The group of Hamiltonian automorphisms of a star product

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Abstract

We deform the group of Hamiltonian diffeomorphisms into a group of Hamiltonian automorphisms, $\text{Ham}(M, \ast)$, of a formal star product $\ast$ on a symplectic manifold $(M, \omega)$. We study the geometry of that group and deform the Flux morphism in the framework of deformation quantization.

Keywords: Deformation quantization, Automorphisms of star product, Flux morphism, Hamiltonian automorphisms group.

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1 Introduction

We define the group of Hamiltonian automorphisms of a star product. It is a deformation of the group of Hamiltonian diffeomorphisms of a symplectic manifold in the framework of formal deformation quantization. We also study the geometric properties of this group.
The group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms is a normal subgroup of $\text{Symp}_0(M, \omega)$ the connected component of the group of all symplectomorphisms of a symplectic manifold $(M, \omega)$. Banyaga [1] showed that the group $\text{Ham}(M, \omega)$ is the kernel of a morphism defined on $\text{Symp}_0(M, \omega)$ with values in $H^1_{dR,c}(M)/\Gamma(M, \omega)$ where $H^1_{dR,c}(M)$ is the first de Rham cohomology group with compact support and $\Gamma(M, \omega)$ is the so-called Flux group. He used this characterisation to prove that the group $\text{Ham}(M, \omega)$ is simple when the manifold is closed, extending the results of Thurston on volume preserving diffeomorphisms to the symplectic case. Observing that the group $\Gamma(M, \omega)$ is at most countable, he proved that the Lie algebra of $\text{Ham}(M, \omega)$ is the space of compactly supported Hamiltonian vector fields. In 2006, Ono [13] showed that $\Gamma(M, \omega)$ is discrete when the manifold $(M, \omega)$ is closed; this proved the famous Flux conjecture which states that $\text{Ham}(M, \omega)$ is $C^1$-closed in $\text{Symp}_0(M, \omega)$.

A similar approach in the framework of deformation quantization ($*$ product) on a symplectic manifold leads as a first step to the study of the group of Hamiltonian automorphisms of a star product.

To avoid technical difficulties we will assume throughout the paper that $(M, \omega)$ is a closed symplectic manifold. Let $*$ be a star product on $(M, \omega)$. Hamiltonian automorphisms are the solutions of the Heisenberg equation

$$\frac{d}{dt} A^H_t = D_{H_t} A^H_t := \frac{1}{\nu} [H_t, A^H_t], \quad \text{with initial condition } A^H_0 = \text{Id}, \quad (1)$$

where $D_{H_t}$ is a smooth family of quasi-inner derivations. We then set

$$\text{Ham}(M, *) := \{ A = A^H_t \text{ for such } D_{H_t} \}. \quad (2)$$

Our first observation is that $\text{Ham}(M, *)$ is a normal subgroup of $\text{Auto}_0(M, *)$ the group of automorphisms of the star product deforming $\text{Symp}_0(M, \omega)$. The group $\text{Ham}(M, *)$ comes with an anti-epimorphism $\text{Cl} : \text{Ham}(M, *) \rightarrow \text{Ham}(M, \omega)$.

We prove that $\text{Ham}(M, *)$ is the kernel of a morphism defined on $\text{Auto}_0(M, *).

More precisely, we define a formal version of the flux morphism denoted by $\text{Flux}^*$ and we obtain :

**Theorem 1.** There is a short exact sequence of groups

$$1 \rightarrow \text{Ham}(M, *) \rightarrow \text{Auto}_0(M, *) \xrightarrow{\mathcal{F}} \frac{H^1_{dR}(M)[[\nu]]}{\Gamma(M, *)} \rightarrow 1,$$

where $\mathcal{F}(A) := \text{Flux}^*((A))$ for any smooth path in $\text{Auto}_0(M, *)$ joining $A$ to the identity and $\Gamma(M, *) := \text{Flux}^*(\pi_1(\text{Auto}_0(M, *)))$ where $\pi_1(\text{Auto}_0(M, *))$ is the subgroup of $\text{Auto}_0(M, *)$ consisting of classes of smooth loops of automorphisms.

Next, we observe that $\Gamma(M, *)$ is the image of a morphism

$$\text{Flux}_{def}^* : \pi_1(\text{Symp}_0(M, \omega)) \rightarrow H^1_{dR}(M)[[\nu]].$$

The values of $\text{Flux}_{def}^*$ only depend on the equivalence class of the star product. We give a condition on the group $\text{Flux}_{def}^*(\pi_1(\text{Ham}(M, \omega)))$ which ensures that the Lie algebra of $\text{Ham}(M, *)$ is the space of quasi-inner derivations. We gives examples of this situation.

In section 4 using the Fedosov construction of star product, we give an explicit expression of the deformed flux in terms of the characteristic 2-from.
parametrizing the star product. This works on nice elements of \( \pi_1(\text{Symp}_0(M, \omega)) \). Consider \(*_{\Omega, \nabla}\) a Fedosov’s star product obtained with the help of a symplectic connection \( \nabla \) and a series of closed 2-forms \( \Omega \), we obtain:

**Theorem 2.** Let \( \{ \varphi_t \} \) be a loop of symplectomorphisms generated by the symplectic vector field \( X_t \) such that \( \varphi_t^* \Omega = \Omega \) and \( \varphi_t^* \nabla = \nabla \). Then, the deformed flux of \( \{ \varphi_t \} \) defined with the star product \(*_{\Omega, \nabla}\) is

\[
\text{Flux}^{*_{\Omega, \nabla}}_{\text{def}}(\{ \varphi_t \}) = \int_0^1 \left[ i(X_t)\omega \right] dt - \left[ \int_0^1 \varphi_t^* i(X_t)\Omega dt \right].
\]

(3)

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### 2 Star products, derivations and automorphisms

In this section, we recall the definitions and basic properties of star products on symplectic manifolds that we need in this paper.

Let \((M, \omega)\) be a symplectic manifold. The space \( C^\infty(M) \) of real valued smooth functions is naturally endowed with a Poisson bracket:

\[
\{ F, G \} = -\omega(X_F, X_G), \quad \forall F, G \in C^\infty(M)
\]

where \( X_F \) is a Hamiltonian vector field on \( M \), that is \( i(X_F)\omega := df \).

A **star product** on \((M, \omega)\) is a \( R[[\nu]]\)-bilinear associative law on the space \( C^\infty(M)[[\nu]] \) of formal series of smooth functions:

\[
*: (C^\infty(M)[[\nu]])^2 \rightarrow C^\infty(M)[[\nu]], \quad (H, K) \mapsto H * K := \sum_{r=0}^{\infty} \nu^r C_r(H, K)
\]

where the \( C_r \)'s are bidifferential operators null on constants such that for all \( H, K \in C^\infty(M)[[\nu]] \):

\[
C_0(H, K) = HK \quad \text{and} \quad C_1(H, K) - C_1(K, H) = \{ H, K \}.
\]

It is a result of De Wilde-Lecomte [5], Fedosov [6] and Omori-Yoshioka-Maeda [12] that all symplectic manifolds admit a star product. Two star products \(*\) and \(*'\) are **equivalent** if there exists a formal power series \( T \) of \( R[[\nu]]\)-linear differential operators

\[
T = Id + \sum_{r=1}^{\infty} \nu^r T_r : C^\infty(M)[[\nu]] \rightarrow C^\infty(M)[[\nu]]
\]

such that for all \( F, G \in C^\infty(M)[[\nu]] \), we have:

\[
T(F * G) = TF *' TG, \quad \text{Star products on symplectic manifolds are classified up to equivalence by } H^2_{dR}(M)[[\nu]],
\]

see for example [4].
2.1 Derivations

At the classical level, a derivation of the Poisson algebra of a symplectic manifold is a symplectic vector field, that is, a vector field $X$ on $M$ such that $L_X\omega = 0$.

Now, we fix a star product $*$ on the symplectic manifold $(M,\omega)$. A derivation of the star product is a $\mathbb{R}[[\nu]]$-linear map $D : C^\infty(M)[[\nu]] \to C^\infty(M)[[\nu]]$, such that

$$D(F \ast G) = DF \ast G + F \ast DG.$$ 

We denote by $\text{Der}(M,*)$ the space of derivations. It is a Lie algebra for the commutator of derivations.

A derivation is called quasi-inner if it is of the form

$$D_H(F) := \frac{1}{\nu}[H,F]_*,$$

for all $F \in C^\infty(M)[[\nu]]$, for some $H \in C^\infty(M)[[\nu]]$. We denote by $\text{qInn}(M,*)$ the space of quasi-inner derivations. It is an ideal of $\text{Der}(M,*)$.

