All the separabilities of N-qubit Greenberger-Horne-Zeilinger states in white noise

Xiao-yu Chen$^a$, Li-zhen Jiang$^a$, Zhu-an Xu$^b$

$^a$College of Information and Electronic Engineering, Zhejiang Gongshang University, Hangzhou, Zhejiang 310018, China
$^b$Department of Physics, Zhejiang University, Hangzhou, Zhejiang 310027, China

Abstract

We demonstrate the necessary and sufficient criterion for an N-qubit Greenberger-Horne-Zeilinger (GHZ) state in white noise when it is divided into N − j parties with N ≥ 2j + 1 for arbitrary N and j. The criterion covers more than a half of all the separabilities of N-qubit Greenberger-Horne-Zeilinger states in white noise. For the rest of multipartite separable problems, we present a method to obtain the sufficient conditions of separability using linear programming.

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1 Introduction

The experimental preparations of multipartite entangled states $^1$ $^2$ $^3$ are typically the N-qubit GHZ states. The imperfection and noise in the preparation is usually described by white noise. The resultant experiment state then is the mixture of N qubit GHZ state with white noise (noisy GHZ state).

Although the bipartite separable condition and the fully separable condition for N qubit noisy GHZ state are known $^4$ $^5$ $^6$, there are still many other partitions in between the bipartite and the N partite especially when N is large. The so called K-separable of a multipartite quantum state is that the state can be written in the form of the probability mixture of K-partite separable states, while in each term of the probability mixture the partition can be different from the other terms. For the case that a four qubit system is divided into three parties, if we denote the qubit as A, B, C, D, we have the partitions $AB|C|D$, $AC|B|D$, $AD|B|C$, $A|B|CD$, $A|C|BD$, $A|D|BC$. The tripartite separable state for partition $AB|C|D$ is $\rho_{AB|C|D} = \sum_i p_i \rho^{(i)}_{AB} \otimes \rho^{(i)}_{C} \otimes \rho^{(i)}_{D}$. The tripartite separable state of the four qubit system is the probability mixture of $\rho_{AB|C|D}$, $\rho_{AC|B|D}$, $\rho_{AD|B|C}$, $\rho_{A|B|CD}$, $\rho_{A|C|BD}$, $\rho_{A|D|BC}$. Hence the multipartite separable (K-separable) of a multipartite quantum state is much involved. However, a GHZ state is highly symmetric. The partition of an N qubit GHZ state is greatly simplified by the symmetry. In fact we just count the number of qubits in a party without considering which qubits are in the party. We can deal with a particular partition then extend the result to its symmetric partitions by permutations. The above six partitions of tripartite four qubit system are symmetric for the noisy GHZ state, we will denote them with partition $1|1|2$ (or $1^2|2$ for short). The digital numbers describe the number of qubit in the parties. A relatively easy but still highly non-trivial case is the N − 1 separable problem (We denote it as sub-full separable comparing to the full separable problem with N parties for the N qubit state). We will prove the necessary and sufficient conditions for the N − 1 separability of N qubit noisy GHZ states in section III. The basic tool of detecting various kinds of entanglement is the entanglement witness $^7$ $^8$, which is described in section II. We will use the characteristic form of witness operator to find the optimal witness. In sections IV and V, the sufficient and necessary conditions are demonstrated for N − j separability of N qubit noisy GHZ states when N ≥ 2j + 1. The sufficient conditions of all the separabilities for N ≤ 12 will be shown in section VI as examples and the method is applicable to all the separable problems of N qubit noisy GHZ states.

2 Entanglement Witness

The witness of entanglement is an operator $W$ such that $tr(\rho W) ≥ 0$ for all separable state $\rho$ and $tr(\rho W) < 0$ for at least one entangled state $\rho$. Denote $W$ as $W = \lambda I - Q$, where $I$ is the identity operator of the quantum system, thus we may define

$$\lambda = \max_{\rho \in SEP} tr(\rho_0 Q),$$

(1)

such that the witness operator is tight enough, where $SEP$ is the set of separable states for the specified partition(s). The noisy N qubit GHZ state is

$$\rho_{ghzN} = \rho |GHZ_N \rangle \langle GHZ_N | + \frac{1 - \rho}{2^N} I_{2^N}$$

(2)
Where $|\text{GHZN}=\frac{1}{\sqrt{2}}(|0\rangle\otimes N + |1\rangle\otimes N)$. In the following, we will denote $|\chi\rangle\langle\chi|$ as $|\chi\rangle\langle\chi|$ for short. The stabilizer of the $|\text{GHZN}\rangle$ is the operator Abel group created by the generators $\{K_1, \ldots, K_N\}$ such that $K|\text{GHZN}\rangle = |\text{GHZN}\rangle$ for any $K = K_1^{x_1} \cdots K_N^{x_N}$ with $k_j = 0, 1$. The generators are $K_1 = XX \cdots X, K_2 = ZIZ \cdots I, K_3 = ZIZIZ \cdots I, \ldots, K_N = ZIZIZ \cdots IZ$, with $X = \sigma_1, Y = \sigma_2, Z = \sigma_3$ being Pauli matrices and $I = 0_0$ is the $2 \times 2$ identity matrix. The characteristic function of $\rho_{\text{ghz}}\otimes\sigma_i \otimes \sigma_j \otimes \cdots \otimes \sigma_i\rangle\langle\sigma_i\rangle$. Hence $R_{00...0} = 1, R_{00...0333} = p, R_{00...03333} = p$ and $R_{00...03333} = p$ with even number of $3$ in the subscript. All the terms obtained by subscript permutations are equal. This comes from the fact that $tr(\rho K) = p \langle \text{GHZN}| K |\text{GHZN}\rangle + \frac{1}{2}p tr(K) = p + (1 - p)\delta_{k_0 \cdots \delta_{k_N}}$ for $K = K_1^{x_1} \cdots K_N^{x_N}$. The other nonzero terms are $R_{11...122} = -p, R_{11...1222} = p$ and $R_{11...12...2} = (-1)^2 p$ for $2j$ number of $2$ in the subscript. Also the subscript permutations do not change the values. This is due to $tr(\rho K) = p \langle \text{GHZN}| K |\text{GHZN}\rangle = p$ for $K = K_1^{x_1} \cdots K_N^{x_N}$, total number of nonzero $R_{11...12...2} = 2^N$. All the other products of Pauli matrices are no longer in the stabilizer, they are in the cosets of the stabilizer. Applying an element in the coset of the stabilizer to the GHZ state will lead to an orthogonal state of the GHZ state. Thus the characteristic function values for coset operators are $0$.

