Weight structures in localizations (revisited) and the weight lifting property

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Abstract

The goal of the current paper is to study in detail the behavior of weight structures under localizations. Let \( \mathcal{D} \) be a triangulated subcategory of a triangulated category \( \mathcal{C} \) endowed with a weight structure \( w \); assume that any object of \( \mathcal{D} \) admits a weak weight decomposition inside \( \mathcal{D} \) (i.e., \( \mathcal{D} \) is compatible with \( w \) in a certain sense). Then we prove that the localized category \( \mathcal{C}/\mathcal{D} \) possesses a weight structure \( w_{\mathcal{C}/\mathcal{D}} \) such that the localization functor \( \mathcal{C} \to \mathcal{C}/\mathcal{D} \) is weight-exact (i.e., "respects weights"). Suppose moreover that \( \mathcal{D} \) is generated by \( B = \text{Cone}(S) \), where \( S \subset \text{Mor}(\mathcal{H}w) \) is a class of morphisms satisfying the Ore condition. In this case we give a certain explicit description of those \( X \in \text{Obj}\mathcal{C} \) such that \( L(X) \) has non-positive weights in \( \mathcal{C}/\mathcal{D} \). This yields a new simplified description of \( w_{\mathcal{C}/\mathcal{D}} \). We apply this result to the setting of Tate motives and prove some new properties of their weight 0 motivic cohomology. This result also will be applied to the study of the so-called Chow-weight homology.

Furthermore, we ask whether there exists a description as above for objects having non-positive weights in \( \mathcal{C}/\mathcal{D} \) for \( \mathcal{D} \) being an arbitrary subcategory of \( \mathcal{C} \) admitting weak weight decompositions. Under certain assumptions we find a necessary and sufficient condition in terms of \( K_0(\mathcal{H}w) \) for having such a description (and we give an example where such a description is not possible).

Contents

1 Introduction 2

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1 Introduction

Let $X$ be a finite spectrum (i.e., $X = (\Sigma^\infty A)[m]$ for some finite CW-complex $A$ and an integer $m$). Assume that the stable homotopy groups $\pi_i(X)$ are torsion for $i \leq n$ for some $n \in \mathbb{Z}$. It is natural to ask whether $X$ belongs to the extension-closure of the union of the class of finite torsion spectra with the class of finite $n$-connected spectra. Below we will establish this result in the following strong form: there exists a triangle $T \to X \to M \to T[1]$ in $SH^{fin}$ such that $T$ is torsion and $M$ is an $n$-connected spectrum.

Now let $F$ be a field of characteristic 0 (one can also work in arbitrary characteristic but we do not consider the case). Denote by $DMT^{eff}_{gm}(F)$ the category of effective geometric Tate motives over $F$ with coefficients in $\mathbb{Z}$; let $M$ be a Tate motive. Assume that the motivic cohomology groups $H^i_M(M, \mathbb{Z}/m) = DMT^{eff}_{gm}(F)(M, \mathbb{Z}/m[i])$ (of weight zero) vanish for all $i \geq -n$ and $m \in \mathbb{Z}$. Certainly, if $M = P(1)$ for some effective $P$ (i.e., $M$ is 1-effective in the sense of Definition 2.2.2 of [BoS14]) then the groups $H^i_M(M, \mathbb{Z}/m) = DMT^{eff}_{gm}(F)(P(1), \mathbb{Z}/m[i])$ are zero for any $i, m \in \mathbb{Z}$. Besides, if $M$ belongs to the extension-closure of the set $\{\mathbb{Z}(k)[j+2k]\}_{j \geq n+1, k \in \mathbb{Z}}$ (which we denote by $DMT^{eff}_{gm,w \geq n+1}$) then the groups $H^i_M(M, \mathbb{Z}/m)$ vanish for $i \geq -n$, $m \in \mathbb{Z}$. Now let $M$ be an object of $DMT^{eff}_{gm}$ such that $H^i_M(M, \mathbb{Z}/m) = 0$ for $i \geq -n$, $m \in \mathbb{Z}$. Similarly to the previous setting, we establish the existence of a triangle $E \to M \to M' \to E[1]$, where $E$ is an 1-effective motive and $M'$ is an element of $DMT^{eff}_{gm,w \geq n+1}$. 
Also we prove that for any \( M \in \text{Obj} \text{DMT}_{t}\) such that \( H^i_M(M, \mathbb{Q}) = 0 \) for \( i \geq -n \) there exists a triangle \( T \rightarrow M \rightarrow M' \rightarrow T[1] \), where \( T \) is a torsion motive and \( H^i_M(M, \mathbb{Z}/m) = 0 \) for \( i \geq -n, m \in \mathbb{Z} \).

Now we will ask a question that generalizes the aforementioned problems. For this purpose we will use the language of weight structures; this notion was introduced by Bondarko and independently by Pauksztello (under the name of co-\( t \)-structures). Weight structures are important counterparts of \( t \)-structures. Similarly to \( t \)-structures, weight structures on a triangulated category \( \mathcal{C} \) are defined in terms of classes of objects \( \mathcal{C}_{w \geq 0} \) and \( \mathcal{C}_{w \leq 0} \) satisfying certain axioms. We will call the objects belonging to the first (resp., to the second) class of non-negative (resp., of non-positive) weights.

Let \( \mathcal{C} \) be a triangulated category endowed with a weight structure \( w \), \( D \) be its subcategory such that the localized category \( \mathcal{C}' = \mathcal{C}/D \) possesses a weight structure \( w' \) such that the localization functor \( L \) is weight-exact (i.e., preserves objects of non-positive and of non-negative weights). Let \( X \) be an object of \( \mathcal{C} \). What can be said about \( X \) if we know the weights of \( L(X) \)? More precisely, the question can be formulated as follows.

**Question 1.0.1.** Assume \( L(X) \) belongs to \( \mathcal{C}'_{w' \geq 0} \). Does \( X \) belong to the extension-closure of the class \( \mathcal{C}_{w \geq 0} \cup \text{Obj} D \)?

Since \( L \) is weight-exact, we have \( \mathcal{C}'_{w' \geq 0} = \text{Kar}_{\mathcal{C}'}(L(\mathcal{C}_{w \geq 0})) \) and \( \mathcal{C}'_{w' \leq 0} = \text{Kar}_{\mathcal{C}'}(L(\mathcal{C}_{w \leq 0})) \) (by Proposition 8.1.1 of [Bon10]; here \( \text{Kar}_{\mathcal{C}'}(A) \) denotes the class of \( \mathcal{C}' \)-retracts of elements for a class \( A \subset \text{Obj} \mathcal{C}' \)). Certainly if \( L(\mathcal{C}_{w \geq 0}) \) is not Karoubi-closed in \( \mathcal{C}' \) then there exist objects of \( \mathcal{C} \) whose image belongs to \( \mathcal{C}'_{w' \geq 0} \) and which cannot be constructed via extensions starting from elements of \( \mathcal{C}_{w \geq 0} \cup \text{Obj} D \).

So, to answer Question 1.0.1 we should first determine whether we can avoid the Karoubi-closures when describing the weight structure on \( \mathcal{C}' \).

We will say that \( L \) satisfies the left weight lifting property if for any \( Y \in \mathcal{C}'_{w' \geq 0} \) there exists \( X \in \mathcal{C}_{w \geq 0} \) such that \( L(X) \cong Y \). Dually, \( F \) is said to satisfy the right weight lifting property if for any \( Y \in \mathcal{C}'_{w' \leq 0} \) there exists \( X \in \mathcal{C}_{w \leq 0} \) such that \( L(X) \cong Y \).

**Question 1.0.2.** Let \( X \) be an element of \( \mathcal{C}_{w \geq 0} \). Do retracts of \( L(X) \) in \( \mathcal{C}/D \) lift to elements of \( \mathcal{C}_{w \geq 0} \)?

When the answer to this question is positive, we also have a positive answer to Question 1.0.1 i.e. the class of those \( X \) in \( \mathcal{C} \) such that \( L(X) \in (\mathcal{C}/D)_{w \geq 0} \) is the extension-closure of the class \( \mathcal{C}_{w \geq 0} \cup \text{Obj} D \) in \( \mathcal{C} \). Besides, any element of this extension-closure is "often" an extension of an element of \( \mathcal{C}_{w \geq 0} \) by an object of \( D \); in this situation we say that the strong left weight lifting property is valid.
Now let $\mathcal{D}$ be the full triangulated subcategory of $\mathcal{C}$ generated by cones of some set $S$ of $Hw$-morphisms. Then the category $\mathcal{C}/\mathcal{D}$ (also) possesses a weight structure such that $L$ is weight-exact (see Theorem 4.2.2 of [BoS13]). Below we will prove that $L$ (in this case) satisfies the strong left weight lifting property whenever $S$ satisfies the left Ore condition (see Theorem 3.2.5 below). Thus in this setting the answers to questions 1.0.1 and 1.0.2 are both positive. This statement certainly can be applied to the stable homotopy category and to Tate motives (see Remark 3.2.8); we obtain the statements mentioned in the beginning of this section as particular cases.

Another useful corollary is as follows: any object of $Hw_{C/D}$ lifts to an object of $Hw_C$ if $\mathcal{D}$ is a bounded subcategory of $\mathcal{C}$ such that $w$ restricts to it. The latter statement was used in the recent paper [BoIv15]. Furthermore, our theorem becomes very useful in the study of certain Chow-weight homology groups (see Theorems 3.2.3 and 4.2.5 of [BoS14]).

Another corollary of our result is Proposition 3.3.1, where we compare the weights of $X \in \text{Obj} \mathcal{C}$ in certain localizations $\mathcal{C}/\mathcal{D}_i$ with its weights in the categories $(\mathcal{C}/\mathcal{D}_i) \otimes \mathbb{Q}$ (where $\mathcal{D}_i$ are some triangulated subcategories on which $w$ restricts). It turns out that there exists a triangle $T \to X \to X' \to T[1]$, where $T$ is a torsion object and the weights of $X'$ in all $\mathcal{C}/\mathcal{D}_i$ "are almost the same" the weights of $X$ in $(\mathcal{C}/\mathcal{D}_i) \otimes \mathbb{Q}$. We note that it is impossible to obtain such comparison if the weight lifting property is not satisfied (see Remark 3.3.2). We apply this result to the setting of Tate motives and their weight 0 motivic homology (see Remark 3.3.3(3)). In Theorem 4.2.5 of [BoS14] we apply this proposition to the case $C = DM_{gm,Z[1/p]}^{\text{eff}}$ and $\mathcal{D}_i = DM_{gm,Z[1/p]}^{\text{eff}}(i)$; this yields quite non-trivial properties of the so-called Chow-weight homology groups.

We also study those subcategories $\mathcal{D}$ of $\mathcal{C}$ whose objects admit weak weight decompositions inside $\mathcal{D}$. We prove that for such a subcategory there exists a weight structure on $\mathcal{C}/\mathcal{D}$ compatible with $w$ (see Theorem 4.1.3). This result generalizes all the known cases when $w$ yields a weight structure on $\mathcal{C}/\mathcal{D}$. Under certain restrictions we obtain necessary and sufficient conditions for the corresponding localization functor to satisfy the weight lifting property. More precisely, we prove the following statement: if $w$ is bounded below then the surjectivity of the map $K_0(Hw) \to K_0(Hw_{C/D})$ is equivalent to the strong right weight lifting property (see Proposition 4.3.1). The author hopes that this result will be useful for dealing with $K$-theory of additive categories (especially with $K_{-1}$).

Now let’s give some details on the organization of the paper. In Section 2 we recall some basics on weight structures. In particular,
we recall certain results about localizations of triangulated categories with weight structures (see Propositions 2.3.2 and 2.3.3).

Section §3 is dedicated to the proof of Theorem 3.2.5 (which is the central result of this paper). It states that the localization functor $C \rightarrow C/D$ satisfies the left weight lifting property whenever $D$ is generated by cones of some set $S \subset \text{Mor } Hw$ satisfying the left Ore condition. In subsection 3.1 we introduce some general statements in triangulated categories. Next, in subsection 3.2 we prove two technical lemmas and prove the main result. We also give some concrete examples of usage of our theorem (see Remark 3.2.8). In subsection 3.3 we describe certain applications and corollaries of our theorem. In particular, we apply it to the study of weight 0 motivic cohomology of Tate motives (see Remark 3.3.3(3)).