Derivations of the star product are in bijection with formal series of symplectic vector fields on $M$, the last space will be denoted by $\mathfrak{S}ymp(M,\omega)[[\nu]]$:

$$p : \text{Der}(M,*) \to \mathfrak{S}ymp(M,\omega)[[\nu]] : D \mapsto p(D),$$

such that on a contractible open set $U$:

$$i(p(D))\omega|_U = dH_U \text{ with } DF|_U = \frac{1}{\nu}[H_U,F]_*.$$

for some $H_U \in C^\infty(U)[[\nu]]$.

**Proposition 2.1.** Let $(M,\omega)$ be a symplectic manifold endowed with a star product $*$. Then,

1. $p(D) = D_0 + \nu(\ldots)$ where $D = \sum_{i=0}^{\infty} \nu^i D_i \in \text{Der}(M,*)$. Moreover, $D = D_F \in \text{qInn}(M,*)$ for $F \in C^\infty(M)[[\nu]]$ if and only if $p(D) = X_F$ with $i(X_F)\omega = dF$ formally in $\nu$.

2. Let $D,D' \in \text{Der}(M,*)$, then $[D,D'] \in \text{qInn}(M,*)$.

3. If $D_F$ and $D_G$ are quasi-inner derivations, we have $[D_F,D_G] = D_{\frac{1}{2}[F,G]}$.

2.2 Automorphisms

From now on, we will assume that the symplectic manifold $(M,\omega)$ is closed.

At the classical level, smooth one parameter families of symplectic vector fields integrate to families of symplectomorphisms. Recall that a symplectomorphism is a diffeomorphism $\phi : M \to M$ such that $\phi^*\omega = \omega$.

We indicate below how to “exponentiate” a family of derivations into a family of automorphisms. This is analogous to what have been done in [7] and [16].

An automorphism of the star product is a $\mathbb{R}[[\nu]]$-linear bijection $A : C^\infty(M)[[\nu]] \to C^\infty(M)[[\nu]]$, such that

$$A(F \ast G) = AF \ast AG.$$

The group $\text{Aut}(M,*)$ of all automorphisms of the star product projects onto the group of symplectomorphisms, denoted by $\text{Symp}(M,\omega)$. Indeed, if $A =$
\[
\sum_{r=0}^{\infty} \nu^r A_r \in \text{Aut}(M,\ast), \text{ then } A_0 := \varphi^* \text{ for some } \varphi \in \text{Symp}(M,\omega). \text{ Hence, the map classical limit defined by } \]
\[
\text{Cl} : \text{Aut}(M,\ast) \to \text{Symp}(M,\omega) : A \mapsto \varphi
\]
is an anti-homomorphism of group. If \( A \in \text{Cl}^{-1}(\text{Id}) \), then there exists \( D \in \nu \text{Der}(M,\ast) \) such that \( A = \exp(D) \).

**Definition 2.2.** The subgroup \( \text{Aut}_0(M,\ast) \) of \( \text{Aut}(M,\ast) \), is defined to be \( \text{Cl}^{-1}(\text{Symp}_0(M,\omega)) \), where \( \text{Symp}_0(M,\omega) \) is the identity component (for the compact-open \( C^\infty \) topology) of the group of symplectic diffeomorphisms.

**Definition 2.3.** Let \( I \) be an interval in \( \mathbb{R} \). A one-parameter family of derivations \( D_t = \sum_{r=0}^{\infty} \nu^r D_{r,t} \in \text{Der}_0(M,\ast) \) for \( t \in I \) is called smooth if for all \( F \in C^\infty(M) \) we have \( D_t(F) \in C^\infty(I \times M)[[\nu]] \).

**Remark 2.4.** Using the bijection \( p : \text{Der}(M,\ast) \to \text{Symp}(M,\omega)[[\nu]] \) defined in the above Subsection 2.1, one sees that a one-parameter family of derivations \( D_t \) is smooth if the coefficients of \( p(D_t) \) are smooth one parameter families of symplectic vector fields, where \( \text{Symp}(M,\omega) \) is endowed with the compact-open \( C^\infty \) topology.

**Proposition 2.5.** Let \( D_t = \sum_{r=0}^{\infty} \nu^r D_{r,t} \) be a smooth one-parameter family of derivations. Then there exists a unique family of automorphisms \( t \mapsto A_t \) defined for all \( t \) such that
\[
\frac{d}{dt} A_t H = D_t A_t H, \quad \forall H \in C^\infty(M)[[\nu]],
\]
with the initial condition \( A_0 H = H \) for all \( H \in C^\infty(M)[[\nu]] \).

Moreover, if the derivation \( D \) does not depend on the time \( t \), then the family \( A_t \) is a one-parameter subgroup of automorphisms and \( D \circ A_t = A_t \circ D \).

Finally, when \( D_t \in \nu \text{Der}_0(M,\ast) \), then the solution \( A_t \) of the equation (3) is the formal exponential \( A_t = \exp(\int_0^t D_s ds) \), the integral \( \int_0^t D_s ds \) is the derivation \( F \mapsto \int_0^t D_s(F) ds \).

**Proof.** Let \( H \in C^\infty(M)[[\nu]] \), we will show that there exists a unique family \( H(t) \in C^\infty(M)[[\nu]] \) such that
\[
\frac{d}{dt} H(t) = D_t H(t)
\]
and \( H(0) = H \).

Let \( \varphi_t \) be the one-parameter family of symplectomorphisms generated by \(-D_{t,0} \), the opposite of the zeroth order term of \( D_t \). It means \( \frac{d}{dt} \varphi_t = -\varphi_t^* D_{t,0} \) and \( \varphi_0 = \text{Id} \). Then, if \( H(t) \) satisfies (5), we have
\[
\frac{d}{dt} \varphi_t^* H(t) = -\varphi_t^* D_{t,0} H(t) + \varphi_t^* D_t H(t) = \varphi_t^* \tilde{D}_t H(t). \]
where \( \tilde{D}_t := D_t - D_{t,0} \). After integration with respect to \( t \), we get
\[
\varphi_t^* H(t) = H + \int_0^t \varphi_s^* \tilde{D}_s H(s) ds.
\]
Now the equation (7) can be solved by induction on the degree in $\nu$ and the solution is unique. Set $A_t : C^\infty(M)[[\nu]] \to C^\infty(M)[[\nu]] : H \mapsto H(t)$.

It remains to show that $A_t$ is an automorphism of star product. For this, consider the two expressions $A_t H \ast A_t K$ and $A_t (H \ast K)$. They are equal for $t = 0$ and are both solutions of equation (5) for all $t$. Then, by uniqueness of the solutions of the equation (5), we have $A_t H \ast A_t K = A_t (H \ast K)$.

The fact that $A_t$ is a one-parameter subgroup when the derivation is autonomous is again a consequence of the uniqueness of $A_t$ (see Remark 2.6).

The last statement is checked by differentiating $\exp[\int_0^t D_s ds] H$ for $H \in C^\infty(M)[[\nu]]$.

**Remark 2.6.** The solution $A_t$ of the equation (3) starts at order 0 in $\nu$ by $(\varphi_t^{-1})^*$, where $\varphi_t$ is the flow of $-D_{t,0}$. In general, $\varphi_t^{-1}$ is NOT the flow of $D_{t,0}$.

**Definition 2.7.** Let $I$ be an interval in $\mathbb{R}$. A one-parameter family of automorphisms $A_t = \varphi_t^* + \sum_{r=1}^\infty \nu^r A_t$ for $t \in I$ is called smooth if for all $F \in C^\infty(M)$ we have $A_t(F) \in C^\infty(I \times M)[[\nu]]$.

**Remark 2.8.** Using the Weinstein tubular neighbourhood, one defines a chart $W : U \subset \text{Symp}_0(M,\omega) \to V \subset \text{Symp}(M,\omega)$ from a neighbourhood $U$ of 0 in $\text{Symp}_0(M,\omega)$ and a neighbourhood $V$ of 0 in $\text{Symp}(M,\omega)$. Together with Proposition 2.5, one sees that $\text{Cl}^{-1}(U)$ is in bijection with $W(U) \times \nu \text{Der}(M,\ast)$ or equivalently with $W(U) + \nu \text{Symp}(M,\Omega)[[\nu]]$. A one-parameter family of automorphisms $A_t$ in $\text{Cl}^{-1}(U)$ is smooth if the coefficients of its image in $W(U) + \nu \text{Symp}(M,\Omega)[[\nu]]$ are smooth one-parameter families of symplectic vector fields.

**Corollary 2.9.** If $D_t$ is a smooth one parameter family of derivation, then the solution $A_t$ of Equation (4) is smooth.

