Quasilocal mass in scalar–tensor gravity: spherical symmetry

Andrea Giusti and Valerio Faraoni

Department of Physics and Astronomy, Bishop’s University, 2600 College Street, Sherbrooke, Québec, J1M 1Z7, Canada

E-mail: agiusti@ubishops.ca and vfaraoni@ubishops.ca

Received 18 May 2020, revised 18 July 2020
Accepted for publication 22 July 2020
Published 9 September 2020

Abstract

A recent generalization of the Hawking–Hayward quasilocal energy to scalar–tensor gravity is adapted to general spherically symmetric geometries. It is then applied to several black hole and other spherical solutions of scalar–tensor and $f(R)$ gravity. The relations of this quasilocal energy with the Abreu–Nielsen–Visser gauge and the Kodama vector are discussed.

Keywords: quasilocal mass, scalar–tensor gravity, alternative theories of gravity

1. Introduction

Einstein’s theory of general relativity (GR) predicts spacetime singularities where it breaks down and clashes with quantum mechanics in the ultraviolet regime since it cannot be quantized in any standard way. Therefore, GR is expected to be modified at high energy. The attempts to quantum-correct GR produce, in the low-energy limit, higher derivative equations or extra fields that couple explicitly to the spacetime curvature. For example, the low-energy limit of string theories contains a dilaton very similar to the scalar field of Brans–Dicke gravity and, in this limit, bosonic string theory reduces to a Brans–Dicke theory [1].

In the infrared regime, compelling motivation for the study of alternative gravity comes from cosmology. The standard model of cosmology, the Λ-cold dark matter (ΛCDM) model, fits into GR the current accelerated expansion of the Universe discovered with high redshift supernovae only at the price of introducing an extremely fine-tuned cosmological constant Λ or a completely ad hoc dark energy [2]. To avoid invoking either one of those, cosmologists consider very seriously the possibility that gravity departs from GR at large (cosmological) scales or low densities. The most popular class of theories studied for this purpose is $f(R)$ gravity [3], which is the subject of a large literature [4, 5]. This is a subclass of scalar–tensor gravity. Scalar–tensor theories [6], which generalize the original Brans–Dicke theory [7], are minimal modifications of GR in the sense that they introduce only a scalar degree of freedom.
\(\phi\) in addition to the spin two field represented by the metric tensor \(g_{ab}\) of GR. However, they still exhibit a rich phenomenology. Considerable theoretical and experimental effort is being put into testing gravity at all scales to either detect or constrain deviations from GR in the study of cosmology, black holes, or astrophysics [8, 9], including the search for scalar hair [10].

The (Jordan frame) action of scalar–tensor gravity is\(^2\)
\[
S_{\text{ST}} = \frac{1}{16\pi} \int \! d^4x \sqrt{-g} \left[ \phi \mathcal{R} - \frac{\omega(\phi)}{\phi} \nabla^a \phi \nabla_a \phi - V(\phi) \right] + S^{(m)}, \tag{1.1}
\]
where \(\mathcal{R}\) is the Ricci scalar of the metric \(g_{ab}\) with determinant \(g\), the positive Brans–Dicke scalar \(\phi\) is approximately equivalent to the inverse of the effective gravitational coupling strength,
\[
G_{\text{eff}} = \phi^{-1}, \tag{1.2}
\]
\(\omega(\phi)\) (a constant parameter in the original Brans–Dicke theory [7]) is the ‘Brans–Dicke coupling’, and \(V(\phi)\) is a scalar field potential. \(S^{(m)} = \int \! d^4x \sqrt{-g} L^{(m)}\) is the matter action.

The variation of the action (1.1) with respect to \(g^{ab}\) and \(\phi\) produces the (Jordan frame) field equations [6, 7]
\[
R_{ab} - \frac{1}{2} g_{ab} \mathcal{R} = \frac{8\pi}{\phi} T_{ab}^{(m)} + \frac{\omega}{\phi^2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla_c \phi \nabla^c \phi \right) + \frac{1}{\phi^2} (\nabla_a \nabla_b \phi - g_{ab} \nabla^c \phi \nabla_c \phi) - \frac{V}{2\phi} g_{ab}, \tag{1.3}
\]
\[
\Box \phi = \frac{1}{2\omega + 3} \left( \frac{8\pi T^{(m)}}{\phi} + \phi \frac{dV}{d\phi} - 2V - \frac{d\omega}{d\phi} \nabla^c \phi \nabla_c \phi \right), \tag{1.4}
\]
where \(T^{(m)} \equiv g^{ab} T_{ab}^{(m)}\) is the trace of the matter stress–energy tensor \(T_{ab}^{(m)}\).

(Metric) \(f(\mathcal{R})\) gravity is described by the action
\[
S_{f(\mathcal{R})} = \frac{1}{16\pi} \int \! d^4x \sqrt{-g} f(\mathcal{R}) + S^{(m)}, \tag{1.5}
\]
where \(f(\mathcal{R})\) is a nonlinear function of the Ricci scalar and \(S^{(m)}\) is again the matter action. The action \(S_{f(\mathcal{R})}\) is equivalent to that of a Brans–Dicke gravity with Brans–Dicke field \(\phi = f'(\mathcal{R})\), coupling \(\omega = 0\), and the rather complicated scalar field potential [5]
\[
V(\phi) = \mathcal{R} f'(\mathcal{R}) - f(\mathcal{R}) \big|_{\mathcal{R} = \mathcal{R}(\phi)}, \tag{1.6}
\]
where \(\mathcal{R}\) is now a function of the scalar field \(\phi = f'(\mathcal{R})\) and a prime denotes differentiation with respect to the curvature scalar. The relation \(\mathcal{R} = \mathcal{R}(\phi)\) is not explicitly invertible in general, and the potential \(V(\phi)\) remains an implicit function of \(\phi\).

The field equations are of fourth order,
\[
f'(\mathcal{R}) R_{ab} - \frac{f(\mathcal{R})}{2} g_{ab} = 8\pi T_{ab}^{(m)} + \nabla_a \nabla_b f'(\mathcal{R}) - g_{ab} \Box f'(\mathcal{R}), \tag{1.7}
\]
\(^2\)We follow the notation of reference [11] and use units in which Newton’s constant \(G\) and the speed of light \(c\) are unity, but sometimes we restore \(G\) for convenience.
and can be written as the effective Einstein equations \[5\]

$$\mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R} = 8\pi \left( \frac{f^{(m)}_{ab}}{f'(\mathcal{R})} + \mathcal{T}^{(\text{eff})}_{ab} \right), \tag{1.8}$$

where

$$\mathcal{T}^{(\text{eff})}_{ab} = \frac{1}{8\pi f'(\mathcal{R})} \left[ \nabla_a \nabla_b f'(\mathcal{R}) - g_{ab} \square f'(\mathcal{R}) + \frac{f(\mathcal{R}) - \mathcal{R} f'(\mathcal{R})}{2} g_{ab} \right]. \tag{1.9}$$

In general, minimal requirements on a \( f(\mathcal{R}) \) theory of gravity are that \( \phi = f'(\mathcal{R}) > 0 \) in order for the graviton to carry positive kinetic energy, and \( f''(\mathcal{R}) > 0 \) to avoid the notorious Dolgov–Kawasaki instability that makes the scalar \( \phi \) tachyonic \([5, 12, 13]\).

