Abstract

We study the algebraic effects and handlers as a way to support decision-making abstractions in functional programs, whereas a user can ask a learning algorithm to resolve choices without implementing the underlying selection mechanism, and give a feedback by way of rewards. Differently from some recently proposed approach to the problem based on the selection monad \cite{2}, we express the underlying intelligence as a reinforcement learning algorithm implemented as a set of handlers for some of these algebraic operations, including those for choices and rewards. We show how we can in practice use algebraic operations and handlers — as available in the programming language \texttt{EFF} — to clearly separate the learning algorithm from its environment, thus allowing for a good level of modularity. We then show how the host language can be taken as a \textit{\lambda}-calculus with handlers, this way showing what the essential linguistic features are. We conclude by hinting at how type and effect systems could ensure safety properties, at the same time pointing at some directions for further work.

1 Introduction

Machine learning is having, and will likely have more and more, a tremendous impact on the way computational problems (e.g. classification or clustering) are solved. Learning techniques, however, turn out to be very fruitful when solving control problems, too. There, in fact, an agent’s goal is to learn how to maximize its reward while interacting with the environment rather than while computing a mere function. From this point of view, the so-called reinforcement learning techniques \cite{34} are proving to be particularly appropriate in many contexts where exploration and optimization have to be interleaved.

Prompted by that, in the past decade there has been an incredible effort to develop programming languages and programming language techniques oriented to the design of machine learning systems. The outcome of such an effort is well-known and gave birth to new programming language paradigms, such as Bayesian \cite{18,39} and differentiable programming \cite{1,25,33}, as well as to the flourishing field of programming languages for inference \cite{1}. Despite the incredible strides made, machine learning support from programming languages is still in its infancy and its deliverables mostly consist of a set of general-purpose programming language libraries (such as \texttt{Theano} \cite{7}, \texttt{TensorFlow} \cite{3}, and \texttt{Edward} \cite{36,37}, just to mention but a few) and domain-specific languages (such as \texttt{Anglican} \cite{35}, \texttt{Pyro} \cite{9}, and \texttt{Stan} \cite{11}, just to mention but a few).

In this work, we deal with a further programming language paradigm oriented to machine learning: choice-based programming. The latter moves from the observation that many machine learning systems — especially those pertaining the realm of reinforcement learning — can be described in terms of choices, costs, and rewards. Prompted by that, choice-based programming languages\cite{2} extend traditional programming languages with high-level decision-making abstractions that allow for the modular design of programs in terms of choices and rewards. While in a probabilistic language programs are structured in terms of sampling and observing — leaving the actual inference process to the interpreter — in a choice-based language the code is structured by specifying where a choice should be made and what its associated cost (or, dually, its reward) is, leaving the actual decision-making process to the interpreter.

All of that considerably changes the way one writes and thinks about software, at the same time raising new — and challenging — questions both from the point of view of programming language theory and of machine learning. In the former case, in fact, we have to deal with the introduction of

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1 See, e.g., the dedicated POPL Workshop LAFl \url{https://popl21.sigplan.org/home/lafi-2021}

2 Such as SmartChoice \cite{10}.
decision-making abstractions, their implementation, and their semantics. In the latter case, instead, we have just begun to realise how to tackle modularity of machine learning systems using programming language-based techniques.

The latter point is the main topic of this paper. We move from the recent work by Abadi and Plotkin \[2\], where a monadic \[22\] approach to choice-based programming is developed. There, the authors show how the so-called selection monad \[14, 16, 15\] can be used to model (and to give semantics to) programs written in a choice-based language as effectful programs, and how it is possible to manage choices as algebraic operations \[27, 26, 28\], delegating the task of solving these choices to those who implement the selection mechanism. In this work, we are concerned with further developing this idea, although in a different direction. In fact, even if we still deal with monadic and algebraic approaches to choice-based programming, we explore the use of the state — rather than selection — monad in the framework of reinforcement learning systems. In particular, we show how the use of the state monad allows for a high level of modularity in the construction of systems based on reinforcement learning.

The reader, at this point, may wonder why one should consider a further kind of monadic programming — viz. one based on the state monad — when dealing with reinforcement learning systems. The next two sections are dedicated to answer this question. There, we will study a simple example coming from the reinforcement learning literature, highlighting some drawbacks of the selection monad, on the one hand, and the strengths of our state-based approach, on the other hand. That will also allow us to illustrate the practical advantages of functional programming techniques in reinforcement learning modelling.

For the moment, we simply remark that at the very heart of our approach lies a clear separation between the part of the program dealing with the learning algorithm and the one that handles the interaction with the environment. Such a separation can be naturally structured in the form of a state monad with choice and reward operations acting as algebraic operations. However — and here comes the main difference between our approach and the one based on the selection monad — the action of such operations is not determined by the monad (as de facto happens when working with the selection monad), but it is ultimately given by the reinforcement learning algorithm used. As a consequence, one can model reinforcement learning systems modularly as monadic, state-based programs written using choice and reward operations, and then view reinforcement learning algorithms as algebraic interpretations of such operations. Concretely, that means that we can view (and implement) reinforcement learning algorithms as handlers giving interpretations to choice and reward operations. Different learning algorithms then give different handlers — and thus different interpretations of choices and rewards — so that it is possible to instantiate the very same program with different learning algorithms by simply changing the way its algebraic operations are handled.

Summing up, the main contributions of this paper are the following:

- The development of a modular approach to reinforcement learning systems throughout functional programming language techniques. Among such techniques, the main role is played by the state monad and its associated algebraic operations for performing choices and rewards. Crucially, the use of the state monad (as opposed to the selection monad) allows us to consider different interpretations of choices and rewards, such interpretations ultimately being the reinforcement learning algorithm used. We make use of handlers to implement these interpretations in a modular fashion.

- The analysis of our approach in a core \(\lambda\)-calculus with handlers and algebraic operations. This allows us to isolate the exact features a language needs to have in order to support our functional approach to reinforcement learning systems.

- A preliminary study of how to use semantic-based techniques (notably polymorphic and graded type systems) to ensure correct behaviours of reinforcement learning systems when implemented in a functional way.

2 Structuring Reinforcement Learning Applications Through Functional Programming

To begin with, we are going to illustrate our approach on a simple running example, common in the reinforcement learning literature \[34\], namely the so-called multi-armed bandit problem, denoted MAB.
In this setting, a gambler faces a row of \( k \) slot machines, each of which distributes rewards in the form of winnings according to a probabilistic model unknown to the gambler. The gambler, it goes without saying, wants to maximize its gains over a fixed number of rounds. At each round, the gambler chooses a machine and obtains a reward depending on its reward distribution. On the one hand, then, the gambler wants to maximize her gains, but on the other she knows nothing about the inner working of the machines. This example, indeed, expresses the need of a trade-off between exploration and optimization: the gambler wants to play on the best machine as much as possible, but in order to find this best machine, she needs to try every machine. Moreover, since rewards are random, playing once on each machine, then playing only on the one with the best observed reward, may not be optimal in the long run. Reinforcement learning indeed focuses in offering techniques which explores this trade-off in a meaningful way.

What we are interested in doing here is deriving a strategy for the agent in the aforementioned problem using reinforcement learning techniques, and with the help of functional programming. We are not interested at devising new algorithms, but rather at showing that functional programming can help in giving structure to programs which comprise both the proper learning algorithm, but also the interaction with the environment.

Conceptually, it is natural to see a program solving MAB by way of reinforcement learning as structured into three parts:

- The first one, the environment, serves as an interface with the outside world, making it visible to the program. Here, this part comprises the \( k \) slot machines, possibly through a system of sensors and functions querying those sensors.

- The second one, the learner, which provides generic reinforcement learning algorithms, possibly through libraries. There are many algorithms the literature offers, for example gradient learning, Q-learning, expected Sarsa or other TD-methods [34].

- The user, namely some code letting the environment and the learner exchange information, thus acting as a bridge between the two. Here, the user is supposed to turn events and data coming from the environment into a form which can be understood by the learner.

