On Supermembrane in $D=4$, multiple M0–brane in $D=11$ and Supersymmetric Higher Spin Theories

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Submitted by Carlos Meliveo García for the degree of Doctor of Physics
A mi familia.
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El modelo estándar de la física de partículas elementales ha demostrado tener la habilidad de describir una multitud de fenómenos que tienen lugar en la naturaleza, siendo ampliamente respaldado por experimentos realizados en aceleradores de partículas que confirman sus predicciones. El ejemplo más reciente es el anuncio del descubrimiento del bosón de Higgs por parte de las colaboraciones ATLAS [1] y CMS [2].

El modelo estándar está basado en la simetría gauge $SU(3) \times SU(2) \times U(1)$ [3,4] donde $SU(2) \times U(1)$ es la simetría de la teoría de campos gauge electrodébiles $W^{\pm}_{\mu}, Z_{\mu}, A_{\mu}$ que provee la descripción unificada de la interacción electromagnética y la débil. Fue propuesta por Sheldon L. Glashow [5], Steven Weinberg [6] y Abdus Salam [7].

La simetría de gauge $SU(2) \times U(1)$ está espontáneamente rota hasta un subgroupo $U(1)$ "diagonal" de $SU(2) \times U(1)$. Como resultado, los bosones $W^{\pm}_{\mu}, Z_{\mu},$ que corresponden a los generadores de las simetrías rotas, obtienen una masa gracias al mecanismo de Englert–Brout–Guralnik–Hagen–Kibble–Higgs [8–11]. En cambio, el campo electromagnético $A_{\mu}$ se corresponde al generador de la simetría preservada y, entonces, describe una partícula sin masa, el fotón.

La unificación de las interacciones electromagnéticas y débiles no es la primera en la historia de la física. Recordemos que en el siglo XIX James C. Maxwell demostró que fenómenos eléctricos y magnéticos, que anteriormente eran considerados como no relacionados, podían ser descritos por un conjunto único de ecuaciones que a día de hoy llevan su nombre. Estas ecuaciones describen la teoría del electromagnetismo y realizan la unificación de los fenómenos eléctricos y magnéticos [12].

La idea de unificación puede ser interpretada como un camino hacia un entendimiento más profundo de la naturaleza, incluso como un principio filosófico de unidad de la naturaleza: la naturaleza es única y el entendimiento más profundo puede –y tiene que– revelar la explicación común de fenómenos aparentemente diferentes.

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1 Galardonados con el Premio Nobel en 1979.
Además de teoría de interacciones electrodébiles, el modelo estándar contiene la teoría de campos gauge \( A_\mu^r, r = 1, ..., 8 \) del grupo \( SU(3) \), que se conoce como Cromodinámica Cuántica (QCD por sus siglas en inglés). Los cuantos de este campo, que describe las interacciones fuertes, se llaman “gluones”.

En el modelo estándar la QCD está aparte de la interacción electrodébil, incluso la constante de acoplamiento de los gluones es uno de los parámetros independientes. Eso se refleja en la aparición del grupo \( SU(3) \) en la simetría gauge del modelo estándar en producto directo con \( SU(2) \times U(1) \) y, en la falta de un mecanismo de mezcla con este último. (Por otro lado, en la teoría electrodébil la mezcla de \( SU(2) \) con \( U(1) \) aparece como el resultado de una rotura espontánea de simetría con preservación del subgroupo \( U(1) \) diagonal de \( SU(2) \times U(1) \).)

Ya en los años 70 se empezó la búsqueda de la teoría que unificara más estas interacciones tomando como base teorías de campos gauge de grupos semisimples que contengan los grupos \( SU(3), SU(2) \) y \( U(1) \) como subgrupos. Entre estas teorías, que se conocen como Teorías de Gran Unificación (GUT por sus siglas en inglés), las más investigadas son las que tienen grupos de simetría gauge \( SU(5), SO(10) \) y \( E_6 \).

Todas las GUT tienen en común ciertas propiedades, entre ellas:

- La existencia de una única constante de acoplamiento, en vez de las tres del modelo estándar: la separación de efectos de interacciones a bajas energías provee de los mecanismos dinámicos.

- En todas las teorías GUT el protón es inestable [13]: se pueden producir decaimientos del tipo \( p \rightarrow \pi^0 e^+ \) en los cuales se viola la conservación del número de bariones. Esto permite estudiar la viabilidad de estas teorías contrastando sus predicciones del tiempo de vida medio del protón con el límite encontrado en el experimento super-Kamiokande

La realización posible del programa de unificación en el marco de las teorías GUT es aparentemente incompleta: no pretenden unificar las interacciones fuertes y electrodébiles con la gravedad, que se puede describir en términos del campo de la métrica del espaciotiempo \( g_{\mu\nu} \). La base natural para buscar la unificación de todas las interacciones fundamentales \( W^\pm_\mu, Z_\mu, A_\mu^r, g_{\mu\nu} \) está relacionada con la supersimetría [14–17] y la teoría de cuerdas (hoy en día también conocida como teoría M). Además estas podrían dar cuenta de otros problemas. Por ejemplo, algunos candidatos a partícula de materia oscura son partículas que surgen de manera natural en teorías supersimétricas. Mencionamos el neutralino, la partícula hipotética, eléctricamente neutra, que sólo interacciona a través de la interacción gravitatoria y la débil y que en algunos modelos aparece como mezcla (combinación lineal) de supercompañeras (“superpartners”) de partículas del modelo estándar, como el ”Higgsino” (del Higgs), los ”Winos” (de \( W^\pm_\mu \)) y ”Zinos” (de \( Z_\mu \)).

La gravedad clásica se describe mediante la Teoría de la Relatividad General. El grupo gauge se puede identificar como difeomorfismos, es decir, transformaciones generales de
coordenadas o en el formalismo de tetradas ("moving frame") con el grupo de Lorentz $SO(1,3)$. El campo gauge (de espín 2) es la métrica $g_{\mu\nu}$, o las tetradas ("vielbeins") $e^a_\mu$, que componen la métrica $g_{\mu\nu} = e^a_\mu \eta^{ab} e^b_\nu$. La teoría también contiene la conexión de Christoffel $\Gamma^\rho_\mu_{\nu}(g)$ o/y un campo gauge para la simetría de Lorentz, la conexión de spin $\omega^{ab}_\mu(e)$, que son campos compuestos construidos de la métrica o de las tetradas, respectivamente.

Resumiendo, la Relatividad General es una teoría gauge para las simetrías del espacio-tiempo [18].

Como se ha mencionado anteriormente, el modelo estándar también es una teoría de campos gauge, pero con grupo $SU(3) \times SU(2) \times U(1)$ de simetrías internas. Es decir, es una teoría de campos gauge del tipo de Yang-Mills [19].

El teorema de Coleman y Mandula [20] establece que el álgebra de Lie más general de las simetrías de la matriz $S$ de una teoría cuántica de campos contiene el operador energía–momento $P_m$, los generadores de rotaciones de Lorentz $M_{mn}$ y un número finito de operadores $B_j$ que son escalares del grupo de Lorentz. Además estos últimos deben pertenecer al álgebra de Lie de un grupo compacto de Lie.

Se puede tratar este como un teorema no–go que prohíbe unificar las simetrías del espacio-tiempo con las simetrías internas.

La supersimetría evita las restricciones del teorema de Coleman y Mandula [21] generalizando la noción de álgebra de Lie. Sus generadores $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ son fermiónicos y satisfacen unas relaciones de anticonmutación, y no conmutación como es el caso de álgebras de Lie habituales. Estas nuevas "álgebras" que, involucran tanto conmutadores como anticonmutadores, se pueden considerar como construidas con generadores bosónicos y fermiónicos, se llaman superalgebras o álgebras de Lie graduadas.

Todas las representaciones lineales de la supersimetría contienen el mismo número de estados bosónicos y fermiónicos. Esta es la razón de que la búsqueda de la supersimetría en la naturaleza se relacione con la búsqueda de las supercompañeras de las partículas elementales conocidas. En caso de supersimetría preservada tienen las mismas características que sus análogas "habituales" pero la estadística opuesta: las supercompañeras de los bosones son fermiones y viceversa. Los datos experimentales sugieren claramente que en la naturaleza la supersimetría tiene que ser una simetría rota, presumiblemente de manera espontánea. La diferencia de masa entre supercompañeras tiene entonces que estar relacionada con la escala de energía a la que la supersimetría se rompe.

Las teorías supersimétricas tienen unas propiedades muy atractivas. A nivel cuántico algunos modelos supersimétricos exhiben un mejor comportamiento ultravioleta. Por ejemplo, ya en el modelo propuesto por Wess y Zumino en 1974 [17] debido a las cancelaciones entre las contribuciones fermiónicas y bosónicas no aparecen divergencias a un lazo ("loop"). Pero sin duda el mejor ejemplo es la teoría extendida de super–Yang–Mills (SYM) con supersimetría extendida $N = 4$, que nos provee con el único ejemplo conocido de teoría de campos 4–dimensionales que es finita a todos los órdenes de la teoría de perturbaciones [22].
Como se había notado ya en uno de los primeros artículos de supersimetría\cite{15,16}, invariancia bajo supersimetría local ("gauge") implica invariancia bajo difeomorfismos, y entonces, es una teoría de gravedad. En efecto, el álgebra de supersimetría sencilla
\[
\{Q_\alpha, Q_\beta\} = 0, \quad \{ar{Q}_\dot{\alpha}, \bar{Q}_\dot{\beta}\} = 0, \\
\{Q_\alpha, \bar{Q}_\dot{\alpha}\} \sim \sigma_{\alpha\dot{\alpha}} T_a P_a,
\]
implica que dos transformaciones de supersimetría producen una traslación. Entonces, el álgebra de supersimetría local se cierra en la traslación local, es decir en transformaciones de coordenadas locales (invertibles) del tipo más general, que se llaman difeomorfismos.

En los años setenta se creía que la gravedad supersimétrica (supergravedad)\cite{23-28} podía resolver la tensión aparente que existía entre la teoría de la Relatividad General de Einstein y la mecánica cuántica. La esperanza original fue que la supersimetría extendida máxima, $\mathcal{N} = 8$, podría prohibir los contratérminos y hacer que la supergravedad $\mathcal{N} = 8$\cite{29} no tuviera divergencias\cite{30}. Investigaciones más detalladas mostraron la existencia de posibles contratérminos en lazos más altos\cite{31-35}. Curiosamente, una tímida esperanza en la cancelación de todas las divergencias en supergravedad $\mathcal{N} = 8$ ha reaparecido recientemente\cite{36-43}.

Aún así, la mayoría de científicos creen esperan más que la supergravedad cuántica necesita una completitud ultravioleta, es decir que unos grados de libertad adicionales tienen que manifiestarse en procesos de alta energía para corregir propiedades de las amplitudes. La teoría de cuerdas supersimétrica se considera actualmente como dicha completitud ultravioleta, siendo su límite de bajas energías la supergravedad. Lo que hoy en día conocemos como teoría de supercuerdas o teoría M es, de un lado, el resultado de incluir supersimetría en la teoría de cuerdas bosónicas, incorporando así fermiones y mejorando su espectro de estados cuánticos al eliminar el estado taquiónico característico de la cuerda bosónica.

De otro lado, la teoría M apareció como unificación de los diferentes modelos de supercuerda. En los años ochenta del siglo pasado, tras la primera revolución de supercuerdas, se habían desarrollado cinco modelos consistentes de cuerdas supersimétricas:

- Supercuerda tipo I
- Supercuerda tipo IIA
- Supercuerda tipo IIB
- Supercuerda heterótica $SO(32)$
- Supercuerda heterótica $E_8 \otimes E_8$.

Todas estas teorías no contienen ningún estado taquiónico (a diferencia del modelo de cuerdas bosónicas) y además son consistentes en 10 dimensiones, es decir, tienen dimensión
critica $D = 10$ en vez de las 26 del modelo bosónico. El hecho de que fueran cinco las teorías consistentes se podía considerar como un problema si lo que se pretende es encontrar una teoría única.

Investigaciones más extensas resultaron en el descubrimiento de las dualidades que conectan los diferentes modelos de cuerdas, y las supergravedades que aparecen en sus límites de bajas energías, entre ellos y con supergravedad once-dimensional [44].

Así por ejemplo, las teorías de tipo IIA y IIB, tras realizar una compactificación de una de las dimensiones del espaciotiempo en un círculo, son equivalentes. Las transformaciones que las identifican se conocen como dualidad T. Curiosamente, si el radio de la dimensión compacta de la teoría IIA es $R$, el radio del círculo en la teoría equivalente del tipo IIB es $\alpha'/R$, donde $\alpha'$ es la pendiente de las trayectorias de Regge $J = \hbar \alpha' M^2 + \alpha$ en las que se sitúan los estados cuánticos de la cuerda con espín $J$ y masa $M$.

La otra dualidad, conocida como dualidad S, es una generalización de la dualidad que existe en la electrodinámica clásica entre el campo eléctrico y el magnético. A diferencia de la dualidad T, que se ve en teoría de perturbaciones de la cuerda, la dualidad S es no perturbativa. Se puede observar su manifestación como una simetría $SL(2, \mathbb{R})$ de la supergravedad del tipo IIB. En la teoría cuántica de la cuerda IIB la simetría $SL(2, \mathbb{Z})$ está rota hasta su subgrupo $SL(2, \mathbb{Z})$ debido a la existencia de otros objetos supersimétricos extendidos, super–p–branas de diferentes tipos, y por la quantización de sus tensiones $T_p$ [46] como consecuencia de una generalización de la ”condición de cuantización de Dirac” [47]. Este último, en su forma clásica, resulta en la cuantización de la carga eléctrica en el caso de la presencia de al menos un monopolo magnético [45]. La unidad de todas las dualidades de la teoría de cuerdas fue apreciada en [44] y resulta en la noción de dualidad $U$.

Estos y otros descubrimientos [49–52] sugirieron la conjetura de la teoría M, una teoría hipotética que reúne los cinco modelos de cuerdas y la supergravedad once–dimensional y tiene las dualidades que relacionan los modelos de cuerdas como sus simetrías. Las cinco teorías de supercuerdas aparecen como distintos límites perturbativos de esta teoría subyacente (”underlying”) que ”vive” en 11 dimensiones [53, 54]; y en otro límite de bajas energías se reduce a la supergravedad en 11 dimensiones formulada por E. Cremmer, B. Julia y J. Scherk en 1978 [55].

Tenemos que mencionar que son once las dimensiones ”elegidas” por la teoría M ya que once dimensiones es el número máximo que puede tener una teoría supersimétrica sin que en su reducción a 4 dimensiones aparezcan partículas con espín mayor que 2.

En este aspecto cabe subrayar que la cuerda sí que contiene campos de espín altos en su espectro de estados cuánticos y que, en relación con este hecho, existen especulaciones sobre que las teorías de cuerdas pueden ser una fase rota de un modelo más simétrico de campos conformes de espines altos sin masa [56].

En el estudio del régimen no perturbativo de la teoría de cuerdas\teoría M la noción de super–p–brana, objeto supersimétrico extendido con dimensión de volumen–mundo $d = p + 1$,
es fundamental. Son estados BPS (Bogomol’nyi-Prasad-Sommerfield), es decir son estables porque saturan un cota BPS ("BPS bound") que, a su vez, significa que su energía (o densidad de energía) tiene un valor minimal entre los estados con valores fijos de algunas cargas. La cota BPS, es decir, la restricción por abajo del valor de la energía por el valor de una carga topológica, suele aparecer en la teoría de cuerdas como resultado de preservar (una parte de) la supersimetría. En teoría de cuerdas, teoría M los estados de p–branas BPS pueden ser descritos por soluciones supersimétricas de las ecuaciones de supergravedad en 10 y 11 dimensiones. La estabilidad mencionada anteriormente, hace que sean similares a soluciones solitónicas de las famosas ecuaciones no lineales de KdV (Korteweg–de Vries) y Sine–Gordon [57].

La relación de la estabilidad de los estados BPS con la topología y con la preservación de supersimetría sugieren que las super–p–branas descritas por soluciones supersimétricas de las ecuaciones de supergravedad clásica [48, 58] son objetos de la teoría completa, es decir de la hipotética teoría M cuántica.

Las soluciones supersimétricas de la teoría de supergravedad en $D$ dimensiones [48, 58] describen los estados fundamentales de las super–p–branas. La dinámica de las excitaciones sobre estos estados se describe mediante acciones efectivas de super–p–branas [59, 69], similares a la acción de Green y Schwarz para la supercuerda [70]. Los límites bosónicos de estas acciones se pueden considerar como fuentes de la teoría de la Relatividad General y de los límites bosónicos de supergravedad en $D$ dimensiones.

La descripción dinámica completa de los sistemas de super–p–branas en interacción con supergravedad dinámica es un problema más complicado y no resuelto de una forma completa.\footnote{Véase [71–74] para un progreso parcial en esta dirección.}

En esta tesis estudiamos, entre otros, sistemas en interacción de supergravedad dinámica y super–p–brana en superespacio simple 4–dimensional. El estudio de este tipo de sistemas es importante ya que pueden ayudarnos a comprender sistemas más complicados en espaciotiempo de 10 y 11 dimensiones de la teoría de cuerdas/teoría M.

**Supergravedad en interacción con super–p–brana**

La acción para la M2–brana en 11 dimensiones se contruyó en [61]. Más aún, en [61] se mostró que la consistencia del acoplo de la supermembrana en un fondo ("background") de supergravedad (es decir, la existencia de simetría–$\kappa$ en espacio curvo) impone al fondo un conjunto de ligaduras del superespacio que resultan en las ecuaciones del movimiento para los campos físicos de supergravedad en 11 dimensiones, del mismo modo en que las condiciones de consistencia para acoplar una supercuerda a supergravedad en supercampos en 10 dimensiones produce las ecuaciones del movimiento para supergravedad $D = 10$ [75]. Es decir, la consistencia del modelo en superespacio curvo de 11 dimensiones requiere que la curvatura y la torsión de este obedezcan las ligaduras de supergravedad que a su vez, resultan en las ecuaciones del movimiento de supergravedad. En este sentido se puede decir que la dinámica de la supergravedad está gobernada por la M2–brana.
Poco después de los artículos pioneros [61] se estudió [76] el homólogo no trivial más simple de la M2–brana, la supermembrana $D = 4 \mathcal{N} = 1$. Su autoconsistencia en superespacio curvo también necesita de un conjunto de ligaduras del superespacio. Sin embargo, al contrario que en el caso de 11 dimensiones estas ligaduras $D = 4 \mathcal{N} = 1$ son *off-shell* en el sentido de que como consecuencia suya no se producen ecuaciones del movimiento. Esto implica que es posible construir la descripción Lagrangiana manifiestamente covariante supersimétrica en supercampos del sistema en interacción de supergravedad y supermembrana $D = 4 \mathcal{N} = 1$. Curiosamente este sistema no se había construido, por lo que será parte del estudio de esta tesis.

Como se ha mencionado anteriormente, los estados fundamentales de las super–p–branas en $D$ dimensiones están descritos por las soluciones supersimétricas de la teoría de supergravedad en $D$ dimensiones [48, 58]. A pesar de que estas soluciones son puramente bosónicas preservan un medio de la supersimetría local característica de la teoría de supergravedad. Esto significa que hay una rotura espontánea parcial de supersimetría debido a la presencia de una super–p–brana [59, 60]. La relación entre las supersimetrías preservadas $(1/2)$ por la solución de la $p$–brana de supergravedad con la simetría–$\kappa$ de la acción en volumen–mundo para la correspondiente super–p–brana se puso de manifiesto en [77].

En [71] se mostró que el límite puramente bosónico de la acción de la supermembrana, donde el fermión de Goldstone de la supermembrana (el homólogo en el volumen–mundo del Goldstonion de Volkov y Akulov [15, 16]) es puesto a cero, $\hat{\theta}^{\beta}(\xi) = 0$, sigue preservando un medio de la supersimetría local de supergravedad. Esta parte preservada $(1/2)$ de la supersimetría del espaciotiempo está en correspondencia uno a uno con la simetría–$\kappa$ de la acción completa de la supermembrana [61] y es la simetría *gauge* del sistema en interacción descrito por la suma de la acción bosónica para la membrana y la acción para supergravedad en 11 dimensiones sin campos auxiliares.

El origen de esta propiedad poco evidente (que no está restringido únicamente a las acciones de supergravedad–supermembrana en interacción en 11 dimensiones, si no que es válido para una larga colección de sistemas dinámicos de super–p–branas y supergravedad) se aclaró en [72, 73] donde se mostró que la suma del límite puramente bosónico de la acción de la super–p–brana y la acción en componentes espaciotemporales para supergravedad se puede obtener fijando un *gauge* en la acción completa en supercampos para el sistema en interacción de supergravedad y super–p–brana. Esta última acción es la suma de la acción completa de la super–p–brana y la acción en supercampos para supergravedad (cuando esta existe y es conocida).

Es por esto que la descripción del sistema en interacción de supergravedad–super–p–brana mediante la suma del límite puramente bosónico de la acción de la $p$–brana y la acción en componentes espaciotemporales de supergravedad sin campos auxiliares [71–74] se llama descripción "completa pero con *gauge* fijo", donde "completa" hace referencia al hecho de que reproduce todas las ecuaciones del sistema en interacción, aunque en su versión con un *gauge* fijo. Incluso reproduce el límite $\hat{\theta}^{\beta}(\xi) = 0$ de la ecuación para el fermión de

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5Véase capítulo 5.
Goldstone de la super–p–brana [72,74]. La descripción de un sistema dinámico de este tipo para supermembrana \( D = 4 \, \mathcal{N} = 1 \) en interacción con supergravidad y multipletes de materia se desarrolló en términos de campos ”componentes” en [78].

Así pues el sistema en interacción de supergravidad dinámica y super–p–brana puede ser estudiado en el marco de la descripción covariante pero con gauge fijo incluso cuando no se conoce la formulación en supercampos para supergravidad.

Sin embargo, el estudio en dimensiones más bajas, en particular en \( D = 4 \), de las ecuaciones en supercampos para los sistemas en interacción de supergravidad y superbranas, cuando es posible, tiene también gran interés ya que podría proveer nuevos datos sobre las propiedades de sistemas más complicados de la teoría M. En particular, como veremos en el capítulo 3 ayuda a comprender el papel de los campos auxiliares de supergravidad en algunos sistemas en interacción de supergravidad y super–p–branas. Además estos sistemas son de interés si mismos ya que podrían servir de base para construir modelos fenomenológicos de supergravidad cuadridimensional. En el capítulo 3 de esta tesis presentamos un estudio completo de la descripción Lagrangiana en supercampos del sistema en interacción de supergravidad minimal \( D = 4 \, \mathcal{N} = 1 \) y supermembrana. Este estudio puede ser considerado como el desarrollo de la línea de investigación de [72,73] y [79].

La descripción Lagrangiana en supercampos para el sistema dinámico de supergravidad \( D = 4 \, \mathcal{N} = 1 \) y superpartícula se desarrolló en [72]. Las ecuaciones en supercampos para el sistema dinámico de supergravidad, supercuerda y supermultiplete tensorial se obtuvieron en [73].

Ambos modelos se han usado para estudiar el origen así como las propiedades de la descripción Lagrangiana completa pero con un gauge fijo del sistema de supergravidad y super–p–brana. Esta descripción propuesta y desarrollada en [71,74] puede ser usada también en sistemas de supergravidad más brana(s) en interacción en dimensiones más altas.

Debido a que el sistema de ecuaciones en supercampos para el sistema de supergravedad, supercuerda y supermultiplete tensorial obtenido en [73] resulta ser demasiado complicado para ser práctico, probablemente se podría obtener un sistema de ecuaciones menos complicado si se usara la formulación en supercampos de la llamada supergravidad minimal ”nueva” [25,80] en vez de la supergravidad minimal ”vieja” [81,82] usada en [73]. Esta suposición está relacionada con el hecho de que la formulación minimal ”nueva” incluye un tensor antisimétrico auxiliar que tiene un acoplo natural al modelo de cuerda por lo que no es necesario introducir a mano, como en [73], un multiplete tensorial además del de supergravedad.

Por otro lado, se pueden usar los resultado de [73] para extraer las ecuaciones en supercampos para la supercuerda en interacción con el multiplete tensorial en superespacio plano \( D = 4 \, \mathcal{N} = 1 \). La existencia de esa interacción no trivial está relacionada con el hecho de que, de acuerdo con [83], se puede usar el multiplete tensorial para construir una 3–forma supersimétrica y cerrada en superespacio plano \( D = 4 \, \mathcal{N} = 1 \). Es natural empezar
estudiando las ecuaciones en supercampos para el sistema formado por la supercuerda y el multiplete tensorial antes de pasar al estudio de supergravedad en interacción con supercuerda pero el sistema de ecuaciones en supercampos en interacción resulta ser incluso más sencillo si lo escribimos para un objeto supersimétrico extendido en interacción con un supermultiplete escalar.

No es posible hacerlo con la supercuerda pero sí con la supermembrana ya que no es posible construir una 3–forma field strength a partir del multiplete escalar, pero una 4–forma sí. Así que como primer paso hacia el estudio de la supermembrana en interacción con supergravedad, estudiamos en el capítulo 2 la descripción Lagrangiana y obtenemos las ecuaciones del movimiento en supercampos para el sistema en interacción de supermembrana $D = 4$ $\mathcal{N} = 1$ y multiplete escalar.

El estudio de la descripción Lagrangiana en supercampos del sistema de supermembrana $D = 4$ $\mathcal{N} = 1$ y supergravedad comenzó en [84], donde se desarrolló un formalismo del tipo Wess–Zumino para la supergravedad minimal especial de Grisaru–Siegel–Gates–Ovrut–Waldram [85–87] y se encontraron las expresiones para las corrientes (en supercampos) asociadas a la supermembrana que aparecen en el lado derecho de las ecuaciones en supercampos de supergravedad.

Como veremos en el capítulo 3 las ecuaciones en supercampos de supergravedad con contribuciones de la membrana son muy complicadas pero se simplifican drásticamente en un gauge especial que llamamos gauge "WZ $\hat{\theta} = 0$". Se llega a este gauge fijando el gauge usual de Wess–Zumino (WZ) para supergravedad y después usando un medio de las supersimetrías locales del volumen–mundo de la supermembrana para fijar el gauge en el cual el fermión de Goldstone de la supermembrana se hace cero, $\hat{\theta}^3(\xi) = 0$.

En este gauge resolvemos las ecuaciones para los campos auxiliares y mostramos que hay tres tipos de contribuciones provinientes de la supermembrana que aparecen en las ecuaciones de Einstein del sistema en interacción. Además de los términos singulares relacionados con el volumen–mundo de la supermembrana $W^3$, la supermembrana produce dos términos regulares que pueden ser considerados como contribuciones a la constante cosmológica en las dos regiones del espaciotiempo separadas por el volumen–mundo de la membrana.

La primera de estas contribuciones, conocida por el estudio [87] (y anteriormente en [88–91], ver [84] para más referencias y discusión sobre ella), es la constante cosmológica generada dinámicamente. La segunda contribución no singular de la supermembrana, cambia el valor de la constante cosmológica en una de las dos regiones de espaciotiempo haciendo que el valor de la constante cosmológica en las dos ramas del espaciotiempo $M^4_+$ y $M^4_-$, separadas por el volumen mundo de la supermembrana sea, en general, distinto. Este efecto, que puede ser llamado shift o renormalización de la constante cosmológica debida a las contribuciones de la supermembrana, fue discutido en [88] y [92] en una perspectiva puramente bosónica.

Así pues genéricamente el estado fundamental de nuestro sistema en interacción describe a la supermembrana separando dos espacios de Anti-deSitter con diferentes valores de la
constante cosmológica. En el contexto puramente bosónico las soluciones de ese tipo fueron estudiadas (además de [88] y [92]) en [93–95]. La solución particular en la que ambas constantes cosmológicas tienen el mismo valor se puede encontrar en [87]. En esta tesis también se presenta una discusión de posibles soluciones supersimétricas del sistema en interacción con una supermembrana separando dos espacios asintóticamente Anti–deSitter con diferentes valores de la constante cosmológica.

Teorías de campos de alto espín

Es conocido que las teorías conformes de campos libres de alto espín en \( D=4 \) pueden ser formuladas como una teoría de campos en un espacio tensorial de diez dimensiones, \( \Sigma^{(10|0)} \), parametrizado por 10 coordenadas bosónicas \((x^m, y^{mn})\) [96–103],

\[
X^{\alpha \beta} = X^{\beta \alpha} = \frac{1}{4} x^m \gamma_m^{\alpha \beta} + \frac{1}{8} y^{mn} \gamma_{mn}^{\alpha \beta} \quad \alpha, \beta = 1, 2, 3, 4, \quad m, n = 0, 1, 2, 3.
\]
y en un superespacio tensorial \( N = 1, \Sigma^{(10|4)} \) con coordenadas \((X^{\alpha \beta}, \theta^\alpha)\) [97–103].

Este espacio tensorial bosónico se propuso como base natural para construir teorías de campos conformes de alto espín en \( D = 4 \) en [96]. En [104, 105], [96–102] y [103, 107, 108] se han presentado espacios tensoriales más generales de dimensión \( \frac{n(n+1)}{2} \) en el sentido de matrices \( n \times n \) simétricas \( X^{\alpha \beta} (\alpha, \beta = 1, \ldots, n) \) que, para \( n \) par \((n = 4, 8, 16 y 32)\), determinan la extensión del espaciotiempo estándar de dimensión \( D=4, 6, 10 (D = \frac{n}{2} + 2) \) y \( D = 11 \).

Añadiendo \( n \) coordenadas fermiónicas \( \theta^\alpha \) se pueden obtener los superespacios tensoriales "simples" \( (N = 1) \Sigma^{(\frac{n(n+1)}{2}|n)} \),

\[
\Sigma^{(\frac{n(n+1)}{2}|n)}: \quad Z^M := (X^{\alpha \beta}, \theta^\alpha), \quad \begin{cases} \alpha, \beta = 1, \ldots, n, \\ X^{\alpha \beta} = X^{\beta \alpha} . \end{cases}
\]

que, en su versión plana, también tienen estructura de supergrupo.

Tomar \( n \) par no es una restricción si uno piensa en el origen espinorial que subyace en los índices \( \alpha; \) es más esto motiva la restricción a \( n = 2^k = 2, 4, 8, 16, \ldots \) suponiendo que \( \theta^\alpha \) son espinores. Aunque las coordenadas fermiónicas en \( \Sigma^{(\frac{n(n+1)}{2}|n)} \) se suponen reales normalmente, en el caso \( n = 4 D = 4 \) es conveniente considerar \( \theta^\alpha \) como un espinor de Majorana en la realización de Weyl de las matrices de Dirac por lo que \( \theta^\alpha = (\theta^A, \bar{\theta}^A) \).

Para \( n = 2 \) las coordenadas espín-tensoriales \( X^{\alpha \beta} \) están expresadas en términos de las coordenadas del 3–vector espaciotemporal, \( X^{\alpha \beta} \propto x^a \gamma^\alpha_a \) así que \( \Sigma^{(3|2)} \) es simplemente el superespacio usual \( D = 3 \mathcal{N} = 1 \). El caso \( n = 32 \) da como resultado la extensión del superespacio de 11 dimensiones \( \Sigma^{(528|32)} \), importante en el contexto de la hipótesis de
los preones BPS [107, 109] y también en el análisis de la estructura gauge oculta de la supergravedad en \( D = 11 \) [106, 108].

En discusiones de las teorías de espines altos, incluso en el capítulo 4 de esta tesis, se consideran los casos \( n = 4, 8, 16 \) que se usan para describir teorías conformes de campos de alto espín sin masa en \( D = 4, 6, 10 \). Casi todas nuestras ecuaciones en el capítulo 4 serán válidas para esas dimensiones, aunque haremos especial énfasis en el caso \( n = 4 \) que corresponde a \( D = 4 \).

El primer sistema mecánico en el superespacio tensorial \( \mathcal{N} = 1 \) \( D = 4 \) \( \Sigma^{(10|4)} \) y sus generalizaciones de dimensiones más altas \( \Sigma^{\frac{n(n+1)}{2}|n} \) con \( n > 4 \) se propusieron en [110], donde se observó que el estado fundamental de ese modelo de superpartícula describe un estado BPS que preserva todas las supersimetrías excepto una. El posible papel como ”constituyentes” de estos estados en la teoría de cuerdas/M se introdujo y se discutió en general en [109], donde se les llamó ”preones BPS” (véase también [107]). Así que desde ese punto de vista, la superpartícula en [110] podría ser llamada ”preónica”. Su cuantización se desarrolló en [97], donde se mostró que el espectro de la superpartícula preónica \( n = 4 \) cuantizada se puede describir con una torre de campos conformes de alto espín sin masa de todas las helicidades posibles y se presentaron evidencias de que los modelos con \( n = 8 \) y \( n = 16 \) describen teorías conformes de alto espín en espaciotiempos de 6 y 10 dimensiones.

En [98, 99] se presentó y estudió una forma elegante de las ecuaciones de alto espín bosónicas y fermiónicas en el espacio tensorial \( \Sigma^{(10|0)} \). Mientras que en [103] se dio la forma explícita de las ecuaciones conformes de alto espín en espacios tensoriales con \( D = 6, 10 \). Las ecuaciones en supercampos para los supermultipletes de campos conformes sin masa de dimensión \( D = 6, 10 \) en los superespacios tensoriales \( \mathcal{N} = 1 \) \( \Sigma^{(36|8)} \) y \( \Sigma^{(136|16)} \),

\[
D = 10 \quad \Sigma^{(136|16)} : \quad Z^M = (X^{\alpha\beta}, \theta^\alpha), \quad \begin{cases} 
\alpha, \beta = 1, \ldots, 16, \\
X^{\alpha\beta} = \frac{1}{16} x^m \tilde{\sigma}^{\alpha\beta}_m + \frac{1}{2 \cdot 5!} y^{m_1 \ldots m_5} \tilde{\sigma}^{\alpha\beta}_{m_1 \ldots m_5}, \\
m = 0, 1, \ldots 9 
\end{cases}
\]

fueron propuestas en [102].

En particular, los supermultipletes \( \mathcal{N} = 1 \) de campos conformes de alto espín en \( D = 4, 6, 10 \) están descritos por supercampos escalares en los correspondiente superespacios \( n = 4, 8, 16 \) \( \Sigma^{\frac{n(n+1)}{2}|n} \),

\[
\Phi(X^{\alpha\beta}, \theta^\gamma) = b(X) + f_\alpha(X) \theta^\alpha + \sum_{i=2}^{n} \phi_{\alpha_1 \ldots \alpha_i}(X) \theta^{\alpha_1} \ldots \theta^{\alpha_i},
\]

obedeciendo la ecuación [102]

\[
D_{[\alpha} D_{\beta]} \Phi(X, \theta) = 0.
\]
Aquí,

\[ D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i \theta^\beta \partial_{\beta\alpha}, \quad D_{\alpha\beta} = \partial_{\alpha\beta} := \frac{\partial}{\partial X_{\alpha\beta}}, \]

son derivadas covariantes en el superespacio tensorial rígido \( \Sigma^{(\frac{n(n+1)}{2}|n)} \) que satisfacen

\[ \{D_\alpha, D_\beta\} = 2i\delta_{\alpha\beta} \]

donde se puede ver que exhiben la estructura de extensión central de las superálgebras de los superespacios tensoriales \([111]\) (véase \([112]\)).

Las ecuaciones de alto espín en superespacio tensorial con supersimetría \( \mathcal{N} \)-extendida, han sido estudiadas en \([113, 114]\) que constituyen la base del capítulo 4 de esta tesis.

En el capítulo 4 presentamos las ecuaciones supersimétricas conformes de alto espín libres con \( \mathcal{N} = 2, \mathcal{N} = 4 \) y \( \mathcal{N} = 8 \) en superespacios tensoriales \( \mathcal{N} \)-extendidos \( \Sigma^{\frac{n(n+1)}{2}|\mathcal{N}n} \). Entre otros, vamos a mostrar que las ecuaciones conformes de alto espín de los supermultipletes \( \mathcal{N} = 2 \) de \( D = 4, 6, 10 \) están descritas por supercampos escalares quirales \( \Phi(X^{\alpha\beta}, \Theta^\gamma, \bar{\Theta}^\gamma) \) en el superespacio tensorial \( \mathcal{N} = 2 \Sigma^{\frac{n(n+1)}{2}|2n} \), que obedecen el siguiente conjunto de ecuaciones lineales en supercampos,

\[ \bar{D}_\alpha \Phi = 0, \quad D_{[\alpha} D_{\beta]} \Phi = 0, \]

donde

\[ D_\alpha = \frac{\partial}{\partial \Theta^\alpha} + i \bar{\Theta}^\beta \partial_{\beta\alpha} = \frac{1}{2} (D_{\alpha1} + i D_{\alpha2}), \]

\[ \bar{D}_\alpha = \frac{\partial}{\partial \Theta^\alpha} + i \Theta^\beta \partial_{\beta\alpha} = -(D_\alpha)^*, \quad D_{\alpha\beta} = \partial_{\alpha\beta} := \frac{\partial}{\partial X_{\alpha\beta}}, \]

son las derivadas covariantes en el superespacio rígido \( \mathcal{N} = 2 \)-extendido \( \Sigma^{\frac{n(n+1)}{2}|2n} \), que obedecen el superálgebra

\[ \{D_\alpha, D_\beta\} = 0, \quad \{D_\alpha, \bar{D}_\beta\} = 2i\delta_{\alpha\beta}, \quad \{\bar{D}_\alpha, \bar{D}_\beta\} = 0. \]

Presentamos también las ecuaciones en supercampos para superespacios tensoriales extendidos con \( \mathcal{N} \) par y mayor que 2 que generalizan las ecuaciones de \( \mathcal{N} = 2 \).

También presentaremos la generalización supersimétrica extendida \( \mathcal{N} > 1 \) del modelo de superpartícula preónica de \([97]\) y mostraremos como se pueden obtener las ecuaciones de los supercampos de alto espín cuantizando el modelo de superpartícula en \( \Sigma^{\frac{n(n+1)}{2}|\mathcal{N}n} \) para \( \mathcal{N} \geq 2 \) con \( \mathcal{N} \) par.

\[ ^6 \text{Vease} \ [115, 116] \text{para una descripción de las ecuaciones supersimétricas de alto espín libres en superespacio usual y} \ [117, 118] \text{para la versión en supercampos de las ecuaciones de campos de alto espín de Vasiliev en interacción} \ [119, 121]. \]
Aunque nuestras ecuaciones son válidas para cualesquiera \( N \) y \( n \) par, elaboraremos en detalle los casos \( N = 2, 4, 8 \), que, para \( n = 4 \), corresponden a los supermultipletes sin masas de campos de alto espín en \( D = 4 \), los cuales están en correspondencia clara con las teorías estándar de campos de "bajo espín". Estas son el hipermultiplete para \( N = 2 \), el supermultiplete supersimétrico de Yang–Mills para \( N = 4 \) y el multiplete de supergravedad maximal para \( N = 8 \), el cual en su versión linearizada se puede describir mediante supercampos escalares en superespacios estándar extendidos en \( D = 4 \ Σ(4|4N) \) con \( N = 2, 4, 8 \).

Una de las razones de nuestro interés en sistemas \( N \)-extendidos supersimétricos de teorías de alto espín venía de la observación de que la supersimetría \( N \)-extendida con \( N = 4 \) unifica los campos gauge escalares y vectoriales. Por otro lado, todas las ecuaciones de alto espín han sido formuladas en términos de campos escalares y espinoriales en espacio tensorial así que el estudio de supersimetrías \( N \)-extendidas podría ser interesante para buscar posibles generalizaciones de las ecuaciones de Maxwell y Einstein en superespacios tensoriales.

De hecho, en cierto punto de nuestro estudio de las ecuaciones en supercampos en superespacio tensorial \( N = 4 Σ^{(10|16)} \) aparecen los análogos de las ecuaciones de Maxwell en espacio tensorial. Sin embargo, un análisis más detallado muestra que estos campos (espín)–tensoriales se pueden expresar como derivadas de otros campos escalares en superespacio tensorial así que, por ejemplo, el sector bosónico del multiplete de alto espín conforme \( N = 4 \) está expresado por dos campos escalares complejos en el espacio tensorial, \( φ \) y \( ˜φ \).

Del mismo modo, cuando estudiamos las ecuaciones en supercampos en superespacio tensorial \( N = 8, Σ^{(10|32)} \), a pesar de que en cierta etapa aparecen las generalizaciones en espacio tensorial de las ecuaciones de (super)gravedad conforme, se demuestra que se reducen a las ecuaciones para campos escalares (y espinoriales) en espacio tensorial presentadas por primera vez por Vasiliev [98]. En cierto sentido se puede decir que el resultado de aumentar \( N \), es que aparecen más campos escalares y espinoriales.

Sin embargo, para \( N = 4 \) aparece un nuevo fenómeno. Como el nuevo campo escalar aparece en la teoría únicamente a través de campos de tipo de Maxwell, es decir bajo derivadas bosónicas \( \partial_\alpha\beta \), la teoría se hace invariante bajo desplazamientos constantes de ese campo bosónico que hace que el segundo campo escalar sea similar a los axiones (para los cuales dicha simetría se llama simetría de Peccei–Queen [122]). En el multiplete \( N = 8 \) reformulado en términos de campos, la simetría de Peccei–Queen se hace más complicada para los campos escalares e incluso está presente en los campos espinoriales que entran en el modelo bajo la acción de una derivada simulando la estructura de los campos de Rarita-Schwinger.

\begin{footnote}{7}Dado que en la teoría de cuerdas de tipo IIB y en supergravedad IIB el axión aparece como un miembro de la familia de los campos gauge RR, su simetría de Peccei–Queen puede ser considerada como el homólogo de la simetría gauge característica de los potenciales gauge RR altos.\end{footnote}
Sistema de múltiples M0–branas

En [50] se motivó que una descripción aproximada de un sistema de p–branas de Dirichlet (Dp–branas) quasi coincidentes viene dada por una teoría de Yang–Mills maximal supersimétrica (SYM) con grupo de gauge $U(N)$, la cual puede ser obtenida mediante reducción dimensional de la teoría de SYM en $D=10$ con $U(N)$ a $d = p + 1$. Esta incluye $D-p-1$ matrices Hermitas con campos escalares cuyos elementos diagonales describen las diferentes posiciones de las Dp-branas mientras que los elementos no diagonales dan cuenta de las cuerdas que unen las diferentes Dp-branas.

Como es sabido que una sola Dp-brana está descrita por la suma de la acción supersimétrica de Dirac–Born–Infeld, proporcionando una generalización no lineal de la acción de Yang–Mills con $U(1)$, y un término de Wess-Zumino (véase [63–67, 123, 124]), era natural buscar una generalización no lineal de la acción de SYM no–abeliana que proporcione una descripción no lineal más completa del sistema de Dp-branas quasi coincidentes. Para el límite bosónico de múltiples Dp-branas quasi coincidentes (sistema mDp) la descripción más popular está dada por la acción de ”branas dieléctricas” de Myers [125]. Esta se obtuvo a través de una serie de transformaciones de dualidad T a partir de la 10D acción no–abeliana de Born–Infeld con traza simétrica, propuesta por Tseytlin [126] para el límite puramente bosónico del sistema de múltiples D9–branas (sistema mD9) que llenan todo el espaciotiempo. Ambas acciones [125] y [126] han resistido todos los intentos de construir sus generalizaciones supersimétricas durante muchos años. Además, la acción de Myers no tiene simetría de Lorentz.

La descripción supersimétrica y covariante Lorentz del sistema de mDp se obtuvo en [127] en el marco del llamado ”enfoque de fermiones de frontera”. Sin embargo, esta descripción viene dada en lo que se llama ”cuantización en nivel menos uno” que significa que para llegar a una descripción del sistema de mDp similar al del de Dp–branas (como por ejemplo el de [65]), se tiene que realizar una cuantización del sistema dinámico. Esta tarea no es trivial y no se ha resuelto de manera completa aún, lo que ha motivado un gran número de intentos de obtener una descripción aproximada pero covariante bajo el grupo de Lorentz y supersimétrico del sistema de mDp que vaya más allá de la aproximación de SYM (véase por ejemplo [128]). Sólo para el caso del sistema de mD0 existe un candidato no lineal, supersimétrico e invariante Lorentz en $D = 10$ para la acción de mD0 [129, 130].

Debido a que las Dp–branas con $p = 0, 2, 4$ pueden ser obtenidas mediante una reducción dimensional de las branas M0, M2 y M5 en $D = 11$, es lógico suponer que se pueda obtener el sistema de mDp de su respectivo sistema de mMp.

Sin embargo, para el caso del sistema de mM5 incluso la cuestión de cuál es el análogo de la descripción aproximada de muy bajas energías de SYM aún no se conoce a ciencia cierta (véase por ejemplo [131–133] para estudios relacionados y referencias). Para el caso del sistema de mM2 branas dicho problema ha permanecido sin solución muchos años pero recientemente el $d = 3 \ N = 8$ modelo de Bagger, Lambert y Gustavsson (BLG) supersimétrico [134] basado en 3–álgebras (véase [135] y referencias allí) en vez de álgebras
de Lie y un modelo de Aharony, Bergman, Jafferis y Maldacena (ABJM) más convencional \[136\] (con simetría de gauge $SU(N) \times SU(N)$ y sólo $\mathcal{N} = 6$ supersimetrías manifiestas) han sido propuestos para dicho papel.

Si se trata del sistema de múltiples M0 branas, se construyó un candidato puramente bosónico en \[137, 138\] generalizando la acción de Myers de la D0–brana dieléctrica de 11 dimensiones. Por otro lado, se obtuvieron las ecuaciones del movimiento aproximadas pero supersimétricas e invariantes bajo el grupo de Lorentz para el sistema de mM0 en \[139\] en el marco del enfoque de superembedding (véase \[140, 141\] así como \[142, 143\] y referencias allí).

La generalización de esas ecuaciones para el sistema de mM0 en superespacio curvo de supergravidad 11 dimensional que describe la generalización de la "teoría de M(atrices)" \[144\] (véase \[145\]) para el caso de su interacción con un fondo de supergravedad arbitrario, se presentó y estudió en \[146\]. En \[147\] se mostró que en el caso de que el fondo fuera de ondas pp (pp–wave), esas ecuaciones reproducen (en cierta aproximación) el llamado modelo BMN de matrices propuesto para ese fondo por Berenstein, Maldacena y Nastase en \[148\].

Este resultado confirma que las ecuaciones de \[146\] describen la teoría de Matrices interaccionando con un fondo de supergravedad. Sin embargo, debido al origen de superespacio de dichas ecuaciones, sus aplicaciones incluso en un fondo de supergravedad puramente bosónico son extremadamente complicadas. Para este fin se necesita por lo primero encontrar la solución completa en supercampos de las ligaduras de supergravedad 11D \[149\] que representan la solución bosónica supersimétrica de las ecuaciones de supergravedad en el espaciotiempo. Esto hizo que fuera deseable encontrar una acción que reprodujera las ecuaciones del modelo de Matrices de \[146\] o sus generalizaciones.

Para el caso del sistema de mM0 en superespacio plano esta acción se propuso en \[150\], donde se mostró que posee supersimetría local $\mathcal{N} = 16$ 1d. En el capítulo 5 se derivarán y estudiarán las ecuaciones del movimiento del sistema de mM0 descrito por dicha acción. Se estudiarán las soluciones supersimétricas de dichas ecuaciones mostrando que su sector relacionado con el centro de energía es similar a la solución de las ecuaciones para una única M0 brana y se presentarán también dos ejemplos de soluciones no supersimétricas con diferentes propiedades del movimiento del centro de energía.

**Contenido de la tesis**

El primer capítulo contiene una introducción al superespacio así como a los supercampos y superformas, ya que a lo largo de la tesis se hará uso de dichos conceptos. Tras esta breve exposición se explicará la geometría de los superfespacios plano y curvo, para pasar a describir la supergravedad. Se da una descripción de la supergravedad minimal, su acción en superespacio, así de como obtener sus ecuaciones del movimiento a partir de esta acción en supercampos. Aunque a priori no es trivial encontrar las variaciones de los *supervielbeins,*
ya que estos están restringidos por las ligaduras de supergravedad, es posible encontrar variaciones admisibles que preservan dichas ligaduras [72, 82]. Terminamos el capítulo 1 presentando estas variaciones admisibles para la supergravedad minimal, que usaremos en el capítulo 3.

En el capítulo 2 presentamos la acción en supercampos para el sistema dinámico de supermembrana $D = 4 \mathcal{N} = 1$ en interacción con multiplete escalar y la usamos para obtener las ecuaciones del movimiento en supercampos de este sistema.

Estas incluyen las ecuaciones de la supermembrana que coinciden formalmente con las ecuaciones de esta en un fondo de campo escalar (off-shell) y las ecuaciones para el supercampo quiral especial con fuente producida por la supermembrana.

En el caso en el cual la parte de la acción correspondiente al supermultiplete escalar contiene únicamente el término cinético más simple, extraemos las ecuaciones en componentes a partir de las ecuaciones en supercampos y las resolvemos a orden principal en la tensión de la supermembrana.

También se discute la inclusión de un superpotencial no trivial y su relación con las soluciones supersimétricas del tipo domain wall conocidas.

El capítulo 3 está dedicado a obtener el conjunto completo de ecuaciones del movimiento para el sistema en interacción de supermembrana y supergravedad dinámica $D = 4 \mathcal{N} = 1$ variando la acción completa en supercampos. Una vez obtenidas, escribimos estas ecuaciones en supercampos en el gauge especial “WZ $\hat{\theta} = 0$” donde el campo de Goldstone de la supermembrana se ha puesto a cero ($\hat{\theta} = 0$) y además está fijado el gauge de Wess–Zumino en los supercampos de supergravedad.