In GR, the concept of mass of a relativistic gravitating system has been scrutinized intensely. Gravitational energy cannot be localized as a consequence of the equivalence principle and research has turned to quasilocal notions, i.e., to the energy enclosed by a compact spacelike two-surface. Several definitions of quasilocal energy have been studied, see \([14]\) for a review and reference \([15]\) for a review of the isolated horizon formalism containing more recent energy definitions. A common feature of quasilocal energies is that they tend to remain the domain of mathematical physics with no practical applications. Recently, we have applied the Hawking–Hayward quasilocal construct \([16–18]\) to cosmology, gravitational lensing, and black holes \([19]\), and we focus on the Hawking–Hayward quasilocal energy here.

The knowledge of quasilocal energy is important in other areas of research: it appears in the first law of thermodynamics for gravity. Black hole thermodynamics is a well developed theoretical subject, while the thermodynamics of gravity and spacetime (e.g., \([20]\)) is much more speculative and still under development. In black hole thermodynamics, the Hawking–Hayward quasilocal energy is usually assumed to be the internal energy of the black hole. Spacetime thermodynamics usually extends the range of theories of gravity beyond GR. Since alternative gravity is so prominent in all the areas of research mentioned, it is essential for their progress to know whether the quasilocal energy construct extends to these theories, and we begin with the simplest and most popular alternative, scalar–tensor gravity (reference \([21]\) extends the Hawking–Hayward construct for spherical symmetry to \(n\)-dimensional Lovelock gravity). There are a few quasilocal prescriptions in scalar–tensor gravity, and they all disagree with each other to some extent \([22–28]\). The prescriptions of \([27, 28]\) agree only \textit{in vacuo}; those of references \([22–26]\) are limited by severe restrictions, including \( f(\mathcal{R}) \) gravity only; spherical symmetry only; special spacetime geometries only, or given only at black hole horizons. These prescriptions are obtained using spacetime thermodynamics and the first law \([22–26]\), but there is much uncertainty on the correct thermodynamical quantities to use (temperature, entropy, work density, and heat supply vector), which reflect in some arbitrariness in any definition of quasilocal energy based on the first law. Moreover, the horizon temperature is a semiclassical concept involving difficult calculations in quantum field theory on curved spacetime which are hard to complete (thus far, only the tunneling method seems to deliver definite results in non-stationary black hole geometries). The prescription of \([27]\) is not restricted to \( f(\mathcal{R}) \) gravity nor to special metrics, spherical symmetry, or asymptotic flatness and is obtained purely classically and independent of thermodynamics by writing the scalar–tensor field equations as effective Einstein equations and using the geometric derivation of the Hawking–Hayward mass in this ‘effective GR’ context.
Here we develop the prescription for a generalization of the Hawking–Hayward quasilocal mass to scalar–tensor (including $f(R)$ gravity) given in reference [27]. In view of future applications, we provide a general formula for spherical symmetry and we apply it to several spherical solutions of scalar–tensor and $f(R)$ gravity. As a first test, the new quasilocal mass of [27] reproduces [29] the monopole term in the multipole expansion of asymptotically flat solutions of scalar–tensor gravity [30].

2. Spherical symmetry in scalar–tensor gravity

In Einstein’s theory, the Hawking–Hayward quasilocal mass is defined [16, 17] on an embedded spacelike, compact, and orientable two-surface $S$ with induced two-metric $h_{ab}$ and induced Ricci scalar $R(h)$. Consider ingoing ($-$) and outgoing ($+$) null geodesic congruences from $S$ and let $\theta^a_{(+)}$ and $\theta^a_{(-)}$ be the expansions and shear tensors of these congruences, respectively. Let $\omega^a$ be the projection of the commutator of the null normal vectors to $S$ onto $S$ (the anholonomicity [17]). Let $\mu$ denote the volume two-form on $S$ and let $A$ be the area of $S$. Then, the Hawking–Hayward quasilocal energy is defined as [16, 17]

$$M_{HH} = \frac{1}{8\pi G \sqrt{A}} \int_S \mu \left( R(h) + \theta^a_{(+)} \theta^a_{(-)} - \frac{1}{2} \sigma^{ab}_{(+)} \sigma_{(+)}^{ab} - 2 \omega^a \omega^a \right).$$  \hspace{1cm} (2.1)

It can be shown that the quasilocal mass has a Newtonian character because, for an observer with four-velocity parallel to the normal to the two-surface $S$, only the electric part of the Weyl tensor contributes to $M_{HH}$ [31].

The contracted Gauss equation [17]

$$R(h) + \theta^a_{(+)} \theta^a_{(-)} - \frac{1}{2} \sigma^{ab}_{(+)} \sigma_{(+)}^{ab} = h_{ab} h^{bd} R_{dcd}$$ \hspace{1cm} (2.2)

is useful to compute the first three terms in the integral and was used in [27].

The scalar–tensor mass prescription of [27] is

$$M_{ST} = \frac{1}{8\pi G \sqrt{A}} \int_S \mu \left[ h^{ab} h^{cd} C_{abcd} - 2 \omega^a \omega^a + \frac{8\pi}{\phi} h^{ab} T_{ab} - \frac{16\pi T}{3\phi} \frac{h^{ab} \nabla_a \nabla_b \phi}{\phi} + \frac{\omega^a}{G^{\phi^2}} \left( h^{ab} \nabla_a \phi \nabla_b \phi - \frac{1}{3} \nabla^a \phi \nabla_a \phi \right) + \frac{V}{3\phi} \right],$$ \hspace{1cm} (2.3)

where the $\phi$ factor in the first term on the right-hand side is introduced by the replacement $G \rightarrow G_{\text{eff}}$.

