CONJECTURES ON THE REPRESENTATIONS OF MODULAR LIE ALGEBRAS

KIM YANGGON

Abstract. We have already seen simple representations of modular Lie algebras of $A_l$-type and $C_l$-type. We shall further investigate simple representations of $B_l$ type, which turn out to be very similar in methodology as those types except for roots. So we may consider some conjectures relating to the representations of classical type modular Lie algebras.

1. INTRODUCTION

The representation theories of simple Lie algebras for characteristic zero of the ground field $F$ are almost done in case $F$ is algebraically closed, whereas those of simple Lie algebras for prime characteristic are under way.

The former ones belong to nonmodular representation theory and the latter ones belong to modular representation theory of Lie algebras.

We intend to investigate in section 2 of this paper the representations of modular Lie algebras of $B_l$-type following the ways of $A_l$-type and $C_l$-type.
Next in section 3, we give some definitions such as Lee’s basis, Park’s Lie algebra, and Hypo-Lie algebra, which are related to the representation of modular Lie algebras.

Afterwards we pose some conjectures for simple Lie algebras of classical types in the last section 4 because those three kinds of representations mentioned above are very similar each other in methodology.

We assume throughout that the ground field $F$ is algebraically closed unless otherwise stated.

2. SIMPLE NONRESTRICTED REPRESENTATIONS OF $B_l$-TYPE LIE ALGEBRA

We are well aware that the orthogonal Lie algebra of $B_l$-type of rank $l$, i.e., the $B_l$-type Lie algebra over $\mathbb{C}$ has its root system $\Phi=\{\pm\epsilon_i (\text{of squared length } 1); \pm (\epsilon_i \pm \epsilon_j) (\text{of squared length } 2), 1 \leq i \neq j \leq l \}$, where $\epsilon_i, \epsilon_j$ are linearly independent unit vectors in $\mathbb{R}^l$ with $l \geq 2$. The base of $\Phi$ equals $\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \cdots, \epsilon_{l-1} - \epsilon_l, \epsilon_l\}$.

Let $L$ be a $B_l$-type simple Lie algebra over an algebraically closed field $F$ of characteristic $p \geq 7$.

**Proposition 2.1.** Let $\chi$ be a character of any irreducible $L$-module with $\chi(x_\alpha) \neq 0$ for some $\alpha \in \Phi$, where $x_\alpha$ is an element in the Chevalley basis of $L$ such that $Fx_\alpha + Fh_\alpha + Fx_{-\alpha} = \mathfrak{sl}_2(F)$ with $[x_\alpha, x_{-\alpha}] = h_\alpha$. 
We then have that any irreducible $L$-module with character
\[ \chi \] is of dimension $p^n = p^{\frac{\dim L}{2}}$, where $n = \dim L = 2m + l$ for a
CSA $H$ with $\dim H = l$.

**Proof.** We meet with 2 cases of root length.

(I) Suppose that $\alpha$ is a short root. Since all roots of a given
length are conjugate under the Weyl group of $\Phi$, we may put
$\alpha = \epsilon_1$ without loss of generality.

Let us put $B_i = b_{i1}h_{\epsilon_1-\epsilon_2} + b_{i2}h_{\epsilon_2-\epsilon_3} + \cdots + b_{i,l-1}h_{\epsilon_{l-1}-\epsilon_l} + b_{il}h_{\epsilon_l}$ for $i = 1, 2, \cdots, 2m$, where
$(b_{i1}, \cdots, b_{il}) \in F^l$ are chosen so that arbitrary $(l + 1) - B_i$'s
are linearly independent in $\mathfrak{m}^l(F)$, the $\mathfrak{B}$ below becomes an $F$
linearly independent set in $U(L)$ if necessary and $x_\alpha B_i \neq B_i x_\alpha$
with $\alpha = \epsilon_1$.

We search for a basis of $U(L)/\mathfrak{M}_\chi$, where $\mathfrak{M}_\chi$ is the ker-
nel of the irreducible representation $\rho_\chi : L \to \mathfrak{gl}(V)$ of the
restricted Lie algebra $(L, [p])$ associated with any given irre-
ducible $L$-module $V$ with a character $\chi$.

