Recent Developments
in Fractional Superstrings

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Abstract

Fractional superstrings experience new types of “internal projections” which alter or deform their underlying worldsheet conformal field theories. In this talk I summarize some recent results concerning both the worldsheet theory which remains after the internal projections have acted, and the spacetime statistics properties of its various sectors.

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1 Introduction

Over the past two years there has been considerable activity in a possible new class of string theories known as fractional superstrings [1–10]: these are non-trivial generalizations of superstrings and heterotic strings, and have the important property that their critical spacetime dimensions are less than ten. This reduction in the critical dimension is accomplished by replacing the worldsheet supersymmetry of the traditional superstring or heterotic string by a $K$-fractional supersymmetry: such symmetries relate worldsheet bosons not to worldsheet fermions, but rather to worldsheet $\mathbb{Z}_K$ parafermions $\epsilon$ of fractional spin $2/(K + 2)$. One then finds that the corresponding critical spacetime dimension of the theory is given by

$$D_c = 2 + \frac{16}{K}, \quad K \geq 2.$$  

Thus while the choice $K = 2$ reproduces the ordinary $D_c = 10$ superstring (with $\mathbb{Z}_2$ “parafermions” reducing to ordinary Majorana fermions), the choices $K = 4, 8,$ and $16$ yield new theories with $D_c = 6, 4,$ and $3$ respectively.

For $K > 2$, the $\mathbb{Z}_K$ parafermion conformal field theory (CFT) is non-linear: the appearance of fractional-spin fields implies that their operator-product expansions contain cuts rather than poles, and indeed these fields have non-trivial (and often non-abelian) braiding relations. It is primarily due to such complications on the worldsheet that fractional superstrings appear to exhibit qualitatively new features in spacetime, as compared to the usual superstrings and heterotic strings. Understanding these new features is thus of paramount importance, not only for demonstrating the internal consistency of the fractional superstring, but also as a means of shedding further light on the general but as yet poorly understood relationship between worldsheet string symmetries and spacetime physics.

While the low-lying states of the fractional superstring are closely analogous those of the ordinary superstring, the non-linearity of the fractional-superstring worldsheet theory becomes manifest at higher mass levels. In particular, two fundamentally new features emerge whose spacetime interpretations have thus far remained unresolved. The first is the appearance of new massive sectors which are not of the standard Ramond or Neveu-Schwarz (NS) variety, and which contain spacetime particles whose physical roles (and spacetime statistics properties) are unclear. The second is the appearance of new so-called “internal projections” which, unlike the traditional GSO projection, appear to change or deform the parafermionic conformal field theories upon which the fractional superstring is built, leaving behind worldsheet CFT’s whose properties are as yet unknown. We shall here provide a short review of recent developments in these areas, referring the reader to Ref. [11] for a non-technical overview of fractional superstrings, to the original papers (Refs. [1–4]) for more information concerning the basic ideas behind fractional superstrings, and to Refs. [7, 8] for further details concerning these new results.
2 Massive Sectors

In order to specify the sense in which the fractional superstring contains new types of sectors, it proves instructive to recall the case of the ordinary superstring in $D = 10$. The underlying light-cone worldsheet CFT of the usual superstring has central charge $c = 12$, and consists of a tensor product of eight free bosons and eight Ising models, one copy of each per transverse spacetime dimension. Each of these Ising-model CFT’s contains three fields: the identity $1$, the Majorana fermion $\psi$, and the spin-field $\sigma$. There are thus a variety of combinations of CFT sectors which could potentially contribute states to the superstring spacetime spectrum. However, as is well-known, only five sectors actually contribute to the spectrum: these are the four NS sectors with positive $G$-parity $1^7\psi, 1^5\psi^3, 1^3\psi^5$, and $1^7\psi$, and the single Ramond sector $\sigma^8$. Since the spin-field $\sigma$ introduces a cut on the worldsheet which changes the boundary conditions of the worldsheet fermion $\psi$, we find in the usual way that states in the NS sectors are spacetime bosons, and states in the Ramond sector are spacetime fermions. The crucial observation, however, is that no “mixed” $1/\sigma$, $\psi/\sigma$, or $1/\psi/\sigma$ combinations contribute to the physical spectrum of states of the ordinary $D = 10$ superstring.

