COMPOSITE BERNSTEIN CUBATURE

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ABSTRACT. We consider a sequence of composite bivariate Bernstein operators and the cubature formula associated with them. The upper-bounds for the remainder term of cubature formula are described in terms of moduli of continuity of order two. Also we include some results showing how non-multiplicative the integration functional is.

1. Introduction

We reconsider (composite) bivariate Bernstein approximation and the corresponding cubature formulae. This is motivated by a recent series of articles by Barbosu et al. (see [2]-[5]). However, some of these papers contain rather misleading statements and claims which can hardly be verified. The present is written with the intention to clean up some of the bugs, to optimize and generalize certain estimates, and thus to further describe the situation at hand.

Our present contribution is a continuation of [8]. Historically the origin of the method discussed seems to be in the article [13] by D.D. Stancu and A. Vernescu.

2. A general result

We first introduce some notation which will be needed to formulate the general result.

Definition 2.1. Let $I$ and $J$ be compact intervals of the real axis and let $L : C(I) \to C(I)$ and $M : C(J) \to C(J)$ be discretely defined operators, i.e.,

$$L(g; x) = \sum_{e \in E} g(x_e)A_e(x), \ g \in C(I), \ x \in I,$$

where $E$ is a finite index set, the $x_e \in I$ are mutually distinct and $A_e \in C(I)$, $e \in E$.

Analogously,

$$M(h; y) = \sum_{f \in F} h(y_f)B_f(y), \ h \in C(J), \ y \in J.$$ 

If $L$ is of the form above, then its parametric extension to $C(I \times J)$ is given by

$$xL(F; x, y) = L(F_y; x) = \sum_{e \in E} F_y(x_e)A_e(x) = \sum_{e \in E} F(x_e, y)A_e(x).$$

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2010 Mathematics Subject Classification. Primary 41A36; Secondary 41A15, 65D30.

Key words and phrases. Composite Bernstein operators, composite quadrature formulas, modulus of continuity.
Here $F_y, y \in J$, denote the partial functions of $F$ given by $F_y(x) = F(x, y), x \in I$. Similarly,$$
yM(F; x, y) = \sum_{f \in F} F(x, y_f)B_f(y).
$$The tensor product of $L$ and $M$ (or $M$ and $L$) is given by$$(xL \circ y M)(F; x, y) = \sum_{e \in E} \sum_{f \in F} F(x_e, y_f)A_e(x)B_f(y).$$The theorem below is given in terms of so-called partial moduli of smoothness of order $r$, given for the compact intervals $I, J \subset \mathbb{R}$, for $F \in C(I \times J)$, $r \in \mathbb{N}_0$ and $\delta \in \mathbb{R}_+$ by$$\omega_r(F; \delta, 0):=\sup \left\{ \left| \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} F(x+\nu h, y) \right| : (x, y), (x+rh, y) \in I \times J, |h| \leq \delta \right\}$$and symmetrically by$$\omega_r(F; 0, \delta):=\sup \left\{ \left| \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} F(x+\nu h, y) \right| : (x, y), (x, y+rh) \in I \times J, |h| \leq \delta \right\}.$$The total modulus of smoothness of order $r$ is defined by$$\omega_r(F; \delta_1, \delta_2):=\sup \left\{ \left| \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} F(x+\nu h_1, y+\nu h_2) \right| : (x, y), (x+rh_1, y+rh_2) \in I \times J, |h_1| \leq \delta_1, |h_2| \leq \delta_2 \right\}.$$We now formulate and prove a simplified form of Theorem 37 in [6].

**Theorem 2.1.** Let $L$ and $M$ be discretely defined operators as given above such that$$|(g-Lg)(x)| \leq \sum_{\rho=0}^{r} \Gamma_{\rho, L}(x)\omega_{\rho}(g; \Lambda_{\rho, L}(x)), g \in C(I), x \in I,$$and$$|(h-Mh)(y)| \leq \sum_{\sigma=0}^{s} \Gamma_{\sigma, M}(y)\omega_{\sigma}(h; \Lambda_{\sigma, M}(y)), h \in C(J), y \in J.$$Here $\omega_{\rho}, \rho = 0, \ldots, r$, denote the moduli of order $\rho$, and $\Gamma$ and $\Lambda$ are bounded functions. Analogously for $M$. Then for $(x, y) \in I \times J$ and $F \in C(I \times J)$ the following hold:

$$|[F - (xL \circ y M)F](x, y)| \leq \sum_{\rho=0}^{r} \Gamma_{\rho, L}(x)\omega_{\rho}(F; \Lambda_{\rho, L}(x), 0) + \|L\| \sum_{\sigma=0}^{s} \Gamma_{\sigma, M}(y)\omega_{\sigma}(F; 0, \Lambda_{\sigma, M}(y)),$$

where $\|L\|$ denotes the operator norm of $L$, which is finite due to the form of $L$. 

