Separably injective C*-algebras

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Abstract. We show that a C*-algebra is a 1-separably injective Banach space if, and only if, it is linearly isometric to the Banach space $C_0(\Omega)$ of complex continuous functions vanishing at infinity on a substonean locally compact Hausdorff space $\Omega$.

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1. Introduction

It is well-known that there are few examples of 1-injective Banach spaces. These are Banach spaces $V$ for which every continuous linear map $T : Y \to V$ on a Banach space $Y$ admits a norm preserving extension to a super space $Z \supset Y$, equivalently, contractive linear maps $T : Y \to V$ extend to contractive ones on $Z$. Indeed, a Banach space $V$ is 1-injective if and only if it is linearly isometric to the continuous function space $C(\Omega)$ on some stonean space $\Omega$ [5, 9]. If the 1-injectivity condition is relaxed to requiring that each continuous linear map $T : Y \to V$ extends to a continuous one on $Z \supset Y$, then it is unclear what constitutes the larger class of Banach spaces, called $\lambda$-injective or $P_\lambda$ spaces, satisfying this condition. However, one can consider the class of $\lambda$-separably injective Banach spaces to which only continuous linear maps on separable spaces are extendable to separable super spaces. Of particular interest is the subclass of 1-separably injective Banach spaces to which contractive linear maps on separable spaces admit contractive extension on separable super spaces. While $c_0$ is the only $\lambda$-separably injective space among infinite dimensional separable Banach spaces [14], it has been shown recently in [1] that among nonseparable real Banach spaces, there are indeed many interesting examples of $\lambda$-separably injective spaces. In particular, the Banach space $C(\Omega, \mathbb{R})$ of real continuous functions on a compact Hausdorff space $\Omega$ is 1-separably injective if, and only if, $\Omega$ is an F-space. It is natural

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to ask if this result also holds for the space $C_0(S)$ of continuous functions vanishing at infinity on a locally compact space $S$. This case has not been discussed in [11] and in fact, the example of $c_0$ provides a negative answer since $\mathbb{N}$ is an $F$-space, but $c_0$ is not 1-separably injective although it is 2-spearably injective.

In this paper, we give a complete answer to the above question and prove, more generally, that a C*-algebra is 1-separably injective if, and only if, it is linearly isometric to the Banach space $C_0(S)$ of complex continuous functions vanishing at infinity on a substonean locally compact Hausdorff space $S$. Particularly, abelian monotone sequentially complete C*-algebras are 1-separably injective. This example may be of interest as the class of monotone complete C*-algebras is closely related to generic dynamics [13].

2. Separably injective Banach spaces

The concept of a separably injective real Banach space, considered in [11], can be extended naturally to that for a complex Banach space.

**Definition 2.1.** A complex (resp. real) Banach space $V$ is said to be 1-
separably injective if for every complex (resp. real) separable Banach space $Z$ and every closed subspace $Y \subset Z$, every bounded linear operator $T : Y \to V$ extends to a bounded linear operator $\tilde{T} : Z \to V$ with $\|\tilde{T}\| = \|T\|$.

Given a locally compact Hausdorff space $\Omega$, we will denote by $C_0(\Omega)$ the abelian C*-algebra of complex continuous functions on $\Omega$ vanishing at infinity. If $\Omega$ is compact, then we omit the subscript 0 and denote by $C(\Omega, \mathbb{R})$ the Banach space of real continuous functions on $\Omega$.

**Definition 2.2.** Let $\Omega$ be a locally compact Hausdorff space. It is called an $F$-space if for each real continuous function $f$ on $\Omega$, there is a real continuous function $k$ on $\Omega$ such that $f = k|f|$ (cf. [3, 14.25]). Following [4], we call $\Omega$ substonean if any two disjoint open $\sigma$-compact subsets of $\Omega$ have disjoint compact closures.

The compact substonean spaces are exactly the compact $F$-spaces. However, infinite discrete spaces are F-spaces without being substonean. We refer to [8, Example 5] for an example of a substonean space which is not an F-space.

