A LAGRANGIAN FORMULATION OF

2-DIMENSIONAL TOPOLOGICAL GRAVITY

AND ČECH-DE RHAM COHOMOLOGY

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ABSTRACT

We present a very simplified analysis of how one can overcome the Gribov problem in a non-abelian gauge theory. Our formulae, albeit quite simplified, show that possible breakdowns of the Slavnov-Taylor identity could in principle come from singularities in space of gauge orbits. To test these ideas we exhibit the calculation of a very simple correlation function of 2-dimensional topological gravity and we show how in this model the singularities of the moduli space induce a breakdown of the Slavnov-Taylor identity. We comment on the technical relevance of the possibility of including the singularities into a finite number of cells of the moduli space.

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1. Introduction

The aim of this report is to discuss to what extent a functional integral approach to 2-dimensional topological gravity [1] is relevant to the study of gauge theories in the presence of Gribov horizons [2]. This is the situation in which the Faddeev-Popov determinant is not positive definite and the corresponding measure becomes singular.

Due to the Gribov phenomenon the Feynman functional integral of a gauge theory cannot be defined as a unique integral over a single connected domain. In a general situation [3], if the gauge freedom has been fixed according to the Landau background gauge prescription, one has to choose a suitable set of background fields \( \{V^i\} \) and a corresponding set of cells, whose characteristic functions are \( \{\chi_i(A)\} \), covering the whole field functional space and such that for some suitable, positive, \( \{a^i\} \):

\[
\chi_i(A) \equiv \chi_i(A) \Theta \left[ a^i - \int d^4x \sum_\mu (V^i_\mu - A_\mu)^2 \right],
\]

where \( \Theta \) is the Heavyside function.

In the \( i^{th} \) cell the gauge fixing condition is:

\[
\nabla_{V^i, \mu} (A - V^i)_\mu = 0,
\]

where the sum over the repeated indices is understood and:

\[
\nabla_{V, \mu} \equiv \partial_\mu - g V^{i}_\mu \wedge,
\]

is the covariant derivative corresponding to the connection \( V \) and the wedge product \( \wedge \) is defined by the gauge group structure constants.

Then the question comes on how should one define the Feynman functional \( Z \). Naively, for any cell, one would construct the action \( S_i \) given by:

\[
S_i \equiv S^{(inv)} + s \int d^4x \left( \bar{c} \cdot \nabla_{V^i, \mu} (A - V^i)_\mu \right),
\]

where \( S^{(inv)} \) is gauge invariant and independent of \( V^i \) and \( s \) is the usual BRS operator acting only on the quantized fields, and one would write:

\[
Z \equiv \sum_i \int d[\Phi] \chi_i e^{-S_i},
\]

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However in general this expression is sick since, for two neighbouring cells $i$ and $j$:

$$e^{-S_i} \neq e^{-S_j},$$

and therefore $Z$ is affected by small deformations of the boundaries of the cells. Indeed, introducing the interpolating action:

$$S_{ij} \equiv t S_i + (1-t) S_j ,$$

one has:

$$e^{-S_j} - e^{-S_i} = s \left[ \int d^4x (V^i - V^j) \mu(x) \cdot \nabla A_{\mu} \bar{c}(x) \int_0^1 dt \ e^{-S_{ij}(t)} \right] .$$  \hspace{1cm} (1)

It is also apparent that the naive definition of the functional integral violates the Slavnov-Taylor identity. Let us remember that this identity is the faithful translation of the gauge invariance of the theory in terms of the Green functional generator. It asserts the vanishing of the expectation value of any $s$-exact operator, that is of any operator of the form $sX$. This vanishing follows from the possibility of performing functional integrations by parts disregarding boundary contributions. This is however not possible in the present situation.

If the whole functional space were covered by only two cells (1 and 2), and hence:

$$\chi_1 + \chi_2 = 1 ,$$

Eq.(1) would suggest the alternative definition:

$$Z \equiv \sum_{i=1}^{2} \int d[\Phi] \chi_i e^{-S_i}$$

$$- \int d[\Phi] s \chi_1 \int d^4x (V^1 - V^2) \mu(x) \cdot \nabla A_{\mu} \bar{c}(x) \int_0^1 dt \ e^{-S_{12}(t)}$$

$$= \sum_{i=1}^{2} \int d[\Phi] \chi_i e^{-S_i}$$

$$- \int d[\Phi] \int d^4y \nabla A_{\mu} \bar{c}(y) \cdot \frac{\delta \chi_1}{\delta A_{\mu}(y)} \int d^4x (V^1 - V^2) \mu(x) \cdot \nabla A_{\mu} \bar{c}(x) \int_0^1 dt e^{-S_{12}(t)} ,$$

that is not affected anymore by the above mentioned sickness.

It is rather clear that the second term in the right-hand side of this formula does not vanish except for very special choices of the observables that appear in the invariant part.
of the action. Therefore the last term in the right-hand side of Eq.(2) gives a relevant contribution to the definition of the Feynman vacuum functional.

