Abstract. Shape theory works nice for (Hausdorff) paracompact spaces, but for spaces with no separation axioms, it seems to be quite poor. However, for finite and locally finite spaces their weak homotopy type is rather rich, and is equivalent to the weak homotopy type of finite and locally finite polynedra, respectively. In the paper there is proposed a variant of shape theory called quasi-shape, which suits both paracompact and locally finite spaces, i.e. the quasi-shape is isomorphic to the weak homotopy type for locally finite spaces, and is \( \natural \)-equivalent to the ordinary shape in the case of paracompact spaces.

1. Main construction

1.1. The connected component functor \( \pi \). We need an appropriate definition of

\[
\pi : \text{TOP} \to \text{SETS}
\]

where \( \text{TOP} \) and \( \text{SETS} \) are the categories of topological spaces and sets, respectively. Neither the usual functor \( \pi_0 \) (the set of pathwise connected components) nor \( \pi'_0 \) (the set of connected components) is suitable for our purposes. We will introduce instead the following functor

\[
\pi : \text{TOP} \to \text{pro-SETS}:
\]

\[
\pi(X)_\mathcal{U} := \mathcal{U}
\]

for any open partition of \( X \) (i.e., a partition into open subsets). We say that \( \mathcal{U} \leq \mathcal{V} \) if \( \mathcal{V} \) refines \( \mathcal{U} \). The set \( \text{Part}(X) \) of all open partitions of \( X \) is clearly directed, and we obtain an inverse system of sets by defining

\[
p_{\mathcal{U} \leq \mathcal{V}} : \pi(X)_\mathcal{V} \to \pi(X)_\mathcal{U}
\]

where

\[
p_{\mathcal{U} \leq \mathcal{V}}(Y) = Y', \\
Y \in \mathcal{V}, \\
Y' \in \mathcal{U},
\]

and \( Y' \) is the unique element of \( \mathcal{U} \), containing \( Y \).

Let now

\[
f : X \to Y
\]
be a continuous mapping. Define
\[ \pi(f) : \pi(X) \rightarrow \pi(Y) \]
by the following. Let \( U \) be an open partition of \( Y \), and let
\[ \mathcal{V} = \xi(U) = \xi_f(U) := \{ Y' = f^{-1}(Y) : Y \in U, Y' \neq \emptyset \} . \]
\( \mathcal{V} \) is clearly an open partition of \( X \), and we have just defined a mapping
\[ \xi = \xi_f : Part(Y) \rightarrow Part(X) . \]
There can be defined also a mapping
\[ f_U : \pi(X)_\mathcal{V} \rightarrow \pi(Y)_U \]
by
\[ f_U(Y') := Y \]
where
\[ \emptyset \neq Y' = f^{-1}(Y) . \]
It can be easily checked that the pair
\[ (\xi_f : Part(Y) \rightarrow Part(X), (f_U : \pi(X)_{\xi_f(\mathcal{V})} \rightarrow \pi(Y)_U : U \in Part(Y))) \]
gives a well-defined morphism
\[ \pi(f) : \pi(X) \rightarrow \pi(Y) \]
in the category pro-SETS, and the correspondence \( f \mapsto \pi(f) \) defines a functor
\[ \pi : TOP \rightarrow pro-SETS. \]

**Proposition 1.1.** Let \( X \) be a locally connected space. Then \( \pi(X) \) is isomorphic in the category pro-SETS to the set \( \pi'_0(X) \) of connected components of \( X \).

**Proof.** The set \( \pi'_0(X) \) is an open partition of \( X \) which refines any other open partition. Therefore, \( Part(X) \) has a maximal element \( \pi'_0(X) \), and \( \pi(X) \) is isomorphic to the trivial pro-set \( \pi'_0(X) \) indexed by a one-point index set, i.e. to the set \( \pi'_0(X) \). \( \square \)

1.2. **Quasi-shape.** Let \( Cov(X) \) be the set of open coverings on \( X \), pre-ordered by the refinement relation. Analogously to \( Part(X) \), \( Cov(X) \) is a directed pre-ordered set, while \( Part(X) \) is a directed ordered set. Let
\[ U = (U_*, d_*, s_*) \]
be a hypercovering on \( X \) (see [AM86], Definition 8.4), i.e. a simplicial space with an augmentation
\[ \varepsilon : U \rightarrow X , \]
and the following properties:

**Hyper**\(_0\):
\[ \varepsilon_0 : U_0 \rightarrow X \]
is an open covering;

**Hyper**\(_n\):
\[ U_{n+1} \rightarrow (\text{Cosk}_n U_*)_n \]
are open coverings, \( n \geq 0 \).
If \( \mathcal{U} \) is an open covering, one can define the corresponding Čech hypercovering by

\[
U_n = \coprod_{U_i \in \mathcal{U}} (U_0 \cap U_1 \cap ... \cap U_n)
\]

with the evident face \((d_*)\) and degeneracy \((s_*)\) mappings, where \( \coprod \) is the coproduct in the category of topological spaces. For the Čech hypercovering, the mappings

\[
U_{n+1} \longrightarrow (\cosk_n U_n)_{n+1}, n \geq 0,
\]

are homeomorphisms.

