On the Low-energy Effective Action of $N = 2$ Supersymmetric Yang-Mills Theory

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Abstract

We investigate the perturbative part of Seiberg’s low-energy effective action of $N = 2$ supersymmetric Yang-Mills theory in Wess-Zumino gauge in the conventional effective field theory technique. Using the method of constant field approximation and restricting the effective action with at most two derivatives and not more than four-fermion couplings, we show some features of the low-energy effective action given by Seiberg based on $U(1)_R$ anomaly and non-perturbative $\beta$-function arguments.

1. Introduction

One cannot but be impressed by the steady increase in our knowledge of the dynamics of supersymmetric gauge theories ever since their invention. The rate of progress has been very rapid in recent years, following the seminal contribution by Seiberg and Witten [1], combining the ideas of holomorphy [2] and duality [3].

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The web of arguments leading to the explicit results consists of a skilful combination of perturbative and nonperturbative arguments, formal considerations and physical reasoning. It should be checked by explicit computations, whenever possible, that no unexpected failure of these arguments occurs. This paper is a modest contribution in that direction.

The subject of our study is that well-studied object, namely the low-energy effective action of an \( N = 2 \) super-Yang-Mills theory with the gauge group \( SU(2) \):

\[
\Gamma = \frac{1}{16\pi} \ln \int d^4xd^2\theta d^2\bar{\theta} \left[ \frac{1}{2} \tau \Psi^2 + \frac{i}{2\pi} \Psi^2 \ln \frac{\Psi^2}{\Lambda^2} + \sum_{n=1}^{\infty} A_n \left( \frac{\Lambda^2}{\Psi^2} \right)^{2n} \right],
\]

(1)

where \( \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \) is the modular parameter and \( \Psi \) the \( N = 2 \) superfield describing the light-field degrees of freedom. The logarithmic term represents the one-loop perturbative result and was first obtained by Di Vecchia et al. \[4\] in a calculation where they coupled the gauge superfield to an \( N = 2 \) matter supermultiplet and integrated out the latter. Subsequently, Seiberg \[2\] used the anomalous transformation behaviour under \( U(1)_R \) and holomorphicity to argue that the full low-energy effective action should take the form (1), where the infinite series arises from nonperturbative contributions (instantons). The Seiberg-Witten solution \[1\] gives the explicit form of this part of \( \Gamma \).

The form (1) has been confirmed by calculations in \( N = 1 \) superspace and in harmonic superspace, extending the result to nonleading terms in the number of derivatives \[5–9\]. Independent confirmation has been obtained from \( M \)-theory \[10\].

In this paper we set out to check the perturbative part of the effective action by a very down-to-earth, conventional calculation. In the Higgs phase of the theory, the \( SU(2) \) gauge symmetry breaks down to \( U(1) \), and the super-Higgs mechanism splits the supermultiplet into a massive one and a massless one. The effective action of the massless fields should be obtained by integrating out the heavy fields. Thus our approach is very close in spirit to \[4\], although the actual computations are different.

Even this modest programme we cannot carry out fully. We report here the computation of heavy fermion determinant. Reassuringly, we find that the form (1) is reproduced. Although no unexpected surprises were unearthed by our calculation, we still hope that it
has some pedagogical value in showing explicitly how the effective action arises.

In section 2, we describe the model and exhibit the Higgs mechanism. Section 3 contains the computation of the heavy fermion determinant using the constant field approximation. The detailed calculations of the fermion eigenvalues and their degeneracies, which contain some subtle points, are given in Appendix B. In section 4 we present a discussion of the results. In the pedagogical vein of this paper, we give in Appendix A the component form of the low-energy effective action.

2. Super-Higgs Mechanism and Splitting of $N = 2$ Supermultiplet

The classical action of $N = 2$ supersymmetric Yang-Mills theory with gauge group $SU(2)$ reads as follows [11],

$$S = \int d^4x \left[ -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a} + D_\mu \varphi^t a D^\mu \varphi^a + i\bar{\psi}^a \gamma^\mu D^a_\mu \psi^b \\
+ \frac{ig}{\sqrt{2}} \epsilon^{abc} \bar{\psi}^c \left[ (1 - \gamma_5) \varphi^a + (1 + \gamma_5) \varphi^t a \right] \psi^b + \frac{g^2}{2} \epsilon^{abc} \epsilon^{ade} \varphi^b \varphi^c \varphi^d \varphi^t e \right], \quad (2)$$

where

$$G^a_{\mu\nu} = \partial_\mu K^a_\nu - \partial_\nu K^a_\mu - g\epsilon^{abc} K^b_\mu K^c_\nu, \quad D_\mu \varphi^a = \partial_\mu \varphi^a - g\epsilon^{abc} K^b_\mu \varphi^c, \quad \varphi^a = \frac{1}{\sqrt{2}}(S^a + iP^a), \quad \varphi^t a = \frac{1}{\sqrt{2}}(S^a - iP^a), \quad a = 1, 2, 3.$$

The bosonic part of the action (2) is similar to the Georgi-Glashow model in the Bogomol’nyi-Prasad-Sommerfield (BPS) limit. In addition to the fermionic term and Yukawa interaction term, this action has the scalar potential

$$V(\varphi) = -\frac{g^2}{2} \epsilon^{abc} \epsilon^{ade} \varphi^b \varphi^c \varphi^d \varphi^t e \equiv g^2 \text{Tr} \left( [\varphi, \varphi^t] \right)^2. \quad (3)$$

The unbroken supersymmetry requires that in the ground state the scalar potential must vanish, i.e.

$$[\varphi, \varphi^t] = 0. \quad (4)$$

(4) means that $\varphi^t$ and $\varphi$ should commute. Owing to the gauge freedom, one can choose \[\varphi^t\]
\[ \langle S^a \rangle = v \delta^{a3}, \quad \langle P^a \rangle = 0, \] (5)

where \( v \) is a real constant. For \( v \neq 0 \) the theory is in the Higgs phase and exhibits gauge symmetry breaking. In an unitary gauge we have

\[ S^T = (0, 0, S + v), \] (6)

and the classical Lagrangian can be written as follows,

\[ \mathcal{L} = \mathcal{L}_V + \mathcal{L}_S + \mathcal{L}_P + \mathcal{L}_F + \mathcal{L}_Y, \] (7)

where \( \mathcal{L}_V, \mathcal{L}_S, \mathcal{L}_P, \mathcal{L}_F \) and \( \mathcal{L}_Y \) denote respectively the vector field, the scalar field, the scalar potential, the fermionic and the Yukawa part,

\[ \mathcal{L}_V = -\frac{1}{4} (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{2} (\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)(\partial^\mu W^- - \partial^\nu W^-) \]

\[ - ig[(\partial^\mu W^+ W^- - \partial^\mu W^- W^+_\mu) A^\nu + (\partial_\nu W^- W^+ - \partial_\nu W^+ W^-) A^\mu] \]

\[ + (\partial^\mu A^\nu - \partial^\nu A^\mu) W^+ W^- + g^2 (-W^+ W^- A^\mu A^\nu + W^+ W^- A^\mu A^\nu) \]

\[ + \frac{g^2}{2} W^+ W^- W^- (W^+ W^- - W^- W^+); \] (8)

\[ \mathcal{L}_S = \frac{1}{2} \partial_\mu P \partial^\mu P + \frac{1}{2} \partial_\mu S \partial^\mu S + \partial_\mu P^+ \partial_\mu P^- + ig A^\mu (\partial_\mu P^- P^+ - \partial_\mu P^+ P^-) \]

\[ + ig P (\partial^\mu P^+ W^- - \partial^\mu P^- W^+) + ig \partial^\mu P (W^+ P^- - W^- P^+) + g^2 P^2 W^+ W^- \]

\[ + g^2 (S + v)^2 W^+ W^- + g^2 A^\nu A_\mu P^- P^- - g^2 (W^+ P^- - W^- P^+) A^\mu P \]

\[ - \frac{g^2}{2} (W^+ P^- - W^- P^+)^2. \] (9)

\[ \mathcal{L}_P = g^2 (S + v)^2 P^+ P^-, \] (10)

where

\[ W^+_\mu \equiv \frac{1}{\sqrt{2}} (K^1_\mu - i K^2_\mu), \quad W^-_\mu \equiv \frac{1}{\sqrt{2}} (K^1_\mu + i K^2_\mu), \quad K^3_\mu \equiv A_\mu; \]

\[ P^+ \equiv \frac{1}{\sqrt{2}} (P^1 - i P^2), \quad P^- \equiv \frac{1}{\sqrt{2}} (P^1 + i P^2), \quad P^3 \equiv P. \] (11)
The above Lagrangians clearly show that $W_{\mu}^\pm$ and $P^\pm$ become massive with mass $m \equiv |gv|$ while $A_\mu$, $S$ and $P$ remain massless.

