The recently proved ‘no short hair’ theorem asserts that, if a spherically-symmetric static black hole has hair, then this hair (the external fields) must extend beyond the null circular geodesic (the “photonsphere”) of the corresponding black-hole spacetime: \( r_{\text{field}} > r_{\text{null}} \). In this paper we provide compelling evidence that the bound can be violated by non-spherically symmetric hairy black-hole configurations. To that end, we analytically explore the physical properties of cloudy Kerr-Newman black-hole spacetimes – charged rotating black holes which support linearized stationary charged scalar configurations in their exterior regions. In particular, for given parameters \( \{M, Q, J\} \) of the central black hole, we find the dimensionless ratio \( q/\mu \) of the field parameters which minimizes the effective lengths (radii) of the exterior stationary charged scalar configurations (here \( \{M, Q, J\} \) are respectively the mass, charge, and angular momentum of the black hole, and \( \{\mu, q\} \) are respectively the mass and charge coupling constant of the linearized scalar field). This allows us to prove explicitly that (non-spherically symmetric non-static) composed Kerr-Newman-charged-scalar-field configurations can violate the no-short-hair lower bound. In particular, it is shown that extremely compact stationary charged scalar ‘clouds’, made of linearized charged massive scalar fields with the property \( r_{\text{field}} \to r_H \), can be supported in the exterior spacetime regions of extremal Kerr-Newman black holes (here \( r_{\text{field}} \) is the peak location of the stationary scalar configuration and \( r_H \) is the black-hole horizon radius). Furthermore, we prove that these remarkably compact stationary field configurations exist in the entire range \( s \equiv J/M^2 \in (0, 1) \) of the dimensionless black-hole angular momentum. In particular, in the large-mass limit they are characterized by the simple dimensionless ratio \( q/\mu = (1 - 2s^2)/(1 - s^2) \).

I. INTRODUCTION.

Wheeler’s famous conjecture that “black holes have no hair” \[1, 2\] predicts a simple and universal fate for all dynamical black-hole spacetimes \[3\]: the matter fields outside the horizon are expected to be swallowed by the black hole or to be scattered away to infinity, thus leaving behind a stationary “bald” Kerr-Newman black hole \[4–6\]. The no-hair conjecture therefore suggests that, within the framework of classical general relativity, black holes are fundamental objects which possess only three conserved physical parameters: mass \( M \), charge \( Q \), and angular momentum \( J \).

The no-hair conjecture predicts, in particular, that asymptotically flat black holes cannot support static matter configurations in their exterior regions. Early studies of the coupled Einstein-matter equations have indeed ruled out the existence of regular black-hole solutions with static scalar hair \[7\], static spinor hair \[8\], and static massive vector hair \[9\]. These early no-hair theorems have therefore supported the simple physical picture suggested by the no-hair conjecture \[1, 2\].

However, the somewhat surprising discovery of regular black-hole solutions \[10\] to the Einstein-Yang-Mills equations \[11\] has revealed that coupled Einstein-matter systems may exhibit a more complex behavior. The numerical discovery of these ‘colored’ black holes \[11\], which provided the first genuine counterexample to the no-hair conjecture, has motivated many researches to search for other types of non trivial hairy black-hole configurations. In fact, it is by now well established that black holes can support various types of non-linear matter fields (that is, matter fields with self-interaction terms) in their exterior regions \[11, 22\].

The hairy black-hole solutions discovered numerically in \[11, 22\] provide compelling evidence that the no-hair conjecture, in its original formulation \[11, 22\], may be violated \[23\]. Accepting the fact that hairy black-hole configurations do exist in general relativity, it is natural to consider the following interesting question: What are the generic features of hairy black-hole spacetimes?

This question was partially addressed in \[24\], where a ‘no short hair’ theorem for spherically-symmetric static black holes was proved. This no-short-hair theorem asserts that, if a spherically-symmetric static black hole has hair, then this hair (i.e., the external matter fields) must extend beyond the null circular geodesic (the “photonsphere”) of the corresponding black-hole spacetime:

\[
r_{\text{field}} > r_{\text{null}}.
\]

It is worth noting that, within the static sector of spherically-symmetric hairy black-hole spacetimes, the no-short-hair lower bound \[11\] is universal in the sense that it is independent of the parameters of the exterior matter fields
One may therefore regard the no-short-hair relation (1) as a more modest alternative (at least within the spherically-symmetric sector of hairy black-hole configurations) to the original [1, 2] no hair conjecture.

