Recently Tallon and Loram critically examined the existing experimental data on specific heat, photoemission, magnetic susceptibility and optical conductivity of the cuprate superconductors [1]. They argued that these data are consistent with the existence of two competing energy gaps, a pseudogap and the superconducting gap, both of which disperse according to d-wave symmetry. The pseudogap exists in the normal state of the underdoped cuprates while the superconducting gap opens up in the BCS fashion at temperature $T_c$. Tallon and Loram also argue that the data indicates the closing of the pseudogap at a doping level $x \approx 19\%$.

This observation motivated Chakravarty, Laughlin, Morr and Nayak (CLMN) to propose the “d-density-wave” (DDW) scenario [2]. According to this proposal a staggered long-range order in the orbital magnetic moment, as firstly suggested by Hsu, Marston and Affleck in another context, [3], exists at low temperatures when the doping is less than a critical value $x_c \approx 0.19$. Such order gives rise to a non-superconducting gap (the pseudogap) with the same $(\cos k_x - \cos k_y)$ dispersion as the superconducting (SC) gap. Moreover, the DDW order competes with superconductivity and causes $T_c(x)$ to trace out the familiar “superconducting dome” in the doping-temperature phase diagram of the cuprates. So far there is no direct experimental evidence for the DDW order. Neither is there evidence for the expected Ising-like phase transition into such a state. CLMN argue that disorder gets in the way of a sharp phase transition and turns it into a cross-over.

In addition to offering a possible new phase for high-$T_c$ cuprates, there are a number of attractive features in CLMN’s proposal. Since the DDW metal (i.e. the normal state in the presence of DDW order) is a doped band-insulator one expects the Drude weight in the normal state [3], and the superfluid density in the superconducting state to be both proportional to the doping density $\rho_s$, as indeed found in the experiments. The closing of the DDW gap at $x = 0.19$ can easily explain the observed kink in the jump of the T-linear specific heat coefficient [3]. In addition the small hole pockets centered around the nodes of the DDW gap could be the progenitor of the Fermi arcs observed in angle-resolved photoemission [7]. From a technical point of view a doped band insulator has more resemblance to a doped Mott insulator than a large-Fermi-surface metal, hence is a better starting point for description of the underdoped cuprates. Finally the CLMN proposal is crisp and in principle falsifiable. For all the above reasons we feel that it is worthwhile to further check the prediction of this proposal against existing experiments.

The theory presented in Ref. [2] is mean-field in nature. For such a theory, quasiparticles are sharply defined. Since a much better case can be made for the existence of quasiparticles in the superconducting states [8], we confine our calculations to low temperatures where SC order parameter exist.

Among various superconducting properties we focus on the superfluid density $\rho_s$ because of its rather unconventional doping and temperature dependence, which has been the focus of many theories [5]. Experimentally it is established that, for a fairly wide range of doping, $dp_s/dT$ is nearly doping independent at low temperatures [10,12]. In contrast the extrapolated zero temperature superfluid density changes significantly with doping [11]. The goal of this paper is to work out the DDW theory’s prediction for $\rho_s(T, x)$ (in the absence of disorder).

Even if the quasiparticles are well-defined in the underdoped superconducting states, the Mott constraint can substantially modify the results of the free, mean-field theory. Therefore along the mean-field prediction we also present the results of the “projected DDW model” where the electron occupation constraint is taken into account. More specifically, the no-double-occupancy constraint is implemented by introducing the slave bosons (holons) plus the gauge fields which couple to both bosons and fermions (spinons). The strict occupation constraint is reflected in the absence of the Maxwell term for the gauge fields, i.e. the coupling constant is infinity. In a recent paper, one of us looked into such a gauge theory where the underlying mean-field vacuum is the $d$-wave RVB state of Kotliar and Liu [12]. It was shown that the gauge field can be integrated out exactly in the continuum approximation of the lattice theory [13,14]. This continuum...
theory describes the (correlated) density and current fluctuations of holons and spinons above the length scale of the inter-holon distance $\lambda_h$. The physics below such a length scale is entirely summarized by a few parameters in the effective action. In the absence of more accurate estimates for these parameters, the author of Ref. took the mean-field prediction of them.

In the following treatment of the projected DDW model we shall follow the same path taken in Ref. while replacing the Kotliar-Liu RVB mean-field vacuum by the CLMN mean-field vacuum in the spinon sector. We expect the program carried out in Ref. to work well in the presence of DDW order, because the DDW metal-a doped band insulator has very little spinon density and current fluctuations below the length scale $\lambda_h$. The fluctuations above this length scale are already captured by the analysis of Ref. 

