Generalization of Higuchi’s fractal dimension for multifractal analysis of time series with limited length

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Abstract  There exist several methodologies for the multifractal characterization of nonstationary time series. However, when applied to sequences of limited length, these methods often tend to overestimate the actual multifractal properties. To address this aspect, we introduce here a generalization of Higuchi’s estimator of the fractal dimension as a new way to characterize the multifractal spectrum of univariate time series or sequences of relatively short length. This multifractal Higuchi dimension analysis (MF-HDA) method considers the order-\(q\) moments of the partition function provided by the length of the time series graph at different levels of subsampling. The results obtained for different types of stochastic processes, a classical multifractal model, and various real-world examples of word length series from fictional texts demonstrate that MF-HDA provides a reliable estimate of the multifractal spectrum already for moderate time series lengths. Practical advantages as well as disadvantages of the new approach as compared to other state-of-the-art methods of multifractal analysis are discussed, highlighting the particular potentials of MF-HDA to distinguish mono- from multifractal dynamics based on relatively short sequences.

Keywords Fractal dimension · Higuchi method · Multifractal spectrum · Partition function · Stochastic processes · Word lengths

1 Introduction

Since Benoît Mandelbrot applied the concept of fractality for the first time to real-world time series, different approaches have been proposed to detect fractal and multifractal scaling properties in diverse signals from complex systems [1–6] (for recent reviews, see [7,8]). In many real-world cases, this scaling behavior reflects the fact that embedded variability components cover a wide range of time scales despite the absence of a dominant periodicity [9,10]. For monofractal time series, a single scaling exponent is sufficient to completely characterize the fractal properties of the dynamics displayed by the analyzed sequence. By contrast, for multifractal signals a variety of scaling exponents are necessary to fully describe the dynamics.

The multifractal formalism was originally introduced in the context of turbulence studies [4,11–14] and velocity fluctuations [4]. Later, Mandelbrot [1,15] described the multifractal properties of geo-
metric objects, where the diversity of local scaling exponents characterizes the multifractality (broad spectrum of exponents). Nowadays, multifractal analysis has become a universal technique for data analysis, with a great variety of applications across a multitude of fields [2,13,14,16–20].

In the context of complex time series, the two standard methods to detect multifractality are based on either the construction of a partition function or the concept of extended self-similarity associated with the notion of structure functions (for details, see [3,8]). Both approaches provide direct procedures to obtain the spectrum of multifractal scaling exponents involved in the multifractal measure. While the partition function method is mainly rooted in the box-counting approach [3] and may therefore require relatively long time series for proper evaluation, structure functions rely on the nonlinear scaling behaviors of increments bridging increasingly large time differences [21,22]. A more sophisticated method for multifractal analysis, the wavelet transform modulus maxima (WTMM) method, has been introduced by Arneodo et al. [23,24] and applied to a great variety of problems from various disciplines. Here, the wavelet transform is used to obtain the sequence of local maximum values over a range of scales, which also accounts for the widespread nonstationarity of real-world time series. In a similar spirit, a conceptually related approach based on time scale decomposition by means of a more data-adaptive technique (the empirical mode decomposition) has been developed more recently [25,26].

Another state-of-the-art approach of multifractal analysis has been introduced by Kantelhardt et al. who proposed a generalization of detrended fluctuation analysis [27] to obtain an estimate of the multifractal spectrum of nonstationary time series. This multifractal detrended fluctuation analysis (MF-DFA), together with various algorithmic variants thereof differing in the way of how local time series detrending is performed, provides a robust way to determine the multifractal scaling characteristics in terms of generalized Hurst exponents \( h(q) \), where \( q \) represents the order of the statistical moment under study. However, both WTMM and MF-DFA exhibit certain practical challenges in determining the corresponding multifractal spectra from relatively short time series [28,29], which may lead to spurious identification of multifractality.

As a possible alternative, this paper introduces a generalization of Higuchi’s fractal dimension (HFD) estimator for univariate time series [30,31], which has been widely used as a technique for quantifying monofractal scaling characteristics in nonstationary time series from different areas of science, ranging from geophysics [30,32–35] to biomedicine [36–39]. Specifically, we introduce the concept of multifractal Higuchi dimension analysis (MF-HDA), which is applicable to comparatively short sequences and provides very stable estimates of scaling exponents for the construction of the multifractal spectrum.

The remainder of this paper is organized as follows. In Sect. 2, we briefly review some essential details on the HFD method and introduce a thorough modification of the original approach to obtain the corresponding scaling exponent. Section 3 describes the multifractal generalization of the monofractal dimension estimator. Its application to simulated time series from two illustrative stochastic model systems and a subsequent comparison of the obtained performance with that of MF-DFA are presented in Sects. 4 and 5. In Sect. 6, we then apply our method to some real-world examples of world length sequences in English fictional texts. Finally, a discussion and some conclusions and general remarks are provided in Sects. 7 and 8, respectively.