We choose the witness operator such that

$$Q = \sum_{k_1, \ldots, k_N} L_{k_1 \cdots k_N} K_{1}^{k_1} \cdots K_N^{k_N}.$$ (3)

Let $k = \sum_{j=2}^{N} 2^j$, and we choose $L_{0k_2 \cdots k_N}$ with $k = 1, 2$ to be equal and denoted with $M_1$, choose $L_{0k_2 \cdots k_N}$ with $k = 3, 4$ to be equal and denoted with $M_2$, choose $L_{0k_2 \cdots k_N}$ with $k = 2i - 1, 2i$ to be equal and denoted with $M_i$. Let $L_{0000} = M_0 = 0, L_{1k_2 \cdots k_N} = 1$, then we have

$$tr(\rho_{\text{ghz}}Q) = p(\sum_{i=1}^{N/2} M_iC_i^2 + 2^{N-1}).$$ (4)

with $C_i^2 = \frac{N!}{(N-2i)!2^i i!}$. If $tr(\rho_{\text{ghz}}W) < 0$, namely $tr(\rho_{\text{ghz}}Q) > \lambda$, then $\rho_{\text{ghz}}$ is entangled. Hence the necessary condition for the separability of $\rho_{\text{ghz}}$ with respect to some partitions is

$$p \leq \frac{\lambda}{(\sum_{i=1}^{[N/2]} M_iC_i^2 + 2^{N-1})}.$$ (5)

For the $N-1$ separability, we set

$$M_i = 4iN - \frac{N}{N-2}.$$ (6)

with $i = 1, \ldots, [N/2]$. Notice that $\sum_{i=1}^{[N/2]} M_iC_i^2 = \frac{2}{N-2} \sum_{i=1}^{[N/2]} 2iC_i^2 - \frac{N}{N-2} \sum_{i=1}^{[N/2]} C_i^2 = \frac{N^2}{N-2}$, we have the necessary condition

$$p \leq \frac{\lambda}{2^{N-1} + \frac{N}{N-2}}.$$ (7)

3 $N - 1$ separability

3.1 Necessary condition of $N - 1$ separability

A separable state can always be written as the probability mixture of the pure separable states. For the maximum of $tr(\rho_Q)$, we only need to consider the pure $N - 1$ separable states. If the last two qubits are classified as a party, the $N - 1$ separable pure state is $\rho_s = \delta_1 \otimes \delta_2 \cdots \otimes \delta_{N-2} \otimes \delta_{N-1, N}$ where $\delta_i = \frac{1}{2}(I + x_iX + y_iY + z_iZ)$ is the pure state of $ith$ qubit, with $x_i^2 + y_i^2 + z_i^2 = 1$. For the operator $Q$ in (3), we denote it as $Q = Q_0 + Q_1$, where $Q_0$ is the $k_1 = 0$ part and $Q_1$ is the $k_1 = 1$ part. We first consider the terms in $Q_1$, they are products of $X$ and $Y$ operators. We have

$$\begin{align*}
tr(\rho_Q) &= tr(\rho_s \sum_{k_2 \cdots k_N} K_1^{k_2} \cdots K_N^{k_N}) \\
&= \sum_{k_2 \cdots k_N = 0}^{1} (-1)^{k + k'} x_i^{1-k} y_i^{k} \prod_{i=2}^{N-2} x_i^{1-k} \ y_i^{k} \\
&\otimes tr(\rho_{N-1, N}(X^{1-k_{N-1}}Y^{k_{N-1}})) \\
&\otimes (X^{1-k_{N}}Y^{k_{N}})),
\end{align*}$$ (8)

where $k' = mod(k, 2)$. Denote $x_i = \sin \theta_i \cos \varphi_i, y_i = \sin \theta_i \sin \varphi_i$, and $c = \prod_{i=1}^{N-2} \sin \theta_i \cos(\sum_{j=1}^{N-2} \varphi_j), d = \prod_{i=1}^{N-2} \sin \theta_i \sin(\sum_{j=1}^{N-2} \varphi_j)$, then

$$\begin{align*}
tr(\rho_Q) &= tr(\rho_{N-1, N}[c(XX - YY)] \\
&\quad - d(YY + XX)).
\end{align*}$$ (9)

The operator $Q_0$ is the summation of terms that are products of $I$ and $Z$. We have

$$\begin{align*}
tr(\rho_Q) &= tr(\rho_s \sum_{k_2 \cdots k_N} L_{0k_2 \cdots k_N} K_2^{k_2} \cdots K_N^{k_N}) \\
&= \sum_{k_2 \cdots k_N = 0}^{1} M_{[k/2]} z_1^{k'_{N-1}} \prod_{i=2}^{N-2} z_i^{k_i} \\
&\otimes tr(\rho_{N-1, N}Z^{k_{N-1}}Z^{k_N}).
\end{align*}$$ (10)

Hence

$$\begin{align*}
tr(\rho_Q) &= a_0 + a_1 tr(\rho_{N-1, N}ZZ) \\
&\quad + btr(\rho_{N-1, N}(IZ + ZI)).
\end{align*}$$ (11)
Lemma 1 The maximum of $\lambda_1$ is

$$\max_{z_1, \ldots, z_{N-2}} \lambda_1 = \frac{N}{N - 2}. \tag{17}$$

Proof: This is equivalent to $a + \sqrt{b^2 + c^2 + d^2} \leq 0$. We should show that (i) $a \leq 0$, (ii) $a^2 - b^2 \geq c^2 + d^2$.

Denote $S_i = \sum_{j< i} z_j$, then $a = \sum_{i=1}^{N-2} M_i[0, 1] - 1$, $b = \sum_{i=1}^{N-2} M_i S_{2i-1} - 1$, $v = \frac{1}{N-2} \sum_{j=1}^{N-2} \prod_{n=1}^{N-2} [1 + (-1)^{\delta_{j,n}} z_n]$, and $v = \frac{1}{N-2} \sum_{j=1}^{N-2} \prod_{n=1}^{N-2} [1 + (-1)^{\delta_{j,n}} z_n]$. A direct calculation shows that $u = 1 + \sum_{i=1}^{N-2} (1 - \frac{2i}{N-2}) S_i$, $v = 1 + \sum_{i=1}^{N-2} (1 - \frac{2i}{N-2}) S_i$. Hence $\frac{1}{2}(u + v) = 1 - \sum_{i=1}^{N-2} \frac{2i}{N-2} S_i$, and $\frac{1}{2}(u - v) = \sum_{i=1}^{N-2} (1 - \frac{2i}{N-2}) S_i$. We arrive at (i) $a \leq 0$, $u \geq 0$ and $v \geq 0$ due to $z_n \in [-1, 1]$ for all $n$. We further have $a^2 - b^2 = \sum_{i=1}^{N-2} (1 - z_i)^2 (1 + z_i)^2 \prod_{n=1, n \neq i, j} (1 - z_n^2)$, meanwhile $c^2 - d^2 = \prod_{i=1}^{N-2} (1 - z_i^2)$, thus we arrive at (ii)

$$a^2 - b^2 - c^2 - d^2 = \frac{8}{(N-2)^2} \sum_{1 \leq i, j \leq N-2} (z_i - z_j)^2 \prod_{n=1, n \neq i, j} (1 - z_n^2) \geq 0. \tag{20}$$

