Electromagnetic space-time crystals. III. Dispersion relations for partial solutions

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Partial solutions of the Dirac equation describing an electron motion in electromagnetic crystals created by plane waves with linear and circular polarizations are treated. It is shown that the electromagnetic crystal formed by circularly polarized waves possesses the spin birefringence.

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I. INTRODUCTION

In the band theory of solids, the substitution of the Bloch function into the single-electron steady-state Shr"{o}dinger equation, due to the periodic nature of a crystal lattice, results in an infinite system of linear homogeneous equations relating scalar Fourier amplitudes of this wave function [1, 2]. The existence condition of nontrivial solutions of the system provides the dispersion relation for the wave function \( \Psi(x) \) in an energy band. Because the system is infinite, the dispersion relation is derived by a method of successive approximations [1, 2]. A similar approach is applied in [3] to find the band structure of the fundamental solution to an infinite set of independent finite systems of interrelated equations [fractal clusters of equations (FCE)]. It can be described as a 4d lattice of such clusters. The aggregation scheme for FCE, presented in [3], is devised to simplify computations and to minimize volumes of data files at calculating the corresponding projection operators. It makes possible to expand FCEs to finite models of ESTC [4] of any desired size and to obtain families of approximate partial solutions of the Dirac equation. To compare in accuracy various approximate solutions of the Dirac equation, obtained in the framework of these models, we use the criterion suggested in [4], i.e., the relative residual \( R \) at the substitution of an approximate solution into the Dirac equation. The way in which its application reveals, in particular, dispersion relations is illustrated in this paper on several examples. We apply the general technique developed in the preceding papers [4, 5] to two types of electromagnetic space-time crystals, denoted ESTC1 and ESTC2, formed by plane waves with linear and circular polarizations, respectively. In section II we discuss in detail the interrelation between the free space solutions of the Dirac equation and the approximate solutions which describe an electron in ESTC at limiting process to the vanishing field. ESTC1 and ESTC2 are treated in sections III and IV, respectively. For the problem under consideration, the Dirac equation reduces to an infinite system of matrix equations, where the interconnections between equations are defined [4] by 12 matrix functions and 56 scalar coefficients. Appendix gives the expressions for them in an explicit form.

II. FREE SPACE SOLUTION AS A LIMIT CASE

In the absence of electromagnetic field, the Dirac equation for the wave function \( \Psi(x) = \exp(i K \cdot x) c_0 \) reduces to the form [see Eq. (37) in [4] with \( n = n_0 = (0, 0, 0) \)]:

\[
P_0 c_0 = 0, \quad P_0 = \frac{1}{2} U - \frac{1}{2q_4} \left( \alpha_4 + \sum_{k=1}^{3} q_k \alpha_k \right).
\]

(1)

Here, \( \Psi \) and \( c_0 \) are the bispinors, \( x = (r, i ct) \), \( K = (k, i\omega/c) \) is the four-dimensional wave vector, \( U \) is the unit \( 4 \times 4 \) matrix, \( \alpha_j \) are the Dirac matrices, and we use
dimensionless parameters
\[ Q = (q, iq_4) = K/\kappa_c, \]
\[ q = q_1 e_1 + q_2 e_2 + q_3 e_3 = \frac{h k}{m_e c}, \quad q_4 = \frac{\hbar \omega}{m_e c^2}, \]  
where \( \kappa_c = m_c c/\hbar, \) \( c \) is the speed of light in vacuum, \( h \) is the Planck constant, \( e \) is the electron charge, \( m_e \) is the electron rest mass. The existence condition \( |P_0| = 0 \) of a nontrivial solution \( c_0 \neq 0 \) results in the dispersion relation
\[ q_4^2 = 1 + q_4^2, \]  
which, in terms of the energy \( E = m_e c^2 q_4 = \hbar \omega \) and the momentum \( p = m_e c q = \hbar k \), takes the familiar form [6]
\[ E^2 = c^2 (m_e c^2 + p^2). \]

Once this condition is satisfied, i.e.,
\[ q_4 = \pm q_{40}, \quad q_{40} = \sqrt{1 - q_4^2}, \]  
Eq. (11) splits into two independent equations for positive \( (q_4 = q_{40}) \) and negative \( (q_4 = -q_{40}) \) frequency domains
\[ P_+ c_+ = 0, \quad P_- c_- = 0, \]  
where
\[ P \pm = \frac{1}{2}U \pm \frac{1}{2q_{40}} \left( \alpha_4 + \sum_{\kappa=1}^{3} q_\kappa \alpha_\kappa \right) \]  
are the Hermitian projection matrices specifying the two-dimensional subspaces of solutions at these domains \( c_\pm = P_\pm c_0 \) for any \( c_0 \) and satisfying the relations
\[ P_+ \dagger = P_+^2 = P_\pm, \quad P_\pm P_\mp = 0, \]
\[ P_+ + P_- = U, \quad tr(P_\pm) = 2. \]  

