NON-ANCIENT SOLUTION OF THE RICCI FLOW

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Abstract. For any complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature, we provide the necessary and sufficient condition for non-ancient solution to the Ricci flow in this paper.

1. Introduction

Let M be any complex n-dimensional complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature, in this paper we always assume M satisfies this condition. The Ricci flow

\[ \frac{\partial}{\partial t} g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}}, \quad g_{\alpha\bar{\beta}}(x,0) = g_{\alpha\bar{\beta}}(x) \]

evolves Kähler metric \( g_{\alpha\bar{\beta}} \) on M by its Ricci tensor \( R_{\alpha\bar{\beta}} \). It was first introduced by Hamilton (1), in (2) he divided a solution to the Ricci flow into three types, the first two types was called ancient solution, we call the last type by non-ancient solution, which is said that the Ricci flow has long time existence \( 0 \leq R(x,t) \leq C(1+t) \), where \( R(x,t) \) denotes the scalar curvature at time t and C is a positive constant. Chen-Zhu (3) proved non-ancient solution exists if M is a complete noncompact complex two-dimensional Kähler manifold with positive and bounded holomorphic bisectional curvature, its geodesic balls have Euclidean volume growth and its scalar curvature decays to zero at infinity in the average sense. They used this result to study the uniformization conjecture by Yau (4), which says that a complete noncompact Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to a complex Euclidean space. They partially confirmed this conjecture in the case of \( n=2 \). Recently Ni and Tam (5) studied the Ricci flow by solving the Poincare-Lelong equation, and got some nice results.

In this paper, we combined with Ni and Tam’s results to obtain the necessary and sufficient condition for the non-ancient solution to the Ricci flow. That is:

Theorem 1.1. Let M be above assumption, then the Ricci flow (1.1) has non-ancient solution if and only if

\[ \int_0^r sk(x,s)ds \leq C\log(2+r) \]

for some constants \( C > 0, \forall x \in M, r \geq 0 \), where

\[ k(x,s) = \frac{1}{V(x,s)} \int_{B(x,s)} R(x)dx \]

\( V(x,s) \) is the volume of the geodesic ball \( B(x,s) \) centered at \( x \in M \) with radius \( s \), \( R(x) \) is the scalar curvature of M.

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2. The necessary condition for the solution

For the convenience, we suppose $C > 0$ be a various constant depending only on $n$ in the following.

**Lemma 2.1.** Let $M$ be above assumption, then there exists a constant $C$ such that for $\forall x, y \in M$,

\[
\frac{d(x, y)^2}{CV(x, d(x, y))} \leq G(x, y) \leq \frac{Cd(x, y)^2}{V(x, d(x, y))}
\]

(2.1)

\[
|\nabla G(x, y)| \leq \frac{Cd(x, y)}{V(x, d(x, y))}
\]

(2.2)

where $d(x, y)$ denotes the distance between $x$ and $y$ with respect to $g_{\alpha \bar{\beta}}(x)$, $\nabla$ is the covariant derivative with respect to $g_{\alpha \bar{\beta}}(x)$, $G(x, y)$ is the positive Green’s function on $M$ with respect to $g_{\alpha \bar{\beta}}(x)$.

**Theorem 2.2.** Let $M$ be above assumption, and the necessary condition holds, then there exists a constant $C > 0$ such that for $\forall x \in M$, $r > 0$,

\[
\int_{B(x, r)} \frac{R(y)d(x, y)^2}{V(x, d(x, y))}dy \leq C \log(2 + r)
\]

Proof. Let $g_{\alpha \bar{\beta}}(x, t)$ be the solution of the Ricci flow (1.1) with $g_{\alpha \bar{\beta}}(x)$ as the initial metric. From the necessary condition, we know that the solution exists for all the times and satisfies

\[
0 \leq R(x, t) \leq \frac{C}{1 + t} \text{ on } M \times [0, +\infty)
\]

(2.3)