In GR, in spherical symmetry, the Hawking–Hayward quasilocal energy (2.1) reduces [18] to the better known Misner–Sharp–Hernandez mass [32]

$$M_{MSH} = \frac{R}{2G} \left( 1 - \nabla^a R \nabla_a R \right),$$ \hspace{1cm} (2.4)

where $R$ is the areal radius. Let us consider now scalar–tensor gravity: assuming spherical symmetry and the surface $S$ to be a two-sphere of symmetry with areal radius $R$ and induced metric $h_{ab}$, the line element can always be diagonalized as

$$ds^2 = g_{00} dt^2 + g_{11} dR^2 + R^2 dl_2^2$$

$$= I_{\mu\nu} dx^\mu dx^\nu + h_{\mu\nu} dx^\mu dx^\nu$$ \hspace{1cm} (2.5)
in spherical coordinates \((t, R, \theta, \varphi)\). Here \(I_{\mu\nu} = \text{diag}(g_{00}, g_{11}, 0, 0)\), \(h_{\mu\nu} = \text{diag}(0, 0, R^2, R^2 \sin^2 \theta)\), and \(d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\varphi^2\) is the metric on the unit two-sphere. Equation (2.3) then simplifies to [27]

\[
M_{\text{ST}} = \frac{\phi R^3}{4} \left[ h^{\mu\nu}h^{\rho\sigma}C_{\mu\nu\rho\sigma} + \frac{8\pi}{\phi} h^{\mu\nu}T_{\mu\nu} - \frac{16\pi T}{3\phi} \right] \\
+ \frac{\omega}{\phi^2} \left( h^{ab}\nabla_a\phi\nabla_b\phi - \frac{1}{3} \nabla^c\phi\nabla_c\phi \right) + \frac{h^{ab}\nabla_a\nabla_b\phi}{\phi} + \frac{V}{3\phi} .
\]  

(2.6)

The scalar–tensor quasilocal mass of spheres in Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes, given in reference [27], follows immediately from equation (2.6). However, here we want to provide a simple formula for the scalar–tensor quasilocal mass valid for any spherically symmetric metric and equation (2.6) is not the most convenient starting point. Let us return instead to the starting point used in [27]) to obtain equation (2.6), that is, the expression

\[
M_{\text{ST}} = \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int_S \mu \phi \left( R^{(k)} + \theta_{(+)\theta_{(-)} - \frac{1}{2} \sigma_{(+)\rho\sigma_{(-)} - 2\omega_{id}\omega^d} \right) .
\]  

(2.7)

If \(S\) is a two-sphere of areal radius \(R\) (denoted by \(S_R\)), and assuming that the scalar field and the metric components in the gauge (2.5) depend only on \(t\) and \(R\) to respect spherical symmetry, then \(\phi(t, R)\) can be extracted from the sign of integration and the integral reduces to the usual Hawking–Hayward mass, so that

\[
M_{\text{ST}}(t, R) = \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int_{S_R} \mu \phi(t, R) \left( R^{(k)} + \theta_{(+)\theta_{(-)} - \frac{1}{2} \sigma_{(+)\rho\sigma_{(-)} - 2\omega_{id}\omega^d} \right) \\
= G\phi(t, R)M_{\text{HH}}(t, R) = G\phi M_{\text{MSH}} .
\]  

(2.8)

Therefore, the sought for formula for the quasilocal mass in scalar–tensor gravity and spherical symmetry is simply

\[
M_{\text{ST}} = \frac{\phi R}{2} (1 - \nabla^cR\nabla_cR) .
\]  

(2.9)

This expression could \textit{a priori} have been guessed by replacing \(G\) with \(G_{\text{eff}} = 1/\phi\) in the expression of the Misner–Sharp–Hernandez mass (2.4). One has

\[
1 - \frac{2M_{\text{ST}}}{\phi R} = \nabla^cR\nabla_cR = g_{RR} 
\]  

(2.10)

therefore, in the gauge (2.5) using the areal radius \(R\) as the radial coordinate, it is always \(g_{RR} = \left(1 - 2M_{\text{ST}}/(\phi R)\right)^{-1}\). Moreover, if the geometry admits horizons, these are located by the roots of the equation \(\nabla^cR\nabla_cR\) (in any coordinate system, since this is a scalar equation) [32–35]. It follows that, on a horizon, it is

\[
R_h = \frac{2M_{\text{ST}}(R_h)}{\phi(R_h)} ,
\]  

(2.11)
generalizing the well known relation between mass and radius of the Schwarzschild horizon. Equation (2.11) applies to black hole horizons, wormhole horizon throats, and cosmological horizons, whether they are static or time-dependent (i.e., apparent) horizons.

Let us come now to \( f(R) \) gravity. In this class of theories, the quasilocal mass in spherical symmetry becomes

\[
M_f(R) = \frac{f'(R)R}{2} \left(1 - \nabla^c R \nabla_c R \right).
\]

As a consequence of the fact that now the effective Brans–Dicke scalar \( \phi = f'(R) \) multiplies the Misner–Sharp–Hernandez mass known from GR, the usual condition \( f'(R) > 0 \) for the gravitational coupling to be positive and the graviton to carry positive kinetic energy corresponds to the non-negativity of the quasilocal mass.

3. Abreu–Nielsen–Visser gauge and Kodama vector

A spherical metric can always be written in a diagonal gauge employing the areal radius \( R \) as the radial coordinate, as in equation (2.5). We have reached the conclusion, with equation (2.10), that we can write

\[
g_{11} = \left( 1 - \frac{2M_{ST}}{\phi R} \right)^{-1} = \left( 1 - \frac{2GM_{MSH}}{R} \right)^{-1}. \tag{3.1}
\]

Nobody forbids to write \( g_{00} < 0 \) as

\[
g_{00} = -e^{-2\phi} \left( 1 - \frac{2GM_{MSH}}{R} \right)
\]

with an appropriate function \( \Phi(t, R) \), so we can always use the Abreu–Nielsen–Visser gauge\(^3\)

\[
ds^2 = -e^{-2\phi} \left( 1 - \frac{2GM_{MSH}}{R} \right) dt^2 + \left( 1 - \frac{2GM_{MSH}}{R} \right)^{-1} dR^2 + R^2 d\Omega^2. \tag{3.3}
\]

The Kodama vector is always defined geometrically in the presence of spherical symmetry and, in this gauge, it is given by

\[
K^a = \frac{1}{\sqrt{-g_{00} g_{11}}} \left( \frac{\partial}{\partial t} \right)^a = e^{\phi} \left( \frac{\partial}{\partial t} \right)^a \tag{3.4}
\]

From this vector one can then construct the Kodama four-current

\[
J^a \equiv G^{ab} K_b, \tag{3.5}
\]

which is a covariantly conserved vector. Indeed, in the Abreu–Nielsen–Visser gauge one has that

\[
J^a = G^{a\mu} K_{\mu} = e^{\phi} G^{a}_0 = \frac{2G e^{\phi}}{R^2} \left( -M'_{\text{MSH}}, M_{\text{MSH}}, 0, 0 \right), \tag{3.6}
\]

\(^3\) Although we use the name Abreu–Nielsen–Visser gauge, this kind of parametrization was used before, without name, in the black hole context (e.g., [36]).
from which it follows that
\[ \nabla_\mu J^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} J^\mu) = 0, \] (3.7)
with \( \sqrt{-g} = e^{-\Phi} R^2 \sin \theta \), as in [33].
A special situation occurs if \( \Phi = 0 \), or
\[ g_{00} g_{11} = -1, \] (3.8)
which covers many spherically symmetric geometries \(^4\). This condition was studied in reference [39], with the result that it is equivalent to the requirement that the double projection \( R_{ab} \ell^a \ell^b \) of the Ricci tensor onto radial null vectors \( \ell^a \) vanishes. Equivalently, the restriction of the Ricci tensor to the \((t, R)\) subspace is proportional to the restriction of the metric \( g_{ab} \) to this subspace [39]. Or, the areal radius \( R \) constitutes an affine parameter along radial null geodesics [39].
In this case, the Kodama vector is not just parallel, but it coincides with the time direction. If, further, the metric is static, the Kodama vector is also the timelike killing vector (while, in general, when the latter exists, the former is only parallel to it).

