The random forced Navier-Stokes equation can be obtained as a variational problem of a proper action. In virtue of incompressibility, the integration over transverse components of the fields allows to cast the action in the form of a large deviation functional. Since the hydrodynamic operator is nonlinear, the functional integral yielding the statistics of fluctuations can be practically computed by linearizing around a physical solution of the hydrodynamic equation. We show that this procedure yields the dimensional scaling predicted by K41 theory at the lowest perturbative order, where the perturbation parameter is the inverse Reynolds number. This result is valid over a finite spatio-temporal domain, where the physical solution can be considered as stationary.

I. INTRODUCTION

A field theoretic approach to the study of the random stirred Navier-Stokes equation (NSE) can be traced back to the seminal paper by Martin, Siggia and Rose [1]. This was the starting point for the application of many perturbative strategies, e.g. diagramatic expansions and renormalization group methods [2]. The many technical difficulties encountered in developing such approaches avoided to gather conclusive achievements. For recent developments and applications the reader can be addressed to [3]. In this paper we show that one step forward along this field-theoretic approach allows one to cast the action associated with the NSE into the form of a large deviation functional. Recently, large-deviation theory has scored sensible success in describing fluctuations in stationary non-equilibrium regimes of various microscopic models [4]. This approach is mainly based on the extension of the time-reversal conjugacy property introduced by Onsager and Machlup [5] to stationary non-equilibrium states. In practice, thermal fluctuations in irreversible stationary processes can be traced back to a proper hydrodynamic description derived from the microscopic evolution rules. The action functional has the quadratic form

\[ I_{[t_1,t_2]}(\rho) = \frac{1}{2} \int_{t_1}^{t_2} dt \langle W, K(\rho) W \rangle \]  

where \(\rho(t, \vec{x})\) represents in general a vector of thermodynamic variables depending on time \(t\) and space variables \(\vec{x}\). The symbol \(\langle \cdot, \cdot \rangle\) denotes the integration over space variables. \(W\) is a hydrodynamic evolution operator acting on \(\rho\): it vanishes when \(\rho\) is equal to the stationary solution \(\bar{\rho}\), which is assumed to be unique. The positive kernel \(K(\rho)\) represents the stochasticity of the system at macroscopic level. According to the large deviation-theory the entropy \(S\) of a stationary non-equilibrium state is related to the action functional \(I\) as follows:

\[ S(\rho) = \inf_{\bar{\rho}} I_{[-\infty,0]}(\bar{\rho}) \]  

where the minimum is taken over all trajectories connecting \(\bar{\rho}\) to \(\rho\).

For our purposes it is enough to consider that the action functional \(I\) provides a natural measure for statistical fluctuations in non-equilibrium stationary states, so that, formally, any statistical inference can be obtained from \(I\). Indeed, from the very beginning we have to deal with a hydrodynamic formulation, namely the random stirred NSE: in the next Section we will argue that an action functional of the form (1) can be obtained by field-theoretic analytic calculations.

In particular, explicit integration over all longitudinal components of the velocity field and over the associated auxiliary fields can be performed. This allows to obtain a hydrodynamic evolution operator \(W\) which depends only on the transverse components of the velocity field \(v(t, \vec{x})\). Moreover, the positive kernel \(K\) amounts to the inverse correlation function of the stochastic source, while any dependence on the form of the pressure tensor and of the noise does not enter in the determination of \(W\). Accordingly, many of the technical difficulties characterizing standard perturbative methods and diagramatic expansions have been removed.
On the other hand, we have to face with new difficulties. The hydrodynamic operator appearing in the large deviation functional is nonlinear, so that functional integration is unfeasible. One should identify a stationary solution \( \bar{\nu}_T(t, \bar{x}) \) of the associated hydrodynamic equation and linearize the hydrodynamic operator around such a solution. Then, functional integration could be performed explicitly on the “fluctuation” field. In order to be well defined this approximate procedure would demand the uniqueness of the stationary solution of the nonlinear hydrodynamic equation. Conversely, it can be easily verified that it admits several solutions. This notwithstanding, we have found only one solution that does not yield unphysical divergences in the long time and large space limits (see Section III). Accordingly, we have assumed that fluctuations can be meaningfully estimated only with respect to this solution. Specifically, our statistical non-equilibrium measure is constructed by considering “trajectories” of the transverse deviation functional is nonlinear, so that functional integration is unfeasible. One should identify a stationary solution

\[
\frac{\partial}{\partial x^\alpha} \nu^\alpha(\bar{x}, t) = 0.
\]

Here, the field \( f^\alpha \) represents a source/sink of momentum necessary to maintain velocity fluctuations. Customarily [7], we assume \( f^\alpha \) to be a white-in-time zero-mean Gaussian random force with covariance

\[
\langle f^\alpha(\bar{x}, t) f^{\beta}(\bar{x}', t') \rangle = F^{\alpha\beta}(\bar{x} - \bar{x}') \delta(t - t').
\]

A standard choice for \( F \) is

\[
F(\bar{x}) = \frac{D_0 L^3}{(2\pi)^3} \int d^3 p \ e^{i\bar{\nu} \bar{x} (L p)} e^{-(L p)^2},
\]

where \( D_0 \) is the power dissipated by the unitary mass, \( L \) is the integral scale and the exponent \( s \) is an integer number of order one (a typical value is \( s = 2 \)). Due to constraint (4), the field \( \nu^\alpha \) depends only on the transverse degrees of freedom of \( f^\alpha \). Without prejudice of generality we can also assume divergence-free forcing yielding the additional relation

\[
\frac{\partial}{\partial x^\alpha} F^{\alpha\beta}(\bar{x} - \bar{x}') = \frac{\partial}{\partial x^\beta} F^{\alpha\beta}(\bar{x} - \bar{x}') = 0.
\]

Following the Martin-Siggia-Rose formalism [1] we introduce the Navier-Stokes density of Lagrangian

\[
L(v, w, P, Q, f) = w^\alpha(\bar{x}, t) \left[ \left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) \nu^\alpha(\bar{x}, t) + \nu^\beta(\bar{x}, t) \frac{\partial}{\partial x^\beta} \nu^\alpha(\bar{x}, t) + \frac{1}{\rho} \frac{\partial}{\partial x^\alpha} P(\bar{x}, t) - f^\alpha(\bar{x}, t) \right] + \frac{1}{\rho} Q(\bar{x}, t) \frac{\partial}{\partial x^\alpha} \nu^\alpha(\bar{x}, t),
\]
where the field \( w^\alpha \) is the conjugate variable to the velocity field \( v^\alpha \) and the field \( Q \) is the Lagrangian multiplier related to constraint (4). In a similar way the pressure field \( P \) acts as a Lagrangian multiplier of the solenoidal constraint for the auxiliary field \( w^\alpha \). The generating functional is given by the integral

\[
W(P, Q, J) = \int \mathcal{D}v \mathcal{D}w \mathcal{D}f \exp \left\{ i \int dt \, d^3x \left[ \mathcal{L}(v, w, P, Q, f) + J_\alpha v^\alpha \right] \right. \\
- \frac{1}{2} \int dtd^3x d^3y f^\alpha F_{\alpha \beta}^{-1} f^\beta \right\}
\]

