Synchronized First-Passages in a Double-Well System
Driven by Asymmetric Periodic Field

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Abstract

We perform Langevin dynamic numerical simulation on a double-well potential system subjected to an asymmetric saw-tooth type external time varying field and white noise forces. The hysteresis loss calculated from the first-passage time distribution obtained shows asymmetric behaviour with respect to the asymmetry in the field sweep. The hysteresis loss, in our model, being a measure of the synchronized passages from one well to the other, indicates asymmetric "correlated" passages in the two opposing directions when driven by a temporally asymmetric external field in the presence of white noise (fluctuating) forces. The implication of our results on the phenomena of predominantly unidirectional motion of a Brownian particle in a symmetric periodic (nonratchet-like) potential is discussed.
Recently, there has been some increased activity on the asymmetric transport, without any obvious applied bias, in periodic systems. Many physical models have been proposed for possible explanations of such phenomenon.[1-5] These are theoretical models and need not be the theories of asymmetric transport observed in nature (for example, motion of macromolecules along microtubules). Yet, they provide impetus to our present understanding of transport phenomena in nonequilibrium systems. Understandably, these models incorporate the ideas of nonlinearities and stochasticities involved; the ingredients of the models in most of the cases are: the nature of periodic potential (ratchet- or nonratchet-like, time independent or fluctuating, etc.), the driving field, and the fluctuating force(s). In the present work we explore possible asymmetric passages from one well to the other in a symmetric two-well system under the influence of additive white noise fluctuating forces when the system is externally driven by a temporally asymmetric but a zero-time averaged field. It is to be noted, however, that the effect of the temporally asymmetric field has already been explored,[6] but mostly for deterministic systems. Also, very recently, the role of temporally asymmetric fluctuations towards the operation of correlation ratchets has been investigated.[7] We, in this work, do arrive at the results not directly but through the study of one of the very important properties of nonequilibrium systems, namely, the hysteresis.

Hysteresis is an important property of inhomogeneous macroscopic systems. These inhomogeneities could be due to the presence of first-order phase boundaries, domain-walls, bulk, surface, or point defects, etc., and inhibit the process of fluctuation-assisted homogenization, in the time scales of experimental observation, of the macroscopic system in the absence of externally applied fields. The response of the system parameters characterizing the state of the macroscopic system to external fields generally involves time delays resulting in the accumulation and subsequent sudden release of strains lead-
ing to frictional losses. The frictional losses cause irreversibility with respect to the response to external field reversals. This irreversibility or the hysteretic property of the macroscopic system depends on the external field sweep rates in an important way. In the last one decade or so there has been a good amount of work on the kinetic aspect of hysteresis in various model systems including the double-well potential system\cite{8}; for example, in Ref.8 the scaling behaviour of hysteresis loss with respect to the external field parameters is discussed and, moreover, the hysteresis loss is shown to exhibit stochastic resonance property with respect to the strength of the external noise (fluctuations). In the present work we show that the hysteresis loss, which is a measure of the noise-aided coherent passages across the potential barrier in a two well system,\cite{8} exhibits asymmetric behaviour with respect to the temporal asymmetry of the external field sweep.

In this article, we present a simple model to understand synchronized motion of a particle in a two-well system subjected to fluctuating (white noise) forces and to a periodic (saw tooth type) external field,\cite{Fig.1} with zero time-averaged force, making the two wells move up and down synchronously. A particle in one of the two minima, consequently, sees a potential barrier for passage to the second well vary synchronously to the external field. However, if we put the particle in the second well it too sees its barrier vary periodically, as the external field is varied, but with a phase difference with respect to the first. Thus if the variation of the external field is symmetric (within a period) in time, for example a sinusoidal field or a symmetric saw tooth type field, the net time averaged (over a large number of cycles) particle flux across the potential barrier will be zero, when the two (equally deep) wells are taken to be equally populated in the beginning. We, now, ask the question: What would happen if the external saw tooth field is asymmetric, \emph{i.e.} the \(\pm\) slopes are different in magnitude? One can easily see, from Fig.1, that, in this case too, each of the particle will experience a zero-time-averaged externally applied force-
field. Yet, we expect a difference in the synchronization behaviour of passage of the two particles.

