Non-universal tunneling resistance at the quantum critical point of mesoscopic SQUIDs array

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We calculate the tunnelling resistance at the quantum critical point of a mesoscopic SQUIDs array in the presence of magnetic flux. We find the analytical relation between the magnetic flux induced dissipation strength and the Luttinger liquid parameter of the system. While the experimental finding for the system is around 40 – 50 mK, we find the behavior of the system even at lower temperatures through the analysis of renormalization group. Apart from the length scale dependent superconductor-insulator transition, we also predict the evidence of length scale independent metallic state. This study also emphasizes the importance of Co-tunnelling effect.

Introduction: Josephson junction arrays have attracted considerable interest in the recent years owing to their interesting physical properties. Currently such arrays can be fabricated in restricted geometries both in one and two dimensions [1,10]. There are a few experimental findings in the field of mesoscopic SQUIDs array. The authors of Ref. (2) and Ref. (8) have fabricated the arrays of SQUIDs junctions of different numbers in a single chip. They have studied the current voltage characteristics of the mesoscopic SQUIDs array in the presence of a magnetic field. They have used an external magnetic field (B) that tunes the effective Josephson coupling (Ed) between the nearest neighbor superconductors by the following relations Ed = Ed0|cos(2πJd)|, where Φ is the external flux and Φ0 is the flux quantum. They have found magnetic field induced superconductor-insulator (SI) quantum phase transition at around 40–50 mK. When the applied magnetic flux is less than the critical value the system shows constant saturated resistance that may arise from the source of quantum phase slip centers (QPS) [11]. When the magnetic flux exceeds the critical value the system turns into the insulating phase due to the flux induced Coulomb blockade of Cooper pair tunneling. In this letter, we study the behavior of the system at very low temperature where there is no experimental findings, i.e., around few mili Kelvin [6,8]. The experimental result shows length scale dependent superconductor-insulator (SI) transition at the quantum resistance but this is not the whole picture, we find the length scale independent metallic phase. We calculate the tunneling resistance at the quantum critical point and also show explicitly the importance of Co-tunnelling effect to get the correct physical behavior of the system.

Analytical relation between the flux induced dissipative strength and the Luttinger liquid parameter: Here we derive analytical expression of magnetic flux induced dissipative strength (α) in terms of the interactions of the system. At first we derive the dissipative action/partition function of a quantum impurity system. We will see that the analytical structure of this dissipative action is identical with the mesoscopic SQUIDs array.

Here we consider that the impurity is present at the origin where the fermions scatter from the left to the right and vice versa. The Hamiltonian describing this process is 

\[ H = V_0(R(0) L(0) + h.c) = V_0 \int dx \delta(x) \cos(\theta(x)). \]

The total Hamiltonian of the system \( H = H_0 + V_0 \int dx \delta(x) \cos(\theta(x), \tau) \)

\[ H_0 = \frac{1}{2 \pi K} \int dx \delta(x) \cos(\theta(x), \tau)^2 + \frac{1}{K}(\partial_x \phi(x, \tau))^2, \]

corresponding Lagrangian of the system is

\[ L = \frac{1}{2 \pi K} \int \frac{1}{u} (\partial_t \phi(x, \tau))^2 + u(\partial_x \phi(x, \tau))^2 + V_0 \int dx \delta(x) \cos(\theta(x, \tau)) = L_0 + L_1 \] (1)

where \( L_0 \) and \( L_1 \) are the non-interacting and the interacting part of the Lagrangian and \( K \) is the Luttinger liquid parameter of the system. The only non-linear term in this Lagrangian is expressed by the field \( \theta(x, \tau) \). We would like to express the action of the system as an effective action by integrating the field \( \theta(x, \tau) \). Therefore one may consider \( \theta(x, \tau) \) as a heat bath, which yields the source of dissipation in the system. The constraint condition for the integration is \( \theta(\tau) = \theta(x = 0, \tau) \). We can write the partition function.

\[ Z = \int D\theta(x, \tau) e^{-\int_0^L L \theta \tau} \]

Here we use the standard trick of introducing the Lagrange multiplier with auxiliary field \( \lambda(\tau) \).

\[ Z = \int D\theta(x, \tau) \int D\theta(\tau) e^{-\int_0^L (L_0 + L_1) \tau} e^{i \int_0^\beta \delta \lambda(\tau) (\theta(0, \tau) - \theta(0, \tau))} \]

\[ Z = \int D\theta(\tau) e^{-\int_0^\beta L_1 \tau} \int D\lambda(\tau) e^{-i \lambda(\tau) \theta(\tau)} \]

\[ \int D\theta(x, \tau) e^{\int_0^\beta (-L_0 + i \lambda(\tau) \theta(0, \tau)) \tau} \]

