The Kakeya conjecture on local fields of positive characteristic

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INTRODUCTION

A Besicovitch set or Kakeya set is defined as a compact set $E \subseteq \mathbb{R}^n$ that contains a unit line segment in every direction. The Kakeya conjecture states that every Besicovitch set in $\mathbb{R}^n$ has Hausdorff dimension $n$. This problem has been thoroughly researched due to its connections with harmonic analysis, differential equations, arithmetic combinatorics and number theory, cf. [2, 21, 24]. The conjecture remains open for $n \geq 3$. Let $d(n)$ denote the least possible value for the Hausdorff dimension of a Besicovitch set in $\mathbb{R}^n$. Currently the best lower bounds for $n = 3, 4$ are established in [15] and [16], where the authors show that $d(3) \geq \frac{5}{2} + \varepsilon_0$ and $d(4) \geq 3.059$, respectively. For $n = 6$, the best known bound is $d(n) \geq 4$, proved by Wolff in [23]. For the best known results in higher dimensions, see [15] and [25]. With regard to Minkowski dimension, Katz and Tao [15] showed that every Besicovitch set in $\mathbb{R}^n$ has Minkowski dimension of at least $\frac{n}{\alpha} + \frac{\alpha - 1}{\alpha}$ where $\alpha \approx 1.675$.

In view of the difficulty of this conjecture, Wolff proposed in [26] a finite field analogue of the Kakeya conjecture as a toy model for the real case. He asked whether it was possible to find constants $C_n > 0$ such that if $F_q$ is the finite field with $q$ elements, then every Kakeya set $E \subseteq F_q^n$ has at least $C_n q^n$ elements. This was solved affirmatively by Dvir in his influential paper [8] using the polynomial method. His original proof gave the value $C_n = \frac{1}{n!}$, which was subsequently improved to $C_n = 2^{-n}$ in [9] and to $C_n = 2^{1-n}$ in [3]. However, it was noted in [11] that the analogy between...
the classical and finite field Kakeya conjectures is flawed in two ways. Firstly, the bound obtained in [8] is too strong, as it says that all Kakeya sets in $F_q^n$ have, in a sense, positive measure. In contrast, there are Besicovitch sets in $\mathbb{R}^n$ which have Lebesgue measure zero. Another deficiency of $F_q$ is that it does not admit ‘multiple scales’: That is, there is no natural notion of distance on this ring that is non-trivial. These problems are related to each other, since the existence of multiple scales in $\mathbb{R}^n$ is what allows for the construction of Besicovitch sets with measure zero. For this reason, in [11], Ellenberg, Oberlin and Tao proposed analogues of the Kakeya conjecture for metric rings that have multiple scales, specifically local fields, their rings of integers and quotient rings. The question of proving local field variants of the Kakeya conjecture had already been investigated in the 1990s by J. Wright, who gave a series of lectures in the University of New South Wales on this topic.

In [7], Dummit and Habilcek constructed Kakeya sets of measure zero in $F_q((t))^n$ for all $n \geq 2$, and they also showed that Kakeya sets in $F_q[t]^2$ or $\mathbb{Z}_p^2$ have Minkowski dimension 2. Fraser [12] constructed Kakeya sets of measure zero in $L^n$, where $L$ is any non-archimedean local field with finite residue field. Hickman and Wright [15] studied the Kakeya conjecture on $\mathbb{Z}/N\mathbb{Z}$ in relation with the discrete restriction conjecture, and they gave a simpler construction of a Kakeya set in $\mathbb{Z}_p^n$ of measure zero. Caruso [4] has shown that almost all Kakeya needle sets have measure zero in the non-archimedean case. These results show that non-archimedean local fields behave in a similar way to $\mathbb{R}$ in regard to Kakeya sets.

Recently Arsovski [1] proved the Kakeya conjecture for the $p$-adic field $\mathbb{Q}_p$ using the polynomial method. This result implies the Kakeya conjecture over any finite field extension $L/\mathbb{Q}_p$, so the Kakeya conjecture holds for any non-archimedean local field of characteristic zero. In this article, we will adapt Arsovski’s proof to the case of local fields of positive characteristic. The main result of this article is the following:

**Theorem 1.** All $c$-Kakeya sets in $F_q((t))^n$ have Hausdorff dimension $n$.

In order to do this, one needs to find an appropriate completely ramified extension of $F_q((t))$ that is analogous to the extension of $\mathbb{Q}_p$ given by adjoining a primitive $p^k$-th root of unity. This is achieved using well-known properties of Lubin–Tate extensions.

### 1.1 Notation

We write $A \gtrsim B$ if there is a constant $C > 0$ (which may or may not depend on $n$ or $q$) such that $A \gtrsim CB$. We may also add $n$, $q$ or another variable as a subscript in order to specify that the implicit constant depends on those variables. Throughout this article, none of the implicit constants will depend on $k$ (or on $\delta = q^{-k}$).

$H^s(\cdot)$ denotes the $s$-dimensional Hausdorff content, which is defined as

$$H^s(E) = \inf \left\{ \sum_{j=1}^\infty (\text{diam } U_j)^s : E \subseteq \bigcup_{j=1}^\infty U_j \right\},$$

where diam $U_j$ stands for the diameter of $U_j$. The Hausdorff dimension of a set $E$ in a metric space is then defined as the smallest real number $d = \text{dim}_H(E)$ such that $H^s(E) = 0$ for all $s < d$. 
For any ring $R$, let $R[[t]]$ be the ring of formal power series
\[ a(t) = c_0 + c_1 t + \cdots + c_n t^n + \cdots, \tag{1} \]
where all the coefficients $c_j$ are elements of $R$. We shall denote by $K = F_q((t))$ the field of formal Laurent series over $F_q$, which is the quotient field of $A = F_q[[t]]$. Since every power series $a \in A$ with non-zero constant term has a multiplicative inverse, it follows that every element of $K$ can be written as $a(t) = t^{n_0}(c_0 + c_1 t + \cdots)$ with $c_0 \neq 0$ and $n_0 \in \mathbb{Z}$. Given a non-zero element $a \in K$, we define $v_t(a)$ as the smallest integer $n_0$ such that the coefficient of $t^{n_0}$ in $a(t)$ is non-zero. We also set $v_t(0) = -\infty$. It is easy to show that the function $|a| = q^{-v_t(a)}$ is an absolute value that satisfies the ultrametric inequality
\[ |a + b| \leq \max\{|a|, |b|\}, \tag{2} \]
for all $a, b \in K$. Therefore, it induces a metric $d(x, y) = |x - y|$ on $K$. We also equip $K^n$ with the supremum norm
\[ |(a_1, \ldots, a_n)| = \max_{1 \leq j \leq n} |a_j|. \]
With this norm $K^n$ becomes a complete, locally compact metric group with Hausdorff dimension $n$. Since $K^n$ is locally compact, it has a Haar measure, which can be normalized so that $|A| = 1$. We shall also use the notation $|\cdot|$ to denote the cardinality of a finite set, however this is unlikely to generate confusion.

