FUNCTIONS OF PERTURBED OPERATORS

ALEKSEI ALEKSANDROV AND VLADIMIR PELLER

Abstract. We prove that if $0 < \alpha < 1$ and $f$ is in the Hölder class $\Lambda_\alpha(\mathbb{R})$, then for arbitrary self-adjoint operators $A$ and $B$ with bounded $A - B$, the operator $f(A) - f(B)$ is bounded and $\|f(A) - f(B)\| \leq \text{const} \|A - B\|^{\alpha}$. We prove a similar result for functions $f$ of the Zygmund class $\Lambda_1(\mathbb{R})$: $\|f(A + K) - 2f(A) + f(A - K)\| \leq \text{const} \|K\|$, where $A$ and $K$ are self-adjoint operators. Similar results also hold for all Hölder-Zygmund classes $\Lambda_\alpha(\mathbb{R})$, $\alpha > 0$. We also study properties of the operators $f(A) - f(B)$ for $f \in \Lambda_\alpha(\mathbb{R})$ and self-adjoint operators $A$ and $B$ such that $A - B$ belongs to the Schatten–von Neumann class $S_p$. We consider the same problem for higher order differences. Similar results also hold for unitary operators and for contractions.

Fonctions d’opérateurs perturbés

Résumé. Nous montrons que si $0 < \alpha < 1$ et $f$ appartient à la classe de Hölder $\Lambda_\alpha(\mathbb{R})$, alors pour tous les opérateurs $A$ et $B$ auto-adjoints dont la différence est bornée on a: $\|f(A) - f(B)\| \leq \text{const} \|A - B\|^{\alpha}$. Nous obtenons un résultat similaire pour les fonctions de la classe de Zygmund $\Lambda_1(\mathbb{R})$: $\|f(A + K) - 2f(A) + f(A - K)\| \leq \text{const} \|K\|$, où $A$ et $K$ sont des opérateurs auto-adjoints. Un résultat similaire est aussi vrai pour toutes les classes de Hölder–Zygmund $\Lambda_\alpha(\mathbb{R})$, $\alpha > 0$. Nous étudions aussi les propriétés des opérateurs $f(A) - f(B)$ si $f \in \Lambda_\alpha(\mathbb{R})$ et $A$ et $B$ sont des opérateurs auto-adjoints dont la différence appartient à la classe de Schatten–von Neumann $S_p$. Nous considérons le même problème pour les différences d’ordre arbitraire. On peut obtenir des résultats similaires pour les opérateurs unitaires et pour les contractions.

Version française abrégée

Il est bien connu qu’il y a des fonctions $f$ lipschitziennes sur la droite réelle $\mathbb{R}$ qui ne sont pas lipschitziennes opératorielles, c’est-à-dire la condition

$$|f(x) - f(y)| \leq \text{const} |x - y|, \quad x, y \in \mathbb{R},$$

n’implique pas que pour tous les opérateurs auto-adjoints $A$ et $B$ l’inégalité

$$\|f(A) - f(B)\| \leq \text{const} \|A - B\|$$

soit vraie.

Il se trouve que la situation change dramatiquement si l’on considère les fonctions de la classe $\Lambda_\alpha(\mathbb{R})$ de Hölder d’ordre $\alpha$, $0 < \alpha < 1$. Nous montrons que si $A$ et $B$ sont des opérateurs auto-adjoints dans un espace hilbertien et $f \in \Lambda_\alpha(\mathbb{R})$ (c’est-à-dire $|f(x) - f(y)| \leq \text{const} |x - y|^\alpha$), alors

$$\|f(A) - f(B)\| \leq \text{const} \|A - B\|^\alpha.$$

On peut considérer un problème similaire pour la classe de Zygmund $\Lambda_1(\mathbb{R})$ de fonctions continues sur $\mathbb{R}$ telles que

$$|f(x + t) - 2f(x) + f(x - t)| \leq \text{const} |t|, \quad x, t \in \mathbb{R}.$$

Nous établissons que dans ce cas $f$ doit satisfaire à l’inégalité

$$\|f(A + K) - 2f(A) + f(A - K)\| \leq \text{const} \|K\|$$

où $A$ et $K$ sont des opérateurs auto-adjoints.

