Linear fractional Galton-Watson processes in random environment and perpetuities

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Abstract Linear fractional Galton-Watson branching processes in i.i.d. random environment are, on the quenched level, intimately connected to random difference equations by the evolution of the random parameters of their linear fractional marginals. On the other hand, any random difference equation defines an autoregressive Markov chain (a random affine recursion) which can be positive recurrent, null recurrent and transient and which, as the forward iterations of an iterated function system, has an a.s. convergent counterpart in the positive recurrent case given by the corresponding backward iterations. The present expository article aims to provide an explicit view at how these aspects of random difference equations and their stationary limits, called perpetuities, enter into the results and the analysis, especially in quenched regime. Although most of the results presented here are known, we hope that the offered perspective will be welcomed by some readers.

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1 Introduction

Let us begin with a disclaimer. This work about Galton-Watson branching processes with linear fractional offspring distributions in i.i.d. random envi-
ronment is neither a research paper nor a survey. It is rather meant as an attempt to provide, via a collection of selected results, some (hopefully) new vantage points of the interesting and rather explicit connections between this class of branching processes when studied in quenched regime and random difference equations and their stationary limits, called perpetuities. The latter have attracted a lot of interest in the last two decades, not at least due to their appearance in various other fields of probability theory. We refer to the recent monograph by Buraczewski, Damek and Mikosch [11] for further information and literature. Another recent monograph by Kersting and Vatutin [18] provides an excellent account of the current state-of-the-art of branching processes in random environment and is here especially recommended in places where this text remains terse on accounting for relevant references.

To keep the presentation at reasonable length while stressing our particular perspective, we restrict ourselves to a collection of results of moderate technical level, and mostly in quenched regime, that nicely illustrate the interplay between linear fractional branching in random environment and random difference equations. Owing to the same constraint, only the subcritical case is considered at greater length, while keeping the sections on supercritical and critical processes relatively short which we deem sufficient for our purposes.

As a motivation, consider the classical Galton-Watson branching process (GWP) \((Z_n)_{n \geq 0}\) with \(Z_0 = 1\) and offspring generating function (g.f.)

\[
f(s) = \sum_{n \geq 0} p_n s^n
\]

for \(s \in [0, 1]\). Then \(f^n\), the \(n\)-fold iteration of \(f\), equals the g.f. of \(Z_n\) for each \(n \in \mathbb{N}_0\), where \(\mathbb{N}_0 = \{0, 1, 2, \ldots\}\) and \(f^0(s) := s\). Among the very few examples that allow to compute all \(f^n\) in closed form from \(f\), the presumably most prominent one is the linear fractional case when

\[
\frac{1}{1 - f(s)} = \frac{a}{1 - s} + b
\]

for parameters \(a, b \in \mathbb{R}_+ = (0, \infty), a + b \geq 1, \) and \(s \in [0, 1]\). Let us write \(LF(a, b)\) for the associated distribution on \(\mathbb{N}_0\) which, as can be seen from (2), is a mixture of \(\delta_0\), the point mass at 0, and a geometric distribution on the positive integers \(\mathbb{N}\) which has parameter \(p\) and is denoted \(Geom_+(p)\). It is in fact a pure geometric law with parameter \(a\) iff \(a + b = 1\), so

\[
LF(a, 1 - a) = Geom_+(a).
\]

Simple computations show that

\[
f(s) = p_0 + (1 - p_0) \frac{ps}{1 - (1 - p)s}
\]
with \((p_0, p)\) determined by the equations

\[
a = \frac{p}{1 - p_0} \quad \text{and} \quad b = \frac{1 - p}{1 - p_0}.
\]

(3)

Conversely, \(p = a(a + b)^{-1}\) and \(p_0 = (a + b - 1)(a + b)^{-1}\). Therefore, by what has been stated above,

\[
LF(a, b) = \frac{a + b - 1}{a + b} \delta_0 + \frac{1}{a + b} \text{Geom+} \left( \frac{a}{a + b} \right).
\]

(4)

We further note that the offspring mean \(m\), say, satisfies

\[
m := f'(1) = \sum_{n \geq 1} np_n = a^{-1},
\]

that \(f''(1) = 2a^{-2}b\), and that the extinction probability equals \(q = b^{-1}(a + b - 1)\) in the supercritical case \(m > 1\). For further details, the reader may consult the classical monograph by Athreya and Ney [8, Sect. A.4]. Finally, we point out for later use that, if \(a + b = 1\) and thus \(f\) belongs to the geometric law \(\text{Geom+}(1)\) on \(\mathbb{N}\), then, for any \(\gamma \in (0, 1]\), the g.f.

\[
\frac{f(\gamma s)}{f(\gamma)} = \frac{(1 - (1 - a)\gamma)s}{1 - (1 - a)\gamma s}, \quad s \in [0, 1]
\]

(5)

belongs to the geometric law \(\text{Geom+}(1 - (1 - a)\gamma)\).