Given $D_t$ a smooth family of derivations, we say that the solution path $A_t$ of the equation (4) is generated by $D_t$. One has:

**Proposition 2.10** (Computation rules). Let $A_t$ and $A'_t$ be smooth paths of automorphisms generated by $D_t$ and $D'_t \in \text{Der}(M,\ast)$. Then,

1. the path $A_t A'_t$ is generated by the derivation $D_t + A_t D'_t A_t^{-1}$,
2. the path $A_t^{-1}$ is generated by the derivation $-A_t^{-1} D_t A_t$,
3. we have $A_t D'_t A_t^{-1} = D'_t + D_{F_t}$ for a family $F_t \in C^\infty(M)[[\nu]]$.
4. Let $D, D' \in \nu \text{Der}(M,\ast)$, then $\exp(D) \exp(D') = \exp(D + D' + D_F)$ for some $F \in C^\infty(M)[[\nu]]$.

**Proof.** To prove point 1, it suffices to differentiate the path $A_t A'_t$. Indeed,

$$\frac{d}{dt} A_t A'_t = (D_t + A_t D'_t A_t^{-1}) \circ A_t A'_t.$$

Point 2 is obtained by applying 1 to the path $Id = A_t A_t^{-1}$.

For point 3, we compute $A_t D'_t (A_t)^{-1} = D'_t + \int_0^t \frac{d}{ds} A_t D'_t (A_s)^{-1} ds$. Applying point 2, we compute $\frac{d}{ds} A_t D'_t (A_s)^{-1} ds = [D_s, A_t D'_t (A_s)^{-1}]$. Now, the commutator of two derivations is quasi-inner (by Proposition 2.5).

Point 4 is obtained by applying points 1 and 3 to the path $\exp(tD) \exp(tD')$. \qed
3 The group of Hamiltonian automorphisms

We integrate the Lie algebra $\mathfrak{qlnn}(M,*)$ of quasi-inner derivations to produce the group $\text{Ham}(M,*)$.

Consider smooth one-parameter families of derivations of the form

$$D_{H_t} := \frac{1}{t} [H_t, ]_*, \in \mathfrak{qlnn}(M,*) .$$

By Proposition 2.5 there exists a one-parameter family of automorphisms $A_t^H$ such that $\frac{d}{dt} A_t^H = D_{H_t} A_t^H$. We say that $A_t^H$ is generated by the time-dependent Hamiltonian $H_t$.

**Definition 3.1.** The set of Hamiltonian automorphisms is the set

$$\text{Ham}(M,*) := \{A \in \text{Aut}(M,*) \mid A = A_t^H \text{ for such } D_{H_t} \in \mathfrak{qlnn}(M,*) \}. \quad (8)$$

The map $\text{Cl}$ restricts to a surjection $\text{Cl} : \text{Ham}(M,*) \mapsto \text{Ham}(M,\omega)$.

**Lemma 3.2.** For all $A \in \text{Ham}(M,*)$, $D_G \in \mathfrak{qlnn}(M,*) : AD_G A^{-1} = D_{AG}$.

**Proof.** The proof is a direct computation.

**Theorem 3.3.** Let $*$ be a star product on a symplectic manifold $(M,\omega)$, then $\text{Ham}(M,*)$ is a normal subgroup of $\text{Aut}_0(M,*)$.

There is an anti-epimorphism $\text{Cl} : \text{Ham}(M,*) \mapsto \text{Ham}(M,\omega)$.

**Proof.** Let $A, B \in \text{Ham}(M,*)$, we show that $AB \in \text{Ham}(M,*)$. Write $A_t^H$ and $B_t^G$, the one-parameter families generated by $H_t$ and $G_t$ respectively such that $A = A_1^H$ and $B = B_1^G$. Using Proposition 2.10 and Lemma 3.2 we see that $A_1^H B_1^G$ is generated by the Hamiltonian $K_t := H_t + A_1^H G_t$. So, $AB = A_1^H B_1^G$ is in $\text{Ham}(M,*)$.

Let $A \in \text{Ham}(M,*)$. Since $A = A_1^H$ for some $D_{H_1} \in \mathfrak{qlnn}(M,*)$, then $A^{-1} = (A_1^H)^{-1}$. Using the computation rules 2.10 and Lemma 3.2 we know that $(A_1^H)^{-1}$ is generated by $-(A_1^H)^{-1} H_t$.

The fact that $\text{Ham}(M,*)$ is a normal subgroup of $\text{Aut}_0(M,*)$ is a consequence of the following identities. Let $A \in \text{Aut}_0(M,*)$ and $A_t^H$ the family of Hamiltonian automorphisms generated by $H_t \in C^\infty(M)[[\nu]]$ then

$$\frac{d}{dt} AA_t^H A^{-1} = AD_{H_t} A_t^H A^{-1} = D_{AH_t} AA_t^H A^{-1}. \quad (9)$$

We immediately have that the projection $\text{Cl}$ is an anti-epimorphism of group.

**Proposition 3.4.** Let $*$ and $*'$ be two equivalent star products on $C^\infty(M)[[\nu]]$. Denote by $T : (C^\infty(M)[[\nu]],*) \rightarrow (C^\infty(M)[[\nu]],*')$ an equivalence of star product. Then the map $C_T : \text{Ham}(M,*) \mapsto \text{Ham}(M,*') : A \mapsto TAT^{-1}$ is an isomorphism of group.

**Proof.** Let $A_t^H \in \text{Ham}(M,*)$ generated by $H_t \in C^\infty(M)[[\nu]]$, then $C_T(A_t^H) = T A_t^H T^{-1}$ is generated by $TH_t$. So, $C_T(A_t^H) \in \text{Ham}(M,*')$. The map $C_T$ is clearly invertible and it is a morphism of group.
4 The formal flux morphism

The goal of this section is to describe \( \text{Ham}(M, \ast) \) as the kernel of a morphism on \( \text{Aut}_0(M, \ast) \); as in the classical case, where the group of Hamiltonian diffeomorphisms is the kernel of the flux morphism [1].

At the level of Lie algebras of derivations. The algebra \( \mathfrak{q} \text{Im}(M, \ast) \) is the kernel of the epimorphism

\[
F : D \in \text{Der}_0(M, \ast) \mapsto [i(p(D))\omega] \in H^1_{\text{dR}}(M)[[\nu]]
\]

where we endow \( H^1_{\text{dR}}(M)[[\nu]] \) with the trivial Lie bracket.

To produce the formal flux morphism we will integrate the morphism \( F \) to the group \( \text{Aut}_0(M, \ast) \). For this, we will consider smooth paths in \( \text{Aut}_0(M, \ast) \). In the sequel, all the paths considered will be parametrized by \( t \in I := [0,1] \).

All the proofs are similar to the one developped by Banyaga [1], see also [2].

Consider \( \{A_t\} \) a smooth path in \( \text{Aut}_0(M, \ast) \) starting at the identity. Set \( D_t \) the derivation defined by \( \frac{d}{dt} A_t = D_t A_t \). We set

\[
\text{Flux}^\ast(\{A_t\}) := \int_0^1 [i(p(D_t))\omega] dt \in H^1_{\text{dR}}(M)[[\nu]]. \tag{11}
\]

We decorate the flux by a * to recall the underlying star product. We say that two paths \( \{A_t\} \) and \( \{A_t'\} \) are homotopic with fixed endpoints if there exists a smooth map \( A : I \times I \to \text{Aut}_0(M, \ast) \) such that \( A_{t0} = A_t \) and \( A_{t1} = A_t' \) for all \( t \), \( A_{0s} = A_0 \) and \( A_{1s} = A_1 \) for all \( s \).

**Proposition 4.1.** \( \text{Flux}^\ast(\{A_t\}) \) only depends on the homotopy class with fixed endpoints of the path \( \{A_t\} \).

Let \( A_{ts} \) be a homotopy with fixed endpoints of a path \( \{A_t\} \) starting at the identity. There is two different ways to define a derivation :

\[
D_{ts} := \left( \frac{d}{dt} A_{ts} \right) \circ A_{ts}^{-1} \text{ and } \tilde{D}_{ts} := \left( \frac{d}{ds} A_{ts} \right) \circ A_{ts}^{-1}.
\]

**Lemma 4.2.** \( \frac{d}{dt} D_{ts} = \frac{d}{dt} \tilde{D}_{ts} + [\tilde{D}_{ts}, D_{ts}] \).

**Proof.** Like in the classical case [1], the proof relies on the computations of \( \frac{d}{dt} D_{ts} \) and \( \frac{d}{dt} \tilde{D}_{ts} \) using point 2 of Proposition 2.10.