In §4 we study the weight lifting property in the setting of arbitrary localizations (i.e., not necessarily generated by some set of "short" objects). First we introduce subcategories with weak weight decompositions. In Theorem 4.1.3 we prove that the localized category $C/D$ possesses a weight structure compatible with the weight structure on $C$ for a subcategory with weak weight decompositions $D$. In §4.2 we describe some constructions of subcategories $D$ of this type and apply Theorem 4.1.3 to the particular cases given by these constructions. In §4.3 under certain assumptions we find necessary and sufficient conditions for the corresponding localization functor to satisfy the strong left weight lifting property. Using the results of §4.2 we give an explicit example of a localization functor which doesn’t satisfy the weight lifting property (see Remark 4.3.2).

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2 Preliminaries

2.1 Some (categorical) notation and lemmas

- For $a \leq b \in \mathbb{Z}$ we will denote by $[a, b]$ the set $\{i \in \mathbb{Z} : a \leq i \leq b\}$.

- Given a category $C$ and $X, Y \in \text{Obj } C$, we denote by $C(X, Y)$ the set of morphisms from $X$ to $Y$ in $C$.

- For categories $C', C$ we write $C' \subset C$ if $C'$ is a full subcategory of $C$. 
• Given a category \(C\) and \(X, Y \in \text{Obj } C\), we say that \(X\) is a retract of \(Y\) if \(\text{id}_X\) can be factored through \(X\) (if \(C\) is triangulated or abelian, then \(X\) is a retract of \(Y\) if and only if \(X\) is its direct summand).

• An additive subcategory \(H\) of additive category \(C\) is called Karoubi-closed in \(C\) if it contains all retracts of its objects in \(C\). The full subcategory \(\text{Kar}_C(H)\) of additive category \(C\) whose objects are all retracts of objects of a subcategory \(H\) (in \(C\)) will be called the Karoubi-closure of \(H\) in \(C\).

• The Karoubization \(\text{Kar}(B)\) (no lower index) of an additive category \(B\) is the category of “formal images” of idempotents in \(B\). So, its objects are pairs \((A, p)\) for \(A \in \text{Obj } B, \ p \in B(A,A), \ p^2 = p\), and the morphisms are given by the formula

\[
\text{Kar}(B)((X,p), (X',p')) = \{ f \in B(X,X') : p' \circ f = f \circ p = f \}. \tag{1}
\]

The correspondence \(A \mapsto (A, \text{id}_A)\) (for \(A \in \text{Obj } B\)) fully embeds \(B\) into \(\text{Kar}(B)\). Besides, \(\text{Kar}(B)\) is Karoubian, i.e., any idempotent morphism yields a direct sum decomposition in \(\text{Kar}(B)\). Equivalently, \(B\) is Karoubian if (and only if) the canonical embedding \(B \rightarrow \text{Kar}(B)\) is an equivalence of categories. Recall also that \(\text{Kar}(B)\) is triangulated if \(B\) is (see [BaS01]).

• \(C\) below will always denote some triangulated category; usually it will be endowed with a weight structure \(w\).

• For any \(A, B, C \in \text{Obj } C\) we will say that \(C\) is an extension of \(B\) by \(A\) if there exists a distinguished triangle \(A \rightarrow C \rightarrow B \rightarrow A[1]\).

• For any \(E, E' \subset \text{Obj } C\) we denote by \(E \ast E'\) the class of all extensions of elements of \(E'\) by elements of \(E\).

• A class \(D \subset \text{Obj } C\) is said to be extension-closed if it is closed with respect to extensions and contains \(0\). We will call the smallest extension-closed subclass of objects of \(C\) that contains a given class \(B \subset \text{Obj } C\) the extension-closure of \(B\).

• Given a class \(D\) of objects of \(C\) we denote by \(\langle D \rangle\) the smallest full triangulated subcategory of \(C\) containing \(D\). We will also call \(\langle D \rangle\) the triangulated category generated by \(D\) (yet note that in some other papers of the authors a somewhat distinct definition was used).
• Suppose that $C$ contains all coproducts (resp., products). An object $X$ of $C$ is called compact (resp., cocompact) if for any family of objects $X_i \in \text{Obj} C$ the natural map \(\prod_i C(X, X_i) \to C(\prod_i X_i, X)\) (resp., \(\prod_i C(X_i, X) \to C(\prod_i X_i, X)\)) is an isomorphism.

• For $X, Y \in \text{Obj} C$ we will write $X \perp Y$ if $C(X, Y) = \{0\}$. For $D, E \subset \text{Obj} C$ we write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. Given $D \subset \text{Obj} C$ we denote by $D^\perp$ the class
\[
\{ Y \in \text{Obj} C : X \perp Y \ \forall X \in D \}.
\]
Dually, $^\perp D$ is the class \(\{ Y \in \text{Obj} C : Y \perp X \ \forall X \in D \}\).

• Given $f \in C(X, Y)$, where $X, Y \in \text{Obj} C$, we will call the third vertex of (any) distinguished triangle $X \to Y \to Z \to X$ a cone of $f$ (recall that different choices of cones are connected by non-unique isomorphisms).

• For an additive category $B$ we denote by $K(B)$ the homotopy category of (cohomological) complexes over $B$. Its full subcategory of bounded (resp., bounded below) complexes will be denoted by $K^b(B)$ (resp., $K^-(B)$). We will write $X = (X^i)$ if $X^i$ are the terms of the complex $X$.

2.2 Weight structures: basics

Definition 2.2.1. A pair of subclasses $C_{w \leq 0}, C_{w \geq 0} \subset \text{Obj} C$ will be said to define a weight structure $w$ for a triangulated category $C$ if they satisfy the following conditions.

(i) $C_{w \geq 0}, C_{w \leq 0}$ are Karoubi-closed in $C$ (i.e., contain all $C$-retracts of their elements).

(ii) Semi-invariance with respect to translations.
\[
C_{w \leq 0} \subset C_{w \leq 0}[1], \quad C_{w \geq 0}[1] \subset C_{w \geq 0}.
\]

(iii) Orthogonality.
\[
C_{w \leq 0} \perp C_{w \geq 0}[1].
\]

(iv) Weight decompositions.
For any $M \in \text{Obj} C$ there exists a distinguished triangle
\[
X \to M \to Y \to X[1]\tag{2}
\]
such that $X \in C_{w \leq 0}$, $Y \in C_{w \geq 0}[1]$. 
II. The category $Hw \subset \mathcal{C}$ whose objects are $C_{w=0} = C_{w \geq 0} \cap C_{w \leq 0}$ and morphisms are $Hw(Z, T) = C(Z, T)$ for $Z, T \in C_{w=0}$, is called the heart of $w$.

III. $C_{w \geq i}$ (resp. $C_{w \leq i}$, resp. $C_{w=i}$) will denote $C_{w \geq 0}[i]$ (resp. $C_{w \leq 0}[i]$; resp. $C_{w=0}[i]$).

IV. We denote $C_{w \geq i} \cap C_{w \leq j}$ by $C_{[i,j]}$ (so it equals $\{0\}$ for $i > j$).

Elements of $C_{[i,j]}$ are said to have length at most $j - i$. The length of an object $X \in \mathcal{C}$ is the minimal $n \in \mathbb{N} \cup \{0\}$ such that $X$ has length less than or equal to $n$.

$\mathcal{C}^b \subset \mathcal{C}$ will be the category whose object class is $\bigcup_{i, j \in \mathbb{Z}} C_{[i,j]}$.

V. We will say that $(\mathcal{C}, w)$ is bounded if $\mathcal{C}^b = \mathcal{C}$ (i.e., if $\bigcup_{i \in \mathbb{Z}} C_{w \leq i} = \text{Obj} \mathcal{C} = \bigcup_{i \in \mathbb{Z}} C_{w \geq i}$). For any full subcategory $\mathcal{D}$ of $\mathcal{C}$ we will use the notation $\mathcal{D}^b$ for the full subcategory of $\mathcal{C}$ whose objects are $\text{Obj} \mathcal{C}^b \cap \text{Obj} \mathcal{D}$.

VI. Let $\mathcal{C}$ and $\mathcal{C}'$ be triangulated categories endowed with weight structures $w$ and $w'$, respectively; let $F: \mathcal{C} \rightarrow \mathcal{C}'$ be an exact functor.

$F$ is said to be weight-exact (with respect to $w, w'$) if it maps $C_{w \leq 0}$ into $C'_{w' \leq 0}$ and maps $C_{w \geq 0}$ into $C'_{w' \geq 0}$.

VII Let $\mathcal{D}$ be a full triangulated subcategory of $\mathcal{C}$.

We will say that $w$ restricts to it if the classes $\text{Obj} \mathcal{D} \cap C_{w \leq 0}$ and $\text{Obj} \mathcal{D} \cap C_{w \geq 0}$ give a weight structure $w_\mathcal{D}$ on $\mathcal{D}$.

Remark 2.2.2. 1. A simple (and yet quite useful) example of a weight structure comes from the stupid filtration on $K^b(B)$ for an arbitrary additive category $B$. In this case $K^b(B)_{w \leq 0}$ (resp. $K^b(B)_{w \geq 0}$) will be the class of complexes that are homotopy equivalent to complexes concentrated in degrees $\geq 0$ (resp. $\leq 0$).

The heart of this weight structure is the Karoubi-closure of $B$ in $K^b(B)$.

One can also define weight structures on $K^-(B)$ and on $K(B)$ in the similar way.

2. A weight decomposition (of any $M \in \text{Obj} \mathcal{C}$) is (almost) never canonical.

Yet for $m \in \mathbb{Z}$ we will often need some choice of a weight decomposition of $M[-m]$ shifted by $[m]$. So we consider a distinguished triangle

$$w_{\leq m}M \rightarrow M \rightarrow w_{\geq m+1}M$$

(3)

for some $w_{\geq m+1}M \in C_{w \geq m+1}, w_{\leq m}M \in C_{w \leq m}$.

We will often use this notation below (though $w_{\geq m+1}M$ and $w_{\leq m}M$ are not canonically determined by $M$).

3. In the current paper we use the “homological convention” for weight structures; it was previously used in [Wil09] and in [Bon13], whereas in
the "cohomological" convention was used. In the latter convention
the roles of $C^w \leq 0$ and $C^w \geq 0$ are interchanged, i.e., one considers $C^w \leq 0 = C^w \geq 0$ and $C^w \geq 0 = C^w \leq 0$.

Let us recall some basics on weight structures.

**Proposition 2.2.3.** Let $\mathcal{C}$ be a triangulated category, $n \geq 0$; we will assume that $w$ is a fixed weight structure on it.

1. The axiomatics of weight structures is self-dual, i.e., for $D = \mathcal{C}^{\text{op}}$ (so $\text{Obj} D = \text{Obj} \mathcal{C}$) there exists the (opposite) weight structure $w'$ for which $D^w \leq 0 = C^w \geq 0$ and $D^w \geq 0 = C^w \leq 0$.

2. If $M \in C^w \geq -n$ then $w_{\leq 0} M \in C_{\{-n, 0\}}$.

3. Assume $w$ is bounded below. Then $C^w \leq 0$ is the extension-closure of $\bigcup_{i \leq 0} C^w_{i-1}$ in $\mathcal{C}$.

Assume $w$ is bounded above. $C^w \geq 0$ is the extension-closure of $\bigcup_{i \geq 0} C^w_{i+1}$ in $\mathcal{C}$.

4. $C^w \geq 0 = (C^w \leq -1)^{\perp}$ and $C^w \geq -1 = \perp C^w \leq 1$.

5. Assume that $w'$ is a weight structure for a triangulated category $\mathcal{C}'$.

Then an exact functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is weight-exact if and only if $F(C^w = 0) \subset C_{w' = 0}$.

**Proof.** All of these statements can be found in [Bon10] (pay attention to Remark 2.2.2(3))!

2.3 Weight structures in localizations (reminder)

**Definition 2.3.1.** Let $\mathcal{C}$ be a category with a weight structure $w$ and let $\mathcal{E}$ be some triangulated category with a functor $F$ from $\mathcal{C}$ to $\mathcal{E}$. We will say that $\mathcal{E}$ admits a weight structure compatible with $w$ if there is a weight structure $w_F$ on $\mathcal{E}$ such that $F$ is $w$-exact.