Nous considérons aussi les espaces $\Lambda_\alpha(\mathbb{R})$ pour tous les $\alpha > 0$. Si $\alpha > 0$, la classe $\Lambda_\alpha(\mathbb{R})$ consiste en fonctions $f$ continues telles que

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kt) \leq \text{const} |t|^\alpha \quad (\text{ici } n - 1 \leq \alpha < n).$$
Nous montrons que dans ce cas
\[ \left\| \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(A + jK) \right\| \leq \text{const} \|K\|^\alpha, \quad n > \alpha, \]
pour chaque opérateurs $A$ et $K$ auto-adjoints.

De plus, on peut généraliser les résultats ci-dessus pour les opérateurs $A$ pas nécessairement bornés (le théorème 4.2 dans la version anglaise).

On peut considérer les mêmes problèmes pour les opérateurs unitaires et pour les contractions.

Nous considérons aussi le cas de modules de continuité arbitraires. Soit $\omega$ une fonction croissante continue sur $[0, \infty)$ telle que $\omega(0) = 0$ et $\omega(x + y) \leq \omega(x) + \omega(y)$, $x, y \geq 0$. Nous définissons la fonction $\omega^*$ par
\[ \omega^*(x) = x \int_x^\infty \frac{\omega(t)}{t^2} dt, \quad x \geq 0. \]

Notons par $\Lambda_\omega(\mathbb{R})$ l’espace de fonctions $f$ sur $\mathbb{R}$ telles que
\[ |f(x) - f(y)| \leq \text{const} \omega(|x - y|), \quad x, y \in \mathbb{R}. \]

Nous démontrons que si $A$ et $B$ sont des opérateurs auto-adjoints dont la différence $A - B$ est bornée et $f \in \Lambda_\omega$, alors
\[ \|f(A) - f(B)\| \leq \text{const} \omega^* (\|A - B\|). \]

On peut obtenir un résultat similaire pour les modules de continuité d’ordre arbitraire. Nous avons aussi obtenu les mêmes résultats pour les opérateurs unitaires et pour les contractions.

Maintenant nous considérons le problème suivant. Rappelons que $S_p$ est la classe de Schatten–von Neumann constituée des opérateurs $T$ dans un espace hilbertien pour lesquels les nombres singuliers $s_n(T)$ appartiennent à l’espace $\ell^p$.

Supposons que $f \in \Lambda_\alpha(\mathbb{R})$, $0 < \alpha < 1$, et $p > 1$. Nous montrons que pour chaque opérateurs auto-adjoints $A$ et $B$ dont la différence $A - B$ appartient à $S_p$, on a
\[ f(A) - f(B) \in S_\infty^\alpha \quad \text{et} \quad \|f(A) - f(B)\|_{S^{\infty}_\infty} \leq \text{const} \|A - B\|^\alpha. \]

Si $p = 1$ et $f \in \Lambda_\alpha(\mathbb{R})$, $0 < \alpha < 1$, la condition $A - B \in S_1$ implique que
\[ f(A) - f(B) \in S^{\infty}_{\infty}, \]

où l’espace $S_q,\infty$ est formé des opérateurs dont les nombres singuliers satisfont à la condition
\[ \sup_{n \geq 0} (1 + n)^{1/q} s_n(T) < \infty. \]

Nous avons aussi obtenu des analogues des résultats ci-dessus pour les différences d’ordre arbitraire. On peut obtenir des résultats similaires pour les opérateurs unitaires et pour les contractions.

