Mean Field Analysis of Deep Neural Networks

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Abstract

We analyze multi-layer neural networks in the asymptotic regime of simultaneously (A) large network sizes and (B) large numbers of stochastic gradient descent training iterations. We rigorously establish the limiting behavior of the multilayer neural network output. The limit procedure is valid for any number of hidden layers and it naturally also describes the limiting behavior of the training loss. The ideas that we explore are to (a) sequentially take the limits of each hidden layer and (b) characterizing the evolution of parameters in terms of their initialization. The limit satisfies a system of integro-differential equations.

1 Introduction

Neural network models in machine learning have achieved immense practical success, revolutionizing fields such as image, text, and speech recognition. Neural networks are also increasingly being used in engineering, medicine, and finance. In particular, deep learning, which uses multi-layer neural networks, has transformed the field of machine learning. However, despite their success in practice, there is currently limited mathematical understanding of deep neural networks. This has motivated recent mathematical research on deep learning models such as [22], [23], [24], [25], [26], [21], [31], [32], [27], and [30].

Neural networks are nonlinear statistical models whose parameters are estimated from data using stochastic gradient descent (SGD) methods. Deep learning uses neural networks with many layers (i.e., “deep” neural networks), which produces a highly flexible, powerful and effective model in practice, see for example [13]. Applications of deep learning include image recognition (see [17] and [13]), facial recognition [41], driverless cars [3], speech recognition (see [17], [2], [18], and [42]), and text recognition (see [44] and [39]). Neural networks also find increasing more applications in engineering, robotics, medicine, and finance (see [19], [20], [40], [46], [49], [52], [51], and [50]).

In this paper we characterize multi-layer neural networks (i.e., “deep neural networks”) in the asymptotic regime of large network sizes and large numbers of stochastic gradient descent iterations. We rigorously prove the limit of the neural network output as the number of hidden units increases to infinity. The proof relies upon weak convergence analysis for stochastic processes. The result can be considered a “law of large numbers” for neural networks when both the network size and the number of stochastic gradient descent steps grow to infinity.

Recently, law of large numbers and central limit theorems have been established for neural networks with a single hidden layer [31], [32], [27], and [30]. The analysis of multi-layer neural networks requires a different approach than in the single hidden layer case. For a single hidden layer, one can directly study the weak convergence of the empirical measure of the parameters. However, in a neural network with multiple layers, there is a closure problem when studying the empirical measure of the parameters (which is explained in Section 3.3). Consequently, the law of large numbers for a multi-layer network is not a straightforward extension of the single-layer network result and the analysis involves unique challenges which require novel ideas.

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To illustrate the idea, we consider a multi-layer neural network with two hidden layers:

\[
g^{N_1,N_2}_\theta(x) = \frac{1}{N_2} \sum_{i=1}^{N_2} C^i \sigma \left( \frac{1}{N_1} \sum_{j=1}^{N_1} W^{2,i,j} \sigma \left( W^{1,j} \cdot x \right) \right).
\] (1.1)

As we will see in Section 3.2, the limit procedure can be extended to neural networks with three layers and subsequently to neural networks with any fixed number of hidden layers.

Notice now that (1.1) can be also written as

\[
H^{1,j}(x) = \sigma(W^{1,j} \cdot x), \quad j = 1, \ldots, N_1,
\]
\[
Z^{2,i}(x) = \frac{1}{N_1} \sum_{j=1}^{N_1} W^{2,i,j} H^{1,j}(x), \quad i = 1, \ldots, N_2,
\]
\[
H^{2,i}(x) = \sigma(Z^{2,i}(x)),
\]
\[
g^{N_1,N_2}_\theta(x) = \frac{1}{N_2} \sum_{i=1}^{N_2} C^i H^{2,i}(x).
\] (1.2)

where \(C^i, W^{2,i,j} \in \mathbb{R}\) and \(x, W^{1,j} \in \mathbb{R}^d\). The neural network model has parameters

\[
\theta = (C^1, \ldots, C^{N_2}, W^{2,1,1}, \ldots, W^{2,N_1,N_2}, W^{1,1}, \ldots, W^{1,N_1}),
\]

which must be estimated from data. The number of hidden units in the first layer is \(N_1\) and the number of hidden units in the second layer is \(N_2\). The multi-layer neural network (1.2) includes a normalization factor of \(\frac{1}{N_1}\) in the first hidden layer and \(\frac{1}{N_2}\) in the second hidden layer.

The loss function is

\[
L^{N_1,N_2}(\theta) = \frac{1}{2} \mathbb{E}_{Y,X} \left[ (Y - g^{N_1,N_2}_\theta(X))^2 \right],
\] (1.3)

where the data \((X,Y) \sim \pi(dx,dy)\). The goal is to estimate a set of parameters \(\theta\) which minimizes the objective function (1.3).

The stochastic gradient descent (SGD) algorithm for estimating the parameters \(\theta\) is, for \(k \in \mathbb{N}\),

\[
C^{i}_{k+1} = C^i_k + \frac{\alpha^{N_1,N_2}}{N_2} (y_k - g^{N_1,N_2}_\theta(x_k)) H^{2,i}_k(x_k),
\]
\[
W^{1,j}_{k+1} = W^{1,j}_k + \frac{\alpha^{N_1,N_2}}{N_1} (y_k - g^{N_1,N_2}_\theta(x_k)) \left( \frac{1}{N_2} \sum_{i=1}^{N_2} C^i_k \sigma'(Z^{2,i}_k(x_k)) W^{2,i,j}_k \right) \sigma'(W^{1,j}_k \cdot x_k)x_k,
\]
\[
W^{2,i,j}_{k+1} = W^{2,i,j}_k + \frac{\alpha^{N_1,N_2}}{N_1 N_2} (y_k - g^{N_1,N_2}_\theta(x_k)) C^i_k \sigma'(Z^{2,i}_k(x_k)) H^{1,j}_k(x_k),
\]
\[
H^{1,i}_k(x_k) = \sigma(W^{1,i}_k \cdot x_k),
\]
\[
Z^{2,i}_k(x_k) = \frac{1}{N_1} \sum_{j=1}^{N_1} W^{2,i,j}_k H^{1,j}_k(x_k),
\]
\[
H^{2,i}_k(x_k) = \sigma(Z^{2,i}_k(x_k)),
\]
\[
g^{N_1,N_2}_\theta(x_k) = \frac{1}{N_2} \sum_{i=1}^{N_2} C^i_k H^{2,i}_k(x_k).
\] (1.4)

where \(\alpha^{N_1,N_2}, \alpha^{N_1,N_2}, \alpha^{N_1,N_2}\) are the learning rates. The learning rates may depend upon \(N_1\) and \(N_2\). The parameters at step \(k\) are \(g^{N_1,N_2}_\theta(x_k) = (C^1_k, \ldots, C^{N_2}_k, W^{2,1,1}_k, \ldots, W^{2,N_1,N_2}_k, W^{1,1}_k, \ldots, W^{1,N_1}_k)\). \((x_k, y_k)\) are samples of the random variables \((X,Y)\).
The goal of this paper is to characterize the limit of an appropriate rescaling of the multi-layer neural network output \( g_{\theta_k}^{N_1,N_2}(x) \) as both the number of hidden units \((N_1, N_2)\) and the stochastic gradient descent iterates \( k \) increase. This is the topic of Theorem 2.3. The idea is to first take \( N_1 \to \infty \) with \( N_2 \) fixed. In Lemma 2.2 we prove that the empirical measure of the parameters converges to a limit measure as \( N_1 \to \infty \) (with \( N_2 \) fixed) which satisfies a measure evolution equation. This naturally implies a limit for the neural network output \( g^{N_1,N_2} \) as \( N_1 \to \infty \). The next step is to take \( N_2 \to \infty \). Theorem 2.3 proves that the limiting distribution can be represented via a system of ODEs.

The rest of the paper is organized as follows. Our main result, which characterizes the asymptotic behavior of a neural network with two hidden layers when the number of hidden units becomes large, is presented in Section 2. The result can be easily extended to an arbitrary number of hidden layers. Section 3 discusses the theoretical results and, as an example, presents the limit for a three-layer neural network. The proof of the convergence theorem is in Section 4. The uniqueness of a solution to the limiting system is established in Section 5. The proof of the limit of the first layer, i.e., the proof of Lemma 2.2 is provided in Appendix A.

2 Main Results

Let us start by presenting our assumptions, which will hold throughout the paper. We shall work on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which all the random variables are defined. The probability space is equipped with a filtration \(\mathcal{F}_t\) that is right continuous and \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-negligible sets.

Assumption 2.1. We assume the following conditions throughout the paper.

- \( \sigma(\cdot) \in C_b^2 \), i.e., it is twice continuously differentiable and bounded.
- The distribution \( \pi(dx, dy) \) has compact support, i.e., the data \((x_k, y_k)\) takes values in the compact set \( \mathcal{X} \times \mathcal{Y} \).
- The initialization of the parameters, i.e. \( \{C_0^i\}_i, \{W_o^{2,i,j}\}_{i,j}, \{W_o^{1,j}\}_j \), are i.i.d. and take values in compact sets \( \mathcal{C}, \mathcal{W}_1 \), and \( \mathcal{W}_2 \).
- The probability distributions of the initial parameters \((C_0^i, W_o^{2,i,j}, W_o^{1,j})_{i,j}\) admit continuous probability density functions.

We denote \( \mu_c(dc) \), \( \mu_{W^2}(du) \), and \( \mu_{W^1}(dw) \) as the probability distributions of \( \{C_0^i\}_i, \{W_o^{2,i,j}\}_{i,j}, \) and \( \{W_o^{1,j}\}_j \) respectively.

As it was done in the single hidden layer case, see [31], one expects that the compactness assumption on \( \mathcal{X} \times \mathcal{Y} \) and \( \mathcal{C}, \mathcal{W}_1 \), and \( \mathcal{W}_2 \) can be relaxed to appropriate moment type conditions. For presentation purposes, we do not explore this further here.

For reasons that will become clearer later on, we shall choose the learning rates to be

\[
\alpha_{C_1,N_2}^{N_1,N_2} = \frac{N_2}{N_1}, \quad \alpha_{W,1}^{N_1,N_2} = 1 \quad \text{and} \quad \alpha_{W,2}^{N_1,N_2} = N_2.
\]

Define the empirical measure

\[
\tilde{\gamma}_k^{N_1,N_2} := \frac{1}{N_1} \sum_{j=1}^{N_1} \delta_{\gamma_k^{N_1,N_2}},
\]

The neural network’s output can be re-written in terms of the empirical measure:

\[
g_{\theta_k}^{N_1,N_2}(x) = \frac{1}{N_2} \sum_{i=1}^{N_2} \left\langle c_i, \tilde{\gamma}_k^{N_1,N_2} \right\rangle \sigma \left( \left\langle w^{2,i} \sigma(w^{1} \cdot x), \tilde{\gamma}_k^{N_1,N_2} \right\rangle \right).
\]

\( \langle f, h \rangle \) denotes the inner product of \( f \) and \( h \). For example, we write

\[
\left\langle w^{2,i} \sigma(w^{1} \cdot x), \tilde{\gamma}_k^{N_1,N_2} \right\rangle = \int w^{2,i} \sigma(w^{1} \cdot x) \tilde{\gamma}_k^{N_1,N_2}(dw^1, dw^2, dc).
\]
Let us next define, the time-scaled empirical measure
\[ \gamma_{t}^{N_{1}, N_{2}} := \gamma_{\lfloor t N_{1} \rfloor}^{N_{1}, N_{2}}, \]
and the corresponding time-scaled neural network output is
\[ g_{t}^{N_{1}, N_{2}}(x) := g_{\lfloor t N_{1} \rfloor}^{N_{1}, N_{2}}(x) = \frac{1}{N_{2}} \sum_{i=1}^{N_{2}} \langle c_{i}, \gamma_{t}^{N_{1}, N_{2}} \rangle \sigma \left( \langle w^{2,i} \sigma(w^{1} \cdot x), \gamma_{t}^{N_{1}, N_{2}} \rangle \right). \]

At any time \( t \), \( \gamma_{t}^{N_{1}, N_{2}} \) is measure-valued. The scaled empirical measure \( (\gamma_{t}^{N_{1}, N_{2}})_{0 \leq t \leq T} \) is a random element of \( D_{E}(\mathbb{R}; E) \) with \( E = \mathcal{M}(\mathbb{R}^{d+2N_{2}}) \).

We study convergence using iterated limits. We first let \( N_{1} \to \infty \) where the number of units in the first layer is \( N_{1} \) and the number of stochastic gradient descent steps is \( [TN_{1}] \). Then, we let the number of units in the second layer \( N_{2} \to \infty \).

We begin by letting the number of hidden units in the first layer \( N_{1} \to \infty \).

**Lemma 2.2.** The process \( \gamma_{t}^{N_{1}, N_{2}} := (\gamma_{t}^{N_{1}, N_{2}})_{0 \leq t \leq T} \) converges in distribution to \( \gamma_{t}^{N_{2}} \) in \( D_{E}(\mathbb{R}; E) \) as \( N_{1} \to \infty \). For every \( f \in C^{2}_{b}(\mathbb{R}^{d+2N_{2}}) \), \( \gamma_{t}^{N_{2}} \) satisfies the measure evolution equation
\[ \langle f, \gamma_{t}^{N_{2}} \rangle - \langle f, \gamma_{0}^{N_{2}} \rangle = \int_{0}^{t} \int_{X \times Y} (y - g_{s}^{N_{2}}(x)) \left( H_{s}^{N_{2}}(x) \cdot \nabla_{c} f, \gamma_{s}^{N_{2}} \right) \pi(dx, dy) ds \]
\[ + \int_{0}^{t} \int_{X \times Y} (y - g_{s}^{N_{2}}(x)) \left( \sigma(w^{1} \cdot x) \sigma'(Z_{s}(x)) \odot c \cdot \nabla_{w^{2}} f, \gamma_{s}^{N_{2}} \right) \pi(dx, dy) ds \]
\[ + \int_{0}^{t} \int_{X \times Y} (y - g_{s}^{N_{2}}(x)) \frac{1}{N_{2}} \sum_{i=1}^{N_{2}} \langle c_{i}, \gamma_{s}^{N_{2}} \rangle H_{s}^{i, N_{2}}(x), \]

where
\[ Z_{s}^{i, N_{2}}(x) = \langle w^{2,i} \sigma(w^{1} \cdot x), \gamma_{s}^{N_{2}} \rangle, \]
\[ H_{s}^{i, N_{2}}(x) = \sigma(Z_{s}^{i, N_{2}}(x)), \]
\[ g_{s}^{N_{2}}(x) = \frac{1}{N_{2}} \sum_{i=1}^{N_{2}} \langle c_{i}, \gamma_{s}^{N_{2}} \rangle H_{s}^{i, N_{2}}(x), \]
\[ \gamma_{0}^{N_{2}}(dw^{1}, dw^{2}, dc) = \mu_{W^{1}}(dw^{1}) \times \mu_{W^{2}}(dw^{2}) \times \cdots \times \mu_{W^{2,N_{2}}} \times \delta_{C_{j}^{1}}(dc^{1}) \times \cdots \times \delta_{C_{j}^{N_{2}}}(dc^{N_{2}}). \]

**Proof.** The proof of this lemma is related to the limit in the first layer as the number of hidden units in the first layer grows with the number of hidden units in the second layer hold fixed. The proof is analogous to the proof in [11] and the details are presented for completeness in the Appendix [A].