(9)

where \( J_\alpha \) is an external source. By performing successive integrations over the statistical measure, \( \mathcal{D}f e^{-\int \mathcal{L}^{-1} f} \), on the auxiliary field \( Q \), on the longitudinal components of \( w^\alpha \) and on the pressure field, all the longitudinal degrees of freedom can be eliminated and we end up with the effective functional for the transverse components (here denoted by \( v_{T \alpha} \) and \( w_{T \alpha} \))

\[
W(J) = \int \mathcal{D}v_T \mathcal{D}w_T \exp \left\{ i \int dt \, d^3x \left[ w_{T \alpha} \left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) v_{T \alpha} \right. \right.
\\
+ w_{T \alpha}^\beta v_{T \beta} \partial_\beta v_{T \alpha} + J_\alpha v_{T \alpha}^\alpha (t, \vec{x}) - \left. \int dt \, d^3x d^3y w_{T \alpha}^\alpha(t, \vec{x}) F_{\alpha \beta}(\vec{x} - \vec{y}) w_{T \beta}^\beta(t, \vec{y}) \right\}.
\]

(10)

Diagramatic strategies are usually applied at this level. We want to point out that a completely different point of view can be followed by observing that also the transverse components of the auxiliary field \( w_{T \alpha}^\alpha \) can be integrated out and one finally obtains:

\[
W(J) = \int \mathcal{D}v_T e^{-\frac{1}{2} I(v_T)} \int dt d^3x J_\alpha v_{T \alpha}^\alpha
\]

(11)

where the action functional \( I \) is given by

\[
I(v_T) = \int dtd^3x d^3y \left[ \left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) v_{T \alpha}^\alpha(t, \vec{x}) + v_{T \alpha}^\alpha(t, \vec{x}) \partial_\beta v_{T \beta}^\beta(t, \vec{x}) \right]
\\
- \frac{1}{F_{\alpha \beta}(\vec{x} - \vec{y})} \left[ \left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) v_{T \alpha}^\alpha(t, \vec{y}) + v_{T \alpha}^\alpha(t, \vec{y}) \partial_\lambda v_{T \lambda}^\lambda(t, \vec{y}) \right].
\]

(12)

This expression links the functional representation of the NSE to the large deviation theory developed in \([8,4]\)). In particular the entropy is related to the functional \( I(v_T) \) by \([4]\)

\[
S(v_T) = \frac{1}{2} \inf_\tilde{v} I(\tilde{v})
\]

(13)

Where the minimum is taken over all trajectories connecting a steady-state at time \( t = -\infty \) with the velocity field \( v_{T \alpha}^\alpha(0) \).

III. A QUASI-STEADY SOLUTION AND ITS STABILITY

We consider the equation of the extremal condition for the functional \( W(J) \) at \( J^\alpha = 0 \). In terms of the functional \( I(v_T) \) it reads:

\[
\frac{\delta I(v_T)}{\delta v_{T \alpha}^\alpha(t, \vec{x})} = 2 \int d^3y \left[ -\delta_{\alpha}^\alpha \left( \frac{\partial}{\partial t} + \nu \nabla^2 \right) v_{T \alpha}^\alpha(t, \vec{x}) \right. \\
- \delta_{\alpha}^\alpha v_{T \alpha}^\alpha(t, \vec{x}) \partial_\beta \left( \frac{1}{F_{\alpha \beta}(\vec{x} - \vec{y})} \right) \left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) v_{T \beta}^\beta(t, \vec{y}) \\
+ \left. v_{T \alpha}^\alpha(t, \vec{y}) \partial_\lambda v_{T \lambda}^\lambda(t, \vec{y}) \right] = 0.
\]

(14)

We observe that every solution of the equation
is solution of (14) as well. This is because of the presence of the projector on the transverse degrees of freedom in $F^\alpha\beta(\{\vec{x} - \vec{y}\})$. In particular the solutions of equation (15) are forcing independent. Indeed this equation is the usual unforced NSE with $\Phi \equiv \frac{1}{\rho} P$. We consider now the particular case where $\partial^\beta \Phi = 0$, that corresponds to search for transverse solutions satisfying the relation

$$\partial^\beta \left( v_T^\beta(t, \vec{x}) \partial^\alpha v_T^\alpha(t, \vec{x}) \right) = 0 .$$

(16)

In order to investigate the statistics of fluctuations, the determination of the hydrodynamic trajectory minimizing the entropy functional has to be performed explicitly. Unfortunately, existence and uniqueness theorems for the solution of eq. (15) are not available and the criterion for choosing a suitable solution can rely only upon physical considerations. Actually, we have found several different solutions: among them, the only one unaffected by divergences in space and time is the following:

$$V^\alpha(t, \vec{x}) = \frac{U^\alpha}{2} \left\{ 1 + e^{-\frac{\nu^2}{4\sqrt{\nu^2 U^2}^2 t}} \sin \left( \frac{2U^\alpha}{\sqrt{\nu^2 U^2 - (U^\alpha \vec{b})^2}} \frac{\delta \Phi(t, \vec{x})}{4\sqrt{\nu^2 R}} \right) \right\} , \text{ with } t > 0 ; \quad V^\alpha(t, \vec{x}) = \frac{U^\alpha}{2} , \quad \text{ with } t < 0 .$$

(17)

Notice that this solution decays exponentially in time with a rate $\nu/\sqrt{\nu^2 U^2}$, where $\tau_D$ is the diffusion time scale. Moreover, for $|x| \ll L$ this solution approximates a linear shear flow. This is well known to produce small scale instabilities for sufficiently large Reynolds number.

Due to the invariance under Galileo transformations, solution (17) is determined up to two vector parameters; the velocity $U^\alpha$ and the rotation axis $\vec{b}^\alpha$. These constants are related to the energy and to the enstrophy at the time $t$. Condition (16) is trivially satisfied, indeed

$$V^\beta(t, \vec{x}) \partial^\beta V^\alpha(t, \vec{x}) = 0 .$$

(18)

In other words, eq.(17) is also solution of the diffusion equation $(\partial_t - \nu^2 \nabla^2) u^\alpha(t, \vec{x}) = 0$. Obviously (17) is not a steady solution. It actually decays with a typical time $\tau = \frac{4\nu R^2}{U^2}$, which increases with $R$. We assume to consider sufficiently large values of the Reynolds number (or, equivalently, sufficiently small values of $U^2$) so that (17) can be viewed as a quasi-steady solution.

Small fluctuations around this quasi-steady solution can be analyzed by introducing the fluctuation field $u^\alpha(t, \vec{x})$:

$$v^\alpha(t, \vec{x}) = V^\alpha(t, \vec{x}) + u^\alpha(t, \vec{x})$$

(19)

and by studying the linearized equation

$$\frac{\partial}{\partial t} u^\alpha(t, \vec{x}) - \nu^2 u^\alpha(t, \vec{x}) + V^\beta(t, \vec{x}) \frac{\partial}{\partial \vec{x}^\beta} u^\alpha(t, \vec{x}) + u^\beta(t, \vec{x}) \frac{\partial}{\partial \vec{x}^\beta} V^\alpha(t, \vec{x}) = 0 .$$

(20)

For sufficiently small initial perturbations, linear analysis can be applied up to a time scale where the nonlinear terms are kept small with respect to the linear ones. An upper bound for this time scale is computed in Appendix A:

$$0 < t \lesssim \frac{8\nu R}{U^2} .$$

(21)

This upper bound turns out to be consistently smaller than $\tau_D$. Moreover, a sufficient condition for the stability of solution (17) with respect to small perturbations has been also derived in Appendix A:

$$\frac{8\nu^2 R}{U^2} k^2 > 1 .$$

(22)

This condition indicates that for high Reynolds numbers only large wave-numbers are stable.
IV. PERTURBATIVE ANALYSIS OF THE GENERATING FUNCTIONAL

By exploiting the translational invariance of the functional measure, (11) can be rewritten in the form

\[ \mathcal{W}(J) = \int D\eta e^{-\frac{1}{2} \mathcal{W}(\eta) + i \int dt d^3 x J(x) u_\eta(x)} . \]

(23)

We recall that we aim at obtaining explicit expressions for the structure functions of the perturbation field around the quasi-steady solution by performing derivatives of the functional generator with respect to the currents \(J^\alpha\).

