Accuracy on eigenvalues
for a Schrödinger operator with a degenerate potential
in the semi-classical limit

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Abstract
We consider a semi-classical Schrödinger operator
$-\hbar^2 \Delta + V$ with a degenerate potential $V(x,y) = f(x) \ g(y)$.
$g$ is assumed to be a homogeneous positive function of $m$ variables and
$f$ is a strictly positive function of $n$ variables, with a strict minimum.
We give sharp asymptotic behaviour of low eigenvalues bounded by
some power of the parameter $\hbar$, by improving Born-Oppenheimer approxi-

1 Introduction

In our paper [MoTr] we have considered the Schrödinger operator on
$L^2(\mathbb{R}^n_x \times \mathbb{R}^m_y)$

$$\hat{H}_\hbar = \hbar^2 D^2_x + \hbar^2 D^2_y + f(x)g(y)$$

(1.1)

with $g \in C^\infty(\mathbb{R}^m \setminus \{0\})$ homogeneous of degree $a > 0$,

$$g(\mu y) = \mu^a g(y) > 0, \ \forall \mu > 0 \text{ and } \forall y \in \mathbb{R}^m \setminus \{0\}.$$  

(1.2)
$h > 0$ is a semiclassical parameter we assume to be small.

We have investigated the asymptotic behavior of the number of eigenvalues less than $\lambda$ of $\hat{H}_h$, 

$$N(\lambda, \hat{H}_h) = \text{tr}(\chi_{[-\infty, \lambda]}(\hat{H}_h)) = \sum_{\lambda_k(\hat{H}_h) < \lambda} 1 . \quad (1.3)$$

$(\text{tr}(P)$ denotes the trace of the operator $P$).

If $P$ is a self-adjoint operator on a Hilbert space $\mathcal{H}$, we denote respectively by $\text{sp}(P)$, $\text{sp}_{\text{ess}}(P)$ and $\text{sp}_d(P)$ the spectrum, the essential spectrum and the discret spectrum of $P$.

When $-\infty < \inf \text{sp}(P) < \inf \text{sp}_{\text{ess}}(P)$, we denote by $(\lambda_k(P))_{k>0}$ the increasing sequence of eigenvalues of $P$, repeated according to their multiplicity:

$$\text{sp}_d(P) \bigcap \left( \left] -\infty, \inf \text{sp}_{\text{ess}}(P) \right[ \right] = \{ \lambda_k(P) \} . \quad (1.4)$$

In this paper we are interested in a sharp estimate for some eigenvalues of $\hat{H}_h$. We make the following assumptions on the other multiplicative part of the potential:

$$f \in C^\infty(\mathbb{R}^n), \quad \forall \alpha \in \mathbb{N}^n, \quad (|f(x)| + 1)^{-1}\partial^\alpha_x f(x) \in L^\infty(\mathbb{R}^n)$$

$$0 < f(0) = \inf_{x \in \mathbb{R}^n} f(x)$$

$$f(0) < \liminf_{|x| \to \infty} f(x) = f(\infty)$$

$$\partial^2 f(0) > 0$$

(1.5)

$\partial^2 f(a)$ denotes the hessian matrix:

$$\partial^2 f(a) = \left( \frac{\partial^2}{\partial x_i \partial x_j} f(a) \right)_{1 \leq i,j \leq n} .$$

By dividing $\hat{H}_h$ by $f(0)$, we can change the parameter $h$ and assume that

$$f(0) = 1 . \quad (1.6)$$

Let us define $h = \hbar^{2/(2+a)}$ and change $y$ in $y\hbar$; we can use the homogeneity of $g$ to get:

$$\text{sp}(\hat{H}_h) = h^a \text{ sp}(\hat{H}_h) , \quad (1.7)$$

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\[
\hat{H}^h = \hbar^2 D_x^2 + D_y^2 + f(x)g(y) = \hbar^2 D_x^2 + Q(x,y,D_y) :
\]
\[
Q(x,y,D_y) = D_y^2 + f(x)g(y). 
\]

Let us denote the increasing sequence of eigenvalues of \(D_y^2 + g(y)\) , (on \(L^2(\mathbb{R}^m)\) ) , by \((\mu_j)_{j>0}\).
The associated eigenfunctions will be denoted by \((\varphi_j)_j\):

\[
D_y^2 \varphi_j(y) + g(y)\varphi_j(y) = \mu_j \varphi_j(y)
\]
\[
\langle \varphi_j \mid \varphi_k \rangle = \delta_{jk} \tag{1.8}
\]
and \((\varphi_j)_j\) is a Hilbert base of \(L^2(\mathbb{R}^m)\).

By homogeneity \(Q_x(y,D_y) = D_y^2 + f(x)g(y)\), for a fixed \(x\), are given by the sequence \((\lambda_j(x))_{j>0}\), where:

\[
\lambda_j(x) = \mu_j \frac{f^2}{2+2a}(x).
\]

So as in \([\text{MoTr}]\) we get :

\[
\hat{H}^h \geq \left[ \hbar^2 D_x^2 + \mu_1 \frac{f^2}{2+2a}(x) \right]. \tag{1.9}
\]

This estimate is sharp as we will see below.

