Computations on Sofic $S$-gap Shifts

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1 Introduction

Let $F$ be the collection of all forbidden blocks on a finite set of alphabet $A$ and define $X_F$ to be the subset of sequences in $A^\mathbb{Z}$ not containing any word in $F$. If $|F| < \infty$, then $X_F$ is called shift of finite type (SFT) which are amongst the most studied systems. A factor of a SFT is called sofic and the collection of sofic shifts is the smallest collection of shift spaces containing SFTs and closed under taking factors. All the sofic systems are accompanied by a matrix called adjacency matrix where almost all the system’s
specifications can be read from that. In particular, this matrix gives the characteristic polynomial and various groups which carry substantial information about the behavior of the system.

On the other hand on the classical examples of symbolic dynamical systems, $S$-gap shifts are easy to define: fix a nonempty increasing subset $S$ of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $S$ is finite, define $X(S)$, the space of $S$-gap shift associated to $S$ to be the set of all binary sequences for which 1’s occur infinitely often in each direction and such that the number of 0’s between successive occurrences of a 1 is an integer in $S$. When $S$ is infinite, we need to allow points that begin or end with an infinite string of 0’s. In general sofic systems are rich. In the space of $S$-gap shifts, they can be easily stated and this makes the sofic $S$-gap shifts a valuable source in application such as coding of data and in theory as a source for constructing examples to identify the universal behaviors in symbolic dynamics.

In this paper, we consider the sofic $S$-gap shifts and we will introduce the notations and some backgrounds in Sect. 2, then in Sect. 3 we will introduce the function

$$f_S(x) = 1 - \sum_{s_n \in S} \frac{1}{x^{s_n + 1}}$$

(1.1)

and the relation between this function and the characteristic polynomial will be revealed. In fact, $f_S(x)$ is a map that is used to compute the entropy of $X(S)$ [8]. Also we will compute the zeta function of sofic $S$-gap shifts in terms of entropy function. The last section is devoted to compute the Bowen–Franks groups and stating the related results. There are four Bowen–Franks groups. For $S$-gap shifts, one of the groups has been given in [5] and we give the remaining groups.

2 Background and Notations

The notations has been taken from [6] and the proofs of the claims in this section can be found there. Let $\mathcal{A}$ be an alphabet, that is a nonempty finite set. The full $\mathcal{A}$-shift denoted by $\mathcal{A}^\mathbb{Z}$, is the collection of all bi-infinite sequences of symbols from $\mathcal{A}$. A block (or word) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. The shift function $\sigma$ on the full shift $\mathcal{A}^\mathbb{Z}$ maps a point $x$ to the point $y = \sigma(x)$ whose $i$th coordinate is $y_i = x_{i+1}$.

Let $B_n(X)$ denote the set of all admissible $n$ blocks. The Language of $X$ is the collection $B(X) = \bigcup_{n=0}^{\infty} B_n(X)$. A word $v \in B(X)$ is synchronizing if whenever $uv$ and $vw$ are in $B(X)$, we have $uvw \in B(X)$.

An edge shift, denoted by $X_G$, is a shift space which consist of all bi-infinite walks in a directed graph $G$.

A labeled graph $\mathcal{G}$ is a pair $(G, \mathcal{L})$ where $G$ is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$. A sofic shift $X_G$ is the set of sequences obtained by reading the labels of walks on $G$,

$$X_G = \{\mathcal{L}_\infty(\xi) : \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$
We say $G$ is a presentation of $X_G$. Every SFT is sofic, but the converse is not true.

A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is right-resolving if for each vertex $I$ of $G$ the edges starting at $I$ carry different labels. A minimal right-resolving presentation of a sofic shift $X$ is a right-resolving presentation of $X$ having the fewest vertices among all right-resolving presentations of $X$. Any two minimal right-resolving presentations of an irreducible sofic shift must be isomorphic as labeled graphs [6, Theorem 3.3.18]. So we can speak of “the” minimal right-resolving presentation of an irreducible sofic shift $X$, which it is called the Fischer cover of $X$.

Let $X$ be a shift space and $w \in \mathcal{B}(X)$. The follower set $F(w)$ of $w$ is defined by $F(w) = \{ v \in \mathcal{B}(X) : vw \in \mathcal{B}(X) \}$. A shift space $X$ is sofic if and only if it has a finite number of follower sets [6, Theorem 3.2.10]. In this case, we have a labeled graph $\mathcal{G} = (G, \mathcal{L})$ called the follower set graph of $X$. The vertices of $G$ are the follower sets and if $wa \in \mathcal{B}(X)$, then draw an edge labeled $a$ from $F(w)$ to $F(wa)$. If $wa \notin \mathcal{B}(X)$ then do nothing.

