ON CATEGORY OF LIE ALGEBRAS

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Abstract. In this paper we describe the the category of Lie algebras of group algebras and the category of Plesken Lie algebras and explore the categorical relations between them. Further we provide the examples of the Lie algebra of the group algebra of subgroups of Heisenburg group and the Plesken Lie algebra of subgroups of Heisenburg group.

1. INTRODUCTION

The concept of Lie algebras was introduced by Sophus Lie to solve problems in Lie groups with some ease. Here we introduce some class of Lie algebras like Lie algebras of group algebras and Plesken Lie algebras of groups. In recent times category theory established itself as a practical tool in dealing with mathematical structures. In particular a categorical approach enables to extract more insight into the interactions between Lie groups and Lie algebras. Here we discuss the category of some class of Lie algebras, functorial relation that exists between them with some examples.

2. PRELIMINARIES

In the following we briefly recall all basic definitions and the elementary concepts needed in the sequel. In particular we recall the definitions of Lie algebra, group algebra, Plesken Lie algebra, categories, functors and discusses some interesting properties of these structures.

Definition 1. (cf. [3]) A category \( \mathcal{C} \) consists of the following data:

1. A class called the class of vertices or objects \( \nu \mathcal{C} \).
2. A class of disjoint sets \( \mathcal{C}(a, b) \) one for each pair \( (a, b) \in \nu \mathcal{C} \times \nu \mathcal{C} \).

An element \( f \in \mathcal{C} \) is called a morphism from \( a \) to \( b \), written \( f : a \to b \); \( a = \text{dom} \ f \) called the domain of \( f \) and \( b = \text{cod} \ f \) called the codomain of \( f \).

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(3) For \(a, b, c, \in \nu C\), a map
\[
\circ : C(a, b) \times C(b, c) \rightarrow C(a, c)
\]
\((f, g) \rightarrow g \circ f\)
o is called the composition of morphisms in \(C\).

(4) for each \(a \in \nu C\), a unique \(1_a \in C(a, a)\) is called the identity morphism on \(a\).

These must satisfy the following axioms:

- \((cat1)\) The composition is associative: for \(f \in C(a, b), g \in C(b, c)\) and \(h \in C(c, d)\), we have
  \[f \circ (g \circ h) = (f \circ g) \circ h\]

- \((cat 2)\) for each \(a \in \nu C\), \(f \in C(a, b)\) and \(g \in C(c, a)\),
  \[1_a \circ f = f \quad \text{and} \quad g \circ 1_a = g\]

Clearly \(\nu C\) can be identify as a subclass of \(C\) and with this identification it is possible to regard categories in terms of morphisms alone. The category \(C\) is said to be small if the class \(C\) is a set. A morphism \(f \in C(a, b)\) is said to be an isomorphism if there exists \(f^{-1} \in C(b, a)\) such that \(ff^{-1} = 1_a = e_a\), domain identity and \(f^{-1}f = 1_b = f_b\), range identity.

**Example 1.** A group \(G\) can be regarded as a category \(C\) with the object set of \(C\) say \(\nu C = G\), and morphisms \(C(G, G) = G\) and composition in \(C\) is the binary operation in \(G\). Identity element in the group will be the identity morphism on the vertex \(G\).

**Definition 2.** [1] For categories \(C\) and \(D\) a functor \(T : C \rightarrow D\) with domain \(C\) and codomain \(D\) consists of two functions: the object function \(Tc\), which assigns to each object \(c\) of \(C\) an object \(Tc\) of \(D\) and the arrow function which assigns to each arrow \(f : c \rightarrow c'\) of \(C\) an arrow \(Tf : Tc \rightarrow Tc'\) of \(D\), in such a way that \(T(1_c) = 1_{Tc}\) and \((Tg \circ f) = T(g \circ f)\).

