Generating the Mapping Class Group of a Nonorientable Surface by Two Elements or by Three Involutions

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Abstract
We prove that, for \( g \geq 19 \) the mapping class group of a nonorientable surface of genus \( g \), \( \text{Mod}(N_g) \), can be generated by two elements, one of which is of order \( g \). We also prove that for \( g \geq 26 \), \( \text{Mod}(N_g) \) can be generated by three involutions.

Keywords Mapping class groups · Nonorientable surfaces · Involutions

Mathematics Subject Classification 57K20 · 20F38 · 20F05

1 Introduction

The mapping class group \( \text{Mod}(N_g) \) of closed connected nonorientable surface \( N_g \) is defined to be the group of the isotopy classes of all self-diffeomorphisms of \( N_g \). In this paper, we are interested in finding generating sets for \( \text{Mod}(N_g) \) consisting of least possible number of elements. Since this group is not abelian, a generating set must contain at least two elements. Szepietowski (2006) proved that \( \text{Mod}(N_g) \) is generated by three elements for all \( g \geq 3 \). Our first result (see Theorem 3.1) answers Problem 3.1(a) in Farb (2006, p. 91) (cf. Problem 5.4 in Korkmaz 2012).

Theorem A For \( g \geq 19 \), the mapping class group \( \text{Mod}(N_g) \) is generated by two elements.
The next aim of the paper is to find an answer to Problem 3(b) in Farb (2006, p. 91). Szepietowski showed that Mod(N_g) can be generated by involutions (Szepietowski 2004) and later he showed that Mod(N_g) can be generated by four involutions if g ≥ 4 (Szepietowski 2006). One can deduce that it can be generated by three involutions by the work of Birman and Chillingworth (1972) if g = 3. It is known that any group generated by two involutions is isomorphic to a quotient of a dihedral group. Thus the mapping class group Mod(N_g) cannot be generated by two involutions, since it contains nonabelian free groups which is not the case for a quotient of a dihedral group. This implies that any generating set consisting only of involutions must contain at least three elements. In this direction, we get the following result (see Theorems 4.1 and 4.2):

**Theorem B** For g ≥ 26, the mapping class group Mod(N_g) can be generated by three involutions. Let us also point out that Mod(N_g) admits an epimorphism onto the automorphism group of H_1(N_g; Z_2) preserving the (mod 2) intersection pairing (McCarthy and Pinkall 1985) and this group is isomorphic to (see Korkmaz 1998; Szepietowski 2014)

\[
\begin{cases}
  Sp(2h; Z_2) & \text{if } g = 2h + 1, \\
  Sp(2h; Z_2) \rtimes \mathbb{Z}_{2}^{2h+1} & \text{if } g = 2h + 2.
\end{cases}
\]

Hence, the action of mapping classes on H_1(N_g; Z_2) induces an epimorphism from Mod(N_g) to Sp(2[\frac{g-1}{2}]; Z_2), which immediately implies the following corollary:

**Corollary C** The symplectic group Sp(g − 1; Z_2) can be generated by two elements for every odd g ≥ 19 and also by three involutions for every odd g ≥ 27. Similarly, the group Sp(g − 2; Z_2) \rtimes \mathbb{Z}_{2}^{g-1} can be generated by two elements for every even g ≥ 20 and also by three involutions for every even g ≥ 26.

**2 Preliminaries**

Let N_g be a closed connected nonorientable surface of genus g. Note that the genus for a nonorientable surface is the number of projective planes in a connected sum decomposition. We use the model for the surface N_g as a sphere with g crosscaps represented by shaded disks in all figures of this paper. Note that a crosscap is obtained by deleting the interior of such a disk and identifying the antipodal points on the resulting boundary. The mapping class group Mod(N_g) of the surface N_g is the group of the isotopy classes of self-diffeomorphisms of N_g. We use the functional notation for the composition of two diffeomorphisms; if f and g are two diffeomorphisms, the composition fg means that g is applied first.