Lemma 2.2 studies the limit of the empirical measure \( \gamma_{t}^{N_{1}, N_{2}} \) as \( N_{1} \to \infty \) with \( N_{2} \) fixed. The limit is characterized by the stochastic evolution equation (2.2)\( - \) (2.3). Notice that Lemma 2.2 immediately implies that
\[ \lim_{N_{1} \to \infty} g_{t}^{N_{1}, N_{2}}(x) = g_{t}^{N_{2}}(x), \]
as \( N_{1} \to \infty \), in probability.

The next step is to study the limit as \( N_{2} \to \infty \). To do so, we study the limit of the random ODE as \( N_{2} \to \infty \) whose law is characterized by (2.2)\( - \) (2.3). Our main goal is the characterization of the limit neural network output \( g_{t}^{N_{1}, N_{2}}(x) \). The following convergence result characterizes the neural network output \( g_{t}^{N_{1}, N_{2}}(x) \) for large \( N_{1} \) and \( N_{2} \).

\( ^{1}D_{S}(\mathbb{R}; E) \) is the set of maps from \([0, T]\) into \( S \) which are right-continuous and which have left-hand limits.
Theorem 2.3. For any \( t \in [0,T] \) and \( x \in \mathcal{X} \),

\[
\lim_{N_2 \to \infty} \lim_{N_1 \to \infty} g_t^{N_1,N_2}(x) = g_t(x),
\]

in probability, where we have that

\[
g_t(x) = \int_{\mathcal{C}} \tilde{C}_t^c \tilde{H}_t^{2,c}(x) \mu_c(dc),
\]

with

\[
\begin{align*}
d\tilde{C}_t^c &= \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t(x)) \tilde{H}_t^{2,c}(x) \pi(dx,dy)dt, \quad \tilde{C}_0^c = c, \\
d\tilde{W}_t^{1,w} &= \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t(x)) V_t^w \sigma(\tilde{W}_t^{1,w} \cdot x) \pi(dx,dy)dt, \quad \tilde{W}_0^{1,w} = w, \\
d\tilde{W}_t^{2,c,w,u} &= \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t(x)) \tilde{C}_t^c \sigma^2(\tilde{Z}_t^c) \tilde{H}_t^{1,w}(x) \pi(dx,dy)dt, \quad \tilde{W}_0^{2,c,w,u} = u, \\
\tilde{H}_t^{1,w}(x) &= \sigma(\tilde{W}_t^{1,w} \cdot x), \\
\tilde{Z}_t^c(x) &= \int_{\mathcal{W}_1} \int_{\mathcal{W}_2} \tilde{W}_t^{2,c,w,u} \tilde{H}_t^{1,w}(x) \mu_{W_2}(du) \mu_{W_1}(dv), \\
\tilde{H}_t^{2,c}(x) &= \sigma(\tilde{Z}_t^c(x)), \\
V_t^w(x) &= \int_{\mathcal{C}} \tilde{C}_t^c \sigma^2(\tilde{Z}_t^c(x)) \left( \int_{\mathcal{W}_2} \tilde{W}_t^{2,c,w,u} \mu_{W_2}(du) \right) \mu_c(dc).
\end{align*}
\] (2.4)

System 2.4 has a unique solution. In addition, letting \( g_t^{N_2}(x) \) defined through Lemma 2.2 we have the following rate of convergence

\[
\sup_{x \in \mathcal{X}} \mathbb{E} \left| g_t^{N_2}(x) - g_t(x) \right| \leq KN_2^{-1/2},
\]

for some constant \( K < \infty \).

Notice that we can also write that \( g_t(x) \) satisfies

\[
g_t(x) = \int_{\mathcal{C}} \tilde{C}_t^c \sigma \left( \int_{\mathcal{W}_1} \int_{\mathcal{W}_2} \tilde{W}_t^{2,c,w,u} \sigma(\tilde{W}_t^{1,w} \cdot x) \mu_{W_2}(du) \mu_{W_1}(dv) \right) \mu_c(dc).
\] (2.5)

The proof of Theorem 2.3 is given in Section 4. Section 5 discusses some of its consequences as well as challenges that come up in the study of the limiting behavior of multi-layer neural networks as the number of the hidden units grows.

3 Discussion on the limiting results and extensions to multi-layer networks with more depth

In Subsection 3.1 we discuss some of the implications of our theoretical convergence results. In Section 3.2 we show that the procedure can be extended to treat deep neural networks with more than two hidden layers. General challenges in the study of multi-layer neural networks are explored Subsection 3.3.

3.1 Discussion on the limiting results

It is instructive to notice that the results of this paper recover the results of [31], see also [27, 30] if we restrict attention to the one-layer case. Indeed, let us set in 1.1-1.2, \( N_2 = 1 \), \( N_1 = N \), \( C^i = 1 \) and \( H^{2,i} = Z^{2,i} \), and get the single layer neural network

\[
g_0^N(x) = \frac{1}{N} \sum_{j=1}^N W^{2,j} \sigma(W^{1,j} \cdot x),
\]
with the corresponding empirical measure of the parameters becomes
\[
\gamma^N_t = \gamma^N_{\lfloor Nt \rfloor}, \quad \text{where} \quad \gamma^N_{\lfloor Nt \rfloor} = \frac{1}{N} \sum_{j=1}^{N} \delta_{W^1_{k,j}, W^2_{k,j}}.
\]

(3.1)

In that case notice that we can simply write
\[
g^N_{\theta_{\lfloor Nt \rfloor}}(x) = \langle w^2\sigma(w^1 \cdot x), \gamma^N_{\lfloor Nt \rfloor} \rangle.
\]

Then, it is relatively straightforward to notice that the result of Lemma 2.2 boils down to the one layer convergence results of [31], see also [27, 30]. Namely, if we write \( \gamma_t \) for the limit in probability of \( \gamma^N_{\lfloor Nt \rfloor} \) we get that
\[
\lim_{N \to \infty} g^N_{\theta_{\lfloor Nt \rfloor}}(x) = \langle w^2\sigma(w^1 \cdot x), \gamma_t \rangle.
\]

(3.2)

It is instructive to compare the limits of the neural network output in the one layer and two layer case, (3.2) and (2.5) respectively. It is clear that the two layer case is more involved, partially explaining the increased complexity of deep neural networks when compared to shallow neural networks.

In addition, Theorem 2.3 gives us the limiting behavior of the objective function \( L^{N_1,N_2}(\theta) \) from (1.3) after proper rescaling. Indeed, we have that
\[
\lim_{N_2 \to \infty} \lim_{N_1 \to \infty} L^{N_1,N_2}(\theta_{\lfloor N_1t \rfloor}) = \lim_{N_2 \to \infty} \lim_{N_1 \to \infty} \frac{1}{2} \mathbb{E}_{Y,X} \left[ (Y - g^N_{\theta_{\lfloor N_1t \rfloor}}(X))^2 \right]
\]
\[
= \frac{1}{2} \int_{X \times Y} \left[ (y - g_t(x))^2 \right] \pi(dx, dy)
\]

where \( g_t(x) \) is given by (2.4).

In particular, Theorem 2.3 shows that under appropriate choice of the learning rates, first taking the limit of the hidden units in the first layer to infinity and then the limit of the hidden units in the second layer to infinity, leads to a well-defined limit for the neural network output and as a consequence for the objective function as well.

We also remark here that the parametrization of the learning rates indicates that one should be using larger learning rates for the weights that connect the different layers, the \( W^{2,i,j} \) in this case, as opposed to the weights that are specific to the different layers, in this case \( C^i \) and \( W^{1,j} \). Notice that this is also the case for the three layer case outlined in Subsection 3.2 below.

3.2 Extension to deep neural networks with more layers

The procedure developed in this paper, naturally extends to deep neural networks with more layers. For brevity, let us present the result in the case of three layers. The situation for more layers is the same, albeit more complicated algebra. A deep neural network with three layers takes the form
\[
g^N_{\theta_{N_1,N_2,N_3}}(x) = \frac{1}{N_3} \sum_{i=1}^{N_3} C^i \sigma \left( \frac{1}{N_2} \sum_{j=1}^{N_2} W^{3,i,j} \sigma \left( \frac{1}{N_1} \sum_{\nu=1}^{N_1} W^{2,i,j,\nu} \sigma \left( W^{1,\nu} \cdot x \right) \right) \right),
\]

(3.3)
which can be also written as

\[ H_{1,\nu}(x) = \sigma(W_{1,\nu} \cdot x), \quad \nu = 1, \ldots, N_1, \]

\[ Z_{2,i,j}(x) = \frac{1}{N_1} \sum_{\nu=1}^{N_1} W_{2,i,j,\nu} H_{1,\nu}(x), \quad i = 1, \ldots, N_3 \text{ and } j = 1, \ldots, N_2 \]

\[ H_{2,i,j}(x) = \sigma(Z_{2,i,j}(x)), \]

\[ Z_{3,i}(x) = \frac{1}{N_2} \sum_{j=1}^{N_2} W_{3,i,j} H_{2,i,j}(x), \quad i = 1, \ldots, N_3 \text{ and } j = 1, \ldots, N_2 \]

\[ H_{3,i}(x) = \sigma(Z_{3,i}(x)), \]

\[ g_{\theta}^{N_1,N_2,N_3}(x) = \frac{1}{N_3} \sum_{i=1}^{N_3} C^i H_{3,i}(x). \quad (3.4) \]

where \( C^i, W_{2,i,j,\nu}, W_{3,i,j} \in \mathbb{R} \) and \( x, W_{1,\nu} \in \mathbb{R}^d \). The neural network model has parameters

\[ \theta = (C^1, \ldots, C^{N_3}, W_{2,1,1,1}, \ldots, W_{2,N_3,N_2,N_1}, W_{3,1,1,1}, \ldots, W_{3,N_3,N_2}, W_{1,1,1,1}, \ldots, W_{1,N_1}), \]

which must be estimated from data. The number of hidden units in the first layer is \( N_1 \), the number of hidden units in the second layer is \( N_2 \) and the number of hidden units in the third layer is \( N_3 \). Naturally, the loss function now becomes

\[ L_{N_1,N_2,N_3}(\theta) = \frac{1}{2} \mathbb{E}_{Y,X} \left[ (Y - g_{\theta}^{N_1,N_2,N_3}(X))^2 \right], \]

where the data \((X,Y) \sim \pi(dx, dy)\).

The stochastic gradient descent (SGD) algorithm for estimating the parameters \( \theta \) is, for \( k \in \mathbb{N}, \nu = \... \)
1, \ldots, N_1, i = 1, \ldots, N_2 and j = 1, \ldots, N_2 is

\[
C_{k+1}^i = C_k^i + \frac{\alpha_{C}^{N_1,N_2,N_3}}{N_3} (y_k - g_{\theta_k}^{N_1,N_2,N_3}(x_k)) H_k^{3,i}(x_k),
\]

\[
W_{k+1}^{1,\nu} = W_k^{1,\nu} + \frac{\alpha_{W,1}^{N_1,N_2,N_3}}{N_1} (y_k - g_{\theta_k}^{N_1,N_2,N_3}(x_k)) \left( \frac{1}{N_3} \sum_{i=1}^{N_3} C_k^i \sigma'(Z_k^{3,i}(x_k)) \left( \frac{1}{N_2} \sum_{j=1}^{N_2} W_k^{2,i,j}(x_k) W_k^{2,i,j,\nu} \right) \right) \times
\]

\[
\sigma'(W_k^{1,\nu} \cdot x_k)x_k,
\]

\[
W_{k+1}^{3,i,j} = W_k^{3,i,j} + \frac{\alpha_{W,3}^{N_1,N_2,N_3}}{N_1N_3} (y_k - g_{\theta_k}^{N_1,N_2,N_3}(x_k)) C_k^i \sigma'(Z_k^{3,i}(x_k)) W_k^{3,i,j} H_k^{3,i,j}(x_k),
\]

\[
W_{k+1}^{2,i,j,\nu} = W_k^{2,i,j,\nu} + \frac{\alpha_{W,2}^{N_1,N_2,N_3}}{N_1N_3} (y_k - g_{\theta_k}^{N_1,N_2,N_3}(x_k)) C_k^i \sigma'(Z_k^{3,i}(x_k)) W_k^{2,i,j,\nu} H_k^{1,\nu}(x_k),
\]

\[
H_k^{1,\nu}(x_k) = \sigma(W_k^{1,\nu} \cdot x_k), \quad \nu = 1, \ldots, N_1,
\]

\[
Z_k^{2,i,j}(x_k) = \frac{1}{N_1} \sum_{\nu=1}^{N_1} W_k^{2,i,j,\nu} H_k^{1,\nu}(x_k),
\]

\[
H_k^{2,i,j}(x_k) = \sigma \left( Z_k^{2,i,j}(x_k) \right),
\]

\[
Z_k^{3,i}(x_k) = \frac{1}{N_2} \sum_{j=1}^{N_2} W_k^{3,i,j} H_k^{2,i,j}(x_k),
\]

\[
H_k^{3,i}(x_k) = \sigma \left( Z_k^{3,i}(x_k) \right),
\]

\[
g_{\theta_k}^{N_1,N_2,N_3}(x_k) = \frac{1}{N_3} \sum_{i=1}^{N_3} C_k^i H_k^{3,i}(x_k).
\]

where \( \alpha_{C}^{N_1,N_2,N_3}, \alpha_{W,1}^{N_1,N_2,N_3}, \alpha_{W,2}^{N_1,N_2,N_3} \) and \( \alpha_{W,3}^{N_1,N_2,N_3} \) are the learning rates. The parameters at step \( k \) are

\[
\theta_k^{N_1,N_2,N_3} = (C_k^1, \ldots, C_k^{N_3}, W_k^{2,1,1,1}, \ldots, W_k^{2,N_2,N_3}, W_k^{3,1,1}, \ldots, W_k^{3,N_2,N_3}, W_k^{1,1}, \ldots, W_k^{1,N_1}).
\]

\((x_k, y_k)\) are samples of the random variables \((X, Y)\). We assume a condition analogous to Assumption 2.1.

