A Vishniac type contribution to the polarisation of the CMBR?

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Abstract

Radiation which has a quadrupole component of anisotropy, can get polarized by Thomson scattering from charged particles. In the cosmological context, the microwave background photons develop significant quadrupole anisotropy as they free stream away from the epoch of standard recombination. Reionization in the post recombination era can provide free electrons to Thomson scatter the incident anisotropic CMBR photons. We compute the resulting polarisation anisotropy on small (arc-minute) angular scales. We look for significant non-linear contributions, as in the case of Vishniac effect in temperature anisotropy, due to the coupling of small-scale electron density fluctuations, at the new last scattering surface, and the temperature quadrupole. We show that, while, in cold dark matter type models, this does not lead to very significant signals ($\sim 0.02 - 0.04 \mu K$), a larger small angular scale polarization anisotropy, ($\sim 0.1 - 0.5 \mu K$), can result in isocurvature type models.

I. INTRODUCTION

The cosmic microwave background radiation has a wealth of information about the parameters which govern the dynamics and the physical processes in the universe. Three important aspects of the CMBR in which the information about the universe is encoded are its temperature anisotropy, spectral distortion and anisotropy in polarization. A study of these aspects will allow one to constrain the evolution of both the background universe and the large scale structures.

Polarization of the CMBR arises from the Thomson scattering of radiation from free electrons. An analysis of polarization properties of the CMBR can reveal valuable information about the ionization history of the universe. Much of the current work on polarization anisotropy is in the context of the linear perturbation theory. The expectation generally is that higher order terms are much smaller. However, for the temperature anisotropy, on very small angular scales, of order of arcminutes, it was shown by Vishniac that nonlinear effects can also make significant contributions [1]. These arise through mode coupling of the electron density perturbations on small scales with source terms which vary over larger
scales. The Vishniac effect is especially important in models where there is significant early re-ionisation ([5], [8]).

Zaldarriaga [4] studied the effects of such early reionisation on first order polarisation anisotropy, in a semi-analytical fashion. It turns out that the CMBR can develop a significant quadrupole, by the epoch of re-ionisation, due to the free streaming of the monopole at recombination. The Thomson scattering of this quadrupole, off the electrons at the re-ionised epoch, can lead to additional polarisation signals. In this paper we wish to follow suit and examine whether this quadrupole, coupling to fluctuations in the electron density, at the new last scattering surface, can also lead to significant, Vishniac type second order effects. And result in a polarization anisotropy of the CMBR at small angular scales.

In the next section, we give the basic equations and their formal solution. Section III presents an analytical estimate of the Vishniac type contribution to the small scale polarization anisotropy, in re-ionised models. Section IV gives numerical values for some illustrative models of structure formation. We summarise our conclusions in Section V.

II. BASIC EQUATIONS AND THEIR FORMAL SOLUTION

The equations governing the evolution of polarization perturbation $\Delta P(x, \gamma, \tau)$ and temperature perturbation $\Delta_T(x, \gamma, \tau) = \Delta T/T$, for scalar modes can be derived from the moments of the Boltzman equation for photons. In the conformal Newtonian gauge, they are given by

$$\dot{\Delta}_P + \gamma_i \partial_i \Delta_P = n_e \sigma_T a(\tau)(-\Delta_P + \frac{1}{2}[1 - P_2(\mu)]\Pi)$$  \hspace{1cm} (2.1)

$$\dot{\Delta}_T + \gamma_i \partial_i \Delta_T = \dot{\phi} - \gamma_i \partial_i \psi + n_e \sigma_T a(\tau)(-\Delta_T + \Delta_{T0} + \gamma_i v_i - \frac{1}{2}P_2(\mu)\Pi)$$ \hspace{1cm} (2.2)

Here $x$ is the comoving co-ordinate, $\tau$ is conformal time, $n_e$ the electron density, $v$ the fluid velocity, $\gamma$ is the direction of photon propagation, $\phi$ and $\psi$ the conformal Newtonian potentials, and a dot represents derivative with respect to conformal time. We have also defined

$$\Pi(x, \tau) = -\Delta_{T2}(x, \tau) - \Delta_{P2}(x, \tau) + \Delta_{P0}(x, \tau).$$ \hspace{1cm} (2.3)

with $\Delta_{T0}$, $\Delta_{P0}$ the monopole, and $\Delta_{T2}$, $\Delta_{P2}$ the quadrupole temperature and polarisation perturbations, respectively. We define these angular moments by

$$\Delta_{P,T}(x, \gamma, \tau) = \sum_l (2l + 1) P_l(\mu) \Delta_{P,Tl}(x, \tau) \hspace{1cm} \Delta_{Pl,Tl} = \int \frac{d\mu}{2} P_l(\mu) \Delta_{P,T}(\mu)$$ \hspace{1cm} (2.4)

[Here, we make the usual assumption that after Fourier transformation, the evolution equations for temperature and polarisation perturbations, depend on $\gamma$, only in the combination $\gamma \cdot k$, where $k$ is the wavevector; so we define $\mu = \gamma \cdot k/k$, where $k = |k|$ (cf. ref. [2]).]

In order to treat inhomogenieties in the electron density, we take $n_e(x, \tau) = \bar{n}_e(\tau)[1 + \delta_e (x, \tau)]$, where $\delta_e$ is the fractional perturbation of electron density about the space averaged
mean. In general we will have $\delta_e \ll 1$. We may note at this point that spatial perturbations in the number density of the electrons is precisely the feature that gives rise to Vishniac effect in second order temperature perturbations. We will investigate a similar effect for polarization perturbations.

