Geometric theta-lifting for the dual pair $\text{GSp}_{2n}$, $\text{GO}_{2m}$

Sergey Lysenko

To cite this version:

Sergey Lysenko. Geometric theta-lifting for the dual pair $\text{GSp}_{2n}$, $\text{GO}_{2m}$. 2008. hal-01227203

HAL Id: hal-01227203
https://hal.archives-ouvertes.fr/hal-01227203
Preprint submitted on 10 Nov 2015
Geometric theta-lifting for the dual pair \( \text{GSp}_{2n}, \text{GO}_{2m} \)

Sergey Lysenko

Abstract
Let \( X \) be a smooth projective curve over an algebraically closed field of characteristic \( > 2 \). Consider the dual pair \( H = \text{GO}_{2m}, G = \text{GSp}_{2n} \) over \( X \), where \( H \) splits over an étale two-sheeted covering \( \pi: \tilde{X} \to X \). Write \( \text{Bun}_G \) and \( \text{Bun}_H \) for the stacks of \( G \)-torsors and \( H \)-torsors on \( X \). We show that for \( m \leq n \) (respectively, for \( m > n \)) the theta-lifting functor \( F_G: D(\text{Bun}_H) \to D(\text{Bun}_G) \) (respectively, \( F_H: D(\text{Bun}_G) \to D(\text{Bun}_H) \)) commutes with Hecke functors with respect to a morphism of the corresponding \( L \)-groups involving the \( \text{SL}_2 \) of Arthur. In two particular cases \( n = m \) and \( m = n + 1 \) this becomes the geometric Langlands functoriality for the corresponding dual pair.

As an application, we prove a particular case of the geometric Langlands conjectures. Namely, we construct the automorphic Hecke eigensheaves on \( \text{Bun}_{\text{GSp}_4} \) corresponding to the endoscopic local systems on \( X \).

1. Introduction

1.1 The classical theta correspondence for the dual reductive pair \( (\text{GSp}_{2n}, \text{GO}_{2m}) \) is known to satisfy a version of strong Howe duality (cf. [12]). In this paper, which is a continuation of [7], we develop the geometric theory of theta-lifting for this dual pair in the everywhere unramified case.

The classical theta-lifting operators for this dual pair are as follows. Let \( X \) be a smooth projective geometrically connected curve over \( \mathbb{F}_q \) (with \( q \) odd). Let \( F = \mathbb{F}_q(X) \), \( A \) be the adèles ring of \( X \), \( \mathcal{O} \) the integer adèles. Write \( \Omega \) for the canonical line bundle on \( X \). Pick a rank \( 2n \)-vector bundle \( M \) with symplectic form \( \wedge^2 M \to A \) with values in a line bundle \( A \) on \( X \). Let \( G \) be the group scheme over \( X \) of automorphisms of the \( \text{GSp}_{2n} \)-torsor \((M, A)\).

Let \( \pi: \tilde{X} \to X \) be an étale two-sheeted covering with Galois group \( \Sigma = \{1, \sigma\} \). Let \( \mathcal{E} \) be the \( \sigma \)-anti-invariants in \( \pi_* \mathcal{O}_X \). Fix a rank \( 2m \)-vector bundle \( V \) on \( X \) with symmetric form \( \text{Sym}^2 V \to \mathcal{C} \) with values in a line bundle \( \mathcal{C} \) on \( X \) together with a compatible trivialization \( \gamma: \mathcal{C}^{-m} \otimes \text{det} V \cong \mathcal{E} \). This means that \( \gamma^2: \mathcal{C}^{-2m} \otimes (\text{det} V)^2 \cong \mathcal{O} \) is the trivialization induced by the symmetric form. Let \( \tilde{H} \) be the group scheme over \( X \) of automorphisms of \( V \) preserving the symmetric form up to a multiple and fixing \( \gamma \). This is a form of \( \text{GO}_{2m} \), where \( \text{GO}_{2m} \) is the connected component of unity of the split orthogonal similitude group. Assume given an isomorphism \( A \otimes \mathcal{C} \cong \Omega \).

Let \( G_{2nm} \) the group scheme of automorphisms of \( M \otimes V \) preserving the symplectic form \( \wedge^2 (M \otimes V) \to \Omega \). Write \( G\tilde{H} \subset G \times \tilde{H} \) for the group subscheme over \( X \) of pairs \((g, h)\) such that \( g \otimes h \) acts trivially on \( A \otimes \mathcal{C} \). The metaplectic cover \( \tilde{G}_{2nm}(\mathbb{A}) \to G_{2nm}(\mathbb{A}) \) splits naturally after restriction under \( G\tilde{H}(\mathbb{A}) \to G_{2nm}(\mathbb{A}) \). Let \( S \) be the corresponding Weil representation.
of \(G\tilde{H}(\mathbb{A})\). The space \(S^{G\tilde{H}(\mathcal{O})}\) has a distinguished nonramified vector \(v_0\). If \(\theta : S \to \mathbb{Q}_\ell\) is a theta-functional then \(\phi_0 : G\tilde{H}(F)\backslash G\tilde{H}(\mathbb{A})/G\tilde{H}(\mathcal{O}) \to \mathbb{Q}_\ell\) given by \(\phi_0(g,h) = \theta((g,h)v_0)\) is the classical theta-function. The theta-lifting operators

\[
F_G : \text{Funct}(\tilde{H}(F)\backslash \tilde{H}(\mathbb{A})/\tilde{H}(\mathcal{O})) \to \text{Funct}(G(F)\backslash G(\mathbb{A})/G(\mathcal{O}))
\]

and

\[
F_{\tilde{H}} : \text{Funct}(G(F)\backslash G(\mathbb{A})/G(\mathcal{O})) \to \text{Funct}(\tilde{H}(F)\backslash \tilde{H}(\mathbb{A})/\tilde{H}(\mathcal{O}))
\]

are the integral operators with kernel \(\phi_0\) for the diagram of projections

\[
\text{Funct}(\tilde{H}(F)\backslash \tilde{H}(\mathbb{A})/\tilde{H}(\mathcal{O})) \quad \quad \quad \quad \text{Funct}(G(F)\backslash G(\mathbb{A})/G(\mathcal{O}))
\]

\[
\begin{array}{c}
\uparrow q \\
\tilde{H}(F)\backslash \tilde{H}(\mathbb{A})/\tilde{H}(\mathcal{O}) \\
\downarrow p \\
G(F)\backslash G(\mathbb{A})/G(\mathcal{O})
\end{array}
\]

The following statement would be an analog of a theorem of Rallis [11] for similitude groups (the author have not found its proof in the literature). If \(m \leq n\) (resp., \(m > n\)) then \(F_G\) (resp., \(F_{\tilde{H}}\)) commutes with the actions of global Hecke algebras \(\mathcal{H}_G, \mathcal{H}_{\tilde{H}}\) with respect to certain homomorphism \(\mathcal{H}_G \to \mathcal{H}_{\tilde{H}}\) (resp., \(\mathcal{H}_{\tilde{H}} \to \mathcal{H}_G\)). We prove a geometric version of this result (cf. Theorem [11]). Its precise formulation in the geometric setting involves the SL\(_2\) of Arthur (or rather its maximal torus). In the particular case \(n = m\) (resp., \(m = n + 1\)) the SL\(_2\) of Arthur disappears, and the corresponding morphisms of Hecke algebras come from morphisms of L-groups \(H^L \to G^L\) (resp, \(G^L \to H^L\)).

Our methods extend those of [7], the global results are derived from the corresponding local ones. Remind that \(S \xrightarrow{\cong} \otimes_{x \in X} S_x\) is the restricted tensor product of local Weil representations. Let \(F_x\) be the completion of \(F\) at \(x \in X\), \(\mathcal{O}_x \subset F_x\) the ring of integers. The geometric analog of the \(G\tilde{H}(F_x)\)-representation \(S_x\) is the Weil category \(W(\mathcal{L}_d(W_0(F_x)))\) (cf. Sections 3.1-3.2). Informally speaking, we work rather with the geometric analog of the compactly induced representation

\[
\tilde{S}_x = c\text{-ind}_{G\tilde{H}(F_x)}^{G\tilde{H}(\mathbb{A})} S_x
\]

Its manifestation is a family of categories \(D_{\mathcal{T}_x}(\mathcal{L}_d(W_a(F_x)))\) indexed by \(a \in \mathbb{Z}\) (cf. Section 4.2).

Our main local result is Theorem [3] In classical terms, it compares the action of Hecke operators for \(G\) and \(\tilde{H}\) on the natural nonramified vector in \(\tilde{S}_x\). As a byproduct, we also obtain some new results at the classical level of functions (Propositions A.1 and A.2). For \(a\) even they reduce to a result from [10], but for \(a\) odd they are new and amount to a calculation of \(K \times \text{SO}(\mathcal{O}_x)\)-invariants in the Weil representation of \((\text{Sp}_{2n} \times \text{SO}_{2m})(F_x)\), where \(K\) is the nonstandard maximal compact subgroup of \(\text{Sp}_{2n}(F_x)\).

1.2 The most striking application of our Theorem [1] is a proof of the following particular case of the geometric Langlands conjecture for \(G = \text{GSp}_{2n}\). Let \(E\) be an irreducible rank 2 smooth \(\mathbb{Q}_\ell\)-sheaf on \(X\) equipped with an isomorphism \(\pi^* \chi \cong \det E\), where \(\chi\) is a smooth \(\mathbb{Q}_\ell\)-sheaf on \(X\) of rank one. Then \(\pi_*(E^*)\) is equipped with a natural symplectic form \(\wedge^2(\pi_* E^*) \cong \chi^{-1}\), so can be viewed as a \(\tilde{G}\)-local system \(E_{\tilde{G}}\) on \(X\), where \(\tilde{G}\) is the Langlands dual group over \(\mathbb{Q}_\ell\). We
construct the automorphic sheaf $K$ on $\text{Bun}_G$, which is a Hecke eigensheaf with respect to $E_\mathcal{G}$ (cf. Corollary 1).

**Acknowledgements.** I am grateful to V. Lafforgue for regular and stimulating discussions.

2. **Main results**

2.1 **Notation** From now on $k$ denotes an algebraically closed field of characteristic $p > 2$, all the schemes (or stacks) we consider are defined over $k$ (except in Section 4.8.7.2).

Fix a prime $\ell \neq p$. For a scheme (or stack) $S$ write $D(S)$ for the bounded derived category of $\ell$-adic étale sheaves on $S$, and $P(S) \subset D(S)$ for the category of perverse sheaves. Set $\text{DP}(S) = \oplus_{i \in \mathbb{Z}} P(S)[i] \subset D(S)$. By definition, we let for $K, K' \in P(S), i, j \in \mathbb{Z}$

$$\text{Hom}_{\text{DP}(S)}(K[i], K'[j]) = \begin{cases} \text{Hom}_P(K, K'), & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

Since we are working over an algebraically closed field, we systematically ignore Tate twists (except in Section 4.8.7.2, where we work over a finite subfield $k_0 \subset k$. In this case we also fix a square root $\bar{Q}_\ell(\frac{1}{2})$ of the sheaf $\bar{Q}_\ell(1)$ over Spec $k_0$). Fix a nontrivial character $\psi : \mathbb{F}_p \to \bar{Q}_\ell^\times$ and denote by $\mathcal{L}_\psi$ the corresponding Artin-Shreier sheaf on $\mathbb{A}^1$.

If $V \to S$ and $V^* \to S$ are dual rank $n$ vector bundles over a stack $S$, we normalize the Fourier transform $\text{Four}_\psi : D(V) \to D(V^*)$ by $\text{Four}_\psi(K) = (p_{V^*})_!(\xi^*\mathcal{L}_\psi \otimes p_V^* K)[n]([\frac{1}{2}])$, where $p_V, p_{V^*}$ are the projections, and $\xi : V \times_S V^* \to \mathbb{A}^1$ is the pairing.

For a sheaf of groups $G$ on a scheme $S$, $\mathcal{F}_G^0$ denotes the trivial $G$-torsor on $S$. For a representation $V$ of $G$ and a $G$-torsor $\mathcal{F}_G$ on $S$ write $V_{\mathcal{F}_G} = V \times^G \mathcal{F}_G$ for the induced vector bundle on $S$. For a morphism of stacks $f : Y \to Z$ denote by $\dim_{\text{rel}}(f)$ the function of connected component $C$ of $Y$ given by $\dim C - \dim C'$, where $C'$ is the connected component of $Z$ containing $f(C)$.

2.2 **Hecke operators** Let $X$ be a smooth connected projective curve. For $r \geq 1$ write $\text{Bun}_r$, for the stack of rank $r$ vector bundles on $X$. The Picard stack $\text{Bun}_1$ is also denoted $\text{Pic}_X$. For a connected reductive group $G$ over $k$, let $\text{Bun}_G$ be the stack of $G$-torsors on $X$.

Given a maximal torus $T \subset B \subset G$, we write $\Lambda_G$ (resp., $\hat{\Lambda}_G$) for the coweights (resp., weights) lattice of $G$. Let $\Lambda^+_G$ (resp., $\hat{\Lambda}^+_G$) denote the set of dominant coweights (resp., dominant weights) of $G$. Write $\check{\rho}_G$ (resp., $\rho_G$) for the half sum of the positive roots (resp., coroots) of $G$, $w_0$ for the longest element of the Weyl group of $G$.

Set $K = k(X)$. For a closed point $x \in X$ let $K_x$ be the completion of $K$ at $x$, $\mathcal{O}_x \subset K_x$ be its ring of integers.

The following notations are borrowed from [7]. Write $\text{Gr}_{G,x}$ for the affine grassmanian $G(K_x)/G(\mathcal{O}_x)$. This is an ind-scheme classifying a $G$-torsor $\mathcal{F}_G$ on $X$ together with a trivialization $\beta : \mathcal{F}_G|_{X-x} \xrightarrow{\sim} \mathcal{F}_G^0|_{X-x}$. For $\lambda \in \Lambda^+_G$ write $\overline{\text{Gr}}_{G,x}^{\lambda} \subset \text{Gr}_{G,x}$ for the closed subscheme classifying $(\mathcal{F}_G, \beta)$ for which $V_{\mathcal{F}_G}(\langle \lambda, \check{\lambda} \rangle) \subset V_{\mathcal{F}_G}$ for every $G$-module $V$ whose weights are $\leq \check{\lambda}$. The unique dense open $G(\mathcal{O}_x)$-orbit in $\overline{\text{Gr}}_{G,x}^{\lambda}$ is denoted $\text{Gr}_{G,x}^{\lambda}$. 

3
For \( \theta \in \pi_1(\mathbb{G}) \) the connected component \( \text{Gr}_G^\theta \) of \( \text{Gr}_G \) classifies pairs \((\mathcal{F}_G, \beta)\) such that \( V_{\mathcal{F}_G}(\theta, \lambda) \sim V_{\mathcal{F}_G} \) for every one-dimensional \( \mathbb{G} \)-module with highest weight \( \lambda \).

Denote by \( \mathcal{A}_G^\lambda \) the intersection cohomology sheaf of \( \text{Gr}_G^\lambda \). Write \( \tilde{\mathbb{G}} \) for the Langlands dual group to \( \mathbb{G} \), this is a reductive group over \( \mathbb{Q}_\ell \) equipped with the dual maximal torus and Borel subgroup \( \tilde{\mathbb{T}} \subset \tilde{\mathbb{B}} \subset \tilde{\mathbb{G}} \). Write \( \text{Sph}_G \) for the category of \( \mathbb{G}(\mathcal{O}_x) \)-equivariant perverse sheaves on \( \text{Gr}_{G,x} \). This is a tensor category, and one has a canonical equivalence of tensor categories \( \text{Loc} : \text{Rep}(\tilde{\mathbb{G}}) \sim \text{Sph}_G \), where \( \text{Rep}(\tilde{\mathbb{G}}) \) is the category of finite-dimensional representations of \( \tilde{\mathbb{G}} \) over \( \mathbb{Q}_\ell \) (cf. [9]).

For the definition of the Hecke functors

\[
\mathbb{H}_G^{-\lambda}, \mathbb{H}_G^{\lambda} : \text{Sph}_G \times D(\text{Bun}_G) \rightarrow D(X \times \text{Bun}_G)
\]

we refer the reader to ([7], Section 2.2.1). Write \( * : \text{Sph}_G \sim \text{Sph}_G \) for the covariant equivalence induced by the map \( \mathbb{G}(\mathcal{O}_x) \rightarrow \mathbb{G}(\mathcal{O}_x), g \mapsto g^{-1} \). In view of Loc, the corresponding functor \( * : \text{Rep}(\tilde{\mathbb{G}}) \sim \text{Rep}(\tilde{\mathbb{G}}) \) sends an irreducible \( \tilde{\mathbb{G}} \)-module with h.w. \( \lambda \) to the irreducible \( \mathbb{G} \)-module with h.w. \( -w_0(\lambda) \). For \( \lambda \in \mathcal{A}_G^\lambda \) we also write \( \mathbb{H}_G^{\lambda}(\cdot) = \mathbb{H}_G^{-\lambda}(\mathcal{A}_G^\lambda \cdot) \).

Set

\[
D\text{Sph}_G = \bigoplus_{r \in \mathbb{Z}} \text{Sph}_G[r] \subset D(\text{Gr}_G)
\]

As in ([7], Section 2.2.2), we equip it with a structure of a tensor category in such a way that the Satake equivalence extends to an equivalence of tensor categories \( \text{Loc}^x : \text{Rep}(\tilde{\mathbb{G}} \times \mathbb{G}_m) \sim D\text{Sph}_G \).

Now let \( \pi : \tilde{X} \rightarrow X \) be a finite étale Galois covering with Galois group \( \Sigma \). Given a homomorphism \( \Sigma \rightarrow \text{Aut}(\mathbb{G}) \), let \( G \) be the group scheme on \( X \) obtained as the twisting of \( \mathbb{G} \) by the \( \Sigma \)-torsor \( \pi : \tilde{X} \rightarrow X \). Set \( \tilde{K} = k(\tilde{X}) \). For a closed point \( \tilde{x} \in \tilde{X} \) write \( K_{\tilde{x}} \) for the completion of \( \tilde{K} \) at \( \tilde{x} \), \( \mathcal{O}_{\tilde{x}} \subset K_{\tilde{x}} \) for its ring of integers, and \( \text{Gr}_{G,\tilde{x}} \) for the affine grassmanian \( \mathbb{G}(\mathcal{O}_{\tilde{x}})/\mathbb{G}(\mathcal{O}_{\tilde{x}}) \).

Write \( \text{Bun}_G \) for the stack of \( G \)-torsors on \( X \). One defines Hecke functors

\[
\mathbb{H}_G^{-r}, \mathbb{H}_G^{r} : \text{Sph}_G \times D(\text{Bun}_G) \rightarrow D(\text{Bun}_G)
\]

as follows. Write \( \mathcal{H}_G \) for the Hecke stack classifying \( G \)-torsors \( \mathcal{F}_G, \mathcal{F}_G' \) on \( X \) and an isomorphism \( \mathcal{F}_G \sim \mathcal{F}_G' \mid_{X - \pi(\tilde{x})} \). We have a diagram

\[
\text{Bun}_G \xrightarrow{h^{-r}} \mathcal{H}_G \xrightarrow{h^{-r}} \text{Bun}_G,
\]

where \( h^{-r} \) (resp., \( h^{-r} \)) sends \((\mathcal{F}_G, \mathcal{F}_G', \tilde{x})\) to \( \mathcal{F}_G \) (resp., to \( \mathcal{F}_G' \)). Set \( D_{\tilde{x}} = \text{Spec} \mathcal{O}_{\tilde{x}} \). Let \( \text{Bun}_{G,\tilde{x}} \) be the stack classifying \( \mathcal{F}_G \in \text{Bun}_G \) together with a trivialization \( \mathcal{F}_G \mid_{D_{\tilde{x}}} \sim \mathcal{F}_G^0 \). Write \( \text{id}^l, \text{id}^r \) for the isomorphisms

\[
\mathbb{H}_G^{-r} \cong \text{Bun}_{G,\tilde{x}} \times \mathbb{G}(\mathcal{O}_{\tilde{x}}) \text{Gr}_{G,\tilde{x}}
\]

such that the projection to the first factor corresponds to \( h^{-r}, h^{-r} \) respectively. To \( S \in \text{Sph}_G \), \( K \in D(\text{Bun}_G) \) one attaches their twisted external product \((K \boxtimes S)^l \text{ and } (K \boxtimes S)^r \) on \( \mathcal{H}_G \), they are normalized to be perverse for \( K, S \) perverse. The functors \( \mathbb{H}_G^{-r} \) are defined by

\[
\mathbb{H}_G^{-r}(S, K) = h^{-r}(K \boxtimes S)^r \quad \text{and} \quad \mathbb{H}_G^{-r}(S, K) = h^{-r}(K \boxtimes S)^l
\]
We have canonically $z \mathcal{H}^{-}_{G}(\ast \mathcal{S}, K) \cong z \mathcal{H}^{-}_{G}(\mathcal{S}, K)$. Letting $\hat{x}$ move along $\hat{X}$, one similarly defines Hecke functors

$$\mathcal{H}^{-}_{G}, \mathcal{H}^{+}_{G} : \text{Sph}_{G} \times \text{D(Bun}_{G} \rightarrow \text{D(}\hat{X} \times \text{Bun}_{G})$$

They are compatible with the tensor structure on $\text{Sph}_{G}$ and commute with the Verdier duality (cf. [3, 7]). The group $\Sigma$ acts on $\text{Gr}_{\Sigma, \hat{x}}$, hence also on $\text{Sph}_{G}$ by transport of structure, and for $\sigma \in \Sigma$ we have isomorphisms of functors $(\sigma \times \text{id})^{*} \circ \mathcal{H}^{-}_{G}(\mathcal{S}, \cdot) \cong \mathcal{H}^{-}_{G}(\sigma^{*} \mathcal{S}, \cdot)$.

Assume that $T$ is $\Sigma$-invariant then $\Sigma$ acts on the root datum $\mathcal{R} = (\Lambda_{G}, R, \hat{\Lambda}_{G}, \hat{R})$ of $(G, \mathcal{T})$, here $R$ and $\hat{R}$ stand for coroots and roots of $G$ respectively. Given an action of $\Sigma$ on $(\hat{G}, \hat{T})$ such that the composition $\Sigma \rightarrow \text{Aut}(\hat{G}, \hat{T}) \rightarrow \text{Out}(\hat{G})$ coincides with $\Sigma \rightarrow \text{Aut}(G, T) \rightarrow \text{Out}(G)$, we form the semi-direct product $G^{L} := \hat{G} \ltimes \Sigma$ included into an exact sequence $1 \rightarrow G \rightarrow \hat{G} \rightarrow \Sigma \rightarrow 1$. This is a version of the $L$-group associated to $G_{F}$. Here $G_{F}$ denotes the restriction of the group scheme $G$ to the generic point $\text{Spec } F \in X$ of $X$ (cf. [5]).

2.3 Theta-lifting functors The following notations are borrowed from [5]. Write $\Omega$ for the canonical line bundle on $X$. For $k \geq 1$ let $G_{k}$ denote the sheaf of automorphisms of $\mathcal{O}_{X}^{k} \oplus \Omega^{k}$ preserving the natural symplectic form $\wedge^{2}(\mathcal{O}_{X}^{k} \oplus \Omega^{k}) \rightarrow \Omega$. The stack $\text{Bun}_{G_{k}}$ of $G_{k}$-torsors on $X$ classifies $M \in \text{Bun}_{2k}$ equipped with a symplectic form $\wedge^{2} M \rightarrow \Omega$. Write $\mathcal{A}_{G_{k}}$ for the line bundle on $\text{Bun}_{G_{k}}$ with fibre $\text{det } R\Gamma(X, M)$ at $M$, we view it as $\mathbb{Z}/2\mathbb{Z}$-graded of parity zero. Let $\text{Bun}_{G_{k}} \rightarrow \text{Bun}_{G_{k}}$ denote the $\mu_{2}$-gerb of square roots of $\mathcal{A}_{G_{k}}$. Write $\text{Aut}$ for the perverse theta-sheaf on $\text{Bun}_{G_{k}}$ (cf. also [6]).

Let $n, m \in \mathbb{N}$ and $G = G = \text{GSp}_{2n}$. Pick a maximal torus and Borel subgroup $T_{G} \subset B_{G} \subset G$. The stack $\text{Bun}_{G}$ classifies $M \in \text{Bun}_{2n}, A \in \text{Bun}_{n}$ with symplectic form $\wedge^{2} M \rightarrow A$. Write $\mathcal{A}_{G}$ for the $\mathbb{Z}/2\mathbb{Z}$-graded line bundle on $\text{Bun}_{G}$ with fibre $\text{det } R\Gamma(X, M)$ at $(M, A)$.

Write $\omega_{0}$ for the character of $G$ such that $A$ is obtained from $(M, A)$ by the extension of scalars $\omega_{0} : G \rightarrow \mathbb{G}_{m}$. Write $\text{Sp}_{G} \subset \text{Sph}_{G}$ for the full subcategory of objects that vanish off the connected components $\text{Gr}_{G}^{0}$ satisfying $\langle \theta, \omega_{0} \rangle = -a$.

Let $\pi : \hat{X} \rightarrow X$ be an étale degree 2 covering with Galois group $\Sigma = \{\text{id}, \sigma\}$. Let $\mathcal{E}$ be the $\sigma$-anti-invariants in $\pi_{*} \mathcal{O}$, it is equipped with a trivialization $\mathcal{E}^{2} \cong \mathcal{O}_{X}$.

Let $\mathbb{H} = \mathbb{GO}_{2m}^{0}$ be the connected component of unity of the split orthogonal similitude group $\mathbb{GO}_{2m}$ over $k$. Pick a maximal torus and Borel subgroup $T_{\mathbb{H}} \subset B_{\mathbb{H}} \subset \mathbb{H}$. Pick $\hat{\sigma} \in \mathcal{O}_{2m}(k)$ with $\hat{\sigma}^{2} = 1$ such that $\hat{\sigma} \notin \mathbb{SO}_{2m}(k)$. We assume in addition that $\hat{\sigma}$ preserves $T_{\mathbb{H}}$ and $B_{\mathbb{H}}$, so for $m \geq 2$ it induces the unique $\mathbb{Z}$-nontrivial automorphism of the Dynkin diagram of $\mathbb{H}$. For $m = 1$ we identify $\mathbb{H} \cong \mathbb{G}_{m} \times \mathbb{G}_{m}$ in such a way that $\hat{\sigma}$ permutes the two copies of $\mathbb{G}_{m}$.

Realize $\mathbb{H}$ as the subgroup of $\text{GL}_{2m}$ preserving up to a multiple the symmetric form given by the matrix

$$
\begin{pmatrix}
0 & E_{m} \\
E_{m} & 0
\end{pmatrix},
$$

where $E_{m} \in \text{GL}_{m}$ is the unity. Take $T_{H}$ to be the maximal torus of diagonal matrices, $B_{H}$ the Borel subgroup preserving for $i = 1, \ldots, m$ the isotropic subspace generated by the first $i$ base

\footnote{except for $m = 4$. The group $\mathbb{GO}_{8}$ also has trilitarian outer forms, we do not consider them.}
isomorphism $B_{\text{un}}$ be the map sending a point as above to $V$ where the map $B_{\text{un}} H(k)$ is isomorphic to the classifying stack $B$ given by $(E \in C)$ such that the corresponding trivialization $(C, V)$ yields a map $\tilde{X}$ and $B_{\text{un}} V$ is the isomorphism induced by $V \cong V^* \otimes C$.

