CONTROLLABILITY OF FRACTIONAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH INFINITE DELAY

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Abstract. In this paper we study the controllability of fractional neutral stochastic functional differential equations with infinite delay driven by fractional Brownian motion in a real separable Hilbert space. The controllability results are obtained by using stochastic analysis and a fixed-point strategy. Finally, an illustrative example is provided to demonstrate the effectiveness of the theoretical result.

Keywords: Controllability, fractional neutral functional differential equations, fractional powers of closed operators, infinite delay, fractional Brownian motion.

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1. INTRODUCTION

Fractional Brownian motion (fBm) \( \{B^H(t) : t \in \mathbb{R}\} \) is a Gaussian stochastic process, which depends on a parameter \( H \in (0, 1) \) called Hurst index, for additional details on the fractional Brownian motion, we refer the reader to [20]. This stochastic process has self-similarity, stationary increments, and long-range dependence properties. It is known that fractional Brownian motion is a generalization of Brownian motion and it reduces to a standard Brownian motion when \( H = \frac{1}{2} \). Fractional Brownian motion is not a semimartingale if \( H \neq \frac{1}{2} \) (see Biagini al. [3]), the classical Itô theory cannot be used to construct a stochastic calculus with respect to fBm.

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of physics, finance, electrical engineering, telecommunication networks, and so on. There has been a significant development in fractional differential equations. Some authors have considered fractional stochastic equations, we refer to Ahmed [1], El-Bori [10], Cui and Yan [8], Sakthivel et al. [25] [26]. The perturbed terms of these fractional equations are Wiener processes. For more details, one can see the monographs of Kilbas et al. [11], Zhou [28], and Zhou et al. [29] and the references therein.

In many areas of science, there has been an increasing interest in the investigation of the systems incorporating memory or aftereffect, i.e., there is the effect of delay on state equations. Therefore, there is a real need to discuss stochastic evolution systems with delay. In many mathematical models the claims often display long-range memories, possibly due to extreme weather, natural disasters, in some cases, many stochastic dynamical systems

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depend not only on present and past states, but also contain the derivatives with delays. Neutral functional differential equations are often used to describe such systems.

Moreover, control theory is an area of application-oriented mathematics which deals with basic principles underlying the analysis and design of control systems. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability plays a crucial role in a lot of control problems, such as the case of stabilization of unstable systems by feedback or optimal control [12, 13]. The controllability concept has been studied extensively in the fields of finite-dimensional systems, infinite-dimensional systems, hybrid systems, and behavioral systems. If a system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability. For more details the reader may refer to [13, 23, 24] and the references therein. In this paper, we study the controllability of fractional neutral functional stochastic differential equations of the form

\[
\begin{aligned}
\frac{d}{dt}J_t^{1-\alpha}(x(t) - g(t,x_t) - \varphi(0) + g(0,\varphi)) &= [Ax(t) + f(t,x_t) + Bu(t)]dt \\
&\quad + \sigma(t)dB^H(t), \quad t \in [0,T],
\end{aligned}
\]

\[x(t) = \varphi(t) \in L^2(\Omega, B_h), \text{ for a.e. } t \in (-\infty,0],\]

where \(\frac{1}{2} < \alpha < 1\), \(J_t^{1-\alpha}\) is the \((1-\alpha)\)-order Riemann-Liouville fractional integral operator, \(A\) is the infinitesimal generator of an analytic semigroup of bounded linear operators, \((S(t))_{t \geq 0}\), in a Hilbert space \(X\); \(B^H\) is a fractional Brownian motion with \(H > \frac{1}{2}\) on a real and separable Hilbert space \(Y\); and the control function \(u(\cdot)\) takes values in \(L^2([0,T], U)\), the Hilbert space of admissible control functions for a separable Hilbert space \(U\); and \(B\) is a bounded linear operator from \(U\) into \(X\).

The history \(x_t : (-\infty,0] \to X, x_t(\theta) = x(t+\theta),\) belongs to an abstract phase space \(B_h\) defined axiomatically, and \(f,g : [0,T] \times B_h \to X,\) and \(\sigma : [0,T] \to L^2_2(Y,X),\) are appropriate functions, where \(L^2_2(Y,X)\) denotes the space of all \(Q\)-Hilbert-Schmidt operators from \(Y\) into \(X\) (see section 2 below).

For potential applications in telecommunications networks, finance markets, biology and other fields [7, 14], stochastic differential equations driven by fractional Brownian motion have attracted researcher’s great interest. Especially, we mention here the recent papers [15, 16, 17, 22]. Moreover, Dung studied the existence and uniqueness of impulsive stochastic Volterra integro-differential equation driven by fBm in [9]. Using the Riemann-Stieltjes integral, Boufoussi et al. [4] proved the existence and uniqueness of a mild solution to a related problem and studied the dependence of the solution on the initial condition in infinite dimensional space. More recently, Li [18] investigated the existence of mild solution to a class of stochastic delay fractional evolution equations driven by fBm. Caraballo et al. [6], and Boufoussi and Hajji [5] have discussed the existence, uniqueness and exponential asymptotic behavior of mild solutions by using the Wiener integral.

To the best of the author’s knowledge, an investigation concerning the controllability for fractional neutral stochastic differential equations with infinite delay of the form (1.1) driven by a fractional Brownian motion has not yet been conducted. Thus, we will make the first attempt to study such problem in this paper. Our results are motivated by those in [15, 17] where the controllability of mild solutions to neutral stochastic functional integro-differential equations driven by fractional Brownian motion with finite delays are studied.
The outline of this paper is as follows: In the next section, some necessary notations and concepts are provided. In Section 3, we derive the controllability of fractional neutral stochastic differential systems driven by a fractional Brownian motion. Finally, in Section 4, we conclude with an example to illustrate the applicability of the general theory.

2. Preliminaries

We collect some notions, concepts and lemmas concerning the Wiener integral with respect to an infinite dimensional fractional Brownian, and we recall some basic results which will be used throughout the whole of this paper.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space. A standard fractional Brownian motion \((fBm)\) with Hurst parameter \(H \in (0, 1)\) is a zero mean Gaussian process with continuous sample paths such that

\[
R_H(t, s) = \mathbb{E}[\beta_H(t)\beta_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad s, t \in \mathbb{R}. \tag{2.1}
\]

Let \(X\) and \(Y\) be two real, separable Hilbert spaces and let \(\mathcal{L}(Y, X)\) be the space of bounded linear operator from \(Y\) to \(X\). For the sake of convenience, we shall use the same notation to denote the norms in \(X, Y\) and \(\mathcal{L}(Y, X)\).

Let \(Q \in \mathcal{L}(Y, Y)\) be an operator defined by \(Qe_n = \lambda_n e_n\) with finite trace \(\text{tr}Q = \sum_{n=1}^{\infty} \lambda_n < \infty\), where \(\lambda_n \geq 0 \ (n = 1, 2, \ldots)\) are non-negative real numbers and \(\{e_n\} \ (n = 1, 2, \ldots)\) is a complete orthonormal basis in \(Y\).