If such a weight structure is unique we will say that $w$ induces a weight structure on $\mathcal{E}$.

We call a category $\mathcal{A}/\mathcal{B}$ the factor of an additive category $A$ by its full additive subcategory $B$ if $\text{Obj}(\mathcal{A}/\mathcal{B}) = \text{Obj} A$ and $(\mathcal{A}/\mathcal{B})(X, Y) = A(X, Y)/(\sum_{Z \in \text{Obj } B} A(Z, Y) \circ A(X, Z))$. 
Proposition 2.3.2. Let $D \subset C$ be a triangulated subcategory of $C$; suppose that $w$ restricts to $D$ (see Definition 2.2.1(VII)). Denote by $l$ the localization functor $C \to C/D$ (the latter category is the Verdier quotient of $C$ by $D$).

Then the following statements are valid.

1. $w$ induces a weight structure on $C/D$, i.e., the Karoubi-closures of $l(C_{w \leq 0})$ and $l(C_{w \geq 0})$ give a weight structure for $C/D$.

2. The heart $H(C/D)$ of the weight structure $w_{C/D}$ obtained is the Karoubi-closure of (the natural image of) $\frac{Hw}{Hw_{\text{add}}}$ in $C/D$.

3. If $(C, w)$ is bounded, then $(C/D, w_{C/D})$ also is.

Proof. All the assertions were proved in §8.1 of [Bon10].

If $w$ is bounded then all subcategories $D \subset C$ on which $w$ restricts come from additive subcategories of $Hw$ (see Corollary [3.1.3(2)]). Yet to ensure that there exists a weight structure for $C/D$ such that the localization functor is weight-exact it actually suffices to assume that $D$ is generated by some set of elements of $C_{[0,1]}$.

To be more precise, we recall the following statement.

Proposition 2.3.3. Let $D \subset C$ be a triangulated subcategory of $C$ generated by some set $B$ of elements of $C_{[0,1]}$. Denote by $L$ the localization functor $C \to C/D$ (the latter category is the Verdier quotient of $C$ by $D$).

For any $i, j \in \mathbb{Z} \cup \{-\infty, +\infty\}$ we denote by $B_{[i,j]}$ the extension-closure of $\bigcup_{i \leq k \leq j} B[k]$. We will also use the notation $B_{\geq i} = B_{[i, +\infty]}$ and $B_{\leq j} = B_{[-\infty, j]}$.

Then the following statements are valid.

1. For any element $T$ of $B_{[m,n]}$ there exists a distinguished triangle $T_1 \to T \to T_2 \to T_1[1]$, where $T_1 \in B_{[m,0]}$ and $T_2 \in B_{[1,n]}$. We will call this triangle a weak weight decomposition for $T$.

2. $B_{[i,j]} \subset C_{[i,j+1]}$

3. $w$ induces a weight structure on $C/D$, i.e., the Karoubi-closures of $L(C_{w \leq 0})$ and $L(C_{w \geq 0})$ give a weight structure for $C/D$. Its heart is equivalent to $\text{Karb}_C(Hw[S^{-1}]_{\text{add}})$, where $Hw[S^{-1}]_{\text{add}}$ is a certain additive localization of the category $Hw$ that was defined in [BoS13].

Proof. All the assertions were proved in [BoS13]: see Proposition 2.1.3(1), Remark 2.1.4(1), and Theorem 4.2.2 of the paper, respectively.

Remark 2.3.4. If there is no ambiguity we will denote by $w$ the weight structure on $C/D$ corresponding to a weight structure $w$ on $C$. 


3 The weight lifting property

In the section we will treat the Question 1.0.2 for different localization functors. Then we give some corollaries and applications. First we need some preliminaries on extensions in triangulated categories.

3.1 Properties of (weak) weight decompositions and extensions

In this subsection we list and prove several rather simple properties of weak weight decompositions that we will use later. We will also propose some useful lemmas about general extensions in triangulated categories.

Definition 3.1.1. We call a triangle \( X_1 \rightarrow X \rightarrow X_2 \rightarrow X_1[1] \) a weak weight decomposition of \( X \) if \( X_1 \in \mathcal{C}_{w \leq 0} \) and \( X_2 \in \mathcal{C}_{w \geq 0} \).

Usually we will consider weak weight decompositions in some triangulated subcategory \( D \) of \( C \). This means that \( X_1 \) and \( X_2 \) are objects of \( D \). Using this definition, the assertion 1 of Proposition 2.3.3 can be reformulated as follows: any element of \( D \) possesses a weak weight decomposition inside \( D \).

Proposition 3.1.2. Let \( X, Y \in \text{Obj} \mathcal{C} \) and let \( E \in X \star Y \). Assume \( X_1 \rightarrow X \rightarrow X_2 \rightarrow X_1[1] \) and \( Y_1 \rightarrow Y \rightarrow Y_2 \rightarrow Y_1[1] \) are weight decompositions (resp., weak weight decompositions) of \( X \) and \( Y \), respectively. Then \( E \) has a weight decomposition (resp., a weak weight decomposition) \( E_1 \rightarrow E \rightarrow E_2 \rightarrow E_1[1] \) such that \( E_1 \in X_1 \star Y_1 \) and \( E_2 \in X_2 \star Y_2 \).

Proof. By the 3 \( \times \) 3-Lemma (see [BBD82], Proposition 1.1.11) there exists a diagram

\[
\begin{array}{ccc}
X_1 & \rightarrow & X \\
\downarrow & & \downarrow \\
E_1 & \rightarrow & E \\
\downarrow & & \downarrow \\
Y_1 & \rightarrow & Y \\
\end{array}
\]

whose rows and columns are distinguished triangles. Since the classes \( \mathcal{C}_{w \geq 0} \) and \( \mathcal{C}_{w \leq 0} \) are extension-closed, the triangle \( E_1 \rightarrow E \rightarrow E_2 \rightarrow E_1[1] \) yields a weight decomposition (resp., a weak weight decomposition) of \( E \).

Corollary 3.1.3. 1. Let \( D \) be the triangulated subcategory of \( C \) generated by \( D \). Then any object of \( D \) possesses a weight decomposition (resp., a weak weight decomposition) in \( D \) whenever any object of \( \bigcup_{i \in \mathbb{Z}} D[i] \) does.
2. Conversely, if $D$ is a subcategory of $C_{\text{b}}$ such that any object of $D$ possesses a weight decomposition (resp., a weak weight decomposition) inside $D$ then $D$ is generated by some set of objects of length 0 (resp., of length 1).

**Proof.** 1. Suppose $X \in \text{Obj} C$. Using induction we can assume that $X$ is an extension of objects of $\bigcup_{i \in \mathbb{Z}} D[i]$; hence it is an extension of objects having weight decompositions (resp., weak weight decompositions). Then Proposition [3.1.2] yields the result.

2. It suffices to prove that any $X \in (\text{Obj} D) \cap C_{[0,n]}$ belongs to $\langle (\text{Obj} D) \cap C_{[0,0]} \rangle$ (resp., to $\langle (\text{Obj} D) \cap C_{[0,1]} \rangle$). We will prove the statement using induction on $n$.

The base of induction is given by the following fact. If $X \in (\text{Obj} D) \cap C_{[0,0]}$ (resp., $X \in (\text{Obj} D) \cap C_{[0,1]}$) then $X \in \text{Obj} (\langle (\text{Obj} D) \cap C_{[0,0]} \rangle)$ (resp., $X \in \text{Obj} (\langle (\text{Obj} D) \cap C_{[0,1]} \rangle)$).

Now we will describe the inductive step. Suppose that any $Y \in (\text{Obj} D) \cap C_{[0,n-1]}$ belongs to $\langle (\text{Obj} D) \cap C_{[0,0]} \rangle$. Let $X \in (\text{Obj} D) \cap C_{[0,n]}$. Consider a weight decomposition (resp., a weak weight decomposition) of $X$, i.e., a triangle

$$X_1 \rightarrow X \rightarrow X_2 \rightarrow X_1[1]$$

such that $X_2 \in C_{w \geq 1}$ and $X_1 \in C_{w \leq 0}$ (resp., $X_1 \in C_{w \leq 1}$). Since $X_1 \in \text{Obj} (\langle (\text{Obj} D) \cap C_{[0,0]} \rangle)$ (resp., $X_1 \in \text{Obj} (\langle (\text{Obj} D) \cap C_{[0,1]} \rangle)$) and $X_2[-1]$ satisfy the inductive assumption, we obtain that $X \in \text{Obj} (\langle (\text{Obj} D) \cap C_{[0,1]} \rangle)$.

Let $R$ be a commutative ring, and suppose that $C$ is an $R$-linear triangulated category. For any set $\Sigma \subset R$ we define $S$ to be the class of multiplications by elements of $\Sigma$ on objects of $C$. Denote by $K$ the Karoubi-closure of the triangulated subcategory category generated by cones of elements of $S$.

The following two lemmas treat this simple case.

**Lemma 3.1.4.** For any $Y \in \text{Obj} K$ there exists an element $\lambda$ which is a product of elements of $\Sigma$ such that $\lambda \text{id}_Y = 0$. So, there exists $\lambda \in \Sigma$ if $\Sigma$ is multiplicatively-closed.

**Proof.** It suffices to prove the following. Let $\lambda_1 \text{id}_{Y_1} = 0$ and $\lambda_1 \text{id}_{Y_2} = 0$ for some $Y_1, Y_2 \in \text{Obj} C$. Then for any distinguished triangle $Y_1 \rightarrow Y \rightarrow Y_2 \rightarrow Y_1[1]$ the equality $\lambda_1 \lambda_2 \text{id}_Y = 0$ holds. Indeed, $q \lambda_2 \text{id}_Y = \lambda_2 \text{id}_{Y_2} q = 0$, so $\lambda_2 \text{id}_Y$ factors through $Y_1$, i.e., $\lambda_2 \text{id}_Y = q \circ t$ for some $t \in C(Y,Y_1)$. Multiplying the both parts of equality by $\lambda_1$ we obtain the result.

**Lemma 3.1.5.** For any $U, X \in \text{Obj} C$ the following conditions are equivalent.

---

12
1. There exists a distinguished triangle \( U \to X \to Y \to U[1] \) in \( C \) such that \( Y \) belongs to \( K \).

2. There exists a distinguished triangle \( Y' \to X \to U \to Y'[1] \) in \( C \) such that \( Y' \) belongs to \( K \).

**Proof.** It suffices to prove \( 1 \Rightarrow 2 \) because the implication \( 1 \Rightarrow 2 \) in the opposite category yields the implication \( 2 \Rightarrow 1 \) in \( C \) and the axiomatics of triangulated categories is self-dual.

Let \( Y \in D \). By Lemma 3.1.4 we may assume that \( \lambda \text{id}_Y = 0 \) where \( \lambda \) becomes invertible in \( C/K \).

We note that \( d \) factors through \( U/\lambda \), where \( U/\lambda \) denotes a cone of \( \lambda \text{id}_U \). Denote the corresponding morphism from \( U/\lambda \) to \( U[1] \) by \( r \), and denote the morphism from \( Y \) to \( U/\lambda \) by \( q \). Certainly, \( U/\lambda \) belongs to \( K \).

The octahedron axiom applied to the commutative triangle \( (q, r, r \circ q) \) (which we specify by its arrows) yields the existence of a distinguished triangle \( \text{Cone}(q)[-1] \to X \to U \to \text{Cone}(q) \). Note that \( \text{Cone}(q)[-1] \) belongs to \( K \) since it is a cone of a morphism between objects of \( K \). So we obtain the result.

**Corollary 3.1.6.** Let \( X, Y, Z \) be subclasses of \( C \).

Then the following statements are valid:

1. \( X \ast (\text{Obj} \, K) = (\text{Obj} \, K) \ast X \).
2. \( (X \ast Y) \ast Z = (X \ast Y) \ast Z \)

**Proof.** The first assertion is immediate from Lemma 3.1.5. For the proof of second assertion see Lemma 1.3.10 of [BBD82].

Now assume \( C \) is endowed with a weight structure \( w \). Denote by \( K' \) the triangulated subcategory of \( C \) generated by the set \( \{ \text{Cone}(X \to X) : X \in \text{Obj} \, Hw, \lambda \in \Sigma \} \).