It should be emphasized that, in the case of a nonvanishing field, all projection operators \( \rho_0(n) = P(n) \) (see Eq. (16) in [4]) have the trace \( tr[P(n)] = 4 \), and the fundamental solution \( S \) is obtained in [4] without recourse to any dispersion relation. To explain the interrelation between the two problems, it is sufficient to assume that the potential of the electromagnetic field is small and to use the most simple finite model of ESTC, 0-model described in [5]. In the frame of this model, we obtain the following relations:
\[ S' = U - \rho_0(n_0), \]  
\[ S(n) = U \delta(n - n_0) - R_0(n, n_0, n_0), \]
\[ n = \{n_1, n_2, n_3, n_4\} \in S_{13}, \]  
\[ \Psi(x) = \sum_{n \in S_{13}} c(n) e^{i\varphi_n(x)}, \quad c(n) = S(n)c_0, \]  
\[ \varphi_n(x) = |K + G(n)| \cdot x \]
\[ = (k + k_0 n) \cdot r - (\omega + \omega_0 n_4)t \]
\[ = 2\pi[(n + q/\Omega) \cdot r' - (n_4 + q_4/\Omega) X_4], \]  
\[ \Omega = \frac{\hbar \omega_0}{m_e c^2}, \]  
where \( S' \) is the fundamental solution of equation \( P(n_0)c_0 = 0, \) \( U \) is the unit operator, \( \delta(n - n_0) \) is the Kronecker delta, matrix \( R_0(n, n_0, n_0) \) is defined in [4], \( n = n_1 e_1 + n_2 e_2 + n_3 e_3, \omega \) is the frequency of electromagnetic field, \( k_0 = \omega_0/c = 2\pi/\lambda_0 \) is the wave number, \( r' = r/\lambda_0, X_4 = c t/\lambda_0 \) are the dimensionless coordinates. The spectral expansion of the matrix \( S(n_0) \) has the form
\[ S(n_0) = S_+ P_+ + S_- P_- , \]  
where
\[ S_\pm = \frac{I_A}{I_A + (q_4 + q_{40})^2} \]  
are the eigenvalues, \( P_\pm \) are given by Eq. (17), and the parameter \( I_A \) specifies the intensity of the electromagnetic field creating ESTC (see Eq. (21) in [4]).

Let us consider the family \( \Psi(x, c_0, q_4) \) of functions \( \Psi(11) \) at given vector \( q_4 \). First, as the initial approximation, called below 0'-model, we treat its truncated form
\[ \Psi'(x, c_0, q_4) = e^{iK \cdot x} S(n_0)c_0. \]  
Then Eqs. (40)–(43) in [5] give
\[ R(c_0, q_4) = \sqrt{I_A + (q_4 + q_{40})^2} \]  
If \( c_0 = c_4 \pm \) is an eigenvector of \( S(n_0) \), Eq. (16) reduces to the relation
\[ R_\pm(q_4) = \sqrt{I_A + (q_4 + q_{40})^2} \]  
which, at \( q_4 = \pm q_{40} \), gives
\[ R_0 = R_+(q_{40}) = R_-(q_4) = \sqrt{I_A}, \]
\[ R_+(-q_40) = R_-(q_{40}) = \sqrt{I_A + 4q_{40}^2} \]
\[ = 2\sqrt{1 + q_4^2} + I_A/4. \]  
Thus, in this approximation, the free space solutions \( \Psi'(x, c_\pm, \pm q_{40}) \) provide the minimum value \( R_0 \) for the relative residual parameter \( R \). Let now \( I_A \) tends to zero. The function \( \Psi' \) can be treated as an approximate solution if, and only if \( R(c_0, q_4) \ll 1 \), i.e., \( c = c_+ \) and \( q_4 \ll 1 \), alternatively, \( c = c_- \) and \( q_4 + q_{40} \ll 1 \). If \( |q_4| \neq q_{40} \), one obtains only the trivial solution \( S(n_0) = 0 \) as the limiting case at \( I_A \to 0 \). The two physically relevant exact solutions, described by
\[ S_+ = 1, \quad S_- = 0, \quad S(n_0) = P_+, \quad R_+(q_4) = 0, \]  
\[ S_+ = 0, \quad S_- = 1, \quad S(n_0) = P_-, \quad R_-(q_4) = 0, \]  
\[ S_+ = \frac{I_A}{I_A + (q_4 + q_{40})^2} \]  
are the eigenvalues, \( P_\pm \) are given by Eq. (17), and the parameter \( I_A \) specifies the intensity of the electromagnetic field creating ESTC (see Eq. (21) in [4]).
arise as limiting cases \((I_A \to 0)\) of Eqs. (13), (14) and (17) at \(q_4 = q_{40} \) and \(q_4 = -q_{40} \), respectively.

