Analytic Determination of the Critical Coupling for Oscillators in a Ring

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We study a model of coupled oscillators with bidirectional first nearest neighbours coupling with periodic boundary conditions. We show that a stable phase-locked solution is decided by the oscillators at the borders between the major clusters, which merge to form a larger one of all oscillators at the stage of complete synchronization. We are able to locate these four oscillators as well as the size of major clusters in the vicinity of the stage of full synchronization which we show to depend only on the set of initial frequencies. Using the method presented here, we are able to obtain an analytic form of the critical coupling, at which the complete synchronization state occurs.

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Weakly coupled oscillators in the chaotic state have been known to represent many physical systems, as well as chemical, biological, neurological and so on. These systems synchronize in frequency under the influence of coupling. Knowing beforehand the value of the coupling constant and the dynamical behavior of the individual oscillators for complete synchronization to occur is an important source of information for real applications. This paper is a continuation of previous theoretical results for these systems. Here, we derive relationships that allow us to determine the oscillators which first lock in phase and drag the whole system into the synchronized state as well as the size of the two existing clusters before the transition.

I. INTRODUCTION

In recent years we have seen oscillators coupled through nearest neighbors interactions to be used to understand the behavior of systems in physics, chemistry, biology, neurology as well as other disciplines, to model several phenomena such as: Josephson junction arrays, multimode lasers, vortex dynamics in fluids, biological information processes, neurodynamics. These systems have been observed to synchronize themselves to a common frequency, when the coupling strength between these oscillators is increased. In spite of the diversity of the dynamics, the synchronization features of many of the above mentioned systems might be described using a simple model of weakly coupled phase oscillators such as the Kuramoto model, as well as its variations to adapt it for finite range interactions which are more realistic to mimic many physical systems. Difficulties arise since finite range coupled systems are difficult to analyze and to solve analytically. In spite of that, in order to figure out the collective phenomena when finite range interactions are considered, it is of fundamental importance to study and to understand the nearest neighbour interactions, which is the simplest form of the local interactions. In this context, a simplified version of the Kuramoto model with nearest neighbour coupling in a ring topology, which we shall refer to as locally coupled Kuramoto model (LCKM), is a good candidate to describe the dynamics of coupled systems with local interactions. Several reports exist where the LCKM has been used to represent the dynamics of a variety of systems, such as Josephson junctions, coupled lasers, neurons, chains with disorders, multi-cellular systems in biology and in communication systems. It has also been shown that the equations of the resistively shunted junction which describe a ladder array of overdamped, critical-current disordered Josephson junctions that are current biased along the rungs of the ladder can be expressed by a LCKM. For nearest neighbours coupled Rössler oscillators the phase synchronization can be described by the LCKM, as well as locally coupled lasers, where local interactions are dominant. LCKM can also be used to model the occurrence of travelling waves in neurons. In communication systems, unidirectionally coupled Kuramoto model can be used to describe an antenna array. Such unidirectionally coupled Kuramoto models can be considered as a special case of the LCKM and it often mimics the same behaviour. Therefore, LCKM can provide a way to understand phase synchronization in coupled systems in general.

While in the Kuramoto model for long range interactions one has to rely on average quantities, in a mean field approximation or by means of an order parameter, etc., in the local model it is necessary to study the behaviour of individual oscillators in order to understand the collective dynamics. Therefore, due to the difficulty in applying standard techniques of statistical mechanics, one should look for a simple approach to understand the coupled system with local interactions by means of numerical study of a temporal behaviour of the individual
oscillators. Such analysis is necessary in order to obtain a close picture of the effect of the local interactions on synchronization. In this case, numerical investigations can assist to figure out the mechanism of interactions at the stage of complete synchronization which in turn helps to get an analytic solution. Earlier studies on the LCKM show several interesting features including tree structures with synchronized clusters, phase slips, bursting behaviour and saddle node bifurcation and so on. It has also been shown that neighbouring elements share dominating frequencies in their time spectra, and that this feature plays an important role in the dynamics of formation of clusters in the local model that the order parameter, which measures the evolution of the phases of the nearest neighbour oscillators, becomes maximum at the partial synchronization points inside the tree of synchronization and a scheme has been developed based on the method of Lagrange multipliers to estimate the critical coupling strength for complete synchronization in the local Kuramoto model with different boundary conditions.

Very recently, we identified two oscillators which are responsible for dragging the system into full synchronization, and the difference in phase for this pair is ±π/2. In this work we develop a method to obtain an analytic solution for the value of the critical coupling at which full synchronization occurs, once a set of initial conditions for the frequencies of the N oscillators is assigned. This method will allow us not only to calculate the analytic form of the critical coupling but also to determine the number of oscillators at the major clusters in the vicinity of the critical coupling as well as to determine which is the pair of oscillators that has a phase difference ±π/2 at the stage of full synchronization.

