An Interpretation of Noncommutative Field Theory in Terms of a Quantum Shift

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Abstract

Noncommutative coordinates are decomposed into a sum of geometrical ones and a universal quantum shift operator. With the help of this operator, the mapping of a commutative field theory into a noncommutative field theory (NCFT) is introduced. A general measure for the Lorentz-invariance violation in NCFT is also derived.
1 Introduction

NCFT is characterized by the NC space-time coordinates $\hat{x}_\mu$ satisfying the commutation relations (CR)

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} ,$$

(1.1)

where $\theta_{\mu\nu}$ is an antisymmetric constant matrix (for reviews, see, e.g., [1, 2, 3]). In accordance with the contemporary wisdom we may assume that the time-space noncommutativity is absent (due to the violation of unitarity and causality [4, 5]),

$$\theta_{0j} = 0 , \quad j = 1, 2, 3 .$$

(1.2)

Then, in order to offer a physical interpretation of (1.1), we shall decompose $\hat{x}_\mu$ as follows:

$$\hat{x}_\mu = x_\mu + \hat{o}_\mu ,$$

(1.3)

where $x_\mu$ denotes the classical geometrical coordinates and $\hat{o}_\mu$ a quantum-mechanical fluctuation, to be referred to as a quantum shift operator hereafter.

Then the CR (1.1) can be reproduced by postulating a more universal one,

$$[\hat{o}_\mu, \hat{o}_\nu] = i\theta_{\mu\nu} .$$

(1.4)

Because of the uniformity of space-time we may assume that this quantum shift is common to all the classical positions, so that we may also decompose $\hat{y}_\mu$, different from $\hat{x}_\mu$, as:

$$\hat{y}_\mu = y_\mu + \hat{o}_\mu .$$

(1.5)
Then let us consider a function of \( x_\mu \), say \( f(x) \), and quantize \( x \) as \( f(x) \rightarrow f(\hat{x}) \), and we may express this process as follows by making use of the Taylor expansion:

\[
f(x) \rightarrow f(\hat{x}) = f(x + \hat{\alpha}) = e^{\hat{\alpha} \cdot \partial} f(x) , \quad (1.6)
\]

where

\[
\hat{\alpha} \cdot \partial = \hat{\alpha}_\mu \partial^\mu . \quad (1.7)
\]

This operation is essentially a translation in space by \( \hat{\alpha} \), the quantum shift.

We have to point out that for defining an expression such as (1.6), \( f(x) \) has to be a linear combination of the eigenfunctions of \( \partial_\mu \). This happens to be the Fourier representation of \( f(x) \) and we can utilize the relation

\[
e^{\hat{\alpha} \cdot \partial} e^{ikx} = e^{\hat{\alpha} \cdot k} e^{ikx} = e^{ik(x+\hat{\alpha})} = e^{ik\hat{x}} . \quad (1.8)
\]

When \( f(x)g(x) = h(x) \) we generally get

\[
f(\hat{x})g(\hat{x}) \neq g(\hat{x})f(\hat{x})
\]

and

\[
f(\hat{x})g(\hat{x}) \neq h(\hat{x})
\]

as we will show in Section 3.

## 2 The Weyl-Moyal product

In quantum mechanics we introduce a Hilbert space \( \mathcal{H}_{\text{phys}} \), but the quantum mechanical operator \( \hat{\alpha}_\mu \) implies the introduction of a new independent Hilbert space \( \mathcal{H}_s \), so that we are led to an extended Hilbert space \( \mathcal{H} \), defined by the
direct product of these two spaces:

\[ \mathcal{H} = \mathcal{H}_{\text{phys}} \otimes \mathcal{H}_s . \]  

(2.9)

The quantum shift is realized, as it has been shown in (1.6) by applying the unitary operator

\[ e^{\hat{o} \cdot \partial} \]  

(2.10)

to the wave function. This operation is a sort of phase transformation representing a parallel translation in space, so that it cannot be recognized for a single-particle system.