It is also easy to see that the Slavnov-Taylor identity is recovered by the introduction of the boundary term. Indeed owing to Eq.(2) we get:

\[ <sX> = \frac{1}{Z} \left[ \sum_{i=1}^{2} \int d[\Phi] \chi_i e^{-S_i} sX \right. \]

\[ - \int d[\Phi] s\chi_1 \int d^4x \left( V^1 - V^2 \right)_{\mu}(x) \cdot \nabla_{A,\mu} \bar{c}(x) \int_0^1 dt \ e^{-S_{12}(t)} sX \] ,

We can compute the numerator in the right-hand side of this equation using the identity:

\[ s\chi_1 + s\chi_2 = 0 \] and integrating by parts, getting:

\[ Z <sX> = - \sum_{i=1}^{2} \int d[\Phi] s\chi_i e^{-S_i} X \]

\[ - \int d[\Phi] s\chi_1 \left[ \int d^4x \left( V^1 - V^j \right)_{\mu}(x) \cdot \nabla_{A,\mu} \bar{c}(x) \int_0^1 dt \ e^{-S_{ij}(t)} \right] X \]

\[ = \int d[\Phi] s\chi_1 \left[ e^{-S_2} - e^{-S_1} \right] X \]

\[ - \int d[\Phi] s\chi_1 \left[ \int d^4x \left( V^1 - V^j \right)_{\mu}(x) \cdot \nabla_{A,\mu} \bar{c}(x) \int_0^1 dt \ e^{-S_{ij}(t)} \right] X = 0 .\]

The natural question is now how Eq.(2) generalises to the case of a more complex covering, an answer to this question is known in the case in which the covering involves only a finite number of cells. Indeed in this case the natural generalisation of Eq.(2) is given by the construction of a Čech-de Rham cocycle. In the case in which the covering requires an infinite number of cells to our knowledge the generalisation of Eq.(2) is not known. We imagine that in the general framework involving local observables the number of cells is infinite and hence the whole construction becomes singular.

The next sections of this report present in some details the application of this construction to the case of two dimensional topological gravity. The situation in this model is different from the gauge theory case, since, strictly speaking, in the topological case any field configuration lies on the Gribov horizon that has carefully been avoided in the gauge case. However, by means of a second Faddeev-Popov procedure that leads to the dynamical interpretation of the moduli of the Riemann surface playing the role of the world-sheet of the model, one is led to a problem that, although finite dimensional, presents interesting analogies with the gauge case. In particular the configuration space, that is the moduli
space, is topologically non-trivial. The Feynman integral over it can be computed using a finite decomposition in cells that however contain singular points of the moduli space. We shall see that these singular points are responsible for the non-trivial properties of the model that originate from a breakdown of the Slavnov-Taylor identity.

The presence of worse singularities in the infinite dimensional situation leaves open the possibility of analogous phenomena.

2. Two-dimensional topological gravity

Two dimensional topological gravity [4],[5] turns out to be a particularly interesting toy model to study the role of Gribov horizons and in particular of the singularities of the moduli space, which is the space of the physically relevant configurations.

We shall see in a moment that in our model the Faddeev-Popov determinant vanishes all over the gauge orbit space, and hence the Gribov horizon coincides with the space of moduli. Therefore a second Faddeev-Popov procedure is needed in order to define a non-degenerate functional measure and for this the action of BRS operator has to be extended to the global quantum mechanical variables that coincide with the moduli and with their supersymmetric partners, the supermoduli.

The new Faddeev-Popov measure, including both the local fields and the global quantum mechanical variables, is generically non-degenerate. The Gribov horizon associated with it has codimension one on moduli space. This implies that the functional integral defines correlators of observables which are local closed top-forms on the moduli space. In order that the correlators have physical meaning, however, such locally defined forms must be local restrictions of forms which are globally defined.

In our case, functional averages of BRS closed operators are not in general globally defined. We will demonstrate this by deriving the Ward identities associated to finite reparametrizations of the background gauge. This phenomenon is originated by the dependence of the observables on derivatives of the super-ghost field.

We shall see that even if functional averages of BRS closed operators are not globally defined, it is still possible to associate to them globally defined forms by resorting to the Čech-De Rham cohomology [6] if the operators correspond to elements of the equivariant BRS cohomology [7].

In a previous paper we have derived from our “anomalous” Ward identities a chain of descendant identities defining a local cocycle of the Čech-De Rham complex of the moduli
space. A well-known construction of cohomology theory leads from this local cocycle to a globally defined form. The integral of the globally defined form receives contributions not only from the original local top-form (which vanishes in the superconformal gauge), but also from the tower of local forms of lower degree that solve the chain of Ward identities.

A very important point that emerges from our analysis is that in the great majority of situations the correlators, when computed in the superconformal gauge, receive non-trivial contributions only from the singularities of moduli space. Indeed, these singularities, which correspond to nodes of the punctured Riemann surface that is identified with the world-sheet of our model, lie generally at the interior of Gribov domains. The correlators turn out to vanish, and hence to be locally integrable, in the neighbourhoods of the singularities of moduli space. These singularities, however, determine the lack of global definition of the local correlators which in fact turn out to correspond to non-trivial elements of Čech-De Rham cohomology.

We now come to the description of the model. Being a field theory model it involves a set of local variables that are identified with the two-dimensional metric $g_{\mu\nu}$, the gravitino field $\psi_{\mu\nu}$, the ghost field $c^\mu$ associated with the local gauge degrees of freedom, that is with diffeomorphisms, and its superpartner $\gamma^\mu$. The theory is defined by its invariance under the nilpotent BRS transformations [8],[9],[10]:

$$s g_{\mu\nu} = \mathcal{L}_c g_{\mu\nu} + \psi_{\mu\nu}$$
$$s c^\mu = \frac{1}{2} \mathcal{L}_c c^\mu + \gamma^\mu$$
$$s \psi_{\mu\nu} = \mathcal{L}_c \psi_{\mu\nu} - \mathcal{L}_c g_{\mu\nu}$$
$$s \gamma^\mu = \mathcal{L}_c \gamma^\mu$$

Notice that the presence of the gravitino and of the superghost field makes the cohomology of the $s$-operator trivial and hence determines the topological nature of the model.