Then using the same kind of estimate as \((1.9)\), one can see that

\[
\inf \ sp_{\text{ess}}(\hat{H}^h) \geq \mu_1 f^{2/(2+a)}(\infty). \tag{1.10}
\]

We are in the Born-Oppenheimer approximation situation described by A. Martinez in \([\text{Ma}]\) : the ”effective” potential is given by \(\lambda_1(x) = \mu_1 f^{2/(2+a)}(x)\), the first eigenvalue of \(Q_x\), and the assumptions on \(f\) ensure that this potential admits one unique and nondegenerate well \(U = \{0\}\), with minimal value equal to \(\mu_1\). Hence we can apply theorem 4.1 of \([\text{Ma}]\) and get :

**Theorem 1.1** Under the above assumptions, for any arbitrary \(C > 0\), there exists \(h_0 > 0\) such that, if \(0 < h < h_0\) , the operator \((\hat{H}^h)\) admits a finite number of eigenvalues \(E_k(h)\) in \([\mu_1,\mu_1 + Ch]\), equal to the number of the eigenvalues \(e_k\) of \(D_x^2 + \frac{\mu_1}{2+a} < \partial^2 f(0) x, x \geq 0\) such that :

\[
E_k(h) = \lambda_k(\hat{H}^h) = \lambda_k \left( \hbar^2 D_x^2 + \mu_1 f^{2/(2+a)}(x) \right) + O(h^2). \tag{1.11}
\]

More precisely \(E_k(h) = \lambda_k(\hat{H}^h)\) has an asymptotic expansion

\[
E_k(h) \sim \mu_1 + h \left( e_k + \sum_{j \geq 1} \alpha_{kj} h^{j/2} \right). \tag{1.12}
\]
If $E_k(\hbar)$ is asymptotically non degenerated, then there exists a quasimode

$$\phi_k^{\hbar}(x, y) \sim \hbar^{-m_k} e^{-\psi(x)\hbar} \sum_{j \geq 0} \hbar^{j/2} a_{kj}(x, y),$$

(1.13)

satisfying

$$C_0^{-1} \leq \|\hbar^{-m_k} e^{-\psi(x)\hbar} a_{k0}(x, y)\| \leq C_0$$
$$\|\hbar^{-m_k} e^{-\psi(x)\hbar} a_{kj}(x, y)\| \leq C_j$$
$$\|\left(\hat{H}^\hbar - \mu_1 - \hbar \epsilon_k - \sum_{1 \leq j \leq J} \alpha_k \hbar^{j/2}\right)\hbar^{-m_k} e^{-\psi(x)/\hbar} \sum_{0 \leq j \leq J} \hbar^{j/2} a_{kj}(x, y)\| \leq C_J \hbar^{(J+1)/2},$$

(1.14)

The formula (1.12) implies

$$\lambda_k(\hat{H}^\hbar) = \mu_1 + \hbar \lambda_k \left(D_x^2 + \frac{\mu_1}{2 + a} < \partial^2 f(0) x , x > \right) + O(\hbar^{3/2}),$$

(1.15)

and when $k = 1$, one can improve $O(\hbar^{3/2})$ into $O(\hbar^2)$. The function $\psi$ is defined by : $\psi(x) = d(x, 0)$, where $d$ denotes the Agmon distance related to the degenerate metric $\mu_1 f^{2/(2+a)} dx^2.$

2 Lower energies

We are interested now with the lower energies of $\hat{H}^\hbar$. Let us make the change of variables

$$(x, y) \rightarrow (x, f^{1/(2+a)}(x) y).$$

(2.1)

The Jacobian of this diffeomorphism is $f^{m/(2+a)}(x)$, so we perform the change of test functions : $u \rightarrow f^{-m/(2+a)}(x) u$, to get a unitary transformation.

Thus we get that

$$\text{sp} (\hat{H}^\hbar) = \text{sp} (\tilde{H}^\hbar)$$

(2.2)

where $\tilde{H}^\hbar$ is the self-adjoint operator on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ given by

$$\tilde{H}^\hbar = \hbar^2 L^*(x, y, D_x, D_y) L(x, y, D_x, D_y) + f^{2/(2+a)}(x) \left(D_y^2 + g(y)\right),$$

(2.3)

with

$$L(x, y, D_x, D_y) = D_x + \frac{1}{(2 + a)f(x)} [(yD_y) - i \frac{m}{2} \nabla f(x)].$$
We decompose $\tilde{H}^\hbar$ in four parts:

$$\tilde{H}^\hbar = \hbar^2 D_x^2 + \frac{f^2/(2+a)}{(2+a)}(x) (D_y^2 + g(y)) + \hbar^2 \frac{2}{(2+a)f(x)}(\nabla f(x) D_x)(y D_y) + i\hbar^2 \frac{1}{(2+a)f(x)}(\nabla f(x))^2( y D_y)^2 - \frac{\hbar^2}{(2+a)^2 f(x)}|\nabla f(x)|^2[(y D_y)^2 + \frac{m^2}{4}]$$

(2.4)

Our goal is to prove that the only significant role up to order $2$ in $\hbar$ will be played by the first operator, namely:

$$\tilde{H}^\hbar_1 = \hbar^2 D_x^2 + \frac{f^2/(2+a)}{(2+a)}(x) (D_y^2 + g(y)) .$$

Let us denote by $\nu_{j,k}^\hbar$ the eigenvalues of the operator $\hbar^2 D_x^2 + \mu_j f^2/(2+a)(x)$ and by $\psi_{j,k}^\hbar$ the associated normalized eigenfunctions.