2 The fractal dimension of irregular time series

2.1 Higuchi’s estimator of the fractal dimension

Consider a time series \( X(t) \) with \( t = 1, 2, \ldots, N \). First, we construct a set of subsampled time series

\[
X^m_k = \left( X(m), X(m + k), X(m + 2k), \ldots, X(m + \left\lfloor \frac{N - m}{k} \right\rfloor \cdot k) \right)
\]

(1)

at different scales (determined by \( k \)) with \( m = 1, 2, \ldots, k \), where \( \lfloor \cdot \rfloor \) denotes the (lower) integer part of a real number. Here, \( k \) and \( m \) are integers which describe a time lag and the initial time index, respectively. Next, the length of the curve associated with a given sequence \( X^m_k \) is defined as [30]

\[
L_m(k) = \left\{ \left( \sum_{j=1}^{j_{\max}} \Delta X^m_k(j) \right) \cdot \frac{N - 1}{j_{\max} \cdot k} \right\} / k,
\]

(2)
where $\Delta X^m_k(j) = |X(m + j k) - X(m + (j - 1) \cdot k)|$ are the absolute increments of the subsampled series, $j_{\text{max}} = \lfloor N - m \rfloor$ is the total number of increments at scale $k$, and $(N - 1)/(j_{\text{max}} \cdot k)$ represents a normalization factor.

We notice that the quantity $\Delta X^m_k$ represents the differences corresponding to each $k$-lag value, which when aggregated and normalized by $j_{\text{max}}$ gives the average of the increments for that scale $k$. For the purpose of the present study, consider for example $X(t)$ to represent the position of a one-dimensional random walk at time $t$. Then, $\Delta X^m_k$ is the absolute displacement within a time step $k$ with $m = 1, 2, \ldots, k$, where the best possible temporal resolution corresponds to $k = 1$, the resolution decreases by one half for $k = 2$, and so on. Thus, the mean absolute displacement of $X(t)$ starting at $m$ at a resolution $k$ is given by

$$\langle \Delta X^m_k \rangle = \frac{1}{j_{\text{max}}} \sum_{j=1}^{j_{\text{max}}} \Delta X^m_k(j), \tag{3}$$

and the total mean displacement is calculated as $\ell_m(k) = \frac{N - k}{N} \langle \Delta X^m_k \rangle$, where $\frac{N - k}{N}$ represents the typical number of displacements available at a resolution $k$. In order to make these total mean displacements obtained for different resolutions $k$ mutually comparable, the total length of the curve describing the time series graph coarse-grained at scale $k$ starting at $m$ is defined as $L_m(k) = \ell_m(k)/k$, and the total mean curve length at scale $k$ is given as $L(k) = \langle L_m(k) \rangle$ [30,31].

In case of a self-affine (i.e., fractal) time series, it has been demonstrated that

$$L(k) \sim k^{-D_A}, \tag{4}$$

where $D_A \in (1, 2)$ is an estimate of the fractal dimension of the original time series graph and the subscript $A$ stands for averaging [30,31].

2.2 Modified HFD estimator

For the purpose of generalizing the HFD method to a multifractal formalism, it is necessary to first introduce an alternative formulation of an estimator of the fractal dimension that is based on the same rationale as Higuchi’s original method. Here, our modified approach replaces the averaging procedure for calculating the mean absolute displacement by an expectation value in terms of probabilities of increment sizes $\Delta X^m_k$ (Eq. 3).

We first recall that the number of increments $\Delta X^m_k$ at scale $k$ is $\lfloor (N - m)/k \rfloor$, and the total number is $\sum_{m=1}^{k} \lfloor (N - m)/k \rfloor$, which equals $N - k$ [40]. For numerically evaluating the corresponding expectation values, we follow here a simple histogram based approach (other more sophisticated density estimators may be used as well, yet we focus here on this simple strategy and outline corresponding follow-up studies on this aspect as parts of future work). Let us consider $P_n(\bullet)$ representing the discrete probabilities of increments falling into disjoint intervals $\Delta X_n(k)$, where $n$ represents the $n$-th interval of an equiprobable partition of the entire range of increment values with $N_b$ bins in increasing order, that is, each interval (bin) contains approximately a fraction of $1/N_b$ of the $N - k$ increments, i.e., $P_n(\Delta X) \approx 1/N_b$. Note that in practice, the total number of increments $\Delta X$ may not be an integer multiple of $N_b$, which results in minor deviations among class frequencies that become gradually less important as $N_b/N \to \infty$. For the purpose of the present work, we employ numerical estimates of the respective $n/N_b$ quantiles ($n = 1, \ldots, N_b - 1$) to define mutually disjoint intervals with approximately the same population.