It is convenient to decompose the metric into the corresponding reduced metric (complex structure) and the Liouville field according:

$$g_{\mu\nu}(x) \equiv \sqrt{g} g_{\mu\nu}(x) \equiv e^\varphi \hat{g}_{\mu\nu}(x)$$

with:

$$\det(\hat{g})_{\mu\nu} = 1$$

We also introduce the traceless gravitino field:

$$\hat{\psi}^{\mu\nu} \equiv \sqrt{g} (\psi^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \psi^\sigma)$$
The conformal background gauge prescription corresponds to the conditions:

\[ g_{\mu\nu}(x) = \eta_{\mu\nu}(x;m) \rightarrow \hat{g}_{\mu\nu} = \hat{\eta}_{\mu\nu}, \quad \varphi = \bar{\varphi} \]

\[ \hat{\psi}_{\mu\nu} = \hat{f}_{\mu\nu} \]

that are implemented under the functional integral by a system of Lagrange multipliers and antighost fields, whose BRS transformations are given by:

\[ sb_{\mu\nu} = \Lambda_{\mu\nu} \quad s\Lambda_{\mu\nu} = 0 \]

\[ s\beta_{\mu\nu} = L_{\mu\nu} \quad sL_{\mu\nu} = 0 \]

\[ s\hat{\chi} = \mathcal{L}_c\chi + \pi \quad s\pi = \mathcal{L}_c\pi - \mathcal{L}_\gamma\chi \]

Due to the above mentioned triviality of the \( s \)-cohomology, the Lagrangian of the model coincides with the gauge fixing term [10],[11]:

\[ \mathcal{L} = s\left[ \frac{1}{2}b_{\mu\nu}(\hat{g}^{\mu\nu} - \hat{\eta}^{\mu\nu}) + \frac{1}{2}\beta_{\mu\nu}(\hat{\psi}^{\mu\nu} - \hat{f}^{\mu\nu}) + \chi\partial_{\mu}(\hat{g}^{\mu\nu}\partial_{\nu}(\varphi - \bar{\varphi})) \right] \]

This completes the local description of our theory.

Concerning its global properties, we identify the world-sheet with a punctured Riemann surface and we limit our study to the observables that correspond to local operators sitting on fixed points of the surface. It is obvious that, the theory being a gauge theory, these operators have to be \( s \)-closed.

It is well known that if the surface has genus \( g \) and contains \( n \) fixed points ( we take \( 3g - 3 + n > 0 \) ) our Lagrangian is degenerate since the field \( b \) has \( 3g - 3 + n > 0 \) zero-modes. Indeed this is the number of independent deformations of the background metric that do not correspond to coordinate transformations. Let :

\[ \int d^2x b_{\mu\nu}\hat{\eta}_{i\mu\nu} \quad \text{for} \quad i = 1, \ldots, 3g - 3 + n \]

define these zero modes, the \( \hat{\eta}_{i\mu\nu} \)'s are interpreted as derivatives of the background complex structure with respect to the moduli of our theory. Fixing these zero modes automatically introduces the moduli into the dynamics of our theory.

We also introduce a set of Grassmann variables \( \{p^i\} \), that we call supermoduli and that we assume to transform as 1-forms under moduli space reparametrizations; we choose the gravitino background as follows

\[ \hat{f}^{\mu\nu} = d_p \hat{\eta}^{\mu\nu} \quad d_p \equiv p^i \frac{\partial}{\partial m^i} . \]
In order to fix the $b$-field zero modes we have to introduce a further set of Lagrange multipliers that are not local field but global variables. Then, to keep the original $s$-exact structure of the Lagrangian, we extend the definition of the coboundary operator $s$ on these new parameters \[12] , \[11] :

\[
s m^i = C^i \quad s C^i = 0
\]
\[
s p^i = - \Gamma^i \quad s \Gamma^i = 0
\]

With the new definition of the coboundary operator the Lagrangian becomes \[11] :

\[
\mathcal{L} = \frac{1}{2} \Lambda_{\mu \nu} (\hat{g}^{\mu \nu} - \hat{\eta}^{\mu \nu}) + \frac{1}{2} L_{\mu \nu} (\hat{\psi}^{\mu \nu} - d_p \hat{\eta}^{\mu \nu}) - \frac{1}{2} b_{\mu \nu} \mathcal{L}_c \hat{g}^{\mu \nu} - \frac{1}{2} \beta_{\mu \nu} \mathcal{L}_c \hat{g}^{\mu \nu}
\]
\[
+ \frac{1}{2} \hat{\psi}^{\mu \nu} [(\mathcal{L}_c \beta)_{\mu \nu} + b_{\mu \nu} + 2 \partial_\mu \chi \partial_\nu (\varphi - \bar{\varphi})] + \pi \partial_\mu (\hat{g}^{\mu \nu} \partial_\nu (\varphi - \bar{\varphi})) - \chi \partial_\mu (\hat{g}^{\mu \nu} \partial_\nu \psi')
\]
\[
+ \frac{1}{2} \beta_{\mu \nu} d_\Gamma \hat{g}^{\mu \nu} + \frac{1}{2} b_{\mu \nu} d_\Gamma \hat{\eta}^{\mu \nu} + \frac{1}{2} \beta_{\mu \nu} d_\Gamma \hat{\eta}^{\mu \nu} + \chi \partial_\mu (\hat{g}^{\mu \nu} \partial_\nu d_\Gamma \bar{\varphi}),
\]

\tag{2.1}

with:

\[
\psi' \equiv \bar{D}_\sigma c^\sigma + \frac{1}{2} \psi^\sigma.
\]

It is apparent that the eighth term in the right-hand side of (2.1) fixes the zero modes of the $b$ field.