The Fourier transform of the first term of Eq.(2) is
\[ L_0 = \sum_{i} \sum_{\omega_n} \frac{\omega_n^2 + v^2 \omega_n^2}{2 K} \theta(q_i, \omega_n) \theta(-q_i, -i \omega_n) \]

At first we would like to calculate the integral: \[ \int_{0}^{\beta} d\tau[L_0 - i\lambda(\tau)\theta(0, \tau)], \] we can write this term as \[ \sum_{\omega} \frac{\omega^2 + v^2 \omega_q^2}{2 K} \theta(q, \omega) \theta(-q, -i \omega) - \frac{i}{4 \pi} \lambda(i \omega) \theta(-q, -i \omega) + \lambda(-i \omega) \theta(q, i \omega). \] This integral appears in the integral \[ \theta(x, \tau). \] This integral is quadratic in \( \theta \). Now we would like to perform the Gaussian integration by completing the square. We can write the result as \[ \frac{1}{4 \pi} \sum_{\omega} \omega^2 + v^2 \omega_q^2 = \int \frac{d\omega}{2 \pi} \omega^2 + v^2 \omega_q^2 = \frac{\pi}{4 \omega}. \] Now we would like to append this result of integration in the second integral of \( Z \), i.e., the integral over \( \lambda \). One can write the integrand as \[ \sum_{\omega} \frac{\omega^2}{2 K} \lambda(\omega) \lambda(-i \omega) + \frac{i}{4 \pi} \lambda(i \omega) \theta(-q, -i \omega) + \lambda(-i \omega) \theta(q, i \omega). \] This integral is again the quadratic integral of \( \lambda \), therefore the Gaussian integral can be performed by completing the square. We get after the integration \[ \sum_{\omega} \frac{\omega^2}{2 K} \theta(\omega) \theta(-\omega). \] From these analytical expression, we obtain the effect of bath on \( \theta(\tau) \). The appearance of the factor \( \omega_n \) signifies the dissipation. Therefore the effective action reduces to

\[ S = \sum_{\omega_n} \frac{\omega_n}{\pi K} \theta(\omega_n) \theta(-\omega_n) + \int dxV_0 \cos\theta(\tau) \quad (3) \]

The above action implies that a single particle moving in the potential \( V_0 \cos \theta(\tau) \) subject to dissipation with friction constant \( \frac{\pi}{4 \omega} \).

Now we calculate the dissipative action of mesoscopic SQUIDs array. We have already proved in Ref. 14 that the strong coupling phase of the system is consistent with the experimental findings. Here we calculate the effective partition function of our system in the strong coupling phase. Our starting point is the Caldeira-Leggett 11 formalism. Following reference we write the action as

\[ S_1 = S_0 + \frac{\alpha'}{4 \pi T} \sum m |\theta_m|^2. \]

Here, \( S_1 \) is the standard action for the system with tiled wash-board potential 12,13 to describe the dissipative physics for low dimensional superconducting tunnel junctions, \( S_0 \) is the action for non-dissipative part, \( \alpha' = \frac{K_0}{\hbar \tau_0 \cos \frac{\pi}{20}} \) (the extra cosine factor which we consider in \( \alpha' \) is entirely new in the literature to probe the effect of an external magnetic flux and is also consistent physically), the Matsubara frequency \( \omega_m = \frac{2 \pi}{\tau_0} \) and \( R_Q = 6.45 \Omega \) is the quantum resistance and \( R_s \) is the tunnel junction resistance, \( \beta = \tau_0^{-1} \) is the inverse temperature.

In the strong potential, tunneling between the minima of the potential is very small. In the imaginary time path integral formalism, tunneling effect in the strong coupling limit can be described in terms of instanton physics. In this formalism, it is convenient to characterize the profile of \( \theta \) in terms of its time derivative,

\[ \frac{d\theta(\tau)}{d\tau} = \sum_{i} \epsilon_i h(\tau - \tau_i), \quad (5) \]