It is important to stress the fact that the formal proof provided in [24] for the existence of the no-short-hair property (1) is restricted to the relatively simple case of spherically-symmetric static hairy black-hole spacetimes. The main goal of the present paper is to challenge the validity of this no-short-hair relation (1) beyond the regime of spherically-symmetric static black holes. To that end, we shall study analytically the physical properties of non-spherically symmetric non-static Kerr-Newman black holes linearly coupled to stationary (rather than static) charged massive scalar fields.

II. COMPOSED BLACK-HOLE-SCALAR-FIELD CONFIGURATIONS.

It is important to emphasize that while existing no-hair theorems [7] rigorously rule out the existence of static regular hairy black-hole-scalar-field configurations, they do not rule out the existence of non-static scalar field configurations in the exterior regions of black-hole spacetimes.

In fact, recent analytical [27] and numerical [28] explorations of the Einstein-scalar and Einstein-Maxwell-scalar equations have revealed that stationary configurations of massive scalar fields (with or without electric charge) can be supported in the exterior spacetime regions of non-spherically symmetric rotating black holes.

These non-static spatially regular black-hole-scalar-field configurations [27, 28] owe their existence to the well-established phenomenon of superradiant scattering [29–31] of bosonic fields in black-hole spacetimes [32]. In particular, these exterior stationary field configurations have azimuthal frequencies $\omega_{\text{field}}$ which coincide with the critical (threshold) frequency $\omega_c$ for superradiant scattering in the charged rotating black-hole spacetime [33]:

$$\omega_{\text{field}} = \omega_c = m\Omega_H + q\Phi_H$$

where [4–6, 34]

$$\Omega_H = \frac{a}{r_+^2 + a^2} \quad \text{and} \quad \Phi_H = \frac{Qr_+}{r_+^2 + a^2}$$

are the angular velocity and the electric potential of the Kerr-Newman black hole, and $\{m, q\}$ are respectively the azimuthal harmonic index and charge coupling constant of the scalar field [35].

The resonance condition (2) guarantees that the orbiting scalar field is not absorbed by the black hole [27, 28, 36]. In addition, for an asymptotically flat black hole to be able to support stationary (that is, non-decaying) field configurations in its exterior region, the bounded fields must be prevented from radiating their energy to infinity. For massive fields the required confinement mechanism is naturally provided by the mutual gravitational attraction between the central black hole and the orbiting massive bosonic configuration. In particular, bound-state (that is, asymptotically decaying) eigenfunctions of a scalar field of mass $\mu$ [37] are characterized by low frequency modes in the regime [see Eq. (17) below]

$$\omega^2 < \mu^2.$$  

As discussed above, the no-short-hair property (1) was rigorously proved in [24] for the particular case of spherically-symmetric static hairy black-hole configurations. The main goal of the present paper is to challenge the validity of this relation beyond the regime of spherically-symmetric static black holes. To that end, we shall here study analytically the physical properties of the (non-static non-spherically symmetric) composed Kerr-Newman-charged-massive-scalar-field configurations [27, 28] in the large-mass regime [38]

$$M\mu \gg 1.$$  

Before proceeding, it is worth emphasizing that the composed black-hole-scalar-field configurations that we shall analyze here are not genuine hairy black-hole configurations. In particular, the exterior charged massive scalar fields will be treated at the linear level. We shall therefore use the term ‘clouds’ to describe these linearized exterior scalar configurations (the term ‘hair’ usually describes non-linear exterior fields). As we shall show below, the main advantage of the present approach lies in the fact that the composed black-hole-linearized-scalar-field system is amenable to an analytical treatment [39].
III. DESCRIPTION OF THE SYSTEM.

We study a physical system which is composed of a charged massive scalar field \( \Psi \) linearly coupled to an extremal charged rotating Kerr-Newman black hole of mass \( M \), electric charge \( Q \), and angular-momentum per unit mass \( a \) \(^{[40]} \). Using the Boyer-Lindquist coordinate system, the spacetime metric is described by the line element \(^{[4–6]} \)

\[
\mathrm{d}s^2 = \frac{-\Delta}{\rho^2} (\mathrm{d}t - a \sin^2 \theta \, \mathrm{d}\phi)^2 + \frac{\rho^2}{\Delta} \, \mathrm{d}r^2 + \rho^2 \, \mathrm{d}\theta^2 + \sin^2 \theta \left[ \mathrm{d}t - (r^2 + a^2) \, \mathrm{d}\phi \right]^2 ,
\]  

(6)

where \( \Delta \equiv r^2 - 2Mr + a^2 + Q^2 \) and \( \rho \equiv r^2 + a^2 \cos^2 \theta \). Extremal Kerr-Newman black holes are characterized by the relation

\[
r_{\text{H}} = M = \sqrt{a^2 + Q^2} ,
\]  

(7)

where \( r_{\text{H}} \) is the radius of the degenerate black-hole horizon.