Now we shall proceed to a two-particle system represented by the wave function \( f(x_1)g(x_2) \). Then its quantum shift is realized as

\[
\begin{align*}
f(x_1)g(x_2) & \rightarrow f(\hat{x}_1)g(\hat{x}_2) \\
& = e^{\hat{o} \cdot \partial_1} e^{\hat{o} \cdot \partial_2} f(x_1)g(x_2)
\end{align*}
\]

(2.11)

in an obvious notation. The application of the Baker-Campbell-Haussdorff formula in this case yields

\[
e^{\hat{o} \cdot \partial_1} e^{\hat{o} \cdot \partial_2} = e^{\hat{o} (\partial_1^1 + \partial_2^2)} e^{\frac{1}{4} (\theta \partial_1 \partial_2)} ,
\]

(2.12)

where

\[
\theta \partial_1 \partial_2 \equiv \theta_{\mu \nu} \partial_1^\mu \partial_2^\nu .
\]

(2.13)

Thus we arrive at the following relationship

\[
f(\hat{x}_1)g(\hat{x}_2) = e^{\hat{o} (\partial_1^1 + \partial_2^2)} f(x_1) \star g(x_2) ,
\]

(2.14)

where

\[
f(x_1) \star g(x_2) = e^{\frac{1}{4} (\theta \partial_1 \partial_2)} f(x_1)g(x_2) .
\]

(2.15)
This generalizes the Weyl-Moyal $\star$-product, which is obtained for $x_1 = x_2 = x$:

$$f(x) \star g(x) = e^{\frac{i}{2} \theta \partial_1 \partial^2} f(x_1)g(x_2) |_{x_1 = x_2 = x} .$$  \hfill (2.16)

In (2.14) the front factor corresponds to an unobservable phase transformation depicting a parallel translation of the centre of mass by $\hat{o}$ and the Weyl-Moyal product reflects the fluctuation of the relative coordinates of the two particles of this system. Eq. (2.15) can be generalized to a product of $n$ functions:

$$f_1(\hat{x}_1) ... f_n(\hat{x}_n) = e^{\hat{o} (\partial_1 + ... + \partial_n)} f_1(x_1) ... \star f_n(x_n)$$

$$= e^{\hat{o} (\partial_1 + ... + \partial_n)} e^{D} f_1(x_1) ... f_n(x_n) ,$$  \hfill (2.17)

where

$$D = \frac{i}{2} \sum_{a < b} \theta^{\mu \nu} \partial_\mu \hat{o}^b_\nu .$$  \hfill (2.18)

Next, we introduce the space integral of a product of functions of $\hat{x}$ and assume that the integral is translationally invariant in $x$. Then we have

$$\int d^3 x f_1(\hat{x}) ... f_n(\hat{x}) = \int d^3 x e^{\hat{o} \partial_x} f_1(x) ... \star f_n(x)$$

$$= \int d^3 x f_1(x) ... \star f_n(x) + \int d^3 x \partial_\mu (f_1(x) ... \star f_n(x)) + ...$$

$$= \int d^3 x f_1(x) ... \star f_n(x) .$$  \hfill (2.19)

We realize that the final result is independent of the quantum shift operator $\hat{o}$.

The space-integral of Moyal products has various properties relevant to the formulation of NCFT.
1) The integral (2.19) is invariant under cyclic permutations of the \( n \) complex-valued functions

\[
\int d^3x f_1(x) \star ... \star f_n(x) = \int d^3x f_2(x) \star ... \star f_n(x) \star f_1(x).
\]  \hspace{1cm} (2.20)

2) In the Moyal product we can replace under the integral one of the \( \star \)-products by a usual (dot) product:

\[
\int d^3x f_1(x) \star ... \star f_n(x) = \int d^3x f_1(x) \cdot (f_2(x) \star ... \star f_n(x))
\]
\[
= \int d^3x (f_1(x) \star f_2(x)) \cdot (f_3(x) \star ... \star f_n(x))
\]
\[
= ... \hspace{1cm} (2.21)
\]

We skip the proofs since they follow directly from (2.17).

3  Noncommutative Field Theory

Let us assume that \( f(x) \cdot g(x) = h(x) \). Then we readily recognize the following mismatch:

\[
f(\hat{x})g(\hat{x}) = e^{i\hat{p} \cdot \hat{x}} f(x) \star g(x)
\]
\[
\ne^{i\hat{p} \cdot \hat{x}} f(x) \cdot g(x) = h(\hat{x}).
\]  \hspace{1cm} (3.22)

Thus, it is important to specify uniquely the factorization of a given function before introducing the quantum shift.