where \( h(\tau - \tau_i) \) is the time derivative at time \( \tau \) of one instanton configuration. \( \tau_i \) is the location of the i-th instanton, \( \epsilon_i = 1 \) and \( -1 \) is the topological charge of instanton and anti-instanton respectively. Integrating the function over \( h \) from \(-\infty \) to \( \infty \), \[ \int_{-\infty}^{\infty} d\tau h(\tau) = \theta(\infty) - \theta(-\infty) = 2\pi. \] It is well known that the instanton (anti-instanton) is almost universally constant except for a very small region of time variation. In the QPS process the amplitude of the superconducting order parameter is zero only in a very small region of space as a function of time and the phase changes by \( \pm 2\pi \). So our system reduce to a neutral system consisting of equal number of instanton and anti-instanton. One can find the expression for \( \theta(\omega) \), after the Fourier transform to the both sides of Eq. 5 which yields \( \theta(\omega) = \frac{1}{\pi} \sum \epsilon_i h(\omega) e^{i\omega \tau_i} \). Now we substitute this expression for \( \theta(\omega) \) in the second term of Eq.4 and finally we get this term as \[ \sum_{ij} F(\tau_i - \tau_j) \epsilon_i \epsilon_j, \]

\[ F(\tau_i - \tau_j) = \frac{2 \pi}{\beta} \sum_{m} \frac{1}{|\omega_m|} e^{i\omega_m (\tau_i - \tau_j)} \approx \ln(\tau_i - \tau_j). \]

We obtain this expression for very small values of \( \omega \) (\( \rightarrow 0 \)). So \( F(\tau_i - \tau_j) \) effectively represents the Coulomb interaction between the instanton and anti-instanton. This term is the main source of dissipation physics of the system. Following the standard prescription of imaginary time path integral formalism, we can write the partition function of the system as

\[ Z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \sum \epsilon_i \int_{0}^{\beta} d\tau_n \int_{0}^{\tau_n-1} d\tau_{n-1}... \int_{0}^{\tau_2} d\tau_1 e^{-F(\tau_i - \tau_j) \epsilon_i \epsilon_j}. \]

We would like to express the partition function in terms of integration over auxiliary field, \( q(\tau) \). After some extensive analytical calculations, we get

\[ Z = \int Dq(\tau) e^{(\sum \omega_n \frac{2 \pi}{\hbar} \theta(\omega_n) q(-i \omega_n))+(2\pi \int_{0}^{\beta} d\tau \cos(q(\tau)))}. \]

Thus by comparing the first term of the action of Eq. (3) and the first term of exponential of Eq. (17), we conclude that the dissipative strength \( \alpha \) and the Luttinger liquid parameter of the system are related by the relation, \( K = 4\alpha \).

Quantum field theoretical study of model Hamiltonian of the system and explicit derivation of dissipative strength:

In our previous study, we have shown explicitly that the mesoscopic SQUIDs array is equivalent to the array of superconducting quantum dots (SQD) with modulated Josephson coupling. We first write the model Hamiltonian of SQD with nearest neighbor (NN) Josephson coupling and also with the presence of the on-site and NN
charging energy between SQD, 
\[ H = H_{J1} + H_{EC0} + H_{EC1}. \] (8)

Now we would like to recast our basic Hamiltonians in the spin language. This is valid when \( E_{C0} >> E_{J1} \). It is also observed from the experiments that the quantum critical point exists for larger values of the magnetic field, when the magnetic field induced Coulomb blockade phase is more prominent than the \( E_J \) induced SC phase. Thus our theoretical model is consistent with the experimental findings. During this mapping process we follow Ref. (2–4).

We would also like to express the fermionic fields in terms of bosonic and the left-moving fermions respectively. We would follow Ref. (2–4).

\[ H_{J1} = -2 E_{J1} \sum_i (S_i^+ S_{i+1}^- + h.c), \]
\[ H_{EC0} = E_{C0} \sum_i S_i^Z, \]
\[ H_{EC1} = 4 E_{Z1} \sum_i S_i^Z S_{i+1}^Z, \]
\[ E_{J1} = E_{J10} |\cos(\frac{\theta}{4})| \].

At the Coulomb blocked regime, the higher order expansion leads to the virtual state with energies exceeding \( E_{C0} \). In this second order process, the effective Hamiltonian reduces to the subspace of charges 0 and 2, and takes the form \( 2–4 \),

\[ H_C = -\frac{3E_{J1}}{4E_{C0}} \sum_i S_i^Z S_{i+1}^Z - \frac{E_{J1}}{E_{C0}} \sum_i (S_i^+ S_i^- + h.c). \] (9)