Having so specified a non-degenerate action for our model we come to the selection of observables. It is clear that the usual prescription that the observables be elements of the $s$-cohomology does not work here since this cohomology is empty; we have therefore to enlarge our study and consider $s$-exact operators. Furthermore we consider a natural assumption that observables be independent of the Lagrange multipliers. That is:

\[
\frac{\delta \Omega}{\delta \Lambda_{\mu \nu}} = \frac{\delta \Omega}{\delta L_{\mu \nu}} = 0.
\]

\tag{2.2}

In the standard situation this hypothesis is not necessary since (2.2) is automatically verified within the $s$-cohomology elements, here (2.2) is assumed since otherwise the observables would affect the gauge fixing. We assume furthermore that our observable algebra be generated by local operators sitting on the punctures of the world-sheet. Therefore a generic element of it is a linear combination of operators of the form:

\[
\Omega = \prod_k O_k (P_{i_k})
\]

\tag{2.3}
Under the above assumptions one can directly substitute $\hat{g}^{\mu\nu} \rightarrow \hat{\eta}^{\mu\nu}$ and $\hat{\psi}^{\mu\nu} \rightarrow d_p \hat{\eta}^{\mu\nu}$ into the Lagrangian, that becomes:

$$
L' = \frac{1}{2} \left[ -b_{\mu\nu} L_c \hat{\eta}^{\mu\nu} - \beta_{\mu\nu} L_c \hat{\eta}^{\mu\nu} + d_p \hat{\eta}^{\mu\nu} (L_c \beta)_{\mu\nu} \\
+ b_{\mu\nu} (d_C \hat{\eta}^{\mu\nu} - d_p \hat{\eta}^{\mu\nu}) + \beta_{\mu\nu} (d_C \hat{\eta}^{\mu\nu} + \beta_{\mu\nu} d_p d_C \hat{\eta}^{\mu\nu}) \right].
$$

(2.4)

If, furthermore, the observables are independent of antighost fields (a sufficient, weaker hypothesis would exclude only the antighost field zero modes):

$$
\frac{\delta \Omega}{\delta b_{\mu\nu}} = \frac{\delta \Omega}{\delta \beta_{\mu\nu}} = 0,
$$

the last term in the right-hand side of (2.4) does not contribute to the Feynman functional integral.

Let us now consider the Feynman integral of our model. The correlation functions should be computed by performing the functional integration over the local fields $g, \psi, c, \gamma, b, \beta, L, \Lambda$, that we collectively label by $\Phi$. However, the presence of zero modes requires that we also integrate over the Lagrange multipliers $C$ and $\Gamma$. With this definition of the Feynman integral the correlators are functions of the moduli $m$ and of the supermoduli $p$. Remembering that the coboundary operator does not leave invariant these variables it is natural to include them among the set of the integration variables; for a moment, however, it is interesting to refrain from doing this.

It is a natural question whether the Slavnov-Taylor identities, that in the standard situation insure the vanishing of the expectation values of $s$-exact operators, hold true in our case. We therefore consider a generic $b, \beta, \Lambda, L, m, p, C$, and $\Gamma$ independent operator that we call admissible and we notice first of all that, if $\Omega$ is admissible:

$$
\int [d\Phi] e^{-S(\Phi;m^i,p^i)} \Omega = \prod_i \delta(C_i - p^i) \prod_j \delta(\Gamma^j) \langle \Omega \rangle_{m,p}.
$$

(2.5)

Indeed the $\delta$ functions are generated by the integral over the $b$ and $\beta$ zero modes.

Now we come to the Slavnov-Taylor identities considering a $s$-exact admissible operator $sX$ and we get:

$$
\langle sX(\Phi) \rangle_{m,p} \equiv \int \prod_i dC_i d\Gamma^i \int [d\Phi] e^{-S(\Phi;m^i,p^i)} sX
$$

$$
= \int \prod_i dC_i d\Gamma^i \int [d\Phi] (s_L + C_i \partial_{m_i} - \Gamma^i \partial_{p^i}) e^{-S(\Phi;m^i,p^i)} X
$$

$$
= \int \prod_i dC_i d\Gamma^i (C_i \partial_{m_i} - \Gamma^i \partial_{p^i}) \int [d\Phi] e^{-S(\Phi;m^i,p^i)} X
$$

$$
= \int \prod_i dC_i d\Gamma^i p^i \partial_{m_i} \int [d\Phi] e^{-S(\Phi;m^i,p^i)} X = d_p \langle X(\Phi) \rangle_{m,p},
$$

(2.6)
where we have introduced $s_L$, i.e. the restrictions of the coboundary operator $s$ to the
clocal fields, and we have used the fact that its integral over $\Phi$ vanishes after integration
by parts. We have also used (2.5) for $X$.

This equation shows that the Slavnov-Taylor identity is violated by a locally exact
form on the moduli space. If the breaking were a globally exact form, by integrating the
correlators over the moduli superspace, that is, by integrating over the moduli space the
top form corresponding to the coefficient of the term of degree $6g - 6 + 2n$ in $p$, one would
recover the unbroken Slavnov-Taylor identity. This, however, would imply the triviality
of our construction since, due to the triviality of $s$-cohomology, all the correlators of our
model would vanish.

It is therefore essential to understand if the above form is globally defined. For this
we have to examine the structure of the moduli space $\mathcal{M}_{g,n}$.

The background metric $\eta_{\mu\nu}(x; m)$ cannot be chosen to be a everywhere continuous
function of the moduli space. In fact $\eta_{\mu\nu}(x; m)$ is a section of the gauge bundle over
$\mathcal{M}_{g,n}$ defined by the space of two-dimensional metrics on a surface of given genus and $n$
punctures. This bundle is non-trivial and therefore does not admit a global section. It
follows that $\eta_{\mu\nu}(x; m)$ must be a local section of the bundle of two-dimensional metrics. Let
$\{\mathcal{U}_a\}$ be a covering of the moduli space. The background gauge is defined by a collections
$\{\eta_{\mu\nu}^a(x; m)\}$ of two-dimensional metrics, with each $\eta_{\mu\nu}^a(x; m)$ defined, as a function of $m$,
on $\mathcal{U}_a$.

