Critical behaviors and phase transitions of black holes in higher order gravities and extended phase spaces

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We consider the critical behaviors and phase transitions of Gauss Bonnet-Born Infeld-AdS black holes (GB-BI-AdS) for \( d = 5, 6 \) and the extended phase space. We assume the cosmological constant, \( \Lambda \), the coupling coefficient, \( \alpha \), and the BI parameter \( \beta \) to be thermodynamic pressures of the system. Having made these assumptions, the critical behaviors are then studied in the two canonical and extended phase spaces. We find "reentrant and triple point phase transitions" (RPT-TP) and "multiple reentrant phase transitions" (multiple RPT) with increasing pressure of the system for specific values of the coupling coefficient \( \alpha \) in the canonical ensemble. Also, we observe a reentrant phase transition (RPT) of GB-BI-AdS black holes in the grand canonical ensemble and for \( d = 6 \). These calculations are then expanded to the critical behavior of Born-Infeld-AdS (BI-AdS) black holes in the third order of Lovelock gravity and in the grand canonical ensemble to find a Van der Waals behavior for \( d = 7 \) and a reentrant phase transition for \( d = 8 \) for specific values of potential \( \phi \) in the grand canonical ensemble. Furthermore, we obtain a similar behavior for the limit of \( \beta \to \infty \), i.e charged-AdS black holes in the third order of the Lovelock gravity. Thus, it is shown that the critical behaviors of these black holes are independent of the parameter \( \beta \) in the grand canonical ensemble.

I. INTRODUCTION

Black holes are indeed known as the thermodynamic objects that can be described by a physical temperature and an entropy [1,8]. Black hole thermodynamics continues to be one of the most important subjects in gravitational physics. The first attempts to explain the instabilities of anti-de Sitter (AdS) black holes are due to Hawking and Page in 1983 [4]. They explored the existence of a specific phase transition in the phase space of the Schwarzschild AdS black hole. Thermodynamics of black holes in the AdS space has attracted a lot of attention for many years due to the AdS/CFT correspondence [5,7]. It has been shown that the properties and critical behaviors of black holes in the AdS space are different from those of the black holes in an asymptotically flat spacetime [8-34]. The critical behaviors of the black hole in the AdS space have been studied in [35,43] by including the cosmological constant as a thermodynamic pressure in the first law of black hole thermodynamics. In this approach, the black hole mass \( M \) is replaced by enthalpy rather than by internal energy. Recent studies have shown the analogy between charged black holes in an AdS space and the Van der Waals fluid in an extended phase space [35,90,45]. Also, the phase diagrams of rotating black holes with single and multiple spinnings are similar to those of reentrant phase transition and triple point phenomena, respectively [20,62].

The low energy effective action of the string theory contains both Einstein Hilbert and higher order of curvature terms. In this condition, the Gauss-Bonnet and third order Lovelock theory are the most important higher order terms in the theory of gravity. The black hole solutions and their thermodynamic quantities of the third order Lovelock theory have been investigated in [53]. Recently, the static black hole solutions of Gauss Bonnet-Born Infeld gravity and the third order of Lovelock-Born-Infeld gravity in the AdS space were investigated in [53,54]. There have also been efforts to explore the critical behavior and phase transitions of black holes in higher order gravities. The critical behavior of charged AdS-Gauss-Bonnet black holes for \( d = 6 \) demonstrated the possibility of triple point phenomena in the canonical ensemble [55]. Also, the critical behavior of higher order gravities including the Gauss-Bonnet and the third order Lovelock gravity have been investigated in [56,61].

This paper investigates the critical behaviors and phase transitions of GB-BI-AdS black holes in the canonical ensemble for \( d = 5, 6 \). We enlarge the phase space by considering the BI parameter and the coupling coefficient as thermodynamic pressures. We observe a new critical behavior dependent on the coupling coefficient \( \alpha \) in the canonical ensemble. It is found that for \( 0 \leq \alpha < 13 \), the system behaves similar to the standard liquid/gas of the Van der Waals fluid. For \( 13 \leq \alpha < 16 \) and \( 18 \leq \alpha < 40 \), the black hole admits a reentrant large/small/large black hole phase transition. For \( 16 \leq \alpha < 18 \), a reentrant phase transition occurs for a special range of pressures while we also observe a triple point phenomenon as the pressure increases. For \( \alpha > 40 \), there is no phase transition. We study the critical pressures with respect to the coupling coefficient \( \alpha \) for these black holes.

The Van der Waals behavior is investigated in the BI-AdS black holes for \( d = 5, 6 \) in the canonical ensemble. Moreover, the critical behavior of both BI-AdS and charged-AdS black holes is investigated in the third order of Lovelock gravity in the grand canonical ensemble. We
find the Van der Waals behavior for \( d = 7 \) and a RPT for \( d = 8 \) for the special values of potential \( \phi \) in the grand canonical ensemble. Thus, the critical behaviors of these black holes are independent of the coupling coefficient \( \beta \) in the grand canonical ensemble.

The outline of this paper is as follows: In Sec. II, the critical behavior and phase transitions of GB-BI-AdS black holes are examined in the canonical and grand canonical ensembles for \( d = 5, 6 \). Also, we study the BI-AdS black holes for \( d = 5, 6 \) and in the canonical ensemble. In Sec. III, the critical behavior and phase transitions of BI-AdS and charged-AdS black holes in the third order of the Lovelock gravity are investigated for \( d = 7, 8 \) in the grand canonical ensemble.

II. GAUSS-BONNET-BORN-INFELD-ADS BLACK HOLES

The action of the Einstein Gauss-Bonnet gravity in the presence of a nonlinear Born-Infeld electromagnetic field with a negative cosmological constant reads as follows [54]:

\[
I = \frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left[ R - 2\alpha + \Lambda \mathcal{L}_{GB} + \mathcal{L}_F \right],
\]

where, \( \alpha \) is the Gauss-Bonnet coefficient and \( \Lambda = -\frac{n(n-1)}{2}\pi \) is a negative cosmological constant.

The Gauss-Bonnet Lagrangian and \( \mathcal{L}_F \) are given by

\[
\mathcal{L}_{GB} = R_{\mu\nu\delta\gamma}R^{\mu\nu\delta\gamma} - 4R_{\mu\nu}R^{\mu\nu} + R^2,
\]

\[
\mathcal{L}_F = 4\beta^2 \left( 1 - \frac{F_{\mu\nu}F^{\mu\nu}}{2\beta^2} \right),
\]

where, the constant \( \beta \) is the BI parameter and the Maxwell field strength is defined by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) with \( A_\mu \) as the vector potential. Let us consider the following metric:

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dx^2 + r^2 h_{ij} dx^i dx^j,
\]

where, \( h_{ij} \) is a \( (n-1) \)-dimensional hypersurface. The metric coefficient \( f(r) \) for static GB-BI-AdS black holes is given by

\[
f(r) = 1 + \frac{\alpha}{\sqrt{2\alpha}} (1 - \sqrt{g(r)}),
\]

where, \( g(r) \) is

\[
g(r) = 1 - \frac{4\alpha + 4\alpha m}{r^n} - \frac{16\alpha r^2}{n(n-1)} - \frac{8(n-1)\alpha q^2}{n r^{2n-2}} \times 2F_1 \left[ \frac{n - 2}{2n - 2} ; \frac{3n - 4}{2n - 2} - \frac{(n-1)(n-2)q^2}{2\beta^2 r^{2n-2}} \right] + \frac{8\sqrt{2}\alpha \beta}{n(n-1)r^{n-1}} \sqrt{\frac{\alpha q^2}{r^{2n-2}} + (n-1)(n-2)q^2},
\]

where,

\[
\alpha = (n-2)(n-3)\bar{\alpha},
\]

\[
Q = q \frac{\sum_{n-1}}{4\pi} \left( \frac{(n-1)(n-2)}{2} \right),
\]