We thus arrive at the conclusion of

$$\lambda \equiv \max_{\rho \in SEP_{N-1}} tr(\rho \{Q\}) = \frac{N}{N - 2}. \tag{21}$$

Where $SEP_{N-1}$ is the set of $N - 1$ partite separable states of $N$ qubits. The necessary condition for $N - 1$ partite separability of $N$ qubit noisy GHZ state is

$$p \leq \frac{1}{1 + \frac{N}{N - 2}} \tag{22}$$

For 3 qubit GHZ state, we have $p \leq \frac{4}{9}$, which is known for biseparability of the noisy GHZ state.

### 3.2 Sufficient condition of the $N - 1$ separability

The way of $tr(\rho_0 \{Q\})$ achieving its maximum $\lambda = \frac{N}{N - 2}$ hints the sufficient condition. We consider the case that the maximum value of $tr(\rho_0 \{Q\})$ is achieved by the sub-full separable state with $z_i = 0$ for $i = 1, \ldots, N - 2$ such that every term in the second line of (23) is zero. Hence the pure state of the $i$th qubit $\rho_i = \frac{1}{2}(I + x_i X + y_i Y)$ with $x_i^2 + y_i^2 = 1$. We may assume $x_i = \cos \varphi_i$, $y_i = \sin \varphi_i$, then $\rho_i = |\alpha_i\rangle \langle \alpha_i|$, where $|\alpha_i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi_i} |1\rangle)$. The state of the last two qubits is the eigenvector of $M_1$ corresponding to eigenvalue $\lambda_1$. The eigenvector is

$$|\psi_{N-1, N}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\sum_{j=1}^{N-1} \varphi_j} |1\rangle). \tag{23}$$

The sub-full separable pure state is $|\Omega\rangle = \frac{1}{\sqrt{2}} \prod_{i=1}^{N-2} (|0\rangle + e^{i\varphi_i} |1\rangle)$, and $e^{-i\sum_{j=1}^{N-1} \varphi_j} |11\rangle)$. Thus we have the sub-full separable state $\rho_{\Omega} = \frac{1}{2} |\Omega\rangle \langle \Omega| + \frac{1}{2} |\omega\rangle \langle \omega|$, which is

$$\rho_{\Omega} = \frac{1}{2} \sum_{k \neq 0, k_1, \ldots, k_{N-2} = 0}^{N-2} (|k| \langle k| + |k_1 \cdots k_{N-2}| \langle k_1 \cdots k_{N-2}|). \tag{24}$$
where the binary string \( \mathbf{k} = (k_1, k_2, \ldots, k_{N-2}) \), \( \overline{k}_i = 1 - k_i \). Averaging over all the cases of qubit permutations we arrive at a sub-full separable state

\[
\rho_{s1} = \frac{1}{2N-1} [2 |GHZ_N\rangle\langle | + \frac{N-2}{N} (T_1 + T_{N-1}) \\
+ \sum_{i=2}^{N-2} C_N^{-2} + C_N^{i-2} T_i],
\]

(25)

where \( T_i = \sum_{k_1 + k_2 + \ldots + k_i = i} |k_1 k_2 \ldots k_i \rangle \langle | \). Notice that for \( i = 2, \ldots, N - 2 \), we have \( N > i + 1 \). The inequality \( N > i + 1 \) leads to the inequality \( C_N^{k-2} + C_N^{\bar{k}-2} < \frac{1}{N^2} \). Denote \( \tilde{\rho}_s = \frac{1}{2N-1} \sum_{i=2}^{N-2} \left( \frac{1}{N^2} - \frac{C_N^{k-2} + C_N^{\bar{k}-2}}{C_N^{k-2} + C_N^{\bar{k}-2}} \right) T_i + \frac{N-2}{N} (T_0 + T_N) \), where \( T_0 = |0^\otimes N\rangle\langle | \), \( T_N = |1^\otimes N\rangle\langle | \). Then \( \tilde{\rho}_s \) is an unnormalized full separable state. We have \( \rho_s = \rho_{s1} + \tilde{\rho}_s = \frac{1}{2N} \sum_{i=2}^{N-2} \left( (2 |GHZ_N\rangle\langle | + \frac{N-2}{N^2} I_{2N} \right) \) being an unnormalized sub-full separable state.

The normalized sub-full separable state is

\[
\rho_s = \frac{\tilde{\rho}_s}{\text{tr}(\tilde{\rho}_s)} = \frac{|GHZ_N\rangle\langle |}{1 + \frac{N-2}{N^2} 2N-1} + \frac{\frac{N^2}{2N} N-2}{1 + \frac{N-2}{N^2} 2N-1} I_{2N}.
\]

(26)

Hence, the noisy \( N \) qubit GHZ state is sub-full separable iff

\[
p \leq \frac{1}{1 + \frac{N-2}{N^2} 2N-1}.
\]

(27)

### 4 Sufficient conditions for \( N - j \) partite separability

#### 4.1 \( N - 2 \) partite separability

For \( N - 2 \) partite separability of the \( N \) qubit system, there are two kind of partitions. One is that the first \( N - 3 \) parties have \( N - 3 \) qubits, each party has one qubit, the last party has three qubits, we denote the partition as \( 11[1] \ldots [1]3 \) or \( 1(1\ldots(1N-3)^{3}) \) for short. The other partition is that \( N - 4 \) parties have \( N - 4 \) qubits, each party has one qubit, the last two parties have two qubits, each has two qubits, we denote it as \( 1(N-4)^2 |2 \). For the first partition, with the similar method of the last section, it can be shown that the sufficient condition of \( N - 2 \) partite separability of \( N \) qubit noisy GHZ state is

\[
p \leq (1 + \frac{N-2}{N^2} 2N-1)^{-1}.
\]

