HOW MANY LATIN RECTANGLES ARE THERE?

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Abstract. Until now the problem of counting Latin rectangles $m \times n$ has been solved with an explicit formula for $m = 2, 3$ and 4 only. In the present paper an explicit formula is provided for the calculation of the number of Latin rectangles for any order $m$. The results attained up to now become particular cases of this new formula. Furthermore, putting $m = n$, the number of Latin squares of order $n$ can also be obtained in an explicit form.

0. Introduction

A Latin rectangle $m \times n$ is a matrix with $n$ rows and $n$ columns the elements of which are chosen in $[n] = \{1, \ldots, n\}$ so that two elements are never the same, neither on the same row nor on the same column. It is said that such a Latin rectangle has order $m$. From the definition it follows that $m \leq n$ and that each row of a Latin rectangle is a permutation of $[n]$.

Furthermore it is clear that it is always possible to standardize the first row making it the same as the permutation $12\ldots n$. In such a case we say that we are dealing with a “reduced Latin rectangle”.

If we call the number of Latin rectangles $m \times n$ with $L(m, n)$ and the number of reduced Latin rectangles with the same dimension with $K(m, n)$, it is clear that $L(m, n) = n!K(m, n)$.

The problem of counting Latin rectangles has engaged several generations of mathematicians, but the results reached up to now, as we will see later, are limited to certain special cases.

With this paper we intend to finally supply an explicit formula for the calculation of $K(m, n)$ for any value of the order $m$.

The partial results attained up to now will result special cases of such a formula.

The result obviously also allows the calculation of the number of Latin squares of order $n$, which represent the special case of $m \times n$ Latin rectangles in which $m$ takes on its maximum admissible value $n$. 
A Latin rectangle consists of \( m \) permutations of \([n]\) which, taken two by two, don’t have fixed points. It is from this point of view that the problem was initially studied by Montmort, Euler and Lucas.

It seems that the solution in the simplest case \( m = 2 \), known as “derangement problem”, can be found going back to Montmort 1713 [6]. It consists of the number \( D_n \) of the permutations of \([n]\) without fixed points given by:

\[
D_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}
\]

and equivalent to \( K(2, n) \).

In 1891 Lucas expounded the famous “ménage problem” which consists of counting the ways of arranging \( n \) couples at a round table so that men and women alternate and no husband and wife are adjacent to one another. The problem, examined since 1878 by Tate, was also studied by Cayley and Muir, however no satisfactory results were reached.

The solution to ménage problem is equivalent to the enumeration of all the permutations of \([n]\) which are discordant with both the permutations \( 12\ldots n \) and \( 23\ldots n1 \). The generalisation of the above mentioned problem — known as “the cyclical Touchard problem of index \( m \)” or “problem of \( m \)-discordant permutations” — proposes the counting of all the permutations \( \sigma \) of \([n]\) so that:

\[
\forall i \in [n], \sigma(i) - i \not\equiv 0, 1, \ldots, m - 1 \pmod{n}.
\]

It is also said that this problem is equivalent to the enumeration of the “very reduced Latin rectangles” of order \( m + 1 \), the number of which is indicated with \( V(m, n) \), that have the first \( m \) permutations \( \sigma_j \), with \( j \in [m] \), in the canonical form: \( \sigma_j(i) = i + j - 1 \pmod{n} \).

The solution in the simplest case \( m = 2 \) was found by Touchard in 1934 [15] and consists of Touchard’s famous numbers \( U_n \), equivalent to \( V(2, n) \), expressed by:

\[
U_n = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!.
\]

For the subsequent case \( V(3, n) \) recursive algorithms have been obtained by Riordan [12] and by Yamamoto [18]. However an explicit formula was only provided in 1967 by Moser [7]. In the case \( V(4, n) \), the biggest yet dealt with, there is only one recursive result by Whitehead in 1979 [16] and possibly an explicit formula by Nechvatal [8] also in 1979.

However let us return to the more general and more complex problem of the calculation of \( K(m, n) \), which represents the aim of this paper.

The first attempts at the calculation of \( K(3, n) \) go back to Jacob and to Kera-wa-\(1941 \) [5], found a recursive formula. The following tidy explicit
formula for \( K(3, n) \) is, on the other hand, attributed to Yamamoto (see [1] and [10]):

\[
K(3, n) = n! \sum_{a+b+c=n} (-1)^b 2^c \frac{a!}{c!} \binom{3a+b+2}{b}.
\]

Furthermore, in 1944, Riordan obtained an expression of \( K(3, n) \) in terms of Touchard’s numbers \( U_n \) and subsequently, in 1946 [11], the well known formula:

\[
K(3, n) = \left[ \frac{n}{2} \right] \sum_{k=0}^{n} \binom{n}{k} D_k D_{n-k} U_{n-2k}
\]

(with \( U_0 = 1 \)) which expresses \( K(3, n) \) in terms of \( D_k \) and \( U_k \).

It is necessary to say that until now, in this line of research, no other progress has been achieved since, for \( m > 3 \), it has not been possible to obtain \( K(m, n) \) in terms of \( K(i, n) \) and \( V(j, n) \) with \( i, j < m \).

The case of \( K(4, n) \), which is the most complex yet to be dealt with successfully, was only solved with an explicit formula in 1979, this was achieved independently by Nechvatal [8] and by Athreya, Pranesachar and Singhi [1].

Subsequently, in 1980, Pranesachar [10] and Nechvatal [9], by different means, found a way to express \( K(m, n) \) for any value of \( m \) by means of the Möbius function of the lattice of partitions of a set. The limit of these research works is that they take the calculation of \( K(m, n) \) back to the enumeration of other combinatorial objects, such as the partitions of an integer, for which no explicit formulas are known, and thus don’t allow an explicit formula for \( K(m, n) \) to be obtained. A further tidy result of this type was achieved by Gessel in 1987 [3].

Until now, then, no explicit formula is known which permits the calculation of \( K(m, n) \) whatever the value of \( m \).

We would like to conclude this section remembering that another interesting line of research tried to get asymptotic expressions of \( K(m, n) \). The first significant paper of this kind is attributed to Erdös and Kaplansky [2] in 1946, subsequent results were obtained by Yamamoto in 1951 [17] and by Stein in 1978 [14].

Finally, in recent years, Godsil and McKay [4] achieved an asymptotic valuation of \( V(m, n) \).

2. Notation and preliminaries

Very useful concepts in the study of permutations without fixed points are those of board and of rook polynomial.

A board is a nonempty subset of \( \mathbb{P} \times \mathbb{P} \) (\( \mathbb{P} = \text{set of positive integers} \)), the elements of board are called squares. Considering a board \( C \) it is usually indicated with \( r_k(C) \) the number of different ways of placing \( k \) non-attacking rooks on it. The rook polynomial of \( C \) in the symbolic variable \( x \), that we’ll call \( R(C) \), is given by \( \sum_k r_k(C) x^k \).
If we write $S_n$ for the set of all permutations of $[n]$, then every $\sigma \in S_n$ can be thought of as a board, called the “graph” of $\sigma$, the squares of which are the couples $(i, \sigma(i)) \forall i \in [n]$. It is furthermore obvious that $m$ permutations, two by two without fixed points, make up a board of $m \cdot n$ squares, if in such a board the first permutation is the identical one $\sigma(i) = 1$, it will hereon be indicated with $C^{(m)}$.

We furthermore state, to have a greater number of symbols, that a number put up to the right of a symbol doesn’t denote a raising to a power of the same, but it acts as a new symbol (therefore $a^2$ it isn’t the square of $a$). When we want to indicate $a$ raised to $m$, we write $(a)^m$.

We will call $C^g$ the board, included in $C^{(m)}$, formed by the $g$-th permutation of $C^{(m)}$ $(g \in [m])$. Since, as has been mentioned previously, to speak about $C^{(m)}$ it is the same as to speak about a reduced Latin rectangle of order $m$, we will refer often to $C^g$ as the $g$-th line of $C^{(m)}$ (which is not to be confused with the $g$-th row or column of $C^{(m)}$ like a board that are different things). We can even say that a subset of $C^g$ has “grade” $g$.

