On a set of numbers arising in the dynamics of unimodal maps

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Abstract

In this paper we initiate the study of the arithmetical properties of a set numbers which encode the dynamics of unimodal maps in a universal way along with that of the corresponding topological zeta function. Here we are concerned in particular with the Feigenbaum bifurcation.

1 Preliminaries.

We start by reviewing some basic ideas of (a version of) the kneading theory for unimodal maps. For related approaches and/or more details see [CE], [Dev], [deMvS].

Definition 1.1 A smooth map $f : [0, 1] \to [0, 1]$ is called unimodal if it has exactly one critical point $0 < c_0 < 1$ and moreover $f(0) = f(1) = 0$.

For unimodal maps the orbit of the critical point $c_0$ determines in a sense the complexity of any other orbit. To be more precise, given $x \in [0, 1]$ we call itinerary of $x$ with $f$ the sequence $i(x) = s_1s_2s_3\ldots$ where $s_i = 0$ or 1 according to $f^{i-1}(x) < c_0$ or $f^{i-1}(x) \geq c_0$. An important point is that such symbolic representation is in fact ‘faithful’, that is if $s(x) = s(x')$ then $x = x'$. Differently said, the partition of $[0, 1]$ in the two semi-intervals $P_0 = [0, c_0)$ e $P_1 = [c_0, 1)$ is generating for a unimodal map $f$ with critical point $c_0$.

It is clear that if $s = i(x)$ is a sequence obtained as above then $i(f(x)) = \sigma(s)$ where $\sigma$ denotes the left-shift: if $s = s_1s_2s_3\ldots$ then $\sigma(s) = s_2s_3s_4\ldots$. The itinerary of the point $c_1 = f(c_0)$ is called kneading sequence $K(f)$ of $f$. We say moreover that a given sequence $s$ of 0 and 1 is admissibile for $f$ if there is $x \in [0, 1]$ such that $i(x) = s$. A nice way to decide whether or not a given sequence is admissible amounts to establish an ordering on the itineraries which corresponds to ordering of the real line. In this way, the admissible sequences are those which never become greater than the kneading sequence.

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sequence when shifted. To this end, let us associate to a sequence \( s = s_1s_2s_3 \ldots \) the number \( \tau(s) \in [0, 1] \) defined as

\[
\tau = 0.t_1t_2t_3\ldots = \sum_{k=1}^{\infty} \frac{t_k}{2^k}, \quad t_k = \sum_{i=1}^{k} s_i \mod 2.
\] (1.1)

Equivalently, if we set

\[
\epsilon_k = (-1)^{\sum_{i=1}^{k} s_i}
\] (1.2)

then \( t_k \) and \( \epsilon_k \) are related by

\[
t_k = \frac{1 - \epsilon_k}{2}, \quad \epsilon_k = 1 - 2t_k.
\] (1.3)

**Lemma 1.1** Given \( x, y \in [0, 1] \) we have

1. If \( \tau(i(x)) < \tau(i(y)) \) then \( x < y \);

2. If \( x < y \) then \( \tau(i(x)) \leq \tau(i(y)) \).

**Remark 1** The equality in 2) cannot be removed. Indeed, the existence of an attracting periodic orbit typically implies the existence of an interval of points with the same itinerary. On the other hand, a theorem due to Guckenheimer (see [deMvS]) says that a unimodal map \( f \) has an attracting periodic orbit if and only if \( K(f) \) is periodic. Vice versa, if \( K(f) \) is not periodic then implication 2) becomes: if \( x < y \) then \( \tau(i(x)) < \tau(i(y)) \). This has important consequences. First of all: if \( K(f) \) is not periodic and \( K(f) = K(g) \) then \( f \) and \( g \) are topologically conjugated.

**Proof of Lemma 1.1.** Let us show the first part. Set \( i(x) = s_1s_2\ldots, i(y) = s'_1s'_2\ldots \) and let \( n = \min\{i \geq 1 : s_i \neq s'_i\} \) be the discrepancy between \( i(x) \) and \( i(y) \). We proceed by induction in \( n \). If \( n = 1 \) the result is clear. Suppose it is true for sequences with discrepancy \( n - 1 \). We have \( i(f(x)) = s_2s_3\ldots \) and \( i(f(y)) = s'_2s'_3\ldots \). Two cases are possible: either \( s_1 = 0 \) or \( s_1 = 1 \). If \( s_1 = 0 \) then \( \tau(i(f(x))) < \tau(i(f(y))) \) because applying \( f \) we don’t modify the number of 1’s before the discrepancy. Using the induction we then have that \( f(x) < f(y) \). But since \( f \) is increasing on \([0, c_0] \) we also have \( x < y \). If \( s_1 = 1 \) then \( \tau(i(f(x))) > \tau(i(f(y))) \) because there is a 1 less among the symbols \( s_2\ldots s_n \). Therefore by the induction we get \( f(x) > f(y) \) and since \( f \) is decreasing on \((c_0, 1] \) we see that \( x < y \). The second assertion follows similarly. \( \square \)

