Remarks on 2-Groups

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March 29, 2022

Abstract

A 2-group is a ‘categorified’ version of a group, in which the underlying set $G$ has been replaced by a category and the multiplication map $m: G \times G \to G$ has been replaced by a functor. A number of precise definitions of this notion have already been explored, but a full treatment of their relationships is difficult to extract from the literature. Here we describe the relation between two of the most important versions of this notion, which we call ‘weak’ and ‘coherent’ 2-groups. A weak 2-group is a weak monoidal category in which every morphism has an inverse and every object $x$ has a ‘weak inverse’: an object $y$ such that $x \otimes y \cong 1 \cong y \otimes x$. A coherent 2-group is a weak 2-group in which every object $x$ is equipped with a specified weak inverse $\bar{x}$ and isomorphisms $i_x: 1 \to x \otimes \bar{x}$, $e_x: \bar{x} \otimes x \to 1$ forming an adjunction. We define 2-categories of weak and coherent 2-groups and construct an ‘improvement’ 2-functor which turns weak 2-groups into coherent ones; using this one can show that these 2-categories are biequivalent. We also internalize the concept of a coherent 2-group. This gives a way of defining topological 2-groups, Lie 2-groups, and the like.

1 Introduction

Group theory has proven to be a powerful tool not only in mathematics, but also in physics, chemistry and other sciences. In recent times it has become evident that in many contexts where we are tempted to use groups, it is actually more natural to use a richer sort of structure, namely a kind of ‘higher-dimensional’ group. One might also call this a ‘categorified’ group, since the underlying set $G$ of a traditional group has been replaced by a category and the multiplication function $m: G \times G \to G$ has been replaced by a functor. To hint at a sequence of further generalizations where we use $n$-categories and $n$-functors, we call this sort of thing a ‘2-group’.
There are various different ways to make the concept of 2-group more precise. Some can already be found in the mathematical literature, but unfortunately they often remain implicit in work that focuses on more general concepts. Indeed, while the basic facts about 2-groups are familiar to most experts in category theory, it is impossible for beginners to find a unified presentation of this material with all the details provided. The present paper tries to start filling this gap.

Whenever one categorifies a mathematical concept, there are some choices involved. For example, one might define a 2-group simply to be a category $G$ equipped with functors describing multiplication, inverses and the identity, satisfying the usual group axioms ‘on the nose’ — that is, as equations between functors. We call this a ‘strict’ 2-group. Strict 2-groups have been applied in a variety of contexts, including homotopy theory [3, 4], topological quantum field theory [19], and gauge theory [1]. Part of the charm of strict 2-groups is that they can be defined in a large number of equivalent ways, including:

1. a strict monoidal category in which all objects and morphisms are invertible,
2. a strict 2-category with one object in which all 1-morphisms and 2-morphisms are invertible,
3. a group object in Cat (also called a ‘categorical group’)
4. a category object in Grp,
5. a crossed module.

There is an excellent review article by Forrester-Barker that explains most of these notions and why they are equivalent [8].

However, as the notion of a group takes on this higher dimensional form, the exact definition most suited for a given task becomes less obvious. For instance, rather than imposing the group axioms as equational laws, we could instead require that they hold up to specified isomorphisms satisfying laws of their own. This leads to the concept of a ‘coherent 2-group’.

For example, given objects $x, y, z$ in a strict 2-group we have

$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

where we write multiplication as $\otimes$. In a coherent 2-group, we instead specify an isomorphism called the ‘associator’:

$$a_{x,y,z}: (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z).$$

Similarly, we replace the left and right unit laws

$$1 \otimes x = x, \quad x \otimes 1 = x$$

by isomorphisms

$$\ell_x: 1 \otimes x \xrightarrow{\sim} x, \quad r_x: x \otimes 1 \xrightarrow{\sim} x.$$
and replace the equations

\[ x \otimes x^{-1} = 1, \quad x^{-1} \otimes x = 1 \]

by isomorphisms called the ‘unit’ and ‘counit’.

Next, in order to manipulate these isomorphisms with some of the same facility as with equations, we require that they satisfy conditions known as ‘coherence laws’. The coherence laws for the associator and the left and right unit laws were developed by Mac Lane [14] in his groundbreaking work on monoidal categories, while those for the unit and counit are familiar from the definition of an adjunction in a monoidal category [11]. Putting these ideas together, one obtains Ulbrich and Laplaza’s definition of a ‘category with group structure’ [13, 18]. Finally, a ‘coherent 2-group’ is a category \( G \) with group structure in which all morphisms are invertible. This last condition ensures that there is a covariant functor

\[ \text{inv}: G \to G \]

going from each object \( x \in G \) to its weak inverse \( \bar{x} \); otherwise there will only be a contravariant functor of this sort.

In this paper we compare this sort of 2-group to a simpler sort, which we call a ‘weak 2-group’. This is a weak monoidal category in which every morphism has an inverse and every object \( x \) has a ‘weak inverse’: an object \( y \) such that \( y \otimes x \cong 1 \) and \( x \otimes y \cong 1 \). Note that in this definition, we do not specify the weak inverse \( y \) or the isomorphisms from \( y \otimes x \) and \( x \otimes y \) to 1; nor do we impose any coherence laws upon them. Instead, we merely demand that they exist. Nonetheless, it turns out that any weak 2-group can be improved to become a coherent one! While this follows immediately from a theorem of Laplaza [13], it seems worthwhile to give an expository account here, and to formalize this process as a 2-functor

\[ \text{Imp}: \text{W2G} \to \text{C2G} \]

where \( \text{W2G} \) and \( \text{C2G} \) are suitable 2-categories of weak and coherent 2-groups, respectively.

