Some results on multi vector space

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Abstract
In the present paper, a notion of M-basis and multi dimension of a multi vector space is introduced and some of its properties are studied.

1 Introduction

Theory of Multisets is an important generalization of classical set theory which has emerged by violating a basic property of classical sets that an element can belong to a set only once. Synonymous terms of multisets viz. list, heap, bunch, bag, sample, weighted set, occurrence set and fireset (finitely repeated element set) are used in different contexts but conveying the same idea. It is a set where an element can occur more than once. Many authors like Yager [23], Miyamoto [15, 17], Hickman [9], Blizard [1], Girish and John [7, 8], Chakraborty [2] etc. have studied the properties of multisets. Multisets are very useful structures arising in many areas of mathematics and computer science such as database queries, multicriteria decision making, knowledge representation in data based systems, biological systems and membrane computing etc. [5, 6, 11, 12, 13, 17, 18, 20]. Again the theory of vector space is one of the most important algebraic structures in modern mathematics and this has been extended in different setting such as fuzzy vector space [10, 14, 22], intuitionistic fuzzy vector space, soft vector space [4] etc. In [3], we introduced a notion of multi vector space and studied some of its basic properties. As a continuation of our earlier paper [3], here we have attempted to formulate the concept of basis and dimension of multi vector space and to study their properties.

2 Preliminaries

In this section definition of a multiset (mset in short) and some of its properties are given. Unless otherwise stated, \( X \) will be assumed to be an initial universal
set and $MS(X)$ denotes the set of all mset over $X$.

**Definition 2.1** [8] An mset $M$ drawn from the set $X$ is represented by a count function $C_M : X \to N$ where $N$ represents the set of non negative integers.

Here $C_M(x)$ is the number of occurrence of the element $x$ in the mset $M$. The presentation of the mset $M$ drawn from $X = \{x_1, x_2, ..., x_n\}$ will be as $M = \{x_1/m_1, x_2/m_2, ..., x_n/m_n\}$ where $m_i$ is the number of occurrences of the element $x_i$, $i = 1, 2, ..., n$ in the mset $M$.

Also here for any positive integer $\omega$, $\lbrack X \rbrack^\omega$ is the set of all msets whose elements are in $X$ such that no element in the mset occurs more than $\omega$ times and it will be referred to as mset spaces.

The algebraic operations of msets are considered as in [8].

**Definition 2.2** [19] Let $M$ be a mset over a set $X$. Then a set $M_n = \{x \in X : C_M(x) \geq n\}$, where $n$ is a natural number, is called $n-$ level set of $M$.

**Proposition 2.3** [19] Let $A, B$ be msets over $X$ and $m, n \in \mathbb{N}$.

1. If $A \subseteq B$, then $A_n \subseteq B_n$;
2. If $m \leq n$, then $A_m \supseteq A_n$;
3. $(A \cap B)_n = A_n \cap B_n$;
4. $(A \cup B)_n = A_n \cup B_n$;
5. $A = B$ iff $A_n = B_n, \forall n \in \mathbb{N}$.

**Definition 2.4** [19] Let $P \subseteq X$. Then for each $n \in \mathbb{N}$, we define a mset $nP$ over $X$, where $C_{nP}(x) = n, \forall x \in P$.

**Definition 2.5** [19] Let $X$ and $Y$ be two nonempty sets and $f : X \rightarrow Y$ be a mapping. Then

1. the image of a mset $M \in \lbrack X \rbrack^\omega$ under the mapping $f$ is denoted by $f(M)$ or $f[M]$, where

\[
C_{f[M]}(y) = \begin{cases} 
\lor f(x) = y & C_M(x) \text{ if } f^{-1}(y) \neq \phi \\
0 & \text{otherwise}
\end{cases}
\]

2. the inverse image of a mset $N \in \lbrack Y \rbrack^\omega$ under the mapping $f$ is denoted by $f^{-1}(N)$ or $f^{-1}[N]$ where $C_{f^{-1}(N)}(x) = C_N[f(x)]$.

The properties of functions, which are used in this paper, are as in [19].

**Definition 2.6** [3] Let $A_1, A_2, ..., A_n \in \lbrack X \rbrack^\omega$. Then we define $A_1 + A_2 + ... + A_n$ as follows:

\[
C_{A_1 + A_2 + ... + A_n}(x) = \lor\{C_{A_1}(x_1) \land C_{A_2}(x_2) \land ... \land C_{A_n}(x_n) : x_1, x_2, ..., x_n \in X \text{ and } x_1 + x_2 + ... + x_n = x\}.
\]

Let $\lambda \in K$ and $B \in \lbrack X \rbrack^\omega$. Then $\lambda B$ is defined as follows:
\[ C_{\lambda B}(y) = \vee \{ C_B(x) : \lambda x = y \}. \]

**Lemma 2.7** \[3\] Let \( \lambda \in K \) and \( B \in [X]^\omega \). Then
(a) For \( \lambda \neq 0 \), \( C_{\lambda B}(y) = C_B(\lambda^{-1}y), \forall y \in X \).
(b) For all scalars \( \lambda \in K \) and for all \( x \in X \), we have \( C_{\lambda B}(\lambda x) \geq C_B(x) \).

**Definition 2.8** \[3\] A multiset \( V \) in \([X]^\omega\) is said to be a multi vector space or multi linear space (in short mvector space) over the linear space \( F \) of \( \lambda \).

**Remark 2**

- \( \{0\} \subseteq V \subseteq \lambda V \), for every scalar \( \lambda \).
- We denote the set of all multi vector space over \( X \) by \( MV(X) \).