and \( m \) is an integration constant which is related to mass

\[
M = \frac{(n-1)\sum_{n-1}}{16\pi} - \frac{1}{16\pi} \left[ ak^2 r^{-4} + kr^{-2} 
+ \frac{8Q^2 r^{-2n}}{n(n-2)} \times F_1 \left[ \frac{1}{2} ; \frac{n-2}{2n-2} ; \frac{3n-4}{2n-2} \right] - \frac{16\pi^2 Q^2 r^{-2n}}{\beta^2} \right]
+ \frac{2r^4}{n(n-1)} \left( -2\beta^2 \left[ \frac{2Q^2 r^{-2n}}{\beta^2} + \frac{1}{2} + 2\beta^2 + 8\pi P \right] \right),
\]

where, \( \sum_{n-1} \) exhibits the volume of the constant curvature hypersurface described by \( h_{ij} dx^i dx^j \) and \( r_+ \) is the horizon radius of the black hole determined by the largest real root of \( f(r_+) = 0 \). In the following calculations, we consider \( \sum_{n-1} = 1 \) and the specific case \( k = 1 \) for simplicity. The thermodynamic quantities of these black holes are:

\[
T = \frac{1}{12\pi r_+ \left( 2\alpha + r_+^4 \right)} \left( 3\alpha(n-4) + \frac{48\pi Pr_+^4}{n-1} \right),
\]

\[
+ \frac{1}{12r_+^4} \left( \frac{1}{2} - \frac{2Q^2 r_+^{-2n}}{\beta^2} + 1 \right) + 3(n-2)r_+^2 \right),
\]

\[
S = \frac{(n-1)r_+^{n-5}}{4} \left( \frac{2\alpha r_+^2}{n-3} + \frac{r_+^4}{n-1} \right),
\]

\[
\phi = \frac{4\pi Q (n-2)r_+^{n-2}}{(n-2)r_+^{n-2}} \times F_1 \left[ \frac{1}{2} ; \frac{n-2}{2n-2} ; \frac{3n-4}{2n-2} - \frac{16\pi^2 Q^2 r_+^{-2n}}{\beta^2} \right],
\]

where, \( n = d - 1 \) and we set \( P = -\frac{1}{\pi} \Lambda = \frac{(d-1)(d-2)}{16\pi^2} \). The first law of thermodynamics and The Smarr relation for GB-BI black holes take the following form:

\[
dH = TdS + \Phi dQ + VdP + A\alpha + B\beta,
\]

\[
H = \frac{d-2}{d-3} TdS + \Phi \frac{2}{d-3} PV \left( \frac{2}{d-3} \alpha A (13) \right)
- \frac{1}{d-3} \beta B,
\]

where, \( H = M \) is the enthalpy of the gravitational system [85, 86, 87].

The parameters \( V, A, \) and \( B \) are the thermodynamic quantities conjugating to pressure \( P \), Gauss Bonnet coupling coefficient \( \alpha \), and Born-Infeld parameter \( \beta \), respectively. These parameters are determined from either the
first law of thermodynamics or the Smarr relation:

\[ V = \frac{r_+^{d-1}}{d - 1} = \frac{1}{d - 1}\left((d - 2)v\right)^{d-1}, \tag{14} \]

\[ A = \frac{(d - 2)r_+^{d-6}}{16\pi} - \frac{d-2}{2(d-4)}Tr_+^{d-4}, \tag{15} \]

\[ B = \frac{r_+^{d-1}}{2\pi\beta(d - 1)}\left(\beta^2 - \beta^3\sqrt{\frac{2Q^2r_+^{4-2d}}{\beta^2}} + 1\right) + \frac{Q_2r_+^{4-d_2-2}}{2\pi} \text{e_1}\left(\frac{d}{2}, \frac{d-3}{2d-4}, \frac{7-3d}{4-2d}, \frac{2Q_2r_+^{4-2d}}{\beta^2}\right). \tag{16} \]

Here, \( v \) is the effective specific volume.

In the following section, we will investigate the critical behaviors and phase transitions of the GB-BI-AdS black hole in the canonical and grand canonical ensembles.

A. Critical behavior in the canonical ensemble

The phase transitions and critical exponents of the BI-AdS black holes for \( d = 4 \) were calculated in the canonical and grand canonical ensembles [12–17]. Also, the stability analysis of five dimensional GB-BI black holes in the AdS space were studied in [54].

Now, let us investigate the critical behavior of GB-BI-AdS black holes in the canonical ensemble. We adopt fixed values for \( \beta, Q \), and \( \alpha \) and consider the \( P - v \) extended phase space. Also, we expand our calculations for GB-BI-AdS black holes in the grand canonical ensemble when \( \beta, \varphi, \) and \( \alpha \) are thermodynamic variables.

Using Eq. (9), we obtain the following equation of state for the black hole system in the canonical ensemble:

\[ P = \frac{T}{v} - \frac{d - 3}{(d - 2)\pi v^2} + \frac{32T\alpha}{(d - 2)^2v^3} \]

\[ - \frac{16\alpha(d - 5)}{(d - 2)^3\pi v^4} + \frac{\beta^2}{4\pi} \sqrt{\frac{16\pi^2Q^2(d - 2)^4 - 2d_4 - 2d}{\beta^2}} + 1 \]

\[ - \frac{\beta^2}{4\pi}. \tag{17} \]

Thus, the critical points can be determined by using the following conditions

\[ \frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0. \tag{18} \]

In the canonical ensemble, the Gibbs free energy from Eqs. (8), (9), and (10) for fixed values of \( \beta, Q, \) and

\[ \begin{align*}
G &= M - TS \\
&= \frac{3r_+^3}{240\pi^3\left(2\alpha + r_+^2\right)}\left(20\alpha^2 - 5\alpha r_+^2 - 4\pi r_+^6\right) \\
&- 48\pi \alpha P r_+\left(\beta^2 r_+^2 + 12\alpha^2 r_+^4\right)\sqrt{\frac{2Q^2}{\beta^2 r_+^8}} + 1 - \beta^2 r_+^4 \\
&- 12\alpha^2 r_+^4 + 11r_+^4 + \frac{32Q^2}{240\pi^3 r_+^2} \text{e_1}\left(\frac{3}{8}, \frac{3}{2}, \frac{11}{8}, \frac{2Q^2}{r_+^2}\right). \tag{19} \end{align*} \]

The critical behavior of the Gibbs free energy with respect to temperature depends on the values of the coupling coefficients \( \beta \) and \( \alpha \) in the canonical ensemble.

In what follows, we discuss in detail some interesting features of the critical behavior of GB-BI-AdS black holes depending on the coupling coefficient \( \alpha \).

1. Van der Waals behaviour

For \( 0 \leq \alpha < 13 \), the critical behavior of the Gibbs free energy with respect to temperature is depicted in Fig. 1 for fixed values of \( \beta, Q, \) and \( \alpha = 6 \).

In this case, we observe one critical point at \( P = P_c \) and the swallowtail behavior, i.e., a first order phase transition between small and large black holes, for \( P < P_c \). Thus, a “standard liquid/gas” Van der Waals phase transition occurs in this limit. The corresponding \( P - v \) diagram is displayed in Fig. 2.

Also, we consider the limit of \( \alpha = 0 \), i.e., the Born-Infeld-AdS black holes. In this condition, a Van der Waals behavior for \( d = 5,6 \) and \( Q = 1 \) and all the values of \( \beta \) is investigated which is similar to that in Fig. 1.

Furthermore, repeating our calculations for GB-BI-AdS
Gibbs free energy is discontinuous and we have two separat
large/small/large black hole phase transition. This critical
behavior admits a reentrant triple point phenomena in the
system for the fixed values of \( \beta, \alpha, Q \) at different pressures.

