Black hole formation from colliding bubbles

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Abstract

Some indication of conditions that are necessary for the formation of black holes from the collision of bubbles during a supercooled phase transition in the early universe are explored. Two colliding bubbles can never form a black hole. Three colliding bubbles can refocus the energy in their walls to the extent that it becomes infinite.

98.80Cq, 04.60.+n, 97.60.Lf
I. INTRODUCTION

A small number of black holes being produced in the early stages of universe would have important cosmological consequences, both from their contribution to the density of the universe and to the high energy cosmic ray background \[1 \text{–} 3\]. For this to have occurred there would have to have been density perturbations with large amplitudes because small amplitude perturbations do not grow in size in the primordial plasma. The isotropy of the universe suggests to us that the universe was fairly homogeneous on the length scales corresponding to galaxies or larger objects, but comparatively little is known about the homogeneity of the early universe on small scales, for example scales much less than the horizon scale at nucleosynthesis which encompassed a mass of around \(3 \times 10^{36} \text{Kg}\).

One situation in which a substantial amount of inhomogeneity may have arisen could be at a first order phase transitions where a gauge symmetry was broken. The possibility of black hole formation was raised very early in the theory of these phase transitions \[4, 5\].

First order phase transitions are characterised by tunnelling from the old to the new phase. For spontaneous symmetry breaking the phases are distinguished by the values of a Higgs field. The tunnelling is modeled by the instantaneous nucleation and expansion of a bubble of the broken–symmetry phase \[3, 7\]. In the supercooled case, where the tunnelling is smaller than the expansion rate of the universe, it is possible for the thermal energy of the old phase to be surpassed by the energy of the vacuum, then the bubble walls accelerate to speeds close to the speed of light and build up considerable amounts of energy.

The collision of these bubbles and the possibility of forming black holes was considered in reference \[3\]. We found that when two bubbles collide the bubble walls could continue through one another unchanged or the energy could convert into the phase of the Higgs field. In the latter case, most of the energy emerged from the collision region as phase waves moving at the speed of light. We also argued on energy grounds that eight colliding bubbles could form a black hole.

It is possible to model the collision of two thin–walled bubbles by distributonal gravitational sources \[8\]. The bubbles start out with flat Minkowski inside and de Sitter space outside. Each bubble is invariant under Lorentz boosts and rotations \[9\], therefore the two–bubbles with a preferred axis have the symmetry \(O(2,1)\) \[5\]. This has as many killing fields as spherical symmetry and Wu was able to demonstrate that there is a modified form of Birkoff’s theorem that can be used to find the metric.

The rapid acceleration of the bubble walls up to luminal speeds indicates that the collision can be simplified by replacing the bubble walls with null surfaces. When the energy from the walls is converted into phase waves, as often happens, the energy also leaves the collision on null surfaces. The collision of two bubbles becomes the collision of two null shock waves, a problem whose solution is known \[10, 11\].

By making these reasonable simplifications it becomes possible to include a third or a fourth bubble in the collision. Under suitable conditions we shall see that the energy of the bubble walls is sufficient to refocus itself into a singularity and therefore form a black hole or a naked singularity.
II. O(2,1) INVARIANT METRICS

The group O(2,1) is the symmetry group of the pseudosphere $H_2$. We can obtain a spacetime metric which is invariant under O(2,1) by replacing the sphere by the pseudosphere in the usual O(3) invariant metric,

$$g = -S(s,x)ds^2 + X(s,x)dx^2 + s^2 \left(d\theta^2 + \sinh^2 \theta d\phi^2\right).$$

(2.1)

Substituting this form of the metric into the vacuum Einstein equations (with a cosmological constant $3/\alpha^2$), we can deduce easily that $X = 1/S = f$, where

$$f = 1 - \frac{2M}{s} + \frac{s^2}{\alpha^2}.$$  

(2.2)

The parameter $M$ is a constant but it does not have the same physical associations of a mass. More details about these metrics can be found in reference [12].