For the more general fractional superstrings with $K > 2$, this is no longer the case: there are a variety of fundamentally new sectors which contribute states to the spacetime spectrum and which must therefore be considered. These can be described as follows. In analogy to the superstring, the light-cone worldsheet CFT of the $K$-fractional superstring consists of $D_c - 2 = 16/K$ free bosons tensored together with $16/K$ copies of the $\mathbb{Z}_K$ parafermion theory; these $\mathbb{Z}_K$ parafermion theories are fractional-spin generalizations of the Ising model. Thus, since each $\mathbb{Z}_K$ parafermion theory has central charge $c_K = (2K - 2)/(K + 2)$, the light-cone worldsheet CFT of the $K$-fractional superstring has total central charge $c = 48/(K + 2)$:

$$K \geq 2 : \quad \left( c = \frac{48}{K + 2} \right) \text{CFT} = \left\{ \frac{D_c - 2 - 16/K}{\mu = 1} X^\mu \right\} \otimes \left\{ \frac{D_c - 2 - 16/K}{\mu = 1} (\mathbb{Z}_K \text{ PF})^\mu \right\} . \quad (2)$$

Like the Ising model, however, each of these $\mathbb{Z}_K$ parafermion theories contains a variety of primary fields, and these can be grouped into three classes: analogues of the Ising-model fields $1$ and $\psi$ (producing spacetime bosonic states), analogues of $\sigma$ (producing spacetime fermionic states), and additional parafermionic fields (to be collectively denoted $\phi$) which have no analogues in the Ising model. There are thus sectors of the forms $(1, \psi)^{16/K}$ and $(\sigma)^{16/K}$ which are respectively the fractional-superstring analogues of the superstring NS and Ramond sectors; these are the so-called “$A$-sectors”, and they contain all of the massless states (including, for example, the supergravity multiplet). There are, however, two other types of sectors which contribute to the fractional-superstring spectrum. The first (the so-called “$B$-sectors”) all have the equally mixed form $(\{1\}\{\sigma\})^{8/K}$, and contain only states with masses $m^2 > 0$ (i.e., states at the Planck scale). The second (the so-called “$C$-sectors”) in-
stead take the form $(\phi)^{16/K}$, and also contain only Planck-scale states. It is these two groups of sectors which are the unusual “massive sectors” whose physical properties have thus far remained elusive.

3 Internal Projections

The second fundamental issue which has remained unresolved concerns the appearance of new types of so-called “internal projections” which remove states from the fractional-superstring spectrum. Whereas the GSO projection in the ordinary superstring removes only entire towers of states (projecting out, for example, the odd $G$-parity NS sectors $1^i\psi^{8-i}$ with $i \in 2\mathbb{Z}$), these new internal projections project away only some of the states in each individual tower, leaving behind a set of states which therefore cannot be interpreted as the complete Fock space of the original underlying worldsheet CFT in Eq. (2). On the face of it, this would seem to render the spacetime spectra of the fractional superstrings hopelessly inconsistent with any underlying worldsheet-theory interpretation. Remarkably, however, evidence suggests that the residual states which survive the internal projections in each tower precisely recombine to fill out the complete Fock space of a different underlying conformal field theory. Thus, whereas the GSO projection merely removed certain highest-weight sectors of the worldsheet conformal field theory, these new internal projections appear to actually change the underlying conformal field theory itself. In fact, since the central charges of the new (post-projection) CFT’s are smaller than those of the original tensor-product parafermion theories, these internal projections must clearly remove exponentially large numbers of states from each of the mass levels of the original Fock space. Such a drastic projection clearly has no analogue in the ordinary superstring, and perhaps more closely resembles the BRST projection which enables unitary minimal models with $c < 1$ to be constructed from free $c = 1$ bosons in the Feigin-Fuchs construction.