The dynamics of a scalar field \( \Psi \) of mass \( \mu \) and charge coupling constant \( q \) \(^{[37]} \) in the Kerr-Newman black-hole spacetime is governed by the Klein-Gordon wave equation \(^{[37]} \)

\[
\left[ (\nabla^\nu - iqA^\nu)(\nabla_\nu - iqA_\nu) - \mu^2 \right] \Psi = 0 ,
\]  

(8)

where \( A^\nu \) is the electromagnetic potential of the charged black-hole spacetime. Substituting the field decomposition \(^{[41]} \)

\[
\Psi(t, r, \theta, \phi) = \sum_{l,m} e^{im\phi} S_{lm}(\theta; m, a\sqrt{\mu^2 - \omega^2}) R_{lm}(r; M, Q, a, \mu, q, \omega)c^{-i\omega_c t}
\]  

(9)

into the Klein-Gordon wave equation (8) and using the Kerr-Newman metric components (6), one obtains two coupled ordinary differential equations \([\text{see Eqs. (10) and (15) below}]\) for the radial and angular components, \( R_{lm}(r) \) and \( S_{lm}(\theta) \), of the scalar eigenfunction \( \Psi \).

The angular equation for the familiar spheroidal harmonic eigenfunctions \( S_{lm}(\theta; m, s\epsilon) \) is given by \([42–47]\)

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS_{lm}}{d\theta} \right) + \left[ K_{lm} + (s\epsilon)^2 \sin^2 \theta - \frac{m^2}{\sin^2 \theta} \right] S_{lm} = 0 .
\]  

(10)

Here

\[
s \equiv \frac{a}{M}
\]  

(11)

is the dimensionless spin (angular-momentum) of the Kerr-Newman black hole, and the dimensionless physical quantity

\[
e \equiv M \sqrt{\mu^2 - \omega_c^2}
\]  

(12)

measures the deviation of the scalar field mass \( \mu \) from the resonant oscillation frequency \( \omega_c \) [see Eq. (2)] of the field mode.

The discrete set of angular eigenvalues \( \{K_{lm}(s\epsilon)\} \) is determined from the regularity requirement of the corresponding angular eigenfunctions (spheroidal harmonics) \( S_{lm}(\theta; s\epsilon) \) at the two poles \( \theta = 0 \) and \( \theta = \pi \). We shall henceforth consider equatorial scalar modes in the eikonal regime

\[
l = m \gg 1 ,
\]  

(13)

in which case the angular eigenvalues are given by \([48–50]\)

\[
K_{mm}(s\epsilon) = m^2 + \sqrt{m^2 + (s\epsilon)^2} - (s\epsilon)^2 + O(1) .
\]  

(14)

The radial eigenfunctions \( R_{lm} \) are determined by the Teukolsky radial equation \([42, 43]\)

\[
\frac{d}{dr} \left( \frac{\Delta}{r^2} \frac{dR_{lm}}{dr} \right) + \left[ \frac{H^2}{\Delta} + 2ma_\omega - \mu^2(r^2 + a^2) - K_{lm} \right] R_{lm} = 0 ,
\]  

(15)

where

\[
H \equiv (r^2 + a^2)\omega_c - am - qQr .
\]  

(16)
Note that the angular eigenvalues \( \{K_{lm}(se)\} \) couple the radial equation (15) to the angular equation (10).

The bound-state resonances of the composed black-hole-charged-massive-scalar-field system are characterized by exponentially decaying (bounded) radial eigenfunctions at asymptotic infinity \[^{27, 28, 52}\]

\[
R(r \to \infty) \sim e^{-\epsilon r/r_H} \tag{17}
\]

with \( \epsilon^2 > 0 \). In addition, regular scalar configurations are characterized by finite radial eigenfunctions. In particular,

\[
R(r = r_H) < \infty . \tag{18}
\]

The physically motivated boundary conditions (17) and (18), together with the resonance condition (2), single out a discrete set of eigen-field-masses \[^{54}\] \( \{\mu_n(M, Q, a, l, m, q)\}_{n=0}^{\infty} \) (along with the associated radial eigenfunctions) which characterize the stationary bound-state resonances of the composed Kerr-Newman-charged-massive-scalar-field system.