Up to some total derivative term, the bosonic part of the Lagrangian can be written as following form,

$$\mathcal{L}_B = \mathcal{L}_V + \mathcal{L}_S + \mathcal{L}_P$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu P \partial^\mu P + \frac{1}{2}\partial_\mu S \partial^\mu S + \frac{1}{2}W^{\mu+}[\eta_{\mu\nu}D^{\alpha}D_\alpha - D^{\nu}_\mu D_\mu - igF_{\mu\nu}]W^{-\nu}$$

$$+ \frac{1}{2}W^{-\mu}[\eta_{\mu\nu}D^{\alpha}D^{\dagger}_\alpha - D^{\nu}_\mu D^{\dagger}_\mu + igF_{\mu\nu}] + g^2[P^2 + (S + v)^2]W^{\mu+}W^{-\nu}$$

$$+ \frac{1}{2}P^+(-\partial^\mu \partial_\mu + 2igA_\mu \partial^\mu + g^2A_\mu A^\mu)P^+ + \frac{1}{2}P^-(\partial^\mu \partial_\mu - 2igA_\mu \partial^\mu + g^2A_\mu A^\mu)P^+$$

$$+ \frac{1}{2}P^+(-2ig\partial^\mu P - ig\partial^\mu P - g^2A_\mu P)W^+$$

$$+ \frac{1}{2}P^{-}(2ig\partial^\mu P - ig\partial^\mu P - g^2A_\mu P)W^- + \frac{1}{2}W^-(\partial^\mu P + ig\partial^\mu P - g^2A_\mu P)P^+$$

$$+ \frac{g^2}{2}W^{\mu+}W^{-\nu}(W^+_{\mu}W^-_{\nu} - W^-_{\mu}W^+_{\nu}) - \frac{g^2}{2}(W^{+}_{\mu}P^- - W^-_{\mu}P^+)^2$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu \phi^* \partial^\mu \phi + \frac{1}{2}W^{\mu+}\Delta_{\mu\nu}W^{-\nu} + \frac{1}{2}W^{-\mu}\Delta^{\dagger}_{\mu\nu}W^{+\nu} + \frac{1}{2}P^+\Delta P^-$$

$$+ \frac{1}{2}P^+\Delta^{\dagger} P^+ + \frac{1}{2}W^{\mu+}\Delta_{\mu}P^- + \frac{1}{2}P^-\Delta^{\dagger}_{\mu}W^{+\mu} + \frac{1}{2}P^+\Delta^{\dagger}_{\mu}W^{-\mu} + \frac{1}{2}W^{-\mu}\Delta^{\dagger}_{\mu}P^+$$

$$+ \frac{g^2}{2}W^{\mu+}W^{-\nu}(W^+_{\mu}W^-_{\nu} - W^-_{\mu}W^+_{\nu}) - \frac{g^2}{2}(W^{+}_{\mu}P^- - W^-_{\mu}P^+)^2,$$ \hspace{1cm} (12)

where

$$\Delta_{\mu\nu} \equiv \eta_{\mu\nu}D^{\dagger}D_\alpha - D^{\dagger}_\nu D_\mu - igF_{\mu\nu} + g^2|\sqrt{2}\phi + v|^2 \eta_{\mu\nu},$$

$$\Delta^{\dagger}_{\mu\nu} \equiv \eta_{\mu\nu}D^{\dagger}D_\alpha - D^{\dagger}_\nu D_\mu + igF_{\mu\nu} + g^2|\sqrt{2}\phi + v|^2 \eta_{\mu\nu},$$

$$\Delta_{\mu} \equiv -ig\partial^\mu P - ig\partial^\mu P - g^2A_\mu P, \Delta^{\dagger}_{\mu} \equiv i\partial^\mu P + (2-i)g\partial^\mu P - g^2A_\mu P,$$

$$\Delta_{\mu} \equiv g\partial^\mu P + (2-i)g\partial^\mu P - g^2A_\mu P, \Delta^{\dagger}_{\mu} \equiv -2ig\partial^\mu P + (2-i)g\partial^\mu P - g^2A_\mu P,$$

$$\Delta = -\partial^\mu \partial_\mu + 2igA_\mu \partial^\mu + g^2A_\mu A^\mu, \Delta^{\dagger} = -\partial^\mu \partial_\mu - 2igA_\mu \partial^\mu + g^2A_\mu A^\mu,$$

$$D_\mu = \partial_\mu - igA_\mu, D^{\dagger}_\mu = \partial_\mu + igA_\mu, \phi \equiv \frac{1}{\sqrt{2}}(S + iP).$$ \hspace{1cm} (13)

To show that the spinor fields split massive and massless parts explicitly, we make some operations on $\mathcal{L}_F$ and $\mathcal{L}_Y$. The spinor part is
\[ \mathcal{L}_F = i\bar{\psi}^1 \gamma^\mu \partial_\mu \psi^1 + i\bar{\psi}^2 \gamma^\mu \partial_\mu \psi^2 + i\bar{\psi}^3 \gamma^\mu \partial_\mu \psi^3 \]
\[ + \frac{g}{\sqrt{2}} \bar{\psi}^1 (W^+_{\mu} - W^-_{\mu}) \gamma^\mu \psi^3 + ig \bar{\psi}^1 A_{\mu} \gamma^\mu \psi^2 \]
\[ + ig \bar{\psi}^1 A_{\mu} \gamma^\mu \psi^2 + \frac{ig}{\sqrt{2}} \bar{\psi}^2 \gamma^\mu (W^+_{\mu} + W^-_{\mu}) \psi^3 \]
\[ - \frac{ig}{\sqrt{2}} \bar{\psi}^3 \gamma^\mu (W^+_{\mu} + W^-_{\mu}) \psi^2 - \frac{g}{\sqrt{2}} \bar{\psi}^3 \gamma^\mu (W^+_{\mu} - W^-_{\mu}) \psi^1. \] (14)

As for the Yukawa part, we first write it in chiral spinors,
\[ \mathcal{L}_Y = i\sqrt{2}gf^{abc} \bar{\psi}^c_L \psi^a_R + i\sqrt{2}gf^{abc} \bar{\psi}^c_R \psi^a_L, \]
(15)
where \( \psi_L = \frac{1}{2}(1 + \gamma_5)\psi \) and \( \psi_R = \frac{1}{2}(1 - \gamma_5)\psi \). In the unitary gauge (13) becomes
\[ \mathcal{L}_Y = ig(\sqrt{2}\phi + v)(\bar{\psi}^2_L \psi^1_R - \bar{\psi}^1_L \psi^2_R) + ig(\sqrt{2}\phi^* + v)(\bar{\psi}^2_R \psi^1_L - \bar{\psi}^1_R \psi^2_L) \]
\[ + \frac{g}{\sqrt{2}}(P^+ + P^-) \left[ (\bar{\psi}^3_R \psi^2_L - \bar{\psi}^2_R \psi^3_L) - (\bar{\psi}^3_L \psi^2_R - \bar{\psi}^2_L \psi^3_R) \right] \]
\[ + \frac{ig}{\sqrt{2}}(P^+ - P^-) \left[ (\bar{\psi}^1_R \psi^3_L - \bar{\psi}^3_R \psi^1_L) - (\bar{\psi}^1_L \psi^3_R - \bar{\psi}^3_L \psi^1_R) \right]. \] (16)

Denoting
\[ \Psi_1 \equiv \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2), \quad \Psi_2 \equiv \frac{1}{\sqrt{2}}(\psi^1 - i\psi^2), \quad \Psi \equiv \psi^3, \]
(17)
we write \( \mathcal{L}_F \) and \( \mathcal{L}_Y \) in terms of these new fields,
\[ \mathcal{L}_Y = -g \bar{\Psi}_1 \left[ \frac{1}{\sqrt{2}}(1 - \gamma_5)\phi + \frac{1}{\sqrt{2}}(1 + \gamma_5)\phi^* + v \right] \Psi_1 \]
\[ + g \bar{\Psi}_2 \left[ \frac{1}{\sqrt{2}}(1 - \gamma_5)\phi + \frac{1}{\sqrt{2}}(1 + \gamma_5)\phi^* + v \right] \Psi_2 \]
\[ - igP^+ \bar{\Psi}_5 \Psi_1 + igP^- \bar{\Psi}_5 \Psi_2 - ig\bar{\Psi}_1 \gamma_5 P^+ \Psi - ig\bar{\Psi}_2 \gamma_5 P^- \Psi. \]
(18)
\[ \mathcal{L}_F = i\bar{\Psi}_1 \gamma^\mu \partial_\mu \Psi_1 + i\bar{\Psi}_2 \gamma^\mu \partial_\mu \Psi_2 + i\bar{\Psi} \gamma^\mu \partial_\mu \Psi \]
\[ + g \bar{\Psi}_1 \gamma^\mu A_{\mu} \Psi_1 - g \bar{\Psi}_2 \gamma^\mu A_{\mu} \Psi_2 \]
\[ + g \bar{\Psi}_2 \gamma^\mu W^+_{\mu} \Psi - g \bar{\Psi}_1 \gamma^\mu W^-_{\mu} \Psi \]
\[ - g \bar{\Psi} \gamma^\mu W^+_{\mu} \Psi_1 + g \bar{\Psi} \gamma^\mu W^-_{\mu} \Psi_2. \]
(19)
So now the whole classical action is given by the Lagrangian