Verifying that the internal projection in fact leaves behind a self-consistent Fock space is a difficult task, and to date the evidence for this has been obtained only through an analysis of the fractional-superstring partition functions. Indeed, even the existence of these internal projections can at present be deduced only through this partition-function approach, and there does not currently exist any internal-projection operator constructed out of worldsheet fields which would enable us to analyze these projections at the level of individual states. Therefore the partition functions remain the primary tool for analyzing these internal projections, and we shall see that this approach is sufficient to determine the central charges, highest weights, fusion rules, and characters of the worldsheet conformal field theories which survive the internal projections in all of the fractional superstring sectors. Moreover, by exploiting the similarity of these CFT’s with those of free compactified bosons, we shall even be able to gain valuable information concerning the spacetime statistics of the surviving states. Actually constructing a suitable representation for this
conformal field theory in terms of worldsheet fields still remains an open question, however, and we shall discuss some of the difficulties at the end of this talk.

4 Fractional Superstring Partition Functions

Let us now review the partition-function evidence for these extra sectors and internal projections. Given the worldsheet light-cone CFT’s of the $K$-fractional superstrings indicated in Eq. (2), it is a straightforward matter to construct their corresponding modular-invariant one-loop partition functions. Such partition functions are of course constructed as linear combinations of products of the characters corresponding to each individual free coordinate boson plus $\mathbb{Z}_K$ parafermion system; these characters are the so-called “string functions”, and we indicate via the schematic notation $\chi_1$, $\chi_\sigma$, and $\chi_\phi$ those groups of string functions which correspond respectively to the Neveu-Schwarz-like, Ramond-like, or “other” parafermion fields of each $\mathbb{Z}_K$ theory. It can then be shown \cite{1–3} that demanding the presence of a massless sector and the absence of physical tachyons leads to the following unique partition functions $\mathcal{Z}_K$ for each relevant value of $K \geq 2$:

$$
\begin{align*}
\mathcal{Z}_2 &= (\text{Im } \tau)^{-4} |A^b_2 - A^f_2|^2 \\
\mathcal{Z}_4 &= (\text{Im } \tau)^{-2} \left\{ |A^b_4 - A^f_4|^2 + 3 |B_4|^2 \right\} \\
\mathcal{Z}_8 &= (\text{Im } \tau)^{-1} \left\{ |A^b_8 - A^f_8|^2 + |B_8|^2 + 2 |C_8|^2 \right\} \\
\mathcal{Z}_{16} &= (\text{Im } \tau)^{-1/2} \left\{ |A^b_{16} - A^f_{16}|^2 + |C_{16}|^2 \right\}
\end{align*}
\tag{3}
$$

where the expressions $A^b_K$, $B_K$, and $C_K$ respectively represent the contributions from the $A$, $B$, and $C$ sectors, and take the forms

$$
\begin{align*}
A^b_K &\sim \sum g_i^{(A^b)} (\chi_1)^{D_{c-2}} , & A^f_K &\sim \sum g_i^{(A^f)} (\chi_\sigma)^{D_{c-2}} , \\
B_K &\sim \sum g_i^{(B)} (\chi_1 \chi_\sigma)^{(D_{c-2})/2} , & C_K &\sim \sum g_i^{(C)} (\chi_\phi)^{D_{c-2}} .
\end{align*}
\tag{4}
$$

Here the $g_i$ indicate various coefficients in the above linear combinations. Thus, whereas we can immediately interpret $A^b_K$ and $A^f_K$ as corresponding to spacetime bosonic (NS) and fermionic (Ramond) states respectively, the interpretation of the states in the $B_K$ and $C_K$ sectors is not as clear: they are evidently built upon unusual vacuum states which have no analogues in the ordinary superstring, and their spacetime interpretations are unknown. It turns out, however, that $A^b_K$ and $A^f_K$ are precisely equal as functions of $\tau$, suggesting that the fractional superstring $A$-sectors enjoy a spacetime supersymmetry; indeed, for each value of $K$ these sectors contain a massless $N = 2$ supergravity multiplet. Thus, for consistency, the $B_K$- and $C_K$-sectors must also be individually spacetime supersymmetric, and indeed we find that $B_K = C_K = 0$ as well. This suggests that we should also be able to write each
$B_K$ and $C_K$ as the difference of matching bosonic and fermionic contributions:

$$B_K \equiv B^b_K - B^f_K \quad \text{and} \quad C_K \equiv C^b_K - C^f_K .$$

(5)