One of the key ingredients of our proposal consists in structuring these three parts and their interaction by stipulating that the user proceeds by invoking some algebraic operations provided by both the environment and the learner. This is schematized in Figure 1. The main algebraic operations are the four mentioned in the figure, two of them provided by the environment and two provided by the learner. The two algebraic operations provided by the environment are \( \text{observe} \) and \( \text{do} \). The former, having type \( 1 \leadsto O_E \) allows the user to retrieve some data from the environment. In our multi-armed bandit example, it corresponds to observing the gain on the current slot machine. The operation \( \text{do} : A_E \leadsto 1 \) executes an action in the environment. In our running example, an action would be the choice of one of the \( k \) slot machines. Those operations depend the set of possible observations and the set of possible actions in the environment, those sets being represented by two types \( O_E \) and \( A_E \) in the user code. In our example, the observations are the gains, thus we can give the type \( O_E = \text{Real} \). As for actions, they stand for choices of a slot machine, and thus \( A_E = \{1, \ldots, k\} \).
Then, there are two other operations for the learner, namely choice and reward. The operation choice : 1 \rightarrow A_E asks the learner to take an action for us. In our setting it means that a call to choice returns an integer between 1 and k designing the slot machine the user should play on. The reward : Real \rightarrow 1 operation allows the user to give a feedback to the learner depending on its previous choice. In our example, the typical reward would be the gain obtained from the slot machine chosen by the learner. There is also an additional bridge between the user and the learner, that we call the abstract interface: the learner may need more information than the feedback given by reward, and this information will be transmitted using the abstract interface by way of some additional algebraic operations invoked by the learner but handled by the user.

One advantage of the just described approach is to show that this learner part of the program, i.e., the handler for the operations choice and reward, can be implemented independently from the details of the environment, once and for all. One could even conceive to have a library of handlers, where each set of handlers corresponds to a reinforcement learning algorithm, in such a way that the user could choose one of these handlers for each of the environments he has access to, implementing the abstract interface but without delving into the details of the underlying algorithms. Similarly, implementing any new RL algorithm would boil down to just write a new set of handlers for choice and reward.

In which order should the user invoke the various algebraic operations? A typical sequence of interactions would be choice \rightarrow do \rightarrow observe \rightarrow reward, meaning intuitively that it asks for a choice, and then does explicitly this action in the environment, observes the results of this action, and finally produces a reward. As we will soon see, it is preferable that the user indeed respects this order, e.g., when handling the reward operation, the learner may need to observe the environment (abstracted by the interface) since a valuation function typically depends on the state reached after an action.

In order to have a learner independent from the environment, it must not make explicit reference to the types A_E and O_E, as we saw on MAB. Thus, we introduce two finite sets O_{RL} and A_{RL} corresponding to the abstract sets of observations and actions, respectively. Except for their sizes, those finite sets must be totally independent from the environment. It means that in practice, the learner only has access to some elements of those sets, and should make choices in A_{RL} depending only on the abstract observations from O_{RL} it receives from the user. Typically, both the policy to choose actions and the learning part are built by constructing and updating a value function mapping pairs in O_{RL} \times A_{RL} to a reward estimation, for example an element of Real.

### 2.1 A Naive RL Algorithm in EFF

We present our approach practically with a very basic algorithm for reinforcement learning, which works by only remembering an evaluation function keeping track of the expectation of the immediate rewards it receives for any of the actions, and chooses an action by way of a so-called \(\epsilon\)-greedy policy \cite{34}: with probability \((1 - \epsilon)\), it makes the best possible choice based on the current valuation, and with probability \(\epsilon\) it explores by choosing an action uniformly at random. To implement this algorithm, we use the language EFF \cite{31}, an OCAML-based language for effects and handlers. The syntax of EFF should be understandable to anyone with some basic knowledge on OCAML, effects and handlers. We use lists for the sake of simplicity, although other kinds of data structures would enable better performances. The source code as well as other examples can be found in \cite{13}.

**Declaring The Abstract Interface.** The first step towards implementing the RL algorithm consists in declaring the sets O_{RL} and A_{RL} together with the abstract interface which will allow us to recover some information on the sets O_{RL} and A_{RL}:

- type rl_act
- type rl_obs
- effect rl_observe : unit \rightarrow rl_obs
- effect rl_getavailableact : rl_obs \rightarrow rl_act list

Here, the first effect allows the learner to observe the environment (after the abstraction) and to get, for each observation, the list of available actions it has to choose from. As we stated before, the learner does not have access to the interpretation of those two effects nor to the actual types, and only knows that those sets of abstract actions and observations are finite, this being enough to implement the reinforcement learning algorithm.
Handling Choices and Rewards. The handler makes essential use of the state monad, where the state represents the memory of the learner. For the specific RL algorithm we are targeting now, this memory consists of an element of this type:

```plaintext
type memory = ((rl_obs*(rl_act*int*float) list)) list*int*int
```

The left type of the internal memory corresponds to a value function, for each pair of an observable and an action, we give an estimation of the immediate reward we can obtain. It is computed as the average reward, and in order to do this average incrementally, it is common to also remember the number of times a choice has been made. Then, the internal memory also remembers the last choice made using a pair of integers denoting indexes in the evaluation function (we usually denote na the index of an action and no the index of an observation). Then, we can implement the $\varepsilon$-greedy policy of the learner (we only describe the important functions and not the simple intermediate one).

```plaintext
let greedypolicy ($\epsilon$,l) = 
  if ( (randomfloat 1.) <= $\epsilon$) then 
    (* Uniform choice in this case *)
    begin 
      let na = randomint (list_length l) in 
      (* Find the action with index na *)
      let a = findact l na in 
      (na,a)
    end 
  else 
    (* Select the action with the maximal estimated reward *)
    argmax l
end
```

And with this, and some other auxiliary functions, we can define the handler for the basic RL algorithm. The only non-standard clause for the handler is the last one, starting with finally, in this setting with a state monad, it should be understood as the initial state for a computation.

```plaintext
let rl_naive $\epsilon$ v = handler 
  (* The first input is the probability of exploring, 
  the second is the initial estimation *)
  | y ↔ (fun (_:memory) ↔ y)
  | effect Choice k ↔ fun (l,_,_) ↔ 
    (* use the interface to get an observation *)
    let o = rl_observe () in 
    (* extract the index no for o, with its list of estimations q *)
    let (l',no,q) = getstateestimate l o v in 
    (* select the action a with the greedy policy *)
    let (na,a) = greedypolicy ($\epsilon$,q) in 
    (* update the estimations*)
    let l'' = updatestate l' no (fun ll ↔ updatechoice ll na) in 
    (* return action a, with the updated memory*)
    (continue k a) (l'',no,na)
  | effect (Reward r) k ↔ fun (l,no,na) ↔ 
    (* update estimations for the previous choice (no,na) *)
    let l' = updatestate l no (fun q ↔ updatereward q na r) in 
    (* give the new memory to the continuation *)
    (continue k ()) (l'',no,na)
  (* initial memory*)
  | finally f ↔ f ([],0,0)
```

Note that we initialize the estimations lists only when we see an element of $O_{RL}$ for the first time. This is standard in RL because there could be an extremely large set of states, and it may well be that not all of them are reached — it may be better to give an initial estimation to a state only when we actually see
this state (in this program, this is done by the \texttt{geststateestimate} function). We do not give the code of the update functions, which is anyway easy to write. The nice thing of this handler is that it \textit{does not depend} on the environment, it can of course interact with its environment (using \texttt{rl\_observe}) but it does so in a modular way, so that this program can be used in \textit{any environment} in which we would like to experiment this (admittedly naive) algorithm.