Let us now choose the learning rates to be

\[
\alpha_{C}^{N_1,N_2,N_3} = \frac{N_3}{N_1}, \quad \alpha_{W,1}^{N_1,N_2,N_3} = 1, \quad \alpha_{W,2}^{N_1,N_2,N_3} = \frac{N_2N_3}{N_1}, \quad \alpha_{W,3}^{N_1,N_2,N_3} = N_2N_3
\]

Similarly to before, define the empirical measure

\[
\gamma_k^{N_1,N_2,N_3} := \frac{1}{N_1} \sum_{\nu=1}^{N_1} \delta_{W_k^{1,\nu}, W_k^{2,1,1,\nu}, \ldots, W_k^{2,N_2,N_3}, W_k^{3,1,1}, \ldots, W_k^{3,N_2,N_3}, C_k^{1}, \ldots, C_k^{N_3}}.
\]

set the time-scaled empirical measure to be

\[
\gamma_\ell^{N_1,N_2,N_3} := \gamma_k^{N_1,N_2,N_3},
\]

and the corresponding time-scaled neural network output to be \( g_{\ell}^{N_1,N_2,N_3}(x) = g_{\theta_k^{N_1,N_2,N_3}}(x) \).

Then following the same procedure as for the two layer case, yields the following limit that describes
\[ g_t(x), \text{ the limit of } g_t^{N_1,N_2,N_3}(x) \text{ as first } N_1 \to \infty, \text{ then } N_2 \to \infty \text{ and then } N_3 \to \infty, \]

\[
d\tilde{C}_t^c = \int_{X \times Y} (y - g_t(x)) \tilde{H}_t^{3,c}(x) \pi(dx, dy) dt, \quad \tilde{C}_0^c = c,
\]

\[
d\tilde{W}_t^{1,w} = \int_{X \times Y} (y - g_t(x)) V_t^w \sigma'(W_t^{1,w} \cdot x) x \pi(dx, dy) dt, \quad \tilde{W}_0^{1,w} = w,
\]

\[
d\tilde{W}_t^{3,c,v} = \int_{X \times Y} (y - g_t(x)) \tilde{C}_t^c \sigma'(\tilde{Z}_t^{3,c}) \tilde{W}_t^{3,c,v} \tilde{H}_t^{2,c,v}(x) \pi(dx, dy) dt, \quad \tilde{W}_0^{3,c,v} = v,
\]

\[
d\tilde{W}_t^{2,c,w,u,v} = \int_{X \times Y} (y - g_t(x)) \tilde{C}_t^c \sigma'(\tilde{Z}_t^{3,c}) \tilde{W}_t^{3,c,v} \sigma'(\tilde{Z}_t^{2,c,v}) \tilde{W}_t^{2,c,w,u,v} \tilde{H}_t^{1,w}(x) \pi(dx, dy) dt, \quad \tilde{W}_0^{2,c,w,u,v} = u,
\]

\[
\tilde{H}_t^{1,w}(x) = \sigma(\tilde{W}_t^{1,w}.x),
\]

\[
\tilde{Z}_t^{2,c,v}(x) = \int_{W_1} \int_{W_2} \tilde{W}_t^{2,c,w,u,v} \tilde{H}_t^{1,w}(x) \mu_{W_2}(du) \mu_{W_1}(dw),
\]

\[
\tilde{H}_t^{2,c,v}(x) = \sigma(\tilde{Z}_t^{2,c,v}(x)),
\]

\[
\tilde{Z}_t^{3,c}(x) = \int_{W_3} \tilde{W}_t^{3,c,v} \tilde{H}_t^{2,c,v}(x) \mu_{W_3}(dv),
\]

\[
\tilde{H}_t^{3,c}(x) = \sigma(\tilde{Z}_t^{3,c}(x)),
\]

\[
V_t^w(x) = \int_c \tilde{C}_t^c \sigma'(\tilde{Z}_t^{3,c}(x)) \left( \int_{W_3} \tilde{W}_t^{3,c,v} \sigma'(\tilde{Z}_t^{2,c,v}) \left( \int_{W_2} \tilde{W}_t^{2,c,w,u,v} \mu_{W_2}(du) \mu_{W_3}(dv) \right) \mu_c(dc) \right)
\]

\[
g_t(x) = \int_c \tilde{C}_t^c \tilde{H}_t^{3,c}(x) \mu_c(dc), \quad (3.6)
\]

In other words, we can write that the neural networks output is given by.

\[
g_t(x) = \int \tilde{C}_t^c \left( \sigma \left( \int_{W_3} \tilde{W}_t^{3,c,v} \sigma \left( \int_{W_2} \tilde{W}_t^{2,c,w,u,v} \sigma(\tilde{W}_t^{1,w}.x) \mu_{W_2}(du) \mu_{W_1}(dw) \right) \mu_{W_3}(dv) \right) \right) \mu_c(dc).
\]

### 3.3 Challenges in the analysis of multi-layer neural networks

Let us now discuss some of the challenges that come up in the study of systems like the multi-layer neural network that we study in this paper.

A standard approach for analyzing (1.4) as \(N_1, N_2 \to \infty\) would be to construct an empirical measure \(\rho_{N_1,N_2}^k\) of the parameters \(\theta_k\) at step \(k\), as for example \(\gamma_{N_1,N_2}^k\). Then, we could study the behavior of \(\rho_{N_1,N_2}^k\) as \(N_1, N_2 \to \infty\). This empirical measure \(\rho_{N_1,N_2}^k\) needs to be designed such that the dynamics of \(\rho_{N_1,N_2}^k\) can be written in terms of \(\rho_{N_1,N_2}^k\) itself and the data \((x_k, y_k)\) (plus martingale and remainder terms). That is, the dynamics of \(\rho_{N_1,N_2}^k\) are closed.

This is straightforward for single-layer neural networks (see [31]), but it is challenging to do for multi-layer neural networks. In the case of single layer networks, the empirical measure is simply given by (3.1) and its analysis has been successfully carried on in [31]. One is tempted to do the same thing for the multi-layer case, i.e. study the limit of the empirical measure defined in (2.1). The problem that one faces with this formulation is that, in contrast to the single layer case, the dimension of the space in which the empirical measure takes values on also increases, with \(N_2\) in this case.

An alternative second way is to define the candidate empirical measure as an appropriately normalized double sum over both indices corresponding to the two hidden layers and then consider the limit of this measure as \(N_1, N_2 \to \infty\). However, when doing that one quickly runs into a closure problem for the dynamics, similar to the situation that we will elaborate now below for another natural formulation in terms of nested measures.

As discussed, yet another alternative way, which is also natural in this case, is to try and create nested-measures, sometimes called multi-level measure valued processes in mathematical biology, see [10] [11] and the review paper [5].

Let us demonstrate how one can potentially think of doing so for (1.4) and the problems that come up. In order to simplify notation, let \(N_1 = N_2 = N\) and \(\rho_{N_1,N_2}^k = \rho_k^N\) in the following example. Considering \(N_1 \neq N_2\) and taking subsequent limits does not alter the conclusions below.
• Let’s first examine the parameter $W^{1,j}$ in the first layer. The $j$-th unit in the first layer (i.e., $H^{1,j} = \sigma(W^{1,j} \cdot x)$) is connected to all of the hidden units in the second layer (i.e., $H^{2,i}$) via the weights $W^{2,i,j}$. Therefore, there is a measure associated with each $W^{1,j}$ which must track $\{W^{2,i,j}, Z^i, C^i\}_{i=1}^N$. $W^{2,i,j}$ and $Z^i$ are required for calculating the SGD update for $W^{1,j}$ (see 1.4). This measure is $\nu_k^{N,j} = \frac{1}{N} \sum_{i=1}^{N} \delta_{W^{2,i,j}, Z^i, C^i}$. 

• Let’s next examine the parameter $C^i$ in the second layer. The $i$-th unit in the second layer is connected to all of the hidden units in the first layer via the weights $W^{2,i,j}$. Therefore, for each $C^i$, we must track $\{W^{2,i,j}, W^{1,j}\}_{j=1}^N$ in order to calculate the SGD update for $C^i$ (see 1.4). Furthermore, updating $C^i$ requires tracking $W^{1,j}$, and updating $W^{1,j}$ requires tracking $\nu^j$. Therefore, updates to $C^i$ require the empirical measure $\mu_k^{N,i} = \frac{1}{N} \sum_{j=1}^{N} \delta_{W^{2,i,j}, W^{1,j}, \nu_k^{N,j}} \in \mathcal{M}(\mathbb{R} \times \mathbb{R}^d \times \mathcal{M}(\mathbb{R}))$. 

• Finally, the entire network at iteration $k$ is specified by the empirical measure 

$$
\rho_k^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{C_k^i, \nu_k^j} \in \mathcal{M}(\mathbb{R} \times \mathcal{M}(\mathbb{R} \times \mathbb{R}^d \times \mathcal{M}(\mathbb{R}))),
$$

(3.7)

where $\mathcal{M}(E)$ is the space of measures on the metric space $E$. Notice that the process (3.7) involves nested measures (sometimes called “multi-level processes”). The process $\rho_k$ takes values in a space of nested measures $\mathcal{M}(\mathcal{M}(\mathcal{M}(\cdots)))$.

Careful inspection of $\rho_k^N$ identifies a crucial problem: its dynamics are not closed. The evolution of $\nu_k^{N,j}$ (the innermost measure in the nested measures) cannot be written in terms of $\rho_k^N$. In particular, updating $\nu_k^{N,j}$ requires also updating $(W^{2,i,j}, Z^i, C^i)_{i=1}^N$. This would in fact require knowledge of $(W^{2,i,j}, Z^i, C^i, \rho_k^{N,i})_{i=1}^N$, i.e. we would have to re-define $\nu_k^{N,j}$ as $\frac{1}{N} \sum_{i=1}^{N} \delta_{W^{2,i,j}, Z^i, C^i, \eta_k^{N,i}}$, where $\eta_k^{N,i} = \frac{1}{N} \sum_{m=1}^{N} W^{2,i,m}_k$. This leads to re-defining $\rho_k^N$ as taking values in $\mathcal{M}(\mathbb{R} \times \mathcal{M}(\mathbb{R} \times \mathbb{R}^d \times \mathcal{M}(\mathbb{R} \times \mathbb{R}^d)))$, i.e. the space of 4-nested measures (before it was 3-nested measures). However, the closure problem remains, since the evolution of $\eta_k^{N,i}$ cannot be written in terms of $\rho_k^N$. In fact, there does not seem to exist any finite number of nested measures for which the empirical measure $\rho_k^N$’s dynamics are closed and despite our best efforts we have not managed to find one.

The discussion in this section highlighted some of the problems that come up in the analysis of multi-layer neural networks. Such problems are not present in the analysis of single-layer neural networks. Therefore, a different approach is required for the asymptotic analysis of multi-layer neural networks and this paper is a first step towards this direction. The problems that we describe above led us to the current formulation. In particular, the approach in Section 2 first studies the limit of the empirical measure as the number of hidden units in the first layer grows to infinity. This is similar to [31]. The limit is a solution to an evolution equation and it is the law of a system of random ODEs. We then make the crucial observation that one can characterize the resulting system in terms of the initialization for the stochastic gradient descent iterates. This means that we can reformulate the limiting system of the first layer into an equivalent system of random ODEs and then consider the limit of the second layer. This allows us to obtain the limit of the output of the neural network as the hidden unit of all layers grow to infinity by studying the limit of the random ODE, in Theorem 2.3.

4 Proof of Theorem 2.3 Characterization of the limit

In preparation for the proof of Theorem 2.3 we first re-express the result from Lemma 2.2.
Corollary 4.1. Consider the particle system:

\[ dC_t^i = \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t^{N_2}(x)) H_t^{2,i}(x) \pi(dx, dy) dt, \quad C_0^i = C_0^i, \quad i = 1, \ldots, N_2, \]

\[ dW_t^1 = \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t^{N_2}(x)) \left( \frac{1}{N_2} \sum_{i=1}^{N_2} C_t^i \sigma'(Z_t^i(x)) W_t^{2,i} \right) \sigma'(W_t^1 \cdot x) \pi(dx, dy) dt, \quad W_0^1 \sim \mu_{W_1}(dw), \]

\[ dW_t^{2,i} = \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t^{N_2}(x)) C_t^i \sigma'(Z_t^i) H_t^1(x) \pi(dx, dy) dt, \quad W_0^{2,i} \sim \mu_{W_2}(du), \quad i = 1, \ldots, N_2, \]

\[ H_t^1(x) = \sigma(W_t^1 \cdot x), \]

\[ Z_t^i(x) = \mathbb{E} \left[ W_t^{2,i} H_t^1(x) \left| C_t^1, \ldots, C_t^{N_2} \right. \right], \]

\[ H_t^{2,i}(x) = \sigma(Z_t^i(x)), \quad i = 1, \ldots, N_2, \]

\[ g_t^{N_2}(x) = \frac{1}{N_2} \sum_{i=1}^{N_2} C_t^i H_t^{2,i}(x). \quad (4.1) \]

Let \( \nu_{t,(c_1, \ldots, c_{N_2})} \) be the conditional Law of \((W_t^1, W_t^{2,1}, \ldots, W_t^{2,N_2}, C_t^1, \ldots, C_t^{N_2})_{0 \leq t \leq T} \) given \((C_0^1, \ldots, C_0^{N_2}) = (c_1, \ldots, c_{N_2}) \). Then, \( \nu_{t,(c_1, \ldots, c_{N_2})} \) is the unique solution to the evolution equation \( [24] \text{ and } [25] \).

