A SURVEY ON THE CONVERGENCE OF MANIFOLDS WITH BOUNDARY

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ABSTRACT. This survey reviews precompactness theorems for classes of Riemannian manifolds with boundary. We begin with the works of Kodani, Anderson-Katsuda-Kurylev-Lassas-Taylor and Wong. We then present new results of Knox and the author with Sormanii.

1. Introduction

Given a sequence of Riemannian manifolds, one can say that the sequence converges if the manifolds are resembling more and more some metric space. This survey reviews theorems that are able to tell when some sequences of manifolds have subsequences that converge, using a variety of notions of convergence presented in Section 2. Section 3 is devoted to those theorems that require smooth boundary conditions and Section 4 states theorems for open manifolds for which the boundary is defined to be \( \partial M := \overline{M} \setminus M \).

Section 2 reviews the definitions of Lipschitz, \( C^{1,\alpha} \) and Gromov-Hausdorff distance. In addition, Cheeger-Gromov, Gromov and Anderson precompactness theorems for Riemannian manifolds without boundary are stated (Theorems 2.3, 2.7, 2.9, 2.14). Although there are interesting theorems proven both for manifolds with and without boundary for the intrinsic flat distance [17, 18] and for ultralimits [19, 8], these are not discussed here.

Section 3 surveys results concerning Riemannian manifolds that have smooth boundaries.

In 1990, Kodani [12] proves a precompactness theorem for Riemannian manifolds with boundary using the Lipschitz topology. He assumes that the manifolds have uniformly bounded sectional curvature, nonnegative second fundamental forms uniformly bounded from above, and a uniform lower bound for the volume of the manifolds. See Subsection 3.1. In 2004, Anderson-Katsuda-Kurylev-Lassas-Taylor [11] proves a precompactness theorem using a different approach under completely different hypotheses. They use \( C^{1,\alpha} \) convergence (\( \alpha < 1 \)) instead of Lipschitz convergence. They assume uniform bounds on the norm of the Ricci tensor of the manifolds and their boundaries, the mean curvatures, the diameter of the manifolds, and on three different radii: injectivity, interior and boundary. See Subsection 3.2. In 2008, Wong [20] uses Alexandrov spaces to prove two precompactness theorems in the Gromov-Hausdorff topology with the same conditions as Kodani, except that the volume bound is replaced by a diameter bound. He assumes a lower bound on sectional curvatures of the manifolds, second fundamental forms and diameter of the manifolds bounded above. Furthermore, Wong also proves that the sectional curvature bound can be replaced by a Ricci curvature bound. See Subsection 3.3. Recently, Knox with the same technique used by Anderson-Katsuda-Kurylev-Lassas-Taylor proves a precompactness theorem in the \( C^1, \alpha < 1 \), and \( L^{1,\alpha} \) topologies with hypothesis analogous to

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their. He assumes bounds on the secional curvatures of the manifolds and their boundaries, the mean curvatures, the diameter of the manifolds, and a uniform lower bound in the volume of the boundaries. This theorem appears in [4].

Section 4 surveys Gromov-Hausdorff precompactness theorems concerning classes of open manifolds where no smoothness of the boundary is required. These results appear in work with Sormani with the author [13]. Rather than proving that a sequence of manifolds converges, we study regions within the manifolds, take the Gromov-Hausdorff limits of those inner regions and then glue the limits together to create a glued limit space. We assume conditions on the Ricci curvature, volume bounded below and above, and diameter of the inner regions uniformly bounded.

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2. Types of convergence

2.1. Lipschitz Convergence. For details about Lipschitz convergence see [10] and [3].

Definition 2.1. Let \((X,d_X)\) and \((Y,d_Y)\) be two metric spaces. The dilation of a Lipschitz map \(f : X \to Y\) is defined by

\[
dil(f) = \sup_{x \neq x' \in X} \frac{d_Y(f(x), f(x'))}{d_X(x, x')}
\]

A function \(f : X \to Y\) is called bi-Lipschitz if both \(f : X \to Y\) and \(f^{-1} : f(X) \to Y\) are Lipschitz maps.

Definition 2.2. The Lipschitz distance between two metric spaces \((X,d_X)\) and \((Y,d_Y)\) is defined by

\[
d_L(X, Y) = \inf_{f : X \to Y} \log(\max\{\text{dil}(f), \text{dil}(f^{-1})\})
\]

where the infimum is taken over all bi-Lipschitz homeomorphisms \(f : X \to Y\).