Until recently, however, it has not been known how to achieve this splitting, and this has impeded progress in understanding these sectors.

The second remarkable feature in these partition functions is the fact that for $K > 2$, some of the coefficients $g_i^{(ab,f)}$ in Eq. (4) turn out to be negative. This indicates that the contributions of certain NS and Ramond sectors are subtracted rather than added to their respective bosonic or fermionic Fock spaces, or more specifically that there exists a new type of projection between different parafermionic towers of states which has the net effect of removing large numbers of states from the physical spectrum. This is the internal projection discussed above. It can easily be verified that despite this internal projection, the numbers of states remaining at each mass level are still positive, and thus it is reasonable to ask whether these remaining states fill out the Fock space corresponding to some new worldsheet conformal field theory. Mathematically, this would mean that the net expressions $A_{K}^{b,f}$ should themselves be interpreted as the characters $\chi'_i$ of the highest-weight sectors of some new conformal field theory:

$$A_{K}^{b,f} \equiv \chi'_i .$$

(6)

Determining the properties of these smaller post-projection CFT’s is of course crucial for ultimately demonstrating the consistency of these internal projections.

5 Recent Developments: The Post-Projection CFT

We shall now summarize some of the recent progress that has been made in determining the various properties of the effective worldsheet CFT’s which survive these internal projections.

Given the original expressions $A_{K}^{b,f}$ in Eq. (4), it turns out that we can determine the central charges, highest weights, fusion rules, and complete set of characters of the corresponding post-projection CFT’s. The method is relatively straightforward. As indicated in Eq. (6), we would like to regard each expression $A_{K}^{b,f}(\tau)$ as a character $\chi'_i(\tau)$ in a corresponding post-projection CFT. These expressions are not modular-invariant by themselves, however, and by taking modular transformations we can construct the complete sets of characters $\{\chi'_i(\tau)\}$ which are eigenfunctions of $T : \tau \rightarrow \tau + 1$ and closed under $S : \tau \rightarrow -1/\tau$, so that

$$\chi'_i(-1/\tau) = \sum_j S_{ij} \chi'_j(\tau) , \quad \chi'_i(\tau + 1) = \exp\{2\pi i \ell_i\} \chi'_i(\tau)$$

(7)

where $S_{ij}$ is the $S$-mixing matrix and $\ell_i$ is a parameter denoting the phase accrued by $\chi'_i$ under $T$. We can then simply expand any of these characters $\chi'_i(\tau)$ as a power
series in \( q \equiv \exp(2\pi i \tau) \):

\[
\chi'_i(\tau) = q^{\ell_i} \sum_{n=0}^{\infty} a_n^{(i)} q^n ,
\]

whereupon the effective central charge \( c_{\text{eff}} \) of the post-projection CFT can be determined by analyzing the growth in the level degeneracies \( a_n^{(i)} \) as a function of \( n \):

\[
a_n^{(i)} \sim n^{-3/4} \exp \left\{ 4\pi \sqrt{\frac{c_{\text{eff}} n}{24}} \right\} \quad \text{as} \quad n \to \infty .
\]

Similarly, the complete spectrum of highest weights in this CFT can be determined by scanning the quantities \( \ell_i \) in Eqs. (7) and (8), for the highest weight \( h_i \) corresponding to a given character \( \chi'_i \) is given in general by

\[
h_i = \ell_i + \frac{c_{\text{eff}}}{24} .
\]

Likewise, if we interpret each of the characters \( \chi'_i \) as corresponding to a certain unique primary field \( \phi_i \) of highest weight \( h_i \) in the post-projection CFT, then the fusion rules of this CFT

\[
[\phi_i] \times [\phi_j] = \sum_k N_{ijk} [\phi_k]
\]

can be obtained from the matrices \( S_{ij} \) in Eq. (7) via the Verlinde formula

\[
N_{ijk} = \sum_n \frac{S_{in} S_{jn} S_{nk}}{S_{0n}}.
\]

Here \( i = 0 \) corresponds to the identity field (or vacuum sector) with \( h_0 = 0 \).