**IV. THE EFFECTIVE BINDING POTENTIAL OF THE BLACK-HOLE SPACETIME**

Before solving the radial Teukolsky equation (15), it proves useful to analyze the spatial behavior of the effective radial potential which binds the charged massive scalar field to the charged rotating Kerr-Newman black hole. To that end, it is convenient to define the new radial function

\[
\psi = xR , \tag{19}
\]

in terms of which the radial Teukolsky equation (15) can be written in the form of a Schrödinger-like wave equation

\[
\frac{d^2\psi}{dx^2} - V\psi = 0 , \tag{20}
\]

where

\[
x \equiv \frac{r - M}{M} \tag{21}
\]

is a dimensionless radial coordinate. The effective radial potential in (20) is given by

\[
V = V(x; M, Q, a, l, m, \mu, q) = \epsilon^2 - \frac{2\kappa \epsilon}{x} + \frac{\beta^2 - \frac{1}{4}}{x^2} , \tag{22}
\]

where

\[
\kappa \equiv \frac{M \omega_c (2M \omega_c - qQ) - (M \mu)^2}{\epsilon} \tag{23}
\]

and

\[
\beta^2 \equiv K + \frac{1}{4} - 2M ms \omega_c - (2M \omega_c - qQ)^2 + (M \mu)^2 (1 + s^2) . \tag{24}
\]

The boundary condition (18) together with the relation (19) dictate

\[
\psi(x = 0) = 0 \tag{25}
\]

at the black-hole horizon. Thus, the effective radial potential (22) of the Schrödinger-like wave equation (20) must be infinitely repulsive at \( x = 0 \): \[^{55}\]

\[
V(x \to 0) \to +\infty . \tag{26}
\]

This implies that the stationary bound-state resonances of the charged massive scalar fields in the extremal Kerr-Newman black-hole spacetime are characterized by the inequality \[^{55, 56}\] \[\text{see Eqs. (22) and (26)}\]

\[
\beta \geq \frac{1}{2} . \tag{27}
\]

In the case (27), and provided \( \kappa > 0 \), the effective radial potential (22) takes the form of a trapping potential well which can support the stationary bound-state resonances of the composed black-hole-charged-massive-scalar-field system. In particular, in this case the binding potential (22) has one minimum which is located at

\[
x_{\min} = \frac{\beta^2 - \frac{1}{4}}{\kappa \epsilon} . \tag{28}
\]
V. THE STATIONARY BOUND-STATE RESONANCES OF THE COMPOSED KERR-NEWMAN-CHARGED-MASSIVE-SCALAR-FIELD SYSTEM.

In the present section we shall derive a (remarkably simple) analytical formula for the discrete spectrum of field masses, \( \{ \mu_n(M,Q,a,l,m,q) \}_{n=0}^{\infty} \), which characterize the bound-state resonances (the stationary charged scalar clouds) of the composed Kerr-Newman-charged-massive-scalar-field system.

The solution of the radial equation (15) can be expressed in the simple form [27, 46]:

\[
R(x) = C_1 \times x^{-\frac{1}{2}+\beta} e^{-\epsilon x} M\left(\frac{1}{2} + \beta - \kappa, 1 + 2\beta, 2\epsilon x\right) + C_2 \times (\beta \to -\beta),
\]

(29)

where \( M(a,b,z) \) is the confluent hypergeometric function [46] and \( \{C_1, C_2\} \) are normalization constants. The notation \( (\beta \to -\beta) \) in (29) means “replace \( \beta \) by \( -\beta \) in the preceding term.”

In order to obtain the resonance equation which characterizes the bound-state resonances of the composed Kerr-Newman-charged-massive-scalar-field system, we shall now examine the asymptotic spatial behaviors of the radial eigenfunction \( R(x) \) in the limits \( x \to 0 \) and \( x \to \infty \):

1. The behavior of the radial eigenfunction (29) in the near-horizon \( x \ll 1 \) region is given by [46]

\[
R(x \to 0) \to C_1 \times x^{-\frac{1}{2}+\beta} + C_2 \times x^{-\frac{1}{2}-\beta}.
\]

(30)

The boundary condition (18), when applied to the near-horizon behavior (30) of the radial eigenfunction, implies that regular bound-state scalar configurations are characterized by the relations

\[
C_2 = 0 \quad \text{and} \quad \beta \geq \frac{1}{2}.
\]

(31)

Note that (31) is consistent with our previous conclusion (27).

2. The asymptotic \( x \to \infty \) behavior of the radial eigenfunction (29) at spatial infinity is given by [46]

\[
R(x \to \infty) \to C_1 \times (-2\epsilon)^{-\frac{1}{2}-\beta+\kappa} \frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\beta+\kappa)} x^{-1+\kappa} e^{-\epsilon x} + C_1 \times (2\epsilon)^{-\frac{1}{2}-\beta-\kappa} \frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\beta-\kappa)} x^{-1-\kappa} e^{\epsilon x}.
\]

(32)

The boundary condition (17), when applied to the asymptotic behavior (32) of the radial eigenfunction, implies that the coefficient of the exploding exponent \( e^{\epsilon x} \) in (32) should vanish. This yields the characteristic resonance equation [57]

\[
\frac{1}{2} + \beta - \kappa = -n \quad \text{with} \quad n = 0, 1, 2, \ldots.
\]

(33)

for the stationary bound-state resonances of the composed Kerr-Newman-charged-massive-scalar-field system.