In $U(L)/\mathfrak{M}_\chi$ we give a basis as $\mathfrak{B} := \{(B_1 + A_{\epsilon_1})^{i_1} \otimes (B_2 + A_{-\epsilon_2})^{i_2} \otimes (B_3 + A_{\epsilon_1-\epsilon_2})^{i_3} \otimes (B_4 + A_{-(\epsilon_1-\epsilon_2)})^{i_4} \otimes \cdots \otimes (B_{2l} + A_{-(\epsilon_{l-1}-\epsilon_l)})^{i_{2l}} \otimes (B_{2l+1} + A_{\epsilon_l})^{i_{2l+1}} \otimes (B_{2l+2} + A_{-\epsilon_l})^{i_{2l+2}} \otimes \mathfrak{m}^{i_{2l+3}} (B_j + A_{\alpha_j})^{i_j} \}$ for $0 \leq i_j \leq p - 1$,

where we put
\[
A_{\epsilon_1} = x_{\epsilon_1},
A_{-\epsilon_1} = c_{-\epsilon_1} + (h_{\epsilon_1} + 1)^2 + 4x_{-\epsilon_1} x_{\epsilon_1},
A_{-\epsilon_1 \pm \epsilon_2} = x_{\epsilon_1 \pm \epsilon_2} (c_{-\epsilon_1 \pm \epsilon_2} \pm x_{-\epsilon_1 \pm \epsilon_2} x_{-(\epsilon_1 \pm \epsilon_2)} \pm x_{\epsilon_1 \pm \epsilon_2} x_{-(\epsilon_1 \pm \epsilon_2)}),
\]
\[ A_{-\varepsilon_1 \pm \varepsilon_j} = x_{-\varepsilon_2 \pm \varepsilon_j} \left( c_{\varepsilon_1 \pm \varepsilon_j} + x_{(\varepsilon_j - \varepsilon_1)} x_{-(\varepsilon_j - \varepsilon_1)} \right) \pm x_{\varepsilon_1 \pm \varepsilon_j} x_{-(\varepsilon_j \pm \varepsilon_j)} \],
\[ A_{\varepsilon_1 \pm \varepsilon_2} = x_{\varepsilon_3 \pm \varepsilon_2} \left( c_{\varepsilon_1 \pm \varepsilon_2} + x_{\varepsilon_2} x_{-\varepsilon_2} \pm x_{\varepsilon_1 + \varepsilon_2} x_{-(\varepsilon_1 + \varepsilon_2)} \right) \pm x_{\varepsilon_1 \pm \varepsilon_2} x_{\varepsilon_1 - \varepsilon_2} \],
\[ A_{\varepsilon_j} = x_{\varepsilon_2 \pm \varepsilon_j} \left( c_{\varepsilon_j} + x_{\varepsilon_j} x_{-\varepsilon_j} \pm x_{\varepsilon_1 + \varepsilon_j} x_{-(\varepsilon_1 + \varepsilon_j)} \right) \pm x_{\varepsilon_j - \varepsilon_1} x_{\varepsilon_1 - \varepsilon_j} \],
\[ A_{-\varepsilon_j} = x_{\varepsilon_2 - \varepsilon_j} \left( c_{-\varepsilon_j} + x_{-\varepsilon_j} x_{\varepsilon_j} \pm x_{\varepsilon_1 - \varepsilon_j} x_{-(\varepsilon_1 - \varepsilon_j)} \right) \pm x_{-\varepsilon_j - \varepsilon_1} x_{\varepsilon_1 + \varepsilon_j} \],

with the sign chosen so that they commute with \( x_\alpha \) and with \( c_\beta \in F \) chosen so that \( A_{-\varepsilon_1} \) and parentheses( ) are invertible.

For any other root \( \beta \), we put \( A_\beta = x_\beta^2 \) or \( x_\beta^3 \) if possible. Otherwise we make use of the parentheses( ) again used for designating \( A_{-\beta} \). So in this case we put \( A_\beta = x_\gamma^2 \) or \( x_\gamma^3 \) attached to these ( ) so that \( x_\alpha \) may commute with \( A_\beta \).