**Remark 2.9** \[3\] For a multi vector space \( V \) in \([X]^\omega\), \( V + V + \ldots + v \) times = \( V \).

**Remark 2.10** \[3\] If \( V \in MV(X) \) with \( \dim X = m \), then \( |C_V(X)| \leq m + 1 \), where \( |C_V(X)| \) represents the cardinality of \( C_V(X) \).

**Proposition 2.11** (Representation theorem) Let \( V \in MV(X) \) with \( \dim X = m \) and range of \( C_V = \{ n_0, n_1, \ldots, n_k \} \subseteq \{0, 1, 2, \ldots, \omega\}, k \leq m \), \( n_0 = C_V(\theta) \) and \( \omega \geq n_0 > n_1 > \ldots > n_k > 0 \). Then there exists a nested collection of subspaces of \( X \) as \( \{\theta\} \subseteq V_{n_0} \subsetneq V_{n_1} \subsetneq V_{n_2} \subsetneq \ldots \subsetneq V_{n_k} = X \) such that \( V = n_0 V_{n_0} \cup n_1 V_{n_1} \cup \ldots \cup n_k V_{n_k} \). Also
1. If \( n, m \in (n_i+1, n_i] \), then \( V_n = V_m = V_{n_i} \).
2. If \( n \in (n_i+1, n_i] \) and \( m \in (n_i, n_i-1] \), then \( V_n \nsubseteq V_m \).

**Definition 2.12** \[3\] Let \( X \) be a finite dimensional vector space with \( \dim X = m \) and \( V \in MV(X) \). Consider Proposition 2.11. Let \( B_{n_i} \) be a basis on \( V_{n_i} \), \( i = 0, 1, \ldots, k \) such that
\[ B_{n_0} \subsetneq B_{n_1} \subsetneq B_{n_2} \subsetneq \ldots \subsetneq B_{n_k} \ldots \text{(iii)}\]
Define a multi subset \( \beta \) of \( X \) by
\[ C_{\beta}(x) = \begin{cases} \vee \{ n_i : x \in B_{n_i} \} \\ 0, \text{ otherwise} \end{cases} \]
Then \( \beta \) is called a multi basis of \( V \) corresponding to \( \text{iii} \). We denote the set of all multi bases of \( V \) by \( B_M(V) \).

**Corollary 2.13** \[3\] Let \( \beta \) be a multi basis of \( V \) obtained by \( \text{iii} \). Then
1. If \( n, m \in (n_i+1, n_i] \), then \( \beta_n = \beta_m = B_{n_i} \).
2. If \( n \in (n_i+1, n_i] \) and \( m \in (n_i, n_i-1] \), then \( \beta_n \nsubseteq \beta_m \).
3. \( \beta_n \) is a basis of \( V_n \), for all \( n \in \{1, 2, \ldots, \omega\} \).
3 Some results on multi vector space

Lemma 3.1 Let $s, t \in \mathbb{R}$ and $A, A_1$ and $A_2$ be multisets on a vector space $X$. Then

1. $s \cdot (t \cdot A) = t \cdot (s \cdot A) = (st) \cdot A$
2. $A_1 \leq A_2 \Rightarrow t \cdot A_1 \leq t \cdot A_2$.

Proposition 3.2 Let $V \in MV(X)$. Then $x \in X$, $a \neq 0 \Rightarrow C_V(ax) = C_V(x)$.

Proposition 3.3 Let $V \in MV(X)$ and $u, v \in X$ such that $C_V(u) > C_V(v)$. Then $C_V(u + v) = C_V(v)$.

Proposition 3.4 Let $V \in MV(X)$ and $v, w \in X$ with $C_V(v) \neq C_V(w)$. Then $C_V(v + w) = C_V(v) \wedge C_V(w)$.

4 Multi linear independence

Definition 4.1 Let $V \in MV(X)$ and $\dim X = n$. We say that a finite set of vectors $\{x_i\}_{i=1}^n$ is multi linearly independent in $V$ if and only if $\{x_i\}_{i=1}^n$ is linearly independent in $X$ and for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$ with $a_i \neq 0$, $C_V(\sum_{i=1}^n a_i x_i) = \bigwedge_{i=1}^n C_V(a_i x_i)$.

The following example shows that every linearly independent set is not multi linearly independent.

Example 4.2 Let $X = \mathbb{R}^2$ and $\omega = 4$. We define a multi vector space $C_V : X \to \mathbb{N}$ by

$$C_V(x) = \begin{cases} 4, & \text{if } x = (0, 0) \\ 2, & \text{if } x = (0, a), a \neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

If we take the vectors $x = (1, 0)$ and $y = (-1, 1)$, then they are linearly independent but not multi linearly independent. As here $C_V(x) = C_V(y) = 1$, but $C_V(x + y) = 2 > (C_V(x) \wedge C_V(y)) = 1$.

Proposition 4.3 Let $V \in MV(X)$ and $\dim X = m$. Then any set of vectors $\{x_i\}_{i=1}^N (N \leq m)$, which have distinct counts is linearly and multi linearly independent.

Proof. The proof follows by method of induction.
Note 4.4 Converse of the above proposition is not true. Let $X = \mathbb{R}^2$ and $\omega = 6$. We define a multi vector space $C_V : X \to N$ by

$$C_V(x) = \begin{cases} 6, & \text{if } x = (0,0) \\ 1, & \text{otherwise.} \end{cases}$$

Then we have $\{\theta\} = V_6 \subset V_1 = \mathbb{R}^2$. Let $e_1 = (1,0), e_2 = (0,1)$. Then $\{e_1,e_2\}$ are multi linearly independent in $V$, although, $C_V(e_1) = C_V(e_2)$.