3. Reentrant phase transitions

For \( 18 \leq \alpha < 25 \), the Gibbs free energy admits one
critical point with a negative pressure of \( P_{c1} \) and three
critical points with positive pressures of \( P_{c2}, P_{c3} \) and
\( P_{c4} \). For \( \alpha = 20 \), we have one critical point with a
negative pressure of \( P_{c1} = -0.032 \) and three critical
points with the positive pressures of \( P_{c2} = 0.000981, P_{c3} = 0.0009897 \) and
\( P_{c4} = 0.001972 \), two of which (\( P = P_{c1} \) and \( P = P_{c4} \)) minimize \( G \). Also, we
observe a large/small/large reentrant phase transition in the
specific range of \( P_r = 0.0006 < P < P_2 = 0.00064 \).
At \( P = P_{c1} < 0 \) we have the thermodynamically
unstable branch with a negative \( C_P \) (Fig. 4a). For
\( P_{c3} < P \leq 0 \), two second order phase transitions which
globally minimize the Gibbs free energy at \( T_1 \) and \( T_0 \)
occur at the stable branch. In this range, the phases
\( \text{LBH/IBH/SBH} \) are connected by two zero-order phase
transitions at \( T_1 \) and \( T_0 \) (Fig. 4i). For \( 0 < P < P_r \), we find a new branch of large thermo-
dynamically stable black holes. Thus, only one phase of
the large black holes exists.

At \( P = P_r \), the swallowtail in the unstable branch
touches the lower stable branch. We observe the reentrant
large/small/large black hole phase transition for
\( P_r < P < P_2 \) in Fig. 4j.

Increasing pressure, we observe two swallowtails. One of
them starts from \( P = P_{c2} \) and terminates at \( P = P_{c3} \).
These critical points do not globally minimize the Gibbs
free energy (Fig. 4j1). In this situation, the system only
sees the swallowtail that occurs at \( P = (P_{c1}, P_{c4}) \). In
this situation, the first order phase transition appears at
the negative value of pressures \( P > P_{c1} \) and terminates
at \( P < P_{c4} \). (Fig. 4j e, f). Above the critical point
\( P = P_{c4} \), the system displays no phase transitions.

For \( 13 \leq \alpha < 16 \) and \( 32 \leq \alpha < 40 \), we have the
reentrant phase transition. For \( 13 \leq \alpha < 16 \), the

\[
\text{FIG. 2: The P-v diagram of GB-BI-AdS black hole for } \alpha = 12, \beta = 1, \text{ and } d = 6. \text{ The temperature of isotherms increase from bottom to top. The upper (blue) solid line correspond to the ideal gas, we have one phase for } T > T_c, \text{ the critical isotherm } T = T_c \text{ is denoted by the dashed (red) line. The lower dot-dashed (black) line corresponds to temperatures smaller than the critical temperature } T < T_c.\]
FIG. 3: Reentrant-Triple point phase transition for $d = 6$, $Q = 1$, $\beta = 1$ and $\alpha = \frac{33}{27}$ of GB-BI-AdS black holes. Here, the Gibbs free energy is displayed with respect to temperature for various values of pressures $P = \{-0.061, 0, 0.0003, 0.00037, 0.0004, 0.001181, 0.001188, 0.00122, 0.0013\}$ (from top to bottom). We consider Solid (blue)/dashed (red) lines corresponding to positive and negative $C_P$, respectively. Picture a) shows the thermodynamically unstable branch of $P_c = -0.061$ with a negative $C_P$. In picture b), for $-0.061 < P \leq 0$, we have two second order phase transitions at $T_1 = 0.0183$ and $T_0 = 0.027$, which globally minimize the Gibbs free energy. In this case, the minimum value of Gibbs free energy is discontinuous and we have three separate phases of large, intermediate, and small size black holes. These phases LBH/IBH/SBH are connected by a jump in the Gibbs free energy $G$, or two zero-order phase transitions at $T_1$ and $T_0$, respectively. In Picture c), for $0 < P \leq P_r = 0.00037$, we find a new branch of large thermodynamically stable black holes, at which two zero-order phase transitions at $T_1$ and $T_0$ do not minimize the Gibbs free energy. Thus, only one phase of the large black holes exists. In Picture d), at $P = P_r = 0.00037$, the swallowtail in the unstable branch touches the lower stable branch, and the reentrant phase transition appears. In Picture e), for $P_r < P < P_z = 0.000415$, the global minimum of Gibbs free energy is discontinuous. In this range of pressure, we have two phases of intermediate and small sizes black holes. These phases are connected by a jump in $G$, or a zero-order phase transition. This critical behavior admits a reentrant large/small/large black hole phase transition. At $P = P_z = 0.000415$, the RPT disappears but the first order phase transition is still present. Increasing pressure, in Picture f), at the triple point (TP), $P = P_t = 0.001181$, we observe two swallowtails in the stable phase of black holes. In Picture g, h), one of these swallowtails starts from $P = P_{c1} = -0.061$ and terminates at $P = P_{c3} = 0.001197$ and the other occurs at $P = P_{c2} = 0.00115$ and disappears at $P = P_{c4} = 0.00123$ although the critical point $P = P_{c2}$ does not minimize $G$ and it is an unphysical critical point. In Picture i), above the critical point $P = P_{c4}$ the system displays no phase transitions.
FIG. 4: Reentrant phase transition for $d = 6$, $Q = 1$, $\beta = 1$ and $\alpha = 20$ of the GB-BI-AdS black holes. Here, the Gibbs free energy is displayed with respect to temperature for various values of pressures $P = \{-0.032, 0, 0.00064, 0.000986, 0.0009897, 0.00197\}$ (from top to bottom). We observe that the Solid (blue)/dashed (red) lines correspond to positive/negative $C_P$ respectively. At $P_{c1} = -0.032 < 0$, we have the thermodynamically unstable branch with a negative $C_P$. When pressure is increased, for $P = 0$, two second order phase transitions appear which globally minimize the Gibbs free energy at $T_1 = 0.0218$ and $T_0 = 0.0253$. In this situation, three phases LBH/IBH/SBH are connected by two zero-order phase transitions at $T_1$ and $T_0$, respectively. Similar to Fig. 3, at $P_r = 0.0006$, the swallowtail in the unstable branch touches the lower stable branch, and we observe the reentrant large/small/intermediate black hole phase transition for $P_r < P < P_z$. At $P_r = 0.00064$, the reentrant phase transition disappearing but the first order phase transition being still present. Increasing pressure, we observe two swallowtails. One which does not globally minimize the Gibbs free energy starts from $P_{c2} = 0.000981$ and terminates at $P_{c3} = 0.0009897$. Another that occurs at $P = P_{c1}$ and disappears at $P_{c4} = 0.00197$ minimize the Gibbs free energy. We, therefore, have only two critical points. Above the critical point $P = P_{c4}$, the system displays no phase transitions.
Gibbs free energy has two critical points with positive pressures, \( P = P_{c1} \) and \( P = P_{c2} \) for the stable black holes. At \( P = P_{c1} > 0 \), we have the thermodynamically unstable branch with negative \( C_P \) (Fig. 5b). Also, we observe a large/small/large reentrant phase transition in the specific range of \( P_1 < P < P_2 \) and only one swallowtail occurs in the range of \( P_2 < P < P_{c2} \) (Fig. 5b, c). At \( P = P_{c2} \) we have the second order phase transition (Fig. 5h).

For \( 32 \leq \alpha < 40 \), although the black hole system admits two critical points at \( P = P_{c1} \) and \( P = P_{c2} \) but it is only the second critical point that minimizes the Gibbs free energy.

4. Multiple-Reentrant phase transitions

For \( 25 \leq \alpha < 32 \), Gibbs free energy has four critical points and we observe two reentrant phase transitions. For the range of \( 25 \leq \alpha < 28 \), we have \( P_{c1} > 0 \) with a thermodynamically unstable branch or a negative \( C_P \). For \( P_{c1} < P \leq 0 \), two second order phase transitions occur which globally minimize the Gibbs free energy at \( T_1 \) and \( T_0 \). In this range of pressures, the phases LBH/IBH/SBH are connected by two zero-order phase transitions at \( T_1 \) and \( T_0 \) respectively.