To analyze the solution \(\Psi(x)\) for dependence on the amplitude \(c_0\), one can use any basis of the four-dimensional bispinor space. In particular, it is convenient to use the orthonormal basis

\[
\begin{align*}
c_1 &= \frac{1}{\delta} \begin{pmatrix} 1 + q_40 \\ q_3 \\ q_1 + iq_2 \\ 0 \end{pmatrix}, \\
c_2 &= \frac{1}{\delta} \begin{pmatrix} 1 + q_40 \\ q_1 - iq_2 \\ -q_3 \\ 0 \end{pmatrix}, \\
c_3 &= \frac{1}{\delta} \begin{pmatrix} q_3 \\ q_1 + iq_2 \\ -1 + q_40 \\ 0 \end{pmatrix}, \\
c_4 &= \frac{1}{\delta} \begin{pmatrix} q_1 - iq_2 \\ -q_3 \\ -1 + q_40 \\ 0 \end{pmatrix},
\end{align*}
\]

where \(\delta = \sqrt{2q_{40}(1 + q_{40})}\), and \(P_{\pm}\) can be written as

\[
P_+ = c_1 \otimes c_1^\dagger + c_2 \otimes c_2^\dagger, \quad P_- = c_3 \otimes c_3^\dagger + c_4 \otimes c_4^\dagger.
\]

The notations \(c_+\) and \(c_-\) denote below any orthonormal bispinors from the two-dimensional subspaces defined by the projection matrices \(P_+\) and \(P_-\), respectively, i.e., \(P_\pm c_\pm = c_\pm, c_\pm \otimes c_\mp = 1, c_\pm^\dagger c_\mp = 0\).

Let us now take into account all 13 Fourier amplitudes \(c(n)\) of \(\Psi(11)\). As example, we treat here ESTC1 composed of six linearly polarized waves with the amplitudes (see Eq. (2) in [4])

\[
\begin{align*}
A_1 &= -A_4 = A_m e_2, \\
A_2 &= -A_5 = A_m e_3, \\
A_3 &= -A_6 = A_m e_1,
\end{align*}
\]

where \(A_m\) is a real scalar amplitude, \(I_A = 12A_m^2\). In this numerical example, we assume \(\Omega = 0.1, q_1 = q_2 = 0, q_3 = 0.02\) [see Eqs. (2) and (12)]. At given \(c_0 = c_{\pm}\), \(c_j \) and \(c_0 = c_{\pm}\), Eq. (43) in [3] and Eq. (11) give functions \(\mathcal{R}(c_j, q_4)\) and \(\mathcal{R}(c_{\pm}, q_4)\).

It follows from the results of our numerical evaluations that in the vicinity of \(q_{40}\) (see Fig. 1) \(\mathcal{R}(c_+, q_4) = \mathcal{R}(c_-, q_4) \approx \mathcal{R}(c_+, q_4) \approx \mathcal{R}(c_-, q_4) \ll 1\), whereas

\[
\mathcal{R}_-(q_4) \equiv \mathcal{R}(c_-, q_4) = \mathcal{R}(c_+, q_4) = \mathcal{R}(c_-, q_4) > 0.143 \quad \text{i.e., in the positive frequency domain} \quad |q_4 - q_{40}| \ll 1
\]

the set of approximate solutions with the best accuracy has the two-dimensional amplitude subspace defined by the projection matrix \(P_+\). By contrast, at \(|q_4 - q_{40}| \ll 1\) the projection matrix \(P_-\) specifies the amplitude subspace, because in this negative frequency domain \(\mathcal{R}_-(q_4) \ll 1\), but \(\mathcal{R}_+(q_4) > 0.143\). The graphic illustration of \(\mathcal{R}_-(q_4)\) for \(q_4 < 0\) can be obtained by the transformation \(\mathcal{R}_+(q_4) = \mathcal{R}_-(q_4)\).

Thus, for both \(\Psi(11)\) and \(\Psi'(15)\) the amplitude subspace remain the same as for the free space solution. Nonetheless, sharp distinctions do exist. In free space, at any given \(q\), there is the discrete spectrum of \(q_4\) values, namely, \(q_{40}\) and \(-q_{40}\) [5]. In the electromagnetic crystal under consideration, it is replaced by the continuous spectrum with two narrow domains in the vicinity of \(\pm q_{40}\), which specify the family of approximate partial solutions with reasonable exactness. The rough initial approximation \(\Psi'\) is sufficient to obtain the free space solution with its major features, the dispersion relation and the two-dimensional amplitude subspace, as the limiting case of vanishing field \(I_A \to 0\). However, the function \(\Psi\) provides a more accurate and detailed description of this limiting process (see Fig. 1). The described above solutions domains with small values of \(\mathcal{R}(c_{\pm}, q_4)\) are very narrow and can be conveniently described in terms of the small variable

\[
\xi = q_4 - q_{40} = \frac{\hbar \omega}{me^2} - \sqrt{1 + \left(\frac{\hbar k}{mc}\right)^2}
\]