We can draw Penrose diagrams of the surfaces that are orthogonal to the pseudospheres. The case with vanishing cosmological constant, which will be called pseudo–Schwartzchild, is shown in figure [4]. This is the usual Penrose diagram of Schwartzchild rotated through 90°. Surfaces of constant coordinate $s$ are spacelike in $s > 2M$ and the metric approaches flat space (Milne universe form) in the limit that $s \to \infty$.

The metric with $M = 0$ is de Sitter space. It will be useful to have the embedding of the metric into a five dimensional Minkowski spacetime with cordinates $X = (X,Y,Z,W,V)$, where de Sitter space is defined by the hyperboloid

$$X \cdot X = X^2 + Y^2 + Z^2 + W^2 - V^2 = \alpha^2.$$  

(2.3)

The two de Sitter coordinate systems are related by

$$X = s \sinh \theta \cos \phi,$$

$$Y = s \sinh \theta \sin \phi,$$

$$Z = (\alpha^2 + s^2)^{1/2} \sin(x/\alpha),$$

$$W = (\alpha^2 + s^2)^{1/2} \cos(x/\alpha),$$

$$V = s \cosh \theta.$$  

(2.4) (2.5) (2.6) (2.7) (2.8)

During a supercooled phase transition the universe becomes locally similar to de Sitter space. Bubbles of the true vacuum phase (Minkowski space) appear instantaneously. Their walls move along trajectories with O(3,1) symmetry,

$$(X - X_0) \cdot (X - X_0) = r_0^2.$$  

(2.9)

The radius $r_0$ is the initial bubble radius and is fixed by physical parameters. For phase transitions at temperatures far below the Planck temperature of $10^{19}GeV$, this radius is far smaller than the cosmological horizon size $\alpha$.

If $r_0 = 0$ we can replace the bubble surface by a null surface. The generators of this surface are light rays

$$X - X_0 = \lambda P,$$  

(2.10)
where \( \lambda \) is an affine distance along the ray and the constant momenta \( \mathbf{P} \) satisfy \( \mathbf{P} \cdot \mathbf{P} = 0 \). This is the approximation that will be used henceforth. The light rays can be also be parameterised in an \( O(2,1) \) invariant form by the substitution \( \mathbf{X}(s, x, \theta, \phi) \). Differentiating with respect to the affine parameter gives

\[
\dot{s} = s^{-1} \left( Z P^Z + WP^W \right), \tag{2.11}
\]
\[
\dot{x} = (sf)^{-1} \left( WP^Z - Z P^W \right), \tag{2.12}
\]
\[
\dot{\phi} = s^{-2} \text{cosh}^2 \theta \left( XP^Y - Y P^X \right), \tag{2.13}
\]
where the components of \( \mathbf{P} \) are denoted by \( (P_X, P_Y, P_Z, P_W, P_V) \). The values of \( \dot{s} \) etc. are the components of the tangent vectors in the \( O(2,1) \) coordinate frame, with the latter two quantities in brackets being the conserved linear and angular momenta respectively.

### III. TWO–BUBBLE COLLISIONS

In this section we will take a look at the problem of two intersecting null shells, being relevant to the collision of two bubbles when the energy from the bubble walls leaves the collision region at the speed of light. Numerical studies have shown that this is a good approximation when there is a large difference of phase of the Higgs field in each of the bubbles.

Consider a null hypersurface with null geodesic generator \( \mathbf{l} \) and affine parameter \( u \). It is convenient to introduce a ‘pseudonormal’ null vector \( \mathbf{n} = \partial/\partial v \), normalised by \( \mathbf{l} \cdot \mathbf{n} = -1 \) and parallel propagated along \( \mathbf{l} \). The remaining tangent vectors \( \mathbf{e}_i, i = 1, 2 \), define a two dimensional surface with induced metric

\[
h_{ab} = g_{ab} + l_a n_b + n_a l_b \tag{3.1}
\]
and it is convenient to have them parallel propagated along \( \mathbf{l} \).