Given a field $F$ with an absolute value $|\cdot|$ satisfying (2), we define $\mathcal{O}_F = \{x \in F : |x| \leq 1\}$, which is called the valuation domain, and $m_F = \{x \in F : |x| < 1\}$. It is easy to show that $\mathcal{O}_F$ is a subring of $F$ and that $m_F$ is an ideal of $\mathcal{O}_F$. Note that for every $x \in \mathcal{O}_F$ with $|x| = 1$, its inverse $x^{-1} \in F$ is also contained in $\mathcal{O}_F$. Since every non-zero element of the quotient ring $\mathcal{O}_F/m_F$ is represented by an element $x \in \mathcal{O}_F$ of absolute value 1, it follows that $\mathcal{O}_F/m_F$ is a field, and it is called the residue field of $F$. In our case of interest $F = K$, we have $\mathcal{O}_K = A$ and $m_K = tA$, so the residue field is $A/tA \cong F_q$. A non-archimedean local field can be defined as a field $F$ with an absolute value $|\cdot|$ satisfying (2), that is complete as a metric space and such that its residue field $\mathcal{O}_F/m_F$ is a finite field. Thus, $F_q((t))$ is a local field for every prime power $q$. The other main example of a local field is the field $\mathbb{Q}_p$ of $p$-adic numbers, where $p$ is a prime number. It is defined as the quotient field of the ring
\[ \mathbb{Z}_p = \mathbb{Z}[[t]]/(t - p), \]
which is an integral domain. The elements of $\mathbb{Z}_p$ are called $p$-adic integers. It can be shown (cf. [24, Chapter 1] or [16, Theorem 9.16]) that every non-archimedean local field is a finite extension of either $F_p[[t]]$ or $\mathbb{Q}_p$, for some prime $p$. Conversely, for every finite separable extension $L$ of either $F_p[[t]]$ or $\mathbb{Q}_p$, it is possible to extend the absolute value of the base field uniquely so that $L$ becomes a local field (cf. [14, §9.8]).

Given $0 < c \leq 1$, we define a $c$-Kakeya set in $K^n$ as a compact set $E \subseteq K^n$ such that for any direction $w \in F_q[[t]]^n$ there is a set $J_w \subseteq F_q[[t]]$ of measure $\geq c$ and $b_w \in K^n$ such that $b_w + J_w \cdot w \subseteq E$. 


2 | OVERVIEW OF THE PROOF

Both Arsovki’s article and this one use Dvir’s polynomial method, which can be succinctly described in the following way: Given a Kakeya set $E$ of small size, find a non-zero polynomial $g$ of low degree that vanishes on $E$; then use the fact that $E$ contains lines in many different directions to show that either $g$ or a related polynomial is zero, arriving at a contradiction.

In order to find the polynomial, Dvir’s original argument used the following lemma whose proof is based on dimension counting:

**Lemma 2.** Let $k, n, d > 0$ be integers. Let $F$ be a field, and let $S \subseteq F^n$ be a set with less than $\binom{n+d}{n}$ elements. Then there is a non-zero polynomial $g \in F[X_1, \ldots, X_n]$ of degree $\leq d$ such that $g(s) = 0$ for all $s \in S$.

Afterwards, a method of multiplicities [9] was developed, which requires a polynomial that vanishes with high multiplicity at each point of $E$, and obtains better lower bounds in the finite field case. However, the proof of Theorem 1 does not require a stronger result than Lemma 2.

It is possible to use the polynomial method for local fields since the problem of bounding Hausdorff dimension can be discretized. As we shall see in the next section, the proof of Theorem 1 essentially boils down to showing that every Kakeya set $E \subseteq (A/t^kA)^n$ has at least $q_{kn}/u(k, n)$ elements, where $u(k, n) \lesssim q^{kn}$ for every $\varepsilon > 0$. By treating $R = A/t^kA$ as an $F_q$-vector space of dimension $k$ and using lower bounds from the finite field case, one can show that $|E| \geq 2^{-kn}q^{kn}$. However, this is essentially the best bound obtainable by applying the polynomial method over finite fields, since there are examples (cf. [20]) of Kakeya sets in $F_q^{kn}$ with $\gtrsim 2^{-kn}q^{kn}$ elements. One problem with this approach is that polynomials with coefficients in $F_q$ normally do not distinguish between an $R$-line and an arbitrary affine subspace of dimension $k$ over $F_q$. A subexponential bound for $u(k, n)$ was proved for $n = 2$ in [7], but the argument seemingly cannot be generalized to higher dimensions.

Arsovki’s innovation consists of working with a primitive $p^k$-th root of unity $\zeta$ in an algebraic closure of $\mathbb{Q}_p$ and taking advantage of the group isomorphism $\zeta^Z \cong Z/p^kZ$ to map the Kakeya set into a subset of $(\zeta^Z)^n \subseteq \mathbb{Q}_p(\zeta)^n$. Arsovki then applies the polynomial method on $\mathbb{Q}_p(\zeta)^n$. This solves the aforementioned problem, and has the added benefit of preserving some of the algebraic structure of $E$. This is because every line in $R^n$ gets mapped into a translate of a subgroup isomorphic to $Z/p^kZ$.

This strategy is not viable over fields of characteristic $p > 0$, since the only $p$-th root of unity in these fields is 1. However, the multiplicative group $\zeta^Z$ can be reinterpreted as the orbit of $\zeta$ under the action of the Galois group $\text{Gal}(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p)$, and this idea also works in positive characteristic. The Galois group needs to be isomorphic to $R = A/t^kA$. Fortunately, the Existence Theorem in Local Class Field Theory tells us that for every open subgroup $H \subseteq K^\times$ of finite index there is a corresponding abelian extension $L/K$ with Galois group $\text{Gal}(L/K) \cong K^\times/H$. Since the additive group $A/t^kA$ can be realized as a quotient of $K^\times/H$ (for instance, taking $H = (1 + t^kA)t^\mathbb{Z}$), this gives us the desired extension $L/K$. It is easy to show that $L$ must necessarily be totally ramified. For this type of abelian extensions, Lubin–Tate theory gives a very concrete description of the Galois action. More precisely, this action can be realized as the quotient of an $A$-action on $L$ by power series. Moreover, there is a polynomial $f_k$ whose splitting field is $L$, such that its sets of roots $\Lambda_k$ is an additive group, and the elements of $A$ act as group endomorphisms,
which makes $\Lambda_k$ an $A$-module. Addition allows for ‘lines’ in $\Lambda^n_k$ passing through the origin to be translated.

