Two-Photon Scattering in One Dimension by Localized Two-Level System

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We study two-photon scattering in a one-dimensional coupled resonator arrays (CRA) by a two-level system (TLS), which is localized as a quantum controller. The $S$-matrix is analytically calculated for various two-photon scattering processes by TLS, e.g., one photon is confined by TLS to form a bound state while the other is in the scattering state. It is discovered from the poles of the $S$-matrix that there exist two kinds of three-body bound states for describing two bound photons localized around TLS.

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Introduction.—For the architecture of a new generation all-optical quantum devices, we need to study various physical mechanisms of the coherent photon transports in the low dimensional confined structure with sub-wavelength scale. Recently, the single photon transmissions in a waveguide and one dimensional (1D) coupled resonator arrays (CRA) with some controllers, e.g., a two-level system (TLS), have been extensively studied [1,2,3,4,5]. It is worthy to notice that, in such CRA hybrid system, there usually exist bound states of single photon around the localized TLS [3,4], thus TLS can behave as a quantum controller to realize a quantum switch [6] for photon transports.

We emphasize that most of investigations in this area focused on the cases with single photon, and two-photon case was only considered for the linear dispersion relation of waveguides [1,4,7]. In this Letter, we develop a two-photon scattering approach for the general case with arbitrarily given nonlinear dispersion relation. Actually, the effects of TLS on the quantum natures of multi-photon statistics, such as photon bunching and anti-bunching, have not been investigated systematically for the nonlinear dispersion relations. However, a comprehensive understanding for fundamental processes of multi-photon scattering is necessary to both practical applications and theoretical explorations. In experiments, the practical processes may concern two or more photons, which obviously affect on the efficiency of single photon creation and transfer. In theoretical studies, the hybrid system with many photons is related to the Lee model [8] in the sector with high excitations.

Our hybrid system consists of the CRA system (Fig. 1b) with a nonlinear dispersion relation [2,3,4] and a TLS localized in one of the resonators. It has been shown that in some circumstances there exist two single-photon bound states, and energy band structure for a scattered photon, which are displayed in Fig. 1b. With this illustration, we address our main results: (a) when scattered by TLS, which simultaneously binds another photon to form a bound state, the incident photon is elastically scattered under a condition we will discuss as follows; (b) when two incident photons are scattered by TLS, the photon correlation is induced, and the nonlinear dispersion results in richer quantum statistical characters of outgoing two photons, which can be controlled by TLS; (c) there exist two kinds of three-body bound states for describing two bound photons localized around TLS.

Model.—The model Hamiltonian of our hybrid system reads

$$H = \Omega |e\rangle\langle e| + \int dk \varepsilon_k a_k^\dagger a_k + V \int \frac{dk}{\sqrt{2\pi}} (a_k^\dagger \sigma^- + \text{H.c.}), \quad (1)$$

where the operator $\sigma^- = |g\rangle \langle e|$ denotes the flip from the ground state (GS) $|g\rangle$ to the excited state $|e\rangle$ with the energy level spacing $\Omega$. The nonlinear dispersion relation of photons in CRA is $\varepsilon_k = \omega_0 - 2J \cos k$, where $\omega_0$ is the eigen-frequency of each cavity, $J$ is the hopping constant characterizing the inter-cavity coupling in the tight-binding approximation, $k$ is the momentum of photon, and the inter-cavity distance is taken as unity. Here, $a_k^\dagger$ ($a_k^\dagger$) is the annihilation (creation) operator for the photon with momentum $k$, and $V$ is the coupling constant of TLS and photon.