The entropy of a shift space $X$ is defined by $h(X) = \lim_{n \to \infty} (1/n) \log |\mathcal{B}_n(X)|$.

### 3 Entropy Function, Characteristic Polynomial and Zeta Function

Let $S = \{s_n\}_n$ be an increasing subset in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $d_1 = s_1$ and $\Delta(S) = \{d_n\}_n$ where $d_n = s_n - s_{n-1}$. In [1], it has been proved that $X(S)$ is a subshift of finite type (SFT) if and only if $S$ is finite or cofinite and it is sofic if and only if $\Delta(S)$ is eventually periodic.

#### 3.1 Adjacency Matrix for Fischer Cover

Let $X(S)$ be a sofic shift. It is an easy exercise to see that the Fischer cover of $X(S)$ is the labeled subgraph of the follower set graph consisting of only the follower sets of synchronizing words. For all $S$-gap shifts, 1 is a synchronizing word. Thereby, for all $i \in \mathbb{N}$ such that $10^i \in \mathcal{B}(X(S))$, $10^i$ is also a synchronizing word [6, Lemma 3.3.15]. On the other hand, if $v \in \mathcal{B}(X(S))$ is synchronizing and $uv \in \mathcal{B}(X(S))$, then $F(\mu v) = F(v)$. Now let $w \in \mathcal{B}(X(S))$. Then $w = 0^p 10^{s_{i_1}} 10^{s_{i_2}} \cdots 10^{s_r} 10^q$ where $s_{i_j} \in S$, $1 \leq j \leq r$ and $p, q \geq 0$ and so $F(w) = F(10^q)$. Furthermore, $0^t, t \in \mathbb{N}$, is not a synchronizing word. Therefore, if $S = \{s_1, s_2, \ldots, s_k\}$ is a finite subset of $\mathbb{N}_0$, then the follower sets of synchronizing words are $F(10^t)$, $0 \leq t \leq s_k$. But for $0 \leq t < u \leq s_k$, $0^{sk-t}1 \in F(10^t) \setminus F(10^u)$ which means that $F(10^t) \neq F(10^u)$. So the Fischer cover of $X(S)$ will be as in Fig. 1 and the adjacency matrix is a $(s_k + 1) \times (s_k + 1)$ matrix

$$A = \begin{pmatrix}
a_{11} & 1 & 0 & 0 & \cdots & 0 \\
a_{21} & 0 & 1 & 0 & \cdots & 0 \\
a_{31} & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
a_{sk1} & 0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix} \quad (3.1)$$
There is always an edge labeled 1 starting at $F(10^k)$ and terminating at $F(1)$. So $a_{i1} = 1$ when $i \in \{s_1 + 1, \ldots, s_k + 1\}$ and 0 otherwise. The value of other entries are easy to see from Fig. 1.

When $S$ is infinite, $\Delta(S)$ is eventually periodic [1] and we set

$$\Delta(S) = \{d_1, d_2, \ldots, d_k, g_1, g_2, \ldots, g_l\} \quad (3.2)$$

where $g_i = s_{k+i} - s_{k+i-1}, 1 \leq i \leq l$. Here $k$ and $l$ are the least integers such that (3.2) holds and we use $g_1, g_2, \ldots, g_l$ to indicate that $g_1, g_2, \ldots, g_l$ repeat forever. Again by starting at $F(1)$ as a vertex we will construct the Fischer cover. By the above reasoning for $|S| < \infty$, the vertices will be $F(10^i)$ for some $0 \leq i \leq n(S)$. We will define $n(S)$ explicitly in 3.1.1, 3.1.2 and 3.1.3 later. This $n(S)$ will be determined by $\Delta(S)$. Suppose $F(10^i) = F(10^j), 0 \leq i < j \leq n(S)$. Then either both $i, j \in S$ or both are not in $S$. This is because $F(10^i)$ has one outer edge labeled 0 unless $i \in S$ where then it has two edges, one labeled 0 and the other 1. If $i, j \in S$, then let $i = s_{i_0}$ and $j = s_{j_0}$ and $F(10^{s_{i_0}}) = F(10^{s_{j_0}})$. This implies $F(10^{s_{i_0}+r}) = F(10^{s_{j_0}+r}), r \in \mathbb{N}$. Then either $k$ or $l$ cannot be the least integer. In fact, if $s_{i_0} < s_{j_0} \leq s_k$ or $s_{i_0} < s_k < s_{j_0}$, then $k$ is not minimum and otherwise $l$ is not minimum. Hence

$$F(10^{s_i}) \neq F(10^{s_j}), \quad s_i, s_j \in S; \quad 1 \leq i < j \leq k+l-1. \quad (3.3)$$

