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DUAL SPHERES HAVE THE SAME GIRTH

By J.C. Álvarez Paiva

Abstract. Symplectic and Finsler geometry are used to settle a conjecture of Schäffer stating that the girth of a normed space—the infimum of the lengths of all closed, rectifiable, centrally symmetric curves on its unit sphere—equals the girth of its dual.

1. Introduction. A remark that deserves to be better known is that the projectivized of a normed space inherits a natural metric that plays an important role in the study of its geometric properties. If the distance between two points on the unit sphere $S$ of a normed space $X$ is defined as the infimum of the length of all rectifiable curves on $S$ that join them, this distance function, being invariant under the antipodal map, induces a metric on the projectivized space $PX$. Since, in general, the metric is a low-regularity Finsler metric, this functorial construction opens the way for the use of metric and differential geometry in the study of affine invariants of convex bodies and the geometry of normed spaces. Indeed, any metric or Finsler invariant of $PX$ is an affine invariant of the convex hypersurface $S$ and an isometry invariant of $X$.

An especially interesting invariant of $PX$ is its systole: the infimum of the lengths of all noncontractible rectifiable curves. The resulting invariant of isometry classes of normed spaces was extensively studied in [9] by J. J. Schäffer, who used the term girth for twice the systole of $PX$. Among other results, Schäffer found estimates for the girth of an $n$-dimensional normed space, showed that the girth is a continuous function on the Banach-Mazur compactum of isometry classes of $n$-dimensional normed spaces, and posed two beautiful conjectures:

CONJECTURE. The girth of a normed space equals the girth of its dual.

CONJECTURE. The girth of a normed space of dimension greater than two is less than or equal to $2\pi$. Equality holds if and only if the norm comes from an inner product.

This paper settles the first of these conjectures by presenting a “proof from the book” that uses only some elementary symplectic and Finsler geometry. Thanks to Schäffer’s work it suffices to verify the conjecture when $X$ is finite-dimensional.
and the metric induced on \( S \), and \( \mathbb{P}X \), is a smooth Finsler metric. In this case, Schäffer’s conjecture may be restated as the length of the shortest centrally symmetric closed geodesic on the unit sphere \( S \) equals the length of the shortest centrally symmetric closed geodesic on the dual unit sphere \( S^* \). In addition, we shall also see that the length spectrum—the set of lengths of all periodic geodesics—of \( S \) equals the length spectrum of \( S^* \subset X^* \) and that the length spectrum of \( \mathbb{P}X \) equals the length spectrum of \( \mathbb{P}X^* \). The paper ends by showing that the Cauchy-Crofton formula on normed spaces also follows from the same symplectic constructions.

Given that the techniques in the proof are somewhat distant from the usual techniques in convex geometry and the theory of normed spaces, it is not surprising that there are almost no previous results related to Schäffer’s conjecture. The only previous reference seems to be a paper by T. Landes [7] where it is shown that a possible counterexample proposed by Schäffer himself [9] is not really a counterexample.

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2. Short review of symplectic and Finsler geometry. Throughout the paper we shall be studying the geometry of a certain class of hypersurfaces in cotangent bundles. While many of the definitions and notions reviewed below are familiar to symplectic geometers, a short exposition is included as a courtesy to readers with other backgrounds and in order to fix notation. The references for this section are the books [8] and [3], as well as the thesis of Cieliebak, [4].

Let \( M \) be an \( n \)-dimensional smooth manifold and let \( \pi : T^*M \rightarrow M \) be its cotangent bundle. It is well known that \( T^*M \) carries a tautological or canonical 1-form, \( \alpha \), whose value at a vector \( \xi \in T_\xi T^*M \) is \( \xi(D\pi(\xi)) \). The two-form \( \omega := -d\alpha \) is the canonical or symplectic two-form.

A compact hypersurface \( H \subset T^*M \) is said to be of contact-type if the pullback of \( \alpha \) to \( H \) is a contact form (i.e., the pullback of \( \alpha \wedge (d\alpha)^{n-1} \) to \( H \) never vanishes). The hypersurfaces of contact-type that we shall consider in this paper are all of the following form: \( M \) is a compact manifold and for every \( x \in M \) the intersection \( H \cap T^*_x M \) is a convex hypersurface of \( T^*_x M \) enclosing the origin.

