A J-Spectral Factorization Condition for the Physical Realizability of a Transfer Function Matrix with only Direct Feedthrough Quantum Noise

Rebecca TY Thien, Shanon L. Vuglar and Ian R. Petersen

Abstract—This paper gives a J-spectral factorization condition for the implementation of a strictly proper transfer function matrix as a physically realizable quantum system using only direct feedthrough quantum noise. A necessary frequency response condition is also presented. Examples are included to illustrate the main results.

I. INTRODUCTION

Quantum linear systems are a class of quantum systems whose dynamics take the specific form of a set of linear quantum stochastic differential equations (QSDEs). Such linear quantum systems are common in the area of quantum optics [1], [2], [3]. Generally, a set of linear QSDEs need not correspond to a physically meaningful quantum system as they must satisfy additional constraints to represent a physical quantum system. The laws of quantum mechanics dictate that quantum systems evolve unitarily, implying that (in the Heisenberg picture) certain canonical commutation relations (CCR) are satisfied at all times. The notion of a physically realizable quantum linear stochastic system can be seen in [4] where the authors also derive a necessary and sufficient characterization for such systems.

The authors in [5] provided a condition given in terms of a non-standard algebraic Riccati equation for physically realizing a given transfer function matrix by only introducing direct feedthrough quantum noises. It is well-known [6] that there is a relationship between spectral factorization and the solution of an algebraic Riccati equation. This motivates us to consider a J-spectral factorization approach to physically realize a given transfer function matrix.

In this work, we present a condition for realizing a given transfer function matrix in terms of a J-spectral factorization problem [7], [8]. This also leads to a necessary frequency response condition.

The remainder of the paper proceeds as follows. In Section II, we describe the quantum linear system models under consideration and define the corresponding notion of physical realizability. This section also gives some preliminary results. Then, in Section III, we present our main results. Examples are given in Section IV followed by a conclusion and future work in Section V.

II. BACKGROUND AND PRELIMINARY RESULTS

A. Quantum Linear Systems

The linear quantum systems considered here can be described by the following linear quantum stochastic differential equations (LQSDSs) [4], [5], [9], [10], [11], [12]:

\[ dx(t) = Ax(t) dt + B dw(t); \]
\[ dy(t) = Cx(t) dt + Dw(t); \]

where \( A, B, C \) and \( D \) are real matrices in \( \mathbb{R}^{n \times n}, \mathbb{R}^{n \times n}, \mathbb{R}^{m \times n} \) and \( \mathbb{R}^{n_y \times n} \) \((n, m, n_y)\) are even positive integers, respectively. Moreover, \( x(t) = [x_1(t) \ldots x_n(t)]^T \) is a column vector of self-adjoint, possibly non-commutative, system variables.

Equations (1) must also preserve certain commutation relations as follows:

\[ [x_j(t), x_k(t)] = x_j(t)x_k(t) - x_k(t)x_j(t) = 2i\Theta_{jk} \]

where \( \Theta \) is a real skew-symmetric matrix with components \( \Theta_{jk} \) where \( j, k = 1, \ldots, n \) and \( i = \sqrt{-1} \) in order to represent the dynamics of a physically meaningful quantum system. The commutation relations (2) are said to be canonical (i.e., the system is fully quantum) if

\[ \Theta_m = \text{diag}(J, J, \ldots, J) \]

where \( J \) denotes the real skew-symmetric \( 2 \times 2 \) matrix

\[ J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \]

and the “\text{diag}” notation indicates a block diagonal matrix assembled from the given entries. Here \( m \) denotes the dimension of the matrix \( \Theta_m \).

The vector quantity \( w \) describes the input signals and is assumed to admit the decomposition

\[ dw(t) = \beta_w(t) dt + d\tilde{w}(t) \]

where the self-adjoint, adapted process \( \beta_w(t) \) is the signal part of \( dw(t) \) and \( d\tilde{w}(t) \) is the noise part of \( dw(t) \) [9], [10], [11]. The noise \( \tilde{w}(t) \) is a vector of self-adjoint quantum noises with Itô table

\[ d\tilde{w}(t) d\tilde{w}^T(t) = F_\tilde{w} \]

where the self-adjoint, adapted process \( F_\tilde{w} \) is the signal part of \( dw(t) \) and \( d\tilde{w} \) is the noise part of \( dw(t) \) [9], [10] with \( S_\tilde{w} \) and \( T_\tilde{w} \) are real and imaginary, respectively. In this paper, we will assume \( F_\tilde{w} \) is of the form \( F_\tilde{w} = I + i\Theta \) where \( \Theta \) is of the form (3).
In this work, we consider a special case of (1):
\[
dx(t) = Ax(t) dt + Bu(t) du(t) + Bv(t) dv(t);
\]
\[
dy(t) = Cy(t) dt + v(t);
\]
see also [4, 5]. Here, \(du(t)\) from (1) has been partitioned into the signal input, \(dv(t)\) (a column vector with \(n_v\) components) and the direct feed through quantum vacuum noise input, \(dv(t)\). We could regard such a quantum system as a coherent controller in a coherent quantum feedback control system; e.g., see [4, 5].