Proof. We have
\[ |[F - (x L \circ y M) F] (x,y)| = |[(Id - x L) + x L \circ (Id - y M)] (F; x,y)| \]
\[ \leq |(Id - x L)(F; x,y)| + |x L \circ (Id - y M)(F; x,y)| \]
\[ = E_1(x,y) + E_2(x,y). \]

Now, for \( x \in I \),
\[ E_1(x,y) = |(Id - L)(F_y; x)| \leq \sum_{\rho=0}^r \Gamma_{\rho,L}(x) \cdot \omega_\rho(F_y; \Lambda_{\rho,L}(x)) \]
\[ \leq \sum_{\rho=0}^r \Gamma_{\rho,L}(x) \cdot \omega_\rho(F; \Lambda_{\rho,L}(x), 0). \]

Furthermore, with \( G := (Id - y M) F \), we have
\[ E_2(x,y) = |x L(G; x,y)| = |L(G_y; x)| \leq \|L(G_y)\|_{\infty, x \in I}. \]

Here again \( G_y \in C(I) \) for all \( y \in J \). By our assumption on \( L \) we have for any \( g \in C(I) \) that
\[ \|Lg\|_\infty \leq \left( 1 + \sum_{\rho=0}^r 2^\rho \cdot \|\Gamma_{\rho,L}\|_\infty \right) \cdot \|g\|_\infty. \]

Hence \( \|L\| < \infty \).

In the situation at hand we have
\[ \|G_y\|_\infty = \|[Id - y M] F_y(\cdot)\|_\infty = \|Id - y M F(\cdot, y)\|_\infty = \|[Id - y M] F(x,y)\|_{\infty, x \in I} \]
\[ \leq \sum_{\sigma=0}^s \Gamma_{\sigma,M}(y) \cdot \omega_\sigma(F_x; \Lambda_{\sigma,M}(y)) \leq \sum_{\sigma=0}^s \Gamma_{\sigma,M}(y) \cdot \sup_{x \in I} \omega_\sigma(F_x; \Lambda_{\sigma,M}(y)) \]
\[ = \sum_{\sigma=0}^s \Gamma_{\sigma,M}(y) \cdot \omega_\sigma(F; 0, \Lambda_{\sigma,M}(y)). \]

Hence
\[ E_1(x,y) + E_2(x,y) \leq \sum_{\rho=0}^r \Gamma_{\rho,L}(x) \cdot \omega_\rho(F; \Lambda_{\rho,L}(x), 0) \]
\[ + \|L\| \cdot \sum_{\sigma=0}^s \Gamma_{\sigma,M}(y) \cdot \omega_\sigma(F; 0, \Lambda_{\sigma,M}(y)). \]

\[ \square \]

3. Application to bivariate Bernstein operators

Example 3.1. If we take \( L = B_{n_1} \) and \( M = B_{n_2} \) with two classical Bernstein operators mapping \( C[0,1] \) into \( C[0,1] \), then for \( F \in C([0,1] \times [0,1]) \) and \( (x,y) \in [0,1] \times [0,1] \)
\[ (xB_{n_1} \circ y B_{n_2})(F; x,y) = \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} F \left( \frac{i_1}{n_1}, \frac{i_2}{n_2} \right) p_{n_1,i_1}(x)p_{n_2,i_2}(y). \]
where \( p_{n,i}(x) = \binom{i}{n} x^i (1-x)^{n-i}, x \in [0,1] \), and

\[
||F - (x B_{n_1} \circ y B_{n_2}) F] (x,y)|| \leq 3 \left[ \omega_2 \left( F; \frac{x(1-x)}{n_1}, 0 \right) + \omega_2 \left( F; 0, \frac{y(1-y)}{n_2} \right) \right]
\]

\[
\leq 3 \left[ \|F^{(2,0)}\|_{\infty} \frac{x(1-x)}{n_1} + \|F^{(0,2)}\|_{\infty} \frac{y(1-y)}{n_2} \right], \quad F \in C^{2,2}([0,1] \times [0,1]).
\]

**Proof.** We apply Theorem 2.1 with \( r = s = 2 \), \( \Gamma_{0,B_n} = \Gamma_{1,B_n} = 0 \), \( \Lambda_{2,B_n} = \frac{3}{2} \), \( \Lambda_{2,B_n}(z) = \sqrt{\frac{z(1-z)}{n}} \), for \( n \in \{n_1, n_2\} \). The latter two choices are possible due to a well-known result of Păltănea (see [11]) showing that for the univariate Bernstein operators one has

\[
|f(x) - B_n(f;x)| \leq \frac{3}{2} \omega_2 \left( f; \frac{x(1-x)}{n} \right).
\]

\( \square \)

**Remark 3.1.** From the last inequality we get

\[
|f(x) - B_n(f;x)| \leq \frac{3}{2} \|f''\|_{\infty} \frac{x(1-x)}{n}, \quad f \in C^2[0,1].
\]

This is worse than the known inequality

\[
|f(x) - B_n(f;x)| \leq \frac{1}{2} \|f''\|_{\infty} \frac{x(1-x)}{n}.
\]

Our inequality was obtained from the more general statement in terms of \( \omega_2 \) and well-known properties of the modulus.

However, we can use instead Theorem 1 in [7] (take \( p = q = 2 \), \( p' = q' = 0 \), \( r = s = 0 \), \( \Gamma_{0,0,B_n} = \frac{1}{2} \cdot \frac{x(1-x)}{n_1} \) and \( \Gamma_{0,0,B_n} = \frac{1}{2} \cdot \frac{y(1-y)}{n_2} \)) to arrive at

\[
||F - (x B_{n_1} \circ y B_{n_2}) F] (x,y)|| \leq \frac{1}{2} \frac{x(1-x)}{n_1} \|F^{(2,0)}\|_{\infty} + \frac{1}{2} \frac{y(1-y)}{n_2} \|F^{(0,2)}\|_{\infty}
\]

\[
+ \frac{1}{4} \frac{x(1-x)y(1-y)}{n_1n_2} \|F^{(2,2)}\|_{\infty}
\]

\[
\leq \frac{1}{8n_1} \|F^{(2,0)}\|_{\infty} + \frac{1}{8n_2} \|F^{(0,2)}\|_{\infty} + \frac{1}{64n_1n_2} \|F^{(2,2)}\|_{\infty}.
\]

An estimate of this kind can be found in Theorem 2.3 of [2].