5 M-basis

Definition 5.1 A M-basis for a multi vector space $V \in MV(X)$ is a basis of $X$ which is multi linearly independent in $V$.

We denote the set of all M-bases of $V$ by $\mathcal{B}(V)$.

Lemma 5.2 If $V \in MV(X)$ and $Y$ is a proper subspace of $X$, then for any $t \in X \setminus Y$ with $C_V(t) = sup[C_V(X \setminus Y)], C_V(t + y) = C_V(t) \wedge C_V(y)$, for all $y \in Y$.

Proof. Since $\omega$ is finite, such a $t$ exists. Let $y \in Y$. If $C_V(y) \neq C_V(t)$ then by Proposition 3.4, $C_V(t + y) = C_V(t) \wedge C_V(y)$. If $C_V(y) = C_V(t)$ then by Definition 2.8, $C_V(t + y) \geq C_V(t) \wedge C_V(y)$. Since $t + y \in X \setminus Y$ and $C_V(t) = sup[C_V(X \setminus Y)]$, we must have $C_V(t + y) \leq C_V(t) = C_V(y)$ and thus $C_V(t + y) = C_V(t) \wedge C_V(y)$.

Lemma 5.3 Let $V \in MV(X), Y$ be a proper subspace of $X$ and $C_V |_Y = C_V$. If $B^*$ is a M-basis for $V'$, then there exists $t \in X \setminus Y$ such that $B^+ = B^* \cup \{t\}$ is a M-basis for $W$, where $C_W = C_V |_{<B^+,>}$ and $<B^+,>$ is the vector space spanned by $B^+$.

Proof. Pick $t \in X \setminus Y$ such that $C_V(t) = sup[C_V(X \setminus Y)]$. Then by Lemma 5.2, $B^+ = B^* \cup \{t\}$ is a multi linearly independent and hence a M-basis for $W$, where $C_W = C_V |_{<B^+,>}$.

Proposition 5.4 All multi vector spaces $V \in MV(X)$ with $dim X = m$ have M-basis.

Proof. The proof follows by mathematical induction.

Proposition 5.5 Let $V \in MV(X)$ where $dim X = m$ and $C_V(X \setminus \{\theta\}) = \{n_0,n_1,n_2,\ldots,n_k\}, k \leq m$. Then a basis $B = \{e_1,e_2,\ldots,e_m\}$ of $X$ is a M-basis for $V$ if and only if $B \cap V_{n_i}$ is a basis of $V_{n_i}$ for any $i = 0,1,\ldots,k$.

Proof. Let $\omega \geq n_0 > n_1 > \ldots > n_k \geq 0$. Then $\{\theta\} \nsubseteq V_{n_0} \nsubseteq V_{n_1} \nsubseteq V_{n_2} \nsubseteq \ldots \nsubseteq V_{n_k} = X$. Let $B_{n_i} = B \cap V_{n_i}, \ i = 0,1,\ldots,k$. 

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First suppose that $B \cap V_{n_i} = B_{n_i}$ is a basis of $V_{n_i}$ for any $i = 0, 1, \ldots, k$. Then $B_{n_0} \subseteq B_{n_1} \subseteq \ldots \subseteq B_{n_k} = B$. Let $B_{n_0} = \{e_{n_0}^1, e_{n_0}^2, \ldots, e_{n_0}^j\}$, $j \leq m$. Then

$$CV\left(\sum_{i=1}^{j} a_i e_{n_0}^i\right) \geq \bigwedge_{i=1}^{j} CV(e_{n_0}^i) = n_0.$$ Since $n_0$ is the highest count,

$$CV\left(\sum_{i=1}^{j} a_i e_{n_0}^i\right) = n_0 = \bigwedge_{i=1}^{j} CV(e_{n_0}^i).$$ Hence $B_{n_0}$ is multi linearly independent.

Next let $B_{n_1} = B_{n_0} \cup \{e_{n_1}^1, e_{n_1}^2, \ldots, e_{n_1}^s\}, j + s \leq m$. Consider the sum

$$\sum_{i=1}^{j} b_i e_{n_0}^i + \sum_{i=1}^{s} c_i e_{n_1}^i,$$ where some $c_i \neq 0$. Then $CV\left(\sum_{i=1}^{j} b_i e_{n_0}^i + \sum_{i=1}^{s} c_i e_{n_1}^i\right) \geq \bigwedge_{i \in J_1} CV(e_{n_0}^i) \wedge \bigwedge_{i \in J_2} CV(e_{n_1}^i), [\text{where } J_1 = \{i : b_i \neq 0\}, J_2 = \{i : c_i \neq 0\}] = n_1.$$

If $CV\left(\sum_{i=1}^{j} b_i e_{n_0}^i + \sum_{i=1}^{s} c_i e_{n_1}^i\right) > n_1$, then

$$CV\left(\sum_{i=1}^{j} b_i e_{n_0}^i + \sum_{i=1}^{s} c_i e_{n_1}^i\right) = n_0 \Rightarrow \sum_{i=1}^{j} b_i e_{n_0}^i + \sum_{i=1}^{s} c_i e_{n_1}^i \in V_{n_0} \Rightarrow c_i = 0,$$ for all $i = 1, 2, \ldots, s$, a contradiction. Thus $B_{n_1}$ is multi linearly independent. Proceeding in the similar way it can be proved that $B_{n_k} = B$ is multi linearly independent and hence a $M$-basis for $V$.