Putting \(\varphi(s) := (1 - s)^{-1}\) for \(s \in [0, 1]\), which is a bijection from \([0, 1]\) to \([1, \infty)\), and \(g(s) := as + b\) for \(s \in \mathbb{R}\), Eq. (1) may be restated as

\[
\varphi \circ f(s) = g \circ \varphi(s)
\]

or, equivalently,

\[
f(s) = \varphi^{-1} \circ g \circ \varphi(s)
\]

for \(s \in [0, 1]\). Therefore the iterations of \(f\), viewed as a dynamical system on \([0, 1]\), coincide up to conjugation with respect to \(\varphi\), with the iterations of \(g\), viewed as a dynamical system on \([1, \infty)\), giving

\[
f^n(s) = \varphi^{-1} \circ g^n \circ \varphi(s)
\]

for each \(n \in \mathbb{N}_0\). Using \(g^n(s) = a^n s + b(a^{n-1} + \ldots + a + 1)\), we thus find

\[
\frac{1}{1 - f^n(s)} = \frac{a^n}{1 - s} + b(a^{n-1} + \ldots + a + 1).
\]

and therefore \(\mathcal{L}(Z_n) = LF(a^n, b(a^{n-1} + \ldots + 1))\), where \(\mathcal{L}(X)\) means the law of \(X\). This conjugation argument or, equivalently, the use of (1) rather than (2) to find the iterations of \(f\) and thus the laws of all \(Z_n\) is easier than the approach described in [8]. It may also be found in [18, p. 3ff].
The last observation becomes even more striking in the situation when the offspring laws are still linear fractional but varying with respect to an i.i.d. random environment, thus leading to a Galton-Watson process in random environment (GWPRE). More precisely, let \( \mathbf{e} := (A_n, B_n)_{n \geq 1} \) be a sequence of i.i.d. random vectors (the environment) with generic copy \((A, B)\) such that
\[
P(A > 0, B > 0, A + B \geq 1) = 1. \tag{6}
\]
Put \( \mathbf{e}_n := (A_n, B_n) \), \( \mathbf{e}_{1:n} := (\mathbf{e}_1, \ldots, \mathbf{e}_n) \) and \( \mathbf{e}_{\geq n} := (\mathbf{e}_n, \mathbf{e}_{n+1}, \ldots) \), so \( \mathbf{e} = \mathbf{e}_{\geq 1} \). Define the random g.f. \( f_n = f(\mathbf{e}_n, \cdot) \) by
\[
\frac{1}{1 - f_n(s)} = \frac{A_n}{1 - s} + B_n
\]
for \( n \in \mathbb{N} \). The pertinent random linear fractional distribution is denoted by \((P_{n,k})_{k \geq 0}, \) with generic copy \((P_k)_{k \geq 0} = LF(A, B)\). Suppose that, conditioned upon \( \mathbf{e}_{1:n} \), the members of the \((n - 1)\)th generation produce offspring in accordance with the linear fractional distribution \((P_{n,k})_{k \geq 0} = LF(A_n, B_n)\) having g.f. \( f_n \). Then
\[
f_{1:n} := f_1 \circ \ldots \circ f_n
\]
equals the g.f. of (the quenched law of) \( Z_n \) given \( \mathbf{e}_{1:n} \), and also given \( \mathbf{e} \). It satisfies
\[
\varphi \circ f_{1:n}(s) = \frac{1}{1 - f_{1:n}(s)} = \frac{\Pi_n}{1 - s} + R_n = g_{1:n} \circ \varphi(s) \tag{7}
\]
for \( s \in [0, 1) \), where
\[
g_n(x) = g(\mathbf{e}_n, x) := A_n x + B_n, \quad \Pi_n := \prod_{k=1}^{n} A_k \quad \text{and} \quad R_n := \sum_{k=1}^{n} \Pi_{k-1} B_k
\]
for \( n \in \mathbb{N} \). We thus see that all quenched laws are linear fractional, namely
\[
\mathcal{L}(Z_n | \mathbf{e}_{1:n}) = \mathcal{L}(Z_n | \mathbf{e}) = LF(\Pi_n, R_n) \tag{8}
\]
with g.f. (see also (6))
\[
f_{1:n}(s) = \frac{\Pi_n + R_n - 1}{\Pi_n + R_n} + \frac{1}{\Pi_n + R_n} \cdot \frac{\Pi_n s}{\Pi_n + R_n(1 - s)} \tag{9}
\]
for each \( n \in \mathbb{N} \), and that, up to conjugation, \( (f_{1:n})_{n \geq 0} \) equals the sequence of backward iterations of the i.i.d. affine linear random maps \( g_1, g_2, \ldots \). The corresponding forward iterations \( g_{n:1}(x) := g_n \circ \ldots \circ g_1(x) \), called iterated function system (IFS), form a Markov chain on \([1, \infty)\) with initial state \( x \). This chain has been extensively studied in the literature, see e.g. [19, 20, 16, 6] and especially the recent monographs [11, 17], and we will review some of its essential properties in the next section. In view of these observations it
appears to be natural to study properties of the linear fractional GWRE
as just introduced by drawing on results about iterations of the \( g_n \). As an
immediate consequence of (7), we have that