**Lemma 3.1.7.** In the assumptions above \( K \) coincides with \( \text{Kar}_C(K') \)

**Proof.** Let \( U \) be an object of \( C \). We need to show that \( U/\lambda = \text{Cone}(U \to U) \) belongs to \( \text{Kar}_C(K') \). Note that \( t(L(\lambda \text{id}_U)) \) is invertible in \( K_w(Hw[S^{-1}]_{\text{add}}) \), where \( L \) denotes the localization functor \( C \to C/K \) and \( t \) is the weight complex functor \( C/D \to K_w(Hw[S^{-1}]_{\text{add}}) \) (see section 3 of [Bon10] for the definition). Now the "conservativity" part of Theorem 3.3.1 of [Bon10] implies the result.
3.2 The Ore conditions and the strong weight lifting property

Definition 3.2.1. A class of morphisms \( S \subset \text{Mor}(A) \) (for a category \( A \)) is said to satisfy the right Ore condition if for any pair of morphisms \( s \in A(c, d) \cap S \), and \( f \in A(b, d) \) there exist \( a \in \text{Obj } A, g \in A(a, b) \), and \( u \in A(a, c) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
a & \xrightarrow{g} & b \\
\downarrow{u} & & \downarrow{f} \\
c & \xrightarrow{s} & d
\end{array}
\] (4)

The left Ore condition is the categorical dual to the right Ore condition.

Note that we do not include the usual "cancellation" property into our Ore conditions (so, one may say that we are only interested in one of the two Ore conditions).

Till the end of the section we will always assume that \((C, w)\) is a triangulated category with a bounded weight structure, \( S \subset \text{Mor}(Hw) \) is a set of morphisms, \( D \subset C \) is the triangulated subcategory of \( C \) generated by the set \( B \) of cones of elements of \( S \) (unless stated otherwise). By Proposition 2.3.3 there exists a weight structure on \( C/D \) such that the localization functor \( L \) is weight-exact. We will denote this weight structure by \( w_{C/D} \).

Lemma 3.2.2. Suppose \( S \) satisfies the left Ore condition. Then for any \( s \in S, X \in C_{w \geq 0} \), and \( u \in C(\text{Cone}(s)[−1], X) \) there exists an element \( X' \) of \( C_{w \geq 0} \) along with a morphism \( f \in C(X, X') \) such that \( \text{Cone}(f) \in \text{Obj } D \cap C_{[0, 1]} \) and the composition morphism \( f \circ u \) from \( \text{Cone}(s) \) to \( X' \) is zero.

Proof. Let \( w_{<0}X \xrightarrow{i} X \to w_{\geq 1}X \) be a weight decomposition of \( X \). The morphism \( u \) factors through \( w_{<0}X \) by the orthogonality axiom for \( w \). Denote the corresponding map from \( \text{Cone}(s)[−1] \) to \( w_{<0}X \) by \( u' \). The left Ore condition yields that there exist \( Y \in C_{w=0} \) and \( g \in C(w_{\geq 0}X, Y) \) such that \( g \circ u = 0 \) and \( \text{Cone}(g) \in \text{Obj } D \). Denote by \( v \) the corresponding morphism from \( \text{Cone}(g)[−1] \) to \( w_{<0}X \).

Applying the octahedron axiom to the commutative triangle \((v, u, u \circ v)\) we obtain a morphism \( g \) from \( X \) to some \( X' \) whose cone belongs to \((\text{Obj } D) \cap \)
\( C_{[0,1]} \) as demonstrated by the following diagram:

\[
\begin{array}{ccc}
\text{Cone}(g)[-1] & \longrightarrow & \text{Cone}(g)[-1] \\
\downarrow^v & & \downarrow \\
w_{\leq 0}X & \longrightarrow & X & \longrightarrow & w_{\geq 1}X \\
\downarrow^g & & \downarrow & & \downarrow \\
Y & \longrightarrow & X' & \longrightarrow & w_{\geq 1}X
\end{array}
\]

(5)

Certainly, the corresponding morphism from \( \text{Cone}(s)[-1] \) to \( X' \) is zero since it factors through \( g \circ u' = 0 \).

\[\square\]

**Lemma 3.2.3.** Let \( X \) be an object of \( \underline{C} \) and let \( w_{\leq -1}X \xrightarrow{i} X \rightarrow w_{\geq 0}X \) be a weight decomposition of \( X \). Let \( f \) be a morphism from \( X \) to \( X' \) with \( \text{Cone}(f) = E \in (\text{Obj} \, D) \cap \underline{C}_{[0,n]} \) such that \( f \circ i \) factors through an object \( T \in (\text{Obj} \, D) \cap \underline{C}_{[0,n]} \) for some \( n \in \mathbb{N} \cup \{+\infty\} \).

Then there exists a morphism from \( X' \) to some \( X'' \in \underline{C}_{w\geq 0} \) whose cone belongs to \( (\text{Obj} \, D) \cap \underline{C}_{[-1,n-1]} \).

**Proof.** Certainly, we have the following commutative square.

\[
\begin{array}{ccc}
w_{\leq -1}X & \longrightarrow & T \\
\downarrow & & \downarrow \\
w_{\leq -1}X' & \longrightarrow & X'
\end{array}
\]

(6)

By Proposition 1.1.11 of [BBD82], it can be completed to the following diagram (whose rows and columns are distinguished triangles):

\[
\begin{array}{ccc}
w_{\leq -1}X & \longrightarrow & T & \longrightarrow & C_1 \\
\downarrow & & \downarrow & & \downarrow \\
w_{\leq -1}X' & \longrightarrow & X' & \longrightarrow & w_{\geq 0}X' \\
\downarrow & & \downarrow^{u_1} & & \downarrow \\
E & \longrightarrow & C_2 & \xrightarrow{u_2} & X''
\end{array}
\]

(7)

The extension-closedness of \( \underline{C}_{[0,n]} \) yields that \( C_1 \) belongs to \( \underline{C}_{[0,n]} \); hence \( X'' \) belongs to \( \underline{C}_{w\geq 0} \). Moreover, \( T'[1] = \text{Cone}(u_1) \) belongs to \( (\text{Obj} \, D) \cap \underline{C}_{[-1,n-1]} \), and \( E[1] = \text{Cone}(u_2) \) also does. So the composition \( u_2 \circ u_1 \) satisfies the conditions desired. \[\square\]
Definition 3.2.4. Let $F : C \to C'$ be a triangulated functor between triangulated categories which is weight-exact with respect to some weight structures $w$ and $w'$ (on $C$ and $C'$, respectively). We say that $F$ satisfies the strong left weight lifting property if for any object $X$ of $C$ such that $F(X) \in C'_{w' \geq 0}$ there exists a triangle

$$T \to X \to X_{\geq 0} \to T[1],$$

where $F(T) = 0$ and $X_{\geq 0} \in C_{w \geq 0}$.

Using the octahedron axiom and weak weight decompositions of $T$ one can show that the existence of a triangle as above is equivalent to the existence of a distinguished triangle

$$T' \to X \to X'_{\geq 0} \to T[1],$$

where $T' \in (\text{Obj } D) \cap C_{w \leq 0}$ and $X'_{\geq 0} \in C_{w \geq 0}$.

We will say that $F$ satisfies strong right weight lifting property if the corresponding functor on the opposite triangulated categories satisfies the strong left weight lifting property.

Theorem 3.2.5. Suppose that $w$ is bounded below and $S$ satisfies the left Ore condition. Then $L$ satisfies the strong left weight lifting property.

Proof. Let $X$ be an object of $C$ such that $L(X) \in (C/D)_{w \geq 0}$. We wish to prove that there exists an element $X' \in C_{w \geq 0}$ along with a morphism $f \in C(X, X')$ such that $\text{Cone}(f)[-1] \in (\text{Obj } D) \cap C_{w \leq 0}$.

We will use induction on the minimal $n \in \mathbb{Z}$ such that $X \in C_{w \leq -n}$.

Fix a (shifted) weight decomposition of $X$:

$$w \leq -1 \xrightarrow{i} X \xrightarrow{j} w_{\geq 0} \xrightarrow{k} X[1].$$

Suppose $X \in C_{w \leq -1}$. The orthogonality axiom for the weight structure $w_{C/D}$ yields $L(i) = 0$ (since $L(X) \in (C/D)_{w \geq 0}$). By a well-known property of Verdier localization, there exists a factorization of $i$ through some object $M$ of $D$, i.e., there exist $g \in C(w_{< -1} X, M)$ and $f \in C(M, X)$ such that $f \circ g = i$. By Proposition 2.1.3(3) of [BeS13], we can assume that $M \in (\text{Obj } D) \cap C_{[-1, 1]}$.

Let $M_1 \xrightarrow{s_1} M \xrightarrow{s_2} M_2$ be a weak weight decomposition for $M$ (in the sense of Proposition 2.3.3(1)). By Lemma 3.2.2, there exists a morphism $r \in C(X, X')$ such that $r \circ f \circ s_1 = 0$ and $\text{Cone}(r) \in (\text{Obj } D) \cap C_{[-1, 0]}$. So, $r \circ f$ factors through $M_2$.

By Lemma 3.2.3, in this situation we have a morphism $u$ from $X'$ to $X''$ such that $X'' \in C_{w \geq 0}$ and $\text{Cone}(u) \in (\text{Obj } D) \cap C_{[-1, 0]}$. Thus the composition $u \circ r$ yields the morphism desired from $X$ to an object whose cone belongs to $(\text{Obj } D) \cap C_{[-1, 0]}$. 

16
Now we describe the inductive step. If \( X \in \mathcal{C}_{w \geq -n} \), then one can apply the inductive assumption to \( X[n-1] \). It yields a distinguished triangle of the form \( M \to X \xrightarrow{\rho} X' \to M[1] \), where \( M \in (\text{Obj } D) \cap \mathcal{C}_{w \leq -(n-1)} \) and \( X' \in \mathcal{C}_{w \geq -(n-1)} \). By the inductive assumption, there also exists a distinguished triangle \( M' \to X' \xrightarrow{\gamma} X'' \to M'[1] \), where \( M' \in (\text{Obj } D) \cap \mathcal{C}_{w \leq 0} \) and \( X'' \in \mathcal{C}_{w \geq 0} \). Applying the octahedron axiom to the commutative triangle of morphisms \((p, q, q \circ p)\) we obtain a triangle of the type desired for \( X \).

\[ \Box \]

**Remark 3.2.6.** Suppose that \( S \) consists of morphisms of the form \( A \to 0 \) for certain objects \( A \) of \( Hw \), i.e., \( w \) restricts to \( D \) (in the sense of Definition 2.2.1(VII)). Applying Theorem 3.2.5 to this setting we obtain the following: for any object \( X \in \text{Obj } C \) such that \( L(X) \in (C/D)_{w \geq 0} \) there exist \( X' \in \mathcal{C}_{w \geq 0} \) and \( g \in \mathcal{C}(X, X') \) such that \( \text{Cone}(g)[-1] \in (\text{Obj } D) \cap \mathcal{C}_{w \leq 0} = D_{w \leq 0} \). Denote the corresponding morphism from \( \text{Cone}(g)[-1] \) to \( X \) by \( j \). Note that \( \text{Cone}(g)[-1] \) admits a weight decomposition \( C_1 \xrightarrow{i} \text{Cone}(g)[-1] \to C_2 \) where \( C_1 \in D_{w \leq -1}, C_2 \in D_{w \geq 0} \). Applying the octahedron axiom to the commutative triangle \((i, j, j \circ i)\) we obtain that \( X \) admits a weight decomposition \( X_1 \to X \to X_2 \) such that \( X_1 \) belongs to \( D_{w \leq -1} \) and \( X_2 \) belongs to \( \mathcal{C}_{w \geq 0} \).

**Remark 3.2.7.** Theorem 3.2.5 is also valid for unbounded weight structures. If \( w \) is an unbounded weight structure then one can also find a morphism from \( X \) to \( X' \in \mathcal{C}_{w \geq 0} \) whose cone lies in \( D \) if \( L(X) \in (C/D)_{w \geq 0} \). Indeed, suppose \( L(X) \in \mathcal{C}_{w \geq 0} \). Then the morphism \( w_{\leq -1} X \xrightarrow{\rho} X \) factors through some \( T \in D \cap \mathcal{C}_{w \geq N} \) for some \( N \in \mathbb{Z} \). Note that the morphism \( w_{\leq N-1} X \to X \) is zero since it factors through \( T \); hence \( X \in \mathcal{C}_{w \geq N} \). Therefore one can apply Theorem 3.2.5 to \( X \) to get the result.