En este gauge resolvemos las ecuaciones para los campos auxiliares y discutimos el efecto de la generación dinámica de la constante cosmológica en la ecuación de Einstein del sistema en interacción y su renormalización debida a contribuciones regulares de la supermembrana. Estos dos efectos (descritos por primera vez en los años 70 y 80 en un contexto bosónico y en la literatura de supergravedad) resultan en que la solución describe en el caso genérico, dos espaciotiempos con distintas constantes cosmológicas separados por el volumen–mundo de la supermembrana.

Continuamos en el capítulo 4 proponiendo las ecuaciones en supercampos en superespacios tensoriales $\mathcal{N}$-extendidos para describir las generalizaciones supersimétricas con $\mathcal{N} = 2, 4, 8$ de las teorías conformes libres de alto espín. Describimos también como se puede obtener estas ecuaciones cuantizando un modelo de superpartícula en superespacios tensoriales $\mathcal{N}$-extendidos.

Mostramos que los supermultipletes de alto espín $\mathcal{N}$-extendidos contienen únicamente campos escalares y espinoriales en el espacio tensorial, así que a diferencia del enfoque estándar de supercampos en superespacios habituales, no aparecen generalizaciones no triviales de las ecuaciones de Maxwell y Einstein cuando $\mathcal{N} > 2$. Para $\mathcal{N} = 4, 8$ las componentes de tipo espín-tensor más altas del supercampo tensorial extendido se expresan a través de campos escalares y espinoriales adicionales que obedecen las mismas ecuaciones libres de alto espín, pero estos son del tipo axión en el sentido de que poseen simetrías de tipo Peccei–Quinn.
En el último capítulo estudiamos las propiedades de la acción covariante supersimétrica y con simetría–κ de N M0–branas quasi coincidentes (sistema mM0) en superespacio plano de once dimensiones y obtenemos las ecuaciones supersimétricas para este sistema dinámico.

A pesar de que una única M0–brana se corresponde con la superpartícula sin masa en 11 dimensiones, el movimiento del centro de energía del sistema de mM0 está caracterizado por una masa M no negativa. Esta masa está construida a partir de los campos matriciales que describen el movimiento relativo de los constituyentes del sistema de mM0.

Mostramos que cualquier solución bosónica del sistema de mM0 puede ser supersimétrica si y sólo si esta masa efectiva se anula, \( M^2 = 0 \), y que todas las soluciones bosónicas supersimétricas preservan un medio de las supersimetrías de once dimensiones. Presentamos también unas soluciones no supersimétricas con \( M^2 \neq 0 \) y discutimos unas propiedades peculiares del sistema de mM0.
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Chapter 1

Introduction

In field theory supersymmetry transforms bosonic fields into fermionic fields and vice versa. Hence, the irreducible representations of supersymmetry, supermultiplets, involve some number of bosonic and fermionic fields. A compact and elegant way to describe supermultiplets is provided by superfield approach.

Superspace is an extension of the ordinary spacetime including, besides the usual spacetime coordinates $x^\mu$, extra anticommutative (or fermionic) coordinates. In the case of flat $\mathcal{N}$-extended $D = 4$ superspace these fermionic coordinates are collected in $\mathcal{N}$ two-component Weyl spinors $\theta^\alpha = (\bar{\theta}^\dot{\alpha})^*$. Chapters 2, 3 are devoted to the study of the $D = 4$ $\mathcal{N} = 1$ supermembrane, which is a membrane moving in simple ($\mathcal{N} = 1$) $D = 4$ superspace, so let us begin reviewing some properties of this superspace.

1.1. Flat $D = 4$ $\mathcal{N} = 1$ superspace

We denote the coordinates of flat $D = 4$ $\mathcal{N} = 1$ superspace $\Sigma^{(4|4)}$ by

$$ z^M = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \quad (1.1) $$

where $\mu = 0, 1, 2, 3$ is a 4-vector index, $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$ are spinorial indices; the "supervector" index $M$ represents both types of indices, $M = (\mu, \alpha, \dot{\alpha})$.

The spinorial coordinates anticommute, $\theta^\alpha \bar{\theta}^{\dot{\alpha}} = -\bar{\theta}^{\dot{\alpha}} \theta^\alpha$, $\theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha$, $\bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = -\bar{\theta}^{\dot{\beta}} \bar{\theta}^{\dot{\alpha}}$, while the bosonic coordinates commute among themselves, $x^\mu x^\nu = x^\nu x^\mu$, and also with fermionic coordinates $x^\mu \theta^\alpha = \theta^\alpha x^\mu$. These properties can be collected in

$$ z^M z^N = (-)^{\epsilon(N)\epsilon(M)} z^N z^M \quad (1.2) $$

where $\epsilon(M) \equiv \epsilon(z^M)$ is the so-called Grassmann parity, $\epsilon(\mu) = 0$, $\epsilon(\alpha) = 1 = \epsilon(\dot{\alpha})$. 
1.2. Superfields on $D = 4 \mathcal{N} = 1$ superspace

Superfields are functions defined on superspace which means that superfields depend on both bosonic and fermionic coordinates. As far as fermionic coordinates anticommute among themselves, they are nilpotent, $\theta \theta' = -\theta' \theta = 0$. This property implies that the series expansion of superfield in fermionic coordinates contains a finite number of terms. The coefficients in that series, called superfield components, are the ordinary spacetime fields

$$F(z) = F(x, \theta, \bar{\theta}) = f(x) + \theta \phi(x) + \bar{\theta} \lambda(x) + \theta \bar{\theta} n(x) + \theta \sigma^m \nu_m(x) + \bar{\theta} \tilde{\lambda}(x) + \bar{\theta} n(x) + \theta \theta m(x) + \theta \bar{\theta} \bar{\theta} \tilde{d}(x).$$

The statistics (Grassmann parity) of the component fields appearing as a coefficient for even and odd powers of $\theta$, $\bar{\theta}$ are different (bosonic versus fermionic or vice versa) and depends on the statistics (Grassmann parity) of the superfield.

The rigid $D = 4 \mathcal{N} = 1$ supersymmetry is realized in superspace as constant translations of the fermionic coordinate, $\delta \theta = \epsilon$, $\delta \bar{\theta} = \bar{\epsilon}$, supplemented with the following transformations of bosonic coordinates, $\delta x^\mu = -i \epsilon \sigma^\mu \bar{\theta} + i \theta \sigma^\mu \bar{\epsilon}$ (see [15, 16]), to resume,

$$\delta x^\mu = -i \epsilon \sigma^\mu \bar{\theta} + i \theta \sigma^\mu \bar{\epsilon},$$
$$\delta \theta^a = \epsilon^a,$$
$$\delta \bar{\theta}^\dot{a} = \bar{\epsilon}^\dot{a}. \tag{1.4}$$

In general a superfield is a highly reducible representation of the supersymmetry. To extract an irreducible representation one usually needs to impose on superfield some equations in terms of fermionic covariant derivatives (see bellow) called constraints. One distinguishes on–shell and off–shell constraints. The on–shell constraints restrict the field content of superfield to physical fields and impose on these equations of motion. The off–shell constraints do not impose on physical fields equations of motion and also leave nonvanishing not only physical component fields of the superfield, but also so–called auxiliary fields, the presence of which provides off-shell closure of the (super)algebra of supersymmetry transformations.

1.3. Differential Superforms

The convenience of differential forms is that they are manifestly invariant under coordinate transformations. Let us introduce differentials of superspace coordinates, $dz^M = (dx^\mu, d\theta^a, d\bar{\theta}^\dot{a})$ and the exterior product of these

$$dz^M \wedge dz^N = -(-)^{(\epsilon(N)+1)(\epsilon(M)+1)} dz^N \wedge dz^M = (-)^{\epsilon(N)\epsilon(M)+1} dz^N \wedge dz^M,$$
$$dz^M z^N = -(-)^{\epsilon(N)\epsilon(M)} z^N dz^M, \tag{1.5}$$

with $\epsilon(N)$ defined in (1.2). This implies

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu, \quad d\theta^a \wedge dx^\mu = -dx^\mu \wedge d\theta^a,$$
$$d\theta^a \wedge d\theta^b = +d\theta^b \wedge d\theta^a, \quad etc. \tag{1.6}$$
Chapter 1. Introduction

The differential p–form in superspace has then the following structure

$$\Omega_p = \frac{1}{p!} dz^M_1 \wedge \ldots \wedge dz^M_p \Omega_{M_p \ldots M_1} (z) \quad (1.7)$$

where $\Omega_{M_p \ldots M_1} (z)$ is a superfield carrying supervector indices. Being contracted with exterior product of $dz^M$, $\Omega_{M_p \ldots M_1} (z)$ has to be graded–antisymmetric, $\Omega_{\{M_p \ldots M_1\}} (z)$, which implies

$$\Omega_{\ldots \mu \ldots \nu \ldots} (z) = - \Omega_{\ldots \nu \ldots \mu \ldots} (z),$$

$$\Omega_{\ldots \mu \ldots \beta \ldots} (z) = - \Omega_{\ldots \beta \ldots \mu \ldots} (z),$$

$$\Omega_{\ldots \alpha \ldots \beta \ldots} (z) = + \Omega_{\ldots \beta \ldots \alpha \ldots} (z). \quad (1.8)$$

If the form $\Omega_p$ is bosonic, the superfields $\Omega_{M_p \ldots M_1} (z)$ with odd number of spinorial indices will have a fermionic behavior while those with even number will have a bosonic one.

With these definitions the exterior product of differential superforms obey

$$(c_1 \Omega_p + c_2 \Sigma_p) \wedge \Omega_q = c_1 \Omega_p \wedge \Omega_q + c_2 \Sigma_p \wedge \Omega_q$$

$$\Omega_p \wedge \Omega_q = (-)^{pq+r+\varepsilon(\Omega_p)\varepsilon(\Omega_q)} \Omega_q \wedge \Omega_p$$

$$\Omega_p \wedge (\Omega_q \wedge \Omega_r) = (\Omega_p \wedge \Omega_q) \wedge \Omega_r. \quad (1.9)$$

Once defined the superforms we introduce the exterior derivative, an operator which maps p-forms into (p+1)-forms

$$d\Omega_p = \frac{1}{p!} dz^M_1 \wedge \ldots \wedge dz^M_p \wedge dz^N \frac{\partial}{\partial z^N} \Omega_{M_p \ldots M_1} (z) =$$

$$= \frac{1}{(p + 1)!} dz^M_1 \wedge \ldots \wedge dz^{M_p+1} (\partial_{M_{p+1}} \Omega_{M_p \ldots M_1} (z) + (-)^{\varepsilon(N)\varepsilon(M)+1} \text{cyclic permutations}). \quad (1.10)$$

Some useful properties involving exterior derivative one

$$dd = 0,$$

$$d(c_1 \Omega_p + c_2 \Sigma_p) = c_1 d\Omega_p + c_2 d\Sigma_p,$$

$$d(\Omega_p \wedge \Sigma_q) = \Omega_p \wedge d\Sigma_q + (-)^q d\Omega_p \wedge \Sigma_q. \quad (1.11)$$

1.4. Vielbein

Supervielbeins are 1-forms which define a supersymmetric generalization of local reference frame. We denote the bosonic and fermionic supervielbein one forms of $\Sigma^{(4|4)}$ by

$$E^a = dZ^M E^a_M (Z), \quad E^\alpha = dZ^M E^\alpha_M (Z), \quad \bar{E}^{\dot{a}} = dZ^M \bar{E}^{\dot{a}}_M (Z), \quad \quad (1.12)$$

$$a = 0, 1, 2, 3; \quad \alpha = 1, 2; \quad \dot{\alpha} = 1, 2.$$
1.4. Vielbein

Sometimes it is convenient to collect them in

$$E^A = (E^a, E^\alpha) = (E^a, E^\alpha, \bar{E}^{\dot{\alpha}}) = dZ^M E^A_M(Z),$$

(1.13)

where \(\alpha = 1, 2, 3, 4\) can be understood as Majorana spinor index.

In superspace the torsion 2–forms are defined as the covariant exterior derivatives of the bosonic and fermionic supervielbein forms

$$T^a := DE^a = dE^a - E^b \wedge w^a_b = \frac{1}{2} E^B \wedge E^C T^B_C^a$$

(1.14)

$$T^\alpha := DE^\alpha = dE^\alpha - E^\beta \wedge w^\alpha_\beta = \frac{1}{2} E^B \wedge E^C T^B_C^\alpha, \quad w^\alpha_\beta := \frac{1}{4} w^{ab} \sigma^{ab}_\alpha \beta,$$

(1.15)

$$T^{\dot{\alpha}} := DE^{\dot{\alpha}} = dE^{\dot{\alpha}} - E^{\dot{\beta}} \wedge w^{\dot{\alpha}}_\dot{\beta} = \frac{1}{2} E^B \wedge E^C T^B_C^{\dot{\alpha}}, \quad w^{\dot{\alpha}}_\dot{\beta} := \frac{1}{4} w^{ab} \tilde{\sigma}^{ab}_\dot{\alpha} \dot{\beta},$$

(1.16)

where \(w^{ab} = -w^{ba} = dZ^M w^{ab}_M(Z)\) is the spin connection 1-form , \(\sigma^{ab}_\beta_\alpha = \sigma^{[a\beta b]}\) and \(\tilde{\sigma}^{ab}_\beta_\alpha = \tilde{\sigma}^{[a\sigma b]}\) are antisymmetrized products of the relativistic Pauli matrices (see Appendix A) and \(d\wedge\) are exterior derivative and exterior product of differential forms previously defined in section 1.3. This later is antisymmetric for bosonic one forms, \(E^a \wedge E^b = -E^b \wedge E^a\), symmetric for two fermionic one forms, \(E^\alpha \wedge E^\beta = E^\beta \wedge E^\alpha\), and again antisymmetric for the product of bosonic and fermionic one forms, \(E^a \wedge E^\alpha = -E^\alpha \wedge E^a\) (see 1.9).

The torsion 2-forms obey the Bianchi identities

$$DT^a + E^b \wedge R^a_b = 0, \quad DT^\alpha + E^\beta \wedge R^\alpha_\beta = 0, \quad DT^{\dot{\alpha}} + E^{\dot{\beta}} \wedge R^{\dot{\alpha}}_{\dot{\beta}} = 0,$$

(1.17)

where

$$R^{ab} = (dw - w \wedge w)^{ab} = \frac{1}{2} E^B \wedge E^C R^B_C^{ab}$$

(1.18)

is the curvature 2-form. Its Bianchi identities read \(DR^{ab} = 0\).

1.4.1. Flat Superspace

In flat \(D = 4\) \(\mathcal{N} = 1\) \(\Sigma^{(4|4)}\) superspace, supervielbeins obey the constraints

$$T^a := dE^a = -2iE \wedge \sigma^a \tilde{E}, \quad T^\alpha := dE^\alpha = 0, \quad T^{\dot{\alpha}} := d\bar{E}^{\dot{\alpha}} = 0,$$

(1.19)

which can be solved by

$$E^a = dx^a - id\theta^a \sigma^a \tilde{E} + i\theta^a \sigma^a d\tilde{E}^{\dot{\alpha}}, \quad E^\alpha = d\theta^\alpha, \quad \bar{E}^{\dot{\alpha}} = d\bar{\theta}^{\dot{\alpha}},$$

(1.20)

expressing the supervielbein in terms of superspace coordinates (1.1). Decomposing the exterior derivative on the supervielbein basis

$$d = E^a D_a + \bar{E}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} + E^a D_a,$$

(1.21)
we obtain the expressions for supersymmetric covariant derivatives,

\[ D_a = \partial_a, \quad D_\alpha = \partial_\alpha + i(\sigma^a \bar{\theta})_\alpha \partial_a, \quad \bar{D}_\dot{\alpha} = \bar{\partial}_{\dot{\alpha}} + i(\theta \sigma^a)_{\dot{\alpha}} \partial_a = -(D_\alpha)^\ast. \]  

(1.22)

These obey the superalgebra with only one nontrivial (anti-)commutation relation,

\[ \{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i\sigma^a_{\alpha \dot{\alpha}} \partial_a, \]  

(1.23)

while the other (anti-)commutators vanish

\[ \{D_\alpha, D_\beta\} = 0 \quad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \]

\[ [\partial_\mu, D_\alpha] = 0 = [\partial_\mu, \bar{D}_{\dot{\alpha}}] \quad [\partial_\mu, \partial_\nu] = 0. \]  

(1.24)

1.5. Supergravity

Supergravity, a supersymmetric version of Einstein’s gravity, has local supersymmetry as gauge symmetry. As mentioned earlier in this chapter, supersymmetry can be realized on–shell or off–shell which means the supersymmetry algebra closes with/without using the equations of motion. This also applies to local supersymmetry. There are several different off–shell formulations of supergravity which differ by choices of auxiliary field sector. In this thesis we will use the so–called minimal supergravity. In chapter 3 we will use minimal \( N = 1 \ D = 4 \) off–shell supergravity in its superspace formulation which we are going to describe now.

1.5.1. Minimal Supergravity

The list of superspace constraints of minimal supergravity \([82, 151–153]\) is

\[ T^{\alpha \beta}_{\gamma} = -2i\sigma^a_{\alpha \beta}, \]

\[ T_{\alpha \beta}^A = 0 = T^{\alpha \beta}_{\dot{\alpha} \dot{\beta}}, \]

\[ T^{\alpha \beta}_{\dot{\gamma}} = 0, \]

\[ T^{ab}_{\dot{\gamma}} = 0, \]

\[ R^{\alpha \beta}_{\dot{\gamma}} = 0. \]  

(1.25)

With these constraints, the identities (1.17) express the torsion and curvature forms through the set of main superfields

\[ G_\alpha := 2i(T_{\alpha \beta}^{\gamma} - T^{\gamma}_{\alpha \beta}), \]  

(1.26)

\[ \bar{R} := -\frac{1}{3}R^{\alpha \beta}_{\gamma} = (R)^\ast, \]  

(1.27)

\[ W^{\alpha \beta \gamma} := 4i\tilde{\sigma}^{\gamma \dot{\gamma}} R^{\alpha \beta}_{\gamma} = W^{(\alpha \beta \gamma)} = (\tilde{W}^{(\alpha \beta \gamma)})^\ast. \]  

(1.28)

\(^1\)A minimal complete set of superspace constraints for the minimal supergravity multiplet \([81]\) can be found, e.g., in \([154, 156]\); see \([91, 157]\) for a discussion of the algebraic origin of the supergravity constraints.
The final expressions for the superspace torsion 2–forms are,
\[ T^a = -2i\sigma^a_{\alpha\beta} E^\alpha \wedge \bar{E}^\beta - \frac{i}{2} E^b \wedge E^c \epsilon^a_{\alpha\beta\gamma} G^d , \]  \( (1.29) \)
\[ T^\alpha = \frac{i}{8} E^c \wedge E^\beta (\sigma_c \sigma_d)_{\beta\alpha} G^d - \frac{i}{8} E^c \wedge \bar{E}^\beta \epsilon_{\beta\alpha} \sigma_{c\beta} R + \frac{i}{2} E^c \wedge E^b T_{bc} \gamma , \]  \( 1.30 \)
\[ T^{\dot{\alpha}} = \frac{i}{8} E^c \wedge E^\beta \epsilon_{\beta\dot{\alpha}} \sigma_{c\dot{\beta}} \bar{R} - \frac{i}{8} E^c \wedge \bar{E}^\beta (\bar{\sigma}_c \sigma_d)_{\beta\dot{\alpha}} G^d + \frac{i}{2} E^c \wedge E^b T_{bc} \dot{\alpha} . \]  \( 1.31 \)

The Bianchi identities also imply the following equations for the main superfields,
\[ \mathcal{D}_a \bar{R} = 0 , \quad \bar{\mathcal{D}}_a R = 0 , \]  \( 1.32 \)
\[ \mathcal{D}_a W^{\alpha\beta\gamma} = 0 , \quad \mathcal{D}_a \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} = 0 , \]  \( 1.33 \)
\[ \bar{\mathcal{D}}^\dot{\alpha} G_{a\dot{\alpha}} = \mathcal{D}_a R , \quad \bar{\mathcal{D}}^\dot{\alpha} G_{a\dot{\alpha}} = \bar{\mathcal{D}}_a \bar{R} , \]  \( 1.34 \)
\[ \mathcal{D}_\gamma W^{\alpha\beta\gamma} = \bar{\mathcal{D}}_\gamma D^{(\alpha} G^{\beta)\gamma} , \]  \( 1.35 \)
\[ \bar{\mathcal{D}}_\gamma \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} = \mathcal{D}_\gamma \bar{D}^{(\dot{\alpha}} (G^{\beta)\gamma} . \]  \( 1.36 \)

The superspace Riemann curvature 2–form can be decomposed on the anti-self-dual and self-dual parts enclosed in symmetric spin–tensor 2–forms \( R^{\alpha\beta} \) and \( R^{\dot{\alpha}\dot{\beta}} = (R^{\alpha\beta})^\ast \). As a consequence of the constraints \( (1.25) \)
\[ R^{\alpha\beta} = dw^{\alpha\beta} - w^{\alpha\gamma} \wedge w_\gamma \beta \equiv \frac{1}{4} R_{ab} (\bar{\sigma}_a \sigma_b)^{\alpha\beta} = \]  \[ = -\frac{i}{2} E^\alpha \wedge E^\beta \bar{R} - \frac{i}{8} E^c \wedge E^d \epsilon_{\alpha\beta} (\bar{\sigma}_c \sigma_d) \mathcal{D}_a \mathcal{D}_b \bar{R} + \]  \[ + \frac{i}{8} E^c \wedge E^\gamma (\sigma_c \bar{\sigma}_d)_{\gamma(\beta} D_{\alpha)} G^d - \]  \[ - \frac{i}{8} E^c \wedge \bar{E}^\beta \epsilon_{\beta\gamma} W^{\alpha\beta\gamma} + \frac{i}{2} E^d \wedge E^c R_{cd}^{\alpha\beta} . \]  \( 1.36 \)

In our conventions the spinor covariant derivatives are defined by the following decomposition of the covariant differential \( D \)
\[ D := E^A \mathcal{D}_A = E^a \mathcal{D}_a + E^\alpha \mathcal{D}_\alpha = \]  \[ = E^a \mathcal{D}_a + E^\alpha \mathcal{D}_a + E^{\dot{\alpha}} \bar{\mathcal{D}}_\dot{\alpha} . \]  \( 1.37 \)

(hence, \((D_\alpha)^\ast = -\bar{\mathcal{D}}_\alpha\)). The algebra of covariant derivatives \( D_A \), Eq. \( (1.37) \), is encoded in the Ricci identities
\[ DDV_A = R_A^B V_B \leftrightarrow \begin{cases} DDV_a = R_a^b V_b \, , \\ DDV_\alpha = R_\alpha^\beta V_\beta \, , \\ DDV_{\dot{\alpha}} = R_{\dot{\alpha}}^{\dot{\beta}} V_{\dot{\beta}} \, . \end{cases} \]  \( 1.38 \)

where \( V_A = (V_a, V_\alpha, V_{\dot{\alpha}}) \) is an arbitrary supervector with tangent superspace Lorentz indices. If we decompose them on the basic 2–forms \( E^A \wedge E^B \), one finds (see \([154, 155]\))
\[ [\mathcal{D}_A, \mathcal{D}_B] V_C = -T_{AB}^D \mathcal{D}_D V_C + R_{ABCD} V_D . \]  \( 1.39 \)
When the constraints (1.29), (1.30), (1.31), (1.32), (1.33), (1.34), (1.36) are taken into account, Eqs. (1.38) (or (1.39)) imply
\[
\{D_\alpha, D_\beta\} V_\gamma = -\bar{R}e^{\gamma(\alpha} V_{\beta)} , \quad (1.40)
\]
\[
\{D_\alpha, \bar{D}_\beta\} V_\gamma = -\bar{R}V_{(\alpha\delta)\beta}^\gamma , \quad (1.41)
\]
\[
\{D_\alpha, \bar{D}_\beta\} = 2i\sigma_{\alpha\beta}\bar{D}_a \equiv 2i\bar{D}_{\alpha\beta} , \quad \text{etc.} \quad (1.42)
\]

The Bianchi identities also allow to find the expression for superfield generalization of the gravitino field strength,
\[
T_{bc\alpha}(Z) = \epsilon_{abcd} T_{bc}(\bar{d}_a\sigma^d\delta_{ab}) = -\frac{1}{8}\epsilon_{abcd} \bar{D}((\delta_{ab}) G_{\gamma(\alpha}) - \frac{1}{8}\epsilon_{abcd}(W_{\alpha\beta\gamma} - 2\epsilon_{\gamma(\alpha} D_{\beta)} R) . \quad (1.43)
\]

As a result, the superfield generalization of the left hand side of the supergravity Rarita–Schwinger equation reads
\[
\epsilon_{abcd} T_{bc}(\bar{d}_a\sigma^d\delta_{ab}) = \frac{i}{8}\bar{d}_{\alpha\beta}\bar{D}(\delta_{ab} G_{\gamma(\alpha}) - \frac{1}{8}\epsilon_{abcd}(W_{\alpha\beta\gamma} - 2\epsilon_{\gamma(\alpha} D_{\beta)} R) . \quad (1.44)
\]

The superfield generalization of the Ricci tensor is
\[
R_{bc} = \frac{1}{32}(D^\delta\bar{D}((\delta_{ab})) - \bar{D}D((\delta_{ab} G_{\alpha})) - \frac{1}{8}\sigma_{\alpha\beta}\bar{D}_a^2 R) . \quad (1.45)
\]

This suggests that superfield supergravity equation should have the form
\[
G_a = 0 , \quad (1.46)
\]
\[
R = 0 , \quad \bar{R} = 0 . \quad (1.47)
\]

1.5.2. Superfield supergravity action and admissible variation of constrained supervielbein

The superfield action of the minimal off-shell formulation of \( D = 4, N = 1 \) supergravity [158] is given by the superdeterminant (or Berezinian) of the matrix of supervielbein coefficients, \( E_A^M(Z) \) in (1.13), which obey the set of supergravity constraints (1.29), (1.30), (1.31),
\[
S_{SG} = \int d^8Z E := \int d^4x d^4\theta sdet(E_A^M) . \quad (1.48)
\]

One can obtain the supergravity superfield equations (1.46), (1.47) by varying the superspace action (1.48). This is not straightforward because the supervielbein superfields are restricted by the constraints (1.29), (1.30), (1.31). There are two basic ways to solve this problem. The first consists in solving the superspace supergravity constraints in terms of unconstrained superfields— pre-potentials [89, 151, 153, 159] – the set of which in the case of minimal supergravity can be restricted to the axial vector superfield \( \mathcal{H}^\mu(Z) = \mathcal{H}^\mu(x, \theta, \bar{\theta}) \)
1.5. Supergravity

[151] and the chiral compensator superfield $\Phi(x + i\mathcal{H}(Z), \theta)$ [159].

Another way, called the Wess–Zumino approach to superfield supergravity [82] [155, 158, 160, 161], does not imply to solve the constraints but rather to solve the equations which appear as a result of requiring that the variations of the supervielbein and spin connection

$$\delta E_A^M(Z) = E_B^M B^A_M(\delta), \quad \delta w_{ab}^M(Z) = E_{ab}^C w_C^M(\delta),$$

(1.49)

preserve the superspace supergravity constraints [158], Eqs. (1.29), (1.30), (1.31).

Actually this procedure of finding admissible variations of the supervielbein and superspace spin connection is a linearized but covariantized version of the constraint solution used in pre-potential approach. The independent parameters of admissible variations clearly reflect the pre-potential structure of the off-shell supergravity. In the case of minimal supergravity their set contains [158] (see also [72]) $\delta \mathcal{H}^a$, corresponding to the variation of the axial vector pre-potential of [151], and complex scalar variations $\delta U$ and $\delta \bar{U}$ entering (1.49) only under the action of chiral projectors (as $(\mathcal{D} \mathcal{D} - \bar{R})\delta U$ and $(\mathcal{D} \mathcal{D} - R)\delta \bar{U}$) and thus corresponding to the variations of complex pre-potential of the chiral compensator of the minimal supergravity. The admissible variations of supervielbein read [72, 158]

$$\delta E^a = E^a(\Lambda(\delta) + \bar{\Lambda}(\delta)) - \frac{1}{4} E_b^a \hat{\sigma}^{\dot{\alpha}}_{\dot{\alpha}} [D_{a}, \bar{D}_{\dot{a}}] \delta H^a + i E^a D_{a} \delta H^a - i E^a \bar{D}_{\dot{a}} \delta H^a,$$

(1.50)

$$\delta E^a = E^a \Xi^a(\delta) + E^a \Lambda(\delta) + \frac{1}{8} E^a R \sigma_{\dot{a}} \delta H^a,$$

(1.51)

where

$$2\Lambda(\delta) + \bar{\Lambda}(\delta) = \frac{1}{4} \hat{\sigma}^{\dot{a}}_{\dot{a}} D_{a} \bar{D}_{\dot{a}} \delta H^a + \frac{1}{8} G_a \delta H^a + 3(\mathcal{D} \mathcal{D} - \bar{R})\delta U$$

(1.52)

and the explicit expression for $\Xi^a(\delta)$ in (1.51) can be found in [72]. Neither that nor the explicit expressions for the admissible variations of spin connection in (1.49) will be needed for our discussion along this thesis.

Indeed, the variation of superdeterminant of the supervielbein of the minimal supergravity superspace can be calculated using (1.50), (1.51) only and reads (see [158])

$$\delta E = E[-\frac{1}{12} \hat{\sigma}^{\dot{a}}_{\dot{a}} [D_{a}, \bar{D}_{\dot{a}}] \delta H^a + \frac{1}{6} G_a \delta H^a +$$

$$+2(\mathcal{D} \mathcal{D} - \bar{R})\delta U + 2(\mathcal{D} \mathcal{D} - R)\delta \bar{U}],$$

(1.53)

Taking into account the identity [158]

$$\int d^8 Z E (D_{a} \xi^A + \xi^B T_{BA})(-1)^A \equiv 0,$$

(1.54)

one finds the variation of the minimal supergravity action (1.48)

$$\delta S_{SG} = \int d^8 Z E \left[ \frac{1}{8} G_a \delta H^a - 2 R \delta U - 2 \bar{R} \delta \bar{U} \right].$$

(1.55)
This clearly produces the superfield supergravity equations of the form (1.46), (1.47) which result in the Rarita–Schwinger equation and Einstein equation without cosmological constant

$$
\epsilon^{abcd} T_{bc}^\alpha \sigma_{d\alpha\dot{\alpha}} = 0 , \quad \text{(1.56)}
$$

$$
R_{bc}^{ac} = 0 . \quad \text{(1.57)}
$$
In this chapter we will present the action for the dynamical $D = 4$ $\mathcal{N} = 1$ supermembrane in interaction with a dynamical scalar multiplet. We will show that this action is invariant under fermionic transformations called $\kappa$–symmetry reflecting that its ground state preserves a part of target space supersymmetry. The possibility for constructing such an action is related to the existence of a nontrivial Chevalley-Eilenberg 3–cocycle (i.e a supersymmetric invariant closed 4–form \cite{162, 163}) constructed from the scalar supermultiplet which enters the Wess-Zumino part of the action allowing us to couple the $D = 4$ $\mathcal{N} = 1$ supermembrane to scalar supermultiplet. To this end we will review the well known description of the scalar supermultiplet by chiral superfield in superspace. We present the supermembrane action in the off–shell scalar supermultiplet background and find the equations of motion. Then we discuss the special scalar supermultiplet contracted from real (instead complex) prepotential, which allows a dual description by 3–form potential $C'_3$ in flat $D = 4$ $\mathcal{N} = 1$ superspace. Just this special scalar supermultiplet can be coupled to supermembrane (beyond the background field approximation) as far as the above $C'_3$ pulled–back to supermembrane worldvolume $W^3$ serves to construct the Wess–Zumino term of the supermembrane action.

We describe the dynamical system of the special scalar supermultiplet interacting with supermembrane and obtain the equations of motion with supermembrane source. In the simplest case in which the scalar multiplet part of the action contains only the simplest kinetic term we also extract the equations of motion for the physical fields. We solve these dynamical field equations for the physical fields at leading order in supermembrane tension. We conclude this chapter by discussing the inclusion of nontrivial superpotential and the relation with known domain wall solutions.
2.1. Free supermembrane in flat $D = 4\,\mathcal{N} = 1$ superspace

The possibility to construct the $D = 4,\,\mathcal{N} = 1$ supermembrane action is related to that in $D = 4,\,\mathcal{N} = 1$ superspace there exists the following supersymmetric invariant closed 4-form\footnote{Recall that in our notation the exterior derivative acts from the right, so that for any $p$-form $\Omega_p$ and $q$-form $\Omega_q$, $d(\Omega_p \wedge \Omega_q) = \Omega_p \wedge d\Omega_q + (-)^q d\Omega_p \wedge \Omega_q$. See [11]}:

$$h_4 = dc_3 := \frac{i}{4} E^b \wedge E^e \wedge E^a \wedge E^\beta \sigma_{a\lambda} + \frac{i}{4} E^b \wedge E^e \wedge E^a \wedge \tilde{E}^\alpha \wedge \tilde{E}^\beta \sigma_{a\lambda}. \tag{2.1}$$

This describes a 3-cocycle which is nontrivial in Chevalley-Eilenberg (CE) cohomology [162, 163], which implies that $h_4$ is a supersymmetric invariant closed four form, $dh_4 = 0$ and, despite it can be expressed as an exterior derivative of a 3-form, $h_4 = dc_3$ (and, hence, is trivial cocycle of de Rahm cohomology), the corresponding 3-form $c_3$ is not invariant under supersymmetry.

The action for a free supermembrane in $D = 4,\,\mathcal{N} = 1$ superspace reads [76]

$$S_{p=2} = \frac{1}{2} \int d^3 \xi \sqrt{g} - \int W^3 = -\frac{1}{6} \int W^3 * \hat{E}_a \wedge \hat{E}^a - \int W^3 \hat{c}_3, \tag{2.2}$$

where, in the first line $g = det(g_{mn})$ is the determinant of the induced metric,

$$g_{mn} = \hat{E}_m \eta_{ab} \hat{E}_n^b, \quad \hat{E}_m^a := \partial_m \hat{x}^a - i \partial_m \hat{\theta}^a \sigma_{a\alpha} \hat{\theta}^\alpha + i \hat{\theta}^a \sigma_{a\alpha} \partial_m \hat{\theta}^\alpha, \tag{2.3}$$

$W^3$ is the supermembrane worldvolume the embedding of which into the target superspace $\Sigma^{[4|4]}$ is defined parametrically by the coordinate functions $\hat{z}^M(\xi) = (\hat{x}^a(\xi), \hat{\theta}^a(\xi), \hat{\theta}^\alpha(\xi))$; $\xi^m = (\xi^0, \xi^1, \xi^2)$ are local coordinates on $W^3$,

$$W^3 \subset \Sigma^{[4|4]} : \quad z^M = \hat{z}^M(\xi) = (\hat{x}^a(\xi), \hat{\theta}^a(\xi), \hat{\theta}^\alpha(\xi)). \tag{2.4}$$

Finally,

$$\hat{c}_3 := \frac{1}{3!} \hat{E}^{a_3} \wedge \hat{E}^{a_2} \wedge \hat{E}^{m_1} c_{a_1 a_2 a_3}(\hat{Z}) = \frac{1}{3!} d\xi^{m_3} \wedge d\xi^{m_2} \wedge d\xi^{m_1} \hat{c}_{m_1 m_2 m_3} = -\frac{1}{6} d^3 \xi \hat{c}_{m_1 m_2 m_3} \hat{c}_{m_1 m_2 m_3} \tag{2.5}$$

is the pull–back of the 3-form defined in Eq. (2.1) to $W^3$, so that the second, Wess–Zumino part of the action can be written in the form of (see [76])

$$\int W^3 \hat{c}_3 = -\frac{1}{6} \int d^3 \xi \hat{c}_{m_1 m_2 m_3} \hat{c}_{m_1 m_2 m_3}.$$

Here we consider only the case of closed supermembrane so that the worldvolume $W^3$
has no boundary, \( \partial W^3 = 0 \), and \( \int W^3 d(...) = 0 \). Then we do not need in the explicit form of \( \hat{e}_{m_1 m_2 m_3} \) in \eqref{2.5} as far as variation of its integral in \eqref{2.2} can be calculated (using the Lie derivative formula, \( \delta c_3 = i_\delta dc_3 + di_\delta c_3 \)) through its exterior derivative, the pull–back \( \hat{h}_4 := h_4(\hat{Z}) \) of the CE cocycle \eqref{2.1}, \( h_4 = dc_3 \).

In the second line of Eq. \eqref{2.2} we have written the first, Nambu-Goto term of the action as an integral of a differential three form. This is constructed from the pull–back of the bosonic vielbein form

\[
\hat{E}^a = d\xi^m \hat{E}^a_m, \quad \hat{E}^a_m := \partial_m \hat{x}^a - i\partial_m \hat{\theta}^\alpha \sigma^a_{\alpha\dot{\alpha}} \hat{\theta}^\dot{\alpha} + i\hat{\theta}^a \sigma^a_{\alpha\dot{\alpha}} \partial_m \hat{\theta}^\alpha ,
\]

using the worldvolume Hodge star operation,

\[
* \hat{E}^a := \frac{1}{2} d\xi^m \land d\xi^n \sqrt{g} \epsilon_{mnk} g^{kl} \hat{E}^a_l.
\]

The action \eqref{2.2} is invariant under the local fermionic \( \kappa \)–symmetry transformations. These have the form of

\[
\delta_\kappa x^\mu = i\kappa^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \theta^\dot{\alpha} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha} , \quad \delta_\kappa \theta^\alpha = \kappa^\alpha , \quad \delta_\kappa \bar{\theta}^\dot{\alpha} = \bar{\kappa}^\dot{\alpha} ,
\]

where the spinorial fermionic parameter \( \kappa^\alpha = \kappa^\alpha(\xi) = (\bar{\kappa}^\dot{\alpha})^* \) has actually only two independent components because it obeys the equations

\[
\bar{\kappa}^\dot{\alpha} = \kappa^\beta \bar{\gamma}^\beta\dot{\alpha} \quad \Leftrightarrow \quad \kappa^\alpha = \bar{\kappa}^\dot{\alpha} \bar{\gamma}^\dot{\beta}^\alpha
\]

with

\[
\bar{\gamma}^\beta^\dot{\alpha}\dot{\beta}^\alpha = \delta^\alpha_\beta
\]

which makes two equations in \eqref{2.9} equivalent.

To prove the \( \kappa \)--symmetry one has to use the identities

\[
\frac{1}{2} \hat{E}^c \land \hat{E}^h \land \hat{E}^a \sigma_{bc\alpha} = * \hat{E}_a \land \hat{E}^a(\sigma^a_0 \hat{Z})_{\alpha\beta}
\]

which allows to present the variation of the kinetic, Nambu-Goto type, and the Wess–Zumino terms in similar form.

It is convenient to write the \( \kappa \)--symmetry transformations in the form of

\[
i_\kappa \hat{E}^a := \delta_\kappa \hat{Z}^M E_M^a(\hat{Z}) = 0 , \quad \left\{ \begin{array}{l}
i_\kappa \hat{E}^\alpha := \delta_\kappa \hat{Z}^M E_M^\alpha(\hat{Z}) = \kappa^\alpha = \bar{\kappa}^\dot{\alpha} \bar{\gamma}^\dot{\beta}^\alpha , \\
i_\kappa \hat{E}^\dot{\alpha} := \delta_\kappa \hat{Z}^M E_M^\dot{\alpha}(\hat{Z}) = \bar{\kappa}^\dot{\alpha} = \kappa^\beta \bar{\gamma}^\beta^\dot{\alpha} . \end{array} \right.
\]
2.2. Scalar supermultiplet as described by chiral superfield

In this section we review the well known description of scalar supermultiplet by chiral superfield in superspace [154, 155, 164].

As noted in section 1.2 superfields are highly reducible representations of the supersymmetry algebra but it is possible to extract irreducible representations from them by imposing suitable covariant constraints. The simplest irreducible representation of the $D = 4, \mathcal{N} = 1$ supersymmetry, the scalar supermultiplet, is described by the chiral superfield, this is to say by complex superfield obeying the so-called chirality equation

$$\bar{D}_\alpha \Phi = 0 \ . \quad (2.14)$$

The complex conjugate (c.c.), $\bar{\Phi} = (\Phi)^*$, obeys

$$D_\alpha \bar{\Phi} = 0 \quad (2.15)$$

and is called anti–chiral superfield. The free equations of motion for the physical fields of a massless scalar supermultiplet ($\phi(x) = \Phi|_{\theta=0}$ and $i\bar{\psi}_\alpha(x) = D_\alpha \Phi|_{\theta=0}$) are collected in the superfield equation

$$DD\Phi := D^\alpha D_\alpha \Phi = 0 \ . \quad (2.16)$$

This equation and its c.c. can be derived from the action

$$S_{\text{kin}} = \int d^8z \, \Phi \bar{\Phi} \ , \quad (2.17)$$

where the superspace integration measure $d^8z = d^4x d^2\theta d^2\bar{\theta}$ is normalized as

$$d^8z = d^4x \bar{D}\bar{D} \, DD := d^4x \bar{D}\bar{D} D^\alpha D_\alpha \ . \quad (2.18)$$

Indeed, the variation of this functional reads $\delta S_{\text{kin}} = \int d^8z \, (\Phi \delta \bar{\Phi} + \bar{\Phi} \delta \Phi)$. As far as the variation of chiral superfield should be chiral, $\bar{D}_\alpha \delta \bar{\Phi} = 0$, and $D_\alpha \delta \Phi = 0$, we can equivalently write the action variation as $\delta S_{\text{kin}} = \int d^4x \, \bar{D}_\alpha \bar{D} \alpha (DD\Phi) \, \delta \bar{\Phi}) + \text{c.c.}$, which results in the equations of motion $DD\Phi = 0$.

The most general selfinteraction of the scalar supermultiplet is described by the superfield action

$$S_{\text{int}}[\Phi; \bar{\Phi}] = \int d^8z \, \mathcal{K}(\Phi, \bar{\Phi}) + \int d^6\zeta_L W(\Phi) + \int d^6\zeta_R \bar{W}(\bar{\Phi}) = \int d^4x \, \bar{D}\bar{D} \, DD \, \mathcal{K}(\Phi, \bar{\Phi}) + \int d^4x \, DVW(\Phi) + \int d^4x \, \bar{D}\bar{D} \, \bar{W}(\bar{\Phi}) \ . \quad (2.19)$$

Notice that, although the r.h.s. of this equation is not manifestly hermitian, its imaginary part is integral of complete derivative (as far as $DDDD = DDD\bar{D} - 4i\bar{\sigma}^A \alpha_\alpha \partial_a [D_\alpha, D_a]$) and, as such, can be ignored in our discussion.
where $\mathcal{K}(\Phi, \bar{\Phi})$ is an arbitrary function of chiral superfield and its complex conjugate called \textit{Kähler potential} and $W(\Phi) = (\bar{W}(\Phi))^*$ is an arbitrary holomorphic function of the complex scalar superfield $\Phi$ called \textit{superpotential}. This latter is chiral, $\bar{D}_\alpha W(\Phi) = W'(\Phi) \bar{D}_\alpha \Phi = 0$, and hence is integrated with chiral measure defined by $d^6\zeta_L = d^4\zeta DD$ (and $d^6\zeta_R = d^4\zeta \bar{D}D$).

To have the standard kinetic term for the scalar field of the supermultiplet, the Kähler potential is usually chosen to obey

$$K''_\Phi(\varphi, \bar{\varphi}) := \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \bar{\varphi}} K(\varphi, \bar{\varphi}) \neq 0 .$$

(2.20)

The superfield equations of motion following from the action $S_{s-int}[\Phi; \bar{\Phi}]$ (2.19) are

$$\bar{\mathcal{E}} := DD K'_\Phi + \bar{W}'_\Phi = 0 ,$$

(2.21)

$$\mathcal{E} := \bar{D} \bar{D} K'_\Phi + W'_\Phi = 0 ,$$

(2.22)

where prime denotes the derivative with respect to argument, $K'_\Phi := \frac{\partial K(\Phi, \bar{\Phi})}{\partial \Phi}$, $K''_\Phi := \frac{\partial^2 K(\Phi, \bar{\Phi})}{\partial \Phi \partial \bar{\Phi}}$, etc. These equations can be obtained by solving the chirality conditions (2.14) and (2.15) in terms of prepotential, generic complex superfield $P (= (\bar{P})^*)$

$$\Phi = \bar{D} \bar{D} P , \quad \bar{\Phi} = D D \bar{P} ,$$

(2.23)

and vary with respect to this prepotential and its complex conjugate,

$$\mathcal{E} = \frac{\delta S_{s-int}[\Phi; \bar{\Phi}]}{\delta P} , \quad \bar{\mathcal{E}} = \frac{\delta S_{s-int}[\Phi; \bar{\Phi}]}{\delta \bar{P}} .$$

(2.24)

### 2.2.1. Four form field strength constructed from the scalar supermultiplet

Having a chiral superfield $K$,

$$\bar{D}_\alpha K = 0 , \quad D_\alpha \bar{K} = 0 ,$$

(2.25)

one can construct the following supersymmetric invariant closed four form (CE cocycle) in flat $D = 4, \mathcal{N} = 1$ superspace [83]

$$F_4 = dC_3 := \frac{1}{4} E^b \wedge E^a \wedge E^\alpha \wedge E^\beta \sigma_{\alpha \beta \gamma} K + \frac{1}{4} E^b \wedge E^a \wedge \bar{E}^\alpha \wedge \bar{E}^\beta \bar{\sigma}_{\alpha \beta \gamma} K + \frac{1}{4!} E^c \wedge E^b \wedge E^a \epsilon_{abcd} \sigma^d_{\alpha \beta} \bar{D} \bar{D} \bar{K} + \frac{1}{4!} E^c \wedge E^b \wedge E^a \epsilon_{abcd} \bar{D} \bar{D} \bar{K} - \frac{1}{4!} E^d \wedge E^c \wedge E^b \wedge E^a \epsilon_{abcd} \left( \bar{D} \bar{D} \bar{K} - D D K \right) .$$

(2.26)
2.3. Supermembrane action in the scalar multiplet background

Notice that we intentionally have not used the notation $\Phi$ for chiral superfield to stress that, e.g. having a free chiral superfield of Eqs. (2.14) satisfying equations of motion (2.16), one can construct the three form using some holomorphic functions $K = K(\Phi)$, $\bar{K} = \bar{K}(\bar{\Phi})$ which obey $DDK = K''(\Phi)$, $D\alpha \Phi D\alpha \bar{\Phi}$ instead of (2.16).

Interestingly enough, $F_4$ in (2.26) can be considered as real part of the complex closed form $\mathcal{F}^L_4$,

$$F_4 = \Re(\mathcal{F}^L_4) := \frac{1}{2} (\mathcal{F}^L_4 + \mathcal{F}^R_4) , \quad (2.27)$$

$$\mathcal{F}^L_4 = \frac{1}{16} E^b \wedge E^a \wedge E^c \wedge E^\alpha \wedge E^\beta \sigma_{ab} \sigma_{\alpha \beta} \bar{K} + \frac{1}{16} E^b \wedge E^a \wedge E^\alpha \wedge E^\beta \sigma_{ab} \sigma_{\alpha \beta} \bar{K} + \frac{1}{16} \epsilon_{abcd} \bar{K} \bar{D}^a K +$$

$$+ \frac{1}{16} \epsilon_{abcd} \bar{K} \bar{D}^a K , \quad (2.28)$$

$$\mathcal{F}^R_4 = \frac{1}{16} E^b \wedge E^a \wedge \bar{E}^\alpha \wedge \bar{E}^\beta \sigma_{ab} \sigma_{\alpha \beta} K + \frac{1}{16} E^b \wedge E^a \wedge \bar{E}^\alpha \wedge \bar{E}^\beta \sigma_{ab} \sigma_{\alpha \beta} K +$$

$$+ \frac{1}{16} \epsilon_{abcd} \bar{K} \bar{D}^a K , \quad (2.29)$$

The fact that these forms are closed as a consequence of (2.25),

$$d\mathcal{F}^L_4 = 0 \quad \Leftrightarrow \quad D\alpha K = 0 , \quad (2.30)$$

$$d\mathcal{F}^R_4 = 0 \quad \Leftrightarrow \quad \bar{D}\alpha \bar{K} = 0 , \quad (2.31)$$

suggests the existence of the complex 3-form potentials $C^L_3$ and $C^R_3 = (C^L_3)^*$ such that $\mathcal{F}^L_4 = dC^L_3$ and $\mathcal{F}^R_4 = dC^R_3$.

To study a supermembrane in the background of scalar multiplet, which will be the subject of the next section, we do not need an explicit expression for $C_3$. However, we do need it to obtain the equations for the scalar multiplet fields with a source from supermembrane, so that we will come back to discussing the problem of constructing potentials in section 2.4.