Conversely, let $B$ be a $M$-basis for $V$. Then either $B_{n_i} = \emptyset$ or $B_{n_i} \neq \emptyset$. Let $B_{n_i} = \emptyset$ and $x \in V_{n_i}$. Then obviously $B_{n_j} = \emptyset, j < i$. Since $B$ is a basis of $X$, there exists some $B' \subseteq B$ such that $x = \sum_{e_j \in B} b_j e_j, b_j \neq 0$. Then

$$CV(x) = \bigwedge_{e_j \in B} CV(e_j) \leq n_{i+1}, \text{ a contradiction. So, } B_{n_i} \neq \emptyset.$$

Then $B_{n_0} \subseteq B_{n_1} \subseteq \ldots \subseteq B_{n_k} = B$. Let $x \in V_{n_i}$ and $B_{n_i}$ is not a basis of $V_{n_i}$.

Choose $x = \sum_{e_j \in B_{n_i}} a_i e_i + \sum_{e_j \notin B_{n_i}} b_i e_i$, for all $b_i \neq 0$.

Now, $CV(x) = CV\left(\sum_{e_j \in B_{n_i}} a_i e_i + \sum_{e_j \notin B_{n_i}} b_i e_i\right)$

$$= \bigwedge_{e_j \in B_{n_i}} CV(e_i) \wedge \bigwedge_{e_j \notin B_{n_i}} CV(e_i)^{'}, \text{ [where } B_{n_i}' = \{e_i \in B_{n_i} : a_i \neq 0\}]$$

$$= \bigwedge_{e_j \notin B_{n_i}} CV(e_i)^{'}, < n_i, \text{ a contradiction to the fact that } x \in V_{n_i}.$$ Thus $x = \sum_{e_j \in B_{n_i}} a_i e_i$ and $B_{n_i}$ is a basis of $V_{n_i}$.

Hence proved.

**Proposition 5.6** Let $V$ be a multi vector space over $X$ where $dim X = m$. Then there is an one-to-one correspondence between $\mathcal{B}_M(V)$ and $\mathcal{A}(V)$.

**Proposition 5.7** Let $V \in MV(X)$ with $dim X = m$ and range of $CV(X \setminus \{\theta\}) = \{n_0, n_1, \ldots, n_k\} \subseteq \{0, 1, 2, \ldots, \omega\}, k \leq m$. If a basis $B = \{e_1, e_2, \ldots, e_m\}$ of $X$ is a $M$-basis, then $CV(B) = \{n_0, n_1, \ldots, n_k\}$.

**Remark 5.8** Converse of the above *Proposition* is not true. For example,
suppose $X = \mathbb{R}^4$, $\omega = 5$. Define multi vector space $V$ with count functions $C_V$ as follows:

\[ C_V((0,0,0,0)) = 5; C_V((0,0,0,\mathbb{R} \setminus \{0\})) = 5; C_V((0,0,\mathbb{R} \setminus \{0\},\mathbb{R})) = 5, \]

\[ C_V((0,\mathbb{R} \setminus \{0\}),\mathbb{R},\mathbb{R})) = 2; C_V(\mathbb{R}^4 \setminus (0,\mathbb{R},\mathbb{R},\mathbb{R})) = 2. \]

Then $B = \{(0,0,0,1),(-1,1,1,1),(1,-1,1,1),(1,1,-1,1)\}$ is a basis of $\mathbb{R}^4$ and $C_V(B) = \{2,5\} = C_V(\mathbb{R}^4)$. But $B$ is not a M-basis as $B$ is not multi linearly independent. In fact, $C_V((-1,1,1,1)) = C_V(1,-1,1,1)) = 2$. But $C_V((-1,1,1,1) + (1,1,1,1)) = C_V((0,0,1,1)) = 5 > |C_V((-1,1,1,1)) \cap C_V((-1,1,1,1))| = 2$.

**Definition 5.9** Let $V \in MV(X)$ with $\dim X = m$, range of $C_V(X \setminus \{\theta\}) = \{n_0,n_1,...,n_k\} \subseteq \{0,1,2,...,\omega\}$, $k \leq m$ and $B_0$ be any M-basis for $V$. Then $C_V(B_0) = \{n_0,n_1,...,n_k\}$. We define multi index of a multi M-basis $B_0$ with respect to $V$ by $[B_0]_M = \{r_i : r_i$ is the number of element of $B_0$ taking the value $n_i\}$.

**Proposition 5.10** For a multi vector space $V$, multi index of M-basis with respect to $V$ is independent of M-basis.

**Proof.** Let $V \in MV(X)$ with $\dim X = m$, range of $C_V(X \setminus \{\theta\}) = \{n_0,n_1,...,n_k\} \subseteq \{0,1,2,...,\omega\}$, $k \leq m$ and $\omega \geq n_0 > n_1 > ... > n_k \geq 0$. Then for any two M-bases $B_0,B'_0$ of $V$, $C_V(B_0) = C_V(B'_0) = \{n_0,n_1,...,n_k\}$. Let $[B_0]_M = \{r_i\}$ and $[B'_0]_M = \{r'_i\}$. Now, $|B_0 \cap V_{n_i}| = \sum_{j=0}^{i} r_j$ and $|B'_0 \cap V_{n_i}| = \sum_{j=0}^{i} r'_j$, for $i = 0,1,2,...,k$. As $B_0 \cap V_{n_i}$ and $B'_0 \cap V_{n_i}$ are both basis of $V_{n_i}$, $|B_0 \cap V_{n_i}| = |B'_0 \cap V_{n_i}|$, for all $i = 0,1,2,...,k$. Thus $[B_0]_M = [B'_0]_M$.

**Note 5.11** As multi index of M-basis with respect to a multi vector space $V$ is independent of M-basis, we can use only the term multi index of a multi vector space $V$.

