First order parent formulation for generic gauge field theories

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\textbf{Abstract.} We show how a generic gauge field theory described by a BRST differential can systematically be reformulated as a first order parent system whose spacetime part is determined by the de Rham differential. In the spirit of Vasiliev’s unfolded approach, this is done by extending the original space of fields so as to include their derivatives as new independent fields together with associated form fields. Through the inclusion of the antifield dependent part of the BRST differential, the parent formulation can be used both for on and off-shell formulations. For diffeomorphism invariant models, the parent formulation can be reformulated as an AKSZ-type sigma model. Several examples, such as the relativistic particle, parametrized theories, Yang-Mills theory, general relativity and the two dimensional sigma model are worked out in details.

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1 Introduction

When dealing with the types of gauge field theories that are of interest in theoretical high energy physics, it is often useful to produce equivalent formulations that are local, make rigid symmetries manifest or allow for an action principle (see e.g. [1, 2, 3] in the context of higher spin fields). For instance, there has been a lot of focus on a first order “unfolded” form [4, 5, 6, 7, 8, 9] of the equations of motion in the context of higher spin interactions. In this approach the equations of motion are represented as a free differential algebra (FDA). The latter structure was originally introduced in mathematics [10] and independently in the context of supergravity [11, 12].

The characteristic feature of the formulation discussed in the present paper is that spacetime derivatives enter exclusively through the de Rham differential acting on form fields. Our aim is the systematic construction of such a first order formulation for generic gauge theories.

In the linear case [13, 14], this problem has been solved by using a BRST first quantized approach in combination with a version of Fedosov quantization [15]. Various equivalent formulations, including the unfolded one, are then reached by reductions that correspond to the elimination of cohomologically trivial pairs on the first quantized level. Through related techniques, generalized symmetries of bosonic singletons of arbitrary spin have been classified [16] and concise formulations of mixed symmetry higher spin gauge fields on Minkowski and AdS spaces have been constructed [17, 18]. In this context, let us also mention recent progress within the usual unfolded formalism in describing free mixed-symmetry fields [19, 20, 21, 22].

For the non linear case treated in the present paper, we take as an input the antifield dependent BRST differential of a starting point interacting theory, which encodes the equations of motion, Noether identities, gauge symmetries and their compatibility conditions [23, 24, 25, 26, 27] (see also [28, 29] for reviews). The parent theory is then constructed in the form of an extended BRST differential by introducing the derivatives of the starting point fields as new independent fields together with associated form fields in such a way that all additional fields form generalized auxiliary fields.

Various reduced forms can then be obtained from the parent formulation by eliminating one or another set of generalized auxiliary fields. In particular, such sets can be related not only to trivial pairs of the original BRST differential but also to trivial pairs for the extension of the original BRST differential by the horizontal differential, such as those studied in [30, 31, 32]. The associated generalized tensor calculus and “Russian formulas” (see [33] for the original derivation) give rise to corresponding geometrical structures in the reduced formulations.

Besides the obvious connection with unfolding, free differential algebras and Fedosov
quantization, the parent theory can also be interpreted as a generalized AKSZ sigma model [34] originally proposed in the context of the Batalin-Vilkovisky formulation for topological field theories (see also [35, 36, 37, 38, 39, 40, 41, 42, 43, 44] for further developments). More precisely, the parent differential for non-topological theories contains an extra term that can however be absorbed by a field redefinition in the diffeomorphism invariant case.

Although in this paper we restrict ourselves to constructing a parent formulation for a given gauge field theory for which the interactions are already known, ultimately our aim is to use the techniques of the parent formalism to built new interacting models. Indeed, the usefulness of the parent approach comes from the fact that it combines in a unified framework the control over the underlying geometry and the manifest realization of global symmetries of the unfolded approach [8, 9] with the cohomological control on gauge symmetries provided by BRST theory which leads to systematic supergeometrical and deformation theoretical techniques [45, 34, 46].

The paper is organized as follows. In Section 2.1 we quickly review the local field theory set-up and the BRST differential that describes the gauge system. We then introduce the necessary additional fields and operators and provide the parent form of the differential in Sections 2.2 and 2.3. Technical details on conventions are relegated to an Appendix. How the parent theory relates to the AKSZ approach and what it looks like in the particular case of linear theories is discussed in the next two sections. General aspects of reductions including cohomological tools are discussed in Section 2.6. We then illustrate various features of our analysis on concrete models: we start with non-degenerate systems, show how the parent formulation for a relativistic particle reduces to its Hamiltonian formulation, discuss parent and unfolded formulations of Yang-Mills theory, produce both off and on-shell versions of parent gravity and finally show how the parent formulation of the Polyakov string gives rise to a gauge theory for the Virasoro algebra.

2 Parent theory

2.1 Original BRST differential

The BRST formulation involves bosonic and fermionic fields \( z^\alpha(x) \). The set of fields is graded by an integer degree \( g \), the ghost number \( \text{gh}(\cdot) \). Parity is denoted by \( | \cdot | \). The physical fields are among the ghost number zero fields, while ghosts and antifields of the minimal sector are typically in positive and negative ghost numbers respectively.

Besides the local coordinates on the spacetime manifold, denoted by \( x^\mu \) with \( \mu = 0, \ldots, n-1 \) and \( \text{gh}(x^\mu) = 0 \), the \( z^\alpha \) and their derivatives are local coordinates on the
fiber of an associated jet-bundle in an algebraic approach (see e.g. [47, 48, 49, 50, 51] for reviews). The $z^\alpha$ and their derivatives are denoted by $z^\alpha_{(\mu)}$ with $(\mu) = \mu_1 \ldots \mu_k$ a symmetric multi-index. Local functions are functions that depend on $x^\mu$, $z^\alpha$ and a finite number of their derivatives. The total derivative is defined as the vector field

$$\partial_\mu = \frac{\partial}{\partial x^\mu} + z^\alpha \frac{\partial}{\partial z^\alpha} + z^\alpha_{\mu\rho} \frac{\partial}{\partial z^\rho_{\mu}} + \cdots \equiv \frac{\partial}{\partial x^\mu} + \frac{\partial F}{\partial x^\mu}, \quad (2.1)$$

where $\frac{\partial F}{\partial x^\mu}$ denotes the action of the total derivative on the fields $z^\alpha$ and their derivatives.

For later use, we note that if we collect the jet coordinates as the coefficients of a Taylor expansion,

$$z^\alpha(x) = \sum_{k=0}^1 \frac{1}{k!} z^\alpha_{\mu_1 \ldots \mu_k} x^{\mu_1} \ldots x^{\mu_k} \equiv z^\alpha_{(\mu)} x^{(\mu)}, \quad (2.2)$$

the part of the total derivative that acts on the jet-coordinates is also uniquely defined through the relation

$$\frac{d}{d x^\mu} z^\alpha(x) = \frac{\partial F}{\partial x^\mu} z^\alpha(x). \quad (2.3)$$

The dynamics and gauge symmetries of the theory are determined by a nilpotent BRST differential $s$ of ghost number one defined through

$$sz^\alpha = S^\alpha [x, z], \quad [s, \partial_\mu] = 0, \quad (2.4)$$

where $S^\alpha [x, z]$ are local functions. The second equations determines the “prolongation” of $s$ on the spacetime derivatives of the fields. A standard field theoretic way to represent the BRST differential is through functional derivatives or using the condensed DeWitt notation (see e.g. [52]),

$$s = \int d^n x S^\alpha \frac{\delta}{\delta z^\alpha(x)} = S^\alpha \frac{\delta}{\delta z^\alpha}, \quad sx^\mu = 0 \quad (2.5)$$

where $S^\alpha$ is taken as a function of $z^\alpha(x)$ and its usual derivatives, $d^n x = dx^0 \wedge \cdots \wedge dx^{n-1}$, $a = (\alpha, x^\mu)$ and the summation convention includes integration over $x^\mu$.

The horizontal complex consists of the exterior algebra of $dx^\mu$ with coefficients that are local functions. Elements of this algebra are denoted by $\omega [x, dx, z]$, with $dx^\mu$ considered as Grassmann odd, i.e., as anticommuting with all odd fields, $\text{gh}(dx^\mu) = 0$. The horizontal differential is $d_H = dx^\mu \partial_\mu$. We assume that horizontal forms can be decomposed into field/antifield independent and dependent parts, $\omega [x, dx, z] = \omega [x, dx, 0] + \tilde{\omega} [x, dx, z]$. The bi-complex involving the latter is denoted by $\hat{\Omega}^{\ast, \ast}$. A standard result is then the “algebraic Poincaré lemma”,

$$H^k (d_H, \hat{\Omega}) = 0 \quad \text{for} \quad 0 \leq k < n,$$

$$\omega^n = d_H \eta^{n-1} \iff \frac{\delta \omega^n}{\delta z^\alpha} = \partial_\mu \frac{\partial \omega^n}{\partial z^\rho_{\mu}} + \cdots = 0. \quad (2.6)$$
The space of local functionals \( \hat{\mathcal{F}} \) is defined as \( \hat{\Omega}^{* n}/d_H \hat{\Omega}^{* n-1} \). Important information on physical properties of the system is contained in \( H^q(s, \hat{\mathcal{F}}) \), the local BRST cohomology groups in ghost number \( g \) (see e.g. [45] and references therein).

When considering the total differential of the bi-complex, \( \tilde{s} = s + d_H \) with degree the sum of the ghost number and the form degree, another standard result is the isomorphism

\[
H^q(s, \hat{\mathcal{F}}) \cong H^{q+n}(\tilde{s}, \hat{\Omega}) ,
\]

(2.7)

where the representative of \( H^q(s, \hat{\mathcal{F}}) \) is obtained by extracting the component of top form degree \( n \) from a representative of \( H^{q+n}(\tilde{s}, \hat{\Omega}) \).

### 2.2 Extended space of fields and basic operations

If \( \Psi^A(x) \) denote the fields of the original formulation, the fields of the parent formulation are given by \( \Psi^A_{(\lambda)[\nu]}(x) \), where \( (\lambda) \) denotes a symmetric multi-index and \( [\nu] \) a skew-symmetric one. The fields without indices are identified with the fields of the original formulation, \( \Psi^A_{(\lambda)[\nu]}(x) \equiv \Psi^A(x) \). By introducing additional Grassmann even variables \( y^\lambda \), \( gh(y^\lambda) = 0 \) and Grassmann odd variables \( \theta^\nu \), \( gh(\theta^\nu) = 1 \), these fields can be collected in a generalized superfield as follows

\[
\Psi^A(x, y, \theta) = \sum_{k=0}^{l} \sum_{\nu_1...\nu_k} \frac{1}{k!!} \Psi^A_{\lambda_1...\lambda_k|\nu_1...\nu_k}(x) \theta^{\nu_1}...\theta^{\nu_k} y^{\lambda_k}...y^{\lambda_1}
\]

(2.8)

Ghost numbers and parities of \( \Psi^A_{(\lambda)[\nu]}(x) \) are then assigned so that the total ghost number and parity of \( \Psi^A(x, y, \theta) \) is equal to that of \( \Psi^A(x) \) by taking the degrees of \( \theta^\nu \) into account, \( gh(\Psi^A_{\lambda_1...\lambda_k|\nu_1...\nu_k}(x)) = gh(\Psi^A(x)) - l \). In the algebraic approach described in the previous section, the \( x^\mu \) dependence of the fields is replaced by considering the jet-bundle coordinates \( \Psi^A_{(\lambda)[\nu]}(x^\mu) \) and

\[
\Psi^A(x, y, \theta) = \Psi^A_{(\mu)(\lambda)[\nu]}(x^\mu) \theta^{[\nu]} y^{(\lambda)}
\]

(2.9)

In the parent formulation, the algebra of local functions is taken as the algebra of functions in \( x^\mu, \Psi^A_{(\mu)(\lambda)[\nu]}, \) where each function depends on a finite number of \( x^\mu \) derivatives, i.e., there is no dependence on fields with index \( \mu_1...\mu_k \) with \( k \) strictly greater than some integer.

Consider then the algebra \( \mathcal{A} \) of differential operators acting from the right in the space of functions in \( x^\mu, y^\lambda, \theta^\nu \). By identifying \( \Psi^A_{(\mu)(\lambda)[\nu]} \) as elements of the basis dual to \( x^\mu \theta^{[\nu]} y^{(\lambda)} \), one naturally makes linear functions in \( \Psi^A_{(\mu)(\lambda)[\nu]} \) into a left \( \mathcal{A} \) module. Explicitly, if \( \mathcal{O} \) is the linear operator acting on \( x^\mu \theta^{[\nu]} y^{(\lambda)} \) and \( \mathcal{O}^F \) the associated linear
operator acting on $\Psi^A_{(\mu)(\lambda)[\nu]}$, we have

$$
\mathcal{O}^F \Psi^A(x, y, \theta) = (-1)^{|\mathcal{O}|} |A| \Psi^A(x, y, \theta) \mathcal{O},
$$

$$
\mathcal{O}_1^F \mathcal{O}_2^F \Psi^A(x, y, \theta) = (-1)^{|(\mathcal{O}_1^\dagger + |\mathcal{O}_2|)|} |A| \Psi^A(x, y, \theta) \mathcal{O}_1 \mathcal{O}_2.
$$

(2.10)

where $|\mathcal{O}|$ and $|A|$ is the Grassmann parity of $\mathcal{O}$ and $\Psi^A(x, y, \theta)$ respectively. One then extends this $\mathcal{A}$-action to generic functions in $\Psi^A_{(\mu)(\lambda)[\nu]}$ through the graded Leibnitz rule as a vector field acting from the left, which we continue to denote by $\mathcal{O}^F$.