Theorem 2.3 (Cheeger-Gromov). The class of connected closed \(m\)-dimensional Riemannian manifolds \(M\) satisfying:

\[
|\sec(M)| \leq K, \ Vol(M) \geq v \ and \ Diam(M) \leq D,
\]

is precompact in the Lipschitz topology.

This theorem follows from a Gromov’s precompactness theorem (Theorem 8.25 in [9]), in which positive uniformly bounded injectivity radii is needed, and Cheeger’s doctoral dissertation [4], which proves that the class of manifolds satisfying [11] have positive uniformly bounded injectivity radii.

2.2. Gromov-Hausdorff Convergence. More about Gromov-Hausdorff convergence can be find in [10] and [3]. Gromov’s embedding theorem appears in [9]. Examples and pictures about Hausdorff converging sequences can be found in [15].

Definition 2.4 (Hausdorff). Let \((Z,d_Z)\) be a metric space, the Hausdorff distance between two subsets, \(A_1, A_2 \subset Z\), is defined as

\[
d_H(A_1, A_2) = \inf \{ r : A_1 \subset T_r(A_2), A_2 \subset T_r(A_1) \}
\]

where the tubular neighborhood, \(T_r(A) = \{ x \in Z : d_Z(x, A) < r \} \).
Meanwhile Gromov-Hausdorff distance avoids fixing a metric space by considering isometric embeddings of two metric spaces into a common metric space:

**Definition 2.5.** A function \( \varphi : (X, d_X) \rightarrow (Z, d_Z) \) between metric spaces is an isometric embedding if

\[ d_Z(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2), \]

for all \( x_1, x_2 \in X \)

**Definition 2.6 (Gromov).** Let \((X_1, d_{X_1})\) and \((X_2, d_{X_2})\) be two compact metric spaces. The Gromov-Hausdorff distance between them is defined as

\[ d_{GH}(X_1, X_2) = \inf \{ d_Z(\varphi_1(x_1), \varphi_2(x_2)) : \varphi_i : X_i \rightarrow Z \} \]

where the infimum is taken over all isometric embeddings \( \varphi_i : X_i \rightarrow Z \) and all metric spaces \( Z \).

Gromov proved that the Gromov-Hausdorff distance is a distance on the space of isometry classes of compact metric spaces. In general, elements of the classes are used and the class to which they belong is never mentioned.

The most general Gromov’s precompactness theorem says:

**Theorem 2.7.** [Gromov] Let \( D > 0 \) and \( N : (0, D] \rightarrow \mathbb{N} \) a function. Then the collection \( \mathcal{M}^{D,N} \) of compact metric spaces \((X, d_X)\) with \( \text{Diam}(X) \leq D \) that can be covered by \( N(\epsilon) \) balls of radius \( \epsilon \) is precompact with respect to the Gromov-Hausdorff distance.

Given \( \epsilon \) and a metric space \( X \), the minimum number of of \( \epsilon \)-balls needed to cover \( X \) is the same as the maximum number of pairwise disjoint \( \epsilon/2 \)-balls in \( X \). Then \( N \) can be thought as a function that bounds the maximum number of pairwise disjoint balls of compact metric spaces inside a certain class.

The converse to Theorem 2.7 also holds.

**Theorem 2.8** (Gromov). Suppose \((X_j, d_j)\) are compact metric spaces. If there exists \( \epsilon_0 > 0 \) such that \( X_j \) contains at least \( j \) disjoint balls of radius \( \epsilon_0 \), then no subsequence of the \( X_j \) has a Gromov-Hausdorff limit.

Thus proving precompactness theorems with respect to the Gromov-Hausdorff distance of a certain class of compact metric spaces is “reduced” to finding a function \( N \) and uniform upper diameter bound \( D \). For sequences of compact Riemannian manifolds with no boundary, Gromov applied the Bishop-Gromov Volume Comparison Theorem (Theorem 2.18 [15]) to control the volume of the balls and obtain the following precompactness theorem.

**Theorem 2.9** (Gromov). Given \( m \in \mathbb{N} \) and \( D > 0 \), let \( \mathcal{M}^{m,D} \) be the class of compact \( m \) dimensional Riemannian manifolds \( M \) with

\[ \text{Ricci}(M) \geq 0 \quad \text{and} \quad \text{Diam}(M) \leq D. \]

Then \( \mathcal{M}^{m,D} \) is precompact with respect to the Gromov-Hausdorff distance.

