Data-based determination of the isospin-limit light-quark-connected contribution to the anomalous magnetic moment of the muon

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We describe how recent determinations of exclusive-mode contributions to $a_{\mu}^{\text{LO,HVP}}$, the leading-order hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon, can be used to provide, up to small electromagnetic (EM) corrections accessible from the lattice, a data-based dispersive determination of $a_{\mu}^{\text{LO,HVP}}$, the isospin-limit, light-quark-connected contribution to $a_{\mu}^{\text{LO,HVP}}$. Such a determination is of interest in view of the existence of a number of lattice results for this quantity, emerging evidence for a tension between lattice and dispersive determinations of $a_{\mu}^{\text{LO,HVP}}$, and the desire to clarify the source of this tension. Taking as input for the small EM correction that must be applied to the purely data-driven dispersive determination the result $-0.93(58) \times 10^{-10}$ obtained in a recent BMW lattice study, we find $a_{\mu}^{\text{LO,HVP}}$ to be $635.0(2.7) \times 10^{-10}$ if the results of Keshavarzi, Nomura and Teubner are used for the exclusive-mode contributions and $638.4(4.1) \times 10^{-10}$ if instead those of Davier, Höcker, Malaescu and Zhang are used.
I. INTRODUCTION

As is well known, the recent FNAL E989 result [1] for the anomalous magnetic moment of the muon, \(a_\mu = (g - 2)/2\), is in good agreement with the earlier BNL E821 result [2], producing an updated world average which is 4.2\(\sigma\) larger than the Standard Model (SM) expectation detailed in the \(g - 2\) Theory Initiative assessment [3], and based on the work of Refs. [4–27].

In determining SM contributions to \(a_\mu\), a particularly important role is played by the leading-order, hadronic vacuum polarization contribution, \(a_\mu^{\text{LO,HVP}}\), the uncertainty on which dominates that on the SM expectation for \(a_\mu\). The assessment of \(a_\mu^{\text{LO,HVP}}\) arrived at in Ref. [3] is based on the results of two analyses [10, 11] using the standard dispersive representation [28–30] with current \(e^+ e^- \rightarrow \text{hadrons}\) cross-section data as input, supplemented by additional input from Refs. [12, 13] for \(\pi \pi\) contributions below 1 GeV\(^2\) and 3\(\pi\) contributions. The SM expectation for \(a_\mu^{\text{LO,HVP}}\) can, however, also be obtained on the lattice using the alternate, weighted Euclidean integral representation [31–33] and there has been intense recent activity in the lattice community aimed at reducing lattice errors to a level sufficient to make lattice results competitive with the dispersive determination [34–61]. The most recent lattice result from the BMW collaboration [50] comes close to this goal, reaching, for the first time, sub-percent precision. This result, however, is in tension with the dispersive value for \(a_\mu^{\text{LO,HVP}}\). This tension is also seen, at an enhanced (up to 3.9\(\sigma\)) level, in comparisons of dispersive [50, 62] and lattice [48, 50, 51, 55–58, 60, 61] results for the intermediate window quantity, \(a_\mu^{W}\), introduced by the RBC/UKQCD collaboration [40]. Similar tensions, reaching as much as 3.7\(\sigma\), are also found for the one-sided windows considered in Ref. [59].

Lattice results for \(a_\mu^{\text{LO,HVP}}\) are typically quoted as sums of isospin-limit light-, strange-, charm- and bottom-quark connected contributions, the disconnected contribution, and strong-isospin-breaking (SIB) and electromagnetic (EM) contributions, the latter two receiving both connected and disconnected contributions. By far the largest of these contributions is the isospin-limit light-quark-connected one, denoted \(a_\mu^{lqc; \text{IL}}\) in what follows. Although continuum/dispersive estimates exist for the SIB contribution [63, 64], and, from our recent work [65], for the sum of the strange-quark-connected and full disconnected contributions, no dispersive result exists for \(a_\mu^{lqc; \text{IL}}\). In this paper, we remedy this situation, performing an almost-completely data-based determination of \(a_\mu^{lqc; \text{IL}}\). Our result confirms the existence of a tension between the dispersive determination and the most precise of the lattice determinations [50]. We also indicate how the same analysis strategy can, in principle, be employed to obtain isospin-limit, light-quark-connected contributions to analogous reweighted/window observables such as the RBC/UKQCD intermediate window quantity, \(a_\mu^{W}\) [40], the one-sided window quantities of Ref. [59], and the window quantities of Ref. [66], based on \(s\)-dependent weights designed to produce results for the associated dispersive determinations focused on more limited regions in \(s\). Work on the reweighted exclusive-mode integrals needed to complete such analyses is in progress, and will be reported on elsewhere. In the present paper we focus on the quantity \(a_\mu^{lqc; \text{IL}}\), where existing results for exclusive-mode contributions to \(a_\mu^{\text{LO,HVP}}\) already make a dispersive determination possible.

The rest of the paper is organized as follows. In Section II we detail how, in the isospin limit, dispersive results for the exclusive-mode contributions to \(a_\mu^{\text{LO,HVP}}\) can be used to provide a dispersive determination of \(a_\mu^{lqc; \text{IL}}\). In Section III we implement this analysis, ignoring for the moment isospin-breaking (IB) corrections, using as input for the exclusive-mode contributions those determined in Refs. [10, 11]. In Section IV we discuss (and
evaluate) IB corrections to the results obtained in Sec. II. This section also contains our final results for $a_{\mu}^{\text{LO, HVP}}$. Finally, in Section III we provide a brief summary and discuss how, using more detailed exclusive-mode information, in the form of the $s$-dependence of exclusive-mode contributions to the $R$-ratio, $R(s)$, analogous results for the isospin-limit, light-quark-connected contributions to differently weighted integrals over $R(s)$, such as the window quantities noted above, can also be obtained.