**Remark 1.2.** The Čech hypercoverings are used in the definition of ordinary shape of a topological space, see [Mar00].

**Definition 1.3.** Let \( X \) be a topological space. The shape of \( X \) is the following pro-space. Given a normal (i.e. admitting a partition of unity) covering \( \mathcal{U} \), let \( N\mathcal{U} \) (the Čech nerve of \( \mathcal{U} \)) be a simplicial set with

\[
(N\mathcal{U})_n = \{(U_0, U_1, ..., U_n) : (U_i \in \mathcal{U}) \& (U_0 \cap U_1 \cap ... \cap U_n \neq \emptyset)\}
\]

with the evident face \((d_*)\) and degeneracy \((s_*)\) mappings. If \( \mathcal{V} \) refines \( \mathcal{U} \), there exists a unique (up to homotopy) mapping

\[
p_{\mathcal{U} \leq \mathcal{V}} : N\mathcal{V} \longrightarrow N\mathcal{U}.
\]

The correspondence

\[
\Upsilon \mapsto |N\Upsilon|
\]

where \( \Upsilon \) runs over all normal coverings on \( X \), and \( |N\Upsilon| \) is the geometric realization of \( N\Upsilon \), defines an object \( SH(X) \) in \( \text{pro-} H(\text{TOP}) \) which is called the shape of \( X \).

Let \( HCov(X) \) be the following category: the objects are hypercoverings on \( X \), and the morphisms from \( \mathcal{U} \) to \( \mathcal{V} \) are homotopy classes of simplicial mappings

\[
\Upsilon \longrightarrow \mathcal{V}.
\]

This category is co-filtering. Given a hypercovering \( \Upsilon \), let

\[
\Gamma (\Upsilon, \pi) = |\pi(\Upsilon)|
\]

where \( |\pi(\Upsilon)| \) is the geometric realization of the simplicial pro-set \( \pi(\Upsilon) \). Varying \( \Upsilon \), one gets an object

\[
\Upsilon \mapsto |\pi(\Upsilon)|
\]

in \( \text{pro-} H(\text{pro-TOP}) \). Finally, applying the canonical functor

\[
\text{pro-} H(\text{pro-TOP}) \longrightarrow \text{pro-} (\text{pro-} H(\text{TOP})) \longrightarrow \text{pro-} H(\text{TOP}),
\]

one gets an object \( QSH(X) \) in \( \text{pro-} H(\text{TOP}) \) which will be called the quasi-shape of \( X \).

**Theorem 1.4.** The correspondence above gives a well-defined functor

\[
QSH : \text{TOP} \longrightarrow \text{pro-} H(\text{TOP}),
\]

which factors through the homotopy category \( H(\text{TOP}) \):

\[
QSH : \text{TOP} \longrightarrow H(\text{TOP}) \longrightarrow \text{pro-} H(\text{TOP}).
\]

**Remark 1.5.** The functor from \( H(\text{TOP}) \) to \( \text{pro-} H(\text{TOP}) \) will be denoted \( QSH \) as well.
2. Comparison

Let \( X \) be a locally finite space (see [McC66], p. 466). It means that every point has a finite neighborhood. Due to [McC66], Theorem 2, there exists a simplicial set \( K(X) \), functorially dependent on \( X \), and a weak homotopy equivalence

\[ |K(X)| \longrightarrow X. \]

Let us consider the functor above as a functor to \( \text{pro-} H(\text{TOP}) \):

\[ X \longmapsto |K(X)| : LF-\text{TOP} \longrightarrow \text{TOP} \subseteq \text{pro-} H(\text{TOP}) \]

where \( LF-\text{TOP} \) is the full subcategory of locally finite spaces.

**Example 2.1.** Let \( X \) be a so called 4-point circle, i.e. a space with four points \( \{a, b, c, d\} \) and the following topology

\[ \tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b, c\}, \{a, d, c\}, \{a, c\}\}. \]

Then \( |K(X)| \) is homeomorphic to an ordinary circle \( S^1 \).

**Theorem 2.2.** On the category

\[ LF-\text{TOP} \subseteq \text{TOP}, \]

there exists a natural isomorphism

\[ |K(X)| \approx QSH(X). \]

**Remark 2.3.** The shape of a locally finite (even a finite) space differs significantly from \( |K(X)| \). Say, the space from Example 2.1 has the shape of a point.