**Example 2.3.** Let $\Omega$ be a compact Hausdorff space. Using the results in [6, 11], it has been shown in [11, Proposition 4.2] that the real continuous function space $C(\Omega, \mathbb{R})$ is 1-separably injective if, and only if, $\Omega$ is an F-space. This result remains true if we replace $C(\Omega, \mathbb{R})$ by the complex continuous function space $C(\Omega)$. Indeed, if $C(\Omega, \mathbb{R})$ is 1-separably injective and given a contractive complex linear operator $T : Y \to C(\Omega)$, where $Y$ is a closed subspace of a separable complex Banach space $Z$, the real part $\text{Re} T : y \in Y \mapsto \text{Re} T(y) \in \mathbb{R}$
$C(\Omega, \mathbb{R})$ extends to a real linear contraction $T_x : Z \to C(\Omega, \mathbb{R})$ which, as in the proof of [5, Theorem 2], gives a complex linear contraction

$$z \in Z \mapsto T_x(z) - iT_x(iz) \in C(\Omega)$$

extending $T$ since $T(y) = \text{Re} T(y) - i \text{Re} T(iy)$ for $y \in Y$. Hence $C(\Omega)$ is 1-separably injective. Conversely, if $C(\Omega)$ is 1-separably injective, it will follow from Theorem 3.5 that $C(\Omega, \mathbb{R})$ is 1-separably injective.

However, as noted earlier, the above result is not valid for the space $C_0(S)$ of continuous functions vanishing at infinity on a locally compact Hausdorff space $S$. Separable injectivity of $C_0(S)$ has not been considered in [1]. A topological criterion for 1-separable injectivity of $C_0(S)$ follows from Theorem 3.5.

Let $V$ be a complex 1-separably injective Banach space and let $T : Y \to V$ be a bounded linear operator on a closed subspace $Y$ of a complex Banach space $X$. The arguments in the proof of [1, Proposition 3.5 (a)] for real 1-separably injective spaces can be extended to the complex case and one can show that $T$ has a norm preserving extension $\tilde{T} : X \to V^{**}$. A further application of the arguments in the proof of [10, Theorem 2.1, (9) \Rightarrow (1)], which are also valid for complex spaces, gives the following result. The result for real 1-separably injective spaces has been shown in [1].

**Lemma 2.4.** Let $V$ be a 1-separably injective complex Banach space. Then the bidual $V^{**}$ is 1-injective.

### 3. Separably injective C*-algebras

We characterize 1-separably injective C*-algebras in this section. We begin with a simple lemma.

**Lemma 3.1.** A 1-separably injective C*-algebra is abelian.

**Proof.** Let $A$ be a 1-separably injective C*-algebra. By Lemma 2.4, the bidual $A^{**}$ is 1-injective and hence linearly isometric to a continuous function space $C(\Omega)$ on some stonean space $\Omega$ [5]. The linear isometry between the C*-algebras $A^{**}$ and $C(\Omega)$ preserves the Jordan triple product

$$\{a, b, c\} := \frac{1}{2} (ab^*c + cb^*a) \quad (a, b, c \in A^{**})$$

by a well-known result of Kadison [7] (see also [2, Theorem 3.1.7]). Since $C(\Omega)$ is an abelian algebra, we must have, via the isometry between $C(\Omega)$ and $A^{**}$,

$$\{a, b, \{c, 1, 1\}\} = \{a, \{b, c, 1\}, 1\}$$

for $a, b, c \in A^{**}$, where $1$ denotes the identity in $A^{**}$. Let $p$ be a projection in $A^{**}$ and $a \in A^{**}$. A simple computation gives

$$\frac{1}{2} (pa + ap) = \{p, p, \{a, 1, 1\}\} = \{p, \{p, a, 1\}, 1\}$$

$$= \frac{1}{4} (pa + ap + 2pap)$$

and \(pa + ap = 2pap\), which implies \(pa = ap\). Hence \(A^{**}\) is abelian since it is generated by projections. In particular, \(A\) itself is abelian.

We have the following result readily.

**Proposition 3.2.** Let \(A\) be a von Neumann algebra. The following conditions are equivalent.

(i) \(A\) is 1-separably injective.

(ii) \(A\) is 1-injective.

**Proof.** (i) \(\Rightarrow\) (ii). By the above observation, the unital algebra \(A\) is abelian and hence linearly isometric to a continuous function space \(C(\Omega)\) on some compact Hausdorff space \(\Omega\). Since \(A\) has a predual, \(\Omega\) must be hyperstonean and therefore \(C(\Omega)\) is 1-injective by [5].

