Explicit formulas for determinantal representations of the Drazin inverse solutions of some matrix and differential matrix equations.

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Abstract
The Drazin inverse solutions of the matrix equations $AX = B$, $XA = B$ and $AXB = D$ are considered in this paper. We use both the determinantal representations of the Drazin inverse obtained earlier by the author and in the paper. We get analogs of the Cramer rule for the Drazin inverse solutions of these matrix equations and using their for determinantal representations of solutions of some differential matrix equations, $X' + AX = B$ and $X' +XA = B$, where the matrix $A$ is singular.

Keywords: Drazin inverse, matrix equation, Drazin inverse solution, Cramer rule, differential matrix equation

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1. Introduction
In this paper we shall adopt the following notation. Let $\mathbb{C}^{m \times n}$ be the set of $m$ by $n$ matrices with complex entries and $I_m$ be the identity matrix of order $m$. Denote by $a_{ij}$ and $a_i$ the $j$th column and the $i$th row of $A \in \mathbb{C}^{m \times n}$, respectively. Let $A_{ij}(b)$ denote the matrix obtained from $A$ by replacing its $j$th column with the vector $b$, and by $A_i(b)$ denote the matrix obtained from $A$ by replacing its $i$th row with $b$.

Let $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. Then $|A_{\alpha\beta}|$ denotes the minor of $A$ determined by the rows indexed by $\alpha$ and the columns indexed by $\beta$. Clearly, $|A_{\alpha\alpha}|$ be a principal minor determined by the rows and columns indexed by $\alpha$. For $1 \leq k \leq n$, denote by

$$L_{k,n} := \{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k), 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n\}$$

the collection of strictly increasing sequences of $k$ integers chosen from the set $\{1, \ldots, n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let

$$I_{k,n}\{i\} := \{\alpha : \alpha \in L_{k,n}, i \in \alpha\}, \quad J_{k,n}\{j\} := \{\beta : \beta \in L_{k,n}, j \in \beta\}.$$
Matrix equation is one of the important study fields of linear algebra. Linear matrix equations, such as

\[ AX = B, \]  

(1)

\[ XA = B, \]  

(2)

and

\[ AXB = D, \]  

(3)

play an important role in linear system theory therefore a large number of papers have presented several methods for investigating these matrix equations (for example, see [1]-[6]).

In some situations, however, people pay more attention to the Drazin inverse solutions of singular linear systems and matrix equations [7]-[13]. Moreover, Xu Zhao-liang and Wang Guo-rong in [14] proved that the Drazin inverse solutions of the matrix equations (1), (2) and (3) with some restricts are their unique solutions. The Cramer rule for the Drazin inverse solution of the restricted system of linear equations are used in [15]-[17]. The Cramer rules for solutions of the restricted matrix equations (1), (2) and (3), in particular for the Drazin inverse solution, are established in [18]-[21].

In this paper, we obtain explicit formulas for determinantal representations of the Drazin inverse solutions of the matrix equations (1), (2) and (3) and using their for determinantal representations of solutions of some differential matrix equations. The paper is organized as follows. We start with some basic concepts and results about the Drazin inverse in Section 2. We use the determinantal representation of the Drazin inverse obtained in [22] and also another determinantal representation is obtained in this section. In Section 3, we derive explicit formulas for determinantal representations of the Drazin inverse solutions for the matrix equations (1), (2) and (3). These formulas generalize the well-known Cramer rule. In Section 4, we demonstrate their using for determinantal representations of solutions of some differential matrix equations, \( X' + AX = B \) and \( X' +XA = B \), where the matrix \( A \) is singular. In Section 5, we show numerical examples to illustrate the main results.
2. Determinantal representations of the Drazin inverse

For any matrix $A \in \mathbb{C}^{n \times n}$ with $IndA = k$, where a positive integer $k = \min \{k \in \mathbb{N} \cup \{0\} \mid \text{rank} A^{k+1} = \text{rank} A^k\}$, the Drazin inverse is the unique matrix $X$ that satisfies the following three properties

1) $A^{k+1}X = A^k$;
2) $XAX = X$;
3) $AX =XA$.

It is denoted by $X = A^D$.

In particular, when $IndA = 1$, then the matrix $X$ in (4) is called the group inverse and is denoted by $X = A^g$.

If $IndA = 0$, then $A$ is nonsingular, and $A^D \equiv A^{-1}$.

Remark 2.1. Since the equation 3) of (4), the equation 1) can be replaced by follows

1a) $XA^{k+1} = A^k$.

The Drazin inverse can be represented explicitly by the Jordan canonical form as follows.

**Theorem 2.1.** [25] If $A \in \mathbb{C}^{n \times n}$ with $IndA = k$ and

$$A = P \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix} P^{-1}$$

where $C$ is nonsingular and rank $C = \text{rank} A^k$, and $N$ is nilpotent of order $k$, then

$$A^D = P \begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

We use the following theorem about the limit representation of the Drazin inverse.

**Theorem 2.2.** [25] If $A \in \mathbb{C}^{n \times n}$, then

$$A^D = \lim_{\lambda \to 0} \left(\lambda I_n + A^{k+1}\right)^{-1}A^k,$$

where $k = IndA$, $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.

The following theorem can be obtained by analogy to Theorem 2.2.

**Theorem 2.3.** If $A \in \mathbb{C}^{n \times n}$, then

$$A^D = \lim_{\lambda \to 0} A^k \left(\lambda I_n + A^{k+1}\right)^{-1},$$

where $k = IndA$, $\lambda \in \mathbb{R}_+$, and $\mathbb{R}_+$ is a set of the real positive numbers.
Denote by $a_{j}^{(k)}$ and $a_{i}^{(k)}$ the jth column and the ith row of $A^k$ respectively.

**Lemma 2.1.** ([22], Lemma 3.1) If $A \in \mathbb{C}^{n \times n}$ with $\text{Ind } A = k$, then for all $i, j = 1, \ldots, n$
\[
\text{rank } A^{k+1} \left( a_{j}^{(k)} \right) \leq \text{rank } A^{k+1}.
\]

Using Theorem 2.2 and Lemma 2.1 we obtained in [22] the following determinantal representations of the Drazin and group inverses and the identity $A^D A$ on $R(A^k)$.

**Theorem 2.4.** ([22], Theorem 3.3) If $\text{Ind } A = k$ and $\text{rank } A^{k+1} = \text{rank } A^k = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then the Drazin inverse $A^D = (a_{ij}^D) \in \mathbb{C}^{n \times n}$ possess the following determinantal representations:
\[
a_{ij}^D = \frac{\sum_{\beta \in J_{r,n} \{i\}} \left| \left( A^{k+1} \left( a_{j}^{(k)} \right) \right)^{\beta} \right|}{\sum_{\beta \in J_{r,n}} \left| (A^{k+1})^{\beta} \right|},
\]
for all $i, j = 1, \ldots, n$.

