Coherent Superposition States as Quantum Rulers

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Abstract

We explore the sensitivity of an interferometer based on a quantum circuit for coherent states. We show that its sensitivity is at the Heisenberg limit. Moreover we show that this arrangement can measure very small length intervals.

There exists a well known isomorphism between interferometers and basic quantum processing circuits. In particular the circuit comprising a Hadamard gate followed by a phase gate and then a second Hadamard gate is equivalent to a single photon, optical interferometer with a phase shift in one arm (see Fig.1). Historically this observation has helped to identify candidate quantum circuits. An alternative viewpoint is to consider the efficacy of such quantum circuits in performing more traditionally interferometric tasks [1–3].

We have recently proposed an efficient quantum computation scheme based on a coherent state qubit encoding [4]. In this paper we investigate how sensitively distance measurements can be made using the equivalent of the circuit in Fig.1(a) when realized using this new scheme. We find that its sensitivity to small perturbations in length is at the Heisenberg limit. Further more we find that its sensitivity in measuring small length intervals is also at the Heisenberg limit. We refer to this effect as a quantum ruler.

Our logical qubits are encoded as follows: the zero state is the vacuum, \(|0\rangle_L = |0\rangle\), and the one state is the coherent state of amplitude \(\alpha\), \(|1\rangle_L = |\alpha\rangle\). We assume that the coherent amplitude is real and that \(\alpha >> 1\). Note that this qubit encoding is distinct from other quantum circuit [5,6] and interferometric [7] proposals. We begin by investigating the sensitivity of the idealized circuit of Fig.1(a) using our coherent state qubit encoding and comparing this with the sensitivity of a standard interferometer with a squeezed vacuum input. We then introduce a physical realization of the quantum circuit and consider some more practical issues.

Consider the case of the logical zero state, ie the vacuum, entering the first Hadamard gate. The effect of a Hadamard gate is to produce the following transformations in the logical basis:
Thus the state of the optical field after the first Hadamard gate is

\[ \frac{1}{\sqrt{2}} (|0\rangle + |\alpha\rangle) \]  

(2)

Now consider small changes in path length (i.e., phase shifts) around an integer number of wavelengths (\(\lambda\)) between the two Hadamard gates. Propagation over a distance \(\Delta\) can be modeled by the unitary operator \(\hat{U}(\theta) = \exp(i\theta \hat{a}^\dagger \hat{a})\) where \(\theta = \frac{2\pi \Delta}{\lambda}\). The effect of propagation on an arbitrary qubit \(|\beta\rangle\), where \(\beta = 0\) or \(\alpha\), is obtained by examining the overlap

\[ \langle \beta | \hat{U}(\theta) | \beta \rangle = \langle \beta | \beta (\cos \theta + i \sin \theta) \]

\[ = \exp[-\beta^2 (1 - \cos \theta - i \sin \theta)] \]

\[ \approx \exp[i \theta \beta^2] \]  

(3)

where the approximate final result is true in the limit that the length is small enough that \(\theta^2 \alpha^2 \ll 1\) but that \(\alpha\) is sufficiently large that \(\alpha^2 \theta\) is of order 1. Eq. 3 implies that under these conditions \(\hat{U}(\theta) | \beta \rangle \approx \exp[i \theta \beta^2] | \beta \rangle\) and thus propagation over short distances constitutes a phase gate for this system:

\[ \hat{U}(\theta) (|0\rangle + |\alpha\rangle) \approx |0\rangle + e^{i \theta \alpha^2} |\alpha\rangle \]  

(4)

Hence the effect of propagation through the entire circuit is given by

\[ |\phi\rangle_{\text{out}} = \hat{H} \hat{U}(\theta) \hat{H} |0\rangle \]

\[ \approx \frac{1}{\sqrt{2}} ((1 + e^{i \theta \alpha^2}) |0\rangle + (1 - e^{i \theta \alpha^2}) |\alpha\rangle) \]  

(5)

Clearly the output state is changed as a function of the propagation distance between the Hadamard gates. We now calculate the sensitivity to that change.