An immediate consequence is the following

**Theorem 1.1** Every sequence \( s \) such that

\[
\tau(\sigma(K(f))) \leq \tau(\sigma^m(s)) \leq \tau(K(f)), \quad m \geq 0
\]

is admissible and is the itinerary of a point in \([f(c_1), c_1]\).
In particular,
\[ \tau(\sigma^m(K(f))) \leq \tau(K(f)), \quad m \geq 0. \]

A sequence \( K \) with this property is said maximal. If moreover we consider a one-parameter family of unimodal maps \( f_r \) so that \( r \to f_r \) is continuous on some real interval with respect to the \( C^1 \) topology, then we can reformulate a theorem of Metropolis et al. (see [deMvS]) by saying that every maximal sequence \( K \) such that
\[ \tau(K(f_{r_a})) \leq \tau(K) \leq \tau(K(f_{r_b})) \]
is the kneading sequence of \( f_r \) for some \( r_a \leq r \leq r_b \). Notice that for \( f_r([0,1]) \subseteq [0,1] \) one needs that \( f_r(c_0) \leq 1 \). In particular if \( r = r_b \) then \( f_{r_b}(c_0) = 1 \) and \( K(f_{r_b}) = 10 \) (where \( s_1 \ldots s_l \) indicates the unended repetition of the word \( s_1 \ldots s_l \)), to which it corresponds the number \( \tau(K(f_{r_b})) = 1 \). At the other end-point we have \( \tau(K(f_{r_a})) = 0 \) (when \( f^m(c_0) \) converges monotonically to zero). We finally observe that given \( q \in [0,1] \) we have
\[ \tau(q) = q \Rightarrow \tau(\sigma(s)) = T(q) \]
where \( T : [0,1] \to [0,1] \) is the tent map given by
\[ T(x) = \begin{cases} 2x & \text{if } x < 1/2, \\ 2(1-x) & \text{if } x \geq 1/2. \end{cases} \]

Putting together these observations we obtain the following representation [IP]:

- The subset \( \Lambda \subset [0,1] \) defined as
  \[ \Lambda = \{ \tau \in [0,1] : T^m(\tau) \leq \tau, \forall m \geq 0 \} \]
represents a universal encoding for the dynamics of unimodal maps: those having the same parameter \( \tau \) have identical topological properties. In particular, every 0 in the binary expansion of \( \tau \in \Lambda \) corresponds to a ‘forbidden word’ in the associated dynamics: let \( K(f) = s_1s_2 \ldots \) and \( \tau(K(f)) = 0.t_1t_2 \ldots \), then if \( t_j = 0 \) the word \( s_1 \ldots \hat{s}_j \) (with \( \hat{s}_j = 1 - s_j \)) is a forbidden word. Let \( A = \{0,1\} \) be the alphabet and \( A^* = \bigcup_{n \in \mathbb{N}} A^n \) the set of all possible finite words written in the alphabet \( A \). A word \( u \in A^* \) of length \( |u| = n \) is said f-admissibile if there is \( x \in [0,1] \) whose itinerary with \( f \) up to the \( n \)-th letter coincides with \( u \). The set \( \mathcal{L} \subseteq A^* \) defined as
  \[ \mathcal{L} = \{ u \in A^* , u \text{ is f-admissibile} \} \]
is the language generated by \( f \). The function
  \[ p(n) = \#\{ u \in \mathcal{L}, |u| = n \} \quad (1.4) \]
is called the complexity function of \( \mathcal{L} \) and the limit
  \[ h = \lim_{n \to \infty} \frac{1}{n} \log p(n) \quad (1.5) \]
is the topological entropy. To summarize, the parameter \( \tau \) furnishes a universal encoding in the sense that all unimodal maps with the same \( \tau \) determine the same language \( \mathcal{L} = \mathcal{L}(\tau) \) and, in particular, have the same topological entropy \( h = h(\tau) \).
Remark 2 It is plain that the extremal situation in which $T^m(\tau) = \tau$ for some $m > 0$ is that in which $\tau$ is a periodic point for the tent map $T$. In this case the kneading sequence $K$ is periodic and so is the corresponding attractor. This suggests that isolated points as well as ‘holes’ in $\Lambda$ have to be related to periodic attractors. In particular, there is a one-to-one correspondence between the holes in $\Lambda$ and the periodic windows in the bifurcation diagram of unimodal maps, namely intervals in parameter space where the topological entropy is constant [IP].