At this point let us remember a classic result which joins the rook polynomials to the permutations without fixed points. We consider the permutations as boards and, taking the board $B \subseteq [n] \times [n]$, we indicate with $N_s(B)$ the number of permutations of $[n]$ which have exactly $s$ squares in common with $B$. And so giving us the following tidy relation:

$$N_s(B) = \sum_{s}^{n}(-1)^{k-s} \binom{k}{s}(n-k)!r_k(B)$$

for the proof of this see [13] chapter 2.3.

Now let us introduce some conventions in the use of symbols. The sets are always indicated with capital letters and the number of the elements which make them up with the corresponding lower case letter (therefore $a = |A|$). $C(A)$ will be the complementary of the set $A$ in the universe set.

$T^{(m)}$ will represent a generic system of independent rooks — that is which don’t attack each other — put on $C^{(m)}$ and $T^g$, with $g \in [m]$, will be the part of the system contained on the $g$-th line of $C^{(m)}$ ($T^g = T^{(m)} \cap C^g$) and thus it will be: $T^{(m)} = T^1 \cup T^2 \cup \ldots \cup T^m$.

Furthermore, if $A \subseteq C^{(m)}$, we will say that $R(A)$ is the projection for rows of $A$ in $C^{(m)}$, consisting of all the squares of $C^{(m)}$ which have any square of $A$ in their own row (obviously $A \subseteq R(A)$). We will also say that $R_g(A)$ is the projection for rows of $A$ on the $g$-th line of $C^{(m)}$ and we will put $R_g(A) = R(A) \cap C^g$, with the consequence that: $R(A) = R_1(A) \cup \ldots \cup R_m(A)$. Similarly, speaking of columns instead of rows, we can define $C(A)$ and $C_g(A)$.

Finally we define the set $I(A) = R(A) \cap C(A)$ as the “impression” of $A$ and the set $O(A) = R(A) \cup C(A)$ as the “shadow” of $A$. Thus $O(T^{(m)})$ will be the set of all the squares of $C^{(m)}$ subject to the attack of any rook of $T^{(m)}$ and which therefore can’t contain other independent rooks from those of $T^{(m)}$. 
To indicate the number of ways in which the set $A$ can generally be arranged, considering the restrictions which have been imposed on it, we will write $\pi(A)$. So we shall obtain that $K(m, n) = \pi(C(m))$.

As it is known, a partition in $k$ blocks of a set $A$, with $i \in [k]$, two by two disjoint and such that $\bigcup_1^k A_i = A$. We will indicate with $\prod(A)$ the set of the partitions of $A$; if $\pi \in \prod(A)$ and $\pi$ has $k$ blocks, we say that $|\pi| = k$; finally we put $\prod_n = \prod([n])$.

Let us also remember that the refinement of two partitions $\pi_1$ and $\pi_2$, with $\pi_1, \pi_2 \in \prod(A)$, is the partition of $A$ consisting of all the non empty intersections of some block of $\pi_1$ with some block of $\pi_2$.

If $X = \{x_1, \ldots, x_s\}$ is a set of variables, we put, for economy of space:

$$\left(\begin{array}{c} n \\ X \end{array}\right) = \left(\begin{array}{c} n \\ x_1, \ldots, x_s \end{array}\right)$$

and furthermore:

$$\sum X = \sum_{x_i \in X} x_i ; \prod X = \prod_{x_i \in X} x_i ; \prod X! = \prod_{x_i \in X} x_i!$$

and, ranging each $x_i$ within its own domain:

$$\sum X = \sum_{x_1} \cdots \sum_{x_s} .$$

We will indicate with $(n)_k = n(n-1)\ldots(n-k+1)$ the falling factorial of $n$ and with $\langle n \rangle_k = n(n+1)\ldots(n+k-1)$ the raising factorial of $n$. Now, given that $(n)_n = n!$, we intend to put $\langle n \rangle_n = n!$ even if in other literature this symbolism has been used to indicate the subfactorial of $n$.

Let us conclude this section with some recalls relative to the permutations of $[n]$.

If $\sigma \in \mathfrak{S}_n$ and $\sigma(i) = a_i \forall i \in [n]$, we can also say that $\sigma$ corresponds to the word $a_1 \ldots a_n$. It is well known, see [13] pages 17 and following, that $\sigma$ can be shared in an unambiguous way in the product of disjoint cycles on the elements of $[n]$ and that it can have a “standard representation”, which we will indicate with $s_1 \ldots s_n$, writing a) the elements of each cycle with the largest element first and b) arranging the cycles in increasing order of their largest element. In such a case each cycle will start with a left–to–right maximum i. e. with an element $s_i$ so that $s_i > s_j$ for each $j < i$.

Another way to describe $\sigma$ can be achieved by indicating with $b_i$ the number of elements $j$ of its standard representation on the left of $i$ with $j > i$ and defining $I(\sigma) = (b_1, \ldots, b_n)$ the “inversion table” of $\sigma$. In fact it can be easily proved, see [13] Proposition 1.3.9, that between the $\sigma \in \mathfrak{S}_n$ and the $I(\sigma)$ there is a bijection and furthermore that: $0 \leq b_i \leq n - i, \forall i \in [n]$.
3. The Associated Partitions

In this section we want to show how the computation of \( K(m, n) \) can be taken back to the enumeration of a double system of partitions of \([n]\).

The first step in the argument is to reiterate the method of calculation by means of systems of independent rooks expressed by the formula (2.1).

Let us suppose that we want to determine the number \( K(m, n) \) of all the possible \( C(m) \) and that we have already counted all the possible arrangements of the first \( m - 1 \) lines \( C(m-1) \), the number of ways in which \( C^m \) can be arranged can be obtained easily, by means of the formula (2.1), once \( r_k \left( C(m-1) \right) \), for \( k = 0, \ldots, m - 1 \), are known.

In fact, if \( T_{1/m-1}^m = T_{1/m-1}^1 \cup \cdots \cup T_{1/m-1}^{m-1} \) is the independent generic system of rooks on \( C(m-1) \) and therefore \( t_{1/m-1}^{(m-1)} = t_{1/m-1}^1 + \cdots + t_{1/m-1}^{m-1} = k \), we will have that:

\[
(3.1) \quad \pi(C^m) = \sum_{l=1}^{n} (-1)^{l} \frac{t_{1/m-1}^{(m-1)}}{(l-1)!} \left( n - t_{1/m-1}^{(m-1)} \right)! \pi \left( T_{1/m-1}^{(m-1)} \right).
\]

Now we are trying to calculate \( \pi(C^{m-1}) \) with the same assumptions. Now \( \pi(C^{m-1}) = \pi(T_{1/m-1}^{m-1}) \pi \left( C^{m-1} - T_{1/m-1}^{m-1} \right) \) and, if we consider the generic \( T_{1/m-2}^{m-2} \subseteq T_{1/m-1}^{m-1} \) and \( T_{2/m-2}^{m-2} \subseteq T_{2/m-2}^{m-2} \), we will have that:

\[
(3.2) \quad \pi(C^{m-1}) = \sum_{l=1}^{m-2} \sum_{l=1}^{m-2} (-1)^{l} \frac{t_{1/m-2}^{(m-2)}}{(l-1)!} \left( t_{1/m-1}^{m-1} - t_{1/m-2}^{m-2} \right)! \pi \left( T_{1/m-2}^{(m-2)} \right) (-1)^{l} \frac{t_{1/m-2}^{(m-2)}}{(l-1)!} \left( n - t_{1/m-1}^{m-1} - t_{2/m-2}^{m-2} \right)! \pi \left( T_{2/m-2}^{(m-2)} \right).
\]

