Axioms for Automated Market Makers: A Mathematical Framework in FinTech and Decentralized Finance

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Abstract

Within this work we consider an axiomatic framework for Automated Market Makers (AMMs). By imposing reasonable axioms on the underlying utility function, we are able to characterize the properties of the swap size of the assets and of the resulting pricing oracle. In providing these general AMM axioms, we define a novel measure of price impacts that can be used to quantify those costs between different constructions. We have analyzed many existing AMMs and shown that the vast majority of them satisfy our axioms. We have also considered the question of fees and divergence loss. In doing so, we have proposed a new fee structure so as to make the AMM indifferent to transaction splitting. Finally, we have proposed a novel AMM that has nice analytical properties and provides a large range over which there is no divergence loss.

Keywords: Decentralized Finance, FinTech, Decentralized Exchange, Automated Market Makers, Divergence Loss, Blockchain.

1 Introduction

Decentralized Finance (DeFi) is a new paradigm for finance which replaces traditional intermediaries with innovative financial technologies based on blockchain. DeFi companies are providing services in areas such as lending and borrowing, insurance underwriting, and trading without working with the traditional financial intermediaries. Within this work, we focus on the use of DeFi
for constructing markets to trade financial instruments; specifically, Automated Market Makers (AMMs) are a decentralized approach for creating financial markets. The key idea of an AMM is to create a (liquidity) pool of assets against which a trader can transact; a swap is executed on an AMM by balancing the reserves according to mathematical formulas. A key benefit of DeFi and AMMs is that any individual can pool resources into an AMM so that he or she can share in the benefits from providing these key services in the financial system.

The organization of this paper is as follows. The motivation for studying DeFi and, more specifically, AMMs is provided within Section 1.1. A review of the literature on AMMs is provided within Section 1.2. The introduction is then concluded by an overview of the primary contributions and results of this work in Section 1.3. In Section 2, the desirable economic and financial properties for markets made by AMMs are provided. The mathematical construction of a generic AMM is presented within Section 3. With that discussion, the properties of the realized swap value and the pricing oracle derived from the AMM are provided and compared with those desirable properties in Section 2. In Section 4 these axioms and properties are then validated against AMMs that exist in practice as well as a new mathematical structure that can be used for generating new AMMs. Finally, in Section 5 we present a discussion on how AMMs can collect fees along with the possible risks that are involved with becoming a liquidity provider (i.e., the divergence or impermanent loss). We summarize and conclude in Section 6. The proofs of all results are provided within Appendix B.

1.1 Motivation

At the most recent peak for cryptocurrency valuation on Nov 8, 2021, the total market capitalization of various DeFi projects was nearly $180B. Since then, and especially after the onset of the so-called “crypto winter” in May 2022, the risk of various DeFi projects has come into full view of the public. To highlight just a few DeFi failures: the demise of the algorithmic stablecoin TerraUSD – which was triggered by a run on the Terra Luna coin – wiped out billions of dollars in wealth in a single week; separately, the DeFi lending platform Celsius Networks, which had approximately $12B in assets under management prior to the crypto winter, filed for bankruptcy in July 2022. With the onset of the crypto winter and the crash in cryptocurrencies, the total value locked in to

1 https://tradingview.com/markets/cryptocurrencies/global-charts/
AMMs has dropped precipitously as well with, e.g., Curve falling from a market capitalization of almost $3B in early January 2022 to just over $300M in June 2022. With the irrational exuberance subsiding, now is the perfect time to explore the viability and risks associated with DeFi projects. Specifically, for the purposes of this work, we wish to quantify the risks, and understand the pitfalls, of AMMs while highlighting the benefits for investors and liquidity providers.

One beneficial aspect of AMMs, and a defining property of it, is the decentralization. While much has been written on the decentralization of transaction verification through blockchain, AMMs also allow for decentralization in rewards. This is because anybody can become a liquidity provider in, and therefore a shareholder of, the AMM. Due to this democratization of market making, AMMs lower the barriers for listing new securities (or tokens) considerably.

Of course, with any type of investment also comes risk; as such, understanding the downsides for liquidity providers (i.e., investors in an AMM) is vital. We tackle this problem by formalizing the axioms that AMMs operate under. For example, we prove that the typical constructions of AMMs in practice (without fees) always have a divergence loss – a detrimental result for liquidity providers. In fact [28] states “this “loss” only disappears when the current proportions of the pool assets equal exactly those at liquidity provision, which is rarely the case.” As such, fees are necessary to provide a cushion for the investors against this loss and, thereby, to encourage investment in AMMs; however, not every fee structure works well. For example, an AMM can charge fees on any of its quoted assets either before or after the verification of a trade. While any combination of fees can intuitively work, some fee structures can lead to, e.g., the AMM charging smaller fees for bulk trades. As far as we are aware, the implication of fees on optimal investor behavior has not previously been studied.

Due to these (and other) conceptual benefits and risks for AMMs, in this work we postulate (intuitive) axioms for AMMs to follow and deduce their implications for traders and liquidity providers. Providing such a mathematical foundation for AMMs permits an exploration of the fundamental properties of these DeFi products; this includes both investigating real-world AMMs and proposing new AMMs that satisfy the appropriate axioms. As touched on above, studying these AMMs mathematically allows us to consider the implications of different fee structures and, importantly, propose a new fee structure for AMMs which is ambivalent to trade execution (i.e.,

https://coinmarketcap.com/currencies/curve-dao-token/
between trading in bulk or splitting a transaction).

1.2 Literature review

AMMs for cryptocurrencies and other digital assets have existed since, at least, 2018 with the launch of Uniswap. As summarized by the defining whitepapers [29, 2], this initial AMM follows a constant product rule for swapping assets. Briefly, and explained more in depth within Section 4.1 and Appendix A.1 if the pool holds two types of assets then the product of the reserves of those assets must be the same before and after any swap is realized. This notion of keeping the value of a function (e.g., the product) of asset reserves invariant to swaps was later generalized into the idea of constant function market making.

The basic construction of constant function market makers was formalized in [7]. These structures were studied for use with making foreign exchange markets for digital assets in [23]. Typically for simplicity, and as is taken within this work, these AMMs are presented for markets with two assets only. In [6], trading in multi-asset AMM structures was considered. Alternatively, trading against multiple AMMs was presented within [18]. [11] provides a generalized structure for the interactions that can occur between investors and the AMM. We refer the interested reader to [28] for a summary of terminology and structures that are currently used in practice in this field. [9] studies specific widely-used AMM structures to investigate the implications that this parameterized AMM has on price stability to determine scenarios in which different structures may be most appropriate.

The profitability of AMMs in an economic framework with different investor classes was studied in [12]. The risks, and the appropriate hedging of those risks, for specific AMM structures have been studied in numerous works; we highlight the study of the divergence loss of Uniswap V2 [3] and Uniswap V3 [17]. (These constant product market makers were studied in a number of other works as well, e.g., [15, 5].) The costs of being a liquidity provider were further analyzed within [24]. Conversely, [10] proposes a method to construct AMMs which replicate the payoff of a financial derivative allowing for the study of derivative pricing to inform AMM valuation.

In a separate context, AMMs for prediction markets were first proposed in [21]. Such a structure is fundamentally different from the constant function market makers considered herein insofar as a prediction market includes a terminal time at which bets are realized. As summarized above,
constant function market making has no terminal time but rather is focused merely on the spot market between two (or more) assets.

Though most prior works on AMMs have focused on specific market structures such as the constant product market maker of Uniswap V2, select literature has considered the generalized AMM construction from a mathematical perspective. For instance, [12] introduces specific sufficient conditions for the constant function construction for the results provided in that work. Similarly to this work, [19, 25] explicitly propose axioms for AMMs as well though the former only loosely imposes conditions and the latter focuses on the relation to prediction markets. Additionally, [8] provides geometric axioms for AMMs in a similar vein to those taken herein. We refer the interested reader to Appendix C for a detailed overview of the axioms assumed in these other works and how they compare with the results presented herein.

1.3 Primary contributions

In light of the aforementioned growth and contraction of DeFi, and keeping the specific motivations provided within Section 1.1 in mind, the primary contributions of this paper are as follows.

• We construct an axiomatic definition for AMMs and characterize the economic implications of these various axioms on the swap amount and pricing oracle within Section 3. As highlighted in Section 1.2, prior studies of AMMs have imposed strict requirements on the structure of the AMM for mathematical simplicity. As discussed in Section 4.1, some of these strict mathematical structures are not satisfied by many widely used AMMs, whereas the axioms proposed herein are satisfied.

• Beyond providing axioms and properties for AMMs, we consider (as far as the authors are aware) a novel fee structure for an AMM in Section 5. This framework, e.g., imposes a logical indifference to trade execution which is lost by other constructions presented within the literature. Along with the construction of fees, we provide a mathematical description and consideration of the divergence loss – also called impermanent loss – which defines the risk a liquidity provider assumes by pooling with the AMM.

• Of particular use for practitioners, we mathematically characterize widely-used real-world AMMs in Section 4 and expanded upon in Appendix A. By characterizing these
AMMs, we are able to demonstrate which axioms are satisfied and violated by these structures.

We then propose some new AMM constructions which extend real-world AMM constructions in novel ways satisfying the desired mathematical and financial properties discussed within this work.

2 Market design

When creating a new financial market, there are certain properties that are desirable for the efficient use of liquidity. As the purpose of this work is to study markets made by AMMs, we will first review the financial properties we deem desirable for these novel markets. Specifically, we consider properties of two related constructs: the quoted price and the execution price. As far as the authors are aware, no other work has compiled as complete a list of properties for markets to satisfy.

In order to consider these constructs, we need to introduce a few notions of AMMs which will be more formally defined below. AMMs function by holding reserves of liquidity in all of the assets that it trades. One of these assets, typically, acts as the numéraire ("cash") against which all prices are quoted. The AMM uses these reserves in order to execute swaps desired by external traders. The AMM quotes prices on these assets in such a way so as to be consistent with the swaps that it permits, i.e., via the cost of a marginal swap.

With these ideas, first, we wish to consider two potential properties on the quoted price. These relate to the dependence on liquidity and the ability to quote any price. These two properties can be formally defined via:

(PML) **Monotone in liquidity**: As the liquidity of a single asset increases, while that of all others remains unchanged, its price drops. Conversely, as the liquidity of only the numéraire asset increases, the price of all other assets grow.

(AP) **Surjective in price**: Every positive price can be obtained from trading.

Second, in regards to the execution price (i.e., the value of a swap), there are numerous important properties, chief among which is a no arbitrage argument. In addition to that, we want, e.g., to guarantee liquidity is available in the market, that all liquidity is usable by the market, and the AMM is indifferent to how a large trade is split over time. We want to fully consider the
following properties on the cost of buying (respectively selling) assets for the numéraire:

(NA) **No arbitrage**: Buying and, immediately, selling (resp. selling and, immediately, purchasing) assets results in no more value at the end than at the beginning of the transaction.

(0) **0 cost of 0**: No amount of assets are recovered when no cash is paid (resp. no cash is paid when 0 assets are sold).

(IL) **Infinite liquidity**: The market can act as a counterparty for any trade, i.e., assets are always available for purchase or sale.

(I) **Strictly increasing**: More assets are purchased when a larger price is paid (resp. more cash is recovered when more assets are sold).

(NW) **No wasted liquidity**: The entire market liquidity can be purchased for an infinite cost (resp. recovered from an infinite-sized sale).

(CC) **Concave and continuous**: The marginal number of assets recovered from a purchase is nonincreasing (resp. the marginal amount of value recovered from selling assets is nonincreasing).

(SA) **Subadditive**: The number of assets purchased in a single large trade is no larger than sum total from placing multiple small trades (that sum to the large transaction) when all such trades are assessed at the current market snapshot, i.e., current asset reserve levels (resp. the total cash recovered from a large liquidation is bounded by that which is recovered from small transactions).

(PI) **Path independent**: The number of assets purchased in a single large trade is equal to the sum total from placing multiple small trades in rapid succession (that sum to the large transaction) (resp. the total cash recovered from a large liquidation is bounded by that which is recovered from small transactions).

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3 This property was independently defined in [20] as BoundedReserves. We also wish to note that this is similar to Liquidation as defined therein except that that property assumes that the market maker can pay out infinite amounts of any asset for a high enough price.

4 This property was independently defined in [20] as NoDominatedTrades.

5 This property is similar to DemandResponsiveness in [20].

6 This property was independently defined in [20] as PathIndependence.
The (PI) property differs from (SA) in that (SA) requires the market conditions to be frozen at the current levels, whereas the market conditions evolve as trades are placed for (PI).

(ML) **Monotone in liquidity:** The number of assets that can be purchased for a fixed cost is nonincreasing in the size of AMM’s numéraire reserves (resp. the amount of cash recovered from selling a fixed number of assets is nonincreasing in the size of the asset reserves. Similarly, the number of assets that can be purchased for a fixed cost is nondecreasing w.r.t. asset reserves (resp. the cash recovered from selling a fixed number of assets is nondecreasing in the size of the numéraire reserves).

Finally, in addition to the properties introduced for the quoted and execution prices, we want to directly consider the impacts of increasing AMM reserves on market liquidity. This property is encoded within:

(PL) **Pooling increases liquidity:** Larger asset reserves, starting from the same initial price, lead to lower execution costs for buying and a higher recovery value for selling assets with the AMM.

In the following section, we will formally introduce the mathematical structure utilized by AMMs and introduce a number of axioms on that framework in order to guarantee the aforementioned properties. The relation between these axioms and desired properties is summarized within Table 1 (with axioms defined in Definition 3.1). A simple review of Table 1 provides a clear indication that some properties are unnecessary to guarantee any of these desirable market properties and, thus, may not be needed in practice.

## 3 Constant function market maker

Constant function market makers are the most prevalent AMMs that exist in practice; for instance, the most prominent AMM – Uniswap – is a constant function market maker. Due to this prominence, we will often equate the concepts of AMMs and constant function market makers within this work. Fundamentally, a constant function market maker is a multivariate utility function which codifies the value placed on a portfolio by the liquidity providers. This utility function has the dual task of providing liquidity to the market via swaps and acting as a pricing oracle to quote spot
Table 1: Summary of main results and necessary axioms. (1) and (2) denote alternative conditions where either all axioms in (1) must be satisfied or axiom (2) is satisfied.

| Prop. | Details | \([UfB]\) | \([UfA]\) | \([SM]\) | \((C)\) | \((QC)\) | \((SI)\) | \((I+)\) | \((SC)\) | \((3.3)\) | \((3.5)\) | \((3.6)\) |
|-------|---------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| [NA]  | L. 3.10 | X         | X         |           |           |           |           |           |           |           |           |           |
| [T]   | T. 3.7(a) |           |           |           |           |           |           |           |           |           |           |           |
| [IL]  | T. 3.7(b) | X         |           |           |           |           |           |           |           |           |           |           |
| [I]   | T. 3.7(c) | X         | X         |           |           |           |           |           |           |           |           |           |
| [NW]  | T. 3.7(d) |           | X         | X         |           |           |           |           |           |           |           |           |
| [CC]  | T. 3.7(e) |           |           |           | (1)       | (1)       | (2)       |           |           |           |           |           |
| [SA]  | T. 3.7(f) |           |           |           | (1)       | (1)       | (1)       | (2)       |           |           |           |           |
| [PI]  | T. 3.7(g) | X         |           | X         |           |           |           |           |           |           |           |           |
| [ML]  | T. 3.9(a) |           |           |           |           |           |           |           |           |           |           |           |
| [PML] | P. 3.11(1) | X         |           |           |           |           |           |           |           |           |           |           |
| [AP]  | P. 3.11(2)-(3) | X         | X         | X         | X         |           |           |           |           |           |           |           |
| [PL]  | T. 3.12   | X         |           | X         |           | X         | X         |           |           |           |           |           |

prices on the assets. The construction and axioms of such a utility function are provided within Section 3.1. For simplicity, we will assume the AMM only permits swaps between two assets \(A\) and \(B\), although similar axioms can be developed for multi-asset swaps as well. Simply put, and as expressed within e.g. [7], a trader exchanging quantity \(x\) of asset \(A\) with a constant function market maker \(u\) receives quantity \(Y(x)\) of asset \(B\) such that \(u(a + x, b - Y(x)) = u(a, b)\) where the market maker has initial reserves \((a, b)\) in the two assets; this is explored within Section 3.2. The pricing oracle provides exactly the swap amounts for a marginally small trade, i.e., \(Y'(0)\); this is explored in depth within Section 3.3. Recall that the proofs for all results are provided within Appendix B.

### 3.1 Axioms

As expressed in the introduction of this section, an AMM is a multivariate (herein always taken to be bivariate) utility function which codifies the value placed on a portfolio by the liquidity providers. This utility function defines the price impacts of swaps and acts as a pricing oracle to quote spot prices on the assets. Within the following definition, we encode the axioms on these AMMs we impose at various points within this work. To correspond more closely to the typical utility theory (see, e.g., [27]), we note that the functions provided for AMMs in practice (see Section 4.1 and Appendix A) are often characterized as the exponential of the formulations given herein. We will clarify this point in Example 3.2 below. To simplify notation, throughout we use subscripts to denote partial derivatives; in line with the nomenclature for our two assets, subscript \(A\) denotes
the partial derivative w.r.t. the first input and $B$ w.r.t. the second input.

**Definition 3.1.** An AMM is a utility function $u : \mathbb{R}^2_+ \to \mathbb{R} \cup \{-\infty\}$ that may satisfy the following properties:

(UfB) **Unbounded from below:** $u(x, 0) = u(0, y) = -\infty$ for every $x, y \geq 0$ and $u(z) > -\infty$ for every $z \in \mathbb{R}^2_{++}$.

(UfA) **Unbounded from above:** $\lim_{\bar{x} \to \infty} u(\bar{x}, y) = \lim_{\bar{y} \to \infty} u(x, \bar{y}) = \infty$ for every $x, y > 0$.

(SM) **Strictly monotonic:** $u(z) > u(\bar{z})$ if $z - \bar{z} \in \mathbb{R}^2_{++}$.

(C) **Continuous:** $u$ is continuous.

(QC) **Quasiconcave:** $u$ is quasiconcave.

(SI) **Scale invariant:** if $u(z) \geq u(\bar{z})$ for $z, \bar{z} \in \mathbb{R}^2_{++}$ then $u(tz) \geq u(t\bar{z})$ for any $t \geq 0$.

