Degenerations of 8-Dimensional 2-step Nilpotent Lie Algebras

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Abstract
In this work, we consider degenerations between 8-dimensional 2-step nilpotent Lie algebras over \( \mathbb{C} \) and obtain the geometric classification of the variety \( \mathcal{N}_8^2 \).

Keywords Nilpotent Lie algebras · Variety of Lie algebras · Degenerations

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1 Preliminaries

The algebraic classification of Lie algebras is a wild problem. Lie algebras are classified up to dimension 6 (see for instance [33] for a list of indecomposable Lie algebras of dimension \( \leq 6 \) over \( \mathbb{C} \) and \( \mathbb{R} \)). In the class of nilpotent Lie algebras, there are classifications up to dimension 7 over algebraically closed fields and \( \mathbb{R} \) (see for instance [14] or [29]). In dimension 8, there are only classifications of 2-step nilpotent and filiform Lie algebras over \( \mathbb{C} \) (see [34] and [11] respectively). A related problem is the one concerning the geometric classification of Lie algebras, their degenerations, rigid elements and irreducible components. Regarding this problem in the variety of Lie algebras we can mention [2, 6–8, 10, 12, 17, 19, 27, 28, 31, 32, 35]. Moreover, the study of the geometric classification for varieties of different structures is an active research field, several results have been obtained recently in different directions regarding nilpotent algebras (see for instance [1, 3, 4, 13, 15, 16, 20–26]).

In this work we obtain degenerations between 2-step nilpotent Lie algebras of dimension 8 over \( \mathbb{C} \) and provide the irreducible components of the variety \( \mathcal{N}_8^2 \), which turn out to be the orbit closures of three rigid Lie algebras.
1.1 The Variety of Lie Algebras

Let $V$ be a complex $n$-dimensional vector space with a fixed basis $\{e_1, \ldots, e_n\}$, and let $\mathfrak{g} = (V, [\cdot, \cdot])$ be a Lie algebra with underlying vector space $V$ and Lie product $[\cdot, \cdot]$. The set of Lie algebra structures on the space $V$ is an algebraic variety in $\mathbb{C}^{n^3}$ in the following sense: Every Lie algebra structure on $V$, $\mathfrak{g}$, can be identified with its set of structure constants $\{c_{i,j}^k\} \in \mathbb{C}^{n^3}$, where $[e_i, e_j] = \sum_{k=1}^n c_{i,j}^k e_k$. This set of structure constants satisfies the polynomial equations given by the skew-symmetry and the Jacobi identity, i.e.

$$c_{i,j}^k + c_{j,i}^k = 0$$

and

$$\sum_{l=1}^n (c_{j,k}^l c_{i,l}^r + c_{k,i}^l c_{j,l}^r + c_{l,i}^j c_{k,l}^r) = 0.$$ We will denote by $L_n$ the algebraic variety of Lie algebras of fixed dimension $n$. The group $G = \text{GL}(n, \mathbb{C})$ acts on $L_n$ via change of basis:

$$g \cdot [X, Y] = g (g^{-1} X, g^{-1} Y), \quad X, Y \in \mathfrak{g}, \; g \in \text{GL}(n, \mathbb{C}).$$

Also, one can define the Zariski topology on $L_n$.

Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, we say that $\mathfrak{g}$ degenerates to $\mathfrak{h}$, and denoted by $\mathfrak{g} \to \mathfrak{h}$, if $\mathfrak{h}$ lies in the Zariski closure of the $G$-orbit $O(\mathfrak{g})$. A degeneration $\mathfrak{g} \to \mathfrak{h}$ is called proper if $\mathfrak{g} \not\cong \mathfrak{h}$. An element $g \in L_n$ is called rigid, if its orbit $O(g)$ is open in $L_n$. Since each orbit $O(g)$ is a constructible set, its closures relative to the Euclidean and the Zariski topologies are the same (see [30], 1.10 Corollary 1, p. 84). As a consequence the following is obtained:

**Lemma 1.1** Let $\mathbb{C}(t)$ be the field of fractions of the polynomial ring $\mathbb{C}[t]$. If there exists an operator $g_t \in \text{GL}(n, \mathbb{C}(t))$ such that $\lim_{t \to 0} g_t \cdot g = \mathfrak{h}$, then $\mathfrak{g} \to \mathfrak{h}$.

