Polya’s inequalities, global uniform integrability and the size of plurisubharmonic lemniscates

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Abstract First we prove a new inequality comparing uniformly the relative volume of a Borel subset with respect to any given complex euclidean ball $B \subset \mathbb{C}^n$ with its relative logarithmic capacity in $\mathbb{C}^n$ with respect to the same ball $B$. An analogous comparison inequality for Borel subsets of euclidean balls of any generic real subspace of $\mathbb{C}^n$ is also proved.

Then we give several interesting applications of these inequalities. First we obtain sharp uniform estimates on the relative size of plurisubharmonic lemniscates associated to the Lelong class of plurisubharmonic functions of logarithmic singularities at infinity on $\mathbb{C}^n$ as well as the Cegrell class of plurisubharmonic functions of bounded Monge-Ampère mass on a hyper-convex domain $\Omega \subset \mathbb{C}^n$.

Then we also deduce new results on the global behaviour of both the Lelong class and the Cegrell class of plurisubharmonic functions.

1 Introduction

Local uniform integrability and estimates on the size of sublevel sets of plurisubharmonic functions in terms of capacities or various measures have been studied earlier in several works (cf. [Cu-Dr-Lu], [Ki], [Ko 2], [Ze 2], [Ze 3], [Pl], [Be-Je]). Such estimates turn out to be useful in many areas of Complex Analysis as Pluripotential Theory, Padé Approximation and Complex Dynamics (cf. [Ki], [Ko 1], [Ko 2], [Cu-Dr-Lu], [Fa-Gu]).

Our aim here is to generalize the classical Polya’s inequality to subsets of any generic subspace of $\mathbb{C}^n$ and to give several new applications to the study of the global behaviour of two important classes of plurisubharmonic functions.

More precisely, given a generic subspace $\mathcal{G} \subset \mathbb{C}^n$, we prove a new inequality estimating from above the relative volume in $\mathcal{G}$ of a Borel subset $B \subset \mathcal{G}$ in terms of its relative logarithmic capacity in $\mathbb{C}^n$ with respect to the same ball $B$, up to a multiplicative numerical constant which depends only on the dimension of $\mathcal{G}$ but not on the "condenser" considered.

*This work was partially supported by the programmes PARS MI 07 and ALMA 180
Formulated in this way, Polya’s inequalities turn out to play an important role in applications, implying interesting results which improve significantly earlier results obtained by several authors (cf. [Cu-Dr-Lu], [Ko 2], [Ze 1], [Ze 2]).

Indeed, first we easily deduce new estimates on the relative volume with respect to balls in a generic subspace of $\mathbb{C}^n$ of the plurisubharmonic lemniscates associated to the Lelong class of plurisubharmonic functions with logarithmic singularities at infinity on $\mathbb{C}^n$ as well as the Cegrell class of plurisubharmonic functions with bounded Monge-Ampère mass on a bounded hyperconvex domain of $\mathbb{C}^n$.

Then we give estimates on global uniform integrability of the Lelong class of plurisubharmonic functions with logarithmic singularities at infinity on $\mathbb{C}^n$ with respect to the Lebesgue measure on any generic subspace. These estimates can be considered as precise quantitative versions for the Lelong class of the well known John-Nirenberg inequalities for BMO—functions on $\mathbb{R}^n$ (cf.[St]).

In particular we prove that restrictions to any generic subspace $G \subset \mathbb{C}^n$ of plurisubharmonic functions with logarithmic singularities at infinity on $\mathbb{C}^n$ are in $\text{BMO}(G)$ with a uniform explicit bound on their $\text{BMO}(G)$—norms depending only on the dimension of $G$.

Finally we give a general sufficient condition for uniform integrability of a given class of plurisubharmonic functions on some domain in terms of the behaviour of the relative Monge-Ampère capacity of their sublevel sets with respect to this domain. In particular, we deduce a new global uniform integrability result for the Cegrell class of plurisubharmonic functions of uniformly bounded Monge-Ampère masses on a bounded hyperconvex domain.

2 Preliminaries

Let us recall the classical Polya’s inequality (cf. [Ra], [Ts]). For any compact subset $K \subset \mathbb{C}$,

\begin{equation}
\lambda_2(K) \leq \pi \cdot c(K)^2,
\end{equation}

with equality for a disc, where $\lambda_2$ is the area measure on $\mathbb{C} = \mathbb{R}^2$ and $c(K)$ is the logarithmic capacity of $K$.

Besides this inequality, there is a corresponding inequality for sets of the real line $\mathbb{R} \subset \mathbb{C}$. Namely, for any compact subset $K \subset \mathbb{R}$,

\begin{equation}
\lambda_1(K) \leq 4 \cdot c(K),
\end{equation}

with equality for an interval, where $\lambda_1$ is the length measure on $\mathbb{R}$.
Recall that the logarithmic capacity \( c(K) \) of a compact subset \( K \subset \mathbb{C} \) coincides with its Chebychev constant (cf. [Ra], [Ts]), so that the following formula holds
\[
c(K) = \inf_{d \geq 1} \left( \inf \{ ||P||_{K}^{1/d} : P \in \hat{P}_d \} \right),
\]
where \( \hat{P}_d \) is the set of monic polynomials of degree \( d \) and \( ||P||_{K} := \sup_{z \in K} |P(z)| \).

In \( \mathbb{C}^n \), it is more convenient to consider the following Chebyshev constant associated to a compact subset \( K \subset \mathbb{C}^n \) (cf. [Al-Ta], [Si 2])
\[
T_B(K) := \inf_{d \geq 1} \left( \inf \{ ||P||_{K}^{1/d} : P \in \mathbb{C}[z], \deg(P) = d, ||P||_{B} = 1 \} \right),
\]
where \( B \) is any regular compact subset of \( \mathbb{C}^n \) and \( ||P||_{B} := \max_{z \in B} |P(z)| \).

If \( n = 1 \), it is easy to prove that the two constants \( c \) and \( T_B \) are equivalent as we shall see below.

The constant defined by (2.3) is related to the pluricomplex Green function with logarithmic singularities at infinity on \( \mathbb{C}^n \), which we will recall below. Its definition is based on the usual Lelong class of plurisubharmonic functions of logarithmic growth at infinity on \( \mathbb{C}^n \) defined as follows
\[
L(\mathbb{C}^n) := \{ u \in PSH(\mathbb{C}^n) ; \sup\{u(z) - \log^+ |z| ; z \in \mathbb{C}^n\} < +\infty \}.
\]
The global extremal function with logarithmic growth at infinity associated to a Borel subset \( K \subset \mathbb{C}^n \) is defined by
\[
V_K(z) := \sup\{u(z) ; u \in L(\mathbb{C}^n), \ u|K \leq 0 \}, z \in \mathbb{C}^n
\]
and its upper semi-continuous regularization \( V^*_K \) in \( \mathbb{C}^n \) is the pluricomplex Green function with logarithmic singularities at infinity associated to \( K \) (see [Za], [Si 1]).

It is well known that \( V_K \) is locally bounded on \( \mathbb{C}^n \) if and only if \( K \) is non pluripolar in \( \mathbb{C}^n \) (see [Si 1], [Si 2]).

By a theorem of Siciak ([Si 2]), we know that if \( K \subset \mathbb{C}^n \) is a compact set, then
\[
T_B(K) = \exp\left( -\max_{\mathbb{C}^n} V^*_K \right)
\]
The formula (2.6) allows us to extend the definition of the set function \( T_B(.) \) to Borel subsets of \( \mathbb{C}^n \). Moreover the extended set function is a generalized Choquet capacity on any bounded domain in \( \mathbb{C}^n \), which is inner regular and outer regular (see [Si 2]). The constant \( T_B(K) \) will be called here the relative logarithmic capacity of \( K \) with respect to \( B \) in \( \mathbb{C}^n \).

It is also well known that the null sets for this capacity are precisely the pluripolar subsets of \( \mathbb{C}^n \) (see [Si 2]).