**Proof of Proposition 4.1.** Consider \( \{A_{ts}\} \) a homotopy of paths with fixed endpoints. Then, for each \( s \), we can compute the flux of the path \( \{A_{ts}\} \). We show that \( \text{Flux}^\ast(A_{t0}) = \text{Flux}^\ast(A_{t1}) \).

\[
\frac{d}{ds} \text{Flux}^\ast(A_{ts}) = \int_0^1 [i(p(\frac{d}{ds} D_{ts}))\omega] dt,
\]

\[
= \int_0^1 [i(p(\frac{d}{dt} D_{ts}))\omega] dt + \int_0^1 [i(p([D_{ts}, D_{ts}]))\omega] dt.
\]

Since the homotopy is with fixed endpoints, \( \tilde{D}_{ts} \) and \( \tilde{D}_{0s} \) vanishes. It means that the \( \text{Flux}^\ast(A_{ts}) \) does not depend on \( s \). Then \( \text{Flux}^\ast(A_{t0}) = \text{Flux}^\ast(A_{t1}) \).
Define \( \widehat{\text{Aut}}_0(M,*) \) to be the set of smooth homotopy classes with fixed endpoints of smooth paths \( A_t \) of automorphisms of the star product starting at the identity. The group structure on \( \widehat{\text{Aut}}_0(M,*) \) is defined as follows. Let \( \{A_t\} \) and \( \{B_t\} \in \widehat{\text{Aut}}_0(M,*) \), we set \( \{A_t\},\{B_t\} := \{A_tB_t\} \).

**Theorem 4.3.** The map

\[
\text{Flux}^* : \widehat{\text{Aut}}_0(M,*) \to H^1_{dbR}(M)[[\nu]] : \{A_t\} \mapsto \text{Flux}^*([A_t]) \tag{12}
\]

is a surjective group morphism.

*Proof.* By Proposition 4.1, the map \( \text{Flux}^* \) is well defined on \( \widehat{\text{Aut}}_0(M,*) \).

We prove that \( \text{Flux}^* \) is a group morphism. Let \( \{A_t\} \) and \( \{B_t\} \in \widehat{\text{Aut}}_0(M,*) \) generated by \( D_t \) and \( D'_t \) respectively. Then the path \( \{A_tB_t\} \) is generated by the path \( D_t + A_tD'_t(A_t)^{-1} \), by the computation rules 2.10. Again by Proposition 2.10, \( A_tD'_t(A_t)^{-1} = D'_t + D_{F_t} \), for some \( F_t \in C^\infty(M)[[\nu]] \). Moreover, \( D_{F_t} \) is a smooth family in \( \mathfrak{q}\text{Inn}(M,*) \).

Now, we compute the flux.

\[
\text{Flux}^*([A_tB_t]) = \int_0^1 \left[ i(p(D_t + A_tD'_t(A_t)^{-1}))\omega \right] dt
\]

\[
= \int_0^1 \left[ i(p(D_t))\omega \right] dt + \int_0^1 \left[ i(p(D'_t))\omega \right] dt + \int_0^1 \left[ i(p(D_{F_t}))\omega \right] dt
\]

\[
= \text{Flux}^*([A_t]) + \text{Flux}^*([B_t]).
\]

We can now characterize Hamiltonian automorphisms using the formal flux morphism.

**Theorem 4.4.** Let \( A \in \text{Aut}_0(M,*) \).

Then \( A \in \text{Ham}(M,*) \) if and only if there exists a smooth path \( A_t \) of automorphisms, with \( A_0 = 1d \) and \( A_1 = A \), such that \( \text{Flux}^*([A_t]) = 0 \).

Moreover, the path \( A_t \) can be homotoped with fixed endpoints to a path of the form \( A_t^H \) generated by some \( H_t \in C^\infty(M)[[\nu]] \).

*Proof.* Assume \( A \in \text{Ham}(M,*) \). Then \( A = A_t^H \) for some smooth family \( D_{H_t} \in \mathfrak{q}\text{Inn}(M,*) \). Then, \( \text{Flux}^*([A_t^H]) = \int_0^1 [i(p(D_{H_t}))\omega] dt = 0 \), because \( p(D_{H_t}) \) is a Hamiltonian vector field.

Conversely, assume there exists a smooth path \( \{A_t\} \) of automorphisms connecting the identity to \( A \) which has vanishing Flux. This means that there exists a series \( F \in C^\infty(M)[[\nu]] \) such that \( \int_0^1 [i(p(D_t))\omega] dt = [dF] \). We want to prove that \( A \in \text{Ham}(M,*) \).

We first observe that we can assume that \( \int_0^1 [i(p(D_t))\omega] dt = 0 \). Indeed, consider the path of automorphisms \( C_t := A_tA_t^{-1}_tA^t \). Then \( \{C_t\} \) is generated by \( D_t - DF \) and \( \int_0^1 [i(p(D_t - DF))\omega] = 0 \). Now, since \( \text{Ham}(M,*) \) is a group, it is sufficient to prove the theorem for \( C_t \).

So, suppose our smooth path \( \{A_t\} \) satisfies \( \int_0^1 [i(p(D_t))\omega] dt = 0 \). Define the family of derivations \( D'_t := -\int_0^t D_0 du \). For each \( t \), it generates a one-parameter group of automorphisms \( Q^*_t \), such that \( \tilde{\Phi}^*_{Q^*_t} := D'_tQ^*_t \). Remark that, since
$D_0' = D_1' = 0$, we get $Q_0^1 = Q_1^1 = Id$. It implies that $A_{t,s} := Q_s^1 A_t$ is a homotopy of path with fixed endpoints. We conclude by showing that $A_{t,1} = Q_1^1 A_t$ is generated by some series of functions. We compute

\[
\text{Flux}^* (\{A_{t}\}_{0 \leq t \leq T}) = \text{Flux}^* (\{Q_1^1\}_{0 \leq t \leq T}) + \text{Flux}^* (\{A_t\}_{0 \leq t \leq T}) \\
= \text{Flux}^* (\{Q_1^1\}_{0 \leq t \leq 1}) + \text{Flux}^* (\{A_t\}_{0 \leq t \leq T}) \\
= \int_0^1 [i(p(D_t')) \omega] dt + \int_0^T [i(p(D_t)) \omega] dt = 0
\]

So, if we write $\overline{D}_t$ the derivation generating the path $A_{t,1}$, then we have proved that $\int_0^T i(p(\overline{D}_t)) \omega dt = dF_T$, for all $T \in [0,1]$. Then, $i(p(\overline{D}_t)) \omega = d(\frac{d}{dt} F_t)$. So, $\overline{D}_t$ is a quasi-inner derivation, which means that $A_{t,1} = A_t^\nu$ for the family $G_t := \frac{d}{dt} F_t$. This finishes the proof.

The above Theorem 4.4 implies that $\text{Flux}^*$ descends to a morphism on $\text{Auto}_0(M, \ast)$ whose kernel is $\text{Ham}(M, \ast)$, as we have stated in Theorem 1.

**Theorem 1.** There is a short exact sequence of groups

\[1 \rightarrow \text{Ham}(M, \ast) \rightarrow \text{Auto}_0(M, \ast) \xrightarrow{\mathcal{F}} H_{dR}^1(M)[\nu] \rightarrow \Gamma(M, \ast) \rightarrow 1,\]

where $\mathcal{F}(A) := \text{Flux}^* (\{A_t\})$ for any smooth path in $\text{Auto}_0(M, \ast)$ joining $A$ to the identity and $\Gamma(M, \ast) := \text{Flux}^* (\pi_1 (\text{Auto}_0(M, \ast)))$ where $\pi_1 (\text{Auto}_0(M, \ast))$ is the subgroup of $\tilde{\text{Auto}}_0(M, \ast)$ consisting of classes of smooth loops of automorphisms.

**Proof.** First, let us check the map $\mathcal{F}$ is well-defined. Let $A \in \text{Auto}_0(M, \ast)$ and consider $A_t$ and $A_t'$ two smooth paths joining $A$ to $Id$. Then, the classes $\{A_t\}$ and $\{A_t'\}$ differ from an element in $\pi_1 (\text{Auto}_0(M, \ast))$. Hence,

\[
\text{Flux}^* (\{A_t\}) - \text{Flux}^* (\{A_t'\}) \in \text{Flux}^* (\pi_1 (\text{Auto}_0(M, \ast))).
\]

Because we quotiented $H_{dR}^1(M)[\nu]$ by $\Gamma(M, \ast)$, the map $\mathcal{F}$ is well-defined.

The map $\mathcal{F}$ is a morphism because $\text{Flux}^*$ is a morphism.

It remains to verify that $\text{Ham}(M, \ast)$ is the kernel of $\mathcal{F}$. Clearly, if $A \in \text{Ham}(M, \ast)$, then $\mathcal{F}(A) = 0$. Now, suppose $\mathcal{F}(A) = 0$. By definition, when we take a smooth path connecting $A$ to $Id$, we have

\[
\text{Flux}^* (\{A_t\}) \in \text{Flux}^* (\pi_1 (\text{Auto}_0(M, \ast))).
\]

Then, one can choose a loop $\{B_t\} \in \pi_1 (\text{Auto}_0(M, \ast))$ so that

\[
\text{Flux}^* (\{A_t\}) = \text{Flux}^* (\{B_t\}).
\]

Now, this means that $\text{Flux}^*$ vanishes on the path $\{A_t B_t^{-1}\}$. Then, by Theorem 4.4 above, it means that its extremity $A$ is a Hamiltonian automorphisms.

