Research on the Sampling Procedures for Inspection by Variables When Quality Characteristics Follow the Chi-Square Distribution

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Abstract. In this paper sampling procedures for inspection by variables are discussed under the assumption that product quality characteristics are distributed by chi-square distribution. In the sampling procedure, due to the randomness of sampling, the judgment result may be inconsistent with the actual situation of product quality. That is, when the actual quality of the product is qualified, it is judged as unacceptable (at this time, a type of error in hypothesis testing is made, and we control it to make the probability of making a type of error less than or equal to 0.05). When the actual quality of the product is unqualified, it may be judged as acceptable (at this time, another type of error in hypothesis testing is made, and we control it to let the probability of making a type of error less than or equal to 0.1). In this paper the idea of nonlinear programming is adopted to determine the sampling procedure.

1. Introduction

It is well-known that most of the sampling procedures for inspection by variables are designed under the assumption that product quality characteristics are normal distribution, such as ISO 3951-4 [1], GB/T 6378.4-2008[2] and so on. ISO 3951-4 and GB/T 6378.4-2008 are both procedures for assessment of declared quality levels. The difference lies in that ISO 3951-4 is based on non-conforming rate of product and GB/T 6378.4-2008 is based on product quality mean. However, in real application, there are some cases that some of product quality characteristics don’t follow normal distribution. In this paper sampling procedures for inspection by variables are discussed under the assumption that product quality characteristics follow chi-square distribution. For illustration, we mainly discuss the upper specification limit case. Other cases are similar so we omitted it here.

2. Method

First we define some symbols. Let product quality characteristic is \(X\); product quality characteristic mean is \(\mu\); \(\alpha\) is type-one error; \(\beta\) is type-two error [3]. Assume \(X \sim \chi^2(\mu)\). \(\mu_0\) is the bound for good quality, \(\mu_1\) is the bound for bad quality. For the upper specification limit case, we hope that when product quality population mean \(\mu \leq \mu_0\), we accept the lot with the higher probability (not lower than \(1 - \alpha\)).
When product quality population mean $\mu \geq \mu_1$, we accept the lot with the lower probability (not higher than $\beta$). $\alpha$ is also the risk of supplier; $\beta$ is also the risk of purchaser. In other words, for given $\mu_0, \mu_1, \alpha, \beta$, we hope

$$
\begin{cases}
P(\mu) \geq 1 - \alpha, \mu \leq \mu_0 \\
P(\mu) \leq \beta, \mu \geq \mu_1
\end{cases}
$$

(1)

Where, $\mu_0 < \mu_1$, $P(\mu)$ is the probability of acceptance when product quality mean is $\mu$.

Here, we use sample mean $\bar{x}$ as the measure of acceptance criteria in our procedure. For the upper specification limit, acceptance criteria is $\bar{x} \leq \lambda$ and sampling plan is $(n, \lambda)$, where $\lambda$ is some constant. Figure 1 is the graphical presentation for two chi-square distribution with this sampling plan.

Assume $X_1, X_2, \ldots, X_m$ are independent and identically distributed as $X$. Because $X \sim \chi^2(\mu)$, according to the additivity of chi-square distribution, $Y = \sum_{i=1}^{m} X_i = m\bar{x} \sim \chi^2(m\mu)[4,5]$. For sampling plan $(m, \lambda)$, the probability of acceptance is

$$
P(\mu) = P(\bar{x} \leq \lambda) = P(\gamma \leq m\lambda)
$$

That is

$$
P(\mu) = \int_{0}^{m\lambda} \frac{1}{\Gamma\left(\frac{m\mu}{2}\right)2^{\frac{m\mu}{2}}} y^{\frac{m\mu}{2}-1} e^{-\frac{y}{2}} dy
$$

(2)

It is well known that when we implement the sampling test, the type one error is $P(\gamma > m\lambda | \mu \leq \mu_0)$ and the type two error is $P(\gamma \leq m\lambda | \mu \geq \mu_1)$.

Because $P(\gamma > m\lambda | \mu \leq \mu_0)$ is increasing function of $\mu$, so we have

$$
P(\gamma > m\lambda | \mu \leq \mu_0) \leq P(\gamma > m\lambda | \mu = \mu_0)
$$

(3)

So, when $\mu \leq \mu_0$, we only let $P(\gamma > m\lambda | \mu = \mu_0) \leq \alpha$ and then $P(\mu) \geq 1 - \alpha$ in formula (1) can be satisfied.

In the same way, because $P(\gamma \leq m\lambda | \mu \geq \mu_1)$ is decreasing function of $\mu$, so we have

$$
P(\gamma \leq m\lambda | \mu \geq \mu_1) \leq P(\gamma \leq m\lambda | \mu = \mu_1)
$$

(4)
So, when $\mu \geq \mu_1$, we only let $P(y \leq m\lambda|\mu = \mu_1) \leq \beta$ and then $P(\mu) \leq \beta$ in formula (1) can be satisfied.