We define the infinite dimensional fBm on \(Y\) with covariance \(Q\) as

\[
B^H(t) = B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t),
\]

where \(\beta^H_n\) are real, independent fBm’s. This process is Gaussian, it starts from 0, has zero mean and covariance:

\[
E(B^H(t), x)(B^H(s), y) = R(s, t)(Q(x), y) \quad \text{for all} \ x, y \in Y \ \text{and} \ t, s \in [0, T].
\]

In order to define Wiener integrals with respect to the \(Q\)-fBm, we introduce the space \(\mathcal{L}_2^Q := \mathcal{L}_2(Y, X)\) of all \(Q\)-Hilbert-Schmidt operators \(\psi : Y \rightarrow X\). We recall that \(\psi \in \mathcal{L}(Y, X)\) is called a \(Q\)-Hilbert-Schmidt operator, if

\[
\|\psi\|_{\mathcal{L}^2_2}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} e_n\|^2 < \infty,
\]

and that the space \(\mathcal{L}_2^Q\) equipped with the inner product \((\varphi, \psi)_{\mathcal{L}_2^2} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle\) is a separable Hilbert space.

Let \(\phi(s) : s \in [0, T]\) be a function with values in \(\mathcal{L}_2^Q(Y, X)\), such that \(\sum_{n=1}^{\infty} \|K^* \phi Q^* e_n\|_{\mathcal{L}_2^2}^2 < \infty\). The Wiener integral of \(\phi\) with respect to \(B^H\) is defined by

\[
\int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n dB^H_n(s). \tag{2.2}
\]

Now, we end this subsection by stating the following result which is fundamental to prove our result. It can be proved by similar arguments as those used to prove Lemma 2 in [6].
Lemma 2.1. If \( \psi : [0, T] \to L^2_0(Y, X) \) satisfies \( \int_0^T \| \psi(s) \|_{L^2_0}^2 \, ds < \infty \), then the above sum in \((2.2)\) is well defined as a \( X \)-valued random variable and we have
\[
E\left\| \int_0^t \psi(s) dB^H(s) \right\|_2^2 \leq 2Ht^{2H-1} \int_0^t \| \psi(s) \|_{L^2_0}^2 \, ds.
\]

It is known that the study of theory of differential equation with infinite delays depends on a choice of the abstract phase space. We assume that the phase space \( B_h \) is a linear space of functions mapping \((-\infty, 0]\) into \( X \), endowed with a norm \( \| \cdot \|_{B_h} \). We shall introduce some basic definitions, notations and lemma which are used in this paper. First, we present Lemma 2.2.

The following lemma is a common property of phase spaces.

Lemma 2.2. \( \psi \in B_h \) is well defined as a \( X \)-valued random variable and we have
\[
E\left\| \int_0^t \psi(s) dB^H(s) \right\|_2^2 \leq 2Ht^{2H-1} \int_0^t \| \psi(s) \|_{L^2_0}^2 \, ds.
\]

We define the abstract phase space \( B_h \) by
\[
B_h = \{ \psi : (\infty, 0] \to X \text{ for any } \tau > 0, \ (E\| \psi \|^2)^\tau \text{ is bounded and measurable function on } [-\tau, 0] \text{ and } \int_{-\infty}^0 h(s) ds < +\infty \}.
\]

If we equip this space with the norm
\[
\| \psi \|_{B_h} := \int_{-\infty}^0 h(s) \sup_{t \in [-s, 0]} (E\| \psi(s) \|^2)^\frac{1}{2} \, ds,
\]
then it is clear that \((B_h, \| \cdot \|_{B_h})\) is a Banach space.

Next, we consider the space \( B_T \), given by
\[
B_T = \{ x : x \in C((\infty, T], X) \text{ with } x_0 = \varphi \in B_h \},
\]
where \( C((\infty, T], X) \) denotes the space of all continuous \( X \)-valued stochastic processes \( \{ x(t), t \in (\infty, T] \} \). The function \( \| \cdot \|_{B_T} \) to be a semi-norm in \( B_T \), it is defined by
\[
\| x \|_{B_T} = \| x_0 \|_{B_h} + \sup_{0 \leq t \leq T} (E\| x(t) \|^2)^\frac{1}{2}.
\]

The following lemma is a common property of phase spaces.

Lemma 2.3. Suppose \( x \in B_T \), then for all \( t \in [0, T] \), \( x_t \in B_h \) and
\[
\left\| \int_0^t \psi(s) dB^H(s) \right\|_2^2 \leq \| x_t \|_{B_h} \leq t \sup_{0 \leq s \leq t} (E\| x(s) \|^2)^\frac{1}{2} + \| x_0 \|_{B_h},
\]
where \( l = \int_{-\infty}^0 h(s) ds < \infty \).

Let us give the following well-known definitions related to fractional order differentiation and integration.

Definition 2.3. The Riemann-Liouville fractional integral of order \( \alpha > 0 \) of a function \( f : \mathbb{R}^+ \to X \) is defined by
\[
J^\alpha_x f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,
\]
where \( \Gamma(.) \) is the Gamma function.

Definition 2.4. The Riemann-Liouville fractional derivative of order \( \alpha \in (0, 1) \) of a function \( f : \mathbb{R}^+ \to X \) is defined by
\[
D^\alpha_x f(t) = \frac{d}{dt} J^{1-\alpha}_t f(t).
\]
The Caputo fractional derivative of order $\alpha \in (0,1)$ of $f : \mathbb{R}^+ \rightarrow X$ is defined by

$$^C D_t^\alpha f(t) = D_t^\alpha (f(t) - f(0)).$$

For more details on fractional calculus, one can see [11].

We suppose $0 \in \rho(A)$, the resolvent set of $A$, and the semigroup, $(S(t))_{t \geq 0}$, is uniformly bounded. That is, there exists $M \geq 1$ such that $\|S(t)\| \leq M$ for every $t \geq 0$.

Then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in $X$, and the expression

$$\|h\|_\alpha = \|(-A)^\alpha h\|$$
defines a norm in $D(-A)^\alpha$. If $X_\alpha$ represents the space $D(-A)^\alpha$ endowed with the norm $\|\cdot\|_\alpha$, then the following properties hold (see [21], p. 74).

**Lemma 2.6.** Suppose that $A$, $X_\alpha$, and $(-A)^\alpha$ are as described above.

(i) For $0 < \alpha \leq 1$, $X_\alpha$ is a Banach space.

(ii) If $0 < \beta \leq \alpha$, then the injection $X_\alpha \rightarrow X_\beta$ is continuous.

(iii) For every $0 < \alpha \leq 1$, there exists $M_\alpha > 0$ such that

$$\|(-A)^\alpha S(t)\| \leq M_\alpha t^{-\alpha}e^{-\lambda t}, \quad t > 0, \quad \lambda > 0.$$

3. CONTROLLABILITY RESULT

Before starting and proving our main result, we introduce the concepts of a mild solution of the problem (1.1) and the meaning of controllability of fractional neutral stochastic functional differential equation.

**Definition 3.1.** An $X$-valued process \{\(x(t) : t \in (-\infty, T]\)\} is a mild solution of (1.1) if

(1) $x(t)$ is continuous on $[0, T]$ almost surely and for each $s \in [0, t)$ and $\alpha \in (0,1)$ the function $(t-s)^{\alpha-1}AS_\alpha(t-s)g(s,x_s)$ is integrable,

(2) for arbitrary $t \in [0, T]$, we have

$$x(t) = T_\alpha(t)(\varphi(0) - g(0, \varphi)) + g(t, x_t)$$

$$+ \int_0^t (t-s)^{\alpha-1} AS_\alpha(t-s)g(s,x_s)ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)f(s,x_s)ds$$

$$+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)Bu(s)ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\sigma(s)dB^H(s), \quad \mathbb{P} - a.s.$$  

(3.1)

(3) $x(t) = \varphi(t)$ on $(-\infty, 0]$ satisfying $\|\varphi\|^2_{B_h} < \infty$

where

$$T_\alpha(t)x = \int_0^\infty \eta_\alpha(\theta)S(t^\alpha \theta)xd\theta, \quad t \geq 0, \quad x \in X.$$  

$$S_\alpha(t)x = \alpha \int_0^\infty \theta \eta_\alpha(\theta)S(t^\alpha \theta)xd\theta, \quad t \geq 0, \quad x \in X,$$

where

$$\eta_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \omega_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\omega_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{\alpha-\frac{1}{\alpha}} \Gamma(n\alpha + 1) \frac{\Gamma(n\alpha + \frac{1}{\alpha})}{n!} \sin(n\alpha \pi), \quad \theta \in ]0, \infty[,$$
\( \eta_\alpha \) is a probability density function defined on \((0, \infty)\).