Our results are as follows. For $x_l < x < x_u$ and at low temperatures, the mean-field DDW theory predicts a superfluid density that behaves as

$$\rho_s(T, x) = \rho_{ddw}(0, x) - \alpha_{ddw}(x)T. \quad (1)$$

Results for $\rho_{ddw}(0, x)$ are shown in the main panel, Fig. 1(a). Note that the zero-temperature superfluid density is non-zero at $x = x_l, x_u$. An infinitesimal temperature will, however, destroy superfluidity because the pairing gap vanishes there. The temperature gradient of $\rho_s$ is given by $\alpha_{ddw}(x) \sim t / \Delta(x)$ where $t$ is the hopping integral and $\Delta(x)$ is the maximum $d$-wave superconducting gap. It diverges at two points, $x = x_l$, and $x = x_u$ because $\Delta(x)$ vanishes there. In general the doping dependence of $\Delta(x)$ is reflected in $\alpha_{ddw}(x)$ as shown in the insert of Fig. 1(a).

The projected DDW model predicts

$$\rho_s(T, x) = z_J(x)\rho_{ddw}(0, x) - z_J(x)^2 \alpha_{ddw}(x)T, \quad (2)$$

where $z_J(x)$ is the current renormalization factor to be explained later. The doping dependence of $z_J(x)\rho_{ddw}(0, x)$ (main panel) and $z_J(x)^2 \alpha_{ddw}(x)$ (insert) are shown in Fig.1(b). Unlike the mean-field result, $d\rho_s/dT$ has a sharp variation near $x_c$ which originates from the kink in the mean-field superfluid density $\rho_{ddw}(x, 0)$.

Let us compare the above theoretical results with existing data on $d\rho_s(T, x)/dT$. In the oxygen-depleted YBCO thin films reported in Ref. 11, the transition temperature $T_c$ ranges from 90K at the optimal doping to 38K on the underdoped side. Meanwhile, $|d\rho_s/T|$ by about 15% from its maximal value at optimal doping. In contrast, the mean-field DDW theory predicts an increase of $|d\rho_s/T|$ by 140%, provided $T_c$ scales with $\Delta_0(x)$. After projection, the dependence of $|d\rho_s/T|$ on $x$ is about 100%. It should be pointed out that in the DDW theory $|d\rho_s/T|$ is not universal, and is subject to quantitative change if the doping and temperature dependences of the pairing amplitude are different from what we assumed in this paper. It is not out of question that the experimental data happens to span the plateau regime in Fig.1(b) (insert). In this case, the present theory predicts a rapid increase of $|d\rho_s/T|$ when the doping level is decreased even further. In the following we report the details of the calculations.

The mean-field DDW theory Following CLMN, we adopt the following mean-field Hamiltonian

$$H_{DDW} = \sum_{k\sigma}(X_k - \mu)\hat{c}_{k\sigma}^\dagger \hat{c}_{k\sigma} + \sum_{k\sigma}(iD_k\hat{c}_{k+Q\sigma}^\dagger \hat{c}_{k\sigma} + \text{h.c.})$$

$$- \sum_k \Delta_k(c_{k+}c_{-k}^\dagger + \text{h.c.}). \quad (3)$$

Here $Q = (\pi, \pi)$, and $X_k = -2(t(\cos k_x + \cos k_y))$ is the nearest-neighbor tight-binding dispersion relation, $D_k = (\cos k_x - \cos k_y)$ is the momentum space DDW order parameter, and $\Delta_k = \Delta(x)(\cos k_x - \cos k_y)$ is the $d$-wave superconducting gap function. In the above and the rest of the paper we shall assume that the lattice constant is unity.