Because both the functions in $x, y, \theta$ and those in $\Psi^A_{(\mu)(\lambda)[\nu]}$ are modules for $\mathcal{A}$ considered as a Lie algebra, the map $\mathcal{O} \mapsto \mathcal{O}^F$ respects the graded commutator,

$$
[\mathcal{O}_1^F, \mathcal{O}_2^F] = [\mathcal{O}_1, \mathcal{O}_2]^F.
$$

(2.11)

Here $\Psi^A(x, y, \theta)[\mathcal{O}_1, \mathcal{O}_2] = (\Psi^A \mathcal{O}_1) \mathcal{O}_2 - (-1)^{|\mathcal{O}_1|} (\Psi^A \mathcal{O}_2) \mathcal{O}_1$. Some details on the origin of these conventions are given in the Appendix.

### 2.3 BRST differential of parent theory

If $s \Psi^A = S^A[x, \Psi]$ with $[s, \partial_\mu] = 0$ defines the BRST differential of the original theory, which involves only the coordinates referring to $x^\mu$ derivatives, the action of $\bar{s}$ on $\Psi^A$, $\bar{s} \Psi^A = \bar{S}^A[x, \Psi]$ is defined by replacing in $S^A[x, \Psi]$ the indices corresponding to $x^\mu$ derivatives with the same indices corresponding to $y^\lambda$ derivatives. It thus follows that $\bar{S}^A$ depends on $x^\mu$ and $\Psi^A$'s with indices $\lambda$ corresponding to $y^\lambda$ variables, but no $\mu$ nor $\nu$ indices. The action of $\bar{s}$ is then extended to $\Psi^A_{(\mu)(\lambda)[\nu]}$ by requiring that

$$
[\bar{s}, \partial_\mu] = 0, \quad [\bar{s}, \frac{\partial^F}{\partial x^\nu}] = 0, \quad [\bar{s}, \frac{\partial^F}{\partial y^\lambda}] = 0.
$$

(2.12)

Let $d = \theta^\mu \frac{\partial}{\partial x^\mu}$ and $\sigma = \theta^\nu \frac{\partial}{\partial y^\nu}$. If their action from the right is defined by $\Psi^A d = \Psi^A \frac{\partial}{\partial x^\mu} \theta^\mu$ and $\Psi^A \sigma = \Psi^A \frac{\partial}{\partial y^\nu} \theta^\mu$, where the arrow denotes right derivatives, (2.10) implies:

$$
d^F \Psi^A = d \Psi^A, \quad \sigma^F \Psi^A = \sigma \Psi^A.
$$

(2.13)

In particular, when acting on $\Psi^A_{(\mu)(\lambda)[\nu]}$, $d^F$ and $\sigma^F$ remove an index from the collection of $\nu$ indices and add it to the $\mu$ respectively the $\lambda$ indices,

$$
dl^F \Psi^A_{\mu_1 \ldots \mu_k(\lambda)[\nu_1 \ldots \nu_p]} = (-1)^A p \Psi^A_{\mu_1 \ldots \mu_k(\nu_1 \ldots \nu_{p+1})},
$$

$$
\sigma^F \Psi^A_{(\mu)[\lambda_1 \ldots \lambda_k(\nu_1 \ldots \nu_p)} = (-1)^A p \Psi^A_{(\mu)[\lambda_1 \ldots \lambda_k(\nu_1 \ldots \nu_{p-1})}. \quad (2.14)
$$

The BRST differential of the parent theory is the ghost number 1 operator defined through

$$
s^F = d^F - \sigma^F + \bar{s}.
$$

(2.15)
In order to show that $s$ is nilpotent, one only needs to show that
\[ [dF - \sigma F, \bar{s}] = 0, \] (2.16)
since both $dF - \sigma F$ and $\bar{s}$ are nilpotent by construction. Let us first note that by definition of $dF$ and $\sigma F$, one obviously gets
\[ [dF - \sigma F, \bar{s}]\Psi^A_{(\mu)(\lambda)[\nu]} = 0, \] (2.17)
if there are no antisymmetric $\nu$ indices. Furthermore,
\[ \frac{\partial F}{\partial \theta^\mu}, [dF - \sigma F, \bar{s}] = \frac{\partial F}{\partial y^\nu}, \bar{s} = 0, \] (2.18)
where one has used (2.11), the second of (2.12) and finally the difference between the first and the third relation of (2.12). It then follows that $[dF - \sigma F, \bar{s}]$ vanishes on all $\Psi^A_{(\mu)(\lambda)[\nu]}$.

To connect to other formulations, let us recall how physical fields, equations of motion and gauge symmetries are encoded in the parent formulation (see [13, 53] for more details). The physical fields are among the ghost number 0 fields where one in general also finds auxiliary fields and pure gauge degrees of freedom. In particular, if $gh(A_l)$ denotes the ghost degree of $\Psi^A$, then there are no physical fields associated to $\Psi^A$ if $gh(A_l) < 0$, and for $gh(A_l) = l \geq 0$, there are some among the ghost number 0 fields $\Psi^A_{(\lambda)(\nu_1...\nu_l)}$. Denoting all the parent formulation fields at ghost degree $l$ by $\Psi^{\alpha_l}$ the equations of motion and gauge transformations for the ghost number 0 fields can be written as
\[ \left. (s^P \Psi^{\alpha_l - 1}) \right|^{\Psi^{\alpha_l} = 0 \text{ for } k \neq 0} = 0, \quad \delta \Psi^{\alpha_l} = \left. (s^P \Psi^{\alpha_0}) \right|^{\Psi^{\alpha_l} = 0 \text{ for } k \neq 0, 1}, \] (2.19)
where for the former, all fields with ghost number different from zero are put to zero, while for the latter, one keeps in addition to the ghost number 0 fields, those in ghost number 1 which are replaced by gauge parameters. In a similar way, reducibility relations for equations of motion and for gauge transformations can be read off from $s^P$ in the sector of fields of higher positive and negative ghost numbers.

### 2.4 AKSZ-type sigma model

The structure of the parent theory BRST differential (2.15) is very similar to that defining AKSZ sigma models [34]. More precisely, the BRST differential of the non-Lagrangian version of an AKSZ sigma model is defined by
\[ s^{AKSZ}\Psi^a(x, \theta) = d\Psi^a(x, \theta) + Q^a(\Psi(x, \theta)) , \] (2.20)
where $Q^a$ are the components of an odd nilpotent vector field on the space with coordinates $\Psi^a$. 
It is then straightforward to see that an AKSZ sigma model corresponds to a parent theory for which all fields with \( \lambda \) indices related to \( y^\lambda \) derivatives vanish and \( \sigma^F \) is absent. Furthermore, \( \bar{s} \) is required not to involve the space-time coordinates explicitly. In other words, \( \bar{s}\Psi^a \) is restricted to be a function of \( \Psi^b \) alone.

Whereas one can freely change coordinates \( x^\mu \) of the base space of an AKSZ sigma model without affecting the differential provided the \( \theta \) prolongations of the fields \( \Psi^a \) transform tensorially, this is no longer true for a generic parent differential due to the \( \sigma^F \) term. Note however that a parent differential, for which this term is absent and there is no explicit \( x^\mu \) dependence, is of AKSZ-type if one takes the indices \( a \) to be given by the collection \( A(\lambda) \). As we will show below, this is the case for diffeomorphism invariant theories, after a suitable field redefinition. It thus follows that changes of coordinates in the base space together with the associated tensorial transformation laws for the fields, do not affect this type of parent differentials either.

### 2.5 Linear theories and first quantized description

Let us illustrate the construction for linear theories and connect to the formulation given in [13]. As a first step, one introduces an auxiliary superspace \( \mathcal{H} \) whose basis elements \( e_A \) are associated to the fields \( \Psi^A \) and defines \( gh(e_A) = -gh(\Psi^A) \), \( |e_A| = -|\Psi^A| \). The space of \( \mathcal{H} \)-valued space-time functions can be then regarded as the space of states of a BRST first-quantized system. Indeed, using the string field \( \Psi(x) = \Psi^A(x)e_A \), one can define the first quantized BRST operator acting from the right according to

\[
    s\Psi = \Psi\Omega, \quad (\phi^Ae_A)\Omega = \phi^A(x)e_B^B(\frac{\partial}{\partial x}, x)e_B.
\]

The nilpotency of \( s \) then implies the nilpotency of the operator \( \Omega \) and vice-versa. In addition \( \Omega \) carries a unit ghost number and hence determines a first quantized BRST system. The starting point field theory then appears as the gauge field theory associated to this first-quantized system [54].

In first quantized terms, the parent theory is obtained by extending \( \mathcal{H} \) to \( \mathcal{H}^T \) through tensoring with the Grassmann algebra generated by variables \( \theta^\mu \) and formal power series in \( y^\mu \). On the \( \mathcal{H}^T \)-valued functions, the BRST operator that gives rise to the parent differential \( s^P \) through (2.21) is defined by

\[
    \Omega^T = d - \sigma + \bar{\Omega},
\]

where \( \bar{\Omega} \) denotes the starting point BRST operator \( \Omega \) extended to act on \( \mathcal{H}^T \) and with \( \frac{\partial}{\partial x^\mu} \) replaced with \( \frac{\partial}{\partial y^\mu} \) and \( x^\mu \) with \( x^\mu + y^\mu \).
2.6 Reductions

The usefulness of the parent theory has to do with the possibility to arrive at other equivalent formulations just by eliminating one or another set of generalized auxiliary fields. A practical way to identify such fields is obtained by relating them to a part of the parent differential \( s^P \) using standard homological techniques.

2.6.1 Generalized auxiliary fields and algebraically trivial pairs

Let us briefly recall the notion of generalized auxiliary fields at the level of equations of motion. Suppose that, after an invertible change of coordinates possibly involving derivatives, the set of fields \( z^\alpha \) splits into \( \varphi^i, w^a, v^a \) such that the equations \( s w^a |_{w^a=0} = 0 \), understood as algebraic equations in the space of fields and their derivatives, are equivalent to \( v^a = V^a[\varphi] \) in the sense that they can be algebraically solved for fields \( v^a \). Fields \( w, v \) are then called generalized auxiliary fields. In the Lagrangian framework, fields \( w, v \) are in addition required to be second-class constraints in the antibracket sense. In this context, generalized auxiliary fields were originally proposed in [55]. Generalized auxiliary fields comprise both standard auxiliary fields and pure gauge degrees of freedom as well as their associated ghosts and antifields.

As explained in section 3.2 of [13], there is a reduced differential associated to the surface defined by the equations

\[
  w^a = 0, \quad v^a - V^a[\varphi] = 0. \tag{2.23}
\]

This reduced differential is defined on the space of fields \( \varphi^i \) and their derivatives through

\[
  s_R \varphi^i = s \varphi^i |_{w^a=0, v^a=V^a[\varphi]}.
\]

By following the reasoning in the proof of proposition 3.1 of [13], it can then be shown that there exists an invertible change of fields from \( z^\alpha \) to \( w^a, s w^a, \varphi^i_R \) such that \( s \varphi^i_R = S^i_R[\varphi_R] \). If in addition this change of variables is local, the fields \( w^a, s w^a \) are called algebraically trivial pairs. Conversely, it follows directly from the form of the differential in these variables that \( w^a \) and \( s w^a \) are generalized auxiliary fields. Finally, it can easily be shown that \( \varphi^i_R \) and \( \varphi^i \) differ only by terms that vanish when \( w^a, s w^a \) vanish and then that \( s_R \varphi^i_R \) and \( s_R \varphi^i \) agree when \( w^a \) and \( s w^a \) vanish. In other words, in this case the reduced theories are identical and the concepts of algebraically trivial pairs and generalized auxiliary fields are the same.

If we restrict to local generalized auxiliary fields, the BRST cohomology, both the standard and the one modulo the horizontal differential \( d_H \), of \( s \) involving all the fields \( z^\alpha \) is isomorphic to the one involving the fields \( \varphi^i_R \) alone, or, what is the same, to the one of \( s_R \) involving \( \varphi^i \) alone. Since these cohomology classes contain relevant physical information (see e.g. [45] for details and references), it is natural to consider as equivalent the original and the reduced theories.
In many cases the new variables $\varphi^i_R$ are indeed local functions. This includes for instance all linear systems but it does not need to be so in general. Indeed, the $\varphi^i_R$ are constructed as power series in the variables $w^a, v^a$ which do not necessarily terminate or sum up to local functions. The elimination of generalized auxiliary fields is then not a strictly local procedure and can in principle affect the local BRST cohomology groups. Except for the case of parametrized theories discussed in Section 2.8, we only consider local generalized auxiliary fields. The generalized auxiliary fields relating the parametrized and non-parametrized formulations of the same theory are manifestly nonlocal. This is so because through parametrization, one reformulates the theory in terms of constants of motion which are by construction nonlocal expressions of the original variables (see e.g. [28]).

2.6.2 Degrees

By using appropriate degrees, generalized auxiliary fields can be identified by focusing on only a part of the BRST differential.

A technical assumption satisfied in all models of interest is that the original BRST differential and thus also the parent differential $s_P$ does not contain constant terms, or in other words, that there is no term of degree $-1$ in an expansion in terms of homogeneity in the fields.