By a slight abuse of notation, if \( H \subset T^*M \) is a hypersurface of contact-type, the pullback of \( \alpha \) to \( H \) will still be denoted as \( \alpha \). The geometry of \( (H, \alpha) \) is quite rich: first of all \( H \) can be given an orientation by specifying that \( \alpha \wedge (d\alpha)^{n-1} \) be a (positive) volume form. We can then define the volume of \( (H, \alpha) \) by the formula

\[
\text{vol}(H, \alpha) := \frac{1}{n!} \int_H \alpha \wedge (d\alpha)^{n-1}.
\]

The integral of \( \alpha \) along a curve \( \sigma : [a, b] \rightarrow H \) is called the action of \( \sigma \).
The fact that the form $\alpha \wedge (d\alpha)^{n-1}$ is a volume form is equivalent to the following statement: for each $\xi \in H$ the kernel of the form $\omega_\xi = -d\alpha|_{T_\xi H}$,

$$\text{Ker} \ \omega_\xi := \{ \xi \in T_\xi H : \omega_\xi(\xi, \cdot) = 0 \},$$

is a one-dimensional subspace. Moreover, this line can be oriented by saying that $\xi$ points in the positive direction if $\alpha(\xi) > 0$. The line field $\xi \mapsto \text{Ker} \ \omega_\xi$ is called the characteristic line field. Its positively oriented integral curves are the characteristics of $(H, \alpha)$.

The set of the actions of all periodic characteristics is called the action spectrum. Note that since a periodic characteristic can be traversed $k$ times, $k$ a positive integer, if a number $a$ is in the action spectrum, then so is $ka$.

**Definition 2.1.** Let $(H_1, \alpha_1)$ and $(H_2, \alpha_2)$ be two hypersurfaces of contact-type. A diffeomorphism $\phi : H_1 \to H_2$ is said to be an exact contactomorphism if either $\phi^* \alpha_2 - \alpha_1$ or $\phi^* \alpha_2 + \alpha_1$ is an exact 1-form.

**Proposition 2.1.** Two exact contactomorphic hypersurfaces of contact-type have the same volume and action spectrum.

**Proof.** Let $\phi$ be a diffeomorphism between two $(2n - 1)$-dimensional hypersurfaces of contact-type $(H_1, \alpha_1)$ and $(H_2, \alpha_2)$ that satisfies $\phi^* \alpha_2 = -\alpha_1 + df$, where $f$ is a smooth function on $H_1$.

The equality $\text{vol}(H_1, \alpha_1) = \text{vol}(H_2, \alpha_2)$ follows from the fact that $H_1$ is closed and that $\phi^* \alpha_2 \wedge (d\alpha_2)^{n-1}$ equals $(-1)^n \alpha_2 \wedge (d\alpha_2)^{n-1}$ plus an exact form.

It is also easy to see that $\phi$ maps the characteristic line field in $(H_1, \alpha_1)$ to the characteristic line field in $(H_2, \alpha_2)$. However, since we defined characteristics to be positively oriented, we cannot say that $\phi$ maps characteristics to characteristics. What we can say is that if $\sigma : [a, b] \to H_1$ is a closed characteristic, then the curve $t \mapsto \phi(\sigma(-t))$ is a closed characteristic with the same action.

This is enough to show that the action spectra of $(H_1, \alpha_1)$ and $(H_2, \alpha_2)$ coincide.

If the map $\phi$ satisfies $\phi^* \alpha_2 = \alpha_1 + df$, then it does take closed characteristics to closed characteristics with the same action, and the proof is only easier.

For the rest of the paper all hypersurfaces of contact-type that we shall consider will be unit co-sphere bundles of Finsler metrics, reversible and non-reversible. Before reviewing Finsler metrics, let us describe these hypersurfaces in simple terms:

**Definition 2.2.** A smooth hypersurface in a finite-dimensional real vector space $V$ is said to be quadratically convex if its principal curvatures are positive for any auxiliary Euclidean structure on $V$. Equivalently, a smooth hypersurface is quadratically convex if and only if its osculating quadrics are all ellipsoids.
A hypersurface $H \subset T^*M$ is said to be *optical* if in each cotangent space $T^*_x M$ the intersection $H \cap T^*_x M$ is a quadratically convex hypersurface enclosing the origin. When the manifold $M$ is compact, optical hypersurfaces of $T^*M$ are of contact-type.