To do this, we start in Section 2 by defining weak 2-groups and the 2-category \( \text{W2G} \). In Section 3 we define coherent 2-groups and the 2-category \( \text{C2G} \). In Section 4 we show that the concept of ‘coherent 2-group object’ can be defined in any 2-category with finite products. This allows us to define notions such as ‘coherent topological 2-group’, ‘coherent Lie 2-group’ and the like. While this may seem a bit of a digression, it serves as an excellent excuse to introduce the technique of string diagrams [15], which turn out to be crucial for constructing the 2-functor \( \text{Imp}: \text{W2G} \to \text{C2G} \). We construct this 2-functor in Section 5. Together with the forgetful 2-functor \( \text{F}: \text{C2G} \to \text{W2G} \), this sets up a ‘biequivalence’ between \( \text{W2G} \) and \( \text{C2G} \).

In other words, the 2-category of weak 2-groups and the 2-category of coherent 2-groups are ‘the same’ in a suitably weakened sense. Thus there is not really too much difference between weak and coherent 2-groups: we can freely
translate theorems about one into theorems about the other using the 2-functors Imp: $W2G \to C2G$ and $F: C2G \to W2G$.

**Note:** in all that follows, we write the composite of morphisms $f: x \to y$ and $g: y \to z$ as $fg: x \to z$.

## 2 Weak 2-groups

Before we define a weak 2-group, recall that a **weak monoidal category** consists of:

(i) a category $M$,

(ii) a functor $m: M \times M \to M$, where we write $m(x, y) = x \otimes y$ and $m(f, g) = f \otimes g$ for objects $x, y, \in M$ and morphisms $f, g$ in $M$,

(iii) an ‘identity object’ $1 \in M$,

(iv) natural isomorphisms

\[ a_{x,y,z}: (x \otimes y) \otimes z \to x \otimes (y \otimes z), \]

\[ \ell_x: 1 \otimes x \to x, \]

\[ r_x: x \otimes 1 \to x, \]

such that the following diagrams commute for all objects $x, y, z, w \in M$:

\[
\begin{array}{ccc}
(x \otimes y) (z \otimes w) & \xrightarrow{a_{x,y,z,w}} & ((x \otimes y) \otimes z) \otimes w \\
& \xleftarrow{a_{x,y,z,w}} & x \otimes (y \otimes (z \otimes w)) \\
(x \otimes (y \otimes z)) \otimes w & \xrightarrow{a_{x,y,z,w}} & x \otimes ((y \otimes z) \otimes w) \\
& \xleftarrow{a_{x,y,z,w}} & (x \otimes y) (z \otimes w)
\end{array}
\]

A **strict monoidal category** is the special case where $a_{x,y,z}, \ell_x, r_x$ are all identity morphisms. In this case we have

\[(x \otimes y) \otimes z = x \otimes (y \otimes z),\]
As mentioned in the Introduction, a **strict 2-group** is a strict monoidal category where every morphism is invertible and every object $x$ has an inverse $x^{-1}$, meaning that $x \otimes x^{-1} = 1, \quad x^{-1} \otimes x = 1$.

Following the principle that it is wrong to impose equations between objects in a category, we can instead start with a weak monoidal category and require that every object has a 'weak' inverse. With these changes we obtain the definition of 'weak 2-group':

**Definition 1.** If $x$ is an object in a weak monoidal category, a **weak inverse** for $x$ is an object $y$ such that $x \otimes y \sim 1$ and $y \otimes x \sim 1$. If $x$ has a weak inverse, we call it **weakly invertible**.

**Definition 2.** A **weak 2-group** is a weak monoidal category where all objects are weakly invertible and all morphisms are invertible.

Weak 2-groups are the objects of a 2-category $\mathbf{W2G}$; now let us describe the morphisms and 2-morphisms in this 2-category. Notice that the only structure in a weak 2-group is that of its underlying weak monoidal category; the invertibility conditions on objects and morphisms are only properties. With this in mind, it is natural to define a morphism between weak 2-groups to be a weak monoidal functor. Recall that a **weak monoidal functor** $F: C \to C'$ between monoidal categories $C$ and $C'$ consists of:

(i) a functor $F: C \to C'$,

(ii) a natural isomorphism $F_2: F(x) \otimes F(y) \to F(x \otimes y)$, where for brevity we suppress the subscripts indicating the dependence of this isomorphism on $x$ and $y$,