If no perturbation of the length around an integer number of wavelengths occurs then the output state will certainly be the vacuum. Thus the signal strength corresponds to the probability of finding the output in the state \(|\alpha\rangle\). The measurement noise is the probability that we none-the-less obtain the vacuum state, \(|0\rangle\) at the output. The signal to noise ratio for measuring small fluctuations in length is hence given by

\[ \frac{S}{N} = \frac{|\langle \alpha | \phi \rangle_{\text{out}}|^2}{|\langle 0 | \phi \rangle_{\text{out}}|^2} \approx \frac{V_\theta \alpha^4}{4} \]

\[ = V_\theta \bar{n}^2 \]  

(6)

Here \(V_\theta\) is the power in the distance fluctuations and \(\bar{n}\) is the average photon number in the cat state between the Hadamard gates (\(\bar{n} = \alpha^2/2\)).
We now compare the sensitivity of the coherent state quantum circuit to that of a standard interferometer using a squeezed light input. We consider the scheme originally proposed by Caves [8]. A beam in a coherent state with a real amplitude $\beta$ is injected into one input port of an interferometer while a phase squeezed vacuum is injected into the other input port. We assume the interferometer is balanced (equal path lengths in each arm) and consider the null output port. Small length fluctuations couple into the phase quadrature of this port. Thus we perform balanced homodyne detection of the phase quadrature, $X^-$, of the null output port. For small length fluctuations we obtain

$$X^- \approx X^+_n \frac{\theta}{2} + X^-_b$$

(7)

where $X^-_b$ is the phase (i.e., the squeezed) quadrature of the squeezed vacuum and $X^+_n$ is the amplitude quadrature of the coherent input. The signal to noise is then given by

$$S/N = \frac{(\beta^2 + 1)V_\theta}{4V^-_b}$$

$$\approx \frac{V_\theta \bar{n}^2}{4}$$

(8)

where $V^-_b$ is the noise power in the squeezed quadrature of the squeezed vacuum. In obtaining the final result in terms of the average photon number we have assumed that there is equal power in the coherent beam and the squeezed vacuum and that the squeezed vacuum is strongly squeezed ($V^-_b << 1$).

We see that the signal to noise’s scale in the same way as a function of photon number for the two systems. This corresponds to an amplitude sensitivity which scales as $1/\bar{n}$, i.e., the Heisenberg limit. Thus both systems perform at the ideal limit set by the uncertainty relations [4]. The factor of four increase in signal to noise achieved by the quantum circuit may not be significant. When we examine a physical realization later in this paper, and keep track of all resources required, we will find this advantage disappears.

On the other hand there is a significant difference in the way the increased sensitivity is reached in the two systems which makes the quantum circuit more versatile. In the squeezed state interferometer the increase in sensitivity arises from the decrease in background noise in the measurement. However in the coherent state circuit the increase is due to a decreasing fringe spacing as the amplitude of the cat is increased. This means, that as $\alpha$ is increased, smaller and smaller length intervals can be resolved with a sensitivity at the Heisenberg limit. This effect is similar to that recently proposed for increasing lithographic resolution [1] and earlier interferometric proposals [10] based on number state superpositions. Increasing the power in the cat state is effectively the same as increasing the frequency of the light in a standard interferometer, and thus decreasing the fringe spacing. We believe this effect could have important applications.

We now consider a physical implementation of our quantum circuit. This is shown schematically in Fig.2. A coherent state phase reference beam is divided at a 50:50 beamsplitter. One of the beams is sent to a “cat-state maker” of some kind, which produces the state given by Eq.2, in phase with the reference beam. Such a device is not trivial of course, though some limited success has been achieved in making such devices experimentally [11]. Also the cat-state maker need not necessarily be deterministic. In principle one
could imagine building up a resource of the required cat-states which are then fed into the interferometer when the measurement is required. A number of non-deterministic schemes for producing cat states have been proposed [12].

The cat-state maker performs the role of the first Hadamard gate in the idealized circuit (Fig.1(a)). The cat-state beam is then passed along the path whose distance is to be measured. In order to implement the second Hadamard gate we use the scheme proposed in Ref. [4]. A second cat state, identical to the first, and phase locked to the second coherent reference beam, is weakly mixed with the beam at a highly reflective beam splitter. A surprising result from Ref. [4] is that such a beamsplitter, with reflectivity $\cos^2 \phi$ where $\phi^2 \alpha^2 << 1$ but $\phi \alpha = \pi/2$, will act as a control sign gate [13] for our coherent state qubits.

As a result if output state $c$ in Fig.2 is measured in the “cat-basis” (see below) and is found to be in the same cat-state as was injected, then the required Hadamard transformation is implemented onto beam $d$. Alternatively if the output is found in the (near) orthogonal state $\frac{1}{\sqrt{2}}(\ket{0} - \ket{\alpha})$, then the output state is a bit flipped version of the Hadamard gate. Homodyne amplitude measurements are performed on output $d$ and the data is collected in two bins according to the results of the cat-basis measurements.

