Knot Theory and Quantum Gravity
in Loop Space: A Primer

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Abstract

These notes summarize the lectures delivered in the V Mexican School of Particle Physics, at the University of Guanajuato. We give a survey of the application of Ashtekar’s variables to the quantization of General Relativity in four dimensions with special emphasis on the application of techniques of analytic knot theory to the loop representation. We discuss the role that the Jones Polynomial plays as a generator of nondegenerate quantum states of the gravitational field.

1 Quantum Gravity: why and how?

I wish to thank the organizers for inviting me to speak here. This may well be a sign of our times, that a person generally perceived as a “General Relativist” would be invited to speak at a Particle Physics School. It just reflects the higher degree of interplay these two fields have enjoyed over the last years. In these lectures we will see more reasons for this enhanced interplay. We will see several notions from Gauge Theories, as Wilson Loops for instance, playing a central role in gravitation. An even greater interplay takes place with Topological Field Theories. We will see the important role that the Chern-Simons form, the Jones Polynomial and other notions of knot theory seem to play in General Relativity.

The quantization of General Relativity is a problem that has defied resolution for the last sixty years. In spite of the long time that has been invested in trying to solve it, we believe that several people do not necessarily fully appreciate the reasons of our failure and the magnitude of the problem. It is a general perception—especially among particle physicists—that “General Relativity is nonrenormalizable” and that is the basic problem with the theory. This statement is misleading in three ways:

a) The fact that a theory is nonrenormalizable does not necessarily mean that the theory has an intrinsic problem or is “bad” in any way. It merely says that perturbation

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theory does not apply to the problem in question. As we will see in c) there are actually good reasons to believe that ordinary perturbation theory should fail for General Relativity.

b) Deciding if a theory is or is not renormalizable can be quite tricky. The prime example is 2+1 dimensional gravity, which most people thought to share the renormalizability pathologies of 3+1 gravity until Witten [1] pointed out that it could be exactly solved. A posteriori it was of course found that the theory is in fact renormalizable [2].

c) There actually are very good reasons why we should expect General Relativity to have “problems” (we would rather call them “subtleties” or “challenges”) in quantization. Prime among them is the issue of diffeomorphism invariance, which in turn implies other problems as the lack of observables for the theory. This last problem clearly reflects the nature of the issue: even if we were somehow able to make General Relativity renormalizable, we would not know what to compute with such a perturbatively well behaved theory. Even if we were able to compute something we would not know how to interpret it.

Due to these and other arguments, we believe that perturbative quantization of General Relativity may well be a red herring. So too may also be the idea of abandoning General Relativity in favour of other theories that present some particular better behaviour (usually only apparent) when perturbatively quantized. We repeat: even if we had a perfectly renormalizable theory of quantum gravity, it is little what we could actually do with it until we address the fundamental questions of what kind of physics we can do in a diffeomorphism invariant context.

If one is interested in these kind of questions, in particular the issue of diffeomorphism invariance, nonperturbative quantization seems the way to go. There are several options if one wants to attempt a nonperturbative quantization of gravity, ranging from quite radical to very conservative ones. Of all these, probably the most conservative is Canonical Quantization. After all, this was the first method of quantization ever invented and is the one most physicists feel comfortable with. Canonical Quantization therefore seems an attractive approach to study the issues that arise in the quantization of General Relativity. If the resulting theory makes sense, one expects that other quantization techniques would in the end give the same results.

We will discuss in these lectures the Canonical Quantization of General Relativity in four dimensions. As we have argued, the use of other theories or number of dimensions seems at the moment superfluous. We do not even understand—for good reasons—arguably the simplest theory (General Relativity). Therefore it seems to make little sense for our purposes to embark on the study of more complicated theories. It may well be that at some point it becomes apparent that General Relativity does not furnish a suitable base for a theory of Quantum Gravity. Until that point is reached we think it is useful as the simplest theory of gravity that has all the desired features one would expect in such a theory.

We will therefore proceed with a very conservative Canonical Quantization scheme but we will pursue a slight variant from the traditional approaches. We will use a new set of canonical variables for the treatment of Hamiltonian General Relativity, the Ashtekar Variables [3]. These variables have the advantage of casting General Relativity in a fashion that closely resembles Yang-Mills theories. This will allow us to introduce several useful techniques from Yang-Mills theories into General Relativity.
The plan of these lectures is as follows: in section 2 we discuss the traditional canonical formulation of General Relativity. In section 3 we introduce the Ashtekar New Variables and discuss the classical theory. In section 4 we discuss the quantum theory in the connection representation, and point out the role that Wilson Loops and the Chern-Simons form play in the theory. In section 5 we present the Loop Representation and develop technology for dealing with the constraints and the wavefunctions written in terms of loops. In section 6 we discuss various aspects of Knot Theory. In section 7 we make use of the results of sections 5 and 6 to construct a family of nondegenerate physical states of Quantum Gravity in terms of knot invariants. We end in section 8 with some final remarks and a general discussion of the present status of the program.

2 Brief Summary of General Relativity and Canonical Quantization

2.1 Classical General Relativity

General Relativity is a theory of gravity in which the gravitational interaction is accounted for by a deformation of spacetime. The fundamental variable for the theory is the spacetime metric $g_{ab}$. The action for the theory is given by,

$$S = \int d^4x \sqrt{-g} R(g_{ab}) + \int d^4x \sqrt{-g} \mathcal{L}(\text{matter})$$  \hspace{1cm} (1)

where $g$ is the determinant of $g_{ab}$, $R(g_{ab})$ is the curvature scalar and we have included also a term to take into account possible couplings to matter, although in these lectures we will largely concentrate on the theory in vacuum. The equations of motion for this action, obtained by varying the action with respect to $g_{ab}$ are,

$$R_{ab} - \frac{1}{2} g_{ab} R = \frac{\delta S_{\text{matter}}}{\delta g_{ab}}$$  \hspace{1cm} (2)

and are called Einstein Equations. The theory is invariant under diffeomorphisms on the four manifold (coordinate transformations). This means it has a symmetry. The Einstein Equations are in principle ten equations (all the tensors are symmetric and therefore only have ten independent components). However, due to the presence of the diffeomorphism symmetry, several of the equations are redundant. This issue is best seen in the canonical formalism. We will briefly set it up in the next two subsections. The reader wanting a more detailed treatment can find it in references [4, 5].

2.2 Canonical formulation, the 3+1 split

To cast General Relativity in a canonical form, we need to split space-time into space and time. Without a notion of time, we do not have a notion of evolution, and therefore no notion of “Hamiltonian”. This may seem strange at first. One of the great accomplishments of Relativity was to put space and time into an equal footing, and now we seem to be destroying it. We will show that this is not the case. Although the Canonical formalism manifestly breaks the covariance of the theory by singling out a particular time direction, in the end the formalism tells us that it really did not matter which time direction we took. That is, the covariance is recovered implicitly in the theory, and the
Figure 1: 3+1 foliation of spacetime and variables of the canonical formalism

time picked is only a "fiducial" one for constructional purposes. We will see more details of this fairly soon.

So we foliate spacetime $^4M$ into a spatial manifold $^3\Sigma$ and a time direction $t^a$, as shown in figure 1. We now decompose the time direction into components normal and perpendicular to the three surface,

$$t^a = Nn^a + N^a$$  \hspace{1cm} (3)

where $n^a$ is the normal to the three surfaces and $N^a$ is tangent to the three surfaces, and is called the shift vector. The scalar $N$ is called the lapse function. Given the metric $g_{ab}$ and the timelike vector $n^a$ one can define a positive-definite (Euclidean) metric on the three surface $q^{ab} = g^{ab} + n^a n^b$. Another important quantity on the three surface is its extrinsic curvature, defined by $K_{ab} = q^{mn} q^{a}_{m} \nabla_{m} n_{n}$, where $\nabla$ is the covariant derivative compatible with $g_{ab}$. To clarify the role of the extrinsic curvature it is enough to compute the "time derivative" of the three metric, given by its Lie derivative with respect to $t^a$ (exercise),

$$\dot{q}_{ab} = \mathcal{L}_{\vec{t}} q_{ab} = 2N K_{ab} + \mathcal{L}_{\vec{N}} q_{ab}$$  \hspace{1cm} (4)

So we see that the role of the extrinsic curvature is roughly that of "time derivative" of the three-metric, giving an idea of how the three dimensional surface is deformed with respect to the ambient four dimensional spacetime. One can rewrite the action of General Relativity in terms of these quantities in the following form,

$$S = \int dt \ L(q, \dot{q})$$  \hspace{1cm} (5)

$$L(q, \dot{q}) = \int d^3x N \sqrt{q}(\frac{3}{2}R + K_{ab} K^{ab} - K^2)$$  \hspace{1cm} (6)

where $t$ is a parameter along the integral curves of the vector $t^a$. $^3R$ is the scalar curvature of the three dimensional surface. To achieve this form of the action one needs to neglect surface terms. Along these lectures we will always assume that the spatial three-surface is compact, as in the case of some cosmologies. One could treat the asymptotically-flat case (which includes for instance, stars and black-holes) by imposing appropriate boundary conditions at infinity. This can be done in a reasonable straightforward manner, although we will not discuss it here for the sake of brevity.
We now have the action of General Relativity in a reasonable form to formulate a canonical analysis. We have expressed it in terms of variables that are functions of “space” (functions of the three-surface) and that “evolve in time”. This is the usual setup for doing canonical formulations.

We pick as a canonical variable the three metric $q^{ab}$ and compute its conjugate momentum,

$$\tilde{\pi}^{ab} = \frac{\delta L}{\delta \dot{q}^{ab}} = \sqrt{q}(K_{ab} - K q_{ab})$$  \hspace{1cm} (7)

(throughout these lectures we will denote tensor densities of weight +1 with a tilde and those of weight -1 with an undertilde). We see that the “conjugate momentum” to $q^{ab}$ is roughly related to the extrinsic curvature (“time derivative”).

We are now in position to perform the Legendre transform and obtain the Hamiltonian of the theory,

$$H(\pi, q) = \int d^3x (\tilde{\pi}^{ab} \dot{q}^{ab} - L)$$  \hspace{1cm} (8)

and replacing $\dot{q}$ in terms of $\tilde{\pi}$, we get (exercise),

$$H(\pi, q) = \int d^3x (N(-q^{1/2}R + q^{-1/2}(\tilde{\pi}^{ab} \tilde{\pi}_{ab} - \frac{1}{2} \tilde{\pi}^2)) - 2N^b D_a \tilde{\pi}^a_b)$$  \hspace{1cm} (9)

where $D_a$ is the covariant derivative on the three surface compatible with $q_{ab}$.

2.3 The constraints

Having done this, let us step back a minute and analyze the formalism we built. We started from a four dimensional metric $g^{ab}$ and we now have in its place the three dimensional $q^{ab}$ and the “lapse” and “shift” functions $N$ and $N^a$. We defined a conjugate momentum for $q_{ab}$. However, notice that nowhere in the formalism does a time derivative of the lapse or shift appear. That means their conjugate momenta are zero. That is, our theory has constraints. In fact, if we rewrite the action using the expression for the Hamiltonian we give above, we get,

$$S = \int dt \int d^3x (\tilde{\pi}^{ab} \dot{q}^{ab} + N(-q^{1/2}R + q^{-1/2}(\tilde{\pi}^{ab} \tilde{\pi}_{ab} - \frac{1}{2} \tilde{\pi}^2)) - 2N^b D_a \tilde{\pi}^a_b)$$  \hspace{1cm} (10)

and if we vary it with respect to $N$ and $N^a$ in order to get their respective equations of motion, we get four expressions, functions of $\tilde{\pi}$ and $q$ which should vanish identically, and are usually called $\tilde{C}^a$ and $\tilde{C}$,

$$\tilde{C}_a(\pi, q) = 2D_b \tilde{\pi}^b_a$$  \hspace{1cm} (11)

$$\tilde{C}(\pi, q) = -\tilde{q}R + (\tilde{\pi}^{ab} \tilde{\pi}_{ab} - \frac{1}{2} \tilde{\pi}^2)$$  \hspace{1cm} (12)

For calculational simplicity, these equations are usually “smoothed out” with arbitrary test fields on the three manifold, $C(\tilde{N}) = \int d^3x N^a \tilde{C}_a$, $C(\tilde{N}) = \int d^3x \tilde{N} \tilde{C}$. (Notice that the notation is unambiguous. One can write the constraints now as $C(\tilde{N}) = 0$ and $C(\tilde{N}) = 0$, due to the arbitrariness of the test fields).

Notice that these equations are “instantaneous” laws, i.e. they must be satisfied on each hypersurface. They tell us that if we want to prescribe data for a gravitational field, not every pair of $\tilde{\pi}$ and $q$ will do, eqs. (11, 12) should be satisfied. (Notice that
there are six degrees of freedom in $q^{ab}$, subjected to four constraints, this leaves us with two degrees of freedom for the gravitational field, as expected).