A simple closed curve on a nonorientable surface N_g is one-sided if its regular neighbourhood is a Möbius band and two-sided if it is an annulus. If a is a two-sided simple closed curve on N_g, to define the Dehn twist t_a about the curve a, we need to choose one of two possible orientations of its regular neighbourhood (as we did for
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Fig. 1 The curves \( a_1, a_2, b_i, c_i, \alpha_i, \beta_i \) and \( \gamma_i \) on the surface \( N_g \), where \( g = 2r + 1 \) or \( g = 2r + 2 \). Note that we do not have the curve \( c_r \) when \( g \) is odd.

Fig. 2 The homeomorphisms \( u \) and \( y = Au \)

the curves in Fig. 1). Throughout the paper, the right-handed Dehn twist \( t_a \) about the curve \( a \) will be denoted by the corresponding capital letter \( A \). In our notation, both the curves on \( N_g \) and self-diffeomorphisms of \( N_g \) shall be considered up to isotopy. In the following we shall make repeated use of some basic relations in \( \text{Mod}(N_g) \): for two-sided simple closed curves \( a \) and \( b \) on \( N_g \) and for any \( f \in \text{Mod}(N_g) \),

- \textit{Commutativity:} If \( a \) and \( b \) are disjoint, then \( AB = BA \).
- \textit{Conjugation:} If \( f(a) = b \), then \( fA f^{-1} = B^\epsilon \), where \( \epsilon = \pm 1 \) depending on the orientation of a regular neighbourhood of \( f(a) \) with respect to the chosen orientation.

Consider the Klein bottle \( K \) with a hole in Fig. 2. We define a \textit{crosscap transposition} \( u \) as the isotopy classes of a diffeomorphism interchanging two consecutive crosscaps as shown on the left hand side of Fig. 2 and equal to the identity outside the Klein bottle with one hole \( K \). The effect of the diffeomorphism \( y = Au \) on the interval \( c \) as in Fig. 2 can be also constructed as sliding a Möbius band once along the core of another one and keeping each point of the boundary of \( K \) fixed. This is a \textit{Y-homeomorphism} (Lickorish 1963) (also called a \textit{crosscap slide}; Korkmaz 2002). Note that \( A^{-1}u \) is a \textit{Y-homeomorphism} i.e. the other choice of the orientation for a neighbourhood of the curve \( a \) also gives a \textit{Y-homeomorphism}. We also note that \( y^2 \) is a Dehn twist about \( \partial K \).

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It is known that $\text{Mod}(N_g)$ is generated by Dehn twists and a $Y$-homeomorphism (one crosscap slide) (Lickorish 1963). We remark that crosscap transpositions can be used instead of crosscap slides since a crosscap transposition equals to the product of a Dehn twist and a crosscap slide.

Before we finish Preliminaries, let us state a theorem which is used in the proofs of following theorems. We work with the model in Fig. 4 in such a way that the surface is obtained from the 2-sphere by deleting the interiors of $g$ disjoint disks which are in a circular position and identifying the antipodal points on the boundary. Moreover, note that the rotation $T$ by $\frac{2\pi}{g}$ about the $x$-axis maps the crosscap $C_i$ to $C_{i+1}$ for $i = 1, \ldots, g - 1$ and $C_g$ to $C_1$. Let us denote by $u_i$ the crosscap transposition supported on the one holed Klein bottle whose boundary is the curve $\alpha_i$ shown in Fig. 1. Note that the rotation $T$ takes $\alpha_i$ to $\alpha_{i+1}$ and the crosscap $C_i$ to $C_{i+1}$, which implies that $Tu_i T^{-1} = u_{i+1}$.

**Theorem 2.1** For $g \geq 7$, the mapping class group $\text{Mod}(N_g)$ can be generated by the elements $T, A_1 A_2^{-1}, B_1 B_2^{-1}$ and $u_{g-1}$.

**Proof** Let $G$ be the subgroup of $\text{Mod}(N_g)$ generated by the set $\{T, A_1 A_2^{-1}, B_1 B_2^{-1}\}$. Szepietowski (2006, Theorem 3) showed that $A_1, A_2, B_i$ and $C_i$ as shown in Fig. 1, together with the element $u_{g-1}$ generate $\text{Mod}(N_g)$. Therefore, it is enough to prove that the elements $A_1, A_2, B_i$ and $C_i$ are contained in $G$ for $i = 1, \ldots, r$.