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \phi^* \partial^\mu \phi + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi + \frac{1}{2} W^{+\mu} \Delta_{\mu\nu} W^{-\nu} + \frac{1}{2} W^{-\mu} \Delta_\mu^{\dagger} W^{+\nu}
\]

\[
+ \frac{1}{2} P^+ \Delta P^- + \frac{1}{2} P^- \Delta_\mu^{\dagger} P^+ + \frac{1}{2} P^{+\mu} \Delta_\mu P^- + \frac{1}{2} P^- \Delta_\mu W^{+\mu}
\]

\[
+ \frac{1}{2} P^+ \Delta_\mu W^{-\mu} + \frac{1}{2} W^{+\mu} \Delta^{\dagger}_\mu P^+ + W_1 \Delta_F \Psi_1 + \bar{W}_2 \Delta_F \Psi_2
\]

\[- ig P^+ \bar{\Psi}_1 \gamma_5 \Psi_1 + ig P^- \bar{\Psi}_2 \gamma_5 \Psi_2 - ig \bar{W}_1 \gamma_5 \Psi P^- + ig \bar{W}_2 \gamma_5 \Psi P^+
\]

\[
+ g \bar{\Psi}_2 \gamma^\mu W^+_\mu \Psi - g \bar{\Psi}_1 \gamma^\mu W^-_\mu \Psi - g \bar{\Psi} \gamma^\mu W^+_\mu \Psi_1 + g \Psi \gamma^\mu W^-_\mu \Psi_2
\]

\[
+ \frac{g^2}{2} W^{+\mu} W^{-\nu}(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu) - \frac{g^2}{2}(W^+_\mu P^- - W^-_\mu P^+)^2, \tag{20}
\]

where

\[
\Delta_F = i \gamma^\mu D_\mu - \frac{g}{\sqrt{2}} (1 - \gamma_5) \phi - \frac{g}{\sqrt{2}} (1 + \gamma_5) \phi^* - g v,
\]

\[
\tilde{\Delta}_F = i \gamma^\mu D^{\dagger}_\mu + \frac{g}{\sqrt{2}} (1 - \gamma_5) \phi + \frac{g}{\sqrt{2}} (1 + \gamma_5) \phi^* + g v. \tag{21}
\]

3. Low-energy Effective Action: Calculation of the Fermionic Determinant in Constant Field Approximation

The low-energy effective action is defined as follows,

\[
\exp \left\{ i \Gamma_{\text{eff}} [A_\mu, \phi, \Psi, \bar{\Psi}] \right\} \equiv \int D W^+_\mu D W^-_\mu D \bar{\Psi}_1 D \bar{\Psi}_2 D \Psi_1 D \Psi_2 D P^+ D P^- \exp \left[ i \int d^4 x \mathcal{L} \right]. \tag{22}
\]

At tree level

\[
\Gamma_{\text{eff}}^{(0)} = S_{\text{tree}} = \int d^4 x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \phi^* \partial^\mu \phi + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi \right]. \tag{23}
\]

At one-loop level, the integration over the heavy modes will lead to the determinant of the dynamical operators, in practice we cannot evaluate the determinant exactly. We shall employ a technique called constant field approximation to compute the determinant, which was first proposed by Schwinger \[12\] and later was used in in \[4\] and \[13\] to extract the anomaly term in $N = 2$ supersymmetric Yang-Mills theory and the one-loop effective action.
of supersymmetric $CP^{N-1}$ model. To apply this method we first write the classical action as following form,

\[ S = \int d^4 x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial^\mu \phi^* \partial_\mu \phi + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi \right] \]

\[ + \int d^4 x \frac{1}{2} (W^+, W^-) \]

\[ + \int d^4 x d^4 x \frac{1}{2} (\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_1, \bar{\Psi}_2) \]

\[ + \frac{1}{2} (\bar{\Psi}_1, \bar{\Psi}_2, \bar{\Psi}_1, \bar{\Psi}_2) \]

\[ + \frac{1}{2} (W^{+\mu}, W^{-\mu}, P^+, P^-) \]

\[ + \text{quartic terms of massive modes}. \quad (24) \]

Denoting

\[ \Phi = \begin{pmatrix} W^- \\ W^+ \\ P^- \\ P^+ \end{pmatrix}, \quad \bar{\Psi} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad (25) \]

we obtain
\[ S = S_{\text{tree}} + \int d^4x \left( \Phi^\dagger M_{bb} \Phi + \bar{\Psi} M_{fb} \bar{\Phi} + \Phi^\dagger M_{bf} \bar{\Psi} + \bar{\Psi} M_{ff} \bar{\Psi} \right) \]  

(26)

with that

\[ M_{bb} = \frac{1}{2} \begin{pmatrix} \Delta_{\mu\nu} & 0 & \Delta_{\mu} & 0 \\ 0 & \Delta_{\mu\nu}^\dagger & 0 & \Delta_{\mu}^\dagger \\ \tilde{\Delta}_{\mu}^\dagger & 0 & \Delta & 0 \\ 0 & \tilde{\Delta}_{\nu} & 0 & \Delta^\dagger \end{pmatrix}, \]

\[ M_{fb} = \frac{1}{2} \begin{pmatrix} -g\gamma_{\mu} \bar{\Psi} & 0 & -ig\gamma_5 \bar{\Psi} & 0 \\ 0 & g\gamma_{\mu} \bar{\Psi} & 0 & ig\gamma_5 \bar{\Psi} \\ -g\gamma_{\mu} \bar{\Psi} & 0 & -ig\gamma_5 \bar{\Psi} & 0 \\ 0 & g\gamma_{\mu} \bar{\Psi} & 0 & ig\gamma_5 \bar{\Psi} \end{pmatrix}, \]

\[ M_{bf} = \frac{1}{2} \begin{pmatrix} -g\bar{\Psi} \gamma_{\mu} & 0 & -g\bar{\Psi} \gamma_{\mu} & 0 \\ 0 & g\bar{\Psi} \gamma_{\mu} & 0 & g\bar{\Psi} \gamma_{\mu} \\ -ig\bar{\Psi} \gamma_5 & 0 & -ig\bar{\Psi} \gamma_5 & 0 \\ 0 & ig\bar{\Psi} \gamma_5 & 0 & ig\bar{\Psi} \gamma_5 \end{pmatrix}, \]

\[ M_{ff} = \frac{1}{2} \begin{pmatrix} \Delta_F & 0 & 0 & 0 \\ 0 & \tilde{\Delta}_F & 0 & 0 \\ 0 & 0 & \Delta_F & 0 \\ 0 & 0 & 0 & \tilde{\Delta}_F \end{pmatrix}. \]  

(27)

Using the standard formulas

\[ I = \int \mathcal{D}b^\dagger \mathcal{D}b \mathcal{D}\bar{\Phi} \mathcal{D}\bar{\Psi} \exp \left[ \int (dx) \left( b^\dagger M_{bb} b + \bar{\Phi} M_{fb} \bar{\Psi} + b^\dagger M_{bf} \bar{\Psi} + \bar{\Phi} M_{ff} \bar{\Psi} \right) \right] \]

\[ = \int \mathcal{D}b^\dagger \mathcal{D}b \mathcal{D}\bar{\Phi} \mathcal{D}\bar{\Psi} \exp \left\{ \int (dx) \left[ b^\dagger (M_{bb} - M_{bf} M_{ff}^{-1} M_{fb}) b \\
+ (\bar{\Phi}^\dagger b M_{bf} M_{ff}^{-1} M_{fb} + \bar{\Psi} ) \right] \right\} \]

\[ = \det M_{ff} \det^{-1} (M_{bb} - M_{bf} M_{ff}^{-1} M_{fb}), \]  

(28)

and

\[ \det M = \exp \text{Tr} \ln M, \]  

(29)
where \( b \) and \( f \) represent the general bosonic and fermionic fields, respectively, we obtain

\[
Z[A, \phi, \Psi, \bar{\Psi}] = \exp \left\{ i \Gamma_{\text{eff}} [A_{\mu}, \phi, \Psi, \bar{\Psi}] \right\} = \int D\Psi \bar{D}\bar{\Psi} D\Psi \bar{D}\bar{\Psi} \exp [iS]
\]

\[
= \exp [iS_{\text{tree}}] \det M_{ff} \det^{-1}(M_{bb} - M_{bf}M_{ff}^{-1}M_{fb})
\]

\[
= \exp \left[ iS_{\text{tree}} + \text{Tr} \ln M_{ff} - \text{Tr} \ln (M_{bb} - M_{bf}M_{ff}^{-1}M_{fb}) \right];
\]

\[
\Gamma_{\text{eff}} = S_{\text{tree}} - i \left[ \text{Tr} \ln M_{ff} - \text{Tr} \ln (M_{bb} - M_{bf}M_{ff}^{-1}M_{fb}) \right].
\]  

(30)

Now we evaluate above determinants. Let us first see the fermionic part, since \( M_{ff} \) in the form of reducible matrix,

\[
\det M_{ff} = \frac{1}{16}(\det \Delta_F)^2(\det \tilde{\Delta}_F)^2 = \frac{1}{16}\exp[2(\text{Tr} \ln \Delta_F + \text{Tr} \ln \det \tilde{\Delta}_F)].
\]  

(31)

We use the constant field approximation to find the eigenvalues and eigenvectors of the above operators and hence evaluate the determinant. As in [4], we choose only the third components of the electric and magnetic fields to be the constants different from zero,

\[-E_3 = F^{03} \neq 0, \quad B_3 = F^{12} \neq 0,\]

(32)

and \( \phi \) is the non-vanishing constant field. Correspondingly the potential will be

\[A^1 = -F^{12}x_2, \quad A^3 = -F^{30}x_0, \quad A^0 = A^2 = 0.\]

(33)

In order to looking for the eigenvalues of the operators, it is necessary to rotate to Euclidean space,

\[x^4 = x_4 = -ix^0, \quad \partial_0 = \frac{\partial}{\partial x^0} = i \frac{\partial}{\partial x^4},\]

\[f^{34} = f_{34} = iF^{30}, \quad f^{12} = f_{12} = F^{12}.\]

(34)