**Definition 5.12** Let $V \in MV(X)$ with $\dim X = m$, $C_V(X) = \{n_0,n_1,...,n_k\} \subseteq \{0,1,2,...,\omega\}$, $k \leq m$ and $B$ be any basis for $X$. We define index of a basis $B$ with respect to $V$ by $[B] = \{r_i : r_i$ is the number of element of $B$ taking the value $n_i\}$.

**Proposition 5.13** Let $V \in MV(X)$ with $\dim X = m$, $C_V(X \setminus \{\theta\}) = \{n_0,n_1,...,n_k\} \subseteq \{0,1,2,...,\omega\}$, $k \leq m$ and $B$ be any basis of $X$ with $C_V(B) = \{n_0,n_1,...,n_k\}$. If index $[B]$ of $B$ with respect to $V$ is equal to the multi index of $V$, then $B$ becomes a M-basis.

**Proof.** Let us assume that $\omega \geq n_0 > n_1 > ... > n_k \geq 0$. Then $\{\theta\} \subseteq V_{n_0} \subseteq V_{n_1} \subseteq V_{n_2} \subseteq ... \subseteq V_{n_k} = X$. Suppose that $[B]_M = \{r_i : i = 0,1,2,...,k\}$. Then $\dim V_{n_i} = \sum_{j=0}^{i} r_j = |B \cap V_{n_i}|$, for all $i = 0,1,2,...,k$. Hence, $B \cap V_{n_i}$ becomes a
basis for $V_n$, for each $i = 0, 1, 2, ..., k$. Thus by Proposition 5.5, $B$ is a M-basis for $V$.

6 Dimension of multi vector space

**Definition 6.1** We define the dimension of a multi vector space $V$ over $X$ by

$$\dim(V) = \sup_{B \text{ a basis for } X} \left( \sum_{x \in B} C_V(x) \right).$$

Clearly $\dim$ is a function from the set of all multi vector spaces to $\mathbb{N}$.

**Proposition 6.2** Let $V \in MV(X)$ where $\dim X = m < \infty$. Then if $B$ is a M-basis for $V$ and $B^*$ is any basis for $X$ then

$$\sum_{x \in B} C_V(x) \leq \sum_{x \in B^*} C_V(x).$$

**Proposition 6.3** If $V$ is a multi vector space over a finite dimensional vector space $X$, then $\dim(V) = \sum_{x \in B} C_V(x)$, where $B$ is any M-basis for $V$.

**Note 6.4** If $V$ is a multi vector space over a finite dimensional vector space $X$, then $\dim(V)$ is independent of M-basis for $V$. It follows from Proposition 5.5 and Proposition 5.7.

**Proposition 6.5** Let $X$ be any finite dimensional vector space and $V, W \in MV(X)$ such that $C_V(\theta) \geq \sup\{C_W(X \setminus \{\theta\})\}$ and $C_W(\theta) \geq \sup\{C_V(X \setminus \{\theta\})\}$. Then there exists a basis $B$ for $X$ which is also a M-basis for $V$, $W$, $V \cap W$ and $V + W$. In addition, if $A_1 = \{x \in X : C_V(x) < C_W(x)\}$, $A_2 = X \setminus A_1$, then for all $v \in B \cap A_1$,

$$(C_V|_W)(v) = C_V(v) \text{ and } C_{V+W}(v) = C_W(v)$$

and for all $v \in B \cap A_2$,

$$(C_V|_W)(v) = C_W(v) \text{ and } C_{V+W}(v) = C_V(v).$$

**Proof.** We prove this by induction on $\dim X$. In case $\dim X = 1$ the statement is clearly true.

Now suppose that the theorem is true for all the multi vector space with dimension of the underlying vector space equal to $n$.

Let $V$ and $W$ be two multi vector spaces over $X$ with $\dim X = n + 1 > 1$. Let $B_1 = \{v_i\}_{i=1}^{n+1}$ be any M-basis for $V$. We may assume that $C_V(v_i) \leq C_V(\theta)$ for all $i = \{2, 3, ..., n+1\}$. Let $H = \langle \{v_i\}_{i=2}^{n+1} \rangle$. Since $n + 1 > 1$, $H \neq \{\theta\}$. Clearly $\dim H = n$. Define the following two multi vector spaces: $V_1$ with count function $C_{V_1} = C_V|_H$ and $W_1$ with the count function $C_{W_1} = C_W|_H$. By inductive hypothesis the exists a basis $B^*$ for $H$ which is also a M-basis for $V_1$, $W_1$, $V_1 \cap W_1$ and $V_1 + W_1$. Also for all $v \in B^* \cap A_1$,

$$(C_{V_1|W_1})(v) = C_{V_1}(v) \text{ and } C_{V_1+W_1}(v) = C_{W_1}(v)$$

and for all $v \in B^* \cap A_2$,
$(C_{V+W})(v) = C_W(v)$ and $C_{V_1+W_1}(v) = C_{V_1}(v)$.

We shall now show that $B^*$ can be extended to $B$ such that $B$ is a M-basis for $V$, $W$, $V \cap W$ and $V + W$. Furthermore, for all $v \in B \cap A_1$,

$(C_{V+W})(v) = C_V(v)$ and $C_{V+W}(v) = C_W(v)$

and for all $v \in B \cap A_2$,

$(C_{V+W})(v) = C_W(v)$ and $C_{V+W}(v) = C_V(v)$.