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Thus if $K \subset \mathbb{C}^n$ is non pluripolar then $- \log T_B(K) = \max_B V_K^*(< +\infty)$ is the best constant for which the following Bernstein-Wals h inequality holds

$$(2.7) \quad \sup_B u \leq \sup_K u - \log T_B(K), \quad \forall u \in \mathcal{L}(\mathbb{C}^n).$$

There is another relative capacity defined using the Monge-Ampère operator (see [Be-Ta 1]). Here we choose a normalisation of the usual differential operators on $\mathbb{C}^n$ so that

$$dd^c := \frac{i}{\pi} \partial \overline{\partial}. $$

Let $\Omega \Subset \mathbb{C}^n$ be an open set and $K \subset \Omega$ a compact subset. Then the relative Monge-Ampère capacity of the condenser $(K, \Omega)$ is defined by the formula (see [Be-Ta 1])

$$(2.8) \quad \text{cap}(K; \Omega) := \sup \{ \int_K (dd^c u)^n; u \in PSH(\Omega), -1 \leq u \leq 0 \}.$$

This capacity is related to the so called plurisubharmonic measure associated to the condenser $(K, \Omega)$ defined by

$$(2.9) \quad h_K(z) := \sup \{ u(z); u \in PSH(\Omega), u \leq 0, u|_K \leq -1 \}, z \in \Omega.$$

Then if $\Omega \Subset \mathbb{C}^n$ is a hyperconvex open set and $K \subset \Omega$ is a compact subset, it follows from ([Be-Ta 1]) that

$$(2.10) \quad \text{cap}(K; \Omega) = \int_K (dd^c h_K^n) = \int_\Omega (dd^c h_K^*)^n.$$

We will need the following Alexander and Taylor’s comparison inequality (see [Al-Ta]). For a fixed bounded domain $\Omega \Subset \mathbb{C}^n$ and a fixed euclidean ball $\mathbb{B} \subset \mathbb{C}^n$ such that $\Omega \subset \mathbb{B}$,

$$(2.11) \quad T_\mathbb{B}(E) \leq \exp(-\text{cap}(E; \Omega)^{-1/n}).$$

for any Borel subset $E \subset \Omega$.

We will also need to define the Cegrell class of plurisubharmonic functions. Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex open set. Denote by $\mathcal{F}(\Omega)$ the class of negative plurisubharmonic functions $\varphi$ on $\Omega$ such that there exists a decreasing sequence $(\varphi_j)$ of bounded plurisubharmonic functions on $\Omega$ with boundary values 0 which converges to $\varphi$ on $\Omega$ and satisfies $\sup_j \int_\Omega (dd^c \varphi_j)^n < +\infty$.

By Cegrell ([Ce 2]), for $\varphi \in \mathcal{F}(\Omega)$, the Monge-Ampère measure $(dd^c \varphi)^n$ is a well defined Borel measure of finite mass on $\Omega$ as the weak limit of the sequence of measures $(dd^c \varphi_j)^n$, where $(\varphi_j)$ is any decreasing sequence converging to $\varphi$ on $\Omega$ and satisfying all the requirements of the definition.
3 Relative Polya’s inequalities

Here we want to compare the relative Lebesgue measure on a generic subspace $G \subset \mathbb{C}^n$ with respect to a real euclidean ball in $G$ with the relative logarithmic capacity in $\mathbb{C}^n$ with respect to the same ball.

First recall some definitions. A real subspace $G \subset \mathbb{C}^n$ is said to be a generic subspace if $G + JG = \mathbb{C}^n$, where $J$ is the complex structure on $\mathbb{C}^n$. We denote by $G^c := G \cap JG$ the maximal complex subspace of $\mathbb{C}^n$ contained in $G$ and set $m := \dim C G^c$, which will be called the complex dimension of $G$. Then it is clear that $\dim R G = n + m$.

If $m = 0$ which means that $G^c = (0)$, the subspace $G$ is said to be totally real. If $m = n$ then $G = \mathbb{C}^n$.

It is easy to see that $G \subset \mathbb{C}^n$ is a generic subspace of complex dimension $m$ if and only if there is a unitary automorphism $U : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $U(G) = \mathbb{C}^m \times \mathbb{R}^{n-m} \subset \mathbb{C}^m \times \mathbb{C}^{n-m} = \mathbb{C}^n$.

Observe that the subspace $G \subset \mathbb{C}^n$ is non pluripolar in $\mathbb{C}^n$ precisely when $G$ is a generic subspace.

The subspace $G \subset \mathbb{C}^n$ will be endowed with the induced euclidean structure and the corresponding Lebesgue measure which will be denoted by $\lambda_{n+m}$.

Now we can state our version of Polya’s inequality which is the main result of this section.

**Theorem 3.1** 1) For any complex euclidean closed ball $B \subset \mathbb{C}^n$ and any Borel subset $K \subset B$,

\[
\frac{\lambda_{2n}(K)}{\lambda_{2n}(B)} \leq c_n \ T_B(K)^2.
\]

where

\[
c_n := \frac{4^n (n!)^2}{(2n-1)!}.
\]

2) Let $G \subset \mathbb{C}^n$ be a generic real subspace of complex dimension $0 \leq m \leq n-1$. Then for any real euclidean closed ball $B \subset G$ and any Borel subset $K \subset B$,

\[
\frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)} \leq 8(n + m) \ T_B(K).
\]

For the proof of relative Polya’s inequalities, we start to look to the simplest case where $n = 1$.

**Lemma 3.2** 1) For any closed disc $D \subset \mathbb{C}$ and any Borel subset $K \subset D$,

\[
\frac{\lambda_2(K)}{\lambda_2(D)} \leq 4 \ T_D(K)^2.
\]
2) For any real closed interval $I \subset \mathbb{R}$ and any Borel subset $K \subset I$

\begin{equation}
\frac{\lambda_1(K)}{\lambda_1(I)} \leq 4 T_1(K).
\end{equation}

We don’t know if 4 is the best constant in these inequalities.

Proof: 1) By regularity of the Lebesgue measure and the relative logarithmic capacity in $\mathbb{C}$, we can assume that $K$ is a non polar compact subset. We can also assume that $\mathbb{C} \setminus K$ is connected since $\lambda_2(K) \leq \lambda_2(\hat{K})$ and $T_D(K) = T_D(\hat{K})$. Then the extremal function $V_K^*$ is a subharmonic function on $\mathbb{C}$ which coincides with the Green function of $\mathbb{C} \setminus K$ with a pole at infinity. Therefore it can be represented by the formula

$$V_K^*(z) = \int_K \log |z - \zeta| d\mu(\zeta) - \log c(K), \quad \forall z \in \mathbb{C},$$

where $\mu := (1/2\pi)\Delta V_K^*$ is the normalized equilibrium measure of $K$. From this representation formula, we get the estimate

$$\max_{\mathbb{D}} V_K^* \leq \log(2R) - \log c(K),$$

where $R$ is the radius of the disc $\mathbb{D} \subset \mathbb{C}$. This inequality implies that

\begin{equation}
\lambda_2(K) \leq 4 \lambda_2(\mathbb{D}) T_D(K)^2,
\end{equation}

which is the required estimate.

2) In the real case we prove in the same way that

$$c(K) \leq 2R T_1(K),$$

where $R$ is the radius of the interval $I$. Therefore using the inequality (2.2), we get

$$\lambda_1(K) \leq 4 \lambda_1(I) T_1(K),$$

which is the required inequality. ▶

To prove our theorem, we need the following elementary slicing lemma.

**Lemma 3.3** 1) Let $B \subset \mathbb{C}^n$ be any complex euclidean closed ball, $K \subset B$ a Lebesgue measurable subset and $a \in \partial B$. Then there exists a complex line $L_a \subset \mathbb{C}^n$ passing through the point $a$ such that $\lambda_2(B \cap L_a) > 0$ and

\begin{equation}
\frac{\lambda_2(K)}{\lambda_2(B)} \leq c_n \frac{\lambda_2(K \cap L_a)}{\lambda_2(B \cap L_a)}.
\end{equation}
where \( c'_n = c_n/A = 4^{n-1}(n!)^2 \).

2) Let \( B \subset \mathbb{R}^N \) any euclidean ball, \( K \subset B \) any Lebesgue measurable subset and \( a \in B \). Then there exists a real line \( l_a \subset \mathbb{R}^N \) passing through the point \( a \) such that \( \lambda_1(B \cap l_a) > 0 \) and

\[
\frac{\lambda_N(K)}{\lambda_N(B)} \leq 2N \frac{\lambda_1(K \cap l_a)}{\lambda_1(B \cap l_a)}.
\]

Observe that \( c_n \sim 2\sqrt{\pi n^{3/2}} \) as \( n \to \infty \), we conjecture that the inequality (3.7) is true with the constant \( c_n = n \). The inequality (3.8) could be deduced from ([BG], lemma 3) with the constant \( N \) but the proof given there is not clear for us. So we decided to give proof which uses the same idea of symmetrization but leads to the constant \( 2N \) instead of \( N \), unless the point \( a \) in the lemma coincides with the center of the ball \( B \).

Proof: 1) We can of course assume that \( n \geq 2 \). Since our inequality is invariant under translation, we can also assume that \( a = 0 \in \partial B \) is the origin and \( \lambda_2(K) > 0 \).