There is also a slight modification in the last part of the argument which arrives at a contradiction, in relation to the finite field case. In Dvir’s argument, after homogenizing the polynomial so that it can be evaluated in the projective space $\mathbb{P}^n(F_q)$, one uses degree considerations to conclude that the homogenization vanishes at the hyperspace at infinity, which corresponds to the set of directions of affine lines in $F_q$. In terms of the original polynomial, this means that its homogeneous component of highest degree vanishes, so applying the Schwartz–Zippel lemma yields a contradiction. In Arsovski’s method, one works over a finite extension $L$ of the local field. In this setting, we shall reduce the coefficients of the polynomial modulo the maximal ideal $\mathfrak{p}_L$ (which can be understood as ‘evaluating at $t = 0$’) to obtain a polynomial $g \in B[z_1, \ldots, z_n]$, where $B = F_q[X]$, that takes small values at many points $y \in C^n$, where $C^n \subseteq B^n$. Thus, $C^n$ is an analogue of the hyperspace at infinity, and the Schwartz–Zippel lemma will be replaced by Lemma 7.

3 REDUCTION TO A COVERING THEOREM

Let $A = F_q[[t]]$. A vector $w \in A^n$ is called primitive or unitary if $|w| = 1$, this is equivalent to $w$ having some coordinate that is not a multiple of $t$. The set of primitive vectors in $A^n$ will be denoted by $S^{n-1}(A)$. Likewise, if $R = A/t^kA$, a vector $w \in R^n$ is primitive if $w \notin (tR)^n$, and we define $S^{n-1}(R) = R^n \setminus (tR)^n$, the set of primitive vectors in $R^n$.

As in [8] and [1], we define an $(\varepsilon, \nu)$-Kakeya set to be a set $E \subseteq K^n$ for which there is a set $\Omega \subseteq S^n(A)$ of directions with measure $|\Omega| \geq \nu$, such that for all $w \in \Omega$ there are $b_w \in K^n$ and a subset $J_w \subseteq A$ of measure $\geq \varepsilon$ such that $b_w + wJ_w \subseteq E$. Similarly, we also say that a subset $E \subseteq R^n$ is an $(\varepsilon, \nu)$-Kakeya set modulo $t^k$ if there is a set $\Omega \subseteq S^{n-1}(R)$ with at least $\nu q^{n^k}$ elements, such that for each $w \in \Omega$ there is a line $\ell_w \subseteq R^n$ in the direction of $w$ with $|\ell_w \cap E| \geq \varepsilon q^k$.

Theorem 1 will be deduced from the following result:

**Theorem 3.** Let $n, k > 0$ be integers, and let $\varepsilon, \nu \in (0, 1)$. Then, an $(\varepsilon, \nu)$-Kakeya set in $F_q[[t]]^n$ cannot be covered by less than

$$\left(\frac{\nu \varepsilon q^{k-1}}{k n} + n\right)$$

closed balls of radius $q^{-k}$.

Theorem 3 also implies the Kakeya maximal conjecture for $F_q[[t]]$. As in the real case, this conjecture can be stated in several equivalent ways. Here it is preferable to state it in terms of the Kakeya maximal operator $K_{\varepsilon, \nu}$, which is defined below.

Let $\delta = q^{-k}$, where $k > 0$ is an integer. A unit line segment in $A^n$ is a set of the form $b + Aw$, where $b, w \in A^n$ and $w$ is primitive. We define a $\delta$-tube to be the closed $\delta$-neighborhood of a unit line segment, that is, a set of the form

$$T_\delta = T_\delta(b, w) = b + Aw + (t^k A)^n,$$
where $w$ is a primitive vector. As one might expect, these sets have measure $\delta^{n-1}$. Given an integrable function $\phi : A^n \to \mathbb{R}$, we define its Kakeya maximal function $\mathcal{K}_\delta \phi : S^{n-1}(A) \to \mathbb{R}$ as

$$\mathcal{K}_\delta \phi(w) = \sup_{b \in A^n} \frac{1}{\delta^{n-1}} \int_{T_\delta(b, w)} |\phi|.$$  \hspace{1cm} (3)

**Theorem 4** (Kakeya maximal conjecture). Let $k > 0$ be an integer, let $\delta = q^{-k}$ and let $\phi : A^n \to \mathbb{R}$ be a measurable function. Then,

$$\|\mathcal{K}_\delta \phi\|_{L^n(S^{n-1}(A))} \lesssim_{n,q} k^{n+2} \|\phi\|_{L^n(A^n)}.$$  \hspace{1cm} (4)

The constant depends logarithmically on $\delta = q^{-k}$, which is also expected to happen in the real case. Similarly to the estimate (4) will be deduced from a distributional estimate

$$|\{\omega \in S^{n-1}(R) : \mathcal{K}_\delta \phi(\omega) \geq \lambda\}| \lesssim_{n,q} k^{n+1} \lambda^{-n} \|\phi\|_{L^n(A^n)},$$  \hspace{1cm} (5)

which will be shown to hold for all $\lambda > 0$. The proof of (5) exploits the fact that there are only finitely many tubes of a given size. Indeed, if $w' = w + at^k$ for some $a \in A$, then $T_\delta(b, w') = T_\delta(b, w)$. This means that $\mathcal{K}_\delta \phi$ is constant on each coset $b + (t^k A)$, and so it determines a well-defined function on $S^{n-1}(R)$. Moreover, each $\delta$-tube is a disjoint union of closed balls of radius $\delta$. In fact, if $\pi : A^n \to \mathbb{R}^n$ denotes the projection to the quotient, the image of any $\delta$-tube $T = T_\delta(b, w)$ is a line $l = \pi(b) + R\pi(w)$ and $T = \pi^{-1}(l)$. Hence,

$$\frac{1}{\delta^{n-1}} \int_T |\phi| = q^{-k} \sum_{v \in l} \frac{1}{|B_v|} \int_{B_v} |\phi|$$

for any function $\phi$, where $B_v = \pi^{-1}\{v}\}$. Therefore, the values of $\mathcal{K}_\delta \phi$ only depend on the averages $|B_v|^{-1} \int_{B_v} |\phi|$. In view of these considerations, the problem can be discretized as follows: Given any function $\phi : \mathbb{R}^n \to \mathbb{R}$, let $\phi^* : S^{n-1}(R) \to \mathbb{R}$ be the function

$$\phi^*(w) = \sup_{\ell \parallel w} \frac{1}{q^k} \sum_{v \in \ell} |\phi(v)|,$$

where the supremum is taken over all lines $\ell \subseteq \mathbb{R}^n$ that point in the direction of $w$. Then the bound (5) is equivalent to the following:

**Theorem 5** (Kakeya distributional estimate). Let $k > 0$ be an integer, let $R = A/t^k A$ and let $\phi : \mathbb{R}^n \to \mathbb{R}$ be any function. Then,

$$|\{w \in S^{n-1}(R) : \phi^*(w) \geq \lambda\}| \lesssim_{n,q} k^{n+1} \lambda^{-n} \|\phi\|_{L^n(A^n)}.$$  \hspace{1cm} (6)

for all $\lambda > 0$.