To make the link between optical hypersurfaces and Finsler metrics, we extend slightly the notion of duality in normed spaces:

Let $V$ be a finite-dimensional real vector space and let $S \subset V$ be a quadratically convex hypersurface enclosing the origin. If $q$ is a point in $S$, there is a unique covector $\xi \in V^*$ such that the hyperplane $\xi = 1$ is tangent to $S$ at $q$ and the half-space $\xi \leq 1$ contains $S$. The map that sends $q$ to $\xi$ is called the Legendre transform and will be denoted by

$$\mathcal{L} : S \rightarrow V^*.$$ 

The image of $S$ under $\mathcal{L}$, the *dual* of $S$, is again a quadratically convex hypersurface that encloses the origin. It will be denoted by $S^*$. It is well-known, and easy to verify, that $S$ is also the dual of $S^*$.

**Definition 2.3.** Let $M$ be a smooth manifold and let $TM \setminus 0$ denote its tangent bundle with the zero section deleted. A *Finsler metric* on $M$ is a smooth function

$$\varphi : TM \setminus 0 \rightarrow \mathbb{R}$$

that is positively homogeneous of degree one and such that on every tangent space $T_x M$ the hypersurface

$$S_x M := \{v \in T_x M : \varphi(v) = 1\}$$

is quadratically convex and encloses the origin. The Finsler metric will be called *reversible* if $S_x M$ is symmetric with respect to the origin (i.e., if and only if $\varphi(v) = \varphi(-v)$).

We follow the standard terminology in Finsler geometry and define a *Minkowski space* to be a normed space whose unit sphere is quadratically convex. Clearly, any submanifold of a Minkowski space inherits a natural reversible Finsler metric. In this paper we are mainly concerned with the geometry of the Finsler metric inherited by the unit sphere of a Minkowski space.

A reversible Finsler metric $\varphi$ on a manifold $M$ allows us to define the length of a smooth curve $\gamma : [a, b] \rightarrow M$ by the equation

$$\text{length of } \gamma := \int_a^b \varphi(\dot{\gamma}(t))\,dt.$$ 

Using this, we can define the distance between two points $x$ and $y$ in $M$ as the infimum of the lengths of all smooth curves joining $x$ and $y$. As usual, a geodesic
on a Finsler manifold is a curve that locally minimizes length. The quadratic convexity of the tangent unit spheres guarantees the local uniqueness of geodesic segments. When the Finsler metric is non-reversible, we must settle for oriented versions of the notions of length, distance, and geodesic.

Notice that the restriction of Schäffer’s conjecture for Minkowski spaces states that the length of the shortest centrally symmetric closed geodesic on the unit sphere of a Minkowski space equals the length of the shortest centrally symmetric closed geodesic on the unit sphere of its dual.

A Finsler metric on a manifold \( M \) defines an optical hypersurface on its cotangent bundle: if \( S^*_x M \subset T^*_x M \) denotes the dual of \( S_x M \), then the unit co-sphere bundle of the Finsler manifold \( (M, \varphi) \),

\[
S^* M := \bigcup_{x \in M} S^*_x M \subset T^* M ,
\]

is an optical hypersurface. Many of the geometric quantities and objects on \( (M, \varphi) \) can be read from the geometry of \( (S^* M, \alpha) \).

Let us denote by \( L : SM \to S^* M \) the map that takes a unit vector \( v \in S_x M \) and sends it to \( L_x(v) \), where \( L_x : S_x M \to T^*_x M \) is the Legendre transform at the point \( x \). If \( \gamma \) is a smooth curve on \( M \) parameterized by arc-length, the curve \( L \circ \dot{\gamma} \) is a smooth curve on \( S^* M \), and it is easy to see that the length of \( \gamma \) equals the action of \( L \circ \dot{\gamma} \):

\[
\text{length of } \gamma := \int_{\gamma} \varphi = \int_{L \circ \dot{\gamma}} \alpha . \tag{1}
\]

It is also well known that a curve \( \gamma : [a, b] \to M \) is a geodesic if and only if the curve \( L \circ \dot{\gamma} \) is a characteristic of \( (S^* M, \alpha) \). This, together with equation (1), shows that the length spectrum of a Finsler manifold — the set of lengths of its periodic geodesics — equals the action spectrum of its unit co-sphere bundle.