We begin by showing that $B_i C_j^{-1}$ is contained in $G$ for all $i, j$. By the definition of $G$ and because $T(b_1, b_2) = (c_1, c_2)$ we have $C_1 C_2^{-1} \in G$ (here, we use the notation $f(a, b)$ to denote $(f(a), f(b))$). Also, by conjugating $C_1 C_2^{-1}$ with powers of $T$, one can conclude that $G$ contains the elements $B_i B_{i+1}^{-1}$ and $C_i C_{i+1}^{-1}$. It follows that $G$ contains the elements $B_i B_{j}^{-1}$ and $C_i C_{j}^{-1}$ for all $i$ and $j$. To start with, we have $B_2 B_3^{-1} \in G$ and it is easy to verify that

$$B_2 B_3^{-1} A_2 A_1^{-1}(b_2, b_3) = (a_2, b_3),$$

so that $A_2 B_3^{-1} \in G$. Then, we have

$$(A_1 A_2^{-1})(A_2 B_3^{-1})(B_3 B_2^{-1}) = A_1 B_2^{-1} \in G,$$

since $G$ contains each of the factors. Thus, $T(a_1, b_2) = (b_1, c_2)$ implies that $B_1 C_2^{-1}$ is also in $G$. Moreover, $G$ contains the element

$$B_1 C_1^{-1} = (B_1 C_2^{-1})(C_2 C_1^{-1}).$$

Thus, $B_i C_i^{-1} \in G$ by conjugating with powers of $T$ for all $i = 1, \ldots, r - 1$. Again, it follows that $B_i C_{i}^{-1} \in G$. Note that, we have

- $(A_1 B_2^{-1})(B_2 C_1^{-1}) = A_1 C_1^{-1} \in G,$
- $(C_1 A_1^{-1})(A_1 A_2^{-1}) = C_1 A_2^{-1} \in G$ and
- $(C_2 C_1^{-1})(C_1 A_1^{-1}) = C_2 A_1^{-1} \in G$.

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from which it follows that the elements $A_1C_1^{-1}, C_1A_2^{-1}$ and $C_2A_1^{-1}$ are all in $G$. It can also be verified that

$$(A_1B_2^{-1})(A_1C_1^{-1})(A_1C_2^{-1})(A_1B_2^{-1})(a_2, a_1) = (d_2, a_1)$$

so that $D_2A_1^{-1} \in G$. Also, the element $D_2C_2^{-1} = (D_2A_1^{-1})(A_1C_2^{-1})$ is in $G$. It can also be shown that

$$(C_2B_1^{-1})(C_2A_1^{-1})(C_2C_1^{-1})(C_2B_1^{-1})(d_2, c_2) = (d_1, c_2),$$

which implies that $G$ contains $D_1C_2^{-1}$. Thus, $G$ contains the element

$$D_1A_1^{-1} = (D_1C_2^{-1})(C_2A_1^{-1})$$

(here, the curves $d_1$ and $d_2$ are shown Fig. 3). By similar arguments as in the proof of Baykur and Korkmaz (2021, Lemma 5), for $g \geq 7$ the lantern relation (see Fig. 3) implies that

$$A_3 = (A_2C_1^{-1})(D_1C_2^{-1})(D_2A_1^{-1}).$$

Since $G$ contains each factor on the right hand side, $A_3 \in G$. It follows from the fact that the diffeomorphism $A_3(B_3B_1^{-1})$ maps the curve $a_3$ to $b_3$ that

$$B_3 = A_3(B_3B_1^{-1})A_3(B_1B_3^{-1})A_3^{-1} \in G.$$  

By conjugating $B_3$ with the powers of $T$, we conclude that $A_1, B_1, C_1, \ldots B_{r-1}, C_{r-1}$ and $B_r$ are all in $G$. Moreover,

$$A_2 = (A_2A_1^{-1})A_1 \in G.$$  

Therefore, the Dehn twist generators are contained in $G$. This finishes the proof.  

\[ \square \]
3 A Generating Set for $\text{Mod}(N_g)$

In this section, we work with the model in Fig. 4.