(iii) an isomorphism $F_0: 1' \to F(1)$, where 1 is the unit object of $C$ and $1'$ is the unit object of $C'$,

such that the following diagrams commute for all objects $x, y, z \in C$:

\[
\begin{array}{ccc}
(F(x) \otimes F(y)) \otimes F(z) & \xrightarrow{F_2 \otimes 1} & F(x \otimes y) \otimes F(z) & \xrightarrow{F_2} & F((x \otimes y) \otimes z) \\
 \text{\small $\alpha_{F(x), F(y), F(z)}$} & & & & \text{\small $F(\alpha_{x, y, z})$} \\
F(x) \otimes (F(y) \otimes F(z)) & \xrightarrow{1 \otimes F_2} & F(x) \otimes F(y \otimes z) & \xrightarrow{F_2} & F(x \otimes (y \otimes z)).
\end{array}
\]

\[
\begin{array}{ccc}
1' \otimes F(x) & \xrightarrow{\ell_{F(x)}} & F(x) \\
\text{\small $F_0 \otimes 1$} & & \text{\small $F(\ell_x)$} \\
F(1) \otimes F(x) & \xrightarrow{F_2} & F(1 \otimes x)
\end{array}
\]
A weak monoidal functor preserves all the structure of a weak monoidal category up to specified isomorphisms. Moreover, if $C$ and $C'$ are weak 2-groups, a weak monoidal functor $F: C \rightarrow C'$ also preserves weak inverses:

**Proposition 3.** If $F: C \rightarrow C'$ is a monoidal functor between weak 2-groups $C$ and $C'$ and $y \in C$ is a weak inverse of $x \in C$, then $F(y)$ is a weak inverse of $F(x)$ in $C'$.

**Proof.** Since $y$ is a weak inverse of $x$, there must exist isomorphisms $\gamma: x \otimes y \rightarrow 1$ and $\xi: y \otimes x \rightarrow 1$. The proposition is then established by composing the following isomorphisms:

\[
\begin{array}{ccc}
F(x) \otimes 1' & \xrightarrow{r(x)} & F(x) \\
1 \otimes F_0 & \downarrow & F(r_x) \\
F(x) \otimes F(1) & \xrightarrow{F_2} & F(x \otimes 1).
\end{array}
\]

We thus make the following definition:

**Definition 4.** A homomorphism $F: C \rightarrow C'$ between weak 2-groups is a weak monoidal functor.

The composite of weak monoidal functors is again a monoidal functor [7], and composition satisfies associativity and the unit laws. Thus, 2-groups and the homomorphisms between them form a category.

Although no direct counterpart can be found in traditional group theory, it is natural in this categorified context to also consider ‘2-homomorphisms’ between homomorphisms. Since a homomorphism between weak 2-groups is just a weak monoidal functor, it makes sense to define 2-homomorphisms to be weak monoidal natural transformations. Recall that if $F, G: C \rightarrow C'$ are weak monoidal functors, then a weak monoidal natural transformation $\theta: F \Rightarrow G$ is a natural transformation such that these diagrams commute for all $x, y \in C$:

\[
\begin{array}{ccc}
F(x) \otimes F(y) & \xrightarrow{\theta_x \otimes \theta_y} & G(x) \otimes G(y) \\
F_2 & \downarrow & G_2 \\
F(x \otimes y) & \xrightarrow{\theta_{x \otimes y}} & G(x \otimes y)
\end{array}
\]
commute. Thus we make the following definitions:

**Definition 5.** A 2-homomorphism \( \theta : F \Rightarrow G \) between homomorphisms \( F, G : C \to C' \) of weak 2-groups is a weak monoidal natural transformation.

**Definition 6.** Let \( W2G \) be the 2-category consisting of weak 2-groups, homomorphisms between these, and 2-homomorphisms between those.

There is a 2-category \( \text{MonCat} \) with monoidal categories as objects, weak monoidal functors as 1-morphisms and weak monoidal natural transformations as 2-morphisms [7]. Thus \( W2G \) a 2-category, since it is a full and 2-full sub-2-category of \( \text{MonCat} \).

### 3 Coherent 2-groups

In this section we explore another notion of 2-group. Rather than requiring that objects be weakly invertible, we will require that every object be equipped with a specified adjunction. Recall that an adjunction is a quadruple \( (x, \bar{x}, i_x, e_x) \) where \( i_x : 1 \xrightarrow{\sim} x \otimes \bar{x} \) (called the unit) and \( e_x : \bar{x} \otimes x \xrightarrow{\sim} 1 \) (called the counit) are morphisms such that the following diagrams commute. For reasons that will become apparent in the sections to come we refer to these diagrams as the first and second zig-zag identities, respectively.

An adjunction \( (x, \bar{x}, i_x, e_x) \) for which the unit and counit are invertible is called an adjoint equivalence. In this case \( x \) and \( \bar{x} \) are weak inverses. Thus, specifying an adjoint equivalence for \( x \) ensures that \( \bar{x} \) is weakly invertible — but it does so by providing \( x \) with extra structure, rather than merely asserting a property of \( x \). We now make the following definition:
**Definition 7.** A **coherent 2-group** is a weak monoidal category \(C\) in which every morphism is invertible and every object \(x \in C\) is equipped with an adjoint equivalence \((x, \bar{x}, i_x, e_x)\).

As noted in the Introduction, a coherent 2-group is the same as a category with group structure \([13, 18]\) in which all morphisms are invertible. It is also the same as an ‘autonomous monoidal category’ \([11]\) with all morphisms invertible, or a ‘bigroupoid’ \([9]\) with one object.