Repeating the argument for the subsequent line \( C^{m-2} \), we see that:

\[
(3.3) \quad C^{m-2} = T_{1/m-2}^{m-2} \cup \left( T_{1/m-1}^{m-2} - T_{2/m-2}^{m-2} \right) \cup \left( T_{1/m-1}^{m-2} \cap T_{2/m-2}^{m-2} \right)
\]

\[
\cup \left( T_{2/m-2}^{m-2} - T_{1/m-1}^{m-2} \right) \cup \left( C^{m-2} - (T_{1/m-1}^{m-2} \cup T_{2/m-2}^{m-2}) \right)
\]

and, calling: \( T_{1/m-3}^{(m-3)}, T_{2/m-3}^{(m-3)}, T_{3/m-3}^{(m-3)}, T_{4/m-3}^{(m-3)}, T_{5/m-3}^{(m-3)} \) the generic \( T^{(m-3)} \) included in the impression of each of the five components of the union of which at
(3.3), we will be able to say, referring always to (2.1), that:

\[
\pi(C^{m-2}) = \pi \left( T_{1/m-2}^{m-2} \right) \pi \left( T_{1/m-1}^{m-2} - T_{2/m-2}^{m-2} \right) \\
\cdot \pi \left( T_{1/m-1}^{m-2} \cap T_{2/m-2}^{m-2} \right) \pi \left( T_{2/m-2}^{m-2} - T_{1/m-1}^{m-2} \right) \pi \left( C^{m-2} - T_{1/m-1}^{m-2} \right) \\
\cup T_{1/m-2}^{m-2} \cup T_{2/m-2}^{m-2} \right) = \sum_{t_i^{(m-3)}} (-1)^{t_i^{(m-3)}} \left( t_{1/m-2}^{m-2} - t_{1/m-3}^{m-3} \right)!
\]

Continuing in this way in order to count all the possible arrangements of the line \(l\), we should take all the possible subsets of rooks which are situated on it, considering the partition refinement of \(C^l\) consisting of all their non empty intersections \(T_j^l\), \(j = 1, \ldots, p_l\), and of the complementary of their union \(T_0^l = C^l - \bigcup_j T_j^l\) and finally choosing \(p_l + 1\) systems of independent rooks \(T_{j/l-1}^{(l-1)}\), \(j = 0, 1, \ldots, p_l\), with \(T_{j/l-1}^{(l-1)} \subseteq I(T_j^l)\).

Then we will have:

\[
\pi(C^l) = \prod_{0}^{p_l} \pi(T_j^l) = \sum_{t^{(l-1)}_{j/l-1}} (-1)^{t^{(l-1)}_{j/l-1}} \prod_{0}^{p_l} \left( t_j^{(l-1)} - t_{j/l-1}^{(l-1)} \right)! \pi(T_j^{(l-1)})
\]

and we can conclude that:

\[
K(m, n) = \prod_2^{m} \pi(C^l) = \sum_{t^{(l-1)}_{j/l-1}} (-1)^{t^{(l-1)}_{j/l-1}} \prod_{2}^{m} \prod_{0}^{p_l} \left( t_j^{(l)} - t_{j/l-1}^{(l-1)} \right)! \pi(T_j^{(l-1)})
\]

The situation, therefore, seems to be somewhat complex, but an idea which allows us to control it is that of identifying any subset of \(C^{(m)}\) by means of its two projections, for rows and for columns, on the main diagonal \(C^1 = \{(i, i) \mid i \in [n]\}\).

Given a set of grade \(g A^g\), we consider in fact \(R_1(A^g)\) and \(C_1(A^g)\). Now, if \(g = 1\), \(R_1(A^1) = C_1(A^1) = A^1\) and projections and set coincide. If, on the other
hand, $g > 1$, then the set $A^g$ determines $\mathcal{R}_1(A^g)$ and $\mathcal{C}_1(A^g)$ in one way only. Vice versa given $A^g_1, A^g_2 \subseteq C^1$ with $|A^g_1| = |A^g_2| = p$, $A^g$ will be one of the $p!$ permutations of the square board $\mathcal{R}(A^g_1) \cap C(A^g_2)$. Furthermore, if $B$ is a board of forbidden positions for $A^g$ and we set $\overline{B} = B \cap \mathcal{R}(A^g_1) \cap C(A^g_2)$, we will have that:

$$\pi(A^g) = \sum_{k=0}^{p} (-1)^k (p-k)!r_k(\overline{B}) .$$

In the light of this new approach, a system of independent rooks $T^{(l)} = T^1 \cup \ldots \cup T^l$ determines the subsets $R^i = \mathcal{R}(T^i)$ and $C^i = \mathcal{C}(T^i)$, for $i \in [l]$, and:

$$R^0 = \mathcal{R}(\mathcal{C}(T^{(l)})) = C \left( \bigcup_i R^i \right)$$

and $C^0 = \mathcal{C}(\mathcal{C}(T^{(l)})) = C \left( \bigcup_i C^i \right)$. Thus it characterizes two partitions, each one with $-m$ must coincide, is equivalent, from what has been said, to calculating the product of the various blocks of the set of columns.

Consequently, if we put together all the elements of grade $i$ of the various partitions $R^i_{j/l-1}$ and we set $R^i_l = \bigcup_{j=0}^{l} R^i_{j/l-1}$, we will obtain that $\{R^i_l\}$, with $i = 0, \ldots, l - 1$, is a partition in $l$ blocks of the set of the rows of our board. Similarly we can construct the partition $\{C^i_j\}$, with $i = 0, \ldots, l - 1$, of the set of columns. Repeating this for each line from the $m$-th to the second, we will eventually have $m - 1$ couples of partitions of the set of the rows and of that of the columns: $\{R^i_j\}$ and $\{C^i_j\}$, with $j = 2, \ldots, m$ and $i = 0, 1, \ldots j - 1$.

Intersecting each of these partitions with $C^1$ we obtain as many partitions of $C^1 = \{(i,i)\}$ with $i \in [n]$. In such partitions the projections for rows and for columns of the sets of grade 1, which belong to $C^1$, obviously coincide.

Calculating the number of all the possible arrangements of these $2(m-1)$ partitions, respecting the condition that the projections of the sets of grade 1 must coincide, is equivalent, from what has been said, to calculating the product of the various blocks of the set of columns:

$$\prod_{j=0}^{m} \prod_{i=0}^{p} \pi(T^{(l-1)}_{j/l-1})$$

which appears in (3.6).

To do this it is natural to consider the partition refinement of the $m - 1 \{R^i_j \cap C^1\}$ partitions and that of the $m - 1 \{C^i_j \cap C^1\}$ partitions. Putting by analogy $R^1 = C^0 = C^1$, the blocks of the refinement partitions will be given from:

$$R_{\alpha_m, \ldots, \alpha_1} = R_{\alpha_m}^{\alpha_m} \cap R_{\alpha_{m-1}}^{\alpha_{m-1}} \cap \cdots \cap R_{\alpha_1}^{\alpha_1} \cap \cdots \cap R_{\alpha_1}^{\alpha_1}, \text{with } 0 \leq \alpha_j \leq j - 1 \forall j \in [m],$$

and from:

$$C_{\beta_m, \ldots, \beta_1} = C_{\beta_m}^{\beta_m} \cap C_{\beta_{m-1}}^{\beta_{m-1}} \cap \cdots \cap C_{\beta_1}^{\beta_1} \text{with similar limitations on the indices } \beta_j .$$

We will then have:

$$\prod_{j=0}^{m} \prod_{i=0}^{p} \pi(T^{(l-1)}_{j/l-1}) = \prod_{j=0}^{m-1} \pi(R_{\alpha_m, \ldots, \alpha_1}) \prod_{j=0}^{m-1} \pi(C_{\beta_m, \ldots, \beta_1}) .$$
We will also say that the pair of partitions \( \{R_{\alpha_m, \ldots, \alpha_1}\} \) and \( \{C_{\beta_m, \ldots, \beta_1}\} \) of \([n]\) is “associated” with the collection of systems of independent rooks \( T_{j/l-1}^{(l-1)} \) with \( l = 2, \ldots, m \) and \( j = 0, \ldots, p_l \).