The embeddings of the null surface are described by extrinsic curvature tensors \( k_{ab} \) and \( m_{ab} \), where

\[
k_{ab} = h_a^c h_b^d n_{c,d} \tag{3.2}
\]
\[
m_{ab} = h_a^c h_b^d l_{c,d}. \tag{3.3}
\]

From the condition that \( \mathbf{e}_i \cdot \mathbf{l} = \mathbf{e}_i \cdot \mathbf{n} = 0 \), it is also possible to write

\[
k_{ab} e_i^a e_j^b = -n_i (\nabla_l \mathbf{e}_j), \tag{3.4}
\]
\[
m_{ab} e_i^a e_j^b = -l_i (\nabla_l \mathbf{e}_j) \tag{3.5}
\]

Their contractions will be denoted by \( k \) and \( m \). It is easy to show that \( m \) is also the expansion of the null geodesic congruence, that is \( m = \nabla \cdot \mathbf{l} \).

A distributional source of stress–energy on the shell would take the form

\[
T_{ab} = \sigma l_a l_b \delta(v). \tag{3.6}
\]
Junction conditions across shell can be obtained from integrating the Einstein field equations. (See ref [13] for example, but note that \( l \) and \( n \) have the opposite meaning.) These force the metric to be continuous and allow the choice of \( l \), \( n \) and \( e_i \) to be continuous. Furthermore, they relate the extrinsic curvatures of the surfaces to the stress energy. If the surface is moving to the right from a region \( C \) into another region \( B \) then,

\[
8\pi G \sigma = k_B - k_C, \quad \text{and} \quad m_B = m_C. \tag{3.7}
\]

where \( G \) is Newton’s constant.

Two null hypersurfaces divide spacetime into four regions, labeled as in the diagram [2]. The null geodesic generators of the surfaces \( i = 1 \ldots 4 \) are labeled by \( l_i \) and the extrinsic curvatures will be indexed accordingly. In the vicinity of the intersection pseudosphere, the junction conditions imply that

\[
8\pi G \sigma_1 = k^1_C - k^1_D, \quad 8\pi G \sigma_2 = k^2_C - k^2_B \tag{3.8}
\]

\[
8\pi G \sigma_3 = k^3_B - k^3_A, \quad 8\pi G \sigma_4 = k^4_D - k^4_A \tag{3.9}
\]

It is also possible to obtain normal vectors from the \( l \)'s, for example

\[
n_1 = -(l_1 \cdot l_2)^{-1}l_2 \quad \text{and} \quad n_2 = -(l_1 \cdot l_2)^{-1}l_1. \tag{3.10}
\]

These relations are specific to the regions in which the vectors are defined, shown in fig. [2], but the normal vectors \( n_i \) are themselves continuous across the \( i \)'th surface. The continuity equations for the \( m \)'s then imply the following equations for the \( k \)'s,

\[
(l_1 \cdot l_2)k^1_C = (l_2 \cdot l_3)k^3_B, \quad (l_4 \cdot l_1)k^1_D = (l_4 \cdot l_3)k^3_A, \tag{3.11}
\]

\[
(l_3 \cdot l_2)k^2_B = (l_3 \cdot l_4)k^4_A, \quad (l_1 \cdot l_2)k^2_C = (l_1 \cdot l_4)k^4_D. \tag{3.12}
\]

Finally, we allow the affine parameterisations on either side of the collision region to differ by a constant scaling,

\[
l_3 = \gamma l_1, \quad l_4 = \beta l_2. \tag{3.13}
\]

The continuity equations can be written in a much more useful form. Introduce

\[
F_C = (l_1 \cdot l_2)k^2_Ck^2, \tag{3.14}
\]

using extrinsic curvatures of surfaces adjacent to region \( C \), and similarly for each of the regions in turn. Then,

\[
F_CF_A = (l_1 \cdot l_2)(l_3 \cdot l_4)k^1_Ck^2_Ck^3_Ak^4_A, \tag{3.15}
\]

Using the continuity relations gives,

\[
F_CF_A = (l_1 \cdot l_4)(l_2 \cdot l_3)k^3_Bk^4_Dk^1_Bk^2_B = F_BF_D. \tag{3.16}
\]

This will be referred to as ‘the pseudo–DTR relation’ [10,11].

