Values of the $F$-pure threshold for homogeneous polynomials

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Abstract
We find a formula, in terms of $n$, $d$, and $p$, for the value of the $F$-pure threshold for the generic homogeneous polynomial of degree $d$ in $n$ variables over an algebraically closed field of characteristic $p$. We also show that in every characteristic $p$ and for all $d \geq 4$ not divisible by $p$, there always exist reduced polynomials of degree $d$ in $k[x, y]$ whose $F$-pure threshold is a truncation of the base $p$ expansion of $2^d \deg(J)$ at some place; in particular, there always exist reduced polynomials $f$ whose $F$-pure threshold is strictly less than $2^d \deg(J)$. We provide an example to resolve, negatively, a question proposed by Hernandez, Núñez-Betancourt, Witt, and Zhang, as to whether a list of necessary restrictions they prove on the $F$-pure threshold of reduced forms are “minimal” for $p \gg 0$. On the other hand, we also provide evidence supporting and refining their ideas, including identifying specific truncations of the base $p$ expansion of $2^d \deg(J)$ that are always $F$-pure thresholds for reduced forms of degree $d$, and computations that show their conditions suffice (in every characteristic) for degrees up to eight and several other situations.

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VALUES OF THE F-PURE THRESHOLD FOR HOMOGE Neous POLYNOMIALS

1 | INTRODUCTION

Fix an algebraically closed field $k$ of prime characteristic $p > 0$.

Let $f$ be a homogeneous polynomial of degree $d$ over $k$. The $F$-pure threshold is an invariant of the degree $d$ hypersurface in projective space cut out by $f$. It can be understood as a measure of how far the hypersurface is from being Frobenius split, analogously to how the log canonical threshold measures the failure of a divisor to be log canonical at a point.

For homogeneous polynomials in $n$ variables of degree $d$, it is known that the $F$-pure threshold satisfies

$$\text{fpt } f \leq \min \left( \frac{n}{d}, 1 \right).$$

However, for specific values of $n$, $d$ (and $p$), this bound is not sharp. In this paper, we establish a sharp upper bound for every value of $n$ and $d$ — indeed, we show the maximum value of the $F$-pure threshold is either $\lambda = \min \left( \frac{n}{d}, 1 \right)$ or a specific truncation of a base $p$ expansion of $\lambda$; see Theorems 4.2 and 4.8.

Our results can be framed in terms of the $F$-pure threshold function on the parameter space $\mathbb{P}(\text{Sym}^d(k^n)^*)$ of all polynomials of degree $d$ in $n$ variables (up to scalar multiple). This function is constant on locally closed sets, and defines a finite stratification of $\mathbb{P}(\text{Sym}^d(k^n)^*)$, as we explain in Section 3. Our Theorem 4.2 provides a formula for the $F$-pure threshold of the polynomials in the largest stratum — a dense open subset of $\mathbb{P}(\text{Sym}^d(k^n)^*)$ — in terms of $p$, $d$, and $n$. In particular, our formula shows that the $F$-pure threshold of a generic degree $d$ polynomial over field of characteristic $p$ is often strictly smaller than the log canonical threshold of a generic degree $d$ polynomial over $\mathbb{C}$, which is always precisely $\min \left( \frac{n}{d}, 1 \right)$.

Our work is inspired by work of Hernandez, Nuñez-Betancourt, Witt, and Zhang [10], who investigated the $F$-pure threshold of homogeneous polynomials cutting out a smooth projective hypersurface. In two variables, they prove that the $F$-pure threshold of a reduced polynomial is either $\frac{2}{d}$ or some truncation of the nonterminating base $p$ expansion of $\frac{2}{d}$; they also list some necessary conditions on the place of the truncation and raise the issue as to what extent these might be sufficient. We show that while their conditions are not sufficient for all $d$ and $p$ (see Example 5.2), there always exist reduced polynomials of degree $d$ (provided that $p$ does not divide $d$ and $d \geq 4$) whose $F$-pure threshold is some truncation of $\frac{2}{d}$; see Theorem 5.6.

Ideally, we would like to understand, for each $n$, $d$, and $p$, the distinct values of the $F$-pure threshold as we range over all reduced polynomials in $\mathbb{P}(\text{Sym}^d(k^n)^*)$ (or over those defining a reduced, or a smooth, projective hypersurface). We can easily describe equations for the locally closed set $X_\lambda$ of polynomials whose $F$-pure threshold is exactly $\lambda$ (see §3), but it may be difficult to determine whether or not a particular $X_\lambda$ is nonempty, and if so, whether its intersection with the locus of reduced polynomials is nonempty. We make progress toward this task by showing that when $p$ does not divide $d$, there is always a nonempty stratum of reduced polynomials of degree $d$ whose $F$-pure threshold is strictly less than $\frac{2}{d}$; see Theorem 5.3. In Conjecture 6.1, we speculate more precisely that there is always a reduced polynomial of degree $d$ whose $F$-pure threshold is exactly the $e$th truncation of $\frac{2}{d}$, for $e$ the order of $p$ in the multiplicative group $(\mathbb{Z}/d\mathbb{Z})^\times$ (when $d$

\footnote{Note that a homogeneous polynomial in two variables over an algebraically closed field defines a smooth subscheme of $\mathbb{P}^1$ if and only if it has no repeated factors, that is, is reduced.}
is odd) or \((\mathbb{Z}/d\mathbb{Z})^\times\) (when \(d\) is even). We prove this conjecture when \(e = 1\) or 2; see Remark 6.3 and Theorem 6.4.

In Section 7, we determine the \(F\)-pure thresholds for all homogeneous polynomials in two variables in degrees up to eight, in each and every characteristic \(p\). These computations provide further support for Conjecture 6.1.

Lower bounds on the \(F\)-pure threshold of reduced polynomials are also of interest. Proposition 8.1 provides a lower bound for the \(F\)-pure threshold of a reduced degree \(d\) polynomial as the first nonvanishing truncation of the base \(p\) expansion of \(\frac{2}{d}\). This raises the question: what is the \(F\)-pure threshold for the smallest stratum containing a reduced polynomial in the stratification on \(\mathbb{P}(\text{Sym}^d(k^n)^*)\) by \(F\)-pure threshold in terms of \(n, d,\) and \(p\)? Of course, it is also of interest to ask the same about polynomials defining a smooth hypersurface. We make some specific speculations about the answers in the final section of the paper.

2 | PRELIMINARIES ON \(F\)-PURE THRESHOLD

Fix a field \(k\) of positive characteristic \(p\). For an ideal \(I\) in a ring of characteristic \(p\), the notation \(I^{[p^e]}\) denotes the ideal generated by the \(p^e\)th powers of the elements (equivalently, any set of generators) of \(I\).

**Definition 2.1.** The \(F\)-pure threshold of \(f \in k[x_1, \ldots, x_n]\) (at the maximal ideal \(m = \langle x_1, x_2, \ldots, x_n \rangle\)) is the real number

\[
\text{fpt}(f) = \sup \left\{ \frac{N}{p^e} \left| f^N \notin m^{[p^e]} \right\} \right\} = \inf \left\{ \frac{N}{p^e} \left| f^N \in m^{[p^e]} \right\} \right\},
\]

where \(m^{[p^e]}\) denotes the Frobenius power \(\langle x_1^{p^e}, \ldots, x_n^{p^e} \rangle\) of \(m\).

Originally, the \(F\)-pure threshold was defined somewhat differently by Takagi and Watanabe [18] as the “threshold” \(c\) beyond which the pair \((S, f^c)\) fails to be \(F\)-pure (see also [7]). The reformulation above has evolved through [15] and [3]. See also [1].

The \(F\)-pure threshold should be viewed as a “characteristic \(p\) analog” of the log canonical threshold, with smaller values of the \(F\)-pure threshold corresponding to “more singular” objects. Many of its properties are familiar from corresponding statements about the log canonical threshold. Some properties are summarized below in our setting.

**Proposition 2.2.** Let \(f\) be a homogeneous form of degree \(d > 0\) over a field \(k\) of characteristic \(p > 0\). Then

1. \(\text{fpt}(f)\leq 1\).
2. For a monomial \(f = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\), \(\text{fpt}(f) = \min\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right)\).
3. \(\text{fpt}(f)\in \mathbb{Q}\).
4. If \(f\) is in \(n\) variables, then \(\frac{1}{d} \leq \text{fpt}(f) \leq \min(1, \frac{n}{d})\).
5. For any \(r \geq 1\), we have \(\text{fpt}(f^r) = \frac{\text{fpt}(f)}{r}\).

†Without restricting to reduced polynomials, the minimal \(F\)-pure threshold is \(\frac{1}{d}\), achieved by \(x^d\), regardless of the characteristic or number of variables.
(6) \( f^n \in m^{p^e} \) if and only if \( \text{fpt}(f) \leq \frac{n}{p^e} \).

(7) For any field extension \( k' \) of \( k \), the \( F \)-pure threshold is independent of whether we view \( f \) as a polynomial over \( k \) or over \( k' \).

Proof. The first property is an immediate consequence of the definition, and the second follows easily as well. The third can be found in [3]. The fourth and fifth are straightforward to check (see [14, 2.2]) and the sixth follows by combining (5) with [14, 4.4]. For a proof of the final statement, see [14, 3.4]. \( \square \)

In light of (7) in Proposition 2.2, there is little loss of generality in studying \( F \)-pure threshold over an algebraically closed field. We therefore assume that our ground field \( k \) is algebraically closed throughout this paper.

2.1 Basics on truncations

Fix a positive real number \( \lambda \leq 1 \), and consider a base \( p \) expansion

\[
\lambda = \frac{\alpha_1}{p} + \frac{\alpha_2}{p^2} + \frac{\alpha_3}{p^3} + \ldots
\]

where the “digits” \( \alpha_i \) are integers satisfying \( 0 \leq \alpha_i \leq p - 1 \) for all \( i \in \mathbb{N} \). If \( \lambda \) is a rational number of the form \( \frac{a}{p^e} \) with \( a \in \mathbb{Z} \), then there are two such expansions (2), one terminating after the \( e \)th place, and the other nonterminating, with all digits equal to \( p - 1 \) after the \( e \)th place. Otherwise, the expression (2) for \( \lambda \) is unique, with the sequence of digits \( \{\alpha_i\} \) neither eventually zero nor eventually the constant sequence \( \{p - 1\} \).

Thus, every real number has a unique nonterminating base \( p \) expansion, in which the sequence \( \{\alpha_i\} \) of digits is not eventually zero. We will use the notation \( \langle \lambda \rangle_e \) for the truncation at the \( e \)th spot of the nonterminating base \( p \) expansion:

\[
\langle \lambda \rangle_e := \frac{\alpha_1}{p} + \frac{\alpha_2}{p^2} + \frac{\alpha_3}{p^3} + \ldots + \frac{\alpha_e}{p^e} = \frac{N_e}{p^e}.
\]

We gather some facts about such truncations for future reference. The proofs are left as easy exercises.

Lemma 2.4. With notation as in (3),

(1) Each truncation \( \langle \lambda \rangle_e \) is strictly less than \( \lambda \);

(2) The sequence of truncations \( \{\langle \lambda \rangle_e\} \) is a nondecreasing sequence converging to \( \lambda \);

(3) The numerator \( N_e \) of \( \langle \lambda \rangle_e = \frac{N_e}{p^e} \) is the unique integer \( M \) such that

\[
0 < \lambda - \frac{M}{p^e} \leq \frac{1}{p^e};
\]

(4) For every \( e \geq 1 \), \( N_{e+1} = pN_e + \alpha_e \) where \( 0 \leq \alpha_e \leq p - 1 \);

(5) For all positive integers \( j \leq e \), the “round-down” \( \lfloor \frac{N_j}{p^{e-j}} \rfloor \) equals \( N_j \).
Lemma 2.5. Let $\lambda = \frac{n}{d}$ be a rational number where $d, n \in \mathbb{N}$, and let $\frac{N}{p^L}$ be the truncation of the nonterminating base $p$ expansion of $\lambda$. Then provided $p^L \lambda \notin \mathbb{Z}$, when we divide the integer $p^L n$ by $d$,

- the integer $N_L$ is the quotient and
- the integer $n p^L - d N_L$ is the remainder (in particular, $1 \leq n p^L - d N_L < d$).

If $p^L \lambda \in \mathbb{Z}$, then the quotient of $n p^L$ by $d$ is the integer $N_L + 1$ (with no remainder).

3 | THE STRATIFICATION BY $F$-PURE THRESHOLD

Fix an algebraically closed ground field $k$ of characteristic $p$.

The $F$-pure threshold can be viewed as a function assigning to each polynomial $f \in k[ x_1, \ldots, x_n ]$ the rational number $\text{fpt}(f)$. In this section, we prove some basic facts about this function that are likely known to experts but hard to find in the literature. At the same time, we set up notation needed in the subsequent sections.

Let $\text{Sym}^d(k^n)^*$ denote the $k$ vector space of degree $d$ homogeneous polynomials in $n$ variables, and $\mathbb{P}(\text{Sym}^d(k^n)^*)$ the projective space of such polynomials up to nonzero scalar multiple. For a positive real number $\lambda$, define the following two subsets of $\mathbb{P}(\text{Sym}^d(k^n)^*)$:

$$
X_\lambda = \{ f \mid \text{fpt}(f) = \lambda \} \quad \text{and} \quad X_{\leq \lambda} = \{ f \mid \text{fpt}(f) \leq \lambda \}.
$$

Note that $X_\lambda \subseteq X_{\leq \lambda}$ and that $X_{\leq \lambda_1} \subseteq X_{\leq \lambda_2}$ whenever $\lambda_1 < \lambda_2$.

Proposition 3.1. The set $X_{\leq \lambda}$ is a Zariski closed subset of $\mathbb{P}(\text{Sym}^d(k^n)^*)$.

Before proving Proposition 3.1, we record the following obvious corollary.

Corollary 3.2. The set $X_{\leq \lambda}$ is the locally closed set of $\mathbb{P}(\text{Sym}^d(k^n)^*)$

$$
X_\lambda = X_{\leq \lambda} \cap \left( \bigcup_{e \in \mathbb{N}} \mathbb{P}(\text{Sym}^d(k^n)^*) \setminus X_{\leq \langle \lambda \rangle_e} \right),
$$

where $\langle \lambda \rangle_e$ is the truncation of the nonterminating base $p$ expansion of $\lambda$ at the $e$th spot.