2.3. Supermembrane action in the scalar multiplet background

The action of supermembrane in the background of a scalar multiplet can be written in the form

$$S_{p=2} = \frac{1}{2} \int d^3 \xi \sqrt{K \bar{K} \sqrt{g}} - \int_{W^3} \mathcal{C}_3 ,$$

$$= -\frac{1}{6} \int_{W^3} *\hat{E}_a \wedge \hat{E}^a \sqrt{K \bar{K}} - \int_{W^3} \mathcal{C}_3 , \quad (2.32)$$

where $\mathcal{C}_3$ is the pull–back of the $C_3$ potential defined by Eq. (2.26) involving the chiral superfields $K$ and $\bar{K}$ (2.25). For simplify we omit the hat symbol from the pull–backs of superfields here and below in the places where this cannot produce a confusion.
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The action (2.32) is invariant under the \( \kappa \)-symmetry transformations

\[
i_\kappa \hat{E}^\alpha = 0, \quad i_\kappa \hat{E}^\alpha = \kappa^\alpha, \quad i_\kappa \hat{E}^{\dot{\alpha}} = \bar{\kappa}_{\dot{\alpha}}, \tag{2.33}\]

with \( i_\kappa d\hat{Z}^M := \delta_\kappa \hat{Z}^M \), similar to ones in (2.13) and (2.8) but with the spinorial parameter obeying the reducibility conditions

\[
\bar{\kappa}_{\dot{\alpha}} = -i\kappa^\beta \bar{\gamma}_{\beta\dot{\alpha}} \sqrt{K/\bar{K}} \quad \Leftrightarrow \quad \kappa^\alpha = i\bar{\kappa}_{\dot{\alpha}} \tilde{\gamma}^{\dot{\alpha}} \sqrt{K/\bar{K}} \tag{2.34}
\]

defined by a projector which differs from the one in (2.9) by a (super)field dependent phase factor \( i\sqrt{K/\bar{K}} \).

Notice that, if we write the counterpart of the action (2.32) with an arbitrary function \( S(K, \bar{K}) \) instead of \( \sqrt{K/\bar{K}} \) and perform the fermionic variation (2.33) of such an action, we find that the local fermionic \( \kappa \)-symmetry parameter should obey the equations \( \kappa^\alpha \partial S/\partial K = i/2 \bar{\kappa}_{\dot{\alpha}} \tilde{\gamma}^{\dot{\alpha}} \) and \( \bar{\kappa}_{\dot{\alpha}} \partial S/\partial \bar{K} = \kappa^\alpha \tilde{\gamma}^{\alpha} \). This system of equations has a nontrivial solution when \( \partial S/\partial \bar{K} = 1/4 \partial S/\partial K \). This latter equation is solved by \( S(K, \bar{K}) = \sqrt{K/\bar{K}} \) so that the action (2.32) for scalar multiplet in supergravity background can be constructed from the requirement of the \( \kappa \)-symmetry.

### 2.3.1. Equations of motion for supermembrane in a background of an off-shell scalar supermultiplet

The supermembrane equations of motion can be obtained by varying the action (2.32) with respect to coordinate functions \( \hat{Z}^M(\xi) \), so that we can write them in the form of

\[
\frac{\delta S_{p=2}}{\delta \hat{Z}^M(\xi)} = 0. \tag{2.35}
\]

The convenient form of the bosonic and fermionic equations can be extracted by multiplying this on the inverse supervielbein, \( E^M_\alpha(\hat{Z}) \frac{\delta S_{p=2}}{\delta \hat{Z}^M(\xi)} = 0 \) and \( E^M_\alpha(\hat{Z}) \frac{\delta S_{p=2}}{\delta \hat{Z}^M(\xi)} = 0 \).

These combinations appear as the coefficients for \( i_\delta \hat{E}^\alpha := \delta \hat{Z}^M(\xi) E^M_\alpha(\hat{Z}) \), \( i_\delta \hat{E}^{\dot{\alpha}} := \delta \hat{Z}^M(\xi) \bar{E}^M_{\dot{\alpha}}(\hat{Z}) \) and \( i_\delta \hat{E}^\alpha := \delta \hat{Z}^M(\xi) E^M_{\dot{\alpha}}(\hat{Z}) \) in the integrand of the action variation. This implies the possibility to write the formal expression for supermembrane equations of motion in the form

\[
\frac{\delta S_{p=2}}{i_\delta \hat{E}^\alpha(\hat{Z}) \delta \hat{Z}^M(\xi)} = 0, \quad \frac{\delta S_{p=2}}{i_\delta \hat{E}^{\dot{\alpha}}(\hat{Z}) \delta \hat{Z}^M(\xi)} = 0, \quad \frac{\delta S_{p=2}}{i_\delta \hat{E}^\alpha(\hat{Z}) \delta \hat{Z}^M(\xi)} = 0. \tag{2.36}
\]
2.4. Superfield equations for the dynamical system of special scalar supermultiplet interacting with supermembrane

The straightforward calculation gives the following explicit form of these equations of motion

\[ *\hat{E}_a \wedge \left( i \hat{E}^\alpha \sigma_{a\dot{a}}^\alpha \sqrt{K \bar{K}} - \hat{E}_\beta (\bar{\sigma}^\dot{a} \gamma^\dot{a})_\alpha \hat{B} K \right) + \frac{1}{12} * \hat{E}_a \wedge \hat{E}^a \left( \sqrt{K \bar{K}} D_\alpha K - i \gamma_{\alpha \dot{a}} \bar{D}^\dot{a} \hat{K} \right) = 0 , \]

(2.37)

\[ *\hat{E}_a \wedge \left( i \hat{E}^\alpha \sigma_{a\dot{a}}^\alpha \sqrt{K \bar{K}} - \hat{E}_\beta (\bar{\sigma}^\dot{a} \gamma^\dot{a})_\alpha \hat{B} K \right) + \frac{1}{12} * \hat{E}_a \wedge \hat{E}^a \left( \sqrt{K \bar{K}} D_\alpha K - i D^\alpha K \gamma_{\alpha \dot{a}} \right) = 0 , \]

(2.38)

\[ D(*\hat{E}_a) = \frac{1}{6} * \hat{E}_b \wedge \hat{E}^b \left( D_\alpha \ln \hat{K} + D_\alpha \ln \hat{K} \right) + \frac{1}{2} * \hat{E}_a \wedge (d \ln \hat{K} + d \ln \hat{K}) - \]

\[ - \frac{i}{12 \sqrt{K \bar{K}}} \hat{E}^d \wedge \hat{E}^c \wedge \hat{E}^b \epsilon_{abcd} (\bar{D} \bar{D} \hat{K} - D D K) - \]

\[ - \frac{1}{4 \sqrt{K \bar{K}}} \hat{E}^c \wedge \hat{E}^b \wedge \epsilon_{abcd} \sigma_{a\dot{a}}^d (\hat{E}^\alpha \bar{D} \hat{\alpha} \hat{K} + \hat{E}^\hat{\alpha} D^\alpha \hat{K}) - \]

\[ \hat{E}^b \wedge \hat{E}^a \wedge \hat{E}^3 \sigma_{ab\dot{a} \dot{b}} \sqrt{K \bar{K}} - \hat{E}^b \wedge \hat{E}^3 \sigma_{ab\dot{a} \dot{b}} \sqrt{K \bar{K}} = 0 . \]

(2.39)

Notice that the above equations of motion are not independent. According to the second Noether theorem, the gauge symmetries of a dynamical system result in the so-called Noether identities relating the left-hand sides of equations of motion of this system. The supermembrane possesses a number of gauge symmetries, including the local fermionic \( \kappa \)-symmetry (2.33), (2.34). This is reflected by the fact that contracting our fermionic equation (2.37) with \( i \sqrt{K \bar{K}} \gamma^{\dot{a} \dot{b}} \) we arrive at Eq. (2.38). Denoting the left hand sides of equations (2.37) and (2.38) by \( \Psi_\alpha \) and \( \bar{\Psi}_\dot{\alpha} \), respectively, we can write the above described Noether identity for the \( \kappa \)-symmetry in the form of

\[ \gamma^{\dot{a} \dot{b}} \Psi_\alpha \equiv - i \sqrt{K \bar{K}} e^{\beta \dot{\alpha}} \bar{\Psi}_\dot{\alpha} . \]

(2.40)

2.4. Superfield equations for the dynamical system of special scalar supermultiplet interacting with supermembrane

2.4.1. Special scalar multiplet and its dual three form potential

In our discussion below we will be considering not generic but special scalar multiplet described by the chiral superfield constructed from the real prepotential \( V = (V)^* \),

\[ \Phi = \bar{D} \bar{D} V , \quad \bar{\Phi} = D D V . \]

(2.41)
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On the level of auxiliary fields the distinction of this special case is that one of the real auxiliary scalars of the generic scalar multiplet is replaced in it by a divergence of a real vector, \( \partial_\mu k^\mu \) or, equivalently, by the field strength of a three form potential \( k_{\mu\nu\rho} = k^\mu \epsilon_{\mu\nu\rho} \) (in this latter form it was described in \([83]\) and, as one of "variant superfield representations", in \([86]\)).

Indeed, the complex prepotential \( P \) of the generic chiral multiplet, \( \Phi = \bar{D}D\bar{P}, \Phi = DD\bar{P}, \) is defined up to the gauge transformations, \( \bar{P} \mapsto \bar{P} + \bar{D}_\alpha \Xi^\alpha \). These imply that the imaginary part of the generic prepotential is transformed by \( \Im m \bar{P} := (\bar{P} - \bar{P})/2i \mapsto \Im m \bar{P} + (\bar{D}_\alpha \Xi^\alpha - \bar{D}_\alpha \Xi^\alpha)/2i \). Hence not-pure gauge parts of the superfield parameter \( \Im m \bar{P} \) are the ones which do not have their exact counterparts in the composed superfield \( (\bar{D}_\alpha \Xi^\alpha - \bar{D}_\alpha \Xi^\alpha)/2i \). One can check that the superfield parameter \( \bar{D}_\alpha \Xi^\alpha - \bar{D}_\alpha \Xi^\alpha \) has all the components but one having contributions of different independent functions without derivatives. The only exception is the highest component in its decomposition which reads \( -4i\theta \bar{\theta} \bar{\theta} \partial_\alpha (k^\alpha + \bar{k}^\alpha) \) and includes the divergence of the real part of the complex vector \( k_\alpha = \bar{\sigma}^{\alpha\dot{\alpha}}(\bar{D}_\alpha \Xi_{\dot{\alpha}})_{\theta=0} - (\bar{k}_\alpha)^{\dot{\alpha}} \) versus an arbitrary function in a generic real scalar superfield, like \( \bar{P} \). Then one can guess that the equations of motion for the special scalar supermultiplet will differ from the set of equations for a generic scalar multiplet, Eqs. (2.22) and (2.21), are, respectively, anti-chiral and chiral, \( \bar{D}_\alpha \bar{E} = 0 \) and \( \bar{D}_\alpha \bar{E} = 0 \), Eq. (2.42) implies that the imaginary part of the complex equation (2.22) is equal to a constant,\n
\[ \bar{E} + \bar{\bar{E}} = 0 \quad \Rightarrow \quad \partial_\alpha (\bar{E} - \bar{\bar{E}}) = 0 \quad \Rightarrow \quad \partial_\alpha (\bar{E} - \bar{\bar{E}}) = 0 . \quad (2.43) \]

Hence the only effect of the use of the special chiral superfields (2.41) instead of the generic scalar superfield (2.23) is that the equation \( \bar{E} = 0 \) is replaced by \( \bar{E} = -ic \) where \( c \) is an arbitrary real constant.

Equations of motion of special scalar multiplet

The variation of the general action (2.19) for the special chiral superfields (2.41) with respect to real prepotential \( \bar{V} \) apparently produces only the real part of the complex equation (2.22),

\[ \frac{\delta S_{s-int}[\bar{D}D\bar{V}; DD\bar{V}]}{\delta \bar{V}} = 0 \quad \Rightarrow \quad \bar{E} + \bar{\bar{E}} : = \bar{D}D\bar{K}_\bar{\Phi} + \bar{D}D\bar{K}_\Phi^* + W^\prime_\Phi + W_\Phi^* = 0 . \quad (2.42) \]

However, as far as the left hand sides of the equations of motion for generic scalar multiplet, Eqs. (2.22) and (2.21), are, respectively, anti-chiral and chiral, \( \bar{D}_\alpha \bar{E} = 0 \) and \( \bar{D}_\alpha \bar{E} = 0 \), Eq. (2.42) implies that the imaginary part of the complex equation (2.22) is equal to a constant,

\[ \bar{E} + \bar{\bar{E}} = 0 \quad \Rightarrow \quad \partial_\alpha (\bar{E} - \bar{\bar{E}}) = 0 \quad \Rightarrow \quad \partial_\alpha (\bar{E} - \bar{\bar{E}}) = 0 . \quad (2.43) \]

Hence the only effect of the use of the special chiral superfields (2.41) instead of the generic scalar superfield (2.23) is that the equation \( \bar{E} = 0 \) is replaced by \( \bar{E} = -ic \) where \( c \) is an arbitrary real constant.
2.4. Superfield equations for the dynamical system of special scalar supermultiplet interacting with supermembrane

On spontaneous supersymmetry breaking

The presence of this arbitrary constant in the right hand side of the superfield equations of motion, \( \mathcal{E} = -ic \), actually suggests a possible spontaneous supersymmetry breaking in the theory of special chiral multiplet. To clarify this, let us discuss the simple case of a free massless special scalar multiplet, in which the action reads \( \int d^8 z \bar{\Phi} \Phi = \int d^8 z D D V \bar{D} D V \) so that \( \bar{\mathcal{E}} = D D \Phi \) and the equations of motion (2.42) simplify to

\[
D D \Phi + \bar{D} \bar{D} \Phi = 0 .
\] (2.44)

As it has been discussed above (Eq. (2.43)) these equations lead to \( D_\alpha (D D \Phi - \bar{D} \bar{D} \Phi) = 0 \) and \( \partial_\alpha (D D \Phi - \bar{D} \bar{D} \Phi) = 0 \). Algebraically all this set of equations is solved by \( D D \Phi = -ic \) with the above mentioned arbitrary real constant \( c \),

\[
D D \Phi + \bar{D} \bar{D} \Phi = 0 \quad \Rightarrow \quad D D \Phi = -ic , \quad c = \text{const} .
\] (2.45)

In particular, this constant enters the solution of auxiliary field equations which now reads

\[
D D \Phi |_0 = -ic , \quad c = \text{const} .
\] (2.46)

As the on-shell supersymmetry transformations of the fermionic fields \( \psi_\alpha = -i D_\alpha \Phi |_0 \) are obtained from the off-shell ones, \( \delta \psi_\alpha = \frac{1}{2} \varepsilon_\beta (D D \Phi |_0 + 2 (\sigma^a \varepsilon)_\alpha \partial_\alpha \phi) \), by inserting the above solution of the auxiliary field equations, they read

\[
\delta \psi_\alpha = \frac{c}{2} \varepsilon_\beta + 2 (\sigma^a \varepsilon)_\alpha \partial_\alpha \phi .
\] (2.47)

Hence, for nonvanishing value of \( c \), the on-shell supersymmetry transformations of \( \delta \psi_\alpha \) contains the additive contribution of supersymmetry parameter \( \varepsilon_\beta \) characteristic of the transformation rules of the Volkov-Akulov Goldstone fermion [15] [16], the presence of which may be considered as an indication of the spontaneous supersymmetry breaking.

However, studying more carefully the case of free special scalar multiplet, one finds that such a spontaneous symmetry breaking actually does not occur if nontrivial boundary conditions are not introduced. Indeed, the constant in the superfield equations (2.45) can be reproduced from the generic scalar supermultiplet action which includes the superpotential linear in chiral superfield, \( W(\Phi) = -ic \Phi \). As it was observed already in [165], such a term can be removed from the action by a field redefinition. However, the boundary term contribution may change the situation; this role can be also played by supermembrane contribution. Further discussion on spontaneous supersymmetry breaking in the interacting system of scalar multiplet and supermembrane goes beyond the scope of this thesis. We turn to the three form potential presentation of the special chiral supermultiplet.
Chapter 2. \(D = 4\, N = 1\) Supermembrane interaction with dynamical Scalar Superfield

### Dual three form potential

The four form field strength constructed with the use of special scalar multiplet (2.41) is obtained from (2.26) by substituting

\[
\bar{K} = \bar{\Phi} = \bar{D}\bar{D}V, \quad K = \Phi = \bar{D}\bar{D}V.
\]  

It reads

\[
F_4 = dC_3' = \frac{1}{4} E^c \wedge E^a \wedge E^\alpha \wedge \sigma_{\alpha\beta} D^\beta D^\alpha V + \frac{1}{4} E^c \wedge E^a \wedge E^\alpha \wedge \E^\beta \tilde{\sigma}_{\alpha\beta} \bar{D}\bar{D}V + \frac{1}{4!} E^c \wedge E^a \wedge E^\alpha \wedge \epsilon_{\alpha\beta\gamma} \left( D^\beta \bar{D}\bar{D}D^\gamma V - \bar{D}\bar{D}D^\alpha \right).
\]  

The corresponding 3-form potential \(C_3'\) can be written in terms of the real prepotential as follows

\[
C_3' = 2i E^c \wedge E^\alpha \wedge \tilde{E}^\beta \sigma_{\alpha\beta} V + \frac{1}{2} E^c \wedge E^\alpha \wedge \sigma_{\beta\alpha} \bar{D}_{\beta} V - \frac{1}{2} E^c \wedge E^\alpha \wedge \tilde{\sigma}_{\alpha\beta} \bar{D}_{\alpha} V - \frac{1}{4!} E^c \wedge E^\alpha \wedge \epsilon_{\alpha\beta\gamma} D_{\beta} [D_{\alpha}, \bar{D}^\gamma] V.
\]  

Of course, this expression can be changed on an equivalent one using gauge transformations \(\delta C_3 = d\alpha_2\). These do not change the field strength (2.49) and are responsible for the possibility to do not have the lower dimensional form contributions (\(\propto E^\alpha \wedge E^\beta \wedge E^\gamma\) etc.) in the above \(C_3'\).

The existence of this simple three form \(C_3'\) giving a dual description of the special chiral supermultiplet (2.41) is the main reason to restrict our discussion below by this special case.

#### 2.4.2. Superfield equations of motion for interacting system

Let us consider the most general interaction of the special scalar supermultiplet with supermembrane as described by the action (2.32) with \(K = \Phi = \bar{D}_{\alpha} D^\alpha V\) and \(\bar{K} = \bar{\Phi} = \bar{D}\bar{D}V\) as in (2.48), i.e.

\[
S = \int d^8z K(\Phi, \bar{\Phi}) + \int d^6\xi L W(\Phi) + c.c. + \frac{1}{2} \int d^8\xi \sqrt{g} \sqrt{\Phi \bar{\Phi}} - \int \hat{\mathcal{C}}_3' = \int d^4x \bar{D}\bar{D}D\mathcal{K}(\Phi, \bar{\Phi}) + \int d^4x (D\bar{D}W(\Phi) + c.c.) - \frac{1}{6} \int \sqrt{\hat{E}^a \wedge \hat{E}^\alpha \sqrt{\Phi \bar{\Phi}}} - \int \hat{\mathcal{C}}_3' \]  

with special chiral superfield (2.41),

\[
\Phi = \bar{D}_{\alpha} D^\alpha V, \quad \bar{\Phi} = D^\alpha D_{\alpha} V.
\]  

The variation of the interacting action (2.51) with respect to supermembrane variables
2.4. Superfield equations for the dynamical system of special scalar supermultiplet interacting with supermembrane

gives formally the same equations of motion as for the supermembrane in the background, (2.37)–(2.39), but with \( K = \Phi = D D V \).

\[
\begin{align*}
* \hat{E}_a \wedge \left( i \hat{E}^a \sigma^a_{\alpha \dot{\alpha}} \sqrt{\Phi \Phi} - \hat{E}_\beta (\hat{\gamma}^\beta \hat{\sigma}^a_{\alpha \dot{\alpha}}) \hat{\Phi} \right) & + \frac{1}{12} * \hat{E}_a \wedge \hat{E}^a \left( \sqrt{\Phi / \Phi} D_\alpha \hat{\Phi} - i \gamma_{\alpha \dot{\alpha}} \hat{D}^a \hat{\Phi} \right) = 0, \\
* \hat{E}_a \wedge \left( i \hat{E}^a \sigma^a_{\alpha \dot{\alpha}} \sqrt{\Phi \Phi} - \hat{E}_\beta (\hat{\gamma}^\beta \hat{\sigma}^a_{\alpha \dot{\alpha}}) \hat{\Phi} \right) & + \frac{1}{12} * \hat{E}_a \wedge \hat{E}^a \left( \sqrt{\Phi / \Phi} D_\alpha \hat{\Phi} - i D^a \Phi \hat{\gamma}_{\alpha \dot{\alpha}} \right) = 0,
\end{align*}
\]

(2.53)

\[
\begin{align*}
D(* \hat{E}_a) & - \frac{1}{6} * \hat{E}_b \wedge \hat{E}_c \left( D_a \ln \Phi + D_a \ln \bar{\Phi} \right) + \frac{1}{2} * \hat{E}_a \wedge (d \ln \Phi + d \ln \bar{\Phi}) - \\
& - \frac{i}{12 \sqrt{\Phi \bar{\Phi}}} \hat{E}^d \wedge \hat{E}^c \wedge \hat{E}^b \epsilon_{\alpha \beta \gamma \delta} (\hat{D} \hat{D} \hat{\Phi} - D D \Phi) - \\
& - \frac{1}{4 \sqrt{\Phi \bar{\Phi}}} \hat{E}^c \wedge \hat{E}^b \wedge \epsilon_{\alpha \beta \gamma \delta} \hat{\sigma}^a_{\alpha \dot{\alpha}} \left( \hat{E}^a \hat{D}^a \hat{\Phi} + \hat{E}_a \hat{D}^a \hat{\Phi} \right) - \\
& - \hat{E}_b \wedge \hat{E}_c \wedge \hat{E}^d \sigma_{\alpha \beta} \sqrt{\Phi / \Phi} - \hat{E}_b \wedge \hat{E}_c \wedge \hat{E}^d \hat{\sigma}_{\alpha \beta} \sqrt{\Phi / \Phi} = 0.
\end{align*}
\]

(2.54)

(2.55)

However, the target superspace superfields the pull–backs of which enter these equations have to be the solutions of interacting equations with the source terms from the supermembrane. These superfield interacting equations read

\[
\mathcal{E} + \bar{\mathcal{E}} = J,
\]

(2.56)

where

\[
\mathcal{E} = \hat{D} \hat{D} \hat{\Phi} \mathcal{K}_{\Phi \Phi}''(\Phi, \bar{\Phi}) + D_\alpha \hat{\Phi} \hat{D}^a \hat{\Phi} \mathcal{K}_{\Phi \Phi \Phi}'''(\Phi, \bar{\Phi}) + W'(\Phi)
\]

(2.57)

(see (2.22) and (2.42)) and

\[
J(z) = - \frac{\delta S_{p=2}}{\delta V(z)}
\]

(2.58)

is the current superfield from the supermembrane. The problem of obtaining the complete set of interacting equations for the dynamical system of supermembrane and special chiral supermultiplet is now reduced to the problem of calculating this supermembrane current.

\[3\text{To simplify the expressions, we omitted the hat symbol from the pull–backs of superfields and their derivatives in equations (2.54) and (2.55), but, in contrast, left all the pull–back symbols in equation (2.53) so that one can appreciate simplification comparing this with its complex conjugate Eq. (2.54).} \]
2.4.3. Supermembrane current

The supermembrane current is split naturally on the contributions from the Nambu–Goto and the Wess–Zumino terms of the action \(2.32\) with \(2.41\)

\[
J(z) = J^{NG}(z) + J^{WZ}(z) = -\frac{\delta S_{p=2}}{\delta V(z)}
\]

(2.59)

The Nambu-Goto part of the current

\[
J^{NG}(z) := -\frac{\delta}{\delta V(z)} \frac{1}{2} \int d^3\xi \sqrt{g} \sqrt{DDV \bar{D}DV}
\]

(2.60)

\(DDV := D^\alpha D_\alpha V(z)|_{z^M = \hat{z}^M(\xi)}\) is calculated by first using the properties of the superspace delta function

\[
\delta^8(z) := \frac{1}{16} \delta^4(x) \theta \theta \bar{\theta} \bar{\theta}, \quad \int d^8z \delta^8(z-z') f(z) = f(z')
\]

(2.61)

Then the calculation reduces to using the definition of variation \(\delta V(z') = \delta^8(z' - z)\) and performing the superspace integration. In such a way one arrives at

\[
J^{NG}(z) = -\frac{\delta}{\delta V(z)} \frac{1}{2} \int d^3z' \sqrt{DDV(z')} \bar{D}DV(z') \int d^3\xi \sqrt{g} \delta^8(z' - \hat{z}) .
\]

(2.62)

2.5. Simplest equations of motion for spacetime fields interacting with dynamical supermembrane

Having the superfield equations with supermembrane current contributions, the next stage is to extract the equations of motion for the physical fields of the supermultiplet. We will do this for the simplest case when the special scalar multiplet part of the interacting action...
2.5. Simplest equations of motion for spacetime fields interacting with dynamical supermembrane

is given by the kinetic term (2.19) only, this is to say for the interacting system described by the action

\[ S = S_{\text{kin}} + S_{p=2} = \int d^8 z \Phi \bar{\Phi} + \frac{1}{2} \int d^3 \xi \sqrt{\hat{\Phi} \hat{\Phi}} \sqrt{\hat{g}} - \int_{W^3} \hat{C}_3' \]  

(2.65)

where \( \Phi = \bar{D} \bar{D} V, \bar{\Phi} = D D V \) (2.52) and \( \hat{C}_3' \) is the pull–back to \( W^3 \) of the 3-form \( C'_3 \) defined in (2.50). The interacting equations of motion for the bulk superfields, Eqs. (2.57), in this case simplifies to

\[ D D \Phi + \bar{D} \bar{D} \bar{\Phi} = J(z) \]  

(2.66)

where the current \( J(z) \) is given by (2.59), (2.63) and (2.64).

2.5.1. General structure of the simplest special scalar multiplet equations with a superfield source

Superfield equation (2.66) encodes the dynamical equations for the physical fields of the scalar multiplet, \( \phi(x) = \Phi|_0 \) and \( \psi_{\alpha}(x) = -i (D_{\alpha} \Phi)|_0 \), as well as algebraic equations for auxiliary fields \( D D \Phi|_0 \) and \( \bar{D} \bar{D} \bar{\Phi}|_0 \). These latter include the leading component of the real superfield equation (2.66)

\[ D D \Phi|_0 + \bar{D} \bar{D} \bar{\Phi}|_0 = J(z)|_0 \]  

(2.67)

as well as the first order equation

\[ \partial_a (D D \Phi|_0 - \bar{D} \bar{D} \bar{\Phi}|_0) = -i \frac{1}{4} \zeta_a [D_{\alpha}, \bar{D}_{\dot{\alpha}}] J(z)|_0 . \]  

(2.68)

The set of dynamical field equations include the Dirac (actually Weyl) equation with the source from supermembrane,

\[ \sigma_{\alpha \dot{\alpha}} \partial_a \psi^a := -i \partial_{\alpha \dot{\alpha}} D_{\alpha} \Phi|_0 = \frac{1}{4} \bar{D}_{\dot{\alpha}} J(z)|_0 , \]  

(2.69)

and the Klein–Gordon equation, also with the source,

\[ \Box \phi(x) := \Box \Phi|_0 = -\frac{1}{16} \bar{D} \bar{D} J(z)|_0 . \]  

(2.70)

Now, to specify supermembrane contributions to the scalar multiplet field equations we have to calculate the derivatives of the supermembrane current.

2.5.2. Dynamical scalar multiplet equations with supermembrane source contributions
Chapter 2. \( D = 4 \mathcal{N} = 1 \) Supermembrane interaction with dynamical Scalar Superfield

Auxiliary field equations

The leading components \( J|_0 \) of the current \( J \) in Eq. (2.67) is the sum of

\[
J^{NG}|_0 = \frac{1}{16} \sqrt{\phi} \frac{1}{H} \int d^3\xi \sqrt{g} \hat{\theta} \hat{\theta} \delta^4(x - \hat{x}) + \frac{1}{16} \sqrt{\phi} \frac{1}{H} \int d^3\xi \sqrt{g} \hat{\theta} \tilde{\theta} \delta^4(x - \hat{x}) \tag{2.71}
\]

and

\[
J^{WZ}(Z)|_0 = \frac{1}{48} \int_{W^3} \hat{E}^a \wedge \hat{E}^b \wedge \hat{E}^c \hat{\epsilon}_{abc} \hat{\theta} \sigma^d \hat{\theta} \delta^4(x - \hat{x}) + \mathcal{O}(f^4) \tag{2.72}
\]

where \( \mathcal{O}(f^4) \) denotes the terms of the fourth order in fermions (in this case, these are worldvolume fermionic fields \( \hat{\theta}, \hat{\tilde{\theta}} \) and their worldvolume derivatives, \( \partial_m \hat{\theta} := \partial \hat{\theta}/\partial x^m \) and c.c.); the explicit form of these one can find in the Appendix B (Eq. (B.10)).

Substituting the above expressions into Eq. (2.67), one finds that the real part of the auxiliary fields of the chiral multiplet has quite a complex form in terms of supermembrane variables

\[
DD \Phi|_0 + \tilde{D} \tilde{D} \Phi|_0 = \frac{1}{16} \sqrt{\phi} \frac{1}{H} \int d^3\xi \sqrt{g} \hat{\theta} \hat{\tilde{\theta}} \delta^4(x - \hat{x}) + \frac{1}{16} \sqrt{\phi} \frac{1}{H} \int d^3\xi \sqrt{g} \hat{\theta} \tilde{\theta} \delta^4(x - \hat{x}) + \frac{1}{48} \int_{W^3} \hat{E}^a \wedge \hat{E}^b \wedge \hat{E}^c \hat{\epsilon}_{abc} \hat{\theta} \sigma^d \hat{\theta} \delta^4(x - \hat{x}) + \mathcal{O}(f^4) \tag{2.73}
\]

where \( \mathcal{O}(f^4) \) are the same as in Eq. (2.72) (and thus can be read off Eq. (B.10)).

The second auxiliary field equation, Eq. (2.68), reads

\[
\partial_a (DD \Phi|_0 - \tilde{D} \tilde{D} \Phi|_0) = -\frac{i}{8} \cdot \frac{4!}{4} \int_{W^3} \hat{E}^d \wedge \hat{E}^c \wedge \hat{E}^b \hat{\epsilon}_{abc} \delta^4(x - \hat{x}) + \mathcal{O}(f^2) \tag{2.74}
\]

where the terms of higher order in fermions, \( \mathcal{O}(f^2) \) can be found in Eqs. (B.18) and (B.19) of Appendix B (multiplying the expressions presented there by \( -\frac{1}{3} \sigma^{abc}_a \)). On the first look it might seem that Eq. (2.74) imposes additional restrictions on the supermembrane motion. Such possible restrictions might come from the selfconsistency condition of Eq. (2.74); at zero order in fermions that reads \[^4\]

\[
\partial_a [\epsilon_{b|c_1c_2c_3} \int_{W^3} d\hat{x}^{c_3} \wedge d\hat{x}^{c_2} \wedge d\hat{x}^{c_1} \delta^4(x - \hat{x})] = 0 \tag{2.75}
\]

However, one can check that this equation is satisfied identically. Indeed, using the identity \( \epsilon_{c_1c_2c_3} [\partial_b] \equiv -\frac{2}{3} \epsilon_{ab[c_1c_2} \partial_{c_3]} \) one can write the l.h.s. of Eq. (2.75) in the form of

\[
-\frac{2}{3} \epsilon_{abc_1c_2} \int_{W^3} d\hat{x}^{c_2} \wedge d\hat{x}^{c_1} \wedge d\delta^4(x - \hat{x}) = -\frac{2}{3} \epsilon_{abc_1c_2} \int_{W^3} d(d\hat{x}^{c_2} \wedge d\hat{x}^{c_1} \delta^4(x - \hat{x}))
\]

which

\[^4 \int_{W^3} \hat{E}^d \wedge \hat{E}^c \wedge \hat{E}^b \hat{\epsilon}_{abc} \delta^4(x - \hat{x}) = \int_{W^3} d\hat{x}^{c_2} \wedge d\hat{x}^{c_1} \wedge d\hat{x}^{c_1} \delta^4(x - \hat{x}) + \text{fermionic contributions.} \]
vanishes as an integral of total derivative in the case of closed supermembrane which we are studying here (\(\partial W^3 = 0 \Rightarrow \int_{W^3} d(...) = 0\)).

As we have discussed in section 2.4.1, the on-shell transformations are obtained from the off-shell ones, \(\delta \psi_\alpha = 2(\sigma^a \bar{\epsilon})_\alpha \partial_a \phi + \frac{i}{2} \epsilon_{\beta \gamma} \partial \Phi |_0\) for the case of fermions, by substituting the solution of the equations for the auxiliary fields. This implies that the on-shell supersymmetry transformation of fermions will be quite complicated due to the complicated structure of the auxiliary field equations (2.73) and (2.68). As the on-shell fermionic supersymmetry transformations can be used to extract BPS conditions for the supersymmetric solutions, their further study in our simple system might lead to useful suggestions for the investigation of the backreaction of D=10,11 super-p-branes on the BPS solutions of supergravity equations.

**Dynamical field equations**

Fortunately, the dynamical equations for the physical fields of the special scalar multiplet following from the simplest interacting action Eq. (2.65) do not obtain contributions from the auxiliary fields of the scalar multiplet, which on the mass shell are expressed by quite complicated Eqs. (2.73) and (2.68).

Specifying the current contributions to (2.69) and (2.70), we find the massless Dirac equation with the supermembrane contributions,

\[
\sigma^{a}_{\alpha \dot{\alpha}} \partial_a \psi^\alpha = \frac{1}{32} \sqrt{\frac{\phi}{\bar{\phi}}} \int d^3 \xi \sqrt{\tilde{g}} \tilde{\theta}^\dagger \delta^4(x - \hat{x}) - \frac{1}{8 \cdot 4!} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd} (\tilde{\theta} \sigma^d)_{\dagger \dot{\alpha}} \delta^4(x - \hat{x}) + \mathcal{O}(f^3) ,
\]

and the Klein-Gordon equation, also with the source from supermembrane,

\[
\Box \phi(x) = \frac{1}{64} \sqrt{\frac{\phi}{\bar{\phi}}} \int d^3 \xi \sqrt{\tilde{g}} \delta^4(x - \hat{x}) - \frac{1}{64} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \tilde{\theta} \delta^4(x - \hat{x}) - \frac{i}{64} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd} (\tilde{\theta} \sigma^d)^2 \partial^d \delta^4(x - \hat{x}) +
\]

\[
+ \frac{i}{64} \int d^3 \xi \sqrt{\tilde{g}} (\tilde{\theta} \sigma^a \tilde{\theta}) \sqrt{\frac{\phi}{\bar{\phi}}} \partial_a \delta^4(x - \hat{x}) + \mathcal{O}(f^4) .
\]

The explicit form of the terms of higher order in fermions, \(\mathcal{O}(f^4)\) in (2.77) and \(\mathcal{O}(f^3)\) in (2.76), can be extracted from the Eqs. (B.7) and (B.16) in Appendix B.
2.5.3. Simplest solution of the dynamical equations at leading order in supermembrane tension

The above equations can be formally solved by

\[
\psi^a = \psi_0^a + \frac{1}{32} \int d^3 \xi \sqrt{g} \sqrt{\frac{\phi}{\phi}} (\hat{\theta} \bar{\sigma}^a)^a \partial_a G_0(x - \hat{x}) + \\
+ \frac{1}{8 \cdot 4!} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd} (\hat{\theta} \bar{\sigma}^d \bar{\sigma}^c)^a \partial_a G_0(x - \hat{x}) + O(f^3),
\]

(2.78)

\[
\phi(x) = \phi_0(x) + \frac{1}{64} \int d^3 \xi \sqrt{g} \sqrt{\frac{\phi}{\phi}} \left(G_0(x - \hat{x}) + i(\hat{\theta} \bar{\sigma}^a \hat{\theta}) \partial_a G_0(x - \hat{x})\right) - \\
- \frac{1}{64} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge d\bar{\sigma}_{\alpha} \bar{\sigma} \sigma_{\alpha} G_0(x - \hat{x}) - \frac{i}{64} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd} (\hat{\theta} \bar{\sigma}^d \bar{\sigma}^c)^a \partial_a G_0(x - \hat{x}) + \\
+ O(f^4),
\]

(2.79)

where \(\psi_0^a\) and \(\phi_0(x)\) are solutions of the free equations and \(G_0(x - \hat{x})\) is the Green function of the free \(D = 4\) Klein-Gordon operator \(\Box := \partial_a \partial^a\),

\[
\Box \phi_0(x) = 0, \quad \sigma_{\alpha \beta} \partial_a \psi_0^a = 0 \quad \Box G_0(x - \hat{x}) = \delta^4(x - \hat{x}). \quad (2.80)
\]

Eqs. (2.78) and (2.79) give only formal solutions as far as the pull–back of the phase of the complex scalar superfield enters their r.h.s. through \(\sqrt{\frac{\phi}{\phi}}\) multipliers in the integrands.

Assuming the solution of the homogeneous equation to be real, \(\phi_0(x) = (\phi_0(x))^*\) one can solve Eqs. (2.76) and (2.77) in the first order in the supermembrane tension \(T\) (this is set to unity in our equations above and below, but can be easily restored by \(\int d^3 \xi \sqrt{g} \mapsto T \int d^3 \xi \sqrt{g}\) and \(\int \mapsto T \int\)). This reads (setting back \(T = 1\))

\[
\psi^a = \psi_0^a + \frac{1}{32} \int d^3 \xi \sqrt{g} (\hat{\theta} \bar{\sigma}^a)^a \partial_a G_0(x - \hat{x}) + \\
+ \frac{1}{8 \cdot 4!} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd} (\hat{\theta} \bar{\sigma}^d \bar{\sigma}^c)^a \partial_a G_0(x - \hat{x}) + O(f^3),
\]

(2.81)

\[
\phi(x) = \phi_0(x) + \frac{1}{64} \int d^3 \xi \sqrt{g} \left(G_0(x - \hat{x}) + i(\hat{\theta} \bar{\sigma}^a \hat{\theta}) \partial_a G_0(x - \hat{x})\right) + \\
- \frac{1}{64} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge d\bar{\sigma}_{\alpha} \bar{\sigma} \sigma_{\alpha} G_0(x - \hat{x}) - \frac{i}{64} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd} (\hat{\theta} \bar{\sigma}^d \bar{\sigma}^c)^a \partial_a G_0(x - \hat{x}) + \\
+ O(f^4), \quad \phi_0(x) = (\phi_0(x))^*.
\]

(2.82)

The contribution of higher order in string tension would include the product of distributions (of the type \(G_0(x - \hat{x}(\xi_1)) \delta^4(x - \hat{x}(\xi_2))\)) and their accounting requires a careful study of a classical counterpart of the renormalization procedure, similar to the one developed for the
2.5. Simplest equations of motion for spacetime fields interacting with dynamical supermembrane

radiation reaction problem [166] and for general relativity [167]. The generalization of such a technique for the case of \( p \)-brane has been developed in very recent [168].

2.5.4. Superfield equations with nontrivial superpotential

As a first stage in searching for solution of the superfield equations with a nontrivial superpotential, let us consider the relation with known domain wall solutions of the Wess-Zumino model [169,170]. To this end let us consider our dynamical system with nontrivial superpotential and the simplest kinetic term. This is described by the interacting action

\[
S = \int d^8 z \Phi \Phi + \int d^6 \zeta L W(\Phi) + \int d^6 \zeta_R \bar{W}(\bar{\Phi}) + \frac{1}{2} \int d^3 \xi \sqrt{\Phi \bar{\Phi}} \sqrt{g} - \int \hat{C}_3'. \tag{2.83}
\]

with \( \Phi \) and \( \bar{\Phi} \) and \( \hat{C}_3' \) expressed in terms of real pre-potential \( V(z) \) by Eqs. (2.41) and (2.50).

The superfield equations of motion (2.66) acquire now the superpotential contributions,

\[
DD\Phi + \bar{D}\bar{\Phi} + W'_\Phi(\Phi) + \bar{W}'_\Phi(\bar{\Phi}) = J(z) \tag{2.84}
\]

The auxiliary field equations read

\[
DD\Phi|_0 + \bar{D}\bar{\Phi}|_0 + W'_\phi(\phi) + \bar{W}'_\phi(\bar{\phi}) = J(z)|_0 , \tag{2.85}
\]

\[
\partial_a (DD\Phi|_0 + \bar{W}'_\phi(\bar{\phi}) - \bar{D}\bar{\Phi}|_0 - W'_\phi(\phi)) = -\frac{i}{4} \sigma^a_{\bar{\alpha}} [D_a, \bar{D}_\bar{\alpha}] J(z)|_0 , \tag{2.86}
\]

and the dynamical field equations are

\[
\sigma^a_{\bar{\alpha}} \partial_a \psi^\alpha + \frac{i}{4} \bar{\psi}_\bar{\alpha} \bar{W}'_{\phi\phi}(\bar{\phi}) = \frac{1}{4} \bar{D}_\bar{\alpha} J(z)|_0 , \tag{2.87}
\]

\[
\square \phi(x) - \frac{1}{16} \bar{D}\bar{D} \Phi|_0 \bar{W}'_{\phi\phi}(\bar{\phi}) + \frac{1}{16} \bar{\psi}_\bar{\alpha} \bar{\psi}^\alpha \bar{W}'_{\phi\phi}(\bar{\phi}) = -\frac{1}{16} \bar{D} \bar{D} J(z)|_0 \tag{2.88}
\]

with the same supermembrane current (2.59), (2.63), (2.64).

In the absence of supermembrane current, \( J = 0 \), the auxiliary field equations are solved by \( DD\Phi|_0 = -\bar{W}'_\phi(\bar{\phi}) + ic, \bar{D}\bar{D}\bar{\Phi}|_0 = -W'_\phi(\phi) - ic \) and the constant \( c \) can be removed by redefining superpotential \( W'_\phi(\phi) \mapsto W'_\phi(\phi) + ic \). Thus, without lost of generality, one can simplify notation and substitute \( -\bar{W}'_\phi(\bar{\phi}) \) for \( DD\Phi|_0 \) in the dynamical equations (2.88). Domain wall ansatz of [169,170] implies that all the fields are static and depend on only one spatial coordinate which we chose to be \( x^2 = y \). Then Eqs. (2.87) and (2.88) with \( J = 0 \) becomes

\[
\sigma^a_{\bar{\alpha}} \partial_y \psi^\alpha(y) + \frac{i}{4} \bar{\psi}_\bar{\alpha}(y) \bar{W}'_{\phi\phi}(\bar{\phi}(y)) = 0 , \tag{2.89}
\]

\[
\partial_y^2 \phi(y) - \frac{1}{16} W'_\phi(\phi) \bar{W}'_{\phi\phi}(\bar{\phi}(y)) - \frac{1}{16} \bar{\psi}_\bar{\alpha} \bar{\psi}^\alpha \bar{W}'_{\phi\phi}(\bar{\phi}(y)) = 0 \tag{2.90}
\]
Notice that Eq. (2.89) split into the pair of equations for $\psi_1, \bar{\psi}_1$ and $\psi_2, \bar{\psi}_2$,

$$\partial_y \psi_1(y) + \frac{1}{4} \bar{\psi}_1(y) W^\mu_{\phi\phi}(\bar{\phi}(y)) = 0, \quad \partial_y \psi_2(y) + \frac{1}{4} \bar{\psi}_2(y) W^\mu_{\phi\phi}(\bar{\phi}(y)) = 0,$$

(2.91)

such that the solution of the second can be constructed from the solution of the first as $\psi_2 = \psi_1, \bar{\psi}_2 = \bar{\psi}_1$. For such a solution of the fermionic equation $\psi_{\alpha} \bar{\psi}^\alpha = 0$ and the bosonic equation simplifies to $\partial_y^2 \phi(y) - \frac{1}{16} W^I_\phi(\phi) W^I_{\phi\phi}(\bar{\phi}) = 0$. This, in its turn, is solved by any solution of the following first order BPS equations \[169, 170\]

$$\partial_y \phi(y) - \frac{e^{i\alpha}}{4} W^I_\phi(\bar{\phi}(y)) = 0, \quad \partial_y \bar{\phi}(y) - \frac{e^{-i\alpha}}{4} W^I_\phi(\phi(y)) = 0.$$

(2.92)

Abraham and Townsend \[169\] studied the intersecting domain wall solutions of (2.92) with $W = \Phi^a - 4\Phi$. The generalization of the above equations for the case of supergravity was studied in \[170\]. For $W = a^2\Phi - \frac{\Phi^3}{3}$ Eq. (2.92) has kink solution $\phi = a \tanh(ya)$ \[170\]. Notice that in the case of special chiral multiplet such a potential would be deformed by the contribution of an arbitrary imaginary constant, $a^2 \mapsto a^2 + ic$.

When the BPS equations (2.92) are satisfied, the solution of fermionic equations can be written in the form \[169\]

$$\psi_1 = 2\chi e^{-i\alpha/2} \partial_y \phi(y) = \psi_2, \quad \bar{\psi}_1 = 2\chi e^{i\alpha/2} \partial_y \phi(y) = \bar{\psi}_2$$

(2.93)

with a real Grassmann (fermionic) constant $\chi$,

$$\chi = \chi^*, \quad \chi \chi = 0.$$

(2.94)

An interesting problem is to study the influence of the supermembrane source on the above discussed nonsingular domain wall solutions. First observation is that, to maintain the general structure of the solution (2.93) of the fermionic equations, the source contribution in the r.h.s. of (2.87), $\sigma_{\alpha\dot{\alpha}} \partial_y \psi^\alpha(y) + \frac{3}{4} \bar{\psi}_\alpha(y) W^\mu_{\phi\phi}(\bar{\phi})(y) = \frac{1}{4} D_\alpha J(z)|_0$, should be proportional to the same real Grassmann constant $D_\alpha J(z)|_0 \propto \chi$. This in its turn suggests the following ansatz for the fermionic coordinates functions

$$\hat{\theta}^\alpha(\xi) = u^\alpha(\xi) \chi, \quad \hat{\theta}^{\dot{\alpha}}(\xi) = \bar{u}^{\dot{\alpha}}(\xi) \chi$$

(2.95)

with some bosonic functions $u^\alpha(\xi) = (\bar{u}^{\dot{\alpha}}(\xi))^*$. Such an ansatz results in that $\hat{\theta}^\alpha \hat{\theta}^{\dot{\alpha}} = 0 = \hat{\theta}^\alpha \hat{\theta}^{\dot{\alpha}}$ and, hence, in that the pull–back of bosonic vielbein simplifies to $\hat{E}^a = d\hat{\alpha}$. Furthermore, assuming that the normal to the supermembrane worldvolume cannot be orthogonal to the $y = \hat{x}^2$ axis, we can chose the ‘static gauge’ where $\hat{x}^0(\xi) = \xi^0 = \tau, \hat{x}^1(\xi) = \xi^1, \hat{x}^3 = \xi^2$ so that the only nontrivial bosonic coordinate function (supermembrane Goldstone field) is identified with $\hat{x}^2 = \hat{y}(\xi)$. In this gauge

$$\hat{E}^0 = d\hat{\xi}^0, \quad \hat{E}^1 = d\hat{\xi}^1, \quad \hat{E}^2 = d\hat{y}(\xi) = d\hat{y}(\xi^0, \xi^1, \xi^2), \quad \hat{E}^3 = d\xi^2.$$

(2.96)

Furthermore, with such an ansatz $J|_0 = 0$ and all the (quite complicated) components of current superfield (see appendix B) simplify drastically reducing to their leading terms. Then
2.5. Simplest equations of motion for spacetime fields interacting with dynamical supermembrane

the problem of finding (particular) solutions of the system of interacting equations looks manageable.
In this chapter we study the interacting system of supermembrane and $D = 4 \, \mathcal{N} = 1$ dynamical supergravity. We obtain the complete set of equations of motion for this system by varying its complete superfield action. These include the supermembrane equations, which are formally the same as in the case of supermembrane in supergravity background, and the superfield supergravity equations with supermembrane contributions. The existence of a three form potential $C_3$ allowing for a Wess–Zumino term in the supermembrane part of the action imposes a restriction on the prepotential structure of minimal supergravity, making its chiral compensator being constructed from a real rather than complex prepotential. We will develop the Wess–Zumino type approach for this special minimal supergravity and present its basic variations which are characterized, besides $\delta H^a$, by one real variation $\delta V$ instead a complex ones $\delta U, \delta \bar{U}$. The most important consequence of this modification in the prepotential structure is that the right hand side of the Einstein equation acquires a term proportional to an integration constant, a dynamically generated cosmological constant. We also present the supergravity superfield equations with the supermembrane contributions obtained by varying the action with respect to the superfields of special minimal supergravity and write these resulting superfield equations in the special ”WZ$_{\hat{\theta} = 0}$” gauge, which is the standard Wess–Zumino gauge completed by the condition that the supermembrane Goldstone field is set to zero $(\hat{\theta} = 0)$. We solve the auxiliary field equations and show that these result in the effect of dynamical generation of cosmological constant and its ’renormalization’(due to supermembrane contributions)
such a way that the cosmological constant values in the branches of spacetime separated by the supermembrane worldvolume are generically different.

