Mirostat: A Perplexity-Controlled Neural Text Decoding Algorithm

Sourya Basu*, Govardana Sachitanandam Ramachandran†, Nitish Shirish Keskar†, and Lav R. Varshney*,†
*University of Illinois at Urbana-Champaign
†Salesforce Research

Abstract

Neural text decoding is important for generating high-quality texts using language models. To generate high-quality text, popular decoding algorithms like top-$k$, top-$p$ (nucleus), and temperature-based sampling truncate or distort the unreliable low probability tail of the language model. Though these methods generate high-quality text after parameter tuning, they are ad hoc. Not much is known about the control they provide over the statistics of the output, which is important since recent reports show text quality is highest for a specific range of likelihoods. Here, first we provide a theoretical analysis of perplexity in top-$k$, top-$p$, and temperature sampling, finding that cross-entropy behaves approximately linearly as a function of $p$ in top-$p$ sampling whereas it is a nonlinear function of $k$ in top-$k$ sampling, under Zipfian statistics. We use this analysis to design a feedback-based adaptive top-$k$ text decoding algorithm called mirostat that generates text (of any length) with a predetermined value of perplexity, and thereby high-quality text without any tuning. Experiments show that for low values of $k$ and $p$ in top-$k$ and top-$p$ sampling, perplexity drops significantly with generated text length, which is also correlated with excessive repetitions in the text (the boredom trap). On the other hand, for large values of $k$ and $p$, we find that perplexity increases with generated text length, which is correlated with incoherence in the text (confusion trap). Mirostat avoids both traps: experiments show that cross-entropy has a near-linear relation with repetition in generated text. This relation is almost independent of the sampling method but slightly dependent on the model used. Hence, for a given language model, control over perplexity also gives control over repetitions.

I. INTRODUCTION

Generative language models have received recent attention due to their high-quality open-ended text generation ability for tasks such as story writing, making conversations, and question answering [1, 2]. These models are trained in an unsupervised manner over large amounts of human-written text data. Generation of texts from these models usually relies on different forms of random sampling methods. Pure sampling from trained models often leads to incoherent and low-quality texts [3]. On the other hand, greedy decoding leads to excessive repetitions resulting in low-quality texts. Hence, choosing the right decoding algorithm is important to generate high-quality texts with controlled attributes [4–6].

In this work, we introduce a neural text decoding algorithm called mirostat, which actively controls the generative process to maintain the perplexity of generated text at a certain desirable value. Mirostat uses an adaptive top-$k$ sampling algorithm to tune the value of $k$ which helps maintain the overall perplexity of the text. Trained language models often have an unreliable tail in their probability distribution; hence a number of recently proposed sampling methods focus on suppressing this unreliable tail [3, 9, 10]. Top-$k$ sampling [3, 9] is where the next word is sampled from the top $k$ most probable choices. Top-$p$ sampling [10], also known as nucleus sampling, is where the next word is chosen from the top $x$ probable choices, where $x$ is the smallest integer such that their cumulative probability mass is at least $p$. While top-$k$ sampling involves choosing from a fixed number of most probable choices, top-$p$ sampling involves a dynamic number of most probable words based on a predefined value of $p$ and shows better performance on a number of different statistical and human evaluated tests. For small values of $k$ and $p$, these sampling methods tend to show repetitions in the generated texts, thereby yielding poor quality. This can be handled by penalizing repetitions and using appropriate temperature values [11] or adding diversity to the generated text [5, 12]. On the other hand, large values of $k$ and $p$ can lead to incoherent texts similar to pure sampling. Although choosing appropriate values of $p$ or $k$ in these two sampling methods can help us avoid problems such as repetition and incoherence, this involves ad hoc tuning of parameters. We also observe that these methods do not give good control over the statistics of the generated text. Even for a fixed value of $p$ or $k$, the generated text can have varying statistical properties. Intriguingly, we show that small value of a certain perplexity statistic of generated texts, which we call surprise (see Def. 1), is closely linked to repetitions and large values of surprise are linked to incoherence. Consider the following example that illustrates that for a fixed value of $p$, repetition positively correlates with small surprise.

**Example 1.** We generate four samples of texts with 200 tokens (which are words or subwords that are part of the vocabulary of the model) using GPT-2 model with 117M parameters [2]. For the first two samples we use top-$p$ sampling with $p = 0.4$, for the third sample we use top-$p$ sampling with $p = 1.0$, and for the fourth sample we use our proposed mirostat sampling algorithm. The fifth example is human-generated, taken from the same corpus as the context. The context used is as follows.

**Context:** “For two months early in 1943, Shannon came into contact with the leading British mathematician Alan Turing. Turing had been posted to Washington to share with the U.S. Navy’s cryptanalytic service the methods used by the British

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1The word mirostat is derived from mirum which is Latin for surprise and stat meaning control.

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to surprise values of each token, while the red plot corresponds to average surprise values over a window of size 10 at each token index. Note that the surprise values drop drastically in Fig. 1a as the repetitions increase in Ex. 1.1. Similarly, in Fig. 1b, we observe a dip in surprise values wherever there is a repetition in Ex. 1.2. Clearly, there is a correlation between small

Figure 1 shows plots of surprise values against indices of tokens in each of the samples in Ex. 1. The blue plot corresponds to surprise values of each token, while the red plot corresponds to average surprise values over a window of size 10 at each token index. Note that the surprise values drop drastically in Fig. 1a as the repetitions increase in Ex. 1. Similarly, in Fig. 1b, we observe a dip in surprise values wherever there is a repetition in Ex. 1. Clearly, there is a correlation between small
average surprise values and repetitions. Further, in Fig. 1a note that the generating model seems to get trapped into a small surprise repetition region. We call this region of small surprise as *boredom trap*. We observe that these models tend to fall into a boredom trap for small values of $p$. Figure 1c corresponds to Ex. 1.3 where we choose $p = 1.0$ and illustrate that for large values of $p$, the average surprise value of the generated text tends to increase with the number of generated tokens, which leads to incoherence. We call this region of large surprise a *confusion trap*. Figure 1d shows surprise values corresponding to Ex. 1.4 which is generated using our proposed sampling algorithm, mirostat. We can observe in Fig. 1d that mirostat has the ability to increase the surprise value when when falling into a boredom trap and, thereby maintaining the average surprise value. By doing so, it not only helps generate high-quality text with predetermined average surprise value, but also helps avoid small surprise repetition regions and large surprise incoherent regions. In Fig. 1e we show the surprise values in human-generated text that followed this context as shown in Ex. 1.5. We observe that human-generated text has average surprise value that is
between values using top-$p$ sampling for $p = 0.4$ and $p = 1.0$. More importantly, human-generated text does not seem to fall into either of the traps described above.

Perplexity is a statistical metric used to evaluate quality of neural text generation. It is closely related to average surprise described in Ex. 1 as will be formally defined in Sec. III-C. A large-scale human experiment in [5] showed that human-based quality evaluation is closely related to the likelihood of the generated text for fixed number of tokens. In particular, it was found that reducing perplexity increases quality till some point before the quality of generated text starts dropping with decrease in perplexity. This implies that good control over perplexity of the generated text would give us direct control over the quality of generated text (as evaluated by humans). Hence, generating texts with an appropriately chosen target perplexity value may maximize the quality of generated text. Unfortunately, existing decoding methods such as top-$k$ or top-$p$ sampling do not provide good control over the average surprise in generated text, as was observed in Ex. 1. On the other hand, our proposed mirostat algorithm can generate high-quality text by controlling the perplexity.

Now we summarize the key contributions of this work.

- In Sec. IV, we show theoretically how cross-entropy and hence perplexity grows in top-$k$ and top-$p$ sampling as a function of $k$ and $p$ respectively, which was previously unknown.
- In Sec. V-B we introduce mirostat sampling, which outputs texts with predetermined target perplexity value. Although perplexity may not be the best metric to measure the quality of generated text [14], much literature discusses its correlation to quality [5]. Hence, our algorithm that controls perplexity would help generate high-quality text.
- In Sec. VI-A we show experimentally that there is much fluctuation in cross-entropy rates in top-$k$ and top-$p$ sampling as a function of their input parameters; hence they cannot provide good control over the perplexity of the output text.
- In Sec. VI-B we observe that repetition is closely related to perplexity of the generated texts and mostly independent of the sampling method but slightly dependent on the model used.
- In Sec. VI-C we experimentally show that mirostat sampling is able to avoid both boredom and confusion traps for a wide range of target perplexity values.