Proof. This follows immediately from Proposition 3.1 and the fact that the sequence $\langle \lambda \rangle_e$ converges to $\lambda$ with every truncation $\langle \lambda \rangle_e$ strictly less than $\lambda$. \qed

Notation 3.3. Let $I_d$ be the set of $n$-tuples $\mathbf{i} := (i_1, \ldots, i_n)$ with $i_1, \ldots, i_n$ nonnegative integers such that $i_1 + \cdots + i_n = d$. The set of all degree $d$ monomials

$$
\{ x_1^{i_1} \cdots x_n^{i_n} \mid (i_1, \ldots, i_n) \in I_d \}
$$

is a basis for $\text{Sym}^d(k^n)^*$, the $k$-vector space of all degree $d$ homogeneous polynomials in $\{ x_1, \ldots, x_n \}$. We can thus write each $f \in \text{Sym}^d(k^n)^*$ uniquely as

$$
f = \sum_{\mathbf{i} \in I_d} a_\mathbf{i} x_1^{i_1} \cdots x_n^{i_n},
$$
where \( a_i \in k \), and view the \( a_i \) as the coordinates for the space \( \operatorname{Sym}^d(k^n)^* \). Thus, the polynomial ring

\[
\mathcal K := k[\{ a_i | i \in I_d \}]
\]
is the coordinate ring of \( \operatorname{Sym}^d(k^n)^* \), as well as the homogenous coordinate ring of the projective space \( P(\operatorname{Sym}^d(k^n)^*) \) of polynomials up to scalar multiple.

Consider the \textit{generic} polynomial of degree \( d \) in \( n \) variables:

\[
f = \sum_{i \in I_d} a_i x_i^1 \cdots x_i^n \in \mathcal K[x_1, ..., x_n]. \tag{4}
\]

For each positive integer \( N \), we have

\[
f^N = \sum_{j = (j_1, ..., j_n) \in I_{Nd}} C_{Nj} x_1^{j_1} \cdots x_n^{j_n} \tag{5}
\]

with \( C_{Nj} \in \mathcal K \) denoting the coefficient of \( x_1^{j_1} \cdots x_n^{j_n} \) in the expansion of \( f^N \). Each \( C_{Nj} \) is a (possibly zero) polynomial in the coordinates \( a_i \) of degree \( N \). The following description of the polynomials \( C_{Nj} \), which will not be needed until \( \S 4 \), is easy to verify by expanding out \( f^N \) and gathering together the \( x_1^{j_1} \cdots x_n^{j_n} \) terms.

Formula 3.4. Fix a positive integer \( N \) and \( j = (j_1, ..., j_n) \in I_{Nd} \). For the polynomial \( C_{Nj} \in \mathcal K \) as defined in (5), we have

\[
C_{Nj} = \sum_k \left( \frac{N!}{\prod_{i \in I_d} k_i!} \right) \prod_{i \in I_d} a_i^{k_i} \tag{6}
\]

where the summation is over all choices of \( |I_d| \)-tuples \( k = (k_i | i \in I_d) \) satisfying

\[
\sum_{i = (i_1, ..., i_n) \in I_d} k_i = N, \quad \text{and} \tag{7}
\]

\[
\sum_{i = (i_1, ..., i_n) \in I_d} k_i i_1 = j_1, \quad \sum_{i = (i_1, ..., i_n) \in I_d} k_i i_2 = j_2, \ldots \sum_{i = (i_1, ..., i_n) \in I_d} k_i i_n = j_n. \tag{8}
\]

Equation (8) can be viewed as an expression of homogeneity of \( C_{Nj} \) with respect to certain nonstandard gradings on the polynomial ring \( \mathcal K \). For example, the first equality in (8) says that \( C_{Nj} \) is homogeneous of degree \( j_1 \) with respect to the grading where \( a_i \) (for \( i = (i_1, ..., i_n) \in I_d \)) has degree \( i_1 \); this grading tracks the degree of \( x_1 \) in the expression (5).

Proof of Proposition 3.1. Fix \( \lambda \in \mathbb{R}_{>0} \). Note that the sequence

\[
[p^e \lambda] \left\langle \frac{p^e}{p^e} \right\rangle e \in \mathbb{N} \tag{9}
\]
is a nonincreasing sequence of rational numbers converging to \( \lambda \) from above. It follows immediately that

\[
X_{\leq \lambda} \supsetneq X_{\leq \lambda} \supsetneq X_{\leq \lambda} \supsetneq \cdots \supsetneq \bigcap_{e \in \mathbb{N}} X_{\leq \lambda} = X_{\leq \lambda}. \tag{10}
\]

So, to show that \( X_{\leq \lambda} \) is closed, it suffices to show that this is so when \( \lambda \) has the form \( \frac{N}{p^e} \).
Recall that $\text{fpt}(f) \leq \frac{N}{p^E}$ if and only if $f^N \in m^{[p^E]}$ where $m = (x_1, \ldots, x_n)$ (Proposition 2.2 (6)). Thus, we need only observe that the inclusion $f^N \in m^{[p^E]}$ is a closed condition on $\mathbb{P}^d(k^n)^*$. Such statements are well known, but we do so by finding explicit defining polynomials in $\mathcal{K}$ because we will need to refer to them later.

For a polynomial $\sum_{i\in I_d} a_i x_1^{i_1} \cdots x_n^{i_n} \in \text{Sym}^d(k^n)^*$, we consider when

$$f^N = \sum_{j \in I_{Nd}} C_{Nj} x^j \in (x_1^{p^E}, \ldots, x_n^{p^E}). \quad (11)$$

Since $(x_1^{p^E}, \ldots, x_n^{p^E})$ is a monomial ideal, the inclusion $(11)$ holds if and only if the polynomial

$$C_{Nj} x^j \in (x_1^{p^E}, \ldots, x_n^{p^E}) \quad \forall j \in I_{Nd}. \quad (12)$$

Thus, $X \leq N_{\frac{p^E}{p}}$ is the closed set of $\mathbb{P}^d(k^n)^*$ defined by the polynomials $(13)$.

$$\Box$$

For future reference, we highlight the final sentence of the proof of Theorem 3.1.

**Corollary 3.5.** With notation as above, the locus in $\text{Sym}^d(k^n)^*$ of polynomials $f$ with $\text{fpt}(f) \leq \frac{N}{p^E}$ is the Zariski closed set

$$\bigcup \{ C_{Nj} \mid j = (j_1, \ldots, j_n) \in I_{Nd} \text{ with } j_k < p^E \forall k \} \subseteq \text{Sym}^d(k^n)^*. \quad (14)$$

In particular, $X \leq N_{\frac{p^E}{p}} = \mathbb{P}(\text{Sym}^d(k^n)^*)$ if and only if all $C_{Nj}$ whose indices $j = (j_1, \ldots, j_n)$ satisfy $j < p^E$ component-wise are zero as polynomials in $\mathcal{K}$.

In light of Sato’s proof of the ACC conjecture for $F$-pure threshold (see [4, 4.4]), we see that the parameter space $\mathbb{P}(\text{Sym}^d(k^n)^*)$ of degree $d$ polynomials in $n$ variables over a fixed field $k$ of characteristic $p$ admits a finite stratification by $F$-pure threshold.

**Corollary 3.6.** There is a finite chain of closed sets of $\mathbb{P}(\text{Sym}^d(k^n)^*)$,

$$X_{\leq \lambda_1} \subset X_{\leq \lambda_2} \subset \cdots X_{\leq \lambda_T} = \mathbb{P}(\text{Sym}^d(k^n)^*), \quad (15)$$

where each $X_{\leq \lambda_i} \setminus X_{\leq \lambda_{i-1}}$ consists of polynomials of $F$-pure-threshold exactly $\lambda_i$.

**Proof.** The point is that there are only finitely many distinct values of $F$-pure thresholds for a polynomial of degree $d$ in $k[x_1, \ldots, x_n]$, in which case the statement follows immediately from Proposition 3.1. The finiteness holds because the set of $F$-pure-thresholds for polynomials in $\mathbb{P}(\text{Sym}^d(k^n)^*)$ satisfies both the ascending and the descending chain conditions (ACC and DCC): there is no strictly ascending, nor any strictly descending sequence, of $F$-pure thresholds among polynomials (in $n$ variables) of degree $d$ over $k$. The ACC condition is [17, Thm 1.2], whereas the DCC condition follows from the Noetherianness of $\mathbb{P}(\text{Sym}^d(k^n)^*)$. $\Box$
Corollary 3.6 implies that there is a **maximal** value for the $F$-pure threshold of a homogeneous polynomial of degree $d$ in $n$ variables, and that the set of polynomial achieving this maximal $F$-pure threshold is a dense open set of $\mathbb{P}(\text{Sym}^d(k^n)^*)$. We call this maximal $F$-pure threshold the **generic** $F$-pure threshold for $\mathbb{P}(\text{Sym}^d(k^n)^*)$. There are always reduced polynomials (and even polynomials defining a smooth projective hypersurface) achieving the generic $F$-pure threshold, since these loci are also dense and open in $\mathbb{P}(\text{Sym}^d(k^n)^*)$.

## 4 | THE LARGEST STRATUM

Fix an algebraically closed base field $k$ of characteristic $p$.

A basic question is: **what is the $F$-pure-threshold of the generic polynomial of degree $d$ in $n$ variables over $k$?** That is, what is the maximal $F$-pure threshold among all polynomials in $\mathbb{P}(\text{Sym}^d(k^n)^*)$?

In this section, we answer this question, showing that the generic $F$-pure threshold will always be either $\min(1, \frac{n}{d})$ or a truncation of its nonterminating base $p$ expansion at some particular spot. When $n \geq d$, this is easy.

**Proposition 4.1.** For $n \geq d$, a generic polynomial in $\mathbb{P}(\text{Sym}^d(k^n)^*)$ has $F$-pure threshold $1$.

**Proof.** By Proposition 2.2, $\text{fpt}(f) \leq 1$ for all $f \in \mathbb{P}(\text{Sym}^d(k^n)^*)$, whereas the monomial $x_1 x_2 \cdots x_d$ has $F$-pure threshold $1$. □

The case where $d > n$ is covered by the next result.

**Theorem 4.2.** Fix a degree $d \geq n$. The value of the $F$-pure threshold of a generic element in $\mathbb{P}(\text{Sym}^d(k^n)^*)$ — or equivalently, the maximum value of the $F$-pure threshold of any homogenous polynomial of degree $d$ in $n$ variables — is the truncation $\left\langle \frac{n}{d} \right\rangle_L$ at the $L$th place of $\frac{n}{d}$, where $L$ is the smallest positive number such that

(a) $p^L \frac{n}{d}$ is not an integer and
(b) the remainder when $np^L$ is divided by $d$ is strictly less than $n$.

If no such $L$ exists, then the generic value of the $F$-pure threshold is $\frac{n}{d}$.

**Remark 4.3.** Note that condition (a) in Theorem 4.2 holds automatically **unless**, writing $\frac{n}{d}$ in lowest terms as $\frac{a}{b}$, we have that $b = p^e$ with $e \leq L$.

For polynomials in two variables, Theorem 4.2 simplifies as follows.

**Corollary 4.4.** Fix a degree $d \geq 2$. The maximum $F$-pure threshold of a polynomial in $\text{Sym}^d(k^2)^*$ is equal to the truncation $\left\langle \frac{2}{d} \right\rangle_e$ of the nonterminating base $p$ expansion of $\frac{2}{d}$, where $e$ is the smallest positive value of $e$ such that $2p^e \equiv 1 \mod d$.

If no such $e$ exists, then the maximum $F$-pure threshold is $\frac{2}{d}$.

**Example 4.5.** The $F$-pure threshold of a generic polynomial of even degree in two variables is $\frac{2}{d}$. Indeed, if $d$ is even, then there is no $e$ such that $2p^e \equiv 1 \mod d$, so this is a consequence of Corollary 4.4.
Remark 4.6. More generally, if \( p \equiv 1 \mod d \) (and \( d \leq n \)), Theorem 4.2 implies that the generic polynomial of degree \( d \) in \( n \) variables over a field of characteristic \( p \) has \( F \)-pure threshold \( \frac{n}{d} \). In particular, fixing \( n \) and \( d \) and letting \( p \) vary, there are infinitely many values of \( p \) for which the \( F \)-pure threshold of the generic polynomial of degree \( d \) in \( n \) variables is \( \frac{n}{d} \). This is expected, since the generic log canonical threshold over \( \mathbb{C} \) in this case is \( \frac{n}{d} \), and a long-standing open conjecture predicts that for every fixed polynomial over \( \mathbb{C} \) with log canonical threshold \( \lambda \), there should be infinitely primes such that the reduction modulo \( p \) has \( F \)-pure threshold equal to \( \lambda \); see [15, Conj 3.6] and [9].

Remark 4.7. Theorem 4.2 can also be deduced by combining [11, Thm B] and [12, Thm 3.6], but the argument is different.

To prove Theorem 4.2, we first restate it as follows.

**Theorem 4.8.** Fix a degree \( d \geq n \). For each natural number \( L \), let \( \langle \frac{n}{d} \rangle_L = \frac{N_L}{p^L} \) be the truncation of the nonterminating base \( p \) expansion of \( \frac{n}{d} \) at the \( L \)-th spot. Then

(i) If \( dN_L \leq np^L - n \) for all \( L \), then the maximum value of the \( F \)-pure threshold of any polynomial in \( n \) variables of degree \( d \) is \( \frac{n}{d} \).

(ii) Otherwise, the maximal \( F \)-pure threshold is equal to the smallest value of \( \langle \frac{n}{d} \rangle_L \) such that \( dN_L \geq np^L - n + 1 \).

**Proof of the Equivalence of Theorems 4.2 and 4.8.** If \( pL \cdot \frac{n}{d} \) is not an integer, condition (b) in Theorem 4.2 is an alternative way to write condition (ii) from Theorem 4.8 (see Lemma 2.5). On the other hand, if \( pL \cdot \frac{n}{d} \) is an integer, then in lowest terms \( \frac{n}{d} = \frac{a}{p^e} \), where \( L \geq e \), so

\[
\frac{n}{d} = \frac{N_L}{p^L} + \frac{1}{p^L}.
\]

In particular, multiplying by \( p^L d \), we have \( dN_L = p^L n - d \leq p^L n - n \) for all \( L \geq e \). So, (i) in Theorem 4.8 holds for all \( L \geq e \) in any case. \( \square \)

We now turn to the proof of Theorem 4.8. The heart is the following nonvanishing result for the coefficient polynomials \( C_{Nj} \) as defined in (5).

