Strong convergence rates for explicit space-time discrete numerical approximations of stochastic Allen-Cahn equations

Sebastian Becker · Benjamin Gess · Arnulf Jentzen · Peter E. Kloeden

Abstract

Strong convergence rates for fully discrete numerical approximations of space-time white noise driven SPDEs with superlinearly growing nonlinearities, such as the stochastic Allen–Cahn equation with space-time white noise, are shown. The obtained strong rates of convergence are essentially sharp.

Keywords  Stochastic partial differential equation · SPDE · Stochastic Allen–Cahn equation · Numerical method · Numerical approximation · Strong convergence

Arnulf Jentzen
ajentzen@cuhk.edu.cn; ajentzen@uni-muenster.de; arnulf.jentzen@sam.math.ethz.ch

Sebastian Becker
sebastian.becker@math.ethz.ch

Benjamin Gess
bgess@mis.mpg.de; bgess@math.uni-bielefeld.de

Peter E. Kloeden
kloeden@math.uni-frankfurt.de

1 RiskLab, Department of Mathematics, ETH Zurich, 8092 Zurich, Switzerland
2 Max Planck Institute for Mathematics in the Sciences, 04103 Leipzig, Germany
3 Faculty of Mathematics, University of Bielefeld, 33615 Bielefeld, Germany
4 School of Data Science and Shenzhen Research Institute of Big Data, The Chinese University of Hong Kong, Shenzhen, 518172 Shenzhen, China
5 Applied Mathematics: Institute of Analysis and Numerics, University of Münster, 48149 Münster, Germany
6 Seminar for Applied Mathematics, Department of Mathematics, ETH Zurich, 8092 Zurich, Switzerland
7 Mathematics Institute, Goethe University Frankfurt, 60325 Frankfurt am Main, Germany
1 Introduction

In this article we are interested in strong convergence rates for fully discrete numerical approximations of space-time white noise driven SPDEs with superlinearly growing nonlinearities, such as stochastic Allen–Cahn equations of the form

\[
\frac{\partial}{\partial t} X_t(x) = \frac{\partial^2}{\partial x^2} X_t(x) + m_0^2 X_t(x) - \lambda^2 (X_t(x))^3 + \frac{\partial}{\partial t} W_t(x)
\]

for \( x \in (0, 1), t \in [0, T], \lambda, m_0 \geq 0, \) where \( T \in (0, \infty) \) is a real number and \((W_t)_{t \in [0,T]}\) is an \( \text{Id}_{L^2((0,1); \mathbb{R})}\)-cylindrical Wiener process. Note that \((W_t)_{t \in [0,T]}\) is the space derivative of a Brownian sheet and \( \frac{\partial}{\partial t} W_t(x), x \in (0, 1), t \in [0, T], \) is usually referred to as space-time white noise.

SPDEs of the type (1) occur in several applications, including, for example, the quantization of the Euclidean \( \phi^4 \) quantum field theory [13, Section 0.7 and Section 13.7], fluctuations in reaction diffusion equations [13, Section 0.7] and in neurophysiology in terms of the Nagumo equation [13, Section 0.8]. In these applications, the case of fixed coefficients \( \lambda, m_0 \geq 0 \) is relevant, while in the analysis of binary phase separation, particular emphasis lies on the sharp interface regime \( \lambda = m_0 \uparrow \infty \) (cf., e.g., [14,41], and the references therein). The focus of the present work lies on the case of fixed parameters \( \lambda, m_0 \geq 0, \) addressing the difficulties arising from the irregularity of the noise in (1).

The literature contains a number of numerical approximation results for SPDEs with superlinearly growing nonlinearities (cf., e.g., [4,6,8,9,15–17,21,24,25,27,29]). The articles [15–17,21] establish the strong convergence of numerical approximations for such SPDEs, without estimates on the speed of strong convergence. Such estimates have been developed in the contributions [4,25,27,29], in the following sense: The article [4] establishes strong convergence rates for semi-discrete temporal numerical approximations of space-time white noise driven SPDEs with superlinearly growing nonlinearities such as stochastic Allen–Cahn equations. The papers [25,27,29] prove strong convergence rates for fully discrete (temporal and spatial discrete) numerical approximations for SPDEs with superlinearly growing nonlinearities in the case of the more regular trace class noise.

This leaves open the problem of essentially sharp strong convergence rates for fully discrete numerical approximation schemes for space-time white noise driven SPDE with a superlinearly growing nonlinearity, such as the stochastic Allen–Cahn equation (1). This open problem is solved in the present work, by providing essentially sharp estimates on strong rates of convergence.

A key difficulty in the case of fully discrete numerical approximations for space-time white noise driven SPDEs with superlinearly growing nonlinearities is to derive appropriate uniform a priori moment bounds for the numerical approximation processes. In this article we overcome this difficulty (cf. (9)–(10) below for our approach to this challenge) and establish essentially sharp strong convergence rates for fully discrete numerical approximations of space-time white noise driven SPDEs with superlinearly growing nonlinearities such as stochastic Allen–Cahn equations with space-time white noise; see Theorem 5.5 in Sect. 5 below for the main convergence
rate result in this work. To illustrate Theorem 5.5, we now present in Theorem 1.1 below the specialization of Theorem 5.5 to the case of stochastic Allen–Cahn equations.

**Theorem 1.1** Let $T \in (0, \infty)$, $(H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H) = (L^2((0, 1); \mathbb{R}), \langle \cdot, \cdot \rangle_{L^2((0, 1); \mathbb{R})})$, $a_0, a_1, a_2 \in \mathbb{R}, a_3 \in (-\infty, 0)$, $(e_n)_{n \in \mathbb{N}} \subseteq H$, $(P_n)_{n \in \mathbb{N}} \subseteq L(H), F : L^6((0, 1); \mathbb{R}) \rightarrow H$ satisfy for all $n \in \mathbb{N}, v \in L^6((0, 1); \mathbb{R})$ that $e_n(\cdot) = \sqrt{2} \sin(n\pi \cdot), F(v) = \sum_{k=0}^{3} a_k v^k, P_n(v) = \sum_{k=1}^{n} (e_k, v)_H e_k$, and $a_2 \| \cdot \|_0(\cdot) = 0$, let $A : D(A) \subseteq H \rightarrow H$ be the Laplacian with Dirichlet boundary conditions on $H$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(W_t)_{t \in [0, T]}$ be an $\text{Id}_H$-cylindrical Wiener process, let $\xi \in D((-A)^{1/2}), \gamma \in (1/6, 1/4), \chi \in (0, \gamma/3 - 1/18)$, and let $\Sigma_{m,N}^{M,N} : [0, T] \times \Omega \rightarrow P_N(H), M, N \in \mathbb{N}$, and $\mathcal{X}_{m,N}^{M,N} : [0, T] \times \Omega \rightarrow P_N(H), M, N \in \mathbb{N}$, be stochastic processes which satisfy that for all $M, N \in \mathbb{N}$, $m \in \{0, 1, 2, \ldots, M - 1\}$, $t \in (mT/M, (m+1)T/M)$ it holds $\mathbb{P}$-a.s. that

$$\Sigma_0^{M,N} = \mathcal{X}_{0}^{M,N} = P_N \xi, \quad \Sigma_t^{M,N} = e^{(t-mT/M)A} \left[ \Sigma_{mT/M}^{M,N} + \int_{mT/M}^{t} P_N dW_s \right].$$ (2)

and

$$\mathcal{X}_t^{M,N} = e^{(t-mT/M)A} \mathcal{X}_{mT/M}^{M,N} + \Sigma_t^{M,N} - e^{(t-mT/M)A} \Sigma_{mT/M}^{M,N} + P_N A^{-1} (e^{(t-mT/M)A} - \text{Id}_H) \mathbb{I}_{\|(-A)^{\gamma} \mathcal{X}_{mT/M}^{M,N} \|_H + \|(-A)^{\gamma} \Sigma_{mT/M}^{M,N} \|_H \leq (M/T)^{\chi}} F(\mathcal{X}_{mT/M}^{M,N}).$$ (3)

Then

(i) we have that there exists an up to indistinguishability unique stochastic process $X : [0, T] \times \Omega \rightarrow L^6((0, 1); \mathbb{R})$ with continuous sample paths which satisfies for all $t \in [0, T], p \in (0, \infty)$ that $\sup_{s \in [0, T]} \mathbb{E}[\|X_s\|_{L^6((0, 1); \mathbb{R})}] < \infty$ and

$$\mathbb{P}\left( X_t = e^{A} \xi + \int_{0}^{t} e^{(t-s)A} F(X_s) \, ds + \int_{0}^{t} e^{(t-s)A} \, dW_s \right) = 1.$$ (4)

(ii) we have for all $p \in (0, \infty)$ that $\sup_{t \in (-\infty, \gamma]} \sup_{M,N \in \mathbb{N}} \sup_{r \in [0, T]} \mathbb{E}[\|(-A)^{\gamma} \mathcal{X}_r^{M,N} \|_H^p] < \infty$, and

(iii) we have for all $p, \varepsilon \in (0, \infty)$ that there exists a real number $C \in \mathbb{R}$ such that for all $M, N \in \mathbb{N}$ it holds that

$$\sup_{t \in [0, T]} \left( \mathbb{E}[\|X_t - \mathcal{X}_t^{M,N} \|_H^p] \right)^{1/p} \leq C(M^{(\varepsilon^{-1/4})} + N^{(\varepsilon^{-1/2})}).$$ (5)

Theorem 1.1 follows from Corollary 6.10 which, in turn, follows from our main result, Theorem 5.5 below. Theorem 5.5 also proves strong convergence rates for fully discrete numerical approximations of a more general class of SPDEs than Theorem 1.1 above.
Next we would like to point out that the numerical approximation scheme (3) has been proposed in Hutzenthaler et al. [21] and has there been referred to as a nonlinearity-truncated approximation scheme (cf. [21, (3) in Section 1] and, e.g., [17–20, 24, 25, 35, 36, 38, 40] for further research articles on explicit approximation schemes for stochastic differential equations with superlinearly growing nonlinearities). Moreover, note that Theorem 1.1, in particular, implies that the fully discrete numerical approximations in (3) converge for every $\varepsilon \in (0, 1/4)$ strongly to the solution of the stochastic Allen–Cahn equation (4) with the spatial rate of convergence $1/2 - \varepsilon$ and the temporal rate of convergence $1/4 - \varepsilon$. We can choose $M = N^2$ to balance the temporal and spatial error terms on the right hand side of (5) in Theorem 1.1. More specifically, observe that (5) implies that for all $p, \varepsilon \in (0, \infty)$ there exists a real number $C \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ it holds that

$$\sup_{t \in [0, T]} \left( \mathbb{E} \left[ \| X_t - X^{N^2, N}_t \|_H^p \right] \right)^{1/p} \leq C N^{(\varepsilon - 1/2)}. \quad (6)$$

We observe that for all $M, N \in \mathbb{N}$ we have that $MN$ realizations of standard normal random variables are used to calculate one realization of $X^{M, N}_T$. This shows that for all $N \in \mathbb{N}$ we have that $N^3$ realizations of standard normal random variables are used to calculate one realization of $X^{N^2, N}_T$. Combining (6) with the computational effort $N^3$ illustrates that for all $\varepsilon \in (0, 1/6)$ we have that the approximation scheme in (2)–(3) converges with the overall rate $1/6 - \varepsilon$ with respect to the number of used independent standard normal random variables.

We also would like to point out that the strong convergence rates established in Theorem 1.1 can, in general, not essentially be improved. More formally, [3, Corollary 2.7] proves in the case where $\sum_{i=0}^3 |a_i| = 0$ and $\xi = 0$ in the framework of Theorem 1.1 that there exist real numbers $c, C \in (0, \infty)$ such that for all $M, N \in \mathbb{N}$ we have that

$$c M^{-1/4} \leq \lim_{n \to \infty} \sup_{t \in [0, T]} \left( \mathbb{E} \left[ \| X_t - X^{M, N}_t \|_H^p \right] \right)^{1/p} \leq C M^{-1/4} \quad (7)$$

and

$$c N^{-1/2} \leq \lim_{m \to \infty} \sup_{t \in [0, T]} \left( \mathbb{E} \left[ \| X_t - X^{m, N}_t \|_H^p \right] \right)^{1/p} \leq C N^{-1/2}. \quad (8)$$

Inequalities (7) and (8) thus show that the spatial rate $1/2 - \varepsilon$ and the temporal rate $1/4 - \varepsilon$ established in Theorem 1.1 can essentially not be improved. Further related lower bounds for strong approximation errors in the linear case $\sum_{i=0}^3 |a_i| = 0$ can, e.g., be found in Müller-Gronbach et al. [32, Theorem 1], Müller-Gronbach and Ritter [31, Theorem 1], Müller-Gronbach et al. [33, Theorem 4.2], Conus et al. [10, Lemma 6.2], and Jentzen and Kurniawan [23, Corollary 9.4].

Finally, we would like to add some comments on the proof of Theorem 1.1 above and Theorem 5.5 below, respectively. The main difficulty to prove Theorem 1.1 is to obtain uniform a priori moment bounds for the space-time discrete numerical approximations (3) (see Sects. 2 and 5.4 below). Once the uniform a priori moment bounds
time dependent PDE (with the time variable entering only through the path
Galerkin approximation of the Ornstein–Uhlenbeck process is nothing else but a
the stochastic Allen–Cahn equation (4) and the
spatial spectral Galerkin approximation of the subtracted equation associated to
incorporates the dissipative dynamics of the stochastic Allen–Cahn equation (4) and
which on the other hand respects the spatial spectral Galerkin approximations used for
the spatial discretization of (4). More formally, a key contribution of this work is to
first subtract the noise process from (3) as it is often done in the literature. The key
idea that we use to derive uniform a priori bounds for the subtracted equation is then
to establish uniform a priori moment bounds for the numerical approximations. We
have been established, we exploit the fact that the nonlinearity of the stochastic Allen–
Cahn equation satisfies a global monotonicity property (see (180)–(182) in the proof
of Lemma 6.9 in this article below and, e.g., [4, Lemma 6.7]) to prevent that the local
discretization errors accumulate too quickly. It thus remains to sketch our procedure
to establish uniform a priori moment bounds for the numerical approximations. We
first subtract the noise process from (3) as it is often done in the literature. The key

is an appropriate path dependent Lyapunov-type function for the system of the
N-dimensional spatial spectral Galerkin approximation of the subtracted equation
associated to the stochastic Allen–Cahn equation (4) (variable \( v \in P_N(H) \)) and the
N-dimensional spatial spectral Galerkin approximation of the Ornstein–Uhlenbeck
process (variable \( w \in C([0, T], L^\infty((0, 1); \mathbb{R})) \)). It is crucial that the Lyapunov-
type function (9) does not only depend on \( w_T \) but on the whole path \( w_t, t \in [0, T] \), of \( w \). Observe that for every realization of the Ornstein–Uhlenbeck process
\( \int_0^T e^{(t-s)A} dW_s, t \in [0, T] \), we have that the systems of the N-dimensional spatial spectral Galerkin approximation of the subtracted equation associated to
the stochastic Allen–Cahn equation (4) and the N-dimensional spatial spectral Galerkin approximation of the Ornstein–Uhlenbeck process is nothing else but a
time dependent PDE (with the time variable entering only through the path \( w \in C([0, T], L^\infty((0, 1); \mathbb{R})) \)) of the Ornstein–Uhlenbeck process. In that sense it is natural
that the Lyapunov-type function which we propose and verify in this article
(see (9) above) depends on the whole path \( w_t, t \in [0, T] \). Details can be found in
the proof of Lemma 6.1 below. Applying the fundamental theorem of calculus to (9)
results, roughly speaking, in the coercivity type condition that there exist real num-
bers \( \varepsilon \in [0, 1) \), \( c \in (0, \infty) \) and \( B(C([0, T], L^\infty((0, 1); \mathbb{R})))/B([0, \infty)) \)-measurable mappings \( \phi, \Phi : C([0, T], L^\infty((0, 1); \mathbb{R})) \rightarrow [0, \infty) \) such that for every \( N \in \mathbb{N}, \)
\( v \in P_N(H) \), \( w \in C([0, T], L^\infty((0, 1); \mathbb{R})) \) we have that

\[
\sup_{t \in [0, T]} \left(\left( (-A)^{1/2} v, (-A)^{1/2} P_N F(v + w_t) \right)_{H} + \phi(w)(v, F(v + w_t))_{H}\right) \\
\leq \varepsilon \| Av \|^2_H + (c + \phi(w)) \left( (-A)^{1/2} v \right)^2_H + c\phi(w)\| v \|^2_H + \Phi(w).
\]

Essentially, the coercivity type condition (10) appears as one of our assumptions
of Theorem 5.5 below (see (84) in Sect. 5.1 below for details). The functions
\( \phi, \Phi : C([0, T], L^\infty((0, 1); \mathbb{R})) \rightarrow [0, \infty) \) in the coercivity type condition (10)
in the case of (4) above can be chosen to satisfy that for all $\epsilon \in (0, 1)$, $c \in \left[ \frac{32}{\epsilon} \max \left\{ \frac{|a_2|^2}{|a_3| + 1}, |a_3| \right\}, \infty \right)$, $w \in C([0, T], H_1)$ it holds that there exists a real number $C \in [0, \infty)$ such that

$$
\phi(w) = C \left[ \sup_{t \in [0, T]} \| w_t \|_{L^\infty(\lambda(0,1); \mathbb{R})}^4 + 1 \right]
$$

and

$$
\Phi(w) = c \left[ \sup_{t \in [0, T]} \| w_t \|_{L^\infty(\lambda(0,1); \mathbb{R})}^8 + 1 \right]
$$

(11)

(see Lemma 6.1 in Sect. 6 and (188)–(189) in the proof of Lemma 6.9 in Sect. 6 below). Our proposal for this specific Lyapunov-type function is partially inspired by the arguments in Section 4 in Bianchi et al. [5] (cf. [5, Theorem 4.1 and Lemma 4.4]).

As mentioned above, in our previous article [4] we establish strong convergence rates for semi-discrete temporal numerical approximations of space-time white noise driven SPDEs with superlinearly growing nonlinearities such as stochastic Allen–Cahn equations. It is, however, unclear how and whether the arguments in [4] can be extended to analyze fully discrete numerical approximation schemes as considered in this article. In [4] the desired regularity is achieved by means of employing an appropriate polynomial Lyapunov-type function. Such polynomial Lyapunov-type functions allow to establish a priori moment bounds for the substracted equation associated to the stochastic Allen–Cahn equation (4) and the Ornstein–Uhlenbeck process, but such polynomial Lyapunov-type functions do not provide a priori moment bounds for its finite dimensional spatial spectral Galerkin approximation. Roughly speaking, the techniques in [4] cannot be applied in this situation as polynomials do not commute with spatial spectral Galerkin projection operators. In this work we overcome this issue by bringing appropriate path dependent Lyapunov-type functions according to (9)–(11) above into play.

After the preprint version of this work has appeared, a series of research articles related to this work have appeared. We refer to Bréhier and Goudenège [8,9], Bréhier et al. [7], Liu and Qiao [28], Wang [39], Qi and Wang [34], and Cui and Hong [11] for details.

The remainder of this article is structured as follows. Section 2 establishes suitable a priori bounds for the numerical approximations. In Sect. 3 the error analysis for the considered nonlinearity-truncated approximation schemes is carried out in the path-wise sense and in Sect. 4 we perform the error analysis for these numerical schemes in the strong $L^p$-sense. In Sect. 5 we combine the results from Sect. 4 with appropriate uniform a priori moment bounds for the numerical approximation processes (see Sect. 2) to establish Theorem 5.5 which is the main result of this article. In Sect. 6 we finally verify that the assumptions of Theorem 5.5 are satisfied in the case of stochastic Allen–Cahn equations.

1.1 Notation

Throughout this article the following notation is used. For every measurable space $(A, \mathcal{A})$ and every measurable space $(B, \mathcal{B})$ we denote by $\mathcal{M}(A, B)$ the set of all $A/B$-measurable functions. For every set $A$ we denote by $\#_A \in \{0, 1, 2, \ldots \} \cup \{\infty\}$ the
number of elements of $A$, we denote by $\mathcal{P}(A)$ the power set of $A$, and we denote by $\mathcal{P}_0(A)$ the set given by $\mathcal{P}_0(A) = \{B \in \mathcal{P}(A): \#B < \infty\}$. For every set $A$ and every set $\mathcal{A}$ with $\mathcal{A} \subseteq \mathcal{P}(A)$ we denote by $\sigma_\mathcal{A}(A)$ the smallest sigma-algebra on $A$ which contains $\mathcal{A}$. For every topological space $(X, \tau)$ we denote by $\mathcal{B}(X)$ the set given by $\mathcal{B}(X) = \sigma_X(\tau)$. For every natural number $d \in \mathbb{N}$ and every set $A \in \mathcal{B}(\mathbb{R}^d)$ we denote by $\lambda_A: \mathcal{B}(A) \to [0, \infty]$ the Lebesgue-Borel measure on $A$. We denote by $[\cdot]_h: \mathbb{R} \to \mathbb{R}$, $h \in (0, \infty)$, the functions which satisfy for all $h \in (0, \infty)$, $t \in \mathbb{R}$ that $[t]_h = \max\{0, h, -h, 2h, -2h, \ldots\} \cap (-\infty, t]$. For every measure space $(\Omega, \mathcal{F}, \nu)$, every measurable space $(S, \mathcal{S})$, every set $R$, and every function $f: \Omega \to R$ we denote by $[f]_{\nu, \mathcal{S}}$ the set given by $[f]_{\nu, \mathcal{S}} = \{g \in \mathcal{M}(\mathcal{F}, \mathcal{S}): (\exists A \in \mathcal{F}: v(A) = 0 \text{ and } \{\omega \in \Omega: f(\omega) \neq g(\omega)\} \subseteq A\}$. For every set $\Omega$ and every set $A$ we denote by $\mathbbm{1}_A^{\Omega}: \Omega \to \mathbb{R}$ the function which satisfies for all $x \in \Omega$ that

$$
\mathbbm{1}_A^{\Omega}(x) = \begin{cases}
1 & : x \in A \\
0 & : x \notin A.
\end{cases}
$$

2 A priori bounds for the numerical approximation

In this section we establish in (19) below pathwise a priori bounds for the difference of the processes $\mathcal{X}: [0, T] \to P(H)$ and $\mathcal{O} \in \mathcal{C}([0, T], P(H))$ in Lemma 2.2 below. In the proof of Lemma 5.4 in Sect. 5.4 below we will employ the pathwise a priori bounds in (19) to establish a priori moment bounds for the stochastic processes $\mathcal{X}^{M, I}: [0, T] \times \Omega \to H_\gamma$, $M \in \mathbb{N}$, $I \in \mathcal{D}$, in (86) in Sect. 5.1 below.

Loosely speaking, the process $\mathcal{O} \in \mathcal{C}([0, T], P(H))$ in Lemma 2.2 below corresponds to one sample path of $\mathcal{O}^{M, N}: [0, T] \to P_N(H)$, $M, N \in \mathbb{N}$, in (2) and the process $\mathcal{X}: [0, T] \to P(H)$ corresponds to one sample path of $\mathcal{X}^{M, N}: [0, T] \times \Omega \to P_N(H)$, $M, N \in \mathbb{N}$, in (3) above. Roughly speaking, in the example of the stochastic Allen–Cahn equation in Theorem 1.1 above and Sect. 6.4 below the functions $\phi: \mathcal{C}([0, T], P(H)) \to [0, \infty)$ and $\Phi: \mathcal{C}([0, T], P(H)) \to [0, \infty)$ on the right hand side of (19) in Lemma 2.2 below correspond to polynomial powers of appropriate norms (see (188)–(189) in the proof of Lemma 6.9 in Sect. 6 below for details). This and the fact that the processes in (2) are Gaussian allow us to uniformly bound the right hand side of (19) in Lemma 2.2 below (see the proof of Lemma 6.9 in Sect. 6 below for details).