We may also define the volume of an \( n \)-dimensional Finsler manifold \( M \) as the volume of \( (S^* M, \alpha) \) divided by the volume of the Euclidean \( n \)-dimensional unit ball. This definition was first introduced from a convex-geometric viewpoint by Holmes and Thompson (see [6] and [11]), and is usually referred to as the Holmes-Thompson volume of \( M \).

If we define the unit co-disc bundle, \( D^* M \), of a Finsler manifold \( M \) to be the open set bounded by \( S^* M \) and containing the zero section of \( T^* M \), then, by Stokes theorem, the Holmes-Thompson volume of \( M \) equals the symplectic volume of \( D^* M \) divided by the volume of the Euclidean \( n \)-dimensional unit ball.

3. A geometric construction. The solution of Schäffer’s conjecture and the other results in this paper depend on a simple geometric construction:

Let \( V \) be a finite-dimensional real vector space and let \( S^*_1 \subset V \) and \( S^*_2 \subset V^* \) be two quadratically convex hypersurfaces. If \( q \in S^*_1 \) and \( P \in V^* \), let us view
We define $\phi$ denoted by a morphism. Let us now consider the sets

$$D_1^+(S_1, S_2^*):=\{p_q \in T_q^* S_1 : q \in S_1, \ p \in S_2^* \} \subset T^* S_1,$$

$$D_2^+(S_2^*, S_1):=\{q_p \in T_p^* S_2^* : q \in S_1, \ p \in S_2^* \} \subset T^* S_2^*$$

together with their interiors, $D^+(S_1, S_2^*)$ and $D^+(S_2^*, S_1)$, and their boundaries, $S^*(S_1, S_2^*)$ and $S^*(S_2^*, S_1)$. We define $\phi(q, p) := (p, q_p) \in T_p S_2^*$. Note that it is not yet obvious that $\phi(q, p)$ belongs to $S^*(S_2^*, S_1)$.

That this is true follows at once from the following general statement: a point $P \in S_2^*$ is in the shadow boundary of the projection $X \mapsto X_q$, $q \in S_1$, from $S_2^*$ to $T_q^* S_1$ if and only if the point $q$ is in the shadow boundary of the projection $x \mapsto x_p$ from $S_1$ to $T_p^* S_2^*$. Indeed, the condition that $P$ be in the shadow boundary of the projection $X \mapsto X_q$ is equivalent to the degeneracy of the bilinear form given by the restriction of the dual pairing in $V^* \times V$ to $T_p S_2^* \times T_q S_1$. This is the same condition for $q$ to be in the shadow boundary of the projection $x \mapsto x_p$.

The map $\phi$ is obviously invertible, and the quadratic convexity of both $S_1$ and $S_2^*$ implies that both it and its inverse are smooth.

The map $\phi$ is an exact contactomorphism. To see that $\phi^* \alpha_2 + \alpha_1$ is an exact 1-form, we borrow an old trick from classical mechanics and consider the graph of $\phi$ as a submanifold of $S^*(S_1, S_2^*) \times S^*(S_2^*, S_1)$:

$$\Gamma_\phi := \{ (q, p; P, Q) \in S^*(S_1, S_2^*) \times S^*(S_2^*, S_1) : q_p = Q \text{ and } P_q = p \}.$$

We may consider the 1-form

$$\alpha_2 + \alpha_1 = Q \cdot dP + p \cdot dq$$
as a form on $S^*(S_1, S_2^*) \times S^*(S_2^*, S_1)$. The pullback of this form to $\Gamma_\phi$ becomes

$$q \cdot dP + P \cdot dq = d(P \cdot q),$$

and, therefore, $\phi$ is an exact contactomorphism. \hfill \square

**Lemma 3.2.** There exists an anti-symplectic diffeomorphism between the open sets $D^*(S_1, S_2^*) \subset T^*S_1$ and $D^*(S_2^*, S_1) \subset T^*S_2^*$. 

