Analyzing the equilibrium states of a quasi-neutral spatially inhomogeneous system of charges above a liquid dielectric film based on the first principles of quantum statistics

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Abstract

A theory of quasi-neutral equilibrium states of charges above a liquid dielectric surface is developed. This theory is based on the first principles of quantum statistics for systems comprising many identical particles. The proposed approach involves applying the variational principle, modified for the considered systems, and the Thomas–Fermi model. In the terms of the developed theory self-consistency equations are obtained. These equations provide the relation between the main parameters describing the system: the potential of the static electric field, the distribution function of charges and the surface profile of the liquid dielectric. The equations are used to study the phase transition in the system to a spatially periodic state. The proposed method can be applied in analyzing the properties of the phase transition in the system in relation to the spatially periodic states of wave type. Using the analytical and numerical methods, we perform a detailed study of the dependence of the critical parameters of such a phase transition on the thickness of the liquid dielectric film. Some stability criteria for the new asymmetric phase of the studied system are discussed.

Keywords: variation principle, phase transition, surface electrons, helium film, perturbation theory

(Some figures may appear in colour only in the online journal)
1. Introduction

Although they have been studied for over 40 years [1–4], phenomena concerned with the formation of spatially periodic states in a system of charged particles above a dielectric surface are still a relevant subject of research. The possibility of spatially periodic ordering in three-dimensional (3D) systems of charges (electrons in metals) was predicted by Wigner [5]. It is from this paper that the ‘Wigner crystallization’ (WC) term in the scientific literature was drawn. Many years after this was demonstrated, the phase transition in such system to a three-dimensional spatially periodic state formation could be predicted in another way [6]. For decades the experimental realization of WC in 3D systems failed. However, stable spatially periodic states of charges were experimentally realized in systems of electrons near the boundary separating two media. Examples of such systems are: electrons above the surface of a liquid helium film in an external clamping electric field [7, 8]; electrons on the surface of solid hydrogen and neon [9, 10]; as well as a system of electrons at the interface between the semiconductor p–n junction [11]. The detection of stable systems, convenient for experimental research of spatially periodic structures of charges near interfaces between media, greatly increased the interest in this kind of research. The chronology of research in this field is described in the books [12, 13] and review articles [14–16].

The available works describe the effects, concerning two-dimensional (2D) Wigner crystals [7], as macroscopic dimple lattices [8]. The theoretical works, describing the corresponding experiments, are usually based on the concept of the energy spectrum of a single (or ‘levitating’) electron above the dielectric surface. This concept considers a single electron located above flat dielectric surface, with its electrostatic image as an analogue of a hydrogen atom with the corresponding energy spectrum [17]. Obviously, in the case of many particle system, such an approach faces not only mathematical difficulties, but also the ‘philosophical’ or methodological ones. Let us recall that the electrostatic image method can be treated only as a mathematical ‘trick’ in the case of a single charge, that is ‘fixed’ above the metal or dielectric surface. This method allows one to avoid the consecutive solving of the Poisson equation. The mentioned difficulties vanish in the case of microscopic theory of the system described. Such a theory should consider the studied system as a quantum mechanical many-body system [18–21] and take into account the external electric clamping field. This field plays an important role in the formation of such systems, because the attracting field, generated by charges in the dielectric, is insufficient to hold them near the surface.

The basics of the development of such a microscopic theory were formulated in [18]. The method is based on the variation principle, modified for the considered system, and the Thomas–Fermi model. This variation principle takes into account the external electrostatic clamping field. This approach allows one to obtain self-consistent equations relating the parameters of the system description (the potential of the electric field, the distribution function of charges, and the surface profile of the liquid dielectric). As an application of the developed theory, here the authors study phase transitions to spatially periodic states in the system of charges above the surface of the liquid dielectric. The approach to solving the self-consistency equation system is outlined and the parameters of the phase transition are obtained. In addition, the period of the reciprocal lattice of spatially periodic structures is obtained. As conformation of the developed theory, its results are compared to experimental data [8] and qualitative agreement of the theoretical and experimental data was obtained.

Let us emphasize the essential feature of [18]. This paper was devoted to describing a system with a fixed number of charges above the dielectric surface (as was [20] as well). In other words, the system is considered to be charged, but not quasi-neutral. The system is charged in the case when the number of charges above the dielectric surface is not defined by the value
of the external clamping electric field [10, 11, 18–20]. This fact allows one to consider the clamping field and the number of charges as independent parameters during the study of the phase transitions in the system to spatially periodic states. Traditionally the system is considered to be quasi-neutral if the number of charges above the dielectric surface is not fixed and is determined by the external field. In this case the number of charges is exactly that needed to compensate the external clamping electrostatic field. For this reason the clamping field vanishes far from the dielectric surface [2–6, 12–16, 21]. Thus, among these two parameters—the number of charges and the external clamping field—the latter is the only external control parameter of the phase transition to spatially periodic states in a quasi-neutral system. This fundamental difference between charged and quasi-neutral systems leads to essentially different behavior of the system characteristics, see [3, 7, 10, 11]. Such a conclusion comes from the comparison of studies on charged [18–20] and quasi-neutral [21] systems. All these papers considered a non-degenerate gas of charges obeying Boltzmann statistics. In particular, the authors obtained the analytical expressions for the equilibrium distributions of charge density, electric field, and potential in the region above the dielectric surface. The dependencies of these expressions on the distance from the dielectric surface showed a much faster decay in the charged case compared to the quasi-neutral one. For example, the particle density of a charged system decreases exponentially [18] while the density of a quasi-neutral system decreases according to a power law [21]. Some of these features can only be predicted by describing the system using the microscopic theory approach (see [21]).

Experimental studies performed in both charged and quasi-neutral (in the above sense) systems are used. The current paper is motivated by the frequent experimental realization of the quasi-neutrality conditions, see [12–16]. The mentioned difference between the system types requires making substantial corrections to the charged system description [18] when we consider the quasi-neutral system. The basics of the necessary corrections were introduced in [21].

As in the case of charged systems [18, 20], the capabilities provided by the theory proposed in this paper are demonstrated through a study of the phase transition in quasi-neutral systems to the state with spatially periodic ordering. However, the main motivation of the current paper is to focus on the influence of liquid dielectric film thickness on the physical characteristics of such phase transitions. Moreover, the current approach provides the ability to describe the system beyond the scope of Boltzmann statistics. ‘Thick’ and ‘thin’ helium films are taken as the limit cases. In the case of thick helium films (e.g. so-called bulk helium [22, 23]) the main contribution to the force acting on the unit of liquid dielectric volume is made by gravitational attraction. On the other hand, in the case of thin helium films the gravitational attraction is negligibly small comparing to the van der Waals interaction forces between helium atoms and the substrate substance [24, 25]. The results obtained in this paper are compared to existing data obtained in other description models and experimental studies. The following sections show that the available experimental data [12–16] provide a large testbed for confirmation of the proposed theory.

The basics of the proposed theory can be used for the description of other systems, such as heavy ions in a gravitational field above a dielectric surface. The above mentioned system can serve as a model for the study of the spatial distribution of ‘levitating’ radiation dust above dielectric surfaces. This fact is relevant for ecological research, such as problems related to Chernobyl, for example.

It should also be noted that a statistical approach to the description of spatially inhomogeneous states in systems of particles with Coulomb interaction (including the electronic system on the surface of liquid helium) was used in [26–29]. The methodology of these studies was based on using the modified electrostatic potential of a single electron and methods of functional integration to calculate the grand partition functions.
In our opinion, among the relatively recent publications considering various dynamic aspects in the behavior of charges above dielectric surface, it is worth pointing out [30, 31]. They demonstrate the relentless interest in the scientific world toward the study of such systems. The analysis of these publications allows us to hope that the microscopic approaches developed in [18–21] will enable further development and application to the description of dynamic processes in the system of charges above a liquid dielectric surface in external fields.

2. Self-consistency equations for the system of charges above a liquid dielectric surface

Unlike in a charged system, in a quasi-neutral system the total number of charges is determined by the external field. In other words, if the external clamping field changes, the number of charges held above the dielectric surface changes as well. However, the difference between the system types does not affect the basic formulations of the statistical approach to their description [21]. The difference between the quasi-neutral and charged systems appears at the level of boundary condition formulation for the self-consistency equation system. For this reason, this paper does not contain a detailed description of the theoretical basics. Following [21], we briefly annotate the terminology and formulations that result in the self-consistent equations.

Let us consider a system of identical particles, having charge $Q$, mass $m$, spin $S_Q$, momentum $p$, and energy $\varepsilon_p = \frac{p^2}{2m}$. The charges are placed in a vacuum above the surface of a liquid dielectric film with thickness $d$. The liquid dielectric has a dielectric permittivity $\varepsilon$ and a surface tension coefficient $\alpha$. We assume that the liquid dielectric film is located on a flat solid substrate, with dielectric permittivity $\varepsilon_d \gg \varepsilon$. The surface profile of the liquid dielectric film is described by $\xi(\rho) \equiv \xi(x, y)$ function, where $\rho \equiv \{x, y\}$ is the radius vector in $z = 0$ plane of the Cartesian coordinate system $\{z, x, y\}$. The boundaries between the ‘1’–‘3’ regions (see figure 1) in the direction of the $\rho$ coordinates are considered as unlimited. To avoid issues concerned with the repulsion of like-charged particles along $\rho$, the system is assumed to be located in a vessel with walls at $\rho \to \infty$. These walls forbid charges to leave the system along the undisturbed flat surface of the liquid dielectric.
The charges are acted on by an external clamping electric field $E^{(o)}_{1}$, directed along the $z$-axis. It is also assumed that there is a potential barrier prohibiting the entry of the charges into the liquid dielectric film. All physical quantities related to the $z > \xi(\rho)$ region are marked by ‘the index 1’; the physical quantities related to the liquid dielectric film ($\xi(\rho) > z > -d$) by ‘2’; and the physical quantities related to the dielectric solid substrate ($z < -d$) by ‘3’.

Let us introduce the parameters describing the system. In region ‘1’ the system is completely described by the distribution function of charges $f_{p}(r)$, the potential of the electric field generated by the system of charges $\varphi^{(i)}_{1}(r)$, the potential of external clamping electrostatic field $\varphi^{(e)}_{1}(r)$, and the surface profile of liquid dielectric $\xi(\rho)$. In region ‘2’ the system is characterized by the profile of liquid dielectric $\xi(\rho)$ and by the total electric field potential. ‘Total’ means the sum of an external electric field in the liquid dielectric and the field, induced by charges of region ‘1’. Region ‘3’ is characterized by the total electric field potential in the solid substrate. To obtain the self-consistency equations relating the equilibrium values of basic parameters $f_{p}(r)$, $\xi(\rho)$, and $\varphi^{(i)}_{1}(r)$ describing the system, it is necessary to solve the problem on obtaining the maximum of the system entropy $S$

$$S = -g_{S} \int \frac{d\rho d\rho'}{(2\pi \hbar)} \left( f \ln f + (1 - f) \ln (1 - f) \right),$$

$$f = (2\pi \hbar)^{3} g_{S}^{-1} f_{p}(r), \quad g_{S} = 2S_{Q} + 1$$

under the following conditions. First, for a fixed external clamping field the total number of particles in the system $N = \int d\rho d\rho' f_{p}(r)$, the total energy of the system

$$E_{v} = \int_{V_{1}} d\rho \int d\rho' \varepsilon_{\rho} + \frac{Q}{2} \int_{V_{1}} d\rho \varepsilon_{\rho} + \frac{Q}{2} \int_{V_{1}} d\rho \varepsilon_{\rho} + \int_{V_{1}} d\rho \left( \frac{\nabla \varphi^{(e)}_{1}}{8\pi} \right)^{2}$$

$$+ \int_{V_{2}} d\rho \left( \frac{\nabla \varphi^{(i)}_{1}}{8\pi \varepsilon_{1}} \right)^{2} + \int_{V_{2}} d\rho \left( \frac{\nabla \varphi^{(i)}_{1}}{8\pi \varepsilon_{1}} \right)^{2} + \frac{\alpha}{2} \int dS \left( \nabla \xi \right)^{2} + \frac{\alpha}{2} \int dS \left( \kappa \xi \right)^{2}$$

and its total momentum remains constant. Assuming that the system stays at rest as a whole, its total momentum must have a value of zero. The first and the second terms in (2) are the kinetic energy of the system of charges and their Coulomb interaction energy, respectively. The third term in equation (2) is the energy of charges in an external electric field, and the fourth term is the energy of the external field in the region above the liquid dielectric. The fifth and sixth terms are the energies of the total electric field in the liquid dielectric film and in the solid substrate, respectively. The seventh term is the energy of the perturbed liquid dielectric surface, and the eighth is the energy of the dielectric volume acted on by external forces (see, e.g. [32]). In addition to the force of the external electric field acting on the liquid dielectric film with thickness $d$, these forces are gravitational attraction and van der Waals forces (acting on this film from the side of the solid substrate). Let us emphasize that the total energy of the system (2) describes the static state of the liquid dielectric, so it does not contain any kinetic terms explicitly depending on viscosity, surface tension, and other fluid characteristics. However, the surface tension coefficient is implicitly contained in the first five terms of (2), due to the integration on $V_{1}$ and $V_{2}$

$$\int_{V_{1}} d\rho ... = \int d\rho \int_{\xi(\rho)}^{+\infty} d\zeta ..., \quad \int_{V_{2}} d\rho ... = \int d\rho \int_{-d}^{\xi(\rho)} d\zeta ....$$
Below it will be shown that the level $\xi(r)$ of the dielectric surface self-consistently changes under the action of the external electric field and the field of charges, and this change depends on the surface tension coefficient.