Although the procedure outlined above is completely general, numerous subtleties appear for CFT’s with central charges \( c \geq 1 \): in these cases there are an infinite number of primary fields, and the characters \( \chi'_i \) typically correspond not to a single primary field with highest weight \( h_i \), but rather to all of those primary fields with highest weights \( H \) satisfying \( H = h_i \) (mod 1). The fusion rules obtained must then be interpreted accordingly, and one requires additional quantum numbers in order to individually distinguish each of these primary fields and its corresponding tower of states. Such an analysis can nevertheless be performed, however, and the details can be found in Ref. [8].

The result we find is as follows. Whereas the original light-cone worldsheet CFT of the fractional superstring is given in Eq. (2) with central charge \( c = 48/(K + 2) \), we find that in the \( A \)-sectors the internal projections effectively reduce this theory down to one with \( c_{\text{eff}} = 24/K \):

\[
K \geq 2 : \quad \text{new CFT} = \left\{ \frac{D_{c=2} = 16/K}{\mu = 1} X^\mu \right\} \otimes \left\{ \left( c = \frac{8}{K} \right) \text{theory} \right\} ;
\]
moreover, this $c = 8/K$ component theory surprisingly turns out to be completely isomorphic to a tensor product of $8/K$ bosons compactified on circles of radius $R = 1$. Specifically, this means that for each relevant value of $K \geq 2$, the $c = 8/K$ post-projection theories in Eq. (13) have the same central charges, highest weights, fusion rules, and characters as those of $8/K$ free compactified bosons — even though (as we shall discuss below) these post-projection CFT’s cannot ultimately be represented in this simple manner as a tensor product of free bosonic worldsheet fields for $K > 2$.

This close relationship between the A-sectors of our fractional superstrings and free-boson theories implies that the characters $A_K^{b,f}$, which are originally obtained as differences of parafermionic string functions as in Eq. (4), should also be expressible directly in terms of ordinary Dedekind $\eta$-functions and Jacobi $\vartheta$-functions, and indeed we find \[ A_K^{b,f}(\tau) = \left( D_c - 2 \right) \frac{\vartheta_2(\tau)}{2 \eta^3(\tau)} \right)^{(D_c-2)/2}. \] (14)

Given these results for the A-sectors, it turns out that we can make similar progress for the B- and C-sectors. Recall that the stumbling block for these sectors in Eq. (5) was the fact that we had no guidance as to how these expressions were to be separated into their separate bosonic and fermionic contributions. However, if these sectors are to be consistent with the A-sectors, then their individual bosonic and fermionic components must also experience analogous internal projections which reduce their effective central charges from $c = 48/(K+2)$ to $c_{\text{eff}} = 24/K$. We thus simply demand a splitting as in Eq. (5) such that when the individual components $B_K^{b,f}$ and $C_K^{b,f}$ are $q$-expanded as in Eq. (8), their level degeneracies each grow as in Eq. (9) with this value of $c_{\text{eff}}$. It turns out that this yields a unique splitting in each case \[ B_K^{b,f}(\tau) = \left( D_c - 2 \right) \frac{\vartheta_2(\lambda \tau)}{2 \eta^3(\tau)} \right)^{(D_c-2)/2}. \] (16)