Now assume by contradiction that the inequality (3.7) is not true. Then we will have

\[
\lambda_2(K \cap L) < \frac{\lambda_2(K)}{c'_n \lambda_2(B)} \lambda_2(B \cap L),
\]

for any complex line \( L \) passing through the origin \( a = 0 \) such that \( \lambda_2(B \cap L) > 0 \).

Since relative volume and relative area are invariant under non singular affine transformations, we can assume that \( B = \{ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n ; |z_1 - R|^2 + |z_2|^2 + \cdots + |z_n|^2 < R^2 \} \) and \( L_w = \{ \zeta, w ; \zeta \in \mathbb{C} \} \) where \( w = (w_1, \ldots, w_n) \in S^{2n-1} \). Then \( L_w \cap B = \{ \zeta \cdot w ; |\zeta|^2 < 2R, \Re(\zeta w_1) \} \) is the disc centred at \( Rw_1 \) of radius \( R \cdot |w_1| \) which by the last inequality leads to

\[
\lambda_2(K \cap L_w) < \frac{\lambda_2(K)}{c'_n \lambda_2(B)} \pi R^2 |w_1|^2, \forall w \in S^{2n-1},
\]

Now, integrating in polar coordinates and using the invariance of the sphere \( S^{2n-1} \) by rotation, we obtain the formula

\[
\lambda_2(K) = \frac{1}{2\pi} \int_{S^{2n-1}} \int_{|\zeta| < R} |\zeta|^{2n-2} \chi_K(\zeta \cdot w) d\lambda_2(\zeta) d\sigma_{2n-1}(w)
\]

\[
\leq \frac{2^{2n-2}R^{2n-2}}{2\pi} \int_{S^{2n-1}} |w_1|^{2n-2} \int_{|\zeta| \leq R|w_1|} \chi_K(\zeta \cdot w) d\lambda_2(\zeta) d\sigma_{2n-1}(w),
\]

where \( \chi_K \) is the characteristic function of the set \( K \).

Using inequality (3.10), we deduce from the last inequality that

\[
\lambda_2(K) < 2^{2n-2}R^{2n} \frac{\lambda_2(K)}{2c'_n \lambda_2(B)} \int_{S^{2n-1}} |w_1|^{2n} d\sigma_{2n-1}(w)
\]
Now, an elementary computation using spherical coordinates leads to the formula

\[(3.12) \quad \int_{S^{2n-1}} |w_1|^2 d\sigma_{2n-1}(w) = \frac{4n(n!)^2}{(2n)!} \tau_{2n}\]

where \(\tau_{2n}\) is the volume of the euclidean unit ball in \(\mathbb{R}^{2n}\).

The last formula \((3.12)\) combined with \((3.11)\) leads finally to the inequality

\[\lambda_{2n}(K) < \frac{2^{2n-2} 2^{2n} \lambda_{2n}(K)}{2 c_n \lambda_{2n}(B)} \frac{(n!)^2}{(2n)!} \tau_{2n} = \lambda_{2n}(K), \]

which yields a contradiction.

2) As in the complex case, we assume that \(a = 0\) is the origin, \(\lambda_N(K) > 0\) and the ball \(B\) of radius 1.

First, observe that \(\lambda_1(B \cap l_a) \leq 2\) for any real line \(l_a\) passing through the point \(a\), then to show \((3.8)\) it is enough to prove that

\[\frac{1}{N} \frac{\lambda_N(K)}{\lambda_N(B)} \leq \lambda_1(K \cap l_a)\]

for some real line \(l_a\).

Assume by contradiction that the last inequality is not true. Then we will have

\[(3.13) \quad \lambda_1(K \cap l) < \frac{1}{N} \frac{\lambda_N(K)}{\lambda_N(B)}\]

for any real line \(l\) passing through the origin \(a = 0\).

Let \(\tilde{K}\) be the annulus with the same center \(x_0\) as \(B\) and of radii \(r\) and 1 \((r < 1)\) such that \(\lambda_N(\tilde{K}) = \lambda_N(K)\)

then

\[r = \left(1 - \frac{\lambda_N(K)}{\lambda_N(B)}\right)^{1/N}.\]

Denote by \(e(\tilde{K}) := 1 - r\) the depth of the annulus \(\tilde{K}\), then

\[e(\tilde{K}) = 1 - r\]

\[= 1 - \left(1 - \frac{\lambda_N(K)}{\lambda_N(B)}\right)^{1/N}\]

\[\geq \frac{1}{N} \frac{\lambda_N(K)}{\lambda_N(B)}.\]

The last inequality together with \((3.13)\) lead to

\[(3.14) \quad e(\tilde{K}) > \lambda_1(K \cap l)\]

for any real line \(l\) passing through \(a\).

Now, observe that, if \(l\) any real line passing through the origin such that
\( l \cap B(x_0, r) \neq \emptyset \), then \( \lambda_1((K \cap l)) \geq 2e(K) \) and hence from (3.14) we derive the inequality

\[
(3.15) \quad \lambda_1(\tilde{K} \cap l) > 2\lambda_1(K \cap l),
\]

for all real line passing through the origin \( a = 0 \).

Now, following ([Br-Ga]), we construct a set \( K^{(s)} \) in the following way: On each real line \( l \) passing through the point \( a = 0 \), we choose the best far segment of \( \tilde{K} \cap l \) of length \( \lambda_1(K \cap l) \).

Then from the inequality \( (3.15) \) we get \( K^{(s)} \subset \tilde{K} \) and therefore

\[
(3.16) \quad \lambda_N(K^{(s)}) < \lambda_N(\tilde{K}).
\]

On the other hand, by the construction of the set \( K^{(s)} \), if \( \tau \in K \cap l \setminus K^{(s)} \) and \( t \in (K^{(s)} \cap l) \setminus K \) then \( |\tau| \leq |t| \) and since \( \lambda_1(K \cap l) = \lambda_1(K^{(s)} \cap l) \) then

\[
\int_{K \cap l} |\tau|^{N-1}d\tau \leq \int_{K^{(s)} \cap l} |t|^{N-1}dt.
\]

Now, integrating in polar coordinates and using the last inequality, we obtain

\[
\lambda_N(K) = \frac{1}{2} \int_{S^{N-1}} \int_{\mathbb{R}} |\tau|^{N-1} \chi_K(\tau \cdot w) d\tau d\sigma_{N-1}(w)
\]

\[
= \frac{1}{2} \int_{S^{N-1}} \int_{K \cap l_w} |\tau|^{N-1} d\tau d\sigma_{N-1}(w)
\]

\[
\leq \frac{1}{2} \int_{S^{N-1}} \int_{K^{(s)} \cap l_w} |t|^{N-1} dt d\sigma_{N-1}(w)
\]

\[
\leq \frac{1}{2} \int_{S^{N-1}} \int_{\mathbb{R}} |t|^{N-1} \chi_{K^{(s)}}(t \cdot w) dt d\sigma_{N-1}(w)
\]

\[
\leq \lambda_N(K^{(s)})
\]

where \( \chi_K \) is the characteristic function of the set \( K \) and \( l_w = \{t \cdot w : t \in \mathbb{R}\} \), which contradicts the inequality \( (3.16) \).

Now we are ready for the proof of the Theorem.

**Proof of the theorem:** 1) By interior and exterior regularity of the Lebesgue measure and the relative logarithmic capacity, we can assume that \( K \subset \overline{B} \) is a compact set of non empty interior in \( \mathbb{C}^n \) so that \( \lambda_2(K) > 0 \) and \( T_\overline{B}(K) > 0 \). Moreover, considering \( \varepsilon \)-neighbourhoods of \( K \) in \( \mathbb{C}^n \), we can assume that \( K \) is regular in the sense that \( V_K \) is continuous on \( \mathbb{C}^n \). Therefore \( V_K \in \mathcal{L}(\mathbb{C}^n) \) and there exists \( a \in \partial \overline{B} \) such that \( V_K(a) = \sup_{\overline{B}} V_K \). By translation, we can assume that \( a = 0 \) is the origin in \( \mathbb{C}^n \).

By the complex slicing lemma, there exists a complex line \( L \subset \mathbb{C}^n \) passing through the point \( a \) such that \( \lambda_2(K \cap L) > 0 \) and

\[
(3.17) \quad \frac{\lambda_{2n}(K)}{\lambda_{2n}(\overline{B})} \leq c_n \frac{\lambda_2(K \cap L)}{\lambda_2(\overline{B} \cap L)}.
\]
Since $a \in L$ and $V_K(a) = \max_\mathbb{B} V_K$, it follows that $T_{\mathbb{B} \cap L}(K \cap L) \leq T_{\mathbb{B}}(K)$ and then from (3.17) and (3.19) we deduce that

$$\lambda_{2n}(K) \over \lambda_{2n}(\mathbb{B}) \leq 4c'_n T_{\mathbb{B}}(K)^2,$$

which is exactly the required inequality (3.1).