\[
q_n(e_{1:n}) := P(Z_n = 0|e_{1:n}) = f_{1:n}(0) = 1 - \frac{1}{I_n + R_n} \quad \text{a.s.} \tag{10}
\]

for all \( n \geq 1 \), and we let

\[
q(e) := \lim_{n \to \infty} P(Z_n = 0|e_{1:n})
\]

denote the quenched extinction probability of \((Z_n)_{n \geq 0}\) given \( e \).

**Reversing the environment.** The trivial fact that \( e_{1:n} \overset{d}{=} e_{n:1} \) for each \( n \in \mathbb{N} \)
allows us to study the given GWRE up to any time \( n \) under the time-
reversed environment \( e_{n:1} \) without changing its (annealed) law. On the other
hand, the quenched laws are naturally different, but the fact that they have
the same distribution (as random measures, see (11) below) will facilitate
assertions about quenched asymptotic behavior that are more tangible than
those without time-reversal. Let us define

\[
P := P(\cdot|e), \quad P^{(1:n)} := P(\cdot|e_{1:n}) \quad \text{and} \quad P^{(n:1)} := P(\cdot|e_{n:1})
\]

with corresponding expectations \( E, E^{(1:n)} \) and \( E^{(n:1)} \). Note that

\[
P^{(1:n)}((Z_0, \ldots, Z_n) \in \cdot) \overset{d}{=} P^{(n:1)}((Z_0, \ldots, Z_n) \in \cdot) \tag{11}
\]

for each \( n \in \mathbb{N} \), in particular,

\[
P^{(n:1)}(Z_n \in \cdot) = LF(I_n, \sum_{k=1}^{n} I_k^{-1} B_k)
\]

(12)

and

\[
q_n(e_{n:1}) := P(Z_n = 0|e_{n:1}) = 1 - \frac{1}{I_n(1 + \sum_{k=1}^{n} I_k^{-1} B_k)} \quad \text{a.s.} \tag{13}
\]

as one can easily check (see (8) and (10)).

For GWP’s in i.i.d. random environment, the usual distinction between
subcritical, critical and supercritical case is based on the asymptotic behavior
of the logarithm of the quenched mean log \( \log E Z_n \). Provided this quantity is
a.s. finite for all \( n \), it constitutes an ordinary random walk, denoted \((S_n)_{n \geq 0}\)
throughout, with generic increment log \( f'(1) \) which in the present situation
equals \( -\log A \). In fact, if \( Z_0 = 1 \), then

\[
\log E Z_n = -\log I_n = -\sum_{k=1}^{n} \log A_k =: S_n \quad \text{a.s.}
\]
for all $n \geq 0$. Depending on the fluctuation-type of this walk, namely