Without any further approximation, such a program seems to be prohibitive. Some simplifications have to be introduced, yielding a structure of the functional integral which involves Gaussian integrations.

As a first step in this direction we replace the original action in the functional (23) by the bilinear action

\[ I_V(u) = \int dt d^3 x d^3 y \left[ (\partial_t - \nu \nabla^2) u_T^\alpha(\hat{x}) + V^\alpha(\hat{x}) \partial_\rho u_T^\rho(\hat{x}) + u_T^\alpha(\hat{x}) \partial_\rho V^\rho(\hat{x}) \right] \frac{1}{F^{\alpha\beta}(|\hat{x} - \hat{y}|)} \times \]

\[ \left[ (\partial_t - \nu \nabla^2) u_T^\beta(\hat{y}) + V^\beta(\hat{y}) \partial_\lambda u_T^\lambda(\hat{y}) + u_T^\beta(\hat{y}) \partial_\lambda V^\lambda(\hat{y}) \right] + O(u^3) . \]

(24)

which comes from the linearized equations (20) and where we have used the notation \( \hat{x} \equiv (t, \hat{x}) \). Note that the extreme solution \( V^\alpha \), around which fluctuations are computed, appears explicitly in the functional \( I_V(u) \). We have already observed that the solution \( V^\alpha \) is not unique. Nevertheless, if the stability conditions of this solution with respect to small perturbations are fulfilled we can conclude that it cannot be influenced by other possible solutions.

One further simplification can be introduced by observing that solution (17) naturally suggests a perturbative expansion in integer powers of \( \frac{1}{R} \). Indeed, the Fourier transform of action (24) up to the first order in the expansion parameter \( \frac{1}{R} \), becomes

\[ I_V(u) = \int \frac{d^4 p}{(2\pi)^4} u^\alpha(-\hat{p}) M^\alpha_\zeta(-\hat{p}) \frac{1}{F^{\alpha\beta}(\hat{p})} M_\zeta^\beta(\hat{p}) u^\zeta(\hat{p}) + O\left( \frac{1}{R^2} \right) , \]

(25)

where the hydrodynamic evolution term \( M^\beta_\zeta(\hat{p}) u^\zeta(\hat{p}) \) has the form

\[ M^\beta_\zeta(\hat{p}) u^\zeta(\hat{p}) = \left\{ \delta^{\beta}_{\zeta} \left[ i \left( p_0 + \frac{1}{2} \hat{p} \cdot \hat{U} \right) + i p^2 - C 4 \hat{p} \cdot \hat{U} \frac{\hat{b} \wedge \hat{U}}{4\nu R} \partial_{p_i} \right] - C 4 \hat{U}^\beta \frac{\hat{b} \wedge \hat{U}}{4\nu R} \right\} u^\zeta(\hat{p}) . \]

Here \( C = \frac{2U}{\sqrt{\nu^2 (t^2 - \hat{U}^2)^2}} \) and \( \hat{p} \equiv (p_0, \hat{p}) \) where \( p_0 \) and \( \hat{p} \) are the conjugate variables of \( t \) and \( \hat{x} \), respectively.

In order to evaluate the functional integral in (23), the diagonalization of the matrix \( M^\alpha_\zeta(-\hat{p}) \frac{1}{F^{\alpha\beta}(\hat{p})} M_\zeta^\beta(\hat{p}) \) is required. Since by definition the factor \( |F^{\alpha\beta}(\hat{p})|^{-1} \) is proportional to the identity operator in the space of the transverse solutions it remains to diagonalize only the matrix \( M^\beta_\zeta(\hat{p}) \).

The computation of the eigenvalues, \( \lambda \), of \( M^\beta_\zeta(\hat{p}) \) can be accomplished by a standard procedure, which, however, requires lengthy and tedious calculations: they are sketched in Appendix B. We report hereafter the final form taken by the generating functional:

\[ \mathcal{W}(\eta) = \int \mathcal{J}(H) D\phi_T e^{-\frac{1}{2} \int \phi_T \mathcal{F}(\eta, \phi_T) \phi_T + i \int \eta \mathcal{J}(\phi_T) \phi_T} \]

(27)

Here \( H \) is the matrix that diagonalizes \( M^\alpha_\zeta(-\hat{p}) \frac{1}{F^{\alpha\beta}(\hat{p})} M_\zeta^\beta(\hat{p}) \); \( \mathcal{J}(H) \) is the Jacobian of the basis transformation \( u \rightarrow \phi, J \rightarrow \eta \) engendered by \( H \).

By performing a Gaussian integration we obtain the normalized functional in term of the \( \eta^\alpha \) source

\[ \mathcal{W}(\eta) = e^{-\frac{1}{2} \int \eta \mathcal{J}(\eta) \eta^\alpha \hat{J}_C^\alpha(\eta) + i \int \eta \mathcal{J}(\eta) \hat{J}_C^\alpha(\eta)} , \]

(28)

By returning to the representation in the original basis this equation can be rewritten as:

\[ \mathcal{W}(J) = e^{-\frac{1}{2} \int J_C^\alpha(\eta) (H \hat{J}_C^\alpha(\eta))_{\eta^\alpha} J_C^\alpha(\eta)} . \]

(29)

The functional (29) is the starting point for the calculation of all correlation functions (and structure functions), that can be obtained by derivation with respect to the \( J_C^\alpha \) currents. The procedure to achieve this goal is the subject of the next Section.
V. SHORT-DISTANCE BEHAVIOR OF THE SECOND ORDER STRUCTURE FUNCTION

The expression derived for the generating functional (29), contains all the statistical information on the fluctuations around the basic solution $V_\alpha$. Here we will perform analytic calculations for the particular class of fluctuations captured by the lowest, nontrivial, integer moment of velocity differences between points separated by a distance $r$. For the velocity field, $u^\alpha$, this is the second-order structure function defined as

$$S_2 = \langle |u(t, \vec{r} + \vec{x}) - u(t, \vec{x})|^2 \rangle = \langle |(u^\alpha(t, \vec{r} + \vec{x}) - u^\alpha(t, \vec{x}))(u_\alpha(t, \vec{r} + \vec{x}) - u_\alpha(t, \vec{x}))|\rangle,$$

where the brackets denote averages on the forcing statistics. By assuming isotropy and homogeneity of the velocity field, expression (30) is expected to assume the typical form of scale invariant functions

$$S_2(r) = r^{\zeta_2} F_2 \left( t, \frac{r}{L} \right)$$

(31)

Here $r = |\vec{r}|$ and $L$ is the large spatial scale associated with the noise source. It is worth stressing that, at variance with fully developed turbulent regimes, here the assumption of isotropy and homogeneity have to be taken as a plausible hypothesis allowing for analytic computations.