2.2 The Multi-Armed Bandit in \textit{EFF}

We show how to make use of the RL algorithm described in the previous section on the environment coming from MAB. The first step consists in declaring the types for $O_E$ and $A_E$. Since rewards are earnings, and our actions are nothing more than a choice of a specific machine, we can proceed as follows:

\begin{verbatim}
type env_obs = float  
type env_act = int
\end{verbatim}

Modeling the Environment We model MAB as a program, where we take a very simple distribution of rewards for the sake of the example. As stated before, the environment correspond to the handling of \texttt{observe} and \texttt{do}.

\begin{verbatim}
type env_state = float

(* The random reward of the machine a. In a real case, this reward should be obtained by observing the result of the slot machine *)
let getreward (a:act) = (float_of_int a) /. (randomfloat 10.) ;;

(*max corresponds to the number of slot machines *)
let MAB_handler max = handler |
  y -> (fun (_,env_state) -> y)  |
  effect (Do a) k -> fun _ ->
  (* When seeing a valid action a, we compute the gain for the slot machine a and store the result in the memory *)
  if (a > 0) && (a <= max) then (continue k ()) (getreward a)
  else raise "This action is not available! \n"
  (*An observation corresponds to showing the stored result *)
  | effect Observe k -> fun r -> (continue k r) r
  (* Initial environment, no rewards observed *)
  | finally f -> f 0.
;;
\end{verbatim}

Implementing the Abstract Interface The abstract interface is a handler that implements the types $A_{RL}$ and $O_{RL}$, and handles the algebraic operations declared by the learner:

\begin{verbatim}
(* Abstractions Types. The type \texttt{unit} for \texttt{observations}
means that the learner has no information on the environment *)
type rl_obs = unit ;;
type rl_act = int ;;
effect rl_observe : rl_obs
effect rl_getavailableact : rl_obs -> rl_act list
(*Transform a standard observation into an abstract one*)
let abstractobs (o : env_obs) :obs = () ;;

(*The handler describes:
  - the abstraction of observations
  - the actions available to the learner *)
let abs_MAB max =
let l = listEnumerate 1 (max + 1) in
handler
\end{verbatim}
The Main Program

With this, we have everything we need to handle the four main operations. In order to use the learner described above, we can just open the file in which the RL algorithm is defined and use it as a handler, with the interface described above. For example, we can write the main program:

```haskell
#use "MAB_Environment.eff" ;; #use "RL_Naive.eff";;

(* Multi-Armed Bandit with 6 machines *)
with (MAB_handler 6) handle
(* Provides the interface to the learner *)
with (abs_MAB 6) handle
(* Call the Basic RL algorithm described previously *)
with (rl naive 0.05 10.) handle
(* Start writing your program with algebraic operations *)
Here, we do 500 rounds *)
    let rec run n r =
        if n = 0 then r else
            let a = choice () in (do a);
            let r' = observe () in
                reward r'; run (n-1) (r ++ r')
in run 500 0.
;;
```

And the point is that if we want to use another learner we only have to load a different handler for the learner, and this naive learner can be used in any environment as long as the abstract interface is handled.

3 The Selection Monad, and Why it is Not an Answer

To gain a better understanding of the differences between our state-based approach and the selection-based approach by Abadi and Plotkin [2], let us illustrate some of the main drawbacks exhibited by the selection monad when applied to MAB. To do so, let us first shortly recap the underlying mechanism behind such a monad.

The selection monad is defined through the functor $S(X) = (X \to R) \to X$, where $R$ is a (usually ordered) set of rewards. Intuitively, a computation in $S(X)$ takes in input a reward function $f : X \to R$ associating to each element in $X$ a reward in $R$ (we can think about $f(x)$ as a measure of the goodness of $x$), and chooses an element in $X$ that, intuitively, is optimal for $f$. By its very definition, even if the set $R$ of rewards is a parameter of the selection monad, the latter does not have direct access to it: the only way to interact with rewards is through a reward function, meaning that the selection monad, by itself, does not handle rewards directly. Abadi and Plotkin [2] overcome this problem by combining the selection monad with another monad $T$ giving direct access to rewards, this way obtaining (a monad whose carrier is) the functor

$$S_T(X) = (X \to R) \to T(X).$$

Here, $T$ can for example be the $R$-based writer monad $T(X) = R \times X$ — this way, rewards can be found in the first component of the result whenever a choice is made. One can even go beyond that and take $T(X) = D_{\text{fin}}(R \times X)$, where $D_{\text{fin}}$ is the finite distribution monad, this way modeling stochastic rewards.

Using this strategy — i.e. working with $S_T$ rather than with $S$ alone — we can indeed model (stochastic) rewards as prescribed by the multi-armed bandit problem. For that, we just consider the monad $(X \to R) \to D_{\text{fin}}(R \times X)$. The last ingredient needed to attempt a selection monad-based model to MAB is to give choice operations, which ultimately constitute the way to construct selection computations. As the reader may guess, this is the most delicate point.
The selection monad comes with a choice operation \texttt{choice} which given two selection computations \(a\) and \(b\) in \(S(X)\), returns a new computation \(\texttt{choice}(a, b)\) belonging to \(S(X)\) working as follow: given a reward function \(f : X \rightarrow R\), \(\texttt{choice}(a, b)\) passes \(f\) to both \(a\) and \(b\), this way obtaining two candidate optimal elements \(a(f)\) and \(b(f)\), and then chooses between the latter on the basis of \(f\). More precisely, given \(a(f)\) and \(b(f)\), we can obtain a reward for each of them as the elements \(f(a(f))\) and \(f(b(f))\) in \(R\). Assuming \(R\) to come with a binary relation \(\preceq\) ranking rewards, we then let \(\texttt{choice}(a, b)(f)\) to return the best one between \(a(f)\) and \(b(f)\) according to \(\preceq\).

At this point, we can already start perceiving the main issue behind choice operations: to choose between two computations, we have to simulate both of them, and then take the optimal one. In any case, this concerns the selection monad \(S\) alone. What about the monad \(S_T\), i.e. the combination of the selection monad with a monad \(T\) giving access to rewards? Here, the situation is slightly more complicated, but the basic mechanism behind choice operations is essentially the same one for \(S\). Proceeding as for the latter, we obtain elements \(a(f)\) and \(b(f)\) belonging to \(T(X)\): this time, however, we cannot directly apply the reward function \(f\) on them. The solution proposed by Abadi and Plotkin [2] is to require to have a \(T\)-algebra \(\alpha : T(R) \rightarrow R\), so that we can apply \(T(f)\) — rather than \(f\) — on \(a(f)\) and \(b(f)\), this way obtaining monadic rewards \(T(f)(a(f))\), \(T(f)(b(f))\) in \(T(R)\); map them into \(R\) using \(\alpha\); and then choose the optimal one according to the order \(\preceq\) on \(R\). For instance, taking real numbers as rewards, we can take, e.g., \(\alpha : R \times R \rightarrow R\) as real-number by addition.

Notice that dealing with choice operations this way, we not only still have the same problem seen for \(S\) — namely that choosing between computations require to simulate all of them — but we also have to compute monadic applications of \(f\). That is, the application of \(T(f)\) to an element \(\phi \in T(X)\) usually requires to compute the reward \(f(x)\) for each element \(x\) which is, intuitively, part of \(\phi\). For instance, if we think about \(\phi\) as a distribution, then computing \(T(f)(\phi)\) requires to compute the reward of each element in the support of \(\phi\). All of that does not fit well with the very essence of reinforcement learning, as we are going to see.

Let us now come to reinforcement learning systems and to the multi-armed bandit example, highlighting the main problem behind the semantic mechanism of choice operations. First, let us notice that the selection monad seems to provide an adequate setting for reinforcement learning. In fact, taking the monad \((X \rightarrow R) \rightarrow D_{\mathbb{R}}(\mathbb{R} \times X)\), we obtain choices and (stochastic) rewards, and the former can in principle depend on the latter. In such a setting, the choice operation is defined parametrically with respect to an algebra \(\alpha : D_{\mathbb{R}}(\mathbb{R} \times \mathbb{R}) \rightarrow \mathbb{R}\), which can be naturally defined as follows: given a distribution \(\phi\), we first apply addition to all the elements in the support of \(\phi\), and then compute the resulting expectation. Given such an algebra \(\alpha\), to perform a choice operation \texttt{choose} between two computations \(a\) and \(b\) with a reward function \(f\), we first compute the probability distributions \(a(f)\) and \(b(f)\), and then compute the expected reward associated to each such a distribution, which means first computing the reward of each element in its support.