For the second partition, we have a pure separable state \( |\Omega\rangle = \frac{1}{\sqrt{2N}} \sum_{j=1}^{N-1} (|0\rangle + e^{i\phi_j} |1\rangle) (|00\rangle + e^{i\phi_j} |11\rangle) (|00\rangle + e^{i\phi_j} |11\rangle) \). The integral over the phase angles will lead to an \( N - 2 \) partite separable state \( \rho_{s3} = \int |\Omega\rangle \langle \Omega| \frac{d\phi_j}{2\pi} \prod_{j=1}^{N-1} \frac{d\phi_j}{2\pi} \), which is

\[
\rho_{s3} = \frac{1}{2N-2} [2 |GHZ_N\rangle\langle | + \sum_{k\neq0; k_1 \ldots k_{N-4}, k_1 = 0} \langle k_1 \ldots k_{N-4} k_1 k_0 00 | + \langle k_1 \ldots k_{N-4} k_1 k_2 11 |],
\]

(28)

where the binary string \( \mathbf{k} = (k_1, k_2, \ldots, k_{N-4}, k_1) \).

Averaging over qubit permutations, we have the \( N - 2 \) partite separable state \( \rho_{s2} = \frac{1}{2N-1} [2 |GHZ_N\rangle\langle | + \frac{N-4}{N} (T_1 + T_{N-1}) + C_N^{2} + C_N^{2} T_1 + C_N^{-2} + C_N^{-2} T_1] \). The inequality \( C_N^{2} + C_N^{2} T_1 < \frac{N-4}{N} \) for \( N > i + 3 \) can be derived from the inequality \( C_N^{2} + C_N^{2} T_1 < \frac{N-2}{N} \) for \( N > i + 1 \). We also have \( \frac{N-4}{N} \geq \frac{C_N^{2}}{N^2} + C_N^{2} T_1 \) and \( \frac{N-4}{N} \geq \frac{C_N^{2}}{N^2} + 2C_N^{2} T_1 \) for \( N \geq 5 \). Hence for \( N \geq 5 \), we add some full separable states to \( \rho_{s2} \) to make an unnormalized \( N - 2 \) partite separable state \( \rho_{s2} = \frac{1}{2} |GHZ_N\rangle\langle | + \frac{N-4}{N} I_{2N} \). Then the \( N - 2 \) partite separable state is

\[
p \leq \frac{1}{1 + \frac{N-4}{N^2} 2N-1}.
\]

(29)

#### 4.2 \( N - j \) partite separability

For \( N - 3 \) partite separability of \( N \) qubit system, there are three kinds of partitions: \( 1(N-4)|4, 1(N-5)|2|3 \) and \( 1(N-6)|2|3 \). The partition \( 1(N-4)|4 \) yields \( N - 3 \) separable condition \( p \leq (1 + \frac{N-4}{N^2} 2N-1)^{-1} \). The partition \( 1(N-5)|2|3 \) yields condition \( p \leq (1 + \frac{N-4}{N^2} 2N-1)^{-1} \). The partition \( 1(N-6)|2|3 \) yields condition \( p \leq (1 + \frac{N-4}{N^2} 2N-1)^{-1} \). The procedure of derivation is similar to that in deriving the \( N - 2 \) partite separable condition. We see that the condition can be written as \( p \leq p_s = (1 + \frac{N-4}{N^2} 2N-1)^{-1} \) (see Appendix), where \( N - L \) is the number of parties with single qubit. The less the number \( N - L \), the larger the \( p_s \), the better the sufficient condition. If the number of parties is fixed to be \( N - j \), the smallest number of \( N - L \) is \( N - 2j \). For \( N - 3 \) partite separability, we have sufficient condition \( p \leq (1 + \frac{N-4}{N^2} 2N-1)^{-1} \) with \( N \geq 7 \). For \( N - j \) partite separability, we have sufficient condition

\[
p \leq \frac{1}{1 + \frac{N-4}{N^2} 2N-1},
\]

(30)

where \( N \geq 2j + 1 \). The sufficient condition is realized by the partition \( 1(N-2)|2|j \). When \( N = 2j + 1 \), we have a partition \( 1|2|j \). For this partition, there is a pure separable state (in the sense of \( N - j = j + 1 \) partite separable) \( |\Omega\rangle = \frac{1}{\sqrt{2j+1}} (|0\rangle + e^{-i \sum_{j=1}^{j} \phi_j} |1\rangle) \prod_{j=1}^{j} (|00\rangle + e^{i\phi_j} |11\rangle) \). The integral over the phase angles leads to an \( N - j \) separable state \( \rho_{s3} = \)
\[ \int |\Omega\rangle \langle \Omega| \prod_{i=1}^{j} \frac{d\theta_i}{2\pi}, \] which is

\[ \rho_{s5} = \frac{1}{2^{j+1}} \left[ 2 |GHZ_{2j+1}\rangle \langle | \right] + \sum_{k \neq 0, k_1, \ldots, k_j = 0}^{1} \left( |0k_1k_2 \ldots k_j\rangle \langle | \right) + |1k_1k_2 \ldots k_j\rangle \langle 1 \right]. \tag{31} \]

where the binary string \( k = (k_1, k_2, \ldots, k_j) \). Averaging over qubit permutations yields the \( j + 1 \) separable state

\[ \rho_{s6} = \frac{1}{2^{j+1}} \left[ 2 |GHZ_N\rangle \langle | + \frac{1}{2^{j+1}} \left( T_1 + T_2 \right) \right] \]

\[ + T_2 + T_{2j-1} + \sum_{i=3}^{2j-2} \frac{C_i^{1/2}}{C_{i+1}^{1/2}} T_i \]. \tag{32} \]

It can be proven that \( \frac{C_i^{1/2}}{C_{i+1}^{1/2}} \leq \frac{1}{2^{j+1}} \), so that we obtain a \( j + 1 \) separable state (unnormalized) state \( \rho_{s6} = 2 |GHZ_{2j+1}\rangle \langle | + \frac{1}{2^{j+1}} I_{2j+1} \) by adding some full separable to \( \rho_{s6} \). The state \( \rho_{ghzN} \) with \( N = 2j + 1 \) is a \( j + 1 \) separable state if

\[ p \leq \frac{1}{1 + \frac{1}{2^{j+1}}} \]

\[ = \frac{1}{1 + \frac{1}{2^{j+1}} 2^j}. \tag{33} \]

In figure 1, we display the \( N - j \) partite separable conditions for \( j = 1, 2, 3, 4, 5 \).