**Theorem 3.1** For $g \geq 19$, the mapping class group $\text{Mod}(N_g)$ is generated by \{T, $u_{g-1}\Gamma_{10}C_2^{-1}$\}.

**Proof** Let $F_1 = u_{g-1}\Gamma_{10}C_2^{-1}$ and let us denote by $G$ the subgroup of $\text{Mod}(N_g)$ generated by $T$ and $F_1$. It follows from Theorem 2.1 that it suffices to prove that the subgroup $G$ contains the elements $A_1A_2^{-1}$, $B_1B_2^{-1}$ and $u_{g-1}$ to prove that $G = \text{Mod}(N_g)$.

Let $F_2$ denote the conjugation of $F_1$ by $T^{-4}$. It follows from the fact that $T^{-4}$ maps the curves $(\alpha_{g-1}, \gamma_{10}, c_2)$ to $(\alpha_{g-5}, \gamma_6, a_1)$ that

$$F_2 = T^{-4}F_1T^4 = u_{g-5}\Gamma_6A_1^{-1}$$

is contained in $G$. Let $F_3$ denote the element $(F_2F_1^{-1})F_2(F_2F_1^{-1})^{-1}$ that is contained in $G$. Hence

$$F_3 = (F_2F_1^{-1})F_2(F_2F_1^{-1})^{-1} = u_{g-5}C_2A_1^{-1}.$$

Since we have similar cases in the remaining parts of the paper, let us give some details before we proceed. It can be verified that the diffeomorphism $F_2F_1^{-1}$ send the curves

![Fig. 4](image-url)  
**Fig. 4** The rotation $T$ and the curves $c_2, \gamma_{10}$ and $\alpha_{g-1}$
In the first part of this section, where the genus of the surface $4 \text{ Involution Generators for Mod}(\alpha g)$

Thus, we have the elements $F_2 F_3^{-1} = \Gamma_6 C_2^{-1}$ and $T^4 (\Gamma_6 C_2^{-1}) T^{-4} = \Gamma_1 C_4^{-1}$, which are both contained in $G$.

Moreover, we have the following elements

$$F_4 = (C_4 \Gamma_1)^{-1} F_1 = u_{g-1} C_4 C_2^{-1},$$

$$F_5 = T^{-1} F_4 T = u_{g-2} B_4 B_2^{-1} \quad \text{and}$$

$$F_6 = (F_4 F_5) F_3 (F_4 F_5)^{-1} = u_{g-5} B_2 A_1^{-1},$$

all of which are contained in the subgroup $G$. From this, we get the element $F_6 F_3^{-1} = B_2 C_2^{-1} \in G$. Also, we have $T (B_2 C_2^{-1}) T^{-1} = C_2 B_3^{-1} \in G$, which gives rise to

$$B_2 B_3^{-1} = (B_2 C_2^{-1}) (C_2 B_3^{-1}) \in G.$$ 

This implies that $T^{-2} (B_2 B_3^{-1}) T^2 = B_1 B_2^{-1}$ is in $G$. We also have the elements

$$T^2 (C_2 B_3^{-1}) T^{-2} = C_3 B_4^{-1} \in G \quad \text{and}$$

$$T^{-2} (\Gamma_1 C_4^{-1}) T^2 = \Gamma_8 C_3^{-1} \in G,$$

implying that $\Gamma_8 B_4^{-1} = (\Gamma_8 C_3^{-1}) (C_3 B_4^{-1}) \in G$. The conjugation of the element $\Gamma_8 B_4^{-1}$ by $T^{-7}$ is the element $\Gamma_1 A_1^{-1} = A_2 A_1^{-1}$ which is contained in $G$. By the proof of Theorem 2.1, the subgroup $G$ contains the elements $A_1$, $A_2$, $B_i$ and $C_i$ for $i = 1, \ldots, r$. Then, in particular we have the elements $T^9 A_2 T^{-9} = \Gamma_1 \in G$ and $C_2 \in G$. We conclude that $u_{g-1} = F_1 (C_2 \Gamma_1^{-1}) \in G$, which completes the proof. \(\square\)

### 4 Involution Generators for Mod$(N_g)$

In the first part of this section, where the genus of the surface $N_g$ is even, we refer to Fig. 5 for the involution generators $\rho_1$ and $\rho_2$ of $N_g$. The elements $\rho_1$ and $\rho_2$ are reflections about the indicated planes in Fig. 5 in such a way that the rotation $T$, depicted in Fig. 4, is given by $T = \rho_2 \rho_1$.