Now let $i, j \notin S$. Then either $s_{i_1} < i < j < s_{i_1+1}, 1 \leq i_1 \leq k+l$ or $s_{i_1} < i < s_{i_1+1}$ and $s_{j_1} < j < s_{j_1+1}$ for some $1 \leq i_1 < j_1 \leq k+l$. In the former case, $0^{s_{i_1}+1} - j_1 \in F(10^i) \setminus F(10^j)$ and in the latter $F(10^{s_{i_1}+1}) = F(10^{s_{j_1}+1})$; so by setting $i = i_1 + 1, j = j_1 + 1$, then statement (3.3) does not hold which is absurd. So the Fischer cover (Fig. 2) has $m = n(S) + 1$ vertices accompanied by a $m \times m$ adjacency matrix.
where $a_{i1} = 1$ for $i \in \{s_1 + 1, \ldots, s_{k+l-2} + 1\}$. Also $a_{(sk+l+1)1} = 1$ except the case 3.1.1 (1) where in that situation it equals 2, any other $a_{i1}$ is 0. Also $a_{m(p+1)} = 1$ due to an edge from $F(10^n(S))$ to $F(10^p)$, $0 \leq p \leq n(S)$. The value of $n(S)$ has been sorted out below. Let $g = \sum_i g_i$.

3.1.1 If $k = 1$ and $g_l > s_1$, then

1. For $g_l = s_1 + 1$, $F(10^{s_1+1}) = F(1)$. So $p = 0$, $n(S) = s_1$, $a_{m1} = 2$ and $a_{mj} = 0$, $2 \leq j \leq m$.
2. For $g_l > s_1 + 1$, $F(10^g) = F(1)$. So $p = 0$, $n(S) = g - 1$, $a_{m1} = 1$ and $a_{mj} = 0$, $2 \leq j \leq m$.

3.1.2 For $k \neq 1$, if $g_l > d_k$, $F(10^{s_k+1}) = F(10^{s_k+1})$. So $p = s_k + 1$, $n(S) = g + s_k - 1$ and $a_{m(s_k+2)} = 1$.

3.1.3 For $k \in \mathbb{N}$, if $g_l \leq d_k$, then $F(10^{s_k+1}) = F(10^{s_k-1})$. So $p = s_k - g_l + 1$, $n(S) = s_k + l - 1$ and $a_{m1} = a_{m(s_k-g_l+2)} = 1$. This includes the case when $k = 1$ and $g_l \leq s_1$.

3.2 Characteristic Polynomial in Terms of Entropy Function

The entropy of a $S$-gap shift is $\log \lambda$ where $\lambda$ is a unique nonnegative solution of the $\sum_{x \in S} x^{-(n+1)} = 1$ [8]. Call $f_S(x)$ in (1.1) the entropy function and note that $f_S(2^{h(X(S)}) = 0$ where $h(X(S))$ is the entropy of $X(S)$. Suppose $X(S)$ is a sofic shift with the adjacency matrix $A$. Our goal is to compute the characteristic polynomial of $A$, $\chi_A$, explicitly by using elements of $S$.

The following lemma computes the determinant of a special matrix which will be used several times. The proof is elementary and straightforward; one needs only compute the determinant on the expansion of the first row.

**Lemma 3.1** Let $E$ be a $n \times n$ matrix with $e_{i1} = -1$, $i \in \{\ell_1, \ell_2, \ldots, \ell_r\} \subseteq \{1, 2, \ldots, n\}$. Also, for $i \neq j$ and $j \neq 1$,

$$e_{ij} = \begin{cases} -1, & j = i + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then if $\ell_i < n$,

$$\det E = -\prod_{\ell_i + 1}^n e_{ii} - \prod_{\ell_2 + 1}^n e_{i1} - \cdots - \prod_{\ell_r + 1}^n e_{i1}.$$
If $\ell_t = n$, then the last term above will be $-1$, or equivalently

$$
\det E = -\prod_{\ell_1 + 1} e_{ii} - \prod_{\ell_2 + 1} e_{ii} - \cdots - \prod_{\ell_t + 1} e_{ii} - 1.
$$

**Theorem 3.2** Let $X(S)$ be a sofic shift with the adjacency matrix $A$.

1. Suppose $X(S)$ is a SFT shift.
   
   (a) If $S = \{s_1, s_2, \ldots, s_k\}$ is a finite subset of $\mathbb{N}_0$, then
   
   $$
   \chi_A(x) = x^{s_k+1} f_S(x) = x^{s_k+1} - x^{s_k-s_1-1} - \cdots - x^{s_k-s_l-1} - 1.
   $$
   
   (b) If $S$ is cofinite (let $\Delta(S) = \{d_1, d_2, \ldots, d_k, 1\})$, then
   
   $$
   \chi_A(x) = x^{s_k} (x - 1) f_S(x)
   = (x - 1) (x^{s_k} - x^{s_k-s_1-1} - \cdots - x^{s_k-s_l-1}) - 1.
   $$

2. Suppose $X(S)$ is strictly sofic (then $\Delta(S)$ is eventually periodic [1]). Let

   $$
   \Delta(S) = \{d_1, d_2, \ldots, d_k, g_1, g_2, \ldots, g_l\}
   $$

   where $g_i = s_{k+i} - s_{k+i-1}$, $1 \leq i \leq l$ and set $g = \sum_{i=1}^l g_i$.
   