### 3.1. Superfield supergravity and supermembrane in curved D=4, \(N=1\) superspace

#### 3.1.1. Superfield supergravity action, superspace constraints and equations of motion

The superfield action of the minimal off-shell formulation of \(D=4, N=1\) supergravity \[158\]

\[
S_{SG} = \int d^8Z \ E := \int d^4x \tilde{d}^4\theta \ sdet(E^A_M), \tag{3.1}
\]

is given by the superdeterminant (or Berezinian) of the matrix of supervielbein coefficients, \(E^A_M(Z)\) in (1.13), which obey the set of supergravity constraints. These can be collected together with their consequences in the following expressions for the superspace torsion 2-forms (1.14), (1.15) (see [72] and refs. therein)

\[
T^a = -\frac{2i}{8} \sigma^a_{\bar{a}a} E^a \wedge \bar{E}^\bar{a} - \frac{1}{8} E^b \wedge E^c \epsilon^{ab} \epsilon_{bcd} G^d, \tag{3.2}
\]

\[
T^\alpha := (T^{\bar{\alpha}})^\ast = \frac{i}{8} E^c \wedge E^\beta (\sigma_c \tilde{\sigma}_d)_{\bar{\beta}} \bar{G}^d - \frac{i}{8} E^c \wedge \bar{E}^\bar{\beta} \epsilon^{a\bar{\alpha}} \epsilon_{c\bar{\beta}d} \bar{R} + \frac{1}{2} E^c \wedge E^b T^a_{bc} \alpha. \tag{3.3}
\]

The main superfields, real vector \(G_a = (G_a)^\ast\) and complex scalar \(R = (\bar{R})^\ast\), entering (3.2) and (3.3), obey

\[
\mathcal{D}_\alpha \bar{R} = 0, \quad \bar{\mathcal{D}}_\bar{\alpha} R = 0, \quad (3.4)
\]

\[
\mathcal{D}^\alpha G_{a\bar{a}} = -\mathcal{D}_{\bar{a}} \bar{R}, \quad \bar{\mathcal{D}}^\bar{\alpha} \bar{G}_{a\bar{a}} = -\bar{\mathcal{D}}_\bar{\alpha} \bar{R}. \tag{3.5}
\]

These relations can be obtained by studying the Bianchi identities (1.17), which also allow to find the expression for superfield generalization of the gravitino field strength, \(T^a_{bc}(Z)\),

\[
T_{a\bar{a}\beta\bar{\beta}\gamma} = \sigma^a_{\bar{a}a} \sigma^b_{\beta\bar{\beta}} \epsilon_{\gamma\delta} T_{ab}^\delta = -\frac{1}{8} \epsilon_{\alpha\beta} \bar{D}_{(\bar{a}i} G_{\gamma]_{\beta\bar{\beta}} - \frac{1}{8} \epsilon_{\alpha\beta} \bar{W}_{\alpha\beta\gamma} = 2\epsilon_{\gamma(\alpha} \mathcal{D}_{\beta)} \bar{R} \quad (3.6)
\]

involving one more main superfield, \(W_{a\beta\gamma} = W_{(a\beta\gamma)} =: (\bar{W}_{a\beta\gamma})^\ast\). This obeys

\[
\bar{D}_\bar{a} W^{\alpha\beta\gamma} = 0, \quad \bar{D}_\alpha \bar{W}^{\alpha\beta\gamma} = 0, \quad (3.7)
\]

\[
D_\gamma W^{\alpha\beta\gamma} = \bar{D}_\bar{a} D^{(a} G^{\beta)\gamma} \quad (3.8)
\]

Studying the Bianchi identities with the constraints (3.2), (3.3) one also finds that the superfield generalization of the left hand side of the supergravity Rarita–Schwinger equation
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reads

\[ \epsilon^{abcd} T_{bc}^\alpha \sigma_{da\dot{\alpha}} = \frac{i}{8} \sigma_{\alpha\dot{\beta}} D(\beta|G^\gamma) + \frac{3i}{8} \sigma_d^\alpha D^\beta R, \]  

(3.9)

and the superfield generalization of the Ricci tensor is

\[ R_{bc} = \frac{1}{32} (D^d \bar{D} \sigma^a \sigma_{\alpha\dot{\beta}} (G^\gamma |D^\beta |G^\delta) - \bar{D}^d D^\beta (G^\gamma |D^\alpha |G^\delta)) \sigma_{\alpha\dot{\alpha}} \sigma_{\beta\dot{\beta}} - \frac{3}{64} (\bar{D} \bar{D} R + D D R - 4 R R) \delta_5^a. \]  

(3.10)

This suggests that superfield supergravity equation should have the form\(^1\)

\[ G_a = 0, \]  

(3.11)

\[ R = 0, \quad \bar{R} = 0. \]  

(3.12)

Eqs. (3.11) and (3.12) can be obtained by varying the action (3.1) with respect to supervielbein obeying the supergravity constraints (3.2), (3.3) \[158\]. Such admissible variations are expressed through a vector parameter \( \delta H^a \) and complex scalar parameter \( \delta U = (\delta \bar{U})^\ast \) which enter the variation of the supervielbein and spin connection under the symbol of the chiral projector \( (D^a D_n - R) \) (see \[72, 84\] for more detail). They correspond to the variations of the so-called prepotentials, unconstrained superfields which appear in the general solution of the supergravity constraints. The minimal supergravity constraints are solved in terms of the axial vector superfield \( H^\mu \) \[151\] and chiral compensator \( \Phi \) \[159\]. This latter obeys \( D_\alpha \Phi = 0 \) and, hence, can be expressed as \( \Phi = (D_\alpha D^\alpha - R) U \) with a complex unconstrained superfield \( U \).

Thus the set of minimal supergravity prepotentials includes \( H^\mu, U \) and \( \bar{U} = (U)^\ast \) which are in one to one correspondence with the set of three independent variations \( \delta H^a, \delta \bar{U} \) and \( \delta U = (\delta \bar{U})^\ast \) of the Wess–Zumino approach to supergravity \[82, 158\] producing the three superfield equations (3.11) and (3.12). In short, as it had been known already from \[158\],

\[ \delta S_{SG} = \int d^8 Z E \left[ \frac{1}{6} G_a \delta H^a - 2 R \delta \bar{U} - 2 \bar{R} \delta U \right]. \]  

(3.13)

3.1.2. Supermembrane action in minimal supergravity background

As it is well known, the supermembrane action \[61, 76\] \( \text{(cf. chapter 2)} \) is given by the sum of the Dirac–Nambu–Goto and the Wess–Zumino term,

\[ S_{p=2} = \frac{1}{2} \int d^3 \xi \sqrt{g} - \int W^3 = -\frac{1}{6} \int \hat{E}^a \wedge \hat{E}^a - \int \hat{C}^3. \]  

(3.14)

The former is given by the volume of \( W^3 \) defined as integral of the determinant of the induced metric, \( g = \text{det}(g_{mn}) \),

\[ g_{mn} = \hat{E}^a_m \eta_{ab} \hat{E}^b_n, \quad \hat{E}^a_m := \partial_m \hat{Z}^M (\xi) E^a_M (\hat{Z}). \]  

(3.15)

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\(^1\)See \[72, 84\] for more detail on the superfield description of minimal supergravity in the present notation.
3.1. Superfield supergravity and supermembrane in curved D=4, $\mathcal{N}=1$ superspace

Here $\xi^m = (\tau, \sigma^1, \sigma^2)$ are local coordinates on $W^3$ and $\hat{Z}^M(\xi)$ are coordinates functions which determine the embedding of $W^3$ as a surface in target superspace $\Sigma^{(4|4)}$,

$$W^3 \subset \Sigma^{(4|4)} : \quad Z^M = \hat{Z}^M(\xi) = (\hat{x}^\mu(\xi), \hat{\theta}^\alpha(\xi)) .$$

In the second equality of (3.14) the Dirac–Nambu–Goto term is written as an integral of the wedge product of the pull–back of the $\Sigma^{(4|4)}$ bosonic supervielbein form $E^a$ to $W^3$,

$$\hat{E}^a = d\xi^m \hat{E}^a_m = d\hat{Z}^M(\xi) E^a_M(\hat{Z}) ,$$

and of its Hodge dual two form defined with the use of the induced metric (3.15) and its inverse $g^{mn}$,

$$* \hat{E}^a := \frac{1}{2} d\xi^m \wedge d\xi^n \sqrt{g} \epsilon_{mnk} \hat{g} \hat{E}^a_l .$$

The second, Wess–Zumino term of the supermembrane action (3.14) describes the supermembrane coupling to a 3–form gauge potential $C_3$ defined on $\Sigma^{(4|4)}$,

$$C_3 = \frac{1}{3!} dZ^M \wedge dZ^N \wedge dZ^K C_{KNM}(Z) = \frac{1}{3!} \hat{E}^C \wedge \hat{E}^B \wedge \hat{E}^A C_{ABC}(\hat{Z}) .$$

Thus, to write a supermembrane action, one has to construct the 3–form gauge potential $C_3$ in the target superspace $\Sigma^{(4|4)}$ and take its pull–back to the supermembrane worldvolume

$$\hat{C}_3 = \frac{1}{3!} d\hat{Z}^M \wedge d\hat{Z}^N \wedge d\hat{Z}^K C_{KNM}(\hat{Z}(\xi)) = \frac{1}{3!} \hat{E}^C \wedge \hat{E}^B \wedge \hat{E}^A C_{ABC}(\hat{Z}) =$$

$$= \frac{1}{3!} d\xi^m \wedge d\xi^n \wedge d\xi^k \hat{C}_{kmn} = d^3 \xi \epsilon_{mnk} \hat{C}_{kmn} .$$

Actually, to study supermembrane in supergravity background, it is sufficient to know the field strength of the above 3–form potential, $H_4 = dC_3$. This should be closed, $dH_4 = 0$, and supersymmetric invariant 4–form. In flat superspace such a form exists and represents a nontrivial Chevalley–Eilenberg cohomology of the $\mathcal{N} = 1$ supersymmetry algebra [162, 163, 171] (cf. chapter 2).

The minimal supergravity superspace allows for existence of two closed 4-forms

$$H_{4L} = -\frac{i}{4} E^b \wedge E^a \wedge E^\alpha \wedge E^\beta \sigma_{ab, \alpha \beta} - \frac{1}{128} E^d \wedge E^e \wedge E^b \wedge E^a \epsilon_{abcd} R , \quad dH_{4L} = 0$$

and its complex conjugate $H_{4R} = (H_{4L})^*$ (see [84]). Its real part,

$$H_4 := dC_3 = \frac{1}{4!} E^{A_4} \wedge ... \wedge E^{A_4} H_{A_1...A_4}(Z) = H_{4L} + H_{4R} ,$$

is also closed and provides the 4–form field strength associated to the Wess–Zumino (WZ) term of the supermembrane action in the minimal supergravity background [87], $\int_{W^4} C_3$ in (3.14). Indeed, the WZ term can also be defined as an integral of the closed 4 form $H_4$, related to $C_3$ by $H_4 = dC_3$, over some four dimensional space $W^4$ the boundary of which is
given by the supermembrane worldvolume $W^3$,

$$\int_{W^3} C_3 = \int_{W^4} H_4. \quad (3.23)$$

The condition that the form $H_4$ is closed, $dH_4 = 0$, guarantees that the integral $\int_{W^4} H_4$ is independent of the choice of $W^4$ and, thus, is related to the supermembrane worldvolume $W^3$.

The fact that the knowledge of $H_4$ is completely sufficient for studying the properties of closed supermembrane in a supergravity background is related to that in this case the only dynamical variables are the supermembrane coordinate functions $\hat{Z}^M(\xi)$, that the action is written in term of pull–back of differential forms to $W^3$, and that the variation of the differential form with respect to the coordinates can be calculated with the use of the Lie derivative formula, in particular

$$\delta_{\delta Z} C_3 = i_{\delta Z} H_4 + d i_{\delta Z} C_3 = \frac{1}{3!} E^A_1 \wedge \ldots \wedge E^A_4 \delta Z^M E^A_M H_{A_1 \ldots A_4}(Z) + d(1/2 E^C \wedge E^B \delta Z^M E^A_M C_{ABC}(Z)).$$

The supermembrane equations of motion in the minimal supergravity background, which are obtained by varying the action (3.14) with respect to the coordinate functions $\delta \hat{Z}^M(\xi)$,

$$\delta S_{p=2} = \int_{W^3} \left( \frac{1}{2} \mathcal{M}_{3a} E_M^a(\hat{Z}) + i\Psi_{3a} E_M^a(\hat{Z}) + i\tilde{\Psi}_{3a} E_M^a(\hat{Z}) \right) \delta \hat{Z}^M(\xi), \quad (3.24)$$

read

$$\mathcal{M}_{3a} := \mathcal{D} \ast \hat{E}_a + i\hat{E}^b \wedge \hat{E}^{\alpha} \wedge \hat{E}^{\beta} \sigma_{ab\beta} - i\hat{E}^b \wedge \hat{E}^{\alpha} \wedge \hat{E}^{\beta} \tilde{\sigma}_{ab\beta} - \frac{1}{8} \hat{E}^{bc} \wedge \hat{E}^d \epsilon_{abcd}(R + \bar{R}) = 0 \quad (3.25)$$

and

$$\tilde{\Psi}_{3a} := \ast \hat{E}_a \wedge \left( \hat{E}^\alpha \sigma_{a\alpha} - (\hat{\sigma}^a)_{\beta\gamma} \hat{E}^{\beta} \right) = 0, \quad (3.26)$$

$$\Psi_{3a} := \ast \hat{E}_a \wedge \left( \sigma_{a\alpha} \hat{E}^\alpha + \hat{E}^{\beta} (\sigma^a_{\beta})_{\alpha\gamma} \right) = 0, \quad (3.27)$$

where the matrix $\tilde{\gamma}_{\beta\gamma}$ is defined by

$$\tilde{\gamma}_{\beta\gamma} := \epsilon_{\beta\alpha} \epsilon_{\alpha\beta} = \frac{i}{3! \sqrt{g}} \sigma_{\beta\alpha} \epsilon_{abcd} \epsilon^{mnk} \hat{E}_m^b \hat{E}_n^c \hat{E}_k^d = -(\tilde{\gamma}_{\alpha\beta})^* \quad (3.28)$$

and obeys

$$\tilde{\gamma}_{\beta\gamma} \tilde{\gamma}^{\beta\alpha} = \delta^{\alpha}_{\beta}, \quad \tilde{\gamma}^{\alpha\beta} \tilde{\gamma}_{\alpha\beta} = \delta^{\beta}_{\beta}. \quad (3.29)$$

---

For closed supermembrane $\partial W^3 = \emptyset$ so that $\int_{W^3} d\alpha_2 = \int_{\partial W^3} \alpha_2 = 0$ for any 2-form $\alpha_2$, including for $\alpha_2 = i_{\delta Z} C_3$. 

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3.1. Superfield supergravity and supermembrane in curved D=4, N=1 superspace

Some identities involving the above matrix are

\begin{align*}
\bar{\gamma} \tilde{\sigma}^a &= -\sigma^a \bar{\gamma} + \frac{i}{3!} \sqrt{g} \epsilon^{abcd} \hat{E}_a \hat{E}_b \hat{E}_c \hat{E}_d, \\
\epsilon \hat{E}_a \gamma^a \tilde{\sigma}^a &= \epsilon \hat{E}_a \sigma^a, \\
\frac{1}{2} \hat{E}^b \wedge \hat{E}^a \wedge \hat{E}^\beta \sigma_{ab\beta} &= \epsilon \hat{E}_a \wedge \hat{E}^\beta (\sigma^a \bar{\gamma})_{\beta}, \\
\frac{1}{2} \hat{E}^b \wedge \hat{E}^a \wedge \hat{E}^{\beta \dot{\beta}} \bar{\sigma}_{ab\dot{\beta}} &= -\epsilon \hat{E}_a \wedge \hat{E}^{\beta \dot{\beta}} (\bar{\sigma}^a \gamma)_{\beta\dot{\beta}}.
\end{align*}

They are useful, in particular, to show that the fermionic equations of motion obey the Noether identity

\begin{equation}
\delta_\kappa \hat{Z}^M = \kappa^\alpha (\xi) (E^M_\alpha (\hat{Z}) + \bar{\gamma}_{\alpha\dot{\alpha}} \bar{\gamma}^\alpha E^M_{\dot{\alpha}} (\hat{Z}))
\end{equation}

with the local fermionic "parameter" \( \kappa^\alpha (\xi) = (\bar{\kappa}^\alpha)^* \) obeying

\begin{equation}
\kappa^\alpha (\xi) = -\bar{\kappa}_{\dot{\alpha}} (\xi) \bar{\gamma}^{\dot{\alpha} \alpha} \iff \bar{\kappa}_{\dot{\alpha}} (\xi) = -\kappa^\alpha (\xi) \bar{\gamma}_{\alpha\dot{\alpha}}.
\end{equation}

The relation of the supermembrane \( \kappa \)-symmetry in curved superspace with the minimal supergravity constraints was discussed in \[87\]. The flat superspace limit of our equations reproduces the equations of the seminal paper \[76\].

3.1.3. 3–form potential in the minimal supergravity superspace.

Special minimal supergravity

Thus, as we have seen in the previous subsection, to find the equations of motion of supermembrane in supergravity background as well as to study its symmetries it is sufficient to know the closed 4-form \( H_4 = dC_3 \) in the background superspace.

However, to calculate the supermembrane current(s) describing the supermembrane contribution(s) to the supergravity (super)field equations, one needs to vary the Wess–Zumino term \( \int \hat{C}_3 \) of the supermembrane action with respect to the supergravity (super)fields. Thus one arrives at a separate problem of finding the variation

\begin{equation}
\delta C_3 = \frac{1}{3!} E^C \wedge E^B \wedge E^A \beta_{ABC}(\delta)
\end{equation}

such that \( d\delta C_3 = \delta H_4 \) reproduces the variation of \( H_4 \) from \(3.22\), \(3.21\), written in terms of the basic supergravity variations (we refer to Appendix C for the explicit expression of \( \delta H_4 \)).

Studying such a technical problem we have found that it imposes a restriction on the independent variations of the supergravity prepotentials, or equivalently, on the independent parameters of the admissible supervielbein variations, thus transforming the generic minimal supergravity into a special minimal supergravity. This off–shell supergravity formulation had been described for the first time in \[85\], further discussed in \[86\] (see also latter \[172\]) and elaborated in \[87\] using the elegant combination of superfield results and the component
Chapter 3. Supermembrane interaction with dynamical D=4 N=1 supergravity. Superfield Lagrangian description and spacetime equations of motion

'tensor calculus' approach on the line of [155].

In [84] we described this special minimal supergravity in the complete Wess–Zumino superfield formalism. Referring to that paper for technical details, we only notice that the existence of the 3-form potential imposes a restriction on the prepotential structure of minimal supergravity which in our approach manifests itself in that the basic complex variations \( \delta U \) and \( \delta \bar{U} = (\delta U)^* \) are expressed in terms of one real variation \( \delta V \), essentially

\[
\delta U = \frac{i}{12} \delta V , \quad \delta \bar{U} = -\frac{i}{12} \delta V .
\]  

(3.37)

As a result, the variation of the special minimal supergravity action is essentially (see Appendix C)

\[
\delta S_{SG} = \frac{1}{6} \int d^8 Z E \left[ G_a \delta H^a + (R - \bar{R}) i \delta V \right] .
\]  

(3.38)

Hence the set of superfield equations of special minimal supergravity still includes the vector superfield equation (3.11),

\[
G_a = 0 ,
\]  

(3.39)

but instead of the complex scalar superfield equations (3.12), valid in the case of generic minimal supergravity, in the case of special minimal supergravity we have only the real scalar equation

\[
R - \bar{R} = 0 .
\]  

(3.40)

Clearly, due to chirality of \( R, \ D_a R = 0 \), and anti-chirality of \( \bar{R}, \ \bar{D}_a \bar{R} = 0 \), the above Eq. (3.40) also implies that \( d(R + \bar{R}) = 0 \) so that on the mass shell the complex superfield \( R \) is actually equal to a real constant,

\[
R = 4c , \quad \bar{R} = 4c , \quad c = const = c^* .
\]  

(3.41)

Using (3.10), one finds that the superfield equation (3.40) results in Einstein equation with cosmological constant

\[
R_{bc}^{\ a\ c} = 3c^2 \delta_b^a .
\]  

(3.42)

The value of the cosmological constant is proportional to the square of the above arbitrary constant \( c \), which has appeared as an integration constant, so that the special minimal supergravity is characterized by a cosmological constant generated dynamically.

The above mechanism of the dynamical generation of cosmological constant in special minimal supergravity is the same as was observed by Ogievetski and Sokatchev [89] in their theory of axial vector superfield. In the language of component spacetime approach to supergravity the dynamical generation of cosmological constant in the special minimal supergravity was described in [87] and before it, in purely bosonic perspective in [88, 91, 92] and in the context of spontaneously broken \( N = 8 \) supergravity in [90].
3.2. Superspace action and superfield equations of motion for the interacting system of dynamical supergravity and supermembrane

The action for interacting system of dynamical supergravity and supermembrane reads

\[ S = S_{SG} + T_2 S_{p=2} = \int d^8 Z E(Z) + \frac{T_2}{2} \int d^8 \xi \sqrt{g} - T_2 \int \hat{C}_3, \tag{3.43} \]

where \( S_{p=2} \) is the same as in Eq. (3.14), the supervielbein (1.13) and the 3-form potential (3.20) are assumed to be restricted by the minimal supergravity constraints (3.2), (3.3), (3.22), (3.21). Furthermore, as we have discussed in previous sections (and in more details in Appendix C), the existence of the 3-form potential imposes the restrictions (3.37) on the prepotentials of minimal supergravity or equivalently on the basic supergravity variations.

As a result, the superfield equations which appear as a result of variation of the interacting action (3.43) read

\[ G_a = T_2 J_a, \tag{3.44} \]

and

\[ R - \bar{R} = -iT_2 \chi \tag{3.45} \]

where \( J_a \) and \( \chi = (\chi')^\ast \) are supermembrane scalar superfields. Roughly speaking, they are obtained as a result of varying the supermembrane action with respect to the prepotentials of the special minimal supergravity, this is to say as \( \delta S_{p=2}/\delta H^a \) and \( \delta S_{p=2}/\delta V \), and have the form

\[ J_a = \int \frac{3}{E} \hat{E}^b \wedge \hat{E}^\alpha \wedge \hat{E}^\beta \sigma_{abc} \sigma_{\alpha \beta} \delta^8 (Z - \hat{Z}) - \]

\[ - \int \frac{3i}{E} \left( * \hat{E}_a \wedge \hat{E}^\alpha + \frac{i}{2} \hat{E}_b \wedge \hat{E}_c \wedge \hat{E}_d \epsilon_{abcd} \right) D_\alpha \delta^8 (Z - \hat{Z}) + c.c - \]

\[ - \int \frac{i}{8E} \hat{E}_c \wedge \hat{E}_d \epsilon_{abcd} \left( D D - \frac{1}{2} \bar{R} \right) \delta^8 (Z - \hat{Z}) + c.c. + \]

\[ + \int \frac{1}{4E} * \hat{E}_b \wedge \hat{E}_c G_a \delta^8 (Z - \hat{Z}) - \]

\[ - \int \frac{1}{4E} * \hat{E}_c \wedge \hat{E}_b \sigma^{\delta \alpha \alpha} \left( 3 \delta^\alpha_a \delta^\delta_b - \delta^\delta_a \delta^\alpha_b \right) \left[ D_\alpha, \bar{D}_\alpha \right] \delta^8 (Z - \hat{Z}), \tag{3.46} \]
and

\[ X = \frac{6i}{E} \int \hat{E}^a \wedge \hat{E}^a \wedge \hat{E}^{\dot{a}} \sigma^{a\dot{a}} \delta^8(Z - \dot{Z}) - \]

\[ -\frac{3}{2} \int \frac{\hat{E}^b \wedge \hat{E}^c \wedge \hat{E}^d}{8\hat{E}} \epsilon_{abcd} \sigma^{a\dot{a}\alpha\dot{\alpha}} [D_a, \bar{D}_{\dot{a}}] \delta^8(Z - \dot{Z}) + \]

\[ + \int \frac{\hat{E}^b \wedge \hat{E}^c \wedge \hat{E}^d}{4\hat{E}} \epsilon_{abcd} \delta^8(Z - \dot{Z}) + \text{c.c.} + \]

\[ + i \int \frac{\hat{E}^a}{4\hat{E}} (D \ddot{D} - \bar{R}) \delta^8(Z - \dot{Z}) + \text{c.c.} + \]

\[ + \int \frac{1}{4\hat{E}} \hat{E}^b \wedge \hat{E}^c \wedge \hat{E}^d \epsilon_{abcd} G^a \delta^8(Z - \dot{Z}). \]  

(3.47)

Notice that, as a consequence of (3.5), the supermembrane current superfields obey

\[ \bar{D}^{a\dot{a}} J_{a\dot{a}} = iD_a X, \quad D^a J_{a\dot{a}} = -i\bar{D}_{\dot{a}} X. \]  

(3.48)

Although at first glance these relations look different from any of listed in [173, 174], they can be reduced to the Ferrara–Zumino multiplet [175] if one takes into account Eq. (3.45). Indeed, this states that the real superfield \( X \) in the r.h.s. of Eq. (3.48) is the sum of chiral superfield (equal to \( iR \)) and its complex conjugate, so that only the first (second) one contributes to the r.h.s. of the first (second) equation in (3.48).

### 3.3. Spacetime component equations of the \( D = 4 \)
\( \mathcal{N} = 1 \) supergravity–supermembrane interacting system

#### 3.3.1. Wess–Zumino gauge plus partial gauge fixing of the local spacetime supersymmetry (WZ\( \theta = 0 \) gauge)

The structure of the current superfields (3.46), (3.47) is quite complicated. So is the structure of their components. To simplify the supercurrent components which contribute to the equations of physical, spacetime component fields, we use the superspace general
3.3. Spacetime component equations of the $D = 4 \ N = 1$ supergravity–supermembrane interacting system

Coordinate invariance to fix the Wess–Zumino (WZ) gauge on supergravity superfields,

$$i \theta E^\alpha := \theta \hat{\partial}^\alpha E^\alpha = \theta \alpha , \quad i \theta E_\dot{\alpha} := \theta \bar{\partial}^\dot{\alpha} E_\dot{\alpha} = \bar{\theta} \dot{\alpha} ,$$  \hspace{1cm} (3.49)

$$\theta ^\alpha := \theta \hat{\delta}^\alpha \delta ^\alpha _\beta , \quad \bar{\theta} ^\dot{\alpha} := \bar{\theta} \bar{\delta}^\dot{\alpha} \delta ^\dot{\alpha} _\beta ,$$  \hspace{1cm} (3.50)

$$i \theta E^a := \theta \hat{\partial}^a E^a = 0 ,$$  \hspace{1cm} (3.51)

$$i \theta w^{ab} := \theta \hat{\partial}^{ab} w^{ab} = 0 ,$$  \hspace{1cm} (3.52)

(see [72] for references and more detail) and the (pull–back to $W^3$ of the) local spacetime supersymmetry to set to zero the fermionic Goldstone field of the supermembrane,

$$\hat{\theta} ^\alpha (\xi) = 0 \ \Leftrightarrow \ \hat{\theta} ^\alpha (\xi) = 0 , \ \hat{\theta} ^\dot{\alpha} (\xi) = 0 .$$  \hspace{1cm} (3.53)

A detailed discussion on this "WZ $\hat{\theta} = 0$" gauge can be found in [71–74]. We notice only few of its properties.

Firstly, in the WZ gauge (3.49), (3.51) the leading component of supervielbein matrix has a triangular form,

$$E^A |_\theta = 0 = \left( \begin{array}{cc} e^a _\nu (x) & \psi ^a _\nu (x) \\ 0 & \delta ^\alpha _\beta \end{array} \right) \Rightarrow E^A |_\theta = 0 = \left( \begin{array}{cc} e^a _\nu (x) & - \psi ^a _\nu (x) \\ 0 & \delta ^\alpha _\beta \end{array} \right) ,$$  \hspace{1cm} (3.54)

which implies, in particular, the following relation between the leading component of $T_{ab} ^\alpha$ and the true gravitino field strength $D [\mu \psi ^a _\nu (x) ]$

$$T_{ab} ^\alpha |_\theta = 0 = 2 e^a _\alpha e^b _\beta D [\mu \psi ^a _\nu (x) ] - \frac{1}{4} (\psi ^a _{[\mu} \sigma _{\nu]} \delta ^\alpha \beta ) _\theta = 0 - \frac{i}{4} (\bar{\psi} ^a _{[\mu} \sigma _{\nu]} \delta ^\alpha \beta ) ^\alpha R |_{\theta} = 0 .$$  \hspace{1cm} (3.55)

Secondly, we would like to comment on symmetries leaving Eqs. (3.49)–(3.53) invariant. The WZ gauge (3.49), (3.51), (3.52) is preserved by spacetime diffeomorphisms, local Lorentz symmetry and supersymmetry. Fixing further the gauge (3.53) we break 1/2 of the local supersymmetry on the worldvolume of the supermembrane. The only restriction on the parameter of the local spacetime supersymmetry $\epsilon ^\alpha (x)$ is the condition that its pull–back to $W^3$, $\hat{\epsilon} ^\alpha := \epsilon ^\alpha (\hat{x} (\xi))$, and its complex conjugate $\hat{\epsilon} ^\dot{\alpha} := \epsilon ^{\dot{\alpha}} (\hat{x} (\xi))$ are related by

$$\hat{\epsilon} ^\alpha = \hat{\epsilon} ^\alpha \hat{\gamma} ^\dot{\alpha} \alpha ,$$  \hspace{1cm} (3.56)

where $\hat{\gamma} ^\dot{\alpha} \alpha$ is the supermembrane $\kappa$–symmetry projector (3.28) calculated with $\hat{\theta} (\xi) = 0$. Eq. (3.56) is tantamount to saying that the pull–back of the local supersymmetry parameter to $W^3$ is expressed through the $\kappa$–symmetry parameter of the supermembrane. There are no restrictions on the local supersymmetry parameter outside the supermembrane worldvolume so that the equations (3.56) can be understood as the boundary condition imposed on the supersymmetry parameter on the domain wall $W^3$. 


3.3.2. Current superfields in the WZ_{\theta=0} gauge. Current prepotentials and Rarita–Schwinger equation

In the gauge (3.49)–(3.56),

\[ \hat{E}^a = \hat{e}^a = d\hat{x}^\mu e_{\mu}^a(\hat{x}), \quad \hat{E}^\alpha = \hat{\psi}^\alpha = d\hat{x}^\mu \psi_{\mu}^\alpha(\hat{x}) , \] (3.57)

and

\[ D_\alpha \delta^8(Z - \hat{Z}) = \frac{1}{8} \theta_\alpha \bar{\theta} \delta^4(x - \hat{x}) + \propto \theta^\Lambda^4 , \]
\[ \overline{D}_\dot{\alpha} \delta^8(Z - \hat{Z}) = -\frac{1}{8} \bar{\theta} \theta \delta^4(x - \hat{x}) + \propto \theta \bar{\theta} \bar{\theta} , \] (3.58)

\[ D^\alpha \overline{D}_{\dot{\alpha}} \delta^8(Z - \hat{Z}) = -\frac{1}{4} \theta \bar{\theta} \delta^4(x - \hat{x}) + \propto \theta \theta \bar{\theta} , \]
\[ [D_\alpha, \overline{D}_{\dot{\alpha}}] \delta^8(Z - \hat{Z}) = -\frac{1}{2} \theta_{\alpha} \bar{\theta}_{\dot{\alpha}} \delta^4(x - \hat{x}) + \propto \theta \theta \delta^3 , \] (3.59)

where \( \theta^\Lambda^4 := \theta \theta \bar{\theta} \bar{\theta} \) and \( \theta^\Lambda^3 \) denotes terms proportional to either \( \theta \theta \bar{\theta} \) or \( \theta \bar{\theta} \bar{\theta} \) (or both, which implies \( \propto \theta \theta \bar{\theta} \)). Using these relations and introducing the current pre-potential fields

\[ P_{a}^b(x) := \int_{W^3} \frac{1}{\hat{e}} \hat{e}_c \wedge \hat{e}^b \delta^4(x - \hat{x}) , \] (3.61)
\[ P_a(x) := \int_{W^3} \frac{1}{\hat{e}_{abcd}} \hat{e}^b \wedge \hat{e}^c \wedge \hat{e}^d \delta^4(x - \hat{x}) = \]
\[ \hat{e}_a^\mu(x) \int_{W^3} \epsilon_{\mu
u\rho\sigma} \hat{d} \hat{x}^\nu \wedge \hat{d} \hat{x}^\rho \wedge \hat{d} \hat{x}^\sigma \delta^4(x - \hat{x}) , \] (3.62)

we find that the vector and scalar current superfields (3.46), (3.47) have the form

\[ J_{\alpha a}|_{\theta=0} = \frac{\theta_{\beta} \bar{\theta}_{\dot{\beta}}}{8} (3P_{a}^b(x)\sigma_{\alpha\sigma}^a \bar{\sigma}_{\dot{b}}^\beta - 2\delta_{\alpha}^\beta \delta_{\dot{\alpha}}^\beta P_{b}^a(x)) - i \left( \frac{\theta \theta - \bar{\theta} \bar{\theta}}{32} \right) \sigma_{\alpha a}^a P_a(x) + \propto \theta^\Lambda^3 \] (3.63)

and

\[ X|_{\theta=0} = -\frac{\theta \sigma^a \bar{\theta}}{16} P_a + i \left( \frac{\theta \theta - \bar{\theta} \bar{\theta}}{16} \right) P_a^a(x) + \propto \theta^\Lambda^3 . \] (3.64)

Using (3.63) and (3.64) one can easily check that Eqs. (3.48) are satisfied at lowest order in \( \theta \).

One also sees that there is no explicit supermembrane contributions to the Rarita–Schwinger equations of the supergravity–supermembrane interacting system which thus
3.3. Spacetime component equations of the $D = 4 \ N = 1$ supergravity–supermembrane interacting system

reads

$$
\epsilon^{\mu\nu\rho\sigma} \epsilon^a_\mu(x) \mathcal{D}_\rho \psi_a^\alpha(x) \sigma_{a\alpha\dot{\alpha}} = 0 .
$$ (3.65)

However, such a contribution is actually present in (3.65) implicitly, hidden inside the covariant derivative. Indeed, as indicated by Einstein equation, the bosonic vielbein and the spin connection do contain some contributions from supermembrane.

3.3.3. Einstein equation of the supergravity–supermembrane interacting system in the WZ $\hat{\theta} = 0$ gauge

The Einstein equation with supermembrane current contributions can be obtained as leading term in the decomposition of Eq. (3.10), i.e.

$$
R_{bc}^{ac} |_{\theta = 0} = \frac{1}{32} (\mathcal{D}^\beta \mathcal{D}^\alpha j^{(\dot{\alpha})}) |_{\theta = 0} \sigma_\alpha^a \sigma_\beta^b - \frac{3i}{64} (\mathcal{D} \mathcal{D} \chi - \mathcal{D} \mathcal{D} \chi) |_{\theta = 0} \delta^a_b + \frac{3}{16} (R \bar{R}) |_{\theta = 0} \delta^a_b .
$$ (3.66)

The first two terms in the $r.h.s.$ of Eq. (3.66) can be easily calculated from Eqs. (3.63), (3.64), while the last term, in the light of the scalar superfield equation has the form of Eq. (3.45), is expressed in terms of $(R + \bar{R})^2 |_{\theta = 0}$ and requires a separate study. As an intermediate resume let us fix that

$$
R_{bc}^{ac} |_{\theta = 0} = \frac{3}{32} T_2 \left( P_b^\alpha(x) - \frac{1}{2} \delta^a_b P_e^c(x) \right) + \frac{3}{64} (R + \bar{R})^2 |_{\theta = 0} \delta^a_b .
$$ (3.67)

The last term is the square of $(R + \bar{R}) |_{\theta = 0}$ which, as a result of (3.45), obeys the equation

$$
\hat{\partial}_\mu (R + \bar{R}) |_{\theta = 0} = \frac{T_2}{16} \int \frac{d^3 \hat{x}}{W^3} \epsilon_{\mu\nu\rho\sigma} \delta^4(x - \hat{x}) .
$$ (3.68)

The solution of this equation can be written in the form

$$
R(x) + \bar{R}(x) = 8c + \frac{T_2}{16} \int \frac{d^3 \hat{x}}{x_0} \int \frac{d^3 \hat{x}}{W^3} \epsilon_{\mu\nu\rho\sigma} \delta^4(x - \hat{x}) ,
$$ (3.69)

where $c$ is an arbitrary constant which corresponds to the value of $(R + \bar{R})$ at the spacetime point $x_0^\mu$ providing the lower limit of the integral in the second term, $c = (R(x_0) + \bar{R}(x_0))/8$.

One can easily check that

$$
\Theta(x, x_0 | \hat{x}) := \int_{x_0}^x \frac{d^3 \hat{x}}{W^3} \epsilon_{\mu\nu\rho\sigma} \delta^4(x - \hat{x}) ,
$$ (3.70)
Chapter 3. Supermembrane interaction with dynamical D=4 N=1 supergravity. Superfield Lagrangian description and spacetime equations of motion

entering the second term in the r.h.s. of \((3.69)\), obeys

\[
\partial_\mu \Theta(x, x_0|\hat{x}) = \int_{W^3} \epsilon_{\mu\nu\rho\sigma} d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma \delta^4(x - \hat{x}).
\]  

(3.71)

Furthermore, using a convenient local frame in the neighborhood of the worldvolume, one can check that \(\Theta(x, x_0|\hat{x})\) vanishes if the points \(x^\mu\) and \(x_0^\mu\) are on the same side of spacetime with respect to the domain wall provided by the supermembrane worldvolume while it is equal to \(\pm 1\) if these points belongs to the different branches of the spacetime separated by this domain wall. This is to say that Eq. \((3.70)\) defines a counterpart of the Heaviside step function associated to the direction orthogonal to the supermembrane worldvolume. The last statement about association implies that \(\Theta(x, x_0|\hat{x})\) is a functional of the supermembrane coordinate function \(\hat{x}^\mu(\xi)\). Furthermore, as in the case of the standard Heaviside step function \(\Theta(y)\), we can use

\[
(\Theta(x, x_0|\hat{x}))^2 = \Theta(x, x_0|\hat{x}).
\]

Thus our solution of the auxiliary field equations \((3.69)\) can be written as (cf. \[88\])

\[
R(x) + \tilde{R}(x) = 8c + \frac{T_2}{16} \Theta(x, x_0|\hat{x})
\]  

(3.72)

and implies that Eq. \((3.67)\) reads

\[
R_{bc}^{ac}(x) = -\frac{3T_2}{32} \left( P_b^{a}(x) - \frac{1}{2} \delta^a_b P_c^{c}(x) \right) + 3\delta^a_b \left( c^2 + \left( \frac{T_2}{128} + c \right)^2 - c^2 \right) \Theta(x, x_0|\hat{x}),
\]  

(3.73)

where \(P_b^{a}(x)\) is the singular contribution defined in \((3.62)\).

3.3.4. Cosmological constant generation in the interacting system and its “renormalization” due to supermembrane

Let us analyze the supermembrane contribution to the Einstein equations. These can be separated in two classes, one containing singular contributions and the other containing regular contributions proportional to \(T_2\).

Being a bit more provocative one can say about three classes, counting also the contribution proportional to square of the arbitrary integration constant \(c\), as far as this comes from the auxiliary field sector of the special minimal supergravity, the off–shell formulation which is ‘elected’ by the supermembrane. As we have already commented in sec. 3.1.3, the supermembrane can exist in a background of a generic minimal supergravity, however the

\[
\text{In the case of standard standard Heaviside step function this is equivalent to setting the indefinite value } \Theta(0) \text{ equal to } 1/2. \text{ Indeed, calculating the derivative } \partial_y (\Theta(y)\Theta(y)) = 2\Theta(y)\delta(y) = 2\Theta(0)\delta(y) \text{ we find that this coincides with } \partial_y \Theta(y) = \delta(y) \text{ when } \Theta(0) = 1/2. \text{ In our case } \Theta(x, x_0|\hat{x}) \text{ is the counterpart of either } +\Theta(y) \text{ or } -\Theta(y) \text{ so that } (\Theta(x, x_0|\hat{x}))^2 = \pm \Theta(x, x_0|\hat{x}). \text{ However, by a suitable choice of the location of the point } x_0 \text{ with respect to } W^3 \text{ one can always arrive at the situation with nonnegative } \Theta(x, x_0|\hat{x}). \text{ Below for simplicity we assume this choice is made.}
\]
supermembrane interaction with dynamical supergravity requires this to be special minimal supergravity. This in its turn, even in the absence of any matter (neither of the field theoretical type nor of branes), produces Einstein equations with a cosmological constant generated dynamically. Then this cosmological constant proportional to the square of the above arbitrary integration constant \(c\) should also be considered as an indirect contribution of the supermembrane to the Einstein equation.

To be more concrete, Eq. (3.73) can be written in the form

\[
R_{ab}^c(x) = \eta_{ab} 3c^2 + T_2 \left( \mathcal{T}^{\text{sing}}(x) + \mathcal{T}^{\text{reg}}(x) \right) .
\]

(3.74)

When \(T_2\) is set to zero, it contains a nonvanishing cosmological constant contribution with \(\Lambda = -3c^2\). This (AdS-type) cosmological constant is generated dynamically as far as it is proportional to the (minus) square of the arbitrary integration constant \(c\) which is inevitable in the special minimal supergravity equations due to its auxiliary field structure (see [87] and [84] for references and more discussion). In its turn, special minimal supergravity, and not generic minimal supergravity can be included into the action of the supergravity–supermembrane interacting system. In this sense the cosmological constant generated dynamically is the first 'relict' contribution from the supermembrane to the Einstein equation of the interacting system.

The second type of the supermembrane contributions to the r.h.s. of the Einstein equation are singular terms \(\propto \mathcal{P}_c^d(x)\) (3.62),

\[
\mathcal{T}^{\text{sing}}(x) = \frac{-T_2}{32} \left( \mathcal{P}_{ba}(x) - \frac{1}{2} \eta_{ba} \mathcal{P}_c^c(x) \right) = \frac{-3T_2}{32} \int_{W^3} \frac{1}{\hat{e}} \hat{e}_a \wedge \hat{e}_b \delta^4(x - \hat{x}) + \frac{3T_2}{64} \eta_{ba} \int_{W^3} \frac{1}{\hat{e}} \hat{e}_c \wedge \hat{e}_c \delta^4(x - \hat{x})
\]

(3.75)

which are expected when (super)gravity interacts with supermembrane.

In the third type we collect the regular supermembrane contributions which are proportional to the supermembrane tension,

\[
\mathcal{T}^{\text{reg}}(x) = \eta_{ab} \mathcal{T}^{\text{reg}}(x) , \quad \mathcal{T}^{\text{reg}}(x) = +\frac{3T_2}{64} \left( \frac{T_2}{256} + c \right) \Theta(x, x_0|\hat{x}) .
\]

(3.76)

To appreciate the role of this contribution it is instructive to consider the Einstein equation in two pieces of the spacetime separated by the supermembrane worldvolume. Let us denote the half-space where \(\Theta(x, x_0|\hat{x}) = 1\) by \(M_+^4\) and the half-space where \(\Theta(x, x_0|\hat{x}) = 0\) by \(M_-^4\). Then the singular terms (3.75) do not contribute and the Einstein equation reads

\[
M_+^4 : \quad R_{abc}^c(x) = 3\eta_{ab} \left( \frac{T_2}{128} + c \right)^2 .
\]

\[
M_-^4 : \quad R_{abc}^c(x) = 3\eta_{ab} c^2 .
\]

(3.77)

(3.78)

An evident observation is that, in the general case, the cosmological constants in different branches of spacetime separated by the worldvolume \(W^3\) are different.
One also notices that the cosmological constants in $M_4^+$ and $M_4^-$ coincide if $c = -\frac{T_2}{256}$. However, as far as $c$ is an arbitrary integration constant, fixing its value is equivalent to imposing a kind of boundary conditions and we do not see any special reason to chose such boundary conditions in such a way that $c = -\frac{T_2}{256}$. Rather we should allow a generic value of $c$ and thus accept that the cosmological constant takes different values in the branches of spacetime separated by the supermembrane worldvolume.

Notice that the solution of the Einstein equation describing membranes separating two $AdS_5$ spaces with different values of cosmological constants were studied in [93], as a Brane world alternative to the dark energy, and [94] in relation with the hypothesis on possible change of signature in the Brane World models. See also [95] for the related studies. In the bosonic perspective the appearance of different cosmological constants on the different sides of a domain wall interacting with gravity and a 3–form gauge field was known from [88], where it was used as a basis for a bag model for hadrons, and from [92] where this effect was proposed as a mechanism for damping the cosmological constant. Our present study indicates that the result on the different values of cosmological constant on the different sides of the supermembrane domain wall is an imminent consequence of the dynamics of the supersymmetric interacting system of the supermembrane and dynamical $D = 4 \, N = 1$ supergravity.

### 3.4. On supersymmetric solutions of the interacting system equations

When searching for purely bosonic supersymmetric solutions, setting $\psi_\mu^\alpha = 0$, one studies the Killing spinor equations, which appears as the conditions of supersymmetry preservation, $\delta_\epsilon \psi_\mu^\alpha = 0$. When starting from superfield formulation of supergravity, $\delta_\epsilon \psi_\mu^\alpha$ can be calculated with the use of superspace Lie derivative, this is to say $\delta_\epsilon \psi_\mu^\alpha = D_\mu \epsilon^\alpha + (E^C_\mu \epsilon^\beta T_\beta C^\alpha)|_{\theta=0}$. Hence, in a generic off-shell $D = 4 \, N = 1$ minimal supergravity background the Killing equations are

$$
D_\epsilon^\alpha + \frac{i}{8} \epsilon^c (\epsilon \sigma_c \tilde{\sigma}_d)^\beta G^d |_{\theta=0} + \frac{i}{8} \epsilon^c (\tilde{\epsilon} \tilde{\sigma}_c)^\alpha R |_{\theta=0} = 0
$$

(3.79)

and the complex conjugate equation. Using the superfield equations of motion (3.44), (3.45), the explicit form of the current superfields in the WZ $\theta=0$ gauge, Eqs. (3.63), (3.64), and Eq. (3.72), we find that the Killing equation (3.79) reads

$$
D_\epsilon^\alpha + \frac{i}{2} \epsilon^a (\tilde{\epsilon} \tilde{\sigma}_a)^\alpha \left( c + \frac{T_2}{128} \Theta(x, x_0|\hat{x}) \right) = 0
$$

(3.80)
3.4. On supersymmetric solutions of the interacting system equations

We can split this on two Killing equations valid in two different branches of spacetime separated by the supermembrane worldvolume,

\[ M_4^- : \quad D \epsilon^a + \frac{i}{2} e^a (\bar{\epsilon} \sigma_a)^\alpha c = 0 \, , \quad (3.81) \]

\[ M_4^+ : \quad D \epsilon^a + \frac{i}{2} e^a (\bar{\epsilon} \sigma_a)^\alpha \left( c + \frac{T_2}{128} \right) = 0 \, . \quad (3.82) \]

The supersymmetry parameter should also obey the boundary conditions (3.56) on the worldvolume \( W^3 \), which is the common boundary of \( M_4^- \) and \( M_4^+ \),

\[ W^3 = \pm \partial M_4^\pm : \quad \hat{\epsilon}^\alpha = \hat{\epsilon}_\alpha \hat{x}^{\hat{\alpha}} \, , \quad \hat{\epsilon}^\alpha := e^\alpha(\hat{x}(\xi)) \, , \quad \hat{\epsilon}_\alpha := \bar{\epsilon}_\alpha(\hat{x}(\xi)) \, . \quad (3.83) \]

The detailed study of these system of Killing spinor equations and the search for the supersymmetric solutions of the interacting system equations on their basis is an interesting subject for future. An intriguing question is whether the supersymmetric solutions of the equations of the interacting system exist in the generic case of arbitrary \( c \) corresponding to different values of cosmological constants on different sides of the supermembrane worldvolume, or supersymmetry selects some particular values of the constant \( c \). Presently we can state that if obstructions existed, they would occur due to the singular terms with support on the worldvolume \( W^3 \), while the mere fact of different values of cosmological constant on the branches of spacetime situated on the different sides of \( W^3 \) does not prohibit supersymmetry. Indeed, let us study the integrability conditions for the Killing spinor equations in \( M_4^\pm \). Applying the exterior covariant derivatives to Eqs. (3.81) and (3.82) and using the Ricci identities \( DD \epsilon^\alpha = -\frac{1}{4} R^{ab}_\epsilon \epsilon^\beta \sigma_{ab\beta}^\alpha \) and the equations complex conjugate to (3.81) and (3.82), we find

\[ M_4^- : \quad R^{ab}_\epsilon \epsilon^\beta \sigma_{ab\beta}^\alpha = \frac{1}{4} |c|^2 e^d \wedge e^c e^\beta \sigma_{cd\beta}^\alpha \, , \quad (3.84) \]

\[ M_4^+ : \quad R^{ab}_\epsilon \epsilon^\beta \sigma_{ab\beta}^\alpha = \frac{1}{4} \left| c + \frac{T_2}{128} \right|^2 e^d \wedge e^c e^\beta \sigma_{cd\beta}^\alpha \, . \quad (3.85) \]

If we search for a purely bosonic solution preserving all the supersymmetry in \( M_4^- \) and \( M_4^+ \), Eqs. (3.84) and (3.85) should be obeyed for an arbitrary \( \epsilon^\alpha \). This implies

\[ M_4^- : \quad R^{ab}_{cd} = \frac{1}{2} |c|^2 \delta_{[c}^a \delta_{d]}^b \, , \quad (3.86) \]

\[ M_4^+ : \quad R^{ab}_{cd} = \frac{1}{2} \left| c + \frac{T_2}{128} \right|^2 \delta_{[c}^a \delta_{d]}^b \, , \quad (3.87) \]

i.e. that \( M_4^\pm \) are AdS spaces with apparently different cosmological constants. One can easily check that (3.86) and (3.87) solve our equations of motion (3.78) and (3.77) and thus describe the completely supersymmetric solution of the system of the supergravity equations of the interacting system (at least) when these are considered modulo singular terms with the support on \( W^3 \).