Taking cognizance of Eqs. (2), (12) and (23), one can write \( \kappa \) in the form

\[
\kappa = \frac{\alpha}{\epsilon} - \epsilon,
\]

(34)

where [58]

\[
\alpha \equiv m^2 \cdot \frac{(s^2 + \gamma)(1 - \gamma)}{(1 + s^2)^2}
\]

(35)

and [59]

\[
\gamma \equiv \frac{\eta Q s}{m}.
\]

Likewise, taking cognizance of Eqs. (2), (12), (13), and (24), one can write \( \beta \) in the eikonal \( m \gg 1 \) regime in the form

\[
\beta = \sqrt{\beta_0^2 + \epsilon^2 + \sqrt{m^2 + (s\epsilon)^2}},
\]

(37)
where
\[ \beta_0^2 \equiv m^2 \cdot \frac{1 - 3s^2 - 4\gamma(1 - s^2) + \gamma^2(3 - s^2)}{(1 + s^2)^2}. \] (38)

Substituting (34) and (37) into (33), one finds that the characteristic resonance condition can be written as a (rather complicated) equation for the dimensionless physical parameter \( \epsilon \):
\[ \frac{1}{2} + \frac{\beta_0^2 + \epsilon^2 + \sqrt{m^2 + (s\epsilon)^2}}{\epsilon} - \frac{\alpha}{\epsilon} + \epsilon = -n. \] (39)

From (39) one deduces that \( \epsilon = \epsilon(n) \) is a decreasing function of the resonance parameter \( n \).

It proves useful to write the resonance equation (39) in the form
\[ (\beta_0^2 + 2\alpha)\epsilon^2 - \alpha^2 = (n + 1/2)[2\epsilon^3 + (n + 1/2)\epsilon^2 - 2\alpha\epsilon] - m\sqrt{1 + (s\epsilon/m)^2}\epsilon^2. \] (40)

Note that in the eikonal regime,
\[ m \gg n + 1/2, \] (41)

the l.h.s of (40) is of order \( O(m^4) \), whereas the r.h.s of (40) contains terms of order \( O[(n + 1/2)m^3] \) and of order \( O((n + 1/2)^2m^2) \) (see Eqs. (35), (38), and (45) below). Thus, in the eikonal regime (41) one can use an iteration scheme in order to solve the characteristic resonance equation (40). The zeroth-order resonance equation is given by
\[ \epsilon_0 = \epsilon \cdot (1 + \delta_n), \] (43)

where
\[ \delta_n = -\frac{\beta_0^2 + \alpha}{(\beta_0^2 + 2\alpha)^{3/2}} \cdot (n + 1/2) - \frac{\sqrt{m^2 + (s\epsilon)^2}}{2(\beta_0^2 + 2\alpha)} + O\left[\left(\frac{n + 1/2}{m}\right)^2\right] \ll 1 \] (44)
is a small correction factor [see Eq. (46) below].

Substituting the relations (35) and (38) into (42) and (44), one finds (61, 62)
\[ \bar{\epsilon} = m \cdot \frac{(s^2 + \gamma)}{(1 + s^2)\sqrt{1 - s^2}} \] (45)

and
\[ \delta_n = -\frac{(1 + s^2)[1 - 2s^2 - 3\gamma(1 - s^2) + \gamma^2(2 - s^2)]}{(1 - \gamma)^3(1 - s^2)^{3/2}} \cdot \frac{n + 1/2 - \frac{(1 + s^2)\sqrt{1 + s^2 - s^4 + 2s^2\gamma + s^2\gamma^2}}{2(1 - \gamma)^2(1 - s^2)^{3/2}}}{m} \cdot \frac{1}{m} = O\left(\frac{n + 1/2}{m}\right) \ll 1. \] (46)

Taking cognizance of Eqs. (2), (12), (13), and (15), one finally finds the discrete spectrum \( \{\mu_n(m, qQ, s)\}_{n=0}^{\infty} \) of eigen field-masses which characterize the stationary bound-state charged scalar clouds of the extremal charged rotating Kerr-Newman black-hole spacetime:
\[ \mu_n = \bar{\mu} \cdot (1 + s^2\delta_n); \quad n = 0, 1, 2, \ldots (n \ll m), \] (47)

where (63)
\[ M\bar{\mu}(m, qQ, s) = m \cdot \frac{s^2 + \gamma}{s(1 + s^2)\sqrt{1 - s^2}}. \] (48)
VI. EFFECTIVE LENGTHS (RADIIS) OF THE STATIONARY BOUND-STATE CHARGED SCALAR CLOUDS.

