BESOV SPACES FOR SCHRÖDINGER OPERATORS WITH BARRIER POTENTIALS

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Abstract. Let $H = -\Delta + V$ be a Schrödinger operator on the real line, where $V = \varepsilon^2 \chi_{[-1,1]}$. We define the Besov spaces for $H$ by developing the associated Littlewood-Paley theory. This theory depends on the decay estimates of the spectral operator $\varphi_j(H)$ in the high and low energies. We also prove a Mikhlin-Hörmander type multiplier theorem on these spaces, including the $L^p$ boundedness result. Our approach has potential applications to other Schrödinger operators with short-range potentials, as well as in higher dimensions.

1. Introduction

Let $H = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}$, where the potential $V$ is real-valued and belongs to $L^1 \cap L^2$. $H$ is the Hamiltonian in the corresponding time-dependent Schrödinger equation

\[
i \frac{\partial}{\partial t} \psi = H \psi, \]
\[
\psi(0, x) = f(x) \in \mathcal{D}(H),
\]

where the solution is uniquely determined by the initial state: $\psi(t, x) = e^{-itH} f(x), \ t \geq 0$.

In [21] Jensen and Nakamura introduced Besov spaces associated with $H$ on $\mathbb{R}^d$ and showed that $e^{-itH}$ maps $B^{s+2,2}_p(H)$ into $B^{s,q}_p(H)$ if $s \geq 0, 1 \leq p, q \leq \infty$ and $\beta > d|\frac{1}{2} - \frac{1}{p}|$, under the condition that $V = V_+ - V_-$ so that $V_+ \in K^{loc}_d$ and $V_- \in K_d$, $K_d$ being the Kato class. In this paper we generalize the definition of Besov spaces to $\alpha \in \mathbb{R}$, $0 < p, q < \infty$ and show, in the case of barrier potential, that such a definition is independent of the choice of the dyadic system $\{\Phi, \varphi_j\}$.

\textbf{Date:} February 18, 2022.

2000 \textit{Mathematics Subject Classification.} Primary: 42B25; Secondary: 35P25.

\textit{Key words and phrases.} Besov spaces, Schrödinger operator, Littlewood-Paley theory.

The research of the first author is supported in part by NSF/ONR. The second author supported in part by DARPA.
For $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, define the quasi-norm for $f \in L^2$ as

$$
\|f\|_{B^\alpha,q_p} := \|f\|_{B^\alpha,q_p(H)} = \|\Phi(H)\|_p + \left\{ \sum_{j=1}^{\infty} (2^{j\alpha}\|\varphi_j(H)f\|_p)^q \right\}^{1/q}.
$$

The Besov spaces associated with $H$, denoted by $B^\alpha,q_p(H)$, is defined to be the completion of the subspace $L^2_0 := \{ f \in L^2 : \|f\|_{B^\alpha,q_p} < \infty \}$ of $L^2$.

As in the Fourier case and the Hermite case \cite{39, 40, 10, 11} we address the Besov space theory associated with $H$ by considering the Schrödinger operator $H = -\triangle + V$, where $V = \varepsilon^2 \chi_{[-1,1]}, \varepsilon > 0$ (called the barrier potential) is one of the simplest discontinuous potential models in quantum mechanics.

Peetre’s maximal operator plays an important role in the theory of function spaces. In order to establish a Peetre type maximal inequality for $H$, we need the decay estimates of the kernel of $\varphi_j(H)$ as well as of its derivative. Based on an integral expression of this kernel we obtain the decay estimates by exploiting the analytic behavior of the eigenfunctions $e(x, \xi)$ as $\xi$ approaches $\infty$ (high energy) and $0$ (low energy) in various cases. When the support of $\Phi$ contains the origin, we are in the so-called “local energy” case, which usually is harder to deal with for general potentials. We use certain “matching” method to put together integrals of the “same type”, so that each of the resulting integrals is the Fourier transform of a Schwartz function. This method seems interesting and may have applications to other potentials.

Our first main result (Theorem 3.7) is an equivalence theorem for $B^\alpha,q_p(H)$, which tells that $\|f\|_{B^\alpha,q_p}$ and $\|f\|_{B^\alpha,q_p}$ are equivalent quasi-norms on $B^\alpha,q_p(H)$, where $\{\phi_j\}$, $\{\psi_j\}$ are two given smooth dyadic systems.

Using functional calculus, Jensen and Nakamura \cite{21, 22} obtained smooth multiplier results for general potentials. For barrier potential we prove a sharp spectral multiplier theorem on $B^\alpha,q_p(H)$ (Theorem 6.5 and Theorem 6.6).

The remaining part of the paper is organized as follows. In §2 we give explicit solutions to the eigenfunction equation. In §3 we give norm characterization of $B^\alpha,q_p(H)$ using Peetre type maximal functions. The proof is based on the decay estimates for the kernel of $\varphi_j(H)$. A detailed proof of these decay estimates are included in §4 and §5. In §6 we prove a Hörmander type multiplier theorem for $H$. In §7, we give identifications of these new spaces $B^\alpha,q_p(H)$ with the ordinary Besov spaces for certain range of parameters $\alpha, p, q$. 

Acknowledgment The authors would like to thank A. Jensen for his useful comments on the identification of Besov spaces.

2. Preliminaries

2.1. Kernel formula for the spectral operator

Let \( e_+(x, \xi) \) and \( e_-(x, \xi) \) be two solutions of the equation
\[
He(x, \xi) = \xi^2 e(x, \xi)
\]
with asymptotic behavior for \( \xi > 0 \) and \( \xi < 0 \) respectively,
\[
e_\pm(x, \xi) \to \begin{cases} T_\pm(\xi)e^{i\xi x} & x \to \pm \infty \\ e^{i\xi x} + R_\pm(\xi)e^{-i\xi x} & x \to \mp \infty. \end{cases}
\]

Then \( e_\pm(x, \xi) \) are unique for \( \xi \in \mathbb{R} \), and equation (3) together with condition (4) is equivalent to the integral equation
\[
e(x, \xi) = e^{i\xi x} + (2i|\xi|)^{-1} \int e^{i|\xi||x-y|}V(y)e(y, \xi)dy.
\]

These generalized eigenfunctions have a physical interpretation in quantum mechanics, where \( \xi^2 \) is viewed as an energy parameter; they represent the transmission and reflection waves when a particle passes through the potential. The coefficients \( T, R \) are called the transmission coefficient and the reflection coefficient (cf. [17], p. 4179, also [14]).

Under the condition that \( V \) is in \( L^1 \cap L^2 \), we show in [41, 42] that,
a) The essential spectrum of \( H \) is \([0, \infty)\); more precisely, \( H \) has only absolutely continuous spectrum (the singular continuous spectrum being empty); the discrete spectrum of \( H \) is at most countable. Hence if denoting \( L^2 \) by \( \mathcal{H} \) we have \( \mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{pp} \).
b) Define the generalized Fourier transform \( \mathcal{F} \) on \( L^2 \):
\[
\mathcal{F}f(\xi) := \text{l.i.m.} \frac{1}{\sqrt{2\pi}} \int f(x)\overline{e(x, \xi)}dx.
\]

Then \( \mathcal{F} \) is a unitary operator from \( \mathcal{H}_{ac} \) onto \( L^2 \) and its adjoint is given by for \( g \in L^2 \)
\[
\mathcal{F}^*g(x) := \text{l.i.m.} \frac{1}{\sqrt{2\pi}} \int g(\xi)e(x, \xi)d\xi.
\]

Therefore \( \varphi(H)|_{\mathcal{H}_{ac}} = \mathcal{F}^*\varphi(\xi^2)\mathcal{F} \). If \( H \) has no point spectrum and all generalized eigenfunctions of \( H \) are uniformly bounded in \( x \) and \( \xi \), then for \( f \in L^2 \),
\[
\varphi(H)f(x) = \int \varphi(H)(x, y)f(y)dy,
\]
where
\[
\varphi(H)(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\xi^2)e(x,\xi)\varphi(y,\xi)d\xi.
\]

A variant of the formula (6) can be found in [17] for short-range potentials defined as a measure. In 3D similar formula is used by Tao [36] in a scattering problem.

Since the barrier potential \( V = \varepsilon^2 \chi_{[-1,1]} \) is in \( L^1 \cap L^2 \) and the eigenfunctions of \( H \) are uniformly bounded (see subsection 2.3), the formula (6) is valid for \( V \). Note that the corresponding point spectrum is empty.

2.2. Dyadic system and Besov spaces

Let \( \Phi, \varphi, \Psi, \psi \) be \( C^\infty \) smooth functions, satisfying

i) \( \text{supp} \ \Phi, \text{supp} \ \Psi \subset \{ |\xi| \leq 1 \}; |\Phi(\xi)|, |\Psi(\xi)| \geq c > 0 \text{ if } |\xi| \leq \frac{1}{2}; \)

ii) \( \text{supp} \ \varphi, \text{supp} \ \psi \subset \{ \frac{1}{4} \leq |\xi| \leq 1 \}; |\varphi(\xi)|, |\psi(\xi)| \geq c > 0 \text{ if } \frac{3}{8} \leq |\xi| \leq \frac{7}{8}; \)

iii) \( \Phi(\xi)\Psi(\xi) + \sum_{j=1}^{\infty} \varphi(2^{-j}\xi)\psi(2^{-j}\xi) = 1, \forall \xi \in \mathbb{R}. \)

The existence of such functions can be justified by consulting e.g., [16]. The almost orthogonal relation (iii) for the system \( \varphi_j(x) := \varphi(2^{-j}x), \psi_j(x) := \psi(2^{-j}x) \) allows us to give an expansion of \( f \) in \( L^2 \) as
\[
f = \Phi(H)\Psi(H)f + \sum_j \varphi_j(H)\psi_j(H)f.
\]

As in the Fourier case, let \( 0 < p, q \leq \infty, \alpha \in \mathbb{R} \) we define the \( B_{p,q}^\alpha \) quasi-norm as in (2) for \( f \in L^2 \). Note that when \( 0 < p < 1 \) or \( 0 < q < 1 \), we can always define a metric \( d \) on \( B_{p,q}^\alpha \), so that the metric space \( (B_{p,q}^\alpha, d) \) is topologically isomorphic to the quasi-normed space. In fact, Lemma 3.10.1 in [5] tells that every quasi-normed linear space is metrizable.

2.3. Generalized eigenfunctions of \( H \)

We now determine eigenfunctions of \( H = -\Delta + V \), where \( V = \varepsilon^2 \chi_{[-1,1]} \), also see, e.g., [14].

First \( e(x,\xi) \) must have the following form. If \( \xi > \varepsilon \), then
\[
e(x,\xi) = \begin{cases} 
A_+e^{i\xi x} + A'_+e^{-i\xi x} & x < -1 \\
B_+e^{iKx} + B'_+e^{-iKx} & |x| \leq 1 \\
C_+e^{i\xi x} + C'_+e^{-i\xi x} & x > 1,
\end{cases}
\]
where \( K = \sqrt{\xi^2 - \varepsilon^2} \); and if \( 0 < \xi < \varepsilon \), then
\[
e(x,\xi) = \begin{cases} 
A_+e^{i\xi x} + A'_+e^{-i\xi x} & x < -1 \\
B_+e^{\rho x} + B'_+e^{-\rho x} & |x| \leq 1 \\
C_+e^{i\xi x} + C'_+e^{-i\xi x} & x > 1,
\end{cases}
\]
where \( \rho = \sqrt{\varepsilon^2 - \xi^2} \).
The Lippman-Schwinger equation (5) requires that \( e(x, \xi) \) is differentiable in \( x \). By the \( C^1 \) condition at \( \pm 1 \) we can obtain the precise values of the coefficients \( A, A', B, B', C, C' \) as follows.