1.1 Topological zeta function

A great deal of information on the set of periodic points of a given map $f : [0,1] \to [0,1]$ can be stored into the topological zeta function of Artin and Mazur, defined as

$$\zeta(f, z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \#\text{Per}_n(f). \quad (1.6)$$

For a unimodal map $f$ with parameter $\tau$ the numbers $\#\text{Per}_n(f)$ are uniquely determined by the value of $\tau$ and we therefore write $\zeta(\tau, z)$. The series converges absolutely and uniformly for $|z| < e^{-h}$ and $z = e^{-h}$ is a singular point (e.g. a pole) of $\zeta(\tau, z)$. This function can also be written as an Euler product noting that

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \#\text{Per}_n(f) = \sum_{p=1}^{\infty} N(p) \sum_{k=1}^{\infty} \frac{z^{kp}}{k} = \log \prod_{p=1}^{\infty} (1 - z^p)^{-N(p)},$$

where $N(p)$ is the number of distinct periodic orbits of prime period $p$. Hence we have

$$\zeta(\tau, z) = \prod_{p=1}^{\infty} (1 - z^p)^{-N(p)}. \quad (1.7)$$

The combinatorial features of the set of periodic orbits of a given map $f$ reflects onto the analytic properties of $\zeta(\tau, z)$ in the complex plane.

More specifically, it is not difficult to realize that from the work of Milnor and Thurston ([MT], Lemma 4.5 and Corollary 10.7) one can extract the following result

**Proposition 1.1** Let $\Lambda \ni \tau = 0.t_1t_2t_3\ldots$. If the sequence $t_1t_2t_3\ldots$ is eventually periodic or aperiodic then

$$\zeta(\tau, z) = \frac{1}{(1 - z)(1 + \sum_{k=1}^{\infty} \epsilon_k z^k)}. \quad (1.8)$$

If instead the kneading sequence is periodic and $\tau = 0.t_1\ldots t_n$ then

$$\zeta(\tau, z) = \frac{1}{(1 - z)(1 + \sum_{k=1}^{n-1} \epsilon_k z^k)}, \quad (1.9)$$

where the numbers $\epsilon_k$ are defined in (1.2)-(1.3).
We now list some examples in which the zeta function can be written in closed form by means of Lemma 1.1.

- The number \( \tau = 1 = 0.1\overline{1} \) corresponds to the situation where the critical point gets mapped to the origin in two steps, and yields
  \[
  \zeta(1, z) = \frac{1}{1-2z}, \quad h(1) = \log 2.
  \]

- The number \( \tau = 5/6 = 0.110 \) corresponds to the situation where the critical point gets mapped to the fixed point in three steps (band merging). In this case we find
  \[
  \zeta(5/6, z) = \frac{1+z}{(1-z)(1-2z^2)}, \quad h(5/6) = \log \sqrt{2}.
  \]

- The number \( \tau = 6/7 = 0.1\overline{110} \) corresponds to the opening of the period three window. The last orbit in the Sarkovskii order settles down and thus there are periodic orbits of any period. Here we get
  \[
  \zeta(6/7, z) = \frac{1}{(1-z)(1-z-z^2)}, \quad h(6/7) = \log \frac{\sqrt{5}-1}{2}.
  \]

In the examples above the number \( \tau \) was always rational. In the next Section we show a situation leading to a transcendental irrational \( \tau \). A systematic study of the arithmetical properties of the numbers in \( \Lambda \), along with their relation with the dynamics, is far from being reached. In particular, the question of what is the most irrational \( \tau \) (and to which chaotic state it corresponds) is open. In Section 2 we shall study the above quantities for the Feigenbaum bifurcation but in order to get a self-contained exposition we first recall some standard notions (for details see [Dev]).

