Hjorth Analysis of General Polish Group Actions

Ohad Drucker (Hebrew U.)

January 28, 2014
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A subspace of a Polish space is Polish if and only it is $G_δ$.

The product of a countable collection of Polish spaces is Polish. In particular, $ω^ω$ and $2^ω$ are both Polish.
A Polish Group is a topological group whose topology is polish.
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A continuous action of a Polish group $G$ on a Polish space $X$ is called a *Polish action*. We will denote by $E^X_G$ the induced orbit equivalence relation on $X$. 

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A continuous action of a Polish group $G$ on a Polish space $X$ is called a *Polish action*. We will denote by $E^X_G$ the induced orbit equivalence relation on $X$.

The orbit equivalence relation $E^X_G$ is analytic, but not always Borel.
Let $\mathcal{L}$ be a countable relational language, $\mathcal{L} = (R_i)_{i \in \omega}$, for $R_i$ an $n_i$-ary relation.
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Let $\text{Mod}(\mathcal{L})$ be the collection of countable $\mathcal{L}$ models.
Logic Action

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Let $Mod(\mathcal{L})$ be the collection of countable $\mathcal{L}$ models.

$Mod(\mathcal{L})$ inherits the Polish topology of $\Pi_{i \in \omega} 2^{\omega n_i}$.

This is exactly the topology generated by

$$A_{\phi, \bar{a}} = \{ \mathcal{M} : \mathcal{M} \models \phi(\bar{a}) \}.$$
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- The induced orbit equivalence relation is $\simeq_{\mathcal{L}}$. 

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Hjorth Analysis of General Polish Group Actions
Definition

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- $(\mathcal{M}, \bar{a}) \equiv_0 (\mathcal{N}, \bar{b})$ if for every $\phi(\bar{x})$ atomic, $\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{b})$.

- For $\lambda$ limit, $(\mathcal{M}, \bar{a}) \equiv \lambda (\mathcal{N}, \bar{b})$ if for every $\alpha < \lambda$, $(\mathcal{M}, \bar{a}) \equiv \alpha (\mathcal{N}, \bar{b})$.
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- $(\mathcal{M}, \bar{a}) \equiv_{\alpha+1} (\mathcal{N}, \bar{b})$ if for every $c \in \omega$ there is $d \in \omega$ s.t. $(\mathcal{M}, \bar{a} \frown c) \equiv_{\alpha} (\mathcal{N}, \bar{b} \frown d)$ and for every $d \in \omega$ there is $c \in \omega$ s.t. $(\mathcal{N}, \bar{b} \frown d) \equiv_{\alpha} (\mathcal{M}, \bar{a} \frown c)$. 
Scott Analysis

**Definition**

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\( M \equiv_\alpha N \) if \((M, \emptyset) \equiv_\alpha (N, \emptyset)\).

- Given \( M \in \text{Mod}(\mathcal{L}) \), there is \( \alpha < \omega_1 \) such that if \((M, \bar{a}) \equiv_\alpha (M, \bar{b})\) then \((M, \bar{a}) \equiv_{\alpha+1} (M, \bar{b})\).
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\( (\mathcal{M}, \bar{a}) \equiv_{\alpha+1} (\mathcal{M}, \bar{b}). \)

Definition

For \( \mathcal{M} \in \text{Mod}(\mathcal{L}) \), \( \delta(\mathcal{M}) \), the Scott rank of \( \mathcal{M} \), is the least such \( \alpha \).
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4. Given $\mathcal{M} \in Mod(\mathcal{L})$, for every $\mathcal{N} \in Mod(\mathcal{L})$: 

$$\mathcal{M} \equiv_{\mathcal{L}} \mathcal{N} \iff \delta(\mathcal{M}) = \delta(\mathcal{N}).$$
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4. Given \( \mathcal{M} \in \text{Mod}(\mathcal{L}) \), for every \( \mathcal{N} \in \text{Mod}(\mathcal{L}) \):

\[ \mathcal{N} \equiv_{\delta(\mathcal{M}) + \omega} \mathcal{M} \implies \mathcal{M} \simeq \mathcal{N} \]
Scott Analysis