(I+) **Inada+:** $u$ is differentiable with:

- $\lim_{\bar{x} \to \infty} u_A(\bar{x}, y) = \lim_{\bar{y} \to \infty} u_B(x, \bar{y}) = 0$,
- $\lim_{\bar{x} \to \infty} u_B(\bar{x}, y) \in (0, \infty)$, $\lim_{\bar{y} \to \infty} u_A(x, \bar{y}) \in (0, \infty)$,
- $\lim_{\bar{x} \to 0} u_A(\bar{x}, y) = \lim_{\bar{y} \to 0} u_B(x, \bar{y}) = \infty$,
- $\lim_{\bar{x} \to 0} u_B(\bar{x}, y) < \infty$, $\lim_{\bar{y} \to 0} u_A(x, \bar{y}) < \infty$

for every $x, y > 0$.

(SC) **Single crossing:** $u$ is twice continuously differentiable with $u_A(z), u_B(z) > 0$, $u_B(z)u_{AA}(z) \leq u_A(z)u_{AB}(z)$, and $u_A(z)u_{BB}(z) \leq u_B(z)u_{AB}(z)$ for every $z \in \mathbb{R}^2_{++}$.

**Example 3.2.** The most commonly reported AMM is the constant product market maker with function $F(x, y) = xy$ which is utilized by, e.g., Uniswap V2. As noted above, to correspond more closely with the literature on utility functions, herein we formalize Uniswap V2 as the logarithmic utility $u(x, y) := \log(F(x, y)) = \log(x) + \log(y)$ instead. We expand on this AMM in Section 4.1 and Appendix A.1. Notably, and as can be trivially verified, the logarithmic utility function satisfies all axioms proposed within Definition 3.1.
Remark 1. We wish to note that some of the axioms of Definition 3.1 can imply others. For instance, \((SC)\) implies \((SM)\) and \((C)\) and, additionally, \((QC)\) if at least one of the defining inequalities is strict; this condition is further implied by \((SM)\) and a monotone differences property \((u_{AA}, u_{BB} \leq 0 \text{ and } u_{AB} \geq 0)\) that is explicitly given in Assumption 1 of \([12]\) for AMMs. Within our discussion of some real-world AMMs (see Section 4.1), we note that this monotone difference condition is not satisfied in many cases. However, the relevant mathematical properties from monotone differences of an AMM – as shown in \([12]\) – appear to be satisfied under the weaker \((SC)\) condition proposed herein. A full description of the axioms used within \([12]\) (as well as other works) and their relation to those provided within Definition 3.1 is provided in Appendix C.

Remark 2. We refer to condition \((I^+)\) as the Inada+ condition as the usual Inada conditions (i.e., \(\lim_{\bar{x} \to 0} u_A(\bar{x}, y) = \lim_{\bar{y} \to 0} u_B(x, \bar{y}) = \infty\) and \(\lim_{\bar{x} \to \infty} u_A(\bar{x}, y) = \lim_{\bar{y} \to \infty} u_B(x, \bar{y}) = 0\) for any \(x, y > 0\) ) are satisfied; in addition to these properties, \((I^+)\) also provides the strict monotonicity of \(u(x, y)\) in \(y\) as \(x \to \infty\) (and vice versa for monotonicity of \(x\) as \(y \to \infty\) ) as well as behavior of the derivatives at 0.

The first task of an AMM, encoded by the utility function \(u : \mathbb{R}_+^2 \to \mathbb{R} \cup \{-\infty\}\), is to act as a liquidity provider to facilitate swaps between the two assets.

Definition 3.3. Let \(u : \mathbb{R}_+^2 \to \mathbb{R} \cup \{-\infty\}\) be an AMM. Let \((a, b) \in \mathbb{R}_+^2\) be the size of the reserves of asset A and B within the AMM respectively. The swap values \(\mathcal{Y} : \mathbb{R}_+^3 \to \mathbb{R}_+\) and \(\mathcal{X} : \mathbb{R}_+^3 \to \mathbb{R}_+\) are defined for any \(x, y \geq 0\) as:

\[
\mathcal{Y}(x; a, b) := \sup\{y \in [0, b] \mid u(a + x, b - y) \geq u(a, b)\},
\]

\[
\mathcal{X}(y; a, b) := \sup\{x \in [0, a] \mid u(a - x, b + y) \geq u(a, b)\}.
\]

Where clear, we will drop the explicit dependence of \(\mathcal{Y}, \mathcal{X}\) on the pool sizes \((a, b) \in \mathbb{R}_+^2\).

Remark 3. Though Definition 3.3 differs slightly from the typical form of a constant function market maker (see, e.g., \([7]\) ) in that the “constant function” can be an inequality here. Provided the equality is attained (see, e.g., the conditions in Theorem 3.7(2) below), i.e., \(u(a + x, b - \mathcal{Y}(x)) = u(a, b)\), we can recover a notion of the indifference price for the quantity \(x\) (see, e.g., \([13]\) for AMMs).
Example 3.4. Consider again the Uniswap V2 logarithmic utility function introduced within Example 3.2. The resulting swap functions \( \mathcal{Y}, \mathcal{X} : \mathbb{R}_+^3 \to \mathbb{R}_+ \) have analytical forms 

\[
\mathcal{Y}(x; a, b) = \frac{bx}{a + x} \quad \text{and} \quad \mathcal{X}(y; a, b) = \frac{ay}{b + y}
\]

for any \( x, y \geq 0 \) and \( a, b > 0 \).

The second task of an AMM, encoded by the utility function \( u : \mathbb{R}_+^2 \to \mathbb{R} \cup \{-\infty\} \), is to act as a pricing oracle \( P : \mathbb{R}_+^2 \to \mathbb{R}_+ \). Throughout this work we will consider the price \( P \) of asset \( A \) denominated in units of asset \( B \). For \( B \) in terms of \( A \), the reciprocal \( 1/P \) is taken instead; further considerations of the change of numéraire and a bid-ask spread are presented within Sections 3.3 and 3.5. For the purposes of the definition of the pricing oracle, we will assume that the swap function \( \mathcal{Y} \) is differentiable at \((0; a, b)\); we will use the above axioms on the utility \( u \) to guarantee this derivative exists within Section 3.2 below.

Definition 3.5. The pricing oracle \( P : \mathbb{R}_+^2 \to \mathbb{R}_+ \) provides the marginal units of asset \( B \) obtained from selling a marginal number of units of asset \( A \), i.e., \( P(a, b) := \mathcal{Y}'(0; a, b) \) for any \( a, b > 0 \).

Example 3.6. Consider again the Uniswap V2 logarithmic utility function introduced within Example 3.2. The resulting pricing oracle \( P : \mathbb{R}_+^2 \to \mathbb{R}_+ \) is provided by the ratio of the reserves, i.e., \( P(a, b) = b/a \) for any \( a, b > 0 \).

3.2 Swaps

Herein we wish to study the swap values \( \mathcal{Y}, \mathcal{X} \) when transferring between the assets using an AMM. To reduce redundancy, throughout this section, we will focus on the properties of the swap of units of \( A \) for \( B \) only, i.e., properties of \( \mathcal{Y} \) as given in Definition 3.3. By symmetry of the AMM, comparable results can be provided for the swap of units of \( B \) for \( A \) (i.e., \( \mathcal{X} \)). We will, specifically, investigate the implications of the axioms of AMMs on: (i) the impact of the swap amount \( x \) (Theorem 3.7); (ii) the impact of the pool size \( a, b \) (Theorem 3.9); and (iii) a no round-trip arbitrage condition (Lemma 3.10).

Theorem 3.7. Consider an AMM \( u : \mathbb{R}_+^2 \to \mathbb{R} \cup \{-\infty\} \). Fix the initial AMM reserves \( a, b > 0 \) and swap amounts \( x, x_1, x_2 \geq 0 \).
1. If \([C]\) then \(u(a + x, b - Y(x)) \geq u(a, b)\), i.e., market utility never drops.

2. If \([C]\) and \(Y(x) \neq b\) then \(u(a + x, b - Y(x)) = u(a, b)\), i.e., the constant function market maker structure is satisfied.

3. If \([UfB]\) and \([C]\) then \(Y(x) < b\), i.e., \([IL]\).

4. If \([SM]\) then \(Y(0) = 0\), i.e., \([0]\).

5. If \([SM]\) then \(Y\) is nondecreasing in \(x\). If, additionally, \([UfB]\) and \([C]\) then \(Y\) is strictly increasing in \(x\), i.e., \([I]\).

6. If \([UfA], [SM]\) and \([C]\) then \(\lim_{x \to \infty} Y(x) = b\), i.e., \([NW]\).

7. If \([C]\) then \(Y\) is upper semicontinuous.

8. If \([C]\) and \([QC]\) or \([SC]\) then \(Y\) is concave, continuous and a.e. differentiable, i.e., \([CC]\).

9. If \([C]\) and \([SI]\) then \(Y\) is positive homogeneous in \((x; a, b)\), i.e., \(Y(tx; ta, tb) = tY(x; a, b)\) for any \(t > 0\).

10. If \([SM], [C]\) and \([QC]\) or \([SC]\) then \(Y\) is subadditive, i.e., \([SA]\).

11. If \([UfB], [SM]\) and \([C]\) then \(Y(x_1 + x_2; a, b) = Y(x_1; a, b) + Y(x_2; a + x_1, b - Y(x_1; a, b))\), i.e., \([PI]\).

Remark 4. Theorem 3.7(11) implies an investor is indifferent to trade execution, i.e., splitting a transaction over time between completing it in one trade or splitting it into smaller trades. However, as will be discussed in Remark 15 below, naively assessing fees to transactions can result in this equality no longer holding in general. Theorem 3.7(10) implies splitting a transaction with AMM recovery receives a higher value.

Remark 5. Assume \([C]\) and \([QC]\) or \([SC]\). It follows from Theorem 3.7(8) that \(Y\) and \(X\) are concave. Therefore, sub-differentials of \(Y\) and \(X\) are always well defined, and will a.s. coincide with the derivative. This allows us to give the following definition.

Definition 3.8. Denote \(Y'(x; a, b) := \lim_{\varepsilon \downarrow 0} \frac{Y(x + \varepsilon; a, b) - Y(x; a, b)}{\varepsilon}\), and similarly for \(X'(x; a, b)\) if they exist. If the AMM is differentiable (with nonzero partial derivatives) then \(Y'(x; a, b) = \frac{u_A(a + x, b - Y(x; a, b))}{u_B(a + x, b - Y(x; a, b))}\).
by implicit differentiation (similarly for $X'(x; a, b)$). Furthermore, assuming differentiability of the AMM, denote $Y_a(x; a, b) = \lim_{\varepsilon \to 0} \frac{Y(x; a + \varepsilon, b) - Y(x; a, b)}{\varepsilon}$, and similarly for $Y_b(x; a, b), X_a(x; a, b), X_b(x; a, b)$, which can all be formulated via implicit differentiation as well.

Within the next theorem, we want to study the implications of changing the pool sizes $a, b$ for the AMM. Specifically, how altering the pool composition changes the value of a swap and the limiting behavior as pool sizes shrink to 0 or grow to infinity.

**Theorem 3.9.** Assume $\text{(UfB)}$ and $\text{(SC)}$

1. $Y_a(x; a, b) \leq 0$ and $Y_b(x; a, b) \in [0, 1)$ for any $x \geq 0$ and $a, b > 0$, i.e., $\text{(ML)}$.

2. $\lim_{\bar{a} \to 0} Y(x; \bar{a}, b) = b$ and $\lim_{\bar{b} \to 0} Y(x; a, \bar{b}) = 0$ for every $a, b, x > 0$ with the latter limit converging uniformly in $a, x$.

3. If additionally $\text{(QC)}$ and $\text{(I+)}$ then $\lim_{\bar{a} \uparrow \infty} Y(x; \bar{a}, b) = 0$ and $\lim_{\bar{b} \uparrow \infty} Y(x; a, \bar{b}) = \infty$ for every $a, b, x > 0$.

Though distinct, jointly (2) and (3) are related to $\text{(AP)}$ in that the value of the assets can vary between 0 and $\infty$ as the reserves hit their own extreme values.

**Remark 6.** Theorem 3.9 provides the interpretation of $\text{(SC)}$ as providing a single crossing condition for both $f^A((x, y), \delta) := u(a + \delta + x, b - y)$ and $f^B((x, y), \delta) := u(a + x, b + \delta - y)$ for any $a, b > 0$. Specifically, for any feasible $(x', y') \geq (x'', y'')$ and $\delta' \geq \delta''$,

\[
\begin{align*}
  f^A((x', y'), \delta') \geq f^A((x'', y''), \delta'') & \Rightarrow f^A((x', y'), \delta'') \geq f^A((x'', y''), \delta'') \\
  f^B((x', y'), \delta'') \geq f^B((x'', y''), \delta'') & \Rightarrow f^B((x', y'), \delta') \geq f^B((x'', y''), \delta').
\end{align*}
\]

Similarly, the results can be shown for subtracting $x$ and adding $y$.

We conclude our discussion of the properties of swaps without fees by considering round-trip arbitrage opportunities. That is, to study if swapping $A$ to $B$ and then back to $A$ (or vice versa of $B$ to $A$ to $B$) ever provides risk-free profits. In particular, as found in Lemma 3.10 below, a round-trip swap will always leave the trader back with (at most) their initial portfolio. We study the implications of fees within Section 5 below.
Lemma 3.10. If \((SM)\) and \((C)\) then \((NA)\) i.e., \(X(Y(x;a,b);a + x, b - Y(x;a,b)) \leq x\) and \(Y(x;a,b);a - X(y;a,b), b + y \leq y\) for any \(x, y \geq 0\) and \(a, b > 0\). If, additionally, \((UfB)\) then these inequalities hold as equalities for every \(x, y \geq 0\) and \(a, b > 0\).

3.3 Pooling and the pricing oracle

Recall from Definition 3.5 that the pricing oracle \(P : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+\) provides the marginal units of asset \(B\) obtained from selling a marginal number of units of asset \(A\), i.e., \(P(a, b) := Y'(0; a, b)\) for any \(a, b > 0\).

Remark 7. Following Lemma 3.10, if \((SM)\) and the AMM \(u\) is differentiable then there does not exist a bid-ask spread in the pricing oracle for strictly positive reserves, i.e., \(P(a, b) = Y'(0; a, b) = \frac{1}{X'(0; a, b)}\) for any \(a, b > 0\).

Proposition 3.11. Assume \((SC)\)

1. The pricing oracle \(P\) is differentiable with \(P_A \leq 0\) and \(P_B \geq 0\), i.e., \((PML)\)

2. If, additionally, \((UfB), (QC)\) and \((I+)\) then for any \(a > 0\), \(b \mapsto P(a, b)\) is nondecreasing and surjective on \(\mathbb{R}_+\).

3. If, additionally, \((UfB), (QC)\) and \((I+)\) then for any \(b > 0\), \(a \mapsto P(a, b)\) is nonincreasing and surjective on \(\mathbb{R}_+\).

4. If, additionally, \((SI)\) then \(P\) is scale invariant, i.e., \(P(ta, tb) = P(a, b)\) for any reserves \(a, b > 0\) with scaling \(t > 0\).

Jointly (2) and (3) imply \((AP)\)

As opposed to more traditional market makers, an AMM allows investors to join the liquidity pool and capture a fraction of the profits (see Section 5 below for a discussion of fee structures). As presented in, e.g., [6], pooling should be accomplished so that the price is unaffected by the injection of additional liquidity into the market. That is, adding \((\alpha, \beta) \in \mathbb{R}_+^2 \setminus \{0\}\) to a pool of size \((a, b) \in \mathbb{R}_+^2\) so that \(P(a + \alpha, b + \beta) = P(a, b)\). Within the following theorem, we consider a sufficient condition on the pricing oracle so that increasing the size of the reserves actually increases
market liquidity (PL), i.e., pooling \((\alpha, \beta) \in \mathbb{R}^2_+ \setminus \{0\}\) results in

\[
\mathcal{V}(x; a + \alpha, b + \beta) \geq \mathcal{V}(x; a, b), \quad \forall x \geq 0, \quad (3.1)
\]

\[
\mathcal{X}(y; a + \alpha, b + \beta) \geq \mathcal{X}(y; a, b), \quad \forall y \geq 0. \quad (3.2)
\]

**Theorem 3.12.** Consider a thrice continuously differentiable AMM \(u : \mathbb{R}^2_+ \to \mathbb{R} \cup \{-\infty\}\), if (UFB), (QC), (I+), (SC), and

\[
P_B(z)P_{AA}(z) - (P(z)P_B(z) + P_A(z))P_{AB}(z) + P(z)P_A(z)P_{BB}(z) \geq 0, \quad \forall z \in \mathbb{R}^2_+ \quad (3.3)
\]

then (PL) pooling (i.e., injecting \((\alpha, \beta) \in \mathbb{R}^2_+ \setminus \{0\}\) to the initial pool \((a, b) \in \mathbb{R}^2_+\) such that \(P(a + \alpha, b + \beta) = P(a, b)\) increases market liquidity as defined in (3.1) and (3.2).

**Remark 8.** Maintaining a constant price is a necessary condition for increased liquidity as \(P(a, b) = \mathcal{V}'(0; a, b) = \frac{1}{\mathcal{X}'(0; a,b)}\). Under (SI), this reduces to the simpler “proportional to current reserves” rule-set that is often assumed instead (see, e.g., [12]).

**Remark 9.** As far as the authors are aware, there is no meaningful financial condition that implies (3.3). This is merely a mathematical argument on the pricing oracle \(P\) that provides sufficient conditions for the desired increasing liquidity property of (3.1) and (3.2). As highlighted in Section 4 below, all AMMs studied within this work satisfy this condition, though as demonstrated in Section 4.2 this curvature condition is independent of the other axioms on AMMs.

Inspired by Proposition 4.1 below, it can be shown that (3.3) follows from (SC) along with \(u_A, u_B\) log-convex (with \(u\) thrice differentiable) and \(u_{AB} \geq 0\); we wish to recall from Remark 4 that Assumption 1 of [12] imposes the second derivative condition \(u_{AB} \geq 0\) though this stronger condition is not satisfied for every AMM utilized in practice as highlighted in Section 4.1.

**Example 3.13.** Consider again the Uniswap V2 logarithmic utility function introduced within Example 3.2. As this utility is such that \(u_A(a, b) = 1/a\), \(u_B(a, b) = 1/b\) are log-convex and \(u_{AB} = 0\) is non-negative, as provided in Remark 4, Uniswap V2 satisfies (3.3). We wish to note that the product formulation of this AMM is such that \(F_A(a, b) = b\) and \(F_B(a, b) = a\) are not log-convex; even though (3.3) is invariant to monotonic transformations of the utility function, the use of the logarithmic transformation can assist in verifying these essential properties.
We conclude this section on pooling and the impacts of liquidity by quantifying a measure of price impacts. We consider this in three parts: first, we determine the effect that pooling has on price impacts from executing a swap; second, we define a (approximating) price impact oracle that only depends on the current AMM reserves; finally, we find conditions so that this price impact oracle can be used to bound the actualized price impacts from any swap. In studying this measure we are able to compare AMM constructions as well as measure the level of liquidity impacts.