Let $F(x, t) = \frac{\log det(g_{\alpha \bar{\beta}}(x, t))}{\log det(g_{\alpha \bar{\beta}}(x, 0))}$, since $-\partial_{\alpha} \partial_{\bar{\beta}} \log det(g_{\alpha \bar{\beta}}(x, t)) = R_{\alpha \bar{\beta}}(x, t) - R(x, 0)$, after taking trace with $g_{\alpha \bar{\beta}}(x, 0)$, we get

\[
R(x, 0) = \Delta F(x, t) + g^{\alpha \bar{\beta}}(x, 0)R_{\alpha \bar{\beta}}(x, t)
\]

(2.4)

where $\Delta$ is the Laplace operator of the metric $g_{\alpha \bar{\beta}}(x, 0)$.

For any fixed $x_0 \in M$ and any $\alpha > 0$, we denote

\[
\Omega_{\alpha} = \{x \in M | G(x_0, x) \geq \alpha\}
\]

combining (2.1) with Shi’s technique on $M \times C^2(M)$, it is not hard to see that there exist a number $d(\alpha) \geq 1$ such that

\[
\frac{d(\alpha)^2}{V(x_0, d(\alpha))} = \alpha
\]

(2.5)

and a constant $C > 0$ such that

\[
B(x_0, C^{-1}d(\alpha)) \subset \Omega_{\alpha} \subset B(x_0, Cd(\alpha))
\]

(2.6)

Recall that $F(x, t)$ evolves by

\[
\frac{\partial F(x, t)}{\partial t} = -R(x, t)
\]

Combining with (2.3), we obtain

\[
0 \geq F(x, t) \geq -C \log(1 + t)
\]

(2.8)

Multiplying (2.7) by $G(x_0, x) - \alpha$ and integrating over $\Omega_{\alpha}$, we have
From the coarea formula, we have that
\[
(2.9) \quad \int_{\Omega_\alpha} R(x, 0)(G(x_0, x) - \alpha)\,dx
\]

\[
= \int_{\Omega_\alpha} (\Delta F(x, t))(G(x_0, x) - \alpha)\,dx + \int_{\Omega_\alpha} g^{\alpha\beta}(x, 0)R_{\alpha\beta}(x, t)(G(x_0, x) - \alpha)\,dx
\]

\[
= -\int_{\partial\Omega_\alpha} F(x, t)\frac{\partial G}{\partial \nu} \,d\sigma - F(x_0, t) + \int_{\Omega_\alpha} g^{\alpha\beta}(x, 0)R_{\alpha\beta}(x, t)(G(x_0, x) - \alpha)\,dx
\]

\[
\leq C(1 + \int_{\partial\Omega_\alpha} \frac{\partial G}{\partial \nu} \,d\sigma \log(1 + t) + \int_{\Omega_\alpha} g^{\alpha\beta}(x, 0)R_{\alpha\beta}(x, t)G(x_0, x)\,dx
\]

Here we use (2.8) and denote \(C \in C^2\).

Integrating (2.9) and denote \(\nu\) by the outer unit normal of \(\partial\Omega_\alpha\).

Integrating (2.9) from \(\alpha\) to \(2\alpha\) and multiplying by \(\frac{1}{\alpha}\), we get

\[
(2.10) \quad \int_{\Omega_{2\alpha}} R(x, 0)(G(x_0, x) - 2\alpha)\,dx
\]

\[
\leq C(1 + \frac{1}{\alpha} \int_{\alpha}^{2\alpha} \int_{\partial\Omega} \frac{\partial G}{\partial \nu} \,d\sigma d\gamma \log(1 + t) + \int_{\Omega_\alpha} g^{\alpha\beta}(x, 0)R_{\alpha\beta}(x, t)G(x_0, x)\,dx
\]

It is easy to see that

\[
(2.11) \quad \frac{1}{2} \int_{\Omega_{2\alpha}} R(x, 0)G(x_0, x)\,dx \leq \int_{\Omega_{2\alpha}} R(x, 0)(G(x_0, x) - 2\alpha)\,dx
\]