4. FLRW geometry

The scalar–tensor quasilocal energy for FLRW universes sourced by perfect fluids was derived from equation (2.3) in [27], but it follows immediately from equation (2.9). Given the importance of the FLRW geometry, we recall briefly the results of reference [27]. The FLRW line element
\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - Kr^2} + r^2 d\Omega_2 \right), \] (4.1)
where \( K = 0, \pm 1 \) is the normalized curvature index, is spherically symmetric about every spatial point and the areal radius is \( R(t, r) = a(t)r \). The prescription (2.9) then gives
\[ M_{ST}(R) = \frac{\phi R^3}{2} \left( H^2 + \frac{K}{a^2} \right) \]
\[ = \frac{H^2 R^3 \phi}{2} = \frac{4\pi R^3}{3} \left( \rho + \rho_\phi \right), \] (4.2) (4.3)
where in the last line the Hamiltonian constraint
\[ H^2 = \frac{8\pi \rho}{3\phi} - H \frac{\dot{\phi}}{\phi} + \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{V}{6\phi} = \frac{8\pi (\rho + \rho_\phi)}{3\phi} \] (4.4)
was used. In reference [27], instead, the expression (4.3) was obtained from the more involved equation (2.3).

In metric \( f(R) \) gravity, where \( \phi = f'(R) \), the Hamiltonian constraint reads [5]

\(^4\) Early work on this class of geometries includes references [37, 38].
\[ H^2 = \frac{1}{3f'} \left[ 8\pi\rho + \frac{\mathcal{R}f' - f}{2} - 3H(f')' \right], \]  
and we obtain [27]

\[ M_f(R) = \frac{H^2R^3\phi}{2} = \frac{4\pi R^3}{3} \rho + \frac{R^3}{2} \left( \frac{\mathcal{R}f' - f}{6} - Hf'' \right) \]

which is, of course, equivalent to equation (2.12).

5. General spherical, static, and asymptotically flat solution of Brans–Dicke theory

Let us consider the original Brans–Dicke theory with a constant coupling parameter \( \omega \) and a scalar field \( \phi \) without mass or potential [7]. Imposing that the solution be static, spherically symmetric, and asymptotically flat, Hawking has proved that all black holes reduce to the Schwarzschild black hole and the scalar field \( \phi \) becomes constant outside the Schwarzschild event horizon (the statement is more general, as it includes all stationary, asymptotically flat black holes of this theory, which then reduce to Kerr [40]). The theorem has been generalized to arbitrary scalar–tensor theories in which the scalar field does not have singularities or zeros on or outside the horizon, and to scalar field potentials with minima that allow states of stable equilibrium for \( \phi \), the exceptions being physically pathological [41–43]. Then, the spherical, static, asymptotically flat black hole solution of scalar–tensor gravity which is physically relevant is Schwarzschild with a constant \( \phi \) and the scalar–tensor quasilocal prescription (2.9) trivially reduces to the Misner–Sharp–Hernandez mass (2.4).

If \( V(\phi) \equiv 0 \), the most general static, spherically symmetric, and asymptotically flat solution of Brans–Dicke theory that is not a black hole is also known [44–48]:

\[ ds^2 = -e^{(\alpha+3)/r}dr^2 + e^{(\beta-\alpha)/r} \left( \frac{\gamma/r}{\sinh(\gamma/r)} \right)^2 d\Omega^2, \]

\[ \phi(r) = \phi_0 e^{-\beta/r}, \quad \beta = \frac{\sigma}{\sqrt{2\omega + 3}} \]

if \( \gamma \neq 0 \). Here \( \alpha, \beta, \) and \( \gamma \) are parameters satisfying the relations

\[ \beta = \frac{\sigma}{\sqrt{2\omega + 3}} \]

where \( \sigma \) is a scalar charge and

\[ 4\gamma^2 = \alpha^2 + 2\sigma^2 \]

if \( \sigma \neq 0 \) (if \( \sigma \) vanishes, both \( \alpha \) and \( \gamma \) must vanish, but this cannot be seen in this notation and one needs to revert to a form of the metric previously used by Wyman [49]). This solution
is conformal to the Fisher–Buchdal–Janis–Newman–Winicour–Wyman solution of general relativity with a free scalar field [49, 50] and, in a certain coordinate chart (of limited validity) [47], takes the Campanelli–Lousto form [51]. The electrovacuum generalization was found by Bronnikov [45], while special cases were found in [52] for $\alpha = \beta$, $\alpha = (2 \omega + 3) \beta$, and $\alpha = -(\omega + 1) \beta$. An exhaustive investigation of the general solutions of the Bergmann–Wagoner class of scalar–tensor theories was given in [53].

If the parameter $\gamma = 0$, the Jordan frame solution is the Brans class IV geometry [54]

$$ds^2 = -e^{-2B/r} dr^2 + e^{2B(C+1)/r} (dr^2 + r^2 d\Omega_2^2),$$

$$\phi = \phi_0 e^{-BC/r},$$

where

$$B = -\frac{(\alpha + \beta)}{2}, \quad C = -\frac{2\beta}{\alpha + \beta}.$$  \hfill (5.7)

Consider now the solution for $\gamma \neq 0$; the areal radius is

$$R(r) = \gamma \frac{e^{\frac{\alpha + \beta}{2}}}{\sinh(\gamma/r)}. \hfill (5.8)$$

When they exist, apparent horizons are the roots of the equation [32, 33] $\nabla^c R \nabla^c R = 0$. A single root describes a black hole horizon, while a double root describes a wormhole throat, and no roots means no horizons. The physical nature of the solutions (5.1)–(5.7) was discussed in [47]. To summarize, for $\gamma \neq 0$ the equation for the apparent horizons becomes

$$g^{rr} \left( \frac{dR}{dr} \right)^2 = \sinh^2(\gamma/r) \left[ \frac{\alpha - \beta}{2 \gamma} + \frac{1}{\tanh (\gamma/r)} \right]^2 = 0; \hfill (5.9)$$

if $(\beta - \alpha)/\gamma > 0$ a double root exists, corresponding to a wormhole throat at

$$r_H = \frac{2 \gamma}{\ln \left( \frac{\beta - \alpha + 2 \gamma}{\beta - \alpha - 2 \gamma} \right)} = \frac{\gamma}{\ln \left( \frac{2 \gamma - \beta}{\beta - \alpha} \right)}. \hfill (5.10)$$