Observe that in Lemma 2.1 below we assume that $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, is a family of interpolation spaces associated to $-A$ (cf., e.g., [37, Section 3.7]). This ensures that for all $r \in (0, \infty)$ it holds that $H_r = D((-A)^r) = \{v \in H: \sum_{h \in \mathbb{N}} \|(-\mu_h)^r(h, v)_{H}\|^2 < \infty\}$. In particular, we have that $H_0 = H$, $H_1 = D(A)$, $H_2 = D(A^2)$, $\ldots$. Moreover, observe that in (16) we assume that for all $u, v \in P(H)$ we have that $\|F(u) - F(v)\|^2_{H} \leq C \max\{1, \|u\|^2_{H_\rho}, \|u - v\|^2_{H_\rho} + C\|u - v\|^{2+\varphi}_{H_\rho}\}$. Note that this is equivalent to the condition that for all $u, v \in P(H)$ we have that $\|F(u) - F(v)\|^2_{H} \leq C \max\{1, \min\{\|u\|^2_{H_\rho}, \|v\|^2_{H_\rho}\}\}\|u - v\|^2_{H_\rho} + C\|u - v\|^{2+\varphi}_{H_\rho}$. The following well known lemma is used frequently throughout this article.
Lemma 2.1 Let $\langle H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H \rangle$ be a separable $\mathbb{R}$-Hilbert space, let $H \subseteq H$ be a non-empty orthonormal basis of $H$, let $x \in [0, 1]$, $s \in [0, \infty)$, $\mu: H \rightarrow \mathbb{R}$ satisfy $\sup_{h \in H} \mu_h < 0$, let $A: D(A) \subseteq H \rightarrow H$ be the linear operator which satisfies $D(A) = \{ v \in H: \sum_{h \in H} |\mu_h \langle h, v \rangle_H|_H^2 < \infty \}$ and $\forall v \in D(A): Av = \sum_{h \in H} \mu_h \langle h, v \rangle_Hh$, and let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \| \cdot \|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (cf., e.g., [37, Section 3.7]). Then we have that

$$\|(-sA)^{-x} (e^{sA} - \text{Id}_H) \|_{L(H)} \leq 1$$

and

$$\|(-sA)^{x} e^{sA} \|_{L(H)} \leq 1.$$  \hfill (13) \hfill (14)

Lemma 2.2 Consider the notation in Sect. 1.1, let $\langle H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H \rangle$ be a separable $\mathbb{R}$-Hilbert space, let $H \subseteq H$ be a non-empty orthonormal basis of $H$, let $T$, $\varphi$, $c \in (0, \infty)$, $C \in (0, \infty)$, $\epsilon, \kappa, \rho \in [0, 1)$, $\gamma \in (\rho, 1)$, $\chi \in (0, (\gamma - \rho)/(1 + \gamma)) \cap (0, (1 - \rho)/(1 + \gamma))$, $M \in \mathbb{N}$, $\mu: H \rightarrow \mathbb{R}$ satisfy $\sup_{h \in H} \mu_h < 0$, let $A: D(A) \subseteq H \rightarrow H$ be the linear operator which satisfies $D(A) = \{ v \in H: \sum_{h \in H} |\mu_h \langle h, v \rangle_H|_H^2 < \infty \}$ and $\forall v \in D(A): Av = \sum_{h \in H} \mu_h \langle h, v \rangle_Hh$, and let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \| \cdot \|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (cf., e.g., [37, Section 3.7]), let $I \in \mathcal{P}_0(H)$, $P \in L(H)$ satisfy for all $v \in H$ that $P(v) = \sum_{h \in I} \langle h, v \rangle_Hh$, and let $X: [0, T] \rightarrow P(H)$, $\mathcal{O} \subseteq \mathcal{O}([0, T], H)$, $F \subseteq \mathcal{C}(\mathcal{P}(H), H)$, $\phi, \Phi: \mathcal{C}([0, T], P(H)) \rightarrow [0, \infty)$ satisfy for all $u, v \in P(H)$, $w \in \mathcal{C}([0, T], P(H))$, $t \in [0, T]$ that

$$\|F(u)\|_H^2 \leq C \max\{1, \|u\|_{H^2}^{(2+\gamma)}\},$$

$$\|F(u) - F(v)\|_H^2 \leq C \max\{1, \|u\|_{H^2}^{2\gamma}, \|v\|_{H^2}^{2\gamma}\} \|u - v\|_H^{2\gamma} + C \|u - v\|_{H^{2\gamma}}^{(2+\gamma)},$$

$$\langle v, PF(v + w_t) \rangle_{H_{1/2}} + \phi(w)(v, F(v + w_t))_H \leq \epsilon \|v\|_{H_{1/2}}^2 + \kappa c \phi(w) \|v\|_{H_{1/2}}^2 + \Phi(w),$$

and

$$\mathcal{X}_t = \int_0^t P e^{(t-s)A} \mathbb{I}_{[0,(M/T)\chi]}(\|\mathcal{X}_s\|_{H^2} + \|\mathcal{O}_{[s]T/M} \|_{H^2}) F(\mathcal{X}_{[s]T/M}) ds + \mathcal{O}_t.$$  \hfill (17) \hfill (18)

Then

(i) we have that the function $[0, T] \ni t \mapsto \mathcal{X}_t - \mathcal{O}_t \in P(H)$ is continuous and

(ii) we have that

$$\sup_{t \in [0, T]} \left( \|\mathcal{X}_t - \mathcal{O}_t\|_{H_{1/2}}^2 + \phi(\mathcal{O}) \|\mathcal{X}_t - \mathcal{O}_t\|_H^2 \right) \leq \frac{\epsilon 2^{2cT}}{c} \left( \Phi(\mathcal{O}) + \frac{\max\{1, \phi(\mathcal{O})\} C(c + 1)}{2(1 - \epsilon)(1 - \kappa)} \left[ \frac{\max\{1, T\}(1 + \sqrt{2})}{(1 - \rho)} \right]^{2+\gamma} \right).$$  \hfill (19)
Proof of Lemma 2.2 Throughout this proof assume w.l.o.g. that \( I \neq \emptyset \) and let \( \hat{X} : [0, T] \to P(H) \) and \( Z : [0, T] \to \{0, 1\} \) be the functions which satisfy for all \( s \in [0, T] \) that

\[
\hat{X}_s = X_s - \mathcal{D}_s \quad \text{and} \quad Z_s = \frac{1}{\int_{[0, (M/T)\tau]}(\|X_s\|_{H^1} + \|\mathcal{D}_s\|_{H^1})}.
\]

Observe that, e.g., Lemma 2.4 in [26] (with \( v \mapsto \|v\|_H^2 \in [0, \infty), T = T, \eta = 0, A = P(H) \ni v \mapsto Av \in P(H), \forall = P(H) \ni v \mapsto \|v\|^2_{H_{1/2}} + \phi(\mathcal{D})\|v\|_H^2 \in \mathbb{R}, Z = [0, T] \times \Omega \ni (t, \omega) \mapsto Z_t(\omega) \in \mathbb{R}, Y = \mathcal{X}, O = \mathcal{D}, \Omega = \mathcal{D}, F = P(H) \ni v \mapsto PF(v) \in P(H), \phi = V \ni v \mapsto 2c \in \mathbb{R}, f = V \ni v \mapsto 0 \in \mathbb{R}, h = T/M \) in the notation of Lemma 2.4 in [26]) implies that item (i) holds and that for all \( t \in [0, T] \) it holds that

\[
e^{-2ct} \left[ \|\hat{X}_t\|_{H_{1/2}}^2 + \phi(\mathcal{D})\|\hat{X}_t\|_H^2 \right] - 2c \int_0^t e^{-2cs} \left[ \|\hat{X}_s\|_{H_{1/2}}^2 + \phi(\mathcal{D})\|\hat{X}_s\|_H^2 \right] ds.
\]

The fact that \( P \in L(H) \) is symmetric and the fact that \( V_s \in [0, T] : \hat{X}_s \in P(H) \) imply for all \( t \in [0, T] \) that

\[
e^{-2ct} \left[ \|\hat{X}_t\|_{H_{1/2}}^2 + \phi(\mathcal{D})\|\hat{X}_t\|_H^2 \right] - 2c \int_0^t e^{-2cs} \left[ \|\hat{X}_s\|_{H_{1/2}}^2 + \phi(\mathcal{D})\|\hat{X}_s\|_H^2 \right] ds.
\]
Moreover, note that the triangle inequality implies that for all \( s \in \mathbb{R} : xy \leq x^2/2 + y^2/2 \) prove that for all \( t \in [0, T] \) we have that

\[
e^{-2ct} \left[ \| \vec{x}_t \|^2_{H^{1/2}} + \phi(\mathcal{D}) \| \vec{x}_t \|^2_H \right]
\leq -2 \int_0^t e^{-2cs} \left[ \| \vec{x}_s \|^2_{H^{1/2}} + (c + \phi(\mathcal{D})) \| \vec{x}_s \|^2_{H^{1/2}} + c \phi(\mathcal{D}) \| \vec{x}_s \|^2_H \right] ds
+ 2 \int_0^t e^{-2cs} \left[ \langle \vec{x}_s, P F(\vec{x}_s + \mathcal{D}_{[s]}^{TM}) \rangle_{H^{1/2}} + \phi(\mathcal{D}) \langle \vec{x}_s, F(\vec{x}_s + \mathcal{D}_{[s]}^{TM}) \rangle_H \right] ds
+ \int_0^t e^{-2cs} \left[ 2(1 - \epsilon) \| \vec{x}_s \|^2_{H^{1}} + \frac{1}{2(1 - \epsilon)} \| P \|_{L(H)}^2 \| F(\mathcal{X}_{[s]}^{TM}) - F(\vec{x}_s + \mathcal{D}_{[s]}^{TM}) \|_H^2 \right] ds.
\]

The fact \( \| P \|_{L(H)} \leq 1 \) therefore shows for all \( t \in [0, T] \) that

\[
\| \vec{x}_t \|^2_{H^{1/2}} + \phi(\mathcal{D}) \| \vec{x}_t \|^2_H
\leq -2 \int_0^t e^{2c(t-s)} \left[ e Z_{[s]}^{TM} \| \vec{x}_s \|^2_{H^{1}} + (c + \phi(\mathcal{D})) \| \vec{x}_s \|^2_{H^{1/2}} + c \phi(\mathcal{D}) \| \vec{x}_s \|^2_H \right] ds
+ 2 \int_0^t e^{2c(t-s)} \left[ \langle \vec{x}_s, P F(\vec{x}_s + \mathcal{D}_{[s]}^{TM}) \rangle_{H^{1/2}} + \phi(\mathcal{D}) \langle \vec{x}_s, F(\vec{x}_s + \mathcal{D}_{[s]}^{TM}) \rangle_H \right] ds
+ \left[ \frac{1}{2(1 - \epsilon)} + \frac{\phi(\mathcal{D})}{2c(1 - \kappa)} \right] \int_0^t e^{2c(t-s)} Z_{[s]}^{TM} \| F(\mathcal{X}_{[s]}^{TM}) \| ds.
\]

Moreover, note that the triangle inequality implies that for all \( s \in [0, T] \) we have that

\[
Z_{[s]}^{TM} \| \mathcal{X}_s - \mathcal{X}_{[s]}^{TM} \|_{H^p}
= Z_{[s]}^{TM} \| \mathcal{X}_s - \mathcal{D}_s \| - (\mathcal{X}_{[s]}^{TM} - \mathcal{D}_{[s]}^{TM}) \|_{H^p}
\leq Z_{[s]}^{TM} \| (e^{(s-[s])^{TM}A} - \text{Id}_H)(\mathcal{X}_{[s]}^{TM} - \mathcal{D}_{[s]}^{TM}) \|_{H^p}
+ Z_{[s]}^{TM} \| (\mathcal{X}_s - \mathcal{D}_s) - e^{(s-[s])^{TM}A}(\mathcal{X}_{[s]}^{TM} - \mathcal{D}_{[s]}^{TM}) \|_{H^p}
\leq Z_{[s]}^{TM} \| (A)^{-(y-p)}(e^{(s-[s])^{TM}A} - \text{Id}_H) \|_{L(H)} \| \mathcal{X}_{[s]}^{TM} - \mathcal{D}_{[s]}^{TM} \|_{H^y}
+ Z_{[s]}^{TM} \int_0^s \| Pe^{(s-u)^{TM}}Z_{[u]}^{TM} F(\mathcal{X}_{[u]}^{TM}) \|_{H^p} du.
\]

\( \square \) Springer
Lemma 2.1, the triangle inequality, the fact that \( \|P\|_{L(H)} \leq 1 \), and (15) hence ensure for all \( s \in [0, T] \) that

\[
Z_{|s|T/M} \| \tilde{\mathcal{X}}_s - \tilde{\mathcal{X}}_{|s|T/M} \|_{H_\rho} \\
\leq (s - |s|T/M)^{(\gamma - \rho)} Z_{|s|T/M} \left( \| \mathcal{X}_{|s|T/M} \|_{H_\rho} + \| \mathcal{O}_{|s|T/M} \|_{H_\rho} \right) \\
+ Z_{|s|T/M} \int_{|s|T/M}^s \| P\|_{L(H)} \| (-A)^{\rho} e^{(s-u)A} \|_{L(H)} \| F(\mathcal{X}_{|s|T/M}) \|_{H_\rho} \, du \\
\leq \frac{|T/M|^{(\gamma - \rho)}}{|M/T|^{\chi}} + \sqrt{\mathcal{C}} \int_{|s|T/M}^s (s-u)^{-\rho} Z_{|s|T/M} \max \left\{ 1, \| \mathcal{X}_{|s|T/M} \|_{H_\rho}^{(1+\psi/2)} \right\} \, du \\
\leq \frac{|T/M|^{(\gamma - \rho - \chi)}}{|M/T|^{\chi}} + \sqrt{\mathcal{C}} \max \left\{ 1, \frac{|M/T|^{(1+\psi/2)}}{|T/M|^{1 - \rho}} \right\} \int_{|s|T/M}^s (s-u)^{-\rho} \, du. \tag{26}
\]

Combining this and the fact that \( \forall \chi \in (0, \infty) : \max \{ x^{(\gamma - \rho - \chi)} , \max \{ x^{(1 - \rho)} , x^{(1+\psi/2)} \} \} \leq \max \{ x, x^{(\gamma - \rho - \chi)} , x^{(1+\psi/2)} \} \) shows that for all \( s \in [0, T] \) we have that

\[
Z_{|s|T/M} \| \tilde{\mathcal{X}}_s - \tilde{\mathcal{X}}_{|s|T/M} \|_{H_\rho} \\
\leq \frac{|T/M|^{(\gamma - \rho - \chi)}}{|M/T|^{\chi}} + \sqrt{\mathcal{C}} \max \left\{ 1, \frac{|M/T|^{(1+\psi/2)}}{|T/M|^{1 - \rho}} \right\} \frac{(s - |s|T/M)^{(1-\rho)}}{(1 - \rho)} \\
\leq \frac{1}{(1 - \rho)} \left[ \frac{|T/M|^{(\gamma - \rho - \chi)}}{|M/T|^{\chi}} + \sqrt{\mathcal{C}} \max \left\{ |T/M|^{(1 - \rho)} , |T/M|^{(1 - \rho - (1+\psi/2))} \right\} \right] \tag{27}
\]

Next observe that (16) ensures for all \( s \in [0, T] \) that

\[
Z_{|s|T/M} \| F(\mathcal{X}_{|s|T/M}) - F(\tilde{\mathcal{X}}_s + \mathcal{O}_{|s|T/M}) \|_H^2 \\
\leq C Z_{|s|T/M} \left[ \max \{ 1, \mathcal{X}_{|s|T/M} \|_{H_\rho}^{\psi} \} \| \tilde{\mathcal{X}}_{|s|T/M} - \tilde{\mathcal{X}}_s \|_{H_\rho}^2 + \| \tilde{\mathcal{X}}_{|s|T/M} - \tilde{\mathcal{X}}_s \|^{(2+\psi)}_{H_\rho} \right] \\
\leq C Z_{|s|T/M} \| \tilde{\mathcal{X}}_{|s|T/M} - \tilde{\mathcal{X}}_s \|_{H_\rho}^2 \left[ \max \{ 1, \frac{|M/T|^{\psi/2}}{|T/M|^{1 - \rho}} \} + Z_{|s|T/M} \| \tilde{\mathcal{X}}_{|s|T/M} - \tilde{\mathcal{X}}_s \|_{H_\rho} \right] \\
\leq 2C Z_{|s|T/M} \| \tilde{\mathcal{X}}_{|s|T/M} - \tilde{\mathcal{X}}_s \|_{H_\rho}^2 \left[ \max \{ 1, \frac{|M/T|^{\psi/2}}{|T/M|^{1 - \rho}} \} + Z_{|s|T/M} \| \tilde{\mathcal{X}}_{|s|T/M} - \tilde{\mathcal{X}}_s \|_{H_\rho} \right] \\
= 2C Z_{|s|T/M} \| \tilde{\mathcal{X}}_{|s|T/M} - \tilde{\mathcal{X}}_s \|_{H_\rho}^2 \left[ \max \{ 1, \frac{|M/T|^{\psi/2}}{|T/M|^{1 - \rho}} \} + Z_{|s|T/M} \| \tilde{\mathcal{X}}_{|s|T/M} - \tilde{\mathcal{X}}_s \|_{H_\rho} \right] \tag{28}
\]
This together with (27) proves for all $s \in [0, T]$ that

\[
Z_{[s]_{T/M}} \| F(\mathcal{X}_{[s]_{T/M}}) - F(\tilde{\mathcal{X}}_s + \mathcal{O}_{[s]_{T/M}}) \|_H^2 \\
\leq \frac{2C(1 + \sqrt{C})^{2+\varphi}}{(1 - \rho)^{2+\varphi}} \left| \max \left\{ \frac{T/M}{|T/M|}, \frac{|T/M| \gamma - \rho - \chi, |T/M| (1 - \rho - (1 + \varphi/2) \chi) \right\} \right|^2 \\
\cdot \left| \max \left\{ 1, |M/T|, |T/M| \gamma - \rho - \chi, |T/M| (1 - \rho - (1 + \varphi/2) \chi) \right\} \right|^\varphi.
\]

(29)

In addition, note that the assumption that $\chi \in (0, (\gamma - \rho)/(1 + \varphi/2)] \cap (0, (1 - \rho)/(1 + \varphi)]$ ensures that

\[
\gamma - \rho - \chi \in (0, 1), \quad 1 - \rho - (1 + \varphi/2) \chi \in (0, 1),
\]

(30)

and

\[
\min \{ \gamma - \rho - (1 + \varphi/2) \chi, 1 - \rho - (1 + \varphi) \chi \} \in [0, 1).
\]

(31)

This implies that for all $h \in (0, 1]$ we have that

\[
\left| \max \left\{ h, h^{(1 - \rho - (1 + \varphi/2) \chi)} \right\} \right|^2 \left| \max \left\{ h, h^{(1 - \rho - (1 + \varphi/2) \chi)} \right\} \right|^\varphi \\
= h^2 \min \{ \gamma - \rho - (1 + \varphi/2) \chi, 1 - \rho - (1 + \varphi) \chi \} \leq 1.
\]

(32)

Moreover, observe that (30) shows for all $h \in (1, \infty)$ that

\[
\left| \max \left\{ h, h^{(1 - \rho - (1 + \varphi/2) \chi)} \right\} \right|^2 \left| \max \left\{ h, h^{(1 - \rho - (1 + \varphi/2) \chi)} \right\} \right|^\varphi \\
= h^{2+\varphi}.
\]

(33)

Combining (29) with (32) and (33) yields that for all $s \in [0, T]$ we have that

\[
Z_{[s]_{T/M}} \| F(\mathcal{X}_{[s]_{T/M}}) - F(\tilde{\mathcal{X}}_s + \mathcal{O}_{[s]_{T/M}}) \|_H^2 \leq 2C \left[ \max \{1, |T/M| \} \frac{1 + \sqrt{C}}{(1 - \rho)} \right]^{2+\varphi}.
\]

(34)

Furthermore, note that (17) and the assumption that $\mathcal{O} \in C([0, T], P(H))$ hence guarantee for all $s \in [0, T]$ that

\[
\hat{\text{Springer}}
\]
Hence, we obtain that the proof of Lemma 2.2 is thus completed. \(\square\)
3 Pathwise error estimates

3.1 Setting

Consider the notation in Sect. 1.1, let \((H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)\) be a separable \(\mathbb{R}\)-Hilbert space, let \(\mathbb{H} \subseteq H\) be a non-empty orthonormal basis of \(H\), let \(T, c, \varphi \in (0, \infty), C \in [0, \infty), M \in \mathbb{N}, \mu : \mathbb{H} \to \mathbb{R}\) satisfy \(\sup_{h \in \mathbb{H}} \mu_h < 0\), let \(A : D(A) \subseteq H \to H\) be the linear operator which satisfies \(D(A) = \{v \in H : \sum_{h \in \mathbb{H}} \mu_h \langle h, v \rangle_H h \|^2 < \infty\}\) and \(\forall v \in D(A) : Av = \sum_{h \in \mathbb{H}} \mu_h \langle h, v \rangle_H h\), let \((V, \|\cdot\|_V)\) be an \(\mathbb{R}\)-Banach space with \(D(A) \subseteq V \subseteq H\) continuously and densely, and let \(O, D, X, \mathcal{X} : [0, T] \to V\) and \(V : V \times V \to [0, \infty)\) be functions, and let \(X \in \mathcal{C}([0, T], V), F \in \mathcal{C}(V, H), I \in \mathcal{P}_0(\mathbb{H}), P \in L(H)\) satisfy the following elementary result, \([2, \text{Section 3.3}]\).

\[ P(\varphi) = \sum_{h \in I} \langle h, \varphi \rangle_H h, \quad \langle v - w, Av + F(v) - Aw - F(w) \rangle_H \leq c \|v - w\|^2_H, \]

\[ \|F(v) - F(w)\|^2_H \leq C \|v - w\|^2_H (1 + \|v\|^2_V + \|w\|^2_V), \]

\[ X_t = \int_0^t e^{(t-s)A} F(X_s) ds + O_t, \quad \mathcal{X}_t = \int_0^t P e^{(t-s)A} \mathbb{E}_{[0, M]_T} (\mathcal{V}(\mathcal{X}_{[s]}_T M, \mathcal{D}_{[s]}_T M)) F(\mathcal{X}_{[s]}_T M) ds + \mathcal{O}_t. \]

3.2 On the separability of a certain Banach space

The next elementary lemma, Lemma 3.1, ensures that the \(\mathbb{R}\)-Banach space \((V, \|\cdot\|_V)\) in Sect. 3.1 is separable. Lemma 3.1 is well-known in the literature. A proof of Lemma 3.1 can, e.g., be found in the extended arXiv version of this article \([2, \text{Section 3.2}]\).

**Lemma 3.1** Let \((V, \|\cdot\|_V)\) be a separable \(\mathbb{R}\)-Banach space and let \((W, \|\cdot\|_W)\) be an \(\mathbb{R}\)-Banach space with \(V \subseteq W\) continuously and densely. Then \((W, \|\cdot\|_W)\) is a separable \(\mathbb{R}\)-Banach space.

3.3 Analysis of the error between the Galerkin projection of the exact solution and the Galerkin projection of the semilinear integrated version of the numerical approximation

In our error analysis in Lemma 3.3 below we employ the following elementary result, Lemma 3.2 below, on mild solutions of certain semilinear evolution equations. A proof of Lemma 3.2 can, e.g., be found in the extended arXiv version of this article \([2, \text{Section 3.3}]\).

\[ \text{We think of the function } [0, T] \ni t \mapsto O_t \in V \text{ in (41)} \text{ as one fixed trajectory of the stochastic convolution process driving the exact solution of the SPDE under consideration and we think of } [0, T] \ni t \mapsto \mathcal{O}_t \in V \text{ in (41)} \text{ as one fixed realization of a numerical approximation process of the function } [0, T] \ni t \mapsto O_t \in V \text{ in (41).} \]
Lemma 3.2 Consider the notation in Sect. 1.1, let \((V, \| \cdot \|_V)\) be a separable \(\mathbb{R}\)-Banach space, and let \(A \in L(V)\), \(T \in (0, \infty)\), \(Y : [0, T] \to V, Z \in C([0, T], V)\) satisfy for all \(t \in [0, T]\) that \(Y_t = \int_0^t e^{(t-s)A} Z_s \, ds\). Then

(i) we have that \(Y\) is continuously differentiable,
(ii) we have for all \(t \in [0, T]\) that \(Y_t = \int_0^t (AY_s + Z_s) \, ds\), and
(iii) we have for all \(t \in [0, T]\) that \(\frac{d}{dt}Y_t = AY_t + Z_t\).

Lemma 3.3 Assume the setting in Sect. 3.1 and let \(\kappa \in (2, \infty), t \in [0, T]\). Then

\[
\|PX_t - X_t\|_H^2 \leq \frac{Ce^{\kappa T}}{(\kappa - 2) c} \int_0^t \left[ \|X_s - PX_s\|_V + \|X_s - X_{[s]T,M}\|_V \right]^2 \cdot \left( 1 + \|X_s\|_V^p + \|PX_s\|_V^p + \|X_s\|_V^q + \|X_{[s]T,M}\|_V^q \right) \, ds.
\] (43)

Proof of Lemma 3.3 Throughout this proof assume w.l.o.g. that \(t \in (0, T)\) (otherwise the proof is clear). Observe that Lemma 3.2 (with \(V = P(H), A = (P(H) \ni v \mapsto Av \in P(H)), T = T, Y = (0, T) \ni s \mapsto P(X_s - O_s) \in P(H)\), \(Z = (0, T) \ni s \mapsto PF(X_s) \in P(H)\)) in the notation of Lemma 3.2 shows that the function \([0, T) \ni s \mapsto P(X_s - O_s) \in P(H)\) is continuously differentiable and that for all \(s \in [0, T]\) we have that

\[
P(X_s - O_s) = \int_0^s AP(X_u - O_u) + PF(X_u) \, du.
\] (44)

Next note that Lemma 2.1 in Hutzenthaler et al. [21] (with \(V = H, A = A, T = T, h = T/M, Y = (0, T) \ni s \mapsto X_s - PO_s \in H, Z = (0, T) \ni s \mapsto PF(X_s) \in H\)) in the notation of Lemma 2.1 in Hutzenthaler et al. [21]) implies that for all \(s \in [0, T]\) we have that \(X_s - PO_s \in D(A)\), that the function \([0, T) \ni s \mapsto X_s - PO_s \in D(A)\) is continuous, that the function \([0, T) \setminus \{0, \frac{T}{M}, \frac{2T}{M}, \ldots\} \ni s \mapsto X_s - PO_s \in H\) is continuously differentiable, and that for all \(s \in [0, T]\) we have that

\[
X_s - PO_s = \int_0^s AP(X_u - PO_u) + PF(X_{[u]T/M}) \, du.
\] (45)

This, (44), the fundamental theorem of calculus, and the fact that

\[
\forall s \in [0, T] : APX_s, AX_s \in P(H)
\] (46)

prove that

\[
e^{-\kappa t} \|PX_t - X_t\|_H^2 = e^{-\kappa t} \|P(X_t - O_t) - (X_t - PO_t)\|_H^2
\]
\[
= 2 \int_0^t e^{-\kappa s} \langle PX_s - X_s, AP(X_s - O_s) + PF(X_s) - A(X_s - PO_s) - PF(X_{[s]T/M}) \rangle \, ds
\]
\[
- \kappa c \int_0^t e^{-\kappa s} \|PX_s - X_s\|_H^2 \, ds
\]

\[\square\]
\[= 2 \int_0^t e^{-\kappa c s} \left( PX_s - X_s, APX_s + PF(X_s) - AX_s - PF(\mathcal{X}_{[s]T/M}) \right)_H ds \]
\[-\kappa c \int_0^t e^{-\kappa c s} ||PX_s - X_s||_H^2 ds \]
\[= 2 \int_0^t e^{-\kappa c s} \left( PX_s - X_s, APX_s + F(X_s) - AX_s - F(\mathcal{X}_{[s]T/M}) \right)_H ds \]
\[-\kappa c \int_0^t e^{-\kappa c s} ||PX_s - X_s||_H^2 ds. \quad (47)\]

The fact that \( P \in L(H) \) is symmetric together with the fact that
\[\forall s \in [0, T]: PX_s, X_s \in P(H) \quad (48)\]

therefore ensures that
\[e^{-\kappa c t} ||PX_t - X_t||_H^2 \]
\[\leq 2c \int_0^t e^{-\kappa c s} ||PX_s - X_s||_H^2 ds - \kappa c \int_0^t e^{-\kappa c s} ||PX_s - X_s||_H^2 ds \]
\[+ 2 \int_0^t e^{-\kappa c s} ||PX_s - X_s||_H ||F(X_s) - F(PX_s) + F(X_s) - F(\mathcal{X}_{[s]T/M})||_H ds \]
\[= (2 - \kappa) c \int_0^t e^{-\kappa c s} ||PX_s - X_s||_H^2 ds + 2 \int_0^t e^{-\kappa c s} \left[ \sqrt{(\kappa - 2) c} ||PX_s - X_s||_H \right] ds \]
\[\leq (2 - \kappa) c \int_0^t e^{-\kappa c s} ||PX_s - X_s||_H^2 ds + \int_0^t e^{-\kappa c s} \left[ (\kappa - 2) c ||PX_s - X_s||_H^2 \right] ds. \quad (50)\]
The triangle inequality, (40), the assumption that \( D(A) \subseteq V \) densely, and the assumption that \( F \in C(V, H) \) therefore yield that

\[
e^{-\kappa c t} \| PX_t - X_t \|_H^2 \leq \frac{1}{(\kappa - 2) c} \int_0^t e^{-\kappa c s} \left[ \| F(X_s) - F(PX_s) \|_H + \| F(X_s) - F(\mathcal{X}_{[s]T/M}) \|_H \right]^2 ds \leq \frac{C}{(\kappa - 2) c} \int_0^t e^{-\kappa c s} \left[ \| X_s - PX_s \|_V \sqrt{1 + \| X_s \|_V^2 + \| PX_s \|_V^2} \right. \\

+ \left. \| X_s - \mathcal{X}_{[s]T/M} \|_V \sqrt{1 + \| X_s \|_V^2 + \| \mathcal{X}_{[s]T/M} \|_V^2} \right]^2 ds. \tag{51}
\]

This completes the proof of Lemma 3.3. \( \square \)

### 3.4 Analysis of the error between the numerical approximation and the Galerkin projection of the semilinear integrated version of the numerical approximation

For the formulation of the next lemma we recall that Setting 3.1 ensures that \( \mathcal{V} : V \times V \to [0, \infty) \) is a function from \( V \times V \) to \( [0, \infty) \).