### 1.2 Kneading theory and renormalization

Let \( f : [0, 1] \to [0, 1] \) be a unimodal map with a unique fixed point \( b \) in the interval \((c_0, 1)\), so that \( f'(b) < 0 \). Let \( a \) the (unique) point in \((0, c_0)\) such that \( f(a) = b \) and set \( J = [a, b] \). Consider the linear map \( L \) defined by

\[
L(x) = \frac{1}{a-b} (x-b).
\]  

(1.10)

It expands \( J \) to \([0, 1]\) reversing its orientation. The inverse map is

\[
L^{-1}(x) = (a-b)x + b.
\]  

(1.11)

The renormalization operator \( \mathcal{R} \) is thus defined as

\[
(\mathcal{R} f)(x) = L \circ f^2 \circ L^{-1}(x).
\]  

(1.12)

Plainly \( (\mathcal{R} f)(0) = (\mathcal{R} f)(1) = 0 \) and \( c_0 \) is the only critical point of \( \mathcal{R} f \). Moreover, 2-periodic points for \( f \) become fixed points of \( \mathcal{R} f \).

Now let \( K(f) = s_1s_2s_3 \ldots \) be the kneading sequence of \( f \). The following properties are easily verified (see [Dev]):
1. If $\mathcal{R}f$ is defined and unimodal then $s_{2k+1} = 1, \forall k \geq 0$;

2. $K(\mathcal{R}f) = \hat{s}_2 \hat{s}_4 \hat{s}_6 \ldots$. In other words, one can define a renormalization operator on sequences acting as (with slight abuse we keep using the same symbol):

$$\mathcal{R}(s_1 s_2 s_3 \ldots) = \hat{s}_2 \hat{s}_4 \hat{s}_6 \ldots$$

(1.13)

3. If both $\mathcal{R}f$ and $\mathcal{R}^2 f$ are unimodal then $s_{4k+2} = 0$;

4. If $\mathcal{R}^lf$ is unimodal for $l \leq n$ then all symbols $s_j$ with $j = 2^nk + 2^{n-1}$ are determined;

5. Since the numbers $2^nk + 2^{n-1}$ exhaust all even numbers as $n$ varies in $\mathbb{N}$ it follows that if $\mathcal{R}^nf$ is unimodal for each $n \geq 1$ then all symbols of $K(f)$ are determined.

How $K(f)$ looks like for an infinitely renormalizable unimodal map, that is a map $f$ such that $\mathcal{R}f = f$?

Set

$$K_1 = \overline{1}$$
$$K_2 = \overline{10}$$
$$K_3 = \overline{1011}$$
$$K_4 = \overline{10111010}$$
$$K_5 = \overline{1011101010111011}$$

and more generally $K_{j+1}$ is obtained from $K_j$ by applying one of the following equivalent procedures:

- duplicating the repeating sequence and reversing the last symbol;
- doubling all indices, reversing the resulting symbols (all with even index) and inserting a 1 at each position with odd index;
- applying the Feigenbaum substitution $1 \rightarrow 10$ and $0 \rightarrow 11$ (the symbol 1 being the prefix) to the repeating sequence.

By construction $K_j$ has period $2^j$ with an odd number of 1’s. We also have that

$$\mathcal{R}(K_{j+1}) = K_j, \quad j \geq 1. \quad (1.14)$$

Therefore the limit sequence

$$K_\infty = \lim_{j \to \infty} K_j = 1011 1010 1011 1011 1010 1011 1010 \ldots \quad (1.15)$$

is aperiodic and invariant under renormalization (the latter can be interpreted as a self-similarity property):

$$\mathcal{R}(K_\infty) = K_\infty. \quad (1.16)$$
For all \( j \geq 0 \), \( K_j \) is a prefix of \( K_\infty \). Finally, one easily verifies that \( K_j \) is the kneading sequence of a unimodal map having a periodic attractor of period \( 2^j \), whereas \( K_\infty \) is that of an infinitely renormalizable map.

Inspection of the sequences \( K_n \) suggests that the asymptotic frequencies of the symbols 0 and 1 appearing in \( K_\infty \) are \( 1/3 \) and \( 2/3 \) respectively. To check this, we shall use a standard technique in the theory of substitution (see [PF]): let \( \phi \) be the substitution \( \phi(1) = 10 \) and \( \phi(0) = 11 \) considered above and \( N_i(\phi(j)) \) be the number of occurrences of the symbol \( i = 0, 1 \) in the word \( \phi(j) \). The asymptotic frequency of \( i \) in \( K_\infty \) is then given by

\[
  f_i = \lim_{n \to \infty} \frac{N_i(\phi^n(1))}{2^n}, \quad i = 0, 1, \quad (1.17)
\]

where we have used the fact that \( |K_n| = 2^n \). To compute \( f_i \) we construct the matrix

\[
  M = [N_i(\phi(j))]_{i,j \in \{0,1\}}. \quad (1.18)
\]

