Bodies with mirror surface invisible from two points

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Received 16 October 2013, revised 10 January 2014
Accepted for publication 21 February 2014
Published 7 May 2014

Recommended by D Dolgopyat

Abstract

We consider a setting where a bounded set with a piecewise smooth boundary in Euclidean space is identified with a body with a mirror surface, and the billiard in the complement of the set is identified with the dynamics of light rays outside the body in the framework of geometric optics. We show that in this setting it is possible to construct a body invisible from two points.

Keywords: billiard invisibility, optical camouflage, invisibility from two points
Mathematics Subject Classification: 37D50, 78A05

(Some figures may appear in colour only in the online journal)

1. Introduction

The problems studied in the framework of billiard invisibility involve the mathematical design of bodies with well-defined surfaces whose scattering map preserves certain trajectories of a flow of elastic particles. The main practical application of this study is optical shielding: by surrounding an object by a specially designed mirror surface, it is possible to create an illusion of invisibility from given points or directions.

From the practical viewpoint, this approach is a low-tech alternative to the major modern attempts at achieving invisibility, which are primarily focussed around the design of metamaterials that allow the bending of electromagnetic waves around the concealed object. The idea was first suggested in [15], and tangible results in this direction were achieved by research teams at Duke University, first in 2006, when an imperfect prototype of a microwave-range cloaking device was created [9], and then in 2012, when remarkable results were achieved completely cloaking a centimetre-thin board, even though it was in one direction only. Another
development in this direction [17] is a successful concealing of a nanoscale object under a carpet cloak made of layers of silicon oxide and silicon nitride arranged in a special way.

Another approach to invisibility was recently suggested in [18] and lead to the development of a working invisibility cloak based on calcite crystals (also see [3]). The cloak works only under one light polarization: it is essentially two-dimensional (2D), although it works at all angles.

The first work that targets the problem of designing a body invisible in a direction in the framework of mirror invisibility appears in [1] and is motivated by the problem of constructing a nonconvex body or zero resistance. The authors demonstrated that there exists a (connected and even simply connected) body invisible in one direction: if this body is manufactured out of perfectly reflective mirrors, a laser beam sent through this construction in the direction of invisibility would leave the body along the same trajectory. Remarkably, in [8] it is shown that this body is also invisible in acoustic waves.

This research led to several intriguing mathematical problems. Some of them, proposed by Tabachnikov [16], ask whether it is possible to design a body with a mirror surface invisible in two directions or a body invisible from a point. The former problem was solved in [11]: it was demonstrated that a parabolic construction can be used to produce a body invisible in two directions in a three-dimensional (3D) case. This body consists of two connected components. It is possible to construct 2D and 3D infinitely connected fractal bodies invisible in two and three directions, respectively [12]. Note that it is impossible to construct a body invisible in all directions [11], and at least in the 2D case there is no piecewise smooth body invisible in a countable number of directions [13]. A somewhat related development that should be mentioned here is the phenomena best known under the name of the digital sundial. It is possible to construct fractal bodies such that their projections on almost all planes can be prescribed up to a set of zero measure (see [5]). In this framework, Burdzy and Kulczycki [2] showed that there are 2D bodies that consist of an infinite number of linear segments, and which are almost invisible in almost all directions.

In addition to the fascinating research problems discussed above, billiard invisibility is linked to some fundamental properties of dynamical systems. As is pointed out in [10] (see also [7]), billiard invisibility is related to the long-standing Ivrii’s conjecture. The Ivrii’s conjecture states that the measure of periodic trajectories in a piecewise smooth planar billiard is zero, and, in particular, for any natural number $n$, the measure of periodic trajectories of period $n$ is zero. The conjecture has been proved for several important special cases (the most recent breakthrough is [6], where the authors proved the nonexistence of billiards with an open set of periodic quadrilateral orbits).

Recently we have identified bodies (connected but not simply connected) invisible from a point [10,14]. In this paper we provide a construction of a body invisible from two points. The body has infinitely many connected components. To our best knowledge, this construction demonstrates a new property of confocal conics. Thus, if you look with two eyes open from a fixed position, the body disappears—it becomes completely invisible. The previous result (on a body invisible from a point) meant that you needed to look at the body with one eye open and the other closed. The eye(s) should be placed at the point(s) from which the body is invisible.

The main result of the paper is the following theorem 1.