If $(\beta - \alpha)/\gamma < 0$, instead, there is a naked singularity at $R = 0$ (the general solution (5.1) has a spacetime singularity there [47]) since, for both signs of $\gamma$, the Ricci scalar

$$\mathcal{R} = \frac{\omega \beta^2}{16 \gamma^4} e^{(\alpha - \beta + 4\gamma)/r} \hfill (5.11)$$

diverges as $r \to 0$ for $\beta - \alpha < 4\gamma$ or for $\alpha - \beta > 4\gamma$, respectively. The scalar–tensor quasilocal mass on a sphere of radius $r$ is

$$M_{ST}(r) = \frac{\gamma \phi_0}{2} \frac{e^{\frac{\alpha + \beta}{2}}}{\sinh (\gamma/r)} \left( 1 - \sinh^2 \left( \frac{\gamma}{r} \right) \left[ \frac{\alpha - \beta}{2 \gamma} + \frac{1}{\tanh (\gamma/r)} \right]^2 \right). \hfill (5.12)$$

In the case $(\beta - \alpha)/\gamma > 0$, the quasilocal mass evaluated at the wormhole throat is
\[ M_{ST}(r) = \frac{\gamma \phi_0}{2} e^{\frac{\alpha + \beta}{r_H}} \sinh \left( \gamma/r_H \right) \]
\[ = \frac{\gamma \phi_0}{2} \sqrt{(\beta - \alpha + 2\gamma)(\beta - \alpha - 2\gamma)} \left( \frac{\beta - \alpha - 2\gamma}{\beta - \alpha + 2\gamma} \right)^{\frac{\alpha + \beta}{2\gamma}} \]
\[ = \frac{\phi_0}{4} \left( \frac{\beta - \alpha - 2\gamma}{\beta - \alpha + 2\gamma} \right)^{\frac{\alpha + \beta}{2\gamma}}, \quad (5.13) \]

where in the middle line we used (5.10) and the identity
\[ \sinh(x) = \frac{e^{2x} - 1}{2e^x}, \]
which implies
\[ \sinh(\gamma/r_H) = \frac{2\gamma}{\sqrt{(\beta - \alpha + 2\gamma)(\beta - \alpha - 2\gamma)}}. \]

For the general spherical, static, and asymptotically flat solution of Brans–Dicke gravity, the scalar–tensor mass (2.9) reproduces [29] the monopole term found in a multipole expansion of the scalar–tensor metric [30].

Let us examine now the \( \gamma = 0 \) case giving the Brans class IV solution. The areal radius is
\[ R(r) = e^{\frac{B(C+1)}{r}} r \quad (5.14) \]
and the equation locating the apparent horizons reduces to
\[ \nabla^c R^a R = \left[ 1 - \frac{B(C + 1)}{r} \right]^2 = 0, \quad (5.15) \]
which has a double root \( r_H = B(C + 1) = (\beta - \alpha)/2 \) corresponding to a wormhole throat if \( \beta > \alpha \) and to a central naked singularity otherwise [47, 55]. The quasilocal mass (2.9) in a sphere of radius \( r \) is
\[ M_{ST}(r) = \frac{\phi_0 r}{2} e^{\frac{\beta}{r}} \left[ 1 - \left( 1 - \frac{B(C + 1)}{r} \right)^2 \right] \]
\[ = \phi_0 e^{\frac{\beta}{r}} r_H \left( 1 - \frac{r_H}{r} \right). \quad (5.16) \]

In the case of the naked central singularity, \( M_{ST}(r) \) is negative in the central region \( 0 < r < r_H/2 \) (as is common for naked singularities, for example for the Schwarzschild solution of GR with negative mass) and positive for \( r > r_H/2 \). When there is a wormhole throat (i.e., for \( \beta > \alpha \), the quasilocal mass on the throat is
\[ M_{ST}(r_H) = \frac{\phi_0 r_H}{2} e^{\frac{\beta}{r_H}} = \frac{\phi_0 (\beta - \alpha)}{4} e^{\frac{\alpha + \beta}{2\gamma}} \quad (5.17) \]
and it is positive.
6. Static geometry but time-dependent mass

Situations can arise in which the geometry is static but the quasilocal mass is time-dependent because $\phi$ is not static. As an example consider the special solution of Brans–Dicke theory with $\omega = -1$ (the value of the Brans–Dicke parameter corresponding to the low-energy limit of bosonic string theory [1]) and linear potential $V(\phi) = V_0 \phi$ found in [56, 57] in the Campanelli–Lousto [51] form

$$ds^2 = -dt^2 + A(r)^{-\sqrt{2}}dr^2 + A(r)^{1-\sqrt{2}}r^2d\Omega_2^2,$$  \hspace{1cm} (6.1)

$$\phi(t, r) = \phi_0 e^{2at} A(r)^{1/\sqrt{2}},$$  \hspace{1cm} (6.2)

where $A(r) = 1-2m/r$ and $a$ and $m$ are parameters.

The areal radius is

$$R(r) = \left(1 - \frac{2m}{r}\right)^{1/\sqrt{2}}r$$ \hspace{1cm} (6.3)

and the equation locating the apparent horizons is [56]

$$\nabla^c R \nabla_c R = g^{rr} \left(\frac{dR}{dr}\right)^2 = A(r)^{-1} \left[1 - \left(1 + \sqrt{2}\right) \frac{m}{r}\right]^2 = 0.$$ \hspace{1cm} (6.4)

For $m > 0$ there is always a double root, corresponding to a wormhole throat at $r_H = (1 + \sqrt{2})m$ or proper radius $R_H = (1 + \sqrt{2}) m \simeq 3.48 m$. The quasilocal mass at this throat is

$$M_{ST}(R_H) = \phi(R_H)R_H = \frac{m \phi_0 e^{2at}}{2}.$$  \hspace{1cm} (6.5)

Naively, one would expect the ‘mass’ to be $m$ and to be constant but, although the wormhole throat at $R_H$ does not change in time, the quasilocal mass $M_{ST}(R_H)$ depends on time through $\phi(R_H)$.

The situation in which the scalar field does not share the symmetries of the spacetime geometry is known to generate stealth solutions and violate the no-hair theorems in Horndeski and generalized Horndeski theories. One possibility is to introduce a linearly time-dependent scalar field profile [42, 58]. If such a solution is found in the more conventional scalar–tensor theory (1.1), then, through $\phi(t)$, the quasilocal mass (2.9) will be time-dependent even though the geometry is stationary.