At the end of this section, we will use Theorem 3 to prove Theorem 5 followed by the proof of Theorem 4. One can deduce Theorem 1 from the maximal conjecture in an analogous way to the real case. However, it is worth mentioning that Theorem 1 can also be deduced directly
from Theorem 3 using the following well-known argument based on $q$-adic decomposition.\footnote{This argument can be found in \cite{26}, for instance.} Let $A \subseteq K^n$ be a set of representatives for the (countable) quotient group $(K/A)^n$. We may write any $c$-Kakeya set $E$ as a disjoint union

$$E = \bigcup_{a \in A} E(a),$$

where $E(a) = E \cap (a + A^n)$. Afterwards we can form the set $E' = \bigcup_{a \in A} (E(a) - a)$. Clearly $\dim_H(E') \leq \dim_H(E)$ and it is easy to see that $E'$ is also a $c$-Kakeya set. Hence we may assume that $E \subseteq A^n$. To prove that $\dim_H(E) = n$, we must show that $H^s(E) > 0$ for all $s < n$.

Fix $s < n$. Let $\{U_j\}_{j=1}^\infty$ be a covering of $E$ by bounded sets of diameter $\leq 1$. We cover each $U_j$ by a ball $D_j = B(y_j, r_j)$ of radius $r_j = \text{diam } U_j$. Note that each $r_j$ is an integral power of $q$ or zero. For $k \in \mathbb{N}$, let

$$\Sigma_k = \{ j \in \mathbb{N} : r_j = q^{1-k} \},$$

let $\nu_k = |\Sigma_k|$ and $E_k = E \cap (\bigcup_{j \in \Sigma_k} D_j)$. For each $w \in A^n$, there exists $a_w \in E$ and a set $J_w \subseteq A$ with $|J_w| \geq c$ such that $a_w + wJ_w \subseteq E$. Let $\phi_w : J_w \to E$ be the map $\phi_w(x) = a_w + wx$. By the pigeonhole principle, we may find an integer $k = k_w \geq 1$ such that $|\phi_w^{-1}(E_k)| \geq c1/k^2$, where $c_1 = 6/\pi^2$. If we denote by $\Omega_m$ the set of points $w \in A^n$ such that $k_w = m$, applying the pigeonhole principle again we can find $k \in \mathbb{N}$ such that $|\Omega_k| \geq c_1/k^2$. This implies that $E_k$ is an $(\epsilon, \delta)$-Kakeya set, where $\epsilon = \delta = c_2k^2$. By Theorem 3 we have

$$\nu_k \geq \left(\frac{\delta \epsilon}{kn}q^{k-2} + n\right) \geq \frac{\delta \epsilon^n q^{(k-2)n}}{(nk)^n n!} \geq_{k,n} q^{kn} k^{5n}.$$

Then,

$$\sum_{j=1}^\infty r_j^n \geq \nu_k q^{(1-k)s} \geq_{n,q} q^{k(n-s)k^{-5n}} \geq_{n,q} 1.$$

The implicit constant does not depend on $k$, so this shows $H^s(E) > 0$ for all $s < n$, and therefore $\dim_H(E) = n$.

We now turn to the proof of Theorem 5. We first prove it for characteristic functions. Note the slightly improved dependence on $k$.

**Proposition 6.** For every subset $E \subseteq \mathbb{R}^n$ the following inequality holds:

$$|\{ w \in S^{n-1}(\mathbb{R}) : 1^*_E(w) \geq \lambda \} | \leq k^n \lambda^{-n} |E|,$$

where $1^*_E$ denotes the characteristic function of $E$.

**Proof.** Fix $\lambda > 0$ and let $\Omega = \{ w \in S^{n-1}(\mathbb{R}) : \phi^*(w) \geq \lambda \}$. First assume that $|\Omega| \geq \frac{1}{2} q^{kn}$. For each $w \in \Omega$, there is a line $\ell_w$ in the direction of $w$ such that $|E \cap \ell_w| = \phi^*(w) q^k \geq \lambda q^k$. Hence $E$ is a
\((\lambda, \alpha)\)-Kakeya set modulo \(t^k\), where \(\alpha = q^{-kn}|\Omega| \geq \frac{1}{2}\). Applying Theorem 3 we get that

\[
|E| \geq \left(\left\lfloor \frac{\lambda}{4kn} q^{k-1} \right\rfloor + n \right) \geq n \frac{\lambda^n}{kn} q^{(k-1)n} \geq q_n \frac{\lambda^n}{kn} |\Omega|.
\]

For a general set, we use a random rotation trick. Let \(J = |\Omega|\) and let \(m > 0\) be the least integer such that \(mJ \geq \frac{1}{2} q^{kn}\). Pick \(m\) random invertible matrices \(R_1, ..., R_m \in \text{GL}_n(\mathbb{R})\) that are independent and set \(\Omega' = \bigcup_{i=1}^m R_i(\Omega), E' = \bigcup_{i=1}^m R_i(E)\). By independence, each \(w \in S^{n-1}(R)\) is contained in \(\Omega'\) with probability

\[
1 - \left(1 - \frac{1}{|S^{n-1}(R)|} \right)^m \geq 1 - \left(1 - \frac{2}{m} \right)^m \geq 1.
\]

By linearity of expectation, it follows that \(\mathbb{E}|\Omega'| \geq |S^{n-1}(R)|\), so we can choose specific values for \(R_1, ..., R_m\) so that \(|\Omega'| \geq |S^{n-1}(R)|\). Fixing those matrices, we also have that \(E'\) contains a proportion of at least \(\frac{1}{2}\) of the line \(\ell_w\) for every \(w \in \Omega'\). By the previous part, we get

\[
m|E| \geq |E'| \geq \frac{\lambda^n}{kn} |\Omega'| \geq \frac{\lambda^n m}{kn} |\Omega|,
\]

which becomes (7) after rearranging the terms.