**Theorem 1.** Given two different points in Euclidean space $\mathbb{R}^d$, $d = 2, 3, \ldots$, there exists a body in $\mathbb{R}^d$ invisible from these points.

We also show that mixed-type invisibility, where two points are substituted with a point and a direction, also holds true.
Theorem 2. Given a point $A \in \mathbb{R}^d$ and a vector $v \in S^{d-1}$, there exists a body in $\mathbb{R}^d$ invisible simultaneously from the point $A$ and in the direction $v$.

The rest of the paper is devoted to the proof of these results.

2. Invisibility from two points

We begin with a reminder of the relevant definitions, then explain our construction and prove that it is invisible from two points.

Definition 1. A body is a bounded set with finitely or countably many connected components, where each component is either a domain with a piecewise smooth boundary, or a piece of smooth hypersurface, and at least one component is a domain.

Definition 2. A body $B \subset \mathbb{R}^d$ is said to be invisible from a point $O \in \mathbb{R}^d \setminus B$, if for almost all $v \in S^{d-1}$ the billiard particle in $\mathbb{R}^d \setminus B$ emanating from $O$ with the initial velocity $v$, after a finite number of reflections from $\partial B$ will eventually move freely with the same velocity $v$ along a straight line containing $O$.

If the point $O$ is infinitely distant, we get the notion of a body invisible in a direction. Namely, we have the following definition.

Definition 3. A body $B$ is invisible in a direction $v \in S^{d-1}$, if for almost all straight lines in $\mathbb{R}^d$ with the director vector $v$, the billiard particle in $\mathbb{R}^d \setminus B$ that initially moves along this line with velocity $v$, after a finite number of reflections from $\partial B$ will finally move freely along the same line with the same velocity $v$.

2.1. Construction of a body invisible from two points

In this section, we construct a 2D body invisible from two different points $A_1$ and $A_2$. An invisible body in three or higher dimensions is obtained by rotating the 2D body around the axis $A_1A_2$.

Let $s$ be the symmetry axis of the system $\{A_1, A_2\}$, i.e. $s$ is a straight line perpendicular to the segment $A_1A_2$. Choose two different arbitrary points on $s$ such that they both lie on the same side of the segment $A_1A_2$ (see figure 1(a)). Denote the point nearest to $A_1A_2$ by $L$ and the other one by $K$. Let $C_1$ be the intersection of $A_1K$ with the line through $A_2$ and $L$, and let $C_2$ be the symmetric intersection of $A_2K$ with the line through $A_1$ and $L$ (see figure 1(a)). Choose a point, $O$, on $LK$. Let $D_2$ be the intersection of $A_2K$ with the line through $A_1$ and $O$, and let $B_1$ be the intersection of the segments $A_1D_2$ and $C_1L$. The points $D_1$ and $B_2$ are constructed symmetrically (see figure 1(b)).

Draw two confocal ellipses with foci $A_1$ and $B_1$ through the points $C_1$ and $L$ and two confocal hyperbolas with foci $A_2$ and $D_1$ through the points $C_1$ and $K$. Repeat this symmetrically on the other side of $s$ (see figure 1(b)). Thus, we have eight curves of second order through the points $C_1$, $K$, $C_2$, $L$; each curve is either an ellipse or a hyperbola; there are two curves through each point. In the sequel by hyperbola we mean the branch of the hyperbola that passes through the corresponding point. Each curve (an ellipse or a branch of a hyperbola) bounds a convex set; by the exterior of an ellipse or a hyperbola we mean the exterior of the corresponding set.

Observe that the curves through each of the points $C_1$, $K$, $C_2$, $L$ are tangent to each other at these points, and hence intersect only at these points. This is due to the property shared by ellipses and hyperbolas: the tangent line at a point on the curve always makes equal angles.
with the lines drawn through this point and foci. Hence it is not difficult to see that all of the eight curves bisect the relevant angles.

Take $H_1$ on the bisector of the angle $B_1C_1D_1$ inside the quadrangle $B_1C_1D_1O$, and denote by $H_2$ the point symmetric to $H_1$ with respect to $KL$. Furthermore, let $N$ be the point of intersection of $A_1H_2$ with $A_2H_1$, and let $M$ be the point of intersection of $A_1H_1$ with $A_2H_2$. Obviously, $N$ belongs to the quadrangle $OB_2LB_1$ and $M$ belongs to the quadrangle $OD_1KD_2$, and both points belong to the line $KL$, which is the bisector of the angles $B_1LB_2$ and $D_1KD_2$. Since the chosen points lie on the bisectors, they belong to the exterior of the corresponding pairs of curves.