4. The blocks of the associated partitions

Before being able to develop the calculation of (3.6) using (3.7), we must examine closely the meaning of the indices \( \alpha_m, \ldots, \alpha_1 \) and \( \beta_m, \ldots, \beta_1 \) which respectively mark the blocks \( R_{\alpha_m, \ldots, \alpha_1} \) and \( C_{\beta_m, \ldots, \beta_1} \).

In order to do this, we first of all define the concept of “covering”. Taking the index \( \alpha_i \), with \( \alpha_i > 0 \), we say that \( \alpha_i \) “covers” \( \alpha_{\alpha_i} \), and we write \( \alpha_i \vdash \alpha_{\alpha_i} \) or \( \kappa(\alpha_i) = \alpha_{\alpha_i} \).

From the definition it immediately follows that:

a) if \( \alpha_i = 0 \) it doesn’t cover any other index, since \( \alpha_0 \) doesn’t exist;

b) if \( \alpha_i \vdash \alpha_s \) then \( s < i \).

Thus, if \( \alpha_s > 0 \), applying to it more times the function of covering \( \kappa \), you will always arrive at an index of value 0.

On the contrary, taking an index \( \alpha_l \) of value 0, we can consider all the indices \( \alpha_s \) which have the property \( \alpha_s \vdash \alpha_l \) (that is \( \kappa^{-1}(\alpha_l) \)). Repeating this procedure more times, we get all the indices which, with a finite number of applications of the function \( \kappa \), finish in \( \alpha_l \).

If we suppose that \( \alpha_h = 0 \) and we put \( \kappa^0(\alpha_s) = \alpha_s \), we will be able to define \( Z_h = \{\alpha_j | \alpha_h = 0 \text{ and } \exists k \in \mathbb{N} \text{ so that } \kappa^k(\alpha_j) = \alpha_h\} \) which we will call the “component \( h \)” of the indices \( \alpha_m, \ldots, \alpha_1 \). Furthermore, as \( \alpha_1 = 0 \), we will have that \( Z_1 \neq \emptyset \).

In this way, we obtain a partition of the set of indices \( \{\alpha_m, \ldots, \alpha_1\} \) in blocks made up from \( Z_h \), with \( h \in [m] \).

We will say that such a subdivision represents the structure of the indices \( \alpha_m, \ldots, \alpha_1 \) and we will write that \( \sigma(\alpha_m, \ldots, \alpha_1) = Z_1 \cup Z_{z_2} \cup \cdots \cup Z_{z_a} \) with \( 1 < z_2 < \cdots < z_a \leq m \). We will also write \( \sigma_l(\alpha_m, \ldots, \alpha_1) = Z_l \) and \( \zeta(\alpha_m, \ldots, \alpha_1) = \)number of the \( \alpha_j \) which are equal to zero. It is clear that, if \( Z = \{\alpha_{i_1}, \ldots, \alpha_{i_a}\} \) is a component, then \( \alpha_{i_a} = 0 \).

Moreover we will put, to be brief \( R_{\alpha_j} = R_{\alpha_m, \ldots, \alpha_1} \), \( C_{\beta_j} = C_{\beta_m, \ldots, \beta_1} \), \( \alpha_j = \alpha_m, \ldots, \alpha_1 \) and \( \beta_j = \beta_m, \ldots, \beta_1 \).

It is necessary to pay attention to the fact that \( Z_h \) is not only a subset \( I \subseteq [m] \), but a subset of the indices \( \alpha_j \), for \( j \in I \), each with its own value.

The following result allows to count the number of the \( R_{\alpha_j} \) at the base of the structure of their indices.

4.1 Proposition. Let \( \sigma(\alpha_m, \ldots, \alpha_1) = Z_1 \cup Z \) and \( Z = Z_{z_2} \cup \cdots \cup Z_{z_a} \), then:

a) if we suppose \( Z \) to be variable, the number of the possible sets of indices will be: \( (m-1)!z! \).

b) if, on the other hand, we keep \( Z \) constant, then the possible \( \alpha_j \) will be \( (m-1-z)! \).
In fact, to determine \( Z \) we will have, first of all, have to choose the \( z \) places of its indices in the set \( \{2, \ldots, m\} \) and this can be done in \( \binom{m-1}{z} \) ways. Furthermore, if the selected indices are \( \alpha_{j_1}, \ldots, \alpha_{j_z} \) (with \( j_1 > \cdots > j_z \)), it can be seen that \( \alpha_{j_z} \) must be equal to 0, \( \alpha_{j_{z-1}} \) can assume the values 0 and \( \alpha_{j_z} \) and so on. Therefore the last index has only one possible value, the penultimate two values etc., thus all the possible ways to attribute a value to \( \alpha_{j_1}, \ldots, \alpha_{j_z} \) are \( 1 \cdot 2 \cdot \ldots \cdot z = z! \). And this proves a).

If, on the other hand, \( Z \) is fixed, the places of the indices of \( Z_1 \) are also fixed. Now the last index of \( Z_1 \) on the right \( \alpha_1 \) can only have the value 0, the penultimate only 1 and thus, for an argument identical to the previous, the number of possible values of the \( m - z \) indices is equal to \( 1 \cdot 1 \cdot 2 \cdot \ldots \cdot (m - 1 - z) = (m - 1 - z)! \). And thus b) too is proved.

It is also possible to calculate the number of possible \( \alpha_j \), in terms of the data of singular components with the following result, which we shall just state.

**4.2 Proposition.** If \( \sigma(\alpha_m, \ldots, \alpha_1) = Z_1 \cup Z_{z_2} \cup \cdots \cup Z_{z_a} \), then \( |Z_1| \) and \( |Z_{z_i}| \), with \( i = 2, \ldots, a \), constitute a partition of the integer \( m \) in which the number of parts equal to \( s \) will be \( \lambda_s \). The number of possible \( \alpha_j \) with this structure will therefore be the same as:

\[
\binom{m!}{1^{\lambda_1}2^{\lambda_2}\cdots m^{\lambda_m}\lambda_1!\lambda_2!\cdots\lambda_m!}
\]

So, if \( \alpha_s \vdash \alpha_l \), we have that \( R_{\alpha_j} \) is a subset of \( \mathcal{R}_1\left(\bigcup_x T_{x/l-1}^{\alpha_1}\right) \), with \( x \) that ranges in a subset of \( \{0, 1, \ldots, p_l\} \), and thus it lies in the projection for rows on the first line of a set of rooks of grade \( \alpha_l \) included in the impression of a set of rooks of grade \( \alpha_s = l \). If instead \( \alpha_s \) doesn't cover \( \alpha_l \), then the set of rooks of grade \( \alpha_l \) is not included in the impression of the set of rooks of grade \( \alpha_s \) and so the number of elements in their intersection varies according to the variation of the projection for rows or for columns.

From this, it follows that if, using the symbolism of section 3, we take \( \mathcal{R}_1(T_j^l) - \mathcal{R}_1\left(\bigcup_i T_{j/l-1}^i\right) \), we see that it will be composed of the union of all the \( R_{\alpha_j} \) with the same \( Z_l \) component. Viceversa, if we fix the \( Z_l \) component and make the other \( \alpha_j \) vary in all possible ways, we obtain a collection of sets \( R_{\alpha_j} \) the union of which will be equal to \( \mathcal{R}_1(T_j^l) - \mathcal{R}_1(T_{j/l-1}^{l-1}) \) for some \( j \). Furthermore, if \( l = 1 \), since \( T_{j/l-1}^{l-1} \) doesn't exist, the union of all the \( R_{\alpha_j} \) with the same \( Z_1 \) will be given by \( \mathcal{R}_1(T_j^1) \) for some \( j \).

Naturally the same argument is true for the sets \( C_{\beta_j} \) and the components of the indices \( \beta_j \).
5. The enumeration of Latin rectangles

Now let us try, applying the contents of the previous section, to give an explicit form to (3.6) in terms of the data of the two associated partitions \( \{ R_{\alpha_j} \} \) and \( \{ C_{\beta_j} \} \), and that is in terms of the sets of variables \( R = \{ r_{\alpha_j} \} \) and \( C = \{ c_{\beta_j} \} \).