B. Physical Realizability

In [4], the notion of physical realizability based around the concept of an open quantum harmonic oscillator is introduced. The following formally defines physical realizability.

**Definition 1.** The system (1) is said to be physically realizable if \(\Theta\) is canonical and there exists a quadratic Hamiltonian operator \(\mathscr{K} = (1/2)x(0)^T R x(0)\), where \(R\) is a real symmetric \(n \times n\) matrix, and a coupling operator \(\mathscr{L} = Ax(0)\), where \(A\) is a complex valued \(\frac{r}{2} \times n\) coupling matrix such that matrices \(A, B, C,\) and \(D\) are defined by

\[
A = 2\Theta(R + \Im(A^T A)) \quad (5a)
\]
\[
B = 2\Theta(-\Lambda^T A^T) \quad (5b)
\]
\[
C = P^T \begin{bmatrix} \Sigma_n & 0 \\ 0 & \Sigma_n \end{bmatrix} \begin{bmatrix} \Lambda + \Lambda^\dagger \\ -i\Lambda + i\Lambda^\dagger \end{bmatrix} \quad (5c)
\]
\[
D = [I_{n_y \times n_y} 0_{n_y \times (n_u - n_y)}]. \quad (5d)
\]

Here
\[
\Gamma = P_{n_u} \text{diag}_{n_u}(M);
\]
\[
M = \begin{bmatrix} 1 & i \\ 2 & -i \end{bmatrix};
\]
\[
\Sigma_n = [I_{n_y \times n_y} 0_{n_y \times (n_u - n_y)}];
\]
\[
P_{n_u}(a_1, a_2, ..., a_{2n_u})^T = (a_1, ..., a_{2n_u - 1}, a_2, ..., a_{2n_u})^T;
\]
and \(\text{diag}(M)\) is an appropriately dimensioned square block diagonal matrix with each diagonal block equal to the matrix \(M\). Note that the permutation matrix \(P\) has the unitary property \(PP^T = P^TP = I\) and \(N_u = n_u/2\) and \(N_v = n_v/2\).

The following theorem [4] gives necessary and sufficient conditions for the physical realizability of our system (4).

**Theorem 1.** [4, Theorem 3.4] The system (4) is physically realizable if and only if
\[
A\Theta_n + \Theta_n A^T + B_v \Theta_n B_v^T + B_u \Theta_n B_u^T = 0;
\]
\[
B_v \begin{bmatrix} I_{n_y \times n_y} \\ 0_{(n_u - n_y) \times n_y} \end{bmatrix} = \Theta C^T \text{diag}(J);
\]
where \(\Theta_n, \Theta_v,\) and \(\Theta_u\) are defined as in (3) but may be of different dimensions.

Here \((\cdot)^T\) denotes the complex conjugate transpose of a matrix while \((\cdot)^\dagger\) denotes the complex conjugate of a matrix.

We consider a strictly proper \(n_y \times n_u\) transfer function matrix \(G(s)\) with McMillan degree [13] \(n\) where \(n, n_u\) and \(n_v\) are all even. \(G(s)\) is said to be physically realizable with only direct feedthrough quantum noise if there exists a minimal realization \(G(s) = C(sI - A)^{-1}B_u\) and a matrix \(B_v\) such that the system (4) is physically realizable.

The following theorem from [5] gives a state-space condition in the form of a non-standard algebraic Riccati equation (NSARE) under which a strictly proper transfer function can be implemented as a physically realizable quantum system, which only introduces direct feedthrough quantum noise.

**Theorem 2.** Consider an \(n_y \times n_u\) strictly proper transfer function matrix \(G(s)\) of McMillan degree \(n\) with minimal state-space realization
\[
G(s) = C(sI - A)^{-1}B_u \quad (6)
\]
where \(n, n_u\) and \(n_v\) are all even. This transfer function matrix is physically realizable with only direct feedthrough quantum noise if and only if the algebraic Riccati equation
\[
A^T X + XA - XB_u \Theta_n B_u^T X + C^T \Theta_n C = 0 \quad (7)
\]
has a non-singular, real, skew-symmetric solution \(X\).