As mentioned earlier, the Brownian particle in a well sees a potential barrier for its passage to the other well. The potential barrier changes periodically as the external periodic saw tooth field in time (Fig.1) is applied. The passage is expected to take place mostly around the time when the potential barrier height is the minimum depending on the fluctuating force strength and how fast the external field is being changed, etc. A passage may take many cycles of the external field sweep before it is completed. An appropriate choice of system parameters may, however, yield synchronized (with respect to the applied field) passage of the particle (Fig.2). This synchronized passage is linked to the hysteretic behaviour of the two-well system.\[8\] The higher is the passage synchronized the larger is the hysteresis loop area. Or in other words, the larger is the hysteresis loop area the higher is the efficiency of correlated passages. We thus perform, following an earlier work on hysteresis[8], a numerical experiment on the two-well (Landau) potential:

\[ \Phi(m) = U(m) - h(t)m, \quad (1a) \]

with the system potential field,

\[ U(m) = -\frac{a}{2}m^2 + \frac{b}{4}m^4, \quad (1b) \]

where \( m \) is the order parameter, for example, the displacement, \( a \) and \( b \) are constants and \( h(t) \) is the external time dependent field.

We proceed in the following way. We take \( h(t) \) to have a saw tooth behaviour, as shown in Fig. 1, with amplitude \( h_0 < |h_c| \), where \( h_c \) is the critical field at which one of the two wells of the potential (1a) disappears. The field \( h(t) \) peaks with value \( h_0 \) at \( t = nT_0 \), \( n = 0, 1, 2, \ldots \) and acquires minimum value \(-h_0\) at \( t = (T_1 + nT_0) \). The asymmetry of \( \pm \) slopes is obtained by taking \( T_1 \neq T_0/2 \) and measure as \( \Delta = \frac{(T_1 - T_0/2)}{T_0/2} \).
We prepare our system, at $t = 0$, at the minimum of, say, the right-side well when $h(0) = h_0$ (we always take $h_0 = 0.7h_c$ so that the potential barrier never vanishes) and let it evolve in time by the combined effect of the potential $\Phi(m)$ (with $a = 2$ and $b = 1$) and a fluctuating force $f(t)$. The evolution of the coordinate $m$ is described by the overdamped Langevin equation

$$\dot{m} = -\frac{\delta \Phi(m)}{\delta m} + f(t),$$

with

$$\langle f(t) \rangle = 0,$$  \hspace{1cm} (3a)

and

$$\langle f(t)f(t') \rangle = 2D\delta(t-t').$$  \hspace{1cm} (3b)

The averaging $\langle ... \rangle$ is done over all possible realizations of the random force $f(t)$. The experiment involves recording the first passage times $\tau$ to the other (left) well minimum of the system for the same initial conditions but with different realizations of the random force $f(t)$[8]. The records give the distribution $\rho(\tau)$ of $\tau$ (Fig.2). $\rho(\tau)$ depends on various parameters including $D$, $h_0$, $T_1$, $T_0$, etc. It may spread over many cycles of $h(t)$ and generally peaks around $t = T_1 + nT_0$, such that $h(t) = h(T_1) = -h_0$. However, $\rho(\tau)$ can be converted into $\rho(h_J)$, where $h_J = h(\tau)$. From $\rho(h_J)$ we calculate[8] the upper half of the hysteresis loop $M(h_J)$:

$$\frac{M(h_J)}{h_c} = 1 - \frac{2}{h_c} \int_{h_J}^{h_0} \rho(h_J')dh_J',$$

with the saturation value of $\frac{M}{h_c} = \pm1$. The lower half is completed by symmetry, meaning thereby that the system is initially prepared at the minimum of the left well at $h(t = 0) = -h_0$ and drive the field in the reverse direction in time to obtain $\rho(\tau)$ for passage to the right well, and then calculate the hysteresis loop area. The hysteresis loop
area, as mentioned earlier, is a measure of synchronized passages; for maximum possible synchroniziation we have the rectangular hysteresis loop with maximum area of $4h_0 h_c$, corresponding to $\rho(\tau) = \frac{1}{N} \sum \delta(\tau - (nT_0 + T_1))$, the summation being over $N \to \infty$ set of $\tau$ calculations.

The experiment is repeated for various values of $\Delta$, the asymmetry in the field sweep rates, for given $D$, $h_0$, etc. The plot of hysteresis loop area versus $\Delta$ is shown in Fig. 3. The asymmetry of the hysteresis loop area about $\Delta = 0$ is quite revealing and becomes more pronounced for lower $D$ values. The result shows that for $\Delta < 0$, passage from right well to the left is more synchronized than for $\Delta > 0$. With a little thought it can be stated, in other words, as the passage from the right well to the left well is more synchronized and hence more correlated than the passage from left well to the right well for $\Delta < 0$, and vice versa.