The metrics that are of interest to us have the \( \text{O}(2,1) \) symmetry. Continuity of the pseudosphere sections across the null hypersurfaces implies that the \( \theta \) and \( \phi \) coordinates are
continuous, and also the $s$ coordinate because it sets the area element, but the $x$ coordinate will be different on either side of the surfaces.

The null generators of the surfaces are then

$$ l_1 = e_s - f^{-1}e_x, \quad l_2 = e_s + f^{-1}e_x. \quad (3.17) $$

Using the connection components of the metric gives,

$$ -k^1_C = k^2_C = f_C/s. \quad (3.18) $$

Therefore the pseudo–DTR relation reads

$$ f_C f_A = f_B f_D. \quad (3.19) $$

This is the junction condition that determines the metric in any region if the metric is known in the other three. It is also possible to get the surface energy density from 3.9,

$$ \sigma_3 = \frac{f_A - f_B}{8\pi G s}. \quad (3.20) $$

For the collision of two bubbles, the initial region $A$ is de Sitter space,

$$ f_A = 1 + s^2/\alpha^2. \quad (3.21) $$

Regions $B$ and $D$ are inside the bubbles, where the metric is flat. The energy on the bubble surface follows from equation (3.20). Noting that the affine length along the surface is $\lambda = s$, this can be written

$$ \sigma = (8\pi G \alpha^2)^{-1} \lambda. \quad (3.22) $$

The increase in surface energy represents the latent energy of the false vacuum that the bubble absorbs as it expands.

The intersection region $C$ is in the true vacuum where there is no cosmological constant and the metric is pseudo–Schwartzchild,

$$ f_C = 1 - 2M/s. \quad (3.23) $$

The collision takes place at a particular value of $s$, say $s_c$. From the pseudo–DTR relation we have

$$ \left(1 + s_c^2/\alpha^2\right) \left(1 - 2M/s_c\right) = 1. \quad (3.24) $$

This gives

$$ 2M = \frac{s_c^3}{\alpha^2 + s_c^2}. \quad (3.25) $$

In particular, we see that $M$ is always positive and we always get $s_c > 2M$. This is the condition for there to be no singularity anywhere in the spacetime. Therefore, in this simplified form of the problem, the collision of two bubbles never produces a black hole.
IV. MANY–BUBBLE COLLISION

Although the collision of two bubbles fails to produce a black hole, the collision can come quite close to the limit $s = 2M$ where the spacetime becomes singular. Only a small additional amount of energy from another bubble should be required to tip the balance in favour of a black hole. Unfortunately, introducing a third bubble breaks the symmetry and prevents us writing down the metric explicitly. What we can do instead is aim to find conditions under which singularities are formed.

As a first step we shall consider the trajectories of light rays in the spacetime formed from two colliding bubbles, especially their refraction from the bubble walls (fig. 3). We shall write the momentum vector $p$ of a general ray in terms of the generators of one of the bubble walls,

$$ p = p^s l + p^n n + p^1 e_1 + p^2 e_2. $$

If the light ray is itself part of a bundle from the same initial point, then the expansion $\dot{\theta}$ of the bundle satisfies Raychaudhuri’s equation,

$$ p(\dot{\theta}) = -R_{ab}p^ap^b - 2\dot{\sigma}^2 - \frac{1}{2}\dot{\theta}^2, $$

where $\dot{\sigma}$ is the shear and $p$ has been identified with its directional derivative following a common practice. (If we define the analogue of $m_{ab}$ for the congruence generated by $p$, then $\dot{\theta}$ and $\dot{\sigma}$ are the trace and trace free parts of $m_{ab}$.) The Einstein equations relate the curvature to the surface energy in the bubble wall,

$$ T_{ab} = \sigma l_a l_b \delta(v), $$

but we also have

$$ p(\dot{\theta}) = p^s \frac{\partial \dot{\theta}}{\partial u} + p^n \frac{\partial \dot{\theta}}{\partial v}. $$

Integrating gives the focusing rule,

$$ \dot{\theta}_B - \dot{\theta}_A = 8\pi G\sigma l \cdot p, $$

for a ray passing from region $A$ into region $B$.