These equations have the same character as the Gauss Law has for electromagnetism, which tells us that any vector field would not necessarily work as an electric field, it must have vanishing divergence in vacuum. As is well known, the Gauss Law appears as a consequence of the $U(1)$ invariance of the Maxwell equations. An analogous situation appears here. To understand this, consider the Poisson bracket of any quantity with the constraint $C(\vec{N})$. It is easy to check that (exercise),

$$\{f(\tilde{\pi}, q), C(\vec{N})\} = L_{\vec{N}}f(\tilde{\pi}, q).$$

Therefore we see that the constraint $C(\vec{N})$ “Lie drags” the function $f(\pi, q)$ along the vector $\vec{N}$. Technically, it is the infinitesimal generator of diffeomorphisms of the three manifold in phases space. As the Gauss law (in the canonical formulation of Maxwell’s theory) is the infinitesimal generator of $U(1)$ gauge transformations, the constraint here is the infinitesimal generator of spatial diffeomorphisms. This clearly shows why we have this constraint in the theory: it is the canonical representation of the fact that the theory is invariant under spatial diffeomorphisms. An analogous situation stands for the constraint $C(\vec{N})$, although we will not discuss it in detail for reasons of space: it is the generator of “time” diffeomorphisms, and it is usually called the Hamiltonian constraint.

We can now work out the equations of motion of the theory either varying the action with respect to $q^{ab}$ and $\tilde{\pi}_{ab}$ or taking the Poisson bracket of these quantities with the Hamiltonian constraint.

## 2.4 Quantization

Having set up the theory in canonical form we can now proceed and attempt a Canonical Quantization. The process can be roughly summarized in the following sequence of steps. The reader may notice that at each step there are many possible choices. Different choices will yield different quantizations.

1. Pick up a complete set of canonical quantities that form an algebra under Poisson brackets. In many systems one simply takes $p$ and $q$, but one can pick other “noncanonical” quantities. We will actually do so in the next chapters.

2. Represent these quantities as operators acting on a space of (wave)functionals of “half” of the elements of the algebra. These operators will in turn give a “quantum” representation of the algebra defined in the previous point. At this point the space of functions is not a Hilbert space. One usual choice is to take functionals of $q$, $\Psi(q)$, and represent $\hat{q}$ as a multiplicative operator and $\hat{p}$ as a functional derivative, $\hat{p}\Psi(q) = \frac{\delta}{\delta q}\Psi(q)$.

3. One wants the wavefunctions to be invariant under the symmetries of the theory. As we saw the symmetries are represented in this language as constraints. The requirement that the wavefunctions be annihilated by the constraints (promoted to operatorial equations) implements this requirement. Not all functionals chosen in the previous steps will be annihilated by the quantum constraint equations. We
call those who are “physical states” (notice that we still do not have a Hilbert space).

4. One introduces an inner product on the space of physical states in order to compute expectation values and make physical predictions. Only at this point does one have an actual Hilbert Space. How to find this inner product is not prescribed by standard canonical quantization (we will discuss this in the next section). Under this inner product the physical states should be normalizable. The expectation values, by the way, only make sense for quantities that are invariant under the symmetries of the theory (quantities that classically have vanishing Poisson brackets with all the constraints). We call them physical observables. For the gravitational case none is known for compact spacetimes (we will return to this issue later). The observables of the theory should be self adjoint operators with respect to the inner product in order to yield real expectation values.

Let us start by step one. As it is said, one can pick various choices of algebras. Let us concentrate on the simplest one, just picking the canonical variables we introduced $q^{ab}$ and $\tilde{\pi}_{ab}$. Their Poisson bracket is, being canonically conjugate variables, 
\[
\{q^{ab}(x), \tilde{\pi}_{cd}(y)\} = \delta(x - y)\delta^a_c \delta^b_d.
\] Step 2 can be fulfilled taking wavefunctions $\Psi(q^{ab})$ and representing $q^{ab}$ as a multiplicative operator and $\tilde{\pi}_{ab}$ as a functional derivative.

It is in step 3 that we run into trouble. We have to promote the constraints we discussed in the last subsection to quantum operators. This in itself is a troublesome issue, since being General Relativity a field theory, issues of regularization and factor ordering appear. One can, —at least formally— find factor orderings in which the diffeomorphism constraint becomes the infinitesimal generator of diffeomorphisms on the wavefunctions. Therefore the requirement that a wavefunction be annihilated by it just translates itself in the fact that the wavefunction has to be invariant under diffeomorphisms. This is not difficult to accomplish (formally!). One simply requires that the wavefunctions not actually be functionals of the three metric $q^{ab}$, but of the “three geometry” (by this meaning the properties of the three geometry invariant under diffeomorphisms). That is, what we are saying is just a restatement of the fact that the functional should be invariant under diffeomorphisms. One can come up with several examples of functionals that meet this requirement. The real trouble appears when we want to make the wavefunctions annihilated by the Hamiltonian constraint. This constraint does not have a simple geometrical interpretation in terms of three dimensional quantities (remember that the idea that it represent “diffeomorphisms in time” does not help here, since we are always talking about equations that hold on the three surface without any explicit reference to time). Therefore we are just forced to proceed crudely: promote the constraint to a wave equation, use some factor ordering (hopefully with some physical motivation), pick some regularization and try to solve the resulting equation. It turns out that this task was never accomplished in general (it was in simplified minisuperspace examples). Among the difficulties that conspire in this direction is the fact that the constraint is a nonpolynomial function of the basic variables (remember it involves the scalar curvature, a nonpolynomial function of the three-metric).

Therefore the program of canonical quantization stalls here. Having been unable to find the physical states of the theory we are in a bad position to introduce an inner product (since we do not know on what space of functionals to act) and actually make
physical predictions. This issue is compounded by the fact that we do not know any observables for the system, which puts us in a more clueless situation with respect to the inner product. This status of affairs was reached already in the work of DeWitt in the 60’s and little improvement was made until recently. We will see in the next chapter that the use of a new set of variables improves the situation with respect to the Hamiltonian constraint, giving hopes of maybe allowing us to attack the problem of the inner product.

3 The Ashtekar New Variables

The following three subsections follow closely the treatment of \[3\]. The reader is referred to it for more detailed explanations.

3.1 Tetradic General Relativity

To introduce the New Variables, we first need to introduce the notion of tetrads. In a nutshell, a tetrad is a vector basis in terms of which the metric of spacetime looks locally flat. Mathematically, 

\[ g_{ab} = e^I_a e^J_b \eta_{IJ} \tag{14} \]

where \( \eta_{IJ} = \text{diag}(-1, 1, 1, 1) \) is the Minkowski metric, and equation (14) simply expresses that \( g_{ab} \) when written in terms of the basis \( e^I_a \), is locally flat. If spacetime were truly flat, one could perform such a transformation globally, integrating the basis vectors into a coordinate transformation \( e^I_a = \frac{\partial x^I}{\partial x^a} \). In a curved spacetime these equations cannot be integrated and the transformation to a flat space only works locally, the flat space in question being the “tangent space”. From equation (14) it is immediate to see that given a tetrad, one can reconstruct the metric of spacetime. One can also see that although \( g_{ab} \) has only ten independent components, the \( e^I_a \) have sixteen. This is due to the fact that eq. (14) is invariant under Lorentz transformations on the indices \( I, J \ldots \). That is, these indices behave as if living in flat space. In summary, tetrads have all the information needed to reconstruct the metric of spacetime but there are extra degrees of freedom in them, and this will have a reflection in the canonical formalism.

3.2 The Palatini action

We now write the Einstein action in terms of tetrads. We introduce a covariant derivative via \( D_a K_I = \partial_a K_I + \omega_a^J K_J \). \( \omega_a^J \) is a Lorentz connection (the derivative annihilates the Minkowski metric). We define a curvature by \( \Omega_{ab}^{IJ} = \partial_{[a} \omega_{b]}^{IJ} + [\omega_a, \omega_b]^{IJ} \), where \([ , ]\) is the commutator in the Lorentz Lie algebra. The Ricci scalar of this curvature can be expressed as \( e^I_a e^J_b \Omega_{ab}^{IJ} \) (indices \( I, J \) are raised and lowered with the Minkowski metric). The Einstein action can be written,

\[ S(e, \omega) = \int d^4x \; e e^I_a e^J_b \Omega_{ab}^{IJ} \tag{15} \]

where \( e \) is the determinant of the tetrad (equal to \( \sqrt{-g} \)).

We will now derive the Einstein Equations by varying this action with respect to \( e \) and \( \omega \) as independent quantities. To take the metric and connection as independent variables in the action principle was first considered by Palatini \[3\].
As a shortcut to performing the calculation (this derivation is taken from [5]), we introduce a (torsion-free) connection compatible with the tetrad via $\nabla_a e^b_I = 0$. The difference between the two connections we have introduced is a field $C_{aIJ}$ defined by

$$C_{aIJ} = (D_a - \nabla_a) e^b_I = \nabla_a [a I] J + C_{aIM} C_{b]M}. \quad (16)$$

The variation with respect to $C_{aIJ}$ is easy to compute: the first term simply does not contain $C_{aIJ}$ so it does not contribute. The second term is a total divergence (notice that $\nabla$ is defined so that it annihilates the tetrad), the last term yields $\epsilon^I_{MN} \delta^K J (I \delta K J) C_{bMN}$. It is easy to check that the prefactor in this expression is nondegenerate and therefore the vanishing of this expression is equivalent to the vanishing of $C_{bMN}$. So this equation basically tells us that $\nabla$ coincides with $D$ when acting on objects with only internal indices. Thus the connection $D$ is completely determined by the tetrad and $\Omega$ coincides with $R$ (some authors refer to this fact as the vanishing of the torsion of the connection).

We now compute the second equation, straightforwardly varying with respect to the tetrad. We get, (after substituting $\Omega_{abIJ}$ by $R_{abIJ}$ as given by the previous equation of motion),

$$e^I R_{cbIJ} - \frac{1}{2} R^{cdMN} e^d_M e^c_N e^I_J. \quad (17)$$

which, after multiplication by $e_{Ja}$ just tells us that the Einstein tensor $R_{ab} - \frac{1}{2} R g_{ab}$ of the metric defined by the tetrads vanishes. We have therefore proved that the Palatini variation of the action in tetradic form yields the usual Einstein Equations.

There is a slight difference between the first order (Palatini) tetradic form of the theory and the usual one. One easily sees that a solution to the Einstein Equations we presented above is simply $e^I_a = 0$. This solution would correspond to a vanishing metric and is therefore forbidden in the traditional formulation since quantities as the Ricci or Riemann tensor are not defined for a vanishing metric. However, the first order action and equation of motion are well defined for vanishing triads. We therefore see that strictly speaking the first order tetradic formulation is a “generalization” of General Relativity that contains the traditional theory in the case of nondegenerate triads. We will see this subtlety playing a role in the future chapters. It should be noticed that the possibility of allowing vanishing metrics in General Relativity is quite attractive since one could evisage the formalism “going through”, say, the formation of singularities. It also allows for topology change [8].

3.3 The self-dual action

Up to now the treatment has been totally traditional. We will now take a conceptual step that allows the introduction of the Ashtekar variables. We will reconstruct the tetradic formalism of the previous subsection but we will introduce a change. Instead of considering the connection $\omega_a^{IJ}$ we will consider its self dual part with respect to the internal indices and we will call it $A_a^{IJ},$ that is $i A_a^{IJ} = \frac{1}{2} \epsilon_{MN}^{IJ} A_a^{MN}$. Now, to really
be able to do this, the connection must be complex (or one should work in an Euclidean signature). Therefore for the moment being we will consider complex General Relativity and we will then specify appropriately how to recover the traditional real theory. The connection now takes values in the (complex) self-dual subalgebra of the Lie algebra of the Lorentz group. We will propose as action,

$$S(e, A) = \int d^4x \ e^a e^b F_{ab}^{JK}$$

where $F_{ab}^{JK}$ is the curvature of the self-dual connection and it can be checked that it corresponds to the self-dual part of the curvature of the usual connection.

We can now repeat the calculations of the previous subsection for the self-dual case. When one varies the self-dual action with respect to the connection $A_a^{IJ}$ again one obtains that this connection should annihilate the triad (if one repeated step by step the previous subsection argument, one now finds that the self-dual part of $C_{a^{IJ}}$ vanishes). The variation with respect to the tetrad goes along very similar lines only that $\Omega_{ab}^{IJ}$ gets everywhere replaced by $F_{ab}^{IJ}$. The final equation one arrives to (exercise) again tells us that the Ricci tensor vanishes. Remarkably, the self-dual action leads to the (complex) Einstein Equations. This essentially be understood in the fact that the two actions differ by boundary terms. We postpone the issue of how to recover the real theory to subsection 3.6.