at \(q_4 > 0\) (see Fig. 1) and \(\xi = q_4 + q_{40}\) at \(q_4 < 0\). For \(\Psi(11)\) at \(q_4 > 0\), the minimum value \(\mathcal{R}_0\) of \(\mathcal{R}_+\) and its position \(\xi_0\) at the \(\xi\)-axis can be evaluated as

\[
\mathcal{R}_0 \approx 0.25A_m, \quad \xi_0 \approx 0.5I_A,
\]

where \(I_A = 12A_m^2\), for ESTC1, and the width of the \(\xi\)-domain satisfying the condition \(\mathcal{R}_+(\xi) \leq \sqrt{\xi}A\) is approximately equal \(I_A\).

III. SPECTRAL CURVE OF APPROXIMATE SOLUTIONS

Let us considerably enhance the amplitude \(A_m\) (23), in comparison to the values treated above, up to the value \(A_m = 5 \times 10^{-4}\) \((I_A = 3 \times 10^{-6})\). In this case, it is necessary to use more elaborate finite p-models of ESTC1 described in [3]. As before, we assume \(\Omega = 0.1, q_1 = q_2 = 0, q_3 = 0.02\) and treat families of functions [3]

\[
\Psi(x) = \sum_{n \in S_d} c(n)e^{i\varphi_n(x)} = \sum_{n \in S_d} e^{i\varphi_n(x)} S(n)c_0
\]
with different values of \( c_0 = c_j, j = 1, 2, \ldots \), where \( S_d \) is the solution domain, i.e., the subset of \( \mathcal{L} \) with nonzero matrices \( S(n) \). However, instead of the basis \( c_j \), we use below the generalized eigenvectors \( c_j \) defined by the equation

\[
U_D c_j = \lambda_j U_E c_j, \tag{27}
\]

where \( U_E \) and \( U_D \) are the Hermitian \( 4 \times 4 \) matrices which define the relative residual \( \mathcal{R} \) as follows (see Eqs. (26), (40) and (43) in [2])

\[
\mathcal{R} = \sqrt{\frac{\xi^2}{c_j^2 U_D c_0}}. \tag{28}
\]

In the case under consideration, the quartic equation

\[
\det(U_D - \lambda U_E) = 0, \tag{29}
\]

which specifies the generalized eigenvalues \( \lambda_j \), has real coefficients and twofold positive roots \( \lambda_1 \) and \( \lambda_2 \), indexed in increasing order of magnitude. The corresponding two-dimensional subspaces of generalized eigenvectors, i.e., bispinor amplitudes \( c_1 \) and \( c_2 \), are defined by the Hermitian projection matrices

\[
\rho_j = U - 2 \frac{U_D - \lambda_j U_E}{\text{tr}(U_D - \lambda_j U_E)}, \quad j = 1, 2. \tag{30}
\]

In the case that the amplitude \( c_0 \) satisfies the condition \( \rho_j c_0 = c_0 \), Eq. (28) gives \( \mathcal{R} = \mathcal{R}_1 \equiv \mathcal{R}_0 \). If \( \mathcal{R}_2 \approx \mathcal{R}_1 \) at any value of \( \xi \). The graphical representation of \( \mathcal{R} = \mathcal{R}_1(\xi) \) will be denoted the spectral curve of approximate solutions. The minimum \( \{\xi_0, \mathcal{R}_0 = \mathcal{R}_1(\xi_0)\} \) of this curve specifies the most accurate approximate solution available in the frame of \( p \)-model under consideration.

The bottom of curve \( \mathcal{R} = \mathcal{R}_1(\xi) \), which is similar to the solid curves depicted in Fig. [3] can be approximated as follows (see the dash curve in Fig. [2])

\[
\mathcal{R}_1^{ap}(\xi) = \sqrt{\mathcal{R}_2^2 + \beta_0^2(\xi - \xi_0)^2}, \tag{31}
\]

where the values of \( \xi_0, \mathcal{R}_0 \) and \( \beta_0 \) for the \( p \)-models applied in this paper are presented in Table [1] for the positive frequency domain \( q_1 > 0 \). This relation gives a rather close approximation of \( \mathcal{R}_1(\xi) \), for example, in 1-model illustrated in Fig. [2] \( \mathcal{R}_1^{ap}(\xi)/\mathcal{R}_1(\xi) - 1 < 0.012 \) at the domain \( 5 \times 10^{-8} < \xi < 7 \times 10^{-8} \), where \( \mathcal{R}_1(\xi) < 0.188 \). Outside this bottom domain the dependence \( \mathcal{R}_1(\xi) \) gradually becomes weak. Let \( \mathcal{R}_0, \mathcal{R}_0, \mathcal{R}_0, \mathcal{R}_0 \), be an available (in \( p \)-model) level of the relative residual \( \mathcal{R} \) and \( \mathcal{R}_0, \mathcal{R}_0, \mathcal{R}_0 \), \( \mathcal{R}_0 \) 1. The half-width \( \delta \xi(\mathcal{R}_0) \) of the solution line, i.e., the

