Resummation of the perturbation series for an interaction of a scalar field with a quantized metric

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Abstract
We consider a scalar field interacting with a quantized metric varying only on a submanifold (e.g. a scalar field interacting with a quantized gravitational wave). We explicitly sum up the perturbation series for the time-ordered vacuum expectation values of the scalar field. As a result we obtain a modified non-canonical short distance behavior.

1 Introduction

We consider a model of a scalar field interacting with a quantized metric. In order to simplify the model we assume that the metric does not depend on all the coordinates. There is a physical model for such a metric tensor: the field of a gravitational wave. Hence, the model would describe an interaction of a scalar particle with the graviton. We have obtained in [1] a path integral representation of the scalar field correlation functions. Then, we assumed an exact scale invariance of the quantum gravitational field. In this paper we insert a flat background metric and consider an expansion around this metric. We show that the path integral formula can be considered as a resummation of the con-
ventional perturbation expansion. Then, the assumption that the perturbation around the flat metric is scale invariant leads to the same conclusions as in [1]: the singular quantum metric substantially changes the short distance behavior of the scalar field (in particular, it can make the scalar field more regular).

2 The conventional perturbation expansion

We consider the Lagrangian for a complex scalar field in $D$-dimensions interacting with gravity

$$\mathcal{L} = \mathcal{L}(g) + g^{AB} \partial_A \phi \partial_B \phi + M^2 \phi \phi$$  \hspace{1cm} (1)

here $\mathcal{L}(g)$ is a gravitational Lagrangian which we do not specify. The metric is assumed to be a perturbation of the flat one

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$$  \hspace{1cm} (2)

where $\mu, \nu = 1, \ldots, D - d$, $h^{\mu\nu}$ is varying only on a $d$-dimensional submanifold and

$$g^{ik} = \eta^{ik}$$  \hspace{1cm} (3)

for $i, k = D - d + 1, \ldots, D$, here $\eta^{AB} = \pm \delta^{AB}$, but we do not specify the signature.

We split the coordinates as $x = (X, \mathbf{x})$ with $\mathbf{x} \in R^d$.

As an example we could consider in $D = 4$ the metric corresponding to a gravitational wave moving in the $x_3$ direction. In such a case the metric is

$$ds^2 = -(dx_4)^2 + (dx_3)^2 + g^{\mu\nu} dx_{\mu} dx_{\nu}$$
where $g^{\mu\nu}(x_3-x_4)$ is an exact solution of the Einstein equations, see e.g. [2] (this case would correspond to $d = 1$). More general solutions are known describing a scattering of gravitational waves and depending on both $x_4 - x_3$ and $x_4 + x_3$ (hence, $d = 2$) (for higher dimensions see [3][4]). The assumption of a weak gravitational field (2) is a realistic approximation. In such a case it can be shown that any solution of linearized Einstein equations by means of a coordinate transformation can be brought to the form where all components of $h^{\mu\nu}$ are equal to zero except of $\mu, \nu = 1, 2$. These components describe the transverse polarization states of the spin 2 field.

Without a self-interaction the $\phi\bar{\phi}$ correlation function averaged over the metric is

$$\langle (\mathcal{A} + \frac{1}{2} M^2)^{-1}(x, y) \rangle$$ (4)

where

$$-\mathcal{A} = \frac{1}{2} \square_D - d + \frac{1}{2} \sum_{\mu=1, \nu=1}^{D-d} h^{\mu\nu}(x) \partial_\mu \partial_\nu + \frac{1}{2} \sum_{k=D-d+1}^{D} \partial_k^2 \equiv \frac{1}{2} \square_D + \frac{1}{2} \sum h^{\mu\nu}(x) \partial_\mu \partial_\nu$$ (5)

and the index at the d’Alambertian denotes its dimensionality. The inverse in eq.(4) is not unique for the pseudo-Riemannian metric (2)-(3). However, we define it in the unique way later on by means of the Feynman proper time representation together with the $i\epsilon$ prescription.