II. BASIC ANALYSIS STRATEGY

A. Notation

We begin with some basic notation and associated isospin decompositions. Key objects in the analysis are the two-point functions, $\Pi_{ab}^{\mu \nu}(q)$, of the flavor-octet, vector currents $V_{\mu}^{a} = \bar{q}^{a} \gamma_{\mu} q$, $a = 1, \cdots, 8$, (where $\lambda^{a}$ are the usual Gell-Mann matrices) together with the associated polarizations, $\Pi^{ab}(Q^{2})$, subtracted polarizations, $\hat{\Pi}^{ab}(Q^{2})$ and spectral functions, $\rho^{ab}(s)$, defined by

$$\Pi_{\mu \nu}^{ab}(q) = (q_{\mu} q_{\nu} - q^{2} g_{\mu \nu}) \Pi^{ab}(Q^{2}) = i \int d^{4}xe^{iq \cdot x} \langle 0 | T \left(V_{\mu}^{a}(x) V_{\nu}^{b}(0)\right) | 0\rangle,$$

$$\hat{\Pi}^{ab}(Q^{2}) = \Pi^{ab}(Q^{2}) - \Pi^{ab}(0),$$

$$\rho^{ab}(s) = \frac{1}{\pi} \text{Im} \Pi^{ab}(Q^{2}), \quad (s = -Q^{2} > 0),$$

where $s = q^{2}$ and $Q^{2} = -q^{2}$.

The decomposition of the $u$, $d$, $s$ part of the EM current, $J_{\mu}^{\text{EM}}$, into isovector ($a = 3$) and isoscalar ($a = 8$) parts,

$$J_{\mu}^{\text{EM}} = V_{\mu}^{3} + \frac{1}{\sqrt{3}} V_{\mu}^{8} \equiv J_{\mu}^{\text{EM,3}} + J_{\mu}^{\text{EM,8}},$$

leads to the following decompositions for the subtracted three-flavor EM vacuum polarization, $\hat{\Pi}_{\text{EM}}(Q^{2})$, and spectral function, $\rho_{\text{EM}}(s)$,

$$\hat{\Pi}_{\text{EM}}(Q^{2}) = \hat{\Pi}_{\text{EM}}^{33}(Q^{2}) + \frac{2}{\sqrt{3}} \hat{\Pi}_{\text{EM}}^{38}(Q^{2}) + \frac{1}{3} \hat{\Pi}_{\text{EM}}^{88}(Q^{2})$$

$$\equiv \hat{\Pi}_{\text{EM}}^{I=1}(Q^{2}) + \hat{\Pi}_{\text{EM}}^{M=1}(Q^{2}) + \hat{\Pi}_{\text{EM}}^{I=0}(Q^{2})$$

$$\rho_{\text{EM}}(s) = \rho^{33}(s) + \frac{2}{\sqrt{3}} \rho^{38}(s) + \frac{1}{3} \rho^{88}(s)$$

$$\equiv \rho_{\text{EM}}^{I=1}(s) + \rho_{\text{EM}}^{M=1}(s) + \rho_{\text{EM}}^{I=0}(s),$$

with the $ab = 33$ parts pure isovector, the $ab = 88$ parts pure isoscalar and the $ab = 38$ parts mixed isospin terms which vanish in the isospin limit. In the isospin limit $\hat{\Pi}_{\text{EM}}^{33}$ is pure light-quark-connected while $\hat{\Pi}_{\text{EM}}^{88}$ contains light-quark-connected, strange-quark-connected and all disconnected contributions.

The leading-order hadronic contribution to $a_{\mu}$, $a_{\mu}^{\text{LO, HVP}}$, is related to $R(s) = 12\pi^{2} \rho_{\text{EM}}(s)$ by the standard “dispersive” representation [28, 30],

$$a_{\mu}^{\text{LO, HVP}} = \frac{\alpha_{\text{EM}}^{2} m_{\mu}^{2}}{9\pi^{2}} \int_{m_{\mu}^{2}}^{\infty} ds \frac{\hat{K}(s)}{s^{2}} R(s),$$

(2.6)
with $\alpha_{\text{EM}}$ the EM fine-structure constant, $R(s)$ determined from the bare inclusive hadronic electroproduction cross section, $\sigma^{(0)}[e^+e^- \to \text{hadrons}(+\gamma)]$, by
\[ R(s) = \frac{3s}{4\pi\alpha^2_{\text{EM}}} \sigma^{(0)}[e^+e^- \to \text{hadrons}(+\gamma)], \tag{2.7} \]
and the kernel $\hat{K}(s)$ an exactly known, slowly (and monotonically) increasing function of $s$ (see, e.g., Ref. [3]). Dispersive determinations of $a_{\mu}^{\text{LO,HVP}}$ are typically obtained by summing (i) exclusive-mode contributions up to just below $s = 4 \text{ GeV}^2$, (ii) narrow charm and bottom resonance contributions, and (iii) contributions evaluated using inclusive $R(s)$ data and/or perturbative QCD (pQCD) in the remainder of the high-$s$ region. The exclusive-mode regions in the analyses of Refs. [10] and [11] we employ below are $s \leq (1.8 \text{ GeV})^2 = 3.24 \text{ GeV}^2$ and $s \leq (1.937 \text{ GeV})^2 = 3.7520 \text{ GeV}^2$, respectively.

The isospin decomposition of $\hat{\Pi}_{\text{EM}}$ in Eqs. (2.5) leads to the related decomposition for the three-flavor $(u, d, s)$ contribution to $a_{\mu}^{\text{LO,HVP}}$,
\[ a_{\mu}^{\text{LO,HVP}} = a_{\mu}^{33} + \frac{2}{\sqrt{3}} a_{\mu}^{38} + \frac{1}{3} a_{\mu}^{88} \equiv a_{\mu}^{I=1} + a_{\mu}^{\text{MI}} + a_{\mu}^{I=0}. \tag{2.8} \]
To first order in $m_d - m_u$ there are no SIB contributions to either $a_{\mu}^{33}$ or $a_{\mu}^{88}$, while SIB is expected to dominate $a_{\mu}^{38}$.

In what follows, we will also denote contributions from an individual exclusive mode, $X$, to $a_{\mu}^{\text{LO,HVP}}$, $a_{\mu}^{I=1}$, $a_{\mu}^{I=0}$ and $a_{\mu}^{\text{MI}}$ by $[a_{\mu}^{\text{LO,HVP}}]_X$, $[a_{\mu}^{I=1}]_X$, $[a_{\mu}^{I=0}]_X$ and $[a_{\mu}^{\text{MI}}]_X$, respectively.