In the present section we shall challenge the validity of the no-short-hair bound (1) beyond the regime of spherically-symmetric static black-hole spacetimes [64]. To this end, we shall now evaluate the effective lengths (radii) of the non-spherically symmetric non-static charged scalar clouds which characterize the extremal charged rotating Kerr-Newman black-hole spacetimes.

Taking cognizance of Eqs. (31) and (33), one can express the radial eigenfunctions (29) which characterize the stationary bound-state charged scalar clouds in the form
\[ R^{(n)}(x) = A x^{\beta - 1/2} e^{-\epsilon x} L_n^{(2\beta)}(2\epsilon x) , \]
where \( L_n^{(2\beta)}(x) \) are the generalized Laguerre polynomials [65] and \( A \) is a normalization constant. In particular, the ground-state \((n = 0)\) radial eigenfunction is given by the compact expression [66]
\[ R^{(0)}(x) = A x^{\beta - 1/2} e^{-\epsilon x} . \]

The radial eigenfunction \((50)\), which characterizes the fundamental (ground-state) charged scalar cloud, has a global maximum at
\[ x_{\text{peak}} = \frac{\beta - \frac{1}{2}}{\epsilon} . \]

Note that the expression \((51)\) for the location of the peak of the ground-state \((n = 0)\) radial eigenfunction \((50)\) agrees with the previously found expression \((28)\) [67] for the location of the minimum of the effective binding potential \((22)\).

Using the relation \((37)\), one can write \((52)\) in the form
\[ x_{\text{peak}} = \sqrt{1 + (\beta_0/\epsilon)^2 + O(m^{-1})} . \]

Substituting \((38)\) and \((45)\) into \((52)\), one finds
\[ x_{\text{peak}}(\gamma; s) = \frac{1 - 2s^2 - \gamma(2-s^2)}{s^2 + \gamma} [1 + O(m^{-1})] . \]

This expression for the peak location of the radial eigenfunction \((50)\) provides a quantitative measure for the characteristic size (length) of the fundamental bound-state charged scalar cloud.

At this point, it is interesting to note that neutral scalar clouds, which are characterized by the simple relation [25, 68]
\[ x_{\text{peak}}(\gamma = 0; s) = \frac{1 - 2s^2}{s^2} [1 + O(m^{-1})] , \]
respect the no-short-hair lower bound \((11)\). To see this, we recall that the radii of the equatorial \((l = m \gg 1)\) null circular geodesics which characterize the charged rotating Kerr-Newman black-hole spacetimes are given by the relation \((4)\)
\[ r^2 - 3Mr + 2Q^2 + 2a(Mr - Q^2)^{1/2} = 0 . \]

This equation can be solved analytically for near-extremal black holes. In particular, one finds
\[ r_{\text{null}} = \begin{cases} 2(M - a) + O(\sqrt{M^2 - a^2 - Q^2}) & \text{for } 0 \leq a/M \leq 1/2 ; \\ M + O(\sqrt{M^2 - a^2 - Q^2}) & \text{for } 1/2 < a/M \leq 1 , \end{cases} \]
which implies [see Eq. \((21)\)]
\[ x_{\text{null}}(s) \to \begin{cases} 1 - 2s & \text{for } 0 \leq s \leq 1/2 ; \\ 0 & \text{for } 1/2 < s \leq 1 \end{cases} \]
in the extremal \(M^2 - a^2 - Q^2 \to 0\) limit.
Taking cognizance of Eqs. (54) and (57), one finds the characteristic inequality
\[
x_{\text{peak}}(\gamma = 0; s) > x_{\text{null}}(s)
\]
for neutral scalar clouds in the entire range \( s \in (0, \frac{1}{\sqrt{1/3}}) \). We therefore conclude that neutral scalar clouds conform to the no-short-hair lower bound [1]. In particular, for the neutral scalar clouds one finds
\[
\min_s \{x_{\text{peak}}(\gamma = 0; s)/x_{\text{null}}(s)\} = 10 + 6\sqrt{3} \approx 20.392 \text{ for } s = (\sqrt{3} - 1)/2 \approx 0.366.
\]