**Theorem 4.1** For $g = 2r + 2 \geq 26$, the mapping class group Mod$(N_g)$ is generated by the involutions $\rho_1$, $\rho_2$ and $\rho_2 A_2 B_r B_{3r+3}$.

**Proof** Consider the surface $N_g$ as in Fig. 5. It follows from

$$\rho_2 (a_2) = a_2 \quad \text{and} \quad \rho_2 (b_r) = b_3$$
and also \( \rho_2 \) reverses the given orientation of a neighbourhood of a two-sided simple closed curve that

\[
\rho_2 A_2 \rho_2 = A_2^{-1} \quad \text{and} \quad \rho_2 B_r \rho_2 = B_3^{-1}.
\]

Since \( \rho_2 u_{r+3} \rho_2 = u_{r+3}^{-1} \), one can verify that the element \( \rho_2 A_2 B_r B_3 u_{r+3} \) is an involution. Let \( H_1 = A_2 B_r B_3 u_{r+3} \) and let \( H \) be the subgroup of \( \text{Mod}(N_g) \) generated by the set

\[
\{ \rho_1, \rho_2, \rho_2 H_1 \}.
\]

It is clear that \( H_1 \) and \( T = \rho_2 \rho_1 \) are contained in the subgroup \( H \). By Theorem 2.1, we need to prove that the subgroup \( H \) contains the elements \( A_1 A_2^{-1}, B_1 B_2^{-1} \) and \( u_{g-1} \). Let \( H_2 \) be the conjugation of \( H_1 \) by \( T^7 \). Thus

\[
H_2 = T^7 H_1 T^{-7} = \Gamma_8 C_2 C_6 u_{r+10} \in H.
\]

Let

\[
H_3 = (H_2 H_1) H_2 (H_2 H_1)^{-1} = \Gamma_8 B_3 C_6 u_{r+10},
\]

which is also in \( H \). From this, we get the element \( H_2 H_3^{-1} = C_2 B_3^{-1} \in H \) implying that \( T(C_2 B_3^{-1}) T^{-1} = B_3 C_3^{-1} \in H \). One can easily see that \( B_i C_i^{-1} \in H \) by conjugating \( B_3 C_3^{-1} \) with powers of \( T \). Also, since \( T(B_3 C_3^{-1}) T^{-1} = C_3 B_4^{-1} \in H \), similarly \( C_i B_{i+1}^{-1} \in H \) by conjugating \( C_3 B_4^{-1} \) with powers of \( T \). Hence, we have the elements

\[
B_i B_{i+1}^{-1} = (B_i C_i^{-1})(C_i B_{i+1}^{-1}).
\]

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which are in \( H \) for all \( i = 1, \ldots, r - 1 \). Moreover, it follows that \( B_1B_j^{-1} \in H \). In particular \( B_1B_2^{-1} \in H \). Now, we have the following elements

\[
H_4 = \begin{cases}
(B_7B_3^{-1})H_1 = A_2B_7B_ru_{r+3}, & \text{if } r \neq 12, 16, 17, 18, \\
(B_8B_3^{-1})H_1 = A_2B_8B_ru_{r+3}, & \text{if } r = 16, \\
(B_9B_3^{-1})H_1 = A_2B_9B_ru_{r+3}, & \text{if } r = 12, 17, 18,
\end{cases}
\]

\[
H_5 = T^6H_4T^{-6} = \begin{cases}
\Gamma_7B_{10}B_2u_{r+9}, & \text{if } r \neq 12, 16, 17, 18, \\
\Gamma_7B_{11}B_2u_{r+9}, & \text{if } r = 16, \\
\Gamma_7B_{12}B_2u_{r+9}, & \text{if } r = 12, 17, 18,
\end{cases}
\]