In the single excitation subspace spanned by the basis $|0\rangle|e\rangle, a_k^\dagger |0\rangle|g\rangle$ [3,4], the system possesses an energy band of width $4J$ centered in $\omega_0$ and two single-photon bound states $|E_s\rangle = \sqrt{Z_s} |0\rangle|e\rangle + V(2\pi)^{-1/2} \int dk (E_s - \varepsilon_k)^{-1} a_k^\dagger |0\rangle|g\rangle$ (s =

![FIG. 1: (Color online) (a) The schematic for the CRA system: the red circle denotes the two level system (TLS), and the blue dots denote the coupled resonators. (b) The band structure of the CRA system for the single excitation case: the continuum displays the single photon scattering states while the blue lines above the band top and below the band bottom represent the single photon bound states.](image-url)
For the Lee model \[8\], the solutions of scattering eigenstates for the secular equation \( H \Psi^{(s)} = E^{(s)} \Psi^{(s)} \) \((s = 1, 2)\) are obtained as

\[
\Phi^{(s)}(k, k') = \frac{V[\Phi^{(s)}_x(k) + \Phi^{(s)}_y(k)']}{2 \sqrt{2\pi E^{(s)} - \epsilon - \epsilon' + i\delta}}
\]

where \( \Phi^{(s)}_x(k) = \sqrt{Z} \delta_{kk_0} + \psi(\epsilon, \epsilon_0) \) contains the incident component \( \delta_{kk_0} \) in the \( k \)-space. The scattering part for single photon reads as

\[
\psi(\epsilon, \epsilon_0) = \frac{-V^2}{2\pi(E^{(s)} - \epsilon)} \left\{ \epsilon_0 - E_s \right\} + \frac{2A_s(\epsilon, \epsilon_0)G_s(\epsilon_0)}{Z_s G_s(\epsilon_0) - A_s(\epsilon_0, \epsilon_0)}
\]

For two-photon scattering, there may exist three possible processes: (a) one incident photon with momentum \( \epsilon_1 \) below the band bottom \( \epsilon_{\min} = \epsilon_0 - 2J \) and \( \epsilon_2 \) above the band top \( \epsilon_{\max} = \epsilon_0 + 2J \) (Fig. 1b); (b) two incident photons scattered by TLS in GS; (c) two photons both bound by TLS form the three-body bound state. We point out that the behaviors of two photons in CRA are essentially different from those of photons propagating in a waveguide, in which the processes (a) and (c) do not exist due to its linear dispersion relation. The processes (a) and (b) are schematically shown in Fig. 2a and 2b, respectively.

Scattering eigen-states.—The scattering eigen-states for the two-photon processes are assumed to be

\[
|\Psi^{(l)}\rangle = \int dk \Phi^{(l)}_x(k)\alpha_k^\dagger |0\rangle |e\rangle + \int dk dk' \Phi^{(l)}(k, k')\alpha_k^\dagger \alpha_{k'}^\dagger |0\rangle |g\rangle
\]

for \( l = 1, 2, 3 \). The corresponding eigenvalues are \( E^{(l)} = E_l + \epsilon k_0 \) for \( l = 1, 2 \) and \( E^{(3)} = \epsilon k_0 + \epsilon k_2 \). Here, the eigenstates \( |\Psi^{(1,2)}\rangle \) describe one photon of momentum \( k_0 \) scattered by TLS in the bound state \( |E_1, 2\rangle \), while \( |\Psi^{(3)}\rangle \) describes the two photons with momenta \( k_1 \) and \( k_2 \) scattered by TLS in GS. For convenient, we set \( \epsilon_0 = \epsilon k_0 \), \( \epsilon_1 = \epsilon k_1 \) and \( \epsilon = \epsilon k_2 \) below.