**Proof of Theorem 5.** We may assume that \(\phi\) is non-negative. Note that \(\phi^*(w) \geq q^{-k(2-1/n)}\|\phi\|_{\ell^n}\) for all \(w \in S^{n-1}(R)\). Indeed, given such \(w\) we can partition \(R^n\) into \(M = q^{k(n-1)}\) parallel lines \(L_1, ..., L_M\) in the direction of \(w\). Then,

\[
\phi^*(w) = q^{-k} \max_{1 \leq j \leq M} \|\phi\|_{\ell^n(L_j)} \geq q^{-k} \left(\frac{1}{M} \sum_{j=1}^M \|\phi\|_{\ell^n(L_j)}\right)^{1/n}
\]

\[
\geq q^{-k} \left(\frac{1}{M} \sum_{j=1}^M \|\phi\|_{\ell^n(L_j)}\right)^{1/n} \geq q^{-k(2-1/n)}\|\phi\|_{\ell^n(R^n)}.
\]

Thus, we may assume that \(\lambda \geq q^{-k(2-1/n)}\|\phi\|_{\ell^n}\) from now on. Consider the set

\[
D = \{v \in R^n : \phi(v) \geq 2q^{-2k}\|\phi\|_{\ell^n}\},
\]

and let \(\phi_D(v) = \phi(v)1_D(v)\). Observe that

\[
\sum_{v \in R^n \setminus D} \phi(v)n < 2^n q^{-2kn} \sum_{v \in R^n} \|\phi\|_{\ell^n}^n = 2^n q^{-kn} \|\phi\|_{\ell^n}^n,
\]

so \(\|\phi - \phi_D\|_{\ell^n} < 2q^{-k}\|\phi\|_{\ell^n}\) and therefore \(\|\phi_D\|_{\ell^n} > \frac{9}{10}\|\phi\|_{\ell^n}\). Similarly, it is easy to show that

\[
\phi^*(w) \leq \phi^*_D(w) + 2q^{-2k}\|\phi\|_{\ell^n},
\]

\[
\|\phi - \phi_D\|_{\ell^n} < 2q^{-k}\|\phi\|_{\ell^n}.
\]

\[
\phi^*(w) \leq \phi_D^*(w) + 2q^{-2k}\|\phi\|_{\ell^n},
\]

\[
\|\phi - \phi_D\|_{\ell^n} < 2q^{-k}\|\phi\|_{\ell^n}.
\]
for every $w \in S^{n-1}(R)$, which implies that
\[
\{w : \phi^*(w) \geq \lambda\} \subseteq \{w : \phi_D^*(w) \geq \frac{9}{10} \lambda\}
\]
since $\lambda \geq q^{-k(2-1/n)}\|\phi\|_{\ell^n}$. Replacing $\phi$ with $\phi_D$ we may assume that the range of $\phi$ is contained in $\{0\} \cup (q^{-2k}\|\phi\|_{\ell^n}, \|\phi\|_{\ell^n}]$. For $0 \leq j < 2k$ define
\[
E_j = \{v \in \mathbb{R}^n : q^{-j-1}\|\phi\|_{\ell^n} < \phi(v) \leq q^{-j}\|\phi\|_{\ell^n}\},
\]
and let $\psi_j = q^{-j}\|\phi\|_{\ell^n}1_{E_j}$. Then the function $\psi = \sum_{j=0}^{2k-1} \psi_j$ satisfies $\frac{1}{q}\psi(v) < \phi(v) \leq \psi(v)$ for all $v$. In particular, $\frac{1}{q}\|\psi\|_{\ell^n} < \|\phi\|_{\ell^n} \leq \|\psi\|_{\ell^n}$ and $\frac{1}{q}\psi^*(w) < \phi^*(w) \leq \psi^*(w)$ for all $w \in S^{n-1}(R)$. For each $j$ let $c_j = \frac{1}{u}\|\psi_j\|_{\ell^n}^{n/(n+1)}$, where
\[
u = \sum_{j=0}^{2k-1} \|\psi_j\|_{\ell^n}^{n/(n+1)}.
\]
Since $\sum_{j=0}^{2k-1} c_j = 1$, we have
\[
\{w : \psi^*(w) \geq \lambda\} \subseteq \bigcup_{j=0}^{2k-1} \{w : \psi_j^*(w) \geq c_j\lambda\},
\]
so applying Proposition 6 to each $\psi_j$ we get
\[
\{|w : \psi^*(w) \geq \lambda| \leq \frac{k^n}{\lambda^n} \sum_{j=0}^{2k-1} \frac{\|\psi_j\|_{\ell^n}^n}{c_j} = \frac{k^n}{\lambda^n} \sum_{j=0}^{2k-1} \|\psi_j\|_{\ell^n}^{n/(n+1)}
\]
\[
= \frac{k^n}{\lambda^n} \left( \sum_{j=0}^{2k-1} \|\psi_j\|_{\ell^n}^{n/(n+1)} \right)^{n+1} \leq \frac{k^n}{\lambda^n} (2k)^{n/(n+1)} \sum_{j=0}^{2k-1} \|\psi_j\|_{\ell^n}^n \leq \frac{k^n}{\lambda^n} \|\psi\|_{\ell^n}^n,
\]
where the penultimate step uses Hölder’s inequality. Finally, the estimate (6) follows from the last inequality since $\{|w : \phi^*(w) \geq \lambda| \leq \{|w : \psi^*(w) \geq \lambda| \}$ and $\|\psi\|_{\ell^n} \leq q\|\phi\|_{\ell^n}$. \qed

**Proof of Theorem 4.** Use the identity
\[
\|K_{\phi}(w)\|_{L^n(S^{n-1}(A))} = n \int_0^\infty \{|w : K_{\phi}(w) \geq \lambda| Representative\lambda^n-1 d\lambda.
\]
Let $C(n, q) > 0$ be the implicit constant in (5). In the proof of Theorem 5, it was shown that $q^{-k(2-1/n)}\|\psi\|_{\ell^n} \leq \psi^*(v) \leq \|\psi\|_{\ell^n}$ for every function $\psi : \mathbb{R}^n \to \mathbb{R}$. Since $K_{\phi}$ can be expressed in terms of a discrete maximal function associated to $\phi$, it follows that the same holds for $K_{\phi}$. Then
we have
\[ \| \mathcal{K}_\delta \phi(w) \|_{L^n(S^{n-1}(A))} = n \int_0^{\| \delta \|} \{ w : \mathcal{K}_\delta \phi(w) \geq \lambda \} \lambda^{n-1} d\lambda \]
\[ \leq n \int q^{-2k} \| \delta \| \lambda^{n-1} d\lambda + n \int \| \delta \| C(n,q)k^{n+1} \| \phi \|^{n} \frac{d\lambda}{\lambda} \]
\[ \leq \| \phi \|^{n} + nC(n,q)k^{n+1} \| \phi \|^{n} \ln(q^{2k}) \]
\[ = (1 + 2n \ln(q)C(n,q)k^{n+2}) \| \phi \|^{n}. \]

\[\square\]