The latter passage is not reasonable from reinforcement learning perspective. In fact, when dealing with reinforcement learning systems, implementing choices operations this way forces us to (i) simulate all actions; (ii) to obtain a perfect knowledge of the environment; (iii) and to make (optimal) choices based on that. In MAB, that means (i) simulating playing on each machine; (ii) observing the whole distributions associated to each machine; (iii) and then select the one with the best expectation, which in turn requires to compute the rewards of each element in the support of the distribution of a machine. All of that is simply too strong to give a reasonable model of reinforcement learning systems. Notice that all of that is essentially independent of the choice of the algebra \(\alpha\), meaning that the problem lies at the very hearth of he selection monad (and its choice operations) rather than on the concrete way one aggregates monadic rewards.

### 4 From Practice to Theory: Effects and Handlers

In this section, we try to inject the ideas we developed in Section 2 into a paradigmatic programming language, so as to be able to isolate the features we deem necessary.
4.1 A Core Language with Effects and Handlers

For the sake of simplicity, we take the core language described in [31] with additional base types corresponding to observations and actions and a simple type system without type effects. We will not talk about the details of the underlying effect, such as randomness, I/O, or exceptions. Those effects are obviously needed in practice, but adding them to the theory would be standard, so for the sake of simplicity we ignore those, and when we write a type \( T \to T' \), this function should be understood as a function that can use those standard effects.

The grammar for terms and types is in Figure 2. In other words, we work with a \( \lambda \)-calculus with pairs base types (ranged over by \( B \)), constants and functions for those base types, denoted by \( c \) and \( f \). Any symbol \( c \in C \) is associated to a base type \( B \), while any symbol \( f \in F \) comes equipped with a function type \( B_1 \times \cdots \times B_n \to B \). Moreover, we have algebraic operations, each operation symbol \( \text{op} \) coming with a type \( \text{op} : T_p \sim T_a \in \Sigma \) where \( T_p \) is the type of parameters and \( T_a \) is the arity. In a computation \( \text{op}(V; x.C) \), the variable \( x \) is bound in \( C \). Those operations are handled by handlers. An handler has the following form

\[
\text{handler} \ \{ \text{return} \ x \mapsto C_r, \text{op}_1(x;k) \mapsto C_1, \ldots, \text{op}_n(x;k) \mapsto C_n \}
\]

Here, the computations \( C_1, \ldots, C_n \) are pieces of code meant to handle the corresponding algebraic operation, while \( C_r \) is meant to handle a return clause. We use the arrow \( \Rightarrow \) to denote handler types, contrary to the function type that uses \( \to \). When one wants to use the aforementioned handler for the purpose of managing some algebraic operations, we do by way of a term in the form

\[
\text{with } V \text{ handle } C
\]

in which \( C \) is executed in a protected environment such that any algebraic operations produced by \( C \) is handled by the handler \( V \), provided it is one among those declared in it.

The typing rules for this language are given in Figure 3. In a computation \( \text{op}(V; x.C) \), \( C \) can be seen as a continuation for the computation, with type \( T_a \to T \), this is why in the typing of handler, the second parameter \( k \) has this type. The typing rule for the handler with simple types looks like a function application, where the handler transforms a computation of type \( T_2 \) to a computation of type \( T_1 \).

Then, the dynamic semantics is given in Figure 4. Algebraic operations can commute with the \text{let} constructor and handlers for other operations. As for handlers, the return computation is handled by the return clause, and an algebraic operation is handled by the corresponding computation in the handler, where the continuation \( k \) is replaced by the actual continuation \( C \) with the same handler.

With this, we have a complete description of a core language with effects, handlers and base types. In practice, we also want other standard constructors for booleans (if then else), lists or other data structures, but this can be easily added to the core language without any theoretical difficulties. It is then relatively standard to prove the subject reduction of this type system, proving some simple kind of safety:

\textbf{Theorem 1} (Subject Reduction). If \( \Gamma \vdash C : T \) and \( C \to C' \) then \( \Gamma \vdash C' : T \)

\textbf{Proof.} The proof is standard. We start by proving a weakening lemma (if \( \Gamma \vdash C : T \) then \( \Gamma, x : T' \vdash C : T \)) and a value substitution lemma (if \( \Gamma, x : T \vdash C : T' \) and \( \Gamma \vdash V : T \) then \( \Gamma \vdash C[V/x] : T' \)) by induction on
For the following rules, we denote \( H = \text{handler} \{ \text{return } x \mapsto C_r, \text{op}_1(x;k) \mapsto C_1, \ldots , \text{op}_n(x;k) \mapsto C_n \} \)

\[
\begin{array}{c}
\text{C} \rightarrow \text{C'} \\
\text{with } H \text{ handle } \text{C} \Rightarrow \text{with } H \text{ handle } \text{C'}
\end{array}
\]

\[
\begin{array}{c}
\text{op} \notin \{ \text{op}_1, \ldots , \text{op}_n \} \\
\text{with } H \text{ handle } \text{op}(V; x; C) \Rightarrow \text{op}(V; x; \text{with } H \text{ handle } C)
\end{array}
\]

\[
\begin{array}{c}
\text{with } H \text{ handle return } V \Rightarrow C_r[V/x] \\
\text{with } H \text{ handle } \text{op}(V; x; C) \Rightarrow C_r[V/x][\lambda x \text{ with } H \text{ handle C/k}]
\end{array}
\]

Figure 4: Semantics
judgment, and then we can prove subject reduction by induction on the relation \( C \rightarrow C' \). The weakening lemma is useful for the cases when an algebraic operation \( \text{op} \) commute with a \texttt{let} or an \texttt{handle}, and the substitution lemma is useful for substitutions, which are always for values as one can see in Figure 4.

However, with those simple types it is not possible to prove that any typable computation reduces to a computation of the shape \texttt{return} \( V \), because a non handled operation \( \text{op} \) cannot be reduced. To have those kind of safety theorems, we need type effects, as we will see in the next section.

### 4.2 Setting the Stage: Types and Algebraic Operations for RL

Let us now instantiate more clearly the set of base types and operations for our approach. We consider that:

- Base types should contain at least \texttt{Bool}, \texttt{Real}, \( A_E \), \( O_E \), \( A_{RL} \), \( O_{RL} \), respectively the types for booleans, real number for rewards, actions, observations and their abstract counterparts.

- Constants for booleans and real numbers are standard. For \( A_E, A_{RL} \) and \( O_{RL} \), we consider finite sets of constants, and \( A_E, A_{RL} \) should have the same size. As for \( O_E \), the set of constants depends on the environment so we have no fixed choice.

- We have standard functions for booleans and real numbers, and equality for all those base types. We may also have additional functions for \( O_E \) depending on the environment.

- For the set of operations symbols, we suppose that we have at least the main four operations described before

  \[
  \begin{align*}
  \text{choice}: \text{Unit} & \rightarrow A_E \\
  \text{reward}: \text{Real} & \rightarrow \text{Unit} \\
  \text{observe}: \text{Unit} & \rightarrow O_E \\
  \text{do}: A_E & \rightarrow \text{Unit}
  \end{align*}
  \]

  And, in order to make the abstraction more formal, we also add

  \[
  \begin{align*}
  \text{choice}_{RL}: \text{Unit} & \rightarrow A_{RL} \\
  \text{reward}_{RL}: \text{Real} & \rightarrow \text{Unit}
  \end{align*}
  \]

  Moreover, we also need to add effects corresponding to the abstract interface. As this interface is not fixed, we chose for the sake of the example the one we used on the naive algorithm. So we add those operations:

  \[
  \begin{align*}
  \text{observe}_{RL}: \text{Unit} & \rightarrow O_{RL} \\
  \text{getactions}_{RL}: O_{RL} & \rightarrow A_{RL} \text{ List}
  \end{align*}
  \]

  This describes all we need in order to formalize our approach. However, in practice, we do not want all the actors (environment, user and learner) to have access to all those operations at all time. A typical example is that the learner must not use \texttt{observe} or have access to \( O_E \) as it would break abstraction, and thus modularity. As a first approach, we define subsets of this language, by defining subsets of base types, functions and algebraic operation, so that we can define clearly which actor has access to which constructors. In the next section on safety, we will formalize this with a type system.