Let us consider the following test functions:

$$u_{j,k}^\hbar(x,y) = \psi_{j,k}^\hbar(x) \varphi_j(y) ,$$

where the $\varphi_j$'s are the eigenfunctions defined in (1.8); we have immediately:

$$\tilde{H}^\hbar_1(u_{j,k}^\hbar(x,y)) = \nu_{j,k}^\hbar u_{j,k}^\hbar(x,y) .$$

We will need the following lemma:

Lemma 2.1. For any integer $N$, there exists a positive constant $C$ depending only on $N$ such that for any $k \leq N$, the eigenfunction $\psi_{j,k}^\hbar$ satisfies the following inequalities: for any $\alpha \in \mathbb{N}^n$, $|\alpha| \leq 2$,

$$\| \hbar_j^{\alpha/2} |D_x^\alpha \psi_{j,k}^\hbar| \| < C$$

and

$$\| \left( \frac{\nabla f(x)}{f(x)} \right) \psi_{j,k}^\hbar \| < \hbar_j^{\alpha/2} C$$

(2.5)

with $\hbar_j = \hbar \mu_j^{-1/2}$.

Proof.

Let us recall that it is well known, (see [He-Sj1]), that

$$\forall k \leq N , \quad \mu_j^{-1} \nu_{j,k}^\hbar = 1 + O(\hbar_j) .$$

It is clear also that

$$\left[ \hbar_j^2 D_x^2 + f^2/(2+a)(x) - \mu_j^{-1} \nu_{j,k}^\hbar \right] \psi_{j,k}^\hbar(x) = 0 .$$

(2.6)
We shall need the following inequality, that we can derive easily from (2.6) and the Agmon estimate (see [He-S]) : \( \forall \varepsilon \in ]0,1[ \),

\[
\varepsilon \int [f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^h]_+ e^{2(1-\varepsilon)^{1/2}d_{j,k}(x)/\hbar_j} |\psi_{j,k}^h(x)|^2 \, dx \leq \int [f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^h]_- |\psi_{j,k}^h(x)|^2 \, dx, \tag{2.7}
\]

where \( d_{j,k} \) is the Agmon distance associated to the metric \( [f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^h]_+ \, dx \).

Let us prove the lemma for \( |\alpha| = 1 \).

As \( \int [\hbar_j^2 D_x \psi_{j,k}^h(x)]^2 + (f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^h)|\psi_{j,k}^h(x)|^2 \, dx = 0 \), \( \mu_j^{-1} \nu_{j,k}^h - 1 = O(\hbar_j) \), and \( f^{2/(2+a)}(x) - 1 > 0 \), we get that \( \hbar_j \| D_x \psi_{j,k}^h(x) \|^2 \leq C \).

Furthermore, we use that \( C^{-1} |\nabla f(x)|^2 \leq f^{2/(2+a)}(x) - 1 \leq C|\nabla f(x)|^2 \), for \( |x| \leq C^{-1} \), the exponential decreasing (in \( \hbar_j \)) of \( \psi_{j,k}^h \) given by (2.7) and the boundness of \( |\nabla f(x)|/f(x) \) to get

\[
\| \frac{|\nabla f(x)|}{f(x)} \psi_{j,k}^h(x) \|^2 \leq C \int [f^{2/(2+a)}(x) - 1] |\psi_{j,k}^h(x)|^2 \, dx \leq \hbar_j C. \tag{2.8}
\]

Now we study the case \( |\alpha| = 2 \).

If \( c_0 \in ]0,1[ \) is large enough and \( |x| \in [\hbar_j^{1/2}c_0, 2c_0] \), then we have

\[
|x|^2/C \leq f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^h \leq C|x|^2 \tag{2.9}
\]

Therefore there exists \( C_1 > 1 \) such that \( C_1^{-1}|x|^2 \leq d_{j,k}(x) \leq C_1|x|^2 \), and then

\[
|x|^2 \leq \hbar_j C e^{d_{j,k}(x)/\hbar_j}. \tag{2.9}
\]

Then the inequality : \( C^{-1}|x| \leq |\nabla f(x)| \leq C|x| \), together with (2.8), (2.9) and (2.7) entail that

\[
\int_{|x|\geq C_0\hbar_j} \frac{|\nabla f(x)|^2}{f(x)} |\psi_{j,k}^h(x)|^2 \, dx \leq \hbar_j C \int \left[ f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^h \right]_+ e^{d_{j,k}(x)/\hbar_j} |\psi_{j,k}^h(x)|^2 \, dx \leq \hbar_j C \int \left[ f^{2/(2+a)}(x) - \mu_j^{-1} \nu_{j,k}^h \right]_- |\psi_{j,k}^h(x)|^2 \, dx \leq \hbar_j^2 C.
\]

It remains to estimate \( \hbar_j^2 \| D^\alpha_x \psi_{j,k}^h(x) \| \) with \( |\alpha| = 2 \).