**Key Lemma 4.9.** Fix a degree \( d \) and number of variables \( n \). Suppose that \( M \) and \( e \) are natural numbers such that \( \frac{M}{p^e} < \frac{n}{d} \leq 1 \). Let

\[
M_j = \left\lfloor \frac{M}{p^e - j} \right\rfloor.
\]

Then the following are equivalent.

(a) \( dM_j \leq np^j - n \) for all positive integers \( j \leq e \).

(b) for some index \( j = (j_1, \ldots, j_n) \in I_{dM} \) with \( j_t \leq p^e - 1 \) for each \( t = 1, \ldots, n \), the polynomial \( C_{Mj} \) is nonzero as a polynomial in the coordinate ring \( \mathcal{K} \) of \( \mathbb{P}(\text{Sym}^d(k^n)^*) \) (see Notation 3.3).

(c) There is a homogeneous polynomial \( g \) of degree \( d \) in \( n \) variables such that \( \text{fpt}(g) > \frac{M}{p^e} \).
Remark 4.10. One situation where we will apply Lemma 4.9 is the following. Let \( \left( \frac{n}{d} \right)_e = \frac{N_e}{p^e} \) for some positive \( e \). Then taking \( M \) to be \( N_e \), it follows that \( M_j = N_j \) for all \( j \leq e \).

We will prove the Key Lemma 4.9 by induction on \( e \). Lemma 4.12 will provide the base case. For both the base case and the inductive step, we make use of the following easy fact.

Lemma 4.11. Given nonnegative integers \( a_1, \ldots, a_n, b \) with \( b \leq \sum_{i=1}^{n} a_i \), there exist nonnegative integers \( b_1, \ldots, b_n \) such that \( b_i \leq a_i \) for all \( i \), and \( \sum_{i=1}^{n} b_i = b \).

Proof of Lemma 4.11. Induce on \( b \). If \( b = 0 \), take \( b_i = 0 \) for all \( i \). If \( b > 0 \), then some \( a_i > 0 \); without loss of generality, say \( a_1 > 0 \). Consider the collection of nonnegative integers \( b-1, a_1 - 1, a_2, \ldots, a_n \). By our inductive assumption, there exist \( b'_1, \ldots, b'_n \) such that \( b'_1 \leq a_1 - 1 \) and \( b'_i \leq a_i \) for all \( i \geq 2 \), with \( \sum_{i=1}^{n} b'_i = b - 1 \). The proof is completed by taking \( b_1 = b'_1 + 1 \) and \( b_i = b'_i \) for \( i \geq 2 \).

Lemma 4.12. With notation as above, if \( p > N \), then the polynomial \( C_{Nj} \in \mathcal{K} \) is nonzero for all \( j \in I_{Nd} \).

Proof of Lemma 4.12. Since \( N < p \), the multinomial coefficients \( \frac{N!}{\prod_{i \in I_d} k_i!} \) are all nonzero, so looking at Formula 3.4, the conclusion follows once we show that for each \( j \in I_{Nd} \), there exist choices of \((k_i | i \in I_d)\) satisfying the requirements (7) and (8). In other words, we need to show that every monomial \( x_{i_1}^{j_1} \cdots x_{i_n}^{j_n} \) with \( j = (j_1, \ldots, j_n) \in I_{Nd} \) appears in the expansion (3.4) of \( f^N \), where \( f \) is the generic polynomial \( \sum_{i \in I_d} a_i x_{i_1}^{j_1} \cdots x_{i_n}^{j_n} \) of degree \( d \) in \( n \) variables.

For this, we induce on \( N \). If \( N = 1 \), the claim is obvious. Let \( N > 1 \), and fix \((j_1, \ldots, j_n) \in I_{Nd} \). From Lemma 4.11, we know that we can choose \((i_1, \ldots, i_n) \in I_d \) such that \((i_1, \ldots, i_n) \leq (j_1, \ldots, j_n) \) component-wise. Since \( \sum_{i=1}^{n} (j_i - i_i) = d(N - 1) \), the inductive assumption implies that the monomial \( x_{i_1}^{j_1-i_1} \cdots x_{i_n}^{j_n-i_n} \) appears with nonzero coefficient in the expansion (3.4) of \( f^N \). Multiplying by \( f \) therefore there is a monomial \( x_{i_1}^{j_1} \cdots x_{i_n}^{j_n} = (x_{i_1}^{j_1} \cdots x_{i_n}^{j_n})(x_{i_1}^{j_1-i_1} \cdots x_{i_n}^{j_n-i_n}) \) appearing in the expansion (3.4) of \( f \cdot f^{N-1} = f^N \). This completes the proof.

Proof of Key Lemma 4.9. First note that (b) and (c) are equivalent by Corollary 3.5 and Proposition 2.2 (6).

We next show that (b) implies (a). If \( dM_j \geq np^j - n + 1 \) for some \( j \leq e \), then \( f^{M_j} \in (x_{1}^{p^j}, \ldots, x_{n}^{p^j}) \) for all polynomials \( f \) of degree \( d \) in \( n \) variables. Raising to the \( p^{e-j} \)-power, also \( f^{p^{e-j}M_j} \in (x_{1}^{p^{e-j}}, \ldots, x_{n}^{p^{e-j}}) \). So, since \( M \geq p^{e-j}M_j \), we have \( f^{M} \in (x_{1}^{p^{e-j}}, \ldots, x_{n}^{p^{e-j}}) \) for all \( f \in \text{Sym}^d(k^n)^* \). But now Corollary 3.5 implies that the polynomials \( C_{Mj} \) are zero for all \( j = (j_1, \ldots, j_n) \in I_{dM} \) with all components \( j_i \leq p^{e-j} - 1 \).

Finally, we prove (a) implies (b) by induction on \( e \). First suppose that \( e = 1 \). In this case, (a) implies \( M_1 = M < p \), so Lemma 4.12 ensures that all \( C_{Mj} \) are nonzero.

For the inductive step, assume that \( e \geq 2 \). Let \( \overline{M} = \lfloor \frac{M}{p} \rfloor \), and \( \overline{M}_j = \lfloor \overline{M} \rfloor \). Observe that

(i) \( \frac{\overline{M}}{p^{e-1}} \leq \frac{M}{p^e} < \frac{n}{d} \leq 1 \),
(ii) \( d\overline{M}_j < d\lfloor \frac{M}{p^{e-1}} \rfloor = dM_j \leq np^j - n \).
Therefore, we can apply the inductive hypothesis to the natural numbers $\overline{M}, e - 1$ to conclude that there is some nonzero polynomial $C_{\overline{M}J} \in K$ with index $J = (J_1, ..., J_n) \in I_{d\overline{M}}$ and all components $J_i \leq p^{e-1} - 1$. Writing $M = p\overline{M} + \alpha$ where $0 \leq \alpha < p$, with this choice of index $J$ therefore the products,

$$(C_{\overline{M}J})^p C_{\alpha k}$$

are nonzero polynomial for all choices of $k = (k_1, ..., k_n) \in I_{d\alpha}$.

Now let us look at the polynomials $C_{M_\ell}$. Fix an index $\ell \in I_{dM}$. Expanding $f^M = (f^{\overline{M}})^p f^\alpha$, we see that

$$C_{M \ell} = \sum_{j \in I_{d\overline{M}}} (C_{\overline{M}J})^p C_{\alpha \ell - pj}.$$  

Here, we use the convention that $C_{\alpha \ell - pj} = 0$ if $\ell - pj$ has a negative component. Note that the nonzero polynomial $(C_{\overline{M}J})^p C_{\alpha \ell - pj}$ as in (16) would be one of the terms in the sum (17), provided that $\ell \geq pJ$ component-wise.

We now claim that there is a choice of $\ell = (L_1, ..., L_n) \in I_{d\overline{M}}$ with every component $L_i \leq p^e - 1$ and $\ell \geq pJ$ component-wise (with $J \in I_{d\overline{M}}$ the special index chosen above). Equivalently, we claim that there is a choice of $K = (K_1, ..., K_n) \in I_{d\alpha}$ with components $K_i \leq p^e - pJ_i - 1$ for each $i$. To see this, note that the assumption that $dM \leq np^e - n$ gives

$$d\alpha = dM - pd\overline{M} \leq np^e - n - \sum_{i=1}^n pJ_i = \sum_{i=1}^n (p^e - 1 - pJ_i).$$

The existence of the claimed index $\ell$ now follows from Lemma 4.11.

We will complete the proof of the Key Lemma by showing that for this choice of $\ell$, the polynomial $C_{M \ell}$ is nonzero. Because the nonzero polynomial $(C_{\overline{M}J})^p C_{\alpha K}$ is one of the terms in the summation (17), it suffices to show that it does not cancel in the sum.

To this end, fix a monomial $\prod_{i \in I} a_i^{l_i}$ that has nonzero coefficient in $C_{\overline{M}J}$, and a monomial $\prod_{i \in I} a_i^{k_i}$ that has nonzero coefficient in $C_{\alpha K}$ (where $K = \ell - pJ$). Then $\prod_{i \in I} a_i^{l_i + k_i}$ has nonzero coefficient in $(C_{\overline{M}J})^p C_{\alpha K}$.

We claim that the monomial $\prod_{i \in I} a_i^{l_i + k_i}$ does not appear in any other term $(C_{\overline{M}J})^p C_{\alpha \ell - pj}$ for $j \neq J$, and therefore, it cannot get canceled in the summation (17). To check this claim, assume, on the contrary, that $\prod_{i \in I} a_i^{l_i + k_i} = \prod_{i \in I} a_i^{l_i'} + k_i'$ where $\prod_{i \in I} a_i^{l_i'}$ is a monomial appearing in $C_{\overline{M}J}$ and $\prod_{i \in I} a_i^{k_i'}$ is a monomial appearing in $C_{\alpha \ell - pj}$. Note that $0 \leq k_i, k_i' \leq p - 1$ for all $i \in I$ (since $\sum_{i \in I} k_i = \sum_{i \in I} k_i' = \alpha \leq p - 1$), so $l_i p + k_i = l_i' p + k_i'$ implies $l_i = l_i'$ and $k_i = k_i'$ for all $i \in I$. We note that the monomial $\prod_{i \in I} a_i^{l_i}$ can only occur as a term in $C_{\overline{M}J}$ if

$$\sum_{i \in I} k_i l_i = j_1, \ldots, \sum_{i \in I} k_i l_n = j_n.$$  

In other words, there is a unique $j$ such that $\prod_{i \in I} a_i^{l_i}$ occurs as a term in $C_{\overline{M}J}$, so we must have $j = J$. This completes the proof of the Key Lemma.
Before proving Theorem 4.8, we need one more lemma.†

**Lemma 4.13.** Fix \( d \geq n \). If the \( F \)-pure threshold of the generic homogeneous polynomial of degree \( d \) in \( n \) variables is not \( \frac{n}{d} \), then it is some truncation of the nonterminating base \( p \) expansion of \( \frac{n}{d} \).

**Proof of Lemma 4.13.** The \( F \)-pure threshold of any \( f \in P(\text{Sym}^d(k^n)^*) \) cutting out a smooth hypersurface in \( \mathbb{P}^{n-1} \) is either \( \frac{n}{d} \) or a smaller rational number with denominator a power of \( p \) by [10, Theorem 3.5]. In particular, we can write the \( F \)-pure threshold of the generic hypersurface in the form

\[
\frac{N_e - E}{p^e}
\]

for some nonnegative integer \( E \) and some truncation \( \frac{N_e}{p^e} = \langle \frac{n}{d} \rangle_e \) of the nonterminating base \( p \) expansion of \( \frac{n}{d} \). We claim that \( E \) must be 0, which will prove the lemma.

For this, note that since \( \frac{N_e}{p^e} < \frac{n}{d} \), we have \( dN_e < np^e \) and so if \( E \geq 1 \),

\[
d(N_e - E) < np^e - dE \leq np^e - d \leq np^e - n.
\]

Let \( M = N_e - E \) and note that

\[
\left\lfloor \frac{M}{p^{e-j}} \right\rfloor = \left\lfloor \frac{N_e - E}{p^{e-j}} \right\rfloor \leq \left\lfloor \frac{N_e}{p^{e-j}} \right\rfloor = N_j \leq np^j - n
\]

for all positive \( j \leq n \). So, by the Key Lemma 4.9, there is a some \( g \in P(\text{Sym}^d(k^n)^*) \) such that \( g^{N_e-E} \notin m[p^e] \). Such \( g \) has \( F \)-pure threshold strictly larger than \( \frac{N_e-E}{p^e} \) by Proposition 2.2 (6). In other words, \( \frac{N_e-E}{p^e} \) cannot be the maximal (generic) \( F \)-pure threshold. This contradiction forces \( E = 0 \). So, the generic \( F \)-pure threshold is \( \frac{N_e}{p^e} \), a truncation of \( \frac{n}{d} \). \( \square \)

We now have all the pieces in place to prove the theorem.

**Proof of Theorem 4.8.** Suppose \( dN_L \leq np^L - n \) for all \( L \). Assume, on the contrary, that the maximal value of \( \text{fpt}(f) \) for \( f \in P(\text{Sym}^d(k^n)^*) \) is strictly less than \( \frac{n}{d} \). Because there are only finitely many values of the \( F \)-pure threshold for polynomials in \( P(\text{Sym}^d(k^n)^*) \), and the truncations \( \frac{N_e}{p^e} \) approach \( \frac{n}{d} \) from below, there must be some \( e > 0 \) such that \( \text{fpt}(f) < \langle \frac{n}{d} \rangle_L \) for all \( f \in P(\text{Sym}^d(k^n)^*) \) and all \( L \geq e \). This says \( f^{N_L} \in m[p^L] \) for all \( f \in P(\text{Sym}^d(k^n)^*) \), and so, the polynomials \( C_{N,L}^j \) are all zero for all large \( L \) and all indices \( j \) component-wise less than \( p^L \) (by Corollary 3.5). This contradicts the Key Lemma 4.9.

Now suppose that \( L \) is minimal such that \( dN_L \geq np^L - n + 1 \). This inequality forces \( f^{N_L} \in m[p^{L-1}] \) for all \( f \in P(\text{Sym}^d(k^n)^*) \), so that the maximal \( F \)-pure threshold is bounded above by \( \frac{N_L}{p^L} \), which is strictly less than \( \frac{n}{d} \). By Lemma 4.13, the maximal \( F \)-pure threshold is therefore some truncation of \( \frac{n}{d} \), so it is either the \( L \)th truncation, or some earlier truncation. By the minimality of \( L \), the Key Lemma ensures that there is some \( g \in P(\text{Sym}^d(k^n)^*) \) such that \( g^{N_{L-1}} \notin m[p^{L-1}] \).