### 3.4 The New Variables

If one took the Palatini action of subsection 3.2 and made a canonical 3+1 decomposition, the formalism basically returns to the traditional one \[9\]. A quite different thing happens if one decomposes the self-dual action. Let us therefore proceed to do the 3+1 split. As before, we introduce a vector $t^a = N n^a + N^a$ (which, since we are actually dealing with complex Relativity, may be complex). Taking the action,

$$S(e, A) = \int d^4x \ e^a e^b F_{ab}^{IJ}$$

and defining the vector fields orthogonal to $n^a$, $E_a^I = q^a_b e^b_I$ (where $q^a_b = \delta^a_b + n^a n_b$ is the projector on the three-surface) we have,

$$S(e, A) = \int d^4x \ (e E^a_I e^b_J F_{ab}^{IJ} - 2 e E^a_I n_d n^b F_{ab}^{IJ}).$$

We now define $\tilde{E}_I^a = \sqrt{q} E_I^a$, which is a density on the three manifold. The determinant of the triad can be written as $e = N \sqrt{q}$. We also introduce the vector in the “internal space” induced by $n^a$, defined by $n_I = e^a_I n_a$. With these definitions, and exploiting the self-duality of $F_{ab}^{IJ}$ to write $F_{ab}^{IJ} = -\frac{i}{2} \epsilon^{IJ MN} F_{ab}^{MN}$, we get,

$$S(e, A) = \int d^4x \ (-\frac{i}{2} N \tilde{E}_I^a e^b_J F_{ab}^{IJ} M N F_{ab}^{MN} - 2 N n^b \tilde{E}_I^a n_J F_{ab}^{IJ}).$$

We now pick a gauge in which $E_I^a = 0$ and $n_I = (1, 0, 0, 0)$. We also exploit the relation between Levi-Civita densities in three and four space, $\epsilon^{IJKLM} = \epsilon^{IJL}$, which in our

\[One can proceed in a more elegant, albeit slightly more complicated, way, see ref. \[9\]
gauge reads $\epsilon^{IJK0} = \epsilon^{IJK}$. Therefore, we have,

$$S(e, A) = \int d^4x \left( -\frac{i}{2} N_{I}^{a} \tilde{E}_{j}^{b} \epsilon^{IJ} M F_{ab}^{MO} - 2 N n_{I}^{b} \tilde{E}_{j}^{a} F_{ab}^{I0} \right). \quad (22)$$

Notice that the indices $I, J \ldots$ in the above expression now only range from 1 to 3. We denote them with lowercase letters to highlight this fact and rename, $A_{a}^{I0} = A_{a}^{i}$ and similarly for $F_{ab}^{I0} = F_{ab}^{i}$. The effect of the presence of the vector in internal space $n_I$ has been to split the two copies of $SU(2)$ present in the Lorentz group in such a way that our new indices $i, j, k$ are $SU(2)$ indices. We now replace in the second term $N n_{I}^{b}$ by $t_{b} - N_{b}$ and use the identity (exercise) $t_{a}^{i} F_{ab}^{i} = L_{t} A_{b}^{i} - D_{b}(t_{a}^{i} A_{a}^{i})$, where $D_{b}$ is the derivative defined by the connection $A_{a}^{i}$, to get,

$$S(e, A) = \int d^4x \left( 2 \tilde{E}_{i}^{a} L_{t} A_{a}^{i} - 2 N_{b}^{b} \tilde{E}_{i}^{a} F_{ab}^{i} - i N \epsilon_{ij}^{a} \tilde{E}_{i}^{a} F_{ab}^{j} F_{ab}^{k} \right). \quad (23)$$

This action is exactly in the form we want. There is a term of the “$p\dot{q}$” form, $(\tilde{E}_{i}^{a} L_{t} A_{a}^{i})$, from which we can read off that the variable canonically conjugate to $A_{a}^{i}$ is $\tilde{E}_{i}^{a}$. The theory also has constraints, given by,

$$\tilde{G}_{i}^{a} = (D_{a} \tilde{E}_{i}^{a})^{i} \quad (24)$$

$$\tilde{C}_{a} = \tilde{E}_{i}^{a} F_{ab}^{i} \quad (25)$$

$$\tilde{C}^{k} = \epsilon^{ij} \tilde{E}_{i}^{a} \tilde{E}_{j}^{b} F_{ab}^{k} \quad (26)$$

The last four equations correspond to the usual diffeomorphism and Hamiltonian constraints of canonical General Relativity. The first three equations are extra constraints that stem from our use of triads as fundamental variables. These equations, which have exactly the same form as a Gauss Law of an $SU(2)$ Yang-Mills theory, are the generators of infinitesimal $SU(2)$ transformations. They tells us that the formalism is invariant under triad rotations, as it should be.

Notice that a dramatic simplification of the constraint equations has occurred. In particular the Hamiltonian constraint is a polynomial function of the canonical variables, of quadratic order in each variable. Moreover, the canonical variables, and the phase space of the theory are exactly those of a (complex) $SU(2)$ Yang-Mills theory. The reduced phase space is actually a subspace of the reduced phase space of the Yang-Mills theory (the phase space modulo the Gauss Law), since General Relativity has four more constraints that further reduce its phase space. This resemblance in the formalism to that of a Yang-Mills theory will be the starting point of all the results we will introduce in the last sections of this paper.

### 3.5 The constraint algebra

When one has a theory with constraints, one needs to check that these are consistent with each other, i.e. that by taking Poisson brackets among the constraints one does not generate new constraints. If this were the case, these secondary constraints should also be enforced. Fortunately, the system we have here is first class, i.e. the Poisson bracket of each two constraints is a combination of the other constraints. Actually, in terms of the New Variables, the structure of the constraints is simple enough for the reader to be able to compute the constraint algebra without great effort (this computation
can also be carried along with the traditional variables and the results are the same). We only summarize the results here. To express them in a simpler form (and to avoid confusing manipulations of distributions while performing the computations), it is again convenient to smooth out the constraints with arbitrary test fields. We denote,

\[ G(N_i) = \int d^3x N_i (D_a \tilde{E}^a)^i \]

\[ C(\tilde{N}) = \int d^3x N^b \tilde{E}^a_i F_{ab} \]

\[ C(\tilde{N}) = \int d^3x \tilde{N} \epsilon^{ij} k \tilde{E}^a_i \tilde{E}^b_j F_{ab} \]

and as before the notation is unambiguous. The constraint algebra then reads,

\[ \{G(N_i), G(N_j)\} = G([N_i, N_j]) \]

\[ \{C(\tilde{N}), C(\tilde{M})\} = C(\mathcal{L}_{\tilde{N}} \tilde{N}) \]

\[ \{C(\tilde{N}), G(N_i)\} = G(\mathcal{L}_{\tilde{N}} N_i) \]

\[ \{C(\tilde{N}), C(M)\} = C(\mathcal{L}_{\tilde{N}} M) \]

\[ \{C(N_i), C(N_j)\} = 0 \]

\[ \{C(N_i), C(M)\} = C(\tilde{K}) - G(A^a_i K^a) \]

where the vector \( \tilde{K} \) is defined by \( K^a = 2 \tilde{E}^a_i \tilde{E}^b_i (\tilde{N} \partial_a \tilde{M} - \tilde{M} \partial_a \tilde{N}) \). Here we clearly see that the constraints are first class. The reader should notice, however, that the algebra is not a true Lie algebra, since one of the structure constants (the one defined by the last equation), is not a constant but depends on the fields \( \tilde{E}^a_i \) (through the definition of the vector \( \tilde{K} \)).

3.6 The evolution equations and the reality conditions

Up to now we have been concerned only with instantaneous relations among the fields (the constraints). However, if one wants to evolve the fields in time, one needs the evolution equations, which are simply obtained taking the Poisson bracket of the fields with the Hamiltonian.

These equations give the “time derivative” of the triad and the connection.

\[ \dot{\tilde{E}}^a_i = \{ \tilde{E}^a_i, H(\tilde{N}) \} = -i \sqrt{2} D_b (\tilde{N} \tilde{E}^b_i \tilde{E}^a_i)_i \]

\[ \dot{A}^a_i = \{ A^a_i, H(\tilde{N}) \} = -i \sqrt{2} [\tilde{N} \tilde{E}^b_i, F_{ab}]^i. \]

From here one can straightforwardly derive the equations of motion for the traditional variables, the metric and the extrinsic curvature.

We now turn our attention to the issue of “reality conditions”. As was mentioned before, the formalism we are dealing with describes complex General Relativity. In fact, the action we are using is complex! If we want to recover the classical theory we must take a “section” of the phase space that corresponds to the dynamics of real Relativity. This can be done. One gives data on the initial surface that corresponds to a real spacetime and the evolution equations will keep these data real through the evolution. Now, strictly speaking, this procedure is not really canonical, since we are imposing
these conditions by hand at the end. That does not mean it is not useful. In fact, one can eliminate the reality conditions and have a canonical theory. However, many of the beauties of the new formulation are lost, in particular the structure of the resulting constraints is basically that of the traditional formalism.

The issue of the reality conditions acquires a different dimension in the quantum theory. A point of view that is strongly advocated, and may turn out to be successful, is the following. Start by considering the complex theory and apply the steps towards canonical quantization that we discussed subsection (2.4). After the space of physical states has been found, when one decides to find an inner product, the reality conditions are used in order to choose an inner product that implements them. That is, the reality conditions can be a guideline to find the appropriate inner product of the theory. One simply requires that the quantities that have to be real according to the reality conditions of the classical theory, become self-adjoint operators under the chosen inner product. This solves two difficulties at once, since it allows us to recover the real quantum theory and the appropriate inner product at the same time. This point of view is strictly speaking a deviation from standard Dirac quantization, and works successfully for several model problems [11]. The success or failure in Quantum Gravity of this approach is yet to be tested and is one of the most intriguing and attractive features of the formalism. (For a critical viewpoint, see [12]).

What are, therefore, the reality conditions? They can be written in several ways, depending on which variables one chooses to express them in. One could simply write them in terms of the three-metric,

$$\hat{q}^{ab} = (\hat{q}^{ab})^* \quad (38)$$
$$\dot{\hat{q}}^{ab} = (\dot{\hat{q}}^{ab})^* \quad (39)$$

Using the expressions we presented above for the time derivatives, we could easily express these equations in terms of the triad and the connection.

How one writes the reality conditions depends largely on what one wants to accomplish. If one, for instance, is interested in pursuing the quantization program outlined above, and using the reality conditions to fix the inner product, one does not necessarily want them written in the above form. The reason for this is that if one wants to write them as conditions of hermiticity under an inner product, one would want to write them in terms of quantities that are observables of the theory, so that taking their expectation value makes sense. Since we do not know any observable for the theory, we cannot write the reality conditions in this way at present.

Let us finish by making a small digression on the issue of observables. The quantities that one wants to observe in a system are quantities that are invariant under the symmetries of the theory. Any other kind of quantity will be gauge dependent and therefore of no physical relevance. In the canonical theory it is easy to define which kind of quantities are observables. Since the constraints are the infinitesimal generators of the symmetries of the theory, any quantity that has vanishing Poisson brackets with the constraints is invariant under the symmetries of the theory. Therefore, if one wanted to find an observable for General Relativity, one has to look for a quantity with vanishing Poisson brackets with the diffeomorphism and Hamiltonian constraint of the theory.

A nontrivial example where it can be worked out to the end is the Bianchi II cosmology [10].
The trouble is that we do not know any single such quantity. This can be a circum-
stantial problem, merely reflecting our ignorance, or it could be fundamental. There are
suggestions that maybe no such quantity exists for General Relativity [13]. This could
be related to the fact that the theory displays chaotic behaviour [15, 16]. The issue of
observables and their relation to quantization (some people argue that since the theory
may have no observables, the whole program of quantization we are pursuing here is
doomed) exceeds the scope and is not in line with the emphasis of these talks so we will
not discuss it here. We just want to make the reader aware that there is potential for
a problem with this issue. For a recent discussion see [17]. From a practical point of
view one could argue that even if the full theory has no exact observable, one could find
some approximation in which the theory has observables (after all, we live in such an
approximation and measure things all the time!). Again, this exceeds the scope of this
treatment. See [18] for more details on this point of view.

4 Quantum Theory: The Connection Representation

4.1 Formulation

Let us suppose we now decide to proceed and apply the canonical quantization program
to the theory as we have it up to now.

We start by picking a polarization. One has many choices. However, let us remem-
ber that our canonical variables are basically similar to those of a Yang-Mills SU(2)
theory. When one quantizes Yang-Mills (and Maxwell) theories a usual choice for the
polarization is to pick wavefunctionals of the connection Ψ(A). We will pursue in this
subsection this treatment for General Relativity. Notice that this is potentially very
different from what one does with the traditional variables, where the more commonly
considered polarization is that in where one takes wavefunctionals of the three metric
Ψ(q). In terms of our variables, this polarization would be closer to choosing wave-
functionals of the triad. We see that the use of these new variables leads us to a new
perspective even at this level.

A representation for the Poisson algebra of the canonical variables considered can
be simply achieved by representing the connection as a multiplicative operator and the
triad as a functional derivative,

\[ \hat{A}_a^i \Psi(A) = A_a^i \Psi(A), \]  \tag{40}

\[ \hat{E}_i^a \Psi(A) = \frac{\delta}{\delta A_a^i} \Psi(A). \]  \tag{41}

If we now want to promote the constraint equations to operatorial equations, we
need first to pick a factor ordering. Two factor orderings have been explored, one with
the triads to the right (we will call it II) and one with the triads to the left (I) (see
[19] for alternatives). Let us stress that all these calculations are only formal until a
regularization is introduced.

\(^3\)All these remarks refer to the cases of compact spacetimes. For the asymptotically flat case observ-
ables as the four momentum and angular momentum are well known. Some non-analytic observables
for the compact case may also be written [14]

\(^4\)The reason for this reversal of notation is to keep it in line with the literature [23].
4.2 Factor ordering II and the role of Wilson Loops

If one orders the triads to the right, the constraints become,

\[ \hat{\mathcal{G}}_i = D_a^i \delta \delta A^i_a \]  \hspace{1cm} (42)

\[ \hat{\mathcal{C}}_a = F_{ab}^i \delta \delta A^i_b \] \hspace{1cm} (43)

\[ \hat{\mathcal{H}} = \epsilon^{ijk} F_{ab}^i \delta \delta A^j_a \delta \delta A^k_b \] \hspace{1cm} (44)

An attractive feature of this ordering is that the Gauss Law becomes the infinitesimal generator of \( SU(2) \) gauge transformations for the wavefunctions and the diffeomorphism constraint becomes the infinitesimal generator of diffeomorphisms on the wavefunctions. This, among other features, attracted the attention of Jacobson and Smolin \[20\] to this ordering. There is a potential awkwardness when one considers the algebra of constraints. Remember that it is not a true algebra, but as we discussed, the commutator of two Hamiltonians has a structure “constant” that depends on one of the canonical variables, the triad. This means that in this ordering such “constant” would have to appear to the right of the resulting commutator, which is not expected, and when one constructs solutions one has to check the consistency of the constraints \[20\].