The cat basis measurements can be made by combining displacements and photon number measurements [4]. The procedure is: first displace by $-\alpha/2$. This transforms our “0”, “\alpha” superposition into “$\alpha/2$”, “$-\alpha/2$” superposition:

$$D(-\alpha/2)1/\sqrt{2}(\ket{0} \pm \ket{\alpha}) = 1/\sqrt{2}(\ket{-\alpha/2} \pm \ket{\alpha/2})$$

These new states are parity eigenstates. Thus if photon number is measured then an even result indicates detection of the state $1/\sqrt{2}(|\alpha/2\rangle + |-\alpha/2\rangle)$ and therefore $1/\sqrt{2}(\ket{0} + \ket{\alpha})$ whilst similarly an odd result indicates detection of $1/\sqrt{2}(\ket{0} - \ket{\alpha})$ as can be confirmed by direct calculation.

Notice our physical implementation requires two cat states as resources. Clearly this other resource should be included in calculating the signal to noise in terms of the photon number. The extra factor of 2 will then make the results for the squeezed state and cat schemes equivalent in this realization.

Having a physical implementation we can now make realistic calculations to confirm the efficacy of the measurement protocol for finite values of $\alpha$. To do this we use the exact solution for the output field for which no assumptions about the magnitude of $\alpha$ have been made. Using the beamsplitter relationship $|\gamma\rangle_a |\beta\rangle_b \to |\cos \theta \gamma + i \sin \theta \beta\rangle_a |\cos \theta \beta + i \sin \theta \gamma\rangle_b$, a straightforward calculation gives:

$$|\text{out}\rangle_\pm = \frac{1}{2 + 2e^{-\alpha^2/2}} \left(A_\pm \ket{0} + B_\pm |i\alpha \sin \phi e^{i\theta}\rangle + C_\pm |\alpha \cos \phi\rangle + D_\pm |\alpha (\cos \phi + i \sin \phi e^{i\theta})\rangle\right)$$

where

$$A_\pm = \frac{1}{\sqrt{2} \pm 2e^{-\alpha^2/2}} (\langle 0| \pm \langle \alpha |) \ket{0}$$

$$B_\pm = \frac{1}{\sqrt{2} \pm 2e^{-\alpha^2/2}} (\langle 0| \pm \langle \alpha |) \alpha \cos \phi e^{i\theta}\rangle$$

$$C_\pm = \frac{1}{\sqrt{2} \pm 2e^{-\alpha^2/2}} (\langle 0| \pm \langle \alpha |) i\alpha \sin \phi\rangle$$
\[ D_\pm = \frac{1}{\sqrt{2} \pm 2 e^{-\alpha^2/2}} (\langle 0 \rangle \pm \langle \alpha \rangle) \alpha (\cos \phi + i \sin \phi e^{i \theta}) \]  

(11)

and \( \phi = \pi/(2\alpha^2) \). The state overlaps can be calculated using the relationship \[ \langle \tau | \alpha \rangle = \exp[-1/2(|\tau|^2 + |\alpha|^2) + \tau^* \alpha]. \]  

The + subscript refers to the situation where a plus cat (ie: \( |0\rangle + |\alpha\rangle \)) is found at output c and the – to the case where a minus cat is found (ie: \( |0\rangle - |\alpha\rangle \)).

We then calculate

\[ P_\pm = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\alpha/2} |\langle x' | out \rangle_\pm|^2 dx' \]  

(12)

where \( \psi_{out} = \langle x' | out \rangle_\pm \) is the amplitude quadrature wave function of the output field and can be calculated using

\[ |\langle x' | \gamma \rangle|^2 = e^{-\frac{1}{2} (x'_2 - x^2)^2} \]  

(13)

Eq.12 gives the probability that a measurement of the amplitude quadrature of output beam \( d \) gives a result lying below \( \alpha/2 \). This we consider a “0” result. When a plus cat is found at output c we label this result \( P_+ \). When a minus cat is found at output c we label the result \( P_- \). The two probabilities show fringes as a function of \( \theta \) but they are \( \pi/2 \) out of phase. Note that this means that without the cat-basis measurements to distinguish the two cases the fringes would be completely washed out.

With the cat basis binning of the results we are able to form the following function: \( (P_- - P_+ + 1)/2 \), which corrects for the bit flip between the results. This is plotted for various values of \( \alpha \) in Fig.3. The width of the middle fringe scales as \( 1/\alpha^2 \) between the three graphs (note changing axis scale). This indicates sensitivity at the Heisenberg limit.