Let $\tilde{\alpha}_0$ for the character of $B$ such that $C$ is the extension of scalars of $(V, C)$ under $\tilde{\alpha}_0 : B \to G_m$. Write $a \text{sph} \subset C$ for the full subcategory of objects that vanish off the connected components $Gr_H^0$ of $Gr_H$ satisfying $(\theta, \tilde{\alpha}_0) = -a$.

Let $R_{\text{cov}}^0$ denote the stack classifying a line bundle $U$ on $X$ together with a trivialization $U^{\otimes 2} \cong O$. Its connected components are indexed by $H^1_m(X, Z/2Z)$, each connected component is isomorphic to the classifying stack $B(\mu_2)$.

Let $B_{\text{un}} H$ be the stack classifying $V \in B_{\text{un}} 2m, C \in B_{\text{un}} 1$ and a symmetric form $\text{Sym}^2 V \to C$ such that the corresponding trivialization $(C^{-m} \otimes \text{det } V)^2 \cong O$ lies in the component of $R_{\text{cov}}^0$ given by $(E, \kappa)$. Note that $B_{\text{un}} H = \text{Spec } k \times_{R_{\text{cov}}^0} B_{\text{un}} H,$

where the map $\text{Spec } k \to R_{\text{cov}}^0$ is given by $(E, \kappa)$. Write $\rho : B_{\text{un}} H \to B_{\text{un}} H$ for the projection.

Let $B_{\text{un}} G$ be the $Z/2Z$-graded line bundle on $B_{\text{un}} H$ with fibre $\text{det } R\Gamma(X, V)$ at $(V, C)$. Set $B_{\text{un}} G, H = B_{\text{un}} H \times_{\text{Pic } X} B_{\text{un}} G,$

where the map $B_{\text{un}} H \to \text{Pic } X$ sends $(V, C, \text{Sym}^2 V \to C)$ to $\Omega \otimes C^{-1}$, and $B_{\text{un}} G \to \text{Pic } X$ sends $(M, \wedge^2 M \to A)$ to $A$. So, we have an isomorphism $C \otimes A \cong \Omega$ for a point of $B_{\text{un}} G, H$. Write $B_{\text{un}} G, H$ for the stack obtained from $B_{\text{un}} G, H$ by the base change $B_{\text{un}} H \to B_{\text{un}} H$. Let

$\tau : B_{\text{un}} G, H \to B_{\text{un}} G, 2n m$

be the map sending a point as above to $V \otimes M$ with the induced symplectic form $\wedge^2 (V \otimes M) \to \Omega$.

By [5], Proposition 2), for a point $(M, A, V, C)$ of $B_{\text{un}} G, H$ there is a canonical $Z/2Z$-graded isomorphism $\text{det } R\Gamma(X, V \otimes M) \cong \frac{\text{det } R\Gamma(X, V, M)^2}{\text{det } R\Gamma(X, A, M)^2}$ (2)

It yields a map $\tilde{\tau} : B_{\text{un}} G, H \to B_{\text{un}} G, 2n m$ sending $(\wedge^2 M \to A, \text{Sym}^2 V \to C, A \otimes C \cong \Omega)$ to $(\wedge^2 (M \otimes V) \to \Omega, B)$. Here $B = \frac{\text{det } R\Gamma(X, V)^n}{\text{det } R\Gamma(X, A)^n}$,

and $B^2$ is identified with $\text{det } R\Gamma(X, M \otimes V)$ via (2).
Definition 1. Set $\text{Aut}_{G,H} = \tilde{\tau}^*\text{Aut}\{\dim \text{rel}(\tau)\}$. For the diagram of projections

$$
\xymatrix{ \text{Bun}_H \ar[r]^p & \text{Bun}_{G,H} \ar[r]^{\tilde{\tau}} & \text{Bun}_G }
$$

define $F_G : D(\text{Bun}_H) \to D(\text{Bun}_G)$ and $F_H : D(\text{Bun}_G) \to D(\text{Bun}_H)$ by

$$
F_G(K) = p!(\text{Aut}_{G,H} \otimes q^*K)[-\dim \text{Bun}_H]
$$

$$
F_H(K) = q!(\text{Aut}_{G,H} \otimes p^*K)[-\dim \text{Bun}_G]
$$

Since $p$ and $q$ are not representable, $F_G$ and $F_H$ a priori may send a bounded complex to a complex, which is not bounded even over some open substack of finite type. Let also $F_{\tilde{H}}$ denote $F_H$ followed by restriction under $\text{Bun}_{\tilde{H}} \to \text{Bun}_H$. Write $\text{Aut}_{G,\tilde{H}}$ for the restriction of $\text{Aut}_{G,H}$ under $\text{Bun}_{G,\tilde{H}} \to \text{Bun}_{G,H}$. By abuse of notation, the composition $F_G \circ (\rho_H)_!$ is also denoted $F_G$.

2.4 Morphism of L-groups For $m \geq 2$ let $i_{\mathbb{H}} \in \text{Spin}_{2m}$ be the central element of order 2 such that $\text{Spin}_{2m}/\{i_{\mathbb{H}}\} \cong \text{SO}_{2m}$. Here $\text{Spin}_{2m}$ and $\text{SO}_{2m}$ denote the corresponding split groups over $\text{Spec} \ k$. For $m \geq 2$ denote by $G\text{Spin}_{2m}$ the quotient of $G_m \times \text{Spin}_{2m}$ by the subgroup generated by $(-1, i_{\mathbb{H}})$. Let us convent that $G\text{Spin}_{2n+1}$, where $G\text{Spin}_{2n+1} := G_m \times \text{Spin}_{2n+1}/\{(-1, i_{\mathbb{G}})\}$. Here $i_{\mathbb{G}} \in \text{Spin}_{2n+1}$ is the nontrivial central element.

Let $V_{\mathbb{H}}$ (resp., $V_{\mathbb{G}}$) denote the standard representation of $\text{SO}_{2m}$ (resp., of $\text{SO}_{2n+1}$).

CASE $m \leq n$. Pick an inclusion $V_{\mathbb{H}} \hookrightarrow V_{\mathbb{G}}$ compatible with symmetric forms. It yields an inclusion $\mathbb{H} \hookrightarrow \tilde{\mathbb{G}}$, which we assume compatible with the corresponding maximal tori. Pick an element $\sigma_{\mathbb{G}} \in \text{SO}(V_{\mathbb{G}}) \cong \text{G}_{\text{ad}}$ normalizing $\tilde{T}_{\mathbb{G}}$ and preserving $V_{\mathbb{H}}$ and $\mathbb{H} \subset \mathbb{B}_{\mathbb{H}}$. Let $\sigma_{\mathbb{H}} \in \mathcal{O}(V_{\mathbb{H}})$ be its restriction to $V_{\mathbb{H}}$. We assume that $\sigma_{\mathbb{H}}$ viewed as an automorphism of $(\mathbb{H}, \tilde{T}_{\mathbb{H}})$ extends the action of $\Sigma$ on the roots datum of $(\mathbb{H}, \tilde{T}_{\mathbb{H}})$ defined in Section 2.3.

In concrete terms, one may take $V_{\mathbb{G}} = \mathbb{Q}_\ell^{2n+1}$ with symmetric form given by the matrix

$$
\begin{pmatrix}
0 & E_n & 0 \\
E_n & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

where $E_n \in \text{GL}_n$ is the unity. Take $\tilde{T}_{\mathbb{G}}$ to be the preimage of the torus of diagonal matrices under $\mathbb{G} \to \text{SO}_{2n+1}$. Let $V_{\mathbb{H}} \subset V_{\mathbb{G}}$ be generated by $\{e_1, \ldots, e_m, e_{n+1}, \ldots, e_{n+m}\}$. Let $\tilde{T}_{\mathbb{H}}$ be the preimage under $\mathbb{H} \to \text{SO}(V_{\mathbb{H}})$ of the torus of diagonal matrices, and $\mathbb{B}_{\mathbb{H}}$ be the Borel subgroup preserving for $i = 1, \ldots, m$ the isotropic subspace generated by $\{e_1, \ldots, e_i\}$. Then one may take $\sigma_{\mathbb{G}}$ permuting $e_m$ and $e_{n+m}$, sending $e_{2n+1}$ to $-e_{2n+1}$ and acting trivially on the other base vectors.

We let $\Sigma$ act on $\mathbb{H}$ and $\tilde{\mathbb{G}}$ via the elements $\sigma_{\mathbb{H}}, \sigma_{\mathbb{G}}$. So, the inclusion $\mathbb{H} \hookrightarrow \tilde{\mathbb{G}}$ is $\Sigma$-equivariant and yields a morphism of the $L$-groups $\tilde{H}^L \to G^L$.

CASE $m > n$. Pick an inclusion $V_{\mathbb{G}} \hookrightarrow V_{\mathbb{H}}$ compatible with symmetric forms. It yields an inclusion $\mathbb{G} \hookrightarrow \mathbb{H}$, which we assume compatible with the corresponding maximal tori. Let $\sigma_{\mathbb{G}}$
be the identical automorphism of $V_G$. Extend it to an element $\sigma_H \in \mathcal{O}(V_H)$ by requiring that $\sigma_H$ preserves $\mathcal{T}_H \subset \mathcal{B}_H$ and $\sigma_H \notin \mathcal{S}(V_H)$, $\sigma_H^2 = \text{id}$.

In concrete terms, take the symmetric form on $V_H = \mathbb{Q}^{2m}$ given by the matrix

$$
\begin{pmatrix}
0 & E_m \\
E_m & 0
\end{pmatrix}
$$

Let $V_G$ be the subspace of $V_H$ generated by $\{e_1, \ldots, e_n; e_{m+1}, \ldots, e_{m+n}; e_{n+1} + e_{m+n+1}\}$. Take $T_H$ to be the preimage under $\mathcal{H} \to \mathcal{S}(V_H)$ of the torus of diagonal matrices, and $\mathcal{B}_H$ the Borel subgroup preserving for $i = 1, \ldots, m$ the isotropic subspace of $V_H$ generated by $\{e_1, \ldots, e_i\}$. Let $T_G$ be the preimage under $\mathcal{G} \to \mathcal{H}$ of $T_H$. Let $\sigma_H \in \mathcal{O}(V_H)$ permute $e_m$ and $e_{2m}$ and act trivially on the orthogonal complement to $\{e_m, e_{2m}\}$. Then $\sigma_H$ lifts uniquely to an automorphism of the exact sequence $1 \to G_m \to \mathcal{H} \to \mathcal{S}(V_H) \to 1$ that acts trivially on $G_m$.

As above, the inclusion $\mathcal{G} \hookrightarrow \mathcal{H}$ is $\Sigma$-equivariant and gives rise to a morphism of the $L$-groups $\mathcal{G} \times \Sigma = G^L \to H^L$.

**Theorem 1.** 1) For $m \leq n$ there is a homomorphism $\kappa : \mathcal{H} \times G_m \to \mathcal{G}$ with the following property. There exists an isomorphism

$$
(\pi \times \text{id})^*H^G_F(S, F_G(K)) \cong (\text{id} \boxtimes F_G)(H^G_F(g \text{Res}^\kappa(S), K))
$$

in $\mathcal{D}(\mathcal{X} \times \text{Bun}_G)$ functorial in $S \in \text{Sph}_G$ and $K \in \mathcal{D}(\text{Bun}_H)$. Here $\pi \times \text{id} : \mathcal{X} \times \text{Bun}_G \to \mathcal{X} \times \text{Bun}_G$, and $\text{id} \boxtimes F_G : \mathcal{D}(\mathcal{X} \times \text{Bun}_G) \to \mathcal{D}(\mathcal{X} \times \text{Bun}_G)$ is the corresponding theta-lifting functor.

2) For $m > n$ there is a homomorphism $\kappa : \mathcal{G} \times G_m \to \mathcal{H}$ with the following property. There exists an isomorphism

$$
H^G_F(S, F_H(K)) \cong (\pi \times \text{id})^*(\text{id} \boxtimes F_H)(H^G_F(g \text{Res}^\kappa(S), K))
$$

in $\mathcal{D}(\mathcal{X} \times \text{Bun}_H)$ functorial in $S \in \text{Sph}_H$ and $K \in \mathcal{D}(\text{Bun}_G)$. Here $\pi \times \text{id} : \mathcal{X} \times \text{Bun}_H \to \mathcal{X} \times \text{Bun}_H$, and $\text{id} \boxtimes F_H : \mathcal{D}(\mathcal{X} \times \text{Bun}_G) \to \mathcal{D}(\mathcal{X} \times \text{Bun}_H)$ is the corresponding theta-lifting functor.

**Remark 1.** If $m = n$ or $m = n + 1$ then the restriction of $\kappa$ to $G_m$ is trivial. The explicit formulas for $\kappa$ are given in Section 4.8.9. If $m \leq n$ then $\kappa$ fits into the diagram

$$
\begin{array}{ccc}
\mathcal{H} \times G_m & \rightarrow & \mathcal{G} \\
\downarrow & & \downarrow \\
\text{SO}_{2m} \times G_m & \rightarrow & \text{SO}_{2n+1}
\end{array}
$$

If $m > n$ then $\kappa$ fits into the diagram

$$
\begin{array}{ccc}
\mathcal{G} \times G_m & \rightarrow & \mathcal{H} \\
\downarrow & & \downarrow \\
\text{SO}_{2n+1} \times G_m & \rightarrow & \text{SO}_{2m},
\end{array}
$$

In both cases $\kappa$ is the map from ([7], Theorem 3).
For $a \in \mathbb{Z}$ let $\text{^aBun}_G, \text{^hBun}_H$ be the stack classifying $\tilde{x} \in \tilde{X}$, $(M,A) \in \text{Bun}_G$, $(V,C,\gamma) \in \text{Bun}_H$, and an isomorphism $A \otimes C \cong \Omega(a\pi(\tilde{x}))$. We have the Hecke functors defined as in Section 2.2.

$$H_G^{-} : -a\text{ Sph}_G \times D(\text{^aBun}_G) \to D(\text{^aBun}_G)$$

and

$$H_H^{-} : -a\text{ Sph}_H \times D(\text{^aBun}_H) \to D(\text{^aBun}_H)$$

Set also $H_H^{-}(S,\cdot) = H_H^{-}(\ast S,\cdot)$. We will derive Theorem 1 from the following Hecke property of $\text{Aut}_G, \text{Aut}_H$.

**Theorem 2.** Let $\kappa$ be as in Theorem 1.

1) For $m \leq n$ there exists an isomorphism

$$H_G^{-}(S, \text{Aut}_G, \text{^hBun}_H) \cong H_H^{-}(\ast \text{gRes}^\kappa(S), \text{Aut}_H)$$

in $D(\text{^aBun}_{G,H})$ functorial in $S \in -a\text{ Sph}_G$.

2) For $m > n$ there exists an isomorphism

$$H_H^{-}(S, \text{Aut}_G, \text{^hBun}_H) \cong H_G^{-}(\text{gRes}^\kappa(\ast S), \text{Aut}_H)$$

in $D(\text{^aBun}_{G,H})$ functorial in $S \in -a\text{ Sph}_H$.

**2.5 Application: Automorphic sheaves on $\text{Bun}_G^{\text{Sp}_4}$.**

Keep the notation of Section 2.3 assuming $m = n = 2$, so $G = \text{Sp}_4$. Let $\tilde{E}$ be an irreducible rank two smooth $\overline{\mathbb{Q}}_\ell$-sheaf on $\tilde{X}$, $\chi$ a rank one local system on $X$ equipped with an isomorphism $\pi^*\chi \cong \det \tilde{E}$. To this data one associates the perverse sheaf $K_{\tilde{E},\chi,H}$ on $\text{Bun}_H$ introduced in ([7], Section 5.1). The local system $\pi_*\tilde{E}^*$ is equipped with a natural symplectic form $\wedge^2(\pi_*\tilde{E}^*) \to \chi^{-1}$, so gives rise to a $\widehat{G}$-local system $E_G$ on $X$. Since $K_{\tilde{E},\chi,H}$ is a Hecke eigensheaf, Theorem 1 implies the following.

**Corollary 1.** The complex $F_G(\rho_H^1 K_{\tilde{E},\chi,H}) \in D(\text{Bun}_G)$ is a Hecke eigensheaf corresponding to the $\widehat{G}$-local system $E_G$.

**Remark 2.** i) We expect that for each open substack of finite type $U \subset \text{Bun}_G$ the restriction of $F_G(\rho_H^1 K_{\tilde{E},\chi,H})$ to $U$ is a bounded complex. We also expect it to be perverse.

ii) If $\tilde{X}$ splits fix a numbering of connected components of $\tilde{X}$. Then $\tilde{E}$ becomes a pair of irreducible rank 2 local systems $E_1, E_2$ on $X$ with the isomorphisms $\det E_1 \cong \det E_2 \cong \chi$.

**3. Local theory**

**3.1 Background on non-ramified Weil category**

Remind the following constructions from [8]. Let $W$ be a symplectic Tate space over $k$. By definition ([2], 4.2.13), $W$ is a complete topological $k$-vector space having a base of neighbourhoods.
of 0 consisting of commensurable vector subspaces (i.e., \(\dim U_1/(U_1 \cap U_2) < \infty\) for any \(U_1, U_2\) from this base). It is equipped with a (continuous) symplectic form \(\wedge^2 W \to k\) (it induces a topological isomorphism \(W \cong W^*\)).

For a \(k\)-subspace \(L \subset W\) write \(L^\perp = \{w \in W \mid \langle w, l \rangle = 0\}\) for all \(l \in L\). Write \(\mathcal{L}_d(W)\) for the scheme of discrete lagrangian lattices in \(W\). For a \(c\)-lattice \(R \subset W\) let \(\mathcal{L}_d(W)_R \subset \mathcal{L}_d(W)\) be the open subscheme of \(L \in \mathcal{L}_d(W)\) satisfying \(L \cap R = 0\).

For a \(k\)-point \(L \in \mathcal{L}_d(W)\) one defines the category \(\mathcal{H}_L\) as in \([S, \text{Section 6.1}]\). Let us remind the definition. For a \(c\)-lattice \(R \subset R^\perp \subset W\) with \(R \cap L = 0\) we have a lagrangian subspace \(L_R := L \cap R^\perp \subset \mathcal{L}(R^\perp/R)\) and the Heisenberg group \(H_R = R^\perp/R \oplus k\). Let \(\mathcal{H}_{L_R}\) be the category of perverse sheaves on \(H_R\), which are \((\bar{L}_R, \chi_{L,R})\)-equivariant under the left multiplication on \(H_R\). Here \(\bar{L}_R = L_R \times \mathbb{A}^1 \subset H_R\) and \(\chi_{L,R}\) is the local system \(\pr^* L^\vee\) for the projection \(\pr : \bar{L}_R \to \mathbb{A}^1\) sending \((l, a)\) to \(a\). Let \(D\mathcal{H}_{L_R} \subset D(H_R)\) be the full subcategory of objects whose all perverse cohomologies lie in \(\mathcal{H}_{L_R}\).

For another \(c\)-lattice \(S \subset R\) we have (an exact for the perverse t-structures) transition functor \(T^I_{S,R} : D\mathcal{H}_{L_R} \to D\mathcal{H}_{L_S}\) (cf. \(\text{loc.cit.}, \text{Section 6.1}\)). Now \(\mathcal{H}_L\) is the inductive 2-limit of \(\mathcal{H}_{L_R}\) over the partially ordered set of \(c\)-lattices \(R \subset R^\perp\) such that \(R \cap L = 0\).

Given a \(c\)-lattice \(M\) in \(W\), we have a \(\mathbb{Z}/2\mathbb{Z}\)-graded line bundle on \(\mathcal{L}_d(W)\), whose fibre at \(L\) is \(\det(M : L)\). Remind that

\[
\det(M : L) = \det(M \oplus L \to W),
\]

where the complex \(M \oplus L \to W\) is placed in cohomological degrees 0 and 1. If \(S \subset M \subset S^\perp\) is a \(c\)-lattice with \(S \cap L = 0\) then \(\det(M : L) \cong \det(M/S) \otimes \det L_S\), where \(L_S := L \cap S^\perp\).

Note that \(\det(M : L) \cong \det(M^\perp : L)\) canonically. If \(M' \subset W\) is another \(c\)-lattice then we have \(\det(M : L) \cong \det(M : M') \otimes \det(M' : L)\) canonically. If \(R' \subset W\) is a lagrangian \(c\)-lattice then, as \(\mathbb{Z}/2\mathbb{Z}\)-graded, \(\det(M : L)\) is of parity \(\dim(R' : M) \mod 2\).

Fix a one-dimensional \(\mathbb{Z}/2\mathbb{Z}\)-graded space \(\mathcal{J}_W\) placed in degree \(\dim(R' : M) \mod 2\). Let \(\mathcal{A}_d\) be the \(\mathbb{Z}/2\mathbb{Z}\)-graded purely of degree zero line bundle on \(\mathcal{L}_d(W)\) with fibre \(\mathcal{J}_W \otimes \det(M : L)\) at \(L\). Let \(\mathcal{L}_d(W)\) be the \(\mu_2\)-gerb of square roots of \(\mathcal{A}_d\).

For \(k\)-points \(N^0, L^0 \in \mathcal{L}_d(W)\) one associates to them in a canonical way a functor \(\mathcal{F}_{N^0,L^0} : D\mathcal{H}_L \to D\mathcal{H}_N\) sending \(\mathcal{H}_L\) to \(\mathcal{H}_N\) (defined as in \([S, \text{Section 6.2}]\)). Let us precise some details. For a \(c\)-lattice \(R \subset R^\perp\) in \(W\) we have the projection

\[
\mathcal{L}_d(W)_R \to \mathcal{L}(R^\perp/R)
\]

sending \(L\) to \(L_R\). Let \(\mathcal{A}_R\) be the \(\mathbb{Z}/2\mathbb{Z}\)-graded purely of degree zero line bundle on \(\mathcal{L}(R^\perp/R)\) whose fibre at \(L_1\) is \(\det L_1 \otimes \det(M : R) \otimes \mathcal{J}_W\). Its restriction to \(\mathcal{L}_d(W)_R\) identifies canonically with \(\mathcal{A}_d\), hence a morphism of stacks

\[
\tilde{\mathcal{L}}_d(W)_R \to \tilde{\mathcal{C}}(R^\perp/R)
\]

where \(\tilde{\mathcal{C}}(R^\perp/R)\) is the gerb of square roots of \(\mathcal{A}_R\). Write \(N^0_R, L^0_R\) for the images of \(N^0, L^0\) under \([S]\). By definition, the enhanced structure on \(L_R\) and \(N_R\) is given by one-dimensional spaces...
Let $G$ be the canonical intertwining functor corresponding to $(N, B)$ (as in loc.cit, Section 6.2). Then $F$ is defined as the limit of the functors $[7]$ over the partially ordered set of $c$-lattices $R \subset R^\perp$ such that $N, R \in \mathcal{L}_d(W)_R$.

The proof of (Theorem 2, [8]) holds through, so for a $k$-point $L^0 \in \tilde{L}_d(W)$ we have the functor $\mathcal{F}_{L^0} : D\mathcal{H}_L \to D(\tilde{L}_d(W))$ exact for the perverse t-structures. For two $k$-points $L^0, N^0 \in \tilde{L}_d(W)$ the diagram is canonically 2-commutative

$$
\begin{array}{ccc}
D\mathcal{H}_L & \xrightarrow{\mathcal{F}_{L^0}} & D(\tilde{L}_d(W)) \\
\downarrow \mathcal{F}_{N^0, L^0} & \nearrow \mathcal{F}_{N^0} & \\
D\mathcal{H}_N &
\end{array}
$$

The non-ramified Weil category $W(\tilde{L}_d(W))$ is defined as the essential image of $\mathcal{F}_{L^0} : \mathcal{H}_L \to P(\tilde{L}_d(W))$ for any $k$-point $L^0 \in \tilde{L}_d(W)$.

3.2 Let $\mathcal{O}$ be a complete discrete valuation $k$-algebra, $F$ its fraction field. Write $\Omega$ for the completed module of relative differentials of $\mathcal{O}$ over $k$. For a free $\mathcal{O}$-module $V$ of finite rank write $V(r) \subset V \otimes F$ for the $\mathcal{O}$-submodule $t^{-r}V$, where $t \in \mathcal{O}$ is any uniformizer.

For $r \in \mathbb{Z}$ let $W_r$ be a free $\mathcal{O}$-module of rank $2n$ with symplectic form $\wedge^2 W_r \rightarrow \Omega(r)$. Then $W_r(F)$ is a symplectic Tate space with the form $\wedge^2 W_r(F) \rightarrow \Omega(F) \xrightarrow{\text{Res}} k$. Set

$$
\mathcal{L}_d^{ex} = \bigsqcup_{r \in \mathbb{Z}} \mathcal{L}_d(W_r(F))
$$

Let $G_{b,a}$ be the set of $F$-linear isomorphisms $g : W_a(F) \rightarrow W_b(F)$ of symplectic $F$-spaces. Let $G_a = \text{Sp}(W_a)$ as a group scheme over $\mathcal{O}$.

Fix a $\mathbb{Z}/2\mathbb{Z}$-graded line $\mathcal{J}_r$ placed in degree $nr$ mod 2. Let $\mathcal{A}_{d,r}$ be the $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero line bundle on $\mathcal{L}_d(W_r(F))$ whose fibre at $L$ is $\mathcal{J}_r \otimes \det(W_r : L)$. Let $\tilde{L}_d(W_r(F))$ be the $\mu_2$-gerb of square roots of $\mathcal{A}_{d,r}$.

Let $\tilde{G}_{b,a}$ be the $\mu_2$-gerb over $G_{b,a}$ classifying $g \in G_{b,a}$, a one-dimensional space $B$ and an isomorphism $B^2 \cong \mathcal{J}_b \otimes \mathcal{J}_a^{-1} \otimes \det(W_b : gW_a)$. The composition $\tilde{G}_{c,b} \times \tilde{G}_{b,a} \rightarrow \tilde{G}_{c,a}$ lifts to a morphism $\tilde{G}_{c,b} \times \tilde{G}_{b,a} \rightarrow \tilde{G}_{c,a}$ sending $(g_2, B_2) \in \tilde{G}_{c,b}, (g_1, B_1) \in \tilde{G}_{b,a}$ to $(g_2g_1, B)$, where $B = B_1 \otimes B_2$.

Consider the action map

$$
\tilde{G}_{b,a} \times \tilde{L}_d(W_a(F)) \rightarrow \tilde{L}_d(W_b(F))
$$

sending $(g, B) \in \tilde{G}_{b,a}$ and $(L, B_L) \in \tilde{L}_d(W_a(F))$ to $(gL, B_1)$, where $B_1 = B \otimes B_L$ is equipped with the induced isomorphism

$$
B_1^2 \cong \mathcal{J}_b \otimes \det(W_b : gL)
$$
In this way \( \tilde{G}^{ex} := \sqcup_{a,b \in \mathbb{Z}} \tilde{G}_{b,a} \) becomes a groupoid acting on

\[ \tilde{L}^{ex}_d := \sqcup_{r \in \mathbb{Z}} \tilde{L}_d(W_r(F)) \]

The gerb \( \tilde{G}_{b,a} \rightarrow G_{b,a} \) has a canonical section over \( G_a(O) \subset G_{b,a} \) sending \( g \in G_a(O) \) to \((g, B = k)\) equipped with \( \text{id} : B^2 \rightarrow \text{det}(W_a : W_b) \). One can define the equivariant derived category \( D_{G_a(O)}(\tilde{L}_d(W_a(F))) \) as in (\[\ref{LL} \], Section 8.2.2).