Let us now return to the more general case of charged scalar clouds. From the expression [53] for \( x_{\text{peak}}(\gamma; s) \) one deduces that, for a given value of the black-hole angular momentum \( s \), \( x_{\text{peak}}(\gamma; s) \) is a decreasing function of the dimensionless physical parameter \( \gamma \). In particular, the expression [53] for \( x_{\text{peak}}(\gamma; s) \) reveals the remarkable fact that the exterior charged scalar clouds can be made arbitrary compact. In particular, one finds [60]
\[
x_{\text{peak}}(s) \to 0 \quad \text{for} \quad \gamma(s) \to \gamma^*(s) = \frac{1 - 2s^2}{2 - s^2}.
\]

It is worth emphasizing that the relation (59) is valid for extremal Kerr-Newman black holes in the entire range \( s \in (0, 1) \) of the black-hole rotation parameter. This fact implies that all charged and rotating extremal Kerr-Newman black holes can support extremely compact charged scalar configurations in their exterior regions [71].

Taking cognizance of Eqs. (57) and (59), one finds the important inequality
\[
x_{\text{peak}}(\gamma = \gamma^*; s) < x_{\text{null}}(s)
\]
for charged scalar clouds in the entire range \( s \in (0, \frac{1}{2}) \). We therefore conclude that charged scalar clouds may violate the no-short-hair lower bound [1] [54, 72].

VII. SUMMARY.

A no-short-hair theorem for spherically-symmetric static black holes was proved in [24]. This theorem reveals that, if a spherically-symmetric static black hole has hair, then this hair (i.e. the external fields) must extend beyond the null circular geodesic (the “photonsphere”) of the corresponding black-hole spacetime [see Eq. (11)].

The main goal of the present study was to challenge the validity of this no-short-hair property beyond the regime of static spherically-symmetric hairy black-hole configurations. To that end, we have studied analytically the physical properties of extremal charged rotating Kerr-Newman black holes linearly coupled to non-spherically symmetric stationary (rather than static) charged massive scalar fields.

In particular, for given parameters \( \{M, Q, J\} \) of the central Kerr-Newman black hole, we have identified the critical value of the field charge coupling constant, \( q = q^*(s) \) [see Eqs. (36) and (59)], which minimizes the effective radial lengths of the exterior stationary charged scalar configurations. This allowed us to prove that the (non-static, non-spherically symmetric) composed Kerr-Newman-charged-massive-scalar-field configurations can violate the no-short-hair lower bound [11]. In particular, it was shown that extremal Kerr-Newman black holes in the entire parameter space \( s \in (0, 1) \) can support extremely compact stationary charged scalar clouds (made of linearized charged massive scalar fields with the property \( r_{\text{field}} \to r_{\text{null}} \)) in their exterior regions. Specifically, these short-range Kerr-Newman-charged-massive-scalar-field configurations are characterized by the simple relation [see Eqs. (48) and (59)]
\[
x_{\text{peak}}(s) \to 0 \quad \text{for} \quad \frac{q}{\mu} \to \frac{1 - 2s^2}{1 - s^2}.
\]

It is interesting to note that, in order to support these extremely compact field configurations, the extremal black hole must have both angular-momentum and electric charge [72].

Finally, it is worth emphasizing again that we have treated here the exterior charged massive scalar fields at the linear level. As we have shown, the main advantage of this approach lies in the fact that the composed Kerr-Newman-linearized-charged-scalar-field system is amenable to an analytical treatment. We believe that it would be highly important to use numerical techniques [28] in order to generalize our results to the regime of non-linear exterior fields.

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rations which are more compact (that is, with smaller spatial extensions) than the corresponding exterior neutral scalar configurations. Thus, as we shall show in the present study [see Eqs. (53) and (54) below], charged scalar clouds impose a greater challenge (as compared with neutral scalar clouds) to the no-short-hair relation (1).

[41] Here $\omega, m,$ and $l \geq |m|$ are respectively the conserved frequency and the harmonic indices (azimuthal and spheroidal) of the field mode [see Eq. (10) below].

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[53] One may choose $\beta > 0$ without loss of generality.

[54] Here $n = 0, 1, 2, \ldots$ is the resonance parameter, see Eq. (39) below.

[55] The limiting case of $\beta^2 = 1/4$ with $V(x \to 0) \sim -1/x$ is also allowed. In this special case the effective binding potential (22) takes the form of a radial Coulomb potential with no centrifugal barrier. Thus, in this special case one finds that the radial function $\psi(x)$ is described by the familiar spherically-symmetric ($l = 0$) eigenfunctions of the hydrogen atom. In particular, as for the spherically-symmetric eigenfunctions of the hydrogen atom, one finds $\psi(x \to 0) \sim x^s$ [or equivalently, $R(x \to 0) \to$ constant, see Eqs. (30) and (31) below] in this special case.

[56] One may choose $\beta > 0$ without loss of generality.