A short reflection yields

\[
  M^n = [N_i(\phi^n(j))]_{i,j=0,1}, \quad (1.19)
\]

and thus, setting \( u = (0,1) \), we get

\[
  f_i = \lim_{n \to \infty} \frac{(M^n u)_i}{2^n}. \quad (1.20)
\]

From Perron-Frobenius theorem we have that \( M \) has a simple positive eigenvalue of maximal modulus \( \lambda \) to which it corresponds an eigenvector with strictly positive components. In our case we find

\[
  M = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \quad (1.21)
\]

whose eigenvalues are 2 and \(-1\). The normalized eigenvector corresponding to the leading eigenvalue is \( v = (1/3, 2/3) \). From (1.20) one deduces that \( f_i = v_i, \ i = 0,1, \) which are the claimed frequencies.

**Remark 3** One may consider the sequence \( K_\infty \) as an element of \( \{0,1\}^\mathbb{N} \) and observe that the continuous injective map \( T : \{0,1\}^\mathbb{N} \to \{0,1\}^\mathbb{N} \) defined as follows: if \( \omega = 111\ldots \) then \( T\omega = 000\ldots \); if \( \omega = 1\ldots 0\ldots \) then \( T\omega = 0\ldots 01\ldots \); if \( \omega = 0\ldots \) then \( T\omega = 1\ldots \), acts a (right) translation on \( K_\infty \). Therefore \( T \) leaves invariant the space \( X = \{T^j K_\infty\}_{j \geq 0} \). The map \( T : X \to X \) is called dyadic adding machine.

## 2 Arithmetics of the Feigenbaum bifurcation

We now look at the values of the parameter \( \tau \) corresponding to the kneading sequences arising in the period doubling scenario discussed in the preceding Section.
Set \( \tau_j = \tau(K_j) \). We find

\[
\begin{align*}
\tau_1 &= 0.\overline{10} \\
\tau_2 &= 0.\overline{1100} \\
\tau_3 &= 0.\overline{11010010} \\
\tau_4 &= 0.\overline{1101001100101100} \\
\tau_5 &= 0.\overline{11010011001011010010110011010010}
\end{align*}
\]

and \( \tau_{j+1} \) is obtained from \( \tau_j \) by applying the rule

\[
\tau_j = 0.t_1 \ldots t_{2j} \implies \tau_{j+1} = 0.t_1 \ldots t_{2j-1}t_2, t_1 \ldots t_{2j-1}t_2,
\]

or, alternatively, by the following substitution: let

\[
a = 00, \quad b = 01, \quad c = 10, \quad d = 11,
\]

then

\[
a \rightarrow ac, \quad b \rightarrow ad, \quad c \rightarrow da, \quad d \rightarrow db.
\]

It is easy to check that \( \tau_j \in \Lambda, \forall j \geq 1 \). They form an increasing sequence:

\[
\tau_1 < \tau_2 < \tau_3 < \cdots
\]

and satisfy

\[
\tilde{R}(\tau_{j+1}) = \tau_j \quad \text{where} \quad \tilde{R}(0.t_1t_2t_3 \ldots) := 0.t_2t_4t_6 \ldots
\]

For each \( j \geq 1 \), \( \tau_j = 0.t_1 \ldots t_{2j} \) is the rational number given by

\[
\tau_j = \frac{2^{2^j}}{2^{2^j} - 1} \sum_{k=1}^{2^j} \frac{t_k}{2^k}
\]

We have

\[
\begin{align*}
\tau_1 &= \frac{2}{3}, \quad \tau_2 = \frac{4}{5}, \quad \tau_3 = \frac{14}{17}, \quad \tau_4 = \frac{212}{257}, \quad \tau_5 = \frac{54062}{65537}, \\
\tau_6 &= \frac{3542953172}{4294967297}, \quad \tau_7 = \frac{1521686801456509742}{18446744073709551617}
\end{align*}
\]