For example, every $n$-dimensional Lie algebra degenerates to the abelian one. The operator defined by $g_t(e_k) = t^{-1} e_k$ for $1 \leq k \leq n$, gives us

$$\lim_{t \to 0} g_t \cdot [e_i, e_j] = \lim_{t \to 0} g_t \left( [g_t^{-1}(e_i), g_t^{-1}(e_j)] \right) = \lim_{t \to 0} g_t \left( [te_i, te_j] \right)$$

$$= \lim_{t \to 0} t^2 \sum_{k=1}^n c_{i,j}^k g_t(e_k) = \lim_{t \to 0} t \left( \sum_{k=1}^n c_{i,j}^k e_k \right) = 0.$$

1.2 The Variety of 2-step Nilpotent Lie Algebras

The variety $\mathcal{N}_2^n$, is the closed subset of $L_n$ given by all at most 2-step nilpotent Lie algebras, i.e. those satisfying $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$.

The geometric classification of the varieties $\mathcal{N}_2^n$ for $n \leq 7$ can be seen from [2], and now can be recovered from this work.

Here we consider the classification of indecomposable 8-dimensional 2-step nilpotent Lie algebras over $\mathbb{C}$ obtained by Yan and Deng in [34]; the classification of indecomposable nilpotent Lie algebras of dimension 7 given by Gong in [14], and the classification of Lie algebras of dimension $\leq 6$ from [33].

**Theorem 1.2** The isomorphism classes of Lie algebras in $\mathcal{N}_8^2$ are given in Table 1.
Degenerations of 8-Dimensional 2-step Nilpotent Lie Algebras

2 Degenerations

In this section, we list all the primary degenerations (those that cannot be obtained by transitivity). Let us consider one example in detail. The other ones in Table 2 are done likewise.
Table 2 Degenerations