- The whole language described above, with all base types and operations is denoted \( \lambda_{\text{eff}}^{I} \). This calculus will be used to define the interface, as an interface should be able to see both the constructors for the environment and for the learner in order to make the bridge.

- We denote by \( \lambda_{\text{eff}}^{E} \) the subset of this whole language without types and operations related to the learner (all types and operations with \( RL \) in the name, such as \( A_{RL}, \text{choice}_{RL}, \ldots \)). This language will be used for the main program and the handler for \texttt{do} and \texttt{observe}, as it is basically the language with no concrete information about the learner.
Dually, we denote by $\lambda_{\text{eff}}^\text{RL}$ the language with only the operations related to the learner. Also, in $\lambda_{\text{eff}}^\text{RL}$, we consider that we do not have any functions nor constants for $O_{\text{RL}}$ and $A_{\text{RL}}$ except equality and the operations in the abstract interface. Indeed, this language will correspond to the learner, and as explained before, we want the learner to be independent from its environment. An important point to make this possible, is that the learner should be modular (or ideally, polymorphic) in the types $O_{\text{RL}}$ and $A_{\text{RL}}$, and so having access to the constants of those types would break this principle. In particular, with this definition of $\lambda_{\text{eff}}^\text{RL}$, changing the size of the finite sets $O_{\text{RL}}$ and $A_{\text{RL}}$ does not modify the language, so whatever the size may be, the learner still uses the same language.

### 4.3 RL Algorithms and Environments as Handlers

We start with the main program. As showed in the previous section, the main program is just an usual functional program that has access to the types $O_E$ and $A_E$ as well as the four main operations choice, reward, observe and do. Thus, it corresponds to the language we denote $\lambda_{\text{eff}}^E$. This is the main focus of our work, to make it possible to program within this language. In order to do this we still need to handle those four operations, and for this we design several handlers.

#### 4.3.1 The Learner

The learner is a handler with the state monad for the operations $\text{choice}_{\text{RL}}$ and $\text{reward}_{\text{RL}}$, written in $\lambda_{\text{eff}}^\text{RL}$. So, formally, the learner is a handler

$$H_{\text{RL}} \equiv \text{handler} \{ \text{return } x \mapsto \lambda m. \text{return } (m, x), \text{choice}_{\text{RL}}(k) \mapsto C_c, \text{reward}_{\text{RL}}(r;k) \mapsto C_r \}$$

such that

$$k : A_{\text{RL}} \to S_M(T) \vdash C_c : S_M(T) \quad r : \text{Real}, k : \text{Unit} \to S_M(T) \vdash C_r : S_M(T)$$

where $T$ is any type and $S_M(T) \equiv M \to (M \times T)$ is the type for the state monad with state $M$, that represents the memory of the learner. With this, we obtain a handler $H_{\text{RL}}$ with type $T \Rightarrow S_M(T)$ for any $T$.

In this handler, we can indeed encode a RL algorithm, as we did for the naive algorithm for MAB, because, informally:

- The memory $M$ can contain a value function, associating an heuristic to pairs of states and actions $(O_{\text{RL}} \times A_{\text{RL}})$. It can also be used to log information if needed, typically the previous choice, the number of times choice was called . . .
- The term for choice $C_c$ has a link with the policy of the learner. Indeed, a policy can be seen as a function of type $M \to A_{\text{RL}} \times M$, where, from an internal memory of the learner, we chose an action in $A_{\text{RL}}$ and we can also modify the memory if needed, for example to log this choice. With this policy, it is easy to obtain a computation $C_c$ with the type described above by composing with the continuation $k$.
- The term for $C_r$ has a link with the update of the value function after a choice. Indeed, learner usually modify their value function after a choice (or a sequence of choices). This can be seen as an update function of type $\text{Real} \times M \to M$, when we modify the memory (and thus the value function) according to the reward. Here, using a log in the memory can come in handy for most reinforcement learning algorithm to remember which choice is being rewarded. It is then easy to see that, from this update function, we can obtain a computation $C_r$ with the type described above by composing with the continuation $k$.

However, the learner will also need additional information from the environment, typically the current observable state or the list of available actions for this state, that is why we can also use the operations of the abstract interface in the computations $C_c$ and $C_r$. 

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4.3.2 Hiding the Memory of the Learner

The concept of this handler for the learner, which is typically a handler with the state monad, is standard but it is not very practical. Indeed, in the type \( T \Rightarrow S_M(T) \) we can see that the handled computation needs an initial memory, and the final memory is returned at the end of the computation. However, this handled computation should be done in the main program, and the user cannot provide an initial memory since it is not supposed to know the actual type of memory. Similarly, there is no reason for the user to have direct access to the memory of the learner, so this type \( M \) should be hidden in the computation type. So, in practice, we want a handler for the learner of type \( T \Rightarrow T \) for any \( T \). Fortunately, it is possible to do this from the previous handler, and it is a standard way to make the state invisible. Suppose that the learner provides an initial memory \( m_i \). Then, it can define the following handler (with the empty set of handled operations):

\[
H_{\text{hide}} \equiv \text{handler } \{ \text{return } f \mapsto \text{let } x = f \text{ } m_i \text{ in } \pi_2(x) \}
\]

with type \( S_M(T) \Rightarrow T \) for any \( T \). So, by composing this handler with the previous one, we obtain a handler of type \( T \Rightarrow T \), and the memory becomes totally hidden from the user. This construction is a way to mimic the \textit{finally} clause of the \textsc{eff} language, that we used in Section 2 using the standard syntax of handler.

4.3.3 The Interface

As the previous handler uses additional algebraic operations (the one from the interface), we need to handle them. Also, the previous handler is for the operations \texttt{choice}_\texttt{RL} and \texttt{reward}\texttt{RL} and we need to make the bridge between those operations and the one for the main program: \texttt{choice} and \texttt{reward}. We do this by defining two handlers in \( \lambda_{\text{eff}}^E \).

The first handler is a simple one, mainly abstracting the set of actions. For this, we need to define a bijection \( f : A_{RL} \rightarrow A_E \), which is easy to do as they are both finite sets with the same size. Then, the handler for abstracting actions is given by:

\[
H_{\text{Act}} \equiv \text{handler } \{ \text{return } x \mapsto \text{return } x, \\
\quad \text{choice}(k) \mapsto \text{choice}\text{RL}(();x.k \ (f \ x)) \\
\quad \text{reward}(r;k) \mapsto \text{reward}\text{RL}(r;x.k \ x) \}
\]

With this handler \( H_{\text{Act}} \), with type \( T \Rightarrow T \) for any \( T \), we can go from the operations from the main program to the operation for the learner. And now the only thing to do in order to successfully use the handler defined by the learner is to define the handler for abstract interface. With the interface we defined for this example, the two operations \texttt{observe}\texttt{RL} and \texttt{getactions}\texttt{RL}, then the interface handler would look like:

\[
H_I \equiv \text{handler } \{ \text{return } x \mapsto \text{return } x, \text{observe}\text{RL}(k) \mapsto C_o, \text{getactions}\text{RL}(a;k) \mapsto C_a \}
\]

with

\[
k : O_{RL} \rightarrow T \vdash C_o : T \quad o : O_{RL}, k : (A_{RL} \texttt{List}) \rightarrow T \vdash C_a : T
\]

so that the handler \( H_I \) has type \( T \Rightarrow T \) for any \( T \). Those computations may use the \texttt{observe} operation, and so with this handler, that can depend on the environment, we can handle the computations coming from the handler for the learner \( H_{RL} \). Now, only two operations remain to be handled \texttt{do} and \texttt{observe}

4.3.4 The Environment

To handle the environment, we only need the types and functions of \( \lambda_{\text{eff}}^E \), without \texttt{reward} and \texttt{choice}. The handler for the environment should have the shape:

\[
H_E \equiv \text{handler } \{ \text{return } x \mapsto C_r, \text{observe}(k) \mapsto C_o, \text{do}(a;k) \mapsto C_a \}
\]

such that this handler is typable with type \( T \Rightarrow F(T) \) for any type \( T \) where \( F(T) \) a transformation of \( T \). The actual computations for this handler depend strongly on the environment and so we cannot give
additional information for the general case. However, in order to illustrate this handler, we show how to implement it in the case where we have a specific model of the environment.