† In two variables, the lemma follows from [10, 4.4].
So, $\text{fpt}(g) > \frac{N_{L-1}}{p^{L-1}}$ by Proposition 2.2 (6). This says that the maximal $F$-pure threshold is strictly larger than the $L - 1$-st truncation. So, the maximal $F$-pure threshold must be $\frac{N_{L}}{p^{L}} = \langle \frac{a}{d} \rangle_{L}$, as claimed.

\section{Nonempty Strata for Reduced Polynomials in Two Variables}

We now take a closer look at the stratification by $F$-pure threshold for reduced polynomials in two variables.

The stratification of Corollary 3.6 can be sharpened for polynomials in two variables by incorporating results due to Hernandez, Núñez-Betancourt, Witt, and Zhang — namely, we see that every $F$-pure threshold of a polynomial defining a smooth\footnote{Since a zero dimensional scheme is smooth over an algebraically closed field if and only if it is reduced, reduced polynomials in two variables are the same as polynomials in two variables defining a smooth hypersurface in $\mathbb{P}^1$.} projective hypersurface is either $\frac{2}{d}$ or some truncation:

\begin{corollary}
Fix a degree $d \geq 2$. There exists a natural number $e$ such that the variety $V_{\text{sm}} \subseteq \mathbb{P}(\text{Sym}^d(k^2)^*)$ of reduced polynomials of degree $d$ in two variables is a disjoint union

$$V_{\text{sm}} = V_{\langle \frac{2}{d} \rangle_1} \bigsqcup V_{\langle \frac{2}{d} \rangle_2} \bigsqcup V_{\langle \frac{2}{d} \rangle_3} \ldots \bigsqcup V_{\langle \frac{2}{d} \rangle_e} \bigsqcup V_{\frac{2}{d}}$$

where $V_{\langle \frac{2}{d} \rangle}$ is the locally closed locus of reduced polynomials whose $F$-pure threshold is precisely $\lambda$.
\end{corollary}

We emphasize that some of the loci $V_{\langle \frac{2}{d} \rangle_L}$ or $V_{\frac{2}{d}}$ in the stratification (18) can be empty. Indeed, Hernandez, Núñez-Betancourt, Witt, and Zhang \cite[Thm 4.4]{10} proved the following necessary conditions on $L$ such that $V_{\langle \frac{2}{d} \rangle_L}$ is nonempty: writing $\frac{2}{d}$ in the lowest terms as $\frac{a}{b}$ and assuming $\langle \frac{2}{d} \rangle_L \neq 0$,

(I) If $p$ does not divide $b$, then $L$ is bounded above by the order of $p$ in the multiplicative group $(\mathbb{Z}/b\mathbb{Z})^\times$.

(II) If $p > b$, then for every natural number $e' < L$, the integer $a$ is less than the remainder when the integer $ap^{e'}$ is divided by $b$.

(III) The remainder when $ap^L$ is divided by $b$ is a positive integer less than or equal to $b - a$.

A natural question\footnote{Here, the notation $\langle \frac{2}{d} \rangle_L$ denotes the truncation of the nonterminating base $p$ expansion of $\frac{2}{d}$ at the $L$th spot; see Subsection 2.3.} is the sufficiency of the above three conditions: given values of $d$ and $L$ satisfying (I)–(III), does there exist a reduced homogeneous polynomial $f \in k[x, y]$ of degree $d$ such that $\text{fpt}(f)$ is the truncation of the nonterminating base $p$ expansion of $\frac{2}{d}$ at the $L$th term? While this

\footnote{See the discussion in \cite{10} about “minimal lists” of conditions; for example, Remark 4.8 and the subsequent examples.}
turns out to be the case for small values of $d$ (see §7), the next example shows that these three conditions are not sufficient in general.

**Example 5.2.** Fix any $p > 7$. Let $d = 2p - 3$. Note that $L = 2$ satisfies conditions (I), (II), and (III) above. The second truncation $\langle \frac{2}{d} \rangle_2$ is $\frac{1}{p} + \frac{1}{p^2}$. However, $\frac{1}{p} + \frac{1}{p^2}$ is in the interval $(\frac{1}{p}, \frac{1}{p-1})$, and hence cannot be the value of any $F$-pure threshold by [4, 4.3].

Indeed, fixing arbitrary values of $n, d,$ and $p$, it is not even obvious whether or not there is any nonempty $U_{\langle \frac{2}{d} \rangle_L}^\ast$. The next result settles this issue.

**Theorem 5.3.** Fix a characteristic $p$ and degree $d \geq 4$, not divisible by $p$. Then there always exist reduced polynomials $f \in k[x, y]$ of degree $d$ whose $F$-pure threshold is a truncation $\langle \frac{2}{d} \rangle_L$ of the nonterminating base $p$ expansion of $\frac{2}{d}$ at the $L$th spot, for some $L$. In particular, at least one $U_{\langle \frac{2}{d} \rangle_L}^\ast$ in the decomposition (18) is nonempty.

When $d < 4$, Theorem 5.3 fails:

**Remark 5.4.** If $d = 2$, then every reduced polynomial has $F$-pure threshold $1 = \frac{2}{d}$: up to linear change of coordinates, such a polynomial is $xy$, so this follows from Proposition 2.2 (b). In this case, the stratification (18) becomes $U_{sm}^\ast = U_{\frac{2}{d}}^\ast$ and all $U_{\langle \frac{2}{d} \rangle_L}^\ast$ are empty.

**Remark 5.5.** If $d = 3$, then every reduced polynomial can be put in the form $xy(x + y)$ after a linear change of coordinates, so all reduced polynomials have the same $F$-pure threshold. In this case, the decomposition (18) has only one stratum

$$U_{sm}^\ast = \begin{cases} U_{\frac{2}{3}}^\ast & \text{if } p = 3 \text{ or } p \equiv 1 \mod 3 \\ U_{\langle \frac{2}{d} \rangle_L}^\ast = U_{\frac{2p-1}{3p}}^\ast & \text{if } p \equiv 2 \mod 3, \end{cases}$$

where the $F$-pure thresholds are computed using Corollary 4.4. (See also [10]).

We will prove Theorem 5.3 by proving the following stronger statement.

**Theorem 5.6.** Fix a characteristic $p$ and degree $d \geq 4$, not divisible by $p$. Let $e$ be the smallest positive integer such that $d$ divides $2p^e - 2$. There exists a reduced homogeneous polynomial in $k[x, y]$ whose $F$-pure threshold is $\langle \frac{2}{d} \rangle_L$ for some $L \leq e$.

Indeed, we conjecture that, under certain obvious necessary conditions, the locus $U_{\langle \frac{2}{d} \rangle_L}^\ast$ itself (with $e$ as in Theorem 5.6) is always nonempty; see Conjecture 6.1.

**Remark 5.7.** If $p$ divides $d$, Theorem 5.3 can fail. For example, if $d = p^t$ or $d = 2p^t$, then no value of $L$ satisfies condition (III) above. In these cases, there is only one stratum in the filtration (18) — every reduced polynomial has $F$-pure threshold equal to $\frac{2}{d}$. 

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**VALUES OF THE $F$-PURE THRESHOLD FOR HOMOGENEOUS POLYNOMIALS**
Theorem 5.6 is only superficially stronger than Theorem 5.3 because condition (I) already gives a bound on the truncation place. Indeed, there are several equivalent ways to think about the bound in condition (I).

**Proposition 5.8.** For a prime $p$ and integer $d \geq 2$ not divisible by $p$, the following are equivalent statements about a positive integer $e$:

1. The integer $e$ is the smallest positive integer such that $d$ divides $2p^e - 2$.
2. Writing the fraction $\frac{2}{d}$ as $\frac{a}{b}$ in the lowest terms, $e$ is the order of $p$ in the multiplicative group $(\mathbb{Z}/b\mathbb{Z})^\times$.
3. The $e$th truncation $\langle \frac{2}{d} \rangle_e$ is the fraction $\frac{2p^{e-2}}{dp^e}$, and no earlier truncation satisfies $\langle \frac{2}{d} \rangle_L = \frac{2p^{L-2}}{dp^L}$.
4. The nonterminating base $p$ expansion of $\frac{2}{d}$ begins repeating with period $e$ at the $e$th place.

**Proof.** We mostly leave this as an exercise, pointing the reader to the basic facts established in Lemmas 2.4 and 2.5. For (4), write $\frac{2}{d} = \frac{N_e}{p^e} + R$ where $R = \frac{1}{p^f} \beta$ for some $\beta \leq 1$ (so $R$ is the remaining part of the base $p$ expansion). Multiplying by $dp^e$, we have $2p^e = dN_e + dp^eR$, so solving for $R$, we see that $R = \frac{1}{p^f} \frac{2}{d}$.

We now turn our attention to the proof of Theorem 5.6. The idea is to show that for $e$ as in the theorem, the closed locus $X \leq N_e \frac{p^e}{p^f}$ is not entirely contained in the closed locus of nonreduced polynomials in $\mathbb{P}(\text{Sym}^d(k^2)^*)$.

**Notation 5.9.** Simplifying the notation defined earlier in (3.3), we let $a_0, \ldots, a_d$ be the affine coordinates for $\text{Sym}^d(k^2)^*$ so that a polynomial of degree $d$ is written

$$a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d.$$ 

It makes sense to also denote the polynomials $C_{N,j_2}$, where $(j_1, j_2) \in I_{dN}$ simply by

$$C_{N,j_2},$$

where $j_2$ is an integer index satisfying $0 \leq j_2 \leq dN$,

as the value of $j_1$ is uniquely determined to be $dN - j_2$ in this case. So, $C_{N,j} \in k[a_0, \ldots, a_d]$ is the coefficient of $x^{dN-j}y^j$ in the expansion of $(a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d)^N$. Note that the degree $d$ is suppressed in our notation but the polynomials $C_{N,j}$ very much depend on $d$.

**The Discriminant.** Recall that the discriminant $\Delta_d \in \mathcal{K} = k[a_0, \ldots, a_d]$ is an irreducible homogeneous polynomial defining the locus of nonreduced polynomials in $\text{Sym}^d(k^2)^*$ (or $\mathbb{P}(\text{Sym}^d(k^2)^*)$). The discriminant $\Delta_d$ is invariant (up to scalar multiple) under the natural $GL_2(k)$ action on $\mathcal{K}$ induced by change of coordinates $x$ and $y$. The reader can consult the first chapter of [5] for background on discriminants.

We make use of the following general lemma.

**Lemma 5.10.** Consider the natural $GL_2(k)$ action by change of coordinates on the affine space $\text{Sym}^d(k^2)^*$ of all polynomials of degree $d$ in two variables. Let $C \subseteq k[a_0, \ldots, a_d]$ be a homogeneous radical ideal stable under this action, and let $R$ be the quotient ring $k[a_0, a_1, \ldots, a_d]/C$. If $\dim(R/(a_{d-1}, a_d)R) = \dim(R) - 2$, then there exist reduced polynomials $f = \sum a_i x^{d-i}y^i \in \mathcal{V}(C)$. Equivalently, the discriminant $\Delta_d \not\in C$. 
Proof of Lemma 5.10. Let $\mathcal{X} \subseteq \text{Sym}^d(k^2)^*$ denote the space of polynomials divisible by $x^2$, so that in the notation (5.9), the set $\mathcal{X}$ is the subvariety of $\text{Sym}^d(k^2)^* \cong k^{d+1}$ cut out by the equations $a_{d-1} = a_d = 0$. Note that $\mathcal{X}$ is a proper closed subset of the closed set $\mathcal{N} = \mathcal{V}(\Delta_d)$ of all nonreduced polynomials of degree $d$.

Consider the regular map

$$
k \times \mathcal{X} \rightarrow \mathcal{N} \quad \text{defined by} \quad \Phi(b, x^2 g(x, y)) = (x + by)^2 g(x + by, y).
$$

This is the action of the subgroup

$$
G = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in k \right\} \subseteq GL_2(k)
$$
on $\mathcal{X}$ by change of coordinates. The image of $\Phi$ is the dense set of nonreduced polynomials divisible by $(x + by)^2$ for some $b \in k$ — indeed, its complement in $\mathcal{N}$ is contained in the proper closed set $\mathcal{V} = \mathcal{V}(a_0, a_1)$ of polynomials divisible by $y^2$. Thus, the closure of $\text{Im}(\Phi)$ is the full set $\mathcal{N}$ of nonreduced polynomials in $\text{Sym}^d(k^2)^* \cong k^{d+1}$.

Because the ideal $\mathcal{V}$ is stable under changes of coordinates, the map $\Phi$ restricts to a regular map $\tilde{\Phi}$ on the closed subsets given by intersection with $\mathcal{V}(C) \subseteq \text{Sym}^d(k^2)^*$:

$$
k \times \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{N}} \quad \text{defined by} \quad (b, x^2 g(x, y)) \mapsto (x + by)^2 g(x + by, y),
$$

where

$$
\tilde{\mathcal{N}} := \mathcal{N} \cap \mathcal{V}(C) \quad \text{and} \quad \tilde{\mathcal{X}} := \mathcal{X} \cap \mathcal{V}(C).
$$

It follows that the image of $\text{Im}(\tilde{\Phi})$ is $\text{Im}(\Phi) \cap \mathcal{V}(C)$, so it is dense in $\tilde{\mathcal{N}}$.

We want to show that the inclusion $\tilde{\mathcal{N}} \subseteq \mathcal{V}(C)$ is strict. For this, it suffices if $\dim \tilde{\mathcal{N}} < \dim \mathcal{V}(C) = \dim R$. Since the image of $\Phi$ is dense in $\tilde{\mathcal{N}}$, it suffices to show $\dim(\text{Im}(\Phi)) < \dim(R)$. But since $\dim(\text{Im}(\Phi)) \leq \dim(k \times \mathcal{X}) = 1 + \dim(\mathcal{X})$, it is enough to show $\dim(\mathcal{X}) \leq \dim(R) - 2$. This follows from our hypotheses, since $\mathcal{X}$ has coordinate ring $R/(a_{d-1}, a_d)R$. □

Proof of Theorem 5.6. Fix $e$ to be the smallest positive integer such that $d$ divides $2p^e - 2$, and observe that $2p^e - 2 = N_e d$, where $N_e$ is the $e$th truncation of $\frac{2}{d}$ (see Lemma 2.5).