1. Introduction

It is well known that a Lipschitz function on the real line is not necessarily operator Lipschitz, i.e., the condition
\[ |f(x) - f(y)| \leq \text{const} |x - y|, \quad x, y \in \mathbb{R}, \]
does not imply that for self-adjoint operators $A$ and $B$ on Hilbert space,
\[ \|f(A) - f(B)\| \leq \text{const} \|A - B\|. \]
The existence of such functions was proved in [4] (see also [5] and [7]). Later in [8] necessary conditions were found for a function $f$ to be operator Lipschitz. Those necessary conditions imply that Lipschitz functions do not have to be operator Lipschitz. It is also well known that a continuously differentiable function does not have to be operator differentiable, see [8] and [9]. Note that the necessary conditions obtained in [8] and [9] are based on the nuclearity criterion for Hankel operators, see [10].
It turns out that the situation dramatically changes if we consider H"older classes $\Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$. In this case such functions are necessarily operator H"older of order $\alpha$, i.e., the condition

$$|f(x) - f(y)| \leq \text{const} |x - y|^\alpha, \quad x, \ y \in \mathbb{R},$$

implies that for self-adjoint operators $A$ and $B$ on Hilbert space,

$$\|f(A) - f(B)\| \leq \text{const} \|A - B\|^\alpha. \quad (1.1)$$

Moreover, a similar result holds for the Zygmund class $\Lambda_1(\mathbb{R})$, i.e., the fact that

$$|f(x + t) - 2f(x) + f(x - t)| \leq \text{const} |t|, \quad x, \ t \in \mathbb{R},$$

and $f$ is continuous implies that $f$ is operator Zygmund, i.e., for self-adjoint operators $A$ and $K$,

$$\|f(A + K) - 2f(A) + f(A - K)\| \leq \text{const} \|K\|. \quad (1.2)$$

We also obtain similar results for the whole scale of H"older–Zygmund classes $\Lambda_\alpha(\mathbb{R})$ for $0 < \alpha < \infty$. Recall that for $\alpha > 1$, the class $\Lambda_\alpha(\mathbb{R})$ consists of continuous functions $f$ such that

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kt) \leq \text{const} |t|^\alpha, \quad \text{where} \quad n - 1 \leq \alpha < n.$$

The same problems can be considered for unitary operators and for functions on the unit circle, and for contractions and analytic functions in the unit disk.

To prove (1.1), we use a crucial estimate obtained for trigonometric polynomials and unitary operators in [8] and for entire functions of exponential type and self-adjoint operators in [9]. We state here the result for self-adjoint operators. It can be considered as an analog of Bernstein’s inequality.

Let $f$ be an entire function of exponential type $\sigma$ that is bounded on the real line $\mathbb{R}$. Then for self-adjoint operators $A$ and $B$ with bounded $A - B$ the following inequality holds:

$$\|f(A) - f(B)\| \leq \sigma \|f\|_{L^\infty(\mathbb{R})} \|A - B\|. \quad (1.3)$$

Inequality (1.3) was proved by using double operator integrals and the Birman–Solomyak formula:

$$f(A) - f(B) = \int \int \frac{f(x) - f(y)}{x - y} dE_A(x)(A - B) dE_B(y),$$

where $E_A$ and $E_B$ are the spectral measures of self-adjoint operators $A$ and $B$; we refer the reader to [1], [2] and [3] for the theory of double operator integrals. Note that $A$ and $B$ do not have to be bounded, but $A - B$ must be bounded.

To estimate the second difference (1.2), we use the corresponding analog of Bernstein’s inequality which was obtained in [11] with the help of triple operator integrals. To estimate higher order differences, we need multiple operator integrals. We refer the reader to [11] for definitions and basic results on multiple operator integrals.

We also consider in this paper the problem of the behavior of functions of operators $f(A)$ under perturbations of $A$ by operators of Schatten–von Neumann class $S_p$ in the case when $f \in \Lambda_\alpha(\mathbb{R})$.

2. Norm estimates for unitary operators

We start with first order differences. We use the notation by $\Lambda_\alpha$, $0 < \alpha < \infty$, for the scale of H"older–Zygmund classes on the unit circle $T$.