Suppose that we can model the environment by a type $E$ and two functions: $\text{Next}_E : A_E \times E \rightarrow E$ and $\text{Observe}_E : E \rightarrow O$. This may seem ad-hoc but it is in fact close to a Markov Decision Process which is a common model for the environment in RL algorithm [34]. Indeed, in this case the $\text{Observe}_E$ function corresponds to observing a reward and the current state of the Markov Decision Process, and the $\text{Next}_E$ function corresponds to moving in the MDP after an action in $A_E$. With those functions, we can define the following handler for the environment:

$$H_E \equiv \text{handler \{ return } x \mapsto \lambda e. \text{return } (e, x),$$

$$\text{observe}(k) \mapsto \lambda e. k(\text{Observe}_E e) e$$

$$\text{do}(a; k) \mapsto k() (\text{Next}_E (a, e))\}$$

with type $T \Rightarrow S_E(T)$ for any $T$. As we saw with the learner, it is possible in this case to hide the type $E$ for the main program. This is what we did for example in the description of the MAB environment in Section 2.

### 4.3.5 The Main Program with Handlers

And now, we can interpret all the four main operations for the main program. Thus, given a computation $C$ in $\lambda_{\text{eff}}^E$, we can handle all those operations with the computation:

$$\text{with } H_E \text{ handle (with } H_I \text{ handle (with } H_{RL} \text{ handle (with } H_{Act} \text{ handle } C)\})$$

With this composition, that we could see in the main program of Section 2, we obtain a handler of type $T \Rightarrow F(T)$ if the learner hides its memory as explained before. And, with a complete model of the environment, by hiding we can then obtain a type $T \Rightarrow T$.

### 5 Type Safety

In the previous section, we have introduced a core simply-typed calculus and used it to implement our running example. The choice of the language was driven by a simple goal: highlighting in the simplest way the essential features needed to implement our functional approach to reinforcement learning systems. The price we have to pay for such a simplicity is the lack of several desirable guarantees on program behavior. In fact, functional programming languages usually come endowed with expressive type systems — such as polymorphic [32], linear [40, 6, 17], and graded [24] type systems — ensuring the validity of nontrivial program properties at static time. Due to its simple type discipline, our calculus offers only but a few guarantees on program correctness, especially if one takes into account that the simplicity of the type system is not reflected at the operational level, which is instead characterised by highly expressive constructs, such as algebraic operations and handlers.

In light of that, it is desirable to strengthen the power of the type system defined in the previous section. Clearly, there are several possible extensions we may look for, and it thus natural to ask what path we should choose. In this section, we first identify three program properties well-known in the field functional programming that we believe to be particularly relevant when modelling reinforcement learning systems functionally, and then outline how such properties could ensured by way of expressive type systems. Let us begin with the target program properties.

1. **Locality of Operations.** As a first property, we would like to ensure (families of) algebraic operations to be used only in specific parts of programs. In MAB, for instance, we would like the code describing the environment, i.e. the slot machines, not to be able to perform the choice operation.

2. **Polymorphism.** Secondly, we would like to ensure code to be as modular as possible, this way enhancing its (re)usability. Concretely, that means having some form of polymorphism at disposal. This way, algorithms may be written just once, the same code being called in many different environments, possibly within the same program. In other words, the learner should use the types
and ARL in a restricted way, the only relevant information about them being that they are finite types, whose values can thus be enumerated. As an example, the naïve algorithm we have presented in Section 2 could be used in any context. In the particular case of MAB, we have defined ORL as the unit type Unit. However, we could very well replace Unit with any other finite type, meaning that our strategy scales to environments with more information.

3. Linearity and Order. Finally, we would like force algebraic operations to be performed in a specified order, this way ensuring the learner to have all the information it needs. Let us clarify this point with an example. In the main program presented in Section 2, we see that the flow of information follows a certain logic: for each iteration, we make a choice; we perform such a choice in the environment; we observe its results; and, finally, we give the reward. Such a logic is reflected in a specific execution order of algebraic operation, and breaking such an order may lead the learner to obtain false information.

Having isolated our target properties, we spend the rest of this section outlining some possible ways to force such properties by means of type systems. In particular, we focus on the use of type and effect systems [23], polymorphism [32], and graded modal types [24].

5.1 Algebraic Operations and Effect Typing

Type and effect systems [23] endowed traditional type systems with annotations giving information on what effects are produced during program execution. For our purposes, we can build upon well-known effect typing systems keeping track of which operations are handled during a computation [19]. Extending the simple type system of Section 4 in this way, we can then ensure well-typed programs to handle all their algebraic operations.

We now define effect typing for our core calculus and show how to take advantage of such a typing in the context of a reinforcement learning problem. We define an effect signature as a collection of operations and annotate the type of handlers with those signatures. Formally, leave the the syntax of terms as in Figure 2, but we replace the one of types as follows:

\[
T ::= B \mid \text{Unit} \mid T \times T \mid T \rightarrow^E T \mid T^{E \Rightarrow^E T}
\]

\[
E ::= \{ \text{op}: T \rightsquigarrow T \} \sqcup E \mid \emptyset
\]

Typing judgments for computations now are of the form \( \Gamma \vdash E \ C : T \), with the informal reading that that \( C \) has type \( T \) in a context where the only unhandled algebraic operations \( C \) can possibly perform are included in \( E \). Typing judgments for values, instead, remain the same (i.e. \( \Gamma \vdash V : T \)), as values not being executed, they cannot perform any algebraic operation at all. The inference system for our new typing judgment is given in Figure 5.

The most important rule in Figure 5 is the one for handlers (cf. the typing rule for open handlers by Kammar et al. [19]): it states that if a handler can handle a subset of the available operations (denoted by \( \{ \text{op}_i : T^o_i \rightsquigarrow T^a_i \mid 1 \leq i \leq n \} \) in Figure 5) and introduce some new operations in their computations (\( E'_f \)), then the final set of free operations contains the unhandled ones in the original set (\( E'_f \)) together with the new operations introduced by the handler.

This new type system satisfies better safety properties than the previous one. In particular, it enjoys preservation (as the type system of previous section) and progress.

**Theorem 2.** Progress and preservation hold for the type system in Figure 5. That is:

1. Preservation. If \( \Gamma \vdash E \ C : T \) and \( C \rightarrow C' \) then \( \Gamma \vdash E \ C' : T \).

2. Progress. Suppose that all functions for base types are total for the constants in the language. If \( \vdash E \ C : T \), then either there exists \( C' \) such that \( C \rightarrow C' \), or \( C \) has the shape \( \text{return} V \) for some \( V \), or \( C \) has the shape \( \text{op}(V, x, C') \) with \( \text{op} \in E \).

