On the Hawking Turok solution to the Open Universe wave function

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Abstract

Hawking and Turok have recently published a solution to the WKB “wave-function for the universe” which they claim leads in a natural way to an open universe as the end point of the evolution for a universe dominated by a scalar field. They furthermore argue that their solution a preferred solution under the rules of the game. This paper will, I hope, clarify their solution and the limits of validity of their argument.

Hawking and Turok [1] (hereafter HT) recently published a fascinating paper in which they claimed to have a solution of the (complex) Einstein equations which satisfy the rules for a WKB solution for the “wave-function of the universe” as laid down by Hartle and Hawking [4], but which has as the future 3-geometry a spatially open hyperbolic homogeneous universe. Thus this “instanton” solutions allows them to calculate the probability that the universe will form as a homogeneous open universe. Using the anthropic arguments, they also claimed to find a preferred value for the present day Ω, the ratio of density to critical density, of 0.01, rather than the value of 1 as usually expected of inflationary models. This of course is “too small” in comparison with experimental evidence, just as the usual prediction of unity may be “too large” [2]. Linde [3] has also criticized their derivation. However, their derivation was somewhat elliptic and clarification of what they calculated seems desirable. This paper will I hope provide such clarification and will also raise some questions about the interpretation of their solution.

Let us first review the rules [4]. The wave function for the universe \( \Psi(\mathcal{G}) \) is a wave function defined on Euclidean non-singular three universes \( \mathcal{G} \). It is defined (at least formally) by means of a path integral, where the path integral is to be taken over all four geometries, including complex four geometries, which are 4 dimensional manifolds whose only boundary is the three geometry of interest. This path integral prescription is poorly defined, and thus its principle application has been via a semi-classical approximation. One chooses as one’s allowed 4 geometries only those which extremize the Einstein-matter action, and have the given three geometry as their only boundary. One hopes that there are only a few, or one, of these solutions. One then takes as the approximation to the wave-function just \( \sum_j e^{iS_j(\mathcal{G})} \) where \( S_j \) is the action for the \( j^{th} \) 4-geometry solution. The real part of \( S_j \) is then the phase of the wave function, while the imaginary part gives the probability for
that particular component of the wave function. Since there is no known normalisation procedure for these wave-functions, such a probability can best be regarded as a relative probability for evaluating various solutions (and perhaps various three geometries, although the normalisation problems remain a difficult problem).

One issue which will arise in the following is the class of solutions to Einstein’s equations which are allowed in this approximation scheme. Einstein’s equations allow singular solutions. Are those singularities to be removed from the manifold? Do the singularities represent a ”boundary”? What sort of singularities should be allowed to be a part of the acceptable solutions? There are clearly a large number of 4-geometries which will interpolate to the given 3-geometry on the boundary. What are the smoothness conditions which one should place on such 4-geometries? It is certainly true that one can find an extremely large number of WKB ”solutions” to the Einstein equations if one simply excises those regions where the Einstein equations are not obeyed and calls them “singularities”. These solutions can, furthermore, have finite action. Does one demand that the ”singularity” must be a curvature singularity? Would for example a negative mass Schwartzschild solution be an acceptable singularity within such an interpolating 4-geometry? Would the existence of such a negative mass Schwartzschild singularity in the initial three geometry be a valid three geometry for which one wishes to determine the probability? Are there criteria by which one could categorise those classes of singularities which are allowed, and those which are not? I raise these concerns, not in order to give a solution (the simplest solutions is clearly to disallow all singularities), but because they arise in considering the HT solutions. The HT solutions contains singularities not only in the interpolating 4 geometry, but also on the three geometry for which one is calculating the wave-function.

These problems with dealing with singularities are especially difficult as the interpolating 4 geometries need not be real at all. One cannot therefor make arguments in which say the space-like of time-like nature of the singularity plays a crucial role, as these concepts are in general inapplicable to a complex geometry.

Let us now examine the Hawking Turok “instanton” in light of this prescription. They choose as their universe a homogeneous solution of Einstein’s equations.

\[ ds^2 = d\sigma^2 + b(\sigma)^2(d\psi^2 + \sin(\psi)^2 d\Omega^2) \] (1)

where \( d\Omega^2 = d\theta^2 + \sin(\theta)^2 d\phi^2 \), the metric on a sphere with coordinates \( \theta, \phi \). \( b(\sigma) \) is to be taken as the scale factor for a Euclidean solution for a homogeneous universe coupled to a homogeneous scalar field. The scalar field and the metric solutions are chosen so that at \( \sigma = 0 \) the factor \( b(\sigma) = \sigma \). This ensures that the point \( \sigma = 0 \) is a regular point in the Euclidean spacetime. In general, the solution will be such that \( b(\sigma) \) will increase as a function of \( \sigma \) and then decrease again to zero, but this time with a singularity where \( b(\sigma) \) goes to zero.