Let us begin with the two-point function (4) and consider the Neumann
\[
\langle(-\Box_D - h\partial\partial + M^2)^{-1}(x, y)\rangle = \langle\sum_{n=0}^{\infty}((Gh\partial\partial)^nG)(x, y)\rangle
\]
\[
= G(x, y) + \partial\partial\partial\partial GGG(x, y)\langle hh \rangle + \partial\partial\partial\partial\partial\partial\partial GGGGG(x, y)\langle hhhh \rangle + ...
\]
(6)

where \( G = (-\Box_D + M^2)^{-1} \) denotes the Feynman propagator for the scalar field of mass \( M \). In eq.(6) we have used an integration by parts in order to move the derivatives to the beginning (this is possible because the correlations of \( h \) depend only on \( R^d \) and the derivatives concern the complementary \( D - d \) variables).

For example, the first non-trivial term in eq.(6) reads

\[
\int d\tau_1 d\tau_2 \partial^\nu \partial^\tau \partial^\sigma \partial^\rho G(x - z_1)G(z_1 - z_2)G(z_2 - y)D^{\mu\nu;\sigma\rho}(z_1 - z_2)
\]
(7)

where

\[
\langle h^{\mu\nu}(z_1)h^{\sigma\rho}(z_2)\rangle = D^{\mu\nu;\sigma\rho}(z_1 - z_2)
\]

In order to compare the conventional expansion with the stochastic one of ref.[1] we apply the proper-time representation for \( G \)

\[
\prod_{k=1}^{n+1} G(v_k) = \frac{1}{2} \prod_{k=1}^{n+1} \int_0^\infty d\tau_k \exp\left(-\frac{i}{2} \tau_k M^2\right)p(\tau_k, v_k)p(\tau_k, V_k)
\]
(8)

where in the dimension \( r \) the Feynman kernel \( p \) is

\[
p(\tau, x) = (2\pi i\tau)^{-\frac{r}{2}} \exp\left(-\frac{x^2}{2i\tau}\right)
\]

and we made use of the property of the Feynman kernel in \( D \)-dimensions that it is a product of the kernels in \( D - d \) and \( d \)-dimensions. Then, in the product

\[
\]
we change the integration variables, so that

\[ G(x - z_n)G(z_n - z_{n-1}) \ldots G(z_1 - y) = \int_0^{\infty} d\tau \int_0^\tau ds_n \int_s^{s_n} \ldots \int_0^{s_2} ds_1 \]

\[ p(\tau - s_n, x - z_n)p(s_n - s_{n-1}, z_n - z_{n-1}) \ldots p(s_1, z_1 - y) \]

(9)

where in eqs. (8)-(9) we introduced new variables

\[ \tau_1 = s_1 \]

\[ \tau_1 + \tau_2 = s_2 \]

\[ \ldots \]

\[ \tau_1 + \ldots + \tau_n = s_n \]

\[ \tau_1 + \ldots + \tau_{n+1} = \tau \]

We write

\[ z_k = x_k + y \]

Then, in the representation (9) we apply many times the Smoluchowski-Kolmogorov equation (the semigroup composition law) for the \( D - d \) dimensional kernels

\[ \int p(s - s', X - Z)p(s', Z)dZ = p(s, X) \] (10)

Performing the integration by parts we can see that the derivatives in the expansion (6) (with the representation (9)) finally act on \( p(\tau, X - Y) \). We may write the result in the momentum representation in the form (which will come out in the stochastic version of the next section)

\[ \partial^X_\mu p(\tau, X - Y) = \int dP \exp(iP(Y - X)) \exp(-\frac{\tau}{2}P^2 - \frac{\tau}{2}M^2)(-iP_\mu) \]
We can compute higher order correlation functions in the Gaussian model of scalar fields. They are expressed by the two-point function. For example, the four-point function expanded around the flat metric reads

\[
\langle \phi(x) \phi(x') \phi(y) \phi(y') \rangle = \langle (A + \frac{1}{2} M^2)^{-1}(x, y) (A + \frac{1}{2} M^2)^{-1}(x', y') \rangle + (x \rightarrow x')
\]

where \( x \rightarrow x' \) means the same expression with the exchanged coordinates. In the next section we sum up the perturbation series (4) and (11). This will allow us to determine the functional dependence of the scalar correlation functions on the metric explicitly.

Let us note that the expectation value over the metric field in eqs. (4) and (11) depends on the scalar determinant. It can involve a complex dependence on the metric. We cannot compute the effective gravitational action explicitly. However, its scaling behavior is sufficient to determine the short distance behavior of the scalar fields.