Proof. See Appendix A \( \Box \)

Due to exchangeability properties and without loss of generality, we can transform \( [4.1] \) into the equivalent particle system:

\[ dC_t^{C_0^i} = \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t^{N_2}(x)) H_t^{2,C_t^i}(x) \pi(dx, dy) dt, \quad C_0^{C_0^i} = C_0^i, \quad i = 1, \ldots, N_2, \]

\[ dW_t^{1,W_0} = \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t^{N_2}(x)) V_t^{N_2,W_0} \sigma'(W_t^{1,W_0} \cdot x) \pi(dx, dy) dt, \quad W_0^{1,W_0} = W_0 \sim \mu_{W_1}(dw) \]

\[ dW_t^{2,C_t^i,W_0,W_0^{2,i}} = \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t^{N_2}(x)) C_t^i \sigma'(Z_t^{C_t^i}) H_t^{1,W_0}(x) \pi(dx, dy) dt, \quad W_0^{2,C_t^i,W_0,W_0^{2,i}} = W_0^{2,i} \sim \mu_{W_2}(du), \]

\[ H_t^{1,W_0}(x) = \sigma(W_t^{1,W_0} \cdot x), \]

\[ Z_t^{C_t^i}(x) = \mathbb{E} \left[ W_t^{2,C_t^i,W_0,W_0^{2,i}} H_t^{1,W_0}(x) \left| C_0^1, \ldots, C_0^{N_2} \right. \right], \quad i = 1, \ldots, N_2, \]

\[ H_t^{2,C_t^i}(x) = \sigma(Z_t^{C_t^i}(x)), \quad i = 1, \ldots, N_2, \]

\[ g_t^{N_2}(x) = \frac{1}{N_2} \sum_{i=1}^{N_2} C_t^{C_0^i} H_t^{2,C_t^i}(x), \]

\[ V_t^{N_2,W_0}(x) = \frac{1}{N_2} \sum_{i=1}^{N_2} C_t^{C_0^i} \sigma'(Z_t^{C_t^i}(x)) W_t^{2,C_t^i,W_0,W_0^{2,i}}. \quad (4.2) \]

Since for all \( i = 1, \cdots, N_2, C_0^i \) have probability density function as described in Assumption [24] we have that

\[ \mathbb{P}(\{C_0^i \neq C_0^j\} : i \neq j, (i, j) = 1, 2, \ldots, N_2) = 1. \]

This allows us to substitute the variable names

\( (\hat{C}_t^i, \hat{W}_t^1, \hat{W}_t^{2,i}, \hat{Z}_t^i, \hat{H}_t^{1,i}, \hat{H}_t^{2,i}) \) for \((C_t^{C_0^i}, W_t^{1,W_0}, W_t^{2,C_t^i,W_0,W_0^{2,i}}, Z_t^{C_t^i}, H_t^{1,W_0}, H_t^{2,C_t^i})\),

for \( i = 1, \ldots, N_2. \)
This produces the system:

\[
d\tilde{C}_t^i = \int_{\mathbb{R}^2} (y - g_t^{N_2}(x)) \tilde{H}_{t}^{2,i}(x) \pi(dx, dy) dt, \quad \tilde{C}_0^i = c_i, \quad i = 1, \ldots, N_2,
\]

\[
d\tilde{W}_t^{1} = \int_{\mathbb{R}^2} (y - g_t^{N_2}(x)) V_t^{N_2, W_0} \sigma'(\tilde{W}_t^{1} \cdot x) x \pi(dx, dy) dt, \quad \tilde{W}_0^{1} = W_0 \sim \mu_{W_1}(dw),
\]

\[
d\tilde{W}_t^{2,i} = \int_{\mathbb{R}^2} (y - g_t^{N_2}(x)) \tilde{C}_t^i \sigma'(\tilde{Z}_t^i) \tilde{H}_t^1(x) \pi(dx, dy) dt, \quad \tilde{W}_0^{2,i} = W_0^{2,i} \sim \mu_{W_2}(du),
\]

\[
\tilde{H}_t^1(x) = \sigma(\tilde{W}_t^{1} \cdot x),
\]

\[
\tilde{Z}_t^i(x) = \mathbb{E}[\tilde{W}_t^{2,i} \tilde{H}_t^1(x) | C_0^i, \ldots, C_0^{N_2}], \quad i = 1, \ldots, N_2,
\]

\[
\tilde{H}_t^2(x) = \sigma(\tilde{Z}_t^i(x)), \quad i = 1, \ldots, N_2,
\]

\[
g_t^{N_2}(x) = \frac{1}{N_2} \sum_{i=1}^{N_2} \tilde{C}_t^i \tilde{H}_t^2(x),
\]

\[
V_t^{N_2, W_0}(x) = \frac{1}{N_2} \sum_{i=1}^{N_2} \tilde{C}_t^i \sigma'(\tilde{Z}_t^i(x)) \tilde{W}_t^{2,i}.
\]

The system (4.3) is exactly the same system as (4.1). Notice also that \( \tilde{C}_t^i \) in (4.3) depends also on \( \{C_0^i \}_{i=1}^{N_2} \) in a symmetric way via \( g_t^{N_2}(x) \). Similarly \( W_0^{1} \) and \( W_0^{2,i} \) depend also on \( \{C_0^i \}_{i=1}^{N_2} \) and on \( \{W_0^{2,i} \}_{i=1}^{N_2} \) symmetrically via \( g_t^{N_2}(x) \) and \( V_t^{N_2, W_0}(x) \). This is also the situation for (4.1). Then independence and identical distribution of the initial conditions together with the aforementioned exchangeability property imply that (4.1) and (4.2) are equivalent.

4.1 Limiting System

The goal is to prove that for any \( t \in [0, T] \) and \( x \in \mathcal{X} \),

\[
\lim_{N_2 \to \infty} g_t^{N_2}(x) = g_t(x),
\]

in \( L^1 \), where \( g_t^{N_2}(x) \) is from (4.2) and the limit \( g_t(x) \) is given by

\[
g_t(x) = \int_{\mathcal{X}} \tilde{C}_t^c \tilde{H}_t^{2,c}(x) \mu_c(dc),
\]

where,

\[
d\tilde{C}_t^c = \int_{\mathbb{R}^2} (y - g_t(x)) \tilde{H}_t^{2,c}(x) \pi(dx, dy) dt, \quad \tilde{C}_0^c = c,
\]

\[
d\tilde{W}_t^{1,w} = \int_{\mathbb{R}^2} (y - g_t(x)) V_t^{w} \sigma'(\tilde{W}_t^{1} \cdot x) x \pi(dx, dy) dt, \quad \tilde{W}_0^{1,w} = w,
\]

\[
d\tilde{W}_t^{2,c,w,u} = \int_{\mathbb{R}^2} (y - g_t(x)) \tilde{C}_t^c \sigma'(\tilde{Z}_t^c) \tilde{H}_t^1(x) \pi(dx, dy) dt, \quad \tilde{W}_0^{2,c,w,u} = u,
\]

\[
\tilde{H}_t^{1,w}(x) = \sigma(\tilde{W}_t^{1,w} \cdot x),
\]

\[
\tilde{Z}_t^c(x) = \int_{\mathbb{R}^2} \tilde{W}_t^{2,c,w,u} \tilde{H}_t^1(x) \mu_{W_2}(du) \mu_{W_1}(dw),
\]

\[
\tilde{H}_t^{2,c}(x) = \sigma(\tilde{Z}_t^c(x)),
\]

\[
V_t^{w}(x) = \int_{\mathcal{X}} \tilde{C}_t^c \sigma'(\tilde{Z}_t^c(x)) \left( \int_{\mathbb{R}^2} \tilde{W}_t^{2,c,w,u} \mu_{W_2}(du) \right) \mu_c(dc).
\]

Before presenting the proof of this result, let us define a quantity that will be of central interest in the sequel. In particular, for \( (c, w, u, x) \in \{(C^c_0, W_0^{1}, W_0^{2,i}), i = 1, \cdots, N_2\} \) and \( x \in \mathcal{X} \), let’s define the error function

\[
E_t^{N_2}(c, w, u, x) := |C_t^c - \tilde{C}_t^c|^2 + \left\| W_t^{1,w} - \tilde{W}_t^{1,w} \right\|^2 + (W_t^{2,c,w,u} - \tilde{W}_t^{2,c,w,u})^2 + (H_t^{2,c}(x) - \tilde{H}_t^{2,c}(x))^2 + (Z_t^c(x) - \tilde{Z}_t^c(x))^2.
\]
Note that we certainly have,
\[ |C_0^c - \tilde{C}_0^c|^2 + \left\| W_0^{1,w} - \tilde{W}_0^{1,w} \right\|^2 + |W_0^{2,c,w,u} - \tilde{W}_0^{2,c,w,u}|^2 + |Z_0^c(x) - \tilde{Z}_0^c(x)|^2 = 0. \]

We will show below that \( \mathbb{E} \left[ E_t^{N_2}(C_0^c, W_0, W_0^{2,i}) \right] \) appropriately converges to zero as \( N_2 \to \infty \) which will then imply that \( g_{t}^{N_2}(x) \) converges to \( g_t(x) \) as indicated.

### 4.2 A Priori Bounds

Let us first establish uniform bounds on the processes \( C_t^c, W_t^{1,w}, W_t^{2,c,w,u} \), and \( g_t^{N_2}(x) \) for the system \( \mathbf{1.2} \). For any \( t \in [0, T] \) and any \( N_2 \in \mathbb{N} \),
\[
\frac{1}{N_2} \sum_{i=1}^{N_2} |C_t^{C_i^c}| \leq \frac{1}{N_2} \sum_{i=1}^{N_2} |C_0^{C_i^c}| + \int_0^t \int_{X \times Y} |y - g_s^{N_2}(x)| \frac{1}{N_2} \sum_{i=1}^{N_2} |H_s^{2,C_i^c}(x)| \pi(dx, dy)ds.
\]

\[ |H_s^{2,C_i^c}(x)| < K \text{ since } \sigma(\cdot) \in C_0^2. \] Then, using the fact that \( X \times Y \) is compact,
\[
\frac{1}{N_2} \sum_{i=1}^{N_2} |C_t^{C_i^c}| \leq \frac{1}{N_2} \sum_{i=1}^{N_2} |C_0^{C_i^c}| + K_t + K \int_0^t \frac{1}{N_2} \sum_{i=1}^{N_2} |C_s^{C_i^c}| ds.
\]

By Gronwall’s inequality, we have that for any \( N_2 \in \mathbb{N} \) and for any \( t \in [0, T] \),
\[
\frac{1}{N_2} \sum_{i=1}^{N_2} |C_t^{C_i^c}| \leq K.
\]

Using the same approach, we can establish uniform bounds on the other processes \( W_t^{1,w} \) and \( W_t^{2,c,w,u} \). Therefore, for any \( N_2 \in \mathbb{N} \), \( t \in [0, T] \), \( x \in X \), and \( (c, w, u) \in \{(C_0^c, W_0^{1}, W_0^{2,i}), i = 1, \cdots, N_2\} \),
\[
\max\{|g_t^{N_2}(x)|, |C_t^{C_i^c}|, |W_t^{2,c,w,u}|, |W_t^{N_2,w}(x)|, |W_t^{1,w}| \} \leq K, \quad (4.5)
\]

### 4.3 Bound for \( \mathbb{E}\left[ V_t^{N_2,W_0}(x) - V_t^{W_0}(x) \right]^2 \)

We have:
\[
\left| V_t^{N_2,W_0}(x) - V_t^{W_0}(x) \right| \leq \left| \frac{1}{N_2} \sum_{i=1}^{N_2} C_t^{C_i^c} \sigma'(Z_t^{C_i^c}(x)) W_t^{2,C_i^c,w_0,W_0^{2,i}} - \int C_t^{C_i^c} \sigma'(Z_t^{C_i^c}(x)) \left( \int_{W_2} \tilde{W}_t^{2,c,w_0,u} \mu_{W_2}(du) \right) \mu_c(dc) \right|
\]
\[
= \left| \frac{1}{N_2} \sum_{i=1}^{N_2} C_t^{C_i^c} \sigma'(Z_t^{C_i^c}(x)) W_t^{2,C_i^c,w_0,W_0^{2,i}} - \frac{1}{N_2} \sum_{i=1}^{N_2} C_t^{C_i^c} \sigma'(Z_t^{C_i^c}(x)) \tilde{W}_t^{2,C_i^c,w_0,W_0^{2,i}} \right|
\]
\[
+ \left| \frac{1}{N_2} \sum_{i=1}^{N_2} C_t^{C_i^c} \sigma'(Z_t^{C_i^c}(x)) \tilde{W}_t^{2,C_i^c,w_0,W_0^{2,i}} - \int C_t^{C_i^c} \sigma'(Z_t^{C_i^c}(x)) \left( \int_{W_2} \tilde{W}_t^{2,c,w_0,u} \mu_{W_2}(du) \right) \mu_c(dc) \right|
\]
\[
:= |\Gamma_t^{V,1,W_0}| + |\Gamma_t^{V,2,W_0}|. \quad (4.6)
\]
The first term in (4.6) can be studied using a decomposition:

\[ \left| \Gamma_t^{V,1,W_0} \right| = \left| \frac{1}{N_2} \sum_{i=1}^{N_2} C_i^{C_i^0} \sigma'(Z_{C_i^0}(x)) W_{t_i}^{2,C_i^0,W_0,W_0^{2,i}} - \frac{1}{N_2} \sum_{i=1}^{N_2} \tilde{C}_i^{C_i^0} \sigma'(\tilde{Z}_{C_i^0}(x)) \tilde{W}_{t_i}^{2,C_i^0,W_0,W_0^{2,i}} \right| \]

\[ = \left| \frac{1}{N_2} \sum_{i=1}^{N_2} \left[ \left( C_i^{C_i^0} - \tilde{C}_i^{C_i^0} \right) \sigma'(Z_{C_i^0}(x)) W_{t_i}^{2,C_i^0,W_0,W_0^{2,i}} \right. \right. \]
\[ + \tilde{C}_i^{C_i^0} \left( \sigma'(Z_{C_i^0}(x)) W_{t_i}^{2,C_i^0,W_0,W_0^{2,i}} - \sigma'(...)ight) \left. \left. \tilde{W}_{t_i}^{2,C_i^0,W_0,W_0^{2,i}} \right] \right| \]

\[ \leq \frac{K}{N_2} \sum_{i=1}^{N_2} \left[ \left| C_i^{C_i^0} - \tilde{C}_i^{C_i^0} \right| + \left| Z_{C_i^0}(x) - \tilde{Z}_{C_i^0}(x) \right| + \left| W_{t_i}^{2,C_i^0,W_0,W_0^{2,i}} - \tilde{W}_{t_i}^{2,C_i^0,W_0,W_0^{2,i}} \right| \right] \]

where the uniform bounds from (4.5) were used. In addition, we also have for some constant \( K < \infty \)

\[ \mathbb{E} \left[ \left( \Gamma_t^{V,2,W_0} \right)^2 \right] = \mathbb{E} \left[ \left( \frac{1}{N_2} \sum_{i=1}^{N_2} \tilde{C}_i^{C_i^0} \sigma'(\tilde{Z}_{C_i^0}(x)) \tilde{W}_{t_i}^{2,C_i^0,W_0,W_0^{2,i}} - \int_\mathcal{C} \tilde{C}_i^{C_i^0} \sigma'(\tilde{Z}_{C_i^0}(x)) \left( \int_{W_2} \tilde{W}_{t_i}^{2,C_i^0,W_0,W_0^{2,i}} \mu(c)(du) \right) \mu_c(dc) \right)^2 \right] \]

\[ = \frac{1}{N_2} \sum_{i=1}^{N_2} \text{Var} \left[ \tilde{C}_i^{C_i^0} \sigma'(\tilde{Z}_{C_i^0}(x)) \tilde{W}_{t_i}^{2,C_i^0,W_0,W_0^{2,i}} \right] \]

\[ \leq \frac{K}{N_2} \]

where we used the assumed independence of \( C_i^0, W_0, \) and \( W_0^{2,i} \) as well as the a-priori bound from (4.5).