**Proof.** Sufficiency follows directly from Theorem 2 of [5]. Necessity is straightforward to verify from Theorem 1 of [5] using a suitable state-space transformation.

Associated with the state-space realization (6) and the Riccati equation (7) is the Hamiltonian matrix
\[
H = \begin{bmatrix} A & -B_u \Theta_n B_u^T \\ -C^T \Theta_n C & -A^T \end{bmatrix} \quad (8)
\]
where \(\Theta_n\) and \(\Theta_v\) are defined as in (3).

**Remark 3.** Note that \(H\) and \(-H^T\) are similar whereby \(\lambda_H\) is an eigenvalue of \(H\) if and only if \(-\lambda_H^T\) is also an eigenvalue of \(H\); e.g., see [14, pp. 327-328].

In this paper, instead of a state-space condition, we want to find a frequency response condition such that a given transfer function matrix is a physically realizable quantum system with only direct feedthrough quantum noise. To achieve this, we use a characterization of the existence of a solution to the NSARE (7) in terms of the J-spectral factorization of a rational matrix following the approach of [6].

III. **Main Result**

In this section, we will show that a given transfer function matrix \(G(s)\) is physically realizable with only direct feedthrough noise if and only if there exists a J-spectral factorization of a certain transfer function matrix \(\Phi_J(s)\). This \(n_u \times n_v\) matrix \(\Phi_J(s)\) is defined as
\[
\Phi_J(s) = \Theta_n - G^{-}(s) \Theta_n G(s) \quad (9)
\]
where \(G^{-}(s) = G(-s)^T\). We now consider some assumptions of \(G(s)\). For a given minimal realization \(G(s) = C(sI - A)^{-1}B_u\), we assume the following
A1. The matrix $A$ is Hurwitz;

A2. The matrix $A$ and the Hamiltonian matrix defined in (8) $H$ have no common eigenvalues.

Note, it follows from the property of Hamiltonian matrices given in Remark 3 that Assumption A2 also implies that the matrix $-A^T$ and the matrix $H$ will have no common eigenvalues.

A. A J-Spectral Factorization Problem

The J-spectral factorization problem considered in this paper is defined as follows:

**Definition 3.** An $n_u \times n_u$ rational matrix $N(s)$ defines a J-spectral factorization of $\Phi_J(s)$ if the following conditions hold:

C1. $\Phi_J(s) = N^-(s)\Theta_nuN(s)$;

C2. $N(s)$ is analytic in $Re \ s \geq 0$;

C3. $N^{-1}(s)$ has no poles in common with $N(s)$;

C4. $\lim_{s \to \infty} N(s) = I$.

**Theorem 4.** Let $G(s)$ be a given $n_y \times n_u$ strictly proper transfer function matrix with McMillan degree $n$ and minimal realization $G(s) = C(sI - A)^{-1}B_u$ where $n$, $n_u$ and $n_y$ are all even. Also, let $\Phi_J(s)$ be defined as in (9) and suppose Assumptions A1-A2 are satisfied. Then $G(s)$ is physically realizable with only direct feedthrough quantum noise if and only if $\Phi_J(s)$ has a J-spectral factorization.

**Proof.** The proof is structured as follows and follows [6]: We prove necessity and sufficiency for the existence of a skew-symmetric solution $X$ to the NSARE (7) and then apply Theorem 2.

Necessity: Suppose $G(s)$ is physically realizable with only direct feedthrough quantum noise. It follows from Theorem 2 that there exists an $X$ which is a skew-symmetric solution of the NSARE (7). Let $N(s)$ be defined as follows

$$N(s) = I + \Theta_{nu} B_u X(sI - A)^{-1} B_u.$$  \hfill (10)

We first show $N(s)$ satisfies C1 in Definition 3. Indeed

$$N^-(s) \Theta_{nu} N(s) = \left[ I + B_u^T (-sI - A^T)^{-1} XB_u \Theta_{nu} \right] \Theta_{nu}$$

$$\times \left[ I + \Theta_{nu} B_u^T X(sI - A)^{-1} B_u \right]$$

$$= \Theta_{nu} - B_u^T X \Theta_{nu} B_u - B_u^T (-sI - A^T)^{-1} XB_u$$

$$- B_u^T (-sI - A^T)^{-1} XB_u \Theta_{nu} B_u^T X(sI - A)^{-1} B_u.$$  \hfill (11)

Also, the NSARE (7) implies

$$- (sI - A^T) X - X(sI - A) + C^T \Theta_{nu} C = XB_u \Theta_{nu} B_u^T X$$

for any $s \in \mathbb{C}$ and hence

$$-X(sI - A)^{-1} (-sI - A^T)^{-1} X$$

$$+ (-sI - A^T)^{-1} C^T \Theta_{nu} C(sI - A)^{-1}$$

$$= (-sI - A^T)^{-1} XB_u \Theta_{nu} B_u^T X(sI - A)^{-1}.$$  

Substituting this result into equation (11), it follows that

$$N^-(s) \Theta_{nu} N(s) = \Theta_{nu} - B_u^T (-sI - A^T)^{-1} C^T \Theta_{nu} C (sI - A)^{-1} B_u = \Phi_J(s).$$

Thus, we have established C1 of Definition 3.