With the above analysis, (1) can be expressed by

\[
\begin{align*}
&\{P(y > m\lambda|\mu = \mu_0) \leq \alpha \} \\
&\{P(y \leq m\lambda|\mu = \mu_1) \leq \beta \}
\end{align*}
\]

(5)

Combine with (2), (5) can be transformed as

\[
\begin{align*}
\int_0^{m\lambda} &\frac{1}{\gamma(\frac{m\mu_0}{2})\frac{m\mu_0}{2}} y^{\frac{m\mu_0}{2}-1} e^{-\frac{y}{2}} dy \geq 1 - \alpha \\
\int_0^{m\lambda} &\frac{1}{\gamma(\frac{m\mu_1}{2})\frac{m\mu_1}{2}} y^{\frac{m\mu_1}{2}-1} e^{-\frac{y}{2}} dy \leq \beta
\end{align*}
\]

(6)

3. Sampling plan

In this paper we use "$\lambda$"-type sampling plan ($m, \lambda$). $m$ is sample size, $\lambda$ is acceptable constant. The key point in this sampling plan is determining the value of $m$ and $\lambda$, which is determined by equation (6). However, we can’t solve the equation directly because it doesn’t have the explicit solution. So here we first obtain the range of $m$ and $\lambda$ according to (6). Then with the help of non-linear programming method we get the smaller value of $m$ and $\lambda$ in that range. As the figure 2 shows the shadow area is the value range of $m$ and $\lambda$. The black point in this figure is the smallest value of $m$ and $\lambda$ in that range.

Let $\alpha_0$ be the probability of falsely contradicting conforming products. Then we have

\[
\alpha_0 = 1 - \int_0^{m\lambda} \frac{1}{\gamma(\frac{m\mu_0}{2})\frac{m\mu_0}{2}} y^{\frac{m\mu_0}{2}-1} e^{-\frac{y}{2}} dy
\]

(7)

The value of $\alpha_0$ is not more than $\alpha$ (i.e. $\alpha_0 \leq 0.05$).
4. The relationship between nonconforming rate $p$ and mean $\mu$

The nonconforming rate for upper specification limit $U$ is

$$p = P(X > U) = 1 - P(X \leq U) = 1 - X_{\mu}^2(U) \quad (8)$$

Where $X_{\mu}^2(U)$ is the cumulative function of $x^2(\mu)$.

When $p$ is known, we can get $\mu$ from (8). When $\mu = \mu_0, p_0 = 1 - X_{\mu_0}^2(U)$; when $\mu = \mu_1, p_1 = 1 - X_{\mu_1}^2(U)$.

5. Real example

The senior management of a bank in a commercial street has declared that the percent ratio of cashier services to more than 5 minutes is no more than 4% [6]. The manager of a bank branch concerned that one of his tellers who are in the probationary period provides a service that is too slow. It has been found from past experience that the service time approximately follows a chi-square distribution. The manager wants to obtain objective evidence that the teller’s service was incompetent, so he investigates a study of her service time to determine if he would be correct to declare that no more than 4% of the service time was more than 5 minutes. We use our method to test.

There are three LQR level in ISO 3951-4 which corresponds to three quality level, which are level I, level II and level III. Now there are 1000 pieces of service time of this teller. Here we use our method to determine $(m, \lambda)$.

Figure 3, figure 4 and figure 5 demonstrate the results of three levels. The abscissa is the sample size $m$, and the ordinate is the accepting constant $\lambda$.

![Figure 3: Range of $(m, \lambda)$ at LQR level I](image)

Note: The position of the black triangle is $(4.2, 2.9)$. 
Figure 4. Range of \((m, \lambda)\) at LQR level II

Note: The position of the black triangle is \((8,2.4)\).

Figure 5. Range of \((m, \lambda)\) at LQR level III

Note: The position of the black triangle is \((9,2.35)\).

Table 1 is the sampling plan in three levels with the help of non-linear programming method and formula (6).
Table 1. Sampling plan in three levels

| LQR level | LQR level I | LQR level II | LQR level III |
|-----------|-------------|--------------|---------------|
| m         | 4           | 8            | 9             |
| λ         | 2.92        | 2.4          | 2.35          |

Now we use the above sampling plan to sample and then conduct the sampling test. In level I we get four samples and the observed values are 6.262899, 7.009562, 4.566664 and 1.501663 (unit is minute). The sample mean $\bar{x}$ is 4.83519, which is larger than $\lambda = 2.92$. So we can conclude the cashier's service is incompetent.

In level II we sample eight times and the observed values of service time are 4.191104, 1.913367, 5.485289, 4.040773, 8.066646, 5.736418, 14.790635 and 4.594166 (unit is minute). The sample mean $\bar{x}$ is 6.1023, which is larger than $\lambda = 2.4$. So we can conclude the cashier's service is incompetent.

In level III we sample nine times and the observed values of service time are 3.955233, 5.499315, 3.174358, 2.473986, 6.292779, 4.220481, 2.813604, 2.375127 and 4.931183 (unit is minute). The sample mean $\bar{x}$ is 3.970674, which is larger than $\lambda = 2.35$. So we can conclude the cashier's service is incompetent.

Based on the above analysis, we found that the cashier service was judged incompetent at all three levels.

At the same time, we can obtain the probability of falsely contradicting conforming products $\alpha_0$, see Table 2.

Table 2. The value of $\alpha_0$

| Level     | $m$  | $\bar{x}$ | $\lambda$ | $\alpha_0$   |
|-----------|------|-----------|-----------|--------------|
| Level I   | 4    | 4.835197  | 2.92      | 0.0471143    |
| Level II  | 8    | 6.1023    | 2.4       | 0.0487198    |
| Level III | 9    | 3.970674  | 2.35      | 0.04658494   |

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