})) \to K_0(\mathbb{Z}[\Gamma])$ and $A_1: H_1(\Gamma, K(\mathbb{Z})) \to K_1(\mathbb{Z}[\Gamma])$. Thus $\text{Wh}(\Gamma) = 0$ if both of these maps are isomorphisms. This last question is studied under the name integral Borel conjecture in algebraic $K$-theory. It is answered positively in [3, 4, 5, 6] whenever (1) the Eilenberg–MacLane space $K(\Gamma,1)$ has the homotopy type of a finite complex, (2) the injectivity of all assembly maps $A_i$ follows from a splitting of the map of nonconnective spectra that model the homology and the $K$-theory, and (3) the group $\Gamma$ is weakly regular coherent, giving therefore the Cartan equivalence from the preceding paragraph.

It is known that (2) holds for groups satisfying (1) and having FDC by a slightly stronger result of Ramras–Tessera–Yu [13]. Condition (3) holds for groups with sFDC and a finite $K(\Gamma,1)$ by the main result of this paper.

This gives the following consequence.

**Theorem 3.11.** If $\Gamma$ is a group with FDC and a finite $K(\Gamma,1)$ then the Whitehead group $\text{Wh}(\Gamma)$ is trivial.

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