Using Shi’s technique on \(M \times C^2\) and combining (2.1) with (2.5), we obtain that there exists a constant \(C > 0\) such that

\[
(2.12) \quad C^{-1}d(\alpha) \leq d(\gamma) \leq Cd(\alpha), \alpha \leq \gamma \leq 2\alpha.
\]

From the coarea formula, we have that

\[
(2.13) \quad d\sigma d\gamma = \frac{\partial G(x_0, x)}{\partial \nu} \,d\sigma d\nu = |\frac{\partial G(x_0, x)}{\partial \nu}| \,d\sigma |d\nu| = |\frac{\partial G(x_0, x)}{\partial \nu}| \,d\nu
\]

Combining (2.2), (2.5), (2.6), (2.12), (2.13) and the standard volume comparison, we have that

\[
(2.14) \quad \frac{1}{\alpha} \int_{\alpha}^{2\alpha} \int_{\partial\Omega} \frac{\partial G(x_0, x)}{\partial \nu} \,d\sigma d\gamma \leq \frac{1}{\alpha} \int_{\alpha}^{2\alpha} \int_{\partial\Omega} |\frac{\partial G(x_0, x)}{\partial \nu}|^2 \,dx
\]

\[
\leq \frac{C}{\alpha} \int_{\alpha}^{2\alpha} \int_{\partial\Omega} \left(\frac{d(\gamma)}{V(x_0, d(\gamma))}\right)^2 \,dx \leq \frac{Cd(\alpha)^2}{\alpha(V(x_0, C^{-1}d(\alpha)))^2} \int_{\Omega_\alpha \setminus \Omega_{2\alpha}} \,dx
\]

\[
\leq \frac{Cd(\alpha)^2}{\alpha(V(x_0, C^{-1}d(\alpha)))^2} V(x_0, Cd(\alpha)) \leq C
\]

Substituting (2.14) and (2.11) into (2.10), we get that

\[
\int_{\Omega_{2\alpha}} R(x, 0)G(x_0, x)\,dx \leq Clog(1 + t) + \int_{\Omega_\alpha} g^{\alpha\beta}(x, 0)R_{\alpha\beta}(x, t)G(x_0, x)\,dx
\]

Integrating this inequality from 0 to \(t\), we see that for any \(t > 0\),

\[
\int_{\Omega_{2\alpha}} R(x, 0)G(x_0, x)\,dx \leq Clog(1 + t) + \frac{1}{t} \int_{\Omega_\alpha} g^{\alpha\beta}(x, 0)R_{\alpha\beta}(x, t)G(x_0, x)\,dx
\]

\[
\leq Clog(1 + t) + \frac{C}{t} \int_{\Omega_\alpha} G(x_0, x)\,dx
\]
Finally, combining this inequality with (2.1) and (2.6), we have that there exists a constant $C > 0$ such that for $\forall x_0 \in M, t > 0$ and $r > 0$

$$
\int_{B(x_0, r)} R(x, 0) \frac{d(x_0, x)}{V(x_0, d(x_0, x))} dx \leq C (\log(1 + t) + \frac{r^2}{t})
$$

Choose $t = r^2$, we can prove Theorem 2.2.

\begin{lemma}
\textbf{Lemma 2.3. (a):} Let $M$ be above assumption, then the Poisson equation $\Delta u(x) = R(x)$ has a solution with $\sup_{x \in B(x_0, r)} |u(x)| \leq C \log(2 + r)$ for a constant $C > 0, \forall r > 0$, if and only if

$$
\int_0^t sk(x, s) ds \leq C' \log(2 + t)
$$

for a constant $C' > 0, \forall t \geq \frac{1}{5} r$.

In the following, we prove the necessary part.

\begin{proof}
From Lemma 2.3, it is sufficient to prove that the Poisson equation $\Delta u(x) = R(x)$ has a solution with $\sup_{x \in B(x_0, r)} |u(x)| \leq C \log(2 + r)$. In fact, if $t \geq \frac{1}{5} r$, then from Lemma 2.3 we can conclude the necessary condition. If $t \leq \frac{1}{5} r$, since the scalar curvature is bounded, then $\int_0^t sk(x, s) ds \leq C'$, so there always exists a constant $C' > 0$ such that $\int_0^t sk(x, s) ds \leq C' \log(2 + t)$ for all $t \geq 0$.

