NEWTON POLYHEDRA, TROPICAL GEOMETRY AND THE RING OF CONDITIONS FOR \((\mathbb{C}^*)^n\)

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ABSTRACT. The ring of conditions defined by C. De Concini and C. Procesi is an intersection theory for algebraic cycles in a spherical homogeneous space. In the paper we consider the ring of conditions for the group \((\mathbb{C}^*)^n\). Up to a big extend this ring can be reduced to the cohomology rings of smooth projective toric varieties. This ring also can be described using tropical geometry. We recall these known results and provide a new description of this ring in terms of convex integral polyhedra.

1. INTRODUCTION

The ring of conditions (see [1]) is an intersection theory for algebraic cycles in a spherical homogeneous space with coefficients in a commutative ring \(\Lambda\). In the paper we consider the ring of conditions \(\mathcal{R}_n(\Lambda)\) for the group \((\mathbb{C}^*)^n\) (which is a spherical homogeneous space with respect to the natural action of the group on itself) with coefficients in \(\Lambda = \mathbb{Z}, \mathbb{R}, \mathbb{C}\). Up to a big extend the ring \(\mathcal{R}_n(\Lambda)\) can be reduced to the cohomology rings of smooth projective toric varieties [1]. Tropical geometry, relates algebraic geometry and piecewise linear geometry (see [2] – [5]). In particular, it studies tropicalization of subvarieties in \((\mathbb{C}^*)^n\) and their intersection theory (see [2]). We remind these results and provide a new description of \(\mathcal{R}_n(\Lambda)\) in terms of convex integral polyhedra. Thus we show that in fact the ring \(\mathcal{R}_n(\Lambda)\) belongs to Newton polyhedra theory.

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2. The ring of conditions \( R_n(\Lambda) \) for the group \((\mathbb{C}^*)^n\)

Two \(k\)-dimensional cycles \(X_1, X_2 \subset (\mathbb{C}^*)^n\) are equivalent \(X_1 \sim X_2\) if for any \((n - k)\)-dimensional cycle \(Y \subset (\mathbb{C}^*)^n\) and for almost all \(g \in (\mathbb{C}^*)^n\) we have \(<X_1, gY> = <X_2, gY>\) where \(<A, B>\) is the intersection index of the cycles \(A\) and \(B\). If \(X_1 \sim X_2\) and \(Y_1 \sim Y_2\) then for almost all \(g_1, g_2 \in (\mathbb{C}^*)^n\) we have \(X_1 \cap g_1Y_1 \sim X_2 \cap g_2Y_2\). The product \(X \ast Y\) of equivalence classes \(X\) and \(Y\) is the equivalence classes of the intersection \(X_1 \cap gY_1\) where \(X_1\) and \(Y_1\) are representatives of \(X\) and \(Y\) and \(g\) is a generic element in \((\mathbb{C}^*)^n\). The ring of conditions \(R_n(\Lambda)\) is the ring of the equivalence classes with the multiplication \(\ast\) and with the tautological addition.

3. Bergman cone

A vector \(k = (k_1, \ldots, k_n) \in \mathbb{Z}^n\) is essential for a variety \(X \subset (\mathbb{C}^*)^n\) if there is a germ of meromorphic map \(f: (\mathbb{C}, 0) \to X \subset (\mathbb{C}^*)^n\) where \(f(t) = at^k + \ldots, a = (a_1, \ldots, a_n) \in (\mathbb{C}^*)^n\) and points \(\ldots\) denote terms \(a_m t^m\) of higher order (i.e. \(m = (m_1, \ldots, m_n)\) where \(m_i \geq k_i\) for \(1 \leq i \leq n\) and \(m \neq k\)).

The Bergman cone \(B(X) \subset \mathbb{R}^n\) of \(X\) is the closure of the set of vectors \(\lambda k \in \mathbb{R}^n\) where \(k\) is essential vector for \(X\) and \(\lambda \geq 0\).

**Theorem 1.** If each irreducible component of \(X\) has complex dimension \(m\) then \(B(X)\) is a finite union of convex rational cones \(|\sigma_i| \subset \mathbb{R}^n\) with \(\dim_{\mathbb{R}} |\sigma_i| = m\). Moreover \(B(X)\) is the support of a fan (defined up to subdivision of \(B(X)\)) of some toric variety.

A first version of theorem 1 appeared in [3]. Now its different versions can be found in many works dedicated to tropical geometry.

4. Good compactification

Toric variety \(M \supset (\mathbb{C}^*)^n\) is a good compactification for a subvariety \(X \subset (\mathbb{C}^*)^n\) with \(\dim X = k\) if its closure \(\overline{X}\) in \(M\) is complete and does not intersect orbits in \(M\) whose codimension is bigger than \(k\).