Hence, we obtain that for some unimportant constant \( K < \infty \)

\[ \mathbb{E} \left| V_t^{N_2,W_0}(x) - V_t^{W_0}(x) \right|^2 \leq \frac{K}{N_2} \sum_{i=1}^{N_2} \left[ \mathbb{E} \left[ E_t^{N_2}(C_i^0,W_0,W_0^{2,i},x) \right] + \frac{K}{N_2} \right]. \quad (4.7) \]

### 4.4 Bound for \( |g_t^{N_2}(x) - g_t(x)| \)

We can write

\[ |g_t^{N_2}(x) - g_t(x)| = \left| \frac{1}{N_2} \sum_{i=1}^{N_2} C_i^{C_i^0} H_{t_i}^{2,C_i^0}(x) - \int_\mathcal{C} \tilde{C}_i^{C_i^0} H_{t_i}^{2,C_i^0}(x) \mu_c(dc) \right| \]

\[ \leq \left| \frac{1}{N_2} \sum_{i=1}^{N_2} C_i^{C_i^0} H_{t_i}^{2,C_i^0}(x) - \frac{1}{N_2} \sum_{i=1}^{N_2} \tilde{C}_i^{C_i^0} H_{t_i}^{2,C_i^0}(x) \right| \]

\[ + \left| \frac{1}{N_2} \sum_{i=1}^{N_2} \tilde{C}_i^{C_i^0} H_{t_i}^{2,C_i^0}(x) - \int_\mathcal{C} \tilde{C}_i^{C_i^0} H_{t_i}^{2,C_i^0}(x) \mu_c(dc) \right| \]

\[ := \Gamma_t^{1,1}(x) + \Gamma_t^{1,2}(x). \quad (4.8) \]

Let’s analyze the first term in (4.8) Using the uniform bounds from (4.5) we have for some unimportant
constant $K < \infty$

\[
\frac{1}{N_2} \sum_{i=1}^{N_2} C_i^c C_i^c H_i^2 C_i^c(x) - \frac{1}{N_2} \sum_{i=1}^{N_2} \tilde{C}_i^c \tilde{H}_i^2 C_i^c(x) \\
= \frac{1}{N_2} \sum_{i=1}^{N_2} (\tilde{C}_i^c - \tilde{C}_i^c) H_i^2 C_i^c(x) + \frac{1}{N_2} \sum_{i=1}^{N_2} \tilde{C}_i^c (H_i^2 C_i^c(x) - \tilde{H}_i^2 C_i^c(x)) \\
\leq K \sum_{i=1}^{N_2} \sqrt{E_{t}^{N_2} (C_i^c, W_0, W_0, x)}. \]

The second term in (4.8) is bounded, as follows,

\[
\mathbb{E} \left[ \left( \frac{1}{N_2} \sum_{i=1}^{N_2} (\tilde{C}_i^c - \tilde{C}_i^c) H_i^2 C_i^c(x) - \int C_i^c H_i^2 C_i^c(x)(\mu_c(\text{dc})) \right)^2 \right] = \frac{1}{N_2} \sum_{i=1}^{N_2} \text{Var} \left[ \tilde{C}_i^c H_i^2 C_i^c(x) \right] \leq \frac{K}{N_2}, \quad (4.9)
\]

where the independence of $C_i^c$ was used. Hence, putting things together we get

\[
\mathbb{E} \left| g_t^{N_2}(x) - g_t(x) \right| \leq \mathbb{E} \left[ \frac{K}{N_2} \sum_{i=1}^{N_2} \sqrt{E_{t}^{N_2} (C_i^c, W_0, W_0, x)} \right] + \frac{K}{\sqrt{N_2}}.
\]

4.5 Bound for $\mathbb{E} \left[ (C_i^c \tilde{C}_i^c - \tilde{C}_i^c)^2 + (W_t \tilde{W}_t - W_0 \tilde{W}_0)^2 + (W_t^2 C_i^c W_0 W_0^2)^2 \right]$.

Let us write for notational convenience $c = C_i^c$. We have

\[
d(C_i^c - \tilde{C}_i^c)^2 = 2(C_i^c - \tilde{C}_i^c) \int_{X \times Y} \left[ (y - g_t^{N_2}(x)) (H_t^2 c(x) - \tilde{H}_t^2 c(x)) + (g_t(x) - g_t^{N_2}(x)) \tilde{H}_t^2 c(x) \right] \pi(dx, dy) dt.
\]

Using Young’s inequality and the uniform bounds from (4.5),

\[
(C_i^c - \tilde{C}_i^c)^2 \leq (C_i^c - \tilde{C}_i^c)^2 + K \int_0^t \int_{X \times Y} \pi(dx, dy) \left[ (C_s^c - \tilde{C}_s^c)^2 + (H_s^2 c(x) - \tilde{H}_s^2 c(x))^2 \right] ds
+ |C_s^c - \tilde{C}_s^c| \cdot |g_s(x) - g_t^{N_2}(x)| \right] ds \]

\[
= K \int_0^t \int_{X \times Y} \pi(dx, dy) \left[ (C_s^c - \tilde{C}_s^c)^2 + (H_s^2 c(x) - \tilde{H}_s^2 c(x))^2 + |C_s^c - \tilde{C}_s^c|(\tilde{H}_s^2 + \tilde{H}_s^2) \right] ds
= K \int_0^t \int_{X \times Y} \pi(dx, dy) \left[ (C_s^c - \tilde{C}_s^c)^2 + (H_s^2 c(x) - \tilde{H}_s^2 c(x))^2 + |C_s^c - \tilde{C}_s^c|(\tilde{H}_s^2 + \tilde{H}_s^2) \right] ds. \quad (4.10)
\]
For the term $|C^c_s - \tilde{C}^c_s| \Gamma^{1,1}_s(x)$, where we recall $c = C^i_o$, we have

$$|C^c_s - \tilde{C}^c_s| \Gamma^{1,1}_s(x) \leq |C^c_s - \tilde{C}^c_s| \frac{K}{N^2} \sum_{j=1}^{N^2} E^N_{s} (C^j_o, W_0^2, W_0^2, x)$$

$$= \frac{K}{N^2} \sum_{j=1}^{N^2} |C^c_s - \tilde{C}^c_s| \sqrt{E^N_{s} (C^j_o, W_0^2, W_0^2, x)}$$

$$\leq KE^N_{s} (C^j_o, W_0^2, W_0^2, x) + \frac{K}{N^2} \sum_{j=1}^{N^2} E^N_{s} (C^j_o, W_0^2, W_0^2, x)$$

$$\leq KE^N_{s} (C^j_o, W_0^2, W_0^2, x) + \frac{K}{N^2} \sum_{j=1}^{N^2} E^N_{s} (C^j_o, W_0^2, W_0^2, x)$$

Therefore, using (4.10) and (4.11) we obtain

$$E \left[ (C^c_t - \tilde{C}^c_t)^2 \right] \leq C \int_0^t \sup_{x \in X} E \left[ E^N_{s} (C^j_o, W_0^2, W_0^2, x) \right] ds$$

$$+ \frac{K}{N^2} \sum_{j=1}^{N^2} \int_0^t \sup_{x \in X} E \left[ E^N_{s} (C^j_o, W_0^2, W_0^2, x) \right] ds$$

Using similar arguments, we can also show, using (4.12), that

$$E \left[ (W^2_t - W_t)^2 \right] \leq C \int_0^t \sup_{x \in X} E \left[ E^N_{s} (C^j_o, W_0^2, W_0^2, x) \right] ds$$

$$+ \frac{K}{N^2} \sum_{j=1}^{N^2} \int_0^t \sup_{x \in X} E \left[ E^N_{s} (C^j_o, W_0^2, W_0^2, x) \right] ds$$

Therefore, we overall get that

$$E \left[ (C^c_t - \tilde{C}^c_t)^2 + (W^2_t, W_t)^2 + (W^2_t, W_t)^2 \right] \leq C \int_0^t \sup_{x \in X} E \left[ E^N_{s} (C^j_o, W_0^2, W_0^2, x) \right] ds + \frac{K}{N^2} \sum_{j=1}^{N^2} \int_0^t \sup_{x \in X} E \left[ E^N_{s} (C^j_o, W_0^2, W_0^2, x) \right] ds$$

4.6 Bound for $E \left[ (Z^c_t - \tilde{Z}^c_t)^2 \right]$.

We next consider $(Z^c_t - \tilde{Z}^c_t)^2$. For the following calculations, we define $\mathcal{F}^N_{c} = (C^1_o, C^2_o, \ldots, C^{N^2}_o)$. 
For \( i = 1, 2, \ldots, N_2 \), we have

\[
(Z_{t, i}^{C_i}(x) - \bar{Z}_{t, i}^{C_i}(x))^2 = \left( \mathbb{E} \left[ W_t^{2, C_i, W_0, W_0^{2, i}} H_t^{1, W_0}(x) \bigg| \mathcal{F}_{C_i}^N \right] - \int_{W_1} \int_{W_2} \bar{W}_t^{2, C_i, w, u, \hat{H}_t^{1, w}}(x) \mu_{W_2}(du) \mu_{W_1}(dw) \right)^2
\]

\[
= \left( \mathbb{E} \left[ (W_t^{2, C_i, W_0, W_0^{2, i}} - \bar{W}_t^{2, C_i, W_0, W_0^{2, i}}) H_t^{1, W_0}(x) \bigg| \mathcal{F}_{C_i}^N \right] \right)^2 \leq \mathbb{E} \left[ (W_t^{2, C_i, W_0, W_0^{2, i}} - \bar{W}_t^{2, C_i, W_0, W_0^{2, i}})^2 \bigg| \mathcal{F}_{C_i}^N \right]
\]

where the uniform bounds from (4.5) were used together with the compactness of the state space assumption from Assumption 2.1.

Therefore, using iterated expectations, we have that

\[
\mathbb{E} \left( Z_{t, i}^{C_i}(x) - \bar{Z}_{t, i}^{C_i}(x) \right)^2 \leq K_1 \frac{1}{N_2} + K_2 \int_0^t \sup_{x \in \mathcal{X}} \mathbb{E} \left[ E_s^{N_2}(C_i, W_0, W_0^{2, i}, x) \right] ds
\]

\[
+ K_3 \frac{1}{N_2} \sum_{j=1}^{N_2} \int_0^t \sup_{x \in \mathcal{X}} \mathbb{E} \left[ E_s^{N_2}(C_i, W_0, W_0^{2, i}, x) \right] ds \tag{4.13}
\]

where (4.11) was used.

Finally, using the assumption that \( \sigma \) is globally Lipschitz,

\[
\mathbb{E} \left[ (H_t^{2, C_i}(x) - \bar{H}_t^{2, C_i}(x))^2 \right] \leq K_0 \mathbb{E} \left[ (Z_{t, i}^{C_i} - \bar{Z}_{t, i}^{C_i})^2 \right] \leq K_1 \frac{1}{N_2} + K_2 \int_0^t \sup_{x \in \mathcal{X}} \mathbb{E} \left[ E_s^{N_2}(C_i, W_0, W_0^{2, i}, x) \right] ds
\]

\[
+ K_3 \frac{1}{N_2} \sum_{j=1}^{N_2} \int_0^t \sup_{x \in \mathcal{X}} \mathbb{E} \left[ E_s^{N_2}(C_i, W_0, W_0^{2, i}, x) \right] ds \tag{4.14}
\]

4.7 Bound for \( \mathbb{E} \left[ E_s^{N_2}(C_i, W_0, W_0^{2, i}, x) \right] \)

Collecting our results from (4.12), (4.13) and (4.14), together with the definition of the error function \( E_s^{N_2} \), we have, for \( i = 1, \ldots, N_2 \), the bound

\[
\sup_{x \in \mathcal{X}} \mathbb{E} \left[ E_s^{N_2}(C_i, W_0, W_0^{2, i}, x) \right] \leq K_1 \frac{1}{N_2} + K_2 \int_0^t \sup_{x \in \mathcal{X}} \mathbb{E} \left[ E_s^{N_2}(C_i, W_0, W_0^{2, i}, x) \right] ds,
\]

\[
+ K_3 \frac{1}{N_2} \sum_{j=1}^{N_2} \int_0^t \sup_{x \in \mathcal{X}} \mathbb{E} \left[ E_s^{N_2}(C_i, W_0, W_0^{2, i}, x) \right] ds
\]
Therefore, by averaging over all \( i = 1, \cdots, N_2 \) and then using Gronwall’s inequality, we get for any \( 0 \leq t \leq T \),

\[
\frac{1}{N_2} \sum_{i=1}^{N_2} \sup_{x \in X} \mathbb{E} \left[ E_t^{N_2}(C^i_0, W_0, W_0^{2,i}, x) \right] \leq \frac{K}{N_2}.
\]

Combining the last two displays we naturally get, again using Gronwall’s inequality, that for \( i = 1, \cdots, N_2 \)

\[
\sup_{x \in X} \mathbb{E} \left[ E_t^{N_2}(C^i_0, W_0, W_0^{2,i}, x) \right] \leq \frac{K}{N_2}, \tag{4.15}
\]

where the constant \( K < \infty \) does not depend on \( i \).