**Lemma 3.4** Assume the setting in Sect. 3.1 and let \( \alpha \in (0, \infty) \), \( \rho \in [0, 1) \), \( t \in [0, T] \) satisfy \( \sup_{s \in [0, T]} s^\rho \| e^{sA} \| _{L(H,V)} < \infty \). Then

\[
\| X_t - \mathcal{X}_t \|_V \leq \frac{T^\alpha}{M^{\alpha}} \left[ \sup_{s \in (0,T)} s^\rho \| e^{sA} \| _{L(H,V)} \right] \left[ \int_0^t (t - s)^{-\rho} \| \mathcal{V}(\mathcal{X}_{[s]T/M}, \mathcal{O}_{[s]T/M}) \|_\alpha \| F(\mathcal{X}_{[s]T/M}) \|_H ds + \| PO_t - \mathcal{O}_t \|_V \right]. \tag{52}
\]

**Proof of Lemma 3.4** Throughout this proof we assume w.l.o.g. that \( t \in (0, T) \) (otherwise the proof is clear). Observe that

\[
\| X_t - \mathcal{X}_t \|_V \leq \int_0^t \| P e^{(t - s)A} \left[ 1 - \mathbb{1}_{[0,M/T]}(\mathcal{V}(\mathcal{X}_{[s]T/M}, \mathcal{O}_{[s]T/M})) \right] F(\mathcal{X}_{[s]T/M}) \|_V ds + \| PO_t - \mathcal{O}_t \|_V \leq \left[ \sup_{s \in (0,T)} s^\rho \| e^{sA} \| _{L(H,V)} \right] \int_0^t (t - s)^{-\rho} \| \mathbb{1}_{(M/T, \infty)}(\mathcal{V}(\mathcal{X}_{[s]T/M}, \mathcal{O}_{[s]T/M})) \|_H ds + \| PO_t - \mathcal{O}_t \|_V \\

\left[ \frac{T}{M} \mathcal{V}(\mathcal{X}_{[s]T/M}, \mathcal{O}_{[s]T/M}) \right]^\alpha \| P \|_{L(H)} \| F(\mathcal{X}_{[s]T/M}) \|_H ds + \| PO_t - \mathcal{O}_t \|_V. \tag{53}
\]
This and the fact that \( \| P \|_{L(H)} \leq 1 \) complete the proof of Lemma 3.4. \( \Box \)

### 3.5 Temporal regularity for the Galerkin projection of the semilinear integrated version of the numerical approximation

**Lemma 3.5** Assume the setting in Sect. 3.1 and let \( \rho \in [0, 1) \), \( \varrho \in [0, 1 - \rho) \), \( t_1 \in [0, T) \), \( t_2 \in (t_1, T] \) satisfy \( \sup_{s \in (0, T)} s^\rho \| e^A_s \|_{L(H, V)} < \infty \). Then

\[
\| X_{t_2} - X_{t_1} \|_V \\
\leq \left[ \sup_{s \in (0, T)} s^\rho \| e^A_s \|_{L(H, V)} \right] \left[ \int_{t_1}^{t_2} (t_2 - s)^{-\rho} \| F(X_{[s,T]M}) \|_H \, ds \right] + 2^{\rho + \varrho} (t_2 - t_1)^\varrho \int_{0}^{t_1} (t_1 - s)^{-(\rho + \varrho)} \| F(X_{[s,T]M}) \|_H \, ds + \| P(O_{t_2} - O_{t_1}) \|_V.
\]

**Proof of Lemma 3.5** Observe that

\[
\| X_{t_2} - X_{t_1} \|_V \\
\leq \int_{t_1}^{t_2} \| P e^{(t_2-s)A} F(X_{[s,T]M}) \|_V \, ds + \int_{0}^{t_1} \| P (e^{(t_2-s)A} - e^{(t_1-s)A}) F(X_{[s,T]M}) \|_V \, ds + \| P(O_{t_2} - O_{t_1}) \|_V
\]

\[
\leq \left[ \sup_{s \in (0, T)} s^\rho \| e^A_s \|_{L(H, V)} \right] \left[ \int_{t_1}^{t_2} (t_2 - s)^{-\rho} \| P F(X_{[s,T]M}) \|_H \, ds \right] + \| P(O_{t_2} - O_{t_1}) \|_V
\]

\[
+ \left[ \sup_{s \in (0, T)} s^\rho \| e^A_s \|_{L(H, V)} \right] \left[ \int_{0}^{t_1} \left[ \frac{2^{\rho + \varrho} (t_1 - s)^{-(\rho + \varrho)}}{t_1 - s} \right]^\rho \| P e^{\frac{1}{2} (t_1-s)A} (e^{(t_2-t_1)A} - \text{Id}_H) F(X_{[s,T]M}) \|_H \, ds \right].
\]

This, Lemma 2.1, and the fact that \( \| P \|_{L(H)} \leq 1 \) prove that

\[
\| X_{t_2} - X_{t_1} \|_V \\
\leq \left[ \sup_{s \in (0, T)} s^\rho \| e^A_s \|_{L(H, V)} \right] \left[ \int_{t_1}^{t_2} (t_2 - s)^{-\rho} \| P \|_{L(H)} \| F(X_{[s,T]M}) \|_H \, ds \right] + \| P(O_{t_2} - O_{t_1}) \|_V
\]

\[
+ 2^\rho \left[ \sup_{s \in (0, T)} s^\rho \| e^A_s \|_{L(H, V)} \right] \left[ \int_{0}^{t_1} (t_1 - s)^{-\rho} \| (-A)^{-\varrho} e^{\frac{1}{2} (t_1-s)A} \|_{L(H)} \| P \|_{L(H)} \| F(X_{[s,T]M}) \|_H \, ds \right]
\]

\[
\cdot \| (-A)^{-\varrho} (e^{(t_2-t_1)A} - \text{Id}_H) \|_{L(H)} \| P \|_{L(H)} \| F(X_{[s,T]M}) \|_H \, ds \right] + \| P(O_{t_2} - O_{t_1}) \|_V
\]

\[
\leq \left[ \sup_{s \in (0, T)} s^\rho \| e^A_s \|_{L(H, V)} \right] \left[ \int_{t_1}^{t_2} (t_2 - s)^{-\rho} \| F(X_{[s,T]M}) \|_H \, ds \right] + \| P(O_{t_2} - O_{t_1}) \|_V
\]

\[
+ 2^{\rho + \varrho} (t_2 - t_1)^\varrho \left[ \sup_{s \in (0, T)} s^\rho \| e^A_s \|_{L(H, V)} \right] \left[ \int_{0}^{t_1} (t_1 - s)^{-(\rho + \varrho)} \| F(X_{[s,T]M}) \|_H \, ds \right].
\]

The proof of Lemma 3.5 is thus completed. \( \Box \)
4 Strong error estimates

4.1 Setting

Consider the notation in Sect. 1.1, let \((H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)\) be a separable \(\mathbb{R}\)-Hilbert space, let \(\mathbb{H} \subseteq H\) be a non-empty orthonormal basis of \(H\), let \(T, c, C, \varphi \in (0, \infty), D \subseteq \mathcal{P}_0(\mathbb{H}), \mu : \mathbb{H} \to \mathbb{R}\) satisfy \(\sup_{h \in \mathbb{H}} \mu_h < 0\), let \(A : D(A) \subseteq H \to H\) be the linear operator which satisfies \(D(A) = \{v \in H : \sum_{h \in \mathbb{H}} |\mu_h \langle h, v \rangle_H|^2 < \infty\}\) and \(\forall v \in D(A) : Av = \sum_{h \in \mathbb{H}} \mu_h \langle h, v \rangle_H H\), let \((V, \| \cdot \|_V)\) be an \(\mathbb{R}\)-Banach space with \(D(A) \subseteq V \subseteq H\) continuously and densely, let \(F \in \mathcal{C}(V, H)\), \((P_I)_{I \in \mathcal{P}(\mathbb{H})} \subseteq L(H)\) satisfy for all \(v, w \in D(A), I \in \mathcal{P}(\mathbb{H})\) that

\[
\langle v - w, Av + F(v) - Aw - F(w) \rangle_H \leq c \|v - w\|_H^2, \tag{57}
\]

\[
\|F(v) - F(w)\|_H^2 \leq C \|v - w\|_V^2(1 + \|v\|_V + \|w\|_V^2), \tag{58}
\]

and \(P_I(v) = \sum_{h \in I} \langle h, v \rangle_H h\), let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, let \(X : [0, T] \times \Omega \to V\) be a stochastic process with continuous sample paths, let \(X_{M,I} : [0, T] \times \Omega \to V, M \in \mathbb{N}, I \in \mathcal{D}\), and \(X_{M,I} : [0, T] \times \Omega \to V, M \in \mathbb{N}, I \in \mathcal{D}\), be stochastic processes, let \(O : [0, T] \times \Omega \to V\) and \(O_{M,I} : [0, T] \times \Omega \to V, M \in \mathbb{N}, I \in \mathcal{D}\), be stochastic processes, let \(V \in \mathcal{M}(\mathcal{B}(V \times V), \mathcal{B}([0, \infty)))\), and assume for all \(t \in [0, T], M \in \mathbb{N}, I \in \mathcal{D}\) that

\[
X_t = \int_0^t e^{(t-s)A} F(X_s) ds + O_t, \quad X_{t,I}^M = \int_0^t P_I e^{(t-s)A} F(X_{[s]T,M}^I) ds + P_I O_t, \quad \tag{59}
\]

and

\[
X_{t,I}^M = \int_0^t P_I e^{(t-s)A} \mathbb{1}_{\Omega\{X_{[s]T,M}^I,M \sim M/T\}} F(X_{[s]T,M}^I) ds + O_{t,I}^M. \quad \tag{60}
\]

4.2 Analysis of the error between the Galerkin projection of the exact solution and the Galerkin projection of the semilinear integrated version of the numerical approximation

Lemma 4.1 Assume the setting in Sect. 4.1 and let \(\kappa \in (2, \infty), t \in [0, T], p \in [2, \infty), M \in \mathbb{N}, I \in \mathcal{D}\) satisfy \(\sup_{s \in (0, T)} \mathbb{E}\left[\|X_s\|_V^p + \|P_I X_s\|_V^p + \|X_{s}^{M,I}\|_V^p + \|X_{[s]T,M}^{M,I}\|_V^p\right] < \infty\). Then

\[
\|P_I X_t - X_{t,I}^M\|_{L^p(\mathbb{P}; H)} \leq \frac{\sqrt{C_T e^{\kappa T}}}{\sqrt{(\kappa - 2) c}} \sup_{s \in (0, T)} \left(\|P_I X_s\|_{L^{2p}(\mathbb{P}; V)} + \|X_{s}^{M,I}\|_{L^{2p}(\mathbb{P}; V)} + \|X_{[s]T,M}^{M,I}\|_{L^{2p}(\mathbb{P}; V)}\right)
\]

\[\square\]
\[
1 + \sup_{s \in (0, T)} \left( \| X_s \|_{L^p(P; V)}^{\psi/2} + \| P_I X_s \|_{L^p(P; V)}^{\psi/2} + \| X_s^{M,I} \|_{L^p(P; V)}^{\psi/2} + \| X_s^{M,I} \|_{L^p(P; V)}^{\psi/2} \right).
\]

(61)

**Proof of Lemma 4.1** Note that Lemma 3.3 and Hölder’s inequality ensure that

\[
\| P_I X_t - X_t^{M,I} \|_{L^p(P; H)}^2 \leq \frac{C e^{\kappa c T}}{(k - 2) c} \int_0^t \left[ \| X_s - P_I X_s \|_V + \| X_s^{M,I} - \tilde{X}_s^{M,I} \|_{\mathcal{L}^p(P; H)}^2 \right] \cdot \left[ 1 + \| X_s \|_V^{\psi} + \| P_I X_s \|_V^{\psi} + \| X_s^{M,I} \|_V^{\psi} + \| \tilde{X}_s^{M,I} \|_V^{\psi} \right] \, ds
\]

\[
\leq \frac{C e^{\kappa c T}}{(k - 2) c} \int_0^t \left[ \| X_s - P_I X_s \|_V + \| X_s^{M,I} - \tilde{X}_s^{M,I} \|_V \right] \cdot \left[ 1 + \| X_s \|_V^{\psi} + \| P_I X_s \|_V^{\psi} + \| X_s^{M,I} \|_V^{\psi} + \| \tilde{X}_s^{M,I} \|_V^{\psi} \right] \, ds.
\]

(62)

This shows that

\[
\| P_I X_t - X_t^{M,I} \|_{L^p(P; H)}^2 \leq \frac{C e^{\kappa c T}}{(k - 2) c} \int_0^t \left[ \| X_s - P_I X_s \|_V + \| X_s^{M,I} - \tilde{X}_s^{M,I} \|_V \right]^2 \cdot \left[ 1 + \sup_{s \in (0, T)} \left( \| X_s \|_{L^p(P; V)}^{\psi} + \| P_I X_s \|_{L^p(P; V)}^{\psi} + \| X_s^{M,I} \|_{L^p(P; V)}^{\psi} + \| \tilde{X}_s^{M,I} \|_{L^p(P; V)}^{\psi} \right) \right] \, ds
\]

\[
\leq \frac{CT e^{\kappa c T}}{(k - 2) c} \int_0^t \left[ \| X_s - P_I X_s \|_V + \| X_s^{M,I} - \tilde{X}_s^{M,I} \|_V \right]^2 \cdot \left[ 1 + \sup_{s \in (0, T)} \left( \| X_s \|_{L^p(P; V)}^{\psi} + \| P_I X_s \|_{L^p(P; V)}^{\psi} + \| X_s^{M,I} \|_{L^p(P; V)}^{\psi} + \| \tilde{X}_s^{M,I} \|_{L^p(P; V)}^{\psi} \right) \right] \, ds.
\]

(63)

Combining this with the fact that \( \forall n \in \mathbb{N}, x_1, \ldots, x_n \in [0, \infty) : \sqrt{x_1 + \ldots + x_n} \leq \sqrt{x_1} + \ldots + \sqrt{x_n} \) completes the proof of Lemma 4.1.

\[\square\]

### 4.3 Analysis of the error between the numerical approximation and the Galerkin projection of the semiflinear integrated version of the numerical approximation

**Lemma 4.2** Assume the setting in Sect. 4.1 and let \( \alpha \in (0, \infty) \), \( \rho \in [0, 1) \), \( t \in [0, T] \), \( p \in [1, \infty) \), \( M \in \mathbb{N} \), \( I \in \mathcal{D} \) satisfy \( \sup_{s \in (0, T)} \| e^{sA} \|_{L(H, V)}^2 \rho < \infty \) and \( \sup_{s \in (0, T)} \mathbb{E} \left[ \| \gamma(X_{t,s}^{M,I}, \xi_{t,s}^{M,I}) \|_{L^2(A)}^2 \right] < \infty. \) Then

\[\text{□ Springer}\]
\[
\|X^M_{t_1} - X^M_{t_2}\|_{\mathcal{L}^p(\mathbb{P}; V)} \\
\leq \frac{T^{(1+\alpha-\rho)}}{(1-\rho)M^\alpha} \left[ \sup_{s \in (0, T)} s^\rho \|e^{sA}\|_{L(H, V)} \right] \\
\left[ \sup_{s \in (0, T)} \left( \|V(\mathcal{X}_s^M, \mathcal{D}_s^M)\|_{L^{2p\alpha}(\mathbb{P}; \mathbb{R})} \|F(\mathcal{X}_s^M)\|_{L^2(\mathbb{P}; H)} \right) \right] \\
+ \|P_1 O_{t_1} - \mathcal{D}_{t_1}^M\|_{\mathcal{L}^p(\mathbb{P}; V)}.
\]

(64)

**Proof of Lemma 4.2** Note that Lemma 3.4 and Hölder’s inequality imply that

\[
\|X^M_{t_1} - X^M_{t_2}\|_{\mathcal{L}^p(\mathbb{P}; V)} \\
\leq \frac{T^{\alpha}}{M^\alpha} \left[ \sup_{s \in (0, T)} s^\rho \|e^{sA}\|_{L(H, V)} \right] \\
\left[ \int_0^t (t-s)^{-\rho} \left( \mathbb{E} \left[ \|V(\mathcal{X}_s^M, \mathcal{D}_s^M)\|^p_{L^{2p\alpha}(\mathbb{P}; \mathbb{R})} \|F(\mathcal{X}_s^M)\|^p_H \right] \right)^{\frac{1}{p'}} \, ds \right]^{\frac{1}{p'}} \\
+ \|P_1 O_t - \mathcal{D}_{t_1}^M\|_{\mathcal{L}^p(\mathbb{P}; V)} \\
\leq \frac{T^{\alpha}}{M^\alpha} \left[ \sup_{s \in (0, T)} s^\rho \|e^{sA}\|_{L(H, V)} \right] \\
\cdot \left[ \int_0^t (t-s)^{-\rho} \left( \mathbb{E} \left[ \|V(\mathcal{X}_s^M, \mathcal{D}_s^M)\|^2_{L^{2p\alpha}(\mathbb{P}; \mathbb{R})} \|F(\mathcal{X}_s^M)\|^2_H \right] \right)^{\frac{1}{2p}} \, ds \right]^{\frac{1}{2p}} \\
+ \|P_1 O_t - \mathcal{D}_{t_1}^M\|_{\mathcal{L}^p(\mathbb{P}; V)}.
\]

This and the fact that \( \int_0^t (t-s)^{-\rho} \, ds = \frac{t^{(1-\rho)}}{(1-\rho)} \) complete the proof of Lemma 4.2. □

### 4.4 Temporal regularity for the Galerkin projection of the semilinear integrated version of the numerical approximation

**Lemma 4.3** Assume the setting in Sect. 4.1 and let \( \rho \in [0, 1), \varrho \in [0, 1 - \rho), t_1 \in [0, T), t_2 \in (t_1, T], p \in [1, \infty), M \in \mathbb{N}, I \in \mathcal{D} \) satisfy \( \sup_{s \in (0, T)} s^\rho \|e^{sA}\|_{L(H, V)} < \infty \). Then

\[
\|X^M_{t_1} - X^M_{t_2}\|_{\mathcal{L}^p(\mathbb{P}; V)} \\
\leq \|P_1 (O_{t_1} - O_{t_2})\|_{\mathcal{L}^p(\mathbb{P}; V)} \\
+ \frac{3T^{(1-\rho-\varrho)}}{(1-\rho-\varrho)} \left[ \sup_{s \in (0, T)} s^\rho \|e^{sA}\|_{L(H, V)} \right] \left[ \sup_{s \in [0, T)} \|F(\mathcal{X}_s^M)\|_{\mathcal{L}^p(\mathbb{P}; H)} \right].
\]

(66)
Proof of Lemma 4.3  Observe that Lemma 3.5 proves that
\[
\|X^{M,I}_{t}\|_{\mathcal{L}^p(\mathbb{P};V)} - \|X^{M,I}_{t}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
\leq \left[ \sup_{s \in (0,T)} s^\rho \|e^{sA}\|_{L(H,V)} \right] \left[ \sup_{s \in [0,T]} \|F(X^{M,I}_s)\|_{\mathcal{L}^p(\mathbb{P};H)} \right] \int_{t_1}^{t_2} (t_2 - s)^{-\rho} ds \\
+ 2(\rho + q) (t_2 - t_1)^\rho \int_{0}^{t_1} (t_1 - s)^{-(\rho + q)} ds \\
+ \|P_t (O_{t_2} - O_{t_1})\|_{\mathcal{L}^p(\mathbb{P};V)} \\
= \left[ \sup_{s \in (0,T)} s^\rho \|e^{sA}\|_{L(H,V)} \right] \left[ \sup_{s \in [0,T]} \|F(X^{M,I}_s)\|_{\mathcal{L}^p(\mathbb{P};H)} \right] \\
\cdot (t_2 - t_1)^{(1-\rho)} (1 - \rho) \\
\cdot \frac{2(\rho + q) (t_2 - t_1)^\rho |t_1|}{(1 - \rho - q)} \\
+ \|P_t (O_{t_2} - O_{t_1})\|_{\mathcal{L}^p(\mathbb{P};V)}. \\
(67)
\]

This completes the proof of Lemma 4.3.

4.5 Analysis of the error between the exact solution and the numerical approximation

Proposition 4.4 Assume the setting in Sect. 4.1 and let \( \alpha \in (0, \infty) \), \( \rho \in [0, 1) \), \( \varphi \in [0, 1 - \rho) \), \( \kappa \in (2, \infty) \), \( p \in [\max\{2, \varphi\}, \infty) \), \( M \in \mathbb{N} \), \( I \in \mathcal{D} \). Then
\[
\sup_{t \in [0,T]} \|X_t - \bar{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
\leq 4^{(2+\varphi)} \max\{1, T^{(\gamma/2 + \alpha - \rho + \varphi/2 - \rho\varphi/2)}\} \max\{1, C^{(1+\varphi/2)} \sqrt{c_T} \min\{1, \sqrt{c} (\kappa - 2)\} (1 - \rho - \varphi)^{(1+\varphi/2)} \}
\]
\[
+ \sup_{t \in (0,T)} \|P_t (O_t - O_{[1]T,M})\|_{\mathcal{L}^2_p(\mathbb{P};V)} + \sup_{t \in [0,T]} \|P_t O_t - \Omega^{M,I}_t\|_{\mathcal{L}^2_p(\mathbb{P};V)} \\
\cdot \max\left\{1, \sup_{t \in V \setminus [0,T]} \frac{\|v\|_H}{\|v\|_V} \right\}
\]
\[
\cdot \left[ 1 + \sup_{s \in (0,T)} \|X_{s}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \sup_{s \in (0,T)} \|P_s \bar{X}_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \sup_{s \in [0,T)} \|\bar{X}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \\
+ \sup_{s \in (0,T)} \|P_s O_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \right] \\
\cdot \left[ 1 + \sup_{s \in (0,T)} \|\nabla(\bar{X}_s^{M,I}, \Omega^{M,I}_s)\|_{\mathcal{L}^{p\varphi}(\mathbb{P};\mathbb{R})} \right] \max\left\{1, \sup_{s \in (0,T)} s^\rho \|e^{sA}\|_{L(H,V)} \right\}^{(1+\varphi/2)} \\
\cdot \left[ \max\left\{1, \sup_{s \in [0,T]} \|\bar{X}_s^{M,I}\|_{\mathcal{L}^{(1+\varphi/2)\max\{2,\varphi\}}(\mathbb{P};V)} \right\} + \|F(0)\|_H^{(1+\varphi/2)} \right]. \\
(68)
\]
Proof of Proposition 4.4 Throughout this proof assume w.l.o.g. that
\[
\begin{aligned}
\sup_{s \in (0,T)} & \left( s^\rho \| e^{sA} \|_{L^\infty(H,V)} + \EE \left[ \| P_1 O_s \|_{V}^{p\rho} + \| P_1 X_s \|_{V}^{p\rho} + \| X_s \|_{H}^{p\rho} \right] \right) \\
+ & \sup_{s \in [0,T)} \EE \left[ |\{ \chi^M_s \}_{s \in [0,T]} \}_{s \in [0,T]} |^{4\rho_\alpha} + \| \chi^M_s \|_{V}^{p(1+\rho/2)} \right]^{\max\{4, \varphi\}} < \infty. \quad (69)
\end{aligned}
\]

Note that the fact that
\[
\forall v, w \in V : \| F(v) - F(w) \|_{H}^2 \leq C \| v - w \|_{V}^2 (1 + \| v \|_{V}^\varphi + \| w \|_{V}^\varphi) \quad (70)
\]
and the fact that \( \forall x, y \in [0, \infty) : \sqrt{x+y} \leq \sqrt{x} + \sqrt{y} \) imply for all \( s \in [0, T] \) that
\[
\begin{aligned}
\| F(\chi^M_s) \|_{H} & \leq \| F(\chi^M_s) - F(0) \|_{H} + \| F(0) \|_{H} \\
& \leq \sqrt{C} \| \chi^M_s \|_{V}^2 \| (1 + \| \chi^M_s \|_{V}^{\varphi/2}) + \| F(0) \|_{H} \\
& \leq \sqrt{C} \| \chi^M_s \|_{V} + \| \chi^M_s \|_{V}^{(1+\varphi/2)} + \| F(0) \|_{H}
\end{aligned} \quad (71)
\]

Hence, we obtain for all \( q \in [1, \infty) \) that
\[
\begin{aligned}
\sup_{s \in (0,T)} \| F(\chi^M_s) \|_{L^q(P;H)} & \leq 2 \sqrt{C} \max \left\{ 1, \sup_{s \in [0,T)} \| \chi^M_s \|_{V}^{(1+\varphi/2)} \right\} + \| F(0) \|_{H}. \quad (72)
\end{aligned}
\]