In particular, the evolution of the spacetime is determined by the constraint equation

\[ 6b(\dot{b})^2 - 6b + b^3 \dot{\phi} + b^3 V(\phi) = 0 \] (2)

and the equation for \( \phi \), the scalar field

\[ \frac{1}{b^3} \partial_\sigma b^3 \partial_\sigma \phi + V'(\phi) = 0 \] (3)
If we demand that the solution at $\sigma = 0$ be regular, we must have $\dot{\phi} = 0$ there. The real solution for $b$ will in general have at least another point, $\sigma = \sigma_1$ where $b$ goes to zero. As HT analyse the solution, this point is in general singular, but they argue, not sufficiently singular to disallow its use as a WKB solution to the equations. $\phi$ diverges to infinity (logarithmically in $\sigma - \sigma_1$), and $b$ goes to zero. As an example, I will examine the (unphysical) case where the potential is a linear potential. The equation for $\phi$ is then

$$\phi = \frac{1}{6} V' \int \frac{1}{b^3} (\int b^3 dt) dt$$

with $b$ satisfying

$$(\dot{b})^2 = 1 - \frac{1}{6} b^2 (-\dot{\phi}^2 + V_0 + V'(\phi))$$

Thus $\dot{\phi}$ diverges as $\frac{1}{b^3}$. Let $b(\sigma) \propto (\sigma_1 - \sigma)^\alpha$, which gives

$$\dot{\phi} = \frac{A}{(\sigma_1 - \sigma)^{3\alpha}} + B + ...$$

and

$$\phi = -\frac{A}{(3\alpha - 1)} (\sigma_1 - \sigma)^{3\alpha - 1} + C$$

Substituting into the equation for $b$ gives

$$D^2 \alpha^2 (\sigma_1 - \sigma)^{2(\alpha - 1)} + 1 - D^2 (\sigma_1 - \sigma)^{2(\alpha)} \left( \frac{A^2}{(\sigma_1 - \sigma)^{6\alpha}} + V'(\frac{-A}{(3\alpha - 1)(\sigma_1 - \sigma)^{3\alpha - 1}}) \right) = 0$$

Assuming that $\alpha$ lies between 0 and 1, the second term in brackets proportional to $V'$ is smaller than the first term derived from $\dot{\phi}^2$ in the constraint equation. (This would be expected to be true of any potential which increased at a rate slower than exponential). Retaining only lowest order terms, we get $\alpha = 1/3$. The derivative of the scalar field and the curvature thus diverge as $\frac{1}{(\sigma_1 - \sigma)}$, and the scalar field diverges logarithmically, in agreement with HT.

Of interest later will be the behaviour of this solution around $\sigma = 0$. Firstly, in order that the solution give a regular geometry at $\sigma = 0$, we must have $b(\sigma) = \sigma$. The above equations are symmetric under the replacement of $b$ with $-b$. The equations are also symmetric in $\sigma$ around $\sigma = 0$. We can therefore choose (and in fact must if the solution is to be analytic) our solution for $b$ to be antisymmetric about $\sigma = 0$, so that $b(-\sigma) = -b(\sigma)$. The Taylor series expansion of $b(\sigma)$ will thus be a series in odd powers of $\sigma$. The solution for $\phi(\sigma)$ will be even in $\sigma$ since $\phi(0)$ is non-zero while $\dot{\phi}(0)$ is zero. Thus a Taylor series expansion of $\phi$ will be an expansion in even powers of $\sigma$.

Now, how do HT use this solution? They first match this Euclidean solution to a Lorentzian solution across the $\psi = \pi/2$ surface. At this surface the extrinsic curvature of the slice is zero, at least everywhere except at $\sigma = \sigma_1$, the singularity. They match this Euclidean solution to the Lorentzian solution

$$ds^2 = d\sigma^2 + b(\sigma)^2 (-d\tau^2 + cosh(\tau)^2 d\Omega^2)$$
at $\tau = 0$ which is also a slice with zero extrinsic curvature, and with the same 3-geometry,

$$ds^2_{(3)} = d\sigma^2 + b(\sigma)^2(d\Omega^2)$$

allowing a matching without the introduction of delta function stresses at the matching surface. The $\sigma = 0$ surface in the above $\sigma, \tau$ coordinates is a null surface, similar to the horizon in Rindler coordinates or the Schwarzschild solution. Define