Corollary 3.14. Define the price impacts for swapping assets \( A \) for \( B \) and vice versa, respectively, as:

\[
I_Y(x; a, b) = P(a, b)x - \mathcal{Y}(x; a, b), \quad I_X(y; a, b) = y/P(a, b) - \mathcal{X}(y; a, b), \quad (a, b) \in \mathbb{R}^2_+, \quad x, y \geq 0.
\]

Under the conditions of Theorem 3.12, pooling decreases the price impact from swapping. In other words, for any \( x, y \geq 0 \), pooling an additional liquidity \((\alpha, \beta) \in \mathbb{R}^2_+ \setminus \{0\}\) to the initial pool \((a, b) \in \mathbb{R}^2_+ \) such that \( P(a + \alpha, b + \beta) = P(a, b) \) decreases the price impacts

\[
I_Y(x; a, b) \geq I_Y(x; a + \alpha, b + \beta), \quad I_X(x; a, b) \geq I_X(x; a + \alpha, b + \beta). \tag{3.4}
\]

Corollary 3.14 introduce the new notations \( I_X, I_Y \) of price impact from swapping that quantifies the price impact of a swap. The key result shows that this impact decreases as liquidity is added to the pool. This is of course a very desirable conclusion—recall [PL] from the market design list—as this implies that the more liquid the market is, the closer to linear the price is even for large transactions. Note that the linear part \( P(a, b)x \) (and \( y/P(a, b) \) respectively) is invariant to additional liquidity.

Though \( I_X, I_Y \) capture the full price impacts from swapping, these functions explicitly depend on the size of the swap. We now wish to introduce a novel price impact oracle which is independent to the swap size \( x, y \).

Definition 3.15. The price impact oracle \( I : \mathbb{R}^2_+ \to \mathbb{R} \) provides (a multiplier of) the marginal change in the pricing oracle from selling a marginal number of units of asset \( A \), i.e., \( I(z) := \frac{1}{2}[P(z)P_B(z) - P_A(z)] \) for any \( z \in \mathbb{R}^2_+ \).
Proposition 3.16. Consider a thrice continuously differentiable AMM \( u : \mathbb{R}^2_+ \rightarrow \mathbb{R} \cup \{-\infty\} \). Fix the AMM reserves \((a, b) \in \mathbb{R}^2_+\), then the price impacts from swapping can be approximated via the price impact oracle for any \( x, y \geq 0 \) as:

\[
I_Y(x; a, b) = I(a, b)x^2 + O\left(\left(\frac{x}{a}\right)^3\right), \quad I_X(y; a, b) = \frac{I(a, b)}{(P(a, b))^3}y^2 + O\left(\left(\frac{y}{b}\right)^3\right).
\]

Furthermore, pooling decreases the price impact oracle (i.e., \( I(a + \alpha, b + \beta) \leq I(a, b) \)) when pooling \((\alpha, \beta) \in \mathbb{R}^2_+ \setminus \{0\}\) into the AMM with reserves \((a, b)\) if either the conditions of Theorem 3.12 or both \((SC)\) and \((SI)\) hold. Moreover, if \((SC)\) and \((SI)\) then the price impact oracle is positive homogeneous of degree \(-1\) (i.e., \( I(ta, tb) = t^{-1}I(a, b) \) for any \( t > 0 \)).

Proposition 3.16 formally presents how the price impact oracle approximates the actualized price impacts. Indeed, when empirically applied to a Uniswap pool, we find that this is a very close approximation as illustrated in Figure 1. Specifically, Figures 1a and 1b show that the price impacts \( I_Y(x; a, b), I_X(y; a, b) \) are (almost) entirely captured by the quadratic terms \( I(a, b)x^2, \frac{I(a, b)}{(P(a, b))^3}y^2 \), the coefficients of which are independent of the trade size and only depend on the current market liquidity. Furthermore, Figure 1c shows that even the relative errors for these quadratic approximations are empirically very small. Thus we conclude that \( I(a, b) \) is a highly effective measure for price impact.

Finally, to further quantify the accuracy of the price impact oracle, we further investigate its use as an explicit bound on the price impacts in the following lemma.

Corollary 3.17. Consider a thrice continuously differentiable AMM \( u : \mathbb{R}^2_+ \rightarrow \mathbb{R} \cup \{-\infty\} \). Assume \((SC)\) holds, and fix the AMM reserves \((a, b) \in \mathbb{R}^2_+\), then:

1. The pricing oracle bounds the swap amount, i.e., \( I_Y(x; a, b), I_X(y; a, b) \geq 0 \) for any \( x, y \geq 0 \). Additionally, the price impact oracle is nonnegative, i.e., \( I(a, b) \geq 0 \).

2. If, additionally, for every \( z \in \mathbb{R}^2_+ \),

\[
\Psi(z) := P_{AA}(z) - 2P(z)P_{AB}(z) + P(z)^2P_{BB}(z) + (P(z)P_B(z) - P_A(z))P_B(z) \geq 0, \quad (3.5)
\]

then \( I_Y(x; a, b) \leq I(a, b)x^2 \) for any \( x \geq 0 \).
Figure 1: Comparison of the approximation of the price impact and the true price impact using Uniswap data between March 23 and April 02, 2023 from a USDC/W ETH pool on the Polygon blockchain (smart contract 0x45dda9cb7c25131df268515131f647d726f50608).

3. If, additionally, for every $z \in \mathbb{R}^2_{++}$,

$$-P(z)\Psi(z) + 3(P(z)P_B(z) - P_A(z))^2 \geq 0,$$

(3.6)

then $I_{\chi}(y; a, b) \leq \frac{f(a, b)}{P(a, b)} y^2$ for any $y \geq 0$.

Remark 10. [9] introduced a definition for $\mu$-price stability so that no swap $(x, Y(x))$ moves the price by more than $\mu x$ with implicit dependence on the AMM reserves $(a, b) \in \mathbb{R}^2_{++}$. Any AMM satisfying [5,6] is, therefore, $I(a, b)$-price stable. Similarly, when considering swaps $(X(y), y)$, the
pricing oracle is \( \frac{I(a,b)}{P(a,b)} \) price stable if (3.6) holds.

As the above results show, the approximation of \( I_Y(x; a, b) \) via \( I(a,b)x^2 \) (respectively \( I_X(y; a, b) \) via \( I(a,b)/(P(a,b)^3)y^2 \)) is not only accurate empirically, but also is an upper bound on the price impact encountered by the trader and is entirely captured by the quadratic term \( I(a,b) \). For instance, if this term were to equal 0 then \( I_Y(x; a, b) = 0 \) via the results of Corollary 3.17, i.e., no price impacts would be experienced by the trader. Additionally, the quantity \( I(a,b) \) is independent of the trade sizes \( x, y \), which makes the approximation simpler and even more desirable. Therefore, we will use \( I(a,b) \) as a measure of price impact.

**Example 3.18.** Consider again the Uniswap V2 logarithmic utility function introduced within Example 3.2. As the pricing oracle for this AMM is given by \( P(a,b) = \frac{b}{a} \), and the price impact oracle \( I(a,b) = \frac{b}{a^2} \). We can directly check conditions (3.5) and (3.6). In particular, we find that

\[
0 \leq \frac{6z_2}{z_1^3} = \Psi(z) \quad \text{and} \quad 0 \leq \frac{6z_2^2}{z_1^3} = -P(z)\Psi(z) + 3(P(z)P_B(z) - P_A(z))^2
\]

for any \( z \in \mathbb{R}^2_+ \). Therefore, following the results of Corollary 3.17, \( I_Y(x; a, b) \in [0, \frac{bx^2}{a^2}] \) and \( I_X(y; a, b) \in [0, \frac{ay^2}{b^2}] \) for any \( x, y \geq 0 \) and \( (a, b) \in \mathbb{R}^2_+ \).

## 4 Examples of AMMs

Within this section we wish to explore a number of different AMMs. Within Section 4.1 we will summarize a number of AMMs that exist in practice; the detailed mathematical representations of these AMMs are provided within Appendix A. All such results have been empirically validated through reference to the applicable white papers and documentation. A summary of the axioms satisfied by these real-world AMMs is provided within Table 2. Within Section 4.2 we construct a novel generalized AMM structure; we use that structure to provide a new concentrated liquidity swap structure which has strong theoretical stability properties. We wish to remind the reader that many of the AMMs presented below, and detailed in Appendix A, are often characterized as the exponential of the utility functions provided herein; as the definition of the swap values \( X, Y \) are based on relative utilities, this exponentiation does not impact the implementation, only the description, of the AMM.
Table 2: Summary of popular AMMs and the axioms satisfied. †: Under some conditions specified in Proposition 4.1.
*: Axiom verified numerically.

4.1 AMMs in practice

Herein we summarize a number of AMMs that exist in practice. The details for all of these AMMs are provided within Appendix A. We also, refer the reader to Table 2 which summarizes the findings about all example AMMs considered within this work.

Fundamentally, most current AMMs are built based on two structures:

- **Uniswap V2**: As presented in the running example throughout Section 3 (beginning with Example 3.2) and Appendix A.1, Uniswap V2 is the AMM with logarithmic utility function, i.e., \( u(x, y) = \log(x) + \log(y) \). Notably, this simple structure satisfies all axioms presented within this work. However, Uniswap V2 is often criticized as it is subject to high price impacts as it tends to save a lot of liquidity solely for the tail of the price distribution (so as to guarantee infinite liquidity).

- **mStable**: As presented in Appendix A.4, mStable is an AMM that has no price impacts at all, i.e., \( u(x, y) = \log(x + y) \). In order to guarantee the constant price of mStable, this AMM fails to satisfy \((UfB)\) and \((I+)\).

By combining the notions of these AMMs in different ways, a liquidity provider is able to concentrate liquidity, i.e., lower the price impacts from trades, when the pool is “balanced” (i.e., when the asset pools are of comparable size). **StableSwap** \( u(x, y) = \log(C(x + y) + xy) \) for parameter \( C > 0 \); see Appendix A.5.1 considers a linear combination of Uniswap V2 and mStable within the logarithm. By doing so, there is less price impact than Uniswap V2 but at the expense of infinite liquidity.
and positive homogeneity (i.e., $[\text{UB}]$, $[\text{SI}]$ and $[\text{I+}]$ are not satisfied). Such a construction is especially valuable when pairing stablecoins (i.e., assets which are constructed so as to keep a stable price against a reference instrument). Curve is an extension of StableSwap that satisfies $[\text{UB}]$ and $[\text{SI}]$ at the expense of $[3.5]$, $[3.6]$ and is only implicitly defined ($u(x, y) = \log(D(x, y))$ for $D(x, y)^3 + 4(C - 1)xyD(x, y) - 4C(x + y)xy = 0$ with parameter $C \geq 1$; see Appendix A.6). To maintain the infinite liquidity and concentrate liquidity around a price of 1, the price impacts in the tails (i.e., when one of the pools is nearly exhausted) become exceedingly high. Within Appendix A.5.2 we introduce a new AMM structure, as the linear combination of UniSwap V2 and mStable, which we call $L.\text{StableSwap}$ ($u(x, y) = C\log(x + y) + \log(x) + \log(y)$ for parameter $C > 0$). As with Uniswap V2 and Curve, all fundamental axioms are satisfied for this new construction and it is subject to low price impacts near pool balance ($a \approx b$ for reserves $a, b > 0$). However, in contrast to Uniswap V2 (but similar to Curve), $[3.5]$ and $[3.6]$ are not satisfied for this construction as it does not exhibit price stability in the edges of its reserves. Notably, in contrast to Curve, L.\text{StableSwap} is constructed with a simple analytical utility function. Both Curve and L.\text{StableSwap}, due to their low price impacts, are able to provide additional liquidity near the theoretical constant price for stablecoin pairs, but still capable of providing some limited liquidity in the tails of the price distribution. Due to the uniform distribution of liquidity in Uniswap V2, that AMM can serve well for asset pairs with high volatility whereas these more stable AMMs are more appropriate for pairs with low volatility.

**Remark 11.** In each of the aforementioned AMMs, we have described them so that $P(t, t) = 1$ for any $t > 0$. Especially when considering pairs of stablecoins, a different balanced price may be desirable. Consider the desired balanced price $p > 0$, then setting $\tilde{u}^p(x, y) := u(px, y)$ satisfies all of the same axioms as $u$ with pricing oracle $\tilde{P}^p(x, y) = pP(px, y)$; in particular, $\tilde{P}^p(t, pt) = p$ by construction for any $t > 0$. Notably, in this case, the balance occurs not when the reserves of asset $A$ and $B$ are equal but rather when their ratio coincides with the desired price $p$.

Other AMMs are built from the structure of Uniswap V2 directly. Balancer ($u(x, y) = w\log(x) + (1 - w)\log(y)$ for parameters $w \in (0, 1)$; see Appendix A.2) is merely a weighted version of Uniswap V2 so as to scale the asset pools when pricing. Uniswap V3 ($u(x, y) = \log(\alpha + x) + \log(\beta + y)$ for

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7Verified empirically for Curve.
parameters \( \alpha, \beta > 0 \); see Appendix [A.3] introduces “virtual reserves” so as to concentrate liquidity and decrease price impacts. However, the concentrated liquidity introduced in Uniswap V3 comes at the expense of \([\text{UB}]\), \([\text{SI}]\), and \([\text{I+}]\) so that, much like StableSwap, it only has finite liquidity and does not scale linearly.

The final AMM which we consider that is used in practice is \( Dodo \) \( u(x, y) = \log(P\alpha(x, y) + \beta(x, y)) \) for external price \( P > 0 \); see Appendix [A.7]. Dodo is fundamentally different from all other AMMs considered within this work as it uses an \textit{exogenous} pricing oracle (such as a centralized exchange) and does not provide its own pricing oracle. Because there is no endogenous pricing oracle, Dodo permits pooling and withdrawing in any combination of assets (though with the possibility of withdrawal fees so as to guarantee the withdrawal is possible). The utility for Dodo is constructed in such a way to attempt to match the external price.

\textbf{Remark 12.} As noted within Remark [1], [12] assumes AMMs satisfy a monotone differences property \( u_{AA}, u_{BB} \leq 0 \) and \( u_{AB} \geq 0 \). However, the utility functions for mStable, StableSwap, Curve, and Dodo do not satisfy this stronger property.

We conclude this discussion of AMMs used in practice by comparing some of these constructions graphically. Demonstrations of Uniswap V2, StableSwap, L.StableSwap, Curve, and the hyperbolic sine SDAMM (see Example [4.2] below) are provided within Figure [2]. In particular, we compare these AMMs in 4 dimensions: (i) the binding curve \( u(z) = 1 \); (ii) the swap function \( \mathcal{Y}(x; 1, 1) \); (iii) the pricing oracle \( P \); and (iv) the price impact oracle \( I \). These plots make clear that Uniswap V2 has the largest price impact, L.StableSwap and Curve have extremely similar behaviors, and, generally, that there are tradeoffs between price impacts and tail behavior (or nonexistence of possible trades in the case of StableSwap).

\section{4.2 Symmetric Decomposable AMM}

Herein we propose a new class of AMMs which we name \textit{Symmetric Decomposable AMMs} (SDAMMs) that provides a simple analytical structure for the utility function. This is in contrast to the majority of currently existing AMMs which, by and large, are combinations of Uniswap V2 and mStable.

\footnote{As detailed in [1] and provided in Appendix [A.3] the virtual liquidity \( \alpha, \beta \) for Uniswap V3 are dynamic in practice; in fact, they are constructed in such a way that Uniswap V3 satisfies \([\text{SI}]\).}
Specifically, define the SDAMM utility function as

$$u(x, y) = U(x) + U(y)$$

for any $x, y \geq 0$ with the univariate utility function $U : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$. This can be seen as a generalization of Uniswap V2 since taking $U(z) := \log(z)$ for any $z \geq 0$ exactly replicates this well-known AMM.

The following proposition relates the properties of the univariate utility function $U$ to the axioms
of the associated SDAMM. We omit the proof of this proposition as these properties are trivial to verify.

**Proposition 4.1.** Let $U : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ be a thrice differentiable univariate utility function and let $u$ be the associated SDAMM. Immediately, $u$ satisfies $(C)$.

1. If $U(0) = -\infty$ and $U(z) > -\infty$ for any $z > 0$ then $(UfB)$.
2. If $\lim_{z \to \infty} U(z) = \infty$ then $(UfA)$.
3. If $U$ is strictly increasing then $(SM)$.
4. If $U$ is concave then $(QC)$ and $(SC)$.
5. If $\lim_{z \to \infty} U'(z) = 0$ and $\lim_{z \to 0} U'(z) = \infty$ then $(I+)$.
6. If $U'$ is log-convex then (3.3).
7. If $U$ is strictly increasing, concave, and $3U''(z)^2 \geq U'(z)U'''(z) \geq 0$ for any $z > 0$ then (3.5) and (3.6).

Proposition 4.1 provides simple conditions on $U : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ to guarantee the various axioms proposed within this work. As these properties are easily satisfied (see, e.g., Uniswap V2 and Example 4.2), we highlight in Table 2 that SDAMM satisfies these axioms.

With this utility function, the pricing oracle is defined as

$$P(x, y) = \frac{U'(x)}{U'(y)}$$

for $x, y \geq 0$. In contrast to many of the aforementioned AMMs, pooling for SDAMM can be much more complex as it need not satisfy $(SI)$. In fact, to accomplish pooling in such a case, we recommend a scheme in which the liquidity provider supplies the assets in any ratio which the AMM then swaps appropriately to maintain a constant price. For instance, in Example 4.2 below, we consider a specific function $U$ that satisfies Properties (1)-(6) of Proposition 4.1 but is not scale invariant $(SI)$ nor does it satisfy (3.5) and (3.6); this proposed new AMM structure concentrates

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9As proven in [26, Theorem 8], this decomposable structure for SDAMM satisfies $(SI)$ if and only if $U$ is a power utility function.
liquidity around the balanced reserve price of 1. Within Section 5.2 below, we investigate some important implications of dropping the scale invariance (SI) axiom.