**Step - 1**: First we have to show that for all $x \in H$,

$C_{V+W}|_H(x) = C_{V_1+W_1}(x)$. .......(1)

Let $x \in H$. Then we have

$C_{V+W}|_H(x) = \sup \{C_V(x_1) \land C_W(x-x_1) : x_1 \in X\} = \sup \{C_V(x_1) \land C_W(x-x_1) : x_1 \in H\} \lor \sup \{C_V(x_2) \land C_W(x-x_2) : x_2 \in X \setminus H\}$. .......(2)

If $x \in H \setminus \{\theta\}$, we have

$C_V(x) \land C_W(x-x) = C_V(x) \land C_W(\theta) \leq \sup \{C_V(x_1) \land C_W(x-x_1) : x_1 \in H\}$,

$C_V(\theta) \land C_W(x-x) = C_V(\theta) \land C_W(x) \leq \sup \{C_V(x_1) \land C_W(x-x_1) : x_1 \in H\}$.

Since $C_V(\theta) \geq \sup \{C_W(H \setminus \{\theta \})\}$ and $C_W(\theta) \geq \sup \{C\{V(W \setminus \{\theta \})\}$,

$C_V(x) \land C_W(\theta) = C_V(x)$ and $C_V(\theta) \land C_W(x) = C_W(x)$, and this leads to the following inequality:

$C_V(x) \lor C_W(x) \leq \sup \{C_V(x_1) \land C_W(x-x_1) : x_1 \in H\}$. .......(3)

Suppose that $\sup \{C_V(x_1) \land C_W(x-x_1) : x_1 \in H\} < \sup \{C_V(x_2) \land C_W(x-x_2) : x_2 \in X \setminus H\}$. .......(4)

This means that there exists $x' \in X \setminus H$ such that

$\sup \{C_V(x_1) \land C_W(x-x_1) : x_1 \in H\} < C_V(x') \land C_W(x-x')$.

In view of (2) we must have

$C_V(x) \lor C_W(x) < C_V(x') \land C_W(x-x')$. .......(5)

Since $x' \in X \setminus H$ and $C_V(X \setminus H) = C_V(v_1) \leq C_V(v_i)$ for all $i \in \{2, 3, ..., n+1\}$, we must have $C_V(x) \geq C_V(x')$. Thus (4) becomes $C_V(x) \lor C_W(x) < C_V(x) \land C_W(x-x')$. It is not possible (Using the properties of $\lor$, $\land$ and $\lhd$). This means that our assumption (3) is false. Therefore we must have

$\sup \{C_V(x_1) \land C_W(x-x_1) : x_1 \in H\} \geq \sup \{C_V(x_2) \land C_W(x-x_2) : x_2 \in X \setminus H\}$. .......(6)

If $x = \theta$, $C_{V+W}|_H(\theta) = C_V(\theta) \land C_W(\theta) = \sup \{C_V(x_1) \land C_W(\theta-x_1) : x_1 \in H\} \geq \sup \{C_V(x_2) \land C_W(x-x_2) : x_2 \in X \setminus H\}$. .......(7).

Using equations (6) and (7), we have for all $x \in H$,

$\sup \{C_V(x_1) \land C_W(x-x_1) : x_1 \in H\} \geq \sup \{C_V(x_2) \land C_W(x-x_2) : x_2 \in X \setminus H\}$.

From (1), we have for all $x \in H$, $C_{V+W}|_H(x) = \sup \{C_V(x_1) \land C_W(x-x_1) : x_1 \in H\}$

$= \sup \{C_V \land C_W|_H(x-x_1) : x_1 \in H\}

= C_{V+W}(x)$.

This establishes (1).

Since $B^*$ is a M-basis of $V_1 + W_1$, (1) implies that $B^*$ is multi linearly independent in $V + W$.

**Step - 2**: Let $v^* \in X \setminus H$ such that $C_W(v^*) = \sup \{C_W(X \setminus H)\}$. By Lemma 5.2
and Lemma 5.3, $v^*$ is an extension of $M$-basis $B^*$ for $W_1$ to $B = B^* \cup \{v^*\}$ a $M$-basis for $W$.

**Step - 3:** Since $C_V(X \setminus H) = C_V(v_1)$, $C_V(v_1) = C_V(v^*)$ and then $v^*$ is also an extension of $M$-basis $B^*$ for $V_1$ to $B$ a $M$-basis for $V$.

**Step - 4:** Now we shall show that $v^*$ is an extension of $M$-basis $B^*$ for $V_1 \cap W_1$ to $B$ a $M$-basis for $V \cap W$. If $v^* \in A_1$, then $(C_V \cap C_W)(A_1 \cap (X \setminus H)) = C_V(v^*)$ and for all $z \in A_2 \cap (X \setminus H)$, $(C_V \cap C_W)(z) \leq C_V(v^*)$, by definition of $A_2$.

From this we may conclude that if $v^* \in A_1$ then $(C_V \cap C_W)(v^*) = sup[(C_V \cap C_W)(X \setminus H)]$.

If $v^* \in A_2$ then $C_W(v^*) < C_V(v^*)$. Since $C_W(v^*) = sup[C_W(X \setminus H)]$ and $C_V$ is constant on $X \setminus H$ we must have $A_1 \cap (X \setminus H) = \emptyset$. Therefore we have that if $v^* \in A_2$, then $(C_V \cap C_W)(v^*) = sup[(C_V \cap C_W)(X \setminus H)]$. By Lemma 5.3, we may now conclude that $v^*$ extends $M$-basis $B^*$ for $V_1 \cap W_1$ to $B$ a $M$-basis for $V \cap W$.