In order to obtain a stable estimate of the expectation value of the increments, we start with calculating the “local-mean” increment values, i.e., the mean values for each bin, as $\langle \Delta X_n(k) \rangle = \frac{1}{\sum_{m=1}^{k} n \cdot f_n \cdot \Delta X_{n,m}}(k)$, where $f_n$ represents the actual number of increments within the $n$-th bin and $\sum_{m=1}^{k} n = N - k$. Then, the expectation value of the increments at scale $k$ can be estimated as

$$E[\Delta X(k)] = \sum_{n=1}^{N_b} (\Delta X_n(k)) P_n(\Delta X(k)). \tag{5}$$

In analogy with Higuchi’s original method, the total length of the curve at resolution $k$ can then be written as:

$$L(k) = \frac{N - 1}{k^2} E[\Delta X(k)]. \tag{6}$$

Again, we may expect that if the original time series is fractal, then the total length follows a power-law behav-
ior $\mathcal{L}(k) \sim k^{-D_E}$, where $D_E$ is our modified Higuchi-type estimator of the fractal dimension and the subscript $E$ stands for expectation.

In order to verify the consistency of our modified approach with the classical Higuchi’s estimator, we applied both our methodology and the original HFD methodology to some elementary stochastic time series with known properties. Specifically, we first considered realizations of three representative monofractal processes: uncorrelated noise (white noise), intermittent long-term correlated noise ($1/f$ noise, with $f$ denoting the frequency), and classical Brownian motion [41]. The respective results are summarized in Fig. 1. In general, we observe an excellent agreement between our proposed modification and Higuchi’s original method for all three monofractal series.

To further evaluate the agreement between both methodologies, we systematically generated monofractal time series of fractional Gaussian noise with power spectra exhibiting a $1/f^\beta$ behavior with gradually varying spectral scaling exponent $\beta$ [43] and then applied both approaches for estimating the corresponding fractal dimensions. Figure 2 shows the numerical results for different processes with $\beta \in (0, 4)$. Note that we have to expect appropriate estimates to exist only for $\beta \in (1, 3)$ and the finite-sample estimates to become gradually biased as any of the boundaries of the latter interval are approached [44]. This expectation is met by our numerical results, which again demonstrates that both methodologies lead to very similar fractal dimension estimates for monofractal time series.

Fig. 1 Estimation of Higuchi’s fractal dimension for three different types of elementary stochastic processes by using averaging (original method) and numerical approximations of the expectation values (modified approach) of the respective curve lengths. a White noise (spectral exponent $\beta = 0$), b $1/f$ noise ($\beta = 1$), and c Brownian motion ($\beta = 2$). The fractal dimension estimate obtained by the expected curve length is denoted as $D_E$, and that obtained using Higuchi’s original method (average length) as $D_A$. The obtained numbers correspond to averages taken over 50 independent realizations of length $N = 25,000$

Fig. 2 Estimates of the fractal dimension ($D_E$ and $D_A$) for monofractal series of fractional Gaussian noises with prescribed spectral exponent $\beta$. Both methodologies lead to very similar average values for the simulated stochastic processes. The dashed line indicates the theoretical relationship between the fractal dimension and the spectral exponent $\beta$ for $N \to \infty$, which is given by $D = (5 - \beta)/2$ and applies for $1 < \beta < 3$ [31,42]. Error bars represent the numerical standard deviations obtained from 100 independent realizations of length $N = 10,000$

3 Multifractal Higuchi dimension analysis

3.1 Generalization of the modified Higuchi’s estimator

Based on our modification of Higuchi’s fractal dimension estimator as presented in the previous section, our principal interest is the generalization of this monofractal analysis method to a multifractal framework. For this purpose, we consider the $q$-th order moment of the modified Higuchi’s curve length as

$$L(k,q) = \frac{N-1}{k^2} \times \left\{ \sum_{n=1}^{N_k} \left[ \frac{1}{f_n} \sum_{x=1}^{f_n} (\Delta X_{n,x}(k))^q \right] P_n(\Delta X(k)) \right\}^{1/q}. \quad (7)$$

Here, $q$ can take any real number (for a discussion of the special case of $q = 0$, see Appendix A). The behavior of this generalized Higuchi length $L(k, q)$ for different orders of the moment $q$ is indicative of a dependence of the characteristic (fractal) scaling exponents on $q$. More precisely, if the time series under study displays power-law correlations (i.e., long-term memory resulting in fractal scaling characteristics), we can expect to find the power-law scaling behavior
\[ L(k, q) \sim k^{-dq}, \quad (8) \]

where \( dq \) is a generalized fractal dimension associated to the \( q \)-th moment. Notice that for \( q = 1 \), our modification of the standard Higuchi’s fractal dimension is recovered. For Gaussian time series exhibiting an extended self-similarity (multifractal) property, \( dq \) is related to the generalized Hurst exponent \( h_q \) (also known as the Hölder exponent) by the relationship \( h_q = 2 - dq \) [10]. In our case, the singularities are identified by the increments \( (\Delta X_{n,s}(k)) \). For \( q > 0 \), the contributions of fluctuations corresponding to large increments are highlighted, while the opposite situation is obtained for \( q < 0 \), i.e., fluctuations associated to small increments are amplified.