Let

\[
\rho = \rho(\xi) = \begin{cases} 
i K = i\sqrt{\xi^2 - \varepsilon^2} & |\xi| > \varepsilon \\ \sqrt{\varepsilon^2 - \xi^2} & |\xi| \leq \varepsilon. \end{cases}
\]

Then for \( \xi > 0 \),

\[
C_+ = 0, \quad A_+ = 1,
\]

\[
C_+ = \frac{2\rho \xi e^{-2i\xi}}{2\rho \xi \cosh 2\rho + i(\rho^2 - \xi^2) \sinh 2\rho},
\]

\[
A_+ = -i \frac{C_+ \varepsilon^2 \sinh 2\rho}{2\rho \xi \cosh 2\rho + i(\rho^2 - \xi^2) \sinh 2\rho} = -i \frac{\varepsilon^2 \sinh 2\rho e^{-2i\xi}}{2\rho \xi \cosh 2\rho + i(\rho^2 - \xi^2) \sinh 2\rho},
\]

\[
B_+ = \frac{C_+}{2\rho}(\rho + i\xi) e^{\rho + i\xi}, \quad B'_+ = \frac{C_+}{2\rho}(\rho - i\xi) e^{\rho + i\xi}.
\]

For \( \xi < 0 \), we obtain similarly, with the same notation \( \rho = \rho(\xi) \),

\[
e(x, \xi) = \begin{cases} A_- e^{i\xi x} + A'_- e^{-i\xi x} & x < -1 \\ B_- e^{\rho x} + B'_- e^{-\rho x} & |x| \leq 1 \\ C_- e^{i\xi x} + C'_- e^{-i\xi x} & x > 1, \end{cases}
\]

where \( C_- = 1, \quad A'_- = 0, \quad \)

\[
A_- = \frac{2\rho \xi e^{2i\xi}}{2\rho \xi \cosh 2\rho - i(\rho^2 - \xi^2) \sinh 2\rho},
\]

\[
C'_- = i \frac{A_- \varepsilon^2 \sinh 2\rho}{2\rho \xi \cosh 2\rho - i(\rho^2 - \xi^2) \sinh 2\rho} = i \frac{\varepsilon^2 \sinh 2\rho e^{2i\xi}}{2\rho \xi \cosh 2\rho - i(\rho^2 - \xi^2) \sinh 2\rho},
\]

\[
B_- = \frac{A_-}{2\rho}(\rho + i\xi) e^{\rho - i\xi}, \quad B'_- = \frac{A_-}{2\rho}(\rho - i\xi) e^{\rho - i\xi}.
\]

Furthermore, if we define for \( \xi \in \mathbb{R} \setminus \{0\} \)

\[
e(x, \xi) = \begin{cases} e_+(x, \xi) & \xi > 0 \\ e_-(x, \xi) & \xi < 0, \end{cases}
\]

then

\[
e(x, -\xi) = e(-x, \xi), \quad \xi \neq 0,
\]

which follows from the following simple relations between the coefficients:

\[
A_-(\xi) = \overline{A_-}(-\xi) = \overline{C_+}(\xi) = C_+(-\xi)
\]
\( C'_-(\xi) = \overline{C'_-}(\xi) = \overline{A'_+}(\xi) = A'_-(-\xi) \)

and

\( B'_+(\xi) = B'_-(\xi), \quad B'_-(\xi) = B'_+(-\xi). \)

Identity (8) allows us to simplify the estimation in various cases; see §4–6. Some of the above relations can also be found in [17] for general potentials.

3. Peetre type maximal inequality

Let \( \Phi, \varphi, \Psi, \psi \) be \( C^\infty \) smooth functions, satisfying the conditions given in §2. Recall that if \( \phi \in C_0(\mathbb{R}) \), the operator \( \phi(H) \) has the kernel (7). Note that \( e(\cdot, \xi) \in C^1(\xi \neq 0, \pm \epsilon) \) implies \( \phi(H)(x, y) \in C^1(\mathbb{R} \times \mathbb{R}). \)

**Lemma 3.1.** Let \( K_j(x, y) = \varphi(2^{-j}H)(x, y) \). \( \text{supp} \varphi \subset [\frac{1}{4}, 1] \).

a) If \( j > 4 + 2 \log_2 \epsilon \), we have for each \( n \in \mathbb{Z}^+ \),

\[
|K_j(x, y)| \leq C_n \sum_{\ell=0}^{2N} 2^{j/2}(1 + 2^{j/2}|x \pm y \pm 2\ell|)^{-n},
\]

where \( N = \text{the smallest integer} \geq \max\{1, n/4\} \).

b) If \( -\infty < j \leq J := 4 + [2 \log_2 \epsilon] \), then for each \( n \geq 0 \)

\[
|K_j(x, y)| \leq C_n 2^{j/2}(1 + 2^{j/2}|x \pm y|)^{-n}.
\]

**Lemma 3.2.** Let \( K(x, y) = \Phi(H)(x, y) \), \( \text{supp} \Phi \subset [-1, 1] \). We have for each \( n \geq 0 \)

\[
|K(x, y)| \leq C_n (1 + |x - y|)^{-n}.
\]

We also need decay estimates for the derivative of the kernel.

**Lemma 3.3.** Let \( \varphi_j(x) = \varphi(2^{-j}x) \). \( K_j(x, y) = \varphi_j(H)(x, y) \).

a) If \( j > J \), then for each \( n \) there is a constant \( C_n \):

\[
\left| \frac{\partial}{\partial x} K_j(x, y) \right| \leq C_n \sum_{\ell=0}^{2N} 2^j (1 + 2^{j/2}|x \pm y \pm 2\ell|)^{-n},
\]

where \( N \) means the the same as in Lemma 3.1 (a).

b) If \( -\infty < j \leq J \), then for each \( n \) there is a constant \( C_n \):

\[
\left| \frac{\partial}{\partial x} K_j(x, y) \right| \leq C_n 2^j (1 + 2^{j/2}|x \pm y|)^{-n}.
\]
Lemma 3.4. Let $\Phi$ be as in Lemma 3.2. Then for each $n$
\[
\left| \frac{\partial}{\partial x} K(x, y) \right| \leq C_n (1 + |x - y|)^{-n}.
\]

Proofs of Lemma 3.1–3.4 are given in §4 and §5, which are elementary calculus but quite lengthy. These lemmas are essential for us to establish a Peetre type maximal inequality.

Given $s > 0$ define the Peetre maximal functions for $H$ by: if $j > J$,
\[
\varphi^*_j f(x) = \sup_{t \in \mathbb{R}} \frac{|\varphi_j(H)f(t)|}{\min_{\ell, \pm} (1 + 2^{j/2}|x \pm t + 2\ell|)^s},
\]
and
\[
\varphi^{**}_j f(x) = \sup_{t \in \mathbb{R}} \frac{|(\varphi_j(H)f)'(t)|}{\min_{\ell, \pm} (1 + 2^{j/2}|x \pm t + 2\ell|)^s},
\]
where the minimum is taken over $0 \leq \ell \leq 2N$ and $N = \max\{1, \frac{|s|+2}{4}\}$.

Similarly, define for $j \leq J$,
\[
\varphi^*_j f(x) = \sup_{t \in \mathbb{R}} \frac{|\varphi_j(H)f(t)|}{\min_{\ell, \pm} (1 + 2^{j/2}|x \pm t|^s)}, \quad s > 0,
\]
\[
\Phi^* f(x) = \sup_{t \in \mathbb{R}} \frac{|\Phi(H)f(t)|}{(1 + |x - t|)^s},
\]
\[
\varphi^{**}_j f(x) = \sup_{t \in \mathbb{R}} \frac{|(\varphi_j(H)f)'(t)|}{\min_{\ell, \pm} (1 + 2^{j/2}|x \pm t|^s)},
\]
\[
\Phi^{**} f(x) = \sup_{t \in \mathbb{R}} \frac{|(\Phi(H)f)'(t)|}{(1 + |x - t|)^s}.
\]
We have used the abbreviation $\varphi^*_j f := \varphi^*_{j,s} f$. Notice that
\[
(9) \quad \varphi^*_j f(x) \geq |\varphi_j(H)f(x)|.
\]

In the following we slightly abuse the notation $\varphi^*_0 f = \Phi^* f$, etc, in case of no confusion.

Lemma 3.5. For $s > 0$, there exists a constant $c_s > 0$ such that
\[
\varphi^{**}_j f(x) \leq c_s 2^{j/2} \max_{k, \pm} \varphi^*_j f(x \pm 2k),
\]
where the maximum is taken over $0 \leq k \leq 2N$ and both $\pm$.

Proof. From the identity
\[
\varphi_j(H)f(x) = \sum_{\nu=-1}^{1} (\varphi \psi)_{j+\nu}(H) \varphi_j(H)f, \quad f \in L^2
\]
with convention \( \varphi_0 = \Phi \) and \( \varphi_{-1} = 0 \), we derive

\[
\frac{d}{dt}(\varphi_j(H)f)(t) = \sum_{\nu=-1}^{1} \int_{\mathbb{R}} \frac{\partial}{\partial t} K_{j+\nu}(t,y) \varphi_j(H)f(y) \, dy,
\]

where \( K_j \) denotes the kernel of \( (\varphi\psi)(2^{-j}H) \).

Let \( j > J \) first. Apply Lemma 3.3 to get

\[
\frac{|\frac{d}{dt}(\varphi_j(H)f)(t)|}{\min_{k,\pm}(1 + 2^{j/2}|x \pm t \pm 2k|)^s} \leq C_n \sum_{\nu=-1}^{1} \sum_{\ell,\sigma,\mu} \int_{\mathbb{R}} 2^{j+\nu} \frac{|\varphi_j(H)f(y)| \, dy}{(1 + 2^{j/2}|t + \sigma y + \mu 2\ell|)^s} \cdot \frac{|\varphi_j(H)f(y)| \, dy}{\min_{k,\pm}(1 + 2^{j/2}|x \pm t \pm 2k|)^s},
\]

where the inner sum is taken over all \( 0 \leq \ell \leq 2N \) and \( \sigma, \mu \in \{ \pm 1 \} \); similar notation for \( \min_{\ell,\pm} \).

**Claim.**

\[
\frac{|\varphi_j(H)f(y)|}{\min_{\ell,\pm}(1 + 2^{j/2}|x \pm t \pm 2\ell|)^s} \leq \max_{k,\pm} \varphi_j^*(f(x \pm 2k) \min_{\ell,\pm} (1 + 2^{j/2}|t \pm y \pm 2\ell|)^s).
\]

To prove the claim, note that \( \exists \delta, \varepsilon \in \{ \pm 1 \} \) and \( \ell_0 \) such that

\[
\min_{\ell,\pm}(1 + 2^{j/2}|x \pm t \pm 2\ell|) = 1 + 2^{j/2}|x + \delta t + \varepsilon 2\ell_0|
\]

for given \( x, t \). Then for each \( \sigma, \mu \) and \( \ell \) the left hand side is less than or equal to

\[
\frac{|\varphi_j(H)f(y)|}{\min_{k,\pm}(1 + 2^{j/2}|x + \epsilon \cdot 2\ell_0 \pm y \pm 2k|)^s} \cdot \frac{(1 + 2^{j/2}|x + \epsilon \cdot 2\ell_0 + \sigma' y + \mu' 2\ell|)^s}{(1 + 2^{j/2}|x + \delta t + \varepsilon 2\ell_0|)^s} \leq \varphi_j^*(f(x + \epsilon 2\ell_0)(1 + 2^{j/2}| - \delta t + \sigma' y + \mu' 2\ell|)^s \leq \max_{k,\pm} \varphi_j^*(f(x \pm 2k)(1 + 2^{j/2}|t + \sigma y + \mu 2\ell|)^s),
\]

where we put \( \sigma' = -\delta \sigma, \mu' = -\delta \mu \) and used for \( s > 0 \),

\[
(1 + 2^{j/2}|x + \epsilon 2\ell_0 + \sigma' y + \mu' 2\ell|)^s \leq (1 + 2^{j/2}|x + \delta t + \varepsilon 2\ell_0|)^s(1 + 2^{j/2}| - \delta t + \sigma' y + \mu' 2\ell|)^s.
\]

Since \( \sigma, \mu, \ell \) are arbitrary, the claim is proved.
It follows that
\[
\frac{|\frac{d}{dt}(\varphi_j(H)f)(t)|}{\min_{k,\pm}(1 + 2^{j/2}|x \pm t \pm 2k|)^s} \leq C_n \sum_{\nu=-1}^{1} \sum_{k,\pm} \max_{\nu,\pm} \varphi^*_j f(x \pm 2k) \times \\
\int_{\mathbb{R}} \frac{2^{j+\nu}}{(1 + 2^{j+\nu}|t + \sigma y + \mu 2\ell|^n)^{s}} \cdot (1 + 2^{j/2}|t + \sigma y + \mu 2\ell|)^s dy
\]
\[
\leq C_n \max_{0 \leq k \leq 2N} \varphi^*_j f(x \pm 2k) \sum_{\ell=0}^{2N} \int_{\mathbb{R}} \frac{2^{j+n/2}}{(1 + 2^{j/2}|t + \sigma y + \mu 2\ell|^{n-s})^n} dy
\]
\[
\leq C_{n,s}(2N + 1) \max_{0 \leq k \leq 2N} \varphi^*_j f(x \pm 2k)^{2^{j/2}},
\]
provided \(n - s > 1\). Thus one may take \(n = \lfloor s \rfloor + 2\).

