EFFECTIVE EQUIDISTRIBUTION OF CIRCLES IN THE LIMIT SETS OF KLEINIAN GROUPS

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Abstract. Consider a general circle packing $\mathcal{P}$ in the complex plane $\mathbb{C}$ invariant under a Kleinian group $\Gamma$. When $\Gamma$ is convex cocompact or its critical exponent is greater than 1, we obtain an effective equidistribution for small circles in $\mathcal{P}$ intersecting any bounded connected regular set in $\mathbb{C}$; this provides an effective version of an earlier work of Oh-Shah [12]. In view of the recent result of McMullen-Mohammadi-Oh [6], our effective circle counting theorem applies to the circles contained in the limit set of a convex cocompact but non-cocompact Kleinian group whose limit set contains at least one circle. Moreover consider the circle packing $\mathcal{P}(T)$ of the ideal triangle attained by filling in largest inner circles. We give an effective estimate to the number of disks whose hyperbolic areas are greater than $t$, as $t \to 0$, effectivising the work of Oh [10].

1. Introduction

A circle packing in the complex plane $\mathbb{C}$ is simply a countable union of circles (here a line is regarded as a circle of infinite radius). Compared to the conventional definition of a circle packing, our definition is more general as circles are allowed to intersect each other. Given a circle packing $\mathcal{P}$, we seek to estimate the number of small circles intersecting a bounded subset in $\mathbb{C}$ (see Figure 1 for examples).

Assume $\mathcal{P}$ is locally finite, i.e., for any $T > 1$, there are only finitely many circles in $\mathcal{P}$ of Euclidean curvature at most $T$ intersecting any fixed bounded subset in $\mathbb{C}$. For a bounded subset $E$ in $\mathbb{C}$ and $T > 1$, we set

$$N_T(\mathcal{P}, E) := \# \{ C \in \mathcal{P} : C \cap E \neq \emptyset, \text{Curv}(C) < T \},$$

where Curv($C$) denotes the Euclidean curvature of $C$. As $\mathcal{P}$ is locally finite, $N_T(\mathcal{P}, E) < \infty$.

In [12], Oh and Shah considered a very general locally finite circle packing $\mathcal{P}$: suppose $\mathcal{P}$ is invariant under a torsion-free non-elementary geometrically finite Kleinian group $\Gamma < \text{PSL}_2(\mathbb{C})$. They obtained...
an asymptotic estimate to $N_T(P, E)$. In particular, they introduced a locally finite Borel measure $\omega_\Gamma$ on $\mathbb{C}$ determined by $\Gamma$ (Definition 3.10) such that under some further assumption on $\Gamma$, we have

$$\lim_{T \to \infty} \frac{N_T(P, E_1)}{N_T(P, E_2)} = \frac{\omega_\Gamma(E_1)}{\omega_\Gamma(E_2)},$$

where $E_1$ and $E_2$ are any bounded Borel sets in $\mathbb{C}$ satisfying $\omega_\Gamma(\partial(E_1)) = \omega_\Gamma(\partial(E_2)) = 0$.

In this paper, we extend Oh-Shah’s result and provide an effective estimate to $N_T(P, E)$. To apply our theorem, we need to impose a more stringent condition on $E$: we require not only $\omega_\Gamma(\partial(E)) = 0$ but the $\epsilon$-neighborhood of $\partial(E)$ is of small size. Sets satisfying such property will be called regular (Definition 3.14). Denote the critical exponent of $\Gamma$ by $\delta_\Gamma$. We show the following.

**Theorem 1.1.** Assume $P$ is a locally finite circle packing invariant under a geometrically finite Kleinian group $\Gamma$ and with finitely many $\Gamma$-orbits. When $\delta_\Gamma \leq 1$, we assume further that $\Gamma$ is convex cocompact. Then for any bounded connected regular set $E \subset \mathbb{C}$, there exists $\eta > 0$, such that as $T \to \infty$,

$$N_T(P, E) = c \cdot \omega_\Gamma(E) \cdot T^{\delta_\Gamma} + O(T^{\delta_\Gamma - \eta}),$$

where $c > 0$ is a constant depending only on $\Gamma$ and $P$.

Denote by $\Lambda(\Gamma) \subset \mathbb{C} \cup \{\infty\}$ the limit set of $\Gamma$ which is the set of accumulation points of an orbit of $\Gamma$ in $\mathbb{C} \cup \{\infty\}$ under the linear fractional transformation action. When $\Gamma$ is convex cocompact or it has no rank 2 cusps with $\delta_\Gamma > 1$, the measure $\omega_\Gamma(E)$ in Theorem 1.1 equals the $\delta_\Gamma$-dimensional Hausdorff measure of $E \cap \Lambda(\Gamma)$ [17].
Circles in the limit set of a Kleinian group. Suppose $\Gamma$ is convex cocompact. Consider the set of circles contained in $\Lambda(\Gamma)$:

$$\mathcal{I}(\Gamma) := \{C \subset \Lambda(\Gamma)\}.$$  

McMullen, Mohammadi and Oh showed that if $\Lambda(\Gamma) \neq \mathbb{C} \cup \{\infty\}$, there are only finitely many $\Gamma$-orbits of circles in $\mathcal{I}(\Gamma)$, and each such circle arises from a compact $\text{PSL}_2(\mathbb{R})$-orbit (Corollary 11.3 and Theorem B.1 in [6]); this implies that $\mathcal{I}(\Gamma)$ is a locally finite circle packing with finitely many $\Gamma$-orbits and hence Theorem 1.1 applies to $\mathcal{I}(\Gamma)$.

**Corollary 1.2.** Let $\Gamma$ be a convex cocompact Kleinian subgroup with critical exponent $\delta_\Gamma < 2$. Assume that $\mathcal{I}(\Gamma)$ is non-empty. Then for any bounded connected regular set $E \subset \mathbb{C}$, there exists $\eta > 0$, such that as $T \to \infty$,

$$N_T(\mathcal{I}(\Gamma), E) = c \cdot \mathcal{H}^{\delta_\Gamma}(E \cap \Lambda(\Gamma)) \cdot T^{\delta_\Gamma} + O(T^{\delta_\Gamma - \eta}),$$

where $c > 0$ is a constant depending only on $\Gamma$, and $\mathcal{H}^{\delta_\Gamma}(E \cap \Lambda(\Gamma))$ is the $\delta_\Gamma$-dimensional Hausdorff measure of $E \cap \Lambda(\Gamma)$.

Circles in an ideal triangle of $\mathbb{H}^2$. Let $T$ be an ideal triangle in the hyperbolic plane $\mathbb{H}^2$, i.e., a triangle whose sides are hyperbolic lines connecting vertices on the geometry boundary $\partial \mathbb{H}^2$. Such an ideal triangle exists and is unique up to hyperbolic isometries. Consider the circle packing $\mathcal{P}(T)$ in $T$ attained by filling in the largest inner circles (see Figure 2). We give an effective estimate to the number of disks enclosed by circles in $\mathcal{P}(T)$ whose hyperbolic areas are greater than $t$.

Let $\overline{\mathcal{P}(T)}$ be the closure of $\mathcal{P}(T)$. The Hausdorff dimension of $\overline{\mathcal{P}(T)}$, denoted by $\alpha$, equals the residual dimension of an Apollonian circle packing [5]. For $C \in \mathcal{P}(T)$, let $\text{Area}_{\text{hyp}}(C)$ be the hyperbolic area of the disk enclosed by $C$.

**Theorem 1.3.** There exist $c > 0$ and $\eta > 0$, such that as $t \to 0$,

$$\#\{C \in \mathcal{P}(T) : \text{Area}_{\text{hyp}}(C) > t\} = c t^{-\frac{\alpha}{2}} + O(t^{-\frac{\alpha}{2} + \eta}).$$

The asymptotic formula for this counting problem without a rate was obtained by Oh in [10].

**On the proof of Theorems 1.1 and 1.3.** Our proof of Theorem 1.1 is built on the approach employed in [12], while providing an effective statement for each step of their arguments and combining it with the effective equidistribution results in [7].

Theorem 1.3 does not immediately follow from Theorem 1.1 since the ideal triangle is not bounded in the hyperbolic space. In order to prove Theorem 1.3 we need to obtain an effective estimate to the $\alpha$-dimensional Hausdorff measure in hyperbolic metric of neighborhoods.
of the vertices of the ideal triangle, as well as to give an effective estimate to the number of circles in such neighborhoods.

We refer readers to [4] and [19] for effective counting results when $\mathcal{P}$ is an Apollonian circle packing. See also [13] for related counting results.

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2. Preliminaries

2.1. Notations. We let

$$G := \text{PSL}_2(\mathbb{C}), \quad K := \text{PSU}(2),$$

$$M := \{m_\theta := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \},$$

$$H := \text{PSU}(1, 1) \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{PSU}(1, 1).$$

Let $\mathbb{H}^3$ be the hyperbolic 3-space. We use the following coordinates for the upper half space model of $\mathbb{H}^3$:

$$\mathbb{H}^3 = \{z + jy : z \in \mathbb{C}, y \in \mathbb{R}_{>0} \},$$

where $j = (0, 1)$.

The geometric boundary $\partial \mathbb{H}^3$ is the extended complex plane $\hat{\mathbb{C}}$. The group of orientation preserving isometries of $\mathbb{H}^3$ is given by $G$. Noting that $G$ acts transitively on $\mathbb{H}^3$ and $K = \text{Stab}_G(j)$, we identify $\mathbb{H}^3$ with $G/K$ via the map $[g] \mapsto gj$. 

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure2.png}
\caption{Circle packing in ideal hyperbolic triangle}
\end{figure}
Let $T^1(\mathbb{H}^3)$ be the unit tangent bundle of $\mathbb{H}^3$, and $X_0 \in T^1(\mathbb{H}^3)$ the upward unit normal vector based at $j$. Then $T^1(\mathbb{H}^3)$ can be identified with $G/M$ via the map $gX_0 \mapsto [g]$ as $M = \text{Stab}_G(X_0)$.

Any circle $C$ in $\hat{\mathbb{C}}$ determines a unique totally geodesic plane in $\mathbb{H}^3$, denoted by $\hat{C}$. Let $C^\dagger$ be the unit normal bundle of $\hat{C}$. The group $\text{Stab}_G(\hat{C})$ acts transitively on both $\hat{C}$ and $C^\dagger$. In particular, if we denote by $C_0$ the unit circle centered at the origin, then $C_0$ and $C_0^\dagger$ can be identified with $H/H \cap K$ and $H/M$ respectively as $H = \text{Stab}_G(\hat{C}_0)$.

2.2. Measures on $\Gamma \backslash G/M$. Let $\Gamma < G$ be a geometrically finite Kleinian group. A family of finite measures $\{\mu_x : x \in \mathbb{H}^3\}$ is called a $\Gamma$-invariant conformal density of dimension $\delta_x > 0$, if each $\mu_x$ is a non-zero finite Borel measure on $\partial \mathbb{H}^3$ satisfying for any $x, y \in \mathbb{H}^3$, $\xi \in \partial \mathbb{H}^3$ and $\gamma \in \Gamma$,

$$\gamma_* \mu_x = \mu_{\gamma x} \quad \text{and} \quad \frac{d\mu_y}{d\mu_x}(\xi) = e^{-\delta_x \beta_\xi(y,x)}$$

where $\gamma_* \mu_x(F) := \mu_x(\gamma^{-1}(F))$ for any Borel subset $F$ of $\partial \mathbb{H}^3$. Here $\beta_\xi(y, x)$ is the Busemann function given by $\beta_\xi(y, x) = \lim_{t \to \infty} d(\xi_t, y) - d(\xi_t, x)$, where $\xi_t$ is any geodesic ray tending to $\xi$.

We denote by $\{\nu_{T,x} : x \in \mathbb{H}^3\}$ (or simply $\{\nu_x : x \in \mathbb{H}^3\}$) the Patterson-Sullivan density (or PS-density), which is a $\Gamma$-invariant conformal density of dimension $\delta_T$ with $\delta_T$ the critical exponent of $\Gamma$. We will denote the critical exponent of $\Gamma$ simply by $\delta$ when there is no room for confusion. Denote by $\{m_x : x \in \mathbb{H}^3\}$ the Lebesgue density, which is a $G$-invariant conformal density on the boundary $\partial \mathbb{H}^3$ of dimension 2.

Let $\pi : T^1(\mathbb{H}^3) \to \mathbb{H}^3$ be the canonical projection map. For $u \in T^1(\mathbb{H}^3)$, denote by $u^+, u_-$ the forward and the backward endpoints of the geodesic determined by $u$. The Hopf parametrization $u \mapsto (u^+, u^-, s := \beta_u(\langle j, \pi(u) \rangle))$ gives a homeomorphism between $T^1(\mathbb{H}^3)$ and $(\partial \mathbb{H}^3 \times \partial \mathbb{H}^3) \setminus \{(\xi, \xi) : \xi \in \partial \mathbb{H}^3\} \times \mathbb{R}$. Using the identification of $G/M$ with $T^1(\mathbb{H}^3)$, we define the Bowen-Margulis-Sullivan measure $\tilde{m}^\text{BMS}_\Gamma$ and Burger-Roblin measure $\tilde{m}^\text{BR}_\Gamma$ on $G/M$ as follows:

**Definition 2.1.** Set

1. $d\tilde{m}^\text{BMS}_\Gamma(g) = e^{\delta j g X_0^+(j, gj)} e^{\delta j g X_0^-(j, gj)} d\nu_j(gX_0^+) d\nu_j(gX_0^-) ds$;
2. $d\tilde{m}^\text{BR}_\Gamma(g) = e^{\delta j g X_0^+(j, gj)} e^{\delta j g X_0^-(j, gj)} dm_j(gX_0^+) d\nu_j(gX_0^-) ds$.