We emphasize that the increments \( \Delta X_{n,s}(k) \) can be small enough to markedly affect the numerical estimates of the moments of order \( q \), especially for negative \( q \), where reliable estimation becomes more and more difficult. To address this challenge, a suitable numerical regularization procedure may become necessary to allow a feasible calculation of the generalized lengths in Eq. (7). If the local behavior of the generalized length \( L(k, q) \) is found to be unstable (see Appendix B for details), we suggest here to replace the local mean \( \langle \Delta X_n(k) \rangle = \frac{1}{f_n} \sum_{s=1}^{f_n} \Delta X_{n,s}(k) \) in Eq. (7) by a regularized version trimmed at small increment values, \( \langle \Delta X_{n'}(k) \rangle \), where \( n' \) represents the \( n' \)-th interval of a new equiprobable partition in which we have removed all increment values below their empirical \( r \)-th percentile \( p_r \). In this case, the number of increments at scale \( k \) is now given by \( \sum_{n'} f_{n'} = N - k - N_{p_r} = (1 - p_r/100)(N - k) \) with \( N_{p_r} = p_r(N - k)/100 \) denoting the number of elements below the \( r \)-th percentile. In our numerical experiments described in the remainder of this work, we typically set a percentile value within the range \( r \in [1, 15] \) (see Appendix B), while for the number of bins in the partition, we set \( N_b = 16 \) (where for very long time series, finer partitions may also become numerically feasible).

3.2 Partition function and multifractal analysis

Considering the scaling behavior suggested in Eqs. (7) and (8),

\[
N - 1 \left\{ \frac{N_b}{k^2} \left[ \sum_{n=1}^{N_b} \left[ \frac{1}{f_n} \sum_{s=1}^{f_n} (\Delta X_{n,s}(k))^{q} \right] P_n(\Delta X(k)) \right] \right\}^{1/q} \\
\sim k^{-dq}, 
\]

it follows that

\[
\sum_{n=1}^{N_b} \left[ \frac{1}{f_n} \sum_{s=1}^{f_n} (\Delta X_{n,s}(k))^{q} \right] P_n(\Delta X(k)) \sim k^{q(2-d_q)}. 
\]

Using the fact that \( P_n(\Delta X(k)) = f_n/(N - k) \) and that for self-affine time series, \( h_q = 2 - dq \), Eq. (10) can be rewritten as

\[
\frac{1}{N - k} \sum_{i=1}^{N-k} (\Delta X_i(k))^{q} \sim k^{qh_q}, 
\]

which represents the well-known generalized structure function [3]. We notice that asymptotically (i.e., for large \( N \)) \( 1/(N - k) \sim k \), and therefore,

\[
\sum_{i=1}^{N-k} (\Delta X_i(k))^{q} \sim k^{qh_q-1}. 
\]

In the standard multifractal formalism, the partition function \( Z(q, k) \) is used to define the mass exponent \( \tau(q) \), and the following scaling relation is observed:

\[
Z(q, k) \equiv \sum_{i=1}^{N-k} (\Delta X_i(k))^{q} \sim k^{\tau(q)}, 
\]

where we have assumed that the left-hand side of Eq. (12) corresponds to the partition function \( Z(q, k) \). By comparing Eqs. (12) and (13), we obtain the relationship between the scaling exponents as

\[
\tau(q) = qh_q - 1. 
\]

Thus, we have related the exponent \( h_q \) to the fractal scaling exponent \( \tau(q) \), and the requirement of the fractal dimension value (when \( q = 0 \)) for the geometric support of the multifractal measure \( -\tau(0) = 1 = D_f \) is fully satisfied [27]. Moreover, the canonical way to characterize a multifractal signal is the singularity spectrum \( f(\alpha) \), which is obtained by means of \( \tau(q) \) and its first derivative \( \alpha \) as
\[ \alpha = \tau'(q) \Rightarrow f(\alpha) = q\alpha - \tau(q), \] (15)

where \( f(\alpha) \) represents the fractal dimension of the subset of the signal characterized by the local exponent \( \alpha \) [3].

4 Numerical examples: stochastic time series

4.1 Monofractal fractional Gaussian noises

In order to demonstrate the performance of our multifractal Higuchi dimension analysis (MF-HDA) method when applied to simulated monofractal times series, we generated realizations of fractional Gaussian noises with prescribed global Hurst exponents of \( H = 0.3 \), \( H = 0.5 \) and \( H = 0.75 \), respectively, by means of the Fourier transform method [43]. Figure 3 shows the results of the multifractal analysis for all three processes. As it can be seen in Fig. 3a, the generalized curve lengths for \( H = 0.5 \) exhibit a power-law scaling behavior for which well-defined exponents can be estimated (Eq. 8). In Fig. 3b, the exponents \( h_q = 2 - d_q \) are plotted against the moment \( q \). As expected, only a weak dependence of the numerical estimates on \( q \) is observed, indicating that the time series are, in fact, monofractal. Moreover, Fig. 3c shows the behaviors of \( \tau(q) \) versus \( q \), where clearly linear dependencies on \( q \) are observed for all three processes. These results lead to very narrow multifractal spectra estimates (Fig. 3d), which are consistent with the expected monofractal properties of the original time series within the expected limitations of numerical finite-sample estimates. In this context, we should emphasize that a correct estimation of the multifractal spectrum of a signal that by construction is monofractal, as in the case of the stochastic processes described above, presents a practical challenge, and that special care must be taken in the selection of the scales for the associated fit as well as in the choice of the range of the values of the moments \( q \).