Such three-term expressions typically appear if one writes (I denoting the identity)

\[
I - A \circ B = I - A + I - B - (I - A) \circ (I - B) = (I - A) \oplus (I - B),
\]

that is, if one uses the fact that the remainder of the tensor product is the Boolean sum of the errors of the parametric extension. The approach behind the above Theorem 2.1 invokes the decomposition

\[
I - A \circ B = I - A + A \circ (I - B),
\]
and therefore leads to the two-term bound.

4. The Bernstein type cubature formula revisited

In this section we give a new upper bound for the approximation error of cubature formula associated with the bivariate Bernstein operators. The bounds are described in terms of moduli of continuity of order two. The consideration of this cubature formula is motivated by Bărboșu and Pop’s result [3]. It deems necessary to also correct some of the wrong statements made there, in particular those with respect to Boolean sums.

Integrating the bivariate Bernstein polynomials for \( F \in C([0,1] \times [0,1]) \) one arrives at the following cubature formula

\[
\int_0^1 \int_0^1 F(x,y)dxdy = \frac{1}{(n_1 + 1)(n_2 + 1)} \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} F\left(\frac{i_1}{n_1}, \frac{i_2}{n_2}\right) + R_{n_1, n_2}[F],
\]

(4.1)

where the remainder is bounded as follows:

\[
|R_{n_1, n_2}[F]| \leq \frac{1}{12n_1} \|F^{(2,0)}\| + \frac{1}{12n_2} \|F^{(0,2)}\| + \frac{1}{144n_1n_2} \|F^{2,2}\|\infty,
\]

if \( F \in C^{2,2}([0,1] \times [0,1]) \).

This follows from the three-term upper bound of Remark 3.1. See [3] where the same integration error bound can be found.

The two-term bound from Example 3.1 leads to the following

**Theorem 4.1.** For the remainder term of the cubature formula (4.1), \( n_1, n_2 \in \mathbb{N} \) and \( F \in C([0,1] \times [0,1]) \) there holds

\[
|R_{n_1, n_2}[F]| \leq \frac{3}{2} \left[ \int_0^1 \omega_2 \left( F; \sqrt{\frac{x(1-x)}{n_1}}, 0 \right) dx + \int_0^1 \omega_2 \left( F; 0, \sqrt{\frac{y(1-y)}{n_2}} \right) dy \right].
\]

Moreover, if \( F \in C^{2,2}([0,1] \times [0,1]) \), then the above implies

\[
|R_{n_1, n_2}[F]| \leq \frac{1}{4} \left( \frac{1}{n_1} \|F^{(2,0)}\|\infty + \frac{1}{n_2} \|F^{(0,2)}\|\infty \right).
\]

**Proof.** All that needs to be observed is that a function of type \([0,1/2] \ni z \rightarrow \omega_2(F;z,0)\) (with \( F \) fixed and continuous) is continuous, thus integrable. The mixed moduli of smoothness of order \((k,l)\), with \( k, l \in \mathbb{N}_0 \), given for \( \delta_1, \delta_2 \geq 0 \) by

\[
\omega_{k,l}(F; \delta_1, \delta_2) := \sup \left\{ \left| \sum_{\nu=0}^{k} \sum_{\mu=0}^{l} (-1)^{\nu+\mu} \binom{k}{\nu} \binom{l}{\mu} \int F(x + \nu \cdot h_1, y + \mu \cdot h_2) dx \right| : (x, y), (x + kh_1, y + lh_2) \in [0,1]^2, |h_i| \leq \delta_i, i = 1, 2 \right\},
\]

is a positive, continuous and non-decreasing function with respect to both variables (see [9], [14]). For continuous \( F \) these moduli are continuous in \( \delta_1 \) and \( \delta_2 \) and satisfy

\[
\omega_k(F; \delta_1, 0) = \omega_{k,0}(F; \delta_1, \delta_2) \text{ and } \omega_k(F; 0, \delta_1) = \omega_{0,k}(F; \delta_1, \delta_2).
\]

The latter is only relevant to us for \( k = 2 \). □
5. The composite bivariate Bernstein operators

In this section we construct the bivariate composite Bernstein operators and the order of convergence is considered involving the second modulus of continuity. Also, some inequalities of Tchebycheff-Grüss type will be proven. These results are obtained using some general inequalities published in [1], [12]. In order to give the main results of this section, we recall the following facts:

1. For \( a, b \in \mathbb{R}, \ a < b \), and \( f \in \mathbb{R}^{[a,b]} \) the Bernstein polynomial of degree \( n \in \mathbb{N} \) associated to \( f \) is given for \( x \in [a,b] \), by
   \[
   B_n^{[a,b]}(f; x) = \frac{1}{(b-a)^n} \sum_{i=0}^{n} \binom{n}{i} (x-a)^i (b-x)^{n-i} f \left( a + \frac{b-a}{n} \right).
   \]

2. For \( g \in C^2[a,b] \) one has
   \[
   g(x) - B_n^{[a,b]}(g; x) = -\frac{(x-a)(b-x)}{2n} g''(\xi_x), \xi_x \in (a,b).
   \]