As we did with weak 2-groups, we can define homomorphisms between coherent 2-groups. As in the weak 2-group case we can begin with a weak monoidal functor, but now we must consider what additional structure this must have to preserve each adjoint equivalence \((x, \bar{x}, i_x, e_x)\), at least up to a specified isomorphism. At first it may seem that an additional structural map is required. That is, if \(F: C \to C'\) is a weak monoidal functor it may seem that we must include a natural isomorphism
\[ F_{-1}: F(x) \to F(\bar{x}) \]
relating the inverse of the image of \(x\) to the image of the inverse \(\bar{x}\). We shall show this is not the case: \(F_{-1}\) can be constructed from the data already present! Thus we make the following definitions:

**Definition 8.** A **homomorphism** \(F: C \to C'\) between coherent 2-groups is a weak monoidal functor.

**Definition 9.** A **2-homomorphism** \(\theta: F \Rightarrow G\) between homomorphisms \(F, G: C \to C'\) of coherent 2-groups is a weak monoidal natural transformation.

**Definition 10.** Let \(\text{C2G}\) be the 2-category consisting of coherent 2-groups, homomorphisms between these, and 2-homomorphisms between those.

It is clear that \(\text{C2G}\) forms a 2-category since it is actually a full and 2-full sub-2-category of \(\text{MonCat}\).

Now let us describe how to define \(F_{-1}\) in terms of the other data in a coherent 2-group homomorphism \(F: C \to C'\). By analogy with \(F_2\) and \(F_0\), we would expect \(F_{-1}\) to make these diagrams commute:

**H1**

\[
\begin{array}{ccc}
F(x) \otimes F(x) & \xrightarrow{1 \otimes F_{-1}} & F(x) \otimes F(\bar{x}) \\
\downarrow F_{i_x} & & \downarrow F(i_x) \\
1' & \xrightarrow{F_0} & F(1)
\end{array}
\]

**H2**

\[
\begin{array}{ccc}
F(x) \otimes F(x) & \xrightarrow{F_{-1} \otimes 1} & F(\bar{x}) \otimes F(x) \\
\downarrow F_{e_x} & & \downarrow F(e_x) \\
1' & \xrightarrow{F_0} & F(1)
\end{array}
\]
for all $x \in C$. These diagrams say that $F_{-1}$ gets along with units and counits.

Suppose we wish to construct an isomorphism that simultaneously satisfies both of these coherence laws. To do this we can take one of these coherence laws, solve it for $F_{-1}$, and prove that the result automatically satisfies the other coherence law! To do this, we start with the axiom $H1$ expressed in a more suggestive manner:

![Diagram](image)

If we assume this diagram commutes, it gives a formula for

$$1 \otimes F_{-1}: F(x) \otimes \overline{F(x)} \xrightarrow{\sim} F(x) \otimes F(\bar{x}).$$

Next we shall solve for $F_{-1}$ by cancelling the tensor product $F(x) \otimes$. In the arguments that follow we will no longer include subscripts on any of the structural maps $a, \ell, r, e$ or $i$ unless confusion is likely to arise. Tensoring the above morphism by $F(x)$ we obtain

$$1 \otimes (1 \otimes F_{-1}): \overline{F(x)} \otimes (F(x) \otimes \overline{F(x)}) \xrightarrow{\sim} F(x) \otimes F(\bar{x}) \overline{F(x)} \otimes (F(x) \otimes F(\bar{x}))$$

However, the left-hand side is isomorphic to $\overline{F(x)}$ via the following composite

$$\overline{F(x)} \xrightarrow{\ell^{-1}} (\overline{F(x)}) \xrightarrow{\epsilon^{-1} \otimes 1} (F(x) \otimes F(x)) \otimes (F(x) \otimes F(x)) \xrightarrow{a} (F(x) \otimes F(x)) \otimes (F(x) \otimes F(x)).$$

Further, the right hand side is isomorphic to $F(\bar{x})$ via the following composite:

$$\overline{F(x) \otimes (F(x) \otimes F(\bar{x}))} \xrightarrow{a} (F(x) \otimes F(\bar{x})) \otimes (F(x) \otimes F(\bar{x})) \xrightarrow{e \otimes 1} (\overline{F(x) \otimes F(x)}) \otimes (\overline{F(x) \otimes F(x)}) \xrightarrow{\ell} \overline{F(x)}.$$

Stringing together these isomorphisms we obtain an isomorphism from $\overline{F(x)}$ to $F(\bar{x})$ which is none other than $F_{-1}$. In other words, we have a commutative diagram which serves to define $F_{-1}$:
We have derived this diagram \( \mathbf{F1} \) from the assumption \( \mathbf{H1} \); conversely one can show that \( \mathbf{F1} \) implies \( \mathbf{H1} \).

As a side remark, note that in the above diagram could have mapped \( \overline{F(x)} \) directly to \( \overline{F(x)} \otimes 1' \) using \( r^{-1} \). This provides an alternative definition of \( F_{-1} \):

\[ \mathbf{F1'} \]

The assertion that these definitions agree is equivalent to the fact that this
Careful inspection reveals that this diagram is none other than the second zig-zag identity satisfied by the adjoint equivalence \((x, \bar{x}, i_x, e_x)\)! So, there is no problem of conflicting choices here. In fact, this is guaranteed by Ulbrich and Laplaza’s coherence theorem for categories with group structure [13, 18].