We want to point out that any exponent $\zeta_\alpha$ should be independent of the basis chosen for representing the functional $W$. Making use of (28), one obtains:

$$S_2(r) = \langle |\vec{u}(\vec{x} + \vec{r}) - \vec{u}(\vec{x})|^2 \rangle = \left. \left( \frac{\delta}{\delta \eta^\alpha(t, \vec{x} + \vec{r})} - \frac{\delta}{\delta \eta^\alpha(t, \vec{x})} \right) W(\eta) \right|_{\eta = 0}$$

$$= 2 \left( \Delta^\alpha_0(0, \vec{r}) - \Delta^\alpha_0(0) \right) = 2 \int \frac{dp_0 dp^2}{(2\pi)^4} \left( e^{i\vec{p} \cdot \vec{r}} - 1 \right) \left( \Delta_{11}(\vec{p}) + \Delta_{22}(\vec{p}) \right)$$

$$= -2 \int \frac{dp_0 dp^2}{(2\pi)^4} \left( e^{i\vec{p} \cdot \vec{r}} - 1 \right) \sum_{\alpha=1}^{2} \frac{F(p)}{\left( p_0 + \frac{1}{2} \vec{p} \cdot \vec{U} \right)^2 + \left( \nu p^2 + \lambda^\alpha_{(2)}(\vec{p}, \vec{U}, \vec{b}) \right)}^2.$$

(32)

The explicit integration over $p_0$ yields

$$\tilde{S}_2(r) = - \int \frac{d^3p}{(2\pi)^3} \left( e^{i\vec{p} \cdot \vec{r}} - 1 \right) \sum_{\alpha=1}^{2} \frac{F(p)}{\left( p_0 + \frac{1}{2} \vec{p} \cdot \vec{U} \right)^2 + \left( \nu p^2 + \lambda^\alpha_{(1)}(\vec{p}, \vec{U}, \vec{b}) \right) + \ldots}$$

(33)

The expressions of the eigenvalues $\lambda^\alpha_{(1)}$ ($\alpha = 1, 2$) are given in Appendix B. In remains to specify the geometrical structure of the flow. For the sake of simplicity, we assume that the vector $\vec{r}$ corresponds to the polar axis and that the vector $\vec{b}$ is orthogonal to both $\vec{r}$ and $\vec{U}$. It turns out that $S_2(r)$ can be expressed as the sum of two terms: the first one is associated with the null eigenvalue $\lambda^1_{(1)}$, while the second one depends on the nonzero eigenvalue $\lambda^2_{(1)}$. Namely,

$$\tilde{S}_2(r) = - \frac{1}{\nu} (I_1(r) + I_2(r)).$$

(34)

The expression of $I_1(r)$ is derived in Appendix C:

$$I_1(r) = D_0 L^3 \int \frac{d^3p}{(2\pi)^3} \left( e^{i\vec{p} \cdot \vec{r}} - 1 \right) \frac{(Lp)^{s} e^{-Lp^2}}{p^2}$$

(35)

By simple algebraic manipulations, it can be recasted into the form

$$I_1(r) = \frac{D_0}{(2\pi)^2} r^2 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\Gamma \left( \frac{s+3+2n}{2} \right)}{\Gamma (2n+4)} \left( \frac{r}{L} \right)^{2n}.$$

(36)
We can conclude that, for short distance \( r \), the leading contribution in \( I_1( r ) \) is \( r^2 \), that is a dissipative contribution. Some lengthy algebra (see Appendix C for details) provides also an expression for \( I_2( r ) \):

\[
I_2( r ) = D_0 L^3 \frac{32 \nu^2 R}{U^2} \int_0^\infty \frac{p^2 dp}{(2\pi)^2} (Lp)^2 e^{-(Lp)^2} \int_{-1}^1 dx \left( e^{iprx} - 1 \right)
\times \left( \sum_{l=1,2} \left( 1 - x^2 \right) \frac{1}{x^2} \left( \sum_{m=0}^{\infty} s_{lm} F_m \left( x \frac{\nu^2 R}{U^2} p^2 ; \Sigma, \Xi \right) + \frac{1}{2} \sum_{k=0}^{\infty} \left( \frac{3}{1 - x^2} \right)^k \right) \right)
\times \left( \frac{1}{R^2} \right),
\]

(37)

where the coefficients \( s_{ij} \) and the functions \( F_i \) are specified in Appendix C.

The key remark for proceeding in this calculation is that the stochastic measure \( p^2 + s e^{-(Lp)^2} dp \) gives a significant contribution to the first integral in (37) only in a narrow region of wavenumbers close to \( \bar{p} \) where the function \( p^2 + s e^{-(Lp)^2} \) has its maximum, i.e.

\[
\bar{p} = \frac{1}{L} \sqrt{\frac{s + 2}{2}}.
\]

(38)

Notice that the function \( \frac{\nu^2 R}{U^2} p^2 \) thus contributes to the integral by taking values close to \( \frac{4(s+2)}{R} \). Moreover, for \( p = \bar{p} \) the sufficient condition (22) for the stability of small perturbations determines the upper bound

\[
R < \frac{4(s + 2)}{\bar{p}^2}.
\]

(39)

This implies that for sufficiently small Reynolds' numbers the wavenumber \( \bar{p} \) is stable. Under this condition, the leading contribution in (37) can be obtained by performing an expansion in powers of \( \frac{1}{R^2} \).

One finally obtains the complete expression of the structure function (see Appendix C for details)

\[
\bar{S}_2( r ) = -\frac{1}{\nu} (I_1( r ) + I_2( r ))
\sim -\frac{D_0}{(2\pi)^2 \nu} r^2 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2} \left( \frac{s + 2n + 3}{2} \right) \left( \frac{1 + \Xi}{\Gamma(2n + 4)} - \frac{2\nu\Xi}{\Gamma(2n + 6)} \right) \left( \frac{r}{L} \right)^{2n} + O \left( \frac{R}{4(2 + s)} \right),
\]

(40)

for \( 1 < R \ll 4(2 + s) \).

At leading order in the distance \( r \) it is dominated by a dissipative contribution.

We conjecture that this analysis can be extended to the parameter region defined by the condition \( R > 4(2 + s) \), where the statistically relevant wavenumbers can be unstable. As shown in Appendix C, in this case \( I_2( r ) \) gives two contributions: one is again dissipative, while there is another one yielding the nontrivial scaling behavior \( r^{2/3} \). Specifically, the expression of \( S_2( r ) \) for \( R > 4(2 + s) \) is

\[
\bar{S}_2( r ) = -\frac{D_0}{\pi \nu} \left\{ \frac{1 + \frac{s + 2}{2}}{4\pi} r^2 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\Gamma \left( \frac{s + 2n + 3}{2} \right)}{\Gamma(2n + 4)} \left( \frac{r}{L} \right)^{2n} + \frac{R^2}{\Gamma \left( \frac{4}{3} \right)} \left( \frac{\nu}{U} \right)^{4/3} \sum_{n=0}^{\infty} C_n(\Sigma) \Gamma \left( \frac{3s + 3n + 5}{6} \right) \left( \frac{r}{L} \right)^{n} + O \left( \frac{4(2 + s)}{R} \right) \right\}
\]

(41)

This is dominated by the term \( r^{2/3} \) for sufficiently small distances. Indeed, the crossover scale between the \( r^2 \) and the \( r^{4/3} \) terms occurs at

\[
\frac{r}{L} \sim FR^{-\frac{1}{4}},
\]

(42)

In appendix C we evaluate the constant \( F \sim 0.6 \) and we report also the expression of the numerical coefficient \( C_0(\Sigma) \) (the general expression of the coefficients \( C_n(\Sigma) \) appearing in (41), has been skipped, because it has no practical interest).