It will be convenient to express equation 2.1 in terms of the fourier modes as follows,

$$\dot{\Delta} P + i k \mu \Delta P = \bar{n}_e \sigma T a \left[ \frac{1}{2} (1 - P_2(\mu)) \Pi(k, \tau) - \frac{1}{2} (1 - P_2(\mu)) S(k, \tau) - \Delta P \right]$$

(2.5)

We have retained the same symbols for the fourier transformed quantities and defined the mode coupling source term $S(k, \tau)$ by

$$S(k, \tau) = - \int \frac{d^3 p}{(2\pi)^3} \delta_e(k - p, \tau) [\Pi(p, \tau) - \Delta P(p, \tau)]$$

(2.6)

The formal solution of Eq. (2.5) is given by

$$\Delta P(k, \tau) = \int_0^\tau \frac{1}{2} (1 - P_2(\mu)) [\Pi(k, \tau') - S(k, \tau')] g(\tau, \tau') e^{ik\mu(\tau' - \tau)} d\tau'$$

(2.7)

where the $g(\tau, \tau')$ called the visibility function is given by

$$g(\tau, \tau') = \bar{n}_e(\tau') \sigma T a(\tau') e^{-\int_\tau^{\tau'} \bar{n}_e(\tau'') \sigma T a(\tau'') d\tau''}$$

(2.8)

In the above equations, the value of the polarization perturbation at the epoch $\tau$ is determined by the entire history from $\tau' = 0$ to $\tau' = \tau$. The visibility function $g(\tau, \tau')$ determines the probability that a photon last scattered at the epoch $\tau'$ reaches us at the epoch $\tau$. The exact form of the visibility function is determined by the ionization history of the universe. Operationally, the role of the visibility function is to give different weightages for the integrand for different epochs.

In this paper we consider a model in which the universe underwent a phase of standard recombination and got reionized completely at a later epoch, $\tau_*$. We will only be concentrating on the second order polarization perturbations arising from the $S(k, \tau)$ term in Eq. (2.7). The $S(k, \tau)$ contribution to the RHS of Eq. (2.7) involves a convolution in the fourier space, which couples the first order temperature (polarisation) perturbations with the first order perturbations in the electron density. A very similar situation exists in the case of Vishniac effect in second order temperature perturbations. So we expect a Vishniac type effect in second order polarization perturbations as well.

Further, the coupling of $\Delta P$, $\Delta P_0$ and $\Delta P_2$ with $\delta_e$ in $S$, are likely to be much smaller than the coupling of $\delta_e$ and $\Delta T_2$. This is because firstly the temperature perturbations generically dominate the polarisation. Also the quadrupole temperature anisotropy, $\Delta T_2(k, \tau)$, will grow to a larger value, between the epochs of recombination and reionisation, due to free streaming of the monopole at recombination (cf. [4]). We therefore retain in $S$ only the $\Delta T_2$ term and neglect the other terms. We then have

$$S(k, \tau) \approx \int \frac{d^3 p}{(2\pi)^3} \delta_e(k - p, \tau) \Delta T_2(p, \tau)$$

(2.9)

We will adopt this approximate form for the mode coupling term in what follows.
III. VISHNIAC TYPE CONTRIBUTION IN REIONIZED MODELS: ANALYTIC ESTIMATE

The absence of "Gunn-Peterson" dips in the spectra of distant quasars indicates that the universe was probably reionised at some redshift $z = z_s > 5$. The value of $z_s$ is not known observationally, while different theoretical models have different predictions for this redshift. In the model which we consider, the universe underwent standard recombination at $\tau_s$, and was reionised completely at a later epoch $\tau_\ast$. In this case as shown in Ref. [1], the visibility function has two peaks, one around $\tau_s$ and another peak around $\tau_\ast$. We wish to consider here Vishniac type, second order contribution to $\Delta_P$. This comes dominantly from the value of $\tau'$ around the latter peak. It is then convenient to separate the integral over $\tau'$ in Eq. (2.7) in two parts: $0 < \tau' < \tau_s$ and $\tau_s < \tau' < \tau_0$, where $\tau_0$ is the present conformal time. We write $\Delta_P = \Delta_P^a + \Delta_P^b$, with

$$\Delta_P^a(k, \tau) = \int_{0}^{\tau_s} \frac{1}{2} (1 - P_2(\mu)) [\Pi(k, \tau') - S(k, \tau')] g(\tau, \tau') e^{ik\mu(\tau' - \tau)} d\tau'$$  

(3.1)

$$\Delta_P^b(k, \tau) = \int_{\tau_s}^{\tau_0} \frac{1}{2} (1 - P_2(\mu)) [\Pi(k, \tau') - S(k, \tau')] g(\tau, \tau') e^{ik\mu(\tau' - \tau)} d\tau'$$  

(3.2)

The first contribution in (3.1) is simply $\Delta_P^a \equiv \exp(-\kappa_s)\Delta_P^{NR}$, Here $\Delta_P^{NR}$ is the polarisation that would be measured if there was no reionisation and $\kappa_s$ is the optical depth to Thomson scattering between now and recombination. This contribution is reduced by the fact that only a fraction $\exp(-\kappa_s)$ of the photons that arrive at the observer come directly from recombination, without further scattering. (Also the second order contributions from the $S$ term are much smaller than the first order term because the electron density fluctuations at recombination $\delta_e(\tau_r) \sim 10^{-4} - 10^{-3} \ll 1$, for the relevant $k$ values).