**Remark 3.2.8.** Now we are able to prove the results stated in the beginning of Introduction.

1. Let \( SH^{fin} \) be the topological stable homotopy category of finite spectra. As was shown in section 4.6 of \([\text{Bon10}]\) there exists a certain spherical weight structure on this category (that we denote by \( w \)) such that \( SH^{fin}_{w \geq n+1} \) consists of (finite) spectra \( X \) such that the stable homotopy groups \( \pi_i(X) \) are zero for \( i \leq n \).

Let \( S \) be the class of multiplications by non-zero integers on objects of \( Hw \) (that consists of finite coproducts of sphere spectra). Consider the localization functor \( SH^{fin} \xrightarrow{\tilde{L}} SH^{fin}/(\text{Cone}(S)) \). The category \( SH^{fin}/(\text{Cone}(S)) \) is isomorphic to \( SH^{fin} \otimes \mathbb{Q} \cong K^b(\mathbb{Q} - \text{Vect}) \) (see Appendix A.2 of \([\text{Kel13}]\)). Denote by \( w_{\mathbb{Q}} \) the weight structure on the category \( K^b(\text{Vect}(\mathbb{Q})) \) induced by the weight structure on \( SH^{fin} \). Certainly, \( Y \) belongs to \( K^b(\text{Vect}(\mathbb{Q}))_{w_{\mathbb{Q}} \geq n+1} \) and if only if the groups \( K^b(\text{Vect}(\mathbb{Q}))_{w_{\mathbb{Q}} \geq n+1} \end{equation} vanish for \( i \leq n \).
Let $X$ be an object of $\text{SH}^{\text{fin}}$ such that $\pi^i(X)$ is torsion for all $i \leq n$. Certainly this means that $X$ belongs to $\text{SH}_{Q,q,n+1}^{\text{fin}}$. Applying Theorem \ref{thm:existence} to this setting we obtain the following: there exists a triangle $T \to X \to M \to T[1]$ in $\text{SH}^{\text{fin}}$ such that all the homotopy groups of $T$ are torsion and $M$ is an $n$-connected spectrum.

2. Let $F$ be a field of characteristic 0 (one can also consider arbitrary characteristic but we do not treat the case), $R$ be either $\mathbb{Z}$ or $\mathbb{Q}$. Consider the category $\text{DMT}^{e\text{ff}}_g(F; R)$ of effective geometric $R$-linear Tate motives over $F$.

Recall that there is a weight structure $w$ on this category whose heart is the additive hull of $\{R(k)[2k]\}_{k \in \mathbb{N}}$ (see Theorem 2.2.1 of [Bon11]). Certainly it restricts to the full triangulated subcategory of 1-effective Tate motives. It follows that the category $\text{DMT}^0_g(F) = \text{DMT}^{e\text{ff}}_g(F; R)/\text{DMT}^{e\text{ff}}_g(F; R)(1)$ possesses a weight structure $w^0$ such that the localization functor $\text{DMT}^{e\text{ff}}_g(F; R) \overset{L}{\to} \text{DMT}^0_g(F; R)$ is weight-exact. Since the subcategory $\text{DMT}^{e\text{ff}}_g(F; R)(1)$ is orthogonal to $R$, it is also orthogonal to $R/m = \text{Cone}(R \to R/m)$. By Lemma 9.1.5 of [Nee01], $H^0_{\text{eff}}(M, R/m) = \text{DMT}^{e\text{ff}}_g(F; R)(M, R/m[n]) \cong \text{DMT}^0_g(F; R)(L(M), L(R/m)[n])$. Note that $L(R)$ generates the heart of $w^0$. Moreover, $\text{DMT}^0_g(F; R)(L(R), L(R)[n]) \cong 0$ whenever $n$ is not zero. Thus, the category $\text{DMT}^0_g(F; R)$ is equivalent to the category $K^b(\text{Free}(R))$ (where $\text{Free}(R)$ is the category of finite-dimensional free modules over $R$).

Now let $M$ be an object of $\text{DMT}^{e\text{ff}}_g(F; R)$. Assume that the motivic cohomology groups $H^i_M(M, \mathbb{Z}/m)$ are zero for $i \geq -n, m \in \mathbb{Z}$. This means that $K^b(\text{Free}(R))(L(M), L(R/m)[i]) \cong \text{DMT}^0_g(F; R)(L(M), L(R/m)[i]) \cong 0$ for $i \geq -n, m \in \mathbb{Z}$, i.e., $L(M)$ viewed as a complex (i.e., as an object of $K^b(\text{Free}(R))$) is concentrated in degrees less than or equal to $- (n+1)$. Thus, $L(M)$ belongs to $\text{DMT}^0_g(F; R)_{w^0 \geq n+1}$.

Applying Theorem \ref{thm:existence} we obtain the existence of a triangle $E \to M \to M' \to E[1]$, where $E$ is an effective motive and $M'$ is an element of $\text{DMT}^0_g(F; R)_{w^0 \geq n+1}$.

Note that we can consider the vanishing of the groups $H^m_M(-, \mathbb{Q})$ (resp., of $H^m_M(-, \mathbb{Q}/\mathbb{Z})$) instead of the vanishing of $H^m_M(-, R/m)$ for all $m \in \mathbb{Z}$ if $R = \mathbb{Q}$ (resp., $R = \mathbb{Z}$).

### 3.3 Applications

Using Theorem \ref{thm:existence} (and its partial case described in Remark \ref{rem:partial}) we make the following observation. It is applied in [BoS14] to the study of so-called Chow-weight homology theories and related questions.

Consider a system of full triangulated subcategories $D_i$ of $C$ (for $i \geq 0$; we allow $i$ to be equal to $+\infty$) such that $w$ restricts to $D_i$, $D_{i+1} \subseteq D_i$. 

18
\( \mathcal{D}_0 = \mathcal{C} \) and \( \mathcal{D}_{+\infty} = \{0\} \). Let \( J \subset \mathbb{Z} \setminus \{0\} \); let \( S \) be the class of morphisms \( X \xrightarrow{\text{id}_X} X \), where \( p \in J \), \( X \in \text{Obj} \mathcal{H}_w \). Denote by \( K \) the minimal Karoubi-closed triangulated subcategory of \( \mathcal{C} \) generated by \( \text{Cone}(S) \). Denote by \( \mathcal{D}' \) the category \( K \star \mathcal{D} \); note that it is triangulated by Lemma 3.1.5 (combined with Lemma 3.1.7).

Let \( a_i \) be a non-decreasing non-negative integer sequence (we allow \( a_i \) to be equal to \( +\infty \)). Denote by \( L^i \) (resp., \( L^i \)) the corresponding localization functor from \( \mathcal{C} \) to \( \mathcal{C}/\mathcal{D}_i \) (resp., to \( \mathcal{C}/\mathcal{D}'_i \)).

**Proposition 3.3.1.** Let \( X \) be an object of \( \mathcal{C} \) such that \( L^{a_i}(X) \in (\mathcal{C}/\mathcal{D}_n)_{w \geq -i+1} \) for any \( i \in \mathbb{Z} \).

1. Assume \( a_l = 0 \) for some \( l \). Then for any integer \( N \) there exists a triangle \( T \to X \to M \to T[1] \) such that \( T \in \text{Obj} K \) and \( M \) is an extension of an element \( M' \in C_{w \geq -N+1} \) such that \( L^{a_i}(M') \in (\mathcal{C}/\mathcal{D}_n)_{w \geq -i+1} \) for any \( i \in \mathbb{Z} \) by an element of \( (\text{Obj} \mathcal{D}_{N}) \cap C_{w \leq -N} \).

2. For any integers \( N \) and \( N' \) there exists a triangle \( T \to X \to M \to T[1] \) such that \( T \in \text{Obj} K \) and \( M \) belongs to the class \( ((\text{Obj} \mathcal{D}_{N}) \cap C_{w \leq -N}) \star \{M'\} \star C_{w \geq -N} \) where \( M' \) is an element of \( C_{w \geq -N+1} \) such that \( L^{a_i}(M') \in (\mathcal{C}/\mathcal{D}_n)_{w \geq -i+1} \) for any \( i \in \mathbb{Z} \).

**Proof.** By Theorem 3.2.5 and Proposition 3.1.2, \( X \in ((\text{Obj} K) \star (\text{Obj} \mathcal{D}_n)) \star C_{w \geq -N+1} = ((\text{Obj} K) \star ((\text{Obj} \mathcal{D}_n) \cap C_{w \leq -N})) \star C_{w \geq -N+1} \). By Corollary 3.1.6(2), \( X \) belongs to \( (\text{Obj} K) \star ((\text{Obj} \mathcal{D}_n) \cap C_{w \leq -N}) \star C_{w \geq -N+1} = (\text{Obj} K) \star (\text{Obj} \mathcal{D}_n) \cap C_{w \leq -N+1} \) for any integer \( n \).

Hence, for any integer \( n \) there exists a triangle \( T_0 \to X \to X' \to T_0[1] \) where \( T_0 \in \text{Obj} K \) and \( X' \) is an extension of an element \( X'' \) of \( C_{w \geq -N+1} \) by an element \( Y_0 \) of \( (\text{Obj} \mathcal{D}_n) \cap C_{w \leq -N} \). We will use this statement later in the proof.

Now we prove the first assertion.

1. Denote by \( n \) the maximal integer such that \( a_n = 0 \). We will prove the statement using induction on \( N - n \) for all triangulated categories \( \mathcal{C} \) with a weight structure and a filtration by full subcategories \( \mathcal{D}_n \), to which \( w \) restricts.

In the base case \( n \geq N \). By the above there exists a triangle \( T_0 \to X \to X' \to T_0[1] \) such that \( X' \) is an extension of an element \( X'' \) of \( C_{w \geq -N+1} \) by an element \( Y_0 \) of \( (\text{Obj} \mathcal{D}_n) \cap C_{w \leq -N} \). Note that \( L^{a_i}(X'') \in (\mathcal{C}/\mathcal{D}_n)_{w \geq -i+1} \) for all \( i \geq n \) by exactness of the localization functor and \( L^n(X'') = L^0(X'') \in (\mathcal{C}/\mathcal{D}_0)_{w \geq -i+1} \) for \( i < n \) since \( \mathcal{D}_0 = \mathcal{C} \).

Now we describe the inductive step. Let us assume the statement to be true for all triangulated categories with a weight structure and a filtration by full subcategories and \( a_k \) being a sequence such that \( a_n = 0 \) and \( a_k > 0 \).
for $k > n$ for some $n \geq l$. We want to prove the statement for a triangulated category $\mathcal{C}$ with a filtration by $\mathcal{D}_n$ and $a_k$ being a sequence such that $a_{i-1} = 0$ and $a_k > 0$ for $k > l - 1$.

Let $X \in \mathcal{C}$ be such that $L^a_i(X) \in (C/D'_{a_i})_{w \geq l+1}$ for any $i \in \mathbb{Z}$. By the argument above there exists a triangle $T_0 \to X \to X' \to T_0[1]$, where $T_0 \in \text{Obj} \ K$ and $X'$ is an extension of an element $X''$ of $\mathcal{C}_{w \geq l+1}$ by an element $Y_0$ of $(\text{Obj} \ D_n) \cap \mathcal{C}_{w \leq -l}$.

Certainly, $L^a_i(X') \in (C/D'_{a_i})_{w \geq j+1}$ for any integer $j \in \mathbb{Z}$. Moreover, $L^a_i(X'') \in (C/D'_{a_i})_{w \geq j+1}$ for all $j > l$ by the weight exactness of the corresponding localization functors and $L^a_i(X''[-1]) \in (C/D'_{a_i})_{w \geq j+1}$ for all $j \leq l$ since $L^a_i(Y_0) = 0$ for any $j \leq l$. By extension-closedness of the classes $(C/D'_{a_i})_{w \geq j+1}$ we have $L^a_i(Y_0) \in (C/D'_{a_i})_{w \geq j+1}$.