4 | A VARIATION OF THE SCHWARTZ–ZIPPEL LEMMA

Theorem 3 remains to be proved. For this, we shall need Lemma 7, which is an adaptation of Lemma 6 from [1] to our case of interest. In order to use the polynomial method we need to introduce an additional variable $X$. This will be the main variable, while all polynomials and power series in $t$ should be treated as scalars, since they belong to the base ring $A$. Let $B = \mathbb{F}_q[X]$ and $A_k = \mathbb{F}_q \oplus \cdots \oplus \mathbb{F}_q t^{k-1}$. We associate to each $a = a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} \in A_k$ a polynomial $s_a = \sum_{j=0}^{k-1} a_j X^{q^j} \in B$. Let $C = \{ s_a : a \in A_k \}$.

**Lemma 7** (Discrete valuation Schwartz–Zippel lemma). Let $f \in B[z_1, \ldots, z_n]$ a non-zero polynomial whose leading term with respect to the lexicographical ordering is $c_\alpha z^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j < q^k$ for all $j$. Then for all $0 < \theta \leq 1$, the number of elements $y \in C^n$ such that $v_X(f(y)) \geq v_X(c_\alpha) + \theta n q^k$ is less than
\[ \max\{ q^{nk}, |\alpha| k q^{k(n-1)+1} / \theta \}, \]
where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

**Proof.** We proceed by induction on $n$. Starting with the base case $n = 1$, assume for the sake of contradiction that there is a set $L \subseteq A_k$ of size $\geq qk / \theta$ such that $v_X(f(y)) \geq v_X(c_\alpha) + \theta q^k$ for all $y \in C_L$. Let $d = \lfloor \log_q(k / \theta) \rfloor \geq 1$. Then $L$ must intersect at least $q^{-d} |L| > \alpha$ congruence classes modulo $t^{k-d}$, so we may find $L' \subseteq L$ of size $\alpha + 1$ that does not contain a pair of elements that are congruent modulo $t^{k-d}$. By Lagrange Interpolation† we have
\[ \sum_{u \in L'} f(s_u) \prod_{w \in L' \setminus \{u\}} \frac{z_1 - s_w}{s_u - s_w} = f(z_1), \]
so looking at the degree $\alpha$ coefficients we obtain
\[ \sum_{u \in L'} f(s_u) \left( \prod_{w \in L' \setminus \{u\}} (s_u - s_w) \right)^{-1} = c_\alpha. \]

†This identity holds for polynomials over any field, even in positive characteristic, cf. [11, §1.6].
Then there must exist $u_0 \in L'$ such that
\[ v_X \left( \prod_{w \in L' \setminus \{u_0\}} (s_{u_0} - s_w) \right) \geq \theta q^k. \]

However, note that $v_X(s_w - s_u) = v_X(s_{w-u}) = q^{u_0(w-u)}$, so
\[
 v_X \left( \prod_{w \in L' \setminus \{u_0\}} (s_{u_0} - s_w) \right) = \sum_{w \in L' \setminus \{u_0\}} q^{u_0(w-u_0)}.
\]

For $0 \leq j \leq k - d - 1$, define
\[
 n_j = |\{w \in L' \setminus \{u_0\} : v_i(w - u_0) = j\}|,
\]
\[
 N_j = |\{w \in L' \setminus \{u_0\} : v_i(w - u_0) \geq j\}|.
\]

By Abel’s summation formula we have
\[
 \sum_{w \in L' \setminus \{u_0\}} q^{u_0(w-u_0)} = \sum_{j=0}^{k-d-1} n_j q^j = N_0 + \sum_{j=1}^{k-d-1} N_j (q^j - q^{j-1}). \tag{8}
\]

Now note that $N_j \leq q^{k-d-j}$ since otherwise the pigeonhole principle would give us two elements of $L'$ which are congruent modulo $t^{k-d}$. And clearly $N_j \leq \alpha \leq q^r$, where $r = \lceil \log q(\alpha) \rceil$. For $j = 0$ we have $N_0 = \alpha \leq q^{k-d}$, so $r \leq k - d$. Thus, we may bound (8) by
\[
 \sum_{w \in L' \setminus \{u_0\}} q^{u_0(w-u_0)} \leq q^r + \sum_{j=1}^{k-d-r} q^r(q^j - q^{j-1}) + \sum_{j=k-d-r+1}^{k-d} q^{k-d-j}(q^j - q^{j-1})
\]
\[
 = q^{k-d} + \sum_{j=k-d-r+1}^{k-d} q^{k-d-j}(q^j - q^{j-1})
\]
\[
 < q^{k-d} + rq^{k-d} = (r + 1)q^{k-d} \leq kq^{k-d} \leq \theta q^k.
\]

We have reached a contradiction, therefore $|L| < \alpha qk/\theta$. Now assume that $n \geq 2$, and write $f$ as
\[ f = z_1^{\alpha_1} g(z_2, \ldots, z_n) + h(z_1, \ldots, z_n), \]
where $\deg_{z_1}(h) < \alpha_1$. Fix a tuple $y' = (y_2, \ldots, y_n) \in C^{n-1}$ and consider two cases. If
\[ v_X(g(y')) \geq v_X(c_{\alpha}) + \theta(n - 1)q^k, \]
there are $\leq (\alpha_2 + \cdots + \alpha_n)kq^{k(n-2)+1}/\theta$ number of $(n-1)$-tuples for which this can hold, which gives us at most $\leq (\alpha_2 + \cdots + \alpha_n)kq^{k(n-1)+1}/\theta$ tuples $(y_1, \ldots, y_n) \in C^n$. Suppose instead that
\[ v_X(g(y')) < v_X(c_{\alpha}) + \theta(n - 1)q^k. \]
Then \( P(z) = f(z, y') \) is a degree \( \alpha_1 \) polynomial with leading coefficient \( g_2(y') \). The condition
\[
\nu_X(f(z, y')) \geq \nu_X(c_2) + \vartheta(n - 1)q^k
\]
(9)
has less solutions than \( \nu_X(P(z)) \geq \nu_X(g(y')) + \vartheta q^k \). And this last inequality has at most \( \alpha_1 kq/\vartheta \) solutions in \( z \) for each \( y' \). Thus, the number of tuples \((z, y')\) satisfying (9) is \( \leq \alpha_1 kq^{(n-1)+1}/\vartheta \) in the second case. Adding the two bounds gives the result.

\[\square\]

5 LUBIN–TATE THEORY

This section is dedicated to establish some results from Lubin–Tate theory needed to prove Theorem 3, followed by the proof of the said theorem. Lemmas 8 and 10 are special cases of more general theorems in this area. The interested reader may consult [19, chapter 1].