In order to calculate the second contribution, one has to determine the form of the visibility function after the standard recombination epoch, that is $g(\tau_0, \tau')$ for $\tau' > \tau_r$. Using the exact form for $g(\tau_0, \tau')$, to solve for the $\Delta_P$ is not analytically tractable. So we resort to an approximation for $g(\tau_0, \tau')$ in this work, which, while preserving its main features, also allows analytical results to be derived. We will return to a full numerical treatment of the problem elsewhere. In particular, we choose the form of the visibility function after standard recombination, to be a truncated exponential, given by

$$g(\tau_0, \tau') = N \frac{1}{\sigma} e^{-\frac{(\tau' - \tau_0)}{\sigma}} \theta(\tau' - \tau_s)$$  

(3.3)

Here the Heavside $\theta(x)$ function, is zero for $x < 0$ and 1 for $x > 0$. It takes account of the fact that before reionisation, $n_e = 0$. Further, $N$ is a normalisation constant and $\sigma$ gives the spread of the exponential. By appropriately chosing $\sigma$, we can set the width of the reionised last scattering surface. Also note that $g(\tau_0, \tau')$ has the interpretation of probability; so its integral over $\tau'$ from $\tau' = 0$ to $\tau' = \tau_0$ should be normalized to unity. This determines the normalisation factor $N$. For a sufficiently early epoch of reionisation, we generally have $(\tau_0 - \tau_s)/\sigma_2 \gg 1$. In this case, the condition that the integral of $g(\tau_0, \tau')$ over $\tau'$ should be unity implies $N + e^{-\kappa_s} = 1$, or $N = 1 - e^{-\kappa_s}$. So $N$ measures the probability of at least one scattering between $\tau_0$ and $\tau_s$, due to the reionisation. Another feature to note is that in our
approximation, \( \tau_0 \) does not appear at all. This is because, for the models we will consider, the major contribution to the scattering optical depth comes from epochs much before \( \tau_0 \).

In the equation (3.2) for \( \Delta^b_p \), the first order contribution to the polarisation due to a reionised universe has already been discussed in detail by Zaldarriaga [4]. So, here, we concentrate on purely the second order Vishniac type effect, due to \( \tau \) where retaining only the \( \Delta^b_0 \) truncated exponential form of Eq. (3.3).

Evaluating the \( \tau' \) integral in Eq. (3.4) we assume the visibility function to be given by the truncated exponential form of Eq. (3.3).

Let us look at the mode coupling term \( S(k, \tau) \), given by Eq. (2.9), in a little more detail. This term involves a coupling of the quadrupole temperature perturbation at \( \tau > \tau_* \), and the electron density perturbation at the same epoch. Note that the temperature quadrupole at late times, can have a significant contribution due to the free-streaming of the monopole. For example, in a flat universe, at large enough wavelengths, the first order quadrupole temperature perturbation, is related to the temperature perturbations at recombination by [3]:

\[
\Delta_{T2}(p, \tau_e) = [\Delta_{T0}(p, \tau_r) + \psi(p, \tau_r)]j_2[p(\tau - \tau_r)]
\]  

(3.5)

where \( j_2 \) is the second order spherical Bessel function, and \( p = |p| \). (Here we have assumed that \( p \) is small enough that the doppler velocity term makes little contribution to the freestreamed quadrupole cf. [3]). The mode coupling term can then be written as

\[
S(k, \tau) = \int \frac{d^3p}{(2\pi)^3} \delta_e(k - p, \tau)[\Delta_{T0}(p, \tau_r) + \psi(p, \tau_r)]j_2[p(\tau - \tau_r)].
\]  

(3.6)

The spherical Bessel function, \( j_2(\xi) \), is well approximated by a gaussian peaked at \( \xi \sim 3.345 \) and with a spread of \( \sim 1.4 \). So \( j_2[p(\tau - \tau_r)] \) is peaked at wavenumbers around \( p = p_0 \sim 3.345/(\tau - \tau_r) \). Note that for the small scale polarisation anisotropy which we wish to calculate, \( k = |k| >> p_0 \); in general we will have \( k \sim (10\text{Mpc})^{-1} \) to \((1\text{Mpc})^{-1} \), where as \( p_0 \sim (300\text{Mpc})^{-1} \) to \((1000\text{Mpc})^{-1} \), for \( \tau > \tau_* \) (cf. Ref. [3] and see below). So in the mode coupling integral, for a fixed \( k >> p_0 \), the electron density perturbation \( \delta_e((k - p), \tau) \) varies negligibly with \( p \), in the range of \( p \) for which the the bessel function makes significant contribution. So we can evaluate \( \delta_e \) at \( p = p_0 \) and pull it out of the \( p \) integral. Also since \( k >> p_0 \), one can approximate \( (k - p_0) \sim k \). The mode coupling integral, for large \( k >> p_0 \) then simplifies to the uncoupled form,

\[
S(k, \tau) = \delta_e(k, \tau) \int \frac{d^3p}{(2\pi)^3} \Delta_{T2}(p, \tau) \equiv \delta_e(k, \tau)Q_2(\tau)
\]  