**Example 4.2.** Let \( U(z) = \log(\sinh(Cz^q)) \) with \( C > 0 \) and \( q \in (0, 1] \) for any \( z \geq 0 \). This satisfies all the properties considered within Proposition 4.1 except (7) (nor does it satisfy (SI)) if \( q < 1 \); however, if \( q = 1 \), then (I+) is no longer satisfied as well. We use the hyperbolic sine function because it limits to the exponential function as the pool size grows to infinity. As such this AMM limits to mStable when the pool size is large enough (and as \( q \to 1 \)). In this way, this AMM can have extremely stable prices at 1, even more than Curve, and have infinite liquidity. In order to gain this price stability, an (extremely) unbalanced pool has extremely high price impacts in order to remain liquid. Much like Curve and L.StableSwap, these stable AMMs fail to satisfy (3.5) and (3.6) in order to still provide liquidity throughout the price curve.

**5 Fee structures and divergence loss**

Thus far, we have only discussed AMMs without any explicit fees collected by the liquidity providers. Within this section we propose a novel structure for assessing fees on a swap. We will provide an explicit formulation of a bid-ask spread and how this fee structure impacts the value of swaps. Additionally, we will present some implications of this fee structure; in doing so, we will also directly compare this construction to the prior methods of assessing fees discussed within the literature (e.g., [7, 23]). We conclude this section by commenting briefly on the so-called “impermanent loss” or divergence loss [28] for the liquidity providers of the AMM. As before, we will primarily focus on the properties of \( Y \) as the results are symmetric to \( X \).

**5.1 Assessing fees on the marginal price**

Herein we wish to propose a novel structure for assessing fees on a swap. Specifically, we propose that a fee \( \gamma \in [0, 1] \) is assessed to each marginal unit being bought. This is accomplished by assessing the fees directly on the price as given by the pricing oracle \( P \). This interpretation is consistent with a typical financial interpretation of a bid-ask spread (see, e.g., [16]). This structure is compared with prior fee structures for AMMs in Remark 15.
**Definition 5.1.** Let \( u : \mathbb{R}_+^2 \to \mathbb{R} \cup \{-\infty\} \) be an AMM with associated pricing oracle \( P : \mathbb{R}_+^2 \to \mathbb{R}_+ \). Let \( a, b > 0 \) be the size of the reserves of asset \( A \) and \( B \) within the AMM respectively. Let \( \gamma \in [0, 1] \) be the fee level assessed on a transaction. The swap functions with fees \( Y_\gamma : \mathbb{R}_+^3 \to \mathbb{R}_+ \) and \( X_\gamma : \mathbb{R}_+^3 \to \mathbb{R}_+ \) are defined for any \( x, y \geq 0 \) as:

\[
Y_\gamma(x; a, b) := (1 - \gamma) \int_0^x P(a + z, b - Y_\gamma(z; a, b))dz,
\]

\[
X_\gamma(y; a, b) := (1 - \gamma) \int_0^y \frac{1}{P(a - X_\gamma(z; a, b), b + z)}dz.
\]

Where clear, we will drop the explicit dependence of \( Y_\gamma, X_\gamma \) on the pool sizes \((a, b) \in \mathbb{R}_+^2\).

The intuition behind the constructions (5.1) and (5.2) is that the AMM should be indifferent to the size of transactions; a sequence of unidirectional small trades should have the same result for the AMM as a single large trade (assuming nothing happens in between the transactions). In other words, a trade of \( Y_\gamma(x; a, b) \) and then another (infinitesimal) trade \( Y_\gamma(dx; a + x, b - Y_\gamma(x; a, b)) \) should be equivalent to a single trade of \( Y_\gamma(x + dx; a, b) \) from the perspective of the AMM. Charging \( \gamma \) proportion in fees, and using the fact that \( Y_\gamma(dx; a + x, b - Y_\gamma(x; a, b)) \approx Y'_\gamma(0; a + x, b - Y_\gamma(x; a, b))dx = (1 - \gamma)P(a + x, b - Y_\gamma(x; a, b))dx \), we arrive at an equivalent ODE formulation for (5.1):

\[
Y'_\gamma(x) = (1 - \gamma)P(a + x, b - Y_\gamma(x)) =: (1 - \gamma)g(x, Y_\gamma(x)) \quad \forall x \geq 0
\]

with initial value \( Y_\gamma(0) = 0 \). (An ODE can similarly be provided for \( X_\gamma \)).

**Assumption 5.2.** Throughout the remainder of this paper, we will consider AMMs satisfying

\( (\text{UfB}) \quad (\text{QC}) \quad (\text{I+}) \quad \text{and} \quad (\text{SC}) \)

Before continuing with the discussion of these fees, we will prove that the AMM with fees is well-defined in that there exists unique swap functions \( Y_\gamma, X_\gamma \) given by Definition 5.1.

**Lemma 5.3.** Let \( u : \mathbb{R}_+^2 \to \mathbb{R} \cup \{-\infty\} \) be an AMM satisfying Assumption 5.2. There exists unique swap functions \( Y_\gamma, X_\gamma \) for any \( \gamma \in [0, 1] \).

**Remark 13.** By construction of the pricing oracle of an AMM satisfying Assumption 5.2, if the fees are zero \( (\gamma = 0) \), we recover \( Y_0 \equiv Y \) since \( P(a + x, b - Y(x; a, b)) = Y'(0; a + x, b - Y(x; a, b)) = \).

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\( \mathcal{Y}'(x; a, b) \). That is, the AMM recovers the swap amounts \( \mathcal{Y} \) as presented in the prior sections when no fees are assessed.

Before studying the properties of the modified swap functions and pricing oracle, we wish to provide an explicit example of the swap function \( \mathcal{Y}_\gamma \) under a real-world AMM construction.

**Example 5.4.** Consider the Uniswap V2 utility function \( u(x, y) = \log(x) + \log(y) \) as discussed in Example 3.2. For this AMM, the pricing oracle \( P(x, y) = y/x \) is the ratio of the reserves as noted in Example 3.6. The swap for \( x \geq 0 \) with fee level \( \gamma \in [0, 1] \) can be found to be

\[
\mathcal{Y}_\gamma(x; a, b) = b \left( 1 - \frac{a^{1-\gamma}}{(a+x)^{1-\gamma}} \right)
\]

by solving the differential equation (5.3).

We now consider our first property of the AMM with fees. Specifically, as expected, as the fees \( \gamma \) increase then the AMM will pay out less in a swap.

**Proposition 5.5.** Let \( u : \mathbb{R}^2_+ \to \mathbb{R} \cup \{-\infty\} \) be an AMM satisfying Assumption 5.2. For any \( 0 \leq \gamma_1 < \gamma_2 \leq 1 \),

\[
0 \leq \mathcal{Y}_{\gamma_2}(x; a, b) < \mathcal{Y}_{\gamma_1}(x; a, b) \leq \mathcal{Y}(x; a, b), \quad (5.4)
\]

\[
0 \leq \mathcal{X}_{\gamma_2}(y; a, b) < \mathcal{X}_{\gamma_1}(y; a, b) \leq \mathcal{X}(y; a, b),
\]

for every \( x, y > 0 \) and \( a, b > 0 \).

We now turn our attention to the fundamental properties of AMMs that, without fees, were studied previously in Theorem 3.7. Within the following corollary, we find that all relevant properties are satisfied by \( \mathcal{Y}_\gamma \) with strictly positive fees. For this result we restrict ourselves to fee levels \( \gamma \in (0, 1) \); if \( \gamma = 0 \) then we recover the original swap values (as discussed in Remark 13) and if \( \gamma = 1 \) then \( \mathcal{Y}_1 \equiv 0 \) by construction.

**Corollary 5.6.** Consider an AMM \( u : \mathbb{R}^2_+ \to \mathbb{R} \cup \{-\infty\} \) satisfying Assumption 5.2. Fix the fee level \( \gamma \in (0, 1) \), initial AMM reserves \( a, b > 0 \), and swap amounts \( x, x_1, x_2 \geq 0 \).

1. \( u(a + x, b - \mathcal{Y}_\gamma(x)) \geq u(a, b) \) with strict inequality if \( x > 0 \), i.e., market utility never drops.
2. $\gamma$ is strictly increasing, concave, and subadditive in $x$, i.e., (I), (CC), and (SA).

3. If, additionally, (UfA), then $\lim_{x \to \infty} \gamma(x) = b$, i.e., (NW).

4. If, additionally, (SI), then $\gamma(x)$ is positive homogeneous in $(x; a, b)$, i.e., $\gamma(tx; ta, tb) = t \gamma(x; a, b)$ for any $t > 0$.

5. $\gamma(x_1 + x_2; a, b) = \gamma(x_1; a, b) + \gamma(x_2; a + x_1, b - \gamma(x_1; a, b))$, i.e., (PI).

Remark 14. In contrast to the fee-less construction provided within Section 3 (and as expected), introducing a fee $\gamma \in (0, 1)$ immediately introduces a bid-ask spread. Specifically, the bid price is given by $\gamma'(0) = (1 - \gamma)p(a, b)$ and the ask price is given by $\gamma'(0)^{-1} = (1 - \gamma)^{-1}p(a, b)$ for any pool reserves $(a, b) \in \mathbb{R}_+^2$. Note that the pricing oracle $p(a, b)$ is between the bid and ask prices but is not the mid-price.

As a direct consequence of the bid-ask spread discussed in Remark 14, the introduction of fees guarantees a strict no-arbitrage condition.

Corollary 5.7. Consider an AMM $u : \mathbb{R}_+^2 \to \mathbb{R} \cup \{-\infty\}$ satisfying Assumption 5.2. If $\gamma \in (0, 1]$ then (NA), i.e., $\gamma(\gamma(x; a, b); a + x, b - \gamma(x; a, b)) < x$ and $\gamma(\gamma(y; a, b); a - \gamma(y; a, b), b + y) < y$ for any $x, y > 0$ and $a, b > 0$.

We wish to conclude the discussion of properties of the swap amounts with fees by studying the dependence of $\gamma$ on the reserves $(a, b)$ as well as the fees $\gamma$.

Corollary 5.8. Consider an AMM $u : \mathbb{R}_+^2 \to \mathbb{R} \cup \{-\infty\}$ satisfying Assumption 5.2. Fix the fee level $\gamma \in (0, 1]$. Then $\frac{\partial}{\partial a} \gamma(x; a, b) \leq 0$ and $\frac{\partial}{\partial b} \gamma(x; a, b) \in [0, 1)$ for any $x \geq 0$ and $a, b > 0$, i.e., (ML).

Remark 15. The fee structure introduced herein is not the only approach that can be used for an AMM. We wish to highlight two alternate structures which can be viewed as assessing the fees on the assets sold or bought by the investor respectively. With both of these structures we wish to briefly comment on how an investor may want to optimally split a transaction when transacting with the AMM; this lack of indifference to trade splitting leads to strange implications for the AMM (i.e., the liquidity provider should only care about the final state of the AMM rather than how trades are executed).
1. Consider the fees collected on the asset being sold to the AMM. That is, a fraction $\gamma \in [0, 1]$ of $x$ is taken by the AMM to compensate it for acting as a liquidity provider. Mathematically, this is encoded by

$$\tilde{Y}_\gamma(x; a, b) := Y((1 - \gamma)x; a, b).$$

That is, the collected fees are $\gamma x$ of asset $A$ so that the realized pool size, after the swap is completed, is still of the form $(a + x, b - \tilde{Y}_\gamma(x; a, b))$. We wish to note that this is the fee structure considered in, e.g., [7, 23].

Though a simple structure to implement, this fee structure provides a discount to buying in bulk, i.e., $\tilde{Y}_\gamma(x_1 + x_2; a, b) \geq \tilde{Y}_\gamma(x_1; a, b) + \tilde{Y}_\gamma(x_2; a + x_1, b - \tilde{Y}_\gamma(x_1; a, b))$, and aside from extreme cases, leading to strict inequality violating $[PI]$. By imposing costs to an investor who splits her transaction, the AMM is subsidizing large traders.

2. Consider the fees collected on the asset being bought by the trader. That is, a fraction $\gamma \in [0, 1]$ of $Y$ is taken by the AMM to compensate it for acting as a liquidity provider. Mathematically, this is encoded by

$$\tilde{Y}_\gamma^*(x; a, b) := (1 - \gamma)Y(x; a, b).$$

That is, the collected fees are $\gamma Y(x; a, b)$ of asset $B$ so that the realized pool size, after the swap is completed, is still of the form $(a + x, b - \tilde{Y}_\gamma^*(x; a, b))$. In contrast to the fees on the sold asset, imposing fees on the bought asset provides a benefit to an investor who splits her trade over time, i.e., $\tilde{Y}_\gamma^*(x_1 + x_2; a, b) \leq \tilde{Y}_\gamma^*(x_1; a, b) + \tilde{Y}_\gamma^*(x_2; a + x_1, b - \tilde{Y}_\gamma^*(x_1; a, b))$, and aside from extreme cases, leading to strict inequality violating $[PI]$. By encouraging these small transactions, the AMM incentivizes traders to make a series of infinitesimally small transactions; that is, an intelligent trader will, in fact, implement the integral strategy $Y_\gamma$ we propose in Definition 5.1.

### 5.2 Divergence loss

We will conclude our discussion of AMMs by considering the profits and losses observed by liquidity providers. The greater the gains (or smaller the losses), the more incentive there is for users to deposit liquidity in order to capture a portion of the fees. The most commonly reported metric for
the costs to liquidity providers is called the divergence loss or impermanent loss (see, e.g., [3, 17]).

The divergence loss measures the mark-to-market loss from repurchasing the original pooled portfolio after withdrawing the liquidity position (accounting for any asset rebalancing and fee collection), i.e., the difference between the value of a buy-and-hold strategy and of a liquidity position in the AMM. Though conceptually this is a dynamic measure due to the relation between fee collection and price volatility, the divergence loss is traditionally simplified so that it measures the gains or losses to the liquidity position after only a single swap is undertaken. That is, first the liquidity provider invests the funds, second the price moves because of a trade, and finally the liquidity is withdrawn to provide the value of the pooled position; the divergence loss then is the difference between the value of the original liquidity position (under the updated price) and this withdrawn position. Formally, the divergence loss is defined directly below.

**Definition 5.9.** Consider an AMM with reserves \((a, b) \in \mathbb{R}^2_+\) and fee level \(\gamma \in [0, 1]\). Let \((\alpha, \beta) \in \mathbb{R}^2_+ \setminus \{0\}\) such that \(P(a + \alpha, b + \beta) = P(a, b)\). The divergence loss \(\Delta : (0, \infty) \to \mathbb{R}\) depending implicitly on \((a, b, \alpha, \beta)\) is the difference between the mark-to-market value of \((\alpha, \beta)\) being held and being pooled, i.e.,

\[
\Delta(p) := \begin{cases} 
[p\alpha + \beta] - \frac{P(a, b)\alpha + \beta}{P(a, b)\alpha + \beta + [b + \beta]} \left[ p(a + \alpha + x_p) + (b + \beta - \mathcal{Y}_p(x_p; a + \alpha, b + \beta)) \right] & \text{if } p < P(a, b) \\
0 & \text{if } p = P(a, b) \\
[p\alpha + \beta] - \frac{P(a, b)\alpha + \beta}{P(a, b)\alpha + \beta + [b + \beta]} \left[ p(a + \alpha - \mathcal{X}_p(y_p; a + \alpha, b + \beta)) + (b + \beta + y_p) \right] & \text{if } p > P(a, b)
\end{cases}
\]

where \(x_p > 0\) is such that \(p = P(a + \alpha + x_p, b + \beta - \mathcal{Y}_p(x_p; a + \alpha, b + \beta))\) and \(y_p > 0\) is such that \(p = P(a + \alpha - \mathcal{X}_p(y_p; a + \alpha, b + \beta), b + \beta + y_p)\).

**Remark 16.** Note that the domain of \(\Delta\) is \((0, \infty)\) under Assumption 5.2. In other words, for any \(p < P(a, b)\) there exists some \(x_p > 0\) such that \(p = P(a + \alpha + x_p, b + \beta - \mathcal{Y}_p(x_p; a + \alpha, b + \beta))\), and similarly for any \(p > P(a, b)\) there exists some \(y_p > 0\) such that \(p = P(a + \alpha - \mathcal{X}_p(y_p; a + \alpha, b + \beta), b + \beta + y_p)\). To see this, recall that the pricing oracle \(P\) is differentiable with \(P_A \leq 0\) and \(P_B \geq 0\). As \(\mathcal{Y}_p\) is strictly increasing, we find that \(P(a + \alpha + x, b + \beta - \mathcal{Y}_p(x; a + \alpha, b + \beta)) < P(a, b)\) for any \(x > 0\). Furthermore, \(0 \leq \lim_{x \to \infty} P(a + \alpha + x, b + \beta - \mathcal{Y}_p(x; a + \alpha, b + \beta)) \leq \lim_{x \to \infty} P(a + \alpha + x, b + \beta) = 0\). Therefore, together with continuity, this guarantees the existence of such \(x_p\).
Often it is convenient to consider the divergence loss as a function of the swap amounts rather than prices. That is, \( \bar{\Delta} : \mathbb{R} \rightarrow \mathbb{R} \) defined as:

\[
\bar{\Delta}(z) := \begin{cases} 
\Delta(P(a + \alpha + z, b + \beta - Y_\gamma(z; a + \alpha, b + \beta))) & \text{if } z \geq 0, \\
\Delta(P(a + \alpha - X_\gamma(-z; a + \alpha, b + \beta), b + \beta - z)) & \text{if } z < 0,
\end{cases}
\]

so as to remove the explicit dependence of \( \Delta \) on the mappings \( x_p, y_p \).

As noted above, the divergence loss is important as it quantifies a cost for the liquidity provider in pooling his or her assets into the AMM. As stated in [28], the divergence loss \( \Delta(p) \) of well-studied AMMs is strictly positive so long as \( p \neq P(a, b) \) when neglecting the fees. As we will see in Lemma 5.10 and Example 5.12, this result is a direct consequence of \([SI]\). This creates a tradeoff between the easy implementation of the proportional rule for pooling assets as noted in Remark 8 and losses for the liquidity providers.

**Lemma 5.10.** Consider an AMM satisfying Assumption 5.2 and \([SI]\). The divergence loss can be simplified as:

\[
\Delta(p) = \begin{cases} 
\frac{\delta}{1+\delta}[Y_\gamma(x_p; (1+\delta)a, (1+\delta)b) - px_p] & \text{if } p < P(a, b), \\
0 & \text{if } p = P(a, b), \\
\frac{\delta}{1+\delta}[pX_\gamma(y_p; (1+\delta)a, (1+\delta)b) - y_p] & \text{if } p > P(a, b),
\end{cases}
\]

where, implicitly, \( \alpha := \delta a \) and \( \beta := \delta b \) for some \( \delta > 0 \). Furthermore, the sign of the divergence loss can be characterized w.r.t. the fees:

1. if \( \gamma = 0 \) then \( \Delta(p) \geq 0 \) with strict inequality if \( p \neq P(a, b) \);

2. if \( \gamma \in (0, 1] \) then there exist \( p_* < P(a, b) < p^* \) such that \( \Delta(p) < 0 \) for \( p \in (p_*, p^*) \setminus \{p\} \).