Both $\tilde{m}^\text{BMS}_\Gamma$ and $\tilde{m}^\text{BR}_\Gamma$ are left $\Gamma$-invariant by the properties of conformal density. They induce locally finite Borel measures on $\Gamma \backslash G/M$, which we will denote by $\tilde{m}^\text{BMS}$ and $\tilde{m}^\text{BR}$. When there is no room for confusion, we will write $\tilde{m}^\text{BMS}$, $\tilde{m}^\text{BR}$, $\tilde{m}^\text{BMS}$ and $\tilde{m}^\text{BR}$ for simplicity.
2.3. Measures on $\langle \Gamma \cap H \rangle \backslash H \backslash M$ and $\text{sk}_\Gamma(P)$. Denote $\Gamma \cap H$ by $\Gamma_H$ for convenience. Using the PS-density $\{\nu_x\}$, we construct the measure $\tilde{\mu}^{PS}_{\Gamma,H}$ on $H \backslash M = C^+_0$:

$$d\tilde{\mu}^{PS}_{\Gamma,H} := e^{\delta_\beta(hX_0^+)}d\nu_j(hX_0^+).$$

Note that $d\tilde{\mu}^{PS}_{\Gamma,H}$ is left $\Gamma_H$-invariant. Hence it induces a locally finite Borel measure on $\Gamma_H \backslash H \backslash M = \Gamma_H \backslash C^+_0$. We denote the induced measure by $d\mu^{PS}_{H}$. When there is no ambiguity about $\Gamma$, we simply write $d\mu^{PS}_{H}$.

We introduce a measure associated with a circle packing $P$:

**Definition 2.2** (The $\Gamma$-skinning size of $P$). For a circle packing $P$ in $\hat{C}$ consisting of finitely many $\Gamma$-orbits, define $0 \leq \text{sk}_\Gamma(P) \leq \infty$ as follows:

$$\text{sk}_\Gamma(P) := \sum_{i \in I} |\mu^{PS}_{g\gamma_i^{-1}g_{C_i}}|,$$

where $\{C_i : i \in I\}$ is a set of representatives of $\Gamma$-orbits in $P$ and $g_{C_i} \in G$ is an element such that $\gamma C_i(C_0) = C_i$.

**Remark 2.3.** It is shown in [17] that $\{\nu_x : x \in \mathbb{H}^3\}$ is unique up to scalars. Using this property, we can verify that Definition 2.2 does not depend on the choice of $g_{C_i}$.

**Theorem 2.4** (Theorem 2.4 and Lemma 3.2 in [12]). Assume that $\Gamma$ is either convex cocompact or its critical exponent $\delta$ is greater than 1. Let $P$ be a locally finite circle packing in $\hat{C}$ invariant under $\Gamma$ with finitely many $\Gamma$-orbits. Then $\text{sk}_\Gamma(P) < \infty$.

**Proposition 2.5.** For any circle $C$, if $\Gamma(C)$ is a locally finite circle packing consisting of infinitely many circles, then $\text{sk}_\Gamma(\Gamma(C)) > 0$.

**Proof.** As $\Gamma(C)$ consists of infinitely many circles, we have $[\Gamma : \Gamma \cap \text{Stab}_G(C)] = \infty$ (Lemma 3.1 in [12]). The local finiteness of $\Gamma(C)$ implies that the map $\Gamma \cap \text{Stab}_G(C) \backslash \hat{C} \to \Gamma \backslash \mathbb{H}^3$ is proper (Lemma 3.2 in [12]). The proposition follows from Proposition 6.7 in [11].

3. Effective counting for general circle packings

Throughout this section, we set $\Gamma < \text{PSL}_2(\mathbb{C})$ to be a torsion-free non-elementary geometrically finite Kleinian group. Let $P$ be a locally finite circle packing consisting of finitely many $\Gamma$-orbits.

Let

$$N = \left\{ n_z := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\} \quad \text{and} \quad N^- = \left\{ n_w^- := \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} : w \in \mathbb{C} \right\}.$$
For a subset $E \subset \mathbb{C}$, set
\[ N_E := \{ n_z : z \in E \}. \]
Let
\[ A = \{ a_t := \left( e^{\frac{t}{2}} 0 \right) : t \in \mathbb{R} \} \quad \text{and} \quad A^+ = \{ a_t : t \geq 0 \}. \]
For $s > 0$, set
\[ A^+_s := \{ a_t : 0 \leq t \leq s \} \quad \text{and} \quad A^-_s := \{ a_{-t} : 0 \leq t \leq s \}. \]
Let $E \subset \mathbb{C}$ be a bounded Borel set. Our goal in this section is to give an effective estimate of the following counting function:
\[ N_T(P, E) := \# \{ C \in P : C \cap E \neq \emptyset, \ \text{Curv}(C) < T \}, \]
where Curv$(C)$ is the Euclidean curvature of $C$.

### 3.1. Reformulation into orbit counting problem

We built a relation between $N_T(P, E)$ and a counting function of a $\Gamma$-orbit on $H \setminus G$ and prove Proposition 3.3, which is a more precise version of Proposition 3.7 in [12]. To obtain an effective estimate of $N_T(P, E)$, it is important to understand the independence of $m_0$ from $\epsilon$ and the size of $T$ in terms of $\epsilon$, where $m_0$, $T$ and $\epsilon$ are described as Proposition 3.3.

Fix a left invariant Riemannian metric on $G$, which induces the hyperbolic metric on $G/K = \mathbb{H}^3$. For any $\epsilon > 0$, set $U_\epsilon$ to be the symmetric $\epsilon$-neighborhood of $e$ in $G$. For any subset $W \subset G$, denote $W_\epsilon = U_\epsilon \cap W$. Define
\[ (3.1) \quad E^+_\epsilon := \bigcup_{u \in U_\epsilon} uE \quad \text{and} \quad E^-_\epsilon := \bigcap_{u \in U_\epsilon} uE. \]
Note that there exists a constant $c_0 > 0$, independent of $E$, such that for any small $\epsilon > 0$, $E^+_\epsilon$ contains the $c_0\epsilon$-neighborhood of $E$ in the Euclidean metric. We fix this constant $c_0$ in the following.

**Lemma 3.2.** For any $0 < \epsilon < \frac{1}{c_0}$ and for any $T > \frac{1}{c_0 \epsilon}$, we have
\[ N_T(P, E) \leq \# \{ C \in P : \hat{C} \cap N_{E^+_\epsilon} A^{-\log T} j \neq \emptyset \}. \]

**Proof.** It suffices to prove that if $C$ is a circle in $P$ intersecting $E$ with Curv$(C) < T$ for some $T > \frac{1}{c_0 \epsilon}$, then the intersection $\hat{C} \cap N_{E^+_\epsilon} A^{-\log T} j$ is non-empty.

Observe that
\[ N_{E^+_\epsilon} A^{-\log T} j = \{ z + lj : z \in E^+_\epsilon, T^{-1} \leq l \leq 1 \}. \]
Let $C \in P$ be such that $C \cap E \neq \emptyset$ and Curv$(C) < T$ for some $T > \frac{1}{c_0 \epsilon}$. Set $z_0$ and $r_C$ to be the center and the radius of $C$ respectively.
If \( r_C \leq c_0 \epsilon \), then \( z_0 \in E^+_\epsilon \) and \( z_0 + r_C j \) belongs to \( \hat{C} \cap N_{E^+} A_{-T}^{-} \). Now suppose \( r_C > c_0 \epsilon \). Choosing \( z \in C \cap E \), set \( w = (r_C - c_0 \epsilon) \frac{z - z_0}{|z - z_0|} + z_0 \). Then \( w \in E^+_\epsilon \). Let \( r > 0 \) be such that \( w + r j \) lies in \( \hat{C} \). We have

\[
 r = \sqrt{r_C^2 - |w - z_0|^2} > T^{-1}.
\]

As a result, any point lying on the geodesic between \( w + r j \) and \( z \) and of Euclidean height greater than \( T^{-1} \) and less than 1 lies in \( N_{E^+} A_{-T}^{-} j \). This proves the claim.

**Lemma 3.3.** If \( E \) is connected, then there exists a positive integer \( m_0 \), depending on \( E \), such that for any \( T > 0 \),

\[
 N_T(\mathcal{P}, E) \geq \#\{ C \in \mathcal{P} : \hat{C} \cap N_{E} A_{-T}^{-} j \neq \emptyset \} - m_0.
\]

**Proof.** The local finiteness condition on \( \mathcal{P} \) implies that there are only finitely many lines in \( \mathcal{P} \) intersecting \( E \). Denote by \( W_T \) the set of all circles \( C \in \mathcal{P} \) of positive curvature such that \( \hat{C} \cap N_{E} A_{-T}^{-} j \neq \emptyset \) and \( C \cap E = \emptyset \). It suffices to prove that \( \cup W_T \) consists of finitely many circles.

We claim that any circle in \( \cup W_T \) must have radius bigger than \( d_E/2 \), as well as intersect the \( 2/d_E \)-neighborhood of \( E \) where \( d_E \) is the diameter of \( E \).

If \( C \in \cup W_T \), it follows from the connectedness of \( E \) that \( E \) is contained in the open disc enclosed by \( C \), and hence the radius of \( C \) is at least \( d_E/2 \). Picking an arbitrary point \( z + s j \in \hat{C} \cap N_{E} A_{-T}^{-} j \), we let \( w \in C \) be such that \( d(w, z) = d(C, z) \). Then

\[
 d(w, z) = r_C - \sqrt{r_C^2 - s^2} \leq s^2 / r_C.
\]

As mentioned above, \( r_C > d_E/2 \) and hence \( d(w, z) \leq 2/d_E \). Therefore \( C \) intersects the \( 2/d_E \)-neighborhood of \( E \) non trivially, proving the claim. Now the proposition follows from the assumption that \( \mathcal{P} \) is locally finite. \( \square \)

**Definition 3.4** (Definition of \( B_T(E) \)). For \( E \subset C \) and \( T > 1 \), define the subset \( B_T(E) \) of \( H \setminus G \) to be the image of the set

\[
 KA_{-T}^{-} N_{-E} = \{ ka_{-t} n_z : k \in K, 0 \leq t \leq \log T, z \in E \}
\]

under the canonical projection \( G \rightarrow H \setminus G \).

Lemmas 3.2 and 3.3 yield the following estimate:

**Proposition 3.5** (cf. Proposition 3.7 in [12]). Suppose \( E \) is connected. We have, for all small \( 0 < \epsilon < \frac{1}{c_0} \) and for any \( T > \frac{1}{c_0 \epsilon} \),

\[
\#([\epsilon] \Gamma \cap B_T(E)) - m_0 \leq N_T(\Gamma(C_0), E) \leq \#([\epsilon] \Gamma \cap B_T(E^+_\epsilon)).
\]
where $m_0$ is a positive integer only depending on $E$.

3.2. Approximate $B_T(E)$ using $HAN$-decomposition. We study the shape of $B_T(E)$ in an effective way. Note that there exists $c_1 > 0$ depending on the metric of $G$, such that for all small $\epsilon > 0$, the set $K_\epsilon(0) := \{k \cdot 0 : k \in K_\epsilon\} \subset \mathbb{C}$ contains the disk of radius $c_1 \epsilon$ centered at 0. Set

$$T_\epsilon = -\log(c_1 \epsilon).$$

We show the following inclusion:

**Proposition 3.6.** There exists $c > 0$ (independent of $E$) such that for all sufficiently small $\epsilon > 0$, we have for all sufficiently large $T > 1$

$$B_T(E) \subset \bigcup_{0 \leq t \leq T_\epsilon} H \setminus Ha_t K(t) N_{-E} \cup \bigcup_{T_\epsilon - c \epsilon \leq t \leq \log T + c \epsilon} H \setminus Ha_t N_{-E_{c1\epsilon}},$$

where $K(t) = \{k \in K : a_t k \in HKA^+\}$.

We first recall some results in [12].

**Lemma 3.7** (Proposition 4.2 and Corollary 4.3 in [12]). (1) If $a_t \in HKA_s K$ for $s > 0$, then $|t| \leq s$. This in particular implies

$$HKA^+_{\log T} = \bigcup_{0 \leq t \leq \log T} Ha_t K(t) \text{ for any } T > 1.$$

(2) Given any small $\epsilon > 0$, we have

$$\{k \in K : a_t k \in HKA^+ \text{ for some } t > T_\epsilon\} \subset K_\epsilon M.$$

In fact, the second statement of the lemma is a more precise version of Proposition 4.2 (2) in [12]. We add to the original proof the observation that $K_\epsilon(0)$ contains the disk of radius $c_1 \epsilon$ centered at 0 for all small $\epsilon > 0$.

**Lemma 3.8.** There exist $c_2 > 1$ and $t_0 > 1$, such that for any sufficiently small $\epsilon > 0$,

$$a_t km_\theta \in Ha_t A_{c_2 \epsilon} N_{c_2 \epsilon}$$

holds for any $t > t_0$, $k \in K_\epsilon$ and $m_\theta \in M$.