- positive divergence ($S_n \to \infty$ a.s.),
- negative divergence ($S_n \to -\infty$ a.s.),
- oscillation ($\limsup_{n \to \infty} S_n = +\infty$ and $\liminf_{n \to \infty} S_n = -\infty$ a.s.),
- trivial ($S_n = 0$ a.s for all $n \geq 0$),

the process $(Z_n)_{n \geq 0}$ is called subcritical, supercritical, critical or strongly critical, respectively [18, Def. 2.3]. If $E \log A$ exists, this means that

$$(Z_n)_{n \geq 0} \begin{cases} \text{subcritical} & \text{if } E \log A > 0, \\ \text{critical} & \text{if } E \log A = 0 \text{ and } P(A \neq 1) > 0, \\ \text{strongly critical} & \text{if } A = 1 \text{ a.s.}, \\ \text{supercritical} & \text{if } E \log A < 0. \end{cases}$$

Due to the very explicit knowledge of the $f_{1:n}$ in the present situation, a precise description of when each of these cases occurs is possible and in fact provided at the end of the next section after the collection of some relevant facts about iterations of random affine linear functions. Based on this, our definition of subcritical and supercritical processes will be slightly more restrictive as above and thus entail a wider definition of critical processes.

2 Basic results for iterations of random affine linear functions

Let us collect some essential facts about IFS generated by affine linear random functions $g_n(x) = A_n x + B_n$ with i.i.d. positive random coefficients $A_n, B_n$. As already stated, the forward iterations

$$g_{n:1}(x) = H_n x + \sum_{k=1}^{n} \frac{H_n}{H_k} B_k, \quad n \geq 0$$

form a Markov chain with initial state $x$. The corresponding backward iterations

$$g_{1:n}(x) = H_n(x) + \sum_{k=1}^{n} H_{k-1} B_k = H_n x + R_n, \quad n \geq 0,$$

though having the same distribution ($g_{n:1}(x) \overset{d}{=} g_{1:n}(x)$), exhibit a very different behavior and are not Markovian. They are in fact strictly increasing, for $A, B$ are positive. Notice that, if $g_{n:1}(x) = G_n(A_1, B_1, \ldots, A_n, B_n)$ for
a suitable function $G_n$, then $g_{1:n}(x) = G_n(A_n, B_n, \ldots, A_1, B_1)$. Goldie and Maller [16, Thm. 2.1] have provided necessary and sufficient conditions for the stability (positive recurrence) of $(g_{n:1}(x))_{n \geq 0}$, here summarized in the subsequent proposition for the case of positive $A, B$.

**Proposition 2.1** Suppose that $A, B$ are a.s. positive. Then the following assertions are equivalent:

(a) If $J^-(x) := \int_0^x P(- \log A > y) \, dy = E(x \wedge \log^- A)$ for $x > 0$, then

$$
\Pi_n \to 0 \text{ a.s. and } I_- := \int_{[1,\infty)} \frac{\log x}{J^-(x)} \, P(\log B \in dx) < \infty. \quad (14)
$$

(b) The so-called perpetuity

$$
R_\infty := \sum_{k \geq 1} \Pi_{k-1} B_k
$$

is a.s. finite and, for each $x \in \mathbb{R}$, the backward iterations $g_{1:n}(x)$ converge a.s. (monotonically) to $R_\infty$, while the forward iterations $g_{n:1}(x)$ converge in distribution to $R_\infty$.

Conversely, if

$$
P(Ax + B = x) < 1 \text{ for all } x \in \mathbb{R} \quad (15)
$$

and at least one of the conditions in (14) fails to hold, then $R_\infty = \infty$ a.s.

Plainly, the law of $R_\infty$, if a.s. finite, forms the unique stationary distribution of the Markov chain $(g_{n:1}(x))_{n \geq 0}$. Let us further point out that for $(A, B)$ with $A + B \geq 1$ a.s., which is a necessary requirement in the above branching framework, we further have

$$
R_\infty \geq 1 \text{ a.s.} \quad (16)
$$

Namely, if $R_\infty < \infty$ a.s. and thus condition (14) is valid, then $B \geq 1 - A$ a.s. implies

$$
R_\infty \geq \lim_{n \to \infty} \sum_{k=1}^n \Pi_{k-1} (1 - A_k)
$$

$$
= \lim_{n \to \infty} \sum_{k=1}^n (\Pi_{k-1} - \Pi_k) = 1 - \lim_{n \to \infty} \Pi_n = 1.
$$

This actually even shows that $R_\infty > 1$ a.s. unless $A + B = 1$ a.s.