(3.7)

where we have used Eq. (3.3) to rewrite the resulting expressions in terms of \( \Delta_{T2} \) again, and defined for later convenience,

\[
Q_2(\tau) = \int \frac{d^3p}{(2\pi)^3} \Delta_{T2}(p, \tau)
\]  

(3.8)
We now evaluate $\Delta \mathcal{V}$, using Eq. (3.4), and Eq. (3.3). Let us assume that the re-ionisation epoch is late enough, that the electron density perturbation trace the perturbations in total matter density. Then, in a flat universe, the electron density perturbations grow as $a(\tau) \propto \tau^2$. For an open model or one involving a cosmological constant $\Lambda$, the growth law is given as

$$
\delta_e(k, \tau) = \left(\frac{\tau}{\tau_0}\right)^2 \delta_e(k, \tau_0) \frac{f(\Omega(\tau))}{f(\Omega_0)} f^{-1}(\Omega_0),
$$

(3.9)

Here, $\tau_0$ is present conformal time and the functions $f$ takes into account the reduced growth in the open or flat $\Lambda$ models at late times, compared to a flat, dark matter dominated model. In a flat universe $\Omega(z) = \Omega_0 = 1$, $f(1) = 1$ and we recover the $\delta_e \propto \tau^2$ growth law. The expressions for $f$, and $\Omega(z)$ for $\Lambda$ dominated and the open models can be found in ref. [15] and [17], respectively (in these papers, our $f(\Omega)$ is called $g(\Omega)$). For $z \gg 1$, or $\tau/\tau_0 << 1$ and $f(\Omega(z)) \to 1$. So the approximation $\delta_e(k, \tau) = \left(\tau/\tau_0\right)^2 \delta_e(k, \tau_0)$ works very well for these other models as well, at sufficiently early times.

Further, as displayed explicitly in ref. [6], the power in the CMB monopole, per unit logarithmic interval of $p$ space, $p^3\left[\Delta T_0(p, \tau) + \psi(p, \tau)\right]$ is roughly constant on scales $p < (100 \text{ Mpc})^{-1}$. (This reflects the fact that perturbations on scales larger than the Hubble radius at recombination have not evolved, and are laid out with constant power, for a scale invariant initial power spectrum). Recall that the presence of the $j_2$ term, in the integral over $p$, picks out dominantly contributions from $p \sim p_0 < (300 \text{ Mpc})^{-1}$, at any time $\tau > \tau_*$. Now in any realisation, one does expect some variation of the monopole, as $p$ is varied. Nevertheless, because of the constancy of monopole power with $p$, for $p \sim p_0$, one expects the integral term $Q_2(\tau)$ in Eq. (3.7), to vary much slower with $\tau$, than the electron density perturbation $\delta_e(k, \tau)$, or the visibility function. This will especially be so, if the visibility function is sufficiently peaked around the reionisation redshift (for $\sigma$ small enough). So when evaluating $\Delta \mathcal{V}$, we will assume that $Q_2(\tau)$ can be evaluated at some effective $\tau_e \sim \tau_*$, and pulled out of the integral over conformal time $\tau'$.

The remaining integral, which can be done analytically, then gives

$$
\Delta \mathcal{V} = -\frac{1}{2}[1 - P_2(\mu)] \exp^{-ik\mu(\tau - \tau_*)} \frac{NF(k, \mu)}{(1 - i k\mu\sigma)} \delta_e(k, \tau_*) Q_2(\tau_e),
$$

(3.10)

where we have taken $(\tau_0 - \tau_*)/\sigma >> 1$, as before, and defined the factor $F(k, \mu)$

$$
F(k, \mu) = 1 + \frac{2\sigma}{\tau_* (1 - i k\mu\sigma)} + \frac{1}{\tau_*^2 (1 - i k\mu\sigma)^2}.
$$

(3.11)

(We have also assumed $\tau_*/\tau_0$ is small enough that $\Omega(\tau_*) \approx 1$).

We are now in a position to compute the Vishniac type contribution to the polarisation power spectrum. We define this simply by analogy to the temperature power spectrum (cf. [7]). Recall that, for the temperature perturbations, one first expands in spherical harmonics, with

$$
\frac{\Delta T}{T} = \Delta T(\mathbf{x}, \gamma, \tau_0) = \sum_{lm} a_{lm} Y_{lm}(\gamma).
$$

(3.12)
Then the mean square temperature perturbation is

\[
< (\frac{\Delta T}{T})^2 > = \sum_l \frac{(2l + 1)}{4\pi} C_{Tl} \equiv \int Q_T(k) \frac{dk}{k}
\]

(3.13)

where

\[
C_{Tl} = |a_{lm}|^2 = 4\pi \int \frac{k^2 dk}{2\pi^2} < |\Delta T_l(k, \tau_0)|^2 >
\]

(3.14)

and

\[
Q_T(k) = \frac{k^3}{2\pi^2} \int_{-1}^{+1} < |\Delta T| > d\mu.
\]

(3.15)

(Note that, as before, we have taken the normalisation volume, over which periodic boundary conditions are assumed to be \(V = 1\)). \(Q_T(k)\) gives the power in temperature perturbations in any logarithmic interval of \(k\).