7. The BBMB maverick solution for conformal coupling

Nonminimal coupling to the Ricci scalar $\mathcal{R}$ appears when a canonical, minimally coupled test scalar field $\psi$ is quantized on a curved space [59] and also, classically, in the context of radiation problems ([60–62], see also [63–67]). The nonminimal coupling of the scalar $\psi$ has been studied extensively in early Universe inflation ([68] and references therein). When the scalar is allowed to gravitate, one has a scalar–tensor theory [4, 69, 70] with action
\[
S_{\text{NMC}} = \int d^4x \sqrt{-g} \left[ \left( \frac{1}{8\pi G} - \xi \psi^2 \right) \frac{\mathcal{R}}{2} - \frac{1}{2} \nabla^\mu \psi \nabla_\mu \psi - V(\psi) \right],
\]
(7.1)

where \( \xi \) is the dimensionless coupling constant (with \( \xi = 1/6 \) corresponding to conformal coupling [11, 59]), the value of which depends on the nature of the scalar and can often be determined as a running coupling going to an infrared fixed point under a renormalization group flow [63, 71]. The general scalar–tensor action is given by equation (1.1) instead of (7.1), but it is sufficient to write

\[
\phi = \frac{1 - 8\pi G \psi^2}{G},
\]
(7.2)

and use

\[
\begin{align*}
\psi &= \pm \sqrt{\frac{1 - G\phi}{8\pi G \xi}}, \quad (7.3) \\
\nabla_\mu \psi &= \pm \sqrt{\frac{G}{32\pi \xi (1 - G\phi)}} \nabla_\mu \phi, \quad (7.4)
\end{align*}
\]

to reduce (7.1) to the standard form (1.1) with

\[
\omega(\phi) = \frac{G\phi}{4\xi (1 - G\phi)}
\]
(7.5)

Contrary to the Brans–Dicke field \( \phi \), the nonminimally coupled scalar \( \psi \) is not restricted to be positive. However, for \( \xi > 0 \) the scalar \( \psi \) must satisfy \( |\psi| < \psi_c \equiv \frac{1}{\sqrt{8\pi G \xi}} \), while all values of \( \psi \) are admissible if \( \xi < 0 \).

The Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) solution of conformally coupled (\( \xi = 1/6 \)) Einstein-scalar field theory found in [72] was rediscovered in [73]. This is a black hole solution with event horizon and scalar hair, but the scalar field \( \psi \) is singular on the horizon. This property is unphysical [73], making this solution a maverick. The BBMB solution is also unstable with respect to linear perturbations [74].

Following the derivations of [72, 73], Xanthopoulos and Zannias [75] and Klimčík [76] proved explicitly that the BBMB construct is the unique solution of the Einstein-conformal scalar field equations which is static, spherical, asymptotically flat, and does not have constant \( \psi \). A new proof of the uniqueness of the BBMBM solution outside the photon surface (the surface composed of the unstable circular photon orbits) was given recently in reference [77]. The BBMB solution has also been generalized by including a cosmological constant, a quartic potential \( V(\psi) = \lambda \psi^4 \), a Maxwell field, different horizon topologies [78–81], or an accelerating BBMB black hole [82].

In the Abreu–Nielsen–Visser gauge it is \( \Phi = 0 \) and the BBMB solution reads [73]

\[
ds^2 = -\left(1 - \frac{m}{R}\right)^2 dt^2 + \frac{dR^2}{(1 - m/R)^2} + R^2 d\Omega^2_{(2)},
\]
(7.6)

\[
\psi(R) = \sqrt{\frac{3}{4\pi G}} \frac{m}{R - m},
\]
(7.7)

which represents an extremal Reissner–Nordström black hole with horizon at \( R = m \), but the scalar field \( \psi \) is singular there. Correspondingly, the Jordan frame Brans–Dicke-like field given
by equation (7.2) is negative and divergent at $R = m$. The scalar–tensor quasilocal mass of a sphere of radius $R$ is then

$$M_{\text{ST}}(R) = \frac{\phi R}{2} (1 - \nabla^c R \nabla_c R) = \frac{R^2}{2G} \frac{R - 2m}{(R - m)^2} \left[ 1 - \left( 1 - \frac{m}{R} \right)^2 \right],$$

from which one finds

$$\lim_{R \to m} M_{\text{ST}}(R) = -\infty. \quad (7.8)$$

The pathology of the scalar field at the horizon (divergent $\psi$ or negative and divergent $\phi$, which means vanishing gravitational coupling strength) is reflected in this unphysical property of the quasilocal mass. Gravity is repulsive, and $M_{\text{ST}}(R)$ is negative, in the entire region $m < R < 2m$ surrounding the horizon, diverging at $R = m$.

### 8. Black holes in $f(R)$ gravity

Finally, let us examine a class of static, spherically symmetric, and asymptotically flat black holes found recently in $f(R) = R + \beta \sqrt{R}$ gravity [83]. In the Abreu–Nielsen–Visser gauge it is again $\Phi = 0$ and the line element reads

$$ds^2 = -u(R)dt^2 + \frac{dR^2}{u(R)} + R^2 d\Omega^2, \quad (8.1)$$

where

$$u(R) = \frac{1}{2} + \frac{1}{3\beta R} + \frac{\kappa^2}{R^2}, \quad (8.2)$$

$\kappa^2 = Q_E^2 + Q_M^2$, $Q_E$ and $Q_M$ are electric and magnetic charges, respectively, and $\beta$ is a parameter with the dimensions of a mass. For $Q_E = Q_M = 0$, the solution reduces to an uncharged one found in reference [84]. The Ricci scalar is $\mathcal{R} = 1/R^2$ [83] and the Kodama vector coincides with the timelike killing vector.

Requiring the gravitational coupling to be positive and the theory to be locally stable with respect to the Dolgov–Kawasaki (tachyonic) instability [12, 13] implies

$$f'(\mathcal{R}) = 1 + \frac{\beta}{\sqrt{\mathcal{R}}} = 1 + \beta R > 0, \quad (8.3)$$

$$f''(\mathcal{R}) = -\frac{\beta}{2R^{3/2}} = -\frac{\beta R^3}{2} > 0, \quad (8.4)$$

which imply that $\beta < 0$ and $R \leq 1/|\beta|$ (therefore, this solution can only be used as a model in this region).

The quasilocal mass (2.12) is

$$M_{f(\mathcal{R})} = \frac{R}{4} \left( 1 - |\beta| R \right) \left( 1 + \frac{2}{3 |\beta| R} - \frac{2\kappa^2}{R^2} \right). \quad (8.5)$$
At the horizons (when they exist), it is
\[ M_{f(R)}(R_H) = \frac{R_H}{2} \left( 1 - |\beta|R_H \right). \] (8.6)

Horizons correspond to the roots of \( w(R) = 0 \), therefore [83]:

- If \(-\frac{1}{3\sqrt{2}} < \kappa\beta < \frac{1}{3\sqrt{2}}\) there are two (inner and outer) horizons at
  \[ R_{\pm} = \frac{1}{3|\beta|} \left( 1 \pm \sqrt{1 - 18\kappa^2|\beta|^2} \right), \] (8.7)
  with
  \[ 0 < R_- < R_+ < \frac{2}{3|\beta|} \lesssim \frac{1}{|\beta|}. \] (8.8)

The scalar–tensormass (2.12) on the outer horizon is
\[ M_{f(R)}(R_+) = \frac{\left( 1 + 18\kappa^2\beta^2 + \sqrt{1 - 18\kappa^2\beta^2} \right)}{18|\beta|}, \] (8.9)
and is positive. For comparison, the quasilocal mass of [24, 26] is [83]
\[ \bar{M}(R_+) = \frac{\left( 1 + 9\kappa^2\beta^2 + \sqrt{1 - 18\kappa^2\beta^2} \right)}{12|\beta|}. \] (8.10)

- If \( \kappa\beta = \pm \frac{1}{3\sqrt{2}} \) there is a double root corresponding to a wormhole throat at \( R_H = (3|\beta|)^{-1} = \sqrt{2}|\kappa| \). The quasilocal mass at this throat
  \[ M_{f(R)}(R_H) = \frac{\left( 1 + 9\kappa^2\beta^2 + \sqrt{1 - 18\kappa^2\beta^2} \right)}{18|\beta|}, \] (8.11)
is also positive.