For \( 0 < P < P_{r1} \), we find a new branch of large thermodynamically stable black holes. In Fig. 6a, at \( P_{c2} < P = P_{r1} < P_{c3} \), the swallowtail in the unstable branch touches the left stable branch. Thus, we observe the reentrant large/small/large black hole phase transition for \( P_{r1} < P < P_{c1} \) (Fig. 6i). Since the critical point \( P_{c2} \) where this reentrant phase transition appears does not minimize Gibbs free energy, so this critical point is unphysical (Fig. 6c, d). By increasing pressure, this swallowtail disappears and we have another reentrant phase transition for \( P_{r2} < P < P_{c2} \) (Fig. 6f). In this situation, three of the critical points are physical.

For \( 28 \leq \alpha < 32 \), we have the same critical behavior (two reentrant phase transitions) but the critical point at \( P = P_{c1} \) has a positive pressure. In this condition, a new branch of large thermodynamically stable black holes appears at \( P = P_{c1} \) and the two zero-order phase transitions at \( T_1 \) and \( T_0 \) do not minimize the Gibbs free energy. Thus, only one phase of the large black hole exists at \( P = P_{c1} \) and both the critical points at \( P = P_{c1} \) and \( P = P_{c2} \) are unphysical. Similar to the previous case (for \( 25 \leq \alpha < 28 \)), we observe two reentrant phase transitions for \( P_{r1} < P < P_{c1} \) and for \( P_{r2} < P < P_{c2} \) (Fig. 6).

The results are summarized in Table I.

The critical values of pressure with respect to \( \alpha \) for \( \beta = 1 \) and \( Q = 1 \) are presented in Fig. 7. Based on the critical behaviors, we can divide the diagram to different regions. For \( 0 \leq \alpha < 13 \), one critical point exists, the system behaves similar to a "standard liquid/gas", and we observe RPT for \( 13 \leq \alpha < 16 \). We have a minimum critical pressure at \( \alpha = 16 \) in which both reentrant and triple point behaviors appear. For \( 16 \leq \alpha < 18 \) we observe RPT and TP with increasing pressure.

For \( 18 \leq \alpha < 25 \), we observe four critical points two of which minimize the Gibbs free energy. Hence, the GB-BI-AdS black holes experience a reentrant phase transition while we have no triple point phase transitions in this range. In the case of \( 25 \leq \alpha < 32 \), the GB-BI-AdS black holes have multiple-RPT in different pressure ranges.

Also, one critical point occur for the range \( 32 \leq \alpha < 40 \). The system has a reentrant phase transition for the specific range of pressure (Fig. 5). For \( \alpha \geq 40 \), we have two critical points, but neither globally minimizes the Gibbs free energy and they are unphysical critical points. Thus, above \( \alpha = 40 \), only one phase of large black holes exist and there is no phase transition (Fig. 7).
FIG. 6: Multiple-RPT for $d = 6$, $Q = 1$, $\beta = 1$ and $\alpha = 27$ of GB-BI-AdS black holes. Here, the Gibbs free energy is displayed with respect to temperature for various values of pressures $P = \{0.0007343, 0.00073445, 0.0007347, 0.00073512, 0.0039, 0.0051\}$ (from top to bottom). The Solid (blue)/dashed (red) lines correspond to $C_P$ positively and negative, respectively. For $P_{c_1} = -0.00126014 < P < 0$, two second order phase transitions that minimize the Gibbs free energy occur, similar to Fig. 4. In this ranges of pressure, three phases LBH/IBH/SBH are connected by two zero-order phase transitions at $T_1$ and $T_0$, respectively. At $0 < P < P_{r_1} = 0.0007343$, we find a new branch of large thermodynamically stable black holes. At $P = P_{r_1}$, the swallowtail which appears between $(P_{c_2} = 0.000733679, P_{c_3} = 0.000735122)$ in the unstable branch touches the lower stable branch. We observe one of the RPT of black hole for $P_{r_1} = 0.0007343 < P < P_{c_1} = 0.00073445$. Since the critical point $P_{c_2}$ does not minimize Gibbs free energy, it is an unphysical critical point. For $P_{c_1} < P < P_{c_3}$ we have a first order phase transition while disappear at the critical point $P = P_{c_3}$. Increasing pressure, we will have another RPT for $P_{r_2} = 0.00365 < P < P_{r_2} = 0.0041$. Above the critical point $P_{c_4} = 0.00591192$, the system displays no phase transitions.
are thermodynamic variables. Let us define a variable $x$ for $d = 5$ as,

$$x = \frac{256\pi Q}{(3\epsilon)^3}\sqrt{\frac{T}{3}}, \quad (20)$$

$$v = \frac{8\phi}{3\sqrt{3x^2} F(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, \frac{3x^2}{\beta^2})}, \quad (21)$$

where, $v$ is the specific volume. By using Eqs. (11) and (17) and the above definition, we can rewrite the equation of state in terms of the electric potentials $\phi$ and $x$ for $d = 5$ in the following form:

$$P = \frac{1}{32\sqrt{3}\pi \phi^3} \left(36\pi T x \phi^2 \gamma^2 F(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, \frac{x^2}{\beta^2})\right)^{-1}$$

$$+ 54\pi \alpha T x^3 F(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, \frac{3x^2}{\beta^2})^3 - 8\sqrt{3}\beta^2 \phi^3$$

$$- 9\sqrt{3x^2} F(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, \frac{3x^2}{\beta^2})^2 + 8\sqrt{3}\beta^2 \phi^3 \sqrt{\frac{\beta^2 + 3x^2}{\beta^2}}, \quad (22)$$

The critical points can be determined by using the conditions

$$\frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0 \quad (23)$$

The results show that there is only one critical point that depends on the coupling coefficients $\alpha$, $\beta$, and $\phi$ for $d = 5$. The Gibbs free energy in the grand canonical ensemble is given by

$$G(x, \phi, \alpha, \beta) = M - TS - Q\phi. \quad (24)$$

By using the above definitions and Eqs. (8), (10), and (20), the Gibss free energy with respect to temperature is displayed for $d = 5$ in Fig. 8. The GB-BI-AdS black holes behave similar to the Van der Waals fluid for $d = 5$ for all the values of $\beta$, $\alpha$, and $\phi$. Let us expand these calculations to the case with $d = 6$. In this case, we observe a reentrant phase transition with two critical points for the specific ranges of $\frac{6}{10} \leq \alpha \leq \frac{18}{10}$ and $\phi = \frac{9}{10}$ in $d = 6$ (Fig. 9).

We also observe the Van der Waals behavior in the BI-AdS black holes, ($\alpha = 0$), in the grand canonical ensemble and for $d = 5, 6$. The diagram is similar to Fig. 8.

Now, we consider the limit of $\beta \rightarrow \infty$ for the charged-GB-AdS black holes. It is shown that these black holes in the grand canonical ensemble and for $d = 5$ behave similar to the standard liquid/gas of the Van der Waals fluid [30].

Also, one phase of the large black holes for $d \geq 6$ of charged-GB-AdS black holes was investigated in the grand canonical ensemble [31].
III. ADS-BORN-INFELD BLACK HOLES IN THE THIRD ORDER LOVELOCK GRAVITY

The action of the third order Lovelock gravity with a nonlinear Born-Infeld electromagnetic field in the d-dimensional space time is given by

\[ I = \frac{1}{16\pi} \int d^{d+1}x \sqrt{-g}(R - 2\Lambda + \alpha_2\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \mathcal{L}_F), \]

where,

\[ \mathcal{L}_2 = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R, \]

\[ \mathcal{L}_3 = R^3 + 2R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\rho\tau}R^{\rho\tau} - 8R_{\mu\nu\sigma\rho\tau\kappa}R^{\rho\tau\kappa} + 24R^{\mu\nu\sigma\kappa}R^{\sigma\kappa}_{\alpha\beta}R_{\alpha\beta}^{\mu\nu}, \]

and

\[ \mathcal{L}_F = 4\beta^2(1 - \sqrt{1 + \frac{F^{\mu\nu}F_{\mu\nu}}{2\beta^2}}), \]

\[ F_{\mu\nu} \] is the vector potential.