If we divide \([0,1]\) into subintervals \([\frac{k-1}{m}, \frac{k}{m}]\), \( k = 1, \ldots, m \in \mathbb{N} \), then on \( \left[ \frac{k-1}{m}, \frac{k}{m} \right] \) we consider
   \[
   B_{n,k}(f; x) = B_n^{[\frac{k-1}{m}, \frac{k}{m}]}(f; x) = m^n \sum_{i=0}^{n} \binom{n}{i} \left( x - \frac{k-1}{m} \right)^i \left( \frac{k}{m} - x \right)^{n-i} f \left( \frac{kn-n+i}{nm} \right).
   \]

Now we compose the \( B_{n,k} \) to obtain the positive linear operator \( \overline{B}_{n,m} : \mathbb{R}^{[0,1]} \to C[0,1] \),
   \[
   \overline{B}_{n,m}(f; x) = B_{n,k}(f; x), \text{ if } x \in \left[ \frac{k-1}{m}, \frac{k}{m} \right], \ 1 \leq k \leq m.
   \]

From now on (subscripted) symbols \( n... \) will refer to a polynomial degree. (Subscripted) numbers \( m... \) will be related to grids. Each function \( \overline{B}_{n,m}(f) \) is a Schoenberg spline of degree \( n \) with respect to the knot sequence given as follows:
   \[
   0 = \frac{0}{m} \quad (n+1) \text{- fold} \quad 1 = \frac{m}{m} \quad (n+1) \text{- fold} \quad 1 = \frac{m}{m} \quad n \text{- fold} \quad \vdots \quad 1 = \frac{m}{m} \quad n \text{- fold} \quad 1 = \frac{m}{m} \quad (n+1) \text{- fold}
   \]

We renounce to give a precise numbering of the knots since this will not be needed below. Thus \( \overline{B}_{n,m} \) reproduces linear functions, interpolates at \( \frac{k}{m} \), \( 0 \leq k \leq m \) and has operator norm \( ||\overline{B}_{n,m}|| = 1 \).
For \( n_1, n_2, m_1, m_2 \in \mathbb{N} \) we now consider the parametric extension \( x\overline{B}_{n_1, m_1} \) and \( y\overline{B}_{n_2, m_2} \) and their product \( x\overline{B}_{n_1, m_1} \circ y\overline{B}_{n_2, m_2} \). For brevity the latter will be denote by \( \overline{B} \).

For \( (x, y) \in \left[ \frac{k-1}{m_1}, \frac{k}{m_1} \right] \times \left[ \frac{l-1}{m_2}, \frac{l}{m_2} \right] \), it follows

\[
\overline{B}(f; x, y) = m_1^{n_1} \cdot m_2^{n_2} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \binom{n_1}{i} \binom{n_2}{j} \left( x - \frac{k-1}{m_1} \right)^i \left( \frac{k}{m_1} - x \right)^{n_1-i} \left( y - \frac{l-1}{m_2} \right)^j \left( \frac{l}{m_2} - y \right)^{n_2-j} f \left( \frac{k-1}{m_1} + \frac{i}{m_1 n_1}, \frac{l-1}{m_2} + \frac{j}{n_2 m_2} \right)
\]

and

\[
|f(x, y) - \overline{B}(f; x, y)| = \frac{(x - \frac{k-1}{m}) \left( \frac{k}{m_1} - x \right)}{2n_1} \|f^{(2,0)}\|_{\infty} + \frac{(y - \frac{l-1}{m_2}) \left( \frac{l}{m_2} - y \right)}{2n_2} \|f^{(0,2)}\|_{\infty}
\]

\[
+ \frac{(x - \frac{k-1}{m}) \left( \frac{k}{m_1} - x \right) (y - \frac{l-1}{m_2}) \left( \frac{l}{m_2} - y \right)}{4n_1 n_2} \|f^{(2,2)}\|_{\infty}
\]

where \( f \in C^2([0,1] \times [0,1]). \)

Using Theorem 1 again we get

**Theorem 5.1.** For \( f \in C([0,1] \times [0,1]), n_1, n_2, m_1, m_2 \in \mathbb{N} \) and \( (x, y) \in [0,1] \times [0,1] \) there holds

\[
|f(x, y) - \overline{B}(f; x, y)| \leq 3 \left( \int_{[0,1]} \omega_2 \left( f; \sqrt{\frac{(x - \frac{k-1}{m_1}) \left( \frac{k}{m_1} - x \right)}{n_1}}, 0 \right) \right) + \omega_2 \left( f; 0, \sqrt{\frac{(y - \frac{l-1}{m_2}) \left( \frac{l}{m_2} - y \right)}{n_2}} \right)
\]

if \( (x, y) \in \left[ \frac{k-1}{m_1}, \frac{k}{m_1} \right] \times \left[ \frac{l-1}{m_2}, \frac{l}{m_2} \right], 1 \leq k \leq m_1, 1 \leq l \leq m_2. \)

**Proof.** For the univariate case we have

\[
|\overline{B}_{n_1, m_1}(f; x) - f(x)| \leq 3 \left( \int_{[0,1]} \omega_2 \left( f; \sqrt{\frac{(x - \frac{k-1}{m_1}) \left( \frac{k}{m_1} - x \right)}{n_1}} \right) \right),
\]

for \( x \in \left[ \frac{k-1}{m_1}, \frac{k}{m_1} \right], 1 \leq k \leq m_1. \) Here \( \omega_2 \) is the second order modulus over \([0,1].\)

An analogous inequality holds for \( \overline{B}_{n_2, m_2} \).