We first evaluate \( \det \Delta_F \), the eigenvalue equation for \( \Delta_F \) is

\[
\Delta_F \psi(x) = \left[ i\gamma^\mu D_\mu - \frac{g}{\sqrt{2}} (1 - \gamma_5) \phi - \frac{g}{\sqrt{2}} (1 + \gamma_5) \phi^* - gv \right] \psi(x) = \omega \psi_1,
\]

(35)

note that \( \psi \) is four-component spinor wave function. In order to get normalizable eigenstates, we consider the system in a box of finite size \( L \) in the \( x_1 \) and \( x_3 \) directions with periodic boundary conditions, so the eigenvector should be following form,
\[ \psi(x) = \frac{1}{L} e^{ip_1 \cdot x_1} e^{ip_3 \cdot x_3} \chi(x_2, x_4), \]
\[ p_1 = \frac{2\pi l}{L}, \quad p_3 = \frac{2\pi k}{L}, \quad k, l = \text{integers}. \]  

(36)

To find the eigenvalues and eigenvectors, we write the operators and the wave function in two-component forms,

\[ \Delta_F = \begin{pmatrix} -g(\sqrt{2}\phi^* + v)1 & \Delta^- \\ \Delta^+ & -g(\sqrt{2}\phi + v)1 \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \]

(37)

where \( 1 \) is the \( 2 \times 2 \) identity matrix and

\[ \Delta^\pm = \partial_4 \mp i \left[ \sigma_1(p_1 + ig f_{12} x_2) + \sigma_2\partial_2 + \sigma_3(p_3 + ig f_{34} x_4) \right]. \]

(38)

Correspondingly, the eigenvalue equation (35) is reduced to the following set of equations,

\[ -g(\sqrt{2}\phi^* + v)\chi_1 + \Delta^- \chi_2 = \omega \chi_1, \]
\[ \Delta^+ \chi_1 - g(\sqrt{2}\phi + v)\chi_2 = \omega \chi_2, \]

(39)

and now

\[ \Delta^\pm = \partial_4 \mp \left[ \sigma_1(p_1 + g f_{12} x_2) - i\sigma_2\partial_2 + \sigma_3(p_3 + g f_{34} x_4) \right]. \]

(40)

A detailed calculation and discussion on the eigenvalues are collected in Appendix B. We obtain two series of eigenvalues,

\[ \omega_{\pm}(m, n) = -g \left[ \frac{(\phi + \phi^*)}{\sqrt{2}} + v \right] \pm \sqrt{\frac{1}{2}g^2(\phi - \phi^*)^2 - 2mg f_{12} - 2ng f_{34}}, \]

(41)

where for \( m \geq 1, n \geq 1 \) both eigenvalues are doubly degenerate, while \( \omega_{\pm}(m, 0) \) and \( \omega_{\pm}(0, n) \) are simply degenerate, and for \( m = n = 0 \), there exists only the simply degenerate eigenvalue \( \omega_{-}(0, 0) \).

For the eigenvalue equation

\[ \tilde{\Delta}_F \tilde{\psi} = \left[ i\gamma^\mu D^\mu_\mu + \frac{g}{\sqrt{2}} (1 - \gamma_5)\phi + \frac{g}{\sqrt{2}} (1 + \gamma_5)\phi^* + gv \right] \tilde{\psi} = \tilde{\omega} \tilde{\psi}, \]

(42)

we obtain the eigenvalues in a similar way,
\[ \tilde{\omega}_\pm(m, n) = g \left[ \frac{(\phi + \phi^*)}{\sqrt{2}} + v \right] \pm \frac{g^2}{2} (\phi - \phi^*)^2 - 2mgf_{12} - 2ngf_{34}, \]  

(43)

where the degeneracies of \( \tilde{\omega}_\pm(m, n), \tilde{\omega}_\pm(m, 0) \) and \( \tilde{\omega}_\pm(0, n) \) with \( m \geq 1, n \geq 1 \) is the same as those of the \( \omega_\pm \)s. There still only exists the simply degenerate eigenvalue \( \omega_-(0, 0) \).

With above eigenvalues, we are able to evaluate \( \text{Tr} \ln \Delta_F \) and \( \text{Tr} \ln \tilde{\Delta}_F \), in general,

\[ \text{Tr} \ln \Delta_F = \ln \det \Delta_F = \ln \left[ \prod_{l, k} \omega_{\pm(lk)}(m, n) \right] = \sum_{l, k=-\infty}^{\infty} \sum_{m, n=0}^{\infty} r \ln \omega_{\pm(lk)}(m, n), \]

(44)

where \( r \) is the degeneracy of \( \omega_{\pm(m, n)} \). Due to the relation \( x_2 = 2\pi l/(gf_{12}L) \) and \( x_4 = 2\pi k/(gf_{34}L) \), the summation over the momentum \( k \) and \( l \) is actually equivalent to the integration over \( x_2 \) and \( x_4 \), since the fields are constants, this integration will yield only a Euclidean space volume factor, which tends to infinity in the continuous limit \( (L \to \infty) \),

\[ \sum_{l, k} = \frac{L^2}{4\pi^2} g^2 f_{12} f_{34} \int dx_2 dx_4 = \frac{V}{4\pi^2} g^2 f_{12} f_{34} \]  

(45)

Consider the degeneracy of each eigenvalue, we have

\[
\begin{align*}
\text{Tr} \ln \Delta_F &= \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \omega_-(0, 0) + \sum_{m=1}^{\infty} \ln \omega_+(m, 0) + \sum_{n=1}^{\infty} \ln \omega_+(0, n) \\
&\quad + 2 \sum_{m, n=1}^{\infty} \ln \omega_+(m, n) \right\} \\
&\quad + \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \omega_-(0, 0) + \sum_{m=1}^{\infty} \ln[\omega_+(m, 0)\omega_-(m, 0)] \\
&\quad + \sum_{n=1}^{\infty} \ln[\omega_+(0, n)\omega_-(0, n)] + 2 \sum_{m, n=1}^{\infty} \ln[\omega_+(m, n)\omega_-(m, n)] \right\} \\
&= \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln[-g(\sqrt{2}\phi + v)] + \sum_{m=1}^{\infty} \ln[g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12}] \\
&\quad + \sum_{n=1}^{\infty} \ln[g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2ngf_{34}] \\
&\quad + 2 \sum_{m, n=1}^{\infty} \ln[g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12} + 2ngf_{34}] \right\} \tag{46}
\end{align*}
\]

Similarly we have

\[
\begin{align*}
\text{Tr} \ln \tilde{\Delta}_F &= \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \left[ g(\sqrt{2}\phi^* + v) \right] + \sum_{m=1}^{\infty} \ln[g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12}] \\
&\quad + \sum_{n=1}^{\infty} \ln[g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2ngf_{34}] \\
&\quad + 2 \sum_{m, n=1}^{\infty} \ln[g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12} + 2ngf_{34}] \right\} \tag{47}
\end{align*}
\]

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Thus we finally obtain

\[
\text{Tr} \ln \Delta_F + \text{Tr} \ln \tilde{\Delta}_F = \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \left[ \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right] + 2 \sum_{m=1}^{\infty} \ln \left[ g^2 (\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12} \right] + 2 \sum_{n=1}^{\infty} \ln \left[ g^2 (\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2ngf_{34} \right] + 4 \sum_{m,n=1}^{\infty} \ln \left[ g^2 (\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v) + 2mgf_{12} + 2ngf_{34} \right] \right\}. \tag{47}
\]

Using the formula in the proper-time regularization,

\[
\ln \alpha = - \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-as}
\]

with \( \Lambda^2 \) is the cut-off to regularize the infinite sum, we have

\[
\text{Tr} \ln \Delta_F + \text{Tr} \ln \tilde{\Delta}_F = \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \left[ \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right] - 2 \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v)s} \right\}
\]

\[
= \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left\{ \ln \left[ \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right] - 2 \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v)s} \right\}
\]

\[
= \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left[ \ln \left( \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right) - 2 \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v)s} \right]
\]

\[
= \frac{V}{4\pi^2} g^2 f_{12} f_{34} \left[ \ln \left( \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right) - \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v)s} \right] \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v)s} \left\{ \ln \left[ \frac{\sqrt{2}\phi^* + v}{\sqrt{2}\phi + v} \right] - 2 \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-g^2(\sqrt{2}\phi^* + v)(\sqrt{2}\phi + v)s} \right\}
\]

where we have used

\[
\sum_{m=1}^{\infty} e^{-2mt} = \frac{e^{-t}}{2 \sinh t}, \quad \cosh(x+y) = \cosh x \cosh y \pm \sinh x \sinh y. \tag{51}
\]
Rotating back to Minkowski space, we get

$$\text{Tr} \ln \Delta F + \text{Tr} \ln \tilde{\Delta} F = \frac{V}{4\pi^2} g^2 i E_z H_z \left[ \ln \left( \frac{\sqrt{2} \phi^* + v}{\sqrt{2} \phi + v} \right) \right]$$

$$- \int_1^{\infty} \frac{ds}{s^2} e^{-g^2 (\sqrt{2} \phi^* + v)(\sqrt{2} \phi + v)s} c\text{osh}[g(H_z + i E_z)s] + c\text{osh}[g(H_z - i E_z)s] \right]$$

$$= \frac{V}{4\pi^2} g^2 i E_z H_z \left[ \ln \left( \frac{\sqrt{2} \phi + v}{\sqrt{2} \phi^* + v} \right) \right]$$

$$- \int_1^{\infty} \frac{ds}{s^2} e^{-g^2 (\sqrt{2} \phi^* + v)(\sqrt{2} \phi + v)s} c\text{osh}[gXs] + c\text{osh}(gX^*s) \right] c\text{osh}(gXs) - c\text{osh}(gX^*s) \right],$$

where

$$X \equiv H + i E.$$

To extract the divergence, we analyze the small-$s$ behaviour of the integrand of (52). According to the series expansion

$$\frac{c\text{osh}[gXs] + c\text{osh}(gX^*s)}{c\text{osh}[gXs] - c\text{osh}(gX^*s)} = \frac{1}{(X^2 - X^*)^2} \left[ \frac{4}{g^2 s^2} + \frac{2}{3} (X^2 + X^*2) + O(s^2) \right]$$

$$= \frac{1}{F_{\mu\nu} F^{\mu\nu}} \left[ \frac{1}{g^2 s^2} + \frac{2}{3} F_{\mu\nu} F^{\mu\nu} + O(s^2) \right],$$

where we have used

$$i E_z H_z = \frac{1}{4} (X^2 - X^*)^2 = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad H^2 - E^2 = \frac{1}{2} (X^2 + X^*2) = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}.$$