Theorem (Becker - Kechris)

\[ \cong \mathcal{L} \] is Borel if and only if there is an \( \alpha < \omega_1 \) such that for every \( \mathcal{M} \in \text{Mod}(\mathcal{L}) \), \( \delta(\mathcal{M}) < \alpha \)
Problem

*Generalize Scott analysis, or, find a topological version of Scott analysis.*
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- Is there a Scott analysis of Polish actions, which is, for every \((G, X)\) a Polish action:
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- Is there a *Scott analysis of Polish actions*, which is, for every $(G, X)$ a Polish action:
  1. A decreasing sequence $\equiv_\alpha$ of Borel equivalence relations which are invariant under $G$.
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  3. A function $\delta : X \to (\omega_1, <)$ which is Borel and $G$ - invariant.
  4. There is an $\alpha < \omega_1$ such that for every $x \in X$ and for every $y \in X$:
     \[ x \equiv_\delta(x) + \alpha \ y \implies x \ E^X_G y. \]
Better yet, can we find a *Scott analysis of Polish actions* such that:
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**Theorem**

$E^X_G$ is Borel if and only if there is an $\alpha$ such that for every $x \in X$, 
$\delta(x) \leq \alpha$. 
Better yet, can we find a Scott analysis of Polish actions such that:

**Theorem**

$E_G^X$ is Borel if and only if there is an $\alpha$ such that for every $x \in X$, $\delta(x) \leq \alpha$.

**Question (Hjorth)**

Let $\alpha$ be a countable ordinal. Is the following set Borel:

$$A_\alpha = \{ x : [x] \text{ is } \Pi^0_\beta \text{ for } \beta < \alpha + \omega \}$$
Let \((G, X)\) be a general Polish action. Fix \(\mathbb{P}\) the poset of nonempty open subsets of \(G\).
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- For \(\alpha < \omega_1\), we define a relation \(\leq_\alpha\) between pairs of an element of \(x\) and an open subset of \(G\):

\[
\text{Definition}\ (x, U) \leq_\alpha (y, W) \text{ if and only if for every } A \text{ a } \Pi_0^\alpha \text{ set, if } W \vDash g^* y \in A \text{ then } U \vDash g^* x \in A.
\]

\[
\text{Proposition 1}\ (x, U) \leq_1 (y, W) \text{ if and only if } U \cdot x \subseteq W \cdot y.
\]

\(\leq_\alpha\) is reflexive and transitive. The sequence \(\langle \leq_\alpha : \alpha < \omega_1 \rangle\) is decreasing.

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**Proposition**

1. \((x, U) \leq_1 (y, W) \text{ if and only if } \overline{U \cdot x} \subseteq \overline{W \cdot y}.

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3. \(\leq_\alpha\) is Borel.
**Definition**

Let \( x_0, x_1 \) in \( X \), \( \alpha < \omega_1 \). \( x_0 \equiv_\alpha x_1 \) iff for all \( V_1 \subseteq G \) nonempty and open there is \( V_0 \subseteq G \) nonempty and open such that

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(x_0, V_0) \leq_\alpha (x_1, V_1),
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and vice versa:
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Proposition

\( \equiv_\alpha \) is a Borel and \( G \)-invariant equivalence relation.
Proposition

Suppose $A \subseteq X$ is an invariant $\Pi^0_\alpha$ set, and $x \equiv_\alpha y$. Then $x \in A \iff y \in A$.