Let us stress that such a system of equations does contain the supermembrane con-
tributions: not only an indirect, which comes from an arbitrary cosmological constant
generated dynamically due to the structure of the supergravity auxiliary fields imposed by
the supergravity interaction with supermembrane (see [87] and also [88–91]), but also direct,
which is a shift of cosmological constant on one of the sides of the brane worldvolume on
the value which is proportional to the supermembrane tension (see [88, 92]). Furthermore,
although preserving all 4 supersymmetries in $M^4_-$ and $M^4_+$, when considered as a solution
of the equations of interacting system, Eqs. (3.86) and (3.87) describe the 1/2 BPS state, i.e.
the state preserving 1/2 of the supersymmetry. Indeed, when considering the interacting
system we have to restrict the local supersymmetry parameter by the boundary conditions
(3.83) on $W^3$ and these clearly break 1/2 of the supersymmetry on $W^3$. 
CHAPTER 4

CONFORMAL HIGHER SPIN THEORY IN EXTENDED TENSORIAL SUPERSPACE

In this chapter we present the superfield equations in tensorial $\mathcal{N}$–extended superspaces to describe the $\mathcal{N} = 2, 4, 8$ supersymmetric generalizations of free conformal higher spin theories. We obtain them by quantizing a superparticle model in $\mathcal{N}$–extended tensorial superspace. We show that no nontrivial generalizations of Maxwell and Einstein equations to tensorial space appear because $\mathcal{N}$–extended higher spin supermultiplets just contain additional scalar and spinor fields which obey the standard higher spin equations in their tensorial space version. We find also that these additional fields appear in the basic superfield under derivatives so that the theory is invariant under Peccei–Quinn–like symmetries.

4.1. Superparticle in $\mathcal{N}$-extended tensorial superspace

Let us begin by presenting a simple dynamical model in extended tensorial superspace the quantization of which produces the supersymmetric higher spin field equations.

4.1.1. An action for the $\Sigma^{(\frac{n(n+1)}{2})[\mathcal{N}n]}$ superparticle

The action for a superparticle in extended tensorial superspace $\Sigma^{(\frac{n(n+1)}{2})[\mathcal{N}n]}$ has the form

$$S = \int d\tau \mathcal{L} = \int d\tau \left[ \dot{X}^{\alpha\beta}(\tau) - i\dot{\theta}^{\alpha I}(\tau)\tilde{\theta}^{\beta I}(\tau) \right] \lambda_{\alpha}(\tau)\lambda_{\beta}(\tau),$$

$$\{ \alpha, \beta = 1, \ldots, n \}, \quad \{ I = 1, 2, \ldots, \mathcal{N} \}.$$  (4.1)
4.1. Superparticle in \(N\)-extended tensorial superspace

where the \(\lambda_\alpha(\tau)\) are auxiliary \textit{commuting} spinor variables, \(\hat{X}^{\alpha\beta}(\tau) = \hat{X}^{\beta\alpha}(\tau)\) and \(\hat{\theta}^{\alpha I}(\tau)\) are the bosonic and fermionic coordinate functions which define the superparticle worldline

\[
W^1 \in \Sigma^{(n(n+1))/2|N} : Z^M = \hat{Z}^M(\tau) = (\hat{X}^{\alpha\beta}(\tau), \hat{\theta}^{\alpha I}(\tau)),
\]

and the dot denotes derivative with respect to proper time \(\tau\).

It is convenient to write the action (4.1) in the form

\[
S = \int_{W^1} \hat{\Pi}^{\alpha\beta}\lambda_\alpha(\tau)\lambda_\beta(\tau),
\]

where

\[
\hat{\Pi}^{\alpha\beta}(\tau) = d\tau \hat{\Pi}^{\alpha\beta}(\tau) = d\tau (\hat{X}^{\alpha\beta} - i\hat{\theta}^{I(\alpha}\hat{\theta}^{\beta)I}) \quad \left\{ \begin{array}{c}
\alpha, \beta = 1, \ldots, n, \\
I = 1, 2, \ldots, N,
\end{array} \right.
\]

is the pull-back to \(W^1\) of the vielbein \(\Pi^{\alpha\beta}\) of the flat \(N\)-extended tensorial superspace \(\Sigma^{(n(n+1))/2|N}\)

\[
\Pi^{\alpha\beta} = dX^{\alpha\beta} - i d\theta^{I(\alpha}\theta^{\beta)I}.
\]

The superparticle action is manifestly invariant under rigid supersymmetry of the \(N\)-extended tensorial superspace \(\Sigma^{(n(n+1))/2|N}\),

\[
\delta_{\epsilon} X^{\alpha\beta} = i \theta^{I(\alpha} e^{\beta)I}, \quad \delta_{\epsilon} \theta^{I\alpha} = e^{\beta I},
\]

which acts on the worldline fields as

\[
\delta_{\epsilon} \hat{X}^{\alpha\beta} = i \hat{\theta}^{I(\alpha} e^{\beta)I}, \quad \delta_{\epsilon} \hat{\theta}^{I\alpha} = e^{\beta I}; \quad \delta_{\epsilon} \lambda_\alpha = 0.
\]

The action (4.1) is also manifestly invariant under the \(GL(n, \mathbb{R})\) transformations of the \(\alpha, \beta = 1, \ldots, n\) indices, which reduce to the \(n\)-dimensional representation of \(Spin(1, D - 1)\) when these indices are thought of as Lorentz-spinorial ones.

4.1.2. Symplectic supertwistor form of the action

Actually, the \(\Sigma^{(n(n+1))/2|N}\) superparticle action (4.1) is invariant under the larger \(OSp(N|2n)\) supergroup. To make this manifest as well as to determine easily the number of physical degrees of freedom it is convenient to use Leibniz’s rule\(^1\) to rewrite the action (4.1) in the form

\[
S = \int_{W^1} (\lambda_\alpha d\mu^\alpha - \mu^\alpha d\lambda_\alpha - i d\chi^I \chi^I) = \int_{W^1} d\Upsilon^\Sigma \Xi_{\Sigma\Omega} \Upsilon^{\Omega}.
\]

\(^1\)In \([98, 99]\) the counterparts of \(\lambda_\alpha\) were called ‘\(s\)-vectors’ to avoid their immediate identification as \(GL(n, \mathbb{R})\) vectors or \(SO(1, D - 1)\) spinors.

\(^2\)It is sufficient to use \(\lambda_\alpha \lambda_\beta dX^{\alpha\beta} = \lambda_\alpha d(\lambda_\beta X^{\alpha\beta}) - \lambda_\alpha X^{\alpha\beta} d\lambda_\beta\); no integration by parts is needed.
This action is written in terms of the bosonic spinor $\lambda_\alpha(\tau)$, which is present in $\text{(4.1)}$, a second bosonic spinor $\mu^\alpha$ and $\mathcal{N}$ real fermionic variables $\chi^I$; these form the $\mathcal{N}$-extended orthosymplectic supertwistor (see [97, 110] for $\mathcal{N} = 1$)

$$\Upsilon^\Sigma = (\mu^\alpha \lambda_\alpha \chi^I) , \quad \alpha = 1, \ldots, n , \quad I = 1, \ldots, \mathcal{N} . \quad (4.9)$$

This generalizes the Penrose twistors [176] (or conformal $SU(2, 2)$ spinors) and the Ferber-Shirafuji supertwistors [177, 178] (carrying the basic representation of $D=4 \, SU(\mathcal{N}|2, 2)$). The $\Upsilon^\Sigma$’s carry the defining representation of the $OSp(\mathcal{N}|2n)$ supergroup, the transformations of which preserve the $(2n + \mathcal{N}) \times (2n + \mathcal{N})$ orthosymplectic ‘metric’ $\Xi_{\Sigma\Omega}$,

$$\Xi_{\Sigma\Omega} = \begin{pmatrix} 0 & \delta_{\alpha\beta} & 0 \\ -\delta_{\alpha\beta} & 0 & 0 \\ 0 & 0 & -i\delta^{IJ} \end{pmatrix} , \quad \alpha = 1, \ldots, n , \quad I = 1, \ldots, \mathcal{N} . \quad (4.10)$$

In fact, $OSp(1|2n)$ may be considered as a supersymmetric generalization of the superconformal group for $D = \frac{n}{2} + 2$ (see [179] and [97, 111, 180] and refs. therein).

The relations between the supertwistor components and the variables of the action $\text{(4.1)}$ that make the transition between both actions are

$$\delta_\kappa \hat{X}^{\alpha\beta} = i\delta_\kappa \hat{\theta}^{(\alpha} \hat{\theta}^{\beta)} I , \quad \delta_\kappa \hat{\theta}^{I} \lambda_\alpha = 0 , \quad \delta_\kappa \lambda_\alpha = 0 , \quad (4.12)$$

where $\hat{X}^{\alpha\beta}$ and $\hat{\theta}^{I}$ are the fermionic and bosonic variables, respectively, and $\lambda_\alpha$ is the bosonic spinor.

### 4.1.3. Gauge symmetries of the original action

By construction, the actions $\text{(4.8)}$ and $\text{(4.1)}$ describe the same dynamical system. Thus, since the action $\text{(4.1)}$ depends on $\frac{n(n+1)}{2} + n$ bosonic variables and $\mathcal{N}$ fermionic ones, it should possess $n(n-1)/2$ bosonic gauge symmetries and $\mathcal{N}(n-1)$ fermionic ones to reduce the number of degrees of freedom to those of the supertwistors $\Upsilon^\Sigma$ appearing in the action $\text{(4.8)}$. The simplest way to describe these gauge symmetries, called fermionic $\kappa$-symmetry and bosonic $b$-symmetry, is to define restrictions on the basic variations of the bosonic and fermionic coordinate functions (see [97] for the $\mathcal{N} = 1$ superparticle case and [111] for the case of superstring in tensorial superspace.)

$$\delta_\kappa \hat{X}^{\alpha\beta} = i\delta_\kappa \hat{\theta}^{(\alpha} \hat{\theta}^{\beta)} I , \quad \delta_\kappa \hat{\theta}^{I} \lambda_\alpha = 0 , \quad \delta_\kappa \lambda_\alpha = 0 , \quad (4.12)$$

$$\delta_b \hat{X}^{\alpha\beta} \lambda_\alpha = 0 , \quad \delta_b \hat{\theta}^{I} \lambda_\alpha = 0 , \quad \delta_b \lambda_\alpha = 0 . \quad (4.13)$$
4.1. Superparticle in $\mathcal{N}$-extended tensorial superspace

4.1.4. Constraints and their conversion to first class

In the hamiltonian formalism, the $\delta_{\kappa}$ and $\delta_{b}$ gauge symmetries in Eqs. (4.12), (4.13) are generated by first class constraints which may be extracted from the following bosonic and fermionic primary constraints of the model (4.1)

$$d_{\alpha I} := \pi_{\alpha I} + iP_{\alpha \beta} \theta^{\beta I} \approx 0,$$

$$p_{\alpha \beta} := P_{\alpha \beta} - \lambda_{\alpha} \lambda_{\beta} \approx 0, \quad P^{\alpha(\lambda)} \approx 0,$$

where

$$P_{\alpha \beta} := \frac{1}{2} \frac{\delta L}{\delta \dot{X}^{\alpha \beta}}, \quad P^{(\lambda)}_{\alpha} := \frac{\delta L}{\delta \dot{\lambda}^{\alpha}}, \quad \pi_{\alpha I} := \frac{\delta L}{\delta \dot{\theta}^{\alpha I}},$$

are the canonical momenta conjugated to the coordinate functions and to the auxiliary bosonic spinor ($s$-vector). Using the canonical Poisson brackets

$$\{ \hat{X}^{\gamma \delta}, P_{\alpha \beta} \}_{PB} = -[P_{\alpha \beta}, \hat{X}^{\gamma \delta}]_{PB} = \delta_{\alpha}^{(\gamma} \delta_{\beta)}^{\delta)},\quad \{ \lambda_{\beta}, P^{\alpha(\lambda)} \}_{PB} = -[P^{\alpha(\lambda)}, \lambda_{\beta}]_{PB} = \delta_{\beta}^{\alpha},$$

$$\{ \pi_{\alpha I}, \hat{\theta}^{\beta J} \}_{PB} = \{ \hat{\theta}^{\beta J}, \pi_{\alpha I} \}_{PB} = -\delta_{\alpha}^{\beta} \delta_{I}^{J},$$

it follows that the nonvanishing Poisson brackets of the above constraints are

$$\{d_{\alpha I}, d_{\beta J}\}_{PB} = -2iP_{\alpha \beta} \delta_{I}^{J}, \quad [P_{\alpha \beta}, P^{\gamma(\lambda)}]_{PB} = -2\lambda_{(\alpha} \delta_{\beta)}^{\gamma}.$$  \hfill (4.20)

These clearly indicate that the primary constraints above are a mixture of first and second class constraints. Rather than separating them, we use below the so-called ‘conversion procedure’ [97, 181–187] by which a pair of degrees of freedom is added to each pair of second class constraints to modify Eqs. (4.20) in such a way that they form a closed algebra. In this way, these modified constraints become first class ones generating gauge symmetries in the enlarged phase space. In it, all the constraints of the model are first class and account as well for the original second class constraints. These can be recovered by gauge fixing the additional gauge symmetries/first class constraints of the system in the enlarged phase space. For the $\mathcal{N} = 1$ version of (2.1) this was done in [97].

As the bosonic sector of all the superparticle models is the same irrespective of $\mathcal{N}$, we may use the results of [97] for $\mathcal{N} = 1$ and state that the conversion in the bosonic sector is effectively reduced to ignoring the constraints $P^{\alpha(\lambda)} \approx 0$ in the analysis. An easy way to see that this is indeed consistent is to observe that, as far as $\lambda_{\alpha} \neq (0, ..., 0)$ (the usual configuration space restriction for twistor-like variables), the second brackets in (4.20) show that $P^{\alpha(\lambda)} = 0$ is a good gauge fixing condition for $n$ of $n(n + 1)/2$ gauge symmetries generated by the constraints $P_{\alpha \beta}$.

To perform the conversion in the fermionic sector, we introduce the $\mathcal{N}$ fermionic variables $\chi^{I}$ and postulate for them the Clifford-type Poisson brackets

$$\{ \chi^{I}, \chi^{J} \}_{PB} = -2i\delta^{IJ}.$$  \hfill (4.21)
These $\chi^I$ are then used to modify the fermionic constraints to $D_{\alpha I} = d_{\alpha I} + i\chi^I\lambda_\alpha$. Thus, after conversion, the superparticle model (4.1) is described by the following set of first class constraints

$$D_{\alpha I} := \pi_{\alpha I} + iP_{\alpha\beta}\theta^{\beta I} + i\chi^I\lambda_\alpha \approx 0 , \quad P_{\alpha\beta} := P_{\alpha\beta} - \lambda_\alpha\lambda_\beta \approx 0 ,$$

which obey the superalgebra of the rigid supersymmetry of $\mathcal{N}$-extended tensorial superspace $\Sigma(\frac{n(n+1)}{2}|\mathcal{N}n)$,

$$\{D_{\alpha I}, D_{\beta J}\}_{PB} = -2iP_{\alpha\beta}\delta_{IJ} , \quad [P_{\alpha\beta}, D_{\gamma I}]_{PB} = 0 , \quad [P_{\alpha\beta}, P_{\gamma\delta}]_{PB} = 0 .$$

### 4.2. Quantization of the superparticle in $\Sigma(\frac{n(n+1)}{2}|\mathcal{N}n)$ with even $\mathcal{N}$ and conformal higher spin equations

Quantizing the model in its orthosymplectic-twistorial formulation (4.8) is straightforward. The canonical hamiltonian is equal to zero and thus the Schrödinger equation simply states that the wavefunction is independent of $\tau$. Following a procedure similar to that in [97] one can show that, in the $n = 4$ tensorial space corresponding to $D = 4$, the wavefunction of the bosonic limit of the superparticle model (4.8) describes the solution of the free higher spin equations. This means that it can be written in terms of an infinite tower of left and right chiral fields $\phi_{A_1...A_2}(p_\mu)$ and $\dot{\phi}_{\dot{A}_1...\dot{A}_2}(\dot{p}_\mu)$ for all half-integer values of $s$ with $p_\mu\dot{p}^\mu = 0$.

Let $\mathcal{N} > 1$ and even (as it will be henceforth). Quantization à la Dirac of a dynamical system with first class constraints requires imposing them as equations to be satisfied by its wavefunction. In the case of our superparticle model (4.1) this wavefunction can be chosen to depend on the coordinates of $\mathcal{N}$-extended tensorial superspace $(X^{\alpha\beta}, \theta^{\alpha I})$, on the bosonic spinor ($s$-vector) variable $\lambda_\alpha$ and on a half of the fermionic variables $\chi^I$ as far as they are, by (4.21), their own momenta. The separation of a half of the real $\chi^I$ coordinates can be achieved by introducing complex variables $\eta^q, (\eta^q)^* = \bar{\eta}_q, q = 1, \ldots, \mathcal{N}/2$, so that $\chi^I = (\chi^q, \chi^{N/2+q}) = ((\eta^q + \bar{\eta}_q), i(\bar{\eta}_q - \eta^q))$.

$$\eta^q = \frac{\chi^q - i\chi^{N/2+q}}{2} , \quad \bar{\eta}_q = \frac{\chi^q + i\chi^{N/2+q}}{2} , \quad \{\bar{\eta}_q, \eta^p\}_{PB} = -i\delta^p_q .$$

Then, the wavefunction superfield in the coordinates representation depends only on $\eta^q$,

$$\mathcal{W} = \mathcal{W}(X^{\alpha\beta}, \theta^{\alpha I}; \lambda_\alpha; \eta^q) ,$$

the various momenta are given by the differential operators

$$P_{\alpha\beta} = -i\partial_{\alpha\beta} , \quad \pi_{\alpha I} = -i\frac{\partial}{\partial \theta^{\alpha I}} , \quad \bar{\eta}_q = \frac{\partial}{\partial \eta^q} .$$

A separation in pairs of conjugate constraints is used in the Gupta-Bleuler method of quantizing systems with second class constraints as the massive superparticle [188,190].
4.2. Quantization of the superparticle in $\Sigma^{(n(n+1)/2\mathcal{N})}$ with even $\mathcal{N}$ and conformal higher spin equations

and the Poisson brackets become quantum commutators or anticommutators ($\{,\}_{PB} \mapsto i\{,\}$; we take $\hbar = 1$). The quantum constraints operators, to be denoted by the same symbol (although having in mind $D$ classical $\alpha I \mapsto -iD$ quantum $\alpha I$, $P$ classical $\alpha \beta \mapsto -iP$ quantum $\alpha \beta$) are then

$D_{\alpha I} := \partial_{\partial \theta^\alpha I} + i \partial_{\partial \theta^\alpha \beta} \bar{\theta}_I - \chi^I \lambda_\alpha$,  \hspace{1cm} (4.27)

$P_{\alpha \beta} := \partial_{\partial \theta^\alpha \beta} - i \lambda_\alpha \lambda_\beta$,  \hspace{1cm} (4.28)

and have to be imposed on the wavefunction (4.25).

For even $\mathcal{N}$, it is convenient to introduce complex Grassmann coordinates and complex Grassmann derivatives,

$\Theta^q = \frac{1}{2}(\theta^q - i\theta^{q+N/2}) = (\bar{\Theta}_q)^* \Leftrightarrow \bar{\partial}_{\bar{\alpha} q} := \frac{\partial}{\partial \Theta^\alpha q} + \frac{i}{2} \frac{\partial}{\partial \theta^{\alpha(q+N/2)}}  \hspace{1cm} (4.29)$

$q = 1, \ldots, N/2$, and conjugate pairs of fermionic constraints

$\nabla_{\alpha q} := \mathbb{D}_{\alpha q} + i \mathbb{D}_{\alpha(q+N/2)} = \partial_{\partial \theta^\alpha q} + 2i \partial_{\partial \theta^\alpha \beta} \bar{\Theta}_q - 2\lambda_\alpha \frac{\partial}{\partial \eta^q} = \mathbb{D}_{\alpha q} - 2\lambda_\alpha \frac{\partial}{\partial \eta^q}$,  \hspace{1cm} (4.30)

$\bar{\nabla}_p := \mathbb{D}_p + i \mathbb{D}_{p(q+N/2)} = \bar{\partial}_{\partial \theta^\alpha q} + 2i \partial_{\partial \theta^\alpha \beta} \Theta^q - 2\lambda_\alpha \eta^q = \bar{\mathbb{D}}_p - 2\lambda_\alpha \eta^q$.  \hspace{1cm} (4.31)

Since $\{D_{\alpha q}, \bar{D}_p\} = 4i\partial_{\partial \theta^\alpha \beta} \delta_q^p$, the above $\nabla_{\alpha q}$, $\bar{\nabla}_p$ and the bosonic constraint (4.28) determine the superalgebra given by the only nonzero bracket

$\{\nabla_{\alpha q}, \bar{\nabla}_p\} = 4iP_{\alpha \beta} \delta_q^p$  \hspace{1cm} (4.32)

This shows that it is sufficient to impose on the superwavefunction (4.25) the fermionic constraints,

$\nabla_{\alpha q} W := \mathbb{D}_{\alpha q} W - 2\lambda_\alpha \frac{\partial}{\partial \eta^q} W = 0$,  \hspace{1cm} (4.33)

$\bar{\nabla}_p W := \bar{\mathbb{D}}_p W - 2\lambda_\alpha \eta^q W = 0$,  \hspace{1cm} (4.34)

since the mass-shell-like bosonic constraint,

$P_{\alpha \beta} W := (\partial_{\partial \theta^\alpha \beta} - i\lambda_\alpha \lambda_\beta) W = 0$,  \hspace{1cm} (4.35)

will follow as a consistency condition for (4.33), (4.34).

Decomposing the superwavefunction in a finite power series in the complex Grassmann variable $\eta^q$,

$W(X, \Theta^q, \bar{\Theta}_q, \lambda, \eta^q) = W^{(0)}(X, \Theta^q, \bar{\Theta}_q, \lambda) + \sum_{k=1}^{N/2} \frac{1}{k!} \eta^{q_k} \ldots \eta^{q_1} W^{(k)}_{q_1 \ldots q_k}(X, \Theta^q, \bar{\Theta}_q, \lambda)$  \hspace{1cm} (4.36)
we find that Eqs. (4.33), (4.34) imply
\[ D_{\alpha q} W^{(0)} = 2\lambda_\alpha W_{q}^{(1)}, \quad \ldots, \quad D_{\alpha q} W_{q_1 \ldots q_k}^{(k)} = 2\lambda_\alpha W_{q_1 \ldots q_k}^{(k+1)}, \quad \ldots, \quad D_{\alpha q} W_{q_1 \ldots q_{N/2}}^{(N/2)} = 0, \] (4.37)
and
\[ \bar{D}^{q}_{\alpha} W^{(0)} = 0, \quad \bar{D}^{q}_{\alpha} W_{q_1 \ldots q_k}^{(k)} = 2k\lambda_\alpha W_{q_1 \ldots q_{k-1}}^{(k-1)} \delta_{q_k}^{q}, \quad \ldots, \quad \bar{D}^{q}_{\alpha} W_{q_1 \ldots q_{N/2}}^{(N/2)} = \mathcal{N}\lambda_\alpha W_{q_1 \ldots q_{N/2-1}}^{(N/2-1)} \delta^{q}_{q_{N/2}}. \] (4.38)
Eqs. (4.37) show that all the superfields \( W_{q_1 \ldots q_k}^{(k)} \) can be constructed from fermionic derivatives of the superfield \( W^{(0)}(X, \Theta^q, \bar{\Theta}_q, \lambda) \) which is chiral as a consequence of the first equation in (4.38),
\[ D_{\alpha q_{k+1}} \ldots D_{\alpha q_1} W^{(0)} = 2^k \lambda_{\alpha_1} \ldots \lambda_{\alpha_k} W_{q_1 \ldots q_k}^{(k)}, \quad \bar{D}^{q}_{\alpha} W^{(0)} = 0. \] (4.39)
Then, the wavefunction \( \mathcal{W} \) is completely characterized by the chiral superfield \( W^{(0)}(X, \Theta^q, \bar{\Theta}_q, \lambda) \).

As a consequence of (4.35), \( W^{(0)} \) obeys
\[ \mathcal{P}_{\alpha \beta} W^{(0)} := (\partial_{\alpha \beta} - i\lambda_\alpha \lambda_\beta) W^{(0)} = 0, \] (4.40)
which is solved by a planewave in tensorial space,
\[ W^{(0)}(X, \Theta^q, \bar{\Theta}_q, \lambda) = \bar{W}(\lambda, \Theta^q, \bar{\Theta}_q) \exp\{i\lambda_\alpha \lambda_\beta X^{\alpha \beta}\}. \] (4.41)
The chirality of \( W^{(0)} \) and the first equation in (4.37), which now implies \( \partial_{\alpha q} W^{(0)} \propto \lambda_\alpha \), show that the general solution for the superparticle wavefunction is determined by the following chiral plane wave superfield
\[ W^{(0)}(X, \Theta^q, \bar{\Theta}_q, \lambda) = w(\lambda, \Theta^q \lambda) \exp\{i\lambda_\alpha \lambda_\beta (X + 2i\Theta^p \bar{\Theta}_p)_{\alpha \beta}\}, \] (4.42)
where \( \Theta^q \lambda = \Theta^a q_\lambda \) and
\[ w(\lambda, \Theta^q \lambda) = w^{(0)}(\lambda) + \sum_{k=1}^{N/2} \frac{1}{k!} (\lambda \Theta^{q_k}) \ldots (\lambda \Theta^{q_1}) w_{q_1 \ldots q_k}^{(k)}(\lambda). \] (4.43)
We refer to [97] for a discussion on how the arbitrary function \( w^{(0)}(\lambda_\alpha) \) with \( \alpha = 1, 2, 3, 4 \) encodes all the solutions of the massless higher spin equations in \( D = 4 \) and to [97, 103] for the \( D = 6, 10 \) cases. The key point is that \( \lambda_\alpha \) carries the degrees of freedom of a light-like momentum \( (\lambda^a \lambda \propto \lambda_\gamma \lambda \) is light-like in \( D = 4, 6, 10 \) which corresponds to \( n = 4, 8, 16 \) plus those of spin. The d.o.f. of \( \lambda_\alpha \) and those of the lightlike momenta are encoded, both up to a scale factor, in the coordinates of the compact manifolds \( S^{n-1} = S^{2D-5} \) and \( S^2 = S^{D-2} \), respectively. The spheres \( S^{D-2} \) are related to helicity in the \( n = 4, D = 4 \) case and to its

\[ \text{The 'celestial spheres' } S^{D-2} \text{ are the bases } S^{2,4,8} \text{ of the Hopf fibrations } S^{2D-5} \to S^{D-2} (S^{n-1} \to S^2). \]
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multidimensional generalizations for \( D = 6, 10 \) [97] [103].

To obtain a superfield on tensorial superspace describing massless conformal higher spin theories with extended supersymmetry, the wavefunction (4.42) has to be integrated over \( \mathbb{R}^n - \{0\} \sim S^{n-1} \times \mathbb{R}_+ \), parametrized by \( \lambda_\alpha \), with an appropriate measure that we denote by \( d^n\lambda \),

\[
\Phi(X, \Theta^q, \bar{\Theta}^q) = \int d^n\lambda W^{(0)}(X, \Theta^q, \bar{\Theta}^q, \lambda) = \int d^n\lambda \lambda_\alpha \lambda_\beta (X + 2i \Theta^p \bar{\Theta}_p)^{\alpha\beta} .
\]

(4.44)

One can easily check that the superfield \( \Phi \) is chiral

\[
\bar{D}^\alpha \Phi = 0 ,
\]

and satisfies the equation

\[
D_{[\gamma} D_{\beta]} \Phi = 0 .
\]

(4.46)

These are the superfield equations for the wavefunction of the superparticle in \( \mathcal{N} \)-extended tensorial superspace for even \( \mathcal{N} \).

4.3. From the superfield to the component form of the higher spin equations in tensorial space

4.3.1. \( \mathcal{N} = 2 \)

When \( \mathcal{N} = 2 \), Eqs. (4.45) and (4.46) can be written as

\[
\bar{D}^\alpha \Phi = 0 ,
\]

\[
D_{[\gamma} D_{\beta]} \Phi = 0
\]

(4.47)

and reproduce equations from page XIV. It is easy to check that Eqs. (4.47) imply the vanishing of all the components of the 'chiral' superfield \( \Phi(X, \Theta, \Theta) = \Phi(X + 2i \Theta \cdot \Theta, \Theta) \), except the first two,

\[
\Phi(X, \Theta, \Theta) = \phi(X_L) + i \Theta^\alpha \psi_\alpha(X_L) , \quad X_L^{\alpha\beta} = X^{\alpha\beta} + 2i \Theta^{(\alpha} \bar{\Theta}^{\beta)} = X_L^{\beta\alpha} ,
\]

(4.48)

where \( X_L^{\alpha\beta} \) is the analogue of the bosonic coordinates for the chiral basis of standard \( D = 4 \) superspace. The above components are the complex bosonic scalar and the complex
d of \( S^{3,7,15} \), \( (n, D)=(4,4), (8,6), (16,10) \). The fibres \( S^{D-3} = S^{1,3,7} \) of these bundles correspond to the complex, quaternion and octonion numbers of unit modulus. The remaining \( n=2, D=3 \) case corresponds to the first of the four Hopf fibrations, \( S^1 \rightarrow \mathbb{R}P^1 \); its fibre is determined by the reals of unit modulus, \( \mathbb{Z}_2 \), and there are no extra coordinates.
fermionic spinor fields obeying the free higher spin equations in tensorial space form \[98\],
\[
\partial_{\alpha[\gamma} \partial_{\delta] \beta} \phi(X) = 0 \ , \quad \partial_{\alpha[\beta} \psi_{\gamma]}(X) = 0 \ .
\] (4.49)

Let us recall that the \(\mathcal{N} = 1\) supermultiplet contains a real bosonic scalar and a real fermionic spinor field that obey the same equations (4.49). Hence, the \(\mathcal{N} = 2\) supermultiplet of the conformal fields in tensorial superspace is given by the complexification of the \(\mathcal{N} = 1\) supermultiplet.

Clearly, the above results are \(n\)-independent and thus, besides \(n = 4\), they are also valid for the \(n = 8\) and \(n = 16\) cases corresponding to the \(D = 6\) and \(D = 10\) multiplets of massless conformal higher spin fields.

### 4.3.2. \(\mathcal{N} = 4\)

In contrast with the \(\mathcal{N} = 2\) case, spin-tensorial components are present when \(\mathcal{N} > 2\). For \(\mathcal{N} = 4\), the general solution of the superfield equations (4.45) and (4.46) is given by
\[
\Phi(X, \Theta^q, \bar{\Theta}^q) = \phi(X_L) + i \Theta^{\alpha q} \psi_{\alpha q}(X_L) + \epsilon_{pq} \Theta^{\alpha p} \bar{\Theta}^{\beta q} F_{\alpha \beta}(X_L) \ ,
\] (4.50)
\[
X_{\alpha \beta}^L = X_{\alpha \beta} + 2i \Theta^q (\alpha \bar{\Theta}^q \beta) , \quad q = 1, 2 \ ,
\] (4.51)
where, again, the complex scalar and spinor fields obey the standard higher spin equations in their tensorial superspace form,
\[
\partial_{\alpha[\gamma} \partial_{\delta] \beta} \phi(X) = 0 \ , \quad \partial_{\alpha[\beta} \psi_{\gamma]}(X) = 0 \ ,
\] (4.52)
while the complex symmetric bi-spinor (or ‘\(s\)-tensor’ \[99\]) \(F_{\alpha \beta} = F_{\beta \alpha}\) satisfies the tensorial counterpart of the \(D = 4\) Maxwell equations (when these are written in spinorial notation \[176\], see also below),
\[
\partial_{\alpha[\gamma} F_{\delta] \beta}(X) = 0 \ , \quad F_{\alpha \beta} = F_{\beta \alpha} \ .
\] (4.53)

However, one can easily show that the general solution of Eq. (4.53) is expressed through a new complex scalar superfield \(\tilde{\phi}(X)\) satisfying the bosonic tensorial space equation in (4.52),
\[
F_{\alpha \beta} = \partial_{\alpha \beta} \tilde{\phi}(X) \ ,
\] (4.54)
\[
\partial_{\alpha[\gamma} \partial_{\delta] \beta} \tilde{\phi}(X) = 0 \ .
\] (4.55)

\(n = 4\) , \(D = 4\)

To prove this when \(n = 4\) \((\alpha, \beta = 1, 2, 3, 4)\), we begin by decomposing the complex symmetric \(GL(4)\) tensor \(F_{\alpha \beta} = F_{\beta \alpha}\) in \(2 \times 2\) blocks, thus keeping only the \(GL(2, \mathbb{C})\)
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manifest symmetry\(^5\)

\[ n = 4 : \quad F_{\alpha\beta} = \begin{pmatrix} F_{AB} & V_{AB} \\ V_{BA} & F_{\dot{A}\dot{B}} \end{pmatrix}, \quad A, B = 1, 2, \quad \dot{A}, \dot{B} = 1, 2. \quad (4.56) \]

Let us first notice that the block components of Eq. (4.53) which contain the antisymmetric tensors (encoded in the symmetric spin-tensors \(F_{AB}\) and \(F_{\dot{A}\dot{B}}\)) only,

\[ \partial_{\dot{A}[B} F_{C]D} = 0, \quad \partial_{A[B} F_{\dot{C}]D} = 0, \quad (4.57) \]

are equivalent to the Maxwell equations for the complex selfdual field \(F_{ab} = \frac{i}{2} \epsilon_{abcd} F^{cd} \propto \sigma_{ab}^{CD} F_{CD}\) i.e., they imply \(\partial^a F_{ab} = 0\) and \(\partial_{[a} F_{bc]} = 0\).

Consider now the components of Eq. (4.53) which contain the complex vector \(V_{AB} = \sigma_{AB}^a V_a\) only, namely

\[ \partial_{\dot{A}[B} V_{C]D} = 0, \quad \partial_{A[B} V_{\dot{C}]D} = 0 \quad (4.58) \]

and

\[ \partial_{\dot{A}[B} V_{C]D} = 0, \quad \partial_{A[B} V_{\dot{C}]D} = 0. \quad (4.59) \]

Eqs. (4.58) imply \(\partial_a V_b = 0\) and \(\partial^a V_a = 0\). The first is solved by \(V_a = \partial_a \tilde{\phi}\) and the second implies that the scalar field \(\tilde{\phi}\) obeys the Klein-Gordon equation \(\partial^a \partial_a \tilde{\phi} = 0\). In the spin-tensor notation these read

\[ V_{\dot{A}\dot{B}} = \partial_{\dot{A}\dot{B}} \tilde{\phi}, \quad \partial_{A[B} \partial_{C]D} \tilde{\phi} = 0. \quad (4.60) \]

Next, the components of Eq. (4.53) which contain both vector and antisymmetric tensor components, \(\partial_{AB} F_{CD} - \partial_{AC} V_{DB} = 0\) and \(\partial_{AB} F_{\dot{C}D} - \partial_{\dot{C}D} V_{AB} = 0\), can be written in the form

\[ \partial_{AB}(F_{CD} - \partial_{CD} \tilde{\phi}) = 0, \quad \partial_{\dot{A}\dot{B}}(F_{\dot{C}D} - \partial_{\dot{C}D} \tilde{\phi}) = 0, \quad (4.61) \]

the only covariant solution of which is given by

\[ F_{CD} = \partial_{CD} \tilde{\phi}, \quad F_{\dot{C}D} = \partial_{\dot{C}D} \tilde{\phi}. \quad (4.62) \]

Keeping in mind the Maxwell equations (4.57), one finds that the scalar field \(\tilde{\phi}(X)\) satisfies, besides the Klein-Gordon equation in (4.60), also the remaining components of Eq. (4.55),

\[ \partial_{\dot{A}[B} \partial_{C]D} \tilde{\phi} = 0, \quad \partial_{A[B} \partial_{\dot{C}]D} \tilde{\phi} = 0. \quad (4.63) \]

\(^5\)Notice that the \(SL(2, \mathbb{C})\) indices \(A, B = 1, 2, \dot{A}, \dot{B} = 1, 2\) are denoted by \(\alpha, \beta\) and \(\dot{\alpha}, \dot{\beta}\) in chapters 2 and 3.
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Proof for arbitrary $n$

We now prove that Eqs. (4.54), (4.55) provide the general solution of the Maxwell-like equation in tensorial space, Eq. (4.53), for any $n$. The Fourier transform of Eq. (4.53) is

$$p_\alpha [\gamma F_{\delta \beta}] (p) = 0.$$  

(4.64)

The solution of this equation is nontrivial iff the matrix of the generalized momentum has rank one, this is to say when $p_{\alpha \beta} = \lambda_\alpha \lambda_\beta$ for arbitrary $\lambda_\alpha \neq (0, ..., 0)$ or, equivalently, when this matrix obeys $p_\alpha [\gamma p_{\delta \beta}] = 0$. The general solution is characterized by $F_{\alpha \beta} (\lambda) = \lambda_\alpha \lambda_\beta \phi (\lambda)$ and can be equivalently written in the form $F_{\alpha \beta} (p) = p_{\alpha \beta} \tilde{\phi} (p)$ if $p_\alpha [\gamma p_{\delta \beta}] \tilde{\phi} (p) = 0$ of Eqs. (4.53). As far as set of equations

$$F_{\alpha \beta} (p) = p_{\alpha \beta} \tilde{\phi} (p), \quad p_\alpha [\gamma p_{\delta \beta}] \tilde{\phi} (p) = 0$$  

(4.65)

provide the Fourier transforms of Eqs. (4.54), (4.55), these describe the general solution.

Peccei-Quinn-like symmetry

Thus, the $\mathcal{N} = 4$ higher spin supermultiplet actually contains two complex scalar fields and two Dirac spinor fields in tensorial space, $\phi (X), \psi_\alpha^1 (X), \psi_\alpha^2 (X), \tilde{\phi} (X)$, which satisfy the free bosonic and fermionic higher spin equations, Eqs. (4.52), (4.55). They appear in the on-shell scalar superfield decomposition as

$$\Phi (X, \Theta^q, \bar{\Theta}_q) = \phi (X_L) + i \Theta^aq \psi_aq (X_L) + \epsilon_{pq} \Theta^aq \bar{\Theta}_b \partial_\alpha \beta \tilde{\phi} (X_L), \quad q, p = 1, 2.$$  

(4.66)

However, as the second complex scalar field $\tilde{\phi}$ enters the original superfield with a derivative, its zero mode is not fixed. In other words, this scalar is axion-like: it possesses the Peccei-Quinn-like symmetry

$$\tilde{\phi} (X) \mapsto \tilde{\phi} (X) + \text{const}.$$  

(4.67)

4.3.3. $\mathcal{N} = 8$

For higher $\mathcal{N} > 4$ the general solution of the set of superfield equations (4.45) and (4.46) is given by

$$\Phi (X, \Theta^q, \bar{\Theta}_q) = \phi (X_L) + i \Theta^aq \psi_aq (X_L) + \sum_{k=2}^{\mathcal{N}/2} \frac{1}{k!} \Theta^aq_k \ldots \Theta^aq_{\mathcal{N} \frac{1}{2}} F_{\alpha_1 \ldots \alpha_k q_1 \ldots q_k} (X_L);$$  

(4.68)

$$F_{\alpha_1 \ldots \alpha_k q_1 \ldots q_k} (X_L) = F_{(\alpha_1 \ldots \alpha_k) [q_1 \ldots q_k]} (X_L), \quad X^\alpha_{\beta L} = X^\alpha_{\beta} + 2i \Theta^q (\alpha \bar{\Theta}_q), \quad q = 1, ..., \mathcal{N}/2.$$  

(4.69)

More formally, the solution of this equation is given by a distribution with support on the subspace of tensorial momentum space defined by the condition $p_\alpha [\gamma p_{\delta \beta}] = 0$, so that $\tilde{\phi} (p) \propto \delta (p_\alpha [\gamma p_{\delta \beta}])$. 

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where \( \phi(X_L) \) and \( \psi_{a q}(X_L) \) obey the standard higher spin equations (4.49) while the higher components satisfy

\[
\partial_{\alpha[\gamma} \mathcal{F}_{\delta]\beta_2...\beta_k q_1...q_k}(X_L) = 0 , \quad \mathcal{F}_{\alpha_1...\alpha_q} = \mathcal{F}_{(\alpha_1...\alpha_q)} . \tag{4.70}
\]

For instance, for \( \mathcal{N} = 8 \) the superfield solution of the higher spin equations (4.45) and (4.46) reads

\[
\Phi(X, \Theta^a, \bar{\Theta}_a) = \phi(X_L) + i \Theta^{a q} \psi_{a q}(X_L) + \frac{1}{2} \Theta^{a_2 q_2} \Theta^{a_1 q_1} \mathcal{F}_{a_1 a_2 q_1 q_2}(X_L) + \\
\frac{i}{3!} \Theta^{a_3 q_3} \Theta^{a_2 q_2} \Theta^{a_1 q_1} \epsilon_{q_1 q_2 q_3 q_4} \psi_{a_1 a_2 a_3}(X_L) + \\
\frac{1}{4!} \epsilon_{q_1 q_2 q_3 q_4} \Theta^{a_4 q_4} ... \Theta^{a_1 q_1} \mathcal{F}_{a_1...a_4}(X_L) . \tag{4.71}
\]

Its two lowest components obey Eqs. (4.49), while its higher order nonvanishing field components satisfy Eqs. (4.70),

\[
\partial_{\alpha[\gamma} \mathcal{F}_{\delta]\beta q_1 q_2}(X) = 0 , \tag{4.72}
\]

\[
\partial_{\alpha[\gamma} \psi_{\delta]\beta_2 q_3 q_4}(X) = 0 , \tag{4.73}
\]

\[
\partial_{\alpha[\gamma} \mathcal{F}_{\delta]\beta_2 \beta_3 \beta_4}(X) = 0 . \tag{4.74}
\]

It is tempting to identify (4.73) with the tensorial space generalization of the Rarita-Schwinger equations and Eq. (4.74) with that of the linearized conformal gravity equation imposed on Weyl tensor. However, similarly to the \( \mathcal{N} = 4 \) case in Sec. 4.3.2 it is possible to show that the general solutions of Eqs. (4.72), (4.73) and (4.74) are expressed in terms of a sextuplet of scalar fields \( \phi_{q_1 q_2}(X) = \phi_{q_1 q_2}(X) \), a quadruplet of spinorial fields \( \psi_{a q}^\alpha \) and a singlet of scalar field \( \phi(X) \) obeying the standard tensorial space fermionic and bosonic higher spin equations (4.49),

\[
\mathcal{F}_{\alpha \beta q_1 q_2}(X) = \partial_{\alpha \beta} \phi_{q_1 q_2}(X) , \tag{4.75}
\]

\[
\psi_{a_1 a_2 a_3}^q(X) = \partial_{a_1 a_2} \psi_{a_3}^q(X) , \tag{4.76}
\]

\[
\mathcal{F}_{a_1...a_4}(X) = \partial_{a_1 a_2} \partial_{a_3 a_4} \phi(X) . \tag{4.77}
\]

Summarizing, the \( \mathcal{N} = 8 \) supermultiplet of free higher spin fields is described by a set of two scalar fields, a sextuplet of scalar fields, a spinor field and a quadruplet of spinorial fields, all in tensorial superspace, which obey the usual type free higher spin equations

\[
\partial_{\alpha[\gamma} \partial_{\delta]\beta \phi(X) = 0 , \quad \partial_{\alpha[\beta} \psi_{\gamma]}(X) = 0 , \quad \partial_{\alpha[\gamma} \partial_{\delta]\beta} \phi_{a q}(X) = 0 , \quad \partial_{\alpha[\beta} \partial_{\gamma]} \tilde{\psi}_{a q}(X) = 0 . \quad \tag{4.78}
\]

Thus, we conclude that, in tensorial superspace, at least all the free field dynamics is carried by the scalar and spinor fields. However, the ‘higher’ scalar and spinor fields appear in the basic superfield under the action of one or two derivatives and, hence, the model is invariant.
under the following generalized bosonic and fermionic Peccei-Quinn-like symmetries

\[
\phi_{qp}(X) \mapsto \phi_{qp}(X) + a_{qp}, \\
\tilde{\psi}_{\alpha q}(X) \mapsto \tilde{\psi}_{\alpha q}(X) + \beta_{\alpha q}, \\
\tilde{\phi}(X) \mapsto \tilde{\phi}(X) + a + X^{\alpha\beta}a_{\alpha\beta},
\] (4.79)

with constant bosonic parameters \( a_{qp} = -a_{pq}, a, a_{\alpha\beta} \) and constant fermionic parameter \( \beta_{\alpha q} \). Note that the non-constant shift in \( \tilde{\phi}(X) \) is allowed by the presence of two derivatives in Eq. (4.77).
In this chapter we present and study the covariant supersymmetric and $\kappa$–symmetric action for a system of $N$ nearly coincident $M^0$–branes (mM0 system) in flat eleven dimensional superspace and the equations of motion obtained from this action. As far as the mM0 action is written with the use of moving frame and spinor moving frame variables, we begin by describing their use in a simpler model of single $M^0$–brane. We obtain the complete set of mM0 equations of motion and study the symmetries of the mM0 action the most relevant of which is the reminiscence of the $K_9$ symmetry of the single $M^0$–brane. This symmetry allows us to write the bosonic equations of motion for the mM0 center of energy variables in their final form. The center of energy motion is characterized by a nonnegative constant mass $M$ which is contracted from the matrix fields which describe the relative motion of the mM0 constituents. We show that all supersymmetric solutions of the mM0 equations preserve 16 of 32 supersymmetries i.e. describe $\frac{1}{2}$ BPS states, and are characterized by vanishing center of energy mass $M^2 = 0$. We also present two examples of non supersymmetric solutions with $M^2 \neq 0$. 
5.1. Single M0–brane in spinor moving frame formulation

5.1.1. Twistor–like spinor moving frame action and its irreducible κ–symmetry

The spinor moving frame action of M0–brane reads (see [191] and also [192–195])

\[ S_{M0} = \int_{W^1} \rho^\# \hat{E}^a = \int_{W^1} \rho^\# u_a^- E^a(\hat{Z}) = \frac{1}{16} \int_{W^1} \rho^\# (v_q^- \Gamma_a v_q^-) \hat{E}^a. \]  

(5.1)

(5.2)

In the first line of this equation, (5.1), \( \rho^\#(\tau) \) is a Lagrange multiplier, \( \hat{E}^a := E^a(\hat{Z}) = d\hat{Z}^M(\tau) E^a_M(\hat{Z}) \) is the pull–back of the bosonic supervielbein of the 11D target superspace (\( a = 0, 1, \ldots, 10 \)), \( E^a = E^a(Z) = dZ^M E^a_M(Z) \), to the worldline \( W^1 \) parametrized by proper time \( \tau \). In the case of flat target superspace the supervielbein can be chosen in the form

\[ E^a = \Pi^a = dx^a - id\theta^\Gamma^\alpha \theta, \quad E^a = d\theta^a. \]  

(5.3)

(5.4)

Finally, \( \hat{E}^a = \hat{E}^a u_a^- \) and \( u_a^- = u_a^- (\tau) \) is a light–like 11D vector, \( u^a u_a^- = 0 \).

One can write the action (5.1) in a probably more conventional from, extracting \( d\tau \) measure from the pull–back of the supervielbein 1–form (see (5.3))

\[ S_{M0} = \int_{W^1} d\tau \rho^\# \hat{E}^a = \int_{W^1} d\tau \rho^\# \partial_\tau \hat{Z}^M(\tau) E^a_M(\hat{Z}(\tau)) u_a^- (\tau). \]  

(5.5)

We however, prefer to hide \( d\tau \) inside of differential form, define the Lagrangian 1–form by \( \mathcal{L}_1 = d\tau \mathcal{L}_\tau \), and write our actions as integral of this 1–form over the worldline, \( \int_{W^1} \mathcal{L}_1 \), rather than as an integral over \( d\tau \) of a density, \( \int d\tau \mathcal{L}_\tau \).

If we were stoping at this stage, one can easily observe that the action (5.1) can be

1The action (5.1), (5.2) makes sense when supervielbein \( E^a = dZ^M E^a_M(Z) \) obeys the 11D superspace supergravity constraints [149]. In this chapter we will restrict ourselves by the case of flat target superspace, described by Eqs. (5.4).

2We use the (real) matrices \( \Gamma^a_{\alpha\beta} = \Gamma^a_{\beta\alpha} = \Gamma^\alpha_{\gamma\beta} C_{\gamma\beta} \) and \( \Gamma^a_{\beta} = \hat{\Gamma}^a_{\alpha} = C_{\alpha\gamma} \Gamma^\gamma_{\beta} \) constructed as a product of 11D Dirac matrices \( \Gamma^\alpha_{\beta} \) (obeying \( \Gamma^\alpha T^b + \Gamma^b T^\alpha = 2\eta^{ab} I_{32 \times 32} \)) with, respectively, the 11D charge conjugation matrix \( C_{\gamma\beta} = -C_{\beta\gamma} \) and its inverse \( C^{\alpha\beta} = -C^{\beta\alpha} \). Both \( \Gamma^a_{\gamma} \) and \( C_{\beta\gamma} \) are pure imaginary in our mostly minus notation \( \eta^{ab} = \text{diag}(1, -1, \ldots, -1) \).
obtained from the first order form of 11D version of the Brink–Schwarz action,

\[ S_{BS} = \int_{\mathcal{W}} \left( p_a \hat{E}^a - \frac{e}{2} p_a p^a d\tau \right), \tag{5.6} \]

by solving the constraints \( p_a p^a = 0 \) (equations of motion for Lagrange multiplier \( e(\tau) \)) and substituting them back to the action. Furthermore, one might wonder why the solution \( p_a = \rho^u(\tau) \) is written with a multiplier \( \rho^u(\tau) \) instead of just stating that it has the form of \( S = \int_{\mathcal{W}} p_a \hat{E}^a \) with \( p_a \) constrained by \( p_a p^a = 0 \). We will answer that question a bit later, just announcing now that \( \rho^u \) is a kind of Stückelberg variable allowing to introduce an \( SO(1,1) \) gauge symmetry; although looking artificial at this stage, this symmetry allows to clarify the group theoretical meaning of \( u_a^= \) and also of the set of 16 constrained spinors appearing in the second representation of \( S_{M0} \), Eq. (5.2).

