MARKED NODAL CURVES WITH VECTOR FIELDS

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Abstract. We discuss two operations on nodal curves with (logarithmic) vector fields, which resemble the ‘stabilization’ construction in Knudsen’s proof that \( \overline{\mathcal{M}}_{g,n+1} \) is the universal curve over \( \overline{\mathcal{M}}_{g,n} \). We prove that both operations work in families (commute with base change). We construct inverse operations under suitable assumptions, which allow us to prove a technical result quite similar to Knudsen’s, in the case of curves with vector fields.

As an application, we prove that the Losev–Manin compactification of the space of configurations of \( n \) points on \( \mathbb{P}^1 \setminus \{0, \infty\} \) modulo scaling degenerates isotrivially to a compactification of the space of configurations of \( n \) points on \( \mathbb{A}^1 \) modulo translation, and the natural group actions fit together globally.

1. Introduction

1.1. Bubbling up in the presence of vector fields. One direction in Knudsen’s proof that \( \overline{\mathcal{M}}_{g,n+1} \) is the universal curve over \( \overline{\mathcal{M}}_{g,n} \) amounts to the procedure called stabilization in [Kn83]. In the case of a single curve, stabilization works as follows. (We are discussing this procedure out of context; perhaps ‘prestabilization’ would be a better name in our setup.) Given a projective nodal curve \( C \) over an algebraically closed field \( K \), with several nonsingular distinct marked points \( w_1, \ldots, w_m \in C(K) \), imagine inserting a new marked point at \( x \in C(K) \). If \( x \) is singular or \( x = w_i \) for some \( i \), then we deem this configuration ‘degenerate’. In this situation, stabilization inserts a new component \( \Sigma \cong \mathbb{P}^1 \) at \( x \) and produces non-degenerate data \( C', w'_1, \ldots, w'_m, x' \), by placing \( x' \) (and \( w'_i \), if \( x = w_i \)) on \( \Sigma \). (In the main body of the paper, we will instead use the notation \( C^\#, w_1^\#, \ldots, w_m^\#, x^\# \), but for now the prime notation is adequate.) Here are two related reasons why stabilization is important:

- it works in families: it can be generalized to families of curves in a manner which commutes with base change (and this commutativity satisfies the suitable ‘cocycle conditions’); and
- iterating the stabilization construction in families suitably intertwined with base changes produces important moduli spaces. For instance, in this way it is possible to obtain \( \overline{\mathcal{M}}_{0,n} \) (or even \( \overline{\mathcal{M}}_{g,n} \) given \( \overline{\mathcal{M}}_g \)), or the Fulton-MacPherson configuration spaces of curves.

In this paper, we will discuss a related (now two-step) procedure when a logarithmic vector field on the nodal curve is given. A logarithmic vector field on a nodal curve is a global section of the dual of the dualizing sheaf, possibly with imposed

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zeroes at some points. The adjective logarithmic will typically be omitted, and we will write just field or vector field.

**Knudsen stabilization with vector fields.** The first step of the procedure operates in the same setup as Knudsen’s stabilization, with the extra data of a vector field \( \phi \) on \( C \), which vanishes at \( w_1, \ldots, w_m \). The meaning of degenerate is the same as in the usual Knudsen stabilization, and fixing it entails inserting a \( \mathbb{P}^1 \) component under the same circumstances. It can be shown that \( \phi \) lifts uniquely to a vector field \( \phi' \) on the curve \( C' \) obtained by Knudsen stabilization, if we require \( \phi'(w'_i) = 0 \) for \( i = 1, \ldots, m \). (This boils down to the fact that, given \( r \in \mathbb{K}^X \), there exists a unique meromorphic 1-form on \( \mathbb{P}^1 \) with poles with residues \( r \) at 0 and \(-r \) at \( \infty \).)

**Inflating at zero vector.** The second operation is applied after the first, so it presumes milder degeneracy to begin with: the point \( x \) is nonsingular and distinct from \( w_1, \ldots, w_m \). We consider such a situation ‘degenerate’ if \( \phi(x) = 0 \). Then we insert a \( \mathbb{P}^1 \) component at \( x \), and fix the degeneracy by placing \( x' \) on \( \mathbb{P}^1 \) and lifting \( \phi \) to a field \( \phi' \) on \( C' \) such that \( \phi'(w'_i) = 0 \), but \( \phi'(x') \neq 0 \).

**Remark 1.1.**
1. Indeed, \( x \) behaves differently from \( w_1, \ldots, w_m \). In applications, we will get around this by thinking of \( x \) as part of a different set of markings \( x_1, \ldots, x_n \).
2. The properties above don’t determine \( \phi' \) uniquely, although they do determine it up to automorphisms of \( C' \) that fix all the rest of the data and the map \( C' \to C \) which contracts the new component, if there is a new component. Still, a canonical choice exists in a certain sense.\(^1\)
3. The new field \( \phi' \) depends nonlinearly on the old field \( \phi \). For instance, \( \phi = 0 \) everywhere is possible, but \( \phi'(x') \neq 0 \) forces \( \phi' \neq 0 \) even in such cases.

**Theorem 1.2.** ‘Knudsen stabilization with vector fields’ and ‘Inflating at zero vector’ work in families: they can be generalized to families in a manner which commutes with base change.

1.2. **Bubbling down and an analogue of Knudsen’s theorem.** Knudsen’s original stabilization admits an inverse operation called contraction. Similarly, we will show that under suitable circumstances, our two operations admit inverse operations (Theorem 4.3), which will allow us to prove a technical result (Theorem 1.4) similar to Knudsen’s theorem that \( \overline{\mathcal{M}}_{g,n+1} \) is the universal curve over \( \overline{\mathcal{M}}_{g,n} \).

We start by introducing some notation.

**Definition 1.3.** Let \( g, m, n \geq 0 \) integers. An object of \( \mathbf{V}^+_{g,m,n} \) consists of
- a (noetherian, cf. Conventions subsection) scheme \( S \);
- a prestable curve \( \pi : C \to S \) of genus \( g \);
- sections \( w_1, \ldots, w_m : S \to C \) and \( x_1, \ldots, x_n : S \to C \) of \( \pi \); and
- an \( \mathcal{O}_C \)-module homomorphism \( \phi : \omega_{C/S} \to \mathcal{O}_C \),

satisfying the following conditions
1. \( \pi \) is smooth at \( w_i(s) \) and \( x_j(s) \), for all \( s \in S \);
2. \( w_i(s) \neq w_j(s) \) if \( i \neq j \), and \( w_i(s) \neq x_j(s) \), for all \( s \in S \);
3. \( w_i^* \phi = 0 \) as homomorphism \( w_i^* \omega_{C/S} \to \mathcal{O}_S \), for all \( i \);

\(^1\)A good analogy is with the following question: in Knudsen stabilization, where do we place \( x' \) on the new \( \mathbb{P}^1 \) component? It seems that all points on \( \mathbb{P}^1 \) except two are equally good, yet, in [Kn83, §2], there is a definite formula for \( x' \).
(4) $x_j^* \phi : x_j^* \omega_{C/S} \to \mathcal{O}_S$ is an isomorphism for all $j$.

We will sometimes write $x$ and $w$ instead of $(x_1, \ldots, x_n)$ and $(w_1, \ldots, w_m)$. Arrows in $V_{g,m,n}^+$ are pullbacks: an arrow $(S', C', \pi', \varphi') \to (S, C, \pi, \varphi)$ is a pair of morphisms $(h : S' \to S, r : C' \to C)$ such that the following diagrams

$$
\begin{array}{ccc}
C' & \xrightarrow{r} & C \\
\downarrow \pi & & \downarrow \pi \\
S' & \xrightarrow{h} & S
\end{array} \quad \begin{array}{ccc}
C' & \xrightarrow{r} & C \\
\downarrow x_j & & \downarrow x_j \\
S' & \xrightarrow{h} & S
\end{array} \quad \begin{array}{ccc}
C' & \xrightarrow{r} & C \\
\downarrow w_i & & \downarrow w_i \\
S' & \xrightarrow{h} & S
\end{array}
$$

are commutative and the first one is cartesian, and $\varphi'$ corresponds to the pullback of $\varphi$ under the isomorphism $C' \to C_{S'}$ induced by the first diagram.

An object of $C_{g,m,n}^+$ is defined verbatim the same as an object of $V_{g,m,n}^+$, with the sole exception that, as part of the data, we consider an additional section $x : S \to C$ of $\pi$. No conditions involve this section in any way. Arrows are pullbacks again (as above, and another commutative diagram saying $rx' = xh$).

Let $V_{g,m,n}$ (respectively $C_{g,m,n}$) be the full subcategory of $V_{g,m,n}^+$ (respectively $C_{g,m,n}^+$) consisting of objects for which $\omega_{C/S} (w_1 + \cdots + w_m + 2x_1 + \cdots + 2x_n)$ is $\pi$-ample (by condition 1, $x_j$ and $w_i$ are Cartier divisors on $C$).

Note that we allow $x_i \cap x_j \neq \emptyset$ in Definition 1.3.

**Theorem 1.4.** If $2g + 2n + m \geq 3$, then the (fibered) categories $V_{g,m,n+1}$ and $C_{g,m,n}$ are equivalent.

1.3. **Application: configurations on a line modulo scaling or translation.**

Consider the following open ended (and seemingly unrelated) problems:

1. compactifying the space of configurations of $n$ not necessarily distinct points on a punctured line modulo scaling; and
2. compactifying the space of configurations of $n$ not necessarily distinct points on a line modulo translation.

Below are two arguably optimal answers to these problems. We first discuss the constructions concretely over an algebraically closed field $K$, and then over $Z$.

**The Losev–Manin space.** An $n$-marked Losev-Manin string is a genus 0 nodal curve $C$ over $K$ whose dual graph is a chain, with $n + 2$ smooth points $p_0, p_\infty, x_1, \ldots, x_n \in C(K)$, such that $p_0$ and $p_\infty$ live on components at opposite ends of the chain, $p_0 \neq p_\infty$ (if there is just one component), $x_i \neq p_0, p_\infty$ for all $i$, and all components contain at least one of the points $x_1, \ldots, x_n$. (However, $x_i = x_j$ is allowed. The concentric circles in Figure 1 represent overlapping markings.) Note that there is a unique $\mathbb{G}_m$-action on $C$ which fixes $p_0$ and $p_\infty$, and acts with weight 1 on each irreducible component.

![Figure 1. A 12-marked Losev-Manin string.](image)
The Losev-Manin space $\mathcal{T}_{n, K}$ is a variety which parametrizes the $n$-marked Losev-Manin strings. It is shown in [LM00] that it is a smooth irreducible toric projective variety of dimension $n-1$. There is an action of $G_m^n$ on $\mathcal{T}_{n, K}$ such that, on $\mathbb{K}$-points,

$$(c_1, \ldots, c_n) \cdot (C, p_0, p_\infty, x_1, \ldots, x_n) = (C, p_0, p_\infty, c_1 \cdot x_1, \ldots, c_n \cdot x_n),$$

where · on the right hand side is the $G_m^n$-action on $C$ above. The action restricts to a trivial one on the diagonal $G_m \hookrightarrow G_m^n$, and thus restricts to the toric $G_m^{n-1}$-action on $\{1\} \times G_m^{n-1} \subset G_m^n$. Moreover, the open stratum $L_{n, K} \simeq G_m^n/G_m \hookrightarrow \mathcal{T}_{n, K}$ where the string has a single component can be thought of as the space of configurations of $n$ points (not necessarily distinct) on $\mathbb{P}^1 \backslash \{0, \infty\}$ modulo scaling.

The construction of the Losev-Manin space in [LM00, §1.3 and §2.1] clearly goes through over Spec $\mathbb{Z}$. From now on, we will typically denote this finite type scheme over $\mathbb{Z}$ by $\mathcal{T}_n$, and the projective variety above is then $\mathcal{T}_{n, K} = \text{Spec } \mathbb{K} \times \text{Spec } \mathbb{Z} \mathcal{T}_n$.

A modular equivariant compactification of $G_a^n/G_a$. A remarkable compactification of the space of configurations of $n$ distinct points on $\mathbb{A}^1$ modulo translation is the moduli space $\mathcal{Q}_n$ of ‘stable scaled marked curves’ constructed as a projective variety by Mau and Woodward in [MW10], after Ziltener discovered it in a symplectic setting [Zi06, Zi14]. The moduli space $\mathcal{Q}_n$ plays an important role in the context of gauged stable maps [Wo15, GSW17, GSW18].

We will construct a related space $\mathcal{P}_n$ which compactifies the space of configurations of $n$ not necessarily distinct points on $\mathbb{A}^1$ modulo translation. This space will turn out to be an equivariant compactification of $G_a^{n-1}$ in the sense of [HT99]. Besides this additional feature and simplicity, Problem 7.1 (discussed at the end of the paper) suggests that the version in which the points may coincide is the ‘right’ object to consider in certain applications.

An $n$-marked $G_a$-rational tree is a connected projective curve $C$ over $\mathbb{K}$ of arithmetic genus 0 with at worst nodal singularities, with a $G_a$-action which operates trivially or ‘by translation’ on each irreducible component of $C$ (i.e. $a \cdot [X : Y] = [X + aY : Y]$ in suitable coordinates), and $n+1$ nonsingular points $p_\infty, x_1, \ldots, x_n \in C(\mathbb{K})$, such that $p_\infty$ is fixed by the $G_a$-action on $C$, but $x_1, \ldots, x_n$ are not. (The condition that $G_a$ acts trivially or by translation on each component is actually automatic in characteristic 0.) The $n$-marked $G_a$-rational tree is stable if any irreducible component of $C$ which doesn’t contain any of the points $x_1, \ldots, x_n$, either intersects at least 3 other irreducible components of $C$, or contains $p_\infty$ and intersects at least 2 other irreducible components of $C$. Note that all marked points $x_1, \ldots, x_n$ live on leaves of the dual tree.

There exists an (irreducible) projective $G_a^{n-1}$-variety $\mathcal{P}_{n, K}$ which parametrizes stable $n$-marked $G_a$-rational trees, cf. Remark 6.11. We use $G$-variety in the sense of [HT99, Definition 2.1]. The $G_a^n$-action on $\mathcal{P}_{n, K}$ comes from the $G_a$-action on $\mathcal{P}_n$.

![Figure 2. A stable 13-marked $G_a$-rational tree.](image-url)
the stable \(n\)-marked \(\mathbb{G}_a\)-rational trees in precisely the same way the \(\mathbb{G}_m^n\)-action on \(\mathcal{T}_{n,K}\) comes from the \(\mathbb{G}_m\)-action on \(n\)-marked Losev-Manin strings. Again, we may restrict to \(\{0\} \times \mathbb{G}_a^{n-1} \subset \mathbb{G}_a^n\) to mimic modding out the trivial diagonal action. Similarly to \(\mathcal{Q}_n\), \(\overline{P}_{n,K}\) is mildly singular for \(n \geq 4\). The open stratum \(\overline{P}_{n,K} \simeq \mathbb{G}_a^n/\mathbb{G}_a \hookrightarrow \overline{P}_{n,K}\) where the rational trees have a single component is the space of configurations of \(n\) points on \(\mathbb{A}^1\) modulo translation.

A similar space can be defined over \(\text{Spec} \mathbb{Z}\), and one of our main goals is to show that this space is a fine moduli space (as opposed to accomplishing just some naive parametization). In this context, the additional structure on the curves will be given by a vector field rather than a \(\mathbb{G}_a\)-action\(^2\).

**Theorem 1.5.** Let \(F\) be the functor which associates to each noetherian scheme \(S\) the set of all collections of data as follows, modulo isomorphism:

- a genus 0 prestable curve \(\pi : C \to S\);
- smooth sections \(x_1, \ldots, x_n, p_\infty : S \to C\) of \(\pi\) (possibly not disjoint); and
- an \(\mathcal{O}_C\)-module homomorphism \(\phi : \omega_{C/S} \to \mathcal{O}_C\),

such that:

1. \(\phi\) factors through the inclusion \(\mathcal{O}_C(2p_\infty(S)) \to \mathcal{O}_C\);
2. \(x_i^*\phi : x_i^*\omega_{C/S} \to \mathcal{O}_S\) is an isomorphism for \(i = 1, \ldots, n\); and
3. the natural stability condition holds: for any geometric point \(\bar{\pi} \to S\),
   
   (a) with the possible exception of the component which contains \(p_\infty, \bar{\pi}\), no irreducible component of \(C_{\bar{\pi}}\) intersects exactly two other components;
   
   (b) any irreducible component of \(C_{\bar{\pi}}\) which intersects exactly one other irreducible component contains at least one of the points \(x_1, \bar{\pi}, \ldots, x_n, \bar{\pi}\) but not the point \(p_\infty, \bar{\pi}\).

Then \(F\) is represented by a projective local complete intersection flat geometrically integral scheme (over \(\text{Spec} \mathbb{Z}\)).

By geometrically integral, we simply mean that all geometric fibers are integral (equivalently, all fibers are geometrically integral in the sense of [SP22, Tag 020H]).

From now on, \(\overline{\mathcal{T}}_n\) will denote the moduli space in Theorem 1.5, and the projective variety above will turn out to be \(\overline{P}_{n,K} = \text{Spec} \mathbb{K} \times_{\text{Spec} \mathbb{Z}} \overline{P}_{n}\).

Our main result is that the two compactification problems discussed above are in fact related (or at least, the proposed answers are): we will show that \(\mathcal{T}_n\) degenerates isotrivially to \(\overline{P}_{n}\), and the actions fit together globally.

First, let’s review the fact that \(\mathbb{G}_m^n\) degenerates isotrivially to \(\mathbb{G}_a\), e.g. [KM78, 3.1]. Consider the commutative cocommutative Hopf \(\mathbb{Z}[t]\)-algebra \(H = \mathbb{Z}[t, x]_{1 + tx}\) with the structure described below, where the maps are always uniquely determined by the property on the right and the requirement that they are \(\mathbb{Z}[t]\)-algebra homomorphisms, and \(x\) is always the element \(x \in H\).

\[
\begin{align*}
(\text{multiplication}) & \quad H \otimes_{\mathbb{Z}[t]} H \to H & x \otimes 1 \mapsto x & \text{and} & 1 \otimes x \mapsto x \\
(\text{comultiplication}) & \quad H \to H \otimes_{\mathbb{Z}[t]} H & x \mapsto x \otimes 1 + 1 \otimes x + tx \otimes x \\
(\text{unit}) & \quad \mathbb{Z}[t] \to H & x \mapsto 0 \\
(\text{counit}) & \quad H \to \mathbb{Z}[t] & x \mapsto -\frac{x}{1 + tx} \\
(\text{antipode}) & \quad H \to H & x \mapsto -x \\
\end{align*}
\]

\(^2\)In fact, it is true that \(\overline{P}_{n}\) is the fine moduli space of stable \(n\)-marked \(\mathbb{G}_a\)-rational trees, though the only proof I was able to put together is unreasonably long and technical, and will be omitted.
The verification of the axioms is a tedious exercise left to the patient reader. Recall that for any ring $R$, $R[y, y^{-1}]$ has a natural $R$-Hopf algebra structure with comultiplication characterized by $y \mapsto y \otimes y$. Note that

$$(1) \quad \mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}[t]} H = \mathbb{Z}[t, t^{-1}, x]_{1+tx} \simeq \mathbb{Z}[t, t^{-1}, y, y^{-1}]$$

as $\mathbb{Z}[t, t^{-1}]$-Hopf algebras. Indeed, the isomorphism is given by $y \mapsto 1 + tx$.

Then $G = \text{Spec } H$ with projection $\gamma : G \to \text{Spec } \mathbb{Z}[t]$ via the unit in $H$ is a flat group scheme over $\text{Spec } \mathbb{Z}[t]$. Note that $G_{\mathbb{Z}[t, t^{-1}]} \simeq \mathbb{G}_m, \mathbb{Z}[t, t^{-1}]$ by (1), and that

$$G_\mathbb{Z} \simeq \begin{cases} \mathbb{G}_m, \mathbb{Z} & \text{if } t \not\in z, \\ \mathbb{G}_a, \mathbb{Z} & \text{if } t \in z, \end{cases}$$

for any geometric point $z \to \text{Spec } \mathbb{Z}[t]$ with the corresponding usual point denoted by $z \in \text{Spec } \mathbb{Z}[t]$. We write $G^k_{\mathbb{Z}[t]} = \mathbb{G} \times G \times \cdots \times G$ for any integer $k \geq 0$.

**Theorem 1.6.** For any positive integer $n$, there exist a flat projective geometrically integral local complete intersection morphism $\xi : X \to \text{Spec } \mathbb{Z}[t]$, and an action of $G^m_{\mathbb{Z}[t]}$ on $X$ over $\text{Spec } \mathbb{Z}[t]$ such that for any geometric point $\xi \to \text{Spec } \mathbb{Z}[t]$ (with the corresponding point denoted by $z \in \text{Spec } \mathbb{Z}[t]$),

- if $t \not\in z$, then $X_{\xi z}$ is isomorphic to $T_{n, z}$, and the induced action of $G^m_{\xi z}$ on $X_{\xi z}$ is isomorphic to the canonical action of $G^m_{\xi, z}$ on $T_{n, z}$;
- if $t \in z$, then $X_{\xi z}$ is isomorphic to $P_{n, z}$, and the induced action of $G^m_{\xi z}$ on $X_{\xi z}$ is isomorphic to the canonical action of $G^m_{\xi, z}$ on $P_{n, z}$.

The case $n = 3$ of Theorem 1.6 is sketched in Figure 3 below.