II. RELATED WORK

- **Sampling from a distorted probability distribution:** Pure sampling from language models often leads to incoherent text and greedy decoding leads to repetitions. It was found that distorted probability distributions obtained using top-$k$, top-$p$, or temperature sampling help improve the quality of generated text [8], [9], [10]. Proper tuning of parameters in these methods generate high-quality text, but these methods are ad hoc and do not provide good control over the statistics of the output. Our method uses the statistics of the previously generated tokens as an input to generate the next token by distorting the probability distribution in a way that helps control the overall statistics of the generated text. The advantage our method provides is control over the perplexity of the output which is not guaranteed in previously proposed decoding methods. This, when used with the relation between perplexity and human-based quality evaluation observed in [5], can provide us text that has better controlled quality.
- **Controllable text generation:** Controllable text generation has commonly focused on controlling semantics of the output text while our approach is purely statistical. We try to solve the statistical problems associated with pure sampling or greedy decoding by guiding the decoder along a desired statistical path. A new model with 1.63 billion parameters, CTRL, was trained to generate text based on a control word [11]. On the other hand, sampling algorithms like Plug and Play Language Model (PPLM) and Constrained sentence Generation by Metropolis-Hastings (CGMH) work at the inference stage on top of a pretrained language model to control certain attributes of the generated text. PPLM shows that using attribute classifiers on top of pretrained language models helps control text generation [14]. CGMH uses Metropolis-Hastings sampling to generate text with certain constraints like appearance of multiple keywords [15].
- **Quality-diversity tradeoff:** Distorting probability distributions for decoding using top-$k$, top-$p$, or temperature sampling with low temperatures improves the quality of the text, however, it also reduces the diversity in text. Certain applications like question-answering demand high-quality generation whereas open-ended tasks such as story generation demand diversity in addition to quality. However, there is a tradeoff between quality and diversity as observed in [5]. Further, it was observed that diversity is closely related to entropy while quality is found to be maximized in a certain range of observed likelihood values for fixed length sentences. Our algorithm provides a very good control over observed cross-entropy, which is equal to observed likelihood per token of the generated text. Hence, by maintaining the observed cross-entropy of the text in a certain range, we can ensure high-quality text generation.
- **Repetitions:** Greedy decoding from language models often lead to texts with excessive repetitions both at token level and at sentence level. Several techniques have been proposed in the literature to solve this problem. Token loss dynamic reweighting (TLDR) hypothesizes that there are some tokens which are more difficult to learn than others [16]. Hence, they propose reweighting the tokens while learning to balance out this effect and thus reduce repetitions. CTRL uses a repetition penalty in the decoding process to reduce the occurrence of repetitive tokens [11]. It was suggested in [17] that the cause for repetitions is in fact due to a flaw in the training objective itself and used a new objective that gives less probability to unlikely sequence including texts with high repetitions. Variants of beam search have been proposed
in [12, 18, 19] to induce diversity in the generated text, which in turn alleviates the problem of repetition. Variants of top-k sampling and repetition penalty used in [11] were also used in [20] for reducing repetitions.

III. SURPRISE, ENTROPY, AND PERPLEXITY

In this section we first briefly discuss language modeling. Later, we formally define the notions of surprise, entropy, and perplexity.

Language modeling is an unsupervised task of learning the probability distribution \( p(x) \) from a set of examples of the form \( x = (x_1, \ldots, x_n) \) where each \( x_i \in \mathcal{V} \) and \( \mathcal{V} \) is a finite vocabulary. Since \( x \) is a sequence, it can be written as a product of conditional probabilities using the chain rule as follows [21]:

\[
p(x) = \prod_{i=1}^{n} p(x_i|x_{<i}).
\]  

(1)

Current state-of-the-art methods [2, 22] train a model with parameter \( \theta \) minimizing the loss function

\[
\mathcal{L}(T) = -\frac{|T|}{n} \sum_{k=1}^{n} \sum_{i=1}^{n} \log p_{\theta}(x_i^k|x_{<i}^k)
\]  

(2)

over dataset \( T = \{x^1, \ldots, x^{|T|}\} \).

A trained model \( p_{\theta} \) can be used for generating the \( i \)th word from the previous words by sampling from the distribution \( p_{\theta}(x_i|x_{<i}) \).

A. Surprise

For a random variable \( X \in \mathcal{X} \) distributed as \( P \), the information content or surprisal associated with an instance \( x \) of \( X \) is defined as \(-\log P(x)\) [23]. Hence, less probable instances are more surprising than more probable instances. Extending this definition to conditional random variables, we next define the surprise value associated with tokens and sentences with respect to generated text for a fixed model distribution \( P_M \).

**Definition 1.** We define the surprise value of a token \( X \) with respect to generated text \( X_{<i} \) and model distribution \( P_M \) for some fixed model \( M \) as

\[
\mathcal{S}_M(X|X_{<i}) = -\log P_M(X|X_{<i}).
\]  

(3)

As we will soon see, this quantity is directly related to perplexity. Now we define the average surprise for a sentence \( X \) with \( n \) tokens.

**Definition 2.** For a sentence \( X^n = (X_1, \ldots, X_n) \) with \( n \) tokens, we define surprise rate with respect to a probability distribution \( P_M \) for some model \( M \) as

\[
\overline{\mathcal{S}}_M(X^n) = -\frac{1}{n} \sum_{i=1}^{n} \log P_M(X_i|X_{<i}).
\]  

(4)

B. Entropy

The entropy of a discrete random variable \( X \in \mathcal{X} \) with probability distribution \( P_M \) is given by

\[
H(P_M) = -\sum_{x \in \mathcal{X}} P_M(x) \log P_M(x)
= \mathbb{E}_{P_M}[\mathcal{S}_M(X)],
\]  

i.e., \( H(P_M) \) is the expected value of \( \mathcal{S}_M(X) \) with respect to \( P_M \). The entropy rate of a stochastic process \( \mathcal{X} = \{X_i\} \) is defined as

\[
\mathcal{H}(P_M) = \lim_{n \to \infty} \mathbb{E}_{P_M}[\overline{\mathcal{S}}_M(X^n)],
\]  

(6)

when the limit exists. Now, if the data is being sampled from \( P_M \) and if \( P_M \) is a stationary ergodic source, then by the Shannon-McMillan-Breiman theorem [24, Thm. 16.8.1], we have

\[
\lim_{n \to \infty} \overline{\mathcal{S}}_M(X^n) = \mathcal{H}(P_M),
\]  

(7)

when the limit exists. For experiments, we will approximate \( \mathcal{H}(P_M) \) by \( \overline{\mathcal{S}}_M(X^n) \) for a sentence of length of \( n \) generated by model \( M \).
C. Perplexity

A popular metric used in natural language processing to measure the quality of a generated text is perplexity [25], [26]. Perplexity is closely related to cross-entropy and hence we define the notion of cross-entropy first.

The cross-entropy of a discrete random variable $X \in \mathcal{X}$ distributed as $P_M$ with respect to a discrete random variable $Y \in \mathcal{Y}$ distributed as $P_N$ such that $\mathcal{Y} \subseteq \mathcal{X}$ is given by

$$H(P_N, P_M) = - \sum_{y \in \mathcal{Y}} P_N(y) \log P_M(y) = \mathbb{E}_{P_N}[\mathcal{S}_M(Y)].$$

The cross-entropy rate of a stochastic process $X = \{X_i\}, X_i \in \mathcal{X}$ distributed as $P_M$ with respect to a stochastic process $Y = \{Y_i\}, Y_i \in \mathcal{Y}$ distributed as $P_N$ and $\mathcal{Y} \subseteq \mathcal{X}$ is defined as

$$\mathcal{H}(P_N, P_M) = \lim_{n \to \infty} \mathbb{E}_{P_N}[\mathcal{S}_M(Y^n)],$$

when the limit exists. Further, if $Y^n$ is sampled from $P_N$ and if $P_N$ is a stationary ergodic source, then by the Shannon-McMillan-Breiman theorem [24 Thm. 16.8.1], we have

$$\lim_{n \to \infty} \mathcal{S}_M(Y^n) = \mathcal{H}(P_N, P_M),$$

when the limit exists. Now, the perplexity corresponding to $\mathcal{H}(P_N, P_M)$ is simply defined as

$$\text{PPL}(P_N, P_M) = 2^{\mathcal{H}(P_N, P_M)},$$

For experiments, when the text is generated using $P_N$, we will approximate $\mathcal{H}(P_N, P_M)$ by $\mathcal{S}_M(Y^n)$ for a sentence of length of $n$. Perplexity denotes how close $P_N$ is to $P_M$. The lower the perplexity, the closer the distributions $P_N$ and $P_M$.