| g → h | Parameterized basis |
|-------|----------------------|
| $n_{5,3} \oplus C^3 \rightarrow n_{3,1} \oplus C^5$ | $x_1 = e_1$, $x_5 = te_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$, $x_{10} = e_{10}$, $x_{11} = e_{11}$ |
| $n_{5,1} \oplus C^3 \rightarrow n_{3,1} \oplus C^5$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_5 = e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$ |
| $(17) \oplus C \rightarrow n_{5,3} \oplus C^3$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = e_4$, $x_5 = e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$ |
| $n_{6,1} \oplus C^2 \rightarrow n_{5,1} \oplus C^3$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = e_4$, $x_5 = e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$ |
| $(37A) \oplus C \rightarrow n_{5,1} \oplus C^3$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = e_4$, $x_5 = e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$ |
| $n_{6,2} \oplus C^2 \rightarrow n_{5,1} \oplus C^3$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = e_4$, $x_5 = e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$ |
| $n_{6,2} \oplus C^2 \rightarrow n_{5,3} \oplus C^3$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = e_4$, $x_5 = e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$ |
| $n_{3,1} \oplus n_{3,1} \oplus C^2 \rightarrow n_{6,2} \oplus C^2$ | $x_1 = -\frac{1}{2} e_4$, $x_2 = e_1 + e_4$, $x_3 = e_6$, $x_4 = e_2 - e_6$, $x_5 = e_5$, $x_6 = \frac{1}{2} e_5 + e_3$, $x_7 = 2 e_1$, $x_8 = e_7$ |
| $(37C) \oplus C \rightarrow n_{3,1} \oplus n_{3,1} \oplus C^2$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = e_4$, $x_5 = e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$ |
| $(37C) \oplus C \rightarrow (37A) \oplus C$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = e_4$, $x_5 = e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$ |
| $(37C) \oplus C \rightarrow n_{6,1} \oplus C^2$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = e_4$, $x_5 = e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$ |
| $(27A) \oplus C \rightarrow n_{3,1} \oplus n_{3,1} \oplus C^2$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = e_4$, $x_5 = e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$ |
| $g \rightarrow h$           | Parameterized basis                              |
|---------------------------|--------------------------------------------------|
| $N_4^{8,2} \rightarrow (27A) \oplus C$ | $x_1 = e_4$, $x_2 = e_5$, $x_3 = e_1$, $x_4 = -e_3$, |
|                           | $x_5 = e_2$, $x_6 = e_8$, $x_7 = e_7$, $x_8 = te_6$. |
| $N_4^{8,2} \rightarrow (17) \oplus C$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = e_4$, |
|                           | $x_5 = e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = \frac{1}{7} e_8$. |
| $(37B) \oplus C \rightarrow (37C) \oplus C$ | $x_1 = e_3$, $x_2 = e_1 + i \frac{1}{\sqrt{3}} e_2 + i \frac{1}{\sqrt{7}} e_3 - \frac{1}{7} e_4$, $x_3 = -\frac{1}{\sqrt{3}} e_3 + e_4$, $x_4 = e_2 + \frac{1}{7} e_3$, |
|                           | $x_5 = -\frac{1}{7} e_7$, $x_6 = \frac{1}{7} e_6$, $x_7 = e_5 + \frac{1}{7} e_7$, $x_8 = e_8$. |
| $(27B) \oplus C \rightarrow (27A) \oplus C$ | $x_1 = e_3$, $x_2 = e_4$, $x_3 = te_1$, $x_4 = -e_2$, |