From now on let \( f(X) = tX + X^q \). This is an additive polynomial, as it satisfies the equation \( f(X + Y) = f(X) + f(Y) \). The polynomial method will be applied on the set of roots of an iterate of \( f \). But first we shall construct the power series \([a]_f\) which will determine the action of \( R = A/t^kA \) on the set of roots. Given two power series \( a(X), b(X) \) with constant term 0, one can define its composition as follows: If \( a = \sum_{j=1}^{\infty} c_j X^j \), then
\[
a \circ b(X) = \sum_{j=1}^{\infty} c_j b^j(X).
\]

Although this is an infinite sum, the computation of each coefficient in the composition only requires a finite number of operations since for any integer \( m \geq 0 \), the coefficient of \( X^m \) in \( b^j(X) \) becomes 0 once \( j > m \). We shall denote by \( b^{\circ m} \) the \( m \)-fold composition of \( b \) with itself.

Lemma 8.

(a) For every \( a \in A \), there is a unique power series \([a]_f \in A[[X]]\) such that \([a]_f(X) \equiv aX \pmod{X^2} \) and \([a]_f \circ f = f \circ [a]_f \). Moreover, \([a]_f \) is additive.
(b) \([a + b]_f = [a]_f + [b]_f \) and \([ab]_f = [a]_f \circ [b]_f \) for all \( a, b \in A \). Moreover, \([c]_f = cX \) for all \( c \in \mathbb{F}_q \) and \([t^m]_f = f^{\circ m} \) for every integer \( m > 0 \).
(c) If \( a = c_0 + c_1 t + \cdots + c_{k-1} t^{k-1} \in A_k \) with \( c_j \in \mathbb{F}_q \), then
\[
[a]_f = \sum_{0 \leq j < k} c_j f^{\circ j}(X)
\]
and \([a]_f \equiv s_a(X) \pmod{t} \).

Proof.

(a) We define a sequence \((a_m)_{m \geq 0}\) recursively. Its first term is \( a_0 = a \), and for \( m > 0 \) we set
\[
a_m = \frac{a_{m-1}^q - a_{m-1}}{t^{a_{m-1}} - t}.
\]
Note that all the elements of the sequence belong to \( A \). Indeed, since \( t^{q^m-1} - 1 \) is a unit in \( A \), it suffices to show that \( a^q_{m-1} - a_{m-1} \) is a multiple of \( t \). If \( c_0 \) is the degree-0 term of \( a_{m-1} \), then the degree-0 term of \( a^q_{m-1} - a_{m-1} \) is \( c_0^q - c_0 = 0 \) since \( c_0 \in \mathbb{F}_q \), which proves the claim.

We now claim that \( [a]_f = \sum_{m=0}^\infty a_m X^{q^m} \) has the desired properties. The first congruence in (a) holds since \( a_0 = a \). With regard to the second one, we have

\[
[a]_f \circ f = \sum_{m=0}^\infty a_m (tX + X^q)^{q^m} = \sum_{m=0}^\infty a_m \left( t^{q^m} X^{q^m} + X^{q^{m+1}} \right)
= a_0 tX + \sum_{m=1}^\infty \left( a_m t^{q^m} + a_{m-1} \right) X^{q^m},
\]

whereas

\[
f \circ [a]_f = \sum_{m=0}^\infty t a_m X^{q^m} + \sum_{m=0}^\infty a^q_m X^{q^{m+1}} = a_0 tX + \sum_{m=1}^\infty \left( t a_m + a^q_{m-1} \right) X^{q^m}.
\]

These two power series are equal as a consequence of the recursive equation (11) defining \( (a_n)_n \). Finally, \( [a]_f \) is additive since \( X^{q^m} \) is an additive polynomial for all \( m \geq 0 \).

(b) The power series \( \phi(X) = [a]_f + [b]_f \) satisfies \( \phi(X) \equiv aX + bX \pmod{X^2} \) and

\[
\phi \circ f = [a]_f \circ f + [b]_f \circ f = f \circ [a]_f + f \circ [b]_f = f \circ \phi,
\]

since \( f \) is an additive polynomial. Hence, the uniqueness of \( [a + b]_f \) implies that \( [a + b]_f = [a]_f + [b]_f \). The proof of the equation \( [ab]_f = [a]_f \circ [b]_f \) is similar. If \( c \in \mathbb{F}_q \) then \( f(cX) = tcX + c^qX^q = tcX + cX^q = cf(X) \), so \( [c]_f = cX \). Finally, note that for every integer \( m > 0 \), the polynomial \( f^{\circ m}(X) \) commutes with \( f \) and its linear coefficient is \( t^m \). It follows that \( [t^m]_f = f^{\circ m} \).

(c) Equation (10) is a direct consequence of (b). The last congruence follows then from the fact that \( f^{\circ i}(X) \equiv X^{q^i} \pmod{t} \) for every integer \( i \geq 0 \). \[\square\]

For general \( a \in A \), define \( P_a(X) = [\pi_k(a)]_f \), where \( \pi_k(a) \in A_k \) is the remainder of \( a \) modulo \( t^k \). Let \( K^a \) be the algebraic closure of \( K \). Given an integer \( k > 0 \), let \( \Lambda_k \subseteq K^a \) be the set of roots of \( f^{\circ k} \) and let \( L = K(\Lambda_k) \) be the field generated by these roots. Note that \( \Lambda_k \) is an additive subgroup of \( L \), since \( f^{\circ k} \) is an additive polynomial. Recall that it is possible to extend the absolute value of \( K \) to \( L \) in a unique way. Given a power series \( h = a_0 + a_1X + \cdots \in A[[X]] \) and an element \( x_0 \in L \) with absolute value \( |x_0| < 1 \), it is possible to evaluate \( h \) at \( x_0 \) since the infinite sum \( h(x_0) = \sum_{j=0}^\infty a_j x_0^j \) converges.