- If \( \kappa\beta < -\frac{1}{3\sqrt{2}} \) or \( \kappa\beta > \frac{1}{3\sqrt{2}} \) there are no real roots of \( \nabla^c R \nabla_c R = 0 \) and the geometry contains a naked singularity at \( R = 0 \), where the Ricci scalar \( \mathcal{R} = 1/R^2 \) diverges. The quasilocal mass of a sphere of radius \( R \) is
  \[ M_{f(R)}(R) = \frac{R \left( 1 - |\beta|R \right)}{2} \left( \frac{1}{2} + \frac{1}{3|\beta|R - \kappa^2 R^2} \right). \] (8.12)

In the region \( R < 1/|\beta| \), this mass is negative when \( 3|\beta|R^2 + 2R - 6|\beta|\kappa^2 < 0 \), which corresponds to \( R_1 < R < R_2 \), where
\[ R_{1,2} = \frac{1 \pm \sqrt{1 + 18\kappa^2|\beta|^2}}{3|\beta|}. \]

Limiting ourselves to the physical region \( R > 0 \), the quasilocal mass is negative in the region
\[ 0 < R < R_2 = \frac{\sqrt{1 + 18\kappa^2|\beta|^2} - 1}{3|\beta|} \] (8.13)
surrounding the naked singularity. Note that \( R_2 \) can potentially exceed \( R = 1/|\beta| \), in which case the quasilocal mass is negative everywhere.
9. Conclusions

There is little doubt that the mass–energy of a system is one of the most basic concepts in physics and astrophysics, yet GR is ambiguous in this regard, offering several different quasilocal energy prescriptions [14]. Moreover, the concept of quasilocal energy seems to have remained confined to the realm of formal mathematical physics while, to be useful, it should become part of relativistic astrophysics and cosmology. The application of the Hawking–Hayward quasilocal prescription [16–18] to cosmology and astrophysics has been started in [19]. Since there is currently much motivation, especially from cosmology, to explore alternative theories of gravity theoretically and observationally, it is useful to extend the quasilocal energy construct of [16–18] to the prototypical alternative to GR, scalar–tensor gravity (which includes the subclass of \( f(R) \) theories nowadays very popular in cosmology [3, 5]). The most straightforward prescription of quasilocal energy in these theories (in the sense that it is based simply on writing the field equations as effective Einstein equations and is independent of thermodynamics of spacetime, black hole thermodynamics, and the subsequent restriction to black hole horizons) was given recently in [27]. In this work we have discussed this prescription for spherically symmetric geometries, which are the simplest situations occurring in the modelling of systems of interest in astrophysics and cosmology. The particularly convenient Abreu–Nielsen–Visser metric gauge has been discussed, together with its relation with the Kodama vector used in black hole thermodynamics. As the case of FLRW cosmology shows, it is much more convenient to derive the quasilocal mass of spherical systems from the simple formula (2.9) than from the general prescription (2.3). These developments will be be used in future work related to black hole thermodynamics and astrophysics in scalar–tensor and \( f(R) \) gravity.

Acknowledgments

This work is supported, in part, by the Natural Sciences and Engineering Research Council of Canada (Grant No. 2016-03803 to VF). The work of AG has also been carried out in the framework of activities of the National Group of Mathematical Physics (GNFM, INdAM).

ORCID iDs

Andrea Giusti https://orcid.org/0000-0003-0329-2726
Valerio Faraoni https://orcid.org/0000-0002-2601-1870

References

[1] Callan C G, Friedan D, Martinez E J and Perry M J 1985 Nucl. Phys. B 262 593
[2] Fradkin E S and Tseytlin A A 1985 Nucl. Phys. B 261 1
[3] Amendola L and Tsujikawa S 2010 Dark Energy, Theory and Observations (Cambridge: Cambridge University Press)
[4] Capozziello S, Carloni S and Troisi A 2003 Recent Res. Dev. Astron. Astrophys. 1 625 arXiv:astro-ph/0303041
[5] Carroll S M, Duvvuri V, Trodden M and Turner M S 2004 Phys. Rev. D 70 043528
[6] Capozziello S and Faraoni V 2010 Beyond Einstein Gravity (Berlin: Springer)
[7] Sotiriou T P and Faraoni V 2010 Rev. Mod. Phys. 82 451
De Felice A and Tsujikawa S 2010 *Living Rev. Relat.* **13** 3
Nojiri S and Odintsov S D 2011 *Phys. Rep.* **505** 59

[6] Bergmann P G 1968 *Int. J. Theor. Phys.* **1** 25
Wagoner R V 1970 *Phys. Rev.* **D 1** 3209
Nordvedt K 1970 *Astrophys. J* **161** 1059
Nordvedt K 1968 *Phys. Rev.* **169** 1017

[7] Brans C H andDicke R H 1961 *Phys. Rev.* **124** 925

[8] Psaltis D, Perrodin D, Dienes K R and Mocioiu I 2008 *Phys. Rev. Lett.* **100** 091101
Psaltis D, Perrodin D, Dienes K R and Mocioiu I 2008 *Phys. Rev. Lett.* **100** 119902 (erratum)
Clifton T, Ferreira P G, Padilla A and Skordis C 2012 *Phys. Rep.* **513** 1
Berti E, Cardoso V, Gualtieri L, Horbatsch M and Sperhake U 2013 *Phys. Rev. D* **87** 124020
Baker T, Psaltis D and Skordis C 2015 *Astrophys. J.* **802** 63

[9] Berti E et al 2015 *Class. Quantum Grav.* **32** 243001

[10] Jacobson T 1999 *Phys. Rev. Lett.* **83** 2699
Horbatsch M W, Burgess C P andCosmol J 2012 *Astropart. Phys.* **1205** 010
Cardoso V, Carucci I P, Fani P and Sotiriou T P 2013 *Phys. Rev. Lett.* **111** 111101
Herdeiro C A R and Radu E 2014 *Phys. Rev. Lett.* **112** 221101
Sotiriou T P and Zhou S-Y 2014 *Phys. Rev. Lett.* **112** 251102
Bhattacharya S, Dialektopoulos K F, Romano A E and Tomaras T N 2015 *Phys. Rev. Lett.* **115** 181104
Faraoni V 2017 *Phys. Rev. D* **95** 124013