For \(j \leq J\) similarly we obtain the following inequalities, using Lemma 3.3(b) and Lemma 3.4 in place of Lemma 3.3(a),
\[
\varphi^*_j f(x) \leq C 2^{j/2} \varphi_j f(x)
\]
and
\[
\Phi^*_f(x) \leq C \Phi f(x).
\]
This proves Lemma 3.5.

We are ready to show Peetre maximal inequality for \(H\). Let \(M\) be the Hardy-Littlewood maximal operator:
\[
Mf(x) = \sup_{x \in I} |I|^{-1} \int_I |f(u)| du,
\]
where the supreme is taken over all intervals \(I\) containing \(x\).

**Lemma 3.6.** Let \(0 < r < \infty\). There exists a constant \(C > 0\) independent of \(0 < \varepsilon \leq 1\) such that
\[
\varphi^*_j f(x) \leq C\varepsilon \sum_{\ell=0}^{2N} \varphi^*_j f(x \pm 2\ell) + C\varepsilon^{-1/r} \sum_{\ell=0}^{2N} [M(|\varphi_j(H)f|^r)]^{1/r}(\pm x \pm 2\ell),
\]
where \(\varepsilon > 0\) can be chosen arbitrarily small.

**Remark 1.** It is well known that \(M\) is bounded on \(L^p, 1 < p < \infty\). Lemma 3.6 implies that if \(s = 1/r\), then
\[
\|\varphi^*_j f\|_p \leq c \|\varphi_j(H)f\|_p, \quad 0 < p \leq \infty
\]
by taking \(\varepsilon\) small enough and \(0 < r < p (s = 1/r > 1/p)\).
Remark 2. For $j \leq J$, the inequality in Lemma 3.6 takes a simpler form

$$\varphi_j^* f(x) \leq C_s \epsilon^{-s} [M(\varphi_j(H)f)^r]^1/r (\pm x),$$

$$\Phi^* f(x) \leq C_s \epsilon^{-s} [M(\Phi(H)f)^r]^1/r (x).$$

Compare the analogue in Fourier case [39] and Hermite case [10].

**Proof.** Let $g(x) \in C^1(\mathbb{R})$. As in [39], the mean value theorem gives for $z_0 \in \mathbb{R}, \delta > 0$

$$|g(z_0)| \leq 2\delta \sup_{|z-z_0| \leq \delta} |g'(z)| + (2\delta)^{-1/r} \left( \int_{|z-z_0| \leq \delta} |g|^r dz \right)^{1/r}.$$ 

Put $g(z) = \varphi_j(H)f(\pm x \mp 2\ell - z) \in C^1$ to get, with $0 < \delta \leq 1, 0 \leq \ell \leq 2N$,

$$\frac{\varphi_j(H)f(\pm x \pm 2\ell - z)}{(1 + 2^{j/2}|z|)^{1/r}} \leq 2\delta \sup_{|u| \leq |\delta|} \frac{(1 + 2^{j/2}|u|)^{1/r} |x|^{2j/2} (\varphi_j(H)f)(\pm x \pm 2\ell - u)}{(1 + 2^{j/2}|z|)^{1/r} (1 + 2^{j/2}|u|)^{1/r}}$$

$$+ (2\delta)^{-1/r} \left( \int_{|u| \leq |\delta|} |\varphi_j(H)f(\pm x \pm 2\ell - u)|^r du \right)^{1/r}$$

$$\leq 2\delta (1 + 2^{j/2}\delta)^{1/r} \sup_{u \in \mathbb{R}} |\frac{\partial}{\partial x} (\varphi_j(H)f)(\pm x \pm 2\ell - u)|$$

$$+ (2\delta)^{-1/r} \left( \int_{|u| \leq |\delta|} |\varphi_j(H)f(\pm x \pm 2\ell - u)|^r du \right)^{1/r}$$

$$= 2\delta (1 + 2^{j/2}\delta)^{1/r} \varphi_j^{**} f(x) + \delta^{-1/r} \left( \frac{|z| + \delta}{1 + 2^{j/2}|z|} \right)^{1/r} [M(\varphi_j(H)f)^r(\pm x \pm 2\ell)]^{1/r}$$

$$= C \epsilon \sum_{\ell=0}^{2N} \varphi_j^* f(x \pm 2\ell) + (1 + \epsilon^{-1})^{1/r} \sum_{\ell=0}^{2N} [M(\varphi_j(H)f)^r]^1/r (\pm x \pm 2\ell),$$

by taking $\delta = 2^{-j/2}\epsilon$ and using Lemma 3.5. This proves the lemma. \(\square\)

A direct consequence of Lemma 3.6 is the Peetre maximal function characterization of the spaces $B^{\alpha,q}_p(H)$.

**Theorem 3.7.** Let $\alpha \in \mathbb{R}, 0 < p, q \leq \infty$. If $\varphi_j^* f$ and $\Phi^* f$ are defined with $s > 1/p$, we have for $f \in L^2$

$$\|f\|_{B^{\alpha,q}_p} \approx \|\Phi^* f\|_p + \left( \sum_{j=1}^{\infty} 2^{jaq} \|\varphi_j^* f\|_p^q \right)^{1/q}.$$ 

(11)
Furthermore, $B_p^{α,q}$ is a quasi-Banach space (Banach space if $p \geq 1, q \geq 1$) and it is independent of the choice of $\{Φ, ϕ_j\}_{j≥1}$.

**Proof.** In view of (9), it is sufficient to show that

$$\|Φ^* f\|_p + \left( \sum_{j=1}^{∞} 2^{jαq} \|ϕ_j^* f\|_p^q \right)^{1/q} \leq C \|f\|_{B_p^{α,q}}$$

but this follows from (10) immediately.

Next we show that $B_p^{α,q}$ is independent of the generating functions, i.e., given two systems $\{ϕ_j, ψ_j\}$ and $\{ϕ̃_j, ψ̃_j\}$, then $\|f\|_{B_p^{α,q}}^ϕ$ and $\|f\|_{B_p^{α,q}}^{ϕ̃}$ are equivalent quasi-norms on $B_p^{α,q}$.

Write $ϕ_j(H) = \sum_{ν=1}^{1} ϕ_j(H)(ϕ̃_ν)_{j+ν}(H)$ by the identity $ϕ_j(x) = ϕ_j(x) \sum_{ν=1}^{1} (ϕ̃_ν)_{j+ν}(x), \forall x$. We have by Lemma 3.1,

$$|ϕ_j(H) f(x)| \leq \sum_{ν=1}^{1} \sum_{ℓ,±} \int_{R} \frac{2j/2}{(1 + 2j/2 |x ± y ± 2ℓ|)^α} |ϕ_j+ν(H) \tilde{f}(y)| dy$$

$$\leq C \sum_{ν=1}^{1} \tilde{ϕ}_{j+ν}^* f(x) \sum_{ℓ,±} \int_{R} \frac{2j/2}{(1 + 2j/2 |x ± y ± 2ℓ|)^α} \min(1 + 2j/2 |x ± y ± 2k|) dy$$

$$\leq C_N \sum_{ν=1}^{1} \tilde{ϕ}_{j+ν}^* f(x),$$

provided $n - s > 1$. Thus for $f ∈ L^2$

$$\|f\|_{B_p^{α,q}}^ϕ = \|2^{jα} ϕ_j(H) f\|_p \|2^{jα} \tilde{ϕ}_j^* f\|_p \leq C_α \|2^{jα} \|\tilde{ϕ}_j^* f\|_p \|f\|_{B_p^{α,q}}^ϕ \approx \|f\|_{B_p^{α,q}}^ϕ.$$ 

This concludes the proof of Theorem 3.7 (That $B_p^{α,q}$ are quasi-Banach spaces follows directly from the definition).

As expected from Lemma 3.6 we can define the homogeneous Besov spaces and obtain a maximal function characterization as well.

Let $ϕ, ψ ∈ C^∞$ satisfy

i) supp $ϕ$, supp $ψ ⊂ \{ \frac{1}{4} ≤ |ξ| ≤ 1 \}$;

$|ϕ(ξ)|, |ψ(ξ)| ≥ c > 0$ if $\frac{3}{8} ≤ |ξ| ≤ \frac{7}{8};$

ii) $\sum_{j=-∞}^{∞} ϕ(2^{-j} ξ) ψ(2^{-j} ξ) = 1$, $\forall ξ \neq 0$.

**Definition.** The homogeneous Besov space $\dot{B}_p^{α,q}(H)$ associated with $H$ is the completion of the set $\{ f ∈ L^2 : \|f\|_{B_p^{α,q}}^ϕ < ∞ \}$ with respect to the norm $\|f\|_{\dot{B}_p^{α,q}} $, where

$$\|f\|_{\dot{B}_p^{α,q}} = \left( \sum_{j=-∞}^{∞} (2^{jα} \|ϕ_j(H) f\|_p)^q \right)^{1/q}.$$
Theorem 3.8. Let $\alpha \in \mathbb{R}, 0 < p, q \leq \infty$. If $\varphi_j^* f$ is defined for $j \in \mathbb{Z}$ with $s > 1/p$, then for $f \in L^2$

$$
\|f\|_{\dot{B}_{p,q}^\alpha} \approx \left( \sum_{j=-\infty}^{\infty} 2^{j\alpha q} \|\varphi_j^* f\|_p^q \right)^{1/q}.
$$

Furthermore, $\| \cdot \|_{\dot{B}_{p,q}^\alpha}$ and $\| \cdot \|_{\tilde{\dot{B}}_{p,q}^\alpha}$ are equivalent norms on the quasi-Banach space $\dot{B}_{p,q}^\alpha$ for any given two systems $\{\varphi_j\}$ and $\{\tilde{\varphi}_j\}$.

The proof is completely implicit in that of Theorem 3.7 and hence omitted.

Moreover, like in the Fourier case and Hermite case [39], [10], Peetre maximal inequality enables us to define and characterize Triebel-Lizorkin spaces; see [42].

4. High and low energy estimates

We give proofs of Lemma 3.1 and Lemma 3.3 for the decay estimates of the kernel $\varphi_j(H)(x,y)$ and $\frac{\partial}{\partial y} \varphi_j(H)(x,y)$. Recall that if $H = \int \lambda dE_\lambda$ is the spectral resolution of $H$, then $\varphi_j(H) = \int \varphi(2^{-j}\lambda) dE_\lambda = \mathcal{F}^{-1} \varphi_j(\xi^2) \mathcal{F}$ with supp $\varphi_j \subset [2^{j-2}, 2^j]$, which means that the spectrum of $\varphi_j(H)$ is bounded away from 0.

When $j > J = 4 + [2 \log_2 \varepsilon]$, we treat $K_j(x,y)$, the kernel of the operator $\varphi_j(H)$, as an oscillatory integral as $\xi \to \infty$. When $j \leq J$, we use the asymptotic property (as $\xi \to 0$) of eigenfunctions $e(x,\xi)$ to get estimates for the kernel.

Since $e(x,\xi)$ has different expressions as $x > 1$, $|x| \leq 1$ and $x < -1$, the estimates are divided into nine cases, namely,

1a. $x > 1, y > 1$; 1b. $x > 1, |y| \leq 1$; 1c. $x > 1, y < -1$

2a. $|x| \leq 1, y > 1$; 2b. $|x| \leq 1, |y| \leq 1$; 2c. $|x| \leq 1, y < -1$

3a. $|x| < -1, y > 1$; 3b. $|x| < -1, |y| \leq 1$; 3c. $x < -1, y < -1$.

By virtue of the relation $e(x,-\xi) = e(-x,\xi)$ and the trivial conjugation relation $\varphi(\lambda^2 H)(x,y) = \overline{\varphi}(\lambda^2 H)(y,x) = \varphi(\lambda H)(-x,-y)$, we see, however, these cases reduces to the following four cases: 1a, 1b, 1c, 2b.