### 4.8 Convergence of neural network prediction

The bound (4.15) of course proves the (uniform) convergence in probability of the neural network output \( g_t^{N_2}(x) \rightarrow g_t(x) \). Recall that

\[
\mathbb{E}|g_t^{N_2}(x) - g_t(x)| \leq \mathbb{E} \left| \Gamma_t^{g,1}(x) \right| + \frac{K}{\sqrt{N_2}},
\]

where

\[
\left| \Gamma_t^{g,1}(x) \right| \leq \frac{K}{N_2} \sum_{i=1}^{N_2} \sqrt{E_t^{N_2}(C^i_0, W_0, W_0^{2,i}, x)}.
\]

Then, using the Cauchy-Schwartz inequality and (4.15),

\[
\mathbb{E} \left| \Gamma_t^{g,1}(x) \right| \leq \frac{K}{N_2} \sum_{i=1}^{N_2} \mathbb{E} \left[ \sqrt{E_t^{N_2}(C^i_0, W_0, W_0^{2,i}, x)} \right] \\
\leq \frac{K}{N_2} \sum_{i=1}^{N_2} \mathbb{E} \left[ \sup_{x \in X} E_t^{N_2}(C^i_0, W_0, W_0^{2,i}, x) \right] \\
\leq \frac{K}{\sqrt{N_2}}.
\]

Therefore, for \( 0 \leq t \leq T \), and for some unimportant constant \( K < \infty \)

\[
\sup_{x \in X} \mathbb{E} \left[ |g_t^{N_2}(x) - g_t(x)| \right] \leq \frac{K}{\sqrt{N_2}},
\]

concluding the identification of the limit in Theorem 2.3.

### 5 Proof of Theorem 2.3 Uniqueness of the limit

**Lemma 5.1.** The solution to the limiting system (2.4) is unique in \( C([0,T], C \times W^1 \times W^2 \times X) \).

**Proof.** Suppose there are two solutions to (2.4). Let’s denote the first solution as \( (\hat{W}_t^w, \hat{W}_t^{2,c,w,u}, \hat{C}_t, \hat{Z}_t(x), \hat{V}_t^w(x), \hat{g}_t(x)) \) and the second solution as \( (\bar{W}_t^w, \bar{W}_t^{2,c,w,u}, \bar{C}_t, \bar{Z}_t(x), \bar{V}_t^w(x), \bar{g}_t(x)) \). Define the function

\[
Q_t = \sup_{(e,w,u,x) \in C \times W^1 \times W^2 \times X} \left\{ (\hat{W}_t^{2,c,w,u} - \bar{W}_t^{2,c,w,u})^2 + \| \hat{W}_t^{1,w} - \bar{W}_t^{1,w} \|^2 + (\hat{V}_t^w(x) - \bar{V}_t^w(x))^2 + (\hat{Z}_t(x) - \bar{Z}_t(x))^2 \\
+ (\hat{C}_t - \bar{C}_t)^2 + (\hat{g}_t(x) - \bar{g}_t(x))^2 \right\}
\]

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Note that $Q_0 = 0$.

We next study the evolution of $Q_t$ for $t > 0$.

Using the same approach as in Section 4.2, we can show that any $\tilde{W}_{t}^{2,c,w,u}, \tilde{C}_{t}, \tilde{H}_{t}^{1,w}(x), \tilde{Z}_{t}, V_{t}$, and $g_{t}(x)$ which solve (2.4) are uniformly bounded on $\mathcal{C} \times W^{1} \times W^{2} \times \mathcal{X} \times [0, T]$.

We can then prove the inequality

$$
(\tilde{W}_{t}^{2,c,w,u} - \bar{W}_{t}^{2,c,w,u})^2 = 2 \int_{0}^{t} \int_{\mathcal{X} \times \mathcal{Y}} (\tilde{W}_{s}^{2,c,w,u} - \bar{W}_{s}^{2,c,w,u}) \left( (y - \tilde{g}_{s}(x)) \tilde{C}_{s}^{c}(\tilde{Z}_{s}(x)) \tilde{H}_{s}^{1,w}(x) - (y - \bar{g}_{s}(x)) \tilde{C}_{s}^{c}(\bar{Z}_{s}(x)) \bar{H}_{s}^{1,w}(x) \right) \pi(dx, dy) ds
$$

where we have also used Young’s inequality and the fact that $\mathcal{X} \times \mathcal{Y}$ is compact. Therefore,

$$
\sup_{(c, w, u, x) \in \mathcal{C} \times W^{1} \times W^{2} \times \mathcal{X}} (\tilde{W}_{t}^{2,c,w,u} - \bar{W}_{t}^{2,c,w,u})^2 \leq K \int_{0}^{t} Q_{s} ds.
$$

Similarly,

$$
\sup_{(c, w, u, x) \in \mathcal{C} \times W^{1} \times W^{2} \times \mathcal{X}} \left\| \tilde{W}_{t}^{1,w} - \bar{W}_{t}^{1,w} \right\|^2 \leq K \int_{0}^{t} Q_{s} ds.
$$

Next, using the Cauchy-Schwarz inequality and Young’s inequality,

$$
(\bar{Z}_{t}^{c}(x) - \tilde{Z}_{t}^{c}(x))^2 = \left( \int_{W^{1}} \int_{W^{2}} \tilde{W}_{t}^{2,c,w,u} \tilde{H}_{t}^{1,w}(x) \mu_{W^{2}}(du) \mu_{W^{1}}(dw) - \int_{W^{1}} \int_{W^{2}} \bar{W}_{t}^{2,c,w,u} \bar{H}_{t}^{1,w}(x) \mu_{W^{2}}(du) \mu_{W^{1}}(dw) \right)^2
$$

$$
\leq K \int_{W^{1}} \int_{W^{2}} \left[ (\tilde{W}_{t}^{2,c,w,u} - \bar{W}_{t}^{2,c,w,u}) + (\tilde{W}_{t}^{1,w} - \bar{W}_{t}^{1,w}) \right]^2 \mu_{W^{2}}(du) \mu_{W^{1}}(dw)
$$

$$
\leq K \sup_{(c, w, u, x) \in \mathcal{C} \times W^{1} \times W^{2} \times \mathcal{X}} \left[ (\tilde{W}_{t}^{2,c,w,u} - \bar{W}_{t}^{2,c,w,u}) + (\tilde{W}_{t}^{1,w} - \bar{W}_{t}^{1,w}) \right] \leq K \int_{0}^{t} Q_{s} ds.
$$
Using a similar approach,

\[(\hat{V}_t^w(x) - \bar{V}_t^w(x))^2 = \left( \int_c \int_{W^2} \hat{C}_t^c \sigma' (\hat{Z}_t^c(x)) \hat{W}_t^{2,c.w,u} \mu_{W^2}(du) \mu_c(dc) \right. \]
\[\left. - \int_c \int_{W^2} \bar{C}_t^c \sigma' (\bar{Z}_t^c(x)) \bar{W}_t^{2,c,w,u} \mu_{W^2}(du) \mu_c(dc) \right) \]
\[\leq \int_c \int_{W^2} \left( \hat{C}_t^c \sigma' (\hat{Z}_t^c(x)) \hat{W}_t^{2,c,w,u} - \bar{C}_t^c \sigma' (\bar{Z}_t^c(x)) \bar{W}_t^{2,c,w,u} \right) \mu_{W^2}(du) \mu_c(dc) \]
\[\leq K \int_c \int_{W^2} \left[ (\hat{C}_t^c - \bar{C}_t^c)^2 + (\hat{Z}_t^c(x) - \bar{Z}_t^c(x))^2 + (\hat{W}_t^{2,c,w,u} - \bar{W}_t^{2,c,w,u})^2 \right] \mu_{W^2}(du) \mu_c(dc) \]
\[\leq K \sup_{(c,w,u,x) \in C \times W^1 \times W^2 \times X} \left[ (\hat{C}_t^c - \bar{C}_t^c)^2 + (\hat{Z}_t^c(x) - \bar{Z}_t^c(x))^2 + (\hat{W}_t^{2,c,w,u} - \bar{W}_t^{2,c,w,u})^2 \right] \]
\[\leq K \int_0^t Q_s ds. \]

Therefore,

\[\sup_{(c,w,u,x) \in C \times W^1 \times W^2 \times X} (\hat{Z}_t^c(x) - \bar{Z}_t^c(x))^2 \leq K \int_0^t Q_s ds, \]
\[\sup_{(c,w,u,x) \in C \times W^1 \times W^2 \times X} (\hat{V}_t^w(x) - \bar{V}_t^w(x))^2 \leq K \int_0^t Q_s ds. \]

Finally,

\[(\hat{g}_t(x) - \bar{g}_t(x))^2 = \left( \int_c \hat{C}_t^c \hat{H}_t^{2,c}(x) \mu_c(dc) - \int_c \bar{C}_t^c \bar{H}_t^{2,c}(x) \mu_c(dc) \right)^2 \]
\[\leq \int_c \left[ \hat{C}_t^c \hat{H}_t^{2,c}(x) \mu_c(dc) - \bar{C}_t^c \bar{H}_t^{2,c}(x) \right]^2 \mu_c(dc) \]
\[\leq K \int_c \left[ (\hat{C}_t^c - \bar{C}_t^c)^2 + (\hat{Z}_t^c(x) - \bar{Z}_t^c(x))^2 \right] \mu_c(dc) \]
\[\leq K \int_0^t Q_s ds. \]

Consequently,

\[\sup_{(c,w,u,x) \in C \times W^1 \times W^2 \times X} (\hat{g}_t(x) - \bar{g}_t(x))^2 \leq K \int_0^t Q_s ds. \]

Collecting our results,

\[Q_t \leq K \int_0^t Q_s ds, \]
\[Q_0 = 0. \]

Therefore, by Gronwall’s inequality,

\[Q_t = 0, \]

for all \( t \in [0, T] \), completing the proof of the lemma.
A Limit of First Layer, Lemma 2.2

In this appendix we prove Lemma 2.2 which is about the limit of the first layer. The proof is analogous to [31]. Hence, instead of repeating all the arguments we will only present the general outline empathizing the differences and refer the interested reader to [31] when the arguments are the same.

We let $N_1 \to \infty$ (with $N_2$ fixed). We want to prove that for each $N_2 \in \mathbb{N}$, $\gamma^{N_1,N_2} \overset{d}{\to} \gamma^{N_2}$ in $D_2([0,T])$. $\gamma^{N_2}_t$ is a random measure-valued process which, for every $f \in C^0_b(\mathbb{R}^{d+2N_2})$, $\gamma^{N_2}_t$, satisfies the evolution equation (2.2)–(2.3).

A.1 Relative Compactness

Let’s first establish a bound on $C^{i}_k$. Recall that $\sigma(\cdot)$ is bounded and $\alpha_{C^{N_1,N_2}} = \frac{N_2}{N_1}$. In the following calculations, the unimportant constants, $K, K_0, K_1,$ and $K_2$ may change from line to line.

For $k = 0, 1, \ldots, [TN_1]$, this implies that

$$C^{i}_{k+1} = C^{i}_k + \frac{\alpha_{C^{N_1,N_2}}}{N_2} \left(y_k - \frac{1}{N_2} \sum_{m=1}^{N_2} C^{i,m}_k H^{2,m}_k\right) H^{2,i}_k,$$

Since $\sigma(\cdot)$ is bounded, $|H^{2,i}_k| < K$. Therefore,

$$|C^{i}_{k+1}| \leq |C^{i}_k| + \frac{1}{N_1} \left( K_1 + \frac{1}{N_2} \sum_{m=1}^{N_2} |C^{i,m}_k| \right).$$

This yields

$$\frac{1}{N_2} \sum_{i=1}^{N_2} |C^{i}_{k+1}| \leq \frac{1}{N_2} \sum_{i=1}^{N_2} |C^{i}_k| + \frac{1}{N_1} \left( K_1 + \frac{1}{N_2} \sum_{i=1}^{N_2} |C^{i}_k| \right).$$

This implies that

$$\frac{1}{N_2} \sum_{i=1}^{N_2} |C^{i}_{k+1}| \leq \frac{1}{N_2} \sum_{i=1}^{N_2} |C^{i}_0| + K_1 \frac{k}{N_1} + K_2 \frac{k}{N_1} \sum_{j=1}^{k} \frac{1}{N_2} \sum_{i=1}^{N_2} |C^{i}_k|$$

$$\leq K_0 + K_1 \frac{k}{N_1} + K_2 \frac{k}{N_1} \sum_{j=1}^{k} \frac{1}{N_2} \sum_{i=1}^{N_2} |C^{i}_k|$$

By Gronwall’s inequality, for $k \leq [TN_1]$,

$$\frac{1}{N_2} \sum_{i=1}^{N_2} |C^{i}_k| \leq K \exp(KT).$$

Then, we also have that

$$|C^{i}_{k+1}| \leq |C^{i}_k| + \frac{K}{N_1}$$

$$\leq \frac{K}{N_1}.$$

This immediately yields that for $k \leq [TN_1]$,

$$|C^{i}_k| < KT.$$

Let’s now address the parameters $W^{2,i,j}_k$. Recall that $\alpha_{W^{2,i,j}_N} = N_2$. Then,

$$|W^{2,i,j}_{k+1}| \leq |W^{2,i,j}_k| + \frac{\alpha_{W^{2,i,j}_N}}{N_1N_2} \left| (y_k - g_{b_k}(x_k)) C^{i}_k \sigma'(Z^{1,i}_k) H^{1,i}_k \right|$$

$$\leq |W^{2,i,j}_k| + \frac{K}{N_1}.$$
Therefore, for $k \leq \lfloor TN_1 \rfloor$,
\[ |W_{k+1}^{2,i,j}| \leq KT. \]

Similarly, we have
\[
|W_{k+1}^{1,j}| \leq |W_k^{1,j}| + \frac{1}{N_1} \left| (y_k - g_k^N(x_k)) \left( \frac{1}{N_2} \sum_{i=1}^{N_2} C_k^i \sigma'(Z_k^i) W_k^{2,i,j} \right) \right| \
\leq |W_k^{1,j}| + \frac{K}{N_1}.
\]

Therefore, we obtain
\[ |W_k^{1,j}| \leq KT. \]

Collecting our results, for all $k/N_1 \leq T$ and for all $j = 1, \cdots, N_2$, we have the uniform bound
\[
|C_k^i| + \left| W_k^{1,j} \right| + |W_k^{2,i,j}| < K. \tag{A.1}
\]

We now prove relative compactness of the family \{\(\gamma_{N_1,N_2}\)\}_{N_1 \in \mathbb{N}} in \(D_E([0,T])\) where \(E = M(\mathbb{R}^{d+2N_2})\). It is sufficient to show compact containment and regularity of the \(\gamma_{N_1,N_2}\)'s (see for example Chapter 3 of [12]).