Second, if the charges above liquid dielectric film are absent, the surface profile cannot be deformed. Third, the Poisson equation takes place in all three regions of the described system. In (2), $v_j$, $j = 1, 2, 3$ denotes the volumes of regions ‘1’–‘3’, respectively. The following denotations are also made: $dS = d^2\rho \sqrt{1 + (\nabla_\rho (\xi (\rho)))^2}$, $\nabla_\rho \equiv \partial / \partial \rho$. $\varphi_j = \varphi_j^{(l)} + \varphi_j^{(e)}$. The meaning of $\kappa$ is given below. In addition, the following definition of the particle density is used

$$n(r) = \int dp f_\rho (r).$$

The problem of determining the conditional maximum of entropy can be reduced to the problem of the unconditional minimum determining the thermodynamical potential $\tilde{\Omega}$ (see [18] for details):

$$\tilde{\Omega} = - S + \sum_0 Y_0 E + Y_1 p_i + Y_2 N + \int dp \lambda_\rho (\rho) (\xi (\rho))|_{\nu=0}$$

$$+ \int dr \lambda (r) \left[ \Delta \varphi (r) + 4\pi Qn (r) \right],$$

where $Y_0, Y_1, Y_4, \lambda (r), \lambda_\rho (\rho)$ are the corresponding Lagrange multipliers to the above conditions. $Y_0, Y_1, Y_4$ are related to the temperature $T$, the average velocity $v_i$ of charge system as a whole, and the chemical potential $\mu$ by the formulae

$$T = \frac{1}{Y_0}, \quad -\frac{Y_1}{Y_0} = v_i, \quad -\frac{Y_4}{Y_0} = \mu.$$ 

As we consider the case $v_i = 0$, then $Y_1 = 0$. $\lambda (r), \lambda_\rho (\rho)$ can be found by solving the variation problem on obtaining the minimum of the thermodynamic potential $\tilde{\Omega}$. The chemical potential $\mu$ must also depend on the electric field. At least this follows from the normalizing condition for the distribution function $f_\rho (r)$ to the total number of particles $N$ (see below). We should remember that in systems similar to those studied in this paper, the so-called electrochemical potential (defined below by (32)) is the more ‘useful’ quantity, because the chemical potential in the general case cannot be isolated from the electrochemical one. This fact will also follow from equations (33)–(36), and it was also discussed earlier in [18].

The solution of such a variation problem in [18] has the following equation

$$\left( \frac{gs}{\alpha} \right) \int dp \frac{\ln (1 - f)}{(2\pi h)^3} + \frac{\varepsilon}{8\pi} \left( \left( \nabla \varphi_2^{(e)} \right)^2 - \left( \nabla \varphi_2^{(e)} \right)^2 \right)_{z=\xi}$$

$$= \kappa^2 \xi \sqrt{1 + (\nabla \xi)^2} - \nabla \left( \xi \frac{2 + \kappa^2 \xi^2 + 3(\nabla \xi)^2}{2\sqrt{1 + (\nabla \xi)^2}} \right),$$

where the distribution function of charges $f_\rho (r)$ is given by

$$f_\rho (r) = \theta (z - \xi (\rho)) \frac{gs}{(2\pi h)^3} \left[ 1 + \exp \left( \xi (\rho) - \xi (\rho) \right) \right]^{-1}.$$ 

Thetas are the Heaviside step function, and the $\kappa$ function in equation (5) is defined by the expression

$$\theta(z)$$.
\[\kappa(d) = \sqrt{\rho \alpha^{-1} (g + f(d))}, \]  
(7)

where \(g\) is gravity acceleration, \(\alpha\) is the surface tension of the liquid dielectric, \(\rho\) is its density, and \(f \sim d^{-4}\) is the van der Waals constant, which in the case of a massive liquid dielectric \((d \to \infty)\) is negligible compared to \(g\) (see below). In the case of a thin dielectric film the gravity force acting on atoms of the liquid dielectric becomes negligibly small compared to van der Waals forces (see [12–16] and references therein). For example, such a situation takes place for liquid helium films thinner than \(d \sim 10^{-4}\) cm [33].

Equations (5) and (6) together with the equations for the electric field potentials, both external \(\varphi_{j}^{(e)}\) and induced by charges \(\varphi_{j}^{(i)}\) in all three regions of the system,

\[\Delta \varphi_{1}^{(i)}(r) + 4\pi Q n(r) = 0, \quad \Delta \varphi_{2}^{(i)}(r) = 0, \quad \Delta \varphi_{3}^{(i)}(r) = 0,\]
\[\Delta \varphi_{j}^{(o)}(r) = 0, \quad j = 1, 2, 3\]
(8)

form a system of self-consistent equations. Let us also note that the first equation in (8)—containing \(\varphi_{1}^{(i)}(r)\) in \(n(r)\) through a Wigner distribution function of charged fermions (see equations (3) and (6))—is also called a Thomas–Fermi equation. The self-consistent equation system must be supplemented by the boundary conditions for the electric fields and their potentials at boundaries \(z = \xi(\rho)\) and \(z = -d\). For the purpose of convenience this is performed in the next section.

3. Scenario of the phase transition resulting in the formation of spatially periodic structures and the boundary conditions for the electric fields

The scenario of the phase transition resulting in the transformation of the surface of the liquid dielectric film is assumed to be as follows. The external electric field, attracting charges to the flat surface of the liquid dielectric film, causes its subsidence within the area of this field action [12–16, 18]. Moreover, the bottom of this deflection remains flat. Therefore, the deformation of the liquid dielectric surface, leaving the bottom deflection flat, can be characterized by a single parameter \(\xi\) (subsidence depth). If the flat surface of the undeformed dielectric is described by the \(z = 0\) plane, the value of \(\xi\) should be negative, \(\xi < 0\). Further increasing the external electric field increases the absolute value of \(\xi\) and the bottom surface of the deformation remains flat up to a certain critical value of the total electric field \(E_c\) on the dielectric surface,

\[E_c = \left| \frac{\partial \varphi_1(z, \rho)}{\partial z} \right|_{z=\xi}. \]
(9)

Naturally, in this case the inequality \(|\xi| < d\) takes place, if the liquid dielectric is a film with thickness \(d\) and located on a solid substrate.

Further increasing the clamping electric field can result in the formation of a periodic structure on the surface profile of the formed deflection bottom. Hence, the phase transition to spatially periodic structures in this system occurs on the background of a flat structure of the liquid dielectric. It should be noted that the control parameter for this phase transition can be not only the external electric field, but also the temperature. The described scenario can take place not only in the case of charged system, but also in the case of a quasi-neutral system. In the latter case the density of charges above the liquid dielectric surface is determined by the value of the external field, as mentioned above. Thus, the density of charges above the
dielectric surface may be excluded from the control parameters of the phase transition (unlike the charged system [18]). Consequently, at the phase transition these two parameters (external electric field and temperature) are related by an equation describing a certain curve. This curve is obtained below.

According to the above scenario of the phase transition, the surface profile of the liquid dielectric in a phase with lower symmetry may be represented as [18, 20]

$$ \xi(\rho) = \tilde{\xi} + \xi(\rho), \quad \text{(10)} $$

where $\tilde{\xi}(\rho)$ is the spatially inhomogeneous surface profile, formed as the result of the phase transition on the background of the flat bottom surface $z = \tilde{\xi}$. Thus, the surface profile $\xi(\rho)$ is the order parameter of the considered phase transition. In the symmetric phase this quantity has a zero value, in the asymmetric phase it describes the spatially periodic structure of the surface. So, near the critical point from the asymmetric phase, the inequality

$$ |\tilde{\xi}| \gg |\xi(\rho)| \quad \text{(11)} $$

takes place. Let us recall, that in the theory of phase transitions term ‘asymmetric phase’ means the phase formed as the result of a phase transition, and this phase has lower symmetry than the initial one. In this case the initial phase is called symmetric. Let us also note that in the case of equation (11) taking place in the neighborhood of the phase transition point, and the zero value of the order parameter at the point, the second order phase transition occurs [32].

To describe the phase transitions associated with the transformation of the liquid dielectric surface and the formation of spatially periodic structures in the researched system, we must obtain the quantities $\tilde{\xi}, \xi(\rho)$, and the distributions of charges and fields in the system as the result of the phase transition. For this purpose we use equations (5)–(8), supplemented by the boundary conditions for the characteristics of the electric field at the boundaries between the three regions.

The boundary conditions for the potentials $\varphi_j$ on the boundaries $z = \xi(\rho)$ and $z = -d$ have the form:

$$ \varphi_1(\xi, \rho) = \varphi_2(\xi, \rho), \quad \varphi_2(-d, \rho) = \varphi_3(-d, \rho), $$

$$ \left( n(\rho) \cdot \nabla \right) \{ \varepsilon \varphi_2(z, \rho) - \varphi_1(z, \rho) \} \big|_{z = \xi} = 0, $$

$$ \varphi_1^{(e)}(\xi, \rho) = \varphi_2^{(e)}(\xi, \rho), \quad \varphi_2^{(e)}(-d, \rho) = \varphi_3^{(e)}(-d, \rho), $$

$$ \left( n(\rho) \cdot \nabla \right) \left( \varphi_1^{(e)}(z, \rho) - \varepsilon \varphi_2^{(e)}(z, \rho) \right) \big|_{z = \xi} = 0, $$

$$ \left\{ \frac{\varepsilon}{\varepsilon_d} \frac{\partial \varphi_2(z, \rho)}{\partial z} - \frac{\partial \varphi_3(z, \rho)}{\partial z} \right\} \big|_{z = -d} = 0, $$

$$ \left\{ \frac{\varepsilon}{\varepsilon_d} \frac{\partial \varphi_2^{(e)}(z, \rho)}{\partial z} - \frac{\partial \varphi_3^{(e)}(z, \rho)}{\partial z} \right\} \big|_{z = -d} = 0, \quad \text{(12)} $$

where $n(\rho)$ is the normal to the surface with profile $\xi(\rho)$ at point $\rho$

$$ n(\rho) = \sigma \left\{ \frac{\partial \xi}{\partial x} - \frac{\partial \xi}{\partial y}, 1 \right\}, \quad \sigma = \left( 1 + (\nabla \xi)^2 \right)^{-1/2}. \quad \text{(13)} $$

These boundary conditions correspond to the case where the surface charges on the boundaries are absent. Equation (12) must be also supplemented by the limit conditions of the fields at infinity.
The potentials of the external and the total fields can be given as

\[
\left. \frac{\partial \varphi_1}{\partial z} \right|_{z \to +\infty} < +\infty, \quad \left. \frac{\partial \varphi_3}{\partial z} \right|_{z \to -\infty} < +\infty,
\]

\[
\left. \frac{\partial \varphi_1^{(e)}}{\partial z} \right|_{z \to +\infty} < +\infty, \quad \left. \frac{\partial \varphi_3^{(e)}}{\partial z} \right|_{z \to -\infty} < +\infty.
\]  \tag{14}

Further on we consider the surface profile which differs slightly from the flat one and show the change of equations (5)–(8) in this case. In [18] it was shown that if the surface profile slowly changes along the coordinate, we have

\[
\left| \frac{\partial \xi}{\partial x} \right| \ll 1, \quad \left| \frac{\partial \xi}{\partial y} \right| \ll 1.
\]  \tag{15}

If equations (11)–(15) take place, we can expect the distribution of charges and fields in the system to be a little different from the distributions taking place in the case of flat dielectric surface \( z = \xi \). Then, the potentials of the external and the total fields can be given as

\[
\varphi_j(z, \rho) = \bar{\varphi}_j(z) + \tilde{\varphi}_j(z, \rho),
\]

\[
\varphi_j^{(e)}(z, \rho) = \bar{\varphi}_j^{(e)}(z) + \tilde{\varphi}_j^{(e)}(z, \rho), \quad j = 1, 2, 3,
\]  \tag{16}

where \( \bar{\varphi}_j(z) \) and \( \tilde{\varphi}_j^{(e)}(z) \) are the potentials of the total and external electric fields, respectively, in all three regions of the system (but not on the boundaries!) in the case of a flat liquid dielectric surface \( z = \bar{\xi} \). Potentials \( \tilde{\varphi}_j(z, \rho) \) and \( \tilde{\varphi}_j^{(e)}(z, \rho) \) describe small potential perturbations in all three regions due to the surface inhomogeneity with profile \( \xi(\rho) \). As the potential perturbations are assumed to be weak, the following inequalities take place:

\[
|\tilde{\varphi}(z)| \gg |\tilde{\varphi}(z, \rho)|, \quad |\varphi_j^{(e)}(z)| \gg |\tilde{\varphi}_j^{(e)}(z, \rho)|.
\]  \tag{17}

Let us further assume that the initially flat surface profile deformed after the phase transition \( \xi(\rho) \) is spatially periodic. In the case of \( \xi(\rho) \) periodicity (see (18)) and (10) leads to

\[
\bar{\xi} \equiv \langle \xi(\rho) \rangle, \quad \xi(\rho) = \xi(\rho) - \langle \xi(\rho) \rangle,
\]  \tag{19}

where \( \langle \cdots \rangle \) is averaging over the period.