Jacobson and Smolin set out to find solutions to the constraint equations in this formalism. If one starts by considering the Gauss Law, one would like the wavefunctionals to be invariant under \( SU(2) \) gauge transformations. A well known infinite parameter family of gauge invariant functionals of a connection are the Wilson Loops,

\[ W(A, \gamma) = \text{Tr} \left( \text{Pexp} \oint ds \dot{\gamma_a}(s) A_a(\gamma(s)) \right), \] \hspace{1cm} (45)

defined by the trace of the path-ordered exponential of the line integral along a loop \( \gamma \) (parametrized by \( s \)) of the connection. In fact, up to some extent any gauge invariant function of a connection can be expressed as a combination of Wilson loops \[21, 22\].

In view of this, one can consider Wilson loops as an infinite family of wavefunctions in the connection representation parametrized by a loop \( \Psi_\gamma(A) = W(\gamma, A) \) that forms an (overcomplete) basis of solutions to the quantum Gauss Law constraint. So, we have managed to find solutions to the first set of constraints, even perhaps a basis of solutions.

What happens to the diffeomorphism constraint? Evidently Wilson loops are not solutions. When a diffeomorphism acts on a Wilson loop, it gives as a result a Wilson loop with the loop displaced by the diffeomorphism performed. Therefore they are not annihilated by the diffeomorphism constraint and cannot become candidates for physical states of Quantum Gravity. In spite of that, they are worthwhile exploring a bit more. Remember they form an overcomplete basis in terms of which any physical state should be expandable (since any physical state has to be Gauge invariant). We will therefore explore what happens when we act with the Hamiltonian constraint on them. To perform this calculation we only need the formula for the action of a triad on a Wilson Loop,

\[ \hat{E}_i^a(x) \Psi_\gamma(A) = \frac{\delta}{\delta A^i_a(x)} \Psi_\gamma(A) = \oint ds \delta^{\alpha}(x - \gamma(s)) \hat{\gamma}^a(s) \text{Tr}(U(0, s) \tau^i U(s, 1)) \] \hspace{1cm} (46)
where we denote by $U(s_1, s_2) = \int_{s_1}^{s_2} dt \gamma^a(t) A_a(\gamma(t))$ the holonomy from the point $\gamma(s_1)$ to the point $\gamma(s_2)$; $\tau^i$ denotes a Pauli matrix. Note that this expression involves a line integral of a three-dimensional delta function. Using some notational latitude, we can rewrite it as,
\[
\hat{E}_i(x) \Psi_\gamma(A) = \hat{\gamma}^a(x) \text{Tr}(U(0, s(x)) \tau^i U(s(x), 1)).
\] (47)

The notational latitude consists in the fact that we have “cancelled” a three dimensional Dirac Delta with a one dimensional integral (which hides a regularization problem) and we have denoted by $s(x)$ the parameter value $s$ for which the loop is at the point $x$ (one has to be careful with this notation if the loop multiply traverses such a point, as in the case of an intersection). The point for this notational deviation is to make more transparent the following result (which actually goes through even regularizing with some care [20], up to the extent that that is possible in this context!). Let us evaluate the action of the Hamiltonian constraint on a Wilson Loop. Using the previous formulae we get,
\[
\hat{H}(x) \Psi(A) = \epsilon^{ijk} F_{iab}(x) \frac{\delta}{\delta A^a_b(x)} \frac{\delta}{\delta A^b_a(x)} \Psi(A) = F_{ab}^i \hat{\gamma}^a(x) \hat{\gamma}^b(x) \text{Tr}(U(0, s(x)) \tau^i U(s(x), 1))
\] (48)

Notice that in this expression we have an antisymmetric tensor, $F_{ab}^i(x)$, contracted with a symmetric tensor $\hat{\gamma}^a(x) \hat{\gamma}^b(x)$. Therefore, the expression vanishes! We have just proved that a Wilson loop formed with the Ashtekar connection is a solution of the Hamiltonian constraint of Quantum Gravity. This is a remarkable fact. Notice that up to this discovery no solution of this constraint was known in a general case (without making minisuperspace approximations). Historically, this discovery fostered the interest for loops in this context and led to the use of the loop representation.

There are some drawbacks to this result. An obvious one is that although we solved the Hamiltonian constraint and the Gauss Law, we did not solve the diffeomorphism constraint, therefore these wavefunctions are not states of Quantum Gravity. The second point is that for the above result to hold, we need the tensor $\hat{\gamma}^a(x) \hat{\gamma}^b(x)$ to be symmetric. This is true if we have a smooth loop. If the loop has kinks or intersections, this is not any longer true and the Wilson loops stop to be solutions (at this naive level). Why care about loops with intersections? Why not just restrict ourselves to smooth loops? The problem appears when we try to get some sort of understanding of what these wavefunctionals are. The first question that comes to mind (of a prejudiced relativist at least) is what is the metric for such a state. This in principle is a meaningless question, since the metric is not an observable, but let us ask it anyway to see where it leads. The metric acting on one of these states, gives,
\[
\hat{\tilde{q}}^{ab}(x) \Psi_\gamma(A) = \delta \frac{\delta}{\delta A^a_b} \frac{\delta}{\delta A^b_a} \Psi_\gamma(A) = \hat{\gamma}^a(x) \hat{\gamma}^b(x) \Psi_\gamma(A)
\] (49)

So Wilson Loops constructed with smooth loops are eigenstates of the metric operator. First, notice that the metric only has support distributionally along the loop (our sloppy notation may not make this totally transparent, but notice that the quantity $\hat{\gamma}^a(x)$ only can be nonvanishing along the loop). Then, notice that the metric has only one nonvanishing component, the one along the loop. Therefore it is a degenerate metric.
Now, this statement is still meaningless in a diffeomorphism invariant context, but it actually can be given a rigorous meaning with a little elaboration. Consider the quantum operator obtained by computing the (square root of the) determinant of the three metric. In terms of the new variables it is given by $\epsilon^{ijk}\epsilon_{gbc}\tilde{E}_i^a\tilde{E}_j^b\tilde{E}_k^c$. It is very easy to see, as is expected for a degenerate metric, that this operator vanishes when applied to a Wilson Loop. The problem with this is made clear when we consider General Relativity with a cosmological constant. The only thing that changes in the canonical formalism is that the Hamiltonian constraint gains an extra term,

$$H_{\Lambda} = H_0 + \Lambda \det q$$

(50)

where $H_0$ is the vacuum Hamiltonian constraint and $\Lambda$ is the cosmological constant. The extra term is given by the determinant of the three metric. Now consider our Wilson Loop state. Since it is annihilated by the vacuum Hamiltonian constraint and the determinant of the three metric, this means it is a state for an arbitrary value of the cosmological constant! That spells serious trouble. General Relativity with and without a cosmological constant are totally different theories, and one does not expect them to share a common set of states, except for special situations, as for degenerate metrics.

It turns out one can improve the situation a little using intersections. One can find some solutions to the Hamiltonian constraint even for the intersecting case by combining holonomies in such a way that the contributions at the intersection cancel [20, 23, 24]. However, unexpectedly, this is not enough to construct nondegenerate solutions. All the solutions constructed in this fashion, if they satisfy the Hamiltonian constraint, are also annihilated by the determinant of the metric [24]. This, plus the fact that they do not satisfy the diffeomorphism constraint, shows that these solutions are of little physical use in this context. They were, however, very important historically as motivational objects for the study of loops. We will show later on how, when one works in the loop representation, it is possible to generate solutions to all the constraints that, although still based on loops, do not have this degeneracy problem.

4.3 Factor ordering I: The role of the Chern-Simons form

If one orders the constraints with the triads to the left, there is potential for a problem: as we said, apparently in this factor ordering the diffeomorphism constraint fails to generate diffeomorphisms on the wavefunctions. For many of us, this would be a reason to abandon this ordering altogether. However, by considering a very generic regularized calculation one can prove that the diffeo constraint actually generates diffeomorphisms, so this is not a problem [25]. Besides, there is the advantage that when one considers the constraint algebra, one obtains (these are only formal unregulated results) the correct closure [3].

In this ordering, Wilson Loops do not solve the Hamiltonian constraint anymore. However, there is a very interesting and rich solution one can construct. Consider the following state, function of the Chern-Simons form built with the Ashtekar connection,

$$\Psi_\Lambda[A] = \exp(-\frac{6}{\kappa} \int \epsilon^{abc}Tr[A_a \partial_b A_c + \frac{2}{3}A_a A_b A_c])$$

(51)

This functional has the property that the triad (you can view it as an electric field)
equals the magnetic field formed from the Ashtekar connection.

$$\frac{\delta}{\delta A^a_i} \Psi_A[A] = \frac{6}{\Lambda} \eta^{abc} F_{bc}^i \Psi_A[A]$$  \hspace{1cm} (52)

Besides, it is well known that this functional is invariant under (small) gauge transformations and diffeomorphisms. One can check that it is annihilated by the corresponding constraints. What may come as a surprise is that it is actually annihilated by the Hamiltonian constraint with a cosmological constant. This is easy to see, simply consider the constraint,

$$\hat{H} = \epsilon_{ijk} \frac{\delta}{\delta A^a_i} \frac{\delta}{\delta A^b_j} \frac{\delta}{\delta A^c_k} F_{ab}^k - \frac{\Lambda}{6} \epsilon_{ijk} \eta^{abc} \frac{\delta}{\delta A^a_i} \frac{\delta}{\delta A^b_j} \frac{\delta}{\delta A^c_k}$$  \hspace{1cm} (53)

and notice that the rightmost derivative of the determinant of the metric simply reproduces the term on the left when acting on the wavefunction. Notice that the result holds without even considering the action of the other derivatives, and therefore is very robust vis a vis regularization. This result was independently noticed by Ashtekar [26] and Kodama [27]. A nice feature of this result is that the metric is nondegenerate in the sense that we discussed in the previous section. The metric is just given by the trace of the product of two magnetic fields. Such property holds classically for spaces of constant curvature. This has lead some authors to suggest this wavefunction as a “ground state” for a DeSitter geometry [28].

Now, this does not seem a very impressive feat. First of all, it is only one state. Secondly a similar state is present in Yang-Mills theory (this is easy to see, since the Hamiltonian is \(E^2 + B^2\) and adjusting constants one gets for the corresponding state \(E = iB\)) and is known to be nonphysical since it is nonnormalizable. This is true, but it is also true that the nature of a theory defined on a fixed background as Yang Mills theory is expected to be radically different from that of a theory invariant under diffeomorphisms, as General Relativity. Therefore normalizability under the inner product of one theory does not necessarily imply or rule out normalizability under the inner product of the other. It is remarkable that the Chern-Simons form, which is playing such a prominent role in particle physics nowadays, should have such a singular role in General Relativity. It is the only state in the connection representation that we know that may have something to do with a nondegenerate geometry!!

There are more things one could say about the connection representation. There is the compelling work of Ashtekar, Balachandran and Jo [9] concerning the CP violation problem and the partial success (in the linearized theory) of Ashtekar [29] in addressing the issue of time. We do not have space here to make justice to these pieces of work and we refer the reader to the relevant literature.

5 The Loop Representation

5.1 Motivation

A rigorous formulation of a loop representation for a quantum field theory is elaborate and involves many delicate details. However, one can get a very simple intuitive grasp of the idea if one ignores subtleties, and that is exactly what we will do here. We will mention some of the problems, but the reader who wants to get an idea of the subtleties is encouraged to see references [30, 31].
The way to easily view the concept of a Loop Representation is to compare it to the Momentum Representation of ordinary Quantum Mechanics. In the latter, one starts from a wavefunction in the Position Representation $\Psi(\vec{x})$, and convolutes it with an (infinite) basis of functions parametrized by a continuous parameter $\vec{k}$, $\exp(i\vec{k} \cdot \vec{x})$. The result depends on the continuous parameter $\vec{k}$ and we call it a wavefunction in the momentum representation,

$$\Psi(\vec{k}) = \int d^3 x \exp(i\vec{k} \cdot \vec{x})\Psi(\vec{x}).$$  \hspace{1cm} (54)$$

Now, in the Loop Representation, we start from a wavefunctional in the connection representation $\Psi(A)$. We convolute it with an infinite basis of (gauge invariant) wavefunctions parametrized by a continuous parameter, a loop $\gamma$, the Wilson Loops $W_{\gamma}(A)$. The result depends on the parameter $\gamma$ and is a wavefunction in the Loop Representation,

$$\Psi(\gamma) = \int ''dA'' W(A, \gamma)\Psi(A).$$  \hspace{1cm} (55)$$

The reader may have several reservations at this point. First of all, what is the measure to perform this integral? We do not know and we denoted that by the quotation marks. Secondly, up to what extent is this transform complete or how faithfully can it represent the wavefunction space. Again we do not quite know the answer. These technical mathematical issues have been addressed up to some extent in the literature and we will not discuss them here. For us to work, it suffices to notice that this transform works for several model problems, as 2+1 gravity [32], Maxwell theory [33] and Chern-Simons theory [34]. Better yet, it works for Yang-Mills on a lattice [35, 36, 37], which includes the phase space of Gravity in terms of the New Variables. For the case of real nonabelian connections, the transform can be given rigorous meaning [30].

Historically, the first construction of a loop representation for Quantum Gravity based on Ashtekar’s new variables is due to Rovelli and Smolin [38, 39], immediately following the results with Wilson Loops in the connection representation by Jacobson and Smolin [20]. In the context of gauge theories, and even gravity in terms of traditional variables, loop representation had been considered before by Gambini and Trias [41].