We establish (68) by an application of (61) in Lemma 4.1 above (cf. (76)) and, thereafter, by estimating the right hand side of (61) (cf. (79)). To estimate the right hand side of (63) in Lemma 4.1 above we now establish upper bounds for the quantity \( \sup_{s \in (0,T)} \| X^M_s \|_{L^p(P;V)} \) (cf. (73) below) and the quantity \( \sup_{s \in (0,T)} \| X^M_s \|_{L^p(P;V)} \) (cf. (73) below). More formally, observe that the triangle inequality, Lemma 4.2, Lemma 4.3, and Hölder’s inequality imply that
\[
\begin{aligned}
\sup_{s \in (0,T)} \| X^M_s \|_{L^p(P;V)} \leq & \sup_{s \in (0,T)} \| X^M_s \|_{L^p(P;V)} + \sup_{s \in [0,T)} \| X^M_s \|_{L^p(P;V)} \\
\leq & \frac{3 T(1-\rho)}{(1-\rho - \varphi) M^\rho} \left[ \sup_{s \in (0,T)} s^\rho \| e^{sA} \|_{L(H,V)} \right] \left[ \sup_{s \in [0,T)} \| F(\chi^M_s) \|_{L^2(P;H)} \right] \\
+ & \frac{T(1+\alpha-\varphi)}{(1-\rho) M^\varphi} \left[ \sup_{s \in (0,T)} s^\rho \| e^{sA} \|_{L(H,V)} \right] \left[ \sup_{s \in [0,T)} \| F(\chi^M_s) \|_{L^4(P;H)} \right].
\end{aligned}
\]
\[ + \sup_{s \in (0, T)} \| P_t (O_s - O_{[s,T]}^J) \|_{L^2(P; V)} + \sup_{s \in [0, T)} \| P_t O_s - \mathcal{D}_s^{M,I} \|_{L^2(P; V)} \]

\[ \leq \frac{3 \max \{1, T^{(1+\alpha-\rho)}\}}{(1 - \rho - \varrho) M \min(\alpha, \varrho)} \left[ \sup_{s \in (0, T)} \| \mathcal{D}_s^{M,I} \|_{L^p(H, V)} \right] \left[ \sup_{s \in (0, T)} \| \mathcal{D}_s^{M,I} \|_{L^4_p(P; \mathbb{R})} \right] \]

\[ \cdot \left[ 1 + \sup_{s \in [0, T)} \| F(\mathcal{X}_s^{M,I}, \mathcal{D}_s^{M,I}) \|_{L^4_p(P; H)} \right] + \sup_{s \in (0, T)} \| P_t (O_s - O_{[s,T]}^J) \|_{L^2(P; V)} \]

\[ + \sup_{s \in (0, T)} \| P_t O_s - \mathcal{D}_s^{M,I} \|_{L^2(P; V)}. \] (73)

Moreover, note that (72) and the fact that \( \| P_t \|_{L(H)} \leq 1 \) yields that

\[ \sup_{s \in (0, T)} \| X_s^{M,I} \|_{L^p(P; V)} \]

\[ \leq \sup_{s \in (0, T)} \left[ \int_0^s \| P_t e^{(s-u)A} F(\mathcal{X}_u^{M,I}) \|_{L^p(P; V)} du \right] + \sup_{s \in (0, T)} \| P_t O_s \|_{L^p(P; V)} \]

\[ \leq \sup_{s \in (0, T)} \left[ \int_0^s \| e^{(s-u)A} \|_{L(H, V)} \| P_t \|_{L(H)} \| F(\mathcal{X}_u^{M,I}) \|_{L^p(P; H)} du \right] \]

\[ + \sup_{s \in (0, T)} \| P_t O_s \|_{L^p(P; V)} \]

\[ \leq \left[ \sup_{s \in (0, T)} s^\rho \| e^{sA} \|_{L(H, V)} \right] \left[ \sup_{s \in [0, T)} \| F(\mathcal{X}_s^{M,I}) \|_{L^p(P; H)} \right] \left[ \sup_{s \in [0, T)} \int_0^s (s-u)^{-\rho} du \right] \]

\[ + \sup_{s \in (0, T)} \| P_t O_s \|_{L^p(P; V)} \]

\[ \leq \frac{T^{(1-\rho)}}{(1 - \rho)} \left[ \sup_{s \in (0, T)} s^\rho \| e^{sA} \|_{L(H, V)} \right] \left[ \sup_{s \in [0, T)} \| F(\mathcal{X}_s^{M,I}) \|_{L^p(P; H)} \right] \]

\[ + \sup_{s \in (0, T)} \| P_t O_s \|_{L^p(P; V)} \]

(74)

Combining this, (73), and the fact that

\[ \forall x, y \in [0, \infty): (x + y)^{\varphi/2} \leq 2^{\max\{0, \varphi/2-1\}} (x^{\varphi/2} + y^{\varphi/2}) \]

(75)

with Lemma 4.1 proves that

\[ \sup_{t \in [0, T]} \| P_t X_t - X_t^{M,I} \|_{L^p(P; H)} \]

\[ \leq \frac{2^{\max\{0, \varphi/2-1\}} \sqrt{CT} e^{cT}}{\sqrt{(k - 2)c}} \left( \sup_{s \in (0, T)} \| P_{[s,T]} X_s \|_{L^2(P; V)} \right) \]

\[ + \sup_{s \in (0, T)} \| P_t (O_s - O_{[s,T]}^J) \|_{L^2(P; V)} \]
\[
\begin{align*}
+ \sup_{s \in [0,T]} \| P_I O_s - \mathcal{D}^M_s \|_{L^2(\mathbb{P}; V)} &+ \frac{3 \max\{1, T^{(1+\alpha-\rho)}\}}{(1-\rho-\varrho)M^{\min[\alpha, \varrho]}} \left[ \sup_{s \in [0,T]} s^\rho \| e^{IA} \|_{L(H, V)} \right] \\
\cdot \left[ \sup_{s \in [0,T]} \| \mathcal{V}(\mathcal{X}^M_s, \mathcal{D}^M_s) \|_{L^2(\mathbb{P}; \mathbb{R})} \right] &\left[ \sup_{s \in [0,T]} \| F(\mathcal{X}^M_s) \|_{L^4(\mathbb{P}; H)} \right] \\
\cdot \left( 1 + \sup_{s \in [0,T]} \| X_s \|_{L^p(\mathbb{P}; V)}^{\psi/2} + \sup_{s \in [0,T]} \| P_I X_s \|_{L^p(\mathbb{P}; V)}^{\psi/2} + \sup_{s \in [0,T]} \| \mathcal{X}^M_s \|_{L^p(\mathbb{P}; V)}^{\psi/2} \right) &+ \frac{T^{(1-\rho)/2}}{(1-\rho)\psi/2} \sup_{s \in [0,T]} s^\rho \| e^{IA} \|_{L(H, V)}^{\psi/2} \\
+ \sup_{s \in [0,T]} \| P_I O_s \|_{L^p(\mathbb{P}; V)}^{\psi/2}. 
\end{align*}
\]

Hence, we obtain that

\[
\begin{align*}
\sup_{t \in [0,T]} \| P_I X_t - \mathcal{X}^M_t \|_{L^p(\mathbb{P}; H)} &\leq \frac{3 \cdot 2\max\{\psi/2-1, \psi/2\} \max\{1, T^{(1/2+\alpha-\rho+\psi/2-\rho/2)}\} \sqrt{C_{\psi, \alpha}}}{(1-\rho-\varrho)\sqrt{(\kappa-2)c}} \left[ \sup_{s \in [0,T]} \| P_I X_s \|_{L^2(\mathbb{P}; V)} \right] \\
&\quad + \sup_{s \in [0,T]} \| P_I (O_s - O(s)_{[0, T]}) \|_{L^2(\mathbb{P}; V)} \\
&\quad + \sup_{s \in [0,T]} \| P_I O_s - \mathcal{D}^M_s \|_{L^2(\mathbb{P}; V)} + M^{-\min[\alpha, \varrho]} \left[ 2 + \sup_{s \in [0,T]} \| X_s \|_{L^p(\mathbb{P}; V)}^{\psi/2} + \sup_{s \in [0,T]} \| P_I X_s \|_{L^p(\mathbb{P}; V)}^{\psi/2} + \sup_{s \in [0,T]} \| \mathcal{X}^M_s \|_{L^p(\mathbb{P}; V)}^{\psi/2} \right] \\
&\quad \cdot \sup_{s \in [0,T]} \max\left\{ 1, \sup_{s \in [0,T]} s^\rho \| e^{IA} \|_{L(H, V)}^{\psi/2} \right\} \cdot \max\left\{ 1, \sup_{s \in [0,T]} \| F(\mathcal{X}^M_s) \|_{L^p(\mathbb{P}; H)}^{(1+\psi/2)} \right\}. 
\end{align*}
\]

In the next step observe that the triangle inequality implies that

\[
\begin{align*}
\sup_{t \in [0,T]} \| X_t - \mathcal{X}^M_t \|_{L^p(\mathbb{P}; H)} &\leq \sup_{t \in [0,T]} \left[ \| X_t - P_I X_t \|_{L^p(\mathbb{P}; H)} + \| P_I X_t - \mathcal{X}^M_t \|_{L^p(\mathbb{P}; H)} \right. \\
&\quad + \| \mathcal{X}^M_t - \mathcal{X}^M_t \|_{L^p(\mathbb{P}; H)} \right] \\
&\leq \sup_{t \in [0,T]} \| P_I X_t - \mathcal{X}^M_t \|_{L^p(\mathbb{P}; H)} \\
&\quad + \left[ \sup_{v \in V \setminus \{0\}} \| v \|_{H} \right] \sup_{t \in [0,T]} \left[ \| P_I X_t \|_{L^p(\mathbb{P}; V)} + \| \mathcal{X}^M_t - \mathcal{X}^M_t \|_{L^p(\mathbb{P}; V)} \right].
\end{align*}
\]
\[
\begin{aligned}
\sup_{t \in [0,T]} \|X_t - \mathcal{X}^M_t\|_{L^p(P;H)} & \leq \left[ 3 \cdot 2^{\max\{1,\psi/2\}} + 1 \right] \max\{1, T^{(3/2+\alpha+\rho+\psi/2)}\} \max\{1, \sqrt{C e^{\kappa T}}\} \\
& \cdot \min\{1, \sqrt{(\kappa - 2) c}(1 - \rho - \theta)^{(1+\psi/2)}\} \\
& \cdot \sup_{t \in [0,T]} \|P H \setminus X_t\|_{L^2(P;V)} + M - \min\{\alpha, c\} + \sup_{t \in (0,T)} \|P_t(O_t - O_{t\mid T})\|_{L^2(P;V)} \\
& + \sup_{t \in [0,T]} \|P \mathcal{P}_t - \mathcal{D}^M_t\|_{L^2(P;V)} \\
& \cdot \left[ 1 + \sup_{s \in (0,T)} \|X_s\|_{L^{p\nu}(P;V)} + \max\{1, \psi\} \sup_{s \in (0,T)} \|P \mathcal{P}_s\|_{L^{p\nu}(P;V)} \right] \\
& \cdot \left[ 1 + \sup_{s \in (0,T)} \|\mathcal{V}(\mathcal{X}^M_s, \mathcal{D}^M_s)\|_{L^{4\nu}(P;E)} \right] \max\{1, \sup_{s \in (0,T)} \|s^\alpha V^{sA} \|_{L(H,V)}\}^{(1+\psi/2)} \\
& \cdot \max\{1, \sup_{s \in [0,T]} \|F(\mathcal{X}^M_s)\|_{L^p}\}^{(1+\psi/2)} \max\{1, \sup_{v \in V \setminus \{0\}} \|v\|_V\}.
\end{aligned}
\]  

(78)

Next note that (69), (72), and the fact that

\[ \forall x, y \in [0, \infty): (x + y)^{(1+\psi/2)} \leq 2^{\psi/2} (x^{(1+\psi/2)} + y^{(1+\psi/2)}) \]

ensure that

\[ \max\{1, \sup_{s \in [0,T]} \|F(\mathcal{X}^M_s)\|_{L^p(4,\psi)(P;H)}\} \]

\[ \leq 2 \max\{1, \sqrt{C}\} \max\{1, \sup_{s \in [0,T]} \|\mathcal{X}^M_s\|_{L^p(1+\psi/2)\max(4,\psi)(P;V)}\} + \|F(0)\|_H \]

\[ \leq 2^{(1+\psi)} \max\{1, C^{(1/2+\psi/4)}\} \left[ \max\{1, \sup_{s \in [0,T]} \|\mathcal{X}^M_s\|_{L^p(1+\psi/2)\max(4,\psi)(P;V)}\} \right] + \|F(0)\|_H \]

\[ < \infty. \]  

(81)
Combining this with (79) and the fact that
\[
\left[ 3 \cdot 2^{\max\{1,\phi/2\}} + 1 \right] \cdot 2^{(1+\phi)} \leq 3 \cdot 2^{\max\{1,\phi/2\}} \left[ 1 + \frac{1}{6} \right] \cdot 2^{(1+\phi)} = 7 \cdot 2^{\max\{1+\phi,3\phi/2\}} \leq 4^{(2+\phi)}
\] (82)
completes the proof of Proposition 4.4. \(\square\)

5 Main result

5.1 Setting

Consider the notation in Sect. 1.1, let \((H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)\) be a separable \(\mathbb{R}\)-Hilbert space, let \(\mathbb{H} \subseteq H\) be a non-empty orthonormal basis of \(H\), let \(T, c, \varphi \in (0, \infty), \varepsilon \in [0,1], \rho \in [0,1/2), \gamma \in (\rho, 1/2), \chi \in (0, (\gamma-\rho)/(1+\phi/2)] \cap (0, (1-\rho)/(1+\phi)]\), \(D \subseteq \mathcal{P}_0(\mathbb{H}) \setminus \{0\}, \mu : \mathbb{H} \rightarrow \mathbb{R}\) satisfy \(\sup_{h \in \mathbb{H}} \mu_h < 0\), let \(A : D(A) \subseteq H \rightarrow H\) be the linear operator which satisfies that \(D(A) = \{v \in H : \sum_{h \in \mathbb{H}} |\mu_h (h, v) H|^2 < \infty\}\) and \(\forall v \in D(A) : Av = \sum_{h \in \mathbb{H}} \mu_h (h, v) H h\), let \((H_r, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H), r \in \mathbb{R}\), be a family of interpolation spaces associated to \(-A\) (cf., e.g., [37, Section 3.7]), let \((V, \| \cdot \|_V)\) be an \(\mathbb{R}\)-Banach space with \(H_\rho \subseteq V \subseteq H\) continuously and densely, let \(\phi, \Phi : \mathcal{C}([0, T], H_1) \rightarrow [0, \infty)\) be \(\mathcal{B}(\mathcal{C}([0, T], H_1))/\mathcal{B}([0, \infty))\)-measurable functions, let \(F \in \mathcal{C}(V, H)\), \((P_t)_{t \in \mathcal{P}_0(\mathbb{H})} \subseteq L(H)\) satisfy for all \(I \in \mathcal{P}_0(\mathbb{H}), u \in H, v, w \in P_I(H), x \in \mathcal{C}([0, T], H_1)\) that
\[
P_I(u) = \sum_{j \in I} (h_j, u) H h, \quad \langle v - w, Av + F(v) - Aw - F(w) \rangle_H \leq c \|v - w\|_H^2,
\] (83)
\[
\sup_{t \in [0, T]} \left( \langle v, P_I F(v + x_t) \rangle_{H_{1/2}} + \phi(x) \langle v, F(v + x_t) \rangle_H \right) \leq \varepsilon \|v\|_H^2 + (c + \phi(x)) \|v\|_{H_{1/2}}^2 + c \phi(x) \|v\|_H^2 + \Phi(x),
\] (84)
and \(\|F(v) - F(w)\|_H^2 \leq c \|v - w\|_V^2 (1 + \|v\|_V^6 + \|w\|_V^6), \) (85)
let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, let \(X, O : [0, T] \times \Omega \rightarrow V\) and \(\mathcal{D}^{M,I} : [0, T] \times \Omega \rightarrow H_1, M \in \mathbb{N}, I \in D, \) be stochastic processes with continuous sample paths, let \(\mathcal{X}^{M,I} : [0, T] \times \Omega \rightarrow H_{\gamma}, M \in \mathbb{N}, I \in D, \) be stochastic processes, and assume for all \(t \in [0, T], M \in \mathbb{N}, I \in D\) that \(\mathcal{D}^{M,I}([0, T] \times \Omega) \subseteq P_I(H)\) and
\[
\mathbb{P}\left( X_t = \int_0^t e^{(t-s)A} F(X_s) ds + O_t \right) = \mathbb{P}\left( \mathcal{X}^{M,I}_t = \int_0^t P_I e^{(t-s)A} \mathbb{I}_{\mathbb{H}_1} M \mathcal{D}^{M,I} ||X_t||_{H_{\gamma}} + ||\mathcal{D}^{M,I}||_{H_{\gamma}} \leq (M/T)x \right) F(\mathcal{X}^{M,I}_{[x]}_{T/M}) ds + \mathcal{D}^{M,I}_t = 1.
\] (86)
5.2 Comments on the setting

The next two results, Lemmas 5.1 and 5.2 below, disclose a few elementary consequences of the framework in Sect. 5.1. Proofs of Lemmas 5.1 and 5.2 can, e.g., be found in the extended arXiv version of this article [2, Section 5.2].

**Lemma 5.1** Assume the setting in Sect. 5.1 and let \( r \in [0, \infty) \). Then \( \text{span}(H) \subseteq H_r \).

**Lemma 5.2** Assume the setting in Sect. 5.1. Then

(i) we have that \( \text{span}(H) = \bigcup_{I \in \mathcal{P}_{0}(\mathbb{H})} P_I(H) \),

(ii) we have for all \( r \in [0, \infty) \) that \( \bigcup_{I \in \mathcal{P}_{0}(\mathbb{H})} P_I(H) \subseteq H_r \),

(iii) we have for all \( v, w \in H_1 \) that \( \langle v-w, Av + F(v) - Aw - F(w) \rangle_H \leq c \|v-w\|^2_H \),

and (iv) we have for all \( v, w \in V \) that \( \|F(v) - F(w)\|^2_H \leq c \|v-w\|^2_V (1 + \|v\|^\phi_V + \|w\|^\phi_V) \).

5.3 On the measurability of a certain function

In our proof of Theorem 5.5 (the main result of this article) we employ the following well-known result, Lemma 5.3 below. Lemma 5.3 is, e.g., proved as Lemma 5.3 in the extended arXiv version of this article [2, Section 5.3].

**Lemma 5.3** Consider the notation in Sect. 1.1, let \((V, \|\cdot\|_V)\) be a separable \( \mathbb{R} \)-Banach space, let \((W, \|\cdot\|_W)\) be an \( \mathbb{R} \)-Banach space with \( V \subseteq W \) continuously and densely, let \((S, \mathcal{S})\) be a measurable space, let \( s \in S \), let \( \psi : V \to S \) be a \( \mathcal{B}(V)/\mathcal{S} \)-measurable function, and let \( \Psi : W \to S \) be the function which satisfies for all \( v \in W \) that

\[
\Psi(v) = \begin{cases} 
\psi(v) & : v \in V \\
\mathcal{S} & : v \in W \setminus V.
\end{cases}
\]

(87)

Then we have that \( \Psi : W \to S \) is a \( \mathcal{B}(W)/\mathcal{S} \)-measurable function.

5.4 A priori moment bounds for the numerical approximation

**Lemma 5.4** Assume the setting in Sect. 5.1, let \( p \in [1, \infty), \sigma \in [0, \gamma] \), and assume that

\[
\sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \mathbb{E} \left[ \|\mathcal{D}^{M,I}_t\|_{H_\sigma}^{2p} + |\Phi(M,I)|^p + |\phi(M,I)|^p \right] < \infty. \tag{88}
\]

Then

\[
\sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \mathbb{E} \left[ \|\mathcal{X}^{M,I}_t\|_{H_\sigma}^{2p} \right] < \infty. \tag{89}
\]
**Proof of Lemma 5.4.** Throughout this proof let $\chi_I : C([0, T], P_I(H)) \to C([0, T], H_I)$, $I \in \mathcal{D}$, be the functions which satisfy for all $I \in \mathcal{D}$, $w \in C([0, T], P_I(H))$, $t \in [0, T]$ that

$$ (\chi_I(w))(t) = w(t), \quad (90) $$

let $\kappa \in (0, 1)$ be a real number, let $\tilde{X}^{M,I} : [0, T] \times \Omega \to P_I(H)$, $M \in \mathbb{N}$, $I \in \mathcal{D}$, be the functions which satisfy for all $M \in \mathbb{N}$, $I \in \mathcal{D}$, $t \in [0, T]$ that

$$ \tilde{X}^{M,I}_t = \int_0^t P_I e^{(t-s)A} \mathbb{P}_{\tilde{X}^{M,I}_s} \mathbb{P}_{H_I + \mathbb{D}^{M,I}_s} dF(\tilde{X}^{M,I}_s) d s + \mathbb{D}^{M,I}_t, \quad (91) $$

and let $C, K \in [0, \infty)$ be given by

$$ K = \sqrt{c} \max \left\{ 1, \sup_{v \in (V \cap H_{\rho}) \setminus [0]} \|v\|_{H_{\rho}}^{(1+\varphi/2)} \right\}, \quad (92) $$

$$ C = \max \left\{ 3K^2(1 + 2\max\{0, \varphi-1\}) \left[ 1 + \sup_{v \in H_{\rho} \setminus [0]} \|v\|_{H_{\rho}}^{\varphi} \right] \right\} \max \left\{ 1, \sup_{v \in H_{\rho} \setminus [0]} \|v\|_{H_{\rho}}^{(2+\varphi)} \right\}, \quad (93) $$

Moreover, observe that in (91) we assume that the processes $\tilde{X}^{M,I} : [0, T] \times \Omega \to P_I(H)$, $M \in \mathbb{N}$, $I \in \mathcal{D}$, satisfy (91) for every trajectory while in (86) we assume that the processes $X^{M,I} : [0, T] \times \Omega \to H_{\gamma}$, $M \in \mathbb{N}$, $I \in \mathcal{D}$, satisfy this relation with probability 1. Note that the fact that $H_{\gamma} \subseteq H_{\rho} \subseteq V$ continuously ensures\(^2\) that $C, K \in [0, \infty)$. In the next step observe that item (iv) in Lemma 5.2 and the fact that $\forall x, y, z \in [0, \infty) : \sqrt{x+y+z} \leq \sqrt{x} + \sqrt{y} + \sqrt{z}$ imply for all $v, w \in H_{\gamma}$ that

$$ \|F(v) - F(w)\|_H \leq \sqrt{c} \|v-w\|^2_V \left( 1 + \|v\|_V^{\varphi} + \|w\|_V^{\varphi} \right) $$

\begin{align*}
&\leq \sqrt{c} \left[ \sup_{u \in H_{\rho} \setminus [0]} \|u\|_V \right] \|v-w\|_{H_{\rho}} \left( 1 + \left[ \sup_{u \in H_{\rho} \setminus [0]} \|u\|_V \right] \|v\|_{H_{\rho}}^{\varphi} + \|w\|_{H_{\rho}}^{\varphi} \right)^{1/2} \\
&\leq K \|v-w\|_{H_{\rho}} \left( 1 + \|v\|_{H_{\rho}}^{\varphi/2} + \|w\|_{H_{\rho}}^{\varphi/2} \right).
\end{align*} \quad (94)

Combining this with Lemma 2.4 in Hutzenhaler et al. [21] (with $(V, \|\cdot\|_V) = (H_{\gamma}, \|\cdot\|_{H_{\rho}})$, $(V, \|\cdot\|_{\gamma}) = (H_{\rho}, \|\cdot\|_{H_{\rho}})$, $(W, \|\cdot\|_W) = (H, \|\cdot\|_{H})$, $(\mathcal{W}, \|\cdot\|_{\gamma}) = \mathcal{W}$)

\(^2\) In Corollary 6.10 we apply the findings from Sect. 5 to the stochastic Allen–Cahn equation with $V = L^6(\omega_{(0,1)}; \mathbb{R})$ (see Sect. 6 below for details).
\((H, \|\cdot\|_H), \epsilon = K, \theta = C, \varepsilon = \varphi/2, \vartheta = \varphi, F = H_y \ni v \mapsto F(v) \in H\) in the notation of Lemma 2.4 in Hutzenthaler et al. \([21]\) implies that for all \(u, v \in H_y\) we have that

\[
\|F(u)\|_H^2 \leq C\max\{1, \|u\|_{H_y}^{(2+\varphi)}\}
\]

(95) and

\[
\|F(u) - F(v)\|_H^2 \leq C\max\{1, \|u\|_{H_y}^{\varphi}\}\|u - v\|_{H_y}^2 + C\|u - v\|_{H_y}^{(2+\varphi)}.
\]

(96) Moreover, observe that (84) and (90) ensure\(^3\) that for all \(I \in \mathcal{D}, v \in P_I(H), w \in \mathcal{C}([0, T], P_I(H)), t \in [0, T]\) we have that

\[
\langle v, P_I F(v + w_t) \rangle_{H_{1/2}} + \phi(\chi_I(w))(v, F(v + w_t))_H
\]

\[
\leq \epsilon \|v\|_{H_{1/2}}^2 + (c_\kappa + \phi(\chi_I(w)))\|v\|_{H_{1/2}}^2 + c\phi(\chi_I(w))\|v\|_H^2 + \Phi(\chi_I(w)).
\]