Due to the breaking of translation symmetry by the DDW order, the Brillouin zone is half of its original size. For $\Delta(x) = 0$ there are two bands given by $\varepsilon_{k\pm} = \pm \sqrt{X_k^2 + D_k^2} - \mu$. In the presence of pairing, the quasi-particle dispersion becomes $E_{k\pm} = \sqrt{\varepsilon_{k\pm}^2 + |\Delta_k|^2}$. In order to study the doping dependence of the superfluidity it is necessary to specify the dependence of $D(x)$ and $\Delta(x)$ on $x$. Following CLMN, we use the solution of $\partial E(D, \Delta) / \partial D = 0$ and $\partial E(D, \Delta) / \partial \Delta = 0$ to parametrize $D(x)$ and $\Delta(x)$ where

$$E(D, \Delta) = a_D(x - x_d)D^2 + a_{D\Delta}(x - x_a)\Delta^2 + b_D D^4 + b_{D\Delta} \Delta^4 + w D^2 \Delta^2. \quad (4)$$

Here $a_D$, $a_{D\Delta}$, $b_D$, $b_{D\Delta}$, and $w$ are (positive) material-dependent constants, while $x_d$ ($x_a$) is the doping level below which DDW (SC) order parameter become nonzero in the absence of the other. In the presence of pairing the onset of DDW happens at $x_c < x_d$ while the SC order

![FIG. 1. (a) The doping dependence of $\rho_{s,ddw}(0, x)$ (main panel) and $\alpha_{ddw}(x)$ (inset). (b) The doping dependence of $\rho_s(0, x) = z_J(0)\rho_{s,ddw}(0, x)$ and $\alpha(x) = z_J(x)^2 \alpha_{ddw}(x)$ using the same notation as in (a). The parameters used are $\Delta_0 = 0.2t$, $D_0 = t$, and $t_b = 2t$.](image)
parameter begins its decline precisely at \(x_c\) and vanishes at \(x_l\). The functional form for the order parameters are given by
\[
D(x) = D_0 \sqrt{(x_c - x)/x_c}
\]

The superfluid density measures the free energy increase caused by the phase twist of the superconducting order parameter \[16\]. After some lengthy but straightforward algebra we obtain
\[
\rho_s^{ab} = \frac{1}{L^2} \sum_{k,\nu=\pm} \left( 1 - \frac{E_{k\nu}}{E_{k\nu}} \tanh \frac{\beta E_{k\nu}}{2} \right) \partial_a \partial_b \varepsilon_{k\nu} - \frac{\beta}{2L^2} \sum_{k,\nu=\pm} \partial_a \varepsilon_{k\nu} \partial_b \varepsilon_{k\nu} \text{sech}^2 \frac{\beta E_{k\nu}}{2} + \frac{4}{L^2} \sum_{k,\nu=\pm} \frac{\nu |\Delta|^2}{E_{k+}^2 - E_{k_-}^2} \tanh(\beta E_{k\nu}/2) \left( X_k \partial_a D_k - D_k \partial_a X_k \right) \left( X_k \partial_b D_k - D_k \partial_b X_k \right) \frac{1}{X_k^2 + D_k^2},
\]
where \(\partial_a\) denotes the momentum derivative in the \(a\)-direction, \(L\) is the linear dimension of the system. By symmetry, the tensor is diagonal and direction-independent. The last term is non-zero only if the DDW and superconducting order coexist. It is a consequence of the time-reversal symmetry breaking. A similar expression, without the last term, has been derived in Ref. \[13\]. For the entire doping range considered, the third term has a negligible contribution to the superfluid density compared to others. For self-consistency, one can check that the superfluid density vanishes in the absence of pairing due to the fact that 1) the third term vanishes identically when \(\Delta(x) = 0\). 2) The first two terms combine to give \[\sum_{k\nu} \partial_a \left\{ [1 - \tanh(\beta \varepsilon_{k\nu}/2)] \partial_b \varepsilon_{k\nu} \right\} = 0\]. At temperatures much smaller than the maximum superconducting gap, the suppression of superfluid density comes from the thermal excitation of nodal quasiparticles. The second term in Eq. \[6\] gives a \(T\)-linear suppression, while all the other terms contribute higher order temperature corrections. As a result, the superfluid density takes the form of Eq. \[6\].

We find \(\alpha_{ddw}(x) = (4 \ln 2)/\pi \Delta(x)\), independent of the DDW order parameter. As to \(\rho_{ddw}(0, x)\) in Eq. \[3\], we were only able to compute it numerically, and the result is shown in Fig.1(a).

The general trend of the doping dependence of \(\rho_{ddw}(0, x)\) shown in Fig.1(a) is consistent with experimental data. However the same cannot be said about \(\alpha_{ddw}(x)\). Due to the competition between DDW order and superconductivity in the DDW theory, \(\Delta(x)\) is suppressed by the emergence of the DDW order, forming a dome. Thus \(\Delta(x) \to 0\) as \(x \to x_l, x_u\) which implies that \(\alpha_{ddw}(x)\) diverges as \(x \to x_l, x_u\). Such \(x\)-dependent \(d\rho_s/dT\) has not been observed experimentally.