2) Now assume that $G \neq \mathbb{C}^n$ is a generic subspace of complex dimension $1 \leq m \leq n - 1$ (the totally real case $m = 0$ is treated in the same way).

By the invariance of the Lebesgue measure and the relative capacity $T_B$ by unitary transformations, we can assume that $G = \mathbb{C}^m \times \mathbb{R}^{n-m}$. By regularity properties of the Lebesgue measure and the relative capacity $T_B$, we can assume that $K \subset B$ is a compact subset of non empty interior in $G$ so that $\lambda_{n+m}(K) > 0$. Let us prove that $T_B(K) > 0$. Indeed, since $K$ is a compact subset of non empty interior in $G$, there exists an interval $I \subset \mathbb{R}$ of positive length and a disc $D \subset G$ of positive radius such that $D^m \times I^{n-m} \subset K$. Then by the product property of the extremal function (cf. [Si 1]), we get

$$V_K(z, \zeta) \leq \max_{1 \leq i \leq n, 1 \leq j \leq n-m} \{V_D(z_i), V_I(\zeta_j)\},$$

for any $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and $\zeta \in \mathbb{C}^{n-m}$. Therefore $V_K$ is locally bounded on $\mathbb{C}^n$ and $\zeta \in \mathbb{C}^{n-m}$. Therefore $V_K$ is locally bounded on $\mathbb{C}^n$ and then $T_B(K) > 0$. Considering $\varepsilon$-neighbourhoods of $K$ in $G$, we can assume by regularity that $K$ is a regular compact set in the sense that $V_K$ is continuous in $\mathbb{C}^n$.

Then $V_K \in L(\mathbb{C}^n)$ and there exists $a \in B$ such that $V_K(a) = \sup_B V_K$.

By translation we may assume that $a = 0$ is the origin in $G$. Then by the real slicing lemma, there exists a real line $l \subset G$ passing through the point $a = 0$ such that $\lambda_1(K \cap l) > 0$ and

$$\lambda_{n+m}(K) \over \lambda_{n+m}(B) \leq 2(n + m) \lambda_1(K \cap l) \over \lambda_1(B \cap l).$$

Let $L := l + i \cdot l$ be the complex line in $\mathbb{C}^n$ generated by the real line $l$. Since $a = 0 \in l$ and $V_K(a) = \sup_B V_K$, it follows that $T_{B \cap l}(K \cap l) \leq T_B(K)$ and then from (3.1) and (3.19) we deduce that

$$\lambda_{n+m}(K) \over \lambda_{n+m}(B) \leq 8(n + m) T_B(K),$$

which is exactly the required inequality (3.3). ▶

It is interesting to observe that from the formula (3.6) it follows that our relative Polya’s inequalities leads to the following quantitative versions of Bernstein-Walsh inequalities.

**Corollary 3.4** 1) For any closed complex euclidean ball $\mathbb{B} \subset \mathbb{C}^n$, any Borel subset $K \subset \mathbb{B}$ and any function $u \in L(\mathbb{C}^n)$,

$$\sup_{\mathbb{B}} u \leq \sup_K u + \frac{1}{2} \log(c_n) - \frac{1}{2} \log \lambda_{2n}(K) \over \lambda_{2n}(\mathbb{B}),$$

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where $c_n$ is the constant given by (3.2).

2) Let $G \subset \mathbb{C}^n$ be any generic subspace of complex dimension $m \leq n - 1$. Then for any closed real euclidean ball $B \subset G$, any Borel subset $K \subset B$ and any function $u \in L(\mathbb{C}^n)$,

\begin{equation}
(3.22) \quad \sup_B u \leq \sup_K u + \log \left(8(n + m)\right) - \log \frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)}.
\end{equation}

Let us mention that in the totally real case $G = \mathbb{R}^n$, inequalities like (3.22) where obtained earlier by A. Brudnyi (cf. [B 1]).

From relative Polya’s inequalities (3.1), (3.3) and Alexander-Taylor’s inequality (2.11), we deduce the following interesting comparison inequalities between relative volumes and the relative Monge-Ampère capacity. These inequalities show that the Lebesgue measure on any generic subspace of a hyperconvex domain $\Omega \subset \mathbb{C}^n$ is dominated by capacity in a strong sense and then by a result of S. Kolodziej, it belongs the image of the complex Monge-Ampère operator acting on the class of bounded plurisubharmonic functions on $\Omega$ (see [Ko 1], [Ko 2], [Ce 1]).

**Corollary 3.5** 1) For any complex euclidean ball $B \subset \mathbb{C}^n$ and any Borel subset $K \subset B$,

\begin{equation}
(3.23) \quad \frac{\lambda_{2n}(K)}{\lambda_{2n}(B)} \leq c_n \exp\left(-2 \text{cap}(K; B)^{-1/n}\right),
\end{equation}

where $c_n$ is the constant given by (3.3).

2) Let $G \subset \mathbb{C}^n$ be a generic real subspace of complex dimension $0 \leq m \leq n - 1$. Then for any euclidean ball $B \subset G$ and any Borel subset $K \subset B$,

\begin{equation}
(3.24) \quad \frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)} \leq 8(1 + \sqrt{2}) (n + m) \exp\left(- \text{cap}(K; B)^{-1/n}\right),
\end{equation}

where $B$ is the euclidean ball in $\mathbb{C}^n$ such that $B \cap G = B$.

Proof: 1) The inequality (3.23) is a direct consequence of (2.11) and (3.1).

2) Let us prove the inequality (3.24). Since both the relative volume and the relative capacity are invariant under non singular affine transformations, we can assume that $G = \mathbb{C}^m \times \mathbb{R}^{n-m}$, $B$ is the unit real euclidean ball in $G$ and $B$ is the unit complex euclidean ball in $\mathbb{C}^n$. Then by (3.3), we have

\begin{equation}
(3.25) \quad \frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)} \leq 8(n + m)T_B(K).
\end{equation}

On the other hand, by (2.11), we have

$$T_B(K) \leq \exp(-\text{cap}(K; B)^{-1/n}).$$
So to prove the inequality (3.24), it remains to estimate $T_B(K)$ from above by $T_B(K)$. Indeed, from the definition of the extremal function $V_B$, it follows that

$$V_K(z) \leq \max_B V_K + V_B(z), \forall z \in \mathbb{C}^n.$$ 

Therefore, we get

(3.26) \quad $T_B(K) \leq e^{\max_B V_B} T_B(K).$

It remains to estimate $\max_B V_B$. Since $\mathbb{R}^n \subset G$, the euclidean unit ball $B$ in $G$ contains the euclidean unit ball $D$ of $\mathbb{R}^n$ and then $V_B \leq V_D$ on $\mathbb{C}^n$, which implies that $\max_B V_B \leq \max_B V_D$. Now by Lundin’s formula (cf. [Lu],[Sa 2], [Kl]), we have

(3.27) \quad $V_D(z) = \max \{\log |h(\xi \cdot z)| ; \xi \in S^{n-1}\}, z \in \mathbb{C}^n,$

where $h(\zeta) := \zeta + \sqrt{\zeta^2 - 1}$ for $\zeta \in \mathbb{C}$, with the right branch of the square root, $S^{n-1} = \partial D$ is the euclidean unit sphere of $\mathbb{R}^n \subset \mathbb{C}^n$ and $\xi \cdot z = \sum_{1 \leq j \leq n} \xi_j \cdot z_j$. It is easy to see from the formula (3.27) that

$$\max_B V_D = \max_{|z|=1} V_D(z) = \max_{|\zeta|=1} |\log h(\zeta)| = \log(1 + \sqrt{2})$$

and then $\exp(\max_B V_B) \leq \exp(\max_B V_D) = 1 + \sqrt{2}$, which by the inequality (3.26) and (3.25) implies the required inequality (3.24).

Remarks: 1) Polya’s inequalities (3.1) and (3.3) can be stated in one formula as follows. Given a generic subspace $G \subset \mathbb{C}^n$ of complex dimension $0 \leq m \leq n$, then for any euclidean ball $B \subset G$ and any Borel subset $K \subset B$, we have

(3.28) \quad $\frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)} \leq c_{n,m} T_B(K)^{1+[m/n]},$

where $c_{n,m} := 8(n + m)$ if $0 \leq m \leq n - 1$ and $c_{n,n} := c_n$.