**Remark 1.7.** (1) In fact, $X \setminus X_{(t)} \simeq \text{Spec } \mathbb{Z}[t, t^{-1}] \times \overline{T}_n$ over $\text{Spec } \mathbb{Z}[t, t^{-1}]$, and the restriction of the $G^m_{\mathbb{Z}[t]}$ action to $X \setminus X_{(t)}$ is the pullback of the action on $\overline{T}_n$ along $X \setminus X_{(t)} \simeq \text{Spec } \mathbb{Z}[t, t^{-1}] \times \overline{T}_n \to \overline{T}_n$.

(2) Although $\overline{T}_{n, C}$ and $\overline{T}_{n, C}$ are homeomorphic only for $n \leq 3$ since $\overline{T}_{n, C}$ is singular for $n \geq 4$. Theorem 1.6 still shows that $\overline{T}_n$ and $\overline{T}_n$ are related topologically. For instance, their Hilbert polynomials relative to suitable polarizations coincide.

(3) The Losev-Manin space $\overline{T}_n$ is the moduli space of weighted pointed stable curves [Ha03] for weight $(1, 1, \epsilon, \ldots, \epsilon)$. However, $\overline{T}_n$ doesn’t resemble any of Hassett’s moduli spaces. I would like to thank Valery Alexeev for raising this issue by email, which prompted me to include it here.

**Example 1.8.** Let $K$ be an algebraically closed field. Recall that $\overline{T}_{3, K}$ is the blowup of $\mathbb{P}^2$ at 3 non-collinear points. It can be checked that $\overline{T}_{3, K}$ is the blowup of $\mathbb{P}^2$ at 3 collinear points. The degeneration is illustrated in Figure 3. The graphs in the top row are strata of $\overline{T}_{3, K}$, the graphs in the bottom row are strata of $\overline{T}_{3, K}$. The vertical arrows $\Sigma \to \sigma_1 + \cdots + \sigma_4$ indicate that the fiber over 0 of the closure relative to $X_K$ (the base of $t_4$) of $\Sigma \times (A^1 \setminus \{0\})$ is set theoretically $\sigma_1 + \cdots + \sigma_4$.

Another application in the style of Theorem 1.6 is discussed very briefly in §5.2.5 and §5.2.6. In §7, we conjecture (Problem 7.1) that this application may be indirectly relevant to the study of curves on abelian surfaces and K3 surfaces. In fact, this ‘conjecture’ was the original motivation for this work.
Conventions. All schemes in this paper, including those in the definitions of moduli functors or fibered categories, are assumed noetherian. In fact, this assumption can be either purged throughout or proved unnecessary a posteriori, but both options require more technical work, which I’ll skip. We write $\mathbb{P}F = \text{Proj} \mathcal{O}_X \text{Sym} \mathcal{F}$, for any coherent $\mathcal{O}_X$-module $\mathcal{F}$. A prestable curve is a proper flat morphism whose geometric fibers are connected curves with at worst nodal singularities, e.g. [BM96, Definition 2.1]. We say that it is of genus $g$ if all geometric fibers have arithmetic genus $g$. If $\pi : C \to S$ is a prestable curve and $x : S \to C$ is a section, we will often abuse notation by writing $x$ instead of $x(S)$ for the scheme-theoretic image of $x$ (sections of separated morphisms are closed immersions [GD60, 5.4.6]).

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2. Rational contractions of prestable curves

Definition 2.1. Let $\pi : X \to S$ and $\rho : Y \to S$ be prestable curves, and $f : X \to Y$ an $S$-morphism. We say that $f$ is a rational contraction if $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism and $R^1 f_* \mathcal{O}_X = 0$, and these hold universally, that is, $f_{S'}^\# : \mathcal{O}_{Y_{S'}} \to f_{S'}* \mathcal{O}_{X_{S'}}$ is an isomorphism and $R^1 f_{S'}* \mathcal{O}_{X_{S'}} = 0$ for all $S' \to S$.

Lemma 2.2. If $\pi : X \to S$ and $\rho : Y \to S$ are prestable curves, and $f : X \to Y$ is an $S$-morphism, then $f$ is a rational contraction if and only if the property in Definition 2.1 holds on the geometric fibers, i.e. $f_{\overline{s}}^\# : \mathcal{O}_{Y_{\overline{s}}} \to f_{\overline{s}}* \mathcal{O}_{X_{\overline{s}}}$ is an isomorphism and $R^1 f_{\overline{s}}* \mathcal{O}_{X_{\overline{s}}} = 0$ for all geometric points $\overline{s} \to S$.

Proof. The ‘only if’ direction is trivial. [SP22, Tag 0E88] reduces the ‘if’ direction to the special case when $S$ is the spectrum of a field. For the lack of a reference, we explain this elementary case in some detail. Let $S = \text{Spec} K$, and $\alpha : X_{\overline{K}} \to X$
and \( \beta : Y_\pi \to Y \) the natural projections. The base change map \( \beta^* R^i f_* O_X \to R^i f_{\pi*}(\alpha^* O_X) = R^i f_{\pi*} O_{X_\pi} \) is an isomorphism for all \( i \) by the cohomology and flat base change theorem. Then \( \beta^* R^1 f_* O_X = 0 \), so \( R^1 f_* O_X = 0 \). Moreover, \( \beta^* f^# : \beta^* O_Y \to \beta^* f_* O_X \) is an isomorphism because it fits in a commutative diagram with \( \beta^* O_Y = O_{Y_{\pi\pi}} f_{\pi\pi}^# \), and \( \beta^* f_* O_X \to f_{\pi*} O_{X_\pi} \), which are all isomorphisms. It follows that \( f^# \) is an isomorphism. \( \square \)

The phrase ‘rational contraction’ is nonstandard, but it is useful to give a name to such frequently occurring objects. In standard language, ‘rational contractions’ are obtained by repeatedly contracting rational tails and rational bridges. Some remarkable properties of rational contractions are proved in [SP22, Tag 0E7B].

Remark 2.3. In the situation of Definition 2.1, the map \( L \to f_* f^* L \) is an isomorphism, for any \( L \in \text{Pic}(Y) \). This is actually true for any morphism \( f \) for which \( f^# \) is an isomorphism.

Lemma 2.4. In the situation of Definition 2.1, let \( Y^\circ \subseteq Y \) consist of all \( y \in Y \) such that \( f^{-1}(y) \) is a singleton set-theoretically, and \( X^\circ = f^{-1}(Y^\circ) \). Then:

1. \( Y^\circ \) is open in \( Y \), and \( X^\circ \) is open in \( X \). (In particular, we endow \( X^\circ \) and \( Y^\circ \) with their natural open subscheme structures.)
2. \( Y \setminus Y^\circ \) is finite over \( S \).
3. The restriction of \( f \) to \( X^\circ \) induces an isomorphism \( X^\circ \to Y^\circ \). Moreover, if \( U \subseteq Y \) is open, then the restriction of \( f \) to \( f^{-1}(U) \) induces an isomorphism \( U \to f^{-1}(U) \) if and only if \( U \subseteq Y^\circ \).
4. The formation of \( Y^\circ \) and \( X^\circ \) commutes with base change.

Proof. The fibers of \( f \) are connected by Zariski’s connectedness theorem [Gr61b, Corollaire (4.3.2)] and hence they are either positive dimensional or singletons set-theoretically, since they are of finite type over a field. Item 1 then follows from Chevalley’s theorem [Gr66, Corollaire (13.1.5)]. Item 4 is set-theoretic and clear, since the set underlying the preimage of an open subscheme is the set-theoretic preimage of the underlying set. Item 4 reduces item 2 to the case when \( S \) is the spectrum of a field, when it is clear – quasi-finiteness suffices, since \( Y \setminus Y^\circ \) is proper over \( S \). For item 3, the general observation is that, if \( f : X \to Y \) is a proper morphism such that \( f^# : O_Y \to f_* O_X \) is an isomorphism, then \( f \) is an isomorphism and only if it is a bijection. Indeed, \( f \) is a continuous closed bijection, hence a homeomorphism, and we are assuming that \( f^# \) is an isomorphism. \( \square \)

The practical use of Lemma 2.4 is that it often allows us to easily check on \( X^\circ \) or \( Y^\circ \) nontrivial statements about \( f \) on \( X \) or \( Y \), and then ‘bootstrap’ to \( X \) or \( Y \), using tricks such as the following.

Lemma 2.5. If \( \pi : X \to S \) is a prestable curve, and \( U \subseteq X \) is an open whose complement \( X \setminus U \) is finite over \( S \), then the restriction map \( \Gamma(V, \mathcal{L}) \to \Gamma(U \cap V, \mathcal{L}) \) is injective for any invertible \( \mathcal{L} \), and any open \( V \subseteq X \).

Proof. It suffices to prove the lemma for \( \mathcal{L} = O_X \). By [SP22, Tag 0B3L], it suffices to prove that all associated points of \( V \) are in \( U \cap V \). If \( S \) is the spectrum of a field, then \( X \) is reduced and hence \( V \) is reduced, so all associated points of \( V \) are generic points of irreducible components of \( X \) by [SP22, Tag 05AL] and the well-known fact that noetherian reduced schemes have no embedded components, and the claim follows in this special case. In the general case, if \( x \) is an associated point of \( V \),
then it is also an associated point of \((\pi|_Y)^{-1}(\pi(x))\) by [SP22, Tag 05DB], which boils down the claim to the special case in the previous sentence, completing the proof. \(\square\)

The main goal of \S 2 is to discuss the ‘logarithmic’ differentials of rational contractions. It turns out that duality provides precisely what is needed, with no need for true logarithmic geometry. The basic properties of the relative dualizing sheaf of a prestable curve are discussed, for instance, in [Kn83, §1].

**Lemma 2.6.** In the situation of Definition 2.1, if \(L\) is an invertible \(\mathcal{O}_Y\)-module, then \(f^!L \simeq \omega_{X/S} \otimes f^*\omega_{Y/S}^\vee \otimes f^*L\) regarded as a complex in degree 0.

**Proof.** We have \(\pi^!\mathcal{O}_S = \omega_{X/S}[1]\) and \(\rho^!\mathcal{O}_S = \omega_{Y/S}[1]\). Then

\[
\omega_{X/S}[1] = \pi^!\mathcal{O}_S = f^!(\rho^!\mathcal{O}_S) \quad \text{by [SP22, Tag 0ATX]}
\]

\[
= f^!(\omega_{Y/S}[1]) = f^!(\omega_{Y/S}[1] \otimes_{\mathcal{O}_Y} L) \\
= Lf^*\omega_{Y/S}[1] \otimes_{\mathcal{O}_X} f^!\mathcal{O}_Y \quad \text{by [SP22, Tag 0A9T]}
\]

(noetherian ensures quasi-compact and quasi-separated, and \(f^!\) is the right adjoint of \(Rf_*\), because \(f\) is proper), so \(f^!\mathcal{O}_Y \simeq \omega_{X/S} \otimes f^*\omega_{Y/S}^\vee\) as a complex in degree 0. However, \(f^!L = Lf^*L \otimes_{\mathcal{O}_X} f^!\mathcal{O}_Y\) by another application of [SP22, Tag 0A9T], and the lemma follows. \(\square\)

By Lemma 2.6, we may think of \(f^!L\) as an invertible sheaf rather than an object of the derived category. Lemma 2.6 also shows that the formation of \(f^!L\) commutes with base changes \(S' \to S\), since the formation of \(\omega_{X/S}\) and \(\omega_{Y/S}\) commutes with base change [SP22, Tag 0E6R].

**Situation 2.7.** Let \(\pi : X \to S\) and \(\rho : Y \to S\) be prestable curves over \(S\), and \(f : X \to Y\) a rational contraction. Let \(x_1, \ldots, x_n : S \to X\) be sections of \(\pi\), and \(y_i = f x_i : S \to Y\) the corresponding sections of \(\rho\), \(i = 1, \ldots, n\). Assume that:

1. \(\pi\) is smooth at \(x_i(s)\) and \(\rho\) is smooth at \(y_i(s)\), for all \(s \in S\), and \(i = 1, \ldots, n\);
2. the sections \(x_1, \ldots, x_n\) respectively \(y_1, \ldots, y_n\) are pairwise disjoint.

In Situation 2.7, \(x_i \subset X\) and \(y_i \subset Y\) (cf. Conventions subsection) are effective Cartier divisors.

**Proposition 2.8.** In Situation 2.7, the following hold.

1. The isomorphism \(f^# : \mathcal{O}_Y \to f_*\mathcal{O}_X\) restricts to an isomorphism

\[
\mathcal{O}_Y(-y_1 - \cdots - y_n) \simeq f_*\mathcal{O}_X(-x_1 - \cdots - x_n).
\]

Moreover, \(R^1 f_*\mathcal{O}_X(-x_1 - \cdots - x_n) = 0\).

2. For any invertible \(\mathcal{O}_S\)-module \(L\), there exists an \(\mathcal{O}_X\)-module homomorphism

\[
\phi : f^*(L(y_1 + \cdots + y_n)) \to (f^!L)(x_1 + \cdots + x_n)
\]

such that for any open \(U \subset Y\) on which \(f\) restricts to an isomorphism \(f^{-1}(U) \simeq U\), \(\phi|_{f^{-1}(U)}\) is the isomorphism \(f^*(L(y_1 + \cdots + y_n))|_{f^{-1}(U)} \simeq (f^!L)(x_1 + \cdots + x_n)|_{f^{-1}(U)}\) induced by \((f|_U)^* = (f|_U)^!\).

**Proof.** Let \(D = \sum_{i=1}^n x_i\) and \(E = \sum_{i=1}^n y_i\). If \(0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \bigoplus_{i=1}^n x_i^*\mathcal{O}_S \to 0\) is pushed forward along \(f\), we obtain a (solid arrow) commutative diagram
Then \( f_*O_X \to f_* \bigoplus_{i=1}^n x_{i,*}O_S \) is surjective as \( O_Y \to \bigoplus_{i=1}^n y_{i,*}O_S \) is surjective. We claim that

\[
R^j f_*O_X(-D) = 0
\]

for all \( j > 0 \). First, \( R^1 f_* x_{i,*}O_S = 0 \) because \( R^1 y_{i,*}O_S = 0 \) (as \( y_i \) is finite) and \( 0 \to R^1 f_* x_{i,*}O_S \to R^1 y_{i,*}O_S \) is the beginning of the five-term sequence of the Grothendieck spectral sequence. Second, in the piece

\[
R^{j-1} f_*O_X \to R^{j-1} f_* \bigoplus_{i=1}^n x_{i,*}O_S \to R^j f_*O_X(-D) \to R^j f_*O_X
\]

of the top row, the first map is surjective (if \( j = 1 \), we’ve shown above that it is surjective; if \( j = 2 \), we’ve shown above that the second term is 0; if \( j \geq 3 \), the second term is trivially 0 [Gr61b, Corollaire (4.2.2)], and the last term is 0 by [Gr61b, Corollaire (4.2.2)] again or Definition 2.1, completing the proof of (4).

In particular, there exists a unique \( O_Y \)-module homomorphism \( \gamma : O_Y(-E) \to f_*O_X(-D) \) that can play the role of the dashed arrow and make the diagram commute, and this \( \gamma \) is an isomorphism, completing the proof of part 1. Combining with (4), we obtain a quasi-isomorphism

\[
R f_*O_X(-D) \simeq O_Y(-E).
\]

To explain the last step in more detail, let \( 0 \to O_X(-D) \to J^0 \to J^1 \to \cdots \) be an injective resolution of \( O_X(-D) \) in \( \text{Qcoh}(X) \). We have \( H^0(f_*J^\bullet) = O_Y(-E) \) and \( H^1(f_*J^\bullet) = 0 \) for \( j \neq 0 \), and hence the map of complexes \( O_Y(-E) \to f_*J^\bullet = \cdots \to f_*J^0 \to f_*J^1 \to \cdots \), with \( O_Y(-E) \) in degree 0, which maps \( O_Y(-E) \cong \ker(f_*J^0 \to f_*J^1) \) to \( f_*J^0 \) in the natural way, is a quasi-isomorphism. Since \( f \) is proper, \( R f_* \) and \( f^! \) are adjoint, so there is a bijection

\[
\text{Hom}_{D+(\text{Qcoh}(Y))}(R f_*O_X(-D), O_Y(-E)) \simeq \text{Hom}_{D+(\text{Qcoh}(X))}(O_X(-D), f^! O_Y(-E)),
\]

and let \( O_X(-D) \to f^! O_Y(-E) \) correspond to (5) under this bijection. Taking \( H^0 \), we may think of this as a map of \( O_X \)-modules, cf. Lemma 2.6. Twisting by \( O_X(D) \otimes f^!(\mathcal{L}(E)) \) on both sides and keeping Lemma 2.6 in mind, we obtain the desired homomorphism \( \phi : f^!(\mathcal{L}(E)) \to (f^!\mathcal{L})(D) \). The fact that \( \phi \) restricts to the ‘obvious’ isomorphism on each open subset \( U \subseteq Y \) on which \( f \) induces an isomorphism \( f^{-1}(U) \simeq U \) follows from the fact that duality behaves naturally relative to restricting to open subschemes.

\[\square\]

**Lemma 2.9.** In Situation 2.7, assume that \( \omega_E^* \omega \gamma_{\bar{\pi}}(y_{l,\bar{\pi}} + \cdots + y_{n,\bar{\pi}}) \simeq \omega_{\bar{\pi}}(x_{1,\bar{\pi}} + \cdots + x_{n,\bar{\pi}}) \) for all geometric points \( \bar{\pi} : \mathbb{P} \to S \). Then there exists a unique homomorphism \( \phi \) as in part 2 of Proposition 2.8, and this \( \phi \) is an isomorphism. Moreover, its formation commutes with base change.
Proof. Let $D = \sum_{i=1}^{n} x_i$ and $E = \sum_{i=1}^{n} y_i$. By Lemma 2.6, we have an isomorphism
\begin{equation}
\text{Hom}(f^*(L(E)), (f^!L)(D)) \simeq \text{Hom}(f^*(\omega_{Y/S}(E)), \omega_{X/S}(D)).
\end{equation}
Let $\mathcal{H}$ be the left hand side of (7), so that $\phi \in \Gamma(X, \mathcal{H})$. We have $\mathcal{H}_{\Sigma} \simeq \mathcal{O}_{X_{\Sigma}}$ for all geometric points $\Sigma \rightarrow S$, by (7) and the assumption, hence $\phi_{\Sigma} \in \Gamma(X_{\Sigma}, \mathcal{H}_{\Sigma})$ is nowhere vanishing since it isn’t identically 0 (for instance, by Lemma 2.4), and the fact that $X_{\Sigma}$ is connected and proper. Then $\phi$ is a nowhere vanishing section of $\mathcal{H}$, thus an isomorphism, proving one of the claims. Uniqueness follows from Lemmas 2.4 and 2.5. Commutativity with respect to base change follows from uniqueness, combined with items 3 and 4 in Lemma 2.4. \hfill \Box

**Definition 2.10.** Let $f : X \rightarrow Y$ be a rational contraction, and let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Then define $f^!\mathcal{L} = (f^!\mathcal{L}')\vee$.

We’ve seen earlier that the formation of $f^!\mathcal{L}'$ commutes with base change, and hence so does the formation of $f^!\mathcal{L}$, since $f^!\mathcal{L}'\vee$ is invertible by Lemma 2.6.

**Proposition 2.11.** In Situation 2.7, there exists a unique homomorphism
\begin{equation}
\psi : f_*((f^!\mathcal{L})(-x_1 - \cdots - x_n)) \rightarrow \mathcal{L}(-y_1 - \cdots - y_n).
\end{equation}
such that, for any open $U \subseteq Y$ above which $f$ restricts to an isomorphism $f^{-1}(U) \simeq U$, $\psi|_U$ is the natural isomorphism induced by $f^!\mathcal{L}|_U \cong f^*\mathcal{L}|_U$.

If $f^*_x \omega_{Y_{\Sigma}}(y_1, \ldots, y_n, \Sigma) \simeq \omega_{X_{\Sigma}}(x_1, \ldots, x_n, \Sigma)$ for all geometric points $\Sigma \rightarrow S$, then $\psi$ is an isomorphism.

Moreover, the formation of $\psi$ is compatible with base change in the following sense: if $h : S' \rightarrow S$ is a morphism, $X', Y', \ldots$ is the pullback along $h$ of the data in Situation 2.7, $m : Y' = S' \times_S Y \rightarrow Y$ is the projection map, and $\psi'$ is the analogue of (8), then the diagram
\[ \begin{array}{ccc}
m^*f_*((f^!\mathcal{L})(-x_1 - \cdots - x_n)) & \xrightarrow{m^*\psi} & m^*\mathcal{L}(-y_1 - \cdots - y_n) \\
\downarrow & & \downarrow \\
f'_*((f'^!\mathcal{L})(-x'_1 - \cdots - x'_n)) & \xrightarrow{\psi'} & L'(-y'_1 - \cdots - y'_n) \end{array} \]
in which the left vertical map comes from the standard base change map $m^*f_*(-) \rightarrow f'_*p^*(-)$ (where $p : X' = S' \times_S X \rightarrow X$ is the projection map) and the remark before the statement of the proposition, is commutative.