4.2 Binomial multifractal cascade

The binomial multifractal cascade (BMC) is one of the most representative models for which the multifractal features can be derived analytically, allowing a direct comparison with the numerical results of our method (for details on the BMC model, see [3]). In the following, we explain only briefly the essential details of the model [3,45,46].

In order to generate a series \( z_j \), with \( j = 1, \ldots, 2^{m_{\text{max}}} \), we consider the following expression:

\[ z_j = a^n(j-1)(1-a)^{m_{\text{max}}-n(j-1)}, \] (16)

where \( a \in (0.5, 1) \) is a parameter, \( n(j-1) \) is the number of digits equal to 1 in the binary representation of the index \( j - 1 \), and \( m_{\text{max}} \) represents the number of iterations. One of the advantages of studying binomial cascades is that the key expressions of the multifractal properties can be obtained analytically in a straightforward manner as [27]

\[ \tau(q) = -\frac{\ln[a^q + (1-a)^q]}{\ln(2)}, \] (17)

and

\[ \alpha = -\frac{[a^q \ln(a) + (1-a)^q \ln(1-a)]}{\ln(2)[a^q + (1-a)^q]}, \] (18)
from which the multifractal spectrum can be obtained [3].

The numerical results of our MF-HDA method for BMC series with parameter \( a = 0.75 \) and \( N = 2^{m_{\text{max}}} = 2,048 \) are shown in Fig. 4. The behavior of the generalized Higuchi’s curve length \( L(k, q) \) versus \( k \) for \(-5 \leq q \leq 5\) is depicted in Fig. 4a, where evident changes in the slope (i.e., the generalized fractal dimensions \( d_q \)) are present as \( q \) varies. The corresponding \( h_q \) exponents are shown in Fig. 4b. The fact that \( h_q \) varies with \( q \) evidences the multifractal property of the BMC series. The mass function \( \tau(q) \) (Fig. 4c) presents a clearly nonlinear behavior with \( q \), as expected for multifractal series [3]. Finally, the resulting multifractal spectrum is shown in Fig. 4d, which was obtained from \( h_q \) and \( \tau(q) \) through the Legendre transform. We observe a very good agreement with the analytical predictions of the BMC model, except for an absence of the largest \( \alpha \) values which may lead to an underestimation of the multifractal width most likely occurring due to the short length of the considered sequence. We should also mention that the determination of the multifractal spectrum for the BMC entails some complications because for negative moments \( q \), the estimation of the length of the curve is limited due to the presence of instabilities for different scales \( k \), which leads to the larger values of \( \alpha \) not being estimated correctly.

5 Comparison of MF-HDA with MF-DFA

In the previous section, we have already presented results on the multifractal properties for selected time series with prescribed lengths and moments within a predefined interval of \( q \). In the following, we will be interested in comparing our new MF-HDA method with the established MF-DFA approach, which presents one of the most widely recognized and applied methodologies to assess the presence of multifractality in non-stationary time series [8,27,28,47,48]. For a detailed study on the applicability of MF-DFA and the WTMM method as another benchmark approach in multifractal analysis, the reader is referred to [29].

5.1 Multifractal detrended fluctuation analysis

Given a time series \( X(t) \), we compute the associated profile [27]

\[
Y(t) = \sum_{i=1}^{t} (X(i) - \bar{X})
\]

with \( t = 1, \ldots, N \), where \( N \) is the number of values in the time series and \( \bar{X} \) denotes the corresponding mean. Next, the integrated series \( Y(t) \) is divided into \( N_s \) intervals of equal size \([N/s]\). The local trend \( Y_r(t) \) is calculated separately for data from each of the segments \( r = 1, \ldots, N_s \) by least square fitting of a polynomial of prescribed degree (i.e., the order of detrending) and removed from the profile. Then, the mean square fluctuation of the detrended profile in each segment is given as

\[
F^2(s, r) = \frac{1}{s} \sum_{i=1}^{s} (Y((r-1)s + i) - Y_r(i))^2.
\]

The resulting (squared) detrended fluctuation function \( F^2(s) \) is defined as the average mean square fluctuation taken over all segments \( r \). For monofractal series, its square-root behaves like \( F(s) \sim s^H \) with \( H \) being an estimate of the process’ characteristic Hurst exponent.