The periodic structure of \( \xi(q) \) allows one to search the potentials \( \tilde{\varphi}_j(z, \rho) \) (see equation (16)) in the form:

\[
\tilde{\varphi}_j(z, \rho) = \sum_{q \neq 0} \tilde{\varphi}_{jq}(z)e^{iq\rho}, \quad \tilde{\varphi}_{jq}(z) = \int \frac{d\rho}{(2\pi)^2} \tilde{\varphi}_j(z, \rho)e^{-iq\rho},
\]

\[
\tilde{\varphi}_j^{(e)}(z, \rho) = \sum_{q \neq 0} \tilde{\varphi}_{jq}^{(e)}(z)e^{iq\rho}, \quad \tilde{\varphi}_{jq}^{(e)}(z) = \int \frac{d\rho}{(2\pi)^2} \tilde{\varphi}_j^{(e)}(z, \rho)e^{-iq\rho}.
\]  \tag{20}

Taking into account equations (16) and (20), we easily see that

\[
\tilde{\varphi}_j(z) \equiv \langle \tilde{\varphi}_j(z, \rho) \rangle, \quad \langle \tilde{\varphi}_j(z, \rho) \rangle = 0,
\]

\[
\tilde{\varphi}_j^{(e)}(z) \equiv \langle \tilde{\varphi}_j^{(e)}(z, \rho) \rangle, \quad \langle \tilde{\varphi}_j^{(e)}(z, \rho) \rangle = 0.
\]  \tag{21}
To describe the phase transition on the scenario, described in the beginning of this section, it is necessary to determine the order parameter $\xi(\rho)$. Considering the second order phase transition, we are able to obtain the order parameter $\tilde{\xi}(\rho)$ as a function of the control parameters $T, E, n_s$ near the critical values $T_c, E_c, n_{sc}$ using the perturbation theory in small parameters $\xi(\rho), \tilde{\varphi}_j(z, \rho)$, and $\tilde{\varphi}_j^{(e)}(z, \rho)$.

Taking into account (15)–(18) after substituting (15) and (16) into (5)–(8) and keeping the terms linear in $\xi(\rho), \tilde{\varphi}_j(z, \rho)$, $\tilde{\varphi}_j^{(e)}(z, \rho)$, and $T - T_c, E - E_c, n - n_{sc}$, we obtain the equations describing the spatial structure of the liquid dielectric surface and the distribution of charges and fields in the asymmetric phase near the critical surface (see the note above). Let us write the equations describing the system above the surface of the liquid dielectric film $z = \tilde{\xi}$, i.e. in the region ‘1’. These equations are the main approximation of the described perturbation theory. Subsequently, the charges above the liquid dielectric surface are considered to be electrons, and therefore in the corresponding formulas we denote the charge of an electron $Q = -e$ instead of $Q$. Due to the periodicity of small quantities $\xi(\rho), \tilde{\varphi}_j(z, \rho)$, and $\tilde{\varphi}_j^{(e)}(z, \rho)$, see equations (18) and (20), the main approximation is obtained by averaging over the period of the self-consistent equations (5)–(8). The components, representing the averaged values of the terms that are quadratic in $\xi(\rho), \tilde{\varphi}_j(z, \rho)$, and $\tilde{\varphi}_j^{(e)}(z, \rho)$, are small compared to the main approximation, therefore they can be omitted. Then, the Poisson equation in the first region in the main approximation has the form

$$\tilde{\varphi}_j''(z) = 4\pi n(z), \quad n(z) = \int d^3 p f_p(z),$$

$$f_p(z) = \frac{\theta (z - \tilde{\xi}) g_S}{(2\pi \hbar)^3} \left( 1 + e^{-(\varphi_j(z) + \varphi_j^{(e)}(z))/\hbar} \right)^{-1}. \quad (22)$$

The main order approximation of equation (5) in the mentioned parameters gives the equation to determine $\xi$

$$\frac{g_s T}{(2\pi \hbar)^3} \int d\varphi \ln \left( 1 - \frac{(2\pi \hbar)^3}{g_s} f_p(z) \right) \bigg|_{z = \tilde{\xi}} = \frac{\alpha}{8\pi} \left( \tilde{\varphi}_2''(z) \right)^2 \bigg|_{z = \tilde{\xi}} + \alpha \kappa^2 \tilde{\xi} \quad (23)$$

Due to the absence of charges in regions ‘2’ and ‘3’, the equations for $\tilde{\varphi}_2(z)$ and $\tilde{\varphi}_3(z)$ potentials have the form:

$$\tilde{\varphi}_2''(z) = 0, \quad \tilde{\varphi}_3''(z) = 0. \quad (24)$$

Let us recall that the external field potentials $\varphi_j^{(e)}(z)$ in all three regions are described by the same equations (Laplace equations):

$$\varphi_j^{(e)''}(z) = 0, \quad j = 1, 2, 3. \quad (25)$$

To make the system of equations (22)–(25) self-contained, the same averaging procedure is applied to the boundary condition equation (12). As a result, we obtain the relation between the total potentials $\varphi_j(z)$ and the potentials of external electric field $\varphi_j^{(e)}(z)$ on the boundaries of three regions:
\( \tilde{\varphi}_1 (\tilde{\xi}) = \tilde{\varphi}_2 (\tilde{\xi}), \quad \tilde{\varphi}_1' (z = \tilde{\xi}) = \varepsilon \tilde{\varphi}_2' (z = \tilde{\xi}), \quad \tilde{\varphi}_2 (d) = \tilde{\varphi}_3 (d), \quad (\varepsilon \tilde{\varphi}_2' - \varepsilon d \tilde{\varphi}_3')_{z = -d} = 0, \quad \tilde{\varphi}_1' (\tilde{\xi}) = \tilde{\varphi}_2' (\tilde{\xi}), \quad (\varepsilon d \tilde{\varphi}_3' - \varepsilon \tilde{\varphi}_2')_{z = -d} = 0, \quad \tilde{\varphi}_2' (d) = \tilde{\varphi}_3' (d), \quad (\tilde{\varphi}_1' - \varepsilon \tilde{\varphi}_2')_{z = \tilde{\xi}} = 0. \) (26)

It is easy to see that the solution of the problem on the phase transition description starts from solving the equations of the main approximation (22)–(25) with the boundary conditions (26). This procedure allows one to obtain the distribution of charges and fields in the system in the case of a flat surface of the liquid dielectric, which is given by the equation \( z = \tilde{\xi}. \) The value of \( \tilde{\xi}. \) obtained from equation (23), determines the subsidence level of the flat dielectric surface due to the pressure of charges (electrons) on it.

To obtain the critical parameters of the considered phase transition, the higher orders of perturbation theory must be involved. Below, we formulate the system of self-consistent equations (5)–(8) in the first order of this theory. To simplify the further calculations, we assume the resulting periodic structure to be one-dimensional (1D) with the period along the \( x \)-axis equal to \( a, \) so \( q = q_s = \frac{2\pi}{a}. \) So, from here, in place of vector \( q \) directed along the \( x \)-axis, we write its corresponding projection \( q. \) Let us consider the quantities \( \tilde{\xi}_q, \tilde{\varphi}_{jq} (z), \) and \( \tilde{\varphi}_{jq}^{(e)} (z) \) to have the following form:

\[ \tilde{\xi}_q (z) = \sum_{l=1}^{\infty} \tilde{\xi}_q^{(l)} (z), \quad \tilde{\varphi}_{jq} (z) = \sum_{l=1}^{\infty} \tilde{\varphi}_{jq}^{(l)} (z), \quad \tilde{\varphi}_{jq}^{(e)} (z) = \sum_{l=1}^{\infty} \tilde{\varphi}_{jq}^{(e)(l)} (z), \] (27)

where

\[ \tilde{\xi}_q^{(1)} = \tilde{\xi}_{q_0}^{(1)} (\Delta (q - q_0) + \Delta (q + q_0)), \quad \tilde{\xi}_q^{(2)} = \tilde{\xi}_{q_0}^{(2)} (\Delta (q - 2q_0) + \Delta (q + 2q_0)), \] (28)

and the values of \( \tilde{\varphi}_{jq}^{(1)} (z), \tilde{\varphi}_{jq}^{(2)} (z), \tilde{\varphi}_{jq}^{(e)(1)} (z), \) and \( \tilde{\varphi}_{jq}^{(e)(2)} (z) \) are related in a similar way to equation (28). In these formulas \( \Delta (q) \) is the Kronecker symbol

\[ \Delta (q) = \begin{cases} 0, & q \neq 0 \\ 1, & q = 0 \end{cases}. \]

In (27) we assume the appeared periodic structure to be 1D with a period along the \( x \)-axis equal to \( a, \) so \( q = q_s = \frac{2\pi}{a}. \)

We also assume that \( \tilde{\varphi}_{jq} (z) = \tilde{\varphi}_{j-\pi} (z) \) and \( \tilde{\xi}_q = \tilde{\xi}_{-q}, \) thereby considering the real values of these quantities, so

\[ \tilde{\xi} (x) = 2 \sum_{l=1}^{+\infty} \tilde{\xi}_q^{(l)} \cos lq_0 x, \quad \tilde{\varphi}_j (x, z) = 2 \sum_{l=1}^{+\infty} \tilde{\varphi}_{jq}^{(l)} (z) \cos lq_0 x, \quad \tilde{\varphi}_j^{(e)} (x, z) = 2 \sum_{l=1}^{+\infty} \tilde{\varphi}_{jq}^{(e)(l)} (z) \cos lq_0 x. \]

Then, the linear approximation for equations (5), (8), and (12) in small values of the first harmonics of \( \tilde{\xi}_q \) and \( \tilde{\varphi}_{jq} (z) \) has the following form:
\[
\frac{\partial^2 \tilde{\varphi}_1^{(1)}}{\partial z^2} - q_0 \varphi_1^{(1)} = 4\pi \epsilon^2 \frac{\partial n}{\partial \mu} \tilde{\varphi}_1^{(1)}, \quad \frac{\partial^2 \tilde{\varphi}_2^{(1)}}{\partial z^2} - q_0 \varphi_2^{(1)} = 0, \\
\frac{\partial^2 \tilde{\varphi}_1^{(1)}}{\partial z^2} - q_0 \varphi_3^{(1)} = 0, \quad \left( \tilde{\varphi}_1^{(1)} - \tilde{\varphi}_3^{(1)} \right)_{z=-d} = 0, \\
\left( (\bar{\varphi}_1^{(1)} - \bar{\varphi}_2^{(1)}) \xi^{(1)} + \bar{\varphi}_1^{(1)} - \bar{\varphi}_2^{(1)} \right)_{z=\xi} = 0, \quad \left( \bar{\varphi}_1^{(1)} \xi^{(1)} + \frac{\partial^2 \tilde{\varphi}_1^{(1)}}{\partial z^2} - \epsilon \frac{\partial \tilde{\varphi}_1^{(1)}}{\partial z} \right)_{z=\xi} = 0, \\
\left( \epsilon \frac{\partial \varphi_3^{(1)}}{\partial z} - \frac{\partial \varphi_3^{(1)}}{\partial z} \right)_{z=-d} = 0, \quad \frac{\epsilon}{4\pi} \left( \varphi_2^{(1)} \frac{\partial \varphi_2^{(1)}}{\partial z} - \varphi_3^{(1)} \frac{\partial \varphi_3^{(1)}}{\partial z} \right)_{z=\xi} = 0, \\
+ \left( e_n \left( \bar{\varphi}_1^{(1)} + \bar{\varphi}_3^{(1)} \right) \right)_{z=\xi} + \alpha \xi^{(1)} \left( \kappa^2 + q_0^2 \left( 1 + \frac{\kappa^2 \xi^2}{2} \right) \right) = 0. \tag{29}
\]

Similarly, the first approximation of the considered perturbation theory for equation (5) for the external potential, supplemented by the appropriate boundary conditions, has the form

\[
\frac{\partial^2 \bar{\varphi}_j^{(o)(1)}}{\partial z^2} - q_0 \bar{\varphi}_j^{(o)(1)} = 0, \quad j = 1, 2, 3, \\
\left( (\bar{\varphi}_1^{(o)(1)} - \bar{\varphi}_2^{(o)(1)}) \xi^{(1)} + \bar{\varphi}_1^{(o)(1)} - \bar{\varphi}_2^{(o)(1)} \right)_{z=\xi} = 0, \quad \left( \bar{\varphi}_2^{(o)(1)} - \bar{\varphi}_3^{(o)(1)} \right)_{z=-d} = 0, \\
\left( \frac{\partial \bar{\varphi}_1^{(o)(1)}}{\partial z} - \epsilon \frac{\partial \bar{\varphi}_1^{(o)(1)}}{\partial z} \right)_{z=\xi} = 0, \quad \left( \epsilon \frac{\partial \bar{\varphi}_2^{(o)(1)}}{\partial z} - \frac{\partial \bar{\varphi}_2^{(o)(1)}}{\partial z} \right)_{z=-d} = 0. \tag{30}
\]

Equations (29) and (30) allow one to solve the above problem of describing the phase transition with the formation of spatially periodic structures above the liquid dielectric film surface in the studied system. We also note that the values of \(T\) and \(E\) included in the coefficients, multiplied by small deviations \(\xi^{(1)}, \bar{\varphi}_j^{(o)(1)}, \tilde{\varphi}_i^{(1)}\), are related by the already mentioned critical surface curve. The linear approximation of the discussed equations does not contain the terms proportional to \(T - T_c\) and \(E - E_c\) because they have a higher order of smallness. This fact causes one to obtain the higher orders of the perturbation theory to calculate the dependence of the order parameter \(\xi^{(1)}\) on the control parameters \(T, E\) near the critical surface. The following sections are devoted to solving this problem, and to solving the system of equations (22)–(26), (29), and (30).