We also see from Raychudhuri’s equation that the expansion is finite, suggesting that the momentum vector should be continuous. Nevertheless, there is some refraction of the ray in the $O(2,1)$ coordinate system, due to the discontinuity of the coordinates themselves. Consider again the ray that passes from region $A$ to region $B$, but now using the explicit forms of the bubble generators.

$$ p = \frac{1}{2}(\dot{s} - f_A \hat{x})l_3 + \frac{1}{2}(\dot{s} + f_A \hat{x})l_4 + \dot{\theta}e_\theta + \dot{\phi}e_\phi. $$

In region $B$ we use the vector $l_2$, related to $l_4$ by equation,

$$ l_4 = (f_B/f_A)l_2. $$
Restoring the $s$ and $x$ coordinates gives

$$\begin{pmatrix} \dot{s} \\ \dot{x} \end{pmatrix}_B = \frac{1}{2} \begin{pmatrix} 1 + \beta & \beta - 1 \\ \beta - 1 & 1 + \beta \end{pmatrix} \begin{pmatrix} \dot{s} \\ \dot{x} \end{pmatrix}_A,$$

(4.8)

where $\beta = f_B/f_A$. Similarly, for the passage from $B$ to $C$,

$$\begin{pmatrix} \dot{s} \\ \dot{x} \end{pmatrix}_C = \frac{1}{2} \begin{pmatrix} 1 + \gamma & 1 - \gamma \\ 1 - \gamma & 1 + \gamma \end{pmatrix} \begin{pmatrix} \dot{s} \\ \dot{x} \end{pmatrix}_B,$$

(4.9)

where $\gamma = f_C/f_B$.

A particular case of interest for us later on is where the ray passes close to the collision region, and the values of $s$ in both of the refraction relations are the same. Then the combination gives

$$\begin{pmatrix} \dot{s} \\ \dot{x} \end{pmatrix}_C = \begin{pmatrix} \beta \\ 0 \end{pmatrix} \begin{pmatrix} \dot{s} \\ \dot{x} \end{pmatrix}_A.$$

(4.10)

and

$$\hat{\theta}_C = \hat{\theta}_A - 8\pi G \sigma f_A^{-1} \left( \dot{s}^2 - f^2 \dot{x}^2 \right)_A.$$

(4.11)

Now we turn to the collision of three bubbles in a symmetrical arrangement with centres equally spaced a proper distance $d$ apart, as shown in figure 4. The cartesian coordinates can be chosen so that the bubbles all begin at the same value the time coordinate $V$ and in the same plane. We therefore take the centres to be at,

$$X_1 = (0, 0, Z_1, W_1, 0),$$

(4.12)

$$X_2 = (0, 0, -Z_1, W_1, 0),$$

(4.13)

$$X_3 = (X_3, 0, 0, W_3, 0),$$

(4.14)

where

$$Z_1 = \alpha \sin(d/2\alpha)$$

(4.15)

and

$$X_3^2 = \frac{3\alpha^2 Z_1^2 - 4Z_1^2}{\alpha^2 - Z_1^4}.$$

(4.16)

We also have the condition that $X_1 \cdot X_i = \alpha^2$ to fix $W_1$ and $W_3$.

Bubbles 1 and 2 nucleate at $s = 0$ in the O(2,1) coordinate system. The bubble surfaces are given by the ray equation 2.10 and collide on a surface of constant coordinate $s$,

$$s_c = \frac{\alpha Z_1}{\sqrt{\alpha^2 - Z_1^2}}.$$

(4.17)

Some elementary geometry shows that the generators of bubble 3 intersect bubble 2 after an affine length
\[ \lambda_p = \frac{\alpha^2 - X_2 \cdot X_3}{X_2 \cdot P}. \]  
(4.18)

Substituting for \( X_3 \) gives
\[ \lambda_p = \frac{2Z_1^2}{W_1 P W + Z_1 P Z}. \]  
(4.19)

This also gives a value of the coordinate \( s \) for the intersection,
\[ s_p = \frac{\alpha Z_1 W_1 P W - Z_1 P Z}{W_1 W_1 P W + Z_1 P Z}. \]  
(4.20)