Let us more generally assume that the space of fields carries a suitable degree such that the degree of the independent fields is bounded from above and that the decomposition of the BRST differential has a lowest degree $s_p$, which we take for definiteness to be $p = -1$,

$$s = s_{-1} + s_0 + \ldots \quad \deg s_k = k, \tag{2.24}$$

where $s_{-1}$ commutes with the total derivative $\partial_\mu$. Note that the considerations below remain true for different values of $p$. We then have:

**Proposition 2.1.** Algebraically trivial pairs for $s_{-1}$ are generalized auxiliary fields of the theory determined by $s$.

**Proof.** Indeed, by assumption there is a coordinate system $(w^a, v^a, \varphi^i_R)$, such that $s_{-1}w^a = v^a$ and $s_{-1}\varphi^i_R = S^i_R[\varphi_R]$. It follows that

$$sw^a = v^a + \sum_{k=0} s_k w^a. \tag{2.25}$$

Equations $sw^a|_{w^a=0} = 0$ can be uniquely solved with respect to $v^a$. To see this, suppose that $v^{a_m}$ are the variable(s) having the maximal degree $m$ among $v^a$-variables. Variables $w^{a_m}$ have degree $m + 1$ while the terms $\sum_{k=0} s_k w^{a_m}$ have degree $m + 1$ or higher and hence the linear part of these terms cannot involve $v$-variables. Repeating the argument...
for degree $m - 1$ and lower shows that $su^a|_{w^a=0}=0$ can be solved to linear order. In the space of formal series in variables $w, v$ the equation $su^a|_{w^a=0}=0$ can be then uniquely solved with respect to $v^a$ order by order.

The theory determined by $s$ can thus be reduced to the one involving the fields $\varphi^i_R$ alone by eliminating generalized auxiliary fields. Suppose now that $s_{-1}$ takes the form $s_{-1} = v^a \frac{\partial}{\partial w^a}$. If in addition the cohomology of $s_{-1}$ is concentrated in degree 0, the BRST differential of the reduced theory has the particularly simple form

$$s_R \varphi^i_R = (s \varphi^i_R)|_{v=0} = (s_0 \varphi^i_R)|_{v=0}. \quad (2.26)$$

Indeed, in this case, the equation $su^a|_{w^a=0}=0$ is solved by $v^a = 0$. This is so because its solution, which has the form $v^a = V^a[\varphi_R]$ for some local functions $V^a$, implies that those $v^a$ that have nonvanishing degree vanish as the degree of all the $\varphi^i_R$ is zero. Let now $v^{a_0}$ denote those $v^a$ that have vanishing degree. Equation $su^{a_0}|_{w=0}=0$ gives $-v^{a_0} = (s_0u^{a_0} + s_1u^{a_0} + \ldots)|_{w^a=0}$. But all the terms on the right hand side have positive degree and hence cannot be local functions of $\varphi^i_R$ unless they vanish.

Note that it can be useful not to identify and eliminate the variables $v^{a_0}$ with vanishing degree explicitly and to keep them in the reduced theory. Indeed, $s_0 v^{a_0}$ can only depend on variables $\varphi^i_R$ and $v^{a_0}$. By eliminating all variables $w, v$ but $v^{a_0}$, the reduced theory is determined by the BRST differential $s_0$ and the constraints $v^{a_0} = 0$.

### 2.6.3 Target space reductions

For the case of the parent theory, consider the negative of the target-space ghost number, i.e., the prolongation to the parent theory of the original ghost number that does not take into account the number of $\nu$ indices. It follows that $s^P = s^P_{-1} + s^P_0$, where $s^P_{-1} = \bar{s}$ and $s^P_0 = dF - \sigma F$. Using this degree which is bounded from below, it follows in particular that:

**Proposition 2.2.** Algebraically trivial pairs for the original BRST differential $s$ give rise in the parent formulation to a family of generalized auxiliary fields comprising all their descendants obtained through $\frac{\partial F}{\partial \theta^\nu}$ and $\frac{\partial F}{\partial y^\lambda}$ derivatives.

In this case, the reduction of the fields $\Psi^A_{(\mu)(\lambda)[\nu]}$ thus involves only the $A$ indices and the reduced theory is simply the parent extension of the reduction $s_R$ for the original BRST differential $s$.

Furthermore, consider trivial pairs for the BRST differential $s$ that are not necessarily algebraic. In other words, the separation of the jet-coordinates into trivial pairs and the remaining coordinates does not respect the differential structure encoded in $\partial_\mu$. In terms of the parent theory, the same trivial pairs for $\bar{s}$ in terms of $y^\lambda$ derivatives are now algebraic as they do not involve the $x^\mu$ derivatives at all. Of course, the separation now does not
respect the differential structure with respect to the $y^\lambda$ derivatives, but still does for the $\theta^\nu$ derivatives. We thus also have:

**Proposition 2.3.** Trivial pairs for $s$ that are not necessarily algebraic give rise to a family of generalized auxiliary fields of the parent theory which comprises all their descendants obtained through $\frac{\partial F}{\partial \theta^\nu}$-derivatives.

### 2.6.4 Going on-shell

Consider now a gauge theory described by an antifield dependent BRST differential that is expanded according to the antifield number,

$$s = \delta + \gamma + s_1 + \ldots,$$

(2.27)

where $\delta$ denotes the Koszul-Tate differential [56] which satisfies standard regularity assumptions (see [57] for the case of local field theories). In this case, $\delta = v^a \frac{\partial}{\partial w^a}$ but the point is that the trivial pairs for $\delta$ are not algebraic. The contracting homotopy $\rho = w^a \frac{\partial}{\partial v^a}$ does not commute with the total derivative $\partial^\mu$ [58]. If one concentrates on the BRST cohomology in the space of local functions or horizontal forms, this is not an issue and the cohomology of $\delta$ is indeed concentrated in degree 0. Not eliminating $v^{a_0}$ explicitly corresponds to the case of considering the weak cohomology of $\gamma$, i.e., the cohomology of $\gamma$ modulo the relations imposed by the equations of motions. Note however that in the case of local functionals there generically is non-trivial cohomology of $\delta$ in strictly negative degree because the trivial pairs are not algebraic, see e.g. [45, 59] for the relation between the cohomology of $s$ and $\gamma$ in this case.

For the parent theory, using the degree given by minus the target-space ghost number in a first stage, and then the extension of the antifield number to the parent theory in a second stage, the piece $\bar{\delta}$ of $\bar{s}$ that corresponds to the prolongation of $\delta$ is in lowest degree. Moreover the degree of fields is bounded from above. In terms of a local field theory in $x$-space, the elimination of the trivial pairs for its cohomology is now algebraic, but of course does not respect the $\lambda$ indices in the sense that it does not commute with $\frac{\partial F}{\partial y^\lambda}$ in the case of non trivial equations.

Since the cohomology of $\bar{\delta}$ is concentrated in antifield number zero, neither the $\mu$ nor the $\nu$ indices are affected, and the reduced theory has a differential whose part involving space-time derivatives $\frac{\partial F}{\partial x^\mu}$ is still $d^F$ alone, the remainder of the differential is of antifield number 0 and no fields of antifield number different from zero remain. The theory is reduced to the prolongation of the stationary surface and the variables $v^{a_0}$ are precisely the lhs of the equations of motion expressed in terms of $y^\lambda$ derivatives. Keeping them in the formulation allows one not to choose explicitly independent coordinates on the stationary surface. Moreover, with these variables kept, the reduced differential becomes simply $d^F - \sigma^F + \bar{\gamma}$. 
2.6.5 Equivalence of parent and original theory

The main statement that justifies the introduction of the parent theory is:

**Proposition 2.4.** The parent theory determined by $s^P$ can be reduced to the starting point theory through the elimination of the generalized auxiliary fields.

**Proof.** The proof very closely follows the one for linear systems in [13]. Let $N_{\partial x}$ be the operator that counts the number of $x^\mu$ derivatives on the fields in the original theory,

$$N_{\partial x} = \sum_{k=0}^k k\partial_{x^\mu_1...x^\mu_k} \frac{\partial}{\partial \Psi^{A\mu_1...\mu_k}}.$$  

(2.28)

By assumption, the original differential $s$ is local in the sense that $s$ involves a finite total number of derivatives, or in other words, the decomposition of $s$ into homogeneous components of $-N_{\partial x}$ is bounded from below, say by $-T$. It then follows that the same is true for $\bar{s}$ in the parent theory in terms of $y^\lambda$ derivatives on the fields counted by the operator

$$N_{\partial y} = \sum_{k=0}^k k\partial_{y^\lambda_1...y^\lambda_k} \frac{\partial}{\partial \Psi^{A(\mu)(\nu)}}.$$  

(2.29)

The grading is then chosen as $-N_{\partial y} + T\times$ (target-space ghost number). It follows that the lowest part of $s^P$ is $-\sigma^F$ in degree $-1$, while $d^F$ is in degree 0 and $\bar{s}$ contains terms that are of degree greater or equal to zero. For the lowest part, all additional fields of the parent formulation form algebraically trivial pairs. Indeed, if $\rho = \partial_{y^\mu} y^\mu$ and $N_{y,\theta} = \partial_{y^\mu} y^\mu + \partial_{\theta^\mu} \theta^\mu$, we have $[\sigma, \rho] = N_{y,\theta}$ which implies the corresponding relation for the prolongation of these operators acting on the space of fields due to (2.11). The result then follows by using the standard homotopy formula.

According to subsection 2.6.2, the additional fields of the parent theory are thus generalized auxiliary fields. We still have to show that the reduced differential coincides with the starting point differential $s$. In order to do so, the fields $\Psi_A^{(\mu)(\nu)} (x)$ are split as follows: $\psi^i (x) = \Psi_A^{(\mu)(\nu)} (x) \equiv \Psi_A(x)$, the fields $w^a(x)$ which form a basis of the image of $\rho^F$ acting on the space of fields, and the fields $v^a = \sigma^F w^a$ which by construction form a basis of the image of $\sigma^F$. The fields $w^a(x), v^a(x)$ can be expressed in terms of suitable Young tableaux involving the $\lambda$ and $\nu$ indices, but explicit expressions are not needed for the proof.

To compute the reduced differential, we have in a first step to solve the equations

$$s^P w^a \big|_{w=0} = 0$$  

(2.30)

with respect to $v^a$. Consider the degree which counts the number of skew-symmetric $\nu$ indices,

$$N_{\partial y} = \sum_{l=0}^l l\Psi_A^{(\mu)(\nu_1...\nu_l)} \frac{\partial}{\partial \Psi^{A(\mu)(\nu_1...\nu_l)}}.$$  

(2.31)
and split the $w^a, v^a$ according to their degree. In particular, $\sigma^F$ and $d^F$ lower the degree by 1, $\rho^F$ raises it by 1, while $\bar{s}$ is of degree 0, as can be seen from (2.12). It also follows that the $v^a$’s have non-negative degree, while the $w^a$’s have strictly positive degree. When acting on $w^a$’s of lowest degree 1, $\Psi^A_{(\rho^F)\lambda_1}, \Psi^A_{(\lambda_1|\rho^F)}$, …, (2.30) gives the sequence of equations
\[
(\partial_{\lambda_1} \Psi^A_{\lambda_1|\cdot} + (-1)^{|A|} s \Psi^A_{\lambda_1})\big|_{w=0} = 0, \\
(\partial_{\lambda_1} \Psi^A_{\lambda_2|\cdot} + (-1)^{|A|} s \Psi^A_{\lambda_1})\big|_{w=0} = 0, \\
\ldots
\]
Since $\bar{s}$ is of degree 0, in each of these equations, the last term on the left hand side is necessarily proportional to $v^a$’s of degree 1 which implies that the above equations can successively be solved for the $v^a$ of degree 0 as $\Psi^A_{\lambda_1\ldots\lambda_k} = \partial_{\lambda_1\ldots\lambda_k} \Psi^A + O(1)$, where $O(k)$ denotes terms that are proportional to $v^a$’s of degree $k$.

Let us split the variables $v^a, w^a$ with respect to both degrees $\mathcal{N}_{\theta_0}$ and $\mathcal{N}_{\bar{\theta}_0}$ such that $\mathcal{N}_{\theta_0} v^a_{k,l} = kv^a_{k,l}$ and $\mathcal{N}_{\bar{\theta}_0} v^a_{k,l} = lv^a_{k,l}$ and analogously for $w$. Note that there are neither $w^a_{k,0}$ nor $v^a_{0,l}$, $v^a_{k,n}$, where $n$ is the space-time dimension. Working in the space of polynomials in $v^a_{k,l}$ with $l > 0$, let us consider the equations
\[
((d^F - \sigma^F + \bar{s}) w^a_{k,m})\big|_{w=0} = 0.
\]
For $m = n$, at linear order in $v^a_{k,l}$, the last term necessarily involves $v^a_{r,n}$ with $r \geq k$ and hence vanishes. The equation then expresses $v^a_{k+1,n-1}$ through the derivatives of $v^a_{k,n-1}$. Because there are no $v^a_{0,l}$ induction in $k$ shows that $v^a_{k,n-1}$ all vanish at linear order. Repeating the argument for $m = n - 1$ and so on shows that all $v^a_{k,l}$ with $l > 0$ vanish at the linear order. Higher order corrections are necessarily proportional to $v^a_{k,l}$ with $l > 0$ this remains true to all orders. This shows that $\Psi^A_{\lambda_1\ldots\lambda_k} = \partial_{\lambda_1\ldots\lambda_k} \Psi^A$.