IV. THEORETICAL ANALYSIS OF SAMPLING METHODS

In this section, we will analyze top-$k$ and top-$p$ sampling methods theoretically. We will note that the log of perplexity or cross-entropy of generated text increases in a nonlinear manner with increase in $k$ in top-$k$ sampling. In particular, perplexity seems to grow faster with increase in $k$ for small values of $k$ compared to large values of $k$. On the other hand, in top-$p$ sampling, we note that log of perplexity increases essentially linearly with increase in $p$. Hence, it is easier to control perplexity using top-$p$ sampling because of this linear growth in cross-entropy.

A. Zipf’s law

Zipf’s law states that the frequency of occurrence of any word in the vocabulary is inversely proportional to its rank in the frequency table [27], [28]. More precisely, for a vocabulary of size $N = |\mathcal{V}|$ the frequency of the $i$th most probable word is given by

$$p(i; s, N) = \frac{1}{i^s H_{N,s}},$$

where $s$ is an exponent characterizing the distribution and $H_{N,s} = \sum_{n=1}^{N} \frac{1}{n^{s}}$ is the $N$th generalized harmonic number.

Further, it is known that for human languages, words have a very heavy-tailed probability distributions; hence the exponent $s$ is very close to 1. Hence, when required, we write $s = 1 + \epsilon$, where $\epsilon$ is a small positive constant. For all the theoretical analysis in this work, we will assume that the sampled words follow Zipf’s law. Proofs to Prop. 1 and all the theorems in this section are provided in the appendix.

B. Surprise and cross-entropy in top-$k$ sampling

Now we will analyze top-$k$ sampling. Thm. 1 gives the expression for $\mathcal{S}(k)$ and its growth with increase in $k$.

**Theorem 1.** If words are sampled from the Zipf’s distribution given by (12), then the surprise value of a word with rank $k$ and its rate of increase are given by

$$\mathcal{S}(k) = s \log k + \log H_{N,s},$$

$$\frac{d\mathcal{S}(x)}{dx} = \frac{s}{x}$$

respectively, where $\mathcal{S}(x)$ is a continuous function with the same expression as $\mathcal{S}(k)$ with a continuous domain.

From Fig. 2 we note that $\mathcal{S}(x)$ is highly sensitive to change in $x$ for small values of $x$ and its sensitivity to $x$ decreases drastically with increase in $x$. Now, we analyze how cross-entropy varies with $k$. Let $P_M$ be the model distribution. Top-$k$
sampling works by truncating the tail of the distribution \( P_M \) and samples from the most probable \( k \) tokens. Let the truncated distribution be denoted by \( P_{M_k} \). In Prop. [1] we provide an expression for \( H(P_{M_k}, P_M) \).

**Proposition 1.** Let \( P_M \) be the model distribution satisfying (12) with vocabulary of size \( N \) and let \( P_{M_k} \) be the model distribution obtained using top-\( k \) sampling. Then \( H(P_{M_k}, P_M) \) is given by

\[
H(P_{M_k}, P_M) = \frac{s}{H_{k,s}} \sum_{i=1}^{k} \frac{\log i}{i^s} + \log H_{N,s}.
\]

It is difficult to get an intuition about the behavior of \( H(P_{M_k}, P_M) \) directly from (15). Thus, in Thm. [2] we obtain an approximation to \( H(P_{M_k}, P_M) \) that shows \( H(P_{M_k}, P_M) \) is essentially of the form \( c_1(1 - c_2 \ln k + \frac{1}{k} + c_3) + c_4 \) for \( 0 < \epsilon < \frac{1}{\ln 2} \), where \( c_1, c_2, c_3, c_4 \) are some constants. Hence we observe that \( H(P_{M_k}, P_M) \) grows fast with small values of \( k \) and slows down for large values of \( k \).

**Theorem 2.** Let \( P_M \) be the model distribution satisfying (12) with vocabulary of size \( N \) and let \( P_{M_k} \) be the model distribution obtained using top-\( k \) sampling. Then, for \( 1 < s \leq \frac{1}{\ln 2} \), \( H(P_{M_k}, P_M) \) can be approximated as

\[
H(P_{M_k}, P_M) \approx \frac{b_1\epsilon}{b_3} \left( 1 - \frac{b_2b_3\ln k + \frac{1}{k}}{b_1(b_3k^\epsilon - 1)} \right) + \log H_{N,s},
\]

where \( b_1 = \frac{\log 2}{\log \epsilon} + \frac{\log 3}{\log \epsilon} + \frac{\log 4}{\log \epsilon} \), \( b_2 = \frac{a}{1 + \epsilon} \), and \( b_3 = 1 + 0.7\epsilon \) are constants.

In Fig. 3 we plot the approximation obtained for \( H(P_{M_k}, P_M) \) in Thm. [2] and observe that the obtained approximation is very close to the actual value, hence provides a good estimate of \( H(P_{M_k}, P_M) \). Note that the value of \( H(P_{M_k}, P_M) \) does not grow much beyond \( k = 2000 \) and hence it makes sense to tune \( k \) between 1 to 2000 to get a desired cross-entropy.

**C. Surprise and cross-entropy in top-\( p \) sampling**

Now we provide a theoretical analysis for top-\( p \) sampling. In Thm. [3] we provide approximate expressions for \( \mathcal{S}(p) \) and \( \frac{d\mathcal{S}(p)}{dp} \) that shows that \( \mathcal{S}(p) \) grows essentially linearly with \( p \).

**Theorem 3.** If words are sampled from the Zipf’s distribution given by (12). If \( \epsilon > 0 \) is a small constant, then \( \mathcal{S}(p) \) and the rate of change of \( \mathcal{S}(p) \) with respect to \( p \) is given by

\[
\mathcal{S}(p) \approx \frac{(1 + \epsilon)}{b\ln 2} H_{N,s,p} = \frac{(1 + \epsilon)}{\epsilon} \log b + \log H_{N,s},
\]

\[
\frac{d\mathcal{S}(p)}{dp} \approx \frac{(1 + \epsilon)H_{N,s}}{b}\left(1 + \frac{H_{N,s,p}}{b}\right),
\]

where \( b = 1 + 0.7\epsilon \).

In Fig. 4a we plot the approximate expression for \( \mathcal{S}(p) \) obtained in Thm. [3] which is a linear function in \( p \) and has a slope approximately 10 for \( s = 1.07 \) and \( N = 50,000 \). In Fig. 4b we plot the approximate expression for \( \frac{d\mathcal{S}(p)}{dp} \) from Thm. [3] which
is also a linear function of $p$. This tells us that even though $\mathcal{G}(p)$ can be approximated as essentially a linear function of $p$, it has a slightly increasing slope. Further, unlike the plot of $\frac{d\mathcal{G}(x)}{dx}$ in Fig. 2b which is decreasing with $k$, $\frac{d\mathcal{G}(p)}{dp}$ in Fig. 4b has a positive slope.

As pointed out earlier, $\mathcal{G}(p)$ gives the surprise values as a function of $p$ for a fixed vocabulary size, while the observations made in Fig. 6b for top-$p$ sampling had a varying vocabulary size. To estimate the change in average surprise with change in the value of $p$ in top-$p$ sampling, Thm. 4 provides approximate expressions for $H(P_{M_p}, P_{M})$ which shows that $H(P_{M_p}, P_{M})$ grows approximately linearly with $p$. This is in contrast to top-$k$ sampling where the growth of average surprise $H(P_{M_k}, P_{M})$ was highly nonlinear.