We use that \( -\hbar_j^2 \Delta \psi_{j,k}^h(x) = [-f^{2/(2+a)}(x) + \mu_j^{-1} \nu_{j,k}^h] \psi_{j,k}^h(x) \), and that we have proved that \( \| [-f^{2/(2+a)}(x) + \mu_j^{-1} \nu_{j,k}^h] \psi_{j,k}^h(x) \| \leq \hbar_j C \); so \( \| D^\alpha_x \psi_{j,k}^h(x) \| \leq C/\hbar_j \) if \( |\alpha| = 2 \).

We will need the following result.
Proposition 2.2 Let \( V(y) \in C^\infty(\mathbb{R}^m) \) such that

\[
\exists s > 0, \ C_0 > 0 \ s.t. \quad -C_0 + |y|^s/C_0 \leq V(y) \leq C_0(|y|^s + 1)
\]
\[
\forall \alpha \in \mathbb{N}^m, \ (1 + |y|^{2s-|\alpha|})/2 \partial^\alpha_y V(y) \in L^\infty(\mathbb{R}^m).
\]

If \( u(y) \in L^2(\mathbb{R}^m) \) and \( D^2_y u(y) + V(y)u(y) \in S(\mathbb{R}^m) \),
then \( u \in S(\mathbb{R}^m) \). \( S(\mathbb{R}^m) \) is the Schwartz space.

The proof comes from the fact that there exists a parametrix of \( D^2_y + V(y) \)
in some class of pseudodifferential operator: see for the more general case in [Hor], or for this special case in Shubin book [Shu].

Theorem 2.3.

Under the assumptions (1.2) and (1.5), for any fixed integer \( N > 0 \),
there exists a positive constant \( h_0(N) \) verifying: for any \( h \in ]0, h_0(N)[ \), for any \( k \leq N \) and any \( j \leq N \) such that

\[
\mu_j < \mu_j f^{2/(2+a)}(\infty),
\]

there exists an eigenvalue \( \lambda_{jk} \in \text{sp}_d(\hat{H}^h) \) such that

\[
|\lambda_{jk} - \lambda_k (\hat{h}^2 D^2_x + \mu_j f^{2/(2+a)}(x))| \leq \hat{h}^2 C.
\]

Consequently, when \( k = 1 \), we have

\[
|\lambda_{j1} - \left[ \mu_j + \hat{h}(\mu_j)^{1/2} \text{tr}\left((\partial^2 f(0))^{1/2}\right)(2 + a)^{1/2}\right] | \leq \hat{h}^2 C.
\]

Proof.

The first part of the theorem will follow if we prove that:

\[
\| (\hat{H}^h - \hat{H}_1^h)(u_{j,k}^h(x,y)) \| = \| (\hat{H}^h - \nu_{j,k}^h)u_{j,k}^h(x,y)\| = O(h^2).
\]

Let us consider a function \( \chi \in C^\infty(\mathbb{R}) \) such that

\( \chi(t) = 1 \) if \( |t| \leq 1/2 \) and
\( \chi(t) = 0 \) if \( |t| > 1 \).

Then \( (D^2_y g(y))(1 - \chi(|y|))\varphi_j(y) \in S(\mathbb{R}^m) \),
and Proposition 2.2 shows that \( (1 - \chi(|y|))\varphi_j(y) \in S(\mathbb{R}^m) \).
As \( D^2_y \varphi_j(y) = (\mu_j - g(y))\varphi_j(y) \), we get that

\[
\forall k \in \mathbb{N}, \quad (1+|y|)^k|\varphi_j(y)|^2 + |D_y \varphi_j(y)|^2 + |D^2_y \varphi_j(y)|^2 \in L^1(\mathbb{R}^m).
\]
The quantity \((\hat{H}^h - \hat{H}^h)(u_{j,k}^h(x,y))\) is, by \((2.4)\), composed of 3 parts. According to Lemma 2.1 and the estimate \((2.13)\), the two last parts are bounded in \(L^2\)-norm by \(h^2C\), \((\mu_j \leq C)\).