For \( g \in \tilde{G}_{b,a} \) and a c-lattice \( R \subset R_{\perp} \subset W_a(F) \) we have \( (gR)^\perp = g(R^\perp) \), and \( g \) induces an isomorphism of symplectic spaces

\[ g : R^\perp / R \cong (gR)^\perp / (gR) \] (8)

If \( L \in L_d(W_a(F))_R \) then \( g \) yields an equivalence \( H_{L_R} \cong H_{gL_R} \) sending \( K \) to \( g_K \) for the map \( g : H_R \rightarrow H_{gR} \). Passing to the limit by \( R \), we further get an equivalence \( g : H_L \cong H_{gL} \).

**Proposition 1.** Let \( a, b \in \mathbb{Z} \), \( \tilde{g} \in \tilde{G}_{b,a} \) over \( g \in G_{b,a} \) and \( L^0 \in \tilde{L}_d(W_a(F)) \) be \( k \)-points. Then the diagram is canonically 2-commutative

\[
\begin{array}{ccc}
D H_L & \xrightarrow{\mathcal{F}_{L^0}} & D(\tilde{L}_d(W_a(F))) \\
\downarrow g & & \downarrow \tilde{g} \\
D H_{gL} & \xrightarrow{\mathcal{F}_{gL^0}} & D(\tilde{L}_d(W_b(F)))
\end{array}
\] (9)

**Proof.** Let \( R \subset R^\perp \subset W_a(F) \) be a c-lattice with \( R \cap L = 0 \). We get an equivalence \( g : H_{L_R} \cong H_{gL_R} \). Let \( \mathcal{A}_R \) be the line bundle on \( \tilde{L}(R^\perp / R) \) whose fibre at \( L_1 \) is

\[ J_a \otimes \text{det}(W_a : R) \otimes \text{det} L_1 \]

Let \( \tilde{L}(R^\perp / R) \) be the \( \mu_2 \)-gerb of square roots of \( \mathcal{A}_R \). We have the projection

\[ \tilde{L}_d(W_a(F))_R \rightarrow \tilde{L}(R^\perp / R) \]

sending \( L^0 \) to \( L^0_R \). As in (\[\ref{S} \], Section 6.4), we have the functors \( \mathcal{F}_{L^0} : H_{L_R} \rightarrow \mathcal{P}(\tilde{L}(R^\perp / R)) \). It suffices to show that the diagram is canonically 2-commutative

\[
\begin{array}{ccc}
\mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0}} & \mathcal{P}(\tilde{L}(R^\perp / R)) \\
\downarrow g & & \downarrow \tilde{g} \\
\mathcal{H}_{gL_R} & \xrightarrow{\mathcal{F}_{gL^0}} & \mathcal{P}(\tilde{L}((gR)^\perp / (gR)))
\end{array}
\] (9)

The above expression \( \tilde{g}L^0_{gR} \) is the image of \( \tilde{g}(L^0) \) under \( \tilde{L}_d(W_b(F))_{gR} \rightarrow \tilde{L}((gR)^\perp / (gR)) \). Note that \( \tilde{g}L^0_{gR} = \tilde{g}(L^0_R) \), where

\[ \tilde{g} : \tilde{L}(R^\perp / R) \cong \tilde{L}((gR)^\perp / (gR)) \]

sends \((L_1, B) \) to \((gL_1, B \otimes B_0) \). Here \( \tilde{g} = (g, B_0) \).
Remind that \( H_R \) denotes the Heisenberg group \( R^\perp / R \times \mathbb{A}^1 \). For the isomorphism
\[
\tilde{g} : \tilde{L}(R^\perp / R) \times \tilde{L}(R^\perp / R) \times H_R \cong \tilde{L}((gR)^\perp / gR) \times \tilde{L}((gR)^\perp / gR) \times H_{gR}
\]
we have \( \tilde{g}^* F \cong F \) canonically, where \( F \) is the CIO sheaf on each side (introduced in [3], Theorem 1). The \( 2 \)-commutativity of [3] follows. \( \square \)

By Proposition 1, each \( \tilde{g} \in G_{b,a} \) yields an equivalence \( \tilde{g} : W(\tilde{L}_d(W_a(F))) \cong W(\tilde{L}_d(W_b(F))) \).

3.3 Now assume that we are given for each \( a \in \mathbb{Z} \) a decomposition \( W_a = U_a \oplus U_a^* \otimes \Omega(a) \), where \( U_a \) is a free \( \mathcal{O} \)-module of rank \( n \), \( U_a^* \otimes \Omega(a) \) are lagrangians, and the form \( \omega : \wedge^2 W_a \to \Omega(a) \) is given by \( \omega(u, u^*) = \langle u, u^* \rangle \) for \( u \in U_a, u^* \in U_a^* \otimes \Omega(a) \), where \( \langle \cdot, \cdot \rangle \) is the canonical pairing between \( U_a \) and \( U_a^* \otimes \Omega(a) \).

**Remark 3.** If \( U_1 \) is a free \( \mathcal{O} \)-module of finite rank and \( U_2 \subset U_1(F) \) is a \( \mathcal{O} \)-lattice then there is a canonical \( \mathbb{Z}/2\mathbb{Z} \)-graded isomorphism \( \det(U_2 : U_1)^* \cong \det(U_1^* \otimes \Omega : U_2^* \otimes \Omega) \).

For \( a, b \in \mathbb{Z} \) let \( U_{b,a} \) be the set of \( F \)-linear isomorphisms \( U_a(F) \to U_b(F) \). We have an inclusion \( U_{b,a} \hookrightarrow G_{b,a} \) given by \( g \mapsto (g, (t^x)^{-1}) \). Here \( t^x \in \text{GL}(U^* \otimes \Omega)(F) \) is the adjoint operator. By Remark 3 for \( g \in U_{b,a} \) we have canonically
\[
\det(W_b : gW_a) \cong \det(U_b : gU_a)^2 \otimes (\det U_{a,x})^a \otimes (\det U_{b,x})^{-b} \otimes \det(\mathcal{O}(-b) : \mathcal{O}(-a))^{|n(b-a)|}
\]

Assume in addition that \( n \) is even. Assume given a one-dimensional \( \mathbb{Z}/2\mathbb{Z} \)-graded purely of degree zero vector space \( J_{U,a} \) equipped with \( J_{U,a}^2 \cong J_a \otimes \det(U_{a,x})^{-a} \). This yields a section \( \rho_{b,a} : U_{b,a} \to G_{b,a} \) defined as follows. We send \( g \in U_{b,a} \) to \( (g, B) \), where
\[
B = J_{U,b} \otimes J_{U,a}^{-1} \otimes \det(U_b : gU_a) \otimes \det(\mathcal{O}(-b) : \mathcal{O}(-a))^{n/2}
\]
is equipped with the induced isomorphism
\[
B^2 \cong J_b \otimes J_a^{-1} \otimes \det(W_b : gW_a)
\]
The section \( \rho \) is compatible with the groupoid structures on \( \tilde{G}^{ex} \) and \( \mathcal{U}^{ex} = \sqcup_{a,b} U_{b,a}. \) We let \( \mathcal{U}^{ex} \) act on \( \tilde{L}_d^{ex} \) via \( \rho \).

**Proposition 2.** For \( a \in \mathbb{Z} \) there is a canonical functor \( \mathcal{F}_{U_a(F)} : \text{D}(U_a^* \otimes \Omega(F)) \to \text{D}(\tilde{L}_d(W_a(F))) \) exact for the perverse t-structures. For \( g \in U_{b,a} \) and \( \tilde{g} = \rho_{b,a}(g) \) the diagram is canonically 2-commutative
\[
\begin{array}{ccc}
\text{D}(U_a^* \otimes \Omega(F)) & \xrightarrow{\mathcal{F}_{U_a(F)}} & \text{D}(\tilde{L}_d(W_a(F))) \\
\downarrow g & & \downarrow \tilde{g} \\
\text{D}(U_b^* \otimes \Omega(F)) & \xrightarrow{\mathcal{F}_{U_b(F)}} & \text{D}(\tilde{L}_d(W_b(F)))
\end{array}
\]  

(10)
Proof

Step 1. Let $R_1 \subset R_2 \subset U_a(F)$ be c-lattices. Write $\langle \cdot , \cdot \rangle_a$ for the symplectic form on the Tate space $W_a(F)$. For a c-lattice $R \subset U_a(F)$ set $R' = \{ w \in U_a^* \otimes \Omega(F) \mid \langle w, r \rangle_a = 0 \text{ for all } r \in R \}$, this is a c-lattice in $U_a^* \otimes \Omega(F)$.

Set $R = R_1 \oplus R_2$ then $R^\perp = R_2 \oplus R_1^\perp$. Let $U_R = R_2/R_1$ then $U_R \in \mathcal{L}(R^\perp/R)$. Set $U_R^0 = (U_R, \mathcal{B})$ equipped with the canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$B^2 \cong J_\alpha \otimes \det(U_R) \otimes \det(W_a : R),$$

where $\mathcal{B} = J_{U,a} \otimes \det(U_a : R_1) \otimes \det(O(-a) : O)^{n/2}$.

Remind the line bundle $\mathcal{A}_R$ on $\mathcal{L}(R^\perp/R)$ with fibre $J_\alpha \otimes \det L_1 \otimes \det(W_a : R)$ at $L_1$. Let $\tilde{\mathcal{L}}(R^\perp/R)$ be the gerb of square roots of $\mathcal{A}_R$. So, $U_R^0 \in \tilde{\mathcal{L}}(R^\perp/R)$.

Write $H_R$ for the Heisenberg group $\mathbb{R}^2/\mathbb{R} \times \mathbb{R}^2$ and $\mathcal{H}_{U_R}$ for the corresponding category of $(\bar{U}_R, \chi_{U,R})$-equivariant perverse sheaves on $H_R$. Here $\bar{U}_R = \bar{U}_R \times \mathbb{A}^1$ and $\chi_{U,R}$ is the local system $pr^* \mathcal{L}_\psi$ on $\bar{U}_R$, where $pr : \bar{U}_R \to \mathbb{A}^1$ is the projection.

Let $\mathcal{F}_{U_R^0} : D\mathcal{H}_{U_R} \to D(\tilde{\mathcal{L}}(R^\perp/R))$ be the corresponding functor (defined as in [8], Section 3.6). The lattice $gR \subset W_b(F)$ satisfies the same assumptions, so we have $U_{gR} = gR_2/gR_1 \in \mathcal{L}(gR^\perp/gR)$, and $g(R^\perp) = (gR)^\perp$. Further, $U_{gR}^0 = (U_{gR}, B_1)$ with

$$B_1 = J_{U,b} \otimes \det(U_b : gR_1) \otimes \det(O(-b) : O)^{n/2}$$

equipped with the canonical isomorphism $B_1^2 \cong J_\beta \otimes \det(U_{gR}) \otimes \det(W_b : gR)$.

We have $\tilde{g} = (g, B_0)$, where

$$B_0 = J_{U,b} \otimes J_{U,a}^{-1} \otimes \det(U_b : gU_a) \otimes \det(O(-b) : O(-a))^{n/2}$$

is equipped with $\mathcal{B}_0^2 \cong J_\beta \otimes J_{U,a}^{-1} \otimes \det(W_b : gW_a)$. It follows that $\tilde{g}(U_R^0) \cong U_{gR}^0$ canonically.

Further, $g$ yields an equivalence $g : D\mathcal{H}_{U_R} \cong D\mathcal{H}_{U_{gR}}$, and the diagram is canonically 2-commutative

$$\begin{array}{ccc}
D\mathcal{H}_{U_R} & \xrightarrow{\mathcal{F}_{U_R^0}} & D(\tilde{\mathcal{L}}(R^\perp/R)) \\
\downarrow g & & \downarrow \tilde{g} \\
D\mathcal{H}_{U_{gR}} & \xrightarrow{\mathcal{F}_{U_{gR}^0}} & D(\tilde{\mathcal{L}}(gR^\perp/gR))
\end{array}$$

(11)

Indeed, this is a consequence of the following isomorphism. We have

$$\tilde{g} : \tilde{\mathcal{L}}(R^\perp/R) \times \tilde{\mathcal{L}}(R^\perp/R) \times H_R \cong \tilde{\mathcal{L}}(gR^\perp/gR) \times \tilde{\mathcal{L}}(gR^\perp/gR) \times H_{gR},$$

and for this isomorphism $\tilde{g}^* F \cong F$ canonically, where $F$ on both sides is the corresponding CIO sheaf (introduced in [8], Theorem 1).

Step 2. Given c-lattices $S_1 \subset R_1 \subset R_2 \subset S_2$ in $U_a(F)$, similarly define $S = S_1 \oplus S_2'$ and $U_S^0 \in \tilde{\mathcal{L}}(S^\perp/S)$ for $S \subset S^\perp \subset W_a(F)$. We have a canonical transition functor $T_{S,R} : D\mathcal{H}_{U_R} \to D\mathcal{H}_{U_S}$
defined as in ([8], Section 6.6). Let $j : \mathcal{L}(S^\perp/S)_{R} \subset \mathcal{L}(S^\perp/S)$ be the open subscheme of $L$ satisfying $L \cap (R/S) = 0$. We have a projection

$$p_{R/S} : \tilde{\mathcal{L}}(S^\perp/S)_{R} \to \tilde{\mathcal{L}}(R^\perp/R)$$

sending $(L, \mathcal{B}_S)$ to $(L_R, \mathcal{B}_S)$, where $L_R := L \cap R^\perp$. It is understood that $\mathcal{B}_S$ is equipped with

$$\mathcal{B}_S^2 \simeq \mathcal{J}_a \otimes \text{det} L \otimes \text{det}(W_a : S),$$

and we used the canonical isomorphism $\text{det} L \otimes \text{det}(W_a : S) \simeq \text{det} L_R \otimes \text{det}(W_a : R)$.

Set $P_{R/S} = p_{R/S}^*[\dim \text{rel}(p_{R/S})]$. Then the following diagram is canonically 2-commutative

$$\begin{array}{c}
\mathcal{D} \mathcal{H}_{U_{R}} \xrightarrow{\mathcal{F}_{U_{a}(F)}} \mathcal{D}(\tilde{\mathcal{L}}(R^\perp/R)) \\
\downarrow \mathcal{T}_{S,R}^U \\
\mathcal{D} \mathcal{H}_{U_{S}} \xrightarrow{\mathcal{F}_{U_{a}(F)}} \mathcal{D}(\tilde{\mathcal{L}}(S^\perp/S))
\end{array}$$

Define $R\mathcal{F}_{U_{a}(F)}$ as the composition

$$\begin{array}{c}
\mathcal{D} \mathcal{H}_{U_{R}} \xrightarrow{\mathcal{F}_{U_{a}(F)}} \mathcal{D}(\tilde{\mathcal{L}}(R^\perp/R)) \\
\downarrow \mathcal{T}_{S,R}^U \\
\mathcal{D} \mathcal{H}_{U_{S}} \xrightarrow{\mathcal{F}_{U_{a}(F)}} \mathcal{D}(\tilde{\mathcal{L}}(S^\perp/S))
\end{array}$$

where the second arrow is the restriction (exact for the perverse t-structures) with respect to the projection $\tilde{\mathcal{L}}_d(W_a(F))_R \to \tilde{\mathcal{L}}(R^\perp/R)$.

The above diagram shows that the following diagram is also 2-commutative

$$\begin{array}{c}
\mathcal{D} \mathcal{H}_{U_{R}} \xrightarrow{R\mathcal{F}_{U_{a}(F)}} \mathcal{D}(\tilde{\mathcal{L}}_d(W_a(F)))_R \\
\downarrow \mathcal{T}_{S,R}^U \\
\mathcal{D} \mathcal{H}_{U_{S}} \xrightarrow{\mathcal{F}_{U_{a}(F)}} \mathcal{D}(\tilde{\mathcal{L}}_d(W_a(F)))_S,
\end{array}$$

where $j_{S,R} : \tilde{\mathcal{L}}_d(W_a(F))_R \subset \tilde{\mathcal{L}}_d(W_a(F))_S$ is the natural open immersion.

So, define

$$\mathcal{F}_{U_{a}(F),R} : \mathcal{D} \mathcal{H}_{U_{R}} \to \mathcal{D}(\tilde{\mathcal{L}}_d(W_a(F)))$$

as the functor sending $K_1$ to the following object $K_2$. For c-lattices $S_1 \subset R_1 \subset R_2 \subset S_2$ as above and $S = S_1 \oplus S'_2$ declare the restriction of $K_2$ to $\tilde{\mathcal{L}}_d(W_a(F))_S$ to be

$$(S\mathcal{F}_{U_{a}(F)} \circ T_{S,R}^U)(K_1)$$

The corresponding projective system (indexed by such $S$) defines an object $K_2 \in \mathcal{D}(\tilde{\mathcal{L}}_d(W_a(F)))$.

Further, passing to the limit by $R$ (of the above form) the functors $\mathcal{F}_{U_{a}(F),R}$ yield the desired functor $\mathcal{F}_{U_{a}(F)} : \mathcal{D}(U^*_a \otimes \Omega(F)) \to \mathcal{D}(\tilde{\mathcal{L}}_d(W_a(F)))$.

Finally, the commutativity of (10) follows from the commutativity of (11).
Remark 4. We could also argue differently in Proposition\textsuperscript{2}. For each $a \in \mathbb{Z}$ and $L^0 \in \tilde{L}_d(W_a(F))$ we could first construct an equivalence $\mathcal{F}_{U_a(F),L^0} : D(U^a_\ast \otimes \Omega(F)) \sim D \mathcal{H}_L$ as in (\textsuperscript{2}, Proposition 5) such that for any $g \in U_{b,a}$ the diagram is 2-commutative

$$
\begin{array}{ccc}
D(U^a_\ast \otimes \Omega(F)) & \xrightarrow{\mathcal{F}_{U_a(F),L^0}} & D \mathcal{H}_L \\
\downarrow g & & \downarrow g \\
D(U^b_\ast \otimes \Omega(F)) & \xrightarrow{\mathcal{F}_{U_b(F),\delta(L^0)}} & D \mathcal{H}_{bL}
\end{array}
$$

with $\tilde{g} = \rho_{b,a}(g)$. Here $\tilde{g}(L^0) \in \tilde{L}_d(W_b(F))$. Then we could define $\mathcal{F}_{U_a(F)}$ as the composition

$$
D(U^a_\ast \otimes \Omega(F)) \xrightarrow{\mathcal{F}_{U_a(F),L^0}} D \mathcal{H}_L \xrightarrow{\mathcal{F}_{L^0}} D(\tilde{L}_d(W_a(F)))
$$

The resulting functor would be (up to a canonical isomorphism) independent of $L^0 \in \tilde{L}_d(W_a(F))$.

4. Dual pair $\text{GSp}_{2n}, \text{GO}_{2m}$

4.1 As in Section 3.2, let $\mathcal{O}$ be a complete discrete valuation $k$-algebra, $F$ its fraction field, $\Omega$ the completed module of relative differentials of $\mathcal{O}$ over $k$. For a free $\mathcal{O}$-module $M$ we write $M_x = M \otimes \mathcal{O} k$ for its geometric fibre.

Fix free $\mathcal{O}$-modules $M_a$ of rank $2n$, $V_a$ of rank $2m$, and $A_a, C_a$ of rank one with symplectic form $\wedge^2 M_a \to A_a$, a nondegenerate symmetric form $\text{Sym}^2 V_a \to C_a$, and a compatible trivialization $\det V_a \simeq C_a^m$. Assume also given an isomorphism $A_a \otimes C_a \simeq \Omega(a)$ for each $a \in \mathbb{Z}$.

Set $W_a = M_a \otimes V_a$, it is equipped with the symplectic form $\wedge^2 W_a \to \Omega(a)$. For $a \in \mathbb{Z}$ set $J_a = C_a^{-anm}$, which is of parity zero as $\mathbb{Z}/2\mathbb{Z}$-graded. Define $\tilde{L}_d(W_a(F))$, $\mathcal{G}_{b,a}$, $G_a$ and $\tilde{G}_{b,a}$ as in Section 3.2.

Let $G = \text{GSp}_{2n}$ be the symplectic similitude group over $k$ of semisimple rank $n$. Let $H$ be the connected component of unity of the split orthogonal similitude group $\text{GO}_{2m}$ of semisimple rank $m$. We may view $(M_a, A_a)$ (resp., $(V_a, C_a)$) as a $G$-torsor (resp., $H$-torsor) on $\text{Spec} \mathcal{O}$.

Let $\mathcal{G}_{b,a}$ be the set of isomorphisms $M_a(F) \to M_b(F)$ of $G$-torsors over $\text{Spec} F$. Let $\mathcal{H}_{b,a}$ be the set of isomorphisms $V_a(F) \to V_b(F)$ of $H$-torsors over $\text{Spec} F$. Let $T_{b,a}$ be the set of pairs $g = (g_1, g_2)$, where $g_1 \in \mathcal{G}_{b,a}$, $g_2 \in \mathcal{H}_{b,a}$ such that $g \in \mathcal{G}_{b,a}$. That is, the composition

$$
\Omega(F) \to A_a \otimes C_a(F) \xrightarrow{g_1 \otimes g_2} A_b \otimes C_b(F) \to \Omega(F)
$$

must equal to the identity. The natural composition map $T_{c,b} \times T_{b,a} \to T_{c,a}$ makes $T = \sqcup_{a,b} T_{b,a}$ into a groupoid.

Lemma 1. Let $M_1, V$ be a free $\mathcal{O}_x$-modules of finite rank, where $M_2 \subset \text{M}_1(F_x)$ is a $\mathcal{O}_x$-lattice. Set $\dim(M_1 : M_2) = \dim(M_1 / R) - \dim(M_2 / R)$ for any $\mathcal{O}_x$-lattice $R \subset M_1 \cap M_2$. Then we have a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism\textsuperscript{2}

$$
det(M_1 \otimes V : M_2 \otimes V) \simeq \det(M_1 : M_2)^{rk V} \otimes (\det V_x)^{\dim(M_1 : M_2)[\dim(M_1 : M_2) \cdot rk V]}
$$

\textsuperscript{2}there may be sign problems, the corresponding isomorphism is well defined at least up to a sign
Proof. Pick a $O_x$-lattice $R \subset M_1 \cap M_2$. It suffices to establish a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism
\[
\det(M_1 \otimes V : R \otimes V) \cong \det(M_1/R)^{\text{rk} V} \otimes \det(V_x)^{\dim(M_1/R) \cdot \text{rk} V}
\]
To do so, it suffices to pick a flag $R = R_0 \subset R_1 \subset \ldots \subset R_a = M_1$ of $O_x$-lattices with $\dim(R_i/R_{i-1}) = 1$. □

For $e \in \mathbb{Z}$ set $G_{b,a}^e = \{ g \in G_{b,a} \mid gA_a = A_b(e) \}$ and $\mathbb{H}_{b,a}^e = \{ g \in \mathbb{H}_{b,a} \mid gC_a = C_b(e) \}$.

Let us construct a canonical section $\tau_{b,a} : T_{b,a} \to \tilde{G}_{b,a}$ compatible with the groupoids structures. Let $g = (g_1, g_2) \in T_{b,a}$ with $g_1 \in G_{b,a}^e$, $g_2 \in \mathbb{H}_{b,a}^e$, so $e + c = a - b$. Using Lemma 1 we get a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism
\[
\det(M_b \otimes V_b : (g_1M_a) \otimes (g_2V_a)) \cong \det(M_b : g_1M_a)^{2n} \otimes \det(V_b : g_2V_a)^{2n} \otimes \det(V_x)^{\dim(M_b \cdot g_1M_a) \otimes (\det M_a)^{\dim(V_b \cdot g_2V_a) \cong} \det(M_b : g_1M_a)^{2n} \otimes \det(V_b : g_2V_a)^{2n} \otimes C^{-mne}_{b,x} \otimes A^{-mne}_{a,x} \cong \det(M_b : g_1M_a)^{2n} \otimes \det(V_b : g_2V_a)^{2n} \otimes C^{-mne}_{b,x} \otimes A^{-mne}_{a,x} \otimes \mathcal{O}(\mu(1 + c))^{n}
\]
We used that $\dim(M_b : g_1M_a) = -ne$, $\dim(V_b : g_2V_a) = -mc$. Identifying further $C_a \cong C_b(c)$, we get
\[
J_b \otimes J_a^{-1} \otimes \det(W_b : gW_a) \cong \det(M_b : g_1M_a)^{2n} \otimes \det(V_b : g_2V_a)^{2n} \otimes C^{2nm}_{b,x} \otimes \mathcal{O}(\mu(1 + c))^{nm}
\]
Let $\nu_{b,a}(g) = (g, \beta)$, where
\[
\beta = \det(M_b : g_1M_a)^{m} \otimes \det(V_b : g_2V_a)^{n} \otimes C^{nm}_{b,x} \otimes \mathcal{O}(\mu(1 + c)/2)^{nm}
\]
is equipped with the induced isomorphism $\mathcal{B}^{2} \cong J_b \otimes J_a^{-1} \otimes \det(W_b : gW_a)$.

We let $\mathcal{T}$ act on $\tilde{T}^{ex}$ via $\nu$.

4.2 Let $G_a = GSp(M_a)$ and $\mathbb{H}_a = G\Omega^0(V_a)$, the connected component of unity of the group scheme $G\Omega(V_a)$ over Spec $\mathcal{O}$. Set
\[
\mathcal{D}_a = \{ (g_1, g_2) \in (G_a \times \mathbb{H}_a)(\mathcal{O}) \mid g_1 \otimes g_2 \text{ acts trivially on } A_a \otimes C_a \}
\]
The line bundle on $\mathcal{L}d(W_a(F))$ with fibre $J_a \otimes \det(W_a : L)$ at $L$ is naturally $\mathcal{D}_a$-equivariant (we underline that $\mathcal{D}_a$ acts nontrivially on $\mathcal{J}_a$). So, it can be seen as a line bundle on the quotient stack $^a\mathcal{X}L := \mathcal{L}d(W_a(F))/\mathcal{D}_a$. We also have the corresponding $\mu_2$-gerb
\[
^a\tilde{\mathcal{X}L} := \tilde{\mathcal{L}}(W_a(F))/\mathcal{D}_a
\]
of square roots of this line bundle. The derived category $D_{\mathcal{D}_a}(\tilde{\mathcal{L}}d(W_a(F)))$ is defined as in (7, Section 8.2.2).
The stack $\mathcal{X}_L$ classifies: a $G$-torsor $(M, A)$ over Spec $O$, a $H$-torsor $(V, C)$ over Spec $O$ (so, we have a compatible isomorphism $det V \cong C^m$), an isomorphism $A \otimes C \cong \Omega(a)$, and a discrete lagrangian subspace $L \subset M \otimes V(F)$.

Let $^{a}A_{\mathcal{X}_L}$ be the line bundle over $^{a}\mathcal{X}_L$ whose fibre at $(M, A, V, C, L)$ is $C_{x}^{-anm} \otimes det(M \otimes V : L)$. It is of parity zero as $\mathbb{Z} / 2\mathbb{Z}$-graded. Then $^{a}\mathcal{X}_L$ is the $\mu_2$-gerb of square roots of $^{a}A_{\mathcal{X}_L}$.