**Proof.** Fix $k \in K_\epsilon$ and $m_\theta \in M$. The product map $N^- \times A \times M \times N \to G$ is a diffeomorphism at a neighborhood of $e$, in particular, bi-Lipschitz. Hence there exists $l_1 > 1$ such that for all small $\epsilon > 0$,

$$K_\epsilon \subset N_{l_1 \epsilon} A_{l_1 \epsilon} M_{l_1 \epsilon} N_{l_1 \epsilon}.$$

So $k$ can be written as

$$k = n_1^{-1} a_{\epsilon_1} m_\theta n_1.$$
where \( n_1^- \in N_{l_1}^- \), \( a_{s_1} \in A_{l_1} \), \( m_{\theta_1} \in M_{l_1} \) and \( n_1 \in N_{l_1} \). Then

\[
\tag{3.9}
a_{t} k m_{\theta} = a_{t} n_1^- a_{s_1} m_{\theta_1} n_1 m_{\theta} \\
= (a_{t} n_1^- a_{-t}) a_{t+s_1} m_{\theta_1} n_1 m_{\theta}.
\]

Set \( t_0 \) to be a constant such that for all \( t > t_0 \), \( a_{t} n_1^- a_{-t} \in N_{l_{-}}^- \). Due to the \( H \times A \times N \) product decomposition of \( G_{\epsilon} \), there exists \( l_2 > 1 \) such that for all small \( \epsilon > 0 \),

\[ N_{l_{-}}^- \subset H_{l_2} A_{l_2} N_{l_2}. \]

This inclusion and (3.9) yield

\[
a_{t} k m_{\theta} = h_2 a_{s_2} n_2 a_{t+s_1} m_{\theta_1} n_1 m_{\theta} \\
= h_2 m_{\theta_1} a_{t+s_1} (m_{-\theta_1} a_{-(t+s_1)} n_2 a_{t+s_1} m_{\theta_1}) (m_{-\theta_1} n_1 m_{\theta_1}) \\
\in H a_{t} A_{(l_1+l_2)} N(l_1+l_2),
\]

where \( h_2 \in H_{l_2} \), \( a_{s_2} \in A_{l_2} \) and \( n_2 \in N_{l_2} \). Setting the constant \( c_2 = l_1 + l_2 \), we have

\[
a_{t} k m_{\theta} \in H a_{t} A_{c_2} N_{c_2}. \]

\[\square\]

**Proof of Proposition 3.6.** For all sufficiently large \( T > 1 \), it follows from Lemmas 3.7 and 3.8 that

\[
B_T(E) = \bigcup_{0 \leq t \leq T_{\epsilon}} H \setminus H a_t K(t) N_{-E} \cup \bigcup_{T_{\epsilon} \leq t \leq \log T} H \setminus H a_t K(t) N_{-E} \\
\subset \bigcup_{0 \leq t \leq T_{\epsilon}} H \setminus H a_t K(t) N_{-E} \cup \bigcup_{T_{\epsilon} \leq t \leq \log T} H \setminus H a_t K_{t} M N_{-E} \\
\subset \bigcup_{0 \leq t \leq T_{\epsilon}} H \setminus H a_t K(t) N_{-E} \cup \bigcup_{T_{\epsilon} - c_2 \leq t \leq \log T + c_2} H \setminus H a_t N_{c_2} N_{-E}. \]

\[\square\]

### 3.3. On the measure \( \omega_{\Gamma} \).

**Definition 3.10.** Define a locally finite Borel measure \( \omega_{\Gamma} \) on \( \mathbb{C} \) as follows: fixing \( x \in \mathbb{H}^3 \), for \( \psi \in C_c(\mathbb{C}) \),

\[
\omega_{\Gamma}(\psi) = \int_{z \in \mathbb{C}} e^{\delta_{\Gamma,\beta}(x,z+j)} \psi(z) d\nu_{\Gamma,x}(z).
\]

The definition of \( \omega_{\Gamma} \) is independent of the choice of \( x \in \mathbb{H}^3 \) by the conformal properties of \( \nu_{\Gamma,x} \).
Lemma 3.11 (Lemma 5.2 in [12]). For any \( x = p + rj \in \mathbb{H}^3 \), and \( z \in \mathbb{C} \), we have
\[
\beta_z(p + rj, z + j) = \log \frac{|z - p|^2 + r^2}{r}.
\]

This lemma provides another formula for \( \omega_G \): for any \( \psi \in C_c(\mathbb{C}) \),
\[
\omega_G(\psi) = \int_{z \in \mathbb{C}} (|z|^2 + 1) \delta \psi(z) d\nu_j(z) = \int_{z \in \mathbb{C}} (|z|^2 + 1) \delta \psi(z) d\nu_j(z).
\]

3.3.1. Relation between \( \omega_G \) and \( m^{BR} \). For a bounded Borel set \( E \subset \mathbb{C} \), let \( E^+ \) and \( E^- \) be the sets defined as (3.1). For small \( \epsilon > 0 \), let \( \psi \in C_c(\mathbb{C}) \) supported in \( U_\epsilon \) with integral one. Set \( \Psi_\epsilon \in C^\infty_c(\Gamma \setminus G) \) to be the \( \Gamma \)-average of \( \psi \):
\[
\Psi_\epsilon(\Gamma g) := \sum_{\gamma \in \Gamma} \psi(\gamma g).
\]

For a bounded Borel subset \( E \subset \mathbb{C} \), let
\[
h_E = \max_{z \in E} |z| + 1.
\]

Proposition 3.12 (cf. Lemma 5.7 in [12]). There exists \( c > 0 \) independent of \( E \) such that for all small \( \epsilon > 0 \),
\[
(1 - c \cdot h_E \cdot \epsilon) \cdot \omega_G(E^-) \leq \int_{z \in E} m^{BR}(\Psi_\epsilon^-) d\nu_j(z) \leq (1 + c \cdot h_E \cdot \epsilon) \cdot \omega_G(E^+),
\]
where \( \Psi_\epsilon^- \in C^\infty_c(\Gamma \setminus G)^M \) is given by \( \Psi_\epsilon^- = \int_{m \in M} \Psi(\gamma mn_{-z}) d\nu_m \) with \( d\nu_m \) the probability measure on \( M \).

Lemma 3.13 (Lemma 5.5 in [12]). If \( (m_{11}a_{11}n_{11}z_1)(m_{12}a_{12}n_{12}z_1) = m_{11}a_{11}n_{11}n_{12}n_{12} \) in the \( MAN^{-N} \) coordinates, then
\[
t_0 = t + t_1 + 2 \log(|1 + e^{-t_1 - 2i\theta_1} w_1 z|),
\]
\[
z_0 = e^{-t_1 - 2i\theta_1} z_1 (1 + e^{-t_1 - 2i\theta_1} w_1 z)^{-1} + z_1.
\]

Proof of Proposition 3.12. Consider the following function on \( MAN^{-N} \subset G \):
\[
\mathcal{R}_E(ma_{11}n_{11}z_1) = e^{-\delta t} \chi_E(-z).
\]

We may regard \( \mathcal{R}_E \) as a function defined on \( G \). It is shown in Lemma 5.7 in [12] that
\[
\int_{z \in E} m^{BR}(\Psi_\epsilon^-) d\nu_j(z) = \int_{g \in U_\epsilon} \psi(g) \int_{k \in K} \mathcal{R}_E(k^{-1}g) d\nu_j(k(0)) dg.
\]
Write $k^{-1} = m_0 a_t n_{\gamma_1} n_z$ and $g = m_0 a_t n_{\gamma_0} n_{z_1} \in U_\epsilon$. By Lemma 3.13, we have $k^{-1} g = m_0 a_t n_{\gamma_0} n_{z_1}$ with $t_0 = t + t_1 + 2 \log(|1 + e^{-t_1 - 2\theta_1 w_1 z}|)$. Noting that $R_E(k^{-1} g) = e^{-\delta t_0} \chi_E(g^{-1} k(0))$, we have for any $g \in U_\epsilon$

$$\int_{k \in K/M} R_E(k^{-1} g) d\nu_j(k(0)) = (1 + O(h_E \cdot \epsilon)) \int_{k(0) \in E_\epsilon^\pm} e^{-\delta t} d\nu_j(k(0)).$$

Using Proposition 5.4 in [12], we have

$$\int_{k(0) \in E_\epsilon^\pm} e^{-\delta t} d\nu_j(k(0)) = \omega_T(E_\epsilon^\pm),$$

which yields the proposition.

3.3.2. Regularity criterion for $\omega_T$. Fix a bounded Borel set $E \subset \mathbb{C}$. To apply the effective result in our paper, it is important to understand the difference between $\omega_T(E_\epsilon^+ - E^-)$ and $\omega_T(E)$.

**Definition 3.14** (Regularity condition). We call a bounded Borel subset $E \subset \mathbb{C}$ regular if there exists $0 < p < 1$ such that for all small $\epsilon > 0$,

$$(3.15) \quad \omega_T(E_\epsilon^+ - E^-) = O(\epsilon^p),$$

where the implied constant depends only on $E$.

**Proposition 3.16** ([15]). Suppose $\Gamma$ is convex cocompact and Zariski dense. For any bounded set $E \subset \mathbb{C}$, if $\partial E$ is a finite union of proper subvarieties, then $E$ is regular in the above sense.

Denote by $\Lambda_p(\Gamma)$ the set of parabolic limit points. For $\xi \in \Lambda_p(\Gamma)$, the rank of $\xi$ is the rank of the abelian subgroup of $\Gamma$ which fixes $\xi$. As $\Gamma \subset \text{PSL}_2(\mathbb{C})$, then rank($\xi$) is either 1 or 2. We provide a regularity criterion for $\Gamma$ with $\Lambda_p(\Gamma) \neq \emptyset$.

**Proposition 3.17.** Let $k_0 := \max_{\xi \in \Lambda_p(\Gamma)} \text{rank}(\xi)$. If the critical exponent of $\Gamma$ satisfies

$$\delta > \max\{1, \frac{k_0 + 1}{2}\},$$

then any bounded Borel set $E \subset \mathbb{C}$ with $\partial E$ a finite union of rectifiable curves is regular.

**Proof.** The proof is adapted from the proof of Proposition 7.10 in [7]. For $\xi \in \partial \mathbb{H}^3$, denote by $s_\xi = \{\xi_t : t \in [0, \infty)\}$ the geodesic ray emanating from $j$ toward $\xi$ and let $S(\xi_t) \subset \mathbb{H}^3$ be the unique 2-dimensional geodesic plane which is orthogonal to $s_\xi$ at the point $\xi_t$. Denote by $b(\xi_t)$ the projection from $j$ onto $\partial \mathbb{H}^3$ of $S(\xi_t)$, that is,

$$b(\xi_t) = \{\xi' \in \partial \mathbb{H}^3 : s_{\xi'} \cap S(\xi_t) \neq \emptyset\}.$$
It is shown in [16] and [17] that there exists a $\Gamma$-invariant collection of pairwise disjoint horoballs $\{H_\xi : \xi \in \Lambda_p(\Gamma)\}$ for which the following holds: there exists a constant $c > 1$ such that for any $\xi \in \Lambda(\Gamma)$ and for any $t > 0$,
\[ e^{-t} e^{d(\xi, \Gamma(\gamma))(k(\xi)-\delta)} \leq \nu_j(b(\xi)) \leq ce^{-t} e^{d(\xi, \Gamma(\gamma))(k(\xi)-\delta)}, \]
where $k(\xi)$ is the rank of $\xi'$ if $\xi \in H_\xi'$ for some $\xi' \in \Lambda_p(\Gamma)$ and $\delta$ otherwise. Using $0 \leq d(\xi_t, \Gamma(\gamma)) \leq t$, we have for any $\xi \in \Lambda(\Gamma)$ and $t > 1$,
\[ \nu_j(b(\xi)) \ll \begin{cases} e^{-(2\delta+k(\xi))t} & \text{if } k(\xi) \geq \delta \\ e^{-\delta t} & \text{otherwise.} \end{cases} \]

By standard computation in hyperbolic geometry, for any bounded set $E \subset \mathbb{C}$, there exist $c' > 1$ and $0 < r_0 < 1$ such that for any $t > -\log r_0$ and $\xi \in E$,
\[ B(\xi, e^{-t}/c') \subset b(\xi_t) \subset B(\xi, c'e^{-t}) \]
where $B(\xi, r)$ is the closed Euclidean ball in $\partial \mathbb{H}^3$ of radius $r$. Since $d\omega_T = (|z|^2 + 1)^{\delta_T} d\nu_T$, setting $k_0 := \max_{\xi' \in \Lambda_p(\Gamma)} \text{rank}(\xi')$, it follows from (3.18) that for all small $\epsilon > 0$ and $\xi \in \Lambda(\Gamma) \cap E$
\[ \omega_T(B(\xi, \epsilon)) \ll \nu_j(B(\xi, \epsilon)) \ll \epsilon^{\delta + 2k_0}. \]

Suppose $\partial E$ is a finite union of rectifiable curves. The collection
\[ \mathcal{F} = \{B(\xi, \epsilon) : \xi \in \partial E\} \]
forms a Besicovitch covering for $\partial E$. By the Besicovitch covering theorem, there exist $d \in \mathbb{N}$ (independent of $\epsilon$ and $E$) and a countable subcollection of $\mathcal{F}$, say $\{B(\xi_n, \epsilon)\}$, such that
\[ \partial E \subset \bigcup B(\xi_n, \epsilon), \]
and the multiplicity of the cover $\bigcup B(\xi_n, \epsilon)$ is at most $d$. Since $\partial E$ is a finite union of rectifiable curves, we have
\[ \sum l(B(\xi_n, \epsilon) \cap \partial E) \ll l(\partial E) < \infty, \]
where $l(\cdot)$ denotes the length function. This implies
\[ \#\{B(\xi_n, \epsilon)\} = O(\epsilon^{-1}). \]

Now cover the $\epsilon$-neighborhood of $\partial E$ by $\{B(\xi_n, 2\epsilon)\}$. Note that $\nu_j$ is supported on $\Lambda(\Gamma)$. For every $B(\xi_n, 2\epsilon)$, if $\xi_n \notin \Lambda(\Gamma)$ but $B(\xi_n, 2\epsilon)$ intersects $\Lambda(\Gamma)$ nontrivially, replace it by a ball of radius $4\epsilon$ with center in $\Lambda(\Gamma)$. We obtain the estimate
\[ \omega_{\Gamma}(\epsilon\text{-nbhd of } \partial E) = O(\epsilon^{\delta-1} + \epsilon^{2k_0-1}), \]
where the constant depending only on $E$. Therefore, if $\delta > \max\{1, \frac{\kappa_0 + 1}{2}\}$, then $E$ is regular.