**Corollary 2.1.** ([22], Corollary 3.1) If $\text{Ind } A = 1$ and $\text{rank } A^2 = \text{rank } A = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then the group inverse $A^g = (a_{ij}^g) \in \mathbb{C}^{n \times n}$ possess the following determinantal representation:
\[
a_{ij}^g = \frac{\sum_{\beta \in J_{r,n} \{i\}} \left| \left( A^{2} \left( a_{j} \right) \right)^{\beta} \right|}{\sum_{\beta \in J_{r,n}} \left| (A^{2})^{\beta} \right|},
\]
for all $i, j = 1, \ldots, n$.

**Corollary 2.2.** ([22], Corollary 3.2) If $\text{Ind } A = k$ and $\text{rank } A^{k+1} = \text{rank } A^k = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then the matrix $A^D A = (p_{ij}) \in \mathbb{C}^{n \times n}$ possess the following determinantal representation:
\[
p_{ij} = \frac{\sum_{\beta \in J_{r,n} \{i\}} \left| \left( A^{k+1} \left( a_{j}^{(k+1)} \right) \right)^{\beta} \right|}{\sum_{\beta \in J_{r,n}} \left| (A^{k+1})^{\beta} \right|},
\]
for all $i, j = 1, \ldots, n$.

Using Theorem 2.3 we can obtain another determinantal representation of the Drazin inverse. At first we consider the following auxiliary lemma similar to Lemma 2.1.
Lemma 2.2. If $A \in \mathbb{C}^{n \times n}$ with $\text{Ind} A = k$, then for all $i, j = 1, \ldots, n$

$$\text{rank} A^k \begin{pmatrix} a_{ij} \end{pmatrix} \leq \text{rank} A^k.$$

PROOF. The matrix $A^k \begin{pmatrix} a_{ij} \end{pmatrix}$ may be represent as follows

$$A^k \begin{pmatrix} a_{ij} \end{pmatrix} = \begin{pmatrix} \sum_{s=1}^{n} a_{is} a_{s1} & \ldots & \sum_{s=1}^{n} a_{is} a_{sn} \\ \vdots & \ddots & \vdots \\ \sum_{s=1}^{n} a_{ns} a_{s1} & \ldots & \sum_{s=1}^{n} a_{ns} a_{sn} \end{pmatrix} \begin{pmatrix} a_{ij} \end{pmatrix}.$$

Let $P_{l}(-a_{lj}) \in \mathbb{C}^{n \times n}$, $(l \neq i)$, be a matrix with $-a_{lj}$ in the $(l, i)$ entry, 1 in all diagonal entries, and 0 in others. It is a matrix of an elementary transformation. It follows that

$$A^k \begin{pmatrix} a_{ij} \end{pmatrix} \cdot \prod_{l \neq i} P_{l}(-a_{lj}) = \begin{pmatrix} \sum_{s=1}^{n} a_{is} a_{s1} & \ldots & \sum_{s=1}^{n} a_{is} a_{sn} \\ \vdots & \ddots & \vdots \\ \sum_{s=1}^{n} a_{ns} a_{s1} & \ldots & \sum_{s=1}^{n} a_{ns} a_{sn} \end{pmatrix} \begin{pmatrix} a_{ij} \end{pmatrix}.$$

The obtained above matrix has the following factorization.

$$\begin{pmatrix} \sum_{s=1}^{n} a_{is} a_{s1} & \ldots & \sum_{s=1}^{n} a_{is} a_{sn} \\ \vdots & \ddots & \vdots \\ \sum_{s=1}^{n} a_{ns} a_{s1} & \ldots & \sum_{s=1}^{n} a_{ns} a_{sn} \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & 0 & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ a_{n1} & \ldots & 0 & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}.$$

Denote the first matrix by

$$\tilde{A} := \begin{pmatrix} a_{11} & \ldots & 0 & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ a_{n1} & \ldots & 0 & \ldots & a_{nn} \end{pmatrix}.$$
The matrix \( \tilde{A} \) is obtained from \( A \) by replacing all entries of the \( i \)th row and the \( j \)th column with zeroes except for 1 in the \((i, j)\) entry. Elementary transformations of a matrix do not change its rank. It follows that rank \( A_{i}^{k+1} (a_{ij}^{(k)}) \leq \min \{ \text{rank} A^{k}, \text{rank} \tilde{A} \} \). Since rank \( \tilde{A} \geq \text{rank} A^{k} \) the proof is completed.

**Theorem 2.5.** If \( \text{Ind} A = k \) and rank \( A^{k+1} = \text{rank} A^{k} = r \leq n \) for \( A \in \mathbb{C}^{n \times n} \), then the Drazin inverse \( A^{D} = (a_{ij}^{D}) \in \mathbb{C}^{n \times n} \) possess the following determinantal representations:

\[
a_{ij}^{D} = \sum_{\alpha \in I_{r,n}(j)} \left| \left( \lambda I + A^{k+1} \right)^{\alpha}_{\lambda} \right| \left( (A^{k+1})^{\alpha}_{\alpha} \right)_{\alpha}^{-1},
\]

for all \( i, j = 1, \ldots, n \).

**Proof.** If \( \lambda \in \mathbb{R}^{+} \), then rank \((\lambda I + A^{k+1}) = n \). Hence, there exists the inverse matrix

\[
(\lambda I + A^{k+1})^{-1} = \frac{1}{\det (\lambda I + A^{k+1})} \begin{pmatrix}
R_{11} & R_{21} & \cdots & R_{n1} \\
R_{12} & R_{22} & \cdots & R_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
R_{1n} & R_{2n} & \cdots & R_{nn}
\end{pmatrix},
\]

where \((R)_{ij}\) is a cofactor in \((\lambda I + A^{k+1})\) for all \( i, j \). By Theorem 2.3, \( A^{D} = \lim_{\lambda \to 0} \left[ (\lambda I + A^{k+1})^{-1} \right]^{\alpha} \). So that

\[
A^{D} = \lim_{\lambda \to 0} \frac{1}{\det (\lambda I + A^{k+1})} \begin{pmatrix}
\sum_{s=1}^{n} a_{1s}^{(k)} R_{1s} & \cdots & \sum_{s=1}^{n} a_{1s}^{(k)} R_{ns} \\
\vdots & \ddots & \vdots \\
\sum_{s=1}^{n} a_{ns}^{(k)} R_{1s} & \cdots & \sum_{s=1}^{n} a_{ns}^{(k)} R_{ns}
\end{pmatrix}
\]

\[
\lim_{\lambda \to 0} \begin{pmatrix}
\frac{\det (\lambda I + A^{k+1}) \left( a_{i}^{(k)} \right)}{\det (\lambda I + A^{k+1})} & \cdots & \frac{\det (\lambda I + A^{k+1}) \left( a_{n}^{(k)} \right)}{\det (\lambda I + A^{k+1})} \\
\vdots & \ddots & \vdots \\
\frac{\det (\lambda I + A^{k+1}) \left( a_{i}^{(k)} \right)}{\det (\lambda I + A^{k+1})} & \cdots & \frac{\det (\lambda I + A^{k+1}) \left( a_{n}^{(k)} \right)}{\det (\lambda I + A^{k+1})}
\end{pmatrix}
\]