(97) This together with (91), (95), and (96) allows us to apply Lemma 2.2 (with \(H = H_y\), \(\mathbb{H} = \mathbb{H}, T = T, \varphi = \varphi, c = c/\kappa, C = C, \epsilon = \epsilon, \kappa = \kappa, \rho = \rho, \gamma = \gamma', \chi = \chi, M = M, \mu = \mu, A = A, I = I, P = P_I, \mathfrak{F} = [0, T] \ni t \mapsto \mathcal{F}^M_I(\omega) \in P_I(H), \mathcal{D} = [0, T] \ni t \mapsto \mathcal{D}^M_I(\omega) \in P_I(H), F = P_I(H) \ni v \mapsto F(v) \in H, \phi = \mathcal{C}([0, T], P_I(H)) \ni v \mapsto \phi([0, T] \ni t \mapsto v(t) \in H_1) \in [0, \infty), \Phi = \mathcal{C}([0, T], P_I(H)) \ni v \mapsto \Phi([0, T] \ni t \mapsto v(t) \in H_1) \in [0, \infty)\) for \(I \in \mathcal{D}, M \in \mathbb{N}, \omega \in \Omega\) in the notation of Lemma 2.2) to obtain that for every \(M \in \mathbb{N}, I \in \mathcal{D}, \omega \in \Omega\) we have that the function \([0, T] \ni t \mapsto \mathcal{F}^M_I(\omega) - \mathcal{D}^M_I(\omega) \in P_I(H)\) is continuous and that

\[
\begin{align*}
\sup_{t \in [0, T]} \left(\|\mathcal{F}^M_I(\omega) - \mathcal{D}^M_I(\omega)\|_{H_{1/2}}^2 + \phi(\mathcal{D}^M_I(\omega))\|\mathcal{F}^M_I(\omega) - \mathcal{D}^M_I(\omega)\|_H^2\right)
\leq \frac{\kappa}{c} \exp\left(\frac{2cT}{\kappa}\right) \left(\phi(\mathcal{D}^M_I(\omega)) + \frac{\max\{1, \phi(\mathcal{D}^M_I(\omega))\}C(c + \kappa)}{2(1 - \epsilon)(1 - \kappa)c}\right)^{(2+\varphi)}.
\end{align*}
\]

(98) This, in particular, implies for all \(M \in \mathbb{N}, I \in \mathcal{D}\) that

\[
\left(\Omega \ni \omega \mapsto \sup_{t \in [0, T]} \|\mathcal{F}^M_I(\omega) - \mathcal{D}^M_I(\omega)\|_{H_{1/2}} \in \mathbb{R}\right) \in \mathcal{M}(\mathcal{F}, \mathcal{B}(\mathbb{R})).
\]

(99) The assumption that \(p \geq 1\), the fact that \(\forall x, y \in [0, \infty)\): \(\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}\), and (98) hence ensure for all \(M \in \mathbb{N}, I \in \mathcal{D}\) that

\footnote{Observe that for all \(I \in \mathcal{D}, w \in \mathcal{C}([0, T], P_I(H)), s \in [0, T]\) we have that \(P_I(w_s) = w_s \in P_I(H) \subseteq H_1\).}

\(\square\) Springer
\[
\left\| \sup_{t \in [0, T]} \left| \tilde{X}^M_{t,j} - \Sigma^M_{t,j} \right| \right\|_{\mathcal{L}^2(\mathbb{P}; \mathbb{R})} \\
\leq \frac{\kappa}{c} \exp \left( \frac{2cT}{\kappa} \right) \left( \Phi(\Theta^M) + \frac{\max\{1, \Phi(\Theta^M)\} C(c + \kappa)}{2(1 - \epsilon)(1 - \kappa)c} \cdot \left[ \frac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)} \right]^{(2+\varphi)} \right)^{1/2} \\
\leq \frac{\sqrt{\kappa}}{\sqrt{c}} \exp \left( \frac{cT}{\kappa} \right) \cdot \left( \|\Phi(\Theta^M)\|_{\mathcal{L}^2(\mathbb{P}; \mathbb{R})} + \frac{\|\Phi(\Theta^M)\|_{\mathcal{L}^2(\mathbb{P}; \mathbb{R})} + 1}{2(1 - \epsilon)(1 - \kappa)c} \cdot \left[ \frac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)} \right]^{(2+\varphi)} \right)^{1/2} \\
\leq \frac{\sqrt{\kappa}}{\sqrt{c}} \exp \left( \frac{cT}{\kappa} \right) \|\Phi(\Theta^M)\|_{\mathcal{L}^2(\mathbb{P}; \mathbb{R})}^{1/2} \\
+ \left( \|\Phi(\Theta^M)\|_{\mathcal{L}^2(\mathbb{P}; \mathbb{R})} + 1 \right) \frac{\sqrt{C(c + \kappa)} \exp \left( \frac{cT}{\kappa} \right) \left[ \frac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)} \right]^{(1+\varphi)/2}}{c\sqrt{2(1 - \epsilon)(1/\varphi - 1)}} \\
\cdot \left[ \frac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)} \right]^{(1+\varphi)/2}.
\]

(100)

In addition, observe that the fact that for every \( r \in \mathbb{R}, I \in \mathcal{D} \) we have that \( P_I(H) \subseteq H_r \) continuously and, e.g., Andersson et al. [1, Lemma 2.2] (with \( V_0 = H_r, \|\cdot\| V_0 = \|\cdot\| H_r, \) \( V_1 = P_I(H), \|\cdot\| V_1 = P_I(H) \) \( \forall v \mapsto \|v\|_H \in [0, \infty) \) for \( I \in \mathcal{D}, r \in \mathbb{R} \) in the notation of Andersson et al. [1, Lemma 2.2]) prove that for all \( I \in \mathcal{D} \) we have that

\[
\mathcal{B}(P_I(H)) = \left\{ S \in \mathcal{P}(P_I(H)) : (\exists B \in \mathcal{B}(H_r) : S = B \cap P_I(H)) \right\} \subseteq \mathcal{B}(H_r).
\]

(101)

The hypothesis that \( \Theta^M_{t,j} : [0, T] \times \Omega \to H_1, M \in \mathbb{N}, I \in \mathcal{D}, \) are stochastic processes therefore demonstrates that for every \( M \in \mathbb{N}, I \in \mathcal{D} \) we have that \( \tilde{X}^M_t : [0, T] \times \Omega \to P_I(H) \) is a stochastic process. Combining this with (101) shows that for all \( B \in \mathcal{B}(H_r), \) \( t \in [0, T], M \in \mathbb{N}, I \in \mathcal{D} \) we have that

\[
\left( \Omega \ni \omega \mapsto \tilde{X}^M_{t,j}(\omega) \in H_{1r} \right)^{-1}(B) = \left( \Omega \ni \omega \mapsto \tilde{X}^M_{t,j}(\omega) \in H_{1r} \right)^{-1}(B \cap P_I(H)) \\
= \left( \Omega \ni \omega \mapsto \tilde{X}^M_{t,j}(\omega) \in P_I(H) \right)^{-1}(B \cap P_I(H)) = \left( \tilde{X}^M_{t,j} \right)^{-1}(B \cap P_I(H)) \in \mathcal{F}.
\]

(102)

Hence, we obtain for all \( t \in [0, T], M \in \mathbb{N}, I \in \mathcal{D} \) that

\[
\{ \omega \in \Omega : \tilde{X}^M_{t,j}(\omega) = \tilde{X}^M_{t,j}(\omega) \} \in \mathcal{F}.
\]

(103)
The assumption that for every $t \in [0, T]$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

\[
\mathbb{P}\left( \mathcal{X}_{t, I}^{M, I} = \int_0^t P_I e^{(t-s)A} \mathbb{E}\left[ \Omega_{[\cdot]_{I,M}} \| \mathcal{X}_{s, I}^{M, I} + \| \Omega_{[\cdot]_{I,M}} \| H_{\gamma} \leq (M/T)^{\gamma} \right] F(\mathcal{X}_{s, I}^{M, I}) \, ds + \Omega_{[\cdot]_{I,M}} \right) = 1
\]

(104)

therefore implies that for all $t \in [0, T]$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that $\mathbb{P}(\mathcal{X}_{t, I}^{M, I} = \tilde{\mathcal{X}}_{t, I}^{M, I}) = 1$. This and the triangle inequality assure that

\[
\begin{align*}
\sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \| \mathcal{X}_{t, I}^{M, I} \|_{\mathcal{L}^2(\mathcal{P}; H_{\sigma})} & = \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \| \tilde{\mathcal{X}}_{t, I}^{M, I} \|_{\mathcal{L}^2(\mathcal{P}; H_{\sigma})} \\
& \leq \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \left[ \| \tilde{\mathcal{X}}_{t, I}^{M, I} - \Omega_{t, I}^{M, I} \|_{\mathcal{L}^2(\mathcal{P}; H_{\sigma})} + \| \Omega_{t, I}^{M, I} \|_{\mathcal{L}^2(\mathcal{P}; H_{\sigma})} \right] \\
& \leq \left[ \sup_{v \in \mathcal{H}_{1/2} \setminus \{0\}} \| v \|_{\mathcal{H}_{1/2}} \right] \left[ \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \| \tilde{\mathcal{X}}_{t, I}^{M, I} - \Omega_{t, I}^{M, I} \|_{\mathcal{H}_{1/2}} \right] + \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \| \Omega_{t, I}^{M, I} \|_{\mathcal{L}^2(\mathcal{P}; \mathbb{R})}.
\end{align*}
\]

(105)

The assumption that

\[
\sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \mathbb{E}\left[ \| \tilde{\mathcal{X}}_{t, I}^{M, I} \|_{H_{\sigma}}^{2p} + | \Phi(\Omega_{t, I}^{M, I}) |^p + | \phi(\Omega_{t, I}^{M, I}) |^p \right] < \infty,
\]

(106)

the fact that $\mathcal{H}_{1/2} \subseteq H_{\sigma}$ continuously, and (100) hence prove that

\[
\begin{align*}
\sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \| \mathcal{X}_{t, I}^{M, I} \|_{\mathcal{L}^2(\mathcal{P}; H_{\sigma})} & \leq \left[ \sup_{v \in \mathcal{H}_{1/2} \setminus \{0\}} \| v \|_{\mathcal{H}_{1/2}} \right] \left[ \frac{\sqrt{\kappa}}{\sqrt{c}} \exp\left( \frac{cT}{\kappa} \right) \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \| \Phi(\Omega_{t, I}^{M, I}) \|_{\mathcal{L}^p(\mathcal{P}; \mathbb{R})} \right] \\
& \quad + \left( \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \| \phi(\Omega_{t, I}^{M, I}) \|_{\mathcal{L}^p(\mathcal{P}; \mathbb{R})} + 1 \right) \frac{\sqrt{C(c + \kappa)} \exp\left( \frac{cT}{\kappa} \right)}{c\sqrt{2(1 - \epsilon)(1/\kappa - 1)}} \\
& \quad \cdot \left[ \max\{1, T\} \left( \frac{1 + \sqrt{C}}{1 - \rho} \right) \right]^{(1 + \rho/2)} \\
& \quad + \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \| \Omega_{t, I}^{M, I} \|_{\mathcal{L}^2(\mathcal{P}; H_{\sigma})} < \infty.
\end{align*}
\]

(107)

This completes the proof of Lemma 5.4.
5.5 Main result

Theorem 5.5 Assume the setting in Sect. 5.1, let \( \vartheta \in (0, \infty) \), \( p \in [\max\{2, \frac{1}{\varphi}\}, \infty) \), \( \varphi \in [0, 1 - \rho) \), and assume that

\[
\sup_{t \in [0, T]} \sup_{I \in \mathcal{D}} \mathbb{E} \left[ \| X_t \|_V^{p(1+\varphi/2)\max\{2, \varphi\}} + \| P_t O_I \|_{V^p} \right] \\
+ \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \mathbb{E} \left[ \| \mathcal{S}_t^{M,I} \|_{H_Y}^2 \right] \\
+ \Phi(\mathcal{S}_t^{M,I}) + \phi(\mathcal{S}_t^{M,I}) \left( p \max\{2\vartheta, 2+\varphi, (1+\varphi/2)\} \right) < \infty.
\]

Then we have

(i) that \( \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \mathbb{E} \left[ \| X_t \|^p | M \|_{H_Y} \right] < \infty \) and

(ii) that there exists a real number \( C \in (0, \infty) \) such that for all \( M \in \mathbb{N}, I \in \mathcal{D} \) it holds that

\[
\sup_{t \in [0, T]} \left( \mathbb{E} \left[ \| X_t - \mathcal{X}_t^{M,I} \|_H^p \right] \right)^{1/p} \leq C \left[ M^{- \min\{\vartheta, \varphi\} + \| (-A)\varphi (I_0 - P_I) \|_{L(H)} \right] \\
+ \sup_{t \in [0, T]} \left( \mathbb{E} \left[ \| (I_0 - P_I) O_I \|_V^{2p} + \| P_I O_I - \mathcal{S}_t^{M,I} \|_V^{2p} \right] \\
+ \Phi(\mathcal{S}_t^{M,I}) \left( 2p \right) \right)^{1/2p} \right].
\]

Proof of Theorem 5.5 Throughout this proof let \( \kappa \in (2, \infty) \) be a real number, let \( (\mathbb{P}_I)_{I \in \mathcal{P}(\mathbb{H})} \subseteq L(H) \) be the linear operators which satisfy for all \( I \in \mathcal{P}(\mathbb{H}), v \in H \) that

\[
\mathbb{P}_I(v) = \sum_{h \in I} (h, v)_H h,
\]

let \( \mathcal{V}: V \times V \rightarrow [0, \infty) \) be the function which satisfies for all \( v, w \in V \) that

\[
\mathcal{V}(v, w) = \begin{cases} \| v \|_{H_Y} + \| w \|_{H_Y} & \text{if} \ (v, w) \in H_Y \times H_Y \\ \| v \|_V + \| w \|_V & \text{if} \ (v, w) \in (V \times V) \setminus (H_Y \times H_Y) \end{cases}
\]

(cf. (58)–(59) in Setting 4.1), let \( \tilde{\mathcal{X}}^{M,I}: [0, T] \times \Omega \rightarrow H_Y, M \in \mathbb{N}, I \in \mathcal{D} \), be the functions which satisfy for all \( M \in \mathbb{N}, I \in \mathcal{D}, t \in [0, T] \) that

\[
\tilde{\mathcal{X}}^{M,I}_t = \int_0^t P_I e^{(t-s)A} \mathcal{L}_I \mathcal{S}^{M,I}_{[x]} \|_{H_Y} + \| \mathcal{S}_t^{M,I} \|_{H_Y \leq (M/T) x} dM \widetilde{\mathcal{P}}^{M,I}_{t/M} ds + \mathcal{S}^{M,I}_{t/M},
\]

Observe that the function (see (110)) \( \mathcal{P}(\mathbb{H}) \ni I \mapsto \mathbb{P}_I \in L(H) \) is an extension of the function \( \mathcal{P}_0(\mathbb{H}) \ni I \mapsto P_I \in L(H) \) (see (83)). In particular, note that for all \( I \in \mathcal{P}^1_0(\mathbb{H}) \subseteq \mathcal{P}(\mathbb{H}) \) we have that \( \mathbb{P}_I = P_I \) and \( \mathbb{P}_{\mathbb{H} \cap I} = I_0 - P_I \).
let $\tilde{\Omega} \subseteq \Omega$ be the set given by

$$
\tilde{\Omega} = \left\{ \forall t \in [0, T]: X_t = \int_0^t e^{(t-s)A} F(X_s) \, ds + O_t \right\}
$$

and let $\tilde{X} : [0, T] \times \Omega \to V$ and $\tilde{O} : [0, T] \times \Omega \to V$ be the functions which satisfy for all $t \in [0, T]$ that $\tilde{X}_t = X_t \upharpoonright \tilde{\Omega}$ and

$$
\tilde{O}_t = O_t \upharpoonright \tilde{\Omega} - \left[ \int_0^t e^{(t-s)A} F(0) \, ds \right] \upharpoonright \tilde{\Omega} = O_t \upharpoonright \tilde{\Omega} + A^{-1}(Id_H - e^{tA})F(0) \upharpoonright \tilde{\Omega}.
$$

Next note that the fact that $H_1 \subseteq H_{\gamma}$ continuously and, e.g., Andersson et al. [1, Lemma 2.2] (with $V_0 = H_{\gamma}$, $\| \cdot \|_{V_0} = \| \cdot \|_{H_{\gamma}}$, $V_1 = H_1$, $\| \cdot \|_{V_1} = \| \cdot \|_{H_1}$ in the notation of Andersson et al. [1, Lemma 2.2]) ensure that

$$
B(H_1) = \left\{ S \in \mathcal{P}(H_1): (\exists B \in B(H_{\gamma}): S = B \cap H_1) \right\} \subseteq B(H_{\gamma}).
$$

The hypothesis that $\mathcal{O}^{M,I} : [0, T] \times \Omega \to H_1, M \in \mathbb{N}, I \in \mathcal{D}$, are stochastic processes therefore demonstrates that for every $M \in \mathbb{N}, I \in \mathcal{D}$ we have that $\tilde{\mathcal{X}}^{M,I} : [0, T] \times \Omega \to H_{\gamma}$ is a stochastic process. Hence, we obtain for all $t \in [0, T], M \in \mathbb{N}, I \in \mathcal{D}$ that

$$
\{ \omega \in \Omega: \mathcal{X}_t^{M,I}(\omega) = \tilde{\mathcal{X}}_t^{M,I}(\omega) \} \in \mathcal{F}.
$$

The assumption that for every $t \in [0, T], M \in \mathbb{N}, I \in \mathcal{D}$ we have that

$$
P\left( \mathcal{X}_t^{M,I} = \int_0^t P_t e^{(t-s)A} \left[ \| X_{t+s} \|_{H_1} \right]_{\mathcal{O}^{M,I}(s)} \right) = 1
$$

therefore implies that for all $t \in [0, T], M \in \mathbb{N}, I \in \mathcal{D}$ we have that

$$
P\left( \mathcal{X}_t^{M,I} = \tilde{\mathcal{X}}_t^{M,I} \right) = 1.
$$

Combining this and (108) with Lemma 5.4 demonstrates that

$$
\sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \| \tilde{\mathcal{X}}_t^{M,I} \|_{L^p_{\mathbb{P}}(\mathcal{O}^{M,I}(s))} < \infty.
$$

$\square$ Springer
This establishes item (i). Moreover, note that the assumption that \( \forall t \in [0, T] : \mathbb{P}(X_t = \int_0^t e^{(t-s)A} F(X_s) \, ds + O_t) = 1 \) yields that \( \mathbb{P}(\tilde{\Omega}) = 1 \). Hence, we obtain for all \( t \in [0, T] \) that \( \mathbb{P}(\tilde{X}_t = X_t) \geq \mathbb{P}(\tilde{\Omega}) = 1 \). Combining this with (108) ensures that

\[
\sup_{t \in [0, T]} \| \tilde{X}_t \|_{L^p(1+\psi/2 \max[2, \varphi]) (\mathbb{P}; V)} = \sup_{t \in [0, T]} \| X_t \|_{L^p(1+\psi/2 \max[2, \varphi]) (\mathbb{P}; V)} < \infty. \quad (120)
\]

Furthermore, observe that Lemma 2.1, the triangle inequality, and item (iv) in Lemma 5.2 show for all \( t \in (0, T] \) that

\[
\int_0^t \| (A)^{(\rho+\varphi)} e^{(t-s)A} F(\tilde{X}_s) \|_{L^p \max[2, \varphi]} (\mathbb{P}; H) \, ds
\]

\[
\leq \int_0^t \| (A)^{(\rho+\varphi)} e^{(t-s)A} \|_{L(H)} \| F(\tilde{X}_s) \|_{L^p \max[2, \varphi]} (\mathbb{P}; H) \, ds
\]

\[
\leq \int_0^t (t-s)^{-(\rho+\varphi)} \left[ \| F(\tilde{X}_s) - F(0) \|_{L^p \max[2, \varphi]} (\mathbb{P}; H) + \| F(0) \|_H \right] \, ds
\]

\[
\leq \int_0^t (t-s)^{-(\rho+\varphi)} \left[ \sqrt{c} \| \tilde{X}_s \|_V (1 + \| \tilde{X}_s \|_V^{\psi/2}) \| \tilde{X}_s \|_{L^p \max[2, \varphi]} (\mathbb{P}; V) + \| F(0) \|_H \right] \, ds
\]

\[
\leq \int_0^t (t-s)^{-(\rho+\varphi)} \left[ \sqrt{c} \| \tilde{X}_s \|_{L^p \max[2, \varphi]} (\mathbb{P}; V) + \| \tilde{X}_s \|_{L^p(1+\psi/2 \max[2, \varphi]) (\mathbb{P}; V)} \right] \, ds
\]

\[
+ \| F(0) \|_H \] \quad (121)
\]

Inequality (120) hence implies for all \( t \in (0, T) \) that

\[
\int_0^t \| (A)^{(\rho+\varphi)} e^{(t-s)A} F(\tilde{X}_s) \|_{L^p \max[2, \varphi]} (\mathbb{P}; H) \, ds
\]

\[
\leq \int_0^t (t-s)^{-(\rho+\varphi)} \left[ 2 \sqrt{c} \max \{ 1, \| \tilde{X}_s \|_{L^p(1+\psi/2 \max[2, \varphi]) (\mathbb{P}; V)} \} + \| F(0) \|_H \right] \, ds
\]

\[
\leq \left[ 2 \sqrt{c} \max \{ 1, \sup_{s \in (0, t)} \| \tilde{X}_s \|_{L^p(1+\psi/2 \max[2, \varphi]) (\mathbb{P}; V)} \} + \| F(0) \|_H \right] \left[ \int_0^t (t-s)^{-(\rho+\varphi)} \, ds \right]
\]

\[
\leq \left[ 2 \sqrt{c} \max \{ 1, \sup_{s \in (0, t)} \| \tilde{X}_s \|_{L^p(1+\psi/2 \max[2, \varphi]) (\mathbb{P}; V)} \} + \| F(0) \|_H \right] T^{(1-\rho-\varphi)} \left( \frac{1}{1-\rho-\varphi} \right) < \infty.
\]

(122)

This and the fact that

\[
\forall t \in [0, T] : \tilde{X}_t = \int_0^t e^{(t-s)A} F(\tilde{X}_s) \, ds + \tilde{O}_t \quad (123)
\]

prove that

\[
\sup_{t \in [0, T]} \| \tilde{X}_t - \tilde{O}_t \|_{L^p \max[2, \varphi]} (\mathbb{P}; H(\rho+\varphi)) \leq \sup_{t \in [0, T]} \left( \int_0^t \| e^{(t-s)A} F(\tilde{X}_s) \|_{L^p \max[2, \varphi]} (\mathbb{P}; H(\rho+\varphi)) \, ds \right) < \infty. \quad (124)
\]
In addition, observe that the triangle inequality assures for all $I \in \mathcal{D}$ that

$$
\sup_{t \in [0,T]} \| \mathcal{P}_{H \setminus I} \tilde{X}_t \|_{L^2^p(P;V)} \leq \sup_{t \in [0,T]} \left[ \| \mathcal{P}_{H \setminus I} (\tilde{X}_t - \tilde{O}_t) \|_{L^2^p(P;V)} + \| \mathcal{P}_{H \setminus I} \tilde{O}_t \|_{L^2^p(P;V)} \right]
\leq \left[ \sup_{v \in H_p \setminus [0]} \| v \|_{H_p} \right] \left[ \sup_{t \in [0,T]} \| \mathcal{P}_{H \setminus I} (\tilde{X}_t - \tilde{O}_t) \|_{L^2^p(P;H_p)} \right] + \sup_{t \in [0,T]} \| \mathcal{P}_{H \setminus I} \tilde{O}_t \|_{L^2^p(P;V)}
= \left[ \sup_{v \in H_p \setminus [0]} \| v \|_{H_p} \right] \left[ \sup_{t \in [0,T]} \| (-A)^{-\rho} \mathcal{P}_{H \setminus I} (-A)^{(\rho+\rho)} (\tilde{X}_t - \tilde{O}_t) \|_{L^2^p(P;H)} \right] + \sup_{t \in [0,T]} \| \mathcal{P}_{H \setminus I} \tilde{O}_t \|_{L^2^p(P;V)}.
$$

(125)

The fact that $\forall I \in \mathcal{P}(\mathbb{H})$: $\| \mathcal{P}_{H \setminus I} \|_{L(H)} \leq 1$ hence guarantees for all $I \in \mathcal{D}$ that

$$
\sup_{t \in [0,T]} \| \mathcal{P}_{H \setminus I} \tilde{X}_t \|_{L^2^p(P;V)}
\leq \left[ \sup_{v \in H_p \setminus [0]} \| v \|_{H_p} \right] \left[ \sup_{t \in [0,T]} \| \mathcal{P}_{H \setminus I} (\tilde{X}_t - \tilde{O}_t) \|_{L^2^p(P;H)} \right] \| (-A)^{-\rho} \mathcal{P}_{H \setminus I} \|_{L(H)} + \sup_{t \in [0,T]} \| \mathcal{P}_{H \setminus I} \tilde{O}_t \|_{L^2^p(P;V)}
\leq \left[ \sup_{v \in H_p \setminus [0]} \| v \|_{H_p} \right] \left[ \sup_{t \in [0,T]} \| \tilde{X}_t - \tilde{O}_t \|_{L^2^p(P;H)} \right] \| (-A)^{-\rho} \mathcal{P}_{H \setminus I} \|_{L(H)} + \sup_{t \in [0,T]} \| \mathcal{P}_{H \setminus I} \tilde{O}_t \|_{L^2^p(P;V)}.
$$

(126)

Furthermore, observe that the triangle inequality, the fact that $\forall I \in \mathcal{P}_0(\mathbb{H})$: $\| P_I \|_{L(H)} \leq 1$, the fact that $H_{(\rho+\rho)} \subseteq V$ continuously, the fact that

$$
\forall t \in [0,T]: \mathbb{P}(O_t = \tilde{O}_t) \geq \mathbb{P}(\tilde{\Omega}) = 1,
$$

(127)

the assumption that

$$
\sup_{t \in [0,T]} \sup_{I \in \mathcal{D}} \mathbb{E} \left[ \| P_I O_t \|_{L^p V}^p \right] < \infty,
$$

(128)

and (124) imply that

$$
\sup_{t \in [0,T]} \sup_{I \in \mathcal{D}} \| P_I \tilde{X}_t \|_{L^p V} \leq \sup_{t \in [0,T]} \sup_{I \in \mathcal{D}} \| P_I \tilde{X}_t - P_I \tilde{O}_t \|_{L^p V} + \sup_{t \in [0,T]} \sup_{I \in \mathcal{D}} \| P_I \tilde{O}_t \|_{L^p V} < \infty.
$$

(129)
In the next step we note that the hypothesis that \( H_p \subseteq V \) continuously and Lemma 2.1 ensure that
\[
\sup_{t \in (0, T)} t^\rho \| e^{tA} \|_{L(H,V)} \leq \left[ \sup_{v \in H_p \setminus \{0\}} \| v \|_{H_p} \right] \left[ \sup_{t \in (0, T)} \sup_{v \in H_p \setminus \{0\}} \frac{\| (-tA)^\rho e^{tA} v \|_H}{\| v \|_H} \right] < \infty.
\] (130)