**4.3.1 HECKE OPERATORS** Denote by $^{a,a'}H_{G, XL}$ the stack classifying: a point $(L, M, A, V, C) \in ^{a}\mathcal{X}_L$, a lattice $M' \subset M(F)$ such that for $A' = A(a' - a)$ the induced form is regular and nondegenerate $\wedge^2 M' \rightarrow A'$. We get a diagram

$$^{a}\mathcal{X}_L \xrightarrow{h} ^{a,a'}H_{G, XL} \xrightarrow{h} ^{a'}\mathcal{X}_L,$$

where $h^-$ (resp., $h^-$) sends a point of $^{a,a'}H_{G, XL}$ to $(L, M, A, V, C)$ (resp., to $(L, M', A', V, C)$).

**Lemma 2.** For a point $(L, M, A, M', A', V, C)$ of $^{a,a'}H_{G, XL}$ there is a canonical $\mathbb{Z} / 2\mathbb{Z}$-graded isomorphism

$$C_{x}^{-anm} \otimes det(M' \otimes V : L) \cong C_{x}^{-anm} \otimes det(M \otimes V : L) \otimes det(M' : M)^{2m}$$

Let $^{a,a'}\tilde{H}_{G, XL}$ be map obtained from $h^-$ by the base change $^{a'}\mathcal{X}_L \rightarrow ^{a'}\mathcal{X}_L$. By Lemma 2 we get a diagram

$$^{a}\mathcal{X}_L \xrightarrow{\tilde{h}} ^{a,a'}\tilde{H}_{G, XL} \xrightarrow{\tilde{h}} ^{a'}\mathcal{X}_L$$

Here a point of $^{a,a'}\tilde{H}_{G, XL}$ is given by a collection $(L, M, A, M', A', V, C) \in ^{a,a'}H_{G, XL}$ together with a one-dimensional space $B$ equipped with $B_{1} = B \otimes det(M : M)^{m}$ with the induced isomorphism $B_{1} \cong C_{x}^{-anm} \otimes det(M \otimes V : L)$.

The affine grassmanian $Gr_{G_a} = G_a(F) / G_a(O)$ is the ind-scheme classifying $O$-lattices $R \subset M_a(F)$ such that for some $r \in \mathbb{Z}$ the induced form $\wedge^2 R \rightarrow A_a(r)$ is regular and nondegenerate. Write $Gr_{G_a}$ for the connected component of $Gr_{G_a}$ given by fixing such $r$.

Trivializing a point of $^{a'}\mathcal{X}_L$ (resp., of $^{a}\mathcal{X}_L$) one gets isomorphisms

$$id^r : ^{a,a'}H_{G, XL} \cong (L_d(W_{a'}(F)) \times Gr_{G_{a'}}^{a-a'}/T_{a'}$$

and

$$id^l : ^{a,a'}H_{G, XL} \cong (L_d(W_a(F)) \times Gr_{G_a}^{a'-a}/T_a,$$

where the corresponding action of $T_{a'}$ (resp., of $T_a$) is diagonal. They lift naturally to a $T_{a'}$-torsor

$$\tilde{L}_d(W_{a'}(F)) \times Gr_{G_{a'}}^{a-a'} \rightarrow ^{a,a'}\tilde{H}_{G, XL}$$

and a $T_a$-torsor

$$\tilde{L}_d(W_a(F)) \times Gr_{G_a}^{a'-a} \rightarrow ^{a,a'}\tilde{H}_{G, XL}$$
So, for \( K \in D_{T_a}(\tilde{L}_d(W_a(F))) \), \( K' \in D_{T_{a'}}(\tilde{L}_a(W_{a'}(F))) \), \( S \in \text{Sph}_{G_a} \), \( S' \in \text{Sph}_{G_{a'}} \), we can form their external products

\[(K \boxtimes S)^t, (K' \boxtimes S')^r\]
on \( ^{a,a'}\tilde{H}_{G,XL} \). The Hecke functor

\[H_{\tilde{G}} : \text{Sph}_{G_{a'}} \times D_{T_{a'}}(\tilde{L}_d(W_{a'}(F))) \to D_{T_{a'}}(\tilde{L}_d(W_{a'}(F)))\]
is defined by

\[H_{\tilde{G}}(S', K') = (h^{-1})(K' \boxtimes S')^r\]

It is understood that this informal definition should be made rigorous in a way similar to ([7], Section 4.3).

Write \( \mathfrak{b}_{\text{Sph}_{G_{a'}}} \subset \text{Sph}_{G_{a'}} \) for the full subcategory of objects that vanish off \( \text{Gr}^{\mathfrak{b}}_{G_{a'}} \). The first argument of \( H_{\tilde{G}} \) actually lies in \( \mathfrak{a'}-a \text{Sph}_{G_{a'}} \).

4.3.2 Let \( ^{a,a'}\tilde{H}_{\mathfrak{H},XL} \) be the stack classifying: a point \((L, M, A, V, C) \in \mathcal{X}L\), a lattice \( V' \subset V(F) \) such that for \( C' = C(a' - a) \the induced form \text{Sym}^2 V' \to C' \) is regular and nondegenerate (we also get the isomorphism \( C'^{-m} \otimes \det V' \cong C^{-m} \otimes \det V \cong \mathcal{O} \)). As for \( G \), we get a diagram

\[
\begin{array}{ccc}
^{a, a'}\mathcal{X}L & \xrightarrow{h^{-1}} & ^{a, a'}\tilde{H}_{\mathfrak{H},XL} \\
\downarrow & & \downarrow \\
^{a, a'}\mathcal{X}L & \xrightarrow{h^{-1}} & ^{a, a'}\tilde{H}_{\mathfrak{H},XL}
\end{array}
\]

where \( h^{-1} \) (resp. \( h^{-1} \)) sends \((L, M, A, V, C, V', C') \) to \((L, M, A, V, C) \) (resp., to \((L, M, A, V', C') \)), the vertical arrows are \( \mu_2 \)-gerbs, and the right square is cartesian (thus defining the stack \( ^{a,a'}\tilde{H}_{\mathfrak{H},XL} \)).

A point of \( ^{a,a'}\tilde{H}_{\mathfrak{H},XL} \) is given by \((L, M, A, V, C, V', C') \in ^{a,a'}\tilde{H}_{\mathfrak{H},XL} \) and a one-dimensional space \( \mathcal{B} \) equipped with

\[B^2 \cong (C_x')^{-a'nm} \otimes \det(M \otimes V' : L)\]
The map \( h^{-1} \) sends this point to \((L, M, A, V, C) \in \mathcal{X}L\), the one-dimensional space \( B_1 \) with \( B_1 \cong C_x'^{-anm} \otimes \det(M \otimes V : L) \), where

\[B_1 = B \otimes C_x'^{anm(a'-a)} \otimes \det(V : V')^{anm} \otimes \mathcal{O}(\frac{1}{2}nm(a - a')(a - a' - 1))_x\]

The affine grassmanian \( \text{Gr}_{\mathfrak{H}_a} \) classifies lattices \( V' \subset V_a(F) \) such that the induced symmetric form \( \text{Sym}^2 V' \to C_a(b) \) is regular and nondegenerate for some \( b \in \mathbb{Z} \). Write \( \text{Gr}^{\mathfrak{b}}_{\mathfrak{H}_a} \) for the locus of \( \text{Gr}_{\mathfrak{H}_a} \) given by fixing this \( b \). For \( m \geq 2 \) there is an exact sequence \( 0 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(\mathfrak{H}_a) \to \mathbb{Z} \to 0 \), so if \( m \geq 2 \) then \( \text{Gr}^{\mathfrak{b}}_{\mathfrak{H}_a} \) is a union of two connected components of \( \text{Gr}_{\mathfrak{H}_a} \). Write \( \mathfrak{b}_{\text{Sph}_{\mathfrak{H}_a}} \subset \text{Sph}_{\mathfrak{H}_a} \) for the full subcategory of objects that vanish off \( \text{Gr}^{\mathfrak{b}}_{\mathfrak{H}_a} \).

The Hecke functor

\[H_{\tilde{H}} : \text{Sph}_{\mathfrak{H}_{a'}} \times D_{T_{a'}}(\tilde{L}_d(W_{a'}(F))) \to D_{T_{a'}}(\tilde{L}_d(W_{a}(F)))\]

19
is defined as in Section 4.3.1 using the diagram (14).

For each $a \in \mathbb{Z}$ a trivialization $\alpha$ of the $G$-torsor $(M_a, A_a)$ on Spec $O$ yields an isomorphism $\alpha : \text{Gr}_{G_a} \cong \text{Gr}_G$. The induced equivalences $\alpha^* : \text{Sph}_G \cong \text{Sph}_{G_a}$ are canonically 2-isomorphic for different $\alpha$’s. In what follows we sometimes identify these two categories in this way. Similarly, we identify $\text{Sph}_{\mathbb{H}_a} \cong \text{Sph}_G$.

4.4 Let $S_{W_0(F)} \in P_{T_0}(\mathcal{L}_d(W_0(F)))$ be the theta-sheaf introduced in (S, Section 6.5). This is a $T_0$-equivariant object of the Weil category $W(\mathcal{L}_d(W_0(F)))$. Here is the main result of Section 4.

**Theorem 3.** 1) Assume $m \leq n$. There is a homomorphism $\kappa : \mathbb{H} \times \mathbb{G}_m \to \mathbb{G}$ such that for the corresponding geometric restriction functor $g\text{Res}^\kappa : \text{Sph}_G \to D\text{Sph}_H$ we have an isomorphism in $D_{T_0}(\mathcal{L}_d(W_a(F)))$

$$H_G^-(S, S_{W_0(F)}) \cong H_H^-(g\text{Res}^\kappa(S), S_{W_0(F)})$$

functorial in $S \in -a\text{Sph}_G$.

2) Assume $m > n$. There is a homomorphism $\kappa : \mathbb{G} \times \mathbb{G}_m \to \mathbb{H}$ such that for the corresponding geometric restriction functor $g\text{Res}^\kappa : \text{Sph}_H \to D\text{Sph}_G$ we have an isomorphism in $D_{T_0}(\mathcal{L}_a(W_a(F)))$

$$H_H^-(S, S_{W_0(F)}) \cong H_G^-(g\text{Res}^\kappa(S), S_{W_0(F)})$$

functorial in $S \in -a\text{Sph}_H$.

The proof occupies the rest of Section 4. The explicit formulas for $\kappa$ are found in Section 4.8.9.

4.5 Assume given a decomposition $M_a = L_a \oplus (L_a^* \otimes A_a)$, where $L_a$ is a free $O$-module of rank $n$, $L_a$ and $L_a^* \otimes A_a$ are lagrangians, and the form is given by the canonical pairing between $L_a$ and $L_a^*$. Assume given a similar decomposition $V_a = U_a \oplus (U_a^* \otimes C_a)$ for $V_a$, here $U_a$ is a free $O$-module of rank $m$.

Write $Q(\mathbb{G}_a) \subset \mathbb{G}_a$ and $Q(\mathbb{H}_a)$ for the Levi subgroups preserving the above decompositions. Set

$$QGH_a = \{g = (g_1, g_2) \in Q(\mathbb{G}_a) \times Q(\mathbb{H}_a) \mid g \in T_a\}$$

$$\mathbb{G}QH_a = \{g = (g_1, g_2) \in \mathbb{G}_a \times Q(\mathbb{H}_a) \mid g \in T_a\}$$

$$\mathbb{H}QG_a = \{g = (g_1, g_2) \in \mathbb{H}_a \times Q(\mathbb{G}_a) \mid g \in T_a\}$$

We view all of them as group schemes over Spec $O$. We also pick Levi subgroups $Q(G) \subset G$ and $Q(H) \subset H$ which identify with the above over Spec $O$.

The affine grassmanian $Gr_{Q(\mathbb{G}_a)}$ classifies pairs of lattices $L' \subset L_a(F)$, $A' \subset A_a(F)$. For $b \in \mathbb{Z}$ write $Gr_{Q(\mathbb{G}_a)}^b$ for the locus of $Gr_{Q(\mathbb{G}_a)}$ given by $A' = A_a(b)$. Write $b\text{Sph}_{Q(\mathbb{G}_a)} \subset \text{Sph}_{Q(\mathbb{G}_a)}$ for the full subcategory of objects that vanish off $Gr_{Q(\mathbb{G}_a)}^b$. As in Section 4.4, we identify canonically $\text{Sph}_{Q(G)} \cong \text{Sph}_{Q(\mathbb{G}_a)}$. The geometric restriction $g\text{Res} : \text{Sph}_G \to \text{Sph}_{Q(G)}$ corresponding to the inclusion of the Langlands dual groups $\hat{Q}(G) \hookrightarrow \mathbb{G}$ yields a faithful functor $b\text{Sph}_G \to b\text{Sph}_{Q(G)}$ for each $b$. And similarly for $H$.  

20
For $b,a \in \mathbb{Z}$ write $Q(\mathbb{G}_{b,a})$ for the set of isomorphisms $(L_a(F) \to L_b(F), A_a(F) \to A_b(F))$ of $\text{GL}_n \times \mathbb{G}_m$-torsors over Spec $F$. Let $Q(\mathbb{H}_{b,a})$ be the set of isomorphisms $(U_{a}(F) \to U_{b}(F), C_{a}(F) \to C_{b}(F))$ of $\text{GL}_m \times \mathbb{G}_m$-torsors over Spec $F$. Set

$$Q\mathbb{G}_{b,a} = \{g = (g_1, g_2) \in Q(\mathbb{G}_{b,a}) \times Q(\mathbb{G}_{b,a}) \mid g \in \mathcal{G}_{b,a}\}$$

$$Q\mathbb{H}_{b,a} = \{g = (g_1, g_2) \in \mathbb{G}_{b,a} \times Q(\mathbb{G}_{b,a}) \mid g \in \mathcal{G}_{b,a}\}$$

$$\mathbb{H}Q_{b,a} = \{g = (g_1, g_2) \in H_{b,a} \times Q(\mathbb{G}_{b,a}) \mid g \in \mathcal{G}_{b,a}\}$$

Set $\Upsilon_a = L_a^{*} \otimes A_a \otimes V_a$ and $\Pi_a = U_a^{*} \otimes C_a \otimes M_a$. For $a \in \mathbb{Z}$ and any $L^0 \in \mathcal{E}_d(W_a(F))$ we have the equivalences

$$\mathcal{F}_{L_a \otimes V_a(F), L^0} : D(\Upsilon_a(F)) \simeq D\mathcal{H}_L$$

and $\mathcal{F}_{U_a \otimes M_a(F), L^0} : D(\Pi_a(F)) \simeq D\mathcal{H}_L$ defined as in Remark 4.6.1.

Remind that for free $\mathcal{O}$-modules of finite type $\mathcal{V}, \mathcal{U}$ one has the partial Fourier transform

$$\text{Four}_{\psi} : D(\mathcal{V}(F) \oplus \mathcal{U}(F)) \simeq D(\mathcal{V}^{*} \otimes \Omega(\mathcal{F}) \oplus \mathcal{U}(F))$$

normalized to preserve perversity and purity (cf. [7], Section 4.8 for the definition). Thus, the decompositions

$$\Pi_a \simeq U_a^{*} \otimes C_a \otimes L_a \oplus U_a^{*} \otimes L_a^{*} \otimes \Omega(a)$$

and

$$\Upsilon_a \simeq L_a^{*} \otimes A_a \otimes U_a \oplus U_a^{*} \otimes L_a^{*} \otimes \Omega(a)$$

yield the partial Fourier transform, which we denote

$$\zeta_a : D(\Upsilon_a(F)) \simeq D(\Pi_a(F))$$

One checks that $\zeta_a$ is canonically isomorphic to the functor $\mathcal{F}^{-1}_{U_a \otimes M_a(F), L^0} \circ \mathcal{F}_{L_a \otimes V_a(F), L^0}$ for any $L^0 \in \mathcal{E}_d(W_a(F))$.

4.6.1 It is convenient to denote $\tilde{\Upsilon}_a = L_a \otimes V_a$ and $\tilde{\Pi}_a = U_a \otimes M_a$. For the decomposition $W_a = \tilde{\Pi}_a \oplus \tilde{\Pi}_a^{*} \otimes \Omega(a)$ we define a $\mathbb{Z}/2\mathbb{Z}$-graded line (purely of parity zero)

$$J_{\Pi,a} = \mathcal{O}((1 + a)/2)^{nm} \otimes (\det U_{a,x})^{-na}$$

equipped with a natural $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$J_{\Pi,a}^2 \simeq J_a \otimes (\det \tilde{\Pi}_a)^{-a}$$

It yields a section $\tilde{\Pi}_{\rho_{b,a}} : Q\mathbb{H}_{b,a} \to \mathcal{G}_{b,a}$ defined as in Section 3.3.

For the decomposition

$$W_a = \tilde{\Upsilon}_a \oplus \tilde{\Upsilon}_a^{*} \otimes \Omega(a)$$

21
define a \( \mathbb{Z}/2\mathbb{Z} \)-graded line (purely of parity zero)

\[ \mathcal{J}_{T,a} = C_{a,x}^m \otimes (\det L_{a,x})^{-m} \]

equipped with a natural \( \mathbb{Z}/2\mathbb{Z} \)-graded isomorphism

\[ \mathcal{J}_{T,a}^2 \cong \mathcal{J}_a \otimes (\det \mathcal{Y}_{a,x})^{-a} \]

It yields a section \( \varpi_{\rho_b, a} : \mathbb{H} Q G_{b,a} \to \mathcal{G}_{b,a} \) defined as in Section 3.3.

From definitions one derives the following.

**Lemma 3.** For \( a, b \in \mathbb{Z} \) the following diagrams are canonically 2-commutative

\[
\begin{array}{ccc}
\mathcal{T}_{b,a} & \xrightarrow{\nu_{b,a}} & \mathcal{G}_{b,a} \\
\uparrow & \nearrow \rho_{b,a} & \\
\mathbb{G} Q \mathbb{H}_{b,a} & & \mathbb{H} Q \mathbb{G}_{b,a}
\end{array}
\]

\[ \mathcal{T}_{b,a} \xrightarrow{\nu_{b,a}} \mathcal{G}_{b,a} \]

\[ \uparrow \rho_{b,a} \]

\[ \mathbb{G} Q \mathbb{H}_{b,a} \]

\[ \mathbb{H} Q \mathbb{G}_{b,a} \]

For \( a \in \mathbb{Z} \) we have the functors \( \mathcal{F}_{\mathcal{T}_{a,F}} : D(\mathcal{T}_{a,F}) \to D(\mathcal{G}_{b,a}) \) defined in Proposition 2. Note that the diagram is canonically 2-commutative

\[
\begin{array}{ccc}
D(\mathcal{T}_{a,F}) & \xrightarrow{\mathcal{F}_{\mathcal{T}_{a,F}}(F)} & D(\mathcal{G}_{b,a}) \\
\downarrow \zeta_a & & \downarrow \zeta_b \\
D(\mathcal{T}_{a,F}) & \xrightarrow{\mathcal{F}_{\mathcal{T}_{a,F}}(F)} & D(\mathcal{T}_{b,F})
\end{array}
\]

**Remark 5.** The following structure emerge. For each \( g \in \mathcal{T}_{b,a} \) we get functors that fit into a 2-commutative diagram

\[
\begin{array}{ccc}
D(\mathcal{T}_{b,F}) & \xrightarrow{g} & D(\mathcal{T}_{b,F}) \\
\uparrow \zeta_a & & \uparrow \zeta_b \\
D(\mathcal{T}_{a,F}) & \xrightarrow{g} & D(\mathbb{H} Q \mathbb{G}_{b,a})
\end{array}
\]

They are compatible with the groupoid structure on \( \mathcal{T} \). Indeed, one first defines these functors separately for \( \mathcal{G} Q \mathbb{H}_{b,a} \subset \mathcal{T}_{b,a} \) and for \( \mathbb{H} Q \mathbb{G}_{b,a} \subset \mathcal{T}_{b,a} \) using the models \( \mathcal{Y} \) and \( \mathcal{Y} \) respectively.

This is sufficient because any \( g \in \mathcal{T}_{b,a} \) writes as a composition \( g = g'' \circ g' \) with \( g'' \in \mathbb{H} Q \mathbb{G}_{b,b} \) and \( g' \in \mathcal{G} Q \mathbb{H}_{b,a} \). The arrows in the above diagram are equivalences.

4.7 We have the full subcategories (stable under subquotients)

\[
P_{\mathbb{H} Q \mathbb{G}_{a}(\mathcal{O})}(\mathcal{T}_{a,F}) \subset \mathbb{P} Q_{\mathbb{H} Q \mathbb{G}_{a}(\mathcal{O})}(\mathcal{T}_{a,F}) \subset P(\mathcal{T}_{a,F})
\]

\[
P_{\mathbb{G} Q \mathbb{H}_{a}(\mathcal{O})}(\mathcal{G}_{b,a}) \subset \mathbb{P} Q_{\mathbb{G} Q \mathbb{H}_{a}(\mathcal{O})}(\mathcal{G}_{b,a}) \subset P(\mathcal{G}_{b,a})
\]

and \( \zeta_a \) yields an equivalence \( \zeta_a : \mathbb{P} Q_{\mathbb{G} Q \mathbb{H}_{a}(\mathcal{O})}(\mathcal{T}_{a,F}) \cong \mathbb{P} Q_{\mathbb{G} Q \mathbb{H}_{a}(\mathcal{O})}(\mathcal{T}_{a,F}) \).

**Definition 2.** For \( a \in \mathbb{Z} \) let \( \text{Weil}_a \) be the category of triples \( (\mathcal{F}_1, \mathcal{F}_2, \beta) \), where

\[
\mathcal{F}_1 \in P_{\mathbb{H} Q \mathbb{G}_{a}(\mathcal{O})}(\mathcal{T}_{a,F}), \quad \mathcal{F}_2 \in P_{\mathbb{G} Q \mathbb{H}_{a}(\mathcal{O})}(\mathcal{G}_{b,a})
\]

and \( \beta : \zeta_a(\mathcal{F}_1) \cong \mathcal{F}_2 \) is an isomorphism in \( P_{\mathbb{G} Q \mathbb{H}_{a}(\mathcal{O})}(\mathcal{G}_{b,a}) \).

Write \( D\text{Weil}_a \) for the category obtained by replacing everywhere in the above definition \( P \) by \( DP \).

22
Clearly, $\text{Weil}_a$ is an abelian category, and the forgetful functors $f_H : \text{Weil}_a \to P_{\mathbb{H}G_{\alpha}(O)}(Y_a(F))$ and $f_G : \text{Weil}_a \to P_{\mathbb{H}G_{\alpha}(O)}(\Pi_a(F))$ are full embeddings. By Proposition 2 we get a functor

$$\mathcal{F}_{\text{Weil}_a} : \text{Weil}_a \to P_{T_a}(\tilde{L}_d(W_a(F)))$$

sending $(\mathcal{F}_1, \mathcal{F}_2, \beta)$ to $\mathcal{F}_{\text{Weil}_a}(\mathcal{F}_1)$.

Let $I_0 \in P_{\mathbb{H}G_{\alpha}(O)}(Y_0(F))$ denote the constant perverse sheaf on $Y_0$ extended by zero to $Y_0(F)$. Remind that $\zeta_0(I_0)$ is the constant perverse sheaf on $\Pi_0$ extended by zero to $\Pi_0(F)$. The object $\zeta_0(I_0)$ will also be denoted $I_0$ by abuse of notation. So, $I_0 \in \text{Weil}_0$ naturally.

By definition of the theta-sheaf, we have canonically $\mathcal{F}_{\text{Weil}_a}(I_0) \simeq S_{W_0}(F)$ in $P_{T_a}(\tilde{L}_d(W_0(F)))$.

4.8 More Hecke Operators

4.8.1 For $a \in \mathbb{Z}$ let $\alpha a \Pi$ be the stack classifying: a $\mathbb{G}_m \times \mathbb{G}_m$-torsor $(U, C)$ over $\text{Spec} \, O$, $\mathbb{G}$-torsor $(M, A, \wedge^2 M \to A)$ over $\text{Spec} \, O$, an isomorphism $A \otimes C \iso \Omega(a)$, and a section $s \in U^* \otimes M^* \otimes \Omega(F)$.

Informally, we may view $D_{GQ \mathbb{H}_{\alpha}(O)}(\Pi_a(F))$ as the derived category on $\alpha a \Pi$. For $a, a' \in \mathbb{Z}$ we are going to define a Hecke functor

$$H^\alpha_G : a' - a \, \text{Sph}_G \times D_{GQ \mathbb{H}_{a'}(O)}(\Pi_{a'}(F)) \to D_{GQ \mathbb{H}_{a}(O)}(\Pi_a(F))$$

(15)

To do so, consider the stack $\alpha a' \Pi$ classifying: a point of $\alpha a \Pi$ as above, a lattice $M' \subset M(F)$ such that for $A' = A(a' - a)$ the induced form $\wedge^2 M' \to A'$ is regular and nondegenerate.

We get a diagram

$$\alpha a \Pi \xrightarrow{h^-} \alpha a' \Pi \xrightarrow{h^-} a' \alpha a \Pi,$$

where $h^-$ sends the above collection to $(U, C, M, A, s)$, the map $h^-$ sends the above collection to $(U, C, M', A', s')$, where $s'$ is the image of $s$ under $U^* \otimes M^* \otimes \Omega(F) \iso U^* \otimes M^* \otimes \Omega(F)$.

Trivializing a point of $\alpha a \Pi$ (resp., of $\alpha a' \Pi$), one gets isomorphisms

$$\text{id}^r : \alpha a' \Pi \iso (\Pi_{a'}(F) \times \text{Gr}_{G_{a'}}^{a' - a},)/\mathbb{G} \mathbb{H}_{a'}(O)$$

and

$$\text{id}^l : \alpha a' \Pi \iso (\Pi_a(F) \times \text{Gr}_{G_a}^{a' - a},)/\mathbb{G} \mathbb{H}_{a}(O)$$

So for

$$K \in D_{GQ \mathbb{H}_{a'}(O)}(\Pi_{a'}(F)),$$  

$$K' \in D_{GQ \mathbb{H}_{a'}(O)}(\Pi_{a'}(F)),$$  

$$S \in a' - a \, \text{Sph}_G,$$  

$$S' \in a - a' \, \text{Sph}_G,$$

one can form the twisted exterior products $(K \boxtimes S)^l$ and $(K' \boxtimes S')^r$ on $\alpha a' \Pi$. The functor (15) is defined by

$$H^\alpha_G(S', K') \iso h^-(K' \boxtimes S')^r$$

It is understood that this informal definition should be made rigorous in the same way as in (7), Section 4.3.)
4.8.2 For \( a \in \mathbb{Z} \) let \( {}^a\mathcal{X}\mathcal{Y} \) be the stack classifying: a \( GL_n \times \mathbb{G}_m \)-torsor \((L, A)\) over \( \text{Spec} \, \mathcal{O} \), a \( \mathbb{G}_m \)-torsor \((V, C)\) over \( \mathcal{O} \) (so, we are also given a compatible trivialization \( \det V \sim C^m \), an isomorphism \( A \otimes C \sim \Omega(a) \), and a section \( s \in L^* \otimes V^* \otimes \Omega(F) \).