**Proof.** The idea of the proof is basically the same as that of the previous lemma. There is only the slight complication that if $p \in T_q^*S_1$ is in $D^*(S_1, S_2^*)$, then there exist two points $P^+$ and $P^-$ in $S_2^*$ such that $P^+_q = P^-_q = p$. To distinguish between these points, and define a diffeomorphism $\Phi$ between $D^*(S_1, S_2^*)$ and $D^*(S_2^*, S_1)$, we need to play with the orientations.

Let us fix an orientation on $V$. This induces an orientation on $V^*$ and, using the natural co-orientation of the hypersurfaces $S_1$ and $S_2^*$, on $S_1$ and $S_2^*$. In turn, the orientations on $S_1$ and $S_2^*$ induce orientations on each of their cotangent spaces.

If $(q, p) \in D^*(S_1, S_2^*)$, there is a unique point $P \in S_2^*$ such that $P_q = p$ and such that the differential of the projection $X \mapsto X_q$ from $S_2^*$ to $T_q^*S_1$ at the point $P$ is orientation preserving. We define $\Phi(q, p) := (P, q_p)$.

Exactly as in the proof of the previous lemma, one shows that $\Phi^*\alpha_2 + \alpha_1$ is an exact 1-form, and $\Phi^*d\alpha_2 = -d\alpha_1$. \hfill \square

When the hypersurfaces $S_1 \subset V$ and $S_2^* \subset V^*$ are symmetric about the origin, we can say something more on the above constructions. For example, in this case we can modify the anti-symplectic diffeomorphism $\Phi$ in lemma 3.2 to the symplectic diffeomorphism

$$\Psi : D^*(S_1, S_2^*) \longrightarrow D^*(S_2^*, S_1)$$

$$(q, p) \longmapsto \Phi(q, -p).$$

Likewise, if we use the diffeomorphism $\phi$ of Lemma 3.1 to define

$$\psi : S^*(S_1, S_2^*) \longrightarrow S^*(S_2^*, S_1)$$

$$(q, p) \longmapsto \phi(q, -p),$$

then $\psi^*\alpha_1 - \alpha_2$ is an exact 1-form, and, hence, it takes closed characteristics of $S^*(S_1, S_2^*)$ to closed characteristics of $S^*(S_2^*, S_1)$ while preserving both their orientation and their action.

Moreover, and this will be important in the proof of Schäffer’s conjecture, if $\psi(q, p) = (P, Q)$, then $\psi(-q, -p) = (-P, -Q)$.
4. Solution of Schäffer’s conjecture. Before presenting the proof of Schäffer’s conjecture, let us derive a more direct consequence of Lemma 3.1.

**Theorem 4.1.** Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be two Minkowski norms on a finite-dimensional real vector space $V$, and let $S_1$ and $S_2$ denote their unit spheres. The volume and the length spectrum of the Finsler metric on $S_1$ induced by its embedding into $(V, \| \cdot \|_2)$ are equal, respectively, to the volume and the length spectrum of the Finsler metric on $S_2^*$ induced by its embedding into $(V, \| \cdot \|_1^*)$.

**Proof.** Notice that the unit co-sphere bundle of $S_1 \subset (V, \| \cdot \|_2)$ is $S^*(S_1, S_2^*)$ and that the unit co-sphere bundle of $S_2^* \subset (V^*, \| \cdot \|_1^*)$ is $S^*(S_2^*, S_1)$. By Lemma 3.1, these two hypersurfaces are exact contactomorphic. Therefore, by Proposition 2.1, their volumes and their action spectra coincide. □

Now we are now ready to prove Schäffer’s conjecture:

**Theorem 4.2.** The girth of a normed space equals the girth of its dual.

**Proof.** We shall need two results of Schäffer [9]:

1. It is enough to prove the conjecture for finite-dimensional normed spaces.
2. The girth is an invariant of isometry classes of normed spaces that is continuous with respect to Banach-Mazur topology.

Since (isometry classes of) Minkowski spaces are dense in the Banach-Mazur compactum of isometry classes of normed spaces with a fixed dimension, then it is enough for us to prove that the girth of a Minkowski space equals the girth of its dual.