It is necessary to note that the search for the solutions in the form of 1D spatially periodic structure equations (27) and (28) is caused by two circumstances. First, it simplifies the procedure of obtaining analytical solutions, as the description of 2D spatially periodic structures assumes incredibly cumbersome and hardly controlled calculations. Second, similar one-period structure systems were experimentally observed in [34]. However, according to that study, this structure was not stationary and then evolved to a hexagonal structure. The 'justifications' for the possibility of the existence one-period solutions could be based on the study of both equilibrium and dynamic stability of such structures in the considered system. The same should apply to the 2D periodic structures, however, such problems require a separate solution and analysis because of their complexity.

In this paper the system of charges is not considered to be localized in any plane, unlike in [7, 13–15]. These papers are concerned with 2D hexagonal crystal structures formed by the electrons above a liquid helium surface. The exceptional cases are those where the so-called
4. Distribution of charges and field in an electro-neutral system above the flat surface of liquid dielectric

The solution of equations (22)–(25) is obtained in terms of the methods proposed in [18] (see also [20, 21]), where a similar problem was considered in the case of a non-degenerate gas of charges above a flat solid dielectric boundary. The difference in the system description in the cases of solid and liquid flat dielectric boundaries, is that the $z$ coordinate of the solid flat dielectric surface stays fixed, and the liquid dielectric surface ‘sinks’ under the influence of the additional pressure created by charges (see below). A similar problem was solved in [35], where the authors obtained the distribution of a non-degenerate electron gas inside a flat capacitor with plates, covered by a flat dielectric layer. However, in the present paper the case of general statistics of the Fermi particle equation (6) is considered. To solve the first equation in (22) it is convenient to rewrite it in the following form:

$$
\varphi''_1(z) = 4\pi e\nu \int_0^\infty d\varepsilon \varepsilon^{1/2} \left( 1 + \exp \frac{\varepsilon - \psi}{T} \right)^{-1},
$$

(31)

where the following notations are introduced

$$
\psi(z) \equiv \mu + e\varphi_1(z), \quad \nu = \sqrt{2/\pi} a_0^{-3/2} e^{-3}.
$$

(32)

We also take into account that $S_Q = 1/2$ and $a_0 \equiv \hbar^2/(me^2)$ is the first Bohr radius. $\psi$ is usually called an electrochemical potential.

The order of equation (31) can be lowered (see, e.g. [18])

$$
\varphi'_1(z) = -\left\{ \frac{16\pi}{3} \nu \int_0^\infty \frac{d\varepsilon \varepsilon^{3/2}}{1 + \exp \frac{\varepsilon - \psi}{T}} + C_1 \right\}^{1/2},
$$

(33)

where $C_1$ is an arbitrary integration constant. The sign before the square root in this equation is chosen assuming a force acting on a negative charge at $z > \xi$, pressing it to the dielectric surface. Equation (33) is simplifed in the case of a quasi-neutral system. Indeed, the quasi-neutrality condition assumes the absence of particles at infinity (see equation (22))

$$
\lim_{z \to +\infty} \left( \exp \beta (z - \psi) + 1 \right)^{-1} = 0.
$$

(34)

Taking this fact into account, equation (33) turns to the following form

$$
\lim_{z \to +\infty} \varphi'_1(z) = -\sqrt{C_1}.
$$

(35)

In the case of quasi-neutrality of the system the total electric field vanishes at infinity, so $C_1 = 0$. Taking into account equations (32) and (33), we have

$$
\frac{\partial \chi}{\partial \varepsilon} = -\frac{2^{5/4}}{a_0} \left( \frac{T a_0}{\pi e^2} \right)^{1/4} (-\text{Li}_{5/2}(-e^\chi))^{1/2}, \quad \chi = \psi/T.
$$

(36)
Here we use a special polylogarithmic function $L_i(s)$, related to the Fermi–Dirac integral $I_s(\chi)$ by the following expression

$$I_s(\chi) = -L_i(s+1)(-e^\chi), \quad I_s(\chi) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{x^s dx}{1 + e^{x-\chi}}.$$  \hfill (37)

The plots of polylogarithmic functions of different orders $s$ are presented on figure 2. Integration of equation (36) requires applying numerical methods. In certain cases they are used below. However, a set of important results, such as the relation between the critical parameters of the phase transition, the quasi-neutrality condition, etc, can be expressed analytically without the calculation of the explicit dependence on $z$ of the contained quantities. Nevertheless, the calculus of $E_1(z), n(z)$, and $\chi(z)$ is performed below.

As shown in [20], in the general case the gas of charges can be degenerate near the dielectric surface and non-degenerate at large distance from it. The condition of gas degeneracy is defined by the system temperature, particle density, and the external clamping field value. Unlike [18, 21], the present paper considers the general case of the distribution function equations (6) and (22).

According to equations (22) and (37), the expression for density, as a function of $\chi$, has the following form (see [18, 20]):

$$n = -\left(\frac{T a_0}{\pi e^2}\right)^{3/2} \frac{L_{3/2}(1-e^\chi)}{\sqrt{2a_0}}.$$  \hfill (38)

Equation (38) is the result of normalization of the distribution function equation (22) on the total number of charges $N$ above the surface $S$:

$$\int dx \sum_p f_p(\chi) = \int d\rho \int_0^\infty dz \int \frac{d\nu f_\nu(z)}{(2\pi)^{3/2}} = S \int_0^\infty dzn(z) = N.$$  \hfill (39)
In fact, equation (39) is approximate, because a certain part of the charges belongs to the spatially periodic structure of the liquid dielectric surface. However, according to equations (11) and (17), the number of such charges is small compared to the total number of charges \(N\).

Let us establish the relation between the value of chemical potential \(\mu\), contained in \(\psi(z)\) (see equation (32)), and the number of charges, located above the unit of the flat liquid dielectric surface area \(n_s\):

\[
n_s = \int_{\xi}^{\infty} \, \delta \mathcal{N}(z) = \frac{N}{S} \quad (40)
\]

Taking into account equations (37), (38) we change in equation (40) the integration variable from \(z\) to \(\chi\) and obtain

\[
n_s = \left(\frac{T a_0}{\pi e^2}\right)^{5/4} \frac{(-\text{Li}_{5/2}(-e^{x_0}))^{1/2}}{2^{1/2}a_0^{3/4}}. \quad (41)
\]

Let us emphasize that equation (41) is obtained using the following assumption

\[
\lim_{\chi \to +\infty} \chi = -\infty, \quad (42)
\]

meaning the absence of charges at infinity. Otherwise, the value of \((-\text{Li}_{5/2}(-e^{x_0}))^{1/2}\) and, hence, the distribution charges at infinity is not equal to zero. This fact contradicts the above made assumption on the absence of charges at infinity and the quasi-neutrality condition.

Equation (41) provides the ability to numerically calculate the dependence of \(\chi(z = \xi) = \chi_0\) on the temperature \(T\) and the number of charges above dielectric surface area unit \(n_s\). This dependence is presented in figure 3.

Noting that \(E_1(z) = -\varphi'(z)\) and taking into account equation (36), we obtain

\[
4\pi e n_s = E_1(z = \xi) \equiv E_0. \quad (43)
\]

Equation (43) shows that the value of the electric field on the dielectric surface is equivalent to the value of the electric field inside a plane capacitor with oppositely charged plates with a surface charge density equal in absolute value to \(\sigma = e n_s\).
To solve the equation system (22)–(25) using the boundary conditions (26), we have to obtain the relation between the values of the external electric field \( E^{e}(z) = -\varphi^{(e)}(z) \) and the total electric field \( E_{1}(z) = -\varphi^{(1)}(z) \). If charges are present above the dielectric surface, they make a contribution to the total electric field \( E_{1}(z) = E^{e}(z) + E_{i}^{(i)}(z) \) by inducing their own field \( E_{i}^{(i)}(z) = -\varphi_{i}^{(i)}(z) \). Using the density distribution of charges (38), we can easily calculate this field value. The \( z \)-component of the electric field at the point with the \( z \) coordinate, which is produced by the elementary volume of charges \( d\xi'dy'dz' \), located at the point \((x',y',z')\), has the form:

\[
\frac{dE_{1i}^{(i)}(z)}{E_{1i}^{(i)}(z)} = -\frac{(z - z')en'(z')dx'dy'dz'}{(x'^2 + y'^2 + (z - z')^2)^{3/2}}. \tag{44}
\]

The minus sign before the ratio in equation (44) shows that the particles are negatively charged. Assuming the system is at infinity along the \((x,y)\) coordinates, the integration of the corresponding components of the electric field \( dE_{1i}^{(i)}, dE_{1y}^{(i)} \) in the total \( V_{1} \) volume gives zero. That is why the electric field generated by the charges has only a \( z \)-component. It can be obtained by applying the integration procedure \( E_{1i}^{(i)}(z) = \int_{V_{1}} dE_{1i}^{(i)}(z) \), which results in the following expression:

\[
E_{1i}^{(i)}(z) = -2\pi e \left( \int_{z}^{\xi} n(z')dz' - \int_{\xi}^{\infty} n(z')dz' \right). \tag{45}
\]

Applying numeric integration methods to equation (45), we obtain figure 4, which illustrates the \( E_{1i}^{(i)}(z) \) dependence for three pairs of \( T \) and \( n_{i} \) values. The reason for choosing these values is explained below. Let us give a simple physical interpretation of equation (45) by considering its limit cases. According to equations (40) and (45), we have \( \lim_{z\rightarrow\xi} E_{1i}^{(i)}(z) = 2\pi en_{s} \) and \( \lim_{z\rightarrow-\infty} E_{1i}^{(i)}(z) = -2\pi en_{s} \). These values of the electric field produced by the system...
of charges, are equivalent to the similar values of an infinite plate having a surface charge density \( \sigma = en_s \). According to equations (8), the value of the external electric field in its main approximation is constant, \( E_1(z) = E_1(z) - E_1(z) \equiv E \). The values of the external and total electric fields coincide at a certain point \( z_\sigma \), where \( E = E_1(z_\sigma) \) and \( E_1(z_\sigma) = 0 \). \( z_\sigma \) is evaluated numerically and is presented in figure 5. At fixed value of \( n_s = 5 \cdot 10^8 \text{ cm}^{-2} \) we obtain \( z_\sigma \approx 4.47 \cdot 10^{-7} \text{ cm} \) for \( T = 2.5 \text{ K} \), \( z_\sigma \approx 1.07 \cdot 10^{-7} \text{ cm} \) for \( T = 0.5 \text{ K} \) and \( z_\sigma \approx 6.66 \cdot 10^{-8} \text{ cm} \) for \( T = 0.1 \text{ K} \). However, rather than evaluating \( z_\sigma \) to obtain \( E \), it is convenient to calculate the following limit:

\[
E = \lim_{z \to +\infty} \left( E_1(z) - E_1(z) \right) = E_\infty + 2\pi en_s, \tag{46}
\]

where \( \lim_{z \to +\infty} E_1(z) = E_\infty \). In the considered electro-neutral case, we have \( E_\infty = 0 \). Consequently, equation (46) turns to the following form:

\[
E = 2\pi en_s. \tag{47}
\]