**Theorem 4.** Let $P_M$ be the model distribution satisfying (12) with vocabulary of size $N$, and let $P_{M_k(p)}$ be the model distribution obtained using top-$p$ sampling where $k(p)$ is the minimum value of $k$ satisfying $\sum_{i=1}^{k(p)} \frac{1}{H_{N,s}} \geq p$. Then, for $1 < s \leq \frac{1}{\ln 2}$, $H(P_{M_p}, P_{M})$ can be approximated as

$$H(P_{M_p}, P_{M}) \approx \frac{8}{2\ln 2} \left( pH_{N,s} + \epsilon pH_{N,s}^2 \right) + \log H_{N,s}. \quad (19)$$

Figure 5 shows that the growth of $H(P_{M_p}, P_{M})$ with $p$ is approximately linear.

D. Effect of temperature on top-$k$ and top-$p$ sampling

Temperature is used to distort the original distribution in a suitable way so as to generate samples that avoid the problems associated with pure sampling. In particular, lowering the temperature makes the sampling more greedy. For a given temperature
Fig. 5: Approximation to $H(P_{M_k}, P_M)$ obtained in Thm. 4 for $s = 1.1$, $N = 50,000$. Note that $H(P_{M_k}, P_M)$ grows approximately linearly with $p$, unlike top-$k$ sampling, where $H(P_{M_k}, P_M)$ grew nonlinearly with $k$.

For $T > 0$, the frequency of the $k$th most probable word in (12) is given by

$$p(k; s, N, T) = \frac{1}{k^s H_{N,s}^T},$$

$$= p(k; \frac{s}{T}, N).$$

Hence the effect of temperature in our analysis can be captured simply by modifying $s$ to $\frac{s}{T}$.

V. PERPLEXITY-CONTROLLED TEXT GENERATION

In this section we propose an algorithm that provides control over the cross-entropy rate of the generated text. The algorithm works in two stages for generating each word. First it estimates the value of $s$ assuming that the words follow Zipf’s law. Then, it uses top-$k$ sampling where $k$ is a function of estimated $s$ and the target surprise value of the output text.

A. Estimating $s$ for Zipf’s distribution

We assume that the words follow Zipf’s distribution (12), i.e.

$$p(k; s, N) = \frac{1}{k^s H_{N,s}}.$$ 

Further, we observe the probabilities produced by our language model as $\{p^{\text{obs}}(1), p^{\text{obs}}(2), \ldots, p^{\text{obs}}(N)\}$, where $p^{\text{obs}}(i)$ is the probability associated with the $i$th most probable word for $i \in \{1, 2, \ldots, N\}$. We use minimum mean squared error estimation to find the value of $s$. However, $s$ shows up in $p(k; s, N)$ both as an exponent of $k$ and in $H_{N,s}$ which makes the computation difficult. Hence we estimate by minimizing the mean squared error between logarithm of ratios of subsequent probabilities which eliminates $H_{N,s}$, i.e. we estimate $s$ as

$$\hat{s} = \arg \min_s \frac{1}{T} \sum_{i=1}^{N-1} (st_i - b_i)^2,$$

$$= \frac{\sum_{i=1}^{N-1} t_i b_i}{\sum_{i=1}^{N-1} t_i^2},$$

(21)

where $t_i = \log \frac{i+1}{i}$ and $b_i = \log \frac{p^{\text{obs}}(i)}{p^{\text{obs}}(i+1)}$. When $N$ is large, we estimate $s$ using the most probable $m$ tokens for $m$ around 100 to improve time complexity, which also gives practically good estimation.

B. Algorithm

Here we provide mirostat\(^2\) which can generate texts with predetermined average surprise value as given as Alg. We This algorithm takes as input a target surprise value $\tau$ and initializes a variable $\mu = 2\tau$. Each word is sampled by first estimating $s$ from (21) as $\hat{s}$, then using top-$k$ sampling by approximating $k$ as a function of $\hat{s}$ and $\mu$ by approximating $H_{N,s} \approx \int_1^N \frac{1}{t^\tau} dt = \frac{(1-N^{-1})}{s^{-1}}$ and using (13) to get

\(^2\)Code for mirostat sampling is available at https://github.com/basusourya/mirostat
We observe that mirostat sampling gives very good control over observed surprise value with low variance for surprise values that truncate the low probability words for controlling surprise value and without any truncation the observed surprise is around less than five. For higher target surprise, Alg. 1 does not provide good control on observed surprise value since the algorithm would reduce the quality of the generated text, hence not considered in this work.

Thus, to get better control over observed surprise values, we must truncate some more probable words as well, which top-$k$ sampling. Our theoretical analysis in Sec. IV helps explain nonlinear growth in cross-entropy rate in top-$k$ sampling by the feedback which updates the value of $\mu$. An error term $e$ is computed as the difference between the observed surprise $\mathcal{S}(X)$ of the sampled word $X$ and the $\tau$ is then updated using this error.

**Algorithm 1: Mirostat sampling for perplexity control**

Target cross-entropy $\tau$, maximum cross-entropy $\mu = 2\tau$, learning rate $\eta$, $m = 100$

\begin{algorithmic}
\While {more words to be generated}
\State Compute $\hat{s}$ from (21): $\hat{s} = \frac{\sum_{i=1}^{N-1} k_i b_i}{\sum_{i=1}^{N-1} t_i}$
\State Compute $k$ from (22): $k = \left( \frac{\hat{s}^2 N}{1-N^{-1}} \right)^{\frac{1}{\tau}}$
\State Sample the next word $X$ using top-$k$ sampling
\State Compute error: $e = \mathcal{S}(X) - \tau$
\State Update $\mu$: $\mu = \mu - \eta e$
\EndWhile
\end{algorithmic}

Note that we can use an alternate algorithm to tune $k$ in Alg. 1 by iterating through the most probable tokens to set $k$ corresponding to a token that has a suitable amount of surprise. This alternate algorithm would have varying computational complexity depending on the target surprise value and can be slow for large surprise values. Hence, in Alg. 1 we first estimate $s$ using a fixed number of tokens and then directly use top-$k$ sampling, which is computationally efficient for all target surprise values and provides good control over perplexity.

**VI. Experimental Analysis**

Here we provide experiments for the performance of top-$k$, top-$p$, and mirostat sampling. We use the GPT-2 language model with 117M parameters for all our experiments unless mentioned otherwise. In Sec. VI-A we show how observed cross-entropy varies with change in parameters for all the sampling methods. This illustrates that mirostat provides direct control over the observed cross-entropy of the output text. In Sec. VI-D we demonstrate output texts for different target cross-entropy and their observed cross-entropy.

**A. Cross-entropy rate for different sampling methods**

In Fig. 6a we plot observed cross-entropy in generated texts versus several input parameters for different sampling methods. For each of the plots, we generate four output texts of 200 tokens corresponding to each value of input parameter in each sampling method with same context in each case.

In Fig. 6a we plot the observed surprise values in generated texts versus $k$ in top-$k$ sampling. Note that cross-entropy has a steep increase for small values of $k$ and relatively slow increase in $k$ for high values of $k$. Thus, for small values of $k$, cross-entropy is very sensitive to change in the $k$, but, for large values of $k$, cross-entropy hardly changes. Hence, even though we can clearly see the increase in cross-entropy with increase in top-$k$, it is difficult to control cross-entropy using top-$k$ sampling.

In Fig. 6b we plot the observed surprise values in generated text versus $p$ in top-$p$ sampling. We observe that cross-entropy grow essentially linearly with increase in $p$ unlike top-$k$ sampling.

In Fig. 6c we plot the observed cross-entropy in generated texts versus target cross-entropy in mirostat sampling. Alg. 1 We observe that mirostat sampling gives very good control over observed surprise value with low variance for surprise values less than five. For higher target surprise, Alg. 1 does not provide good control on observed surprise value since the algorithm truncates the low probability words for controlling surprise value and without any truncation the observed surprise is around five. Thus, to get better control over observed surprise values, we must truncate some more probable words as well, which would reduce the quality of the generated text, hence not considered in this work.