This paper is organized as follows. In Sec. III we investigate the LCKM where periodic boundary conditions are used. We derive an analytic form for the critical coupling at the stage of complete synchronization as well and determine the number of oscillators at each cluster in the vicinity of the critical coupling. Finally, in Sec. IV we give a conclusion which is based on a summary of the results.

II. OSCILLATORS IN A RING

The local model of nearest neighbour interactions, or LCKM can be considered as a diffusive version of the Kuramoto model, and it is expressed as

\[ \dot{\phi}_i = \omega_i + K [\sin(\phi_i) - \sin(\phi_{i-1})], \]

with periodic boundary conditions \( \phi_{i+N} = \phi_i \) and for \( i = 1, 2, ..., N \). The set of the initial values of frequencies \( \{\omega_i\} \) are the natural frequencies which are taken from a Gaussian distribution and \( K \) is the coupling strength.

The phase difference is defined as \( \phi_i = \theta_{i+1} - \theta_i \) for \( i = 1, 2, 3, ..., N \). These nonidentical oscillators cluster in time averaged frequency, until they completely synchronize to a common value given by the average frequency \( \omega_0 = \frac{1}{N} \sum_{i=1}^{N} \omega_i \), at a critical coupling \( K_c \). At

\[ K \geq K_c \] the phases and the frequencies are time independent and all the oscillators remain synchronized. In Fig.1 we show the synchronization tree for a periodic system with \( N = 30 \) oscillators, where the elements which compose each one of the major clusters are indicated in each branch. These clusters merge into one at \( K_c \) where all oscillators have the same frequency. The major clusters just before \( K_c \) contain \( N_1 \) and \( N_2 \) oscillators, where \( N = N_1 + N_2 \). It is not necessary for these clusters to have the same numbers of oscillators. At the vicinity of \( K_c \), major clusters of successive oscillators have sets of nearest neighbours at the borders. An interesting fact emerges: the phase-locked solution is always valid for one and only one phase difference, and this phase difference is between two oscillators at the border of the clusters. Thus, for these two neighbouring oscillators, the equation for the phase difference is:

\[ \dot{\phi}_n = \Delta_n - 2K \sin(\phi_n) + K \sin(\phi_{n-1}) + K \sin(\phi_{n+1}), \]

where \( \Delta_n = \omega_{n+1} - \omega_n \). Equation (2) at \( K_c \) has \( \dot{\phi}_n = 0 \), and hence \( \theta_n = \theta_{n+1} = \omega_0 \). It has been found that the phase-locked solution is satisfied when \( \phi_n = \pi/2 \) for the case of \( \omega_{n+1} > \omega_n \) and \( \phi_n = -\pi/2 \) for the reverse. In addition the phase-locked solution exists at \( K_c \).

FIG. 1: Synchronization tree for a system of 30 oscillators, with detailed composition of each cluster before full synchronization.
The value of \( \sin(\phi_N) \) presents a difficulty in determining which one of the phase differences \( \phi_l \) or \( \phi_m \) corresponds to \( \pi/2 \), since the phase-lock condition at the critical coupling will be satisfied either \( K_c = Z_l + C_c \sin(\phi_N) \) or \( K_c = Z_m + C_c \sin(\phi_N) \). We can then rely on numerical findings and characteristics of \( \phi_l \) and \( \phi_m \) to use equations (3) and (4) to obtain an approximate analytic expression for \( K_c \). A detailed numerical investigation of \( \sin(\phi_N) \) shows that it is always small, in comparison to both values \( Z_l \) and \( Z_m \), and that \( K_c \) depends mainly only on these two quantities \( Z_l \) and \( Z_m \). Taking this fact into consideration and since both \( \sin(\phi_l) \) and \( \sin(\phi_m) \) are always opposite in sign as well as the two quantities \( Z_l \) and \( Z_m \), we use equations (3) and (4) to obtain

\[
K_c = \frac{|Z_l|}{2} + \frac{|Z_m|}{2} + \epsilon,
\]

where \( \epsilon \) depends on the difference between both \( |\sin(\phi_l)| \) and \( |\sin(\phi_m)| \). Helping ourselves by numerical studies we find that \( \epsilon \) depends on the quantities \( \Delta_l \) and \( \Delta_m \) and it can be written as \( \epsilon \approx \frac{|\Delta_l| + |\Delta_m|}{2} \). Thus we obtain an approximate expression for \( K_c \), which we call \( K_c^a \), and is given by:

\[
K_c^a \approx \frac{|Z_l| + |Z_m|}{2} + \frac{|\Delta_l| + |\Delta_m|}{2} - \frac{|\Delta_l + \Delta_m|}{16}.
\]

FIG. 2: Selected Values of \( \sin(\phi) \) at \( K = K_c \) for the system of 30 oscillators of Fig.1.