The light–like vector \( u_a^= \) can be considered as a composite of (any of) the 16 spinors \( v_q^= \) provided these are constrained by

\[ v_q^{-\alpha}(\Gamma^a)_{\alpha\beta} v_p^{-\beta} = \delta_{qp} u_a^= \tag{5.7a} \]

\[ 2 v_q^{-\alpha} v_q^{-\beta} = u_a^= \tilde{\Gamma}_a^{\alpha\beta}. \tag{5.7b} \]

Notice that the trace of (5.7a) as well as the \( \Gamma \)–trace of (5.7b) give \( 16 u_a^= = v_q^{-\alpha}(\Gamma^a)_{\alpha\beta} v_q^{-\beta} \) which can be read off (5.2) and (5.1). The set of spinors \( v_q^= \) constrained by (5.7) are called \textit{spinor moving frame variables} (hence the name ‘spinor moving frame’ for the formulation of superparticle mechanics based on the action (5.1), (5.2)). Before discussing their origin and nature (in sec. 5.1.3), in the next sec. 5.1.2 we would like to try to convince the reader in the usefulness of these ‘square roots’ of the light–like vector \( u_a^= \).

\subsection*{5.1.2. Irreducible \( \kappa \)–symmetry of the spinor moving frame action}

The action (5.1), (5.2) is invariant under the following local fermionic \( \kappa \)–symmetry transformations

\[ \delta_{\kappa} \hat{x}^a = -i \hat{\theta} \Gamma^a \delta_{\kappa} \hat{\theta}, \quad \delta_{\kappa} \hat{\theta}^a = \epsilon^{+q}(\tau) v_q^{-\alpha}, \]

\[ \delta_{\kappa} \rho^# = 0, \]

\[ \delta_{\kappa} u_a^= = 0 \iff \delta_{\kappa} v_q^{-\alpha} = 0. \tag{5.8} \]

These symmetry is \textit{irreducible} in the sense of that each of 16 fermionic parameters \( \epsilon^{+q}(\tau) \) acts efficiently on the variables of the theory and can be used to remove some component of fermionic field \( \hat{\theta}^a(\tau) \) thus reducing the number of the degrees of freedom in it to 16 (while \( \alpha = 1, \ldots, 32 \)).

\footnote{The \( \kappa \)–symmetry was discovered in \cite{188,196} and was shown to coincide with the local worldline supersymmetry in \cite{197}. Our notation \( \epsilon^{+q}(\tau) \) for the (irreducible) \( \kappa \)–symmetry parameter is an implicit reference on this later result which will be useful in the discussion below.}
5.1. Single M0–brane in spinor moving frame formulation

In contrast, the $\kappa$–symmetry of the original Brink–Schwarz superparticle action (5.6) is infinitely reducible. It is parametrized by 32 component fermionic spinor function $\kappa_{\beta}(\tau)$ which however is not acting efficiently on the variable of the theory.

The irreducible $\kappa$–symmetry of the spinor moving frame formulation (5.8) can be obtained from the infinitely reducible (5.9) by substituting for $p_a$ the solution $p_a = \rho^\# u_a^\alpha$ of the constraint $p_a p^a = 0$; furthermore, using (5.7), we find

$$\epsilon^+ = 2 \rho^\# v_q^{-\alpha} \kappa_{\alpha}.$$  

(5.10)

Let us stress that this relation, as well as the transformation rules of the irreducible $\kappa$–symmetry (5.8), necessarily involves the constrained spinors $v_q^{-\alpha}$. Thus the covariant irreducible form of the $\kappa$–symmetry is a characteristic property of the spinor moving frame and similar (‘twistor–like’) formulations of the superparticle mechanics.

The importance of the $\kappa$–symmetry is related to the fact that it reflects a part of target space supersymmetry which is preserved by ground state of the brane under consideration thus insuring that it is a BPS state. Its irreducible form, reached in the frame of supersymmetry ([197]), but also for finding the corresponding induced supergravity multiplet spinor moving frame formulation, is useful not only for clarifying its nature as worldline supersymmetry ([71, 77] thus insuring that it is a BPS state. Its irreducible form of the $\kappa$–symmetry (5.8), necessarily involves the constrained spinors $v_q^{-\alpha}$. Let us stress that this relation, as well as the transformation rules of the irreducible $\kappa$–symmetry is a characteristic property of the spinor moving frame and similar (‘twistor–like’) formulations of the superparticle mechanics.

5.1.3. Moving frame and spinor moving frame

To clarify the origin and nature of the set of spinors $v_q^{-\alpha}$ which provide the square root of the light–like vector $u_a^\alpha$ in the sense of Eqs. (5.7), and which have been used to present the $\kappa$–symmetry in the irreducible form (5.8), it is useful to complete the null–vector $u_a^\alpha$ till the moving frame matrix,

$$U_b^{(a)} = \left( \frac{u_a^\alpha + u_b^\alpha}{2}, u_b^\alpha, \frac{u_b^\alpha - u_a^\alpha}{2} \right) \in SO(1,10)$$  

(5.11)

$^4$Roughly speaking, due to the constraint $p_a p^a = 0$, $\kappa_{\alpha}$ and $\kappa_{\alpha} + p_a \tilde{\Gamma}_{\alpha \beta}^{\alpha \beta} \kappa^{(1,\beta)}(\tau)$ produce the same $\kappa$ variation of the Brink–Schwarz superparticle variables. One says that the above transformation has a null–vector $\kappa^{(1,\beta)}(\tau)$ and, hence, the symmetry is reducible. But this is not the end of story. One easily observes that $\kappa^{(1,\beta)}(\tau)$ and $\kappa^{(2,\beta)}(\tau) + p_a \tilde{\Gamma}_{\alpha \beta}^{\alpha \beta} \kappa^{(2,\beta)}(\tau)$, with an arbitrary $\kappa^{(2,\beta)}(\tau)$, makes the same change of the parameter $\kappa_{\alpha}$. This implies that there is a null–vector for null–vector and that the $\kappa$-symmetry possesses at least the second rank of reducibility. Furthermore, one sees that this process of finding higher null–vectors can be continued up to infinity (next stages are completely equivalent to the first two ones) so that one speaks about infinite reducibility of the $\kappa$–symmetry of the Brink–Schwarz superparticle. The number of the fermionic degrees of freedom which can be removed by $\kappa$–symmetry is then calculated as an infinite sum $32 - 32 + 32 - 32 + ... = 32 \cdot (1 - 1 + 1 - 1 + ...) = 32 \cdot \lim_{q \to 1} (1 - q + q^2 - ...) = 32 \cdot \lim_{q \to 1} \frac{1}{1 + q} = 16$.

$^5$Notice that in D=3,4 and 6 dimensions the counterpart of $v_q^{-\alpha}$ can be chosen to be unconstrained spinors; see references in e.g. [191] [192] [194].
The statement that this matrix is an element of the $SO(1,10)$, having been made in (5.11), is tantamount to saying that
\[ U^T \eta U = I, \quad \eta^{ab} = \text{diag}(+1,-1,\ldots,-1), \quad (5.12) \]
which in its turn implies that the moving frame vectors obey the following set of constraints
\[ u_a u^a = 0, \quad u_a u^a = 0, \quad u_a u^{a#} = 2, \quad (5.13) \]
\[ u_a u^{a#} = 0, \quad u_a u^{a#} = 0, \quad u_a u^{a#} = -\delta^{ij}. \quad (5.14) \]

The 11D spinor moving frame variables (appropriate for our case) can be defined as $16 \times 32$ blocks of the $Spin(1,10)$ valued matrix
\[ V_{(\beta)}^\alpha = \left( \begin{array}{c} v^+_{\alpha} \\ v^-_{\alpha} \end{array} \right) \in Spin(1,10) \quad (5.16) \]
double covering the moving frame matrix (5.11). This statement implies that the similarity transformations with the matrix $V$ leaves the 11D charge conjugation matrix invariant and, when applied to the 11D Dirac matrices, produce the same effect as 11D Lorentz rotation with matrix $U$,
\[ V C V^T = C, \quad (5.17) \]
\[ V \Gamma_b V^T = U_b^{(a)} \Gamma_a, \quad (5.18) \]
\[ V^T \tilde{\Gamma}^{(a)} V = \tilde{V}^b U^b_{(a)}. \quad (5.19) \]

The two seemingly mysterious constraints (5.7) appear as a $16 \times 16$ block of the second of these relations, (5.17), and as a component $V^T \tilde{\Gamma}^{(a)} V = \tilde{V}^b u^b_{(a)}$ of the third one, (5.19) (with an appropriate representation of the 11D Gamma matrices (see Appendix E)). The other blocks/components of these constraints involve the second set of constrained spinors,
\[ v^+_{q} \Gamma_a v^+_{p} = u_a^{#} \delta_{qp}, \quad v^-_{q} \Gamma_a v^+_{p} = -u^i_a \gamma^{i}_{qp}, \quad (5.20) \]
\[ 2v^+_{q} \Gamma_{a} v^+_{q} = \tilde{\Gamma}^{a\alpha\beta} u^#_{a}, \quad 2v^-_{q} \Gamma_{a} v^+_{q} = -\tilde{\Gamma}^{a\alpha\beta} u^i_{a}. \quad (5.21) \]

Here $\gamma^{i}_{qp}$ are the 9d Dirac matrices; they are real, symmetric, $\gamma^{i}_{qp} = \gamma^{i}_{pq}$, and obey the Clifford algebra
\[ \gamma^i \gamma^j + \gamma^j \gamma^i = 2 \delta^{ij} I_{16 \times 16}, \quad (5.22) \]
5.1. Single M0–brane in spinor moving frame formulation

as well as the following identities
\[ \gamma^i_q(p_1 \gamma^i_{p_2 p_3}) = \delta_q(p_1 \delta_{p_2 p_3}) , \]
\[ \gamma^{ij}_q(p_1 \gamma^{ij}_{p_2 p_3}) = \gamma^j_q(p_1 \delta_{p_2 p_3} - \delta_{q_1 q_2} \gamma^j_{q_1 q_2}) . \]

Thus \( v_{q^-}^a \) and \( v_{q^+}^a \) can be identified as square roots of the light–like vectors \( u^a_\alpha \) and \( u^a_\alpha \), respectively, while to construct \( u^a_\alpha \) one needs both these sets of constrained spinors.

The first constraint, Eq. (5.17), implies that the inverse spinor moving frame matrix
\[ V^{(a)}(\beta) = (v^{a+}_q, v^{a-}_q) \in Spin(1, 10) , \]
\[ V^{(\beta)} V^{(a)} = \delta^{(\beta)}_{(a)} = \begin{pmatrix} \delta_{qp} & 0 \\ 0 & \delta_{qp} \end{pmatrix} \]
\[ \Leftrightarrow \begin{cases} v_{q^-}^a v_{qp}^+ = \delta_{qp} = v_{q^+}^a v_{qp}^- , \\
v_{q^-}^a v_{qp}^- = 0 = v_{q^+}^a v_{qp}^+ , \end{cases} \]
can be constructed from \( v_{q^+}^a \),
\[ v_{aq}^- = iC_{a\beta v_{q}^-}^{\beta} , \quad v_{aq}^+ = -iC_{a\beta v_{q}^+}^{\beta} . \]

5.1.4. Cartan forms, differentiation and variation of the (spinor) moving frame variables

To vary the action and to clarify the structure of the equations of motion one needs to vary and to differentiate the moving frame and spinor moving frame variables. As these are constrained, at the first glance this problem might look complicated, but, actually, this is not the case. The clear group theoretical structure beyond the moving frame and spinor moving frame variables makes their differential calculus and variational problem extremely simple.

Referring again for the details to [139, 191], let us just state that the derivatives of the moving frame and spinor moving frame variables can be expressed in terms of the \( so(1, 10) \)–valued Cartan forms \( \Omega^{\alpha(b)} = U^{(a)c} dU^{(b)}_c \) the set of which can be split onto the covariant Cartan forms
\[ \Omega^{=} = u^a d u^{a} , \quad \Omega^{#} = u^{#} d u^{a} , \]
providing the basis for the coset \( SO(1, 10) / SO(1,1) \times SO(9) \), and the forms
\[ \Omega^{(0)} = \frac{1}{4} u^a d u^a , \]
\[ \Omega^{ij} = u^{ia} d u^{ja} , \]
which have the properties of the \( SO(1, 1) \) and \( SO(9) \) connection respectively. These can be used to define the \( SO(1, 1) \times SO(9) \) covariant derivative \( D \). The covariant derivative of
the moving frame vectors is expressed in terms of the covariant Cartan forms (5.27)

\[ Du^b_\alpha := du^b_\alpha + 2\Omega^{(0)}u^b_\alpha = u^b_\alpha \Omega = i, \]  
\[ Du^b_\# := du^b_\# - 2\Omega^{(0)}u^b_\# = u^b_\# \Omega = i, \]  
\[ Du^i_\alpha := du^i_\alpha - \Omega^{ij}u^j_\alpha = \frac{1}{2}u^b_\# \Omega^i = \frac{1}{2}u^b_\# \Omega^i. \]  

The same is true for the spinor moving frame variables,

\[ Dv^-_q := dv^-_q + \Omega^{(0)}v^-_q - \frac{1}{4} \Omega^{ij}_q \gamma^{ij} v^-_p = -\frac{1}{2} \Omega^i v^-_p \gamma^i_{pq}, \]  
\[ Dv^+_q := dv^+_q - \Omega^{(0)}v^+_q + \frac{1}{4} \Omega^{ij}_q \gamma^{ij} v^+_p = -\frac{1}{2} \Omega^i v^+_p \gamma^i_{pq}. \]  

The variation of moving frame and spinor moving frame variables can be obtained from the above expression for derivatives by a formal contraction with variation symbol, \( i\delta d = \delta \) (this is to say, by taking the Lie derivatives). The independent variations are then described by \( i\delta \Omega \) contraction of the Cartan forms, \( i\delta \Omega^{(a)(b)} \). Furthermore, \( i\delta \Omega^{(i)} \) and \( i\delta \Omega^{ij} \) are the parameters of the \( SO(1, 1) \) and \( SO(9) \) transformations, which are manifest gauge symmetries of the model. Then the essential variation of the moving frame and spinor moving frame variables, this is to say, variations which produce (better to say, which may produce) nontrivial equations of motion, are expressed in terms of \( i\delta \Omega^i \) and \( i\delta \Omega^i \),

\[ \delta u^b_\alpha = u^b_\alpha i\delta \Omega^i = i, \quad \delta u^b_\# = u^b_\# i\delta \Omega^i = i, \]  
\[ \delta u^i_\alpha = \frac{1}{2}u^b_\# i\delta \Omega^i = \frac{1}{2}u^b_\# i\delta \Omega^i. \]  
\[ \delta v^-_q = -\frac{1}{2} i\delta \Omega^i v^-_p \gamma^i_{pq}, \]  
\[ \delta v^+_q = -\frac{1}{2} i\delta \Omega^i v^+_p \gamma^i_{pq}. \]  

5.1.5. \( K_9 \) gauge symmetry of the spinor moving frame action of the 11D \( M_0 \)-brane

A simple application of the above formulae begins by observing that the parameter \( i\delta \Omega^i \) does not enter the variation of neither \( u^\alpha \) nor \( v^-_\alpha \). However, the \( M_0 \)-brane (5.1), (5.2) involves only these (spinor) moving frame variables. Hence the transformation of the spin moving frame corresponding to \( \tau \) dependent parameters \( k^i = i\delta \Omega^i \) are gauge symmetries
5.1. Single M0–brane in spinor moving frame formulation

of this M0 action. These so–called K9–symmetry transformations

\[ \delta u_b^- = 0, \quad \delta u_b^+ = u_b^i k^{#i}, \quad \delta u_b^i = \frac{1}{2} u_b^- k^{#i}, \]  

\[ \delta v^- = 0, \quad \delta v^+ = -\frac{1}{2} k^{#i} v_p^- \gamma_p^i \]  

(5.39)

(5.40)

should be taken into account when calculating the number of M0 degrees of freedom. Quite interesting remnants of this K9 symmetry survives in the multiple M0 case and will be essential to understand the structure of mM0 equations of motion.

5.1.6. Derivatives and variations of the Cartan forms

One can easily check that the covariant Cartan forms are covariantly constant,

\[ D \Omega^{=i} = 0, \quad D \Omega^{#i} = 0, \]  

(5.41)

where the covariant derivatives include the induced connection (5.28), (5.29) The curvatures of these connections,

\[ F^{(0)} := d \Omega^{(0)} = \frac{1}{4} \Omega^{=i} \wedge \Omega^{#i}, \]  

\[ G^{ij} := d \Omega^{ij} + \Omega^{ik} \wedge \Omega^{kj} = -\Omega^{=i} \wedge \Omega^{#j}, \]  

(5.42)

(5.43)

can be calculated, e.g., from the integrability conditions of Eqs. (5.30)–(5.32),

\[ DD u_a^# = 2 F^{(0)} u_a^#, \quad DD u_a^i = u_a^j G^{ji}. \]  

(5.44)

As in the case of moving frame variables (see sec. 5.1.4), the variations of the Cartan forms can be obtained from the above expressions using the Lie derivative formula. Omitting the transformations of manifest gauge symmetries SO(1,1) and SO(9) (parametrized by \( i_\delta \Omega^{(0)} \) and \( i_\delta \Omega^{ij} \)), we present the essential variations:

\[ \delta \Omega^{#i} = D i_\delta \Omega^{#i}, \quad \delta \Omega^{=i} = D i_\delta \Omega^{=i}, \]  

\[ \delta \Omega^{ij} = \Omega^{=#i} [i_\delta \Omega^{#j} - \Omega^{=#j} i_\delta \Omega^{=]=i}, \]  

\[ \delta \Omega^{(0)} = \frac{1}{4} \Omega^{=#i} i_\delta \Omega^{#i} - \frac{1}{4} \Omega^{=#i} i_\delta \Omega^{=i}. \]  

(5.45)

(5.46)

(5.47)

These equations will be useful to vary the multiple M0–brane action in Sec. 5.4. For deriving the equations of motion of single M0–brane it is sufficient to use Eqs. (5.35), (5.37) and (5.30)–(5.34).

\[ D \Omega^{=i} := d \Omega^{=i} + 2 \Omega^{=i} \wedge \Omega^{(0)} - \Omega^{=j} \wedge \Omega^{ij}, \]  

(5.30)
5.2. Equations of motion of a single $M_0$–brane and induced $\mathcal{N} = 16$ supergravity on the worldline $W^1$

The moving frame matrix $U_{ab}^{(5.11)}$ provides a ‘bridge’ between the 11D Lorentz group and its $SO(9) \otimes SO(1,1)$ subgroup in the sense that it carries one index $(a)$ of $SO(1,10)$ and one index $(b)$ transformed by a matrix from $SO(9) \otimes SO(1,1)$ subgroup of $SO(1,10)$. Contracting the pull–back of the bosonic supervielbein form $\hat{E}_b$ we arrive at

$$\hat{E}^a = \hat{E}^b U_{ba}^{(a)} = (\hat{E}^= , \hat{E}^\# , \hat{E}^i)$$

(5.48)

which is split covariantly in three types of one forms. These are inert under $SO(1,10)$ but carry the nontrivial $SO(9)$ vector index (in the case of $\hat{E}^i$) or $SO(1,1)$ weights (in the cases of $\hat{E}^=$ and $\hat{E}^\#$). The corresponding decomposition of the vector representation of $SO(1,10)$ with respect to its $SO(9) \otimes SO(1,1)$ subgroup,

$$11 \mapsto 1_{-2} + 1_{+2} + 9_0,$$

is even better illustrated by the equation $\hat{E}^{(a)} U_{(a)}^{b} = \hat{E}^b$ which, in more detail, reads

$$\hat{E}^a = \frac{1}{2} \hat{E}^= u^a^+ + \frac{1}{2} \hat{E}^\# u^a^- - \hat{E}^i u^a^i .$$

(5.49)

Thus the moving frame vectors help to split the pull–back of the supervielbein in a Lorentz covariant manner. The $SO(9)$ singlet one form with $SO(1,1)$ weight $-2$, $\hat{E}^= = \hat{E}^b u^b_-$ enters the action (5.1) multiplied by the weight $+2$ worldline scalar field $\rho^\#(\tau)$. This clearly has the meaning of the Lagrange multiplier: its variation results in vanishing of $\hat{E}^= , \hat{E}^\# , \hat{E}^i$

$$\delta \hat{E}^a := \hat{E}^a u^a = 0 .$$

(5.50)

Now, the variation of $\hat{E}^= $ contain two different contributions, $\delta \hat{E}^= = \delta \hat{E}^a u^a^- + \hat{E}^a \delta u^a^-$. The first comes from the variation of the pull–back of the bosonic supervielbein form which in our case of flat target superspace can be easily calculated with the result

$$\delta \hat{E}^a = -i d\theta \Gamma^a \delta \theta + (i d\hat{\theta} \Gamma^a - i d\hat{\theta} \Gamma^a \hat{\theta}) .$$

(5.51)

The second term contains the variation of the light–like vector $u^-_a$ which can be written as in Eq. (5.35), $\delta u^-_a = u^-_a i_\delta \Omega^{-i}$. The corresponding variation of the action (5.1) reads $\delta S_{M0} = \int_{W1} \rho^\# \delta u^-_a \hat{E}^a = \int_{W1} \rho^\# u^-_a \hat{E}^a i_\delta i_\bar{\delta} \Omega^{-i}$ and produce the equation of motion

$$\hat{E}^a := \hat{E}^a u^a = 0 .$$

(5.52)

Using Eq. (5.49) one can collect Eqs. (5.50) and (5.52) in

$$\hat{E}^a := \frac{1}{2} \hat{E}^\# u^a^- .$$

(5.53)
5.2. Equations of motion of a single M0–brane and induced $\mathcal{N} = 16$ supergravity on the worldline $W^1$

This equation shows that the M0–brane worldline $W^1$ is a light–like line in target (super)space, as it should be for the massless superparticle.

Furthermore [5.53] suggests to consider $\hat{E}^\#$ as einbein on the worldline $W^1$; this composite einbein is induced by embedding of $W^1$ into the target superspace. The transformation of $\hat{E}^\#$ under the irreducible $\kappa$–symmetry (5.8) is given by $\delta_{\kappa} \hat{E}^\# = -2i \hat{E}^{+q} e^{+q}$. In the light of the identification of $\kappa$–symmetry with local worldline supersymmetry [197], this equation suggests to consider the covariant $16_+$ projection, $\hat{E}^{+q} = \hat{E}^\alpha v^\alpha_{16}q$, of the pull–back of the fermionic 1–form $E^\alpha$ as induced ‘gravitino’ companion of the induced 1d ‘graviton’ $\hat{E}^\#$. Indeed under the $\kappa$–symmetry (5.8) this set of forms show the typical transformations rules of (1d $\mathcal{N} = 16$) supergravity multiplet,

$$\delta_{\kappa} \hat{E}^{+q} = D e^{+q}(\tau) , \quad \delta_{\kappa} \hat{E}^\# = -2i \hat{E}^{+q} e^{+q} . \quad (5.54)$$

Here $D = d\tau D_\tau$ is the $SO(1, 1) \times SO(9)$ covariant derivative which we will specify below. The connection in this covariant derivative are defined in terms of moving frame variables and, hence, are inert under the $\kappa$–symmetry; in this sense the induced 1d $\mathcal{N} = 16$ supergravity multiplet is described essentially by 1 bosonic and 16 fermionic 1–forms $\hat{E}^\#$ and $\hat{E}^{+q}$. Our action for the mM0 system, which we present in the next section, will contain the coupling of these induced 1d supergravity to the matter describing the relative motion of the mM0 constituents.

The other, $16_-$ projection $\hat{E}^{-q} = \hat{E}^\alpha v^{-q}_\alpha$ of the pull–back of fermionic supervielbein form to $W^1$ vanishes on the mass shell,

$$\hat{E}^{-q} := \hat{E}^\alpha v^{-q}_\alpha = 0 . \quad (5.55)$$

Indeed, varying the coordinate functions in the action (5.1) we arrive at equation $\frac{\delta S_{M0}}{\delta \hat{Z}^M} = 0$ which reads

$$\partial_\tau (\rho^# u^\alpha = E^\alpha_M(\hat{Z})) = 0 . \quad (5.56)$$

In our case of flat target superspace $E^\alpha_M(\hat{Z}) = \delta^\alpha_M - i\delta^\alpha_M(\Gamma^\alpha \hat{\theta})_\alpha$ and one can easily split (5.56) into the bosonic vector and fermionic spinor equations (which we prefer to write with the use of $d = d\tau \partial_\tau$)

$$d(\rho^# u^\alpha) = 0 , \quad (5.57)$$

$$\rho^# u^\alpha (\Gamma^\alpha \partial_\tau \hat{\theta})_\alpha = 0 . \quad (5.58)$$

Using (5.7b) and assuming $\rho^# \neq 0$ we find that (5.58) is equivalent to Eq. (5.55). This implies that the $d\hat{\theta}^\alpha$ can be expressed through the induced gravitino,

$$\hat{E}^\alpha = d\hat{\theta}^\alpha = \hat{E}^{+q} v^{-q}_\alpha . \quad (5.59)$$

Let us come back to the equation for the bosonic coordinate functions, (5.57) (or equivalently, $\partial_\tau (\rho^# u^\alpha) = 0$). Using (5.30) we can write this in the form $0 = D \rho^# u^\alpha + \rho^# u^\alpha_\Omega^{-1}$. Here and below we use the covariant derivatives defined in (5.30), (5.31), (5.32).
Chapter 5. Covariant action and equations of motion for the 11D system of multiple M0-branes

Contracting that equation with $u^{\alpha\#}$ gives us

$$D\rho^{\#} = 0 \ ,$$

(5.60)

while the nontrivial part of the bosonic equations of motion of a single M0–brane, which can be read off from the coefficient for $u^i_a$, states that the covariant Cartan form $\Omega^{\#i}$ vanishes,

$$\Omega^{\#i} = 0 \ .$$

(5.61)

Coming back to Eq. (5.30), we see that Eq. (5.61) can be expressed by stating that the covariant derivative of the light–like vector $u_a^\#$ vanishes,

$$Du_a^\# = 0 \ ,$$

(5.62)

or, equivalently, by

$$Dv_q^{-\alpha} = 0 \ .$$

(5.63)

On the other hand, using

$$D = d\tau D_\tau = \hat{E}^\# D_\# \ ,$$

(5.64)

we can write Eq. (5.62) in the form $D_# u_a^\# = 0$, and, as far as (5.53) implies $u_a^\# = 2\hat{E}_a^\#$, in the following more standard form

$$D_# \hat{E}_a^\# = 0 \ ,$$

(5.65)

or, in more detail,

$$D_# D_# \hat{x}_a = iD_# (D_# \hat{\theta} \Gamma^a \hat{\theta}) \ .$$

(5.66)

Two more observations will be useful below. The first is that Eq. (5.60), $0 = D\rho^{\#} = d\rho^{\#} - 2\rho^{\#} \Omega^{(0)}$, can be solved with respect to the induced $SO(1,1)$ connection,

$$\Omega^{(0)} = \frac{d\rho^{\#}}{2\rho^{\#}} \ .$$

(5.67)

Notice that this is in agreement with the statement that one can always gauge away any 1d connection: using the local $SO(1,1)$ symmetry to fix the gauge $\rho^{\#} = \text{const}$ we arrive at $\Omega^{(0)} = 0$.

The second comment concerns the supersymmetric pure bosonic solutions of the above equations of motion.
5.2.1. All supersymmetric solutions of the M0 equations describe 1/2 BPS states

As far as the fermionic coordinate function $\hat{\theta}^\alpha$ is transformed by both spacetime supersymmetry and by the worldline supersymmetry ($\kappa$-symmetry), $\delta \hat{\theta}^\alpha = -\epsilon^\alpha + \epsilon^{+q}(\tau)v_q^{-\alpha}(\tau)$, the purely bosonic solutions of the M0 equations, having

$$\dot{\theta}^\alpha = 0 \quad \text{(5.68)}$$

may preserve a part of target space supersymmetry. This is characterized by parameter

$$\epsilon^\alpha = \epsilon^{+q}(\tau)v_q^{-\alpha}(\tau) \quad \text{(5.69)}$$

The left hand side of this equation contains a constant fermionic spinor $d\epsilon^\alpha = 0$, so that $d(\epsilon^{+q}v_q^{-\alpha}) = De^{+q}v_q^{-\alpha} + \epsilon^{+q}Dv_q^{-\alpha} = 0$. Furthermore, taking into account that the equations of motion for the bosonic coordinate function, Eq. (5.66), implies (5.63), one finds that the consistency of (5.69) is the covariant constancy of the $\kappa$–symmetry parameter $\epsilon^{+q}(\tau), \quad D\epsilon^{+q} = 0 \quad \text{(5.70)}$.

In 1d system the connection can be gauged away so that this condition can be reduced to the existence of a constant $SO(9)$ spinor $\epsilon^q$. For instance gauging away the $SO(9)$ connection and using Eq. (5.67), we can present (5.70) in the form $d(\epsilon^{+q}/\sqrt{\rho^q}) = 0$ and solve it by $\epsilon^{+q} = \sqrt{\rho^q} \epsilon^q$ with $d\epsilon^q = 0$.

This implies that any purely bosonic solution of the M0 equations preserves exactly 1/2 of the spacetime supersymmetry.

5.3. Covariant action for multiple M0–brane system

5.3.1. Variables describing the mM0 system

Let us introduce the dynamical variables describing the system of multiple M0–branes, which we abbreviate as mM0. Its dimensional reduction is expected to produce the system of N nearly coincident D0–branes (mD0 system) and at very low energy this later is described by the action of 1d $\mathcal{N} = 16$ supersymmetric Yang–Mills theory (SYM) with the gauge group $U(N)$, which is given by dimensional reduction of the 10D $\mathcal{N} = 1$ U(N) SYM down to d=1. Now, the set of fields of the U(N) SYM can be split onto the non-Abelian SU(N) SYM and Abelian U(1) SYM multiplets. Roughly speaking, this later describes the center of energy motion of the mD0 system while the former corresponds to the relative motion of the constituents of the mD0 system. Then it is natural to assume that the relative motion of the mM0 constituents are also described by the fields of SU(N) SYM multiplet.

Now let us turn to the center of energy motion. We begin by noticing that the $U(1)$ SYM fields can be seen in the single D0 brane action (see [65] and refs therein) after fixing the gauge with respect to $\kappa$–symmetry and reparametrization symmetry. Originally the
action of a single D0 brane is written in terms of 10 bosonic and 32 fermionic coordinate functions, worldline fields corresponding to the coordinates of type IIA $D = 10$ superspace. The above gauge fixing reduces the number of fermionic fields to 16 and the number of bosonic coordinate functions to 9. These are the same as the number of physical fields as in 1d reduction of the 10D SYM theory. This also contains the time component of the gauge field which can be gauged away by the U(1) gauge symmetry transformation and do not carry degrees of freedom. The U(1) SYM multiplet describing the center of energy motion of the mD0 system can be obtained by fixing the gauge with respect to $\kappa$-symmetry and reparametrization symmetry on the coordinate functions, the same as in the case of single D0 brane.

In the light of the above discussion, it is natural to describe the center of energy motion of the mM0 system by the 11 commuting and 32 anti-commuting coordinate functions

$$\hat{Z}^M(\tau) = (\hat{x}^\mu(\tau), \hat{\theta}^\alpha(\tau)), \quad \mu = 0, 1, ..., 10; \quad \alpha = 1, 2, ..., 32$$

(5.71)

The bosonic $X^i(\tau)$ carries the index $i = 1, ..., 9$ of the vector representation of $SO(9)$, while the fermionic $\Psi^q$ transforms as a spinor under $SO(9)$, $q = 1, ..., 16$.

5.3.2. First order form of the 1d $\mathcal{N} = 16$ SYM Lagrangian as a starting point to build mM0 action

The standard 1d $\mathcal{N} = 16$ SYM Lagrangian (obtained by dimensional reduction of 10D SYM) can be written in the following first order form

$$d\tau L_{SYM} = tr \left(-\mathbb{P}^i \nabla_\tau X^i + 4i \Psi^q \nabla_\tau \Psi_q\right) + d\tau \mathcal{H}$$

(5.73)

where the Hamiltonian

$$\mathcal{H} = \frac{1}{2} tr \left(\mathbb{P}^i \mathbb{P}^i\right) + \mathcal{V}(X) - 2 tr \left(X^i \psi^i \psi\right)$$

(5.74)
contains the positively definite scalar potential

\[ \mathcal{V} = -\frac{1}{64} tr [X^i, X^j]^2 \equiv +\frac{1}{64} tr [X^i, X^j] \cdot [X^i, X^j]^\dagger, \]  
(5.75)

Eqs. (5.73) and (5.74) involve the auxiliary 'momentum' fields, the nanoplet of traceless \( N \times N \) matrices \( \mathbb{P}^i \), and also the gauge field \( A_\tau(\tau) \) which enters the covariant derivatives \( \nabla = d\tau \nabla_\tau \) of the above bosonic and fermionic fields,

\[ \nabla X^i = dX^i + [A, X^i], \quad \nabla \Psi_q = d\Psi_q + [A, \Psi_q]. \]  
(5.76)

The action with the above Lagrangian are invariant under the following \( d=1 \mathcal{N} = 16 \) supersymmetry transformations with constant fermionic parameter \( \epsilon^q \)

\[ \delta_\epsilon X^i = 4i \epsilon^q (\gamma^i \Psi)_q, \quad \delta_\epsilon \mathbb{P}^i = [\epsilon^q (\gamma^i \Psi)_q, X^j], \]  
(5.77)

\[ \delta_\epsilon \Psi_q = \frac{1}{2} \epsilon_{pq} [\gamma^i \Psi^i] - \frac{i}{16} \epsilon_{pqij} [X^i, X^j], \]  
(5.78)

\[ \delta_\epsilon A = -d\tau \epsilon^q \Psi_q. \]  
(5.79)

The mM0 action should describe the coupling of the above SYM theory to the center of energy variables (5.71). As we have discussed above, such an action should possess the reparametrization symmetry and a 16 parametric local fermionic symmetry, a counterpart of the irreducible \( \kappa \)–symmetry (5.8) of the single M0 action. It is natural also to think on this fermionic gauge symmetry as on the local version of the above rigid \( d=1 \mathcal{N} = 16 \) supersymmetry of the SYM action, Eqs. (5.77)–(5.79).

### 5.3.3. Induced supergravity on the center of energy worldline

The natural way to make a supersymmetry local is to couple it to supergravity multiplet. As a by–product such a coupling should guaranty the reparametrization (general coordinate) invariance. Now it is the time to recall about induced supergravity multiplet on the worldline of the single M0–brane constructed in sec. 5.2. Similarly, we can associate a moving frame (5.11) and spinor moving frame (5.16) to the center of energy motion of the mM0 system and use these together with center of energy coordinate functions (5.71) to build the composite \( d=1 \mathcal{N} = 16 \) supergravity multiplet including the 1d 'graviton' and 'gravitino'

\[ \hat{\mathbb{E}}^\# = \hat{E}^\alpha u^\#_\alpha = (d\hat{x}^\alpha - id\hat{\Theta}^\alpha \hat{\bar{\Theta}})u^\#_\alpha, \]  
(5.80)

\[ \hat{\mathbb{E}}^{+q} = \hat{E}^\alpha v^{+q}_\alpha = d\hat{\Theta}^\alpha v^{+q}_\alpha, \]  
(5.81)

transforming under the local supersymmetry as in (5.54),

\[ \delta_\epsilon \hat{\mathbb{E}}^{+q} = D\epsilon^{+q}(\tau), \quad \delta_\epsilon \hat{\mathbb{E}}^\# = -2i\hat{\mathbb{E}}^{+q}\epsilon^{+q}. \]  
(5.82)

Notice that the use of such a composite supergravity induced by embedding of the center of energy worldline into the flat target 11D superspace implies that the local supersymmetry parameter carries the weight +1 of the \( SO(1,1) \) group transformations defined on the
moving frame variables. This implies the necessity to adjust the $SO(1, 1)$ weight also to the fields describing the relative motion of the mM0 constituents. Following [139 146 150] we define the $SO(1, 1)$ weight of the bosonic and fermionic fields to be $-2$ and $-3$, respectively, so that in a more explicit notation (and using the conventions were the upper $-$ index indicate the same $-1$ weight as the lower $+$ one)\footnote{Such a chose of weight of the basic matrix fields is preferable for the description in the frame of superembedding approach, like developed in [139 146]. Once using the density $\rho^\# = \rho^{++}$ which enters the spinor moving frame action for single M0, we can easily change the weights of the fields multiplying them by corresponding power of $\rho^\#$. However we find more convenient to work with the ‘weighted’ fields (5.83), (5.84).}

$$X^i = X^i_{++} := X^i_{++}, \quad i = 1, \ldots, 9 ,$$ (5.83)

$$\Psi_q = \Psi_{++q} := \Psi_{++q} = \Psi_q^- , \quad q = 1, \ldots, 16 .$$ (5.84)

As in the case of single M0–brane, we expect the $SO(1, 1)$ as well as $SO(9)$ transforation to be a gauge symmetry of our action. This implies the use of covariant derivative with $SO(1, 1)$ and $SO(9)$ connection. As in the case of single M0–brane, we define these connections to be constructed from the moving frame variables

$$\Omega^{(0)} = \frac{1}{4} u^a d u^a , \quad \Omega^{ij} = u^a d u_a$$ (5.85)

(see Eqs. (5.28) and (5.29)), which are now associated to the center of energy motion of the mM0 system. The covariant derivatives of the $su(N)$ valued matrix fields (5.83) are defined by

$$D X^i := d X^i + 2 \Omega^{(0)} X^i - \Omega^{ij} X^j + [A, X^i] ,$$ (5.86)

$$D \Psi_q := d \Psi_q + 3 \Omega^{(0)} \Psi_q - \frac{1}{4} \Omega^{ij} \gamma_{qp} \Psi_p + [A, \Psi_q] .$$ (5.87)

They also involve the $SU(N)$ connection $A = d \tau A_\tau (\tau)$ on the center of energy worldline $W^1$. The anti-Hermitian traceless $N \times N$ matrix gauge field $A_\tau (\tau)$ is an independent variable of our model. Let us stress, however, that, as any 1d gauge field, it can be gauged away and thus does not carry any degree of freedom.

The covariant derivative of the supersymmetry parameter in (5.82) reads

$$D \epsilon^{+q} = d \epsilon^{+q} - \Omega^{(0)} \epsilon^{+q} + \frac{1}{4} \Omega^{ij} \epsilon^{+p} \gamma_{ij}^{pq} ,$$ (5.88)

so that the induced connection (5.85) are also the members of the composite d=1 $\mathcal{N} = 16$ supergravity multiplet.

### 5.3.4. A way towards mM0 action

Now we are ready to present the action for the system of N nearly coincident M0–branes (mM0 system) which was proposed in [150]. It can be considered as a result of ‘gauging’ of rigid d=1 $\mathcal{N} = 16$ supersymmetry (5.77)-(5.79) of the $SU(N)$ SYM action with the
5.3. Covariant action for multiple M\textsubscript{0}–brane system

Lagrangian (5.73) achieved by coupling it to a composite d=1 \( \mathcal{N} = 16 \) supergravity (5.80), (5.81), (5.85) induced by embedding of the center of energy worldline of the mM\textsubscript{0} system into the target 11D superspace.

The natural first step on this way is to make the Lagrangian (5.73) covariant by coupling it to a 1d gravity. This can be reached by just replacing \( d\tau \) in the right hand side of (5.73) by the 1-form \( \hat{E}^\# \) of (5.80). Then, to provide also the \( SO(1,1) \) and \( SO(9) \) gauge symmetries, which play the role of Lorentz and R-symmetries in our induced 1d \( \mathcal{N} = 16 \) supergravity, we should replace the Yang–Mills covariant derivatives in (5.76) by the \( SO(1,1) \times SO(9) \) covariant derivatives defined in (5.86), (5.87), and to multiply the Lagrangian 1-form thus obtained by \( (\rho^\#)^3 \). The next stage is suggested by the fact that setting \( N=1 \) in the action for the system of \( N \) nearly coincident M\textsubscript{0}–brane one should arrive a single M\textsubscript{0}–brane action. As the SU(N) SYM Lagrangian, and all the matrix fields involved in it, vanish when \( N=1 \), this implies the necessity just to add the single M\textsubscript{0} action to the integral of the above described Lagrangian form. Then the coupling to induced gravitino can be restored from the requirement of local supersymmetry invariance of the mM\textsubscript{0} action.

5.3.5. mM\textsubscript{0} action

In such a way we arrive at the mM\textsubscript{0} action proposed in [150]. It reads

\[
S_{mM\textsubscript{0}} = \int_{W^1} \rho^\# \hat{E}^\# + \int_{W^1} (\rho^\#)^3 \left( tr \left( -\hat{P}^i DX^i + 4i\Psi_q D\Psi_q \right) + \hat{E}^\# \mathcal{H} \right) + \int_{W^1} (\rho^\#)^3 \hat{E}^+ a tr \left( 4i(\gamma^i \Psi)_q \hat{P}^i + \frac{1}{2}(\gamma^{ij} \Psi)_q [X^i, X^j] \right),
\]

where \( \mathcal{H} \) is the relative motion Hamiltonian (cf. (5.74))

\[
\mathcal{H} := \mathcal{H}_{\#\#\#\#}(X, \mathcal{P}, \Psi) = \frac{1}{2} tr \left( \hat{P}^i \hat{P}^i \right) + \mathcal{V}(X) - 2 tr \left( X^i \Psi \gamma^i \Psi \right)
\]

including the scalar potential (cf. (5.75))

\[
\mathcal{V} := \mathcal{V}_{\#\#\#\#}(X) = -\frac{1}{64} tr \left[ X^i, X^j \right]^2
\]

\[
= +\frac{1}{64} tr \left[ X^i, X^j \right] \cdot \left[ X^i, X^j \right]^\dagger,
\]

and the Yukawa–type coupling \( tr (X^i \Psi \gamma^i \Psi) \).

The covariant derivatives \( D \) are defined in (5.86), (5.87). Their connection are build from the (spinor) moving frame variables, Eq. (5.85), which are related to the center of energy motion of the mM\textsubscript{0} system. These are also used to construct the composite graviton and
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gravitino 1-forms \( \hat{E}^\# \) and \( \hat{E}^{+q} \), Eqs. (5.80), (5.81). The 1-form \( \hat{E}^- \) is the same as in the case of single M0–brane

\[
\hat{E}^- = \hat{E}^a u^-_a. \tag{5.93}
\]

For the completeness of this section, let us recall that in these equations \( \hat{E}^a \) is the pull–back of the bosonic supervielbein to the center of energy worldline \( W^1 \), Eq. (5.3), (5.4), \( \hat{E}^a = d\hat{\theta}^a(\tau), u^-_a \) and \( u^\#_a \) are light–like moving frame vectors (5.11), (5.13), (5.14), and \( v^\pm_\alpha \) is an element of spinor moving frame (5.16).

Although the first term in (5.89) coincides with the single M0–brane action (5.1), now the Lagrange multiplier \( \rho^\# \) and spinor moving frame variables are also present in the second and third terms. This results in that their equations of motion differ from (5.53), and, as we discuss in the next section, generically, the center of energy motion of the mM0 system is not light-like.

5.3.6. Local supersymmetry of the mM0 action

The action (5.89) is invariant under the transformation of the 16 parametric local worldline supersymmetry

\[
\delta_\epsilon \hat{\theta}^a = \epsilon^+ q(\tau) v_q^- \alpha, \tag{5.94}
\]

\[
\delta_\epsilon \hat{x}^a = -i \hat{\theta} \Gamma^a \delta_\epsilon \hat{\theta} + \frac{1}{2} u^\#_a \epsilon \hat{E}^-, \tag{5.95}
\]

\[
\delta_\epsilon \rho^\# = 0, \tag{5.96}
\]

\[
\delta_\epsilon v^\pm_\alpha = 0 \Rightarrow \delta_\epsilon u^-_a = \delta_\epsilon u^\#_a = \delta_\epsilon u^+_a = 0, \tag{5.97}
\]

\[
\delta_\epsilon X^i = 4i \epsilon^+ \gamma^i \Psi, \quad \delta_\epsilon \Psi^i = [(\epsilon^+ \gamma^i \Psi), X^j], \tag{5.98}
\]

\[
\delta_\epsilon \Psi_q = \frac{1}{2} (\epsilon^+ \gamma^i \Psi^i) - \frac{i}{16} (\epsilon^+ \gamma^{ij})_q [X^i, X^j], \tag{5.99}
\]

\[
\delta_\epsilon \hat{A} = -\hat{E}^\# \epsilon^+ q \Psi_q + \hat{E}^+ \gamma^i \epsilon^+ X^i, \tag{5.100}
\]

where

\[
i_\epsilon \hat{E}^- = 6 (\rho^\#)^2 tr \left( i \Phi^i \epsilon^+ \gamma^i \Psi - \frac{1}{8} \epsilon^+ \gamma^{ij} \Psi [X^i, X^j] \right). \tag{5.101}
\]

The local supersymmetry transformations of the fields describing relative motion of mM0 constituents, (5.98), (5.99) coincide with the SYM supersymmetry (5.77), (5.78) modulo the fact that now the fermionic parameter is an arbitrary function of the center of energy proper time, \( \epsilon^+ q = \epsilon^+ q(\tau) \). The local supersymmetry transformation of the 1d \( SU(N) \) gauge field (5.100) differs from the SYM transformation by additional term involving the composite gravitino.

The transformations of the center of energy variables Eqs. (5.94)–(5.97) describe a deformation of the irreducible \( \kappa \)–symmetry (5.8) of the free massless superparticle. Actually, the deformation touches the transformation rule (5.95) for the the bosonic coordinate function, \( \delta_\epsilon \hat{x}^a \) only. The Lagrange multiplier \( \rho^\# \) and the (spinor) moving frame variables

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are invariant under the supersymmetry, like they are under the $\kappa$-symmetry of a single superparticle.

## 5.4. mM0 equations of motion

In this section we present and study the complete set of equations of motion for the multiple M0–brane system which follow from the action (5.89).

### 5.4.1. Equations of the relative motion

Varying the action with respect to the momentum matrix field $P^i$ gives us the equation

$$D X^i = \hat{E}^i_\# P^i + 4i \hat{E}^+ q^i (\gamma^i \Psi)_q$$

which allows to identify $P^i$, modulo fermionic contribution, with the covariant time derivative of $X^i$,

$$P^i = D_# X^i - 4i \hat{E}^+_\gamma^i \Psi .$$

Here

$$D_# = \frac{1}{E^\#} D_r , \quad \hat{E}^+_q = \frac{1}{E^\#} \hat{E}^+_r .$$

are covariant derivative and the induced 1d gravitino field corresponding to the induced einbein on the worldvolume, $\hat{E}^\# = \hat{E}^a u^a =: d\tau \hat{E}^\#$; in the sense of that

$$D = \hat{E}^\# D_# , \quad \hat{E}^+_q = \hat{E}^\# \hat{E}^+_\# .$$

The variation with respect to the worldline gauge field $A = d\tau A_r$ gives

$$[P^i, X^j] = 4i \{\Psi_q, \Psi_q\}$$

and the variation with respect to $X^i$ results in

$$D P^i = -\frac{1}{16} \hat{E}^\# [X^i, (\gamma^i \Psi)] + 2 \hat{E}^\# \gamma^i \Psi + \hat{E}^+_q \gamma^i \gamma^j \Psi_p [\Psi, X^j] .$$

Using (5.102) we can easily present this equation in the form

$$D_# D_# X^i = -\frac{1}{16} [X^i, X^j] + 2 \gamma^i \Psi + 4i D_\# (\hat{E}^+_\gamma^i \Psi_p) + \hat{E}^+_q \gamma^i \Psi .$$

Finally, the variation with respect to the traceless matrix fermionic field $\Psi_q$ produces

$$D \Psi = \frac{i}{4} \hat{E}^\# [X^i, (\gamma^i \Psi)] + \frac{1}{2} \hat{E}^+ \gamma^i P^i - \frac{i}{16} \hat{E}^+_\gamma^i [X^i, X^j] .$$
5.4.2. A convenient gauge fixing

To simplify the above equations, let us use the fact that 1-dimensional connection can always be gauged away and fix the gauge where the composed $SO(9)$ connection \(\Omega^{ij} = d\tau \Omega^j = 0\), \(\mathbb{A} = d\tau \mathbb{A}_\tau = 0\).

\[\Omega^{ij} = d\tau \Omega^j = 0, \quad \mathbb{A} = d\tau \mathbb{A}_\tau = 0.\] (5.110) (5.111)

This breaks the local $SO(9)$ and $SU(N)$, but the symmetry under the rigid $SO(9) \otimes SU(N)$ transformations remains.