Now we apply the inductive assumption to the object $Y_0$ in $D_{a_i}$ where the filtration on $D_{a_i}$ is induced from $C$ and $a_{i,D_{a_i}} = \max \{a_i - a_k, 0\}$. Obviously, $a_{i,D_{a_i}} = 0$, so the maximal integer $i$ such that $a_{i,D_{a_i}} = 0$ is bigger or equal than $l$, and we can apply the inductive assumption. We obtain that $Y_0$ belongs to the class $(\text{Obj} \ K) \ast (Q \ast Y)$, where $L^a_i \omega(Y) \in (D_{a_i}/D_n)_{w \geq j+1}$ for $j \in \mathbb{Z}$, $Y \in \mathcal{C}_{w \geq -N + 1} \cap (\text{Obj} \ D_n)$, and $Q \in (\text{Obj} \ D_n) \cap \mathcal{C}_{w \leq -N}$. Thus, $X \in (\text{Obj} \ K) \ast (((\text{Obj} \ K) \ast (Q \ast Y)) \ast X'')$. By Corollary 3.1.6(2), $X$ also belongs to $(\text{Obj} \ K) \ast (Q \ast (Y \ast X''))$. This means that there exists a triangle $T \to X \to M \to T[1]$, where $M$ is an extension of $M' \in Y \ast X''$ by $Q \in (\text{Obj} \ D_n) \cap \mathcal{C}_{w \leq -N}$. Note that $L^a_i(Y)$ belongs to $(C/D_{a_i})_{w \geq l+1}$ for $i > l$ by the definition of $a_{i,D_n}$ and $L^a_i(Y)$ belongs to $(C/D_{a_i})_{w \geq j+1}$ for $j \leq l$ since $Y \in \text{Obj} \ D_n$. Since $L^a_i(Y)$ and $L^a_i(X'')$ belong to $(C/D'_{a_i})_{w \geq l+1}$ for $i \in \mathbb{Z}$ and $X'', Y$ belong to $\mathcal{C}_{w \leq -N + 1}$, we obtain the desired properties for the triangle.

2. By the argument in the beginning of the proof there exists a triangle $T_0 \to X \to X' \to T_0[1]$, where $T_0 \in \text{Obj} \ K$ and $X'$ is an extension of an element $X''$ of $\mathcal{C}_{w \geq -N}$ by an element $Y_0$ of $(\text{Obj} \ D_n)_{w \geq -N+1} \cap \mathcal{C}_{w \leq -N-1}$. As in the previous assertion $L^a_i(Y_0) \in (C/D_{a_i})_{w \geq -l+1}$ for any $i \in \mathbb{Z}$. Applying the previous assertion to the object $Y_0$ in $D_{n+1}$ (where the filtration is induced from $C$ and $a_{i,D_{a_i}} = a_i - a_{n+1}$) we obtain that $Y_0 \in (\text{Obj} \ K) \ast (Q \ast M')$, where $L^a_i(M') \in (C/D_{a_i})_{w \geq -l+1}$ for any $i \in \mathbb{Z}$ and $Q \in (\text{Obj} \ D_n) \cap \mathcal{C}_{w \leq -N}$.

We obtain that $X$ belongs to $(\text{Obj} \ K) \ast (((\text{Obj} \ K) \ast (Q \ast M')) \ast X'')$. By Corollary 3.1.6(2), there exists a triangle $T \to X \to M \to T[1]$ such that $T$ is torsion and $M$ belongs to $(((\text{Obj} \ D_n) \cap \mathcal{C}_{w \leq -N}) \ast M') \ast \mathcal{C}_{w \geq -N}$, where $L^a_i(M') \in (C/D'_{a_i})_{w \geq -l+1}$ for any $i \in \mathbb{Z}$.

**Remark 3.3.2.** It is not enough to know that every element of $(C/D'_{a_i})_{w \geq 0}$ is a direct summand of an element of $L^a_i(C_{w \geq 0})$ to prove Proposition 3.3.1.
Indeed, let $X$ be an element of $\mathcal{C}_{w \geq n}$ such that $L'^{a_{n+1}}(X)$ belongs to $(\mathcal{C}/\mathcal{D}'_{w_{n+1}})_{w \geq n+1}$. To make the inductive step we should be able to represent $X$ as an element of the extension-closure of the class $(\text{Obj}\mathcal{D}_{a_{n+1}}) \cup \{X\}$ where $X'$ is some object of smaller length than $X$ (in our proof we have $X' \in \mathcal{C}_{w \geq n}$). However, we can do this for arbitrary $X$ if and only if the localization functor $L'^{a_{n+1}}$ satisfies the weight lifting property.

**Remark 3.3.3.** 1. If $\mathcal{C}$ is $R$-linear, then the conclusion of Proposition 3.3.1 certainly remains valid if we take $J \subset R$.

2. If the sequence $a_i$ stabilizes, i.e. there exists $n$ such that $a_i = a_{i+1}$ for $i \geq N$ (equivalently, $a_i$ is bounded from above) then one obtains the following simplification of Proposition 3.3.1.

For any $M$ such that $L^a(M) \in (\mathcal{C}/\mathcal{D}_a)_{w \geq -i+1}$ there exists a triangle $T \to X \to M \to T[1]$ such that $T \in \text{Obj}\mathcal{K}$ and $L^a(M) \in (\mathcal{C}/\mathcal{D}_a)_{w \geq -i+1}$.

Indeed, by Proposition 3.3.1 we have a triangle $T \to X \to M \to T[1]$ such that $T \in \text{Obj}\mathcal{K}$ and $M$ is an extension of $M'$ such that $L^a(M') \in (\mathcal{C}/\mathcal{D}_a)_{w \geq -i+1}$ by an element $Q$ of $(\text{Obj}\mathcal{D}_a) \cap \mathcal{C}_{w \geq -n+1}$. Since $a_i \leq n$, $L^a(Q) = 0$ and thus $L^a(M) \in (\mathcal{C}/\mathcal{D}_a)_{w \geq -i+1}$.

3. Adopt the notation of Remark 3.2.8(2) (so, $\mathcal{C} = DMT_{gm}^c(f) = DMT_{gm}^c(f;\mathbb{Z})$ and $\mathcal{D} = DMT_{gm}^c(f)(1)$); let $J$ be the set of non-zero integers.

Let $X$ be an object of $DMT_{gm}^c(f)$ such that $H^i_M(X, \mathbb{Q}) = \{0\}$ for $i \geq -n$. Then Remark 3.2.8(2) yields that $L(X \otimes \mathbb{Q})$ belongs to $K^b(\mathbb{Q} - \text{vect})_{w \geq n+1} \cong DMT_{gm,w \geq n+1}^c(f) \otimes \mathbb{Q}$ (here $\mathbb{Q} - \text{vect}$ is the category of finite-dimensional $\mathbb{Q}$-vector spaces). Certainly, Proposition 3.3.1(1) (and also the previous part of the current remark) implies that there exists a triangle $T \to X \to X' \to T[1]$ in $DMT_{gm}^c(f)$ such that $T$ is a torsion motive and the groups $H^i_M(X', \mathbb{Z}/m)$ vanish for $i \leq n, m \in \mathbb{Z}$.

Roughly speaking, this means that one can cut off the "torsion part" of weight 0 motivic cohomology from a Tate motive.

As in Remark 3.2.8(2) we can consider $H^i_M(-, \mathbb{Q}/\mathbb{Z})$ instead of $H^i_M(-, \mathbb{Z}/m)$ for all $m$.

Furthermore, we have the following abstract corollary of our results.

**Proposition 3.3.4.** 1. Suppose $\mathcal{D}$ is generated by a class of objects of $\mathcal{H}_w$. Then any object of $\mathcal{H}_{w_{C}/\mathcal{D}}$ is isomorphic to the image of an object of $\mathcal{H}_w$.

2. Suppose that $\mathcal{D}$ is generated by $\text{Cone}(S)$, where $S$ is a class of morphisms between objects of $\mathcal{H}_w$ satisfying both the left and the right Ore conditions. Then any object of $\mathcal{H}_{w_{C}/\mathcal{D}}$ is isomorphic to the image of an object of $\mathcal{C}_{[-1,0]}$. 

21
Proof. 1. Indeed, by the dual to Theorem 3.2.5 any \( Y \in \text{Hw}_{C/D} \) can be lifted to an object \( X \) of \( C_{w \leq 0} \). By Remark 3.2.6 there exists a weight decomposition \( w_{\leq -1} X \to X \overset{j}{\to} w_{\geq 0} X \) such that \( w_{\leq -1} X \) belongs to \( D \). Hence \( l(w_{\geq 0} X) \cong Y \). Since \( w_{\geq 0} X \in C_{w=0} \), we obtain the result.

2. Applying the dual to Theorem 3.2.5 we obtain that any \( Y \in \text{Hw}_{C/D} \) can be lifted to an object \( X \) of \( C_{w \leq -1} \). Again, by Theorem 3.2.5 we have a distinguished triangle \( X_{\leq -1} \to X \to X_{\geq -1} \to X_{\leq -1}[1] \) such that \( X_{\leq -1} \) belongs to \( C_{w \leq -1} \cap \text{Obj } D \) and \( X_{\geq -1} \) belongs to \( C_{w \geq -1} \). Since \( X_{\geq -1} \) is an extension of objects of \( C_{w \leq 0} \), we have \( X_{\geq -1} \in C_{[-1,0]} \). Certainly, \( l(X_{\geq -1}) \cong l(X) \cong Y \). So, we obtain the result.

Remark 3.3.5. Certainly, Proposition 3.3.4(1) implies Theorem 1.7(b) of [Wil11]. Moreover, this result easily extends to motives with arbitrary coefficients. Note that we do not use any extra properties of our localization setting (such as the existence of an adjoint functor to the localization one) in this proof.

Definition 3.3.6. Let \( A \) be an additive category. Following Definition 4.3.1 of [Bon10], we call the full subcategory of \( \text{Kar}(A) \) consisting of the objects \( Y \) of \( \text{Kar}(A) \) such that \( X \oplus Y \) belongs to \( \text{Obj } A \) for some \( X \in \text{Obj } A \) the small envelope of \( A \).

Keeping in mind this definition we can reformulate part 2 of Proposition 3.3.4 in the following way.

Corollary 3.3.7. Let \( A \) be an additive category and \( B \) be its additive subcategory of \( A \). Then the small envelope of \( \frac{A}{B} \) coincides with \( \frac{A'}{B} \), where \( A' \) is the small envelope of \( A \).

Proof. By Theorem 4.3.2(II) of [Bon10], the heart of the stupid weight structure on \( K^b(A) \) is \( A' \). Moreover, loc. cit. along with Proposition 2.3.2 implies that the heart \( H \) of the localization \( K^b(A)/K^b(B) \) is equivalent to the small envelope of \( \frac{A}{B} \) and also to the small envelope of \( \frac{A'}{B} \).

By Corollary 3.3.4(1), any object of the heart \( H \) lifts to an object of \( A' \). So, \( \frac{A}{B} \) coincides with its small envelope.

4 Localization of triangulated categories with unbounded weight structures

Let \( \underline{C} \) be a triangulated category \( C \) endowed with a weight structure \( w \). In this section we define a notion of subcategory with weak weight decompositions.
In §4.1 we prove that for a subcategory with weak weight decompositions $D$ there is a weight structure $w'$ on $C/D$ such that the localization functor preserves weights.

In §4.2 we study subcategories with weak weight decompositions in more detail.

In §4.3 we answer Question 1.0.2 for these types of localizations. We describe certain necessary and sufficient conditions and also an example that demonstrates their relevance using the constructions from §4.2.

4.1 Localizing by subcategories with weak weight decompositions

In subsection 2.3 we have considered two types of subcategories of a triangulated category $C$ endowed with a weight structure $w$. The subcategories of the first type are the subcategories $D \subset C$ such that the weight structure $w$ restricts to $D$ (see Definition 2.2.1(VII)). The subcategories of the second type are the subcategories $D \subset C$ generated by some subset $B$ of $C_{[0,1]}$.

We have proved that the localization $C/D$ admits a weight structure such that the localization functor preserves weights for $D$ of both types (see Proposition 2.3.2 and Proposition 2.3.3, respectively). If $w$ is a bounded weight structure then the first setting described is a particular case of the second setting (see Corollary 3.1.3(2)). However, it isn’t if $w$ is unbounded. In Theorem 4.1.3 we introduce a natural generalization of our theorems on weight structures in localizations which yields some interesting and useful examples in the case of unbounded weight structures. Although the theorem can be proved directly via the methods used in the proof of Proposition 2.3.3 (in [BoS13]), we will show instead that it easily follows from Proposition 2.3.2 and Proposition 2.3.3.