As a direct consequence of the previous proposition, we can state the following duality result for the case when $\Pi_n \to \infty$ a.s.

**Proposition 2.2** Defining $g_n^{(-1)}(x) := A_n^{-1} x + A_n^{-1} B_n$ for $n \in \mathbb{N}$ (which is not the inverse of $g_n$), the duality relation
\[ \frac{R_n}{H_n} = \frac{g_{1:n}(0)}{H_n} = g_{n:1}(0) \overset{d}{=} \sum_{k=1}^{n} \Pi_k^{-1} B_k =: R_{n}^{(-1)} \quad (17) \]

holds for all \( n \in \mathbb{N} \). Moreover, if \( \Pi_n \to \infty \) a.s. and \( I_+ := \int_{[1,\infty)} \frac{\log x}{J^+(x)} P(\log(B/A) \in dx) < \infty \), then

\[ \frac{R_n}{H_n} \overset{d}{\to} R_{\infty}^{(-1)}. \]

On the other hand, if (15) is valid and at least one of the conditions in (18) fails to hold, then \( R_{\infty}^{(-1)} = \infty \) a.s.

Our standing assumption (6) ensures that \( R_{\infty} \) and \( R_{\infty}^{(-1)} \) always exist as the strictly increasing limits of \( R_n \) and \( R_n^{(-1)} \), respectively. One can also easily verify that these random variables cannot be a.s. finite at the same time. Therefore, the trichotomy

(C1) \( R_{\infty} < \infty = R_{\infty}^{(-1)} \) a.s.
(C2) \( R_{\infty}^{(-1)} < \infty = R_{\infty} \) a.s.
(C3) \( R_{\infty} = R_{\infty}^{(-1)} = \infty \) a.s.

holds, and with the help of the previous two propositions characterization of the three cases in terms of \((A, B)\) is easily provided and leads to the announced classification of the four criticality regimes of \((Z_n)_{n \geq 0}\) that differs slightly from the one based only on the fluctuation type of the random walk \((S_n)_{n \geq 0}\) used in [18].

**Proposition 2.3** Assuming \( A, B > 0 \), we have that

(C1) occurs iff one of the following two sets of conditions holds:

(C1.1) Cond. (15), \( \Pi_n \to 0 \) a.s. and \( I_- < \infty \).
(C1.2) \( B = x(1 - A) \) a.s. for some \( x > 0 \) (\( \Rightarrow \Pi_n \to 0 \) a.s.).

(C2) occurs iff one of the following two sets of conditions holds:

(C2.1) Cond. (15), \( \Pi_n \to \infty \) a.s. and \( I_+ < \infty \).
(C2.2) \( B = x(A - 1) \) a.s. for some \( x > 0 \) (\( \Rightarrow \Pi_n \to \infty \) a.s.).

(C3) occurs iff one of the following four sets of conditions holds:

(C3.1) Cond. (15), \( \Pi_n \to 0 \) a.s. and \( I_- = \infty \).
(C3.2) Cond. (15), \( \Pi_n \to \infty \) a.s. and \( I_+ = \infty \).
(C3.3) \( \liminf_{n \to \infty} \Pi_n = 0 \) and \( \limsup_{n \to \infty} \Pi_n = \infty \) a.s.
Let \( (Z_n)_{n \geq 0} \) be subcritical, thus \( R_n^{(-1)} < \infty = R_\infty \) and \( \Pi_n \rightarrow \infty \) a.s. In order to determine the quasistationary behavior of \( Z_n \) given \( Z_n > 0 \) as \( n \rightarrow \infty \),
put
\[ h_n(s) := E^{(1:n)}(sZ_n | Z_n > 0) \]
for \( n \in \mathbb{N} \). Then
\[
\frac{1}{1 - h_n(s)} = \frac{1 - f_{1:n}(0)}{1 - f_{1:n}(s)} = 1 - \frac{1 - f_{1:n}(s)}{1 - f_{1:n}(0)}
\]
and therefore
\[
\frac{1}{1 - h_n(s)} = 1 - \frac{1 - f_{1:n}(0)}{1 - f_{1:n}(s)} = \frac{\Pi_n (1 - s)^{-1} + R_n}{\Pi_n + R_n} = \frac{\Pi_n}{\Pi_n + R_n} \cdot 1 - \frac{1}{1 - s} + \frac{R_n}{\Pi_n + R_n}
\]
for each \( n \in \mathbb{N} \). In other words,
\[
\mathbb{P}^{(1:n)}(Z_n \in \cdot | Z_n > 0) = \text{Geom}_+ \left( \frac{1}{1 + R_n/\Pi_n} \right)
\]
is a.s. geometric on \( \mathbb{N} \) and thus again linear fractional. However, it fluctuates in accordance with \( R_n/\Pi_n \) which converges only in distribution. The same observation is made for the pertinent quenched survival probability (see also (10))
\[
\Pi_n \mathbb{P}^{(1:n)}(Z_n > 0) = \frac{1}{\mathbb{E}^{(1:n)}(Z_n | Z_n > 0)} = \frac{1}{1 + R_n/\Pi_n} \text{ a.s.}
\]
As already indicated, the situation improves under reversal of the environment at each \( n \) because this means to replace \( R_n/\Pi_n \) by its a.s. convergent counterpart \( R_n(-1) \). We have
\[
\mathbb{P}^{(n:1)}(Z_n \in \cdot | Z_n > 0) = \text{Geom}_+ \left( \frac{1}{1 + R_n(-1)} \right)
\]
and accordingly
\[
\Pi_n \mathbb{P}^{(n:1)}(Z_n > 0) = \frac{1}{\mathbb{E}^{(n:1)}(Z_n | Z_n > 0)} = \frac{1}{1 + R_n(-1)} \text{ a.s.}
\]
Using Prop. 2.2, the following quenched convergence result under (C2) is almost immediate.

**Theorem 3.1** Let \((Z_n)_{n \geq 0}\) be subcritical and \( R_{\infty}^{(-1)} < \infty = R_{\infty} \) a.s. Then
So this law fluctuates in accordance with \( \vartheta \) variation, that is parameter only in law. Reversal of the environment provides the same result with random recall from (10) that
\[
q_{\abla} \text{ geometric as well, the parameter being }
\]
and \( \mathbf{P}^{(n:1)}(Z_n \in \mid Z_n > 0) \) converges a.s. to \( \text{Geom}(1/(1 + R_\infty^{(-1)})) \) in total variation, that is
\[
\mathbf{P}^{(n:1)}(Z_n \in \mid Z_n > 0) - \text{Geom} \left( \frac{1}{1 + R_\infty^{(-1)}} \right) \xrightarrow{n \to \infty} 0 \quad \text{a.s.} \quad (24)
\]
Proof. As \( R_n^{(-1)} \to R_\infty^{(-1)} \) a.s., the assertions follow directly from (21) and (22). \( \square \)

Our second quenched result takes a look at the size of the population at the eve of extinction and must therefore condition on \( \{Z_n > 0, Z_{n+l} = 0\} \) with any fixed \( l \in \mathbb{N} \). Put
\[
h_{n,l}(s) := \mathbb{E}^{(1:n+l)}(Z_n > 0, Z_{n+l} = 0)
\]
and recall from (10) that \( q_n(e) := f_{1:n}(0) = 1 - (R_n + \Pi_n)^{-1} \). Then
\[
h_{n,l}(s) = \frac{\mathbb{E}^{(1:n+l)}(sZ_n \mid Z_n > 0, Z_{n+l} = 0)}{\mathbb{P}^{(1:n+l)}(Z_n > 0, Z_{n+l} = 0)}
= \frac{\mathbb{E}^{(1:n+l)}(s(f_{n+1:n+l}(0))Z_n \mid Z_n > 0)}{\mathbb{E}^{(1:n+l)}(Z_n > 0, Z_{n+l} = 0)}
= \frac{\mathbb{E}^{(1:n+l)}((f_{n+1:n+l}(0))Z_n \mid Z_n > 0)}{\mathbb{E}^{(1:n+l)}((f_{n+1:n+l}(0))Z_n \mid Z_n > 0)}
= \frac{h_n(f_{n+1:n+l}(0)s)}{h_n(f_{n+1:n+l}(0))} = \frac{h_n(q_n(e_{n+1:n+l})s)}{h_n(q_n(e_{n+1:n+l}))},
\]
and since \( h_n \) is the g.f. of the geometric law \( \text{Geom}(\Pi_n(1 - q_n(e_{1:n}))) \) (cf. (19)), we conclude by recalling (14) that \( \mathbb{P}^{(1:n+l)}(Z_n \mid Z_n > 0, Z_{n+l} = 0) \) is a.s. geometric as well, the parameter being
\[
\vartheta_{n,l}(e_{1:n+l}) := 1 - q_l(e_{n+1:n+l})(1 - \Pi_n(1 - q_n(e_{1:n}))).
\]
So this law fluctuates in accordance with \( \vartheta_{n,l}(e_{1:n+l}) \) which again converges only in law. Reversal of the environment provides the same result with random parameter
\[
\vartheta_{n,l}(e_{n+l+1}) = 1 - q_{l}(e_{l+1}) \left( 1 - \frac{\Pi_{l+n}(1 - q_n(e_{n+l+1}))}{\Pi_{l}(1 - q_n(e_{n+l+1}))} \right)
\]
\[ \frac{1}{\Pi_l(1 + R_l^{(-1)})} = \left( 1 - \frac{1}{\Pi_l(1 + R_l^{(-1)})} \right) \frac{1}{1 + R_l^{(-1)}} \]