To generalize this detrended fluctuation analysis to a multifractal analysis framework, we consider the order-\( q \) moments of the mean square fluctuations at scale \( s \)
5.2 Fractional Gaussian noise

As in the previous section, we first consider monofractal time series with prescribed Hurst exponents of $H = 0.3$, $H = 0.5$ and $H = 0.75$ with different lengths $N = 500, 1000, 5000, 15,000$, and $65,000$. We apply both MF-HDA and MF-DFA to the generated series. Figure 5 shows the corresponding estimates of the associated multifractal spectra. Especially for short time series lengths, MF-DFA displays wider spectra of values as compared to those obtained by means of MF-HDA. This behavior is especially noticeable for $N \leq 5000$, where the MF-DFA spectrum is markedly extended toward larger $\alpha$. As the time series length increases, both methods reveal gradually more similar multifractal features, except for the fact that the spectra estimated by means of MF-HDA are less symmetric, due to an absence of the largest $\alpha$ values (corresponding to very negative moment orders $q$), which are most affected by the employed regularization procedure removing the smallest increments.

Concerning the effect of the selected range of $q$ values, we also evaluated the changes in $f(\alpha)$ in terms of different intervals of $q$ for the same monofractal series, here with a fixed length of $N = 130,000$. The corresponding results are shown in Fig. 6. We observe that, for all three monofractal sequences, both methods lead to similar spectra for the interval $|q| \leq 3$, while for intervals $|q| \leq 5$ and $|q| \leq 10$, MF-HDA provides narrower spectra compared to those obtained by means of MF-DFA for the reasons already mentioned above related to the regularization affecting the smallest fluctuations.

5.3 Binomial multifractal cascade

In addition to simple monofractal processes, we also compared the MF-HDA and MF-DFA methods for the BMC model for several time series lengths. Figure 7 shows the spectra estimated with both methods along with the analytical results. We find that the obtained estimates are generally in good agreement with the theoretical spectra. Some minor deviations at small $\alpha$ values (i.e., for large positive $q$ values) are observed with both methods (with MF-DFA generally showing slightly larger deviations from the theoretical values),
strength exponents approaches, we finally computed the absolute differences when using MF-DFA (lacking the shortcomings due to the regularization procedure within our numerical implementation of MF-HDA). For long series, both methods lead to similar results with slightly smaller deviations when using MF-HDA. From these results, we conclude that in the BMC model MF-HDA/MT-FM/AF-H estimates are closer to the theoretical values when $q \leq -2$ and $s < 200$. Symbols and error bars represent the mean and standard deviation from 10 independent realizations.

For a more systematic comparison between both approaches, we finally computed the absolute differences between the theoretical values of the singularity strength exponents $\alpha_T$ (obtained from Eqs. (17) and (18)) and the corresponding numerical estimates $\alpha_E$ obtained using either MF-HDA or MF-DFA. Figure 9 shows these absolute differences (i.e., the estimation bias) as a function of the moments $q$ for several representative time series lengths. For very short series and for $q \geq -1$, the MF-HDA estimates are closer to the theoretical values, while differences between the theoretical and estimated values are smaller for $q \leq -2$ when using MF-DFA. We also tested the dependence of the obtained estimates on the range of $q$ values considered. Figure 8 shows that, as expected, the effect of widening the $q$ interval is to increase the width of the spectrum, with fairly similar behaviors of both methods.

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non-overlapping segments of the same length $N$ have been considered for each book. Then, we have used the following procedure to perform MF-HDA:

(i) For each $q$ and the $m$-th segment, the generalized curve length $L_m(k, q)$ is calculated.

(ii) The average value $L_{av}(k, q) = \frac{1}{M} \sum_{m=1}^{M} L_m(k, q)$ is evaluated, where $M$ denotes the number of segments for each book.

(iii) The value $d_{q, av}$ is estimated from the fit of the scaling behavior $L_{av}(k, q) \sim k^{d_{q, av}}$.

Figure 10 shows the results obtained with both MF-HDA and MF-DFA. We observe that unlike for the simple stochastic model cases studied in the previous section, the spectra $f(\alpha)$ obtained with MF-HDA are broader than those provided by MF-DFA, especially for short sequences. Notoriously, the expected convex shape of multifractal spectra is not well expressed in the MF-DFA results. Moreover, as the length of the considered segments is increased, the width of the spectra decreases but still remains relatively broad for MF-HDA as compared to MF-DFA. This result is remarkable since, as we have noticed in our controlled numerical experiments above, we may have expected MF-DFA tending to rather overestimate the multifractal spectral width for short time series.

As a consistency check, we finally generated shuffled versions of the word length series by applying a random permutation of the individual word lengths, thereby destroying eventual correlations in the sequence. The results of applying both multifractal analysis methods to the accordingly randomized sequences are also presented in Fig. 10. As expected, we obtain narrow spectra centered at $\alpha = 0.5$ for all books and different segment lengths, which do not differ much between the two methods.

7 Discussion

The identification of multifractal properties in time series is a challenge especially when dealing with short sequences, as are present in many real-world cases. The proposed MF-HDA method offers an alternative to assess the multifractal properties of relatively short series by constructing generalized curve lengths, the scaling of which encodes the multifractal characteristics of the signal. The implementation of our methodology is relatively simple, as it is based on the expected $q$-moment values of absolute increment distributions.