Proof. Let $D = \sum_{i=1}^{n} x_i$ and $E = \sum_{i=1}^{n} y_i$, and let $f^*(\mathcal{L}'(E)) \rightarrow (f^!\mathcal{L}'(D))$ obtained by replacing $\mathcal{L}$ with $\mathcal{L}'$ in (3). Dualizing, we obtain $(f^!\mathcal{L})(-D) \rightarrow f^*(\mathcal{L}(-E))$. Pushing the last homomorphism forward along $f$ with $f^* \mathcal{L}(-E)$, we obtain $f_*(f^!\mathcal{L}(-D)) \rightarrow \mathcal{L}(-E)$, which is (8). This proves existence. The isomorphism criterion follows easily from Lemma 2.9 and the construction of $\psi$ above. Both uniqueness and compatibility with base change follow from Lemmas 2.4 and 2.5. Since uniqueness can be regarded as a sort of special case of compatibility with base change, we only explain the latter. Let $Y^0 \subseteq Y$ as in Lemma 2.4. It is clear that the restriction of the square diagram to $m^{-1}(Y^0)$ is commutative. Then the square diagram commutes a fortiori. Indeed, the two images in $\mathcal{L}(-y_1 - \cdots - y_n)$ of any local section of $f'_*(f'^!\mathcal{L})(-x'_1 - \cdots - x'_n)$ above some open $V \subseteq Y$ via the two possible routes restrict to the same local section of $\mathcal{L}(-y'_1 - \cdots - y'_n)$ on $V \cap m^{-1}(Y^0)$, so they must coincide on $V$ by Lemma 2.5, which applies in our situation in light of items 2 and 4 in Lemma 2.4. \hfill \Box
3. Bubbling up

3.1. Knudsen stabilization with sections of line bundles. In §3.1, we generalize to families the operation called ‘Knudsen stabilization with vector fields’ in §1.1. We also allow for sections of arbitrary line bundles instead of vector fields, though this won’t be a major improvement in generality.

**Situation 3.1.** Assume that \( \pi : C \to S \) is a prestable curve, \( x \) is a section of \( \pi \), \( L \) is an invertible \( \mathcal{O}_C \)-module, \( w_1, \ldots, w_m : S \to C \) are disjoint smooth sections of \( \pi \), and \( \sigma \) is a global section of \( L(-w_1 - \cdots - w_m) \).

In §3.1, we consider such data degenerate if \( \pi \) is singular at \( x(s) \) for some \( s \in S \), or \( x(s) = w_i(s) \), for some \( s \in S \) and \( i \in \{1, \ldots, m\} \).

In Situation 3.1, we construct \( S^\# = S, C^\#, \pi^\#, x^\#, L^\#, \sigma^\# \), \( w_1^\#, \ldots, w_m^\# \) such that

1. the new data satisfies the requirements of Situation 3.1;
2. the new data is nondegenerate; and
3. the construction is functorial, i.e. commutes with base change (in a manner which satisfies the natural ‘cocycle conditions’, at least implicitly).

Moreover, the construction will also provide a rational contraction \( f : C^\# \to C \), which satisfies some natural compatibilities: \( fx^\# = x \), \( fw_i^\# = w_i \), and a few others explained below.

If \( L \) and \( \sigma \) weren’t in discussion, this would be nothing else but the famous stabilization procedure of Knudsen [Kn83, §2]. For the reader’s convenience, we very briefly review the construction (but not the proofs) from [Kn83, §2] with our notation, then deal with \( L \) and \( \sigma \).

**Remark 3.2.** We will frequently use the description of the functor of points of projectivizations of coherent modules, as written at the beginning of [Kn83, §2]. This remark serves as an indirect link to the respective property, so, when we say ‘by Remark 3.2’, we actually mean ‘by the remark at the beginning of [Kn83, §2]’.

**Construction 3.3.** In Situation 3.1, let \( \delta : \mathcal{O}_C \to \mathcal{O}_C(w_1 + \cdots + w_m) \oplus \mathcal{I}_{x,C}^\# \) be the homomorphism of \( \mathcal{O}_C \)-modules such that \( \delta(1) = 1 \oplus \iota \), where \( \iota : \mathcal{I}_{x,C} \to \mathcal{O}_C \) is the inclusion. Note that \( \delta \) is injective, and let \( K \) be its cokernel. Then

\[
0 \to \mathcal{O}_C \to \mathcal{O}_C(w_1 + \cdots + w_m) \oplus \mathcal{I}_{x,C}^\# \to K \to 0
\]

is exact. Define

\[
C^\# = \mathbb{P}(K),
\]

and let \( f : C^\# \to C \) be the natural projection, and \( \pi^\# = \pi f \). For the definition of the lifts \( x^\#, w_{1,\#}, \ldots, w_{m,\#} \) of \( x, w_1, \ldots, w_m \), please see [Kn83, §2]. As in [Kn83, §2],

1. the morphism \( \pi^\# : C^\# \to S \) is a prestable curve;
2. for each \( s \in S \), \( x^\#(s) \neq w_{i,\#}(s) \) for all \( i \), and \( \pi^\# \) is smooth at \( x^\#(s) \);
3. \( f \) is a rational contraction cf. Definition 2.1 (by Lemma 2.2 and the functoriality in 5);
4. \( f \) induces an isomorphism \( f^{-1}(U) \simeq U \) if \( U = \pi^{\# s m} \setminus \bigcup_{i=1}^{m} (x \cap w_i) \); and
5. the formation of all new data is functorial.

Let

\[
\mathcal{L}_\# = f^! \mathcal{L},
\]
cf. Definition 2.10. The homomorphism (8) from Proposition 2.11 reads
\[ f_{\ast}(\mathcal{L}(w_{1} - \cdots - w_{m})) \rightarrow L(w_{1} - \cdots - w_{m}) \]
in our case, and in fact it is an isomorphism by Proposition 2.11. Indeed, the
condition on geometric fibers in Proposition 2.11 (or Lemma 2.9) is elementary to
check. Define \( \sigma \) to be the preimage of \( \sigma \) under (12).

Remark 3.4. Here are two remarks on Construction 3.3.

1. The map \( f \) is projective in the sense of [Gr61a, §5.5]. Indeed, \( I^\vee_{x,C} \) is a
finitely generated \( \mathcal{O}_{S} \)-module since the dual of a finitely generated module
over a noetherian ring is finitely generated.

2. The scheme-theoretic vanishing locus of \( x_{\ast}^{\sigma}_{\ast} \) coincides with the scheme-
theoretic vanishing locus of \( x_{\ast}^{\sigma} \). We may argue as follows. Applying
Lemma 2.9 with \( L^\vee \) in the role of \( L \) and dualizing, we obtain an isomor-
phism \( \mathcal{L}(w_{1} - \cdots - w_{m}) \cong f^{\ast}L(w_{1} - \cdots - w_{m}) \) whose adjoint maps
\( \sigma_{\ast} \) to \( \sigma \), by Construction 3.3. Pulling back along \( x_{\ast} \), we obtain an isomor-
phism \( x_{\ast}^{\ast} \mathcal{L}(w_{1} - \cdots - w_{m}) \cong x_{\ast}^{\ast}L(w_{1} - \cdots - w_{m}) \) under which \( x_{\ast}^{\ast} \sigma_{\ast} \)
corresponds to \( x_{\ast}^{\ast} \sigma \).

Proposition 3.5. Construction 3.3 is functorial (commutes with base change).

Proof. For \( C_{\ast}, \pi_{\ast}, f, x_{\ast}, w_{1}, w_{2} \) this is completely analogous to [Kn83, p. 176] (the
key is that \( \mathcal{K} \) is stably reflexive, everything else is clear). For \( L_{\ast} \), this is an easy
consequence of Lemma 2.6, as noted after Definition 2.10 too. For \( \sigma_{\ast} \), this follows
from Lemma 2.9.

When \( S \) is the spectrum of an algebraically closed field, and \( L = \omega_{C}^{\vee} \), the effect
of Construction 3.3 is that stated in §1.1 for ‘Knudsen stabilization with vector
fields’.

Finally, we state an analogue of [Kn83, Lemma 2.5], which will be used later to
construct an inverse of this bubbling up operation, under the suitable circumstances.

Proposition 3.6. Let \( (S, C, \pi, x, w_{1}, \ldots, w_{m}, \sigma) \) and \( (S, Y, \varpi, y, u_{1}, \ldots, u_{m}, \rho) \) sat-
sify the requirements of Situation 3.1, and assume moreover that the second set of
data is nondegenerate, i.e. \( y(s) \neq u_{i}(s) \) for \( i = 1, \ldots, m \) and \( \varpi \) is smooth at \( y(s) \),
for all \( s \in S \). Let \( q : Y \rightarrow C \) be a rational contraction such that:

1. \( x = qy, w_{i} = qw_{i} \), and \( J = q^{\ast}L \);
2. the homomorphism \( q_{\ast}J(-u_{1} - \cdots - u_{m}) \rightarrow L(-w_{1} - \cdots - w_{m}) \) obtained
as a special case of (8) from \( J = q^{\ast}L \) maps \( \rho \) to \( \sigma \);
3. the homomorphism \( q_{\ast}\mathcal{O}_{Y}(y - u_{1} - \cdots - u_{m}) \rightarrow \mathcal{I}_{x,C}^{\vee}(-w_{1} - \cdots - w_{m}) \)
which is analogous to that constructed in the proof of [Kn83, Lemma 2.5] is an iso-
morphism;
4. the positive-dimensional fibers of \( q \) are irreducible, and intersect \( y \).

Then there exists a canonical \( S \)-morphism \( q_{\ast} : Y \rightarrow C_{\ast} \) such that \( q = f_{\ast}q_{\ast} \). Moreover,
if \( q_{\ast} \) is an isomorphism, then it is compatible with all given sections of \( \varpi \) and \( \pi_{\ast} \),
and with \( \sigma_{\ast} \) and \( \rho \).

At least the last condition may be removed at the expense of a longer proof.
Still, Proposition 3.6 seems of no interest in itself, so we’ll take this shortcut.

Proof. The proof of Proposition 3.6 is parallel to the proof of [Kn83, Lemma 2.5].
We merely comment on how to repeat the uses of [Kn83, Corollary 1.5] from the
proof of [Kn83, Lemma 2.5]. The one in the beginning of the proof in loc. cit. can be replaced with Definition 2.1 and part 1 of Proposition 2.8. The one later in the proof still stands, and to this end, we need to know that $O_Y(y)$ satisfies conditions (1) and (2) in [Kn83, Corollary 1.5] relative to $q$. However, this is clear, because assumption 4 implies that all fibers are projective lines over the respective residue fields (the main point is the famous fact that genus 0 curves with rational points are $\mathbb{P}^1$). For the exactness of the top row in the (analogous) diagram we may invoke part 1 of Proposition 2.8 again. Everything involving the vector fields is straightforward, as (12) is an isomorphism. 

3.2. Inflating at nonsingular zero of a section. In §3.2, we generalize the operation ‘inflating at zero vector’ from §1.1 to families (and arbitrary line bundles).

Situation 3.7. Assume that $\pi : C \to S$ is a prestable curve, $x$ is a section of $\pi$, $L$ is an invertible $O_C$-module, and $\sigma$ is a global section of $L$, such that $\pi$ is smooth at $x(s)$, for all $s \in S$.

We deem such data degenerate if $\sigma$ vanishes at $x(s)$ for some $s \in S$ (vanishes at $x(s)$ means $\sigma x(s) \in mL_{x(s)}$, where $m \subset O_{C,x(s)}$ is the maximal ideal).

In Situation 3.7, we will construct $S_2 = S, C_1, \pi_1, x_1, L_1, \sigma_2$ such that:
• they satisfy the respective requirements of Situation 3.7;
• $x_2^*\sigma_2$ is a nowhere vanishing section, i.e. $(S_1, C_2, \ldots)$ is nondegenerate;
• the construction is functorial.

Once again, the construction will also provide a rational contraction $f : C_1 \to C$ which satisfies various compatibilities, similarly to §3.1.

We will first carry out the required calculations in Proposition 3.8 below, and only then return to Situation 3.7 and state the construction.

Proposition 3.8. Let $X$ be a (noetherian) scheme and $\sigma$ a section of a rank 2 locally free $O_X$-module $\mathcal{E}$. Assume that the germ $\sigma_x$ of $\sigma$ at $x$ is nonzero, and not a zero-divisor in the ring $\text{Sym}(\mathcal{E})_x$, for all $x \in X$. Let

$Y = \mathbb{P}^n F$ where $F = \text{Coker}(O_X \xrightarrow{\times \sigma} \mathcal{E})$,

$f : Y \to X$ the projection, and $O_Y(1)$ the twisting sheaf.

(1) (a) $f$ is a projective (in the sense of [Gr61, §5.5]) global lci morphism.

(b) The adjoint $F \to f_*O_Y(1)$ of $f^*\mathcal{F} \to O_Y(1)$ is an isomorphism. Moreover, this isomorphism is compatible with base changes $h : X' \to X$ such that the germ $(h^*\sigma)_{x'}$ is nonzero, and not a zero-divisor in $\text{Sym}(h^*\mathcal{E})_{x'}$, for all $x' \in X'$.

(c) There exists an isomorphism $f'_*O_X \simeq (f^* \text{det } \mathcal{F})(-1)$; in particular, we think of $f'_*O_X$ as a sheaf. Moreover, if $U$ is an open subset of $X$ such that $\sigma$ vanishes at no point of $U$, then $f$ restricts to an isomorphism $V := f^{-1}(U) \simeq U$, and there exists a commutative diagram of isomorphisms

$$(f'_*O_X)|_V \xrightarrow{\sim} O_V \xleftarrow{\sim} ((f^* \text{det } \mathcal{F})(-1))|_V$$

in which the curved arrow is the restriction of the isomorphism above, the left arrow comes from $f'|_V = f^*|_V$, since $f|_V$ is an isomorphism, and the one on the right commutes with any base change $X' \to X$. 

(2) Assume that \( g : X \to T \) is a flat morphism, and that for all morphisms 
\( T' \to T \), the germ \((h^*\sigma)_{x'}\) (where \( h : X' = T' \times_T X \to X \) is the projection) 
is nonzero, and not a zero-divisor in the ring \( \text{Sym}(h^*\mathcal{E})_{x'} \), for all \( x' \in X' \).
Then \( fg \) is flat.

(3) If \( X \) is a prestable curve over the spectrum of an algebraically closed field 
\( K \) and the vanishing locus of \( \sigma \) is a reduced closed subscheme of dimension 0 
contained in the nonsingular locus of \( X \), then \( Y \) is also a prestable curve 
over \( K \), and \( f \) is a rational contraction, cf. Definition 2.1.

Proof. The assumption on the germs of \( \sigma \) implies, in particular, that the map 
\( \mathcal{O}_X \to \mathcal{E} \) which sends 1 to \( \sigma \) is injective, so we have a short exact sequence
\[
0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0. 
\]
Let \( P = \mathbb{P} \mathcal{E} \), and \( p : P \to X \) the projection which exhibits \( P \) as a \( \mathbb{P}^1 \)-bundle over \( X \).
The graded \( \mathcal{O}_X \)-algebras homomorphism \( \text{Sym} \mathcal{E} \to \text{Sym} \mathcal{F} \) induces a morphism over 
\( X \) from an open subset of \( Y \) to \( P \). In fact, since \( \text{Sym}^+ \mathcal{E} \to \text{Sym}^+ \mathcal{F} \) is surjective
(with the notation \( \text{Sym}^+ = \bigoplus_{k>0} \text{Sym}^k \)), it can be checked (e.g. using [Gr61a, (2.8.1)] over open affines on \( X \))
that this open subset is all of \( Y \), and let’s denote the resulting map by \( j : Y \to P \).

Claim 3.9. The morphism \( j \) is a closed immersion. Moreover, if \( \psi \) is the image of 
\( p^*\sigma \) under the map \( p^*\mathcal{E} \to \mathcal{O}_P(1) \), then \( \psi \) is a regular section of \( \mathcal{O}_P(1) \), and \( j(Y) \) is 
the Cartier divisor on \( P \) cut out by \( \psi \).

Proof. We may assume that \( X = \text{Spec} R \) and that \( \mathcal{E} \) is trivial.
Then \( Y = \text{Proj} R[T,S] \), and let \( \sigma \) and \( \psi \) correspond to the linear polynomial 
\( aT + bS \in R[T,S] \).
First, we claim that \( aT + bS \) is a nonzero non-zero-divisor in \( R[T,S] \).
Indeed, the hypothesis implies that this is the case in \( R_p[T,S] \) for all prime ideals \( p \in \text{Spec} R \),
so any \( p \in R[T,S] \) such that \((aT + bS)p(T,S) = 0\) is killed by all homomorphisms 
\( R[T,S] \to R_p[T,S] \), and thus must be equal to 0. Then \( at + b \) and \( a + bs \) are 
nonzero non-zero-divisors in \( R[t] \) and \( R[s] \), so \( \psi \) indeed cuts out a Cartier divisor on \( Y \).
Moreover, by construction, this Cartier divisor is \( \text{Proj} R[T,S]/(aT + bS) \to \text{Proj} R[T,S] \), which is nothing but \( j \). \( \square \)

1a: The \( f = pj \) factorization settles both issues.
1b: Note that \( I_{j(Y),P} = \mathcal{O}_P(-1) \) by Claim 3.9 and \( j_* \mathcal{O}_Y \otimes \mathcal{O}_P(1) = j_* \mathcal{O}_Y(1) \)
by construction. Twisting \( 0 \to I_{j(Y),P} \to \mathcal{O}_P \to j_* \mathcal{O}_Y \to 0 \) by \( \mathcal{O}_P(1) \), we obtain 
the exact sequence
\[
0 \to \mathcal{O}_P \to \mathcal{O}_P(1) \to j_* \mathcal{O}_Y(1) \to 0. 
\]
We have \( R^1p_*\mathcal{O}_P = 0 \), and \( \mathcal{O}_X \to p_*\mathcal{O}_P \) and \( \mathcal{E} \to p_*\mathcal{O}_P(1) \) are isomorphisms 
(the claims are local on \( X \), so they reduce to \( X \) affine and \( \mathcal{E} \) trivial), so we have a commutative diagram
\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_X & \to & \mathcal{E} & \to & \mathcal{F} & \to & 0 \\
\downarrow & \simeq & \downarrow & \simeq & \downarrow & & & \\
0 & \to & p_*\mathcal{O}_P & \to & p_*\mathcal{O}_P(1) & \to & f_*\mathcal{O}_Y(1) & \to & 0 \\
\end{array}
\]
in which the top row is \((13)\), the bottom row is the pushforward of \((14)\) along \( p \), and 
the left and central vertical arrows are isomorphisms. It follows that the right vertical arrow is also an isomorphism, as desired. For compatibility with base change,
we first clarify that the map $h^*f_*\mathcal{O}_Y(1) \to f'_*\mathcal{O}_Y'(1)$ is the usual map discussed in context of ‘cohomology and base change’. It is clear that $h^*\mathcal{F} \cong \mathcal{F}'$, since (13) remains exact after applying $h^*$. The commutativity of the square diagram which expresses that $\mathcal{F} \to f_*\mathcal{O}_Y(1)$ commutes with base change is standard; in particular, $h^*f_*\mathcal{O}_Y(1) \to f'_*\mathcal{O}_Y'$, is a fortiori an isomorphism, since the other three maps in the square are isomorphisms.

1c: Taking determinants in the Euler sequence of $P$, we obtain

$$\omega_{P/X} = p^* \det \mathcal{E} \otimes \mathcal{O}_P(-2).$$

Keeping in mind that $j$ is the immersion of an effective Cartier divisor, we have

$$f!\mathcal{O}_X = j^!p^!\mathcal{O}_X = j^!(\omega_{P/X}[1]) = j^*\omega_{P/X}[1] \otimes j^!\mathcal{O}_P$$

$$= j^*\omega_{P/X}[1] \otimes j^*\mathcal{O}_P(j(Y))[-1] \quad \text{by [SP22, Tag 0B4B]}$$

$$= f^* \det \mathcal{E} \otimes \mathcal{O}_Y(-1) \quad \text{by (15) and } \mathcal{O}_P(j(Y)) \cong \mathcal{O}_P(1),$$

$$= f^* \det \mathcal{F} \otimes \mathcal{O}_Y(-1) \quad \text{since (13) implies } \det \mathcal{F} = \det \mathcal{E},$$
as desired.

For the claim regarding the restriction to $U$, it is alright to assume that $U = X$ since both (15) and (16) obviously commute with restricting to open subschemes of $X$. With this assumption, $\mathcal{F}$ is invertible and isomorphic to $\det \mathcal{E}$, and $Y = \mathbb{P}\mathcal{F} = X$ with $\mathcal{O}_Y(1) = \mathcal{F}$. This provides the isomorphism on the right, and the rest is clear.

2: Clearly, $P \to T$ is flat, because it is a composition of flat morphisms $P \to X \to T$. Recall from Claim 3.9 that the section $\psi$ of $\mathcal{O}_P(1)$ cuts out the effective Cartier divisor $Y$ on $P$. However, the assumption in the statement of 2 implies that the latter remains true after any base change $T' \to T$, so we may conclude by [SP22, Tag 056Y].

3: Indeed, $j(Y) = \Sigma + p^{-1}(\{\sigma = 0\})$ as divisors on $P$ for some section $\Sigma \subset P$ of the $\mathbb{P}^1$-bundle $P \to X$, and the claim follows easily. (Lemma 2.2 is implicitly relied on.)

\[\square\]

Remark 3.10. The condition on the germs of $\sigma$ in Proposition 3.8 is stronger than the condition that $\mathcal{O}_X \xrightarrow{\times} \mathcal{E}$ is injective. For instance, if

$$X = \text{Spec } \mathbb{C}[a, b, c, d]/(ac, ad + bc, bd),$$

$\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X$, and $\sigma = a \oplus b$, then $\mathcal{O}_X \xrightarrow{\times} \mathcal{E}$ is injective, but $(c \oplus d) \cdot \sigma = 0$. Here, $X$ is the union of two affine planes intersecting at a point, with some nonreduced structure at the intersection point.

We can now state the main construction.