We shall restrict attention to the symmetrical rays that pass through the collision points. For these rays, \( \dot{x} = P Z = 0 \), and from equation 2.11,
\[ \dot{s}_p = \frac{P W}{\alpha Z_1} \left( \alpha^2 + Z_1^2 \right). \]  
(4.21)

The surface energy of bubble 3 at the intersection points is given by equation 3.22
\[ \sigma_p = \frac{Z_1^2}{4\pi G \alpha W_1 P W}. \]  
(4.22)

The expansion of the bubble wall is given by
\[ \dot{\theta}_A = \frac{2}{\lambda_p} = \frac{W_1 P W}{Z_1^2}. \]  
(4.23)

Now it is possible to use the focussing rule 4.11,
\[ \dot{\theta}_C = \dot{\theta}_A - 8\pi G \sigma_p f^{-1} \dot{s}_p^2. \]  
(4.24)

This gives,
\[ \dot{\theta}_C = \frac{W_1 P W}{Z_1^2 \alpha^6} \left\{ \alpha^6 - 2Z_1^2 \left( \alpha^2 + Z_1^2 \right)^2 \right\}. \]  
(4.25)

Therefore the expansion of the outgoing bubble wall is negative after the collision when \( Z_1 > z\alpha \), where \( z \approx 0.54512 \) is a root of \( z^3 + z - \sqrt{1/2} = 0 \). If \( Z_1 > \alpha \sqrt{3}/2 \), then the de Sitter geometry does not allow the three bubbles to have a common collision point and there is no refocussing.

Raychaudhuri’s equation tells us that once the expansion becomes negative it remains negative and diverges in a finite affine distance along the ray. When the expansion diverges this forces the surface energy to become infinite. At this point there is a spacetime singularity, in the sense that the curvature is more singular than the distributional curvature with which we began.

The preceding arguments have established that refocussing occurs along a line extending from the collision point in the direction of the symmetrical ray. This line was chosen only for convenience and the refraction rules obtained earlier can be used to examine refocussing anywhere on the bubble wall.
V. CONCLUSION

We have argued that the bubbles nucleating at a supercooled phase transition can be replaced by expanding null shells of energy. With this simplification, the collision of two bubbles never produces a singularity, in agreement with previous results.

When three bubbles collide the surface energy in parts of the walls can be refocused to the extent that it becomes infinite. What happens next depends on general issues of general relativity such as the cosmic censorship conjecture. If the singularities are to be invisible from future infinity of the emerging universe then they would have to be hidden behind an event horizon, and form a black hole. If cosmic censorship fails, then the bubble collision results in a new object, but one whose time–evolution presents problems in general relativity.

The condition for refocusing the energy along a symmetry axis for a symmetrical collision was a proper separation $d > 1.153\alpha$. These three bubbles have a triple collision point provided that $d < 2.094\alpha$. Although the non–symmetrical case is more complicated it is theoretically possible to follow it through using the equations that have been given. In particular, small deviations from the symmetrical case should leave the results unchanged.

If there are four colliding bubbles, then there is a question of how close the boundary of the intersection region $E$ in figure 4 is to being future trapped. The arguments given for three bubbles can be generalised to show that the symmetrical null generators of this surface are converging for large bubble separations. Of course, the surface will never be converging at the (one dimensional) corners because of lack of smoothness. This raises two interesting questions. If the surface where trapped, then would it be possible to generalise the singularity theorems to conclude that the singularity is generic, given that the strong energy condition only holds to the future of the first collision? What is the effect of replacing a smooth trapped surface with a piecewise smooth surface?

ACKNOWLEDGMENTS

I am grateful to Chris Chambers for help with some of the calculations.
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FIGURES

FIG. 1. Penrose diagram of the pseudo-Schwartzchild spacetime with $M > 0$. The $s$ coordinate increases from bottom to top.

FIG. 2. Two intersecting null hypersurfaces.

FIG. 3. Refraction of a light ray with momentum $p$ by two bubbles.

FIG. 4. The collision of three bubbles and a ray from $X_3$ that passes through the bubbles symmetrically.