We may view \( D_{\mathbb{H}G\mathcal{H}_{a}(\mathcal{O})}(\mathbb{Y}_a(F)) \) as the derived category on \( {}^a\mathcal{X}\mathcal{Y} \). For \( a, a' \in \mathbb{Z} \) we define a Hecke functor

\[
H^\mathcal{H}_a : {}^a\mathcal{X}\mathcal{Y} \xleftarrow{h^{-}} {a, a'}\mathcal{H}_{X\mathcal{Y}} \xrightarrow{h^{-}} {}^{a'}\mathcal{X}\mathcal{Y}
\]

as follows. Let \( {a, a'}\mathcal{H}_{X\mathcal{Y}} \) be the stack classifying: a point of \( {}^a\mathcal{X}\mathcal{Y} \) as above, a lattice \( V' \subset V(F) \) such that for \( C' = C(a' - a) \) the induced form \( \text{Sym}^2 V' \to C' \) is regular and nondegenerate (we also get a compatible trivialization

\[
C' - m \otimes \det V' \sim C^{-m} \otimes \det V \sim \mathcal{O},
\]

so \( (V', C') \) is a \( \mathbb{H} \)-torsor over \( \text{Spec} \, \mathcal{O} \).

As in Section 4.8.1, we get a diagram

\[
{}^a\mathcal{X}\mathcal{Y} \xleftarrow{h^{-}} {a, a'}\mathcal{H}_{X\mathcal{Y}} \xrightarrow{h^{-}} {}^{a'}\mathcal{X}\mathcal{Y}
\]

and the desired functor \([16]\).

4.8.3 We need the following lemma. Write \( {}^a\mathcal{X}\mathcal{Y} \) for the stack classifying: a \( GL_n \times \mathbb{G}_m \)-torsor \((U, C)\) over \( \text{Spec} \, \mathcal{O} \), a \( \mathbb{G}_m \)-torsor \((M, A, \wedge^2 M \to A)\) over \( \text{Spec} \, \mathcal{O} \), an isomorphism \( A \otimes C \sim \Omega(a) \), and a section \( s_1 \in U \otimes M(F) \). View \( D_{\mathbb{G}Q\mathcal{H}_{a}(\mathcal{O})}(\Pi_a^* \otimes \Omega(F)) \) as the derived category on \( {}^a\mathcal{X}\mathcal{Y} \).

For \( a, a' \in \mathbb{Z} \) define the Hecke functor

\[
H_{\mathbb{G}} : {}^a\mathcal{X}\mathcal{Y} \xleftarrow{h^{-}} {a, a'}\mathcal{H}_{X\mathcal{Y}} \xrightarrow{h^{-}} {}^{a'}\mathcal{X}\mathcal{Y},
\]

as follows. Let \( {a, a'}\mathcal{H}_{X\mathcal{Y}} \) be the stack classifying: a point of \( {}^a\mathcal{X}\mathcal{Y} \) as above, a lattice \( M' \subset M(F) \) such that for \( A' = A(a' - a) \) the induced form \( \wedge^2 M' \to A' \) is regular and nondegenerate.

As above, we get a diagram

\[
{}^a\mathcal{X}\mathcal{Y} \xleftarrow{h^{-}} {a, a'}\mathcal{H}_{X\mathcal{Y}} \xrightarrow{h^{-}} {}^{a'}\mathcal{X}\mathcal{Y},
\]

where \( h^{-} \) sends the above point to \((U, C, M, A, s_1)\), the map \( h^{-} \) sends the above point to \((U, C, M', A', s'_1)\), where \( s'_1 \) is the image of \( s_1 \) under \( U \otimes M(F) \sim U \otimes M'(F) \). Now \([17]\) is defined in a way similar to \([15]\).

Write \( \text{Four}_\psi : D_{\mathbb{G}Q\mathcal{H}_{a}(\Pi_a^*(F))} \xrightarrow{\sim} D_{\mathbb{G}Q\mathcal{H}_{a'}}(\Pi_{a'}^* \otimes \Omega(F)) \) for the Fourier transform (normalized as in Section 2.1). The following is standard (cf. also \([7\), Lemma 11].

**Lemma 4.** We have a canonical isomorphism in \( D_{\mathbb{G}Q\mathcal{H}_{a}}(\Pi_a^* \otimes \Omega(F)) \)

\[
H_{\mathbb{G}}(\mathcal{S}, \text{Four}_\psi(K)) \xrightarrow{\sim} \text{Four}_\psi H_{\mathbb{G}}(\mathcal{S}, K)
\]

functorial in \( \mathcal{S} \in a, a' \mathbb{S} \mathcal{P} \mathcal{H}_G, K \in D_{\mathbb{G}Q\mathcal{H}_{a'}}(\Pi_{a'}(F)). \) □
4.8.4 Write $P_{\mathbb{H}a} \subset \mathbb{H}_{a}$ (resp., $P_{\mathbb{H}a}^{-} \subset \mathbb{H}_{a}$) for the parabolic subgroup preserving $U_{a}$ (resp., $U_{a}^{-} \otimes C_{a}$). Let $U_{\mathbb{H}a} \subset P_{\mathbb{H}a}$ and $U_{\mathbb{H}a}^{-} \subset P_{\mathbb{H}a}^{-}$ denote their unipotent radicals. We view all of them as group schemes over $\text{Spec} \mathcal{O}$. Then $U_{\mathbb{H}a} \xrightarrow{\sim} \mathbb{C}_{a} \otimes \wedge^{2} U_{a}$ and $U_{\mathbb{H}a}^{-} \xrightarrow{\sim} \mathbb{C}_{a} \otimes \wedge^{2} U_{a}^{-}$ canonically.

Similarly, let $P_{\mathbb{G}_{a}} \subset \mathbb{G}_{a}$ (resp., $P_{\mathbb{G}_{a}}^{-} \subset \mathbb{G}_{a}$) be the parabolic subgroup preserving $L_{a}$ (resp., $L_{a}^{-} \otimes A_{a}$). Write $U_{\mathbb{G}_{a}} \subset P_{\mathbb{G}_{a}}$ and $U_{\mathbb{G}_{a}}^{-} \subset P_{\mathbb{G}_{a}}^{-}$ for their unipotent radicals. All of them are group schemes over $\text{Spec} \mathcal{O}$. We have canonically

$$U_{\mathbb{G}_{a}} \xrightarrow{\sim} A_{a}^{\ast} \otimes \text{Sym}^{2} L_{a}, \quad U_{\mathbb{G}_{a}}^{-} \xrightarrow{\sim} A_{a} \otimes \text{Sym}^{2} L_{a}^{-}$$

View $v \in \Pi_{a}(F)$ as a map $v : C_{a}^{\ast} \otimes U_{a}(F) \to M_{a}(F)$. For $v \in \Pi_{a}(F)$ let $s_{\Pi}(v)$ denote the composition

$$\wedge^{2}(U_{a} \otimes C_{a}^{-1})(F) \xrightarrow{\wedge^{2}v} \lambda^{2} M_{a}(F) \to A_{a}(F)$$

Let $\text{Char}(\Pi_{a}) \subset \Pi_{a}(F)$ denote the ind-subscheme of $v \in \Pi_{a}(F)$ such that $s_{\Pi}(v) : C_{a}^{\ast} \otimes \wedge^{2} U_{a} \to \Omega$ is regular. An object $K \in P(\Pi_{a}(F))$ is $U_{\mathbb{H}a}(\mathcal{O})$-equivariant iff $\zeta_{a}(K)$ is the extension by zero from $\text{Char}(\Pi_{a})$.

View $v \in \mathcal{T}_{a}(F)$ as a map $v : L_{a} \otimes A_{a}^{\ast}(F) \to V_{a}(F)$. For $v \in \mathcal{T}_{a}(F)$ let $s_{\mathcal{T}}(v)$ denote the composition

$$\text{Sym}^{2}(A_{a}^{\ast} \otimes L_{a}) \xrightarrow{\text{Sym}^{2}v} \text{Sym}^{2} V_{a}(F) \to C_{a}(F)$$

Write $\text{Char}(\mathcal{T}_{a}) \subset \mathcal{T}_{a}(F)$ for the ind-subscheme of $v \in \mathcal{T}_{a}(F)$ such that $s_{\mathcal{T}}(v) : A_{a}^{\ast} \otimes \text{Sym}^{2} L_{a} \to \Omega$ is regular. An object $K \in P(\Pi_{a}(F))$ is $U_{\mathbb{G}_{a}}(\mathcal{O})$-equivariant iff $\zeta_{a}^{-1}(K)$ is the extension by zero from $\text{Char}(\mathcal{T}_{a})$.

The next result follows from ([7], Lemma 13).

**Lemma 5.** The full subcategory $P_{\mathbb{H}Q\mathbb{G}_{a}(\mathcal{O})}(\mathcal{T}_{a}(F)) \subset P(\mathcal{T}_{a}(F))$ is the intersection of the full subcategories

$$P_{U_{\mathbb{H}a}(\mathcal{O})}(\mathcal{T}_{a}(F)) \cap P_{U_{\mathbb{G}_{a}}(\mathcal{O})}(\mathcal{T}_{a}(F)) \cap P_{\mathbb{Q}G_{a}(\mathcal{O})}(\mathcal{T}_{a}(F))$$

inside $P(\mathcal{T}_{a}(F))$. □

**Proposition 3.** For $a \in \mathbb{Z}$ the functor $-_a \text{Sph}_{G} \to D_{\mathbb{Q}G_{a}}(\Pi_{a}(F))$ sending $S$ to $\text{H}_{G}^{-}(\mathcal{S}, I_{0})$ factors naturally into

$$-_a \text{Sph}_{G} \to D \text{Weil}_{a} \to D_{\mathbb{Q}G_{a}}(\Pi_{a}(F))$$

For $a \in \mathbb{Z}$ the functor $-_a \text{Sph}_{H} \to D_{\mathbb{Q}G_{a}(\mathcal{O})}(\mathcal{T}_{a}(F))$ sending $S$ to $\text{H}_{G}^{-}(\mathcal{S}, I_{0})$ factors naturally into

$$-_a \text{Sph}_{H} \to D \text{Weil}_{a} \to D_{\mathbb{Q}G_{a}(\mathcal{O})}(\mathcal{T}_{a}(F))$$

**Proof** The argument is similar or both claims, we prove only the first one. For a finite subfield $k' \subset k$ we may pick a $k'$-structure on $\mathcal{O}$. Then $I_{0}$ admits a $k'$-structure and, as such, is pure of weight zero. So, by the decomposition theorem ([1]), one has $\text{H}_{G}^{-}(\mathcal{S}, I_{0}) \in D_{\mathbb{Q}G_{a}(\mathcal{O})}(\Pi_{a}(F))$.

It remains to show that each perverse cohomology sheaf $K$ of $\zeta_{a}^{-1}\text{H}_{G}^{-}(\mathcal{S}, I_{0})$ lies in the full subcategory $P_{\mathbb{H}Q\mathbb{G}_{a}(\mathcal{O})}(\mathcal{T}_{a}(F))$ of $P_{\mathbb{Q}G_{a}(\mathcal{O})}(\mathcal{T}_{a}(F))$.
By definition of the Hecke functors, $H_G^-(S, I_0)$ is the extension by zero from $\text{Char}(\Pi_a)$, so $\zeta_a(K)$ also satisfies this property. This yields a $U_{\text{Fil}}(O)$-action on $K$.

To get a $U_{\text{Fil}}(O)$-action on $K$, consider the commutative diagram of equivalences

$$
P_{Q\text{Fil}}(O)(Y_a(F)) \overset{\zeta_a}{\longrightarrow} P_{Q\text{Fil}}(O)(\Pi_a(F))$$

where $\text{Four}_\psi$ is the complete Fourier transform, and $\zeta_{1,a}$ is the corresponding partial one.

For $v \in \Pi_a^* \otimes \Omega(F)$ write $s_{\Pi}(v)$ for the composition

$$\wedge^2 U_a^*(F) \xrightarrow{\wedge^2 \psi} \wedge^2 M_a(F) \to A_a(F)$$

Write $\text{Char}(\Pi_a^* \otimes \Omega) \subset \Pi_a^* \otimes \Omega(F)$ for the ind-subscheme of $v$ such that $s_{\Pi}(v) : C_a \otimes \wedge^2 U_a^* \to \Omega$ is regular. The $U_{\text{Fil}}(O)$-equivariance of $K$ is equivalent to the fact that $\zeta_{1,a}(K)$ is the extension by zero from $\text{Char}(\Pi_a^* \otimes \Omega)$.

By Lemma 4 we have $\text{Four}_\psi H_G^-(S, I_0) \cong H_G^-(S, I_0)$, where $I_0 := \text{Four}_\psi(I_0)$ is the constant perverse sheaf on $\Pi_a^* \otimes \Omega$ extended by zero to $\Pi_a^* \otimes \Omega(F)$. Clearly, $H_G^-(S, I_0)$ is the extension by zero from $\text{Char}(\Pi_a^* \otimes \Omega)$, and our assertion follows. \[\square\]

According to Proposition 3 in what follows we will write $H_G^-(\cdot, I_0) : -_{a} \text{Sph}_G \to D \text{Weil}_a$ and $H_H^-(\cdot, I_0) : -_{a} \text{Sph}_H \to D \text{Weil}_a$ for the corresponding functors. From Proposition 2 one derives the following.

**Corollary 2.** For $a \in \mathbb{Z}$, $S \in -_{a} \text{Sph}_G$, $T \in -_{a} \text{Sph}_H$ there are canonical isomorphisms

$$F_{\text{Weil}_a} H_G^-(S, I_0) \cong H_G^-(S, S_{W_0}(F))$$

and

$$F_{\text{Weil}_a} H_H^-(T, I_0) \cong H_H^-(T, S_{W_0}(F))$$

in $D P_{\mathcal{L}_d}(\mathcal{L}_a(W_a(F)))$.

Thus, Theorem 3 is reduced to the following.

**Theorem 4.** Let the maps $\kappa$ be as in Theorem 3

1) Assume $m \leq n$. The two functors $-_{a} \text{Sph}_G \to D \text{Weil}_a$ given by

$$S \mapsto H_G^-(S, I_0) \quad \text{and} \quad S \mapsto H_G^-(\ast g\text{Res}^\kappa(S), I_0)$$

are isomorphic.

2) Assume $m > n$. The two functors $-_{a} \text{Sph}_H \to D \text{Weil}_a$ given by

$$T \mapsto H_H^-(T, I_0) \quad \text{and} \quad T \mapsto H_G^-(g\text{Res}^\kappa(T), I_0)$$

are isomorphic.
Remark 6. For $a = 0$ Theorem 4 is nothing but ([7], Theorem 7).

4.8.5 HECKE OPERATORS FOR LEVI SUBGROUPS

For $a \in \mathbb{Z}$ set $Q\Pi_a = U_a^{*} \otimes C_a \otimes L_a \subset \Pi_a$ and $Q\Upsilon_a = L_a^{*} \otimes A_a \otimes U_a \subset \Upsilon_a$.

We are going to define for $a, a' \in \mathbb{Z}$ Hecke functors

$$H_{Q(G)}^{0}: a' - a \text{ Sph}_{Q(G)} \times D_{QGH_{a'}}(Q\Pi_a(F)) \to D_{QGH_a}(Q\Pi_a(F))$$

in a way compatible with the functors defined in Section 4.8.

Let $^a\mathcal{X}Q\Pi$ be the stack classifying a $Q(\mathbb{H})$-torsor $(U, C)$ over $\text{Spec} \, \mathcal{O}$, a $Q(G)$-torsor $(L, A)$ over $\text{Spec} \, \mathcal{O}$, an isomorphism $A \otimes C \to \Omega(a)$, and a section $s \in U^{*} \otimes C \otimes L(F)$.

Informally, we think of $^a\mathcal{X}Q\Pi$ over $\text{Spec} \, \mathcal{O}$ as the derived category on $^a\mathcal{X}Q\Pi$. Consider the stack $^a, a'\mathcal{H}_{XQ\Pi,Q(G)}$ classifying: a point of $^a\mathcal{X}Q\Pi$ as above, a lattice $L' \subset L(F)$, for which we set $A' = A(a' - a)$. We get a diagram

$$^a\mathcal{X}Q\Pi \xrightarrow{h} ^a, a'\mathcal{H}_{XQ\Pi,Q(G)} \xrightarrow{h} ^a\mathcal{X}Q\Pi,$$

where $h$ sends the above collection to $(U, C, L, A, s)$, the map $h$ sends the above collection to $(U, C, L', A', s')$, where $s'$ is the image of $s$ under $U^{*} \otimes C \otimes L(F) \to U^{*} \otimes C \otimes L'(F)$.

Trivializing a point of $^a\mathcal{X}Q\Pi$ (resp., of $^a\mathcal{X}Q\Pi$), one gets isomorphisms

$$\text{id}^r: ^a, a'\mathcal{H}_{XQ\Pi,Q(G)} \xrightarrow{\cong} (Q\Pi_a(F) \times G_{l}^{a'-a'})/QGH_a(\mathcal{O})$$

and

$$\text{id}^l: ^a, a'\mathcal{H}_{XQ\Pi,Q(G)} \xrightarrow{\cong} (Q\Pi_a(F) \times G_{l}^{a'-a})/QGH_a(\mathcal{O})$$

So, for

$$K \in D_{QGH_a}(Q\Pi_a(F)), \quad K' \in D_{QGH_{a'}}(Q\Pi_{a'}(F))$$

and $S \in ^a\text{Sph}_{Q(G)}, \quad S' \in ^a\text{Sph}_{Q(G)}$ one can form their twisted exterior products $(K \boxtimes S)^l$ and $(K' \boxtimes S')^r$ on $^a, a'\mathcal{H}_{XQ\Pi,Q(G)}$. The functor $([8])$ is defined by

$$H_{Q(G)}^0(S', K') = h^{-1}(K' \boxtimes S')^r$$

Let $^a\mathcal{X}Q\Upsilon$ be the stack classifying a $Q(\mathbb{H})$-torsor $(U, C)$ over $\text{Spec} \, \mathcal{O}$, a $Q(G)$-torsor $(L, A)$ over $\text{Spec} \, \mathcal{O}$, an isomorphism $A \otimes C \to \Omega(a)$, and a section $s \in U \otimes A \otimes L^*(F)$. Informally, we think of $D_{QGH_a}(Q\Upsilon_a(F))$ as the derived category on $^a\mathcal{X}Q\Upsilon$. One defines the Hecke functor

$$H_{Q(\mathbb{H})}^0: a' - a \text{ Sph}_{Q(\mathbb{H})} \times D_{QGH_{a'}}(Q\Upsilon_a(F)) \to D_{QGH_a}(Q\Upsilon_a(F))$$

using a similar diagram

$$^a\mathcal{X}Q\Upsilon \xrightarrow{h} ^a, a'\mathcal{H}_{XQ\Upsilon,Q(\mathbb{H})} \xrightarrow{h} ^a\mathcal{X}Q\Upsilon$$

By abuse of notation, we also write $I_0$ for the constant perverse sheaf on $Q\Upsilon_0$ and on $Q\Pi_0$, the exact meaning is easily understood from the context. The next result is a straightforward consequence of ([7], Corollary 4).
**Proposition 4.** 1) Assume $m > n$. The functor
\[ -a \text{Sph}_{Q(G)} \to D_{QGH_0(O)}(Q\Pi_a(F)) \]
given by $S \mapsto H_{Q(H)}^-(S, I_0)$ takes values in $P_{QGH_0(O)}^{ss}(Q\Pi_a(F))$ and induces an equivalence
\[ -a \text{Sph}_{Q(G)} \simeq P_{QGH_0(O)}^{ss}(Q\Pi_a(F)) \]

2) Assume $m \leq n$. The functor
\[ -a \text{Sph}_{Q(H)} \to D_{QGH_0(O)}(Q\Psi_a(F)) \]
given by $S \mapsto H_{Q(H)}^-(S, I_0)$ takes values in $P_{QGH_0(O)}^{ss}(Q\Psi_a(F))$ and induces an equivalence
\[ -a \text{Sph}_{Q(H)} \simeq P_{QGH_0(O)}^{ss}(Q\Psi_a(F)) \]

\[ \square \]

4.8.5.2 For $a, a' \in \mathbb{Z}$ we will use in Section 4.8.9 the following Hecke functor
\[ H_{Q(G)}^{\ast}: a' - a \text{Sph}_{Q(G)} \times D_{QGH_0(O)}(Q\Psi_a(F)) \to D_{QGH_0(O)}(Q\Psi_a(F)) \] (20)

Consider the stack $\mathcal{H}_{\Psi Q\Psi Q(G)}$ classifying: a point $(U, C, L, s)$ of $\mathcal{H}_{\Psi Q\Psi}$ as above, a lattice $L' \subset L(F)$ for which we set $A' = A(a' - a)$. We get a diagram
\[ \mathcal{H}_{\Psi Q\Psi} \xrightarrow{h^-} \mathcal{H}_{\Psi Q\Psi Q(G)} \xrightarrow{\mathcal{H}_{\Psi Q\Psi Q(G)}} \mathcal{H}_{\Psi Q\Psi}, \]
where $h^-$ sends the above collection to $(U, C, L, A, s)$, and $h^-$ sends the same collection to $(U, C, L', A', s')$, where $s'$ is the image of $s$ under $U \otimes A \otimes L' \in U \otimes A' \otimes L'(F)$. The functor (20) is defined as in Section 4.8.5 for the above diagram.

The following is a consequence of (22), Lemma 11.

**Lemma 6.** For $S \in a' - a \text{Sph}_{Q(G)}$ the diagram of functors is canonically 2-commutative
\[ D_{QGH_0(O)}(Q\Psi_a(F)) \xrightarrow{\text{Four}_\psi} D_{QGH_0(O)}(Q\Pi_a(F)) \]
\[ D_{QGH_0(O)}(Q\Psi_a(F)) \xrightarrow{\text{Four}_\psi} D_{QGH_0(O)}(Q\Pi_a(F)) \]

4.8.6 **Weak Jacquet functors**

As in (22), Section 4.7 for each $a \in \mathbb{Z}$ we define the weak Jacquet functors
\[ J_{P_{QH_a}}^-, J_{P_{QH_a}}^+: D_{QGH_0(O)}(\Psi_a(F)) \to D_{QGH_0(O)}(Q\Psi_a(F)) \] (21)
and
\[ J_{P_{QH_a}}^*, J_{P_{QH_a}}^*: D_{QGH_0(O)}(\Pi_a(F)) \to D_{QGH_0(O)}(Q\Pi_a(F)) \] (22)
Both definitions being similar, we recall the definition of (21) only.

For a free \(O\)-module of finite type \(M\) and \(N, r \in \mathbb{Z}\) with \(N + r \geq 0\) write \(N, r M = M(N)/M(-r)\).

For \(N + r \geq 0\) consider the natural embedding \(i_{N,r} : N, r QY_a \hookrightarrow N, r Y_a\). Set

\[
PQG_a = \{ g = (g_1, g_2) \in P_{\mathbb{H}_a} \times Q(G_a) \mid g \in T_a \},
\]

this is a group scheme over \(\text{Spec} \ O\). We have a diagram of stack quotients

\[
PQG_a(\mathcal{O}/t^{N+r}) \backslash (N, r QY_a) \xrightarrow{i_{N, r}} PQG_a(\mathcal{O}/t^{N+r}) \backslash (N, r Y_a) \xrightarrow{\varphi} HQG_a(\mathcal{O}/t^{N+r}) \backslash (N, r Y_a)
\]

where \(t \in \mathcal{O}\) is a uniformizer, \(\varphi\) comes from the inclusion \(P_{\mathbb{H}_a} \subset \mathbb{H}_a\), and \(q\) is the natural quotient map. First, define functors

\[
J_{P_{\mathbb{H}_a}}, J_{P_{\mathbb{H}_a}}^! : D_{HQG_a(\mathcal{O}/t^{N+r})}(N, r Y_a) \to D_{QG_a(\mathcal{O}/t^{N+r})}(N, r QY_a)
\]

by

\[
qu^* \circ J_{P_{\mathbb{H}_a}}^*[\dim.\rel(q)] = i_{N, r}^* p^*[\dim.\rel(p) - rnm]
\]

\[
qu^* \circ J_{P_{\mathbb{H}_a}}^![\dim.\rel(q)] = i_{N, r}^! p^*[\dim.\rel(p) + rnm]
\]

Since

\[
qu^*[\dim.\rel(q)] : D_{QG_a(\mathcal{O}/t^{N+r})}(N, r QY_a) \to D_{PQG_a(\mathcal{O}/t^{N+r})}(N, r QY_a)
\]

is an equivalence (exact for the perverse t-structures), the functors (23) are well-defined. Further, (23) are compatible with the transition functors in the definition of the corresponding derived categories, so give rise to the functors (21) in the limit as \(N, r\) go to infinity. Note that for (21) we get \(\mathbb{D} \circ J_{P_{\mathbb{H}_a}}^* \simeq J_{P_{\mathbb{H}_a}}^! \circ \mathbb{D}\) naturally.

We identify \(\mathbb{H} \simeq \mathbb{H}_0\) and \(Q(\mathbb{H}) \simeq Q(\mathbb{H}_0)\). Let \(\mu = \det V_0\) and \(\nu = \det U_0\) viewed as characters of \(\mathbb{H}\) or, equivalently, as cocharacters of the center \(Z(\hat{Q}(\mathbb{H}))\) of the Langlands dual group \(\hat{Q}(\mathbb{H})\) of \(Q(\mathbb{H})\). Let \(\kappa : \hat{Q}(\mathbb{H}) \times G_m \to \mathbb{H}\) be the map, whose first component is the natural inclusion of the Levi subgroup, and the second one is \(2(\mu - \rho_{Q(\mathbb{H})}) + n(\mu - \nu)\). The corresponding geometric restriction functor is denoted \(g\text{Res}^{\kappa_{\mathbb{H}}}\).

**Lemma 7.** For \(a, a' \in \mathbb{Z}\) and \(S \in a' - a\) \(\text{Sph}_{\mathbb{H}}\), \(K \in D_{HQG_{a'}(\mathcal{O})}(Y_{a'}(F))\) there is a filtration in the derived category \(D_{QG_a(\mathcal{O})}(Y_a(F))\) on

\[
J_{P_{\mathbb{H}_a}}^* H^\pm_{\mathbb{H}}(S, K)
\]

such that the corresponding graded object identifies with \(H^\pm_{Q(\mathbb{H})}(g\text{Res}^{\kappa_{\mathbb{H}}}(S), J_{P_{\mathbb{H}_a}}^*(K))\).
Proof The proof is quite similar to ([7], Lemma 10), we only have to determine the corresponding map \( \kappa \). To do so, it suffices to perform the calculation for a particular \( K \). Let \( I_a' \) be the constant perverse sheaf on \( Y_{a'} \) extended by zero to \( Y_{a'}(F) \). Take \( K = I_a' \).