**Remark 3.2.** (see [27])

\[
\int_0^\infty \theta \eta_\alpha(\theta)d\theta = \frac{1}{\Gamma(1 + \alpha)}.
\]  

(3.2)

The following properties of \( T_\alpha \) and \( S_\alpha \) appeared in [27] are useful.

**Lemma 3.3.** Under the previous assumptions on \( S(t), \ t \geq 0 \) and \( A \), the operators \( T_\alpha(t) \) and \( S_\alpha(t) \) have the following properties:

(i) For any \( x \in X \), \( \|T_\alpha(t)x\| \leq M\|x\| \), \( \|S_\alpha(t)x\| \leq \frac{M}{\Gamma(1 + \delta)}\|x\| \).

(ii) \( \{T_\alpha(t), \ t \geq 0\} \) and \( \{S_\alpha(t), \ t \geq 0\} \) are strongly continuous.

(iii) For any \( t > 0 \), \( T_\alpha(t) \) and \( S_\alpha(t) \) are also compact operators if \( S(t) \) is compact.

(iv) For any \( x \in X \), \( \beta \in (0, 1) \) and \( \delta \in (0, 1] \), we have

\[
A S_\alpha(t)x = A^{1-\beta} S_\alpha A^{\beta}x, \text{ and } \|A^\beta S_\alpha(t)\| \leq \frac{\alpha M_\delta}{\Gamma(1 + \alpha(1-\delta))}, \ t \in (0, T].
\]

**Definition 3.4.** The fractional neutral stochastic functional differential equation (1.1) is said to be controllable on the interval \((\alpha, T] \) if for every initial stochastic process \( \varphi \) defined on \((\alpha, 0] \), there exists a stochastic control \( u \in L^2([0, T], U) \) such that the mild solution \( x(t) \) of (1.1) satisfies \( x(T) = x_1 \), where \( x_1 \) and \( T \) are the preassigned terminal state and time, respectively.

Our main result in this paper is based on the following fixed point theorem.

**Theorem 3.5.** (Karasnoselkii’s fixed point theorem) Let \( V \) be a bounded closed and convex subset of a Banach space \( X \) and let \( \Pi_1, \Pi_2 \) be two operators of \( V \) into \( X \) satisfying:

1. \( \Pi_1(x) + \Pi_2(x) \in V \) whenever \( x \in V \),
2. \( \Pi_1 \) is a contraction mapping, and
3. \( \Pi_2 \) is completely continuous.

Then, there exists \( a \in V \) such that \( a = \Pi_1(a) + \Pi_2(a) \).

In order to establish the controllability of (1.1), we impose the following conditions on the data of the problem:

(\( \mathcal{H}.1 \)) The analytic semigroup, \( (S(t))_{t \geq 0} \), generated by \( A \) is compact for \( t > 0 \), and there exists \( M \geq 1 \) such that

\[
\sup_{t \geq 0} \|S(t)\| \leq M, \quad \text{and} \quad c_1 = \|(-A)^{-\beta}\|.
\]

(\( \mathcal{H}.2 \)) The map \( f : [0, T] \times \mathcal{B}_h \rightarrow X \) satisfies the following conditions:

(i) The function \( t \mapsto f(t, x) \) is measurable for each \( x \in \mathcal{B}_h \), the function \( x \mapsto f(t, x) \) is continuous for almost all \( t \in [0, T] \),

(ii) there exists a nonnegative function \( p \in L^1([0, T], \mathbb{R}^+) \), and a continuous nondecreasing function \( \vartheta : \mathbb{R}^+ \rightarrow (0, +\infty) \) such that for \( \delta > \frac{1}{2\alpha - 1} \),

\[
(\alpha \in (\frac{1}{2}, 1)), \quad \int_0^T (\vartheta(s))^\delta ds < \infty, \quad \lim_{k \rightarrow +\infty} \inf_{k \rightarrow +\infty} \frac{\vartheta(k)}{k} = \gamma < \infty, \quad \text{and}
\]

\[
\|f(t, x)\|^2 \leq p(t)\vartheta(\|x\|^2_{\mathcal{B}_h}), \quad \text{for all } x \in \mathcal{B}_h, \text{ almost surely and for a.e. } t \in [0, T].
\]
(H.3) The function \( g : [0, T] \times \mathcal{B}_h \rightarrow X \) is continuous. For \( \beta \in (0, 1) \), satisfied with \( \alpha \beta > \frac{1}{2} \), the function \( g \) is \( X_\beta \)-valued and there exists positive constant \( M_g \), such that
\[
\|(-A)^\beta g(t, x) - (-A)^\beta g(t, y)\|^2 \leq M_g \|x - y\|_{\mathcal{B}_h}^2, \quad \text{for all } x, y \in \mathcal{B}_h, \quad \text{almost surely and for a.e. } t \in [0, T],
\]
\[
\|(-A)^\beta g(t, x)\|^2 \leq M_g \|x\|_{\mathcal{B}_h}^2 + 1, \quad \text{for all } x \in \mathcal{B}_h, \quad \text{almost surely and for a.e. } t \in [0, T].
\]

(H.4) There exists a constant \( p > \frac{1}{2\alpha - 1} \) such that the function \( \sigma : [0, \infty) \rightarrow L^2(Y, X) \) satisfies
\[
\int_0^T \|\sigma(s)\|^{2p} ds < \infty, \quad \forall T > 0.
\]

(H.5) The linear operator \( W \) from \( U \) into \( X \) defined by
\[
Wu = \int_0^T (T - s)^{\alpha - 1} S_\alpha(T - s) Bu(s) ds
\]
has an inverse operator \( W^{-1} \) that takes values in \( L^2([0, T], U) \setminus \ker W \), where
\[
\ker W = \{ x \in L^2([0, T], U) : Wx = 0 \}
\]
(see [12]), and there exists finite positive constants \( M_b, M_u \) such that \( \|B\|^2 \leq M_b \) and \( \|W^{-1}\|^2 \leq M_u \).

(H.6) Assume the following inequality holds:
\[
2\lambda \left( \lambda^2 \frac{t^{2(\alpha - 1)}}{c_1^2} \right) \left( \lambda^2 \frac{t^{2(\beta + 1)}}{(2\alpha - 1)^2} \right)^2 \left( \frac{(2\alpha - 1)^2}{c_2} + \frac{(2\alpha - 1)^2}{c_1} \right) \int_0^T \|\sigma(s)\|^2 ds + \frac{2\lambda M^2 \alpha M_u T^2 \alpha^2}{(2\alpha - 1)^2} \int_0^T (T - s)^{2\alpha - 2} p(s) ds < 1.
\]
(3.3)

The main result of this chapter is the following.