According to the three-particle scattering approach \[9, 10, 11, 12\] for the Lee model \[8\], the solutions of scattering eigen-

FIG. 2: (Color online) (a) The schematic for the photon scattering by TLS in the single photon bound state: the first line indicates the single photon incident, the second line means the scattering into the out-going state of single photon, and the third line shows that the bound state is broken so that there are two out-going photons; (b) the schematic for two-photon scattering by the TLS in GS: the first line indicates the two photon incident, the second line means the two photon scattered by TLS in the bound state. The scattering part for single photon reads as

\[
|\psi(\epsilon, \epsilon_0)\rangle = \frac{-V^2}{2\pi(E^{(s)} - \epsilon)} \left\{ \epsilon_0 - E_s \right\} + \frac{2A_s(\epsilon, \epsilon_0)G_s(\epsilon_0)}{Z_s G_s(\epsilon_0) - A_s(\epsilon_0, \epsilon_0)}
\]

where \( A_s(\epsilon, \epsilon_0) = Z_s(\epsilon - E_s)G_s(E^{(s)} - \epsilon)G_s(\epsilon - E_s) \). For two-photon scattering by TLS in GS, the two-photon wave-function

\[
\Phi^{(3)}(k, k') = \delta_{kk_0}\delta_{kk_2} + \frac{V[\Phi^{(3)}_x(k) + \Phi^{(3)}_y(k)']}{2 \sqrt{2\pi(E^{(3)} - \epsilon - \epsilon' + i\delta)}}
\]

contains the incident two-photon component \( \delta_{kk_0}\delta_{kk_2} \), and the two-photon wave-functions in the scattering part

\[
\Phi^{(3)}_x(k) = \frac{V}{(2\pi)^{3/2}}[V^2\phi(\epsilon)G_s(\epsilon_1)G_s(\epsilon_2) + 2\pi G_s(\epsilon_3) - \epsilon] \sum_i \delta_{kk_i}
\]

where \( \phi(\epsilon) = 2I(\epsilon)/A - \sum_{\ell=1,2}(\epsilon - \epsilon_{\ell})^{-1} \). Here, \( I(\epsilon) = \sum_{\ell=1,2}Z_s G_s(E^{(3)} - \epsilon) G_s(\epsilon - \epsilon') + \sum_{\ell} d_{\ell}^{\max} G_s(E^{(3)} - \epsilon') \Im G_s(\epsilon') \). The above obtained scattering eigen-states \(|\Psi^{(1,2)}\rangle\) result in the \( S \)-matrix \[11, 12, 13, 14\]. Its element

\[
S_{p_0 k_0}^{(s)} = \frac{\delta_{kk_0}\delta_{kk_2}}{2J|\sin k_0|} Z_s G_s(\epsilon_0) - A_s(\epsilon_0, \epsilon_0)
\]

represents an elastic process “\( \gamma + BS \rightarrow \gamma + BS \)”: a photon \( \gamma \) with momentum \( k_0 \) scattered by TLS in the bound state \(|E_1, 2\rangle\) into a photon \( \gamma \) with momentum \( p_0 \) while the bound state is unchanged. The transmission and reflection coefficients are \( k_0 = 1 + i\delta_{kk_0} \) and

\[
\delta_{kk_0} = \frac{iV^2}{(2\pi)^{3/2}}[V^2\phi(\epsilon)G_s(\epsilon_1)G_s(\epsilon_2) + 2\pi G_s(\epsilon_3) - \epsilon] \sum_i \delta_{kk_i}
\]

where \( \delta_{kk_i} = \frac{\delta_{kk_i}}{2J|\sin k_0|} Z_s G_s(\epsilon_0) - A_s(\epsilon_0, \epsilon_0) \). Another \( S \)-matrix element

\[
S_{p_1 p_2 k_0}^{(s)} = \frac{\delta_{kk_0}\delta_{kk_2}}{2J|\sin k_0|} Z_s G_s(\epsilon_0) - A_s(\epsilon_0, \epsilon_0)
\]

\[
\times \delta_{kk_0}\delta_{kk_2}
\]
means an inelastic process “γ + BS → 2γ”: a photon γ with momentum k₀ scattered by TLS in the bound state |E₁⟩ into two photons with momenta p₁ and p₂.