Moreover, observe that, e.g., Lemma 5.3 (with \( V = H_p \times H_p \), \( W = V \times V \), \( S = \mathcal{B}([0, \infty)) \), \( s = 0 \), \( \psi = H_p \times H_p \ni (v, w) \mapsto \| v \|_{H_p} + \| w \|_{H_p} \)) \( \subseteq \{0, \infty\} \), \( \psi = \mathcal{V} \) in the notation of Lemma 5.3) establishes that
\[
\mathcal{V} \in \mathcal{M}(B(V \times V), B([0, \infty))).
\] (131)

Combining the fact that \( \forall t \in [0, T] : \tilde{X}_t = \int_0^t e^{(t-s)A} F(\tilde{X}_s) \, ds + \tilde{\xi}_t \) (112), (130), item (iii) in Lemma 5.2, item (iv) in Lemma 5.2, and, e.g., Andersson et al. [1, Lemma 2.2] with Proposition 4.4 (with \( H = H \), \( \mathcal{H} = \mathbb{H} \), \( T = T \), \( c = c \), \( \varphi = \varphi \), \( C = C \), \( D = D \), \( \mu = \mu \), \( A = A \), \( V = V \), \( \mathcal{V} = \mathcal{V} \), \( F = F \), \( (P_t)_{t \in [0, T]} \subseteq L(H) = (\mathbb{P}_t)_{t \in [0, T]} \subseteq L(H) \), \( (\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P}) \), \( X_t(\omega) = \tilde{X}_t(\omega), \), \( O_t(\omega) = \tilde{\xi}_t(\omega), \), \( X_t \in M \), \( I = I \) for \( M \in \mathbb{N} \), \( I \in D \), \( t \in [0, T] \), \( \omega \in \Omega \) in the notation of Proposition 4.4), the fact that \( \forall t \in [0, T] : \mathbb{P}(X_t = \tilde{X}_t) \geq \mathbb{P}(\tilde{\Omega}) = 1 \), the fact that \( \forall t \in [0, T] : \mathbb{P}(O_t = \tilde{\xi}_t) \geq \mathbb{P}(\tilde{\Omega}) = 1 \), and (111) hence proves that for all \( M \in \mathbb{N} \), \( I \in D \) we have that
\[
\sup_{t \in [0, T]} \| X_t - \tilde{X}_t \|_{L^p(\mathbb{P}; H)} = \sup_{t \in [0, T]} \| X_t - \tilde{X}_t \|_{L^p(\mathbb{P}; H)} \leq \sqrt{4^{(2+\varphi)} \max\{1, T^{(3/2+\varphi-\rho+\varphi/2-2\rho/2)}\} \max\{1, e^{(1+\varphi/2)}\} \sqrt{\text{ess inf} T}} \\min\{1, \sqrt{c(\kappa-2)}\} (1-\rho-\varphi)^{(1+\varphi/2)}
\]
\[
\cdot \left[ \sup_{t \in [0, T]} \| \mathbb{P}_t X_t \|_{L^2(\mathbb{P}; V)} + M^{\min\{\varphi, \varphi/2\}} \right] \cdot \sup_{t \in [0, T]} \| P_t (O_t - O_t) \|_{L^2(\mathbb{P}; V)} + \sup_{t \in [0, T]} \| P_t O_t - \Sigma_{M,I} \|_{L^2(\mathbb{P}; V)}
\]
\[
\cdot \left[ 1 + \sup_{s \in [0, T]} \| \tilde{X}_s \|_{L^p(\mathbb{P}; V)}^{\varphi/2} + \sup_{s \in [0, T]} \| P_t \tilde{X}_s \|_{L^p(\mathbb{P}; V)}^{\varphi/2} + \sup_{s \in [0, T]} \| \tilde{X}_s \|_{L^p(\mathbb{P}; V)}^{\varphi/2} \right]
\]
\[
\cdot \left[ 1 + \sup_{s \in [0, T]} \left\{ \| \tilde{X}_s \|_{L^2(\mathbb{P}; H_p)} + \| \Sigma_{M,I} \|_{L^2(\mathbb{P}; H_p)} \right\}^{\varphi/2} \right].
\]
\[
\begin{align*}
\max \left\{ 1, \sup_{s \in (0, T)} \left[ S^2 \| \mathcal{e}^A \|_{L(H, V)} \right] \right\} \\
\cdot \max \left\{ 1, \sup_{s \in [0, T)} \| \tilde{X}_s^{M,I} \|_{L^P(\mathbb{P}; H)^{\lfloor (1+\varphi/2)^2 \rfloor}} \right\} + \| F(0) \|_{L^H(\mathbb{P}; H)^{\lfloor (1+\varphi/2) \rfloor}}.
\end{align*}
\]  

The fact that

\[
\forall M \in \mathbb{N}, I \in \mathcal{D}, t \in [0, T]: \mathbb{P}(\mathcal{X}_t^{M,I} = \tilde{X}_t^{M,I}) = 1,
\]  

the fact that

\[
\forall t \in [0, T]: \mathbb{P}(O_t = \tilde{O}_t) \geq \mathbb{P}(\tilde{\Omega}) = 1,
\]  

the fact that \( \forall I \in \mathcal{D}: \| P_I \|_{L(H)} \leq 1 \), (126), and (130) therefore assure that for all \( M \in \mathbb{N}, I \in \mathcal{D} \) we have that

\[
\begin{align*}
\sup_{t \in [0, T]} \| X_t - \tilde{X}_t^{M,I} \|_{L^P(\mathbb{P}; H)} &= \sup_{t \in [0, T]} \| X_t - \tilde{X}_t^{M,I} \|_{L^P(\mathbb{P}; H)} \\
& \leq 4^{(2+\varphi)} \max\{1, T^{(3/2+\varphi)} X - \rho + \varphi / 2 \} \max\{1, c(1+\varphi/4) \sqrt{e^{cT}} \} \\
& \cdot \left[ \sup_{t \in [0, T]} \| \mathbb{P}_{\mathbb{H} \setminus I} O_t \|_{L^2(\mathbb{P}; V)} \\
& \quad + \sup_{t \in [0, T]} \| \tilde{X}_t - \tilde{O}_t \|_{L^2(\mathbb{P}; H(\rho + \varphi))} \| (-A)^{-\theta} \mathbb{P}_{\mathbb{H} \setminus I} \|_{L(H)} + M^{-\min\{\theta, \varphi / 2\}} \right] \\
& \quad + \sup_{t \in (0, T)} \| P_I (O_t - O_{[t], I}) \|_{L^2(\mathbb{P}; H(\rho + \varphi))} + \sup_{t \in [0, T]} \| P_I O_t - \mathfrak{M}_t^{M,I} \|_{L^2(\mathbb{P}; V)} \\
& \cdot \max\left\{ 1, \sup_{v \in V \setminus \{0\}} \| v \|_H \right\} \\
& \cdot \left[ 1 + \sup_{s \in (0, T)} \| \tilde{X}_s \|^{\phi/2}_{L^p(\mathbb{P}; V)} + \sup_{s \in (0, T)} \| P_s \tilde{X}_s \|^{\phi/2}_{L^p(\mathbb{P}; V)} + \sup_{s \in (0, T)} \| \tilde{X}_s^{M,I} \|^{\phi/2}_{L^p(\mathbb{P}; H)} \right] \\
& \quad + \sup_{s \in (0, T)} \| P_s O_s \|^{\phi/2}_{L^p(\mathbb{P}; V)} \right] \\
& \cdot \left[ 1 + \sup_{s \in (0, T)} \left\{ \| \tilde{X}_s^{M,I} \|_{L^4(\mathbb{P}; H)} + \| \mathfrak{M}_s^{M,I} \|_{L^4(\mathbb{P}; H)} \right\}^\theta \right] \\
& \cdot \max\left\{ 1, \sup_{v \in H \setminus \{0\}} \| v \|_{L^2(\mathbb{P}; H)}^{\lfloor (1+\varphi/2)^2 + \varphi/2 \rfloor} \right\}.
\end{align*}
\]
Therefore establish item (ii). The proof of Theorem 5.5 is thus completed.

**Corollary 5.6** Assume the setting in Sect. 5.1, let \( \theta \in [0, \infty) \), \( \vartheta \in (0, \infty) \), \( p \in [\max\{2, 1/\varphi\}, \infty) \), \( \varphi \in [0, 1 - \rho) \), and assume that

\[
\sup_{t \in [0, T]} \mathbb{E}\left[ \|X_t\|_V^{p(1+\varphi/2)\max\{2,\varphi\}} \right] \quad + \quad \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \left( M^\theta \left( \mathbb{E}\left[ \|P_I(O_t - O_{|I\cup M})\|_V^{2p} \right] \right)^{1/p} \quad + \quad \mathbb{E}\left[ \|\Sigma_t^{M,I}\|_H^{2\varphi} \right] \right) \quad + \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \mathbb{E}\left[ \|\Sigma_t^{M,I}\|_{H^\varphi}^2 + \Phi(\Sigma_t^{M,I}) + J(\Sigma_t^{M,I})^{p \max\{2\vartheta,2+\varphi,1+\varphi/2\}} \right] \quad < \quad \infty.
\]

Then we have

(i) that \( \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0, T]} \mathbb{E}\left[ \|X_t^{M,I}\|_H^{p \max\{4\vartheta,4+2\varphi,1+\varphi/2\}} \right] \quad < \quad \infty \) and

(ii) that there exists a real number \( C \in (0, \infty) \) such that for all \( M \in \mathbb{N}, I \in \mathcal{D} \) it holds that

\[
\sup_{t \in [0, T]} \left( \mathbb{E}\left[ \|X_t - X_t^{M,I}\|_H^p \right] \right)^{1/p} \quad \leq \quad C \left[ M^{-\min\{\theta_X, \varphi, \theta\}} + \|(-A)^{-\theta}(\text{Id}_H - P_I)\|_{L(H)} \right] + \sup_{t \in [0, T]} \|\text{Id}_H - P_I\|_{L^2(\mathbb{P}; V)} \right].
\]
Proof of Corollary 5.6} Note that the triangle inequality, (137), and the fact that $H_V \subseteq V$ continuously yield that

$$\sup_{I \in D} \sup_{s \in (0, T)} \| P_I O_s \|_{L^p(P; V)}$$

$$\leq \sup_{M \in \mathbb{N}} \sup_{I \in D} \sup_{s \in (0, T)} \left[ \| P_I O_s - \mathcal{D}_s M, I \|_{L^p(P; V)} + \| \mathcal{D}_s M, I \|_{L^p(P; V)} \right]$$

$$\leq \sup_{M \in \mathbb{N}} \sup_{I \in D} \sup_{s \in (0, T)} \| P_I O_s - \mathcal{D}_s M, I \|_{L^p(P; V)}$$

$$+ \left[ \sup_{v \in H_V \setminus \{0\}} \|v\|_{V} \right] \left[ \sup_{M \in \mathbb{N}} \sup_{I \in D} \sup_{s \in (0, T)} \| \mathcal{D}_s M, I \|_{L^p(P; H_V^c)} \right] < \infty. \quad (139)$$

This together with (137) allows us to apply Theorem 5.5 to obtain that item (i) holds and that there exists a real number $K \in (0, \infty)$ such that for all $M \in \mathbb{N}$, $I \in D$ we have that

$$\sup_{t \in [0, T]} \| X_t - \mathcal{X}_t^M, I \|_{L^p(P; H)} \leq K \left( M^{-\min\{d, \omega, \theta\}} + \|(-A)^{-\theta}(\text{Id}_H - P_I)\|_{L(H)} \right)$$

$$+ \sup_{t \in [0, T]} \left[ \| (\text{Id}_H - P_I) O_t \|_{L^2(P; V)} + \| P_I O_t - \mathcal{D}_t M, I \|_{L^2(P; V)} \right]$$

$$+ \| P_I (O_t - O_{[I, T]} \|_{L^2(P; V)} \right). \quad (140)$$

Hence, we obtain that for all $M \in \mathbb{N}$, $I \in D$ we have that

$$\sup_{t \in [0, T]} \| X_t - \mathcal{X}_t^M, I \|_{L^p(P; H)}$$

$$\leq K \left( \|(-A)^{-\theta}(\text{Id}_H - P_I)\|_{L(H)} + \sup_{t \in [0, T]} \| (\text{Id}_H - P_I) O_t \|_{L^2(P; V)} + M^{-\min\{d, \omega, \theta\}} \right)$$

$$\cdot \left\{ 1 + \sup_{t \in [0, T]} \left[ M^\theta \left[ \| P_I O_t - \mathcal{D}_t M, I \|_{L^2(P; V)} + \| P_I (O_t - O_{[I, T]} \|_{L^2(P; V)} \right] \right] \right\}$$

$$\leq K \left[ 1 + \sup_{N \in \mathbb{N}} \sup_{j \in D} \sup_{t \in [0, T]} \left( N^{\theta} \left[ \| P_J (O_t - O_{[I, T]} \|_{L^2(P; V)} + \| P_J O_t - \mathcal{O}_j N, I \|_{L^2(P; V)} \right] \right) \right]$$

$$\cdot \left[ M^{-\min\{d, \omega, \theta\}} + \|(-A)^{-\theta}(\text{Id}_H - P_I)\|_{L(H)} + \sup_{t \in (0, T)} \| (\text{Id}_H - P_I) O_t \|_{L^2(P; V)} \right]. \quad (141)$$

Combining this with (137) establishes item (ii). The proof of Corollary 5.6 is thus completed. \hfill \Box
6 Stochastic Allen–Cahn equations

6.1 Setting

Consider the notation in Sect. 1.1, let \( T, v \in (0, \infty), a_0, a_1, a_2 \in \mathbb{R}, a_3 \in (-\infty, 0], (H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H) = (L^2(\lambda(0,1); \mathbb{R}), \langle \cdot, \cdot \rangle_{L^2(\lambda(0,1); \mathbb{R})}, \| \cdot \|_{L^2(\lambda(0,1); \mathbb{R})}), (e_n)_{n \in \mathbb{N}} \subseteq H, F: L^6(\lambda(0,1); \mathbb{R}) \to H, (P_n)_{n \in \mathbb{N}} \subseteq L(H) \) satisfy for all \( n \in \mathbb{N}, v \in L^6(\lambda(0,1); \mathbb{R}) \) that \( a_2 \mathbb{I}_{\{0\}}(a_3) = 0, e_n = ([\sqrt{2} \sin(n\pi x)]_{x \in (0,1)}),_{{\lambda(0,1)},B(\mathbb{R})}, F(v) = \sum_{k=0}^{3} a_k v^k, P_n(v) = \sum_{k=1}^{n} (e_k, v)_H e_k, \) let \( A: D(A) \subseteq H \to H \) be the Laplacian with Dirichlet boundary conditions on \( H \) times the real number \( v, \) let \( (H_r, \langle \cdot, \cdot \rangle_{H_r}, \| \cdot \|_{H_r}), r \in \mathbb{R}, \) be a family of interpolation spaces associated to \(-A\) (cf., e.g., [37, Section 3.7]), let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, let \((W_t)_{t \in [0,T]}\) be an \( \text{Id}_H \)-cylindrical Wiener process, and let \((\cdot): \{[v]_{\lambda(0,1),B(\mathbb{R})} \in H: (v \in C((0, 1), \mathbb{R})\text{ is a uniformly continuous function}) \to C((0, 1), \mathbb{R})\) be the function which satisfies for all uniformly continuous functions \( v: (0, 1) \to \mathbb{R} \) that \([v]_{\lambda(0,1),B(\mathbb{R})} = v.\)

6.2 Properties of the nonlinearities of stochastic Allen–Cahn equations

Lemma 6.1 Assume the setting in Sect. 6.1 and let \( \epsilon \in (0,1), c \in \left\{ \frac{32}{9} \right\} \max \left\{ \frac{|a_2|^2}{|a_3|+\|a_3\|_{H_0}}, |a_3| \right\}, \infty \). Then there exists a real number \( C \in (0, \infty) \) such that for all \( N \in \mathbb{N}, v \in P_N(H), w \in \tilde{C}([0, T], H_1), t \in [0, T] \) we have that

\[
\langle v, P_N(F(v + w_t))_H \rangle_{H^{1/2}} + c \left[ \sup_{s \in [0, T]} \| w_s \|_{L^\infty(\lambda(0,1); \mathbb{R})}^4 + 1 \right] \langle v, F(v + w_t) \rangle_H \\
\leq \epsilon \| v \|^2_{H^1} + \left| a_1 \right| + \frac{|a_2|^2}{3|a_3|+\|a_3\|_{H_0}} \left\| v \right\|^2_{H^{1/2}} + c \left[ \sup_{s \in [0, T]} \| w_s \|_{L^\infty(\lambda(0,1); \mathbb{R})}^4 + 1 \right] \left| a_0 \right| + \frac{3|a_1|}{2} \| v \|^2_{H} + C \left[ \sup_{s \in [0, T]} \| w_s \|_{L^\infty(\lambda(0,1); \mathbb{R})}^8 + 1 \right].
\]

(142)

Proof of Lemma 6.1 Throughout this proof let \( \eta: \mathbb{N}_0 \times \mathbb{N}_0 \to (0, \infty) \) be a function which satisfies that

\[
\mathbb{I}_{(-\infty,0)}(a_3) \left( a_3 + \frac{1}{2} \sum_{k=0}^{3} \min\{k,2\} \sum_{j=0}^{k} (k+j)! \frac{|a_3|}{(j+1)!^2} \right) \leq 0.
\]

(143)

Observe that the fact that for every \( N \in \mathbb{N} \) we have that \( P_N \) is symmetric implies that for all \( N \in \mathbb{N}, v \in P_N(H), w \in \tilde{C}([0, T], H_1), t \in [0, T] \) we have that

\[
\langle v, \sum_{k=0}^{3} a_k P_N[v + w_t]^k \rangle_{H^{1/2}} = \sum_{k=0}^{3} a_k \langle (-A)^{1/2} v, (-A)^{1/2} P_N[v + w_t]^k \rangle_H = \sum_{k=0}^{3} a_k \langle (-A) v, P_N[v + w_t]^k \rangle_H
\]
\[
\begin{align*}
&= - \sum_{k=0}^{3} a_k \langle A P_N v, [v + w_t]^k \rangle_H = - \sum_{k=0}^{3} a_k \langle A v, [v + w_t]^k \rangle_H. \\
\end{align*}
\]

This shows that for all \( N \in \mathbb{N} \), \( v \in P_N(H) \), \( w \in \mathcal{C}([0, T], H_1), t \in [0, T] \) we have that
\[
\langle v, \sum_{k=0}^{3} a_k P_N[v + w_t]^k \rangle_{H^{3/2}} = - \sum_{k=0}^{3} a_k \langle v'', [v + w_t]^k \rangle_H \leq - \sum_{k=0}^{3} \sum_{j=0}^{k} \binom{k}{j} a_k \langle v'', v^j w_t^{(k-j)} \rangle_H
\]
\[
= - \sum_{k=1}^{3} \binom{k}{k} a_k \langle v'', v^k \rangle_H - \sum_{k=0}^{3} \sum_{j=0}^{\max(0, k-1)} \binom{k}{j} a_k \langle v'', v^j w_t^{(k-j)} \rangle_H.
\]

Moreover, note that integration by parts and the fact that
\[
\forall x, y \in \mathbb{R}, r \in (0, \infty) : x \sqrt{2r} \cdot y \frac{\sqrt{2r}}{\sqrt{2r}} \leq r x^2 + \frac{y^2}{4r}
\]
prove that for all \( N \in \mathbb{N} \), \( v \in P_N(H) \) we have that
\[
- \sum_{k=1}^{3} \binom{k}{k} a_k \langle v'', v^k \rangle_H = \sum_{k=1}^{3} k a_k \langle v', v^{(k-1)} v' \rangle_H \\
\leq 3 \sqrt{a_3} \int_0^1 |v'(x)|^2 |v(x)|^2 dx + 2v|a_2| \int_0^1 |v'(x)|^2 |v(x)| dx + v|a_1| \|v''\|_H^2 \\
\leq 3 \sqrt{a_3} \int_0^1 |v'(x)|^2 |v(x)|^2 dx + \sqrt{\int_0^1 \left( 3|a_3||v(x)|^2 + \frac{4|a_2|^2}{4(3|a_3| + \frac{R}{(a_3)})} \right)^2 dx} \\
+ v|a_1| \|v''\|_H^2 \\
= v \left( |a_1| + \frac{|a_2|^2}{3|a_3| + \frac{R}{(a_3)}} \right) \|v''\|_H^2 = \left( |a_1| + \frac{|a_2|^2}{3|a_3| + \frac{R}{(a_3)}} \right) \|v''\|_{H^{3/2}}^2.
\]

Furthermore, observe that the fact that
\[
\forall x, y \in \mathbb{R} : xy \leq \epsilon x^2 + \frac{y^2}{4\epsilon}
\]
shows that for all \( N \in \mathbb{N} \), \( v \in P_N(H) \), \( w \in \mathcal{C}([0, T], H_1), t \in [0, T] \) we have that
\[
- \sum_{k=0}^{3} \sum_{j=0}^{\max(0, k-1)} \binom{k}{j} a_k \langle v'', v^j (w_t)^{(k-j)} \rangle_H \\
\leq v \sum_{k=0}^{3} \sum_{j=0}^{\max(0, k-1)} \binom{k}{j} |a_k| \int_0^1 |v''(x)| |v(x)|^j |w_t(x)|^{(k-j)} \, dx
\]
\[
\begin{align*}
&= \int_0^1 v |v''(x)| \left[ \sum_{k=0}^3 \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} |a_k| |v(x)|^j |w_t(x)|^{(k-j)} \right] \, dx \\
&\leq \epsilon v^2 \|v''\|_H^2 + \frac{1}{4\epsilon} \int_0^1 \left[ \sum_{k=0}^3 \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} |a_k| |v(x)|^j |w_t(x)|^{(k-j)} \right]^2 \, dx.
\end{align*}
\]

(149)

The fact that
\[
\forall x_1, x_2, \ldots, x_7 \in \mathbb{R}: (x_1 + x_2 + \ldots + x_7)^2 \leq 7([x_1]^2 + [x_2]^2 + \ldots + [x_7]^2)
\]

(150)

hence assures that for all \( N \in \mathbb{N} \), \( v \in P_N(H) \), \( w \in C([0, T], H_1) \), \( t \in [0, T] \) we have that
\[
\begin{align*}
&- v \sum_{k=0}^3 \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} a_k \langle v'', v^j (w_t)^{(k-j)} \rangle_H \\
&\leq \epsilon v^2 \|v''\|_H^2 + \frac{7}{4\epsilon} \sum_{k=0}^3 \sum_{j=0}^{\max\{0,k-1\}} \left[ \binom{k}{j} |a_k| \right]^2 \int_0^1 \|v(x)|^j |w_t(x)|^{(k-j)-j} \, dx \\
&= \epsilon \|v\|_{H_1}^2 + \frac{7}{4\epsilon} \sum_{k=0}^3 |a_k|^2 \int_0^1 |w_t(x)|^{2k} \, dx \\
&\quad + \frac{7}{4\epsilon} \sum_{k=2}^3 \sum_{j=1}^{k-1} \left[ \binom{k}{j} |a_k| \right]^2 \int_0^1 \|v(x)|^j |w_t(x)|^{2(k-j)} \, dx.
\end{align*}
\]

(151)

This and the fact that
\[
\forall x, y \in \mathbb{R}: xy \leq \frac{x^2}{2} + \frac{y^2}{2}
\]

(152) imply that for all \( N \in \mathbb{N} \), \( v \in P_N(H) \), \( w \in C([0, T], H_1) \), \( t \in [0, T] \) we have that
\[
\begin{align*}
&- v \sum_{k=0}^3 \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} a_k \langle v'', v^j (w_t)^{(k-j)} \rangle_H \\
&\leq \epsilon \|v\|_{H_1}^2 + \frac{7}{\epsilon} \left[ \max_{k \in \{0,1,2,3\}} |a_k|^2 \right] \int_0^1 \max\{|w_t(x)|^6, 1\} \, dx \\
&\quad + \frac{7}{4\epsilon} \left[ \max_{k \in \{2,3\}, j \in \{1,2\}} \binom{k}{j} |a_k| \right]^2 \int_0^1 \left[ |v(x)|^2 |w_t(x)|^2 + |v(x)|^2 |w_t(x)|^4 \right] \, dx \\
&\quad + |v(x)|^4 |w_t(x)|^2 \, dx
\end{align*}
\]
\begin{equation}
\begin{aligned}
\leq \varepsilon \|v\|^2_{H_1} + \frac{7}{\varepsilon} \left[ \max_{k \in \{0, 1, 2, 3\}} |a_k|^2 \right] \left( \|w_t\|^6_{L^6(\lambda(0, 1); \mathbb{R})} + 1 \right) \\
+ \frac{7}{4\varepsilon} \left[ \max_{k \in \{2, 3\}, j \in \{1, 2\}} (k)! |a_k| \right]^2 \int_0^1 \left[ |v(x)|^4 + |w_t(x)|^4 + |\langle x \rangle^4 |w_t(x)|^4 \right] dx.
\end{aligned}
\end{equation}

(153)

Hölder’s inequality therefore ensures that for all \( N \in \mathbb{N} \), \( v \in P_N(H) \), \( w \in C([0, T], H_1) \), \( t \in [0, T] \) we have that

\begin{equation}
\begin{aligned}
- \nu \sum_{k=0}^3 \max_{0 \leq k \leq 1} \sum_{j=0}^{k} \binom{k}{j} a_k \langle v^n, v^j (w_t)^{(k-j)} \rangle_H \\
\leq \varepsilon \|v\|^2_{H_1} + \frac{7}{\varepsilon} \left[ \max_{k \in \{0, 1, 2, 3\}} |a_k|^2 \right] \left( \|w_t\|^6_{L^6(\lambda(0, 1); \mathbb{R})} + 1 \right) \\
+ \frac{63}{4\varepsilon} \left[ \max_{k \in \{2, 3\}} |a_k|^2 \right] \int_0^1 |w_t(x)|^4 dx \\
+ \frac{63}{4\varepsilon} \left[ \max_{k \in \{2, 3\}} |a_k|^2 \right] \int_0^1 |v(x)|^4 \left( 1 + |w_t(x)|^4 \right) dx \\
\leq \varepsilon \|v\|^2_{H_1} + \frac{23}{\varepsilon} \left[ \max_{k \in \{0, 1, 2, 3\}} |a_k|^2 \right] \left( \|w_t\|^6_{L^6(\lambda(0, 1); \mathbb{R})} + 1 \right) \\
+ \frac{16}{\varepsilon} \left[ \max_{k \in \{2, 3\}} |a_k|^2 \right] \|v\|^4_{L^4(\lambda(0, 1); \mathbb{R})} \left( \|w_t\|^4_{L^4(\lambda(0, 1); \mathbb{R})} + 1 \right).
\end{aligned}
\end{equation}