We are going to show that \( \mathfrak{B} \) is a basis in \( U(L)/M_\chi \).
It is not difficult to see that \( \mathfrak{B} \) is a linearly independent set over \( F \) in \( U(L) \) by virtue of P-B-W theorem. Moreover \( A_\beta \notin M_\chi \) for any \( \beta \in \Phi \) (see detailed proof below).

We intend to prove that a nontrivial linearly dependence equation leads to absurdity.

Suppose that we have a dependence equation which is of least degree with respect to \( h_{\alpha_i} \in H \) and the number of whose highest degree terms is also least. If it is conjugated by \( x_\alpha \), then there arises a nontrivial dependence equation of lower degree than the given one, which is absurd.

Otherwise it reduces to one of the following forms:
(i) \( x_{\varepsilon_j} K + K' \in M_\chi \),
(ii) \(x_{-e_j} K + K' \in \mathcal{M}_\chi\),
(iii) \(x_{e_j + e_k} K + K' \in \mathcal{M}_\chi\),
(iv) \(x_{-e_j - e_k} K + K' \in \mathcal{M}_\chi\),
(v) \(x_{e_j - e_k} K + K' \in \mathcal{M}_\chi\),

where \(K\) and \(K'\) commute with \(x_\alpha = x_{e_1}\).

Because (i),(ii) reduce to (iii),(iv) respectively by applying \(ad x_\alpha\), we have only to consider (iii),(iv) and (v).

As for the case (v) we deduce successively, \(x_{-e_j} (x_{e_j - e_k} K + K') \in \mathcal{M}_\chi \Rightarrow (x_{-e_k} + x_{e_j - e_k} x_{-e_j}) K + x_{-e_j} K' \in \mathcal{M}_\chi \Rightarrow \) by \(ad x_{e_1}, (x_{e_1 - e_k} + x_{e_j - e_k} x_{e_1 - e_j}) K + x_{e_1 - e_j} K' \in \mathcal{M}_\chi\) for \(j, k \neq 1\) \(\Rightarrow x_{e_1 - e_k} K + K'' \in \mathcal{M}_\chi\) for some \(K''\) commuting with \(x_\alpha\) and being compared to the start equation.

In this case we deduce \(x_{e_k - e_1} x_{e_1 - e_k} K + x_{e_k - e_1} K'' \in \mathcal{M}_\chi \Rightarrow (x_{e_k - e_1} + x_{e_1 - e_k} x_{e_k - e_1}) K + x_{e_k - e_1} K'' \in \mathcal{M}_\chi \Rightarrow (x_{e_1} + x_{e_1 - e_k} x_{e_k}) K + x_{e_k} K'' \in \mathcal{M}_\chi\) 4 by \(ad x_{e_1} \Rightarrow x_{e_1} K + x_{e_1 - e_k} x_{e_k} K + x_{e_k} K'' \in \mathcal{M}_\chi \Rightarrow \) we may put \(x_{e_1} K + K'' \in \mathcal{M}_\chi\) for some \(K''\) commuting with \(x_\alpha\) modulo \(\mathcal{M}_\chi\) because \(x_{e_1} K\) commute with \(x_\alpha = x_{e_1}\).

We may also deduce a similar form \(x_{e_1} K + K'' \in \mathcal{M}_\chi\) from the types (iii) and (iv). So we have only to prove that such a form is absurd.

From this we get \(x_{-e_1} x_{e_1} K + x_{-e_1} K'' \in \mathcal{M}_\chi\), so we get \(4^{-1}\{w - (h + 1)^2\} K + x_{-e_1} K'' \in \mathcal{M}_\chi\). Here \(w = (h + 1)^2 + 4x_{-e_1} x_{e_1}\) is contained in the center of \(U(\mathfrak{sl}_2(F))\).

If \(x_{-e_1}^{p-1} \equiv c\), then multiplying this last equation by \(x_{-e_1}^{p-1}\) gives rise to \(4^{-1}x_{-e_1}^{p-1}\{w - (h + 1)^2\} K + cK'' \equiv 0\). Multiplying
this equation by \(x_{\epsilon_1}^{p-1}\) we have \(4^{-1}x_{\epsilon_1}^{p-1}x_{-\epsilon_1}^{-1}\{w - (h + 1)^2\}K + c_{x_{\epsilon_1}^{p-1}K''} \equiv 0.\) By making use of \(w\), we have an equation of the form

\[(a \text{ polynomial of degree} \geq 1 \text{ with respect to} h)K + c_{x_{\epsilon_1}^{p-1}K''} \equiv 0.\] Finally consecutive conjugation and subtraction by \(x_{\epsilon_1}\) gives rise to \(K \in \mathcal{M}_x\), which is absurd.