We can deduce from the general relative Polya’s inequality (3.28) analogous inequalities in terms of relative volume and relative logarithmic capacity with respect to balls associated to any fixed real norm on the generic space $G$. Indeed, if we denote by $|.|$ the euclidean norm and we are given another real norm $\|\|\|\|$ on $G$, then there exists two constants $\alpha, \beta > 0$ such that

$$\alpha \|z\| \leq |z| \leq \beta \|z\|, \forall z \in G.$$ 

Then given a ball $B'$ for the norm $\|\||\|$, there exists a ball $B$ for the norm $|.|$ such that $\alpha \cdot B \subset B' \subset \beta \cdot B$. Then it follows easily from (3.28) that for any Borel set $K \subset B'$, we have

(3.29) \quad $\frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B')} \leq c_{n,m}(\beta/\alpha)^{n+m} T_{B'}(K)^{1+[m/n]}.$
2) Observe that relative Polya’s inequalities proved above are optimal as far as the exponents are concerned. Indeed we will use inequality (3.29) for the sup-norm, since in this case, explicit computations can be made using the product formula for the relative logarithmic capacity. Let \( B_1, \ldots, B_n \) be regular sets in \( \mathbb{C} \), \( K_1, \ldots, K_n \) Borel subsets such that \( K_j \subset B_j \) for \( j = 1, \ldots, n \) and set \( K := K_1 \times \ldots \times K_n \) and \( B := B_1 \times \ldots \times B_n \). Then using the product property for the extremal function (cf. [Si 1]), we get the formula

\[
T_B(K) = \min_{1 \leq j \leq n} \{ T_{B_j}(K_j) \}.
\]

In the case where \( G = \mathbb{C}^n \), take \( B' \) to be the closed unit polydisc \( \Delta^n \) in \( \mathbb{C}^n \) and \( K_r := \{ z \in \Delta^n; |z_1| \leq r \} \). Then the relative volume of \( K_r \) with respect to \( \Delta^n \) is \( \lambda_{2n}(K_r)/\lambda_{2n}(\Delta^n) = r^2 \) while its relative logarithmic capacity is \( T_{\Delta^n}(K_r) = r \). Then by (3.29) this proves that the exponent 2 in the complex Polya’s inequality (3.1) is the best possible.

In the totally real case, we can assume that \( G = \mathbb{R}^n \) and consider an analogous example with intervals. Take \( B' \) to be the unit \( n \)-cube \( I^n \), where \( I := [-1, +1] \) is the closed unit real interval and define \( I^n(r) := \{ x \in I^n; |x_1| \leq r \} \). Then it is easy to see that

\[
T_{I^n}(I^n(r)) = \frac{r}{1 + \sqrt{1 - r^2}} \sim \frac{r^2}{2} \quad \text{as } r \to 0,
\]

while the relative \( n \)-volume of \( I^n(r) \) with respect to \( I^n \) is equal to \( r \), which proves by (3.29) that the exponent 1 in Polya’s inequality (3.3) is the best possible in this case.

Now if \( G = \mathbb{C}^m \times \mathbb{R}^{n-m} \) with \( 1 \leq m \leq n-1 \), it is enough to take \( B' = \Delta^m \times I^{n-m} \) and \( K_r := \Delta^m \times I^{n-m}(r) \). Then \( T_{B'}(K_r) \sim r/2 \) as \( r \to 0 \), while \( \lambda_{n+m}(K_r)/\lambda_{n+m}(B') = r \), which proves again by (3.29) that the exponent 1 in Polya’s inequality (3.3) is the best possible in this case.

### 4 Relative size of plurisubharmonic lemniscates

Here we want to deduce from relative Polya’s inequalities an estimate on the relative size of plurisubharmonic lemniscates associated to two important classes of plurisubharmonic functions. Let us start with estimating precisely the size of the lemniscates associated to the Lelong class \( \mathcal{L}(\mathbb{C}^n) \).

**Theorem 4.1** 1) For any complex euclidean closed ball \( B \subset \mathbb{C}^n \) and any \( u \in \mathcal{L}(\mathbb{C}^n) \) with \( \max_{\mathbb{B}} u = 0 \),

\[
\frac{\lambda_{2n}(\{ z \in B; u(z) \leq -s \})}{\lambda_{2n}(B)} \leq c_n e^{-2s}, \quad \forall s > 0,
\]

where \( c_n \) is the constant given by (3.2).

2) Let \( G \subset \mathbb{C}^n \) be a generic real subspace of complex dimension \( m \leq n - 1 \).
Then for any real euclidean closed ball $B \subset \mathbb{G}$ and any $u \in \mathcal{L}(\mathbb{C}^n)$ with $\max_B u = 0$,

\begin{equation}
\frac{\lambda_{n+m}(\{ x \in B ; u(z) \leq -s \})}{\lambda_{n+m}(B)} \leq 8 (n + m) e^{-s}, \ \forall s > 0.
\end{equation}

Proof: 1) Let $B \subset \mathbb{C}^n$ be an arbitrary complex ball and $u \in \mathcal{L}(\mathbb{C}^n)$ with $\max_B u = 0$. Set $E(t) := \{ z \in B ; u(z) \leq t \}$ for $t < 0$. Then $u - t \leq V_{E(t)}$ on $\mathbb{C}^n$ and then $-t = \max_B u - t \leq \max_B V_{E(t)}$. This implies that $T_B(E(t)) \leq e^t$ for any $t < 0$. Now in order to get the estimate (4.1), it is enough to apply the complex Polya's inequality (3.1) to the Borel set $E_t(u)$ with $s = -t$. To prove the estimate (4.2), we proceed in the same way using the real Polya's inequality (3.3).

Observe that estimates of plurisubharmonic lemniscates we re obtained in the complex case earlier by the third author in a more general context but with less precise exponents (cf. [Ze 2], [Ze 3]).

In particular, observing that $(1/d) \log |P| \in \mathcal{L}(\mathbb{C}^n)$ for any polynomial $P \in \mathbb{C}[z]$ with degree $d \geq 1$, we obtain the following precise estimate for polynomial lemniscates.

**Corollary 4.2** 1) For any complex ball $B \subset \mathbb{C}^n$ and any polynomial $P \in \mathbb{C}[z]$ of degree $d \geq 1$ satisfying $\|P\|_B = 1$, we obtain the following precise estimate for polynomial lemniscates.

\begin{equation}
\frac{\lambda_2(\{ z \in B ; |P(z)| \leq \varepsilon^d \})}{\lambda_2(B)} \leq c_n \varepsilon^2, \ \forall \varepsilon \in [0,1],
\end{equation}

where $c_n$ is the constant given by (3.3).

2) Let $\mathbb{G} \subset \mathbb{C}^n$ be a generic subspace of complex dimension $0 \leq m \leq n - 1$. Then for any real euclidean ball $B \subset \mathbb{G}$ and any polynomial $P \in \mathbb{C}[z]$ of degree $d \geq 1$ satisfying $\|P\|_B = 1$, the following estimate holds

\begin{equation}
\frac{\lambda_{n+m}(\{ z \in B ; |P(z)| \leq \varepsilon^d \})}{\lambda_{n+m}(B)} \leq 8(n + m) \varepsilon, \ \forall \varepsilon \in [0,1].
\end{equation}

All these estimates are optimal as far as the exponents are concerned (see Remarks above). The first inequality is an improvement of previous results (see [Cu-Dr-Lu], [Ze 2], [Ze 3]) and answers a question asked by the third author in ([Ze 2]). In the totally real case where $\mathbb{G} = \mathbb{R}^n$, the second inequality appears also in ([Br-Ga]).

Now let us estimate the size of plurisubharmonic lemniscates associated to the Cegrell class $\mathcal{F}(\Omega)$.

**Theorem 4.3** Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex open set. Then for any plurisubharmonic function $\varphi \in \mathcal{F}(\Omega)$ with $\int_\Omega (dd^c \varphi)^n \leq 1$, we have

\begin{equation}
\lambda_{2n}(\{ z \in \Omega ; \varphi(z) \leq -s \}) \leq c_n \tau_{2n}(\Omega) e^{-2s}, \ \forall s > 0,
\end{equation}

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where $\tau_{2n}(\Omega)$ is the volume of the smallest euclidean ball of $\mathbb{C}^n$ containing $\Omega$ and $c_n$ is the constant given by (3.2).

Moreover, if $G \subset \mathbb{C}^n$ is a generic subspace of complex dimension $m \leq n-1$ such that $D := \Omega \cap G \neq \emptyset$, then for any $s > 0$,

\begin{equation}
\lambda_{n+m}(\{z \in D; \varphi(z) \leq -s\}) \leq 8(1 + \sqrt{2}) \ (n + m) \ \tau_{n+m}(D) \ e^{-s},
\end{equation}

where $\tau_{n+m}(D)$ is the volume of the smallest euclidean ball of $G$ containing $D$.

For the proof of this theorem, we will need the following elementary lemma.