It is a remarkable fact that \( S_2 \) can exhibit the scaling behavior predicted by K41 theory, which is assumed to hold when the velocity fluctuations are turbulent in the so-called inertial range of scales. This suggests that hydrodynamic fluctuations in a system at the very initial stage of instability development already contain some properties attributed to the developed turbulence regime.
VI. CONCLUSIONS

In this paper we have exploited field-theoretic calculations to reformulate the random forced Navier–Stokes problem in terms of a quadratic action functional. At a formal level, the latter has the same structure of the action describing thermal fluctuations in irreversible stationary processes. The crucial step for obtaining the hydrodynamic evolution operator which appears in the action functional, is the integration over all longitudinal components of both velocity and associated auxiliary fields. With respect to the standard formulation which is the starting point for diagramatic strategies, we thus perform one more field integration. The positive definite kernel in the action functional appears in the form of the inverse of the forcing correlation function.

In terms of the action functional, the knowledge of the whole velocity statistics reduces to the computation of functional integrals. However, finding an explicit solution for these integrals is quite a difficult task, due to the nonlinear character of the problem. This forces us to introduce some approximations. The starting point is the identification of a stationary solution around which we linearize the evolution operator. We define also a fluctuation field (with respect to the stationary solution) and we are able to compute (perturbatively, in the inverse of the Reynolds number) the functional integrals over such fluctuation field.

In principle, the strategy might be applied to evaluate any velocity multipoint statistical quantity. In order to reduce the complexity of the algebraic manipulations we limited ourselves to the calculation of the two-point second order momentum of velocity. Indeed, we aim at understanding if fluctuations at the early stage of their development (accordingly, we dub them as pre-turbulent fluctuations) already contain some important features of developed turbulence. We are interested, in particular, to scale invariance. In this respect, we find that fluctuations are organized at different scales in a self–similar way. Remarkably, the scaling exponent coincides with the dimensional prediction of the Kolmogorov 1941 theory [6] valid for developed turbulence regimes. Whether or not such exponent is a genuine reminiscence of the developed turbulence phenomenology needs further investigations.

Unfortunately, the complexity of the derivation leading to the K41 scaling law does not allow to identify precisely the very origin of such a dimensional prediction. We can however argue a relationship between the observed dimensional scaling and the conservation laws (for momentum and energy) associated to the two eigenvalues of the matrix appearing in the action functional (25).

Another point to be emphasized is that the pressure term does not play any role in the derivation of the dimensional scaling law. This is just a consequence of the fact that all the longitudinal degrees of freedom can be averaged out from the very beginning of the computation.

We want to conclude by outlining some open problems and perspectives. A first question concerns the relevance to be attributed to the solution around which we linearize the evolution operator. On one side we do not see any rigorous mathematical motivation for invoking the need of a unique solution. Just heuristic arguments based on physical considerations allowed us to identify the selected solution. Indeed, it represents a shear-like solution, which is a well-known generator of instability towards smaller and smaller scales.

Another interesting point concerns the computation of the third-order moment of the velocity correlators. In this case the predictions of our approach could be compared with the $4/5$-law, which is one among the very few exact results of turbulence theories.

Finally, the extension of our results to other classes of transport problems, including passive scalar advection, could provide a better understanding of the basic mechanism at the origin of the observed scaling behaviors.

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APPENDIX A

In this Appendix we study the stability analysis of the solution $V^\alpha$ relative to the linearized equation (20). We have already observed that $V^\alpha$ is a quasi-steady solution for a time $t \ll \tau_D = \frac{4\nu R}{U^2}$. The Fourier transform of eq.(20) with respect to the $\vec{x}$ spatial variable yields:

$$\frac{\partial}{\partial t} \tilde{u}^\alpha(t, \vec{k}) - \nu k^2 \tilde{u}^\alpha(t, \vec{k}) + \frac{i\vec{v}}{2} \cdot \vec{U} \tilde{u}^\alpha(t, \vec{k}) + \frac{1}{4} e^{-\frac{t}{\tau_D}} \left\{ U^\beta k_\beta \left[ \tilde{u}^\alpha(t, \vec{k} - C \frac{\vec{y}}{4\nu R} + \vec{U} \right] \right\}$$
\[-\tilde{u}^\alpha \left( t, \vec{k} + C \vec{b} \land \vec{U} \right) + U^\alpha C \left( \frac{\vec{b} \land \vec{U}}{4\nu R} \right)_\beta \left\{ \tilde{u}^\beta \left( t, \vec{k} - C \vec{b} \land \vec{U} \right) + \tilde{u}^\beta \left( t, \vec{k} + C \vec{b} \land \vec{U} \right) \right\} \]
\[= 0. \quad (A1)\]

By performing a perturbative expansion up to second order in the parameter \( R^{-1} \), one obtains the system of equations

\[
\frac{\partial}{\partial t} \tilde{u}^\alpha_{(0)} \left( t, \vec{k} \right) + \nu k^2 \tilde{u}^\alpha_{(0)} \left( t, \vec{k} \right) + \frac{i}{2} \vec{k} \cdot \vec{U} \tilde{u}^\alpha_{(0)} \left( t, \vec{k} \right) = 0, \quad (A2)
\]

\[
\frac{\partial}{\partial t} \tilde{u}^\alpha_{(1)} \left( t, \vec{k} \right) + \nu k^2 \tilde{u}^\alpha_{(1)} \left( t, \vec{k} \right) + \frac{i}{2} \vec{k} \cdot \vec{U} \tilde{u}^\alpha_{(1)} \left( t, \vec{k} \right) = 1 \frac{\vec{k} \cdot \vec{U} C \left( \vec{b} \land \vec{U} \right)}{4\nu R} \frac{\partial}{\partial k} \tilde{u}^\alpha_{(0)} \left( t, \vec{k} \right), \quad (A3)
\]

\[
\frac{\partial}{\partial t} \tilde{u}^\alpha_{(2)} \left( t, \vec{k} \right) + \nu k^2 \tilde{u}^\alpha_{(2)} \left( t, \vec{k} \right) + \frac{i}{2} \vec{k} \cdot \vec{U} \tilde{u}^\alpha_{(2)} \left( t, \vec{k} \right) = 1 \frac{\vec{k} \cdot \vec{U} C \left( \vec{b} \land \vec{U} \right)}{4\nu R} \frac{\partial}{\partial k} \tilde{u}^\alpha_{(1)} \left( t, \vec{k} \right), \quad (A4)
\]

This system of equations yields the perturbative solution

\[
\tilde{u}^\alpha \left( t, \vec{k} \right) = e^{-\left( \nu k^2 + \frac{i}{2} \vec{k} \cdot \vec{U} \right) t} \left\{ F^\alpha_{(0)} \left( \vec{k} \right) + F^\alpha_{(1)} \left( \vec{k} \right) \right. \]
\[+ C \frac{\vec{k} \cdot \vec{U}}{8\nu R} \left( \vec{b} \land \vec{U} \right) \cdot \vec{k} \left[ F^\alpha_{(0)} \left( \vec{k} \right) t - \frac{U^\alpha}{\vec{k} \cdot \vec{U}} \left( \vec{b} \land \vec{U} \right) \cdot \vec{k} \right] \]
\[- \left( \vec{b} \land \vec{U} \right) \cdot \vec{k} \left[ t F^\alpha_{(0)} \left( \vec{k} \right) \nu t^2 \right] + O \left( \frac{1}{R^2} \right) \left. \right\}. \quad (A5)
\]

For the expansion (A5) to be meaningful, the time \( t \) must be smaller than \( \sim R \). This amounts to impose the condition:

\[
0 < t \lesssim \frac{8\nu R}{U^2} = \frac{2\tau_D}{R}. \quad (A6)
\]

This stability condition implies that, at any time \( t \), the linear term in the curly brackets cannot overtake the exponential factor. Such a requirement can be translated into the following spectral condition

\[
\frac{8\nu^2 R}{U^2} k^2 > 1. \quad (A7)
\]

**APPENDIX B**

In this Appendix we sketch the calculation of the eigenvalues of the matrix \( M^\beta_\zeta \left( \vec{p} \right) \). We exploit a perturbative approach, whose expansion parameter is \( \frac{1}{R} \). The matrix \( M^\beta_\zeta \left( \vec{p} \right) \) acts on the two-dimensional space of the transverse functions and on the one-dimensional space of the longitudinal functions. Only the transverse degrees of freedom are physically meaningful.