Let us make an important remark. The obtained values of electric fields equations (43) and (47) are used in the approach of the present paper. However, comparing these to the experimental data of some papers (e.g. see [13–16]) can lead to some quantitative mismatch. This is caused by the experimental measurement of the potential difference between the plates of the capacitor creating the external field, but not the measurement of this field value. Let us obtain the relation between the potential difference of the capacitor plates and the \( n_s \) parameter in the quasi-neutral case. In our system the solid substrate can be considered as a lower capacitor plate, dived into the liquid dielectric on a \( d \) depth. As far as the upper capacitor plate is concerned, in our system it is located at an infinite distance from the dielectric surface. If the charges between the capacitor plates are absent, the applied voltage has the form \( \Delta U = \varphi_1(+) - \varphi_2(-) - d \). This potential difference produces the electric field \( E \) above the liquid dielectric and \( E/\varepsilon \) below it. Then the charges are injected into the volume above the liquid dielectric until the complete screening of the initial field \( E \) near the upper capacitor plate. As shown above, \( E \) and \( n_s \) are related by equation (47). The charges also increase
the value of total electric field on the dielectric surface up to $E_0$ and $E_0/\varepsilon$ below the surface (see equation (43)). The potential difference between the capacitor plates is now equal to $\Delta U = \varphi_1(+\infty) - \varphi_2(-d)$. In papers (see, e.g. [7, 16, 36]) it is usually considered that the electric field above liquid dielectric is zero. If the electric field above the dielectric is absent, the potentials of the upper capacitor plate and the dielectric surface coincide. Consequently, the voltage drop $\Delta U$ occurs only in the dielectric, but not above it. That is why the electric field inside the liquid dielectric is equal to $\Delta U/d$. On the other hand, as shown above, this field is equal to $E_0/\varepsilon$ and according to equation (43) we have

$$\Delta U/d = 4\pi\varepsilon n_s/\varepsilon. \quad (48)$$

If the dielectric is liquid helium, the permittivity value of which is close to unity, in the literature (see [7, 36]) you can often find a value for the clamping field inside the capacitor equal to $E = 4\pi\varepsilon n_s$. In such cases we compare the values of $n_s$ obtained from our theory to the experimental values of $n_s = E/(4\pi\varepsilon)$. Considering the absence of a total electric field above the liquid dielectric, we can characterize this statement as approximate, but still highly accurate. Indeed, $\Delta U = 4\pi\varepsilon n_s d/\varepsilon + T(\chi(+\infty) - \chi_0)/\varepsilon$, where the second term is the voltage drop between the upper capacitor plate and the surface of the liquid dielectric. Putting $T = 2.5 K$, $n_s = 5 \cdot 10^8 \text{ cm}^{-2}$, and $d = 0.1 \text{ cm}$, we obtain $4\pi\varepsilon n_s d/\varepsilon \approx 10^{-3} \text{ V}$. The voltage drop $T(\chi(z) - \chi_0)/\varepsilon$ at the macroscopic distance from the dielectric surface, e.g. $z = 10 \text{ cm}$, is approximately equal to $7.9 \cdot 10^{-3} \text{ V}$. This fact suggests that using equation (48) provides rather good accuracy.

According to equations (24)–(26), (43), and (47)–(55), the potentials of the total and external electric fields in liquid film and solid dielectric substrate are determined by the expressions:

$$\varphi_2(z) = -\frac{E_0}{\varepsilon}(z - \bar{\xi}) + \varphi_0, \quad \varphi_3(z) = -\frac{E_0}{\varepsilon_d}(z + d) + \frac{E_0}{\varepsilon}(d + \bar{\xi}) + \varphi_0,$$

$$\varphi_1^{(e)}(z) = -\frac{E}{\varepsilon_d}(z - \bar{\xi}) + \varphi_0^{(e)}, \quad \varphi_2^{(e)}(z) = -\frac{E}{\varepsilon}(z - \bar{\xi}) + \varphi_0^{(e)}, \quad \varphi_3^{(e)}(z) = -\frac{E}{\varepsilon_d}(z + d) + \frac{E}{\varepsilon}(d + \bar{\xi}) + \varphi_0^{(e)}, \quad (49)$$

where $\varphi_1 = \varphi_2 \equiv \varphi_0, \varphi_1^{(e)} = \varphi_2^{(e)} \equiv \varphi_0^{(e)}$.

Based on equations (23), (37), (43), (47), and (49), we obtain the value of the dielectric surface subsidence:

$$\bar{\xi} = -\frac{(4\pi\varepsilon n_s)^2}{8\pi\varepsilon_\alpha(\varepsilon d)} \left(1 + \frac{3}{4\varepsilon}\right). \quad (50)$$

According to equation (50), in the case of the absence of charges the value of the dielectric surface subsidence $\bar{\xi}$ is zero. This value is in good agreement with the experimental data of [37]. Equation (50) allows one to impose a natural constraint on the clamping electric field and, as a consequence, on the permissible surface electron density. Indeed, in the system in the equilibrium state, described by the self-consistent equations (5)–(8), the absolute value of the surface subsidence of the liquid dielectric film must be substantially smaller, compared to the thickness of this film (or, at least, several times smaller):

$$|\bar{\xi}| << d. \quad (51)$$

This condition allows one to define the maximum value of the number of charges per dielectric surface unit $n_s^m$ as significantly exceeding the values under consideration:
In the case of macroscopic values of the film thickness (the so-called massive helium case), the value of $\kappa$ is almost independent of $d$, which allows one to estimate the value of $\kappa d$. For example, for a liquid helium film with thickness $d = 0.1$ cm, the value of $\kappa d \approx 2.18 \cdot 10^9$ cm$^{-2}$, which is comparable to the value of $\kappa d \approx 2.2 \cdot 10^9$ cm$^{-2}$ (see [38]), determining the instability condition of a homogeneous electron system above the flat surface of liquid helium. For such system the instability means the appearance of a static deformation of the liquid helium surface with a periodic structure, as the result of forming of standing gravitational waves (see [39]). However, the value of $\kappa$ depends on film thickness $d$. For example, for liquid helium this dependence has the form (see [33]):

$$
\kappa(d) = \sqrt{\frac{\rho}{\alpha}} \left( g + \frac{g_0 d_e}{d^2 (d + d_e)} \left( 3 + \frac{d}{d + d_e} \right) \right),
$$

(53)

where $d_e = 1.65 \cdot 10^{-5}$ cm, and $g_0 = 2.2 \cdot 10^{-14}$ cm$^5$ s$^{-2}$. So, according to (50), in some cases this dependency should be taken into account during the analysis of the stability condition equation (51). A diagram illustrating the satisfaction of the stability condition equation (51) of the system is presented on figure 6 in the $\{d, n_s\}$ plane. Figure 6 takes into account equation (53). In this figure gray marks the region where $|\xi|/d < 1$. In other words, according to our theory, this region is the stability region of the system relative to the surface deformations, as the result of the gas charge pressure. The black line, separating the gray and white regions, corresponds to the equality $|\xi| = d$. Figure 6 shows that helium is considered to be ‘massive’ if $d > 5 \cdot 10^{-2}$ cm. Let us also note that figure 6 shows good agreement with the data of [40]. In this paper the author obtained the relation between the thickness of the dielectric film and the maximum available value of $n_s$, at which the surface of this film stays

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**Figure 6.** Diagram of the system stability against surface deformation of the liquid dielectric film in the $\{d, n_s\}$ plane.
flat. This relation was obtained by solving the problem on the stability of small oscillations in such a system.

The decrease of film thickness decreases the contribution of gravitational force in $\kappa$ and increases the contribution of van der Waals forces. This competitive process between gravitational and van der Waals forces ends at $d \sim d_v$, when the gravitational forces acting on the atoms of the liquid dielectric become negligibly small compared to van der Waals forces.

According to (53) the dependence of $\bar{\xi}$ on $d$ in (50) is contained in the quantity $\kappa(d)$. Equation (50) determines the lowering of the helium surface level, i.e. the film thickness $d$ is reduced by the value $\bar{\xi}$. In fact, the real film thickness becomes equal to $d - |\bar{\xi}|$. If condition (51) takes place, equation (50) can be used correctly. However, when $d$ becomes comparable with $|\bar{\xi}|$, calculated by equation (50), condition (51) breaks and we must replace the film thickness $d$ in (50) (without the account of external electric field and the field of charges) with its real value $d - |\bar{\xi}|$. So in this case the surface level $\bar{\xi}$ must be calculated from the modified equation (50), that takes into account the ‘effective’ film thickness $d - |\bar{\xi}|$:

$$\bar{\xi} = -\frac{(4\pi en_s)^2}{8\pi \varepsilon \kappa (d - |\bar{\xi}|)^2} \left(1 + \frac{3}{4\varepsilon}\right). \tag{54}$$

Taking into account equation (53), it is easily seen that equation (54) has no analytical solution for $\xi$. The numeric solution of equation (54) $\xi_n$ provides a new region $|\xi_n| < d$ of system stability against the deformations of the liquid helium surface. The stability region $|\xi_n| < d$ is marked in figure 7 in gray, and the region $d > |\xi|$ is located below the dashed line. Figure 7 shows that taking into account the effective film thickness is significant for ‘thin’ helium films in the range of film thickness $d < 5 \cdot 10^{-4}$ cm. In the massive helium case this effect vanishes. In the case of thin helium films the region of available values on the $\{n_s, d\}$ plane is much wider than in the case of ignoring this effect (see figure 6).

Let us also note one more important fact. In the case of thin enough films, with thickness $d \sim 10^{-6} \div 10^{-5}$ cm (if the clamping field is absent), and high enough values of $n_s > 5 \cdot 10^{10}$ cm$^{-2}$, the effective film thickness $d - |\xi_n|$ is almost independent of $d$. This
fact is illustrated in figure 8, which is in good agreement with the data of [24, 41]. If
\(n_s > 4 \cdot 10^{11} \text{ cm}^{-2}\), the effective dielectric film thickness reaches a value of about 50 ang-
stroms. According to [24], this fact can lead to electron tunneling through the dielectric film
towards the metal substrate. However, the study of this effect is outside the scope of the pre-
cent paper, which aims to develop a quasi-classical description of a system of charges in terms
of the Wigner distribution function equation (6), simultaneously depending on \(r\) and \(p\). So,
in this approach the description of the quantum mechanical tunneling effect is not possible.
However, using the equation (54) provides numerical estimates comparable with the results
of [41]. For example, for the thickness of helium film \(d < 10^{-6} \text{ cm}\) on a metal substrate, we
obtain \(n_e^m \approx 0.7 \cdot 10^{11} \text{ cm}^{-2}\).

The dependence \(\chi(z)\), related to \(\varphi_1(z)\) by (32) and (36), is obtained by the numeric integra-
tion of equation (36), or

\[
\frac{a_0}{2^{5/4}} \left( \frac{\pi e^2}{T a_0} \right)^{1/4} \int_{x_0}^{\chi} \frac{d\chi'}{\sqrt{-L\xi/2 (-e^{\chi'})}} = \xi - z.
\]  

Figure 9 shows the comparison between \(\chi(z)\) and \(\chi_n(z)\) for exact values of \(T\) and \(n_s\). \(\chi_n(z)\) is
the non-degenerate analog of \(\chi(z)\) function, obtained in [21]:

\[
\chi_n(z) = \chi_n(0) - 2 \ln \left( 1 + \frac{z - \xi}{2z_0} \right),
\]

\[
\chi_n(0) = \ln \left( \frac{n_s a_0}{\sqrt{2z_0}} \left( \frac{\pi e^2}{T a_0} \right)^{5/2} \right), \quad z_0 = \frac{T}{eE_0}.
\]

Three selection cases of the specific pairs of values of \(T\) and \(n_s\) are chosen, based on the prin-
ciple of satisfying the non-degeneracy condition of electron gas (see [21]):

\[
2^{3/2} n_e^2 a_0^4 \left( \frac{\pi e^2}{T a_0} \right)^{5/2} << 1.
\]
In the first case $T = 2.5 \, \text{K}$ and $n_s = 5 \cdot 10^8 \, \text{cm}^{-2}$ and the gas of charges is non-degenerate, as $2^{3/2}n_s^2a_0^4\left(\pi e^2/(T a_0)\right)^{5/2} \approx 0.05$. That is why, in this case, we observe the practical coincidence of the $\chi(z)$ and $\chi_n(z)$ curves. The second case is $T = 0.5 \, \text{K}$ and $n_s = 5 \cdot 10^8 \, \text{cm}^{-2}$. In this case the non-degeneracy condition equation (57) breaks ($2^{3/2}n_s^2a_0^4\left(\pi e^2/(T a_0)\right)^{5/2} \approx 2.79$) and this is observable in figure 9. If the distance from the dielectric surface $z - \bar{\xi}$ increases, the gas density equation (38) decreases (see figure 11), as the distance between the $\chi(z)$ and $\chi_n(z)$ curves does. If $z - \bar{\xi} = 10z_0 \approx 4.45 \cdot 10^{-7} \, \text{cm}$, the ratio $\chi/\chi_n \approx 0.95$. With further increase of the distance from dielectric surface, the gas can be considered as non-degenerate with 95 percent accuracy. This explains the practical coincidence of the $\chi(z)$ and $\chi_n(z)$ curves in this range of $z - \bar{\xi}$. In the third case, $T = 0.1 \, \text{K}$ and $n_s = 5 \cdot 10^8 \, \text{cm}^{-2}$, the condition equation (57) breaks dramatically ($2^{3/2}n_s^2a_0^4\left(\pi e^2/(T a_0)\right)^{5/2} \approx 156.18$), and the gas of charges cannot be considered as non-degenerate. However, unlike the previous case, the gas of charges can be considered as non-degenerate at distances greater than $z - \bar{\xi} = 200z_0 \approx 1.78 \cdot 10^{-6} \, \text{cm}$ with not less than 95 percent accuracy.