$$\Sigma = \int_0^\sigma \frac{d\sigma'}{b(\sigma')}$$

and

$$U = e^{-\tau + \Sigma}$$

$$V = -e^{\tau + \Sigma}.$$  

Since near $\sigma = 0$, $b(\sigma)$ is odd with leading term equal to $\sigma$, $\Sigma$ will be of the form of a leading order divergence of $\ln(\sigma)$ plus a series in even powers of $\sigma$. In $U$, $V$ coordinates we have

$$ds^2 = e^{2\Sigma} \left(-dUdV + \frac{1}{4} (U - V)^2 (d\Omega^2) \right)$$

From the expression for $b(\sigma)$ and $\Sigma$ near $\sigma = 0$, the overall multiplier approaches unity as $\sigma$ goes to zero. We can solve for $\sigma$ as a function of $UV$ by noticing that

$$-UV = e^{2\Sigma(\sigma)} = \sigma^2 f(\sigma^2)$$

where $f(0) = 1$ and $f$ has a power series expansion in powers of $\sigma^2$. I will assume that we can invert this equation so that we can write $\sigma^2$ as a power series in $UV$ with leading term of just $UV$. Thus we will have $\sigma(UV) = \sqrt{-UV}(1 + \text{series in } UV))$. The expression $\Sigma(\sigma) - \ln(b(\sigma))$ will be a power series in $UV$ with lowest order term of unity. These arguments are formal, and may fail if the formal power series has zero radius of convergence. I will assume that this does not occur.

Now define

$$F(UV) = e^{2\Sigma - 2\ln(b(\sigma))} = -\frac{UV}{b(\sigma(\zeta))^2}$$

which, by the above arguments, has a power series expansion in $UV$ and has $F(0) = 1$. The metric is thus

$$ds^2 = F(UV)(-dUdV + \frac{1}{4} (U - V)^2 d\Omega^2)$$

By our assumption that the series solution for $F$ has a non-zero radius of convergence, we can continue $F(UV)$ to positive values of $UV$. $F$ will be a real function for real values of $UV$.

Let me now define new coordinates $t, \chi$ by

$$U = e^{T-\chi}$$

$$V = e^{\chi+T}.$$
This gives
\[ ds^2 = F(e^{2T})e^{2T}(-dT^2 + d\chi^2 + \sinh^2(\chi)d\Omega^2) \] (20)
which is just the equation for an open spatially homogeneous universe with scale factor \( e^T \sqrt{F(e^{2T})} \).

Having examined the geometry, we can also examine the solution for the scalar field \( \phi \). As argued above, \( \phi \) has a Taylor series in even powers of \( \sigma \) and thus has a Taylor series in powers of \( UV \). It can thus also be continued in a regular fashion into the \( T, \chi \) region, where it will be a function only of \( T \), and independent of \( \chi \).

Does this now mean that we have a an interpolation 4-geometry for a homogeneous hyperbolic spatial three-geometry \( T = \text{const} \)? This would be true if we could arrange for this hyperbolic spatial geometry to be the only boundary of the four dimensional spacetime given above. Unfortunately, this is not the case. In particular, there is a space-like boundary at infinity where \( \tau \) goes to infinity for any constant \( \sigma \) (ie, in the region where \( UV \) is negative). 

Ie, in order that this four geometry be regardable as the interpolating geometry for some space-like 3-geometry, we must find such a 3-geometry which can be chosen as the only boundary of this four geometry. To do so the space-like surface will have to penetrate the horizon and enter the region in which the \( \tau, \sigma \) coordinates are defined, and furthermore, it will have to include the singularity at \( \sigma = \sigma_1 \).

Figure 1 shows this solution in \( U, V \) coordinates. The dotted line at \( U=V \) corresponds to \( \chi = 0 \) the origin of the spherically symmetric \( \chi, \theta, \varphi \) coordinate system. The dashed line at \( UV = -e^{\Sigma_1} \) is the singularity where \( \phi \) and \( \dot{\phi} \) go to infinity. I have drawn in a space-like hypersurface which for much of it is a part of the hyperbolic spatially homogeneous geometry. It then ceases to follow that geometry and enters the \( \tau, \sigma \) region finally ending of the singularity. The 4-geometry presented in HT is a valid interpolating geometry for this spatial hypersurface (if we neglect the problem of the singularity). Under the rules of the “wave function of the universe” prescription, this 4-geometry can be used to calculate the probability of this three geometry occurring by looking at the imaginary part of the action, as HT do.