**Theorem 2.1.** Let $0 < \alpha < 1$. Then there is a constant $c > 0$ such that for every $f \in \Lambda_\alpha$ and for arbitrary unitary operators $U$ and $V$ on Hilbert space the following inequality holds:

$$\|f(U) - f(V)\| \leq c \|f\|_{\Lambda_\alpha} \cdot \|U - V\|^\alpha.$$
Theorem 2.2. There exists a constant $c > 0$ such that for every function $f \in \Lambda_1$ and for arbitrary unitary operators $U$ and $V$ on Hilbert space the following inequality holds:

$$
\|f(U) - f(V)\| \leq c \|f\|_{\Lambda_1} \left(2 + \log_2 \frac{1}{\|U - V\|}\right) \|U - V\|.
$$

Note that this result improves an estimate obtained in [4] for Lipschitz functions in the case of bounded self-adjoint operators.

We proceed now to higher order differences.

Theorem 2.3. Let $n$ be a positive integer and $0 < \alpha < n$. Then there exists a constant $c > 0$ such that for every $f \in \Lambda_{\alpha}$ and for an arbitrary unitary operator $U$ and an arbitrary bounded self-adjoint operator $A$ on Hilbert space the following inequality holds:

$$
\left\| \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(e^{ikA}U) \right\| \leq c \|f\|_{\Lambda_{\alpha}} \|A\|^\alpha.
$$

Let us consider now a more general problem. Suppose that $\omega$ is a modulus of continuity, i.e., $\omega$ is a nondecreasing continuous function on $[0, \infty)$ such that $\omega(0) = 0$ and $\omega(x + y) \leq \omega(x) + \omega(y)$, $x, y \geq 0$.

The space $\Lambda_\omega$ consists of functions $f$ on $\mathbb{T}$ such that $$|f(\zeta) - f(\tau)| \leq \text{const} \omega(|\zeta - \tau|), \quad \zeta, \tau \in \mathbb{T}.$$ With a modulus of continuity $\omega$ we associate the function $\omega^*$ defined by $$\omega^*(x) = x \int_x^\infty \frac{\omega(t)}{t^2} dt, \quad x \geq 0.$$

Theorem 2.4. Suppose that $\omega$ is a modulus of continuity and $f \in \Lambda_\omega$. If $U$ and $V$ are unitary operators, then

$$
\|f(U) - f(V)\| \leq \text{const} \|f\|_{\Lambda_{\omega}} \omega^*(\|U - V\|).
$$

In particular, if $\omega^*(x) \leq \text{const} \omega(x)$, then for unitary operators $U$ and $V$

$$
\|f(U) - f(V)\| \leq \text{const} \|f\|_{\Lambda_{\omega}} \omega(\|U - V\|).
$$

We have also proved an analog of Theorem 2.4 for higher order differences.

3. Norm estimates for contractions

We denote here by $(\Lambda_{\alpha})_+$ the set of functions $f \in \Lambda_{\alpha}$, for which the Fourier coefficients $\hat{f}(n)$ vanish for $n < 0$.

Recall that an operator $T$ on Hilbert space is called a contraction if $\|T\| \leq 1$. The following result is an analog of Theorem 2.3 for contractions.

Theorem 3.1. Let $n$ be a positive integer and $0 < \alpha < n$. Then there exists a constant $c > 0$ such that for every $f \in (\Lambda_{\alpha})_+$ and for arbitrary contractions $T$ and $R$ on Hilbert space, the following inequality holds:

$$
\left\| \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f \left( T + \frac{k}{n}(T - R) \right) \right\| \leq c \|f\|_{\Lambda_{\alpha}} \|T - R\|^\alpha.
$$

Note that an analog of Theorem 2.4 also holds for contractions.
4. Norm estimates for self-adjoint operators