With this corrections \( H_{EC1} \) become
\[ H_{EC1} \simeq (4E_{Z1} - \frac{3E_{J1}}{E_{C0}}) \sum_i S_i^Z S_{i+1}^Z \]
One can express spin chain systems as spinless fermions systems through the application of Jordan-Wigner transformation. We have transformed all Hamiltonians in spinless fermions which we have not present in this letter. In order to study the continuum field theory of these Hamiltonians, we recast the spinless fermions operators in terms of field operators by a relation \( 18 \).
\[ \psi(x) = [e^{ikp_x} \psi_R(x) + e^{-ikp_x} \psi_L(x)] \]
where \( \psi_R(x) \) and \( \psi_L(x) \) describe the second-quantized fields of right- and the left-moving fermions respectively. We would like to express the fermionic fields in terms of bosonic field by the relation \( \psi_r(x) = \frac{U}{\sqrt{2\pi\sigma}} e^{-i(\phi(x) - \theta(x))} \), \( r \) is denoting the chirality of the fermionic fields, right (1) or left movers (−1). The operators \( U_r \) preserve the anti-commutativity of fermionic fields. \( \phi \) field corresponds to the quantum fluctuations (bosonic) of spin and \( \theta \) is the dual field of \( \phi \). They are related by the relations \( \phi_R = \theta - \phi \) and \( \phi_L = \theta + \phi \). The total Hamiltonian is
\[ H = H_0 + \int dx : \cos(4\sqrt{K} \phi(x)) : + \frac{E_{C0}}{\pi\alpha} \int (\partial_x \phi(x)) dx \] (10)

\( H_0 \) is the non-interacting part of the Hamiltonian, The dissipative strength \( (\alpha_1) \) of this system in the absence of Co-tunneling effect is
\[ \alpha_1 = \frac{1}{4} \sqrt{\frac{2E_{J1}}{2E_{J1} + 16E_{Z1}}}. \] (11)

If we consider the total effect of Co-tunneling process, the system reduces to Heisenberg spin chain with NN and NNN interactions. In this limit the dissipative strength \( (\alpha_2) \) of the system is
\[ \alpha_2 = \frac{1}{4} \sqrt{\frac{2E_{J1}}{E_{J1} + \frac{1}{4}(4E_{Z1} - \frac{3E_{J1}}{E_{C0}})}} \] (12)

We calculate the dissipation strength by calculating \( K \) for both cases and then we use the relation \( K = 4\alpha \). This is the first analytical derivation of flux induced dissipation strength in terms of the interactions of the system. We consider these two processes to emphasize the importance of Co-tunneling effect for this system.

**Physical Analysis of Renormalization Group Equation and Calculation of Tunneling Resistance at The Quantum Critical Point:**

The RG equation based on the Eq. (7) is,
\[ \frac{dz}{d\ln b} = (1 - \alpha')z \] (13)

Following Ref.\([4]\), we can write fugacity depends on length scale and temperature as, \( z(L) \propto L^{-1-\alpha} \), \( z(T) \propto T^{\gamma - 1} \). The physical explanations based on this RG equation are in order.

In our study, the resistance is evolving due to dissipation effect at very low temperature (few milli Kelvin, less than the superconducting Coulomb blocked temperature). According to our calculations, for large dissipation \( (\alpha' \gg 1) \), \( R(T) \propto R_0 T^{\beta_1} \) and \( \beta_1 > 0 \). Therefore at very low temperature, the system shows stable SC behaviour (region B of Fig. 1). and the system shows no more saturated resistance behaviour (region D of Fig. 1). When \( \alpha' < 1 \), the resistance of system \( R(T) \propto R_0 T^{\gamma_1} \) and \( \gamma_1 > 0 \). So at very low temperature, the resistance of the system shows Kondo-like divergence behavior (region A of Fig. 1). These stable SC and Kondo behaviour are unseen in the experimental findings because they have measured the zero bias resistance at around 40 – 50 mK. According to our calculations, for large dissipation \( (\alpha' \gg 1) \), \( R(T) \propto R_0 L^{-1-\gamma} \) and \( \gamma_1 > 0 \). Therefore the longer array system shows the less resistive state than shorter array in the superconducting phase. When \( \alpha' < 1 \), the resistance of system \( R(T) \propto R_0 L^{\gamma_2} \), where \( \gamma_2 > 0 \) (\( \beta_1, \beta_2, \gamma_1 \) and \( \gamma_2 \) are independent numbers). So the resistance in the insulating state is larger for longer array system than shorter one. We find the dual behavior of the resistance for lower and higher values of magnetic field.

From the knowledge of LL physics, we know that for a particular range of \( K \), there is a metallic state in between the insulating phase and the superconducting phase of the system. The range of \( K \) depends on the nature of the system. Therefore we conclude that there is also a metallic state for a particular range of magnetic flux induced dissipation strength. This prediction is unnoticed in the experimental findings. At very low temperature
around $\alpha \sim 1$ (region E of Fig. 1), the quantum phase slip centres poliferate and as a result a screening appears in the system. A detailed explanation of QPS for this type of system is discussed in Ref. [4]. Physical explanation is as follows.