Another point that may disquiet the reader is how can one know or at least guess that the quantity of information conveyed in a connection on a three manifold is equal to that present in the set of all possible loops on the manifold. Leaving aside technicalities, it turns out that the loop basis is way overcomplete. The practical inconvenience of this fact can be clearly seen, for example, in investigations done with the loop representation on the lattice. Formulating exactly what are the “free” degrees of freedom in the loop representation is an open and difficult problem [41]. There are lots of hidden identities among states in the loop representation. They can all be derived from three basic identities among Wilson loops:

- It is easy to see that reversing the orientation of the loop leaves the Wilson loop invariant. Therefore wavefunctions in the Loop Representation are invariant under reversal of orientation of the loop $\Psi(\gamma) = \Psi(\gamma^{-1})$.

- Due to the fact that the Wilson Loop is the trace of a holonomy and that the traces are cyclic, $\Psi(\gamma_1 \circ \gamma_2) = \Psi(\gamma_2 \circ \gamma_1)$. 

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There is a nontrivial identity satisfied between traces of $SU(2)$ matrices (other groups have similar identities, although there are different from the specific one for $SU(2)$), $\text{Tr}(A)\text{Tr}(B) = \text{Tr}(A \cdot B) + \text{Tr}(A \cdot B^{-1})$, where $A, B$ are $SU(2)$ matrices. This is usually called the Mandelstam identity. In terms of the Loop Representation, this means we can express any wavefunctional of two loops $\Psi(\gamma_1, \gamma_2)$ in terms of wavefunctionals of one loop by $\Psi(\gamma_1, \gamma_2) = \Psi(\gamma_1 \circ \gamma_2) + \Psi(\gamma_1 \circ \gamma_2^{-1})$. From now on therefore, we need only concentrate on wavefunctionals of a single loop, the multiple loop cases always being reducible to the single loop instance.\footnote{The Mandelstam identity obviously holds only for a pair of intersecting loops $\gamma_1, \gamma_2$. We are considering loops that share a common basepoint, that is why we can always reduce a wavefunction to that of one loop.}

Unfortunately, the story does not end here. Many nontrivial combinations of these identities are possible, for example the following identity holds for three loops,

$$\Psi(\gamma_1 \circ \gamma_2 \circ \gamma_3) + \Psi(\gamma_1 \circ \gamma_2 \circ \gamma_3^{-1}) = \Psi(\gamma_2 \circ \gamma_1 \circ \gamma_3) + \Psi(\gamma_2 \circ \gamma_1 \circ \gamma_3^{-1})$$

(56)

(and usually it takes some time to figure out exactly how to derive it from the above identities). The situation only gets worse for higher numbers of loops. An important point, however, is that any wavefunction on loop space that one wants to define (any wavefunction must have some value for a given number of loops) should be consistent with these identities.

There is a very important aspect of the Loop Representation for Quantum Gravity. Since the theory is invariant under diffeomorphisms, this means that the wavefunctionals will be invariant under deformations of the loops. This means the wavefunctionals will be what in the mathematical literature is known as knot invariants\cite{38}. Knot theory is the branch of mathematics that studies properties of knot invariants and it has enjoyed a great deal of activity recently. It is really exciting that this new flourishing branch of mathematics seems to have something to do with Quantum Gravity, since it opens the opportunity for new insights into the field. We will return to these issues at length in the next chapter.

5.2 A more rigorous approach

We said that the introduction of a loop representation via a transform is only a motivational approach since one actually does not know how to perform the functional integral appearing in the transform. It turns out there is a more rigorous way of introducing a loop representation, and this is by quantizing a noncanonical algebra of loop-dependent quantities. This approach was followed for gravity by Rovelli and Smolin in their original article\cite{38}. We will not discuss it in great detail here for reasons of space. The main idea is the following:

- Define a set of classical quantities in the phase space of General Relativity based on loops, called generically T variables. $T^a_0[A]$ is the Wilson Loop, $T^1$ is a vector density obtained by inserting in the expression of the Wilson loop a triad at a given point $T^a(x), T^2$ is a two-vector density obtained by inserting in a Wilson loop triads at two points and so on for higher order T’s. It is possible, by shrinking the loops to points, to represent any classical
quantity in terms of these variables. In particular, one can write the constraints. These quantities close a noncanonical algebra under Poisson Brackets.

- Promote the T variables to a set of operators acting on a space of wavefunctions. The action is obtained by mimicking the action under Poisson brackets of the classical T variables with the Wilson loop. For example, $\hat{T}^0_\gamma \Psi(\eta) = \Psi(\gamma \circ \eta)$ and so on. The quantum operators close an algebra under commutators that reproduces, in the limit $\hbar \to 0$ the classical T algebra.

- Promote any quantity one is interested in (for example the constraints) to quantum operators simply by writing its classical expression in terms of the T variables and then promoting them to operators by the rules given above.

There are subtleties in the definition of the quantum algebra of T variables that have led to the consideration of strips instead of loops to avoid some regularization issues. See Rovelli \cite{42} for details.

### 5.3 Differential operators in loop space

If one accepts that a representation of Quantum Gravity exists in which wavefunctions are functionals of loops, the next question is how to represent the constraint equations. To this end we will introduce a differential operator in Loop Space, the loop derivative. Given a functional of a loop $\Psi(\gamma)$, the loop derivative is defined by,

$$\Psi(\gamma \circ \delta \gamma) = (1 + \sigma^{ab} \Delta_{ab}(P)) \Psi(\gamma)$$

where $\delta \gamma$ is an infinitesimal loop added at the end of the path P as indicated in the figure\textsuperscript{2}. $\sigma^{ab}$ is the area of the infinitesimal loop. The prescription to compute it is, take the wavefunction evaluated on a loop constructing by appending to the original loop an infinitesimal loop $\delta \gamma$ at the end of a path P starting from the basepoint; subtract the wavefunction evaluated on the original loop, and divide the result by the area element $\sigma^{ab}$. The result is the loop derivative of the wavefunction. The loop derivative is present in the work of Mandelstam \textsuperscript{13}, Polyakov \textsuperscript{14} and Makeenko and Migdal \textsuperscript{15}. The work of Gambini and Trias really brought to the forefront the role of this operator in connection with gauge theories \textsuperscript{16}.

![Figure 2: The infinitesimal loop that gives rise to the Loop Derivative](image)

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A fundamental property that can be immediately checked applying the above prescription is,

\[
\Delta_{ab}(\gamma_0)W(\gamma, A) = Tr(U(0, s(x))F_{ab}(x)U(s(x), 1)) = F_{ab}^i Tr(U(0, s(x))\tau_i U(s(x), 1)).
\] (58)

As usual, one should be careful if such a point lies at an intersection. However, in most practical situations, intersections are regulated and derivatives act away from them and are evaluated at the intersection only in the limit.

5.4 The diffeomorphism constraint

In terms of the loop derivative it is straightforward to write the generator of infinitesimal diffeomorphisms. Although we could proceed from geometric considerations, let us derive it from the New Variable formulation. Let us consider the generator of diffeomorphisms acting on both sides of the transform (55),

\[
\hat{C}(\vec{N})\Psi(\gamma) = \int dA W(\gamma, A) \int d^3x N^a(x) \delta \frac{\delta}{\delta A^i_b(x)} F_{ab}^i(x)\Psi(A) \] (59)

we now integrate by parts and apply the constraint on the Wilson Loop,

\[
\hat{C}(\vec{N})\Psi(\gamma) = \int dA \int d^3x N^a(x) F_{ab}^i(x) \delta \frac{\delta}{\delta A^i_b(x)} W(\gamma, A)\Psi(A) \] (60)

We now functionally differentiate the Wilson Loop and use the equations (46) and (58),

\[
F_{ab}^i(x) \delta \frac{\delta}{\delta A^i_b(x)} W(\gamma, A) = F_{ab}^i(x) \epsilon^b(s(x)) Tr(U(0, s(x))\tau U(s(x), 1)) = \epsilon^b(s(x)) Tr(F_{ab}(x)U(x, x)) = \epsilon^b(x) \Delta_{ab}(x) W(\gamma, A) \] (61)

Therefore we can now replace this in the expression of the constraint to find,

\[
\hat{C}(\vec{N}) = \int d^3x N^a(x) \int ds \delta(x - \gamma(s)) \epsilon^b(s) \Delta_{ab}(x) \] (63)

Whenever we drop the dependence in the path of the loop derivative, we assume the path goes from the basepoint of the loop to the point of interest. The expression we arrived to is known to be the generator of infinitesimal deformations of the loops. What we see above is what we anticipated at the end of the last subsection, that the wavefunctions will have to be invariant under smooth deformations of the loops, they have to be knot invariants.

An important point if we are really to consider the above expression as a diffeomorphism constraint is if it reproduces the constraint algebra of diffeomorphisms. The calculation is possible, though we will not perform it here for reasons of space. The interested reader can consult [40, 46].

5.5 The Hamiltonian constraint

We can now use the same kind of reasoning to determine the action of the Hamiltonian constraint. Naturally, it will be more complicated. Again for reasons of space we will not...
perform a derivation here. Careful derivations can be found in [47, 48]. The resulting expression is,

$$\hat{H}(x)\Psi(\gamma) = \oint ds \oint dt \delta(x - \gamma(s))f_\epsilon(\gamma(s), \gamma(t)) \times$$

$$\times \dot{\gamma}^a(s)\dot{\gamma}^b(t)\left(\Delta_{ab}(s)\Psi(\gamma_{st} \circ \gamma_{st0}) + \Delta_{ab}(s)\Psi(\gamma_{ts} \circ \gamma_{ts0})\right)$$  \hspace{1cm} (64) (65)

This expression requires some explanation. Conceptually it can be viewed as “$\dot{\gamma}^a \dot{\gamma}^b \Delta_{ab}$”. However, several details should be taken into account. First of all, the expression is regulated with a regulator satisfying $f_\epsilon(x, y) \rightarrow \delta(x - y)$ when $\epsilon \rightarrow 0$. This means the two tangent vectors effectively are evaluated at the same point. Since they are contracted with the loop derivative, which is antisymmetric, this means that the only possibility for the constraint to be nonvanishing is if it acts at an intersection. This fact we already encountered in the connection representation. The parameter values $s$ and $t$ can therefore be thought as slightly displaced from an intersection (which lies at $x$), but tend to it as we remove the regulator $\epsilon \rightarrow 0$. There is a rerouting in the loops, indicated in the wavefunctions. For instance, this should be read in the following way: $\Psi(\gamma_{st} \circ \gamma_{st0})$ should be understood as the loop composed by traversing the original loop from $s$ to $t$ first and then from $t$ to $s$, but going through the basepoint $0$. This is depicted in the figure 3. The reader may find this expression technically difficult. However, it should be kept in mind that this expression embodies all the dynamical information of the General Theory of Relativity at a quantum level in loop space. When viewed in this way the above expression appears as remarkably simple!

At this point the reader may be puzzled about the kind of derivations we have performed for the constraints. We have made use of the loop transform (55) as if it were an actually well defined expression, instead of the motivational tool we claimed it to be in the introduction. There actually is a way of deriving the form of the constraints in the loop representation without making use at all of the transform (55). One takes the second viewpoint we described in connection with the construction of the loop representation in section 5.2: its elaboration based on a quantization of a noncanonical set of loop-dependent classical quantities (the Rovelli-Smolin T operators). The idea is to express the constraints classically in terms of the T’s. Then, since one has a well defined way of promoting the T operators to quantum operators on a space of loop functionals, one has also a prescription for promoting the constraints to differential operators in loop

Figure 3: The loops $\gamma$, $\gamma_{st} \circ \gamma_{st0}$ and $\gamma_{ts} \circ \gamma_{ts0}$.
space. It is quite reassuring that proceeding in this way and proceeding via the transform as we detailed above, actually leads to the same results for both the Hamiltonian and diffeomorphism constraints [48].

5.6 Coordinates on loop space

We have established analytic expressions for the constraints of Quantum Gravity in the Loop Representation. We now develop technology that will allow us to write wavefunctions. We will discuss specifically the construction of wavefunctions in the next chapter. Here we will just set up the framework that will be used.

Let us return for a minute to the connection representation. We are interested in wavefunctions that are SU(2) invariant. Modulo technicalities, all such functions can be expressed as combinations of Wilson Loops [79, 22]. Therefore we can just concentrate on these latter ones. Let us write the expression for a Wilson Loop explicitly,

\[ W(\gamma, A) = 2 + \sum_{i=1}^{\infty} \text{Tr} \left[ \oint ds_1 \int s_1^{s_2} \cdots \int s_{n-1}^{s_n} ds_n \gamma^{a_1}(s_1) \cdots \gamma^{a_n}(s_n) A_{a_1}(\gamma(s_1)) \cdots A_{a_n}(\gamma(s_n)) \right] \] (66)

We now rearrange this expression in the following way,

\[ W(\gamma, A) = 2 + \sum_{i=1}^{\infty} \int d^3x_1 \cdots \int d^3x_n \text{Tr}(A_{a_1}(x_1) \cdots A_{a_n}(x_n)) \times \] (67)

\[ \times \oint ds_1 \cdots \oint ds_n \Theta(s_1, ..., s_n) \delta^3(x_1 - \gamma(s_1)) \cdots \delta^3(x_n - \gamma(s_n)) \gamma^{a_1}(s_1) \cdots \gamma^{a_n}(s_n) \] (68)

where \( \Theta(s_1, ..., s_n) = 1 \) if \( s_1 < ... < s_n \) is a generalized Heaviside Function. The purpose of this rearrangement is to separate the contributions of the loop and the connection to the Wilson Loop. This allows us to write,

\[ W(\gamma, A) = 2 + \sum_{n=1}^{\infty} \text{Tr}(A_{a_1}x_1 \cdots A_{a_n}x_n) X^{a_1x_1 \cdots a_nx_n}(\gamma) \] (69)

where,

\[ X^{a_1x_1 \cdots a_nx_n}(\gamma) = \oint ds_1 \cdots \oint ds_n \gamma^{a_1}(s_1) \cdots \gamma^{a_n}(s_n) \times \] (70)

\[ \times \Theta(s_1, ..., s_n) \delta^3(x_1 - \gamma(s_1)) \cdots \delta^3(x_n - \gamma(s_n)) \]

where we have assumed a “generalized Einstein convention” meaning repeated \( x_i \) coordinates are integrated over and we treat them as indices. We have isolated all the loop dependence in the quantities \( X \). These quantities behave like multi vector densities (really they are distributions) on the three manifold at the points \( x_i \). They can be viewed as vector densities that have support along the loops associated with the value of the tangent vector to the loop at the point of interest.