**Definition 3.** [3] A vector space \(L\) over a field \(\mathbb{F}\), with an operation \(L \times L \rightarrow L\), denoted by \((x, y) \mapsto [x, y]\) for \(x\) and \(y\) in \(L\) and satisfying the following axioms:

1. The bracket operation is bilinear. For \(x, y, z \in L\), \(a, b \in \mathbb{F}\)
   \[
   [ax + by, z] = a[x, z] + b[y, z] \quad [x, ay + bz] = a[x, y] + b[x, z]
   \]

2. \([x, x] = 0\) for all \(x \in L\)

3. Jacobi identity:
   \[
   [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{for all} \ x, y, z \in L
   \]
is called a bracket product and \((L, [], .)\) is called a Lie algebra over \(\mathbb{F}\).

**Example 2.** \(\text{End}(V)\), the set of all linear transformations on a finite dimensional vector space \(V\) over a field \(\mathbb{F}\) is a Lie algebra with Lie bracket \([x, y] = xy - yx\) for \(x, y \in \text{End}(V)\).

A subspace \(K\) of a Lie algebra \(L\) is called a Lie subalgebra if \([x, y] \in K\) whenever \(x, y \in K\).

**Definition 4.** Let \(G\) be a group and let \(\mathbb{F}\) be \(\mathbb{R}\) or \(\mathbb{C}\). Define a vector space over \(\mathbb{F}\) with elements of \(G\) as a basis, and denote it by \(\mathbb{F}G\). That is; \(\mathbb{F}G = \{ \sum a_i g_i : a_i \in \mathbb{F} \text{ for all } i \}\). The addition and scalar multiplication in \(\mathbb{F}G\) are defined by; for \(\alpha = \sum a_i g_i\) and \(\beta = \sum b_i g_i\) in \(\mathbb{F}G\) and \(k \in \mathbb{F}\),

\[
\alpha + \beta = \sum (a_i + b_i) g_i \quad \text{and} \quad k\alpha = \sum (ka_i) g_i
\]

The vector space \(\mathbb{F}G\), with multiplication defined by

\[
\left( \sum_i a_i g_i \right) \left( \sum_j b_j g_j \right) = \sum_{i,j} a_i b_j (g_i g_j), \quad a_i, b_j \in \mathbb{F}
\]

is called the group algebra of \(G\) over \(\mathbb{F}\).

The group algebra of a finite group \(G\) is a vector space of dimension \(|G|\) which also carries extra structure involving the the product operation on \(G\).

**Example 3.** \(G = C_3 = \langle a : a^3 = e \rangle\) where \(e\) is the identity in \(G\). Then

\[
\mathbb{C}G = \{ \lambda_1 e + \lambda_2 a + \lambda_3 a^2 : \lambda_i \in \mathbb{C} \text{ for } i = 1, 2, 3 \}
\]

\(\mathbb{C}G\) is a group algebra with usual addition and multiplication of series.

**3. Lie algebras of group algebras and Plesken Lie algebras**

Let \(\mathbb{F}G\) be a group algebra over \(\mathbb{F}\). Then \(\mathbb{F}G\) can be regarded as a Lie algebra by defining the Lie bracket \([, ,] : \mathbb{F}G \times \mathbb{F}G \to \mathbb{F}G\) as follows:

\[
[\alpha, \beta] = \alpha \beta - \beta \alpha \quad \text{for } \alpha = \sum a_i g_i \text{ and } \beta = \sum b_i g_i \text{ in } \mathbb{F}G
\]

Clearly \(\mathbb{F}G\) is a Lie algebra with respect to the given Lie bracket and is called the Lie algebra of the group algebra \(\mathbb{F}G\) and we denote it by \(L_{\mathbb{F}G}\).
A linear map between $L_{FG}$ and $L_{FH}$ which preserves the Lie bracket is a homomorphism of Lie algebras of group algebras.

**Proposition 1.** Let $f : G \to H$ be a group homomorphism. Then $\bar{f} : L_{FG} \to L_{FH}$ defined by

$$
\bar{f}(\alpha) = \bar{f}(\sum_i a_i g_i) = \sum_i a_i f(g_i)
$$

is a homomorphism between Lie algebras of group algebras.