The theorem mentioned implies, with \( r = s = 2 \), the inequality claimed. \( \square \)
Remark 5.1. As mentioned earlier, for \( g \in C^2[a, b] \) one has
\[
|g(x) - B^g_{n}^{[a, b]}(g; x)| = \left| -\frac{(x-a)(b-x)}{2n} g''(\xi_x) \right| \leq \frac{(b-a)^2}{8n} \|g''\|_{[a, b], \infty}.
\]
For \([a, b] = \left[\frac{k-1}{m}, \frac{k}{m}\right]\), the last expression equals \( \frac{1}{8m^2n} \|g''\|_{\left[\frac{k-1}{m}, \frac{k}{m}\right], \infty} \).

If \( f \in C^{2, 2}([0, 1] \times [0, 1]) \) and \((x, y) \in [0, 1] \times [0, 1]\), using Theorem 1 in [7], this leads to
\[
|f(x, y) - B(f; x, y)| \leq \frac{1}{8m^2n_1} \|f^{(2, 0)}\|_{\infty} + \frac{1}{8m^2n_2} \|f^{(0, 2)}\|_{\infty} + \frac{1}{64m^2n_1m^2n_2} \|f^{(2, 2)}\|_{\infty}.
\]

For \( m_1 = m_2 = 1 \) this is exactly the inequality in Remark 3.1.

6. A Chebyshev-Grüss inequality

In what follows we present an inequality for the bivariate composite Bernstein operators, expressed in term of least concave majorant of continuity. Let \( C(X) \) be the Banach lattice of real valued continuous functions defined on the compact metric space \((X, d)\).

Definition 6.1. Let \( f \in C(X) \). If, for \( t \in [0, \infty) \), the quantity
\[
\omega_d(f; t) := \sup \{|f(x) - f(y)|, d(x, y) \leq t\}
\]
is the usual modulus of continuity, then its least concave majorant is given by
\[
\tilde{\omega}_d(f, t) = \begin{cases} 
\sup_{0 \leq x < t \leq y \leq d(X)} \frac{(t-x)\omega_d(f, y) + (y-t)\omega_d(f, x)}{y-x}, & 0 \leq t \leq d(X), \\
\omega_d(f, d(X)), & t > d(X),
\end{cases}
\]
and \( d(X) < \infty \) is the diameter of the compact space \( X \).

Denote
\[
\text{Lip}_r = \left\{ g \in C(X) \mid \|g\|_{\text{Lip}_r} := \sup_{d(x, y) > 0} \frac{|g(x) - g(y)|}{d^r(x, y)} < \infty \right\}, \quad 0 < r \leq 1.
\]
\( \text{Lip}_r \) is a dense subspace of \( C(X) \) equipped with the supremum norm \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_{\text{Lip}_r} \) is a seminorm on \( \text{Lip}_r \).

The \( K \)-functional with respect to \((\text{Lip}_r, \| \cdot \|_{\text{Lip}_r})\) is given by
\[
K(t, f; C(X), \text{Lip}_r) := \inf_{g \in \text{Lip}_r} \left\{ \|f - g\|_{\infty} + t\|g\|_{\text{Lip}_r} \right\}, \quad \text{for } f \in C(X) \text{ and } t \geq 0.
\]

Lemma 6.1. [10] Every continuous function \( f \) on \( X \) satisfies
\[
K\left(\frac{t}{2}, f; C(X), \text{Lip}_1\right) = \frac{1}{2} \tilde{\omega}_d(f, t), \quad 0 \leq t \leq d(X).
\]

Let \( H : C(X^2) \to C(X^2) \) be a positive linear operator reproducing constant function and define
\[
T(f, g; x, y) = H(fg; x, y) - H(f; x, y) \cdot H(g; x, y).
\]
In order to give an inequality of Chebyshev-Grüss type we recall a general result given by M. Rusu in [12].
Theorem 6.1. [12] If \( f, g \in C(X^2) \) and \( x, y \in X \) fixed, then the inequality
\[
|T(f, g; x, y) - T(f; x, y) \cdot T(g; x, y)| \leq \frac{1}{4} \tilde{\omega}_d \left( f; 4\sqrt{H(d^2(\cdot, (x, y))); x, y} \right) \cdot \tilde{\omega}_d \left( g; 4\sqrt{H(d^2(\cdot, (x, y))); x, y} \right)
\]
holds, where \( H(d^2(\cdot, (x, y)); x, y) \) is the second moment of the bivariate operator \( H \). We consider here the Euclidian metric \( d \).

Proposition 6.1. For \( f, g \in C(X^2) \) and \( x, y \in X \) fixed, the following Grüss type inequality holds
\[
|\overline{B}(fg; x, y) - \overline{B}(f; x, y) \cdot \overline{B}(g; x, y)| \leq \frac{1}{4} \tilde{\omega}_d \left( f; 2\sqrt{(1/n_1 m_1^2 + 1/n_2 m_2^2)} \right) \cdot \tilde{\omega}_d \left( g; 2\sqrt{(1/n_1 m_1^2 + 1/n_2 m_2^2)} \right)
\]
where \( \Psi(x, y) = \frac{(x - k/m_1) (k/m_1 - x)}{n_1} + \frac{(y - l/m_2) (l/m_2 - y)}{n_2} \) and \( (x, y) \in \left[ \frac{k-1}{m_1}, \frac{k}{m_1} \right] \times \left[ \frac{l-1}{m_2}, \frac{l}{m_2} \right] \).