(II) Suppose that \(\alpha\) is a long root; then we may put \(\alpha = \epsilon_1 - \epsilon_2\) since all roots of the same length are conjugate under the Weyl group of \(\Phi\).

Similarly as in (I), we put \(B_i :=\) the same as in (I) except that \(\alpha = \epsilon_1 - \epsilon_2\) this time instead of \(\epsilon_1\).

We claim we have a basis \(\mathfrak{B} := \{(B_1 + A_{\epsilon_1 - \epsilon_2})^{i_1} \otimes (B_2 + A_{-(\epsilon_1 - \epsilon_2)})^{i_2} \cdots \otimes (B_{2l-2} + A_{-(\epsilon_{l-1} - \epsilon_l)})^{i_{2l-2}} \otimes (B_{2l-1} + A_{\epsilon_l})^{i_{2l-1}} \otimes (B_{2l} + A_{-\epsilon_l})^{i_{2l}} \otimes (\otimes_{j=2l+1}^{2m}(B_j + A_{\alpha_j})^{i_j}), 0 \leq i_j \leq p - 1\},\)

where we put
\[
A_{\epsilon_1 - \epsilon_2} = x_{\alpha} = x_{\epsilon_1 - \epsilon_2},
A_{\epsilon_2 - \epsilon_1} = c_{\epsilon_2 - \epsilon_1} + (h_{\epsilon_1 - \epsilon_2} + 1)^2 + 4x_{\epsilon_2 - \epsilon_1}x_{\epsilon_1 - \epsilon_2},
A_{\epsilon_2 + \epsilon_3} = x_{\epsilon_2 + \epsilon_3} + x_{\epsilon_2 + \epsilon_3}x_{-(\epsilon_2 + \epsilon_3)} \pm x_{\epsilon_2 + \epsilon_3}x_{-(\epsilon_1 + \epsilon_3)}),
A_{\epsilon_2 + \epsilon_k} = x_{\epsilon_2 + \epsilon_k} + x_{\epsilon_2 + \epsilon_k}x_{-(\epsilon_2 + \epsilon_k)} \pm x_{\epsilon_2 + \epsilon_k}x_{-(\epsilon_1 + \epsilon_k)}\) if \(k \neq 1,
A_{\epsilon_2} = x_{\epsilon_2}(c_{\epsilon_2} + x_{\epsilon_2}x_{-\epsilon_2} \pm x_{\epsilon_1}x_{-\epsilon_1}),
A_{-\epsilon_1} = x_{-\epsilon_2}(c_{-\epsilon_1} + x_{-\epsilon_1}x_{\epsilon_1} \pm x_{-\epsilon_2}x_{\epsilon_2}),
A_{-(\epsilon_1 + \epsilon_3)} = x_{-(\epsilon_1 + \epsilon_3)}(c_{-(\epsilon_1 + \epsilon_3)} + x_{\epsilon_2 + \epsilon_3}x_{-(\epsilon_2 + \epsilon_3)} \pm x_{\epsilon_1 + \epsilon_3}x_{-(\epsilon_1 + \epsilon_3)}),
A_{-(\epsilon_1 + \epsilon_k)} = x_{-(\epsilon_1 + \epsilon_k)}(c_{-(\epsilon_1 + \epsilon_k)} + x_{\epsilon_2 + \epsilon_k}x_{-(\epsilon_2 + \epsilon_k)} \pm x_{\epsilon_1 + \epsilon_k}x_{-(\epsilon_1 + \epsilon_k)}),
A_{\epsilon_l} = x_{\epsilon_l}^2,
A_{-\epsilon_l} = x_{-\epsilon_l}^2.
\]
with the signs chosen so that they commute with $x_\alpha$ and with $c_\beta \in F$ chosen so that $A_{\epsilon_2 - \epsilon_1}$ and parentheses are invertible.