**Remark 3.3.** The above proposition is false for unital C*-algebras. Indeed, let \(\beta \mathbb{N}\) be the Stone-Čech compactification of \(\mathbb{N}\). Then \(\beta \mathbb{N}\setminus \mathbb{N}\) is a compact F-space by [3, p. 210]. Hence the C*-algebra \(C(\beta \mathbb{N}\setminus \mathbb{N})\) is 1-separably injective (cf. Example 2.3), but not 1-injective since \(\beta \mathbb{N}\setminus \mathbb{N}\) is not stonean [3, p. 98].

We now determine the class of 1-separably injective C*-algebras. A useful fact noted in [4, Proposition 1.1] is that a locally compact Hausdorff space \(S\) is substonean if, and only if, the following condition holds: given \(f\) and \(g\) in \(C_0(S)\) satisfying \(fg = 0\), there are functions \(f_1, g_1 \in C_0(S)\) such that \(f_1g_1 = 0\), \(f_1f = f\) and \(g_1g = g\). We will need the following definition introduced in [8].

**Definition 3.4.** A nonempty subset \(S_0\) of a topological space \(S\) is called a P-set if it is closed and any \(G_\delta\)-set containing \(S_0\) is a neighborhood of \(S_0\). A point \(p \in S\) is called a P-point if \(\{p\}\) is P-set in \(S\).

It has been remarked in [8] that a nonempty subspace \(S_0\) of \(S\) is a P-set if and only if each real continuous function \(f\) on \(S\), vanishing on \(S_0\), must vanish on a neighborhood of \(S_0\). If \(S\) is a locally compact and noncompact space, then \(S\) is substonean if, and only if, the one-point compactification \(S \cup \{\infty\}\) is an F-space and \(\infty\) is a P-point in \(S \cup \{\infty\}\) (cf. [8, Theorem 1]).

**Theorem 3.5.** Let \(A\) be a C*-algebra. The following conditions are equivalent.

(i) \(A\) is 1-separably injective.

(ii) \(A\) is linearly isometric to the Banach space \(C_0(S)\) of complex continuous functions vanishing at infinity on a substonean locally compact Hausdorff space \(S\).

**Proof.** (i) \(\Rightarrow\) (ii). By Lemma 3.1 \(A\) is abelian and hence linearly isometric to the function space \(C_0(S)\) on some locally compact Hausdorff space \(S\). We show that \(S\) is substonean.

Given any function \(f \in C_0(S)\), we define the cozero set of \(f\) to be the set \(\text{coz}(f) = \{x \in S : f(x) \neq 0\}\). Let \(U\) and \(V\) be two disjoint open \(\sigma\)-compact sets in \(S\). We show that they have disjoint compact closures. It has been
observed in [1] p.125] that one can find, via Urysohn’s lemma, two functions $f, g \in C_0(S)$ with $0 \leq f, g \leq 1$ such that $U = \coz(f)$ and $V = \coz(g)$. We note that $fg = 0$ in this case.

Let $h = \chi_{\coz(f)}$ be the characteristic function of $\coz(f)$. Let $Y$ be the closed linear span of $\{f^{1/n}, g^{1/n} : n = 1, 2, \ldots \}$ in $C_0(S)$, and let $Z = Y + \mathbb{C}h \subset L^\infty(S)$. Since $C_0(S)$ is 1-separably injective, the identity map $\iota : Y \to C_0(S)$ admits a norm preserving extension $\widetilde{\iota} : Z \to C_0(S)$. Write $k = \widetilde{\iota}(h) \in C_0(S)$ and note that $\|k\| \leq 1$.

For each $n \in \mathbb{N}$, we have $\|h - 2f^{1/n}\| \leq 1$ and therefore $\|k - 2f^{1/n}\| = \|\widetilde{\iota}(h) - 2\widetilde{\iota}(f^{1/n})\| \leq 1$. In particular, $\|k(x) - 2f^{1/n}(x)\| \leq 1$ for all $n \in \mathbb{N}$ which, together with $\|k(x)\| \leq 1$, implies $k(x) = 1$ for $x \in \coz(f)$. It follows that $kf = f$.