Let $\lambda = 2^{-j/2}$, then $\lambda^{-1} > 4\varepsilon \iff j > J = [2 \log_2 \varepsilon] + 4$. Recall from (7) that

$$(12) \quad K_j(x,y) = \frac{1}{2\pi} \int \psi(\lambda \xi) e(x,\xi) \pi(y,\xi) d\xi,$$

where $\psi(x) = \varphi(x^2)$, with $\varphi$ satisfying $\varphi \in C^\infty_0, \text{supp } \varphi \subset [1/4, 1] \cup [-1, -1/4]$.
4.1. High energy estimates $j > J$

**Proof of Lemma 3.1(a).** We only show Cases 1a and 2b. Cases 1b and 1c can be shown similarly.

**Case 1a.** $x > 1, y > 1$. Let $I(x, y) = 2\pi K_j(x, y)$. Then by (12)

\[
I(x, y) = \int_{1/2\lambda}^{1/\lambda} \psi(\lambda \xi) C_+ e^{ix\xi} C_+ e^{iy\xi} d\xi \\
+ \int_{-1/\lambda}^{-1/2\lambda} \psi(\lambda \xi) (e^{ix\xi} + C'_- e^{-ix\xi}) e^{iy\xi} + C'_- e^{-iy\xi} d\xi := I^+ + I^-.
\]

Convention. $\int^+ = \int_{1/2\lambda}^{1/\lambda}$, $\int^- = \int_{-1/\lambda}^{-1/2\lambda}$.

We break the estimate of $I^+$ into two parts:

\[
\int^+ = \int_{1/2\lambda}^{1/\lambda} \psi(\lambda \xi)|C_+|^2 e^{i(x-y)\xi} d\xi \\
= \int_{1/2\lambda}^{1/\lambda} \psi(\lambda \xi) \frac{4K^2\xi^2}{4K^2\xi^2 + \varepsilon^4 \sin^2 2K} e^{i(x-y)\xi} d\xi \\
\leq \sum_{p=0}^{N-1} \left| \int_{1/2\lambda}^{1/\lambda} \psi(\lambda \xi) \left( \frac{\varepsilon^4 \sin^2 2K}{4K^2\xi^2} \right)^p e^{i(x-y)\xi} d\xi \right| \\
+ \left| \int_{1/2\lambda}^{1/\lambda} \psi(\lambda \xi) \tilde{O}\left( \frac{\varepsilon^4 \sin^2 2K}{4K^2\xi^2} \right)^N e^{i(x-y)\xi} d\xi \right| := I^+_N + R^+_N,
\]

where we used

\[
\frac{4K^2\xi^2}{4K^2\xi^2 + \varepsilon^4 \sin^2 2K} = \sum_{p=0}^{\infty} (-1)^p \left( \frac{\varepsilon^4 \sin^2 2K}{4K^2\xi^2} \right)^p
\]

because $\frac{\varepsilon^4 \sin^2 2K}{4K^2\xi^2} \leq \frac{\varepsilon^4}{3\xi^2} \leq \frac{1}{3} \left( \frac{1}{2} \right)^4 < 1$, if $|\xi| \geq 1/2\lambda > 2\varepsilon$ ($K^2 \geq 3/4\xi^2$).

**Notation.** $\tilde{O}(\xi^{-m}) := O(\xi^{-m})$ denotes a function whose derivatives of arbitrary order $\geq 0$ has estimates $O(\xi^{-m})$, as $\xi \to \infty$.

Note that

\[
\sum_{p=N}^{\infty} (-1)^p \left( \frac{\varepsilon^4 \sin^2 2K}{4K^2\xi^2} \right)^p = (-1)^N \left( \frac{\varepsilon^4 \sin^2 2K}{4K^2\xi^2} \right)^N / \left( 1 + \frac{\varepsilon^4 \sin^2 2K}{4K^2\xi^2} \right).
\]

If we write $\sin 2K = (2i)^{-1}(e^{i2K} - e^{-i2K})$, the integral in each term of the sum $I^+_N$ is bounded by a linear combination of the form, with $0 \leq \ell \leq 2p, 0 \leq p \leq N - 1$,

\[
\int_{1/2\lambda}^{1/\lambda} \psi(\lambda \xi) \frac{e^{\pm i2K\ell}}{(4K^2\xi^2)^p} e^{i(x-y)\xi} d\xi
\]
\[ = \int^+ \psi(\lambda \xi)(4K^2\xi^2)^{-p} e^{\pm i2(K-\xi)\ell} e^{i(x-y+2\ell)\xi} d\xi. \]

The following estimates will be used often.

\[
\begin{align*}
\frac{d^n}{d\xi^n}[\psi(\lambda \xi)] &= \lambda^n \psi^{(n)}(\lambda \xi) \leq C \lambda^n \\
\frac{d^n}{d\xi^n}[(K^2\xi^2)^{-p}] &= \frac{d^n}{d\xi^n}\left(\frac{1}{\xi^n e^{i\pi/2}}} \right) \\
\frac{d^n}{d\xi^n}[e^{\pm i2(K-\xi)\ell}] &= \frac{d^n}{d\xi^n}\left[\sum_{n=0}^{\infty} \frac{(\pm i2(K-\xi)\ell)^n}{n!} \right] = \sum_{n=0}^{\infty} \frac{(\pm i2\ell)^n}{n!} \frac{d^n}{d\xi^n}(K-\xi)^n \\
\end{align*}
\]

where we estimated

\[
\frac{d^n}{d\xi^n}(K-\xi)^n = \frac{d^n}{d\xi^n}\left(\frac{e^{2n}}{\xi^n} \left[\sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{e^{2}}{\xi^2} \right)^{k-1} \right] \right) \\
= O\left(\frac{1}{\xi^{n+j}}, \ n > 0.\right)
\]

We have

\[
\frac{d^n}{d\xi^n}[\psi(\lambda \xi)(K^2\xi^2)^{-p} e^{\pm i2(K-\xi)\ell}] \\
= \sum_{i+j+k=n} \lambda^i \frac{1}{\xi^{4p+j}} \frac{1}{\xi^{k+1}} + \sum_{i+j+k=n} \lambda^i \frac{1}{\xi^{4p+j}} O(1) \\
\leq C_{n,p,\ell} \lambda^{4p+1+n} + C_n \lambda^{4p+n} \leq \lambda^{4p+n}.
\]

Integration by parts yields

\[
\int^+ = C_{n,\xi} \frac{\lambda^{4p+n-1}}{|x-y\pm 2\ell|^n}.
\]

It follows that

\[
(14) \quad I_N^+ \leq C_{n,\xi} \sum_{p=0}^{N-1} \sum_{\ell=0}^{2p} \lambda^{4p+n-1} \frac{1}{|x-y\pm 2\ell|^n}.
\]

Also,

\[
(15) \quad R_N^+ \leq C \frac{\lambda^{-1}}{|x-y|^n} \lambda^{4N} \leq C \frac{\lambda^{n-1}}{|x-y|^n} \quad (4N \geq n)
\]
follows via integration by parts and the estimates
\[
\begin{align*}
\left\{ \frac{d}{d\xi} \psi(\lambda \xi) \right\} & \leq C \lambda^i, \\
\left\{ \frac{d}{d\xi} (-1)^N \frac{\epsilon^4 \sin^2 2K}{4K^2 \xi^2} \right\} N & = O(\frac{1}{\xi^{4N}}).
\end{align*}
\]

Combining (14) and (15) we obtain
\[
|I^+| \leq I^+_N + R^+_N \leq C_{N, \epsilon} \sum_{\ell=0}^{2N-2} \frac{\lambda^{n-1}}{|x - y \pm 2\ell|^n}.
\]

For \( I^- \), we denote \( \int^- = \int_{-1/\lambda}^{-1/2\lambda} \), then
\[
I^- = \int_{-1/\lambda}^{-1/2\lambda} \psi(\lambda \xi) e^{i(x-y)\xi} d\xi + \int \psi(\lambda \xi) C^-_e e^{-i(x+y)\xi} d\xi
\]
\[
+ \int \psi(\lambda \xi) C'_e e^{-i(x+y)\xi} d\xi + \int \psi(\lambda \xi) |C'_e|^2 e^{-i(x-y)\xi} d\xi
\]
\[
:= I_1^- + I_2^- + I_3^- + I_4^-.
\]

As estimating \( I^+ \) we have
\[
|I^-| \leq C \sum_{\ell=0}^{2N} \frac{\lambda^{n-1}}{|x - y \pm 2\ell|^n}.
\]

Hence we obtain that if \( x > 1, y > 1 \),
\[
2\pi |K_j(x, y)| \leq |I^+| + |I^-| \leq C \sum_{\ell=0}^{2N} \frac{\lambda^{n-1}}{|x - y \pm 2\ell|^n}.
\]

**Case 2b.** \(|x| \leq 1, |y| \leq 1\). Let notation be the same as in Case 1a. by symmetry it is enough to deal with \( I^+ \).

From the expression of \( B_+, B'_+ \) we have
\[
I^+ = \int^+ \psi(\lambda \xi) (B_+ e^{iKx} + B'_+ e^{-iKx}) B_+ e^{iKy} d\xi + B'_+ e^{-iKy} d\xi
\]
\[
= \int^+ \psi(\lambda \xi) |B'_+|^2 e^{iK(x-y)} d\xi + \int^+ \psi(\lambda \xi) B_+ B'_+ e^{iK(x+y)} d\xi
\]
\[
+ \int^+ \psi(\lambda \xi) B' B e^{-iK(x+y)} d\xi + \int^+ \psi(\lambda \xi) |B'_+|^2 e^{-iK(x-y)} d\xi
\]
\[
:= I^+_1 + I^+_2 + I^+_3 + I^+_4.
\]

We estimate these terms separately. For instance,
\[
I^+_2 = \frac{1}{4} \int^+ \psi(\lambda \xi) e^{i(x+y)K} |C_+|^2 e^{-2iK(1 - \xi^2/K^2)} d\xi
\]
Using the identity
\[ |C_+|^2 = \frac{4K^2\xi^2}{4K^2\xi^2 + \varepsilon^4\sin^2 2K} = \sum_{p=0}^{\infty} (-1)^p \left( \frac{\varepsilon^2\sin 2K}{2K\xi} \right)^{2p} \]
\[ = \sum_{p=0}^{N-1} (-1)^p \left( \frac{\varepsilon^2\sin 2K}{2K\xi} \right)^{2p} + \tilde{O}(\xi^{-4N}), \]
we can write
\[ 4I_2^+ = \sum_{p=0}^{N-1} (-1)^p \int^+ \psi(\lambda\xi)e^{i(x+y-2)K} \left( \frac{\varepsilon^2\sin 2K}{2K\xi} \right)^{2p}(1 - \xi^2/K^2)d\xi \]
\[ + \int^+ \psi(\lambda\xi)e^{i(x+y-2)K} \tilde{O}(\xi^{-4N})(1 - \xi^2/K^2)d\xi := I^+_{2,N} + R^+_{2,N}. \]

The integral in each term of the sum \( I^+_{2,N} \) is bounded by a linear combination of the form
\[ \int^+ \psi(\lambda\xi)e^{i(x+y-2)K} e^{\pm i2K\ell}(2K\xi)^{-2p}(1 - \xi^2/K^2)d\xi. \]

Integration by parts gives us
\[ |I_2^+| \leq CN \sum_{\ell=0}^{2N-1} \frac{\lambda^{n-1}}{|x + y \pm 2\ell|^n}. \]

The other terms \( I_1^+, I_3^+, I_4^+ \) also verify the above inequality (possibly with \( x + y \) replaced by \( x - y \)). And so does \( I^+ \) and \( I^- \).

We have
\[ |K_j(x,y)| \leq C \sum_{\ell=0}^{2N-1} \frac{\lambda^{n-1}}{|x \pm y \pm 2\ell|^n} \quad (|x| \leq 1, |y| \leq 1). \]

This completes the proof of Lemma 3.1(a).

**Proof of Lemma 3.3 (a).** Note that \( \frac{\partial}{\partial x} e(x,\xi) \) exist for all \( \xi \neq \pm \varepsilon, 0 \) and are uniformly bounded in \( x \in \mathbb{R} \) and \( \xi \) in any bounded set.

\[ \frac{\partial}{\partial x} K_j(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda^2\xi^2) \frac{\partial}{\partial x} e(x,\xi) \bar{e}(y,\xi) d\xi. \]
Case 1. $x > 1$, $y \in \mathbb{R}$.

\[
\frac{\partial}{\partial x} K_j(x, y) = \frac{\partial}{\partial x} \int_{2\epsilon < 1/2\lambda \leq |\xi| \leq 1/\lambda} \psi(\lambda \xi)(Ce^{ix\xi} + C'e^{-ix\xi})\bar{e}(y, \xi) d\xi \\
= \int i\xi \psi(\lambda \xi)(Ce^{ix\xi} - C'e^{-ix\xi})\bar{e}(y, \xi) d\xi \\
= i\lambda^{-1} \int \delta(\lambda \xi)(Ce^{ix\xi} - C'e^{-ix\xi})\bar{e}(y, \xi) d\xi,
\]

where $\delta(x) = x\psi(x)$ satisfies the same conditions as $\psi(x)$: (i) $\delta \in C^\infty$ (ii) $\text{supp } \delta \subset \{1/2 \leq |\xi| \leq 1\}$ (except for $\psi$ being even, which is unimportant). Thus we obtain, similar to the case for $\psi(\lambda H)(x, y)$

\[
\left| \frac{\partial}{\partial x} \int \right| \leq C_N \sum_{\ell=0}^{2N} \frac{\lambda^{-2}}{(1 + \lambda^{-1}|x \pm y \pm 2\ell|)^n}.
\]

Case 2. $|x| \leq 1$, $y \in \mathbb{R}$.