It can be easily seen from (54) that the integral in (52) has a quadratic and a logarithmic divergence, so the divergence term can be extracted by writing (52) as following form,

$$\text{Tr} \ln \Delta F + \text{Tr} \ln \tilde{\Delta} F = \frac{V}{4\pi^2} \left\{ \frac{1}{4} g^2 F_{\mu\nu} F^{\mu\nu} \ln \left[ \frac{\sqrt{2} \phi^* + v}{\sqrt{2} \phi + v} \right] \right\}$$

$$- \int_1^{\infty} \left( \frac{1}{s^3} + \frac{1}{6} s^2 g^2 F_{\mu\nu} F^{\mu\nu} \right) e^{-g^2 (\sqrt{2} \phi^* + v)(\sqrt{2} \phi + v)s}$$

$$- \int_0^{\infty} \frac{ds}{s^3} e^{-g^2 (\sqrt{2} \phi^* + v)(\sqrt{2} \phi + v)s} \left[ \frac{1}{4} g^2 s^2 F_{\mu\nu} F^{\mu\nu} \frac{c\text{osh}(gXs) + c\text{osh}(gX^*s)}{c\text{osh}(gXs) - c\text{osh}(gX^*s)} \right]$$

$$- \frac{1}{s^3} - \frac{1}{6} s^2 g^2 F_{\mu\nu} F^{\mu\nu} \right\},$$

where the second term is the UV divergent term, so the cut-off $1/\Lambda^2$ is preserved to regularize the integral, while the last term is a finite term and hence the cut-off has been removed.
Now we turn to the bosonic determinant. From (30) we have

\[
M_{ff}^{-1} = 2 \begin{pmatrix}
1/\Delta_F & 0 & 0 & 0 \\
0 & 1/\tilde{\Delta}_F & 0 & 0 \\
0 & 0 & 1/\Delta_F & 0 \\
0 & 0 & 0 & 1/\tilde{\Delta}_F
\end{pmatrix}, \quad M_{bb} - M_{bf} M_{ff}^{-1} M_{fb} =
\]

\[
\frac{1}{2} \begin{pmatrix}
\Delta_{\mu\nu} - 2g^2 \bar{\psi}_\mu \gamma_\mu \frac{1}{\Delta_F} \gamma_\nu \psi & 0 & \Delta_{\mu} - 2ig^2 \bar{\psi}_\mu \gamma_\mu \frac{1}{\Delta_F} \gamma_5 \psi & 0 \\
0 & \Delta_{\mu} - 2g^2 \bar{\psi}_\mu \gamma_\mu \frac{1}{\Delta_F} \gamma_\nu \psi & 0 & \Delta_{\mu}^\dagger - 2ig^2 \bar{\psi}_\mu \gamma_\mu \frac{1}{\Delta_F} \gamma_5 \psi \\
\tilde{\Delta}_{\nu} - 2ig^2 \bar{\psi}_5 \gamma_\mu \frac{1}{\tilde{\Delta}_F} \gamma_\nu \psi & 0 & \Delta + 2g^2 \bar{\psi}_5 \gamma_5 \frac{1}{\tilde{\Delta}_F} \gamma_5 \psi & 0 \\
0 & \tilde{\Delta}_{\nu} - 2ig^2 \bar{\psi}_5 \gamma_\mu \frac{1}{\tilde{\Delta}_F} \gamma_\nu \psi & 0 & \Delta_{\nu}^\dagger + 2g^2 \bar{\psi}_5 \gamma_5 \frac{1}{\tilde{\Delta}_F} \gamma_5 \psi
\end{pmatrix}
\]  

(57)

In constant field approximation, \( \bar{\psi} \) and \( \psi \) can be regarded Grassman numbers, so we can expand the bosonic determinant only to the quartic terms in \( \bar{\psi} \) and \( \psi \). Now the key problem is how to find the eigenvalues and eigenstates of the operator matrix \( M_{bb} - M_{bf} M_{ff}^{-1} M_{fb} \). If they could be worked out, with the eigenvalues and eigenvectors of fermionic operator, we can use the technique developed in [4] to evaluate this determinant. Unfortunately it is seems to us that in the constant field approximation there is no possibility to find the eigenvalues and eigenstates of such a horrible operator matrix. This difficulty needs to be overcome.

Despite the fact that the bosonic part cannot be evaluated, from (30) and (56), the effective Lagrangian associated with fermionic part has already shown the feature of the perturbative part of the low-energy effective action. First we believe that the quadratic divergence will be canceled owing to the nonrenormalization theorem. As for the logarithmic divergence, using

\[
\int_1^\infty e^{-g^2(\sqrt{2}\phi + v)(\sqrt{2}\phi + v)} s \sim - \ln \left[ \frac{g^2(\sqrt{2}\phi + v)(\sqrt{2}\phi + v)}{\Lambda^2} \right],
\]

(58)

we can see that in the Wilson effective action there is one term proportional to

\[
F_{\mu\nu} F^{\mu\nu} \ln \left[ \frac{g^2(\sqrt{2}\phi + v)(\sqrt{2}\phi + v)}{\Lambda^2} \right],
\]

(59)
thus the complete calculation should give the form (1) of the low-energy effective action, one can even guess this from the requirement of supersymmetry since the constant field approximation and the proper-time regularization preserve the supersymmetry explicitly.

There is a finite term proportional to $F\tilde{F}\ln\left(\frac{\sqrt{2}\phi + v}{\sqrt{2}\phi^* + v}\right)$ in (56), as that pointed out in [4], this is the reflect of the axial $U(1)_R$ anomaly in the effective action. This anomaly term had played an important role in Seiberg’s nonperturbative analysis [2].

4. Summary and Conclusion

In summary, we have tried to calculate the perturbative part of the Seiberg-Witten low-energy effective action of $N = 2$ supersymmetric Yang-Mills theory based on the standard effective field theory technique. It is well known that Seiberg-Witten effective action is the cornerstone for all these new developments in supersymmetric gauge theory, and that this effective action has been obtained in a hardly understandable way, so it is worthwhile to explore this effective action in a familiar method. Unfortunately we have confronted a insurmountable difficulty in evaluating bosonic operator in adopting constant field approximation, which prevents us from getting the complete result and comparing with the form of (1). However, the calculation of fermionic determinant has shown some features of the low-energy effective action. This gives a partial verification of the pure symmetry analysis for obtaining the low-energy effective action. The complete calculation presents an interesting problem for further investigation.

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APPENDIX A: THE LOW-ENERGY EFFECTIVE ACTION IN WESS-ZUMINO
GAUGE

To compare our result with that obtained from non-perturbative analysis, in this appendix we write the perturbative part of the Seiberg-Witten low-energy effective action \( (1) \) in the Wess-Zumino gauge. In \( N = 1 \) superfield, \( (1) \) can be written as following form

\[
\Gamma = \frac{1}{16\pi} \text{Im} \int d^4x \left[ \int d^2\theta F''(\Phi) W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta} F'(\Phi) \right],
\]

(A1)

where \( \Phi \) is the \( N = 1 \) chiral superfield

\[
\Phi = \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta^2 F(x),
\]

(A2)

and

\[
F(\Phi) = \frac{1}{2}\tau\Phi^2 + \frac{i}{2\pi}\Phi^2 \ln \frac{\Phi^2}{\Lambda^2}, \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}
\]

(A3)

and

\[
F'(\Phi) = \frac{dF}{d\Phi}, \quad F''(\Phi) = \frac{d^2F}{d\Phi^2}.
\]

(A4)

In Wess-Zumino gauge, the Abelian vector superfield is

\[
V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2(\bar{\theta}\lambda) - i\bar{\theta}^2(\theta\lambda) + \frac{1}{2}\theta^2\bar{\theta}^2D,
\]

(A5)

and the corresponding superfield strength is

\[
W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - i\sigma_\alpha^{\mu\nu}\theta^\nu\partial_\mu A_\nu + \theta^2\sigma_\alpha^{\mu\nu}\partial_\mu\bar{\lambda}_\beta(y),
\]

(A6)

where \( y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}, \quad \sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu) \) and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). Using

\[
F(\Phi) = \frac{1}{2} F''(\phi) \left[ i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta^2 F(x) \right]
\]

\[
+ \frac{1}{2} F''(\phi) \left[ i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta^2 F(x) \right]
\]

\[
\times \left[ i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta^2 F(x) \right]
\]