We now want to consider the divergence loss for two AMMs in detail. First, we revisit Uniswap V2 with fees (Example 5.4) in order to formalize the results of Lemma 5.10 for this well-known AMM. Second, we consider SDAMM with the log of the hyperbolic sine (Example 4.2) to consider the divergence loss when \([SI]\) is not satisfied to highlight the value in dropping this axiom that is widely assumed in practice (see Table 2) and within the literature (see, e.g., [12, 25]).
Example 5.11. Recall the Uniswap V2 AMM with fees from Example 5.4 with initial reserves $(a, b) \in \mathbb{R}^2_{++}$. That is, given the fee level $\gamma \in (0, 1)$, $\mathcal{Y}_\gamma(x; a, b) = b \left( 1 - \left( \frac{a}{a+y} \right)^{1-\gamma} \right)$ and $\mathcal{X}_\gamma(y; a, b) = a \left( 1 - \left( \frac{b}{b+y} \right)^{1-\gamma} \right)$ for any transactions $x, y \geq 0$. As Uniswap V2 satisfies $(SI)$ the results of Lemma 5.10 hold. Furthermore, as noted within Remark 8 and used in Lemma 5.10, we take the liquidity injection to be $\alpha = \delta a$ and $\beta = \delta b$ for some $\delta > 0$. We wish to consider the form of the divergence loss (characterized as $\bar{\Delta}$) and the threshold prices $p_*, p^*$ which construct the interval of divergence gains for the liquidity provider. Let $z \in \mathbb{R}$ then

$$\bar{\Delta}(z) = \begin{cases} \delta b \left( 1 - 2 \left( \frac{(1+\delta)a}{(1+\delta)a+z} \right)^{1-\gamma} + \left( \frac{(1+\delta)a}{(1+\delta)a+z} \right)^{2-\gamma} \right) & \text{if } z \geq 0 \\ \delta b \left( 1 - 2 \left( \frac{(1+\delta)b-z}{(1+\delta)b} \right) + \left( \frac{(1+\delta)b-z}{(1+\delta)b} \right)^{2-\gamma} \right) & \text{if } z < 0. \end{cases}$$

We will now direct our attention to determining the threshold prices $p_*, p^*$. Consider, first, $p_* \leq P(a, b) = b/a$ which involves studying $\bar{\Delta}(z)$ for $z \geq 0$. It can be determined that $\bar{\Delta}(z) < 0$ for $z \geq 0$ if and only if $z \in (0, x_* := \frac{(1+\delta)(1-t_*)}{t_*})$ where $t_* \in (0, 1-\gamma)$ solves $1 - 2t^{1-\gamma} + t^{2-\gamma} = 0$. The lower bound $p_*$ for divergence gains can be determined by finding the price assuming $x_*$ assets were transacted, i.e.,

$$p_* := P((1+\delta)a + x_*, (1+\delta)b - \mathcal{Y}_\gamma(x_*; (1+\delta)a, (1+\delta)b) = \frac{b}{a} t_*^{2-\gamma} < \frac{b}{a} = P(a, b).$$

Following similar arguments, we determine that the divergence loss is negative for $z < 0$ if and only if $z \in (-y_* := -(1+\delta)b(t^* - 1), 0)$ where $t^* \in (1 + \gamma)^{-\frac{1}{2-\gamma}} + (0, 1 - 2(1+\gamma)\frac{1}{(1-\gamma)^{1-\gamma} + (1+\gamma)^{2-\gamma}})$ solves $1 - 2t + t^{2-\gamma} = 0$, i.e., the upper bound $p^*$ for divergence gains is

$$p^* := P((1+\delta)a - \mathcal{X}_\gamma(y^*; (1+\delta)a, (1+\delta)b), (1+\delta)b + y^*) = \frac{b}{a}(t^*)^{2-\gamma} > \frac{b}{a} = P(a, b).$$

Remark 17. Though only provided as a sufficient condition for divergence gain, we hypothesize that Lemma 5.10 can be strengthened insofar as we conjecture that $\Delta(p) > 0$ for $p \notin [p_*, p^*]$ under $(SI)$ with fees $\gamma \in (0, 1)$. We highlight in Example 5.11, that this stronger conjecture is satisfied in that special case. To be proven for a generic AMM, this requires that $\mathcal{Y}_\gamma(x; a, b) - P(a + x, b - \mathcal{Y}_\gamma(x; a, b))x \geq 0$ for $x > x^*$ (wlog suppressing the need for $\delta \geq 0$ in this expression).
Figure 3: Plot of the divergence loss $\bar{\Delta}$ for the hyperbolic sine SDAMM under varied $q \in \{0.8, 0.95, 1\}$.

However, this expression is not necessarily monotonic nor convex in $x$ which makes such a proof highly non-trivial and beyond the scope of this work.

As previously mentioned, as a point of comparison, we will focus on SDAMM with the hyperbolic sine (Example 4.2) to study divergence loss when some axioms may not be satisfied. Specifically, and comparing to Lemma 5.10, (SI) appears to be necessary to the positivity of divergence loss.

Example 5.12. Recall the SDAMM structure from Section 4.2 such that $u(x, y) = U(x) + U(y)$ for univariate utility functions $U$. Specifically, take $U(z) := \log(\sinh(Cx^q))$ for $C > 0$ and $q \in (0, 1]$ as in Example 4.2. Note that this structure does not satisfy (SI) and, therefore, the results of Lemma 5.10 do not hold. We wish to explore two cases herein: the possibility of divergence gains even without fees (and the impact of $q \in (0, 1]$ on the gains/losses) and the impact that fees have on decreasing the divergence loss.

For the first scenario, consider $q \in \{0.8, 0.95, 1\}$ (i.e., satisfying (I+) in the first two settings and not satisfying it in the $q = 1$ setting) without fees $\gamma = 0$. The divergence losses are plotted in Figure 3 with gains plotted in blue and losses in red. Notably, in all three choices of $q$ we have regions in which there are divergence gains even though there are no fees. Though these plots may appear as if the divergence loss is kept negative when trading is in the direction of the pool imbalance, this is only due to the $x$-axis used in these plots; if $|z|$ is large enough then $\Delta(z) \geq 0$ in any choice of $q$. We also wish to note that the divergence loss decreases as we increase $q$. 

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In the second scenario, we focus solely on the $q = 1$ setting but vary the fee level $\gamma \in [0,1)$. In order to study this case, we first note that the swaps with fees can be computed explicitly as:

\[
Y_\gamma(x; a, b) = b - \frac{1}{C} \sinh^{-1}\left( \left( \frac{\sinh(Ca)}{\sinh(C(a + x))} \right)^{1-\gamma} \sinh(Cb) \right),
\]

\[
X_\gamma(y; a, b) = a - \frac{1}{C} \sinh^{-1}\left( \sinh(Ca) \left( \frac{\sinh(Cb)}{\sinh(C(b + y))} \right)^{1-\gamma} \right),
\]

for $x, y \geq 0$ with reserves $(a, b) \in \mathbb{R}_+^2$. The divergence losses are plotted in Figure 4 with gains plotted in blue and losses in red. Not surprisingly, as the fees increase, the divergence loss decreases. However, for $|z|$ large enough, i.e. when the price moves far enough away from the initial price, there is always a divergence loss.

Finally, we want to demonstrate how the divergence loss differs based on the AMM construction used. In Figure 5 we compare the divergence loss $\Delta(p)$ for the same AMMs as considered in Figure 2 (i.e., Uniswap V2, StableSwap, L.StableSwap, Curve, and the hyperbolic sine SDAMM) without fees, i.e., $\gamma = 0$. For this figure, we consider each AMM without fees. Notably, both StableSwap and the hyperbolic sine SDAMM are able to capture divergence gains in a portion of the price range, whereas the other AMMs considered all suffer divergence losses throughout. As provided in Table 2, neither StableSwap nor the hyperbolic sine SDAMM satisfy [SI] whereas the rest of the

\[^{10}\text{Though the } q = 1 \text{ setting does not satisfy Assumption 5.2, the requisite properties for swaps with fees still hold in this case.}\]
Figure 5: The divergence loss $\Delta(p)$ where $u(a, b) = 1$, $P(a, b) = 2$, and $\frac{P(a, b)\alpha + \beta}{P(a, b)(a + \alpha) + (b + \beta)} = \frac{1}{11}$ for Uniswap V2, StableSwap, L.StableSwap, Curve, and the hyperbolic sine SDAMM.

considered AMMs are scale invariant.

**Remark 18.** As discussed above, though (SI) leads to simplifying mathematical properties (see Theorem 3.7 and Proposition 3.11), it introduces significant risks to liquidity providers (see Lemma 5.10). This is especially notable in comparison to AMMs that are utilized in practice that are, frequently, satisfying (SI) as highlighted within Table 2. Furthermore, when reviewing Table 4 we note that (SI) is not necessary for any of the fundamental properties for financial markets.

## 6 Conclusion

Within this work we have considered an axiomatic framework for AMMs. By imposing reasonable axioms on the underlying utility function, we are able to characterize the properties of the swap size of the assets and of the resulting pricing oracle. In addition, we have introduced a novel price impact oracle which quantifies these costs for traders. We have analyzed many existing AMMs and shown that the vast majority of them satisfy our axioms. Finally, we have also considered the question of fees and divergence loss. In doing so, we have proposed a new fee structure so as to make the AMM indifferent to trade execution. Finally, we have proposed a novel AMM that, while it does not satisfy all of our axioms, has nice analytical properties and provides a large range over which there is no divergence loss.

We wish to provide a few extensions for this work. First, and importantly, a rigorous study
of the conjecture provided within Remark 17 would greatly enhance our understanding of the divergence loss and the impacts that scale invariance has on those costs. Second, within this work and throughout the literature on constant function market makers, a single utility function $u : \mathbb{R}_+^2 \rightarrow \mathbb{R} \cup \{-\infty\}$ is always considered; herein we propose considering the generalized AMM with incomplete preference relation $\succeq$ on $\mathbb{R}_+^2$ so that, e.g., $\mathcal{Y}(x; a, b) = \sup \{y \in [0, b] \mid (a + x, b - y) \succeq (a, b)\}$. In this way the fees can be endogenized within the preference relation itself. Related to the first two extensions, we propose further studies on new AMM constructions so as to minimize divergence loss (and maximize revenue for liquidity providers) much as we undertook with the SDAMM example within this work. Finally, within this paper, we only considered the traditional style AMMs, i.e., for swap markets. In order to grow the decentralized finance offerings, a rigorous study of AMMs for derivatives and other complex financial securities needs to be undertaken.

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A Details of Existing Automated Market Makers

A.1 Uniswap V2

Though we refer to this example as Uniswap V2, the same mathematical structure is also utilized within many other AMMs in practice such as SushiSwap, DefiSwap, and Quickswap. Structurally, Uniswap V2 is the logarithmic utility function, i.e.,

$$u(x, y) = \log(x) + \log(y)$$

for $x, y \geq 0$. With this utility function, the pricing oracle and price impact oracle are defined as

$$P(x, y) = \frac{y}{x}, \quad I(x, y) = \frac{y}{x^2}, \quad x, y \geq 0.$$

This AMM permits pooling at the ratio of the current reserves. As highlighted within Table 2, this AMM satisfies every axiom proposed within this work.

A.2 Balancer

Though we refer to this example as Balancer, the same mathematical structure is also utilized within many other AMMs in practice such as Bancor and Loopring. Structurally, Balancer is a weighted version of Uniswap V2, i.e.,

$$u(x, y) = w \log(x) + (1 - w) \log(y)$$

with $w \in (0, 1)$ for $x, y \geq 0$. With this utility function, the pricing oracle and price impact oracle are defined as

$$P(x, y) = \frac{wy}{(1 - w)x}, \quad I(x, y) = \frac{wy}{2(1 - w)^2x^2}, \quad x, y \geq 0.$$

Notably, this AMM always achieves a lower price impact than Uniswap V2 for $w \in (0, \frac{1}{2})$. It also permits pooling at the ratio of the current reserves. As highlighted within Table 2, this AMM satisfies every axiom proposed within this work.

A.3 Uniswap V3

Though we refer to this example as Uniswap V3, the same mathematical structure is also utilized within many other AMMs in practice such as KyberSwap and MooniSwap. The generic structure of all of these AMMs is the same as Uniswap V2 but with “virtual reserves” to provide concentrated liquidity, i.e.,

$$u(x, y) = \log(\alpha + x) + \log(\beta + y)$$
with $\alpha, \beta > 0$ for $x, y \geq 0$. Each of these real-world AMMs select $\alpha, \beta$ in different ways which may be dynamic in time or based on the implemented trades. For instance, Uniswap V3 implements this AMM in such a way that $\alpha, \beta$ are adjusted dynamically so that the AMM maintains a fixed maximum and minimum quoted price. With this utility function, the pricing oracle and price impact oracle are defined as

$$P(x, y) = \frac{\beta + y}{\alpha + x}, \quad I(x, y) = \frac{\beta + y}{(\alpha + x)^2}, \quad x, y \geq 0.$$ 

This AMM permits pooling at the ratio of the current reserves inclusive of the virtual reserves $\alpha, \beta$. As mentioned, these virtual reserves concentrate the liquidity to reduce price impacts from a transaction when the virtual reserves $(\alpha, \beta)$ are close to the true reserves $(a, b)$, but this comes at the cost of unbounded from below. Additionally, as constructed here with static $\alpha, \beta$, Uniswap V3 fails to be scale invariant. As highlighted in Table 2, Uniswap V3 does not satisfy (UfB), (SI), or (I+).

In practice this is undertaken with functional forms for $\alpha, \beta$ so that the liquidity is concentrated within constant upper $P^U$ and lower $P^L$ prices without regard to the amount of physical liquidity provided. Within the actual Uniswap V3 this is formulated via:

$$\alpha(x, y) = \frac{\sqrt{P^L P^U x + y} + \sqrt{(\sqrt{P^L P^U x + y})^2 + 4\sqrt{P^U} (\sqrt{P^U} - \sqrt{P^L})xy}}{2\sqrt{P^U} (\sqrt{P^U} - \sqrt{P^L})}$$

$$\beta(x, y) = \frac{\sqrt{P^L} \left[\sqrt{P^L P^U x + y} + \sqrt{(\sqrt{P^L P^U x + y})^2 + 4\sqrt{P^U} (\sqrt{P^U} - \sqrt{P^L})xy}\right]}{2(\sqrt{P^U} - \sqrt{P^L})}$$

Notably, following this functional form (as $\alpha, \beta$ are positive homogeneous), (SI) is now recovered.

### A.4 mStable

mStable is an AMM constructed to have no price impacts from trading. This is accomplished through the mathematical structure

$$u(x, y) = \log(x + y)$$

for $x, y \geq 0$. This AMM comes with the constant pricing oracle and zero price impacts

$$P(x, y) = 1, \quad I(x, y) = 0, \quad x, y \geq 0.$$ 

Pooling for mStable can, in theory, be accomplished with any combination of assets; traditionally, pooling is done either at the current ratio of assets or so that the pooled assets are in equal proportion. As with Uniswap V3, the ability to reduce price impacts (in this case to 0) comes at the expense of unbounded from
below \( \text{(UfB)} \), in fact, these zero price impacts also cause the AMM to lose \( \text{(I+)} \). Furthermore, though \( \text{(SC)} \) is satisfied, it is only satisfied with an equality (and thus does not guarantee quasiconcavity by itself).

**A.5 Stable swaps**

**A.5.1 StableSwap**

Though we refer to this example as StableSwap, the same mathematical structure is also utilized with many other AMMs in practice such as Saber and Saddle. Much like Uniswap V3, StableSwap aims to concentrate liquidity towards the “balanced” pool. This is accomplished by taking a linear combination of (the exponentials of) Uniswap V2 and mStable, i.e.,

\[
    u(x, y) = \log(C(x + y) + xy)
\]

with \( C > 0 \) for \( x, y \geq 0 \). With this utility function, the pricing oracle and price impact oracle are defined as

\[
    P(x, y) = \frac{C + y}{C + x}, \quad I(x, y) = \frac{C + y}{(C + x)^2}, \quad x, y \geq 0.
\]

This AMM permits pooling at the ratio of the current reserves inclusive of the parameter \( C \). As with Uniswap V3, the ability to reduce price impacts for a (potentially wide) neighborhood around the balanced pool comes at the expense of unbounded from below \( \text{(UfB)} \) and is neither scale invariant \( \text{(SI)} \) nor satisfies \( \text{(I+)} \).

**A.5.2 Liquid StableSwap [L.StableSwap]**

As far as we are aware, this construction has never been implemented before as an AMM in practice. Rather than taking the linear combination of exponentials of Uniswap V2 and mStable, here we take the linear combination directly, i.e.,

\[
    u(x, y) = C \log(x + y) + \log(x) + \log(y)
\]

with \( C > 0 \) for \( x, y \geq 0 \). This concentrates liquidity when the reserves of the AMM are not too far out of balance, but exacerbates price impacts once the reserves of the AMM become too skewed towards one asset. With this utility function, the pricing oracle and price impact oracle are defined as

\[
    P(x, y) = \frac{y[(C + 1)x + y]}{x[x + (C + 1)y]}, \quad I(x, y) = \frac{(C + 2)y(x + y)(C(x^2 + y^2) + (x + y)^2)}{2x^2(Cy + x + y)^3}, \quad x, y \geq 0.
\]
By taking this structure all fundamental axioms proposed within this work are satisfied in comparison to Uniswap V3, mStable and StableSwap. However, in order to have both a stable price near the balanced pool but provide liquidity throughout the price curve, L.StableSwap loses (3.5) and (3.6). Indeed, near the balanced pool, the price impacts are lower than those of Uniswap V2.