**Proposition 9.** Let \( h(X) = a_0 + a_1X + \cdots + a_mX^m \in A[X] \) be a polynomial such that \( |a_0| = q^{-1} \) and \( |a_j| < 1 \) for all \( 1 < j < m \), but \( |a_m| = 1 \). Then any root \( \zeta \in K^a \) of \( h \) has absolute value \( |\zeta| = q^{-1/m} \).

**Proof.** See [20, Proposition 7.55]. \[\square\]
For every \( k > 0 \), the polynomial \( f^{\circ k}(X) \) satisfies the conditions of Proposition 9, so it is possible to evaluate power series at points in \( \Lambda_k \). The map \( \Lambda \times \Lambda_k \to \Lambda_k, \ a \cdot \xi = [a]_f(\xi) \) gives \( \Lambda_k \) an \( A \)-module structure. This follows from Lemma 8.

**Lemma 10.** Let \( K^a \) be the algebraic closure of \( K = \mathbb{F}_q((t)) \), let \( \Lambda_k \subseteq K^a \) be the set of roots of \( f^{\circ k}(X) \) and let \( L = K(\Lambda_k) \). Then

(a) \( \Lambda_k \) is isomorphic to \( A/t^kA \) as an \( A \)-module,

(b) \( L/K \) is a totally ramified extension. That is, if \( \mathcal{O}_L \subseteq L \) is the valuation domain of \( L \) and \( \mathfrak{m}_L \) is its maximal ideal, then \( \mathcal{O}_L/\mathfrak{m}_L \cong \mathbb{F}_q \).

**Proof.** See [21, Proposition 3.4, Theorem 3.6]. \( \square \)

**Proof of Theorem 3.** Let \( E \subseteq \mathbb{F}_q[[t]]^n \) be an \((\varepsilon, \nu)\)-Kakeya set. Note that every closed ball of radius \( q^{-k} \) in \( A^n \) is a coset of the form \( a + t^k A^n, a \in A^n \). Hence the least number of closed balls of radius \( q^{-k} \) needed to cover \( E \) is equal to the cardinality of the projection \( \tilde{E} \) of \( E \) onto \((A/t^kA)^n \). The Kakeya property of \( E \) implies that there is a subset \( \Omega \subseteq (A/t^kA)^n \) of size at least \( \nu q^{kn} \), such that for all \( w \in \Omega \) there exist \( b_w = (b_{w,1}, ..., b_{w,n}) \in (A/t^kA)^n \) and a set \( J_w \subseteq A/t^kA \) of size at least \( \varepsilon q^k \) such that \( b_w + w J_w \subseteq \tilde{E} \). Suppose for the sake of contradiction that \( |\tilde{E}| < (\beta + n)^n \), where \( \beta = \lfloor \varepsilon \nu q^{kn} \rfloor \).

Let \( \zeta_1 \in \Lambda_k \) be a generator of \( \Lambda_k \) as an \((A/t^kA)\)-module, and consider the set

\[ S = \{([s_1]_f(\zeta_1), ..., [s_n]_f(\zeta_1)) : (s_1, ..., s_n) \in E \}. \]

By Lemma 2 there is a non-zero polynomial \( g(z_1, ..., z_n) \in L[z_1, ..., z_n] \) with degree at most \( \beta \) that vanishes on \( S \). We may assume that \( g \in \mathcal{O}_L[z_1, ..., z_n] \) and that its reduction modulo \( \mathfrak{m}_L \) is non-zero. Given \( w \in \Omega \), define \( c_{w,j} = [b_{w,j}]_f(\zeta_1) \) and

\[ h_w(X) = g\left(c_{w,1} + P_{w,1}(X), ..., c_{w,n} + P_{w,n}(X)\right). \]

For all \( a \in J_w \) we have

\[ 0 = g([b_{w,1} + w_{1}a]_f(\zeta_1), ..., [b_{w,n} + w_{n}a]_f(\zeta_1)) \]

\[ = g(c_{w,1} + [w_1]_f(\zeta_a), ..., c_{w,n} + [w_n]_f(\zeta_a)) = h_w(\zeta_a), \]

where \( \zeta_a = [a]_f(\zeta_1) \). Thus, \( \prod_{a \in J_w} (X - \zeta_a) \) \( |h_w(X)| \), which implies that the reduction \( \overline{h}_w(X) \in \mathbb{F}_q[X] \) modulo \( \mathfrak{m}_L \) satisfies \( \nu_X(\overline{h}_w) \geq |J_w| \geq \varepsilon q^k \). Since \( \zeta_1 \in \mathfrak{m}_L \) and \( [b_{w,j}]_f \) is a polynomial with constant term equal to zero, it follows that also \( c_{w,j} \in \mathfrak{m}_L \) for all \( j \). Therefore,

\[ \overline{h}_w(X) = \overline{g}\left(\overline{P}_{w,1}(X), ..., \overline{P}_{w,n}(X)\right), \]

and each \( \overline{P}_{w,j} \in C \). Applying Lemma 7 to \( \overline{g} \) with \( \theta = \varepsilon/n \), we get

\[ \nu q^{kn} \leq |\Omega| < \beta q^{k(n-1)+1} \frac{n}{\varepsilon} \leq \nu q^{kn}, \]

which is absurd. This concludes the proof. \( \square \)
Remark. It is possible to use Lubin–Tate polynomials to prove the Kakeya conjecture for $p$-adic fields, although some adjustments have to be made. In the first place, one can no longer take $f$ to be an additive polynomial, since the only additive polynomials in characteristic zero are the linear polynomials $f(X) = aX$ with constant term equal to zero. However, it can be shown that if $f \in \mathbb{Q}_p[X]$ satisfies $f(X) \equiv pX \pmod{X^2}$ and $f(X) \equiv X^p \pmod{p}$, then there is a unique formal group law $F(X, Y) \in \mathbb{Q}_p[[X, Y]]$ such that $f(F(X, Y)) = F(f(X), f(Y))$. Instead of the additive property, the map $[\cdot]_f$ satisfies $[a + b]_f = F([a]_f, [b]_f)$ for all $a, b \in \mathbb{Z}_p$. Thus, one should choose $f$ so that the corresponding power series $F(X, Y)$ and $[a]_f$ take a relatively simple form. A sensible choice is given by $f(X) = (1 + X)^{p-1}$, whose associated formal group law is $F(X, Y) = X + Y + XY = (1 + X)(1 + Y) - 1$. The roots of $f^{ak} = (1 + X)^{pk} - 1$ are of the form $1 + \zeta^j$, where $0 \leq j < p^k$ and $\zeta$ is a primitive $p^k$-th root of unity. Therefore, using this choice of $f$ one recovers Arsovski’s proof.

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