**B. The basic idea**

The basic idea underlying the analysis is as follows. In the isospin limit, $\hat{\Pi}_{\text{EM}}^{I=1}$ is pure light-quark-connected, while $\hat{\Pi}_{\text{EM}}^{I=0}$ is a sum of light-quark-connected, strange-quark-connected and all disconnected contributions, with light-quark-connected contribution
\[ \left[\hat{\Pi}_{\text{EM}}^{I=0}\right]_{\text{lcq}} = \frac{1}{9} \hat{\Pi}_{\text{EM}}^{I=1}. \tag{2.9} \]
The full light-quark-connected contribution to $\hat{\Pi}_{\text{EM}}$ is thus
\[ \hat{\Pi}_{\text{EM}}^{\text{lcq}} \equiv \frac{10}{9} \hat{\Pi}_{\text{EM}}^{I=1} \tag{2.10} \]
and the corresponding spectral function
\[ \rho_{\text{EM}}^{\text{lcq}}(s) = \frac{10}{9} \rho_{\text{EM}}^{I=1}(s). \tag{2.11} \]
The desired light-quark-connected contribution to $a_{\mu}^{\text{LO,HVP}}$, $a_{\mu}^{\text{lcq:IL}}$, is then given by the following dispersive integral involving the $I = 1$ spectral function:
\[ a_{\mu}^{\text{lcq:IL}} = \frac{\alpha^2_{\text{EM}} m_{\mu}^2}{9 \pi^2} \int_{m_{\mu}^2}^{\infty} ds \frac{\hat{K}(s)}{s^2} \left[ \left( \frac{10}{9} \right) 12 \pi^2 \rho_{\text{EM}}^{I=1}(s) \right]. \tag{2.12} \]
An accurate determination of $a^{lqc; IL}_{\mu}$ is thus possible provided the $I = 1$ contribution to $R(s)$, or equivalently $\rho_{EM}(s)$, can be identified with sufficient precision.

The separation of $I = 1$ and $I = 0$ contributions is straightforward in the higher-$s$ inclusive region, where $R(s)$ is approximated using pQCD and the $I = 1$ part represents $3/4$ of the total. In the lower-$s$ region, where $R(s)$ is obtained as a sum over exclusive-mode contributions, the separation is also straightforward for those exclusive modes having well-defined $G$-parity since states with positive/negative $G$-parity necessarily have $I = 1/1 = 0$. This provides unique isospin assignments for contributions from exclusive modes consisting entirely of narrow and/or strong-interaction-stable states having well-defined $G$-parity ($\pi, \eta, \omega, \phi$), which constitute more than 93% of the total exclusive-mode-region contribution. Further input is needed to separate the $I = 1$ and $I = 0$ components of contributions from exclusive modes which are not eigenstates of $G$-parity, such as those containing at least one $K\bar{K}$ pair. We outline in the next section how this separation is accomplished using experimental input for the $K\bar{K}$ and $K\bar{K}\pi$ exclusive modes. For all other $G$-parity-ambiguous exclusive modes, $X$, we employ a "maximally conservative" assessment in which the $I = 1$ contribution, $[a^{I=1}_{\mu}]_{X}$, is taken to be $50\pm50\%$ of the total $[a^{LO,HVP}_{\mu}]_{X}$. Fortunately, spectral contributions from these additional $G$-parity-ambiguous modes lie at higher $s$ and thus have contributions to $a^{LO,HVP}_{\mu}$, and hence also $I = 1/1$ separation uncertainties, which are strongly numerically suppressed, in spite of their 100% uncertainties.

In Sec. III we will implement the above analysis framework using as input the exclusive-mode results of Refs. [10, 11], neglecting, to begin with, isospin-breaking (IB) corrections. The resulting nominal $a^{lqc; IL}_{\mu}$, which we will denote by $a^{lqc}_{\mu}$, will differ from the desired isospin-limit value by small IB contributions. These IB contributions are taken into account and removed in Sec. IV, which contains our final results for $a^{lqc; IL}_{\mu}$.

### III. A DATA-BASED IMPLEMENTATION IGNORING ISOSPIN-BREAKING EFFECTS

In this section we carry out two determinations of $a^{lqc}_{\mu}$, neglecting IB corrections. These differ in the input used for the exclusive-mode $a^{LO,HVP}_{\mu}$ contributions, one employing the results of Ref. [11] (KNT19), the other those of Ref. [10] (DHMZ). The reader is reminded that the KNT19 and DHMZ exclusive-mode regions are different, the former extending up to $s = 3.7520$ GeV$^2$, the latter up to only $s = 3.24$ GeV$^2$. Contributions from the region above these exclusive-mode endpoints will be obtained using pQCD, with an error component, to be discussed below, designed to take into account possible small duality-violating (DV) contributions. The higher onset in $s$ of the inclusive region for KNT19 increases the expected accuracy of the pQCD result used to represent $\rho_{EM}^{I=1}(s)$ in this region and constitutes an advantage for the determination employing KNT19 input over that employing DHMZ input. Details of the form of the pQCD representation used in the inclusive region, together with our strategy for estimating possible residual DV corrections, are provided in Sec. III A.

The KNT19- and DHMZ-based analyses are outlined in Sec. III B below. In both cases we take advantage of results already worked out in Ref. [65]. This includes results for the

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1 The shorthand “pQCD” refers here, and in what follows, to dimension $D = 0$, mass-independent perturbative OPE contributions. For the $I = 1$ polarization $\Pi^{I=1}_{EM}$ considered this paper, mass-dependent $D = 2$ perturbative corrections are $O\left(m_{u,d}^2\right)$, and numerically negligible.
data-based $I = 0/1$ separation of $K\bar{K}$ and $K\bar{K}\pi$ contributions. For the reader’s benefit, the paragraphs which follow briefly review the treatment of the contributions from these modes. Further details may be found in Secs. IV.B and IV.C of Ref. [65].