**Theorem 2.** 1) For any finite set \(S\) of algebraic subvarieties in \((\mathbb{C}^*)^n\) there is a toric variety \(M \supset (\mathbb{C}^*)^n\) which provides a good compactification for each subvariety from \(S\).

2) Toric variety \(M\) is a good compactification of \(X \subset (\mathbb{C}^*)^n\) if and only if the support of its fan contains the Bergman cone \(B(X)\).

The part 1) of theorem 2 was discovered (in a stronger form) in [1]. It is essential for the theory of rings of conditions. Now different versions of theorem 2 can be found in many works dedicated to tropical
geometry (for example, see [7]). Let \( S_r \) be a set of all subvarieties in \((\mathbb{C}^*)^n\) such that any \( X \) from \( S_r \) can be defined by a system of Laurent polynomials whose Newton polyhedra belong to a ball of radius \( r \). The following more precise version of the part 1) of theorem 2 easily follows from [8].

**Theorem 3.** There is a Newton polyhedron \( \Delta_r \) such that the projective toric variety \( M_{\Delta_r} \) corresponding to \( \Delta_r \) is smooth and it provides a good compactification for any \( X \in S_r \). Bergman set \( B(X) \) of any \( X \in S_r \) is a subfan of the dual fan \( \Delta_r^\perp \) to the polyhedron \( \Delta_r \).

5. The ring \( \mathcal{R}_n(\Lambda) \) and cohomology rings of toric varieties

For a complete smooth toric variety \( M \supset (\mathbb{C}^*)^n \) and for any \( k \)-dimensional cycle \( X = \sum k_iX_i \) one can defined the cycle \( \overline{X} \) in \( M \) as \( \sum k_i\overline{X}_i \), where \( \overline{X}_i \) is the closure in \( M \) of \( X_i \subset (\mathbb{C}^*)^n \). The cycle \( \overline{X} \) defines an element \( \rho(X) \) in \( H^{2(n-k)}(M^n, \Lambda) \) whose value on the closure \( \overline{O}_i \) of an \((n-k)\)-dimensional orbit \( O_i \) in \( M \) is equal to the intersection index \( < \overline{X}, \overline{O}_i > \). A compactification \( M \supset (\mathbb{C}^*)^n \) is good for a cycle \( X = \sum k_iX_i \) in \((\mathbb{C}^*)^n\) if it is good compactification for each \( X_i \).

**Theorem 4** (see [1]). If a smooth toric compactification \( M \) is good for cycles \( X, Y \) and \( Z \) where \( Z = X \ast Y \), then the product \( \rho(X)\rho(Y) \) in the cohomology ring \( H^*(M, \Lambda) \) of the elements \( \rho(X) \) and \( \rho(Y) \) is equal to \( \rho(Z) \).

6. The ring of balanced \( \Lambda \)-enriched fans

6.1. An \( \Lambda \)-enriched \( k \)-fan is a fan \( F \subset \mathbb{R}^n \) of an \( n \)-dimensional toric variety equipped with a weight function \( c : F_k \to \Lambda \) defined on the set \( F_k \) of all \( k \)-dimensional cones from \( F \). The support \( |F| \) of \( F \) is the union of all cones \( |\sigma_i| \subset \mathbb{R}^n \) where \( \sigma_i \in F_k \) and \( c(\sigma_i) \neq 0 \). Two enriched \( k \)-fans \( F_1 \) and \( F_2 \) are equivalent if: 1) \( |F_1| = |F_2| \); 2) the weight functions \( c_1 \) and \( c_2 \) induce the same weight function on every common subdivision of the fans \( F_1 \) and \( F_2 \).

6.2. Let \( F \) be an enriched \( k \)-fan. For a cone \( \sigma_i \in F_k \) let \( L_i^\perp \subset (\mathbb{R}^n)^* \) be the space dual to the span \( L_i \) of \( |\sigma_i| \subset \mathbb{R}^n \). Let \( O \) be an orientation of \( |\sigma_i| \). Denote by \( e_i^+(O) \in \Lambda^{n-k}L_i^\perp \) the \((n-k)\)-vector, such that: 1) the integral volume of \( |e_i^+(O)| \) in \( L_i^\perp \) is equal to one; 2) the orientation of \( e_i^+(O) \) is induced from the orientation \( O \) of \( |\sigma_i| \) and from the standard orientation of \( \mathbb{R}^n \). An enriched \( k \)-fan \( F \) satisfies the balance condition if for any orientation of a \((k-1)\)-dimensional cone \( |\rho| \) where \( \rho \in F_{k-1} \), the relation

\[
(1) \quad \sum e_i^+(O(\rho))c(\sigma_i) = 0
\]
holds, where $c$ is the weight function and summation is taken over all $\sigma_i \in \mathcal{F}_k$ such that $|\rho| \subset \partial|\sigma_i|$ and $O(\rho)$ is such orientation of $|\sigma_i|$ that the orientation of $\partial|\sigma_i|$ agrees with the orientation of $|\rho|$.