Write

\[
B(\xi) = \begin{cases} B_+ & \xi > 0 \\ B_- & \xi < 0. \end{cases}
\]

\[
\frac{\partial}{\partial x} \int = \frac{\partial}{\partial x} \int_{2\epsilon < 1/2\lambda \leq |\xi| \leq 1/\lambda} \psi(\lambda \xi)(Be^{iKx} + B'e^{-iKx})\bar{e}(y, \xi) d\xi \\
= \int_{\mathbb{R}} iK \psi(\lambda \xi)(Be^{iKx} - B'e^{-iKx})\bar{e}(y, \xi) d\xi \\
= i\lambda^{-1} \int (\lambda \xi)\psi(\lambda \xi)(Be^{iKx} - B'e^{-iKx})K/\xi \bar{e}(y, \xi) d\xi \\
= i\lambda^{-1} \int \delta(\lambda \xi)(Be^{iKx} - B'e^{-iKx})K/\xi \bar{e}(y, \xi) d\xi,
\]

where $\delta(x) = x\psi(x)$. Thus we obtain, similar to the case for $\psi(\lambda H)(x, y)$

\[
\left| \frac{\partial}{\partial x} \int \right| \leq C_N \sum_{\ell=0}^{2N} \frac{\lambda^{-2}}{(1 + \lambda^{-1}|x \pm y \pm 2\ell|)^n}.
\]

Case 3. $x < -1$, $y \in \mathbb{R}$. The relation $\varphi(\lambda H)(x, y) = \varphi(\lambda H)(-x, -y)$ implies

\[
\frac{\partial}{\partial x} [\varphi(\lambda H)(x, y)]_{x<-1} = \frac{\partial}{\partial x} [\varphi(\lambda H)(-x, -y)]_{x<-1} \\
= -[\frac{\partial}{\partial x} \phi(\lambda H)](-x, -y)]_{x<-1}.
\]
Therefore, if $x < -1$

$$
\left| \frac{\partial}{\partial x} \varphi(\lambda H)(x, y) \right| = \left| \frac{\partial}{\partial x} \left[ \varphi(\lambda H)(x, y) \right](-x, -y) \right|
\leq C \sum_{\ell=0}^{2N} \frac{\lambda^{-2}}{(1 + \lambda^{-1}| -x + y + 2\ell|)^n}.
$$

This concludes the proof of Lemma 3.3(a). \qed

4.2. Low energy estimates \quad $j \leq J$

4.2.1. Proofs of Lemma 3.1(b). We study the decay of the kernel of $\varphi_j(H)$ as $j \to -\infty$. As in the high energy case, we only need to check four cases 1a, 1b, 1c and 2b. Outline will be given for 1a, 2b only.

Case 1a. $x > 1$, $y > 1$ \quad $(1/\lambda \leq 4\varepsilon)$

$$
2\pi K_j(x, y) = \int_{\mathbb{R}} \psi(\lambda \xi) e(x, \xi) \bar{e}(y, \xi) d\xi = \int^+ + \int^-.
$$

We obtain by integration by parts

$$
\left| \int^+ \right| \leq C \frac{\lambda^{n-1}}{|x - y|^n},
$$

where we used

$$
\left\{ \frac{d^n}{d^n \xi^n} [\psi(\lambda \xi)] = \lambda^n \psi^{(n)}(\lambda \xi) \leq C \lambda^n, \quad \lambda = 2^{-j/2} \to +\infty \quad (j \to -\infty) \right. \left. \right. \right.
$$

$$
\left\{ \frac{d^i}{d^i \xi^i} (|C_+|^2) = \left\{ \begin{array}{ll} O(\xi^2) & i = 0 \\ O(\xi) & i = 1 \\ O(1) & i > 1 \end{array} \right. \right. \right.
$$

We obtain also

$$
\left| \int^- \right| \leq C_n \frac{\lambda^{n-1}}{|x - y|^n},
$$

using

$$
\frac{d^i}{d^i \xi^i} (|C'_-|^2) = \frac{d^i}{d^i \xi^i} \left[ \frac{\varepsilon^4}{(2\rho / \sinh 2\rho)^2 \xi^2 + \varepsilon^1} \right] = O(1), \quad \xi \to 0^-
$$

and \( \frac{d^i}{d^i \xi^i} C'_- = O(1) \), since \( C'_- \) is \( C^{\infty}_{[-4\varepsilon, 0]} \).
Case 2b. \(|x| \leq 1, |y| \leq 1\). Let \(I^+(x, y) := \int^+, I^-(x, y) := \int^-\).

\[
I^+(x, y) = \int^+ \psi(\lambda \xi)(B_+ e^{\rho x} + B'_+ e^{-\rho x})B_+ e^{\rho y} + B'_+ e^{-\rho y} d\xi
\]

\[
= \int^+ |C_+|^2 (\cosh \rho (1 - x) - i \xi / \rho \sinh \rho (1 - x))(\cosh \rho (1 - y) + i \xi / \rho \sinh \rho (1 - y)) d\xi
\]

\[
\leq C \lambda^{-1} \leq 3^n C \frac{\lambda^{-1}}{1 + \lambda^{-1}|x \pm y|^n},
\]

where we note that \(\cosh \rho (1 - x) - i \xi / \rho \sinh \rho (1 - x)\) and \(\cosh \rho (1 - y) + i \xi / \rho \sinh \rho (1 - y)\) are bounded by a constant uniformly in \(|x| \leq 1\) and \(|y| \leq 1\). The term \(I^-(x, y)\) satisfies the same inequality by the relation \(I^-(x, y) = I^+(-x, -y)\).

Proof of Lemma 3.3 (b). The same argument in proving Lemma 3.1(b) is valid for the proof of Lemma 3.3(b). The interested reader can fill in the details.

5. LOCAL ENERGY ESTIMATES

Let \(\Phi \in C^\infty\) have support contained in \(\{ \xi : |\xi| \leq 1\}\). Then the spectrum of \(\Phi(H)\) includes the low energy, a neighborhood of 0. We use the term “local energy” to distinguish from the low energy case, where the support of \(\varphi_j (j \leq J)\) keeps away from 0. Since \(0 \in \text{supp} \Phi\) and \(e(x, \xi)\) is not continuous at the origin \(\xi = 0 (\), we need to treat the corresponding kernel more carefully. The proof is more delicate and requires a “matching” method.

Proof of Lemma 3.2. As in §4, the estimates rely on four cases 1a, 1b, 1c, 2b. We use \(\hat{f}\) and \(\hat{f}\) to denote the ordinary Fourier transform and its inverse, resp.

Case 1a. \(x > 1, y > 1\). Let \(K(x, y) = \Phi(H)(x, y), \Psi(x) = \Phi(x^2)\).

\[
2\pi K(x, y) = \int_0^1 \Psi(\xi)C'_+ e^{ix\xi}C'_+ e^{iy\xi}d\xi + \int_{-1}^0 \Psi(\xi)(e^{ix\xi} + C'_+ e^{-ix\xi}) e^{iy\xi} + C'_+ e^{-iy\xi} d\xi
\]

\[
= I^+ + I^-.
\]
Write

\[ I^- = \int_{-1}^{0} \Psi(\xi)e^{i(x-y)\xi}d\xi + \int_{-1}^{0} \Psi(\xi)C'_-e^{-i(x+y)\xi}d\xi \]
\[ + \int_{-1}^{0} \Psi(\xi)\overline{C'_-}e^{i(x+y)\xi}d\xi + \int_{-1}^{0} \Psi(\xi)|C'_-|^2e^{-i(x-y)\xi}d\xi \]
\[ := I_1^- + I_2^- + I_3^- + I_4^- . \]

The relations \( C'_-(-\xi) = A'_+(\xi) = \overline{C'_-}(\xi) \) and \( |C_+|^2 + |A'_+|^2 = |C_+|^2 + |C'_-|^2 = 1 \) imply that

\[ I^+ + I_1^- + I_4^- = \int_{0}^{1} \Psi(\xi)|C_+|^2e^{i(x-y)\xi}d\xi \]
\[ + \int_{0}^{1} \Psi(\xi)e^{i(x-y)\xi}d\xi + \int_{0}^{1} \Psi(\xi)|C'_-|^2e^{i(x-y)\xi}d\xi \]
\[ = \int_{-1}^{1} \Psi(\xi)e^{i(x-y)\xi}d\xi = \sqrt{2\pi} \Psi^V(x-y) . \]

Also, the relation \( C'_-(-\xi) = \overline{C'_-}(\xi) \) gives

\[ I_2^- + I_3^- = \int_{-1}^{0} \Psi(\xi)C'_-e^{-i(x+y)\xi}d\xi + \int_{0}^{1} \Psi(\xi)C'_-(\xi)e^{-i(x+y)\xi}d\xi \]
\[ = \sqrt{2\pi} \Psi(\xi)C'_-(\xi)^V(x+y) . \]

Since \( \Psi \in C_0^\infty \) and \( C'_- \in C^\infty \), we have for \( x > 1, y > 1 \)

\[ 2\pi |K(x,y)| \leq |I^+ + I_1^- + I_4^-| + |I_2^- + I_3^-| \]
\[ \leq \frac{C_n}{(1 + |x - y|)^n} + \frac{C_n}{(1 + |x + y|)^n} \leq \frac{C_n}{(1 + |x - y|)^n} \]

by the rapid decay for the Fourier transform of \( C_0^\infty \) functions, where

\[ C'_-(\xi) = \frac{\varepsilon^2 \sinh 2\rho}{2\rho \xi} A_- (\xi) = i \frac{\varepsilon^2 e^{2i\xi}}{\xi \cosh 2\rho - i(\rho^2 - \xi^2) \sinh 2\rho/2\rho} \in C^\infty (\mathbb{R}) . \]

**Case 1b.** \( x > 1, |y| \leq 1 \)

Using \( e_+(y, \xi) = C_+e^{i\xi}[\cosh \rho(1-y) - i\xi/\rho \sinh \rho(1-y)] \) and \( A_- = \frac{2\rho \xi e^{2i\xi}}{2\rho \xi \cosh 2\rho - i(\rho^2 - \xi^2) \sinh 2\rho} \).
\[2\pi K(x, y) = \int_{0}^{1} \Psi(\xi)e^{i(x-1)\xi}|C_+|^2 \text{Re} + i\text{Im}[\cosh \rho(1 - y) + i\xi/\rho \sinh \rho(1 - y)]d\xi + \int_{-1}^{0} \Psi(\xi)e^{i(x-1)\xi} \text{Re} + i\text{Im} \frac{2\rho \xi[\cosh \rho(1 + y) - i\xi/\rho \sinh \rho(1 + y)]}{2\rho \cosh 2\rho + i(\rho^2 - \xi^2) \sinh 2\rho} d\xi + \int_{-1}^{0} \Psi(\xi)e^{-i(x-1)\xi} \text{Re} + i\text{Im} \frac{\epsilon^2 \sinh 2\rho \cdot 2\rho \xi}{4\rho^2 \xi^2 + \epsilon^4 \sinh^2 2\rho} [\cosh \rho(1 + y) - i\xi/\rho \sinh \rho(1 + y)]d\xi := \text{"Re" } + i\epsilon\text{"Im"},\]

where we break each of the above three integrals into two parts; then let \text{"Re"} be the sum of the three integrals involving real parts only, and let \text{"Im"} be the sum of the three integrals involving imaginary parts only.