In terms of the \( \frac{1}{R} \) expansion we have

\[
M = M_{(0)} + M_{(1)} + ... \quad (B1)
\]

where
\[ M_{(0)}^\alpha = \delta_\beta^\alpha \left[ i \left( p_0 + \frac{1}{2} \bar{\gamma} \cdot \bar{U} \right) + \nu \bar{p}^2 \right], \]
\[ M_{(1)}^\alpha = -\delta_\beta^\alpha \frac{C}{4} \bar{\gamma} \cdot \bar{U} \frac{\left( \bar{b} \wedge \bar{U} \right) \gamma}{4\nu R} \partial_{p_\gamma} - \frac{C}{4} \frac{U^\alpha}{4\nu R} \bar{v}_p. \]

A complete orthonormal basis in \( R^3 \) is given by the vectors
\[ \Pi_1^\alpha = \frac{\left( \bar{b} \wedge \bar{p} \right)^\alpha}{\sqrt{f(p)}}, \]
\[ \Pi_2^\alpha = \frac{g(p) \left( \bar{b} \wedge \bar{p} \right)^\alpha - f(p) \left( \bar{U} \wedge \bar{p} \right)^\alpha}{\sqrt{f(p)\sqrt{f(p)h(p) - g^2(p)}}}, \]
\[ \Pi_3^\alpha = \frac{\bar{p}^\alpha}{p}, \]

where we have defined
\[ f(p) = b^2 p^2 - (\bar{b} \cdot \bar{p})^2, \quad g(p) = (\bar{b} \cdot \bar{U}) p^2 - (\bar{b} \cdot \bar{p}) (\bar{U} \cdot \bar{p}), \quad h(p) = U^2 p^2 - (\bar{U} \cdot \bar{p})^2. \]

Here, \( \Pi_1^\alpha \) and \( \Pi_2^\alpha \) span the transverse subspace, while \( \Pi_3^\alpha \) spans the longitudinal one. Likewise, the eigenvalues can be represented in terms of a perturbative expansion as
\[ \lambda^a = \lambda^a_{(0)} + \lambda^a_{(1)} + \ldots \quad \text{where} \quad a = 1, 2, 3. \]

The zero-order eigenvalues \( \lambda^a_{(0)} \) are degenerate and have the form
\[ \lambda^a_{(0)} = \left( i \left( p_0 + \frac{1}{2} \bar{\gamma} \cdot \bar{U} \right) + \nu \bar{p}^2 \right), \]

The evaluation of the first order corrections \( \lambda^a_{(1)} \) requires the diagonalization of the matrix with elements \( M_{(1)ij} = \langle \Pi_i, M_{(1)} \Pi_j \rangle, \quad (i, j = 1, 2, 3) \). After some simple but lengthy calculations we find
\[ \lambda^1_{(1)} = \frac{1}{2} \left( M_{(1)11} + M_{(1)22} - \sqrt{(M_{(1)11} + M_{(1)22})^2 + 4M_{(1)21}M_{(1)12}} \right), \]
\[ \lambda^2_{(1)} = \frac{1}{2} \left( M_{(1)11} + M_{(1)22} + \sqrt{(M_{(1)11} + M_{(1)22})^2 + 4M_{(1)21}M_{(1)12}} \right), \]
\[ \lambda^3_{(1)} = M_{(1)33}, \]

and
\[ M_{(1)11} = \frac{C}{16\nu R} \frac{\left( \bar{b} \wedge \bar{U} \right) \bar{p} \cdot \bar{p}}{f(p) \sqrt{g(p)h(p) - g^2(p)}} w(p) , \]
\[ M_{(1)22} = -\frac{C}{16\nu R} \frac{\left( \bar{b} \wedge \bar{U} \right) \bar{p} \cdot \bar{p} \cdot \bar{b} \cdot \bar{p} g(p)}{f(p) \sqrt{g(p)h(p) - g^2(p)}} \left[ \left( \bar{b} \cdot \bar{U} \right) w(p) + (\bar{b} \cdot \bar{U}) g(p) - U^2 f(p) \right] , \]
\[ M_{(1)12} = -\frac{C}{16\nu R} \frac{\left( \bar{b} \wedge \bar{U} \right) \bar{p} \cdot \bar{p} \left( \bar{b} \cdot \bar{p} \right)}{f(p) \sqrt{g(p)h(p) - g^2(p)}} \left[ \left( \bar{b} \cdot \bar{U} \right) g(p) + 2 \left( \bar{b} \cdot \bar{U} \right) w(p) - U^2 f(p) \right] , \]
\[ M_{(1)21} = -\frac{C}{16\nu R} \frac{\left( \bar{b} \wedge \bar{U} \right) \bar{p} \cdot \bar{p} \left( \bar{b} \cdot \bar{p} \right)}{f(p) \sqrt{g(p)h(p) - g^2(p)}} \left( \bar{b} \cdot \bar{U} \right) \left[ \left( \bar{b} \cdot \bar{p} \right) g(p) - \left( \bar{p} \cdot \bar{U} \right) f(p) \right] , \]
\[ M_{(1)33} = -\frac{C}{16\nu R} \frac{\left( \bar{b} \wedge \bar{U} \right) \bar{p} \cdot \bar{p}}{\bar{p}^2} \left( \bar{b} \wedge \bar{U} \right) \bar{p} , \]

where we have defined
\[ w(p) = b^2 (\vec{p} \cdot \vec{U}) - (\vec{b} \cdot \vec{p}) (\vec{b} \cdot \vec{U}) \, . \]

With the particular choice performed in Section V, the two physically relevant eigenvalues are

\[ \lambda_{(1)}^1 = 0 \, , \]
\[ \lambda_{(1)}^2 = \frac{U^2}{16 \nu R} \left\{ \sin \theta_U \cos \theta_U \left[ \cos^2 \phi_U + \cos (2(\phi_U - \phi)) \right] \sin^2 \theta \ight. 
\left. + \cos^2 \theta_U \sin 2\theta \cos (\phi_U - \phi) \right\} \, . \]  

(B10)

APPENDIX C

This Appendix contains the essential steps necessary for computing the second order structure functions.