Let now \( X \) be a Hausdorff paracompact space. We will simply call such spaces paracompact. Remind that a \( \natural \)-equivalence between pro-spaces is a mapping

\[ f : X \longrightarrow Y \]

in \( \text{pro-} H(\text{TOP}) \) inducing an isomorphism of pro-sets

\[ \pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y), \]

and isomorphisms of pro-groups

\[ \pi_n(f) : \pi_n(X, f^{-1}(y)) \longrightarrow \pi_n(Y, y), n \geq 1, \]

for any point \( y \rightarrow Y \). It is known [AM86] that the canonical morphism

\[ X_\alpha \longrightarrow (\text{Cosk}_\alpha X_\alpha) \]

is a \( \natural \)-equivalence between pro-spaces

\[ X \longrightarrow \text{Cosk}(X). \]

**Theorem 2.4.** Let \( X \) be a paracompact space. Then \( QSH(X) \) is naturally \( \natural \)-equivalent to the ordinary shape \( SH(X) \) of \( X \).
3. Proofs

3.1. Proof of Theorem 1.4

Proof. The crucial step is the following. Given two homotopic mappings
\[ f, g : X \rightarrow Y, \]
the corresponding morphisms:
\[ QSH(f) = QSH(g) : QSH(X) \rightarrow QSH(Y) \]
are equal in the category \( \text{pro-H(TOP)} \). This, in turn is proved using compactness of the unit interval and the technique of Proposition (8.11) from [AM86]: given a hypercovering \( U \) on \( Y \), one constructs a sequence of truncated hypercoverings on \( X \), resulting in a hypercovering \( V \) on \( X \), which refines both \( f^{-1}(U) \) and \( g^{-1}(U) \), and such that the corresponding morphisms
\[ \Gamma(V, \pi) \rightarrow \Gamma(U, \pi) \]
are equal in the category \( \text{pro-H(TOP)} \). \( \square \)

3.2. Proof of Theorem 2.2

Proof. Introduce the following pre-order on \( X \) (see [McC66], p. 468):
\[ x \leq y \iff V_y \subseteq V_x \]
where \( V_x \) is the minimal (finite) open neighborhood of \( x \). Let now \( U \) be the following hypercovering:
\[ U_n = \coprod_{x_0 \leq x_1 \leq \ldots \leq x_n} V_{x_n} \]
with the evident face and degeneracy mappings. This hypercovering is clearly an initial object in the category \( HCov(X) \). All spaces \( V_x \) are connected, therefore, for each \( n \), \( \pi(U_n) \) is a set (i.e. a trivial pro-set). Finally, \( QSH(X) \) is a space (i.e. a trivial pro-space) \( |K(X)| \) where \( K(X) \) is the following simplicial set:
\[ K(X)_n = \{x_0 \leq x_1 \leq \ldots \leq x_n\}. \]
The latter simplicial set is exactly the simplicial set \( \mathcal{K}(X) \) from [McC66], Theorem 2. It follows that
\[ QSH(X) \simeq |\mathcal{K}(X)| \]
(homotopy equivalent) while
\[ |\mathcal{K}(X)|_{\text{weak}} \approx X \]
(weak homotopy equivalent). \( \square \)

3.3. Proof of Theorem 2.4

Proof. There exists [AM86] a natural \( \sharp \)-equivalence
\[ QSH(X) \rightarrow \text{Cosk}(QSH(X)). \]
Let now construct a homotopy equivalence
\[ Sh(X) \rightarrow \text{Cosk}(QSH(X)). \]
Let \( U \in HCov(X) \), let \( n \in \mathbb{N} \) and let
\[ V = \text{Cosk}_n(QSH(X)) = \text{Cosk}_n(\Gamma(U, \pi)). \]
Consider the following open covering $\mathcal{U}$ on $X$:

$$\mathcal{U} = (d_0)^n : U_n \rightarrow X.$$ 

Let us now consider the open partitions $\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_n$, of $U_0, U_1, \ldots, U_n$, involved in the construction of pro-sets $\pi(U_0), \pi(U_1), \ldots, \pi(U_n)$. Finally, since $X$ is paracompact, there exists a normal open covering $\mathcal{V}$ on $X$, refining $\mathcal{U}$ and all coverings $(d_0)^i W_i$, $i = 0, 1, \ldots, n$.

Denote the correspondence $(U, n, W_i) \mapsto \mathcal{V}$ by

$$\xi(U, n, W_i) = \mathcal{V}.$$ 

Given $V \in \mathcal{V}$, there exist unique elements $W_i$ from $\mathcal{W}_i$ such that

$$V \subseteq (d_0)^i W_i.$$ 

This gives a well-defined mapping from the Čech nerve

$$\varphi(U, n, W_i) : NV \rightarrow \text{Cosk}_n(\Gamma(U, \pi)).$$ 

Finally, the pair $(\xi, \varphi)$ gives the desired equivalence

$$SH(X) \rightarrow \text{Cosk}(QSH(X))$$ 

in $pro-H(TOP)$.

\[\square\]

**References**

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Institute of Maths & Stats, Univ. of Tromsø, Norway

E-mail address: andrei.prasolov@uit.no

URL: http://www.math.uit.no/users/andreip/Welcome.html