We define therefore, a corresponding Vishniac type contribution to the polarisation power spectrum \(Q_P(k)\), given by (cf. [8]),

\[
Q_P^V(k) = \frac{k^3}{2\pi^2} \int_{-1}^{+1} < |\Delta V^p| > d\mu.
\]

(3.16)

We calculate \(Q_P^V\) below. For this, first take the ensemble average of \(|\Delta V^p|^2\). We get

\[
< |\Delta V^p|^2 > = \frac{1}{4} [1 - P_2(\mu)]^2 \frac{N^2 |F|^2}{1 + k^2 \mu^2 \sigma^2} P_e(k, \tau_*) \left[ \frac{1}{5} \left( \frac{\Delta T}{T} \right)^2 Q(\tau_e) \right]
\]

(3.17)

In the above we have assumed that the \(\delta_e\) and \(\Delta T_2\) are uncorrelated with each other, defined the power spectrum of electron density fluctuation as, \(P_e(k, \tau_*) \equiv < |\delta_e(k, \tau_*)|^2 >\) and also defined

\[
\left( \frac{\Delta T}{T} \right)^2 Q(\tau_e) \equiv \frac{5C_{T2}(\tau_e)}{4\pi} = \frac{5}{4\pi} \int \frac{d^3p}{(2\pi)^3} < |\Delta T_2(p, \tau_e)|^2 >
\]

(3.18)

Here, \(\left( \frac{\Delta T}{T} \right)^2 Q(\tau_e)\) is the quadrupole temperature anisotropy as seen by an observer at the conformal time \(\tau_e\).

Now turn to the integral of \(< |\Delta V^p|^2 >\) over \(\mu\). The factor, \(1 - P_2\) can be expressed as a sum of even powers in \(\mu\).

\[
[1 - P_2]^2 = \frac{9}{4} (1 + \mu^4 - 2\mu^2)
\]

(3.19)

We are generally interested in the behaviour of the power spectrum for large values of \(k\), or small angular scales. In this case the dominant contribution to the integral over \(\mu\), in determining \(Q_P^V\), will come from the vicinity of \(\mu = 0\). It then suffices to retain only the first term in the above expression for \((1 - P_2)^2\). The integral over \(\mu\) can be done analytically to give the remarkably simple expression
\[ Q^V_p(k, \tau_0) = \frac{9\pi N^2}{160} G_0 \left( \frac{\sigma}{\tau_s} \right) \left( \frac{\Delta^2(k, \tau_s)}{k\sigma} \right) \left( \frac{\Delta T}{T} \right)_Q \left( \tau_e \right). \]  

(3.20)

In doing the integral we have assumed that \( k\sigma \gg 1 \) is large enough that \( \tan^{-1}(k\sigma) \approx \pi/2 \), and defined the factor \( G_0(y) = 1 + 2y + 3y^2 + 5y^3/2 + 35y^5/32 \), where \( y = \sigma/\tau_s \). Further,

\[ \Delta^2_e(k, \tau_s) = \frac{k^3 P_e(k, \tau_s)}{2\pi^2} = \Delta^2_e(k, \tau_0) \left( \frac{\tau_s}{\tau_0} \right)^4 \left( \frac{f(\Omega(z_*))}{f(\Omega_0)} \right)^2 \]  

(3.21)

is the power per until logarithmic interval in \( k \) space, of the electron density perturbations, at the epoch \( \tau = \tau_* \).

We see that the contribution to polarisation anisotropy, due to the second order Vishniac type effect, for re-ionised models, is basically proportional to the product of the temperature quadrupole, and the power in electron density perturbations, at last scattering. For small angular scales, or large \( k \), \( Q^V_p \) is suppressed because of the finite thickness of the last scattering surface (\( \sigma \)), by a factor \( k\sigma \). We note that this suppression is much milder than estimated in Ref \[8\], essentially because in that paper, the first order temperature quadrupole contribution due to free-streaming of the monopole at recombination, was not included. The power spectrum of electron density perturbations of course depends on the model for structure formation. Also the parameters \( \sigma, \tau_s, N \) depend on the re-ionisation history. However the power in the temperature quadrupole at \( \tau_* \approx \tau_s \) is likely to be of order the observed quadrupole; for a large enough \( \tau_e \) and if it arises due to the free-streaming of the monopole at recombination. This is again because of the slow variation of \( Q_2(\tau) \) mentioned previously. We now use Eq. (3.20) to make numerical estimates of the polarisation due to Vishniac type effect in CDM and other models of structure formation.

IV. NUMERICAL ESTIMATES IN DIFFERENT MODELS

A. CDM and variants

Consider first the case of a standard CDM model (SCDM), with matter density equal to critical density (\( \Omega_0 = 1 \)), a baryonic contribution \( \Omega_b = 0.05 \), and a Hubble constant \( h = (H_0/100\text{km}\text{s}^{-1}\text{Mpc}^{-1}) = 0.5 \). The optical depth to Thomson scattering in a fully ionised, matter dominated, flat universe is given by \( \kappa(z) = 0.0418 \Omega_B h[(1+z)^{3/2} - 1] \). In general, for a universe with matter density \( \Omega_0 \), and assuming \( z_* \gg 1 \), an optical depth \( \kappa_* \) is obtained at a redshift

\[ z_* \approx 97.1 \kappa_*^{2/3} \Omega_0^{1/3} \left( \frac{\Omega_B h}{0.025} \right)^{-2/3}. \]  

(4.1)

So to have \( \kappa_* = 1 \), in standard CDM, we need \( z_* \approx 97.1 \), and \( \kappa_* = 0.2 \) needs \( z_* \approx 33.2 \). Also in a flat matter dominated universe, the conformal time is related to the redshift by \( \tau = \tau_0/(1+z)^{1/2} \), where \( \tau_0 = 2H_0^{-1} = 6000h^{-1}\text{Mpc} \). (Note we adopt units with \( c=1 \)). So given \( z_* \), this fixes \( \tau_* \); for \( \kappa_* = 1 \), we have \( \tau_* \approx 605.8h^{-1}\text{Mpc} \), while for \( \kappa_* = 0.2 \), we have \( \tau_* \approx 1025.8h^{-1}\text{Mpc} \).