The projected DDW model The lattice action for the projected DDW model is given by
\[
L = L_b + L_{DDW} - i \sum_i a_{0i}
\]
\[
L_b = \sum_i \left[ \tilde{b}_i \left( \partial_0 + ia_{0}\omega \right) - iA_{0}\omega \mu \right] b_i - \sum_{ij} \left[ e^{i(a_{ij} - A_{ij})} \tilde{b}_i b_j + \text{h.c.} \right] + U_b \sum_i \tilde{b}_i b_i \tilde{b}_i b_i
\]
\[
L_{DDW} = \sum_i \left[ \tilde{f}_{i\sigma} \left( \partial_0 + iA_{0}\omega \right) f_{i\sigma} \right] - \sum_{ij} \left[ \left( t + iD_{ij} \right) e^{i\phi_{ij}} \tilde{f}_{i\sigma} f_{j\sigma} + \text{h.c.} \right] - \sum_{ij} \left( \Delta_{ij} e^{i\phi_{ij}} \epsilon_{\sigma\sigma'} \tilde{f}_{i\sigma} \tilde{f}_{j\sigma'} + \text{h.c.} \right]
\]
In the above \(b_i\) and \(f_{i\sigma}\) are the holon and spinon fields respectively. With \(\phi_{ij}\) set to zero, \(L_{DDW}\) is the real space equivalent of Eq. \[3\]. In Eq. \[7\] \(a_{0i}\) is the gauge field that enforces the constraint \(b_i^\dagger b_i + f_{i\sigma}^\dagger f_{i\sigma} = 1\). The DDW mean-field theory describes a doped band insulator, consequently spinon density fluctuations only occurs above the length scale \(\lambda_h\). Obviously the same is true for the holon density fluctuation. As a result it should be adequate to project out the spinon and holon density fluctuations above the length scale \(\lambda_h\).

The effective action above such cutoff length is
\[
\mathcal{L} = \mathcal{L}_b + \mathcal{L}_{fp} + \mathcal{L}_{Dirac} + \mathcal{L}_j
\]
\[
\mathcal{L}_b = \frac{K_b}{2} |\phi_b^* (\nabla + ia) \phi_b|^2 + \frac{U_b}{2} \delta \phi_b^2
\]
\[
\mathcal{L}_{sp} = \frac{K_{sp}}{2} |\phi_{sp}^* (\nabla + 2ia) \phi_{sp}|^2 + \frac{1}{2U_{sp}} (\phi_{sp}^* \partial_0 \phi_{sp} + 2i\omega_0)^2
\]
\[
\mathcal{L}_j = i \delta \rho_b (\phi_b^* \partial_0 \phi_b + iA_0) + J_{jpp}^p (\phi_{sp}^* \partial_\mu \phi_{sp} + 2i\omega_\mu)
\]
\[
\quad + \bar{\rho} (\phi_b^* \partial_0 \phi_b - \frac{1}{2} \phi_{sp}^* \partial_0 \phi_{sp}) - i \rho A_0 - i \mathbf{j}_0 \cdot \mathbf{A}.
\]
Here $\bar{\rho}$ is the doping density, and $j_0$ is the transverse ground-state current produced by the DDW order. $\phi_0$ and $\phi_{sp}$ are the U(1) phase factors associated with the holon field and the spinon pair-field respectively. In addition $L_{\text{Dirac}}$ is the Dirac action for the spinon quasiparticles near the d-wave gap nodes, $J^{\text{pp}} = \frac{1}{2}(\sum_\alpha \psi_{1\alpha}^\dagger \tau_2 \psi_{1\alpha} + i v_F \psi_{1\alpha}^\dagger \psi_{2\alpha} - i v_F \psi_{2\alpha}^\dagger \psi_{1\alpha})$ is their 3-current ($\tau_2$ is the third component of the Pauli matrices, and $\psi_{1\alpha}$ is the spinon Nambu spinor associated with the nth d-wave gap node). $K_b = t_b x$ is the holon superfluid density, $4K_{sp}$ is the spinon zero-temperature superfluid density given by Eq. (10). $U_b$ and $U_{sp}$ also depend on the parameters in Eq. (9), however their values are not important for the following discussion. Given Eqs. (8) the gauge field $a_\mu$ can be integrated out straightforwardly to yield the final effective action of a correlated DDW superconductor

$$L = K \frac{1}{2} \phi^* \nabla \phi - 2i A^2 + \frac{1}{2U}(\phi^* \partial_\phi - 2i A_0)^2$$

$$- \frac{1}{2} \phi^* \partial_\phi \phi - i \bar{\rho} A_0 - i j_0 \cdot A + L_{\text{Dirac}}' \quad (9)$$