Let $\hat{\eta}_{\mu\nu}^a$ and $\hat{\eta}_{\mu\nu}^0$ be two gauge-equivalent reduced metrics (complex structures):

$$
\hat{\eta}_{\mu\nu}^a(x_a; m) = \frac{1}{\det(\frac{\partial x_a}{\partial x_0})} \frac{\partial x_a}{\partial x_0} \frac{\partial x_a}{\partial x_0} \frac{\partial}{\partial x_0^\rho} \hat{\eta}_{\sigma\rho}^0(x_0; m),
$$

(2.7)

related by a diffeomorphism which may in general depend on $m$:

$$
x_a \rightarrow x_0(x_a; m).
$$

(2.8)

From (2.7) one derives the transformation law for $\frac{\partial \hat{\eta}_{\mu\nu}^a}{\partial m}$:

$$
\partial_i \hat{\eta}_{\mu\nu}^a(x_a; m) = \frac{1}{\det(\frac{\partial x_a}{\partial x_0})} \frac{\partial x_a}{\partial x_0} \frac{\partial x_a}{\partial x_0} \left( \partial_i \hat{\eta}_{\sigma\rho}^0(x_0; m) + (\mathcal{L}_{v_a} \hat{\eta}_{\rho})(x_0; m) \right),
$$

(2.9)

where $v_a$ is the vector field defined by the equation

$$
v_{\mu}^a = \partial_i x_0^\mu(x_a; m)|_{x_a=x_a(x_0; m)}.
$$

(2.10)
We also define:
\[ \hat{v}_\mu^a \equiv p^i v_{ia} \quad (2.11) \]

The action of our model is invariant under background coordinate transformations that are independent of the moduli; if the observables are admissible and transform like scalars under world-sheet coordinate transformations, the whole effect of the transition from \( \hat{\eta}_0^{\mu\nu} \) to \( \hat{\eta}_a^{\mu\nu} \) is reduced to the substitution: \( \partial_i \hat{\eta}_a^{\mu\nu}(x_a; m) \rightarrow \partial_i \hat{\eta}_0^{\sigma\rho}(x_0; m) + (L_{v_a} \hat{\eta}_0)^{\sigma\rho}(x_0; m) \), and \( c \rightarrow c + \hat{v}_a \). This result is a consequence of (2.5).

Therefore we restrict, from now on, the admissible local observables to those that transform as scalars under coordinate transformations. Under this condition we can compute the correlation functions in the \( \hat{\eta}_a^{\mu\nu} \) background in terms of those in the \( \hat{\eta}_0^{\mu\nu} \) one. Labelling by \( S_0 \) the action corresponding to \( \hat{\eta}_0^{\mu\nu} \), we get:

\[
\langle \Omega \rangle_{\hat{\eta}_a} = \int \prod_i dC^i d\Gamma^i \int d[\Phi] e^{-S_0} \Omega(c, \ldots) = \int \prod_i dC^i d\Gamma^i \int d[\Phi] e^{-S_0 + s} \int d^2 x \beta_{\mu\nu} (L_{v_a} \hat{\eta}_0)^{\mu\nu} \Omega(c + \hat{v}_a, \ldots). \tag{2.12}
\]

We consider now the correlation functions of the same observable \( \Omega \) in two background metrics \( \hat{\eta}_a^{\mu\nu} \) and \( \hat{\eta}_b^{\mu\nu} \) that are gauge-equivalent to \( \hat{\eta}_0^{\mu\nu} \). We have:

\[
\langle \Omega \rangle_{\hat{\eta}_b} = \int \prod_i dC^i d\Gamma^i \int d[\Phi] e^{-S_0 + s} \int d^2 x \beta_{\mu\nu} (L_{v_b} \hat{\eta}_0)^{\mu\nu} \Omega(c + \hat{v}_b, \ldots). \tag{2.13}
\]

In order to compare the correlation functions (2.12) and (2.13) we introduce the interpolating action:

\[
S_{ab}(t) = S_0 - s \int d^2 x \beta_{\mu\nu} \left[ t(L_{\tilde{v}_b} \hat{\eta}_0)^{\mu\nu} + (1 - t)(L_{\tilde{v}_a} \hat{\eta}_0)^{\mu\nu} \right], \tag{2.14}
\]

in terms of which we can compute the difference:

\[
\langle \Omega \rangle_{\hat{\eta}_b} - \langle \Omega \rangle_{\hat{\eta}_a} = \langle \delta \langle \Omega \rangle \rangle_{ab}
\]

\[
= \int_0^1 dt \int \prod_i dC^i d\Gamma^i \int d[\Phi] \frac{d}{dt} \left[ e^{-S_{ab}(t)} \Omega \left( c + t\hat{v}_b + (1 - t)\hat{v}_a, \gamma, \tilde{g}, \psi \right) \right]
\]

\[
= - \int_0^1 dt \int \prod_i dC^i d\Gamma^i \int d[\Phi] \frac{dS_{ab}(t)}{dt} e^{-S_{ab}(t)} \Omega \left( c + t\hat{v}_b + (1 - t)\hat{v}_a, \gamma, \tilde{g}, \psi \right)
\]

\[
+ \int_0^1 dt \int \prod_i dC^i d\Gamma^i \int d[\Phi] e^{-S_{ab}(t)} \int d^2 x \left( \hat{v}_b^\mu - \hat{v}_a^\mu \right) \frac{\partial}{\partial \hat{v}_a^\mu} \Omega \left( c + t\hat{v}_b + (1 - t)\hat{v}_a, \gamma, \tilde{g}, \psi \right). \tag{2.15}
\]
Taking into account the structure of the interpolating action and (2.6) it is clear that the first term in the right-hand side of this equation corresponds to a locally exact form; this is however not true for the second term. Thereby we see that the restriction of the observable algebra to that generated by the admissible scalar operators is not sufficient to guarantee that the difference of the local top forms corresponding to the correlators in two different backgrounds be a locally exact form on moduli space. Remember that, as noticed at the beginning of this report, if this condition is not verified, the correlators depend on the particular choice of the particular covering chosen for \( \mathcal{M}_{g,n} \).