7. A cubature formula based on \( \overline{B} \)

In this section some upper-bounds of the error of cubature formula associated with the bivariate Bernstein operators are given. In [4] D. Bărbosu, D. Miclăuş introduced the following cubature formula:
\[
\int_0^1 \int_0^1 f(x, y)dxdy = \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} \int_{k/m_1}^{(k+1)/m_1} \int_{l/m_2}^{(l+1)/m_2} f(x, y)dxdy
\]
\[
\approx \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} \int_{k/m_1}^{(k+1)/m_1} \int_{l/m_2}^{(l+1)/m_2} \overline{B}(f; x, y)dxdy = \int_0^1 \int_0^1 \overline{B}(f; x, y)dxdy := \mathcal{L}(f).
\]
It follows
\[
\int_{k/m_1}^{(k+1)/m_1} \int_{l/m_2}^{(l+1)/m_2} \overline{B}(f; x, y)dxdy
\]
\[
= m_1^{n_1} m_2^{n_2} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \binom{n_1}{i} \binom{n_2}{j} \int_{k/m_1}^{(k+1)/m_1} \left( x - \frac{k-1}{m_1} \right)^i \left( \frac{k}{m_1} - x \right)^{n_1-i} dx
\]
\[
\cdot \int_{l/m_2}^{(l+1)/m_2} \left( y - \frac{l-1}{m_2} \right)^j \left( \frac{l}{m_2} - y \right)^{n_2-j} dy f \left( \frac{k-1}{m_1} + \frac{i}{m_1 n_1}, \frac{l-1}{m_2} + \frac{j}{n_2 m_2} \right)
\]
\[
= \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} A_{n_1, n_2, m_1, m_2} f \left( \frac{k-1}{m_1} + \frac{i}{m_1 n_1}, \frac{l-1}{m_2} + \frac{j}{n_2 m_2} \right),
\]
where \( A_{n_1, n_2, m_1, m_2} = \frac{1}{m_1 m_2 (n_1 + 1)(n_2 + 1)} \).
Theorem 7.1. For $f \in C^{2,2}([0, 1] \times [0, 1])$ it follows
\[
\left| \int_0^1 \int_0^1 f(x, y) \, dx \, dy - I(f) \right| \leq \frac{1}{12n_1m_1^3} \|f^{(2,0)}\|_\infty + \frac{1}{12n_2m_2^3} \|f^{(0,2)}\|_\infty + \frac{1}{144n_1n_2m_1^3m_2^3} \|f^{(2,2)}\|_\infty.
\]

Proof. We have
\[
\left| \int_0^1 \int_0^1 f(x, y) \, dx \, dy - I(f) \right| = \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} \int_{k/m_1}^{(k+1)/m_1} \int_{l/m_2}^{(l+1)/m_2} f(x, y) \, dx \, dy - \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} \int_{k/m_1}^{(k+1)/m_1} \int_{l/m_2}^{(l+1)/m_2} B(f; x, y) \, dx \, dy
\]
\[
\leq \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} \int_{k/m_1}^{(k+1)/m_1} \int_{l/m_2}^{(l+1)/m_2} \left| f(x, y) - B(f; x, y) \right| \, dx \, dy
\]
\[
= \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} \int_{k/m_1}^{(k+1)/m_1} \int_{l/m_2}^{(l+1)/m_2} \left| \frac{x - k/m_1}{2n_1} \left( \frac{k/m_1 - x}{m_1} \right) \|f^{(2,0)}\|_\infty + \frac{y - l/m_2}{2n_2} \left( \frac{l/m_2 - y}{m_2} \right) \|f^{(0,2)}\|_\infty \right| \, dx \, dy
\]
\[
= \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} \left[ \frac{1}{12n_1m_1^3m_2} \|f^{(2,0)}\|_\infty + \frac{1}{12n_2m_2^3m_1} \|f^{(0,2)}\|_\infty + \frac{1}{144n_1n_2m_1^3m_2^3} \|f^{(2,2)}\|_\infty \right]
\]
\[
= \frac{1}{12n_1m_1^2} \|f^{(2,0)}\|_\infty + \frac{1}{12n_2m_2^2} \|f^{(0,2)}\|_\infty + \frac{1}{144n_1n_2m_1^3m_2^3} \|f^{(2,2)}\|_\infty.
\]

One further estimate is given in

Theorem 7.2. For $f \in C^{2,2}([0, 1] \times [0, 1])$ it follows
\[
\left| \int_0^1 \int_0^1 f(x, y) \, dx \, dy - I(f) \right| \leq \frac{1}{4} \left\{ \frac{1}{m_2^3n_1} \|f^{(2,0)}\|_\infty + \frac{1}{m_2^3n_2} \|f^{(0,2)}\|_\infty \right\}.
\]

Proof. Integrating the error given in Theorem 5.1 leads to
\[
\left| \int_0^1 \int_0^1 f(x, y) \, dx \, dy - I(f) \right|
\]
\[
\leq \frac{3}{2} \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} \left\{ \frac{1}{m_2^2} \int_{k/m_1}^{(k+1)/m_1} \omega_2 \left( f; 0, \sqrt{\frac{x - k/m_1}{m_1}, 0} \right) \right\} dx
\]
\[
+ \frac{1}{m_1} \int_{l/m_2}^{(l+1)/m_2} \omega_2 \left( f; 0, \sqrt{\frac{y - l/m_2}{m_2}, 0} \right) dy \right\}.
\]
Since $f \in C^{2,2}([0,1] \times [0,1])$ leads to
\[
\left| \int_0^1 \int_0^1 f(x, y)dxdy - \overline{I}(f) \right| \leq \frac{3}{2} \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} \left\{ \frac{1}{m_2} \|f^{(2,0)}\|_{\infty} \int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}} \left( x - \frac{k-1}{m_1} \right) \left( \frac{k}{m_1} - x \right) dx \right. \\
+ \frac{1}{m_1} \|f^{(0,2)}\|_{\infty} \int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}} \left( y - \frac{l-1}{m_2} \right) \left( \frac{l}{m_2} - y \right) dy \right\} \\
= \frac{3}{2} \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} \left\{ \frac{1}{6m_1^2m_2n_1} \|f^{(2,0)}\|_{\infty} + \frac{1}{6m_1m_2^2n_2} \|f^{(0,2)}\|_{\infty} \right\} \\
= \frac{1}{4} \left\{ \frac{1}{m_1^2n_1} \|f^{(2,0)}\|_{\infty} + \frac{1}{m_2^2n_2} \|f^{(0,2)}\|_{\infty} \right\}.
\]