5 Necessary conditions of \( N - j \) separability

5.1 An \( N - L + 1 \) Partition

We consider a special partition \( 1^{(N-L)} |L \) as the basis of necessary condition of \( N - j \) separability. The partition can be seen as the generalization of \( N - 1 \) partition (the case of \( L = 2 \)) we have discussed in previous sections. With the partition \( 1^{(N-L)} |L \), an \( N - L + 1 \) separable pure state is \( \rho_t = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_{N-L} \otimes \rho_{N-L+1}, \ldots, N \), where \( \rho_i \) \((i = 1, \ldots, N - L)\) is the pure state of \( i-th \) qubit, \( \rho_{N-L+1}, \ldots, N \equiv |\psi\rangle \langle \psi| \) is the pure state for the last \( j \) qubits. In the computational basis, \( |\psi\rangle = \sum_{i_1, i_2, \ldots, i_L = 0}^{N-1} \alpha_{i_1, i_2, \ldots, i_L} |i_1, i_2, \ldots, i_L\rangle \). Then

\[ tr(\rho_s Q_0) = \prod_{i=1}^{N-L} \sum_{k_i=0}^{1} \alpha_{i_1, i_2, \ldots, i_L} M_{i_1, i_2, \ldots, i_L} \]

\[ = \prod_{i=1}^{N-L} |i\rangle |Z_{N-L+1}^{i} \ldots Z_{N}^{i} | \psi\rangle, \]

which can be written as

\[ tr(\rho_s Q_0) = \sum_{j=1}^{[N/2]} M_j \sum_{n_1, n_2} S_{2i-n_1} q_n, \tag{34} \]

where \( n_1 = \max \{0, 2i + L - N\} \), \( n_2 = \min \{L, 2i\} \), and

\[ q_n = \sum_{k_{N-L+1} + \cdots + k_n = n} \langle \psi| Z_{N-L+1}^{k_{N-L+1}} \ldots Z_{N}^{k_n} | \psi\rangle. \]

The summation of \( j \) is from \( \max \{0, n + l - L\} \) to \( \min \{n, l\} \). However, we do not write the lower and upper bound of summation explicitly as they are implied by the definition of combinatorial numbers \( C_{m-1}^{n-1} \) and \( C_l^{l} \).

For the QI part of witness, we have

\[ tr(\rho_s Q_1) = \sum_{i=1}^{N-L} \sum_{k_i=0}^{1} \alpha_{i_1, i_2, \ldots, i_L}^2 \]

\[ \prod_{i=1}^{N-L} |i\rangle |X_{N-L+1}^{i} \ldots X_{N}^{i} | \psi\rangle, \]

where \( \psi = \sum_{i=1}^{N-L} \alpha_{i_1, i_2, \ldots, i_L} |i_1, i_2, \ldots, i_L\rangle \). Then

\[ tr(\rho_s Q_1) = \langle \psi| c\mathcal{X} - d\mathcal{Y} | \psi\rangle, \tag{36} \]

\[ = \prod_{i=1}^{N-L} \sin \theta_i \cos(\sum_{j=1}^{N-L} \varphi_j), d = \prod_{i=1}^{N-L} \sin \theta_i \sin(\sum_{j=1}^{N-L} \varphi_j). \]

and we have

\[ \langle \psi| \mathcal{X} | \psi\rangle = \sum_{i=1}^{N-L} \alpha_{i_1, i_2, \ldots, i_L}^2 \]

\[ + \sum_{i=1}^{N-L} \alpha_{i_1, i_2, \ldots, i_L}^2 \]

\[ \langle \psi| \mathcal{Y} | \psi\rangle = \sum_{i=1}^{N-L} \alpha_{i_1, i_2, \ldots, i_L}^2 \]

\[ + \sum_{i=1}^{N-L} \alpha_{i_1, i_2, \ldots, i_L}^2. \tag{37} \]

Figure 1: The entanglement properties of \( N \)-qubit GHZ states mixed with white noise. \( \rho_N = p |GHZ_N\rangle \langle GHZ_N| + (1-p) I_N \). It was known before [4, 5] that these are fully separable if \( p \leq \frac{1}{1 + 2^{L-1}} \). They are shown with dot lines, up dot line for biseparability and down dot line for full separability. Our results show that these states are \( N - 1 \) separable if \( p \leq \frac{1}{1 + 2^{L-1}} \). They are shown by solid lines for \( N - j \) separability with \( j = 1, 2, 3, 4, 5 \) from bottom up. The dash lines are for \( j \) separability with \( j = 3, 4, 5 \) top down.
where $\alpha_{0L} = \alpha_{00\ldots0}. \alpha_{1L} = \alpha_{11\ldots1}$. We have used that $\langle 0^{L-1}| X |1^{L-1}\rangle = 0$ if $l \neq 0$ and $l \neq L$. This is due to fact that for each term in $X$ if there are odd number of $Y$ in the last $l$ qubits, then there are odd number of $Y$ in the first $L - l$ qubits too, so that a new factor of $-1$ emerges for such a term. The probabilities of odd and even number of $Y$ in the last $l$ qubits are equal. The null result is also true for $Y$ when $l \neq 0$ and $l \neq L$.

We may write $tr(\rho Q) = tr(\rho Q_1) + tr(\rho Q_1^c) = \langle \psi | M | \psi \rangle$, where $M$ is a $2^L \times 2^L$ matrix with all of its non-diagonal entries nullified except the entries of $|0^L\rangle \langle 1^L|$ and $|1^L\rangle \langle 0^L|$. The maximum of $tr(\rho Q)$ then is the maximal eigenvalue of $M$. The diagonal part of $M$ comes from $tr(\rho_{Q_0})$, which is shown in equation 34, and can be further written as $tr(\rho_{Q_0}) = S_0 \sum_{i=1}^{[L/2]} M_i q_{2i} + \sum_{m=1}^{[\frac{(N-L)/2}]} S_{2m-1} \left( \frac{[L/2]}{2} \right) M_i q_{2i-1} + \sum_{m=1}^{[\frac{(N-L)/2}]} S_{2m} M_i q_{2i}$ by changing the order of summations. Denote $tr(\rho_{Q_0}) = \sum_{l=0}^{L} \sum_{i_1+i_2+\ldots+i_{2l}=l} |\alpha_{i_1i_2\ldotsi_{2l}}|^2 \Gamma_l$, the diagonal elements of matrix $M$ are $M_{ii}$, then $M_{ii} = \Gamma_i$, with $|i| = \sum_{j=1}^{L} i_j$. Then we have the diagonal elements