**Lemma A.1.** For each \(\eta > 0 \text{ and } t \geq 0\), there is a compact subset \(K\) of \(E\) such that
\[
\sup_{N_1 \in \mathbb{N}, 0 \leq t \leq T} \mathbb{P}[\gamma_{t+N_1,N_2} \notin K] < \eta.
\]

**Proof.** This uniform bound \(\text{(A.1)}\) actually implies the stronger statement of compact support. In particular, notice that the set \([-K, K]^{d+2N_2}\) is compact, and define
\[ K = \{ \omega \in M(\mathbb{R}^{d+2N_2}) : \omega([0,T]) = 0 \}. \]

Then \(K \subset M(\mathbb{R}^{d+2N_2})\), and \(\mathbb{P}\text{-a.s. } \gamma_{t,N_1,N_2} \in K\) for all \(N_1 \in \mathbb{N}\) and \(t \in [0,T]\). This concludes the proof.

We now establish regularity of the \(\gamma_{N_1,N_2}\)'s. Define the function \(q(z_1, z_2) = \min\{|z_1 - z_2|, 1\}\) where \(z_1, z_2 \in \mathbb{R}\).

**Lemma A.2.** For any \(p \in (0,1)\), there is a constant \(K < \infty\) such that for \(0 \leq u \leq \delta, 0 \leq v \leq \delta \land t, t \in [0,T]\),
\[
\mathbb{E} \left[ q(\left\langle f, \gamma_{t,u+N_1,N_2}^N \right\rangle, \left\langle f, \gamma_{t,u}^N \right\rangle) q(\left\langle f, \gamma_{t,u}^N \right\rangle, \left\langle f, \gamma_{t-v}^N \right\rangle) \left| F_t^N \right| \right] \leq C \delta^p + \frac{K}{N_1}.
\]

**Proof.**
\[
|C_{[N_1 \cdot]} - C_{[N_1 \cdot]}^t| = \left| \sum_{k=[N_1 \cdot]}^{(N_1 \cdot) - 1} (C_k^t - C_k) \right| \
\leq \sum_{k=[N_1 \cdot]}^{(N_1 \cdot) - 1} \frac{1}{N_1} \left| (y_k - g_{\theta_k}^N(x_k)) H_k^{2,i} \right| \
\leq \frac{1}{N_1} \sum_{k=[N_1 \cdot]}^{(N_1 \cdot) - 1} K \leq K_1(t-s) + \frac{K}{N_1} \
\leq K(t-s)^p 1_{t-s<1} + K(t-s)^p T^{1/p} 1_{t-s \geq 1} + \frac{K}{N_1} \
\leq K(t-s)^p + \frac{K}{N_1},
\]
Using the same approach, we can establish similar bounds for the other parameters:

\[
|W_{[N_1t]}^{1,j} - W_{[N_1s]}^{1,j}| \leq K(t-s)^p + \frac{K}{N_1} \\
|W_{[N_1t]}^{2,i,j} - W_{[N_1s]}^{2,i,j}| \leq K(t-s)^p + \frac{K}{N_1}.
\]

The desired result then follows.

We conclude this section now with the required relative compactness of the sequence \(\{\gamma^{N_1,N_2}\}_{N_1 \in \mathbb{N}}\). This implies that for each fixed \(N_2\), every subsequence \(\gamma^{N_1,N_2}\)'s has a convergent sub-subsequence as \(N_1 \to \infty\).

**Lemma A.3.** The sequence of probability measures \(\{\gamma^{N_1,N_2}\}_{N_1 \in \mathbb{N}}\) is relatively compact in \(D_E([0,T])\).

**Proof.** Given Lemmas A.1 and A.2, Theorem 8.6 and Remark 8.7 B of Chapter 3 of [12], gives the statement of the lemma.

\[ \square \]

### A.2 Identification of the Limit

We consider the evolution of the empirical measure \(\gamma_t^{N_1,N_2}\) via test functions \(f \in C_b^2(\mathbb{R}^{d+2N_2})\). Using a Taylor expansion based argument similarly to [31], we can show that the scaled empirical measure satisfies

\[
\left\langle f, \gamma_t^{N_1,N_2} \right\rangle - \left\langle f, \gamma_0^{N_1,N_2} \right\rangle = \int_0^t \int_{X \times Y} (y - g_s^{N_1,N_2}(x)) \left( H_s^{N_1,N_2}(x) \cdot \nabla_c f, \gamma_s^{N_1,N_2} \right) \pi(dx,dy)ds \\
+ \frac{1}{N_2} \sum_{i=1}^{N_2} \langle c_i \sigma' (Z_s^{i,N_1,N_2}(x)) w^2, \sigma'(w \cdot x) x \cdot \nabla_{w^2} f, \gamma_s^{N_1,N_2} \rangle \pi(dx,dy)ds \\
+ M_{N_1,N_2}(t) + O(1/N_1),
\]

(A.2)

where \(M_{N_1,N_2}(t)\) is a martingale term and

\[
Z_s^{i,N_1,N_2}(x) = \left\langle w^{2,i} \sigma(w \cdot x), \gamma_s^{N_1,N_2} \right\rangle, \\
H_s^{i,N_1,N_2}(x) = \sigma(Z_s^{i,N_1,N_2}(x)), \\
g_s^{N_1,N_2}(x) = \frac{1}{N_2} \sum_{i=1}^{N_2} H_s^{i,N_1,N_2}(x) \langle c_i, \gamma_s^{N_1,N_2} \rangle.
\]

We define \(H_s^{N_1,N_2}\) as the vector \((H_1^{1,N_1,N_2}, \ldots, H_1^{2,N_1,N_2}, \ldots)\), \(c\) as the vector \((c_1, \ldots, c_{N_2})\), and \(Z_s^{N_1,N_2}\) as the vector \((Z_1^{1,N_1,N_2}, \ldots, Z_1^{2,N_1,N_2}, \ldots)\). The martingale term \(M_{N_1,N_2}(t)\) converges to 0 in \(L^2\) as \(N_1 \to \infty\). That is,

\[
\lim_{N_1 \to \infty} \mathbb{E} \left[ \left( M_{N_1,N_2}(t) \right)^2 \right] = 0.
\]

(A.3)

The proof for (A.3) follows as in Lemma 3.1 of [31] and thus it is omitted. The limit point of \(\gamma^{N_1,N_2}\), as \(N_1 \to \infty\) and for a fixed \(N_2\), will satisfy the evolution equation

\[
\left\langle f, \gamma_t^{N_2} \right\rangle - \left\langle f, \gamma_0^{N_2} \right\rangle = \int_0^t \int_{X \times Y} \alpha(y - g_s^{N_2}(x)) \left( H_s^{N_2}(x) \cdot \nabla_c f, \gamma_s^{N_2} \right) \pi(dx,dy)ds \\
+ \frac{1}{N_2} \sum_{i=1}^{N_2} \langle c_i \sigma' (Z_s^{i,N_2}(x)) w^2, \sigma'(w \cdot x) x \cdot \nabla_{w^2} f, \gamma_s^{N_2} \rangle \pi(dx,dy)ds \\
+ \frac{1}{N_2} \sum_{i=1}^{N_2} \langle c_i \sigma' (Z_s^{i,N_2}(x)) w^2, \sigma'(w \cdot x) x \cdot \nabla_{w^2} f, \gamma_s^{N_2} \rangle \pi(dx,dy)ds,
\]

(A.4)
where

\[ Z_s^{i,N_2}(x) = \langle w^{2,i}\sigma(w^1 \cdot x), \gamma_{s}^{N_2} \rangle, \]

\[ H_s^{i,N_2}(x) = \sigma(Z_s^{i,N_2}(x)), \]

\[ g_s^{N_2}(x) = \frac{1}{N_2} \sum_{i=1}^{N_2} H_s^{i,N_2}(x) \langle c_i, \gamma_{s}^{N_2} \rangle, \]

and

\[ \gamma_0^{N_2}(dw^1, dw^2, dc) = \mu_{W^1}(dw^1) \times \mu_{W^2}(dw^{2,1}) \times \cdots \times \mu_{W^2}(dw^{2,N_2}) \times \delta_{C_2} (dc^1) \times \cdots \times \delta_{C_N} (dc^{N_2}). \]

Let \( \pi^{N_1,N_2} \) be the probability measure of \( (\gamma^{N_1,N_2})_{0 \leq t \leq T} \). Each \( \pi^{N_1,N_2} \) takes values in the set of probability measures \( \mathcal{M}(D_E([0,T])) \). Relative compactness, proven in Section A.1, implies that there is a subsequence \( \pi^{N_{1_k},N_2} \) which weakly converges. We must prove that any limit point \( \pi^{N_2} \) of a convergent subsequence \( \pi^{N_{1_k},N_2} \) will satisfy the evolution equation \( \text{(A.4)} \).

**Lemma A.4.** Let \( \pi^{N_{1_k},N_2} \) be a convergent subsequence with a limit point \( \pi^{N_2} \). Then, \( \pi^{N_2} \) is a Dirac measure concentrated on \( \gamma^{N_2} \in D_E([0,T]) \) and \( \gamma^{N_2} \) satisfies the measure evolution equation \( \text{(A.4)} \).

**Proof.** We define a map \( F(\gamma) : D_E([0,T]) \rightarrow \mathbb{R}_+ \) for each \( t \in [0,T], f \in C_b^2(\mathbb{R}^{d+2N_2}), g_1, \cdots, g_p \in C_b(\mathbb{R}^{d+2N_2}) \) and \( 0 \leq s_1 < \cdots < s_p \leq t \).

\[
F(\gamma) = \left| \left( \langle f, \gamma_t \rangle - \langle f, \gamma_0 \rangle - \int_0^t \int_{X \times Y} \alpha(y - g_s^{N_2}(x)) \langle H_t^{N_2}(x) \cdot \nabla c f, \gamma_s^{N_2} \rangle \pi(dx, dy)ds \right. \right.
\]
\[
- \left. \int_0^t \int_{X \times Y} \alpha(y - g_s^{N_2}(x)) \langle \sigma(w^1 \cdot x)(\sigma'(Z_s) \circ c) \cdot \nabla w^2 f, \gamma_s^{N_2} \rangle \pi(dx, dy)ds \right.
\]
\[
- \left. \int_0^t \int_{X \times Y} \alpha(y - g_s^{N_2}(x)) \frac{1}{N_2} \sum_{i=1}^{N_2} \langle c_i, \sigma'(Z_i^{N_2}(x))w^{2,i}(w^1 \cdot x) \cdot \nabla w^2 f, \gamma_s^{N_2} \rangle \pi(dx, dy)ds \right) \times \langle g_1, \gamma_{s_1} \rangle \times \cdots \times \langle g_p, \gamma_{s_p} \rangle \right|.
\]

Then,

\[
E_{\pi^{N_1,N_2}}[F(\gamma)] = E[F(\gamma_{N_1,N_2})]
\]
\[
= E \left( M_{N_1,N_2}(t) + O(N_1^{-1}) \right) \prod_{i=1}^{p} \langle g_i, \gamma_{s_i}^{N_1,N_2} \rangle
\]
\[
\leq E[|M_{N_1,N_2}(t)|] + O(N^{-1})
\]
\[
\leq E[(M_{N_1,N_2}(t))^2]^{1/2} + O(N^{-1})
\]
\[
\leq K\left( \frac{1}{\sqrt{N}} + \frac{1}{N} \right).
\]

Therefore,

\[
\lim_{N_1 \to \infty} E_{\pi^{N_1,N_2}}[F(\gamma)] = 0.
\]

Since \( F(\cdot) \) is continuous and \( F(\gamma_{N_1,N_2}) \) is uniformly bounded (due to the uniform boundedness results of Section A.1),

\[
E_{\pi^{N_2}}[F(\gamma)] = 0.
\]

Since this holds for each \( t \in [0,T], f \in C_b^2(\mathbb{R}^{d+2N_2}) \) and \( g_1, \cdots, g_p \in C_b(\mathbb{R}^{d+2N_2}) \), \( \gamma^{N_2} \) satisfies the evolution equation \( \text{(A.4)} \). \( \square \)

It remains to prove that the evolution equation \( \text{(A.4)} \) has a unique solution. This is the content of Section A.3.

24
A.3 Uniqueness

Lemma A.5. There exists a unique solution to the evolution equation (A.4).