|                           | $x_5 = \frac{1}{7} e_5$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = e_8$. |
| $N_5^{8,3} \rightarrow (37B) \oplus C$ | $x_1 = -e_4$, $x_2 = e_1$, $x_3 = e_2$, $x_4 = e_5$, |
|                           | $x_5 = e_8$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = te_3$. |
| $N_5^{8,3} \rightarrow (27A) \oplus C$ | $x_1 = e_1$, $x_2 = e_4$, $x_3 = e_2$, $x_4 = e_3$, |
|                           | $x_5 = e_5$, $x_6 = e_8$, $x_7 = e_7$, $x_8 = \frac{1}{7} e_6$. |
| $(37D) \oplus C \rightarrow (37B) \oplus C$ | $x_1 = e_2$, $x_2 = e_4$, $x_3 = -e_3$, $x_4 = te_1$, |
|                           | $x_5 = e_7$, $x_6 = e_5$, $x_7 = te_6$, $x_8 = e_8$. |
| $N_2^{8,4} \rightarrow (37B) \oplus C$ | $x_1 = e_3$, $x_2 = e_2$, $x_3 = e_1$, $x_4 = e_4$, |
|                           | $x_5 = -e_7$, $x_6 = -e_5$, $x_7 = e_8$, $x_8 = \frac{1}{7} e_6$. |
| $n_5,3 \oplus n_3,1 \rightarrow (27A) \oplus C$ | $x_1 = e_5$, $x_2 = e_3$, $x_3 = e_7$, $x_4 = -e_4$, |
|                           | $x_5 = e_8$, $x_6 = e_1$, $x_7 = e_2$, $x_8 = \frac{1}{7} (e_6 - e_2)$. |
| $N_2^{8,2} \rightarrow (27B) \oplus C$ | $x_1 = e_1$, $x_2 = e_2 - e_6$, $x_3 = e_4$, $x_4 = e_5$, |
|                           | $x_5 = e_3$, $x_6 = e_7$, $x_7 = e_8$, $x_8 = te_6$. |
| $N_2^{8,3} \rightarrow N_5^{8,3}$ | $x_1 = t^{-1}(e_2 + e_3)$, $x_2 = te_4$, $x_3 = -te_1$, $x_4 = -te_2$, |
|                           | $x_5 = t^{-1}e_5$, $x_6 = e_8$, $x_7 = e_6$, $x_8 = e_7$. |
| $N_8^{8,3} \rightarrow (27B) \oplus C$ | $x_1 = e_2$, $x_2 = e_3$, $x_3 = -e_1$, $x_4 = -e_4$, |
|                           | $x_5 = e_5$, $x_6 = e_8$, $x_7 = e_7$, $x_8 = \frac{1}{7} e_6$. |
| \( g \to h \) | Parameterized basis |
|----------------|------------------|
| \( N_8^{8.3} \to N_5^{8.3} \) | \( x_1 = e_1 \), \( x_2 = e_2 \), \( x_3 = e_3 \), \( x_4 = \frac{1}{7} e_4 \). |
| \( N_9^{8.3} \to (37D) \oplus C \) | \( x_1 = e_1 \), \( x_2 = e_3 \), \( x_3 = \frac{1}{7} e_2 \), \( x_4 = te_5 - e_2 \). |
| \( n_{5,1} \oplus n_{3,1} \to N_5^{8.3} \) | \( x_1 = e_5 + e_7 \), \( x_2 = -e_4 + e_8 \), \( x_3 = ie_4 \), \( x_4 = -e_3 \). |
| \( N_1^{8.3} \to N_2^{8.3} \) | \( x_1 = -ie_1 \), \( x_2 = e_2 + e_4 \), \( x_3 = e_3 \), \( x_4 = e_1 + e_5 \). |
| \( N_1^{8.3} \to n_{5,1} \oplus n_{3,1} \) | \( x_1 = e_7 \), \( x_2 = ie_9 \), \( x_3 = e_2 \), \( x_4 = ie_1 \). |
| \( N_3^{8.4} \to (37D) \oplus C \) | \( x_1 = e_1 \), \( x_2 = e_3 \), \( x_3 = e_4 \), \( x_4 = -e_2 \). |
| \( N_3^{8.4} \to N_2^{8.4} \) | \( x_1 = e_1 \), \( x_2 = e_2 \), \( x_3 = e_3 \), \( x_4 = ie_4 \). |
| \( N_5^{8.2} \to N_2^{8.2} \) | \( x_1 = \frac{1}{7} e_2 \), \( x_2 = -ie_1 \), \( x_3 = ie_3 \), \( x_4 = e_5 \). |
| \( N_5^{8.2} \to n_{5,3} \oplus n_{3,1} \) | \( x_1 = e_8 \), \( x_2 = e_2 \), \( x_3 = e_4 \), \( x_4 = e_3 \). |
| \( N_5^{8.2} \to N_4^{8.2} \) | \( x_1 = \frac{1}{7} e_1 \), \( x_2 = ie_2 \), \( x_3 = e_3 \), \( x_4 = e_4 \). |
| \( N_7^{8.3} \to N_2^{8.3} \) | \( x_1 = \frac{1}{7} e_5 \), \( x_2 = -ie_1 \), \( x_3 = e_3 \), \( x_4 = e_4 \). |
| \( N_9^{8.3} \to N_8^{8.3} \) | \( x_1 = e_1 \), \( x_2 = -\frac{1}{7} e_3 + e_4 \), \( x_3 = e_2 \), \( x_4 = -e_5 \). |
| \( yN_7^{8.3} \to N_8^{8.3} \) | \( x_1 = e_1 \), \( x_2 = -\frac{1}{7} e_3 + e_4 \), \( x_3 = e_2 \), \( x_4 = -e_5 \). |
### Table 2  (continued)