*Note that a single $R = 1$ boson can be fermionized, yielding two copies of the $c = 1/2$ Ising model. Thus, in the $K = 16$ case, this “tensor product of $8/K$ bosons” refers to the Ising model. Likewise, for the $K = 2$ case of the ordinary superstring, there is no internal projection: the initial and final central charges are equal, and since each $Z_2$ “parafermion” theory in Eq. (4) is nothing but the Ising model, the CFT’s in Eqs. (8) and (13) are indeed equivalent.
Note that while the internal projections in the $A$-sectors seem to remove all traces of our original worldsheet $Z_K$ parafermion theory, this $B$-sector scaling factor $\lambda$ is in fact the inverse of the spin of the original parafermion $\epsilon$ for each value of $K$. For the $C$-sectors, on the other hand, we find a somewhat different story: the only splitting consistent with the internal projections yields expressions $C^K_{b,f}$ which each separately vanish. Thus the internal projections actually remove all $C$-sector states from the physical spectrum, and the $C$-sectors play no role in the post-projection worldsheet CFT.

Taken together, then, these results suggest that the internal projections act in an internally consistent manner, with the surviving states recombining to precisely fill out all of the momentum and winding-mode sectors appropriate to compactified-boson worldsheet theories. Indeed, the only difference between the $A$-sectors and the $B$-sectors is an apparent change in the compactification radius of these isomorphic bosons, and this indicates that although the $B$-sectors appear very different from the $A$-sectors from the pre-projection (or parafermionic) point of view, they turn out to closely resemble the $A$-sectors after the internal projections have acted. This isomorphism between the post-projection CFT’s and the compactified-boson theories does not imply, however, that the former can ultimately be represented in this manner — i.e., in terms of free bosonic worldsheet fields. Indeed, as we shall now discuss, such a simple representation would not yield the correct spacetime statistics properties for the various sectors of our post-projection CFT’s.

## 6 Lattices and Spacetime Statistics

This isomorphism between the post-projection theories and the compactified-boson theories enables us to go one step further, in fact, and actually examine the individual states which comprise the various post-projections sectors of the fractional superstring. This occurs because the compactified-boson theories furnish us with an additional quantum number — namely the $U(1)$ charge $\alpha$ — according to which the infinite numbers of primary fields in these $c \geq 1$ theories may be distinguished and placed on a lattice. As we shall see, this proves to be of great importance in describing the spacetime statistics of the various surviving states, and thereby demonstrating that a simple free-boson representation of our post-projection CFT’s is unsuitable for $K > 2$.

Let us first recall some features of the (chiral) compactified-boson CFT. This theory contains primary fields $e^{i\alpha \phi}$ of conformal dimensions $\alpha^2/2$, where $\phi(z)$ indicates the boson field and where $\alpha$ turns out to be the charge of the primary field with respect to the $U(1)$ current $i\partial \phi$. This charge is conserved under fusion:

$$[e^{i\alpha \phi}] \times [e^{i\beta \phi}] = [e^{i(\alpha+\beta)\phi}].$$

(17)

If $\phi$ is compactified on a circle of radius $R$, so that $\phi \approx \phi + 2\pi R$, then $\alpha$ is restricted to the values $n/(2R)$ with $n \in \mathbb{Z}$. Thus the set of allowed $\alpha$-values forms a one-
dimensional lattice with lattice spacing $1/2R$. Each lattice site corresponds to a
different vacuum state $e^{i\alpha_0(0)}|0\rangle$, and gives rise to an infinite tower of states reached
by bosonic mode excitations. Thus, a tensor product of $8/K$ chiral bosons consists of
states $\vec{\alpha}$ which fill out an $8/K$-dimensional lattice: the corresponding highest weights
in the full $8/K$-boson theory are given by $h = \vec{\alpha} \cdot \vec{\alpha}/2$, and its fusion rules are
equivalent to vector addition for $\vec{\alpha}$. Such a (left-moving) $8/K$-dimensional lattice $\Lambda_L$
must of course be tensored with a corresponding (right-moving) $8/K$-dimensional lattice $\Lambda_R$
in order to fully describe the spectrum of states in a closed string theory.