By (2.22) and (2.25) the following recursive law is in force:

\[
\tau_j = \frac{p_j}{q_j} \implies \tau_{j+1} = \frac{p_{j+1}}{q_{j+1}} = \frac{2 + p_j(q_j - 2)}{2 + q_j(q_j - 2)}
\]

where all fractions are in lowest terms. From the above it follows \( q_{j+1} - 1 = (q_j - 1)^2 \) and thus \( q_j = 2^{2^{j-1}} + 1 \). Note that the above recursion can be written in the form

\[
p_1 = 2, \quad q_1 = 3, \quad p_{j+1} = 2 + (2^{2^{j-1}} - 1)p_j, \quad q_{j+1} = 2 + (2^{2^{j-1}} - 1)q_j, \quad j \geq 1
\]

This yields

\[
q_j - p_j = (2^{2^{j-2}} - 1)(q_{j-1} - p_{j-1}) = \cdots = \prod_{k=0}^{j-2} (2^{2^{k}} - 1)
\]
and recalling that $q_j = 2^{2j-1} + 1$ we get $p_j = 2^{2j-1} + 1 - \prod_{k=0}^{j-2}(2^{2k} - 1)$. We thus find the expression

$$
\tau_j = 1 - \frac{\prod_{k=0}^{j-2}(2^{2k} - 1)}{2^{2j-1} + 1} = 1 - \frac{\prod_{k=0}^{j-1}(1 - 2^{-2k})}{2(1 - 2^{-2j})}
$$

(2.29)

and

$$
\tau_{j+1} - \tau_j = \left(\frac{2}{2^{2j} + 1}\right) \tau_j.
$$

(2.30)

The number

$$
\tau_{\infty} = \lim_{j \to \infty} \tau_j = 1 - \frac{1}{2} \prod_{k=0}^{\infty} (1 - 2^{-2k}) = 0.11010011 00101101 00101100 11010011 \ldots
$$

(2.31)

satisfies $\tau_{\infty} = \tau(K_{\infty})$ and is plainly irrational (since $K_{\infty}$ is aperiodic). One easily recognizes the Thue-Morse sequence beginning in 0, that is the fixed point of the substitution 0 → 01 and 1 → 10 with prefix 01. It enjoys the invariance property

$$
\hat{R}(\tau_{\infty}) = \tau_{\infty}
$$

(2.32)

which can also be expressed in the form

$$
\tau_{\infty} = \sum_{k=1}^{\infty} \frac{t_k}{2^k} = \sum_{k=1}^{\infty} \frac{t_{2^l k}}{2^k}, \quad \forall l \geq 0.
$$

(2.33)

Thus, for instance, $t_k = 1$ whenever $k = 2^\ell$ for some $\ell \geq 0$. More specifically, we have

**Proposition 2.3** For an integer $p \geq 1$ set

$$
s(p) = \sum_{i \geq 0} n_i (\text{mod } 2) \quad \text{if} \quad p = \sum_{i \geq 0} n_i 2^i, \quad n_i \in \{0, 1\}.
$$

Let $\tau_{\infty} = 0.t_1 t_2 \ldots$. Then $t_k = s(p)$ whenever $k = p \cdot 2^\ell$ for some $\ell \geq 0$ and $p \geq 1$ odd.

**Proof.** Due to (2.33) it will suffice to show by induction over $r$ the following property: $P_r = \{p \text{ odd and } p \leq 2^r \Rightarrow t_k = s(p)\}$. Note that $P_0$ is obvious. Consider an odd $p'$ such that $2^r < p' \leq 2^{r+1}$. Then $p' = 2^r + p$ with $1 \leq p \leq 2^r$ and $p$ odd. Then $s(p') = s(p) + 1 (\text{mod } 2)$ and by the above $P_r \Rightarrow P_{r+1}$. □

Furthermore, from (2.23) we see at once that the symbols 0 and 1 both appear in $\tau_{\infty}$ with frequency 1/2. One may wonder if $\tau_{\infty}$ is a normal number, in the sense of Borel. That means that in its dyadic expansion (2.31) the asymptotic frequency of any word of length $n$ is $2^{-n}$. On the other hand, reasoning as for the sequence $K_{\infty}$ (and using the substitutions (2.23)) it is not difficult to verify that the frequency of the pairs 00, 11 cannot both be 1/2.

By the way, we have shown the following result:

**Proposition 2.2** Let $\xi : \{0, 1\}^N \to \{0, 1\}^N$ be the map defined as $(\xi s)_k = \sum_{i=1}^{k} s_i (\text{mod } 2)$. Let $u$ be the fixed point of the Feigenbaum substitution 1 → 10 and 0 → 11 with prefix 1 and $w$ be the fixed point of the Thue-Morse substitution 0 → 01 and 1 → 10 with prefix 0. Then $0 \xi(u) = \xi(0u) = w$. 