| $g \rightarrow h$ | Parameterized basis |
|-------------------|----------------------|
| $N^{8.4}_{1} \rightarrow N^{8.4}_{3}$ | $x_1 = -e_1 + \frac{1}{\sqrt{2}} e_2$, $x_2 = -\frac{1}{\sqrt{2}} e_3 + e_4$, $x_3 = e_3 + \sqrt{2} e_4$, $x_4 = \sqrt{2} e_1 + e_2$, $x_5 = -\frac{1}{\sqrt{2}} e_6 + e_8$, $x_6 = \frac{1}{\sqrt{2}} e_6 + \sqrt{2} e_8$, $x_7 = -2e_7$, $x_8 = -2e_5$. |
| $N^{8.2}_{3} \rightarrow N^{8.2}_{5}$ | $x_1 = -t^2 e_6$, $x_2 = -\frac{1}{t} e_1 + \frac{1}{t} e_3 - \frac{1}{t} e_5$, $x_3 = te_4$, $x_4 = e_1 + e_3$, $x_5 = e_2 + e_4 + e_6$, $x_6 = -te_1$, $x_7 = -te_8$, $x_8 = e_7 + e_8$. |
| $N^{8.3}_{3} \rightarrow N^{8.3}_{7}$ | $x_1 = e_2 + e_4$, $x_2 = te_1 + te_5$, $x_3 = e_1 + e_3$, $x_4 = -te_2$, $x_5 = -t^2 e_8$, $x_6 = -te_6 + te_7$, $x_7 = -e_6 + e_7 - e_8$, $x_8 = -t^2 e_7$. |
| $N^{8.3}_{3} \rightarrow n^{8.1}_{5,1} \oplus n^{3.1}_{3}$ | $x_1 = e_6$, $x_2 = e_7$, $x_3 = e_1$, $x_4 = -e_3$, $x_5 = e_2$, $x_6 = te_8$, $x_7 = e_4$, $x_8 = t(e_5 - e_3)$. |
| $N^{8.2}_{1} \rightarrow N^{8.2}_{3}$ | $x_1 = e_1$, $x_2 = e_2$, $x_3 = te_3$, $x_4 = e_4 - e_6$, $x_5 = e_5 + e_3$, $x_6 = te_6$, $x_7 = e_7$, $x_8 = te_8$. |
| $N^{8.3}_{6} \rightarrow N^{8.3}_{3}$ | $x_1 = e_5$, $x_2 = e_4$, $x_3 = e_2$, $x_4 = e_4$, $x_5 = e_3$, $x_6 = -e_7$, $x_7 = te_1$, $x_8 = -te_6$. |
| $N^{8.3}_{6} \rightarrow N^{8.3}_{1}$ | $x_1 = -te_1$, $x_2 = e_3$, $x_3 = e_2$, $x_4 = e_4$, $x_5 = -tes$, $x_6 = -te_7$, $x_7 = te_8$, $x_8 = -t^2 e_6$. |
| $N^{8.3}_{10} \rightarrow N^{8.3}_{3}$ | $x_1 = e_5$, $x_2 = e_2 + 2e_4$, $x_3 = e_1 + e_3$, $x_4 = e_1$, $x_5 = e_5$, $x_6 = -2e_8$, $x_7 = \frac{1}{\sqrt{2}} (e_6 - e_7)$, $x_8 = -\frac{1}{\sqrt{2}} (e_6 + e_7)$. |
| $N^{8.3}_{11} \rightarrow N^{8.3}_{10}$ | $x_1 = e_3$, $x_2 = e_4$, $x_3 = te_2$, $x_4 = e_1$, $x_5 = e_2 + e_3$, $x_6 = e_6$, $x_7 = e_7$, $x_8 = te_6$. |
| $N^{8.3}_{11} \rightarrow N^{8.3}_{6}$ | $x_1 = e_2 + e_3$, $x_2 = e_1 + e_4$, $x_3 = i\sqrt{2}(e_2 - e_3)$, $x_4 = i\sqrt{2}(e_4 - e_1)$, $x_5 = -e_5$, $x_6 = -e_6 + e_8$, $x_7 = -2i\sqrt{2}e_7$, $x_8 = i\sqrt{2}(e_6 + e_8)$. |
| $N^{8.3}_{4} \rightarrow N^{8.3}_{11}$ | $x_1 = e_1 + e_2 + e_4$, $x_2 = 2\sqrt{2}(e_2 - e_1)$, $x_3 = -e_4 + e_5$, $x_4 = -2e_4$, $x_5 = \sqrt{2}(e_3 + e_5)$, $x_6 = 4\sqrt{2}e_6$, $x_7 = 2\sqrt{2}(-e_7 + e_8)$, $x_8 = 2(e_8 + e_7)$. |
| $N^{8.3}_{9} \rightarrow N^{8.3}_{4}$ | $x_1 = e_4$, $x_2 = e_5$, $x_3 = te_4$, $x_4 = \frac{1}{7}e_5$, $x_5 = te_3$, $x_6 = e_6$, $x_7 = e_8$, $x_8 = e_7$. |
Consider the Lie algebra $N_5^{8,2}$ and the base change