**Rule 1**

*In NCFT we decompose a given operator into a product of primitive factors or a linear combination of such products and make the replacement \( x \to \hat{x} \) in the primitive factors, thereby identifying incoming fields with primitive factors.*
In what follows we shall confine ourselves to a neutral scalar theory, and \( \phi(x) \) and \( \Phi(x) \) shall denote the incoming field and the Heisenberg field, respectively. Then we may write

\[
\phi(\hat{x}) = \phi(x + \hat{o}) = e^{\hat{o} \cdot \partial} \phi(x).
\]

(3.23)

For the Heisenberg field, however, eq. (3.23) is not valid and it should be modified as follows:

\[
\Phi(\hat{x}) = \Phi(x + \hat{o}) = e^{\hat{o} \cdot \partial} \Phi_{\theta}(x),
\]

(3.24)

in accordance with the Rule 1. As we shall see later, \( \Phi_{\theta}(x) \) denotes the NC field corresponding to \( \Phi \), and it satisfies a field equation obtained by modifying the one for the commutative field \( \Phi \).

4 Action principle

The Lagrangian density for the neutral scalar field with \( \Phi^4 \) interaction, in the conventional commutative field theory, is given by

\[
\mathcal{L} = -\frac{1}{2} \left( \partial^\mu \Phi \partial_\mu \Phi + m^2 \Phi^2 \right) - \frac{\lambda}{4!} \Phi^4.
\]

(4.25)

The action principle for commutative field theory is given by

\[
\delta \int d^4x \mathcal{L}(x) = 0.
\]

(4.26)

The corresponding action principle for the NCFT is given by

\[
\delta \int d^4x \mathcal{L}(\hat{x}) = 0,
\]

(4.27)
with the same postulated functional form of the Lagrangian density as in (4.26).

In deriving the field equation we assume that the field is not yet quantized and is treated as a complex function. Inside the Lagrangian density we make the replacement (3.24). Then, with the help of eqs. (2.20) and (2.21), we find

\[
\int d^4x \mathcal{L}(\hat{x}) = -\frac{1}{2} \int d^4x \left( \partial^\mu \Phi_\theta \partial_\mu \Phi_\theta + m^2 \Phi_\theta^2 \right) - \frac{\lambda}{4!} \int d^4x \Phi_\theta(x) \star \Phi_\theta(x) \star \Phi_\theta(x) \star \Phi_\theta(x).
\]

(4.28)

In taking the variation, the second integrand has a rather unfamiliar form, so that we handle it separately.

\[
\delta \int d^4x \Phi_\theta(x) \star \Phi_\theta(x) \star \Phi_\theta(x) \star \Phi_\theta(x) = \int d^4x \left\{ \delta \Phi_\theta(x) \star \Phi_\theta(x) \star \Phi_\theta(x) \star \Phi_\theta(x) \\
+ \Phi_\theta(x) \star \delta \Phi_\theta(x) \star \Phi_\theta(x) \star \Phi_\theta(x) + \ldots \right\} \\
= 4 \int d^4x \delta \Phi_\theta(x) \cdot (\Phi_\theta(x) \star \Phi_\theta(x) \star \Phi_\theta(x)) ,
\]

(4.29)

where use has been made of (2.20) and (2.21). Thus the field equation for \( \Phi_\theta \) turns out to be given by

\[
(\Box - m^2)\Phi_\theta(x) - \frac{\lambda}{3!} \Phi_\theta(x) \star \Phi_\theta(x) \star \Phi_\theta(x) = 0 .
\]

(4.30)

It is clear the \( \Phi_\theta \) satisfies the NC version of the neutral scalar theory.

5  The S-matrix

The S-matrix is a functional of the incoming field \( \phi(x) \), relating the \( |\text{in}\rangle \) and \( |\text{out}\rangle \) states

\[
|\text{out}\rangle = S|\text{in}\rangle
\]

(5.31)
and we have the NC version defined by

\[ S[\phi(\hat{x})] = S_\theta[\phi(x)] , \]  

(5.32)

where \( S_\theta \) denotes the S-matrix in the NCFT.