We have

\[\text{"Re" } = \int_{0}^{1} \Psi(\xi)|C_+|^2 e^{i(x-1)\xi} \cosh \rho(1 - y)d\xi + \int_{-1}^{0} \Psi(\xi)e^{i(x-1)\xi} \text{Re} \left[\frac{2\rho \xi[\cosh \rho(1 + y) - i\xi/\rho \sinh \rho(1 + y)]}{2\rho \cosh 2\rho + i(\rho^2 - \xi^2) \sinh 2\rho}\right] d\xi + \int_{-1}^{0} \Psi(\xi)e^{-i(x-1)\xi} \text{Re} \left[\frac{\epsilon^2 \sinh 2\rho \cdot 2\rho \xi}{4\rho^2 \xi^2 + \epsilon^4 \sinh^2 2\rho}\right] [\cosh \rho(1 + y) - i\xi/\rho \sinh \rho(1 + y)]d\xi = \int_{0}^{1} \Psi(\xi)e^{i(x-1)\xi} \frac{4\rho^2 \xi^2 \cosh 2\rho \cosh(1 - y) + 2\epsilon^2 \xi^2 \sinh 2\rho \sinh(1 + y)}{4\rho^2 \xi^2 + \epsilon^4 \sinh^2 2\rho} d\xi.
\]

Noting that \(\rho^2 - \xi^2 = 2\rho^2 - \epsilon^2\), and \(\cosh 2\rho \cosh(1 + y) - \sinh 2\rho \sinh(1 + y) = \cosh \rho(1 - y)\) we obtain

\[(16)\]

\[\text{"Re" } = \sqrt{2\pi} \left[\Psi(\xi)e^{-i\xi} \frac{4\rho^2 \xi^2 \cosh(1 - y) + 2\epsilon^2 \xi^2 \cosh(1 - y) + 2\epsilon^2 \xi^2 \sinh 2\rho \sinh(1 + y)}{4\rho^2 \xi^2 + \epsilon^4 \sinh^2 2\rho}\right]^x\]
For

\[ "\mathcal{I}m" = \int_{0}^{1} \Psi(\xi)\vert C_{+} \vert^{2} e^{i(x-1)\xi} / \rho \sinh \rho (1-y) d\xi \]

\[ + \int_{-1}^{0} \Psi(\xi) e^{i(x-1)\xi} \Im \left\{ \frac{2\rho \xi [\cosh \rho (1+y) - i\xi / \rho \sinh \rho (1+y)]}{2\rho \xi \cosh 2\rho + i(\rho^{2} - \xi^{2}) \sinh 2\rho} \right\} d\xi \]

\[ + \int_{-1}^{0} \Psi(\xi) e^{-i(x-1)\xi} \frac{\varepsilon^{2} \sinh 2\rho \xi}{4\rho^{2} \xi^{2} + \varepsilon^{4} \sinh^{2} 2\rho} \cosh \rho (1+y) d\xi \]

\[ = \int_{-1}^{1} \Psi(\xi) e^{i(x-1)\xi} \frac{2\rho \xi}{4\rho^{2} \xi^{2} + \varepsilon^{4} \sinh^{2} 2\rho} \left\{ 2\xi^{2} \sinh \rho (1-y) - \varepsilon^{2} \sinh 2\rho \cosh \rho (1+y) \right\} d\xi + \int_{-1}^{0} \Psi(\xi) e^{i(x-1)\xi} \frac{2\rho \xi}{4\rho^{2} \xi^{2} + \varepsilon^{4} \sinh^{2} 2\rho} \left\{ -2\xi^{2} \cosh 2\rho \sinh (1+y) - (\rho^{2} - \xi^{2}) \sinh 2\rho \cosh (1+y) \right\} d\xi \]

Noting that \( \rho^{2} - \xi^{2} = \varepsilon^{2} - 2\xi \), and \( \sinh 2\rho \cosh \rho (1+y) - \cos 2\rho \sinh (1+y) = \sinh (1-y) \), we obtain (17)

\[ "\mathcal{I}m" = \sqrt{2\pi} [\Psi(\xi) e^{-i\xi} \frac{2\rho \xi}{4\rho^{2} \xi^{2} + \varepsilon^{4} \sinh^{2} 2\rho} \left\{ 2\xi^{2} \sinh \rho (1-y) - \varepsilon^{2} \sinh 2\rho \cosh \rho (1+y) \right\}]^{\vee}(x). \]

Since the functions in the square brackets of (16) and (17) are \( C^{\infty} \), it follows that for \( x > 1, |y| \leq 1 \),

\[ \frac{C_{n}}{1 + |x|^{n}} \leq \frac{C_{n}'}{(1 + |x-y|^{n})} \]

where \( C_{n} \) can be taken to be independent of \( |y| \leq 1 \).

Case 1c. \( x > 1, y < -1 \). The proof is similar to that of Case 1a and hence omitted.

Case 2b. \( |x| \leq 1, |y| \leq 1 \). Since \( |e(x, \xi)| \leq C_{\varepsilon} \), for all \( x, \xi \), the result is trivial:

\[ |K(x, y)| \leq C_{\varepsilon} \sim \frac{C_{n}}{(1 + |x-y|^{n})} \]

whenever \( |x| \leq 1, |y| \leq 1 \). This concludes the proof of Lemma 3.2. \( \square \)

**Proof of Lemma 3.4.** With the convention \( \int_{1} = \int_{0}^{1}, \int_{-1} = \int_{-1}^{0} \),

\[ 2 \pi \frac{\partial}{\partial x} K (x, y) = \frac{\partial}{\partial x} \int_{-1}^{1} \Psi(\xi) e(x, \xi) \overline{e(y, \xi)} d\xi \]

\[ := \frac{\partial}{\partial x} \int_{1} + \frac{\partial}{\partial x} \int_{-1} . \]

The function \( \xi \mapsto \frac{\partial}{\partial x} e(x, \xi) \) is discontinuous at \( \xi = 0 \). As suggested by the treatment of \( K(x, y) \) we want to properly “match” different parts of
the above integrals so $\frac{\partial}{\partial x}K(x, y)$ can be written as a linear combination of the Fourier transform of $C^\infty_0$ functions.

We only need to check five cases 1a, 1b, 1c, 2a, 2b. Estimates for the other cases follow readily from the relation $\frac{\partial}{\partial x}K(x, y) = \frac{\partial}{\partial x}[K(-x, -y)] = -(\frac{\partial}{\partial x}K)(-x, -y).$ We outline the proofs for 1a, 1b and 2b only, since 1c and 2a can be dealt with similarly.

**Case 1a.** $x > 1, y > 1$. Let $\Delta(\xi) = i\xi \Psi(\xi) \in C^\infty_0$.

$$\frac{\partial}{\partial x} \int^+ \Psi(\xi) i\xi |C_+|^2 e^{i(x-y)\xi} d\xi = \int^+ \Delta(\xi) |C_+|^2 e^{i(x-y)\xi} d\xi.$$

$$\frac{\partial}{\partial x} \int^- = \int^- i\xi \Psi(\xi)(e^{ix\xi} - C'_- e^{-ix\xi})e^{iy\xi} + C'_- e^{-iy\xi} d\xi$$

$$\Delta(\xi)e^{i(x-y)\xi} d\xi - \int^- \Delta(\xi) |C'_-|^2 e^{-i(x-y)\xi} d\xi$$

$$+ \int^- \Delta(\xi) C'_- e^{i(x+y)\xi} d\xi - \int^- \Delta(\xi) C'_- e^{-i(x+y)\xi} d\xi$$

$$= \int^- \Delta(\xi)e^{i(x-y)\xi} d\xi + \int^+ \Delta(\xi) |C'_-|^2 e^{i(x-y)\xi} d\xi$$

$$- \int^+ \Delta(\xi) C'_- e^{-i(x+y)\xi} d\xi - \int^- \Delta(\xi) C'_- e^{-i(x+y)\xi} d\xi,$$

where we note $\Delta(\xi)$ is odd and the relation $C'_-(\xi) = \overline{C'_+(\xi)}$.

We have, by the relation $|C_+|^2 + |C'_-|^2 = 1$,

$$\frac{\partial}{\partial x} \int^+ + \frac{\partial}{\partial x} \int^- = \int \Delta(\xi) e^{i(x-y)\xi} d\xi - \int \Delta(\xi) C'_- e^{-i(x+y)\xi} d\xi$$

$$\sqrt{2\pi} |\Delta(\xi)|^\gamma (x - y) - \sqrt{2\pi} |\Delta(\xi) C'_-|^\gamma (x + y).$$

Since $\Delta \in C^\infty_0, C'_-(\xi) \in C^\infty$, the inequality in Lemma 3.4 holds for $x > 1, y > 1$.

**Case 1b.** $x > 1, |y| \leq 1$. Let notation be as in Case 1a.

$$\frac{\partial}{\partial x} \int^+ = \int^+ \Delta(\xi)|C_+|^2 e^{i(x-1)\xi}(\Re i\Im)\cosh \rho(1 - y) + i\xi/\rho \sinh \rho(1 - y)]d\xi$$

$$\frac{\partial}{\partial x} \int^- = \int^- \Delta(\xi)e^{i(x-1)\xi}(\Re + i\Im)\left[\frac{2\rho\xi(\cosh \rho(1 + y) - i\xi/\rho \sinh \rho(1 + y)}{2\rho\xi \cosh \rho + i(\rho^2 - \xi^2) \sinh 2\rho}\right]d\xi$$

$$- \int^- \Delta(\xi)e^{-i(x-1)\xi}(\Re + i\Im)\left[\frac{i\epsilon^2 \sinh 2\rho \cdot 2\rho\xi}{4\rho^2 \xi^2 + \epsilon^4 \sinh^2 2\rho}\right]d\xi.$$

As in the case for $K(x, y)$, we split each integral into two parts and let “Re” and “Im” denote the sum of integrals involving only these
“reals” and “imaginaries” respectively. As a result,
\[ 2\pi \frac{\partial}{\partial x}K(x, y) = \Re e + i\Im m, \]
where we find, by noting that $\Delta$ is odd, $\Re e$ and $\Im m$ have the same expressions as in (16) and (17) resp., except that $\Psi$ should be replaced by $\Delta$. Case 1b is so verified.

Finally, the decay estimate for Case 2b ($|x|, |y| \leq 1$) follows trivially from the fact that $e(y, \xi) \in L_\infty^\infty(\mathbb{R} \times \mathbb{R})$ and $\frac{\partial}{\partial x} e(x, \xi) \in L_\infty^\infty_{\text{loc}}(\mathbb{R} \times \mathbb{R})$

\[ |2\pi K(x, y)| = \left| \int_{|\xi| \leq 1} \Psi(\xi) \frac{\partial}{\partial x} e(x, \xi) \bar{e}(y, \xi) d\xi \right| \]

\[ \leq C \leq \frac{C_n'}{(1 + |x - y|)^n}, \]

where
\[ \frac{\partial}{\partial x} e_+(x, \xi) = C_+ e^{i\xi} [-\rho \sinh \rho (1 - x) + i\xi \cosh (1 - x)], \]
\[ \frac{\partial}{\partial x} e_-(x, \xi) = A_- e^{-i\xi} [\rho \sinh \rho (1 + x) + i\xi \cosh (1 + x)]. \]

This completes the proof of Lemma 3.4.

\section{Spectral multipliers for $L^p$ and $B^\alpha_{p, q}(H)$}

The operator $m(H)$ can be defined using functional calculus: $m(H) = \int_0^\infty m(\lambda) dE(\lambda)$, where $H = \int_0^\infty \lambda dE(\lambda)$ is the spectral resolution of $H$. From \[10\] \[11\], \[12\], we know $m(H)$ also has the expression $m(H) = \mathcal{F}^{-1} m(\xi^2) \mathcal{F}$ if $m$ is bounded.

We shall prove that under the same differentiability condition on $m \in L^\infty$ as in the Fourier case, $m(H)$ has a bounded extension on $L^p$ from $L^p \cap L^2$, by showing that the kernel $m(H)(x, y)$ satisfies a Hörmander type condition: (compare the Fourier case \[Stein 93\]):

\[ (18) \quad \int_{z > 2|y - \bar{y}|} |m(H)(x, y) - m(H)(x, \bar{y})| dx \leq A, \]

where $z = \min_{\pm} (|x \pm \bar{y}|)$.

An immediate question arises: what is the kernel expression for $m(H)$? Since $m$ may not necessarily have compact support, the answer is not so immediate. Let $\{ \delta_j \}_{-\infty}^{\infty}$ be a smooth dyadic resolution of unit and $m_j(x) = m\delta_j(x)$. Then for $f \in L^2$, $m(H)f = \sum_{-\infty}^{\infty} m_j(H)f$ in
$L^2$. This suggests that $m(H)(x, y)$ may have the (pointwise) expression $\sum_{-\infty}^{\infty} K_j(x, y)$, where $K_j$ denotes the kernel of $m_j(H)$. Our next lemma shows that this is true in an appropriate sense.