First of all, we observe that, if \( l > 1 \), \( t_j^l - t_j^{(l-1)} = |T_j^l| - |T_j^{(l-1)}| = |R_1(T_j^l)| - |R_1(T_j^{(l-1)})| = |R_1(T_j^l) - R_1(T_j^{(l-1)})| \) but, following what was said before, \( R_1(T_j^l) - R_1(T_j^{(l-1)}) \) is formed, in such a case, from the union of all the sets \( R_{\alpha_j} \) with the same \( Z_1 \) component and viceversa.

Therefore, if we put, \( \forall l \in [m], Q(Z_l) = \{ r_{\alpha_j} \mid \sigma_l(\alpha_j) = Z_l \}, \tilde{Q}(Z_l) = \{ c_{\beta_j} \mid r_{\beta_j} \in Q(Z_l) \} \) and \( q(Z_l) = \sum Q(Z_l) \), thus we have that, if \( l > 1, j \in \{0, \ldots, p_l\} \) exists so that:

\[
q(Z_l) = t_j^l - t_j^{(l-1)}
\]

and viceversa.

Now, we will compute the product \( \prod_{l=2}^{m} \prod_{j=0}^{p_l} \pi(T_j^{(l-1)}) \) that, for (3.7), is the same as \( \prod_{\alpha_j} \prod_{\beta_j} \pi(R_{\alpha_j}) \pi(C_{\beta_j}) \).

The first partition \( \{ R_{\alpha_j} \} \) can be chosen in a completely arbitrary way and thus the number of its possible arrangements is given by the multinomial coefficient \( \binom{n}{R} \). The second partition is, on the other hand, subject to some restrictions.

First of all, for \( l > 1 \), \( T_{j/l-1} \subseteq T_j^l \) and so \( |C_1(T_j^l) - C_1(T_j^{(l-1)})| = |R_1(T_j^l) - R_1(T_j^{(l-1)})| = q(Z_l) \) and since, following the same reasoning as we have already done, \( |C_1(T_j^l) - C_1(T_j^{(l-1)})| = \sum \tilde{Q}(Z_l) \) we have that:

\[
\sum \tilde{Q}(Z_l) = q(Z_l).
\]

Furthermore, taking a generic set of rooks of grade \( l \) it is clear that \( |C_1(T_l)| = |R_1(T_l)| \).

Now, if \( l = 1 \), then \( T_l \subseteq C_1 \) and even \( C_1(T_l) = R_1(T)_l \), but, for the reasons stated in section 4, \( R_1(T)_l \) is made up of the union of all the \( R_{\alpha_j} \) with the same \( Z_1 \), and the same argument is valid for \( C_1(T_l) \), therefore:

\[
\bigcup_{\sigma_1(\alpha_j) = Z_1} R_{\alpha_j} = \bigcup_{\sigma_1(\beta_j) = Z_1} C_{\beta_j}
\]

and:

\[
\sum \tilde{Q}(Z_1) = q(Z_1).
\]
Furthermore, the restrictions (5.2) and (5.4) imposed on $c_{\beta_j}$ imply that, for $T_j^l$, with $j = 0, \ldots, p_l$, $|C_1(T_j^l)| = |R_1(T_j^l)|$.

This can be easily proved for complete induction on $l$ considering that, if $l = 1$, the result has already been expressed by (5.4), while, if $l > 1$ and we suppose that we have already proved this $\forall j \in [l - 1]$, it follows from the consideration that:

$$|T_j^l| = \sum_{i=1}^{l-1} |T_{j/l-1}^i| + |R_1(T_j^l) - R_1(T_{j/l-1}^l)|.$$

Therefore there are no other restrictions on $c_{\beta_j}$, apart from those expressed by (5.2) and (5.4).

If we now group the $C_{\beta_j}$ sets on the basis of the value of their component $Z_1$, (5.3) allows us to state that:

$$\prod_{\beta_j} \pi(C_{\beta_j}) = \prod_{Z_1} \left( \frac{q(Z_1)}{Q(Z_1)} \right)$$

on the condition, however, that the $C$ variables also respect the restrictions imposed by (5.2).

Let us finally examine the $t_{j/l-1}^{(l-1)}$ which appear in (3.6) as exponents of $-1$.

Now, for (5.1), if $l > 1$, $t_{j/l-1}^{(l-1)} = t_j^l - q(Z_l)$ for any $Z_l$ and so $\sum_0^{p_l} t_{j/l-1}^{(l-1)} = \sum_j t_j^l - \sum_{Z_l} q(Z_l) = n - \sum_{Z_l} q(Z_l)$. Therefore, being $l = 2, \ldots, m$, the exponent of $-1$ in (3.6) will be the same as $n(m - 1) - \sum_{Z_l} q(Z_l)$. Furthermore, since, as we have already seen, $n = \sum Z_l q(Z_1)$, adding and subtracting $n$ it can be expressed by: $nm - \sum_{Z_l} q(Z_l)$.

Finally, set $W = \{ r_{\alpha_j} \mid \zeta(\alpha_j) \text{ odd} \}$ and considering that every $r_{\alpha_j}$ variable compares $\zeta(\alpha_j)$ times in $\sum_{Z_l} q(Z_l)$, we will have that the exponent of $-1$ can be substituted by $nm - \sum W$, since the even multiples of $r_{\alpha_j}$ can obviously be omitted.

Using all these results in (3.6), we obtain the following remarkable result:
5.1 Theorem.

\[(5.6) \quad K(m, n) = \sum_R \sum_C (-1)^{n+m+1} \sum_{\tilde{Q}(Z_i)=q(Z_i)} (n R)^m \prod_{l} \prod_{Z_l} q(Z_l)!
\]

\[\cdot \prod_{Z_l} \left( q(Z_1) \right) = \sum_{R=n} \sum_{C} (-1)^{m+n+1} \sum_{\tilde{Q}(Z_i)=q(Z_i)} \prod_{l=0}^{m} \prod_{Z_l} (-q(Z_l))^{i} \prod_{R! \prod C!}
\]

\[= (-1)^{m(n-1)} \sum_{R=n} \sum_{C} \prod_{l=0}^{m} \prod_{Z_l} (-q(Z_l))^{i} \prod_{R! \prod C!}
\]

\[= (-1)^{m} \sum_{R} \sum_{C} \left( n R \prod_{l=1}^{m} \prod_{Z_l} (-1)^{q(Z_l)} \left( q(Z_l) \right) \tilde{Q}(Z_l) \prod_{C} c_{\beta_j}^{c_{\beta_j}^{-1}} \right).
\]

Where, by analogy with the preceding symbolism, we have set \(Z_0 = \emptyset\) since \(\alpha_0\) doesn't exist. Thus \(q(Z_0) = \sum R = n\) since the elements of \(Q(Z_0)\), not being subject to any restrictions, are all the elements of \(R\).

So (5.6) is an explicit formula for the computation of \(K(m, n)\) in \(2m!\) variables \(R\) and \(C\), while \(q(Z_l)\) with \(l \in [m]\) and \(\sum W\) are sums of particular subsets of \(R\).

This therefore represents the result which we proposed to achieve with the present paper.

The \(C\) variables, in contrast to the \(R\) variables, are not, however, between their independent since they must be subject to the restrictions \(\sum \tilde{Q}(Z_i) = q(Z_i)\) for \(l \in [m]\).

If we want to limit ourselves to considering only independent variables we can proceed as follows.

For each \(Z_l\) component we indicate with \(d(Z_l)\) the \(c_{\beta_j}\) variable with \(\sigma_l(\beta_j) = Z_l\) and all the indices \(\beta_j\) which are different from those of \(Z_l\) equal to zero, and we put \(D = \{d(Z_l)\}\).

Now \(d(Z_l) = q(Z_l) - \sum_{c_{\beta_j} \in \tilde{Q}(Z_l) - D} c_{\beta_j}\) and thus the variables of \(D\) can be obtained from those of \(C - D\).