Proof. Assume $x \in A$ for $A$ a $\Pi^0_\alpha$ invariant set. As $A$ is invariant, $G \models g^* \cdot x \in A$. Since $x \equiv_\alpha y$, there is a non-empty and open $W$ such that $(y, W) \leq_\alpha (x, G)$. By the definition and the above, $W \models g^* \cdot y \in A$. In particular, there is a $g$ such that $g \cdot y \in A$. By the invariance of $A$, $y$ must be in $A$. 
Proposition

Suppose \( A \subseteq X \) is an invariant \( \Pi_0^\alpha \) set, and \( x \equiv_\alpha y \). Then \( x \in A \iff y \in A \).

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Suppose $A \subseteq X$ is an invariant $\prod_0^\alpha$ set, and $x \equiv_\alpha y$. Then $x \in A \iff y \in A$.

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Hjorth Analysis of General Polish Group Actions
So far...

1. A decreasing sequence \( \equiv_\alpha \) of Borel equivalence relations which are invariant under \( G \).

2. \( E^X_G = \bigcap_{\alpha < \omega_1} \equiv_\alpha \).

3. A function \( \delta : X \to (\omega_1, \lt) \) which is Borel and \( G \)-invariant.

4. There is an \( \alpha < \omega_1 \) such that for every \( x \in X \) and for every \( y \in X \):

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x \equiv_{\delta(x) + \alpha} y \implies x E^X_G y.
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**Definition**

For $x \in X$, let $\delta(x)$ be the least $\alpha$ such that for every $U, V \subseteq G$ open and nonempty, and every $\alpha < \omega_1$:

Proposition

Hjorth rank is $G$-invariant and Borel. In fact:

For every countable ordinal $\alpha$:

$$\{x : \delta(x) \leq \alpha\}$$

is $\Pi^0_{\alpha+k}(\alpha)$, for $k(\alpha) \in \omega$. 

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Scott’s Isomorphism Theorem

**Proposition**

If $\delta(x_0), \delta(x_1) \leq \delta$ and $x_0 \equiv_{\delta+1} x_1$, then $x_0$ and $x_1$ are orbit equivalent.
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Theorem
For every $x \in X$ there is a natural number $m$ such that $[x] = \{ y : y \equiv_{\delta(x)+m} x \}$. 
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**Proof.**

- The set $\{z : \delta(z) \leq \delta(x)\}$ is $\Pi^0_{\delta(x)+m}$ for some $m \in \omega$. 
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**Proof.**

- The set $\{z : \delta(z) \leq \delta(x)\}$ is $\Pi^0_{\delta(x)+m}$ for some $m \in \omega$.
- So if $y \equiv_{\delta(x)+m} x$ then $\delta(y) \leq \delta(x)$. 
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If \( \delta(x_0), \delta(x_1) \leq \delta \) and \( x_0 \equiv_{\delta+1} x_1 \), then \( x_0 \) and \( x_1 \) are orbit equivalent.

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\]

**Proof.**

- The set \( \{ z : \delta(z) \leq \delta(x) \} \) is \( \Pi^0_{\delta(x)+m} \) for some \( m \in \omega \).
- So if \( y \equiv_{\delta(x)+m} x \) then \( \delta(y) \leq \delta(x) \).
- Hence if \( x \) and \( y \) are \( \delta(x) + m + 1 \) equivalent, they are orbit equivalent.
1 A decreasing sequence \( \equiv_{\alpha} \) of Borel equivalence relations which are invariant under \( G \).

2 \( E^X_G = \bigcap_{\alpha < \omega_1} \equiv_{\alpha} \).

3 A function \( \delta : X \to (\omega_1, <) \) which is Borel and \( G \)-invariant.

4 There is an \( \alpha < \omega_1 \) such that for every \( x \in X \) and for every \( y \in X \):

\[
x \equiv_{\delta(x) + \alpha} y \implies x \ E^X_G y.
\]

In our case, \( \alpha = \omega \).
What about the boundedness principle?