The whole point of this construction is that these quantities embody all the information of the loops that is needed to build any wavefunction of interest. This transcends the connection representation and means we can use them in the loop representation. If we are to write functions of the \( X \)'s as candidates for physical states of gravity, we will need to study the action of the constraints on the \( X \)'s. We do not have space here to
give a detailed account of this, but we will work out an example just to show how this works. The reader should be able to generalize to the other needed cases.

Let us consider the multitangent of order one, $X^a y^b$, and let us study its loop derivative. We start by the definition (57), appending an infinitesimal parallelogram loop (at the point $z$) with edges $d u^a$, $d v^a$ to the definition of the $X$ of order one (70),

$$X^a y^b (\gamma \circ \delta \gamma _z) = \int _0 ^z ds \dot {\gamma} ^a (s) \delta (\gamma (s) - x) + d u ^a \delta (z - x) + d v ^a \delta (z + d u - x) -
-d u ^a \delta (z + d u + d v - x) - d v ^a \delta (z + d v - x) + \int _z ^1 ds \dot {\gamma} ^a (s) \delta (\gamma (s) - x)$$

where we now expand to first order assuming $d u$ and $d v$ are infinitesimal (for instance, $\delta (z + d u - x) = \delta (z - x) + d u ^a \partial _a \delta (z - x)$, and we also recombine the first and last terms to give a loop integral and get,

$$X^{a x} (\gamma \circ \delta \gamma _z) = X^{a x} (\gamma) + (d u ^a d v ^b - d v ^a d u ^b) \partial _b \delta (x - z).$$

We now read off from the definition of the loop derivative (57), taking into account that the area element of the loop in question is given by $d \sigma ^{a b} = d u ^a d v ^b - d v ^a d u ^b$,

$$\Delta _{e b} (z) X^{a x} (\gamma) = \delta ^a _c \partial _d \delta (x - z)$$

Similar computations can be carried along for the higher order $X$’s, for instance for a second order one,

$$\Delta _{c d} (z) X^{a x b y} = X^{b y} \delta ^a _c \partial _d \delta (x - z) + X^{a x} \delta ^b _d \partial _c \delta (y - z) + \delta ^a _c \delta ^b _d \delta (x - z) \delta (x - y)$$

Generic formulae for an $X$ of any order are given in [49].

The reader may be surprised by the fact that we use the word “coordinates” to describe the multitangents. A “coordinate” should be an object that one prescribes freely. How do we know we can freely prescribe the $X$’s to any order? Actually we cannot. The $X$’s satisfy a series of identities, both algebraic and differential. Examples of these identities are,

$$\partial _a X^{a x} = 0$$

$$\partial _a X^{a x b y} = \delta (x - y) X^{b y}$$

$$X^{a x b y} + X^{b y a x} = X^{a x} X^{b y}$$

The algebraic identities, like the last one, stem from identities of the generalized Heaviside Function. The differential identities ensure that the resulting Wilson loop is gauge invariant. Therefore the $X$’s are not coordinates, since they cannot be freely specified.

A remarkable fact, however, is that one can actually solve the aforementioned identities for the “freely specifiable part” of the loop coordinates (in reference [49], they are referred to as $Y$’s). These objects really work as coordinates in loop space. The importance of this last fact cannot be overstressed. Unfortunately, for reasons of space we are unable to present a thorough account of this construction here. The reader is referred to the literature for more details [49]. We will keep on to loosely refer to the $X$’s as loop coordinates.

Another important point connected with the loop coordinates is that they suggest a way to obtain a quantum representation that goes beyond the loop representation.
Consider a generic multivector density on a three manifold, X, not necessarily associated with any particular loop. If this quantity satisfies the family of identities mentioned above, one could use it in expression (69) and obtain as a result, not a Wilson loop anymore, but a gauge invariant function of the connection parametrized by the vector density given. This would allow, for example, to use completely smooth vector densities, that have potential for removing several of the regularization difficulties associated with the loop representation. We call the resulting representation, where wavefunctions are functionals of the coordinates,

\[ \Psi(X^1, X^2, \ldots), \]

“coordinate representation” (some authors call it “form factor representation”). At least for the Maxwell case it has been proven that this provides a viable quantum representation [33]. For the nonabelian case, work is in progress to assess the feasibility of this representation.

6 Knot Theory

6.1 Knots

Knot theory studies the properties of knots in three dimensions that are invariant under smooth deformations of the knots (diffeomorphisms). It was largely set up by the attempts in the last century by P. G. Tait [50] and others to explain the properties of atoms using knotted vortices of “aether” (this was before special Relativity or Quantum Mechanics were invented. A discussion of several appealing aspects of this theory of atomic spectra can be found in the book by Atiyah [51]). We will use the term “knot” loosely to refer either to a single knotted curve or to several curves linked and/or knotted. A central issue in knot theory is to distinguish and classify inequivalent knots. A useful tool for accomplishing this is the use of knot invariants, that is, numbers associated with knots which are invariant under deformations of the knots. At the moment, however, there is no “canonical” or definite set of invariants that would allow us to completely classify knots. The simplest example of such an invariant is the linking number, first considered by Gauss. Consider two curves as in figure 4, the linking number is one if they are linked (4a) and zero if they are not linked (4b). It is evidently (at least for these cases) invariant under diffeomorphisms. However, how does one compute it in general,

Figure 4: The knots in (b) and (c) (The Whitehead Link) are not linked
say for the curves in $4c$. There is a procedure for these cases. One should traverse one of the curves and add $1/2$ for every crossing of the type shown in figure $5a$ and $-1/2$ for the type $5b$. This in particular shows that the curves in figure $4c$ have vanishing linking number in spite of being obviously linked (this is called the Whitehead link). This clearly shows that we will need other, more complicated, invariants to be able to classify linkings.

It should be apparent to the reader that these concepts are both elegant and clearly related to the issues of Quantum Gravity we discussed in the previous chapter. However, they do not seem quite geared to a direct application. Say we wanted to consider the Gauss linking number as a candidate for a wavefunction of the gravitational field in the loop representation (this actually fails since it is not consistent with the Mandelstam identity). We meet an important requirement in the fact that it is diffeomorphism invariant. But in the form we have casted it up to now, we cannot compute much further. For instance, we do not know how to take an area derivative of this quantity.

It would be convenient to have an analytic expression for the knot invariants. For the Gauss Linking number there actually is one. It is easy to see (for a demonstration see Maxwell’s 1873 Treatise on Electricity and Magnetism! Volume II page 419-423) that the following expression,

$$GL(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint_{\gamma_1} ds \oint_{\gamma_2} dt \frac{\gamma_1^a(s)\gamma_2^b(t)\epsilon_{abc}(\gamma_1^c(s) - \gamma_2^c(t))}{|\gamma_1(s) - \gamma_2(t)|^3}$$

(79)

gives an analytic expression for the Linking number. A very exciting fact of the recent work on Chern-Simons theories is that it has provided similar analytic expressions for several other knot invariants.

We can actually write the above invariant in terms of the loop coordinates we introduced in the last chapter. The expression is,

$$GL(\gamma_1, \gamma_2) = \frac{1}{4\pi} X^{ax}(\gamma_1) X^{by}(\gamma_2) \epsilon_{abc} \frac{(x^c - y^c)}{|x - y|^3}$$

(80)

where we again assume that repeated $x, y$ indices are integrated over the whole manifold. In spite of its appearance (it involves a local chart of coordinates and even a distance!) this expression is actually diffeomorphism invariant. The fact that we can express it in terms of the loop coordinates enormously facilitates the application of the constraints to these expressions. For instance, one can easily prove that the diffeomorphism constraint
actually annihilates $\text{GL}(\gamma_1, \gamma_2)$, in accordance with the fact that this expression is diffeomorphism invariant. (We will see soon what happens when one applies the Hamiltonian constraint).

There are many other things to be said about knot theory. We will stop here for reasons of space. The reader is encouraged to enjoy the books by Lou Kauffman \[52, 53\] on the subject for many other amusing and important properties of knots.

### 6.2 Knot Polynomials

A very important step towards the construction of knot invariants, and indirectly towards the classification problem, has been the invention of the knot polynomials. These are polynomials in an arbitrary variable, usually called $t$, uniquely associated with a given knot. For each knot there is a given finite order polynomial, although the order may be different for another knot. The important point is that the polynomials are invariant under diffeomorphisms, therefore each coefficient is a knot invariant. Examples of such polynomials are those of Alexander-Conway, The Kauffman Bracket, Jones and HOMFLY. The work on Chern-Simons theories has given rise to even newer polynomials \[54\] and to previously unknown relationships among the known ones.

Knot polynomials are usually prescribed by a set of implicit relations, known as skein relations. These relations are enough to find out the particular polynomial associated with a given knot (sometimes the task is not totally trivial!). Let us see an example. Consider the Conway Polynomial $C(\gamma)[t]$, defined by the Skein Relation that appears in figure 6.

This relation should be interpreted in the following way: for a given knot, project it on a plane and focus on a single crossing. Cut out the crossing and leave four incoming threads. The value of the polynomial for the knot with the crossing drawn glued into the threads left out is related to that of the polynomial with the other crossings drawn via the skein relation. This plus the fact that the polynomial on the unknot is normalized to one is enough for computing the polynomial for an arbitrary knot.

Let us work out an example, evaluating the Conway Polynomial for the trefoil knot given in figure 7. We focus on the crossing encircled by the dotted line and apply the Skein Relation of figure 6. We see in figure 8 that it relates the value of the polynomial evaluated on the knot of interest (our final result) to the value of the polynomial for two other knots. For the left one, one can immediately see that since the knot is homotopic to the unknot, the value of the polynomial on it is 1. For the link on the right we need to do some more work. We again apply the skein relation focusing on the encircled crossing, to evaluate the value of the polynomial on the link of interest. Again we see in figure 8 that one of the polynomials is one and the other is evaluated on two unlinked
Figure 7: The Trefoil knot

Figure 8: Evaluation of the Conway Polynomial for the trefoil knot
curves. Applying again the skein relation we can see that the value of the polynomial on such a link is vanishing. Therefore, substituting back, one sees that the value of the Conway Polynomial for the knot of interest is $1 + t^2$.

So we see that the innocent looking skein relations actually contain all the information needed to associate a series of knot invariants to a knot (the reader can verify that by deforming the knot in question and applying the skein relations, one gets the same result).

Again, having a knot polynomial written as a skein relation is not very useful for our purposes. We would like to have a more analytic kind of expression for the knot polynomials to, say, apply the Hamiltonian constraint of Quantum Gravity to it and see if it is a quantum state of the gravitational field. We will see in the following subsections that Chern-Simons theory has been extremely useful to find such analytic expressions for the knot polynomials.

6.3 Chern-Simons theory and knot polynomials

As the reader must have noticed from previous sections, Chern-Simons theories seem to play a crucial role concerning Knots. Dozens of articles have been written studying various aspects of these theories and this brief subsection can certainly not be anything else but a gross oversimplification. The reader interested in gaining a good understanding is encouraged to explore the relevant papers.

In a nutshell, Chern-Simons theories are gauge theories defined in 2+1 dimensions having as action,

$$S_{CS} = k \int d^3x \, \text{Tr}(A_a \partial_b A_c + \frac{2}{3} A_a A_b A_c) \epsilon^{abc}.$$  \hspace{1cm} (81)

where the connection $A$ takes value on some group, let us fix for our interests $SU(2)$. $k$ is the coupling constant of the theory. Notice that no use of a spacetime metric was needed to write this action (as one needs, for instance to write the Yang Mills action when one raises or lowers indexes in $F_{ab} F^{ab}$). This is therefore a topological field theory (although what is meant by this is quite context dependent). It is invariant under diffeomorphisms. The classical equations of motion for these theories simply state that the theory is $SU(2)$ invariant and that the connection is flat ($F_{ab}$ constructed out of $A_a$ is zero). Wilson Loops are actually observables in the theory. Since the connection is flat, when one deforms the loop with a diffeomorphism, the value of the Wilson loop does not change. Notice that this does not happen for gravity.