**Proof.** We prove progress by induction on \( C \). The use of typed effects is important to keep track of what are exactly the set of operations that can appear, and the commutation of \( \text{op} \) with \( \text{let} \) and \( \text{handler} \) is also essential in this proof. The hypothesis on base type functions ensures that an application of a base
The learner, seen by the environment, the interface used by the learner. These are, respectively, the set of effects for the environment, the learner seen by the environment, the abstract interface, the initial memory, and the learner seen by the environment. Notice that in this example upon. Let us define the following sets of effects:

- $E_{E} = \{\text{observe} \cdot \text{Unit} \leadsto O_{E} \cdot \text{do} : A_{E} \leadsto \text{Unit}\}$
- $E_{RL} = \{\text{choice} \cdot \text{Unit} \leadsto A_{RL} \cdot \text{reward} : \text{Real} \leadsto \text{Unit}\}$
- $E_{RL}^{Abs} = \{\text{choice}_{RL} \cdot \text{Unit} \leadsto A_{RL} \cdot \text{reward}_{RL} : \text{Real} \leadsto \text{Unit}\}$
- $E_{i}^{Abs} = \{\text{observe}_{RL} \cdot \text{Unit} \leadsto O_{RL} \cdot \text{get_actions} : O_{RL} \leadsto A_{RL} \cdot \text{List} ; \cdots\}$

These are, respectively, the set of effects for the environment, the learner seen by the environment, the learner seen by the learner, and the interface used by the learner.

**The Learner**  The handler $H_{RL}$ for the learner should have the type

$$\vdash H_{RL} : T \overset{E_{RL}}{\leadsto} E_{RL}^{Abs} S_{M}(T),$$

where $T$ is any type and $S_{M}(T) \equiv M \rightarrow M \times T$. This means that the learner handles the operations $\text{choice}_{RL}$ and $\text{reward}_{RL}$ using only the operations in the interface $E_{RL}^{Abs}$. Thus, with this type, we ensure that the learner does not have a direct access to the environment, and it only sees effects for the abstract types $A_{RL}$ and $O_{RL}$ and not the concrete types $O_{E}$ and $A_{E}$. Then, with hiding, we can obtain a handle of type $T \overset{E_{RL}}{\leadsto} E_{RL}^{Abs} T$ as long as the learner provides an initial memory. Notice that in this hiding handler of type $T \overset{E_{RL}}{\leadsto} E_{RL}^{Abs} T$, the initial memory can depend on the abstract interface, this being useful in those cases in which some parameters in the initial memory have to be exposed.
The Interface  The interface is separated into two handlers: $H_{Act}$ and $H_I$. This first handler is only a bridge between abstract actions and concrete actions. It is not difficult to see that the former should have the type:

\[ \vdash H_{Act} : T^{E_{RL} \not\subseteq} E_{Abs}_{RL} T, \]

for any type $T$, thus handling the operations choice and reward seen from the exterior to the internal one of the learner. As for the interface, it should be a bridge between the environment and learner abstracted by the user, so it should have the type:

\[ \vdash H_I : T^{E_{Abs}_{I} \subseteq} E_{E} T, \]

where the fixed set of operations ($E_f$ in the rule) would be $E_E$.

The Environment  As for the environment, the handler only has to handle the two operations observe and do, without using the learner nor the interface. So, it should have the type:

\[ \vdash H_E : T^{E_E} \Rightarrow \emptyset F(T). \]

5.1.1 Composition of Handlers and Main Program

Now that we have types for each of the four handlers needed, we compose them. An important point of our effect typing is that it supports weakening: in the handler rule, we can always take a larger set $E_f$, so that we are licensed to use the handler in different contexts.\(^3\) In our case, it means that we can give the following types to handlers:

\[ \vdash H_{RL} : T^{E_{RL} T \not\subseteq} E_{Abs}_{RL} T \]
\[ \vdash H_{Act} : T^{E_{RL} T \not\subseteq} E_{Abs}_{RL} T \]
\[ \vdash H_I : T^{E_{Abs}_{I} \subseteq} E_{E} T \]
\[ \vdash H_E : T^{E_E} \Rightarrow \emptyset F(T). \]

Consequently, we can compose them to obtain:

\[ H_E \circ H_I \circ H_{RL} \circ H_{Act} : T^{E_{RL} T} \Rightarrow \emptyset F(T). \]

When we have a model of the environment, with hiding, we have

\[ H_E \circ H_I \circ H_{RL} \circ H_{Act} : T^{E_{RL} T} \Rightarrow \emptyset T. \]

This shows that the main program can use the four main effects, as expected. Without hiding, we would have a type which is essentially the composition of the two state monads on $E$ and $M$, meaning that the program should be understood in a context with both an environment and a memory for the learner.

5.2 Polymorphism

The typing rule for handlers allows us to assign handlers arbitrary types $T$. To ensure such types not to be inspected by programs, and to ensure uniqueness of typing derivation of handlers, it is natural to extend our type system with polymorphic types. Moreover, the base types we presented for the language, $A_{RL}, O_{RL}, A_{E}$ and $O_{E}$ all depend on the actual environment, whereas we expect the handler $H_{RL}$ for the learner not to do so. Consequently, we may rely on polymorphism to enforce $A_{RL}$ and $O_{RL}$ not to be inspected, this way giving a unique polymorphic type to the handler $H_{RL}$ and thus ensuring the latter to be usable in any environment.

Languages, however, can be polymorphic in many ways: especially in presence of effects. We now outline what kind of polymorphism is needed for our purposes. As a first requirement, we need to be able to declare polymorphic handlers; write their implementation once and for all; and then use them in different contexts. To do that, we believe having handlers as first-class citizen of the language, as we

\(^3\)This is one of the main reasons why this rule is very useful to achieve polymorphism.
presented in Figure 2 is necessary. Secondly, we also want to declare polymorphic effects, so to give a sense to those polymorphic handlers.

We take as an inspiration the language by Biernacki et al. [8]. However, since the aforementioned language does not consider handlers as first-class citizens, we cannot directly rely on that. In the rest of this section, we informally outline a possible polymorphic type assignment for our language, leaving its formal definition and analysis (such as type soundness) as future work.

We consider different kinds, starting from the base kinds of types, effects, and rows, and building complex kinds using a functional arrow. Formally, kinds are generated by the following grammar:

$$\kappa := T \mid E \mid R \mid \kappa \rightarrow \kappa.$$

A row $\rho$ is a sequence of effects $(E_1 \mid \rho')$, with intuitively the same meaning as effect signature in our last type system, the main difference being that we can have polymorphism on row, and an effect can appear several in a row (but possibly with different instantiation of their polymorphic types). We would like to write, as in [8], the following computation:

$$\text{effect } E_{RL}^{Abs} = \forall \alpha_A :: T.\{\text{choice}_{RL} : \text{Unit} \twoheadrightarrow \alpha_A; \text{reward}_{RL} : \text{Real} \twoheadrightarrow \text{Unit}\} \in \cdots$$

declaring a new effect, called $E_{RL}^{Abs}$, waiting for a type $\alpha_A$ representing the type (of kind $T$) of actions, and then declare the types of those two algebraic operations. The kind of $E_{RL}^{Abs}$ would then be $T \rightarrow E$, so given a type of kind $T$ this indeed gives an effect of kind $E$. Similarly, the abstract interface would be declared with:

$$\text{effect } E_I^{Abs} = \forall \alpha_A :: T, \alpha_O :: T.\{\text{observe}_{RL} : \text{Unit} \twoheadrightarrow \alpha_O; \text{getactions}_{RL} : \alpha_O \twoheadrightarrow \alpha_A \text{ List;} \cdots\}$$

And then, in this context, the type of the handler for the learner, $H_{RL}$ would be, after hiding:

$$H_{RL} : \forall \alpha_A :: T, \alpha_O :: T, \alpha_T :: T, \rho :: R.\alpha_T \langle E_{RL}^{Abs} \alpha_A | \rho \rangle \Rightarrow \langle E_I^{Abs} \alpha_A \alpha_O | \rho \rangle \alpha_T$$

meaning that for any type of actions $\alpha_A$, any type of observations $\alpha_O$, any type of computation $\alpha_T$ and any effect environment $\rho$, the handler for the learner has an identity type on the computation, handling the operations in $E_{RL}^{Abs} \alpha_A$ and introducing the new operations of $E_I^{Abs} \alpha_A \alpha_O$. In practice, we would like $\alpha_A$ and $\alpha_O$ to always represent base types, and especially finite sets, so we would need some more restriction on this polymorphism. But informally, this is the shape of type we would like for the handler written by the learner, as it ensures that it can be used in any environment without modifications. Notice that in order to do this, we want handlers as first-class citizens but we do not need the full power of polymorphism since the algebraic operations themselves do not need to be polymorphic. This kind of polymorphism that we need differs slightly from the polymorphism mainly studied in the literature: usually, it allows the programmer to use different handlers for the same operation, however what we want in our case is to use a unique handler for different contexts.