The quantum ruler effect is also clear. As \( \alpha \) increases, a number of high visibility, narrowly spaced fringes emerge, which could enable very short length intervals to be accurately measured. As an example suppose our laser wavelength is 1\( \mu \)m. In a standard interferometer this would enable length intervals of 0.5\( \mu \)m to be stepped off. The use of squeezing would increase the precision of our measurements but would not change the length scale. However using the cat-state interferometer with an \( \alpha \) of 20 (Fig.3(c)) leads to the fringe separation being reduced to 1.25\( \mu \)m.

We have introduced an interferometer based on a recently introduced quantum circuit for coherent states and their superposition. We have shown that this arrangement has a sensitivity at the Heisenberg limit and also displays a quantum ruler effect which could be used to resolve precisely very small length intervals. As well as possible applications in metrology the experiments suggested here may serve as an initial testing ground for coherent state quantum circuits.

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REFERENCES

[1] A. N. Boto, P. Kok, D. S. Abrams, S. L. Braunstein, C. P. Williams and J. P. Dowling, Phys. Rev. Lett. 85, 2733 (2000).
[2] V. Giovannetti, S. Lloyd, L. Maccone, and F. N. C. Wong, Phys. Rev. Lett. 87, 117902 (2001).
[3] W. J. Munro, K. Nemoto, G. J. Milburn, S. L. Braunstein, quant-ph/0109049 (2001).
[4] T. C. Ralph, W. J. Munro and G. J. Milburn, submitted to Nature (2001) preprint available on request.
[5] P. T. Cochrane, G. J. Milburn and W. J. Munro, Phys. Rev. A 59, 2631 (1999)
[6] H. Jeong and M. S. Kim, quant-ph/0109077 (2001).
[7] D. A. Rice, G. Jaeger and B. C. Sanders, Phys. Rev. A 62, 012101 (2000).
[8] C. M. Caves, Phys. Rev. D 23, 1693 (1981).
[9] W. Heitler, The Quantum Theory of Radiation (Oxford University Press, Oxford, 1954).
[10] M. J. Holland and K. Burnett, Phys. Rev. Lett. 71, 1355 (1993).
[11] C. Monroe, D. M. Meekhof, B. E. King and D. J. Wineland, Science 272, 1131 (1996), M. Brune, E. Hagley, J. Dreyer, X. Maitre, A. Maali, C. Wunderlich, J. M. Raimond and S. Haroche, Phys. Rev. Lett. 77, 4887 (1996), Q. A. Turchette, C. J. Hood, W. Lange, H. Mabuchi and H. J. Kimble, Phys. Rev. Lett. 75, 4710 (1995).
[12] S. Song, C. M. Caves and B. Yurke, Phys. Rev. A 41, 5261 (1990), M. Dakna, T. Anhut, T. Opatrny, L. Knill and D. G. Welsch, Phys. Rev. A 55, 3184 (1997).
[13] A control sign gate produces the following logic table: $|0\rangle_L|0\rangle_L \rightarrow |0\rangle_L|0\rangle_L$, $|0\rangle_L|1\rangle_L \rightarrow |0\rangle_L|1\rangle_L$, $|1\rangle_L|0\rangle_L \rightarrow |1\rangle_L|0\rangle_L$ but $|1\rangle_L|1\rangle_L \rightarrow -|1\rangle_L|1\rangle_L$.
[14] D. F. Walls and G. J. Milburn, Quantum Optics (Springer-Verlag, Berlin, 1994).
FIGURES

FIG. 1. Schematics of quantum circuit (a) and optical interferometer (b). If a single photon is incident on the interferometer then the description of the path of the photon is mathematically equivalent to the description of the state of the qubit in the quantum circuit with the beamsplitters (BS) playing the role of the Hadamards and the phase shift that of the phase gate.

FIG. 2. Schematic of a physical realization of the quantum circuit of Fig.1(a) using coherent state encoding. Solid lines are used to indicate coherent beams whilst dashed lines are beams that in general are in superposition states.

FIG. 3. Probability of obtaining “0” result as a function of the phase shift/distance shift in the interferometer. The coherent amplitude is varied between the three graphs. In (a) $\alpha = 5$, in (b) $\alpha = 10$ and in (c) $\alpha = 20$. Note that the scale on the horizontal axis of each graph is scaled by $1/\alpha$. 
FIGURE 1
FIGURE 2
Figure 3