**Step - 5:** Now we shall show that $v^*$ is also an extension of $B^*$ a $M$-basis for $V_1 + W_1$ to $B$ a $M$-basis for $V + W$. Suppose that there exists $z \in X \setminus H$ such that $C_{V_1 + W_1}(v^*) < C_{V_1 + W_1}(z)$. Clearly vector $z$ can be written in the form $z = a(v^* + v)$ where $a \neq 0$ and $v \in H$. Therefore we have

$$C_{V_1 + W_1}(v^*) < C_{V_1 + W_1}(z) = C_{V_1 + W_1}(a(v^* + v)) = C_{V_1 + W_1}(v^* + v).$$

This means that there exists $x_1 \in X$ such that for all $x' \in X$,

$$C_V(x') \cap C_W(v^* - x') < C_V(x_1) \cap C_W(v^* + v - x_1). \quad ...(8)$$

In particular this is true for $x' = \theta$, i.e.

$$C_V(\theta) \cap C_W(v^*) < C_V(x_1) \cap C_W(v^* + v - x_1).$$

But since $C_V(\theta) \geq sup[C_W(X \setminus \{\theta\})]$ we have

$$C_W(v^*) < C_V(x_1) \cap C_W(v^* + v - x_1) \quad ...(9)$$

If $x_1 \in H$ then since $v \in H$ we must have $v - x_1 \in H$. Again $v^* \in X \setminus H$. So, by Lemma 5.2, $C_W(v^* + v - x_1) = C_W(v^*) \cap C_W(v - x_1)$ and so (9) becomes

$$C_W(v^*) < C_V(x_1) \cap C_W(v^* + v - x_1),$$

which is impossible. Thus $x_1 \in X \setminus H$.

Let $x' = v^*$ in (5). Since $C_W(\theta) \geq sup[C_W(X \setminus \{\theta\})]$ we have

$$C_V(v^*) < C_V(x_1) \cap C_W(v^* + v - x_1) \quad ...(10)$$

Recall that $C_V(X \setminus H) = C_V(v_1)$ and thus $C_V(v_1) = C_V(v^*) = C_V(x_1)$, as $x_1 \in X \setminus H$.

This again means that the inequality (10) is false. This means that for all $z \in X \setminus H$, $C_{V_1 + W_1}(v^*) \geq C_{V_1 + W_1}(z)$. Therefore by Lemma 5.3, $v^*$ is an extension of $B^*$ a $M$-basis for $V_1 + W_1$ to $B$ a $M$-basis for $V + W$.

**Step - 6:** Now we shall show that if $v^* \in A_1$ then $C_{V_1 + W_1}(v^*) = C_W(v^*)$ and if $v^* \in A_2$ then $C_{V_1 + W_1}(v^*) = C_V(v^*)$.

From the definition we have:

$$C_{V_1 + W_1}(v^*) = sup[C_V(x_1) \cap C_W(v^* - x_1) : x_1 \in X].$$

Let $x'$ be such that

$$sup[C_V(x_1) \cap C_W(v^* - x_1) : x_1 \in X] = C_V(x') \cap C_W(v^* - x').$$

By substituting $x_1 = \theta$ and then $x_1 = v^*$ and recalling that $C_V(\theta) \geq sup[C_W(X \setminus \{\theta\})]$ and $C_W(\theta) \geq sup[C_W(X \setminus \{\theta\})]$, we obtain

$$C_V(v^*) \cup C_W(v^*) \leq C_V(x') \cap C_W(v^* - x').$$
Suppose that
\[ C_V(v^*) \lor C_W(v^*) < C_V(x') \land C_W(v^* - x'). \quad \ldots \ldots (11) \]
If \( x' \in H \) then by Lemma 5.2 (as \( B = B^* \cup \{v^*\} \) is a M-basis for \( W \)), (11) becomes
\[ C_V(v^*) \lor C_W(v^*) < C_V(x') \land C_W(v^*) \land C_W(x'). \]
This is never true, and thus \( x' \in X \setminus H \). But now since \( C_V(v^*) = C_V(x') \) the inequality (11) never holds, and so,
\[ C_V(v^*) \lor C_W(v^*) = C_V(x') \land C_W(v^* - x') = C_{V+W}(v^*). \quad \ldots \ldots (12) \]
From equation (12), we have \( v^* \in A_1 \) then \( C_{V+W}(v^*) = C_W(v^*) \) and if \( v^* \in A_2 \) then \( C_{V+W}(v^*) = C_V(v^*) \).
This completes the proof.

**Corollary 6.6** If \( V \) and \( W \) are two multi vector spaces over \( X \) such that the dimension of \( X \) is finite and \( C_V(\theta) \geq \sup(C_W(X \setminus \{\theta\})) \) and \( C_W(\theta) \geq \sup(C_V(X \setminus \{\theta\})) \), then
\[ \dim(V + W) = \dim V + \dim W - \dim (V \cap W). \]

**Example 6.7** Suppose \( X = \mathbb{R}^2 \), \( \omega = 6 \). Define two multi vector spaces \( V \) and \( W \) with count functions \( C_V \) and \( C_W \) respectively as follows:
\[ C_V((0,0)) = 5; C_V((0, \mathbb{R} \setminus \{0\})) = 3; C_V(X \setminus \mathbb{R}) = 1, \]
\[ C_W((0,0)) = 6; C_W(\{(x, x) : x \in \mathbb{R} \setminus \{0\}\}) = 2; C_W(X \setminus \{(x, x) : x \in \mathbb{R}\}) = 1. \]
It is easily checked that \( V \) and \( W \) are multi vector spaces and \( C_V(\theta) \geq \sup(C_W(X \setminus \{\theta\})) \) and \( C_W(\theta) \geq \sup(C_V(X \setminus \{\theta\})) \). It is also easy to check that
\[ C_{V \cap W}((0,0)) = 5; C_{V \cap W}((x, x) : x \in \mathbb{R} \setminus \{0\}) = 1; C_{V \cap W}(X \setminus \{(x, x) : x \in \mathbb{R}\}) = 1, C_{V+W}(0,0) = 5; C_{V+W}(0, \mathbb{R} \setminus \{0\}) = 3; C_{V+W}(X \setminus (0, \mathbb{R})) = 2 \]
and \( B = \{(0,1), (1,1)\} \) is a M-basis for \( V, W, V \cap W \) and \( V + W \). Thus
\[ \dim (V + W) = 3 + 2 = 5, \dim(V \cap W) = 1 + 1 = 2, \]
\[ \dim V = 3 + 1 = 4, \dim W = 2 + 1 = 3. \]
So, \( \dim V + \dim W - \dim (V \cap W) = 4 + 3 - 2 = 5 = \dim (V + W) \).