**Theorem**

$E^X_G$ is Borel if and only if there is an $\alpha$ such that for every $x \in X$, $\delta(x) \leq \alpha$. 
Complexity of $B \cdot x$

Let $B \subseteq G$ be a Borel set, $x \in X$. What is the complexity of $B \cdot x$?
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**Proposition**

$B \cdot x$ is Borel if and only if $B \cdot G_x$ is Borel. In particular, $U \cdot x$ is Borel, for $U$ open.
Complexity of $B \cdot x$

**Proposition**

If $G \cdot x$ is $\Pi^0_{\alpha+1}$ for $\alpha \geq 1$ then for every open $U$, $U \cdot x$ is $\Pi^0_{\alpha+1}$. 
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**Sketch of proof**

- $\alpha = 1$: $G \cdot x$ is $G_\delta$. 

Ohad Drucker (Hebrew U.)

Hjorth Analysis of General Polish Group Actions
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- By a theorem of Effros, the canonical bijection $G/G_x \to G \cdot x$ is a homeomorphism.
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**Sketch of proof**

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- By a theorem of Effros, the canonical bijection $G/G_x \to G \cdot x$ is a homeomorphism.
- Then $U \cdot x$ is open in $G \cdot x$, hence $G_\delta$ in $X$. 
Sketch of proof ( ctd. )

For arbitrary $\alpha$, $G \cdot x = \bigcap_{n \in \omega} B_n$. for $\langle B_n : n \in \omega \rangle$ $\Sigma^0_\alpha$ sets.
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- We then apply a Theorem of Hjorth to refine the topology of $X$ to a topology in which $G \cdot x$ is $G_\delta$. 
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- For arbitrary $\alpha$, $G \cdot x = \bigcap_{n \in \omega} B_n$. for $\langle B_n : n \in \omega \rangle \Sigma^0_\alpha$ sets.
- We then apply a Theorem of Hjorth to refine the topology of $X$ to a topology in which $G \cdot x$ is $G_\delta$.
- Using the case $\alpha = 1$, $U \cdot x$ is $G_\delta$ in the new topology, and hence $U \cdot x$ was $\Pi^0_{\alpha+1}$ in the original topology.
The Boundedness Theorem

**Theorem**

Let \((G, X)\) be a Polish action. Then \(E^X_G\) is Borel if and only if there is an \(\alpha\) such that for every \(x\), \(\delta(x) \leq \alpha\).
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**Proof.**

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- For all \(U \subseteq G\) open, \(U \cdot x\) is \(\Pi^0_{\alpha+1}\).
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- If \(E^X_G\) is Borel, there is an \(\alpha < \omega_1\) such that all orbits are \(\prod^0_{\alpha+1}\).
- For all \(U \subseteq G\) open, \(U \cdot x\) is \(\prod^0_{\alpha+1}\).
- It turns out that in this case, \(\delta(x) \leq \alpha + 1\).
The Decomposition Theorem

Theorem (Decomposition of Polish actions)

Let $X$ be a Polish $G$-Space. There is a sequence $\{A_\zeta\}_{\zeta<\omega_1}$ of pairwise disjoint Borel subsets of $X$ such that:

1. $A_\zeta$ is invariant, and
2. $\bigcup_{\zeta<\omega_1} A_\zeta = X$.
3. (Boundedness) If $A \subseteq X$ is Borel invariant and $E_{X,G} A$ is Borel, then $A \subseteq \bigcup_{\zeta<\alpha} A_\zeta$ for some $\alpha<\omega_1$.

Proof.

$A_\zeta = \{x : \delta(x) = \zeta\}$
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Proof.

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Theorem

For $\alpha$ countable, the set

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This set is in fact $\{ x : \delta(x) < \alpha + \omega \}$. 
Hjorth’s question

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For $\alpha$ countable, the set

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This set is in fact $\{ x : \delta(x) < \alpha + \omega \}$.

**Corollary**

For every countable $\alpha$, there are either countably many or perfectly many orbits that are $\Pi^0_\beta$, for $\beta < \alpha + \omega$. 