$$\Gamma_l = \frac{[L/2]}{2} S_0 \sum_{i=1}^{[L/2]} M_i w_{2i,1} + \sum_{m=1}^{[\frac{(N-L)/2}]} S_{2m-1} \left( \frac{[L/2]}{2} \right) M_i w_{2i-1,1}$$

$$+ \sum_{m=1}^{[\frac{(N-L)/2}]} S_{2m} M_i w_{2i,1,1}.$$  

For $l = 0, L$, we have $w_{l0} = C_{L}^0 \cdot w_{iL} = (-1)^i C_{L}^i$. Suppose we choose

$$M_i = \frac{4i - N}{N - L}.$$  

Then

$$\sum_{i=1}^{[L/2]} M_i C_{L}^{2i} = \frac{N}{N - L} - 2^{L-1}, \sum_{i=0}^{[L/2]} M_i w_{2i,1} C_{L}^{2i} = \frac{2^{L-1}}{N - L} [4m - (N - L)]; \sum_{i=1}^{[L/2]} M_i w_{2i-1,1} C_{L}^{2i} = \frac{a^2 - 1}{N - L} [4m - 2 - (N - L)].$$

The submatrix of $M$ for the computational basis $|0^L\rangle$ and $|1^L\rangle$ is

$$M_0 = \begin{bmatrix} \Gamma_0 & \frac{2^{L-1}}{N - L} (c - id) \\ \frac{2^{L-1}}{N - L} (c + id) & \Gamma_L \end{bmatrix}.$$  

with $\Gamma_0 = \frac{N}{N - L} + 2^{L-1} (a + b), \Gamma_L = \frac{N}{N - L} + 2^{L-1} (a + b), \Gamma_0 = -1 + \sum_{m=1}^{[\frac{(N-L)/2}]} \frac{4m - (N - L)}{N - L} S_{2m}, b = \sum_{m=1}^{[\frac{(N-L)/2}]} \frac{4m - 2 - (N - L)}{N - L} S_{2m-1}$. Due to almost the same proof in Lemma 1, the maximal eigenvalue of $M_0$ is

$$\lambda_1 = \frac{N}{N - L}.$$  

The other eigenvalues of $M$ are just the other diagonal elements. Hence we should deal with the case of $l \neq 0, L$. From the definition of $w_{n,l}$ in 39, it is not difficult to show that

$$\sum_{i=0}^{[L/2]} w_{2i,l} = 0, \sum_{i=1}^{[L/2]} w_{2i-1,l} = 0,$$  

$$\sum_{i=1}^{[L/2]} iw_{2i,l} = 0, \text{ for } l = 1, L - 1,$$  

$$\sum_{i=1}^{[L/2]} iw_{2i-1,l} = 0, \text{ for } l = 1, L - 1.$$  

Using 40, we have

$$\sum_{i=1}^{[L/2]} M_i w_{2i,l} = \frac{N}{N - L} - 2^{L-1} (\delta_{l,1} + \delta_{l,L-1}), \sum_{i=0}^{[\frac{(N-L)/2}]} S_{2m} M_i q_{2i-1} = \frac{2^{L-1}}{N - L} (\delta_{l,1} + \delta_{l,L-1}), \sum_{i=0}^{[\frac{(N-L)/2}]} S_{2m} M_i q_{2i} = \frac{2^{L-1}}{N - L} (\delta_{l,1} + \delta_{l,L-1}).$$

From 45, for $1 < l < L - 1$, we have $\Gamma_i = \frac{N}{N - L}$. For $l = 1$, we arrive at $\Gamma_1 = \frac{N}{N - L} - 2^{L-1} \sum_{m=1}^{[\frac{(N-L)/2}]} \frac{N - m - 1}{N - L} \Pi_{m=1}^{N-L} (1 + z_m) \leq \frac{N}{N - L}$, and $\Gamma_L = \frac{N}{N - L} - 2^{L-1} \sum_{m=1}^{[\frac{(N-L)/2}]} \frac{N - m - 1}{N - L} \Pi_{m=1}^{N-L} (1 + z_m) \leq \frac{N}{N - L}$. Thus we have proved that all the eigenvalues of $M$ are up bounded by $\frac{N}{N - L}$. The up bound is achievable. Hence for our choice of the witness in 40, the biggest eigenvalue of the matrix $M$, thus the maximum of $tr(\rho Q)$ is $\frac{N}{N - 2} = 2^{L-1} (\delta_{l,1} + \delta_{l,L-1})$, for the partition $1^{|(N-L)|}$. The $M_i$ in 40 yields

$$\sum_{i=1}^{[\frac{(N-L)/2}]} M_i C_{L}^{2i} = \frac{N}{N - 2}.$$ 

The necessary condition of separability is

$$p \leq \frac{\frac{\sum_{i=1}^{[\frac{(N-L)/2}]} M_i C_{L}^{2i} + 2^{L-1}}}{1 + \frac{N - 2}{N - L} 2^{L-1}}.$$  

for the partition $1^{|(N-L)|}$.  

### 5.2 Further splits of the last $L$ qubits

We consider further splits of the last $L$ qubits in the partition $1^{|(N-L)|}$. Keep in mind that we have exhausted all the single qubit partite into the $1^{|(N-L)|}$, the split of the last $L$ qubits do not produce new single qubit partite. The smallest piece from the further split of the last $L$ qubits is a two qubit partite. One the other hand, we have noticed that the maximum of $tr(\rho Q)$ do not increase by the split of the last $L$ qubits. This is due to the fact that when we split the pure state of the last $L$ qubits $|\psi\rangle$ into a product of pure states, the matrix $M$ does not change since we only change the last part of $\rho$ and do not change the state of the first $N - L$ qubits and the operator $Q$. The maximal eigenvalue of $M$ is achieved when $|\psi\rangle$ is the corresponding eigenvector, namely $\langle \psi | M | \psi \rangle \geq \langle \psi' | M | \psi' \rangle$ for all the other state $|\psi'\rangle$ including the product state produced by the split of the last $L$ qubits.