To obtain a bound for the first part, we integrate by parts to get that
\[
\|\nabla f(x)D_x\psi_{j,k}^h\| \leq C \left[ \|D_x^2\psi_{j,k}^h\| \times \left\| \frac{\nabla f(x)^2}{f(x)} \psi_{j,k}^h \right\| + \|D_x\psi_{j,k}^h\| \times \left\| \frac{|\nabla f(x)|}{f(x)} \psi_{j,k}^h \right\| \right],
\]
and then we use again Lemma 2.1. Thus:
\[
\|\nabla f(x)D_x\psi_{j,k}^h\| \leq C.
\]

According to estimate \((2.13)\) we have finally
\[
\|\nabla f(x)D_x\psi_{j,k}^h\| \leq C.
\]

3 Middle energies

We are going to refine the preceding results when \(a \geq 2\) and \(f(\infty) = \infty\). It is possible then to get sharp localization near the \(\mu_j\)'s for much higher values of \(j\)'s. More precisely we prove :

**Theorem 3.1**. We assume \((1.5)\) with \(f(\infty) = \infty\), \((1.2)\) with \(a \geq 2\) and with \(g \in C^\infty(\mathbb{R}^m)\).

Let us consider \(j\) such that \(\mu_j \leq h^{-2}\); then for any integer \(N\), there exists a constant \(C\) depending only on \(N\) such that, for any \(k \leq N\), there exists an eigenvalue \(\lambda_{jk} \in \text{sp}_d(\hat{H}^h)\) verifying
\[
|\lambda_{jk} - \lambda_k (h^2D_x^2 + \mu_jf^{2/(2+a)}(x))| \leq C\mu_jh^2. \tag{3.1}
\]

Consequently, when \(k = 1\), we have
\[
|\lambda_{j1} - \left[ \mu_j + h(\mu_j)^{1/2}\frac{\text{tr}((\partial^2f(0))^{1/2})}{(2+a)^{1/2}} \right] | \leq C\mu_jh^2. \tag{3.2}
\]

**Proof** : Let us define the class of symbols \(S(p^s(y,\eta))\), \(s \in \mathbb{R}\), with \(p(y,\eta) = |\eta|^2 + g(y) + 1\). 

\[
q(y,\eta) \in S(p^s(y,\eta)) \iff q(y,\eta) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)
\]

and for any \(\alpha\) and \(\beta\) \(\in \mathbb{N}^m\),
\[
p^{-s}(y,\eta)(|\eta|+1)^{-|\alpha|}(|y|+1)^{-|\beta|}D_\alpha^\beta q(y,\eta) \in L^\infty(\mathbb{R}^{2m}).
\]
For such a symbol \( q(y, \eta) \in S(p^s(y, \eta)) \), we define the operator \( Q \) on \( S(\mathbb{R}^m) \):

\[
Qf(y) = (2\pi)^{-m} \int_{\mathbb{R}^{2m}} q(\frac{y+z}{2}, \eta) e^{i(y-z)\eta} f(z) dz d\eta .
\]

We will say that \( Q \in OPS(p^s(y, \eta)) \).

It is well known, (see [Hor]) that \( (D_y^2 + g(y))^s \in OPS(p^s(y, \eta)) \).

As \( a \geq 2 \), we get that \( yD_y \in OPS(p(y, \eta)) \), and then that \( yD_y(D_y^2 + g(y))^{-1} \in OPS(1) \).

Therefore \( yD_y(D_y^2 + g(y))^{-1} \) and \( (yD_y)^2(D_y^2 + g(y))^{-2} \) are bounded operator on \( L^2(\mathbb{R}^m) \), and we get as a consequence the following bound:

\[
\mu_j^{-1} \| yD_y \varphi_j \| + \mu_j^{-2} \| (yD_y)^2 \varphi_j \| \leq C .
\]

As in the proof of Theorem 2.3, using (3.3) instead of (2.13), we get easily that

\[
\| (\hat{H} - \tilde{H}) u_{j,k}^h \| \leq C[h^2 \mu_j + h^3 \mu_j^{3/2}] \leq Ch^2 \mu_j ,
\]

and then Theorem 3.1 follows.

### 4 An application

We consider a Schrödinger operator on \( L^2(\mathbb{R}^d) \) with \( d \geq 2 \),

\[
P^h = -h^2 \Delta + V(z) \tag{4.1}
\]

with a real and regular potential \( V(z) \) satisfying

\[
V \in C^\infty(\mathbb{R}^d ; [0, +\infty]) \quad \liminf_{|z| \to \infty} V(z) > 0 \quad \Gamma = V^{-1}(\{0\}) \text{ is a regular hypersurface.}
\]

By hypersurface, we mean a submanifold of codimension 1. Moreover we assume that \( \Gamma \) is connected and that there exist \( m \in \mathbb{N}^* \) and \( C_0 > 0 \) such that for any \( z \) verifying \( d(z, \Gamma) < C_0^{-1} \)

\[
C_0^{-1} d^{2m}(z, \Gamma) \leq V(z) \leq C_0 d^{2m}(z, \Gamma) \tag{4.3}
\]

( \( d(E,F) \) denotes the euclidian distance between \( E \) and \( F \)).
We choose an orientation on $\Gamma$ and a unit normal vector $N(s)$ on each $s \in \Gamma$, and then, we can define the function on $\Gamma$,

$$f(s) = \frac{1}{(2m)!} \left( N(s) \frac{\partial}{\partial s} \right)^{2m} V(s), \quad \forall s \in \Gamma.$$  \hfill (4.4)

Then by (4.2) and (4.6), $f(s) > 0$, $\forall s \in \Gamma$.