6.3. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be balanced $k$- and $(n-k)$-fans. Cones $\sigma^1_i \in \mathcal{F}_1$, $\sigma^2_j \in \mathcal{F}_2$ with $\dim \sigma^1_i = k$, $\dim \sigma^2_j = n-k$ are admissible for a vector $a \in \mathbb{R}^n$ if $|\sigma^1_i| \cap (|\sigma^2_j| + a) \neq \emptyset$. Let $C_{i,j}$ be the index of $\Lambda_i \bigoplus \Lambda_j$ in $\mathbb{Z}^n$ where $\Lambda_i = L^1_i \cap \mathbb{Z}^n$, $\Lambda_j = L^2_j \cap \mathbb{Z}^n$ and $L^1_i$, $L^2_j$ are linear spaces spanned by $|\sigma^1_i|$, $|\sigma^2_j|$. The intersection number $c(0)$ of $\mathcal{F}_1$ and $\mathcal{F}_2$ is

\[
\sum C_{i,j} c_1(\sigma^1_i)c_2(\sigma^2_j),
\]

where the sum is taken over all admissible couples $\sigma^1_i, \sigma^2_j$ for any generic vector $a \in \mathbb{R}^n$ (one can show that if the fans $\mathcal{F}_1, \mathcal{F}_2$ satisfy the balance condition (1) then the sum (2) is independent of the choice of generic vector $a$). The product $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ is a 0-fan $\mathcal{F} = \{0\}$ with the weight $c(0)$ equal to the intersection number.

6.4. Consider a $k$-fan $\mathcal{F}_1$ and a $m$-fan $\mathcal{F}_2$ from the set $\mathcal{T}\mathcal{R}_n(\Lambda)$ of all balanced $\Lambda$-enriched fans. Let $d$ be $n - (k + m)$. If $d < 0$ then $\mathcal{F}_1 \times \mathcal{F}_2 = \emptyset$. If $d = 0$ the fan $\mathcal{F}_1 \times \mathcal{F}_2$ is defined above. Let us define the $d$-fan $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ for $d > 0$. Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are subfans of a complete fan $\mathcal{G}$. Then $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ also is a subfan of $\mathcal{G}$. The weight $c(\delta)$ of a cone $\delta$ from $\mathcal{G}$ with $\dim \delta = d$ is defined as follows. Let $L$ be a space spanned by the cone $|\delta|$ and let $(\mathcal{F}_1)_\delta$ and $(\mathcal{F}_2)_\delta$ be the enriched subfans of $\mathcal{F}_1$ and of $\mathcal{F}_2$ consisting of all cones from these fans containing the cone $\delta$. The weight $c(\delta)$ of the cone $\delta$ in $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ is equal to the intersection number of the images under the factorization of $(\mathcal{F}_1)_\delta$ and $(\mathcal{F}_2)_\delta$ in the factor space $\mathbb{R}^n/L$ equipped with the factor lattice $\mathbb{Z}^n/L \cap \mathbb{Z}^n$.

7. Tropicalization of the ring $\mathcal{R}_n(\Lambda)$

Let $\Delta^\perp$ be a fan of a smooth complete projective toric variety $M^\perp_{\Lambda}$. Let $\mathcal{T}\mathcal{R}_n(\Lambda, \Delta)$ be a ring of balanced $\Lambda$-enriched fans equal to $\Lambda$-linear combination of cones from the fan $\Delta^\perp$. The following theorems 5,6 are proved in [2].

**Theorem 5** (see [2]). The ring $\mathcal{T}\mathcal{R}_n(\Lambda, \Delta)$ is isomorphic to the intersection ring $H_*(M_{\Delta}, \Lambda)$. The component of $\mathcal{T}\mathcal{R}_n(\Lambda, \Delta)$ consisting of $k$-fans under this isomorphism corresponds to the component $H_{2k}(M_{\Delta}, \Lambda)$.