Since the Wilson loop is an observable, one can ask what is its quantum expectation value. In the language of Path Integrals,

$$\langle W(\gamma) \rangle = \int dA \, e^{i S_{CS}} \, W(A, \gamma)$$  \hspace{1cm} (82)

Now this expectation value should be invariant under diffeomorphisms, i.e. it should be a knot invariant, parametrized by the coupling constant $k$. In the following subsubsections we will explore in some detail what sort of knot invariant this quantity is. We will first derive a skein relation for it and then, taking advantage of the fact that perturbation theory can be used to evaluate (82) (remember this is Chern-Simons theory, not Quantum Gravity!) we will present an analytic expression for the knot invariant.
6.3.1 A skein relation for \( < W(\gamma) > \)

It turns out that one can prove that \( < W(\gamma) > \) satisfies the Skein relations of the knot polynomial known as Kauffman Bracket, which is intimately related to the Jones Polynomial. This was an important discovery due to Witten [55]. Witten obtained his result nonperturbatively by recurring to conformal field theories. It turns out that the result can be obtained also in a perturbative fashion with a more modest machinery [56, 57]. The result also holds when the loops have intersections [24]. We here sketch part of the perturbative proof in order to give the reader a flavor of these calculations and also to illustrate the usefulness in this context of some of the techniques we introduced in section 5.

In order to establish a skein relation for the expectation value of the Wilson loop we basically need to relate an under and an upper crossing. This can be done since, as illustrated in figure 9, one can obtain an upper (or under, depending on the orientation of the small loop) crossing by adding a small loop at a given point of the loop. It turns out that we have already developed a technique for evaluating the change in a function of a loop when one adds a small loop: it is the loop derivative. Therefore what we basically need to compute is the loop derivative of \( < W(\gamma) > \).

The change in \( < W(\gamma) > \) due to the addition of a small loop of area \( \sigma^{ab} \) is given by formula (57). When combined with formula (58), it gives rise to,

\[
\sigma^{ab} \Delta_{ab}(x) \Psi[\gamma] = \int dA \sigma^{ab} F^{ab}_i(x) \text{Tr}[\tau^i U(\gamma^x_x)] \exp(S_{\text{CS}}).
\]

Using the relation (52), integrating by parts, and applying (47) one obtains:

\[
2k \int dA \sigma^{ab} \epsilon_{abc} \int dy^c \delta(x - y) \text{Tr}[\tau^i U(\gamma^y_y) \tau^i U(\gamma^x_x)] \exp(S_{\text{CS}})
\]

The integral depends on the volume factor

\[
\sigma^{ab} \epsilon_{abc} dy^c \delta(x - y)
\]

which depending on the relative orientation of the two-surface \( \sigma^{ab} \) and the differential \( dy^c \) (which is tangent to \( \gamma \)), can lead to \( \pm 1 \) or zero. (This expression should really be regularized. We have absorbed appropriate extra factors in the definition of the coupling

Figure 9: Under and upper crossings created by adding an oriented small loop

\[\text{(83)}\]

\[\text{(84)}\]

\[\text{(85)}\]
c constant so to normalize the volume to \( \pm 1 \). Consequently, depending on the value of the volume there are three possibilities:

\[
\begin{align*}
\delta \Psi[\gamma] &= 0 \quad \text{(86)} \\
\delta \Psi[\gamma] &= \pm 2k \Psi[\gamma] \quad \text{(87)}
\end{align*}
\]

These equations can be diagrammatically interpreted in the following way:

\[
\Psi[\hat{L}_x] - \Psi[\hat{L}_0] = \pm 2k \Psi[\hat{L}_0] \quad \text{(88)}
\]

and coincide with part of the skein relations of a known knot polynomial, the Kauffman Bracket \([52, 53]\). So we see that \( \langle W(\gamma) \rangle \) is actually an analytic expression for the Kauffman Bracket in the variable \( k \).

It is interesting to notice how helpful, in order to perform this calculations, were the notion of an area derivative and of its properties. The original derivations \([57]\) did not use these concepts (although the treatment is fully equivalent) and therefore the proof we give here is much more economical. This is a good example where techniques developed for gravity are of use in a particle physics problem. We will see more of this happening in the next subsection.

The reader may be confused by figure 9. In the past we have considered these kinds of “curls” in knots as removable. We will see the meaning of this in the next section.

### 6.3.2 Perturbative calculation of the Kauffman Bracket

Being Chern-Simons theory a renormalizable theory, one could compute the expectation value of the Wilson Loop perturbatively. One gets as a result a polynomial in the variable \( k \) (the coupling constant of the theory), which should provide analytic expressions for the coefficients of the Kauffman Bracket. In this language therefore a coefficient of the Kauffman Bracket becomes a sum of Feynmann diagrams for the perturbative expansion of \( \langle W(\gamma) \rangle \). We will sketch here the derivation. The reader should be aware that the proofs given here are very schematic. They ignore, for instance, the presence of ghosts (it can actually be seen that ghosts do not contribute to the order of perturbation we are going to discuss). The complete treatment can be seen in \([58]\) and a rigorous mathematical derivation in \([59]\).

In terms of the expression of the Wilson loop written as a function of the Loop Coordinates \((69)\), we can write for the expectation value,

\[
\langle W(\gamma) \rangle = 2 + \langle A_{ax} A_{by} \rangle X^{ax by}(\gamma) + \langle A_{ax} A_{by} A_{cz} \rangle X^{ax by cz}(\gamma) + \langle A_{ax} A_{by} A_{cz} A_{dw} \rangle X^{ax by cz dw}(\gamma) + \ldots \quad \text{(89)}
\]

which corresponds to a diagrammatic expansion given in figure 10, where we have represented the propagator by a wavy line and the loop coordinates of nth order as a circle with n insertions of propagators.

The Chern-Simons theory has a propagator given by,

\[
g_{ax by} = k \, \epsilon_{abc} \frac{(x - y)^c}{|x - y|^3} \quad \text{(90)}
\]
Figure 10: Diagrammatic expansion of the Wilson Loop in a Chern-Simons theory and a vertex (contracted with three propagators),

\[ h_{axbycz} = k^2 \int d^3w \epsilon^{def} g_{axdw} g_{byew} g_{czfw} \]  

(91)

One simply has to contract them with the propagators and vertices to get the expression for the Feynmann diagram.

The Feynmann diagram of first order in \( k \) is therefore simply given by,

\[ < W(A, \gamma) >^{(1)} = X^{axby} g_{axby} \]  

(92)

If we expand this out, remembering the expressions for \( X \) and \( g \) we get,

\[ < W(A, \gamma) >^{(1)} = \oint ds \oint dt \dot{\gamma}^a(s) \dot{\gamma}^b(t) \epsilon_{abc} \frac{(\gamma^c(s) - \gamma^c(t))}{|\gamma(s) - \gamma(t)|^3} = \text{GSL}(\gamma) \]  

(93)

The reader may recognize in this expression the Gauss Linking number, except for the fact that instead of having two knots, we have only one. We are actually computing the linking of the knot with itself! This sometimes is called “Gauss self-linking number” and we denote it as \( \text{GSL}(\gamma) \). This presents a small difficulty. The expression seems to be ill defined when \( s = t \) since the denominator vanishes. This is actually not true since the numerator also vanishes, and faster. But there is a problem with this expression, if one computes it carefully, one finds out it not only depends on the loop but on the definition of an arbitrary normal vector to the loop [60, 55, 57]. This means that in the calculation diffeomorphism invariance has been broken (mildly). A simple solution for it is to “frame” the loop, i.e. convert it to a ribbon and compute the self-linking number as the linking number of the two loops on the sides of the ribbon. This however is not a univocous prescription since one can add twists to the ribbon and change its value. Moreover, as we see in the figure [11], two equivalent loops may yield inequivalent ribbons, so diffeomorphism invariance in the loop sense is lost when one generalizes to ribbons. We will see how one can deal with this problem of the loss of diffeomorphism invariance later on.

The order \( k^2 \) Feynmann diagram is the sum of two terms. These terms can be rearranged into three contributions. One of them is the square of the self-linking number. The other two terms together give an analytic representation for another well known knot invariant. Details can be seen in reference [57].

\[ < W(\gamma) >^{(2)} = k^2 (\text{GSL}(\gamma)^2 + A_2(\gamma)) \]  

(94)
where $A_2$ is related to the second coefficient of the Conway Polynomial (it is actually $(A_2 + \frac{1}{12})$) and is also related to the second coefficient in an expansion of the Jones Polynomial. From now on we will loosely refer to it as “the second coefficient of the Jones Polynomial”. Notice that this invariant is truly diffeomorphism invariant, i.e. it is framing-independent.

This construction goes on at higher orders, each coefficient of the Kauffman Bracket breaks up into a framing dependent portion function of the coefficients of lower order plus a new knot invariant, which is framing independent. The third order expression in $k$ is

$$< W(\gamma) > ^{(3)} = k^3 (\text{GSL}(\gamma)^3 + \text{GSL}(\gamma) A_2(\gamma) + A_3(\gamma))$$

(95)

where $A_3(\gamma)$ is another framing independent knot invariant related to the third coefficient in the expansion of the Jones Polynomial. We do not have space to discuss it here, but crucial in the recombination of the terms to forming expressions of the type (95) is the use of the free part of the loop coordinates, discussed in section 5.5. If one uses the free parts of the loop coordinates it is immediate to find expressions for the framing independent knot invariants that appear at each order. Although this decomposition can be done “by hand” (and that was the way it was done in ref. [62]), it is much more economical to perform it using the loop coordinates. At order three and higher, the use of loop coordinates is almost mandatory due to the complexity of the expressions involved, and actually the first calculation of the third order terms was done in that way. The reader is referred to references [49, 61] for more details on the use of loop coordinates to generate the expansions discussed.

Therefore, as a consequence of this analysis we have the following analytic expression for the Kauffman Bracket,

$$\text{Kauffman Bracket}(\gamma)[k] = 1(\gamma) + \text{GSL}(\gamma)k + (\text{GSL}(\gamma)^2 + A_2(\gamma))k^2 +$$

$$+(\text{GSL}(\gamma)^3 + \text{GSL}(\gamma) A_2(\gamma) + A_3(\gamma))k^3 + ...$$

(96)

What can we do about the framing problem? Clearly we cannot use the Kauffman Bracket as a candidate for a state of gravity because it is not invariant under deformations of the loops (remember figure 11). However, we notice that “buried” inside each coefficient of the Kauffman Bracket is present a framing independent knot invariant (a
quantity that is really invariant under deformations of the loops). What we are seeing is the perturbative emergence of the exact relation, known to mathematicians,

\[ \text{Kauffman Bracket}(\gamma)[k] = e^{(k \text{GSL}(\gamma))} \text{Jones Polynomial}(\gamma)[k] \] (97)

so we see that all the framing dependence can be concentrated in the “phase factor” \( \exp(k \text{GSL}(\gamma)) \). We will see that this result allows us to construct real diffeomorphism invariant states of gravity in the next section.

7 Knot theory and quantum states of gravity

The reader may seem surprised by the rather lengthy detour into Chern-Simons theory in the last section. The reason for this will become apparent here.

In section 4.3 we saw that there existed an exact solution to all the constraints of Quantum Gravity in the connection representation (with cosmological constant) given by,

\[ \Psi^{CS}_\Lambda(A) = \exp(-\frac{6}{\Lambda} \int \varepsilon^{abc} Tr[A_a \partial_b A_c + \frac{2}{3} A_a A_b A_c]]). \] (98)

An interesting question would be: what is the counterpart of this state in the loop representation? In general such a question goes unanswered, since we do not how to perform the integral in the loop transform,

\[ \Psi^{CS}_\Lambda(\gamma) = \int dA' W(A, \gamma) \Psi^{CS}_\Lambda(A). \] (99)

However, the reader may notice that if one replaces in (99) the value for \( \Psi^{CS}_\Lambda(A) \) given by (98), one gets back the expression for the expectation value of a Wilson loop in a Chern Simons theory we derived last section!,

\[ < W(\gamma) > = \int dA \ e^{iS_{CS}} W(A, \gamma) \] (100)

where the role of the coupling constant of the theory \( k \) of last section is now played by \( \frac{6}{\Lambda} \). So we see that for the particular wavefunction (98) we can actually compute the transform into the loop representation, and we already know the answer, it is the Kauffman Bracket!. Again we should stress that we cannot consider this knot polynomial strictly as a state of Quantum Gravity since it is not diffeomorphism invariant due to the issue of framing. It is still remarkable that we can find an analogue of state (98) in the loop representation.

If all the formalism works, the Kauffman bracket should be a solution of the Hamiltonian constraint of Quantum Gravity with a cosmological constant in the loop representation. Can we check this fact? We actually can. That is what all the technology of the loop representation we developed in section 5 is good for. We have a wavefunction in the loop representation (the Kauffman Bracket), we can write it in terms of the loop coordinates, (as we saw in the last section) and therefore we can apply to it the constraints of Quantum Gravity to see if it is a solution.

Let us therefore apply the Hamiltonian constraint of Quantum Gravity (with a cosmological constant) in the Loop Representation \( \hat{H}_\Lambda \), to the Kauffman Bracket \( \Psi^{CS}_\Lambda(\gamma) \),

\[ \hat{H}_\Lambda \Psi^{CS}_\Lambda(\gamma) = (\hat{H}_0 + \Lambda \det q)(1(\gamma) + \Lambda \text{GSL}(\gamma) + \Lambda^2(\text{GSL}(\gamma) + A_2(\gamma)) + \ldots) \] (101)
where \( \hat{H}_0 \) is the vacuum Hamiltonian constraint.