We give a nice geometric interpretation of the group $\Gamma(M, \ast)$. There is a natural way to lift a loop in $\text{Symp}_0(M, \omega)$ into a path $B_t$ (not necessarily a loop) of automorphisms of the star product. Elements in $\Gamma(M, \ast)$ can be used to measure what is needed to close the path $B_t$ into a loop.
Consider a loop \( \varphi_t \in \text{Symp}_0(M, \omega) \) generated by the smooth time dependent symplectic vector field \( X_t \). Then, consider the unique solution \( B_t^{-1} \) of the equation

\[
\frac{d}{dt} B_t^{-1} = -p^{-1}(X_t) B_t^{-1}, \quad \text{with} \quad B_0^{-1} = \text{Id}.
\]

(13)

Now, the path \( B_t \) is a lift of \( \varphi_t \) in the sense that \( \text{Cl}(B_t) = \varphi_t \).

Since \( \varphi_t \) is a loop, \( B_t = \exp(D) \) for some \( D \in \nu \text{Der}_0(M, \ast) \). Then the above path \( B_t \) can be closed into the loop \( \exp(-tD)B_t \). Because \( \text{Cl}^{-1}(Id) \) is in bijection with the vector space \( \nu \text{Der}_0(M, \ast) \), there is a well-defined isomorphism

\[
q : \{ \varphi_t \} \in \pi_1(\text{Symp}_0(M, \omega)) \mapsto \{ \exp(-tD)B_t \} \in \pi_1(\text{Aut}_0(M, \ast)).
\]

The formal flux morphism induces a morphism

\[
\text{Flux}^*_{deff} : \pi_1(\text{Symp}_0(M, \omega)) \to H^1_{dR}(M)[[\nu]] : \{ \varphi_t \} \mapsto \text{Flux}^*(q(\{ \varphi_t \})).
\]

We call \( \text{Flux}^*_{deff} \) the deformed flux morphism. Its image is \( \Gamma(M, \ast) \), because the map \( q \) is an isomorphism.

**Proposition 4.5.** If \( \varphi_t \) is a loop in \( \text{Symp}_0(M, \omega) \) generated by \( X_t \), write \( B_t^{-1} \) the solution of equation (13) and \( D \in \nu \text{Der}_0(M, \ast) \) such that \( B_1 = \exp(D) \), then

\[
\text{Flux}^*_{deff}(\{ \varphi_t \}) = \int_0^1 [i(X_t \omega)]dt - [i(p(D))\omega].
\]

(14)

If \( \ast \) and \( \ast' \) are two equivalent star products, then \( \text{Flux}^*_{deff}(\{ \varphi_t \}) = \text{Flux}^{\ast'}(\{ \varphi_t \}) \) for all \( \{ \varphi_t \} \in \pi_1(\text{Symp}_0(M, \omega)) \).

**Proof.** By construction \( q(\{ \varphi_t \}) = \{ \exp(-tD)B_t \} \). Now, we can compute

\[
\begin{align*}
\text{Flux}^*_{deff}(\{ \varphi_t \}) &= \text{Flux}^*(\{ \exp(-tD)B_t \}) \\
&= \text{Flux}^*(\{ B_t \}) + \text{Flux}^*(\{ \exp(-tD) \}) \\
&= \int_0^1 [i(X_t \omega)]dt - [i(p(D))\omega].
\end{align*}
\]

Now, let \( \ast \) and \( \ast' \) be two equivalent star products on \( C^\infty(M)[[\nu]] \). Consider

\[
T = \text{Id} + \sum_{r=1}^\infty \nu^r T_r : (C^\infty(M)[[\nu]], \ast) \to (C^\infty(M)[[\nu]], \ast')
\]

an equivalence of star product. We want to prove

\[
\text{Flux}^*_{deff}(\{ \varphi_t \}) = \text{Flux}^{\ast'}_{deff}(\{ \varphi_t \}),
\]

(15)

for all \( \{ \varphi_t \} \in \pi_1(\text{Symp}_0(M, \omega)) \). To do that, we will decorate by a \( \ast' \) all the objects consider in the hypothesis but corresponding to the star product \( \ast' \). We will use the bijection \( p' \) between derivations of \( \ast' \) and series of symplectic vector fields. We denote by \( (B'_t)^{-1} \) the path generated by \(- (p')^{-1}(X_t) \). We set \( D' \) the derivation of \( \ast' \) such that \( B'_t = \exp(D') \). To prove equation (15), we will show \( p'(D') = p(D) + X_F \) for some \( F \in C^\infty(M)[[\nu]] \).

First, we check that \( TB_tT^{-1} = B'_t \exp(D'_K) \) for some \( D_K \in \nu \text{q} \text{Im}(M, \ast) \).

For this, consider a good cover \( \mathcal{U} \) on \( M \). On \( U \in \mathcal{U} \), \( p^{-1}(X_t)|_U = \frac{1}{\nu}[H^U_t, \ast] \) and \( (p')^{-1}(X_t)|_U = \frac{1}{\nu}[H^U_t, \ast'], \) for some \( H^U_t \in C^\infty(U)[[\nu]] \). Then we compute

\[
Tp^{-1}(X_t)T^{-1}|_U = \frac{1}{\nu}[H^U_t, \ast'] + \frac{1}{\nu}\sum_{r=1}^\infty \nu^r T_r[H^U_t, \ast'].
\]

(16)
Now, the function \( \tilde{H} \) defined by \( \tilde{H}|_U := \sum_{r=1}^{\infty} \nu^r T_r(H^U) \) for all \( U \in \mathcal{U} \) is globally defined. So that, we have

\[ TB_1 T^{-1} = B_1'(A_1' B^r \tilde{H}), \]

because the two paths are generated by the same family of derivations. This means \( TB_1 T^{-1} = B_1' \exp(D'_K) \) for \( K = -\int_0^1 B_1' \tilde{H} dt \in \nu C^\infty(M)[[\nu]]. \)

Now, by definition \( TB_1 T^{-1} = \exp(TDT^{-1}) \). Writing locally on \( U \in \mathcal{U} \) the derivation \( TDT^{-1} \), we get \( p'(TDT^{-1}) = p(D) + X_G \). Since,

\[ (B_1')^{-1} = \exp(-T^{-1}DT) \exp(D'_K), \]

by applying the computation rules [2,10] we obtain \( p'(D') = p(D) + X_G + X_K + X_{\tilde{K}} \) for some \( \tilde{K} \in C^\infty(M)[[\nu]] \). The proof is over.

**Example 4.6.** Consider a symplectic surface \((\Sigma_g, \omega)\) of genus \( g \geq 2 \). Then one knows \( \pi_1(\text{Symp}(\Sigma_g, \omega)) = \{0\} \), see [14]. Consequently, for all star product \( * \) on \((\Sigma_g, \omega)\), we have \( \Gamma(\Sigma_g, * ) = \{0\} \). The exact sequence of Theorem [11] writes

\[ 1 \to \text{Ham}(\Sigma_g, *) \to \text{Aut}_0(\Sigma_g, *) \to \mathbb{R}^{2g}[[\nu]] \to 1. \]

**Proposition 4.7.** The set of de Rham classes arising at order 0 and 1 in \( \nu \) in elements of \( \Gamma(M, *) \) is at most countable.

**Proof.** For our computations, we select a particular star product \( * \) on a given equivalence class. Let \( \Omega = \nu \Omega_1 + \nu^2 \ldots \in \Omega^2(M)[[\nu]] \) a series of closed 2-forms that represents the characteristic class parametrizing \( * \). Up to equivalence, we can assume that \( C_1(\ldots) = \frac{1}{2} \{ \ldots \} \) and \( C_2^- (F, G) := C_2(F, G) - C_2(G, F) = -\Omega_1(X_F, X_G) \).

Let \( \{ \varphi_t \} \) be a path of symplectomorphisms generated by \( X_1 \in \text{Symp}(M, \omega) \). We compute its deformed flux at order 1 in \( \nu \). For this we consider the path \( B_1^{-1} \) of automorphisms generated by \(-p^{-1}(X_1)\). We compute

\[ B_1^{-1}(H) = H + \nu \int_0^1 \varphi_t^*(\Omega_1(X_t, \varphi_t^{-1} H)) dt + \nu^2(\ldots) \]

So that, \( \text{Flux}^*_\nu(\{ \varphi_t \}) = \int_0^1 [i(X_1)\omega] dt + \nu [i(Y_1)\omega] + \nu^2(\ldots) \) for the symplectic vector field \( Y_1 = \int_0^1 \varphi_t^*(\Omega_1(X_t, \varphi_t^{-1} H)) dt \). Moreover, we have \( i(Y_1)\omega = -\int_0^1 \varphi_t^*(i(X_t)\Omega_1) dt \). Then, we conclude

\[ \text{Flux}^*_\nu(\{ \varphi_t \}) = \int_0^1 [i(X_1)\omega] dt - \nu \left[ \int_0^1 \varphi_t^*(i(X_t)\Omega_1) dt \right] + \nu^2(\ldots). \] (17)

So that, the set of the Rham classes arising at order 0 and 1 in \( \nu \) of elements of \( \Gamma(M, *) \) is at most countable.