### 2.3. \( C^{1,\alpha} \) Convergence

For a more detailed account on \( C^{k,\alpha} \) convergence, consult [12].

**Definition 2.10.** Let \( \{(M_i, g_i)\} \) be a sequence of \( m \)-dimensional Riemannian manifolds. The sequence converges in the \( C^{1,\alpha} \) topology to a \( C^{1,\alpha} \) manifold \((M, g)\) if \( M \) is a \( C^\infty \) manifold such that for some fixed \( C^{1,\alpha} \) atlas on \( M \) compatible with its \( C^\infty \) structure, \( g \) is \( C^{1,\alpha} \), and there are diffeomorphisms \( \varphi_i : M_i \rightarrow M \), \( i = 1, 2, 3, \ldots \), for which \( \varphi_i^* g_i \rightarrow g \) with the \( C^{1,\alpha} \)-norm.
Remark 2.11. Theorem 2.3 also holds for $C^{1,\alpha}$ convergence.

$C^{1,\alpha}$ precompactness theorems for manifolds with or without boundary have been proved using the notion of $(r, N, C^{1,\alpha})$ harmonic coordinate atlas and harmonic radius.

Definition 2.12. Let $(M, g)$ be an m-dimensional compact Riemannian manifold. $(M, g)$ has an adapted harmonic coordinate atlas $(r, N, C^{1,\alpha})$ if there exist $C > 1$ and $\{B(x_k, r)\}_{k=1}^N$ that cover $M$, $\{B(x_k, r/4)\}_{k=1}^N$ is pairwise disjoint, and for each $k$ there is an harmonic coordinate chart $u = (u_1, ..., u_m) : B(x_k, 10r) \to \mathbb{R}^m$ with

\[(6)\]  
$C^{-1}\delta_{ij} \leq g_{ij} \leq C\delta_{ij}$

and

\[(7)\]  
$r^{1+\alpha}\|g_{ij}(x)\|_{C^{1,\alpha}} \leq C$

for all $x \in B(x_k, 10r)$, where $g_{ij} = g(\nabla u_i, \nabla u_j)$ and $\nabla$ is the Levi-Civita connection.

Definition 2.13. Let $(M, g)$ be a compact Riemannian manifold. For $x \in M$, the $C^{1,\alpha}$ harmonic radius at $x$, $r_h(x)$, is the largest radius of a geodesic ball centered at $x$ for which there is a constant $C > 1$ and a coordinate chart $v : B(x, r) \to \mathbb{R}^m$ that satisfy

\[(8)\]  
$C^{-1}\delta_{ij} \leq g_{ij} \leq C\delta_{ij}$

and

\[(9)\]  
$r^{1+\alpha}\|g_{ij}(x)\|_{C^{1,\alpha}} \leq C$,  

where $g_{ij} = g(\nabla v_i, \nabla v_j)$.  

Theorem 2.14 (Anderson). The class of compact, connected Riemannian m-manifolds $M$ satisfying

$|Ricci(M)| \leq R, \quad \text{injrad}(M) \geq i \quad \text{and} \quad \text{Diam}(M) \leq D,$

is precompact in the $C^{1,\alpha}$ topology.

The theorem is proven by Anderson \[2\] by showing first that the harmonic radii for manifolds in this class is uniformly bounded below. Then, for given $r, N$ and $C$, he uses the fact that the class of compact Riemannian manifolds with $(r, N, C^{1,\alpha})$ atlases is precompact in the $C^{1,\alpha'}$ topology for all $0 < \alpha' < \alpha$.

3. PRECOMPACTNESS THEOREMS FOR MANIFOLDS WITH SMOOTH BOUNDARY CONDITIONS

3.1. Kodani's Precompactness Theorem. In 1990, Kodani \[12\] proves a theorem in the same line as Theorem 2.3 except that the Riemannian manifolds that Kodani considers have boundary and the diameter bound is replaced by bounds on the second fundamental forms of the boundaries. See examples of neccessity of this replacement in \[12\].

Theorem 3.1 (Kodani). Given a positive integer $m$, and numbers $K, \lambda, v > 0$, the class $M(m, K, \lambda, v)$ of connected m-dimensional Riemannian manifolds, $M$, with boundary that satisfy

\[(10)\]  
$|\text{sec}(M)| \leq K, \quad 0 \leq II \leq \lambda, \quad \text{and} \quad \text{Vol}(M) \geq v,$

where $II$ stands for the second fundamental form of $\partial M$, is precompact in the Lipschitz topology.