Let us analyze the physical processes described by Eqs. (7) and (9). When ε₀ < εₙ₁ = 2εₘᵢₓ − E₁, Sₙ₁(pₙ₅k₀) vanishes, thus the incident photon is elastically scattered by TLS in the bound state |E₁⟩. When ε₀ > εₙ₁, the bound state |E₁⟩ is stimulated by the incident photon, and the out-going two photons emerge simultaneously. The incident photon with energy ε₀ > εₙ₁ = 2εₘᵢₓ − E₁ is elastically scattered by TLS in the bound state |E₂⟩ while the bound state |E₁⟩ is not affected. When ε₀ < εₙ₁ the two photons emerge.

As shown in Fig. 3a and Fig. 3b, the reflective probabilities |Pₙ₁|² associated to the bound states are suppressed in comparison with that for the single photon scattering in Ref. [2]. Here, the probabilities pₙ₁ = |Pₙ₁|² + |Pₙ₂|² are also displayed in Fig. 3a for s = 1 and Fig. 3b for s = 2: when ε₀ < εₙ₁ (ε₀ > εₙ₁), the probability pₙ₁ = 1 means the elastic scattering by TLS in the bound state |E₁⟩ (|E₂⟩); when ε₀ > εₙ₁ (ε₀ < εₙ₁), the probability pₙ₂ < 1 means the inelastic scattering by TLS in the bound state |E₁⟩ (|E₂⟩), and in this case the two photons emit. The probabilities |Φₙ₁(x₁, x₂)|² for two-photon creation in the out-put, which is defined by

\[ \Phiₙ₁(x₁, x₂) = \int \frac{dp₁ dp₂}{4\pi} Sₙ₁(pₙ₅k₀)(e^{ipₙ₁ + ipₙ₂} + e^{ipₙ₂ + ipₙ₁}), \]

(10)

are plotted in Fig. 4a and Fig. 5a. The numerical calculations show that the out-going two photons prefer to occupy around TLS.

**Two-photon scattering by one TLS.**—For the two incident photons both not bound by TLS, the out-going state (out) generally contains the single-photon part (one) = \[ \sum_{p₁,p₂} e^{ip₁} aₙ₁ aₙ₂ |E₁⟩ |0⟩ |g⟩ \] and the two-photon part (two) = \[ \sum_{p₁,p₂} S_{p₁,p₂,k₁} aₙ₁ aₙ₂ |0⟩ |g⟩ \]. The Lippmann-Schwinger approach for |Ψ(0)⟩ gives the S-matrix explicitly, which elements are \[ S_{p₁,p₂,k₁} = −i V^3 W(2\pi Zₜ) e^{1/2} δ(E₁ + E_p + E_p - Eₜ) \] for the inelastic process “γ + BS” and \[ S_{p₁,p₂,k₁} = S_{m₁} + Rδ(Eₜ - E_p + E_p) \] for the elastic process “2γ → 2γ". Here, \[ W = Gₜ(G_p)Gₜ(Gₜ)Gₜ(A) \], and the first term \[ S_{m₁} = S_{p₁,p₂,k₁} S_{p₁,k₂} S_{p₂,k₁} S_{p₂,k₁} \] describes the factorization of the two-photon scattering, where \[ S_{m₁} = kₙδ(pₙ₁ + pₙ₂) \] contains the transmission coefficient \[ kₙ = 1 + rₙ \] and reflection one

\[ rₙ = -\frac{iv²}{2J|\sin k| (εₙ₁ + Ω) - iv²}. \]

(11)

They describe the single photon scattering for TLS in GS [2]. The correlated photon scattering induced by TLS is described
by the second term of $S_{p_1p_2k_i}$:

$$R = -iV^4 \pi \int [G_{s}(r_e)G_{s}(r_{p_i})]$$

which exhibits the background fluorescence of two photons [1, 4, 7].