However, this 3-geometry, together with the interpolating 4-geometry, is a singular geometry. At \( \sigma = \sigma_1 \ (UV = \Sigma_1) \), the scalar field and the curvature diverge. Note that this time-like singularity does not engulf the whole spacetime, as it does in the Vilenkin model with a similar naked time-like singularity in an asymptotically flat space-time [3].
The global structure of the Hawking Turok Space-time

This analysis thus answers a frequently asked question regarding the HT solution, namely how can a finite compact Euclidean instanton create an infinite open universe? The answer is that it doesn’t. The instanton creates a finite bounded universe, of which a part is a part of a homogeneous hyperbolic geometry. Note that one can make the open universe section as large a part of the spatial section as one desires by allowing the curve representing the three geometry to follow the open universe for as long as one wishes. Eventually however one must close off the universe with the singularity.

The singularity is a time-like attractive singularity (i.e., geodesics fall toward and are not repelled away from the singularity) as we can see by looking at the path $\sigma, \theta, \varphi$ all constant, which corresponds to the particle remaining a constant proper distance from the singularity (even though the circumferential distance is increasing exponentially). The unit tangent vector is

$$v^\mu = [0, 1/b(\sigma), 0, 0].$$

Near $\sigma = \sigma_1$, the solution for $b$ is $b(\sigma) = A(\sigma_1 - \sigma)^{\frac{3}{2}}$ where $A$ is some constant. The acceleration vector for this path is

$$\frac{Dv^a}{Ds} = \left[ \frac{A^2}{3(\sigma_1 - \sigma)}, 0, 0, 0 \right]$$

which is an outward (away from the singularity) directed acceleration. I.e., in order to keep the particle at a constant $\sigma$ (constant proper distance from the singularity) a force directed away from the singularity must be applied to the particle. A geodesic will thus tend to decrease its proper distance from the singularity, which is why I call it attractive. The singularity is also spherically symmetric, and $\tau$-time dependent, although the scalar field is $\tau$-time independent.

The question one must now raise is whether or not this solution is actually what is wanted? Does one want a solution which singular? Does one want a 3-geometry which
is singular? Is the probability that for producing an open universe bubble, or that for producing the singularity, and how would one answer this question?

In short, HT calculate the probability for forming a closed universe which contains a part of a hyperbolic spatially homogeneous universe, and also contains a naked, time-like, attractive singularity. Whether or not this thus corresponds to the creation of a closed universe is open to question.

The chief problem with time-like singularities is that they have no boundary conditions on the singularity which disallows matter from streaming out of the singularity. Let \( \Phi \) be some massless conformally invariant scalar field (not related to \( \phi \), the inflaton field). The metric is conformally flat metric (as can be easily seen if we define \( t = (U + V)/2 \) and \( r = (V - U)/2 \). The equation for \( \Phi \) in the \( U, V \) coordinates for a spherically symmetric solution is

\[
\partial_U((U - V)^2 \partial_V(\sqrt{F}\Phi)) + \partial_V((U - V)^2 \partial_U(\sqrt{F}\Phi)) = 0 \tag{23}
\]

(The \( \sqrt{F} \) term comes through the conformal transformation of \( \Phi \) and the metric to the flat metric.) We can now choose initial conditions on a \( U + V = \text{const} \) hypersurface such that \( \sqrt{F}\Phi \) and \( \partial_{U+V}\sqrt{F}\Phi \) are zero for all \( U < U_i \) on this surface. Since the characteristics of this equation are just the \( U \) or \( V \) constant lines, the complete solution will be zero for \( U < U_i \) everywhere, (at least until the \( U = U_i \) line hits the \( U = V \) origin of the spherical symmetry).

Furthermore, we note that

\[
(UV) = e^{2\Sigma} = e_1^\Sigma(1 + C(\sigma_1 - \sigma)^{2/3} + ...) \tag{24}
\]

or

\[
(UV) - e^{2\Sigma_1} \propto (\sigma_1 - \sigma)^{2/3} \tag{25}
\]

Thus

\[
F(UV) = UV/b(\sigma^2) \propto \frac{e^{2\Sigma_1}}{(UV - e^{2\Sigma_1})} \tag{26}
\]

Thus, since the function \( \sqrt{F}\Phi \) can be chosen to be regular near the singularity, \( \Phi \) will go to zero at the singularity as \( \sqrt{UV - e^{2\Sigma_1}} \). I.e., the solution will be such that the the wave \( \Phi \) is emitted by the singularity, and that value of \( \Phi \) is zero at the time-like singularity (although its derivative does diverge). This of course does not take into account the back reaction of this wave on the geometry of the singularity, but it is difficult to see how that would rule out such solutions. I.e, the time-like singularity can act as the source for matter streaming out into the universe.

In conclusion, the HT solution is not the calculation of the wave-function at a homogeneous open universe, but rather of a universe with an arbitrarily large section of such an open universe, but then inhomogeneously closed off by a naked time-like singularity. Whether or not such a solution is a valid one for the wave function of the universe is an open question.
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