Finally we assume that the function $f$ achieves its minimum on $\Gamma$ on a finite number of discrete points:

$$\Sigma_0 = f^{-1}(\{\eta_0\}) = \{s_1, \ldots, s_{\ell_0}\}, \quad \text{if } \eta_0 = \min_{s \in \Gamma} f(s),$$  \hfill (4.5)

and the hessian of $f$ at each point $s_j \in \Sigma_0$ is non degenerated:

$$\exists \eta_1 > 0 \quad \text{s.t.} \quad \frac{1}{2} \langle d(\langle df ; w \rangle) ; w \rangle(s_j) \geq \eta_1 \lvert w(s_j) \rvert^2, \quad \forall w \in T\Gamma, \forall s_j \in \Sigma_0.$$  \hfill (4.6)

If $g = (g_{ij})$ is the riemannian metric on $\Gamma$, then $\lvert w(s) \rvert = (g(w(s), w(s)))^{1/2}$.

The hessian of $f$ at each $s_j \in \Sigma_0$, is the symmetric operator on $T_{s_j} \Gamma$, $Hess(f)_{s_j}$, associated to the two-bilinear form defined on $T_{s_j} \Gamma$ by:

$$\langle v, w \rangle \in (T_{s_j} \Gamma)^2 \rightarrow \frac{1}{2} \langle d(\langle df ; \tilde{v} \rangle) ; \tilde{w} \rangle(s_j),$$  \hfill (4.7)

$$\forall (\tilde{v}, \tilde{w}) \in (T\Gamma)^2 \text{ s.t. } (\tilde{v}(s_j), \tilde{w}(s_j)) = (v, w).$$

$Hess(f)_{s_j}$ has $d - 1$ non negative eigenvalues

$$\rho_1^2(s_j) \leq \ldots \leq \rho_{d-1}^2(s_j), \quad (\rho_j(s_j) > 0).$$

In local coordinates, those eigenvalues are the ones of the symmetric matrix

$$\frac{1}{2} G^{1/2}(s_j) \left( \frac{\partial^2}{\partial x_k \partial x_\ell} f(s_j) \right)_{1 \leq k, \ell \leq d-1} G^{1/2}(s_j), \quad (G(x) = (g_{k,\ell}(x))_{1 \leq k, \ell \leq d-1}).$$

The eigenvalues $\rho_k^2(s_j)$ do not depend on the choice of coordinates. We denote

$$Tr^+(Hess(f(s_j))) = \sum_{\ell=1}^{d-1} \rho_\ell(s_j).$$  \hfill (4.8)

We denote by $(\mu_j)_{j \geq 1}$ the increasing sequence of the eigenvalues of the operator $-\frac{d^2}{dt^2} + t^{2m}$ on $L^2(\mathbb{R})$,

and by $(\varphi_j(t))_{j \geq 1}$ the associated orthonormal Hilbert base of eigenfunctions.
Theorem 4.1 Under the above assumptions, for any \( N \in \mathbb{N}^* \), there exist \( h_0 \in [0,1] \) and \( C_0 > 0 \) such that, if \( \mu_j < h^{-m/(m+1)(2m+3)} \), and if \( \alpha \in \mathbb{N}^{d-1} \) and \( |\alpha| \leq N \), then \( \forall \ s_\ell \in \Sigma_0 \), \( \exists \lambda_{j_\ell}^h \in sp_d(P^h) \) s.t.
\[
\left| \lambda_{j_\ell}^h - h^{2m/(m+1)} \left[ \eta_0^{1/(m+1)} \mu_j + h^{1/(m+1)} \mu_j^{1/2} \mathcal{A}_\ell(\alpha) \right] \right| \leq h^2 \mu_j^{2+3/2m} C_0 ;
\]
with \( \mathcal{A}_\ell(\alpha) = \frac{1}{\eta_0^{m/(2m+2)}(m+1)^{1/2}} \left[ 2\alpha \rho(s_\ell) + \text{Tr}^+(\text{Hess}(f(s_\ell))) \right] \).
\( (\alpha \rho(s_\ell) = \alpha_1 \rho_1(s_\ell) + \ldots + \alpha_d \rho_d(s_\ell) \).