For any other root $\beta$, we put $A_\beta = x_\beta^2$ or $x_\beta^3$ if possible.

Otherwise we make use of the parentheses( ) again used for designating $A_{-\beta}$. So in this case we put $A_\beta = x_\gamma^2$ or $x_\gamma^3$ attached to these ( ) so that $x_\alpha$ may commute with $A_\beta$.

We intend to show that $\mathfrak{B}$ is a basis in $U(L)/\mathfrak{M}_\chi$. We may see easily that $\mathfrak{B}$ is a linearly independent set over $F$ in $U(L)$ by virtue of P-B-W theorem. Furthermore for any $\beta \in \Phi$, $A_\beta \notin \mathfrak{M}_\chi$ (see detailed proof below).

We shall show that a nontrivial linearly dependence equation leads to absurdity. We assume that we have a dependence equation which is of least degree with respect to $h_{\alpha_j} \in H$ and the number of whose highest degree terms is also least.

If it is conjugated by $x_\alpha = x_{\epsilon_1 - \epsilon_2}$, then we get a nontrivial dependence equation of lower degree than the given one contravening our assumption.

Otherwise it reduces to one of the following forms:

(i) $x_{\epsilon_j} K + K' \in \mathfrak{M}_\chi$,
(ii) $x_{-\epsilon_j} K + K' \in \mathfrak{M}_\chi$,
(iii) $x_{\epsilon_j + \epsilon_k} K + K' \in \mathfrak{M}_\chi$,
(iv) $x_{-\epsilon_j - \epsilon_k} K + K' \in \mathfrak{M}_\chi$,
(v) $x_{\epsilon_j - \epsilon_k} K + K' \in \mathfrak{M}_\chi$,
(vi) $x_{\epsilon_k - \epsilon_j} K + K' \in \mathfrak{M}_\chi$,
where $K$ and $K'$ commute with $x_\alpha = x_{\epsilon_1 - \epsilon_2}$.

As for the case (i),(ii), these may easily be changed to the forms from (iii) to (vi). Moreover the cases (iii),(iv),(vi) are analogous to the case (v).

Hence it suffices to consider only the case (v). However the proof leading to absurdity is very similar in methodology as that of the case (I) above or that of the case for $C_l$-type Lie algebra in proposition 4.1 [3]. □

3. Definitions

We give here some definitions related to some conjectures in the next section.

**Definition 3.1.** Let $L$ be a finite dimensional Lie algebra over an algebraically closed field $F$ of characteristic $p > 0$ and let $H$ be a Cartan subalgebra (abb. CSA) of $L$.

We shall call $B$ a Lee’s basis of the algebra $U(L)/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$ of $U(L)$ if $B$ satisfies the following properties:

(i) $\dim_F U(L)/\mathfrak{m} = [Q(U(L)) : Q(Z)] = p^{2m}$ for the center $Z$ of the universal enveloping algebra $U(L)$ of $L$, where $Q(U(L))$ and $Q(Z)$ are quotient algebras of $U(L)$ and $Z$ respectively;
(ii) \( B \) is a basis of \( U(L)/\mathfrak{M} \) over \( F \) and is of the form \( \{ (B_1 + A_1)^{i_1} \otimes (B_2 + A_2)^{i_2} \otimes \cdots \otimes (B_{2m} + A_{2m})^{i_{2m}} \} \), where \( 0 \leq i_j \leq p - 1 \) and \( B_i \)'s are elements in the subalgebra of \( U(L) \) generated by 1 and \( H \), and \( A_i \)'s are some elements in \( U(L) \).

**Definition 3.2.** Suppose that \( L \) is an indecomposable Lie algebra over a field \( F \) of nonzero characteristic \( p \).

If \( U(L)/\mathfrak{M} \) has a Lee’s basis for all maximal ideals \( \mathfrak{M} \) except for a finite number of them up to isomorphism, we shall call \( L \) a Park’s Lie algebra.

**Definition 3.3.** By a Hypo - Lie algebra, we shall mean a sub-Lie algebra of some simple Lie algebra which also becomes a Park’s Lie algebra.