**Proof.** Let $G = \{g_1, g_2, g_3, \ldots \}$ and $L_{FG} = \{\sum_i a_i g_i : a_i \in \mathbb{F} \text{ for all } i\}$.

Then for $\alpha = \sum_i a_i g_i, \beta = \sum_j b_j g_j \in L_{FG}$,

$$
\bar{f}([\alpha, \beta]) = \bar{f}(\alpha \beta - \beta \alpha)
$$

$$
= \bar{f}(\sum_i a_i g_i \sum_j b_j g_j - \sum_j b_j g_j \sum_i a_i g_i)
$$

$$
= \bar{f}(\sum_{i,j} a_i b_j g_i g_j - \sum_{j,i} b_j a_i g_j g_i)
$$

$$
= \sum_{i,j} a_i b_j f(g_i g_j) - \sum_{j,i} b_j a_i f(g_j g_i)
$$

$$
= \sum_i a_i f(g_i) \sum_j b_j f(g_j) - \sum_j b_j f(g_j) \sum_i a_i f(g_i)
$$

$$
= \bar{f}(\alpha) \bar{f}(\beta)
$$

Hence, $\bar{f}$ is a homomorphism between Lie algebras of group algebras. 

Next we proceed to describe Plesken Lie algebra. Let $G$ be a group and $\mathbb{F}G$ be its group algebra over $\mathbb{F}$, then for each $g \in G$, $g - g^{-1} \in \mathbb{F}G$, denote it by $\hat{g}$, then the linear span of $\hat{g}$ admits a Lie algebra structure as explained below.

**Definition 5.** [2] Plesken Lie algebra $\mathcal{L}(G)$ of a group $G$ over $\mathbb{F}$ is the linear span of elements $\hat{g} \in \mathbb{F}G$ together with the Lie bracket

$$
[\hat{g}, \hat{h}] = \hat{gh} - \hat{hg}
$$

That is, for any group $G = \{g_1, g_2, g_3, \ldots \}; \{\sum_i a_i \hat{g}_i : a_i \in \mathbb{F} \text{ for all } i\}$ together with the Lie bracket defined above is the Plesken Lie algebra $\mathcal{L}(G)$. 

Lemma 1. The Plesken Lie algebra $L(G)$ over $\mathbb{F}$ is a Lie subalgebra of the Lie algebra $L_{\mathbb{F}G}$.

Proof. Let $G = \{g_1, g_2, g_3, \ldots\}$ be a group. Then the Lie algebra of the group algebra $\mathbb{F}G$ is $L_{\mathbb{F}G} = \{ \sum a_i g_i : a_i \in \mathbb{F} \text{ for all } i \}$ and the Plesken Lie algebra is $L(G) = \{ \sum a_i \hat{g}_i : a_i \in \mathbb{F} \text{ for all } i \}$. Since $L(G)$ is the linear span of $\hat{g}$ and $\hat{g} \in L_{\mathbb{F}G}$, $L(G)$ is a subset of $L_{\mathbb{F}G}$.

Let $\hat{\alpha} = \sum a_i \hat{g}_i$ and $\hat{\beta} = \sum b_i \hat{g}_i$ in $L(G)$,

$$\hat{\alpha} + \hat{\beta} = \sum a_i \hat{g}_i + \sum b_i \hat{g}_i = \sum (a_i + b_i) \hat{g}_i \in L(G)$$

and $k(\hat{\alpha}) = k \sum a_i \hat{g}_i = \sum (ka_i) \hat{g}_i \in L(G)$.

Thus, $L(G)$ is a subspace of $L_{\mathbb{F}G}$.