(8)

Similar to the characteristic polynomial, we have

\[
\det (\lambda I + A^{k+1}) = \lambda^{n} + d_{1} \lambda^{n-1} + d_{2} \lambda^{n-2} + \ldots + d_{n},
\]

where \( d_{s} = \sum_{\alpha \in I_{r,n}} |(A^{k+1})^{\alpha}_{\alpha}| \) is a sum of the principal minors of \( A^{k+1} \) of order \( s \), for all \( s = 1, n-1, \) and \( d_{n} = \det A^{k+1} \). Since rank \( A^{k+1} = r \), then \( d_{n} = d_{n-1} = \ldots = d_{r+1} = 0 \) and

\[
\det (\lambda I + A^{k+1}) = \lambda^{n} + d_{1} \lambda^{n-1} + d_{2} \lambda^{n-2} + \ldots + d_{r} \lambda^{n-r}.
\]

(9)
Similarly we have for all $i, j = 1, n$

$$\det (\lambda I + A^{k+1})_{ij} \left( a_{ij}^{(k)} \right) = l_1^{(ij)} \chi^{n-1} + l_2^{(ij)} \chi^{n-2} + \ldots + l_r^{(ij)},$$

where for all $s = 1, n - 1$,

$$l_s^{(ij)} = \sum_{\alpha \in I_{s,n}(j)} \left| \left( A^{k+1} a_{ij}^{(k)} \right)_{\alpha} \right|,$$

and $l_n^{(ij)} = \det A_{ij}^{k+1} \left( a_{ij}^{(k)} \right)$. By Lemma 2.2, rank $A_{ij}^{k+1} \left( a_{ij}^{(k)} \right) \leq r$, so that if $s > r$, then for all $\alpha \in I_{s,n}(j)$ and for all $i, j = 1, n$,

$$\left| \left( A^{k+1} a_{ij}^{(k)} \right)_{\alpha} \right| = 0.$$

Therefore if $r + 1 \leq s < n$, then for all $i, j = 1, n$,

$$l_s^{(ij)} = \sum_{\alpha \in I_{s,n}(j)} \left| \left( A^{k+1} a_{ij}^{(k)} \right)_{\alpha} \right| = 0,$$

and $l_n^{(ij)} = \det A_{ij}^{k+1} \left( a_{ij}^{(k)} \right) = 0$. Finally we obtain

$$\det (\lambda I + A^{k+1})_{ij} \left( a_{ij}^{(k)} \right) = l_1^{(ij)} \chi^{n-1} + l_2^{(ij)} \chi^{n-2} + \ldots + l_r^{(ij)} \chi^{n-r}. \quad (10)$$

By replacing the denominators and the nominators of the fractions in the entries of the matrix $(8)$ with the expressions $(9)$ and $(11)$ respectively, finally we obtain

$$A^D = \lim_{\lambda \to 0} \begin{pmatrix}
\frac{l_1^{(11)} \chi^{n-1} + \ldots + l_1^{(11)} \chi^{n-r}}{\lambda^n + d_1 \lambda^{n-1} + \ldots + d_r \lambda^{n-r}} & \ldots & \frac{l_1^{(1n)} \chi^{n-1} + \ldots + l_1^{(1n)} \chi^{n-r}}{\lambda^n + d_1 \lambda^{n-1} + \ldots + d_r \lambda^{n-r}} \\
\ldots & \ldots & \ldots \\
\frac{l_1^{(n1)} \chi^{n-1} + \ldots + l_1^{(n1)} \chi^{n-r}}{\lambda^n + d_1 \lambda^{n-1} + \ldots + d_r \lambda^{n-r}} & \ldots & \frac{l_1^{(nn)} \chi^{n-1} + \ldots + l_1^{(nn)} \chi^{n-r}}{\lambda^n + d_1 \lambda^{n-1} + \ldots + d_r \lambda^{n-r}}
\end{pmatrix} = \begin{pmatrix}
\frac{l_1^{(11)}}{d_{e_1}} & \ldots & \frac{l_1^{(1n)}}{d_{e_1}} \\
\ldots & \ldots & \ldots \\
\frac{l_1^{(n1)}}{d_{e_1}} & \ldots & \frac{l_1^{(nn)}}{d_{e_1}}
\end{pmatrix},$$

where for all $i, j = 1, n$,

$$l_r^{(ij)} = \sum_{\alpha \in I_{r,n}(j)} \left| \left( A^{k+1} a_{ij}^{(k)} \right)_{\alpha} \right|.$$

This completes the proof.
Using Theorem 2.5 we evidently can obtain another determinantal representation of the group inverse and the following determinantal representation of the identity $AA^D$ on $R(A^k)$.

**Corollary 2.3.** If $\text{Ind} A = 1$ and $\text{rank } A^2 = \text{rank } A = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then the group inverse $A^g = (a^g_{ij}) \in \mathbb{C}^{n \times n}$ possess the following determinantal representations:

$$a^g_{ij} = \frac{\sum_{\alpha \in I_{r,n}(j)} \left| (A^2_j, (a_{i \cdot}))^\alpha \right|}{\sum_{\alpha \in I_{r,n}} \left| (A^2)^\alpha \right|^2},$$

(11)

for all $i, j = 1, n$.

**Corollary 2.4.** If $\text{Ind} A = k$ and $\text{rank } A^{k+1} = \text{rank } A^k = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then the matrix $AA^D = (q_{ij}) \in \mathbb{C}^{n \times n}$ possess the following determinantal representation

$$q_{ij} = \frac{\sum_{\beta \in J_{r,n}(i)} \left| (A^{k+1}_{\cdot i}, (\hat{b}_{\cdot j}))^\beta \right|}{\sum_{\beta \in J_{r,n}} \left| (A^{k+1})^\beta \right|^2},$$

(12)

for all $i, j = 1, n$.

3. Cramer’s rule of the Drazin inverse solutions of some matrix equations

Consider a matrix equation

$$AX = B,$$

(13)

where $A \in \mathbb{C}^{n \times n}$ with $\text{Ind} A = k$, $B \in \mathbb{C}^{n \times m}$ are given and $X \in \mathbb{C}^{n \times m}$ is unknown.