In a general $K = 2$ superstring or heterotic string, not all of these potential
lattice sites $\vec{\alpha} = (\vec{\alpha}_\text{left}|\vec{\alpha}_\text{right})$ actually contribute states to the physical spectrum, for
most suffer GSO projections and only a few sites $\vec{\alpha}$ remain. Indeed, those which
remain form not a lattice but rather a “shifted lattice”: this means that there exists
a constant “shift vector” $\vec{S}$ such that the set $\{\vec{\alpha} - \vec{S}\}$ forms a true lattice, and such
that the spacetime statistics of the states in each tower $\vec{\alpha}$ can be determined by
computing the inner product $(\vec{\alpha} - \vec{S}) \cdot \vec{S}$:

$$(\vec{\alpha} - \vec{S}) \cdot \vec{S} \in \begin{cases} \mathbb{Z} & \text{bosonic} \\ \mathbb{Z} + 1/2 & \text{fermionic} \end{cases}.$$  

(18)

For example, for the ordinary $K = 2$ Type IIA superstring, the four-dimensional
shifted lattices $\Lambda_{L,R}$ of GSO-surviving Ramond and NS states are

$$\Lambda_L = \Lambda_R = \left\{ n_1, n_2, n_3, n_4 \right\} \oplus \left\{ n_1 - \frac{1}{2}, n_2 - \frac{1}{2}, n_3 - \frac{1}{2}, n_4 - \frac{1}{2} \right\}$$  

(19)

with $n_i \in \mathbb{Z}$ and $\sum n_i = \text{odd}$; the full shifted lattice is then given by $\Lambda_2 \equiv \Lambda_L \otimes \Lambda_R$,
and the shift vector $\vec{S}_{K=2}$ can be taken to be $\vec{S}_{K=2} = (1,0,0,0|1,0,0,0)$. This implies
that the partition function of the ordinary $K = 2$ superstring can be expressed in
terms of the lattice $\Lambda_2$ of surviving states in the usual manner:

$$Z_2 = (\text{Im } \tau)^{-4}\left| \eta \right|^{-24} \sum_{\vec{\alpha} \in \Lambda_2} q^{(\vec{\alpha}_\text{left})^2/2} \overline{q}^{(\vec{\alpha}_\text{right})^2/2} \exp \left[ 2\pi i (\vec{\alpha} - \vec{S}) \cdot \vec{S} \right].$$  

(20)

Note that Eq. (18) insures that states contribute to $Z_2$ with the proper statistics
factor $(-1)^F$. Indeed, in either chiral (left-moving or right-moving) sector of the
theory, the NS states $\vec{\alpha}^b$ appear on lattice sites with integer components $\alpha_i \in \mathbb{Z}$, while
the Ramond states $\vec{\alpha}^f$ have half-integer components $\alpha_i \in \mathbb{Z} + 1/2$. The equality of
the respective numbers of these states at each highest weight $h = \vec{\alpha} \cdot \vec{\alpha}/2$ is consistent
with the spacetime supersymmetry of the $K = 2$ superstring.

Remarkably, a similar situation exists for the $K$-fractional superstring. Those
states which survive the internal and GSO projections in the $A$-sectors again fill out
an $(8/K + 8/K)$-dimensional shifted lattice $\Lambda_K$:

$$K > 2 : \quad \Lambda_K \equiv \left\{ n_1 \pm \frac{1}{2}, ..., n_{8/K} \pm \frac{1}{2} \right\} \otimes \left\{ n_1 \pm \frac{1}{2}, ..., n_{8/K} \pm \frac{1}{2} \right\}$$  