For the $N - j$ ($N \geq 2j + 1$) partite separability, we first distribute each partite with one qubit and we remain $j$ qubits at hand. Then we have many strategies.
to distribute the remain qubits. The best way to decrease the number of single qubit parties is to distribute the remain $j$ qubits to $j$ parties, the resultant partition is $1^k(N-2j)[2]$. The maximum of $tr(\rho_s Q)$ for the partition $1^k(N-2j)|2|j$ with the witness (40) is less than or equal to the maximum of $tr(\rho_s Q)$ for the partition $1^k(N-2j)|2|j$, the later is $\frac{1}{N-2j}$. The necessary condition of separability for partition $1^k(N-2j)|2|j$ then is $p \leq p_0$ where $p_0 = (1 + \frac{N-2j}{N}2^{-N-1})^{-1}$. However, we have shown in the previous section that for partition $1^k(N-2j)|2|j$ the state $\rho_{ghz}^N$ with $p \leq (1 + \frac{N-2j}{N}2^{-N-1})^{-1}$ is separable. Hence $p_0 = (1 + \frac{N-2j}{N}2^{-N-1})^{-1}$. The necessary and sufficient condition of separability of $\rho_{ghz}^N$ for partition $1^k(N-2j)|2|j$ is

$$p \leq \frac{1}{1 + \frac{N-2j}{N}2^{-N-1}}.$$  

(47)

Which is also the necessary and sufficient condition of $N-j$ ($N \geq 2j+1$) partite separability of the white noisy $N$ qubit GHZ state $\rho_{ghz}^N$.

6 Other $K$-separability

There are many $K$-separability noisy GHZ states that can not be fit into the former regime of $N-j$ separability or biseparability. The first case is the triseparability of noisy GHZ states for $N \geq 6$. The partitions are $1|2|3$ and $2^3$ for $N = 6$. These partitions give rise to the tripartite states: $\rho_{1|2|3}, \rho_{2^3} = \frac{1}{3}(2|GHZ\rangle \langle 2| + \sum_{i=1}^{3} v_i T_i \langle 4|)$, where $v = (1, 1, 2, 1, 1)$ for $\rho_{1|2|3}$ and $v = (0, 3, 0, 3, 0)$ for $\rho_{2^3}$, respectively. Consider the mixture of this two kinds of partitions, the mixed state is $\rho_{x0} = q \rho_{1|2|3} + (1-q)\rho_{2^3}$. We then have $\rho_{x0} = \frac{1}{2}(2|GHZ\rangle \langle 2| + \sum_{i=1}^{3} u_i T_i \langle 4|)$, where $u = (q, q+3(1-q), 2q, q+3(1-q), q)$. Let $u_1/C_1^2 = u_2/C_2^2 \geq u_3/C_3^2$, then $q = \frac{2}{7}$. Denote the coefficient of $T_1$ as $\frac{1}{7}$, then $\tau = 9$. The tripartite separable sufficient condition is

$$p \leq \frac{1}{1 + 2^5/\tau} = \frac{9}{41}.$$  

(48)

To find the mixed state that has maximal $\tau$, we need to solve the linear programming problem. This is because the mixture probability is positive and we let $u_1/C_1^2 = u_2/C_2^2 = \cdots = u_i/C_i^2 \geq u_{i+j}/C_{i+j}^2$ to optimize $\tau$, for some $i \leq \left[ \frac{N}{2} \right]$ and all positive $j \leq \left[ \frac{N}{2} \right] - i$.

For $N \leq 12$, we list all these mixed optimal states in Table I.

| $N$ | $K$ partitions | $\tau \cdot$ prob. | $\tau$ | $p_s$ |
|-----|----------------|-------------------|--------|------|
| 6   | $3$            | $2^3, 1|2|3$       | 3,6    | 9    |
| 7   | $3$            | $2^3[3, 1]|3|^2$   | 10,5   | 7    |
| 8   | $3$            | $2^3[4, 2]|3|^2, 2^2|4$ | 8,24   | 34   |
| 4   | $4$            | $2^3[3, 2]^2$     | 11     | 1169 |
| 9   | $3$            | $2^3[2, 3]|4^3, 3^3$ | 16,36  | 61   |
| 4   | $4$            | $2^3[2|3, 2|^3 |3|^3$ | 9,9    | 14   |
| 10  | $3$            | $2^3[2, 4^2, 2^2|4$ | 10,45  | 60   |
| 10  | $4$            | $2^3[3, 2]^2|3|^2$ | 10,22  | 32,5 |
| 10  | $5$            | $2^3[3, 2]^5$     | 13,75  | 4,6  |
| 11  | $3$            | $3|4^2, 2|4|5$    | 111    | 1375 |
| 11  | $4$            | $3|2^3[4, 2]^3, 3^3$ | 4,46,2 | 61,6 |
| 11  | $5$            | $3|2^3[3, 2|^3|3|^3$ | 11     | 22   |
| 12  | $3$            | $4^3, 3|4|5, 2^5|2, 1|5|6$ | 220, 220 | 1169 |
| 12  | $4$            | $4|3^4, 2^2|4^2, 1|3|^2  | 55,33, 100 | 25   |
| 12  | $5$            | $2^3[3, 2]^3|2|^3|2$ | 12,18  | 30   |
| 12  | $6$            | $2|4^3, 3|^2|6$    | 12,3   | 15   |

7 Conclusion

The mixture an $N$ qubit GHZ state with white noise is almost the simplest model for the evolution of a graph state hence quantum stabilizer code passing through a quantum channel, also it might be the actual state realized when preparing multi-qubit GHZ state. We have strictly proven the necessary and sufficient condition for $N-j$ partite separability of an $N$ qubit GHZ state for arbitrary $N$ when $N \geq 2j+1$. The necessary and sufficient condition is achieved by a partition (up to qubit permutations) with $j$ qubit pairs in $j$ parties and $N-2j$ qubits in $N-2j$ parties. The $N-j$ separability condition is $p \leq (1 + \frac{N-2j}{N}2^{-N-1})^{-1}$, where $p$ is the percentage of pure GHZ state and $1 - p$ is the white noise percentage in the noisy GHZ state. This gives rise to the necessary and sufficient conditions for more than a half of the $K$-separability of multi-qubit noisy GHZ states. For all the other $K$-separable problems of noisy multi-qubit GHZ states, we find that the sufficient conditions are achieved by the mixtures of the partitions with the same $K$ (number of parties) but different qubit number distributions. We display all the separabilities of $N$ qubit noisy GHZ states for $N \leq 12$. The problem for sufficient conditions is solved up to a linear programming problem.