which has obviously the a.s. limit

\[ \vartheta_l := \frac{1}{\Pi_l(1 + R_l^{(-1)})} + \left( 1 - \frac{1}{\Pi_l(1 + R_l^{(-1)})} \right) \frac{1}{1 + R_l^{(-1)}} \]

as \( n \to \infty \), where

\[ R_{l,n}^{(-1)} := \sum_{k=1}^{n} \frac{\Pi_l}{\Pi_{k+l}} B_{k+l} = \Pi_l \left( R_{n+l}^{(-1)} - R_l^{(-1)} \right) \]

for \( n \in \mathbb{N} \cup \{\infty\} \) and \( R_{l,n}^{(-1)} \) for all \( l \) and \( n \) should be noted. The first two assertions of the subsequent theorem are now immediate.

**Theorem 3.2** Let \( (Z_n)_{n \geq 0} \) be subcritical and \( l \in \mathbb{N} \). Then

\[ \left\| \mathbb{P}^{(n+1)}(Z_n \in \cdot | Z_n > 0, Z_{n+l} = 0) - \text{Geom}_+(\vartheta_l) \right\| \xrightarrow{n \to \infty} 0 \quad \text{a.s.,} \quad (25) \]

\[ \mathbb{E}^{(n+1)}(Z_n | Z_n > 0, Z_{n+l} = 0) \xrightarrow{n \to \infty} \vartheta_l^{-1} \quad \text{a.s.} \quad (26) \]

and

\[ \frac{\Pi_{n+l}}{\Pi_l} \mathbb{P}^{(n+1)}(Z_n > 0, Z_{n+l} = 0) \xrightarrow{n \to \infty} \frac{1 - (\Pi_l + R_l)^{-1}}{1 + R_{l,\infty}^{(-1)} (\Pi_l + R_l)^{-1}} \cdot \frac{1}{1 + R_{l,\infty}^{(-1)}} \quad \text{a.s.} \quad (27) \]

**Proof.** By (21),

\[ \mathbb{P}^{(n+1)}(Z_n \in \cdot | Z_n > 0) = \text{Geom}_+ \left( \frac{1}{1 + R_{l,n}^{(-1)}} \right) \quad \text{a.s.} \]

and this conditional law is the same under \( \mathbb{P}^{(n+1)} \). Hence

\[ \mathbb{E}^{(n+1)}(s^Z_n | Z_n > 0) = \frac{(1 + R_{l,n}^{(-1)})^{-1}s}{1 - (1 + R_{l,n}^{(-1)})^{-1} R_{l,n}^{(-1)}s} \]

We further have

\[ \frac{\Pi_{n+l}}{\Pi_l} \mathbb{P}^{(n+1)}(Z_n > 0) = \frac{1}{1 + R_{l,n}^{(-1)}} \quad \text{a.s.} \]

and so
\[
\frac{\Pi_{n+l}}{\Pi_l} P^{(n+l:1)}(Z_n > 0, Z_{n+l} = 0)
\]

\[
= \frac{\Pi_{n+l}}{\Pi_l} E^{(n+l:1)} \left( q_I(e_{l+1}) Z_n I_{\{Z_n > 0\}} \right)
\]

\[
= E^{(n+l:1)} \left( \left( \frac{\Pi_l + R_l - 1}{\Pi_l + R_l} \right) Z_n \left| Z_n > 0 \right. \right) \frac{\Pi_{n+l}}{\Pi_l} P^{(n+l:1)}(Z_n > 0)
\]

\[
= \frac{1 - (\Pi_l + R_l)^{-1}}{1 + R_{l,n}^{(-1)}(\Pi_l + R_l)^{-1}} \cdot \frac{1}{1 + R_{l,n}^{(-1)}} \text{ a.s.}
\]