Recall some definitions.

**Definition 4.1.1.** Elements of $\bigcap_{n \in \mathbb{Z}} C_{w \leq n}$ (resp. of $\bigcap_{n \in \mathbb{Z}} C_{w \geq n}$) are called right degenerate (resp. left degenerate).

We will say that $w$ is right non-degenerate (resp. left non-degenerate) if $\bigcap_{n \in \mathbb{Z}} C_{w \leq n} = \{0\}$ (resp. $\bigcap_{n \in \mathbb{Z}} C_{w \geq n} = \{0\}$).

**Proposition 4.1.2.** 1. The full subcategory $C_{\text{deg},-}$ (resp. $C_{\text{deg},+}$) of $C$ consisting of right degenerate (resp. above) objects is triangulated.

Besides, $w$ restricts to it as well as to all of its full triangulated subcategories (in the sense of Definition 2.2.1(VII)).

2. Let $G_-$ be some set of right degenerate objects and $G_+$ be a set of left degenerate objects.
Denote by $D$ (resp., by $D_-$, by $D_+$) the triangulated category generated by $G_-$ $\cup$ $G_+$ (resp., by $G_-$, by $G_+$). Then $w$ restricts to $D$. Moreover, any object of $D$ is an extension of an object of $D_+$ by an object of $D_-$.

3. In the notation of assertion 2 consider the localized category $C/D$. There exists a weight structure on $C/D$ such that the localization functor $C \xrightarrow{L} C/D$ is weight-exact. The heart of this weight structure is equivalent to $\text{Kar}_{C/D}(Hw_C)$.

4. Let $X, Y \in C^b$. Then $C(X, Y) \to C/D(L(X), L(Y))$ is an isomorphism.

Proof. 1. Certainly, this category is closed under translations and under extensions. Note that the pair of subclasses $D_{w\leq 0} = D$, $D_{w\geq 0} = \{0\}$ yields a weight structure on $D$ for any triangulated category $D$. Since $C_{w\leq 0} \cap \text{Obj}^{D}_{\text{deg}, -} = \text{Obj}^{D}_{\text{deg}, -}$ and $C_{w\geq 0} \cap \text{Obj}^{D}_{\text{deg}, -} = 0$, we obtain the result.

2. Since $D_- \perp D_+$ (by the orthogonality axiom of weight structures), any extension of an object of $D_-$ by an object of $D_+$ splits. Hence any such extension is also an extension of an object of $D_+$ by some object of $D_-$. Hence any object $X$ of $D$ is an extension of an object of $D_+$ by an object of $D_-$, i.e., there is a triangle $X_\leq 0 \to X \to X_\geq 0 \to X_\leq 0[1]$, where $X_\leq 0 \in (\text{Obj}^{D}_C) \cap C_{w\leq 0}$ and $X_\geq 0 \in (\text{Obj}^{D}_C) \cap C_{w\geq 0}$.

Now we are able to prove the main result of this subsection. It is a natural generalization of Theorem 4.2.2 of [BoS13].

Let $D$ be a full triangulated subcategory of $C$ such that any object of $D$ possesses a weak weight decomposition inside $D$, i.e., for any $X \in \text{Obj}^{D}_D$ there exists a triangle $X_{\leq 0} \to X \to X_{\geq 0} \to X_{\leq 0}[1]$, where $X_{\leq 0} \in (\text{Obj}^{D}_C) \cap C_{w\leq 0}$ and $X_{\geq 0} \in (\text{Obj}^{D}_C) \cap C_{w\geq 0}$.
By Corollary 3.1.3(2), $B = (\text{Obj } D) \cap C_{[0,1]}$ is the extension-closure of the set of cones of some set $S$ of $Hw$-morphisms.

**Theorem 4.1.3.** In the notation as above the following statements are valid.

1. There exists a weight structure $w_{C/D}$ on $C/D$ such that the localization functor is weight-exact.

2. The natural functor from $Hw[S^{-1}]_{\text{add}}$ to $Hw_{C/D}$ is a full embedding, and every object of $Hw_{C/D}$ is a direct summand of an object in the image of the functor.

**Proof.** Proposition 2.3.3(3) yields that there exists a weight structure on $C/D$ such that the localization functor is weight-exact.

Denote by $D'$ the full triangulated subcategory of $C/D$ whose objects are the images of the objects of $D$. Certainly, it is generated by $G_- \cup G_+$, where $G_-$ is the image of $(\text{Obj } D) \cap C_{w<0}$ and $G_+$ is the image $(\text{Obj } D) \cap C_{w>0}$ in $C/D$. The elements of $G_-$ are right degenerate and the elements of $G_+$ are left degenerate. So, by 4.1.2 the weight structure $w_{C/D}$ restricts to $D'$ (in the sense of Definition 2.2.1(VII)).

By Proposition 4.1.2(3) there exists a weight structure on $(C/D)^b$ such that the localization functor $C/D^b \to \mathcal{C}/D^b$ is weight-exact. Moreover, the heart of $(C/D)^b/D'$ is equivalent to the Karoubi-closure in $(C/D)^b/D'$ of the heart of $C/D$. The latter is equivalent to $\text{Kar}(C/D)^b/D'(Hw[S^{-1}])$ by Proposition 2.3.3(3). Denote by $e$ the natural equivalence $(C/D^b)/D' \cong C/D$ such that $e \circ L_2 \circ L_1 = L$. Consider the corresponding weight structure on $C/D$ such that $e$ preserves it. Since the functors $L_1, L_2,$ and $e$ are weight-exact, we obtain that $L$ is weight-exact also.

So, we obtain the weight structure desired and its heart is equivalent to $\text{Kar}(C/D^b)/D'(Hw[S^{-1}]_{\text{add}})$.

\[\square\]

**Remark 4.1.4.** Certainly, Theorem 4.1.3 generalizes Proposition 2.3.3. Besides, in the case $C$ is unbounded and $D$ is bounded the the corresponding weight structure on $C/D$ can be degenerate. For example, it is so if $\prod T[i]$ exists in $C$ for some $T \in B$ (certainly, such a product exists for $C = K^{-}(A)$ or $K(A)$ for any $T \in B$). Indeed, $\prod_{i>0} T[i] \cong \prod_{i \geq n} T[i] \times \prod_{0<i<n} T[i]$. The first summand $\prod_{i>n} T[i]$ belongs to $C_{w \geq n}$ and the second summand is zero in the localization. Since $\prod_{i>0} T[i] \notin \text{Obj } D$, we obtain a degenerate object in $C/D$.
4.2 Constructing subcategories with weak weight decompositions

In this subsection we will give some non-trivial examples of subcategories with weak weight decompositions.

**Lemma 4.2.1.** Let \( \{X_i\} \subset C_{w \geq 0} \) be a set indexed by a set \( I \). Suppose that the product \( \prod_{i \in I} X_i \) exists in \( C \). Then \( \prod_{i \in I} X_i \) belongs to \( C_{w \geq 0} \).

*Proof.* See Proposition 1.3.3(5) of [Bon10].

**Remark 4.2.2.** In general the product \( \prod_{i \in I} X_i \) of objects \( X_i \in C_{w \leq 0} \) doesn’t belong to \( C_{w \leq 0} \).

However, the latter is true, for example, for cocompactly cogenerated weight structures on triangulated categories (i.e., for those constructed via the method described in Theorem 2.2.6 of [Bon13]). The property is also satisfied for the homotopy category of complexes.

We will say that small products preserve \( C_{w \leq 0} \) if \( \prod_{i \in I} X_i \in C_{w \leq 0} \) for any set \( \{X_i\} \subset C_{w \leq 0} \) for which \( \prod_{i \in I} X_i \) exists in \( C \).

Now suppose \( C \) is bounded below and \( \prod_{i \geq 0} X_i[i] \) exists in \( C \) for any \( X_i \in C_{w \geq 0} \). In particular, it is satisfied for the bounded below homotopy category of complexes \( K^-(\mathbb{A}) \). Let \( D \) be the minimal full triangulated subcategory of \( C \) containing \( \text{Cone}(S) \) and the product \( \prod_{i \geq 0} X_i[i] \) of any sequence of objects \( X_i \in (\text{Obj } D) \cap C_{w \geq 0} \). One can construct \( \text{Obj } D \) as the union \( \bigcup_{i \in \mathbb{N}} D_i \), where

\[
E_0 = \text{Obj}(\text{Cone}(S)) \quad \text{and} \quad D_i = \text{Obj}(\theta(D_{i-1})).
\]

Here \( \theta(D_{i-1}) = \{ \prod_{i \geq 0} X_i[i] : X_i \in (\text{Obj } D_{i-1}) \cap C_{w \geq 0} \} \).

In particular, \( D \) is closed under the operation \( X \rightarrow \prod_{i \geq 0} X[2i] \).

**Lemma 4.2.3.** In the notation above there exist weak weight decompositions for objects of \( D \).

Being more precise, assume that \( T \) is an object of \( D \). Then there exists a distinguished triangle \( T_1 \rightarrow T \rightarrow T_2 \rightarrow T_1[1] \), where \( T_1 \in (\text{Obj } D) \cap C_{w \leq 0} \) and \( T_2 \in (\text{Obj } D) \cap C_{w \geq 0} \).

*Proof.* By construction, \( \text{Obj } D = \bigcup_{i \in \mathbb{N}} D_i \), where \( D_0 = \text{Obj}(\text{Cone}(S)) \) and \( D_i = \text{Obj}(\theta(D_{i-1})) \).

We will prove that there is a weak weight decomposition for any element of \( D_i \) using induction on \( n \).
The base of induction follows from Proposition 2.3.3(1).

Now we will describe the inductive step. We assume that the statement is valid for the elements of $D_{n-1}$. By Corollary 3.1.3(1) it suffices to check that elements from $\bigcup_{i \in \mathbb{Z}} \theta(D_{n-1})[i]$ have weak weight decompositions in $D_n$. Since any element of $\bigcup_{i \geq 0} \theta(D_{n-1})[i]$ possesses a weak weight decomposition, it suffices to show that any element of $\theta(D_{n-1})[-j]$ for $j \in \mathbb{N}$ satisfies this property also.

Let $X_i \in D_{n-1} \cap C_{w \geq -j}$. We need to show that $\prod_{i \geq j} X_i[i]$ has a weak weight decomposition. Since $X[j]$ belongs to $C_{w \geq 0}$, we obtain that $\prod_{i \geq j} X_i[i]$ belongs to $D_n \cap C_{w \geq 0}$ (see Lemma 4.2.1). Thus, there is a trivial weak weight decomposition for $\prod_{i \geq j} X_i[i]$: $0 \to \prod_{i \geq j} X_i[i] \to \prod_{i \geq j} X_i \to 0$.

For $i < j$ we have weak weight decompositions $(X_i[i])_{\leq 0} \to X_i[i] \to (X_i[i])_{\geq 0} \to (X_i[i])_{\leq 0}[1]$ by the inductive assumption. Taking the product of all the triangles above we obtain a triangle

$$
\prod_{j > i \geq 0} (X_i[i])_{\leq 0} \to \prod_{i \geq 0} X_i[i] \to (\prod_{j > i \geq 0} (X_i[i])_{\geq 0}) \times (\prod_{i \geq j} X_i[i]) \to \prod_{j > i \geq 0} (X_i[i])_{\leq 0}[1]
$$

which is certainly a weak weight decomposition for $\prod_{i \geq 0} X_i[i]$.

\[\square\]

The following proposition is a generalization of Theorem 3.1 of [Schn11]. Although all the arguments are precisely the same as in ibid.; we give a proof here just for the convenience of the reader.

**Proposition 4.2.4.** Now suppose $\mathcal{C}$ is any triangulated category. Let $F$ be an exact functor from $\mathcal{C}$ into itself such that $F \cong F[2] \oplus \text{id}_{\mathcal{C}}$. Then $\mathcal{C}$ is Karoubian.

In particular, if $\mathcal{C}$ admits a weight structure $w$ then $Hw$ is Karoubian.