**Theorem 3.6.** Suppose that (H.1) – (H.6) hold. Then, the system [11] is controllable on \((-\infty, T]\).

**Proof.** Transform the problem [13] into a fixed-point problem. To do this, using the hypothesis (H.5) for an arbitrary function \( x(\cdot) \), define the control by
\[
u(t) = W^{-1} \{ x_1 - T_\alpha(T) [\varphi(0) - g(0, x_0)] - g(T, x_T) \}
- \int_0^T (T - s)^{\alpha - 1} AS_\alpha(T - s) g(s, x_s) ds - \int_0^T (T - s)^{\alpha - 1} S_\alpha(T - s) f(s, x_s) ds
- \int_0^T (T - s)^{\alpha - 1} S_\alpha(T - s) \sigma(s) dB^H(s)(t), \quad t \in [0, T].
\]
(3.4)

To formulate the controllability problem in the form suitable for application of the fixed point theorem, put the control \( \nu(\cdot) \) into the stochastic control system [3.1] and obtain a non
linear operator \( \Pi \) on \( B_T \) given by

\[
\Pi(x)(t) = \left\{ \begin{array}{cl}
\varphi(t), & \text{if } t \in (-\infty, 0], \\
T_\alpha(t)(\varphi(0) - g(0, \varphi)) + g(t, x_t) + \int_0^t (t-s)^{\alpha-1} A S_\alpha(t-s)g(s, x_s)ds \\
+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)f(s, x_s)ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)Bu(s)ds \\
+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\sigma(s)dB^H(s), & \text{if } t \in [0, T].
\end{array} \right.
\]

Then it is clear that to prove the existence of mild solutions to equation (1.1) is equivalent to find a fixed point for the operator \( \Pi \). Clearly, \( \Pi x(T) = x_1 \), which means that the control \( u \) steers the system from the initial state \( \varphi \) to \( x_1 \) in time \( T \), provided we can obtain a fixed point of the operator \( \Pi \) which implies that the system is controllable.

Let \( y : (-\infty, T] \rightarrow X \) be the function defined by

\[
y(t) = \left\{ \begin{array}{cl}
\varphi(t), & \text{if } t \in (-\infty, 0], \\
S(t)\varphi(0), & \text{if } t \in [0, T],
\end{array} \right.
\]

then, \( y_0 = \varphi \). For each function \( z \in B_T \), set

\[
x(t) = z(t) + y(t).
\]

It is obvious that \( x \) satisfies the stochastic control system (3.1) if and only if \( z \) satisfies \( z_0 = 0 \) and

\[
z(t) = g(t, z_t + y_t) - T_\alpha(t)g(0, \varphi) + \int_0^t (t-s)^{\alpha-1} A S_\alpha(t-s)g(s, z_s + y_s)ds \\
+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)f(s, z_s + y_s)ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)Bu(s)ds \\
+ \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\sigma(s)dB^H(s),
\]

where \( u_{z+y}(t) \) is obtained from (3.4) by replacing \( x_t = z_t + y_t \).

Set

\[
B^0_T = \{ z \in B_T : z_0 = 0 \};
\]

for any \( z \in B^0_T \), we have

\[
\|z\|_{B^0_T} = \|z_0\|_{B_h} + \sup_{t \in [0, T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t \in [0, T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}}.
\]

Then, \((B^0_T, \|\cdot\|_{B^0_T})\) is a Banach space. Define the operator \( \hat{\Pi} : B^0_T \rightarrow B^0_T \) by
and Π prove that then $z$ is defined on $B$, but $\hat{\Pi}t = 0$ if $t \in (-\infty, 0]$, $0 \leq t \leq T$.

For the sake of convenience, the proof will be given in several steps.

**Step 1.** We claim that there exists a positive number $k$, such that $\Pi_1(x) + \Pi_2(x) \in B_k$ whenever $x \in B_k$. If it is not true, then for each positive number $k$, there is a function $z^k(.) \in B_k$, but $\Pi_1(z^k) + \Pi_2(z^k) \notin B_k$, that is $\|\Pi_1(z^k)(t) + \Pi_2(z^k)(t)\|^2 > k$ for some...
\( t \in [0, T] \). However, on the other hand, we have

\[
k < \mathbb{E}\|\Pi_1(z^k(t)) + \Pi_2(z^k(t))\|^2 \leq 6\{\mathbb{E}\|T_\alpha(t)g(0, \varphi)\|^2 + \mathbb{E}\|g(t, z^k_t + y_t)\|^2
\]

\[
+ \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} AS_\alpha(t-s)g(s, z^k_s + y_s)ds \|^2
\]

\[
+ \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)f(s, z^k_s + y_s)ds \|^2
\]

\[
+ \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)Bu_z + y(s)ds \|^2
\]

\[
+ \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\sigma(s)dB^H(s) \|^2 \}\}
\leq 6 \sum_{i=1}^6 I_i.
\]

By (H.3), \( (i) \) of Lemma 3.3, we have

\[
I_1 \leq \mathbb{E}\|T_\alpha(t)g(0, \varphi)\|^2
\]

\[
\leq M^2\|(-A)^{-\beta}\|^2\|(-A)\beta g(0, \varphi)\|^2
\]

\[
\leq M^2c_2^2M_g\|\varphi\|_{B_h}^2 + 1.
\]

By (H.3), \( (3.7) \), we have

\[
I_2 \leq \|(-A)^{-\beta}\|^2\mathbb{E}\|(-A)\beta g(t, z^k_t + y_t)\|^2
\]

\[
\leq c_2^2M_g\|z^k_t + y_t\|_{B_h}^2 + 1
\]

\[
\leq c_2^2M_g[4l^2(k + M^2\mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{B_h}^2 + 1].
\]

By \((iv)\) of Lemma 3.3 \( (H.3) \), Hölder inequality, we have

\[
I_3 \leq \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} AS_\alpha(t-s)g(s, z^k_s + y_s)ds \|^2
\]

\[
\leq \mathbb{E}\|(\int_0^t (t-s)^{\alpha-1}(-A)^{1-\beta}S_\alpha(t-s)(-A)\beta g(s, z^k_s + y_s)ds\|^2
\]

\[
\leq \mathbb{E}(\int_0^t (t-s)^{\alpha-1}\|(-A)^{1-\beta}S_\alpha(t-s)(-A)\beta g(s, z^k_s + y_s)\|ds)^2
\]

\[
\leq \frac{\alpha^2M^2\|t^{\alpha-1}\|^2T^{\alpha+1}(\alpha\beta + 1)}{T^2(\alpha\beta+1)}\mathbb{E}(\int_0^t (t-s)^{\alpha-1}\|(-A)^{1-\beta}S_\alpha(t-s)(-A)\beta g(s, z^k_s + y_s)\|ds)^2
\]

\[
\leq \frac{\alpha^2M^2\|t^{\alpha-1}\|^2T^{\alpha+1}(\alpha\beta + 1)}{T^2(\alpha\beta+1)}\int_0^t (t-s)^{2\alpha\beta-2}ds \int_0^t \mathbb{E}\|(-A)\beta g(s, z^k_s + y_s)\|^2ds
\]

\[
\leq \frac{T^{2\alpha\beta-1}M^2\|t^{\alpha-1}\|^2T^{\alpha+1}(\alpha\beta + 1)}{(2\alpha\beta-1)T^2(\alpha\beta+1)}M_g(4l^2(k + M^2\mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{B_h}^2 + 1)ds
\]

\[
\leq \frac{T^{2\alpha\beta-1}M^2\|t^{\alpha-1}\|^2T^{\alpha+1}(\alpha\beta + 1)}{(2\alpha\beta-1)T^2(\alpha\beta+1)}M_g[4l^2(k + M^2\mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{B_h}^2 + 1].
\]