   (a) For $k = 1$, $g_l > s_1$,
   
   $$
   \chi_A(x) = (x^g - 1) f_S(x) = x^g - x^{g-s_1-1} - \cdots - x^{g-s_l-1} - 1.
   $$
   
   (b) For $g_l \leq d_k$,
   
   $$
   \chi_A(x) = x^{s_k+l-1-s_l+1} (x^g - 1) f_S(x)
   = (x^g - 1) (x^{s_k-g_l+1} - x^{s_k-g_l-s_1} - \cdots - x^{s_k-g_l-s_k-1})
   - (x^{s_k+l-1-s_k} + \cdots + x^{s_k+l-1-s_k+l-2} + 1).
   $$
   
   (c) For $k \neq 1$, $g_l > d_k$,
   
   $$
   \chi_A(x) = x^{s_k-1+1} (x^g - 1) f_S(x)
   = (x^g - 1) (x^{s_k-1+1} - x^{s_k-1-s_1} - \cdots - x^{s_k-1-s_k-2} - 1)
   - (x^{g+s_k-1-s_k} + \cdots + x^{g+s_k-1-s_k+l-1}).
   $$

**Proof** To compute the characteristic polynomial of $A$, we consider different cases and in all cases, we use the expansion with the first row of

$$
B = x \text{Id} - A
$$

(3.5)
to obtain the determinant. Let $B_{ij}$ be the first minor matrix associated to $(i, j)$, that is the sub-matrix obtained by deleting $i$th row and $j$th column of $B$. 
Suppose $X(S)$ is a SFT shift and $S = \{s_1, s_2, \ldots, s_k\}$. Then

$$f_S(x) = 1 - \left( \frac{1}{x^{s_1+1}} + \cdots + \frac{1}{x^{s_k+1}} \right).$$

Let $A$ be as in (3.1). We have

$$\chi_A(x) = b_{11} \det(B_{11}) + \det(B_{12}) = b_{11}(x^{s_k}) + \det(B_{12}),$$

where $b_{11}$ equals $x - 1$ if $0 \in S$ and $x$ otherwise. Now let $r = 2$ if $0 \in S$ and $r = 1$ otherwise. Applying Lemma 3.1 for $E = B_{12}$ gives the conclusion for the case $|S| < \infty$. That is,

$$\chi_A(x) = b_{11}(x^{s_k}) + (-x^{s_k-s_r} - \cdots - x^{s_k-s_{k-1}} - 1) = x^{s_k+1} f_S(x).$$

In all remaining cases $|S| = \infty$ and in our computations we use geometrical series which are only convergent for $|x| > 1$. Therefore, we first show that the identities are valid for $|x| > 1$; then an application of the identity theorem for analytic functions justifies the validity for all $x$. So from now on let $|x| > 1$.

Suppose $S$ is cofinite, that is $\Delta(S) = \{d_1, d_2, \ldots, d_k, \bar{T}\}$. We have

$$f_S(x) = 1 - \frac{1}{x^{s_1+1}} - \cdots - \frac{1}{x^{s_{k-1}+1}} - \frac{1}{x^{s_k+1}} \left( \frac{x}{x - 1} \right).$$

Apply 3.1.3 to have adjacency matrix $A$ in (3.4) for $m = s_k + 1$ and $a_{mm} = 1$. Then by setting $E = B_{12}$ and $\ell_i = s_i + 1$ in Lemma 3.1 we will have

$$\chi_A(x) = b_{11}(x^{s_k-1}(x - 1)) + (-x^{s_k-s_r} - \cdots - x^{s_k-s_{k-1}-1})(x - 1) - 1
= x^{s_k}(x - 1) f_S(x).$$

Now suppose $X(S)$ is a strictly sofic shift. Then

$$f_S(x) = 1 - \left( \frac{1}{x^{s_1+1}} + \cdots + \frac{1}{x^{s_k+1}} \right) - \left( \frac{1}{x^{s_k+1}} + \cdots + \frac{1}{x^{s_{k+1}+1}} \right) \left( \frac{x^g}{x^g - 1} \right).$$