Dyson’s formula for the S-matrix in commutative field theory is:

\[ S[\phi(x)] = T \exp \left[ i \int_{-\infty}^{\infty} dt \int d^3x L^{\text{int}}(\vec{x}, t) \right] , \]  

(5.33)

where \( L^{\text{int}} \) denotes the interaction Lagrangian density in commutative field theory and is given by (4.25) as

\[ L^{\text{int}} = -\frac{\lambda}{4!} \phi^4(x) . \]  

(5.34)

Now we make the replacement \( x \to \hat{x} \) in (5.34) and integrate over the space coordinates

\[ \int d^3x L^{\text{int}}(\hat{x}) = -\frac{\lambda}{4!} \int d^3x \phi(x) \ast \phi(x) \ast \phi(x) \ast \phi(x) \]  

= \int d^3x L^{\text{int}}_\theta (x) . \]  

(5.35)

Hence

\[ S_\theta[\phi(x)] = S[\phi(\hat{x})] \]  

= \[ T \exp \left[ i \int_{-\infty}^{\infty} dt \int d^3x L^{\text{int}}_\theta(x, t) \right] . \]  

(5.36)

This S-matrix is obtained by replacing \( L^{\text{int}} \) by \( L^{\text{int}}_\theta \).

6 The Heisenberg Field

The Heisenberg field \( \Phi(x) \) in commutative field theory can be expanded in powers of the coupling constant as

\[ \Phi(x) = \phi(x) + i \int_{-\infty}^{t} dt' \left[ \phi(x), \int d^3x' L^{\text{int}}(\vec{x}', t') \right] \]
In NCFT we replace \( x \) by \( \hat{x} \) and obtain
\[
e^{\hat{\partial} \cdot \hat{\partial}} \Phi_\theta(x) = e^{\hat{\partial} \cdot \hat{\partial}} \{ \phi(x) + i \int_{-\infty}^{t} dt' \left[ \phi(x), \int d^3x' \mathcal{L}_{\text{int}}(\vec{x}', t') \right] \}
+ i^2 \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \left[ \phi(x), \int d^3x' \mathcal{L}_{\text{int}}(\vec{x}', t') \right] \int d^3x'' \mathcal{L}_{\text{int}}(\vec{x}'', t'') \]
+ \ldots \tag{6.38}
\]

Apparently \( \Phi_\theta(x) \) denotes the Heisenberg field in the NCFT characterized by the interaction Lagrangian density \( \mathcal{L}_{\text{int}} \).

## 7 Violation of Lorentz Invariance

NC QFT violates Lorentz invariance, however, it possesses twisted Poincaré invariance [6]. Consequently the individual elementary fields are in the representations of the usual Poincaré group, i.e., in the case of the scalar field:
\[
[\phi(x), P_\mu] = \mathcal{P}_\mu \phi(x) , \quad \mathcal{P}_\mu = \frac{1}{i} \frac{\partial}{\partial x^\mu} , \]
\[
[\phi(x), M_{\mu\nu}] = \mathcal{M}_{\mu\nu} \phi(x) , \quad \mathcal{M}_{\mu\nu} = \frac{1}{i} \left( x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} \right) . \tag{7.39}
\]

In commutative field theory, the invariance of the \( S \)-matrix under the generators of the Lorentz group is expressed through the commutator:
\[
[S, M_{\mu\nu}] = 0 . \tag{7.40}
\]

In the noncommutative case,
\[
[S_\theta, M_{\mu\nu}] \neq 0 \tag{7.41}
\]
where $S_\theta$ is given by (5.36), i.e.

$$S_\theta = T \exp \left[ i \int d^4x \mathcal{L}_\theta^{\text{int}}(x) \right].$$

The nonvanishing expression which represents the commutator (7.41) gives the amount of the violation of Lorentz invariance in NC QFT and can be found as follows.

Explicitly, the commutator (7.41) can be written as:

$$[S_\theta, M_{\mu\nu}] = T \left[ i \int d^4y \mathcal{L}_\theta^{\text{int}}(y), M_{\mu\nu} \right] \exp \left( i \int d^4x \mathcal{L}_\theta^{\text{int}}(x) \right), \quad (7.42)$$

One can chose for $\mathcal{L}_\theta^{\text{int}}(x)$ any $n$-linear form:

$$\mathcal{L}_\theta^{\text{int}}(x) = \sum_{i_1...i_n} f_{i_1...i_n} \phi_{i_1}^1(x) \star ... \star \phi_{i_n}^n(x)$$

$$= e^D \sum_{i_1...i_n} f_{i_1...i_n} \phi_{i_1}^1(x_1)...\phi_{i_n}^n(x_n)|_{x_1=...=x_n=x}$$

$$= e^D \mathcal{L}(x_1,...,x_n)|_{x_1=...=x_n=x}, \quad (7.43)$$

with $i, j = 1, ..., n$ standing for spinorial or tensorial indices and the coefficients $f_{i_1...i_n}$ chosen such as to make the combination

$$\mathcal{L}(x_1,...,x_n) = \sum_{i_1...i_n} f_{i_1...i_n} \phi_{i_1}^1(x_1)...\phi_{i_n}^n(x_n) \quad (7.44)$$

in the local limit ($x_1 = ... = x_n = x$), a Lorentz scalar.