In the above action, \( \beta, \alpha_2, \) and \( \alpha_3 \) are the Born-Infeld parameters, the second and third order Lovelock coefficients, respectively. Let us consider the following case

\[ \alpha_2 = \frac{\alpha}{(d-3)(d-4)}, \]

\[ \alpha_3 = \frac{\alpha^2}{72(d-3)}, \]

The metric of the d-dimensional static solution is as follows:

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2h_{ij}dx^idx^j, \]

Here, \( h_{ij} \) denotes the line element of (\( d-2 \))-dimensional hypersurface. Setting \( k = 1 \), we have

\[ f(r) = 1 + \frac{r^2}{\alpha}(1 - g(r)^{1/3}), \]

where,

\[ g(r) = 1 + \frac{3\alpha m}{r^{d-1}} - \frac{12\alpha\beta^2}{(d-1)(d-2)} \left( 1 - \sqrt{1 + \frac{(d-2)(d-3)q^2}{2\beta^2} + \frac{8\pi P}{\beta^2} + \frac{(d-2)(d-3)q^2}{2\beta^2}2F(r) } \right), \]

and \( 2F(r) \) is the hypergeometric function

\[ 2F(r) = 2F_1 \left[ \frac{1}{2}, \frac{d-3}{2d-4}, \frac{3d-7}{2d-4}, \frac{(d-2)(d-3)q^2}{2\beta^2} \right], \]

where,

\[ Q = \frac{q\Sigma_{d-2}}{4\pi} \sqrt{\frac{(d-2)(d-3)}{2}}. \]

The ADM mass of the black holes is

\[ M = \frac{(d-2)\Sigma_{d-2}}{16\pi} m \]

\[ = \frac{(d-2)\Sigma_{d-2} r_+^{d-1}}{48\pi\alpha} \left( \frac{r_+^2 + \alpha}{r_+^2 + \alpha \frac{r_+^2}{1 + \frac{12\alpha\beta^2}{(d-1)(d-2)} \left( 1 + \frac{8\pi P}{\beta^2} - \frac{1 + \eta_+ + \frac{(d-2)(d-3)q^2}{d-3}}{d-3} \right) } } \right), \]

where, \( \Sigma_{d-2} \) denotes the volume of (\( d-2 \))-dimensional hypersurface and we set \( \Sigma_{d-2} = 1 \) for simplicity. Also, \( r_+ \) is calculated from \( f(r_+) = 0 \). The thermodynamic
The quantities are given by

\[ T = \frac{1}{12(d-2)(r_+^4 + \alpha)^2} \left( 12r_+^6 \beta^2 + 6\pi P r_+^6 \right) \]  
\[ - 12r_+^6 \beta^2 \sqrt{1 + \eta_+ + (d-2) \left( 3(d-3)r_+^4 + 3(d-5)\alpha r_+^2 + (d-7)\alpha^2 \right)} \]  
\[ + 3(d-5)\alpha r_+^2 + (d-7)\alpha^2 \right) \right), \]  
\[ S = \frac{(d-2)r_+^{d-6}}{4} \left( \frac{r_+^4}{d-2} + \frac{2\alpha^2}{d-4} + \frac{\alpha^2}{d-6} \right), \]  
\[ \phi = \frac{4\pi Q}{(d-3)r_+^{d-3} 2F_1(r_+)} \]  

Here, \( \eta_+ = \frac{(d-2)(d-3)Q^2}{2\beta^2 r_+^{d-1}} \) and the above thermodynamic quantities are valid for \( d \geq 7 \).

The pressure is proportional to the cosmological constant. We identify the pressure of the black hole in the extended phase space with the following form

\[ P = -\frac{\Lambda}{8\pi} \]  

Since the cosmological constant is considered as the thermodynamic pressure, we replace the ADM mass of the black hole by the enthalpy. So, the first law of thermodynamics and its relevant quantities read as follow

\[ dM = TdS + \phi dQ + VdP + A d\alpha + B d\beta \]

The parameters \( V, A, \) and \( B \) are the thermodynamic quantities conjugating to the pressure \( P, \) the Gauss Bonnet coefficient \( \alpha, \) and the Born Infeld parameter \( \beta, \) respectively

\[ V = \frac{r_+^{d-1}}{d-1} = \frac{1}{d-1} \left( \frac{(d-2)\phi}{4} \right)^{d-1}, \]  
\[ A = \frac{(d-2)r_+^{d-7}}{48\pi} \left( 3r_+^2 + 2\alpha \right) \]  
\[ - \frac{1}{2} \left( \frac{(d-2)r_+^{d-6}T}{d-4} + \frac{\alpha}{d-6} \right), \]  
\[ B = \frac{r_+^{d-1}}{2\pi \beta (d-1)} \left( \beta^2 - \beta^2 \sqrt{\frac{2Q^2 r_+^{4-2d}}{\beta^2} + 1} \right) + \frac{Q^2 r_+^2}{r_+^{2d-2}} 2F_1 \left( \frac{1}{2}, \frac{d-3}{2d-4}, \frac{7-3d}{4-2d}, -\frac{2Q^2 r_+^{4-2d}}{\beta^2} \right) \]

In next section, we may assume the critical behaviors and phase transitions of BI-AdS black holes in the third order of Lovelock gravity and in the grand canonical ensemble.

A. Critical behavior in the grand Canonical ensemble

The phase transitions and critical behaviors of charged-AdS black holes in the third order of Lovelock gravity have been investigated in the canonical ensemble [56, 61]. Also, the third order Lovelock-BI-AdS black holes have been studied in the canonical ensemble and \( d = 7 \) in [58].

Now, let us assume the third order Lovelock-BI-AdS black holes in the grand Canonical ensemble in which the electric potentials \( \phi, \alpha, \) and \( \beta \) are considered as thermodynamic variables.
G\text{second order phase transition at the critical point}

\begin{equation}
G = M - TS - Q\phi. \tag{49}
\end{equation}

One is able to obtain the above Gibbs free energy by using Eqs. (36), (37), (38), and (45). We plot the diagram of Gibbs free energy with respect to temperature for these black holes for \(d = 7\) in Fig. 10. The results show that the system has the Van der Waals behavior for all given values of the parameters \(\alpha, \beta,\) and \(P\) for \(d = 7\) (Fig. 10). For \(\phi > 0.42\), we have one critical point with a positive value of the Gibbs free energy. By decreasing \(\phi\), we observe that one critical point for \(\phi < 0.42\) occurs with the negative values of the Gibbs free energy.