8. Non-multiplicativity of the cubature formula

In this section we will give some results which suggest how non-multiplicative the functional $\overline{I}(f) = \int_0^1 \int_0^1 \overline{B}(f; (x, y))dxdy$ is.

Let $(X, d)$ be a compact metric space and $L : C(X) \to \mathbb{R}$ be a positive linear functional reproducing constant. We consider the positive bilinear functional
\[
D(f, g) := L(fg) - L(f)L(g).
\]

**Theorem 8.1.** If $f, g \in C(X)$, $(X, d)$ a compact metric space, then the inequality
\[
|D(f, g)| \leq \frac{1}{4} \omega_d \left( f; 2\sqrt{L^2(d^2(\cdot, \cdot))} \right) \omega_d \left( g; 2\sqrt{L^2(d^2(\cdot, \cdot))} \right)
\]
holds.

**Proof.** Let $f, g \in C[a, b]$ and $r, s \in Lip_1$. Using the Cauchy-Schwarz inequality for positive linear functional gives
\[
|L(f)| \leq L(||f||) \leq \sqrt{L(f^2) \cdot L(1)} = \sqrt{L(f^2)},
\]
so we have
\[
D(f, f) = L(f^2) - L(f)^2 \geq 0.
\]
Therefore, $D$ is a positive bilinear form on $C(X)$. Using the Cauchy-Schwarz inequality for $D$ it follows
\[
|D(f, g)| \leq \sqrt{D(f, f)D(g, g)} \leq ||f||_{\infty} ||g||_{\infty}.
\]
Since $L$ is a positive linear functional we can represent as follows
\[
L(f) := \int_X f(t)d\mu(t),
\]
where $\mu$ is a Borel probability measure on $X$, i.e., $\int_X d\mu(t) = 1$. For $r \in Lip_1$, it follows

$$D(r, r) = L(r^2) - L(r)^2 = \int_X r^2(t)d\mu(t) - \left(\int_X r(u)d\mu(u)\right)^2$$

$$= \int_X \left(r(t) - \int_X r(u)d\mu(u)\right)^2 d\mu(t) = \int_X \left(\int_X (r(t) - r(u))d\mu(u)\right)^2 d\mu(t)$$

$$\leq \int_X \left(\int_X (r(t) - r(u))^2d\mu(u)\right) d\mu(t)$$

$$\leq |r|^2_{Lip_1} \int_X \left(\int_X d^2(t, u)d\mu(u)\right) d\mu(t)$$

$$= |r|^2_{Lip_1} L^1 \left[ L \left( d^2(t, \cdot) \right) \right] = |r|^2_{Lip_1} L^2 \left( d^2(\cdot, \cdot) \right).$$

For $r, s \in Lip_1$ we have

$$|D(r, s)| \leq \sqrt{D(r, r)D(s, s)} \leq |r|_{Lip_1}|s|_{Lip_1} L^2 \left( d(\cdot, \cdot) \right).$$

Moreover, for $f \in C(X)$ and $s \in Lip_1$, we have the estimate

$$|D(f, s)| \leq \sqrt{D(f, f)D(s, s)} \leq \|f\|_{\infty}|s|_{Lip_1} \sqrt{L^2 \left( d(\cdot, \cdot) \right)}.$$

In a similar way, if $r \in Lip_1$ and $g \in C(X)$, we have

$$|D(r, g)| \leq \sqrt{D(r, r)D(g, g)} \leq \|g\|_{\infty}|r|_{Lip_1} \sqrt{L^2 \left( d(\cdot, \cdot) \right)}.$$

Let $f, g \in C(X)$ be fixed and $r, s \in Lip_1$ arbitrary, then

$$|D(f, g)| = |D(f - r + r, g - s + s)|$$

$$\leq |D(f - r, g - s) + |D(f - r, s) + |D(r, g - s) + |D(r, s))|$$

$$\leq \|f - r\|_\infty \cdot \|g - s\|_\infty + \|f - r\|_\infty \cdot \|s\|_{Lip_1} \sqrt{L^2 \left( d^2(\cdot, \cdot) \right)}$$

$$+ \|g - s\|_\infty \cdot |r|_{Lip_1} \sqrt{L^2 \left( d^2(\cdot, \cdot) \right)} + |r|_{Lip_1} \|s\|_{Lip_1} L^2 \left( d^2(\cdot, \cdot) \right)$$

$$= \left\{\|f - r\|_\infty + |r|_{Lip_1} \sqrt{L^2 \left( d^2(\cdot, \cdot) \right)}\right\} \left\{\|g - s\|_\infty + |s|_{Lip_1} \sqrt{L^2 \left( d^2(\cdot, \cdot) \right)}\right\}.$$
Corollary 8.1. If \( f, g \in C([0, 1] \times [0, 1]) \), then
\[
|\mathcal{I}(fg) - \mathcal{I}(f)\mathcal{I}(g)| \leq \frac{1}{4} \tilde{\omega}_{d_2} \left( f; 2 \sqrt{\frac{1}{3} \left( 1 + \frac{1}{n_1 m_1^2} + \frac{1}{n_2 m_2^2} \right)} \right)
\]
\[
\cdot \tilde{\omega}_{d_2} \left( g; 2 \sqrt{\frac{1}{3} \left( 1 + \frac{1}{n_1 m_1^2} + \frac{1}{n_2 m_2^2} \right)} \right)
\]