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01, 10 and 11 are \( \frac{1}{5} \), \( \frac{1}{6} \), \( \frac{1}{6} \) and \( \frac{1}{7} \). Therefore \( \tau_\infty \) is not a normal number. In fact \( \tau_\infty \) is transcedental, as is shown by Mahler in [Ma] (see also [Dek], [AZ], [FM])

We end this digression with some partial insight into the structure of the continued fraction expansion of \( \tau_\infty \).

Recall that any number \( \tau \in [0, 1] \) can be expanded as

\[
\tau = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} \equiv [a_1, a_2, a_3, \ldots] \tag{2.34}
\]

where the \( a_i \)'s are integers. Successive truncations of this expansion yields a sequence of rational numbers

\[
\frac{r_n}{s_n} = [a_1, a_2, a_3, \ldots, a_n] \tag{2.35}
\]

which are called convergents of \( \tau \) (see [Kh]).

Now, the problem we are interested in is the following: are the continued fraction expansions of the numbers \( \tau_j \) predictable (i.e. have a definite pattern) as their binary expansions do? The expansions of the first eight \( \tau_j \)'s are

\[
\begin{align*}
\tau_1 &= [1, 2] \\
\tau_2 &= [1, 4] \\
\tau_3 &= [1, 4, 1, 2] \\
\tau_4 &= [1, 4, 1, 2, 2, 6] \\
\tau_5 &= [1, 4, 1, 2, 2, 6, 2, 1, 2, 9, 1, 2] \\
\tau_6 &= [1, 4, 1, 2, 2, 6, 2, 1, 2, 9, 1, 2, 2, 1, 21, 1, 10, 2, 1, 1, 1, 5] \\
\tau_7 &= [1, 4, 1, 2, 2, 6, 2, 1, 2, 9, 1, 2, 2, 1, 21, 1, 10, 2, 1, 1, 1, 4, 1, 2, 29, 1, 24, 1, 1, 7, 11, 3, 2, 5, 1, 1, 1, 89] \\
\tau_8 &= [1, 4, 1, 2, 2, 6, 2, 1, 2, 9, 1, 2, 2, 1, 21, 1, 10, 2, 1, 1, 1, 4, 1, 2, 29, 1, 24, 1, 1, 7, 11, 3, 2, 5, 1, 1, 1, 88, 1, 1, 6, 1, 1, 33, 2, 6, 1, 24, 1, 5, 212, 2, 1, 1, 10, 1, 3, 11, 2, 1, 2, 1, 10, 1, 1, 2, 3, 2549, 1, 2]
\end{align*}
\]

A direct inspection suggests that there is a subsequence \( n_j \) of the integers so that if \( \tau_j = [a_1, \ldots, a_{n_j}] \) then

\[
\tau_{j+1} = \begin{cases} 
[a_1, \ldots, a_{n_j} - 1, b_{n_j+1}, \ldots, b_{n_{j+1}}] & \text{if } n_j \text{ is odd,} \\
[a_1, \ldots, a_{n_j}, b_{n_j+1}, \ldots, b_{n_{j+1}}] & \text{if } n_j \text{ is even,}
\end{cases} \tag{2.36}
\]

for some \( b_{n_j+1}, \ldots, b_{n_{j+1}} \). The sequence \( n_j \) for \( 1 \leq j \leq 12 \) is

\[
2, 2, 4, 6, 12, 23, 39, 71, 121, 253, 528, 1129
\]

Unfortunately we are not able to say much more. In particular it is not clear what kind of relation could be established between the \( \tau_j \)'s and the convergents of \( \tau_\infty \). Note
that Shallit obtained in [S] a rather complete description of the patterns arising for irrational numbers of the type \( \sum_{k \geq 0} u^{-2^k} \), \( u \) an integer. On the other hand, a high-temperature-like expansion of the product appearing in (2.31) yields the expression

\[
\tau_\infty = 1 - \frac{1}{2} \left[ 1 - \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell!} \sum_{k_1 \neq k_2 \neq \cdots \neq k_\ell} 2^{-\sum_{i=1}^\ell 2k_i} \right] \tag{2.37}
\]

of which the numbers studied by Shallit are just the first order (\( \ell = 1 \)) term with \( u = 2 \).

We conclude with a brief description of the topological zeta functions arising in this situation.