3.4. **Conclusion.** We keep the notations from Section 2. One of our main theorems is the following:

**Theorem 3.21.** Assume $\Gamma$ is either convex cocompact or its critical exponent $\delta$ is greater than 1. Let $\mathcal{P}$ be a locally finite circle packing in $\hat{\mathbb{C}}$ invariant under $\Gamma$ with finitely many $\Gamma$-orbits. For any bounded connected regular set $E$ in $\mathbb{C}$, there exists $\eta > 0$, such that as $T \to \infty$, we have

$$N_T(\mathcal{P}, E) = \frac{\text{skf}(\mathcal{P})}{\delta |m_{\text{BMS}}|^T} \omega_T(E) + O(T^{\delta - \eta}).$$

The rest of the section is devoted to the proof of Theorem 3.21.

For $\psi \in C^\infty(\Gamma \setminus G)$ and $l \in \mathbb{N}$, we consider the following $L^2$-Sobolev norm of degree $l$:

$$S_l(\psi) = \sum \|X(\psi)\|_2,$$

where the sum is taken over all monomials $X$ in a fixed basis of the Lie algebra of $G$ of order at most $l$ and $\|X(\psi)\|_2$ is the $L^2(\Gamma \setminus G)$-norm of $X(\psi)$. For $\varphi \in C^\infty(\Gamma_H \setminus H)$, $S_l(\varphi)$ is defined similarly.

The key ingredient in the proof of Theorem 3.21 is the following effective equidistribution result:

**Theorem 3.22 ([7]).** Assume $\Gamma$ is convex cocompact or its critical exponent $\delta$ is greater than 1. Suppose the natural projection $\Gamma_H \setminus C_0^1 \to \Gamma \setminus G/M$ is proper. Then there exist $\eta_0 > 0$ (depending on the spectral gap data for $\Gamma$) and $l \in \mathbb{N}$ such that for any compact subset $\Omega \subset \Gamma \setminus G/M$, any $\Psi \in C^\infty(\Omega)$ and any bounded $\phi \in C^\infty(\Gamma_H \setminus C_0^1)$, as $t \to \infty$, we have

$$e^{(2-\delta)t} \int_{h \in \Gamma_H \setminus C_0^1} \Psi(ha_t)\phi(h)dh = \frac{\mu_{\text{PS}}(\phi)}{|m_{\text{BMS}}|^t} m_{\text{BR}}(\Psi) + O(S_l(\Psi) \cdot S_l(\phi) e^{-\eta_0 t}),$$

where the implied constant depends only on $\Omega$.

**Remark 3.23.**

(1) If $\delta > 1$, then $\Gamma$ is Zariski dense (Lemma 2.11 in [8]).

(2) Strictly speaking, Theorem 3.22 in [7] is shown using the exponential mixing of geodesic flow on $T^1(\Gamma \setminus \mathbb{H}^3)$. Such an exponential mixing is provided in [15] and [16] for $\Gamma$ convex cocompact and in [7] for $\Gamma$ with critical exponent greater than 1.
For every \( T > 1 \), and \( 0 < \epsilon < 1 \), denote

\[
V_{T,\epsilon}^{+}(E_{2\epsilon}) := \{a \in \mathbb{Z}^{-z} : z \in E_{2\epsilon}, -\epsilon \leq s \leq \log T + \epsilon\},
\]

\[
V_{T,\epsilon}^{-}(E_{2\epsilon}) := \{a \in \mathbb{Z}^{-z} : z \in E_{2\epsilon}, \epsilon \leq s \leq \log T - \epsilon\}.
\]

**Lemma 3.24** (Lemma 6.3 in [12]). There exists \( c_3 > 0 \), such that for all \( T > 1 \) and for all small \( \epsilon > 0 \),

\[
KA_{\log T}^{+}N_{-E_{\epsilon}^{+}} U_{c_3 \epsilon} \subset KV_{T,\epsilon}^{+}(E_{2\epsilon}),
\]

\[
KV_{T,\epsilon}^{-}(E_{2\epsilon}) U_{c_3 \epsilon} \subset KA_{\log T}^{-} N_{-E}.
\]

For simplicity, denote the subsets:

(3.25) \( W_{T,\epsilon}^{+} := H \backslash HKV_{T,\epsilon}^{+}(E_{2\epsilon}) \) and \( W_{T,\epsilon}^{-} := H \backslash HKV_{T,\epsilon}^{-}(E_{2\epsilon}) \).

Define the counting functions \( F_{T,\epsilon}^{\pm}(g) \) on \( \Gamma \backslash G \):

\[
F_{T,\epsilon}^{+}(g) := \sum_{\gamma \in \Gamma \backslash \Gamma} \chi_{W_{T,\epsilon}^{+}}([e]\gamma g) \quad \text{and} \quad F_{T,\epsilon}^{-}(g) := \sum_{\gamma \in \Gamma \backslash \Gamma} \chi_{W_{T,\epsilon}^{-}}([e]\gamma g).
\]

Let \( c_3 \) be as Lemma 3.24. The following lemma can easily be deduced from Proposition 3.5 and Lemma 3.24.

**Lemma 3.26** (cf. Lemma 6.4 in [12]). Given any small \( \epsilon > 0 \), we have, for all \( g \in U_{c_3 \epsilon} \) and \( T > \frac{1}{c_0 \epsilon} \),

(3.27) \( F_{T,\epsilon}^{-}(g) - m_0 \leq N_{T}(\Gamma(C_0), E) \leq F_{T,\epsilon}^{+}(g) \),

where \( m_0 \) is a positive integer depending only on \( E \).

**Proof of Theorem 3.21, Step 1:** We first prove the theorem for \( \mathcal{P} = \Gamma(C_0) \).

For \( \epsilon > 0 \), let \( \psi^{\epsilon} \) be a non-negative function in \( C_{c}^{\infty}(G) \) supported in \( U_{c_3 \epsilon} \) with integral one. Denote by \( \Psi^{\epsilon} \in C_{c}^{\infty}(\Gamma \backslash G) \) the \( \Gamma \)-average of \( \psi^{\epsilon} \).

For \( T > \frac{1}{c_0 \epsilon} \), integrating (3.27) against \( \Psi^{\epsilon} \), we obtain

\[
\langle F_{T,\epsilon}^{-}, \Psi^{\epsilon} \rangle - m_0 \leq N_{T}(\Gamma(C_0), E) \leq \langle F_{T,\epsilon}^{+}, \Psi^{\epsilon} \rangle.
\]

By abusing notation, we use \( dh \) to denote the Haar measures on \( H \) and \( H/M \). We require that these two measures are compatible with the probability measure \( dm \) on \( M \). The following defines a Haar measure on \( G \): for \( g = ha, k \in HA^{+}K \),

\[
dg = 4 \sinh r \cdot \cosh rdhdrdm_j(k),
\]

where \( dm_j(k) := dm_j(kX_0^{+}) \). Denote by \( d\lambda \) the unique \( G \)-invariant measure on \( H \backslash G \) which is compatible with \( dg \) and \( dh \).
For $\langle F_{T,\epsilon}^+, \Psi^\epsilon \rangle$, we have

\begin{equation}
\langle F_{T,\epsilon}^+, \Psi^\epsilon \rangle = \int_{\Gamma \setminus \Gamma_H \setminus \Gamma} \sum_{\gamma \in \Gamma} \chi_{W_{T,\epsilon}^+}([e]\gamma g) \Psi^\epsilon(g) dg
\end{equation}

\begin{equation}
= \int_{g \in \Gamma \setminus \Gamma_H \setminus \Gamma} \chi_{W_{T,\epsilon}^+}([e]g) \Psi^\epsilon(g) dg
\end{equation}

\begin{equation}
= \int_{g \in W_{T,\epsilon}^+} \int_{h \in \Gamma \setminus \Gamma_0} \int_{m \in M} \Psi^\epsilon(hmg) dm dh d\lambda(g).
\end{equation}

Consider the set $W_{T,\epsilon}^+$ defined in (3.25). We can rewrite it in the following form by Lemma 3.7 (1)

\begin{equation}
W_{T,\epsilon}^+ = \bigcup_{0 \leq s \leq \log T + \epsilon} H \setminus H a_s K(s) N_{-E_{2\epsilon}^+},
\end{equation}

where the set $K(\cdot)$ is defined as Proposition 3.6. Applying Proposition 3.6, we get:

\begin{equation}
W_{T,\epsilon}^+ \subset \bigcup_{0 \leq t \leq T_\epsilon} H \setminus H a_t K(t) N_{-E_{2\epsilon}^+} \cup \bigcup_{T_\epsilon - \rho_1 \epsilon \leq t \leq \log T + \rho_1 \epsilon} H \setminus H a_t N_{-E_{2\epsilon}^+},
\end{equation}

\begin{equation}
:= V_1 \cup V_2,
\end{equation}

where $T_\epsilon = -\log(c_1 \epsilon)$ and $\rho_1 > 0$ is some constant.

Notice that the measure $e^{2t} dt dn$ is a right invariant measure of $AN$ and $[e]AN$ is an open subset in $H \setminus G$. Hence $d\lambda(a_t n)$ (restricted to $[e]AN$) and $e^{2t} dt dn$ are constant multiplies of each other. It follows from the formula of $dg$ that $d\lambda(a_t n) = e^{2t} dt dn$. Besides, observe that the local finiteness of $\Gamma(C_0)$ implies the map $\Gamma_H \setminus C_0^1 \to \Gamma \setminus G / M$ is proper. Using Theorem 3.22, we have

\begin{equation}
\int_{V_2} \int_{\Gamma_H \setminus C_0^1} \int_{M} \Psi^\epsilon(hmg) dm dh d\lambda(g)
\end{equation}

\begin{equation}
= \int_{z \in E_{T,\epsilon}^+} \int_{T_\epsilon - \rho_1 \epsilon}^{\log T + \rho_1 \epsilon} e^{2s} \int_{\Gamma_H \setminus C_0^1} \int_{M} \Psi^\epsilon(hma_s n_{-z}) dm dh ds dn_{-z}
\end{equation}

\begin{equation}
\leq \int_{z \in E_{T,\epsilon}^+} \int_{T_\epsilon - \rho_1 \epsilon}^{\log T + \rho_1 \epsilon} \frac{|\mu^{|PS|}_H|}{|m^{BMS}|} m^{BR}(\Psi^\epsilon_{-z}) e^{\delta s} + \rho_2 \cdot (S_l(\Psi^\epsilon_{-z}) e^{(\delta - \eta_0) s}) ds dn_{-z}
\end{equation}

\begin{equation}
\leq \frac{|\mu^{|PS|}_H|}{\delta |m^{BMS}|} T^3 (1 + \rho_3 \cdot \epsilon) \int_{z \in E_{T,\epsilon}^+} m^{BR}(\Psi^\epsilon_{-z}) dn_{-z} + \rho_3 \cdot (S_l(\Psi^\epsilon) T^{\delta - \eta_0}),
\end{equation}
where $\Psi^{\epsilon}(g) = \int_{m \in M} \Psi^{\epsilon}(gmnz)dm$ and $\rho_2, \rho_3 > 0$ are some constants. To obtain the last inequality above, we use the estimate that $\sup_{z \in E_{\rho_1}} S_t(\Psi^{\epsilon}(z)) < S_t(\Psi^{\epsilon})$ because for any monomial $X$ of order 1, we have $X(\Psi^{\epsilon}(z))(g) = \int_{m \in M} \text{Ad}_{n,m} X(\Psi^{\epsilon})(gmnz)dm$.

Since $E$ is bounded and regular, Proposition 3.12 implies

$$\int_{z \in E_{\rho_1}^{+}} m^{BR}(\Psi^{\epsilon})dnz \leq (1 + \rho_4 \cdot \epsilon) \cdot \omega_T(E^{+}_{\rho_6 \epsilon})$$

$$\leq (1 + \rho_5 \cdot \epsilon^p) \omega_T(E),$$

where $\rho_4, \rho_5 > 0$ are some constants and $p$ is the constant appearing in the definition of the regularity of $E$ (Definition 3.14).

Therefore,

$$\int_{V_1} \int_{H \setminus \hat{C}_0} \int_{M} \Psi^{\epsilon}(hmg)dmdhd\lambda(g)$$

$$\leq \frac{|\mu_H^{PS}|}{\delta |m^{BMS}|} T^{T \delta} \omega_T(E) + \rho_6 \cdot (\epsilon^p T^{T \delta} + \epsilon^{\delta(i) T^{T \delta - \eta_0}})$$

for some $\rho_6 > 0$,

where we use the estimate $S_t(\Psi^{\epsilon}) = O(\epsilon^{-(3+i)})$ since $\text{dim } G = 6$.