**Remark 4.8.** The study of \( \pi_1(\text{Symp}(M, \omega)) \) through a lifting procedure to loops of automorphisms of star product was also suggested in [11].
5 Paths of Hamiltonian automorphisms

Definition 5.1. The Lie algebra $\mathfrak{Ham}(M,\cdot)$ of $\text{Ham}(M,\cdot)$ is the set of derivations $D$ of $\cdot$ such that there exists a smooth path $A$ $\epsilon^{-1} \epsilon \mapsto \text{Aut}_0(M,\cdot)$ for $\epsilon \in \mathbb{R}$ such that $A_t \in \text{Ham}(M,\cdot)$ for all $t \in [-\epsilon,\epsilon]$ and $\frac{d}{dt}|_0 A_t = D$, with Lie bracket given by the commutator of derivations.

One checks $(\mathfrak{Ham}(M,\cdot),[\cdot,\cdot])$ is indeed a Lie algebra, using the computation rules of Proposition 2.10.

Question 5.2. Is it true that $(\mathfrak{Ham}(M,\cdot),[\cdot,\cdot]) \cong (\mathfrak{qInn}(M,\cdot),[\cdot,\cdot])$?

By construction, the algebra $\mathfrak{Ham}(M,\cdot)$ contains the algebra $\mathfrak{qInn}(M,\cdot)$. When $H^1_{\text{dR}}(M) = 0$, a derivation is always of the form $D_H$ for $H \in C^\infty(M)[[\nu]]$. Then, $(\mathfrak{Ham}(M,\cdot),[\cdot,\cdot]) \cong (\mathfrak{qInn}(M,\cdot),[\cdot,\cdot])$. However when $H^1_{\text{dR}}(M) \neq 0$, we will see the answer is not trivial and depends on the image of $\text{Flux}_{\text{def}}$.

The above question 5.2 is equivalent to the following question:

Question 5.3. Is any smooth path $A_t \in \text{Ham}(M,\cdot)$ generated by a time-dependent Hamiltonian $H_t \in C^\infty(M)$?

In the classical case, Banyaga [11] shows that every path in $\text{Ham}(M,\omega)$ is generated by a time dependent Hamiltonian $H_t \in C^\infty(M)$.

Theorem 5.4. Any smooth paths of Hamiltonian automorphisms is generated by a Hamiltonian $H_t \in C^\infty(M)[[\nu]]$ if there is no non constant smooth paths in $\text{Flux}_{\text{def}}(\pi_1(\text{Ham}(M,\omega))) \subset H^1_{\text{dR}}(M)[[\nu]]$, where $\pi_1(\text{Ham}(M,\omega))$ is viewed as a subgroup of $\pi_1(\text{Symp}_0(M,\omega))$ using the canonical inclusion.

Remark 5.5. In the above Theorem 5.3 a path in $H^1_{\text{dR}}(M)[[\nu]]$ is smooth if and only if its coefficients are smooth paths.

Proof. Because $\text{Ham}(M,\cdot)$ and $\text{Flux}_{\text{def}}(\pi_1(\text{Ham}(M,\omega)))$ are groups, it is enough to consider paths starting at the neutral element.

Now, consider $\{A_t\}$ a path of Hamiltonian automorphisms starting at the identity. Then, $\text{Cl}(\{A_t\})$ is a path of Hamiltonian diffeomorphisms which is generated by some $F_t \in C^\infty(M)$. So, $\{A_t^{-1} F_t A_t\}$ is a path in $\text{Cl}^{-1}(\text{Id}) \cap \text{Ham}(M,\cdot)$. Then it suffices to prove the theorem for paths in $\text{Cl}^{-1}(\text{Id}) \cap \text{Ham}(M,\cdot)$.

Consider a smooth path $\{A_t := \exp(D_t)\}_{t \in [0,1]} \subset \text{Cl}^{-1}(\text{Id}) \cap \text{Ham}(M,\cdot)$ starting at the identity. Then the images of the partial paths $\{A_t\}_{t \in [0,s]}$ for $0 \leq s \leq 1$ by Flux gives a smooth path $t \mapsto [i(p(D_t))\omega]$ in $H^1_{\text{dR}}(M)[[\nu]]$. Because $A_t \in \text{Ham}(M,\cdot)$, the path $t \mapsto [i(p(D_t))\omega]$ is in $\text{Flux}_{\text{def}}(\pi_1(\text{Ham}(M,\omega)))$. By hypothesis, the path is constant and $[i(p(D_t))\omega] = [i(p(D_0))\omega] = 0$ for all $t$, then $D_t \in \mathfrak{qInn}(M,\cdot)$.

Example 5.6. Let $(\Sigma_g,\omega)$ be a closed orientable surface of genus $g \geq 1$ equipped with an area form $\omega$. One can show $\pi_1(\text{Ham}(\Sigma_g,\omega)) = 0$ for all $g \geq 1$ (see [14] for an outline of the proof). Let $\ast$ be a star product on $(\Sigma_g,\omega)$. Then, by Theorem 5.3, every path $\{A_t\} \subset \text{Ham}(\Sigma_g,\cdot)$ is generated by a time-dependent Hamiltonian $H_t \in C^\infty(\Sigma_g)[[\nu]]$ and then

$(\mathfrak{Ham}(\Sigma_g,\cdot),[\cdot,\cdot]) \cong (\mathfrak{qInn}(\Sigma_g,\cdot),[\cdot,\cdot])$. 

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**Proposition 5.7.** Assume \((M, \omega)\) is a closed symplectic manifold equipped with a star product \(*\). Let \(A_t\) be a path of Hamiltonian automorphisms, then there exists \(H_t := H_t^0 + \nu H_t^1 \in C^\infty_0(M)[[\nu]]\) such that \(A_t^H = A_t \mod O(\nu^2)\).

**Proof.** In Proposition 4.7 we showed that at order 0 and 1 in \(\nu\) the group \(\Gamma(M, +)\) is at most countable. Since \(\text{Flux}_{d\ell f}(\pi_1(\text{Ham}(M, \omega))) \subset \Gamma(M, +)\), there is no non constant smooth path at order 1 in \(\nu\) in \(\text{Flux}_{d\ell f}(\pi_1(\text{Ham}(M, \omega)))\). This is enough to guarantee that paths of Hamiltonian automorphisms are generated by a formal function modulo terms in \(O(\nu^2)\).

### 6 Computation using Fedosov’s star products

In this section we give a nice expression of the deformed flux for nice loops of symplectomorphisms. In order to make concrete computation we will use a Fedosov’s star product. This is not a restriction in view of Proposition 4.5.

Let \(\Omega \in \nu \Omega^2(M)[[\nu]]\) a formal serie of closed 2-forms and \(\nabla\) a symplectic connection on \((M, \omega)\) (i.e. a torsion free connection such that \(\nabla \omega = 0\)). Through this section we will denote by \(*_{\Omega, \nabla}\) the star product obtained via the Fedosov construction with respect to \(\Omega\) and \(\nabla\).

**Theorem 6.1.** Let \(\{\varphi_t\}\) be a loop of symplectomorphisms generated by the symplectic vector field \(X_t\) such that \(\varphi_t^* \Omega = \Omega\) and \(\varphi_t^* \nabla = \nabla\) for all \(t\). Then, the deformed flux of \(\{\varphi_t\}\) defined with the star product \(*_{\Omega, \nabla}\) is

\[
\text{Flux}_{d\ell f}^{*_{\Omega, \nabla}}(\{\varphi_t\}) = \int_0^1 i(X_t)\omega dt - \left[ \int_0^1 \varphi_t^* i(X_t)\Omega dt \right].
\]

### 6.1 Fedosov construction of star product

We recall the Fedosov construction [6, 7]. This construction of star product is obtained by identifying \(C^\infty(M)[[\nu]]\) with the algebra of flat sections of the Weyl bundle \(W\) endowed with a flat connection.