If the symmetric configuration of points is slightly changed, one can also choose the points $H_1$, $M$, $H_2$, $N$ possessing the mentioned properties.

Now, let $l^1$ be the arc of the ellipse with foci $A_1$ and $B_1$ with the endpoints at $L$ and at the point of intersection of the ellipse with the line $A_1N$, and $l^2$ be the arc of the ellipse with foci $A_2$ and $B_2$ with the endpoints at $L$ and at the point of intersection of the ellipse with the line $A_2N$. It follows from the construction that these points of intersection lie on the segments $A_1N$ and $A_2N$, respectively, and the arcs $l^1$ and $l^2$ belong to the quadrangle bounded by the straight lines $A_1N$, $A_2N$, $C_1L$, $C_2L$. Let $QL$ be the curvilinear quadrangle bounded by $l^1$, $l^2$, and by segments of the lines $A_1N$, $A_2N$. In a similar way, define the curvilinear quadrangles $QC_1$, $Q_C$, and $Q_K$; each of them is bounded by two segments of lines $A_1M$, $A_2M$, $A_1N$, $A_2N$ and by two arcs. There are eight arcs in total (four of them: $l^1$, $l^2$, $c_\lambda$, $c_k$, are shown in figure 2); each of them generates an infinite sequence of arcs to be defined below—eight sequences in total.

Each sequence is totally contained in one of eight quadrangles that are shown in the figure and determined by their diagonals: $NB_1$, $NB_2$, $B_1H_1$, $B_2H_2$, $H_1D_1$, $H_2D_2$, $MD_1$, $MD_2$ (see figure 3). Let us describe the sequences $l_0$, $l_1$, $l_2$, ... and $c_0$, $c_1$, $c_2$, ... generated by the arcs $l^i$ and $c_k$, respectively; the other sequences are defined analogously.

The sequence $l_0$, $l_1$, $l_2$, ... of arcs of ellipses with foci at $A_2$ and $B_2$ is uniquely defined by the following conditions. (1) $l_0$ coincides with $l^2$. (2) The endpoints of the arc $l_i$, $i = 0, 1, 2, ...$ are denoted by $\lambda_i$ and $\nu_i$; $\lambda_i$ lies on the segment $A_1B_2$ and $\nu_i$ lies on the segment $B_2A_2$, (3) $\lambda_0$ lies on the segment $B_2A_2$ (see figure 5).

Similarly, the sequence $c_0$, $c_1$, $c_2$, ... of arcs of hyperbolas with foci at $A_2$ and $D_1$ is uniquely defined by the following conditions. (1) $c_0$ coincides with $c_k$. (2) The endpoints of the arc $c_i$, $i = 0, 1, 2, ...$ are denoted by $\sigma_i$ and $\chi_i$; $\sigma_i$ lies on the segment $A_1D_1$.
and $\chi_i$ lies on the segment $\chi_0D_1$. (3) $\sigma_{i+1}$ lies on the extension of the segment $\chi_iA_2$ (see figure 5).

Below, we will need the following statement.

**Lemma 1.**

(a) Consider two different points $F_1$ and $F_2$, and let two rays from $F_1$ intersect two rays from $F_2$ at four points: $e_1$, $e_2$, $h_1$, $h_2$ (see figure 4). Draw two ellipses $E_1$, $E_2$ and two hyperbolas $H_1$, $H_2$ with foci $F_1$ and $F_2$ through $e_1$, $e_2$, $h_1$, $h_2$, respectively. (Recall that by hyperbola we mean the corresponding branch of the hyperbola.) We claim that the point of intersection of $E_1$ and $H_1$, the point of intersection of $E_2$ and $H_2$, and $F_2$ are collinear.
(b) Consider a ray from $F_2$ and two points $u_1$, $u_2$ on it. Draw the ellipse $\mathcal{E}_1$ and the hyperbola $\mathcal{H}_1$ with foci $F_1$, $F_2$ through $u_1$, and the ellipse $\mathcal{E}_2$ and the hyperbola $\mathcal{H}_2$ with the same foci $F_1$, $F_2$ through $u_2$. Take a ray from $F_1$ intersecting $\mathcal{E}_1$ and $\mathcal{H}_1$ at $e_1$ and $h_1$, respectively. Let the rays $F_2 e_1$ and $F_2 h_1$ intersect $\mathcal{E}_2$ and $\mathcal{H}_2$ at $e_2$ and $h_2$. We claim that the points $e_2$, $h_2$, and $F_1$ are collinear.