We start from the expression of \( \bar{S}_2 \)

\[ \bar{S}_2(r) = - \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p} \cdot \vec{r}} - 1}{\nu} \sum_{\alpha=1}^{2} \frac{F(p)}{\nu^2 + \frac{\lambda_{(1)}^\alpha (\vec{p}, \vec{U}, \vec{b})}{\nu}} \]
\[ = - \frac{1}{\nu} (I_1(r) + I_2(r)) \, . \]  

(C1)

By considering the explicit expressions of the statistical function \( F(p) \) and of the eigenvalues \( \lambda_{(1)}^\alpha \) (see eq. (B10)), one has

\[ I_1(r) = D_0 L^3 \int \frac{d^3p}{(2\pi)^3} \left( e^{i\vec{p} \cdot \vec{r}} - 1 \right) \frac{(Lp)^s e^{-(Lp)^2}}{p^2} \, , \]  

(C2)

\[ I_2(r) = D_0 L^3 \int \frac{d^3p}{(2\pi)^3} p^2 + \frac{U^2}{16 \nu R} \left[ 2 \sin \theta_U \cos \theta_U \sin^2 \theta \cos^2 \phi + \cos^2 \theta_U \sin 2\theta \cos \phi \right] \, . \]  

(C3)

By exploiting translational invariance, in (C3) we have applied the transformation \( \phi_U - \phi \to -\phi \). The evaluation of (C2) follows from a standard procedure:

\[ I_1(r) = D_0 L^3 \int \frac{d^3p}{(2\pi)^3} \left( e^{i\vec{p} \cdot \vec{r}} - 1 \right) \frac{(Lp)^s e^{-(Lp)^2}}{p^2} \]
\[ = D_0 L^2 \frac{\nu}{2 \pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)! (2n + 1)} \left( \frac{r}{L} \right)^{2n} \int_0^\infty d\zeta \, \zeta^{s+2n} e^{-\zeta^2} \]
\[ = D_0 \frac{\nu}{(2\pi)^2} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\Gamma \left( \frac{s+2n}{2} \right)}{\Gamma (2n + 4)} \left( \frac{r}{L} \right)^{2n} \, . \]  

(C4)

Concerning the term \( I_2(r) \) we first perform the integration in the \( \phi \) variable. Namely,

\[ I_2(r) = D_0 L^3 \int_0^\infty \frac{p^2 dp}{(2\pi)^3} (Lp)^s e^{-(Lp)^2} \int_{-1}^{+1} d(\cos \theta) \left( e^{ipr \cos \theta} - 1 \right) I_0 \]  

(C5)

where

\[ I_0 = \int_0^{2\pi} \frac{d\phi}{p^2 + \frac{U^2}{16 \nu R} \left[ 2 \sin \theta_U \cos \theta_U \sin^2 \theta \cos^2 \phi + \cos^2 \theta_U \sin 2\theta \cos \phi \right]} \]
\[ = - \frac{32 \nu^2 R}{U^2} \int \frac{dz}{a z^4 + b z^3 + c z^2 + b z + a} \, , \]  

(C6)

with \( z = e^{i\phi} \) and the integration is on the unitary circle. The coefficients \( a, b, c \) are given by
\[ a = \sin \theta_U \cos \theta_U \sin^2 \theta, \quad b = 2 \cos^2 \theta_U \sin \theta \cos \theta \]
\[ c = \frac{32 \nu^2 R}{U^2} \rho^2 + 2 \sin \theta_U \cos \theta_U \sin^2 \theta . \] (C7)

The evaluation of the integral (C6) requires the knowledge of the root of a fourth degree algebraical equation. By exploiting the Euler method [9] we end up with the expression

\[
z_i = z_i \left( x, \frac{8 \nu^2 R}{U^2} \rho^2, \Sigma, \Xi \right)
= \frac{x^{\frac{1}{6}}}{(1 - x^2)^{\frac{1}{6}}}
\left[ \sum_{l=0}^{2} s_{il} F_l \left( x, \frac{8 \nu^2 R}{U^2} \rho^2, \Sigma, \Xi \right) + \frac{1}{2} \sum_{\xi} \frac{x^{\frac{1}{6}}}{(1 - x^2)^{\frac{1}{6}}} \right]
\quad i = 1, 2, 3, 4 .
\]

The following definition has been adopted:

\[
F_l = F_l \left( x, \frac{8 \nu^2 R}{U^2} \rho^2, \Sigma, \Xi \right)
= \left( \frac{\Sigma^\frac{2}{3}}{12} \left[ \frac{81}{4} \Sigma^4 \frac{x^4}{(1 - x^2)^2} + \frac{81}{2} \Sigma^2 \frac{x^2}{1 - x^2} - 90 \right]
\frac{64}{\Sigma^2 x^2} + \frac{8 \nu^2 R}{U^2} \rho^2 \left( \frac{189 \Sigma^2}{\Xi (1 - x^2)} + \frac{382}{\Xi (1 - x^2)} - 120 \frac{\Sigma^2}{\Xi x^2} \right)
+ \left( \frac{8 \nu^2 R}{U^2} \rho^2 \right)^2 \left( \frac{504}{\Xi^2 (1 - x^2)^2} + \frac{47 \Sigma^2}{\Xi^2 x^2 (1 - x^2)} \right) + \left( \frac{8 \nu^2 R}{U^2} \rho^2 \right)^3 \frac{32 \Sigma^2}{\Xi^3 x^2 (1 - x^2)^2} \right)^{\frac{1}{2}}
\times \left( \epsilon^l \left( 1 + 4 \times 27 \frac{h}{2} \right)^{\frac{1}{4}} \epsilon^l \left( 1 - 4 \times 27 \frac{h}{2} \right)^{\frac{1}{4}} \right) + \frac{1}{2} \frac{\Sigma^\frac{2}{3} x^\frac{1}{3}}{\Xi} \right) ^{\frac{1}{2}},
\]

with

\[ x = \cos \theta, \quad \Xi = \sin \theta_U \cos \theta_U, \quad \Sigma = \cot \theta_U , \quad (C9) \]
\[ s_{il} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}. \]

(C10)

Here \( \epsilon \) is the cubic root of the unity: \( \epsilon = -\frac{1 + i \sqrt{3}}{2} \). Finally the function \( h \) is

\[
h = \left[ 16 + 30 \Sigma^2 \frac{x^2}{1 - x^2} + \frac{111}{4} \Sigma^4 \frac{x^4}{(1 - x^2)^2} + 16 \frac{8 \nu^2 R}{U^2} \rho^2 \frac{17 \Sigma^2 x^2 + 3 (1 - x^2)}{(1 - x^2)^2} \right]
+ \left[ 48 \left( \frac{8 \nu^2 R}{U^2} \rho^2 \right)^2 \left[ \frac{1}{(1 - x^2)^2} \right]^3 \times \left[ -128 + 81 \Sigma^4 \frac{x^4}{(1 - x^2)^2} + \frac{81}{2} \Sigma^6 \frac{x^6}{(1 - x^2)^3} \right]
- 180 \Sigma^2 \frac{x^2}{1 - x^2} + \frac{8 \nu^2 R}{U^2} \rho^2 \left( \frac{378 \Sigma^4 x^4}{(1 - x^2)^3} + 764 \Sigma^2 x^2}{(1 - x^2)^2} - 240 \frac{1}{1 - x^2} \right)
+ \left( \frac{8 \nu^2 R}{U^2} \rho^2 \right)^2 \left( 1008 \frac{\Sigma^2 x^2}{(1 - x^2)^2} + 94 \frac{1}{(1 - x^2)^2} \right) + 64 \left( \frac{8 \nu^2 R}{U^2} \rho^2 \right)^3 \frac{1}{(1 - x^2)^3} \right]^{-2}. \] (C11)