The observation on different growth rate of surprise values in top-$k$ and top-$p$ sampling in Fig. 6 is not very intuitive on its own. Our theoretical analysis in Sec. IV helps explain nonlinear growth in cross-entropy rate in top-$k$ sampling and essentially linear growth in cross-entropy rate in top-$p$ sampling. Note that our theoretical analysis in Sec. IV deals with cross-entropy while our experiments deal with cross-entropy rate. However, for practical purposes cross-entropy helps us give an intuition about cross-entropy rate in different sampling methods. Further, the fluctuation in cross-entropy rate in Fig. 6a and Fig. 6b is likely because 200 tokens is not sufficient for the convergence in (9). There is not much fluctuation in cross-entropy rate in Fig. 6c because we use feedback to control the cross-entropy rate more accurately, which gives accurate results even for a small number of tokens.
Fig. 6: Cross-entropy rate in different sampling methods. Note that cross-entropy rate shows nonlinearity in top-\( k \) sampling and near-linearity in top-\( p \) sampling. We also note that for a fixed input parameter in top-\( k \) and top-\( p \) sampling, observed cross-entropy rate shows much fluctuations. Note that mirostat shows negligible fluctuation in observed cross-entropy rate.

**B. Perplexity and repetitions**

Here, we present some experimental observations for percentage of repeated tokens across different sampling methods and language models. In Fig. 7, we generate texts with 200 tokens using different sampling methods and models with varying relevant input parameters such as \( k, p, \) or target surprise values, \( \tau \). We also consider the percentage of \( n \)-gram repetitions for different values of \( n \) for a fixed sampling method. We define percentage \( n \)-gram repetition as

\[
\text{percentage } n\text{-gram repetition} = \left( 1 - \frac{\text{number of distinct } n\text{-gram tokens}}{\text{total number of } n\text{-gram tokens}} \right) \times 100, \tag{23}
\]

where an \( n \)-gram token simply means concatenation of \( n \) contiguous tokens. Hence, for \( n = 1 \), \( n \)-gram repetitions capture word-level repetitions, whereas larger values of \( n \) capture sentence-level repetitions. For \( n = 1 \), we refer to percentage 1-gram repetition simply as percentage repetition.

In Fig. 7a, we fix the language model to GPT-2 with 117M parameters and observe that percentage repetition decreases with increase in cross-entropy and more importantly, for a fixed model, this relation is independent of the sampling method.

In Fig. 7b, we fix the language model as GPT-2 with 117M parameters and use top-\( k \) sampling with varying temperature values. We observe that repetitions for different temperature values and \( k \) follow the same curve as in Fig. 7a. This implies that cross-entropy controls the percentage repetitions in generated texts. Moreover, it implies that once the model and cross-entropy are fixed, percentage repetition is not affected by the considered sampling methods.

In Fig. 7c, we fix the language model as GPT-2 with 117M parameters and capture \( n \)-gram repetitions for varying cross-entropy rate and different values of \( n \). We note from Fig. 7c that for small values of \( n \), which captures word-level repetitions, the percentage \( n \)-gram repetitions drop almost linearly with increase in cross-entropy; whereas for larger values of \( n \), which captures sentence-level repetitions, the percentage \( n \)-gram repetitions is very close to zero for cross-entropy greater than 3. This indicates that sentence-level repetitions disappear after a threshold of cross-entropy whereas word-level repetitions continue to appear for larger values of cross-entropy. Also, note that in human-generated text data, there are often common pronouns and conjunctions that are essential and are often repeated, hence we do not expect a good sampling algorithm to have absolutely
Observed cross-entropy rate  
Percentage repetition  

(a) Percentage repetition vs. observed cross-entropy rate for different sampling methods.

(b) Percentage repetition vs. observed cross-entropy rate for different temperature values, $T$.

(c) Percentage repetition vs. observed cross-entropy for $n$-gram tokens and different sampling methods.

(d) Percentage repetition vs. observed cross-entropy rate for different language models.

Fig. 7: Percentage repetition vs. observed cross-entropy rate. Note that repetitions vary heavily with observed cross-entropy rate in the generated text and mostly independent of sampling method used. Further, when we vary language models, large models like GPT-2-XL with 1558M parameters seem to have slightly less repetitions compared to small models like GPT-2 with 117M parameters for the same cross-entropy rate.

zero 1-gram repetitions. But, we do expect a good sampling algorithm to have minimum sentence-level repetitions, which all the sampling seems to show beyond a threshold of cross-entropy, which seems to be around 2.5 for GPT-2 language model with 117M parameters.

In Fig. 7(a) we plot percentage repetition versus cross-entropy for different language models using top-$p$ sampling for varying values of $p$. We observe that larger language models such as GPT-2-XL with 1558M parameters have slightly less repetitions for a fixed value of cross-entropy compared to smaller models such as GPT-2 with 117M parameters.

From these observations, we conclude that in order to control percentage repetition in generation we need to control the cross-entropy of the output text. Hence, mirostat can help provide a good control over repetitions in generated texts.

C. Boredom and confusion traps

Here we show that due to lack of control over statistics in top-$k$ and top-$p$ sampling, these methods get trapped into generating low-quality texts when the number of tokens generated are high for a wide range of $k$ and $p$. We generated 10 samples of texts on the same context that are 900 tokens long and averaged their observed cross-entropy at various points of generation for each of the plots in Fig. 8 except for the human-generated text for which only one sample was used (the tokens following the context in the corpus).

In Fig. 8(a) we illustrate the boredom trap. We observe that for small values of $k$ and $p$, both top-$k$ and top-$p$ sampling methods fall into these low-cross-entropy regions—boredom traps—which results in increase in repetitions as the length of the generated text increases, as illustrated in Sec. [VI-B]. Hence, lack of control over output statistics in these methods leads to degradation of quality in generated texts for longer texts.

In Fig. 8(b) we illustrate the confusion trap. Here we observe that for high values of $k$ and $p$ in top-$k$ and top-$p$ sampling methods respectively, the observed cross-entropy of the generated texts increases with the length of generated texts. This leads to increase in incoherence in the text as the token index increases.

3For more information on various GPT models, refer to [https://huggingface.co/transformers/pretrained_models.html](https://huggingface.co/transformers/pretrained_models.html)
Fig. 8: Cross-entropy rate vs. number of tokens generated for different sampling methods. We observe that small values of input parameter in top-k or top-p sampling falls into boredom trap, whereas for large values these methods fall into confusion trap. Mirostat shows control over cross-entropy over varying lengths of texts and large range of input parameter values.

In Fig. 8c we choose certain values of $k$ and $p$ in an ad hoc manner and generate texts using top-k and top-p sampling methods respectively to observe that for these values of $k$ and $p$, the generated texts tend to have cross-entropy that seems to be converging to some limiting value with increase in text length and not fall into either boredom or confusion trap. We also illustrate how the observed cross-entropy varies with increase in text length in the human-generated text corresponding to the tokens following the context used for these experiments. We find that the human-generated text converges to some limiting value of cross-entropy when the generated text is long enough and does not fall into either boredom or confusion.

Finally, in Fig. 8d we show the cross-entropy for the generated texts using Alg. 1 converge to their target cross-entropy value within a few tokens and continue to maintain the desired value of cross-entropy for long generated texts.

**D. Text generation and compression**

Here we will look at texts generated for various target surprise values using mirostat sampling with GPT-2 with 117M. We also observe the well-known relation between cross-entropy and data compression [24, Ch. 5], [29]. In particular, it is known that when the actual probability distribution of the generated text is not known, then the minimum lossless compression rate achievable is equal to the cross-entropy of the assumed distribution, which is the language model here, with respect to the actual unknown distribution, which is obtained from adaptive top-k sampling here.

We generate 200 tokens for different values of target surprise values using the GPT-2 model with 117M parameters to show the quality of the text generated using Alg. 1 for different target surprise values. We also observe the compression rates obtained using arithmetic coding [30], [31] with the language model as the probability distribution. So, in a way, mirostat can generate text that has a predetermined minimum lossless compression rate for a given model.