Furthermore, if \(\sigma_l(\beta_j) = Z_l\), then \(c_{\beta_j} \in \tilde{Q}(Z_l)\) and so \(c_{\beta_j} \leq q(Z_l)\). Thus, if we put \(\mu_{\beta_j} = \min_{\sigma_l(\beta_j) \neq \emptyset} (q(\sigma_l(\beta_j)))\), then \(\forall c_{\beta_j} \in C - D\), we will have that \(c_{\beta_j} \leq \mu_{\beta_j}\) and such a restriction guarantees that \(d(Z_l) \geq 0\).

Using this new symbolism (5.6) can be rewritten like this:

\[(5.7) \quad K(m, n) = \sum_{\alpha, \beta}^{n} \sum_{c_{\beta_j} \in C - D} (-1)^{n+m+1} \sum_{l=0}^{m} \prod_{Z_l} q(Z_l)!
\]

\[\prod_{R}^{1} \prod_{(C-D)!}^{1} \prod_{D!}^{1}
\]
with $r_{\alpha_j} \in R$ and $c_{\beta_j} \in C - D$.

Now if $|Z_l| = s$, for the Proposition 4.1, the possible $d(Z_l)$ are $\binom{m}{s}(s-1)!$. Furthermore, if $s = 1$, all the $d(Z_l)$ coincide with the $c_{\beta_j}$ which has all the indices at 0 and so, in such a case, instead of $\binom{m}{1}0! = m$ we only have one distinct element and $|D| = \sum_{s=1}^{m} \binom{m}{s}(s-1)! - (m-1)$.

Thus, in (5.7), other than the $m!$ independent variables $R$, there are the $m! + m - 1 - \sum_{s=1}^{m} \binom{m}{s}(s-1)!$ independent variables $C - D$.

6. Simplifications of the Formula

We have seen that (5.7) needs $2m! + m - 1 - \sum_{s=1}^{m} \binom{m}{s}(s-1)!$ independent variables for the computation of $K(m, n)$.

It is possible, though, to effect two types of elimination among these parameters which allow us to reduce their number considerably, even though this fact makes (5.7) lose its symmetry. This is obviously important when we would like to calculate concretely $K(m, n)$ for $m$ and $n$ prefixed.

Let us therefore examine the two possible reductions of the independent variables $R$ and $C - D$.

A) We consider $r_{\alpha_j}$ and $c_{\alpha_j}$ with $\zeta(\alpha_j) = 1$ and so with $\sigma(\alpha_j) = Z_1$. In such an assumption $q(\sigma(\alpha_j))$ contains a unique element and so, for (5.4), $c_{\alpha_j} = r_{\alpha_j}$ and, in (5.7), $c_{\alpha_j}$! is simplified with $q(\sigma(\alpha_j))! = r_{\alpha_j}$!. As far as $r_{\alpha_j}$ is concerned instead, if we put: $F_0 = \{r_{\alpha_j} | \zeta(\alpha_j) = 1\}$, $f_0 = \sum F_0, \overline{Q}_0 = R - F_0$ and $\tilde{F}_0 = \{c_{\alpha_j} | r_{\alpha_j} \in F_0\}$, we will have that the variables of $F_0$ don’t appear in any set $Q(Z_l)$ with $l > 1$ and that:

\[
\sum_{R} \binom{n}{R} = \sum_{F_0} \sum_{\overline{Q}_0} \binom{n}{F_0, \overline{Q}_0} = \sum_{F_0} \sum_{\overline{Q}_0} \binom{n}{f_0} \binom{n - f_0}{F_0, \overline{Q}_0} = \sum_{\overline{Q}_0} \sum_{f_0} ((m-1)!)f_0 \frac{n!}{f_0! \prod \overline{Q}_0!}
\]

since, for the Proposition 4.1, $|F_0| = (m-1)!$. Furthermore the $F_0$ appear among the exponents of $-1$ with their total $f_0$. The $2(m-1)!$ variables of $F_0$ and of $\tilde{F}_0$ can therefore be substituted by $f_0$.

B) Let us now consider the $r_{\alpha_j}$ and $c_{\alpha_j}$ with $\sigma(\alpha_j) = Z_h \cup Z_1$ (and so $\zeta(\alpha_j) = 2$) and $\min(z_1, z_h) = 1$ and put, $\forall s \in [m]$: $F_s = \{r_{\alpha_j} | \zeta(\alpha_j) = 2 \text{ and } |\sigma(\alpha_j)| = 1\}$, $f_s = \sum F_s, \overline{Q}(Z_s) = Q(Z_s) - F_s, \tilde{F}_s = \{c_{\alpha_j} | r_{\alpha_j} \in F_s\}$ and $\overline{Q}(Z_s) = q(Z_s) - f_s$. First we observe that, if $r_{\alpha_j} \in F_s$, it doesn’t appear among the exponents of $-1$ since $\zeta(\alpha_j)$ is even. Now, if $\sigma(\alpha_j) = Z_s \cup Z_v$, $|Z_v| = m - 1$ and so $q(Z_v)$ has only one element and, for (5.2) and (5.4), $c_{\alpha_j} = r_{\alpha_j}$. Therefore in (5.7), if $c_{\alpha_j} \in \tilde{F}_s$, $c_{\alpha_j}$!
is simplified with \( q(Z_v)! = r_{\alpha_j}! \). Moreover, \( \forall s \in [m] \):

\[
\frac{\sum_{F_s} q(Z_s)!}{\prod_{F_s}!} = \frac{\sum_{F_s} (f_s + \overline{q}(Z_s))!}{\prod_{F_s}!}
\]

\[
= \sum_{F_s} \frac{(f_s + \overline{q}(Z_s))!}{f_s!} \left( \frac{f_s}{f_s} \right) = ((m-2)!)^{f_s} \frac{(f_s + \overline{q}(Z_s))!}{f_s!}
\]

since \( |F_s| = (m-2)! \), and so also the \( F_s \) and the \( \tilde{F}_s \) are eliminated and substituted by \( f_s \). We must, however, be careful because, if \( m = 2, F_1 \) and \( F_2 \) are equal.

In conclusion, putting \( \mathcal{R} = R - \bigcup_{i=0}^{m} F_i, \mathcal{C} = \{ c_{\alpha_j} \mid r_{\alpha_j} \in \mathcal{R} \} \) and \( \mathcal{D} = \{ d(\sigma_l(\alpha_j)) \mid c_{\alpha_j} \in \mathcal{C} \} \) we have that (5.7) transforms itself into:

\[
K(m, n) = \sum_{f_0} \sum_{f_s} \sum_{0}^{n} \sum_{\pi}^{\mu_{\beta_j}} \sum_{\mathcal{C} - \mathcal{D}} (-1)^{\pi} \sum_{W}^{m} \left( (m-1)! \right)^{f_0} \left( (m-2)! \right)^{f_s} \cdot \frac{1}{\prod(\mathcal{C} - \mathcal{D})! \prod \mathcal{D}!} \left( f_0, \ldots, f_m, R \right) \prod_{1}^{m} (f_s + \overline{q}(Z_s))! \prod_{1}^{m} \prod_{|z_l|} \overline{q}(Z_l)! .
\]

It is, however, possible to accomplish a further step to simplify (6.3). In fact, putting \( f = \sum_{1}^{m} f_s \), we have that:

\[
\sum_{f_1} \ldots \sum_{f_m} \prod_{1}^{m} \frac{(f_s + \overline{q}(Z_s))!}{f_s!} = \sum_{f_1} \ldots \sum_{f_m} \prod_{1}^{m} \overline{q}(Z_s)! \left( f + \sum_{1}^{m} \overline{q}(Z_s) + m - 1 \right)
\]

and so (6.3) becomes:

\[
K(m, n) = \sum_{f_0} \sum_{f} \sum_{0}^{n} \sum_{\pi}^{\mu_{\beta_j}} \sum_{\mathcal{C} - \mathcal{D}} (-1)^{\pi} \sum_{W}^{m} \left( (m-1)! \right)^{f_0} \cdot \frac{1}{\prod(\mathcal{C} - \mathcal{D})! \prod \mathcal{D}!} \left( f_0, f, R \right) \prod_{1}^{m} \prod_{|z_l|} \overline{q}(Z_l)! .
\]
From the independent variables $R$ we have so eliminated the $(m-1)!$ of $F_0$ and the $(m-2)!$ of each $F_s$, with $s \in [m]$, and therefore $|\overline{R}| = m! - (m-1)! - m(m-2)! = m! - (2m-1)(m-2)!$.