The second term in the right-hand side of (2.15) in general does not vanish. However, recalling that the local operators in (2.3) are sitting on fixed points of the Riemann surface and hence the vector fields \( \hat{v} \) vanish at these points, we see that the unwanted term vanishes if the observable does not depend on the derivatives of the \( c \) field. That is, if the observable depends on the \( c \) field only through its value at the punctures. It is shown in [7] that this further restriction reduces the local observables to the elements of the *equivariant cohomology* of \( s \).

With this final restriction of the observables we obtain:

\[
(\delta \langle \Omega \rangle)_{ab} = dp \int_0^1 dt \int \prod_i dC_i \, d\Gamma_i \int d[\Phi] e^{-S_{a,b}(t)} (I_{\hat{v}_b} - I_{\hat{v}_a}) \Omega \quad (2.16)
\]

with:

\[
I_{\hat{v}} \equiv - \int d^2 x \hat{v}^\mu(x) \frac{\delta}{\delta \gamma^\mu(x)}
\]

The possibility of generalising this equation to the case of many overlapping covers is a crucial result of our analysis that allows a complete characterisation of the correlators as elements of Čech-De Rham cohomology [1]. Indeed, one extends iteratively the definition of the difference operator \( \delta \) to the case of many intersecting covers as follows:

\[
(\delta \langle \Omega \rangle)_{a_0...a_{q+1}} = \sum_{l=0}^{q+1} (-1)^l (\langle \Omega \rangle)_{a_0...\tilde{a}_l...a_{q+1}} , \quad (2.17)
\]

where the check mark above \( a_l \) means that this symbol should be omitted. Then, using exactly the same method as above, one finds:

\[
(\delta \langle \Omega \rangle)_{a_0...a_q} = dp (\langle \Omega \rangle)_{a_0...a_q} . \quad (2.18)
\]
In (2.17) and (2.18) one has introduced:

\[
(\langle \Omega \rangle)_{a_0 \ldots a_q} \equiv \sum_{k=0}^q (-1)^k \int_0^1 \prod_{l=0}^q dt_l \delta \left( \sum_{j=0}^q t_j - 1 \right) \int \prod_i d\mathcal{C}_i d\Gamma_i \int d[\Phi] e^{-S_{a_0 \ldots a_q}(t_0, \ldots, t_q)} I_{a_0}, \ldots, \tilde{I}_{a_k}, \ldots, I_{a_q} \Omega, \tag{2.19}
\]

and the interpolating action is given by:

\[
S_{a_0 \ldots a_q}(t_0, \ldots, t_q) = S_0 - s \sum_{k=0}^q t_k \int d^2 x \beta_{\mu \nu} (\mathcal{L}_{\partial_k} \hat{\eta}_0)^{\mu \nu}. \tag{2.20}
\]

A detailed discussion of the structure of a correlator as element of the Čech-De Rham cohomology can be found in [1]. For the purposes of the present report it is sufficient to discuss the simplest non-trivial application of our method.

However it is useful to mention that, given a Čech-De Rham cocycle, the corresponding moduli space integral is computed as follows. Let \( \{ \mathcal{C}_a \} \) be a cell decomposition of \( \mathcal{M}_{g,n} \), with \( \mathcal{C}_a \subset \mathcal{U}_a \), and let \( \mathcal{C}_{a_0 a_1 \ldots a_q} \) of codimension \( q \) in \( \mathcal{M}_{g,n} \) oriented in such a way that the boundary of a cell \( \partial \mathcal{C}_{a_0 a_1 \ldots a_q} \) satisfies:

\[
\partial \mathcal{C}_{a_0 a_1 \ldots a_q} = \bigcup_b \mathcal{C}_{a_0 a_1 \ldots a_q} b, \tag{2.21}
\]

where we have introduced the convention that \( \mathcal{C}_{a_0 a_1 \ldots a_q} \) is antisymmetric in its indices in the sense that it changes orientation when exchanging a pair of indices. We have defined in this way \( q \)-chains of cells of codimension \( q \) that are adjoint to the \( q \)-cochains defined by \( (\langle \Omega \rangle)_{a_0 \ldots a_q} \). Given a \( q \)-chain and a \( q \)-cochain, we can define the integral:

\[
\int_{\mathcal{C}_q} \langle \Omega \rangle \equiv \sum_{a_0 < a_1 < \ldots < a_q} \int_{\mathcal{C}_{a_0 a_1 \ldots a_q}} (\langle \Omega \rangle)_{a_0 a_1 \ldots a_q}. \tag{2.22}
\]

To study explicitly an example we still have to find a suitable set of local observables satisfying our constraints, that is corresponding to non-trivial classes of the \( s \)-equivariant cohomology.