$$x_1 = e_8, \ x_2 = e_2, \ x_3 = e_4, \ x_4 = e_3, \ x_5 = e_5, \ x_6 = \frac{1}{t}e_7, \ x_7 = e_1 + \frac{1}{t}e_5, \ x_8 = e_6.$$ 

Then the new product is given by:

$$[x_2, x_4] = x_1, \quad [x_2, x_7] = -tx_6, \quad [x_3, x_4] = -tx_6, \quad [x_3, x_5] = x_1, \quad [x_3, x_7] = tx_1, \quad [x_5, x_8] = tx_6, \quad [x_7, x_8] = x_6.$$ 

When $t \to 0$ we obtain the Lie product of $n_5,3 \oplus n_3,1$, therefore $N_5^{8,2} \to n_5,3 \oplus n_3,1$.

### 3 Invariants for Lie Algebras and Non-Degenerations

The techniques used in this work are the usual for obtaining non-degenerations arguments. There are several invariants for the orbit closure of a Lie algebra that have been successfully applied in previous works. Among them, we mention the ones that we use in this work.

**Lemma 3.1** Let $\mathfrak{g}, \mathfrak{h} \in \mathcal{L}_n$. If $\mathfrak{g} \to \mathfrak{h}$, then the following relations must hold:

(a) $\dim O(\mathfrak{g}) > \dim O(\mathfrak{h})$.
(b) $\dim \mathfrak{z}(\mathfrak{g}) \leq \dim \mathfrak{z}(\mathfrak{h})$, where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$.
(c) $\dim [\mathfrak{g}, \mathfrak{g}] \geq \dim [\mathfrak{h}, \mathfrak{h}]$.
(d) $\dim H^k(\mathfrak{g}) \leq \dim H^k(\mathfrak{h})$ for $0 \leq k \leq n$, where $H^k(\mathfrak{g})$ is the $k$-th trivial cohomology group for $\mathfrak{g}$.
(e) $\mathfrak{a}(\mathfrak{g}) \leq \mathfrak{a}(\mathfrak{h})$, where $\mathfrak{a}(\mathfrak{g}) = \max \{\dim W : W \text{ is an abelian subalgebra of } \mathfrak{g}\}$.

**Proof** The first relation follows from the Closed Orbit Lemma (see [5], I. Lemma 1.8, p. 53). Items (b) and (c) follow by proving that the corresponding sets are closed (see [9], §3 Theorem 2, p. 14). The proofs of (d) and (e) can be found in [7] and [17], respectively. 

We explain the use of these arguments with an example:

$N_1^{8,3} \not
\rightarrow (37D) \oplus C$ because $\dim H^4(N_1^{8,3}) = 30 > 28 = \dim H^4((37D) \oplus C)$ contradicting Lemma 3.1 (d).

Moreover, if $\dim O(\mathfrak{g}_1) > \dim O(\mathfrak{g}_2)$ and there are no $\mathfrak{g}_3$ and $\mathfrak{g}_4$ such that the following conditions hold:

- $\mathfrak{g}_3 \to \mathfrak{g}_1$,
- $\mathfrak{g}_2 \to \mathfrak{g}_4$,
- $\mathfrak{g}_3 \not
\rightarrow \mathfrak{g}_4$ and
- $\mathfrak{g}_3 \to \mathfrak{g}_1$ or $\mathfrak{g}_2 \to \mathfrak{g}_4$ is a proper degeneration,

then $\mathfrak{g}_1 \not
\rightarrow \mathfrak{g}_2$ is called a **primary non-degeneration**.

For instance, consider the Lie algebras $n_{6,1} \oplus \mathbb{C}^2, n_{5,1} \oplus \mathbb{C}^3$ and $n_{5,3} \oplus \mathbb{C}^3$. Then we have:

(a) $n_{6,1} \oplus \mathbb{C}^2 \to n_{5,1} \oplus \mathbb{C}^3$ by the previous section,
(b) $n_{6,1} \oplus \mathbb{C}^2 \not
\rightarrow n_{5,3} \oplus \mathbb{C}^3$ (as we will see in the following table),

then we must have $n_{5,1} \oplus \mathbb{C}^3 \not
\rightarrow n_{5,3} \oplus \mathbb{C}^3$ (else contradicting (b)). Moreover, $n_{5,1} \oplus \mathbb{C}^3 \not
\rightarrow n_{5,3} \oplus \mathbb{C}^3$ is not a primary non-degeneration, and $n_{6,1} \oplus \mathbb{C}^2 \not
\rightarrow n_{5,3} \oplus \mathbb{C}^3$ is a primary non-degeneration.
We will use the previous Lemma to summarize most primary non-degenerations in Table 3.

We continue in obtaining the primary non-degenerations. The next lemma can be found, for instance, in [17] or [32]:

**Lemma 3.2** Let $B$ be a Borel subgroup of $GL(n, \mathbb{C})$ and let $S \in \mathcal{L}_n$ be a closed subset which is $B$-stable. If $g \rightarrow \mathfrak{h}$ and $g \in S$ then there exists a Lie algebra $\overline{\mathfrak{h}} \in S$ such that $\overline{\mathfrak{h}} \simeq \mathfrak{h}$.

Let $B$ be the set of lower triangular matrices and consider the following set:

$$S = \left\{ g = (e_{i,j}^k) \in \mathcal{L}_8 \mid \begin{array}{l}
c_{r,s}^2 = \lambda c_{r,s}^6, \text{ for } 4 \leq s \leq 5, 1 \leq r < s \\
c_{r,s}^8 = \mu c_{r,s}^6, \text{ for } 4 \leq s \leq 5, 1 \leq r < s \\
c_{r,s}^8 = 0, \text{ for } 6 \leq s \leq 8, 1 \leq r < s \\
\lambda, \mu \in \mathbb{C}.
\end{array} \right\}$$

It is not difficult to check that this set is $B$-stable. Moreover, $N_1^{8,3} \subset S$ but there is no $g \in GL(8, \mathbb{C})$ such that $g \cdot [(27B) \oplus \mathbb{C}] \subset S$, therefore $N_1^{8,3} \not\subset (27B) \oplus \mathbb{C}$.