In order to establish C2 of the definition, note that Assumption A1 implies that the $N(s)$ is analytic in $Re \ s \geq 0$.

To show condition C3, note that $\Phi_J(s)$ in (9) can be rewritten in the form

$$\Phi_J(s) = \Theta_{nu} + \left[ B_u^T \begin{bmatrix} sI - A & 0 \\ -C^T \Theta_{nu} C & sI + A^T \end{bmatrix}^{-1} B_u \right].$$

Taking the determinant of the above equation and using the determinant relation from [15, p. 135, Fact 2.14.13], it follows that

$$\det \left( \begin{bmatrix} sI - A & 0 \\ -C^T \Theta_{nu} C & sI + A^T \end{bmatrix} \right) \det(\Phi_J(s))$$

$$= \det(\Theta_{nu})$$

$$\times \det \left( \begin{bmatrix} sI - A & 0 \\ -C^T \Theta_{nu} C & sI + A^T \end{bmatrix} + \left[ B_u \right] \Theta_{nu}^{-1} \left[ B_u^T \right] \right),$$

and hence

$$\det(sI + A^T) \det(sI - A) \det(\Phi_J(s))$$

$$= \det(\Theta_{nu}) \det \left( \begin{bmatrix} sI - A & -B_u \Theta_{nu} B_u^T \end{bmatrix} \right),$$

where $\det(\Theta_{nu}) = 1$. Now, using the Hamiltonian matrix (8), we get

$$\det(\Phi_J(s)) = \frac{\det(sI - H)}{\det(sI + A^T) \det(sI - A)}. \hfill (12)$$

Furthermore, using the fact that there exist a skew-symmetric solution $X$ of the NSARE (7), it follows that $\Phi_J(s)$ can be represented as

$$\Phi_J(s) = \Theta_{nu} - B_u^T (-sI - A^T)^{-1} C^T \Theta_{nu} C (sI - A)^{-1} B_u$$

$$= \Theta_{nu} - B_u^T \left[ X(sI - A)^{-1} + (-sI - A^T)^{-1} X \right]$$

$$+ (-sI - A^T)^{-1} \Theta_{nu} B_u^T X(sI - A)^{-1} B_u$$

$$= \Theta_{nu} - B_u^T X(sI - A)^{-1} B_u - B_u^T (-sI - A^T)^{-1} XB_u$$

$$- B_u^T (-sI - A^T)^{-1} XB_u \Theta_{nu} B_u^T X(sI - A)^{-1} B_u$$

$$= \left[ I + B_u^T (-sI - A^T)^{-1} XB_u \Theta_{nu} \right] \Theta_{nu}$$

$$\times \left[ I + \Theta_{nu} B_u^T X(sI - A)^{-1} B_u \right].$$

This implies

$$\det(\Phi_J(s)) = \det \left( I + (-B_u^T)(sI + A^T)^{-1} \Theta_{nu} \right)$$

$$\times \det(\Theta_{nu}) \det(sI - A)^{-1} B_u). \hfill (13)$$

Since $\det(\Theta_{nu}) = 1$, then

$$\det(\Phi_J(s)) = \det \left( I + (-B_u^T)(sI + A^T)^{-1} \Theta_{nu} \right)$$

$$\times \det(sI - A)^{-1} B_u). \hfill (14)$$
Now, considering the right-hand side of this equation using the determinant relation from [15, p. 135, Fact 2.14.13], it follows that
\[
\det(sI + A^T)\det(I + (-B_u^T)(sI + A^T)^{-1}XB_u\Theta_nu) \\
= \det(I)\det(sI + A^T - XB_u\Theta_nuB_u^T) \\
= \det(sI + A^T - XB_u\Theta_nuB_u^T).
\]
and hence
\[
\det(I + (-B_u^T)(sI + A^T)^{-1}XB_u\Theta_nu) \\
= \frac{\det(sI + A^T - XB_u\Theta_nuB_u^T)}{\det(sI + A^T)}.
\] (14)