\section{The stochastic representation}

Let us recall a representation of the correlation functions (6) and (11) by means of the Feynman integral \cite{1}\cite{5}. In this paper we work with a real time. It makes no difference for a perturbation expansion whether we work in Minkowski or in Euclidean space. However, the non-perturbative Euclidean version may not exist (e.g. this is the case when the Hamiltonian is unbounded from below). For this reason we consider here a representation with an indefinite metric al-
ternative to the Euclidean version of ref.[1].

We represent the scalar field two-point function by means of the proper time method

\[(\mathcal{A} + \frac{1}{2}M^2)^{-1}(x, y) = i \int_0^\infty d\tau \exp(-\frac{i}{2}M^2\tau) (\exp(-i\mathcal{A}) (x, y)\] (12)

The kernel \((\exp(-i\mathcal{A})) (x, y)\) can be expressed by the Feynman integral

\[K_\tau(x, y) \equiv (\exp(-i\mathcal{A})) (x, y) = \int Dx \exp(i \frac{1}{2} \int \frac{dx}{dt} + \frac{i}{2} \int (h^{\mu\nu}(x) + \eta^{\mu\nu}) \frac{dX_\mu}{dt} \frac{dX_\nu}{dt}) \delta(x(0) - x) \delta(x(\tau) - y)\] (13)

In the Feynman integral (13) we make a change of variables \((x \to b)\) defined by the solution of the Stratonovitch stochastic differential equation \([6]\)

\[dx_\Omega(s) = e^\Omega_A(x(s)) db^A(s)\] (14)

where for \(\Omega = 1, \ldots, D - d\)

\[e_\Omega^\mu e_\Omega^\nu = \eta^{\mu\nu} + h^{\mu\nu}\]

and \(e^\Omega_A = \delta^\Omega_A\) if \(\Omega > D - d\).

The change of variables (14) transforms the functional integral (13) into a Gaussian integral with the covariance

\[E[b_A(t)b_C(s)] = i\delta_{AC} \min(s, t)\]

It can be interpreted as an analytic continuation of the standard Wiener integral.

This means that on a formal level

\[E[F] = \int Db \exp \left( \frac{i}{2} \int ds \left( \frac{db}{ds} \right)^2 \right) F(b)\]
The solution $q_\tau$ of eq.(14) is defined by two vectors $(Q,q)$ where

$$q(\tau,x) = x + b(\tau) \quad (15)$$

and $Q$ has the components (for $\mu = 1, \ldots, D-d$)

$$Q^\mu(\tau,X) = X^\mu + \int_0^\tau e_\mu^\alpha(q(s,x)) dB^\alpha(s) \quad (16)$$

The kernel (13) can be expressed by the solution (15)-(16)

$$K_\tau(x,y) = E[\delta(y - q_\tau(x))]$$

$$= E[\delta(y - x - b(\tau)) \prod_\mu \delta(Y^\mu - Q^\mu(\tau,X))]$$

When the $\delta$-function is defined by its Fourier transform then the kernel $K_\tau$ takes the form

$$K_\tau(x,y) = (2\pi)^{-D+d} \int dP \exp(iP(Y - X))$$

$$E[\delta(y - x - b(\tau)) \exp\left(-i \int P^\mu e_\mu^\alpha(q(s,x)) dB^\alpha(s)\right)] \quad (17)$$

We can perform the $B$-integral. The random variables $b$ and $B^\alpha$ are independent. We can use the formula [6]

$$E[\exp\int f_\alpha(b) dB^\alpha] = E[\exp(-\frac{i}{2} \int f_\alpha f_\alpha ds)]$$

Then, the Feynman kernel is expressed by the metric tensor

$$K_\tau(x,y) = (2\pi)^{-D+d} \int dP \exp(iP(Y - X)) \exp(-i \frac{1}{2} P^2)$$

$$E[\delta(y - x - b(\tau)) \exp\left(-\frac{i}{2} \int_0^\tau P^\mu h^\mu_\nu(q(s,x)) P^\nu_s ds\right)] \quad (18)$$

Note that in eq.(18) instead of Feynman paths we may use the Brownian paths $\tilde{b}$ [5] (the $\delta$-function can be taken away by replacing the Brownian motion by the Brownian bridge as in [1])

$$\tilde{b} = \exp(-\frac{\pi}{4})b \quad (19)$$
Then, the expectation value really is an average over the random process.