Construction 3.11. In Situation 3.7, consider the section $(-\sigma, 1)$ of $\mathcal{L} \oplus \mathcal{O}_C(x)$. (Note the sign!) Because $x$ is a smooth section of $\pi$, it is a Cartier divisor on $C$ and it follows that for all points $z \in C$, the germ $(-\sigma, 1)$ of $(-\sigma, 1)$ is not a zero divisor in $\text{Sym}(\mathcal{L} \oplus \mathcal{O}_C(x))_z$. (The general commutative algebra statement is the following; if $R$ is a commutative ring, $a, b \in R$ such that $b \neq 0$ and $b$ is not a zero divisor in $R$, then $aX + bY$ is not a zero divisor in $R[X, Y]$. We leave this exercise to the reader.) In conclusion, we are in a situation covered by Proposition 3.8, with $C$ in the role of $X$, $\mathcal{L} \oplus \mathcal{O}_C(x)$ in the role of $\mathcal{E}$, and $(-\sigma, 1)$ in the role of $\sigma$.

Let $\gamma : \mathcal{O}_C \to \mathcal{L} \oplus \mathcal{O}_C(x)$ be the homomorphism such that $\gamma(1) = (-\sigma, 1)$. Then

$$0 \to \mathcal{O}_C \xrightarrow{\gamma} \mathcal{L} \oplus \mathcal{O}_C(x) \xrightarrow{\xi} \mathcal{K} \to 0$$
is exact, where $\mathcal{K}$ is the cokernel of $\gamma$. Define
\begin{equation}
C_2 = \mathbb{P}(\mathcal{K}),
\end{equation}
and let $f : C_2 \to C$ be the natural projection, and $\pi_2 = \pi f$. The complex $\mathcal{O}_S \xrightarrow{x^*\gamma} x^*\mathcal{L} \oplus x^*\mathcal{O}_C(x) \to x^*\mathcal{O}_C(x) \to 0$ is exact at $x^*\mathcal{O}_C(x)$, and we get a surjective homomorphism $x^*\mathcal{K} \to x^*\mathcal{O}_C(x)$ by the universal property of cokernels, since $x^*$ is right exact. This map induces a lift
\begin{equation}
x_2 : S \to C_2
\end{equation}
of $x : S \to C$ by Remark 3.2. It is clear that all constructions so far commute with base change.

In particular, part 3 of Proposition 3.8 implies that the geometric fibers of $\pi_2$ are curves with at worst nodal singularities. On the other hand, part 2 of Proposition 3.8 implies that $\pi_2$ is flat, since, once more, our constructions are functorial and the non-zero-divisor condition holds universally just as well as it holds in the given case – simply repeat the first paragraph in this construction after the base change. Thus $\pi_2$ is a prestable curve. Part 3 of Proposition 3.8 again, the functoriality of the construction so far, and Lemma 2.2 imply that $f$ is a rational contraction. Let
\begin{equation}
L_t = f^*\mathcal{L},
\end{equation}
cf. Definition 2.10 and the remark thereafter that $f^*\mathcal{L}$ is an invertible sheaf. Some preliminary calculations are needed before we can define $\sigma_2$. We have
\begin{equation}
f^*\mathcal{L} = f^*\mathcal{L} \otimes f^!\mathcal{O}_C \quad \text{by the analogous property for } f^!
\end{equation}
\begin{equation}
= f^*\mathcal{L} \otimes f^*\mathcal{L}'(-x) \otimes \mathcal{O}_{C_1}(1) \quad \text{by part 1c of Proposition 3.8}
\end{equation}
\begin{equation}
= f^*\mathcal{O}_C(-x) \otimes \mathcal{O}_{C_1}(1)
\end{equation}
\begin{equation}
= \mathcal{O}_{\mathbb{P}(\mathcal{K})}(-x)(1) \quad \text{by a well-known fact},
\end{equation}
and it follows that
\begin{equation}
f_*f^*\mathcal{L} = f_*(f^*\mathcal{O}_C(-x) \otimes \mathcal{O}_{C_1}(1)) \quad \text{by (21)}
\end{equation}
\begin{equation}
= \mathcal{O}_C(-x) \otimes f_*\mathcal{O}_{C_1}(1)
\end{equation}
\begin{equation}
= \mathcal{K}(-x) \quad \text{by part 1b of Proposition 3.8}.
\end{equation}
If we combine the composition $\mathcal{O}_C \xrightarrow{\text{id} \otimes \text{id}} \mathcal{L}(-x) \oplus \mathcal{O}_C \xrightarrow{\kappa \otimes \text{id}} \mathcal{K}(-x)$ with (22), we obtain an $\mathcal{O}_C$-module homomorphism $\mathcal{O}_C \to f_*f^*\mathcal{L} = f_*L_2$. Let
\begin{equation}
\sigma_2 \in \Gamma(C, L_2)
\end{equation}
be the image of $1 \in \Gamma(C, \mathcal{O}_C)$ under the last homomorphism.

**Remark 3.12.** Here are some technical remarks on Construction 3.11.

1. The composition $\mathcal{L}|_{C \setminus x} \xrightarrow{\text{id} \otimes \text{id}} (\mathcal{L} \oplus \mathcal{O}_C(x))|_{C \setminus x} \xrightarrow{\kappa} \mathcal{K}|_{C \setminus x}$ is an isomorphism $\mathcal{L}|_{C \setminus x} \simeq \mathcal{K}|_{C \setminus x}$. The section $1 \in \Gamma(C \setminus x, \mathcal{O}_C)$ is mapped to $\sigma|_{C \setminus x}$ under the composition
   \begin{equation}
   \mathcal{O}_C|_{C \setminus x} = \mathcal{O}_C(x)|_{C \setminus x} \xrightarrow{\text{id} \otimes \text{id}} (\mathcal{L} \oplus \mathcal{O}_C(x))|_{C \setminus x} \to \mathcal{K}|_{C \setminus x} \simeq \mathcal{L}|_{C \setminus x},
   \end{equation}
   which begins to explain the sign convention in Construction 3.11.

2. The map $f$ restricts to an isomorphism $f^{-1}(C \setminus x) \simeq C \setminus x$ since $\mathcal{L}|_{C \setminus x} \simeq \mathcal{K}|_{C \setminus x}$, cf. item 1.
The adjoint of (22) is the homomorphism $f^*\mathcal{K}(-x) \to f^!\mathcal{L}$ which induces (via Remark 3.2) the isomorphism $C^*_q \cong \mathbb{P}\mathcal{K}(t) = \mathbb{P}\mathcal{K}$.

If the data $(S, C, \ldots)$ from Situation 3.7 is nondegenerate, then the output data $(S_q = S, C_q, \ldots)$ is canonically isomorphic to the input data $(S, C, \ldots)$. We also note for future use that (17) takes the form $0 \to \mathcal{O}_C \to \mathcal{L} \oplus \mathcal{O}_C(x) \to \mathcal{L}(x) \to 0$ in this situation.

First, let’s check that Construction 3.11 is functorial.

**Proposition 3.13.** The formation of $C_q$, $\pi_q$, $x_q$, $L_q$, $\sigma_q$, and $f$ is functorial.

**Proof.** The upshot is that the relevant constructions visibly commute with base changes $S' \to S$, with the exception of those involving duality (as customary, we will not check the ‘cycloc conditions’ for compositions $S'' \to S' \to S$ for the isomorphisms which express commutativity with respect to base change). Thus functoriality of $C_q$, $\pi_q$, $f$, $x_q$ is straightforward. The fact that $L_q$ commutes with base change comes from $L_q = f^*\mathcal{L} \otimes f^*\omega_{C/S} \otimes \omega_{C_q/S}$ (by (20), Definition 2.10, and Lemma 2.6) and the fact that the formation of the relative dualizing sheaf of prestable curves commutes with base change, e.g. [SP22, Tag 0E6R].

The functoriality of most ingredients involved in the functoriality of $\sigma_q$ is quite straightforward and left to the reader, with one exception: the application of part 1c of Proposition 3.8. Specifically, given a base change $h : S' \to S$, let $C', \pi', x'_q, L', \sigma'_q$ be the pullback of the data in Situation 3.7 along $h$, if $q : C'_q \to C_q$ is the induced morphism, then we need the diagram

\[
\begin{array}{ccc}
q^*f^!\mathcal{O}_C & \longrightarrow & q^*(f^*\mathcal{L}(x) \otimes \mathcal{O}_{C_q}(1)) \\
\downarrow & & \downarrow \\
f'^*\mathcal{O}_{C'} & \longrightarrow & f'^*\mathcal{L}'(x') \otimes \mathcal{O}_{C'_q}(1)
\end{array}
\]

in which all sheaves are invertible $\mathcal{O}_{C_q}$-modules, and all homomorphisms are isomorphisms (the horizontal ones come from the application of part 1c of Proposition 3.8, the left vertical one comes from $f^!\mathcal{O}_C = \omega_{C_q/S} \otimes f^*\omega_{C/S}$ and the fact that the formation of relative dualizing sheaves commutes with base change) to commute. Luckily, we can circumvent the issue of base change in Grothendieck duality [Co00] using the following trick. The commutativity of the square boils down to a statement of the form that an automorphism (say, $\alpha$) of $q^*f^!\mathcal{O}_C$ is the identity (compose all arrows around the square, two reversed). We have

$Aut(q^*f^!\mathcal{O}_C) \cong \mathcal{O}_{C_q}^\times$ (the sheaf of nowhere vanishing functions on $C_q$)

since $q^*f^!\mathcal{O}_C$ is invertible. However, by the second half of part 1c of Proposition 3.8, it is clear that the restriction of the square diagram above to $(f')^{-1}(C' \setminus x')$ commutes, so the invertible section corresponding to $\alpha$ restricts to 1 on $(f')^{-1}(C' \setminus x')$. It then suffices to check that the restriction map on sections of the structure sheaf of $C'_q$ from global sections to sections over $C'_q \setminus (f')^{-1}(x')$ is injective. Since $f'^!$ is an isomorphism, this boils down to the statement that the restriction map on sections of $\mathcal{O}_{C'}$ from global sections to sections over $C' \setminus x'$ is injective, which follows from Lemma 2.5.

Second, let’s check that $\sigma_q$ is indeed a lift of $\sigma$. 


**Lemma 3.14.** In the situation of Construction 3.11, the homomorphism (8) reads

\[ f_* \mathcal{L}_t = f_* f^* \mathcal{L} \to \mathcal{L}. \]

Then the map \( \Gamma(C_\sharp, \mathcal{L}_\sharp) \to \Gamma(C, \mathcal{L}) \) on global sections maps \( \sigma_\sharp \mapsto \sigma \).

**Proof.** It suffices to check that \( f_* \mathcal{L}_t \to \mathcal{L} \) maps the restriction of \( \sigma_\sharp \) to \( f^{-1}(C \setminus x) \) to the restriction of \( \sigma \) to \( C \setminus x \). Indeed, \( \Gamma(C \setminus x, \mathcal{O}_C) \to \Gamma(C \setminus x, \mathcal{L}) \) is injective by Lemma 2.5, so this suffices. We have \( \mathcal{K}(-x)|_{C \setminus x} \cong \mathcal{L}|_{C \setminus x} \) by item 1 in Remark 3.12 and \( f_* f^* \mathcal{L}|_{C \setminus x} \cong \mathcal{L}|_{C \setminus x} \) by item 2 in Remark 3.12 compatibly with the restriction of (22), that is, the resulting triangle is commutative. To justify this compatibility, it is necessary to revisit (22) and (21), but everything is clear. Then the diagram

\[ O_S \to x^* \mathcal{K}(-x) \to x^* f^* f_* \mathcal{L} \]

in which the curved arrow is the restriction of the homomorphism \( O_C \to f_* f^* \mathcal{L} \) used in Construction 3.11 is obviously commutative. It remains to see where \( 1 \in \Gamma(C \setminus x, \mathcal{O}_C) \) ends up in the diagram above. On one hand, its image in \( \mathcal{L}|_{C \setminus x} \) is \( \sigma|_{C \setminus x} \) by item 1 in Remark 3.12. On the other hand, its image in \( f_* f^* \mathcal{L}|_{C \setminus x} \) is the restriction of \( \sigma_\sharp \) to \( f^{-1}(C \setminus x) \) by Construction 3.11. Then the claim in the beginning of this proof follows, since the diagram is commutative, and \( f_* f^* \mathcal{L} \to \mathcal{L} \) restricts on \( C \setminus x \) to the isomorphism in the diagram, cf. part 2 of Proposition 2.8. \( \Box \)

**Remark 3.15.** If \( (S, C, \ldots) \) satisfies Situation 3.7 with the output of Construction 3.11 denoted \( (S, C_\sharp, \ldots) \) as above, and \( \alpha \in \Gamma(S, \mathcal{O}_S^\alpha) \), then \( (S, C, \pi, x, \mathcal{L}, \pi^* \alpha \cdot \sigma) \) also satisfies Situation 3.7, and the output of Construction 3.11 on this data is \( (S, C_\sharp, \pi_\sharp, x_\sharp, \mathcal{L}_\sharp, \pi_\sharp^* \alpha \cdot \sigma_\sharp) \). This is true in general, but requires yet another technical verification which we’d rather skip. For our needs, it will suffice to note that the claim follows from Lemma 3.14 in the special case when \( C \) is integral.

Finally, let’s check that Construction 3.11 eliminated the degeneracy.

**Lemma 3.16.** In Construction 3.11, \( x_\sharp^* \sigma_\sharp \) is a nowhere vanishing global section of \( x_\sharp^* \mathcal{L}_\sharp^* \).

**Proof.** Let’s first check that \( \mathcal{L}_\sharp = f^* \mathcal{L} \) is trivial along \( x_\sharp \). We have

\[ x_\sharp^* f^! \mathcal{L} = x_\sharp^* \mathcal{O}_{\mathbb{P}^1_{K(-x)}}(1) \quad \text{by \text{(21)}} \]

\[ = x^* \mathcal{O}_C(-x) \otimes x_\sharp^* \mathcal{O}_{\mathbb{P}^1_{K(-x)}}(1) \quad \text{taking a step back in \text{(21)}} \]

\[ = x^* \mathcal{O}_C(-x) \otimes x^* \mathcal{O}_C(x) = \mathcal{O}_S \quad \text{by the definition of \text{x_\sharp}} \]

as desired. Consider the following diagram of \( \mathcal{O}_S \)-modules

\[ \mathcal{O}_S \to x^* \mathcal{K}(-x) \to x_\sharp^* f^* f_* \mathcal{O}_{\mathbb{P}^1_{K(-x)}}(1) \]

\[ \downarrow \text{id}_{\mathcal{O}_S} \]

\[ \mathcal{O}_S \to x_\sharp^* \mathcal{O}_{\mathbb{P}^1_{K(-x)}}(1) \]

\[ \downarrow \text{id}_{\mathcal{O}_S} \]

\[ \mathcal{O}_S \to x_\sharp^* f^! \mathcal{L} \]

\[ \downarrow \text{id}_{\mathcal{O}_S} \]
in which the map \( \mathcal{O}_S \to x^*\mathcal{K}(-x) \) is the composition \( \mathcal{O}_S \to x^*(\mathcal{L}(-x) \oplus \mathcal{O}_C) \to x^*\mathcal{K}(-x) \), the map \( x^*\mathcal{K}(-x) \to \mathcal{O}_S \) is a twist of the map \( x^*\mathcal{K} \to x^*\mathcal{O}_C(x) \) used in Construction 3.11 to define \( x_\sharp \), the map \( x^*\mathcal{K}(-x) \to x_\sharp^*\mathcal{O}_{\mathcal{K}(-x)}(1) \) also comes from the definition of \( x_\sharp \), and the isomorphism \( x^*\mathcal{K}(-x) = x_\sharp^*f^*f_*\mathcal{O}_{\mathcal{K}(-x)}(1) \) comes from \( x_\sharp^*f^* = x^* \) and

\[
f_*\mathcal{O}_{\mathcal{K}(-x)}(1) = f_*((\mathcal{O}_{\mathcal{K}}(1) \otimes f^*\mathcal{O}_C(-x)) = f_*\mathcal{O}_{\mathcal{K}}(1) \otimes \mathcal{O}_C(-x)
\]

similarly to some steps in (22). It is not hard to check that the diagram is commutative.

Consider the image of \( 1 \in \Gamma(S, \mathcal{O}_S) \) in \( \Gamma(S, x_\sharp^*f^*\mathcal{L}) \). On one hand, if we take the upper route in the diagram, we see that this image is \( x_\sharp^*\sigma \) by revisiting Construction 3.11. Indeed, the composition of (25) with the pushforward of (21) along \( f \) is (22), hence the image of 1 in \( \Gamma(S, x_\sharp^*f^*f_*\mathcal{L}) \) is \( x_\sharp^*f^*\sigma = x^*\sigma \), and the claim follows. On the other hand, the lower route is clearly an isomorphism, so the upper route is also an isomorphism. Hence \( x_\sharp^*\sigma \) is nowhere vanishing.

When \( S = \text{Spec} \, \mathbb{K} \) with \( \mathbb{K} \) algebraically closed, \( \mathcal{L} = \omega_C^\vee \), and \( \sigma \) is a vector field (required to vanish at some smooth points \( w_1, \ldots, w_m \neq x \)), the construction has all the features of ‘Inflating at zero vector’ in §1.1. We leave it to the reader to think through the details, but the required work is already done.

At this point, we consider Theorem 1.2 proved.

However, we are not quite done with this section yet. We will prove a technical proposition which contains most of the work needed later to prove that the bubbling down operation constructed in §4 is inverse to bubbling up. The reason we are writing this here is that the calculations use more the details in the construction of bubbling up than those in the construction of bubbling down.

**Proposition 3.17.** Let \((S, \mathcal{C}, \pi, x, \mathcal{L}, \sigma)\) and \((S, \mathcal{Y}, \varpi, y, \mathcal{J}, \rho)\) both satisfy the requirements of Situation 3.7, and assume in addition that \( y^*\rho \) is a nowhere vanishing section of \( y^*\mathcal{J} \). Let \( q : Y \to C \) be a rational contraction such that:

1. \( x = qy \), \( \mathcal{J} = q^*\mathcal{L} \) and \( q_*\mathcal{J} = q_*q^*\mathcal{L} \to \mathcal{L} \), cf. (8), maps \( \rho \mapsto \sigma \);
2. the homomorphism \( q^*(\mathcal{L}^\vee(x)) \to (q^!\mathcal{L}^\vee)(y) \) from Proposition 2.8 is an isomorphism; and
3. \( q \) induces an isomorphism \( q^{-1}(C \setminus x) \simeq C \setminus x \).

Then there exists a canonical morphism \( q_2 : Y \to C_2 \) such that \( q = f q_2 \) and \( x_2 = qx_2 \).

If \( q_2^* \) is an isomorphism, then \( q_2^*\sigma_2 \mapsto \rho \) under the isomorphism \( q_2^*f^!\mathcal{L} \simeq q^!\mathcal{L} \).

**Proof.** By item 1 in Proposition 2.8, we have an isomorphism \( \mathcal{O}_C(-x) \simeq q_*\mathcal{O}_Y(-y) \).

We also have

\[
q_*[(q^!\mathcal{L}(-y))] = q_*q^*\mathcal{L}(-x) = \mathcal{L}(-x)
\]

by assumption 2 and Remark 2.3. We claim that the following diagram

\[
\begin{array}{ccc}
\mathcal{O}_C(-x) & \longrightarrow & \mathcal{L}(-x) \\
\simeq \downarrow & & \vert (26) \\
q_*\mathcal{O}_Y(-y) & \longrightarrow & q_*(q^!\mathcal{L})(-y)
\end{array}
\]
in which the top row is induced by \( \sigma \) and the bottom one by \( \rho \) is commutative. Indeed, the claim amounts to a statement of the form that two elements of \( \text{Hom}(\mathcal{O}_C(-x), \mathcal{L}(-x)) = \Gamma(C, \mathcal{L}) \) coincide. However, it is clear that they coincide over \( C \setminus \{ x \} \) by assumptions 1 and 3 and part 2 of Remark 3.12, so they must coincide on \( C \) by Lemma 2.5.

Since \( (S, Y, \ldots) \) is nondegenerate in the sense of the current section, part 4 of Remark 3.12 gives a short exact sequence \( 0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{J} \oplus \mathcal{O}_Y(y) \rightarrow \mathcal{J}(y) \rightarrow 0 \).

We have a (only solid arrow, for now) diagram with exact rows

\[
0 \rightarrow q_* \mathcal{O}_Y(-y) \rightarrow q_*[q^i(\mathcal{L}(-y))] \oplus q_* \mathcal{O}_Y \rightarrow q_*q^i\mathcal{L} \rightarrow 0,
\]

since \( R^1q_* \mathcal{O}_Y(-y) = 0 \) by item 1 in Proposition 2.8 once more, since \( qy = x \) is a smooth section. We have a (only solid arrow, for now) diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{O}_C(-x) & \rightarrow & \mathcal{L}(-x) \oplus \mathcal{O}_C & \rightarrow & \mathcal{K}(-x) & \rightarrow & 0 \\
0 & \rightarrow & q_* \mathcal{O}_Y(-y) & \rightarrow & q_*[q^i(\mathcal{L}(-y))] \oplus q_* \mathcal{O}_Y & \rightarrow & q_*q^i\mathcal{L} & \rightarrow & 0 \\
\end{array}
\]

The top row is \( (17) \) twisted by \( \mathcal{O}_C(-x) \), the bottom row is \( (27) \). This diagram is commutative, by the commutativity of the first diagram in this proof, and part 1 of Proposition 2.8. A trivial diagram chase shows that there exists a (unique) homomorphism \( \mathcal{K}(-x) \rightarrow q_*q^i\mathcal{L} \) that makes the diagram above commute if assigned as the dashed arrow and moreover, it is an isomorphism \( \mathcal{K}(-x) \cong q_*q^i\mathcal{L} \). We claim that its adjoint

\[
q^*\mathcal{K}(-x) \rightarrow q^i\mathcal{L}
\]

is surjective. Consider the ‘adjoint’ diagram of the 10-term diagram above.

\[
\begin{array}{cccccc}
q^*\mathcal{O}_C(-x) & \rightarrow & q^*\mathcal{L}(-x) \oplus \mathcal{O}_Y & \rightarrow & q^*\mathcal{K}(-x) & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{O}_Y(-y) & \rightarrow & (q^i\mathcal{L})(-y) \oplus \mathcal{O}_Y & \rightarrow & q^i\mathcal{L} & \rightarrow & 0 \\
\end{array}
\]

The central vertical map is still an isomorphism by assumption 2. The adjoint map \( q^*\mathcal{K}(-x) \rightarrow q^i\mathcal{L} \) we are interested in is the right vertical map in the diagram, and it is surjective by the snake lemma. Twisting, we obtain a surjection \( q^*\mathcal{K} \rightarrow q^*\mathcal{O}_C(x) \oplus q^i\mathcal{L} \). By Remark 3.2, this induces an \( S \)-morphism \( q_! : Y \rightarrow C \) such that \( f q_! = q \).