$K\bar{K}$ contributions to $a_{\mu}^{LO,HVP}$ are expected to be dominated by the $I = 0$ contribution of the $\phi$ resonance, with a much smaller $I = 1$ contribution. This qualitative expectation can be quantified by combining the electroproduction-based result for the sum of $I = 1$ and 0 contributions with recent BaBar results [67] for the differential $\tau^- \to K^- K^0 \nu_\tau$ decay distribution, which provides an experimental determination of the charged $I = 1$ vector current spectral function, and, via the Conserved Vector Current (CVC) relation, up to numerically negligible IB corrections, an experimental determination of the $I = 1 K\bar{K}$-mode contribution to $\rho_{EM}(s)$. In the region where it is rather precise (up to $s = 2.7556 \text{ GeV}^2$, well above the $\phi$ peak), the BaBar $\tau$ data can thus be used to provide a direct determination of the $I = 1 K\bar{K}$ contribution to $a_{\mu}^{LO,HVP}$. For the contributions from the parts of the KNT19 and DHMZ exclusive-mode regions above $2.7556 \text{ GeV}^2$, we employ the maximally conservative separation treatment of KNT19 results for the $K\bar{K}$ contributions to $R(s)$. The contributions from this region turn out to be much smaller than those from the region covered by the BaBar $\tau$ data. Note that KNT19 $K\bar{K}$ input is used for the determination of this higher-$s$ $K\bar{K}$ exclusive-mode contribution for both the KNT19- and DHMZ-based versions of the analyses. The reason is that the full $K\bar{K}$ exclusive-mode data and covariances are publicly available only in the KNT19 case. Numerical details for the KNT19- and DHMZ-based analyses are provided below.

The exclusive-mode-region, $I = 1$, $K\bar{K}\pi$-mode contributions to $a_{\mu}^{LO,HVP}$ are obtained using the Dalitz-plot-based $I = 1/0$ separation of $K\bar{K}\pi$ cross-sections performed by BaBar [68], an analysis made possible by the observed saturation of the cross sections in the region of interest for this paper by $K\bar{K}^*$ contributions.

A. pQCD and DV corrections in the inclusive region

For the pQCD expression used to represent $\rho_{EM}^{I=1}(s)$ in the inclusive region, we employ the standard five-loop, $n_f = 3$ pQCD result [69, 70] with PDG2020 input for $\alpha_s$ [71]. $D = 2$, perturbative quark-mass-squared corrections are completely negligible. In the region from just below $\sqrt{s} = 2 \text{ GeV}$ up to charm threshold, $n_f = 3$ perturbative expectations for $R(s)$ are compatible within errors with the experimental determinations of BES [72, 73] and KEDR [74] (especially those of KEDR [74]), but lie slightly below recent BESIII results [75]. Small residual DV contributions to $\rho_{EM}(s)$ may thus be present even in this relatively large-$s$ region, making it important to estimate the impact of possible DV corrections to $\rho_{EM}^{I=1}(s)$ in the inclusive region of our analyses as well. While the whole of the KNT19 inclusive region lies in the region of agreement with BES and KEDR, the lower part of the DHMZ inclusive region extends to lower $s$, where the deviation of the perturbative expectation for $R(s)$ from the experimental sum-of-exclusive-mode-contributions determination is larger. An estimate of possible DV corrections to the pQCD approximation is thus of even more importance for the DHMZ-based analysis.

We investigate possible DV corrections using recent results for DV contributions to the charged $I = 1$ vector current spectral function, $\rho_{ud,V}(s)$, measured in hadronic $\tau$ decays. $\rho_{ud,V}(s)$ is related to the conventionally normalized EM isovector spectral function, $\rho_{EM}^{I=1}(s)$ by the CVC relation, $\rho_{EM}^{I=1}(s) = \frac{1}{2} \rho_{ud,V}(s)$. In Ref. [76], finite-energy sum rule (FESR) analyses of weighted integrals of a recently improved version of $\rho_{ud,V}(s)$ were carried out
using the large-s ansatz

\[ [\rho_{ud;V}]_{DV}(s) = \exp(-\delta_1 - \gamma_1 s) \sin(\alpha_1 + \beta_1 s), \]

for the DV contribution to \( \rho_{ud;V}(s) \). As detailed in Ref. [77], this ansatz follows for massless quarks from large-\( N_c \) and Regge arguments. The DV parameters \( \delta_1, \gamma_1, \alpha_1 \) and \( \beta_1 \), were obtained as part of the FESR fits. A range of different fits was considered, characterized by the choice of \( s_{\text{min}} \), the minimum \( s \) for which the DV ansatz was to be employed. Of these, ten, with \( s_{\text{min}} \) lying between 1.4251 and 1.7256 GeV\(^2\), show excellent \( p \)-values, good stability of the DV parameter results between the different fits, and reasonably controlled DV parameter errors. With all these \( s_{\text{min}} \) lying below the onset of both the KNT19 and DHMZ inclusive regions, it is thus possible to consider integrated DV contributions to \( a^{\mu}_{\text{lc}} \) in the inclusive region using any of these fits. We evaluate the DV contributions and associated errors for each of these fits. We then take as the central value of our estimate of the DV contribution the midpoint of the range spanned by these results and their errors, and as the uncertainty on that estimate half of that range. Numerical details for the KNT19 and DHMZ cases are provided below.

**B. \( a^{\mu}_{\text{lc}} \) using KNT19 or DHMZ exclusive-mode input**

We now turn to KNT19- and DHMZ-based determinations of \( a^{\mu}_{\text{lc}} \), neglecting small IB corrections, which will be dealt with in the next section.

We begin with the KNT19-based analysis. Reference [65] provides the following results for the various KNT19-based contributions to \( a^{\mu}_{\text{LO,HVP}} \).

The sum of all KNT19 \( G \)-parity-positive exclusive-mode contributions to \( a^{\mu}_{\text{LO,HVP}} \) from the region \( s \leq 3.7520 \) GeV\(^2\) is

\[ \left[ a^{I=1}_{\mu} \right]_{G=+} = 543.21(2.09) \times 10^{-10}. \]  

(3.2)

The breakdown, showing the modes which contribute and the KNT19 contribution from each, may be found in Table I of Ref. [65].