The wanted set of observables can be constructed [9],[13],[14],[15],[10],[16] starting from the Euler two-form

\[
\sigma^{(2)} = \frac{1}{8\pi} \sqrt{g} R \epsilon_{\mu \nu} dx^\mu \wedge dx^\nu, \tag{2.23}
\]
where $R$ is the two-dimensional scalar curvature and $\epsilon_{\mu\nu}$ is the antisymmetric numerical tensor defined by $\epsilon_{12} = 1$. Since $s$ and the exterior differential $d$ on the two-dimensional world-sheet anti-commute, the two-form in Eq. (2.23) gives rise to the descent equations:

$$
\begin{align*}
\sigma^{(2)} &= d\sigma^{(1)} \\
\sigma^{(1)} &= d\sigma^{(0)} \\
\sigma^{(0)} &= 0.
\end{align*}
$$

The 0-form $\sigma^{(0)}$ and the 1-form $\sigma^{(1)}$ are computed to be

$$
\sigma^{(0)} = \frac{1}{4\pi} \sqrt{g} \epsilon_{\mu\nu} \left[ \frac{1}{2} c^\mu c^\nu R + c^\mu D_\rho (\psi^\nu \psi^\rho - g^{\nu\rho} \psi^\sigma) + D^\mu \gamma^\nu - \frac{1}{4} \psi^\mu \psi^\nu \right] \\
\sigma^{(1)} = \frac{1}{4\pi} \sqrt{g} \epsilon_{\mu\nu} \left[ c^\nu R + D_\rho (\psi^\nu \psi^\rho - g^{\nu\rho} \psi^\sigma) \right] dx^\mu.
$$

$\sigma^{(0)}$ correspond to non-trivial class in the equivariant cohomology of $s$, in particular it is clear that it satisfies our constraints. Therefore we shall choose in general:

$$
\Omega = \prod_i (\sigma^{(0)}(P_i))^{n_i}
$$

### 3. The study of a very simple physical expectation value

Now we give some formulae that are useful for explicit calculations. Identifying $x_0^\mu$ with the isothermal coordinate frame, i.e. $\hat{\eta}_0^{\mu\nu} = \delta^{\mu\nu}$, and given a generic choice of the background metric $\hat{\eta}_a$, we introduce the transition matrix:

$$
(M_a)_{\mu}^\nu \equiv \frac{\partial x_0^\mu}{\partial x_a^\nu}.
$$

We have:

$$
(\hat{\eta}_a)^{\mu\nu} = \det(M_a)(M_a^{-1}\tilde{M}_a^{-1})^{\mu\nu},
$$

where $\tilde{M}_a$ is the transposed of $M_a$. We also have the vector field $\hat{v}_a$ (see (2.10)) given by:

$$
\hat{v}_a^\mu = d_p x_0^\mu (x_a; m)|_{x_a = x_a(x_0; m)}.
$$

and satisfying the following identities:

$$
\frac{\partial \hat{v}_a^\mu}{\partial x_0^\lambda} = (d_p M_a M_a^{-1})^\mu_\lambda,
$$

13
and:
\[ d_p \hat{v}^\mu_a = -\hat{v}^\nu_a \frac{\partial \hat{v}^\mu_a}{\partial x^\nu_0}. \] (3.5)

Now, recalling again that \( \hat{v} \) vanishes at the punctures, it is easy to verify that:
\[ \langle I_{\hat{v}_a} \sigma_1^{(0)} \rangle_{\tilde{\eta}_0} = \text{tr} (\epsilon M_a d_p M_a^{-1}) . \] (3.6)

In the following it will be convenient to choose complex coordinates, thus we introduce the isothermal coordinates \( x_0^\mu = (Z, \bar{Z}) \) and the local ones \( x_a^\mu = (z_a, \bar{z}_a) \). The relation between \( (Z, \bar{Z}) \) and \( (z_a, \bar{z}_a) \) involves the Beltrami differentials \( \mu_a \) and \( \bar{\mu}_a \):
\[ dZ \otimes d\bar{Z} = |\lambda_a|^2 (dz_a + \mu_a d\bar{z}_a) \otimes (d\bar{z}_a + \bar{\mu}_a dz_a). \] (3.7)

Now the matrix \( M_a \) is written
\[ (M_a)_{\mu}^\nu = \left( \begin{array}{cc} \lambda_a & \lambda_a \mu_a \\ \bar{\lambda}_a \bar{\mu}_a & \bar{\lambda}_a \end{array} \right). \] (3.8)

Inserting this expression into (3.6) and taking into account that \( \hat{\eta}_0 \) in complex coordinates corresponds to the matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) we obtain:
\[ \text{tr} (\epsilon d_p M_a M_a^{-1}) = -i \left( d_p \log \frac{\lambda_a}{\bar{\lambda}_a} + \frac{\mu_a d_p \bar{\mu}_a - \bar{\mu}_a d_p \mu_a}{1 - |\mu_a|^2} \right). \] (3.9)

We are now in condition to analyse a particular example. The simplest non-trivial one is the vacuum expectation value of \( \sigma^{(0)}(P) \) when the world-sheet is the sphere with four fixed points. We identify the sphere with the compactified complex plane and the four fixed points with \( 0, 1, P, \infty \).

Our world-sheet is characterised by a single complex modulus \( m \) that identifies the point \( P \) in the isothermal frame. Thus, the moduli space coincides with the complex plane with the three singular points \( 0, 1 \) and \( \infty \) taken out. It can be covered by three open disks centred around the three singularities. It is therefore clear that each disk is not simply connected since its centre is removed. Let us consider for example the disk centred at the origin. When \( m \) belongs to this disk the map \( Z(z_0, \bar{z}_0, m) \) relating the isothermal coordinates to the background ones can be chosen according to the equation:
\[ Z(z, \bar{z}, m) \equiv z \left( \frac{m}{z_P} \right)^{\theta_R(1-|z|^2)}. \] (3.10)
\[\theta_R(x) \text{ is a regularized } C^\infty \text{ step function, interpolating smoothly between 0 and 1:} \]

\[
\theta_R(x) = \begin{cases} 
0, & \text{if } x \leq 0; \\
1, & \text{if } x \geq 1 - C,
\end{cases}
\]  

(3.11)

with \(0 < C < 1\) and \(z_P\) is the coordinate of \(P\).