As far as the $SO(1,1)$ gauge symmetry is concerned, we would not like to fix it but rather use a part \(\frac{1}{2} u^a d\nu^a = 0\) of the equations of motion for the center of energy coordinate functions \(\tilde{x}^a\) (discussed below in full),

\[D\rho^\# = 0,\] (5.112)

to find the explicit form of the induced $SO(1,1)$ connection \(\Omega^{(0)} := \frac{1}{2} u^a d\nu^a\). Indeed, as far as \(D\rho^\# = d\rho^\# - 2\rho^\# \Omega^{(0)}\), Eq. (5.112) implies

\[\Omega^{(0)} = \frac{d\rho^\#}{2\rho^\#}.\] (5.113)

In the gauge \(\Omega^{ij} = d\tau \Omega^j = 0\), \(\mathbb{A} = d\tau \mathbb{A}_\tau = 0\), the set of bosonic gauge symmetries is reduced to the Abelian $SO(1,1)$, \(\tau\)-reparametrization and \(b\)-symmetry (which we describe below in sec. 5.4.5), and the covariant derivatives simplify to

\[DX^i = (\rho^\#)^{-1} d(\rho^\# X^i),\]
\[D\mathbb{P}^i = (\rho^\#)^{-2} d((\rho^\#)^2 \mathbb{P}^i),\]
\[D\Psi_q = (\rho^\#)^{-3/2} d((\rho^\#)^{3/2} \Psi_q).\] (5.114)

As a result, Eqs. (5.108) and (5.109) can be written in the following (probably more transparent) form:

\[\partial_\tau \tilde{\Psi} = \frac{i}{4} e [\tilde{X}^i, (\gamma^i \tilde{\Psi})] + \frac{1}{2\sqrt{\rho^\#}} \dot{\tilde{E}}^+ \gamma^i \tilde{\mathbb{P}}^i - \frac{i}{16\sqrt{\rho^\#}} \dot{\tilde{E}}^+ \gamma^{ij} [\tilde{X}^i, \tilde{X}^j],\] (5.115)

\[\partial_\tau \left( \frac{1}{e} \partial_\tau \tilde{X}^i \right) = -\frac{e}{16} [[\tilde{X}^i, \tilde{X}^j], \tilde{X}^j] + 2 e \tilde{\Psi} \gamma^i \tilde{\Psi} +
+ 4i \partial_\tau \left( \frac{\dot{\tilde{E}}^+ \gamma^i \tilde{\Psi}}{e \sqrt{\rho^\#}} \right) + \frac{1}{\sqrt{\rho^\#}} \dot{\tilde{E}}^+ \gamma^{ij} [\tilde{\Psi}, \tilde{X}^j].\] (5.116)
Writing Eqs. (5.115) and (5.116) we used the redefined fields
\[
\widetilde{X}^i = \rho^# X^i, \quad \widetilde{\Psi}^q = \left(\rho^#\right)^{3/2} \Psi^q, \quad (5.117)
\]
\[
\widetilde{P}^i = \left(\rho^#\right)^2 P^i = 1 e \left(\partial_\tau \widetilde{X}^i - \frac{4i}{\sqrt{\rho^#}} \hat{E}^+_i \gamma^i \widetilde{\Psi}\right), \quad (5.118)
\]
which are inert under the \(SO(1,1)\), and
\[
e(\tau) = \hat{E}^# / \rho^# \quad (5.119)
\]
which has the properties of the einbein of the Brink–Schwarz superparticle action \( (5.6) \).

5.4.3. By pass technical comment on derivation of the equations for the center of energy coordinate functions

This is the place to present some comments on the convenient way to derive equations of motion for the center of energy variables (which was actually used as well when working with single \(M_0\) in Sec 5.2).

To find the manifestly covariant and supersymmetric invariant linear combinations of the equations of motion for the bosonic and fermionic coordinate functions, \( \delta S_{mM0} / \delta \tilde{Z}^a = 0 \) and \( \delta S_{mM0} / \delta \theta^a = 0 \), we introduce the covariant basis \( i_\delta \hat{E}^A \) in the space of variation such that
\[
\delta Z^M S_{mM0} = \int W^1 \left( \delta \hat{Z}^a \frac{\delta S_{mM0}}{\delta \tilde{Z}^a} + \delta \theta^a \frac{\delta S_{mM0}}{\delta \theta^a} \right) = \int W^1 \left( i_\delta \hat{E}^a \frac{\delta S_{mM0}}{i_\delta \hat{E}^a} + i_\delta \hat{E}^a \frac{\delta S_{mM0}}{i_\delta \hat{E}^a} \right). \quad (5.120)
\]
In the generic case of curved superspace \( i_\delta \hat{E}^A = \delta \hat{Z}^M E^A_M (\hat{Z}) \); in our case of flat target superspace this implies
\[
i_\delta \hat{E}^a = \delta \hat{x}^a - i \delta \hat{\theta} \Gamma^a \hat{\theta}, \quad i_\delta \hat{E}^a = \delta \hat{\theta}^a. \quad (5.121)
\]
Furthermore, it is convenient to use the moving frame variables to split covariantly the set of bosonic equations \( \delta S_{mM0} / i_\delta \hat{E}^a = 0 \) into
\[
\delta S_{mM0} / i_\delta \hat{E}^a = \frac{1}{2} \hat{x}^a \frac{\delta S_{mM0}}{i_\delta \hat{E}^a}, \quad \delta S_{mM0} / i_\delta \hat{E}^a = \frac{1}{2} \hat{\theta}^a \frac{\delta S_{mM0}}{i_\delta \hat{E}^a}, \quad \delta S_{mM0} / i_\delta \hat{E}^a = -u_{a}^i \frac{\delta S_{mM0}}{i_\delta \hat{E}^a}, \quad (5.122)
\]
and the set of fermionic equations, \( \frac{\delta S_{mM}}{i\delta E^a} = \frac{\delta S_{mM}}{i\delta \theta^a} \), into

\[
\begin{align*}
\delta S_{mM} = & v_+^a \frac{\delta S_{mM}}{i\delta \theta^a} , \\
\delta S_{mM} = & v_-^a \frac{\delta S_{mM}}{i\delta \theta^a} ,
\end{align*}
\]

(5.123)

To resume,

\[
\delta Z_{mM} = \int W^1 \left( \delta \hat{x}^a - i \delta \hat{\theta} \Gamma^a \hat{\theta} \right) \left( \frac{u_a}{i\delta E^a} \right) + \\
+ \frac{u_a}{i\delta E^a} \frac{\delta S_{mM}}{i\delta E^a} + \frac{u_a}{i\delta E^a} \frac{\delta S_{mM}}{i\delta E^a} + \\
+ \int W^1 \left( v_-^a \frac{\delta S_{mM}}{i\delta E^a} + v_+^a \frac{\delta S_{mM}}{i\delta E^a} \right).
\]

(5.124)

### 5.4.4. Equations for the center of energy coordinate functions

As we have already stated, the bosonic equation \( \frac{\delta S_{mM}}{i\delta E^a} = \frac{1}{2} u_a^\# \frac{\delta S_{mM}}{\delta x^a} = 0 \) results in Eq. (5.112) which is equivalent to (5.113). This observation is useful to extract consequences of the next equation, \( \frac{\delta S_{mM}}{i\delta E^a} = 0 \), which reads

\[
D((\rho^\#)^4 \mathcal{H}) = 0.
\]

(5.125)

Using (5.112) one can write Eq. (5.125) in the form of

\[
d((\rho^\#)^4 \mathcal{H}) = 0.
\]

(5.126)

or, equivalently, \( (\rho^\#)^4 \mathcal{H} = const. \) Due to the structure of \( \mathcal{H} \), Eq. (5.90), this constant is nonnegative. Furthermore, as it has been shown in [150] (see also sec. 5.7.3), it can be identified (up to numerical multiplier) with the mass parameter \( M^2 \) characterizing the center of energy motion,

\[
M^2 = 4(\rho^\#)^4 \mathcal{H} = const \geq 0.
\]

(5.127)

The remaining projection of the equation for the bosonic center of energy coordinate functions, \( \frac{\delta S_{mM}}{i\delta E^a} = -\frac{1}{2} u_a^\# \frac{\delta S_{mM}}{\delta x^a} = 0 \), gives us the relation between covariant Cartan forms (5.27),

\[
\Omega^i = -(\rho^\#)^2 \mathcal{H} \Omega^\#_i = -\frac{M^2}{4(\rho^\#)^2} \Omega^\#_i.
\]

(5.128)
5.4. **mM0 equations of motion**

The nontrivial part of the fermionic equation of the center of energy motion, \( \frac{\delta S_{mM0}}{i\hbar \dot{E}^{-q}} := v^{-\alpha} \frac{\delta S_{mM0}}{i\hbar E^{-q}} = 0 \), reads

\[
\dot{E}^{-q} = - \frac{1}{2} \Omega^{\#i} \gamma_{q}^{i} \nu_{\#q},
\]

(5.129)

where

\[
\nu_{\#q} := (\rho^{\#})^{2} tr \left( (\gamma^{j} \Psi)_{q} \Psi^{j} - \frac{i}{8} (\gamma^{jk} \Psi)_{q} [X^{i}, X^{j}] \right).
\]

(5.130)

### 5.4.5. Noether identities for gauge symmetries. First look.

Actually one can show that Eq. (5.125) is satisfied identically when other equations are taken into account. (To be precise, Eqs. (5.102), (5.106), (5.107), (5.109), (5.112) have to be used). This is the Noether identity for the 'tangent space' copy of the reparametrization symmetry (sometimes it is called 'b-symmetry') with the parameter function \( \dot{E}^{\#} \). Similarly, one can find the Noether identity reflecting the existence of the \( \mathcal{N} = 16 \) 1d gauge supersymmetry (5.94)–(5.101) with the basic parameter \( \epsilon^{+q} = i\hbar \dot{E}^{+q} \). It states the dependence of the one half of the fermionic equations, namely

\[
\frac{\delta S_{mM0}}{i\hbar E^{+q}} := v^{-\alpha} \frac{\delta S_{mM0}}{i\hbar E^{\alpha}} = 0,
\]

which reads

\[
D\nu_{\#q} = \rho^{2} \dot{E}^{+q} \mathcal{H},
\]

(5.131)

or \( D\nu_{\#q} = \rho^{2} \dot{E}^{+q} \mathcal{H} \) in a more complete notation.

### 5.4.6. Equations which follow from the auxiliary field variations and simplification of the above equations

Variation with respect to the Lagrange multiplier \( \rho^{\#} \), \( \frac{\delta S_{mM0}}{\delta \rho^{\#}} = 0 \), expresses the projection \( \dot{E} = \dot{E}^{\alpha} \) of the pull–back \( \dot{E}^{\alpha} \) of the bosonic supervielbein to the center of energy worldline through the relative motion variables,

\[
\dot{E}^{\alpha} := \dot{E}^{\alpha} u^{\alpha} = -3(\rho^{\#})^{2} \mathcal{L}_{\#\#} = 3(\rho^{\#})^{2} tr \left( \frac{1}{2} \Omega^{\#i} \Psi_{q} \Psi^{j} \right).
\]

(5.132)

The \( \dot{E}^{i} = \dot{E}^{\alpha} u^{i}_{\alpha} \) projection of this pull–back is expressed by equations appearing as a result of variation with respect to the spinor moving frame variables. According to Eqs. (5.35)–(5.38), that should appear as coefficients for \( i\hbar \dot{\Omega}^{-i} \) and \( i\hbar \Omega^{\#i} \) in the variation of the action. Equation \( \frac{\delta S_{mM0}}{i\hbar \Omega^{\#i}} = 0 \) reads

\[
\dot{E}^{i} := \dot{E}^{\alpha} u^{i}_{\alpha} = - (\rho^{\#})^{-1} \Omega^{\#j} \left( J^{ij} + \delta^{ij}_{j} \right).
\]

(5.133)
where we have introduced the notation
\[ J^{ij} := (\rho^\#)^3 \text{tr} \left( \mathcal{P}^{[i} \mathcal{X}^{j]} - i \mathcal{P} \gamma^{ij} \mathcal{P} \right) , \]  
\[ J := \frac{(\rho^\#)^3}{2} \text{tr} \left( \mathcal{P} \mathcal{X}^i \right) . \]  
(5.134)  
(5.135)

The \((\rho^\#)^3\) multipliers are introduced to make \(J^{ij}\) and \(J\) inert under the \(SO(1,1)\) transformations.

In this notation, equation \(\delta S_{\text{mM}0} = 0\) reads
\[ (\rho^\#)^3 \mathcal{H} \hat{E}^i = -\Omega^{-j} \left( J^{ij} - \delta^{ij} J \right) - 2i(\rho^\#) \hat{E}^{-q}(\gamma^i \nu^\# q) . \]  
(5.136)

Using (5.128), (5.133), (5.127) and (5.129), one can rewrite Eq. (5.136) as equation for \(\Omega^\# i\),
\[ \Omega^{\# j} \left( M^2 J^{ij} - 2i(\rho^\#)^2 \nu^\# \gamma^{ij} \nu^\# \right) = 0 . \]  
(5.137)

Actually, as we are going to show in the next section 5.4.7, taking into account the remnant of the \(K_9\) gauge symmetry of single \(M0\)–brane (see (5.39) and (5.40)) , which is present in the \(mM0\) action, one can present the above equation in the form of
\[ \Omega^{\# i} = 0 , \]  
(5.138)
or, in terms of component, \(\Omega^{\# \tau j} = 0\). Due to (5.128) Eq. (5.138) implies
\[ \Omega^{-i} = 0 \]  
(5.139)
and (5.133) acquires the same form as in the case of single \(M0\)–brane,
\[ \hat{E}^i := \hat{E}^a u_a^i = 0 . \]  
(5.140)

Furthermore, the fermionic equation of motion (5.129) also becomes homogeneous, of the same form as the equation for single \(M0\)–brane,
\[ \hat{E}^{-q} = 0 . \]  
(5.141)

Eqs. (5.138) and (5.139) also imply that all the moving frame and spinor moving frame variables are covariantly constant,
\[ Du_a^\# = 0 , \quad Du_a^\tau = 0 , \quad Du_a^i = 0 , \quad Dv_q^+ \alpha = 0 , \quad Dv_q^- \alpha = 0 . \]  
(5.142)  
(5.143)

Notice that in the case of single \(M0\)–brane such a form of equations for moving frame variables can be reached after gauge fixing the \(K_9\) gauge symmetry with parameter \(i_d \Omega^{\# i}\). In the \(mM0\) case only a part (remnant) of \(K_9\) symmetry is present so that a part of variations
5.4. mM0 equations of motion

\[ \delta \Omega^{\#i} \] produce nontrivial equations which, together with the above mentioned remnant of \( K_9 \) symmetry, results in Eqs. (5.142), (5.143).

5.4.7. Noether identity, remnant of the \( K_9 \) gauge symmetry and the final form of the \( \Omega^{\#i} \) equation

In this section we present the remnant of \( K_9 \) gauge symmetry leaving invariant the mM0 action and show that, modulo this gauge symmetry, Eq. (5.137) is equivalent to (5.138).

Let us write Eq. (5.137) as

\[ \Omega^{\#j} \mathcal{J}^{ij} = 0, \quad (5.144) \]

where

\[ \mathcal{J}^{ij} = M^2 J^{ij} - 2i (\rho^{\#})^2 \nu^{\#} \gamma^{ij} \nu^{\#}. \quad (5.145) \]

As this \( 9 \times 9 \) matrix is antisymmetric, it has rank 8 or lower, \( \text{rank}(\mathcal{J}^{ij}) \leq 8 \). In other words, it has at least one 'null vector', this is to say a vector \( V^i \) which obey

\[ \exists V^i, i = 1, ..., 9 : \quad \mathcal{J}^{ij} V^j = 0. \quad (5.146) \]

Actually, the matrix \( \mathcal{J}^{ij} \) is constructed from the dynamical variables of our model in such a way (according to Eqs. (5.145) and (5.134)) that the number of its null vectors depends on the configuration of the fields describing the relative motion of the mM0 constituents. However, as one 'null vector' always exists, it is sufficient to consider a configuration with \( \text{rank}(\mathcal{J}^{ij}) = 8 \), and \( \mathcal{J}^{ij} \) having just one 'null vector', at some neighborhood \( \Delta \tau \) of a proper-time moment \( \tau \); the generalization for a more complicated configurations/neighborhoods is straightforward.

Then, on one hand, the solution of Eq. (5.144) in the neighborhood \( \Delta \tau \) is given by \( \Omega^{\#i}_\tau \propto V^i \), or, equivalently,

\[ \Omega^{\#i}_\tau = f V^i, \quad (5.147) \]

where \( f = f(\tau) \) is an arbitrary function of the center of energy proper time \( \tau \). [For configurations/neighborhoods with several 'null vectors' \( V^i_r, r = 1, ..., (9 - \text{rank} \mathcal{J}) \) the solution will be \( \Omega^{\#i}_\tau = f^r V^i_r \) with arbitrary functions \( f^r = f^r(\tau) \).]

On the other hand, the existence of null vector, Eq. (5.146), implies that a part of Eqs. (5.144) is satisfied identically

\[ \Omega^{\#j}_\tau \mathcal{J}^{ij} V^j \equiv 0, \quad (5.148) \]

when some other equations are taken into account. This is the Noether identity reflecting

\[ ^8 \text{This should not be confused with light-like vectors which can exist in the space with indefinite metric. In particular, our 11D moving frame vectors } u^i_a \text{ and } u^{\#}_a \text{ are light-like. To exclude any confusion, in this chapter we never use the name 'null-vectors' for the light-like vectors.} \]
Chapter 5. Covariant action and equations of motion for the 11D system of multiple M0-branes

the existence of the gauge symmetry with the basic variation

\[ i_\delta \Omega^\#_i = \alpha V^i \]  

(5.149)

with an arbitrary function \( \alpha = \alpha(\tau) \). This is clearly a remnant of the \( K_9 \) gauge symmetry (5.39) of the action (5.1) for single M0–brane.

The generic variation of the Cartan 1–form \( \Omega^\#_i \) can be expressed as in Eq. (5.45), which in our 1d case can also be written as

\[ \delta \Omega^\#_i = D_\tau i_\delta \Omega^\#_i . \]  

(5.150)

Applying (5.150) to the variation of the solution (5.147) of Eq. (5.144) under (5.149), we find that

\[ \delta f(\tau) = \partial_\tau \alpha(\tau) . \]  

(5.151)

Hence, one can use the local symmetry (5.149) to set \( f = 0 \) and, thus, to gauge away (to trivialize) the solution (5.147) of Eq. (5.144).

This proves that the gauge fixing version of Eq. (5.144) is given by Eq. (5.138), \( \Omega^\#_i = 0 \).

In sec. 5.7 we give more detailed discussion of the above local symmetry and its Noether identities reproducing independently the above conclusion for the purely bosonic case.

5.5. Ground state solution of the relative motion equations

The natural first step in studying the above obtained mM0 equations is to address the sector of

\[ \Psi_q = 0 . \]  

(5.152)

As far as the fermionic equations of motion have the same form (5.141) as for the single M0–brane, \( \hat{E}^-_q = 0 \), the only possible fermionic contribution to the relative motion equations might come from the induced gravitino \( \hat{E}^+q = d\theta^\alpha \nu^\alpha q \). However, with (5.152), the fermionic equation of the relative motion (5.109) results in

\[ \hat{E}^+ \gamma^i \Psi^i - \frac{i}{8} \hat{E}^+ \gamma^{ij} [\mathcal{X}^i, \mathcal{X}^j] = 0 . \]  

(5.153)

As it will be clear after our discussion below, for \( M^2 > 0 \) this equation has only trivial solution \( \hat{E}^+q = 0 \), while for \( M^2 = 0 \) the 1d gravitino \( \hat{E}^+q \) remains arbitrary.

---

9See sec. 5.7 for more details on these Noether identity and gauge symmetry in the purely bosonic case. Here let us just recall that Eq. (5.137) appears as an essential part of the coefficient for \( i_\delta \Omega^\#_i \) in the variation of the mM0 action.
5.5. Ground state solution of the relative motion equations

5.5.1. Ground state of the relative motion

It is easy to see that a particular configuration of the bosonic fields for which Eq. (5.153) is satisfied is
\[ P^i = 0, \quad [X^i, X^j] = 0. \] (5.154)

Then the fermionic 1-form \( \hat{E}^{+q} \) remains arbitrary (and pure gauge) as it is in the case of single M0–brane.

Together with (5.152), Eqs. (5.154) describe the ground state of the relative motion. For it the relative motion Hamiltonian (5.90) and the center of energy effective mass vanish,
\[ M^2 = 0 \] (5.155)
so that the center of energy motion is light–like. Moreover, when Eqs. (5.152) and (5.154) hold, all the equations of the center of energy motion coincide with the equations for single M0–brane.

The ground state of the mM0 system is thus described by Eqs. (5.152), (5.154) and by a (pure bosonic) ground state solution of the single M0 equations. This preserves all 16 worldline supersymmetries, which corresponds (as we have discussed in sec. 5.2.1) to the preservation of 16 of 32 spacetime supersymmetries.

5.5.2. Solutions with \( M^2 = 0 \) have relative motion in the ground state sector

Curiously enough, being in the ground state of the relative motion is the only possibility for the mM0 system to have the light–like center of energy motion characterized by zero effective mass
\[ M^2 = 0 \iff \mathcal{H} = \frac{1}{2} \text{tr}(P^i P^i) - \frac{1}{64} \text{tr}[X^i, X^j]^2 = 0. \] (5.156)

Indeed, the pure bosonic relative motion Hamiltonian \( \mathcal{H} \) is given by the sum of two terms both of which are traces of squares of Hermitian operators \( \text{tr}[X^i, X^j]^2 \); hence, the sum vanishes, \( \mathcal{H} = 0 \), iff both equations in (5.154) hold.\(^{10}\) \( P^i = D_\# X^i = 0 \) and \([X^i, X^j] = 0\).\(^{11}\)

Thus any nontrivial configuration of the relative motion, with either \( P^i \neq 0 \) or/and \([X^i, X^j] \neq 0\), creates a nonzero effective mass of the center of energy motion, \( M^2 \neq 0 \).

\(^{10}\)We do not discuss here the possible nilpotent contributions, like the possibility to solve the equation \( a^2 = 0 \) for a real bosonic \( a(\tau) \) by \( a = \beta_{\alpha_1...\alpha_{17}} \hat{\theta}^{\alpha_1} \ldots \hat{\theta}^{\alpha_{17}} \) with 17 center of energy fermions \( \hat{\theta}^{\alpha}(\tau) \) contracted with some fermionic \( \beta_{\alpha_1...\alpha_{17}} = [\alpha_{\alpha_1...\alpha_{17}}] \).

\(^{11}\)This is true for finite size matrices. In the \( N \to \infty \) limit (mM0 condensate) one can consider a 'non–commutative plane' solution with \([X^i, X^j] = i \Theta^{ij} \) and c-number valued \( \Theta^{ij} = -\Theta^{ji} \), see for instance, [199]. In the case of finite \( N \) this solution cannot be used as far as the right hand side is assumed to be proportional to the unity matrix, \( I_{N \times N} \) while the trace of the commutator vanishes.
5.6. Supersymmetric solutions of mM0 equations

5.6.1. Supersymmetric solutions of the mM0 equations have $M^2 = 0$

From Eq. (5.99) one concludes that a solution of the mM0 equations with vanishing relative motion fermionic fields, Eq. (5.152), can be supersymmetric if

$$ (e^+\gamma^i)_q \mathbb{P}^i - \frac{i}{8} (e^+\gamma^{ij})_q [X^i, X^j] = 0. $$

(5.157)

All the 16 worldline supersymmetries (1/2 of the target space supersymmetries) can be preserved iff this equation is satisfied for arbitrary $e^+p$. This implies

$$ \gamma^i_{qp} \mathbb{P}^i - \frac{i}{8} \gamma^{ij}_{qp} [X^i, X^j] = 0 $$

(5.158)

the only solution of which is given by the ground state of the relative motion, Eq. (5.154).

Thus all the bosonic solutions of mM0 equations preserving 16 supersymmetries have the trivial relative motion sector described by Eq. (5.154), which is characterized by the light–like center of energy motion, $M^2 = 0$.

This suggests that $M^2 = 0$, is the BPS condition, i.e. the necessary condition for the 1/2 supersymmetry preservation. As we are going to show, this is indeed the case, and, moreover

$$ M^2 = 0 $$

(5.159)

is the BPS equation for preservation of any part of the target space supersymmetry.

Indeed, on one hand, tracing Eq. (5.157) with $\gamma^i\mathbb{P}^i$ and using the properties of $tr$ we find

$$ e^+q tr(\mathbb{P}^i\mathbb{P}^i) = \frac{i}{8} (e^+\gamma^{ijk})_q tr(\mathbb{P}^i[X^j, X^k]). $$

On the other hand, tracing (5.157) with $\frac{i}{8} \gamma^{jk}[X^j, X^k]$ and using the Jacobi identities $[X^i[X^j, X^k]] \equiv 0$ we find

$$ \frac{i}{8} (e^+\gamma^{ijk})_q tr(\mathbb{P}^i[X^j, X^k]) = \frac{1}{32} (e^+q tr([X^j, X^k]^2). $$

Taking the sum of these two equations and using (5.106) (with fermionic fields set to zero) we find $e^+q \mathcal{H} = 0$ which, using (5.127), can be written as $e^+q M^2 = 0$,

$$ e^+q M^2 = 0 \iff e^+q \mathcal{H} = 0. $$

(5.160)

For $M^2 \neq 0$ this implies $e^+q = 0$, so that the supersymmetry is broken. Thus all the supersymmetric solutions of mM0 equation are characterized by $M^2 = 0$.

This fact is very important: it means that the existence of our action does not imply
5.7. On solutions of $mM^0$ equations with $M^2 > 0$

the existence of a new type of supersymmetric solutions of the 11D SUGRA equations. A BPS solution is in correspondence with the ground state of the brane or of the multiple brane system; the ground state of mM$^0$ system is characterized by the vanishing effective mass and with the center of energy motion characteristic for the single M$^0$–brane. Thus a supersymmetric solution of 11D SUGRA equations corresponding to single M-wave also describe the mM$^0$ (multiple M-wave) ground state.

5.6.2. All BPS states of mM$^0$ system are 1/2 BPS

As we have shown, a solution of mM$^0$ equations can preserve some part of the 16 worldline supersymmetries (and some part ($\leq 1/2$) of the target space supersymmetry) if and only if $M^2 = 0$. Now, in the light of the observation in sec. 5.5.2 $M^2 = 0$ implies that the relative motion of the mM$^0$ constituents is in its ground state, Eq. (5.154). This has two consequences. Firstly, as the ground state trivially solves the Killing spinor equation (5.157), it preserves all the supersymmetries allowed by the center of energy motion. Secondly, when the relative motion sector is in its ground state, the center of energy sector of supersymmetric solution is described by the same equations as the motion of single M$^0$–brane (massless 11D superparticle). Now, as we have shown in sec. 5.2.1 the supersymmetric solutions of these M$^0$ equations preserve just 1/2 of the target space supersymmetry.

This proves that all the supersymmetric solutions of the equations of motion of the mM$^0$ system preserve just one half of 32 target space supersymmetries. In other words, all the mM$^0$ BPS states are 1/2 BPS.

5.7. On solutions of mM$^0$ equations with $M^2 > 0$

When $M^2 \neq 0$, Eq. (5.153) has only trivial solutions. (The proof of this fact follows the stages of sec. 5.6.1). This means that (5.152) results in

$$\hat{E}^+ q = 0,$$

(5.161)

so that, when $M^2 > 0$, a configuration with vanishing relative motion fermion is purely bosonic.

\[12\text{Although this statement can be done about the solutions preserving 1/2 of the 11D supersymmetry, as it will be clear in a moment, it is universal as far as a supersymmetric solution of mM$^0$ equations can preserve only 1/2 of the tangent space supersymmetry.} \]
5.7.1. Purely bosonic equations in the case of \( M^2 > 0 \)

The complete list of nontrivial pure bosonic equations for mM0 system with nonvanishing center of energy mass, \( M^2 > 0 \), reads

\[
D\rho^# = 0 \quad \iff \quad \Omega^{(0)} = \frac{d\rho^#}{2\rho^#},
\]

\[
D_\# D_\# X^i = -\frac{1}{16} [[X^i, X^j] X^j],
\]

\[
[D_\# X^i, X^i] = 0,
\]

\[
\hat{E}^\# := d\hat{x}^a u_a^\# = 3\hat{\Omega}^\# \left( (\rho^#)^2 tr(D_\# X^i)^2 - \frac{M^2}{4(\rho^#)^2} \right),
\]

\[
\hat{E}^i := d\hat{x}^a u^i_a = 0,
\]

\[
\Omega^{#i} = 0, \quad \Omega^{\#i} = 0,
\]

where \( \hat{E}^\# = d\hat{x}^a u_a^\# \) and the center of energy mass \( M \) is defined by Eq. (5.127), \( M^2 = 4(\rho^#)^4 \mathcal{H} \), with the relative motion Hamiltonian

\[
\mathcal{H} = tr\left( \frac{1}{2}(D_\# X^i)^2 - \frac{1}{64} [X^i, X^j]^2 \right).
\]

Notice that (as we have discussed in the general case) the currents

\[
J^{ij} = (\rho^#)^3 tr D_\# X^i [X^j],
\]

\[
J = \frac{(\rho^#)^3}{2} tr D_\# X^i X^i
\]

disappear from the final form of equations when one takes into account the presence of the remnants of the \( K_0 \) symmetry. As far as this statement is very important in the analysis of the mM0 equations, we are going to give more detail on this symmetry and gauge fixing now.

But before let us make an observation that the current \( J^{ij} \) is covariantly constant on the mass shell (i.e. when the above equations of motion are taken into account),

\[
D J^{ij} = 0.
\]

In contrast, in the generic purely bosonic configuration the scalar current is not a constant, \( DJ = dJ \neq 0 \).
5.7. On solutions of mM0 equations with $M^2 > 0$

5.7.2. Remnant of $K_9$ symmetry in the bosonic limit of the mM0 action and $\Omega^{\#i}$ equations

The variation of the bosonic limit of the mM0 action (5.89) can be written in the form

$$\delta S^{\text{bosonic}}_{mM0} = \int_{W^1} \mathcal{E}_u^{\#i} i_\delta \Omega^{\#i} + \int_{W^1} \mathcal{E}_u^{\#i} i_\delta \Omega^{\#i} - \int_{W^1} \mathcal{E}_x^{\#i} i_\delta \hat{E}^i + \ldots .$$

(5.172)

where

$$\mathcal{E}_u^{\#i} = \frac{M^2 \hat{E}^i}{4\rho^\#} + \Omega^{\#j} (J^{ij} - \delta^{ij} J),$$

$$\mathcal{E}_u^{\#i} = \rho^\# \hat{E}^i + \Omega^{\#j} (J^{ij} + \delta^{ij} J),$$

$$\mathcal{E}_x^{\#i} = \rho^\# \Omega^{\#i} + \frac{M^2 \Omega^{\#i}}{4\rho^\#},$$

(5.173)

with $J^{ij}$ and $J$ defined in (5.170) and (5.135), and dots denote the terms involving the other basic variations ($\delta \rho^\#, i_\delta \hat{E}^i$ etc.). Furthermore, one can rearrange the terms in (5.172) in the following way:

$$\delta S^{\text{bosonic}}_{mM0} = \int_{W^1} \mathcal{E}_u^{\#i} \left( i_\delta \Omega^{\#i} - \frac{M^2}{4(\rho^\#)^2} i_\delta \Omega^{\#i} \right) - \int_{W^1} \mathcal{E}_x^{\#i} \left( i_\delta \hat{E}^i + (J^{ij} + \delta^{ij} J) \ i_\delta \Omega^{\#j} \right) + \int_{W^1} \frac{M^2}{2(\rho^\#)^2} \ d\tau \ \Omega^{\#i} J^{ij} i_\delta \Omega^{\#j} + \ldots ,$$

(5.174)

In this form it is transparent that the equations of motion corresponding to the $i_\delta \Omega^{\#j}$ variation can be written in the form

$$\Omega^{\#i} J^{ij} = 0 ,$$

(5.175)

which is the bosonic limit of Eq. (5.137). As we have already discussed in the general case, Eq. (5.175) always has a nontrivial solution as far as the antisymmetric $9 \times 9$ matrix $J^{ij} = -J^{ji}$ always has at least one null vector, a non-zero vector $V^i$ such that $V^i J^{ij} = 0$.

Each null–vector generates a nontrivial solution of (5.175), but also a gauge symmetry of the mM0 action. Indeed, as one can easily see from (5.174), the transformations with $\tau$-dependent parameter $i_\delta \Omega^{\#j}$ obeying

$$j^{ij} i_\delta \Omega^{\#j} = 0 ,$$

(5.176)
completed by

\[ i_\delta \Omega^\pm_i = \frac{M^2}{4(\rho^\#)^2} i_\delta \Omega^\pm_i, \]
\[ i_\delta \hat{E}^i = -(J^{ij} + J\delta^{ij}) i_\delta \Omega^\pm_j, \]  \hspace{1cm} (5.177)

leave the action invariant, \( \delta S_{\text{bosonic}}^{mM0} = 0 \), and, thus define the gauge symmetries of the \( mM0 \) action. The transformations of \( \Omega^\pm_i \) under this gauge symmetry are \( \delta \Omega^\pm_i = D_\tau \delta \Omega^\pm_i \) (5.150). As far as in purely bosonic limit \( DJ^{ij} = 0 \) on the mass shell (see Eq. (5.171)),

\[ J^{ij} D_\tau i_\delta \Omega^\pm_j = 0 \] \hspace{1cm} (5.178)

is also obeyed. Furthermore, in 1d case all the connection can be gauged away so that the transformation rules of the nontrivial solution of Eq. (5.175) can be summarized as follows

\[ \delta \Omega^\pm_i = \partial_\tau i_\delta \Omega^\pm_i, \quad \begin{cases} \Omega^\pm_i J^{ij} = 0 \\ J^{ij} i_\delta \Omega^\pm_j = 0 \\ \partial_\tau J^{ij} = 0 \end{cases}, \]  \hspace{1cm} (5.179)

This form makes transparent that any nontrivial solution of Eq. (5.175) can be gauged away using local symmetry (5.176), (5.177). Thus, modulo the gauge symmetry, Eq. (5.175) is equivalent to Eq. (5.167), \( \Omega^\pm_i = 0 \).

### 5.7.3. Center of energy velocity and momentum for \( M^2 \neq 0 \)

Let us notice one property of the center of energy motion of our \( M0 \) system which, on the first glance, might looks strange, and try to convince the reader that it is rather a natural manifestation of the influence of relative motion on the center of energy dynamics.

Using Eqs. (5.165), (5.166) we can easily calculate center of energy velocity of the bosonic limit of our \( mM0 \) system,

\[ \dot{x}^a := \partial_\tau \hat{x}^a = \frac{1}{2} \hat{E}_\tau u^\# a + \frac{1}{2} \hat{E}_\tau u^a - \hat{E}_\tau u^a = \]
\[ = \frac{1}{2} \hat{E}_\tau^\# \left( u^a + 3u^\# a \left( (\rho^\#)^2 tr(D^\# X^i)^2 - \frac{M^2}{4(\rho^\#)^2} \right) \right). \] \hspace{1cm} (5.180)

On the other hand, the canonical momentum conjugate to the center of energy coordinate function \( \hat{x}^a \) is \(^{13}\)

\[ p_a = \frac{\partial L_{mM0}}{\partial \dot{\hat{x}}^a} = \rho^\# \left( u_a^\# + u^\# a \frac{M^2}{4(\rho^\#)^2} \right). \]  \hspace{1cm} (5.181)

This equation justifies our identification of the constant \( M^2 \) as a square of the effective

\(^{13}L_{mM0}^\tau \) is the Lagrangian of the \( mM0 \) action (5.89), \( S_{mM0} = \int d\tau L_{mM0}^\tau \).
5.7. On solutions of $mM_0$ equations with $M^2 > 0$

mass of the $mM_0$ system as it gives

$$p^a p_a = M^2. \tag{5.182}$$

Thus, generically, the center of energy velocity and its momentum are oriented in different directions of 11D spacetime,

$$\dot{x}_a \propto (p_a - A_a), \tag{5.183}$$

$$A_a = u^b_a \left( \frac{M^2}{\rho^b} - 3(\rho^b)^3 tr(D_\# X^i) \right). \tag{5.184}$$

Eq. (5.183) might look strange if one expects the center of energy motion to be similar to the motion of a free particle. However, this relation is characteristic for a charged particle moving in a background Maxwell field (see e.g. [200]). In our case the counterpart (5.184) of the electromagnetic potential $A_a$ is constructed in terms of the relative motion variables. It vanishes when the relative motion is in its ground state.

Thus the seemingly unusual effect of that the $mM_0$ center of energy velocity and momentum are not parallel one to another is just one of the manifestations of the mutual influence of the center of energy and the relative motion in $mM_0$ system. The relative motion variables, when they are not in ground state, generate a counterpart of the 11D background vector potential for the center of energy motion.

5.7.4. An example of non-supersymmetric solutions

Let us fix the gauge (5.110), (5.111), $\Omega^{ij} = 0 = A_\tau$, use the $SO(1,1)$ gauge symmetry to set $\rho^\# = 1$ and the reparametrization symmetry to fix $\dot{\hat{E}}^\#_\tau = 1^{14}$

$$\Omega^{ij}_\tau = 0 = A_\tau, \quad \dot{\hat{E}}^\#_\tau = 1 = \rho^\#. \tag{5.185}$$

Then

$$D_\# = \partial_\tau \tag{5.186}$$

and Eqs. (5.163) simplify to

$$\ddot{X}^i = -\frac{1}{16} [[X^i, X^j] X^j], \tag{5.187}$$

$$[[X^i, X^j] = 0. \tag{5.188}$$

These very well known equations describe the 1d reduction of the 10D $SU(N)$ Yang-Mills gauge theory.

---

14Actually, to be precise, there exists an obstruction to fix such a gauge by $\tau$ reparametrization [201]. The best what one can do is to fix $\partial_\tau \dot{\hat{E}}^\#_\tau = 0$, while the constant value remains indefinite. This is especially important for path integral quantization, where the integration over this constant value (modulus) should be included in the definition of the path integral measure. As here we do not need in this level of precision, we allow ourselves to simplify the formulas by just setting this indefinite constant to unity.
A very simple solution of Eqs. (5.187) and (5.188) is provided by
\[ X^i(\tau) = (A^i\tau + B^i)Y, \]  
(5.189)

where \( Y \) is a constant traceless \( N \times N \) matrix, \( A^i \) and \( B^i \) are constant \( SO(9) \) vectors, and \( \tau \) is the proper time of the m\( M^0 \) center of energy. The center of energy effective mass is defined by the trace of \( Y^2 \) and by the length of vector \( \vec{A} = \{ A^i \} \),
\[ M^2 = 4\mathcal{H} = 2\vec{A}^2 \text{tr} Y^2, \quad \vec{A}^2 := A^i A^i. \]  
(5.190)

Actually, by choosing the initial point of the proper time, \( \tau \mapsto \tau - a \), we can always make the constant \( SO(9) \) vectors \( A^i \) and \( B^i \) orthogonal,
\[ \vec{A} \vec{B} := A^i B^i = 0. \]  
(5.191)

Then the ‘currents’ (5.170) read
\[ J^{ij} = A^i B^j \text{tr} Y^2 = \frac{A^i B^j}{2\vec{A}^2} M^2, \quad J = \frac{\tau}{4} M^2. \]  
(5.192)

Now the equations for the center of energy coordinate functions (5.132), (5.166) and the gauge fixing condition \( \hat{E}^\#_\tau = 1 \) imply
\[ \dot{\hat{x}}^a u_a^- = \frac{3M^2}{4}, \]  
(5.193)
\[ \dot{\hat{x}}^a u_a^i = 0, \]  
(5.194)
\[ \dot{\hat{x}}^a u_a^\# = 1. \]  
(5.195)

With our gauge fixing, Eqs. (5.142), which follow from (5.167), (5.168), implies that moving frame vectors are constant
\[ \dot{u}_a^\# = 0, \quad \dot{u}_a^- = 0, \quad \dot{u}_a^i = 0. \]  
(5.196)

Thus (5.193), (5.194), (5.195) is a simple system of linear differential equations
\[ \dot{\hat{x}}^a u_a^- = 3M^2/4, \]  
(5.197)
\[ \dot{\hat{x}}^i = 0, \]  
(5.198)
\[ \dot{\hat{x}}^\# = 1, \]  
(5.199)

for the variables
\[ \dot{\hat{x}}^a = \dot{\hat{x}}^a u_a^- , \quad \dot{\hat{x}}^i = \dot{\hat{x}}^a u_a^i , \quad \dot{\hat{x}}^\# = \dot{\hat{x}}^a u_a^\#. \]  
(5.200)

This system can be easily solved for the 'comoving frame' coordinate functions (5.200). The
5.7. On solutions of $m M_0$ equations with $M^2 > 0$

solution in an arbitrary frame

$$\dot{x}^\mu(\tau) = \dot{x}^\mu(0) + \frac{\tau}{2} \left( u^-\mu + \frac{3M^2}{4} u^\#\mu \right) \tag{5.201}$$

describe a time-like motion of the center of energy characterized by a nonvanishing effective mass $M^2$. The velocity of this motion,

$$\dot{x}^\mu = \frac{1}{2} \left( u^-\mu + \frac{3M^2}{4} u^\#\mu \right) \tag{5.202}$$

is not parallel to the canonical momentum (see (5.181))

$$p_\mu = u^-\mu + \frac{M^2}{4} u^\#\mu. \tag{5.203}$$

As it was discussed in general case in sec. 5.7.3 this is due to the influence of the relative motion of the $m M_0$ constituents on the center of energy motion and can be considered as an effect of the induction by the relative motion dynamics of a counterpart of the Maxwell background field interacting with the center of energy coordinate functions. In the case under consideration this induced Maxwell field is constant, $A_\mu = -u^\#\mu M^2/2$.

5.7.5. Another non-supersymmetric formal solution

In the case of the system of $2 M_0$-branes, the $2 \times 2$ matrix field $X^i$ can be decomposed on Pauli matrices,

$$X^i = f^i_j(\tau) \sigma^J,$$

$$\sigma^I \sigma^J = \delta^{IJ} I_{2 \times 2} + i \epsilon^{IJK} \sigma^K, \quad I, J, K = 1, 2, 3. \tag{5.204}$$

The simplest ansatz which solves the Gauss constraint (5.188) is $f^i_j(\tau) = \delta^i_j f(\tau)$ so that

$$X^i(\tau) = f(\tau) \delta^i_J \sigma^J, \quad i = 1, ..., 9; \quad I, J, K = 1, 2, 3. \tag{5.205}$$

Eq. (5.187) then implies that this function should obey

$$\dot{f} + \frac{1}{2} f^3 = 0. \tag{5.206}$$

The simplest solution of this equation is given by $f(\tau) = \frac{2i}{\tau}$ which is complex and thus breaks the condition that $X^i$ is a Hermitian matrix. Actually one can consider this solution,

$$X^i(\tau) = \frac{2i}{\tau} \delta^i_J \sigma^J, \quad J = 1, 2, 3. \tag{5.207}$$

as an analog of instanton as far as after Wick rotation $\tau \mapsto i \tau$ restores the Hermiticity properties.

Ignoring for a moment the problem with Hermiticity we can calculate the Hamiltonian and find that it is equal to zero. Thus (5.207) is a solution with vanishing center of energy
mass, $M^2 = 0$.

A configuration (5.205) with nonzero effective center of energy mass can be obtained by observing that (5.206) has a more general solution given by the so-called Jackobi elliptic function \cite{202}. These functions obey

$$\dot{f}^2 = -f^4/4 + C$$

with an arbitrary constant $C$. The above discussed particular solution (5.207) of (5.206) solves (5.208) with $C = 0$ which suggests the relation of $C$ with $M^2$. Indeed, a straightforward calculation shows that $C = M^2/12$ so that the instanton–like solution of the mM0 equations of relative motion is given by 2x2 matrices (5.205) with the function $f(\tau)$ obeying

$$\dot{f}^2 = \frac{M^2 - 3f^4}{12}.$$

The set of equations for the center of energy motion includes (5.198), (5.199) and

$$\dot{x}^\equiv = 3M^2/4 - 9(f(\tau))^4/2.$$

This equation can be solved numerically, but its detailed study goes beyond the scope of this thesis.
In conclusion we list the main results obtained in this thesis.

1. The complete set of the superfield equations of motion for the interacting system of four dimensional supermembrane and dynamical scalar multiplet have been obtained. Our study has provided the first example of superfield equations for interacting system involving matter (not supergravity) superfields and supersymmetric extended object, as well as the first set of superfield equations of motion for a dynamical superfield system including supermembrane. Furthermore
   a) The action of supermembrane in a chiral $\mathcal{N} = 1, D = 4$ superfield background has been presented for the first time.
   b) It was shown that the consistency of the interaction with dynamical scalar multiplet requires this to be special, namely to be described by chiral superfield of special form: expressed through the real (rather than complex) pre-potential superfield.
   c) The equations of motion for spacetime, component fields with supermembrane contributions have been obtained from the bulk superfield equations for the simplest case when the bulk part of the action is given by the free kinetic term (with Kähler potential $\mathcal{K} = \Phi \bar{\Phi}$). A solution of the dynamical equations for physical fields in the leading order on supermembrane tension have been obtained. The effects of inclusion of nontrivial superpotential and relation with known supersymmetric domain wall solutions have been discussed.

2. The complete set of superfield equations of motion for the interacting system of dynamical $D = 4$ $\mathcal{N} = 1$ supergravity and supermembrane has been obtained from the superspace action principle. The supermembrane model is consistent in an arbitrary off-shell minimal supergravity background. However the interaction with supermembrane requires the dynamical supergravity to be the Grisaru–Siegel–Gates–Ovrut–Waldram special minimal supergravity. The chiral compensator superfield of this are constructed from real (rather than complex) prepotential.
a) We have developed the Wess–Zumino type approach to this special minimal supergravity the characteristic property of which is a dynamical generation of the cosmological constant.

b) To see this effect in the interacting system we extract the spacetime component equations from the superfield equations. To this end we have fixed the usual Wess–Zumino gauge in the superfield supergravity equations, supplemented by partial gauge fixing of the local supersymmetry on the supermembrane worldvolume \( (\theta^2(\xi) = 0) \). We have shown that the supermembrane current superfields simplify drastically in this "WZ \( \theta^2 = 0 \) gauge".

c) In the component form of equations obtained in this way it is seen that in the interacting system the supermembrane produces a kind of renormalization of the cosmological constant, making its value different in the branches of spacetime separated by the supermembrane worldvolume.

d) This allowed us to show that configuration describing a domain wall separating two branches of AdS space with different cosmological constants provides a supersymmetric solution of the system of our superfield supergravity equations considered outside the supermembrane worldvolume \( \mathcal{W}^3 \).

3. We have obtained the superfield equations in \( \mathcal{N} = 2, 4 \) and \( 8 \) extended tensorial superspaces \( \Sigma^{(10\mid\mathcal{N})} \), which describe the supermultiplets of the \( D = 4 \) massless conformal free higher spin field theory with \( \mathcal{N} \)-extended supersymmetry.

a) The \( \mathcal{N} = 2 \) supermultiplet of massless conformal higher spin equations is simply given by the complexification of the \( \mathcal{N} = 1 \) supermultiplet.

b) For \( \mathcal{N} = 4, 8 \) no tensorial space generalizations of the Maxwell, Rarita-Schwinger or linearized conformal gravity equations appear. It is shown that \( \mathcal{N} \geq 4 \) supermultiplets are built from the scalar and spinor fields in tensorial space which obey the standard higher spin equations in their tensorial space version. However, some of these appear in the basic superfields under derivatives, so that the \( \mathcal{N} \geq 4 \) supersymmetric theory is invariant under Peccei–Quinn–like symmetries shifting these fields.