Let $\hat{g}, \hat{h} \in L(G)$, then

$$[\hat{g}, \hat{h}] = \hat{gh} - \hat{hg} = (g - g^{-1})(h - h^{-1}) - (h - h^{-1})(g - g^{-1})$$

$$= \hat{gh} - \hat{gh}^{-1} - \hat{g}^{-1}h + \hat{g}^{-1}h^{-1}$$

Thus Lie bracket is closed in $L(G)$. Hence, $L(G)$ is a Lie subalgebra of $L_{\mathbb{F}G}$. \qed

Example 4. Consider the symmetric group $S_3$, then

$L(S_3) = \text{span}\{\sigma - \sigma^{-1} : \sigma \in S_3\}$

$$= \{a_1((1) - (1)) + a_2((1 2) - (1 2)) + a_3((1 3) - (1 3)) + a_4((2 3) - (2 3))$$

$$+ a_5((1 2 3) - (1 3 2)) + a_6((1 3 2) - (1 2 3)) : a_i \in \mathbb{C}\}$

$$= \{a((1 2 3) - (1 3 2)) : a \in \mathbb{C}\}$

is a one dimensional Plesken Lie algebra over $\mathbb{C}$ with Lie bracket

$$[a((1 2 3)), b((1 2 3))] = 0$$

A linear map between two Plesken Lie algebras $L(G)$ and $L(H)$ is a Plesken Lie algebra homomorphism if it preserves the Lie bracket.

Proposition 2. Let $f : G \to H$ be a group homomorphism. Then $\hat{f} : L(G) \to L(H)$ defined by
\[ \hat{f}(\sum_i a_i \hat{g}_i) = \sum_i a_i \hat{f}(g_i) \]

is a Plesken Lie algebra homomorphism.

**Proof.** For \( \sum_i a_i \hat{g}_i, \sum_i b_i \hat{g}_i \in \mathcal{L}(G) \) and \( k \in \mathbb{F} \),

\[
\hat{f}(\sum_i a_i \hat{g}_i + k \sum_i b_i \hat{g}_i) = \hat{f}(\sum_i (a_i + kb_i) \hat{g}_i)
= \sum_i (a_i + kb_i) \hat{f}(g_i)
= \hat{f}(\sum_i a_i \hat{g}_i) + k \hat{f}(\sum_i b_i \hat{g}_i)
\]

thus \( \hat{f} \) is linear.

For \( \sum_i a_i \hat{g}_i, \sum_j b_j \hat{g}_j \in \mathcal{L}(G) \),

\[
\hat{f}(\sum_i a_i \hat{g}_i, \sum_j b_j \hat{g}_j) = \hat{f}(\sum_{i,j} (a_i b_j - b_i a_j) \hat{g}_i \hat{g}_j - \sum_{i,j} (a_i b_j - b_i a_j) g_i g_j^{-1})
- \sum_{i,j} (a_i b_j - b_i a_j) \hat{f}(g_i) \hat{f}(g_j) - \sum_{i,j} (a_i b_j - b_i a_j) \hat{f}(g_i) \hat{f}(g_j^{-1})
- \sum_{i,j} (a_i b_j - b_i a_j) \hat{f}(g_i^{-1}) \hat{f}(g_j) + \sum_{i,j} (a_i b_j - b_i a_j) \hat{f}(g_i^{-1}) \hat{f}(g_j^{-1})
= \sum_i a_i \hat{f}(g_i), \sum_j a_j \hat{f}(g_j)
\]

Hence \( \hat{f} \) preserves Lie bracket and is a Plesken Lie algebra homomorphism. \( \square \)

4. CATEGORY OF LIE ALGEBRAS

Here we describe the category of Lie algebras over \( \mathbb{F} \) whose objects are Lie algebras over \( \mathbb{F} \) and morphisms Lie algebra homomorphisms. Let \( L \) and \( L' \) be two Lie algebras over a field \( \mathbb{F} \). If \( f \) and \( g \) are two morphisms, then \( f \circ g \) exists only when \( f \in \text{hom}(L, L') \) and \( g \in \text{hom}(L', L'') \). Also the identity in \( \text{hom}(L, L) \) is the morphism \( 1_L \).