Consider $V_1$ in (3.29). Fix small $\epsilon'$ such that $\epsilon' > \epsilon$ and it satisfies Proposition 3.6. We decompose $V_1$ into two parts using Proposition 3.6.

$$V_1 \subset \bigcup_{0 \leq t \leq T_1} H \setminus Ha_tK(t)N_{-E_{2+}} \cup \bigcup_{T_1 - \rho \epsilon' \leq t \leq T_1 + \rho \epsilon'} H \setminus Ha_tN_{-E_{\rho \epsilon'}}$$

$$= V_3 \cup V_4,$$

where $T_1 := -\log(c_1 \epsilon')$ and $\rho_7 > 0$ is some constant.

For $V_4$, by the similar way as we get (3.31), we have

$$\int_{V_4} \int_{H \setminus \hat{C}_0} \int_{M} \Psi^{\epsilon}(hmg)dmdhd\lambda(g)$$

$$\leq 8 e^{\delta T \epsilon} \frac{|\mu_H^{PS}|}{\delta |m^{BMS}|} \omega_T(E^{+}_{\rho \epsilon'}),$$

for some constant $\rho_8 > 0$.

Since $E$ is bounded and $\epsilon'$ is fixed, $\omega_T(E^{+}_{\rho \epsilon'}) = O(1)$. As $T \epsilon = -\log(c_1 \epsilon)$, we get

$$\int_{V_4} \int_{H \setminus \hat{C}_0} \int_{M} \Psi^{\epsilon}(hmg)dmdhd\lambda(g) = O(\epsilon^{-\delta}).$$

(3.32)
For $V_3$, reversing the process of translating the circle counting into orbit counting, we have

$$(3.33) \quad \int_{V_3} \int_{\gamma \backslash C_0} \int_M \Psi(hmg) dm dh d\lambda(g) = O(\#\{C \in \Gamma(C_0) : C \cap E_{2\epsilon}^+ \neq \emptyset, \text{Curv}(C) \leq T_1\}) = O(1),$$

as $T_1$ is fixed.

Adding (3.31), (3.32) and (3.33) together, we get

$$\langle F_{\epsilon}^{-}, \Psi \epsilon \rangle \leq \frac{\mu_{PS}}{\delta |m_{BMS}|} T^\delta \omega_T(E) + \rho_9 \cdot (\epsilon^p T^\delta + \epsilon^{-3/2} + \epsilon^{-\eta_0} + \epsilon^{-\delta}),$$

for some constant $\rho_9 > 0$.

Set $\eta' = \min\{\frac{\rho_9}{\delta + p}, \frac{\eta_0}{3 + l + p}\}$ and $\epsilon = \max\{T_{2^{-1}}, T_{\eta'-\eta_0}^{-1}\}$. Then $T > \frac{1}{c_0 \epsilon}$. Hence

$$N_T(\Gamma(C_0), E) \leq \langle F_{\epsilon}^{-}, \Psi \epsilon \rangle \leq \frac{\mu_{PS}}{\delta |m_{BMS}|} T^\delta \omega_T(E) + \rho_9 \cdot T^\delta - \eta'.
$$

For $\langle F_{\epsilon}^{+}, \Psi \epsilon \rangle$, the definition of $W_{T, \epsilon}^{-}$ (3.25) implies the following inclusion:

$$W_{T, \epsilon}^{-} \supset \bigcup_{2^{-1} \leq t \leq T^{-1}} H \backslash H_{a t}N_{-E_{2\epsilon}},$$

where $T_2$ is some large fixed number.

Using similar argument as above, we have

$$(3.34) \quad \langle F_{\epsilon}^{+}, \Psi \epsilon \rangle \geq \frac{\mu_{PS}}{\delta |m_{BMS}|} T^\delta \omega_T(E) + \rho_{10} \cdot T^\delta - \frac{\eta_0}{3 + l + p},$$

for some constant $\rho_{10} > 0$.

Therefore, there exists $\eta > 0$ so that as $T \to \infty$, we have

$$(3.35) \quad N_T(\Gamma(C_0), E) = \frac{\mu_{PS}}{\delta |m_{BMS}|} T^\delta \omega_T(E) + O(T^\delta - \eta).$$

**Step 2:** We prove the theorem for a general circle packing.

Let $C$ be any circle in the circle packing $P$. Denote the radius of $C$ by $r$ and the center of $C$ by $z_0$. Set

$$g_C := n_{z_0} a_{\log r} = \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{r} & 0 \\ 0 & \sqrt{r^{-1}} \end{pmatrix}.$$

Then $g_C(C_0) = C$. And

$$N_T(\Gamma(C), E) = N_{rT}(g_{C}^{-1} \Gamma g_C(C_0), g_C^{-1}(E)).$$
Lemma 3.36 (Lemma 6.5 in [12]).

\[ |m_{gC^{-1} \Gamma gC}^{BMS}| = |m_{\Gamma}^{BMS}|. \]

Lemma 3.37 (Lemma 6.7 in [12]). For any bounded Borel subset \( E \subset \mathbb{C} \),

\[ \omega_{gC^{-1} \Gamma gC}(g_C^{-1}(E)) = r^{-\delta} \omega_{\Gamma}(E). \]

Using the above two lemmas, we obtain

\[ N_T(\Gamma(C), E) = \frac{|\mu_{gC^{-1} \Gamma gC, H}^PS|}{|m_{\Gamma}^{BMS}|} (rT)\omega_{gC^{-1} \Gamma gC}(g_C^{-1}(E)) + O(T^{\delta-\eta}) \]

\[ = \frac{|\mu_{gC^{-1} \Gamma gC, H}^PS|}{|m_{\Gamma}^{BMS}|} T^\delta \omega_{\Gamma}(E) + O(T^{\delta-\eta}). \]

Since \( \mathcal{P} \) consists of finitely many \( \Gamma \)-orbits, (3.38) implies our theorem. □

4. Effective circle count for circle packing in ideal triangle of \( \mathbb{H}^2 \)

We use the upper half plane model for \( \mathbb{H}^2 \):

\[ \mathbb{H}^2 = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \}. \]

For any circle packing \( \mathcal{P} \) contained in \( \mathbb{H}^2 \) and any \( t > 0 \), we define the following counting function:

\[ N_t(\mathcal{P}) := \# \{ C \in \mathcal{P} : \text{Area}_{\text{hyp}}(C) > t \}, \]

where \( \text{Area}_{\text{hyp}}(C) \) is the hyperbolic area of the disk enclosed by \( C \).

Let \( \mathcal{T} \) be the ideal triangle in \( \mathbb{H}^2 \) whose three sides are given by the circles \( \{ x = \pm 1 \} \) and \( \{ z \in \mathbb{C} : |z| = 1 \} \). Let \( \mathcal{P}(\mathcal{T}) \) be the circle packing attained by filling in the largest inner circles. This section is devoted to give an effective estimate of \( N_t(\mathcal{P}(\mathcal{T})) \).

Denote by \( \mathcal{P}_0 \) the Apollonian circle packing generated by the circles \( \{ x = \pm 1 \}, \{ |z| = 1 \} \) and \( \{ |z-2i| = 1 \} \). It is clear that \( \mathcal{P}(\mathcal{T}) \) is a part of \( \mathcal{P}_0 \) (see Figure 3). Let \( C_1, C_2, C_3 \) and \( C_4 \) be the circles \( \{|z-(1+i)| = 1 \}, \{|z-(-1+i)| = 1 \}, \{y = 0\} \) and \( \{y = 2\} \) respectively. Set \( S_i \) to be the reflection with respect to \( C_i \). Denote \( \text{PSL}_2(\mathbb{C}) \cap \langle S_1, S_2, S_3, S_4 \rangle \) by \( \mathcal{A} \). We fix \( \Gamma \) a torsion-free finite index subgroup in \( \mathcal{A} \). The limit set \( \Lambda(\Gamma) \) of \( \Gamma \) is exactly the closure of \( \mathcal{P}_0 \) (Proposition 2.9 in [3]). As a result, the critical exponent of \( \Gamma \) equals the Hausdorff dimension of the residual set of \( \mathcal{P}_0 \), which is denoted by \( \alpha \).
Throughout this section, we let \( \eta \) be a constant in \((0, \frac{1}{4})\) satisfying the constraints described in (4.26), (4.31) and (4.35). For all small \( t > 0 \), set
\[
T(\eta, t) := \{ x + iy \in \mathcal{T} : t^\eta \leq y \leq t^{-\eta} \}.
\]
In fact, an elementary computation in hyperbolic geometry shows that
\[
N_t(\mathcal{P}(T)) = \# \{ C \in \mathcal{P}(T) : C \cap \mathcal{T}(\frac{1}{4}, t) \neq \emptyset, \text{Area}_{\text{hyp}}(C) > t \}.
\]
However, it will be clear from the proof that \( T(\frac{1}{4}, t) \) is not the right region to consider in order to obtain an effective estimate of \( N_t(\mathcal{P}(T)) \).

**Notation 4.2.** For any circle packing \( \mathcal{P} \) in \( \hat{\mathbb{C}} \) and subset \( E \subset \mathbb{H}^2 \), set
\[
\mathcal{P} \cap E := \{ C \in \mathcal{P} : C \cap E \neq \emptyset \}.
\]

### 4.1. Reformulation into orbit counting problem.

Let \( C \) be a hyperbolic circle in \( \mathbb{H}^2 \). Note that \( C \) is also an Euclidean circle. Denote the Euclidean center and the Euclidean radius of \( C \) by \( e_C \) and \( r_C \) respectively. We have the following equivalent relation through a basic computation in hyperbolic geometry:
\[
\text{Area}_{\text{hyp}}(C) > t \iff r_C > \text{Im}(e_C) \cdot \beta(t),
\]
where
\[
\beta(t) = \frac{\sqrt{t(4\pi + t)}}{(2\pi + t)}.
\]

Fix a circle packing \( \mathcal{P} \) contained in \( \mathcal{T} \) in the following. The counting function \( N_t(\mathcal{P}) \) can be reformulated as follows using (4.3):
\[
N_t(\mathcal{P}) = \# \{ C \in \mathcal{P} : r_C > \text{Im}(e_C) \cdot \beta(t) \}.
\]

We introduce the following functions on \( \mathbb{H}^2 \): for every small \( \epsilon > 0 \),
\[
\begin{align*}
h_+^t(z) &= -\log(\beta(t) \text{Im}(z)) \\
h_-^t(z) &= -\log((1 + t)\beta(t) \text{Im}(z)).
\end{align*}
\]
Lemma 4.7. For any small $t > 0$, the following holds

\[ N_t(\mathcal{P} \cap \mathcal{T}(\eta, t)) \leq \# \left\{ C \in \mathcal{P} : \hat{C} \cap B_+(\mathcal{T}(\eta, t)) \neq \emptyset \right\} + n(\mathcal{P}, t), \]

where $n(\mathcal{P}, t) := \# \{ C \in \mathcal{P} : \text{Area}_{\text{hyp}}(C) > t, e_C \notin \mathcal{T}(\eta, t) \}$.

As $B_+(\mathcal{T}(\eta, t)) = \{ z + rj \in \mathbb{H}^3 : z \in \mathcal{T}(\eta, t), \beta(t) \text{Im}(z) < r \leq 1 \}$, this lemma can be easily verified using (4.3).

Lemma 4.8. For any small $t > 0$, we have

\[ N_t(\mathcal{P} \cap \mathcal{T}(\eta, t)) \geq \# \left\{ C \in \mathcal{P} : \hat{C} \cap B_-(\mathcal{T}(\eta, t)) \neq \emptyset \right\}. \]

Proof. Let $C$ be any circle in $\mathcal{P}$ such that $\hat{C} \cap B_-(\mathcal{T}(\eta, t)) \neq \emptyset$. Suppose $e_C = x_0 + iy_0$. We may assume that $x_0 + i(y_0 + r_C \cos \theta_0) + j r_C \sin \theta_0$ is the point at which $\hat{C}$ and $B_-(\mathcal{T}(\eta, t))$ intersect. We claim that $C \cap \mathcal{T}(\eta, t) \neq \emptyset$ and $\text{Area}_{\text{hyp}}(C) > t$. The claim $C \cap \mathcal{T}(\eta, t) \neq \emptyset$ directly follows from the observation that $B_-(\mathcal{T}(\eta, t)) = \{ z + rj \in \mathbb{H}^3 : z \in \mathcal{T}(\eta, t), (1 + t)\beta(t) \text{Im}(z) < r \leq 1 \}$. Now we show the second claim. In view of (4.3), it suffices to show that $r_C > y_0 \beta(t)$.

Consider the function $f$ given by $f(\theta) = r_C \sin \theta - (1 + t)\beta(t)(y_0 + r_C \cos \theta)$ for $\theta \in (0, \pi)$. Since $x_0 + i(y_0 + r_C \cos \theta_0) + j r_C \sin \theta_0$ is in $B_-(\mathcal{T}(\eta, t))$, $f(\theta_0) > 0$. There are two possibilities to discuss.

First suppose $\theta_0 \in \left[ \frac{\pi}{2}, \pi \right)$. Then $\max f|_{\left[ \frac{\pi}{2}, \pi \right)} = r_C(1 + (1 + t)^2 \beta(t)^2)^{1/2} - y_0(1 + t)\beta(t) > 0$. When $t$ is small enough, we have

$$(1 + (1 + t)^2 \beta(t)^2)^{1/2} < 1 + t.$$

Consequently,

$$r_C > \frac{y_0(1 + t)\beta(t)}{(1 + (1 + t)^2 \beta(t)^2)^{1/2}} > y_0 \beta(t).$$

Next we suppose that $\theta_0 \in (0, \frac{\pi}{2})$. Since the derivative of $f$ is always positive on $(0, \frac{\pi}{2}]$, and $f(\theta_0) > 0$, we have $f(\frac{\pi}{2}) > 0$, i.e. $r_C > (1 + t)\beta(t)y_0 > \beta(t)y_0$, completing the proof of the second claim.