### 5.3 Impose an Order on Operations

Our last property of interest focuses on linearity and order of algebraic operations. For instance, we would like to ensure that when a call to $\text{choice}$ is made, it is always followed by a $\text{do}$ and a $\text{reward}$ ($\text{observe}$ can be useful to give the reward but it is not mandatory as it should have no effect on the environment). We now outline how such a goal can be achieved in presence of algebraic operations relying on suitable graded type systems [20]. Such an approach, however, is not readily extendable to effect handlers. As far as we know, graded types have not been extended to handlers.

Informally, we define a monoid on algebraic operations and associate each computation to an element of this monoid, the latter giving information on the operations executed during program evaluation. Sequential composition of computations corresponds to monoid multiplication, basic computations to the unit of the monoid, and algebraic operations to the corresponding elements of the monoid. A typing rule (for computations) then has shape $\Gamma \vdash_E C : T; m$, where $m$ is an element of the monoid with, for example, the following rules:

\[
\begin{align*}
\Gamma \vdash_V V : T ; \Gamma +_E \text{ return } V & : T ; 1 \\
\Gamma \vdash_E T & \quad \Gamma \vdash E \, C_1 : T_1 ; m_1 \quad \Gamma , x : T_1 \vdash_E C_2 : T_2 ; m_2 \\
\Gamma \vdash_E \text{ let } x = C_1 & \text{ in } C_2 : T_2 ; m_1 \cdot m_2 \\
(\text{op} : T_p \rightarrow T_a) & \in E \\
\Gamma \vdash V : T_p \quad \Gamma , x : T_a \vdash_E C : T ; m \\
\Gamma \vdash_E \text{ op}(V ; x , C) & : T ; m_{\text{op}} \cdot m
\end{align*}
\]
where 1 is the unit of the monoid, \( m_1 \cdot m_2 \) is monoid multiplication and \( m_{\text{op}} \) is the element of the monoid corresponding to the operation \( \text{op} \).

In our case, the monoid would be the free monoid on the three element associated to the algebraic operations \( \text{choice}, \text{do} \) and \( \text{reward} \), with \( \text{observe} \) confounded with the unit of the monoid, with the following equation

\[
m_{\text{choice}} \cdot m_{\text{do}} \cdot m_{\text{reward}} = 1
\]

and then, we would like the main program to be associated to an element of this monoid with the shape \( m_{\text{do}} \). With this, we can ensure that, if we forget about \( \text{observe} \) since it has no effect at all, then any \( \text{choice} \) is always followed by exactly one \( \text{do} \) and one \( \text{reward} \). Moreover, only the \( \text{do} \) operation can be done outside of this loop, because essentially the user can make choices without calling the learner, and for the point of view of the learner then the user would just be a part of the transition function of the environment.

However, this method, which is standard for effects, does not generalise well in presence of handlers. Indeed, we need to take into account the potentially new operations introduced by a handler, and intuitively the handler would then represent a map from the new operations to an element of the monoid. For a very simple example, consider the operation \( \text{choicedoandobserve}(k) \mapsto \text{choice}(\text{a.do(a; \_observe(o.k (a, o))})) \)

doing the three operations \( \text{choice}, \text{do} \) and \( \text{observe} \) in sequence. Then, this handler should be typed with the information that the new operation \( \text{choicedoandobserve} \) is associated to the element \( m_{\text{choice}} \cdot m_{\text{do}} \cdot m_{\text{observe}} \).

We believe such a type system should be feasible, but we have no concrete formalization of this. So, as for polymorphism, those kind of type system would be desirable for safety but we leave the formalization to future work.

### 6 Related Work

The interaction between programming language theory and machine learning is an active and flourishing research area. Arguably, the current, most well-established products of such an interaction are the so-called Bayesian \([36, 37, 35, 9, 11]\) and differentiable \([7, 3]\) programming languages, and their associated theories \([18, 39, 1, 25, 33]\).

Choice-based operations are not new in programming language theory, as they first appeared in the context of computational effects (more specifically, choice operations has been first studied as nondeterministic choices \([21]\)). However, their integration with reward constructs as a way to structure machine learning systems as data-driven, decision-making processes is quite recent. As an example of that, the programming language SmartChoice \([10]\) integrates choice and rewards operations with traditional programming constructs. To the best of the authors’ knowledge, the first work dealing with semantic foundations for choice-based programming languages is the recent work by Abadi and Plotkin \([2]\). There, a modular approach to higher-order choice-based programming languages is developed in terms of the selection monad \([14, 16, 15]\) and its algebraic operations \([27, 26, 28]\), both at the level of operational and denotational semantics. As such, that work is the closest to ours.

The results presented in this paper has been obtained starting from an investigation of possible applications of the aforementioned work by Abadi and Plotkin to modelling reinforcement learning systems. We have explained in Section 5 why, when dealing with reinforcement learning, moving from the state to the state monad is practically beneficial. From a theoretical point of view, such a shift can be seen as obtained by looking at comodels \([32]\) of choice operations, which directly leads to view reinforcement learning algorithms as ultimately defining stateful runners \([38, 4]\) for such operations, and thus as generic effects \([29]\) for the state monad. Following this path, we have consequently relied on handlers \([29, 5, 31]\) to define modular implementations of such generic effects, and thus of reinforcement learning algorithms. Even if handlers and algebraic operations are well-known tools in functional programming, to the best of our knowledge the present work is the first one investigating their application to modelling reinforcement learning systems.

Finally, we mention that applications of functional programming techniques in the context of machine learning similar in spirit to ones investigated in this work, as well as in the work by Abadi and Plotkin
have been proposed in terms of induction and abduction constructs, rather than in terms of choices and rewards.

7 Conclusion

In this article, we address the problem of implementing reinforcement learning algorithms within a functional programming language. The starting idea is to manage the interaction between the three involved agents (namely the learner, the user, and the environment) through a set of algebraic operations and handlers for them. This way, a certain degree of modularity is guaranteed.

The path we followed starts from practice, i.e. from the implementation of these ideas in a concrete language such as EFF, and then progressively moves towards theory, i.e. towards the study of a paradigmatic language for effects and handlers in which these ideas can be formalized, thus becoming an object of study. Technically, the most important contribution of this work lies in highlighting where the state of the art is deficient with respect to the type of type safety and code reuse properties that one would like.

Some ideas for future works can be sought precisely in the direction just mentioned and, in particular, in the definition of systems of types sufficiently powerful to guarantee that the algebraic operations involved are actually carried out in the good order, or that they allow the right level of polymorphism.

An alternative to powerful type systems for some safety properties could also be the use of some advanced features of programming language for abstraction. For example, we presented the abstract interface as a handler, with a learner polymorphic in the types of actions and observation. In OCAML, an alternative could be to use modules and signatures: an abstract interface would be represented by a signature, with abstract data types for actions and observations, and additional functions to replace the operations. The learner would then have access to elements of this signature, and the user would need to implement an actual module respecting this signature in order to use the learning algorithm. This way, the fact that the learner can be written independently from the environment would come from the OCAML abstraction, instead of polymorphism in a type and effect system.

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