![FIG. 2. The functions \( \mathcal{R} = \mathcal{R}_1(\xi) \) (solid curve) and \( \mathcal{R} = \mathcal{R}_1^{ap}(\xi) \) (dash curve) for 1-model of ESTC1 at \( A_m = 5 \times 10^{-4} \).](image)

This half-width is a rapidly decreasing function of \( p \). Table [1] presents its values \( \delta \xi = \delta \xi(\mathcal{R}_0) \) at \( \mathcal{R}_0 = \sqrt{T_1} = \sqrt{3} \times 10^{-3} \).

The projection matrix \( \rho_1 \) for ESTC1 is concisely defined by its Dirac set (see appendix in [4])

\[
D_s(\rho_1) = \{0.5, 0, 0, 0, 0.142825632163, 0, 0, 0, 0, -0.47903979645, 0.0040693990718, 0.00926170280712, -0.00527448410786, 0, 0, 0\}. \tag{33}
\]

For any bispinor \( c_a \), substituting \( c_0 = \rho_1 c_a \) in (26) gives a partial solution with the same value of relative residual: \( \mathcal{R} = \mathcal{R}_1 \). In particular, one can use the free space basis \( c_a = c_j \). This yields four different partial solutions with amplitudes \( c_j^{(1)} = \rho_1 c_j, j = 1, 2, 3, 4 \). Of course, only two of them are linearly independent. To compare mean values of Hamiltonian \( \langle H \rangle \), components of kinetic momentum \( \langle p_k \rangle \), probability current density \( \langle j_k \rangle \), and spin \( \langle S_k \rangle \) for these solutions, we substitute \( c_0 = c_j^{(1)} \) and the operators

\[
H = c \sum_{k=1}^{3} \alpha_k p_k + m_c c^2 \alpha_4, \quad p_k = -i\hbar \frac{\partial}{\partial x_k} - \frac{e}{c} A_k, \tag{34}
\]

\[
j_k = c \alpha_k, \quad S_k = \frac{\hbar}{2} \Sigma_k, \quad k = 1, 2, 3, \tag{35}
\]

in Eqs. (24)-(26) [3]. The calculations in the framework of 3-model result in the mean values

\[
\langle H \rangle = \langle H \rangle / m_c c^2 = 1.000201480083165, \tag{36}
\]

\[
\langle P \rangle = \langle p_3 \rangle / m_c c = 0.0200001996504673, \tag{37}
\]

\[
\langle a_3 \rangle = \langle j_3 \rangle / c = 0.0199959711969098, \tag{38}
\]
TABLE I. Parameters of spectral curves for $p$-models of ESTC1.

| $p$ | $\xi_0$ | $R_{1,2}$ | $\beta_0$ | $\delta\xi$ |
|-----|---------|-----------|----------|------------|
| 0   | 0       | 0.00173205| 2.00040  | 1.49871 $\times 10^{-6}$  |
| 2   | 1.492174147536518 $\times 10^{-6}$ | 0.000123775 | 0.143302 | 1152.74 | 757524 |
| 2   | 1.499705741630043 $\times 10^{-6}$ | 0.0000342977 | 2.02580 | 2.28602 $\times 10^{-9}$  |
| 3   | 1.499679217218709 $\times 10^{-6}$ | 1.00113 $\times 10^{-6}$ | 1.00000 | 4.15825 $\times 10^{-11}$  |
| 3   | 1.499679217930120 $\times 10^{-6}$ | 1.72762 $\times 10^{-9}$ | 0.902999 | 5.85538 $\times 10^{-14}$  |

which are the same for these four solutions. However, the solutions have different mean values of spin components

$$\langle \Sigma_j \rangle = \pm 6.4228848693 \times 10^{-11}$$

for $j = 1, 2$, 

$$= \pm 0.017300503410$$

for $j = 3, 4$, 

$$\langle \Sigma_j \rangle = \pm 1.4618089196 \times 10^{-10}$$

for $j = 1, 2$, 

$$= \pm 0.038232859850$$

for $j = 3, 4$, 

$$\langle \Sigma_j \rangle = \pm 0.9999980004$$

for $j = 1, 2$, 

$$= \pm 0.99911672861$$

for $j = 3, 4$. 

(39)

Mean values of $p_{1,2}$ and $j_{1,2}$ are negligibly small: $|\langle P_k \rangle|, |\langle \alpha_k \rangle| < 10^{-20}$ for $k = 1, 2$. Figure 3 illustrates the dependence of probability current density $j_3 = c\alpha_3\Psi^\dagger(x)\alpha_3\Psi(x)$ on the coordinates $X_3$ and $X_1$ at $X_1 = X_2 = 0$.