In the second and last step, we compute the reduced differential,
\[
s_R \Psi^A = (s^F \Psi^A)|_{\sigma^a=0,\nu^a=v^a[\Psi^A]} = (s \bar{s} \Psi^A)|_{\nu^a=0,\nu^a=v^a[\Psi^A]},
\]
which reduces to the original differential $s$ because $\bar{s} \Psi^A$ is by definition $s \Psi^A$ where the $x^a$ derivatives of $\Psi^A$ are replaced with the corresponding $y^A$ derivatives, but the latter are precisely the $v^a$’s of degree 0.

**Remark 1:** Instead of assuming polynomials in $v^a_{k,l}$, i.e., in some of the fields that carry $\nu$ indices, one can repeat the proof assuming polynomials in fields with nonvanishing ghost degree. This assumption can be more natural from the point of view of BRST theory and would allow for nonpolynomial expressions in form-fields.

**Remark 2:** From the above proof, it appears that it is possible to eliminate only a part of the contractible pairs $w, v$ for $\sigma^F$. Namely, one can eliminate only $w^a_{k,l}$ and $v^a_{k,l}$ with $k + l \geq M$ for some $M$. Of course in this case, some $v^a_{k,l}$ with $l > 0$ do not vanish anymore but are expressed through derivatives of the remaining fields. For $M = 0$, one recovers
the original theory while for $M$ sufficiently large, lower order equations are unaffected and remain first order. By such a consistent truncation, one can arrive at a first order formulation with a finite number of fields. This is just the parent theory counterpart of the usual truncation of the infinite jet space to a finite one in the case of equations involving a finite number of derivatives.

2.6.6 Contractible pairs for $\tilde{s}$

Consider now as a starting point the extended BRST differential $\tilde{s} = \theta^\mu \partial_\mu + s$ that acts in the space of local functions with an explicit dependence in $\theta^\mu$. When constructing the associated parent differential, one first has to replace the $x$ derivatives of the fields in $\tilde{s}$ by $y$ derivatives. This gives a differential $\tilde{\tilde{s}} = \theta^\mu (\partial_{x^\mu} + \frac{\partial F}{\partial y^\mu}) + \bar{s}$ acting on the space of $y$-derivatives $\Psi^A_{(\lambda)[\nu]}$ of $\Psi^A$.

Treating $x, \theta$ as independent variables and the $y$-derivatives of $\Psi^A$ as dependent variables, the prolongation of $\tilde{\tilde{s}}\Psi^A$ to the entire jet space (i.e., to the $x$ and $\theta$ derivatives of $\Psi^A_{(\lambda)[\nu]}$) is obtained by using the total $x$ and $\theta$-derivatives, $\partial_\mu = \partial_{\theta^\mu} + \partial_F^{\theta^\mu}$ (see e.g. [47] for details on prolongations):

$$ (\tilde{s})^P = \theta^\mu (\partial_{x^\mu} + \frac{\partial F}{\partial y^\mu}) + dF - \sigma^F + \bar{s}. \quad (2.35) $$

It follows that the standard parent differential for $s$ is related to $\tilde{s}$ through

$$ s^P = (\tilde{s})^P \big|_{\theta=0}. \quad (2.36) $$

In other words, the $dF - \sigma^F$ term of the parent differential is automatically generated from the parent prolongation of the term $\theta^\mu \partial_\mu$ in $\tilde{s}$. This property can be used as follows:

**Proposition 2.5.** Trivial pairs for $\tilde{s}$ give rise to a family of generalized auxiliary fields comprising all descendants obtained through total $\theta$ derivatives at $\theta = 0$.

**Proof.** By assumption, in the original theory there are new independent variables $w = w(x, \theta, \Psi), v = v(x, \theta, \Psi)$ such that $\tilde{s}w = v$. In the expression for $w, v$ we replace $x$-derivatives by $y$-derivatives so that $(\tilde{s})^P w = v$. It then follows from (2.36) that $s^P w|_{\theta=0} = v|_{\theta=0}$. Let $w_{\nu_1...\nu_k} = (\partial^\theta_{\nu_1} \ldots \partial^\theta_{\nu_k} w)|_{\theta=0}$ and $v_{\nu_1...\nu_k} = (-1)^k (\partial^\theta_{\nu_1} \ldots \partial^\theta_{\nu_k} v)|_{\theta=0}$. Using $[\partial^\theta_\mu, s^P] = \frac{\partial}{\partial x^\mu} + \frac{\partial F}{\partial x^\nu}$ one finds

$$ (dF - \sigma^F + \bar{s}) w_\nu = v_\nu + \left( \frac{\partial}{\partial x^\nu} + \frac{\partial F}{\partial x^\nu} \right) w, \quad (2.37) $$

\textsuperscript{1}Strictly speaking the prolongation should be done using $\frac{\partial}{\partial \theta^\mu} - \frac{\partial F}{\partial \theta^\mu}$ as a total derivative. We use here the prolongation modified by a change of signs for the $\theta$-derivatives in order to fit the convention for the parent differential used in the rest of the paper. Alternatively, consistent signs can be achieved by starting with $\bar{s} = -dH + s$ or by exchanging the sign of the $dF - \sigma^F$ term in the parent differential.
and similar formulas for higher $w_{\nu_1...\nu_k}$ and $v_{\nu_1...\nu_k}$. Using as a degree $N \partial \theta$, one observes that the equations $(dF - \sigma F + \bar{s})w_{\nu_1...\nu_k} = 0$ can be algebraically solved for $v_{\nu_1...\nu_k}$, so that $w_{\nu_1...\nu_k}$ and $v_{\nu_1...\nu_k}$ are indeed generalized auxiliary fields for the parent theory. Note in particular that, for $x$-independent $w$’s, equation (2.37) implies that the $w_{\nu_1...\nu_k}$ and $v_{\nu_1...\nu_k}$ are simply contractible pairs for $-\sigma F + \bar{s}$.

### 2.7 Diffeomorphism invariant theories

Suppose that the starting point theory is diffeomorphism invariant and that diffeomorphisms are among the generating set of gauge transformations. By this we mean that there is no explicit $x^\mu$ dependence in the starting point BRST differential, and thus also none in the parent differential. Furthermore, the starting point theory has diffeomorphism ghost fields $\xi^\mu$ (replacing the vector fields parametrizing infinitesimal diffeomorphisms) among the fields $\Psi^A$ and the part of $s\Psi^A$ that involves the undifferentiated $\xi^\mu$ is given by $s'\Psi^A = \xi^\mu \partial_\mu \Psi^A$ for all $\Psi^A$. When suitably prolonged to all derivatives of the fields, this means that $s = s' + s''$ where $s''$ does not depend on the undifferentiated $\xi^\mu$.

At the level of the parent theory, this implies in particular that $\bar{s}$ contains $\xi^\lambda \partial^F \partial y^\lambda$ as the only piece which depends on the undifferentiated diffeomorphism ghosts $\xi^\lambda$. From the prolongation formulas (2.12), it also follows that, when acting on $\Psi^A_{(\mu)\lambda_1...\lambda_l|\nu_1...\nu_k}$, the piece originating from $s'$ and containing no derivatives of the diffeomorphism ghosts but one of type $\theta^\nu$ is given by $(-1)^A k \xi^\lambda_{(0)0|\nu_1} \Psi^A_{(\mu)\lambda_1...\lambda_l|\nu_2...\nu_k}$. Note also that this is the only term in the parent differential that depends on $\xi^\lambda_{(0)0|\nu_1}$.

The piece $-\sigma F$ in the parent differential $s^P$ can then be absorbed through the field redefinition

$$\xi^\lambda_{(0)0|\nu} \rightarrow \xi^\lambda_{(0)0|\nu} + \delta^\lambda_{\nu}.$$  

Since all other terms of the parent differential are unaffected, the parent differential in terms of the new fields takes the form

$$s^P = dF + \bar{s},$$

where $\bar{s} = \bar{s}' + \bar{s}''$ is precisely the prolongation of the original BRST differential. If one regroups the $\lambda$ indices corresponding to the $y^\lambda$ derivatives together with the $A$ indices and considers the $\Psi^A_{(\lambda)}$ as coordinates of a $Q$-manifold, we have:

**Proposition 2.6.** The parent formulation of a diffeomorphism invariant theory is of AKSZ-type with an infinite-dimensional target space that contains all derivatives of the original fields and a $Q$-structure that coincides with the starting point BRST differential.
2.8 Parametrized theories

As we have seen, the parent formulation is simpler if the starting point theory is diffeomorphism invariant. Of course, any theory can be made diffeomorphism invariant through parametrization. This means that the independent variables, the coordinates of space-time, become fields on the same level as the other fields, while new arbitrary parameters are introduced instead of the original independent variables. In this section, we analyze the parent formulation for parametrized theories.

One way to construct the parametrized parent formulation is to first make the theory diffeomorphism invariant and then to construct its parent formulation following the general procedure explained in the previous sections. Another possibility is to parametrize directly in the parent formulation by adding extra fields and gauge symmetries. It turns out to be more economical and instructive to directly build the parametrized parent formulation from scratch.

Suppose that the original gauge theory involves a space-time with coordinates $y^a$, fields $\Psi^A$, and a BRST differential $s$ defined by $s\Psi^A = s^A[\Psi, y]$ and $[\partial_a, s] = 0$, where $\partial_a$ denotes total derivative with respect to $y^a$. As before, we introduce Grassmann odd variables $\xi^a$, $gh(\xi^a) = 1$ standing for $dy^a$ so that horizontal forms become functions of $\Psi^A(y), y^a, \xi^a$. The space of horizontal forms is equipped with the total BRST differential $\tilde{s} = s + \xi^a \partial_a$. Note that we have changed notations with respect to the considerations in 2.1 because we reserve $x^\mu, \theta^\mu$ to denote the space-time coordinates and their differentials after parametrization.

Let us then consider the AKSZ-type sigma model with target space the extended jet space with coordinates $\Psi^A(y), y^a, \xi^a$ equipped with the differential $\tilde{s}$ and source space the extended space-time manifold with coordinates $x^\mu, \theta^\mu$. We call the resulting theory the parametrized parent formulation.

**Proposition 2.7.** The parent formulation as defined in Section 2.3 can be obtained through the elimination of the following generalized auxiliary fields from the parametrized parent formulation:

$$y^a - Y^a(x), \ y^a_{\nu_1...\nu_k}, k > 0 \quad \xi^a, \ \xi^a_{\nu} + \frac{\partial Y^a}{\partial x^\nu}, \ \xi^a_{\nu_1...\nu_k}, k > 1.$$ (2.40)

Here $Y^a(x)$ define an invertible change of space-time coordinates. To obtain both formulations in the same coordinates, one takes $Y^a(x) = \delta^a_\mu x^\mu$.

**Proof.** It is straightforward to check that fields (2.40) can be eliminated by imposing the following constraints

$$\frac{\partial F}{\partial \theta_{\nu_1}} \ldots \frac{\partial F}{\partial \theta_{\nu_k}} (y^a - Y^a(x)) = 0, \quad (dF + \tilde{s}) \frac{\partial F}{\partial \theta_{\nu_1}} \ldots \frac{\partial F}{\partial \theta_{\nu_k}} (y^a - Y^a(x)) = 0.$$ (2.41)
so that they are indeed generalized auxiliary fields. After the reduction, the terms in the reduced differential originating from $\xi^a \frac{\partial}{\partial y^a}$ in $\bar{s}$ give rise to precisely $-\sigma F$ if one in addition takes $Y^a = \delta^a_\mu x^\mu$. Finally, the terms $dF$ and $\bar{s}$ remain intact.

It is important to stress that in contrast to other reductions considered in this paper, the elimination of variables (2.40) is not a strictly local operation. In addition to the explicit space-time dependence of the gauge condition $y^a = Y^a(x)$, the elimination breaks locality in the sense described in Section 2.6. Namely, in the space of local functions, it is impossible to decouple variables (2.40) and the remaining variables $\Psi^A$ and their $\theta$, $y$-descendants. Indeed, looking for a completion $\tilde{\Psi}^A$ such that $(dF + \bar{s})\tilde{\Psi}^A$ is a function of $\tilde{\Psi}^A$ and their descendants, one finds that $\tilde{\Psi}^A$ necessarily involves derivatives of arbitrarily high order (see [32] for an algebraically similar example in the context of local BRST cohomology) and hence such $\tilde{\Psi}^A$ do not exist in the space of local functions.

In spite of this nonlocality, the local BRST cohomology of the parametrized parent formulation is isomorphic to that of the starting point theory. Indeed, the local BRST cohomology of the AKSZ-type sigma model with the target space differential being $\bar{s}$ is isomorphic to $\bar{s}$-cohomology of the target space local functions [44] and hence coincides with that of the starting point theory.

The parametrized parent formulation can be also used as a shortcut to parent formulation for diffeomorphism invariant theories. If the starting point theory is diffeomorphism invariant $\bar{s}$ can be brought to the form $\xi^a \frac{\partial}{\partial y^a} + s$ by redefining the diffeomorphism ghosts by $\xi^a$ (see e.g. [58, 30]). In this case $\xi^a$ and $y^a$ are algebraically trivial pairs and can be eliminated so that the parametrized parent formulation reduces to that of Proposition (2.6).