The \( I = 1 \) part of the \( G \)-parity-ambiguous \( K\bar{K} \) contribution from threshold to 3.7520 GeV\(^2\) is

\[ \left[ a^{I=1}_{\mu} \right]_{K\bar{K}} = 0.85(9) \times 10^{-10}. \]  

(3.3)

The \( I = 1 \) part of the \( G \)-parity-ambiguous \( K\bar{K}\pi \) contribution from threshold to 3.7520 GeV\(^2\) is

\[ \left[ a^{I=1}_{\mu} \right]_{K\bar{K}\pi} = 0.74(12) \times 10^{-10}. \]  

(3.4)

The \( I = 1 \) \( K\bar{K}2\pi \) mode contribution is obtained by first subtracting from the KNT19 \( I = 1 + 0 \) total, \( \left[ a^{\mu}_{\text{LO,HVP}} \right]_{K\bar{K}2\pi} = 1.93(8) \times 10^{-10} \), the purely \( I = 0 \) component, 0.159(10) \( \times 10^{-10} \), resulting from \( e^+e^- \rightarrow \phi\pi\pi \) with the \( \phi \) subsequently decaying to \( K\bar{K} \), and then applying the maximally conservative 50 \pm 50\% assessment of the \( I = 1 \) contribution to the resulting still-ambiguous difference. The result is

\[ \left[ a^{I=1}_{\mu} \right]_{K\bar{K}2\pi} = 0.89(89) \times 10^{-10}. \]  

(3.5)
The $I = 0 \phi \pi \pi$ subtraction was evaluated using the $e^+e^- \rightarrow \phi \pi \pi$ cross sections reported in Ref. [78].

The sum of the total ($I = 1 + 0$) contributions to $a^{\text{LO, HVP}}_{\mu}$ from all remaining $G$-parity ambiguous KNT19 modes, as detailed in the Appendix of Ref. [65], is $0.23(3) \times 10^{-10}$, leading to a maximally conservative estimate of the corresponding $I = 1$ contribution

$$\left[a^{I=1}_{\mu}\right]_{\text{other}} = 0.12(12) \times 10^{-10}. \quad (3.6)$$

The KNT19 inclusive region $I = 1$ pQCD contribution is a factor $9/2$ times the corresponding strange-connected-plus-disconnected contribution reported in Ref. [65], and hence

$$\left[a^{I=1}_{\mu}\right]_{\text{pQCD}} = 28.27(2) \times 10^{-10}. \quad (3.7)$$

The uncertainty reflects that on the input value of $\alpha_s$ and the estimated impact of 5-loop truncation and is much smaller than the size of the estimated DV contribution, obtained as discussed above, \footnote{The DV contributions to $a^{I=1}_{\mu}$ obtained from the ten fits noted above lie between $0.21(8) \times 10^{-10}$ and $0.26(12) \times 10^{-10}$, and hence cover the range from $0.13 \times 10^{-10}$ to $0.38 \times 10^{-10}$. The central value and error of the result quoted in Eq. (3.8) represent, respectively, the midpoint and half the extent of this range.}

$$\left[a^{I=1}_{\mu}\right]_{\text{DV}} = 0.26(12) \times 10^{-10}. \quad (3.8)$$

The total KNT19 inclusive region contribution, including this estimate of the DV correction, is thus

$$\left[a^{I=1}_{\mu}\right]_{\text{incl}} = 28.53(26) \times 10^{-10}, \quad (3.9)$$

where, to be conservative, we have assigned the full central value of the estimated DV correction as an expanded uncertainty.

Adding the above contributions yields the following interim result, prior to applying IB corrections, for the $I = 1$ contribution

$$a^{I=1}_{\mu} = 574.34(2.29) \times 10^{-10} \quad (3.10)$$

and hence the associated interim light-quark-connected result

$$a^{\text{LQ}}_{\mu} = 638.16(2.55) \times 10^{-10}. \quad (3.11)$$

Turning now to the analogous DHMZ-based analysis, once more relying heavily on results already detailed in Ref. [65], we find the following results for the components of the DHMZ-based determination of $a^{I=1}_{\mu}$.

The sum of all DHMZ $G$-parity positive exclusive-mode contributions to $a^{\text{LO, HVP}}_{\mu}$ from the region $s \leq 3.24$ GeV$^2$ is

$$\left[a^{I=1}_{\mu}\right]_{G=} = 542.74(3.39)(1.12)_{\text{lin}} \times 10^{-10}, \quad (3.12)$$

where the first error is the quadrature sum of the statistical and mode-specific, mode-to-mode-uncorrelated systematic errors of Ref. [10], and the second is the 100%-correlated common systematic error. The subscript “lin” is a reminder of the fact that, as specified
in Ref. [10], this error is obtained by summing linearly the corresponding errors on the individual DHMZ exclusive-mode contributions.

The DHMZ contributions from $G$-parity-ambiguous exclusive modes in the region $s \leq 3.24$ GeV$^2$ are

$$a_{I=1}^{\mu} = \begin{bmatrix} a_{\bar{K}K} & a_{\bar{K}K\pi} & a_{\bar{K}K2\pi} & a_{\text{other}} \end{bmatrix} = \begin{bmatrix} 0.83(8) \times 10^{-10} \\ 0.66(11) \times 10^{-10} \\ 0.37(37) \times 10^{-10} \\ 0.00(1) \times 10^{-10} \end{bmatrix},$$

with all results, with the exception of that from the $K\bar{K}2\pi$ mode, given previously in Sec. VI of Ref. [65]. The additional information needed to obtain the $K\bar{K}2\pi$ result is as follows. First, the DHMZ result for the total $I = 1 + 0$ contribution to $a_{\mu}^{\text{LO,HVP}}$ is $0.85(2)(5)(1)_{\text{lin}} \times 10^{-10}$. Second, the $s \leq 3.24$ GeV$^2$, $I = 0 \phi(\to K\bar{K})\pi\pi$ contribution implied by BaBar $e^+e^- \to \phi\pi\pi$ cross-sections [78] is $0.117(8) \times 10^{-10}$. The $G$-parity ambiguous $K\bar{K}2\pi$ remainder is thus $0.73(5)(1)_{\text{lin}} \times 10^{-10}$ and the maximally conservative assessment of the $I = 1$ component thereof $0.37(37) \times 10^{-10}$, where we have ignored all error components other than the strongly dominant maximally conservative separation uncertainty.