**Lemma 6.1.** Let $m$ be bounded and $|m'(\xi)| \leq C|\xi|^{-1}$ for $\xi \in \mathbb{R} \setminus \{0\}$. Then for $f \in L^2_0 = \{f \in L^2 : \text{supp } f \text{ is compact}\}$, $m(H)f$ has the expression

$$m(H)f(x) = \int m(H)(x, y)f(y)dy$$

for a.e. $x \notin \pm \text{supp } f$, where $m(H)(x, y) = \sum_{-\infty}^{\infty} m_j(H)(x, y)$.

**Proof.** Since $\sum_{-\infty}^{\infty} m_j(H)f$ converges to $m(H)f$ in $L^2$, it suffices to show the series $\sum_{-\infty}^{\infty} m_j(H)f(x)$ converges pointwise for each $x \notin \pm \text{supp } f$.

Let $0 < t = \text{the distance from } x \text{ to the set } (\text{supp } f) \cup (-\text{supp } f)$. Then $\text{supp } f \subset \{y : \text{min}(|y + x|, |y - x|) \geq t\}$. By Lemma 6.2 we have for $x \notin \pm \text{supp } f$, picking $J \in \mathbb{Z}$

$$\sum_{-\infty}^{J} \left| \int m_j(H)(x, y)f(y)dy \right| \leq \|f\|_2 \sum_{-\infty}^{J} \|m_j(H)(x, \cdot)\|_2$$

$$\leq C\|f\|_2 \sum_{-\infty}^{J} 2^{j/4} \leq C_J\|f\|_2,$$

and writing $\text{min } |y \pm x| = \text{min}(|y + x|, |y - x|)$

$$\sum_{J+1}^{\infty} \left| \int m_j(H)(x, y)f(y)dy \right| = \sum_{J+1}^{\infty} \left| \int_{\text{min}|y \pm x| > t} m_j(H)(x, y)f(y)dy \right|$$

$$\leq \|f\|_2 \sum_{J+1}^{\infty} \left( \int_{\text{min}|y \pm x| > t} |m_j(H)(x, y)|^2dy \right)^{1/2}$$

$$= C\|f\|_2 t^{-1} \sum_{J+1}^{\infty} 2^{-j/4} \leq C_J\|f\|_2 t^{-1},$$

where we used the inequality $(\int_{\text{min}|y \pm x| > t} |m_j(H)(x, y)|^2dy)^{1/2} \leq Ct^{-1}2^{-j/4}$, by (20). This shows that $\sum m_j(H)f(x)$ converges for all $x \notin \pm \text{supp } f$. \qed
Lemma 6.2. Let $z = \min |x \pm y|$ and $\lambda = 2^{-j/2}$. Then there exists a constant $C$ independent of $y$ so that

\[ \begin{align*}
\|K_j(\cdot, y)\|_2 &\leq C\lambda^{-1/2}, \\
\|zK_j(\cdot, y)\|_2 &\leq C\lambda^{1/2}, \\
\int_{|z| > t} |K_j(x, y)| dx &\leq Ct^{-1/2}\lambda^{1/2}.
\end{align*} \]

Lemma 6.3. Let $z, \lambda$ be as above. Then there exists a constant $C$, independent of $y$ so that

\[ \begin{align*}
\|\frac{\partial}{\partial y}K_j(\cdot, y)\|_2 &\leq C\lambda^{-3/2}, \\
\|z\frac{\partial}{\partial y}K_j(\cdot, y)\|_2 &\leq C\lambda^{-1/2}, \\
\int_{|z| > t} \left|\frac{\partial}{\partial y}K_j(x, y)\right| dx &\leq Ct^{-1/2}\lambda^{-1/2}.
\end{align*} \]

We are ready to verify the Hörmander condition for $m(H)$.

Lemma 6.4. Let $z = \min |x \pm \overline{y}|$, $t = |y - \overline{y}|$ and $\lambda = 2^{-j/2}$. Then

\[ \begin{align*}
\int_{|z| > 2t} |K_j(x, y) - K_j(x, \overline{y})| dx &\leq C \begin{cases} 
 t^{1/2}\lambda^{-1/2} & \text{if } t\lambda^{-1} \leq 1 \\
 t^{-1/2}\lambda^{1/2} & \text{if } t\lambda^{-1} > 1.
\end{cases} \\
\int_{|z| > 2t} |K(x, y) - K(x, \overline{y})| dx &\leq A,
\end{align*} \]

Moreover,

where $K(x, y)$ agrees with a “function” in the sense of Lemma 6.1.

Remark. Compare [35, 11] for Lemma 6.2–6.4.
Proof. Let \( y \in \mathbb{R} + I, I = [-t, t] \). If \( t\lambda^{-1} \leq 1 \), by Lemma 6.3

\[
\int_{\{x-y>2t\} \cap \{x+y>2t\}} |m_j(H)(x, y) - m_j(H)(x, \bar{y})|\,dx
\]

\[
= \int_{z>2t} \left| \int_{\bar{y}}^{y} \frac{\partial}{\partial \xi} m_j(H)(x, \xi)\,d\xi \right|\,dx
\]

\[
\leq \int_{\bar{y}}^{y} \int_{z>2t} \left| \frac{\partial}{\partial \xi} m_j(H)(x, \xi) \right|\,dx
\]

\[
\leq t \sup_{\xi \in \bar{y} + I} \int_{\{x-y>2t\} \cap \{x+y>2t\}} \left| \frac{\partial}{\partial \xi} m_j(H)(x, \xi) \right|\,dx
\]

\[
\leq C \varepsilon t^{1/2} \lambda^{-1/2}, \text{ all } \bar{y},
\]

If \( t\lambda^{-1} > 1 \),

\[
\int_{|z|>2t} |K_j(x, y) - K_j(x, \bar{y})|\,dx
\]

\[
\leq \int_{|z|>2t} |K_j(x, y)|\,dx + \int_{|z|>2t} |K_j(x, \bar{y})|\,dx
\]

\[
\leq \int_{\min |x+y|>t} |K_j(x, y)|\,dx + \int_{\min |x+y|>2t} |K_j(x, \bar{y})|\,dx
\]

\[
\leq Ct^{-1/2} \lambda^{1/2}.
\]

This proves (25).

Now we show (26) using (25). Let \( h = |y - \bar{y}| \). By Lemma 6.1,

\[
\int_{z>2|h-y|} |m(H)(x, y) - m(H)(x, \bar{y})|\,dx
\]

\[
\leq \sum_{|h| \lambda^{-1} \leq 1} \int_{z>2|h-y|} |m_j(H)(x, y) - m_j(H)(x, \bar{y})|\,dx := \sum_{|h| \lambda^{-1} \leq 1} + \sum_{|h| \lambda^{-1} > 1}.
\]

By Lemma 6.4 the first sum is bounded by

\[
C h^{1/2} \sum_{h2^{j/2} \leq 1} 2^{j/4} \leq C
\]

and the second sum by

\[
\sum_{h2^{j/2} > 1} \left( \int_{z>2h} |m_j(H)(x, y)|\,dx + \int_{z>2h} |m_j(H)(x, \bar{y})|\,dx \right)
\]

\[
\leq \sum_{h2^{j/2} > 1} h^{-1/2} 2^{-j/4} \leq C,
\]
where we note that \( \int_{z > 2h} |m_j(H)(x, y)| dx \leq \int_{\min|x \pm y| > h} |m_j(H)(x, y)| dx \), because \( |z| > 2h \) implies that \( |x \pm y| \geq |x \pm \bar{y}| - |y - \bar{y}| > 2h - h = h \). Hence (26) holds.

**Theorem 6.5.** Suppose \( m \in L^\infty : \mathbb{R} \to \mathbb{C} \) satisfies \( |m'(x)| \leq C|x|^{-1} \). Then \( m(H) \) is bounded on \( L^p, 1 < p < \infty \) and of weak type \((1, 1)\).

As a consequence of Theorem 6.5, we shall show that, \( m(H) \) initially defined for \( f \in L^2 \), has a bounded linear extension to the Banach spaces \( B^{\alpha,q}_p(H) \), \( 1 < p < \infty \).

**Theorem 6.6.** Suppose \( m \in L^\infty \) be as above. Then \( m(H) \) extends to a bounded linear operator on \( B^{\alpha,q}_p(H) \) for \( 1 < p < \infty, 0 < q \leq \infty, \alpha \in \mathbb{R} \).

**Proof of Theorem 6.5.** Applying Calderón-Zygmund decomposition and using Lemma 6.4, we can get the weak \((1, 1)\) result for \( m(H) \). Then the \( L^p \) result, \( 1 < p < \infty \), follows via Marcinkiewicz interpolation and duality. For completeness, we give the weak \((1, 1)\) estimation. It is enough to assume \( f \in L^1 \cap L^2 \) by density.

Given \( f \in L^1, s > 0 \), according to Calderón-Zygmund lemma there is a decomposition \( f = g + b \) with \( b = \sum b_k \) and a countable collection of disjoint open intervals \( I_k \) such that

i) \( |g(x)| \leq Cs \) a.e.

ii) Each \( b_k \) is supported in \( I_k, \int b_k dx = 0 \) and

\[
s \leq \frac{1}{|I_k|} \int_{I_k} |b_k| dx \leq 2s
\]

iii) Let \( D_s = \bigcup_k I_k = \bigcup_k (\bar{y}_k - t_k, \bar{y}_k + t_k) \), where \( 2t_k = |I_k| > 0, \bar{y}_k \) is the center of \( I_k \). Then

\[
|D_s| \leq Cs^{-1}\|f\|_1.
\]

iv) \( g \in L^1 \cap L^2, g(x) = f(x) \) if \( x \notin D_s \), and

\[
(27) \quad \|g\|_2^2 \leq Cs\|f\|_1, \quad \|b\|_1 \leq 2\|f\|_1.
\]

Now let \( f \in L^1 \cap L^2 \), then \( b = \sum b_k \) converges both a.e. and in \( L^1 \cap L^2 \), by the definition of \( b_k \) and properties (ii), (iii), where

\[
b_k(x) = \begin{cases} f(x) - \frac{1}{|I_k|} \int_{I_k} f dy & x \in I_k \\ 0 & x \notin I_k. \end{cases}
\]
It follows from Lemma 6.4 and properties (ii) and (iv) that
\[
\int_{\mathbb{R} \setminus D^*_s} |m(H)b(x)|dx \leq \sum_k \int_{\mathbb{R} \setminus D^*_s} |m(H)b_k(x)|dx
\]
(28)
\[
\leq \sum_k \int_{I_k^*} |b_k(y)|dy \int_{\mathbb{R} \setminus I_k^*} |m(H)(x, y) - m(H)(x, \overline{y}_k)|dx
\]
\[
\leq A \sum_k \int |b_k(y)|dy \leq 2A\|f\|_1,
\]
where \( D^*_s = \bigcup_k I_k^* \), with
\( I_k^* = (\overline{y}_k - 2t_k, \overline{y}_k + 2t_k) \cup (-\overline{y}_k - 2t_k, -\overline{y}_k + 2t_k) \). Since \( |D^*_s| \leq 4|D_s| \), from (27) and (28) we obtain the weak \((1, 1)\) estimate.  

\textbf{Proof of Theorem 6.6} For \( g \in L^2 \cap B^{\alpha,q)}_p(H) \),
\[
\|m(H)g\|_{B^{\alpha,q)}_p} = \|\Phi(H)m(H)g\|_p + \left\{ \sum_{j=1}^{\infty} (2^{j\alpha}\|\varphi_j(H)m(H)g\|_p)^q \right\}^{1/q}
\]
\[
= \|\{2^{j\alpha}\varphi_j(H)m(H)g\}\|_{c_0(L^p)}.
\]
Using \( \varphi_j(H) = \sum_{\nu=1}^{1}(\varphi\psi)_{j+\nu}(H)\varphi_j(H) \), with convention \( \phi_0 = \Phi, \phi_{-1} = 0 \), we have
\[
\|\{2^{j\alpha}\varphi_j(H)m(H)g\}\|_{c_0(L^p)} \leq C_{\alpha,p,q} \sum_{j=0}^{\infty} \left\{ \sum_{\nu=1}^{\infty} 2^{j\alpha q}\|m_{j+\nu}(H)\varphi_j(H)g\|_p^q \right\}^{1/q},
\]
where \( m_j = m(\varphi\psi)_j \). Therefore it is sufficient to show that \( m_j(H) \) are uniformly bounded on \( L^p, 1 < p < \infty \). But according to Theorem 6.5 this is true because \( m_j = m\psi_j \) verify the obvious condition
\[
|m_j^{(k)}(x)| \leq C|x|^{-k},
\]
k = 0, 1, with \( C \) independent of \( j \).  