Earlier in 1967, two scholars I.R.Shafarevich and Rudakof observed such a fact regarding irreducible modules of \( L = \mathfrak{sl}_2(F) \) [5]. They proved that for any nonzero character \( \chi \) any irreducible \( L \)-module over \( F \) associated with \( \chi \) is of dimension \( p \), where \( F \) is an algebraically closed field of characteristic \( p \geq 3 \).

Also they proved that for zero character \( \chi = 0 \), the number of nonisomorphic irreducible \( L \)-modules is finite. For this proof the center of \( U(L) \) is crucial according to them.

Now we might as well give a nontrivial example of Hypo-Lie algebra.

Let \( L \) be a Park’s Lie algebra as a simple Lie algebra contained in another classical simple Lie algebra \( L' \) over an algebraically closed field \( F \) of characteristic \( p \geq 7 \). Let \( H, H' \) be CSA’s of \( L, L' \) respectively satisfying \( L \cap H' = H \).
Given a Chevalley basis $\mathfrak{B}_c$ with respect to $H'$ of $L'$, we let $h_\alpha \in (H' - H) \cap \mathfrak{B}_c$ for a certain root $\alpha$. Suppose that for this $\alpha$, $L_H := L \cup \{h_\alpha\}$ is indecomposable.

Then we may see without difficulty that $L_H$ becomes a Hypo - Lie algebra [2].

We are not sure for now whether or not we may extend such a fact to the Cartan type Lie algebras. According to [6], there are only 2 kinds of finite dimensional simple Lie algebras over an algebraically closed field $F$ of characteristic $p \geq 7$, namely Classical type or Cartan type simple Lie algebras.

4. Conjectures

We found out some counter examples to the nonrestricted representation theory of $C_l$-type Lie algebras in [3].

Furthermore a Lie algebraist Jörg Feldvoss reviewed the paper [1] in the affirmative. He agreed tacitly to the fact that any $A_l$ type modular Lie algebra with $l \geq 1$ becomes a Park’s Lie algebra and so a Hypo Lie algebra over an algebraically closed field $F$ of characteristic $p \geq 7$.

In other words, for any nonzero character $\chi$ we have $W_\chi(L) = V_\chi(L)$ for $L = A_l$-type simple Lie algebras except for a finite number of restricted irreducible $L$-modules, where the left hand side is a Weyl module of $L$ and the right hand side is a
Verma module of $L$ associated with a character $\chi$.

So we could give a conjecture as follows, motivated by this fact in [1] and section 2 of this paper and references [2],[3]:

**[CONJECTURE]** Suppose that $L$ is a Lie algebra of classical type over an algebraically closed field $F$ of characteristic $p \geq 7$, and that $L'$ is another finite dimensional simple Lie algebra containing $L$ with a CSA $H'$ such that $H' \cap L$ is a CSA of $L$.

Then (i) $L + H'$ is a subalgebra of $L'$;
(ii) if $L + H'$ is an indecomposable subalgebra of $L'$, then $L + H'$ becomes a Hypo-Lie algebra and the maximal dimension of irreducible $(L + H')$-modules equals that of irreducible $L$-modules.

**References**

[1] Cheol Kim, SeokHyun Koh, and Y.G. Kim *Some study of Lee’s basis and Park’s Lie algebra*, Pragmatic algebra, SAS international Publications(2006), 21-38
[2] Y.G. Kim *Hypo-Lie algebras and their representations*, Journal of Algebras and Discrete Structures vol.8, No.1(2010), pp.1-5
[3] Kim YangGon *Counter examples to the nonrestricted representation theory* (2020) [http://arxiv.org/abs/1912.10849](http://arxiv.org/abs/1912.10849)
[4] A. Premet *Support varieties of nonrestricted modules over Lie algebras of reductive groups*, Journal of the London mathematical society Vol.55, part2, 1997.
[5] A.N. Rudakov and I.R. Shafarevich *Irreducible representations of a simple three dimensional Lie algebra over a field of finite characteristic*, Math. Notes Acad. Sci. USSR 2(1967), pp.760-767
[6] H. Strade and R. Farnsteiner *Modular Lie algebras and their representations*, Marcel Dekker, 1988

emeritus professor Department of Mathematics, Jeonbuk National University, Republic of Korea.

*E-mail address: kyi.chonbuk@hanmail.net*