We want to show that the closed set $X_{\leq N_e}$ in $\mathbb{P}(\text{Sym}^d(k^2)^*)$ is not entirely contained in the locus $\mathcal{V}(\Delta_d)$ of nonreduced polynomials in $\mathbb{P}(\text{Sym}^d(k^2)^*)$. Note that because $N_e d = 2p^e - 2$, the closed set $X_{\leq N_e}$ is defined by just one polynomial in $k[a_0, \ldots, a_d]$, namely, the coefficient $C_{N_e p^e - 1}$ of $(xy)^{p^e - 1}$ (see Corollary 3.5). So, to prove Theorem 5.6, we must show that

$$
\Delta_d \not\in \sqrt{(C_{N_e p^e - 1})}.
$$

(19)

Taking $C$ to be the principal ideal $\sqrt{(C_{N_e p^e - 1})}$, note that $C$ is invariant under linear changes of coordinates in $k[x, y]$, since the condition that $f^{N_e} \in \mathfrak{m}^{1/[p^e]}$ is. So, by Lemma 5.10, it suffices to show that $a_{d-1}, a_d$ forms a regular sequence on the ring $k[a_0, a_1, \ldots, a_d]/(C_{N_e p^e - 1})$. For this, it suffices to show that $a_{d-1}, a_d, C_{N_e p^e - 1}$ is a regular sequence on the polynomial ring $k[a_0, a_1, \ldots, a_d]$. 

So, to prove Theorem 5.6, we can prove that

\[ C_{N_e p^{e-1}} \notin (a_{d-1}, a_d). \]  

But (20) is equivalent to the claim that there is some (nonreduced) polynomial of the form

\[ x^2(a_0x^{d-2} + a_1x^{d-3}y + \cdots + a_{d-2}y^{d-2}), \]  

which has $F$-pure threshold greater than $\frac{N_e}{p^e}$. This is easy if $d$ is even: the polynomial $(xy)^{d/2}$ has $F$-pure threshold $\frac{2}{d}$ by Proposition 2.2, and $\frac{2}{d}$ is larger than any truncation. So, we might as well assume that $d$ is odd.

Note that, for any $N$,

\[
(x^2 y)^N = x^{2N}(a_0x^{d-2} + a_1x^{d-3}y + \cdots + a_{d-2}y^{d-2})^N = x^{2N} \sum_{k=0}^{(d-2)N} \binom{(d-2)N}{k} x^{(d-2)N-k} y^k = \sum_{k=0}^{(d-2)N} C_N k x^{dN-k} y^k,
\]

where the coefficient of the monomial $x^{dN-k} y^k$ can be interpreted either as the polynomial $C_N k \in k[a_0, \ldots, a_{d-2}]$ (arising from expanding out a generic polynomial of degree $d - 2$) or as the image $C_N k$ of $C_N k$ modulo $(a_{d-1}, a_d)$ (arising from expanding out a generic degree $d$ polynomial). In particular, by Lemma 4.12, we see that when $N < p$, the polynomials

\[ C_N k \notin (a_{d-1}, a_d) \quad \text{for all } k \text{ in the range } 0 \leq k \leq (d-2)N. \]  

Now, to prove Theorem 5.6, we must show that $C_{N_e p^{e-1}} \notin (a_{d-1}, a_d)$. First suppose $e = 1$. The assumptions $dN_1 = 2p - 2$ and $d$ odd imply that $N_1$ is even. Consider the degree $d$ polynomial $f = (xy)^{d-1} (x + y)$. We see that

\[ f^{N_1} = (xy)^{\frac{N_1(d-1)}{2}} (x + y)^{N_1} \]

has $(xy)^{p-1}$ term with coefficient the binomial coefficient $\binom{N_1}{\frac{N_1}{2}}$, which is nonzero since $N_1 < p$. This says that $f^{N_1} \notin (x^p, y^p)$, and so, the $F$-pure threshold of $f$ is strictly greater than $\left\langle \frac{2}{d} \right\rangle_1 = \frac{N_1}{p}$ as needed.

Now assume $e \geq 2$. Writing $N_e = pN_{e-1} + \alpha$ where $\alpha \leq p - 1$, we have

\[ C_{N_e p^{e-1}} = \sum_i (C_{N_{e-1}})^i p C_{\alpha p^{e-1} - pi}, \]

where the sum is taken over all nonnegative $i$ satisfying $i \leq dN_{e-1}$ and $0 \leq p^e - 1 - pi \leq d\alpha$. To show that this polynomial is not in $(a_{d-1}, a_d)$, we argue as in the proof of Lemma 4.9: it suffices to find one valid choice of indices $i_0$ and $j_0$ with $pi_0 + j_0 = p^e - 1$ and both

\[ C_{N_{e-1} i_0} \quad \text{and} \quad C_{\alpha j_0} \quad \text{not in} \quad (a_{d-1}, a_d). \]  

Indeed, reducing modulo \((a_{d-1}, a_d)\), we see \((\overline{C_{N_{e-1}i_0}})^p \overline{C_{\alpha j_0}}\) is not the zero polynomial, and it admits some monomial in \(a_0, \ldots, a_{d-2}\) with nonzero coefficient that does not cancel in the summation (24). This implies that \(\overline{C_{N_e p^{e-1}}} \neq 0\) and so \(\overline{C_{N_e p^{e-1}}} \notin (a_{d-1}, a_d)\).

To find the needed indices \(i_0\) and \(j_0\), note that by (23) above, since \(\alpha < p\), the only restriction needed on \(j_0\) is that \(j_0 \leq (d-2)\alpha\). To find \(i_0\), we use the following lemma, whose proof we defer until after finishing the proof of Theorem 5.6.

**Lemma 5.11.** With notation as above, fix \(d \geq 5\) and a positive integer \(j \leq e - 1\). Then there exists an index \(i\) with \(dN_j - p^{j} + 2 \leq i \leq p^j - 1\), such that the polynomial \(\overline{C_{N_j i}} \notin (a_{d-1}, a_d)\).

Using Lemma 5.11, we choose \(i_0\) such that
\[
dN_{e-1} - p^{e-1} + 2 \leq i_0 \leq p^{e-1} - 1 \quad \text{and} \quad \overline{C_{N_{e-1}i_0}} \notin (a_{d-1}, a_d).
\]

Then \((\overline{C_{N_{e-1}i_0}})^p \overline{C_{\alpha p^{e-1}i_0}}\) is one of the terms of the sum (24) and \(\overline{C_{\alpha p^{e-1}i_0}} \notin (a_{d-1}, a_d)\), provided that
\[
0 \leq p^e - 1 - pi_0 \leq (d-2)\alpha. \tag{26}
\]

The first inequality in (26) follows from the fact that \(i_0 \leq p^{e-1} - 1\). To get the second inequality in (26), first use the fact that \(i_0 \geq dN_{e-1} - p^{e-1} + 2\) and \(pN_{e-1} = N_e - \alpha\), to see that
\[

p^e - 1 - pi_0 \leq p^e - 1 - p(dN_{e-1} - p^{e-1} + 2) = 2p^e - dN_e + d\alpha - 2p - 1.
\]

Then using that \(dN = 2p^e - 2\), we see that
\[

p^e - 1 - pi_0 \leq d\alpha - 2p + 1 \leq (d-2)\alpha,
\]

so (26) holds. This completes the proof of Theorem 5.6, once we have proved Lemma 5.11.

**Proof of Lemma 5.11.** Fix \(e > 1\), else the statement is vacuous. We induce on \(j\).

We may assume that \(2p^j \not\equiv 1 \pmod d\) for all \(j \leq d\). For if \(2p^j \equiv 1 \pmod d\), then Corollary 4.4 ensures that the generic polynomial has \(F\)-pure threshold \(\left\langle \frac{j}{d} \right\rangle_{\alpha}\), which is less than or equal to \(\left\langle \frac{j}{d} \right\rangle_{\alpha}\), as needed. Therefore, also \(dN_j \leq 2p^j - 2\) for all \(j \leq e\) (by the opening lines of the proof of Corollary 4.4.) Further, since \(e\) is the smallest natural number such that \(dN_e = 2p^e - 2\), we see also that \(dN_j \leq 2p^j - 3\) for all \(j \leq e - 1\). \(\tag{27}\)

**Base case:** Assume \(j = 1\). Note that \(N_1 \leq p - 1\). So, by (23), we know that \(C_{N_1 i} \notin (a_{d-1}, a_d)\) for all nonnegative values of \(i \leq (d-2)N_1\). We need only make sure that some such value of \(i\) also satisfies
\[
dN_1 - p + 2 \leq i \leq p - 1. \tag{28}
\]

Since \(d > 4\), using (27), we have \(2N_1 < \frac{dN_1}{2} \leq \frac{2p-3}{2}\). Since \(2N_1\) is an integer, this implies \(2N_1 \leq p - 2\). Adding \((d-2)N_1\) to both sides and rearranging, we have
\[
dN_1 - p + 2 \leq (d-2)N_1.
\]
So taking \( i = \min(p - 1, (d - 2)\alpha) \) produces an \( i \) in the desired range, proving the base case.

**Inductive step:** For \( j \geq 2 \), write \( N_j = pN_{j-1} + \alpha \), where \( 0 \leq \alpha \leq p - 1 \). We have \( f^{N_j} = (f^{N_{j-1}})^p f^\alpha \), so

\[
\sum_{k=0}^{dN_j} C_{N_j,k} x^{N_j d-k} y^k = \left( \sum_{i=0}^{dN_{j-1}} C_{N_{j-1},i} x^{(dN_{j-1}-i)p} y^i \right) \left( \sum_{j=0}^{d\alpha} C_{\alpha,j} x^{\alpha d-j} y^j \right),
\]

and therefore,

\[
C_{N_j,k} = \sum_{i,j} (C_{N_{j-1},i})^p C_{\alpha,j},
\]

where the summation is over all the choices of \( 0 \leq i \leq dN_{j-1} \) and \( 0 \leq j \leq d\alpha \) with \( pi + j = k \).

If \( I = pi_0 + j_0 \) for some \( i_0 \) and \( j_0 \) with \( C_{N_{j-1}-i_0,0} \not\in (a_{d-1}, a_d) \), then \( (C_{N_{j-1}-i_0})^p \) \( C_{\alpha,j_0} \not\in (a_{d-1}, a_d) \) because the ideal \( (a_{d-1}, a_d) \) is prime. We claim that in this case, the polynomial \( C'_{N_{j-1}} \) cannot be in the ideal \( (a_{d-1}, a_d) \) either. Indeed, working modulo \( (a_{d-1}, a_d) \), we can argue as in the last part of the proof of Lemma 4.9: picking a monomial \( a_{k_0} \cdots a_{k_{d-2}} \) that has a nonzero coefficient in \( C'_{N_{j-1}} \) and a monomial \( a_{l_0} \cdots a_{l_{d-2}} \) that has nonzero coefficient in \( C'_{\alpha,j_0} \), the monomial \( a_{k_0+l_0} \cdots a_{k_{d-2}+l_{d-2}} \) has nonzero coefficient in \( (C_{N_{j-1}})^p C_{\alpha,j_0} \) and cannot cancel in the summation (29).

From the inductive hypothesis, there is an \( i_0 \) in the range

\[
dN_{j-1} - p^{j-1} + 2 \leq i_0 \leq p^{j-1} - 1
\]

such that \( C_{N_{j-1}-i_0} \not\in (a_{d-1}, a_d) \). We want to find an integer \( j_0 \) such that \( j_0 + pi_0 \) is in the interval \([dN_j - p^j + 2, p^j - 1]\) and the polynomial \( C_{\alpha,j_0} \not\in (a_{d-1}, a_d) \). This last condition is equivalent to \( j_0 \leq (d - 2)\alpha \) by (23). So, we are looking for an integer \( j_0 \) in the intersection of the intervals

\[
[0, (d - 2)\alpha] \quad \text{and} \quad [dN_j - p^j + 2 - pi_0, p^j - pi_0 - 1].
\]

The second interval is nonempty because \( dN_j \leq 2p^j - 3 \). Thus, to ensure that the intervals (31) have nonempty intersection, it suffices if

\[
p^j - pi_0 - 1 \geq 0, \quad \text{and} \quad dN_j - p^j + 2 - pi_0 \leq (d - 2)\alpha.
\]

Multiplying (30) by \( p \), we have

\[
pdN_{j-1} - p^j + 2p \leq pi_0 \leq p^j - p.
\]

Inequality (32) follows from (34) since \( pi_0 \leq p^j - p < p^j - 1 \). To get inequality (33), observe

\[
pi_0 \geq pdN_{j-1} - p^j + 2p \quad \text{from (34)}
\]

\[
= dN_j - d\alpha - p^j + 2p \quad \text{because} \quad pN_{j-1} = N_j - \alpha
\]

\[
\geq dN_j - d\alpha - p^j + 2\alpha + 2 \quad \text{because} \quad \alpha \leq p - 1
\]

\[
= dN_j - (d - 2)\alpha - p^j + 2.
\]

This concludes the proof of Lemma 5.11, and hence of Theorem 5.6. □
6 | THE EXISTENCE OF REDUCED POLYNOMIALS WITH SPECIFIC F-PURE THRESHOLDS

A natural question is whether there is any specific truncation of \( \frac{2}{d} \) that is always the F-pure threshold of a reduced polynomial in two variables.

For example, we speculate the following.

**Conjecture 6.1.** Fix any characteristic \( p \) and degree \( d \geq 4 \) not divisible by \( p \). Let \( e \) be the smallest positive integer\(^\dagger\) such that \( 2p^e \equiv 2 \mod d \). Then there is always a reduced polynomial in \( k[x, y] \) whose F-pure threshold is equal to the truncation

\[
\left\langle \frac{2}{d} \right\rangle_e \quad \text{or equivalently,} \quad \frac{2p^e - 2}{dp^e},
\]

provided that \( 2p^L \not\equiv 1 \mod d \) for any \( L < e \).

**Remark 6.2.** The condition that \( 2p^L \not\equiv 1 \mod d \) for any \( L < e \) is clearly necessary. Indeed, if \( 2p^L \equiv 1 \mod d \) for some value of \( L \), then Corollary 4.4 guarantees there is no polynomial in \( k[x, y] \) with F-pure threshold larger than \( \left\langle \frac{2}{d} \right\rangle_L \).