However, for the simplification of the argument, we shall take as an illustration the NC $\lambda \phi^3$-theory, where

$$\mathcal{L}(x_1, x_2, x_3) = \frac{\lambda}{3!} \phi(x_1)\phi(x_2)\phi(x_3) \quad (7.45)$$

and, according to (2.18)

$$D = \frac{i}{2} \theta^{\alpha\beta} \left[ \partial_\alpha \partial_\beta + \partial_\beta \partial_\alpha - \partial_\alpha \partial_\beta \right]. \quad (7.46)$$
The action of the Lorentz generator on the composite object $\mathcal{L}(x_1, x_2, x_3)$ makes use of the fact that the component fields $\phi(x_i), i = 1, 2, 3$ are scalar representations of the Lorentz group (see eq. (7.39)):

$$
\mathcal{L}(x_1, x_2, x_3), M_{\mu\nu} = (M_{\mu\nu}^x \phi(x_1)) \phi(x_2) \phi(x_3) + \phi(x_1) (M_{\mu\nu}^x \phi(x_2)) \phi(x_3) + \phi(x_1) \phi(x_2) (M_{\mu\nu}^x \phi(x_3)) \equiv M_{\mu\nu}^{x_1 x_2 x_3} \mathcal{L}(x_1, x_2, x_3).
$$

(7.47)

Then we can compute the commutator of $\mathcal{L}^\text{int}(y)$ with the Lorentz generator $M_{\mu\nu}$:

$$
[\mathcal{L}^\text{int}(y), M_{\mu\nu}] = e^D [\mathcal{L}(y_1, y_2, y_3), M_{\mu\nu}]|_{y_1 = y_2 = y_3 = y} = e^D M_{\mu\nu}^{y_1 y_2 y_3} \mathcal{L}(y_1, y_2, y_3)|_{y_1 = y_2 = y_3 = y} = (M_{\mu\nu}^{y_1 y_2 y_3} + [D, M_{\mu\nu}^{y_1 y_2 y_3}]) e^D \mathcal{L}(y_1, y_2, y_3)|_{y_1 = y_2 = y_3 = y}

= M_{\mu\nu}^y \mathcal{L}^\text{int}(y) + [D, M_{\mu\nu}^{y_1 y_2 y_3}] e^D \mathcal{L}(y_1, y_2, y_3)|_{y_1 = y_2 = y_3 = y}
$$

(7.48)

In the expression of the commutator

$$
[D, M_{\mu\nu}^{y_1 y_2 y_3}] = \frac{1}{2} \left[ \sum_{1 \leq a < b \leq 3} \theta^{\alpha\beta} \partial^a_{\alpha} \partial^b_{\beta}, \sum_{k=1}^3 (y^k_{\nu} \partial^k_{\nu} - y^k_{\mu} \partial^k_{\mu}) \right]
$$

(7.49)

one can view the action of $M_{\mu\nu}^{y_1 y_2 y_3}$ on $D$ as changing $\theta_{\nu\beta} \rightarrow \theta_{\mu\beta}$ (first term), $\theta_{\mu\beta} \rightarrow -\theta_{\nu\beta}$ (second term), $\theta_{\alpha\nu} \rightarrow \theta_{\alpha\mu}$ (third term) and $\theta_{\alpha\mu} \rightarrow -\theta_{\alpha\nu}$ (last term). This amounts to introducing a “auxiliary Lorentz generator”, which transforms properly $\theta_{\alpha\beta}$ as a tensor:

$$
[D, M_{\mu\nu}^{y_1 y_2 y_3}] =: M_{\mu\nu}^\theta D,
$$

(7.50)
with

\[
[M^\theta_{\mu\nu}, \theta_{\alpha\beta}] = \frac{1}{i} (\eta_{\mu\alpha} \theta_{\nu\beta} - \eta_{\mu\beta} \theta_{\nu\alpha} - \eta_{\nu\alpha} \theta_{\mu\beta} + \eta_{\nu\beta} \theta_{\mu\alpha}) . \]  

(7.51)