Let’s first check that \( q_!y = x_{2 \ast} \). In light of Remark 3.2, \( q_!y \) corresponds to the surjection \( x^*\mathcal{K} \rightarrow x^*\mathcal{O}_C(x) \oplus y^*q^i\mathcal{L} \). Earlier, \( x_{2 \ast} \) was defined to correspond to the natural map \( x^*\mathcal{K} \rightarrow x^*\mathcal{O}_C(x) \). Since \( y^*\rho \) is nowhere vanishing, the homomorphism \( \mathcal{O}_S \rightarrow y^*\mathcal{J} \) is an isomorphism, and hence so is its twist \( x^*\mathcal{O}_C(x) \rightarrow x^*\mathcal{O}_C(x) \oplus y^*q^i\mathcal{L} \). It remains to check that the triangle with the three homomorphisms among \( x^*\mathcal{K}, x^*\mathcal{O}_C(x), x^*\mathcal{O}_C(x) \oplus y^*q^i\mathcal{L} \) is commutative. To this end, consider the diagram
in which all arrows in the bottom (trapezoidal) face are obtained as pullbacks along $y$ of homomorphisms in the 9-term commutative diagram above, the $x^* \mathcal{K}(-x) \to \mathcal{O}_S$ arrow is a twist of the map $x^* \mathcal{K} \to x^* \mathcal{O}_C(x)$ discussed earlier, the curved arrow is projection to the second factor, and the right vertical arrow is the map discussed earlier which maps $1 \mapsto y^* \rho$. We argue commutativity as follows. First, the top (curved, triangular) face commutes, essentially by construction. Second, the bottom (trapezoidal) face commutes, thanks to the 9-term diagram. Third, the convex hull commutes. Indeed, we may check this separately on the submodule $y^*[(q^i \mathcal{L})(-y)] \oplus \mathcal{O}_S$ and the section $(0,1)$ of $y^*[(q^i \mathcal{L})(-y)] \oplus \mathcal{O}_S$; via either route to $y^* (q^i \mathcal{L})$, the former is mapped to $0$, and the latter to $y^* \rho$. Since $x^* \mathcal{L}(-x) \oplus \mathcal{O}_S \to x^* \mathcal{K}(-x)$ is surjective, the three points above imply that the right triangular face commutes as well. After twisting by $x^* \mathcal{O}_C(x)$, this is precisely what had to be checked.

It remains to check that, if $q_\sharp$ is an isomorphism, then $q_\sharp^* \sigma_\tau \mapsto \rho$ under the isomorphism $q_\sharp^* f^i \mathcal{L} \simeq q^i \mathcal{L}$. This isomorphism can be obtained, for instance, from $q_\sharp^* f^i = q^1 f^i = q^i$. Moreover, we have maps $q^* \mathcal{K}(x) = q_\sharp^* f^i \mathcal{K}(-x) \to q_\sharp^* f^i \mathcal{L}$ coming from (22) and the adjoint property, and $q^* \mathcal{K}(-x) \to q^i \mathcal{L}$, cf. (28). We claim that these 3 homomorphisms fit it a commutative triangle. We will give an indirect argument. The starting point is part 3 in Remark 3.12. Pulling back along $q_\sharp$, we see that $q^* \mathcal{K}(-x) = q_\sharp^* f^i \mathcal{K}(-x) \to q_\sharp^* f^i \mathcal{L}$ corresponds to $q_\sharp : Y \to C_\sharp = \mathbb{P} \mathcal{K}(-x)$ in the sense of Remark 3.2. On the other hand, in view of the same remark, $q_\sharp : Y \to \mathbb{P} \mathcal{K}(-x)$ also corresponds to (28), since (28) is a twist of the morphism defining $q_\sharp$ as a morphism to $\mathbb{P} \mathcal{K}$. It follows that there exists an isomorphism $q_\sharp^* f^i \mathcal{L} \simeq q^i \mathcal{L}$, such that the triangle diagram with this isomorphism and the maps $q^* \mathcal{K}(-x) \to q_\sharp^* f^i \mathcal{L}$ and $q^* \mathcal{K}(-x) \to q^i \mathcal{L}$ commutes. Although there is no obvious formal reason why this isomorphism $q_\sharp^* f^i \mathcal{L} \simeq q^i \mathcal{L}$ coincides with the earlier isomorphism $q^* f^i \mathcal{L} \simeq q^i \mathcal{L}$, this is true, and the reason is the following. Since all geometry becomes trivial over $C \backslash x$, it is straightforward to check that the two isomorphisms above coincide over $q^{-1}(C(x)) \simeq C \backslash x$. However, $\text{Aut}(q^i \mathcal{L}) \simeq \mathcal{O}_Y^\times \text{ and } \Gamma(Y, \mathcal{O}_Y^\times) \to \Gamma(Y \backslash y, \mathcal{O}_Y^\times)$ is injective since $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y \backslash y, \mathcal{O}_Y)$ is injective by Lemma 2.5, so the equalities of the isomorphisms a fortiori extends to all of $Y$. This concludes the proof of the desired commutativity. Let $\tau \in \Gamma(Y, q^* \mathcal{K}(-x))$ be the image of $1 \in \Gamma(Y, \mathcal{O}_Y)$ under the composition $\mathcal{O}_Y \to q^* (\mathcal{L}(-x) \oplus \mathcal{O}_C) \to q^* \mathcal{K}(-x)$. Then the image of $\tau$ in $q_\sharp^* f^i \mathcal{L}$ is $q_\sharp^* \sigma_\tau$ by Construction 3.11, while the image of $\tau$ in $q^i \mathcal{L}$ is $\rho$ by the commutativity of the right square in the 9-term diagram earlier in this proof. Then the commutativity of the triangle above concludes the proof.

4. Bubbling down

4.1. An intermediate class of curves. Recall the classes of curves $V_{g,m,n+1}$ and $C_{g,m,n}$ from Definition 1.3. To prove Theorem 1.4, we introduce a class of curves $C_{g,m,n}^2$ which will serve as an intermediate step between $V_{g,m,n+1}$ and $C_{g,m,n}$.
Theorem 4.3. The functors in (29) restrict to functors
\[ 
\begin{align*}
C^{1+}_{g,m,n} & \to C^{3+}_{g,m,n}, \\
\end{align*}
\]
Indeed, condition 3 in Definition 4.1 may be checked on geometric fibers, and is then elementary.

Remark 4.2. Obviously, \( V_{g,m,n+1} \) is isomorphic to \( C^{3}_{g,m,n} \) and \( C_{g,m,n} \) to \( C^{1}_{g,m,n} \).

Proposition 4.4. Let \( g, m, n, c \geq 0 \) be integers such that \( 2g + 2n + m + c \geq 4 \) and \( c \in \{1, 2\} \). Let \( \mathbb{K} \) be an algebraically closed field, \( (S = \text{Spec } \mathbb{K}, C, \ldots) \) an object of \( C^{c}_{g,m,n}(\mathbb{K}) \) with notation as in Definition 4.1, and
\[ \mathcal{L} = \omega_{C/S}(w_1 + \cdots + w_m + 2x_1 + \cdots + 2x_n + (c-1)x). \]

Then:
\[ \begin{align*}
&\text{(1) } \mathcal{L}^\otimes k \text{ is generated by global sections if } k \geq 2; \\
&\text{(2) } H^1(C, \mathcal{L}^\otimes k) = 0 \text{ if } k \geq 2; \\
&\text{(3) } \mathcal{L}^\otimes k \text{ is normally generated (e.g. } [\text{Kn83, Definition 1.7}]) \text{ if } k \geq 3; \\
&\text{(4) } \mathcal{L}^\otimes k(-p) \text{ is generated by global sections for any smooth } p \in C(\mathbb{K}), \text{ if } k \gg 0. \\
\end{align*}\]

A constant lower bound for \( k \) in item 4 is possible, but we will simply not use it so we leave it to the interested reader. Item 4 in our form is trivial.

4.2. Bubbling down. In §4.2, we construct inverses to the two functors in (30) using the fundamental techniques in [Kn83, BM96]. Artificial as it may be, it is often feasible to treat the two cases simultaneously.
Proof. First, let’s prove the proposition in the case when all components of \( C \) on which \( \phi \) is not identically zero are rational tails. Let \( T_1, \ldots, T_a \) (\( a \geq 0 \)) be these components. Let \( Y \) be the nodal curve obtained by contracting \( T_1, \ldots, T_a \); then we have both a contraction \( C \to Y \), and an immersion \( j : Y \hookrightarrow C \). Note that
\[
\text{(31)} \quad w_1, \ldots, w_m \notin T_1 \cup \cdots \cup T_a
\]
unless \( a = 1 \) and \( C = T_1 \) because \( \phi \) must vanish at the point where \( T_a \) is attached to the rest of the curve, and hence cannot vanish anywhere else on \( T_a \), as \( \deg \omega_C^{\vee}|_{T_a} = 1 \). We assume (31) holds since the alternative is trivial. We have \( x_1, \ldots, x_n \in T_1 \cup \cdots \cup T_a \) since \( \phi \) cannot vanish at \( x_1, \ldots, x_n \). Let \( y_1, \ldots, y_r \) be the following points on \( Y \):

- \( w_1, \ldots, w_m \) (or \( j^{-1}(w_1), \ldots, j^{-1}(w_m) \) to be exceedingly accurate);
- the images of the tails \( T_1, \ldots, T_a \) under the contraction \( C \to Y \); and
- \( x \), but only in case \( c = 2 \) and \( x \notin T_1 \cup \cdots \cup T_a \).

Thus, \( r = m + a + 1 \) if \( c = 2 \) and \( x \notin T_1 \cup \cdots \cup T_a \), and \( r = m + a \) otherwise. Note that \( y_1, \ldots, y_r \) are smooth points of \( Y \), and that \( \omega_Y(y_1 + \cdots + y_r) \simeq j^* \mathcal{L} \). Then \( (Y, y_1, \ldots, y_r) \in \mathcal{M}_{g,r}(\mathbb{K}) \) since \( \mathcal{L} \) is ample by condition 3 in Definition 4.5 and the fact that ample restricts to ample. The analogues of items 1 – 3 above for \( Y, y_1, \ldots, y_r \) hold by [Kn83, Theorem 1.8]. Items 1 – 3 follow inductively, by reattaching the tails \( T_1, \ldots, T_a \) back to \( Y \) one by one. The inductive step is essentially the argument in the last paragraph in the proof of [Kn83, Theorem 1.8].

It remains to consider the case when \( \phi \) is not identically 0 on some component \( \Sigma \) of \( C \) which is not a rational tail. Such cases occur extremely rarely. Recall from \( \S 1.2 \) that if \( \Sigma_1 \neq \Sigma_2 \) are irreducible components of \( C \) such that \( \Sigma_1 \cap \Sigma_2 \neq \emptyset \), \( \phi|_{\Sigma_1} = 0 \), and \( \phi|_{\Sigma_2} \neq 0 \), then \( \Sigma_2 \) is a rational tail intersecting \( \Sigma_1 \). However, this scenario with \( \Sigma_2 = \Sigma \) is ruled out, so the only possible conclusion is that \( \phi \) doesn’t vanish identically on any irreducible component of \( C \). The only nodal curves \( C \) which admit (logarithmic) vector fields which don’t vanish identically on any irreducible component are smooth genus 1 curves, rational chains, and rational cycles. All items are elementary to check in all these cases. \( \square \)

Definition 4.5. Let \( g, m, n, c \geq 0 \) be integers such that \( 2g + 2n + m + c \geq 5 \) and \( c \in \{2, 3\} \). Let \( \mathbf{o} = (S, C, \pi, \overline{w}, \overline{x}, x, \phi) \) be an object of \( \mathcal{C}_{g,m,n}^c \). A bubbling down of \( \mathbf{o} \) is an object \( \mathbf{o}_\flat = (S, C_\flat, \pi_\flat, \overline{w}_\flat, \overline{x}_\flat, x_\flat, \phi_\flat) \) of \( \mathcal{C}_{g,m,n}^{c-1} \), together with a rational contraction (Definition 2.1) \( h : C \to C_\flat \), such that:

1. \( \pi_\flat h = \pi, hx_j = x_{j,\flat} \) for all \( j \), \( hx = x_\flat \), \( h\overline{w}_i = \overline{w}_{i,\flat} \) for all \( i \); and
2. \( \phi_\flat \) is the image of \( \phi \) under \( h_*\omega_C^{\vee} \to \omega_{C_\flat}^{\vee} \), which is a special case of (8), in light of Definition 2.10 and Lemma 2.6.

Remark 4.6. If \( \phi \) and \( \phi_\flat \) are regarded as sections of \( \omega_C^{\vee}(-w_1 - \cdots - w_n) \) and \( \omega_{C_\flat}^{\vee}(-w_1,\flat - \cdots - w_n,\flat) \), condition 2 in Definition 4.5 is equivalent to asking that the homomorphism \( h_*\omega_C^{\vee}(-\sum_{i=1}^n w_i) \to \omega_{C_\flat}^{\vee}(-\sum_{i=1}^n w_{i,\flat}) \), which is also an example (8), maps \( \phi \) to \( \phi_\flat \).

A base change of a bubbling down is a bubbling down by Proposition 2.11. If \( S = \text{Spec} \mathbb{K} \) with \( \mathbb{K} \) algebraically closed, then bubbling down contracts an irreducible component \( \Sigma \subset C \) if and only if
\[
\text{(32)} \quad \deg(\omega_C(w_1 + \cdots + w_n + 2x_1 + \cdots + 2x_n + (c-2)x))|_{\Sigma} = 0.
\]
(Then $\Sigma \simeq \mathbb{P}^1$.) This may happen only if $x \in \Sigma$, so there exists at most one such component. Following [BM96], we call it the component to be contracted.

**Proposition 4.7.** Let $g, m, n, c$ as in Definition 4.5. The bubbling down of any object of $C_{g,m,n}$ exists and is unique up to unique isomorphism in $C_{g,m,n+1}$ compatible with the rational contractions.

**Proof.** We start by proving uniqueness. First, we claim that $U := h^{-1}(C'_\gamma \setminus x_0) = C \setminus h^{-1}(h(x))$ is independent of the choice of bubbling down. This claim can obviously be checked on geometric fibers. In fact, there are even no nontrivial automorphisms of $C'$, in fact, there are even no nontrivial automorphisms of $C'_s$, which in turn relates to [Kn83, 1]. Let $\phi'_s = (S, C'_s, \pi'_s, \omega'_s, \phi'_s)$ be another bubbling down of $o$. By [BM96, Lemma 2.2], there exists an $S$-morphism $r : C_s \rightarrow C'_s$ such that $rh = h'$. It is straightforward to check that $r$ must be an isomorphism on the geometric fibers, and hence simply an isomorphism by flatness (this trick is used frequently in [Kn83]). The compatibility of $r$ with all sections is elementary. In particular, if $U$ is as above, we have a commutative diagram

\[
\begin{array}{ccc}
C'_s \setminus x'_s & \xrightarrow{r} & C \setminus h^{-1}(h(x)) \\
\downarrow & & \downarrow \quad h'
\end{array}
\]

in which the left triangular face consists of isomorphisms only. It remains to check that the isomorphism $r$ is compatible with $\phi_\gamma$ and $\phi'_\gamma$, that is, that $\phi_\gamma$ is mapped to $\phi'_\gamma$ by the isomorphism $\Gamma(C'_s \setminus x'_s, \omega'^\gamma_{C'_s/S'}) \simeq \Gamma(C'_s \setminus x'_s, \omega'^\gamma_{C'_s/S'})$ induced by $r$. The commutative diagram above clearly induces a commutative diagram on the level of vector fields. The map $\Gamma(C'_s, \omega'^\gamma_{C'_s/S'}) \rightarrow \Gamma(C'_s \setminus x'_s, \omega'^\gamma_{C'_s/S'})$ is injective by Lemma 2.5.

The claim then follows by a simple diagram chase. Uniqueness of the isomorphism is equivalent to triviality of all ‘automorphisms’ of any given bubbling down, and, in fact, there are even no nontrivial automorphisms of $C'_s$ compatible with $h$.

To prove existence, we follow the proof of [BM96, Proposition 3.10] extremely closely (which in turn relates to [Kn83, §1]). Let

\[ L = \omega_{C/S}(w_1 + \cdots + w_m + 2x_1 + \cdots + 2x_n + (c-2)x), \]

and define

\[ C_\gamma = \text{Proj}_S S \quad \text{where} \quad S = \bigoplus_{k \geq 0} \pi_* \mathcal{L}^\otimes k, \]
with structure map \( \pi_3 : C'_g \to S \).

**Claim 4.8.** If \( S = \text{Spec } \mathbb{K} \), where \( \mathbb{K} \) is an algebraically closed field, then:

1. \( \mathcal{L}^\otimes k \) is generated by global sections if \( k \geq 2 \);
2. \( H^1(C, \mathcal{L}^\otimes k) = 0 \) if \( k \geq 2 \);
3. \( \mathcal{L}^\otimes k \) is normally generated if \( k \geq 3 \);
4. \( \mathcal{L}^\otimes k \) is not in a component to be contracted, if \( k \gg 0 \).

**Proof.** Apply Proposition 4.4 to \( \mathcal{C}' = \mathcal{C}_g \), etc. (we can talk about \( \mathcal{C}_g \) since we’re in the \( S = \text{Spec } \mathbb{K} \) case); note that the line bundle from Proposition 4.4 is

\[
\mathcal{L}' = \omega_{C_3}(w_1, \ldots + w_{m,b} + 2x_{1,b} + \cdots + 2x_{n,b} + (c - 2)x_3).
\]

Then the Claim follows from this in the same way [Kn83, Corollary 1.10] follows from [Kn83, Theorem 1.8] using [Kn83, Lemma 1.6]. Please see also Proposition 3.9 and Lemmas 3.11 and 3.12 in [BM96]. \( \square \)

By Claim 4.8, we may simply imitate the arguments in Claims 1–5 inside the proof of [BM96, Proposition 3.10], and obtain the following:

1. the formation of \( \mathcal{C}_g \) commutes with base change;
2. \( \pi_3 \) is flat and projective;
3. using the language of [Gr61a, (3.7.1)], the ‘morphism associated to \( \mathcal{L} \) and \( \pi^* S \to \bigoplus_{k \geq 0} \mathcal{L}^\otimes k \)’ is defined everywhere, proper and surjective – we denote it by \( h \colon \mathcal{C} \to \mathcal{C}_g \);
4. \( h^\#: \mathcal{O}_{\mathcal{C}_g} \to h_* \mathcal{O}_C \) is an isomorphism.

Thus, to check that \( \pi_3 : \mathcal{C}_g \to S \) is a prestable curve, it suffices to check that the singularities of \( \mathcal{C}_g \) are at worst nodal in the case when \( S = \text{Spec } \mathbb{K} \), with \( \mathbb{K} \) algebraically closed. Indeed, in this case, if \( h' : C \to \mathcal{C}_g \) is the contraction of the component to be contracted to a nodal curve \( C'_g \) (constructed by elementary means), then \( h'_* \mathcal{L} \) is invertible and ample, \( h'_* (\mathcal{L}^\otimes k) \simeq (h'_* \mathcal{L})^\otimes k \), and

\[
\mathcal{C}_g = \text{Proj } \bigoplus_{k \geq 0} \Gamma(C, \mathcal{L}^\otimes k) \to \text{Proj } \bigoplus_{k \geq 0} \Gamma(C'_g, (h'_* \mathcal{L})^\otimes k) = \mathcal{C}'_g
\]

by [Gr61a, Proposition (4.6.3)], since \( h'_* \mathcal{L} \) is ample. On the other hand, with the argument in the proof of [BM96, Claim 5 inside the proof of Proposition 3.10] in mind, [Kn83, Corollary 1.5] gives \( R^1 h_* \mathcal{O}_C = 0 \), so, together with points 1 and 4 above, we see that \( h \) is a rational contraction. It is straightforward to construct the rest of the data in the bubbling down. For the sections of \( \pi_3 \), we compose with \( h \), i.e. \( x_3 = hx \), etc., then we simply define \( \phi_3 \) to be the image required by Definition 4.5. It remains to check that the data defines indeed an object of \( \mathcal{C}^{g,m,n}_{g,m,n} \), i.e. that \( \omega_{C_g}(w_1, \ldots + w_{m,b} + 2x_1, + \cdots + 2x_{n,b} + (c - 2)x_3) \) is \( \pi_3 \)-ample, and, if \( c = 3 \), that \( \pi_3 \) is smooth at \( x_3 \) and \( w_{i,b} \cap x_3 = \emptyset \) for all \( i \). Both can be checked on geometric fibers. \( \square \)

A more abstract (but not necessarily shorter) proof of Proposition 4.7 using the results of [SP22, Tag 0E7B] is also possible.