Our goal is to find $\det B = \det(x \text{Id} - A)$. So we consider several cases, and in all cases there is a $(q + g) \times (q + g)$ square sub-matrix $G$ in the lower right corner of $A$:

$$G = \begin{pmatrix}
  x & -1 & 0 & \cdots & 0 & 0 \\
  0 & x & -1 & \cdots & 0 & 0 \\
  0 & 0 & x & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & x & -1 \\
  0 & 0 & 0 & \cdots & -1 & \cdots & x
\end{pmatrix}. $$
Since $\text{det}(G_{12}) = 0$, so $\text{det} \ G = x^q \text{det} \ G_g$ where

$$G_g = \begin{pmatrix} x & -1 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & x \end{pmatrix}$$

is the lower right sub-matrix of $G$ and $g = \sum_{i=1}^{l} g_i$. But $G_g$ is the adjacency matrix of a $S'$-gap shift with $S' = \{g - 1\}$. So

$$\text{det} \ G = x^q(x^g - 1). \quad (3.7)$$

Note that $b_{ii} = x$, $1 < i \leq m$, $b_{j(j+1)} = -1$, $1 \leq j \leq m - 1$ and $b_{(s_1+1)1} = -1$, $1 \leq i < k + l - 1$. Also $b_{(s_1+1)1}$ equals $-1$ except the case 3.1.1 (1) where in that situation it equals $-2$. In the process of computing determinant of sub-matrices if $2 \leq (s_i + 1) \leq m - g$, then the identity (3.7) will be used; otherwise, we apply Lemma 3.1. First let $k = l = 1$.

(1) $g_1 \leq s_1$. Use 3.1.3 with $m = s_1 + 1$ to obtain the adjacency matrix $A$ in (3.4) with $a_{m1} = a_{m(s_1-g_1+2)} = 1$ and $a_{11} = 0$, $1 \leq i \leq m - 1$. The matrix $B_{11}$ associated to $b_{11}$ is the matrix $G$ for $q = s_1 - g_1$. So by (3.7),

$$\chi_A(x) = b_{11}x^{s_1-g_1}(x^{g_1} - 1) + \text{det} \ B_{12}.$$ 

Let $B_{12} = [b'_{i,j}]$ with $b'_{i,1} = 0$ for $i \neq s_1$. Actually, $B_{12}$ is an $s_1 \times s_1$ matrix with $B_{12} = (-1)$ if $s_1 = 1$ an for $s_1 > 1$ of the form

$$B_{12} = \begin{pmatrix} V & L \\ -1 & W \end{pmatrix}$$

where $V$ is an $(s_1 - 1) \times 1$ matrix containing only zeros, $W$ is a $1 \times (s_1 - 1)$ matrix and $L$ an $(s_1 - 1) \times (s_1 - 1)$ lower triangular matrix where there are only $-1$'s on the diagonal. Hence $\text{det}(L) = (-1)^{s_1-1}$ and therefore $\text{det}(B_{12}) = -1$.

As a result,

$$x^{g_1}\chi_A(x) = x^{s_1+1}(x^{g_1} - 1)f_S(x).$$

(2) $g_1 > s_1$. If $g_1 = s_1 + 1$, then use 3.1.1 to have

$$\chi_A(x) = b_{11}x^{s_1} - 2 = (x^{g_1} - 1)f_S(x);$$

and if $g_1 > s_1 + 1$, use 3.1.1 and Lemma 3.1 to have

$$\chi_A(x) = \begin{cases} x^{g_1} - x^{g_1-s_1-1} - 1, & 0 \not\in S, \\ (x - 1)x^{g_1-1} - 1, & 0 \in S; \end{cases}$$
in any case, \( \chi_A(x) = (x^{s_1} - 1) f_S(x) \).

Now suppose \( \Delta(S) = \{ s_1, g_1, g_2, \ldots, g_l \} \). Then by an induction argument on \( l \) if \( g_l \leq s_1 \), we will have

\[
x^g \chi_A(x) = x^{s_1 + 1} (x^g - 1) f_S(x); \]

and if \( g_l > s_1 \),

\[
\chi_A(x) = (x^g - 1) f_S(x). \]

For the general case, suppose \( k > 1 \). Then by an induction on \( k \), the proof will be established. First let \( \Delta(S) = \{ d_1, d_2, g_1, g_2, \ldots, g_l \} \) and consider two cases.