Theorem 4.1. Let $0 < \alpha < 1$ and let $f \in \Lambda_\alpha(\mathbb{R})$. Suppose that $A$ and $B$ are self-adjoint operators such that $A - B$ is bounded. Then $f(A) - f(B)$ is bounded and

$$
\|f(A) - f(B)\| \leq \text{const} \|f\|_{\Lambda_\alpha(\mathbb{R})}\|A - B\|^\alpha.
$$

In this connection we mention the paper [4] where it was proved that for self-adjoint operators $A$ and $B$ with spectra in an interval $[a, b]$ and a function $\varphi \in \Lambda_\alpha(\mathbb{R})$, the following inequality holds:

$$
\|\varphi(A) - \varphi(B)\| \leq \text{const} \|\varphi\|_{\Lambda_\alpha(\mathbb{R})} \left(\log \left(\frac{b-a}{\|A - B\|}\right) + 1\right)^2 \|A - B\|^\alpha
$$

(see also [6] where the above inequality is generalized for general moduli of continuity).

Theorem 4.2. Suppose that $n$ is a positive integer and $0 < \alpha < n$. Let $A$ be a self-adjoint operator and let $K$ be a bounded self-adjoint operator. Then the map

$$
f \mapsto (\Delta_K^n f)(A) \overset{\text{def}}{=} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(A + jK) \tag{4.1}
$$

has a unique extension from $L^\infty \cap \Lambda_\alpha(\mathbb{R})$ to a sequentially continuous operator from $\Lambda_\alpha(\mathbb{R})$ (equipped with the weak-star topology) to the space of bounded linear operators on Hilbert space (equipped with the strong operator topology) and

$$
\|(\Delta_K^n f)(A)\| \leq \text{const} \|f\|_{\Lambda_\alpha(\mathbb{R})}\|K\|^\alpha.
$$

We use the same notation $(\Delta_K^n f)(A)$ for the unique extension of the map (4.1).

We can also prove an analog of Theorem 2.4 for self-adjoint operators.

5. Perturbations of class $S_p$

In this section we consider the behavior of functions of self-adjoint operators under perturbations of Schatten–von Neumann class $S_p$. Similar results also hold for unitary operators and for contractions.

Recall that the spaces $S_p$ and $S_{p,\infty}$ consist of operators $T$ on Hilbert space such that

$$
\|T\|_p \overset{\text{def}}{=} \left(\sum_{n=0}^{\infty}(s_n(T))^p\right)^{1/p} < \infty \quad \text{and} \quad \|T\|_{S_{p,\infty}} \overset{\text{def}}{=} \sup_{n \geq 0} (1 + n)^{1/p} s_n(T) < \infty.
$$

Theorem 5.1. Let $1 \leq p < \infty$, $0 < \alpha < 1$, and let $f \in \Lambda_\alpha(\mathbb{R})$. Suppose that $A$ and $B$ are self-adjoint operators such that $A - B \in S_p$. Then

$$
f(A) - f(B) \in S_{p,\infty} \quad \text{and} \quad \|f(A) - f(B)\|_{S_{p,\infty}} \leq \text{const} \|f\|_{\Lambda_\alpha(\mathbb{R})}\|A - B\|_{S_p}^\alpha.
$$

Note that in Theorem 5.1 in the case $p > 1$ we can replace the condition $A - B \in S_p$ with the condition $A - B \in S_{p,\infty}$.

Using interpolation arguments, we can deduce from Theorem 5.1 the following result.

Theorem 5.2. Let $1 < p < \infty$, $0 < \alpha < 1$, and let $f \in \Lambda_\alpha(\mathbb{R})$. Suppose that $A$ and $B$ are self-adjoint operators such that $A - B \in S_p$. Then

$$
f(A) - f(B) \in S_{p,\infty} \quad \text{and} \quad \|f(A) - f(B)\|_{S_{p,\infty}} \leq \text{const} \|f\|_{\Lambda_\alpha(\mathbb{R})}\|A - B\|_{S_p}^\alpha.
$$

Let us now state similar results for higher order differences.