If \( \mathfrak{o} \) is an object of \( \mathcal{C}^{g,m,n}_{g,m,n} \), and \( \mathfrak{o}_g \) comes from the application of the suitable functor in (30), then \( \mathfrak{o} \) together with the morphism \( f : C_g \to C \) that is also provided by Constructions 3.3 or 3.11, satisfies all the requirements of Definition 4.5. Note that besides minor verifications, this also relies on the quite nontrivial Lemma 3.14.
Corollary 4.9. Let $g, m, n, c \geq 0$ be integers such that $2g + 2n + m + c \geq 4$ and $c \in \{1, 2\}$. The composition $\sharp \circ \sharp$ is naturally isomorphic to the identity on $\mathbb{C}^c_{g,m,n}$, where $\sharp$ is as in (30).

Proving that $\sharp \circ \sharp$ is the identity is harder, but the work has already been done.

Proposition 4.10. Let $g, m, n, c \geq 0$ be integers such that $2g + 2n + m + c \geq 5$ and $c \in \{2, 3\}$. The composition $\sharp \circ \sharp$ is naturally isomorphic to the identity on $\mathbb{C}^c_{g,m,n}$, where $\sharp$ is as in (30).

Proof. We apply Proposition 3.6 if $c = 2$, respectively Proposition 3.17 if $c = 3$, with the current $(\mathbb{C}^c_{\cdot,\cdot,\cdot},\cdot)$, $(\mathbb{C},\cdot,\cdot,\cdot)$, and $h$ in the roles of $(\mathbb{C},\cdot,\cdot,\cdot), (Y,\cdot,\cdot,\cdot)$, and $q$ from the respective propositions. Let’s inspect the conditions in these propositions. As in the proof of [Kn83, Lemma 2.5], condition 3 in Proposition 3.6 may be checked on geometric fibers, and the verification in this case is still left to the reader. Condition 2 in Proposition 3.17 follows from Lemma 2.9. The other conditions are either clear, or elementary to check on geometric fibers. The propositions may thus be applied, and, in fact, it can be checked on geometric fibers in either case that the map denoted $q_\sharp$ is an isomorphism. Then the final sentences in Proposition 3.6 or Proposition 3.17 complete the proof. $\square$

Corollary 4.9 and Proposition 4.10 complete the proof of Theorem 4.3, and hence also of Theorem 1.4.

5. Constructing moduli spaces

5.1. Coresidues. We now turn our attention towards Theorems 1.5 and 1.6.

Definition 5.1. Let $o = (S, C, \ldots)$ be an object of $V_{0,1,n}^+$ or $C_{0,1,n}^+$ (with notation as in Definition 1.3), and regard $\phi$ as a homomorphism $\omega_{C/S} \rightarrow O_S(-w_1)$. The element of $\text{End}_{O_S}(w^*_1\mathcal{O}_C(-w_1)) = \Gamma(O_S)$ defined as the composition (note the sign!)

$$w^*_1\mathcal{O}_C(-w_1) \cong w^*_1\omega_{C/S} \cong w^*_1\omega_{C/S} \xrightarrow{\omega_{C/S}} w^*_1\mathcal{O}_C(-w_1)$$

induces a map $S \rightarrow \text{Spec } \mathbb{Z}[t]$, called the negative coresidue (NCR) morphism of $o$.

Note that the NCR morphism is functorial in the natural sense.

Theorem 5.2. The category $V_{0,1,n}$ admits a terminal object. The NCR morphism of the terminal object is a flat, projective, local complete intersection (lci), geometrically integral morphism of relative dimension $n - 1$ to $\text{Spec } \mathbb{Z}[t]$.

The base of the terminal object in Theorem 5.2 will turn out to be precisely the space of the degeneration of $\overline{L}_n$ to $\overline{P}_n$ from Theorem 1.6. Theorem 5.2 will be proved towards the end of §5.3.

5.2. Iterating bubbling up: examples. Next, we explain how to obtain interesting spaces by iterating Constructions 3.3 and 3.11.

Construction 5.3. Given an object $v_{n_0}$ of $V_{g,m,n}$ (referred to as the initial data), we construct inductively a sequence of objects as follows:
We refer to $v_n$ as the $n$th object, to its base as the $n$th space, etc.

If any of the objects in the sequence belongs to $C^c_{g,m,n}$ or $V_{g,m,n}$ (that is, we can drop the ‘+’), then we can always drop the ‘+’ thereafter. In practice, the ‘+’ is rarely necessary.

Here are some examples of what can be obtained depending on the choice of the initial data. As agreed in §1.3, $\mathcal{L}_n$ denotes the ‘Losev-Manin space over Speck Z’, that is, the space obtained by repeating the construction in [LM00, 1.3 and 2.1] over Speck Z. The first two examples below are artificial, but they are meant to prepare the more interesting examples that follow.

5.2.1. The Losev–Manin space: first artificial construction. Run Construction 5.3 with the following initial data in $V_{0,2,1}$:

$S = \text{Spec } \mathbb{Z}, \quad C = \mathbb{P}^1_\mathbb{Z}, \quad w_1 = [1 : 0], \quad w_2 = [0 : 1], \quad x_1 = [1 : 1], \quad \phi = x \frac{d}{dx},$

in the chart $Y \neq 0$, where $x = X/Y$ and $[X : Y]$ are the projective coordinates on $\mathbb{P}^1_\mathbb{Z}$. The resulting $n$th space is $\mathcal{L}_n$. This is clear because, at all steps, the vector field will only vanish at $w_1$ and $w_2$, so the application of Construction 3.11 (that is, of step 3 in Construction 5.3) will never have any effect by (4) in Remark 3.12. Thus we recover the inductive construction of $\mathcal{L}_n$.

5.2.2. The Losev–Manin space: second artificial construction. Run Construction 5.3 with the following initial data in $V_{0,1,1}$:

$S = \text{Spec } \mathbb{Z}, \quad C = \mathbb{P}^1_\mathbb{Z}, \quad w_1 = [1 : 0], \quad x_1 = [1 : 1], \quad \phi = x \frac{d}{dx},$

with notation and conventions as in 5.2.1. We claim that by applying Construction 5.3, the resulting $n$th space will once again be $\mathcal{L}_n$. We briefly argue this inductively, comparing with 5.2.1. Assume inductively that this is the case for a given $n$, and that the object denoted by $c_{n,1}$ in Construction 5.3 has base $\mathcal{L}_{n+1}$ and prestable curve $\mathcal{L}_{n+1} \times^{\mathcal{L}_n} \mathcal{L}_{n+1} \to \mathcal{L}_{n+1}$, just like in the situation of 5.2.1. The only difference between these objects is that, in 5.2.2, the marking $w_2$ is not given, and that point is just a nameless isolated vanishing point of the vector field, the only one besides the marking $w_1$. We claim that after applying Constructions 3.3 and 3.11 we will obtain essentially the same objects, again with the sole difference that $w_2$ is not considered marked in 5.2.2 (though after applying just Construction 3.3, the curves are truly different). Indeed, this may be checked on the open subset of the base where $x \neq w_1$ and $x$ is smooth, since both Construction 3.3 and Construction 3.11 agree in 5.2.1 and 5.2.2 over a neighbourhood of the complement, such as the open subset of the base where $x \neq w_2$. However, on the former open subset, Construction 3.11 is trivial in 5.2.1 and Construction 3.3 is trivial in 5.2.2, so we just need to compare Construction 3.3 in 5.2.1 with Construction 3.11 in 5.2.2. It is easy to check that the respective cokernels to be projectivized, cf. (9) and (17), agree up to a twist by an invertible sheaf, hence their projectivizations coincide. The other
details are straightforward and skipped. In Figure 4, 2 and 3 refer to the items in Construction 5.3, i.e. Constructions 3.3 and 3.11.

5.2.3. Compactifying the space of configurations modulo translation. Run Construction 5.3 with the following initial data in $V_{0,1,1}$:

$$S = \text{Spec } \mathbb{Z}, \quad C = \mathbb{P}^1_{\mathbb{Z}}, \quad w_1 = [1 : 0], \quad x_1 = [0 : 1], \quad \phi = \frac{d}{dx},$$

with the usual notation (so the vector field vanishes doubly at $w_1$, but nowhere else). We will prove later that the resulting $n$th space is the moduli space $\mathcal{P}_n$ from §1.3. For now, we take this explicit construction to be the definition of $\mathcal{P}_n$, and we will prove later that it represents the functor in Theorem 1.5.

5.2.4. The first glimpse of the degeneration in Theorem 1.6. Run Construction 5.3 with the following initial data in $V_{0,1,1}$:

$$(33) \quad S = \text{Spec } \mathbb{Z}[t], \quad C = \mathbb{P}^1_{\mathbb{Z}[t]}, \quad w_1 = [1 : 0], \quad x_1 = [0 : 1], \quad \phi = (1 + tx) \frac{\partial}{\partial x},$$

with the usual notation, and $\pi$ is the projection $\mathbb{P}^1_{\mathbb{Z}[t]} \to \text{Spec } \mathbb{Z}[t]$. Throughout the application of Construction 5.3, the bases will retain the morphisms to Spec $\mathbb{Z}[t]$.

**Remark 5.4.** All objects that occur throughout the application of Construction 5.3 in this situation have the property that their base (what is usually denoted by $S$) and their curve (what is usually denoted by $C$) are integral. This is straightforward to check inductively. Flatness of the curve over the base guarantees that all generic points of the curve map to generic points of the base. On the other, there exists a nonempty open $U \subset S$ such that $\pi^{-1}(U) \simeq \mathbb{P}^1_U$ over $U$, and Cohen-Macaulay-ness rules out embedded components.

Construction 5.3 commutes with base change, because Constructions 3.3 and 3.11 do, cf. Propositions 3.5 and 3.13. Moreover, it also commutes multiplying the vector field with the pullback of an invertible regular function on the base. We haven’t quite proved this in general, but in our situation it suffices to rely on the justified part of Remark 3.15, thanks to Remark 5.4. Then we can conclude the following regarding our current construction:

1. The fibers over $(t) \in \text{Spec } \mathbb{Z}[t]$ mimic 5.2.3.
(2) After the change of coordinates $X' = tX + Y, Y' = Y$ on $\mathbb{P}^1_{\mathbb{Z}[t, t^{-1}]}$, the preimages of Spec $\mathbb{Z}[t, t^{-1}] \subset$ Spec $\mathbb{Z}[t]$ mimic the direct product of 5.2.2 with Spec $\mathbb{Z}[t, t^{-1}]$, with the sole exception that $\phi$ is multiplied by $t$. (In the new coordinates, $\phi = tx'\frac{\partial}{\partial x'}$ where $x' = X'/Y'$.)

This hints at Theorem 1.6. Indeed, we will see that the $n$th space obtained this way is $X$ from Theorem 1.6. We will return to this example in §5.3.

The next two examples will not be developed fully, but are too interesting to not mention at least.

5.2.5. An example in arbitrary genus. Let $\mathbb{K}$ be an algebraically closed field, and $Y$ a smooth, projective, connected curve over $\mathbb{K}$ of genus $g \geq 1$. Run Construction 5.3 with the following initial data in $V^+_{g, 0, 0}$:

$S = $ Spec $\mathbb{K}$, $C = Y$, $\phi = 0$,

and denote the resulting $n$-th space by $Y\{n\}$. (Note that $Y\{n\}$ is reducible for $n \geq 1$.) Aesthetically, $Y\{n\}$ is some sort of cross between [Wo15, Example 4.2.(e)] and our $P_{n, \mathbb{K}}$. An interpretation of $Y\{n\}$ as a moduli space, very similar to $\mathbb{P}_{n, \mathbb{K}}$ in Theorem 1.5, is possible and can be established using very similar methods. The interested reader should find this exercise slightly tedious, but extremely straightforward after reading all of §5. (There exists a rational contraction $\psi : C \to S \times Y$; when adapting the ‘♭’ functors in §4.2, use [BM96, Lemma 2.2] to obtain $\psi_\ast$.)

![Figure 5. A typical object parametrized by $Y\{n\}$.

5.2.6. A special feature of 5.2.5 in genus 1. Let $\mathbb{K}$ be an algebraically closed field, and $E$ a smooth genus 1 curve over $\mathbb{K}$ with a nonzero vector field denoted rather abusively by $\frac{\partial}{\partial z}$. Run Construction 5.3 with the following initial data in $V^+_{1, 0, 0}$:

$S = $ Spec $\mathbb{K}[t] = \mathbb{A}^1$, $C = \mathbb{A}^1 \times E$, $\phi = t\frac{\partial}{\partial z}$,

and denote the resulting $n$-th space by $W$. As in 5.2.4, the bases retain projective flat morphisms to $\mathbb{A}^1$ throughout the application of Construction 5.3. Then:

- $W_0 \simeq E\{n\}$, cf. 5.2.5.
- $W \setminus W_0 \simeq (\mathbb{A}^1\setminus\{0\}) \times E^n$ over $\mathbb{A}^1\setminus\{0\}$, parametrizing simply $n$-tuples of points on $E$ together with a nonzero vector field on $E$.

The reason for the second point is that, in the complement of the fiber over 0, the applications of Constructions 3.3 and 3.11 inside Construction 5.3 are always trivial – the bubbling up will simply never get off the ground. Thus $E\{n\}$ deforms isotrivially to $E^n$! We will explain in §7 why this might be important.
5.3. A closer look at 5.2.4. Next, we revisit and analyze in depth the construction in 5.2.4.

**Definition 5.5.** Let \( t_n \) (respectively \( t_{n,c} \)) be the objects of \( V_{0,1,n} \) (respectively \( C_{0,1,n}^c \)) constructed in 5.2.4. Let \( \eta_n \) (respectively \( \eta_{n,c} \)) be the morphism from the base of \( t_n \) (respectively \( t_{n,c} \)) to \( \text{Spec } \mathbb{Z}[t] \) constructed in 5.2.4.

We will prove that \( t_n, t_{n,c} \) are terminal objects.

For now, we state some geometric properties that follow from our construction. In the statement and proof of Proposition 5.6 below, we agree to write \( S[\mathfrak{o}], C[\mathfrak{o}] \), etc. for the base, curve, etc. of an object \( \mathfrak{o} \) when there is possibility of confusion, respecting the letters used in Definitions 1.3 and 4.1.

**Proposition 5.6.** (1) Let \( t_* \) be either \( t_n \) or \( t_{n,c} \). Then the schemes \( S[t_*] \) and \( C[t_*] \) are normal and integral, as well as geometrically integral, separated, flat, of finite type, and lci over both \( \text{Spec } \mathbb{Z}[t] \) and \( \text{Spec } \mathbb{Z} \), and projective over \( \text{Spec } \mathbb{Z}[t] \). The geometric generic fiber of \( \pi[t_*] \) is integral.

(2) (a) \( x[t_{n,1}] = w_1[t_{n,1}] \) and \( x[t_{n,1}] \cap w_1[t_{n,1}] \) is integral.

(b) The scheme-theoretic vanishing locus of \( x[t_{n,2}] + \mathfrak{o}[t_{n,2}] \) (viewed as a global section of \( x[t_{n,2}]^n \omega_C[t_{n,2}]/S[t_{n,2}] \)) is an integral (prime) Cartier divisor on \( S[t_{n,2}] \).

(3) The morphism \( \eta_{n,c} \) from Definition 5.5 is the NCR morphism of \( t_{n,c} \), cf. Definition 5.1, and a similar statement holds for \( \eta_n \).

**Proof.**

1: The claims that the bases and curves are integral has already been established in Remark 5.4. The other claims concerning (geometric) integrality follow using the same techniques. The flatness, projectivity, lci-ness, and normality claims can be proved comfortably by induction, following through the construction in 5.2.4, though we quote the relevant results for the reader’s convenience.

**Projective:** Since projective in the sense of [Gr61a] is the same as projective in the sense of [Ha77] if the base is itself quasi-projective over an affine scheme [Ha77, p. 103], and composition of Hartshorne-projective morphisms is Hartshorne-projective, it follows inductively from these observations and Remark 3.4, Constructions 3.3 and 3.11, and item 1a in Proposition 3.8 that ‘everything in sight’ is Hartshorne-projective over \( \text{Spec } \mathbb{Z}[t] \).

**Local complete intersection:** Prestable curves are lci over their base [Ol16, Corollary 13.2.7], compositions of lci morphisms are lci [SP22, Tag 069J], and flat base changes of lci morphisms are lci [SP22, Tag 069I].

**Normal:** We need to check the \( R_1 \) and \( S_2 \) conditions. First, \( S_2 \) is automatic: both the bases and the curves are flat and lci over \( \text{Spec } \mathbb{Z}[t] \), hence Gorenstein over \( \text{Spec } \mathbb{Z}[t] \) [SP22, Tag 0C15], hence Cohen-Macaulay over \( \text{Spec } \mathbb{Z}[t] \) [SP22, Tag 0C06], hence absolutely (over \( \text{Spec } \mathbb{Z} \)) Cohen-Macaulay [SP22, Tag 0C0W], hence satisfy all \( S_5 \) conditions [SP22, Tag 0342]. For \( R_1 \), if \( \pi : C \rightarrow S \) denotes the prestable curve in question, consider the open subset \( U \subset C \) defined as the intersection of \( \pi^{\text{sm}} \) with the preimage under \( \pi \) of the open subset of \( S \) where \( S \rightarrow \text{Spec } \mathbb{Z}[t] \) is smooth. It is clear that \( C \) is regular at all points of \( U \), and moreover, \( C \setminus U \) has codimension at least 2, and the \( R_1 \) condition follows too.

2a: Recall that \( C[t_{n,1}] = C[t_n] \times_S[t_n] C[t_{n,1}] \), \( x[t_{n,1}] \) the diagonal of this fiber square, and \( w_1[t_{n,1}] \) is the pullback of \( w_1[t_n] \). Thus \( x[t_{n,1}] \cap w_1[t_{n,1}] \cong S[t_n] \), which we know to be integral.
2b: We write $\phi_0$ for $\phi$ regarded as a section of $\omega_C^{\vee}/S(-w_1)$ (rather than $\omega_C^{\vee}$) for clarity. Then, if we write $\{ \cdots = 0 \}$ for the vanishing loci, we have
\[
\{ x[t_{n,2}]^*\phi(t_{n,2}) = 0 \} = \{ x[t_{n,2}]^*\phi_0[t_{n,2}] = 0 \} \quad \text{since } x[t_{n,2}] \cap w_1[t_{n,2}] = \emptyset \\
= \{ x[t_{n,1}]^*\phi_0[t_{n,1}] = 0 \} \quad \text{by item 2 in Remark 3.4} \\
\cong \{ \phi_0[t_{n}] = 0 \} \quad \text{by construction},
\]
so it suffices to check that the last one is integral. The restriction to the complement of the central fiber, $\{ \phi_0[t_{n}] = 0 \}\setminus C[t_{n}]_{(t)}$, is a smooth section of the respective restriction of $\pi[t_{n}]$, hence integral. Moreover, $\{ \phi_0[t_{n}] = 0 \}$ doesn’t contain $C[t_{n}]_{(t)}$ (which is irreducible), thus $\{ \phi_0[t_{n}] = 0 \}$ is irreducible and also reduced at all generic points, hence it is reduced everywhere since embedded components are ruled out by Cohen-Macaulay-ness.

3: We proceed inductively. If $y = Y/X = 1/x$, then
\[
\phi(t_1) = (1 + tx) \frac{\partial}{\partial x} = (1 + tx) \frac{dy}{dx} \frac{\partial}{\partial y} = -\frac{1 + tx}{x^2} \frac{\partial}{\partial y} = -y(t + y) \frac{\partial}{\partial y}.
\]
Therefore,
\[
w_1[t_1]^* \left( \frac{\phi(t_1)}{y} \right) = -t \frac{\partial}{\partial y},
\]
so the NCR morphism of $t_1$ is the identity. Inductively, we have the following. If $c = 1$, then the NCR morphism doesn’t change at the next step: the claim is clear over the dense open subset of the base where Construction 3.3 is trivial (density is left to the reader), so it’s true everywhere by reducedness. If $c = 2$, then the NCR morphism doesn’t change at the next step. This is actually obvious because, in Construction 3.11, the sections $w_1, \ldots, w_m$ are always contained in the open subset of the source where the operation is trivial. If $c = 3$, then the NCR morphism at the next step simply pulls back together with the curve and the rest of the data, completing the proof. \(\square\)

Finally, we prove Theorem 5.2 inductively using Theorem 4.3. The only missing ingredient is the base case of the induction.

**Lemma 5.7.** The object $t_1$ (cf. Definition 5.5) is a terminal object of $V_{0,1,1}$. For any object $o = (S, C, \ldots)$ of $V_{0,1,1}$, the morphism $S \to \text{Spec } \mathbb{Z}[t]$ that exhibits $o$ as a pullback of $t_1$ is the NCR morphism of $o$, cf. Definition 5.1.

**Proof.** In this proof, let $t_1 = (S_0, C_0, \ldots)$, and let $o = (S, C, \ldots)$ be an arbitrary object of $V_{0,1,1}$. We claim that $C \simeq \mathbb{P}_S^1$ over $S$. This requires a few steps.