The perturbation expansion in $h$ reads

$$
\langle (A + \frac{i}{2}M^2)^{-1}(x, y) \rangle = i \int_0^\infty d\tau \exp(-\frac{i}{2}M^2)K_\tau(x, y)
$$

$$
= (2\pi)^{-D+d} \int dP \exp(iP \cdot (Y - X)) i \sum_{n=0}^\infty \int d^s d_{s, n} (s, x_{n-1}, \ldots, x_1) \delta(y - x - b(\tau))
$$

$$
\langle P_\rho h^{\alpha\beta} (q(s_{n-1}, x)) \rangle \langle P_{\rho'} \ldots \rangle
$$

(20)

The expectation value over the Feynman paths (the complex Brownian motion) can be computed and we obtain as a result the expectation value depending on the correlation functions of the metric field

$$
\langle (2\pi)^{-D+d} \int dP \exp(iP \cdot (Y - X)) i \sum_{n=0}^\infty \int d^s d_{s, n} (s, x_{n-1}, \ldots, x_1) \delta(y - x - b(\tau))
$$

$$
\langle P_\rho h^{\alpha\beta} (x + x_n) \rangle \langle P_{\beta'} \rangle
$$

(21)

It is easy to see that this formula coincides with eq.(6) where the representation (9) is applied with the simplification concerning the $X$-dependence (that the result is expressed finally by $p(\tau, X - Y)$) discussed at the end of sec.2.

Representing the four-point function by means of the stochastic method we obtain

$$
\langle \phi(x) \phi(x') \rangle = -(2\pi)^{-2D+2d} \int d\tau_1 d\tau_2 \int dPdP'
$$

$$
\exp(-\frac{i}{2}\tau_1 P^2 - \frac{i}{2}\tau_2 P'^2 - \frac{1}{2}\tau_1 M^2 - \frac{1}{2}\tau_2 M^2)
$$

$$
\exp(iP \cdot (Y - X) + iP' \cdot (Y' - X'))
$$

$$
E[\delta(y - x - b(\tau)) \delta(y' - x' - b'(\tau))]
$$

$$
\exp \left( -\frac{i}{2} \int_0^{\tau_1} P_\mu h^{\mu\nu} (q(s, x)) P_\nu ds \right) \exp \left( -\frac{i}{2} \int_0^{\tau_2} P'_\mu h^{\mu\nu} (q'(s, x')) P'_\nu ds' \right)
$$

$$
+ (x \rightarrow x')
$$

(22)
We make a perturbation expansion in the metric

\[ \sum_{n=0}^{\infty} (2i)^{-n} \int_{s_{n-1}}^{s_n} ds \int_{0}^{s} ds_1 P_{\mu} h^{\mu\nu} (q(s, x)) P_{\nu} \]

\[ P_{\mu} h^{\mu\nu} (q(s_{n-1}, x)) P_{\nu} \ldots P_{\alpha} h^{\alpha\beta} (q(s_1, x)) P_{\beta} \] (23)

The paths \( q \) and \( q' \) are independent. We calculate the expectation values according to the formula (21). In the conventional perturbation expansion (11) we make the same change of the proper time variables (9) as in the two-point function (for each two-point function separately). Then, it becomes evident by the same argument as the one at the two-point function that the stochastic formula (23) and the conventional one (11) coincide (no matter what are the correlation functions for the metric field). It is now easy to see that the argument concerning the four-point function can be generalized to any \( 2n \)-point function.

### 4 Ultraviolet behavior of the perturbation series

In order to study the ultraviolet behavior of the perturbation expansion (6) and (11) it is useful to express the functional integral in the momentum space. First, we write down the scalar part of the action (1) in the form

\[ L = \int dx dP (\nabla_x \phi(x, P) \nabla_x \phi(x, P) + P^2 \phi(x, P) \phi(x, P) + M^2 \phi(x, P) \phi(x, P) + h^{\mu\nu} (x) P_{\mu} P_{\nu} \phi(x, P) \phi(x, P)) \] (24)