1. \( g_l \leq s_2 - s_1 \). Then \( m = s_{l+1} + 1 \) and \( A \) will be obtained by 3.1.3. We have \( \det B = b_{11} \det B_{11} + \det B_{12} \). The sub-matrix \( B_{11} \) is the matrix \( G \) with \( q = s_{l+1} - g \). So by (3.7) its determinant is \( x^{s_{l+1} - g} (x^g - 1) \). Also \( \det B_{12} \) is equivalent to determinant of a sub-matrix

\[
C = \begin{pmatrix}
-1 & -1 & 0 & \cdots & 0 & 0 \\
0 & x & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & 0 & \cdots & -1 & \cdots & x
\end{pmatrix}
\]

of \( B \) after deleting rows 1, 2, \ldots, \( s_1 \) and columns 2, 3, \ldots, \( s_1 + 1 \). This means

\[
\det B_{12} = \begin{cases}
-x^{s_{l+1} + 1} - s_{l+1} - s_1 - g (x^g - 1) - (x^{s_{l+1} + 1} - s_{l+1} + s_1) - \cdots - (x^{s_{l+1} + 1} - s_{l+1} + 1), & 0 \not\in S, \\
-(x^{s_{l+1} + 1} - s_{l+1} + s_1) - \cdots - (x^{s_{l+1} + 1} - s_{l+1} + 1), & 0 \in S;
\end{cases}
\]

Therefore,

\[
x^g \chi_A(x) = x^{s_{l+1} + 1} (x^g - 1) f_S(x). \]

2. \( g_l > s_2 - s_1 \) and \( m = g + s_1 + 1 \). The adjacency matrix \( A \) can be derived by 3.1.2. Therefore, similar to the above

\[
\chi_A(x) = b_{11} x^{s_1} (x^g - 1) + \det \begin{pmatrix}
-1 & -1 & \cdots & 0 & 0 \\
0 & x & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & 0 & \cdots & -1 & \cdots \\
0 & -1 & \cdots & -1 & 0 & x
\end{pmatrix}. \tag{3.8}
\]
The matrix is a \((g + 1) \times (g + 1)\) matrix. Note that in this matrix the \(-1\) in the first column appears in the first row and rows \(s_2 - s_1, s_3 - s_1, \ldots \) and \(s_{l+1} - s_1\). Therefore, Lemma 3.1 can be applied to have

\[
\chi_A(x) = \begin{cases} 
  x^{s_1+1}(x^g - 1) - (x^g - 1) - (x^{s_1-s_2} + \cdots + x^{s_1-s_l+1})x^g, & 0 \not\in S, \\
  (x - 1)x^{s_1}(x^g - 1) - (x^{s_1-s_2} + \cdots + x^{s_1-s_l+1})x^g, & 0 \in S;
\end{cases}
\]

or

\[
x^{s_{l+1}-s_1} \chi_A(x) = x^{s_{l+1}+1}(x^g - 1) f_S(x).
\]

Now suppose \(\Delta(S)\) is in the general form, that is \(\Delta(S) = \{d_1, d_2, \ldots, d_k, g_1, g_2, \ldots, g_l\}\). Similar argument gives the following

1. If \(g_l \leq d_k\), then

\[
x^{g} \chi_A(x) = x^{s_{k+l-1}+1}(x^g - 1) f_S(x);
\]

2. If \(g_l > d_k\) \((k \neq 1)\), then

\[
x^{s_{k+l-1}-s_{k-1}} \chi_A(x) = x^{s_{k+l-1}+1}(x^g - 1) f_S(x).
\]

\[\square\]

3.3 Zeta Function in Terms of Entropy Function

For a dynamical system \((X, T)\), let \(p_n(T)\) be the number of periodic points in \(X\) having period \(n\). When \(p_n(T) < \infty\), the zeta function \(\zeta_T\) is defined as

\[
\zeta_T(t) = \exp\left( \sum_{n=1}^{\infty} \frac{p_n(T)}{n} t^n \right).
\]

**Theorem 3.3** Let \(X(S)\) be a sofic shift and \(\zeta_S\) be the zeta function for \(X(S)\). Then \(\zeta_S(t)\) is either \(\frac{1}{f_S(t-1)}\) or \(\frac{1}{1-(t-1)f_S(t^{-1})}\) for \(|S| < \infty\) or \(|S| = \infty\) respectively.

**Proof** If \(A\) is a \(r \times r\) nonnegative integer matrix, then

\[
\zeta_{\sigma A}(t) = \frac{1}{t^r \chi_A(t^{-1})} = \frac{1}{\det(1d - tA)}
\]

[6, Theorem 6.4.6]. Recall that \(X(S)\) is SFT if and only if \(S\) is finite or cofinite [1]. So by (3.9) and Theorem 3.2, if \(S\) is finite, then

\[
\zeta_S(t) = \frac{1}{f_S(t^{-1})} = \frac{1}{1 - t^{s_1+1} - \ldots - t^{s_k+1}}
\]
and if $S$ is cofinite,

$$
\zeta_{\sigma_S}(t) = \frac{1}{(1-t) f_S(t^{-1})} = \frac{1}{(1-t^{s_1+1} - \cdots - t^{s_k+1})(1-t)-t^{s+1}}.
$$