**Example 2. Context:** “For two months early in 1943, Shannon came into contact with the leading British mathematician Alan Turing. Turing had been posted to Washington to share with the U.S. Navy’s cryptanalytic service the methods used by the British Government Code and Cypher School at Bletchley Park to break the ciphers used by the Kriegsmarine U-boats in the north Atlantic Ocean. He was also interested in the encipherment of speech and to this end spent time at Bell Labs.”
In April 1943, Shannon was assigned to the U.S. Navy’s cryptanalytic service, the Naval Intelligence Service. He was assigned to the cryptanalytic service’s cryptanalysis unit, the Naval Intelligence Laboratory. He was also assigned to the cryptanalytic unit’s cryptanalysis unit, the Naval Intelligence Laboratory.

In May 1943, Shannon was assigned to the cryptanalytic unit’s cryptanalysis unit, the Naval Intelligence Laboratory. He was also assigned to the cryptanalytic unit’s cryptanalysis unit, the Naval Intelligence Laboratory.

In June 1943, Shannon was assigned to the cryptanalytic unit’s cryptanalysis unit. He was assigned to the cryptanalytic unit’s cryptanalysis unit, the Naval Intelligence Laboratory.

In July 1943, Shannon was assigned to the cryptanalytic unit’s cryptanalysis unit. He was assigned to the cryptanalytic unit’s cryptanalysis unit, the Naval Intelligence Laboratory.

In August 1943, Shannon was assigned to the British Intelligence Service. He was assigned to the British Intelligence Service’s Intelligence Advisory Board.

In early 1944, Shannon was assigned to the British Intelligence Service’s Intelligence Advisory Board. He was also assigned the task of deciphering the ciphertext of the British Government Code. He was also assigned to the British Intelligence Service’s Cryptanalytic Service. He was assigned to the British Intelligence Service’s Intelligence Advisory Board at the end of 1944.

He was confirmed to work as the Director of the British Intelligence Service’s Information Service at the end of 1944. He was confirmed as an Intelligence Advisory Officer to the United Kingdom’s Intelligence Advisory Committee at the end of the war.

After a short stay at Bell Labs, Shannon was commissioned to report back to the Naval Intelligence Services (NISC) to learn more about his research. He made his way over to the Royal Naval Institute. He had already been working with a number of other cryptanalytic researchers, including the mathematician Alan Turing, for over eight years at Bell Labs, but this research was not yet done. He spent the next two months studying the cipher schemes of the various British cryptanalytic systems. The first step was to find out the actual cipher scheme for the cipher used by the HMS Queen Elizabeth on a submarine.

By the time Shannon arrived at the Royal Naval Institute he had been working for Patagonia Scientific and Technology, the British cryptanalytic company, for eight years. Patagonia received a call at 9 A.D. on the evening of September 4, 1943, asking to check the cipher system for a cipher used by the HMS Queen Elizabeth, which is written as:

Naval Contractors hired Shannon as a shortage specialist at the Navy. So she worked alone with Turing to decipher the British cipher for the northern German submarines. Shannon undertook the work initially on the S.G.F. dual cipher. Shannon spent the whole working days at the Bell Labs lab at Bletchley Park.

After weeks of interrogations Shannon was able to break the great El-Fran jujito with a German accent. Shannon then calculated that this hydrocephalic consciousness alone would allow her to think the same words but without the huge amount
of writing required to produce such a thing, and with millions of hours on board she was able to write some of the same phrases over a twenty-eight hours workweek.

Output: Individualist Hypothesis.

Miranda (alias Charlotte Reagmire, aka "The Lady in the Lake") (1945-2049) (wife of B.G. Lloyd) (Philadelphia, PA)"

Target average surprise value = 6.0  Observed cross-entropy rate = 5.735  Average compressed size (per token) = 5.834  Percentage compression = 85.55%

Generated text: “The CIA trained a small band of cryptanalysts to do the maths again, this time using a UK number generator. A few days after the wars Belgium introduced Bermuda rugby as the appropriate sport for the National Guard. Notwithstanding there being no convention around using English for Rugby at the time, there would be no possible instance in the history of fencing in Europe. Flags for the Hurricanes had evolved recently using a Dutch Italian design called the Crazy Flag. These flag designs come largely of British origin and the date published of its introduction by the Royal Armouries of Lameucers is from 1638. The camouflage was recently added to the new Irish power orange flag. The design is based on the weapons bao mouèret Standard and has two coloured pouches connected to the rifle barrel by two checks along the top of the barrel with protection straps around the barrel to protect the cutouts. NATO hired a team of physicists to do the reconstruction. Readers who want to know more about this new"

In Ex. 2 we can see that low value of surprise value results in repetitions and high value of surprise value results in incoherent generated texts. Moderate surprise values result in good quality, coherent text with no repetition. Also, note that the control does not work well when the target surprise value is greater then 5. This is because without any truncation, the average surprise of pure sampled text comes out to be around 5.4. Thus, in order to attain higher values of average surprise, we need to truncate from both sides of the distribution.

VII. Conclusion

We provide a theoretical understanding of how perplexity varies as a function of input parameters in popular top-\(k\) and top-\(p\) neural text decoding algorithms. In particular, we show that log of perplexity varies nearly linearly as a function of \(p\) in top-\(p\) sampling whereas it is highly nonlinear as a function of \(k\) in top-\(k\) sampling. Building on this analysis, we present mirostat, a neural text decoding algorithm that provides good control over the perplexity of the generated text. Mirostat provides several advantages over other sampling algorithms. While top-\(k\) and top-\(p\) do not provide a good control over the statistics of the output, mirostat can maintain the perplexity of generated text over a wide range of text length. For longer texts and certain range of input parameters, top-\(k\) and top-\(p\) sampling seem to fall into boredom and confusion traps which lead to low-quality texts. Mirostat can avoid both these traps. Further, recent large-scale human-based evaluation of neural generated text suggests that human judgement of text quality is maximized for a certain range of perplexity of the output. Since mirostat provides direct control over the perplexity of the output, it helps generate high-quality text. As a takeaway, we find that our proposed sampling algorithm, mirostat, with a target surprise value around 3.0 produces varying lengths of high-quality texts with minimal repetitions.

Additionally, we also analyze the relation between perplexity and repetitions in text. We find that for fixed model, repetitions vary linearly with perplexity and are independent of the sampling method used. We also find that larger models have less repetitions in them for any fixed amount of perplexity. Future work would include theoretically analyzing repetitions, boredom and confusion traps, and convergence properties of mirostat.

APPENDIX

Proof to Thm. 1 The expression of \(\mathbb{S}(k)\) follows directly from Def. 1 and (12).

Proof to Prop. 1 The distribution \(P_M\) is given by (12) with vocabulary size \(N\), and it is easy to check that the distribution \(P_{M_k}\) corresponding to top-\(k\) sampling is also given by (12) but with vocabulary size \(k\). The rest follows directly from (8).

Proof to Thm. 2 From Prop. 1 we have \(H(P_{M_k}, P_M) = \frac{1}{H_{k,s}} \sum_{i=1}^{k} \log i + \log H_{N,s}\). We start by finding bounds for the expression \(\sum_{i=1}^{k} \log i\).

First note that the function \(\frac{-1}{\epsilon^s} \sum_{i=1}^{k} \log i\) is a decreasing function of \(t\) for \(t > e^k\). Thus, for \(1 \leq s \leq \frac{1}{\ln 2}\), we have the following inequalities

\[
\frac{\log 2}{2^s} + \int_{1}^{k+1} \frac{\log t}{t^s} dt \leq \sum_{i=1}^{k} \log \frac{i}{s} \leq \frac{\log 2}{2^s} + \log 3 + \int_{1}^{k+1} \frac{\log (t-1)}{(t-1)^s} dt.
\]

Solving the above integration for \(1 < s \leq \frac{1}{\ln 2}\) we get

\[
\frac{a_1}{H_{k,s}} - \frac{a_2}{H_{k,s}(k+1)^e} \left( \ln (k+1) + \frac{1}{\epsilon} \right) + \log H_{N,s} \leq H(P_{M_k}, P_M) \leq \frac{b_1}{H_{k,s}} - \frac{b_2}{H_{k,s}k^e} \left( \ln k + \frac{1}{\epsilon} \right) + \log H_{N,s},
\]
where \( a_1 = s \left( \frac{\log 2}{3^2 + \pi^2} \right), a_2 = \frac{s}{e^{\ln 2}}, b_1 = s \left( \frac{\log 2 + \log 3}{3^2 + \pi^2} \right), b_2 = \frac{s}{e^{\ln 2}} \).