Similarly,
\[
\det(sI - A)\det(I + \Theta_nuB_u^TX(sI - A)^{-1}Bu) \\
= \det(I)\det(sI - A + Bu\Theta_nuB_u^TX(sI - A)) \\
= \det(sI - A + Bu\Theta_nuB_u^TX).
\]
and hence
\[
\det(I + \Theta_nuB_u^TX(sI - A)^{-1}Bu) \\
= \frac{\det(sI - A + Bu\Theta_nuB_u^TX)}{\det(sI - A)}.
\] (15)

Substituting equations (14) and (15) into equation (13), it follows that
\[
\det(\Phi_J(s)) \\
= \frac{\det(sI - A + Bu\Theta_nuB_u^TX)}{\det(sI - A + Bu\Theta_nuB_u^TX)\det(sI - A)} \frac{\det(sI + A^T - XB_u\Theta_nuB_u^T)}{\det(sI + A^T)}.
\]

Substituting det(\Phi_J(s)) using (12), it follows that
\[
\frac{\det(sI - H)}{\det(sI + A^T)\det(sI - A)} \\
\times \frac{\det(sI - A + Bu\Theta_nuB_u^TX)}{\det(sI - A)} \times \frac{\det(sI + A^T - XB_u\Theta_nuB_u^T)}{\det(sI + A^T)}.
\] (16)

Now, using the Matrix Inversion Lemma [15], the inverse of \(N(s)\) can be calculated as
\[
N^{-1}(s) = I - \Theta_nuB_u^TX(sI - A + Bu\Theta_nuB_u^TX)^{-1}Bu.
\] (17)

This shows that the poles of \(N(s)\) and \(N^{-1}(s)\) serve as part of the denominator and numerator of equation (16), respectively. Looking at Assumption A2 and considering Remark 3, it follows that there can be no pole-zero cancellation on the left hand side of equation (16). Therefore, no pole-zero cancellation can exist in the right hand side of equation (16). Thus, condition C3 has been shown.

Now observe that condition C4 of Definition 3 follows directly from equation (10). Thus, we have demonstrated that \(N(s)\) defined in (10) indeed defines a J-spectral factorization of \(\Phi_J(s)\).

Sufficiency: Conversely, suppose that \(\Phi_J(s)\) has a J-spectral factorization \(\Phi_J(s) = N^{-1}(s)\Theta_nuN(s)\). We will show there exists a skew-symmetric solution \(X\) of the NSARE (7). Firstly, recall equation (12) and consider Assumptions A1 and A2; it follows that there will be no pole zero cancellations in this expression. Thus, \(\det(\Phi_J(s))\) must be of McMillan degree \(2n\). We now establish some useful claims to aid the proof.

**Claim 1.** The matrix \(N(s)\) is of McMillan degree \(n\).

To establish this claim, first let
\[
\det(N(s)) = \frac{\pi_n(s)}{\pi_l(s)}
\]
where \(\pi_n(s)\) and \(\pi_d(s)\) (relatively prime) are of the same degree. Then, from equation (12) and condition C1 of Definition 3 implies
\[
\det(\Phi_J) = \frac{\det(sI - H)}{\det(sI + A^T)\det(sI - A)}
\]
\[
= \frac{\pi_{n-1}(s)\pi_n(s)}{\pi_{d-1}(s)\pi_d(s)}
\]
with a McMillan degree \(2n\). From C3 of Definition 3, we obtain
\[
\pi_{n-1}(s)\pi_n(s) = \det(sI - H)
\]
\[
\pi_{d-1}(s)\pi_d(s) = \det(sI + A^T)\det(sI - A).
\]
We can write
\[
\pi_{d-1}(s) = n_a,
\]
\[
\pi_d(s) = n_b.
\]
The McMillan degrees are related by [13]
\[
2n \leq n_a + n_b.
\]
Hence, \(n_a = n_b = n\) and \(\pi_d(s)\) will be of degree of \(n\) and thus \(N(s)\) will have McMillan degree \(n\). This completes the proof of the claim.

We now define the matrix \(L\) to be a skew-symmetric solution to the Lyapunov equation
\[
A^TL + LA + C^T\Theta_nuC = 0.
\] (18)

**Claim 2.** The matrix \(\Phi_J\) can be written in the form
\[
\Phi_J(s) = \Theta_nu - B_u^T(-sI - A)^{-1}LB_u - B_u^T(sI - A)^{-1}Bu.
\]

such that the realization of \(-B_u^T(-sI - A)^{-1}LB_u\) and \(-B_u^T(sI - A)^{-1}Bu\) are minimal. Hence, the pair \((A,B_u)\) is controllable.