From \((H.2)\), Hölder inequality, we have
\[ I_4 \leq E \int_0^T (t-s)^{\alpha-1} S_\alpha(t-s) f(s, z^k_s + y_s) ds \]
\[ \leq \frac{M^2T^\alpha}{(\alpha-1)^2} E \int_0^T (t-s)^{\alpha-1} f(s, z^k_s + y_s) ds \]
\[ \leq \frac{M^2T^\alpha}{(\alpha-1)^2} \int_0^T (T-s)^{2\alpha-2} E \| f(s, z^k_s + y_s) \|^2 ds \]
\[ \leq \frac{M^2T^\alpha}{(\alpha-1)^2} \int_0^T (T-s)^{2\alpha-2} p(s) \| z^k_s + y_s \|^2_{\mathcal{B}_h} ds \]
\[ \leq \frac{M^2T^\alpha}{(\alpha-1)^2} \int_0^T (T-s)^{2\alpha-2} p(s) ds \]
(3.14)

From (ii) of (H.2), Holder inequality, it follows that for \( \delta > \frac{1}{2\alpha-1} \),
\[ \int_0^T (T-s)^{2\alpha-2} p(s) ds \leq \left( \int_0^T (T-s)^{\frac{2\alpha-2-\delta}{2}} ds \right)^{\frac{2}{2-\delta}} \left( \int_0^T (p(s))^\delta ds \right)^{\frac{2}{\delta}} \]
\[ \leq T^{\frac{(2\alpha-2-\delta)}{2(\alpha-1)}} \left( \int_0^T (p(s))^\delta ds \right)^{\frac{2}{\delta}} \]
\[ < \infty. \]

From our assumptions, (iv) of Lemma 3.3, using the fact that \( (\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2 \) for any positive real numbers \( a_i, i = 1, 2, ..., n \), we have
\[ E \| u_{z+y} \|^2 \leq 6 M_w \| x_1 \|^2 + M^2 E \| \varphi(0) \|^2 + M^2 c^2 T M_g \| y \|^2_{\mathcal{B}_h} + 1 \]
\[ + \left[ c^2 + \frac{\alpha^2 M^2}{(2\alpha-1)^2} T^{2\alpha-2} \sigma(1) \right] M_g [4 t^2 (k + M^2 E \| \varphi(0) \|^2) + 4 \| y \|^2_{\mathcal{B}_h} ] \int_0^T (T-s)^{2\alpha-2} p(s) ds \]
\[ + \frac{\int_{0}^{T} (T-s)^{2\alpha-2} \sigma(s) \| \sigma(s) \|^2_{\mathcal{B}_h} ds}{T} := \mathcal{G}. \]
(3.15)

For \( p > \frac{1}{2\alpha-1} \), we have
\[ \int_0^T (T-s)^{2\alpha-2} \| \sigma(s) \|^2_{\mathcal{B}_h} ds \leq \left( \int_0^T (T-s)^{\frac{2\alpha-2-\delta}{2}} ds \right)^{\frac{2}{2-\delta}} \left( \int_0^T \| \sigma(s) \|^\delta_{\mathcal{B}_h} ds \right)^{\frac{2}{\delta}} \]
\[ \leq T^{\frac{(2\alpha-2-\delta)}{2(\alpha-1)}} \left( \int_0^T \| \sigma(s) \|^\delta_{\mathcal{B}_h} ds \right)^{\frac{2}{\delta}} \]
\[ < \infty. \]
(3.16)
By (3.15), (i) of Lemma 3.3 Hölder inequality, we have
\[ I_5 \leq \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) Bu_{z+y}(s) ds \|^2 \]
\[ \leq \frac{M^2 M_0 T_{\alpha}}{1-\frac{\alpha}{3}} \int_0^t (t-s)^{2\alpha-2} ds \int_0^t \mathbb{E}\| u_{z+y}(s) \|^2 ds \]
\[ \leq \frac{6 M^2 M_0 T_{\alpha}^2}{(2\alpha-1)T_{\alpha}^2} \{ \| x_1 \|^2 + M^2 \mathbb{E}\| \varphi(0) \|^2 + M^2 c_1^2 M_g \left\| \| y \|_{B_h} + 1 \right\| \} \]
\[ \begin{aligned} &+ \left[ c_2^2 + \frac{\alpha^2 M^2 T_{\alpha}^2}{(2\alpha-1)T_{\alpha}^2} \right] M_g \left[ 4l^2 (k + M^2 \mathbb{E}\| \varphi(0) \|^2) + 4 \| y \|_{B_h}^2 \right] \int_0^T (T-s)^{2\alpha-2} p(s) ds \\
&+ \frac{M^2}{T_{\alpha}^2} \int_0^T (T-s)^{2\alpha-2} \| \varphi(s) \|_{L_2}^2 ds \right\} . \] (3.17)

By Lemma 2.1, Lemma 3.5, (3.16), for \( p > \frac{1}{2\alpha-1} \), we have
\[ I_6 \leq \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s) dB H(s) \|^2 \]
\[ \leq \frac{2 M^2 T_{\alpha}^{2H-1}}{1-\frac{\alpha}{3}} \int_0^T (T-s)^{2\alpha-2} \| \sigma(s) \|_{L_2}^2 ds \]
\[ \leq \frac{2 M^2 T_{\alpha}^{2H-1}}{1-\frac{\alpha}{3}} \left( \int_0^T \| \sigma(s) \|_{L_2}^{2p} ds \right)^{\frac{1}{p}} . \] (3.18)

By (3.10), (3.11), (3.12), (3.13), (3.14), (3.17), (3.18), we have
\[ k < \mathbb{E}\| \Pi_1(z^k)(t) + \Pi_2(z^k)(t) \|^2 \leq \mathbb{K} + 24 l^2 k c_1^2 M_g + 24 l^2 k^2 T_{\alpha}^{2\alpha-2} \left( \frac{T_{\alpha}}{2\alpha-1} \right)^2 T_{\alpha}^{(\alpha+1)} M_g \]
\[ + 6 (1 + \frac{6 M^2 M_0 T_{\alpha}^2}{(2\alpha-1)T_{\alpha}^2}) \frac{M^2}{T_{\alpha}^2} \int_0^T (T-s)^{2\alpha-2} \| \varphi(s) \|^2 ds \\
+ 4 \| y \|_{B_h}^2 \int_0^T (T-s)^{2\alpha-2} p(s) ds \\
+ \frac{144 M^2 M_0 T_{\alpha}^2}{(2\alpha-1)T_{\alpha}^2} \left[ c_2^2 + \frac{\alpha^2 M^2}{(2\alpha-1)T_{\alpha}^2} \right] M_g l^2 k , \]

where
\[ \mathbb{K} = 6 M^2 c_1^2 (M_g \| \varphi \|_{B_h}^2 + 6 c_2^2 M_g \left[ 4l^2 M^2 \mathbb{E}\| \varphi(0) \|^2 + 4 \| y \|_{B_h}^2 \right] + 1] \]
\[ + 6 \frac{T_{\alpha}^{2\alpha-2} M^2}{(2\alpha-1)T_{\alpha}^2} \left( \int_0^T (T-s)^{2\alpha-2} \| \sigma(s) \|^2 ds \right)^{\frac{1}{p}} . \]

Noting that \( \mathbb{K} \) is independent of \( k \). Dividing both sides by \( k \) and taking the lower limit as \( k \to \infty \), we obtain
\[ q' = 4l^2 (k + M^2 \mathbb{E}\| \varphi(0) \|^2) + 4 \| y \|_{B_h} \to \infty \text{ as } k \to \infty , \]
\[
\liminf_{k \to \infty} \frac{\vartheta(q'_k)}{k} = \liminf_{k \to \infty} \frac{\vartheta(q'_k)}{q'_k} = 4l^2 \gamma.
\]