The $C$ variables have undergone the same reduction. However it is necessary to add $f_0$ and $f$ and subtract the $\overline{D}$, which are as many as the components $Z_l$ with $1 < |Z_l| < m-1$, and so equal to $\sum_{h=2}^{m-2} \binom{m}{h} (h-1)!$ plus the $c_{\alpha_j}$ with all the $\alpha_j$ indices equal to zero (which is determined by the $m$ equivalent restrictions $f(\bar{Z}_s) = \sum (\bar{Q}(\bar{Z}_s) - \bar{F}(\bar{Z}_s))$ with $s \in [m]$), and thus:

$$
(6.6) \quad |\overline{C} - \overline{D}| = m! - (m-1)! - m(m-2)! - \left(\sum_{h=2}^{m-2} \binom{m}{h} (h-1)! + 1\right) = m! - (2m-1)(m-2)! - 1 - \sum_{h=2}^{m-2} \binom{m}{h} (h-1)!
$$

The independent parameters of (6.5) are therefore all together: $2m! - (2m-1)(m-2)! + m + 1 - \sum_{h=1}^{m} \binom{m}{h} (h-1)!$.

7. The simplest cases

Let us see what in concrete terms happens calculating the formulas obtained in sections 5 and 6 for the first values of $m = 2, 3, 4$.

A) $m = 2$. $r_{\alpha_j}$ are of $r_{\alpha_2 \alpha_1}$ which can therefore assume the values $r_{10}$ and $r_{00}$. Furthermore $C - D = 0$, $c_{00} = r_{00}$ and $c_{10} = r_{10}$ and so, applying (5.7), we have:

$$
(7.1) \quad K(2, n) = \sum_{0}^{n} r_{10} \sum_{0}^{n} r_{00} (-1)^{2n+r_{10}} r_{00}! \binom{n}{r_{10}, r_{00}} \frac{r_{00}! r_{10}!}{c_{00}! c_{10}!} = \sum_{0}^{n} r_{10} (-1)^{r_{10}} \frac{n!}{r_{10}!}
$$

which, for (1.1), is equivalent to $D_n$.

B) $m = 3$. $r_{\alpha_j}$ are of $r_{\alpha_3 \alpha_2 \alpha_1}$. As seen in section 6, the $F_0 = \{r_{210, r_{110}}\}$, $F_1 = \{r_{100}\}$, $F_2 = \{r_{101}\}$ and $F_3 = \{r_{010}\}$ and the homologous $c_{\alpha_j}$ are eliminated. Furthermore $\overline{\alpha}(Z_1) = \overline{\alpha}(Z_2) = \overline{\alpha}(Z_3) = r_{000}$ and $c_{000} = r_{000}$ and so, applying (6.5), we have that:

$$
(7.2) \quad K(3, n) = \frac{(r_{000})!^3 (f + 3r_{000} + 2)!}{c_{000}! (3r_{000} + 2)!} = \sum_{f_0 + f + r_{000} = n} (-1)^f 2f_0 n! r_{000}! \left(3r_{000} + f + 2\right)
$$
and we find (1.3) again.

C) \( m = 4 \). \( r_{\alpha_j} \) are of \( r_{\alpha_4\alpha_3\alpha_2\alpha_1} \). The \( F_0 = \{r_{3210}, r_{2210}, r_{1210}, r_{1110}, r_{2110}, r_{3110}\} \), \( F_1 = \{r_{3200}, r_{2200}\} \), \( F_2 = \{r_{3100}, r_{1100}\} \), \( F_3 = \{r_{1010}, r_{2010}\} \), \( F_4 = \{r_{0110}, r_{0210}\} \) and the homologous \( c_{\alpha_j} \) are eliminated. Furthermore \( q(Z'_1) = r_{1000} + r_{1200} \), \( q(Z''_1) = r_{2100} + r_{0100} \), \( q(Z'_2) = r_{0010} + r_{3010} \), \( q(Z'_2) = r_{0200} + r_{1200} \), \( q(Z''_2) = r_{2000} + r_{2100} \), \( q(Z_3) = r_{3000} + r_{3100} \), and \( \overline{q}(Z_1) = r_{0000} + r_{3000} + r_{2000} + r_{0200} \), \( \overline{q}(Z_2) = r_{0000} + r_{3000} + r_{0100} + r_{1000} \), \( \overline{q}(Z_3) = r_{0000} + r_{2000} + r_{1000} + r_{0100} \), \( \overline{q}(Z_4) = r_{0000} + r_{0100} + r_{0010} + r_{2000} \). We besides have that: \( c_{1000} = r_{1000} + r_{1200} + c_{1200}, c_{0100} = r_{0100} + r_{2100} + c_{2100}, c_{0010} = r_{0010} + r_{3010} - c_{3010}, c_{0200} = r_{0200} + r_{1200} - c_{1200}, c_{2000} = r_{2000} + r_{2100} - c_{2100}, c_{3000} = r_{3000} + r_{3100} - c_{3100}, c_{0000} = r_{0000} + r_{3000} + r_{2000} + r_{0200} - c_{3000} - c_{2000} - c_{0200}, \) that: \( \overline{R} = \{r_{1000}, r_{1200}, r_{2100}, r_{1010}, r_{0100}, r_{3010}, r_{3020}, r_{3000}, r_{2000}, r_{0000}\} \) and that: \( \overline{C} - \overline{D} = \{c_{1200}, c_{2100}, c_{3100}\} \) and, applying (6.3), we obtain:

\[
(7.3) \quad K(4, n) = \sum_{f_0} \sum_{f_s} \sum_{R} \sum_{\overline{C} - \overline{D}} (-1)^{4n - f_0 - r_{1000} - r_{0100} - r_{0010} - r_{3000} - r_{2000} - r_{0200}} 6^{f_0} 2^{f_1 + f_2 + f_3 + f_4} \binom{n}{f_0, f_1, f_2, f_3, f_4, \overline{R}} (f_2 + r_{0000} + r_{3000} + r_{0100} + r_{1000})! \cdot (f_3 + r_{0000} + r_{2000} + r_{1000} + r_{0010})! (f_4 + r_{0000} + r_{0100} + r_{0010} + r_{0200})! \cdot \frac{(f_1 + r_{0000} + r_{3000} + r_{2000} + r_{0200})!}{c_{0000}! c_{3000}! c_{2000}! c_{0200}!} (r_{0010} + r_{3010})! (r_{0100} + r_{3100})! (r_{1000} + r_{1200})! (r_{2000} + r_{2100})! (r_{1000} + r_{2100})! (r_{1000} + r_{3100})! \]

\[
= \sum_{f_0} \sum_{f_s} \sum_{R = n} \sum_{\overline{C} - \overline{D}} (-1)^{f_0 + r_{1000} + r_{0100} + r_{0010} + r_{3000} + r_{2000} + r_{0200}} 6^{f_0} 2^{\sum_{s} f_s} \frac{n!}{f_0! \prod_s f_s! \prod \overline{R}! c_{0000}!} c_{1200}! (r_{2000} + r_{2100})! (c_{2000}, c_{2100})! c_{2100}! (r_{3000} + r_{3100})! (c_{3000}, c_{3100})! c_{3100}! \]