Since $hg = 0$, we have $\|h + e^{i\theta}g\| \leq 1$ for every $\theta \in [0, 2\pi)$ which gives $\|k + e^{i\theta}g\| = \|\widetilde{\iota}(h) + \iota(e^{i\theta}g)\| \leq 1$. Hence for each $x \in \coz(g)$, one has $|k(x) + e^{i\theta}g(x)| \leq 1$ for all $n \in \mathbb{N}$ and $\theta \in [0, 2\pi)$. This implies $k(x) = 0$ for $x \in \coz(g)$ and therefore $kg = 0$.

We have $\coz(f) \subset \{x \in S : k(x) = 1\}$ and $\coz(g) \subset \{x \in S : k(x) = 0\}$, where $\{x \in S : k(x) = 1\}$ is compact and contained in the open set $\{x \in S : |k(x)| > \frac{1}{2}\}$. Applying Urysohn’s lemma to the compact set $\coz(f) \subset \{x \in S : k(x) = 1\}$, one can find a function $f_1 \in C_0(S)$ such that $0 \leq f_1 \leq 1$, $f_1 = 1$ on $\coz(f)$ and $f_1 = 0$ outside $\{x \in S : |k(x)| > \frac{1}{2}\}$.

Considering the function $h' = \chi_{\coz(g)}$ with similar arguments, we can find another function $k' \in C_0(S)$ with $\|k'\| \leq 1$ such that $\coz(g) \subset \{x \in S : k'(x) = 1\}$ and hence $\coz(g)$ is a compact subset of $S$. Since $\{x \in S : |k(x)| < \frac{1}{2}\}$ is an open subset of $S$ containing $\coz(g)$, Urysohn’s lemma again yields a function $g_1 \in C_0(S)$ such that $0 \leq g_1 \leq 1$, $g_1 = 1$ on $\coz(g)$ and $g_1 = 0$ outside $\{x \in S : |k(x)| < \frac{1}{2}\}$. It follows that $f_1g_1 = 0$ and $f_1f = f, g_1g = g$.

It follows that the closures $\overline{U} = \coz(f) \subset \{x \in S : f_1(x) \geq 1/2\}$ and $\overline{V} = \coz(g) \subset \{x \in S : g_1(x) \geq 1/2\}$ are compact and disjoint.

(ii) $\Rightarrow$ (i). Let $A$ be linearly isometric to $C_0(S)$ where $S$ is a substonean locally compact Hausdorff space. We show that $C_0(S)$ is separably injective. This is true if $S$ is compact, as shown in Example 2.3. Let $S$ be noncompact. Since the one-point compactification $S \cup \{\infty\}$ is an $F$-space, $C(S \cup \{\infty\})$ is 1-separably injective, again by Example 2.3.

Let $Y$ be a closed subspace of a separable Banach space $Z$ and let $T : Y \to C_0(S)$ be a bounded linear operator. We identify $C_0(S)$ with the closed subspace $\{u \in C(S \cup \{\infty\}) : u(\infty) = 0\}$ of $C(S \cup \{\infty\})$. Let $\widetilde{T} : Z \to C(S \cup \{\infty\})$ be a norm preserving extension of $T$.

Let $\{f_n : n \in \mathbb{N}\}$ be a countable dense subset of $T(Y) \subset C_0(S)$. Since $S$ is substonean, $\infty$ is a P-point and there exists an open neighbourhood $U_\infty$ of $\infty$ such that $f_n = 0$ on $U_\infty$.

Moreover, the $G_\delta$-set $\bigcap_n U_n$ contains an open neighbourhood $U$ of $\infty$. Hence Urysohn’s lemma again enables us to choose a function $e \in C_0(S)$ with
∥e∥ = 1 such that e = 1 on S \ U and e(∞) = 0. This gives f_n e = f_n for all n \in \mathbb{N}.

It can now be seen readily that the linear map T_e : Z \to C_0(S) defined by

\[ T_e(z) = \tilde{T}(z)e \quad (z \in Z). \]

is a norm preserving extension of T. \qed

We conclude by mentioning some interesting examples of substonean spaces. For any locally compact, σ-compact Hausdorff space S with Stone-Čech compactification βS, the space βS \ S is a compact F-space \[3, 14.27\]. Locally compact substonean spaces include the Rickart spaces \[4\] which are exactly those locally compact spaces S for which C_0(S) is monotone sequentially complete \[4\]. In particular, abelian monotone sequentially complete C*-algebras are 1-separably injective.

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