**Lemma 4.4** Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex open set. Then for any $\varphi \in \mathcal{F}(\Omega)$,

\begin{equation}
\operatorname{cap}(\{z \in \Omega; \varphi(z) \leq -s\}; \Omega) \leq s^{-n} \int_{\Omega} (dd^c \varphi)^n, \ \forall s > 0.
\end{equation}

Proof: 1) Assume first that $\varphi$ is a bounded plurisubharmonic function on $\Omega$ with boundary values 0 and finite Monge-Ampère mass on $\Omega$. Let $s > 0$ be fixed and $K \subset \Omega(\varphi; s) := \{z \in \Omega; \varphi(z) \leq -s\}$ any fixed regular compact set in the sense that the plurisubharmonic measure $h_K$ the condenser $(K, \Omega)$ is continuous on $\Omega$. Since $h_K$ and $\varphi$ have boundary values 0, from the comparison principle (see [Be-Ta 1], [Kl]) it follows that

\begin{equation}
\operatorname{cap}(K; \Omega) = \int_K (dd^c h_K)^n \leq \int_{\{s^{-1,\varphi<h_K}\}} (dd^c h_K)^n \leq \frac{1}{s^n} \int_{\Omega} (dd^c \varphi)^n.
\end{equation}

Taking an exhaustive sequence of regular compact subsets of the open set $\Omega(s; \varphi)$ and using interior regularity of the capacity we obtain our inequality in this case.

2) Now for an arbitrary given function $\varphi \in \mathcal{F}(\Omega)$, there exists a decreasing sequence $(\varphi_j)$ of bounded plurisubharmonic functions with boundary values 0 which converges to $\varphi$ such that $\int_{\Omega} (dd^c \varphi_j)^n = \lim_{j} \int_{\Omega} (dd^c \varphi_j)^n$ (cf. [Ce 2], [Ce-Ze]). Then the estimate (4.7) follows from the first case and the lemma is proved.

Now we can prove the theorem.

Proof of the theorem: 1) Let $\mathbb{B}$ be the smallest euclidean ball of $\mathbb{C}^n$ containing $\Omega$. Let $\varphi \in \mathcal{F}(\Omega)$ as in the theorem and set $\Omega(\varphi; s) := \{z \in \Omega; \varphi(z) \leq -s\}$ and $c(s) = c_{\Omega}(s, \varphi) := \operatorname{cap}(\Omega(\varphi; s); \Omega)$ for $s > 0$. Then applying inequality (3.23), we obtain

\begin{equation}
\lambda_{2n}(\Omega(\varphi; s)) \leq c_n \lambda_{2n}(\mathbb{B}) \exp(-2c_{\Omega}(s)^{-1/n}), \forall s > 0.
\end{equation}

Now the estimate (4.5) follows from the estimate (4.8) using the estimate (4.7).

The estimate (4.6) is proved in the same way using the inequalities (3.23) and (4.7). £
5 Global behaviour of the Lelong class

The next application of our theorems from the last section will concern the Lelong class of plurisubharmonic functions with logarithmic singularities at infinity defined by the formula (2.4).

The Lelong class of plurisubharmonic functions is known to play an important role in pluripotential theory (cf. [Le 1], [Be-Ta 2], [Si 1], [Si 2], [Sa 1], [Za], [Ze 1], [Ze 2]).

Here we want to prove new general uniform integrability theorems for the Lelong class of plurisubharmonic functions.

Let \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) be an increasing function such that \( g(0) = 0 \) and \( \lim_{t \to +\infty} g(t) = +\infty \). For \( \delta > 0 \), consider the following Riemann-Stieltjes' integral

\[
I_\delta(g) := \delta \int_0^{+\infty} e^{-\delta t} dg(t).
\]

Then we have the following result.

**Theorem 5.1**

1) For any complex euclidean closed ball \( B \subset \mathbb{C}^n \) and any function \( u \in L(\mathbb{C}^n) \)

\[
\frac{1}{\lambda_{2n}(B)} \int_B g(\max_B u - u) d\lambda_{2n} \leq c_n I_2(g),
\]

provided that \( I_2(g) < +\infty \), where \( c_n \) is the constant given by (3.2).

2) Let \( G \subset \mathbb{C}^n \) be a generic real subspace of complex dimension \( m \). Then for any real euclidean closed ball \( B \subset G \) and any function \( u \in L(\mathbb{C}^n) \)

\[
\frac{1}{\lambda_{n+m}(B)} \int_B g(\max_B u - u) d\lambda_{n+m} \leq 8(n + m) I_1(g)
\]

provided that \( I_1(g) < +\infty \).

**Proof:** We can assume \( g \) to be strictly increasing. Let \( \mu \) be any Borel measure on \( \mathbb{C}^n \) and \( K \subset \mathbb{C}^n \) any Borel set. Then for any function \( u \in L(\mathbb{C}^n) \) with \( u|K \leq 0 \), we have

\[
\int_K g(-u) d\mu = \int_0^{+\infty} \mu(K \cap \{g(-u) \geq t\}) dt = \int_0^{+\infty} \mu(K \cap \{u \leq -s\}) dg(s).
\]

1) Assume that \( \mu := 1_B \lambda_{2n} \), where \( B \subset \mathbb{C}^n \) is a complex euclidean closed ball and \( u \in L(\mathbb{C}^n) \) with \( \max_B u = 0 \). Then by (5.4), we get

\[
\int_B g(-u) d\lambda_{2n} = \int_0^{+\infty} \lambda_{2n}(B \cap \{u \leq -s\}) dg(s).
\]
Applying the estimates (4.1) to the formula (5.5), we obtain the following inequality

\[(5.6) \int_B g(-u) d\lambda_{2n} \leq c_n \lambda_{2n}(B) \int_0^{+\infty} e^{-2s} dg(s). \]

If \(I_2(g) < +\infty\), we easily see that \(\lim_{t \to +\infty} g(t)e^{-2t} = 0\) and then by integration by parts, it follows that \(\int_0^{+\infty} e^{-2s} dg(s) = I_2(g)\), which implies the required inequality thanks to the inequality (5.6).

2) Assume that \(\mu := 1_B \lambda_{n+m}\), where \(B \subset \mathbb{G}\) is a real euclidean closed ball and \(u \in \mathcal{L}(\mathbb{C}^n)\) with \(\max_B u = 0\). Then applying the estimates (4.2) to the formula (5.5), we obtain the following inequality

\[(5.7) \int_B g(-u) d\lambda_n \leq 8(n + m) \lambda_{n+m}(B) \int_0^{+\infty} e^{-s} dg(s). \]

If \(I_1(g) < +\infty\), then as in the first case the required inequality follows from the inequality (5.7) by integration by parts. 

From this general result we derive the following corollaries which will be useful later.

**Corollary 5.2** For any complex euclidean ball \(B \subset \mathbb{C}^n\), any function \(u \in \mathcal{L}(\mathbb{C}^n)\) and any \(0 < \alpha < 2\),

\[(5.8) \frac{1}{\lambda_{2n}(B)} \int_B e^{-\alpha u} d\lambda_{2n} \leq \left(1 + c_n \frac{\alpha}{2 - \alpha}\right) e^{-\alpha \max_B u}, \]

where \(c_n\) is the constant given by (3.2).

2) Let \(G \subset \mathbb{C}^n\) be a generic real subspace of complex dimension \(m \leq n - 1\). Then for any real euclidean ball \(B \subset G\), any function \(u \in \mathcal{L}(\mathbb{C}^n)\) and any 
\(0 < \alpha < 1\),

\[(5.9) \frac{1}{\lambda_{n+m}(B)} \int_B e^{-\alpha u} d\lambda_{n+m} \leq \left(1 + 8(n + m) \frac{\alpha}{1 - \alpha}\right) e^{-\alpha \max_B u}. \]

Proof: 1) Indeed, it is enough to apply Theorem 5.1 with the increasing function \(g(t) := e^{\alpha t} - 1\), with \(0 < \alpha < 2\) in the complex case and \(0 < \alpha < 1\) in the real generic case. 

**Corollary 5.3** 1) For any complex euclidean ball \(B \subset \mathbb{C}^n\), any function \(u \in \mathcal{L}(\mathbb{C}^n)\) and any real number \(p > 0\),

\[(5.10) \frac{1}{\lambda_{2n}(B)} \int_B (\max_B u - u)^p d\lambda_{2n} \leq c_n 2^{-p} \Gamma(p + 1), \]

where \(\Gamma(s) := \int_0^{+\infty} t^{s-1} e^{-t} dt, s > 0\) is the Euler function and \(c_n\) is the constant given by (3.2).
2) Let $G \subset \mathbb{C}^n$ be a generic real subspace of complex dimension $m \leq n - 1$. Then for any real euclidean ball $B \subset G$, any function $u \in \mathcal{L}(\mathbb{C}^n)$ and any real number $p > 0$,

$$
\frac{1}{\lambda_{n+m}(B)} \int_B (\max_B u - u)^p d\lambda_{n+m} \leq 8(n + m) \Gamma(p + 1),
$$

where $\Gamma$ is the Euler function.