Without any loss of generality we can choose for example \(z_P = 1/2 + i\sqrt{2}/2\) and \(C = 3/4\). With this choice it is easy to see that:

\[
Z(z_i, \bar{z}_i, m) = z_i \quad \text{for} \quad z_i = 0, 1, \infty, \quad \text{(3.12)}
\]

and:

\[
Z(z_P, \bar{z}_P, m) = m. \quad \text{(3.13)}
\]

It is also easy to verify that for \(0 < |m| < 1\), \(Z(z, \bar{z}, m)\) defines a quasi-conformal map of the sphere with three fixed points into itself. Furthermore, comparing with (3.8) one has:

\[
\mu_0(z_P; m) \equiv 0
\]

(3.14)

in the whole disk and

\[
\lambda_0(z_P; m) = \log \frac{m}{z_P}
\]

(3.15)

in the disk without the origin. It is clear that a completely analogous construction can be repeated for the disks centred at 1 and \(\infty\) obtaining in particular:

\[
\mu_i(z_P; m) \equiv 0,
\]

(3.16)

for \(i = 1, \infty\) and

\[
\lambda_1(z_P; m) = \log \frac{m - 1}{z_P - 1} \quad \text{and} \quad \lambda_\infty(z_P; m) = \log \frac{m - 1/2}{z_P - 1/2}.
\]

(3.17)

Now we see that \(\langle \sigma^{(0)}(P) \rangle\) vanishes as a function of \(m\) in every disk. Indeed the terms of \(\sigma^{(0)}(P)\) depending on \(c\) and \(\gamma\) do not contribute since these fields have no sources, while the last term in (2.25) vanishes due to (3.16). Thus, apparently, the presence of a singularity in the center of each disk does not affect the functional integral.

However, computing the discontinuity of \(\langle \sigma^{(0)}(P) \rangle\) between two and more neighboring disks we have, from (2.16), (2.17), (2.19), (3.6) and (3.9),

\[
\left(\langle \sigma^{(0)}(P) \rangle\right)_{a,b} = \frac{1}{2\pi} d_p \text{ Im } \log \frac{\lambda_a(z_P, m)}{\lambda_b(z_P, m)},
\]

and

\[
\left(\langle \sigma^{(0)}(P) \rangle\right)_{a,b,c} = 0,
\]

where \(a, b, c\) correspond to the three disks around 0, 1 and \(\infty\). Then, using (2.22) we verify that the value of the vacuum expectation value of \(\sigma^{(0)}(P)\) is 1.
4. Conclusions

It is rather clear from this analysis that the singular points of the moduli space, while not affecting the contributions of their neighbourhood to the physical expectation value, determine the discontinuities from which the final, non-trivial result gets its origin.

It is also remarkable that this result implies a breakdown of the Slavnov-Taylor identity since, the $s$-cohomology being trivial, if this identity where verified, all the physical correlators should vanish. It results clearly from our study how the singularities of the moduli space have determined this breakdown. One might speculate about the correspondence between this mechanism and the instabilities discussed by Fujikawa in [17].

The third important point that has to be noted is that our explicit calculation has been possible since we have been able to include the whole neighbourhood of each singularity into a finite number of cells of the moduli space. In this way these singularities do not correspond to any divergence but determine the lack of global definition of the local correlators which therefore correspond to non-trivial elements of Čech-De Rham cohomology. In our opinion this situation is suggestive of an analogous possibility in the case of gauge theories.

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References

[1] C. Becchi and C. Imbimbo, hep-th/9510003, CERN and Genoa preprints CERN-TH 242/95, GEF-Th 8/1995.
[2] V. N. Gribov, Instability of non-abelian gauge theories and impossibility of choice of Coulomb gauge, SLAC Translation 176, (1977).
[3] I. M. Singer, Commun. Math. Phys. 60 (1978) 7.
[4] J. Labastida, M. Pernici and E. Witten, Nucl. Phys. B310 (1988) 611.
[5] D. Montano and J. Sonnenschein, Nucl. Phys. B313 (1989) 258.
[6] R. Bott and L.W. Tu, Differential Forms in Algebraic Topology (Springer-Verlag, New York, 1982).
[7] R. Stora, F. Thullier and J.C. Wallet, preprint 1994, ENSLAPP-A-481/94, IPNO-TH-94/29; R. Stora, Report at this Conference.
[8] R. Brooks, D. Montano and J. Sonnenschein, Phys. Lett. B214 (1988) 91.
[9] R. Myers and V. Periwal, Nucl. Phys. B333 (1990) 536.
[10] E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 457.
[11] C.M. Becchi, R. Collina and C. Imbimbo, hep-th/9406096, CERN and Genoa preprints CERN-TH 7302/94, GEF-Th 6/1994, Symmetry and Simplicity in Theoretical Physics, Proceedings of the Symposium for the 65-th Birthday of Sergio Fubini, Turin, 1994 (World Scientific, Singapore, 1994).
[12] L. Baulieu and M. Bellon, Phys. Lett. B202 (1988) 67.
[13] R. Myers, Nucl. Phys. B343 (1990) 705.
[14] D. Montano and J. Sonnenschein, Nucl. Phys. B324 (1989) 348.
[15] L. Baulieu and I.M. Singer, Comm. Math. Phys. 135 (1991) 253.
[16] C.M. Becchi, R. Collina and C. Imbimbo, hep-th/9311097, Phys. Lett. B322 (1994) 79.
[17] K. Fujikawa, Nucl. Phys. B223 (1983) 218.