[11] Wald R M 1984 *General Relativity* (Chicago, IL: Chicago University Press)

[12] Dolgov A D and Kawasaki M 2003 *Phys. Lett. B* **573** 1

[13] Faraoni V 2006 *Phys. Rev. D* **74** 104017

[14] Szabados L B 2009 *Living Rev. Relat.* **12** 4

[15] Gourgoulhon E andJaramillo J I 2006 *Phys. Rep.* **423** 159

[16] Hawking S 1968 *J. Math. Phys.* **9** 598

[17] Hayward S A 1994 *Phys. Rev. D* **49** 831

[18] Hayward S A 1996 *Phys. Rev. D* **53** 1938

[19] Faraoni V, Lapierre-Léonard M andPrain A 2015 *Phys. Rev. D* **92** 023511
Faraoni V, Lapierre-Léonard M and Prain A 2015 *J. Cosmol. Astropart. Phys.* **10** 013
Lapierre-Léonard M, Faraoni V and Hammad F 2017 *Phys. Rev. D* **96** 083525
Faraoni V andLapierre-Léonard M 2017 *Phys. Rev. D* **95** 023509
Faraoni V, Bellknap-Keet S andLapierre-Léonard M 2016 *Phys. Rev. D* **93** 044042
Casadio R, Giugno A andGiusti A 2017 *Gen. Relat. Gravit.* **49** 32
Casadio R, Giusti A and Rahim R 2018 *Europhys. Lett.* **121** 60004
Rahim R, Giusti A andCasadio R 2018 *Int. J. Mod. Phys. D* **28** 1950021
Giusti A 2019 *Int. J. Geomet. Methods Mod. Phys.* **16** 1950001
Giusti A andFaraoni V 2019 *Phys. Dark Universe* **26** 100353
Faraoni V, Giusti A andCôté J 2020 arXiv:2003.13935[gr-qc]

[20] Jacobson T 1995 *Phys. Rev. Lett.* **75** 1260
Elizalde C, Guedens R and Jacobson T 2006 *Phys. Rev. Lett.* **96** 121301
Hayward S A, Mukohyama S andAshworth M C 1999 *Phys. Lett. A* **256** 347
Mukohyama S and Hayward S A 2000 *Class. Quantum Grav.* **17** 2153
Cai R G andKim S P 2005 *J. High Energy Phys.* JHEP02(2005)050
Cai R G andCao L M 2007 *Phys. Rev. D* **75** 064008
Sheykhi A, Wang B andCai R G 2007 *Phys. Rev. D* **76** 023515
Cai R G, Cao L M and Hu Y P 2008 *J. High Energy Phys.* JHEP08(2008)090
Gong Y andWang A 2007 *Phys. Rev. Lett.* **99** 211301
Wu S F, Wang B, Yang G H andZhang P M 2008 *Class. Quantum Grav.* **25** 235018
Bamba K andGeng C Q 2009 *Phys. Lett. B* **679** 282
Akbar M andCai R G 2007 *Phys. Rev. D* **75** 084003
Padmanabhan T 2002 *Class. Quantum Grav.* **19** 5387
Padmanabhan T 2005 *Phys. Rep.* **406** 49
Paranjape A, Sarkar S andPadmanabhan T 2006 *Phys. Rev. D* **74** 104015
Kothawala D, Sarkar S andPadmanabhan T 2007 *Phys. Lett. B* **652** 338
Chirco G, Haggard H M, Riello A andRovelli C 2014 *Phys. Rev. D* **90** 044044
Charmousis C, Kolyvaris T, Papantonopoulos E and Tsoukalas M 2014 *J. High Energy Phys.* [HEPTh(2014)085]

Babichev E and Esposito-Farese G 2017 *Phys. Rev.* D 95 024020

Motohashi H and Minamitsuji M 2018 *Phys. Lett.* B 781 728

Minamitsuji M and Motohashi H 2018 *Phys. Rev.* D 98 084027

[59] Callan C G Jr, Coleman S and Jackiw R 1970 *Ann. Phys.*, NY 59 42

[60] Chernikov N A and Tagirov E A 1968 *Ann. Inst. Henri Poincaré* A 9 109

[61] DeWitt B S and Brehme R W 1960 *Ann. Phys., NY* 9 220

[62] Sonego S and Faraoni V 1993 *Class. Quantum Grav.* 10 1185

[63] Buchbinder I L, Odintsov S D and Shapiro I L 1992 *Effective Action in Quantum Gravity* (Bristol: IOP Publishing)

[64] Birrell N D and Davies P C 1980 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)

[65] Birrell N D and Davies P C W 1980 *Phys. Rev.* D 22 322

[66] Nelson B and Panangaden P 1982 *Phys. Rev.* D 25 1019

[67] Friedlander F G 1975 *The Wave Equation on a Curved Spacetime* (Cambridge: Cambridge University Press)

[69] Fujii Y and Maeda K 2003 *The Scalar-Tensor Theory of Gravity* (Cambridge: Cambridge University Press)

[70] Faraoni V 2004 *Cosmology in Scalar-Tensor Gravity* (Fundamental Theories of Physics Series) vol 139 (Dordrecht: Kluwer)

[71] Buchbinder I L and Odintsov S D 1983 *Sov. J. Nucl. Phys.* 40 848

[72] Bocharova N M, Bronnikov K A and Melnikov V N 1970 *Vestn. Mosk. Univ. Fiz. Astron.* 6 706

[73] Bekenstein J D 1974 *Ann. Phys., NY* 82 535

[74] Bronnikov K A and Kireyev Y N 1978 *Phys. Lett.* A 67 95

[75] Xanthopoulos B C and Zannias T 1991 *J. Math. Phys.* 32 1875–80

[76] Klimčík C 1993 *J. Math. Phys.* 34 1914
[77] Tomikawa Y, Shiromizu T and Izumi K 2017 Class. Quantum Grav. 34 155004
[78] Martinez C, Troncoso R and Zanelli J 2003 Phys. Rev. D 67 024008
[79] Virbhadra K S and Parikh J C 1994 Phys. Lett. B 331 302
[80] Virbhadra K S and Parikh J C 1994 Phys. Lett. B 331 340265
[81] Martinez C, Staforelli J P and Troncoso R 2006 Phys. Rev. D 74 044028
[82] Charmousis C, Kolyvaris T and Papantonopoulos E 2009 Class. Quantum Grav. 26 175012
[83] Elizalde E, Nashed G G L, Nojiri S and Odintsov S D 2020 Eur. Phys. J. C 80 109
[84] Sebastiani L and Zerbini S 2011 Eur. Phys. J. C 71 1591