and

\[
H_6 = (H_5H_4)H_5(H_5H_4)^{-1} = \begin{cases}
\Gamma_7B_{10}A_2u_{r+9}, & \text{if } r \neq 12, 16, 17, 18, \\
\Gamma_7B_{11}A_2u_{r+9}, & \text{if } r = 16, \\
\Gamma_7B_{12}A_2u_{r+9}, & \text{if } r = 12, 17, 18,
\end{cases}
\]

which are all contained in \( H \). Thus, we get the element \( H_6H_5^{-1} = A_2B_2^{-1} \in H \). On the other hand, since \( C_1B_2^{-1} \) is contained in \( H \), the subgroup \( H \) contains the following elements

\[
T^{-2}(C_1B_2^{-1})T^2 = A_1B_1^{-1},
\]

\[
(A_1B_1^{-1})(B_1B_2^{-1}) = A_1B_2^{-1},
\]

\[
(A_2B_2^{-1})(B_2A_1^{-1}) = A_2A_1^{-1}.
\]

It follows from \( T, A_1A_2^{-1} \) and \( B_1B_2^{-1} \) are in \( H \) that the Dehn twists \( A_1, A_2, B_i \) and \( C_i \) are also in \( H \) for \( i = 1, \ldots, r \). This implies that

\[
u_{r+3} = (B_3^{-1}B_r^{-1}A_2^{-1})H_1 \in H.
\]

By the action of \( T \), we get \( u_i \in H \) for \( i = 1, \ldots, g - 1 \), which finishes the proof. \( \square \)

In the second part of this section, where the genus of the surface \( N_g \) is odd, we refer to Fig. 6 for the involution generators \( \rho_1 \) and \( \rho_2 \) of \( N_g \). Similarly, the elements \( \rho_1 \) and \( \rho_2 \) are reflections about the indicated planes in Fig. 6 such that the rotation \( T \) in Fig. 4 is given by \( T = \rho_2\rho_1 \). In the proof of the following theorem, we use the crosscap transposition supported on the one holed Klein bottle whose boundary is the curve \( \beta_i \) shown in Fig. 1. Let us denote this crosscap transposition by \( v_i \). Note that the rotation \( T \) sends \( \beta_i \) to \( \beta_{i+1} \) and the crosscap \( \mathcal{C}_i \) to \( \mathcal{C}_{i+1} \), which implies that \( Tv_iT^{-1} = v_{i+1} \).

**Theorem 4.2** For \( g = 2r + 1 \geq 27 \), the mapping class group \( \text{Mod}(N_g) \) is generated by the involutions \( \rho_1, \rho_2 \) and \( \rho_2A_2C_{r-1}B_3v_{r+2} \).
Fig. 6 The reflections $\rho_1$ and $\rho_2$ for $g = 2r + 1$

**Proof** We will follow the proof of Theorem 4.1, closely. Let us consider the surface $N_g$ as in Fig. 6. Since

$$\rho_2(a_2) = a_2 \quad \text{and} \quad \rho_2(c_{r-1}) = b_3$$

and also since $\rho_2$ reverses the given orientation of a neighbourhood of a two-sided simple closed curve, we get

$$\rho_2 A_2 \rho_2 = A_2^{-1} \quad \text{and} \quad \rho_2 C_{r-1} \rho_2 = B_3^{-1}.$$

By the fact that $\rho_2 v_{r+2} \rho_2 = v_{r+2}^{-1}$, it is easy to verify that the element $\rho_2 A_2 C_{r-1} B_3 v_{r+2}$ is an involution. Let $E_1 = A_2 C_{r-1} B_3 v_{r+2}$ and let $K$ denote the subgroup of $\text{Mod}(N_g)$ generated by the set

$$\{\rho_1, \rho_2, \rho_2 E_1\}.$$

It is easy to see that $E_1$ and $T = \rho_2 \rho_1$ are in $K$. By Theorem 2.1, we need to show that $K$ contains the elements $A_1 A_2^{-1}, B_1 B_2^{-1}$ and $u_{g-1}$. Let $E_2$ be the following:

$$E_2 = T^7 E_1 T^{-7} = \Gamma_8 C_2 C_6 v_{r+9} \in K.$$

Consider the element

$$E_3 = (E_2 E_1) E_2 (E_2 E_1)^{-1} = \Gamma_8 B_3 C_6 v_{r+9},$$

which belongs to $K$. One can conclude that the element $E_2 E_3^{-1} = C_2 B_3^{-1} \in K$, which implies that $T (C_2 B_3^{-1}) T^{-1} = B_3 C_3^{-1} \in K$. From this, we get the elements $B_i C_i^{-1} \in K$ by conjugating $B_3 C_3^{-1}$ with powers of $T$. Also, since $T (B_3 C_3^{-1}) T^{-1} = \Gamma_6 C_2 C_6 v_{r+9}$.
This implies that

\[ C_3B_4^{-1} \in K, C_iB_{i+1}^{-1} \in K \]

by again conjugating \( C_3B_4^{-1} \) with powers of \( T \). Thus, we get the elements

\[ B_iB_{i+1}^{-1} = (B_iC_i^{-1})(C_iB_{i+1}^{-1}), \]

which belong to \( K \) for all \( i = 1, \ldots, r - 1 \). Also, it follows that \( K \) contains the elements \( B_iB_j^{-1} \). In particular \( B_1B_2^{-1} \in K \). Moreover, we have the elements

\[
E_4 = \begin{cases} 
(B_7B_3^{-1})E_1 = A_2B_7C_{r-1}v_{r+2}, & \text{if } r \neq 13, 16, 17, 18, 19, \\
(B_8B_3^{-1})E_1 = A_2B_8C_{r-1}v_{r+2}, & \text{if } r = 16, 17, \\
(B_0B_3^{-1})E_1 = A_2B_9C_{r-1}v_{r+2}, & \text{if } r = 13, 18, 19,
\end{cases}
\]

\[
E_5 = T^6E_4T^{-6} = \begin{cases} 
\Gamma_7B_{10}B_2v_{r+8}, & \text{if } r \neq 13, 16, 17, 18, 19, \\
\Gamma_7B_{11}B_2v_{r+8}, & \text{if } r = 16, 17, \\
\Gamma_7B_{12}B_2v_{r+8}, & \text{if } r = 13, 18, 19,
\end{cases}
\]

and

\[
E_6 = (E_5E_4)E_5(E_5E_4)^{-1} = \begin{cases} 
\Gamma_7B_{10}A_2v_{r+8}, & \text{if } r \neq 13, 16, 17, 18, 19, \\
\Gamma_7B_{11}A_2v_{r+8}, & \text{if } r = 16, 17, \\
\Gamma_7B_{12}A_2v_{r+8}, & \text{if } r = 18, 19, \\
\Gamma_7C_{12}A_2v_{r+8}, & \text{if } r = 13,
\end{cases}
\]

which are all contained in the subgroup \( K \). Thus, we conclude that the subgroup \( K \) contains the element

\[ E_7 = A_2B_2^{-1} = \begin{cases} 
E_6E_5^{-1}, & \text{if } r \neq 13, \\
E_6E_5^{-1}(B_{12}C_{12}^{-1}), & \text{if } r = 13.
\end{cases}
\]

Since the element \( C_1B_2^{-1} \in K \), as in the proof of Theorem 4.1, one can conclude that the Dehn twists \( A_1, A_2, B_i \) and \( C_j \) are in \( K \) for \( i = 1, \ldots, r \) and \( j = 1, \ldots, r - 1 \). This implies that

\[ v_{r+2} = (B_3^{-1}C_{r-1}A_2^{-1})E_1 \in K. \]

By conjugating the crosscap transposition \( v_{r+2} \) with powers of \( T \), we have the elements \( v_i \in K \) for \( i = 1, \ldots, g - 2 \). Observe that \( T^{-1}(v_{g-2}v_{g-4} \cdots v_3) \) takes \( \beta_1 \) to \( \alpha_{g-1} \) and

\[ T^{-1}(v_{g-2}v_{g-4} \cdots v_3)v_1(v_{g-2}v_{g-4} \cdots v_3)^{-1}T = u_{g-1} \in K, \]

which finishes the proof. \( \square \)

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