Proof: Indeed, it is enough to apply Theorem 5.1 with the increasing function $g(t) := t^p, t \geq 0$. ▷

Now we want to study the global behaviour of the Lelong class $\mathcal{L}(\mathbb{C}^n)$, estimating uniformly the size of the deviation between a function and its mean values on complex or real euclidean balls.

Let us recall the general definition of the space BMO. Let $G$ be a real euclidean space of dimension $k \geq 1$ and let $\lambda_k$ the Lebesgue measure on $G$. For a locally integrable function $f : G \rightarrow \mathbb{R}$ and any euclidean ball $B \subset G$, define the mean value of $f$ on $B$ by

$$
f_B := \frac{1}{|B|_k} \int_B f d\lambda_k,
$$

where $|B|_k = \lambda_k(B)$. Then we say that $f \in \text{BMO}(G)$ if and only if

$$\|f\|_{\text{BMO}(G)} := \sup_B \left\{ \frac{1}{|B|_k} \int_B |f - f_B| d\lambda_k \right\} < +\infty,$$

where the supremum is taken over all the euclidean balls $B \subset G$. Let us first prove the following result which can be considered as a quantitative version for the Lelong class $\mathcal{L}(\mathbb{C}^n)$ of the classical John-Nirenberg inequality for BMO−functions (cf. [St]).

**Theorem 5.4** 1) For any complex euclidean ball $B \subset \mathbb{C}^n$, any function $u \in \mathcal{L}(\mathbb{C}^n)$ and any real number $\alpha < 2$,

$$
\frac{1}{|B|_{2n}} \int_B e^{\alpha|u - u_B|} d\lambda_{2n} \leq (1 + c_n \frac{\alpha}{2 - \alpha}) \exp\left(\frac{\alpha c_n}{2}\right),
$$

where $u_B := (1/|B|_{2n}) \int_B ud\lambda_{2n}$ and $c_n$ is the constant given by (3.2).

2) Let $G \subset \mathbb{C}^n$ be a generic real subspace of complex dimension $0 \leq m \leq n - 1$. Then for any real euclidean ball $B \subset G$, any function $u \in \mathcal{L}(\mathbb{C}^n)$ and any real number $\alpha < 1$,

$$
\frac{1}{|B|_{n+m}} \int_B e^{\alpha|u - u_B|} d\lambda_{n+m} \leq \left(1 + 8(n + m) \frac{\alpha}{1 - \alpha}\right) \exp(8\alpha(n + m)),
$$

where $u_B := (1/|B|_{n+m}) \int_B ud\lambda_{n+m}$.
Proof: 1) From Corollary 5.2, it follows that for a fixed function \(u \in \mathcal{L}(\mathbb{C}^n)\) and any euclidean ball \(B \subset \mathbb{C}^n\),

\[
(5.14) \quad \frac{1}{|B|2n} \int_B e^{\alpha(\max_B u - u)} d\lambda_{2n} \leq 1 + c_n \frac{\alpha}{2 - \alpha}.
\]

Now, from Corollary 5.3, we get

\[
(5.15) \quad \max_B u - u_B \leq \frac{c_n}{2}
\]

Therefore by (5.14) and (5.15) we get

\[
\frac{1}{|B|2n} \int_B e^{\alpha|u - u_B|} d\lambda_{2n} \leq (1 + c_n \frac{\alpha}{2 - \alpha}) e^{c_n \alpha}.
\]

The real case is proved in the same way. ▶

Observe that in the complex case, a better estimate can be obtained using a refined version of the inequality (5.15) due to Lelong (cf. [Le 2], [De], [Si 2]).

From the last theorem we deduce the following result.

**Corollary 5.5** Let \(G \subset \mathbb{C}^n\) be a generic real subspace of complex dimension \(m \leq n\). Then for any function \(u \in \mathcal{L}(\mathbb{C}^n)\), \(u|G \in \text{BMO}(G)\) and

\[
\|u\|_{\text{BMO}(G)} \leq \sigma_{n,m},
\]

In particular, for any polynomial \(P \in \mathbb{C}[z]\), with \(\deg(P) = d \geq 1\),

\[
(5.16) \quad \|\log|P|\|_{\text{BMO}(G)} \leq \sigma_{n,m} \cdot d.
\]

Here \(\sigma_{n,m} := 2 \log(1 + 8(n + m)) + 8(n + m)\) if \(0 \leq m \leq n - 1\) and \(\sigma_{n,n} := \log(1 + c_n) + c_n/2\), where \(c_n\) is the constant given by (3.2).

In the totally real case where \(G = \mathbb{R}^n\), the existence of a uniform bound for the \(\text{BMO}(\mathbb{R}^n)\)–norm of plurisubharmonic functions of logarithmic singularities on \(\mathbb{C}^n\) was proved earlier by A.Brudnyi (cf. [B 1]) with a different proof. Our proof gives a precise quantitative estimate of the uniform bound.

### 6 Global uniform integrability of plurisubharmonic functions

Here we want to give a sufficient condition for global integrability of plurisubharmonic functions in terms of the relative Monge-Ampère capacity of their sublevel sets. Then we will deduce a global integrability theorem for the class of plurisubharmonic functions with uniformly bounded Monge-Ampère masses.
For any \( u \in \text{PSH}^-(\Omega) \) and any Borel subset \( E \subset \Omega \) we define the truncated plurisubharmonic lemniscates associated to \( u \) as \( E(s,u) := \{ z \in E; u(z) < -s \} \), for \( s > 0 \) and the corresponding capacity function

\[
c_E(s,u) = \text{Cap}(E(s,u);\Omega).
\]

Let \( \mathcal{U} \subset \text{PSH}^-(\Omega) \) be a class of plurisubharmonic functions on \( \Omega \) then define

\[
c_E(s,\mathcal{U}) := \sup\{c_E(s,u); u \in \mathcal{U}\}, s > 0.
\]

Let \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) be an increasing function such that \( g(0) = 0 \) and \( \lim_{t \to +\infty} g(t) = +\infty \). As in the last section, consider the following Riemann-Stieltjes’ integral for \( \delta > 0 \),

\[
I_\delta(g) := \int_0^{+\infty} e^{-\delta t} dg(t).
\]

The main result of this section is the following.

**Theorem 6.1** Let \( \mathcal{U} \subset \text{PSH}^-(\Omega) \) be a class of plurisubharmonic functions on \( \Omega \) and \( E \subset \Omega \) a Borel subset such that

\[
\eta = \eta(E;\mathcal{U}) := \sup_{s \geq 0} s \left(c_E(s,\mathcal{U})\right)^{1/n} < +\infty.
\]

Then the following properties hold.

1) For any function \( u \in \mathcal{U} \),

\[
\int_E g(-u) d\lambda_{2n} \leq c_n \tau_{2n}(E) I_{2/\eta}(g),
\]

provided that \( I_{2/\eta}(g) < +\infty \) (see (6.1)), where \( \tau_{2n}(E) \) is the \( 2n \)-volume of the smallest complex euclidean ball of \( \mathbb{C}^n \) containing \( E \) and \( c_n \) is the constant given by (3.2).

2) If \( G \subset \mathbb{C}^n \) is a generic real subspace of complex dimension \( m \leq n - 1 \) such that \( \Omega \cap G \neq \emptyset \) and \( E \subset \Omega \cap G \) then for any function \( u \in \mathcal{U} \),

\[
\int_E g(-u) d\lambda_{n+m} \leq 8(1 + \sqrt{2}) (n + m) \tau_{n+m}(E) I_{1/\eta}(g),
\]

provided that \( I_{1/\eta}(g) < +\infty \) (see (6.1)), where \( \tau_{n+m}(E) \) is the \( (n+m) \)-volume of the smallest euclidean ball in \( G \) which contains \( E \).

Proof: By approximation we can assume that \( g \) is strictly increasing. Let \( \mu \) be any positive Borel measure on \( \Omega \) and \( u \in \text{PSH}^-(\Omega) \). Then

\[
(6.2) \quad \int_{\Omega} g(-u) d\mu = \int_0^{+\infty} \mu(\Omega(u;s)) dg(s).
\]
Now let $\mu = 1_E \lambda_{2n}$ and $B$ be a complex euclidean ball of $\mathbb{C}^n$ containing $E$. Then by (3.23) we get

$$\lambda_{2n}(E(u; s)) \leq c_n \lambda_{2n}(\mathbb{B}) \exp(-2c_E(s, u)^{-1/n}).$$

Therefore from (6.2) we conclude that

$$\lambda_{2n}(E(u; s)) \leq c_n \lambda_{2n}(B) \exp(-2c_E(s, u)^{-1/n})d\mathcal{B}(s).$$

From the estimate (6.3) and the hypothesis, we deduce that

$$\lambda_{2n}(E(u; s)) \leq c_n \lambda_{2n}(B) \int_{\mathbb{B}} \exp(-2s/\eta) d\mathcal{B}(s),$$

which proves the required estimate. The real generic case is proved in the same way. \hfill \Box

From this result we can deduce the following corollaries.