Finally, for the DHMZ inclusive region pQCD, DV and total contributions, we find

$$a_{I=1}^{\mu} = \begin{bmatrix} a_{\mu}^{\text{pQCD}} & a_{\mu}^{\text{DV}} & a_{\mu}^{\text{incl}} \end{bmatrix} = \begin{bmatrix} 32.74(3) \times 10^{-10} \\ -0.19(31) \times 10^{-10} \\ 32.55(31) \times 10^{-10} \end{bmatrix},$$

We note that the uncertainty on the DV contribution in this case is significantly larger than that obtained for the KNT19 case above.\(^3\)

Adding the above contributions yields the following interim DHMZ-based result

$$a_{\mu}^{I=1} = 577.15(3.43)(1.12)_{\text{lin}} \times 10^{-10}$$

and hence the associated interim light-quark-connected result

$$a_{\mu}^{\text{HQ}} = 641.28(3.81)(1.24)_{\text{lin}} \times 10^{-10}.$$\(^3\)

The final step required to obtain the desired isospin-limit version, $a_{\mu}^{\text{HQ};\text{IL}}$, of $a_{\mu}^{\text{HQ}}$ is to apply EM and SIB corrections to the interim results \((3.11)\) and \((3.21)\). This step is discussed in the next section.

**IV. EM AND SIB CORRECTIONS**

In this section we consider EM and SIB corrections to the results above. In assessing EM corrections, where it is not feasible to carry out a dispersive determination, we take advantage

\(^3\) The DV contributions to $a_{\mu}^{I=1}$ here lie between $-0.34(16) \times 10^{-10}$ and $-0.04(16) \times 10^{-10}$, covering the range from $-0.50 \times 10^{-10}$ to $0.12 \times 10^{-10}$. The central value and error of the result quoted in Eq. \((3.18)\) represent, respectively, the midpoint and half the extent of this range.
of the results of the recent BMW lattice study \[50\], which provides the first determination of all EM contributions to \(a_{\mu}^{\text{LO},\text{HVP}}\). This (unavoidable) introduction of lattice EM input into an otherwise purely dispersive determination of \(a_{\mu}^{\text{QCD},\text{IL}}\) turns out to constitute a rather minor "deviation," since the lattice result for the EM correction is, in fact, rather small.

The separation of the IB sum of EM and SIB contributions into separate EM and SIB parts is, as is well known, ambiguous at \(O(\alpha_{\text{EM}}(m_d + m_u))\).\(^4\) The separation scheme used by BMW \[50\] in determining the EM corrections we employ below is defined such that the EM contributions to the masses of the purely connected neutral pseudoscalar mesons are zero, \(i.e.,\) such that all such EM contributions are absorbed into the definitions of the quark masses. It is numerically very similar to the widely used GRS scheme \[80\]. By using BMW EM results, we are working in the BMW separation scheme.\(^5\)

To perform the desired EM and SIB corrections, one needs to identify and subtract EM and SIB contributions present in the experimental versions of the nominal \(I = 1\) contribution \(a_{\mu}^{I=1}\) determined in the previous section. These are of two types: those present in the physical \(a_{\mu}^{I=1}\) contribution itself, and those associated with nominally \(G\)-parity positive contributions which are actually part of the mixed-isospin contribution, \(a_{\mu}^{\text{MI}}\), and which hence "contaminate" the nominal \(a_{\mu}^{I=1}\) results obtained above. The correction for the mixed-isospin contamination cannot be done inclusively since \(\rho_{\text{EM}}(s)\) receives contributions from both nominally \(G\)-parity positive and nominally \(G\)-parity negative exclusive modes, with, \(e.g.,\) \(\rho - \omega\) mixing inducing both mixed-isospin \(\pi^+\pi^-\) and mixed-isospin \(\pi^+\pi^-\pi^0\) contributions, via, respectively, the processes \(e^+e^- \to \omega \to \rho \to \pi^+\pi^-\) and \(e^+e^- \to \rho \to \omega \to \pi^+\pi^-\pi^0\).

To first order in IB, SIB contributions to \(\bar{\Pi}_{\text{EM}}(Q^2)\) and \(\rho_{\text{EM}}(s)\) occur only in the mixed-isospin parts, while EM contributions are present in all of the \(I = 1, I = 0\) and mixed-isospin components. It follows that, to this order, the correction required to convert the physical version of \(a_{\mu}^{33} = a_{\mu}^{I=1}\) to the corresponding isospin-limit version is purely EM in nature. We denote the associated contribution to \(a_{\mu}^{\text{QCD}}\), to be subtracted from the nominal \(a_{\mu}^{\text{QCD}}\) obtained in the previous section, by \(\delta_{\text{EM}}a_{\mu}^{\text{QCD}}\). There is no need for a breakdown of this correction into components associated with individual exclusive modes, and we take as input for this contribution the inclusive lattice result quoted in Ref. \[50\],

\[
\delta_{\text{EM}}a_{\mu}^{\text{QCD}} = -0.93(34)(47) \times 10^{-10},
\]

where the first error is statistical and the second systematic. As noted above, the associated correction represents a very small fraction of the nominal \(a_{\mu}^{\text{QCD}}\) results above.

We now address the mixed-isospin-contamination correction. Evaluating this correction requires identifying and subtracting mixed-isospin contaminations present in each of the individual exclusive-mode contributions summed to obtain the nominal \(a_{\mu}^{\text{QCD}}\) results of the previous section. In contrast to \(a_{\mu}^{I=1}\), which, to first order in IB, receives no SIB contribution, both EM and SIB contributions are present in \(a_{\mu}^{\text{MI}}\), with SIB expected to dominate. In what follows, we rely on experimental input to quantify what should be the dominant exclusive-mode correction and provide conservative bounds on the remaining sub-dominant contributions.

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\(^4\) For an expanded discussion, see, \(e.g.,\) Sections 3.1.1 and 3.1.2 of Ref. \[79\].

\(^5\) This is, for our purposes, a somewhat academic point since \(m_d + m_u\) is only a factor of \(\sim 3\) greater than \(m_d - m_u\), making the \(O(\alpha_{\text{EM}}(m_d + m_u))\) separation ambiguity comparable in size to contributions of \(O(\alpha_{\text{EM}}(m_d - m_u))\) which, being second order in IB, we are neglecting throughout.
contributions. In relying on experimental input, the results for the mixed-isospin corrections are, of course, those for the sum of EM and SIB effects.