**Theorem 3.1.** ([14], Theorem 1) If the range space $R(B) \subset R(A^k)$, then the matrix equation (13) with constrain $R(X) \subset R(A^k)$ has a unique solution $X = A^D B$.

We denote $A^kB =: \hat{B} = (\hat{b}_{ij}) \in \mathbb{C}^{n \times m}$.

**Theorem 3.2.** If $\text{rank } A^{k+1} = \text{rank } A^k = r \leq n$ for $A \in \mathbb{C}^{n \times n}$, then for Drazin inverse solution $X = A^D B = (x_{ij}) \in \mathbb{C}^{n \times m}$ of (13) we have for all $i = 1, n$, $j = 1, m$,

$$x_{ij} = \frac{\sum_{\beta \in J_{r,n}(i)} \left| (A^{k+1}_{\cdot i}, (\hat{b}_{\cdot j}))^\beta \right|}{\sum_{\beta \in J_{r,n}} \left| (A^{k+1})^\beta \right|^2}. $$

(14)
PROOF. By Theorem 2.4 we can represent the matrix \( A^D \) by (5). Therefore, we obtain for all \( i = 1, n, j = 1, m, \)
\[
x_{ij} = \sum_{s=1}^{n} a_{is}^D b_{sj} = \sum_{s=1}^{n} \sum_{\beta \in J_{r,n}} \left| \left( A^{k+1}_i \left( a_s^{(k)} \right) \right) \beta \right| \sum_{\beta \in J_{r,n}} \left| (A^{k+1})_\beta \right| \cdot b_{sj} =
\]
\[
\sum_{\beta \in J_{r,n}} \left( A^{k+1}_i \left( a_s^{(k)} \right) \right) \beta \cdot b_{sj}
\]
Since \( \sum_s a_s^{(k)} b_{sj} = \begin{pmatrix} \sum_s a_{1s}^{(k)} b_{sj} \\ \sum_s a_{2s}^{(k)} b_{sj} \\ \vdots \\ \sum_s a_{ns}^{(k)} b_{sj} \end{pmatrix} = \hat{b}_j, \) then it follows (14).

Corollary 3.1. (42, Theorem 4.2.) If \( \text{Ind} A = k \) and \( \text{rank} A^{k+1} = \text{rank} A^k = r \leq n \) for \( A \in \mathbb{C}^{n \times n}, \) and \( y = (y_1, \ldots, y_n)^T \in \mathbb{C}^n, \) then for Drazin inverse solution \( x = A^D y =: (x_1, \ldots, x_n)^T \in \mathbb{C}^n \) of the system of linear equations
\[
A \cdot x = y,
\]
we have for all \( j = 1, n, \)
\[
x_j = \sum_{\beta \in J_{r,n}} \left| \left( A^{k+1}_j (f) \right) \beta \right| \sum_{\beta \in J_{r,n}} \left| (A^{k+1})_\beta \right|,
\]
where \( f = A^k y. \)

Consider a matrix equation
\[
XA = B, \quad (15)
\]
where \( A \in \mathbb{C}^{m \times m} \) with \( \text{Ind} A = k, \) \( B \in \mathbb{C}^{n \times m} \) are given and \( X \in \mathbb{C}^{n \times m} \) is unknown.

Theorem 3.3. (43, Theorem 2) If the null space \( \text{N}(B) \supset \text{N}(A^k), \) then the matrix equation (15) with constrain \( \text{N}(X) \supset \text{N}(A^k) \) has a unique solution
\[
X = BA^D.
\]

We denote \( BA^k =: \hat{B} = (\hat{b}_{ij}) \in \mathbb{C}^{n \times m}. \)
Theorem 3.4. If rank $A^{k+1} = rank A^k = r \leq m$ for $A \in \mathbb{C}^{n \times m}$, then for Drazin inverse solution $X = BA^D = (x_{ij}) \in \mathbb{C}^{n \times m}$ of (13), we have for all $i = 1, n, j = 1, m$,

$$x_{ij} = \frac{\sum_{\alpha \in I_{r,m}(j)} |(A^{k+1})_{j.} (\tilde{b}_i)_\alpha|}{\sum_{\alpha \in I_{r,m}} |(A^{k+1})_{\alpha}|}. \quad (16)$$

Proof. By Theorem 2.5 we can represent the matrix $A^D$ by (7). Therefore, for all $i = 1, n, j = 1, m$, we obtain

$$x_{ij} = \sum_{s=1}^{m} b_{is} a_{sj} = \sum_{s=1}^{m} b_{is} \cdot \frac{\sum_{\alpha \in I_{r,m}(j)} |(A^{k+1})_{j.} (a_{s})_\alpha|}{\sum_{\alpha \in I_{r,m}} |(A^{k+1})_{\alpha}|} = \frac{\sum_{s=1}^{m} b_{ik} \sum_{\alpha \in I_{r,m}(j)} |(A^{k+1})_{j.} (a_{s})_\alpha|}{\sum_{\alpha \in I_{r,m}} |(A^{k+1})_{\alpha}|}$$

Since for all $i = 1, n$

$$\sum_{s} b_{is} a_{s.}^{(k)} = (\sum_{s} b_{is} a_{s1}^{(k)} \sum_{s} b_{is} a_{s2}^{(k)} \cdots \sum_{s} b_{is} a_{sm}^{(k)}) = \tilde{b}_i,$$

then it follows (16).

Consider a matrix equation

$$AXB = D, \quad (17)$$

where $A \in \mathbb{C}^{n \times n}$ with $Ind A = k_1$, $B \in \mathbb{C}^{m \times m}$ with $Ind B = k_2$ and $D \in \mathbb{C}^{n \times m}$ are given, and $X \in \mathbb{C}^{n \times m}$ is unknown.

Theorem 3.5. (14], Theorem 3) If $R(D) \subset R(A^{k_1})$ and $N(D) \supset N(B^{k_2})$, $k = \max\{k_1, k_2\}$, then the matrix equation (17) with constrain $R(X) \subset R(A^k)$ and $N(X) \supset N(B^k)$ has a unique solution

$$X = A^DDB^D.$$

We denote $A^{k_1}DB^{k_2} =: \tilde{D} = (\tilde{d}_{ij}) \in \mathbb{C}^{n \times m}$.

Theorem 3.6. If rank $A^{k_1+1} = rank A^{k_1} = r_1 \leq n$ for $A \in \mathbb{C}^{n \times m}$, and rank $B^{k_2+1} = rank B^{k_2} = r_2 \leq m$ for $B \in \mathbb{C}^{m \times m}$, then for the Drazin inverse solution $X = A^DDB^D =: (x_{ij}) \in \mathbb{C}^{n \times m}$ of (17) we have

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1,m}(i)} |(A^{k_1})_{i.} (d_{\beta})_\beta|}{\sum_{\beta \in J_{r_1,m}} |(A^{k_1})_{\beta}| \sum_{\alpha \in I_{r_2,m}} |(B^{k_2})_{\alpha}|}. \quad (18)$$
It follows from respectively, and \( B \).