Then, it is easy to see from the functional integral for the correlation functions

\[ \int \mathcal{D}\phi \exp(-L) \phi \ldots \phi \]
that there is a direct correspondence between the model (1) and the trilinear interaction \( P_\mu P_\nu h^{\mu\nu} \overline{\phi} \phi \) where \( P \) is treated just as a parameter. The Fourier transform over \( P \) is performed in the final formula. For example

\[
\langle \overline{\phi}(x, X)\phi(x', X') \rangle = \int dP dP' \langle \overline{\phi}(x, P)\phi(x', P') \rangle \exp(iPX - iP'X')
\]

\[
= \int dP \langle (-\Delta_d + P^2 + h^{\mu\nu} P_\mu P_\nu)^{-1}(x, x') \rangle \exp(iP(X - X'))
\]

whereas for the four-point function

\[
\langle \overline{\phi}(x, X)\overline{\phi}(y, Y)\phi(x', X')\phi(y', Y') \rangle = \int dP dP' dK dK' \langle \overline{\phi}(x, P)\overline{\phi}(x', P')\phi(y, K)\phi(y', K') \rangle \exp(iPX + iP'X' - iKY - iK'Y')
\]

\[
= \int dP dK \langle (-\Delta_d + P^2 + h^{\mu\nu} P_\mu P_\nu)^{-1}(x, y) \rangle \exp(iP(X - Y)) \exp(iK(X' - Y')) + (x \rightarrow x')
\]

The integration over the \( D - d \) momenta \( P \) does not lead to any additional infinities. It follows that the divergencies depend on the dimension \( d \) and on the singularity of the metric field correlations. Hence, if \( d < 4 \) and \( h \) has the canonical short distance behavior \( |x - y|^{-d+2} \) then there will be no divergencies at all.

5 **Modified short distance behavior**

The correlation functions (12) and (22) have a different scaling behavior in \( x \) and \( X \) directions. We obtain a fixed scaling behavior either setting \( x = y = 0 \) or \( X = Y = 0 \). Assume that \( h^{\mu\nu} \) is scale invariant at short distances

\[
h^{\mu\nu}(x) \simeq \lambda^{2\gamma} h^{\mu\nu}(\lambda x)
\]
Then, by scaling (just as in [1], although $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ is not scale invariant) and using the stochastic formula (18) we obtain

$$\langle A^{-1}(x, y) \rangle \simeq |X - Y|^{-D+2-\frac{2\gamma}{d}}$$

(28)

at short distances. This argument can be extended to all correlation functions showing that

$$\phi(0, X) \simeq \lambda^{\frac{d-2}{2} + \frac{D-d}{2}(1-\gamma)} \phi(0, \lambda X)$$

(29)

where the equivalence means that both sides have the same correlation functions at short distances.

For the behavior in the $x$ direction we let $X = Y = 0$. In such a case just by scaling of momenta in eq.(18) we can bring the propagator to the form

$$\langle A^{-1}(x, y) \rangle = \int_0^\infty d\tau \int dP \exp\left(\frac{1}{2}P^2 \tau^\gamma \lambda^{1-\gamma}(X - Y)\right) \exp\left(-\frac{i}{\tau}\lambda^{-2\gamma}(P^2 + M^2)\right) \sum_n f_n(\tau, P, \lambda(y - x))$$

(33)
We set \( \lambda = |y-x|^{-1} \) if \( X = Y \) and \( \lambda = |X-Y|^{-1} \) if \( x = y \). The formulas (28) and (31) follow under the assumption that setting \( P^2 + M^2 = 0 \) in eq.(33) we obtain a finite result from the integral over \( \tau \) of the sum of \( f_n \). In a similar way we can obtain the short distance behavior of higher order correlation functions. As an example we could consider a dimensional reduction of the gravitational action (a static approximation) from \( d = 4 \) to \( d = 3 \). Then, in the quadratic approximation to the threedimensional quantum gravity we obtain \( \gamma = \frac{1}{4} \) and an explicit finite average over \( h \) in eq.(18).

6 Discussion

We have obtained formulas expressing quantum scalar field correlation functions by quantum gravitational correlation functions in a model where the metric field depends only on a lower dimensional submanifold. Such models can either be considered as an approximation to a realistic theory or they may come from a dimensional reduction of a higher dimensional Einstein gravity [7]- [8]. In either case the non-perturbative phenomena are essential for the short distance behavior. In this way we have confirmed by a resummation of the perturbation series our results [9] based on a formal functional integral. The result is that a singular quantum gravitational field substantially modifies the short distance behavior of the matter fields. In particular, it can make the matter fields more regular.
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