**Remark 6.3.** Another way to think about Conjecture 6.1: there is a reduced polynomial in \( k[x, y] \) whose F-pure threshold is \( \left\langle \frac{2}{d} \right\rangle_e \), where \( e \) is smallest natural number such that \( 2p^e \equiv 1 \) or \( 2 \mod d \).

Theorem 5.6 implies immediately that Conjecture 6.1 holds when the value of \( e \) is one. Our next goal is to prove Conjecture 6.1 holds when \( e \) is two.

**Theorem 6.4.** Fix \( p \) and \( d \geq 4 \) not divisible by \( p \). Then if \( 2p^2 \equiv 2 \mod d \) and \( 2p \not\equiv 1 \mod d \), then there exist reduced \( f \in k[x, y] \) of degree \( d \) such that \( fpt(f) = \left\langle \frac{2}{d} \right\rangle_2 = \frac{2p^2 - 2}{dp^2} \).

The key point in the proof of Theorem 6.4 is the following.

**Theorem 6.5.** Fix a prime \( p \) and degree \( d \geq 4 \) not divisible by \( p \). Assume that the remainder, when \( 2p \) is divided by \( d \) is at least three — that is, we can write \( 2p = Nd + r \) with \( N \) nonnegative and \( 3 \leq r \leq d - 1 \).

Then the closed set \( X \leq \frac{N}{p} \) of polynomials in two variables of degree \( d \) whose F-pure threshold is bounded above by \( \frac{N}{p} \) has codimension at least two in \( \mathbb{P}(\text{Sym}^d(k^2)^*) \).

**Proof that Theorem 6.5 implies Theorem 6.4.** To prove Theorem 6.4, it suffices to show that

\[
X \leq \frac{N_2}{p^2} \not\subseteq X \leq \frac{N_1}{p} \bigcup \forall(\Delta_d),
\]

where \( \Delta_d \) is the degree \( d \) discriminant (defining the locus of nonreduced polynomials in \( \mathbb{P}(\text{Sym}^d(k^2)^*) \)). Note that because \( dN_2 = 2p^2 - 2 \), the closed set \( X \leq \frac{N_1}{p^2} \) is defined by just one

\(^\dagger\) See Proposition 5.8 for some equivalent ways to describe the natural number \( e \).
polynomial, \( C_{N_2} p^{2-1} \), so has dimension \( d - 1 \). So, if
\[
X \subseteq X_{N_2} p^2 \subseteq X_{N_1} p \cup \mathcal{V}(\Delta_d),
\]
then assuming Theorem 6.5, no component of the dimension \( d - 1 \) subvariety \( X \subseteq X_{N_2} p^2 \) can be contained \( X \subseteq X_{N_1} p \), because \( X \subseteq X_{N_1} p \) has strictly smaller dimension. That is, every component of \( X \subseteq X_{N_2} p^2 \) is contained in \( \mathcal{V}(\Delta_d) \), and so \( X \subseteq X_{N_2} p^2 \subseteq \mathcal{V}(\Delta_d) \). But this means that the \( F \)-pure threshold of every reduced polynomial is strictly greater than \( \frac{N_2}{p^2} = \left( \frac{2}{d} \right)_2 \), contrary to Theorem 5.6. Thus, Theorem 6.4 follows from Theorem 6.5.

To prove Theorem 6.5, we will use the following technical lemma.

**Lemma 6.6.** Fix a prime \( p \) and degree \( d \geq 4 \). Write \( 2p = dN + r \) where \( N \) is nonnegative and \( 3 \leq r \leq d - 1 \). Suppose that there exist indices \( i, j \) such that
\[
• \ 0 \leq i, j < \frac{p}{N},
• \ d - 1 \leq i + j \leq d - 2.
\]
Then for any degree \( d \) polynomial of the form \( f = x^i y^j g \) with \( g \) not divisible by \( x \) or \( y \), we have \( f^N \notin (x^p, y^p) \).

To prove Lemma 6.6, we need the following “folklore” fact.

**Lemma 6.7.** Let \( h_1, \ldots, h_n \) and \( g_1, \ldots, g_n \) be homogeneous regular sequences in the polynomial ring \( k[x_1, \ldots, x_n] \). Then assuming that \( D := \sum_{i=1}^n \deg g_i - \sum_{i=1}^n \deg h_i > 0 \),
\[
(g_1, \ldots, g_n) : (h_1, \ldots, h_t) \subseteq (g_1, \ldots, g_n) + (x_1, \ldots, x_n)^D.
\]

**Proof.** Set \( d = \sum_{i=1}^n \deg g_i \). We first claim that for all \( N \in \mathbb{N} \),
\[
(g_1, \ldots, g_n) : (x_1, \ldots, x_n)^N \subseteq (g_1, \ldots, g_n) + (x_1, \ldots, x_n)^{d-n+1-N}. \tag{35}
\]
Indeed, when \( N = 1 \), this follows from the fact that
\[
k[x_1, \ldots, x_n] / (g_1, \ldots, g_n) \text{ and } k[x_1, \ldots, x_n] / (x_1^{\deg g_1}, \ldots, x_n^{\deg g_n})
\]
have the same Hilbert function, and hence the same socle degree, \( d - n \). For \( N \geq 2 \), the inclusion (35) follows by induction on \( N \).

Next, set \( d' = \sum_{i=1}^n \deg h_i \). Note that \( (x_1, \ldots, x_n)^{d' - n + 1} \subseteq (h_1, \ldots, h_t) \), since, again, the socle degree of the quotient \( k[x_1, \ldots, x_n] / (h_1, \ldots, h_t) \) is \( d' - n \). So,
\[
(g_1, \ldots, g_n) : (h_1, \ldots, h_t) \subseteq (g_1, \ldots, g_n) : (x_1, \ldots, x_n)^{d' - d' + 1} \subseteq (x_1, \ldots, x_n)^{d - d'},
\]
with the second inclusion obtained by applying (35) with \( N = d' - n + 1 \).

**Proof of Lemma 6.6.** Note that our assumptions imply that \( g \) is either degree one or two.
Assume $f^N \in (x^p, y^p)$, or, equivalently, $g^N \in (x^{p-N}, y^{p-N})$. Let $N'$ denote the smallest positive integer satisfying $g^{N'} \in (x^{p-N}, y^{p-N})$. The assumption that $i, j < \frac{p}{N}$ guarantees that $(x^{p-N}, y^{p-N})$ is not the unit ideal.

**Case 1: $g$ reduced.** If $g$ is reduced and not divisible by $x$ or $y$, the reader will easily verify that $x^{\frac{p}{2}} \frac{\partial}{\partial x}, y^{\frac{p}{2}} \frac{\partial}{\partial y}$ form a regular sequence consisting of forms of degree equal to $\deg g$, which is $d - (i + j)$.

Applying the operators $x^{\frac{p}{2}} \frac{\partial}{\partial x}, y^{\frac{p}{2}} \frac{\partial}{\partial y}$ to the equation $g^{N'} \in (x^{p-N}, y^{p-N})$ yields

$$g^{N'-1} \in (x^{p-N}, y^{p-N}) : \left( x^{\frac{p}{2}} \frac{\partial g}{\partial x}, y^{\frac{p}{2}} \frac{\partial g}{\partial y} \right),$$

which is contained in $(x^{p-N}, y^{p-N}) + m^{2p-(i+j)N-2d+2(i+j)}$ by Lemma 6.7. Since $g^{N'-1} \notin (x^{p-N}, p^{p-N})$, this implies

$$\deg(g^{N'-1}) \geq \deg(g^{N'-1}) \geq 2p -(i+j)N - 2d + 2(i+j).$$

Thus, $(N-1)(d-(i+j)) \geq 2p -(i+j)N - 2d + 2(i+j)$, which simplifies to

$$Nd \geq 2p - d + (i+j).$$

Recalling that $Nd = 2p - r$, we obtain $i + j \leq d - r \leq d - 3$, which contradicts our hypothesis.

**Case 2: $g$ not reduced.** In this case, $g$ has degree two, and changing coordinates, we can assume that $f = x^i y^j (x+y)^2$. Assume, on the contrary, that $f^N \in (x^p, y^p)$, then $(x+y)^{2N} \in (x^{p-N}, y^{p-N})$. Because all binomial coefficients $\binom{2N}{\ell}$ are nonzero (note $2N < p$), this implies that every monomial $x^\alpha y^\beta$ with $\alpha + \beta = 2N$ is in $(x^{p-N}, y^{p-N})$. Taking $(\alpha, \beta) = (p-iN-1, 2N-(p-iN-1))$, it follows that $2N - p + iN + 1 \geq p - jN$. Rearranging, we get $(2+i+j)N \geq 2p - 1$, and hence $dN = dN + r - 1$, contradicting our assumption on $r$.

We are now ready to prove Theorem 6.5, and hence complete the proof of Theorem 6.4.

**Proof of Theorem 6.5.** We know that the closed set $X_{\leq N/p}$ is defined by the polynomials

$$C_{Np-1}, C_{Np-2}, \ldots, C_{Np-(r-1)}$$

(36)

in $k[a_0, a_1, \ldots, a_d]$, using the notation of Subsection 5.9. If the ideal generated by the polynomials (36) has height two or more, we are done: $X_{\leq N/p}$ has codimension two in $P(\text{Sym}^d(k^2)^*)$. So, assume the ideal has height one, in which case the polynomials (36) have a common factor, since all height one primes in a polynomial ring are principal. Let $C$ be the product of the irreducible common factors, and observe that $C$ has the following properties:

(A) $C$ is homogeneous under the standard grading on $k[a_0, \ldots, a_d]$.

(B) $C$ is homogeneous under the grading on $k[a_0, \ldots, a_d]$ assigning degree $i$ to $a_i$.

(C) $C$ is invariant under interchanging $a_i$ and $a_{d-i}$ for all $i = 0, 1, \ldots, d$.

Indeed, (A) and (B) hold because each $C_N j$ is homogeneous under the respective grading (see the discussion immediately following (8)), hence so are the irreducible divisors of each, as well as their product $C$. And (C) holds because the condition $f^N \in (x^p, y^p) = m[|p|]$ is clearly invariant

\footnote{A similar trick is used in [10] as well as in [2].}
under any linear change of coordinates on \(k[x, y]\), including the action of swapping \(x\) and \(y\); such a swap induces the action on \(k[a_0, \ldots, a_d]\) that interchanges \(a_i\) and \(a_{d-i}\) for all \(i\) and swaps \(C_{N,j} \) with \(C_{N,dN-j}\) for all \(j\). This says that the greatest common square-free divisor \(C\) is invariant under swapping \(a_i\) and \(a_{d-i}\) for all \(i\).

To prove that the closed set \(X_{\leq N} \subseteq \mathbb{P}(\text{Sym}^d(k^2)^*)\), it suffices to show that we can find a projective linear space \(\Lambda\) in \(\mathbb{P}(\text{Sym}^d(k^2)^*)\) such that

\[ X_{\leq N} \cap \Lambda \]

is codimension two or more in \(\Lambda\). For this, define, for any fixed index \(j\), the ideal

\[ \mathcal{A}_j = (a_0, \ldots, a_{j-1}, \hat{a}_j, a_{j+1}, a_{j+2}, a_{j+3}, \ldots, a_d) \]  

(37)

Letting \(\Lambda_j = \mathcal{V}(\mathcal{A}_j) \subseteq \mathbb{P}(\text{Sym}^d(k^2)^*)\), we will show we can choose \(j\) (depending on whether \(d\) is even or odd), so that \(X_{\leq N} \cap \Lambda_j\) has codimension two or more in \(\Lambda_j \cong \mathbb{P}^2\).

**The case when \(d\) is even:** Let \(j = \frac{d-2}{2}\). We claim that

\[ X_{\leq N} \cap \mathcal{V}(\mathcal{A}_j) \subseteq \mathcal{V}(a_j, a_{j+1}) \cup \mathcal{V}(a_{j+1}, a_{j+2}), \]  

(38)

which will complete the proof in the case where \(d\) is even because it implies each component of \(X_{\leq N} \cap \mathcal{V}(\mathcal{A}_j)\) is contained in either \(\mathcal{V}(a_j, a_{j+1})\) or \(\mathcal{V}(a_{j+1}, a_{j+2})\), and so has codimension two or more in \(\Lambda_j\). To prove (38), observe that it is equivalent to stating that if

\[ f = x^{d-2-j} y^j \quad g = (xy)^{\frac{d-2}{2}} \quad (a_j x^2 + a_{j+1} xy + a_{j+2} y^2 + \cdots) \]

satisfies \(\text{fpt}(f) \leq \frac{N}{p}\), then either \(a_j = a_{j+1} = 0\) or \(a_{j+1} = a_{j+2} = 0\). Note that \(\frac{d}{2} < \frac{p}{N}\) (since reciprocally, \(\frac{2}{d} > \frac{N}{p}\)), so we can apply Lemma 6.6 with \((i, j) = (\frac{d-2}{2}, \frac{d}{2})\) to see immediately that \(a_j = 0\) or \(a_{j+2} = 0\). Without loss of generality, say that \(a_j = 0\). In this case, we can write

\[ f = x^{\frac{d-2}{2}} y^j (a_{j+1} x + a_{j+2} y), \]

and we can again apply Lemma 6.6, this time with \((i, j) = (\frac{d-2}{2}, \frac{d}{2})\) to see that either \(a_{j+1} = 0\), in which case (38) follows, or \(a_{j+2} = 0\). But in this latter case, we have \(a_j = a_{j+2} = 0\), so that \(f = a_{j+1} (xy)^{\frac{d}{2}}\). But then \(\text{fpt} f = \frac{2}{d} > \frac{N}{p}\). This establishes (38), and hence the proof of Theorem 6.5 for even \(d\).

**The case when \(d\) is odd:** Let \(j = \frac{d-3}{2}\). We first claim that

\[ X_{\leq N} \cap \mathcal{V}(\mathcal{A}_j) \subseteq \mathcal{V}(a_{j+2}) \cup \mathcal{V}(a_j, a_{j+1}), \]

(39)
or equivalently, that all polynomials of the form

\[ f = x^{d-2-j} y^j g = x^{\frac{d-1}{2}} y^{\frac{d-3}{2}} \left( a_j x^2 + a_{j+1} xy + a_{j+2} y^2 \right) \]

with \( \text{fpt}(f) \leq \frac{N}{p} \) must satisfy either \( a_{j+2} = 0 \) or both \( a_j = a_{j+1} = 0 \).