**Example 5.** Consider a group \( G \) and all its subgroups \( H_i \). The group algebras \( \mathbb{F}H_i \) of each subgroup \( H_i \) of \( G \) together with a Lie bracket
\[ [x,y] = xy - yx \text{ for } x = \sum_i a_i h_i, y = \sum_j b_j h_j \in F H_i \text{ is the Lie algebra } L_{FH}. \] The collection of all such Lie algebras of group algebras of subgroups of \( G \) form the category \( L_{FG} \) whose morphisms are \( \{ \bar{f}_{ij} : L_{FH_i} \to L_{FH_j} | \bar{f}(\sum_i a_i g_i) = \sum_i a_i f(g_i) \text{ where } f : H_i \to H_j \text{ is the group homomorphism } \} \).

\textbf{Example 6.} Consider the Heisenberg group \( H(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \) and its non-isomorphic subgroups:

\[ H_1 = \{ I_{3\times3}, \text{ the identity matrix } \}, H_2 = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\} \]

\[ H_3 = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : b, c \in \mathbb{R} \right\}, H_4 = H(\mathbb{R}) \]

The group algebras \( F H_i \) over \( F \) of each subgroups \( H_i \) of \( H(\mathbb{R}) \) together with a Lie bracket \( [X,Y] = XY - YX \) where \( X = \sum_i \lambda_i A_i, Y = \sum_i \mu_i B_i \in F H_i \) is the Lie algebra \( L_{FH_i} \) of the group algebra \( F H_i \). Consider \( L_{FH(\mathbb{R})} \) whose objects are

\[ L_{FH_1} = \{ \lambda I : \lambda \in F \} = \text{ the Lie algebra of scalar matrices} \]

\[ L_{FH_2} = \{ \sum_i \lambda_i A_i : A_i \in H_2, \lambda_i \in F \} = \left\{ \begin{pmatrix} \sum_i \lambda_i & 0 & \sum_i \lambda_i b_i \\ 0 & \sum_i \lambda_i & 0 \\ 0 & 0 & \sum_i \lambda_i \end{pmatrix} \right\} \]

\[ L_{FH_3} = \{ \sum_i \lambda_i A_i : A_i \in H_3, \lambda_i \in F \} = \left\{ \begin{pmatrix} \sum_i \lambda_i & 0 & \sum_i \lambda_i b_i \\ 0 & \sum_i \lambda_i & \sum_i \lambda_i c_i \\ 0 & 0 & \sum_i \lambda_i \end{pmatrix} \right\} \]
\[
L_{\mathbb{F}H_4} = \{ \sum_i \lambda_i A_i : A_i \in H_4, \lambda_i \in \mathbb{F} \} = \left\{ \begin{pmatrix} \sum_i \lambda_i & \sum_i \lambda_i a_i & \sum_i \lambda_i b_i \\ 0 & \sum_i \lambda_i & \sum_i \lambda_i c_i \\ 0 & 0 & \sum_i \lambda_i \end{pmatrix} \right\}
\]

and morphisms: \( \text{hom}(L_{\mathbb{F}H_i}, L_{\mathbb{F}H_j}) = \{ f_{ij} : L_{\mathbb{F}H_i} \rightarrow L_{\mathbb{F}H_j} | \bar{f}(\sum_i a_i g_i) = \sum_i a_i f(g_i) \text{ where } f : H_i \rightarrow H_j \text{ is the group homomorphism} \} \) is the category \( L_{\mathbb{F}(H(\mathbb{R}))} \) of Lie algebras of the group algebras of subgroups of \( H(\mathbb{R}) \).

Next we consider the category whose objects are Plesken Lie algebras \( L(G) \) over a field \( \mathbb{F} \) and morphisms are Plesken Lie algebra homomorphisms which we denote it by \( C_{P(LG)} \).