In the inelastic scattering of two photons, a part of incident photon energy is absorbed by TLS, and thus the outgoing state (one) describes that one photon forms a bound state while another emits. Another out-going state (two) means that the two photons are elastically scattered by TLS without any energy loss. When $E_1 + \varepsilon_{\text{max}} < E^{(3)}_2 < E_2 + \varepsilon_{\text{min}}$ the inelastic process is forbidden and only the elastic scattering takes place. The second order correlation function [4] for two photons is

$$G(x_1, x_2) = \int \frac{dp_1 dp_2}{4\pi} S_{p_1p_2k_i} (e^{ip_1x_1 + ip_2x_2} + e^{ip_1x_2 + ip_2x_1})$$

is the out-going wave-function in the coordinate representation.

Two correlation functions $G(x_1, x_2)$ are drawn in Fig. 4a for $\varepsilon_1 = \varepsilon_2 = \Omega$ and in Fig. 4b for $\varepsilon_1 = 3$ and $\varepsilon_2 = 5$. In fact, due to the nonlinear dispersion relation, the correlation function $G(x_1, x_2)$ not only depends on the center-of-mass coordinate $x_c = (x_2 + x_1)/2$, but also on the relative coordinate $x_r = x_2 - x_1$ of two photons. For the different $x_r$, the correlation functions $G$ are shown in Fig. 4c for $\varepsilon_1 = \varepsilon_2 = \Omega$ and in Fig. 4d for $\varepsilon_1 = 3$ and $\varepsilon_2 = 5$. These figures indeed show the different photon statistics for the different center-of-mass positions of photons.

Three-body bound states.—It follows from the poles of $S$-matrix that there exist two three-body bound states with energies $E_{3} = E_{1} + \beta_{s}$ ($s = 1, 2$), where $\beta_{s}$ are determined by $Z_s G_s(\beta_{s}) = A_s(\beta_{s}, \beta_{s})$. It can be proved that $\beta_{1} < \epsilon_{\text{min}}$ and $\beta_{2} > \epsilon_{\text{max}}$. Using the approach in Ref. [15], we obtain the bound state

$$|B_s\rangle = N\left[\int dk \eta^{(s)}_e(k)|a_k^\dagger|0\rangle|e\rangle + \int dk dk' \eta^{(s)}_e(k, k') a_k^\dagger a_{k'}^\dagger |0\rangle|g\rangle\right]$$

by the secular equation $H|B_s\rangle = B_s|B_s\rangle$. Here, $\eta^{(s)}_e(k) = V^{2}(\beta_{s} - \epsilon)^{-1} A_s(\epsilon, \beta_{s})/2\pi$, $\eta^{(s)}_e(k, k') = V\sqrt{2\pi(\beta_{s} - \epsilon - \epsilon')}|A_s(\epsilon, \beta_{s})|/2\pi$, and $N$ is the normalization constant. The wave-functions $\eta^{(s)}_e(x) = \int dk \eta^{(s)}_e(k) \exp(ikx)/\sqrt{2\pi}$ and

$$\eta^{(s)}(x_1, x_2) = \int \frac{dk_1 dk_2}{2\pi} \eta^{(s)}_e(k_1, k_2) (e^{ip_1x_1 + ip_2x_2} + e^{ip_1x_2 + ip_2x_1})$$

are shown in Fig. 5 for $s = 1, 2$, which both go to zero as $x \rightarrow \infty$.

**Conclusion.**—We have investigated the two-photon scattering in the hybrid CRA system. The two-photon $S$-matrix is calculated for various scattering processes by TLS in the CRA system with nonlinear dispersion relation. We show that the three-body bound states exist for describing the two bound photons localized around TLS. The binding energies are determined explicitly by the poles of $S$-matrix. For the two-photon scattering by TLS in GS, the quantum statistics of scattered photons are analyzed in detail. The similar approach can also be applied to dealing with scattering problems of the TLS-photon hybrid system with more complicated architectures and dispersion relations.

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