We can take advantage that there are four oscillators, now labeled \( l, l+1, m \) and \( m+1 \), at the borders of the major clusters in the vicinity of \( K_c \) from which only one pair will have a phase difference corresponding to the phase-locked solution; i.e., \( |\sin(\phi_n)| = 1 \). As shown in Fig. 2, the values of \( \sin(\phi_l) \) and \( \sin(\phi_m) \) are always the maximum and minimum of the \( \sin(\phi_i) \)'s for all phase differences. From these two phase differences \( \phi_l \) and \( \phi_m \), one of them has a value \( \pm \pi/2 \), while the other would be close to \( \mp \pi/2 \), getting closer as \( N \) increases. This fact has been verified numerically for several realizations of \( N \) and \{\( \omega_i \)\}'s. If we start adding equations of the systems in a ring (1) (adding elements) in a similar way as in reference 3, we generate a sequence \( Z_i \), with \( Z_i = i\omega_0 - \sum_{j=1}^{i-1} \omega_j \) for \( i = 2, 3, ..., N-1 \). After a detailed study of existing correlations, we arrive to a criteria to determine the four oscillators at the borders of the major clusters in the vicinity of \( K_c \). We find that the maximum value of \( Z_i \) refers always to the oscillators at one border while the minimum of \( Z_i \) points to the oscillators at the other border of the given cluster. One of them is always positive and the other is negative. The sign of \( Z_i \) depends on the values of \( \Delta_i = \omega_{i+1} - \omega_i \); for \( \max(Z_i) \), \( \Delta_i > 0 \), while \( \min(Z_i) \) corresponds to \( \Delta_i < 0 \). A thorough study also shows that the value of \( \sin(\phi_i) > 0 \) corresponds to \( \max(Z_i) \) while the \( \sin(\phi_i) < 0 \) corresponds to \( \min(Z_i) \). Thus we can calculate all the \( Z_i \)'s and assign the maximum and minimum values which refer to the integers \( l \) and \( m \). However, we have not yet resolved which one refers to the two oscillators with the phase difference equals \( \pm \pi/2 \).

Since now we know how to point to the four oscillators \( l, l+1, m \) and \( m+1 \) at the borders of major clusters at \( K_c \), it is possible to obtain an analytic form for \( K_c \), within reasonable accuracy. These two quantities \( Z_l \) and \( Z_m \) are related to both values \( \phi_l \) and \( \phi_m \) for a system of coupled oscillators in a ring (1) such that:

\[
K_c \sin(\phi_l) - \sin(\phi_N) = Z_l \tag{3}
\]

\[
K_c \sin(\phi_m) - \sin(\phi_N) = Z_m \tag{4}
\]

Fig.3 summarizes numerical investigations of the determinations of \( K_c \). We plot \( \log K_c \) versus \( \log N \) from numerical simulations of equation (1) (triangles) and from the analytic results given by (4), first by considering \( \epsilon \) going to zero (squares) and then taking its complete dependence on both \( \Delta_l \) and \( \Delta_m \) (circles). The validity of equation (6) is clearly shown for values of \( N \) ranging from 30 to 1000. The dependence of \( K_c \) on both \( |Z_l| \) and \( |Z_m| \) as in equation (4) and as \( N \) increases becomes clear. It can also be inferred that the term which depends on \( \epsilon \) becomes negligible. This is due to the fact that as \( N \) increases the oscillators of indexes \( l \) and \( l+1 \) are becoming closer in frequencies to each other as well as the
two oscillators of indexes $m$ and $m + 1$. We also observe that $K_c$ grows as $\sim O(\sqrt{N})$, in the same limit as found by Strogatz and Mirollo\cite{StrogatzMirollo1991} (for details, see explanation in this reference).

Summarizing, if one knows the set of initial frequencies \{\omega_i\}, it is possible to point at the four oscillators at the borders of the major clusters just below $K_c$ and then the calculation of $K_c$ is performed using equation (6) (thus obtaining $K^o_c$), without the need of computer simulation of system (1), just using the values of $Z_l$ and $Z_m$. If we are interested in determining which phase difference will have a phase-lock condition $\pm \pi/2$, we use the fact that $\sin(\phi_l)$ and $\sin(\phi_m)$ have opposite signs as well as they are maximum and minimum among all values of sine of the phase differences. The sign of the quantity $\sin(\phi_N)$ has the same sign of the quantity $x_1 = -(Z_l + Z_m)/2$, which is taken from the sum of equations (3) and (4) (eliminating for a moment the small difference between $\sin(\phi_l)$ and $\sin(\phi_m)$). Depending on the signs of $Z_l$ and $Z_m$, we know the signs of $\sin(\phi_l)$ and $\sin(\phi_m)$, and hence the sign of $\sin(\phi_N)$. Therefore, we count two quantities $x_2 = \pm K^o_c - Z_l$ and $x_3 = \pm K^o_c - Z_m$, positive sign for $Z > 0$ and negative sign for the reverse. Three cases will exist: first from the quantities $x_2$ and $x_3$, one is positive and the other is negative. Thus depending on the sign of $x_1$ we choose either $x_2$ or $x_3$ to be $K^o_c \sin(\phi_N)$. Second $x_2$ and $x_3$ have the same signs, then we check the minimum between $|x_1 - x_2|$ and $|x_1 - x_3|$ and depending on which one is the minimum, we take either $x_2$ or $x_3$ to be $K^o_c \sin(\phi_N)$. Third $|x_1 - x_2| = |x_1 - x_3|$, then we take the minimum outcome of $x_2$ and $x_3$. Now we know the value of $K^o_c \sin(\phi_N)$ and its sign. Therefore, we know which equation from (3) and (4) will be used to give $K^o_c$. Thus we specify which phase difference of index $l$ or $m$ would have $\pm \pi/2$. We tested this method on the simulations we have done and it matches the outcome of the numerical simulations.