Proof. We have
\[
\mathcal{I} \left( d_2^2 (\cdot, \cdot) \right) = \sum_{k,k_1=1}^{m_1} \sum_{l,l_1=1}^{m_2} \sum_{i,i_1=0}^{n_1} \sum_{j,j_1=0}^{n_2} \frac{1}{m_1^2 m_2^2 (n_1+1)(n_2+1)} \left[ \left( \frac{k_1-1}{m_1} + \frac{i_1}{m_1} - \frac{k-1}{m_1} - \frac{i}{m_1} \right)^2 + \left( \frac{l_1-1}{m_2} + \frac{j_1}{m_2} - \frac{l-1}{m_2} - \frac{j}{m_2} \right)^2 \right]
\]
\[
= \frac{1}{m_1^2 m_2^2 (n_1+1)^2} \sum_{k,k_1=1}^{m_1} \sum_{i,i_1=0}^{n_1} \left( \frac{k_1-k}{m_1} + \frac{i_1-i}{m_1} \right)^2 + \frac{1}{m_2^2 (n_2+1)^2} \sum_{l,l_1=1}^{m_2} \sum_{j,j_1=0}^{n_2} \left( \frac{l_1-l}{m_2} + \frac{j_1-j}{m_2} \right)^2
\]
\[
= \frac{1}{3} \left( 1 + \frac{1}{m_1^3 n_1} + \frac{1}{m_2^3 n_2} \right)
\]
Therefore, using Theorem 8.1 it follows
\[
|\mathcal{I}(fg) - \mathcal{I}(f)\mathcal{I}(g)| \leq \frac{1}{4} \tilde{\omega}_{d_2} \left( f; 2 \sqrt{\frac{1}{3} \left( 1 + \frac{1}{n_1 m_1^2} + \frac{1}{n_2 m_2^2} \right)} \right)
\]
\[
\cdot \tilde{\omega}_{d_2} \left( g; 2 \sqrt{\frac{1}{3} \left( 1 + \frac{1}{n_1 m_1^2} + \frac{1}{n_2 m_2^2} \right)} \right)
\]

\[\square\]

In the following part of this section we will give a Chebyshev-Grüss type inequality which involves oscillations of function. This result is obtained using a general inequality published in [1]. Let \( Y \) be an arbitrary set and \( B(Y^2) \) the set of all real-valued, bounded functions on \( Y^2 \). Take \( a_n, b_n \in \mathbb{R} \), \( n \geq 0 \), such that \( \sum_{n=0}^{\infty} |a_n| < \infty \), \( \sum_{n=0}^{\infty} a_n = 1 \) and \( \sum_{n=0}^{\infty} |b_n| < \infty \), \( \sum_{n=0}^{\infty} b_n = 1 \), respectively. Furthermore, let \( x_n \in Y, n \geq 0 \) and \( y_m \in Y, m \geq 0 \) be arbitrary mutually distinct points. For \( f \in B(Y^2) \) set \( f_{n,m} := f(x_n, y_m) \). Now consider the functional \( L : B(Y^2) \to \mathbb{R}, Lf = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m f_{n,m} \). The functional \( L \) is linear and reproduces constant functions.

Theorem 8.2. [1] The Chebyshev-Grüss-type inequality for the above linear functional \( L \) is given by:
\[
|L(fg) - L(f) \cdot L(g)| \leq \frac{1}{2} \cdot \text{osc}_L(f) \cdot \text{osc}_L(g) \cdot \sum_{n,m,i,j=0, (n,m) \neq (i,j)}^{\infty} |a_n b_m a_i b_j|
\]
where \( f, g \in B(Y^2) \) and we define the oscillations to be:
\[
\text{osc}_L(f) := \sup\{ |f_{n,m} - f_{i,j}| : n, m, i, j \geq 0 \}.
\]

**Theorem 8.3.** [1] In particular, if \( a_n \geq 0, b_m \geq 0, n, m \geq 0 \), then \( L \) is a positive linear functional and we have:
\[
|L(fg) - L(f) \cdot L(g)| \leq \frac{1}{2} \left( 1 - \sum_{n=0}^{\infty} a_n^2 \cdot \sum_{m=0}^{\infty} b_m^2 \right) \cdot \text{osc}_L(f) \cdot \text{osc}_L(g),
\]
for \( f, g \in B(Y^2) \) and the oscillations given as above.

The following result gives us the non-multiplicative of the functional \( \mathcal{I} \) using discrete oscillations. This result is better than (8.1) in the sense that the oscillations of functions are relative only to certain points, while in (8.1) the oscillations, expressed in terms of \( \tilde{\omega} \), are relative to the whole interval \([0, 1]\).

**Corollary 8.2.** If \( f, g \in B([0, 1]^2) \), then
\[
|\mathcal{I}(fg) - \mathcal{I}(f)\mathcal{I}(g)| \leq \frac{1}{2} \left( 1 - \frac{1}{m_1m_2(n_1+1)(n_2+1)} \right) \text{osc}(f)\text{osc}(g).
\]

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