In order to estimate the parameter \( \sigma \), in the model visibility function (3.3), we proceed as follows. Let \( \tau_m = \tau_* + \sigma \). From Eq. (3.3), at epochs after re-ionisation, we
have \( g(\tau_0, \tau_s)/g(\tau_0, \tau_m) = e \). For the exact visibility function, the same ratio is given by \( g(\tau_0, \tau_s)/g(\tau_0, \tau_m) = [a^2(\tau_m)/a^2(\tau_s)] \exp(\kappa_m - \kappa_s) \), where \( \kappa_m = \kappa(\tau_m) \). Equating these two expressions gives an estimate of \( \sigma \). In particular, using \( \kappa \propto (1 + z)^{3/2} \), valid for large \( z \gg 1 \) and \( a \propto t^2 \propto (1 + z)^{−1} \), we have the implicit equation for \( \tau_m/\tau_s, 1 = 4\ln(\tau_m/\tau_s) - \kappa_s[1 - (\tau_s/\tau_m)^3] \). For \( \kappa_s = 1 \), this gives \( \sigma \approx 0.54\tau_s = 327h^{-1}Mpc \), while for \( \kappa_s = 0.2 \), one gets \( \sigma = 0.32\tau_s = 328.3h^{-1}Mpc \).

It remains to fix \( \Delta_\kappa \) and the temperature quadrupole. We take the \( \Delta_\kappa(k, \tau_0) = \Delta(k) \), where the matter power spectrum

\[
\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2} = \left( \frac{k}{H_0} \right)^4 \delta_H^2(k) T^2(k)
\]

with the transfer function \( T(k) \) in the form given by Bardeen et al. [3]

\[
T(q) = \frac{\ln(1 + 2.34q)}{2.34q[1 + 3.89q + (16.1q)^2 + (5.64q)^3 + (6.71q)^4]^{1/4}}
\]

where \( q = k/(h\Gamma) \) (cf. ref. [4]; [5]). The parameter \( \Gamma \) is referred to as the shape parameter, is given in ref. [4] (eq. D28 and E12). It is of order 0.48 for standard CDM. The four-year COBE normalisation gives \( \delta_H(k = H_0) = 1.94 \times 10^{-5} \) (cf. ref. [2]), for a scale invariant initial power spectrum (with \( n = 1 \)). For such initial conditions, the COBE data also give the \( (\Delta T/T)_Q(\tau_0) = 18 \pm 1.6\mu K \) [3]. The value of \( \tau_s > 1000Mpc \), that we generally obtain, is likely to be large enough so that \( (\Delta T/T)_Q(\tau_e) \sim (\Delta T/T)_Q(\tau_0) \), with reasonable accuracy. So we will scale the quadrupole at \( \tau_e \) with the present day observed value. From the above considerations, and normalising all quantities to a value of \( k = k_c = 1hMpc^{-1} \), we get

\[
\sqrt{Q^V_P(k)} = 2.2 \times 10^{-2}hK \left( \frac{\Delta^2(k)/k}{\Delta^2(k_c)/k_c} \right)^{1/2} \left( \frac{(\Delta T/T)_Q(\tau_e)}{18hK} \right) ; \quad \text{for} \quad \kappa_s = 1
\]

In case \( \kappa_s = 0.2 \), one has to replace the numerical value in (4.4) by \( \sqrt{Q^V_P(k_c)} \approx 1.5 \times 10^{-2}hK \). Lower values of \( \kappa_s < 1 \) may already be implied by the observed tentative rise in the CMB anisotropy on degree scales. Note that decreasing \( \kappa_s \), means a decrease in the fraction of photons last scattered from the re-ionised epochs, and so a decrease in \( Q^V_P \). But at the same time since \( z_s \) is decreased, the electron density perturbations at last scattering are larger than the \( \kappa_s = 1 \) case, which partially compensates by increasing \( Q^V_P \). If the power spectrum can be approximated locally as a power law \( \Delta^2(k) \propto k^{3+n} \), then \( Q^V_P(k) \propto k^{2+n} \). Recall that on galactic scales with \( k \sim k_c, n \sim -2 \), while for \( k << k_c, n \sim -1 \) and for \( k >> k_c, n \rightarrow -3 \). So the polarisation anisotropy \( Q^V_P(k) \propto k^{2+n} \), will increase with \( k \) at \( k << k_c \), and decrease for \( k >> k_c \), attaining a maximum at \( k \sim k_c \). This was one of the reasons for normalising our estimate to \( k = k_c \).