Now suppose $X(S)$ is strictly sofic with $\Delta(S)$ as in (3.2). By the Fischer cover of $X(S)$ discussed in Sect. 3.1, if $n = gk$ ($k \in \mathbb{N}$), then every point in $X_G$ of period $n$ is the image of exactly one point in $X_G$ of the same period, except $0^\infty$, which is the image of $g$ points of period $n$. Hence in this case $p_n(\sigma_G) = p_n(\sigma_S) - (g - 1)$. When $g$ does not divide $n$, $0^\infty$ is not the image of any point in $X_G$ with period $n$ and so $p_n(\sigma_G) = p_n(\sigma_S) + 1$. Therefore,

$$
\zeta_{\sigma_S}(t) = \exp \left( \sum_{n=1}^{\infty} \frac{p_n(\sigma_S) + 1}{n} t^n + \sum_{n=1}^{\infty} \frac{p_n(\sigma_S) - (g - 1)}{n} t^n \right)
\quad = \exp \left( \sum_{n=1}^{\infty} \frac{p_n(\sigma_S)}{n} t^n + \sum_{n=1}^{\infty} \frac{t^n}{n} - g \sum_{n=1}^{\infty} \frac{t^n}{n} \right)
\quad = \zeta_{\sigma_A}(t) \times \frac{1-t^g}{1-t}
$$

and by (3.9) and Theorem 3.2,

$$
\zeta_{\sigma_A}(t) = \frac{1}{(1-t^g) f_S(t^{-1})} = \frac{1}{(1-\sum_{i=1}^{k-1} t^{s_i+1})(1-t^g) - \sum_{i=k}^{k+l-1} t^{s_i+1}}.
$$

\qed

**Remark 3.4** One should not expect to arrive at a formula for strictly sofic case by a limiting process on SFT's. This cannot happen even for a SFT when $|S| = \infty$. For instance, consider $S = \{1, 2, \ldots\}$. Then by Theorem 3.3,

$$
\zeta_{\sigma_S}(t) = \frac{1}{1-t-t^2}.
$$

On the other hand, suppose $S_n = \{1, 2, \ldots, n\}$ for all $n \in \mathbb{N}$. Then $S_n \not\supset S$ and

$$
\zeta_{\sigma_{S_n}}(t) = \frac{1}{1-t^2 - \cdots - t^{n+1}}.
$$
With an easy computation for $|t| < 1$,
\[
\zeta_{\sigma_n}(t) = \frac{1}{1 - t^2 - \ldots - t^{n+1}} \rightarrow \frac{1 - t}{1 - t - t^2} \neq \zeta_{\sigma}(t).
\]

However, $\frac{1-t}{1-t-t^2}$ is the zeta function of a mixing almost finite type shift. For let $\Lambda_1 = \{1\}$ and $\Lambda_2 = \{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$. Also for any set $\Lambda \subseteq \mathbb{C}\setminus \{0\}$ define $tr_n(\Lambda) = \sum_{d|n} \mu(\frac{n}{d}) tr(\Lambda^d)$ where $\mu$ is the Möbius function. Then $tr_1(\Lambda_1) = tr_1(\Lambda_2) = 1$, $tr_n(\Lambda_1) = 0$ and $tr_n(\Lambda_2) \geq 0$ for $n \geq 2$. Hence $\Lambda_1$ and $\Lambda_2$ satisfy all the hypotheses of [2, Theorem 6.1].

4 The Bowen–Franks Groups

Let $A$ be a $n \times n$ integer matrix. The Bow–Franks group of $A$ is
\[
BF(A) = \mathbb{Z}^n / \mathbb{Z}^n (\text{Id} - A),
\]
where $\mathbb{Z}^n (\text{Id} - A)$ is the image of $\mathbb{Z}^n$ under the matrix $\text{Id} - A$ acting on the right. In order to compute the Bowen–Franks group, we will use the Smith form defined below for an integral matrix. Define the elementary operations over $\mathbb{Z}$ on integral matrices to be:

1. Exchanging two rows or two columns.
2. Multiplying a row or column by $-1$.
3. Adding an integer multiple of one row to another row, or of one column to another column.

Every integral matrix can be transformed by a sequence of elementary operations over $\mathbb{Z}$ into a diagonal matrix
\[
\begin{pmatrix}
  d_1 & 0 & 0 & \cdots & 0 \\
  0 & d_2 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 0 \\
  0 & 0 & 0 & \cdots & d_n
\end{pmatrix}
\]

where $d_j \geq 0$ and $d_j$ divides $d_{j+1}$. This is called the Smith form of the matrix [7]. If we put $\text{Id} - A$ into its Smith form, then
\[
BF(A) \simeq \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n}.
\]

By our convention, each summand with $d_j = 0$ is $\mathbb{Z}$, while summands with $d_j > 0$ are finite cyclic groups. Since $\mathbb{Z}_1$ is the trivial group, the elementary divisors of $BF(A)$ are the diagonal entries of the Smith form of $\text{Id} - A$ which are not 1.