(A7)

and the similar expansion for \( F'(\Phi) \), we obtain
\[ \Gamma = \frac{1}{16\pi} \text{Im} \int d^4x \left[ -F''(\phi)\phi\partial^2\phi - F''(\phi)\phi\partial^\mu\phi\partial_\mu\phi + 2F''(\phi)\partial_\mu\phi\partial^\mu\phi \\
- \partial^2\phi F'(\phi) + 2iF''(\phi)\partial_\mu\psi\sigma^\mu\bar{\psi} - 2iF''(\phi)\psi\sigma^\mu\partial_\mu\bar{\psi} + 2iF''(\phi)\psi\sigma^\mu\bar{\psi}\partial_\mu\phi \\
- 2F^{(3)}(\phi)F^\dagger\psi\psi + 4F^\dagger F F''(\phi) + 4iF''(\phi)\lambda\sigma^\mu\partial_\mu\bar{\lambda} - 2F''(\phi)D^2 \\
+ 4F''(\phi)(\psi_{\mu\nu} + iF_{\mu\nu}\bar{F}^{\mu\nu}) - 2\sqrt{2}iF^{(3)}(\phi)\psi\lambda D + 2F^{(3)}(\phi)(\lambda\lambda)F \\
- F^{(4)}(\phi)(\psi\psi)(\lambda\lambda) \right]. \quad (A8) \]

From (A3) and (A4),
\[ F'(\phi) = \left( \tau + \frac{i}{\pi} \right) \phi + \frac{i}{\pi} \phi \ln \frac{\phi^2}{\Lambda^2}, \quad F''(\phi) = \tau + \frac{3i}{\pi} + \frac{i}{\pi} \ln \frac{\phi^2}{\Lambda^2}, \]
\[ F^{(3)}(\phi) = \frac{2i}{\pi} \frac{1}{\phi}, \quad F^{(4)}(\phi) = -\frac{2i}{\pi} \frac{1}{\phi^2}, \quad (A9) \]

and rescaling the field \( X \rightarrow gX \) with \( X = (A, \phi, \lambda, \psi) \), we can write (A8) as follows
\[ \Gamma = \int d^4x \left\{ \left[ -8\pi\phi\partial^2\phi + 8\pi\partial_\mu\phi\partial^\mu\phi + 8\pi i\partial_\mu\psi\sigma^\mu\bar{\psi} - 8\pi i\psi\bar{\psi}\mu\partial_\mu\bar{\psi} + 16\pi i\lambda\sigma^\mu\partial_\mu\bar{\lambda} \\
- 4\pi F_{\mu\nu}F^{\mu\nu} \right] + \frac{g^2}{\pi} \left[ -4\phi\partial^2\phi + 4\partial_\mu\phi\partial^\mu\phi + 6i\partial_\mu\psi\bar{\psi}\partial_\mu\bar{\psi} - 6\pi i\psi\bar{\psi}\mu\partial_\mu\bar{\psi} \right] \\
+ 4i\psi\bar{\psi}\partial_\mu\phi\phi + 12i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - 3F_{\mu\nu}F^{\mu\nu} \right] + \frac{g^2}{\pi} \ln \frac{\phi^2}{\Lambda^2} \left[ -2\phi\partial^2\phi + 2\partial_\mu\phi\partial^\mu\phi \right] \\
+ 2i\partial_\mu\psi\bar{\psi}\partial_\mu\bar{\psi} - 2i\psi\bar{\psi}\mu\partial_\mu\bar{\psi} + 4i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - F_{\mu\nu}F^{\mu\nu} \right] + \frac{g^2}{8\pi^2} \frac{1}{\phi^2} (\psi\psi)(\lambda\lambda) \\
- \frac{1}{2} \left( 1 + \frac{3g^2}{4\pi^2} + \frac{g^2}{8\pi^2} \ln \frac{\phi^2}{\Lambda^2} \right) - \frac{\sqrt{2}g^2 i(\psi\lambda)D}{4\pi^2} - \frac{g^2}{4\pi^2} \frac{F^\dagger(\psi\psi)}{\phi} \\
+ F^\dagger F \left( 1 + \frac{3g^2}{4\pi^2} + \frac{g^2}{4\pi^2} \ln \frac{\phi^2}{\lambda^2} \right) + \frac{g^2}{4\pi^2} \frac{(\lambda\lambda)F}{\phi}, \quad (A10) \]

where we set the vacuum angle \( \theta = 0 \). Using the equations of motion \( F, F^\dagger \) and \( D \) derived from (A10)
\[ F \left( 1 + \frac{3g^2}{4\pi^2} + \frac{g^2}{8\pi^2} \ln \frac{\phi^2}{\Lambda^2} \right) - \frac{g^2}{4\pi^2} \frac{\psi\psi}{\phi} = 0, \]
\[ F^\dagger \left( 1 + \frac{3g^2}{4\pi^2} + \frac{g^2}{8\pi^2} \ln \frac{\phi^2}{\Lambda^2} \right) + \frac{g^2}{4\pi^2} \frac{\lambda\lambda}{\phi} = 0, \]
\[ D \left( 1 + \frac{3g^2}{4\pi^2} + \frac{g^2}{8\pi^2} \ln \frac{\phi^2}{\Lambda^2} \right) + i\sqrt{2}g^2 \frac{\psi\lambda}{4\pi^2} = 0, \quad (A11) \]
and the algebraic manipulations

\[
\int d^4x \ln \frac{g^2 \phi^2}{\Lambda^2} \phi \partial^2 \phi = \int d^4x \left\{ \partial_\mu \left[ \ln \frac{g^2 \phi^2}{\Lambda^2} \phi \partial^\mu \phi \right] - \partial_\mu \ln \frac{g^2 \phi^2}{\Lambda^2} \phi \partial^\mu \phi - \ln \frac{g^2 \phi^2}{\Lambda^2} \partial_\mu \phi \partial^\mu \phi \right\}
\]

\[
= -\int d^4x \left[ 2 + \ln \frac{g^2 \phi^2}{\Lambda^2} \right] \partial_\mu \phi \partial^\mu \phi,
\]

\[
\int d^4x \psi \bar{\sigma}^\mu \bar{\psi} \partial_\mu \phi
= \frac{1}{2} \int d^4x \psi \bar{\sigma}^\mu \bar{\psi} \partial_\mu \ln \frac{g^2 \phi^2}{\Lambda^2}
\]

\[
= -\frac{1}{2} \int d^4x \ln \frac{g^2 \phi^2}{\Lambda^2} \left[ \partial_\mu \psi \bar{\sigma}^\mu \bar{\psi} + \psi \bar{\sigma}^\mu \partial_\mu \bar{\psi} \right],
\]

we obtain

\[
\Gamma = \int d^4x \left\{ \left[ \partial_\mu \phi \partial^\mu \phi + \lambda \sigma^\mu \partial_\mu \lambda + i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right]
\]

\[
+ \frac{3g^2}{4\pi^2} \left[ \partial_\mu \phi \partial^\mu \phi + \lambda \sigma^\mu \partial_\mu \lambda + i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right]
\]

\[
+ \frac{g^2}{4\pi^2} \ln \frac{g^2 \phi^2}{\Lambda^2} \left[ \partial_\mu \phi \partial^\mu \phi + i \bar{\lambda} \sigma^\mu \partial_\mu \lambda + i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right] + \frac{g^2}{8\pi^2} \frac{(\lambda \lambda)(\psi \psi)}{\phi^2} \right\}. \quad (A14)
\]

In four-component form

\[
\Psi = \begin{pmatrix} \psi \\ \bar{\lambda} \end{pmatrix}, \quad \bar{\Psi} = (\lambda, \bar{\psi}), \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},
\]

especially using the fact that for \( N = 2 \) Abelian multiplet, \( \Psi \) should be a Majorana spinor, \( \psi = \lambda \) and \( \bar{\psi} = \bar{\lambda} \) and \( (\lambda \lambda)(\psi \psi) = (\psi \psi)^2 = 1/(\bar{\Psi} \Psi)^2 \), thus (A14) can be written as following final form

\[
\Gamma = \int d^4x \left\{ \left[ 1 + \frac{3g^2}{4\pi^2} \right] \left[ \partial_\mu \phi \partial^\mu \phi + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right]
\]

\[
+ \frac{g^4}{32\pi^2} \frac{(\bar{\Psi} \Psi)^2}{g^2 \phi^2} \right\}. \quad (A15)
\]

(A15) is the perturbative part of the low-energy effective action in Wess-Zumino gauge given by Seiberg.
APPENDIX B: CALCULATION OF THE EIGENVALUES OF FERMIONIC OPERATOR

In this appendix we present a detailed calculation on the eigenvalues of fermionic operator $\Delta_F$. This can be regarded as an alternative method to the calculation in ref. [4].