For \( s_1, s_2 \geq 0 \) let \( s_1, s_2 \mathbb{G}_m a \subset \mathbb{G}_m a \) be the closed subscheme of \( h\mathbb{G}_m a(\mathcal{O}) \in \mathbb{G}_m a \) such that

\[
V_a(-s_1) \subset hV_a \subset V_a(s_2)
\]

Assume that \( s_1, s_2 \) are large enough so that \( S \) is the extension by zero from \( s_1, s_2 \mathbb{G}_m a \). Then \( H_{\mathbb{G}_m a}(\mathcal{S}, I_a' \mathcal{O}) \) is as follows. Write \( 0, s_1 Y \times s_1, s_2 \mathbb{G}_m a \) for the scheme classifying pairs

\[
h\mathbb{G}_m a(\mathcal{O}) \in s_1, s_2 \mathbb{G}_m a, \ v \in L_a^* \otimes A_a \otimes (hV_a)/V_a(-s_1)
\]

Let \( \pi : 0, s_1 Y \times s_1, s_2 \mathbb{G}_m a \rightarrow s_2, s_1 Y_a \) be the map sending \((h\mathbb{G}_m a(\mathcal{O}), v)\) to \( v \). By definition,

\[
H_{\mathbb{G}_m a}(\mathcal{S}, I_a') \xrightarrow{\pi} \pi_!(\mathcal{O}_Y/S),
\]

where \( \mathcal{Q}_\ell \mathcal{O} S \) is normalized to be perverse. If \( \theta \in \pi_1(\mathbb{H}) \) then \( 0, s_1 Y \times s_1, s_2 \mathbb{G}_m a \) is a vector bundle over \( s_1, s_2 \mathbb{G}_m a \), of rank \( 2s_1 nm - \langle \theta, n\mathbb{H} \rangle \).

Let \( s_1, s_2 P_{\mathbb{G}_m a} = \{ p \in P_{\mathbb{G}_m a}(F) \mid V_a(-s_1) \subset pV_a \subset V_a(s_2) \} \).

Then

\[
s_1, s_2 \mathbb{G}_m a = (s_1, s_2 P_{\mathbb{G}_m a}(F))/P_{\mathbb{G}_m a}(\mathcal{O})
\]

is closed in \( \mathbb{G}_m a \). The natural map \( s_1, s_2 \mathbb{G}_m a \rightarrow s_1, s_2 \mathbb{G}_m a \) at the level of reduced schemes yields a stratification of \( s_1, s_2 \mathbb{G}_m a \) by the connected components of \( s_1, s_2 \mathbb{G}_m a \). Calculate \([24]\) with respect to this stratification. Denote by \( s_1, s_2 \mathbb{G}_m a \subset \mathbb{G}_m a \) the closed subscheme of \( h\mathbb{G}_m a(\mathcal{O}) \in \mathbb{G}_m a \) satisfying

\[
U_a(-s_1) \subset hU_a \subset U_a(s_2),
\]

write \( t_P : \mathbb{G}_m a \rightarrow \mathbb{G}_m a \) for the natural map. We have the diagram

\[
\begin{array}{ccc}
0, s_1 Y \times s_1, s_2 \mathbb{G}_m a & \xrightarrow{T} & 0, s_1 Y \times s_1, s_2 \mathbb{G}_m a \\
\downarrow \pi_Q & & \downarrow \pi \\
0, s_1 Y \times s_1, s_2 \mathbb{G}_m a & \xrightarrow{T} & 0, s_1 Y \times s_1, s_2 \mathbb{G}_m a
\end{array}
\]

where the square is cartesian. Here \( 0, s_1 Y \times s_1, s_2 \mathbb{G}_m a \) is the scheme classifying pairs

\[
h\mathbb{G}_m a(\mathcal{O}) \in s_1, s_2 \mathbb{G}_m a, \ v \in L_a^* \otimes A_a \otimes (hV_a)/V_a(-s_1),
\]

and \( 0, s_1 Y \times s_1, s_2 \mathbb{G}_m a \) is its closed subscheme given by the condition \( v \in L_a^* \otimes A_a \otimes (hU_a)/U_a(-s_1) \).

By definition, for \( T \in a' \mathbb{S} \mathbb{G}_m a \) we have

\[
H_{\mathbb{G}_m a}(T, I_a') \xrightarrow{\pi_T} \pi_!(\mathcal{O}_Y/S),
\]

where \( \mathcal{Q}_\ell \mathcal{O} T \) is normalized to be perverse. If \( \theta \in \pi_1(Q(H)) \) then \( 0, s_1 Y \times s_1, s_2 \mathbb{G}_m a \) is a vector bundle over \( s_1, s_2 \mathbb{G}_m a \) of rank \( s_1 nm - \langle \theta, n\mathbb{H} \rangle \). Our assertion follows. \( \square \)
We identify $G \cong G_0$, $Q(G) \cong Q(G_0)$. Write $\mu_G = \det M_0$ and $\nu_G = \det L_0$ as cocharacters of the center $Z(Q(G))$ of the Langlands dual group $Q(G)$ of $Q(G)$. Let $\kappa_G : Q(G) \times G_m \to \hat{G}$ be the map whose first component is the natural inclusion of the Levi subgroup, and the second one is $2(\mu_G - \mu_Q(G)) + m(\mu_G - \nu_G)$. The corresponding geometric restriction functor is denoted $gRes^\kappa_G$.

**Lemma 8.** For $a, a' \in \mathbb{Z}$ and $S \in a'-a \text{Sph}_G$, $K \in D_{GQ\mathbb{H}_{a'}(O)}(\Pi_a(F))$ there is a filtration in the derived category $D_{GQ\mathbb{H}_{a'}(O)}(\Pi_a(F))$

$$J_{P_{\Phi_a}}^* \mathbb{H}^-_G(S, K)$$

such that the corresponding graded object identifies with $\mathbb{H}^-_{Q(G)}(gRes^\kappa_G(S), J_{P_{\Phi_{a'}}}^* (K))$. □

We will use Lemmas 7 and 8 in the following form (the proof is as in [7], Corollary 3).

**Corollary 3.** Let $For \ a, a' \in \mathbb{Z}$ and $S \in a'-a$ Sph$_G$. Assume that $K \in P_{GQ\mathbb{H}_{a'}(O)}(\mathcal{T}_{a'}(F))$ admits a $k_0$-structure for some finite subfield $k_0 \subset k$ and, as such, is pure of weight zero. Then $J_{P_{\Phi_{a'}}}^* (K)$ is also pure of weight zero over $k_0$, and there is an isomorphism in $D_{GQ\mathbb{H}_{a'}(O)}(QY_{a}(F))$

$$J_{P_{\Phi_a}}^* \mathbb{H}^-_H(S, K) \sim \mathbb{H}^-_{GQ\mathbb{H}_{a'}}(gRes^\kappa_G(S), J_{P_{\Phi_{a'}}}^* (K))$$

(Similar strengthened version of Lemma 8 also holds.)

### 4.8.7 Action of Sph$_G$

Pick a maximal torus and a Borel subgroup $T_G \subset B_G \subset G$, and similarly for $H$. We assume $T_G \subset Q(G)$ and $T_H \subset Q(H)$. A trivialization of the $G_{a'}$-torsor $(M_a, A_a)$ over Spec $O$ yields a maximal torus and a Borel subgroup in $G_a$, hence also an equivalence Sph$_{G_a} \cong$ Sph$_G$ and a bijection $\Lambda^+_a \cong \Lambda^+_G$ as in Section 4.3.2 (and similarly for $H$ and $H_a$).

Write $\omega_i$ for the h.w. of the fundamental representation of $G_a$ that appear in $\Lambda^+M_a$ for $i = 1, \ldots, n$. All the weights of $\Lambda^+M_a$ are $\leq \omega_i$. Write $\omega_0$ for the h.w. of the $G_a$-module $A_a$.

For $\lambda \in \Lambda^+_G$ set $a = (\lambda, \omega_0)$ then $A^\lambda_a \in -a$ Sph$_G$. By definition, the complex

$$H^\lambda_G(I_0) = H^-_G(A^\lambda, I_0) \in D_{GQ\mathbb{H}_{a}}(\Pi_a(F))$$

is as follows. Set $r = (\lambda, \omega_1)$ and $N = (-\omega_0^G(\lambda), \omega_1)$. Let $0, r \overrightarrow{\times \mathcal{G}^\lambda_{G_a}}$ be the scheme classifying $g \in \mathcal{G}^\lambda_{G_a}$, $x \in U^*_a \otimes C_a \otimes ((gM_a)/M_a(-r))$. Let

$$\pi : 0, r \overrightarrow{\times \mathcal{G}^\lambda_{G_a}} \to N, r \Pi_a \quad (25)$$

be the map sending $(x, g\mathcal{G}_a(O))$ to $x$. Then $H^\lambda_G(I_0) \sim \pi(\mathcal{G}_a \mathcal{G}^\lambda_{G_a})$ canonically (recall that $\mathcal{G}_a \mathcal{G}^\lambda_{G_a}$ is normalized to be perverse).

Define the closed subscheme $\lambda \Pi_a \subset \Pi_a(N)$ as follows. A point $v \in \Pi_a(N)$ lies in $\lambda \Pi_a$ if the following conditions hold:

---

31
C1) $v \in \text{Char}(\Pi_a)$;

C2) for $i = 1, \ldots, n$ the map $\wedge^i v : \wedge^i (U_a \otimes C_a^{-1}) \to (\wedge^i M_a)((-w_0^G(\lambda), \tilde{\omega}_i))$ is regular.

The subscheme $\lambda \Pi_a$ is stable under translations by $\Pi_a(-r)$, so there is a closed subscheme $\lambda \Pi_a \subset N, r \Pi_a$ such that $\lambda \Pi_a$ is the preimage of $\lambda \Pi_a$ under the projection $\Pi_a(N) \to N, r \Pi_a$. Since all the weights of $\wedge^i M_a$ are $\leq \tilde{\omega}_i$, the map (25) factors through the closed subscheme $\lambda \Pi_a \subset N, r \Pi_a$.

For each $v \in \text{Char}(\Pi_a)$ let us define a $\mathcal{O}$-lattice $M_v \subset M_a(F)$ as follows. View $v$ as a map $U_a \otimes C_a^{-1} \to M_a(F)$. For a $\mathcal{O}$-lattice $R \subset M_a(F)$ set

$$R^\perp = \{ m \in M_a(F) \mid \langle m, x \rangle \in A_a(-a) \text{ for all } x \in R \}$$

Consider two cases.

CASE: $a$ is even. For $v \in \text{Char}(\Pi_a)$ set $R_v = v(U_a \otimes C_a^{-1}) + M_a(-\frac{a}{2})$ and $M_v = v(U_a \otimes C_a^{-1}) + R_v^\perp$. Then $R_v^\perp \subset M_v \subset R_v$, and the induced form $\wedge^2 M_v \to A_a(-a)$ is regular and nondegenerate. So, $M_v \in \text{Gr}_{G_a}^{-a}$.

CASE: $a$ is odd. Let $b = (-a - 1)/2$. Note that $(M_a(b))^\perp = M_a(b + 1)$. Set $R_v = v(U_a \otimes C_a^{-1}) + M_a(b + 1)$ and $M_v = v(U_a \otimes C_a^{-1}) + R_v^\perp$. Clearly, the induced form $\wedge^2 M_v \to A_a(-a)$ is regular, but still can be degenerate. We call $v$ generic if the form $\wedge^2 M_v \to A_a(-a)$ is nondegenerate. In this case $M_v \in \text{Gr}_{G_a}^{-a}$.

For $a$ even we get a stratification of $\text{Char}(\Pi_a)$ indexed by $\{ \lambda \in \Lambda_G^+ \mid \langle \lambda, \tilde{\omega}_0 \rangle = a \}$, the stratum $\lambda \text{ Char}(\Pi_a)$ is given by the condition that $M_v \in \text{Gr}_{G_a}^{\lambda}$. This condition is also equivalent to requiring that there is an isomorphism of $\mathcal{O}$-modules

$$R_v/(M_a(-a/2)) \cong \mathcal{O}/t^{a_1 - \frac{a}{2}} \oplus \ldots \oplus \mathcal{O}/t^{a_n - \frac{a}{2}},$$

where $t \in \mathcal{O}$ is a uniformizer.

Clearly, $\lambda \text{ Char}(\Pi_a) \subset \lambda \Pi_a$. There is a unique open subscheme $\lambda \Pi_a^0 \subset \lambda \Pi_a$ whose preimage under the projection $\lambda \Pi_a \to \lambda \Pi_a$ equals $\lambda \text{ Char}(\Pi_a)$.

We say that a morphism of free $\mathcal{O}$-modules $M_1 \to M_2$ is maximal if it does not factor through $M_2(-1) \subset M_2$.

For $a$ odd define $\lambda \text{ Char}(\Pi_a) \subset \lambda \Pi_a$ as the open subscheme given by the condition that each map $\wedge^i v$ in C2) is maximal. Then there is an open subscheme $\lambda \Pi_a^0 \subset \lambda \Pi_a$ whose preimage under the projection $\lambda \Pi_a \to \lambda \Pi_a$ equals $\lambda \text{ Char}(\Pi_a)$. One checks that any $v \in \lambda \text{ Char}(\Pi_a)$ is generic and the corresponding lattice $M_v$ satisfies $M_v \in \text{Gr}_{G_a}^{\lambda}$. Note that for $v \in \lambda \text{ Char}(\Pi_a)$ we have an isomorphism of $\mathcal{O}$-modules

$$R_v/(M_a(b + 1)) \cong \mathcal{O}/t^{a_1 -(a+1)/2} \oplus \ldots \oplus \mathcal{O}/t^{a_n -(a+1)/2}$$

for any uniformizer $t \in \mathcal{O}$.

Write $\text{IC}(\lambda \Pi_a^0)$ for the intersection cohomology sheaf of $\lambda \Pi_a^0$.  

32
Proposition 5. Let $\lambda \in \Lambda^+_G$ with $\langle \lambda, \omega_0 \rangle = a$.
1) The map
\[ \pi : G \times \mathcal{G} \rightarrow \lambda, N \Pi_a \]
is an isomorphism over the open subscheme $\lambda, N \Pi_a^0$.
2) Assume $m > n$ then one has a canonical isomorphism $H^\lambda_G(I_0) \cong IC(\lambda, N \Pi_a^0)$.

Proof. 1) The fibre of $\pi$ over $v \in \lambda, N \Pi_a^0$ is the scheme classifying lattices $M' \in \mathcal{G}$ such that $v(U_a \otimes C_a^{-1}) \subset M'$. Given such a lattice $M'$ let us show that $M_v = M'$.

Consider first the case of $a$ odd. The inclusion $R_v \subset M' + M_a(b + 1)$ must be an equality, because for $M' \in \mathcal{G}$ with $\mu \leq \lambda$ we have
\[ \dim(M' + M_a(b + 1))/(M_a(b + 1)) = \epsilon(\mu) \leq \epsilon(\lambda) = \dim R_v/(M_a(b + 1)) \]

We have denoted here $\epsilon(\mu) = \langle \mu, \omega_a \rangle - \frac{a}{2}(a + 1)$. So, $M_v = v(U_a \otimes C_a^{-1}) + (M' \cap M_a(b)) \subset M'$ is also an equality, because both $M_v$ and $M'$ have symplectic forms with values in $A_a(-a)$.

The case of $a$ even is quite similar to ([7], Lemma 15). Namely, the inclusion $R_v \subset M' + M_a(-\frac{a}{2})$ must be an equality, because for $M' \in \mathcal{G}$ with $\mu \leq \lambda$ we get
\[ \dim(M' + M_a(-a/2))/(M_a(-a/2)) = \epsilon(\mu) \leq \epsilon(\lambda) = \dim R_v/(M_a(-a/2)) \]

Here for $a$ even we have set $\epsilon(\mu) = \langle \mu, \omega_a \rangle - \frac{a}{2}a$. So,
\[ M_v = v(U_a \otimes C_a^{-1}) + (M' \cap (M_a(-a/2))) \subset M' \]
is also an equality. The first assertion follows.

2) For $m \geq n$ the scheme $\lambda, N \Pi_a^0$ is nonempty, so $IC(\lambda, N \Pi_a^0)$ appears in $H^\lambda_G(I_0)$ with multiplicity one. So, it suffices to show that
\[ \text{Hom}(H^\lambda_G(I_0), H^\lambda_G(I_0)) = \mathbb{Q}_\ell, \]
where Hom is taken in the derived category $D^{GQH_a(O)}(\Pi_a(F))$. By adjointness,
\[ \text{Hom}(H^\lambda_G(I_0), H^\lambda_G(I_0)) \cong \text{Hom}(H^{-\omega_0}_G(\lambda), H^\lambda_G(I_0), I_0), \]
where Hom in the RHS is taken in $D^{GQH_0(O)}(\Pi_0(F))$. We are reduced to show that for any $0 \neq \mu \in \Lambda^+_G$ with $\langle \mu, \omega_0 \rangle = 0$ one has
\[ \text{Hom}(H^\mu_G(I_0), I_0) = 0 \]
in $D^{GQH_0(O)}(\Pi_0(F))$. The latter assertion is true for $m > n$, it is proved in ([7], part 2) of Lemma 15). □
Remark 7. For any \(a, b \in \mathbb{Z}\) let us construct an equivalence \(\text{Weil}_a \simeq \text{Weil}_{a+2b}\). Pick isomorphisms of \(\mathcal{O}\)-modules

\[
L_a(b) \simeq L_{a+2b}, \quad A_a(2b) \simeq A_{a+2b}, \quad U_a \simeq U_{a+2b},
\]

They yield isomorphisms \(C_a \simeq C_{a+2b}\), \(V_a \simeq V_{a+2b}\), \(M_a(b) \simeq M_{a+2b}\). Hence, also isomorphisms \(Q(G_a) \simeq Q(G_{a+2b})\), \(G_a \simeq G_{a+2b}\) of group schemes over \(\text{Spec} \mathcal{O}\) (and similarly for \(\mathbb{H}\)). We also get isomorphisms of group schemes over \(\text{Spec} \mathcal{O}\)

\[
QG\mathbb{H}_a \simeq QG\mathbb{H}_{a+2b}, \quad GQ\mathbb{H}_a \simeq GQ\mathbb{H}_{a+2b}, \quad \mathbb{H}QG_a \simeq \mathbb{H}QG_{a+2b}
\]

The isomorphisms (26) also yield \(\Pi_a(b) \simeq \Pi_{a+2b}\) and \(\Upsilon_a(b) \simeq \Upsilon_{a+2b}\). In turn, we get equivalences

\[
P_{\mathbb{H}QG_a}(\Upsilon_a(F)) \simeq P_{\mathbb{H}QG_{a+2b}}(\Upsilon_{a+2b}(F)), \quad P_{GQ\mathbb{H}_a}(\Pi_a(F)) \simeq P_{GQ\mathbb{H}_{a+2b}}(\Pi_{a+2b}(F))
\]

which yield the desired equivalence \(\text{Weil}_a \simeq \text{Weil}_{a+2b}\). The diagram commutes

\[
\begin{array}{ccc}
-a \text{Sph}_G & \rightarrow & \text{Weil}_a \\
\downarrow \epsilon & & \downarrow \iota \\
-a-2b \text{Sph}_G & \rightarrow & \text{Weil}_{a+2b}
\end{array}
\]

where the horizontal arrows are given by \(S \mapsto H_G^-((S, I_0))\), and \(\epsilon\), at the level of representations of \(\hat{G}\), is given by \(V \mapsto V \otimes V^\omega\). Here \(V^\omega\) is the one-dimensional representation of \(\hat{G}\) with h.w. \(\omega\) such that \(\langle \omega, \omega_0 \rangle = 2\). So, the case of \(a\) even in Proposition 5 also follows from (7, Lemma 15).

4.8.7.2 Let \(k_0 \subset k\) be a finite subfield. In this subsection we assume that all the objects of Sections 4 are defined over \(k_0\).

Write \(\text{Weil}_{a,k_0}\) for the category of triples \((\mathcal{F}_1, \mathcal{F}_2, \beta)\) as in Definition 2 of \(\text{Weil}_a\) but with a \(k_0\)-structure and, as such, pure of weight zero. It is understood that the Fourier transform functors are normalized to preserve purity. Note that for any \((\mathcal{F}_1, \mathcal{F}_2, \beta) \in \text{Weil}_a\) the perverse sheaf \(\mathcal{F}_1\) is \(G_m\)-equivariant with respect to the homotheties on \(\Upsilon_a(F)\).

Denote by \(D\text{Weil}_{a,k_0}\) the category of complexes as in the definition of \(D\text{Weil}_a\) but, in addition, with a \(k_0\)-structure and, as such, pure of weight zero. So, for an object of \(D\text{Weil}_{a,k_0}\) its semi-simplification is a bounded complex of the form \(\oplus_{i \in \mathbb{Z}} F_i[i](\frac{i}{2})\) with \(F_i \in \text{Weil}_{a,k_0}\).

Write \(F_0\) for \(k_0\)-valued points of \(F\). For a totally disconnected locally compact space \(Y\) write \(\mathcal{S}(Y)\) for the Schwarz space of locally constant \(\mathbb{Q}_l\)-valued functions on \(Y\) with compact support. Write \(\text{Weil}_a(k_0)\) for the \(\mathbb{Q}_l\)-vector space of pairs \((\mathcal{F}_1, \mathcal{F}_2)\), where \(\mathcal{F}_1 \in \mathcal{S}_{\mathbb{H}QG_a(\mathcal{O})}(\Upsilon_a(F_0))\), \(\mathcal{F}_2 \in \mathcal{S}_{GQ\mathbb{H}_a(\mathcal{O})}(\Pi_a(F_0))\) with \(\zeta_a(\mathcal{F}_1) = \mathcal{F}_2\).

Write \(\mathcal{P}\) for the composition of functors

\[
D\text{Weil}_a \xrightarrow{f_\mathcal{H}} D\mathcal{P}_{\mathbb{H}QG_a(\mathcal{O})}(\Upsilon_a(F)) \xrightarrow{J_{\mathbb{H}QG_a}} D\mathbb{Q}G_{\mathbb{H}a}(\mathcal{O})(Q\Upsilon_a(F)),
\]

where \(f_\mathcal{H}\) sends \((\mathcal{F}_1, \mathcal{F}_2, \beta)\) to \(\mathcal{F}_1\). By abuse of notation, we also write \(\mathcal{P} : D\text{Weil}_{a,k_0} \rightarrow D\mathbb{Q}G_{\mathbb{H}a}(\mathcal{O})(Q\Upsilon_a(F))\) for the similary defined functor over \(k_0\).
Proposition 6. For $i = 1, 2$ let $K_i \in D\text{Weil}_{a,k_0}$. If $\mathcal{P}(K_1) \cong \mathcal{P}(K_2)$ then $K_1 \cong K_2$ in $D\text{Weil}_a$.

Proof. Write $K_{k_0}$ (resp., $DK_{k_0}$) for the Grothendieck group of the category $\text{Weil}_{a,k_0}$ (resp., of $D\text{Weil}_{a,k_0}$). Note that $DK_{k_0} \cong K_{k_0} \otimes \mathbb{Z}[t,t^{-1}]$. Write $\mathcal{Y}K_{k_0}$ for the Grothendieck group of the category of pure complexes of weight zero on $Q\mathcal{Y}_a(F_0)$, whose all perverse cohomologies lie in $\mathcal{P}_{QG\mathcal{H}(\mathcal{O})}(Q\mathcal{Y}_a(F_0))$. The functor $J_{\mathcal{P}_{\mathcal{H}}a}$ yields a homomorphism $J_{\mathcal{P}_{\mathcal{H}}a} : DK_{k_0} \to \mathcal{Y}K_{k_0}$. Let us show that it is injective. Let $F$ be an object in its kernel. For any finite subfield $k_0 \subset k_1 \subset k$ the map $tr_{k_1}$ trace of Frobenius over $k_1$ fits into the diagram

$$
\begin{array}{ccc}
DK_{k_0} & \xrightarrow{J_{\mathcal{P}_{\mathcal{H}}a}} & \mathcal{Y}K_{k_0} \\
\downarrow tr_{k_1} & & \downarrow tr_{k_1} \\
\text{Weil}_a(k_1) & \xrightarrow{J_{k_1}} & S_{QG\mathcal{H}(\mathcal{O})}(Q\mathcal{Y}_a(F_1)),
\end{array}
$$

where $F_1$ denotes the $k_1$-valued points of $F$. By (Lemma 
 Appendix A), $J_{k_1}$ is injective, so $tr_{k_1}(F) = 0$ for any finite extension $k_0 \subset k_1$. By the result of Laumon [4], Theorem 1.1.2) this implies $F = 0$ in $DK_{k_0}$. Finally, if $K_1 = K_2$ in $DK_{k_0}$ then $K_1 \cong K_2$ in $D\text{Weil}_a$. $\square$

The following result will not be used in this paper, its proof is found in Appendix A.

Proposition A.1. Assume $m > n$. The map $K_0(-a\text{Sph}_G) \otimes \hat{Q}_\ell \to \text{Weil}_a(k_0)$ given by $S \mapsto tr_{k_0}H^0_G(S,I_0)$ is an isomorphism of $\hat{Q}_\ell$-vector spaces.

Write $\text{Weil}_{a}^s \subset \text{Weil}_a$ for the full subcategory of semi-simple objects.

Conjecture 1. Assume $m > n$. The functor $-a\text{Sph}_G \to \text{Weil}_{a}^s$ given by $S \to H^0_G(S,I_0)$ is an equivalence of categories.

4.8.8 ACTION OF $\text{Sph}_H$

We write $V^{\lambda}$ for the irreducible $H$-module with h.w. $\lambda$. Assume that $V_0$ is a $2m$-dimensional $k$-vector space with nondegenerate symmetric form $\text{Sym}^2V_0 \to C_0$, and $H$ is the connected component of unity of $G_0(V_0)$. Write $\alpha_0$ for the h.w. of the $H$-module $C_0$. For $0 < i < m$ let $\alpha_i$ denote the h.w. of the irreducible $H$-module $\wedge^iV_0$. Remind that

$$
\wedge^mV_0 \cong V^{\alpha_m} \oplus V^{\alpha_m'}
$$

is a direct sum of two irreducible representations, this is our definition of $\alpha_m, \alpha_m'$. Say that a maximal isotropic subspace $\mathcal{L} \subset V_0$ is $\alpha_m$-oriented (resp., $\alpha_m'$-oriented) if $\wedge^m\mathcal{L} \subset V^{\alpha_m}$ (resp., $\wedge^m\mathcal{L} \subset V^{\alpha_m'}$). The group $H$ has two orbits on the set of maximal isotropic subspaces in $V_0$ given by the orientation.

Remind that $\text{Gr}_H^a$ classifies lattices $V' \subset V_0(F)$ such that the induced form $\text{Sym}^2V' \to C(b)$ is regular and nondegenerate, here $C = C_0(\mathcal{O})$.

Let $\lambda \in \Lambda^+_H$, set $a = (\lambda, \alpha_0)$. Remind that $A^\lambda_H \in \text{Sph}_H$ denotes the IC-sheaf of $\text{Gr}_H^a$, so $A^\lambda_H \in -a\text{Sph}_H$. By definition, the complex

$$
\text{H}_H^0(I_0) = H_H^0(A^\lambda_H,I_0) \in D_{HQG}^a(T_a(F))
$$
is as follows. Set \( r = \langle \lambda, \tilde{\alpha}_1 \rangle \) and \( N = \langle -w_0^H(\lambda), \tilde{\alpha}_1 \rangle \). Let \( 0, r \mathcal{Y} \times \overline{\mathcal{Gr}}^{\lambda}_{\mathbb{H}_a} \) be the scheme classifying \( h \in \overline{\mathcal{Gr}}^{\lambda}_{\mathbb{H}_a}, x \in L_a \otimes A_a \otimes ((hV_a)/(V_a(r)) \). Let

\[
\pi : 0, r \mathcal{Y} \times \overline{\mathcal{Gr}}^{\lambda}_{\mathbb{H}_a} \to N, r \mathcal{Y}_a
\]

be the map sending \((x, h^H_a(\mathcal{O}))\) to \( x \). Then \( H^}_0(I_0) \cong \pi_!(\overline{\mathcal{Q}}_\lambda \otimes \mathbb{A}_\mathbb{H}^A)[b] \) canonically, where \( b \) is the unique integer such that \( \overline{\mathcal{Q}}_\lambda \otimes \mathbb{A}_\mathbb{H}^A \) is perverse.