To complete the discussion of parametrization and to make contact with the literature, let us show how the parametrized parent formulation can be seen as a systematic way to obtain a manifestly diffeomorphism invariant form for theories invariant under some space-time symmetries. Without trying to be exhaustive, let us for simplicity assume that the starting point theory is translation invariant so that the BRST differential is $y^a$ independent for a suitable choice of space-time coordinates. One can then consistently drop the $y^a$-fields in the parametrized parent formulation as these variable are completely decoupled from the rest. In this case, the reduction to the usual parent formulation can be seen as imposing the gauge condition $\xi^a = 0$, $\xi^a = -\frac{\partial Y^a}{\partial x^\nu}$, $\xi^a \ldots \xi^a = 0$ so that the theory itself and its reduction to the usual description can be defined without any reference to fields originating from $y$. The rôle of the $\xi^a$ variables can also be given another interpretation: for a translation invariant theory the starting point $\bar{s}$ can be considered as acting on the truncated jet space that does not involve the $y^a$-variables. Variables $\xi^a$ can then be interpreted as constant ghosts that take the translation symmetry into account in the BRST differential.

This has a straightforward generalization to the case where translations are part of a
larger global space-time symmetry algebra such as the Poincaré, AdS or conformal algebras for instance and results in the formulation where this symmetry algebra is realized in a manifest way. Formulations of this type are extensively used in the context of the unfolded approach (see. [8, 9] and references therein) and were also used in [14, 60, 16] in the context of parent-like formulations.

2.9 Local BRST cohomology

It is instructive to see how the BRST and the local BRST cohomology, which are by construction isomorphic to the ones of the original theory, appear in the parent formulation. Let us begin with the cohomology in the space of local functions. In the case of the parent formulation it is natural to consider functions that are local in the sense that they depend on both $x$ and $y$-derivatives of the fields only up to some finite order. The isomorphism of BRST cohomologies in the space of local function can be seen as follows: take as a degree $N_\theta + N_y$. The lowest order terms in $s^p = d^F$. Its cohomology is given by local functions that do not depend on both $x$ and $\theta$ derivatives of fields. The reduced differential is simply $\bar{s}$ restricted to act on the space of local functions in $x$ and $\Theta_A(x)$. Exchanging the role of $x$ and $y$-derivatives this complex can be identified with the starting point BRST complex.

As briefly explained at the end of section 2.1 in order to compute the local BRST cohomology, one has to compute the cohomology of $\tilde{s}^p = s^p + d_H$ in the space of horizontal forms. When identifying $\theta^\nu \equiv dx^\nu$, this simply amounts to including an explicit $\theta^\nu$ dependence in the space of local functions. To explicitly verify the isomorphism, let us again take as a degree $N_\theta + N_y$ so that the lowest term in $\tilde{s}^p$ is again $d^F$. Identifying its cohomology with functions in $\Psi_A^{(\nu\lambda)}(x,\theta)$ and repeating the steps of the proof of equivalence in section 2.6.5 with the rôle of $x^\mu$ and $y^\lambda$ derivatives exchanged, one finds that the term $\theta^\mu \frac{\partial^F}{\partial x^\mu}$ entering $d_H$ acts in the cohomology as $\theta^\mu \frac{\partial^F}{\partial y^\mu}$, $\sigma^F$ acts trivially, while the action of $\theta^\mu \frac{\partial}{\partial x^\mu}$ is unchanged. Finally the reduced differential is just $\tilde{s}$ with the role of $x$ and $y$-derivatives exchanged. In order to make sure that this indeed gives an isomorphism of cohomologies, let us note that a complete coordinate system can be chosen to contain besides the trivial pairs for $d^F$ and $\theta^\mu$, $x^\mu$, the coordinates

$$\tilde{\Psi}_A^{(\nu\lambda)} = \sum_{l=0}^{1} \frac{1}{l!} \Psi_A^{(\nu\lambda)_{\mu_1...\mu_l}} \theta^\mu_1 ... \theta^\mu_l,$$

which are local functions satisfying $\tilde{s}^p \tilde{\Psi}_A = (\bar{s} + \theta^\mu \frac{\partial^F}{\partial y^\mu}) \tilde{\Psi}_A$ and $\tilde{s}^p x^\mu = \theta^\mu$. In this way one confirms that the reduced differential is indeed $\bar{s}$ with the role of $x$ and $y$ derivatives of the fields exchanged. In terms of representatives, the isomorphism sends functions in $x, \theta, \partial^{(\mu)} \Psi_A$ to the same functions with $\partial^{(\mu)} \Psi_A$ replaced by $\tilde{\Psi}_A^{(\mu)}$.

From the above argument, it follows that if one replaces $\theta^\mu \frac{\partial^F}{\partial x^\mu}$ with $\theta^\mu \frac{\partial^F}{\partial y^\mu}$ in the
expression for $\widetilde{s}^P$, the reduced differential obviously remains intact. Moreover, after this replacement, the extended differential coincides with $(\widetilde{s})^P$ from (2.35) and can be seen as the prolongation of $\widetilde{s}$ with the role of $x$ and $y$ derivatives exchanged, up to the sign conventions discussed in footnote [1].

It appears more natural to consider such a modified differential as the extended BRST differential associated to the parent theory because then the only term that involves $x$-derivatives of the fields is still $dF$.

In the context of the extended parent theory, there are now bona fide $\theta$ dependent combinations of variables and field redefinitions. For instance, when taking as a degree $-N_{\theta_0}$ which is bounded from above, the term $\widetilde{s}$ (with the role of $x$, $y$ derivatives exchanged) is in lowest degree. It follows that

**Proposition 2.8.** Algebraically trivial pairs for $\widetilde{s}$ give rise to a family of generalized auxiliary fields for the extended parent theory involving all descendants obtained through $\partial_\nu^F$ and $\partial_\lambda^F$ derivatives. Trivial pairs for $\widetilde{s}$ that are not necessarily algebraic give rise to a family of generalized auxiliary fields comprising all descendants obtained through $\partial_\nu^\theta$ derivatives.

For instance, for diffeomorphism invariant theories as discussed in subsection 2.7, but now considered in the context of the extended parent formulation, it is most useful to consider the $\theta$ dependent change of variables

$$\xi^\lambda \rightarrow \xi^\lambda - \theta^\lambda,$$  \hfill (2.43)

from the very beginning. Indeed, on the level of $\widetilde{s}$, it allows one to absorb the field dependent part of $d_H$ into the starting point BRST differential. It follows that no $\sigma^F$ appears in the prolongation. This is consistent with the fact that, on the level of the standard parent theory, the prolongation of (2.43) gives rise to the redefinition (2.38) needed to absorb $\sigma^F$. In terms of the new variables, the extended parent differential simply becomes $(\widetilde{s})^P = d^F + \bar{s} + \theta^\mu \frac{\partial}{\partial x^\mu}$. The only term that involves $x^\mu, \theta^\mu$ is the last one. As a consequence, these variables are trivial pairs that can be eliminated. The extended parent theory is then simply described by

$$(\widetilde{s})^P_R = d^F + \bar{s},$$  \hfill (2.44)

acting in the space of $x^\mu, \theta^\mu$ independent local functions.
3 Examples

3.1 Theory without gauge freedom

Suppose we have a theory without gauge freedom. Let $\phi^k$ denote the fields of the theory. In the BV description, there are in addition antifields $\phi^*_a$ and the BRST differential is determined by

$$s\phi^k = 0, \quad s\phi^*_a = L_a, \quad [s, \partial_\mu] = 0,$$

(3.1)

where $L_a[x, \phi^k_{(\mu)}] = 0$ and their prolongations $\partial_{(\mu)} L_a = 0$ are the original dynamical equations determining the so-called stationary surface in the space of fields and their $x$-derivatives.

In the parent theory, the only fields of ghost number zero are the $y$-derivatives of the original fields $\phi^k_{(\lambda)}$ as all antifields carry negative ghost number. The parent theory equations of motion are

$$(\partial_\mu - \frac{\partial F}{\partial y^\lambda}) \phi^k_{(\lambda)} = 0,$$

(3.2)

$$\frac{\partial F}{\partial y^\lambda_1} \cdots \frac{\partial F}{\partial y^\lambda_N} \bar{L}_a = 0,$$

(3.3)

where the equations in the second line determine the equivalent of the stationary surface in the space of $x^\mu$, the fields and their $y$-derivatives. Note that $\frac{\partial F}{\partial y^\mu}$ is a vector field on this space which does not affect $x^\mu$. Equations (3.3) are obviously preserved under the action of $\frac{\partial F}{\partial y^\mu}$ so that the vector field $\frac{\partial F}{\partial y^\mu}$ restricts to this stationary surface. We use $\sigma^\lambda$ to denote this restriction.

Let $x^\mu, Q^\alpha, v^i$ denote a new coordinate system replacing $x^\mu, \phi^k_{(\lambda)}$ such that $Q^\alpha$ can be used as coordinates on the stationary surface, while $v^i$ are complementary coordinates that replace the left hand side of the equations in (3.3). In the $Q^\alpha$ coordinate system one has

$$\sigma^\lambda = \sigma^\lambda_\alpha(Q) \frac{\partial}{\partial Q^\alpha}, \quad \sigma^\alpha_\lambda(Q) = \left[\frac{\partial F}{\partial y^\lambda} Q^\alpha\right]_{v^i=0}.$$

(3.4)

In terms of the new coordinates, Equations (3.3) simply put $v^i$ to zero, while Equation (3.2) take the form of a covariant constancy condition

$$\partial_\mu Q^\alpha - \sigma^\alpha_\mu(Q) = 0.$$

(3.5)

As a simple illustration, let us consider a scalar field on Minkowski space with a cubic interaction. We refer to [7] for a detailed discussion of the unfolded formulation for a scalar field (see also [9] for an off-shell description).

If $\Box^y \phi = \eta^{\lambda_1 \lambda_2} \phi_{\lambda_1 \lambda_2}$, constraints (3.3) are given by

$$\Box^y \phi + g\phi^2 = 0,$$

(3.6)
and its prolongations through \( y \)-derivatives. As coordinates \( Q^\alpha \) one can take the traceless parts \( \phi^T_{\lambda_1 \lambda_2} \) of \( \phi(\lambda) \) while the \( \frac{\partial F}{\partial y^\mu} (\Box \phi + g \phi^2) \) are the coordinates \( v^i \). In order to write down explicitly how \( \sigma_\lambda \) acts on some of the \( Q^\alpha \), we have to use for instance

\[
\phi_{\lambda_1 \lambda_2} = \phi^T_{\lambda_1 \lambda_2} + \frac{1}{n} \eta_{\lambda_1 \lambda_2} \Box^y \phi = \phi^T_{\lambda_1 \lambda_2} + \frac{1}{n} \eta_{\lambda_1 \lambda_2} \left[ (\Box^y \phi + g \phi^2) - g \phi^2 \right],
\]

where \( n \) is a space time dimension. One then finds

\[
\sigma_{\lambda} \phi^T = \left( \frac{\partial F}{\partial y^\lambda} \phi \right)_{\psi=0} = \phi^T_{\lambda}, \quad \sigma_{\lambda_1} \phi^T_{\lambda_2} = \phi^T_{\lambda_1 \lambda_2} - \frac{g}{d} \eta_{\lambda_1 \lambda_2} \phi^2, \quad \ldots,
\]

so that already for \( \phi^T_{\lambda} \), the coefficients of \( \sigma_\lambda \) become nonlinear.

### 3.2 Geodesic motion of point particle

Let us illustrate the construction on the example of a point particle moving along a geodesic in a (pseudo)-Riemannian space-(time). Since the model is diffeomorphism invariant in one dimension, its equations of motion can be brought into an AKSZ form according to our general discussion in section 2.7. In fact, this holds at the level of the master action as well. Indeed by going on-shell, we will show that the target space of the AKSZ parent theory can be reduced to the extended BFV phase space of the model on which the target space differential is induced by the BRST charge. Furthermore, it is known that the BV master action associated to canonical BFV gauge theories with vanishing Hamiltonians are of AKSZ form [36].

Using an auxiliary field \( \lambda \) playing the rôle of an einbein and the notation \( \partial = \frac{\partial}{\partial \tau} \), the action for geodesic motion of a point paritice is given by

\[
S[X, \lambda] = \frac{1}{2} \int d\tau \left[ \lambda^{-1} g_{\mu\nu}(X) \partial X^\mu \partial X^\nu + \lambda m^2 \right] = \int d\tau L.
\]

The gauge symmetry corresponding to infinitesimal reparametrizations of \( \tau \) acts as

\[
\delta X^\mu = \partial X^\mu \epsilon, \quad \delta \lambda = \partial (\lambda \epsilon),
\]

where \( \epsilon \) is the gauge parameter.