To establish this claim, first observe that (18) implies
\[
L(sI - A)^{-1} = (-sI - A)^{-1}C^T\Theta_nuC(sI - A)^{-1}.
\]
Hence, we can write
\[
\Phi_J(s) = \Theta_nu - B_u^T(-sI - A)^{-1}C^T\Theta_nuC(sI - A)^{-1}Bu
\]
\[
= \Theta_nu - B_u^T(L(sI - A)^{-1} - (sI - A)^{-1}LB_u - B_u^T(sI - A)^{-1}Bu
\]
\[
= \Theta_nu - B_u^T(-sI - A)^{-1}LB_u - B_u^T(sI - A)^{-1}Bu
\] (19)
as required. Also, it follows from this expression for \( \Phi_J(s) \) that its McMillan degree satisfies the inequality

\[
\delta(\Phi_J(s)) \leq \delta\left(-B_n^T(sI - A)^{-1}LB_n\right) \\
+ \delta\left(-B_n^T(sI - A)^{-1}Bu\right) \\
\leq n + n.
\]

We know that \( \delta(\Phi_J(s)) = 2n \) and hence equality (19) must hold. Therefore, \( \delta\left(-B_n^T(sI - A)^{-1}LB_n\right) = n \) and \( \delta\left(-B_n^T(sI - A)^{-1}Bu\right) = n \). That is the realizations \(-B_n^T(sI - A)^{-1}LB_n\) and \(-B_n^T(sI - A)^{-1}Bu\) are minimal.

It remains to show that \( L \) is a skew-symmetric solution to (18). Add equations (18) and the transpose of (18)

\[
A^TL + LA + CT\Theta_nC + A^TL^T + L^TA - C^T\Theta_nC = 0 \\
A^T(L + L^T) + (L + L^T)A = 0.
\]

Since \( A \) is Hurwitz according to Assumption A1, this implies that \( L + L^T = 0 \) i.e., \( L^T = -L \). This completes the proof of the claim.

**Claim 3.** There exists a minimal realization of \( N(s) \) of the form \( N(s) = I + N_A(sI - A)^{-1}Bu \).

To establish this claim, first recall condition C4 of Definition 3. Hence, \( N(s) \) will have a minimal realization of the form \( N(s) = I + N_A(sI - E)^{-1}E \) where \( F \) is a \( n \times n \) matrix. Also, since we know \( N(s) \) is analytic in \( \Re s \geq 0 \) therefore the matrix \( F \) will be Hurwitz. We can now define the matrix \( Y \) to be the solution to the Lyapunov equation

\[
F^TY + YF + N_A^T\Theta_nN_A = 0.
\]

Using this equation, it follows that

\[
Y(sI - F)^{-1} + (-sI - F)^{-1}Y \\
= (-sI - F)^{-1}N_A^T\Theta_nN_A(sI - F)^{-1}.
\]

Thus,

\[
\Phi_J(s) = N^-(s)\Theta_nN(s) \\
= (I + N_A(sI - F)^{-1}E)^{-1}\Theta_n(I + N_A(sI - F)^{-1}E) \\
= \Theta_n + \Theta_nN_A(sI - F)^{-1}E + E^T(-sI - F)^{-1}N_A^T\Theta_n \\
+ E^T(-sI - F)^{-1}N_A^T\Theta_n(sI - F)^{-1}E \\
= \Theta_n + \Theta_nN_A(sI - F)^{-1}E + E^T(-sI - F)^{-1}N_A^T\Theta_n \\
+ E^T[Y(sI - F)^{-1} - (-sI - F)^{-1}Y]E \\
= \Theta_n + \Theta_nN_A(sI - F)^{-1}E + E^T(-sI - F)^{-1}N_A^T\Theta_n \\
+ E^TY(sI - F)^{-1}E + E^T(-sI - F)^{-1}YE \\
= \Theta_n + (\Theta_nN_A + E^TY)(sI - F)^{-1}E \\
+ E^T(-sI - F)^{-1}(N_A^T\Theta_n + YE)
\]

(21)

Using this equation, it follows that the McMillan degree of \( F(s) \) satisfies the inequality

\[
\delta(\Phi_J(s)) \leq \delta\left((\Theta_nN_A + E^TY)(sI - F)^{-1}E\right) \\
+ \delta\left(E^T(-sI - F)^{-1}(N_A^T\Theta_n + YE)\right) \\
\leq n + n.
\]

However, we know that \( \delta(\Phi_J(s)) = 2n \) and hence equality must hold. Thus, \( \delta((\Theta_nN_A + E^TY)(sI - F)^{-1}E) = n \) and \( \delta(E^T(-sI - F)^{-1}(N_A^T\Theta_n + YE)) = n \). That is, the realizations \((\Theta_nN_A + E^TY)(sI - F)^{-1}E\) and \(E^T(-sI - F)^{-1}(N_A^T\Theta_n + YE)\) are minimal.