We expand our calculations to the third order Lovelock-BI-AdS black holes for \(d = 8\). We observe a reentrant phase transition for \(\beta = 1, \alpha = 1, \) and \(\phi \leq 0.6\). This RPT appears between the critical points \(P_{c1}\) and \(P_{c2}\) in Fig. 11b. At \(P = P_{c2}\), the second critical point, the Gibbs free energy has negative values (Fig. 11b). The third order Lovelock-BI-AdS black holes has only one phase of the large black holes for \(\phi > 6/10, \beta = 1\) and \(\alpha = 1\). Although the critical values of thermodynamic quantities of these black holes depend on the BI parameter \(\beta\) and the coupling coefficient \(\alpha\), the critical behaviors and the types of phase transitions depend only

\begin{table}[h]
\centering
\caption{Critical behaviors of black holes for \(Q = 1\) and \(\beta = 1\) in the canonical ensemble}
\begin{tabular}{|c|c|c|c|c|}
\hline
Black holes system & \(d\) & Critical behaviors & \hline
\hline
GB – BI – AdS & 5 & Van der Waals (VdW) & \\
\hline
GB – BI – AdS & 6 & \begin{tabular}{c}
\(0 < \alpha < 13\)
\(13 < \alpha < 16, 18 < \alpha < 25\) and \(32 < \alpha < 40\)
\(16 < \alpha < 18\)
\end{tabular} & \begin{tabular}{c}
\(25 < \alpha < 32\)
\end{tabular} & \\
\hline
BI – AdS & 5, 6 & VdW & \\
\hline
\end{tabular}
\end{table}

\begin{align}
G_b &= \frac{2048 \sqrt{\frac{7}{\pi} Q}}{3125 v^5}, \tag{45} \\
v &= \frac{8 \sqrt{\frac{2 \phi}{5}}}{5 x^2 F_1 \left(\frac{2}{5}, \frac{1}{2}, \frac{7}{5}, -\frac{10 x^2}{\beta^2}\right)}, \tag{46}
\end{align}

Here, \(v\) is the specific volume. By using Eq. (37), the equation of state for \(d = 7\) and the grand Canonical ensemble for the third order Lovelock-BI in the AdS space are given by

\begin{align}
P &= \frac{1}{512 \sqrt{10 \pi \phi^5}} \left(1600 \pi T x^3 \phi^4 F_1 (r_+) \right) + 625 \pi \alpha^2 T x^2 F_1 (r_+) + 2000 \pi \alpha T x^3 \phi^2 F_1 (r_+) \right) - 400 \sqrt{10} x^2 \phi \beta F_1 (r_+)^2 - 125 \sqrt{10} \alpha x^4 \phi \beta F_1 (r_+)^4 - 128 \sqrt{10} \beta^2 \phi^5 + 128 \sqrt{10} \beta^2 \phi^5 \sqrt{\frac{\beta^2 + 10 x^2}{\beta^2}}. \tag{47}
\end{align}

The critical points can be determined by using the conditions

\begin{align}
\frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0. \tag{48}
\end{align}
on $\phi$ and the coupling coefficient $\alpha$.
In the next subsection, we consider the limit of $\beta \to \infty$ for the charged AdS black holes in the third order of Lovelock gravity.

**B. The charged-AdS third order Lovelock black holes in the grand canonical ensemble**

Let us consider the limit of $\beta \to \infty$ to concentrate on the charged black holes in the third order of Lovelock gravity. In this case, i.e., $\beta \to \infty$, the Born-Infeld Lagrangian reduces to the Maxwell form and $2F(r+) \to 1$ in Eq. (54). Thus, Eq. (52) reduces to the following form

$$g(r) \to 1 + \frac{3\alpha m}{r^{d-1}} + \frac{6\alpha}{(d-1)(d-2)} - \frac{3\alpha Q^2}{r^{2d-4}}. \quad (50)$$

Now, we define a parameter $x$ in the form

$$x = 4\sqrt{2}\pi \sqrt{\frac{1}{(d-3)(d-2)} Q r^{2-d}}, \quad (51)$$

$$r_+ = \frac{\sqrt{\frac{2(d-1)-4}{d-2}}}{2F\left[\frac{3}{2}, \frac{d}{2(d-1)-4}, \frac{3(1-d)}{2(d-1)-2}, -\frac{(d-3)(d-2)x^2}{2}\right]}.$$

Using Eqs. (50), (52), and (51), the Mass and Hawking temperature of the charged-AdS third order Lovelock black holes in the grand Canonical ensemble turn into the following forms

$$M = \frac{2^{\frac{d-5}{2}} \sqrt{d-2}}{3\pi (d-3)^{7/2} (d-1) \phi^2} x^d \left(6\alpha (d-2)^3 (d^2 - 4d + 3) x^d + 12(d-3)^2 (d^2 - 3d + 2) x^2 \phi^4 + 2(\phi - 6) x^2 \phi^2 + 384\pi (d-3)^3 P \phi^6 + \alpha^2 (d-2)^4 (d-1) x^6\right), \quad (52)$$

$$T = \frac{x}{12\pi \sqrt{\frac{2(d-1)-4}{d-2}} \sqrt{(d-3)^3 \phi^6}} (\frac{384\pi (d-3)^3 P \phi^6}{(d-2)^3 x^6} - \frac{24(d-3)^4 \phi^6}{(d-2)^2 x^4} + (d-2) + \alpha^2 (d-7) + \frac{12(d-3)^3 \phi^4}{(d-2)^2 x^4} + \frac{6\alpha (d-5)(d-3) \phi^2}{(d-2)^2 x^2}). \quad (53)$$

$$S = \frac{2^{\frac{d-5}{2}} x^2 (\sqrt{\frac{2(d-1)-4}{d-2}})}{(d-6)(d-4)(d-3)^3 \phi^6} \left(4(d-3)^2 \right) \times (d^2 - 10d + 24) \phi^4 + \alpha^2 (d^2 - 9d + 18) x^2 \phi^2 + 4\alpha (d-2)^2 (d^2 - 9d + 18) x^2 \phi^2 \quad (54)$$

Thus, the equation of state for these black holes from Eq. (53) in the grand canonical ensemble is given by

$$P = \frac{\sqrt{\frac{d-3}{d-2}} (d-2)^4 x^5 (\alpha + \frac{2(d-3) \phi^2}{(d-2)x^2})^2}{16\sqrt{2}(d-3)^3 \phi^5} \times \left( - \frac{12\sqrt{2} \pi \sqrt{\frac{d-3}{d-2}} \phi (\alpha - 2(d-3) \phi^2)^2}{(d-2)^2 x^4} \right). \quad (55)$$

The critical points should satisfy the following conditions

$$\frac{\partial P}{\partial v} = \frac{\partial^2 P}{\partial v^2} = 0. \quad (56)$$

Considering Eqs. (55) and (56), for $\phi = 0.5$, $\alpha = 1$ and $d = 7$ we have only one critical point with positive pressure at $P_{c1} = 0.022204$. Also, we have the following two critical points for $\phi = 0.5$, $\alpha = 1$ and $d = 8$ with a positive pressures at $P_{c1} = 0.09909895$ and $P_{c2} = 0.0476283$.

The Gibbs free energy in the grand canonical ensemble is given by

$$G = M - TS - Q\phi, \quad (57)$$

We plot the diagram of the Gibbs free energy with respect to temperature using Eqs. (52), (53) and (54), for $\alpha = 1$ and $d = 7$. The critical behaviors of these black holes for a given $\alpha$ shows Van der Waals phenomena similar to the case of the third order of Lovelock-BI-AdS black holes in Fig. 10.

In other words, we have the swallowtail behavior for $P < P_c$ and a second order phase transition occurs at $P = P_c$. For $P > P_c$ there is no phase transition.

We have the critical points with the positive values of the Gibbs free energy for $\phi > 0.65$ (Fig. 10). By decreasing $\phi$, we observe the critical points for $\phi \leq 0.65$ with negative values for the Gibbs free energy. Also we consider these calculations in $d = 8$, we observe a reentrant phase transition for $\alpha = 1$ and $\phi \leq 0.58$. This reentrant phase transition appears between two critical points $P_{c1}$ and $P_{c2}$, with the diagram being similar to the third order of Lovelock-BI-AdS black holes in Fig. 11. There is no phase transition for $\phi > 0.58$ and $\alpha = 1$.

Consequently, we observe that the critical behaviors of the third order of Lovelock-charged-AdS black holes are similar to those of the third order of Lovelock-BI-AdS black holes and that their critical behaviors are independent of the parameter $\beta$ (Table. 1).