Finally, we want to prove the following lemma:

**Lemma 3.3** $n_{5,1} \oplus n_{3,1}$ is not in the orbit closure of $N_7^{8,3}$.

In order to do this, we use the results in [18]. First, denote by $g(b)$ the central extension of $g$ by $\mathbb{C}^r$ defined by the 2-cocycle $b$ and let $\mathbb{B}_0 = \{ b \in Z^2(g, \mathbb{C}^r) : b^r \cap 3(g) = 0 \}$. Then one has:

**Theorem 3.4** ([18], Theorem 2.2) For $b_0, b_1 \in \mathbb{B}_0$, $b_0 \in \overline{O(b_1)}$ if and only if $g(b_0) \in O(g(b_1))$.

With this result we obtain the last primary non-degeneration:

| Table 3 Primary non-degenerations | $g \not\rightarrow \mathfrak{h}$ | Reason |
|----------------------------------|-------------------------------|--------|
| $n_{6,1} \oplus \mathbb{C}^2$ $\not\rightarrow$ $n_{5,3} \oplus \mathbb{C}^3$ | Lemma 3.1 (b) |
| $n_{5,3} \oplus n_{3,1}$ $\not\rightarrow$ $N_4^{8,2}$, (17) $\oplus$ $\mathbb{C}$; | |
| $N_1^{8,4}$ $\not\rightarrow$ $(27A)$ $\oplus$ $\mathbb{C}$, (17) $\oplus$ $\mathbb{C}$; | |
| $N_9^{8,3}$ $\not\rightarrow$ $N_2$, (17) $\oplus$ $\mathbb{C}$, $n_{5,3} \oplus n_{3,1}$; | |
| $(17) \oplus \mathbb{C}$ $\not\rightarrow$ $n_{5,1} \oplus \mathbb{C}^3$; | Lemma 3.1 (c) |
| $N_9^{8,3}$ $\not\rightarrow$ $N_2^{8,4}$; | |
| $N_1^{8,2}$ $\not\rightarrow$ $(37A)$ $\oplus$ $\mathbb{C}$, $n_{6,1} \oplus \mathbb{C}^2$; | |
| $(37A) \oplus \mathbb{C}$ $\not\rightarrow$ $n_{6,1} \oplus \mathbb{C}^2$; | Lemma 3.1 (d) $k = 2$ |
| $N_1^{8,3}$ $\not\rightarrow$ $N_1^{8,3}$ | |
| $N_2^{8,2}$ $\not\rightarrow$ $(17) \oplus \mathbb{C}$; | Lemma 3.1 (d) $k = 4$ |
| $N_1^{8,3}$ $\not\rightarrow$ $(37D)$ $\oplus$ $\mathbb{C}$; | |
| $(37A) \oplus \mathbb{C}$ $\not\rightarrow$ $n_{5,3} \oplus \mathbb{C}^3$; | Lemma 3.1 (e) |
| $N_2^{8,2}$ $\not\rightarrow$ $n_{5,3} \oplus n_{3,1}$; | |
Proof of Lemma 3.3. A base change allow us to write the following products:

| \( n_{5,1} \oplus n_{3,1} \) | \( N_{7}^{8,3} \) | \( g = h_{3} \oplus \mathbb{C}^{3} \) |
|---|---|---|
| \( [e_1, e_2] = e_3 \) | \( [e_1, e_2] = e_3 \) | \( [e_1, e_2] = e_3 \) |
| \( [e_1, e_4] = e_7 \) | \( [e_1, e_4] = e_7 \) | \( [e_1, e_4] = e_7 \) |
| \( [e_5, e_6] = e_8 \) | \( [e_1, e_5] = e_8 \) | \( [e_4, e_6] = e_8 \) |
| \( [e_2, e_6] = e_7 \) | \( [e_5, e_6] = e_7 \) | \( [e_4, e_6] = e_8 \) |