**Proof.**

By Theorems 2.4 and 2.5 the Drazin inverses of the matrices possess the following determinantal representations, respectively,

\[
\begin{align*}
  a^D_{ij} &= \sum_{\beta \in J_{1, n}} \left| \begin{array}{c}
  A^{k_1 + 1}_i \left( \begin{array}{c}
    a_{ij}^{(k_1)}
  \end{array} \right) \beta \\
  B^{k_2 + 1}_j \left( \begin{array}{c}
    b_{ij}^{(k_2)}
  \end{array} \right) \alpha
  \end{array} \right| \\
  b^D_{ij} &= \sum_{\alpha \in I_{r, m}} \left| \begin{array}{c}
  B^{k_2 + 1}_j \left( \begin{array}{c}
    b_{ij}^{(k_2)}
  \end{array} \right) \alpha \\
  A^{k_1 + 1}_i \left( \begin{array}{c}
    a_{ij}^{(k_1)}
  \end{array} \right) \beta
  \end{array} \right|
\end{align*}
\]

Then an entry of the Drazin inverse solution \( X = A^DDB^D =: (x_{ij}) \in C^{n \times m} \) is

\[
  x_{ij} = \sum_{s=1}^{m} \left( \sum_{t=1}^{n} a^D_{it} d_{ts} \right) b^D_{sj}.
\]

Denote by \( \tilde{d}_s \) the \( s \)th column of \( A^{k}D =: \tilde{D} = (\tilde{d}_{ij}) \in C^{n \times m} \) for all \( s = 1, \ldots, m \).

It follows from \( \sum \limits_{t} a^D_{it} d_{ts} = \tilde{d}_s \) that

\[
  \begin{align*}
    \sum_{t=1}^{n} a^D_{it} d_{ts} &= \sum_{t=1}^{n} \sum_{\beta \in J_{1, n}} \left| A^{k_1 + 1}_i \left( \begin{array}{c}
      a_{ij}^{(k_1)}
    \end{array} \right) \beta \\
    &\quad \cdot \left| B^{k_2 + 1}_j \left( \begin{array}{c}
      b_{ij}^{(k_2)}
    \end{array} \right) \alpha \right| \cdot d_{ts} = \\
    &= \sum_{\alpha \in I_{r, m}} \sum_{t=1}^{n} \left| A^{k_1 + 1}_i \left( \begin{array}{c}
      a_{ij}^{(k_1)}
    \end{array} \right) \beta \\
    &\quad \cdot \left| B^{k_2 + 1}_j \left( \begin{array}{c}
      b_{ij}^{(k_2)}
    \end{array} \right) \alpha \right| \\
    &= \sum_{\beta \in J_{1, n}} \sum_{t=1}^{n} \left| A^{k_1 + 1}_i \left( \begin{array}{c}
      \tilde{d}_s
    \end{array} \right) \beta \right| \cdot \left| \left( A^{k_1 + 1}_i \right) \beta \right| \cdot d_{ts}.
  \end{align*}
\]
Substituting (23) and (21) in (22), we obtain

\[
x_{ij} = \sum_{s=1}^{m} \sum_{\beta \in J_{r_1}, n(i)} \left| A_{i,j}^{k_1+1} \left( d_s \right) \right| \sum_{\alpha \in I_{r_2, m(j)}} \left| B_{j}^{k_2+1} \left( b_{it}^{(k_2)} \right) \right|.
\]

Suppose \( e_s \) and \( e_t \) are respectively the unit row-vector and the unit column-vector whose components are 0, except the \( s \)th components, which are 1. Then we have

\[
d_s = \sum_{l=1}^{n} e_l d_{ls}, \quad b_{it}^{(k_2)} = \sum_{s=1}^{m} b_{st}^{(k_2)} e_t, \quad \sum_{s=1}^{m} d_{st} b_{it}^{(k_2)} = \tilde{d}_{it},
\]

then we have

\[
x_{ij} = \sum_{s=1}^{m} \sum_{l=1}^{n} \sum_{\beta \in J_{r_1}, n(i)} \left| A_{i,j}^{k_1+1} \left( e_l \right) \right| \sum_{\alpha \in I_{r_2, m(j)}} \left| B_{j}^{k_2+1} \left( e_t \right) \right|.
\]

Denote by

\[
d_{it}^A := \sum_{\beta \in J_{r_1}, n(i)} \left| A_{i,j}^{k_1+1} \left( \tilde{d}_{it} \right) \right| = \sum_{l=1}^{n} \sum_{\beta \in J_{r_1}, n(i)} \left| A_{i,j}^{k_1+1} \left( e_l \right) \right| \tilde{d}_{it}
\]

the \( j \)th component of a row-vector \( d_{it}^A = (d_{it}^A, ..., d_{im}^A) \) for all \( t = 1, m \). Substituting it in (23), we obtain

\[
x_{ij} = \frac{\sum_{l=1}^{m} d_{lt}^A \sum_{\alpha \in I_{r_2, m(j)}} \left| B_{j}^{k_2+1} \left( e_t \right) \right|}{\sum_{\beta \in J_{r_1}, n(i)} \left| A_{i,j}^{k_1+1} \left( \tilde{d}_{it} \right) \right| \sum_{\alpha \in I_{r_2, m(j)}} \left| B_{j}^{k_2+1} \left( e_t \right) \right|}.
\]

Since \( \sum_{t=1}^{m} d_{it}^A e_t = d_{it}^A \), then it follows (19).

If we denote by

\[
d_{it}^B := \sum_{l=1}^{m} \left| B_{j}^{k_2+1} \left( e_t \right) \right| = \sum_{\alpha \in I_{r_2, m(j)}} \left| B_{j}^{k_2+1} \left( \tilde{d}_{it} \right) \right|
\]

we have

\[
x_{ij} = \frac{\sum_{l=1}^{m} d_{lt}^A \sum_{\alpha \in I_{r_2, m(j)}} \left| B_{j}^{k_2+1} \left( e_t \right) \right|}{\sum_{\beta \in J_{r_1}, n(i)} \left| A_{i,j}^{k_1+1} \left( \tilde{d}_{it} \right) \right| \sum_{\alpha \in I_{r_2, m(j)}} \left| B_{j}^{k_2+1} \left( e_t \right) \right|}.
\]

Since \( \sum_{t=1}^{m} d_{it}^A e_t = d_{it}^A \), then it follows (19).
the $l$th component of a column-vector $d^B_j = (d^B_{1j}, ..., d^B_{jn})^T$ for all $l = 1, n$ and substitute it in (24), we obtain

$$x_{ij} = \sum_{l=1}^{n} \sum_{\beta \in J_{1,n}} \left| A^{k_1+1} (a_{\beta})^\beta d^B_{lj} \right| \sum_{\alpha \in I_{2,m}} \left| (B^{k_2+1} (a_{\alpha})^\alpha \right|.$$

Since $\sum_{l=1}^{n} e_\beta d^B_{lj} = d^B_j$, then it follows (18).