The number of oscillators in each cluster at the vicinity of $K_c$ can be determined, once we assigned the indexes $l$, $l + 1$, $m$ and $m + 1$, which, we remind the reader, are obtained from $Z_l$ and $Z_m$, maximum and minimum values of the sequence $Z_i$. The size of one cluster of $N_l$ oscillators is determined by counting the difference $N_1 = (m + 1) - l$ and the size of the other cluster is determined as $N_2 = N - N_1$. Similarly to the calculation of $Z_l$ and $Z_m$, we can determine other two quantities which are $Y_1 = N_1 \omega_0 - \sum_{i=l+1}^{m} \omega_i$ and $Y_2 = N_2 \omega_0 - \sum_{i=m+1}^{N} \omega_i$, taking into consideration the periodic boundary conditions. It is found that $|Y_1| = |Y_2|$. These quantities are related to $Z_l$ and $Z_m$ by $Y_1 = Z_l - Z_l = -Y_2$. It is easy to show that $K_c = |Y_1^2| + \epsilon = |Y_2^2| + \epsilon$. The two quantities $Y_1$ and $Y_2$ provide a criteria to understand synchronization-desynchronization at $K_c$. If one arrives from above $K_c$ where all oscillators are synchronized and have the same value of frequency, at $K_c$ the oscillators split into two groups of $N_1$ and $N_2$, at $K_c$ depending on these two quantities $Y_1$ and $Y_2$, where $|Y_1| = |Y_2|$. It is not necessary for $N_1$ to be equal to $N_2$. Both quantities $Y_1$ and $Y_2$ have opposite signs since they refer to two groups of oscillators (two clusters) one of them rotates with average frequency over $\omega_0$ and the other has an average frequency lower than $\omega_0$.

Comparing our findings of $K_c$ with the work of Daniels et al.\cite{Danielsetal2019}, our method has the advantage of finding the value of $K_c$ without performing numerical simulations once we know the set of initial frequencies \{\omega_i\}. In addition we get the condition of synchronization-desynchronization at $K_c$ and obtain the number of oscillators in each branch in the vicinity of $K_c$.

### III. Conclusion

We have analyzed the conditions of the phase differences for the onset of complete synchronization at the critical coupling strength in a Kuramoto-like model with nearest neighbour coupling with periodic boundary conditions. Such analysis allows us to determine the four oscillators located at the borders of the major clusters (formed by successive oscillator) which will meet at the critical coupling to form one cluster of all synchronized oscillators. With the help of these findings we derive the analytic expression for the critical coupling when all oscillators will have the same frequency and, phase differences and instantaneous velocities become time independent. In addition, we are able to determine which is the phase difference, that will have a phase-lock solution $\pm \pi/2$. From the derivation we also extract the size of the clusters before complete synchronization. The analytic form of $K_c$ depends only on the initial frequencies, through the quantities $Z_l$ and $Z_m$, where the indexes $l$ and $m$ correspond to the borders of the clusters. The
quantities $Z_l$ and $Z_m$ correspond to the maximum and minimum values of the sequence $Z_i$. These quantities in fact are related to the statistics of the distribution of the set of initial frequencies $\{\omega_i\}$, when this sample is obtained from a Gaussian distribution, as shown by Strogatz and Mirollo. A detailed study within this context could shine light on the behavior of $K_c$ as $N \to \infty$, not just for the case of a Gaussian distribution, but for others. This investigation plus extension of the method to study cluster formation inside the tree will be topics of further analysis. The advantages of the study presented here is that we can determine the value of the coupling constant that will synchronize the system of coupled oscillators without carrying out numerical simulation as well as to determine the sizes of the clusters just before this happens. Generalization of these results to different couplings and boundaries are under investigation and will be presented elsewhere.

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