Alternatively, we could have defined \(F_{-1}\) so that \(H2\) is satisfied. Then this diagram commutes:

Tensoring on the right by \(\bar{F}(x)\) and applying a similar argument as before we obtain two additional possibilities for \(F_{-1}\), which again are actually equal thanks to the second zig-zag identity. We denote this way of defining \(F_{-1}\) simply as \(F2\).
In fact, just as $F_1$ is equivalent to $H_1$, $F_2$ is equivalent to $H_2$. Next, we would like to show that $F_1$ and $F_2$ give the same definition of $F_{-1}$, so that using either to define this isomorphism guarantees that both $H_1$ and $H_2$ are satisfied. To accomplish this we shall assume $H_1$ and use this to establish $F_2$. 
Consider the following diagram:

Squares I, IV, VIII, X, XI, and XII commute from the naturality of
the isomorphisms \( \ell, F_2, a, e_x \). Application of \( F \) to the second zig-zag law gives
II. Diagrams III, VI and VII commute by the definition of weak monoidal
functor, while IX commutes by a well-known property of monoidal categories.
Diagram V is merely \( H_1 \) tensored on the left by \( F(\bar{x}) \). Thus, our choice of
\( F^{-1} \) satisfies both \( H_1 \) and \( H_2 \). It follows that \( F^{-1} \) and its coherence laws are
superfluous to the definition of a coherent 2-group homomorphism; we get them
‘for free’.

4 Internalization

‘Internalization’ is valuable tool for generalizing concepts from the category of
sets to other categories. To internalize a concept, we need to express it in a
purely diagrammatic form. As an example, consider the ordinary notion of
group. We can define this notion using commutative diagrams by specifying:

(i) a set \( G \),
(ii) a multiplication function \( m: G \times G \to G \),

(iii) a unit for the multiplication given by the function \( e: 1 \to G \) where 1 is the terminal object in Set,

(iv) a function \( \text{inv}: G \to G \),

making the following diagrams commute:

where \( \Delta_G \) is the diagonal map. To internalize this concept one replaces the set \( G \) by an object in an arbitrary category \( C \) with finite products, and the functions \( m, e, \) and \( \text{inv} \) by morphisms in \( C \). Making these substitutions in the definition above we arrive at the definition of a group object in \( C \). An ordinary group is the special case where \( C = \text{Set} \). We can also define a strict 2-group to be a group object in \( \text{Cat} \). Similarly, a topological group is a group object in \( \text{Top} \), and a Lie group is a group object in \( \text{Diff} \).

We would like to define a ‘coherent 2-group object’ in a similar manner. To motivate this definition it is helpful to use ‘string diagrams’ [10, 15]. These are Poincaré dual to the globular diagrams previously used in the theory of bicategories. In other words, to obtain a string diagram one draws objects as 2-dimensional regions in the plane, 1-morphisms as 1-dimensional ‘strings’ separating regions, and 2-morphisms as 0-dimensional points (or small balls, if we wish to label them). We will only need these diagrams in the special case of a weak monoidal category, which we will think of as a bicategory with a single object, say \( \bullet \). A morphism \( f: x \to y \) in a weak monoidal category corresponds to a 2-morphism in a bicategory with one object, and we convert the globular picture of this into a string diagram as follows:
where diagrams are read from top to bottom. Composition of morphisms is achieved by placing one of the strings on top of the other. For instance, given $f: x \to y$ and $g: y \to z$, their composite is depicted as the following:

$$f \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow x \quad g \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow z$$

This diagram is Poincaré dual to the globular way of drawing composition of 2-morphisms in a bicategory:

$$x \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow x \quad = \quad x \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow x$$

Tensoring objects in the monoidal category will be written by setting arrows side by side; the unit object will not be drawn in the diagrams, but merely implied. As an example of this, consider how we obtain the string diagram corresponding to $i_x: 1 \to x \otimes \bar{x}$:

$$1 \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow x \quad \leadsto \quad \bar{x} \quad \Rightarrow \quad x \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \bar{x}$$

Note that weak inverse objects are written as arrows ‘going backwards in time’. We will find it beneficial to draw the above diagram in a simpler form:

where it is understood that the downward pointing arrow corresponds to $x$ and the upward pointing arrow to $\bar{x}$. Similarly, we draw the morphism $e_x$ as

In this notation, the zig-zag identities become

$$\bigwedge \bigwedge = x \quad , \quad \bigwedge \bigwedge = x$$

15
which explains their name.