Its proof is given in the appendix. Note that another proof of lemma using an elegant geometric argument is provided by Dolgirev in [4].

Let us now prove by induction that the points $\chi_i$, $\nu_i$ and $A_2$ (and therefore the points $\sigma_{i+1}$, $\chi_i$, $\nu_i$, $\lambda_{i+1}$, $A_2$) are collinear (see figure 5). For $i = 0$ this statement is obvious, since the points $\chi_0$, $\nu_0$, $A_2$ lie on the line $A_2 H_1$. Now assume that $\chi_{i-1}$, $\nu_{i-1}$, and $A_2$ are collinear for $i \geq 1$, and prove that $\chi_i$, $\nu_i$, $A_2$ are collinear. Consider the dilation with the centre at $A_2$ that
Figure 6. The path of a light ray emanating from $A_2$. 

The path of a light ray emanating from $A_2$ takes $D_1$ to $B_2$. This dilation takes $c_i$ to the arcs $c'_i$ of the hyperbolas with foci at $A_2$ and $B_2$. Let the endpoints of $c'_i$ be $\sigma'_i$ and $\chi'_i$; the assumption of induction implies that the points $\sigma'_i$, $\chi'_i$, $v_{i-1}$, $\nu_i$, $\lambda_i$, $A_2$ are collinear. We are going to prove that $\chi'_i$, $\nu_i$, and $A_2$ are collinear.

The rays $A_2\sigma_{i-1}$ and $A_2\sigma_i$ intersect the rays $B_2A_1$ and $B_2\sigma_{i-1}$ at four points: $\lambda_{i-1}$, $\lambda_i$, $\sigma'_{i-1}$, $\sigma'_i$. The ellipses $l_{i-1}$ and $l_i$ contain the points $\lambda_{i-1}$ and $\lambda_i$, and the hyperbolas $c'_{i-1}$ and $c'_i$ contain the points $\sigma'_{i-1}$ and $\sigma'_i$; therefore, according to statement (a) of lemma 1, the point of intersection of $l_{i-1}$ and $c'_{i-1}$, the point of intersection of $l_i$ and $c'_i$, and $B_2$ are collinear. Furthermore, using additionally that the points $\nu_{i-1}$ and $\nu_i$ lie on a single ray from $B_2$, $\chi'_{i-1}$ and $\chi'_i$ lie on another ray from $B_2$, $\nu_{i-1}$ and $\chi'_{i-1}$ lie on a ray from $A_2$, and applying the statement (b) of lemma, we finally conclude that the points $\chi'_i$, $\nu_i$, and $A_2$ are collinear.

The resulting body is the union of sets $Q_L$, $Q_C_1$, $Q_C_2$, $Q_K$, and eight sequences of arcs. Let us show that it is invisible from points $A_1$ and $A_2$.

It is enough to show invisibility from point $A_2$; the invisibility from $A_1$ can be verified in a completely similar way. If the light ray emanating from $A_2$ does not belong to the angle $C_1A_2K$ (which is the union of the angles $C_1A_2O$ and $OA_2K$), it does not hit the body. It remains to consider the cases when it belongs to the angle $C_1A_2O$ and to the angle $OA_2K$.

We consider only the angle $C_1A_2O$, since the case of the angle $OA_2K$ is completely similar.

The light ray reflects from an arc $l_{i-1}$ ($i \geq 1$) and then goes along a straight line containing $B_2$. (In figure 6, the case $i = 1$ is shown.) Next, it reflects from $l_i$ and then goes along a line containing $A_2$. Then after reflection from the arc $c_i$, it goes along a line containing $D_1$, hits $c_{i-1}$ and finally goes along a line containing $A_2$. (The points of reflection are denoted by $P_1$, $P_2$, $P_3$, $P_4$.) It remains to prove that this final line coincides with the initial one.