- All geometric fibers of $\pi$ are irreducible by a simple combinatorial argument using the ampleness of $\omega_C^{\vee}(w_1 + 2x_1)$. Otherwise, a leaf of the dual graph will correspond to a component which doesn’t contain $x_1$, and this is destabilizing.
- The morphism $\pi : C \to S$ is actually a projective $\mathbb{P}^1$-bundle since it is a geometrically integral conic bundle with a section. This is well-known, we leave as hint that $C \cong \mathbb{P}(\pi_*\mathcal{O}_C(x_1))$ in fact.
- The projective bundle $\pi : C \to S$ is the projectivization of the direct sum of two line bundles because $w_1$ and $x_1$ are disjoint sections.
- We have $C \simeq \mathbb{P}_S^1$ by the previous step and $x_1^*\omega_C^{\vee}$ trivial.
If \( o \) is isomorphic to the pullback of \( t_1 \) along a morphism \( S \xrightarrow{\alpha} \text{Spec } \mathbb{Z}[t] \), then \( \alpha \) must be the NCR morphism of \( o \) by the functoriality of NCR maps and item 3 in Proposition 5.6. Conversely, to see that \( o \) is indeed isomorphic to the pullback of \( t_1 \) along \( \eta \), form the fiber product \( S \times_{\mathbb{Z}[t]} C_0 \simeq \mathbb{P}_S^1 \) relative to the NCR morphism \( \eta: S \to \text{Spec } \mathbb{Z}[t] \) of \( o \) and \( \pi_0 : C_0 \to \text{Spec } \mathbb{Z}[t] \), choose an isomorphism \( C \simeq S \times_{\mathbb{Z}[t]} C_0 \) over \( S \) compatible with \( x_1, w_1, x_1.0, w_1.0 \), and note that the remaining \( \mathcal{O}_S \)-ambiguity is precisely what is needed to make the vector fields \( \phi \) and \( \eta^* \phi_0 \) match, since they already both vanish on \( w_1 \) with equal 'coresidues', and both are nowhere vanishing on \( x_1 \). Indeed, after composing with a suitable automorphism which fixes both \( x_1 \) and \( w_1 \), we may arrange so that \( x_1^* \phi = x_1^* \eta^* \phi_0 \), and then \( \phi \) and \( \eta^* \phi_0 \) are a global section of \( \omega^{\nu}_{C/S}(-2w_1-x_1) \), but \( \pi_\ast \omega^{\nu}_{C/S}(-2w_1-x_1) = 0 \) by the cohomology and base change theorem, completing the proof. \( \square \)

**Theorem 5.8.** The object \( t_n \) (respectively \( t_{n,c} \)) from Definition 5.5 is a terminal object of \( \mathbf{V}_{0,1,n} \) (respectively \( \mathbf{C}_{0,1,n} \)).

**Proof.** Follows from Lemma 5.7, Theorem 4.3 and the obvious fact that the pullback of a terminal object of \( \mathbf{C}_{0,1,n-1} \) \( \simeq \mathbf{V}_{0,1,n} \) along the projection from the curve to the base is a terminal object of \( \mathbf{C}_{0,1,n} \simeq \mathbf{C}_{1,1,n} \). \( \square \)

Theorem 5.2 follows from Theorem 5.8 and Proposition 5.6.

**Proof of Theorem 1.5.** We will prove that \( F \) is represented by the space \( \overline{P}_n \) constructed in 5.2.3. Indeed, \( F \) corresponds to the subcategory of \( \mathbf{V}_{0,1,n} \) whose objects have identically 0 NCR morphisms. Since NCR morphisms are functorial, it follows from Theorem 5.2 that \( F \) is represented by the fiber over \( (t) \in \text{Spec } \mathbb{Z}[t] \) of the NCR morphism of \( t_n \). By the remarks in 5.2.4 and Proposition 5.6, this is nothing but \( \overline{P}_n \). The fact that \( \overline{P}_n \) is projective, local complete intersection, flat, and geometrically integral over \( \text{Spec } \mathbb{Z} \) follows from Proposition 5.6. \( \square \)

At this point, we know that \( \mathcal{L}_n \) degenerates isotrivially to \( \overline{P}_n \). To complete the proof of Theorem 1.6, it only remains to deal with the group actions.

### 6. Actions of \( \mathbb{G}_a \) and \( \mathbb{G}_m \) on curves and their moduli

**6.1. Preliminaries.** In this subsection, we review some generalities. These may be skipped and referred back to as necessary.

**6.1.1. Review of Weil divisors and associated reflexive sheaves.** Let \( X \) be an integral normal separated excellent scheme. For any Weil divisor \( D \) on \( X \), let \( \mathcal{O}_X(D) \) be the sheaf associated to the divisor \( D \), that is, \( \Gamma(U, \mathcal{O}_X(D)) = \{ f \in K(X) : (\text{div}(f) + D)|_U \geq 0 \} \). However, in other sections we will reserve the notation \( \mathcal{O}_X(D) \) for the Cartier case, unless specified otherwise. We collect here some basic facts regarding associated sheaves. I have been unable to find a suitable published reference, but the excellent notes [Sch10] contain what is needed. Besides, everything is quite elementary.

1. \( \mathcal{O}_X(D) \) is a rank 1 reflexive coherent \( \mathcal{O}_X \)-module [Sch10, Proposition 3.4];
2. conversely, if \( F \) is a rank 1 reflexive coherent \( \mathcal{O}_X \)-module, then there exists a Weil divisor \( D \) such that \( F \simeq \mathcal{O}_X(D) \) [Sch10, Propositions 3.6 and 3.7];
3. if \( D \) is prime, then \( \mathcal{I}_D = \mathcal{O}_X(-D) \) [Sch10, Proposition 3.4];
4. if \( D \) is prime, then \( \mathcal{I}_D^{\nu} = \mathcal{O}_X(D) \), by 1, 3, and [Sch10, Proposition 3.13.(b)].
Lemma 6.1. Let X and X' be integral normal separated excellent schemes, and \( f : X' \to X \) a flat morphism. Let \( D \subset X \) be a prime Weil divisor such that \( D' = f^{-1}(D) \) is a prime Weil divisor on \( X' \). Then \( f^* \mathcal{O}_X(D) \simeq \mathcal{O}_{X'}(D') \).

Proof. First, we claim that \( f^* \mathcal{I}_{D,X} \simeq \mathcal{I}_{D',X'} \). This is a consequence of flatness, as follows. We have the following solid arrow commutative diagram

\[
\begin{array}{ccc}
0 & \to & f^* \mathcal{I}_{D,X} \\
& \downarrow \simeq & \downarrow \simeq \\
0 & \to & \mathcal{I}_{D',X'}
\end{array}
\]

in which the top row is exact because \( f \) is flat. The right vertical isomorphism comes from the flat cohomology and base change theorem [Ha77, Proposition 9.3]. Then there must exist a dashed arrow which makes the diagram commute, proving \( f^* \mathcal{I}_{D,X} \simeq \mathcal{I}_{D',X'} \). However, as in the proof of [Ha80, Proposition 1.8], \( f^* \) commutes with dualizing, so \( f^* \mathcal{I}_{D,X} \simeq \mathcal{I}_{D',X'} \). Then item 4 completes the proof. \( \square \)

6.1.2. Some calculations on \( \mathcal{T}_n \). The Losev-Manin space has already been reviewed in §1.3 and discussed in §5.2.1 and §5.2.2. As agreed in §1.3, \( \mathcal{T}_n \) is the Losev-Manin space over Spec \( \mathbb{Z} \). [LM00, Theorem 2.2] still holds over Spec \( \mathbb{Z} \), and \( \mathcal{T}_{n+1} \) is the universal curve over \( \mathcal{T}_n \). In particular, \( \mathcal{T}_n \) is smooth over Spec \( \mathbb{Z} \). Indeed, it is clear inductively that it is flat and of finite type, since \( \mathcal{T}_{n+1} \to \mathcal{T}_n \) is flat and of finite type, and the fact that the geometric fibers are regular is stated on page 46 of [LM00] — nothing changes in positive characteristic. (Since no details on smoothness are provided in [LM00], we mention an alternative argument: use the moduli space interpretation and deformation theory. The usual deformation theory of marked curves still applies.)

Moreover, there is a natural action of \( \mathcal{T}_n \times \mathbb{G}_m \) on \( \mathcal{T}_{n+1} \) — on the open subset corresponding to Losev-Manin chains of length 1, this is just the \( \mathbb{G}_m, L_n \)-action on \( \mathbb{P}^{L_n}_m \) fixing 0 and \( \infty \). We denote it by \( \nu : \mathbb{G}_m \times \mathcal{T}_{n+1} \to \mathcal{T}_{n+1} \). This also follows from [LM00], although it is worth noting that we will actually recover this as a byproduct of an inductive argument below (it will take one moment of thought to convince ourselves that there is no logical circularity here).

Let \( Y_n = \mathcal{T}_{n+1} \times_{\mathcal{T}_n} \mathcal{T}_{n+1} \). The action \( \nu : \mathbb{G}_m \times \mathcal{T}_{n+1} \to \mathcal{T}_{n+1} \) pulls back along \( Y_n \to \mathcal{T}_{n+1} \) to an action \( \mu : \mathbb{G}_m \times Y_n \to Y_n \). Let \( \Delta \subset Y_n \) be the diagonal of the fiber product, and \( p_2 : \mathbb{G}_m \times Y_n \to Y_n \) the projection to the second factor.

First, we show that \( Y_n \) admits a small resolution (over Spec \( \mathbb{Z} \)).

Lemma 6.2. Let \( R_n = \mathbb{P} \mathcal{T}'_{\Delta,Y_n} \), and \( \lambda : R_n \to Y_n \) the natural projection. Then \( R_n \) is smooth over Spec \( \mathbb{Z} \), and there exists a closed subscheme \( N \subset Y_n \) such that:

1. \( N \) has relative (over Spec \( \mathbb{Z} \)) codimension 3 in \( Y_n \);
2. \( \lambda^{-1}(N) \) has relative codimension 2 in \( R_n \); and
3. \( \lambda \) restricts to an isomorphism \( R_n \setminus \lambda^{-1}(N) \simeq Y_n \setminus N \).

Proof. Roughly, the point is that \( R_n \) resolves the singularities of \( Y_n \) in the same way \( \mathcal{T}_{n+2} \) does. Let \( y_0, y_{\infty} : \mathcal{T}_{n+1} \to Y_n \) be 0 and \( \infty \) sections, i.e. the pullbacks
of the 0 and $\infty$ sections of $T_{n+1}$ over $T_n$. As reviewed earlier, the Losev-Manin spaces are constructed inductively in [LM00, 1.3 and 2.1] by $T_{n+2} = \mathbb{P}K$, where

$$K = \text{Coker} \left( \mathcal{O}_{Y_n} \xrightarrow{b \mapsto (b, b)} T_{\Delta, Y_n} \oplus \mathcal{O}_{Y_n}(y_0 + y_\infty) \right).$$

Then $T_{n+2}$ and $R_n$ are isomorphic above $Y_n \setminus (y_0 \cup y_\infty)$, and hence $R_n \to \text{Spec } \mathbb{Z}$ is smooth everywhere in $\lambda^{-1}(Y_n \setminus (y_0 \cup y_\infty))$. Let $N \subset \Delta \simeq T_{n+1}$ correspond to the locus where $T_{n+1} \to T_n$ fails to be smooth, taken with, say, the reduced closed subscheme structure. It is clear that $\lambda$ is an isomorphism above $Y_n \setminus N$. Since $T_{n+1} \to \text{Spec } \mathbb{Z}$ is smooth, $Y_n \to \text{Spec } \mathbb{Z}$ is clearly smooth at all points where at least one of the two projection maps $Y_n \to T_{n+1}$ is smooth, and in particular, it is smooth in a neighbourhood of $y_0 \cup y_\infty$. Therefore, $R_n \to \text{Spec } \mathbb{Z}$ is also smooth in a neighborhood of $\lambda^{-1}(y_0) \cup \lambda^{-1}(y_\infty)$ as $N \cap (y_0 \cup y_\infty) = \emptyset$, completing the proof of the smoothness claim. The dimension claims are straightforward. \hfill $\square$

**Proposition 6.3.** $\mu^{-1}(\Delta) \sim p_2^{-1}(\Delta)$ on $\mathbb{G}_m \times Y_n$.

**Proof.** Let $D = p_2^{-1}(\Delta)$ and $D' = \mu^{-1}(\Delta)$, and $E$ and $E'$ the ‘proper transforms’ of $D$ and $D'$ on $\mathbb{G}_m \times R_n$, that is, with notation as in Lemma 6.2, $E$ is the closure of $(\text{id}_{\mathbb{G}_m} \times \lambda)^{-1}(D(N))$ with the reduced structure, and $E'$ is the closure of $(\text{id}_{\mathbb{G}_m} \times \lambda)^{-1}(D'(N))$ with the reduced structure. Let $p_1: \mathbb{G}_m \times Y_n \to \mathbb{G}_m$ and $q_1: \mathbb{G}_m \times R_n \to \mathbb{G}_m$ be the projections to the first factors. Since $\mathbb{G}_m \times R_n$ is regular, $E$ and $E'$ are effective Cartier divisors.

Let $z = (f, p) \in \mathbb{G}_m = \text{Spec } \mathbb{Z}[x, x^{-1}]$ be a closed point, where $p$ is a prime number, and $f \in \mathbb{Z}[x]$ is irreducible mod $p$, $f \not\equiv p \cdot x$. First, we claim that

$$D_z \sim D'_z$$

on $p_1^{-1}(z)$. Let $K = \mathbb{F}_p[x]/(f)$, the residue field of $\mathbb{G}_m$ at $z$. As usual, write $\square_K = \text{Spec } K \times_{\text{Spec } \mathbb{Z}} \square$. Let $w \in \mathbb{G}_m, K(K)$ be a preimage of $z \in \mathbb{G}_m(K)$, and let 1 denote the point $(t^{-1}) \in \mathbb{G}_m, K(K)$. We have $D'_z = D'_{K, w} \sim D'_{K, 1} = D_{K, 1} = D_{K, w} = D_z$ in $p_1^{-1}(z) \cong Y_{n, K}$, where the linear equivalence comes from [Fu98, Proposition 1.6] (the family over $\mathbb{G}_m, K$ may be extended to a family over $\mathbb{P}^1_K$ simply by taking the closure of $D_K$ in $\mathbb{P}^1_K \times Y_{n, K}$), which proves (35). Second, we claim that

$$E_z \sim E'_z$$

on $q_1^{-1}(z)$. Note that $E_z$ and $E'_z$ are the proper transforms of $D_z$ and $D'_z$ under the small resolution $\lambda: q_1^{-1}(z) \to Y_{n, K} \to Y_{n, K} = p_1^{-1}(z)$. (Since $((\text{id}_{\mathbb{G}_m} \times \lambda)^{-1}(D'))_z$ contains $E'_z$ and has a unique component of codimension 1 in $(\mathbb{G}_m \times R_n)_z$ as $D'_z$ is irreducible and $\text{id}_{\mathbb{G}_m, K} \times \lambda_K$ is an isomorphism in codimension 2, and $E'_z$ is cut out by a single equation (line bundle section) on $(\mathbb{G}_m \times R_n)_z$ being the pullback of the Cartier divisor $E$, it follows that $E'_z$ is a fortiori the codimension 1 irreducible component above, and also that it is the proper transform of $D'_z$. Similarly for $E_z$.) By Lemma 6.2 and [Ha77, Proposition 6.5, part b], we have $\text{Cl}(R_{n, K}) \cong \text{Cl}(Y_{n, K})$ compatible with taking proper transforms, so (36) follows from (35).

Let $J = \mathcal{O}_{\mathbb{G}_m \times R_n}(E' - E)$. We will show that $J \simeq \mathcal{O}_{\mathbb{G}_m \times R_n}$ using a standard argument. First, we claim that

$$q_* J \simeq \mathcal{O}_{\mathbb{G}_m}.$$ 

By (36), $\dim_{\mathbb{Z}}(J_z) = 1$, for all closed points $z \in \mathbb{G}_m$. Since all closed and all open subsets of $|\mathbb{G}_m|$ contain at least one closed point, the semicontinuity theorem
[Ha77, III, Theorem 12.8] implies that \( \dim_{κ}(z) \Gamma(J_z) = 1 \) holds for all \( z \in \mathbb{G}_m \), not just the closed points. By Grauert’s Theorem [Ha77, III, Corollary 12.9], \( q_*J \) is invertible. Then (37) follows as \( \text{Pic}(\mathbb{G}_m) \) is trivial. However, \( \dim_{κ}(z) \Gamma(J'_z) = 1 \) for all closed points \( z \in \mathbb{G}_m \) follows equally well from (36), so a completely analogous argument shows that

\[
q_*(J'_z) \simeq O_{\mathbb{G}_m}.
\]

Let \( s_1 \) and \( s_2 \) be the global sections of \( J \) and \( J' \) which correspond to the section 1 of \( O_{\mathbb{G}_m} \) under the isomorphisms (37) and (38). Clearly, \( s_1 \otimes s_2 \neq 0 \), so \( s_1 \otimes s_2 \) is nowhere vanishing as \( J \otimes J' \simeq O_{\mathbb{G}_m \times \mathbb{R}^n} \). Then \( s_1 \) is nowhere vanishing, so \( J \simeq O_{\mathbb{G}_m \times \mathbb{R}^n} \). Hence \( E \sim E' \) on \( \mathbb{G}_m \times R_n \). By Lemma 6.2 and [Ha77, Proposition 6.5, part b], \( \text{Cl}(\mathbb{G}_m \times R_n) \cong \text{Cl}(\mathbb{G}_m \times Y_n) \) compatible with taking proper transforms, and \( D \sim D' \) follows. □

6.1.3. Projectivization of equivariant sheaves. Finally, we review the fact that group actions lift to projectivizations of equivariant sheaves.

**Lemma 6.4.** Let \( G \) be an \( S \)-group scheme, \( Y \) an \( S \)-scheme, and \( \alpha \) an action of \( G \) on \( Y \) relative to \( S \). Let \( F \) be a \( G \)-equivariant coherent \( O_Y \)-module, \( X = \mathbb{P}(F) \), and \( f : X \rightarrow Y \) the natural projection map. Then there exists a \( G \)-action on \( X \) over \( S \), relative to which \( f \) is \( G \)-equivariant.

**Proof.** Given that \( \mathbb{P} \) commutes with base change, this is purely formal and surely well-known. In essence, the discussion in [MFK94, p. 31] still applies – it doesn’t truly matter that we have \( \mathbb{P} = \text{Proj Sym} \) instead of \( \text{Spec Sym} \), and that [MFK94] operates in a more restrictive setup (invertible sheaf, etc.). □

6.2. The \( G \)-action on the universal curve. We return to the main logical thread of the paper. The main goal of §6.2 is to ‘integrate’ the vector field on the universal curve in the terminal object of \( \mathcal{V}_{0,1,n} \) to obtain a group action.

Let \( \gamma : G \rightarrow \text{Spec} \mathbb{Z}[t] \) as in §1.3, \( e : \text{Spec} \mathbb{Z}[t] \rightarrow G \) the identity section, and \( \mathfrak{g} = e^* \mathcal{T}_{G/\mathbb{Z}[t]} \) its Lie algebra, where \( \mathcal{T}_{G/\mathbb{Z}[t]} \) is the relative tangent bundle of \( \gamma \). For simplicity, let \( ℓ = \text{Spec} \mathbb{Z}[t] \) and \( ℓ^* = \text{Spec} \mathbb{Z}[t, t^{-1}] \).

**Definition 6.5.** Let \( o = (S, C, \ldots, φ) \) be an object of \( \mathcal{V}^+_{0,1,n} \) or \( \mathcal{C}^+_{0,1,n} \), with notation as in Definition 1.3. (Below, we use the notation \( X[\varepsilon] = \text{Spec} \mathbb{Z}[\varepsilon]/(\varepsilon^2) \times_{\text{Spec} \mathbb{Z}} X \).

\[
\begin{align*}
\{ \text{actions of } G \times ℓ S \text{ on } C \text{ over } S \} & \xrightarrow{\mathfrak{g} = \frac{∂}{∂x} O_\mathbb{Z}[t]} \{ \text{automorphisms of } C[\varepsilon] \text{ over } S[\varepsilon] \text{ which restrict to } \text{id}_C \text{ on } C \} \\
\phi & \in \text{Hom}(\omega_{C/S}, \mathcal{O}_C) \xrightarrow{[Kn83, \S 1]} \text{Hom}(\Omega_{C/S}, \mathcal{O}_C)
\end{align*}
\]

Consider the pullback \( G \times ℓ S \) of \( G \) along the NCR morphism \( S \rightarrow ℓ \). We say that an action \( α \) of \( G \times ℓ S \) on \( C \) over \( S \) is compatible with \( φ \) if the images of \( φ \) and \( α \) in \( \text{Hom}(\Omega_{C/S}, \mathcal{O}_C) \) in the diagram above coincide. If only the restrictions of these images to an open \( U \subset C \) coincide, we say the compatibility holds on \( U \).

Henceforth, \( G \times ℓ S \), \( G \times ℓ C \), etc. are relative to the NCR morphism \( S \rightarrow ℓ \).
Theorem 6.6. An action compatible with the respective field exists for the terminal object $t_n$ of $V_{0_1,n}$, cf. Theorem 5.8, and fixes the respective section $w_1$.

We recall the following elementary fact which will be used soon.

Remark 6.7. Let $D, E \subset X$ be effective Cartier divisors such that $D|_E = D \cap E$ is Cartier on $E$, that is, the restriction $\mathcal{O}_E \to (\mathcal{O}_X(D))|_E$ of $\mathcal{O}_X \to \mathcal{O}_X(D)$ is injective. Then we have a short exact sequence $0 \to \mathcal{I}_{D+}E, X \to \mathcal{I}_{D, X} \oplus \mathcal{I}_{E, X} \to \mathcal{I}_{D \cap E, X} \to 0$. Indeed, the elements cutting out $D$ and $E$ locally form a regular sequence, and then the exactness of the sequence is essentially the exactness of the Koszul complex.