ZETA FUNCTIONS. For the values \( \tau = \tau_j \) considered above, we get the polynomial zeta function

\[
\frac{1}{\zeta(\tau_j, z)} = (1 - z) \prod_{n=0}^{j} (1 - z^{2^n}), \tag{2.38}
\]

whose zeroes are all on the unit circle \( |z| = 1 \). Moreover we have \( \zeta(\tau_j, z) \to \zeta(\tau_\infty, z) \) when \( j \to \infty \), where

\[
\frac{1}{\zeta(\tau_\infty, z)} = (1 - z) \prod_{n=0}^{\infty} (1 - z^{2^n}). \tag{2.39}
\]

From Sarkovskii theorem (see [BGMY]) it follows that \( h(\tau_j) = 0 \) for all \( j \geq 0 \). Also \( h(\tau_\infty) = 0 \) (but for any \( \tau \in \Lambda \) with \( \tau > \tau_\infty \) we have \( h(\tau) > 0 \)). Put

\[
\Xi(z) = \prod_{n=0}^{\infty} (1 - z^{2^n}). \tag{2.40}
\]

This function satisfies the functional equation

\[
\Xi(z) = (1 - z) \Xi(z^2) \tag{2.41}
\]

from which we see that if \( \Xi(z) = 0 \) then \( |z| = 1 \). In particular, given \( m \geq 1 \) and \( k = 0, 1, \ldots, 2^l - 1 \) all factors of the product defining \( \Xi(z) \) corresponding to \( n \geq m \) vanish at \( z = e^{2\pi ik/2^l} \). Therefore the zeroes of \( \Xi(z) \) are dense on the unit circle. We then have that the radius of convergence of \( \zeta(\tau_\infty, z) \) is equal to 1 and that the unit circle is a (opaque) natural boundary for this function.

Finally, expanding the product (2.40) we get

\[
\Xi(z) = 1 - z - z^2 + z^3 - z^4 + z^5 + z^6 - z^7 - z^8 + z^9 + z^{10} - z^{11} + z^{12} - z^{13} - z^{14} + z^{15} - z^{16} + \ldots
\]

from which we see that if \( \tau_\infty = 0, t_1 t_2 t_3 \ldots \) then the coefficient of \( z^k \) with \( k \geq 1 \) in the above expansion is but the number \( \epsilon_k = 1 - 2 t_k \) defined in (1.3), in agreement with Proposition 1.1. In turn, we notice that the number \( \tau_\infty \) can be written as

\[
\tau_\infty = 1 - \frac{1}{2} \Xi \left( \frac{1}{2} \right). \tag{2.42}
\]
References

[AZ]  J-P Allouche, L Q Zamboni, Algebraic irrational numbers cannot be fixed points of non-trivial constant length or primitive morphisms, *Journal of Number Theory* **70** (1998), 119-124.

[BGMY]  L Block, J Guckenheimer, M Misiurewicz, L S Young, *Periodic points and topological entropy of a one-dimensional map*, Lecture Notes in Mathematics 819, Springer-Verlag, Berlin and New-York (1980), pp. 18-34.

[CE]  Pierre Collet, Jean-Pierre Eckmann, *Iterated maps of the Interval as Dynamical Systems*, Birkhäuser, Boston (1980).

[deMvS]  W de Melo and S van Strien, *One-Dimensional Dynamics*, Springer-Verlag, Berlin Heidelberg (1993).

[Dev]  R Devaney, *An Introduction to Chaotic Dynamical Systems*, The Benjamin Cummings Publ.Co. (1986).

[Dek]  F M Dekking, *Trascendance du nombre de Thue-Morse*, *C. R. Acad. Sci. Paris, Série I* **285** (1977), 157-160.

[FM]  S Ferenczi, C Mauduit, Trascendence of numbers with a low complexity expansion, *Journal of Number Theory* **67** (1997), 146-161.

[IP]  S Isola, A Politi, Universal encoding for unimodal maps, *Journal of Statistical Physics* **61** (1990), 263.

[Kh]  A I Khinchin, *Continued Fractions*, University of Chicago Press 1964.

[Ma]  K Mahler, Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen, *Math. Annalen* **101** (1929), 342-366, Corrigendum **103** (1930), 532.

[MT]  J Milnor, W. Thurston, *Iterated maps of the Interval*, in Lecture Notes in Mathematics 1342, Springer Verlag (1988), p. 465.

[PF]  N Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lecture Notes in Mathematics 1794, Springer Verlag (2002).

[S]  J Shallit, Simple continued fractions for some irrational numbers, *Journal of Number Theory* **11** (1979), 209-217.