**Example 7.** Consider the Heisenberg group \( H(\mathbb{R}) = \{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \} \) and its non-isomorphic subgroups \( H_1, H_2, H_3 \) and \( H_4 \) as given in Example.6. For each \( A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in H_i, \)

\[
\hat{A} = A - A^{-1} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2a & 2b - ac \\ 0 & 0 & 2c \\ 0 & 0 & 0 \end{pmatrix}
\]

Thus \( L(H_i) = \{ \sum_i \lambda_i \hat{A}_i : A_i \in H_i, \lambda_i \in \mathbb{F} \} \) and so

\[
L(H_1) = \{ 0 \}, \quad L(H_2) = \left\{ \begin{pmatrix} 0 & 0 & 2 \sum_i \lambda_i b_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}
\]

\[
L(H_3) = \left\{ \begin{pmatrix} 0 & 0 & 2 \sum_i \lambda_i b_i \\ 0 & 0 & 2 \sum_i \lambda_i c_i \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad L(H_4) = \left\{ \begin{pmatrix} 0 & 2 \sum_i \lambda_i a_i & \sum_i \lambda_i (2b_i - a_i c_i) \\ 0 & 0 & 2 \sum_i \lambda_i c_i \\ 0 & 0 & 0 \end{pmatrix} \right\}
\]

The category whose objects are \( L(H_i), i = 1, 2, 3, 4 \) and morphisms are \( \text{hom}(L(H_i), L(H_j)) = \{ \bar{f} : L(H_i) \rightarrow L(H_j) | \bar{f}(\sum_i \lambda_i \hat{A}_i) = \sum_i \lambda_i \bar{f}(A_i) \} \).
where \( f : H_i \to H_j \) is the group homomorphism \( \) is the category of Plesken Lie algebras of subgroups of \( H(\mathbb{R}) \).

**Theorem 1.** If \( L_{FG} \) is the category of Lie algebras of group algebras and \( C_{PLG} \) is the category of Plesken Lie algebras, then there exists a functor from \( L_{FG} \) to \( C_{PLG} \).

Proof. Define \( T : L_{FG} \to C_{PLG} \) such that \( \nu T : L_{FG} \to L(G_i) \) is defined by

\[
\nu T(\sum_i a_i g_i) = \sum_i (a_i - a'_i)(g_i - g_i^{-1})
\]

and on morphisms, for a group homomorphism \( f : G_i \to G_j, \bar{f} : L_{FG_i} \to L_{FG_j} \) is given by

\[
\bar{f}(\sum_i a_i g_i) = \sum_i a_i f(g_i)
\]

and \( T\bar{f} : T(L_{FG_i}) \to T(L_{FG_j}) \) is defined by

\[
T\bar{f} = \hat{f}
\]

and \( \hat{f} : L(G_i) \to L(G_j) \) is given by

\[
\hat{f}(\sum_i (a_i - a'_i)(g_i - g_i^{-1})) = \sum_i (a_i - a'_i)(f(g_i) - f(g_i)^{-1})
\]

where \( \hat{f} \) is a Plesken Lie algebra homomorphism (Proposition 2).

Let \( \bar{f}_1 : L_{FG_i} \to L_{FG_j} \) and \( \bar{f}_2 : L_{FG_j} \to L_{FG_k} \) be two homomorphisms between Lie algebras of group algebras which are induced from the group homomorphisms \( f_1 : G_i \to G_j \) and \( f_2 : G_j \to G_k \). Then \( T(\bar{f}_1) : T(L_{FG_i}) \to T(L_{FG_j}) \) and \( T(\bar{f}_2) : T(L_{FG_j}) \to T(L_{FG_k}) \) are Plesken Lie algebra homomorphisms. Then their composition \( T(\bar{f}_2) \circ T(\bar{f}_1) : T(L_{FG_i}) \to T(L_{FG_k}) \) is also a Plesken Lie algebra homomorphism. Moreover,

\[
(T(\bar{f}_2) \circ T(\bar{f}_1))(\sum_i (a_i - a'_i)(g_i - g_i^{-1})) = (f_2 \circ \hat{f}_1)(\sum_i (a_i - a'_i)(g_i - g_i^{-1}))
\]

\[
= \sum_i (a_i - a'_i)(f_2(\hat{f}_1(g_i) - \hat{f}(g_i^{-1})))
\]