The following two definitions are needed to explain the proof of Theorem 3.1 and state Theorem 3.5.
Definition 3.2. Let $M$ be a Riemannian manifold with boundary and $p$ a point in its interior. Define the interior injectivity radius of $p$, $i_{\text{int}}(p)$, to be the supremum over all $r > 0$ such that if $\gamma : [0, r] \to M$ is a normal geodesic with $\gamma(0) = p$, then it is minimizing from 0 to $\min\{t_r, r\}$, where $t_r$ is the first time $\gamma$ intersects $\partial M$. The interior injectivity radius of $M$ is defined as $i_{\text{int}}(M) := \inf\{i_{\text{int}}(p) | p \in M\}$.

Definition 3.3. Let $M$ be a Riemannian manifold with boundary and $p$ a point in $\partial M$. Define the boundary injectivity radius of $p$, $i_{\partial}(p)$, to be the supremum over all $r > 0$ such that there is a minimizing geodesic $\gamma : [0, r] \to M$ with $\gamma(0) = p$ normal to $\partial M$. The boundary injectivity radius of $M$ is defined as $i_{\partial}(M) := \inf_{p \in \partial M} i_{\partial}(p)$.

Theorem 3.4 (Kodani). Let $\mathcal{M}(m, K, \lambda, i)$ be the class of connected $m$-dimensional Riemannian manifolds, $M$, with boundary and
\begin{equation}
|\text{sec}(M)| \leq K, \quad |\mathcal{II}| \leq \lambda, \quad i_{\text{int}}(M) \geq i, \quad \text{and} \quad i_{\partial}(M) \geq i,
\end{equation}
where $\mathcal{II}$ stands for second fundamental form of $\partial M$, $i_{\text{int}}$ is the interior injectivity radius and $i_{\partial}$ the boundary injectivity radius. Then
- for all $\epsilon > 0$, there exists $\delta > 0$ for which if $M, N \in \mathcal{M}(m, K, \lambda, i)$ and $d_{\text{GH}}(M, N) < \delta$ then $d_{\text{GH}}(M, N) < \epsilon$. Thus sequences in $\mathcal{M}(m, K, \lambda, i)$ that converge in Gromov-Hausdorff sense also converge in Lipschitz sense.
- $\mathcal{M}(m, K, \lambda, \nu) \subset \mathcal{M}(m, K, \lambda, i)$
- $\mathcal{M}(m, K, \lambda, \nu)$ is precompact in the Gromov-Hausdorff topology.

Proving $\mathcal{M}(m, K, \lambda, \nu) \subset \mathcal{M}(m, K, \lambda, i)$ involves finding lower bounds for $i_{\partial}$ and $i_{\text{int}}$, which is done by looking at the conjugate radius of $M$ and the length of simple closed geodesics in $M$, and looking at the focal radius of $\partial M$ and the length of geodesics whose endpoints are orthogonal to $\partial M$, respectively.

The fact that $\mathcal{M}(m, K, \lambda, \nu)$ is precompact in the Gromov-Hausdorff topology comes from applying volume comparison theorems. First, he shows that for all $M \in \mathcal{M}(m, K, \lambda, \nu)$, $\text{Vol}(M) \leq V$. Second, he shows that if $M \in \mathcal{M}(m, K, \lambda, i)$, $p \in M$ and $\epsilon > 0$ then $\text{Vol}(B(p, \epsilon)) \leq C$ where $C > 0$ is a constant that only depends on $\epsilon$ and $K$.

Wong proves later [20] that $\mathcal{II}$ does not have to be nonnegative and the volume condition can be replaced by a diameter condition. See Subsection 3.3

### 3.2. Anderson-Katsuda-Kurylev-Lassas-Taylor’s Precompactness Theorem

This subsection reviews the precompactness theorem that appear in [1] which extend Theorem 1.1 [2] of Anderson to manifolds with boundary.