We now compare the expression for \( \Phi_J(s) \) obtained in Claim 2 and equation (21). It follows that

\[
\Phi_J(s) = \Theta_n + B_n^T(sI - A)^{-1}LB_n - B_n^T(sI - A)^{-1}Bu \\
= \Theta_n + (\Theta_nN_A + E^TY)(sI - F)^{-1}E \\
+ E^T(-sI - F)^{-1}(N_A^T\Theta_n + YE).
\]

Equating stable and anti-stable terms in this equation, it follows that

\[
B_n^T(sI - A)^{-1}(-LB_n) = E^T(-sI - F)^{-1}(N_A^T\Theta_n + YE) \\
(22)
\]

and

\[
-B_n^T(sI - A)^{-1}Bu = (\Theta_nN_A + E^TY)(sI - F)^{-1}E. \\
(23)
\]

However, both sides of equation (23) are minimal realizations. Thus, we conclude that there exists a minimal realization of \( N(s) \) such that \( F = A \) and \( E = Bu \). In a similar manner to Claim 2, it remains to show that \( Y \) is a skew-symmetric solution to (20). Adding equations (20) and its transpose

\[
F^TY + YF + N_A^T\Theta_nN_A + F^TY^T + Y^TF - N_A^T\Theta_nN_A = 0 \\
F^T(Y + Y^T) + (Y + Y^T)F = 0.
\]

Since \( F \) is Hurwitz, this implies that \( Y + Y^T = 0 \) i.e., \( Y = -Y \). This completes the proof of the claim.

Now, returning to the proof of the theorem, substitute \( F = A \) and \( E = Bu \) into equations (22) and (23). Thus

\[
B_n^T(sI - A)^{-1}(-LB_n) = B_n^T(-sI - A)^{-1}(N_A^T\Theta_n + YBu) \\
and
\]

\[
-B_n^T(sI - A)^{-1}Bu = (\Theta_nN_A + B_n^TY)(sI - A)^{-1}Bu.
\]

However, we know from Claim 2 that equations (22) and (23) are minimal realizations. Therefore, it follows from the result above that

\[
-LBu = N_A^T\Theta_n + YBu \\
-B_n^TY = \Theta_nN_A + B_n^TY.
\]

Defining \( X = L + Y \), then

\[
-XBu = N_A^T\Theta_n \\
-B_n^TX = \Theta_nN_A,
\]

which provides the desired realizations. It now remains to show that \( X \) is a solution of the NSARE (7). Add (18) and (20), this gives

\[
A^TX + XA + C^T\Theta_nC - XBu\Theta_nB_n^TX = 0
\]

Hence, substitute \( X = L + Y \) and the result above gives

\[
A^TX + XA + C^T\Theta_nC - XBu\Theta_nB_n^TX = 0
\]

5823
as required. In addition, from Claim 2 and 3, we know that \( L \) and \( Y \) are skew-symmetric solution. Therefore, \( X \) is a skew-symmetric solution of the NSARE (7). It now follows from Theorem 2 that \( G(s) \) is physically realizable with only direct feedthrough quantum noise.

B. Frequency Response Condition

The following corollary gives a necessary frequency response condition for physical realizability with only direct feedthrough quantum noise.

**Corollary 1.** Suppose \( G(s) \) is physically realizable with only direct feedthrough quantum noise. Then the following frequency response condition

\[
\det(\Theta_{n_u} - G^\dagger(j\omega)\Theta_{n_y}G(j\omega)) > 0
\]

holds for all \( \omega \).

**Proof.** From Theorem 4, it follows that \( \Phi_j(s) \) has a J-spectral factorization. \( \Phi_j(s) \) can also be written as

\[
\Phi_j(s) = \Theta_{n_u} - G^\dagger(s)\Theta_{n_y}G(s).
\]

Now, we let \( s = j\omega \)

\[
\Phi_j(j\omega) = \Theta_{n_u} - G^\dagger(j\omega)\Theta_{n_y}G(j\omega)
\]

and take the determinant of equation (25)

\[
\det(\Phi_j(j\omega)) = \det(\Theta_{n_u} - G^\dagger(j\omega)\Theta_{n_y}G(j\omega)).
\]

First, look at the right-hand side of this expression as \( \omega \to \infty \), and note that given \( G(s) \) is strictly proper

\[
\det(G^\dagger(j\omega)\Theta_{n_y}G(j\omega)) \to 0
\]

as \( \omega \to \infty \). Hence,

\[
\det(\Phi_j(j\omega)) \to \det(\Theta_{n_u}) = 1 > 0
\]

as \( \omega \to \infty \).