Proof. We only sketch the proof as the argument is similar to the uniqueness argument of [31]. Consider the particle system:

\[
\begin{align*}
    dC_i^t &= \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t^{N_2}(x)) H_{2, i}^t(x) \pi(dx, dy) dt, \quad C_i^0 = C_i^0, \quad i = 1, \ldots, N_2, \\
    dW_i^1 &= \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t^{N_2}(x)) \left( \frac{1}{N_2} \sum_{i=1}^{N_2} C_i^t \sigma'(Z_i(x)) W_i^{2, i} \right) \sigma'(W_i^1 \cdot x) \pi(dx, dy) dt, \quad W_0^1 \sim \mu_{W^1} (dw), \\
    dW_i^{2, i} &= \int_{\mathcal{X} \times \mathcal{Y}} (y - g_t^{N_2}(x)) C_i^t \sigma'(Z_i(x)) H_i^t(x) \pi(dx, dy) dt, \quad W_0^{2, i} \sim \mu_{W^2} (dw^2), \quad i = 1, \ldots, N_2, \\
    H_i^t(x) &= \sigma(W_i^1 \cdot x), \\
    Z_i^t(x) &= \mathbb{E} \left[ w^{2, i} H_i^t(x) \right], \\
    H_i^{2, i}(x) &= \sigma(Z_i(x)), \\
    g_t^{N_2}(x) &= \frac{1}{N_2} \sum_{i=1}^{N_2} C_i^t H_i^{2, i}(x). \quad (A.6)
\end{align*}
\]

Let \( \nu_{t, (c_1, \ldots, c_{N_2})} \) be the conditional law of \((W_t^1, W_t^{2, 1}, \ldots, W_t^{2, N_2}, C_t^1, \ldots, C_t^{N_2})_{0 \leq t \leq T}\) given \((C_1^0, \ldots, C_0^{N_2}) = (c_1, \ldots, c_{N_2})\). Similarly to the general results of [16], \( \nu_{t, (c_1, \ldots, c_{N_2})} = \gamma_t^{N_2} \) is a solution to the evolution equation (A.4) and conversely, any solution \( \nu_{t, (C_1^1, \ldots, C_0^{N_2})} \) to the evolution equation (A.4) must also be the law of the solution to (A.6).

Let us next prove that we can write \( Z_i^t(x) = \mathbb{E} \left[ w^{2, i} H_i^t(x) \left| C_1^1, \ldots, C_0^{N_2} \right. \right] \). Recall that \( \mathcal{F}_C^{N_2} = (C_1^0, \ldots, C_0^{N_2}) \) and let us similarly define \( \mathcal{F}_W^{N_2} = (W_0^{2, 1}, \ldots, W_0^{2, N_2}) \). Inspecting (A.6) it becomes clear that the random variables \((W_t^1, W_t^{2, 1}, \ldots, W_t^{2, N_2}, C_t^1, \ldots, C_t^{N_2})_{0 \leq t \leq T}\) depend, in addition to their own initial conditions, also on \( \mathcal{F}_C^{N_2} \) and \( \mathcal{F}_W^{N_2} \) in a symmetric way through the terms \( g_t^{N_2}(x) \) and \( \frac{1}{N_2} \sum_{i=1}^{N_2} C_i^t \sigma'(Z_i(x)) W_i^{2, i} \). In order to make this dependence specific in the notation, we write in particular that

\[
W_t^1 = W_t^{1, W_0, \mathcal{F}_C^{N_2}, \mathcal{F}_W^{N_2}}, \quad W_t^{2, i} = W_t^{2, C_i^0, W_0, \mathcal{F}_C^{N_2}, \mathcal{F}_W^{N_2}} \quad \text{and} \quad C_t^i = C_t^{C_i^0, \mathcal{F}_C^{N_2}}.
\]

This, then leads to the following calculations

\[
\begin{align*}
    Z_i^t(x) &= \mathbb{E} \left[ w^{2, i} H_i^t(x) \right] \\
    &= \mathbb{E} \left[ w^{2, i} \sigma(W_t^1 \cdot x) \right] \\
    &= \mathbb{E} \left[ W_t^{2, C_i^0, W_0, \mathcal{F}_C^{N_2}, \mathcal{F}_W^{N_2}} \sigma(W_t^{1, W_0, \mathcal{F}_C^{N_2}, \mathcal{F}_W^{N_2}} \cdot x) \right] \\
    &= \mathbb{E} \left[ W_t^{2, C_i^0, W_0, \mathcal{F}_C^{N_2}, \mathcal{F}_W^{N_2}} \sigma(W_t^{1, W_0, \mathcal{F}_C^{N_2}, \mathcal{F}_W^{N_2}} \cdot x) \mathcal{F}_W^{N_2} \right] \\
    &= \mathbb{E} \left[ W_t^{2, C_i^0, W_0, \mathcal{F}_C^{N_2}, \mathcal{F}_W^{N_2}} \sigma(W_t^{1, W_0, \mathcal{F}_C^{N_2}, \mathcal{F}_W^{N_2}} \cdot x) \mathcal{F}_C^{N_2} \right] \\
    &= \mathbb{E} \left[ W_t^{2, i} \sigma(W_t^1 \cdot x) \mathcal{F}_C^{N_2} \right].
\end{align*}
\]
The second to the last equality is due to the assumed independence of the initial conditions via Assumption 2.1 together with the fact that the marginal of $\gamma_0^{N_2}$ with respect to $C_0^1, \ldots, C_0^{N_2}$ is a product of delta Dirac distributions.

It remains to show that the solution to (A.6) is unique. We do this via a fixed point argument, completely analogous to Section 4 of [31]. The details are omitted due to the similarity of the argument.

A.4 Convergence Result for First Layer

Let $\pi^{N_1,N_2}$ be the probability measure corresponding to $\gamma^{N_1,N_2}$. Each $\pi^{N_1,N_2}$ takes values in the set of probability measures $M(D_E([0,T]))$. Relative compactness, proven in Section A.1 implies that every subsequence $\pi^{N_{1k},N_2}$ has a further sub-sequence $\pi^{N_{1m},N_2}$ which weakly converges. Section A.2 proves that any limit point $\pi^{N_2}$ of $\pi^{N_{1m},N_2}$ will satisfy the evolution equation (A.4). Section A.3 proves that the solution of the evolution equation (A.4) is unique. Therefore, by Prokhorov’s Theorem, $\pi^{N_1,N_2}$ weakly converges to $\pi^{N_2}$, where $\pi^{N_2}$ is the distribution of $\gamma^{N_2}$, the unique solution of (A.4). That is, $\gamma^{N_1,N_2}$ converges in distribution to $\gamma^{N_2}$.

Bibliography

[1] B. Alipanahi, A. Delong, M. Weirauch, and B. Frey. Predicting the sequence specificities of DNA-and RNA-binding proteins by deep learning. Nature Biotechnology, 33(8): 831, 2015.

[2] S. Arik, M. Chrzanowski, A. Coates, G. Diamos, A. Gibiansky, Y. Kang, X. Li, J. Miller, A. Ng, J. Raiman, S. Sengputa. Deep voice: Real-time neural text-to-speech. [arXiv:1702.07825], 2017.

[3] M. Bojarski, D. Del Test, D. Dworakowski, B. Firnier, B. Flepp, P. Goyal, L. Jackel, M. Monfort, U. Muller, J. Zhang, and X. Zhang. End to end learning for self-driving cars. [arXiv:1604.07316], 2016.

[4] D.A. Dawson. Hierarchical and mean-field stepping stone models, in Progress in Population Genetics and Human Evolution, eds. P. Donnelly and S. Tavare, Springer, IMA Volume in Mathematics and its Applications vol. 87. (1997).

[5] D.A. Dawson. Multilevel mutation-selection systems and set-valued duals, Journal of Mathematical Biology, Vol. 76, Issue 12, (2018), pp 295378.

[6] D.A. Dawson and K.J. Hochberg. Wandering random measures in the Fleming-Viot model, Annals of Probability 10 (1982), pp. 554-580.

[7] D.A. Dawson and K.J. Hochberg. A multilevel branching model, Advances in Applied Probability 23, (1991), pp. 701-715.

[8] D.A. Dawson, K.J. Hochberg and V. Vinogradov. High-density limits of hierarchically structured branching-diusing populations, Stochastic Processes and their Applications, 62, (1996), pp. 191-222.

[9] D.A. Dawson and J. Gärtner, Analytic aspects of multilevel large deviations, in Asymptotic Methods in Probability and Statistics (ed. B. Szyszkowicz), Elsevier, Amsterdam, (1998), pp. 401-440.

[10] D.A. Dawson and A. Greven. Hierarchically interacting Fleming-Viot processes with selection and mutation: multiple space-time scale analysis and quasi equilibria, Electronic J. of Prob., vol. 4, paper 4, (1993), pp. 1-81.

[11] D.A. Dawson and Y. Wu. Multilevel multitype models of an information system, I.M.A. Volume 84, Classical and Modern Branching Processes, eds. K.B. Athreya and P. Jagers, Springer-Verlag, (1996), pp. 57-72.

[12] S. Ethier and T. Kurtz. Markov Processes: Characterization and Convergence. 1986, Wiley, New York, MR0838085.

[13] I. Goodfellow, Y. Bengio, and A. Courville. Deep Learning. Cambridge: MIT Press, 2016.
[14] X. Glorot and Y. Bengio. Understanding the difficulty of training deep feedforward neural networks. In Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics, pp. 249-256, 2010.

[15] S. Gu, E. Holly, T. Lillicrap, and S. Levine. Deep reinforcement learning for robotic manipulation with asynchronous off-policy updates. IEEE Conference on Robotics and Automation, 3389-3396, 2017.

[16] V.N. Kolokoltsov. Nonlinear Markov processes and kinetic equations Vol. 182, Cambridge University Press, 2010.

[17] Y. LeCun, Y. Bengio, and G. Hinton. Deep Learning. Nature, 521(7553), 436, 2015.

[18] Y. Leviathan and Y. Matias. Google Duplex: An AI System for Accomplishing Real-World Tasks Over the Phone. Google, 2018.

[19] J. Ling, A. Kurzawski, and J. Templeton. Reynolds averaged turbulence modelling using deep neural networks with embedded invariance. Journal of Fluid Mechanics, 807, 155-166, 2016.

[20] J. Ling, R. Jones, and J. Templeton. Machine learning strategies for systems with invariance properties. Journal of Computational Physics, 318, 22-35, 2016.

[21] S. Mallat. Understanding deep convolutional neural networks. Philosophical Transactions of the Royal Society A. 374.2065, 20150203, 2016.

[22] D. Li, T. Ding, and R. Sun. Over-Parameterized Deep Neural Networks Have No Strict Local Minima For Any Continuous Activations. arXiv: 1812.11039, 2018.

[23] S. Liang, R. Sun, J. Lee, and R. Srikant. Adding One Neuron Can Eliminate All Bad Local Minima. NIPS, 2018.

[24] S. Liang, R. Sun, Y. Li, and R. Srikant. Understanding the Loss Surface of Neural Networks for Binary Classification. ICML, 2018.

[25] S. Du, J. Lee, H. Li, L. Wang, and X. Zhai. Gradient Descent Finds Global Minima of Deep Neural Networks. arXiv: 1811.03804, 2019.

[26] S. Du, X. Zhai, B. Poczos, and A. Singh. Gradient Descent Provably Optimizes Over-parameterized Neural Networks. arXiv: 1810.02054, 2019.

[27] S. Mei, A. Montanari, and P. Nguyen A mean field view of the landscape of two-layer neural networks 2018, arXiv:1804.06561.

[28] R. Nallapati, B. Zhou, C. Gulcehre, and B. Xiang. Abstractive text summarization using sequence-to-sequence RNNs and beyond. arXiv:1602.06023 2016.

[29] H. Pierson and M. Gashler. Deep learning in robotics: a review of recent research. Advanced Robotics, 31(16): 821-835, 2017.

[30] G. M. Rotskoff and E. Vanden-Eijnden Neural Networks as Interacting Particle Systems: Asymptotic Convexity of the Loss Landscape and Universal Scaling of the Approximation Error. 2018, arXiv:1805.00915.

[31] J. Sirignano and K. Spiliopoulos. Mean Field Analysis of Neural Networks 2018, arXiv:1805.01053.

[32] J. Sirignano and K. Spiliopoulos. Mean Field Analysis of Neural Networks: A central limit theorem 2018, arXiv:1808.09372.

[33] J. Sirignano, A. Sadhwani, and K. Giesecke. Deep Learning for Mortgage Risk. arXiv:1607.02470, 2016.

[34] J. Sirignano and R. Cont. Universal features of price formation in financial markets: perspectives from Deep Learning. arXiv:1803.06917 2018.
[35] J. Sirignano and K. Spiliopoulos, Stochastic gradient descent in continuous time, SIAM Journal on Financial Mathematics, Vol. 8, Issue 1, (2017), pp. 933-961.

[36] J. Sirignano and K. Spiliopoulos, DGM: A deep learning algorithm for solving partial differential equations, Journal of Computational Physics, (2018), Vol. 375, pp. 1339-1364.

[37] A-S. Sznitman. Topics in propagation of chaos. in Ecole d’Eté de Probabilités de Saint-Flour XIX - 1989. series, Lecture Notes in Mathematics, P.-L. Hennequin, Ed. Springer, Berlin Heidelberg. 1464, 165-251, 1991.

[38] A-S. Sznitman. Nonlinear reflecting diffusion processes, and the propagation of chaos and fluctuations associated. Journal of Functional Analysis. 56, 311-336, 1984.

[39] I. Sutskever, O. Vinyals, and Q. Le. Sequence to sequence learning with neural networks. In Advances in Neural Information Processing Systems, 3104-3112, 2014.

[40] N. Sunderhauf, O. Brock, W. Cheirer, R. Hadsell, D. Fox, J. Leitner, B. Upcroft, P. Abbeel, W. Burgard, M. Milford, and P. Corke. The limits and potentials of deep learning for robotics. The International Journal of Robotics Research, 37(4): 405-420, 2018.

[41] Y. Taigman, M. Yang, M. Ranzato, L. Wolf. Deepface: Closing the gap to human-level performance in face verification. In Proceedings of the IEEE conference on computer vision and pattern recognition, 1701-1708, 2014.

[42] Y. Zhang, W. Chan, and N. Jaitly. Very deep convolutional networks for end-to-end speech recognition. In IEEE International Conference on Acoustics, Speech, and Signal Processing. 4845-4849, 2017.

[43] C. Wang, J. Mattingly, and Y. Lu Scaling limit: Exact and tractable analysis of online learning algorithms with applications to regularized regression and PCA. 2017, arXiv:1712.04332.

[44] Y. Wu, M. Schuster, Z. Chen, Q. Le, M. Norouzi, W. Macherey, M. Krikun, Y. Cao, Q. Gao, K. Macherey, and J. Klingner. Google’s neural machine translation system: Bridging the gap between human and machine translation. arXiv:1609.08144, 2016.