Promoting the gauge paramater \( \epsilon \) to a Grassmann odd ghost \( \xi \) and introducing the antifields \( X^*_\mu, \lambda^*, \xi^* \), the complete starting point BRST differential is given by

\[
s X^\mu = \xi^* \partial X^\mu, \quad s \lambda = \partial (\xi \lambda), \quad s \xi = \xi \partial \xi,
\]

\[
s X^*_\mu = \frac{\delta L}{\delta X^\mu} + \partial (\xi X^*), \quad s \lambda^* = \frac{\delta L}{\delta \lambda} + \xi \partial \lambda^*,
\]

\[
s \xi^* = \xi \partial \xi^* + x^*_\mu \partial X^\mu - \lambda \partial \lambda^* + 2\xi^* \partial \xi.
\]

Since the model is diffeomorphism invariant and the BRST transformation of each variable contains the time derivative of this variable, the considerations of the section 2.7 apply and the parent theory can be expressed in AKSZ form.
Let us recall the results of [30]. They state that the variables \( \{ \partial^q \lambda, \partial^q X^*_\mu, \partial^q \xi^*, q = 0, 1, \ldots \} \), where \( \partial \) denotes the \( \tau \) derivatives along with their \( s \) variations, which can be used to replace the variables \( \{ \partial^{q+2} X^\mu, \partial^{q+1} \xi, \partial^{q+1} \lambda^* \} \), form trivial pairs for the extended BRST differential \( \tilde{s} \). The remaining coordinates are chosen as

\[
\begin{align*}
\tau, \theta, & \quad X^\mu, \quad P^\mu = \lambda^{-1} \dot{X}^\mu - (\xi + \theta) g^{\mu\nu} X^*_\nu, \\
\eta = -\lambda (\xi + \theta), & \quad \mathcal{P} = -\lambda^* + \lambda^{-1} (\xi + \theta) \xi^*. 
\end{align*}
\] (3.12)

Note that no \( \tau \) derivatives of the remaining variables appear. Besides \( \tilde{s} \tau = \theta \), the reduced BRST differential reads

\[
\begin{align*}
\tilde{s} X^\mu &= -\eta P^\mu, & \tilde{s} P^\mu &= \eta \Gamma^\mu_{\rho\nu} P^\rho P^\nu, & \tilde{s} \eta &= 0, & \tilde{s} \mathcal{P} &= \frac{1}{2} (P^\mu P_\mu - m^2). 
\end{align*}
\] (3.13)

When using the results of Section 2.6.6 it follows that the parent differential reduces to \( dF \) plus the prolongation of the differential defined by (3.13). Before describing the latter prolongation more explicitly, let us note that \( \tilde{s} \) is the BFV Hamiltonian BRST differential of the model. Indeed, introducing the Poisson bracket by \( \{ X^\mu, P_\nu \} = \delta^\mu_\nu \) and \( \{ \eta, \mathcal{P} \} = 1 \),

\[
\tilde{s} = \{ \Omega, \cdot \}, \quad \Omega = \frac{1}{2} \eta (P^\mu P_\mu - m^2),
\] (3.14)

For the prolongation, one introduces for each of the remaining coordinates \( z^A \equiv (X^\mu, P^\mu, \eta, \mathcal{P}) \) a coordinate of opposite Grassmann parity and ghost number differing by \(-1\) according to

\[
\begin{align*}
\tilde{X}^\mu &= X^\mu + P^\mu \theta, & \tilde{P}_\mu &= P_\mu - X^*_\mu \theta, & \tilde{\eta} &= \eta + \mathcal{P}^* \theta, & \tilde{\mathcal{P}} &= \mathcal{P} + \eta^* \theta.
\end{align*}
\] (3.15)

The notations here are chosen such that \( z^A \) and \( z^*_A \) are conjugated with respect to the antibracket induced by the above Poisson bracket (see [34, 36] for the details on relation of the target space and the field space bracket structures).

It turns out that the parent differential \( s^P = (d^F + \tilde{s}) |_{\theta = 0} \) coincides with the BV differential associated to the first order master action

\[
S = \int d\tau [P_\mu \dot{X}^\mu + \mathcal{P} \dot{\eta} - \{ \Omega, z^A z^*_A \}] = \int d\tau d\theta [d\tilde{X}^\mu \tilde{P}_\mu - d\tilde{\eta} \tilde{\mathcal{P}} - \Omega (\tilde{z})].
\] (3.16)

The associated classical action can be obtained from \( S \) by putting to zero all the variables with nonvanishing ghost degree and is given by:

\[
S_0 = \int d\tau [P_\mu \dot{X}^\mu - \frac{1}{2} \mathcal{P}^* (P^\mu P_\mu - m^2)],
\] (3.17)

where \( \mathcal{P}^* \) is to be identified with the Lagrange multiplier of the Hamiltonian formalism.
3.3 Parametrized mechanics

Consider a system of ordinary differential equations

\[ \dot{\psi}^A = V^A(\psi, t). \]  

(3.18)

In the parametrized version, one considers new fields \( e, t \) and introduces a new independent variable \( \tau \). The equations of motion and gauge symmetries take the form

\[ \frac{\partial}{\partial \tau} \psi^A = e V^A(\psi, t), \quad \frac{\partial}{\partial \tau} t = e, \quad \delta_e \psi^A = e V^A, \quad \delta_e t = \epsilon, \quad \delta_e e = \frac{\partial}{\partial \tau} \epsilon. \]  

(3.19)

In the gauge, \( t = \tau \), they indeed coincide with the starting point system.

Let us now show how \( (3.19) \) can be arrived at through the parametrized parent formulation. As a byproduct, this also shows that \( (3.19) \) in fact defines an AKSZ-type sigma model in 1 dimension. The BRST description of the dynamics \( (3.18) \) is achieved by introducing a ghost \( \xi \) and antifields \( P^A \). Variables on the extended jet space are \( n^\psi^A, n^P^A, t, \xi \) where the superscript \( n \) denotes the order of derivatives, i.e., \( 1^\psi^A = \dot{\psi}^A \). The BRST differential is determined by

\[ \tilde{s} t = \xi, \quad \tilde{s} \xi = 0, \quad \tilde{s} P^A = 1^\psi^A - V^A(\psi, t) + \xi 1^P^A, \quad \tilde{s} \psi^A = \xi 1^\psi^A, \]  

(3.20)

and the condition that it commutes with the total time derivative. According to the general prescription of Section 2.8, the parametrized parent formulation is a 1d AKSZ-type sigma model whose extended space-time has coordinates \( \tau, \theta \) while the target space coordinates are \( n^\psi^A, n^P^A, t, \xi \).

It is easy to see that \( n^P^A \) and \( \psi^A - \partial_n V^A \) for \( n \geq 0 \) enter trivial pairs for \( \tilde{s} \) and can be eliminated. In the reduced theory one stays with just the coordinates \( t, \xi, \psi^A \). The reduced differential is given by

\[ s^{\text{red}} = dF + \bar{Q}, \quad Q = \xi (V^A \frac{\partial}{\partial \psi^A} + \frac{\partial}{\partial t}). \]  

(3.21)

Identifying \( \frac{\partial F}{\partial \theta} \xi \) with the field \( e \) of the starting point formulation, it is straightforward to check that \( s^{\text{red}} \) precisely determines the parametrized system \( (3.19) \).

The fact that parametrized mechanics can be represented as a 1d AKSZ sigma model is not surprising and seems to be known.\(^2\) Indeed, as it was already shown in the case of Hamiltonian/Lagrangian systems that any theory with vanishing Hamiltonian, and thus in particular a parametrized system, can be reformulated as a 1d AKSZ sigma model \(^{36}\) (see also the discussion in \( [53] \)). The above example provides the non-Lagrangian/non-Hamiltonian version of this result and can of course easily be generalized to include systems with a gauge freedom.

\(^2\)In particular, it was independently arrived at by A. Sharapov whom we wish to thank for a related discussion.
3.4 Yang-Mills theory

The set of fields for Yang-Mills theory are the components of a Lie algebra valued 1-form $H_\mu$ and ghost $C$ along with their conjugate antifields $H^*_\mu$ and $C^*_i$, where $i$ is the Lie algebra index. The BRST differential is given by $s = \gamma + \delta$,

$$
\gamma H_\mu = \partial_\mu C + [H_\mu, C], \quad \gamma C = -\frac{1}{2} [C, C].
$$

$$
\gamma H^*_\mu = f^i_{jk} H^*_\mu C^k, \quad \gamma C^*_i = f^k_{ji} C^*_k C^j,
$$

(3.22)

$$
\delta H^*_\mu = \frac{\delta}{\delta H_\mu} L[H], \quad \delta C^*_i = -\partial_\mu H^*_\mu + f^k_{ji} H^*_k H^j_\mu,
$$

where $L[H] = \text{Tr} F_{\mu\nu} F^{\mu\nu}$ is the Lagrangian in terms of the associated curvatures $F_{\mu\nu} = \partial_\mu H_\nu - \partial_\nu H_\mu + [H_\mu, H_\nu]$ and $f^k_{ij}$ are the structure constants. Note that all of the discussion below that does not involve the precise form of the original equations of motion and their parent implementation applies to any regular Lagrangian that is gauge invariant up to a total derivative.

By reducing to the cohomology of $\delta$, the antifields can be eliminated from the parent theory as explained at the end of subsection 2.6.4. The parent theory is then determined by

$$
s^P = dF - \sigma F + \tilde{\gamma},
$$

(3.23)

and the algebraic constraints coming from the equations of motions

$$
\frac{\delta L[H]}{\delta H_\mu} = 0,
$$

(3.24)

with $x$-derivatives replaced by $y$-derivatives, together with all prolongations of these equations obtained by acting multiple times with $\partial_F^{x\mu}$ and $\partial_F^{y\mu}$.

Let us identify explicitly the field content and the equations of motion of this reduced parent theory. At ghost number zero we have the fields $(H_\mu)_{(\lambda)\mu}(x)$ and $(C_{(\lambda)})_{\mu}(x)$. It is useful to keep the $y$ variables and to work in terms of the following generating functions

$$
A(x, y, \theta) = (C_{(\lambda)})_{\mu}(x) y^{(\lambda)} \theta^\mu, \quad B_\mu(x, y, \theta) = (H_\mu)_{(\lambda)\mu}(x) y^{(\lambda)}.
$$

(3.25)

The equations of motion for these fields are given by $s^P \Psi^A = 0$ after having put to zero all fields except for those at ghost number 0. One thus has to act with $s^P$ on $C_{(\lambda)\mu} y^{(\lambda)}(x)$ and $(H_\mu)_{(\lambda)\mu} y^{(\lambda)}$ to find

$$
dA = \sigma A + \frac{1}{2} [A, A], \quad dB_\mu = -\frac{\partial}{\partial y^\mu} A + \sigma B_\mu + [A, B_\mu].
$$

(3.26)

Along with the above algebraic constraints, these equations are equivalent to the ones of the original Yang-Mills theory. In the abelian case, Equations (3.26), reduce to the spin 1 sector of the equations proposed in [13]. In the non-abelian case, they were proposed...
in [9]. Let us also mention a closely related formulation in terms of bi-local fields [61, 62, 63].

The above parent formulation can be reduced further by eliminating from the very start contractible pairs for \( \tilde{\gamma} \) as discussed in [30, 31, 32]. For Yang-Mills theories, these pairs are given for \( k \geq 1 \) by the variables \( \partial(\mu_1 \ldots \partial(\mu_{k-1} H_{\mu_k}) \) and \( \tilde{\gamma}(\partial(\mu_1 \ldots \partial(\mu_{k-1} H_{\mu_k}) \) which substitute for \( \partial(\mu_1 \ldots \partial(\mu_{k-1} C \). At the same time, as remaining variables one uses \( \tilde{C} = C + H \), where \( H = H_\mu \theta^\mu \) and the algebraically independent components of the covariant derivatives of the curvatures, \( D_{\mu_1} \ldots D_{\mu_{k-1}} F_{\mu_k \nu} \), which are given by \( D(\mu_1 \ldots D_{\mu_{k-1}} F_{\mu_k \nu} \) on account of the Bianchi identities. In the approach of [30, 31, 32], the former are known as generalized connections and the latter as generalized tensor fields. By direct computation, it follows that

\[
\tilde{\gamma}\tilde{C} = -\frac{1}{2} [\tilde{C}, \tilde{C}] + F, \quad F = \frac{1}{2} F_{\mu\nu}\theta^\mu\theta^\nu, \tag{3.27}
\]

which is the celebrated “Russian formula” [33]. Furthermore,

\[
\tilde{\gamma}(D_{\mu_1} \ldots D_{\mu_{k-1}} F_{\mu_k \nu}) = \theta^{\mu_0} D_{\mu_0} D_{\mu_1} \ldots D_{\mu_{k-1}} F_{\mu_k \nu} + [D_{\mu_1} \ldots D_{\mu_{k-1}} F_{\mu_k \nu}, \tilde{C}], \tag{3.28}
\]

so that the reduced differential is simply given by (3.27) and (3.28) in terms of the over-complete coordinates \( D_{\mu_1} \ldots D_{\mu_{k-1}} F_{\mu_k \nu} \). The standard equations of motion are imposed by extracting all traces from the independent covariant derivatives of the curvatures [64] : the independent jet-coordinates parametrizing solution space are given by \( [D(\mu_1 \ldots D_{\mu_{k-1}} F_{\mu_k \nu}]^T \), where the superscript \( T \) indicates the trace-free part. Note that in case one does not want to take out these traces, one has to keep the Koszul-Tate differential acting on the antifields and their covariant derivatives.

The field content of the fully reduced parent theory is given by the \( \theta \) prolongation of the Lie algebra valued fields \( \tilde{C}, \tilde{C} + \tilde{C}_\nu \theta^\nu + \frac{1}{2} \tilde{C}_{\mu\nu} \theta^\mu \theta^\nu, \) of which only \( A^i_\nu \equiv -\tilde{C}^i_\nu \) are in ghost number 0. Note that there are no more \( y \) derivatives of these variables. In addition there are the \( \theta \) prolongations of the over-complete set of fields given by the covariant derivatives of the curvatures, \( D^y_{\lambda_1} \ldots D^y_{\lambda_{k-1}} F^y_{\lambda_k \nu} + G_{\lambda_1 \ldots \lambda_{k-1}} \theta^\nu + \ldots, \) of which only the \( \theta \) independent terms are of ghost number 0.