Only the roots \( z_1 \) and \( z_2 \) are included into the unit circle, therefore (C5) becomes
\[ I_2(r) = D_0 L^2 \frac{32\pi^2 R}{U^2} \int_0^\infty \frac{p^2 \, dp}{(2\pi)^2} \left( Lp \right)^s e^{-(Lp)^2} \int_{-1}^1 dx \left( e^{iprx} - 1 \right) \times \sum_{l=1,2} \left( 1 - x^2 \right) \frac{1}{x^4} \sum_{m=0}^{\infty} \frac{s_m F_m (s, 2 \mu R; \Sigma, \Xi)}{\prod_{i \neq l} \left( \sum_{k=0}^{\infty} (s_{ik} - s_{ik}) F_k (x, 2 \mu R; \Sigma, \Xi) \right)} . \] (C12)

As we have already observed in Section V, only the values of the variable \( p \) around \( \bar{p} = \frac{1}{L} \sqrt{\frac{2\pi R}{s\mu^2}} \) give a significant contribution to the integral in (C12). We observe that \( \frac{8\mu R}{s\mu^2} p^2 \to \frac{4(s+2)}{\bar{R}} \) and the stability condition (22) imposes:

\[ 1 < \bar{R} < 4(s+2) . \] (C13)

The evaluation of the leading terms is then possible by performing an expansion in the parameter \( \frac{L^2}{8\mu R} p^{-2} \to \frac{R}{\bar{R}} \zeta^{-2} \) that, by virtue of (C13), is smaller than unity if \( \zeta < \sqrt{\frac{s+2}{2}} \).

For \( \zeta > \sqrt{\frac{s+2}{2}} \) the contribution to the integral rapidly vanishes. For \( 1 < \bar{R} < 4(s+2) \) we obtain:

\[ S_2(r) = -\frac{1}{\nu} (I_1(r) + I_2(r)) \sim -\frac{D_0}{(2\pi)^2 \nu} R \sum_{n=0}^\infty \left\{ (-1)^{n+1} \frac{1 + \Xi}{\Gamma(2n + 4)} \left[ \frac{1 + \Xi}{\Gamma(2n + 4)} - \frac{2n+2}{\Sigma + 2} \frac{2n+4}{\Gamma(2n + 6)} \right] \right\} \] (C14)

By extending the validity of our calculations to \( \bar{R} > 4(s+2) \), we have \( \frac{8\mu R}{s\mu^2} p^2 \to \frac{R}{\bar{R}} \zeta^{-2} < 1 \) for \( \zeta < \sqrt{\frac{s+2}{2}} \). As in the previous case, we expand (C12) in power of the parameter \( \frac{R}{\bar{R}} \zeta^2 < 1 \) and we obtain:

\[ I_2(r) \sim D_0 L^2 \int_0^\infty \frac{d\zeta}{(2\pi)^2} \zeta^s e^{-\zeta^2} \int_{-1}^1 dx \left( e^{i\zeta^2 x} - 1 \right) \left\{ 1 + \frac{8\zeta^2 + \ldots}{2} \right\} \] (C15)

Two different terms, \( I_2(r) = I_2^A(r) + I_2^B(r) \), can be identified in (C15). The evaluation of the first is straightforward and we obtain:

\[ I_2^A(r) = D_0 L^2 \int_0^\infty \frac{d\zeta}{(2\pi)^2} \zeta^s e^{-\zeta^2} \int_{-1}^1 dx \left( e^{i\zeta^2 x} - 1 \right) \left\{ 1 + \frac{8\zeta^2 + \ldots}{2} \right\} \] (C16)

The evaluation of the second term is more cumbersome. The leading term can be recasted in the form:

\[ I_2^B(r) = \frac{8D_0 L^2}{\bar{R}} \int_0^\infty \frac{d\zeta}{(2\pi)^2} \zeta^s e^{-\zeta^2} \int_{-1}^1 dx \left( e^{i\zeta^2 x} - 1 \right) \left( 1 - x^2 \right) \frac{1}{x^4} \sum_{n=0}^\infty A_n(\Sigma) x^{2n} . \] (C17)

The coefficients \( A_i \) are \( \Sigma \)-dependent numerical constants. The first two of them are given by the expressions
\[ A_0(\Sigma) = \frac{1}{16\sqrt{3}(1 - \sin \frac{\pi}{6}) \cos \left( \frac{1}{3}\tan^{-1}\sqrt{26} \right)} \]
\[ A_1(\Sigma) = -\frac{65 \sin \left( \frac{2}{3}\tan^{-1}\sqrt{26} \right)}{512\sqrt{26}\cos^2 \left( \frac{1}{3}\tan^{-1}\sqrt{26} \right)} \Sigma^2, \ldots \]  

The exact form of these coefficients is however irrelevant for our analysis. After some calculations we obtain
\[ I_B^2(r) = \frac{D_0 R_1^{\frac{2}{3}}}{\pi \Gamma \left( \frac{2}{3} \right)} \left( \frac{U}{L} \right)^{\frac{2}{3}} r^{\frac{2}{3}} \sum_{n=0}^{\infty} C_n(\Sigma) \Gamma \left( \frac{3s + 3n + 5}{6} \right) \left( \frac{r}{L} \right)^n, \] 
where the coefficients \( C_n(\Sigma) \) are depend on the constants \( A_i \). For \( n = 0 \) we have
\[ C_0(\Sigma) = \frac{54\sqrt{3} - 74}{27\sqrt{3}} A_0 + \frac{128}{9\sqrt{3}} A_1(\Sigma). \]

Comparing the scaling behavior of the term \( I_B^2(r) \) with the term \( I_A^2(r) \) we find
\[ r \sim \left| 2 \times 8.328\sqrt{\pi} 0.0336 - 0.1127 \cot^2 \theta_U \left( 2 + \sin \theta_U \cos \theta_U \right)^{\frac{1}{3}} \right| R^{-\frac{2}{3}} L. \]

For the expansion in term of \( \frac{1}{R} \) to be meaningful, the parameter \( \theta_U \) must lie in a small region around the value \( \frac{\pi}{2} \). This implies:
\[ r \sim F R^{-\frac{2}{3}} L, \quad \text{with} \quad F \sim 0.6. \]

[1] P. C. Martin, E. D. Siggia, H. A. Rose, *Statistical Dynamics of Classical System*, Phys. Rev. A8 (1973) 423
[2] C. De Dominicis and P.C. Martin, Phys. Rev. A 19, 419 (1979); J. P. Fournier and U. Frisch, Phys. Rev. A 17, 747 (1978); V. Yakhot and S. A. Orszag, J. of Sci. Comp. 1, 3 (1986).
[3] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil’ev, *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (Gordon & Breach, London, 1999).
[4] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, *Fluctuations in Stationary Nonequilibrium States of Irreversible Processes*, Phys. Rev. Lett. 87 (2001) 1
[5] L. Onsager and S. Machlup, Phys. Rev. 91, 1505 (1953).
[6] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR 30 (1941) 9; Dokl. Akad. Nauk SSSR 31 (1941) 538; Dokl. Akad. Nauk SSSR 32 (1941) 16; U. Frisch, *Turbulence; the legacy of A. N. Kolmogorov* (Cambridge U. press 1996)
[7] L.Ts. Adzhemyan, N.V. Antonov and A.N. Vasil’ev, Renormalization Group, Operator Product Expansion, and Anomalous Scaling in a Model of Adveced Passive Scalar, Phys. Rev. E 58 (1998) 1823
[8] C. Kipnis, C. Landim, *Scaling Limits of Interacting Particle Systems* (Springer, New York, 1999).
[9] W. S. Burnside and A. W. Panton, *The Theory of Equations - with an introduction to the theory of binary algebraic forms - V.1* (Dover Publications, inc. New York 1912)