Proof:

Let \( \mathcal{O}_0 \subset \mathbb{R}^d \) be an open neighbourhood of \( s_\ell \in \Sigma_0 \), such that there exists \( \phi \in C^\infty(\mathcal{O}_0 ; \mathbb{R}) \) satisfying
\[
\Gamma_0 = \Gamma \cap \mathcal{O}_0 = \{ z \in \mathcal{O}_0 ; \phi(z) = 0 \} ;
\]
\[
|\nabla \phi(z)| = 1 \, , \, \forall \ z \in \mathcal{O}_0 . \quad (4.9)
\]

After changing \( \mathcal{O}_0 \) into a smaller neighbourhood if necessary, we can find \( \tau \in C^\infty(\mathcal{O}_0 ; \mathbb{R}^{d-1}) \) such that \( \tau(s_\ell) = 0 \) and \( \forall \ z \in \mathcal{O}_0 \),
\[
\nabla \tau_j(z) \cdot \nabla \phi(z) = 0 \, , \, \forall j = 1, \ldots, d-1
\]
\[
\text{rank}\{\nabla \tau_1(z), \ldots, \nabla \tau_{d-1}(z)\} = d-1 . \quad (4.10)
\]

Then \( (x,y) = (x_1, \ldots, x_{d-1}, y) = (\tau_1, \ldots, \tau_{d-1}, \phi) \) are local coordinates in \( \mathcal{O}_0 \) such that
\[
\Delta = |\tilde{g}|^{-1/2} \sum_{1 \leq i,j \leq d-1} \partial_{x_i} \left( |\tilde{g}|^{1/2} \tilde{g}^{ij} \partial_{x_j} \right) + |\tilde{g}|^{-1/2} \partial_y \left( |\tilde{g}|^{1/2} \partial_y \right)
\]
\[
V = y^{2m} \tilde{f}(x,y) \text{ with } \tilde{f} \in C^\infty(\mathcal{V}_0) . \quad (4.11)
\]

\( \mathcal{V}_0 \) is an open neighbourhood of zero in \( \mathbb{R}^d \),
\( \tilde{g}^{ij}(x,y) = \tilde{g}^{ij}(x,y) \in C^\infty(\mathcal{V}_0 ; \mathbb{R}) , \ |\tilde{g}|^{-1} = \text{det} \left( \tilde{g}^{ij}(x,y) \right) > 0 . \)

\( x = (x_1, \ldots, x_{d-1}) \) are local coordinates on \( \Gamma_0 \)
and the metric \( g = (g_{ij}) \) on \( \Gamma_0 \) is given by
\[
(g_{ij}(x))_{1 \leq i,j \leq d-1} = G(x) , \text{ with } (G(x))^{-1} = (\tilde{g}^{ij}(x,0))_{1 \leq i,j \leq d-1} .
\]
If \( w \in C^2_0(O_0) \) then

\[
P^h w = \hat{P}^h u \quad \text{with} \quad u = |\tilde{g}|^{1/4}w \quad \text{and} \quad \hat{P}^h = -h^2 \sum_{1 \leq i,j \leq d-1} \partial_{x_i} (\tilde{g}^{ij} \partial_{x_j}) - h^2 \partial_y^2 + V + h^2 V_0 ,
\]

for some \( V_0 \in C^\infty(\mathbb{R}) \).

Let us write

\[
V(x,y) = y^{2m} f(x) + y^{2m+1} f_1(x) + y^{2m+2} \tilde{f}_2(x, y) : (4.13)
\]

\( f(x) = \tilde{f}(x, 0) \) and \( \tilde{f}_2 \in C^\infty(V_0) \).

We perform the change of variable (2.1) and the related unitary transformation,

\[
(x, y) \rightarrow (x, t) = (x, f^{1/(2(m+1))}(x)y) , \quad u \rightarrow v = f^{-1/(4(m+1))}u ,
\]

to get that

\[
\hat{P}^h u = \hat{Q}^h v \quad \text{with} \quad \hat{Q}^h = Q^h_0 + t^{2m+1} f_1^0(x) + h^2 R_0 + +h^2 t R_1 + t^{2m+2} \tilde{f}_2^0 : (4.14)
\]

\[
Q^h_0 = -h^2 \sum_{1 \leq i,j \leq d-1} \partial_{x_i} (g^{ij} \partial_{x_j}) + f^{1/(m+1)}(x) (-h^2 \partial_t^2 + t^{2m})
\]

and

\[
R_0 = ta(x, t)(\partial_x f(x) \partial_x) \partial_t + b(x, t)t \partial_t + \sum_{i,j} b_{ij}(x, t) \partial_{x_i} f(x) \partial_{x_j} f(x)(t \partial_t)^2 + c(x, t) ,
\]

\[
R_1 = \sum_{1 \leq i,j \leq d-1} \partial_{x_i} \left( \alpha_{ij}(x, t) \partial_{x_j} \right) , \quad \text{all coefficients are regular in a neighbourhood of the zero in } \mathbb{R}^d .
\]

Let \( \mu_j \) be as in the theorem 4.1. We define \( h_j = h^{1/(m+1)} / \mu_j^{1/2} \).

Let \( O'_0 \) be a bounded open neighbourhood of zero in \( \mathbb{R}^{d-1} \) such that \( \overline{O'_0} \subset O_0 \cap \{(x, 0) ; \ x \in \mathbb{R}^{d-1}\} \).