Applying \( \partial^\mu_\nu \partial^\rho_\nu \) to both sides of the Russian formula (3.27) and using Equation (2.37) and its generalizations, one directly gets \((-\sigma^F + \tilde{\gamma})\tilde{C}_{\mu\nu} = [A_\mu, A_\nu] - F^y_{\mu\nu} + \ldots, \) where \( \ldots \) denote terms involving fields with nonvanishing ghost numbers and the curvature involves \( y \) derivatives of \( H \). Using this in \( (d F - \sigma F + \tilde{s})\tilde{C}_{\mu\nu} = 0 \) and putting all the fields of nonvanishing ghost degree to zero gives the first part of the equations of motion. When contracting indices with \( \theta^\nu \)’s, they can be compactly written as

\[
dA + \frac{1}{2} [A, A] = F^y, \tag{3.29}
\]

and express the equality of the parent form of the \( H \) curvature with the \( A \) curvature in terms of \( x \) derivatives as dynamical equations.
In terms of the over-complete set of fields, the derivation of the remaining equations of motion is straightforward. Applying $\partial^\rho$ to both sides of Equation (3.28) and using Equation (2.37) and its generalizations, one gets

$$(-\sigma^F + \bar{s}) G_{\lambda_1...\lambda_k \nu \rho} = -D^y_{\rho} D^y_{\lambda_1} \cdots D^y_{\lambda_{k-1}} F_{\lambda_k \nu} - [D^y_{\lambda_1} \cdots D^y_{\lambda_{k-1}} F_{\lambda_k \nu}, A_\rho],$$

and the corresponding equation of motion reads

$$\left(\partial_\rho + [A_\rho, \cdot]\right) D^y_{\lambda_1} \cdots D^y_{\lambda_{k-1}} F_{\lambda_k \nu} = D^y_\rho D^y_{\lambda_1} \cdots D^y_{\lambda_{k-1}} F_{\lambda_k \nu}. \quad (3.30)$$

These equations merely equate covariant derivatives of the tensor fields with respect to $x^\mu$ using $A_\mu$ with such derivatives with respect to $y^\mu$ using $H_\mu$.

After an algebraic projection of the latter equations on the independent coordinates $D^y_{(\lambda_1} \cdots D^y_{\lambda_{k-1}) \nu}, F^y_{\lambda_k \nu}$, the reduced theory is known as the unfolded form of the theory at the off-shell level and has been constructed in [9]. In the unfolded approach, the ghost number zero fields originating from $C$ are known as the gauge module, while the $D^y_{(\lambda_1} \cdots D^y_{\lambda_{k-1}) \nu}$ form the so-called Weyl module. The completely reduced on-shell system can be arrived at by projecting out the traces of $D^y_{(\lambda_1} \cdots D^y_{\lambda_{k-1}) \nu}$ (see [64] for the explicit structure of the projection). For instance, requiring $D^y_{(\mu} F^y_{\nu)^\rho}$ to be totally traceless obviously imposes the Yang-Mills equations on $A_\mu$ through (3.29), (3.30), and the Bianchi identity. Note however that for higher tensors, this projection brings in further nonlinear terms.

### 3.5 Metric gravity

The BV description of metric gravity involves as fields the inverse metric $g^{ab}$ and a ghost field $\xi^a$ that replaces the vector field parametrizing an infinitesimal diffeomorphism, along with their antifields $g^*_a$ and $\xi^*_a$. The BRST differential decomposes as $s = \delta + \gamma$ where

$$\delta g^*_a = \frac{\delta}{\delta g^a} L[g], \quad \delta \xi^*_a = g^*_a \partial_c g^{ab} + 2 \partial_a (g^{ab} g^*_b) \quad (3.31)$$

and

$$\gamma g^{ab} = L_\xi g^{ab} = \xi^c \partial_c g^{ab} - g^{cb} \partial_c \xi^a - g^{ac} \partial_c \xi^b,$$

$$\gamma \xi^c = \frac{1}{2} [\xi, \xi]^c = \xi^a \partial_a \xi^c,$$

$$\gamma g^*_a = - \partial_c (g^*_a \xi^c) - g^*_a \partial_c \xi^c - g^*_b \partial_c \xi^a,$$

$$\gamma \xi^*_a = \partial_a (\xi^*_a \xi^a) + \xi^a \partial_c \xi^a. \quad (3.32)$$

We leave open the precise choice of the diffeomorphism covariant equations of motion determined by $L$ and only require standard regularity conditions together with $\gamma L = \partial_a j^a$ for some $j^a$. In this way, we allow for gravitational theories with higher curvature and/or gravitational Chern-Simons terms.
For metric gravity, $\gamma X$ contains $\xi^a \partial_a X$ for any field $X$, so that the general discussion of Section 2.7 applies. After the field redefinition, the theory is thus of AKSZ type with target space coordinates $g_{ab}, \xi^a, g^*_{ab}, \xi^*_a$ along with their $y$-derivatives. The parent BRST differential takes the form $s^P = dF + \bar{s}$. It also follows from the discussion in Section 2.4 that one can use generic coordinates $x^\mu$ and $\theta^\mu$ in the source space without affecting the target space. As we are going to see, this gives to the fields of the parent theory a natural geometrical interpretation in terms of vielbeins, connections and their higher analogs.

Equivalent reduced formulations are obtained by eliminating various sets of generalized auxiliary fields. For instance, following Section 2.6.4, the elimination of the antifields $g^*_{ab}, \xi^*_a$ gives rise to the differential

$$s^P = dF + \bar{\gamma},$$

(3.33)

together with the algebraic constraints

$$\left( \left( \frac{\partial}{\partial y^a} \right)^F \ldots \left( \frac{\partial}{\partial \theta^a} \right)^F \ldots \left( \frac{\delta}{\delta g_{ab}(L[g])} \right) \right) = 0.$$  

(3.34)

These constraints can also be understood as constraints in the target space of the AKSZ sigma model. In this case, the $\theta$-derivatives are to be dropped and the target space becomes the stationary surface in the jet-space approach in terms of $y$ derivatives. In other words, the reduced theory is again an AKSZ-type sigma model with a target space that is the submanifold defined by the constraints (3.34) (with $\theta$-derivatives dropped) in the supermanifold with coordinates $g^{ab,(c)}$ and $\xi^a_{(c)}$. The associated odd nilpotent vector field is given by $\gamma$. That $\gamma$ restricts to the submanifold is a consequence of the covariance of the equations of motion expressed through $[\delta, \gamma] = 0$.

In ghost number zero, one finds the 0-form fields $g^{ab}_{(c)}$ and the 1-form fields $A^a_{\mu(c)}$ coming from the component linear in $\theta^\mu$ in the expansion of $\xi^a_{(c)}$. In order to write the equations of motion in terms of generating functions, let us introduce besides $y^a$ additional formal variables $p_b$, and consider the algebra of polynomials in $y, p$ equipped with the standard Poisson bracket $\{p_a, y^b\} = \delta^b_a$. The target space coordinates $g^{ab}_{c_1 \ldots c_l}$ and $\xi^a_{c_1 \ldots c_l}$ can then be encoded in

$$G = \frac{1}{2} g^{ab}_{(c)} y^{(c)} p_a p_b, \quad \Xi = \xi^a_{(c)} y^{(c)} p_a,$$

(3.35)

and the action of $\bar{\gamma}$ on these coordinates can be compactly written as

$$\gamma \Xi = \frac{1}{2} \{\Xi, \Xi\}, \quad \gamma G = \{\Xi, G\}.$$  

(3.36)

Indeed, to lowest order in $y$ these are just formulas (3.32) and then one uses an induction in homogeneity in $y$. In these terms, the nilpotency of $\gamma$ is a consequence of the graded Jacobi identity for the even Poisson bracket. It then follows that $\gamma$ is the Chevalley-Eilenberg differential associated to the Lie algebra of formal vector fields in
the \( y \)-variables with coefficients in formal symmetric bi-vectors. In terms of the parent theory fields, \( \Xi + A + \ldots \) with \( A = A^a_{(c)} y^{(c)} p_a \theta^\mu \) and \( G + \ldots \), the equations for the ghost-number-zero fields determined by the parent differential \( s^P = d^F + \tilde{\gamma} \) take the familiar form
\[
dA + \frac{1}{2} \{ A, A \} = 0, \quad dG + \{ A, G \} = 0, \tag{3.37}
\]
which should be supplemented by the algebraic constraints \( (3.34) \). In this reformulation, metric gravity has turned into a gauge theory for the diffeomorphism group since the gauge field \( A_\mu \) takes values in the Lie algebra of vector fields.

The associated linearized equation for spin 2 gauge fields were derived in [13] from the parent theory perspective. Note that the Poisson bracket can be replaced by the associated \( \ast \)-commutator in \( (3.36) \) and \( (3.37) \) if one allows for a \( p \)-independent component in \( G \). This corresponds to coupling to an extra scalar field. With this replacement, equations \( (3.37) \) are known in the context of Fedosov quantization and were shown in [9] to describe gravity at the off-shell level. More generally, in [9] it was shown that by allowing for all powers in \( p \), one describe the entire set of symmetric higher spin fields at the off-shell level. Let us also mention that understanding gravity as a gauge theory of the diffeomorphism group dates back to [65]. A closely related formulation of gravity was considered in [66].

Finally, let us eliminate additional generalized auxiliary fields originating from contractible pairs for \( \gamma \). In metric gravity, these are all the derivatives of the ghosts of degree 2 or higher and all the symmetrized derivatives of the Christoffel symbol [30]. After this elimination, one stays with the following variables: \( \xi^a, C^a_b = \xi^a_b + \Gamma^a_b_c \xi^c \) at ghost degree 1 and \( g^{ab}, R^d_{abc}, \ldots \) where dots denote independent components of covariant derivatives of the Riemann tensor in terms of \( y \) derivatives, \( D^y_{c_1} \ldots D^y_{c_k} R^b_{a_1 a_2 a_3} \). The action of \( \gamma \) on the ghost variables is given by
\[
\gamma \xi^a = \xi^b C^a_b, \quad \gamma C^a_b = C^a_c C^b_c + \frac{1}{2} \xi^c \xi^d R^b_{acd}, \tag{3.38}
\]
while
\[
\gamma D^y_{c_1} \ldots D^y_{c_k} R^b_{a_1 a_2 a_3} = \xi^c_0 D^y_{c_0} D^y_{c_1} \ldots D^y_{c_k} R^b_{a_1 a_2 a_3} - C^b_d D^y_{c_1} \ldots D^y_{c_k} R^d_{a_1 a_2 a_3} + \ldots + C^d_a D^y_{c_1} \ldots D^y_{c_k} R^a_{a_1 a_2 a_3} \tag{3.39}
\]
If one performs the change of variables \( g^{ab} = \eta^{ab} + h^{ab} \) where \( \eta^{ab} \) is the inverse Minkowski metric and considers formal power series in \( h^{ab} \), one can further eliminate \( h^{ab} \) and the symmetric part of \( C^{ab} \), where the index has been raised with \( \eta^{ab} \). The antisymmetric part \( C^{[ab]} \) are the ghosts associated with the Lorentz algebra.

---

[3] When reasoning in terms of \( \tilde{\gamma} \), the absorption of the \( \sigma^F \) term in the parent differential comes from the redefinition \( \tilde{\xi}^a = \xi^a + \theta^a \) and \( (3.38) \) with \( \gamma, \xi^a \) replaced by \( \tilde{\gamma}, \tilde{\xi}^a \) is the gravitational Russian formula.
The ghost number zero fields of the completely reduced theory are then given by 
\[ e^a_\mu = \frac{\partial e^c}{\partial \theta^c} C^a_\mu \] and \[ \omega^a_\mu = \frac{\partial \omega^b}{\partial \theta^b} C^a_\mu \] and by \[ D^c_{y_1} \cdots D^c_{y_k} R^b_{a_{1} a_{2} a_{3}} \]. In terms of \( e^a = e^a_\mu \theta^\mu \) and \( \omega^{[ab]} = \omega^{[ab]}_\mu \theta^\mu \), the equations of motion of the completely reduced theory take the form

\[ de^a + \omega^{a} e^b = 0, \quad d\omega^a_c + \omega^a_c \omega^b = \frac{1}{2} e^c e^d R^b_{cda}, \quad (3.40) \]

\[ d(D^c_{y_1} \cdots D^c_{y_k} R^b_{a_{1} a_{2} a_{3}}) = e^c_0 D^c_{y_0} D^c_{y_1} \cdots D^c_{y_k} R^b_{a_{1} a_{2} a_{3}} - \omega^b_c D^c_{y_1} \cdots D^c_{y_k} R^d_{c_{a_{1} a_{2} a_{3}}} + \omega^d_{y_1} D^c_{y_0} \cdots D^c_{y_k} R^b_{c_{a_{1} a_{2} a_{3}}} + \cdots + \omega^d_{a_3} D^c_{y_1} \cdots D^c_{y_k} R^b_{a_{1} a_{2} d}, \quad (3.41) \]

to be supplemented by the algebraic constraints coming from taking into account the Bianchi identities to select independent variables among the \( D^c_{y_1} \cdots D^c_{y_k} R^b_{a_{1} a_{2} a_{3}} \) and the algebraic constraints (3.34) (without \( \theta \) derivatives). This completes the systematic derivation starting from the parent formulation of the completely reduced first order gravitational equations. When reformulated in terms of the independent fields, the above equations are known as off-shell unfolded equations and were proposed in [9]. Note that the explicit elimination of the dependent fields and the implementation of the algebraic constraints (3.34) brings in further nonlinear terms.