Theorem 3.5 (Anderson-Katsuda-Kurylev-Lassas-Taylor). Let $\mathcal{M}(m, R, i, H_0, D)$ be the class of compact, connected Riemannian $m$-manifolds with boundary $M$ satisfying
\[|\text{Ricci}(M)| \leq R, \quad |\text{Ricci}(\partial M)| \leq R\]
\[\text{injrad}(M) \geq i, \quad i_{\text{int}}(M) \geq i, \quad i_{\partial}(M) \geq 2i\]
\[\text{Diam}(M) \leq D, \quad |H_{\text{Lip}}(\partial M)| \leq H_0\]
where $i_{\partial}(M)$ denotes the boundary injectivity radius of $M$, and $H$ is the mean curvature of $\partial M$. Then $\mathcal{M}(m, R, i, H_0, D)$ is precompact in the $C^{1,\alpha}$ topology.

Theorem 3.5 is proved by showing that a larger class of manifolds is precompact in the $C^{1,\alpha'}$ topology for each $0 < \alpha' < 1$. The second step is to show that $\mathcal{M}(m, R, i, H_0, D)$ is contained in the larger class. This part relies completely on the use of harmonic coordinates and harmonic radii for manifolds with boundary.
3.3. **Wong’s Precompactness Theorem.** This theorem, which appears in [20], is an improvement of Theorem 3.1. Unlike Theorem 3.3, the hypotheses do not assume any type of injectivity radius, do not require any bound on the Ricci curvature of the boundary, and the condition on the mean curvature vector is replaced by a condition on the second fundamental form.

**Theorem 3.6 (Wong).** The class $\mathcal{M}(m, r^-, \lambda^+, D)$ of $n$-dimensional Riemannian manifolds with boundary with

$$\text{(12)} \quad \text{Ricci}(M) \geq r^-, \quad \lambda^- \leq \lambda^+, \quad \text{and} \quad \text{Diam}(M) \leq D,$$

where $\lambda^+$ denotes the second fundamental form of $M$, is precompact in the Gromov-Hausdorff topology.

The proof consists of applying Theorem 2.7 (Gromov Compactness Theorem). To show that the maximum number of disjoint $\varepsilon$-balls, $N(\varepsilon, M)$, for any $M \in \mathcal{M}(m, r^-, \lambda^+, D)$ and $\varepsilon > 0$ is bounded above by some constant $N(\varepsilon)$. Clearly, $N(r, M) \leq N(c r, M)$ for $c < 1$.

Wong shows that there is an isometric extension $\tilde{M}$ of $M$ that is an Alexandrov space. Then he proves that there are constants $c < 1$ and $\tilde{D} > 0$ such that for all $M \in \mathcal{M}(m, K^-, \lambda^+, D)$, $N(c r, M) \leq N(\varepsilon, \tilde{M})$ and $\text{Diam}(\tilde{M}) \leq \tilde{D}$. Then by volume comparison in $\tilde{M}$, $N(\varepsilon, \tilde{M}) \leq \tilde{N}(\varepsilon)$.

3.4. **Knox’s Precompactness Theorem.** The following precompactness theorem appears in [11]. The approach taken to prove it is similar to Theorem 1 of Anderson-Katsuda-Kurylev-Lassas-Taylor. Unlike Theorem 3.5 there are no conditions on any type of injectivity radius, but the Ricci curvature is replaced by sectional curvature and a lower bound on the volume is added. Note that this theorem is not an extension of an existing theorem for manifolds without boundary because it requires a lower bound on $\text{Vol}(\partial M)$.

**Theorem 3.7 (Knox).** Let $\mathcal{M}(m, K, H_0, D, v_0)$ be the class of compact connected Riemannian $m$-manifolds with connected boundary satisfying

$$\left\{ \begin{array}{l}
|\text{sec}(M)| \leq K, \quad |\text{sec}(\partial M)| \leq K \\
0 < 1/H_0 < H < H_0 \\
\text{Diam}(M) \leq D, \quad \text{Vol}(\partial M) \geq v_0,
\end{array} \right.$$ 

where $H$ is the mean curvature. Then $\mathcal{M}(m, K, H_0, D, v_0)$ is precompact in the $C^\alpha$ and weak $L^1$ topologies, for any $0 < \alpha < 1$ and any $p < \infty$.

Knox notes that if $0 < 1/H_0 < H < H_0$ is replaced by a bound on the Lipschitz norm of $H$, then $C^1$ convergence can also be obtained. The proof of Theorem 3.7 follows once it is shown that $\mathcal{M}(m, K, H_0, D, v_0)$ satisfies the hypothesis of Theorem 3.8.