In order to find a representation of \( M^\theta_{\mu\nu} \) (without the necessity of imposing the antisymmetry constraint on the elements \( \theta_{\alpha\beta} \)), we define the matrix \( \sigma_{\alpha\beta} \) with totally independent components by

\[
\theta_{\alpha\beta} \equiv \sigma_{\alpha\beta} - \sigma_{\beta\alpha} , \]  

(7.52)

and re-express the above changes in terms of the \( \sigma \)-matrix:

\[
\begin{align*}
\sigma_{\nu\beta} & \rightarrow \sigma_{\mu\beta} , \quad \sigma_{\mu\beta} \rightarrow -\sigma_{\nu\beta} \\
\sigma_{\alpha\nu} & \rightarrow \sigma_{\alpha\mu} , \quad \sigma_{\alpha\mu} \rightarrow -\sigma_{\alpha\nu} .
\end{align*} \]

(7.53)

Thus, we can represent \( M^\theta_{\mu\nu} \) as:

\[
M^\theta_{\mu\nu} = -\frac{1}{i} \left( \sigma_{\nu\beta} \frac{\partial}{\partial \sigma_{\mu\beta}} - \sigma_{\mu\beta} \frac{\partial}{\partial \sigma_{\nu\beta}} + \sigma_{\alpha\nu} \frac{\partial}{\partial \sigma_{\alpha\mu}} - \sigma_{\alpha\mu} \frac{\partial}{\partial \sigma_{\alpha\nu}} \right) . \]  

(7.54)

Hence (7.42) yields

\[
[S_\theta, M_{\mu\nu}] = T \left[ i \int d^4 y (M^y_{\mu\nu} + M^\theta_{\mu\nu} D) \mathcal{L}^{\text{int}}_\theta(y) \exp \left( i \int d^4 x \mathcal{L}^{\text{int}}_\theta(x) \right) \right] . \]  

(7.55)

Since

\[
T \left[ i \int d^4 y M^y_{\mu\nu} \mathcal{L}^{\text{int}}_\theta(y) \exp \left( i \int d^4 x \mathcal{L}^{\text{int}}_\theta(x) \right) \right] \\
= i \int d^4 y M^y_{\mu\nu} T \left[ \mathcal{L}^{\text{int}}_\theta(y) \exp \left( i \int d^4 x \mathcal{L}^{\text{int}}_\theta(x) \right) \right] = 0 \]  

(7.56)

and

\[
M^\theta_{\mu\nu} \mathcal{L}^{\text{int}}_\theta(y) = M^\theta_{\mu\nu} e^D \mathcal{L}(y_1, y_2, y_3) \big|_{y_1 = y_2 = y_3 = y} \\
= (M^\theta_{\mu\nu} D) e^D \mathcal{L}(y_1, y_2, y_3) \big|_{y_1 = y_2 = y_3 = y} \]  

13
\[ = (\mathcal{M}_\mu^\theta \, D) \mathcal{L}_\theta^{\text{int}}(y), \quad (7.57) \]

we conclude, by putting together (7.55) with (7.56) and (7.57) and using (7.50), that

\[ [S_\theta, M_{\mu\nu}] = \mathcal{M}_\mu^\theta S_\theta. \quad (7.58) \]

The expression in the r.h.s. of (7.58) represents in general the amount of Lorentz-invariance violation of the $S$-matrix in the case of NC QFT.

8 Conclusions

We may conclude that the NCFT is obtained from the corresponding commutative field theory by a simple mapping resulting from the replacement $x \rightarrow \hat{x} = x + \hat{\theta}$, which amounts to a modification of $\mathcal{L}$ into $\mathcal{L}_\theta$. We should also emphasize the fact that observable quantities depend only on $\theta$ but never on $\hat{\theta}$ explicitly.

The violation of the Lorentz invariance in the action of NCFT has been also found in general, in terms of a newly introduced operator, or "auxiliary Lorentz generator", $\mathcal{M}_\mu^\theta$ (eq. (7.54)), whose role is to transform $\theta_{\alpha\beta}$ as a Lorentz tensor. Therefore, as expected, the theory would be Lorentz invariant, had $\theta_{\alpha\beta}$ transformed properly under Lorentz transformations. Though the expression of the violation was derived in the particular case of the NC $\lambda \phi^3$ theory, it is valid in general, since $\mathcal{M}_\mu^\theta$ acts solely on the $\theta$-variable, i.e. on the $\star$-product, disregarding the actual fields involved in the interaction, as long as the in-fields are in the representations of the Poincaré algebra, which are identical to the ones of the twisted Poincaré.
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