Let us note that the value of the electric field $E_1(z)$ is obtained by the differentiation $E_1 = -\chi'(z)T/e$. This dependence is presented in figure 10 in the three above mentioned cases together with the dependence $E_n(z)$ of non-degenerate gas, obtained in [21]:

$$E_n(z) = E_0/(1 + (z - \bar{\xi})/(2z_0)).$$

(58)

Figure 10 shows that the decreasing rate of the electric field with increasing distance $z - \bar{\xi}$ is defined by the extent of gas non-degeneracy. The closer the gas is to the non-degeneracy state, the slower the decrease of the electric field value with increasing $z - \bar{\xi}$. In figure 10, all curves start from one point at $z = \bar{\xi}$, because all of them have the same value of $n_s = 5 \cdot 10^8 \, \text{cm}^{-2}$, corresponding to the electric field value equation (43).

Based on (55), the $\chi(z)$ dependence is obtained, which allows one to obtain $n(z)$, using equation (38). Figure 11 shows the $n(z)$ curve in the three above mentioned cases of a pair values $T$ and $n_s$. In the first case $T = 2.5 \, \text{K}$ and $n_s = 5 \cdot 10^8 \, \text{cm}^{-2}$ the gas of charges is close
to the non-degenerate state. So, in this case the $n(z)$ curve practically coincides with the $n_0(z)$ curve of non-degenerate gas, obtained in [21, 36]:

$$n_0(z) = \frac{n_s}{2z_0} \left(1 + \frac{z - \bar{\xi}}{(2z_0)}\right)^{-2}. \quad (59)$$

Let us note that the closer the gas is to the non-degeneracy state, the slower its density decreases with distance from the dielectric surface growth. Of particular interest is a gas in state that is close to degeneracy. This state is realized at sufficiently high density and low temperature.
In this case an important role in the inter-particle interaction can be played by the exchange processes. However, the detailed study of this effect is outside the scope of the current paper.

Using the above obtained density function \( n(z) \), we can estimate certain typical distances of the considered system. Let us introduce the distance from the dielectric surface characterizing the volume, containing the major portion of system charges. With this purpose we introduce the following function

\[
\Delta(z) = \frac{1}{n_s} \int_\xi^z n(x) \, dx, \tag{60}
\]

giving the relative percentage of charges, located between the dielectric surface and the distance \( z \) above it. Figure 12 shows \( \Delta(z) \) curves for different temperature values and fixed density value \( n_s = 5 \times 10^8 \text{ cm}^{-2} \). Let us introduce the \( z_m \) distance by the following definition \( \Delta(z_m) \approx 0.95 \). In other words, the number of charges located in the range \( z_m \geq z \geq \xi \) is equal to 95 percent of the total number of charges in the system. Taking this into account, we can consider \( z_m \) as an approximate or 'effective' boundary for the gas of charges. So, for the temperature \( T = 2.5 \text{ K} \), we have \( z_m = 40z_0 \approx 8.9 \times 10^{-6} \text{ cm} \). In the case of \( T = 0.5 \text{ K} \), \( z_m = 40z_0 \approx 1.78 \times 10^{-6} \text{ cm} \) and in the case of \( T = 0.1 \text{ K} \), \( z_m = 57z_0 \approx 5.1 \times 10^{-7} \text{ cm} \). As expected, we can see that in the case of a fixed total number of charges in the gas, its effective boundary decreases if the temperature decreases. Let us emphasize that the distance \( z_m \) has the same order of value at the localization distance above the helium surface that a single electron in the ground state has (see [42, 43]). The problem of obtaining such a localization distance can be reduced to the problem of obtaining the average electron distance from the nucleus in a hydrogen atom in the ground state. The authors of [38] pointed out that in the range \( n_s \sim 10^8 \div 10^9 \text{ cm}^{-2} \) the mean distance between charges was one or two orders greater than the electron localization above the helium surface. For this reason, they considered the gas of electrons as being 2D.

In current paper we consider the case where the value of the above introduced distance \( z_m \) is also more than one order less than \( n_s^{-1/2} \). At first glance this fact justifies considering the studied system as ‘quasi-2D’. Indeed, at certain values of \( T \) and \( n_s \), the mean distance between

![Figure 12. \( \Delta(z) \) for different temperature values.](image-url)
the charge projections on the flat dielectric surface, which is proportional to \( n_s^{-1/2} \), can be two orders greater than \( z_m \). For this reason, charges can be considered as being located almost in one plane with an accuracy up to the small value \( z_m n_s^{1/2} \ll 1 \). However, in the most general case the inequality \( z_m n_s^{1/2} \ll 1 \) breaks. Even if this inequality takes place, it cannot serve as a justification for the considering the system as 2D. Let us prove the last statement by calculating the mean distance from the flat dielectric surface to the charges, located in a volume between two planes \( z = \xi \) and \( z = z_m \). The probability of charge location in the range \( (z, z + dz) \) from the dielectric surface is equal to \( n(z) dz / n_s \). Then, according to equations (36), (38), and (41), the mean distance from the charge to the dielectric surface has the following form:

\[
\langle z - \xi \rangle = n_s^{-1} \int_{\xi}^{z_m} dz n(z) (z - \xi) = z_0 \chi |z_m^{\xi}.
\]  

(61)

Let us estimate this value in the three above mentioned cases of \( T \) and \( n_s \) values. In each pair \( n_s = 5 \cdot 10^9 \) cm\(^{-2} \), and for \( T = 2.5 \) K we have \( \langle z - \xi \rangle \approx 4 \xi_0 = 0.11 z_m \approx 0.9 \cdot 10^{-6} \) cm, for \( T = 0.5 \) K we obtain \( \langle z - \xi \rangle \approx 4.5 \xi_0 = 0.15 z_m \approx 2 \cdot 10^{-7} \) cm, and for \( T = 0.1 \) K - \( \langle z - \xi \rangle \approx 10 \xi_0 = 0.18 z_m \approx 0.9 \cdot 10^{-7} \) cm. So, in the considered region of the \( \{T, n_s\} \) plane, in the volume between the \( z = \xi \) and \( z = z_m \) planes, the mean distance from charges to the dielectric surface \( z = \xi \) is 5–10 times less than the typical distance \( z_m \) of charge localization above the surface of the dielectric film. This is the main obstacle for considering the studied quasi-neutral system of charges above a liquid dielectric as a 2D system. Taking into account the inequality \( z_m n_s^{1/2} \ll 1 \) and the above estimates, the mean distance between charges along the \( z \)-axis \( \langle z \rangle \) is small compared to the mean distance between them in the \( \{x, y\} \) plane, which is proportional to \( \sqrt{n_s^{-1}} \). So, if \( \langle z \rangle \ll \sqrt{n_s^{-1}} \), the mean distance between charges \( l \) also has the order of \( \sqrt{n_s^{-1}} \), according to the estimation \( l \sim \sqrt{n_s^{-1}} + \langle z \rangle^2 \approx \sqrt{n_s^{-1}} \).

This fact allows us to obtain the applicability condition for the quasi-classical approach used in this paper.

To make the corresponding estimations, we calculate the mean thermal de Broglie wavelength \( \langle \lambda \rangle \sim \hbar / \sqrt{\langle p^2 \rangle} \) of charges above the dielectric surface. Taking into account equations (6), (36)–(38), the mean value of the squared momentum has the following form:

\[
\langle p^2 \rangle = \frac{\int d^3r d^3p \langle p^2 \rangle}{\int d^3r d^3p} = \frac{T a_0}{\pi e^2} 5/4 \int_{-\infty}^{\infty} m T \frac{d^4p}{m n_s} \frac{3}{2} \int_{-\infty}^{\infty} \frac{d\chi}{\sqrt{\chi}} \sqrt{\chi} \left( -\frac{\sqrt{\chi}}{\sqrt{\chi} - \sqrt{1 - \chi}} \right)^{1/2}.
\]

This equation allows us to estimate the thermal de Broglie wavelength \( \langle \lambda \rangle \):

\[
\langle \lambda \rangle \sim \frac{\hbar}{\sqrt{\langle p^2 \rangle}} = a_0 \sqrt{n_s} \left( \frac{\pi e^2}{3 T a_0} \right)^{1/2} \frac{2^{7/4} a_0^2}{3 \pi} \frac{d\chi}{\sqrt{\chi}} \left( -\frac{\sqrt{\chi}}{\sqrt{\chi} - \sqrt{1 - \chi}} \right)^{1/2}.
\]

The numeric calculation of the last equation shows that in the ranges \( 10^8 \) cm\(^{-2} \) \( n_s < 2 \cdot 10^9 \) cm\(^{-2} \) and \( 0.1 \) K \( T < 5 \) K, the order of the \( \langle \lambda \rangle \) value is \( 10^{-6} \) cm. This is two orders less than the mean inter-particle distance proportional to \( n_s^{-1/2} \) (see above). This fact allows us to solve the problems of the current paper in terms of the quasi-classical approach, neglecting such quantum type inter-particle interactions as an exchange type. This type of interaction can become significant near the degenerate state of the gas of charges. This state is achieved by decreasing the temperature and increasing the density of charges. As far as the experimental realization of such degenerate states is concerned, we face difficulties in
reaching a sufficiently low temperature range, and the high density range is limited by the
stability criterion equation (51) and (52), see figure 7. However, decreasing the dielectric film
thickness up to the thin films region results in reaching the permitted density region, which is
several orders greater than in the massive dielectric case. The description of such a situation is
outside the scope of the current paper. However, preliminary calculations show the availability
of a theory modification in the case of a degenerate gas of charges above a liquid dielectric
surface. The motivation for such description comes from experiments with thin films [41] and
theoretical papers based on other theoretical approaches [25, 40, 44].

Equations (43)–(55) are the solution for the problem of obtaining the distributions of the den-
sity and the electric field in the quasi-neutral system of charges above a liquid dielectric film in an
external clamping field. These results are used in the next section, which is devoted to the study
of the phase transition concerned with the formation of spatially periodic states in the system.

5. Critical parameters of the phase transition to a spatially periodic
state of the system

The starting point in the study of critical parameters of the phase transition with formation of
spatially periodic structures of dimple type is equation (29). Let us rewrite the first equation in
equation (29) in the following form:

$$\frac{\partial^2 \varphi_1^{(1)}}{\partial \xi^2} = \left( q_0^2 - 2\sqrt{2} \frac{L_{1/2}}{a_0^2} \left( -e^\xi \right) \left( \frac{T a_0}{\pi e^2} \right)^{1/2} \right) \varphi_1^{(1)},$$

(62)

where, according to equations (32), (36)–(38), we take into account the following expression

$$\frac{\partial n}{\partial \mu} = \frac{L_{1/2} \left( -e^\xi \right) \left( T a_0 \right)}{\sqrt{2} \pi a_0^2 e^2} \left( \frac{T a_0}{\pi e^2} \right)^{1/2}.$$  

(63)

Using equation (36), we reduce the derivatives on $z$ to the derivatives on $\chi$ and consider the
case:

$$\frac{2\sqrt{2}}{q_0^2 a_0^2} \left( \frac{T a_0}{\pi e^2} \right)^{1/2} \left| L_{1/2} \left( -e^\xi \right) \right| >> 1,$$

(64)

which significantly simplifies the solution of equation (62). Let us estimate the distance from
the dielectric surface, where the inequality equation (64) starts breaking. At $T = 5$ K and
$n_s = 10^8$ cm$^{-2}$ and maximum possible $q_0 \approx 3 \cdot 10^4$ cm$^{-1}$ at the given $n_s$ (corresponding to
the case of one charge in the lattice node), condition (64) takes place in the range $z = \xi < 10^{-5}$
cm. Further increasing $z$ results in the condition equation (64) breaking. However, we shall use
the solution of equation (62) near the point $z = \xi$ where the condition equation (64) is sure to
take place. Indeed, according to the last equation in (29), in order to obtain the critical param-
eters of the phase transition, we must obtain the solution for (64) at $z = \xi$, as well as the first
derivative on $z$ of this solution at $z = \xi$. So, after calculating the numerical estimates of the
results obtained in this section in approximation (64), we must validate it at $z = \xi$.