6 Real-world application: word length sequences

As an application to empirical data sets with intrinsic (time) order, we consider the sequences of subsequent word lengths (i.e., the numbers of letters forming each word [49]) obtained from a selection of fictional texts in English language, which include mainly novels from different authors (see Table 1 for details). Previous studies have highlighted the presence of positive correlations and multifractality in such word length sequences, which have been attributed to the fact that written language is the conformation of grammatical properties and semantic connotations with the idea of expressing information [50].

In order to allow for a feasible application of the MF-HDA method to the different word length series, the original length sequence has first been integrated to obtain a profile compatible with a fractional Brownian motion regime. Then, we applied both MF-HDA and MF-DFA to word length sequences with different segment lengths ($N = 2500$, $5000$, $10, 000$). In order to improve the reliability of the obtained statistics, several
Table 1  Books used for the multifractal analysis of word length sequences

| Title and author                          | Code  | Words      | $h_{q=1}$ | $\alpha^*$ | $\Delta\alpha$ |
|------------------------------------------|-------|------------|-----------|------------|--------------|
| Alice’s Adventure in Wonderland, L. Caroll | AAW   | 27,330     | 0.56      | 0.61       | 0.08         |
| The Anarchy, W. Dalrymple                 | ANC   | 15,971     | 0.54      | 0.55       | 0.08         |
| Animal Farm, G. Orwell                    | ANF   | 30,384     | 0.60      | 0.61       | 0.15         |
| Around the World in 80 Days, J. Verne     | AWD   | 63,760     | 0.62      | 0.63       | 0.04         |
| The Conquest of Bread, P. Kropotkin       | CQB   | 72,017     | 0.63      | 0.65       | 0.12         |
| On the Origin of Species, C. Darwin       | DRW   | 156,812    | 0.62      | 0.64       | 0.17         |
| The Picture of Dorian Gray, O. Wilde      | PDG   | 80,408     | 0.68      | 0.70       | 0.21         |
| Dracula, B. Stoker                        | DRC   | 162,317    | 0.67      | 0.68       | 0.20         |
| The Great Gatsby, F. S. Fitzgerald        | GTG   | 50,103     | 0.64      | 0.65       | 0.12         |
| Golden State, Ben H. Winters              | GDS   | 27,731     | 0.57      | 0.67       | 0.11         |
| The Grapes of Wrath, J. Steinbeck         | GPW   | 187,579    | 0.74      | 0.76       | 0.21         |
| Gulliver’s Travels, J. Swift              | GLT   | 104,798    | 0.69      | 0.71       | 0.28         |
| Hopscotch, J. Cortázar                     | HPS   | 195,703    | 0.69      | 0.71       | 0.26         |
| The Metamorphosis, F. Kafka               | MTM   | 22,383     | 0.58      | 0.58       | 0.11         |
| Moby Dick, H. Melville                    | MBD   | 218,705    | 0.69      | 0.70       | 0.16         |
| Pierre and Jean, G. de Maupassant         | PRJ   | 46,544     | 0.64      | 0.66       | 0.21         |
| The Idiot, F. Dostoyevsky                 | TID   | 247,953    | 0.69      | 0.70       | 0.21         |
| Three Men in a Boat, J. K. Jerome         | TMB   | 68,805     | 0.67      | 0.69       | 0.19         |
| The Time Machine, H. G. Wells             | TTM   | 32,776     | 0.63      | 0.64       | 0.05         |
| Ulysses, J. Joyce                         | ULY   | 272,416    | 0.74      | 0.77       | 0.36         |
| War and Peace, L. Tolstoy                 | WNP   | 572,628    | 0.66      | 0.67       | 0.14         |
| War of the Worlds, H. G. Wells            | WOW   | 60,897     | 0.68      | 0.70       | 0.22         |

The table lists the title and author, code name, and the total number of words, along with estimates of the global Hurst exponent $h_{q=1}$, the spectral mode $\alpha^*$ and the spectral width $\Delta\alpha$ as obtained using MF-HDA averaged over different segments with length $N = 2500$ (see Fig. 10). Although we have not followed a particular selection strategy for the studied titles beyond just considering well-known books, all estimated values for the global Hurst exponent ($h_{q=1}$) are clearly larger than 0.5, confirming the presence of long-range correlations in the word length series. These values are also in good agreement with previous monofractal analyses reported in [49,51,52]. The $\alpha^*$ values at which the multifractal spectra take their maxima are also consistently larger than 0.5 for all books. Moreover, most of the books are characterized by relatively broad multifractal spectra with $\Delta\alpha \geq 0.1$, except for AAW, ANC, and TTM (highlighted in boldface), for which the spectra exhibit a discontinuous behavior. Our results are consistent with previous analyses based on traditional multifractal procedures [53,54] and provide a better characterization of the complexity displayed in written texts. Interestingly, the book with the largest multifractal spectral width ($\Delta\alpha = 0.36$) corresponds to Joyce’s Ulysses (ULY, underlined), which has been described as a text with particularly great diversity in language [55].

We have described several cases where the MF-HDA method identifies a multifractal scaling behavior of the time series with very similar results as the celebrated MF-DFA approach, which provides a well-established methodology for the analysis of nonstationary signals. For very short series, the results for the MF-HDA indicate a certain gain in accuracy as compared to those obtained using the MF-DFA method, except for very negative moment orders $q$.