View a point of \( \mathcal{Y}_a(F) \) as a map \( L_a \otimes A_a^* \to V_a(F) \). Define a closed subscheme \( \lambda \mathcal{Y}_a \subset \mathcal{Y}_a(N) \) as follows. A point \( v \in \mathcal{Y}_a(N) \) lies in \( \lambda \mathcal{Y}_a \) if the following conditions hold:

C1) \( v \in \text{Char}(\mathcal{Y}_a) \);

C2) for \( 1 \leq i < m \) the map \( \wedge^i v : \wedge^i(L_a \otimes A_a^*) \to (\wedge^iV_a)(\langle -w_0^H(\lambda), \tilde{\alpha}_i \rangle) \) is regular;

C3) the map \((v_m, v_m^*) : \wedge^m(L_a \otimes A_a^*) \to V_a^{\tilde{\alpha}_m}(\langle -w_0^H(\lambda), \tilde{\alpha}_m \rangle) \oplus V_a^{\tilde{\alpha}'_m}(\langle -w_0^H(\lambda), \tilde{\alpha}'_m \rangle) \) induced by \( \wedge^m v \) is regular.

The scheme \( \lambda \mathcal{Y}_a \) is stable under translations by \( \mathcal{Y}_a(-r) \), so there is a closed subscheme \( \lambda, N \mathcal{Y}_a \subset N, r \mathcal{Y}_a \) such that \( \lambda \mathcal{Y}_a \) is the preimage of \( \lambda, N \mathcal{Y}_a \) under the projection \( \mathcal{Y}_a(N) \to N, r \mathcal{Y}_a \). Clearly, the map \( \overline{(28)} \) factors through the closed subscheme \( \lambda, N \mathcal{Y}_a \subset N, r \mathcal{Y}_a \).

For each \( v \in \text{Char}(\mathcal{Y}_a) \) define a \( \mathcal{O} \)-lattice \( V_v \subset V_a(F) \) as follows. For a \( \mathcal{O} \)-lattice \( R \subset V_a(F) \) set

\[
R^\perp = \{ x \in V_a(F) \mid \langle x, y \rangle \in C_a(-a) \text{ for all } y \in R \}
\]

Consider two cases.

CASE: \( a \) is even. For \( v \in \text{Char}(\mathcal{Y}_a) \) set \( R_v = v(L_a \otimes A_a^*) + V_a(-\frac{a}{2}) \) and \( V_v = v(L_a \otimes A_a^*) + R_v^\perp \). Then \( V_v \in \text{Gr}^{\lambda}_{\mathbb{H}_a} \). In this case we get a stratification of \( \text{Char}(\mathcal{Y}_a) \) by locally closed subschemes \( \lambda \text{Char}(\mathcal{Y}_a) \) indexed by \( \{ \lambda \in \Lambda_+^\mathbb{H} \mid \langle \lambda, \tilde{\alpha}_0 \rangle = a \} \). Namely, \( v \in \text{Char}(\mathcal{Y}_a) \) lies in \( \lambda \text{Char}(\mathcal{Y}_a) \) iff \( V_v \in \text{Gr}^{\lambda}_{\mathbb{H}_a} \).

Clearly, \( \lambda \text{Char}(\mathcal{Y}_a) \subset \lambda \mathcal{Y}_a \). There is a unique open subscheme \( \lambda, N \mathcal{Y}_a^0 \subset \lambda, N \mathcal{Y}_a \) whose preimage under the projection \( \lambda \mathcal{Y}_a \to \lambda, N \mathcal{Y}_a \) equals \( \lambda, N \mathcal{Y}_a^0 \).

CASE: \( a \) is odd. Let \( b = (-a - 1)/2 \). We have \( (V_v(b + 1))^\perp = V_v(b) \). Set \( R_v = v(L_a \otimes A_a^*) + V_v(b + 1) \) and \( V_v = v(L_a \otimes A_a^*) + R_v^\perp \). Then the induced form \( \text{Sym}^2 V_v \to C_a(-a) \) is regular, but still can be degenerate. We call \( v \) generic if the form \( \text{Sym}^2 V_v \to C_a(-a) \) is nondegenerate. In this case \( V_v \in \text{Gr}^{\lambda}_{\mathbb{H}_a} \).

For \( a \) odd define an open subscheme \( \lambda \text{Char}(\mathcal{Y}_a) \subset \lambda \mathcal{Y}_a \) as follows. Note that \( \langle w_0^H(\lambda), \tilde{\alpha}_m - \tilde{\alpha}'_m \rangle \neq 0 \). A point \( v \in \lambda \mathcal{Y}_a \) lies in \( \lambda \text{Char}(\mathcal{Y}_a) \) if the following conditions hold:

- the maps in C2) are maximal;
- if \( \langle w_0^H(\lambda), \tilde{\alpha}_m - \tilde{\alpha}'_m \rangle < 0 \) then \( v_m \) in C3) is maximal, otherwise \( v'_m \) in C3) is maximal.
There is a unique open subscheme $\lambda_N \Upsilon^0_a \subset \lambda_N \Upsilon_a$ whose preimage under the projection $\lambda \Upsilon_a \to \lambda_N \Upsilon_a$ equals $\lambda \Char(\Upsilon_a)$.

Write $\IC(\lambda_N \Upsilon^0_a)$ for the intersection cohomology sheaf of $\lambda_N \Upsilon^0_a$.

**Proposition 7.** Let $\lambda \in \Lambda^+_{\mathbb{H}_a}$ with $\langle \lambda, \alpha_0 \rangle = a$.

1) The map
\[ \pi : 0_r \Upsilon \times_{\Gr_{H_0}}^{\lambda} \to \lambda_N \Upsilon_a \]
is an isomorphism over the open subscheme $\lambda_N \Upsilon^0_a$.

2) Assume $m \leq n$ then one has a canonical isomorphism $H_{\mathbb{H}_a}(I_0) \sim \IC(\lambda_N \Upsilon^0_a)$.

**Proof.**

1) Let $v \in \lambda_N \Upsilon^0_a$. The fibre of $\pi$ over $v$ is the scheme classifying lattices $V' \in \Gr_{\mathbb{H}_a}^{\lambda}$ such that $v(L_0 \otimes A^{-1}_a) \subset V'$. Given such a lattice $V'$ let us show that $V' = V$.

In view of Remark 7 the case of $a$ even is reduced to the case $a = 0$, and the latter is done in ([7], Lemma 14).

Consider the case of $a$ odd. The inclusion $R_v \subset V' + V_a(b + 1)$ must be an equality, because for $V' \in \Gr_{\mathbb{H}_a}^{\lambda}$ with $\mu \leq \lambda$ we have
\[
\dim(V' + V_a(b + 1))/(V_a(b + 1)) = \epsilon(\mu) \leq \epsilon(\lambda) = \dim R_v/(V_a(b + 1))
\]
We have denoted here $\epsilon(\mu) = -m(b + 1) + \max\{\langle -w^H_0(\mu), \alpha_m \rangle, \langle -w^H_0(\mu), \alpha'_m \rangle\}$.

It follows that $V_v = v(L_0 \otimes A^{-1}_a) + (V' \cap V_a(b)) \subset V'$. To prove that $V' = V_v$, it suffices to show that $v$ is generic. This follows from the fact that $(v(L_0 \otimes A^{-1}_a) + R_v^+)/R_v^+$ is a maximal isotropic subspace in $R_v/R_v^+$.

2) For $m \leq n$ the scheme $\lambda_N \Upsilon^0_a$ is nonempty, so $\IC(\lambda_N \Upsilon^0_a)$ appears in $H_{\mathbb{H}_a}(I_0)$ with multiplicity one. Now it remains to show that
\[
\Hom(H_{\mathbb{H}_a}(I_0), H_{\mathbb{H}_a}(I_0)) = \mathbb{Q}_\ell,
\]
where $\Hom$ is taken in the derived category $D_{\mathbb{H}_a}(\Upsilon_a(\mathcal{O}))(\Upsilon_a(F))$. By adjointness,
\[
\Hom(H_{\mathbb{H}_a}(I_0), H_{\mathbb{H}_a}(I_0)) \sim \Hom(H_{\mathbb{H}_a}^{-w_0^H}(\lambda)H_{\mathbb{H}_a}(I_0), I_0),
\]
where $\Hom$ in the RHS is taken in $D_{\mathbb{H}_a}(\Upsilon_a(\mathcal{O}))(\Upsilon_a(F))$. We are reduced to show that for any $0 \neq \mu \in \Lambda^+_{\mathbb{H}_a}$ with $\langle \mu, \alpha_0 \rangle = 0$ one has
\[
\Hom(H_{\mathbb{H}_a}^{\mu}(I_0), I_0) = 0
\]
in $D_{\mathbb{H}_a}(\Upsilon_a(\mathcal{O}))(\Upsilon_a(F))$. For $m \leq n$ this is proved in ([7], part 2) of Lemma 14).

As in the case $m > n$, assume for a moment that $k_0 \subset k$ is a finite subfield, and all the objects introduced in Section 4 have a $k_0$-structure. The following result is analogous to Proposition A.1, its proof is omitted.

**Proposition A.2.** Assume $m \leq n$. Then the map $K_0(-\_ \Sph_{\mathbb{H}}) \otimes \mathbb{Q}_\ell \to \text{Weil}_a(k_0)$ given by $\mathcal{S} \mapsto \text{tr}_{k_0} H_{\mathbb{H}_a}^{\mu}(\mathcal{S}, I_0)$ is an isomorphism of $\mathbb{Q}_\ell$-vector spaces.
Conjecture 2. Assume $m \leq n$. The functor $\kappa : \text{Sph}_{\mathbb{H}} \to \text{Weil}_{\mathbb{H}}$ given by $S \mapsto H_{\mathbb{H},m}(S, I_0)$ is an equivalence of categories.

4.8.9 Proof of Theorem 4

Use the notations of Section 4.8.7 and 4.8.8. Assume that $U_0$ is $\bar{\alpha}_m$-oriented. Below we identify $\hat{\omega}_0 : \mathbb{G}_m \sim \bar{\text{GL}}(A_0)$ and $\hat{\bar{\alpha}}_0 : \mathbb{G}_m \sim \bar{\text{GL}}(C_0)$.

For $m > n$ fix a decomposition $U_0 \sim 1U \oplus 2U$ into direct sum of free $O$-modules, where $1U$ is of rank $n$, and $2U$ is of rank $m - n$. Fix also an isomorphism $1U \sim L_0$ of $O$-modules. We assume that these choices are compatible with the maximal tori chosen before. For $m > n$ let $\kappa_0 : \bar{\text{GL}}(L_0) \times \mathbb{G}_m \to \bar{\text{GL}}(U_0)$ be the composition

$$\bar{\text{GL}}(L_0) \times \mathbb{G}_m \xrightarrow{\tau \times \text{id}} \bar{\text{GL}}(L_0) \times \mathbb{G}_m = \bar{\text{GL}}(1U) \times \mathbb{G}_m \xrightarrow{\text{id} \times 2\bar{\rho}_{\text{GL}(2U)}} \bar{\text{GL}}(1U) \times \bar{\text{GL}}(2U) \xrightarrow{\text{Levi}} \bar{\text{GL}}(U_0),$$

where $\tau$ is an automorphism of $\bar{\text{GL}}(L_0)$ inducing the functor $\star : \text{Sph}_{\bar{\text{GL}}(L_0)} \sim \text{Sph}_{\bar{\text{GL}}(L_0)}$.

Let $\kappa_Q : \bar{Q}(\mathbb{G}) \times \mathbb{G}_m \to \bar{Q}(\mathbb{H})$ be the map

$$\bar{\text{GL}}(L_0) \times \bar{\text{GL}}(A_0) \times \mathbb{G}_m \to \bar{\text{GL}}(U_0) \times \bar{\text{GL}}(C_0)$$

given by $(x, y, z) \mapsto (\kappa_0(x, z), y \omega_m(x))$. Here $\omega_m$ is the unique coweight of the center of $\bar{\text{GL}}(L_0)$ such that $\langle \omega_m, \bar{\omega}_1 \rangle = 1$.

Write $\kappa_{Q,ex} : \bar{Q}(\mathbb{G}) \times \mathbb{G}_m \to \bar{Q}(\mathbb{H}) \times \mathbb{G}_m$ for the map $(\kappa_Q, \text{pr})$, where $\text{pr} : \bar{Q}(\mathbb{G}) \times \mathbb{G}_m \to \mathbb{G}_m$ is the projection.

For $m \leq n$ fix a decomposition $L_0 \sim 1L \oplus 2L$ into direct sum of free $O$-modules, where $1L$ is of rank $m$, and $2L$ is of rank $n - m$. Fix also an isomorphism $U_0 \sim 1L$ of $O$-modules. We assume that these choices are compatible with the maximal tori chosen before. For $m \leq n$ let $\kappa_0 : \bar{\text{GL}}(U_0) \times \mathbb{G}_m \to \bar{\text{GL}}(L_0)$ be the composition

$$\bar{\text{GL}}(U_0) \times \mathbb{G}_m = \bar{\text{GL}}(1L) \times \mathbb{G}_m \xrightarrow{\text{id} \times 2\bar{\rho}_{\text{GL}(2L)}} \bar{\text{GL}}(1L) \times \bar{\text{GL}}(2L) \xrightarrow{\text{Levi}} \bar{\text{GL}}(L_0) \xrightarrow{\tau} \bar{\text{GL}}(L_0),$$

here $\tau$ is an automorphism inducing the functor $\star : \text{Sph}_{\bar{\text{GL}}(L_0)} \sim \text{Sph}_{\bar{\text{GL}}(L_0)}$.

Let $\kappa_Q : \bar{Q}(\mathbb{H}) \times \mathbb{G}_m \to \bar{Q}(\mathbb{G})$ be the map

$$\bar{\text{GL}}(U_0) \times \bar{\text{GL}}(C_0) \times \mathbb{G}_m \to \bar{\text{GL}}(L_0) \times \bar{\text{GL}}(A_0)$$

given by $(x, y, z) \mapsto (\kappa_0(x, z), y \alpha_m(x))$. Here $\alpha_m$ is the unique coweight of the center of $\bar{\text{GL}}(U_0)$ such that $\langle \alpha_m, \bar{\alpha}_1 \rangle = 1$.

Define $\kappa_{Q,ex} : \bar{Q}(\mathbb{H}) \times \mathbb{G}_m \to \bar{Q}(\mathbb{G}) \times \mathbb{G}_m$ as $(\kappa_Q, \text{pr})$. The following is a consequence of ([7], Corollary 5).

Proposition 8. 1) For $m > n$ the two functors $-a : \text{Sph}_{\bar{\mathbb{H}}} \to \text{D}_{\bar{\mathbb{G}}\bar{\mathbb{H}}}(\mathbb{O})(QY_a(F))$ given by $T \mapsto H_{\bar{\mathbb{H}}}(T, I_0)$ and $T \mapsto H_{\bar{\mathbb{G}}}(\text{gRes}_{\kappa_Q}(T), I_0)$.
are isomorphic.

2) For \( m \leq n \) the two functors \(-\) Sph\(_Q(G)\) → \( D_{QGH_a(O)}(QY_a(F))\) given by

\[
T \mapsto H_{Q(G)}^{-}(T, I_0) \quad \text{and} \quad T \mapsto H_{Q(H)}^{-}(\text{gRes}^{\kappa_Q}(T), I_0)
\]

are isomorphic. □

As in ([7], Theorem 7), for each \( a \in \mathbb{Z} \) the diagram of functors is canonically 2-commutative

\[
\begin{array}{ccc}
DP_{\mathbb{H}Q_a(O)}(\mathbf{Y}_a(F)) & \xrightarrow{\mathcal{D}W eil_a} & DP_{\mathbb{H}Q_a(O)}(\mathbf{Y}_a(F)) \\
\downarrow J_{\mathcal{P}_{\mathcal{A}_a}} & & \downarrow J_{\mathcal{P}_{\mathcal{A}_a}} \\
D_{QGH_a(O)}(QY_a(F)) & \xrightarrow{\text{Four}_\psi} & D_{QGH_a(O)}(QY_a(F)),
\end{array}
\]

where \( f_\mathbb{H} \) (resp., \( f_\mathbb{G} \)) sends \((\mathcal{F}_1, \mathcal{F}_2, \beta)\) to \( \mathcal{F}_1 \) (resp., to \( \mathcal{F}_2 \)).

Remind the maps \( \kappa_\mathbb{H} : \hat{Q}(\mathbb{H}) \times G_m \rightarrow \hat{H} \) and \( \kappa_\mathbb{G} : \hat{Q}(\mathbb{G}) \times G_m \rightarrow \hat{G} \) from Section 4.8.6. The restriction of \( \kappa_\mathbb{H} \) and of \( \kappa_\mathbb{G} \) to \( G_m \) equals

\[
2(\hat{\beta}_\mathbb{H} - \hat{\beta}_Q(\mathbb{H})) + nm\hat{\alpha}_0 - n\hat{\alpha}_m
\]

and \( 2(\hat{\beta}_\mathbb{G} - \hat{\beta}_Q(\mathbb{G})) + mn\hat{\omega}_0 - m\hat{\omega}_n \), respectively. From definitions one gets

\[
\begin{cases}
2(\hat{\beta}_\mathbb{H} - \hat{\beta}_Q(\mathbb{H})) = (m - 1)\hat{\alpha}_m - \frac{m(m-1)}{2}\hat{\alpha}_0 \\
2(\hat{\beta}_\mathbb{G} - \hat{\beta}_Q(\mathbb{G})) = (n + 1)\hat{\omega}_n - \frac{n(n+1)}{2}\hat{\omega}_0
\end{cases}
\]

Write \( \kappa_{\mathbb{H},ex} : \hat{Q}(\mathbb{H}) \times G_m \rightarrow \hat{H} \times G_m \) for the map, whose first component is \( \kappa_\mathbb{H} \) and the second \( \hat{Q}(\mathbb{H}) \times G_m \rightarrow G_m \) is the projection, and similarly for \( \kappa_{\mathbb{G},ex} \).

By Corollary 8 for \( T \in -a \text{Sph}_H \) and \( S \in -a \text{Sph}_G \) we get isomorphisms

\[
\mathcal{P}(H_{\mathbb{H}}^{-}(T, I_0)) \simeq H_{\mathbb{H}(\mathbb{H})}^{-}(\text{gRes}^{\kappa_{\mathbb{H}}}(T), I_0)
\]

and

\[
\mathcal{P}(H_{\mathbb{G}}^{-}(S, I_0)) \simeq H_{\mathbb{G}(\mathbb{G})}^{-}(\text{gRes}^{\kappa_{\mathbb{G}}}(S), I_0)
\]

in \( D_{QGH_a(O)}(QY_a(F)) \). The Hecke functors in the RHS of (29) and (30) are from \( D_{QGH_0(O)}(QY_0(F)) \) to the category \( D_{QGH_a(O)}(QY_a(F)) \).

CASE \( m > n \). Proposition 8 together with (29) yields an isomorphism

\[
\mathcal{P}(H_{\mathbb{H}}^{-}(T, I_0)) \simeq H_{\mathbb{H}(\mathbb{G})}^{-}(\text{gRes}^{\kappa_Q,ex} \text{gRes}^{\kappa_{\mathbb{H}}}(T), I_0)
\]

We will define an automorphism \( \tau_{\mathbb{H}} \) of \( \mathbb{H} \) inducing \( \ast : \text{Rep}(\mathbb{H}) \rightarrow \text{Rep}(\mathbb{H}) \) and \( \kappa \) making the following diagram commutative

\[
\begin{array}{ccc}
\hat{G} \times G_m & \xrightarrow{\tau_{\mathbb{H}}^{\kappa_{\mathbb{H}}}} & \mathbb{H} \\
\uparrow \kappa_{\mathbb{G},ex} & & \uparrow \kappa_{\mathbb{H}} \\
\hat{Q}(\mathbb{G}) \times G_m & \xrightarrow{\kappa_{Q,ex}} & \hat{Q}(\mathbb{H}) \times G_m
\end{array}
\]

(31)
The above diagram together with (30) yield isomorphisms
\[
P(H_G^- (gRes^\kappa(*T), I_0), \sim H_{Q(G)}^- (gRes^{\kappa_{Q,G}} gRes^\kappa(*T), I_0), \sim H_{Q(G)}^- (gRes^{\kappa_{Q,G,\epsilon}} gRes^\kappa(T), I_0)
\]
Thus, we get an isomorphism
\[
P(H_G^- (gRes^\kappa(*T), I_0)) \sim P(H_{\tilde{H}}^- (T), I_0))
\]
By Proposition 6 it lifts to the desired isomorphism in D Weil.

Note that \( m \geq 2 \). Let \( W_0 = \tilde{Q}_t^m \), let \( W_1 \) (resp., \( W_2 \)) be the subspace of \( W_1 \) spanned by the first \( n \) (resp., last \( m - n \)) base vectors. Equip \( W_0 \oplus W_0^* \) with the symmetric form given by the matrix
\[
\begin{pmatrix}
0 & E_m \\
E_m & 0
\end{pmatrix},
\]
where \( E_m \in GL(\tilde{Q}_0,t) \) is the unity. Let \( i_{\tilde{H}} \in \text{Spin}(W_0 \oplus W_0^*) \) be the unique central element such that
\[
SO(W_0 \oplus W_0^*) \sim \text{Spin}(W_0 \oplus W_0^*) / \{ i_{\tilde{H}} \}
\]
Realize \( \tilde{H} \) as \( GSpin(W_0 \oplus W_0^*) := G_m \times \text{Spin}(W_0 \oplus W_0^*) / \{ (-1, i_{\tilde{H}}) \} \). There is a unique automorphism \( \tau' \) of \( \text{Spin}(W_0 \oplus W_0^*) \) that preserves \( i_{\tilde{H}} \) and induces the automorphism \( g \mapsto t^g \) on \( SO(W_0 \oplus W_0^*) \). The automorphism \( (a, g) \mapsto (a^{-1}, \tau'(g)) \) of \( G_m \times \text{Spin}(W_0 \oplus W_0^*) \) descends to an automorphism of \( \tilde{H} \) that we denote \( \tau_{\tilde{H}} \).

Let \( \tilde{W} \subset W_2 \oplus W_2^* \) be the subspace spanned by \( e_{n+1} + e_{n+1}^* \). Equip \( W_1 \oplus W_1^* \oplus \tilde{W} \) with the induced form. Write \( i_{G} \) for the central element of \( \text{Spin}(W_1 \oplus W_1^* \oplus \tilde{W}) \). Realize \( \tilde{G} \) as
\[
GSpin(W_1 \oplus W_1^* \oplus \tilde{W}) := G_m \times \text{Spin}(W_1 \oplus W_1^* \oplus \tilde{W}) / \{ (-1, i_{\tilde{G}}) \}
\]
There is a unique inclusion \( \epsilon_0 : \text{Spin}(W_1 \oplus W_1^* \oplus \tilde{W}) \hookrightarrow \text{Spin}(W_0 \oplus W_0^*) \) extending the natural inclusion \( SO(W_1 \oplus W_1^* \oplus \tilde{W}) \hookrightarrow SO(W_0 \oplus W_0^*) \) and sending \( i_{\tilde{G}} \) to \( i_{\tilde{H}} \). The map
\[
id \times \epsilon_0 : G_m \times \text{Spin}(W_1 \oplus W_1^* \oplus \tilde{W}) \hookrightarrow G_m \times \text{Spin}(W_0 \oplus W_0^*)
\]
gives rise to an inclusion \( i_{\kappa} : \tilde{G} \hookrightarrow \tilde{H} \). Finally, there is a unique \( \alpha_{\kappa} : G_m \rightarrow \tilde{H} \) such that for \( \kappa := (i_{\kappa}, \alpha_{\kappa}) \) the diagram (31) commutes. The map \( \tau_{\tilde{H}} \circ \kappa : \tilde{T}_G \rightarrow \tilde{T}_{\tilde{H}} \) is uniquely defined by the formulas
\[
\begin{align*}
\tilde{\omega}_i & \mapsto -\tilde{\alpha}_i + i\tilde{\alpha}_0, \quad 1 \leq i \leq n \\
\tilde{\omega}_0 & \mapsto \tilde{\alpha}_0
\end{align*}
\]
Using these formulas, one checks that
\[
\tau_{\tilde{H}}(\alpha_{\kappa}) = 2\hat{\rho}_{GL(2U)} + (m - 1 - n)(\tilde{\alpha}_m - \tilde{\alpha}_n) + \left( \frac{n(n + 1)}{2} - \frac{m(m - 1)}{2} \right) - (n + 1 - m)n\tilde{\alpha}_0
\]
If \( m = n + 1 \) then \( \alpha_{\kappa} \) is trivial.
CASE $m \leq n$. Proposition 8 together with (30) yields an isomorphism

$$\mathcal{P}(H^\cdot_G(S, I_0)) \cong H^\cdot_{Q(\mathbb{H})}(\text{gRes}^{K_Q, ex} \text{gRes}^{K_G}(S), I_0)$$

We will define an automorphism $\tau_\mathbb{H}$ of $\mathbb{H}$ inducing $*: \text{Rep}(\mathbb{H}) \cong \text{Rep}(\mathbb{H})$ and $\kappa$ making the following diagram commutative

$$\begin{array}{c}
\mathbb{H} \times \mathbb{G}_m & \xrightarrow{\tau_\mathbb{H} \times \text{id}} & \mathbb{H} \times \mathbb{G}_m \\
\uparrow_{\kappa_{\mathbb{H}, ex}} & & \uparrow_{\kappa_{\mathbb{G}}} \\
\hat{Q}(\mathbb{H}) \times \mathbb{G}_m & \xrightarrow{\kappa_{Q, ex}} & \hat{Q}(\mathbb{G}) \times \mathbb{G}_m
\end{array} \quad (32)
$$

The above diagram together with (29) yield isomorphisms

$$\mathcal{P}(H^\cdot_{\mathbb{H}}(* \text{gRes}^\kappa(S), I_0)) \cong H^\cdot_{Q(\mathbb{H})}(\text{gRes}^{K_\mathbb{H}}(* \text{gRes}^\kappa(S)), I_0) \cong H^\cdot_{Q(\mathbb{H})}(\text{gRes}^{K_Q, ex} \text{gRes}^{K_G}(S), I_0)$$

Thus, we get

$$\mathcal{P}(H^\cdot_{\mathbb{H}}(* \text{gRes}^\kappa(S), I_0)) \cong \mathcal{P}(H^\cdot_G(S, I_0))$$

By Proposition 8 it lifts to the desired isomorphism in $D \text{Weil}_\alpha$

$$H^\cdot_{\mathbb{H}}(* \text{gRes}^\kappa(S), I_0) \cong H^\cdot_G(S, I_0)$$

For $m = 1$ the map $\kappa_{\mathbb{H}, ex}$ is an isomorphism, so there is a unique $\kappa$ making (32) commutative. Now assume $m > 1$. Let $W_0 = \mathbb{Q}_\ell^n$, let $W_1$ (resp., $W_2$) be the subspace of $W_0$ generated by the first $m$ (resp., last $n - m$) vectors. Equip $W_0 \oplus W_0 \oplus \mathbb{Q}_\ell$ with the symmetric form given by the matrix

$$\begin{pmatrix}
0 & E_n & 0 \\
E_n & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},$$

where $E_n \in \text{GL}_n(\mathbb{Q}_\ell)$ is the unity. Write $i_{\mathbb{G}}$ for nontrivial the central element of $\text{Spin}(W_0 \oplus W_0 \oplus \mathbb{Q}_\ell)$. Realize $\mathbb{G}$ as

$$\text{GSpin}(W_0 \oplus W_0^* \oplus \mathbb{Q}_\ell) := \mathbb{G}_m \times \text{Spin}(W_0 \oplus W_0^* \oplus \mathbb{Q}_\ell)/\{(1, i_{\mathbb{G}})\}$$

Equip the subspace $W_1 \oplus W_1^* \subset W_0 \oplus W_0^* \oplus \mathbb{Q}_\ell$ with the induced symmetric form. Write $i_{\mathbb{H}}$ for the unique central element of $\text{Spin}(W_1 \oplus W_1^*)$ such that $\text{SO}(W_1 \oplus W_1^*) \cong \text{Spin}(W_1 \oplus W_1^*)/\{i_{\mathbb{H}}\}$. Realize $\mathbb{H}$ as

$$\text{GSpin}(W_1 \oplus W_1^*) := \mathbb{G}_m \times \text{Spin}(W_1 \oplus W_1^*)/\{(1, i_{\mathbb{H}})\}$$

There is a unique automorphism $\tau'$ of $\text{Spin}(W_1 \oplus W_1^*)$ preserving $i_{\mathbb{H}}$ and inducing the map $g \mapsto t g^{-1}$ on $\text{SO}(W_1 \oplus W_1^*)$. The automorphism $(a, g) \mapsto (a^{-1}, \tau'(g))$ of $\mathbb{G}_m \times \text{Spin}(W_1 \oplus W_1^*)$ descends to an automorphism of $\mathbb{H}$ that we denote $\tau_{\mathbb{H}}$.