\[
\cdot \left(\frac{r_{1000} + r_{1200}}{c_{1000}, c_{1200}}\right) \left(\frac{r_{2100} + r_{0100}}{c_{2100}, c_{0100}}\right) \left(\frac{r_{0010} + r_{3010}}{c_{0010}, c_{3010}}\right) \left(\frac{r_{0200} + r_{1200}}{c_{0200}, c_{1200}}\right) \cdot (f_1 + r_{0000} + r_{3000} + r_{2000} + r_{0200})! (f_2 + r_{0000} + r_{3000} + r_{0100} + r_{1000})! \cdot (f_3 + r_{0000} + r_{2000} + r_{1000} + r_{0010})! (f_4 + r_{0000} + r_{0100} + r_{0010} + r_{0200})!
\]
apply (6.5), we obtain:

\[
K(4, n) = \sum_{f_0 + f + \sum R = n} (-1)^{f_0 + r_{1000} + r_{0100} + r_{0010} + r_{3000} + r_{2000} + r_{0200}}
\]

\[
6^{f_0} 2^f \frac{n!}{f_0! \prod R! \prod (C - D)!}(r_{0000} + r_{3000} + r_{2000} + r_{0200})!
\]

\[
\cdot (r_{0000} + r_{3000} + r_{0100} + r_{1000})(r_{0000} + r_{2000} + r_{1000} + r_{0010})!
\]

\[
\cdot (r_{0000} + r_{0100} + r_{0010} + r_{0200})(r_{1000} + r_{1200})(r_{2100} + r_{0100})!
\]

\[
\cdot (r_{0010} + r_{3010})(r_{0200} + r_{1200})(r_{2000} + r_{2100})(r_{3000} + r_{3010})!
\]

\[
\cdot \left( f + 4 r_{0000} + 2 (r_{3000} + r_{2000} + r_{0100} + r_{1000} + r_{0200} + r_{0010}) + 3 \right)
\]

Which is an improvement on the results known up to now, since it needs only 15 independent variables (the ten of $\mathcal{T}$, the three of $\overline{C} - \overline{D}$ and the two $f, f_0$) as compared with the 18 of the formula of Pranesachar, Athreya and Singhi.

8. ANOTHER POINT OF VIEW

In conclusion we want to show how the Theorem 5.1 can have another interpretation which sheds light on its combinatorial nature in a more profound way.

The circumstance — which won’t have escaped a careful reader — that the elements of $R$ and of $C$ are as many as those of $\mathfrak{S}_m$, and that is $m!$, is not casual. In fact if we interpret the indices $\alpha_m, \ldots, \alpha_1$ and $\beta_m, \ldots, \beta_1$ as the inversion tables of one of the permutations of $[m]$, putting $\beta_i = \alpha_{m+1-i}$ (or $\beta_i = \beta_{m+1-i}$), we will have two bijective maps between $C$ and $\mathcal{R}$ and $\mathfrak{S}_m$, since $0 \leq \alpha_{m+1-i} \leq m - i$.

Furthermore $\zeta(\alpha_j)$ will be the same as the number of cycles of $\sigma \in \mathfrak{S}_m$ which corresponds in this way to $r_{\alpha_j}$. However it is not true — as could be thought — that the components $Z_i$ of $\alpha_j$ correspond, in some way, to the cycles of the permutation $\sigma$ corresponding to $r_{\alpha_j}$.

To achieve this result we must introduce a new concept. Let us take a $\sigma \in \mathfrak{S}_m$ written in its standard representation and put, $\forall i \in [m] k_i$ equal to $n+1-t$ where $t$ is the element furthest on the right among those to the left of $i$ satisfying $t > i$ (or if $i = s_h$, $k_i = m+1-s_t$ with $s_t > s_h$ and $t$ maximum); moreover we set $k_i = 0$ if there are no elements greater than $i$ on the left of $i$. We say that $K(\sigma) = (k_1, \ldots, k_m)$ is the “covering table” of $\sigma$. It can be proved that the function $K(\sigma)$ is a bijection. Furthermore it is clear that $0 \leq k_i \leq m - i$, $\forall i \in [m]$, and that, if $k_i = 0$, $i$ is a left-to-right maximum of the standard representation of $\sigma$.

Now, if we put $k_i = \alpha_{m+1-j}$, we have that, $\forall i \in [m]$, $0 \leq k_i \leq m - i$ and therefore that $(k_1, \ldots, k_m)$ can be interpreted as the covering table of a $S(r_{\alpha_j}) \in \mathfrak{S}_m$. It can be easily proved that $S(r_{\alpha_j})$ is a bijection between $R$ and $\mathfrak{S}_m$ and that, in this case too, $\zeta(\alpha_j)$ is the number of the cycles of $S(r_{\alpha_j})$. Here however, if $s_h s_{h+1} \ldots s_{h+p}$ are the elements of a cycle of $\sigma$ written in its standard representation
and if we take \( k_{sh}, k_{sh+1}, \ldots, k_{sh+p} \), we have that \( \{\alpha_{m+1-sh}, \alpha_{m+1-sh+1}, \ldots, \alpha_{m+1-sh+p}\} \) constitute a component \( Z_{m+1-sh} \) of \( \alpha_j \).

In the light of this new bijective map, the results obtained previously can be expressed in a new combinatory language. In fact we can now consider the new variables \( r, c, \) the indices of which consist of elements of \( \mathcal{S}_m \) \((\sigma, \vartheta \in \mathcal{S}_m)\) and again indicate their sets with \( R \) and \( C \). Furthermore, writing \( \gamma \mid \sigma \in \mathcal{S}_m \) to say that \( \gamma \) is a cycle of \( \sigma \), we can put \( Q(\gamma) = \{r_\sigma \mid \gamma \mid \sigma\} \) and \( q(\gamma) = \sum Q(\gamma) \); corresponding meaning, going from \( r_\sigma \) to \( c_\vartheta \), will have \( \tilde{Q}(\gamma) \) and \( \tilde{q}(\gamma) \). In this way (5.6) can be reformulated like this:

\[
K(m, n) = \sum R \sum C (-1)^{nm} + \sum W' \prod_{\gamma | \sigma} q(\gamma)! \prod_{R} \prod_{C} \prod_{Z_l} (-q(Z_l))! \prod_{R} \prod_{C} = n! \sum R \sum C \frac{\prod_{\gamma | \sigma} (-q(\gamma))! \prod_{R} \prod_{C} \prod_{Z_l} (-q(Z_l))! \prod_{R} \prod_{C}}{n!}.
\]

where \( W' \) indicates the set of all the \( r_\sigma \) in which \( \sigma \) has an odd number of cycles.

The simplifications of section 6, which conduct us to (6.5), can also be read more clearly now. In fact \( F_0 \) consists of all the \( r_\sigma \) in which \( \sigma \) is made up of only one cycle of order \( m \), while \( F_s \), with \( s \in [m] \), is formed of those \( r_\sigma \) in which \( \sigma \) has a fixed point, made up of the element \( m + 1 - s \), and a cycle of order \( m - 1 \) which permutes the other elements of \([m]\).

Furthermore (8.1) reminds somehow the result attained by Gessel [3].

9. The latin squares

When \( m = n \), we find ourselves facing the Latin squares, much more famous than the Latin rectangles for their applications in various branches of mathematics.

The number of \( n \times n \) Latin squares is usually indicated by \( L(n) \).

If we put \( m = n \) in (5.6) and in (8.1), and, abandoning the condition that the first row is in standard form, we multiply everything by \( n! \), we obtain the following elegant result which allows us to count of the number of Latin squares of any order.

9.1 Theorem.

(9.1) \[
L(n) = n! \sum_{R} \sum_{C} \frac{\prod_{i=0}^{n} \prod_{Z_l} (-q(Z_l))! \prod_{R} \prod_{C}}{\prod_{R} \prod_{C} \prod_{Z_l} (-q(Z_l))! \prod_{R} \prod_{C}} = n! \sum_{R} \sum_{C} \frac{\prod_{\gamma | \sigma} (-q(\gamma))! \prod_{R} \prod_{C} \prod_{Z_l} (-q(Z_l))! \prod_{R} \prod_{C}}{n!}.
\]

In which the \( 2n! \) parameters \( R \) and \( C \), and their totals \( q(Z_l) \) and \( q(\gamma) \) previously defined, appear.
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