The sections of the Weyl bundle are formal series of the form:

\[
a(x, y, \nu) := \sum_{2k+l \geq 0} \nu^k a_{k,i_1...i_l}(x) y^{i_1} \ldots y^{i_l}.
\]

The \(a_{k,i_1...i_l}(x)\) are, in the indices \(i_1, \ldots, i_l\), the components of a symmetric tensor on \(M\) and \(2k+l\) is the degree in \(W\). The space of sections of \(W\), denoted by \(\Gamma W\), has a structure of an algebra defined by the fiberwise product

\[
(a \circ b)(x, y, \nu) := \left. \left( \exp\left( \frac{\nu^2}{2} \sum_i \partial_{y^i} \right) a(x, y, \nu) b(x, z, \nu) \right) \right|_{y = z}
\]

To describe connections on \(W\), we will consider forms with values in the Weyl algebra. Those can be written in local coordinates as

\[
\sum_{2k+l \geq 0} \nu^k a_{k,i_1...i_l,j_1...j_k}(x) y^{i_1} \ldots y^{i_l} dx^{j_1} \wedge \ldots \wedge dx^{j_k}.
\]

The \(a_{k,i_1...i_l,j_1...j_k}(x)\) are, in the indices \(i_1, \ldots, i_l, j_1, \ldots, j_k\), the components of a tensor on \(M\), symmetric in the \(i\)'s and antisymmetric in the \(j\)'s. The space
of such sections, ΓW ⊗ Λ°(M), is endowed with a structure of algebra. For
a ⊗ α and b ⊗ β, we define (a ⊗ α) ◦ (b ⊗ β) := a ◦ b ⊗ α ∧ β. The space of
W-valued forms inherits the structure of a graded Lie algebra from the graded
commutator [s, s'] := s ◦ s' − (−1)^q_1q_2 s' ◦ s, where s is a form of degree q_1 and
s' of degree q_2.

The connection ∂ in W is defined by

\[ \partial a := da + \frac{1}{\nu}[\Gamma, a] \in \Gamma W \otimes \Lambda^1 M. \]

where \( \Gamma := \frac{1}{2} \omega_{lk} \Gamma^k_{ij} y^i y^j dx^l \) with \( \Gamma^k_{ij} \) the Christoffel symbols of a symplectic
connection \( \nabla \) on \( (M, \omega) \). Of course, the connection \( \partial \) extends to a covariant
derivative on all \( \Gamma W \otimes \Lambda^M \) using the Leibniz rule :

\[ \partial (a \otimes \alpha) := (\partial a) \wedge \alpha + a \otimes d\alpha. \]

The curvature of \( \partial \) is denoted by \( \partial \circ \partial \) and is expressed in terms of the
curvature \( R \) of the symplectic connection \( \nabla \).

\[ \partial \circ \partial a := \frac{1}{\nu}[R, a], \]

where \( R := \frac{1}{4} \omega_{ij} R^k_{ij} y^i y^j dx^k \wedge dx^j \).

Define

\[ \delta(a) := dx_k \wedge \partial_{y_k} a = -\frac{1}{\nu}[\omega_{ij} y^i dx^j, a], \]

and

\[ \delta^{-1} a_{pq} := \frac{1}{p + q} y^k i(\partial_{x^k}) a_{pq} \] if \( p + q > 0 \) and \( \delta^{-1} a_{00} = 0, \)

where \( a_{pq} \) is a q-forms with \( p \) y’s and \( p + q > 0 \). We then have the Hodge
decomposition of \( \Gamma W \otimes \Lambda^M : \delta \delta^{-1} a + \delta^{-1} \delta a = a - a_{00}. \)

Now, we recall the construction of a flat connection \( D \) on \( \Gamma W \) of the form

\[ Da := \partial a - \delta a + \frac{1}{\nu}[r, a], \]

where \( r \) is a W-valued 1-form and \( D^2 a = 0. \)

Because,

\[ D^2 a = \frac{1}{\nu} \left[ R + \partial r - \delta r + \frac{1}{2\nu}[r, r], a \right], \]

one choose \( r \) such that

\[ R + \partial r - \delta r + \frac{1}{2\nu}[r, r] = \Omega, \]

for a central 2-form \( \Omega \). Which means that \( Dr = \Omega - R + \frac{1}{2\nu}[r, r] \). Then, for all
closed central 2-form \( \Omega \), there exists a unique solution \( r \in \Gamma W \otimes \Lambda^1 M \) of degree
at least 3 of the equation

\[ r = \delta^{-1}(R + \partial r + \frac{1}{\nu}[r, r] - \Omega), \]

satisfying \( \delta^{-1} r = 0. \)
Then, to solve the equation (4), we build the unique family
\[ A_t \] such that
\[ A_t|_U = D_{H_t^U} \] for all \( U \in \mathcal{U} \).

In the sequel we will need some low degree terms of \( QF \) for \( F \in C^\infty(M) \).
\[ QF = F + \partial_t F y^i + \frac{1}{2} (\nabla_i X_F)_j y^j + (QF)^{\geq 3} =: (QF)^{< 3} + (QF)^{\geq 3}, \] (19)
where \( (\nabla_i X_F)_j = (\nabla_i X_F)^k \omega_{kj} \) and \( (QF)^{\geq 3} \) denotes the term of degree bigger than 3.

### 6.2 Exponentiation of derivations

In this subsection, we write the solution of the equation (4) in the Weyl algebra. This is the first step in the proof of Theorem 6.1. We follow the book [7].

We first translate the equation (4) in the Weyl algebra. Let \( \mathcal{U} \) be a good cover of \( M \), then for all \( U \in \mathcal{U} \) there exists a series \( H_t^U := \sum_{r=0}^\infty \nu^r H_t^U \in C^\infty(U)[[\nu]] \) such that \( D_t|_U = D_{H_t^U} \). We can then consider the local section \( QH_t^U \) of \( W \).

Because two functions \( H_t^U \) and \( H_t^{U'} \) differ on \( U \cap U' \) by a constant, we can define a global section
\[ (QH_t^U - H_t^{U'})(x) := (QH_t^U - H_t^U)(x) \] if \( x \in U \).

Then, to solve the equation (4), we build the unique family \( A_t \) of automorphisms of \( \Gamma W_D \) such that for all \( s \in \Gamma W_D \):
\[ \frac{d}{dt} A_t(s) = \frac{1}{\nu} [QH_t^d - H_t^d, A_t(s)], \] (20)
with initial condition \( A_0 = Id \).

The strategy is the same as in proposition [2.5].

We define the natural pull-back on \( \Gamma W \otimes \Lambda M \) by a symplectomorphism \( \varphi :\)
\[ \varphi_*(a(x, y, \nu) \otimes \alpha) := a(\varphi(x), (\varphi_* y)^{-1}, \nu) \otimes \varphi^* \alpha = \sum_{2k+l \geq 0} \nu^k a_{l+1,...,l}(x)(\partial_{x^i} \varphi)^{i_1} \ldots (\partial_{x^i} \varphi)^{i_l} y^{i_1} \ldots y^{i_l} \otimes \varphi^* \alpha. \]

In the paper [9], Gutt and Rawnsley showed that the Lie derivative satisfies the following Cartan formula. Let \( \varphi_t \) be a symplectic isotopy generated by the time dependent symplectic vector field \( X_t \), then
\[ \frac{d}{dt} \varphi_{t*} = \varphi_{t*} \left( i(X_t) D + D i(X_t) + \frac{1}{\nu} a d_\omega (\omega_j y^j + \frac{1}{2} (\nabla_i X_t)_j y^j y^j - i(X_t) r) \right), \] (21)
where \( \frac{1}{2} (\nabla_i X_t)_j y^j y^j := \frac{1}{2} (\nabla_i X_t)^k \omega_{kj} y^j. \)
Because we recall that $X_t$ such that (21), we have Let $\phi_t$ generated by $\phi$.

Proof of Proposition 6.1. Choose a good cover $U$ and let $\phi_t$ be a loop of symplectomorphisms generated by $\phi_t$. Then, to compute $\int_{\Gamma} \omega_{\phi_t} \wedge \nu_{\phi_t}$ for all $\phi_t$, $\phi_t^* \omega_{\phi_t} \wedge \nu_{\phi_t}$ is generated by $\phi_t$. Next, we solve the equation

$$A_t = \phi_t \exp \left( \frac{1}{\nu} \int_0^t \psi_t^{-1} \left( (QH_{t,0}^{(U)})^{\geq 3} + i(X_t) \nu + H_t \right) ds \right).$$

(23)

6.3 The deformed flux in the Fedosov’s construction

Let $\{\phi_t\}$ be a loop of symplectomorphisms generated by $X_t$. Let us explain how to compute its deformed flux in the Fedosov’s formalism.