If $(V, \| \cdot \|_1)$ and $(V, \| \cdot \|_2)$ are two Minkowski spaces with unit spheres $S_1$ and $S_2$, respectively, we know from the last paragraph of the previous section that there exists an exact contactomorphism $\psi : S^*(S_1, S_2^*) \rightarrow S^*(S_2^*, S_1)$ that takes closed characteristics to closed characteristics while preserving both their orientation and action. Recall also that if $\psi(q, p) = (P, Q)$, then $\psi(-q, -p) = (-P, -Q)$. This means that any characteristic in $S^*(S_1, S_2^*)$ that is invariant under the map $(q, p) \mapsto (-q, -p)$ is sent to a characteristic in $S^*(S_2^*, S_1)$ that is invariant under the map $(P, Q) \mapsto (-P, -Q)$.

Since such characteristics are in one-to-one correspondence to geodesics that are centrally symmetric, we have that the set of lengths of all centrally symmetric closed geodesics in $S_1 \subset (V, \| \cdot \|_2)$ equals the set of lengths of all centrally symmetric closed geodesics in $S_2^* \subset (V^*, \| \cdot \|_1^*)$.

When $\| \cdot \|_1 = \| \cdot \|_2$ this implies that the girth of $(V, \| \cdot \|)$ equals the girth of $(V^*, \| \cdot \|_1^*)$. □

Notice that Lemma 3.2 has the following immediate consequence:

**Theorem 4.3.** Let $(V, \| \cdot \|_1)$ and $(V, \| \cdot \|_2)$ be two Minkowski spaces and let $S_1$ and $S_2$ denote their unit spheres. The unit co-disc bundle of the Finsler metric on
\( S_1 \) induced by its embedding into \((V, \| \cdot \|_2)\) is symplectomorphic to the unit co-disc bundle of the Finsler metric on \( S^*_2 \) induced by its embedding into \((V, \| \cdot \|_1^*)\).

Schäffer has given examples of finite-dimensional normed spaces where the unit sphere and the dual unit sphere have different diameters. These norms can be approximated by Minkowski norms to yield Minkowski spaces with the same property. From the theorem above we deduce that the diameter of a Finsler manifold is not a symplectic invariant of its unit co-disc bundle.

An interesting billiard version of Theorems 4.1 and 4.3 due to S. Tabachnikov and E. Gutkin can be found in their paper [10].

5. Integral geometry. In this section we uncover the symplectic underpinnings of El-Ekhtiar’s generalization of the Cauchy-Crofton formula to hypersurfaces in finite-dimensional normed spaces (see [5]).

Our first observation is that if \((V, \| \cdot \|)\) is a Minkowski space, it is possible to define a natural symplectic structure on the manifold of oriented lines in \( V \), \( H^*_1(V) \), which makes it naturally symplectomorphic to the cotangent bundle of the dual unit sphere, \( T^*S^* \).

Consider the diagram

\[
\begin{aligned}
\pi_1 & : V \times S^* \to H^*_1(V) \\
\pi_2 & : V \times S^* \to T^*S^* \\
\end{aligned}
\]

where \( \pi_1 \) is the map that sends a point \((q, P)\) to the oriented line passing through \( q \) in the direction of \( \mathcal{L}(P) \), the Legendre transform of \( P \), and \( \pi_2 \) is the map \((q, P) \mapsto (P, q_P)\).

If \( i : V \times S^* \to V \times V^* \) is the natural inclusion, and \( \omega \) is the symplectic form on \( V \times V^* \), then define \( \omega_1 \) to be the unique 2-form on \( H^*_1(V) \) such that \( \pi_1^*\omega_1 = i^*\omega \). This is the natural symplectic form on the space of oriented lines in \( V \) seen as the space of geodesics of the Finsler manifold \((V, \| \cdot \|)\) (see [2]). It is not hard to see that if \( \omega_2 \) is the symplectic form on \( T^*S^* \), then \( \pi_2^*\omega_2 = i^*\omega \).

In order to identify \( H^*_1(V) \) and \( T^*S^* \) notice that the fibers of the projections \( \pi_1 \) and \( \pi_2 \) coincide. Indeed, \( \pi_1(q, P) = \pi_1(q', P') \) if and only if \( P = P' \) and \( q - q' \) is a multiple of \( \mathcal{L}(P) \) or, equivalently, if \( q - q' \) vanishes on \( T_P S^* \). This is the same condition for \( \pi_2(q, P) = \pi_2(q', P') \). Thus, we obtain a diffeomorphism from \( H^*_1(V) \) to \( T^*S^* \) by sending a line \( \ell \) to the point \( \pi_2(\pi_1^{-1}\{\ell\}) \). The equation \( \pi_1^*\omega_1 = i^*\omega = \pi_2^*\omega_2 \) implies that this map is a symplectomorphism.