Theorem 5.3. Suppose that $n$ is a positive integer, $\alpha$ is a positive number such that $n - 1 \leq \alpha < n$, $f \in \Lambda_\alpha(\mathbb{R})$, and $n \leq p < \infty$. Let $A$ be a self-adjoint operator and let $K$ be a self-adjoint operator of class $S_p$. Then the operator $(\Delta_K^n f)(A)$ defined in Theorem 4.2 belongs to $S_{p,\infty}^\alpha$ and

$$
\|(\Delta_K^n f)(A)\|_{S_{p,\infty}^\alpha} \leq \text{const} \|f\|_{\Lambda_\alpha(\mathbb{R})}\|K\|_{S_p}^\alpha.
$$
Theorem 5.4. Suppose that $n$ is a positive integer, $\alpha$ is a positive number such that $n - 1 \leq \alpha < n$, $f \in \Lambda_\alpha(\mathbb{R})$, and $n < p < \infty$. Let $A$ be a self-adjoint operator and let $K$ be a self-adjoint operator of class $S_p$. Then the operator $(\Delta_n^K f)(A)$ defined in Theorem 4.2 belongs to $S_p$ and

$$\| (\Delta_n^K f)(A) \|_{S_p} \leq \text{const} \| f \|_{\Lambda_\alpha(\mathbb{R})} \| K \|_{S_p}^\alpha.$$ 

REFERENCES

[1] M.S. Birman and M.Z. Solomyak, Double Stieltjes operator integrals, Problems of Math. Phys., Leningrad. Univ. 1 (1966), 33–67 (Russian). English transl.: Topics Math. Physics 1 (1967), 25–54, Consultants Bureau Plenum Publishing Corporation, New York.

[2] M.S. Birman and M.Z. Solomyak, Double Stieltjes operator integrals. II, Problems of Math. Phys., Leningrad. Univ. 2 (1967), 26–60 (Russian). English transl.: Topics Math. Physics 2 (1968), 19–46, Consultants Bureau Plenum Publishing Corporation, New York.

[3] M.S. Birman and M.Z. Solomyak, Double Stieltjes operator integrals. III, Problems of Math. Phys., Leningrad. Univ. 6 (1973), 27–53 (Russian).

[4] Yu.B. Farforovskaya, The connection of the Kantorovich-Rubinshtein metric for spectral resolutions of selfadjoint operators with functions of operators, Vestnik Leningrad. Univ. 19 (1968), 94–97. (Russian).

[5] Yu.B. Farforovskaya, An estimate of the norm of $| f(B) - f(A) |$ for selfadjoint operators $A$ and $B$, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 56 (1976), 143–162 (Russian).

[6] Yu.B. Farforovskaya and L. Nikolskaya, Modulus of continuity of operator functions, Algebra i Analiz 20:3 (2008), 224–242.

[7] T. Kato, Continuity of the map $S \mapsto | S |$ for linear operators, Proc. Japan Acad. 49 (1973), 157–160.

[8] V.V. Peller, Hankel operators in the theory of perturbations of unitary and self-adjoint operators, Funktsional. Anal. i Prilozhen. 19:2 (1985), 37–51 (Russian). English transl.: Funct. Anal. Appl. 19 (1985) , 111–123.

[9] V.V. Peller, Hankel operators in the perturbation theory of unbounded self-adjoint operators. Analysis and partial differential equations, 529–544, Lecture Notes in Pure and Appl. Math., 122, Dekker, New York, 1990.

[10] V.V. Peller, Hankel operators and their applications, Springer-Verlag, New York, 2003.

[11] V.V. Peller, Multiple operator integrals and higher operator derivatives, J. Funct. Anal. 233 (2006), 515–544.

A.B. Aleksandrov
St-Petersburg Branch
Steklov Institute of Mathematics
Fontanka 27, 191023 St-Petersburg
Russia

V.V. Peller
Department of Mathematics
Michigan State University
East Lansing, Michigan 48824
USA