It follows from the above numerical results that $\xi_0$ converges to a positive limit and $R(\xi_0)$ tends to zero with increasing $p$, i.e., with expansion of a finite subsystem of equations described in [3]. In the limit, $\Psi$ converges to a family of exact solutions with the dispersion relation [see Eq. (24)]

$$\frac{\hbar\omega}{mc^2} = \xi_0 + \sqrt{1 + \left(\frac{\hbar k}{mc}\right)^2}$$

and the two-dimensional amplitude subspace defined by $\rho_1 = U - 2U_D/tr(U_D)$.

IV. ESTC COMPOSED OF CIRCULARLY POLARIZED WAVES: SPIN BIREFRINGENCE

In this section, we treat ESTC2 composed of six circularly polarized waves with the amplitudes (see Eqs. (2) and (4) in [3])

$$A_1 = A_4 = A_m(e_2 + ie_3)/\sqrt{2},$$

$$A_2 = A_5 = A_m(e_3 + ie_1)/\sqrt{2},$$

$$A_3 = A_6 = A_m(e_1 + ie_2)/\sqrt{2},$$

where $A_m = 5 \times 10^{-4}$ and $I_A = 3 \times 10^{-6}$ take the same values as in the case of EmCr1 treated above. The parameters $\Omega = 0.1, q_1 = q_2 = 0$ and $q_3 = 0.02$ are also retain their previous values, so that we change only the polarizations of electromagnetic waves from linear to circular. To study the properties of ESTC2, we apply $p$-models with $p = 0, 1, 2, 3$.

In the case of ESTC2, Eq. (29) has four different real positive roots: $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$, and the generalized eigenvectors $c_j$ satisfy the orthogonality relations

$$c_i^\dagger U_{Ej} = 0, c_i^\dagger U_{Dj} = 0, i \neq j, i, j = 1, 2, 3, 4.$$

The corresponding generalized one-dimensional eigensubspaces are uniquely defined by the Hermitian projection matrices (dyads)

$$\rho_j = \frac{c_j \otimes c_j^\dagger}{c_j^\dagger c_j} = \frac{D_j}{tr(D_j)}, j = 1, 2, 3, 4,$$

FIG. 3. The Hermitian form $\alpha_3\Psi^\dagger(x)\alpha_3\Psi(x)$ at $X_1 = X_2 = 0$ for ESTC1.

where $D_j = U_D - \lambda_j U_E, \overline{D}$ is the adjoint matrix, $D\overline{D} = D\overline{D} = |D|U$. In the limiting case, when $R(\xi_0)$ tends to zero with increasing $p$, $\rho_1 = \overline{U_D}/tr(U_D)$. It is significant that these matrices are the uniquely defined descriptors of the subspaces in contrast to basis elements $c_j$. This provides a convenient means to use real Dirac sets of Hermitian projection matrices for comparative analysis of subspaces.
The solution curve $\mathcal{R} = \mathcal{R}(\xi)$ in ESTC2 splits into two doublet lines called below line $a$ and line $b$ with minimum at $\xi_0 = \xi_{0a}$ and $\xi_0 = \xi_{0b}$, respectively, see Table III and Fig. 4. Although the doublet lines are very close, D-sets of $\rho_{1a} = \rho_1(\xi_{0a})$ and $\rho_{1b} = \rho_1(\xi_{0b})$ considerably differ from one another, in 3-model they are given by

$$D_s(\rho_{1a}) = \{0.25, 0.206495, -0.00688536, -0.00688574, -0.102352, -0.117887, 0.09093974, 0.09093979, 0.179608, -0.220245, -0.0419267, -0.0419338, -0.607254 \times 10^{-6}, 0.84274 \times 10^{-6}, 0.9094123, -0.0994089\},$$

$$D_s(\rho_{1b}) = \{0.25, 0.125694, 0.0236921, 0.023691, -0.119579, -0.200412, -0.0992794, -0.0992784, 0.0512099, -0.145645, 0.116162, 0.116167, -0.30381 \times 10^{-7}, 2.68718 \times 10^{-6}, -0.150963, 0.150959\}. $$

The finite $p$-models of ESTC2 with $p = 0, 1$ are qualified mainly for fast scanning of $\mathcal{R}(\xi)$ in preliminary search of minima. This is necessary because in the frame of $p$-models with $p = 2, 3$, which provide approximate solutions with much better accuracy, the solution domain width $\delta \xi$ becomes very small. Most important of all, $\mathcal{R}_1$ rapidly decreases whereas $\mathcal{R}_2$ increases with increasing $p$ for the both doublet lines, see Table III. Because of this, only line $a$ provides the solution at $\xi = \xi_{0a}$, whereas line $b$ provides the solution at $\xi = \xi_{0b}$.