**Corollary 6.2** Let $U \subset PSH^-(\Omega)$ be a class of plurisubharmonic functions on $\Omega$ and $E \subset \Omega$ be a Borel subset such that

$$\eta = \eta(E; U) := \sup_{s \geq 0} s \left( c_E(s, U) \right)^{1/n} < +\infty.$$

Then the following properties hold.

1) For any function $u \in U$ and any exponent $0 < \alpha < 2/\eta$,

$$\int_E e^{-\alpha u} d\lambda_{2n} \leq \lambda_{2n}(E) + c_n \tau_{2n}(E) \frac{\alpha \eta}{2 - \alpha \eta},$$

where $\tau_{2n}(E)$ is the $2n-$volume of the smallest complex euclidean ball of $\mathbb{C}^n$ containing $E$ and $c_n$ is the constant given by (3.2).

2) Moreover if $G \subset \mathbb{C}^n$ is a generic real subspace of complex dimension $m \leq n - 1$ such that $\Omega \cap G \neq \emptyset$ and $E \subset \Omega \cap G$, for any function $u \in U$ and any real number $\alpha < 1/\eta$,

$$\int_D e^{-\alpha u} d\lambda_{n+m} \leq \lambda_{n+m}(D) + 8(1 + \sqrt{2}) (n + m) \tau_{n+m}(D) \frac{\alpha \eta}{1 - \alpha \eta},$$

where $\tau_{n+m}(D)$ is the $(n + m)-$volume of the smallest euclidean ball of $G$ containing $D$.

From the last result we can easily deduce the following one.

**Corollary 6.3** Let $U \subset PSH^-(\Omega)$ be a class of plurisubharmonic functions on $\Omega$. Then the following properties hold.

1) If

$$\gamma := \limsup_{s \to +\infty} s \left( c_{\Omega}(s, U) \right)^{1/n} < +\infty,$$
then for any exponent $0 < \alpha < 2/\gamma$, there exists a constant $A_{2n} = A_{2n}(\alpha, \delta, \Omega, \mathcal{U}) > 0$ such that
\[
\int_{\Omega} e^{-\alpha u} d\lambda_{2n} \leq A_{2n}, \forall u \in \mathcal{U}.
\]

2) If $G \subset \mathbb{C}^n$ is a generic real subspace of complex dimension $m \leq n - 1$ such that $D := \Omega \cap \mathbb{R}^n \neq \emptyset$ and
\[
\delta := \limsup_{s \to +\infty} s \left( c_D(s, \mathcal{U}) \right)^{1/n} < +\infty,
\]
then for any $\alpha < 1/\delta$, there is a constant $A_{n,m} = A_{n,m}(\alpha, \delta, D, \mathcal{U}) > 0$ such that
\[
\int_{D} e^{-\alpha u} d\lambda_{n+m} \leq A_{n,m}, \forall u \in \mathcal{U}.
\]

Proof: 1) If $\gamma(\mathcal{U}) < +\infty$, for any $\alpha < 2/\gamma(\mathcal{U})$, there is $s_0 > 0$ and $\gamma_0 > 0$ such that $\alpha < 2/\gamma_0$ and
\[
s c_\Omega(s, u)^{1/n} \leq \gamma_0, \forall s \geq s_0, \forall u \in \mathcal{U}.
\]

Then if we define the class $\mathcal{V} := \mathcal{U} + s_0$, it follows that
\[
t c_\Omega(t, v)^{1/n} \leq \gamma_0, \forall t \geq 0, \forall v \in \mathcal{V},
\]
which implies that $\eta := \eta(\Omega, \mathcal{V}) \leq \gamma_0$. Therefore, since $\alpha < 2/\gamma_0 \leq 2/\eta$, we can apply Theorem 6.1 to the class $\mathcal{V}$ and get the estimate
\[
\int_{\Omega} e^{-\alpha u} d\lambda_{2n} \leq \lambda_{2n}(\Omega) + c_n \tau_{2n}(\Omega) \frac{\alpha \eta}{1 - \alpha},
\]
This inequality implies clearly that
\[
\int_{\Omega} e^{-\alpha u} d\lambda_{2n} \leq \lambda_{2n}(\Omega) + c_n \tau_{2n}(\Omega) e^{\alpha s_0} \frac{\alpha \eta}{1 - \alpha}, \forall u \in \mathcal{U},
\]
which proves the first estimate of the theorem. The second estimate is proved in the same way. \(\square\)

Now we will give an application of the corollary 6.2 to the global uniform integrability of the Cegrell class of plurisubharmonic functions of bounded Monge-Ampère mass on a bounded hyperconvex domain.

**Corollary 6.4** 1) For any $\alpha < 2$ and any $\varphi \in \mathcal{F}(\Omega)$ with $\int_{\Omega} (dd^c \varphi)^n \leq 1$,
\[
(6.4) \quad \int_{\Omega} e^{-\alpha \varphi(z)} d\lambda_{2n}(z) \leq \lambda_{2n}(\Omega) + c_n \tau_{2n}(\Omega) \frac{\alpha}{1 - \alpha},
\]
where $c_n$ is the constant given by (3.3).

2) If $G \subset \mathbb{C}^n$ is a generic real subspace of complex dimension $m \leq n - 1$ such that $D := \Omega \cap G \neq \emptyset$, then for any $\alpha < 1$ and any $\varphi \in \mathcal{F}(\Omega)$ with $\int_{\Omega} (dd^c \varphi)^n \leq 1$,
\[
(6.5) \quad \int_{D} e^{-\alpha \varphi(z)} d\lambda_{n+m}(z) \leq \lambda_{n+m}(D) + 8(1 + \sqrt{2}) (n + m) \tau_{n+m}(D) \frac{\alpha}{1 - \alpha}.
\]

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Proof: Consider the class $\mathcal{U}$ of plurisubharmonic functions in $\varphi \mathcal{F}(\Omega)$ such that $\int_{\Omega} (dd^c \varphi)^n \leq 1$. Then by Lemma 4.4, we get the inequality $\eta = \eta(E, \mathcal{U}) \leq 1$ for any Borel subset $E \subset \Omega$. Therefore the results above follows immediately from Corollary 6.2.

A uniform estimate of type (6.4) was obtained recently in ([Ce-Ze]) with a different method but with a non explicit uniform constant, while the estimate (6.5) seems to be new.

As in section 5, from Theorem 6.1 we can deduce uniform $L^p$ estimates for functions from the class $\mathcal{F}(\Omega)$.

Corollary 6.5 1) For any $\varphi \in \mathcal{F}(\Omega)$ and any real number $p > 0$,

$$\int_{\Omega} (-\varphi)^p d\lambda_{2n} \leq c_n \tau_2(\Omega) 2^{-p} \Gamma(p+1) \left( \int_{\Omega} (dd^c \varphi)^n \right)^{p/n},$$

where $c_n$ is the constant given by (3.2).

2) If $G \subset \mathbb{C}^n$ is a generic real subspace of complex dimension $m \leq n - 1$ such that $D := \Omega \cap G \neq \emptyset$, then for any $\varphi \in \mathcal{F}(\Omega)$ and any real number $p > 0$,

$$\int_{D} (-\varphi)^p d\lambda_{n+m} \leq 8(1 + \sqrt{2}) (n + m) \tau_{n+m}(D) \Gamma(p+1) \left( \int_{\Omega} (dd^c \varphi)^n \right)^{p/n}.$$

Proof: Indeed, by Lemma 4.4 the real number $\eta = \eta(E, \mathcal{U})$ for the class $\mathcal{U}$ of plurisubharmonic functions $\varphi \in \mathcal{F}(\Omega)$ such that $\int_{\Omega} (dd^c \varphi)^n \leq 1$ and any subset $E \subset \Omega$ satisfies the inequality $\eta \leq 1$. Since the function $I_\delta(g)$ is decreasing in $\delta$, we easily see that the corollary is an easy consequence of Theorem 6.1 with the function $g(t) = t^p$.

Acknowledgments: We thank Urban CEGRELL for his useful comments on the paper and for pointing out a mistake in our earlier version of the lemma 3.3.

AMS classification: 32U05, 32U15, 32U20, 32W20.

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