**Definition 6.8** Let \( V \) be a multi vector space over \( X \) and \( f : X \to Y \) be a linear map. Then we define \( f(V) \) as
\[ C_{f(V)}(x) = \begin{cases} \sup\{C_V(z) : z \in f^{-1}(x)\} & \text{if } f^{-1}(x) \neq \phi \\ 0 & \text{otherwise} \end{cases} \]
and \( \ker f = (\ker f, C_V \mid_{\ker f}), \im f = (\im f, C_V \mid_{\im f}). \)

**Proposition 6.9** If \( V \) be a multi vector space over \( X \) where \( \dim X \) is finite and \( f : X \to Y \) is a linear map, then
\[ \dim(\ker f) + \dim(\im f) = \dim(V). \]

*Proof.* Suppose that \( \ker f \neq \{\phi\} \). If \( \ker f = \{\phi\} \) then the proof is similar. Now let \( B_{Ker f} \) be a M-basis for \( ker f \) and \( B_{Ex} \) be an extension of \( B_{Ker} \) to a M-basis for \( V \) (this is clearly possible by repeated application of Lemma 5.3). Then \( B_{Ker f} \cup B_{Ex} = B \) is M-basis for \( V \) and \( B_{Ker f} \cap B_{Ex} = \phi \).
We first show that \( f(B_{Ex}) = B_{Im} \) is a M-basis for \( im f \). Clearly \( B_{Im} \) is a basis.
for \textit{imf}. Let \(v_1, v_2, \ldots, v_k \in B_{E_x}\) and \(a_1, \ldots, a_k \in \mathbb{R}\) not all zero. By definition we have
\[
C_{f(V)}(\sum_{i=1}^{k} a_i f(v_i)) = \begin{cases} 
\sup \{C_V(x) : x \in f^{-1}(\sum_{i=1}^{k} a_i f(v_i))\} & \text{if } f^{-1} \left( \sum_{i=1}^{k} a_i f(v_i) \right) \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]
Since \(\sum_{i=1}^{k} a_i f(v_i) \in \text{imf}\) we have
\[
C_{f(V)}(\sum_{i=1}^{k} a_i f(v_i)) = \sup \{C_V(x) : x \in f^{-1}(\sum_{i=1}^{k} a_i f(v_i))\}.
\]
By linearity of \(f\) and by the property of \(f^{-1}\) we get
\[
C_{f(V)}(\sum_{i=1}^{k} a_i f(v_i)) = \sup \{C_V(x) : x \in \ker f + \sum_{i=1}^{k} a_i v_i\}.
\]
If \(x \in \ker f\) then \(x = \theta\) or \(x = \sum_{i=1}^{p} b_i u_i, u_i \in B_{ker f}\) where not all \(b_i\) are zero; so if \(x \in \ker f + \sum_{i=1}^{k} a_i v_i\) then either \(C_V(x) = C_V(\theta + \sum_{i=1}^{k} a_i v_i)\) or
\[
C_V(x) = C_V(\sum_{i=1}^{p} b_i u_i + \sum_{i=1}^{k} a_i v_i) \text{ and thus}
\]
\[
C_V(x) = \min \left( \bigwedge_{i=1}^{p} C_V(b_i u_i), \bigwedge_{i=1}^{k} C_V(a_i v_i) \right).
\]
As \(u_i\) and \(v_i\) are M-basis elements of \(V\),
\[\text{which is clearly smaller than or equal to } C_V(\sum_{i=1}^{k} a_i v_i).\]
Thus
\[
C_{f(V)}(\sum_{i=1}^{k} a_i f(v_i)) = \sup \{C_V(x) : x \in \ker f + \sum_{i=1}^{k} a_i v_i\} = C_V(\sum_{i=1}^{k} a_i v_i) = \bigwedge_{i=1}^{k} C_V(a_i v_i).
\]
By the same argument we get that \(C_{f(V)}(f(v_i)) = C_V(v_i).\) Thus \(C_{f(V)}(\sum_{i=1}^{k} a_i f(v_i)) = \bigwedge_{i=1}^{k} C_{f(V)}(a_i v_i).\)

and therefore \(B_{Im}\) is a M-basis for \(\text{imf}\).

Now by definition of multi dimension we get
\[
\dim(V) = \sum_{v \in B_{ker f} \cup B_{E_x}} C_V(v) = \sum_{v \in B_{ker f}} C_V(v) + \sum_{v \in B_{E_x}} C_V(v).
\]
But by the above we have if \(z \in B_{E_x}\), then \(C_{f(V)}(f(z)) = C_V(z)\), and thus
\[
\dim(V) = \sum_{v \in B_{ker f}} C_V(v) + \sum_{v \in B_{E_x}} C_{f(V)}(f(v)) = \dim(\ker f) + \dim(\text{imf}).
\]
7 Conclusion

There is a future scope of study of infinite dimensional multi vector space and behavior of linear operators in multi vector space context.

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