**Theorem 3.8 (Knox).** If $\{(M_i, g_i)\}$ is a sequence of Riemannian manifolds with boundary such that

$$\text{(13)} \quad r_h(g_i) \geq r_0 \quad \text{and} \quad \text{Diam}(M_i) \leq D,$$

where $r_h(g_i)$ is the $L^{1,p}$ harmonic radius. Then there is a subsequence of $\{(M_i, g_i)\}$ that converges in weak $L^{1,p}$ topology to a manifold with boundary whose metric is in $L^{1,p}$.

The harmonic radius, $r_h(g)$, of a Riemannian manifold with boundary, $(M, g)$, depends on the harmonic radius of points in the interior of $M$ and the harmonic radius of points in $\partial M$. Knox deals with these two cases separately. First, by looking at the volume of cylinders whose base is in $\partial M$, he finds that there is a $c > 0$ that only depends on $\mathcal{M}(m, K, H_0, D, v_0)$ such that

$$\text{(14)} \quad r_h(x) \geq c d_M(x, \partial M)$$
for all \( x \in M \setminus \partial M \), where \((M, g) \in \mathcal{M}(m, K, H_0, D, v_0)\). Then he shows that
\[
(15) \quad r_h(x) \geq r
\]
for all \( x \in \partial M \) where \( r \) is a constant that only depends on \( \mathcal{M}(m, K, H_0, D, v_0) \). Thus, by definition of harmonic radius of a manifold with boundary, \( r_h(g) \) has a lower bound that depends only on the class \( \mathcal{M}(m, K, H_0, D, v_0) \).

4. Precpactness Theorems for Manifolds without Boundary Conditions

This section presents a joint work between the author with Sormani appearing in \([13] \). We make no assumptions on the boundary. In fact, we consider open Riemannian manifolds, \((M^m, g)\), endowed with the length metric, \( d_M \). The boundary of \( M \) is defined as \( \partial M := \bar{M} \setminus M \) where \( \bar{M} \) is the metric completion of \( M \). Here the boundary is avoided by considering \( \delta \) inner regions. Precpactness theorems are proven for these inner regions, then their Gromov-Hausdorff limit spaces are glued together into a single metric space (Theorem 4.4). At the end, the Hausdorff dimension of this single metric space is obtained and it is shown that its Hausdorff measure has positive lower density everywhere. Analogous theorems for constant sectional curvature are proven in \([13] \).

4.1. \( \delta \)-Inner Regions and their Limits. Let \((M^m, g)\) be an open Riemannian manifold and \( \delta > 0 \), the \( \delta \) inner region of \( M \) is defined by \( M^\delta := \{ x \in M : d_M(x, \partial M) > \delta \} \). There are two metrics in \( M^\delta \): the restricted metric \( d_M \), and the induced length metric \( d_M^\delta \). If \( M^\delta \) is not path connected, then the distance between points in two different path components is defined to be infinity. In general, \( d_M(x, y) \leq d_M^\delta(x, y) \) for all \( x, y \in M^\delta \).

**Definition 4.1.** Given \( m \in \mathbb{N} \), \( \delta > 0 \), \( D > 0 \), \( V > 0 \), and \( \theta > 0 \), set \( \mathcal{M}_0^{m, \delta, D, V} \) to be the class of \( m \)-dimensional open Riemannian manifolds, \( M \), with boundary, with
\[
(16) \quad \text{Ricci}(M) \geq 0, \quad \text{Vol}(M) \leq V, \quad \text{and} \quad \text{Diam}(M^\delta, d_M^\delta) \leq D,
\]
that are noncollapsing at a point:
\[
(17) \quad \exists q \in M^\delta \text{ such that } \text{Vol}(B_q(\delta)) \geq \theta \delta^m.
\]

**Theorem 4.2** (P–Sormani). If \((M_j, g_j) \subset \mathcal{M}_0^{m, \delta, D, V} \), then there is a subsequence \( \{M_k\} \) and a compact metric space \((Y^\delta, d)\) such that \((M_k^\delta, d_{M_k^\delta}) \overset{GH}{\to} (Y^\delta, d_Y)\).

Note that even though \( D \) bounds the diameter of the inner regions with respect to the induced length metric, the convergence is guaranteed by adding the inner regions with the restricted metric.

Replacing \( \delta > 0 \) in the above theorem by a decreasing sequence, \( \delta_i \to 0 \), and adding bounds on the diameter of \( \delta_i \)-inner regions the following can be proved.