\textbf{Proof of Lemma 6.2} Assuming \( \|zK_j(\cdot, y)\|_2 \leq C\lambda^{1/2} \), Schwartz inequality gives
\[
\int_{|x|>t} |K_j(x, y)|dx =
\]
\[
\int_{\{x-y>t\} \cap \{|x+y|>t\}} (\min |x \pm y|)^{-1}(\min |x \pm y|)K_j(x, y)|dx
\]
\[
\leq \left( \int_{\{x-y>t\} \cap \{|x+y|>t\}} (\min |x \pm y|)^{-2}dx \right)^{1/2} \|zK_j(\cdot, y)\|_2 \leq Ct^{-\frac{3}{2}}\lambda^\frac{1}{2}.\]
Next we need to show \( \| zK_j(\cdot, y) \|_2 \leq C\lambda^{1/2} \). Clearly,
\[
\| zm_j(H)(\cdot, y) \|_2 \leq \| zm_j(H)(x, y)\chi_{\{x > 1\}} \|_2 \\
+ \| zm_j(H)(x, y)\chi_{\{x \leq 1\}} \|_2 + \| zm_j(H)(x, y)\chi_{\{x < -1\}} \|_2.
\]

We can show that each of these three terms is \( \leq C\varepsilon\lambda^{1/2} \). We shall prove the estimate for the first term only since the other two terms can be proved similarly. The discussion is divided into three cases: \( y > 1 \), \( |y| \leq 1 \) and \( y < -1 \). Again here we indicate the proof for the case \( y > 1 \) only.

Let \( y > 1 \), \( x > 1 \) and consider the high frequency case \( j > J := 4 + \lfloor 2 \log_2 \varepsilon \rfloor \) first. Recall that \( j > J \leftrightarrow \lambda^{-1} > 4\varepsilon \).

\[
2\pi \int_{|\xi| > 2\varepsilon} |x \pm y| m_j(H)(x, y) = 2\pi \int_{|\xi| > 2\varepsilon} |x \pm y| m_j(\xi^2) e(x, \xi) e(y, \xi) d\xi
\]
\[
= \int_{\xi > 2\varepsilon} m_j(\xi^2) |C_+|^2 e^{i(x-y)\xi} d\xi + \int_{\xi < -2\varepsilon} m_j(\xi^2) (e^{i\xi} + C'e^{-i\xi}) d\xi + C'' e^{-i\xi}
\]
\[
:= I^+(x, y) + I^-(x, y).
\]

Integrating by parts we get
\[
|I^+(x, y)| \leq |x - y| \left| \int_{|\xi| > 2\varepsilon} m_j(\xi^2) |C_+|^2 e^{i(x-y)\xi} d\xi \right|
\]
\[
= \left| \int_{|\xi| > 2\varepsilon} \frac{d}{d\xi} (m_j(\xi^2) |C_+|^2) e^{i(x-y)\xi} d\xi \right|
\]
\[
= \sqrt{2\pi} \left| \frac{d}{d\xi} (m_j(\xi^2) |C_+|^2 \chi_{\{\xi > 0\}}) \right|^{\top} (x - y).
\]

By Plancherel formula,
\[
\| I^+(x, y)\chi_{\{x > 1\}} \|_2 \leq \sqrt{2\pi} \| \frac{d}{d\xi} (m_j(\xi^2) |C_+|^2 \chi_{\{\xi > 0\}}) \|_2 \leq C\varepsilon \lambda^{1/2},
\]
where we used if \( 1/2\lambda \leq |\xi| \leq 1/\lambda \),
\[
\begin{aligned}
  m_j(\xi^2) &= O(1), \\
  \frac{d}{d\xi} [m_j(\xi^2)] &= O(1/\xi), \\
  |C_+|^2 &= O(1), \\
  \frac{d}{d\xi} (|C_+|^2) &= O(1/\xi^4).
\end{aligned}
\]

Similarly, one can show that
\[
\| I^-(x, y)\chi_{\{x > 1\}} \|_2 \leq C\varepsilon \lambda^{1/2}.
\]

Combing (29), (30), we get
\[
\| z m_j(H)(x, y)\chi_{\{x > 1\}} \|_2 \leq C\varepsilon \lambda^{1/2}.
\]
Estimation for the low frequency case \( j \leq J \) can be obtained by following the same line (with a suitable modification when necessary) for the high frequency case above, except that we use certain asymptotic properties near the origin instead of \( \infty \), (consult \( \S 4 \)).

We are left with the first inequality \((19)\) concerning the “size” of the kernel. The proof of \((19)\) is similar to but easier than that of \((20)\) and may be left as a dull exercise. This closes the proof of Lemma 6.2.

\[ \square \]

Outline of the proof of Lemma 6.3: Lemma 6.3 can be proved in the same fashion as Lemma 6.2. Assuming \((23)\), apply Schwartz inequality to get for all \( y \)

\[
\int_{|z| > t} \left| \frac{\partial}{\partial y} K_j(x, y) \right| \, dx \leq \\
\left( \int_{\{|x-y| > t\} \cap \{|x+y| > t\}} (\min |x \pm y|)^{-2} \, dx \right)^{1/2} \cdot \left\| z \frac{\partial}{\partial y} K_j(\cdot, y) \right\|_2 \leq C t^{-1/2} \lambda^{-1/2}.
\]

Inequalities \((22), (23)\) measure the \( L^2 \)-norm of \( z \frac{\partial}{\partial y} K_j(\cdot, y) \) and \( z \frac{\partial}{\partial y} K_j(\cdot, y) \), which are derivative analogue of \((19), (20)\) in Lemma 6.2. We only indicate here some points for \((23)\) since \((22)\) is easier to deal with. Consider the high frequency case \( j > J \) first. To prove \((23)\) we break the function \( x \mapsto z \frac{\partial}{\partial y} K_j(x, y) \) into three parts: its restriction to the sets \( \{x > 1\}, \{|x| \leq 1\} \) and \( \{x < -1\} \). As before we are able to show that the \( L^2 \)-norm of these restrictions (in \( x \)) is \( \leq C \lambda^{-1/2} \).

For instance, in the case \( y > 1, x > 1 \), the identities

\[
\begin{align*}
&\frac{\partial}{\partial y} e_+(y, \xi) = i\xi e_+(y, \xi) \\
&\frac{\partial}{\partial y} e_-(y, \xi) = i\xi (e^{i\xi} - C'e^{-i\xi})
\end{align*}
\]

tell that the integral expression of \( z \frac{\partial}{\partial y} K_j(x, y) \) differs from that of \( z K_j(x, y) \) only by a factor \( i\xi \) (up to a \pm sign), for which reason we use the estimate \( \frac{d}{d\xi} |m_j(\xi^2)| = O(1), \xi \to \infty(1/2\lambda \leq |\xi| \leq 1/\lambda) \) in place of the estimate \( \frac{d}{d\xi} |m_j(\xi^2)| = O(1/\xi) \).

The interested reader can check the remaining cases as an exercise. The corresponding inequality is valid for the low frequency case, based on some simple asymptotic estimates as \( \xi \to 0 \).

\[ \square \]

7. Identification of \( B^\alpha_p(\mathcal{H}), 0 < \alpha < 1 \)

Generalized Besov space method was considered in \([20], [23]\) and \([25]\) in the study of perturbation of Schrödinger operators. In application to PDE problems it is of interest to identify these spaces.
The spaces $B_{p,q}^\alpha(H)$ we have defined using (2) and the system $\{\Phi, \psi_j\}$ is essentially of the same type as those defined in [21] for $p,q \geq 1$ and $\alpha \geq 0$. In [21], sufficient conditions are given on $V$ so that $B_{p,q}^\alpha(H)$ can be identified with ordinary Besov spaces. The proof is based on a real interpolation result, where the interpolation spaces are defined via semigroups. The following result is a variant of Theorem 5.1 in [21].

Let $K := \{V : V = V_+ - V_- \text{ so that } V_+ \in K_{loc}^d, V_- \in K_d\}$, where $K_d$ denote the Kato class (see §1, [21] or [32]).

**Theorem 7.1.** Suppose $V \in K$ and $\mathcal{D}(H^m) = W_p^{2m}$ for some $m \in \mathbb{N}$ and $1 \leq p < \infty$. Then for $1 \leq q \leq \infty, 0 < \alpha < m$, $B_{p,q}^\alpha(H) = B_{2\alpha,q}^p(\mathbb{R}^d)$ (with equivalent norms).

Theorem 7.1 can be directly proved by following the proof of Theorem 5.1 in [21] with obvious modifications. Indeed, noting that $\mathcal{D}(H^m) = W_p^{2m}$, the proof is contained in the commutative diagram

$$
\begin{array}{ccc}
B_{p,q}^\alpha(H) & \longrightarrow & (L^p, \mathcal{D}(H^m))_{\theta,q}, \\
\uparrow & & \uparrow \\
B_{p,q}^{2\alpha,q}(\mathbb{R}^d) & \longrightarrow & (L^p, W_p^{2m})_{\theta,q}
\end{array}
$$

with $\theta = \frac{\alpha}{m}$.

Remark 1. For convenience we state the theorem as above. Note that the Besov norm was defined in [21] using 4-adic system, while we have used dyadic system in this paper. By the way in the condition $B(p,m)$ in [21] $W_p^m$ should be $W_p^{2m}$ as above.

Remark 2. Note that the condition of $V$ on the domain of $H^m$ is equivalent to Assumption $B(p,m)$ in [21], which assumes that $(H + M)^{-m}$ is a bounded map from $L^p(\mathbb{R}^d)$ to $W_p^{2m}(\mathbb{R}^d)$ with a bounded inverse.

It is essential to verify the domain condition on $H^m$ or, the assumption $B(p,m)$. In his communication to the second author A. Jensen explained that it is easy to show that if $V$ is bounded relative to $\Delta$ on $L^p(\mathbb{R}^d)$ with relative bound less than one, then the condition $B(p,m)$ is satisfied for $m = 1$. For $m > 1$, the condition $B(p,m)$ is valid for all $m \geq 1$ and all $p$ if $V$ is $C^\infty$ with all derivatives bounded.

In the following let $V$ be the barrier potential defined in §1. Obviously $V << -\Delta$ with relative bound zero, satisfying the conditions in Theorem 7.1. Thus $B_{p,q}^\alpha(H) = B_{p,q}^{2\alpha,q}(\mathbb{R})$ for $1 \leq p < \infty, 1 \leq q \leq \infty, 0 < \alpha < 1$. This, combined with Theorem 6.6 implies the following multiplier result on ordinary Besov spaces.
Proposition 7.2. Suppose \( m \in L^\infty \) be as in Theorem 6.6. Then \( m(H) \) is bounded on \( B_{p}^{\alpha,q}(\mathbb{R}) \) for \( 1 < p < \infty, 1 \leq q \leq \infty, 0 < \alpha < 2 \).

The other interesting result follows from the discussion above for barrier potential and Theorem 5.2 in [21]. Note that in one dimension we can take equality for \( \beta \).

Proposition 7.3. Suppose \( 1 \leq p < \infty, 1 \leq q \leq \infty \) and \( 0 < \alpha < 2 - 2\beta \) with \( \beta = |\frac{1}{2} - \frac{1}{p}| \). Then \( e^{-itH} \) maps \( B_{p}^{\alpha+2\beta,q}(\mathbb{R}) \) continuously to \( B_{p}^{\alpha,q}(\mathbb{R}) \). Moreover, \( e^{-itH} \) maps \( B_{p}^{2\beta,q}(\mathbb{R}) \) continuously to \( L^p \). In both cases the operator norm is less than or equal to \( C\langle t \rangle^\beta \).

For \( m = 2 \), we have good reason in doubting the verification of the domain condition for \( H^m \).

Conjecture. \( B_p^{\alpha,q}(H) \neq B_p^{\alpha,q}(H_0), \quad \alpha = 2. \)

To see the reason we compare \( H^2 \) and \( H_0^2 \). Write \( H^2 = H_0^2 + H_0V + VH_0 + V^2 \). If we take \( A = H_0^2, B = H_0V + VH_0 + V^2 \), the only term that could cause problem is the term \( H_0V \), which involves formally Dirac delta distributions and their first derivatives. On the other hand, Theorem 3.2.2 in [2] tells that the domain of the operator \( H_0 + c_1\delta + c_2\delta' \) consists of functions \( u \in W_2^2(\mathbb{R}\setminus\{0\}) \) with \( u \) satisfying certain boundary condition at the origin. So if \( (H + M)^2 \) is bounded from \( W_4^{4,p} \) to \( L^p, \ p = 2 \), we would have that the domain of the commutator \([V, H_0]\) is \( W_4^{4,p}, \ p = 2 \), which is not the case by the above theorem in [2].

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