In the above $\phi \equiv \phi_{sp} \phi_{0}^2$, and

$$K \equiv K_{sp} K_b / (K_b + 4K_{sp}), \quad U \equiv U_{sp} + 4U_b$$

$$z_j \equiv K_b / (K_b + 4K_{sp}), \quad z_p \equiv U_{sp} / (U_{sp} + 4U_b) \quad (10)$$

and

$$L_{\text{Dirac}}' \equiv L_{\text{Dirac}} + \frac{2|J^{\text{pp}}|^2}{K_b + 4K_{sp}} + \frac{2(\rho^{\text{pp}})^2}{U_b - 1 + 4U_{sp}} \quad (11)$$

Combining the above result with Eq. (1) we obtain Eq. (3) as the superfluid density prediction of the projected DDW theory where

$$z_j(x) = t_b x / [t_b x + \rho_{ddw}(0, x)]. \quad (12)$$

In Fig 1(b) we plot the $x$-dependence of the zero-temperature superfluid density in the main pannel and $d\rho_s / dT$ in the insert. For $x > x_1$ the zero-temperature superfluid density varies with $x$ in roughly linear fashion. As in the mean-field DDW theory, $d\rho_s / dT$ has a strong $x$-dependence.

In conclusion, we showed that in the DDW scenario the zero-temperature superfluid density increases monotonically with doping. Its temperature gradient $|d\rho_s / dT|$, being proportional to $1/\Delta(x)$, also depends sensitively on doping. Such sensitivity to doping is expected in any mean-field theory where another order parameter competes with d-wave superconductivity. It is possible, however, that this result is due to the particular approximation scheme employed in this paper, and that another scheme may well render the temperature gradient insensitive to doping.

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[1] J. L. Tallon and J. W. Loram, cond-mat/0005063.
[2] S. Chakravarty, R. B. Laughlin, D. K. Morr and C. Nayak, cond-mat/0005443.
[3] T. C. Hsu, J. B. Marston, and I. Affleck, Phys. Rev. B 43, 2866 (1991).
[4] J. Orenstein et al, Phys. Rev. B 42, 6342 (1990).
[5] P. A. Lee and X.-G. Wen, Phys. Rev. Lett. 78, 4111 (1997); P. A. Lee et al, Phys. Rev. B 57, 6003 (1998).
[6] J. L. Tallon, et al, Phys. Stat. Sol. B 215, 531 (1999); J. W. Loram, et al, J. Phys. Chem. Solids 59, 2091 (1998).
[7] A. G. Loeser, et al, Science 273, 325 (1996); D. S. Marshall et al, Phys. Rev. Lett. 76, 4841 (1996); J. M. Harris et al, Phys. Rev. Lett. 79, 143 (1997).
[8] A. G. Loeser et al, Phys. Rev. B 56, 14185 (1997); A. V. Fedorov et al, Phys. Rev. Lett. 82, 217 (1999); A. Kaminski et al, Phys. Rev. Lett. 84, 1788 (2000).
[9] Q. Chen et al, Phys. Rev. Lett. 81, 4708 (1998).
[10] D. A. Bonn et al, Czech J. Phys. 46, S6, 3195 (1996).
[11] B. R. Boyce, K. M. Paget and T. R. Lemberger, cond-mat/9907196, and data available at http://www.ftp.ucsb.edu.
[12] G. Kotliar and J. Liu, Phys. Rev. B 38, 5142 (1988).
[13] D.-H. Lee, Phys. Rev. Lett. 84, 2694 (2000).
[14] C. Nayak, Phys. Rev. Lett. 85, 178 (2000).
[15] A two-band model with a d-wave normal state pseudogap has also been considered by Benfatto et al, Eur. Phys. J. 17, 95 (2000). There are two essential differences between this work and Ref. 3: 1) Benfatto et al introduce separate intra- and inter-band d-wave pairing. 2) There is time reversal symmetry breaking in Ref. 3.
[16] See, for example, R. P. Feynman, Statistical Mechanics (W. A. Benjamin, Inc. 1972) Chap. 10.