As with the majority of studies focusing on the detection of multifractality in nonstationary time series, the proposed methodology is subject to certain limitations: (i) dominant scaling exponents of the time series can be located outside of the region where the MF-HDA allows us to detect multifractal scaling in a reliable and robust manner; (ii) the limited sampling rate and numerical precision of the time series makes the expected moments of the generalized curve length quite difficult to estimate (especially for negative moments); (iii) the removal of the $p_r$ percentile of the absolute increments has a regularizing effect but can potentially lead to another bias in the estimated scaling exponents, especially for very negative $q$ values. Further refining the presented numerical procedures to address
MF-HDA, MF-DFA, or WTMM is recommended. We envision to further explore the corresponding potentials and more systematically apply our methodology to a range of real-world and simulated time series in our future work.

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Code availability Numerical implementations and examples for the application of MF-HDA can be found at https://github.com/carrizales90/MF-HDA.

Appendix A: The case \( q = 0 \)

In multifractal analysis, it is common that the scaling exponent \( d_q \) is not well defined when \( q \to 0 \). In our case, this value cannot be directly determined by means of the generalized curve lengths (Eq. (7)) due to the presence of a divergence in the exponent. More formally, we have

\[
\lim_{q \to 0} \mathcal{L}(q, k) = \lim_{q \to 0} \frac{N - 1}{k^2} \left\{ \sum_{n=1}^{N_b} \langle \Delta X_n(k) \rangle^q \right\}^{1/q} \sim \lim_{q \to 0} k^{-d_q}. \tag{19}
\]

Using some algebraic operations applied to the latter equation, which are omitted here for brevity, and applying L'Hôpital's rule, we find that a logarithmic transformation is required in order to determine the scaling exponent \( d_0 \) as

\[
\mathcal{L}(0, k) \equiv \frac{N - 1}{k^2} \exp \left( E[\ln\langle \Delta X(k) \rangle] \right) \sim k^{-d_0}, \tag{20}
\]

where \( E[\ln\langle \Delta X(k) \rangle] = \sum_{n=1}^{N_b} \langle \ln\Delta X(k) \rangle P_n(\Delta X(k)) \).
Appendix B: Regularization effect of removing the lower percentile of absolute increments

As discussed in Sect. 3, numerical instabilities can appear in the evaluation of the moments of the generalized curve length $L(k, q)$ (Eq. 7), especially for $q < -1$. In this case, we have suggested that the local mean could be replaced by a one-sided trimmed version $\langle \Delta X_n^r(k) \rangle > p_r$ in Eq. (7), where $n'$ represents the $n'$-th interval of a new equiprobable partition in which we have removed the $r$-th percentile $p_r$ of the empirical distribution of the absolute increments. We note that numerical experiments with both, one-sided (asymmetrically) and two-sided (symmetrically) trimmed means revealed no qualitative differences in the resulting estimates (not shown), while removing the uppermost percentiles (i.e., very large increments) appears unnecessary since those values have no negative effects on the stability of the numerical estimates of the generalized curve lengths.

To address the problem of selecting a specific percentile to be removed, we focus here on just one statistical property, the confidence interval (CI) of the estimated slope (i.e., the scaling exponent $d_q$ in Eq. (8)) in the linear regression of the double-logarithmic generalized curve length versus scale relationship, at a certain confidence level (here, $\gamma = 0.05$), and for the most negative value of $q$. For two-sided confidence intervals, the CI width (CIW) (measured in units of the associated standard error) is given by $I_{q, p_r} \equiv I_{q, p_r} / S_{dq}$, with $S_{dq}$ being the standard error of the estimated slope $d_q$ [56].

Figure 11 shows the behavior of the rescaled CIW (in units of $I_{p_r=0}$) as a function of the removed percentile $p_r$, for some of the simulated stochastic processes and real-world data sets discussed in Sections 4 and 6, respectively, for $q = -5$. The results show that, as the removed percentile is increased, the CIW decreases in such a way that, for fractional Gaussian noises with $H = 0.3$, $H = 0.5$ and $H = 0.75$, the rescaled CIW has decayed by more than one half of its initial value when $p_r = 1$, while for the word length data (exemplified here by the ULY book) the observed decay is slower. For practical purposes, we suggest that a criterion for selecting the value of the percentile to be removed should consider empirically a value of the percentile for which the rescaled CIW has stabilized, that is, even if higher percentiles are removed, there are no substantial further changes. We observe that $p_r \approx 5$ and $p_r \approx 12$ would be desirable in the cases of the simulated fractional Gaussian noises and word length data, respectively.

While the suggested strategy presents just a first attempt to improving the practical estimation of the generalized Higuchi’s fractal dimensions, we emphasize that there may be cases in which the rescaled CIW may behave in a more unstable way with increasing percentile $p_r$. In such cases, additional numerical tests with larger $p_r$ values may become necessary to determine a reliable value leading to sufficiently stable estimates.

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