Table III presents mean values of operators $H$, $p_\alpha$, $\alpha_\lambda$ and $\Sigma_k$ with respect to the functions $\Psi(x)$ with the amplitude $c_0$ satisfying the conditions $\rho_{1a}c_0 = c_0$ and $\rho_{1b}c_0 = c_0$ at $\xi = \xi_{0a}$ and $\xi = \xi_{0b}$, respectively, calculated for 3-model of ESTC2. The major difference between these two partial solutions for doublet lines manifests itself in spin projections (see Table III and Figs. 5 and 6). In other words, the electromagnetic crystal formed by circularly polarized waves possesses the spin birefringence. It reveals itself as the splitting of Eq. (40) into two dispersion relations with $\xi_0 = \xi_{0a}$ and $\xi_0 = \xi_{0b}$. For a given wave vector $k$ they provide frequencies $\omega_a$ and $\omega_b$, which specify two different partial solutions $\Psi(x)$ with one-dimensional amplitude subspaces defined by $\rho_{1a}$ and $\rho_{1b}$.

V. CONCLUDING REMARKS

The electromagnetic crystals is a family of periodic fields specified by complex vector amplitudes of six plane harmonic waves forming a crystal and the field frequency. These crystals have a specific impact on the motion of electrons, which may result in such interesting effects as spin birefringence. In this paper, we have restricted our consideration to the particular case with the fixed value.
of the wave vector $k$ in Eq. (12). Results of an investigation into the dependence of ESTCs properties on the magnitude and the direction of $k$, in particular, the energy band structure of ESTCs, will be discussed in our subsequent papers.

The fundamental solution of the Dirac equation and the techniques presented in this series of papers provide a means for detailed study of the electron motion in ESTCs. Some of these techniques, in particular, the method \[4\] for calculating the fundamental solution of a system of homogeneous linear equations, the fractal approach \[5\] to expansion of subsystems of equations at calculating approximate solutions, and the use of the relative residual $R_{\alpha}$ \[5\] at the comparative analysis of families of approximate solutions, may be also useful in solving other problems in mathematical and theoretical physics.

### Appendix

As we have shown in Ref. \[4\], the Dirac equation describing the motion of an electron in ESTC reduces to the infinite system of linear equations relating Fourier amplitudes $|\text{bispinors } c(n)|$ of the wave function $\Psi$. The interconnections of equations depend on complex vector amplitudes $A_j, j = 1, 2, \ldots, 6$ of six plane waves forming ESTC, for example, see Eqs. (23) and (41). These interconnections are described by 12 matrix functions $N_1(m,s)$ with $m,s \in L, g_{ad}(s) = 1$ and 56 scalar coefficients $N_2(s)$ with $g_{ad}(s) = 2$. The definitions of $N_1(m,s)$ and $N_2(s)$ are given in Ref. \[4\]. Here, we present these major structural parameters in the explicit form that is

### TABLE II. Parameters of spectral curves for $p$-models of ESTC2.

| $p$ | $\xi_{R_0}, \xi_{A_0}, (\xi_{R_0} - \xi_{A_0})$ | $R_{1, 2, 3, 4}$ | $\beta_0$ | $\delta_\xi$ |
|-----|----------------------------------|-----------------|------------|-------------|
| 0   | 1.451 475 281 655 971 $\times 10^{-6}$ | 0.000 110 130   | 1185.08    | 1.458 59 $\times 10^{-6}$ |
|     | $\delta_\xi$                      |                 |            |             |
|     | $\delta_\xi$                      |                 |            |             |
| 1   | 1.499 696 566 656 439 $\times 10^{-6}$ | 0.000 023 9446  | 1.034 61 $\times 10^6$ | 1.673 95 $\times 10^{-9}$ |
| 2   | 1.499 676 893 270 552 $\times 10^{-6}$ | 1.217 02 $\times 10^{-6}$ | 3.803 69 $\times 10^7$ | 4.553 61 $\times 10^{-11}$ |

### TABLE III. Mean values $\langle H \rangle$, $\langle P_k \rangle$, $\langle \alpha_k \rangle$ and $\langle \Sigma_k \rangle$, $k = 1, 2, 3$ for 3-model of ESTC2.

| quantity | line $a$ | line $b$ |
|----------|-----------|-----------|
| $\langle H \rangle$ | 1.900201393 528 065 | 1.900201366 634 309 |
| $\langle P_1 \rangle = \langle P_2 \rangle$ | 5.936 41978 888 $\times 10^{-10}$ | $-5.925 669 554 \times 10^{-10}$ |
| $\langle \alpha_1 \rangle = \langle \alpha_2 \rangle$ | 5.327 012 83 $\times 10^{-13}$ | 5.646 867 0 $\times 10^{-13}$ |
| $\langle \Sigma_1 \rangle$ | 0.589 431 208 749 64 | $-0.589 431 207 318 02$ |
| $\langle \Sigma_2 \rangle$ | 0.589 431 208 759 0 | $-0.589 431 207 319 07$ |
| $\langle \Sigma_3 \rangle$ | 0.593 586 130 989 99 | $-0.593 586 130 914 24$ |
necessary in any numerical implementation of the general techniques developed in Refs. [4, 5].