4. Applications of the determinantal representations of the Drazin inverse to some differential matrix equations

Consider the matrix differential equation

$$X' + AX = B$$

(25)

where $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times n}$ are given, $X \in \mathbb{C}^{n \times n}$ is unknown. It’s well-known that the general solution of (25) is found to be

$$X(t) = \exp(-At) \left( \int \exp(At) \ dt \right) B.$$

If $A$ is invertible, then

$$\int \exp(At) \ dt = A^{-1} \exp(At) + G,$$

where $G$ is an arbitrary $n \times n$ matrix. If $A$ is singular, then the following theorem gives an answer.

**Theorem 4.1.** (24), Theorem 1) If $A$ has index $k$, then

$$\int \exp(At) \ dt = A^D \exp(At) + (I - AA^D)t \left[ I + \frac{A^2}{2} + \frac{A^3}{3!} t^2 + \ldots + \frac{A^{k-1}}{k!} t^{k-1} \right] + G.$$

Using Theorem 4.1 and the power series expansion of $\exp(-At)$, we get an explicit form for a general solution of (25)

$$X(t) = \left\{ A^D + (I - AA^D)t \left( I - \frac{A^2}{2} t + \frac{A^3}{3!} t^2 - \ldots (-1)^{k-1} \frac{A^{k-1}}{k!} t^{k-1} \right) + G \right\} B.$$

If we put $G = 0$, then we obtain the following partial solution of (25),

$$X(t) = A^D B + (B - A^D AB)t - \frac{1}{k!(l-1)!} (A^{k-1} B - A^D A^k B) t^k.$$

(26)

Denote $A'B = \hat{B}(l) = (\hat{b}^{(l)}_{ij}) \in \mathbb{C}^{n \times n}$ for all $l = 1, 2k$. 

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Theorem 4.2. The partial solution \( X(t) = (x_{ij}) \), possess the following determinantal representation

\[
x_{ij} = \sum_{\beta \in J_{r,n}(i)} \left( \alpha_{\beta} \sum_{\beta \in J_{r,n}(i)} \left| \begin{array}{c} \beta \\ \beta \end{array} \right| \right) t^{\beta} + \left( b_{ij} - \sum_{\beta \in J_{r,n}(i)} \left| \begin{array}{c} \beta \\ \beta \end{array} \right| \right) t^{\beta} \]

for all \( i, j = 1, n \).

PROOF. Using the determinantal representation of the identity \( 1^{\beta} \) we obtain the following determinantal representation of the matrix \( A^D A^m B := (y_{ij}) \),

\[
y_{ij} = \sum_{s=1}^{n} \sum_{t=1}^{n} a_{st}^{(m-1)} b_{ij} = \sum_{\beta \in J_{r,n}(i)} \left( \alpha_{\beta} \sum_{\beta \in J_{r,n}(i)} \left| \begin{array}{c} \beta \\ \beta \end{array} \right| \right) \sum_{t=1}^{n} \left| \begin{array}{c} \beta \\ \beta \end{array} \right| \sum_{\beta \in J_{r,n}(i)} \left| \begin{array}{c} \beta \\ \beta \end{array} \right| \]

for all \( i, j = 1, n \) and \( m = 1, K \). From this and the determinantal representation of the Drazin inverse solution \( 1^{\beta} \) and the identity \( 0^{\beta} \) it follows (27).

Corollary 4.1. If \( \text{Ind} A = 1 \), then the partial solution of \( 2^{\beta} \),

\[
X(t) = (x_{ij}) = A^g B + (B - A^g AB) t,
\]

possess the following determinantal representation

\[
x_{ij} = \beta \sum_{\beta \in J_{r,n}(i)} \left( \sum_{\beta \in J_{r,n}(i)} \left| \begin{array}{c} \beta \\ \beta \end{array} \right| \right) t^{\beta} \]

for all \( i, j = 1, n \).

Consider the matrix differential equation

\[
X' + AX = B \quad (29)
\]
where $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times n}$ are given, $X \in \mathbb{C}^{n \times n}$ is unknown. The general solution of (29) is found to be

$$X(t) = B \exp^{-At} \left( \int \exp^{At} \, dt \right)$$

If $A$ is singular, then an explicit form for a general solution of (29) is

$$X(t) = B \left\{ A^D + (I - AA^D)t - \frac{1}{2!}((BA - BA^2)A) t^2 + \ldots \right\}$$

If we put $G = 0$, then we obtain the following partial solution of (29),

$$X(t) = BA^D + (B - BAA^D)t - \frac{1}{2}((BA - BA^2A^2)A)t^2 + \ldots$$

The partial solution (30), $X(t) = (x_{ij})$, possess the following determinantal representation,

$$x_{ij} = \sum_{\alpha \in \mathfrak{G}_{n,j}(i)} \left| \frac{A_{j+1} \left( b_{1,\alpha}^{(1)} \right)}{\alpha} \right| + \left( b_{ij} - \sum_{\alpha \in \mathfrak{G}_{n,j}(i)} \left| \frac{A_{j+1} \left( b_{1,\alpha}^{(1)} \right)}{\alpha} \right| \right) t$$

$$- \frac{1}{2} \left( \sum_{\alpha \in \mathfrak{G}_{n,j}(i)} \left| \frac{A_{j+1} \left( b_{1,\alpha}^{(2)} \right)}{\alpha} \right| \right) t^2 + \ldots$$

$$\frac{(-1)^k}{k!} \left( \sum_{\alpha \in \mathfrak{G}_{n,j}(i)} \left| \frac{A_{j+1} \left( b_{1,\alpha}^{(k+1)} \right)}{\alpha} \right| \right) t^k$$

for all $i, j = 1, \ldots, n$.

**Corollary 4.2.** If $\text{Ind} A = 1$, then the partial solution of (29),

$$X(t) = (x_{ij}) = BA^g + (B - BAA^g)t,$$

possess the following determinantal representation

$$x_{ij} = \sum_{\alpha \in \mathfrak{G}_{n,j}(i)} \left| \frac{A_{j+1} \left( b_{1,\alpha}^{(1)} \right)}{\alpha} \right| + \left( b_{ij} - \sum_{\alpha \in \mathfrak{G}_{n,j}(i)} \left| \frac{A_{j+1} \left( b_{1,\alpha}^{(1)} \right)}{\alpha} \right| \right) t.$$
5. Examples

In this section, we give examples to illustrate our results.