As is well known, the strong low-s enhancement produced by the dispersive weight $\hat{K}(s)/s^2$ is such that the dispersive determination of $a_{\mu}^{\text{LO.HVP}}$ is dominated by contributions from the region of the lowest-lying (especially $\rho$ and $\omega$) resonances. A similar low-s, resonance-region dominance is expected for $a_{\mu}^{\text{MI}}$, doubly so since IB contributions in this region are subject to enhancements generated by the impact of the very small $\rho-\omega$ mass difference on contributions induced by $\rho-\omega$ mixing. In this region, the mixed-isospin spectral contribution, $\rho_{\text{EM}}^{\text{MI}}(s)$, will appear essentially entirely in the $2\pi$ and $3\pi$ exclusive modes. Prior to implementing the mixed-isospin correction, the IB $2\pi$ and $3\pi$ components appear, respectively, in the nominal $I = 1$ and $I = 0$ sums, and hence represent mixed-isospin contaminations of those sums. In this study, we are interested only in carrying out the mixed-isospin correction for the nominal $I = 1$ sum, and hence focus on the IB contribution to the $2\pi$ distribution. Note that more than 90% of the full exclusive-mode-region contribution to $a_{\mu}^{\text{lqc}}$, in fact, comes from $\pi\pi$ contributions in the region below $s = 1$ GeV$^2$.

The presence of the obvious $\rho-\omega$ interference “shoulder” in the $e^+e^- \rightarrow \pi^+\pi^-$ cross section makes possible an experimental determination of the IB $\rho-\omega$-region contribution from the $\pi\pi$ exclusive-mode. To first order in IB, the associated low-s ($s < 1$ GeV$^2$) contribution to $a_{\mu}^{\text{LO.HVP}}$ lies entirely in $a_{\mu}^{\text{MI}}$. This contribution, which should strongly dominate $[a_{\mu}^{\text{MI}}]_{\pi\pi}$, has recently been determined using the results of a fit to the $e^+e^- \rightarrow \pi^+\pi^-$ cross sections based on a dispersively constrained representation of the timelike $\pi$ form factor incorporating the IB $\rho-\omega$ interference effect $[64]$. The use of rigorous dispersive constraints turns out to produce a rather tightly constrained result,

$$[a_{\mu}^{\text{MI}}]_{\pi\pi} = 3.68(14)(10) \times 10^{-10}, \quad (4.2)$$

where the first error is the fit uncertainty and the second the combination of systematic uncertainties. The associated mixed-isospin $\pi\pi$ contamination of the nominal $a_{\mu}^{\text{lqc}}$ obtained in the last section, which we denote $[\delta_{\text{MI}}a_{\mu}^{\text{lqc}}]_{\pi\pi}$, is thus

$$[\delta_{\text{MI}}a_{\mu}^{\text{lqc}}]_{\pi\pi} = 4.09(16)(11) \times 10^{-10}. \quad (4.3)$$

This contribution must be subtracted from the nominal $a_{\mu}^{I=1}$ (i.e. $a_{\mu}^{\text{lqc}}$) results of the previous section.

Note that, in spite of the strong, narrow resonance enhancement, the IB contribution in Eq. (4.2) represents only 0.7% of the full exclusive-mode-region $\pi\pi$ contribution $[a_{\mu}^{\text{LO.HVP}}]_{\pi\pi}$. Since contributions to the nominal $a_{\mu}^{I=1}$ total from exclusive modes other than $\pi\pi$ are dominated by regions in $s$ for which no analogous narrow interfering resonance enhancements are possible, it should be extremely conservative to assume the magnitudes of contributions to $a_{\mu}^{\text{MI}}$ from all non-$\pi\pi$ exclusive modes are also less than $\sim 1\%$ of the corresponding nominal $a_{\mu}^{I=1}$ exclusive-region contributions. The non-$\pi\pi$ exclusive-mode-region contributions to the nominal $I = 1$ sum $a_{\mu}^{I=1}$ total $41.6 \times 10^{-10}$ in the KNT19 case and $36.7 \times 10^{-10}$ in the DHMZ case. We thus expect the sum of exclusive-mode-region contributions to $a_{\mu}^{\text{MI}}$ from all exclusive modes in that sum other than $\pi\pi$ to be less than $0.42 \times 10^{-10}$ and $0.38 \times 10^{-10}$ in magnitude, respectively, for the KNT19 case and DHMZ cases. Unsurprisingly, these bounds are much smaller than the accurately determined $\pi\pi$ contribution given in Eq. (4.2).
We thus take, as our estimate of the full mixed-isospin contamination present in the nominal $a_{\mu}^{I=1}$ sums of the previous section, the result of Eq. (4.2), adding the non-$\pi\pi$ exclusive-mode bounds as additional systematic uncertainties.

Adding the results of Eqs. (4.1) and (4.3), we find, for the sum of EM+SIB contributions, $\delta_{EM+SIB}^{lc} \equiv \delta_{EM}^{lc} + \delta_{MI}^{lc,\pi\pi}$, to be subtracted from the nominal light-quark-connected results, Eqs. (3.11) and (3.21), to convert to the corresponding isospin-limit values, $a_{\mu}^{lc;IL}$, the results $\delta_{EM+SIB}^{lc} = 3.16(37)(48)(46) \times 10^{-10}$ and $3.16(37)(48)(41) \times 10^{-10}$ for the KNT19 and DHMZ cases, respectively, where the first errors are statistical, the second are systematic, and the third are the uncertainties estimated above for missing mixed-isospin contributions from non-$\pi\pi$ exclusive modes. Combining, for simplicity of presentation, all errors in quadrature, we obtain the following final results for $a_{\mu}^{lc;IL}$:

$$a_{\mu}^{lc;IL} = 635.0(2.7) \times 10^{-10} \quad \text{(KNT19)} \quad (4.4)$$