First eq. (39) implies that

$$\Delta^+ \Delta^- \chi_1(x_2, x_4) = \left[ \omega + g(\sqrt{2}\phi^* + v) \right] \left[ \omega + g(\sqrt{2}\phi + v) \right] \chi_1(x_2, x_4),$$

$$\Delta^- \Delta^+ \chi_2(x_2, x_4) = \left[ \omega + g(\sqrt{2}\phi^* + v) \right] \left[ \omega + g(\sqrt{2}\phi + v) \right] \chi_2(x_2, x_4)$$

with

$$\Delta^+ \Delta^- = \partial_{x_2}^2 - g^2 f_{12}^2(x_2) + \frac{p_1}{g f_{12}} \partial_{x_2}^2 - g^2 f_{34}^2(x_4) + \frac{p_3}{g f_{34}} \partial_{x_4}^2 + g \sigma_3 (f_{12} + f_{34})$$

$$= -H_{12} - H_{34} + g \sigma_3 (f_{12} + f_{34}),$$

$$\Delta^- \Delta^+ = \partial_{x_2}^2 - g^2 f_{12}^2(x_2) + \frac{p_1}{g f_{12}} \partial_{x_2}^2 - g^2 f_{34}^2(x_4) + \frac{p_3}{g f_{34}} \partial_{x_4}^2 + g \sigma_3 (f_{12} - f_{34})$$

$$= -H_{12} - H_{34} + g \sigma_3 (f_{12} - f_{34}),$$

where $H_{12}$ and $H_{34}$ are the Hamiltonian operators of two independent harmonic oscillators,

$$H_{12} = -\partial_{x_2}^2 + g^2 f_{12}^2(x_2) + \frac{p_1}{g f_{12}} = -\partial_{x_2}^2 + \frac{\Omega_{12}^2}{2} \xi_2^2, \quad \xi_2 = x_2 + \frac{p_1}{g f_{12}}, \quad \Omega_{12} = \sqrt{|g f_{12}|};$$

$$H_{34} = -\partial_{x_4}^2 + g^2 f_{34}^2(x_4) + \frac{p_3}{g f_{34}} = -\partial_{x_4}^2 + \frac{\Omega_{34}^2}{2} \xi_4^2, \quad \xi_4 = x_4 + \frac{p_3}{g f_{34}}, \quad \Omega_{34} = |g f_{34}|.$$  \hspace{1cm} (B2)

Eq. (B1) means that the eigenvalue and the eigenvector of $\Delta_F$ must be that of $\Delta^+ \Delta^-$ and $\Delta^- \Delta^+$, while the reverse may be not true. We can make use of the eigenvalue and the eigenvector of $\Delta^+ \Delta^-$ and $\Delta^- \Delta^+$ to find the ones of $\Delta_F$. Like the usual method dealing with harmonic oscillator, defining the destruction and creation operators,

$$a_2 = \frac{1}{\sqrt{2}} \left( \sqrt{\Omega_{12}} \xi_2 + \frac{1}{\sqrt{\Omega_{12}}} \partial \right), \quad a_2^\dagger = \frac{1}{\sqrt{2}} \left( \sqrt{\Omega_{12}} \xi_2 - \frac{1}{\sqrt{\Omega_{12}}} \partial \right), \quad [a_2, a_2^\dagger] = 1,$$

$$a_4 = \frac{1}{\sqrt{2}} \left( \sqrt{\Omega_{34}} \xi_4 + \frac{1}{\sqrt{\Omega_{34}}} \partial \right), \quad a_4^\dagger = \frac{1}{\sqrt{2}} \left( \sqrt{\Omega_{34}} \xi_4 - \frac{1}{\sqrt{\Omega_{34}}} \partial \right), \quad [a_4, a_4^\dagger] = 1,$$

$$[a_2, a_4] = 0, \quad [a_2, a_4^\dagger] = 0.$$  \hspace{1cm} (B4)
we have the Hamiltonian operators and their eigenstates in Fock space,

\[ H_{12} = \Omega_{12} (2a_2 a_2^\dagger + 1), \quad |n_{12}\rangle = \frac{1}{\sqrt{n_{12}!}} (a_2^\dagger)^{n_{12}} |0_{12}\rangle, \]

\[ a_2 |0_{12}\rangle = 0, \quad H_{12} |n_{12}\rangle = \Omega_{12} (2n_{12} + 1) |n_{12}\rangle; \]

\[ H_{34} = \Omega_{34} (2a_2 a_4^\dagger + 1), \quad |n_{34}\rangle = \frac{1}{\sqrt{n_{34}!}} (a_2^\dagger)^{n_{34}} |0_{34}\rangle, \]

\[ a_2 |0_{34}\rangle = 0, \quad H_{34} |n_{34}\rangle = \Omega_{34} (2n_{34} + 1) |n_{34}\rangle. \]  

\[ (B5) \]

Correspondingly, the operators \( \Delta^+ \) and \( \Delta^- \) can be written in terms of the destruction and creation operators,

\[ \Delta^+ = i \sigma_2 \frac{\partial}{\partial \xi_2} - gf_{12} \xi_2 \sigma_1 + \frac{\partial}{\partial \xi_4} - gf_{34} \xi_4 \sigma_3 \]

\[ = \left( \sqrt{\frac{\Omega_{12}}{2}} (a_4 - a_4^\dagger) - \frac{g_{14}}{\sqrt{2\Omega_{12}}} (a_4 + a_4^\dagger) \sqrt{\frac{\Omega_{34}}{2}} (a_2 - a_2^\dagger) - \frac{g_{14}}{\sqrt{2\Omega_{12}}} (a_2 + a_2^\dagger) \right) \]

\[ \left( -\sqrt{\frac{\Omega_{12}}{2}} (a_2 - a_2^\dagger) - \frac{g_{14}}{\sqrt{2\Omega_{12}}} (a_2 + a_2^\dagger) \sqrt{\frac{\Omega_{34}}{2}} (a_4 - a_4^\dagger) - \frac{g_{14}}{\sqrt{2\Omega_{12}}} (a_4 + a_4^\dagger) \right); \]

\[ \Delta^- = -i \sigma_2 \frac{\partial}{\partial \xi_2} + gf_{12} \xi_2 \sigma_1 + \frac{\partial}{\partial \xi_4} + gf_{34} \xi_4 \sigma_3 \]

\[ = \left( \sqrt{\frac{\Omega_{12}}{2}} (a_4 - a_4^\dagger) + \frac{g_{14}}{\sqrt{2\Omega_{12}}} (a_4 + a_4^\dagger) - \sqrt{\frac{\Omega_{34}}{2}} (a_2 - a_2^\dagger) + \frac{g_{14}}{\sqrt{2\Omega_{12}}} (a_2 + a_2^\dagger) \right) \]

\[ \left( -\sqrt{\frac{\Omega_{12}}{2}} (a_2 - a_2^\dagger) + \frac{g_{14}}{\sqrt{2\Omega_{12}}} (a_2 + a_2^\dagger) \sqrt{\frac{\Omega_{34}}{2}} (a_4 - a_4^\dagger) - \frac{g_{14}}{\sqrt{2\Omega_{12}}} (a_4 + a_4^\dagger) \right). \]  

\[ (B6) \]

Four different cases should be considered, respectively,

1. \( gf_{12} > 0, \ gf_{34} > 0; \ \Omega_{12} = gf_{12}, \ \Omega_{34} = gf_{34}; \)

\[ \Delta^+ = \left( \frac{-\sqrt{2\Omega_{12}} a_4^\dagger}{\sqrt{2\Omega_{12}} a_2^\dagger}, \frac{\sqrt{2\Omega_{12}} a_4^\dagger}{\sqrt{2\Omega_{12}} a_2^\dagger} \right), \quad \Delta^- = \left( \frac{\sqrt{2\Omega_{34}} a_4}{\sqrt{2\Omega_{34}} a_2}, \frac{\sqrt{2\Omega_{34}} a_4}{\sqrt{2\Omega_{34}} a_2} \right); \]

\[ (B7) \]

2. \( gf_{12} > 0, \ gf_{34} < 0; \ \Omega_{12} = gf_{12}, \ \Omega_{34} = -gf_{34}; \)

\[ \Delta^+ = \left( \frac{\sqrt{2\Omega_{12}} a_4}{\sqrt{2\Omega_{12}} a_2}, \frac{-\sqrt{2\Omega_{12}} a_4}{\sqrt{2\Omega_{12}} a_2} \right), \quad \Delta^- = \left( \frac{-\sqrt{2\Omega_{34}} a_4^\dagger}{\sqrt{2\Omega_{34}} a_2^\dagger}, \frac{\sqrt{2\Omega_{34}} a_4^\dagger}{\sqrt{2\Omega_{34}} a_2^\dagger} \right); \]

\[ (B8) \]

3. \( gf_{12} < 0, \ gf_{34} > 0; \ \Omega_{12} = -gf_{12}, \ \Omega_{34} = gf_{34}; \)

\[ \Delta^+ = \left( \frac{-\sqrt{2\Omega_{34}} a_4^\dagger}{\sqrt{2\Omega_{34}} a_2^\dagger}, \frac{\sqrt{2\Omega_{12}} a_4^\dagger}{\sqrt{2\Omega_{12}} a_2^\dagger} \right), \quad \Delta^- = \left( \frac{\sqrt{2\Omega_{34}} a_4}{\sqrt{2\Omega_{34}} a_2}, \frac{-\sqrt{2\Omega_{12}} a_4}{\sqrt{2\Omega_{12}} a_2} \right); \]

\[ (B9) \]
Now we look for the eigenvalues of $\Delta F$ with aid of the ones of $\Delta^+\Delta^-$ and $\Delta^-\Delta^+$. Due to Eq.(B1) we have

$$\Delta^+ = \begin{pmatrix} \sqrt{2\Omega_{34}a_4} & \sqrt{2\Omega_{12}a_2} \\ \sqrt{2\Omega_{12}a_2} & -\sqrt{2\Omega_{34}a_4} \end{pmatrix}, \quad \Delta^- = \begin{pmatrix} -\sqrt{2\Omega_{34}a_4} & -\sqrt{2\Omega_{12}a_2} \\ -\sqrt{2\Omega_{12}a_2} & \sqrt{2\Omega_{34}a_4} \end{pmatrix}. \quad (B10)$$