5.1. Example 1

Let us consider the matrix equation

\[ AXB = D, \]  

where

\[
A = \begin{pmatrix} 2 & 0 & 0 \\ -i & i & i \\ -i & -i & -i \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 1 \\ i & -i & i \\ -1 & 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & i & 1 \\ i & 0 & 1 \end{pmatrix}.
\]

We shall find the Drazin inverse solution of (31) by (18). We obtain

\[
A^2 = \begin{pmatrix} 4 & 0 & 0 \\ 2 - 2i & 0 & 0 \\ -2 - 2i & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 8 & 0 & 0 \\ 4 - 4i & 0 & 0 \\ -4 - 4i & 0 & 0 \end{pmatrix},
\]

\[
B^2 = \begin{pmatrix} 1 & i & 3 - i \\ 1 - 1 & 1 + 3i \\ -3 + i & 3 - i & 3 + i \end{pmatrix}.
\]

Since \( \text{rank} A = 2 \) and \( \text{rank} A^2 = \text{rank} A^3 = 1 \), then \( k_1 = \text{Ind} A = 2 \) and \( r_1 = 1 \).

Since \( \text{rank} B = \text{rank} B^2 = 2 \), then \( k_2 = \text{Ind} B = 1 \) and \( r_2 = 2 \). Then we have

\[ \tilde{D} = A^2 DB = \begin{pmatrix} -4 & 4 & 8 \\ -2 + 2i & 2 - 2i & 4 - 4i \\ 2 + 2i & -2 - 2i & -4 - 4i \end{pmatrix}, \]

and

\[
\sum_{\beta \in J_{1,3}} |(A^3)^{\beta}_{\beta}| = 8 + 0 + 0 = 8,
\]

\[
\sum_{\alpha \in I_{2,3}} |(B^2)^{\alpha}_{\alpha}| = \det\left(\begin{array}{cc} -i & i \\ 1 & -1 \end{array}\right) + \det\left(\begin{array}{cc} -1 & 1 + 3i \\ 3 - i & 3 + i \end{array}\right) + \det\left(\begin{array}{cc} -i & 3 - i \\ -3 + i & 3 + i \end{array}\right) = 0 + (-9 - 9i) + (9 - 9i) = -18i.
\]

By (20), we can get

\[
d^B_1 = \begin{pmatrix} 12 - 12i \\ -12i \\ -12 \end{pmatrix}, \quad d^B_2 = \begin{pmatrix} -12 + 12i \\ 12i \\ 12 \end{pmatrix}, \quad d^B_3 = \begin{pmatrix} 8 \\ -12 - 12i \\ -12 + 12i \end{pmatrix}.
\]
Since $A_{3,1} \left( B_{1} \right) = \begin{pmatrix} 12 - 12i & 0 & 0 \\ -12i & 0 & 0 \\ -12 & 0 & 0 \end{pmatrix}$, then finally we obtain

$$x_{11} = \frac{\sum_{\beta \in J_{1,3}(1)} |A_{3,1} \left( B_{1} \right)_{\beta}|}{\sum_{\beta \in J_{1,3}} |(A_{3})_{\beta}|} \frac{12 - 12i}{8 \cdot (-18i)} = \frac{1 + i}{12}. $$

Similarly,

$$x_{12} = \frac{-12 + 12i}{8 \cdot (-18i)} = \frac{-1 - i}{12}, \quad x_{13} = \frac{8}{8 \cdot (-18i)} = \frac{i}{18}, $$

$$x_{21} = \frac{-12i}{8 \cdot (-18i)} = \frac{1}{12}, \quad x_{22} = \frac{12i}{8 \cdot (-18i)} = \frac{-1 - i}{12}, \quad x_{23} = \frac{-12 - 12i}{8 \cdot (-18i)} = \frac{1 - i}{12}, $$

$$x_{31} = \frac{12}{8 \cdot (-18i)} = \frac{-i}{12}, \quad x_{32} = \frac{-12}{8 \cdot (-18i)} = \frac{i}{12}, \quad x_{33} = \frac{-12 + 12i}{8 \cdot (-18i)} = \frac{-1 - i}{12}. $$

Then

$$X = \begin{pmatrix} \frac{1 + i}{12} & \frac{-1 + i}{12} & \frac{i}{12} \\ \frac{-1 + i}{12} & \frac{-1 - i}{12} & \frac{i}{12} \\ \frac{i}{12} & \frac{i}{12} & \frac{1 + i}{12} \end{pmatrix} $$

is the Drazin inverse solution of (31).

5.2. Example 2

Let us consider the differential matrix equation

$$X' + AX = B, $$

(32)

where

$$A = \begin{pmatrix} 1 & -1 & 1 \\ i & -i & i \\ -1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & i & 1 \\ i & 0 & 1 \\ 1 & i & 0 \end{pmatrix}. $$

Since rank $A = \text{rank} A^2 = 2$, then $k = \text{Ind} A = 1$ and $r = 2$. The matrix $A$ is the group inverse. We shall find the partial solution of (32) by (28). We have

$$A^2 = \begin{pmatrix} -i & i & 3 - i \\ 1 & -1 & 1 + 3i \\ -3 + i & 3 - i & 3 + i \end{pmatrix}, \quad \hat{B}^{(1)} = AB = \begin{pmatrix} 2 - 2i & 2i & 0 \\ 1 + 2i & -2 & 0 \\ 1 + i & i & 0 \end{pmatrix}, $$

$$\hat{B}^{(2)} = A^2 B = \begin{pmatrix} 2 - 2i & 2 + 3i & 0 \\ 2 + 2i & -3 + 2i & 0 \\ 1 + 5i & -2 & 0 \end{pmatrix}. $$
and

\[
\sum_{\alpha \in J_{2,3}} \left| (A^2)^{\beta}_{\beta} \right| = \\
\det \begin{pmatrix} -i & i \\ 1 & -1 \end{pmatrix} + \det \begin{pmatrix} -1 & 1+3i \\ 3-i & 3+i \end{pmatrix} + \det \begin{pmatrix} -i & 3-i \\ -3+i & 3+i \end{pmatrix} = \\
0 + (-9 - 9i) + (9 - 9i) = -18i.
\]