(21)
where \( n_i \in \mathbb{Z} \) and where each sign is chosen independently; similarly, the \( B \)-sector states fill out the lattice \( \sqrt{\lambda} \Lambda \). It then turns out \([8]\) that our fractional-superstring partition functions can be rewritten in a manner completely analogous to Eq. (20):

\[
|A^b_K - A^f_K|^2 = 4 |\eta|^{-48/K} \sum_{\vec{\alpha} \in \Lambda_K} q^{(\vec{\alpha} \text{left})^2/2} f^{(\vec{\alpha} \text{right})^2/2} \exp \left[ 2\pi i \left( \vec{\alpha} - \vec{S} \right) \cdot \vec{S} \right] 
\]

\[
|B^b_K - B^f_K|^2 = 4 |\eta|^{-48/K} \sum_{\vec{\alpha} \in \sqrt{\lambda} \Lambda_K} q^{(\vec{\alpha} \text{left})^2/2} f^{(\vec{\alpha} \text{right})^2/2} \exp \left[ 2\pi i \left( \vec{\alpha}/\sqrt{\lambda} - \vec{S} \right) \cdot \vec{S} \right] 
\]

provided the shift vectors \( \vec{S}_K \) are now taken to be \( \vec{S}_4 = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}) \) and \( \vec{S}_8 = (\frac{1}{2}, 0, 0, \frac{1}{2}) \).

This result is in fact consistent with spacetime supersymmetry as well, with equal numbers of spacetime bosonic and fermionic states at each highest weight. Indeed, Eq. (18) now allows us to consistently identify the spacetime statistics of the individual chiral states that survive the internal projections \([8]\):

\[
K = 8 : \quad \begin{cases} \alpha^b = + \frac{1}{2} \sqrt{\lambda} \pmod{2 \sqrt{\lambda}} \\ \alpha^f = - \frac{1}{2} \sqrt{\lambda} \pmod{2 \sqrt{\lambda}} \end{cases} 
\]

\[
K = 4 : \quad \begin{cases} \vec{\alpha}^b = (\pm \frac{1}{2} \sqrt{\lambda}, \pm \frac{1}{2} \sqrt{\lambda}) \pmod{2 \sqrt{\lambda}} \\ \vec{\alpha}^f = (\pm \frac{1}{2} \sqrt{\lambda}, \mp \frac{1}{2} \sqrt{\lambda}) \pmod{2 \sqrt{\lambda}} \end{cases} 
\]

where the rescaling factor \( \lambda \) is understood to be equal to 1 for the \( A \)-sectors. This identification is also consistent with an alternative analysis making use of the so-called “twist current” \([8]\).

Although these lattice results show the great similarity between the \( K \)-fractional superstring and the ordinary \( K = 2 \) superstring, they also clearly demonstrate that we cannot ultimately represent our post-projection CFT’s in terms of free worldsheet bosons for \( K > 2 \), or actually associate each lattice site \( \vec{\alpha} \) with a primary field \( e^{i\vec{\alpha} \cdot \vec{\phi}} \). In the superstring, for example, such a representation poses no problem, for those states with fermionic spacetime statistics are associated with lattice sites \( \vec{\alpha} \) with half-integer components \( \alpha_i \), and the primary fields \( e^{i\phi_i/2} \) are each equivalent to a tensor product of two Ising-model spin fields \( \sigma \) which create the necessary worldsheet cuts to alter the boundary conditions of worldsheet fermions and produce fermionic spacetime statistics. For \( K > 2 \), however, the statistics assignments in Eq. (23) clearly preclude any such free-boson representation, and only the fermionic states in the \( A \)-sectors appear representable in this manner. Therefore an alternative representation for our light-cone worldsheet theory is needed, one which is consistent not only with these spacetime statistics assignments, but more generally with transverse \((D_c - 2 = 16/K)\)-dimensional Lorentz invariance. Such issues are discussed further in Ref. \([8]\).

Thus, the above new results concerning the post-projection worldsheet conformal field theories of the fractional superstring constitute only the first steps in their eventual construction, and many issues remain to be resolved before the consistency of the fractional superstring is demonstrated. Work in all of these areas is continuing.
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Note: Listed above are only those references concerning fractional superstrings: Refs. [1–8,11] deal with the so-called “tensor product formulation” which has been the focus of this review, while Refs. [9,10] deal with an alternative “chiral algebra formulation”. A summary of the possible relation between the two can be found in Ref. [7].