The eigenstates $\chi_1$ and $|\chi_2\rangle$ should be the following form,

$$|\chi_i\rangle \sim \begin{pmatrix} |k, l\rangle \\ |m, n\rangle \end{pmatrix}, \quad i = 1, 2, \quad (B12)$$

where $|k, l\rangle \equiv |k| |l\rangle$, $|m, n\rangle \equiv |m| |n\rangle$, $k, m$ are the quantum numbers of the harmonic oscillator $H_{12}$ and $l, n$ are those of $H_{34}$. We first consider the case 1, since that

$$\Delta^- \Delta^+ \begin{pmatrix} |k, l\rangle \\ |m, n\rangle \end{pmatrix} = \begin{pmatrix} -2k\Omega_{12} - 2(l + 1)\Omega_{34} |k, l\rangle \\ -2(m + 1)\Omega_{12} - 2n\Omega_{34} |m, n\rangle \end{pmatrix},$$

$$\Delta^+ \Delta^- \begin{pmatrix} |k, l\rangle \\ |m, n\rangle \end{pmatrix} = \begin{pmatrix} -2k\Omega_{12} - 2l\Omega_{34} |k, l\rangle \\ -2(m + 1)\Omega_{12} - 2(n + 1)\Omega_{34} |m, n\rangle \end{pmatrix}, \quad (B13)$$

the common eigenstate of $\Delta^- \Delta^+$ and $\Delta^+ \Delta^-$ with eigenvalue $-2m\Omega_{12} - 2n\Omega_{34}$, $m, n \geq 1$ is

$$\begin{pmatrix} |\chi_1\rangle \\ |\chi_2\rangle \end{pmatrix} = \begin{pmatrix} \left( \begin{array}{c} \alpha |m, n - 1\rangle \\ \beta |m - 1, n\rangle \\ \gamma |m, n\rangle \\ \delta |m - 1, n - 1\rangle \end{array} \right) \end{pmatrix}, \quad (B14)$$

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are normalization parameters. With this eigenstate, we come to the eigenvalue equation in Fock space,
we can see that the eigenvalue

Using the fact

this is in fact changed into the ordinary matrix eigenvalue problem,

\[
\begin{pmatrix}
-g(\sqrt{2}\phi^* + v) & 0 & \sqrt{2mn}\Omega_{34} & \sqrt{2m}\Omega_{12} \\
0 & -g(\sqrt{2}\phi^* + v) & \sqrt{2m}\Omega_{12} & -\sqrt{2mn}\Omega_{34} \\
-\sqrt{2mn}\Omega_{34} & -\sqrt{2m}\Omega_{12} & -g(\sqrt{2}\phi + v) & 0 \\
-\sqrt{2m}\Omega_{12} & \sqrt{2m}\Omega_{34} & 0 & -g(\sqrt{2}\phi + v)
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix}
= \omega
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix}.
\] (B16)

Using the fact

\[
\det
\begin{pmatrix}
K & 0 & A & B \\
0 & K & B & -A \\
-A & -B & L & 0 \\
-B & A & 0 & L
\end{pmatrix} = (A^2 + B^2 + KL)^2,
\] (B17)

we can see that the eigenvalue \(\omega\) is determined by the following equation,

\[
[\omega + g(\sqrt{2}\phi^* + v)] [\omega + g(\sqrt{2}\phi + v)] + 2m\Omega_{12} + 2n\Omega_{34} = 0,
\]

\[
\omega_{\pm}(m, n) = -g \left[ \frac{\phi + \phi^*}{\sqrt{2}} + v \right] \pm \frac{1}{2} g^2 (\phi - \phi^*)^2 - 2m\Omega_{12} - 2n\Omega_{34}.
\] (B18)

Eqs. (B16) and (B17) explicitly show that \(\omega_{+}(m, n)\) and \(\omega_{-}(m, n)\) with \(m, n \geq 1\) are doubly degenerate, since for a 4x4 matrix there should exist four eigenvalues. Special attention should be paid to the cases when \(m = 0\) or \(n = 0\) as well as both of them equal to zero, when we will see that the degeneracies of the eigenvalue will change:
• $m \geq 1, n = 0$: in this case the eigenvalue equation (B13) will be reduced to the following form,

$$
\begin{pmatrix}
0 \\
\left[-g(\sqrt{2}\phi^* + v)\beta + \sqrt{2m\Omega_{12}}\gamma\right]|m-1,0\rangle \\
\left[-\sqrt{2m\Omega_{12}}\beta - g(\sqrt{2}\phi + v)\gamma\right]|m,0\rangle \\
0
\end{pmatrix}
= \omega
\begin{pmatrix}
0 \\
\beta|m-1,0\rangle \\
\gamma|m,0\rangle \\
0
\end{pmatrix},
$$

$$
\omega_{\pm}(m,0) = -g \left[\frac{\phi + \phi^*}{\sqrt{2}} + v\right] \pm \sqrt{\frac{1}{2}g^2(\phi - \phi^*)^2 - 2m\Omega_{12}}. \quad (B19)
$$

The eigenvalues $\omega_{\pm}(m,0)$ are obviously simply degenerate.

• $m = 0, n \geq 1$: in this case we have the eigenvalue equation as follows,

$$
\begin{pmatrix}
0 \\
\left[-g(\sqrt{2}\phi^* + v)\alpha + \sqrt{2n\Omega_{34}}\gamma\right]|0,n-1\rangle \\
\left[-\sqrt{2n\Omega_{34}}\alpha - g(\sqrt{2}\phi + v)\gamma\right]|0,n\rangle \\
0
\end{pmatrix}
= \omega
\begin{pmatrix}
\alpha|0,n-1\rangle \\
0 \\
\gamma|0,n\rangle \\
0
\end{pmatrix},
$$

$$
\omega_{\pm}(0,n) = -g \left[\frac{\phi + \phi^*}{\sqrt{2}} + v\right] \pm \sqrt{\frac{1}{2}g^2(\phi - \phi^*)^2 - 2n\Omega_{34}}. \quad (B20)
$$

The eigenvalues $\omega_{\pm}(0,n)$ are also simply degenerate.

• $m = n = 0$: the eigenvalue equation becomes very simple,

$$
\begin{pmatrix}
0 \\
0 \\
-g(\sqrt{2}\phi + v)|0,0\rangle \\
0
\end{pmatrix}
= \omega
\begin{pmatrix}
0 \\
0 \\
|0,0\rangle \\
0
\end{pmatrix},
$$

$$
\omega(0,0) = -g(\sqrt{2}\phi + v) = \omega_{-}(0,0). \quad (B21)
$$

Thus there exists only one $\omega_{-}(0,0)$ and it is simply degenerate.
For the case 2, the common eigenstate of $\Delta^+\Delta^-$ and $\Delta^+\Delta^-$ with eigenvalue $-2m\Omega_{12} - 2n\Omega_{34}$ is

\[
\begin{pmatrix}
|\chi_1\rangle \\
|\chi_2\rangle
\end{pmatrix} = \begin{pmatrix}
\alpha|m, n\rangle \\
\beta|m - 1, n - 1\rangle \\
\gamma|m, n - 1\rangle \\
\delta|m - 1, n\rangle
\end{pmatrix}.
\]

(B22)

In a similar way, one can see that the eigenvalues $\omega_{\pm}(m, n)$, $\omega_{\pm}(m, 0)$, $\omega_{\pm}(0, n)$ with $m, n \geq 1$ and their degeneracies are the same as the case 1; only $\omega(0, 0)$ is different,

\[
\omega(0, 0) = -g(\sqrt{2}\phi^* + v) = \omega_{\pm}(0, 0).
\]

(B23)

As for the cases 3 and 4, the common eigenstates of $\Delta^+\Delta^-$ and $\Delta^+\Delta^-$ with eigenvalue $-2m\Omega_{12} - 2n\Omega_{34}$ are, respectively,

3. \[
\begin{pmatrix}
|\chi_1\rangle \\
|\chi_2\rangle
\end{pmatrix} = \begin{pmatrix}
\alpha|m - 1, n - 1\rangle \\
\beta|m, n\rangle \\
\gamma|m - 1, n\rangle \\
\delta|m - 1, n - 1\rangle
\end{pmatrix}.
\]

(B24)

The eigenvalues $\omega_{\pm}(m, n)$, $\omega_{\pm}(m, 0)$, $\omega_{\pm}(0, n)$ with $m, n \geq 1$ and their degeneracies are the same as the cases 1, 2, but $\omega(0, 0)$'s are, respectively,

3. $\omega(0, 0) = -g(\sqrt{2}\phi^* + v) = \omega_{\pm}(0, 0)$;

4. $\omega(0, 0) = -g(\sqrt{2}\phi + v) = \omega_{-}(0, 0)$.

(B25)

The eigenvalues of $\tilde{\Delta}_F$ can be determined in a similar way; the only difference is $g \rightarrow -g$. 
It should be emphasized that these four cases are not equivalent, since the eigenstates are different with each other. However, they give the identical $\det \Delta_F \det \tilde{\Delta}_F$. 
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