Thus, we have
\[
1 \leq 24l^2 c_2^2 M_g + 24l^2 T^{2\alpha} M_g^2 \frac{T^2(\beta + 1)}{2(\alpha - 1)1^2(\alpha + 1)} M_g
\]
\[+ 24l^2 \gamma (1 + \frac{6M_g^2 M_g T^2}{2(\alpha - 1)1^2(\alpha + 1)}) \frac{T^2}{\alpha} \int_0^T (T - s)^{2\alpha - 2} p(s) ds
\]
\[+ \frac{144M_g^2 M_g T^2}{2(\alpha - 1)1^2(\alpha + 1)} \epsilon_1^2 + \frac{\alpha^2 M_g^2 M_g T^2(\beta + 1)}{2(\alpha - 1)1^2(\alpha + 1)} M_g l^2.
\]

This contradicts (3.3). Hence for some positive \(k\),
\[(\Pi_1 + \Pi_2)(B_k) \subseteq B_k.
\]

**Step 2.** \(\Pi_1\) is a contraction.

Let \(t \in [0, T]\) and \(z^1, z^2 \in B_T^n\)
\[
\mathbb{E}\| (\Pi_1 z^1)(t) - (\Pi_1 z^2)(t) \|^2 \leq 2\mathbb{E}\| g(t, z^1_t + y_t) - g(t, z^2_t + y_t) \|^2
\]
\[+ 2\mathbb{E}\| \int_0^t (t - s)^{\alpha - 1} A S_\alpha (t - s) (g(s, z^1_s + y_s) - g(s, z^2_s + y_s)) ds \|^2
\]
\[\leq 2M_g\| (\alpha - \beta)^2 \| z^1_s - z^2_s \|_{B_n}^2
\]
\[+ 2\int_0^t (t - s)^{\alpha - 1} (-A)^{-\beta} S_\alpha (t - s) (-A)^{\beta} (g(s, z^1_s + y_s) - g(s, z^2_s + y_s)) ds \|^2
\]
\[\leq 2M_g\| (\alpha - \beta)^2 \| z^1_s - z^2_s \|_{B_n}^2
\]
\[+ \frac{2\alpha^2 M_g^2 (T + 1)^2}{T^2(\alpha + 1)} \int_0^t (t - s)^{2\alpha + 2} ds \int_0^t M_g \| z^1_s - z^2_s \|_{B_n}^2 ds
\]
\[\leq 2M_g \left\{ \| (\alpha - \beta)^2 \| + \frac{2\alpha^2 M_g^2 (T + 1)^2}{T^2(\alpha + 1)} \right\} (2l^2 \sup_{0 \leq s \leq T} \mathbb{E}\| z^1(s) - z^2(s) \|^2)
\]
\[\leq \nu \sup_{0 \leq s \leq T} \mathbb{E}\| z^1(s) - z^2(s) \|^2 \quad (\text{since } z^1_0 = z^2_0 = 0)
\]

Taking supremum over \(t\),
\[\| (\Pi_1 z^1)(t) - (\Pi_1 z^2)(t) \|_{B_T^n} \leq \nu \| z^1 - z^2 \|_{B_T^n},
\]
where
\[
\nu = 4M_g (c_2^2 + \frac{2\alpha^2 M_g^2}{T^2(\alpha + 1)} T^{2(\alpha + 1)} - \frac{2\alpha^2 M_g^2}{2(\alpha - 1)1^2(\alpha + 1)} M_g l^2).
\]

By (H.6), we have \(\nu < 1\). Thus \(\Pi_1\) is a contraction on \(B_T^n\).

**Step 3.** \(\Pi_2\) is completely continuous on \(B_T^n\).

**Claim 1.** \(\Pi_2\) is continuous on \(B_T^n\).

Let \(z^n\) be a sequence such that \(z^n \rightarrow z\) in \(B_T^n\). Then, for \(t \in [0, T]\), and thanks to hypothesis (H.2) – (H.3), for each \(t \in [0, T]\), we have
\[f(t, z^n_t + y_t) \rightarrow f(t, z_t + y_t),
\]
\[g(t, z^n_t + y_t) \rightarrow g(t, z_t + y_t).
\]
By the dominated convergence theorem, we obtain continuity of $\Pi_2$

$$
\mathbb{E}\|\Pi_2 z^n(t) - (\Pi_2 z)(t)\|^2 \leq 2\mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B[u_{z+y} - u_{z+y}]ds \|^2 \\
+ 2\mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) [f(s, z^n_s + y_s) - f(s, z_s + y_s)]ds \|^2 \\
\leq \frac{2M^2 M_h}{T^{(\alpha+1)}} \frac{T^{2\alpha-1}}{2\alpha-1} \int_0^T \mathbb{E}\|u_{z+y}(s) - u_{z+y}(s)\|^2 ds \\
+ \frac{2M^2}{T^{(\alpha+1)}} \frac{T^{2\alpha-1}}{2\alpha-1} \int_0^T \mathbb{E}\|f(s, z^n_s + y_s) - f(s, z_s + y_s)\|^2 ds \\
\longrightarrow 0 \text{ as } n \longrightarrow \infty.
$$

Thus, $\Pi_2$ is continuous.

**Claim 2.** $\Pi_2$ maps $B_k$ into equicontinuous family. Let $z \in B_k$ and $|h|$ be sufficiently small, we have

$$
\mathbb{E}\| (\Pi_2 z)(t+h) - (\Pi_2 z)(t)\|^2 \leq \mathbb{E}\| \int_0^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s) B[u_{z+y}(s)] ds \\
+ \int_0^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s) f(s, z_s + y_s) ds \\
- \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) B[u_{z+y}(s)] ds \\
- \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, z_s + y_s) ds\|^2 \\
\leq 6\mathbb{E}\| \int_0^t ((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}) S_\alpha(t+h-s) B[u_{z+y}(s)] ds \|^2 \\
+ 6\mathbb{E}\| \int_t^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s) B[u_{z+y}(s)] ds \|^2 \\
+ 6\mathbb{E}\| \int_0^t (t-s)^{\alpha-1} (S_\alpha(t+h-s) - S_\alpha(t-s)) B[u_{z+y}(s)] ds \|^2 \\
+ 6\mathbb{E}\| \int_t^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s) f(s, z_s + y_s) ds \|^2 \\
+ 6\mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t+h-s) f(s, z_s + y_s) ds \|^2 \\
+ 6\mathbb{E}\| \int_t^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s) f(s, z_s + y_s) ds \|^2 \\
+ 6\mathbb{E}\| \int_0^t (t-s)^{\alpha-1} (S_\alpha(t+h-s) - S_\alpha(t-s)) f(s, z_s + y_s) ds \|^2.
$$

From (iii) of Lemma 3.3 we have $S_\alpha(t)$ is compact for any $t > 0$. Let $0 < \varepsilon < t < T$, and $\delta > 0$ such that $\|S_\alpha(\tau_1) - S_\alpha(\tau_2)\| \leq \varepsilon$ for every $\tau_1, \tau_2 \in [0, T]$ with $|\tau_1 - \tau_2| \leq \delta$. 
From (3.15), (i) of Lemma 3.3, Hölder inequality, it follows that

\[
\begin{align*}
E\| (\Pi_2 z)(t+h) - (\Pi_2 z)(t) \|^2 & \\
\leq & \frac{6M^2M_hG\Gamma}{T(\alpha)} \int_0^t ((t+h-s)^\alpha - (t-s)^\alpha)^2 \, ds \\
+ & 6M^2M_hGh \int_t^{t+h} (t+h-s)^{2\alpha-2} \, ds \\
+ & 6M^2T^2G^2(\alpha) \int_0^t (t+h-s)^\alpha - (t-s)^\alpha \, ds \\
+ & 6M^2T^2G^2(\alpha) \int_0^t (t+h-s)^{2(\alpha-1)} \, ds \\
+ & 6M^2T^2G^2(\alpha) \int_0^t (t+h-s)^{2(\alpha-1)} \, ds.
\end{align*}
\]