\[
(1)
\]

and

\[
T(\bar{f}_2 \circ \bar{f}_1)(\sum_i (a_i - a'_i)(g_i - g_i^{-1})) = (f_2 \circ \hat{f}_1)(\sum_i (a_i - a'_i)(g_i - g_i^{-1}))
\]

\[
= \sum_i (a_i - a'_i)((f_2 \circ \hat{f}_1)(g_i) - ((f_2 \circ \hat{f}_1)(g_i))^{-1})
\]

\[
(2)
\]
(1) and (2) shows that \( T(\bar{f}_2 \circ \bar{f}_1) = T(\bar{f}_2) \circ T(\bar{f}_1) \).

The identity Lie algebra homomorphism, \( 1_{L_{FG_i}} : L_{FG_i} \rightarrow L_{FG_i} \), is induced from the identity group homomorphism \( 1_{G_i} : G_i \rightarrow G_i \). Then \( T(1_{L_{FG_i}}) : T(L_{FG_i}) \rightarrow T(L_{FG_i}) \) is a Plesken Lie algebra homomorphism and

\[
T(1_{L_{FG_i}})(\sum_i (a_i - a'_i)(g_i - g_i^{-1})) = \sum_i (a_i - a'_i)(1_{G_i}(g_i) - (1_{G_i}(g_i))^{-1})
\]

\[
= \sum_i (a_i - a'_i)(g_i - g_i^{-1})
\]

\[
= 1_{T(L_{FG_i})}(\sum_i (a_i - a'_i)(g_i - g_i^{-1}))
\]

That is, \( T(1_{L_{FG_i}}) = 1_{T(L_{FG_i})} \), hence \( T \) is a functor.

\[\square\]

**Corollary 1.** The functor \( T : L_{FG} \rightarrow C_{PLG} \) in Theorem 1 is a full functor.

**Proof.** For the functor \( T : L_{FG} \rightarrow C_{PLG} \) define a map \( T_{L_{FG_i},L_{FG_j}} : \hom(L_{FG_i}, L_{FG_j}) \rightarrow \hom(TL_{FG_i}, TL_{FG_j}) \) by

\[
T_{L_{FG_i},L_{FG_j}}(\bar{f}) = \hat{f}
\]

Now it is enough to prove that \( T_{L_{FG_i},L_{FG_j}} \) is surjective. For \( \hat{f} \in \hom(TL_{FG_i}, TL_{FG_j}) \), a Plesken Lie algebra homomorphism, define \( \bar{f} : L_{FG_i} \rightarrow L_{FG_j} \) by

\[
\bar{f}(\sum_i a_i g_i) = \sum_i a_i f(g_i)
\]

is a homomorphism between Lie algebras of group algebras and \( T\bar{f} : TL_{FG_i} \rightarrow TL_{FG_j} \) given by,

\[
T\bar{f}(\sum_i a_i \hat{g}_i) = \sum_i a_i \hat{f}(\hat{g}_i) = \hat{f}(\sum_i a_i \hat{g}_i)
\]

That is, \( T\bar{f} = \hat{f} \). Hence \( T \) is full. \[\square\]

However, it should be noted that the functor defined above need not be faithful as illustrated in the following example.

**Example 8.** Let \( K_4 = \{e, a, b, c\} \) be the Klein 4- group and \( L_{\mathbb{F}K_4} \) Lie algebra of the group algebra \( \mathbb{F}K_4 \). Consider the identity map \( 1_{L_{\mathbb{F}K_4}} \) and \( \bar{f} : L_{\mathbb{F}K_4} \rightarrow L_{\mathbb{F}K_4} \) given by

\[
\bar{f}(\sum_i a_i g_i) = \sum_i a_i f(g_i)
\]
where $g_i \in K_4$ and $f$ is the trivial homomorphism. Clearly both $T\tilde{f}$ and $T1_{L\oplus K_4}$ are zeroes for the functor $T : LSG \to C_{PGL}$.

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