4. We have obtained and studied the equations of motion of multiple M0–brane system (mM0) which follow from the covariant supersymmetric and \( \kappa \)-symmetric mM0 action proposed in \([150]\).

a) We have found that the mM0 action is invariant under an interesting reminiscent of the so–called \( K_9 \) gauge symmetry which is necessary to find the final form of the bosonic equations of motion for the center of energy coordinate functions.

b) We have found that, generically, there exists the 'backreaction', the influence of the relative motion on the motion of the center of energy, the most important effects of which are that the generic center of energy motion of mM0 system is characterized by a nonvanishing effective mass \( \mathcal{M} \) constructed from the matrix field describing the relative motion. Furthermore, when the relative motion is not
in its ground state, the center of energy velocity and the canonical momentum
conjugate to the center of energy coordinate function are oriented in different
directions of the 11D spacetime. These two effects disappear when $M^2 = 0$.

c) We have shown that all the mM0 BPS states are 1/2 BPS and have the same
properties as BPS states of single M0–brane. In particular, the effective mass
of the center of energy motion vanishes for the BPS states. In other words, all
the supersymmetric purely bosonic solutions of mM0 equations preserve just $\frac{1}{2}$
of 11D supersymmetry, have the relative motion sector in its ground state so
that all the equations of the center of energy motion acquire the same form as
equations for single M0–brane.
APPENDIX A

NOTATION, CONVENTIONS AND SOME USEFUL FORMULAE IN $D = 4$

In chapters 2 3 we use mostly minus Minkowski metric $\eta^{ab} = \text{diag}(1, -1, -1, -1)$ and complex Weyl spinor notation. $D = 4$ vector and spinor indices are denoted by symbols from the beginning of Latin and Greek alphabets, $a, b, c = 0, 1, 2, 3$, $\alpha, \beta, \gamma = 1, 2$, $\dot{\alpha}, \dot{\beta}, \dot{\gamma} = 1, 2$. In particular, the coordinates of flat $D = 4$, $\mathcal{N} = 1$ superspace $\Sigma^{(4|4)}$ are denoted, respectively, by $x^a$ and $\theta^\alpha$, $\bar{\theta}^\dot{\alpha} = (\theta^\alpha)^*$. The contraction the spinorial indices are raised and lowered by the unit antisymmetric tensors $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = i\sigma^2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon_{\dot{\beta}\dot{\alpha}}$. In flat superspace (or within the WZ gauge) this implies $\theta^\alpha = \epsilon_{\alpha\beta}\theta^\beta$, $\bar{\theta}^\dot{\alpha} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^\dot{\beta}$, etc. However, to get $\partial^\alpha \theta^\beta = \delta^\beta_\alpha$ simultaneously with $\partial^\alpha \theta^\alpha = \delta^\beta_\alpha$ we have to assume that for the derivatives over the fermionic variables $\partial^\alpha = -\epsilon^{\alpha\beta}\partial^\beta$ so that, when we rise the spinorial indices of the covariant fermionic derivative (1.22), we arrive at $D^\alpha := \epsilon^{\alpha\beta}D_\beta = -\partial^\alpha - i(\bar{\theta}\tilde{\sigma}^a\theta^a)\partial^\alpha$. In curved superspace we need to introduce world supervector indices $M, N, ...$ The coordinates of curved superspace are denoted by $Z^M = (x^\mu, \theta^\alpha)$ with $\tilde{\alpha} = 1, 2, 3, 4$; clearly beyond the WZ guage these $\tilde{\alpha}$ are not spinorial indices so we do not use their splitting on $\tilde{\alpha}$ and $\dot{\tilde{\alpha}}$ (which may however, be useful in the prepotential approach, see [151]). The star superscript * denotes complex conjugation of bosonic variables and involution on Grassmann algebra (see [164] and refs. therein); in practical terms this implies that $(\theta^\alpha \tilde{\theta}^\beta)^* = \tilde{\theta}^\beta \theta^\alpha = -\tilde{\theta}^\alpha \tilde{\theta}^\beta$. Then, to keep the plus sign in $(\theta^\alpha)^* = \bar{\theta}^\alpha$ with $(\theta)^2 := \theta^\alpha \theta^\alpha$, we have to define $\bar{\theta}^\alpha := \bar{\theta}^\alpha \tilde{\sigma}^a$. The consistency of the Grassmann algebra involution requires that $(\partial^\alpha)^* \equiv (\frac{\partial^\alpha}{\partial \theta^\alpha})^* = -\bar{\partial}^\alpha \equiv -\frac{\partial^\alpha}{\partial \bar{\theta}^\alpha}$. The covariant spinor derivative defined in (1.22) are related by $(\bar{D}^\dot{\alpha})^* = -\bar{D}_{\dot{\alpha}}$, $\bar{D} \bar{D} := \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} = (\bar{D} \bar{D})^*$ where $\bar{D} \bar{D} := D^\alpha D_\alpha$. 

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The list of properties of relativistic Pauli matrices $\sigma^a_{\beta\dot{\alpha}} = \epsilon_{\beta\alpha}\epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\sigma}^{a\dot{\alpha}}$ include

$$\sigma^a\tilde{\sigma}^b = \eta^{ab} + \frac{i}{2}\epsilon^{abcd}\sigma_c\tilde{\sigma}_d,$$
$$\tilde{\sigma}^a\sigma^b = \eta^{ab} - \frac{i}{2}\epsilon^{abcd}\tilde{\sigma}_c\sigma_d,$$  \hspace{1cm} (A.1)

The 3-dimensional worldvolume vector indices are denoted by symbols from the middle of Latin alphabet. In particular, the local coordinates of the supermembrane worldvolume $W^3$ are denoted by $\xi^m$ with $m = 0, 1, 2$. The worldvolume Hodge star (denoted by $\ast$ in the line) operation is defined as in (3.18),

$$\ast \hat{E}^a := \frac{1}{2}d\xi^m \wedge d\xi^n \sqrt{g}\epsilon_{mnk}g^{kl}\hat{E}^a_l.$$  \hspace{1cm} (A.5)

In our conventions $d\xi^m \wedge d\xi^n \wedge d\xi^k = -\epsilon^{mnk}d^3\xi \equiv \epsilon^{knm}d^3\xi$ so that

$$\ast \hat{E}_a \wedge \hat{E}^a = -3d^3\xi\sqrt{g}, \quad \ast \hat{E}_a \wedge \delta \hat{E}^a = -d^3\xi\sqrt{g}\hat{E}_ma\tilde{g}^{mn}\delta \hat{E}^a,$$

and

$$\delta(\ast \hat{E}_a \wedge \hat{E}^a) = 3 \ast \hat{E}_a \wedge \delta \hat{E}^a.$$  \hspace{1cm} (A.6)

The superspace generalization of Dirac delta function reads

$$\delta^8(z) := \frac{1}{16}(\theta^2)(\bar{\theta}^2)\delta^4(x)$$

and obeys

$$\int d^8z \delta^8(z - \hat{z}) f(z) = f(\hat{z}), \quad \int d^8z \delta^8(z) = \int d^4x \bar{D}D\hat{D}D\delta^8(z) = 1.$$
The leading component of the Nambu-Goto current superfield \[ (2.63) \]

\[ J_{NG}(Z) = -\frac{1}{4} \int d^3\xi \sqrt{g} \sqrt{\frac{\Phi}{\Phi}} \, D D \delta^8(z - \hat{z}) - \frac{1}{4} \int d^3\xi \sqrt{g} \sqrt{\frac{\Phi}{\Phi}} \, \bar{D} \bar{D} \delta^8(z - \hat{z}) \] (B.1)

reads

\[ J_{NG} \big|_0 = +\frac{1}{16} \int d^3\xi \sqrt{g} \hat{\theta} \hat{\bar{\theta}} \delta^4(x - \hat{x}) + \frac{1}{16} \int d^3\xi \sqrt{g} \hat{\bar{\theta}} \hat{\theta} \delta^4(x - \hat{x}) \] (B.2)

The general expression for the fermionic derivative of \( J_{NG} \)

\[ \bar{D}_\dot{\alpha} J_{NG} = -\frac{1}{4} \int d^3\xi \sqrt{g} \sqrt{\frac{\Phi}{\Phi}} \, \bar{D}_\dot{\alpha} D D \delta^8(Z - \hat{Z}) . \] (B.3)

so that

\[ \bar{D}_\dot{\alpha} J_{NG} \big|_0 = \frac{1}{8} \int d^3\xi \sqrt{g} \hat{\theta} \hat{\bar{\theta}} \delta^4(x - \hat{x}) - \frac{i}{16} \int d^3\xi \sqrt{g} \sqrt{\frac{\Phi}{\Phi}} (\sigma^a \hat{\theta})_\dot{\alpha} (\hat{\bar{\theta}}) \partial_a \delta^4(x - \hat{x}) . \] (B.4)

Furthermore, as

\[ \bar{D} \bar{D} J_{NG} = -\frac{1}{4} \int d^3\xi \sqrt{g} \sqrt{\frac{\Phi}{\Phi}} \, \bar{D} \bar{D} D D \delta^8(Z - \hat{Z}) \] (B.5)
we find
\[ DD_{J^{NG}}|_0 = \int d^3 \xi \sqrt{g} \sqrt{\phi} \left( -\frac{1}{4} \delta^4(x - \hat{x}) - \frac{i}{4} (\hat{\theta} \sigma^a \hat{\theta}) \partial_a \delta^4(x - \hat{x}) + \frac{1}{16} (\hat{\theta})^2 (\hat{\theta})^2 \square \delta^4(x - \hat{x}) \right) = \]
\[ = -\frac{1}{4} \sqrt{\phi} \int d^3 \xi \sqrt{g} \delta^4(x - \hat{x}) - \]
\[ -\frac{i}{4} \sqrt{\phi} \int d^3 \xi \sqrt{g} \left( \delta^4(x - \hat{x}) + \left( \frac{\partial_a \phi}{2\phi} - \frac{\partial_a \delta^4}{2\phi} \right) \delta^4(x - \hat{x}) \right) + O(f^4). \quad (B.6) \]

One can also write Eq. (B.6) in the equivalent but more compact form of
\[ DD_{J^{NG}}|_0 = -\frac{1}{4} \sqrt{\phi} \int d^3 \xi \sqrt{g} \delta^4(x - \hat{x}) + O(f^2)_{DD_{J^{NG}}} + O(f^4)_{DD_{J^{NG}}}, \quad (B.7) \]
where
\[ O(f^2)_{DD_{J^{NG}}} = -\frac{i}{4} \sqrt{\phi} \int d^3 \xi \sqrt{g} \left( \delta^4(x - \hat{x}) + \left( \frac{\partial_a \phi}{2\phi} - \frac{\partial_a \delta^4}{2\phi} \right) \delta^4(x - \hat{x}) \right). \quad (B.8) \]

The leading term of the second, Wess–Zumino (WZ) contribution to the supermembrane current,
\[ J^{WZ}(Z) = \left( 2i \int_{W^3} \hat{E}^c \wedge \hat{E}^a \wedge \hat{E}^\alpha \sigma_{ca\hline} + \right. \]
\[ + \frac{1}{2} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^\alpha \sigma_{bc\hline}^\beta D_\beta - \frac{1}{2} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^\alpha \sigma_{bc\hline}^\beta \hat{D}_\beta - \]
\[ \left. -\frac{1}{4!} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{ab\hline\alpha} \sigma_{\alpha\hline\delta a\hline}^\beta [D_\alpha, \hat{D}_\alpha] \right) \delta^8(z - \hat{z}) \quad (B.9) \]
reads
\[ J^{WZ}(x) := J^{WZ}(Z)|_0 = \left( \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{ab\hline\alpha} \sigma^\beta \hat{\theta} \delta^4(x - \hat{x}) - \right. \]
\[ -\frac{1}{16} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge d\hat{\theta}^\alpha \sigma_{bc\hline}^\beta \hat{\theta} \delta^4(x - \hat{x}) - \]
\[ -\frac{1}{16} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge d\hat{\theta}^\alpha \sigma_{bc\hline}^\beta \hat{\theta} \delta^4(x - \hat{x}) + \]
\[ +\frac{i}{8} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge d\hat{\theta}^\alpha \sigma_{ca\hline}^\beta \hat{\theta} \delta^4(x - \hat{x}) = \]
\[ = \frac{1}{48} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{ab\hline\alpha} \sigma^\beta \hat{\theta} \delta^4(x - \hat{x}) + O(f^4). \quad (B.10) \]
Its closest fermionic partners are

\[
D_\alpha J^{WZ}|_0 = -\frac{1}{48} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd}(\sigma^d \hat{\theta})_a \delta^4(x - \hat{x}) + \\
+ \frac{1}{16} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \left(2(d\hat{\theta} \sigma_{bc}) \hat{\theta}_a + d\hat{\theta} \sigma_{bc} \alpha(\hat{\theta})^2\right) \delta^4(x - \hat{x}) + \\
+ \frac{i}{96} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd}(\sigma^d \hat{\theta})_a \delta^4(x - \hat{x}) + \\
+ \frac{i}{32} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \left(d\hat{\theta} \sigma_{bc} \sigma^a \hat{\theta}_a \delta^4(x - \hat{x}) =
\right.
\]

\[
\left. - \frac{i}{4} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \sigma_{c\beta} \hat{\theta}_a \delta^4(x - \hat{x}) =
\right. \tag{B.11}
\]

\[
\hat{D}_\alpha J^{WZ}|_0 = \frac{1}{48} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd}(\hat{\theta} \sigma^d)_a \delta^4(x - \hat{x}) - \\
- \frac{1}{16} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \left(2(d\hat{\theta} \sigma_{bc}) \hat{\theta}_a + (d\hat{\theta} \sigma_{bc}) \alpha(\hat{\theta})^2\right) \delta^4(x - \hat{x}) - \\
- \frac{i}{96} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd}(\hat{\theta} \sigma_{bc} \sigma^a \hat{\theta}_a) \delta^4(x - \hat{x}) - \\
- \frac{i}{32} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \left(d\hat{\theta} \sigma_{bc} \sigma^a \hat{\theta}_a \delta^4(x - \hat{x}) + \\
+ \frac{i}{4} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \sigma_{c\beta} \hat{\theta}_a \delta^4(x - \hat{x}) =
\right. \tag{B.12}
\]

Then, as far as

\[
DDJ^{WZ}(Z) = 2i \int_{W^3} \hat{E}^c \wedge \hat{E}^a \wedge \hat{E}^\alpha \sigma_{c\alpha} \hat{D} \delta^8(Z - \hat{Z}) + \\
- \frac{1}{2} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^\alpha \sigma_{c\alpha} \hat{D} \delta^8(Z - \hat{Z}) + \\
+ \frac{i}{6} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd} \delta^8(Z - \hat{Z}) \tag{B.13}
\]

\[
\hat{D}DJ^{WZ}(Z) = 2i \int_{W^3} \hat{E}^c \wedge \hat{E}^a \wedge \hat{E}^\alpha \sigma_{c\alpha} \hat{D} \delta^8(Z - \hat{Z}) + \\
+ \frac{1}{2} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^\alpha \sigma_{c\alpha} \hat{D} \delta^8(Z - \hat{Z}) - \\
+ \frac{i}{6} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd} \delta^8(Z - \hat{Z}) \tag{B.14}
\]
one finds that

\[
DDJ^{WZ}|_0 = \frac{1}{4} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge d\hat{\theta} \hat{\partial}_c \hat{\partial}^4(x - \hat{x}) - \frac{i}{4!} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd}(\hat{\theta})^2 \hat{\partial}^d \delta^4(x - \hat{x}) - \\
- \frac{i}{2} \int_{W^3} \hat{E}^c \wedge d\hat{\theta}^a \wedge d\hat{\theta}^a \sigma_{ca\delta}(\hat{\theta})^2 \delta^4(x - \hat{x}) + \\
+ \frac{i}{8} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge d\hat{\theta}^a (\hat{\theta} \sigma^{ca\delta} \sigma_{ca\delta})(\hat{\theta})^2 \partial_a \delta^4(x - \hat{x}) = \\
= \frac{1}{4} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge d\hat{\theta} \hat{\partial}_c \hat{\partial}^4(x - \hat{x}) - \\
- \frac{i}{4} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd}(\hat{\theta})^2 \hat{\partial}^d \delta^4(x - \hat{x}) + O(f^4). \quad (B.15)
\]

\[
\bar{D} \bar{D}J^{WZ}|_0 = \frac{1}{4} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge d\hat{\theta} \hat{\partial}_c \hat{\partial}^4(x - \hat{x}) + \frac{i}{4!} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd}(\hat{\theta})^2 \hat{\partial}^d \delta^4(x - \hat{x}) - \\
- \frac{i}{2} \int_{W^3} \hat{E}^c \wedge d\hat{\theta}^a \wedge d\hat{\theta}^a \sigma_{ca\delta}(\hat{\theta})^2 \delta^4(x - \hat{x}) - \\
- \frac{i}{8} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge d\hat{\theta} \hat{\partial}_c \hat{\partial}^4(x - \hat{x}) = \\
= \frac{1}{4} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge d\hat{\theta} \hat{\partial}_c \hat{\partial}^4(x - \hat{x}) + \\
+ \frac{i}{4} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd}(\hat{\theta})^2 \hat{\partial}^d \delta^4(x - \hat{x}) + O(f^4). \quad (B.16)
\]

To analyze the structure of the auxiliary field equation one needs also to know

\[
[D_\alpha, \bar{D}_{\beta}]J^{WZ}|_0 = -\frac{1}{4!} \int_{W^3} \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd} \sigma_{\alpha\beta}^d \delta^4(x - \hat{x}) + O(f^2), \quad (B.17)
\]
Appendix B. Supermembrane current superfield which enters the scalar multiplet equations

where

\[ \mathcal{O}(f^2) = \frac{i}{4!} \int \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \epsilon_{abcd} \left( (\sigma^{de}\hat{\theta})_{\alpha} \hat{\theta}_{\beta} + \hat{\theta}_{\alpha} (\hat{\theta} \hat{\sigma}^{de})_{\beta} \right) \partial_e \delta^4(x - \hat{x}) + \]

\[ + \frac{1}{4} \int \hat{E}^c \wedge \hat{E}^b \wedge \left( (\sigma_{bc} d\hat{\theta})_{\alpha} \hat{\theta}_{\beta} - (d\hat{\theta} \hat{\sigma}_{bc})_{\beta} \hat{\theta}_{\alpha} \right) \delta^4(x - \hat{x}) + \mathcal{O}(f^4), \quad (B.18) \]

\[ \mathcal{O}(f^4) = \frac{1}{2 \cdot 4!} \int \hat{E}^c \wedge \hat{E}^b \wedge \hat{E}^a \hat{(\hat{\theta})}^2 \epsilon_{abcd} \left( \sigma^d_{\alpha\beta} \Box \delta^4(x - \hat{x}) - \sigma^e_{\alpha\beta} \partial_e \partial^4 \delta^4(x - \hat{x}) \right) + \]

\[ + \frac{i}{8} \int \hat{E}^c \wedge \hat{E}^b \wedge (d\hat{\theta} \sigma_{bc} \sigma^a)_{\beta} \hat{\theta}_{\alpha} \hat{(\hat{\theta})}^2 \partial_a \delta^4(x - \hat{x}) + \]

\[ + \frac{i}{8} \int \hat{E}^c \wedge \hat{E}^b \wedge (d\hat{\theta} \sigma_{bc} \sigma^a)_{\alpha} \hat{\theta}_{\beta} \hat{(\hat{\theta})}^2 \partial_a \delta^4(x - \hat{x}) - \]

\[ - i \int \hat{E}^c \wedge d\hat{\theta}^7 \wedge d\hat{\theta}^7 \sigma_{c74} \hat{\theta}_{\alpha} \hat{\theta}_{\beta} \delta^4(x - \hat{x}). \quad (B.19) \]
On Admissible Variations of Superfield Supergravity

The admissible variations of supervielbein are the variation preserving the superspace constraints. In the case of minimal supergravity that read [72, 158]

$$
\delta E^a = E^a(\Lambda(\delta) + \bar{\Lambda}(\delta)) - \frac{1}{4} E^b \tilde{\sigma}_b^{\alpha\alpha}[D_\alpha, \bar{D}_\alpha] \delta H^a + i E^\alpha D_\alpha \delta H^a - i \bar{E}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} \delta H^a \ , \quad \text{(C.1)}
$$

$$
\delta E^\alpha = E^\alpha \Xi^\alpha(\delta) + E^\alpha \Lambda(\delta) + \frac{1}{8} \bar{E}^{\dot{\alpha}} R \sigma_{a\dot{\alpha}} \delta H^a \ . \quad \text{(C.2)}
$$

where

$$
2\Lambda(\delta) + \bar{\Lambda}(\delta) = \frac{1}{4} \tilde{\sigma}_a^{\alpha\alpha} D_\alpha \bar{D}_\alpha \delta H^a + \frac{1}{8} G_a \delta H^a + 3(DD - \bar{R}) \delta U \ . \quad \text{(C.3)}
$$

The explicit expression for $\Xi^\alpha(\delta)$ in (C.2), as well as the admissible variations of the spin connection superform, can be found in [72].

The variation of the closed 4–form (3.22), (3.21) reads [84]

$$
\delta H_4 = \frac{1}{2} E^b \wedge E^\alpha \wedge E^\beta \wedge E^\gamma \sigma_{ab(\alpha\beta} D_{\gamma)} \delta H^a - \frac{1}{2} E^b \wedge E^\alpha \wedge E^\beta \wedge \bar{E}^{\dot{\gamma}} \sigma_{a\dot{\alpha}\dot{\beta}} \bar{D}_{\dot{\gamma}} \delta H^a + \text{c.c.} - \frac{i}{2} E^b \wedge E^\alpha \wedge E^\beta \left( \sigma_{ab\alpha\beta} \left( 2\Lambda(\delta) + \bar{\Lambda}(\delta) \right) + \frac{1}{4} \sigma_{c[|a\alpha\beta|\dot{\alpha}\dot{\beta}]\dot{\gamma}} [D_\gamma, \bar{D}_{\dot{\gamma}}] \delta H^c \right) + \text{c.c.} + \frac{i}{16} E^b \wedge E^\alpha \wedge E^\beta \left( R \sigma_{ab\alpha\beta} - \bar{R} \sigma_{a\dot{\alpha}b\alpha} \right) \delta H^c \wedge \wedge E^c \wedge E^b \wedge E^a \ . \quad \text{(C.4)}
$$

The conditions of that $\delta H_4$ can be expressed in terms of the variation of the 3–form potential $\delta C_3$,

$$
\delta H_4 = d(\delta C_3) \quad \text{(C.5)}
$$
with $\delta C_3$ decomposed on the basic covariant 3–forms, as in Eq. (3.36), restrict the set of independent variations by 

$$(\mathcal{D}\mathcal{D} - \bar{R})\delta U = \frac{1}{12}(\mathcal{D}\mathcal{D} - \bar{R}) \left( i \delta V + \frac{1}{2} \mathcal{D}_a \delta \bar{\kappa}^a \right).$$

(C.6)

This is equivalent to

$$\delta U = \frac{i}{12} \delta V + \frac{1}{24} \mathcal{D}_a \delta \bar{\kappa}^a + \frac{i}{24} \mathcal{D}_a \delta \nu^a,$$

(C.7)

where $\delta \nu^a$ is an additional independent variation (which does not contribute to $(\mathcal{D}\mathcal{D} - \bar{R})\delta U$ and, hence, to the variations of supergravity potentials).

Factoring out the gauge transformations, we can write the variation $\delta C_3$, which produces (C.4) through (C.5), in the form (3.36) with

$$\beta_{\alpha\beta\gamma}(\delta) = 0 = \beta_{\alpha\beta}(\delta), \quad \beta_{\alpha\beta\alpha}(\delta) = i\sigma_{\alpha\beta\gamma} \delta V,$$

(C.8)

and

$$\beta_{\alpha\beta\gamma}(\delta) = -\sigma_{ab\alpha\beta} (\delta H^b + \sigma^{b\gamma} D_\gamma \delta \bar{\kappa}_\gamma),$$

$$\beta_{\alpha\alpha\beta}(\delta) = \frac{1}{2} \epsilon_{abcd} \sigma_{c\alpha} \bar{D}^a \delta H^d + \frac{1}{2} \sigma_{ab\alpha\beta} D_\beta \delta V -$$

$$-\frac{i}{4} \sigma_{ab\alpha\beta} \bar{D}_\beta D_\alpha \delta \bar{\kappa},$$

$$\beta_{\alpha\beta\gamma}(\delta) = \frac{i}{8} \epsilon_{abcd} \left( \left( \mathcal{D}\mathcal{D} - \frac{1}{2} \bar{R} \right) \delta H^d - c.c. \right) + \frac{1}{4} \epsilon_{abcd} G^d \delta V +$$

$$+ \frac{1}{8} \epsilon_{abcd} \sigma^{d\gamma} [D_\gamma, \bar{D}_\gamma] \delta V - \frac{i}{16} \epsilon_{abcd} \sigma^{d\gamma} \left( \left( \mathcal{D}\mathcal{D} + \frac{5}{2} \bar{R} \right) \bar{D}_\gamma \kappa_\gamma - c.c. \right).$$

(C.9)

(C.10)

The variation of the special minimal supergravity action reads

$$\delta S_{SG} = \frac{1}{6} \int d^8 Z E \left[ G_a \delta H^a + (R - \bar{R})i \delta V \right] -$$

$$- \frac{1}{12} \int d^8 Z E \left( \mathcal{D}_a \delta \kappa^a + \bar{R} \mathcal{D}_a \delta \bar{\kappa}^a \right).$$

(C.12)

Notice that the variations $\delta \kappa^a$ and $\delta \bar{\kappa}^a$ result in equations $\mathcal{D}_a R = 0$ and $\mathcal{D}_a \bar{R} = 0$, which are satisfied identically due to the minimal (Eq. (3.12)) or special minimal supergravity equations of motion (Eq. (3.40)). In the WZ $\hat{\theta} = 0$ gauge (3.49)–(3.53) it is also relatively easy to check that $\delta \kappa^a$ does not produce any independent equation for the physical fields of the interacting system. This observation has allowed us to simplify the discussion in the main text by neglecting the existence $\delta \kappa^a$ variation.
The variation of the supermembrane action (3.14) with respect to the vector prepotential of supergravity, $\delta H^a$, gives us the vector supercurrent of the form

$$J_a = \int W^3 \hat{E}^b \wedge \hat{E}^\alpha \wedge \hat{E}^\beta \sigma_{a b c d} \delta^8(Z - \hat{Z}) - \int \frac{3}{4} \hat{E}^b \wedge \hat{E}^c \wedge \hat{E}^d \epsilon_{a b c d} \sigma_{\alpha \beta \gamma \delta} \delta^8(Z - \hat{Z}) + c.c.$$

$$- \int \frac{3i}{4} \hat{E}^b \wedge \hat{E}^c \wedge \hat{E}^d \epsilon_{a b c d} \left( \mathcal{D} \mathcal{D} - \frac{1}{2} \mathcal{R} \right) \delta^8(Z - \hat{Z}) + c.c. + \int \frac{1}{4} \hat{E}_b \wedge \hat{E}^b G_a \delta^8(Z - \hat{Z}) - \int \frac{1}{4} \hat{E}_c \wedge \hat{E}^b \sigma^{\alpha \beta} \left( 3\delta^d_{a b} - \delta^d_{a c} \right) \left[ \mathcal{D}_a, \bar{\mathcal{D}}_b \right] \delta^8(Z - \hat{Z}),$$

where $\hat{E} = sdet(E_M^A(\hat{Z}))$ and

$$\delta^8(Z) := \frac{1}{16} \delta^4(x) \theta \bar{\theta} \bar{\theta} \theta,$$

is the superspace delta function which obeys $\int d^8Z \delta^8(Z - Z') f(Z) = f(Z')$ for any superfield $f(Z)$. 

**APPENDIX D**

**SUPERMEMBRANE CURRENT SUPERFIELDS ENTERING THE SUPERFIELD SUGRA EQUATIONS**
The supercurrent \( (D.1) \) enters the r.h.s. of the vector superfield equation

\[
G_a = T_2 J_a \, ,
\]

which follows from the action \((3.14)\) of the supergravity—supermembrane interacting system.

The scalar superfield equation of the interacting system, which is obtained by varying the interacting action \((3.43)\) with respect to the real scalar prepotential of special minimal supergravity, \(\delta S/\delta V = 0\), reads

\[
R - \bar{R} = -iT_2 \mathcal{X}
\]

where

\[
\mathcal{X} = \frac{6i}{E} \int \frac{\hat{E}^a \wedge \hat{E}^\alpha \wedge \hat{\sigma}^{\alpha}_{\alpha \dot{\alpha}} \delta^8 (Z - \hat{Z})}{E} - \frac{3}{2} \int \frac{\hat{E}^b \wedge \hat{E}^a \wedge \hat{E}^\alpha}{E} \sigma_{abc} \delta_\beta \delta^8 (Z - \hat{Z}) + c.c + \]

\[
+ \int \frac{\hat{E}^b \wedge \hat{E}^c \wedge \hat{E}^d}{8E} \epsilon_{abcd} \sigma_{\dot{\alpha} \dot{\beta}} [\mathcal{D}_{\alpha}, \bar{\mathcal{D}}_{\dot{\alpha}}] \delta^8 (Z - \hat{Z}) + \]

\[
+ i \int \frac{\hat{E}_a \wedge \hat{E}^a}{4E} (\mathcal{D} \bar{R} - \bar{D} \mathcal{R}) \delta^8 (Z - \hat{Z}) + c.c. + \]

\[
+ \int \frac{1}{4E} \hat{E}^{\dot{a}} \wedge \hat{E}^c \wedge \hat{E}^d \epsilon_{ab} G^a \delta^8 (Z - \hat{Z}) \] \hspace{1cm} (D.5)

Notice that, as a consequence of \((3.5)\), the supermembrane current superfields obey

\[
\bar{\mathcal{D}}^{\dot{a}} J_{a \dot{\alpha}} = i \mathcal{D}_a \mathcal{X} \, , \quad \mathcal{D}^a J_{a \dot{\alpha}} = -i \bar{\mathcal{D}}_{\dot{\alpha}} \mathcal{X} \, .
\]

In the WZ manner \((3.49)-(3.53)\),

\[
i_{\theta} E^\alpha := \theta^\alpha E_{\dot{\alpha}} = 0 \, , \quad i_{\bar{\theta}} E_{\dot{\alpha}} := \theta_{\dot{\alpha}} E^\alpha = 0 \, ,
\]

\[
\theta^\alpha := \theta^3 \delta_\beta \delta^\alpha_{\beta} \, , \quad \bar{\theta}_{\dot{\alpha}} := \bar{\theta}^\dot{\alpha} \delta^3_{\beta} \delta_{\dot{\alpha}}^\beta \, ,
\]

\[
i_{\theta} u_{ab} := \theta^a \delta^\beta_{\beta} = 0 \, ,
\]

\[
i_{\bar{\theta}} u^{ab} := \delta_{ab} \bar{\theta} \delta^3_{\beta} \delta_{\dot{\alpha}}^\beta = 0 \, ,
\]

these current superfields simply drastically,

\[
J_{a \dot{\alpha}} |_{\theta = 0} = \frac{1}{8} (3 \mathcal{P}^b_a (x) \sigma^{\alpha}_{\alpha \dot{\alpha}} \bar{\sigma}^\beta_{b} - 2 \delta_{a \alpha} \delta_{\dot{\alpha}}^\beta \mathcal{P}^b_a (x)) - \frac{i}{32} \sigma^{\alpha}_{\alpha \dot{\alpha}} \mathcal{P}_a (x) + \propto \delta_{a \alpha} \delta_{\dot{\alpha}}^3
\]

\hspace{1cm} (D.11)
Appendix D. Supermembrane current superfields entering the superfield SUGRA equations

and

$$\mathcal{X}|_{\theta=0} = -\frac{\theta^a \bar{\theta}}{16} P^a + i \frac{(\theta \bar{\theta} - \bar{\theta} \theta)}{16} P^a(x) + \propto \theta^3,$$

(D.12)

where we use the current pre–potential fields defined in (3.62)

$$P^b(x) := \int_{\mathcal{W}^3} \frac{1}{\hat{e}} * \hat{e}_a \wedge \hat{e}_b \delta^4(x - \hat{x}),$$

(D.13)

$$P_a(x) := \int_{\mathcal{W}^3} \frac{1}{\hat{e}} \epsilon_{abcd} \hat{e}_b \wedge \hat{e}_c \wedge \hat{e}_d \delta^4(x - \hat{x}) =
\int_{\mathcal{W}^3} \epsilon^\mu_a(x) \int \epsilon_{\mu\nu\rho\sigma} d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma \delta^4(x - \hat{x}) 
$$

(D.14)

and $\theta^3$ denotes terms proportional to either $\theta \theta \bar{\theta}$ or $\theta \bar{\theta} \bar{\theta}$.
APPENDIX E

USEFUL FORMULAE INVOLVING 11-DIMENSIONAL AND 9–DIMENSIONAL GAMMA MATRICES

We use the mostly minus metric convention so that flat spacetime metric reads $\eta_{ab} = \text{diag}(1, -1, \ldots, -1)$. We choose the following $SO(1,1) \otimes SO(9)$ invariant representation for the 11–dimensional $32 \times 32$ gamma matrices and charge conjugation matrix,

\begin{align}
(\Gamma^a)_{\alpha\beta} &= \left( \frac{1}{2}(\Gamma^# + \Gamma^=), \Gamma^i, \frac{1}{2}(\Gamma^# - \Gamma^=) \right), \quad a = 0, 1, \ldots, 9, 10, \quad i = 1, \ldots, 9, \\
(\Gamma^#)_{\alpha\beta} &= \left( \begin{array}{cc} 0 & 2i\delta_{pq} \\
0 & 0 \end{array} \right), \quad (\Gamma^=)_{\alpha\beta} = \left( \begin{array}{cc} 0 & 0 \\
-2i\delta_{pq} & 0 \end{array} \right), \quad (\Gamma^i)_{\alpha\beta} = \left( \begin{array}{cc} -i\gamma^i_{pq} & 0 \\
0 & i\gamma^i_{pq} \end{array} \right),
\end{align}
(E.1)

which are imaginary due to our mostly minus metric convention. In these representation appear the $16 \times 16$ 9–dimensional Dirac matrices $\gamma^i_{pq}$, which possesses the following properties

\begin{align}
\gamma^{(i\gamma^j)} &= \delta_{ij}^{16 \times 16}, \\
\gamma^i_{pq} &= \gamma^i_{qp} := \gamma^i_{(pq)}, \\
\gamma^i_{(pq)}\gamma^i_{(rs)} &= \delta_{(pq)(rs)}.
\end{align}
(E.3)

We do not distinguish upper and lower $SO(9)$ spinor indices because the 9–dimensional charge conjugation matrix is symmetric allowing us to chose its representation by Kronecher delta symbol $\delta_{pq}$. The matrices $\delta_{pq}, \gamma^i_{pq}$ and $\gamma^i_{pq}$ provide a complete basis for the set of $16 \times 16$ symmetric matrices,

\begin{align}
\delta_{r(s)pq} = \frac{1}{16} \delta_{pq} \delta_{rs} + \frac{1}{16} \gamma^i_{pq} \gamma_{rs} + \frac{1}{16} \cdot 4i \gamma^i_{pq} \gamma^i_{rs}.
\end{align}
(E.4)
In our conventions $\gamma_{qp}^{123456789} = \delta_{qp}$ and, consequently,

$$\gamma_{qp}^{i_1...i_7} = -\frac{1}{2} \varepsilon^{i_1...i_7 j k} \gamma_{qp}^{jk},$$

(E.5)

$$\gamma_{qp}^{i_1...i_5} = \frac{1}{4!} \varepsilon^{i_1...i_5 j_1...j_4} \gamma_{qp}^{j_1...j_4}.$$  

(E.6)

This, together with (E.1) implies that our 11D dirac matrices obey

$$\Gamma^0 \Gamma^1 \ldots \Gamma^9 = \frac{1}{2} \Gamma^\# \Gamma^1 \ldots \Gamma^9 = -i \mathbb{I}_{32 \times 32}.$$ 

(E.7)
MOVING FRAME AND SPINOR MOVING FRAME VARIABLES

Moving frame and spinor moving frame variables are defined as blocks of, respectively, \( SO(1, 10) \) and \( Spin(1, 10) \) valued matrices,

\[
U^{(a)}_b = \left( \frac{u^a_+ + u^a_-}{2}, u^a_b, \frac{u^a_+ - u^a_-}{2} \right) \in SO(1, 10) \tag{F.1}
\]

\((i = 1, \ldots, 9)\) and

\[
V^{(\alpha)}_\beta = \left( v^+_{\alpha}, v^-_{\alpha} \right) \in Spin(1, 10) \tag{F.2}
\]

We also use

\[
V^{(\beta)}_{\alpha} = \left( v^+_{\alpha}, v^-_{\alpha} \right) \in Spin(1, 10) \tag{F.3}
\]

with

\[
v^-_{\alpha} = iC_{\alpha\beta} v^-_{\beta}, \quad v^+_{\alpha} = -iC_{\alpha\beta} v^+_{\beta} \tag{F.4}
\]

obeying

\[
V^{(\beta)}_\gamma V^{(\alpha)}_\beta = \delta^{(\beta)}_\gamma \delta^{(\alpha)}_\epsilon = \begin{pmatrix} \delta_{qp} & 0 \\ 0 & \delta_{qp} \end{pmatrix} \quad \iff \begin{cases} v^-_{\alpha} v^+_{\alpha} = \delta_{qp} = v^+_{\alpha} v^-_{\alpha} \\ v^-_{\alpha} v^-_{\alpha} = 0 = v^+_{\alpha} v^+_{\alpha} \end{cases} \tag{F.5}
\]

The algebraic properties of moving frame and spinor moving frame variables are summarized
as

\begin{align}
&u_a^a u^a = 0, \quad u_a^a u^{ai} = 0, \quad u_a^a u^{a\#} = 2, \quad (F.6) \\
&u_a^\# u^{a\#} = 0, \quad u_a^\# u^{ai} = 0, \quad (F.7) \\
&u_a^a u^{aj} = -\delta^{ij}. \quad (F.8)
\end{align}

\begin{align}
&v_q^{-} \Gamma_a v_p^{-} = u_a^a \delta_{qp}, \quad (F.9) \\
&v_q^{+} \Gamma_a v_p^{+} = u_a^\# \delta_{qp}, \\
&2v_q^{-} v_q^{-\beta} = \tilde{\Gamma}^{a\alpha \beta} u_a^a, \quad 2v_q^{+\alpha} v_q^{+\beta} = \tilde{\Gamma}^{a\alpha \beta} u_a^\#, \\
&2v_q^{- \alpha} v_q^{+\beta} = -\tilde{\Gamma}^{a\alpha \beta} u_a^i. \quad (F.10)
\end{align}

In (F.9) and (F.10) we have used real symmetric $16 \times 16$ Dirac matrices $\gamma_{qp}^i = \gamma_{pq}^i$ which obey Clifford algebra

\begin{equation}
\gamma^i \gamma^j + \gamma^j \gamma^i = 2 \delta^{ij} I_{16 \times 16}, \quad (F.11)
\end{equation}

and

\begin{align}
\gamma^i_{q(p_1 \cdots p_3)} &= \delta_{q(p_1} \delta_{p_2 p_3)}, \\
\gamma^{ij}_{q(q'\cdots p')} + \gamma^{ij}_{p(p'\cdots q')} &= \gamma^{ij}_{q(p'\cdots q')} \delta_{qp} - \delta_{q(p'\cdots q')} \gamma^{ij}. \quad (F.12)
\end{align}

Derivatives of the moving frame and spinor moving frame variables are expressed in terms of covariant $SO(1,10)$ Cartan forms

\begin{equation}
\Omega^=^i = u^=^a d\! u_a^i, \quad \Omega^\#^i = u^\#^a d\! u_a^i, \quad (F.14)
\end{equation}

and induced $SO(1,1) \times SO(9)$ connection

\begin{align}
\Omega^{(0)} &= \frac{1}{4} u^=^a d\! u_a^\#, \\
\Omega^{ij} &= u^i a d\! u_a^j. \quad (F.15) \quad (F.16)
\end{align}

It is convenient to use these latter to define covariant derivative. Then

\begin{align}
Du_b^= &= d\! u_b^= + 2\Omega^{(0)} u_b^= = u_b^i \Omega^=^i, \quad (F.17) \\
Du_b^\# &= d\! u_b^\# - 2\Omega^{(0)} u_b^\# = u_b^i \Omega^\#^i, \quad (F.18) \\
Du_b^i &= d\! u_b^i - \Omega^{ij} u_b^j = \frac{1}{2} u_b^\# \Omega^=^i + \frac{1}{2} u_b^i \Omega^\#^i. \quad (F.19)
\end{align}
Appendix F. Moving frame and spinor moving frame variables

\[ Dv_q^{-\alpha} := dv_q^{-\alpha} + \Omega^{(0)} v_q^{-\alpha} - \frac{1}{4} \Omega^{ij} \gamma_{ij} v_q^{-\alpha} = -\frac{1}{2} \Omega^{\#i} v_p^{-\alpha} \gamma_{pq}, \quad (F.20) \]

\[ Dv_q^{+\alpha} := dv_q^{+\alpha} - \Omega^{(0)} v_q^{+\alpha} - \frac{1}{4} \Omega^{ij} \gamma_{ij} v_q^{+\alpha} = -\frac{1}{2} \Omega^{\#i} v_p^{+\alpha} \gamma_{pq}. \quad (F.21) \]

The Cartan forms obey

\[ D\Omega^{-i} = 0, \quad D\Omega^{\#i} = 0, \quad (F.22) \]

\[ F^{(0)} := d\Omega^{(0)} = \frac{1}{4} \Omega^{-i} \wedge \Omega^{\#i}, \quad (F.23) \]

\[ G^{ij} := d\Omega^{ij} + \Omega^{ik} \wedge \Omega^{kj} = -\Omega^{-[i} \wedge \Omega^{\#j]}. \quad (F.24) \]

Notice that, e.g.

\[ DDu^\#_a = 2 F^{(0)} u^\#_a, \quad DDu^i = u^j G^{ji}. \quad (F.25) \]

The essential variations of moving frame and spinor moving frame variables can be written as

\[ \delta u^\#_b = u^\#_b i_\delta \Omega^{\#i}, \quad \delta u^i = u^i i_\delta \Omega^{-i}, \quad (F.26) \]

\[ \delta u^\#_b = \frac{1}{2} u^\#_b i_\delta \Omega^{\#i} + \frac{1}{2} u^i = i_\delta \Omega^{\#i}, \quad (F.27) \]

\[ \delta v^{-\alpha}_q = -\frac{1}{2} i_\delta \Omega^{-i} v_p^{+\alpha} \gamma_{pq}, \quad (F.28) \]

\[ \delta v^{+\alpha}_q = -\frac{1}{2} i_\delta \Omega^{\#i} v_p^{-\alpha} \gamma_{pq}. \quad (F.29) \]

where \( i_\delta \Omega^{\#i} \) and \( i_\delta \Omega^{-i} \) are independent variations.

The essential variations of the Cartan forms read

\[ \delta \Omega^{\#i} = D_i \delta \Omega^{\#i}, \quad \delta \Omega^{-i} = D_i \delta \Omega^{-i}, \quad (F.30) \]

\[ \delta \Omega^{ij} = -\Omega^{-[i} i_\delta \Omega^{\#j]} - \Omega^{\#[i} i_\delta \Omega^{-j]}, \quad (F.31) \]

\[ \delta \Omega^{(0)} = \frac{1}{4} \Omega^{-i} i_\delta \Omega^{\#i} - \frac{1}{4} \Omega^{\#i} i_\delta \Omega^{-i}. \quad (F.32) \]
APPENDIX G

EQUATIONS OF MOTION FOR A SINGLE M0–BRANE

In this appendix we collect the equations of motion for the single M0–brane obtained in chapter 5 from the spinor moving frame action (5.1), (5.2). They read

\begin{align}
\dot{E}^a &:= \dot{\dot{E}}^a u_a = 0, \quad (G.1) \\
\dot{E}^i &:= \dot{\dot{E}}^a u^i_a = 0, \quad (G.2) \\
D\rho^\# &= 0 \iff \Omega^{(0)} = \frac{d\rho^\#}{2\rho^\#}, \quad (G.3) \\
\Omega^{\alpha i} &= 0 \iff D u_\alpha^a = 0 \iff D v_q^{-\alpha} = 0, \quad (G.4) \\
\dot{E}^{-q} &:= \dot{\dot{E}}^a v^{-q}_a = 0. \quad (G.5)
\end{align}

These equations are formulated in terms of pull–backs of bosonic and fermionic supervielbein forms of flat 11D superspace to the mM0 worldline $W^1$

\begin{align}
\dot{E}^a &= d\dot{x}^a - id\dot{\theta}\Gamma^a\dot{\theta}, \quad a = 0, 1, ..., 10, \quad (G.6) \\
E^a &= d\dot{\theta}^\alpha, \quad \alpha = 1, ..., 32, \quad (G.7)
\end{align}

which are constructed from the coordinate functions $\dot{x}^a(\tau), \dot{\theta}^\alpha(\tau)$ of the proper time $\tau$, and of the moving frame and spinor moving frame variables $u_\alpha^a, u_\beta^i, v_q^{-\alpha}$. The properties of these latter as well as of the Cartan forms $\Omega^\alpha = i, \Omega^{(0)}$ and covariant derivatives $D$ are collected in Appendix F.

In (G.6) and in the main text we have used the real symmetric $32 \times 32$ 11D $\Gamma$–matrices $\Gamma_{\alpha\beta}^a = (\gamma^a C)_{\alpha\beta}$ which, together with $\Gamma^{\alpha\beta} = (C\gamma_5)^{\alpha\beta}$, obey $\Gamma^{(a}\Gamma^{b)} = \eta^{ab} I_{32 \times 32}$.
The mM0 system, which is to say an interacting system of N nearly coincident M0-branes, is described in terms of center of energy variables and the traceless $N \times N$ matrices $X^i$ ($i = 1, \ldots, 9$), $\Psi_q$ ($q = 1, \ldots, 16$). Our action includes also the auxiliary $N \times N$ matrix fields: momentum $P^i$ and the 1d SU(N) gauge field $A_\tau$ ($\dot{A}_\tau = d\tau A_\tau$).

The complete list of equations of motion for the mM0 system splits naturally on the equations for the relative motion variables,

$$D_X^i = \hat{E}^\# P^i + 4i \hat{E}^{+q}(\gamma^i \Psi)_q, \quad [P^i, X^i] = 4i\{\Psi_q, \Psi_q\},$$

$$D_{\hat{E}} = -\frac{1}{16} \hat{E}^\# [[X^i, X^j]X^j] + 2\hat{E}^\# \Psi^i \Psi + \hat{E}^{+q} \gamma_{ij} \Psi_p [\Psi_p, X^j],$$

$$D_{\hat{E}} = \frac{i}{4} \hat{E}^\# [X^i, (\gamma^i \Psi)] + \frac{1}{2} \hat{E}^{+q} P^i - \frac{i}{16} \hat{E}^{+q} [X^i, X^j],$$

and the center of energy equations which can be considered as a deformation of the system of equations for single M0 brane. After fixing the gauge under a reminiscent of the $K_9$ symmetry, these equation read

$$\hat{E}^\# := \hat{E}^\alpha u^\alpha = 3(\rho^\#)^2 tr \left( \frac{1}{2} P^i D_X^i + \frac{1}{64} \hat{E}^\# [X^i, X^j]^2 - \frac{1}{4} (\hat{E}^{+q} \gamma^i \Psi) [X^i, X^j] \right),$$

$$\hat{E}^\# := \hat{E}^\alpha u^\alpha = 0, \quad \hat{E}^{-q} := \hat{E}^\alpha v^{-\alpha} = 0,$$

$$\Omega^{\#} = 0 \quad \Omega^i = 0$$

$$\leftrightarrow \begin{cases} D u^\alpha = 0, & D u^\# = 0, \\ D v^{-\alpha} = 0, & D v^{+\alpha} = 0. \end{cases}$$

$$D \rho^\# = 0 \quad \leftrightarrow \quad \Omega^{(0)} = \frac{d\rho^\#}{2\rho^\#}.$$
As a consequence of the above equation the effective mass $M$ of the $mM_0$ center of energy motion,

$$M^2 = 4(\rho^#)^4\mathcal{H},$$  \hspace{1cm} (H.7)

is a constant

$$dM^2 = 0.$$  \hspace{1cm} (H.8)

Eq. (H.7) expresses $M^2$ in terms of Lagrange multiplier $\rho^#$ and the relative motion Hamiltonian (5.74)

$$\mathcal{H} = \frac{1}{2} tr (\tilde{P}^i \tilde{P}^i) - \frac{1}{64} tr [\tilde{X}^i, \tilde{X}^j]^2 - 2 tr (\tilde{X}^i \gamma^i \tilde{\Psi}).$$  \hspace{1cm} (H.9)

If we fix the gauge where the composed SO(9) connection and also the SU(N) gauge field vanish,

$$\Omega_{ij} = d\tau \Omega_{ij} = 0, \quad A = d\tau A = 0,$$  \hspace{1cm} (H.10)

the equations of relative motion and Eq. (H.2) simplify to

$$\partial_\tau \tilde{\Psi} = \frac{i}{4} e [\tilde{\Psi}, (\gamma^i \tilde{\Psi})] + \frac{1}{2\sqrt{\rho^#}} \hat{E}_r^+ \gamma^i \tilde{\Psi}^i - \frac{i}{16\sqrt{\rho^#}} \hat{E}_r^+ \gamma^{ij} [\tilde{X}^i, \tilde{X}^j],$$

$$\partial_\tau \left( \frac{1}{e} \partial_\tau \tilde{X}^i \right) = -\frac{e}{16} [\tilde{X}^i, \tilde{X}^j] [\tilde{X}^j, \tilde{X}^j] + 2 e \tilde{\Psi} \gamma^i \tilde{\Psi} + 4i \partial_\tau \left( \frac{\hat{E}_r^+ \gamma^i \tilde{\Psi}}{e \sqrt{\rho^#}} \right) + \frac{1}{\sqrt{\rho^#}} \hat{E}_r^+ \gamma^{ij} \tilde{\Psi},$$

$$\partial_\tau \tilde{X}^i = e \tilde{\Psi}^i + \frac{4i}{\sqrt{\rho^#}} \left( \hat{E}_r^+ \gamma^i \tilde{\Psi} \right), \quad [\tilde{P}^i, \tilde{X}^i] = 4i \{\tilde{\Psi}_q, \tilde{\Psi}_q\},$$

$$\rho^# \hat{E}_r^+ = 3 tr \left( \frac{1}{2} \tilde{P}^i \partial_\tau \tilde{X}^i + \frac{1}{64} e [\tilde{X}^i, \tilde{X}^j]^2 - \frac{1}{4\sqrt{\rho^#}} \left( \hat{E}_r^+ \gamma^{ij} \tilde{\Psi} \right) [\tilde{X}^i, \tilde{X}^j] \right).$$  \hspace{1cm} (H.11)

These equations are written in terms of redefined fields,

$$\tilde{X}^i = \rho^# X^i, \quad \tilde{\Psi}_q = (\rho^#)^{3/2} \Psi_q,$$

$$\tilde{P}^i = (\rho^#)^2 P^i = \frac{1}{e} \left( \partial_\tau \tilde{X}^i - \frac{4i}{\sqrt{\rho^#}} \hat{E}_r^+ \gamma^i \tilde{\Psi} \right),$$  \hspace{1cm} (H.12)

and

$$e(\tau) = \hat{E}_r^# / \rho^#.$$  \hspace{1cm} (H.13)
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