**Theorem 6** (see [2]). The ring of conditions $\mathcal{R}_n(\Lambda)$ is isomorphic to the tropical ring $\mathcal{T}\mathcal{R}_n(\Lambda)$ of all balanced $\Lambda$-enriched fans. (The rings $\mathcal{R}_n(\mathbb{Z})$, $\mathcal{R}_n(\mathbb{C})$ have similar descriptions).
8. The graded ring associated to a homogeneous polynomial

To a homogeneous polynomial \( P \) on a \( \mathbb{R} \)-linear space \( \mathcal{L} \), \( \dim \mathcal{L} < \infty \), one can associate the graded commutative ring \( A(\mathcal{L}, P) \) (one can produce a similar constructions for homogeneous polynomials on infinite dimensional spaces over any field and for functions analogues to homogeneous polynomials on free abelian groups). Let \( D(\mathcal{L}) \) be the ring of \( \mathbb{R} \)-linear differential operators on \( \mathcal{L} \) with constant coefficients. It is generated by Lie derivatives \( L_v \) along constant vector fields \( v(x) \equiv v \in \mathcal{L} \) and by multiplications on real constants. Let \( I_P \subseteq D(\mathcal{L}) \) be a set defined by the following condition: \( L \in I_P \iff L(P) \equiv 0 \). It is easy to see that \( I_P \) is a homogeneous ideal. By definition the ring associated to \( P \) is the factor ring \( A(\mathcal{L}, P) = D(\mathcal{L}) / I_P \). One can to see that:

1. \( A(\mathcal{L}, P) \) is a graded ring with homogeneous components \( A_0, \ldots, A_n \) where \( n = \deg P \);
2. \( A_0 = \mathbb{R} \);
3. there is a non-degenerate pairing between \( A_k \) and \( A_{n-k} \) with values in \( A_0 \), thus \( A_k = A^*_{(n-k)} \) and \( A_n \sim \mathbb{R} \).

9. Khovanskii–Pukhlikov ring and the ring \( R_n(\Lambda) \)

Let \( L_\Delta \) be a space of formal differences of convex polyhedra whose support functions are linear on each cone from the fan of a smooth projective toric variety \( M_\Delta \). Let \( n!V \) be the degree \( n \) homogeneous polynomial on \( L_\Delta \) whose value on \( \tilde{\Delta} \in L_\Delta \) is equal to the volume of \( \tilde{\Delta} \) multiplied by \( n! \). The ring \( A(L_\Delta, n!V) \) is called Khovanskii–Pukhlikov ring [9].

**Theorem 7.** The intersection ring \( H_*(M_\Delta, \mathbb{R}) \) is isomorphic (up to a change of the grading) to the Khovanskii–Pukhlikov ring \( A(L_\Delta, n!V) \).

Let \( \mathcal{L}_n \), \( \dim \mathcal{L}_n = \infty \), be the space of formal differences of convex polyhedra \( \Delta \) with rational dual fans \( \Delta^\perp \). Let \( n!V \) be the degree \( n \) homogeneous polynomial on \( \mathcal{L}_n \) whose value on \( \Delta \in \mathcal{L}_n \) is equal to the volume of \( \Delta \) multiplied by \( n! \).

**Theorem 8.** The ring \( R_n(\mathbb{R}) \) is isomorphic to the ring \( A(\mathcal{L}_n, n!V) \). (The rings \( R_n(\mathbb{Z}) \), \( R_n(\mathbb{C}) \) have similar descriptions).

10. The BKK theorem and the ring \( R_n(\Lambda) \)

Let \( \{ \Gamma_i \} \) be a set of \( n \) hypersurfaces in \( (\mathbb{C}^*)^n \) defined by \( P_i = 0 \) where \( P_i \) are Laurent polynomials with Newton polyhedra \( \Delta_i \). Bernstein-Koushniorenko-Khovanskii theorem (BKK theorem [10]-[12]) can be stated in the following two ways:
Theorem 9. The intersection number of the hypersurfaces $\Gamma_i$ in the ring of conditions is equal to the mixed volume of $\Delta_1, \ldots, \Delta_n$ multiplied by $n!$.

Let $\mathcal{F}_i$ be $\mathbb{R}$-enriched $(n - 1)$-fan dual to $\Delta_i$ whose weight function at a cone $\sigma$ dual to a side $\sigma^\perp$ of $\Delta_i$ is equal to the integral length of the $\sigma^\perp$.

Theorem 10. The intersection number of the hypersurfaces $\Gamma_i$ in the ring of conditions is equal to the intersection number of the $\mathbb{R}$-enriched fans $\mathcal{F}_i$ in the ring $T\mathcal{R}_n$.

Thus theorems 8 and 6 could be considered as generalizations of the BKK theorem. Such generalizations are possible because of the following reason: the cohomology ring of a smooth toric variety is generated by elements of degree two.

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