We would like to define a coherent 2-group using only commutative diagrams, so that the groundwork will have been laid for the definition of the more general notion of a ‘coherent group object’ in a 2-category with products. To begin, notice that for any coherent 2-group \( C \) there is a functor \( -1 : C^{\text{op}} \to C \), expressed diagrammatically as

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \\
\uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\( \xrightarrow{\begin{array}{c}
f \\
y
\end{array}} \xrightarrow{\begin{array}{c}
f^{-1} \\
x
\end{array}} \)

and coming from the invertibility of morphisms in a coherent 2-group. There is also a functor \( * : C^{\text{op}} \to C \) sending each object \( x \in C \) to its specified weak inverse \( \bar{x} \), and acting on morphisms as follows:

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \\
\uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\( \xrightarrow{\begin{array}{c}
f \\
y
\end{array}} \xrightarrow{\begin{array}{c}
f^{-1} \\
x
\end{array}} \)

Composing these two contravariant functors \( -1 \) and \( * \) we construct a covariant functor \( \text{inv} : C \to C \) given by

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \\
\uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\( \xrightarrow{\begin{array}{c}
f \\
y
\end{array}} \xrightarrow{\begin{array}{c}
f^{-1} \\
x
\end{array}} \)

In order to prove the functoriality of \( \text{inv} \), consider two composable morphisms \( f : x \to y \) and \( g : y \to z \). The equation \( \text{inv}(fg) = \text{inv}(f)\text{inv}(g) \) becomes the following in string diagram notation:

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \\
\uparrow \uparrow \uparrow \uparrow
\end{array}
\end{array}
\]

\( \xrightarrow{\begin{array}{c}
(fg)^{-1} \\
e_x
\end{array}} \xrightarrow{\begin{array}{c}
f^{-1} \\
e_x
\end{array}} \xrightarrow{\begin{array}{c}
g^{-1} \\
e_y
\end{array}} \)

16
Thus, in order for equality to hold we must have

\[
\begin{array}{c}
\downarrow \\
i_y \\
\uparrow
\end{array}
\quad = 
\begin{array}{c}
\downarrow \\
e_y \\
\uparrow
\end{array}
\]

This diagram is merely the second zig-zag identity!

Now, when all the unit and counit are isomorphisms either of the zig-zag identities implies the other [17]. In this case, assuming that \( \text{inv} \) is a functor implies both zig-zag identities. This observation allows us to give an equivalent definition of ‘coherent 2-group’:

**Definition 11.** A **coherent 2-group** consists of:

(i) a category \( C \),

(ii) a functor \( m: C \times C \to C \), where we write \( m(x, y) = x \otimes y \) and \( m(f, g) = f \otimes g \) for objects \( x, y, \in C \) and morphisms \( f, g \) in \( C \),

(iii) a functor \( \text{id}: I \to C \) where \( I \) is the terminal category, and we write the object in the range of this functor as \( 1 \in C \),

(iv) a functor \( \text{inv}: C \to C \),

(v) natural isomorphisms \( a, \ell, r, e, i \) as follows:
making the following diagrams commute:

A1

\[
\begin{align*}
&(x \otimes y)(z \otimes w) \\
&((x \otimes y) \otimes z) \otimes w \\
&(x \otimes (y \otimes z)) \otimes w
\end{align*}
\]

A2

\[
\begin{align*}
&(x \otimes 1) \otimes y \\
&x \otimes (1 \otimes y)
\end{align*}
\]
We are now almost ready to define a ‘coherent 2-group object’ in an arbitrary 2-category with finite products. Before we can do this, we must be careful to phrase the definition in a way that does not use any properties of a particular 2-category and its objects (in this case Cat). Rather, we must formulate the definition in a purely 2-categorical way. We have not done this yet in Definition 11, since we made explicit use of the fact that $C$ was a category: the commutative diagrams for the natural isomorphisms have objects of $C$ labelling their vertices. We must correct this oversight. The problem, well-known to experts, is that making explicit mention of the objects of $C$ has suppressed one dimension in the coherence diagrams. This can be seen in axiom A5:

The 2-dimensional appearance of this diagram results from mentioning the objects $x, y, z, w \in C$. We can avoid this by working with (for example) the functor $(1 \times 1 \times m) \circ (1 \times m) \circ m$ instead of its value on the object $(x, y, z, w) \in C^4$, namely $x \otimes (y \otimes (z \otimes w))$. If we do this, we see that the diagram is actually 3-dimensional! It is a pentagonal prism, a bit difficult to draw:
where the downwards-pointing single arrows are functors from $C^4$ to $C$, and the horizontal double arrows are natural transformations between these functors, forming a commutative pentagon. Luckily we can also draw this pentagon in a 2-dimensional way again, as follows:

This style of writing coherence laws allows us to make the following definition:

**Definition 12.** A coherent 2-group object in a 2-category $K$ with finite products consists of

(i) an object $C \in K$,

(ii) a morphism $m: C \times C \rightarrow C$,

(iii) a morphism $\text{id}: I \rightarrow C$ where $I$ is the terminal object of $K$,

(iv) a morphism $\text{inv}: C \rightarrow C$, 

20
(v) 2-isomorphisms \( a, \ell, r, e, i \) as follows:

![Diagram](image)

making the following diagrams commute:
Proposition 13. A coherent 2-group object in $\text{Cat}$ is a coherent 2-group.

Proof. We prove this merely by noting that the morphisms in $\text{Cat}$ are functors and the 2-morphisms are natural transformations. With these substitutions Definition 12 becomes Definition 11. □

We can define a topological 2-group to be a coherent 2-group object in $\text{TopCat}$, the 2-category of topological categories. Similarly, we can define a Lie 2-group to be a coherent 2-group object in $\text{DiffCat}$, the 2-category of smooth categories. Here ‘topological categories’ are categories internal to Top, while ‘smooth categories’ are categories internal to Diff; concepts of this sort were first studied by Ehresmann [5, 6] Baez has used strict Lie 2-groups in his work on categorified gauge theory [1], and it should be interesting to extend these ideas to general Lie 2-groups.