(154)

In the next step we combine (145) with (147) and (154) to obtain that for all \( N \in \mathbb{N} \), \( v \in P_N(H) \), \( w \in C([0, T], H_1) \), \( t \in [0, T] \) we have that

\begin{equation}
\begin{aligned}
\langle v, \sum_{k=0}^3 a_k P_N[v + w_t]^k \rangle_{H_{1/2}} \\
\leq \varepsilon \|v\|^2_{H_1} + \left( |a_1| + \frac{|a_2|^2}{3|a_3| + 1} \right) \|v\|^2_{H_{1/2}} \\
+ \left[ \max_{k \in \{0, 1, 2, 3\}} \frac{|a_k|^2}{\varepsilon^2} \right] \left( \|w_t\|^6_{L^6(\lambda(0, 1); \mathbb{R})} + 1 \right) \\
+ \frac{c|a_3|^2}{2} \left( \|w_t\|^4_{L^4(\lambda(0, 1); \mathbb{R})} + 1 \right) \|v\|^4_{L^4(\lambda(0, 1); \mathbb{R})}.
\end{aligned}
\end{equation}

(155)

In addition, note that for all \( N \in \mathbb{N} \), \( v \in P_N(H) \), \( w \in C([0, T], H_1) \), \( t \in [0, T] \) we have that

\begin{equation}
\begin{aligned}
\langle v, \sum_{k=0}^3 a_k [v + w_t]^k \rangle_H \\
= \sum_{k=0}^3 a_k \langle v, [v + w_t]^k \rangle_H = \sum_{k=0}^3 \sum_{j=0}^{k} \binom{k}{j} a_k \langle v, v^j (w_t)^{(k-j)} \rangle_H
\end{aligned}
\end{equation}
\[
= a_3 \|v\|_{L^4(\lambda; \mathbb{R}^d; \mathbb{R})}^4 + \sum_{k=0}^3 \sum_{j=0}^{\min\{k, 2\}} \binom{k}{j} a_k \mathcal{H}^j \mathcal{H}^{j-1} \int_0^1 |v(x)|^j \, dx.
\]

Young’s inequality hence demonstrates that for all \(N \in \mathbb{N}, v \in P_N(H), w \in C([0, T], H_1), t \in [0, T], r \in [0, \infty)\) we have that

\[
\sum_{k=0}^3 \mathcal{I}_{(-\infty, 0)}(a_k) \mathcal{H} \mathcal{H}^{j-1} \int_0^1 |v(x)|^j \, dx
\]

Moreover, note that the fact that

\[
\forall x, y \in [0, \infty) : xy \leq \frac{x^2}{2} + \frac{y^2}{2}
\]

(159)
ensures that for all \( N \in \mathbb{N} \), \( v \in P_N(H) \), \( w \in C([0, T], H_1) \), \( t \in [0, T] \), \( r \in [0, \infty) \) we have that

\[
\begin{align*}
\langle v, \sum_{k=0}^{3} a_k [v + w_t]^k \rangle_{H} & \\
& \leq r \int_0^1 |a_0| |v(x)| + |a_1| |v(x)|^2 + |a_1| |v(x)| |w_t(x)| \, dx \\
& \leq r \int_0^1 |a_0| \left( 1 + |v(x)|^2 \right) + |a_1| |v(x)|^2 + \frac{|a_1|}{2} \left( |v(x)|^2 + |w_t(x)|^2 \right) \, dx \\
& = r \left( |a_0| + \frac{3|a_1|}{2} \right) \|v\|_H^2 + r \left( |a_0| + \frac{|a_1|}{2} \right) \|w_t\|_H^2 \\
& \leq r \left( |a_0| + \frac{3|a_1|}{2} \right) \|v\|_H^2 + r \left( |a_0| + \frac{|a_1|}{2} \right) \left[ \|w_t\|_{L^4(\lambda(0,1);\mathbb{R})}^4 + 1 \right].
\end{align*}
\]

(160)

Hence, we obtain that for all \( N \in \mathbb{N} \), \( v \in P_N(H) \), \( w \in C([0, T], H_1) \), \( t \in [0, T] \), \( r \in [\|w_t\|_{L^4(\lambda(0,1);\mathbb{R})}^4 + 1, \infty) \) we have that

\[
\begin{align*}
\langle v, \sum_{k=0}^{3} a_k P_N[v + w_t]^k \rangle_{H_{1/2}} & + cr \langle v, \sum_{k=0}^{3} a_k [v + w_t]^k \rangle_{H} \\
& = \langle v, \sum_{k=0}^{3} a_k P_N[v + w_t]^k \rangle_{H_{1/2}} + cr \langle v, \sum_{k=0}^{3} a_k [v + w_t]^k \rangle_{H} \|_{\mathbb{R}^R(\infty,0)}(a_3) + cr \langle v, \sum_{k=0}^{3} a_k [v + w_t]^k \rangle_{H} \|_{\mathbb{R}^R(0)}(a_3) \\
& \leq \langle v, \sum_{k=0}^{3} a_k P_N[v + w_t]^k \rangle_{H_{1/2}} + cr \langle v, \sum_{k=0}^{3} a_k [v + w_t]^k \rangle_{H} \|_{\mathbb{R}^R(\infty,0)}(a_3) \\
& + cr \left( |a_0| + \frac{3|a_1|}{2} \right) \|v\|_H^2 + cr \left( |a_0| + \frac{|a_1|}{2} \right) \left[ \|w_t\|_{L^4(\lambda(0,1);\mathbb{R})}^4 + 1 \right].
\end{align*}
\]

(161)

Combining this with (155) and (158) assures that for all \( N \in \mathbb{N} \), \( v \in P_N(H) \), \( w \in C([0, T], H_1) \), \( t \in [0, T] \), \( r \in [\|w_t\|_{L^4(\lambda(0,1);\mathbb{R})}^4 + 1, \infty) \) we have that

\[
\begin{align*}
\langle v, \sum_{k=0}^{3} a_k P_N[v + w_t]^k \rangle_{H_{1/2}} & + cr \langle v, \sum_{k=0}^{3} a_k [v + w_t]^k \rangle_{H} \\
& \leq \epsilon \|v\|_{H_1}^2 + \left( |a_1| + \frac{|a_2|^2}{3|a_3|} + \|_{\mathbb{R}^R(0)}(a_3) \right) \|v\|_{H_{1/2}}^2 + cr \left( |a_0| + \frac{3|a_1|}{2} \right) \|v\|_H^2 \\
& + \frac{cr}{2} \|v\|_{L^4(\lambda(0,1);\mathbb{R})}^4 (a_3 + |a_3|) + \left[ \max_{k \in \{0,1,2,3\}} \frac{5|a_k|}{\sqrt{6}} \right]^2 \left[ \|w_t\|_{L^6(\lambda(0,1);\mathbb{R})} + 1 \right] \\
& + cr \left[ |a_0| + \frac{|a_1|}{2} \right] + \sum_{k=0}^3 \sum_{j=0}^{\min(k,2)} \left( k \right) \left( 3 - j \right) |a_k| \frac{4}{3-j} \left[ \|w_t\|_{L^4(\lambda(0,1);\mathbb{R})} + 1 \right].
\end{align*}
\]

(162)
Hölder’s inequality and the fact that
\[ a_3 + |a_3| = a_3 - a_3 = 0 \] (163)

therefore prove that for all \( N \in \mathbb{N}, v \in P_N(H), w \in C([0, T], H_1), t \in [0, T], r \in \|w_t\|^3_{L^\infty(\lambda(0,1); \mathbb{R})} + 1, \infty \) we have that

\[
\langle v, \sum_{k=0}^3 a_k P_N[v + w_t]^k \rangle_{H^{1/2}} + cr \langle v, \sum_{k=0}^3 a_k [v + w_t]^k \rangle_{H^1}
\leq \epsilon \|v\|_{H^1}^2 + \left( |a_1| + \frac{|a_2|^2}{3|a_3| + \|R \|_0(a_3)} \right) \|v\|_{H^{1/2}}^2 + cr \left( |a_0| + \frac{3|a_1|}{2} \right) \|v\|^2_{H^1}
\]

\[
+ \left[ \max_{k \in \{0,1,2,3\}} \frac{5|a_k|}{\sqrt{\epsilon}} \right]^2 \left\| w_t \right\|^6_{L^\infty(\lambda(0,1); \mathbb{R})} + 1
\]

\[
+ cr^2 \left[ |a_0| + \frac{1}{2} a_1 + \sum_{k=0}^3 \sum_{j=0}^{\min\{k,2\}} \frac{k}{j} \frac{(3-j)|a_k|}{2|\eta(k, j)|^{4/(3-j)}} \right].
\] (164)

The fact that
\[ \forall x, y \in (0, \infty): (x+y)^2 \leq 2(x^2 + y^2) \] (165)
hence implies that for all \( N \in \mathbb{N}, v \in P_N(H), w \in C([0, T], H_1), t \in [0, T] \) we have that

\[
\langle v, \sum_{k=0}^3 a_k P_N[v + w_t]^k \rangle_{H^{1/2}} + c \left[ \sup_{s \in [0,T]} \|w_s\|^4_{L^\infty(\lambda(0,1); \mathbb{R})} + 1 \right] \langle v, \sum_{k=0}^3 a_k [v + w_t]^k \rangle_{H^1}
\leq \epsilon \|v\|_{H^1}^2 + \left( |a_1| + \frac{|a_2|^2}{3|a_3| + \|R \|_0(a_3)} \right) \|v\|_{H^{1/2}}^2
\]

\[
+ c \left[ \sup_{s \in [0,T]} \|w_s\|^4_{L^\infty(\lambda(0,1); \mathbb{R})} + 1 \right] \left( |a_0| + \frac{3|a_1|}{2} \right) \|v\|^2_{H^1}
\]

\[
+ \left[ \max_{k \in \{0,1,2,3\}} \frac{5|a_k|}{\sqrt{\epsilon}} \right]^2 \left\| w_t \right\|^8_{L^\infty(\lambda(0,1); \mathbb{R})} + 1
\]

\[
+ c \left[ 2|a_0| + |a_1| + \sum_{k=0}^3 \sum_{j=0}^{\min\{k,2\}} \frac{k}{j} \frac{(3-j)|a_k|}{2|\eta(k, j)|^{4/(3-j)}} \right] \left[ \sup_{s \in [0,T]} \|w_s\|^8_{L^\infty(\lambda(0,1); \mathbb{R})} + 1 \right].
\] (166)

Hence, we obtain that for all \( N \in \mathbb{N}, v \in P_N(H), w \in C([0, T], H_1), t \in [0, T] \) we have that

\[
\langle v, \sum_{k=0}^3 a_k P_N[v + w_t]^k \rangle_{H^{1/2}} + c \left[ \sup_{s \in [0,T]} \|w_s\|^4_{L^\infty(\lambda(0,1); \mathbb{R})} + 1 \right] \langle v, \sum_{k=0}^3 a_k [v + w_t]^k \rangle_{H^1}
\]
The proof of Lemma 6.1 is thus completed.

Lemma 6.2 Assume the setting in Sect. 6.1 and let \( q \in [6, \infty) \), \( v, w \in L^q(\lambda_{(0,1)}; \mathbb{R}) \). Then

\[
\| F(v) - F(w) \|_H^2 \leq 36 \left[ \max_{j \in \{1,2,3\}} |a_j| \right]^2 \| v - w \|_{L^q(\lambda_{(0,1)}; \mathbb{R})}^4 \left( 1 + \| v \|_{L^q(\lambda_{(0,1)}; \mathbb{R})}^4 \right)
\]

(168)

Proof of Lemma 6.2 Observe that the fundamental theorem of calculus and Jensen’s inequality ensure for all \( k \in \mathbb{N} \), \( x, y \in \mathbb{R} \) that

\[
|x^k - y^k| = \left| \int_0^1 k (y + r (x - y))^{(k-1)} (x - y) \, dr \right|
\]

\[
\leq k |x - y| \int_0^1 |r x + (1 - r) y|^{(k-1)} \, dr
\]

(169)

Combining this and Hölder’s inequality implies that

\[
\| F(v) - F(w) \|_H
\]

\[
= \left\| \sum_{k=0}^3 a_k \left( v^k - w^k \right) \right\|_H \leq \sum_{k=1}^3 |a_k| \left\| v^k - w^k \right\|_H
\]

\[
\leq \sum_{k=1}^3 |a_k| \int_0^1 \| v - w \|_{L^q(\lambda_{(0,1)}; \mathbb{R})} \left( r |v|^{(k-1)} + (1 - r) |w|^{(k-1)} \right) \, dr
\]

\[
\leq \sum_{k=1}^3 |a_k| \int_0^1 \| v - w \|_{L^q(\lambda_{(0,1)}; \mathbb{R})} \left( r |v|^{(k-1)} + (1 - r) |w|^{(k-1)} \right) \| v \|_{L^2 q/(q-2)(\lambda_{(0,1)}; \mathbb{R})} \, dr
\]

\[
\leq \| v - w \|_{L^q(\lambda_{(0,1)}; \mathbb{R})} \left[ |a_1| + \cdots + |a_3| \right]
\]
Again Hölder’s inequality therefore demonstrates that

$$\| F(v) - F(w) \|_H \leq \| v - w \|_{L^q(\lambda(0,1);\mathbb{R})} \left[ |a_1| + \frac{1}{2} \sum_{k=2}^{3} k |a_k| \left( \| v \|_{L^2(\lambda(0,1);\mathbb{R})}^{(k-1)} + \| w \|_{L^2(\lambda(0,1);\mathbb{R})}^{(k-1)} \right) \right]$$

This completes the proof of Lemma 6.2. \(\square\)

### 6.3 Properties of stochastic convolutions

In this subsection we present a few elementary regularity and approximation results for stochastic convolutions; see Lemmas 6.3–6.7 and Corollary 6.8 below. Similar regularity and approximation results for stochastic convolutions can, e.g., be found in Hutzenthaler et al. [21, Lemma 5.6, Corollary 5.8, and Lemma 5.9]. Proofs of Lemmas 6.3–6.7 and Corollary 6.8 can, e.g., be found in the extended arXiv version of this article [2, Section 6.3].

**Lemma 6.3** Assume the setting in Sect. 6.1, let \( \gamma \in [0, 1/4) \), \( \beta \in (1/4, 1/2 - \gamma) \), \( B \in HS(H, H_{-\beta}) \), and let \( \varphi : [0, T] \rightarrow [0, T] \) be a \( B([0, T])/B([0, T]) \)-measurable function which satisfies for all \( t \in [0, T] \) that \( \varphi(t) \leq t \). Then there exists an up to indistinguishability unique stochastic process \( O : [0, T] \times \Omega \rightarrow H \) with continuous sample paths which satisfies for all \( t \in [0, T] \) that

$$[O_t]_{\mathbb{F}, B(H)} = \int_0^t e^{(t-s)A} B \, dW_s. \tag{172}$$
Lemma 6.4 Assume the setting in Sect. 6.1 and let \( p, q \in [2, \infty), \theta \in \left[ \frac{1}{4} - \frac{1}{2q}, \frac{1}{4} \right], \xi \in \mathcal{L}^p(\mathbb{P}; H_{20}) \). Then there exists a stochastic process \( O : [0, T] \times \Omega \to L^q(\lambda_{(0,1)}; \mathbb{R}) \) with continuous sample paths which satisfies

(i) that for all \( t \in [0, T] \) we have that

\[
[O_t - e^{tA}\xi]_{\mathbb{P},B(H)} = \int_0^t e^{(t-s)A} \, dW_s
\]

and

(ii) that

\[
\sup_{N \in \mathbb{N}} \sup_{0 \leq s < t \leq T} \left( \frac{\| P_N(O_t - O_s)\|_{\mathcal{L}^p(\mathbb{P}; L^q(\lambda_{(0,1)}; \mathbb{R}))}}{(t-s)^q} \right) + \sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} (N^{2q} \| O_t - P_N O_t\|_{\mathcal{L}^p(\mathbb{P}; L^q(\lambda_{(0,1)}; \mathbb{R}))}) < \infty. \tag{173}
\]

Lemma 6.5 Assume the setting in Sect. 6.1 and let \( p \in [2, \infty), \theta \in [0, 1/4), \xi \in \mathcal{L}^p(\mathbb{P}; H_{R}) \). Then there exist stochastic processes \( \mathcal{O}^M,N : [0, T] \times \Omega \to P_N(H), M, N \in \mathbb{N}, \) with continuous sample paths which satisfy

(i) that for all \( t \in [0, T], M, N \in \mathbb{N} \) we have that

\[
[\mathcal{O}_t^M,N - P_N e^{tA}\xi]_{\mathbb{P},B(H)} = \int_0^t P_N e^{(t-s)A} \, dW_s
\]

and

(ii) that

\[
\sup_{\gamma \in [0,\theta]} \sup_{M,N \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E}\left[ \left\| \mathcal{O}_t^M,N \right\|^p_{L^\infty(\lambda_{(0,1)}; \mathbb{R})} \right] < \infty. \tag{174}
\]

Then

\[
\sup_{M,N \in \mathbb{N}} \mathbb{E}\left[ \sup_{t \in [0,T]} \left\| \mathcal{O}_t^M,N \right\|^p_{L^\infty(\lambda_{(0,1)}; \mathbb{R})} \right] < \infty. \tag{175}
\]

Lemma 6.6 Assume the setting in Sect. 6.1, let \( p \in [1, \infty), \xi \in \cup_{r \in (1/4, \infty)} \mathcal{L}^p(\mathbb{P}; H_{R}), \) and let \( \mathcal{O}^M,N : [0, T] \times \Omega \to P_N(H), M, N \in \mathbb{N}, \) be stochastic processes with continuous sample paths which satisfy for all \( M, N \in \mathbb{N}, t \in [0, T] \)

\[
[\mathcal{O}_t^M,N - P_N e^{tA}\xi]_{\mathbb{P},B(H)} = \int_0^t P_N e^{(t-s)A} \, dW_s. \tag{176}
\]

Then

\[
\sup_{M,N \in \mathbb{N}} \mathbb{E}\left[ \left( \left\| P_N O_t - \mathcal{O}_t^M,N \right\|^p_{L^q(\lambda_{(0,1)}; \mathbb{R})} \right)^{1/p} \right] < \infty. \tag{176}
\]

Corollary 6.8 Assume the setting in Sect. 6.1 and let \( p, q \in [2, \infty), \theta \in \left[ \frac{1}{4} - \frac{1}{2q}, \frac{1}{4} \right], \xi \in \cup_{r \in (1/4, \infty) \cap [2q, \infty)} \mathcal{L}^p(\mathbb{P}; H_{R}), \) and \( \mathcal{O}^M,N : [0, T] \times \Omega \to L^q(\lambda_{(0,1)}; \mathbb{R}) \) are stochastic processes. Then there exist stochastic processes \( O : [0, T] \times \Omega \to P_N(H), \) \( M, N \in \mathbb{N}, \) with continuous sample paths which satisfy

(i) that for all \( t \in [0, T] \) we have that

\[
[O_t - e^{tA}\xi]_{\mathbb{P},B(H)} = \int_0^t e^{(t-s)A} \, dW_s,
\]
(ii) that for all $M, N \in \mathbb{N}$, $t \in [0, T]$ we have that $[\mathcal{D}_t^{M,N} - P_N e^{tA}\xi]_{\mathbb{P},B(H)} = \int_0^t P_N e^{(t-s)[\gamma]} dW_s$, and

(iii) that for all $\gamma \in [0, 2\theta] \cap [0, 1/4]$ we have that

$$
\begin{align*}
&\sup_{M, N \in \mathbb{N}} \sup_{t \in [0, T]} \left[ M^\theta \left( \mathbb{E} \left[ \| P_N (O_t - O_{[1]T/M}) \|_{L^p(\lambda(0,1); \mathbb{R})}^p \right] \right)^{1/p} \
&\quad + \| P_N O_t - \mathcal{D}_t^{M,N} \|_{L^p(\lambda(0,1); \mathbb{R})}^{1/p} \right] \
&\quad + \sup_{M, N \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} \left[ \| \mathcal{D}_t^{M,N} \|_{H_p}^p \right] + \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} N^{2\theta} \left( \mathbb{E} \left[ \| O_t - P_N O_t \|_{L^p(\lambda(0,1); \mathbb{R})}^p \right] \right)^{1/p} \
&\quad + \sup_{M, N \in \mathbb{N}} \mathbb{E} \left[ \sup_{t \in [0, T]} \| \mathcal{D}_t^{M,N} \|_{L^\infty(\lambda(0,1); \mathbb{R})}^p \right] < \infty.
\end{align*}
$$
(177)

6.4 Strong convergence rates for numerical approximations of stochastic Allen–Cahn equations

Lemma 6.9 Assume the setting in Sect. 6.1, let $p \in [2, \infty)$, $\theta \in (0, \infty)$, $\theta \in [1/6, 1/4)$, $\gamma \in (1/6, 1/4)$, $\chi \in (0, \gamma/3 - 1/18)$, $\xi \in \cup_{r \in (1/4, \infty) \cap [20, \infty)} \mathcal{L}^{16p \max[3, \theta]}(\mathbb{P}; H_r)$, let $X : [0, T] \times \Omega \rightarrow L^6(\lambda(0,1); \mathbb{R})$ be a stochastic process with continuous sample paths which satisfies for all $t \in [0, T]$ that $[X_t - e^{tA}\xi - \int_0^t e^{(t-s)A} F(X_s) ds]_{\mathbb{P},B(H)} = \int_0^t e^{(t-s)A} dW_s$ and $\sup_{s \in [0, T]} \mathbb{E} \left[ \| X_s \|_{L^p(\lambda(0,1); \mathbb{R})}^{12p} \right] < \infty$, and let $\mathcal{X}^{M,N} : [0, T] \times \Omega \rightarrow H_Y$, $M, N \in \mathbb{N}$, and $\mathcal{D}^{M,N} : [0, T] \times \Omega \rightarrow P_N(H)$, $M, N \in \mathbb{N}$, be stochastic processes which satisfy for all $M, N \in \mathbb{N}$, $t \in [0, T]$ that $[\mathcal{D}_t^{M,N} - P_N e^{tA}\xi]_{\mathbb{P},B(H)} = \int_0^t P_N e^{(t-s)[\gamma]} dW_s$ and

$$
\mathbb{P} \left( \mathcal{X}_t^{M,N} = \int_0^t P_N e^{(t-s)[\gamma]} ds \bigg| \mathcal{X}_s^{M,N} \right) = \mathbb{E} \left[ \| \mathcal{X}_t^{M,N} \|_{H_p}^{12p \max[3, \theta]} \right] F(\mathcal{X}_s^{M,N} \cap [1/T]) ds + \mathcal{D}_t^{M,N} = 1.
$$
(178)

Then we have

(i) that $\sup_{M, N \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} \left[ \| \mathcal{X}_t^{M,N} \|_{H_p}^{12p \max[3, \theta]} \right] < \infty$ and

(ii) that there exists a real number $C \in \mathbb{R}$ such that for all $M, N \in \mathbb{N}$ we have that

$$
\sup_{t \in [0, T]} \left( \mathbb{E} \left[ \| X_t - \mathcal{X}_t^{M,N} \|_{H_p}^{12p \max[3, \theta]} \right] \right)^{1/p} \leq C (M^{-\min[\theta, \chi]} + N^{-2\theta}).
$$
(179)

Proof of Lemma 6.9 Throughout this proof let $(V, \| \cdot \|_V) = (L^6(\lambda(0,1); \mathbb{R}), \| \cdot \|_{L^6(\lambda(0,1); \mathbb{R})})$ and let $\epsilon \in (0, 1)$, $C \in [32/\epsilon \max\{3a_3r^3/(a_3+\|a_3\|_0), a_3\}]$, $\infty$ be real numbers. Note that, e.g., [4, Lemma 6.7] proves that for all $v, w \in L^{18}(\lambda(0,1); \mathbb{R})$ with
\[ v - w \in H_1 \text{ we have that} \]
\[ \langle v - w, A(v - w) + F(v) - F(w) \rangle_H \]
\[ \leq 2 \max\{1, \frac{1}{|a_3| + \frac{1}{(a_3^2)|a_3|}}\} \max_{k \in [1,2]} [k |a_k|^2] \|v - w\|_{H}^2. \tag{180} \]

The fact that \( H_1 \subseteq L^{18}(\lambda(0,1); \mathbb{R}) \) therefore implies that for all \( v, w \in H_1 \) we have that
\[ \langle v - w, A v + F(v) - A w - F(w) \rangle_H \]
\[ \leq 2 \max\{1, \frac{1}{|a_3| + \frac{1}{(a_3^2)|a_3|}}\} \max_{k \in [1,2]} [k |a_k|^2] \|v - w\|_{H}^2. \tag{181} \]

Combining this, Lemma 6.1 (with \( \epsilon = \epsilon, c = C \) in the notation of Lemma 6.1) and Lemma 6.2 ensures that there exit a real number \( c \in (0, \infty) \) such that
(a) we have for all \( N \in \mathbb{N}, v, w \in P_N(H) \) that
\[ \langle v - w, A v + F(v) - A w - F(w) \rangle_H \leq c \|v - w\|_{H}^2, \tag{182} \]
(b) we have for all \( N \in \mathbb{N}, v \in P_N(H), w \in C([0, T], H_1), t \in [0, T] \) that
\[ \langle v, P_N F(v + w_t) \rangle_{H_1/2} + C \left[ \sup_{s \in [0, T]} \|w_s\|_{L^{\infty}(\lambda(0,1); \mathbb{R})}^4 + 1 \right] \langle v, F(v + w_t) \rangle_H \]
\[ \leq \epsilon \|v\|_{H_1}^2 + c \|v\|_{H_1/2}^2 + c C \left[ \sup_{s \in [0, T]} \|w_s\|_{L^{\infty}(\lambda(0,1); \mathbb{R})}^4 + 1 \right] \|v\|_{H}^2 \]
\[ + c \left[ \sup_{s \in [0, T]} \|w_s\|_{L^{\infty}(\lambda(0,1); \mathbb{R})}^8 + 1 \right], \tag{183} \]
and
(c) we have for all \( N \in \mathbb{N}, v, w \in P_N(H) \) that
\[ \|F(v) - F(w)\|_{H}^2 \leq c \|v - w\|_{V}^2 \left(1 + \|v\|_{V}^4 + \|w\|_{V}^4\right). \tag{184} \]