1. Dirac sets of matrices $N_1(m, s)$

We present $N_1(m, s)$ and $N_2(s)$ in order of the sequential numbering $i = 0, 1, \ldots$ of points $s = s(i) \in \mathcal{L}$ (see appendix in Ref. [5]). Let $A_{jk}$ be the Cartesian components of $A_j$, $m = (m_1, m_2, m_3, m_4) \in \mathcal{L}$, $w_k = q_k + m_k \Omega$ and $\Omega_\pm = \pm \Omega + 2w_4$. There are 12 points with $g_{4d}(s) = 1$. They are elements (from 2 to 13) of the list $S_{60}$ [6]. The Dirac sets (see appendix in [4]) of matrices $N_1[m, s(i)], i = 1, \ldots, 12$ have the form:

$$D_s\{N_1[m, (0, 0, 0, -1, -1)]\} =$$

$$\{ -2(A_{31}w_1 + A_{32}w_2), 0, iA_{32}\Omega, -iA_{31}\Omega, 0, 0, 0, 0, 0, 0, -A_{31}\Omega_+, -A_{32}\Omega_-, 0, 0, 0 \},$$

$$D_s\{N_1[m, (0, 0, 0, -1, -1)]\} =$$

$$\{ -2(A_{21}w_1 + A_{23}w_3), iA_{21}\Omega, -iA_{23}\Omega, 0, 0, 0, 0, 0, 0, -A_{21}\Omega_+, -A_{23}\Omega_-, 0, 0, 0 \},$$

$$D_s\{N_1[m, (-1, 0, 0, 0, 1)]\} =$$

$$\{ -2(A_{12}w_2 + A_{13}w_3), -iA_{12}\Omega, 0, iA_{13}\Omega, 0, 0, 0, 0, 0, 0, -A_{12}\Omega_+, -A_{13}\Omega_-, 0, 0, 0 \},$$

$$D_s\{N_1[m, (0, 1, 0, 0, -1)]\} =$$

$$\{ -2(A_{42}w_2 + A_{43}w_3), iA_{42}\Omega, 0, -iA_{43}\Omega, 0, 0, 0, 0, 0, 0, -A_{42}\Omega_+, -A_{43}\Omega_-, 0, 0, 0 \},$$

$$D_s\{N_1[m, (0, 1, 0, 0, -1)]\} =$$

$$\{ -2(A_{51}w_1 + A_{53}w_3), -iA_{51}\Omega, iA_{53}\Omega, 0, 0, 0, 0, 0, 0, 0, -A_{51}\Omega_+, -A_{53}\Omega_-, 0, 0, 0 \},$$

$$D_s\{N_1[m, (0, 0, 1, 0, -1)]\} =$$

$$\{ -2(A_{61}w_1 + A_{62}w_2), 0, -iA_{62}\Omega, iA_{61}\Omega, 0, 0, 0, 0, 0, 0, -A_{61}\Omega_+, -A_{62}\Omega_-, 0, 0, 0 \},$$

$$D_s\{N_1[m, (0, 0, 1, 0, -1)]\} =$$

$$\{ -2(A_{41}w_1 + A_{43}w_3), iA_{41}\Omega, -iA_{43}\Omega, 0, 0, 0, 0, 0, 0, 0, -A_{41}\Omega_+, -A_{43}\Omega_-, 0, 0, 0 \}.$$

2. Coefficients $N_2(s)$

There are 56 points $s = s(i) \in \mathcal{L}, i = 13, \ldots, 68$ with $g_{4d}(s) = 2$. They are elements (from 14 to 69) of the list $S_{60}$ [6]. The list of the coefficients $N_2(s)$ has the form

$$\{ N_2[s(i)], i = 13, \ldots, 68 \} =$$

$$\{ -2(A_{12}^* w_2 + A_{13}^* w_3), iA_{12}^* \Omega, -iA_{13}^* \Omega, 0, 0, 0, 0, 0, 0, -A_{13}^* \Omega_+, -A_{12}^* \Omega_-, 0, 0, 0 \},$$

$$\{ -2(A_{21}^* w_1 + A_{23}^* w_3), -iA_{21}^* \Omega, iA_{23}^* \Omega, 0, 0, 0, 0, 0, 0, -A_{23}^* \Omega_+, -A_{21}^* \Omega_-, 0, 0, 0 \},$$

$$\{ -2(A_{31}^* w_1 + A_{32}^* w_2), 0, -iA_{32}^* \Omega, iA_{31}^* \Omega, 0, 0, 0, 0, 0, 0, -A_{31}^* \Omega_+, -A_{32}^* \Omega_-, 0, 0, 0 \}.$$
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