We can apply Lemma 6.6 to the indices \((i, j) = \left( \frac{d-1}{2}, \frac{d-3}{2} \right)\) to see that if \( \text{fpt}(f) \leq \frac{N}{p} \), then \( g \) must be divisible by \( x \) or \( y \). This says that in the expression (40), either \( a_{j+2} = 0 \), in which case (39) follows, or \( a_j = 0 \). In this latter case, when \( a_j = 0 \), we can write

\[ f = (xy)^{\frac{d-1}{2}} (a_{j+1} x + a_{j+2} y), \]

and again apply Lemma 6.6, this time with \( i = j = \frac{d-1}{2} \) and \( g = a_{j+1} x + a_{j+2} y \). Again, we see that because \( \text{fpt} f \leq \frac{N}{p} \), either \( a_{j+1} \) or \( a_{j+2} \) must be zero. In either case, (39) follows.

Now, having established (39), we interpret it as saying that each codimension one component of \( X \leq \frac{N}{p} \cap V(A_j) \) must be contained in \( V(a_{j+2}) \cap V(A_j) \), which means that their union \( V(C) \cap V(A_j) \) is contained in \( V(a_{j+2}) \cap V(A_j) \). In particular, if \( \overline{C} \) denotes the image of \( C \) in the quotient ring \( k[a_0, \ldots, a_d]/A_j = k[a_j, a_{j+1}, a_{j+2}] \), we have

\[ \overline{V(C)} \subseteq \overline{V(a_{j+2})} \]

in the copy of \( \mathbb{P}^2 \) defined by \( A_j \). This says that \( a_{j+2} \in \sqrt{(C)} \), and so some power of \( a_{j+2} = (a_{d+1})^{\frac{1}{2}} \) appears with nonzero coefficient in the homogeneous polynomial \( C \in k[a_0, a_1, \ldots, a_d] \). Now property (C) above implies that the same power of \( a_{d-1} \) appears in \( C \) with the same nonzero coefficient. But this contradicts property (B), since \( a_{d-1} \) and \( a_{d+1} \) have different degrees in the weighted grading. This completes the proof of Theorem 6.5 in the case where \( d \) is odd.

7 | LOW AND SPECIAL DEGREES

In this section, we apply our results to identify the precise lists of rational numbers that occur as the F-pure threshold of a reduced polynomial in two variables over an algebraically closed field in certain cases, including the cases of fixed small degree \( d \) (in terms of the congruence class of \( p \) modulo \( d \)) as well as certain fixed values of \( d \) relative to \( p \).

A list of potential values for the F-pure threshold was described in conditions (I)–(III) in §5 (following [10, Thm 4.4]). While we saw in Example 5.2 that these conditions are not sufficient to identify the rational numbers that are F-pure thresholds of reduced polynomials in \( k[x, y] \) in general, we verify that they are sufficient for every characteristic \( p \) and degree \( d \) up to eight. These computations showcase the use of Theorems 4.8, 5.6, and 6.4 to compute F-pure threshold, as well as a few additional tricks.

We first point out some consequences for certain \( d \) and \( p \).
Proposition 7.1. If \( 2p \equiv 2 \mod d \) and \( d \geq 4 \), then all reduced polynomials of degree \( d \) over a field of characteristic \( p \) have \( F \)-pure threshold either \( \frac{2}{d} \) or its truncation \( \frac{2p - 2}{dp} \) at the first spot. If the field is algebraically closed, then both such values occur for each \( d \).

Proof. By Corollary 4.4, the generic value of the \( F \)-pure threshold is \( \frac{2}{d} \). Note our assumptions on \( d \) imply that \( p \neq 2 \), and so \( p \nmid d \). Thus, Theorem 5.6 implies that there is a reduced polynomial whose \( F \)-pure threshold is the first truncation \( \frac{2p - 2}{dp} \). No other truncation can be the \( F \)-pure threshold of a reduced polynomial by condition (I).

Proposition 7.2. Fix a field \( k \) of characteristic \( p > 3 \). Consider reduced \( f \in k[x, y] \) of degree \( p + 1 \). Then \( \text{fpt}(f) \) is

1. \( \frac{2}{p+1} \) or
2. \( \frac{2p^2 - 2}{p^2(p+1)} \) (which is the truncation at the second spot) or
3. \( \frac{1}{p} \) (which is the truncation at the first spot).

All these cases occur for reduced polynomials when \( k \) is algebraically closed.

Proof. The generic value of the \( F \)-pure threshold is \( \frac{2}{p+1} \) by Corollary 4.4, as there is no natural number \( e \) such that \( 2p^e \equiv 1 \mod (p + 1) \). Next, note that

\[
2p^2 - 2 = 2(p+1)(p-1) \equiv 0 \mod (p + 1),
\]

where as \( p \neq 1 \mod (p + 1) \), so Theorem 6.4 guarantees that the second truncation of \( \frac{2}{p+1} \), \( \frac{2p^2 - 2}{p^2(p+1)} \), occurs as the \( F \)-pure threshold of some reduced polynomial. Finally, the first truncation \( \frac{1}{p} \) occurs for reduced Frobenius forms, such at \( x^{p+1} + y^{p+1} \) by [14, 1.1].

We highlight one more special case, which follows from restriction (III) in Section 5.

Proposition 7.3. Let \( k \) be a field of characteristic \( p > 0 \). Let \( f \) be a reduced homogeneous polynomial of degree \( d \) where \( d = pt \) or \( d = 2pt \) for some \( t \in \mathbb{N} \). Then

\[
\text{fpt}(f) = \frac{2}{d}.
\]

The next several propositions show that the conditions (I)–(III) precisely describe the possible \( F \)-pure thresholds in degrees up to 8. We first dispose of the case \( d = 4 \).

Proposition 7.4. Fix an algebraically closed field \( k \) of characteristic \( p > 0 \). The precise list of all \( F \)-pure thresholds that can and do occur as the \( F \)-pure threshold of a reduced polynomial of degree 4 over \( k \) is:

(a) In every characteristic \( p \), the generic value of the \( F \)-pure threshold is \( \frac{1}{2} \).
(b) If \( p = 2 \), all reduced polynomials have \( F \)-pure threshold \( \frac{1}{2} \).
(c) If \( p \neq 2 \), then there are also reduced polynomials with \( F \)-pure threshold \( \frac{1}{2} \).
Proof. By [10, 4.4] (conditions (I)–(III)), every $F$-pure threshold of a reduced degree four polynomial is one of the types listed above. To see that each actually occurs for a reduced polynomial, note that (b) follows from Proposition 7.3, and (a) and (c), follow from Corollary 4.4 and Theorem 5.6, respectively, as $2p^e \not\equiv 1 \pmod{4}$ for any natural number $e$ while $p^1 \equiv 1 \pmod{2}$ for all odd $p$. □

For degree 5, a list of possible values of the $F$-pure threshold were described by Hara and Monsky (when $p \neq 5$); see [6, 3.9] or [10, 4.4]. In fact, for every $p$, our work shows that all these values do occur as the $F$-pure threshold for a reduced polynomial of degree 5.

**Proposition 7.5.** Fix an algebraically closed field $k$ of characteristic $p > 0$. The complete list of values that can and do occur as the $fpt(f)$ of a reduced form in $k[x, y]$ of degree 5 is:

(a) if $p = 5$: $\frac{2}{5}$;
(b) if $p \equiv 1 \pmod{5}$: $\frac{2}{5}$, $\frac{2p-2}{5p}$;
(c) if $p \equiv 2 \pmod{5}$: $\frac{2p^2-3}{5p^2}$, $\frac{2p^3-1}{5p^3}$ for $p \geq 7$, but only the value $\frac{1}{4}$ when $p = 2$;
(d) if $p \equiv 3 \pmod{5}$: $\frac{2p-1}{5p}$;
(e) if $p \equiv 4 \pmod{5}$: $\frac{2}{5}$, $\frac{2p-3}{5}$, and $\frac{2p^2-2}{5p^2}$.

**Remark 7.6.** It is shown in [10, 4.8], that for each congruence class in Proposition 7.5 and each formula listed above, there exist $p$ and reduced $f$ such that $fpt(f)$ has the given formula. Proposition 7.5 is stronger: it ensures that this holds for every prime $p$.

**Proof.** The existence of reduced polynomials for (a) is immediate from Proposition 7.3 and (b) from Proposition 7.1.

For (c), the generic value of the $F$-pure threshold is the third truncation, which is $\frac{1}{4}$ when $p = 2$ and $\frac{2p^3-1}{5p^3}$ otherwise. The first truncation does not occur by condition (III). The second truncation is again $\frac{1}{4}$ if $p = 2$ and $\langle \frac{2}{5} \rangle_2 = \frac{2p^3-3}{5p^3}$ for larger $p$. This latter value occurs for a reduced polynomial because it is the $F$-pure threshold of $f = x^5 + y^5$. To see this, we observe that $(x^5 + y^5)^N \in (x^{p^2}, y^{p^2})$ where $N = \frac{2p^3-3}{5p^3}$. Indeed, the only monomials of the expansion of $(x^5 + y^5)^N$ that are not a priori in $(x^{p^2}, y^{p^2})$ are $x^{p^2-1}y^{p^2-2}$ and $x^{p^2-2}y^{p^2-1}$, and neither of these has both exponents divisible by 5. (Alternatively, one can compute that $fpt(x^5 + y^5) = \frac{2p^3-3}{5p^3}$ using [8, 3.1].)

Statement (d) follows from Corollary 4.4. In case (e), the values $\frac{2}{5}$ and its second truncation are $F$-pure thresholds of reduced polynomials by Corollary 4.4 and Theorem 6.4, respectively. The first truncation, $\frac{2p-3}{5}$, is the $F$-pure threshold of $x^3 + y^5$, arguing as in case (c) above or using [8, 3.1]. □

**Proposition 7.7.** Fix an algebraically closed field $k$ of characteristic $p > 0$. The precise list of all rational numbers that can and do occur as the $F$-pure threshold of a reduced homogeneous polynomial in two variables of degree 6 over $k$ is:

(a) if $p = 3$: $\frac{1}{5}$;
(b) if $p \equiv 1 \pmod{3}$: $\frac{1}{3}$, $\frac{p-1}{3p}$;
(c) if $p \equiv 2 \pmod{3}$: $\frac{1}{3}$, $\frac{p-2}{3p}$, and $\frac{p^2-1}{3p^2}$ for $p \geq 5$, but only $\frac{1}{5}$ and $\frac{1}{4}$ if $p = 2$. □
Proof. The list above is created by the necessary conditions (I)–(III), as verified in [10, 4.6]. It remains to show that every listed formula is achieved for every valid $p$. Case (a) follows from Proposition 7.3 and (b) from Proposition 7.1. For case (c), the values $\frac{1}{3}$ and its second truncation are $F$-pure thresholds of reduced polynomials by Corollary 4.4 and Theorem 6.4, respectively. The first truncation is zero when $p = 2$. For larger $p$, the first truncation is the $F$-pure threshold of a reduced polynomial of the form $f = x^6 + ax^3y^3 + y^6$. To show this, it suffices to show that there is a value of $a$, not $a = \pm 2$, such that $f^N \in (x^p, y^p)$ where $N = \frac{p-2}{3}$. Expanding out $f^N$ as a sum of monomials $x^\alpha y^\beta$ of degree $6N = 2p - 4$, and using the fact that both $\alpha$ and $\beta$ are multiples of 3, we see that

$$f^N \equiv C(a)(xy)^{p-2} \mod (x^p, y^p),$$

where $C(a)$ is a monic polynomial of degree $N$ in $a$. So, $fpt(f) \leq \frac{N}{p}$ if and only if $C(a) = 0$. Note finally that $\pm 2$ are not roots of $C$, since

$$[(x^3 \pm y^3)^2]^{2N} \equiv \left(\frac{2N}{N}\right)(xy)^N = \left(\frac{2N}{N}\right)(xy)^{p-2} \notin (x^p, y^p)$$

as $2N < p$. So, there is at least value of $a$ for which $f$ is reduced and $C(a) = 0$. For such choice of $a$, $fpt(f) = \frac{N}{p}$. $\square$

The $F$-pure threshold of a reduced polynomial of degree 7 is necessarily included on the list below, using the conditions (I)–(III), and verified in [10, 4.7]. The next proposition shows that, for every $p$, (I), (II), and (III) are sufficient as well as necessary.

**Proposition 7.8.** Fix an algebraically closed field $k$ of characteristic $p > 0$. The complete list of numbers that can and do occur the $F$-pure threshold of a reduced homogeneous polynomial in $k[x, y]$ of degree 7:

(a) if $p = 7$: $\frac{2}{7}$

(b) if $p \equiv 1 \mod 7$: $\frac{2}{7}$ and $\frac{2p-2}{7p}$;

(c) if $p \equiv 2 \mod 7$: $\frac{2p-4}{7p}$ and $\frac{2p^2-1}{7p^2}$ for $p > 2$, but only $\frac{1}{4}$ when $p = 2$;

(d) if $p \equiv 3 \mod 7$: $\frac{2p^2-4}{7p^2}$, $\frac{2p^3-5}{7p^3}$, and $\frac{2p^4-1}{7p^4}$ for $p > 3$,

but only $\frac{23}{81}$ and $\frac{2}{9}$ when $p = 3$;

(e) if $p \equiv 4 \mod 7$: $\frac{2p-1}{7p}$;

(f) if $p \equiv 5 \mod 7$: $\frac{2p-3}{7p}$ and $\frac{2p^2-1}{7p^2}$;

(g) if $p \equiv 6 \mod 7$: $\frac{2}{7}$, $\frac{2p-5}{7p}$, and $\frac{2p^2-2}{7p^2}$.

Proof. We must justify that each listed value occurs as the $F$-pure threshold of a reduced polynomial in $k[x, y]$. Cases (a) and (b) follow from Propositions 7.3 and 7.1, respectively.

For (c), the second truncation occurs by Corollary 4.4. To see that the first truncation occurs, we can then argue as in the proof of Proposition 7.7 (c) to show that there is a value of $a$, not $a = \pm 2$, such that the reduced polynomial $f = x(x^6 + ax^3y^3 + y^6)$ satisfies $f^N \in (x^p, y^p)$, where $N = \frac{2p-4}{7}$. Indeed, expanding out $f^N$ as a sum of monomials $x^\alpha y^\beta$ and noting that $\beta$ must be a
multiple of 3, we see that there is a unique pair \((\alpha, \beta) = (i, j)\) such that

\[
f^N \equiv C(a)x^i y^j \mod (x^p, y^p),
\]

for some polynomial \(C \in k[a]\). Since \(2N < p\), note \(x^N(x^3 \pm y^3) \equiv 2^N \not\in (x^p, y^p)\), so that \(\pm 2\) are not roots of \(C(a)\). So, choosing \(a\) to be any root of \(C\), the reduced polynomial \(f = x(x^6 + ax^3y^3 + y^6)\) satisfies \(\text{fpt}(f) = \frac{N}{p} = \frac{2p-4}{7p}\).