Applying Lemmas 4.7 and 4.8 to $\Gamma(C_0) \cap \mathcal{T}(\eta, t)$, we relate the circle counting function to the following orbit counting functions:
Proposition 4.9. For small $t > 0$, the following inequalities hold:
\[
\# (\{e \in \Gamma \cap H \mid HKP_-(\mathcal{T}(e,t))\}) \leq N_t (\Gamma(C_0) \cap \mathcal{T}(e,t)),
\]
\[
N_t (\Gamma(C_0) \cap \mathcal{T}(e,t)) \leq \# (\{e \in \Gamma \cap H \mid HKP_+(\mathcal{T}(e,t))\}) + n(\Gamma(C_0), t),
\]
where $n(\Gamma(C_0), t) = \# \{C \in \Gamma(C_0) : \text{Area}_{\text{hyp}}(C) > t, e_C \notin \mathcal{T}(e,t)\}$.

4.2. Number of disks in the cuspidal neighborhoods. We estimate the counting function $n(\mathcal{P}_0, t)$ defined in Lemma 4.7.

Proposition 4.10. For all sufficiently small $t > 0$, we have
\[
n(\mathcal{P}_0, t) = O(t^{-\frac{\alpha}{2} + \eta(\alpha - 1)}).
\]

Lemma 4.11. For all sufficiently small $t > 0$, we have
\[
N_t (\mathcal{P}_0 \cap \{ |z| \geq t^{-\eta} \}) = O(t^{-\frac{\alpha}{2} + \eta(\alpha - 1)}).
\]

Proof. Let $C$ be any circle in $\mathcal{P}_0$ such that $C$ intersects $\{ |z| \geq t^{-\eta} \}$. Since $\mathcal{P}_0$ is periodic, we choose $C' \subset \mathcal{P}_0 \cap \{ 0 < \text{Im} z \leq 2 \}$ so that $C = C' + 2k$ for some $k \in \mathbb{N}$. Denote the Euclidean radius of $C'$ by $r_0$. Then
\[
\text{Area}_{\text{hyp}}(C) = \frac{\pi r_0^2}{4k^2} (1 + O(k^{-1})).
\]
It follows from Theorem 3.21 that
\[
N_t (\mathcal{P}_0 \cap \{ |z| \geq t^{-\eta} \})
\ll \sum_{k \geq t^{-\eta}} \# \{ C' \in \mathcal{P}_0 \cap \{ 0 < \text{Im} z \leq 2 \} : \text{Curv}(C) < \sqrt{\frac{\pi}{4k^2t}} \}
\ll \sum_{k \geq t^{-\eta}/2} \left( \frac{\pi}{4k^2t} \right)^{\frac{\alpha}{2}} \ll t^{-\frac{\alpha}{2} + \eta(\alpha - 1)}.
\]
\]

Lemma 4.12. For all sufficiently small $t > 0$, we have
\[
N_t (\mathcal{P}_0 \cap \{ |z| \geq t^{\eta} \}) = O(t^{-\frac{\alpha}{2} + \eta(\alpha - 1)}).
\]

Proof. Let $g_0 := \left( \begin{array}{cc} \frac{1}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{array} \right) \in \text{PSL}_2(\mathbb{R})$. As $g_0$ preserves $\mathcal{P}_0$ as well as the hyperbolic metric, we have
\[
N_t (\mathcal{P}_0 \cap \{ |z| \geq t^{\eta} \}) = N_t (\mathcal{P}_0 \cap \{ |z| \geq c t^{\eta} \}),
\]
where $c > 0$ is some constant. As a result, the estimate of $N_t (\mathcal{P}_0 \cap \{ |z| \geq c t^{\eta} \})$ easily follows from Lemma 4.11. The estimate of $N_t (\mathcal{P}_0 \cap \{ |z| \geq c t^{\eta} \})$ can be verified similarly using $g_0^{-1}$. \qed
In view of the inclusion
\[ \mathcal{T}(\eta, t) \subset \{|z| \geq t^{-\eta}\} \cup \{|z - 1| \leq 2t^\eta\} \cup \{|z + 1| \leq 2t^\eta\}, \]
Proposition 4.10 directly follows from Lemmas 4.11 and 4.12.

4.3. **On the measure** \((\text{Im } z)^{-\alpha} d\omega_{\Gamma}\). We utilize the measure \(d\omega_{\Gamma}\) (Definition 3.10) when we count the number of circles with respect to the Euclidean metric. As now we are counting the number of circles with respect to the hyperbolic metric, we introduce a modification of \(d\omega_{\Gamma}\), \((\text{Im } z)^{-\alpha} d\omega_{\Gamma}\) on \(\mathbb{H}^2\).

4.3.1. **Estimate of the measure of cuspidal neighborhoods.**

**Theorem 4.13** (Theorem 9.3 in [10]). We have
\[ \int_{z \in \mathcal{T}} (\text{Im } z)^{-\alpha} d\omega_{\Gamma}(z) < \infty. \]
Recall that \(\nu_j\) is the PS-density at \(j\).

**Proposition 4.14.** There exists \(N_1 \geq 1\) such that for all \(N \geq N_1\), we have
\[ \nu_j(\mathcal{E}_N) \ll N^{-2\alpha+1}, \]
where \(\mathcal{E}_N = \{z \in \mathbb{C} : |z| \geq N\}\).

**Proof.** Because the limit set \(\Lambda(\Gamma)\) of \(\Gamma\) is exactly the closure of \(\mathcal{P}_0\), we have
\[ \Lambda(\Gamma) \setminus \{\infty\} \subset \{|\text{Re } z| \leq 1\}. \]

The stabilizer of \(\infty\) in \(\Gamma\) is generated by \(\gamma_0 = \begin{pmatrix} 1 & iy_0 \\ 0 & 1 \end{pmatrix}\) for some \(y_0 > 0\). Define the following relatively compact set in \(\mathcal{F}\):
\[ \mathcal{F} = \{z : |\text{Re } z| \leq 1, \text{ and } 0 \leq \text{Im } z < y_0\}. \]
For any \(z \in \Lambda(\Gamma) \setminus \{\infty\}\), there exists a unique \(k \in \mathbb{Z}\) such that \(z \in \gamma_0^k \mathcal{F}\). This yields the estimate
\[ y_0^2 \cdot (|k| + 1)^2 \geq |z|^2 - |\text{Re } z|^2 \geq |z|^2 - 1. \]
Therefore, there exist \(c \geq 1\) and \(N_1 \geq 1\) such that for all \(N \geq N_1\), we have
\[ \mathcal{E}_{cN} \cap \Lambda(\Gamma) \subset \bigcup_{|k| \geq N} \gamma_0^k \mathcal{F}. \]
We continue to prove the proposition. By the above inclusion, for any $N \geq N_1$, we have

\[
\nu_j(\mathcal{E}_{cN}) \leq \sum_{k \geq N} \nu_j(\gamma_{0}^{k}\mathcal{F}) + \sum_{k \geq N} \nu_j(\gamma_{0}^{-k}\mathcal{F}) \\
= \sum_{k \geq N} \nu_{\gamma_{0}^{-k}j}(\mathcal{F}) + \sum_{k \geq N} \nu_{\gamma_{0}^{k}j}(\mathcal{F}) \\
= \sum_{k \geq N} \int_{z \in \mathcal{F}} e^{-\alpha \beta_{s}(\gamma_{0}^{-k}j,j)}d\nu_j(z) + \sum_{k \geq N} \int_{z \in \mathcal{F}} e^{-\alpha \beta_{s}(\gamma_{0}^{k}j,j)}d\nu_j(z) \\
\ll \sum_{k \geq N} k^{-2\alpha} \nu_j(\mathcal{F}) + \sum_{k \geq N} k^{-2\alpha} \nu_j(\mathcal{F}) \quad \text{(by Lemma 3.11)} \\
\ll N^{-2\alpha + 1}.
\]

□

**Proposition 4.15.** There exists $T_0 > 1$ such that for any pair of real numbers $a, b$ with $T_0 < a < b < \infty$ and any $\kappa$ with $\kappa > -2$ and $\kappa \neq -1$, we have

\[
\int_{\{z \in \mathcal{T}: a < \text{Im} z < b\}} (\text{Im } z)^{\kappa} d\omega_{\Gamma}(z) = O \left(\frac{a^{\kappa+1} + b^{\kappa+1}}{|\kappa + 1|}\right).
\]

**Proof.** For $a > 0$ large enough, using Lemma 3.11 we have

\[
\int_{\{z \in \mathcal{T}: a < \text{Im} z < b\}} (\text{Im } z)^{\kappa} d\omega_{\Gamma}(z) = \int_{\{z \in \mathcal{T}: a < \text{Im} z < b\}} (\text{Im } z)^{\kappa}(|z|^2 + 1)^{\alpha} d\nu_j(z) \\
\ll \int_{\{z \in \mathcal{T}: a < \text{Im} z < b\}} (\text{Im } z)^{\kappa + 2\alpha} d\nu_j(z).
\]

Denote by $\pi$ the projection map from points in $\{z \in \mathcal{T}: a < \text{Im} z < b\}$ to their $y$-coordinates. Let $\pi_{\ast} \nu_j$ be the pushforward of $\nu_j$. Letting
\( \mathcal{E}_s = \{ z \in \mathbb{C} : |z| > s \} \) for \( s > 0 \), we have

\[
\int_{\{z \in \mathcal{T} : \text{Im} z < b \}} (\text{Im} z)^{\kappa + 2\alpha} d\nu_j(z)
\]
\[= \int_a^b y^{\kappa + 2\alpha} d\pi_* \nu_j(y)
\]
\[
\ll \int_a^b \int_0^y s^{\kappa + 2\alpha - 1} ds d\pi_* \nu_j(y)
\]
\[= \int_a^b \int_a^b s^{\kappa + 2\alpha - 1} d\pi_* \nu_j(y) ds
\]
\[\leq \int_a^b s^{\kappa + 2\alpha - 1} \nu_j(\mathcal{E}_a) ds + \int_a^b s^{\kappa + 2\alpha - 1} \nu_j(\mathcal{E}_s) ds
\]
\[
\ll \int_a^b s^{\kappa + 2\alpha - 1} \cdot a^{1-2\alpha} ds + \int_a^b s^{\kappa + 2\alpha - 1} \cdot s^{1-2\alpha} ds \quad \text{(by Proposition 4.14)}
\]
\[
\ll (\kappa + 1)^{-1} \cdot (a^{\kappa + 1} + b^{\kappa + 1}).
\]

Now we estimate the size of the cuspidal neighborhoods under the measure \((\text{Im} z)^{-\alpha} d\omega_\Gamma\).

**Corollary 4.16.** For all small \( t > 0 \), we have

\[
\int_{z \in \{w \in \mathcal{T} : \text{Im} w > t^{-\eta} \}} (\text{Im} z)^{-\alpha} d\omega_\Gamma(z) = O(t^{\eta(\alpha - 1)}),
\]

where \( \alpha \) is the critical exponent of \( \Gamma \) and \( \eta \) is defined as (4.1).

The corollary can be proved by setting \( k = -\alpha \) in Proposition 4.15 and taking the limit of \( \int_{\{z \in \mathcal{T} : t^{-\eta} < \text{Im} z < N\}} (\text{Im} z)^{-\alpha} d\omega_\Gamma(z) \) as \( N \to \infty \).

**Corollary 4.17.** For all sufficiently small \( t > 0 \), we have

\[
\int_{\{z \in \mathcal{T} : |z| \leq t^\eta \}} (\text{Im} z)^{-\alpha} d\omega_\Gamma(z) = O(t^{\eta(\alpha - 1)}).
\]

**Proof.** As observed in Lemma 4.12, \( g_0^{-1} \) maps \( \{ z \in \mathcal{T} : |z - 1| \leq t^\eta \} \) to \( \{ z \in \mathcal{T} : \text{Im} z \geq ct^{-\eta} \} \) for some constant \( c > 0 \).

For every \( x \in \mathbb{H}^3 \), set

\[
\tilde{\nu}_x := (g_0)^{*} \nu_{\Gamma, g_0 x},
\]

where \( g_0^* \nu_{\Gamma, g_0(x)}(R) = \nu_{\Gamma, g_0(x)}(g_0(R)) \). It is easy to check that \( \{ \tilde{\nu}_x : x \in \mathbb{H}^3 \} \) is a \( g_0^{-1} \Gamma g_0 \)-invariant conformal density of dimension \( \delta_{g_0^{-1} \Gamma g_0} = \delta_\Gamma = \alpha \).
Using these two observations, we obtain
\[
\int \{ z \in T : |z - 1| \leq t \eta \} (\text{Im} z)^{-\alpha} d\omega_T(z) = \int \{ z \in T : \text{Im} z \geq ct - \eta \} (\text{Im}(g_0 z))^{-\alpha} e^{\alpha \beta g_0 z (j_0 g_0 z + j)} d\nu_{g_0^{-1} \Gamma g_0 g_0^{-1} j} (z).
\]
We can get an estimate of \( d\omega_T \) is a locally finite measure, we obtain the following corollary by setting \( \kappa = 0 \) in Proposition 4.15:

**Corollary 4.18.** For all small \( t > 0 \), we have
\[
\omega_T(T(\eta, t)) = O(t^{-\eta}).
\]

4.3.2. Relate \( m^{BR} \) with \((\text{Im} z)^{-\alpha} d\omega_T\). For small \( \epsilon > 0 \), let \( U_\epsilon \) be the symmetric \( \epsilon \)-neighborhood of \( e \) in \( G \). Recall that we defined \( T(\eta, t)^{\pm} = \bigcup_{u \in U_\epsilon} u T(\eta, t) \) and \( T(\eta, t)^{\pm}_\epsilon = \bigcap_{u \in U_\epsilon} u T(\eta, t) \).