Consider the arcs $c'_{i-1}$, $c'_i$ homothetic to $c_{i-1}$, $c_i$ and extend the segment $P_3P_1$ of the trajectory between $l_i$ and $c_i$ until the intersection $P'_3$ with $c'_i$ (dashed line $P_3P'_3$ in the figure). Let $P'_3$ be the image of $P_3$ and $P_4$, respectively, under the dilation (recall that the dilation is centred at $A_2$ and takes $D_1$ to $B_2$). Next we take the straight line through $P'_3$ and...
Figure 7. Bodies invisible from two points: (a) the 2D case, (b) the 3D case.

B2 and fix the point P′4 of its intersection with c′i−1. The segment P′3P′4 is a dashed line in the figure. We already know that the point of intersection of the hyperbola c′i−1 with the ellipse li−1, the point of intersection of the hyperbola c′i with the ellipse li, and B2 are collinear. The points P2, P′i and A2 are collinear, the points P′i, P′4 and B2 are collinear, and the points P1, P2 and B2 are also collinear. Thus, P′4 lies on A2P1, and therefore, P4 also lies on this line. This implies that the line A2P4 of final motion of the particle coincides with the line A2P1 of initial motion. Invisibility is proved.

The invisible body is shown in figure 7(a). It is the union of four curvilinear quadrangles and eight infinite sequences of curves of vanishing length. Notice that the body lies on one side of the line A1A2.

To obtain a body invisible from two points in a higher dimensional setting, it is sufficient to rotate the 2D construction around the axis A1A2 (see figure 7(b) for the illustration of the 3D case).

More precisely, take two different points A1 and A2 in Rd, and let l be the straight line through these points. Rd is the union of 2D planes through l. Each plane is divided by l into two half-planes. Fix a 2D plane through l and take a body B in this plane invisible from A1 and A2. Take the half-plane Π containing B. There exists a unique mapping Φ : Rd → Π such that the restriction of Φ on any 2D half-plane bounded by l is an isometry with the fixed points A1 and A2. Then the body Φ−1(B) ⊂ Rd is invisible from A1 and A2.

It is not necessary to make a full rotation; for instance, in the 3D case a rotation by a certain angle (not necessary 360°) results in an invisible body. In arbitrary dimension, let a domain A ⊂ Rd be a union of 2D half-planes bounded by l, and let Φ|A be the restriction of Φ on A. Then Φ−1(A) is a body invisible from A1 and A2. We hence proved theorem 1.

Remark 1. The construction of the invisible body does not have to be symmetric. In fact, it is enough to choose four points H1, M, H2, N in the quadrangles OBC1D1, OD1KD2, OD2C2B2, OB2LB1, respectively. We require that the straight lines H1M and H2N contain A1, the lines M H2 and N H1 contain A2, and each of the points H1, M, H2, N belongs to the exterior of both curves through C1, K, C2 or L, respectively. Moreover, we require that each of the segments A1N, A2N intersects only one curve through L and that the analogous conditions related to the other three vertices of LC1KC2 hold.
To demonstrate the existence of a body invisible in one direction and from one point simultaneously, a similar construction may be used. The difference is that here one needs to use four parabolas, two ellipses, and two hyperbolas instead of four ellipses and four hyperbolas. The details of this modified construction are left to the reader.

**Acknowledgments**

This work was supported by Portuguese funds through CIDMA—Center for Research and Development in Mathematics and Applications and FCT—Portuguese Foundation for Science and Technology, within the project PEst-OE/MAT/UI4106/2014, as well as by the FCT research project PTDC/MAT/113470/2009.

**Appendix**

Consider two rays emanating from $F_2$ and a ray emanating from $F_1$. Let the points of intersection of the first two ones with the third one be denoted by $h$ and $e$. Let $\alpha = \angle F_1F_2h$, $\beta = \angle F_1F_2e$, $\gamma = \angle hF_1F_2$. Draw an ellipse and a hyperbola with foci at $F_1$ and $F_2$ through $e$ and $h$, respectively; let $u$ be the point of their intersection and $\varphi = \angle F_1F_2u$ (figure A.1). We are going to show that $\varphi$ depends only on $\alpha$ and $\beta$, and does not depend on $\gamma$ (that is, does not depend on the choice of the ray from $F_1$). Thus statement (a) of lemma 1 will be proved. The proof of statement (b) is completely similar and therefore is omitted here.