### A.6 Curve

Curve is a popular AMM that, much like our newly proposed Liquid StableSwap, extends StableSwap in such a way so as to guarantee infinite liquidity through satisfying [UBS]. For the construction of Curve, let \( D(x, y) \) denote the total number of coins when the reserves \((x, y) \in \mathbb{R}^2_{++}\) are traded into balance from a StableSwap AMM, i.e., \( \log(C(x + y) + xy) = \log(CD(x, y) + D(x, y)^2/4) \). However, in contrast to StableSwap, Curve considers a functional parameter \( C(x, y) \); by dimensional analysis \( C(x, y) \) must have units in number of coins (so that \( C(x, y)(x + y) \) and \( xy \) are both in number of coins squared). Furthermore, with the notion that Curve desires a balanced pool, \( C(x, y) \) is constructed to be proportional to both the number of coins for the balanced pool \((D(x, y))\) and to a unitless measure of pool balance, i.e., \( C(x, y) \propto \frac{xy}{D(x, y)^{3/4}} \) where \( \frac{xy}{D(x, y)^{3/4}} \in [0, 1] \) provides a measure of AMM balance. With this functional parameter \( C(x, y) := \frac{Cxy}{D(x, y)} \) with constant \( C \), the balanced pool size \( D(x, y) \) satisfies the equation

\[
\log(C(x, y)(x + y) + xy) = \log\left(C(x, y)D(x, y) + \frac{D(x, y)^2}{4}\right) \\
\iff D(x, y)^3 + 4(C-1)xyD(x, y) - 4C(x+y)xy = 0. \tag{A.1}
\]

It is this balanced pool size \( D(x, y) \) which defines the utility function for Curve. Specifically,

\[
u(x, y) = \log(D(x, y))\]

for \( x, y \geq 0 \) where \( D(x, y) \) is implicitly defined as the unique nonnegative root of (A.1) for \( C \geq 1 \). Uniqueness of \( D(x, y) \) follows from an application of Descartes’ rule of signs. With this utility function, the pricing oracle and price impact oracle are defined as

\[
P(x, y) = \frac{y[C(2x + y) - (C - 1)D]}{x[C(x + 2y) - (C - 1)D]} \\
I(x, y) = \frac{y[C(x + y) - (C - 1)D]}{x^2[C(x + 2y) - (C - 1)D]^3} \left((C - 1)D^2 - 3C(C - 1)(x + y)D + 3C^2(x^2 + xy + y^2)\right) \]
for $x, y \geq 0$ where, for simplicity, we set $D = D(x, y)$. Despite the complex, implicit, structure of this pricing oracle, Curve permits pooling at the current ratio of reserves because it satisfies [SI] Due to the implicit construction of the Curve AMM, (some of) the axioms presented within this paper can only be verified numerically. Even so, as highlighted within Table 2, all axioms presented are satisfied for Curve either analytically or (for [QC], [SC], and (3.3)) numerically. Similar to L.StableSwap above, in order to have both a stable price near the “balanced” pool but provide liquidity throughout the price curve, Curve loses (3.5) and (3.6).

A.7 Dodo

In contrast to all other AMMs presented herein, Dodo is constructed based on an *exogenous* pricing oracle (e.g., a centralized exchange) and does not provide its own pricing oracle. In doing so, Dodo permits pooling in any combination of assets rather than guaranteeing the pricing oracle is kept constant. To guarantee that the AMM has the requisite liquidity, withdrawal fees may be assessed; these withdrawal fees take the place of the divergence loss (see Section 5.2) of other AMMs. Mathematically, Dodo takes the form

$$u(x, y) = \log(P\alpha(x, y) + \beta(x, y))$$

for exogenous pricing signal $P$ and such that $\alpha, \beta : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ satisfy price matching and equilibrium pooling, i.e.,

- *price matching*: the value of $\alpha(x, y)$ is equal to $\beta(x, y)$, i.e., $P\alpha(x, y) = \beta(x, y)$;

- *equilibrium pooling*: when an endogenized price $P f(x, y)$ (to account for the actual AMM reserves) is used, the value of the portfolio $(x, y)$ should be equivalent to $(\alpha(x, y), \beta(x, y))$, i.e., $P f(x, y) (\alpha(x, y) - x) + (\beta(x, y) - y) = 0$. Within the current construction of Dodo, the price modifier function $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is defined as

$$f(x, y) := \begin{cases} 1 + C \left( \frac{\alpha(x, y)}{x} - 1 \right) & \text{if } Px \leq y, \\ \left[ 1 + C \left( \frac{\beta(x, y)}{y} - 1 \right) \right]^{-1} & \text{if } Px > y, \end{cases}$$

with $C \in [0, 1]$ for any $x, y \geq 0$.

Note that the construction of Dodo has a parameter $C \in [0, 1]$ for the appropriate notion of equilibrium pooling. If $C = 0$ then $u(x, y) = \log(Px + y)$ is equivalent to the mStable AMM; if $C = 1$ then $u(x, y) = \log(2\sqrt{Pxy}) \sim \log(x) + \log(y)$ is equivalent to the Uniswap V2 AMM. In fact, a closed form for the Dodo
construction can be provided for $C \in [0, 1]$ such that

$$u(x, y) = \begin{cases} 
\log \left( 2 \frac{-(1-C)x + \sqrt{(1-C)^2 P^2 x^2 + CPx(1-C)P^2 + y}}{C} \right) & \text{if } Px \leq y, \\
\log \left( 2 \frac{-(1-C)y + \sqrt{(1-C)^2 y^2 + Cy(Px + (1-C)y)}}{C} \right) & \text{if } Px > y.
\end{cases}$$

As highlighted within Table 2, Dodo satisfies all relevant axioms proposed within this work provided $C > 0$; (3.3), (3.5), and (3.6) are not studied for Dodo due to the use of the exogenous pricing oracle.

**B Proofs**

**B.1 Proof of Theorem 3.7**

*Proof.*

1. Let $C(\bar{z}) := \{ z \in \mathbb{R}_+^2 \mid u(z) \geq u(\bar{z}) \}$ denote the set of positions $z \in \mathbb{R}_+^2$ that exceed $\bar{z} \in \mathbb{R}_+^2$ in utility. By (upper semi)continuity, $C(\bar{z})$ is closed for every $\bar{z} \in \mathbb{R}_+^2$. Therefore $\mathcal{Y}(x) = \sup \{ y \mid y \in [0, b], (a + x, b - y) \in C(a, b) \}$ is the supremum over a compact set and thus is attained.

2. Assume $\mathcal{Y}(x) < b$. Then, by construction of $\mathcal{Y}$, $u(a + x, b - [\mathcal{Y}(x) + \epsilon]) < u(a, b)$ for any $\epsilon \in (0, b - \mathcal{Y}(x)]$.

By (lower semi)continuity, $u(a + x, b - \mathcal{Y}(x)) \leq \lim \inf_{\epsilon \downarrow 0} u(a + x, b - [\mathcal{Y}(x) + \epsilon]) \leq u(a, b)$.

3. Assume $\mathcal{Y}(x) = b$ then $u(a + x, b - \mathcal{Y}(x)) = u(a + x, 0) \geq u(a, b)$ contradicts \([\text{UB}]\).

4. By \([\text{SM}]\), $u(a, b) > u(a, b - y)$ for any $y \in (0, b]$, which immediately results in $\mathcal{Y}(0) = 0$.

5. It trivially follows from \([\text{SM}]\) that $\mathcal{Y}(x) \leq \mathcal{Y}(x + \Delta)$ for any $\Delta \geq 0$. To prove the strict monotonicity, by contradiction, let $\Delta > 0$ and assume $\mathcal{Y}(x) = \mathcal{Y}(x + \Delta)$. Therefore, by \([\text{SM}]\),

$$u(a + x + \Delta, b - \mathcal{Y}(x + \Delta)) = u(a + x + \Delta, b - \mathcal{Y}(x)) > u(a + x, b - \mathcal{Y}(x)) \geq u(a, b).$$

From item (3) within this theorem, we can already conclude that $\mathcal{Y}(x), \mathcal{Y}(x + \Delta) < b$. As such, we now obtain a contradiction to item (2). Therefore, it must follow that $\mathcal{Y}(x + \Delta) > \mathcal{Y}(x)$.

6. By the first claim in item (5) within this theorem and monotone convergence, $\mathcal{Y}(x) \nearrow \mathcal{Y}^*$ for some $\mathcal{Y}^* \leq b$. Assume $\mathcal{Y}^* < b$ (and therefore $\mathcal{Y}(x) \neq b$ for every $x > 0$), then

$$u(a, b) = u(a + x, b - \mathcal{Y}(x)) \geq u(a + x, b - \mathcal{Y}^*) \quad \forall x > 0.$$ 

Taking the limit as $x$ tends to infinity implies $\infty > u(a, b) \geq \lim_{a \to \infty} u(a, b - \mathcal{Y}^*) = \infty$ which forms a contradiction, i.e., $\mathcal{Y}^* = b$. 


7. Note that \( \tilde{C}(x) := \{ y \in [0,b] \mid u(a+x, b-y) \geq u(a,b) \} \) is upper continuous by the closed graph theorem \cite[Theorem 17.11]{4}. Thus by \cite[Lemma 17.30]{4}, \( \mathcal{Y}(x) = \sup \{ y \mid y \in \tilde{C}(x) \} \) is upper semicontinuous.

8. First, under axioms \([C]\) and \([QC]\) the hypograph

\[
\text{hypo } \mathcal{Y} = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R} \mid \mathcal{Y}(x) \geq y \} = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R} \mid y \leq b, u(a+x, b-y) \geq u(a,b) \}
\]

is trivially convex. Now, instead, assume \([SC]\) holds. By implicit differentiation, and with all derivatives of the utility \( u \) being taken at \((a+x, b-\mathcal{Y}(x))\),

\[
\mathcal{Y}''(x) = \frac{u_B[u_Bu_{AA} - u_Au_{AB}] + u_A[u_Au_{BB} - u_Bu_{AB}]}{u_B^3} \leq 0.
\]

9. By item \( 1 \) within this theorem, \( u(a+x, b-\mathcal{Y}(x;a,b)) \geq u(a,b) \). An application of \([SI]\) implies \( u(t[a+x], t[b-\mathcal{Y}(x;a,b)]) \geq u(ta, tb) \) for any \( t > 0 \), i.e., \( \mathcal{Y}(tx; ta, tb) \geq t\mathcal{Y}(x; a, b) \) by construction. The reverse inequality \( \mathcal{Y}(tx; ta, tb) \leq t\mathcal{Y}(x; a, b) \) follows trivially by applying this result to \( (tx; ta, tb) \) with scaling factor \( t^{-1} \).

10. By item \( 3 \) within this theorem, \( \mathcal{Y}(0) = 0 \). Further, by item \( 6 \), \( \mathcal{Y} \) is concave. Combining these properties, \( \mathcal{Y}(x_1) \geq \frac{x_1}{x_1 + x_2} \mathcal{Y}(x_1 + x_2) \) and \( \mathcal{Y}(x_2) \geq \frac{x_2}{x_1 + x_2} \mathcal{Y}(x_1 + x_2) \). Therefore, \( \mathcal{Y}(x_1) + \mathcal{Y}(x_2) \geq \mathcal{Y}(x_1 + x_2) \), i.e., \( \mathcal{Y} \) is subadditive.

11. By items \( 2 \) and \( 3 \), it immediately follows that \( u(a+x, b-\mathcal{Y}(x;a,b)) = u(a,b) \) for any \( x \geq 0 \) and \( a,b > 0 \). By \([SM]\) \( u(a+x, b-\mathcal{Y}(x;a,b) - \epsilon) < u(a,b) \) for any \( x \geq 0 \), \( a,b > 0 \), and \( \epsilon \in (0, b-\mathcal{Y}(x;a,b)] \). Therefore, if \( u(a+x, b-y) = u(a,b) \) it must follow that \( \mathcal{Y}(x;a,b) = y \). Using this result we recover the desired property because:

\[
u(a + x_1, x_2, b - \mathcal{Y}(x_1; a, b) - \mathcal{Y}(x_2; a + x_1, b - \mathcal{Y}(x_1; a, b))) = u(a + x_1, b - \mathcal{Y}(x_1; a, b)) = u(a, b).
\]

\( \square \)

### B.2 Proof of Theorem 3.9

**Proof.**

1. By implicit differentiation (and noting \( u_A, u_B > 0 \) by \([SC]\)):

\[
\mathcal{Y}_a(x; a, b) = \frac{u_A(a+x, b-\mathcal{Y}(x; a, b)) - u_A(a,b)}{u_B(a+x, b-\mathcal{Y}(x; a, b))} \quad \mathcal{Y}_b(x; a, b) = \frac{u_B(a+x, b-\mathcal{Y}(x; a, b)) - u_B(a,b)}{u_B(a+x, b-\mathcal{Y}(x; a, b))} < 1.
\]
Therefore, the results hold if \( \frac{\partial}{\partial x} u_A(a + x, b - \mathcal{Y}(x; a, b)) \leq 0 \) and \( \frac{\partial}{\partial x} u_B(a + x, b - \mathcal{Y}(x; a, b)) \geq 0 \).

Consider these derivatives:

\[
\begin{align*}
\frac{\partial}{\partial x} u_A(a + x, b - \mathcal{Y}(x; a, b)) &= u_{AA}(a + x, b - \mathcal{Y}(x; a, b)) - \mathcal{Y}'(x; a, b) u_{AB}(a + x, b - \mathcal{Y}(x; a, b)) \\
&= u_{AA}(a + x, b - \mathcal{Y}(x; a, b)) - \frac{u_A(a + x, b - \mathcal{Y}(x; a, b))}{u_B(a + x, b - \mathcal{Y}(x; a, b))} u_{AB}(a + x, b - \mathcal{Y}(x; a, b)) \\
&\leq 0, \\
\frac{\partial}{\partial x} u_B(a + x, b - \mathcal{Y}(x; a, b)) &= u_{AB}(a + x, b - \mathcal{Y}(x; a, b)) - \mathcal{Y}'(x; a, b) u_{BB}(a + x, b - \mathcal{Y}(x; a, b)) \\
&= u_{AB}(a + x, b - \mathcal{Y}(x; a, b)) - \frac{u_A(a + x, b - \mathcal{Y}(x; a, b))}{u_B(a + x, b - \mathcal{Y}(x; a, b))} u_{BB}(a + x, b - \mathcal{Y}(x; a, b)) \\
&\geq 0
\end{align*}
\]

taking advantage of \( \mathcal{Y} \) strictly increasing and differentiable.

2. (a) By item (1) within this theorem and monotone convergence, \( \mathcal{Y}(x; \bar{a}, b) \nearrow \mathcal{Y}^* \) for some \( \mathcal{Y}^* \leq b \) as \( \bar{a} \searrow 0 \). Assume \( \mathcal{Y}^* < b \), then

\[
u(\bar{a}, b) = u(\bar{a} + x, b - \mathcal{Y}(x; \bar{a}, b)) \geq u(\bar{a} + x, b - \mathcal{Y}^*) \quad \forall \bar{a} > 0.
\]

Taking the limit as \( \bar{a} \) tends to 0 leads to \( -\infty = \lim_{\bar{a} \searrow 0} u(\bar{a}, b) \geq \lim_{\bar{a} \searrow 0} u(\bar{a} + x, b - \mathcal{Y}^*) = u(x, b - \mathcal{Y}^*) > -\infty \) which forms a contradiction, i.e., \( \mathcal{Y}^* = b \).

(b) For every \( \epsilon > 0 \), we have that \( |\mathcal{Y}(x; \bar{a}, \bar{b}) - \mathcal{Y}^*| < \bar{b} \leq \epsilon \) for any \( \bar{b} \in (0, \epsilon] \) by Theorem 5.7.3.

3. (a) By item (1) within this theorem and monotone convergence, \( \mathcal{Y}(x; \bar{a}, b) \searrow \mathcal{Y}^* \) for some \( \mathcal{Y}^* \geq 0 \) as \( \bar{a} \nearrow \infty \). Assume \( \mathcal{Y}^* > 0 \), then

\[
u(\bar{a}, b) = u(\bar{a} + x, b - \mathcal{Y}(x; \bar{a}, b)) \leq u(\bar{a} + x, b - \mathcal{Y}^*) \quad \forall \bar{a} > 0.
\]

By quasiconcavity this inequality implies \( x u_A(\bar{a}, b) \geq \mathcal{Y}^* u_B(\bar{a}, b) \) for every \( \bar{a} > 0 \); in particular, this implies \( 0 = x \lim_{\bar{a} \to \infty} u_A(\bar{a}, b) \geq \mathcal{Y}^* \lim_{\bar{a} \to \infty} u_B(\bar{a}, b) > 0 \) which forms a contradiction due to \( \text{[1+]} \) i.e., \( \mathcal{Y}^* = 0 \).

(b) By item (1) within this theorem and monotone convergence, \( \mathcal{Y}(x; a, \bar{b}) \nearrow \mathcal{Y}^* \) for some \( \mathcal{Y}^* \leq \infty \) (possibly infinite) as \( \bar{b} \nearrow \infty \). Assume \( \mathcal{Y}^* < \infty \), then

\[
u(a, \bar{b}) = u(a + x, \bar{b} - \mathcal{Y}(x; a, \bar{b})) \geq u(a + x, \bar{b} - \mathcal{Y}^*) \quad \forall \bar{b} > \mathcal{Y}^*.
\]

By quasiconcavity this inequality implies \( \mathcal{Y}^* u_B(a + x, \bar{b} - \mathcal{Y}^*) \geq x u_A(a + x, \bar{b} - \mathcal{Y}^*) \); in particular, this implies \( 0 = \mathcal{Y}^* \lim_{\bar{b} \to \infty} u_B(a + x, \bar{b} - \mathcal{Y}^*) \geq x \lim_{\bar{b} \to \infty} u_A(a + x, \bar{b} - \mathcal{Y}^*) > 0 \) which forms a contradiction due to \( \text{[1+] \ i.e., } \mathcal{Y}^* = \infty \).
B.3 Proof of Lemma 3.10

Proof. We will only prove the first (in)equality, the second follows comparably. Denote \( Y := Y(x; a, b) \) and \( X := X(Y; a + x, b - Y) \). Then, by Theorem 3.7(1),
\[
u(a + x - X, b) = \nu(a + x - X, b - Y + Y) \geq \nu(a + x, b - Y) \geq \nu(a, b).
\]
By (SM) this implies \( a + x - X \geq a \), i.e., \( X \leq x \). If, additionally, (UB) then the inequalities above hold as equalities and the result follows comparably.

B.4 Proof of Proposition 3.11

Proof. First, we will prove that \( P \) is differentiable. By explicitly providing the derivatives, monotonicity of the pricing oracle can be proven directly. With this result, we will prove surjectivity of \( P \) by demonstrating that \( \lim_{\bar{a} \to 0} P(\bar{a}, b) = \infty \), \( \lim_{\bar{b} \to 0} P(a, \bar{b}) = 0 \), \( \lim_{\bar{a} \to \infty} P(\bar{a}, b) = 0 \), and \( \lim_{\bar{b} \to \infty} P(a, \bar{b}) = \infty \) for any \( a, b > 0 \).