Figure 13 shows the dependence $q_0 (T, n_s)$, below which the condition (64) takes place for
the reciprocal lattice vectors $q_0$ and $2q_0$:

$$q_0 (T, n_s) = \sqrt{\frac{\left| L_{1/2} \left( -e^\xi \right) \right|}{23/4 5^{1/2} a_0}} \left( \frac{T a_0}{\pi e^2} \right)^{1/4}.$$  

(65)
The necessity of $2q_0$ satisfying equation (64) arises from the further calculation of $\tilde{\xi}^{(1)}$. This procedure requires solving the equation for $\tilde{\xi}^{(2)}$, with a similar form to (62), where $2q_0$ is in the place of the $q_0$ parameter. However, this is a rather cumbersome procedure, and we only briefly describe it in the next section. Here we only emphasize that figure 13 shows the region that is more than sufficient to satisfy equation (64) and sufficient to satisfy a similar condition with $2q_0$ in place of $q_0$. Applying the approximation (64) to (62), we transform it into the following form:

$$2Li_{5/2} (-e^\chi) \frac{\partial^2 \tilde{\varphi}_1^{(1)}}{\partial \chi^2} + Li_{3/2} (-e^\chi) \frac{\partial \tilde{\varphi}_1^{(1)}}{\partial \chi} - Li_{1/2} (-e^\chi) \tilde{\varphi}_1^{(1)} = 0.$$ \hspace{1cm} (66)

Noting the following polylogarithm property

$$Li_{i-1} (-e^\chi) = \frac{d}{d\chi} Li_i (-e^\chi),$$ \hspace{1cm} (67)

we transform equation (66) into the form

$$\frac{\partial^2}{\partial \chi^2} \left( Li_2 (-e^\chi) \tilde{\varphi}_1^{(1)} \right) = \frac{3}{2} \frac{\partial}{\partial \chi} \left( Li_3 (-e^\chi) \tilde{\varphi}_1^{(1)} \right).$$ \hspace{1cm} (68)

By integrating both sides of (68) and further applying simple transformations, it turns to the following form:

$$Li_{5/2} (-e^\chi) \frac{\partial \tilde{\varphi}_1^{(1)}}{\partial \chi} = \frac{1}{2} Li_{3/2} (-e^\chi) \tilde{\varphi}_1^{(1)} + C,$$ \hspace{1cm} (69)

where $C$ is the integration constant. The last equation belongs to linear inhomogeneous type and it is solved by the method of arbitrary constant variation. In this case the solution of equations (66) and (69) is

$$\tilde{\varphi}_1^{(1)} (\chi) = C_1^{(1)} \sqrt{-Li_{3/2} (-e^\chi)} + C_2^{(1)} \sqrt{-Li_{5/2} (-e^\chi)} \int \frac{d\chi}{\left(-Li_{3/2} (-e^\chi)\right)^{1/2}}.$$ \hspace{1cm} (70)

Figure 13. Dependence of maximum available value of reciprocal lattice vector $q_0$ on $T$ and $n_v$. 

The necessity of $2q_0$ satisfying equation (64) arises from the further calculation of $\tilde{\xi}^{(1)}$. This procedure requires solving the equation for $\tilde{\xi}^{(2)}$, with a similar form to (62), where $2q_0$ is in the place of the $q_0$ parameter. However, this is a rather cumbersome procedure, and we only briefly describe it in the next section. Here we only emphasize that figure 13 shows the region that is more than sufficient to satisfy equation (64) and sufficient to satisfy a similar condition with $2q_0$ in place of $q_0$. Applying the approximation (64) to (62), we transform it into the following form:

$$2Li_{5/2} (-e^\chi) \frac{\partial^2 \tilde{\varphi}_1^{(1)}}{\partial \chi^2} + Li_{3/2} (-e^\chi) \frac{\partial \tilde{\varphi}_1^{(1)}}{\partial \chi} - Li_{1/2} (-e^\chi) \tilde{\varphi}_1^{(1)} = 0.$$ \hspace{1cm} (66)

Noting the following polylogarithm property

$$Li_{i-1} (-e^\chi) = \frac{d}{d\chi} Li_i (-e^\chi),$$ \hspace{1cm} (67)

we transform equation (66) into the form

$$\frac{\partial^2}{\partial \chi^2} \left( Li_2 (-e^\chi) \tilde{\varphi}_1^{(1)} \right) = \frac{3}{2} \frac{\partial}{\partial \chi} \left( Li_3 (-e^\chi) \tilde{\varphi}_1^{(1)} \right).$$ \hspace{1cm} (68)

By integrating both sides of (68) and further applying simple transformations, it turns to the following form:

$$Li_{5/2} (-e^\chi) \frac{\partial \tilde{\varphi}_1^{(1)}}{\partial \chi} = \frac{1}{2} Li_{3/2} (-e^\chi) \tilde{\varphi}_1^{(1)} + C,$$ \hspace{1cm} (69)

where $C$ is the integration constant. The last equation belongs to linear inhomogeneous type and it is solved by the method of arbitrary constant variation. In this case the solution of equations (66) and (69) is

$$\tilde{\varphi}_1^{(1)} (\chi) = C_1^{(1)} \sqrt{-Li_{3/2} (-e^\chi)} + C_2^{(1)} \sqrt{-Li_{5/2} (-e^\chi)} \int \frac{d\chi}{\left(-Li_{3/2} (-e^\chi)\right)^{1/2}}.$$ \hspace{1cm} (70)
where $C_1^{(1)}$, $C_2^{(1)}$ are the arbitrary integration constants. The second partial solution in equation (70) increases in absolute value with increasing of $z$. This fact leads to the first condition breaking in equation (17), starting from a certain value of $z$. In this case the used perturbation theory becomes inapplicable and that is why $C_1^{(1)}$ should be set to zero.

So, we have the following expression for $\varphi_1^{(1)}(z)$:

$$\varphi_1^{(1)}(z) = C_1^{(1)} \sqrt{-L_1/2 \, (-e^x)}.$$  \hspace{1cm} (71)

The general solutions of the second and the third equations in equation (28) have the form:

$$\varphi_2^{(1)}(z) = C_1^{(2)} e^{q\sigma z} + C_2^{(2)} e^{-q\sigma z}, \quad \varphi_3^{(1)}(z) = C_1^{(3)} e^{q\sigma z} + C_2^{(3)} e^{-q\sigma z}.$$  \hspace{1cm} (72)

Taking into account the finiteness of the electric field value at $z \to -\infty$, $C_2^{(3)}$ constant in equation (72) should be set to zero, $C_2^{(3)} \equiv 0$. Arbitrary constants $C_1^{(1)}$, $C_1^{(2)}$, $C_2^{(2)}$, and $C_1^{(3)}$ can be obtained from the boundary conditions in equation (29). It is easily seen that these constants are linear in $\xi^{(1)}$. According to equations (18), (27), and (28), $\xi^{(1)}$ is the first harmonic of the Fourier transform of spatially periodic profile perturbation of a liquid dielectric film surface.

$$\varphi_1^{(1)}(z) = E_0 \xi^{(1)} \sqrt{L_1/2 \, (-e^x) / L_1/2 \, (-e^x)} G(q_0),$$

$$\varphi_2^{(1)}(z) = \xi^{(1)} \left( e^{q(z-\xi)} - C e^{q(\xi - z)} \right) E_0 F(q),$$

$$\varphi_3^{(1)}(z) = \xi^{(1)} e^{q(z-\xi)} E_0 F(q) (1 - \delta),$$  \hspace{1cm} (73)

where the following notations are introduced:

$$G(q_0) = \frac{L_1/2 \, (-e^x) \left( 2bq_0 \frac{x}{d} + y_0 (\varepsilon - 1) \right)}{L_1/2 \, (-e^x) \, x y_0 + bL_1/2 \, (-e^x)},$$

$$F(q_0) = \frac{\left( \frac{\varepsilon}{\varepsilon_d} - 1 \right) L_1/2 \, (-e^x) + L_1/2 \, (-e^x) 2q_0 \frac{x}{d}}{(1 + C) \left( L_1/2 \, (-e^x) x y_0 + bL_1/2 \, (-e^x) \right)},$$

$$y_0 = 2q_0 x_0, \quad b \equiv \frac{1 - C}{1 + C}, \quad \delta \equiv \frac{\varepsilon_d - \varepsilon}{\varepsilon_d + \varepsilon}, \quad C = \delta e^{-2q_0(d + \xi)}.$$  \hspace{1cm} (74)

The general solution of equation (29) in the analogous approximation for $\varphi_j^{(e)(1)}$, $j = 1, 2, 3$ has the form:

$$\varphi_1^{(e)(1)}(z) = C_1^{(e)} e^{-q\sigma z}, \quad \varphi_2^{(e)(1)}(z) = C_1^{(e)} e^{q\sigma z},$$

$$\varphi_3^{(e)(1)}(z) = C_1^{(e)} e^{-q\sigma z} + C_2^{(e)} e^{q\sigma z}.$$  \hspace{1cm} (75)

The form of equation (75) is chosen to satisfy the finiteness conditions (14) at $z \to \pm \infty$. The constants in equation (75) can also be obtained from the corresponding linear approximation of boundary conditions in equation (29). So, the expression for $\varphi_j^{(e)(1)}$ has the form:

$$\varphi_1^{(e)(1)}(z) = (\varepsilon + 1) \left( C E \xi^{(1)} F^{(e)}(q) e^{q(z-\xi)} \right),$$

$$\varphi_2^{(e)(1)}(z) = \xi^{(1)} E F^{(e)}(q_0) \left( C e^{q(z-\xi)} - e^{q(z-\xi)} \right),$$

$$\varphi_3^{(e)(1)}(z) = (1 - \delta) \xi^{(1)} E F^{(e)}(q_0) e^{q(z-\xi)} + F^{(e)}(q) \left( 1 - e^{-\varepsilon} \right) / (\varepsilon (1 + C) + 1 - C).$$  \hspace{1cm} (76)
Let us consider the last equation in (29). Taking into account (73), (76), and (19), it can be written in the following form:

\[ \Phi(q_0) \tilde{\xi}(1) = 0, \]  

(77)

where we introduce the following function

\[ \Phi(q_0) \equiv \frac{4\pi\alpha}{E_0} \left( \kappa^2 + q_0^2\beta \right) - \frac{n}{n_s} \left( 1 - G(q_0) \right) \]

\[ - \frac{n}{n_s} (1 + C) y_0 \left( F(q_0) + \frac{F^{(e)}(q_0)}{4} \right), \quad \beta = 1 + \frac{\kappa^2\xi^2}{2}. \]

(78)

It is easily seen that equation (77) has two solutions: \( \tilde{\xi}(1) = 0 \) and \( \Phi(q_0) = 0 \). The first trivial solution \( \tilde{\xi}(1) = 0 \) describes the absence of spatially periodic structures on a liquid dielectric surface. So, in this case the liquid dielectric surface remains flat. In the case of the phase transition to the state with a spatially periodic profile of the liquid dielectric surface, we consider \( \tilde{\xi}(1) \neq 0 \) (see (15), (18), and (19)). So, in this case we choose the second solution

\[ \Phi(q_0) = 0. \]  

(79)

According to (74), the last equation defines the value of the reciprocal lattice vector as a function of the phase transition parameters: temperature \( T_c \), external clamping field \( E_c \) (or electron areal density \( n_{sc} \), see equation (47)), dielectric density \( \rho \), surface tension \( \alpha \), and permittivity \( \varepsilon \), and the solid substrate permittivity \( \varepsilon_d \). In fact, equation (79) defines a certain critical surface \( q_0 = q_0(n_{sc}, T_c) \) of the phase transition. The procedure of obtaining such a surface goes outside the analytical approach and requires numerical calculations. As the result we obtain the dependence presented in figures 14 and 15.

Figure 14 shows curves \( q_0 = q_0(n_{sc}) \) for different fixed temperatures and takes into account equation (54) in the case of liquid helium film with a thickness of \( d = 1 \text{ cm} \). These data are obtained from equation (79), which is in good agreement with the experimental data [8, 23]. According to figure 14, at \( T = 3.5 \text{ K} \) the periodic structures, having reciprocal lattice period \( q_0 \approx 27 \text{ cm}^{-1} \) (corresponding to lattice distance \( a = 2\pi q_0^{-1} \approx 0.28 \text{ cm} \)), start appearing at...
$n_{nc} > 1.2 \times 10^9 \text{ cm}^{-2}$. On the other hand, according to [23], at $T = 3.5$ K the lattice period distance is $a = 0.24$ cm. Again, according to [23], at $T = 2.5$ K the periodic structures appear at the clamping field values, higher than $E_c = 2600 \text{V/cm} = 4\pi e \cdot 1.38 \times 10^9 \text{ cm}^{-2}$.

In our case, as seen from figure 14, at $T = 2.5$ K the critical value of the external clamping field is $E_c \approx 4\pi e \cdot 1.6 \times 10^9 \text{ cm}^{-2}$. According to [8], at $T = 4.2$ K, the lattice period distance is $a = 0.176$ cm, which forms at external clamping field values higher than $E_c \approx 4\pi e \cdot 0.95 \times 10^9 \text{ cm}^{-2}$. According to the calculation of the present paper (see figure 14), the corresponding values are: $T = 4.2 \text{K}, a = 2\pi q_0^{-1} \approx 0.22 \text{cm}$, and $E_c \approx 4\pi e \cdot 1.22 \times 10^9 \text{ cm}^{-2}$.