Two subshifts are flow equivalent if they have topologically equivalent suspension flows [6]. Franks in [3] classified irreducible SFT’s up to flow equivalent by showing
that two nontrivial irreducible SFT’s $X_A$ and $X_B$ are flow equivalent if and only if $BF(A) \simeq BF(B)$ and $\text{sgn}(\det(\text{Id} - A)) = \text{sgn}(\det(\text{Id} - B))$.

**Theorem 4.1** [5] Suppose $X(S)$ is a sofic shift with the adjacency matrix $A$ for its Fischer cover.

(1) Let $X(S)$ be a SFT shift.

(a) If $|S| = k$, then $BF(A) \simeq \mathbb{Z}_{(k-1)}$ and $\det(\text{Id} - A) < 0$. So $X(S)$ is flow equivalent to the full $k$-shift.

(b) If $|S| = \infty$, then $BF(A) \simeq \mathbb{Z}_1$ and $\det(\text{Id} - A) < 0$. So $X(S)$ is flow equivalent to the full 2-shift.

(2) Suppose $X(S)$ is a strictly sofic shift with $\Delta(S)$ as in (3.2). Then $BF(A) \simeq \mathbb{Z}_I$ and $\det(\text{Id} - A) < 0$.

The Bowen–Franks groups consists of four groups called $BF$-groups [4]. The main group $BF(A)$ of a sofic $S$-gap shift was given in the above theorem. To introduce other groups, note that $BF(A)$ is the cokernel of $\text{Id} - A$ acting on the row space $\mathbb{Z}^n$. Now the kernel is another Bowen–Franks group $BF_1(A) := \text{Ker}(\text{Id} - A)$. Similarly, acting on the column space $(\mathbb{Z}^n)^t$, $\text{Id} - A$ defines other two groups as cokernel and kernel, denoted by $BF^t(A)$ and $BF_1^t(A)$ respectively. Next theorem gives these groups.

**Theorem 4.2** Let $X(S)$ be a sofic shift with the adjacency matrix $A$ for its Fischer cover. Then $BF^t(A) \simeq BF(A)$ and $BF_1(A) = BF_1^t(A) = \{0\}$.

**Proof** By definition of matrices in Sect. 3.1, $BF_1(A) = BF_1^t(A) = \{0\}$. Let $A$ be the adjacency matrix with the rows $\{r_1, r_2, \ldots, r_m\}$ and the columns $\{c_1, c_2, \ldots, c_m\}$. For computing $BF^t(A)$, notice that $BF^t(A)$ can be naturally identified with $BF(A^t)$, where $A^t$ is the transpose of $A$. Our method is based on using Smith form applied on $A^t$. Due to similarity of routines, we only compute the case when $X(S)$ is SFT.

If $S = \{s_1, s_2, \ldots, s_k\}$ is finite, the following sequence of elementary operations over $\mathbb{Z}$ for $p = m$ puts $\text{Id} - A^t$ into its Smith form. We do these operations in order, that is, we first do (1) and we apply (2) to the obtained matrix in (1) and so on. In the course of operations, we call the last matrix $D = [d_{ij}]$.

(1) $c_i + c_{i+1} \rightarrow c_i$, $1 \leq i \leq p - 1$. This must be done in order: first $i = 1$, then $i = 2$, ..

(2) Now we have a matrix such that $d_{ii} = 1$, $d_{i(i-1)} = d_{i(i-2)} = -1$, and all other entries are zero except some on the first row. By operation $c_i + c_{i-2} \rightarrow c_{i-2}$, $3 \leq i \leq p$, $d_{51} = -1$ and $c_i + c_{i-2} \rightarrow c_{i-2}$, $5 \leq i \leq p$, $d_{51} = 0$, but $d_{71} = -1$. In general, we let $k = \max\{l : 2^l < m\}$ and do $c_i + c_{i-2l} \rightarrow c_{i-2l}$ for $1 \leq j \leq k$, $2^j + 1 \leq i \leq p$.

(3) $d_{11}r_1 + r_1 \rightarrow r_1$, $2 \leq i \leq p$.

(4) $-c_1 \rightarrow c_1$.

(5) $r_1 \leftrightarrow r_m$.

(6) $c_1 \leftrightarrow c_m$. When $S$ is cofinite, put $p = m - 1$ in the above and carry out with the following extra operations.

(7) $r_i + r_m \rightarrow r_m$, $2 \leq i \leq m - 1$.

(8) $d_{11}r_m + r_1 \rightarrow r_1$. 


(9) $r_1 \leftrightarrow r_m$. This will set $d_{11} = d_{mm} = -1$.
(10) $-c_1 \rightarrow c_1$ and $-c_m \rightarrow c_m$.

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