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Appendix

We consider the separable condition of the state $\rho_{ghzN}$ for partition $1|\tilde{L}$, where $\tilde{L}$ is a set of $N-1$ qubits that splits into $J-1$ ($J \geq 2$) parties. Suppose with respect to the partition $1|\tilde{L}$ there is a separable state $\rho_{s7} = \frac{1}{2^N} |2\rangle \langle GHZ_N| \{+| + \sum_{i=1}^{N-1} \frac{f(i)}{C_N} T_i \}$, with $\sum_{i=1}^{N-1} f(i) = 2J - 2$, $f_1(i) = f_1(N - i)$, $f_1(1) \leq N$ and $0 \leq f_1(i) \leq \frac{f(i)}{C_N}$ for all $i \in [1, N-1]$. We may add some fully separable states to $\rho_{s7}$ to form an unnormalized state $\tilde{\rho}_{s7} = 2 |GHZ_N\rangle \langle \hat{\rho} | + \frac{L_{N-2}}{C_N} L_{N-2}$. The state is a $\rho_{ghzN}$ state, it is separable with respect to the partition $1|\tilde{L}$ when $p \leq (1 + \frac{f(1)}{N} 2^{N-1})^{-1}$. Now we consider the partition $1|\tilde{L}$, which has $N + 1$ qubits. The separable state produced will be $\rho_{s8} = \frac{1}{2^N} |2\rangle \langle GHZ_N| \{+| + \sum_{i=1}^{N-1} \frac{f(i)}{C_N} T_i \}$, where $f_1(i) = f_1(i-1)$ for $i \in [1, N]$ and we assume $f_1(0) = f_1(N) = 1$ for convenience. Then for $i \in [2, N-1]$, we have $f_2(i) \leq \frac{f(i)}{C_N} (C_N + C_{N+1}) = \frac{f(i)}{C_N} C_{N+1} = \frac{L_{N-2}}{C_N} C_{N+1}$. While $f_2(1) = f_2(N) = \frac{f(1)}{C_N} C_{N+1}$. Thus for $i \in [1, N]$, we obtain $f_2(i) \leq \frac{f(1)}{C_N} C_{N+1}$. Hence with respect to $1|\tilde{L}$, we obtain a noisy GHZ state of $N + 1$ qubits, it is separable when $p \leq (1 + \frac{f(1)}{N+1} 2^{N-1})^{-1}$. By recursion, we can obtain the sufficient condition of $1|\tilde{L}$ for $l \geq 2$ from the sufficient condition of partition $1|L$.

Consider the bipartition $1|(N-1)$. Taking integral over the phase angle of the pure separable state $|\Omega\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\varphi} |1\rangle) (|0^{N-1}\rangle + e^{-i\varphi} |1^{N-1}\rangle)$ and averaging over qubit permutations, we will have the separable state $\rho_{s9} = \frac{1}{2^N} |2\rangle \langle GHZ_N| \{+| + \sum_{i=1}^{N-1} \frac{f(i)}{C_N} T_i \}$, where $f_1(1) = f_1(N-1) = 1$ and $f_1(i) = 0$ for $i = 2, \ldots, N-2$. We have $\frac{f(i)}{C_N} \leq \frac{1}{N}$ for $i \in [1, N-1]$, the unnormalized separable state $\tilde{\rho}_{s9} = 2 |GHZ_N\rangle \langle \hat{\rho} | + \frac{L_{N-2}}{N} L_{N-2}$ follows. By recursion, we have the sufficient condition of separability $p \leq (1 + \frac{N-L}{N} 2^{N-2})^{-1}$ for the state $\rho_{ghzN}$ with partition $1|(N-L)|L$.

Consider the tripartition $1|N_1|N_2$ with $1 + N_1 + N_2 = N$ and $2 \leq N_1 \leq N_2$ is assumed, we will have the separable state $\rho_{s9} = \frac{1}{2^N} |2\rangle \langle GHZ_N| \{+| + \sum_{i=1}^{N-1} \frac{f(i)}{C_N} T_i \}$, where (i) $f_1(N_1) = f_1(N_1 + 1)$, (ii) $f_1(N_2) = f_1(N_2 + 1) = 1$ when $N_2 > N_1 + 1$, (iii) $f_1(N_1) = f_1(N_1 + 1) = 2$ when $N_2 = N_1 + 1$, (iv) $f_1(N_1) = f_1(N_1 + 1) = 1$ when $N_2 = N_1 + 1$ and $f_1(1) = f_1(N-1) = 1$ meanwhile $f_1(i) = 0$ for all the other $i \in [2, N-2]$. For all the three cases we can show that $\frac{f(i)}{C_N} \leq \frac{1}{N}$ for $i \in [1, N-1]$. For case (i), we have $f_1(i) \leq 1$ and $C_i \geq N$ when $i \in [1, N-1]$. For case (ii), we have $N_1 = N_2$, $N = 2N_1 + 1$. We need to show $\frac{2^{N_1-2}}{C_N} \leq \frac{1}{N}$. It is true as far as $N \geq 5$. Notice that we have assumed $2 \leq N_1 \leq N_2$, thus $N = 1 + N_1 + N_2 \geq 5$. For case (iii), we have $N_1 = N_2 - 1$, $N = 2N_2$. We should prove $\frac{2^{N_2-2}}{C_N} \leq \frac{1}{N}$. It is true as far as $N \geq 5$. Since $N_1 \geq 2$, $N_2 = N_1 + 1 \geq 3$, thus $N = 1 + N_1 + N_2 \geq 6$.

If we have a partition $1|\tilde{L}$ of $N$ qubit system, then there is separable state $\rho_{s7}$ with proper defined $f_1(i)$. We now consider a partition $1|\tilde{L}|l$ of $N + l$ qubit system with $l \geq 2$. The separable state for this partition can be described by $f_2(i) = f_1(i) + f_1(i-l)$ for $i \in [l, N]$ and $f_2(i) = f_1(i)$ for all the other $i$. We want show that $f_2(i)/C_{N+l} \leq f_2(1)/(N+l)$ for $i \in [2, N + l - 2]$. Apparently, we only need to show for $i \in [l, N]$. We have $f_2(i) = f_1(i) + f_1(i-l) = \frac{f(i)}{N} (C_N + C_{N+i})$. We have $f_2(1) = f_1(1)$ for $l \geq 2$, thus we need to show is

$$\frac{1}{N} (C_N + C_{N+i}) \leq \frac{1}{N+l} C_{N+l}.$$ 

The inequality can be analytically proven for $l = 2, 3$. We have numerically checked that the inequality is true for $4 \leq l \leq 100$. We conclude that for a partition with at least one party with single qubit, the sufficient condition for separability of $\rho_{ghzN}$ is

$$p \leq (1 + \frac{N-L}{N} 2^{N-2})^{-1}$$

for the state $\rho_{ghzN}$ with partition $1|(N-L)|L$, with $\tilde{L}$ being the set of $L$ qubits split into some parties, each party has two or more qubits. $\tilde{L}$ does not contain single qubit partite.

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