Let \( \psi^\epsilon \) be a non-negative function in \( C_c^\infty(G) \) supported in \( U_\epsilon \) with integral one. Set \( \Psi^\epsilon = C_c^\infty(\Gamma \backslash G) \) to be the \( \Gamma \)-average of \( \psi^\epsilon \).

We establish the following relation between \( m^{BR} \) and \((\text{Im} z)^{-\alpha} d\omega_T\):

**Proposition 4.19.** For all small \( \epsilon > 0 \) and \( t > 0 \), we have
\[
\int_{z \in T(\eta, t)} m^{BR}(\Psi^\epsilon - z)(\text{Im} z)^{-\alpha} dn_{-z} = (1 + O(\epsilon t^{-\eta})) \int_{z \in T(\eta, t)^{\pm}} (\text{Im} z)^{-\alpha} d\omega_T(z),
\]
where \( \Psi^\epsilon \in C_c^\infty(\Gamma \backslash G)^M \) is given by \( \Psi^\epsilon(g) := \int_{m \in M} \Psi^\epsilon(gmn_{-z}) dm \).

We first introduce a function on \( G \):

**Definition 4.20.** For \( \psi \in C_c(\mathbb{H}^2) \), define a function \( F_\psi \) on \( MAN^-N \subset G \) by
\[
F_\psi(ma_t n_x n_z) = \begin{cases} e^{-\alpha t} \psi(-z)(-y)^{-\alpha}, & \text{if } z = x + iy \text{ with } y < 0 \\ 0, & \text{otherwise.} \end{cases}
\]

Since the product map \( M \times A \times N^- \times N \to G \) has a diffeomorphic image, \( F_\psi \) can be regarded as a function on \( G \).

**Lemma 4.21.** For any \( \psi \in C_c(\mathbb{H}^2) \),
\[
\int_{z \in \mathbb{H}^2} \psi(z) (\text{Im} z)^{-\alpha} d\omega_T(z) = \int_{k \in K/M} F_\psi(k^{-1}) d\nu_j(k(0)).
\]
Lemma 4.22. For small $\epsilon > 0$ and any $g \in U_\epsilon$,

$$\int_{k \in K/M} \mathcal{F}_{T(\eta,t)}(k^{-1}g) d\nu_j(k(0)) = (1+O(\epsilon t^{-\eta})) \int_{z \in T(\eta,t)^+} (\text{Im } z)^{-\alpha} d\omega_T(z),$$

where $\mathcal{F}_{T(\eta,t)} := \mathcal{F}_{\chi_T(\eta,t)}$ with $\chi_T(\eta,t)$ the characteristic function of $T(\eta,t)$.

Proof. Write $k^{-1} = m_0a_t n_0^\omega n_z$ and $g = m_0a_t n_0^\omega n_{z_1} \in U_\epsilon$. By Lemma 3.13 we have $k^{-1}g = m_0a_t n_0^\omega n_{z_0}$, where $t_0 = t + t_1 + 2 \log(1 + e^{-t_1 - 2\theta_1} |w_1 z|)$ and $z_0 = e^{-t_1 - 2\theta_1} (1 + w_1 e^{-2\theta_1} z)^{-1} + z_1$. We have

$$\int_{k \in K/M} \mathcal{F}_{T(\eta,t)}(k^{-1}g) d\nu_j(k(0))$$

$$= \int_{k \in K/M} e^{-\alpha t_0} \chi_T(\eta,t) (g^{-1}k(0)) \text{Im } (z_0)^{-\alpha} d\nu_j(k(0))$$

$$= \int_{k(0) \in T(\eta,t)^+} e^{-\alpha t_0} \text{Im } (z_0)^{-\alpha} d\nu_j(k(0)).$$

Using the estimate $e^{-\alpha t_0} = (1 + O(|z|\epsilon))e^{-\alpha t}$ and $z_0 = (1 + O(|z| + |z|^{-1}\epsilon))z$,

we reach the conclusion that

$$\int_{k \in K/M} \mathcal{F}_{T(\eta,t)}(k^{-1}g) d\nu_j(k(0)) = (1+O(\epsilon t^{-\eta})) \int_{z \in T(\eta,t)^+} (\text{Im } z)^{-\alpha} d\omega_T(z).$$

With Lemma 4.22 available, Proposition 4.19 follows from the same argument as the proof of Lemma 5.7 in [12].

4.4. Conclusion.

Theorem 4.23. There exists a constant $\rho > 0$, such that as $t \to 0$, we have

$$N_t(P(\mathcal{T})) = \frac{\text{skr}(P_0)}{\alpha |m_1^{\text{BMS}}|} \left( \frac{t}{\pi} \right)^{-\frac{\alpha}{2}} \int_{P(\mathcal{T})} (\text{Im } z)^{-\alpha} dH^\alpha(z) + O(t^{-\frac{\alpha}{2} + \rho}),$$

where $dH^\alpha$ is the $\alpha$-dimensional Hausdorff measure on $P(\mathcal{T})$.

Set $\xi = t^{4\eta}$ for the rest of the paper. For any subset $E \subset \mathcal{T}$ and every small $t > 0$, define

$$Q^+(E) := \{a_s n_{-z} : z \in E \text{ and } -t^\eta \leq s \leq h_t^+(z) + t^\eta\},$$

$$Q_t^-(E) := \{a_s n_{-z} : z \in E \text{ and } t^\eta \leq s \leq h_t^-(z) - t^\eta\},$$
where $h^+_t(\cdot)$ and $h^-_t(\cdot)$ are the functions defined as (4.5). We first show the following lemma:

**Lemma 4.24.** There exists $c_4 > 0$, such that for all sufficiently small $t > 0$, we have

$\quad KP_+(T(\eta,t))U_{c_4} \subset KQ^+_\xi(T(\eta,t)_{\xi_2^T})$, 
$\quad KQ^-_\xi(T(\eta,t)_{\xi_2^T})U_{c_4} \subset KP_-(T(\eta,t))$, 

where $P_+(\cdot)$ and $P_-(\cdot)$ are defined as (4.6).

**Proof.** Up to a uniform Lipschitz constant, we may write 

$\quad U_\xi = N_\xi M_\xi A_\xi N_\xi$. 

It suffices to show

$\quad KP_+(T(\eta,t))M_\xi A_\xi N^-_\xi \subset KQ^+_\xi(T(\eta,t)_{\xi_2^T})$. 

Fix $z \in T(\eta,t)$. For any $m_\theta a_s \in M_\xi A_\xi$, 

$\quad n_\xi w m_\theta a_s = m_\theta a_s n_\xi (z + z')$, 

where $|z'| = O(t^{3\eta})$. Note that 

$\quad h^+_t(z + z') = h^+_t(z) + O(t^{2\eta})$. 

Hence 

$\quad P_+(T(\eta,t))M_\xi A_\xi \subset KQ^+_\xi(T(\eta,t)_{\xi_2^T})$. 

For $n^-_w \in N^-_\xi$ and $a_s$ with $0 \leq s \leq h^+_t(z)$,

$\quad a_s n^-_w = a_s n^-_{w-\frac{1}{1-wz}} \begin{pmatrix} 1 - wz & 0 \\ 0 & (1 - wz)^{-1} \end{pmatrix} n^-_{w-\frac{1}{1-wz}}$. 

Since $n^-_{w-\frac{1}{1-wz}} \in U_\xi$, we have $a_s n^-_{w-\frac{1}{1-wz}} a_s \in U_\xi$. Up to a uniform Lipschitz constant, we may write $a_s n^-_{w-\frac{1}{1-wz}} a_s = k a_{s_1} n_{z_1}$ with $k \in K_\xi$, $a_{s_1} \in A_\xi$ and $n_{z_1} \in N_\xi$. Then 

$\quad a_s n^-_w = k a_{s+s_1} \begin{pmatrix} 1 - wz & 0 \\ 0 & (1 - wz)^{-1} \end{pmatrix} n_-(z + z_2)$, 

with $|z_2| = O(t^{2\eta})$. Therefore 

$\quad P_+(T(\eta,t))N^-_\xi \subset KQ^+_\xi(T(\eta,t)_{\xi_2^T})$. 

The second statement can be proved similarly. 

**Proof of Theorem 4.22. Step 1:** First we consider $N_\xi(\Gamma(C_0) \cap T)$. For simplicity, we set 

$\quad V_+ := H \setminus HKQ^+_\xi(T(\eta,t)_{\xi_2^T})$, 
$\quad V_- := H \setminus HKQ^-_\xi(T(\eta,t)_{\xi_2^T})$. 

For small $t > 0$, define functions $F^\pm_t$ on $\Gamma \backslash G$:

$$F^+_t(g) := \sum_{\gamma \in \Gamma \backslash \Gamma} \chi_{V_+}([e] \gamma g),$$
$$F^-_t(g) := \sum_{\gamma \in \Gamma \backslash \Gamma} \chi_{V_-}([e] \gamma g).$$

We deduce from Propositions 4.9, 4.10 and Lemma 4.24 that for all small $t > 0$ and $g \in U_{c_4 \xi}$ with $\xi = t^{4 \eta}$

$$F^-_t(g) \leq N_t(\Gamma(C_0) \cap T(\eta, t)) \leq F^+_t(g) + \rho_1 \cdot t^{-\frac{\eta}{2} + \eta(\alpha - 1)},$$

where $\rho_1 > 0$ is some constant (cf. Lemma 6.4 in [12]).

Let $\psi^\xi$ be a non-negative function in $C^\infty_c(G)$, supported in $U_{c_4 \xi}$ with integral one. Set $\Psi^\xi \in C^\infty_c(\Gamma \backslash G)$ to be the $\Gamma$-average of $\psi^\xi$. Integrating $F^+_t$ against $\Psi^\xi$, we get

$$\langle F^+_t, \Psi^\xi \rangle = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma \backslash \Gamma} \chi_{V_+}([e] \gamma g) \Psi^\xi(g) dg$$
$$= \int_{g \in \Gamma \backslash G} \chi_{V_+}([e] g) \Psi^\xi(g) dg$$
$$= \int_{g \in V_+} \int_{h \in \Gamma \backslash C^0 \cap} \int_{m \in M} \psi^\xi(hmg) dmdhd\lambda(g).$$

Recall the function $h^+_t(\cdot)$ defined in (4.5). The first constraint for $\eta$ is that

$$0 < \eta < 1/10.$$  

This constraint guarantees that for every $z \in T(\eta, t)^+_{t_{20}}$,

$$- \log(c_1 \xi) < h^+_t(z),$$

where $c_1$ is the constant described at the beginning of Section 3.2.

Set

$$T_0 := - \log(c_1 \xi), \quad \hat{A}(z) := \{ a_s : T_0 - 2t^\eta \leq s \leq h^+_t(z) + 2t^\eta \},$$
$$K(s) := \{ k \in K : a_s k \in HKA^+ \} \text{ for every } s > 0,$$
$$W_1 := \bigcup_{z \in T(\eta, t)^+_{t_{20}}} H \backslash H \hat{A}(z)n_{-z},$$
$$W_2 := \bigcup_{0 \leq s \leq T_0} H \backslash Ha_s K(s)N_{-T(\eta, t)^+_{t_{20}}}. $$
For every $z \in \mathcal{T}(\eta, t)_{12n}^+$, using Lemma 3.7 (1), we get

$$\bigcup_{-\nu \leq s \leq h_1^+(z)+\nu} H \setminus H \mathcal{K} \mathcal{A}_s n_{-z} = \bigcup_{0 \leq s \leq h_1^+(z)+\nu} H \setminus H \mathcal{K} \mathcal{A}(s) n_{-z}.$$  

Applying the same argument as the proof of Proposition 3.6 to the subset $\bigcup_{0 \leq s \leq h_1^+(z)+\nu} H \setminus H \mathcal{K} \mathcal{A}(s) n_{-z}$ for every $z \in \mathcal{T}(\eta, t)_{12n}^+$, we have

$$V_+ \subset W_1 N_{\rho_2 \xi} \bigcup W_2,$$

for some constant $\rho_2 > 0$.

For $W_1 N_{\rho_2 \xi}$, we have

$$\int_{W_1 N_{\rho_2 \xi}} \int_{\Gamma \setminus \mathcal{C}_0^+} \int_{m \in M} \Psi^\xi(hmg) dmdhd\lambda(g)$$

$$\leq \int_{z \in \mathcal{T}_c^+} \int_{T_0-3\nu}^{h_1^+(z)+3\nu} e^{2s} \int_{\Gamma \setminus \mathcal{C}_0^+} \int_{m \in M} \Psi^\xi(mh \mathcal{A}_s n_{-z}) dmdhd\lambda(g)$$

$$= \int_{z \in \mathcal{T}_c^+} \int_{T_0-3\nu}^{h_1^+(z)+3\nu} e^{2s} \int_{\Gamma \setminus \mathcal{C}_0^+} \Psi^\xi(hn_{-z}) dhd\lambda(g),$$

where $\mathcal{T}_c^+ = U_{\rho_2 \xi}^c \cap \Gamma^{T_+}$ and $\Psi^\xi(g) := \int_{m \in M} \Psi^\xi(gmn_{-z}) dm$ for every $g \in \Gamma \setminus G/M$.