For (d), the fourth truncation is the generic \(F\)-pure threshold by Corollary 4.4. When \(p = 3\), the base \(p\) expansion of \(\frac{2}{7} = \frac{2}{3} + \frac{2}{7} + \frac{0}{27} + \ldots\), so the only possible additional value to check is whether or not the truncation \(\frac{2}{3}\) is an \(F\)-pure threshold; that is, because the polynomial \(f = x(x^6 + ax^3y^3 + y^6)\) satisfies \(f^2 \in (x^3, y^3)\) when \(a^2 = -2\). When \(p > 3\), the third truncation is the \(F\)-pure threshold of \(x^7 + y^7\) by Hernandez’s formula [8, Thm 3.1].

In case (d) when \(p > 3\), we claim that there is reduced polynomial \(f\) of the form \(x(x^6 + ax^3y^3 + y^6)\) such that \(f^N \in (x^{p^2}, y^{p^2})\) for \(N = \frac{2p^2-4}{7}\). Indeed, only three monomials of degree \(7N\),

\[x^{p^2-1}y^{p^2-3}, \quad x^{p^2-2}y^{p^2-2}, \quad x^{p^2-3}y^{p^2-1},\]

fail to be in the ideal \((x^{p^2}, y^{p^2})\) but only the third has \(y\) exponent divisible by 3. So, we must show that the coefficient \(C(a) \in k[a]\) of the term \(x^{p^2-3}y^{p^2-1}\) in the expansion of \(f^N\) has some root other than \(\pm 2\). Note that \(C(a)\) is constructed by summing over all choices of one monomial from each of the \(N\)-copies of \((x^6 + ax^3y^3 + y^6)\) so that the exponents on \(y\) sum to \(p^2 - 1\); in particular, the coefficient of \(a^T\) in \(C(a)\) is

\[
\binom{N}{T}^{\# \text{choices}} \binom{N - T}{\frac{p^2 - 3T - 1}{6}}^{\# \text{choices}},
\]

or zero if \(\frac{p^2 - 3T - 1}{6} \not\in \mathbb{N}\). Now suppose \(T = \frac{5p^2 - 17}{21}\). Note first that

\[
\frac{1}{6}(p^2 - 3T - 1) = \frac{1}{6} \left( p^2 - \frac{5p^2 - 17}{7} - 1 \right) = \frac{1}{6} \left( \frac{2p^2 + 10}{7} \right) = \frac{1}{3} \left( \frac{p^2 + 5}{7} \right) \in \mathbb{N}
\]

(since \(p^2 = 1 \mod 3\), so the term \(a^T\) appears in \(C(a)\) with coefficient (41). To compute the coefficient of \(a^T\), note that

\[
N - T = \frac{2p^2 - 4}{7} - \frac{5p^2 - 17}{21} = \frac{p^2 + 5}{21} = \frac{1}{6}(p^2 - 3T - 1),
\]

so the factor \(\binom{N - T}{\frac{p^2 - 3T - 1}{6}}\) in (41) is 1. For the other factor in (41), use the base \(p\) expansions

\[
N = \frac{2p - 6}{7} p + \frac{6p - 4}{7} \quad \text{and} \quad N - T = \begin{cases} \frac{p - 10}{21} p + \frac{10p + 5}{21} & \text{if } p \equiv 1 \mod 3 \\ \frac{p - 17}{21} p + \frac{17p + 5}{21} & \text{if } p \equiv 2 \mod 3, \end{cases}
\]
and Lucas’s theorem\(^\dagger\) to check that \( \binom{N}{N-T} \neq 0 \). So, the coefficient \( \binom{N}{N-T} \) of \( a^T \) in \( C(a) \) is nonzero, and \( C(a) \) must have roots. To check that \( \pm 2 \) are not among the roots of \( C \), observe that

\[
N(N - T) \neq 0.
\]

So, the coefficient \( \binom{N}{N-T} \) of \( a^T \) in \( C(a) \) is nonzero, and \( C(a) \) must have roots. To check that \( \pm 2 \) are not among the roots of \( C \), observe that

\[
\binom{2N}{p^2-1} x^{p^2-3} y^{p^2-1} \mod (x^p, y^p),
\]

and the binomial coefficient \( \binom{2N}{p^2-1} \neq 0 \). Indeed, using the base \( p \) expansions,

\[
2N = \frac{4p - 5}{7} p + \frac{5p - 8}{7} \quad \text{and} \quad \frac{p^2 - 1}{3} = \begin{cases} \frac{p-1}{3} p + \frac{2p-1}{3} & \text{if } p \equiv 1 \mod 3 \\ \frac{p-2}{3} p + \frac{2p-1}{3} & \text{if } p \equiv 2 \mod 3, \end{cases}
\]

Lucas’s theorem implies \( \binom{2N}{p^2-1} \neq 0 \).

For (e), the generic \( F \)-pure threshold is the first truncation, so this is both the maximal and minimal possible \( F \)-pure threshold. For (f), the generic \( F \)-pure threshold is the second truncation by Corollary 4.4, while the first truncation is \( \text{fpt}(x^7 + y^7) \), by the same method as in Proposition 7.5 (c). And for (g), the values \( \frac{2}{7} \) and the second truncation are achieved by Corollary 4.4 and Theorem 6.4, respectively, while the first truncation is \( \text{fpt}(x^7 + y^7) \), as in Proposition 7.5 (c).

**Proposition 7.9.** Fix an algebraically closed field \( k \) of characteristic \( p > 0 \). The complete list of rational numbers that are \( F \)-pure thresholds of a reduced homogeneous polynomial of degree eight in two variables over \( k \) is

(a) if \( p = 2 \): \( \frac{1}{4} \);

(b) if \( p \equiv 1 \mod 4 \): \( \frac{1}{4} \) and \( \frac{p-1}{4p} \);

(c) if \( p \equiv 3 \mod 4 \): \( \frac{1}{4} \), \( \frac{p-3}{4p} \), and \( \frac{p^2-1}{4p^2} \) for \( p > 3 \), but only \( \frac{1}{4} \) and \( \frac{2}{9} \) when \( p = 3 \).

**Proof.** The list of potential \( F \)-pure thresholds is easily constructed from conditions (I)–(III). We must verify that each occurs (for every valid \( p \)). Statement (a) follows from Proposition 7.3 and (b) from Corollary 7.1.

For (c), note that \( \frac{1}{4} \) and the second truncation are the \( F \)-pure thresholds of reduced polynomials by Corollary 4.4 and Theorem 6.4, respectively. Note that when \( p = 3 \), the first truncation of \( \frac{1}{4} \) base 3 is zero, so we have found all possible \( F \)-pure thresholds. For \( p \neq 3 \), it remains to show that there exists a reduced degree 8 polynomial \( f \) such that \( f^N \in (x^p, y^p) \), where \( N = \frac{p-3}{4} \). For \( p \equiv 7 \mod 8 \), one checks that \( \text{fpt}(x^8 + y^8) \) is such a polynomial, using the method of Proposition 7.5 (c). For \( p \equiv 3 \mod 8 \), one checks that there is a reduced polynomial of the form \( f = xy(x^6 + ax^3y^3 + y^6) \) with \( \text{fpt}(f) = \frac{p-3}{4p} \) similarly to the proof of Proposition 7.7 (c).

\[\square\]

8 | LOWER BOUNDS ON THE \( F \)-PURE THRESHOLD

To conclude, we discuss lower bounds on the \( F \)-pure threshold. First, we note that the first nonzero truncation of the base \( p \) expansion of \( \frac{2}{d} \) always gives a lower bound:

\[\dagger\] Which in this case says \( \binom{ap+\beta}{y \in +\delta} = \binom{a}{y \in} \binom{\beta}{\delta} \mod p \), when \( 0 \leq \alpha, \beta, \gamma, \delta \leq p - 1 \).
**Proposition 8.1.** Fix a field \( k \) of characteristic \( p > 0 \). Let \( f \) be a homogeneous reduced polynomial of degree \( d \geq 2 \) in \( k[x_1, \ldots, x_n] \), \( n \geq 2 \). Then

\[
\text{fpt}(f) \geq \left[ \frac{2}{d} \right]_e, \quad \text{where } e \in \mathbb{N} \text{ is minimal such that } \left[ \frac{2}{d} \right]_e \neq 0. \tag{42}
\]

Here, \( \left[ \frac{2}{d} \right]_e \) denotes the \( e \)th truncation of the possibly-terminating base \( p \) expansion of \( \frac{2}{d} \).

Here, by “possibly-terminating” base \( p \) expansion of a positive rational number \( \lambda \), we mean that if \( \lambda \) has more than one base \( p \) expansion, we choose the one that terminates. Writing \( \lambda \) in lowest terms, note that unless the denominator is a power of \( p \), then \( \lambda \) has only one base \( p \) expansion, so the “possibly-terminating” base \( p \) expansion is simply its nonterminating expansion. See Subsection 2.3.

**Proof.** The \( F \)-pure threshold can only decrease after modding out linear forms [14, 3.6]. Modulo \( n - 2 \) general enough forms, we get a reduced polynomial \( \overline{f} \) in two variables whose \( F \)-pure threshold is a lower bound for \( \text{fpt}(f) \). If \( d = p^e \) or \( 2p^e \), the \( F \)-pure threshold of \( \overline{f} \) is \( \frac{2}{d} \) by Proposition 7.3, which has terminating base \( p \) expansion \( \frac{2}{p^e} \) or \( \frac{1}{p^e} \), respectively, showing the lower bound is satisfied. Otherwise, \( \frac{2}{d} \) does not have denominator a power of \( p \), so its possibly-terminating and its nonterminating base \( p \) expansion are the same. Since we know that the \( F \)-pure threshold of the reduced polynomial \( \overline{f} \) in two variables is either \( \frac{2}{d} \) or some truncation of the nonterminating base \( p \) expansion by [10, 4.4], the first nonzero truncation is certainly a lower bound. \( \square \)

**Corollary 8.2.** The bound of Proposition 8.1 is sharp for homogeneous polynomials of degree \( d \) in the range \( p^e + 1 \leq d \leq 2p^e \) in \( k[x_1, \ldots, x_n] \) \((n \geq 2)\). That is, there are reduced polynomials of degree \( d \) whose \( F \)-pure threshold is \( \left[ \frac{2}{d} \right]_e \).

**Proof.** For \( d \) in the stated range, it is easy to compute that \( \left[ \frac{2}{d} \right]_e = \frac{1}{p^e} \) and that all previous truncations are zero. Thus, we only need to find reduced polynomials of degree \( d \) with \( F \)-pure threshold \( \frac{1}{p^e} \). Since we already know that this is a lower bound (Proposition 8.1), we need only show that there are reduced polynomials \( f \) with \( \text{fpt}(f) \leq \frac{1}{p^e} \). But any reduced polynomial \( f \) in \((x_1^{p^e}, \ldots, x_n^{p^e})\) satisfies

\[
\text{fpt}(f) \leq \frac{1}{p^e}, \tag{43}
\]

by definition of \( F \)-pure threshold. So, for such \( f \), the lower bound \( \left[ \frac{2}{d} \right]_e \) of Proposition 8.1 is sharp. \( \square \)

Putting together results proved in earlier sections, we summarize cases where we know that the lower bound is sharp.

**Corollary 8.3.** The bound of Proposition 8.1 is sharp for reduced polynomials of degree \( d \geq 4 \) in \( k[x_1, \ldots, x_n] \) in each of the following cases:

1. \( p^e + 1 \leq d \leq 2p^e \);
2. \( 2p = 1 \mod d \);
3. \( 2p = 2 \mod d \);
4. \( d \leq 8 \).
Proof. Simply note that a reduced polynomial in two variables can be viewed as a reduced polynomial in \( k[x_1, \ldots, x_n] \), and apply Corollary 8.2, Proposition 7.1, and Theorem 6.4, respectively, as well as the computations for degrees \( 4 \leq d \leq 8 \) in Section 7.

Remark 8.4. The lower bound is not sharp in general: in the case where \( d = 3 \) and \( p \equiv 1 \mod p \), no reduced polynomial has \( F \)-pure threshold any truncation of \( \frac{2}{3} \).

In all cases, we know where the lower bound of Proposition 8.1 fails, the reason is that the first nonzero truncation fails one of the conditions (I), (II), or (III) from Section 5 (i.e., from [10, 4.4]). We suspect that the first truncation of the possibly-terminating base \( p \) expansion of \( \frac{2}{d} \) that satisfies (I)–(III) is precisely the minimal possible \( F \)-pure threshold of a reduced polynomial of degree \( d \geq 2 \) in \( k[x_1, \ldots, x_n] \) in general.

We do not know whether the first truncation of the possibly-terminating base \( p \) expansion of \( \min(1, \frac{n}{d}) \), which satisfies (I)–(III) could be the minimal possible \( F \)-pure threshold for polynomials in \( k[x_1, \ldots, x_n] \) defining a smooth hypersurface in \( \mathbb{P}^{n-1} \).

Remark 8.5. In [14], a lower bound of \( \frac{1}{d-1} \) was proved for a reduced homogeneous polynomial of degree \( d \) (in any number of variables); moreover, this bound was shown to be sharp only when \( d = p^e + 1 \) for some \( e \in \mathbb{N} \). This comports with Corollary 8.3 (1) above.

Remark 8.6. A lower bound on the \( F \)-pure threshold of elements \( f \) in more general local rings \( (R, m) \) is proved by Núñez-Betancourt and Smirnov: \( \text{fpt}(f) \geq \frac{1}{e_{HK}(R/fR)} \) where \( e_{HK}(R/fR) \) is the Hilbert–Kunz multiplicity of the quotient local ring \( R/fR \) [16, 2.2]. In our main case of interest, however, this is much weaker than the bound in Proposition 8.1: when \( R = k[x, y] \), the Hilbert–Kunz multiplicity of the one-dimensional ring \( R/(f) \) is the same as its Hilbert–Samuel multiplicity (see, e.g., [13, 3.3]), so the Núñez–Betancourt–Smirnov bound simply recovers Proposition 2.2 (4).

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