Also, it follows from C1 in Definition 3 that

\[
\det(\Phi_j(j\omega)) = \det(N^\dagger(j\omega)\Theta_{n_y}N(j\omega))
\]

for all \( \omega \).

Furthermore, the matrix \( iN^\dagger(j\omega)\Theta_{n_y}N(j\omega) \) is congruent to the matrix \( \Theta_{n_u} \) which has \( \frac{n_u}{2} \) positive eigenvalues and \( \frac{n_u}{2} \) negative eigenvalues. Hence, using the Inertia Theorem [16, pp. 281-282, Definition 4.5.6], it follows that the eigenvalues of \( iN^\dagger(j\omega)\Theta_{n_y}N(j\omega) \) will have this same property. From this, it follows that the \( \det(N^\dagger(j\omega)\Theta_{n_y}N(j\omega)) \) must be real and non-zero for any \( \omega \). Hence, it now follows that

\[
\det(\Phi_j(j\omega)) > 0
\]

for all \( \omega \).

IV. Examples

For illustrative examples demonstrating the application of Theorem 4 and Corollary 1, see [17].

V. CONCLUSION AND FUTURE WORK

A. Conclusion

In [5], the existence of a non-singular, real, and skew-symmetric solution \( X \) to the NSARE (7) guarantees a strictly proper transfer function can be implemented as a physically realizable quantum system with only direct feedthrough noise. In this work, we have shown that the J-spectral factorization of \( \Phi_j(s) \) gives a necessary and sufficient condition for \( G(s) \) to be physically realizable with only direct feedthrough quantum noise. We also present a necessary frequency response condition.

B. Future Work

Future work will consider making the frequency response condition in Corollary 1 to a necessary and sufficient condition by somehow extending the notion of physically realizability with only direct feedthrough noise.

REFERENCES

[1] H. Bachor and T. C. Ralph, A Guide to Experiments in Quantum Optics. Wiley-VCH, 2004.
[2] D. Walls and G. J. Milburn, Quantum Optics. Springer-Verlag Berlin Heidelberg, 2008.
[3] C. Gardiner and P. Zoller, Quantum Noise. Springer-Verlag Berlin Heidelberg, 2004.
[4] M. R. James, H. I. Nurdin, and I. R. Petersen, “H∞ control of linear quantum stochastic systems,” IEEE Transactions on Automatic Control, vol. 53, pp. 1787–1803, 2008.
[5] S. L. Vuglar and I. R. Petersen, “Quantum noises, physical realizability and coherent quantum feedback control,” IEEE Transactions on Automatic Control, vol. 62, no. 2, pp. 998–1003, 2017.
[6] B. P. Molinari, “Equivalence relations for the algebraic Riccati equations,” The Bell System Technical Journal, vol. 42, no. 2, pp. 355–382, 1963.
[7] J. D. Stefanovski, “Strongly (J,J′) lossless rational matrices and H∞ problem,” International Journal of Robust and Nonlinear Control, vol. 28, pp. 4261–4286, 2018.
[8] H. Kimura, Chain-scattering approach to H∞ control. Birkhäuser, 1997.
[9] R. L. Hudson and K. Parthasarathy, “Quantum Ito’s formula and stochastic evolutions,” Communications in Mathematical Physics, vol. 93, pp. 301–323, 1984.
[10] K. Parthasarathy, An Introduction to Quantum Stochastic Calculus. Birkhäuser Basel, 1992.
[11] V. P. Belavkin, “Quantum continual measurements and a posteriori collapse on CCR,” Communications in Mathematical Physics, vol. 146, pp. 611–631, 1992.
[12] R. T. Y. Thien, S. L. Vuglar, and I. R. Petersen, “When do additional quantum noises affect controller performance?” in 2020 59th IEEE Conference on Decision and Control (CDC), 2020, pp. 3855–3859.
[13] R. E. Kalman, “Irreducible realizations and the degree of a rational matrix,” Journal of the Society for Industrial and Applied Mathematics, vol. 13, no. 2, pp. 520–544, 1965.
[14] K. Zhou, J. C. Doyle, and K. Glover, Robust and Optimal Control. New Jersey: Prentice Hall, 1996.
[15] D. Bernstein, Matrix Mathematics: Theory, Facts, and Formulas. Princeton University Press, 2009.
[16] A. Horn and C. R. Johnson, Matrix Analysis. New York: Cambridge University Press, 2013.
[17] R. T. Y. Thien, S. L. Vuglar, and I. R. Petersen, “A J-Spectral Factorization Condition for the Physical Realizability of a Transfer Function Matrix with only Direct Feedthrough Quantum Noise,” in arXiv:2209.01730, 2022.