There is a unique inclusion

$$\epsilon_0 : \text{Spin}(W_1 \oplus W_1^*) \hookrightarrow \text{Spin}(W_0 \oplus W_0^* \oplus \mathbb{Q}_\ell)$$
From this formulas one gets that
\[ \tilde{A} \otimes C \]
for formulas \[ \{ \]
establish isomorphisms (4) and (5) over a on \[ \tilde{X} \]
For a point \[ \in \tilde{X} \]
Lemma 9.
Applying this to \[ M \]
gives rise to an inclusion \[ i \]
Here \[ C \]
By \([5\], Lemma 1\), we get a canonical \[ \tilde{X} \]
is the previous one multiplied by \[ x \]
is the trivialization \[ \tilde{X} \]
is given by a collection: \( (\tilde{x}, x) \), \( x \)
Recall the line bundle \[ E \]
Recall the stack \[ X, \tilde{M} \]
is of parity zero as \( x \)
\[ \det R\Gamma(X, M \otimes V) \otimes C_x^{anm} \leftarrow \frac{\det R\Gamma(X, M)^{2m} \otimes \det R\Gamma(X, V)^{2m}}{\det R\Gamma(X, C)^{2nm} \otimes \det R\Gamma(X, O)^{2nm}} \]
Here \( C_x \) is of parity zero as \( \mathbb{Z}/2\mathbb{Z} \)-graded.

**Proof** By \([5\], Lemma 1\), we get a canonical \( \mathbb{Z}/2\mathbb{Z} \)-graded isomorphism
\[ \det R\Gamma(X, M \otimes V) \leftarrow \frac{\det R\Gamma(X, M)^{2m} \otimes \det R\Gamma(X, V)^{2n}}{\det R\Gamma(X, A^n \otimes \det V)} \times \frac{\det R\Gamma(X, A^n \otimes \det V)}{\det R\Gamma(X, O)^{4nm-1}} \]
Applying this to \( M = O^n \oplus A^n \) with natural symplectic form \( \Lambda^2 M \rightarrow A \), we get
\[ \frac{\det R\Gamma(X, V \otimes A)^n}{\det R\Gamma(X, V)^n} \leftarrow \frac{\det R\Gamma(X, A)^{2nm} \otimes \det R\Gamma(X, A^n \otimes \det V)}{\det R\Gamma(X, A^n) \otimes \det R\Gamma(X, V) \otimes \det R\Gamma(X, O)^{2nm-1}} \]
Since $A \otimes C \simeq \Omega(ax)$ and $V \simeq V^* \otimes C$, the LHS of the above formula identifies with
\[
\det \Gamma(X, V/V(-ax))^{-n} \simeq (\det V_x)^{-an} \otimes \det(\mathcal{O}/\mathcal{O}(-ax))^{-2nm}
\]
We have used a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism
\[
\det(V/V(-ax)) \simeq (\det V_x)^a \otimes (\det(\mathcal{O}/\mathcal{O}(-ax)))^{2m}
\]
Since $\det V_x \simeq C_x^m$, we get
\[
\det \Gamma(X, M) \simeq \frac{\det \Gamma(X, M)^{2m} \otimes \det \Gamma(X, \mathcal{V})^{2m}}{\det \Gamma(X, A)^{2nm} \otimes \det \Gamma(X, \mathcal{O})^{2nm} \otimes \det \Gamma(X, \mathcal{O}(\mathcal{O}(-ax)))^{2nm}} \otimes C_x^{-anm}
\]
To simplify the above expression, note that $\det \Gamma(X, A) \simeq \det R\Gamma(X, C(-ax))$ and
\[
\det \Gamma(X, C) \simeq \det \Gamma(X, \mathcal{O}(\mathcal{O}(-ax))) \otimes C_x^a \otimes \det \Gamma(X, \mathcal{O}/\mathcal{O}(-ax))
\]
Our assertion follows. □

Let $a^x \mathcal{A}$ be the line bundle on $a^x \text{Bun}_{G,H}$ with fibre $\det \Gamma(X, M \otimes V) \otimes C_x^{-anm}$ at $(M, A, V, C)$. We have canonically $(a^x \xi)^*(a^x \mathcal{A}_\mathcal{X} \mathcal{L}) \simeq a^x \mathcal{A}$. Extend $a^x \xi$ to a morphism $a^x \xi : a^x \text{Bun}_{G,H} \to a^x \mathcal{X}\mathcal{L}$ sending $(M, A, V, C)$ to its image under $a^x \xi$ together with the one-dimensional space
\[
B = \frac{\det \Gamma(X, M)^m \otimes \det \Gamma(X, \mathcal{V})^n}{\det \Gamma(X, C)^{nm} \otimes \det \Gamma(X, \mathcal{O})^{nm}}
\]
equipped with the isomorphism $B^2 \simeq \det \Gamma(X, M \otimes V) \otimes C_x^{-anm}$ of Lemma 9.

5.2 Let $a^x \mathcal{H}_{G,H}$ be the stack classifying collections: a point of the Hecke stack $(M, A, M', A', \beta) \in x \mathcal{H}_G$ such that the isomorphism $\beta$ of the $G$-torsors $(M, A)$ and $(M', A')$ over $X - x$ induces an isomorphism $\mathcal{A}(-ax) \simeq \mathcal{A}'$; a $\tilde{H}$-torsor $(V, C) \in \text{Bun}_{\tilde{H}}$, and an isomorphism $\mathcal{A}' \otimes C \simeq \Omega$. We have the diagram
\[
a^x \text{Bun}_{G,H} \xrightarrow{h^-} a^x \mathcal{H}_{G,H} \xrightarrow{h^-} \text{Bun}_{G,H},
\]
where $h^-$ (resp., $h^-$) sends the above point of $a^x \mathcal{H}_{G,H}$ to $(M', A', V, C) \in \text{Bun}_{G,H}$ (resp., to $(M, A, V, C) \in a^x \text{Bun}_{G,H}$).

Restriction to $D_x$ gives rise to the diagram
\[
\begin{array}{c}
\text{a}^x \text{Bun}_{G,H} \\
\xrightarrow{h^-} \text{a}^x \mathcal{H}_{G,H} \\
\downarrow a^x \xi \\
a^x \mathcal{X}\mathcal{L} \\
\end{array} \quad \begin{array}{c}
\xrightarrow{h^-} \text{Bun}_{G,H} \\
\downarrow a^x \xi_G \\
a^x \mathcal{X}\mathcal{L} \\
\end{array}
\]
where the low row is the diagram (12) for $a' = 0$. Now Lemma 9 allows to extend (33) to the following diagram, where both squares are cartesian
\[
\begin{array}{c}
\text{a}^x \text{Bun}_{G,H} \\
\xrightarrow{h^-} \text{a}^x \mathcal{H}_{G,H} \\
\downarrow a^x \xi \\
a^x \mathcal{X}\mathcal{L} \\
\end{array} \quad \begin{array}{c}
\xrightarrow{h^-} \text{Bun}_{G,H} \\
\downarrow a^x \xi_G \\
a^x \mathcal{X}\mathcal{L} \\
\end{array}
\]
\[43\]
and the low row is the diagram (13) for $a' = 0$ from Section 4.3.1. This provides an isomorphism

$$\text{H}^-_G(S, (0\xi)^*K) \cong (0\xi)^*\text{H}^-_G(S, K)$$

functorial in $S \in -\text{Sph}_\mathbb{G}$ and $K \in D_{\mathcal{T}_0}(\tilde{\mathcal{L}}_d(W_0(F_x)))$. Here the functors

$$((0\xi)^* : D_{\mathcal{T}_0}(\tilde{\mathcal{L}}_d(W_0(F_x))) \to D^{(a\check{\text{Bun}}_{G,\check{H}})}$$

are defined as in ([8], Section 7.2).

Let $a\check{H}_{\check{R},G}$ be the stack classifying collections: a point of the Hecke stack $(V, C, V', C', \beta) \in x\check{H}_{\check{R}}$ such that the isomorphism $\beta$ of $\check{H}$-torsors $(V, C)$ and $(V', C')$ over $X - x$ induces an isomorphism $C(-ax) \cong \check{C'}$; a $G$-torsor $(M, A)$ on $X$ and an ismorphism $A \otimes C' \cong \Omega$. As above, we get a diagram

$$a\check{\text{Bun}}_{G,\check{H}} \xrightarrow{\check{h}^-} a\check{\mathcal{H}}_{\check{R},G} \xrightarrow{\check{h}^-} \text{Bun}_{G,\check{H}},$$

where $h^-$ (resp., $h^-$) sends the above point of $a\check{\mathcal{H}}_{\check{R},G}$ to $(M, A, V', C')$ (resp., to $(M, A, V, C)$).

As in the case of the Hecke functor for $G$, we get the diagram, where both squares are cartesian

$$\begin{array}{ccc}
    a\check{\text{Bun}}_{G,\check{H}} & \xrightarrow{\check{h}^-} & a\check{\mathcal{H}}_{\check{R},G} & \xrightarrow{\check{h}^-} & \text{Bun}_{G,\check{H}} \\
    \downarrow \check{\xi} & & \downarrow \check{\xi} & & \downarrow \check{\xi} \\
    a\check{\mathcal{L}} & \xrightarrow{\check{h}^-} & a,0\check{\mathcal{H}}_{\check{R},\check{L}} & \xrightarrow{\check{h}^-} & 0\check{\mathcal{L}},
\end{array}$$

and the low row is the diagram (14) for $a' = 0$ from Section 4.3.2. This provides an isomorphism

$$\text{H}^-_{\check{H}}(S, (0\xi)^*K) \cong (0\xi)^*\text{H}^-_{\check{H}}(S, K)$$

functorial in $S \in -\text{Sph}_\mathbb{G}$ and $K \in D_{\mathcal{T}_0}(\tilde{\mathcal{L}}_d(W_0(F_x)))$. By ([8], Proposition 6), we have $(0\xi)^*S_{W_0(F)} \cong \text{Aut}_{G,\check{H}}$ canonically. Now Theorem [2] from Theorem [3] by applying the functor $(0\xi)^*$. Theorem [2] is proved.

5.3 In this subsection we derive Theorem [1] from Theorem [2]. We give the argument only for $m \leq n$ (the case $m > n$ is completely similar).

Let $a \in \mathbb{Z}$. It suffices to establish the isomorphism (3) for any $S \in -\text{Sph}_\mathbb{G}$. By base change theorem, for $K \in D(\text{Bun}_{\check{H}})$ we get

$$(\pi \times \text{id})^*\text{H}^-_G(S, F_G(K)) \cong (a\mathfrak{p})!(a^{q^*}K \otimes \text{H}^-_G(S, \text{Aut}_{G,\check{H}}))[-\dim \text{Bun}_{\check{H}}],$$

where $a^{q^*} : a\text{Bun}_{G,\check{H}} \to \text{Bun}_{\check{H}}$ and $a^{\mathfrak{p}} : a\text{Bun}_{G,\check{H}} \to \check{X} \times \text{Bun}_G$ send a collection $(\check{x} \in \check{X}, M, A, V, C) \in a\text{Bun}_{G,\check{H}}$ to $(V, C)$ and $(\check{x}, M, A)$ respectively.

By Theorem [2] the latter complex identifies with

$$(a^{\mathfrak{p}})!((a^{q^*}K \otimes \text{gRes}_K(S), \text{Aut}_{G,\check{H}}))[-\dim \text{Bun}_{\check{H}}] \quad (34)$$
Consider the diagram

\[ \Xh \times \text{Bun}_\cH^{-} \overset{\text{supp} \times h^{-}}{\leftarrow} \aH^{-} \overset{h^{-}}{\to} \text{Bun}_\cH \]

\[ \left. \Xh \times \text{Bun}_{\cH,\cG} \right| \overset{\text{supp} \times h^{-}}{\leftarrow} \aH^{-} \overset{h^{-}}{\to} \text{Bun}_{\cH,\cG} \]

\[ \left. \Xh \times \text{Bun}_\cG \right| \overset{\text{id} \times p}{\downarrow} \overset{\text{id} \times q}{\downarrow} \overset{\text{supp} \times h^{-}}{\leftarrow} \aH^{-} \overset{a}{\to} \text{Bun}_{\cG} \]

where \( \aH^{-} \) is the stack classifying \( \tilde{x} \in \Xh, \cH^{-}\)-torsors \((V,C)\) and \((V',C')\) on \( X \) identified via an isomorphism \( \beta \) over \( X - \pi(\tilde{x}) \) so that \( \beta \) yields \( C' \cong C(a\pi(\tilde{x})) \). The map \( \text{supp} \times h^{-} \) (resp., \( h^{-} \)) in the top row sends this point to \((\tilde{x},V,C)\) (resp., to \((V',C')\)).

The stack \( \aH^{-}_{\cG} \) is the above diagram classifies collections: \((\tilde{x},V,C,V',C',\beta) \in \aH^{-}, \aG^{-}\)-torsor \((M,A)\) on \( X \), and an isomorphism \( A \otimes C \cong \Omega \). The map \( \text{supp} \times h^{-} \) (resp., \( \text{supp} \times h^{-} \)) is the middle row sends this collection to \((\tilde{x},M,A,V,C)\) (resp., to \((\tilde{x},M,A,V',C')\)).

By the projection formulas, now (34) identifies with

\[ (\text{id} \times p)!(\text{Aut}_{\cG,H} \otimes (\text{id} \times q)^*H^{-}_c(g\text{Res}^\kappa(S),K))[-\dim \text{Bun}_{\cH}] \]

Theorem \( \square \) is proved.

**APPENDIX A. INVARIANTS IN THE CLASSICAL SETTING**

**A.1** In this appendix we assume that \( k_0 \subset k \) is a finite subfield, and all the objects introduced in Section 4 are defined over \( k_0 \). Write \( F_0 \) for \( k_0 \)-valued points of \( F \). Our purpose is to prove Proposition A.1 formulated in Section 4.8.7.2.

**Lemma 10.** Let \( G \) be a group scheme over \( \text{Spec} \, \cO \), \( P \subset G \) be a parabolic and \( U \subset P \) its unipotent radical. Let \( V \) be a smooth \( \cQ_\ell \)-representation of \( G(F) \). Then the natural map \( V^{G(\cO)} \to V_{U(F)} \) is injective, here \( V_{U(F)} \) denotes the corresponding Jacquet module.

**Proof** The author thanks J.-F. Dat for the following proof communicated to me. Pick a Borel subgroup \( B \subset P \), write \( I \subset G(\cO) \) for the corresponding Iwahori subgroup. It suffices to show that \( V^I \to V_{U(F)} \) is injective.

Let \( v \in V^I \) vanish in \( V_{U(F)} \). Then one may find a semisimple \( t \in B(F) \) such that the characteristic function \( \phi \) of \( ITI \) annihilates \( v \) (it suffices that the action of \( t \) on \( U(F) \) be sufficiently contracting). However, \( \phi \) is invertible in the Iwahori-Hecke algebra of \( (G(F),I) \), so \( v = 0 \). \( \square \)

**Lemma 11.** The maps \( J^*_{P_{\text{Pa}}} : \text{Weil}_a(k_0) \to \cS_{Q\text{H}_{a}(\cO)}(Q\text{Y}_{a}(F_0)) \) and

\[ J^*_{P_{\text{Pa}}} : \text{Weil}_a(k_0) \to \cS_{Q\text{H}_{a}(\cO)}(Q\text{II}_{a}(F_0)) \]

are injective.
Proof Both claims being similar, we prove only the second one. Apply Lemma \[\text{[10]}\] for the parabolic $P_{\mathbb{H}a} \subset \mathbb{H}_a$ and the representation $S(\Pi_a(F))$ of $T_a(F)$. Remind that $T_a = \{(g_1, g_2) \in G_a \times \mathbb{H}_a | (g_1, g_2) \text{ acts trivially on } A_a \otimes C_a\}$, and $U_{\mathbb{H}_a} \subset P_{\mathbb{H}_a}$ is the unipotent radical.

For $v \in \Pi_a(F)$ let $s_{\Pi}(v) : C^*_{\alpha} \otimes \mathbb{H}^2 U_a(F) \to \Omega(F)$ be the map introduced in Section 4.8.4. Write $\text{Cr}(\Pi_a)$ for the space of $v \in \Pi_a(F)$ such that $s_{\Pi}(v) = 0$. By \([\text{[10]}, \text{page 72}]\), the Jacquet module $S(\Pi_a(F))_{U_{\mathbb{H}_a}(F)}$ identifies with the Schwarz space $S(\text{Cr}(\Pi_a))$, and the projection

$$S(\Pi_a(F)) \to S(\Pi_a(F))_{U_{\mathbb{H}_a}(F)}$$

identifies with the restriction map $S(\Pi_a(F)) \to S(\text{Cr}(\Pi_a))$. We learn that the restriction map $\text{Weil}_a(k_0) \to S_G \text{Heil}_{a\mathbb{H}_a}(\mathcal{O})(\text{Cr}(\Pi_a))$ is injective. So, \([\text{[35]}\] is also injective. \[\square\]

Proof of Proposition A.1

For $b \in \mathbb{Z}$ set $b \mathcal{H}_G = K_0(b \text{Sph}_G) \otimes \mathbb{Q}_{\ell}$ and $b \mathcal{H}_{Q(\mathbb{G})} = K_0(b \text{Sph}_{Q(\mathbb{G})}) \otimes \mathbb{Q}_{\ell}$. So,

$$\mathcal{H}_G = \bigoplus_{b \in \mathbb{Z}} b \mathcal{H}_G, \quad \mathcal{H}_{Q(\mathbb{G})} = \bigoplus_{b \in \mathbb{Z}} b \mathcal{H}_{Q(\mathbb{G})}$$

are the Hecke algebras for $G$ and $Q(\mathbb{G})$ respectively. From Proposition \[\text{[4]}\] we learn that the map

$$-a \mathcal{H}_{Q(\mathbb{G})} \to S_G \text{Heil}_{a\mathbb{H}_a\mathbb{O}}(\mathcal{O})(\Pi_a(F_0))$$

given by $S \mapsto \text{tr}_{k_0} H^{a\mathbb{H}_a\mathbb{O}}(S, I_0)$ is an isomorphism of $\mathbb{Q}_{\ell}$-vector spaces. Write $-a W \subset -a \mathcal{H}_{Q(\mathbb{G})}$ for the image of the map \([\text{[35]}\] \). We get a $\mathbb{Z}$-graded subspace $W := \bigoplus_{a \in \mathbb{Z}} a W \subset \mathcal{H}_{Q(\mathbb{G})}$.

For $a, a' \in \mathbb{Z}$ we have the Hecke operators

$$\Pi_{a'} : a' - a \mathcal{H}_G \times S_G \text{Heil}_{a'\mathbb{H}_a\mathbb{O}}\left(\Pi_{a'}(F_0)\right) \to S_G \text{Heil}_{a\mathbb{H}_a\mathbb{O}}\left(\Pi_a(F_0)\right)$$

defined as in Section 4.8.1. We claim that for $S \in a' - a \mathcal{H}_G$ the operator $\Pi_{a'}(S, \cdot)$ sends $\text{Weil}_{a}(k_0)$ to the subspace $\text{Weil}_{a}(k_0) \subset S_G \text{Heil}_{a\mathbb{H}_a\mathbb{O}}\left(\Pi_{a}(F_0)\right)$. This follows from the fact the actions of the groupoids $GQ\mathbb{H}$ and $\mathbb{H}_a\mathbb{O}$ on the spaces $S_G \text{Heil}_{a\mathbb{H}_a\mathbb{O}}(\mathcal{O})(\Pi_a(F_0))$ commute with each other.

More precisely, for $a, b \in \mathbb{Z}$ given $g = (g_1, g_2) \in T_{b,a}$ such that $g_2 : V_a \rightarrow V_b$ is an isomorphism of $Q(\mathbb{H})$-torsors over $\text{Spec} \mathcal{O}$, let $h = (h_1, h_2) \in T_b$ be any element such that $h_1 : M_b \rightarrow M_b$ is a scalar automorphism of the $G$-torsor $M_b$ over $\text{Spec} \mathcal{O}$. Here $h_2$ is an automorphism of the $\mathbb{H}$-torsor $V_b$ over $\text{Spec} \mathcal{O}$. Set $h'_2 = g_2^{-1} h_2 g_2$, so $h'_2$ is an automorphism of the $\mathbb{H}$-torsor $V_a$ over $\text{Spec} \mathcal{O}$. Set $h'_1 = h_1$ then $h' = (h_1, h_2) \in T_{a}$. The equality $gh' = hq$ in $T$ shows that $g : S(\Pi_a(F)) \rightarrow S(\Pi_b(F))$ sends $\mathbb{H}_a(\mathcal{O})$-equivariant objects to $\mathbb{H}_b(\mathcal{O})$-equivariant objects. We have used the action of the groupoid $T$ on the spaces $S(\Pi_a(F))$ obtained as in Remark \[\text{[5]}\]

Thus, $W$ is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring $\mathcal{H}_G$. We also know from \([\text{[5]}, \text{Proposition 2}\] \) that $0 W = 0 \mathcal{H}_G$. Our statement is reduced to Lemma \[\text{[12]}\] below. \[\square\]

Remind that we have picked a maximal torus $T_{\mathbb{G}} \subset Q(\mathbb{G})$. Write $W$ (resp., $W_Q$) for the Weyl group of $(G, T_{\mathbb{G}})$ (resp., of $(Q(\mathbb{G}), T_{\mathbb{G}})$). Then

$$\mathcal{H}_{Q(\mathbb{G})} \rightarrow \mathbb{Q}_{\ell} [T_{\mathbb{G}}] W_Q, \quad \mathcal{H}_G \rightarrow \mathbb{Q}_{\ell} [T_{\mathbb{G}}] W$$
The homomorphism \( \text{Res}^{\kappa}: \mathcal{H}_G \to \mathcal{H}_{Q(G)} \) (cf. Section 4.8.6) comes from the map \( f^{\kappa}_Q: \tilde{T}_G^W \to \tilde{T}_G^W \) obtained by taking the Weil group invariants of the map \( \tilde{T}_G \to \tilde{T}_G, t \mapsto t\nu(q^{1/2}) \), where \( \nu \) is some coweight of the center \( Z(\tilde{Q}(G)) \), and \( q \) is the number of elements of \( k_0 \).

**Lemma 12.** View \( \mathcal{H}_{Q(G)} \) as a \( \mathbb{Z} \)-graded \( \mathcal{H}(G) \)-module via \( \text{Res}^{\kappa}: \mathcal{H}_G \to \mathcal{H}_{Q(G)} \). Let \( W = \bigoplus_{a \in \mathbb{Z}} aW \subset \mathcal{H}_{Q(G)} = \bigoplus_{a \in \mathbb{Z}} a\mathcal{H}_{Q(G)} \) be a \( \mathbb{Z} \)-graded submodule over the \( \mathbb{Z} \)-graded ring \( \mathcal{H}_{G} \). Assume that \( 0W = 0\mathcal{H}_G \). Then \( W = \mathcal{H}_G \).

**Proof** Given \( x \in aW \), pick a nonzero \( h \in -a\mathcal{H}_G \) then \( hx \in 0\mathcal{H}_G \). So, \( x \) is a rational function on \( \tilde{T}_G^W \) which becomes everywhere regular after restriction under \( f^{\kappa}_Q: \tilde{T}_G^W \to \tilde{T}_G^W \). Since \( \tilde{T}_G^W \) is normal by Remark 8 below, and \( x \) is entire over \( \bar{\mathbb{Q}}[\tilde{T}_G^W] \), it follows that \( x \in \bar{\mathbb{Q}}[\tilde{T}_G^W] \). \( \square \)

**Remark 8.** Let \( A \) be an entire normal ring, \( W \) be a finite group acting on \( A \). Assuming that \( A \) is finite over \( A^W \), one checks that \( A^W \) is normal.

**References**

[1] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Asterisque 100 (1982)

[2] A. Beilinson, V. Drinfeld, Quantization of the Hitchin integrable system and Hecke eigensheaves, preprint downloadable from http://www.math.utexas.edu/~benzvi/Langlands.html

[3] A. Braverman, D. Gaitsgory, Geometric Eisenstein series, Inv. Math. 150 (2002), 287-384

[4] G. Laumon, Transformation de Fourier, constantes d’équations fonctionnelles et conjectures de Weil, Publ. IHES, t. 65 (1987), p. 131-210

[5] S. Lysenko, Geometric Waldspurger periods, [math.AG/0510110](http://arxiv.org/abs/math.AG/0510110) to appear in Compositio Math.

[6] S. Lysenko, Moduli of metaplectic bundles on curves and Theta-sheaves, Ann. Scient. Éc. Norm. Sup. 4 série, t.39 (2006), 415-466

[7] S. Lysenko, Geometric theta-lifting for the dual pair \( \text{SO}_{2m}, \text{Sp}_{2n} \) [math/0701170](http://arxiv.org/abs/math/0701170)

[8] V. Lafforgue, S. Lysenko, Geometric Weil representation: local field case, [arXiv:0705.4213](http://arxiv.org/abs/0705.4213)

[9] I. Mirković, K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings, [math.RT/0401222](http://arxiv.org/abs/math.RT/0401222) (to appear in Ann. of Math.)

[10] C. Moeglin, M.-F. Vigneras, J.-L. Waldspurger, Correspondance de Howe sur un corps p-adique, Lecture Notes in Math. 1291 (1987)

[11] S. Rallis, Langlands functoriality and the Weil representation, Amer. J. Math., vol.104, no. 3, p. 469-515 (1982)

47
[12] B. Roberts, The theta correspondence for similitudes, Israel J. Math. 94 (1996), 285–317