$$a_{\mu}^{lc;IL} = 638.1(4.1) \times 10^{-10} \quad \text{(DHMZ)} \quad (4.5)$$

V. CONCLUSIONS AND DISCUSSION

We have shown how recent dispersive results for exclusive-mode contributions to $a_{\mu}^{LO,HVP}$ can be used to provide a determination of the corresponding isospin-limit, light-quark-connected contribution, $a_{\mu}^{lc;IL}$, the precision of which turns out to be of order 0.5%. The determination employs lattice input for a small, 0.15%, EM correction, but is otherwise purely dispersive. The result, of course, depends on the choice of exclusive-mode input, and small differences in the KNT19 and DHMZ assessments of individual exclusive-mode contributions lead to an associated $\sim 0.5\%$ difference between the $a_{\mu}^{lc;IL}$ results obtained using KNT19 and DHMZ input, given in Eqs. (4.4) and (4.5), respectively. This difference is similar in size to the errors on the individual KNT19- and DHMZ-based determinations, and sufficiently small to allow meaningful conclusions to be drawn from a comparison of our dispersive results to those of recent lattice analyses. This comparison is summarized in Table 1 and Figure 1. Our dispersive results lie lower than the majority of central lattice values, though some variability, at the roughly 2$\sigma$ level, remains in the lattice results. Among the lattice results, that of Ref. [20] (BMW 2020) has, at present, by far the smallest error, and would strongly dominate any putative lattice average. Our KNT19- and DHMZ-based dispersive results are in 3.2 and 2.4$\sigma$ tension, respectively, with the BMW 2020 result. Other lattice results, from multiple groups, with similar or smaller errors, are anticipated in the near future, and our dispersive results provide a useful comparison target for such future lattice determinations.

The analysis strategy employed above, though applied there only to the dispersive determination of $a_{\mu}^{lc;IL}$, is readily adapted to determinations of other quantities of interest also having a dispersive representation. One well-known example is the standard intermediate window quantity, $a_{\mu}^{W}$, introduced by RBC/UKQCD [40], and constructed, by design, to be rather precisely determinable on the lattice. The isospin-limit, light-quark-connected component of $a_{\mu}^{W}$, $a_{\mu}^{W,lc;IL}$, has now been determined by a number of lattice groups [40, 48, 50, 51, 53, 55, 58, 60, 61], with updates of earlier ABGP [48], ETMe [53] and RBC/UKQCD [40] results, reported in Refs. [56, 58, 60], bringing results from all groups into excellent agreement. These results are also found to lie significantly higher than alternate mixed “R-ratio+lattice” estimates obtained by subtracting from R-ratio-based dispersive
determinations of $a_{\mu}^{W}$ contributions for all non-light-quark-connected components evaluated on the lattice. Of course, in view of other signs of tension between lattice and dispersive results, one would prefer to compare the rather precise lattice results with a dispersive, rather than mixed lattice-dispersive expectation. This is not currently possible because no purely dispersive $a_{\mu}^{W,\text{lqc};\text{IL}}$ determination exists.

Weighted exclusive-mode and inclusive-region integrals are, however, equally easy to evaluate for any dispersive weight as they are for the weight $\hat{K}(s)/s^2$ which enters the determinations of the exclusive-mode $a_{\mu}^{\text{LO},\text{HVP}}$ contributions of Refs. [10, 11], provided, that is, the relevant exclusive-mode distributions, $[\rho_{\text{EM}}(s)]_X$, required to perform these re-weighted integrals, are publicly available. This information is available for the distributions underlying the KNT19 exclusive-mode results of Ref. [11], and KNT19-based dispersive determinations of, not just $a_{\mu}^{W,\text{lqc};\text{IL}}$, but also other dispersive integral quantities, involving different dispersive weights, are thus also possible using the analysis strategy above.

Quantities of this type likely to be of interest for future investigation include both those naturally formulated in terms of their Euclidean-time ($t$) weightings and those naturally formulated in terms of their dispersive $s$-weightings. Examples of the former include the additional intermediate window quantities of Ref. [40], the one-sided window quantities of Ref. [59], the linear-combinations-of-Euclidean-window quantities of Ref. [62], and the window quantity, $a_{\mu}^{W2}$, introduced in Ref. [56], designed to more strongly weight higher-$t$ lattice contributions and improve the reliability of ChPT-based estimates of lattice finite-volume effects. Examples of the latter are dispersive integrals involving $s$-dependent weights of the type introduced in Ref. [66], designed to emphasize contributions from more limited regions in $s$ and potentially help in obtaining a more detailed understanding of the source of the current dispersive-lattice tensions. Since the sum of isospin-limit strange-quark-connected and full-disconnected contributions to $\rho_{\text{EM}}(s)$ also has a representation as an appropriately weighted difference of nominally $I = 0$ and nominally $I = 1$ exclusive-mode contributions [65], the current analysis strategy can be employed to determine isospin-limit versions of not just light-quark-connected, but also strange-connected-plus-full-disconnected, contri-

| $a_{\mu}^{\text{lqc};\text{IL}} \times 10^{10}$ | Reference |
|------------------------------------------|------------|
| 647.6(19.3) | BMW 2017 [38] |
| 649.7(15.0) | RBC/UKQCD 2018 [40] |
| 629.1(13.7) | ETMC 2018/2019 [41, 49] |
| 673(14) | PACS 2019 [45] |
| 637.8(8.8) | FHM 2019 [46] |
| 674(13) | Mainz 2019 [47] |
| 659(22) | ABGP 2019 [48] |
| 654.5(5.5) | BMW 2020 [50] |
| 657(29) | LM 2020 [51] |
| 646(14) | ABGP 2022 [56] |
| 635.0(2.7) | This work (KNT19-based) |
| 638.1(4.0) | This work (DHMZ-based) |

TABLE 1: Comparison of our dispersive results with recent lattice results for $a_{\mu}^{\text{lqc};\text{IL}}$. The latter are listed above the internal horizontal line, the former below it.
FIG. 1: Comparison of lattice determinations and our dispersive results for $a^\text{q.c.; IL}_\mu$. Lattice results superseded by those from later publications by the same collaboration are plotted in grey. See Table 7 for the corresponding numerical values and references.

butions to all such window quantities. We plan to implement such dispersive determinations of these various window quantities and will report on the results of this work in a future paper.

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