(3.19)

From (ii) of (H.2), Hölder inequality, it follows that for \( \delta > \frac{1}{2\alpha-1} \),

\[
\int_0^t (t-s)^{2\alpha-2} \, ds \leq \left( \int_0^T (t-s)^{\frac{2\alpha-2\delta}{\delta}} \, ds \right) \frac{\delta}{2\alpha-2} \left( \int_0^T (p(s))^{\delta} \, ds \right)^{\frac{1}{\delta}} \\
\leq T^{\frac{2\alpha-2}{2\alpha-1}} \left( \int_0^T (p(s))^{\delta} \, ds \right)^{\frac{1}{\delta}} \\
< \infty.
\]

Similarly, we have

\[
\int_0^t (t+h-s)^{2(\alpha-1)} \, ds < \infty.
\]

By the dominated convergence theorem, we have

\[
\int_0^t ((t+h-s)^\alpha - (t-s)^\alpha)^2 \, ds \rightarrow 0, \text{ as } h \rightarrow 0.
\]

Therefore, for sufficiently small positive number \( \epsilon \), we have from (3.19) that

\[
E\| (\Pi_2 z)(t+h) - (\Pi_2 z)(t) \|^2 \rightarrow 0 \text{ as } h \rightarrow 0.
\]

Thus, \( \Pi_2 \) maps \( B_k \) into an equicontinuous family of functions.

**Claim 3.** \( (\Pi_2 B_k)(t) \) is precompact set in \( X \).

Let \( 0 < t \leq T \) be fixed, and \( \epsilon \) be a number satisfying \( 0 < \epsilon < t \). For \( \delta > 0 \) and \( z \in B_k \), we define
(Π_{2,\epsilon}^{\delta}z)(t) = \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^{\alpha}\theta)f(s, z_s + y_s) d\theta ds

+ \alpha \int_0^t \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^{\alpha}\theta) Bu_{z+y}(s) d\theta ds

= S(\epsilon^\alpha \delta) \alpha \int_0^t \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^{\alpha}\theta - \epsilon^\alpha \delta)f(s, z_s + y_s) d\theta ds

+ S(\epsilon^\alpha \delta) \alpha \int_0^t \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^{\alpha}\theta - \epsilon^\alpha \delta) Bu_{z+y}(s) d\theta ds

From the compactness of \( S(t) \) \((t > 0) \), we obtain that the set \( V_\epsilon^\delta(t) = \{ (\Pi_{2,\epsilon}^{\delta}z)(t) : z \in B_k \} \) is relative compact in \( X \) for every \( \epsilon, 0 < \epsilon < t \) and \( \delta > 0 \). Moreover, for every \( z \in B_k \), we have

\[
E\|\Pi_{2z}(t) - \Pi_{2,\epsilon}^{\delta}z(t)\|^2 \leq 4a^2E \int_0^t \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^{\alpha}\theta)f(s, z_s + y_s) d\theta ds

+ 4a^2E \int_0^t \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^{\alpha}\theta) Bu_{z+y}(s) d\theta ds

+ 4a^2E \int_0^t \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^{\alpha}\theta - \epsilon^\alpha \delta)f(s, z_s + y_s) d\theta ds

+ 4a^2E \int_0^t \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^{\alpha}\theta - \epsilon^\alpha \delta) Bu_{z+y}(s) d\theta ds

= 4 \sum_{i=1}^4 J_i. \tag{3.20}
\]

A similar argument as before, we can show that

\[
J_1 \leq \alpha^2 M^2 T E \int_0^t \| \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) f(s, z_s + y_s) d\theta \|^2 ds

\leq \alpha^2 M^2 T \| \int_0^t \theta \eta_\alpha(\theta) d\theta \|^2 \int_0^t (t-s)^{2\alpha-2} E \| f(s, z_s + y_s) \|^2 ds \tag{3.21}

\leq \alpha^2 M^2 T \vartheta(q') \| \int_\delta^\infty \theta \eta_\alpha(\theta) d\theta \|^2 \int_0^t (t-s)^{2\alpha-2} p(s) ds.
\]

For \( J_2 \), by (3.2), we have

\[
J_2 \leq \alpha^2 M^2 T \vartheta(q') \| \int_\delta^\infty \theta \eta_\alpha(\theta) d\theta \|^2 \int_\delta^t (t-s)^{2\alpha-2} p(s) ds

\leq \frac{\alpha^2 M^2 T \vartheta(q')}{1+\alpha} \int_\delta^t (t-s)^{2\alpha-2} p(s) ds

\leq \frac{\alpha^2 M^2 T \vartheta(q')}{1+\alpha} \left( \int_\delta^t (t-s)^{2\alpha-2} ds \right)^{\frac{1}{2\alpha-2}} \left( \int_\delta^t (p(s))^\delta ds \right)^{\frac{1}{\delta}}

\leq \frac{\alpha^2 M^2 T \vartheta(q')}{1+\alpha} \left( \int_\delta^t (p(s))^\delta ds \right)^{\frac{1}{\delta}}, \tag{3.22}
\]

where \( \delta > \frac{1}{2\alpha-1} \).
For $J_3$, by Hölder inequality, we have

$$J_3 \leq \alpha^2 \mathbb{E} \left( \int_0^t \int_0^s \|\theta(t-s)^{\alpha-1} \eta_s(\theta)S((t-s)^{\alpha} \theta)Bu_{s+y}(s)\|d\theta ds \right)^2 \tag{3.23}$$

$$\leq \alpha^2 M^2 M_0 T \int_0^t \int_0^s \|u_{s+y}(s)\|^2 ds \int_0^s \theta \eta_s(\theta)d\theta \|d\theta\|^2.$$

For $J_4$, by (3.2), we have

$$J_4 \leq \alpha^2 M^2 \left( \int_{t-\epsilon}^t \|u_{s+y}(s)\|^2 ds \right) \left( \int_{t-\epsilon}^t \theta \eta_s(\theta) d\theta \right)^2 \tag{3.24}$$

Put (3.21), (3.22), (3.23), (3.24) into (3.20) to obtain

$$\mathbb{E} \|\Pi_2 z(t) - \Pi_{2,z}^\delta(t)\|^2 \to 0, \quad \text{as } \epsilon \to 0^+, \; \delta \to 0^+.$$

Therefore, there are precompact sets arbitrarily close to the set $V(t) = \{ (\Pi_2 z)(t) : z \in B_k \}$, hence the set $V(t)$ is also precompact in $X$.