Moreover, note that Corollary 6.8 (with \( p = 16 p \max\{3, \theta\}, q = 6, \theta = \theta, \xi = \xi \) in the notation of Corollary 6.8) and the fact that \( 2\theta > \gamma \) imply that there exist stochastic processes \( O : [0, T] \times \Omega \rightarrow V \) and \( \tilde{O}_M^{N, N} : [0, T] \times \Omega \rightarrow P_N(H), M, N \in \mathbb{N} \), with continuous sample paths which satisfy that
\[ \mathcal{S} \text{ Springer} \]
that for all $t \in [0, T]$ we have that
\begin{equation}
[O_t - e^{\Delta \alpha \xi}]_{\mathbb{P}, B(H)} = \int_0^t e^{(t-s)A} \, dW_s, \tag{186}
\end{equation}
and that for all $t \in [0, T]$, $M$, $N \in \mathbb{N}$ we have that
\begin{equation}
[\tilde{\mathcal{D}}_t^{M,N} - P_N \, e^{\Delta \alpha \xi}]_{\mathbb{P}, B(H)} = \int_0^t P_N \, e^{(t-s)A} \, dW_s. \tag{187}
\end{equation}
Observe that (186) and the fact that $\forall t \in [0, T], \forall M, N \in \mathbb{N}: \mathbb{P}(\mathcal{D}_t^{M,N} = \tilde{\mathcal{D}}_t^{M,N}) = 1$ guarantee for all $t \in [0, T]$, $M$, $N \in \mathbb{N}$ that
\begin{equation}
\mathbb{P}
\left(
X_t = \int_0^t e^{(t-s)A} \, F(X_s) \, ds + O_t
\right)
\begin{aligned}
\left\{ \bar{X}_t^{M,N} = \int_0^t P_N \, e^{(t-s)A} \, 1_{\{X_s^{\bar{X},T/M} \leq \|X_s^{\bar{X},T/M} \|_{L^p(\lambda(0,1);\mathbb{R})} \leq (M/T)^{\chi}\}} \, ds + \bar{\tilde{\mathcal{D}}}_t^{M,N} \right\}
\end{aligned}
\end{equation}
Combining (182)–(184) and (185)–(188) allows us to apply Corollary 5.6 (with $H = H$, $\mathbb{H} = \{e_k: k \in \mathbb{N}\}$, $T = T$, $c = c$, $\varphi = 4$, $\epsilon = \epsilon$, $\rho = \frac{1}{6}$, $\gamma = \gamma$, $\chi = \chi$, $\mathcal{D} = \{\{e_1\}, \{e_1, e_2\}, \{e_1, e_2, e_3\}, \ldots\}$, $\mu(e_N) = -\nu \pi^2 N^2$, $A = A$, $H_r = H_r$, $V = V$, $F = F$, $\phi = C([0, T], H_1) \ni w \mapsto C[\sup_{t \in [0, T]} \|w_t\|_{L^\infty(\lambda(0,1);\mathbb{R})} + 1] \in [0, \infty)$, $\Phi = C([0, T], H_1) \ni w \mapsto C[\sup_{t \in [0, T]} \|w_t\|_{L^\infty(\lambda(0,1);\mathbb{R})} + 1] \in [0, \infty)$, $P_t(e_1, e_2, \ldots, e_N)(v) = \sum_{k=1}^N \langle e_k, v \rangle_H \, e_k$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $X = X$, $O = O$, $\bar{\mathcal{D}}_t^{M,N} = \int_0^t \mathbb{1}_{\{X_s^{\bar{X},T/M} \leq \|X_s^{\bar{X},T/M} \|_{L^p(\lambda(0,1);\mathbb{R})} \leq (M/T)^{\chi}\}} \, ds + \bar{\tilde{\mathcal{D}}}_t^{M,N}$, $\theta = \theta$, $\vartheta = \vartheta$, $p = p$, $q = q$ for $N \in \mathbb{N}, r \in \mathbb{R}, v \in H$ in the notation of Corollary 5.6) to obtain that item (i) holds and that there exists a real number $K \in (0, \infty)$ such that for all $M, N \in \mathbb{N}$ we have that
\begin{equation}
\sup_{t \in [0, T]} \|X_t - \bar{X}_t^{M,N} \|_{L^p(\mathbb{P}; H)} \leq K \left[ M^{-\min(\theta, \chi, \theta)} + \|(-A)^{-\theta}(\mathbb{I}_H - P_N)\|_{L(H)} + \sup_{t \in [0, T]} \|O_t - P_N O_t\|_{L^2(\mathbb{P}; V)} \right]. \tag{189}
\end{equation}
The fact that
\begin{equation}
\forall N \in \mathbb{N}: \|(-A)^{-\theta}(\mathbb{I}_H - P_N)\|_{L(H)} = (\nu \pi^2 (N + 1)^2)^{-\theta} \tag{190}
\end{equation}
hence yields that for all \( M, N \in \mathbb{N} \) we have that

\[
\sup_{t \in [0,T]} \|X_t - \tilde{X}_t^{M,N}\|_{\mathcal{L}^p(\mathbb{P}; H)} \\
\leq K \left[ M^{-\min\{\delta, \theta\}} + N^{-2\theta} \left( \frac{N^{2\theta}}{\nu^\theta \pi^{2\theta} (N + 1)^{2\theta}} \right) + \sup_{t \in [0,T]} \left[ N^{2\theta} \|O_t - P_N O_t\|_{\mathcal{L}^2(\mathbb{P}; V)} \right] \right] \\
\leq K \left[ M^{-\min\{\delta, \theta\}} + N^{-2\theta} \left( \frac{1}{\nu^\theta \pi^{2\theta}} + \sup_{t \in [0,T]} \left[ N^{2\theta} \|O_t - P_N O_t\|_{\mathcal{L}^2(\mathbb{P}; V)} \right] \right) \right] \\
\leq K \max \left\{ 1, \frac{1}{\nu^\theta} + \sup_{t \in [0,T]} \left[ N^{2\theta} \|O_t - P_N O_t\|_{\mathcal{L}^2(\mathbb{P}; V)} \right] \right\} \left[ M^{-\min\{\delta, \theta\}} + N^{-2\theta} \right].
\]

(191)

Combining this with (185) completes the proof of Lemma 6.9. \( \square \)

**Corollary 6.10** Assume the setting in Sect. 6.1, let \( \xi \in \bigcap_{p \in [1,\infty]} \mathcal{L}^p(\mathbb{P}; H_{/2}) \), \( \gamma \in (1/6, 1/4) \), \( \chi \in (0, \gamma/3 - \gamma/18) \), and let \( \tilde{X}^M,N : [0, T] \times \Omega \to P_N(H) \), \( M, N \in \mathbb{N} \), and \( \Sigma^M,N : [0, T] \times \Omega \to P_N(H) \), \( M, N \in \mathbb{N} \), be stochastic processes which satisfy for all \( M, N \in \mathbb{N} \), \( t \in [0, T] \) that [\( \Sigma_t^M,N - P_N e^{tA} \xi \] ] = \( \int_0^t P_N e^{(t-s)A} \xi ] \mathbb{P}, B(H) = \int_0^t P_N e^{(t-s)A} ds + \Sigma_t^M,N \]

(192)

Then

(i) we have that there exists an up to indistinguishability unique stochastic process \( X : [0, T] \times \Omega \to L^6(\lambda_{(0,1)}; \mathbb{R}) \) with continuous sample paths which satisfies for all \( t \in [0, T] \), \( p \in (0, \infty) \) that \( \sup_{s \in [0,T]} \mathbb{E} \left[ \|X_s\|_{L^p(\lambda_{(0,1)}; \mathbb{R})} \right] < \infty \) and

\[
[X_t - e^{tA} \xi - \int_0^t e^{(t-s)A} F(X_s) ds]_{\mathbb{P}, B(H)} = \int_0^t e^{(t-s)A} ds.
\]

(193)

(ii) we have for all \( p \in (0, \infty) \) that \( \sup_{t \in [0,T]} \sup_{M,N} \mathbb{E} \left[ \|X_t^{M,N}\|_{H_r}^p \right] < \infty \), and

(iii) we have for all \( p \in (0, \infty) \), \( r \in [0, 1/4] \) that there exists a real number \( C \in \mathbb{R} \) such that for all \( M, N \in \mathbb{N} \) it holds that

\[
\sup_{t \in [0,T]} \left( \mathbb{E} \left[ \|X_t - \tilde{X}_t^{M,N}\|_{H_r}^p \right] \right)^{1/p} \leq C (M^{-r} + N^{-2r}).
\]

(194)
Proof of Corollary 6.10} Note that under the assumptions of Corollary 6.10 it is well known (cf., e.g., [30, Theorem 3.4.1 (ii) in Section 3.4, Lemma 2.4.2 in Section 2, and Definition 2.7 in Section 2], [22, Lemma 28 in Section 3.2], Lemma 6.2, the hypothesis that $\xi \in \cap_{p \in [1, \infty]} L^p(\mathbb{P}; H_{1/2})$, and [12, (A.46) in Section A.5.2]) that there exist stochastic processes $\tilde{X}_q : [0, T] \times \Omega \to L^q(\lambda_{(0,1)}; \mathbb{R})$, $q \in \{6, 7, 8, \ldots\}$, with continuous sample paths which satisfy for all $t \in [0, T]$, $q \in \{6, 7, 8, \ldots\}$ that $\sup_{s \in [0, T]} \mathbb{E}[\|\tilde{X}_{q,s}\|_{L^q(\lambda_{(0,1)}; \mathbb{R})}] < \infty$ and

$$\left[\tilde{X}_{q,t} - e^{tA} \xi - \int_0^t e^{(t-s)A} F(\tilde{X}_{q,s}) \, ds\right]_{\mathbb{P}, \mathcal{B}(H)} = \int_0^t e^{(t-s)A} \, dW_s. \quad (195)$$

Combining this with (192), the assumption that $\xi \in \cap_{p \in [1, \infty]} L^p(\mathbb{P}; H_{1/2})$, and, e.g., Lemma 2.2 in Andersson et al. [1] allows us to apply Lemma 6.9 (with $p = p$, $\vartheta = \theta$, $\gamma = \gamma$, $\chi = \chi$, $X = [0, T] \times \Omega \ni (t, \omega) \mapsto \tilde{X}_{p,t}(\omega)$ $\in L^6(\lambda_{(0,1)}; \mathbb{R})$, $\mathcal{X}^{M,N} = [0, T] \times \Omega \ni (t, \omega) \mapsto \mathcal{X}^{M,N}_t(\omega) \in H_\gamma$, $\mathcal{O}^{M,N} = \mathcal{O}^{M,N}$ for $p \in \{6, 7, 8, \ldots\}$, $\vartheta \in [r, \infty)$, $r \in [1/6, 1/4)$, $M, N \in \mathbb{N}$) to obtain that there exists a function $C : [2, \infty) \times [1/6, 1/4) \to \mathbb{R}$ such that for all $p \in \{6, 7, 8, \ldots\}$, $r \in [1/6, 1/4)$, $M, N \in \mathbb{N}$ we have that

$$\sup_{t \in [0, T]} \|\tilde{X}_{p,t} - \mathcal{X}^{M,N}_t\|_{L^p(\mathbb{P}; H)} \leq C_{p,r} (M^{-r} + N^{-2r}). \quad (196)$$

Next observe that the triangle inequality ensures that for all $p_1, p_2 \in \{6, 7, 8, \ldots\}$ with $p_1 \leq p_2$ we have that

$$\sup_{t \in [0, T]} \|\tilde{X}_{p_1,t} - \tilde{X}_{p_2,t}\|_{L^{p_1}(\mathbb{P}; H)} \leq \limsup_{M \to \infty} \limsup_{N \to \infty} \sup_{t \in [0, T]} \left[\|\tilde{X}_{p_1,t} - \mathcal{X}^{M,N}_t\|_{L^{p_1}(\mathbb{P}; H)} + \|\tilde{X}_{p_2,t} - \mathcal{X}^{M,N}_t\|_{L^{p_1}(\mathbb{P}; H)}\right]. \quad (197)$$

Hölder’s inequality and (196) hence prove that for all $p_1, p_2 \in \{6, 7, 8, \ldots\}$, $r \in (1/6, 1/4)$ with $p_1 \leq p_2$ we have that

$$\sup_{t \in [0, T]} \|\tilde{X}_{p_1,t} - \tilde{X}_{p_2,t}\|_{L^{p_1}(\mathbb{P}; H)} \leq C_{p_1,r} \limsup_{M \to \infty} \limsup_{N \to \infty} \left[M^{-r} + N^{-2r}\right] + C_{p_2,r} \limsup_{M \to \infty} \limsup_{N \to \infty} \left[M^{-r} + N^{-2r}\right] = 0. \quad (198)$$

This implies that for all $q_1, q_2 \in \{6, 7, 8, \ldots\}$, $t \in [0, T]$ we have that $\mathbb{P}(\tilde{X}_{q_1,t} = \tilde{X}_{q_2,t}) = 1$. In the next step let $\tilde{\Omega} \subseteq \Omega$ be the set given by $\tilde{\Omega} = \{\omega \in \Omega : (\forall q_1, q_2 \in \{6, 7, 8, \ldots\}, t \in [0, T] : \tilde{X}_{q_1,t} = \tilde{X}_{q_2,t}\}$ and let $X : [0, T] \times \Omega \to L^6(\lambda_{(0,1)}; \mathbb{R})$ be the function which satisfies for all $t \in [0, T]$, $\omega \in \Omega$ that $X_t(\omega) = \tilde{X}_{6,t}(\omega) \mathbf{1}_{\Omega}(\omega)$. Note that the fact that every $q \in \{6, 7, 8, \ldots\}$ we have that $\tilde{X}_q : [0, T] \times \Omega \to L^q(\lambda_{(0,1)}; \mathbb{R})$ has continuous sample paths shows that
\[ \hat{\Omega} = \left\{ \omega \in \Omega : \left( \forall q_1, q_2 \in \{6, 7, 8, \ldots \}, t \in [0, T] \cap \mathbb{Q}: \tilde{X}_{q_1, t}(\omega) = \tilde{X}_{q_2, t}(\omega) \right) \right\} \\
\quad = \cap_{q_1, q_2 \in \{6, 7, 8, \ldots \}} \cap_{t \in [0, T] \cap \mathbb{Q}} \{ \tilde{X}_{q_1, t}(\omega) = \tilde{X}_{q_2, t}(\omega) \} . \]

(199)

Combining this with the fact that \( P(\tilde{X}_{q_1, t} = \tilde{X}_{q_2, t}) = 1 \) ensures that \( \hat{\Omega} \in \mathcal{F} \) and \( P(\hat{\Omega}) = 1 \). Next observe that the fact that \( \tilde{X} : [0, T] \times \Omega \to L^6(\lambda(0, 1); \mathbb{R}) \) has continuous sample paths demonstrates that \( X \) has continuous sample paths. Moreover, note that the fact that for all \( t \in [0, T], \omega \in \Omega \) we have that \( X_t(\omega) = \tilde{X}_{6, t}(\omega) \) ensures that for all \( q \in \{6, 7, 8, \ldots \} \) we have that \( P(\forall t \in [0, T] : \tilde{X}_{q, t} = X_t) = 1 \). Combining this with (195) demonstrates that for all \( t \in [0, T] \), \( p \in (0, \infty) \) we have that \( \sup_{s \in [0, T]} E[\|X_s\|_{L^p(\lambda(0, 1); \mathbb{R})}^p] < \infty \) and

\[ \left[ X_t - e^{tA} \xi - \int_0^t e^{(t-s)A} F(X_s) \, ds \right]_{\mathbb{P}, \mathcal{B}(H)} = \int_0^t e^{(t-s)A} \, dW_s . \]  

(200)

This, the fact that \( X : [0, T] \times \Omega \to L^6(\lambda(0, 1); \mathbb{R}) \) has continuous sample paths, and again Lemma 6.9 (with \( p = p, \vartheta = \vartheta, \theta = \theta, \xi = \xi, \gamma = \gamma, \chi = \chi, X = X, \mathbb{X}^{M, N} = [0, T] \times \Omega \ni (t, \omega) \mapsto \mathbb{X}_t^{M, N}(\omega) \in H_y, \mathbb{Y}^{M, N} = \mathbb{Y}^{M, N} \) for \( p \in [2, \infty), \vartheta \in [0, \infty), \theta \in [1/6, 1/4), M, N \in \mathbb{N} \) in the notation of Lemma 6.9) ensures that for every stochastic processes \( X : [0, T] \times \Omega \to L^6(\lambda(0, 1); \mathbb{R}) \) with continuous sample paths with \( \forall t \in [0, T], p \in (0, \infty) \) : \( \sup_{s \in [0, T]} E[\|X_s\|_{L^p(\lambda(0, 1); \mathbb{R})}^p] < \infty \) and

\[ \left[ X_t - e^{tA} \xi - \int_0^t e^{(t-s)A} F(X_s) \, ds \right]_{\mathbb{P}, \mathcal{B}(H)} = \int_0^t e^{(t-s)A} \, dW_s \]  

(201)

we have that for all \( p \in (0, \infty) \) it holds that \( \sup_{r \in (-\infty, \gamma]} \sup_{M, N \in \mathbb{N}} \sup_{t \in [0, T]} E[\|X_t^{M, N}\|_{L^p}^p] < \infty \) and we have that for all \( p \in (0, \infty), r \in [0, 1/4) \) there exists a real number \( C \in \mathbb{R} \) such that for all \( M, N \in \mathbb{N} \) it holds that

\[ \sup_{t \in [0, T]} \left( E[\|X_t - X_t^{M, N}\|_{L^p}^p] \right)^{1/p} \leq C (M^{-r} + N^{-2r}) . \]  

(202)

Combining this with (200) yields that \( P(\forall t \in [0, T] : X_t = \tilde{X}_t) = 1 \). This together with (202) establishes item (i), item (ii), and item (iii). The proof of Corollary 6.10 is thus completed. \( \square \)

**Acknowledgements** We thank Dirk Blömker for fruitful discussions and for pointing out his instructive paper Bianchi et al. [5] to us. This work has been partially supported through the SNSF-Research project 200021_156603 “Numerical approximations of nonlinear stochastic ordinary and partial differential equations”. This work has been funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics-Geometry-Structure. B. Gess acknowledges financial support by the DFG through the CRC 1283 “Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications”.

**Funding** Open Access funding enabled and organized by Projekt DEAL.
References

1. Andersson, A., Jentzen, A., Kurniawan, R.: Existence, uniqueness, and regularity for stochastic evolution equations with irregular initial values. J. Math. Anal. Appl. 495(1), 124558 (2021)
2. Becker, S., Gess, B., Jentzen, A., Kloeden, P.E.: Strong convergence rates for explicit space-time discrete numerical approximations of stochastic Allen-Cahn equations. arXiv:1711.02423v1 104 pages (2017)
3. Becker, S., Gess, B., Jentzen, A., Kloeden, P.E.: Lower and upper bounds for strong approximation errors of numerical approximations of linear stochastic heat equations. BIT Numer. Math. 60, 1057–1073 (2020)
4. Becker, S., Jentzen, A.: Strong convergence rates for nonlinearity-truncated Euler-type approximations of stochastic Ginzburg-Landau equations. Stochastic Process. Appl. 129(1), 28–69 (2019)
5. Bianchi, L.A., Blömker, D., Schneider, G.: Modulation equation and SPDEs on unbounded domains. Commun. Math. Phys. 371(1), 19–54 (2019)
6. Blömker, D., Kamrani, M.: Numerically computable a posteriori-bounds for the stochastic Allen–Cahn equation. BIT 59(3), 647–673 (2019)
7. Bréhier, C.-E., Cui, J., Hong, J.: Strong convergence rates of semidiscrete splitting approximations for the stochastic Allen–Cahn equation. IMA J. Numer. Anal. 39(4), 2096–2134 (2019)
8. Bréhier, C.-E., Goudenège, L.: Analysis of some splitting schemes for the stochastic Allen–Cahn equation. Discrete Contin. Dyn. Syst. Ser. B 24(8), 4169–4190 (2019)
9. Bréhier, C.-E., Goudenège, L.: Weak convergence rates of splitting schemes for the stochastic Allen–Cahn equation. BIT 60(3), 543–582 (2020)
10. Conus, D., Jentzen, A., Kurniawan, R.: Weak convergence rates of spectral Galerkin approximations for SPDEs with nonlinear diffusion coefficients. Ann. Appl. Probab. 29(2), 653–716 (2019)
11. Cui, J., Hong, J.: Strong and weak convergence rates of a spatial approximation for stochastic partial differential equation with one-sided Lipschitz coefficient. SIAM J. Numer. Anal. 57(4), 1815–1841 (2019)
12. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and its Applications, vol. 44. Cambridge University Press, Cambridge (1992)
13. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions, 2nd ed., vol. 152 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge (2014)
14. Funaki, T.: Singular limit for stochastic reaction–diffusion equation and generation of random interfaces. Acta Math. Sin. (Engl. Ser.) 15(3), 407–438 (1999)
15. Furihata, D., Kovács, M., Larsson, S., Lindgren, F.: Strong convergence of a fully discrete finite element approximation of the stochastic Cahn–Hilliard equation. SIAM J. Numer. Anal. 56(2), 708–731 (2018)
16. Gyöngy, I., Millet, A.: On discretization schemes for stochastic evolution equations. Potential Anal. 23(2), 99–134 (2005)
17. Gyöngy, I., Sabanis, S., Šiška, D.: Convergence of tamed Euler schemes for a class of stochastic evolution equations. Stoch. Partial Differ. Equ. Anal. Comput. 4(2), 225–245 (2016)
18. Hutzenthaler, M., Jentzen, A.: Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients. Mem. Amer. Math. Soc. 236, 1112, v+99 (2015)
19. Hutzenthaler, M., Jentzen, A., Kloeden, P.E.: Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients. Ann. Appl. Probab. 22(4), 1611–1641 (2012)
20. Hutzenthaler, M., Jentzen, A., Kloeden, P.E.: Divergence of the multilevel Monte Carlo Euler method for nonlinear stochastic differential equations. Ann. Appl. Probab. 23(5), 1913–1966 (2013)
21. Hutzenthaler, M., Jentzen, A., Salimova, D.: Strong convergence of full-discrete nonlinearity-truncated accelerated exponential Euler-type approximations for stochastic Kuramoto–Sivashinsky equations. Commun. Math. Sci. 16(6), 1489–1529 (2018)
22. Jentzen, A.: Taylor Expansions for Stochastic Partial Differential Equations. Johann Wolfgang Goethe University, Frankfurt am Main, Germany, Dissertation (2009)
23. Jentzen, A., Kurniawan, R.: Weak convergence rates for Euler-type approximations of semilinear stochastic evolution equations with nonlinear diffusion coefficients. Found. Comput. Math. 21(2), 445–536 (2021)
24. Jentzen, A., Pušnik, P.: Exponential moments for numerical approximations of stochastic partial differential equations. Stoch. Partial Differ. Equ. Anal. Comput. 6(4), 565–617 (2018)
25. Jentzen, A., Pušnik, P.: Strong convergence rates for an explicit numerical approximation method for stochastic evolution equations with non-globally Lipschitz continuous nonlinearities.IMA J. Numer. Anal. 40(2), 1005–1050 (2020)
26. Jentzen, A., Salimova, D., Welti, T.: Strong convergence for explicit space-time discrete numerical approximation methods for stochastic Burgers equations. J. Math. Anal. Appl. 469(2), 661–704 (2019)
27. Kovács, M., Larsson, S., Lindgren, F.: On the discretisation in time of the stochastic Allen–Cahn equation. Math. Nachr. 291(5–6), 966–995 (2018)
28. Liu, Z., Qiao, Z.: Strong approximation of monotone stochastic partial differential equations driven by white noise.IMA J. Numer. Anal. 40(2), 1074–1093 (2020)
29. Majee, A.K., Prohl, A.: Optimal strong rates of convergence for a space-time discretization of the stochastic Allen–Cahn equation with multiplicative noise. Comput. Methods Appl. Math. 18(2), 297–311 (2018)
30. Manthey, R., Zausinger, T.: Stochastic evolution equations in $L^2(\nu)$. Stochastics Stochastics Rep. 66(1–2), 37–85 (1999)
31. Müller-Gronbach, T., Ritter, K.: Lower bounds and nonuniform time discretization for approximation of stochastic heat equations. Found. Comput. Math. 7(2), 135–181 (2007)
32. Müller-Gronbach, T., Ritter, K., Wagner, T.: Optimal pointwise approximation of a linear stochastic heat equation with additive space-time white noise. In: Monte Carlo and quasi-Monte Carlo methods, vol. 2008, pp. 577–589. Springer, Berlin (2006)
33. Müller-Gronbach, T., Ritter, K., Wagner, T.: Optimal pointwise approximation of infinite-dimensional Ornstein-Uhlenbeck processes. Stoch. Dyn. 8(3), 519–541 (2008)
34. Qi, R., Wang, X.: Optimal error estimates of Galerkin finite element methods for stochastic Allen–Cahn equation with additive noise. J. Sci. Comput. 80(2), 1171–1194 (2019)
35. Sabanis, S.: A note on tamed Euler approximations. Electron. Commun. Probab. 18(47), 1–10 (2013)
36. Sabanis, S.: Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients. Ann. Appl. Probab. 26(4), 2083–2105 (2016)
37. Sell, G.R., You, Y.: Dynamics of Evolutionary Equations. Applied Mathematical Sciences, vol. 143. Springer, New York (2002)
38. Tretyakov, M.V., Zhang, Z.: A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications. SIAM J. Numer. Anal. 51(6), 3135–3162 (2013)
39. Wang, X.: An efficient explicit full-discrete scheme for strong approximation of stochastic Allen–Cahn equation. Stochastic Process. Appl. 130(10), 6271–6299 (2020)
40. Wang, X., Gan, S.: The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients. J. Differ. Equ. Appl. 19(3), 466–490 (2013)
41. Weber, H.: Sharp interface limit for invariant measures of a stochastic Allen–Cahn equation. Commun. Pure Appl. Math. 63(8), 1071–1109 (2010)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.