Lemma 6.8. Let $\mathfrak{o} = (S, C, \pi, w_1, x, \phi)$ be an object of $C^1_{0,1,n}$ such that

1. $C$ and $S$ are integral and separated;
2. $x \neq w_1$, that is, there exists a point $s \in S$ such that $x(s) \neq w_1(s)$; and
3. the scheme theoretic intersection $x \cap w_1$ is integral.

For simplicity, we write $w = w_1$. Let $\alpha$ be a $G \times \ell S$-action compatible with $\phi$ which fixes $w$. Assume that $\mathcal{T}_{x, C}^\vee$ is $G \times \ell S$-equivariant. Let $\mathfrak{o} = (S, C, \ldots)$ cf. (30). Then there exists an action $\alpha_2$ of $G \times \ell S$ on $C_2$ compatible with $\phi_2$, which fixes $w_1$.

Formally, the requirement that $\alpha$ fixes $w$ means that the restriction of $\alpha : G \times S C \to C$ to $G \times S w$ is the composition of the projection $G \times S w \to w$ with the closed immersion $w \hookrightarrow C$.

Proof. We will first show that if some twist $K \otimes \mathcal{J}$ of $K$ (cf. Construction 3.3) by a line bundle admits an equivariant structure relative to $\alpha$, then the conclusion holds. Indeed, Lemma 6.4 then produces an action $\alpha_2$ of $G \times \ell S$ on $C_2 = \mathbb{P}(K \otimes \mathcal{J})$ such that $f : C_2 \to C$ (Construction 3.3) is $G \times \ell S$-equivariant. The compatibility of $\alpha_2$ with $\phi_2$ holds at least over a dense open subset above which $f$ is an isomorphism, and then it holds everywhere a fortiori, since $\Omega_{C_2/S}^\vee$ is torsion free. Moreover, $\alpha_2$ fixes $w_1 \setminus x_2$, and hence it must fix $w_2$.

Since the claim is local on the base, it suffices to analyze separately two situations: $x$ is contained in the open subset of $C$ where $\pi$ is smooth, and $x \cap w = \emptyset$. In the first case, $x$ is an effective Cartier divisor on $C$ and $\mathcal{T}_{x, C} = \mathcal{O}_C(x)$. We have a (solid arrow) commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_C(-x - w) & \longrightarrow & \mathcal{O}_C(-x) \oplus \mathcal{O}_C(-w) & \longrightarrow & \mathcal{I}_{x \cap w, C} \longrightarrow & 0 \\
| & | & | & | & | & | & | & | \\
0 & \longrightarrow & \mathcal{O}_C(-x - w) & \longrightarrow & \mathcal{O}_C(-x) \oplus \mathcal{O}_C(-w) & \longrightarrow & \mathcal{K}(-x - w) & \longrightarrow & 0
\end{array}
$$

in which the top row comes from Remark 6.7, and the bottom row is (9) twisted by $\mathcal{O}_C(-x - w)$. Then there exists a unique dashed isomorphism $\mathcal{I}_{x \cap w, C} \simeq \mathcal{K}(-x - w)$ which makes the diagram commute. However, $\mathcal{I}_{x \cap w, C}$ is $G \times \ell S$-equivariant because $\alpha$ fixes $w$ and hence $x \cap w$ too, and then $\mathcal{I}_{x \cap w, C}$ is the kernel of the homomorphism $\mathcal{O}_C \to (x \cap w \hookrightarrow C)_* \mathcal{O}_{x \cap w}$ in the category of $G \times \ell S$-equivariant coherent $\mathcal{O}_C$-modules. In the second case, $\mathcal{K}(-w) = \mathcal{T}_{x, C}^\vee$, and we are done since we are assuming that $\mathcal{T}_{x, C}$ is equivariant.

Lemma 6.9. Let $\mathfrak{o} = (S, C, \pi, w_1, x, \phi)$ be an object of $C^2_{0,1,n}$ such that

1. $C$ and $S$ are integral and separated; and
Let $w = w_1$. Let $\alpha$ be a $G \times_\ell S$-action on $C$ compatible with $\phi$ which fixes $w$. Let $\mathfrak{g} = (S, C_2, \ldots)$, cf. (30). Then there exists a $G \times_\ell S$-action $\alpha_\phi$ on $C_2$ compatible with $\phi_\mathfrak{g}$, which fixes $w_{1,\mathfrak{g}}$.

Proof. As in the proof of Lemma 6.8, if some twist $K \otimes J$ of $K$ (cf. Construction 3.11) by a line bundle admits an equivariant structure relative to $\alpha$, then we are done. We have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_{C/S}(-x) & \longrightarrow & \mathcal{O}_{C}(-x) \oplus \omega_{C/S} & \longrightarrow & \mathcal{I}_{x(Z), C} & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & \omega_{C/S}(-x) & \longrightarrow & \mathcal{O}_{C}(-x) \oplus \omega_{C/S} & \longrightarrow & K \otimes \omega_{C/S}(-x) & \longrightarrow & 0
\end{array}
\]

in which the top row comes from Remark 6.7, and the bottom row is (17) twisted by $\omega_{C/S}(-x)$. It follows that $K \otimes \omega_{C/S}(-x) \cong \mathcal{I}_{x(Z), C}$.

We claim that $x(Z) \subset C$ is fixed by $\alpha$. Formally, this means that the restriction of $\alpha : G \times_\ell C \rightarrow C$ to $G \times_\ell x(Z)$ is the composition of the projection $G \times_\ell x(Z) \rightarrow x(Z)$ with the closed immersion $x(Z) \hookrightarrow C$. We are thus claiming that two $S$-morphisms $G \times_\ell x(Z) \rightarrow C$ coincide, or equivalently, that a morphism $G \times_\ell x(Z) \rightarrow C \times_{S} C$ factors through the diagonal immersion $C \hookrightarrow C \times_{S} C$. It is elementary to check on (geometric) fibers over $S$ that the image of the restriction of $\alpha$ to $G \times_\ell x(Z)$ is contained in $\pi^{sm} \subset C$, the open subset where $\pi$ is smooth, and it follows that the image of the morphism $G \times_\ell x(Z) \rightarrow C \times_{S} C$ is contained in the open subset $\pi^{sm} \times_{S} \pi^{sm}$. However, the diagonal $\Delta$ of $\pi^{sm} \times_{S} \pi^{sm}$ is a Cartier divisor, and our claim boils down to the statement that the section 1 of $\mathcal{O}_{\pi^{sm} \times_{S} \pi^{sm}}(\Delta)$ pulls back to 0 on $G \times_\ell x(Z)$. However, the claim, and in particular the vanishing of the section above, are elementary to check on geometric fibers over $S$; then they hold on the fibers of $S$, and hence everywhere since $G \times_\ell x(Z)$ is integral as a consequence of the assumption that $x(Z)$ is integral. Then $\mathcal{I}_{x(Z), C} = \text{Ker}(\mathcal{O}_{C} \rightarrow x_{*} \mathcal{O}_{Z})$ is equivariant (as in the proof of Lemma 6.8), which completes the proof. \hfill $\square$

To prove Theorem 6.6, we need to show that the assumption that the dual ideal sheaf in Lemma 6.8 is equivariant holds for the terminal object of $\mathbf{C}^{1}_{0,1,n}$.

Proposition 6.10. Let $t_{n,1} = (S, C, \pi, w_1, \pi, x, \phi)$ be the terminal object of $\mathbf{C}^{1}_{0,1,n}$, cf. Definition 5.5 and Theorem 5.8, and $\alpha$ an action of $G \times_\ell S$ on $C$ over $S$ compatible with $\phi$, cf. Definition 6.5. Then $\mathcal{I}_{x/C}^{\alpha}$ is $\alpha$-equivariant.

Proof. Throughout this proof, we rely (sometimes implicitly) on Proposition 5.6. All of $C$, $G \times_\ell C$, and $G \times_\ell C \times_\ell C$ are dense open in some $K \times C$, hence normal and integral by Proposition 5.6. Clearly, we may think of $\alpha$ as an action of $G$ on $C$ over $\ell$, in view of the identification $G \times_\ell C = (G \times_\ell S) \times_{S} C$. Let $\omega_{2} : G \times_\ell C \rightarrow C$ be the projection to the second factor. First, we will show that

\[
\alpha^{-1}(x) \sim \omega_{2}^{-1}(x).
\]

Let $G^* = \ell^* \times_\ell G \cong \ell^* \times \mathbb{G}_{m}$. With notation as in §6.1.2, $\ell^* \times_\ell C \cong \ell^* \times Y_{n}$ and this isomorphism restricts on $\ell^* \times_\ell x$ to an isomorphism $\ell^* \times_\ell x \cong \ell^* \times \Delta$. With $\mu$ also as in §6.1.2, we have the following commutative diagram.
We briefly explain this claim. A priori, \( \alpha \) restricts to some action of \( \mathbb{G}_m \times (S \setminus S_{(t)}) \) on \( C \setminus C_{(t)} \cong \ell^* \times Y_n \) over \( S \setminus S_{(t)} \). If we restrict to, say, the fiber of \( (t-1) \in \ell^* \), we recover the \( \mathbb{G}_m \times T_{n+1} \) action \( \mu \) on \( Y_n \) in \S 6.1.2. All the \( \mathbb{G}_m \) actions in discussion satisfy various elementary properties (such as having weight 1 on generic fibers and fixing the two suitable sections) that determine them uniquely, so there is no concern that we have obtained a different action. The same type of uniqueness argument establishes commutativity of the right half of the diagram.) Note that \( \text{Cl}(G^* \times \ell C) \cong \text{Cl}(G \times \ell C) \) by [Ha77, Proposition 6.5, part c] and \( C_{(t)} \approx 0 \) (as \( C_{(t)} \) is integral by Proposition 5.6), so (39) may be checked on the restriction to \( G^* \times \ell C \), i.e. the complement of the fiber over \( (t) \in \ell^* \). Then it becomes \( \ell^* \times \mu^{-1}(\Delta) \approx \ell^* \times p_2^{-1}(\Delta) \), and it follows from Proposition 6.3 and the elementary fact that, for any (integral, normal, separated, noetherian) \( X \), \( \text{Cl}(X) \cong \text{Cl}(\ell^* \times X) \) via \([D] \mapsto [\ell^* \times D]\) by e.g. [Ha77, Propositions 6.5c and 6.6] and their proofs.

Let's temporarily reinstate the \( \mathcal{O} \) notation for associated sheaves from \S 6.1.1. We have \( \mathcal{O}_{G^* \times \ell C}(\varpi_2^{-1}(x)) \cong \varpi_2^{-1} \mathcal{O}_C(x) \) and \( \mathcal{O}_{G^* \times \ell C}(\alpha^{-1}(x)) \cong \alpha^* \mathcal{O}_C(x) \) by Lemma 6.1. On the other hand, \( \mathcal{O}_{G^* \times \ell C}(\alpha^{-1}(x)) \cong \mathcal{O}_{G \times \ell C}(\varpi_2^{-1}(x)) \) by (39), so \( \alpha^* \mathcal{O}_C(x) \cong \varpi_2^{-1} \mathcal{O}_C(x) \), or \( \alpha^* \mathcal{T}_{\ell C} \cong \varpi_2^{-1} \mathcal{T}_{\ell C} \) by item 4. Let \( \mathcal{F} = \mathcal{T}_{\ell C} \) for simplicity, so \( \alpha^* \mathcal{F} \cong \varpi_2^{-1} \mathcal{F} \). We will see what we've done so far suffices for proving Proposition 6.10.

First, we claim that we there exists an isomorphism \( \psi : \alpha^* \mathcal{F} \cong \varpi_2^{-1} \mathcal{F} \) which is ‘unitary’, that is, \((\epsilon_G \times \ell \text{id}_C)^* \psi = \text{id}_\mathcal{F} \). Let \( \psi_0 : \alpha^* \mathcal{F} \cong \varpi_2^{-1} \mathcal{F} \) be an isomorphism. Note that \((\epsilon_G \times \ell \text{id}_C)^* \psi_0\) is an automorphism of \( \mathcal{F} \), and then it is easy to check that if \( \psi \) is the composition

\[
\alpha^* \mathcal{F} \xrightarrow{\psi} \varpi_2^{-1} \mathcal{F} \xrightarrow{(\varpi_2^{-1} \epsilon_G \times \ell \text{id}_C)^* \psi_0^{-1}} \varpi_2^{-1} \mathcal{F},
\]

then \( \psi \) is unitary.

We will show that \( \psi \) actually satisfies the ‘cocycle condition’ automatically. The cocycle condition is a statement that two isomorphisms between two sheaves isomorphic to \( q_3^* \mathcal{F} \) coincide, where \( q_3 : G \times \ell G \times \ell C \to C \) is the projection to the third factor. However, since \( \psi \) is unitary, these isomorphisms at least coincide over \( B \times \ell C \), where \( B = G \times \ell \{e_G\} \cup \{e_G\} \times \ell G \subset G \times \ell G \). Then it suffices to prove that the only automorphism of \( q_3^* \mathcal{F} \) which restricts to the identity on \( B \times \ell X \) is the identity. Note that \( q_3^* \mathcal{F} \) is reflexive by [Ha80, Proposition 1.8]. It follows from item 5 in the review of Weil divisors (\S 6.1.1) that \( \text{Aut}(q_3^* \mathcal{F}) \cong \mathcal{O}_C^*_{G \times \ell G \times \ell C} \).

Let \( \text{Spec} \, \mathbb{K} \to C \) be the geometric generic point of \( X \). Because \( C \) is integral and \((G \times \ell G \times \ell C)_\mathbb{K} \simeq \mathbb{G}_m^2 \), to conclude the proof, it suffices to check the following: the only (multiplicatively) invertible regular function on \( \mathbb{G}_m^2 \) which restricts to the constant 1 on \( \mathbb{G}_m^2 \times \{1\} \cup \{1\} \times \mathbb{G}_m^2 \) is the constant 1. Indeed, the invertible regular functions are all monomials, so the claim is clear.

**Proof of Theorem 6.6.** With the notation in Definition 5.5, we will prove by induction that \( \mathfrak{t}_n \) and \( \mathfrak{t}_{n,c} \) admit group actions compatible with their vector fields in the sense of Definition 6.5. For \( \mathfrak{t}_1 \), it can be checked that the standard action of \( G \) on itself extends to an action on the relative compactification \( G \hookrightarrow \ell \times \mathbb{P}^1 \), and
that the extended $G$-action is compatible with the vector field. Going from $t_n$ (or equivalently, $t_{n-3}$) to $t_{n,1}$ is elementary, just pull back the action accordingly. Going from $t_{n,1}$ to $t_{n,2}$ is Lemma 6.8 backed up by Propositions 6.10 and 5.6, while going from $t_{n,2}$ to $t_{n,3}$ is Lemma 6.9 backed up by Proposition 5.6.

**Remark 6.11.** Let $\mathbb{K}$ be an algebraically closed field. It is elementary to see that there is a bijection between the set of stable $n$-marked $G_n$-rational trees (§1.3) up to isomorphism and the set of objects of $V_{0,1,n}(\mathbb{K})$ with 0 NCR morphisms up to isomorphism. It is then obvious from Theorem 1.5 that the fibers over the closed points of $\mathcal{P}_{n,\mathbb{K}}$ are the set of stable $n$-marked $G_n$-rational trees over $\mathbb{K}$, as claimed in the introduction.

### 6.3. The $G^n_\ell$-action on the moduli space

Let $t_n = (S,C,\pi,\pi_1,\phi)$ be the terminal object of $V_{0,1,n}$, cf. Theorem 5.8 and Definition 5.5.

Let’s construct the object $t^n_\ell = (S^n,\pi^n,\ldots)$ of $V_{0,1,n}$ which corresponds to the open stratum of the terminal object where the parametrized curves are integral:

- $S^n = G^n_\ell - 1$, $C^n = G^n_\ell - 1 \times \mathbb{P}^1$, $\pi^n$ is the projection to the first factor;
- $\pi^n_1$ is the constant $[1 : 0]$ section;
- $x^n_i$ is the constant $[0 : 1]$ section, for $i \geq 2$, $x^n_i$ is the graph of the map
  
  $G^n_\ell - 1 \xrightarrow{\text{projection to } (i-1)-\text{st factor}} G \to \ell \times \mathbb{P}^1 \to \mathbb{P}^1$;

- $\phi^n = (1 + tx) \frac{\partial}{\partial x}$, where $[X : Y]$ are the coordinates on $\mathbb{P}^1$ and $x = X/Y$.

The details of the verification that this is indeed the open stratum are skipped. There is a tautological $G^n_\ell - 1$-action on $S^n$, but if we are to think of $t^n_\ell$ more symmetrically, we should extend this to a $G^n_\ell$-action as follows: the composition

$G \xrightarrow{\text{inverse}} G \xrightarrow{\text{diagonal}} G^n_\ell - 1$

and the tautological action produce a $G$-action on $S^n$, and we may combine the latter action and the tautological action into a $G^n_\ell = G \times_\ell G^n_\ell - 1$-action on $S^n$, called the natural action on $S^n$.

**Proposition 6.12.** With notation as above, there exists a $G^n_\ell$-action on $S$ over $\ell$ which extends the natural $G^n_\ell$-action on $S^n$.

**Proof.** Consider the object $y_n$ of $V_{0,1,n}$ which coincides with the pullback $p_2^n \times t_n$, where $p_2 : G^n_\ell \times_\ell S \to S$ is the projection to the second factor, with the sole exception that the $i$-th section (denoted by $x_i$ in Definition 1.3) is the composition

$G^n_\ell \times_\ell S \xrightarrow{(p_2,x_i)} G^n_\ell \times_\ell C \xrightarrow{(q_1,\alpha_i)} G^n_\ell \times_\ell C$,

where $p_1, q_1$ are the projections to the first factors respectively, and $\alpha_i$ uses the $i$-th factor of $G^n_\ell$ to act on $C$ via the action provided by Theorem 6.6. Note that $y_n$ is indeed is an object in $V_{0,1,n}$. By Theorem 5.8, there exists a morphism $\beta : G^n_\ell \times_\ell S \to S$ such that $y_n = \beta^* t_n$.

We only sketch the verification that $\beta$ is a $G^n_\ell$-action on $S$ over $\ell$. Let $pr_1$ be the projection to the $i$th factor of $G^n_\ell \times_\ell G^n_\ell \times_\ell S$, and let $\mu : G \times_\ell G \to G$ and $\mu^n_\ell : G^n_\ell \times_\ell G^n_\ell \to G^n_\ell$ be the group laws on $G$ and $G^n_\ell$. We need to check that

$\beta \circ (\mu^n_\ell, pr_3) = \beta \circ (pr_1, pr^*_1 \beta)$

(40)
as morphisms $G^n \times \ell G^n \times \ell S \to S$. It is straightforward to check that $\beta$ restricts to the natural action on $S^\circ$. Then the restriction of (40) to $G^n \times \ell G^n \times \ell S^\circ$ must hold, so (40) must hold a fortiori since everything in sight is reduced and separated. □

Proposition 6.12 also shows that $\mathcal{P}_{n,K}$ is a $\mathbb{G}_{a,K}^{n-1}$-variety, as claimed in §1.3.

Theorem 1.6 follows from the examples in §§5.2.2, §5.2.3, and §5.2.4, Theorem 5.8, Theorem 1.5 (serving as the definition of $\mathcal{P}_n$), and Proposition 6.12.

7. Questions

We conclude the paper with two questions.

7.1. Primitive linear systems on abelian and K3 surfaces. The first one concerns the constructions sketched in §§5.2.5 and §5.2.6 (which will also reveal how I arrived at the ideas in this paper). Accepting the moduli space interpretation of $E\{n\}$, we may form the quotient $E\{n\}/S_n$ by permuting the markings, and we may even restrict to a fixed linear system $|L|$ for the divisor on $E$ which is the image of the markings. Then the resulting space $E(L)$ deforms isotrivially to $\mathbb{P}^{n-1}$. This is strong indirect evidence for [Za22, Question 4.14] and even for the existence of a degeneration of the primitive linear system with geometry in the style of loc. cit.

Here is a sketch. A suitable ‘slice’ – fiber over a closed point of $F$ – of [Za22, Figure A] is roughly the same as a curve parametrized by $E(L)$, although the ‘simple F-curves’ [Za22, §4.2] in the picture also need to be blown up and moved to Atiyah-type components, and destabilizing chains of $\mathbb{P}^1 \times F$ components should be contracted. Then the data is uniquely determined by what is visible inside the slice, by [Za19, Proposition 2.3]. Thus the ‘moduli space of tidied up [Za22, Figure A]-s’ should be isomorphic to $E(L)$, and above was stated that this can deform to some $\mathbb{P}^N$. However, constructing the desired deformation to the primitive linear system seems quite different and subtle, and requires new ideas that I haven’t found yet.

Thus the ideas in this paper might be relevant to the study of curves on K3 and abelian surfaces, since [Za22, Question 4.14] aims to further refine the techniques of [BL99, BL00, Ch02].

Problem 7.1. Understand the relevance of §§5.2.6 to [Za22, Question 4.14].

7.2. Other toric-to-$G_d^a$ degenerations. Recall the program in [HT99] of classifying $G_d^a$-varieties, and comparing the resulting picture with toric geometry. Although [HT99] makes it clear that no direct analogy with toric geometry holds, Theorem 1.6 raises the following question.

Problem 7.2. Which projective toric varieties of dimension $d$ degenerate isotrivially to $G_d^a$-varieties in a manner compatible with the group actions? Classify all such degenerations.

For instance, the toric variety associated with the permutohedron (the Losev-Manin space) has the property in Problem 7.2, by Theorem 1.6.

References

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