As mentioned above, we will prove differentiability and monotonicity by explicitly providing the partial derivatives of \( P \):
\[
\frac{\partial}{\partial a} P(a, b) = \frac{u_B(a, b)u_{AA}(a, b) - u_A(a, b)u_{AB}(a, b)}{u_B(a, b)^2} \leq 0,
\]
\[
\frac{\partial}{\partial b} P(a, b) = \frac{u_B(a, b)u_{AB}(a, b) - u_A(a, b)u_{BB}(a, b)}{u_B(a, b)^2} \geq 0,
\]
using (SC) and the fact that \( P(a, b) = Y'(0; a, b) = u_A(a, b)/u_B(a, b) \).

Consider now the limits:

- Consider \( \lim_{\bar{a} \to 0} P(\bar{a}, b) \): By concavity of \( Y \), \( P(a, b) = Y'(0; a, b) \geq \frac{Y(x; a, b)}{x} \) for any \( x > 0 \). Therefore, by Theorem 3.7(2), \( \lim_{\bar{a} \to 0} P(\bar{a}, b) = \lim_{\bar{a} \to 0} Y(x; \bar{a}, b)/x = b/x \) for any \( x > 0 \). As this inequality holds for any \( x > 0 \), it must follow that \( \lim_{\bar{a} \to 0} P(\bar{a}, b) = \infty \).

- Consider \( \lim_{\bar{b} \to 0} P(a, \bar{b}) \): Recall that \( P(a, b) = \frac{1}{X'(0; a, b)} \) by the provided assumptions. Therefore \( \lim_{\bar{b} \to 0} P(a, \bar{b}) = 0 \) if \( \lim_{\bar{b} \to 0} X'(0; a, \bar{b}) = \infty \). By symmetry of the assets, this is equivalent to \( \lim_{\bar{a} \to 0} Y'(0; a, \bar{b}) = \infty \) which is proven in the prior case.

- Consider \( \lim_{\bar{a} \to \infty} P(\bar{a}, b) \): As in the prior case, recall that \( P(a, b) = \frac{1}{X'(0; a, b)} \) by the provided assumptions. Therefore \( \lim_{\bar{a} \to \infty} P(\bar{a}, b) = 0 \) if \( \lim_{\bar{a} \to \infty} X'(0; a, \bar{b}) = \infty \). By symmetry of the assets, this is equivalent to \( \lim_{\bar{b} \to \infty} Y'(0; a, \bar{b}) = \infty \) which is proven in the next case.
Consider \( \lim_{\beta \to \infty} \mathcal{Y}(a, b) \). As in the first case, by concavity of \( \mathcal{Y} \), \( P(a, b) = \mathcal{Y}(0; a, b) \geq \frac{\mathcal{Y}(x; a, b)}{x} \) for any \( x > 0 \). Therefore, by Theorem 3.11, \( \lim_{\beta \to \infty} P(a, b) \geq \lim_{\beta \to \infty} \mathcal{Y}(x; a, b)/x = \infty \) for any \( x > 0 \).

Finally, the scale invariance of \( P \) follows directly from the positive homogeneity of \( \mathcal{Y} \) (see Theorem 3.7). Specifically, let \( t > 0 \) then

\[
P(ta, tb) = \lim_{\epsilon \to 0} \frac{\mathcal{Y}(\epsilon; ta, tb)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\mathcal{Y}(\epsilon t; ta, tb)}{t \epsilon} = \lim_{\epsilon \to 0} \frac{t \mathcal{Y}(\epsilon; a, b)}{\epsilon} = P(a, b).
\]

B.5 Proof of Theorem 3.12

Proof. Fix \((a, b) \in \mathbb{R}^2_+\). Define \( \beta : [-a, \infty) \to [-b, \infty) \) such that \( P(a, b) = P(a + \delta, b + \beta(\delta)) \) for any \( \delta \in (-a, \infty) \) which is guaranteed to exist by the surjective property of the pricing oracle \( P \) as provided in Proposition 3.11 (and with \( \beta(-a) = -b \)). This proof could comparably be defined w.r.t. \( \alpha : [-b, \infty) \to [-a, \infty) \) constructed as with \( \beta \) but on the first asset. 

Recall \( P(\hat{a}, \hat{b}) := \mathcal{Y}(0; \hat{a}, \hat{b}) = u_A(\hat{a}, \hat{b})/u_B(\hat{a}, \hat{b}) \) for any reserves \((\hat{a}, \hat{b}) \in \mathbb{R}_+^2\). Therefore, by implicit differentiation,
\[
\beta'(\delta) = \frac{P_A(\hat{a}, \hat{b})}{P_B(\hat{a}, \hat{b})}
\]
where \((\hat{a}, \hat{b}) := (a + \delta, b + \beta(\delta))\) for any \( \delta \in (-a, \infty) \).

We will prove this result only in the case of \( \mathcal{Y} \): the monotonicity of \( \mathcal{X} \) follows similarly. For shorthand, define \( \tilde{\mathcal{Y}}(\delta) := \mathcal{Y}(x; a + \delta, b + \beta(\delta)) \) for fixed \( x > 0 \). Then \( u(a + \delta + x, b + \beta(\delta) - \tilde{\mathcal{Y}}(\delta)) = u(a + \delta, b + \beta(\delta)) \) by construction. Using the same construction of \((\hat{a}, \hat{b}) := (a + \delta, b + \beta(\delta))\) as above, by implicit differentiation

\[
\tilde{\mathcal{Y}}'(\delta) = \frac{u_A(\hat{a} + x, \hat{b} - \tilde{\mathcal{Y}}(\delta)) + \beta'(\delta)u_B(\hat{a} + x, \hat{b} - \tilde{\mathcal{Y}}(\delta))}{u_B(\hat{a} + x, \hat{b} - \tilde{\mathcal{Y}}(\delta))}.
\]

Therefore \( \tilde{\mathcal{Y}}'(\delta) \geq 0 \) if

\[
\frac{\partial}{\partial x} \left[ u_A(\hat{a} + x, \hat{b} - \mathcal{Y}(x; \hat{a}, \hat{b})) + \beta'(\delta)u_B(\hat{a} + x, \hat{b} - \mathcal{Y}(x; \hat{a}, \hat{b})) \right] \geq 0.
\]

\[\]
Explicitly computing the derivative ($\ast$), we recover the equivalent condition:

$$
\beta'(\delta) = \frac{P_A(\hat{a} + x, \hat{b} - Y(x; \hat{a}, \hat{b}))}{P_B(\hat{a} + x, \hat{b} - Y(x; \hat{a}, \hat{b}))}
$$

In particular, this holds if

$$
\frac{\partial}{\partial x} \left( \frac{P_A(\hat{a} + x, \hat{b} - Y(x; \hat{a}, \hat{b}))}{P_B(\hat{a} + x, \hat{b} - Y(x; \hat{a}, \hat{b}))} \right) \geq 0.
$$

Explicitly computing the derivative ($\ast\ast$), we recover the desired monotonicity:

$$(\ast\ast) := \frac{P_B(\hat{z})P_{AA}(\hat{z}) - [P(\hat{z})P_B(\hat{z}) + P_A(\hat{z})]P_{AB}(\hat{z}) + P(\hat{z})P_A(\hat{z})P_{BB}(\hat{z})}{P_B(\hat{z})^2} \geq 0$$

by assumption where $\hat{z} := (\hat{a} + x, \hat{b} - Y(x; \hat{a}, \hat{b}))$.

**B.6 Proof of Corollary 3.14**

*Proof.~* The decrease of the price impact from swapping inequalities, i.e. the inequalities in (3.3) follow immediately from the definition of $I_Y$, $I_X$ and (3.1), (3.2) shown in Theorem 3.12.

**B.7 Proof of Proposition 3.16**

*Proof.~* To show the approximating power of the price impact oracle, we will only investigate the bounds for $Y$; similar arguments can be made for the bounds on $X$. Fix $(a, b) \in \mathbb{R}_+^2$ and let $z(x) := (a + x, b - Y(x; a, b))$. By simple differentiation, we find that

$$
Y'(x) = P(z(x)), \quad (B.1)
$$

$$
Y''(x) = P_A(z(x)) - P(z(x))P_B(z(x)), \quad (B.2)
$$

$$
Y'''(x) = P_{AA}(z(x)) - 2P(z(x))P_{AB}(z(x)) + P(z(x))^2P_{BB}(z(x)) + (P(z(x))P_B(z(x)) - P_A(z(x)))P_B(z(x))
$$

for any $x \geq 0$. Therefore, since $Y$ is thrice continuously differentiable at $x = 0$, we have the Taylor expansion of $Y$ around zero as

$$
Y(x) = Y(0) + ay'(0)\frac{x}{a} + \frac{a^2}{2}y''(0)\left(\frac{x}{a}\right)^2 + O\left(\left(\frac{x}{a}\right)^3\right)
$$

$$
= P(a, b)x - \frac{1}{2}(P(a, b)P_B(a, b) - P_A(a, b))x^2 + O\left(\left(\frac{x}{a}\right)^3\right). \quad (B.3)
$$
For the second part of the proposition, first let the conditions of Theorem 3.12 hold. Let \( \beta : [-a, \infty) \to [-b, \infty) \) be defined as in the proof of Theorem 3.12. To simplify notation, let \((\hat{a}, \hat{b}) := (a + \delta, b + \beta(\delta))\) for \( \delta \in (-a, \infty) \). The result follows by taking the derivative of the price impact oracle w.r.t. the added liquidity \( \delta \) (noting that \( P(a, b) = P(\hat{a}, \hat{b}) \)):

\[
\frac{\partial}{\partial \delta} I(\hat{a}, \hat{b}) = \frac{\partial}{\partial \delta} [P(a, b)P_B(\hat{a}, \hat{b}) - P_A(\hat{a}, \hat{b})] \\
= -\frac{P_B(\hat{a}, \hat{b})P_{AA}(\hat{a}, \hat{b}) - (P(a, b)P_B(\hat{a}, \hat{b}) + P_A(\hat{a}, \hat{b}))P_{AB}(\hat{a}, \hat{b}) + P(a, b)P_A(\hat{a}, \hat{b})P_{BB}(\hat{a}, \hat{b})}{P_B(\hat{a}, \hat{b})} \\
= -\frac{P_B(\hat{a}, \hat{b})P_{AA}(\hat{a}, \hat{b}) - (P(\hat{a}, \hat{b})P_B(\hat{a}, \hat{b}) + P_A(\hat{a}, \hat{b}))P_{AB}(\hat{a}, \hat{b}) + P(\hat{a}, \hat{b})P_A(\hat{a}, \hat{b})P_{BB}(\hat{a}, \hat{b})}{P_B(\hat{a}, \hat{b})} \leq 0.
\]

Now assume \([SC] \) and \([SI] \). Note that, as in Remark 8, \( \beta(\delta) = \delta b/a \) for any \( \delta \in (-a, \infty) \). Therefore, the modified liquidity \((\hat{a}, \hat{b}) = (ta, tb)\) for some \( t > 0 \) and, as such, we can prove the desired monotonicity in liquidity by demonstrating that the price impact oracle is positive homogeneous of degree \(-1\). Fix \( t > 0 \) then, by the positive homogeneity of \( \mathcal{Y} \) (see Theorem 3.7[9]) and the scale invariance of \( P \) (see Proposition 3.11[11]),

\[
I(ta, tb) = -\frac{1}{2} \mathcal{Y}''(0; ta, tb) = -\frac{1}{2} \lim_{\epsilon \to 0} \frac{\mathcal{Y}'(\epsilon; ta, tb) - \mathcal{Y}'(0; ta, tb)}{\epsilon} \\
= -\frac{1}{2} \lim_{\epsilon \to 0} \frac{\mathcal{Y}'(\epsilon; ta, tb) - \mathcal{Y}'(0; ta, tb)}{\epsilon} = -\frac{1}{2} \lim_{\epsilon \to 0} \frac{P(t[a + \epsilon], t[b - \mathcal{Y}(\epsilon; a, b)]) - P(ta, tb)}{\epsilon} \\
= -\frac{1}{2} \lim_{\epsilon \to 0} \frac{P(a + \epsilon, b - \mathcal{Y}(\epsilon; a, b)) - P(a, b)}{\epsilon} = \frac{1}{t} I(a, b).
\]

\(\square\)

B.8 Proof of Corollary 3.17

**Proof.** In the same setting as in the proof of Proposition 3.16, the upper and the lower bounds trivially follow from the Taylor expansion of \( \mathcal{Y} \) in (B.3), and the assumptions of the lemma that the derivatives in (B.1)–(B.2) satisfy \( \mathcal{Y}'(x) > 0, \mathcal{Y}''(x) \leq 0, \) and \( \mathcal{Y}'''(x) \geq 0 \) for any \( x \geq 0 \). The bounds for \( I_X \) can similarly be proven. Finally, by Proposition 3.11[11], \( I(a, b) \geq 0 \) for any \( (a, b) \in \mathbb{R}_{++}^2 \). \(\square\)

B.9 Proof of Lemma 5.3

**Proof.** We will prove the result for \( \mathcal{Y}_\gamma \) only, the proof for \( \mathcal{X}_\gamma \) follows comparably. Consider the ODE representation (5.3) and note the domain \( \text{dom } g := \mathbb{R} \times [0, b) \). By Proposition 5.11[11] \( g \) and \( \frac{\partial}{\partial x} g, \frac{\partial}{\partial y} g \) are continuous, and thus bounded, on this domain. Therefore, by the Picard-Lindelöf Theorem and extension

\(\[\text{As noted within the proof of Theorem 3.12 we can take } \alpha : [-b, \infty) \to [-a, \infty) \text{ instead if } \beta \text{ is not well-defined for every reserve level.}\]\)
theorem (e.g., [22, Theorem II.1.1 and Theorem II.3.1]), there exists a unique solution \( \mathcal{Y}_\gamma(x) \) on some maximal domain \( x \in [0,x^\ast) \) for some \( x^\ast > 0 \). Furthermore, by the extension theorem, if \( x^\ast < \infty \) then \( \lim_{x \to x^\ast} \mathcal{Y}_\gamma(x) \in \{0,b\} \). Trivially, \( \mathcal{Y}_\gamma(x) > 0 \) for \( x > 0 \) by \( g(x,y) > 0 \) for any \( y \in [0,b) \). Assume now that \( x^\ast < \infty \) and \( \lim_{x \to x^\ast} \mathcal{Y}_\gamma(x) = b \). To complete this proof, we will demonstrate that \( \mathcal{Y}_\gamma(x) \leq \mathcal{Y}(x) \) for any \( x \in [0,x^\ast) \), from which it will follow that \( \lim_{x \to x^\ast} \mathcal{Y}_\gamma(x) \leq \lim_{x \to x^\ast} \mathcal{Y}(x) = \mathcal{Y}(x^\ast) < b \) (by Theorem 3.7(3)) to reach a contradiction. To show that \( \mathcal{Y}_\gamma(x) \leq \mathcal{Y}(x) \) for any \( x \in [0,x^\ast) \), assume by contradiction that this inequality is false and take \( x^\dagger := \inf\{x \in [0,x^\ast) \mid \mathcal{Y}_\gamma(x) > \mathcal{Y}(x)\} < x^\ast \). Note that \( \mathcal{Y}_\gamma(x^\dagger) = \mathcal{Y}(x^\dagger) \) by a simple continuity argument, and \( \mathcal{Y}_\gamma(x^\dagger + \epsilon) > \mathcal{Y}(x^\dagger + \epsilon) \) for every \( \epsilon \in (0,\delta) \) for some \( \delta > 0 \). However, this implies \( \mathcal{Y}_\gamma'(x^\dagger) = (1-\gamma)g(x^\dagger, \mathcal{Y}_\gamma(x^\dagger)) \leq g(x^\dagger, \mathcal{Y}(x^\dagger)) = \mathcal{Y}'(x^\dagger) \) which results in a simple contradiction.

\[\Box\]

**B.10 Proof of Proposition 5.5**

**Proof.** As in prior proofs, we will provide the proof of this result for \( \mathcal{Y}_\gamma \) only as the proof for \( \mathcal{X}_\gamma \) follows comparably. First, we wish to note that the non-strict monotonicity encoded in \( (5.4) \) follows as a simple continuity argument, and \( \mathcal{Y}_\gamma(x^\dagger + \epsilon) > \mathcal{Y}(x^\dagger + \epsilon) \) for every \( \epsilon \in (0,\delta) \) for some \( \delta > 0 \). However, this implies \( \mathcal{Y}_\gamma'(x^\dagger) = (1-\gamma)g(x^\dagger, \mathcal{Y}_\gamma(x^\dagger)) \leq g(x^\dagger, \mathcal{Y}(x^\dagger)) = \mathcal{Y}'(x^\dagger) \) which results in a simple contradiction.

\[\Box\]

**B.11 Proof of Corollary 5.6**

**Proof.**

1. Let \( x > 0 \) as the case of equality when \( x = 0 \) is trivial. By Theorem 3.7(2) and Proposition 5.5 and recalling from Remark 13 that \( \mathcal{Y}_0 \equiv \mathcal{Y}_\gamma u(a + x, b - \mathcal{Y}_\gamma(x)) > u(a + x, b - \mathcal{Y}(x)) = u(a, b) \).

2. (a) Strict monotonicity of \( \mathcal{Y}_\gamma \) in \( x \) follows trivially from its integral representation as the pricing oracle is strictly positive on positive pool sizes.

(b) By implicit differentiation of \( (5.3) \) and the monotonicity of the pricing oracle as provided in Proposition 5.11 \( \mathcal{Y}_\gamma''(x) = (1-\gamma)[P_A(a + x, b - \mathcal{Y}_\gamma(x)) - (1-\gamma)P(a + x, b - \mathcal{Y}_\gamma(x))P_B(a + x, b - \mathcal{Y}_\gamma(x))] \).
\[ \mathcal{Y}_\gamma(x) \leq 0. \]

(c) Subadditivity follows from concavity and \( \mathcal{Y}_\gamma(0) = 0 \) as demonstrated in the proof of Theorem 3.4(10).