We consider the Dirichlet operator on \( L^2(O'_0) \), \( H^h_0 \) : 

\[
H^h_0 = -h^2 \sum_{1 \leq k, \ell \leq d-1} \partial_{x_k} (g^{k\ell}(x) \partial_{x_\ell}) + f^{1/(m+1)}(x) . \quad (4.15)
\]
It is well known, (see for example [He1] or [He-Sj1]), that for any $\alpha \in \mathbb{N}^{d-1}$ satisfying the assumptions of the theorem 4.1, one has:

$$\exists \lambda_{j,\alpha}^h \in sp \left( H_{0}^{h_j} \right) \text{ s.t. } |\lambda_{j,\alpha}^h - \eta_0^{1/(m+1)} + h_j A_l(\alpha)| \leq h_j^2 C ;$$

$A_l(\alpha)$ is defined in theorem 4.1 in relation with our $s_l \in \Sigma_0$.

$C$ is a constant depending only on $N$. We will denote by $\psi_{j,\alpha}^h(x)$ any associated eigenfunction with a $L^2$-norm equal to 1. Let $\chi_0 \in C^\infty(\mathbb{R})$ such that

$$\chi_0(t) = 1 \text{ if } |t| \leq 1/2 \text{ and } \chi(t) = 0 \text{ if } |t| \geq 1 .$$

We define the following function :

$$u_{j,\alpha}^h(x,t) = h^{-1/(2m+2)} \chi_0(t/\epsilon_0) \psi_{j,\alpha}^h(x) \left[ \varphi_j(h^{-1/(m+1)} t) - h^{1/(m+1)} F_j^h(x,t) \right] ,$$

with

$$F_j^h(x,t) = f_1^0(x)f^{-1/(m+1)}(x) \varphi_j(h^{-1/(m+1)} t) ,$$

where $\varphi_j \in S(\mathbb{R})$ is solution of :

$$-\frac{d^2}{dt^2} \varphi_j(t) + (t^{2m} - \mu_j) \varphi_j(t) = t^{2m+1} \varphi_j(t) ,$$

and $\epsilon_0 \in [0,1]$ is a small enough constant, but independent of $h$ and $j$.

$\varphi_j$ exists because $\mu_j$ is a non-degenerated eigenvalue and the related eigenfunction $\varphi_j$ (see 1.8) verifies $\int_{\mathbb{R}} t^{2m+1} \varphi_j^2(t) dt = 0$, since it is a real even or odd function.

Using the similar estimates as in chapter 3, one can get easily that

$$\mu_j^{-1} ||t \partial_t \varphi_j||_{L^2(\mathbb{R})} + \mu_j^{-2} ||(t \partial_t)^2 \varphi_j||_{L^2(\mathbb{R})} \leq C$$

and $\forall \ k \in \mathbb{N} , \ \exists \ C_k > 0 \ s.t. \ \mu_j^{-k/2m} ||t^k \varphi_j||_{L^2(\mathbb{R})} \leq C_k$.

It is well known that there exists $\epsilon_1 > 0$ s.t.

$$|\mu_j - \mu_\ell| \geq \epsilon_1 , \ \forall \ \ell \neq j ,$$

then the inverse of $\frac{d^2}{dt^2} + t^{2m} - \mu_j$ is $L^2(\mathbb{R})$-bounded by $1/\epsilon_1$, (on the orthogonal of $\varphi_j$). So in the same way as in chapter 3, we get also that

$$\mu_j^{-2-1/2m} ||t \partial_t \varphi_j||_{L^2(\mathbb{R})} + \mu_j^{-3-1/2m} ||(t \partial_t)^2 \varphi_j||_{L^2(\mathbb{R})} \leq C$$

and $\forall \ k \in \mathbb{N} , \ \exists \ C_k > 0 \ s.t. \ \mu_j^{-1-(k+1)/2m} ||t^k \varphi_j||_{L^2(\mathbb{R})} \leq C_k$.

As in the proof of Theorem 3.1 we get easily that

$$||[\hat{Q}^h - \mu_j \lambda_{j,\alpha}^h] \chi_0(|x|/\epsilon_0) u_{j,\alpha}^h(x,t)||_{L^2(O_0)} \leq h^2 \mu_j^{(4m+3)/2m} C$$

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and
\[ | \| \chi_0(|x|/\epsilon_0) u_{j,\alpha}^h(x,t) \|_{L^2(\Omega_0)} - 1 | = O(h^{1/(m+1)} \mu_j^{(2m+1)/2m}) = o(1). \]
So the theorem 4.1 follows easily.

Remark 4.2 If in Theorem 4.1 we assume that \( j \) is also bounded by \( N \), then, as in \[He-Sj4\], we can get a full asymptotic expansion
\[
\lambda_j^h \sim h^{2m/(m+1)} \sum_{k=0}^{+\infty} c_{jk\alpha} h^{k/(m+1)},
\]
and for the related eigenfunction, a quasimode of the form
\[
u_{j,\alpha}^h(x,t) \sim c(h)e^{-\psi(x)/h^{1/(m+1)}} \chi_0(t/\epsilon_0) \sum_{k=0}^{+\infty} h^{k/(2m+2)} a_{jk\alpha}(x) \phi_{jk}(t/h^{1/(m+1)}).
\]

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