It is clear that \( n_{5,1} \oplus n_{3,1} = g(b_0) \) and \( N_{7}^{8,3} = g(b_1) \) where

\[
\begin{align*}
\begin{array}{c|c}
b_0 & b_1 \\
\hline
b_0(e_1, e_4) = e_7 & b_1(e_1, e_4) = e_7 \\
b_0(e_5, e_6) = e_8 & b_1(e_2, e_6) = e_7 \\
b_0(e_5, e_6) = e_8 & b_1(e_2, e_5) = e_8 \\
b_0(e_5, e_6) = e_8 & b_1(e_4, e_6) = e_8 \\
\end{array}
\end{align*}
\]

By Theorem 3.4, \( n_{5,1} \oplus n_{3,1} \) is in the orbit closure of \( N_{7}^{8,3} \) if and only if \( b_0 \) is in the orbit closure of \( b_1 \).

Suppose \( b_0 \) is in the orbit closure of \( b_1 \). Then by Theorem 1.2 of [18], there is a coordinate ring \( \mathbb{C}[Z] \) for some affine set \( Z \), an element \( g \in \text{GL}(8, \mathbb{C}[Z]) \), and an element \( x \in Z \) such that \( b_0 \) is the evaluation of \( g \cdot b_1 \) at \( x \). The element \( g \) is of the form

\[
g = \begin{pmatrix}
\alpha & 0 \\
u & \Phi
\end{pmatrix},
\]

where \( \alpha^{-1} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
a_{41} & a_{42} & 0 & a_{44} & a_{45} & a_{46} \\
a_{51} & a_{52} & 0 & a_{54} & a_{55} & a_{56} \\
a_{61} & a_{62} & 0 & a_{64} & a_{65} & a_{66}
\end{pmatrix}
\]

and \( \Phi = \begin{pmatrix} p & r \\
q & s
\end{pmatrix} \).

(\( \alpha \in \text{Aut}(g) \)). Then we have

\[
\begin{align*}
g \cdot b_1(e_1, e_4) &= (a_{45}a_{66} - a_{65}a_{46})(re_7 + se_8), \\
g \cdot b_1(e_4, e_5) &= (a_{44}a_{65} - a_{64}a_{45})(re_7 + se_8), \\
g \cdot b_1(e_2, e_6) &= (a_{44}a_{66} - a_{64}a_{46})(re_7 + se_8), \\
g \cdot b_1(e_5, e_6) &= (a_{11}a_{44} + a_{21}a_{64})(pe_7 + qe_8) + (a_{11}a_{54} + a_{41}a_{64})(re_7 + se_8).
\end{align*}
\]

If \( g \cdot b_1 \) evaluated in \( x \) is \( b_0 \) we obtain:

\[
e_8 = (a_{45}(x)a_{66}(x) - a_{65}(x)a_{46}(x))(r(x)e_7 + s(x)e_8), \tag{3.1}
\]

therefore \( r(x) = 0 \) and \( s(x) \neq 0 \). Moreover,

\[
\begin{align*}
0 &= a_{44}(x)a_{65}(x) - a_{64}(x)a_{45}(x), \tag{3.2} \\
0 &= a_{44}(x)a_{66}(x) - a_{64}(x)a_{46}(x), \tag{3.3} \\
1 &= (a_{11}(x)a_{44}(x) + a_{21}(x)a_{64}(x))p(x). \tag{3.4}
\end{align*}
\]

Equations (3.2) and (3.3) imply that

• If \( a_{44}(x) \neq 0 \) then \( a_{65}(x) = \frac{a_{64}(x)a_{45}(x)}{a_{44}(x)} \) and \( a_{66}(x) = \frac{a_{64}(x)a_{46}(x)}{a_{44}(x)} \). This contradicts Eq. (3.1).
• If \( a_{44}(x) = a_{64}(x) = 0 \) this contradicts Eq. (3.4).
• If \( a_{44}(x) = a_{45}(x) = a_{46}(x) = 0 \) this contradicts Eq. (3.1).

Therefore \( n_{5,1} \oplus n_{3,1} \not\in O(N_{7}^{8,3}) \).

Finally, the Hasse diagram of degenerations is given by:
With all this, we obtain:

**Theorem 3.5** The irreducible components of the variety $N^2_8$ are:

- $C_1 = \overline{O(N^{8,2}_1)}$,
- $C_2 = \overline{O(N^{8,3}_9)}$,
- $C_3 = \overline{O(N^{8,4}_1)}$.

Moreover, the Lie algebras $N^{8,2}_1$, $N^{8,3}_9$ and $N^{8,4}_1$ are rigid in $N^2_8$.

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