The result of this calculation is a polynomial in \( \Lambda \). If it is to vanish, it should do so order by order in \( \Lambda \). This leaves us with the following equations,

\[
\begin{align*}
\Lambda^0 : \quad & \hat{H}_0 \ 1(\gamma) = 0 \quad (102) \\
\Lambda^1 : \quad & \text{det} q \ 1(\gamma) + \hat{H}_0 \ \text{GSL}(\gamma) = 0 \quad (103) \\
\Lambda^2 : \quad & \text{det} q \ \text{GSL}(\gamma) + \hat{H}_0 \ \text{GSL}(\gamma)^2 + \hat{H}_0 \ A_2(\gamma) = 0 \quad (104)
\end{align*}
\]

and so on for higher orders.

Equation (102) trivially holds, since the area derivative in the expression of \( \hat{H}_0 \) annihilates the constant. Equation (103) is a bit harder to check, but one can actually show that it holds with minor effort.

Something really interesting happens at order \( \Lambda^2 \), since the terms \( \text{det} q \ \text{GSL}(\gamma) + \hat{H}_0 \ \text{GSL}(\gamma)^2 \) cancel among themselves (this calculation is rather lengthy). That means that for the equation to hold at this order it must happen that,

\[
\hat{H}_0 \ A_2(\gamma) = 0 \quad (105)
\]

That is, the second coefficient of the Jones Polynomial has to be a solution of the vacuum Hamiltonian constraint of Quantum Gravity! Therefore, in order to prove that the Kauffman Bracket is a solution of the Hamiltonian constraint with cosmological constant, it must happen that the second coefficient of the Jones Polynomial has to be a solution of the Hamiltonian constraint without cosmological constant.

Historically, eqs. (103) and (105) were shown to hold previous to this discovery [63, 64]. Actually eq. (105) was quite involved to prove, requiring the use of a complicated computer algebra code. Whereas here we find a very natural argument why it should hold.

What happens to higher orders? At each order a similar decomposition occurs for the coefficients of the Kauffman Bracket. That led us to conjecture [65] that maybe at all orders the same occurred. That is, maybe at all orders “nested” inside the Kauffman Bracket was a state of vacuum Quantum Gravity. Even better, since at each order the portion that is a candidate to be a state of vacuum gravity is a coefficient of an expansion of the Jones Polynomial, we could conjecture that,

\[
\hat{H}_0 \ \text{Jones}_{\Lambda}(\gamma) = 0?? \quad (106)
\]

Unfortunately, computations to try to prove it get more and more involved for higher orders. There is preliminary evidence of a possible proof to all orders involving heavy use of the loop coordinates, but we are unprepared to report about it here [66].

Notice that this state we have found for the vacuum Hamiltonian constraint is framing independent, that is, it is a true knot invariant, and therefore a true state of Quantum Gravity. Therefore the use of framing dependent objects before can be seen as an intermediate artifact in the calculation, as if one used a non-diffeomorphism invariant proof to show that a diffeomorphism identity holds. This would be true for all the conjectured states.

How confident can one be of this result? To put this issue in perspective, we should list the potential points where our argumentation has been weak.
• Framing dependence. As we mentioned, when using the result from Chern-Simons theory to obtain the “loop transform” of the Chern-Simons state, one introduces a framing dependence. Crudely put, this means the “loop transform” of a wave-function that was invariant under diffeomorphisms fails to be invariant. We do not know how to improve this situation. The framing dependence of the Chern-Simons result is well established and is related to spin-statistics in three dimensions. We can just argue that this result was used as an intermediate result and the final result, that the Jones Polynomial coefficient solves the vacuum constraint, is framing independent. Another possible way out of the framing difficulty would be to abandon loops and work directly in the Coordinate Representation mentioned at the end of section 5.6. One would then lose the connection with knot theory but all the expressions involved could be written as well defined functions of smooth vector densities. Unfortunately, many details have to be worked out before we can really claim this is a solution to the problem.

• Regularization. The Hamiltonian constraint we are using involves a regularization and when we claim that something is annihilated by the constraint we really mean it is annihilated at leading order when the regulator is removed. A more careful study of regularization is in order.

• The measure. When we use the Chern-Simons result for the expectation value of the Wilson loop, implicitly we are assuming that the measure used in Chern-Simons theory to perform the path integral is the same as the one to be used in Quantum Gravity. This is by no means obvious. The measure in the Loop Transform should in the end be related to the reality conditions of the formalism and it is clear that the one in Chern-Simons theory is, prima facie, not taking into account this fact. This is related to the next point.

• Reality. In Chern-Simons theory the connection is real, whereas in Quantum Gravity it is complex. This affects our calculation of the expectation value of the Wilson Loop. Clearly if one allows the connection to be complex, formulae like (55) cease to make sense. The integrals basically fail to converge. Even proofs like the one of the skein relation should be taken with care in the complex case since the quantities involved in the skein relations diverge. At the moment, lacking any control about the reality conditions in the loop representation for Quantum Gravity, there is very little we can say about this point. The only hope is that the correct measure, reflecting the reality conditions, could somehow be analytically connected with the Chern-Simons measure, and therefore the results in Chern-Simons theory could be taken as analytic continuations to the purely real case of the gravitational ones. It is evident that this is just a hope and that we cannot say anything else at present.

Given these reservations about our result, do we have any hope that it is correct? We believe there are some supportive elements, that although far from offering a proof, give some reassurance that our result may hold. They are schematically shown in figure 12, and they can be summarized as follows.

• Constraints in connection representation. These constraints were shown to generate the correct diffeomorphism symmetry of the theory 25 and to formally close
the commutator algebra \[3\].

- Equivalence between constraints in both representations. It was shown both using the transform \[17\] and based on the T operators of Rovelli and Smolin \[18\].

- Equivalence between wavefunctions. It was proven using perturbative techniques in loop space \[54, 55\] even for the intersecting case \[25\]. It was also proven non-perturbatively \[53\] and using Feynmann diagrammatics \[58\].

- Constraints in the Loop Representation. Their consistency has been partially proven at the formal level and studies are been done taking into account regularization \[46\].

Figure 12: Redundancies in the calculation offer hope that it may be correct

All this means that if the result is wrong one or more of the previous results should also be wrong. This could well be. For instance, we are implicitly using the same measure to perform the transform of the constraints and of the states. However, even if the result is wrong, one would learn an important lesson about various aspects of the formalism.

This was the main result we wanted to highlight in these lectures, that using this new formalism for Canonical Quantum Gravity one could find for the first time some nondegenerate physical states of the theory, maybe an infinite family of them. Moreover a new branch of mathematics has been brought into contact with Quantum Gravity, Knot Theory, both at a kinematical level as was emphasized by Rovelli and Smolin \[18\] but now also at a dynamical level, due to the role of the Jones Polynomial as a state. It is a remarkable fact that there is a connection between General Relativity and Knot Theory at a dynamical level. After all, the Jones Polynomial was developed without taking into account at all the Einstein Equations. This may just be a coincidence or it may mean that the notions of Knot Theory are deeply intertwined with gravity in a way we do not know at present. Will this mean that the Jones Polynomial is a state of any theory of gravity one proposes? At present we can just offer this as a conjecture.
Assuming the Jones Polynomial is a state, as conjectured, how general a state can it be? Mathematicians seem to agree that the Jones Polynomial is not enough to solve the problem of knot theory, classify inequivalent knots. That means other invariants are to be found in the future that are more powerful. In this view, one would also expect to find states of Quantum Gravity among them, and therefore the conjectured present family of states would be incomplete. A recent trend in mathematics is to consider Vassiliev invariants as more general invariants to classify knots. It is remarkable that these invariants are defined for loops with intersections, exactly the kind of loops that are relevant for gravity. It seems that our generalization of the Jones Polynomial for loops with intersections is not a Vassiliev invariant. However, a more careful study of this aspect is in order.

8 Final Remarks

Due to space limitations these lecture notes can only be taken as a “tourist brochure” of the subject in question. Many oversimplifications have been introduced that allow the reader to quickly view several important results, but may also obscure a detailed understanding of the topics. We urge the readers who want more than a lax overview to consult the appropriate references. We acknowledge that chronologically this may be the first complete account of these findings that sees the light through publication. We urge the interested readers to pay attention, since in the immediate future more detailed accounts of these topics will be published. We would like to finish by referring to some topics that were not even discussed in the text and making some final remarks on the present status and prospect of the subject. Even here we will have to be unfair and leave unmentioned important topics.

In these lectures we have reviewed basically three things.

- The Ashtekar reformulation of General Relativity.
- Some attempts to construct canonical quantizations using these variables.
- The relation of some results from Chern-Simons theories to the Loop Representation of Quantum Gravity.

The first item is clearly of great importance. The Ashtekar variables are finding new applications in Classical Relativity every day and will certainly become a standard tool of analysis for Relativists even if attempts to quantize the theory using them fail. Of the many aspects not even mentioned in these notes concerning this subject, we would like to point out to the reader the following: a) The Capovilla-Dell-Jacobson Lagrangian reformulation of the theory purely in terms of a connection. This was a long-cherished dream of many relativists. The work also presents a novel way of solving the constraints which may have implications for the issue of free data of the theory. b) The work of Samuel and Torre that showed how instantons can be transferred from Yang-Mills theory to General Relativity and their stability analysis showing they can actually be a countable number in some cases. c) The application of the variables to Bianchi cosmologies, offering a new picture of the classical (and maybe quantum) dynamics of these systems. d) The Newman-Rovelli method for solving
the constraint equations using a Hamilton-Jacobi reformulation. d) The possibility that topology change and negative energy may occur in the theory. Summarizing, this is a healthy area of research in which many new and important developments will be studied.

In the second item we count the connection and loop representations. These may or may not succeed in providing a basis for a quantum theory of gravity. Even in the case of failure, it is clear that many lessons have been learnt from their use. We list some important pieces of work on these areas not mentioned in these notes a) The application to two and one killing vector spacetimes, allowing in some cases to find observables in the systems. Recent work also shows that the quantization scheme for one polarization two Killing vector fields may coincide with the usual quantization based on the equivalence with a scalar field. b) The work on 2+1 gravity, which shows for a model system how the connection and loop representations and the loop transform can actually be given a rigorous meaning. b) The application of loop techniques to Gauge theories on the lattice, linearized gravity and Maxwell theory, again offering test cases where the quantization program works to the end. c) The work on C-P violation. d) The discussion of the issue of time in the linearized theory.

An aspect that cannot be overstressed is the development in the loop representation, of techniques for writing differential operators in loop space and to write wavefunctions and knot invariants in analytic form. These findings transcend the area of Quantum Gravity and have immediate application in the quantization of gauge theories (in the continuum and lattice). In fact, we have seen some examples of their application in the sections on Chern-Simons theory. They can also become standard tools of analysis for knot theorists. In fact, understanding in this area is just beginning and we may see even more progress in the near future. Of particular interest is the Coordinate Representation of section 5.2 that could allow for the quantization of diffeomorphism invariant theories without the framing ambiguities of the Loop Representation.

On the final item, we can just say that it is work in progress and that in the end technical difficulties may hamper further development or even disprove the present results. An interesting point seems to be that some of the results seem to survive the inclusion of matter. Another result is that some notions of Knot theory seem to be useful to select an inner product for the theory at least in some toy subsectors. Moreover, the first hint of what a semiclassical interpretation may look like in this context is starting to emerge.

Notice that our treatment has evaded the “big questions” of Quantum Gravity, as what is the inner product, the issue of observables and the issue of time. The only comment we can make is that the fact of being able to explore (tentatively) the space of states of the theory may provide a better framework in which to address these problems in the future.

We do not know what will the outcome of this –at the moment– happy marriage of Knot Theory and Quantum Gravity be. As in any other case where a new mathematical technique is introduced into an area of Physics there is potential for striking new results and also for a lot of red herrings. Only time and much more effort will decide which of these two situations we are actually creating with our work.
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Figure 1: 3+1 foliation of spacetime and variables of the canonical formalism

Figure 2: The infinitesimal loop that gives rise to the Loop Derivative

Figure 3: The loops $\gamma_t, \gamma_{st}$ and $\gamma_{ts}$.
Figure 7: The Trefoil knot

\[ C \left[ \begin{array}{c} \hline \end{array} \right] - C \left[ \begin{array}{c} \hline \end{array} \right] = \tau C \left[ \begin{array}{c} \hline \end{array} \right] \]
\[ = 1 \]

\[ C \left[ \begin{array}{c} \hline \end{array} \right] - C \left[ \begin{array}{c} \hline \end{array} \right] = \tau C \left[ \begin{array}{c} \hline \end{array} \right] \]
\[ = \tau \]

\[ C \left[ \begin{array}{c} \hline \end{array} \right] - C \left[ \begin{array}{c} \hline \end{array} \right] = \tau C \left[ \begin{array}{c} \hline \end{array} \right] \]
\[ = 1 \]

\[ \Rightarrow C \left[ \begin{array}{c} \hline \end{array} \right] = 1 + \tau^2 \]

Figure 8: Evaluation of the Conway Polynomial for the trefoil knot
Figure 4: The knots in (b) and (c) (The Whitehead Link) have vanishing linking number.

Figure 5: Upper and under crossings for the definition of the Gauss Linking number

\[ C [X] - C [\bar{X}] = t \ C [\bar{X}] \]

Figure 6: Skein relation for the Conway Polynomial

Figure 9: Under and upper crossings created by adding an oriented small loop

\[ \langle W \rangle = 2 + \bigotimes_k + \bigotimes_{k^2} + \cdots \]

Figure 10: Diagrammatic expansion of the Wilson Loop in a Chern-Simons theory

Figure 11: Two different framings for a given loop. In one case the self-linking number (computed as the linking number of the two curves defined by the framing) is zero and in the other +1.