Now, we bound \( H_{k,s} \) as follows. Note that \( \frac{1}{t^s} \) is a decreasing function in \( t \) for \( t > 0 \) and \( s > 0 \), hence, we have

\[
\int_1^{k+1} \frac{1}{t^s} dt \leq \sum_{i=1}^{k} \frac{1}{i^s} \leq 1 + \int_2^{k+1} \frac{1}{(t-1)^s} dt
\]

(26)

\[
\frac{1 - (k+1)^{-l}}{l} \leq \sum_{i=1}^{k} \frac{1}{i^s} \leq 1 + \frac{1 - k^{-l}}{l}.
\]

(27)

We empirically observed that \( H_{k,s} \) can be approximated well as

\[
H_{k,s} \approx 0.7 + \frac{1 - k^{-l}}{l},
\]

(28)

which lies between the bounds found in (27). Moreover, we approximate \( H(P_{M,k}, P_M) \) using the upper bound obtained in (24) to get

\[
H(P_{M,k}, P_M) \approx \frac{1}{H_{k,s}} \left( b_1 - \frac{b_2}{l^s} \left( \ln k + \frac{1}{l} \right) \right) + \log H_{N,s}
\]

(29)

\[
\approx \frac{\epsilon}{b_3 (1 - \epsilon/b_3)} \left( b_1 - \frac{b_2}{l^s} \left( \ln k + \frac{1}{l} \right) \right) + \log H_{N,s}
\]

(30)

\[
\approx \frac{b_1 \epsilon}{b_3} \left( 1 - \frac{b_2 b_3 (\ln k + \frac{1}{l}) - b_1}{b_1 b_3 (b_3 - 1)} \right) + \log H_{N,s}.
\]

(31)

where (31) follows by writing \( \frac{1}{(1 - \frac{b_2}{l^s})} \) as an infinite series in (30), then simplifying the expression and writing the infinite series back as a fraction.

**Proof to Thm. 3** The cumulative probability \( p(k) \) for Zipf’s distribution is given by \( p(k) = \frac{H_{k,s}}{H_{N,s}} \). Using the approximation to \( H_{k,s} \) in (28), we have

\[
p(k) = \frac{b - k^{-l}}{\epsilon H_{N,s}}.
\]

(32)

where \( b = 1 + 0.7 \epsilon \).

Now, writing \( k \) as a function of \( p \), we get

\[
k = (b - \epsilon p H_{N,s})^{-\frac{1}{l}}.
\]

(33)

Using (33) in the equation \( \mathcal{G}(x) = s \log x + \log H_{N,s} \) from Thm. 1, we get

\[
\mathcal{G}(p) = -\frac{1}{l} \log (b - H_{N,s} \epsilon p) + \log H_{N,s}
\]

\[
= -\frac{1}{l} \log \left( 1 - \frac{H_{N,s} \epsilon p}{b} \right) - 1 + \frac{\epsilon}{l} \log b + \log H_{N,s}.
\]

(34)

Further, taking \( \epsilon \) small enough, we can approximate \( \log \left( 1 - \frac{H_{N,s} \epsilon p}{b} \right) \approx -\frac{H_{N,s} \epsilon p}{b} \). Thus, we have

\[
\mathcal{G}(p) \approx \frac{(1 + \epsilon)}{b} H_{N,s} p - \frac{(1 + \epsilon)}{l} \log b + \log H_{N,s}.
\]

(35)

Now, \( \frac{d \mathcal{G}(p)}{dp} \) can be directly computed from (34) as

\[
\frac{d \mathcal{G}(p)}{dp} = \frac{H_{N,s}(1 + \epsilon)}{b \ln 2 (b - H_{N,s} \epsilon p)}.
\]

(36)

For \( \epsilon \) small enough, we can use the approximation \( \frac{1}{1 - \frac{H_{N,s} \epsilon p}{b}} \approx 1 + \frac{H_{N,s} \epsilon p}{b} \) which gives

\[
\frac{d \mathcal{G}(p)}{dp} = \frac{H_{N,s}(1 + \epsilon)}{b \ln 2} \left( 1 + \frac{H_{N,s} \epsilon p}{b} \right).
\]

(37)

**Proof to Thm. 4** The cumulative probability \( p(k) \) for (12) can be written as

\[
p(k) = \frac{H_{k,s}}{H_{N,s}}.
\]

(38)
We approximate $\sum_{i=1}^{k} \ln t_i^{\frac{1}{t_i}} = \int_1^k \frac{\ln t}{t} dt$ to get

$$H(P_{M_F}, P_M) \approx \frac{s}{pH_{N,s}} \ln 2 \left( \int_1^k \ln t^s dt \right) + \log H_{N,s},$$

(39)

$$= \frac{s}{pH_{N,s}} \ln 2 \left( \frac{1}{c^2} - \frac{1}{c e^c (\ln k + \frac{1}{e})} \right) + \log H_{N,s},$$

(40)

$$\approx \log H_{N,s},$$

(41)

Approximating $p(k)$ from (38) as $p(k) = \frac{1}{n_{N,s}} \int_1^k \frac{1}{t} dt$, we get

$$k = (1 - \epsilon pH_{N,s})^{-\frac{1}{s}}.$$

(42)

Using (42) in (40), we have

$$H(P_{M_F}, P_M) \approx \frac{s}{pH_{N,s}} \ln 2 \left( \frac{1}{c^2} - \frac{1}{c e^c (\ln k + \frac{1}{e})} \right) + \log H_{N,s},$$

$$= \frac{s}{c^2 pH_{N,s}} \ln 2 \left( 1 + (1 - \epsilon pH_{N,s})(\ln (1 - \epsilon pH_{N,s}) - 1) \right) + \log H_{N,s},$$

$$= \frac{s}{c^2 pH_{N,s}} \ln 2 \left( \ln (1 - \epsilon pH_{N,s}) - \epsilon pH_{N,s} \ln (1 - \epsilon pH_{N,s}) + \epsilon pH_{N,s} \ln (1 - \epsilon pH_{N,s}) + \log H_{N,s} \right),$$

$$\approx \frac{s}{2} \ln 2 \left( pH_{N,s} + \epsilon p^2 H^2_{N,s} \right) + \log H_{N,s},$$

(43)

where (43) is obtained by taking the approximation $\ln (1 - \epsilon pH_{N,s}) \approx -\epsilon pH_{N,s} - \frac{(\epsilon pH_{N,s})^2}{2}$ for sufficiently small $\epsilon pH_{N,s}$. 

REFERENCES

[1] T. B. Brown, B. Mann, N. Ryder, M. Subbiah, J. Kaplan, P. Dhariwal, A. Neelakantan, P. Shyam, G. Sastry, A. Askell, S. Aragwal, A. Herbert-Voss, G. Krueger, T. Henighan, R. Child, A. Ramesh, D. M. Ziegler, J. Wu, C. Winter, H. Chen, E. Sigler, M. Litwin, B. Chess, J. Clark, C. Berner, S. McCandlish, A. Radford, I. Sutskever, and D. Amodei, “Language models are few-shot learners,” arXiv:2005.14165v3 [cs.CL]., Jun. 2020.

[2] A. Radford, J. Wu, R. Child, D. Luan, D. Amodei, and I. Sutskever, “Language models are unsupervised multitask learners,” Unpublished manuscript, Feb. 2019. Available: https://d4mucfpksyvw.cloudfront.net/better-language-models/language-models_unsupervised_multitask_learners.pdf

[3] A. Holtzman, J. Buys, M. Forbes, A. Bosselut, D. Golub, and Y. Choi, “Learning to write with cooperative discriminators,” in Proc. Assoc. Comput. Linguist. Annu. Meet. (ACL 2018), Jul. 2018, pp. 1638–1649. [Online]. Available: https://www.aclweb.org/anthology/P18-1152

[4] D. Ippolito, D. Duckworth, D. Callison-Burch, and D. Eck, “Automatic detection of generated text is easiest when humans are fooled,” in Proc. Assoc. Comput. Linguist. Annu. Meet. (ACL 2020), Jul. 2020, pp. 1808–1822.