Let us emphasize that [8, 23] described not only the case of quasi-neutrality of the system (see (47)), but also the case of charged systems. For this reason it is convenient to relate the formation of spatially periodic structures to the critical value of the clamping field $E_c$, but not to $n_{nc}$. The value of $n_{nc}$ defines the settlement rate of the dielectric surface with dimples. In other words, it defines the number of dimples per unit of dielectric surface area. Indeed, in the case of small $n_s$ values, if the clamping field exceeds the value of $E_c$, it is only possible to observe several dimples (e.g. 2, 8, 20). With increasing $n_s$, the number of dimples increases as well and the helium surface becomes more densely filled with dimples. On reaching the value of $n_s^p \approx 2 \times 10^9 \text{ cm}^{-2}$ the considered periodic structure starts breaking, because the electron clusters from dimples start sinking into the helium in the form of bubbles [45] that move towards the metal substrate electrode, generating the clamping field. In the case of $n_s < n_s^p$, the periodic structures can also be formed in a quasi-neutral system if $n_s^p > n_s > E_c(4\pi e)^{-1}$.

Below we do not consider the formation of periodic dimple structures in a charged system. In the general case this problem has a separate solution and for a non-degenerate gas of charges above a liquid dielectric surface this problem was solved in [18]. An important property of ‘massive’ helium is the dependence of $E_c$ (or $n_{nc}$ in the quasi-neutral system case, see equation (47)) on the temperature. As in experiments [46], in our case $E_c$ decreases with $T$ growth (see figure 14).

Now, let us consider the phase transition in the case of a thin film of liquid dielectric. In this situation the van der Waals forces acting on the dielectric atom predominate over the gravitation force. In such a system the possibility of the theoretical prediction and experimental registration of some interesting effects arises. As seen in figures 6 and 7, thin dielectric films provide values of permitted $n_s$ several orders higher than the massive films do. The numeric estimates show that the quite acceptable $n_s$ values for the phase transition observation are $n_s \sim 10^{12} \text{ cm}^{-2}$ or even higher. It was shown in [24] that helium films with a thickness of several hundred angstroms are stable for all $n_s$ values, and only by reducing the film thickness up to 50 angstroms or less do the electrons begin tunneling through the film towards the solid substrate. In the theoretical paper [47] it was shown that for a helium film with $d = 100$ Å located on a metallic substrate, Wigner crystals were formed in the low temperature range at $n_{nc1} \approx 10^{11} \text{ cm}^{-2}$ or higher. The further increase of the $n_s$ value up to $n_{nc2} \approx 1,37 \times 10^{12} \text{ cm}^{-2}$ leads to the so-called crystal quantum melting effect. An effect similar to the quantum melting effect was registered in [48], however at lower $n_{nc2}$ value than [47] predicted.

In terms of the current approach, the numeric evaluation of equation (79) at $T = 0$, 1 K and $d = 10^{-6}$ cm also shows that in the range $n_{nc1} \leq n_s \leq n_{nc2}$ the periodic structures can exist in a large scope of $q_0$ values. The limit values $n_{nc1} \approx 10^{11} \text{ cm}^{-2}$ and $n_{nc2} \approx 2.4 \times 10^{12} \text{ cm}^{-2}$ can be interpreted as the points of the structures appearing and disappearing (melting) correspondingly. The obtained upper limit value $n_{nc2}$ significantly differ from the analogous value in [47]. However, in the range of high density and low temperature values our approach requires more rigorous clarification, as in this range a significant role can be played by quantum effects, e.g. the exchange interaction. For this reason the given estimates demonstrate only qualitative
agreement with the similar effect evaluations in [47] and cannot provide a valid quantitative agreement. In the end of this section let us also note the qualitative agreement between the \( n_{sc}(d) \) dependence (see figure 15) obtained based on equations (78) and (79), and the experimental data [49].

6. The order parameter of the phase transition to the symmetric phase near the critical point

Before performing the calculation of the amplitude of the studied spatially periodic structures, let us make the following methodological note. As mentioned before, the formation of spatially periodic structures (considering them as 2D!) in the system of charges above a liquid dielectric surface is called Wigner crystallization (WC) [7]. On the other hand, dimple crystals are associated with charges, located in dimples of liquid dielectric surface, forming a periodic structure [8]. It is obvious that the dimple structure is 3D. In the current approach the considered periodic structures are associated with spatially periodic deformation of the liquid dielectric surface. In other words, all the periodic structures described here can be considered as being of dimple type. Thus the proposed approach seems to be available only for dimple crystals description, but not for Wigner crystal description. However, the depth of the dimples can vary in value. In the case of extremely shallow dimples, their small depth does not affect the experimental data concerned with the existence of periodic structures. Therefore, from the experimental point of view these structures are perceived to be 2D. Thus, the approach to considering these periodic structures as 2D or 3D depends on their properties (e.g. dimple depth) and the experimental registration technique. In the macroscopic dimple case the dimples are visible, and in the WC case, the phase transition is registered by indirect parameters concerned with the dynamical properties of the system [7], while the dielectric surface is assumed to be planar. In the last example we will deal with the case of a small deformation of the dielectric surface that has no affect on the experimental measurement process.

To make a theoretical grounding for this assumption let us obtain the value of the order parameter \( \tilde{\xi}(1) \) near the phase transition point. This is a cumbersome procedure, so in this

![Figure 15. Critical curves \( n_{sc}(d) \) in the thin helium film case.](image-url)
paper we do not present the detailed calculations. We only briefly describe the procedure and demonstrate the main results. In [21] the procedure of obtaining the amplitude of spatially periodic states was considered in a more detailed way for the case of non-degenerate gas.

To obtain \( \tilde{\xi}^{(2)} \) we have to make expansion of equations (5) and (8) on small perturbations \( \tilde{\xi}(\rho), \tilde{\varphi}(z, \rho), \tilde{\varphi}^{(e)}(z, \rho) \) and small differences \( T - T_c, E - E_c \) (or \( n_e - n_w \) in quasi-neutrality case (47)). Then, making the Fourier transforms (18) and (20) and taking into account the main approximations (22) and linear approximations (29)–(30) of the considered perturbation theory, we obtain the next non-vanishing approximation at \( q = q_0 \). On making these calculations it becomes obvious that we also have to obtain the relation between the first \( \xi^{(1)} \) and the second \( \xi^{(2)} \) harmonics. For this reason we have to expand equations (5) and (8) on small perturbations \( \tilde{\xi}(\rho), \tilde{\varphi}(z, \rho), \tilde{\varphi}^{(e)}(z, \rho) \), and small differences \( T - T_c, E - E_c \). Then take the Fourier transform of this expansion at \( q = 2q_0 \). Based on the methods developed in the previous section and [21], we obtain the relation between the first and second harmonics of the Fourier transform of the order parameter \( \xi \):

\[
\tilde{\xi}^{(2)} = \frac{\gamma}{\bar{z}} \left( \tilde{\xi}^{(1)} \right)^2, \tag{80}
\]

that leads to the non-linear equation for the amplitude \( \tilde{\xi}^{(1)} \) obtaining:

\[
\left( \tilde{\xi}^{(1)} \right)^3 = \tilde{\xi}^{(1)} \frac{\partial E_c(T_c)}{n} \left( \frac{\partial n}{\partial E} (E - E_c) + \frac{\partial n}{\partial T} (T - T_c) \right). \tag{81}
\]

We do not present here the explicit expressions for the \( \gamma \) and \( E_c(T_c) \) functions because of their cumbersome structure and complicated dependence on \( E_c \) and \( T_c \). Equation (81) has two solutions. The first solution \( \tilde{\xi}^{(1)} = 0 \) is trivial and it does not describe any phase transition. So, for the same reason as on obtaining the critical curve (see (77)–(79)), we do not consider this solution. Below, during the numeric evaluation of the amplitude \( \tilde{\xi}^{(1)} \) value at fixed \( E_c \) and \( T_c \) values, we use the explicit expressions for \( \gamma \) and \( E_c(T_c) \). So as in the case of calculating the first harmonic of the Fourier transform of the density perturbation at \( z = \xi \) (see equations (38), (62), and (71)):

\[
n^{(1)} = -T \frac{\partial n}{\partial \mu} \left( 1 - G(q_0) \right) \frac{\tilde{\xi}^{(1)}}{\bar{z}}. \tag{72}
\]

The numeric estimates show that \( G(q_0) < 1 \) (see equation (72)). According to equation (81), in the case \( \tilde{\xi}^{(1)} \neq 0 \) we have the situation where the charge density maximums are located above the dimples on the liquid dielectric surface. Also, the charge density minimums are located above the ‘hills’ on the dielectric surface. We have a similar situation in the case of the second solution of equation (81):

\[
\tilde{\xi}^{(1)} = \bar{z} \sqrt{\Psi \left( \frac{\partial \ln n}{\partial E} (E - E_c) + \frac{\partial \ln n}{\partial T} (T - T_c) \right)}. \tag{81}
\]

Evaluating this expression at \( T = 2.5 \text{ K}, n_i = 1.4 \cdot 10^9 \text{ cm}^{-2}, d = 0.1 \text{ cm}, \) and \( q_0 = 23 \text{ cm}^{-1} \) parameter values and \( \sqrt{(E/E_c) - 1} \approx 0.1, \sqrt{(T_c/T) - 1} \approx 0.1, \) we obtain \( \tilde{\xi}^{(1)} \approx 4.7 \cdot 10^{-11} \text{ cm} \). This value is in satisfactory agreement with the corresponding estimations of this quantity in [12, 13]. This estimation can hardly have a physical interpretation, because its value is several orders less than the typical atom size \( a_0 \sim 10^{-8} \text{ cm} \). For this reason the dielectric surface can be considered as planar. However, our estimations are made in the region \( E/E_c \sim 1,
$T/T_c \sim 1$, where the proposed perturbation theory takes place. Moving far from the transition point (decreasing temperature, increasing clamping field) can significantly change the surface structure, including appearance of the 3D period structure.

Let us emphasize that in the mentioned range of parameters $T$, $n_s$, and $q_0$ used for numerical estimates in the present and previous sections, the value of the left side of inequality equation (64) at $z = \xi$ has an order not less than $10^7$, which makes the use of equations (79)–(81) quite justified.

Thus, for the purpose of simplicity we demonstrate the benefits of the developed approach by studying the phase transition to the structured state, characterized by a single reciprocal lattice distance $q_0$. However, under the conditions of a real experiment [46], the periodic wavy structure on the dielectric surface is observed as an intermediate state during the phase transition from the homogeneous state to the 2D hexagonal structure. The description of periodic structures, characterized by two independent reciprocal lattice vectors in the plane parallel to the dielectric surface, is a separate problem, waiting for a solution in terms of the developed approach.

7. Conclusion

Summarizing the present paper, we develop a quantum-statistical theory of equilibrium spatially inhomogeneous states of the system of charges above the liquid dielectric surface in an external clamping electric field. The state of the system is considered to be quasi-neutral, i.e. the field, induced by charges, compensates the external electric field at infinity. The theory is developed in the quasi-classical approach, applying the concept of the Wigner distribution function of electrons above the liquid dielectric surface.

Beyond the scope of Boltzmann statistics we obtain self-consistency equations describing the phase transition in the system to the state with spatially periodic structures near the critical point. The benefits of this approach are demonstrated by describing the phase transition with the formation of spatially periodic structures of wave type. Applying the analytical and numeric methods, we analyze the influence of dielectric film thickness on the critical parameters of the studied phase transition. We discuss the criterion on the system stability against the possible quantum tunneling of electrons to the solid substrate. The obtained results are compared to the theoretical and experimental data previously obtained by other authors.

Such a study of the proposed system for different thickness values of the liquid dielectric film is a development of [18–21], which paid little attention to the influence of the dielectric film thickness value on the system properties. In those papers the authors considered only the case of massive films. As for the relevance of the results of the current paper to the experimental data, the authors confirmed that the effective thickness of helium film is almost independent of the number of electrons above a unit of the helium surface (see figure 8), as noted in [24, 41]. In agreement with the data of [40] we also confirm the significant increase of the available values $n_s^{\text{m}}$ for the films with a thickness less than $10^{-4}$ cm (see figure 7) and obtain qualitative agreement between the quantum melting concentration $n_s^{\text{sc}}$ and the data of [47].

As the available $n_s$ values increase significantly and the non-degeneracy condition (57) breaks, the need to go beyond the Boltzmann statistics appears, and the authors of the current paper have realized this.

The present approach requires modification in at least two directions. First, this approach can be generalized to the description of spatially periodic structures characterized by two
independent reciprocal (and direct) lattice vectors. However, as mentioned before, this requires sufficiently detailed information on the thermodynamic and dynamic stability of the phase states with spatially periodic structures in terms of the proposed theory. Second, in order to determine the applicability limits of the developed theory, it must be modified to take quantum effects into account, first of all the exchange interaction. Such a modification is particularly interesting because of the possible success in further experimental registration of quantum melting, which faces some difficulty as mentioned in [30]. At the present time the authors are working on both problems.

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