[57] Here we have used the well-known pole structure of the Gamma functions, see equation 6.1.7 of [16].

[58] Taking cognizance of Eqs. (51) and (53), one finds the necessary requirement $\kappa = (n + 1/2) + \beta > 0$ for the existence of the stationary bound-state charged scalar configurations. This implies that $\alpha > \epsilon^2 > 0$ [see Eq. (54)].

[59] Taking cognizance of Eq. (24), one finds that the requirement $\omega_{\text{field}} = \omega_0 > 0$ [20] implies $m + qQ > 0$, or equivalently $\gamma > -s^2$.

[60] From the relation $n = -\frac{1}{2} - \sqrt{\frac{\beta^2}{4} + \epsilon^2 + \sqrt{\frac{\beta^2}{4} + \epsilon^2 + \left(\frac{\alpha}{\beta} + \frac{\epsilon}{\alpha}\right)^2}} + \frac{\alpha}{\beta} - \frac{\epsilon}{\alpha}$ one finds $dn/\epsilon = -\{(1 + \alpha^2)/2\sqrt{\beta^2 + \alpha^2}\}/\epsilon/\beta + \alpha/\epsilon^2 + 1\}$ which is negative definite (see Eq. (51) and (55)). Thus, $dn/\epsilon < 0$, which implies that $\epsilon = \epsilon(n)$ is a decreasing function of the resonance parameter $n$.

[61] Note that the requirement $\alpha - \epsilon^2 > 0$ (see [58]) implies the inequality $\gamma < (1 - 2\epsilon^2)/(2 - \epsilon^2) + O(m^{-1})$ for the charged scalar clouds [see Eqs. (59) and (54)].

[62] In the opposite regime $n + 1/2 \gg m$ of excited eigenstates, one finds from [59] the simple asymptotic solution $\epsilon_n = \alpha/n \ll 1$, which implies $\mu_n = \omega_n \{1 + m(s(1 - \gamma)/(1 + s^2)n^2)/2\}$ [see Eq. (12)].

[63] Note that the resonance equation (39) forbids $\epsilon = 0$, which implies $s \neq 0$ [The requirement $\epsilon \neq 0$ stems from the physical boundary condition (17). In particular, from (14) one learns that an exponentially decaying radial eigenfunction at asymptotic infinity requires $\epsilon > 0$. Likewise, the resonance equation (39) forbids $\epsilon = \infty$, which implies $s \neq 1$ (The inequality $\epsilon < \infty$ reflects the fact that the allowed scalar field masses $\mu$ are finite).]

[64] It is worth emphasizing again that the no-short-hair lower bound (11) was rigorously proved in [24] for the special case of spherically-symmetric static hairy black-hole configurations.

[65] Here we have used equation 13.6.9 of [16].

[66] Here we have used the fact that $L_0^{(2)}(x)$ is a constant independent of $x$.

[67] To see this agreement, substitute into (28) $\kappa = \frac{1}{2} + \beta$ for the fundamental ground-state $(n = 0)$ mode [see the resonance equation (53)].

[68] Note that the requirement $\alpha - \epsilon^2 > 0$ (see [58]) implies the inequality $s < \frac{1}{\beta} + O(m^{-1})$ for the neutral scalar clouds [see Eqs. (59) and (45)].

[69] This value of $\gamma^s(s)$ corresponds to $Mq(s) \to Mq^s(s) = m \cdot \frac{1 - s^2}{\sqrt{(1 - s^2)(1 + s^2)}}$. Taking cognizance of Eqs. (48) and (59), one finds the dimensionless ratio $Qq^s(s)/\mu(s) = (1 - s^2)/\sqrt{1 - s^2}$. Note that $Qq^s(s)/\mu(s) < 1$ in the entire range $s \in (0, 1)$.}
That is, extremal black holes in the entire range \( s \in (0, 1) \).

This property of the exterior charged scalar fields should be contrasted with the simpler case of exterior neutral scalar fields \([25]\). In the former case, \( x_{\text{peak}}(s) \) can approach zero in the entire range \( s \in (0, 1) \) of the black-hole rotation parameter [see \((59)\)], whereas in the later case the radial location of the peak, \( x_{\text{peak}}(s) \), can approach zero only in the particular case \( s \to 1/\sqrt{2} \) [see \((54)\)].

It is important to emphasize the fact that the results of \([27, 28]\) indicate that scalar clouds with reasonably small values of the angular indices \( \{l, m\} \) and with reasonably small values of the dimensionless quantity \( qQ \) respect the no-short hair conjecture \([11]\).

Thus, the values \( s = 0 \) and \( s = 1 \) for the extremal black-hole angular momentum are excluded \([63]\).