To solve the poisson equation, we first construct a family of approximate solution $u_r$ as follows.

For a fixed $x_0 \in M$ and $\forall r > 0$, define $u_r(x)$ on $B(x_0, r)$ by

$$
u_r(x) = \int_{B(x_0, r)} (G(x_0, y) - G(x, y)) R(y) dy$$

It is clear that $u_r(x_0) = 0$, and $\Delta u_r(x) = R(x)$ on $B(x_0, r)$. For $x \in B(x_0, \frac{r}{4})$, we write

$$
u_r(x) = \left( \int_{B(x_0, r) \setminus B(x_0, 4d(x_0, x))} + \int_{B(x_0, 4d(x_0, x))} \right) (G(x_0, y) - G(x, y)) R(y) dy = I_1 + I_2$$

From Theorem 2.2, we see that

\begin{equation}
|I_2| \leq C \log(2 + d(x_0, x_0)) \text{ on } B(x_0, \frac{r}{4})
\end{equation}

From (2.2), we have that for $y \in B(x_0, r) \setminus 4d(x_0, x)$,

$$|G(x_0, y) - G(x, y)| \leq d(x, x_0) \sup_{z \in B(x_0, d(x_0, x_0))} |\nabla_z G(z, y)|$$

\begin{align*}
&\leq C d(x, x_0) \sup_{z \in B(x_0, d(x_0, x_0))} \frac{d(z, y)}{V(z, d(z, y))} \\
&\leq C \frac{d(x, x_0) d(x_0, y)}{V(x_0, d(x_0, y))}
\end{align*}

Combining this with Theorem 2.2, we have that:

\begin{equation}
|I_1| \leq C d(x, x_0) \int_{B(x_0, r) \setminus 4d(x_0, x)} \frac{R(y) d(x_0, y)}{V(x_0, d(x_0, y))} dy
\end{equation}

\end{proof}
\[ \leq C d(x_0, x) \sum_{k=2}^{\infty} \frac{1}{2^k d(x, x_0)} \int_{B(x_0, 2^k d(x, x_0)) \setminus B(x_0, 2^k d(x, x_0))} \frac{R(y) d(x_0, y)^2}{V(x_0, d(x, y))} dy \]

\[ \leq C \sum_{k=2}^{\infty} \frac{1}{2^k} \log(2 + 2^{k+1} d(x, x_0)) \]

\[ \leq C \log(2 + d(x, x_0)) \]

Hence, from (2.15) and (2.16), we deduce that

\[ |u_r(x)| \leq C \log(2 + d(x, x_0)) \quad \text{for} \quad r \geq 4 d(x_0, x). \]

Therefore, it follows from the Schauder theory of elliptic equations that there exists a sequence of \( r_j \to +\infty \) such that \( u_{r_j}(x) \) converges uniformly on compact subset of \( M \) to a smooth function \( u \) satisfying

\[ \begin{aligned} &u(x_0) = 0 \text{ and } \Delta u = R \text{ on } M \\ &|u(x)| \leq C \log(2 + d(x, x_0)), \text{ for } x \in M \end{aligned} \]

Thus we proved the necessary part of Theorem 1.1. \( \square \)

3. The sufficient condition of the solution

**Lemma 3.1.** ([7]): Let \( M \) be above assumption, if there exists a constant \( C > 0, \varepsilon > 0, \) such that

\[ k(x_0, r) \leq \frac{C}{(1 + r)^{1+\varepsilon}}, \quad \text{for } x_0 \in M, \quad \forall r \geq 0 \]

then the Ricci flow (1.1) has long time existence, \( R(x, t) \) is nonnegative and bounded.