Here, we consider the other options to calculate the heat capacity for mixed ensemble with a fixed electric

$$\text{heat capacity for mixed ensemble with a fixed electric...}$$
Moreover, for an ensemble with fixed electric charges, pressure and \( A \) in the following form

\[
C_{\phi,P,A}(r+, \phi, P, A) = T \left( \frac{\partial S}{\partial T} \right)_{Q,P,A} = 0,
\]

We find that there is no phase transition for the above heat capacity. Corresponding to this heat capacity, we can define a Grand potential \( \Omega_1(r+, \phi, P, A) = H - TS - Q\phi - A\alpha \).

Moreover, for an ensemble with fixed electric charges, \( P \) and \( A \), one can obtain the heat capacity as follows:

\[
C_{Q,P,A}(r+, Q, P, A) = T \left( \frac{\partial S}{\partial T} \right)_{Q,P,A} = 0,
\]

The thermodynamic potential \( \Omega_2(r+, Q, P, A) = H - TS - A\alpha \) corresponds to the above heat capacity. Thus, there is no phase transition in these ensembles.

### IV. CONCLUSION

In this paper, we investigated the critical behaviors of the GB-BI-AdS black holes in the canonical (fixed \( Q \)) and grand canonical (fixed \( \phi \)) ensembles for \( d = 5, 6 \). We assumed the extended phase space with a cosmological constant and the coupling coefficient \( \alpha \) and Born-Infeld parameter \( \beta \) as the thermodynamic pressures of the system.

It was shown that the GB-BI-AdS black holes exhibit interesting thermodynamic phenomena that depend on the coupling coefficient \( \alpha \) in the canonical ensemble. We also observed "reentrant and triple point phase transitions" (RPT-TP) as well as "multiple reentrant phase transitions" (multiple RPT) for different ranges of pressure depending on the coefficient \( \alpha \) in the canonical ensemble. For \( 0 \leq \alpha < 13 \), the system was observed to behave similar to the standard liquid/gas of the Van der Waals fluid. For \( 13 \leq \alpha < 16 \), \( 18 \leq \alpha < 25 \), and \( 32 \leq \alpha < 40 \), the black hole system admitted a reentrant large/small/large black holes phase transition. For \( 16 \leq \alpha < 18 \), a reentrant phase transition was found to occur for a specific range of pressure. Increasing pressure led to a triple point for the special value of pressure. Also, a reentrant phase transition occurred for \( 25 \leq \alpha < 32 \). In other words, a reentrant large/small/large black holes phase transition occurred for a specific range of pressures and another reentrant phase transition happened by increasing pressure. Finally, no phase transition occurred for \( \alpha > 40 \).

We also studied the diagram of critical pressures with respect to the coupling coefficient \( \alpha \) of the GB-BI-AdS black holes. A minimum critical pressure was obtained at \( \alpha = 16 \) at which both the reentrant and triple point behaviors appeared.

GB-BI-AdS black holes were considered in the grand canonical ensemble to find that they behave similar to the standard liquid/gas of the Van der Waals fluid for \( d = 5 \) and to observe a reentrant phase transition for \( d = 6 \) and the specific value of \( \phi \). Extending our calculations to the BI-AdS black holes for \( d = 5, 6 \), we observed the Van der Waals behavior for the given \( d \).

The study of the critical behaviors of BI-AdS black holes in the third order of Lovelock gravity in the grand canonical ensemble revealed a Van der Waals behavior for \( d = 7 \) and a reentrant phase transition for \( \phi \geq 0.6 \) for \( d = 8 \) at a specific range of pressure in the grand canonical ensemble.

Furthermore, the limit of \( \beta \rightarrow \infty \) was considered for these black holes, i.e. charged-AdS black holes in the third order of the Lovelock gravity in the grand canonical ensemble. Similar to the previous case, we observed a Van der Waals behavior for \( d = 7 \) and a reentrant phase transition in \( d = 8 \) for \( \phi \geq 0.56 \). Thus, the critical behaviors were shown to be independent of the value of coefficient \( \beta \). We also extended our calculations to different mixed ensembles and the results showed no phase transitions in these ensembles.

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[1] Hawking, S. W., Gravitational radiation from colliding black holes, *Phys. Rev. Lett.* **26**, 1344 (1971).

[2] Bekenstein, J. D., Black holes and entropy, *Phys. Rev.*
On five-dimensional Thermodynamics of Black Holes in anti-De Sitter Space, Commun.Math. Phys. 87 (1983) 577.

S. Wang, Thermodynamics of de Sitter black holes: First laws and phases of nonextremal charged black holes and their CFT duals, Phys. Rev. D61 (2000) 024014, arXiv:hep-th/9908109.

S. Hawking and H. Reall, Charged and rotating AdS black holes and their CFT duals, Phys. Rev. D61 (2000) 024014, arXiv:hep-th/9908022.

S. W. Hawking, C. J. Hunter, and M. M. Taylor-Smith, Robinson, Rotation and the AdS/CFT correspondence, Phys. Rev. D59 (1999) 064005, [arXiv:hep-th/9811056].

C. Peca and J. Lemos, Thermodynamics of Reissner-Nordstrom anti-de Sitter black holes in the grand canonical ensemble, Phys. Rev. D59 (1999) 124007, [arXiv:gr-qc/9805004].

S. Hawking and D. N. Page, Thermodynamics of Black Holes, Commun. Math. Phys. 87 (1983) 167.

M. Caldarelli, G. Cognola, and D. Klemm, Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories, Class.Quant.Grav. 17 (2000) 399–420, arXiv:hep-th/9908022.

Y. Sekiwa, S. Wang, S.Q. Wu, F. Xie, and L. Dan, The First law of thermodynamics for Kerr-anti-de Sitter black holes, Class.Quant.Grav. 22 (2005) 1503–1526, [arXiv:hep-th/0408217].

S. W. Hawking and D. Robinson, Rotation and the AdS/CFT correspondence, Phys. Rev. D59 (1999) 064005, [arXiv:hep-th/9811056].

S. Hawking and H. Reall, Charged and rotating AdS black holes and their CFT duals, Phys. Rev. D61 (2000) 024014, arXiv:hep-th/9908109.

S. Hawking and H. Reall, Charged and rotating AdS black holes and their CFT duals, Phys. Rev. D61 (2000) 024014, arXiv:hep-th/9908022.

S. Hawking and D. N. Page, Thermodynamics of Black Holes, Commun. Math. Phys. 87 (1983) 167.

M. Caldarelli, G. Cognola, and D. Klemm, Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories, Class.Quant.Grav. 17 (2000) 399–420, arXiv:hep-th/9908022.

Y. Sekiwa, S. Wang, S.Q. Wu, F. Xie, and L. Dan, The First law of thermodynamics for Kerr-anti-de Sitter black holes, Class.Quant.Grav. 22 (2005) 1503–1526, [arXiv:hep-th/0408217].

S. Fernando, Thermodynamics of Born-Infeld-anti-de sitter black holes in the grand canonical ensemble, [arXiv:hep-th/0608040].

S. Wang, S.Q. Wu, F. Xie, and L. Dan, The first laws of thermodynamics of the (2+1)-dimensional BTZ black holes and Kerr-Sitter spacetimes, Chin. Phys. Lett. 23 1096 (2006), arXiv:hep-th/0601117.

S. Wang, Thermodynamics of Schwarzschild de Sitter spacetimes: Variable cosmological constant, arXiv:gr-qc/0606109.

G.L. Cardoso and V. Grass, On five-dimensional nonextremal charged black holes and FRW cosmology, Nucl. Phys. B803 209 (2008), arXiv:0803.2819.

H. Elvang, R. Emparan, and P. Figueras, Phases of five-dimensional black holes, JHEP 0705 (2007) 056, [arXiv:hep-th/0702111].

R. Emparan, T. Harmark, V. Niarchos, N. A. Obers, and M. Rodriguez, The Phase Structure of Higher-Dimensional Black Rings and Black Holes, JHEP 0710 (2007) 110, arXiv:0708.2181.