In the same spirit, one can construct the parent formulation for conformal gravity, for which the relevant tensor calculus has been constructed in [67]. The same applies to gravity with nonvanishing cosmological constants formulated as a gauge theory of the (A)dS group. In the later case the formulation at the off-shell level can be inferred from the spin-2 sector of the off-shell theory proposed in [60] (see also a somewhat related construction in [68]).

### 3.6 2d sigma model

As a final example, let us consider a two dimensional sigma model invariant with respect to both diffeomorphisms and Weyl transformations. The field content of the BRST formulation is given by the two-dimensional metric \( g_{\mu\nu} \), scalar fields \( \varphi^i \), diffeomorphism ghosts \( \xi^a \) and the Weyl ghost \( C \). The BRST differential in the sector of these variables is given by

\[ \gamma g_{\mu\nu} = \xi^0 \partial_\mu g_{0\nu} + \partial_\mu \xi^0 g_{0\nu} + \partial_\nu \xi^0 g_{0\mu} + C g_{\mu\nu}, \]

\[ \gamma \varphi^i = \xi^0 \partial^\mu \varphi^i, \quad \gamma \xi^0 = \xi^0 \partial^\mu \xi^0, \quad \gamma C = \xi^0 \partial^\mu C. \quad (3.42) \]

The action can for instance be taken as

\[ S_0 = \int d^2 x \left[ \frac{1}{2} \sqrt{|g|} g^{\alpha\beta} G_{ij}(\varphi) \partial_\alpha \varphi_i \partial_\beta \varphi_j + \frac{1}{2} B_{ij}(\varphi) e^{\alpha\beta} \partial_\alpha \varphi_i \partial_\beta \varphi_j \right]. \quad (3.43) \]

As before, implementing the corresponding equations with their Noether identities is done through the antifields and the Koszul-Tate part of the BRST differential. We will not discuss this part explicitly below.
Just like in the case of gravity, $\gamma$ contains the $\xi^\mu \partial_\mu$ term so that after the redefinition $\tilde{\xi}^\mu = \xi^\mu + \theta^\mu$, the parent theory is determined by

$$s^P = dF + \tilde{\gamma},$$

(3.44)
to be supplemented by the parent implementation of the original equations of motion.

We are now going to work locally both in the base and in the target space and eliminate the generalized auxiliary fields related to the contractible pairs identified in [69, 30] to construct the reduced off-shell parent theory. One first uses the Beltrami parametrization of the 2d metric and changes the basis for ghosts accordingly:

$$h = \frac{g_{11}}{g_{12} + \sqrt{g}}, \quad \bar{h} = \frac{g_{22}}{g_{12} + \sqrt{g}}, \quad e = \sqrt{g},$$

$$\eta = \xi^1 + \bar{h}\xi^2, \quad \bar{\eta} = \xi^2 + h\xi^1.$$ 

(3.45)

We will use the notation $H$ while the scalar fields and their derivatives $\partial^P \bar{\partial}^P \varphi^i$ are replaced by the tensor fields

$$T^i_{p, \bar{p}} = (L_{-1})^p (\bar{L}_{-1})^{\bar{p}} \varphi^i, \quad p, \bar{p} = -1, 0, 1, \ldots,$$

(3.47)

which satisfy

$$L_q T^i_{p, \bar{p}} = \frac{p!}{(p-q)!} T^i_{p-q, \bar{p}} \quad \text{for} \quad q < p, \quad L_q T^i_{p, \bar{p}} = 0 \quad \text{for} \quad q \geq p,$$

(3.48)

with analogous formulae for $\bar{L}_q T^i_{p, \bar{p}}$ and where $L_p, \bar{L}_p$ for $p, \bar{p} = -1, 0, 1, \ldots$ satisfy

$$[L_p, L_q] = (p-q)L_{p+q}, \quad [\bar{L}_p, \bar{L}_q] = (\bar{p}-\bar{q})\bar{L}_{p+q}, \quad [L_p, \bar{L}_q] = 0.$$

(3.49)

Furthermore,

$$L_{-1} = \frac{1}{1 - h\bar{h}}(\partial - h\bar{\partial} - \sum_{\bar{p} \geq 0} \bar{H}^{\bar{p}} \bar{L}_{\bar{p}} + h \sum_{p \geq 0} H^p L_p).$$

(3.50)

Here $H^p = \frac{1}{(p+1)!} \partial^{p+1} h$ and the corresponding expressions obtained through formal complex conjugation hold for $\bar{L}_{-1}$ and $\bar{H}^{\bar{p}}$. The explicit expressions for the tensor fields are determined by the requirement that the set of variables $(3.46)-(3.47)$ is closed under $\gamma$ (see [69, 30] for details):

$$\gamma \eta^p = \frac{1}{2} \sum_{q=-1}^{p+1} (p-2q)\eta^q \eta^{p-q}, \quad \gamma \bar{\eta}^\bar{p} = \frac{1}{2} \sum_{\bar{q}=-1}^{\bar{p}+1} (\bar{p}-2\bar{q})\bar{\eta}^\bar{q} \bar{\eta}^{\bar{p}-\bar{q}},$$

(3.51)

$$\gamma T^i_{p, \bar{p}} = \sum_{q=-1}^p \eta^q L_q T^i_{p, \bar{p}} + \sum_{\bar{q}=-1}^{\bar{p}} \bar{\eta}^\bar{q} \bar{L}_{\bar{q}} T^i_{p, \bar{p}}.$$

(3.52)
The above relations can be compactly written using extra variables \(z, \bar{z}\), the regular vector fields \(l_p = \frac{\partial}{\partial z} z^{p+1}, \bar{l}_\bar{p} = \frac{\partial}{\partial \bar{z}} \bar{z}^{\bar{p}+1}\), \(p, \bar{p} = -1, 0, 1, \ldots\) satisfying the same algebra as in (3.49) and the generating functions

\[
\Xi = \sum_{p=-1}^{\infty} \eta^p l_p, \quad \bar{\Xi} = \sum_{\bar{p}=-1}^{\infty} \bar{\eta}^{\bar{p}} \bar{l}_{\bar{p}}, \quad T^i = \sum_{p=0}^{\infty} \sum_{\bar{p}=0}^{\infty} \frac{1}{p!\bar{p}!} T^i_{p\bar{p}} z^p \bar{z}^{\bar{p}},
\]

(3.53)

so that \(L_p T^i = T^i l_p\). In these terms, Equations (3.51) and (3.52) take the form

\[
\gamma \Xi = -\frac{1}{2} [\Xi, \Xi], \quad \gamma T^i = T^i \Xi + T^i \bar{\Xi}.
\]

(3.54)

In the reduced parent theory, the ghost number zero fields come from the ghost fields \(\eta^p(x), \bar{\eta}^{\bar{p}}(x)\) which give rise to 1-form gauge fields, \(A^p = A^p_\mu(x) \theta^\mu, \bar{A}^{\bar{p}} = \bar{A}^{\bar{p}}_\mu(x) \theta^\mu\), with \(A^p_\mu \equiv -\eta^p, \bar{A}^{\bar{p}}_\mu \equiv -\bar{\eta}^{\bar{p}}\), and the \(T^i_{p\bar{p}}(x)\) fields which are 0 form fields. In terms of generating functions, \(A(x, z) = \sum_{p=-1}^{\infty} A^p l_p\) and \(\bar{A}(x, \bar{z}) = \sum_{\bar{p}=-1}^{\infty} \bar{A}^{\bar{p}} \bar{l}_{\bar{p}}\) and \(T^i(x, z, \bar{z})\), the equations of motion of the reduced theory take the form

\[
dA + \frac{1}{2} [A, A] = 0, \quad d\bar{A} + \frac{1}{2} [\bar{A}, \bar{A}] = 0, \quad dT^i + T^i A + T^i \bar{A} = 0,
\]

(3.55)

with \(d = \theta^\mu \frac{\partial}{\partial x^\mu}\). These equations can be considered as defining the off-shell system. The on-shell version requires in addition to impose the analog in terms of \(y\) derivatives of the original equations of motion and their prolongations on the \(T^i(x, z, \bar{z})\) fields.

4 Conclusions

In this paper, we have shown how to systematically construct a first order parent theory associated with a generic interacting gauge field theory described by an antifield dependent BRST differential. We have then discussed how to obtain various equivalent formulations through the elimination of generalized auxiliary fields. Our emphasis here has been the case where the various equivalent formulations are local field theories in the sense that all functions depend on the fields and a finite number of their derivatives. Relaxing the locality requirement is crucial for other types of questions, such as for instance the reduction to the light-cone description where physical degrees of freedom are isolated (see e.g. [70] and references therein for a discussion in BRST theoretic terms), or the understanding of the relation of the proper BV master action for BRST first quantized Hamiltonian system and the \(\langle \psi, \hat{Q} \psi \rangle\) master action [54] used in the context of string field theories [71, 72, 73, 74].

Related to this issue, the variables \(y^\lambda\) have been auxiliary in our construction and merely a bookkeeping device for additional fields in the theory. At the same time, all fields were considered as fields on the original space-time with coordinates \(x^\mu\). But as
suggested by the superfield notation used in Section 2.2, one could also consider these fields as fields on a doubled space-time with coordinates \( x^\mu, y^\lambda \), or even more generally as fields on the superspace with coordinates \( x^\mu, y^\lambda, \theta^\nu \). In this context, it would be interesting to try to connect the parent formulation with the recently constructed double field theory [75] or the bi-local fields used for the dual formulation of interacting higher spins on \( AdS_4 \) in [76].

Possible applications of the proposed formalism involve higher spin theories at the interacting level. In this context, the most striking results have been obtained using the unfolded formalism [6, 8] (see also [7, 8, 9, 10, 11] and [83] for a review). We hope to gain a somewhat better control over the theory and to make geometrical structures manifest by phrasing it in parent form. This is supported by a concise formulation of nonlinear higher spin theory at the off-shell level [60] (see also [9]) that can be understood as an appropriate AKSZ-type sigma model.

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**A Conventions**

Let \( \mathcal{H} \) be a graded superspace with basis \( e_\alpha \). Consider the associative superalgebra \( \mathcal{A} \) of linear operators acting on \( \mathcal{H} \) from the right, i.e.

\[
\phi(AB) = (\phi A)B, \quad A, B \in \mathcal{A}. \quad (A.1)
\]

The components are introduced according to

\[
e_\alpha A = A^\beta_\alpha e_\beta \quad (A.2)
\]

so that \( \mathcal{A} \) is a matrix superalgebra with the following multiplication law

\[
(AB)^\beta_\alpha = A^\gamma_\alpha B^\beta_\gamma. \quad (A.3)
\]
Let $\mathcal{H}^*$ be the dual space to $\mathcal{H}$ and $\psi^\alpha$ a dual basis so that
\[ \langle e_\beta, \psi^\alpha \rangle = \delta^\alpha_\beta. \] (A.4)

$\mathcal{H}^*$ is naturally a left module over $\mathcal{A}$ with the module structure defined by
\[ \langle \phi A, f \rangle = \langle \phi, A^F f \rangle, \quad \forall \phi \in \mathcal{H}, \; f \in \mathcal{H}^*. \] (A.5)

The space $\mathcal{H}$ can be identified with the space of linear functions in $\psi^\alpha$ considered as supercommuting variables with parity and grading defined by that of $e_\alpha$, $|\psi^\alpha| = |e_\alpha|$ and $gh(\psi^\alpha) = gh(e_\alpha)$. Under this identification $A^F$ is a linear vector field on the space with coordinates $\psi^\alpha$. In components, it reads as
\[ A^F = \psi^\alpha A^\alpha_\beta \frac{\partial}{\partial \psi^\beta}. \] (A.6)

It can be then extended to the algebra of polynomials in $\psi^\alpha$ through the Leibnitz rule
\[ A^F(f g) = (A^F f) g + (-1)^{|A||f|} f A^F g. \] (A.7)

The above relations can be compactly written by using a distinguished element $\Psi$ of $\mathcal{H}^* \otimes \mathcal{H}$, the latter being understood as a $\mathcal{A}$-bimodule such that $\mathcal{A}$ acts on the first factor from the left and on second from the right. This element corresponds to the identity if one identifies $\mathcal{H}^* \otimes \mathcal{H}$ with $\mathcal{A}$ and is given in components by
\[ \Psi = \psi^\alpha \otimes e_\alpha. \] (A.8)

In what follows we omit the tensor product sign. This is consistent with identifying $\Psi$ as the identity element in the space of functions on $\mathcal{H}$ with values in $\mathcal{H}$.

Finally, the relation between left and right actions can be compactly written as
\[ A^F \Psi = \Psi A, \quad A^F B^F \Psi = \Psi AB. \] (A.9)

To make contact with the main text, let us suppose that instead of $\mathcal{H}$ we started with $\hat{\mathcal{H}} = \mathcal{H} \otimes \mathcal{V}$, for some graded vector space $\mathcal{V}$ with basis $e_A$. The distinguished element is
\[ \Psi = \Psi^A_\alpha (e_\alpha \otimes e_A), \] (A.10)

and in terms of components $\Psi^A = \Psi^A_\alpha e_\alpha$, the action of an element of $\mathcal{A}$ satisfies
\[ B^F \Psi^A = (-1)^{|B||\Psi^A|} \Psi^A B \] (A.11)

if one consistently applies the usual sign rule. If $e_\alpha$ stands for a basis in polynomials in $y, \theta, x$, this gives the definitions used in the main text.
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