3. Note that \( c_B(b) := \sup_{a>0} u_B(\bar{a}, b) < \infty \) for every \( b > 0 \), because \( u_B \) is continuous with limiting behavior \( \lim_{\bar{a} \to 0} u_B(\bar{a}, b), \lim_{\bar{a} \to \infty} u_B(\bar{a}, b) < \infty \). Assume by contradiction that \( \lim_{x \to \infty} \mathcal{Y}_\gamma(x) = \mathcal{Y}_\gamma^* < b \). Recall \( \mathcal{Y}_\gamma \) is strictly monotonic, therefore \( \mathcal{Y}_\gamma(x) < \mathcal{Y}_\gamma^* \) for every \( x \). Therefore,

\[
\lim_{x \to \infty} \mathcal{Y}_\gamma(x) = \lim_{x \to \infty} (1 - \gamma) \int_0^x P(a + y, b - \mathcal{Y}_\gamma(y)) dy \geq (1 - \gamma) \int_0^\infty P(a + y, b - \mathcal{Y}_\gamma^*) dy
\]

\[
= (1 - \gamma) \int_0^\infty \frac{u_A(a + y, b - \mathcal{Y}_\gamma^*)}{u_B(a + y, b - \mathcal{Y}_\gamma^*)} dy \geq \frac{(1 - \gamma) \int_0^\infty u_A(a + y, b - \mathcal{Y}_\gamma^*) dy}{c_B(b - \mathcal{Y}_\gamma^*)}
\]

\[
= \frac{1 - \gamma}{c_B(b - \mathcal{Y}_\gamma^*)} \left( \lim_{x \to \infty} u(a + x, b - \mathcal{Y}_\gamma^*) - u(a, b - \mathcal{Y}_\gamma^*) \right) = \infty.
\]

4. Let \( t > 0 \). From (5.3) it follows that

\[
\frac{\partial}{\partial x} \mathcal{Y}_\gamma(tx; ta, tb) = (1 - \gamma) P(ta + tx, tb - \mathcal{Y}_\gamma(tx; ta, tb)) = (1 - \gamma) P(a + x, b - \frac{1}{t} \mathcal{Y}_\gamma(tx; ta, tb)).
\]

Since for \( x = 0 \), we have that both \( \frac{1}{t} \mathcal{Y}_\gamma(0; ta, tb) = \mathcal{Y}_\gamma(0; a, b) \), we can appeal to the existence and uniqueness result of Lemma 5.3 to recover \( \frac{1}{t} \mathcal{Y}_\gamma(tx; ta, tb) = \mathcal{Y}_\gamma(x; a, b) \), i.e., positive homogeneity.

5. From (5.3) it follows that

\[
\mathcal{Y}_\gamma(x_1 + x_2; a, b) = (1 - \gamma) P(a + x_1 + x_2, b - \mathcal{Y}_\gamma(x_1; a, b) - [\mathcal{Y}_\gamma(x_1 + x_2; a, b) - \mathcal{Y}_\gamma(x_1; a, b)]) \tag{B.5}
\]

Note also that

\[
\mathcal{Y}_\gamma(x_1 + x_2; a, b) = \frac{\partial}{\partial x_2} [\mathcal{Y}_\gamma(x_1 + x_2; a, b) - \mathcal{Y}_\gamma(x_1; a, b)]. \tag{B.6}
\]

Combining (B.5) and (B.6), it follows that

\[
\frac{\partial}{\partial x_2} [\mathcal{Y}_\gamma(x_1 + x_2; a, b) - \mathcal{Y}_\gamma(x_1; a, b)]
\]

\[
= (1 - \gamma) P(a + x_1 + x_2, b - \mathcal{Y}_\gamma(x_1; a, b) - [\mathcal{Y}_\gamma(x_1 + x_2; a, b) - \mathcal{Y}_\gamma(x_1; a, b)])
\]

We also have, by construction of (5.3), that

\[
\mathcal{Y}_\gamma'(x_2; a + x_1, b - \mathcal{Y}_\gamma(x_1; a, b))
\]

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\[(1 - \gamma)P(a + x_1 + x_2, b - \mathcal{Y}_\gamma(x_1; a, b) - \mathcal{Y}_\gamma(x_2; a + x_1, b - \mathcal{Y}_\gamma(x_1; a, b)))\]

Since for \(x_2 = 0\), we have that both \(\mathcal{Y}_\gamma(x_1 + x_2) - \mathcal{Y}_\gamma(x_1) = 0 = \mathcal{Y}_\gamma(x_2; a + x_1, b - \mathcal{Y}_\gamma(x_1; a, b))\), we can appeal to the existence and uniqueness result of Lemma 5.3 to conclude

\[\mathcal{Y}_\gamma(x_1 + x_2; a, b) - \mathcal{Y}_\gamma(x_1; a, b) = \mathcal{Y}_\gamma(x_2; a + x_1, b - \mathcal{Y}_\gamma(x_1; a, b))\]

for any \(x_1, x_2 \geq 0\).

\[\square\]

### B.12 Proof of Corollary 5.7

**Proof.** We will only prove the first inequality, the second follows similarly. By Theorem 3.7(5), Theorem 3.9(1), and Proposition 5.5

\[\mathcal{X}_\gamma(\mathcal{Y}_\gamma(x; a, b); a + x, b - \mathcal{Y}_\gamma(x; a, b)) < \mathcal{X}(\mathcal{Y}_\gamma(x; a, b); a + x, b - \mathcal{Y}_\gamma(x; a, b))\]

\[\leq \mathcal{X}(\mathcal{Y}_\gamma(x; a, b); a + x, b - \mathcal{Y}(x; a, b))\]

\[< \mathcal{X}(\mathcal{Y}(x; a, b); a + x, b - \mathcal{Y}(x; a, b)) = x.\]

\[\square\]

### B.13 Proof of Corollary 5.8

**Proof.** We will prove this result for \(\frac{\partial}{\partial a} \mathcal{Y}_\gamma(x; a, b) \leq 0\) only; the proof for \(\frac{\partial}{\partial b} \mathcal{Y}_\gamma(x; a, b) \in (0, 1]\) follows similarly.

Following the same strategy as in the proof of Proposition 5.5 we can construct the ODE

\[\frac{\partial}{\partial a} \mathcal{Y}_\gamma(x; a, b) = (1 - \gamma)[P_A(a + x + b - \mathcal{Y}_\gamma(x; a, b)) - \frac{\partial}{\partial a} \mathcal{Y}_\gamma(x; a, b)P_B(a + x, b - \mathcal{Y}_\gamma(x; a, b))]\]

with initial condition \(\frac{\partial}{\partial a} \mathcal{Y}_\gamma(0; a, b) = 0\) for every \(a, b > 0\). By solving this ODE, we recover

\[\frac{\partial}{\partial a} \mathcal{Y}_\gamma(x; a, b) = (1 - \gamma) \int_0^x e^{-(1-\gamma) t} P_B(a + u, b - \mathcal{Y}_\gamma(u; a, b)) du \int_0^t P_A(a + t, b - \mathcal{Y}_\gamma(t; a, b)) dt, \ \forall x \geq 0.\]

From Proposition 3.11 it now follows that \(\frac{\partial}{\partial a} \mathcal{Y}_\gamma(x; a, b) \leq 0.\)

\[\square\]
B.14 Proof of Lemma 5.10

Proof. We will prove these results for \( p < P(a, b) \) only; the case for \( p > P(a, b) \) follows similarly. To simplify notation, let \( \mathcal{V}_\gamma(p) := \mathcal{V}_\gamma(x_p; (1+\delta)a, (1+\delta)b) \). Furthermore, recall that \( p = P((1+\delta)a + x_p, (1+\delta)b - \mathcal{V}_\gamma(p)) = \frac{1}{1-\gamma} \mathcal{V}_\gamma'(x_p; (1+\delta)a, (1+\delta)b) \). First, we wish to demonstrate that we recover the simplified version of the divergence loss:

\[
\Delta(p) = \delta [pa + b] - \delta \left[ p((1+\delta)a + x_p) + ((1+\delta)b - \mathcal{V}_\gamma(p)) \right] = \frac{\delta}{1+\delta} [(1+\delta)(pa + b) - p((1+\delta)a + x_p) - ((1+\delta)b - \mathcal{V}_\gamma(p))] = \frac{\delta}{1+\delta} [\mathcal{V}_\gamma(p) - px_p].
\]

Therefore the sign of the divergence loss \( \Delta(p) \) is completely characterized by the sign of \( \mathcal{V}_\gamma(p) - px_p \). As there is a one-to-one relation between \( \Delta \) and \( \mathcal{V}_\gamma \), we will consider the question of the sign of \( \mathcal{V}_\gamma(x; (1+\delta)a, (1+\delta)b) - P((1+\delta)a + x, (1+\delta)b - \mathcal{V}_\gamma(x; (1+\delta)a, (1+\delta)b)) \) as \( x \geq 0 \) varies (corresponding to \( p \leq P(a, b) \)). To simplify notation, as before we will drop the arguments of these functions where the meaning is clear.

Note that \( [\mathcal{V}_\gamma - Px]_{x=0} = 0 \) by construction and \( \frac{\partial}{\partial x} [\mathcal{V}_\gamma - Px] = -\gamma P + x((1-\gamma)PP_B - PA) \). Recall from Proposition 3.11 that \( PA \leq 0 \) and \( PB \geq 0 \). Therefore, at \( \gamma = 0 \), \( \mathcal{V}_\gamma - Px \geq 0 \) for any \( x > 0 \) with equality if and only if \( P((1+\delta)a + x, (1+\delta)b - \mathcal{V}_\gamma(x; (1+\delta)a, (1+\delta)b)) = P(a, b) \), i.e., where \( p = P(a, b) \).

If \( \gamma \in (0, 1) \) then, for \( x \) small enough, \( \frac{\partial}{\partial x} [\mathcal{V}_\gamma - Px] < 0 \); thus there exists some \( x_* \in (0, \infty) \) such that \( \mathcal{V}_\gamma - Px < 0 \) for every \( x \in (0, x^*) \) and \( [\mathcal{V}_\gamma - Px]_{x=x^*} = 0 \). By the relation between \( x \) and \( p \), we can define \( p_* := P((1+\delta)a + x_*, (1+\delta)b - \mathcal{V}_\gamma(x_*; (1+\delta)a, (1+\delta)b)) \). \( \square \)

C Comparison to Prior Axioms and Definitions

Within this section, we wish to highlight four other contemporary papers which, independently, provide generalized definitions of AMMs. Within this section we highlight how the axioms proposed herein relate to the definitions used within [12, 19, 25, 8]. For ease of reference, these results are summarized within Table 3.

First, [12] introduces four properties for the utility function \( u : \mathbb{R}^2_+ \rightarrow \mathbb{R} \cup \{-\infty\} \) of an AMM within Assumption 1 of that work. Briefly, that paper assumes that:

1. Positive derivatives: \( u_A(z), u_B(z) > 0 \) for every \( z \in \mathbb{R}^2_+ \) is a stronger version of \([\text{SM}]\) though, we note, we also assume this property within \([\text{SC}]\).

2. Convexity: \( u_{AA}(z), u_{BB}(z) < 0, u_{AB}(z) > 0 \) for every \( z \in \mathbb{R}^2_+ \) was discussed in more details within Remark 1 and (along with the prior property) implies \([\text{SC}]\).
3. **Homogenous of degree \( l \):** \( \exists l > 0 : \forall c \geq 0, \ c^l u(z) = u(cz) \) for every \( z \in \mathbb{R}^2_+ \) is strictly stronger than \([\text{SI}]\) and

4. **Surjective in price:** \( \lim_{x \to 0} P(x, y) = \infty, \ \lim_{x \to \infty} P(x, y) = 0 \), \\
\( \lim_{y \to 0} P(x, y) = \infty, \ \lim_{y \to \infty} P(x, y) = 0 \) corresponds to the desired property \([\text{AP}]\) which we find is implied by \([\text{UIB}] \ [\text{QC}] \ [\text{I+}] \) and \([\text{SC}]\) within Proposition 3.11.

Second, [19] independently considered utility functions \( u : \mathbb{R}^2_+ \to \mathbb{R} \cup \{-\infty\} \) that were strictly increasing (i.e., \([\text{SM}]\)) and quasiconcave (i.e., \([\text{QC}]\)). Notably, that work makes no further assumptions upon the AMMs under consideration. We highlight in Table I how these two axioms alone are insufficient to guarantee most desired properties for the markets made by an AMM.

Third, [25] introduces a number of properties for the utility functions \( u : \mathbb{R}^2_+ \to \mathbb{R} \cup \{-\infty\} \). Briefly, that paper assumes that:

1. **Existence of marginal prices:** \( u \) is differentiable everywhere implies \([\text{C}]\) and, in turn, is implied by \([\text{SC}]\) where we assume twice-differentiability;

2. **Aversion to permanent loss:** \( \{ z \in \mathbb{R}^2_+ | u(z) \geq u(\bar{z}) \} \) is convex for every \( \bar{z} \in \mathbb{R}^2_+ \) is equivalent to \([\text{QC}]\);

3. **Sufficient funds:** \( u(z) = u(\bar{z}) \) with \( \bar{z} \in \mathbb{R}^2_{++} \) implies \( z \in \mathbb{R}^2_{++} \) provides the same implications as \([\text{UIB}]\) used herein;

4. **Scale invariance:** \( u(z) = u(\bar{z}) \) implies \( u(tz) = u(t\bar{z}) \) for any \( z, \bar{z} \in \mathbb{R}^2_+ \) and \( t > 0 \) is equivalent to \([\text{SI}]\);

5. **Homogeneity in liquidity:** \( u(tz) = tu(z) \) for any \( z \in \mathbb{R}^2_+ \) and \( t > 0 \) is strictly stronger than \([\text{SI}]\);

6. **Translation invariance:** \( u(z) = u(\bar{z}) \) implies \( u(z + t\mathbf{1}) = u(\bar{z} + t\mathbf{1}) \) for any feasible \( t \in \mathbb{R} \) is only applicable for proving the equivalence to scoring rules for prediction markets (as in [14]);

7. **One invariance:** \( u(z + t\mathbf{1}) = u(z) + t \) is a stronger translation invariance property; and

8. **Symmetry:** \( u(a, b) = u(b, a) \) is not explicitly considered herein though we note that all examples in Section 4 satisfy this property.

Finally, [8] considers a geometric representation for AMMs rather than the functional approach taken elsewhere. Specifically, in that work, the authors consider the reachable set \( S_z = \{ \bar{z} \in \mathbb{R}^2 | u(\bar{z}) \geq u(z) \} \)

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13 We wish to note that [25] proposes all of these axioms within a multi-asset market. For ease of comparison, we consider only the two asset case herein.
as the set of positions that can be attained through trading with the AMM (the equivalence to the utility formulation provided is the functional equivalent). Briefly, that paper assumes that:

1. **Non-empty and non-negative reserves:** \( \emptyset \neq S_z \subseteq \mathbb{R}^2_+ \) for any \( z \in \mathbb{R}^2_+ \) is equivalent to \( \text{dom} \ u \subseteq \mathbb{R}^2_+ \);

2. **Closed:** \( S_z \) is closed for any \( z \in \mathbb{R}^2_+ \) is equivalent to the upper semicontinuity of \( u \) (and therefore implied by, e.g., \([C]\));

3. **Convex:** \( S_z \) is convex for any \( z \in \mathbb{R}^2_+ \) is equivalent to \([QC]\) and

4. **Upward closed:** \( S_z + \mathbb{R}^2_+ = S_z \) for any \( z \in \mathbb{R}^2_+ \) is implied by \([SM]\) (and is equivalent to the non-strict monotonicity).

\[\textbf{14} \] We wish to note that \([8]\) proposes all of these axioms within a multi-asset market. For ease of comparison, we consider only the two asset case herein.
### Table 3: Summary of properties defined within [12, 19, 25] and the relation to the axioms provided herein.

| Reference | Property | (UB) | (UA) | (SM) | (C) | (QC) | (SI) | (I+) | (SC) | 3.3 | 3.5 | 3.6 |
|-----------|----------|------|------|------|-----|------|------|------|------|------|------|------|
| [12]      | Positive derivatives \(u_A, u_B > 0\) | ⇒ | ⇒ | | | | | | | | | |
|           | Convexity \(u_{AA}, u_{BB} < 0, u_{AB} > 0\) | | | | | | | | | | | | 3.3 |
|           | Homogeneous of degree \(l\) \(\exists l > 0 : c'u(z) = u(cz)\) | ⇒ | | | | | | | | | | | 3.5 |
|           | Surjective in price \(\lim_{x \to 0} P(x, y) = \infty, \lim_{x \to \infty} P(x, y) = 0, \lim_{y \to 0} P(x, y) = 0, \lim_{y \to \infty} P(x, y) = \infty\) | ≤ | ≤ | ≤ | ≤ | | | | | | | | 3.6 |
| [19]      | Strictly increasing | X | | | | | | | | | | | |
|           | Quasiconcave | | | | | | | | | | | | |
| [25]      | Existence of marginal prices \(u\) is differentiable \(\{z | u(z) \geq u(\bar{z})\}\) convex | ⇒ | | | | | | | | | | | |
|           | Aversion to permanent loss \(u(z) = u(\bar{z})\) then \(z \in \mathbb{R}^2_+ \Rightarrow \bar{z} \in \mathbb{R}^2_+\) | X | | | | | | | | | | | |
|           | Sufficient funds \(u(z) = u(\bar{z})\) implies \(u(tz) = u(t\bar{z})\) | | | | | | | | | | | | |
|           | Scale invariance \(u(z) = u(\bar{z})\) implies \(u(tz) = u(t\bar{z})\) | | | | | | | | | | | | |
|           | Homogeneity in liquidity \(u(tz) = tu(z)\) | | | | | | | | | | | | |
|           | Translation invariance \(u(z) = u(\bar{z}) \Rightarrow u(z + t1) = u(\bar{z} + t1)\) | | | | | | | | | | | | |
|           | One invariance \(u(z + t1) = u(z) + t\) | | | | | | | | | | | | |
|           | Symmetry \(u(a,b) = u(b,a)\) | | | | | | | | | | | | |
| [3]       | Non-empty and non-negative reserves \(\emptyset \neq S_z := \{\bar{z} | u(\bar{z}) \geq u(z)\} \subseteq \mathbb{R}^2_+\) | | | | | | | | | | | | |
|           | Closed \(S_z\) is closed \(S_z\) is convex | | | | | | | | | | | | |
|           | Upward closed \(S_z + \mathbb{R}^2_+ = S_z\) | X | | | | | | | | | | | |

X: Equivalence of axioms.

⇒: The property from the external paper implies our axiom.

⇐: The collection of our axioms imply the property from the external paper.

*: Convexity \((u_{AA}, u_{BB} < 0, u_{AB} > 0)\) implies [SC] if positive derivatives \((u_A, u_B > 0)\) is also assumed.

†: Upward closed \((S_z + \mathbb{R}^2_+ = S_z)\) is equivalent to nondecreasing without requiring the strict monotonicity.