O. Dias, P. Figueras, R. Monteiro, J. E. Santos, and R. Emparan, Instability and new phases of higher-dimensional rotating black holes, Phys. Rev. D80 (2009) 111701, arXiv:0907.2248.

R. Banerjee, S. K. Modak, and S. Samanta, Second Order Phase Transition and Thermodynamic Geometry in Kerr-AdS Black Hole, Phys. Rev. D84 (2011) 064024, arXiv:1005.4832.

C. Niu, Y. Tian, and X.-N. Wu, Critical Phenomena and Thermodynamic Formula of RN-AdS Black Holes, Phys. Rev. D85 (2012) 024017, arXiv:1104.3066.

Y.-D. Tsai, X. Wu, and Y. Yang, Phase Structure of Kerr-AdS Black Hole, Phys.Rev. D85 (2012) 044005, arXiv:1104.0502.

A. Belhaj, M. Chabab, H. El Moumni, and M. Sedra, On Thermodynamics of AdS Black Holes in Arbitrary Dimensions, Chin. Phys. Lett. 29 (2012) 100401, arXiv:1210.4617.

S. Hendi and M. Vahidinia, P-V criticality of higher dimensional black holes with nonlinear source, arXiv:1212.6128.

R.-G. Cai, L.-M. Cao, L. Li, and R.-Q. Yang, P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space, JHEP 1309 (2013) 005, arXiv:1306.6233.

M.-S. Ma, H.-H. Zhao, L.-C. Zhang, and R. Zhao, Existence condition and phase transition of Reissner-Nordström-de Sitter black hole, arXiv:1312.0731.

S. Chen, X. Liu, C. Liu, and J. Jing, P – V criticality of AdS black hole in f(R) gravity, Chin. Phys. Lett. 30 (2013) 060401, arXiv:1301.3234.

R. Zhao, H.-H. Zhao, M.-S. Ma, and L.-C. Zhang, On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes, arXiv:1305.3723.

J.-X. Mo, X.-X. Zeng, G.-Q. Li, X. Jiang, and W.-B. Liu, A unified phase transition picture of the charged topological black hole in Horava-Lifshitz gravity, JHEP 1310 (2013) 056.

D.-C. Zou, S.-J. Zhang, and B. Wang, Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics, arXiv:1311.7299.

M. B. J. Poshteh, B. Mirza, and Z. Sherkatghanad, Phase transition, critical behavior, and critical exponents of Myers-Perry black holes, Phys. Rev. D 88, 024005 (2013), arXiv:1306.4516.

S. A. Hosseini Manousoori, Behrouz Mirza, Correspondence of phase transition points and singularities of thermodynamic geometry of black holes, arXiv:gr-qc/1308.1543v1.

De. Cheng, Zou, Yunqi, Bin. Wang, Critical behavior of charged Gauss-Bonnet AdS black holes in the canonical ensembles, arXiv:1404.5194.

B. Mirza, and Z. Sherkatghanad, Phase transitions of Hairy black holes in massive gravity and thermodynamic behavior of charged-AdS black holes in the extended phase space, Phys.Rev. D90 (2014) 084006, arXiv:1409.6839.

A. Chamblin, R. Emparan, C. Johnson, and R. Myers, Charged AdS black holes and catastrophic holography, Phys.Rev. D60 (1999) 064018, arXiv:hep-th/9902170.

A. Chamblin, R. Emparan, C. Johnson, and R. Myers, Holography, thermodynamics and fluctuations of charged AdS black holes, Phys.Rev. D60 (1999) 104026, arXiv:hep-th/9904197.

D. Kastor, S. Ray, and J. Traschen, Entalphy and the Mechanics of AdS Black Holes, Class.Quant.Grav. 26 (2009) 195011, arXiv:0904.2765.

M. Urano, A. Tomimatsu, and H. Saida, Mechanical First Law of Black Hole Spacetimes with Cosmological Constant and Its Application to Schwarzschild-de Sitter Spacetime, Class.Quant.Grav. 26 (2009) 105010, arXiv:0903.4230.

D. Kastor, S. Ray, and J. Traschen, Smarr Formula and an Extended First Law for Lovelock Gravity, Class.Quant.Grav. 27 (2010) 235014, arXiv:1005.5053.

B. Dolan, The cosmological constant and the black hole
equation of state, Class. Quant. Grav. 28 (2011) 125020, arXiv:1008.5023.

[41] B. P. Dolan, Pressure and volume in the first law of black hole thermodynamics, Class. Quant. Grav. 28 (2011) 235017, arXiv:1106.6260.

[42] B. P. Dolan, The compressibility of rotating black holes in D-dimensions, arXiv:1308.5403.

[43] B. P. Dolan, Where is the PdV term in the fist law of black hole thermodynamics?, arXiv:1209.1272.

[44] N. Goldenfeld, Lectures on Phase Transitions and the Renormalization Group, Westview Press, New York, 1992.

[45] D. Kubiznak and R. B. Mann, P-V criticality of charged AdS black holes, JHEP 1207 (2012) 033, arXiv:1205.0559.

[46] S. Gunasekaran, D. Kubiznak, and R. Mann, Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization, JHEP 1211 (2012) 110, arXiv:1208.6251.

[47] N. Altamirano, D. Kubiznak, and R. Mann, Reentrant Phase Transitions in Rotating AdS Black Holes, Phys. Rev. D88 (2013) 101502, arXiv:1306.5756.

[48] N. Altamirano, D. Kubiznak, R. B. Mann, and Z. Sherkatghadani, Kerr-AdS analogue of tricritical point and solid/liquid/gas phase transition, arXiv:1308.2672.

[49] N. Altamirano, D. Kubiznak, R. B. Mann, and Z. Sherkatghadani, Thermodynamics of rotating black hole and black rings: phase transitions and thermodynamic volume, arXiv:1401.2586.

[50] T. Narayanan and A. Kumar, Reentrant phase transitions in multicomponent liquid mixtures, Physics Reports 249 (1994) 135–218.

[51] V. P. Maslov, Zeroth-order phase transitions, Mathematical Notes 76 (2004), no. 5-6 697–710.

[52] M. Dehghani, N. Alinejadi, S. H. Hendi, Topological Black Holes in Lovelock-Born-Infeld Gravity, arXiv:0802.2637.

[53] D-C. Zou, Z-Y. Yang, R-H. Yue, P. Li Thermodynamics of Gauss-Bonnet-Born-Infeld Black Holes in AdS space, arXiv:1011.3184.

[54] Shao-Wen. Wei, Yu-Xiao. Liu, Triple points and phase diagrams in the extended phase space of charged Gauss-Bonnet Black Holes in AdS space, arXiv:1402.2837.

[55] Hao. Xu, Wei. Xu, L. Zhao, Extended phase space thermodynamics for third order Lovelock black holes in diverse dimensions, arXiv:1405.4143.

[56] W. Xu, H. Xu, L. Zhao, Gauss-Bonnet coupling constant as a free thermodynamical variable and associated criticality, arXiv:1311.3053.

[57] J.-X. Mo and W.-B. Liu, P − V Criticality of Topological Black Holes in Lovelock-Born-Infeld Gravity, arXiv:1401.0785.

[58] S.-W. Wei and Y.-X. Liu, Critical phenomena and thermodynamic geometry of charged Gauss-Bonnet AdS black holes, Phys. Rev. D87 (2013), no. 4 044014, arXiv:1209.1707.

[59] De. Zou, Y. Liu, B. Wang, Critical behavior of charged Gauss-Bonnet AdS black holes in the grand canonical ensemble, Phys. Rev. D87 (2013), no. 4 044014, arXiv:hep-th/1404.51947.

[60] Antonia M. Frassino, David. Kubiznak, Robert B. Mann, Fil. Simovic, Multiple Reentrant Phase Transitions and Triple Points in Lovelock Thermodynamics, arXiv:1406.7015.