**Proof.** Certainly $F$ extends to an endofunctor of $\text{Kar}(\mathcal{C})$. We denote it by the same symbol and the equality $F \cong F[2] \oplus \text{id}_{\text{Kar}(\mathcal{C})}$ holds. Let $M$ be an object of $\mathcal{C}$ such that $M \cong X \oplus Y$ in $\text{Kar}(\mathcal{C})$ for some $X, Y \in \text{Obj Kar}(\mathcal{C})$. Taking the direct sum of the triangles

$$
0 \to F(X) \xrightarrow{\text{id}_F(X)} F \to 0
$$

27
we obtain the triangle

\[ F(M) \to F(M) \to F(X)[1] \oplus F(X) \to F(M)[1]. \]

It yields that \( F(X)[1] \oplus F(X) \) belongs to \( \text{Obj} \mathcal{C} \). Similarly, \( F(Y)[2] \oplus F(Y)[1] \) belongs to \( \text{Obj} \mathcal{C} \). Also \( X \oplus F(X)[2] \oplus F(Y) \cong F(X) \oplus F(Y) \cong F(M) \) belongs to \( \text{Obj} \mathcal{C} \).

Now, taking the direct sum of the objects above we obtain that the object \( H = X \oplus (F(X)[2] \oplus F(Y)[2]) \oplus (F(Y)[1] \oplus F(X)[1]) \oplus (F(X) \oplus F(Y)) \cong X \oplus (F(M)[2] \oplus F(M)[1] \oplus F(M)) \) belongs to \( \mathcal{C} \). Since \( F(M)[2] \oplus F(M)[1] \oplus F(M) \) belongs to \( \text{Obj} \mathcal{C} \), we obtain that \( X \) lies in the small envelope of \( \mathcal{C} \). Now, Theorem 4.3.2(1) of [Bon10] implies that a triangulated category coincides with its small envelope. Thus \( X \) belongs to \( \mathcal{C} \).

**Corollary 4.2.5.** In the assumptions of the Proposition 4.2.4 let \( D \) be a subcategory such that \( F \) maps \( D \) into itself. Then \( \mathcal{C}/D \) is Karoubian.

**Proof.** By the universal property of localization of triangulated categories, \( F \) extends to a functor on \( \mathcal{C}/D \). Moreover, the equality \( F \cong F[2] \oplus \text{id}_{\mathcal{C}/D} \) holds. Hence we can apply Proposition 4.2.4 to the category \( \mathcal{C}/D \).

In the following proposition we describe certain constructions of subcategories with weak weight decompositions and prove some of their properties.

**Proposition 4.2.6.** Let \( S \) be some set of \( Hw \)-morphisms.

I Assume \( \mathcal{C} \) is bounded below and that \( \prod_{i \geq 0} X_i[i] \) exists for any sequence of objects \( X_i \in \mathcal{C}_{w \geq 0} \).

Denote by \( D \) the minimal full triangulated subcategory of \( \mathcal{C} \) containing \( \text{Cone}(S) \) as well as the objects \( \prod_{i \geq 0} X_i[i] \) of all sequences of objects \( X_i \in (\text{Obj} \mathcal{D}) \cap \mathcal{C}_{w \geq 0} \).

Then the following statements are valid.

1. \( \mathcal{C} \) is Karoubian, as well as \( \mathcal{C}/D \).
2. \( w \) induces a weight structure on \( (\mathcal{C}/D) \). The equality \( \langle \text{Cone}(S) \rangle = D^b \) holds. The heart of \( w \) is equivalent to the Karoubi-closure of the additive localization \( Hw[S^{-1}]_{\text{add}} \).

II Assume \( \mathcal{C} \) contains all small products and \( \mathcal{C}_{w \leq 0} \) is closed under products. Denote by \( D \) the minimal triangulated subcategory of \( \mathcal{C} \) that contains \( \text{Cone}(S) \) and is closed under all products.

28
Then the following statements are valid.
1. $\mathcal{C}$ is Karoubian, as well as $\mathcal{C}/D$.
2. $w$ induces a weight structure on $(\mathcal{C}/D)$.

Proof. 

1. Follows from Corollary 4.2.5.

2. Lemma 4.2.3 along with Theorem 4.1.3 yields that there is a weight structure on $\mathcal{C}/D$ and its heart is equivalent to $\text{Kar}(Hw[S^{-1}]_{\text{add}})$, where $S' \subseteq \text{Obj}(D) \cap \mathcal{C}_{[0,1]}$ is the extension-closure of $\text{Cone}(S')$. Moreover, the heart of $\mathcal{C}/D$ is Karoubian by the first assertion of this proposition; hence it is equivalent to $\text{Kar}(Hw[S^{-1}]_{\text{add}})$.

Now we want to prove that the canonical map $\text{Kar}(Hw[S^{-1}]_{\text{add}}) \to \text{Kar}(Hw[S'^{-1}]_{\text{add}})$ is an isomorphism. It suffices to prove that any element of $S'$ becomes invertible in $Hw[S'^{-1}]_{\text{add}}$. For this purpose we show that a cone of any element of $S'$ belongs to $\text{Obj}(\text{Kar}(\langle \text{Cone}(S) \rangle))$.

Let $Y$ be an object of $D^b$. By Lemma 4.2.3 any object $Y$ of $D$ has a weak weight decomposition

$$Y_{\leq n+1} \to Y \to Y_{\geq n+1} \to Y_{\leq n+1}[1]$$

such that $Y_{\leq n+1}$ belongs to $\langle \text{Cone}(S) \rangle$.

Now taking such a weak weight decomposition for $X$ we obtain a triangle

$$X' \to X \xrightarrow{p} X_{\geq n+1} \to X'[1],$$

where $X'$ belongs to $\langle \text{Cone}(S) \rangle$ and $X_{\geq n+1} \subseteq \mathcal{C}_{w_{\geq n+1}}$. The morphism $p$ is zero by the orthogonality axiom for $w$; hence $X$ is a retract of $X' \in \text{Obj}(\langle \text{Cone}(S) \rangle)$.

Since cones of $S'$ belong to $D$, we obtain the result.

II.1. See Proposition 1.6.8 and Corollary 3.2.11 of [Nee01].

2. Certainly, the product of any set of objects of $D$ having weak weight decompositions in $D$ possesses a weak weight decomposition in $D$ also. Indeed, let $X_i$ be a family of object of $D$ and let $(X_i)_{\leq 0} \to X_i \to (X_i)_{\geq 0}$ be some weak weight decompositions. Then $\prod(X_i)_{\leq 0} \to \prod X_i \to \prod(X_i)_{\geq 0}$ is a weak weight decomposition for $\prod X_i$. Extensions of objects having weak weight decompositions has a weak weight decomposition by Proposition 3.1.2. Since $D$ is generated by objects having weak weight decompositions, their extensions and products, we obtain that all the objects of $D$ have weak weight decompositions in $D$.

So, the result follows from Theorem 4.1.3. 

\qed
4.3 On the weight lifting property in the unbounded case

Recall that the goal of this section is to determine whether the localization functor $C \xrightarrow{L} C'$ satisfies the left (resp. the right) weight lifting property. In subsection 3.2 we have studied the case of a localization by a subcategory generated by some set $B \subset C_{[0,1]}$. Yet one may ask whether $L$ satisfies the property in the case of general localizations by a subcategory with weak weight decompositions (introduced in Theorem 4.1.3).

The following proposition answers this question in a very general situation.

**Proposition 4.3.1.** Assume $(\text{Obj } D) \cap C_{[0,1]}$ is the extension-closure of the set $B$ of cones of some set of $Hw$-morphisms satisfying the right Ore condition. Suppose that $C$ is bounded below. The following statements are equivalent:

1. The localization functor $C \xrightarrow{L} C/D$ satisfies the right weight lifting property.
2. The localization functor $C \xrightarrow{L} C/D$ satisfies the strong right weight lifting property.
3. The map $K_0(Hw) \rightarrow K_0(Hw_{C/D})$ is surjective.

**Proof.** First we prove the implication 2 $\Rightarrow$ 3. Assume that the localization functor $C \xrightarrow{L} C/D$ satisfies the strong right weight lifting property. It suffices to show that the functor $C^b \xrightarrow{L'} (C/D)^b$ is surjective on objects. Let $X \in (C/D)^b$. Without loss of generality we can assume that $X \in C_{w \leq 0}$. By our assumptions there exists an element $Y \in C_{w \geq n}$ for some $n \in \mathbb{Z}$ such that $L(Y) = X$. Since the strong right weight lifting property is fulfilled, there exists a triangle

$$T \rightarrow X' \rightarrow X \rightarrow T[1]$$

where $T \in (\text{Obj } D) \cap C_{w \geq 0}$ and $X' \in C_{w \leq 0}$. By the extension-closedness of $C_{w \geq \text{min}(n,0)}$, we obtain that $X'$ belongs to $C_{w \leq 0} \cap C_{w \geq \text{min}(n,0)} \subset C^b$. So, we obtain the result.

Now suppose 3 holds. By Lemma 2.2 of [Tho97] and Theorem 5.3.1 of [Bon10] the functor $G : C^b \rightarrow (C^b/D^b) \rightarrow (C/D)^b$ is surjective on objects. Hence, the functor $(C^b/D^b) \xrightarrow{U} (C/D)^b$ is surjective on objects. Since $U$ is also a full embedding (by Proposition 4.1.2(4)), it is an equivalence. By Theorem 3.2.5 $G$ satisfies the strong right weight lifting property. Since $w$ is bounded below, any element of $(C/D)_{w \leq 0}$ is also an element of $(C/D)^b_{w \leq 0}$. So, we obtain condition 2.
Certainly, 2 implies 1 and 1 implies 3, so the three conditions are equivalent.

Remark 4.3.2. So, we see that in general, the answer to Question 1.0.1 is negative. Indeed, let \( R \) be a (commutative) ring and \( S \) be a set of endomorphisms of \( R \) (considered as a right module over itself) such that \( K_0(\text{Proj}(R)) = K_0(R) \rightarrow K_0(R[S^{-1}]) = K_0(\text{Proj}(R[S^{-1}])_{\text{add}}) \) is not surjective. For example, one can take for \( R \) the ring of real-valued continuous functions on a real plane and denote by \( \sigma \) the set of functions whose zero set is contained in \{0\}. Denote by \( S \) the set of multiplications by elements of \( \sigma \) on projective modules over \( R \). The localized ring \( \sigma^{-1}R \) in this case is isomorphic to the ring of continuous functions on a plane without a point (see Lemma 1.5 of [ReqSun94]). Plane without a point is homotopy equivalent to a circle, so it has a non-trivial vector bundle (corresponding to the map from Mobius strip to the circle), and module of its sections defines a non-trivial projective module over the ring \( \sigma^{-1}R \). Since there are no non-trivial projective modules over \( R \), the map \( K_0(R) \rightarrow K_0(R[S^{-1}]) \) is not surjective. Besides, an example of such a pair \((R, S)\) can be constructed even in the case \( R \) is a Noetherian (non-regular) ring.

Now consider the category \( C = K^-(\text{Free}(R)) \) endowed with the stupid weight structure \( w \). And let \( D \) be the minimal triangulated subcategory containing \( \text{Cone}(S) \) as well as \( \prod_{i \geq 0} X_i[i] \) for any sequence of objects \( X_i \in (\text{Obj } D) \cap C_{w \geq 0} \). By Proposition 4.2.4, \( Hw \) is Karoubian; hence it is isomorphic to \( \text{Proj}(R) \). Moreover, by Proposition 4.2.6, the stupid weight structure \( w \) restricts to \( C/D \) and the heart of this weight structure is isomorphic to \( \text{Proj}(R)[S^{-1}] \). The latter is equivalent to \( \text{Proj}(\sigma^{-1}R) \) (see Section 3.2 of [BoS13]). So, by Proposition 4.3.1 we obtain that neither the right weight lifting property nor the strong right weight lifting property is satisfied.

Remark 4.3.3. It is possible to state an analogue of Proposition 4.3.1 without an assumption on boundedness of \( C \). To do this, one should consider the groups \( K_0(\bigcup_n C_{w \geq n}) \) (in the sense of \( K_0 \) of triangulated category) instead of \( K_0(Hw) \). However, these groups are much harder to deal with. In particular, the surjectivity of the map \( K_0(\bigcup_n C_{w \geq n}) \rightarrow K_0(\bigcup_n (C/D)_{w \geq n}) \) is hard to check.

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