It seems difficult to internalize the notion of a weak 2-group as we have just done for coherent 2-groups. Naively, the definition of a ‘weak 2-group object’
would require that \( \text{inv} \) be a morphism in the ambient 2-category \( K \). In the case \( K = \text{Cat} \) this means that \( \text{inv} \) is a functor. However, requiring that \( \text{inv} \) be a functor implies the zig-zag identities, leading us back to the notion of coherent 2-group.

5 Improvement

In this section we show that any weak 2-group can be improved to a coherent one, using the technique of string diagrams [10, 15]. In a strict monoidal category, one can interpret any string diagram as a morphism in a unique way. With the help of Mac Lane’s coherence theorem [14] we can also do this in a weak monoidal category. To do this, we interpret any string of objects and 1’s as a tensor product of objects where all parentheses start in front and all 1’s are removed. Using the associator and left/right unit laws to do any necessary reparenthesization and introduction or elimination of 1’s, any string diagram then describes a morphism between tensor products of this sort. The fact that this morphism is unambiguously defined follows from Mac Lane’s coherence theorem.

Let \( C \) be a weak 2-group. We will only need string diagrams where all the strings are labelled by \( x \) and \( \bar{x} \), where \( x \) is some fixed object of \( C \) and \( \bar{x} \) is a chosen weak inverse for \( x \). Thus we can omit these labels and just use downwards or upwards arrows on our strings to distinguish between \( x \) and \( \bar{x} \). We fix isomorphisms \( i_x: 1 \xrightarrow{\sim} x \otimes \bar{x} \) and \( e_x: \bar{x} \otimes x \xrightarrow{\sim} 1 \). We draw these just as we did in Section 4; however, they need not satisfy the zig-zag identities. Nonetheless, if we write \( i_x \) as

\[
\begin{array}{c}
\uparrow \downarrow \\
\downarrow \uparrow \\
\end{array}
\]

\( i_x^{-1} \) as

\[
\begin{array}{c}
\downarrow \uparrow \\
\uparrow \downarrow \\
\end{array}
\]

\( e_x \) as

\[
\begin{array}{c}
\downarrow \uparrow \\
\uparrow \downarrow \\
\end{array}
\]

and \( e_x^{-1} \) as

\[
\begin{array}{c}
\downarrow \uparrow \\
\uparrow \downarrow \\
\end{array}
\]

we obtain some rules for manipulating string diagrams just from the fact that these morphisms are inverses of each other. For instance, the equations \( i_x i_x^{-1} = \)}
$1_1$ and $e_x^{-1}e_x = 1_1$ give the rules

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1.png}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram2.png}
\end{array}
\]

These rules mean that in a string diagram, a loop of either form may be removed or inserted without changing the morphism described by the diagram. Similarly, the equations $e_x e_x^{-1} = 1_{x \otimes x}$ and $i_x^{-1} i_x = 1_{x \otimes x}$ give the rules

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram3.png}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram4.png}
\end{array}
\]

Again, these rules mean that in a string diagram we can modify any portion as above without changing the morphism in question. We shall also need another rule, the ‘horizontal slide’:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram5.png}
\end{array}
\]

This follows from general results on string diagrams [11], but it is easy to prove directly. First, write down the corresponding globular diagram:

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{globular_diagram.png}
\end{array}
\]

Then, use the bicategory axioms [2] and Mac Lane’s coherence theorem to manipulate this diagram in the following manner:

\[
\begin{array}{c}
\includegraphics[width=0.7\textwidth]{final_diagram.png}
\end{array}
\]
Rewriting the end result as a string diagram, the result follows. A similar argument proves another version of the horizontal slide:

With the help of these rules we now prove:

**Theorem 14.** Given any weak 2-group $C$, it can be improved to a coherent 2-group $\text{Imp}(C)$ by equipping each object with an adjoint equivalence.

**Proof.** First, for each object $x$ we choose a weak inverse $\bar{x}$ and isomorphisms $i_x: 1 \to x \otimes \bar{x}$, $c_x: \bar{x} \otimes x \to 1$. From this data we construct an adjoint equivalence $(x, \bar{x}, i'_x, c_x)$ where $i'_x$ is defined as the following composite morphism:

$$
1 \xrightarrow{i_x} x\bar{x} \xrightarrow{x,c_x^{-1}} x(1\bar{x}) \xrightarrow{x,c_{\bar{x}x}^{-1} x} x((\bar{x}x)\bar{x}) \xrightarrow{x,a_{\bar{x}x,x}^{-1} x} x(\bar{x}(x\bar{x})) \xrightarrow{a_{x,\bar{x},x}^{-1} x} (x\bar{x})(x\bar{x})
$$

$$
1 \xrightarrow{i^{-1}_x(x\bar{x})} x\bar{x} \xrightarrow{x,c_{\bar{x}x}^{-1} x} x(1\bar{x}) \xrightarrow{x,c_x^{-1} x} x((1\bar{x})\bar{x}) \xrightarrow{x,a_{\bar{x}1,x}^{-1} x} x(\bar{x}(1\bar{x})) \xrightarrow{a_{x,\bar{x},x}^{-1} x} (1\bar{x})(x\bar{x}) \xrightarrow{\ell_x \bar{x}} x\bar{x}.
$$
where we omit tensor product symbols for brevity.