Since \((A^2)_{1} (\hat{b}^{(1)}_{1}) = \begin{pmatrix} 2 - i & i & 3 - i \\ 1 + 2i & -1 & 1+3i \\ 1+i & 3-i & 3+i \end{pmatrix}\) and

\[
(A^2)_{1} (\hat{b}^{(2)}_{1}) = \begin{pmatrix} 2 - 2i & i & 3 - i \\ 2 + 2i & -1 & 1+3i \\ 1+5i & 3-i & 3+i \end{pmatrix},
\]

then finally we obtain

\[
x_{11} = \frac{\sum_{\beta \in J_{2,3}} \left| (A^2)^{\beta}_{\beta} (\hat{b}^{(1)}_{1})^\beta \right|}{\sum_{\beta \in J_{2,3}} \left| (A^2)^{\beta}_{\beta} \right|} + \left( b_{11} - \frac{\sum_{\beta \in J_{2,3}} \left| (A^2)^{\beta}_{\beta} \right|}{\sum_{\beta \in J_{2,3}} \left| (A^2)^{\beta}_{\beta} \right|} \right) t = \\
\frac{3-3i}{-18i} + \left( 1 - \frac{-18i}{18i} \right) t = \frac{1+i}{6}. 
\]

Similarly,

\[
x_{12} = \frac{-3 + 3i}{-18i} + \left( i - \frac{9 + 9i}{-18i} \right) t = \frac{-1 - i}{6} + \frac{1+i}{2} t, \ x_{13} = 0 + (1 - 0) t = t, 
\]

\[
x_{21} = \frac{3 + 3i}{-18i} + \left( i - \frac{-18}{-18i} \right) t = \frac{-1 + i}{6}, 
\]

\[
x_{22} = \frac{-3 - 3i}{-18i} + \left( 0 - \frac{-9 + 9i}{-18i} \right) t = \frac{1 - i}{6} + \frac{1+i}{2} t, \ x_{23} = 0 + (1 - 0) t = t, 
\]

\[
x_{31} = \frac{-12i}{-18i} + \left( 1 - \frac{-18i}{-18i} \right) t = \frac{2}{3}, 
\]

\[
x_{32} = \frac{9 + 3i}{-18i} + \left( i - \frac{-18}{-18i} \right) t = \frac{-1 + 3i}{6}, \ x_{33} = 0 + (0 - 0) t = 0. 
\]

Then

\[
X = \frac{1}{6} \begin{pmatrix} 1 + i & -1 - i + (3 + 3i)t & t \\ -1 + i & 1 - i + (3 + 3i)t & t \\ 4 & -1 + 3i & 0 \end{pmatrix}
\]

is the partial solution of \((32)\).
References

[1] H. Dai, On the symmetric solution of linear matrix equation, Linear Algebra Appl. 131 (1990) 1-7.

[2] F.J. Henk Don, On the symmetric solutions of a linear matrix equation, Linear Algebra Appl. 93 (1987) 1-7.

[3] C.G. Khatri, S.K. Mitra, Hermitian and nonnegative definite solutions of linear matrix equations, SIAM J. Appl. Math. 31 (1976) 578-585.

[4] Y. H. Liu, Ranks of least squares solutions of the matrix equation AXB=C, Comput. Math. Appl. 55 (2008) 1270-1278.

[5] Z.Y. Peng and X.Y. Hu, The generalized reflexive solutions of the matrix equations AX = D and AXB = D, Numer. Math. 5 (2003) 94-98.

[6] W.J. Vetter, Vector structures and solutions of linear matrix equations, Linear Algebra Appl. 9 (1975) 181-188.

[7] G. Chen, X. Chen, A new splitting for singular linear system and Drazin inverse, J. East China Norm. Univ. Natur. Sci. Ed. 3 (1996) 12-18.

[8] A. Sidi. A unified approach to Krylov subspace methods for the Drazin-inverse solution of singular nonsymmetric linear systems. Linear Algebra Appl. 298 (1999) 99-113.

[9] P.S. Stanimirovic. A representation of the minimal P-norm solution, Novi Sad Journal of Mathematics 30 (2000) 177-183.

[10] Israr Ali Khan, Q.W. Wang, The Drazin inverses in an arbitrary semiring, Linear and Multilinear Algebra 59 (9) (2011) 1019-1029.

[11] Y. Wei, A characterization for the W-weighted Drazin inverse and a Cramer rule for the W-weighted Drazin inverse solution, Appl. Math. Comput. 125 (2002) 303-310.

[12] C. Gu, Guorong Wang, Zhaoliang Xu, PCR algorithm for the parallel computation of the solution of a class of singular linear systems, Applied Mathematics and Computation, 176 (2006) 237-244.

[13] C. Gu, A parallel algorithm for computing the solution of restricted matrix equation, Proceedings of The 14th Conference of International Linear Algebra Society, World Academic Press, (2007) 23-26.

[14] Z.Xu, G. Wang, On extensions of Cramer’s rule for solutions of restricted matrix equations, Journal of Lanzhou University 42 (3) (2006) 96-100.

[15] H. J. Werner, On extensions of Cramer’s rule for solutions of restricted linear systems, Linear Multilinear Algebra 15 (1984) 319-330.
[16] G. Wang, Cramer rule for finding the solution of a class of singular equations, Linear Algebra Appl. 161 (1989) 27-34.

[17] R.E.Hartwig, G. Wang, Y.M. Wei, Some additive results on Drazin inverse, Appl. Math. Comput. 322 (2001), 207-217.

[18] G. Wang, S. Qiao, Solving constrained matrix equations and Cramer rule, Appl. Math. Comput. 159 (2004) 333-340

[19] G. Wang, Z.Xu, Solving a kind of restricted matrix equations and Cramer rule, Appl. Math. Comput. 162 (2005) 329-338.

[20] J.Ji, Explicit expressions of the generalized inverses and condensed Cramer rule, Linear Algebra Appl. 404 (2005), 183-192.

[21] C.Gu, G. Wang, Condensed Cramer rule for solving restricted matrix equations, Appl. Math. Comput. 183 (2006) 301-306.

[22] I.I. Kyrchei, Analogs of the adjoint matrix for generalized inverses and corresponding Cramer rules, Linear Multilinear Algebra 56 (2008), 453-469.

[23] I. Kyrchei, Analogs of Cramer’s rule for the minimum norm least squares solutions of some matrix equations, Appl. Math. Comput., 218 (2012) 6375-6384.

[24] X. Liu, G. Zhu, G. Zhou, Y. Yu, An Analog of the Adjugate Matrix for the Outer Inverse $A_{F,S}^2$, Mathematical Problems in Engineering, Volume 2012, Article ID 591256, 14 pages, doi:10.1155/2012/591256.

[25] S. L. Campbell, C. D. Meyer Jr., Generalized Inverse of Linear Transformations, Pitman, London, (1979).

[26] S. L. Campbell, C. D. meyer, JR. and N. J. Rose, Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients, SIAM J. Appl. Math. 31 (1976) 411-425.