**Theorem 5.1.** Let \((V, \| \cdot \|)\) be a Minkowski space and let \( M \subset V \) be a smooth, quadratically convex hypersurface. The unit co-disc bundle for the induced Finsler metric on \( M \) and the set of all oriented lines in \( V \) which pass through the interior of \( M \) are symplectomorphic.
Proof. Applying Lemma 3.2 with $S_1 := M$ and $S_2^* = S^*$, we have an anti-symplectic diffeomorphism $\Phi$ between the unit co-disc bundle of the Finsler metric on $M$ induced from its embedding in $(V, \| \cdot \|)$ and the open set $D^*(S^*, M) \subset T^*S^*$.

Composing $\Phi$ with the involution $(q, p) \mapsto (q, -p)$ on $D^*(M, S^*) \subset T^*M$ (we are using now that $S^*$ is symmetric about the origin), we obtain a symplectomorphism. The theorem now follows from the fact that the natural symplectomorphism between $T^*S^*$ and $H^+_1(V)$ takes a point $(P, q_P) \in T^*S^*$ and sends it to the line passing through $q$ in the direction of the Legendre transform of $P$.

If we define the volume density on the manifold of oriented lines of the $n$-dimensional Minkowski space $V$, $H^+_1(V)$, as $|\omega^{n-1}_1|/(n-1)!$, then the previous theorem immediately implies the Cauchy-Crofton formula for finite-dimensional normed spaces.

**Corollary 5.1.** (El-Ekhtiar, [5]) Let $(V, \| \cdot \|)$ be a normed space of dimension $n$ and let $M \subset V$ be a convex hypersurface. The volume of the set of lines passing through $M$ equals the volume of $M$ times the volume of the Euclidean unit ball of dimension $n - 1$.

**Proof.** If we assume that $M$ and the unit sphere of $(V, \| \cdot \|)$ are quadratically convex, then the result follows from the previous theorem and the definition of volume in the manifold of oriented lines. An approximation argument takes care of the general case.

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**REFERENCES**

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, 2nd ed., Addison-Wesley Publishing Company Inc., Reading, MA, 1978.
[2] J. C. Álvarez Paiva, The symplectic geometry of spaces of geodesics, Ph.D. thesis, Rutgers University, 1995.
[3] V. I. Arnold and A. B. Givental, *Symplectic Geometry, Dynamical Systems IV*, Encyclopaedia of Mathematical Sciences, vol. 4 (V. I. Arnold and S. P. Novikov, eds.), Springer-Verlag, Berlin, 1990.
[4] K. Cieieliebkak, Symplectic boundaries: closed characteristics and action spectra, Ph.D. thesis, ETH Zürich, 1996.
[5] A. El-Ekhtiar, Integral geometry in minkowski space, Ph.D. Thesis, Univ. of California, Davis, 1982.
DUAL SPHERES HAVE THE SAME GIRTH

[6] R. D. Holmes and A. C. Thompson, N-dimensional area and content in Minkowski spaces, Pacific J. Math. 85 (1979), 77–110.

[7] T. Landes, Geometry of spheres: On a conjecture of J.J. Schäffer, Math. Ann. 257 (1981), 367–369.

[8] D. McDuff and D. Salamon, Introduction to Symplectic Topology, 2nd ed., Oxford Mathematical Monographs, Clarendon Press, Oxford, 1998.

[9] J. J. Schäffer, Geometry of Spheres in Normed Spaces, Lecture Notes in Pure and Appl. Math., vol. 20, Marcel Dekker, Inc. VI, New York, 1976.

[10] S. Tabachnikov and E. Gutkin, Billiards in Finsler and Minkowski geometries, J. Geom. Phys. 40 (2002), 277–301.

[11] A. C. Thompson, Minkowski Geometry, Encyclopedia of Math. and its Applications, vol. 63, Cambridge Univ. Press, Cambridge, 1996.