**Lemma 3.2.** ([2]): Let \( M \) be above assumption, then there exists a constant \( C > 0, \) such that for all \( (x_0, t) \in M \times [0, +\infty) \)

\[ -F(x_0, t) \leq C[(1 + \frac{t(1-m(t))}{R^2}) \int_{0}^{R} sk(x_0, s) ds - \frac{tm(t)(1-m(t))}{R^2}] \]

where \( m(t) = \inf_{x \in M} F(x, t) \)

Now we can use Lemma 3.1 and Lemma 3.2 to prove the sufficient part.

**Proof.** From the sufficient condition of Theorem 1.1, we have

\[ \int_{0}^{r} sk(x, s) ds \leq C \log(2 + r) \]

On the other hand,

\[ \int_{0}^{r} sk(x, s) ds \geq \int_{0}^{r} \frac{s}{V(x, s)} \int_{B(x, s)} R(y) dy ds \]

\[ \geq \int \frac{1}{V(x, r)} \int_{\frac{r}{2}}^{r} s \int_{B(x, s)} R(y) dy ds \]

\[ \geq \frac{3r^2}{V(x, r)} \int_{B(x, \frac{r}{2})} R(y) dy \]
From the standard volume comparison, we have that there exists a constant $C$ and $\varepsilon > 0$ such that $$k(x_0, r) \leq \frac{C}{(1 + r)^1 + \varepsilon}, \text{ for } x_0 \in M, \forall r \geq 0.$$ Thus from Lemma 3.1, we know that the Ricci flow has long time solution.

From Lemma 3.2, we have $$-m(t) \leq C\left[(1 + t) R^2 \right] \int_0^2 sk(x_0, s) ds - \frac{tm(t)(1 - m(t))}{R^2},$$ Let $R^2 = 2Ct(1 - m(t))$, from the sufficient condition, we have

(3.1) $$-m(t) \leq C\log(2 + 2R)$$

Since $\frac{\partial F(x, t)}{\partial t} = -R(x, t)$, $R(x, t)$ is bounded and $F(x, 0) = 0$, then $-F(x, t) = \int_0^t R(x, s) ds \leq Ct$, for all $x \in M, t \geq 0$; So

(3.2) $$-m(t) \leq Ct$$

from (3.2), we have $$R^2 = 2Ct(1 - m(t)) \leq Ct^2$$

(3.3) $$R \leq Ct$$

Substituting (3.3) into (3.1), we get

(3.4) $$-m(t) \leq C\log(2 + t)$$

By Li-Yau-Hamilton type inequality [8]: $\frac{\partial R}{\partial t} + \frac{R}{t} \geq 0$, we know $\frac{\partial R(x, t)}{\partial t} \geq 0$, then for all $t \geq T$, we have

$$TR(x, T) \leq tR(x, t)$$

$$TR(x, T) \frac{1}{t} \leq R(x, t)$$

$$\int_T^t TR(x, T) \frac{1}{s} ds \leq \int_T^t R(x, s) ds \leq -F(x, t) \leq C\log(2 + t)$$

$$TR(x, T)(\log t - \log T) \leq C\log(2 + t)$$

Let $t \to +\infty$, we see that $R(x, T) \leq \frac{C}{t}$, for some constant $C > 0$, all $T \geq 0$.

So the Ricci flow has non-ancient solution. \[ \square \]

**Corollary 3.3.** Let $M$ be above assumption, if $\int_0^r sk(x, s) ds \leq C\log(2 + r)$ for all $x \in M$, $r \geq 0$, and $V(x, r) \geq C r^{2n}$ for some constants $C > 0$, $r \geq 0$, then $M$ is diffeomorphic to $R^{2n}$ in case $n \geq 3$.

**Proof.** From Theorem 1.1, we know that the Ricci flow has long time existence and $0 \leq R(r, t) \leq \frac{C}{t}$, $\forall r \geq 0$. Let $t \to +\infty$, we have $R(x, t) \to +\infty$, it means that the Ricci flow will improves the injectivity radius to $\infty$ along the flow. The rest argument is the same as in section 3 of \[4\]. \[ \square \]
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