[5] H. Zhang, D. Duckworth, D. Ippolito, and A. Neelakantan, “Trading off diversity and quality in natural language generation,” arXiv:2004.10450v1 [cs.CL]., Apr. 2020.

[6] D. Ippolito, R. Kriz, M. Kustikova, J. Sedoc, and C. Callison-Burch, “Comparison of diverse decoding methods from conditional language models,” in Proc. Assoc. Comput. Linguist. Annu. Meet. (ACL 2019), Jul. 2019, pp. 3752–3762.

[7] S. G. Krueger, T. Henighan, R. Child, A. Ramesh, D. M. Ziegler, J. Wu, C. Winter, H. Chen, E. Sigler, M. Litwin, B. Chess, J. Clark, C. Berner, S. McCandlish, A. Radford, I. Sutskever, and D. Amodei, “Language models are unsupervised multitask learners,” Unpublished manuscript, Feb. 2019. Available: https://d4mucfpksyvw.cloudfront.net/better-language-models/language-models_unsupervised_multitask_learners.pdf

[8] A. Holtzman, J. Buys, M. Forbes, A. Bosselut, D. Golub, and Y. Choi, “Learning to write with cooperative discriminators,” in Proc. Assoc. Comput. Linguist. Annu. Meet. (ACL 2018), Jul. 2018, pp. 1638–1649. [Online]. Available: https://www.aclweb.org/anthology/P18-1152

[9] D. Ippolito, D. Duckworth, D. Callison-Burch, and D. Eck, “Automatic detection of generated text is easiest when humans are fooled,” in Proc. Assoc. Comput. Linguist. Annu. Meet. (ACL 2020), Jul. 2020, pp. 1808–1822.

[10] H. Zhang, D. Duckworth, D. Ippolito, and A. Neelakantan, “Trading off diversity and quality in natural language generation,” arXiv:2004.10450v1 [cs.CL]., Apr. 2020.

[11] D. Ippolito, R. Kriz, M. Kustikova, J. Sedoc, and C. Callison-Burch, “Comparison of diverse decoding methods from conditional language models,” in Proc. Assoc. Comput. Linguist. Annu. Meet. (ACL 2019), Jul. 2019, pp. 3752–3762.

[12] L. Itti and P. F. Baldi, “Bayesian surprise attracts human attention,” in Proc. 20th Ann. Conf. Neural Inf. Process. Syst. (NIPS), May 2006, pp. 547–554.

[13] L. R. Varshney, “Mathematical limit theorems for computational creativity,” IBM J. Res. Develop., vol. 63, no. 1, pp. 2:1–2:12, Jun. 2019.

[14] A. Funk, M. Lewis, and Y. Dauphin, “Hierarchical neural story generation,” in Proc. Assoc. Comput. Linguist. Annu. Meet. (ACL 2018), Jul. 2018, pp. 889–898. [Online]. Available: https://www.aclweb.org/anthology/P18-1082

[15] A. Holtzman, J. Buys, L. Du, M. Forbes, and Y. Choi, “The curious case of neural text degeneration,” in Proc. 9th Int. Conf. Learn. Represent. (ICLR), May 2020.

[16] N. S. Keskar, B. McCann, L. R. Varshney, C. Xiong, and R. Socher, “CTRL: A conditional transformer language model for controllable generation,” arXiv:1909.05858v2 [cs.CL]., Sep. 2019.

[17] A. K. Vijayakumar, M. Cogswell, R. R. Selvaraju, Q. Sun, S. Lee, D. Crandall, and D. Batra, “Diverse beam search for improved description of complex scenes,” in Proc. 32nd AAAI Conf. Artif. Intell., Apr. 2018.

[18] T. Hashimoto, H. Zhang, and P. Liang, “Unifying human and statistical evaluation for natural language generation,” in Proc. NAACL-HLT 2019, Jun. 2019, pp. 1689–1701.

[19] S. Dathathri, A. Madotto, J. Lan, J. Hung, E. Frank, P. Molino, J. Yosinski, and R. Liu, “Plug and play language models: a simple approach to controlled text generation,” in Proc. 9th Int. Conf. Learn. Represent. (ICLR), Apr. 2020.

[20] N. Miao, H. Zhou, L. Mou, R. Yan, and L. Li, “CGMH: Constrained sentence generation by metropolis-hastings sampling,” in Proc. 33rd AAAI Conf. Artif. Intell., Jul. 2019.

[21] S. Jiang, T. Wolf, C. Monz, and M. de Rijke, “TLDR: Token loss dynamic reweighting for reducing repetitive utterance generation,” arXiv:2003.11963 [cs.CL]., Apr. 2020.

[22] S. Welleck, I. Kulikov, S. Roller, E. Dinan, K. Cho, and J. Weston, “Neural text generation with unlikelihood training,” in Proc. 9th Int. Conf. Learn. Represent. (ICLR), May 2020.

[23] J. Li, W. Monroe, and D. Jurafsky, “A simple, fast, diverse decoding algorithm for neural generation,” arXiv:1611.08562 [cs.CL]., Nov. 2016.

[24] I. Kulikov, A. Miller, K. Cho, and J. Weston, “Importance of search and evaluation strategies in neural dialogue generation,” in Proc. 12th Int. Conf. Natural Language Generation (ICNGL 2019), Oct. 2019, pp. 76–87.

[25] M. E. Foster and M. White, “Avoiding repetition in generated text,” in Proc. Conf. Eleventh European Workshop Natural Language Generation. (ENLG’07), Jun. 2007, pp. 33–40. [Online]. Available: https://www.aclweb.org/anthology/W07-2305

[26] Y. Bengio, R. Ducharme, P. Vincent, and C. Jauvin, “A neural probabilistic language model,” J. Mach. Learn. Res., no. 3, pp. 1137–1155, Feb. 2003.
[22] Z. Dai, Z. Yang, Y. Yang, J. G. Carbonell, Q. Le, and R. Salakhutdinov, “Transformer-XL: Attentive language models beyond a fixed-length context,” in Proc. Assoc. Comput. Linguist. Annu. Meet. (ACL 2019), Jul. 2019, pp. 2978–2988. [Online]. Available: https://www.aclweb.org/anthology/P19-1285

[23] T. S. Han and K. Kobayashi, “Mathematics of information and coding,” Ann. Math. Stat., vol. 203, Mar. 2007.

[24] T. M. Cover and J. A. Thomas, Elements of Information Theory, 2nd ed. New York: John Wiley & Sons, 2006.

[25] P. F. Brown, S. A. D. Pietra, V. J. D. Pietra, J. C. Lai, and R. L. Mercer, “An estimate of an upper bound for the entropy of English,” Comput. Linguist., vol. 18, no. 1, pp. 31–40, 1992.

[26] L. R. Varshney, N. S. Keskar, and R. Socher, “Limits of detecting text generated by large-scale language models,” in Proc. 2020 Inf. Theory Appl. Workshop, Feb. 2020.

[27] G. K. Zipf, Human Behavior and the Principle of Least Effort. Cambridge, MA: Addison-Wesley, 1949.

[28] D. M. W. Powers, “Applications and explanations of Zipf’s law,” in New Meth. Language Process. and Comp. Natural Language Learning, 1998, pp. 151–160.

[29] E. N. Gilbert, “Codes based on inaccurate source probabilities,” IEEE Trans. Inf. Theory, vol. IT-17, no. 3, pp. 304–314, May 1971.

[30] I. H. Witten, R. M. Neal, and J. G. Cleary, “Arithmetic coding for data compression,” Commun. ACM, vol. 30, no. 6, pp. 520–540, Jun. 1987.

[31] J. J. Rissanen and G. G. Langdon, “Arithmetic coding,” IBM J. Res. Develop., vol. 23, no. 2, pp. 149–162, Mar. 1979.