For small angular scales, one can also set up an approximate correspondence between the wavenumber \( k \) and the angular multipole number \( l \), using \( l = kR_\ast \). Here \( R_\ast \) translates co-moving distance at the surface of last scattering, (roughly the epoch when \( \tau = \tau_s \)), to angle on the sky, and for a flat model is given by \( R_\ast = \tau_0 - \tau_s \). In case \( \sigma/\tau_0 << 1 \), this approximation will be reasonable, but for a thick last scattering surface the above correspondence is less accurate. In the model discussed above, where \( \kappa_s = 1 \), a wavenumber \( k = k_c = 1hMpc \), corresponds to \( l \approx 5394 \), and an angular scale \( \sim 1/l \sim 0.64 \) arc minutes.
We briefly discuss the predicted small angular scale polarisation anisotropy \( Q_\nu^V \), for some variations on the standard CDM model, by using the same method of calculation as above. For example, increasing the baryon density to \( \Omega_b = 0.1 \), leads to a smaller redshift of last scattering with a given \( \kappa_\ast \) but also a smaller \( \Gamma \). For \( \kappa_\ast = 1 \), one gets a slightly larger value \( \sqrt{Q_\nu^V(k_c)} \sim 2.9 \times 10^{-2} \mu K \), than in SCDM. Also if the primordial spectrum is tilted to \( n = 1.2 \), the best fit slope determined by COBE [13], (keeping all other parameters of SCDM same), then also one has a larger value \( \sqrt{Q_\nu^V(k_c)} \sim 4.0 \times 10^{-2} \mu K \).

Suppose we adopt a \( \Lambda+ \) CDM type model, as discussed for example in Ref. [14], with \( \Omega_0 = 0.35, \Omega_\Lambda = 0.65, h = 0.7, n = 1, \Omega_b = 0.04 \). Then we get \( z_\ast \sim 63 \) for \( \kappa_\ast = 1 \). Assuming that \( \tau_\ast \) is early enough that \( \sigma \) can be estimated as for the flat universe, we have

\[
\sigma = \frac{6000 h^{-1} \text{Mpc}}{\Omega_0^{1/2} (1 + z_\ast)^{1/2}} \left( \frac{\tau_m}{\tau_\ast} - 1 \right)
\]

So for the \( \Lambda+ \) CDM model, we get \( \sigma \sim 687.5 h^{-1} \text{Mpc} \). For this model \( f^{-1}(\Omega_0) \sim 1.24 \), and adopting a normalisation of the power spectrum as in [12,13] we then get \( \Delta_\ast(k_c, z_\ast) \sim 3.56/(1 + z_\ast) \), which leads to \( \sqrt{Q_\nu^V} \sim 1.9 \times 10^{-2} \mu K((\Delta T/T)_Q(\tau_e)/18 \mu K) \). Note that in this model the integrated Sachs-Wolfe effect, which makes a contribution to the present day quadrupole, will make little contribution at redshifts \( \sim z^\ast \). However, using Eq. (10) of ref. [10], we estimate that this will cause only a 10% reduction in the above value. For an open CDM model (OCDM), with \( \Lambda = 0 \), but all other parameters as for the above \( \Lambda+ \) CDM model, \( z_\ast \) is the same and \( \Delta_\ast(k_c) \) is of the same order, using the power spectrum as determined by [7]. However the quadrupole at last scattering is likely to be smaller; because the integrated Sachs Wolfe effect contributes a larger part of present day quadrupole. So the predicted \( \sqrt{Q_\nu^V(k_c)} \) is likely to be smaller than the for the above models.

### B. Isocurvature type models

Finally, consider as an alternative to the standard models, the isocurvature model recently discussed by Peebles [15]: where density perturbations are provided by CDM that is the remnant of a massive scalar field frozen from quantum fluctuations during inflation. The novel feature of such a picture, as pointed out in [15], is that the primeval CDM mass distribution is proportional to the square of a random Gaussian process; so prominent upward fluctuations are much larger (by factor \( F \sim 3 \)), than for a Gaussian process with the same RMS. The merits of such a picture has been discussed by Peebles [15]. We consider two representative models. Model 1 discussed in [15] adopts \( \Omega_0 = 0.3, \Lambda = 0.7, \Omega_b = 0.05, h = 0.7 \) and, a matter power spectrum, which, on the relevant small scales, can be approximated as \( \Delta_\ast^2(k) = (k/k_0)^{3+m} \), with \( m = -1.8 \), \( k_0 = 0.1 h \text{Mpc}^{-1} \). And for model 2, \( \Omega_0 = 0.1, \Lambda = 0.9, \Omega_b = 0.05, h = 0.7 \) and, \( m = -1.4 \).

Note that in these models, due to early structure formation, re-ionisation is expected to occur at large redshifts. The optical depth to electron scattering, measured from the present epoch could then rise to values larger than unity. However the possible ionisation history in these models is largely unexplored. In order to get a preliminary estimate of the anisotropies in polarisation that could be generated, we simply use Eq. (3.21) (implicitly making the simplifying assumptions which went into its derivation). So the universe after
standard recombination is assumed to be largely neutral, and then re-ionised after an epoch \( \tau = \tau_* \). Again at \( \tau \sim \tau_* \), a quadrupole would arise from the free-streaming of the large scale entropy perturbation at recombination (the isocurvature effect cf. [19]). Further, in Eq. (3.3) we assume \( \tau_* \) to be the epoch with \( \kappa_* = 1 \), but take \( N \sim 1 \) (to reflect the fact that little of the small angular scale anisotropy is due to conventional last scattering at around the re-combination epoch). We hope to return to a better treatment of these models in future work.