The external field sweep $h(t)$ satisfies, as it should be clear from Fig.1, $\int_0^{T_0} h(t) dt = 0$ and the fluctuating force $f(t)$ satisfies eq.(3a). The numerical experiment is performed under two conditions, namely, the initial condition $m(t = 0) = \bar{m}_2(0)$, the minimum of the right well, for example, and the condition that the particle gets absorbed as soon as it reaches the left well minimum $\bar{m}_1(\tau)$. The system is constructed to be dynamic and the time average of $\dot{m}$ in eq.(2) need not be non-zero to have an asymmetric diffusive motion. This is not justified, however, for, say $U(m) = \text{constant}$, or a frictionless flat potential surface considering eq.(2). But, with a flat potential surface, depending on the problem at hand, one would perhaps be wiser to use the full Langevin equation rather than the overdamped eq.(2) and, then, again it will be quite easy to verify that even though the average velocity vanishes one can still have asymmetric average displacement given the form of $h(t)$ with $\Delta \neq 0$. However, the asymmetry of passage with respect to $\Delta$, in our case could be affected by drag-effect which distinguishes the fast processes from the slow
ones; the particle on the right well sees the barrier height to decrease faster (for example for $t < T_1$ during the very first cycle of $h(t)$) than it sees the height increase (for $t > T_1$ in the same cycle) for $\Delta < 0$ and vice versa. This is the result we wanted to confirm.

The numerical experiment performed is computationally expensive even for the two-well system and it is beyond our means to go for extended periodic systems. However, we make some extrapolatory remarks. If we have a spatially periodic potential field, with reflection symmetry about the maxima when the sweeping external field $h(t)$ is zero, no asymmetric motion of the Brownian particle could be expected when a symmetric sweeping field with ($\Delta = 0$) is applied. Let the minima of the potential wells $i$ be at $\bar{m}_i$. Now, we make further idealization to make contact with our experiment: A Brownian particle at the $i$th well feels as though it were in the left well of the potential $\Phi(m)$ for its motion towards the $(i + 1)$th well and similarly for its motion towards the $(i - 1)$th well it feels as though it were in the right well of $\Phi(m)$. Now, $h(t) > 0 (< 0)$ would make the common tangent to all the minima tilt with negative (positive) slope. Thus, a symmetric saw tooth field $h(t)$ ($\Delta = 0$) will make the common tangent change its slope with time at the same uniform rate on either direction (negative slope to positive slope and vice versa). However, $\Delta \neq 0$ makes the rate of flapping ($-\text{slope to } +\text{slope}$) different from ($+\text{slope to } -\text{slope}$), reminiscent of the flagellar strokes of sperm molecules of eukaryotes. (The analogy is, however, not to be taken literally for in our case the three important time scales involved, namely, the relaxation time of the local minima, the $(\frac{h}{\dot{h}_c})^{-1}$ and the mean first passage time $< \tau >$, are required to be comparable.[8]) Given the asymmetric nature of the result we have obtained for the two well system, it is plausible that owing to the application of a zero-time-averaged asymmetric field a preferentially asymmetric motion of the Brownian particle in the periodic system potential field will ensue.[7]

In summary, we perform a (Langevin dynamic) numerical experiment on a two-well
system subjected to Gaussian white noise (fluctuating) force and driven by an external saw-tooth type field and show that the passages of a Brownian particle from one well to the other have a different behaviour than the passages in the reverse direction if the driving field has temporal asymmetry. The conclusion is based on the study of hysteresis loop area calculated for the two-well system. It is important to note that in our calculation, unlike in Ref.6, noise plays a crucial role for the applied field amplitude $h_0$ is restricted to a value lower than the critical value $h_c$ and hence making the passages impossible without noise. We extrapolate to conclude that asymmetric passage or transport is possible in a symmetric (nonratchet-like) periodic potential provided the system is driven by temporally asymmetric driving force (Fig.1) and assisted by (additive) Gaussian white noise. Moreover, our numerical experiment is amenable to real experiment such as the one performed recently by Simon and Libchaber[9] on optical systems with double-well potential.
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Figure Captions

Fig. 1. Depicts the external periodic field $h(t)$. We have taken amplitude $h_0=0.7h_c$, and period $T_0=28.0$.

Fig. 2. Part of the first-passage time distribution $\rho(\tau)$ for the field sweep shown in Fig.1. $\rho(\tau)$ extends to 24 cycles of $h(t)$ in 15000 runs for $D = 0.5$ and $h_0=0.7h_c$.

Fig. 3. Hysteresis loop area versus the asymmetry $\Delta = \frac{(T_1-T_0/2)}{T_0/2}$ in the $\pm$ slopes of $h(t)$ (Fig. 1) for $D = 0.3(\bigcirc)$ and $D = 0.5(\otimes)$. For each data point $\rho(\tau)$ calculated from a minimum of 10000 first-passage-times $\tau$ from the right well to the left well were used. The vertical dashed line at $\Delta = 0$ is drawn only for convenience.