Thus, by Arzela-Ascoli theorem $\Pi_2$ is a compact operator. These arguments enable us to conclude that $\Pi_2$ is completely continuous, and by the fixed point theorem of Krasnoselskii there exists a fixed point $x(.)$ for $\Pi$ on $B_k$. If we define $x(t) = z(t) + y(t)$, $-\infty < t \leq T$, it is easy to see that $x(.)$ is a mild solution of (1.1) satisfying $x_0 = \varphi$, $x(T) = x_1$. Then the proof is complete.

\[ \square \]

4. Example

To illustrate the previous result, we consider the following fractional neutral stochastic partial differential equation with infinite delays, driven by a fractional Brownian motion of the form

$$dJ^\alpha_1 = [\varphi(t, \xi) - g(t, v(t, r, \xi)) - \varphi(0, \xi) + g(0, v(-r, \xi))] dt + \sigma(t) d\mathcal{B}^H(t), \quad 0 \leq t \leq T, \; r > 0, \; 0 \leq \xi \leq 1$$

$$+ f(t, r, \xi) dr + \sigma(t) \frac{d\mathcal{B}^H(t)}{dt}, \quad 0 \leq t \leq T;$$

$$v(t, 0) = v(t, 1) = 0, \quad 0 \leq t \leq T;$$

$$v(s, \xi) = \varphi(s, \xi), \quad -\infty < s \leq 0, \; 0 \leq \xi \leq 1,$$ \tag{4.1}

where $B^H(t)$ is cylindrical fractional Brownian motion, $\varphi : (-\infty, 0] \times [0, 1] \longrightarrow \mathbb{R}$ is a given measurable and satisfies $\|\varphi\|^2_{B_n} < \infty$.

We rewrite (4.1) into abstract form of (1.1). We take $X = Y = U = L^2([0, 1])$. Define the operator $A : D(A) \subset X \longrightarrow X$ given by $A = \frac{d^2}{d\xi^2}$ with

$$D(A) = \{ y \in X : y' \text{ is absolutely continuous}, \; y'' \in X, \; y(0) = y(1) = 0 \},$$

then we get

$$Ax = \sum_{n=1}^{\infty} n^2 < x, e_n >_X e_n, \quad x \in D(A),$$

where $e_n := \sqrt{\frac{2}{n}} \sin nx, \; n = 1, 2, \ldots.$ is an orthogonal set of eigenvector of $-A$. 

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The bounded linear operator $(-A)^{\frac{1}{2}}$ is given by

$$(-A)^{\frac{1}{2}}x = \sum_{n=1}^{\infty} n^\frac{1}{2} < x, e_n > e_n,$$

with domain

$$\mathcal{D}((-A)^{\frac{1}{2}}) = \{x \in X, \sum_{n=1}^{\infty} n^\frac{1}{2} < x, e_n > e_n \in X\}.$$

It is known that $A$ generates a compact analytic semigroup $\{S(t)\}_{t \geq 0}$ in $X$, and is given by (see [21])

$$S(t)x = \sum_{n=1}^{\infty} e^{-nt^2} < x, e_n > e_n,$$

for $x \in X$ and $t \geq 0$. Since the semigroup $\{S(t)\}_{t \geq 0}$ is analytic, there exists a constant $M > 0$ such that $\|S(t)\|^2 \leq M$ for every $t \geq 0$. In other words, the condition (H.1) holds.

If we choose $\alpha \in (\frac{1}{4}, 1)$,

$$S_n(t)x = \int_0^\infty \alpha \theta \eta_n(\theta)S(\theta t^{\alpha}) d\theta, \quad x \in X.$$

Further, the operator $B : \mathbb{R} \rightarrow X$ is a bounded linear operator defined by $Bu(t)(\xi) = c(\xi)u(t)$, $0 \leq \xi \leq 1$, $c(\xi) \in L^2([0, 1])$, and the operator $W : L^2([0, T], U) \rightarrow X$ is given by

$$Wu(\xi) = \int_0^T (T-s)^{\alpha-1}S_n(T-s)c(\xi)u(t) ds, \quad 0 \leq \xi \leq 1,$$

$W$ is linear and by Hölder inequality, we can show that $W$ is bounded operator but not necessarily one-to-one. Let

$$\text{Ker } W = \{x \in L^2([0, T], U), \text{Wx} = 0\}$$

be the null space of $W$ and $[\text{Ker } W]^\perp$ be its orthogonal complement in $L^2([0, T], U)$. Let $\tilde{W} : [\text{Ker } W]^\perp \rightarrow \text{Range}(W)$ be the restriction of $W$ to $[\text{Ker } W]^\perp$. $\tilde{W}$ is necessarily one-to-one operator. The inverse mapping theorem says that $\tilde{W}^{-1}$ is bounded since $[\text{Ker } W]^\perp$ and $\text{Range}(W)$ are Banach spaces. So that $W^{-1}$ is bounded and takes values in $L^2([0, T], U) \setminus \text{Ker } W$. Hypothesis (H.5) is satisfied.

We choose the phase function $h(s) = e^{2s}, s < 0$, then $l = \int_{-\infty}^{0} h(s) ds = \frac{3}{2} < \infty$, and the abstract phase space $B_\theta$ is Banach space with the norm

$$\|\varphi\|_{B_\theta} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s, 0]} (\text{E}\|\varphi(\theta)\|^2)^{\frac{1}{2}} ds.$$

To rewrite the initial-boundary value problem (4.1) in the abstract form (1.1), we assume the following:

For $(t, \varphi) \in [0, T] \times B_\theta$, where $\varphi(\theta)(\xi) = \varphi(\theta, \xi), (\theta, \xi) \in (-\infty, 0) \times [0, 1]$, we put $v(t)(\xi) = v(t, \xi)$. Define $g : [0, T] \times B_\theta \rightarrow X, f : [0, T] \times B_\theta \rightarrow X$ by

$$(-A)^{\frac{1}{2}}g(t, \varphi)(\xi) = \int_{-\infty}^{0} e^{-4\theta} \varphi(\theta)(\xi) d\theta,$$

$$f(t, \varphi)(\xi) = \int_{-\infty}^{0} \mu(t, \xi, \theta) f_1(\varphi(\theta)(\xi)) d\theta,$$

where

(i) the function $\mu(t, \xi, \theta) \geq 0$ is continuous in $[0, T] \times [0, 1] \times (-\infty, 0)$,

$$\int_{-\infty}^{0} \mu(t, \xi, \theta) d\theta = p_1(t, \xi) < \infty, \quad \text{and} \quad \left(\int_{0}^{1} p_1^2(t, \xi)\right)^{\frac{1}{2}} = p(t) < \infty;$$
(ii) the function \( f_1(.) \) is continuous, \( 0 \leq f_1(v(\theta, \xi)) \leq \theta(\|v(\theta, \cdot)\|_{L^2}) \) for \( (\theta, \xi) \in (-\infty, 0) \times (0, 1) \), where \( \theta(.) : [0, \infty) \rightarrow (0, \infty) \) is continuous and nondecreasing.

By the similar method as in Balasubramaniyam and Ntouyas \([2]\), we can show that the assumptions \((H.2) - (H.3)\) are satisfied.

In order to define the operator \( Q : Y := L^2([0, 1], \mathbb{R}) \rightarrow Y \), we choose a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+ \), set \( Qe_n = \lambda_n e_n \), and assume that

\[
\text{tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty.
\]

Define the fractional Brownian motion in \( Y \) by

\[
B^H_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta^H_t e_n,
\]

where \( H \in \left( \frac{1}{2}, 1 \right) \) and \( \{\beta^H_t\}_{n \in \mathbb{N}} \) is a sequence of one-dimensional fractional Brownian motions mutually independent. Let us assume the function \( \sigma : [0, +\infty) \rightarrow L^2_0(L^2([0, 1]), L^2([0, 1])) \) satisfies

\[
\int_0^T \|\sigma(s)\|_{L^2_p}^2 ds < \infty, \quad \text{for some} \ p > \frac{1}{2\alpha - 1}.
\]

Then all the assumptions of Theorem \( 3.6 \) are satisfied. Therefore, we conclude that the system \( (4.1) \) is controllable on \( (-\infty, T] \).

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