For model 1 of Peebles [18], one then gets \( z_* \sim 52 \), and from Eq. (1.3), \( \sigma = 821 h^{-1} Mpc \). Also for \( z_* \gg 1 \), we have \( f(\Omega(z_*)) \rightarrow 1 \), while for \( \Omega_0 = 0.3, \Lambda = 0.7, f^{-1}(\Omega_0) \sim 1.3 \). Putting in all the numerical values, we then get

\[
\sqrt{Q_P^V(k)} \approx 2.7F \times 10^{-2} \mu K \left( \frac{\Delta T/T}{Q(\tau_e)} \right) \left( \frac{k}{1 hMpc^{-1}} \right)^{0.1} \quad (4.6)
\]

Here we have scaled \( (\Delta T/T)Q(\tau_e) \) by a smaller value of \( 10 \mu K \), since these isocurvature models predict a somewhat smaller quadrupole than SCDM models (cf. Fig 1. of [18]). We have also incorporated a factor \( F \), to remind ourselves that the non-gaussian statistics of the density field, may lead to \( F \) times larger prominent upward fluctuations. If we wish to convert \( k \) to \( \ell \) in this model, we again use \( \tau_* = \tau_0 - \tau_* \), with \( \tau_0 \sim 2.17/(\Omega_0^{1/2}H_0) \sim 11,862.2h^{-1} Mpc \) (cf. Eq. (20) of ref. [21]). Also \( \tau_* \sim 1520.6h^{-1} \) and so \( k = 1 hMpc \) corresponds to \( \ell \sim 10,342 \) or an angular scale \( \sim 0.33 \) arc minutes.

A similar analysis for model 2, gives \( z_* \sim 36, \sigma \sim 1710 h^{-1} Mpc \), and a larger polarisation signal, with

\[
\sqrt{Q_P^V(k)} \approx 5.7F \times 10^{-2} \mu K \left( \frac{\Delta T/T}{Q(\tau_e)} \right) \left( \frac{k}{1 hMpc^{-1}} \right)^{0.3} \quad (4.7)
\]

Note that Peebles adopts a cosmological constant for convenience of analysis. If we were to consider open versions of these models, \( z_* \) and \( \sigma \) are nearly the same (since \( z_* \gg 1 \), but the power in electron density perturbation at last scattering is much larger, because of a much larger \( f^{-1}(\Omega_0) \). The numerical value, in Eq. (4.6) and (4.7), at \( k = 1 hMpc^{-1} \) gets increased respectively to \( \sqrt{Q_P^V(k_c)} \sim 4.6F \times 10^{-2} \mu K \) (model 1) and \( \sqrt{Q_P^V(k_c)} \sim 0.17F \mu K \) (model 2). The \( k \) dependence remains the same. In the open models, the density on scales of \( k = k_c \) are already going non-linear at \( z_* \) and so the above numbers provide only a crude estimate. We see in general that these isocurvature models predict much larger polarisation signals compared to the SCDM type models. First, the RMS value is larger. Further, because of the non-gaussian statistics of the density field, one expects prominent upward fluctuations \( F \sim 3 \) times larger than the RMS value (cf. [18]).

We note in passing that the older versions of the baryonic isocurvature models, say with \( \Omega_0 \sim 0.2, \Omega_b = 0.05, h = 0.8, m \sim -0.5 \) cf. [21], leads to even larger signals with \( \sqrt{Q_P^V} \sim 0.3 \mu K((\Delta T/T)Q(\tau_e)/10 \mu K)(k/k_c)^{3/4} \). However these models may already be ruled out by the fact that they result in spectral distortions, larger than the limit implied by the COBE observations [22].
V. CONCLUSIONS

In this paper we have explored the possibility of a Vishniac type contribution to the polarisation anisotropy at small angular scales. It is well known that non-linear effects can make significant contribution to temperature anisotropy on small angular scales, through the Vishniac effect, especially in re-ionised models. This arises due to the mode coupling of large angular scale, first-order velocity perturbations, with small angular scale electron density perturbations. We have considered here whether a similar effect contributes to the polarization anisotropy, by studying the coupling of large angular scale, first-order temperature anisotropy (quadrupole) with small angular scale electron density perturbations, in re-ionized models.

We find that in cold dark matter models and its variants, the Vishniac type effect leads to a fairly small polarisation anisotropy, with \( \sqrt{Q_{P}^{V}} \sim 0.02-0.04 \mu K \), on scales with \( k \sim 1hMpc \) (or angular scales of arc minute or smaller). However in isocurvature type models the Vishniac type contribution can result in much larger signals. For the models of Ref. [18], the anisotropy on small angular scales is non-gaussian, with prominent upward fluctuations of order \( 0.1-0.5 \mu K \), assuming \( F \sim 3 \). This reflects basically the fact that, the isocurvature type models have much more power on small scales and so produce much larger electron density fluctuations. We note in passing, that the suppression factor, due to the finite thickness of the last scattering surface, on the small scale polarisation anisotropy, is much milder than that obtained in ref. [8]. This is because, as mentioned earlier, the first order temperature quadrupole contribution, arising due to the free-streaming of the monopole at recombination, was not included in their analysis.

It is clear that the polarisation signals on arc minute scales predicted by CDM type models will be difficult to detect, but those predicted by isocurvature type models will be much easier. If small scale polarisation anisotropy is eventually detected, it will open up the novel prospect of studying directly, both the quadrupole anisotropy and small scale electron density fluctuations, at high redshifts. As pointed out in ref. [23], in a different context, one can then also reduce the cosmic variance of the quadrupole significantly. In this paper we have made several approximations to analytically estimate the polarization anisotropy. We plan to return to a better numerical analysis in the near future.

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