The above rather cryptic formula for $i_x'$ becomes much clearer if we use pictures. If we think of a weak 2-group as a one-object bicategory and write this formula in globular notation it becomes:

![Globular notation of the formula](image)

If we write it as a string diagram it looks even simpler:

![String diagram](image)

Now let us show that $(x, \bar{x}, i_x', e_x)$ satisfies the zig-zag identities. Recall that these identities say that the following diagrams commute:

\[
\begin{align*}
1 \otimes x \xrightarrow{i_x' \otimes 1} (x \otimes \bar{x}) \otimes x \xrightarrow{a_{x,\bar{x},x}} x \otimes (\bar{x} \otimes x) & \quad x \otimes 1 \xrightarrow{1 \otimes i_x'} \bar{x} \otimes (x \otimes \bar{x}) \xrightarrow{a_{\bar{x},x,\bar{x}}^{-1}} (\bar{x} \otimes x) \otimes \bar{x}.
\end{align*}
\]

Utilizing the observation made regarding Mac Lane’s coherence theorem we can express the zig-zag identities in globular notation as follows:

![Globular notation of the zig-zag identities](image)
If we express $i'_x$ in terms of $i_x$ and $e_x$, these equations become

$$\begin{align*}
\bullet & \longrightarrow \bullet \\
\bar{x} & \rightarrow x \rightarrow i'_x \rightarrow \bar{x} \\
e_x & \rightarrow \tilde{x} \\
\bullet & \longrightarrow \bullet
\end{align*}$$

and

$$\begin{align*}
\bullet & \longrightarrow \bullet \\
\bar{x} & \rightarrow x \rightarrow i'_x \rightarrow \bar{x} \\
i_x & \rightarrow \tilde{x} \\
\bullet & \longrightarrow \bullet
\end{align*}$$

To verify these two equations we use string diagrams. In the calculations that follow, we denote an application of the ‘horizontal slide’ rule by a dashed line connecting the appropriate zig and zag. Dotted lines connecting two parallel strings will indicate an application of the rules $e_x e_x^{-1} = 1_{x \otimes x}$ or $i_x^{-1} i_x = 1_{x \otimes \bar{x}}$. Furthermore, the rules $i_x i_x^{-1} = 1_1$ and $e_x^{-1} e_x = 1_1$ allow us to remove a closed loop any time one appears. The first equation can be proved as follows:

$$\begin{align*}
\bullet & \longrightarrow \bullet \\
\bar{x} & \rightarrow x \rightarrow i'_x \rightarrow \bar{x} \\
e_x & \rightarrow \tilde{x} \\
\bullet & \longrightarrow \bullet
\end{align*}$$
The proof of the second equation is accomplished in a similar manner:
We can now make this ‘improvement’ process into a 2-functor Imp: W 2G → C2G:

**Theorem 15.** There exist a 2-functor Imp: W 2G → C2G which sends any object C ∈ W 2G to Imp(C) ∈ C2G and acts as the identity on morphisms and 2-morphisms.

**Proof.** The proof of this theorem is a trivial consequence of Theorem 14. Obviously all domains, codomains, identities and composites are preserved, since the 1-morphisms and 2-morphisms are unchanged as a result of Definitions 8 and 9.

On the other hand, there is also a forgetful 2-functor F: C2G → W 2G, which forgets the extra structure on objects and acts as the identity on morphisms and 2-morphisms. It is easy to see that improvement followed by this forgetful 2-functor acts as the identity of W 2G. On the other hand, applying F and then Imp to a coherent 2-group C amounts to forgetting the choice of adjoint equivalence for each object x ∈ C and then making a new such choice. We obtain a new coherent 2-group C′, but it has the same underlying weak monoidal category, so the identity functor 1_C: C → C′ is a coherent 2-group homomorphism from C to C′. It should thus not be surprising that:

**Theorem 16.** The 2-functors Imp: W 2G → C2G, F: C2G → W 2G extend to define a biequivalence between the 2-categories W 2G and C2G.

We omit the proof, but refer the reader to Bénabou [2] and Street [15] for a discussion of the concept of biequivalence. The upshot is that we can use either weak or coherent 2-groups, whichever happens to be more convenient at the time, and freely translate results between the two formalisms.

To conclude, let us summarize why weak and coherent 2-groups are not really so different. At first, the choice of a specified adjoint equivalence for each object
seems like a substantial extra structure to put on a weak 2-group. However, Theorem 14 shows that we can always succeed in putting this extra structure on any weak 2-group. Furthermore, while there are many ways to equip a weak 2-group with this extra structure, there is ‘essentially’ just one way, since the remarks at the end of Section 7 show that this structure is automatically preserved up to coherent isomorphism by any homomorphism of weak 2-groups.

Acknowledgements
I thank John Baez and James Dolan for very helpful discussions and correspondence, and also Miguel Carrión Álvarez for his assistance in making string diagrams.

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