Here we consider two QPS with co-ordinates $(x_1, \tau_1)$ and $(x_2, \tau_2)$, we assume that the cores of the QPS centres donot overlap, i.e., $|x_2 - x_1| > x_0(= \xi)$ and $|\tau_2 - \tau_1| > \tau_0(= \frac{1}{3})$. Where $\xi$ and $\Delta$ are the coherence length and SC orderparameter of the system. We have already proven that the topological charges interact with each other logarithmically. Therefore we can write the action of the system as $S_{QPS} = 2S_{core} - \mu e_1 e_2 \ln|x_1 - x_2|$. QPS with opposite topological charge attract each other and same charge repel with each other. Suppose we consider a gas of $n$ QPS and assume that QPS cores do not overlap. When a current is passing through the system, we can write the total action of the system as $S_{QPS}^n = nS_{core} + S_{int}[17]$. Where,

$$S_{int} = -\mu \sum_{i \neq j} e_i e_j \ln\left(\frac{\rho_{ij}}{x_0}\right) + \sum_i \frac{\Phi_0}{c} e_i \tau_i (14)$$

Where $\rho_{ij}$ is the distance between the two QPS at the site $i$ and $j$ and $I$ is the current passing through the system. The grand partition function of the system is represented as a sum of all topological charges and the analytical form is equivalent to Eq.6, $z \sim e^{-S_{core}}$. In the absence of current, Eq. (14) define the model for a 2-dimensional Coulomb gas interacting logarithmically. We consider the very low temperature limit ($T \rightarrow 0$). The RG equations of this 2-dimensional Coulomb gas are the following $\frac{\partial \Phi}{\partial \tau} = -4\pi^2 \mu^2 z^2$, $\Phi = (2 - \mu) z$. In our case we can write the second equation as $\frac{\partial z}{\partial \ln(\Phi)} = (2 - \mu) z$. Finally it yields $z(T) = z(\Delta)\left(\frac{\Phi}{\pi}\right)^{2-\mu}$. When the interaction coefficient is less than 2, then the RG equation diverges at the scale $T^* = (z(\Delta))^\mu$). At the scale larger than $\xi$ the interaction between the QPS is screened and cease to be logarithmic, it becomes an exponentially decaying function ($\sim e^{-2\mu K_0(\sqrt{3\pi+\tau})}$), $K_0$ is the modified Bessel function.

When $\alpha' = 1$, i.e., $\Phi = (\Phi_0/\pi) \cos^{-1}\left(\frac{R_s}{R_0}\right)$, the system has no length scale dependence SI transition at very low temperature. This is the critical behavior of system for a specific value of magnetic field. The analytical expression for tunneling resistance at the quantum critical point can be calculated by comparing the expression of $\alpha_1$ and $\alpha_2$ with the expression of $\alpha'$. $R_s^{(1)} = R_Q\left(\frac{16E_{J1}}{\cos^2\Phi}\right)$ and $R_s^{(2)} = R_Q\left(\frac{5\pi E_{J0}}{4E_{J1}}\right) \sqrt{1 + \frac{277E_{J1}}{E_{CO}}}$ are the tunneling resistance at the quantum critical point in the absence and presence of Co-tunneling effect respectively. It is clear from our analytical derivation that the tunneling resistance at quantum critical point is not $R_Q$ but it is rather non-universal. $R_s^{(2)}$ is proportional to the $E_{CO}$ as one would expect from the physical criteria of the system. Here we discuss the importance of Co-tunneling effect explicitly. In the SC phase when $\alpha > 1$, from the analysis of $\alpha_1$, we get the condition $\frac{16E_{J1}}{\pi} > 30E_{J10}[\cos(\frac{\Phi}{\pi})]$. This condition is unphysical, because the all coupling constants are repulsive. In presence of Co-tunneling process, from the analysis of $\alpha_2$, we achieve the condition of SC $E_{J10}[(\cos(\frac{\Phi}{\pi}))][\frac{192E_{J10}}{E_{CO}}][\cos(\frac{\Phi}{\pi})] > \frac{3}{5}$$. This condition is physically reliable. Similarly one can do the analysis for the insulating phase for both absence and presence of co-tunneling effect.

**Conclusions:** we have found the non-universal tunneling resistance at the quantum critical point of mesoscopic SQUIDs array. We have found stable SC and Kondo phase at very low temperature, which is still unseen experimentally. We have found the length scale independent metallic phase in the system. The importance of Co-tunneling effect has been studied explicitly.

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