**Theorem 4.3** (P–Sormani). Given \( m \in \mathbb{N} \), a decreasing sequence, \( \delta_i \to 0 \), \( D_i > 0 \), \( i = 0, 1, 2, ..., V > 0 \), \( \theta > 0 \). Suppose that \((M_j, g_j) \subset \mathcal{M}_0^{m, \delta_i, D_i, V} \) and
\[
(18) \quad \sup \left\{ \text{Diam} \left( M_j^\delta, d_{M_j^\delta} \right) : j \in \mathbb{N} \right\} < D_i \quad \forall i \in \mathbb{N}.
\]
Then there is a subsequence \( \{M_{j_i}\} \), and there are compact metric spaces \((Y^{\delta_i}, d_{Y^{\delta_i}})\) such that \((M_{j_i}^\delta, d_{M_{j_i}^\delta}) \overset{GH}{\to} (Y^{\delta_i}, d_{Y^{\delta_i}}) \) for all \( i \).
4.2. Constructing a Glued Limit Space. By constructing isometric embeddings between the limit spaces, \( \varphi_{\delta_{i+1}, \delta_i} : Y^{\delta_i} \to Y^{\delta_{i+1}} \), it is possible to define a metric space into which all the limit spaces isometrically embed. Define \( \varphi_{\delta_{i,j}, \delta_i} = \varphi_{\delta_{i,j}, \delta_{i,j-1}} \circ \cdots \circ \varphi_{\delta_{i+1}, \delta_i} \). Define
\[
Y := Y((\delta_i), \{\varphi_{\delta_{i+1}, \delta_i}\}) = Y^{\delta_0} \cup \left( \bigcup_{i=1}^{\infty} \left( Y^{\delta_i} \setminus \varphi_{\delta_{i+1}, \delta_i}(Y^{\delta_i}) \right) \right)
\]
and
\[
d_Y(x, y) := \begin{cases} 
   d_{Y_0}(x, y) & \text{if } x, y \in Y^{\delta_0}, \\
   d_{Y^{\delta_i}}(x, y) & \text{if } x, y \in Y^{\delta_i} \setminus \varphi_{\delta_{i+1}, \delta_i}(Y^{\delta_i}), \\
   d_{Y^{\delta_{i+1}}}(x, \varphi_{\delta_{i+1}, \delta_i}(y)) & \text{if } x \in Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1}, \delta_i}(Y^{\delta_i}), \\
   d_{Y^{\delta_{j+1}}}(x, \varphi_{\delta_{j+1}, \delta_i}(y)) & \text{if } x \in Y^{\delta_{j+1}} \setminus \varphi_{\delta_{j+1}, \delta_i}(Y^{\delta_{j+1}}) \text{ for some } i, j, \in \mathbb{N}, \text{ and } y \in Y^{\delta_j},
\end{cases}
\]

Theorem 4.4 (P–Sormani). Under the hypothesis of Theorem 3.3 There exists a metric space \((Y, d_Y)\) such that for all \( \delta \in (0, \delta_0] \), there is a subsequence of \((\bar{M}^{\beta_i}, d_{M^{\beta_i}})\) that Gromov-Hausdorff converges to some compact metric space \((Y^{\delta}, d_Y)\). For any such \(Y^{\delta}\), there exists an isometric embedding
\[
F_{\delta} = F_{\delta_{i+1}, \delta_i} : Y^{\delta} \to Y.
\]
If \( \delta = \delta_i \) for some \( i \), then
\[
F_{\delta_i}(Y^{\delta_i}) \subset F_{\delta_{i+1}}(Y^{\delta_{i+1}}).
\]
If \( \beta_j \) is any sequence decreasing to 0, then
\[
Y = \bigcup_{i=1}^{\infty} F_{\beta_i}(Y^{\beta_i}).
\]

This glued limit space may exist even when \((M_j, d_j)\) has no Gromov-Hausdorff limit.

Hausdorff measures and topologies of the Gromov-Hausdorff limit spaces of noncollapsing sequences of manifolds have been studied by Cheeger, Colding, Naber, Wei and Sormani (c.f. [3], [6], [17] and [16]). Applying some of these results, we are able to prove the following.

Theorem 4.5 (P–Sormani). Suppose that \( Y \) is a glued limit constructed as in Theorem 4.4 Then \( Y \) has Hausdorff dimension \( m \), \( \mathcal{H}^m(Y) \leq V_0 \) and its Hausdorff measure has positive lower density everywhere.

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