Choose a good cover $U$ of $M$. Then, on $U \in \mathcal{U}$, there exists $H_t^U \in C^\infty(U)$ such that $X_t^{\mid U} = X_{H_t^U}$. Next, we solve the equation

$$\frac{d}{dt} A_t^{-1}(a) = \frac{1}{\nu} \left[ -(QH_t^U - H_t^U), A_t^{-1}(a) \right],$$

for all $a \in \Gamma \omega_D$ with initial condition $A_0^{-1} = Id$. By equation (23), we have

$$A_1 = \exp \left( \frac{1}{\nu} \int_{\Gamma} \varphi_t^{-1} \left( (QH_{t,0}^{(U)})^{\geq 3} + i(X_t) \nu + H_t \right) ds \right).$$

(24)

Then, to compute $\text{Flux}_{\nu}(\{\phi_t\})$ where $Y \in \nu \mathfrak{symp}(M,\omega)$ which writes locally as $Y^{\mid U} = X_{F^U}$ for some $F^U \in \nu C^\infty(U)[[\nu]]$ such that

$$A_1 = \exp \left( \frac{1}{\nu} \int_{\Gamma} (QF^U - F^U) \right).$$

Proof of Proposition 6.1. We assume that $\{\phi_t\}$ preserves $\nabla$ and $\Omega$. Recall that $\{\phi_t\}$ is generated by $X_t$ and we write locally $X_t^{\mid U} = X_{H_t^U}$ for $H_t^U \in C^\infty(U)[[\nu]]$. We consider the automorphism $A_1$ defined in Equation (24). The goal is to prove that $\int_0^1 \varphi_t^* \left( (QH_t^U)^{\geq 3} + i(X_t) \nu + H_t \right) dt = QF^U - F^U$ with $F^U$ satisfying $dF^U = \int_0^1 \varphi_t^* i(X_t) \Omega dt^{\mid U}$.
We compute $D \int_0^1 \varphi_t^* \left( (QH^H_t)^{\geq 3} + i(X_t)r \right) dt$. By assumption, $\varphi_t \circ \partial = \partial \circ \varphi_t$ and $\varphi_t^* r = r$. This imply that $\varphi_t \circ D = D \circ \varphi_t$. Thus, we have

\[
D \int_0^1 \varphi_t^* \left( (QH^H_t)^{\geq 3} + i(X_t)r \right) dt = \int_0^1 \varphi_t^* D \left( (QH^H_t)^{\geq 3} + i(X_t) r \right) dt = \int_0^1 \varphi_t^* \left(-D(QH^H_t)^{<3} + Di(X_t)r \right) dt.
\]

(25)

Where we use the fact that $QH^H_t$ is locally a flat section. Remark that the section $D(QH^H_t)^{<3}$ is globally defined. Since $\varphi_t$ preserves $r$, using (21), we have

\[
\int_0^1 \varphi_t^* D(i(X_t)r) dt
= -\int_0^1 \varphi_t^* \left( i(X_t) D r + \frac{1}{\mu} \omega_{ij} X_t^i y^j \right) dt
= -\int_0^1 \varphi_t^* \left( i(X_t) \Omega - i(X_t) \mathbf{R} + \frac{1}{\mu} \omega_{ij} X_t^i y^j \right) dt.
\]

We have to compare the above expression with

\[
-\int_0^1 \varphi_t^* D(QH^H_t)^{<3} dt = -\int_0^1 \varphi_t^* D \left( H^H_t + \omega_{ij} X_t^i y^j + \frac{1}{2} (\nabla_i X_t)_j y^j y^i \right) dt.
\]

(26)

Remark that the right hand side is globally defined. So, we compute

\[
DH^H_t = dH^H_t = \delta(\omega_{ij} X_t^i y^j).
\]

(27)

Also, we have

\[
\partial(\omega_{ij} X_t^i y^j) = \delta \left( \frac{1}{2} (\nabla_i X_t)_j y^j y^i \right).
\]

(28)

And,

\[
\partial \left( \frac{1}{2} (\nabla_i X_t)_j y^j y^i \right) = \frac{1}{2} (\nabla^2_{ij} X_t)_{ij} \omega_{ik} y^k y^i dx^i.
\]

(29)

Using equations (21) to (29), the equation (26) becomes

\[
\int_0^1 \varphi_t^* \left( D(-(QH^H_t)^{<3}) \right) dt = -\int_0^1 \varphi_t^* \left( \frac{1}{2} (\nabla^2_{ij} X_t)_{ij} \omega_{ik} y^k y^i dx^i \right.
\]

\[
+ \frac{1}{\mu} \left[ r, \omega_{ij} X_t^i y^j + \frac{1}{2} (\nabla_i X_t)_j y^j y^i \right] dt.
\]

Then,

\[
D \int_0^1 \varphi_t^* \left( (QH^H_t)^{\geq 3} + i(X_t)r \right) dt = \int_0^1 \varphi_t^* \left( -i(X_t) \Omega + i(X_t) \mathbf{R} - \frac{1}{2} (\nabla^2_{ij} X_t)_{ij} \omega_{ik} y^k y^i dx^i \right).
\]

Remark that $i(X_t) \mathbf{R} - \frac{1}{2} (\nabla^2_{ij} X_t)_{ij} \omega_{ik} y^k y^i dx^i = \frac{1}{2} (L_{X_t} \nabla)_{ij} y^j y^i dx^i$. Since $\varphi_t$ preserves the symplectic connection, we get

\[
D \int_0^1 \varphi_t^* \left( (QH^H_t)^{\geq 3} + i(X_t)r \right) dt = -\int_0^1 \varphi_t^* (i(X_t) \Omega) dt.
\]

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Consequently, if $F^U \in C^\infty(U)[[\nu]]$ satisfies $dF^U = \int_0^1 \varphi_t^*i(X_t)\Omega dt|_U$, then $QF^U = F^U = \int_0^1 \varphi_t^*((QH^U_t)^{>3} + i(X_t)r)dt$. It means that:

$$\text{Flux}_{\text{def}}^{\ast_{\text{def}}} (\{\varphi_t\}) = \int_0^1 [i(X_t)\omega] dt - \left[ \int_0^1 \varphi_t^* i(X_t)\Omega dt \right].$$

The proof is over.

\[\square\]

**Example 6.2.** Consider the 2-torus $(T^2, d\theta_1 \wedge d\theta_2)$ with usual coordinates $(\theta_1, \theta_2)$. The group $\pi_1(\text{Symp}(T^2, d\theta_1 \wedge d\theta_2))$ is known to be generated by the rotations $\{\varphi_t\}$ and $\{\psi_t\}$ along the symplectic vector fields $\partial_{\theta_1}$ and $\partial_{\theta_2}$, see [13].

Consider Fedosov’s star products of the form $\ast_{\Omega, d}$ with $d$ the flat connection and $\Omega = \sum_{i=1}^{\infty} \nu^i C_i, d\theta_1 \wedge d\theta_2$. Then, all the equivalence classes of star product are represented. Because the 2-form $d\theta_1 \wedge d\theta_2$ and $d$ are preserved by $\{\varphi_t\}$ and $\{\psi_t\}$, we can use Theorem 6.1 to compute the deformed flux. We obtain

$$\text{Flux}_{\text{def}}^{\ast_{\text{def}}} (\{\varphi_t\}) = d\theta_2(1 - \sum_{i=1}^{\infty} \nu^i C_i)$$

and

$$\text{Flux}_{\text{def}}^{\ast_{\text{def}}} (\{\psi_t\}) = d\theta_1(1 - \sum_{i=1}^{\infty} \nu^i C_i).$$

Then, $\Gamma(M, *_{\text{def}}, d) = <d\theta_2(1 - \sum_{i=1}^{\infty} \nu^i C_i), d\theta_1(1 - \sum_{i=1}^{\infty} \nu^i C_i) >_z$.

The exact sequence of Theorem 1 writes

$$1 \to \text{Ham}(T^2, *_{\Omega, d}) \to \text{Aut}_0(T^2, *_{\Omega, d}) \xrightarrow{\varphi} \frac{H^1_{dR, \langle \omega \rangle}(T^2)[[\nu]]}{(1 - \sum_{i=1}^{\infty} \nu^i C_i) < d\theta_2, d\theta_1 >_z} \to 1.$$

**Corollary 6.3.** Consider the symplectic manifold $(T^2, d\theta_1 \wedge d\theta_2)$.

Two star products $\ast$ and $\ast'$ are equivalent if and only if $\Gamma(T^2, \ast) = \Gamma(T^2, \ast')$.

We think our results together with the example of symplectic surfaces motivate a deeper study of the groups $\Gamma(M, \ast)$. In particular, it would be nice to generalize the formula [13] to arbitrary loops in $\text{Symp}_0(M, \omega)$. It would imply $\Gamma(M, \ast)$ is at most countable at any order in $\nu$. That would mean the hypothesis in Theorem 5.3 above is always satisfied.

As a concluding remark we mention that the flux morphism can be defined on certain Poisson manifolds [17][15]. It might be interesting to see how the work of this paper extends to star products on Poisson (non symplectic) manifolds.

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