Denote $a_1 = F_1h$, $a_2 = F_1e$, $b_1 = F_2h$, $b_2 = F_2e$, $f = F_1F_2$, $c = F_2u$. By the focal property of the hyperbola and ellipse we have
\[
 F_1h - F_2h = F_1u - F_2u \quad \text{and} \quad F_1e + F_2e = F_1u + F_2u, 
\]
whence
\[
 a_1 - b_1 = \sqrt{f^2 + c^2 - 2cf \cos \varphi} - c, \quad a_1 + b_1 = \sqrt{f^2 + c^2 - 2cf \cos \varphi} + c, 
\]
and therefore,
\[
 \sqrt{f^2 + c^2 - 2cf \cos \varphi} = \frac{1}{2}(a_1 - b_1 + a_2 + b_2), \quad c = \frac{1}{2}(-a_1 + b_1 + a_2 + b_2). 
\]
Further, by sine law,
\[
\frac{a_1}{\sin \alpha} = \frac{b_1}{\sin \gamma} = \frac{f}{\sin(\alpha + \gamma)}, \quad \frac{a_2}{\sin \beta} = \frac{b_2}{\sin \gamma} = \frac{f}{\sin(\beta + \gamma)}.
\]

hence
\[
a_1 = f \frac{\sin \alpha}{\sin(\alpha + \gamma)}, \quad b_1 = f \frac{\sin \gamma}{\sin(\alpha + \gamma)}, \quad a_2 = f \frac{\sin \beta}{\sin(\beta + \gamma)}, \quad b_2 = f \frac{\sin \gamma}{\sin(\beta + \gamma)},
\]
and substituting these quantities in (1) and (2) one gets
\[
f^2 + c^2 - 2cf \cos \varphi = f^2 \frac{\sin \alpha - \sin \gamma + \sin \beta + \sin \gamma}{\sin \alpha + \gamma} \frac{1}{\sin(\beta + \gamma)},
\]
\[
c = \frac{f}{2} \left[ \frac{\sin \gamma - \sin \alpha}{\sin(\alpha + \gamma)} + \frac{\sin \beta + \sin \gamma}{\sin(\beta + \gamma)} \right],
\]

and thus,
\[
\cos \varphi = \frac{A}{B},
\]
where
\[
B = 2cf = f^2 \left[ \frac{\sin \gamma - \sin \alpha}{\sin(\alpha + \gamma)} + \frac{\sin \beta + \sin \gamma}{\sin(\beta + \gamma)} \right],
\]

\[
A = f^2 + c^2 - \frac{f^2}{4} \left[ \frac{\sin \alpha - \sin \gamma}{\sin(\alpha + \gamma)} + \frac{\sin \beta + \sin \gamma}{\sin(\beta + \gamma)} \right]^2
\]
\[
= f^2 \left[ 1 + \frac{1}{4} \left[ \frac{\sin \gamma - \sin \alpha}{\sin(\alpha + \gamma)} + \frac{\sin \beta + \sin \gamma}{\sin(\beta + \gamma)} \right]^2 \right] - \frac{1}{4} \left[ \frac{\sin \alpha - \sin \gamma}{\sin(\alpha + \gamma)} + \frac{\sin \beta + \sin \gamma}{\sin(\beta + \gamma)} \right]^2 \right].
\]

Thus,
\[
B \frac{\sin(\alpha + \gamma) \sin(\beta + \gamma)}{f^2} = \sin(\beta + \gamma)(\sin \gamma - \sin \alpha) + \sin(\alpha + \gamma)(\sin \beta + \sin \gamma),
\]
\[
A \frac{\sin(\alpha + \gamma) \sin(\beta + \gamma)}{f^2} = \sin(\alpha + \gamma) \sin(\beta + \gamma) + (\sin \gamma - \sin \alpha)(\sin \gamma + \sin \beta).
\]

After some algebra one gets
\[
B \frac{\sin(\alpha + \gamma) \sin(\beta + \gamma)}{f^2} = 2 \sin \gamma \cos \frac{\alpha - \beta}{2} \left[ \sin \left( \frac{\alpha + \beta}{2} \right) + \sin \frac{\beta - \alpha}{2} \right],
\]
\[
A \frac{\sin(\alpha + \gamma) \sin(\beta + \gamma)}{f^2} = 2 \sin \gamma \cos \frac{\alpha + \beta}{2} \left[ \sin \left( \frac{\alpha + \beta}{2} \right) + \sin \frac{\beta - \alpha}{2} \right],
\]

and therefore,
\[
\cos \varphi = \frac{\cos \frac{\alpha + \beta}{2}}{\cos \frac{\alpha - \beta}{2}}.
\]
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