For any $z \in \mathcal{T}_c^+$, we have $\text{supp}(\Psi^\xi_{-z}) \subset \Gamma \setminus \mathcal{A}_n U_1 M/M$, with $U_1$ the 1-neighborhood of identity in $G$. Let $E := \{ x + iy \in \mathbb{C} : -2 \leq x \leq 2, 0 \leq y \leq 4 \}$. We can see $\mathcal{A}_n U_1 K/K \subset \mathcal{A}_n \mathcal{A} N E U_1 K/K$ by applying $S_4 S_3 \in \mathcal{A}$ to $N E U_1 j$ finitely many times. Hence there exists a compact subset $\Omega \subset \Gamma \setminus G/M$ such that $\text{supp}(\Psi^\xi_{-z}) \subset \Omega$. Applying Theorem 3.22 to $\Psi^\xi_{-z}$, we get

$$\int_{W_1 N_{\rho_2 \xi}} \int_{\Gamma \setminus \mathcal{C}_0^+} \int_{m \in M} \Psi^\xi(hmg) dmdhd\lambda(g)$$

$$\leq \int_{z \in \mathcal{T}_c^+} \int_{T_0-3\nu}^{h_1^+(z)+3\nu} \frac{|\mu_{\mathcal{H}}|}{|\mu_{\mathcal{B}}|} m_{\mathcal{BR}}(\Psi^\xi_{-z}) e^{\alpha s} + \rho_3(S_t(\Psi^\xi_{-z}) e^{(\alpha-\eta_0)s}) dsn_{-z}$$

$$\leq \int_{z \in \mathcal{T}_c^+} c \left( \frac{1}{\pi} \right)^{-\frac{\alpha}{2}} m_{\mathcal{BR}}(\Psi^\xi_{-z})(\text{Im } z)^{-\alpha}(1 + \rho_4 \nu) + \rho_4(\text{Im } z)^{2\nu + \eta_0 - \alpha} S_t(\Psi^\xi_{-z}) t^{\frac{\eta_0 - \alpha}{2}} dn_{-z}$$

$$\leq c(1 + \rho_5 \nu)^\frac{\alpha}{2} \int_{z \in \mathcal{T}_c^+} m_{\mathcal{BR}}(\Psi^\xi_{-z})(\text{Im } z)^{-\alpha} dn_{-z}$$

$$+ \rho_5 \left( t^{-\frac{\alpha}{2}} + \frac{\eta_0}{2} - \eta_0 (6\nu + 13 + \eta_0 - \alpha) \right),$$
where $c := \frac{|p_{HMS}|}{\alpha |m_{HMS}|}$, $\eta_0$ is the constant in Theorem 3.22 and $\rho_3, \rho_4, \rho_5 > 0$ are some constants. In fact, to get the second inequality above, we use the fact that there exists $C > 0$ such that $S_i(\Psi^\xi_z) \leq C \cdot (y^{2^i} S_i(\Psi^\xi))$ for $z = x + iy \in T$.

The second constraint for $\eta$ is that

$$\eta_0^2 - \eta(6l + 13 + \eta_0 - \alpha) > \eta.$$  

Applying Proposition 4.19 to the first term in the third inequality of (4.30), we obtain

$$c \left( 1 + \rho_5 \cdot t^{\eta} \right) \left( \frac{t}{\pi} \right)^{-\frac{\alpha}{2}} \int_{z \in T_\epsilon^+} m_{\text{BR}}(\Psi^\xi_z)(\text{Im} z)^{-\alpha} dn_z + \rho_5 \cdot (t^{-\frac{\alpha}{2} + \eta})$$

$$\leq c \left( \frac{t}{\pi} \right)^{-\frac{\alpha}{2}} (1 + \rho_6 \cdot (t^{-\eta} \xi))(1 + \rho_5 \cdot t^{\eta}) \int_{z \in T} (\text{Im} z)^{-\alpha} d\omega_T(z) + \rho_5 \cdot (t^{-\frac{\alpha}{2} + \eta})$$

$$\leq c \left( \frac{t}{\pi} \right)^{-\frac{\alpha}{2}} \int_{z \in T} (\text{Im} z)^{-\alpha} d\omega_T(z) + \rho_7 \cdot (t^{-\frac{\alpha}{2} + \eta}),$$

for some constants $\rho_6, \rho_7 > 0$.

For the set $W_2$ in (4.27), note that

$$W_2 = H \setminus H K A_{T_0}^+ N_{-T(\eta, t)}^+ \subset H \setminus H K A_{T_1}^+ N_{-T(\eta, t)}^+ \cup \bigcup_{T_1 - \rho_8 \leq s \leq T_0 + \rho_8} H \setminus H a_s N_{-T_\epsilon} \quad \text{(by Prop. 3.6)}$$

$$:= W_3 \cup W_4,$$

where $T_1 = -\log(c_1 \epsilon)$ for some fixed small $\epsilon > 0$ such that $T_1 < T_0 = -\log(c_1 \xi)$, $\rho_8 > 0$ is a constant and $T_\epsilon := U_{\rho_8 \epsilon} T(\eta, t)$.

For the set $W_4$, by similar calculation as (3.30), we have

$$\int_{W_4} \int_{\Gamma_H \setminus C_0^+} \int_M \Psi^\xi(hmg) dmdhd\lambda(g)$$

$$\leq 2 c e^{\alpha T_0} \int_{z \in U_{\rho_9 \epsilon} T_\epsilon} m_{\text{BR}}(\Psi^\xi_z) dn_z \quad \text{(} \rho_9 > 0 \text{ some constant)}.$$  

Using Proposition 3.12 and Corollary 4.18 and substituting $T_0$ by $-\log(c_1 \xi)$, we obtain

$$\int_{W_4} \int_{\Gamma_H \setminus C_0^+} \int_M \Psi^\xi(hmg) dmdhd\lambda(g) = O \left( t^{-\frac{\alpha}{2} + (\frac{\alpha}{2} - (4 \alpha + 1) \eta)} \right).$$
The third constraint for $\eta$ is that

\begin{equation}
\frac{\alpha}{2} - (4\alpha + 1)\eta > \eta.
\end{equation}

This yields

\begin{equation}
\int_{W_3} \int_{\Gamma \setminus C_0^*} \int_{M} \Psi^\xi(hmg)dmdhd\lambda(g) = O(t^{-\frac{\eta}{2}} + \eta).
\end{equation}

For the set $W_3$ in (4.33), reserving the process of translating the circle counting with respect to Euclidean metric into orbit counting, we have

\begin{equation}
\int_{W_3} \int_{\Gamma \setminus C_0^*} \int_{M} \Psi^\xi(hmg)dmdhd\lambda(g)
= O(t^{-\eta} \cdot \# \{ C \in \mathcal{P}(T) : C \cap T_{0,2} \neq \emptyset, \text{Curv}(C) \leq T_1 \})
= O \left( t^{-\frac{\eta}{2}} + \frac{\eta}{2} \right),
\end{equation}

since $T_1$ is fixed.

Summing (4.32), (4.36) and (4.37) together, we get

\[\langle F^+_t, \Psi^\xi \rangle \leq c \left( \frac{t}{\pi} \right)^{-\frac{\eta}{2}} \int_{z \in \mathcal{T}} (\text{Im } z)^{-\alpha} d\omega_{\mathcal{T}}(z) + \rho_9 \cdot (t^{-\frac{\eta}{2}} + \eta),\]

for some constant $\rho_9 > 0$. In view of (4.25), we conclude that there exists $\rho' > 0$, such that

\[N_t(\Gamma(C_0) \cap \mathcal{T}) \leq N_t(\Gamma(C_0) \cap \mathcal{T}(\eta, t)) + n(\Gamma(C_0), t)
\leq \langle F^+_t, \Psi^\xi \rangle + \rho_1 \cdot (t^{-\frac{\eta}{2}} + \eta(\alpha - 1)) + n(\Gamma(C_0), t)
\leq c \left( \frac{t}{\pi} \right)^{-\frac{\eta}{2}} \int_{z \in \mathcal{T}} (\text{Im } z)^{-\alpha} d\omega_{\mathcal{T}}(z) + \rho_{10} \cdot (t^{-\frac{\eta}{2}} + \rho'),\]

for some constant $\rho_{10} > 0$.

As for $\langle F^-_t, \Psi^\xi \rangle$ in (4.25), observe that

\[V_- \supset \{ H : z \in \mathcal{T}(\eta, t), t_{2}\eta \leq s \leq h_t^{-}(z) - t\eta \}, \]

where $T_2 > 0$ is some fixed large number and $h_t^{-}(\cdot)$ is defined as (4.5). Using similar argument as above, there exists $\rho'' > 0$ such that

\[\langle F^-_t, \Psi^\xi \rangle \geq c \left( \frac{t}{\pi} \right)^{-\frac{\eta}{2}} \int_{z \in \mathcal{T}} (\text{Im } z)^{-\alpha} d\omega_{\mathcal{T}}(z) + \rho_{11} \cdot (t^{-\frac{\eta}{2}} + \rho''),\]

where $\rho_{11} > 0$ is some constant.

Therefore, setting $\rho = \min \{ \rho', \rho'' \}$, we conclude that

\[N_t(\Gamma(C_0)) = c \left( \frac{t}{\pi} \right)^{-\frac{\eta}{2}} \int_{z \in \mathcal{T}} (\text{Im } z)^{-\alpha} d\omega_{\mathcal{T}}(z) + O(t^{-\frac{\eta}{2}} + \rho).\]
Step 2: We consider other $\Gamma$-orbits in $P_0$. Let $C_1$ be a representative of a $\Gamma$-orbit with Euclidean center $p = p_1 + ip_2 \in \mathbb{C}$ and Euclidean radius $r > 0$. Set
\[
g_0 := \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{r} & 0 \\ 0 & \frac{1}{\sqrt{r}} \end{pmatrix}.
\]
Then $g_0^{-1}(z) = \frac{z-p_2}{r}$ for $z \in \mathbb{C}$ and $g_0^{-1}(C_1) = C_0$.

Setting $\Gamma_0 = g_0^{-1} \Gamma g_0$, we have
\[
N_t(\Gamma(C_1) \cap \mathcal{T}) = \# \{ C \in \Gamma_0(C_0) : C \cap g_0^{-1} \mathcal{T} \neq \emptyset, \text{Area}_{\text{hyp}}(g_0 C) > t \}.
\]

If the Euclidean center and the Euclidean radius of $C$ are $e_C$ and $r_C$ respectively, then the Euclidean center and the Euclidean radius of $g_0 C$ are $re_C + p$ and $rr_C$ respectively. We deduce from (4.3) that
\[
N_t(\Gamma(C_1) \cap \mathcal{T}) = \left\{ C \in \Gamma_0(C_0) : C \cap g_0^{-1} \mathcal{T} \neq \emptyset, r_C > (\text{Im} e_C + \frac{p_2}{r}) \beta(t) \right\},
\]
with $\beta(t) = \frac{\sqrt{t(4\pi + t)}}{2\pi + t}$.

Recall that we introduced the measure $(\text{Im} z)^{-\alpha} d\omega_T$ on $\mathcal{T}$ to estimate $N_t(\Gamma(C_0) \cap \mathcal{T})$. Similarly, we consider the measure $(\text{Im} z + \frac{p_2}{r})^{-\alpha} d\omega_{\Gamma_0}$ on $g_0^{-1}(\mathcal{T})$. We claim that for any Borel set $E \subset \mathcal{T}$,
\[
(4.38) \quad \int_{z \in g_0^{-1} E} (\text{Im} z + \frac{p_2}{r})^{-\alpha} d\omega_{\Gamma_0}(z) = \int_{z \in E} (\text{Im} z)^{-\alpha} d\omega_T(z).
\]

Note that $\{ g_0 \nu_{\Gamma_0,x} : x \in \mathbb{H}^3 \}$ is a family of $\Gamma_0$-invariant conformal density of dimension $\delta_{\Gamma_0} = \delta_T = \alpha$. The claim can be verified using this observation together with Lemma 3.11.

Repeating the process of getting effective estimate of $N_t(\Gamma(C_0) \cap \mathcal{T})$, we can show that there exists $\rho > 0$ such that as $t \to 0$,
\[
N_t(\Gamma(C_1) \cap \mathcal{T}) = \frac{|\mu_{\text{PS}}|}{\alpha |m_{\text{BMS}}^{\Gamma_0}|} \left( \frac{t}{\pi} \right)^{-\frac{\alpha}{2}} \int_{z \in \mathcal{T}} (\text{Im} z)^{-\alpha} d\omega_T(z) + O(t^{-\frac{\alpha}{2} + \rho}).
\]

Note that $|m_{\text{BMS}}^{\Gamma_0}| = |m_{\text{BMS}}^{\Gamma}|$ by Lemma 3.36. Moreover, since $\Gamma$ just has rank 1 cusps, we know that $(\text{Im} z)^{-\alpha} d\omega_T$ agrees with $(\text{Im} z)^{-\alpha} d\mathcal{H}^\alpha$ on its support $\mathcal{P}(\mathcal{T})$ ([17]). In conclusion, there exists $\rho > 0$ such that as $t \to 0$,
\[
N_t(\mathcal{P}(\mathcal{T})) = \frac{\text{sk}_\Gamma(P_0)}{\alpha |m_{\text{BMS}}^{\Gamma_0}|} \left( \frac{t}{\pi} \right)^{-\frac{\alpha}{2}} \int_{\mathcal{P}(\mathcal{T})} (\text{Im} z)^{-\alpha} d\mathcal{H}^\alpha(z) + O(t^{-\frac{\alpha}{2} + \rho}).
\]
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