The asserted limit in (27) is now obvious. \(\square\)

In order to better understand the typical path to extinction, another process of interest in both the subcritical and critical regime is the so-called reduced GWPRE \((Z_{m,n})_{0 \leq m \leq n}\) for any \(n \in \mathbb{N}\) and under \(P^{(1:n)}(\cdot | Z_n > 0)\). Fixing a time horizon \(n\), it accounts for the number of individuals in the population up to that time whose families have not died out by time \(n\). More precisely, \(Z_{m,n}\) equals the number of individuals in generation \(m\) who have descendants in generation \(n\). In random environment, it was studied in a series of papers by Vatutin with varying collaborators, often with a focus on annealed limit laws; see for example [10, 25, 13, 24]. The following lemma is not difficult to prove but we refrain from giving the somewhat tedious technical details regarding its second part.

Lemma 3.3 The reduced process \((Z_{m,n})_{0 \leq m \leq n}\) is nondecreasing and

\[
P^{(1:n)}(Z_{m,n} \in \cdot | Z_n > 0) = \text{Geom}_+ \left( 1 - \frac{R_m}{\Pi_m + R_n} \right).
\] (28)

Moreover, \(Z_{m,n} \overset{d}{=} \sum_{i=1}^{Z_{l,m}} s_{i,m}^{l,m} \) for \(0 \leq l < m \leq n\), where \(s_{i,1}^{l,m}, s_{i,2}^{l,m}, \ldots\) are independent of \(Z_{l,n}\) under \(P^{(1:n)}(\cdot | Z_n > 0)\) and further i.i.d. with common law

\[
\text{Geom}_+ \left( 1 - \frac{R_m - R_l}{\Pi_m + R_n - R_l} \right).
\] (29)

One can interpret \(s_{i,m}^{l,m}\) as the number of individuals in generation \(m\) who have descendants in generation \(n\) and are stemming from individual \(i\) in generation \(l\) (who has therefore descendants in generation \(n\) as well).

Proof. As for (28), it suffices to note that \(Z_{m,n}\) is obtained from \(Z_m\) by tagging each individual in generation \(m\) which has offspring in generation \(n\). Formally, \(Z_{m,n} = \sum_{k=1}^{Z_m} I_k\) where \(I_1, I_2, \ldots\) are i.i.d. Bernoulli variables under \(P^{(1:n)}\) with parameter

\[
1 - q_{m-m}(e_{m+1:n}) = \left( \frac{\Pi_n}{\Pi_m} + \frac{R_n - R_m}{\Pi_m} \right)^{-1} = \frac{\Pi_m}{\Pi_n + R_n - R_m}.
\]
Since $\mathbf{P}^{(1:n)}(Z_m \in \cdot) = LF(\Pi_m, R_m)$, the law of $Z_{m,n}$ under $\mathbf{P}^{(1:n)}$ equals

$$LF\left(\frac{\Pi_m}{1 - q_{n-m}(e_{m+1:n})}, R_m\right) = LF(\Pi_n + R_n - R_m, R_m).$$

Now (28) follows easily when additionally conditioning on the event $\{Z_n > 0\}$ which is the same as $\{Z_{m,n} > 0\}$. ⊓⊔

We also note the conditional law of $Z_{m,n}$ when reversing the environment as in the previous results, thus under $\mathbf{P}^{(n:1)}(\cdot|Z_n > 0)$. As one can readily check,

$$\mathbf{P}^{(n:1)}(Z_{m,n} \in \cdot|Z_n > 0) = \text{Geom}_+\left(\frac{1 + R_{n-m}^{(-1)}}{1 + R_n^{(-1)}}\right),$$

and also that the two random parameters appearing in the geometric laws in (28) and (30), and also those in (29) and (31) are identically distributed as they must.

**Theorem 3.4** (a) Let $(m_n)_{n \geq 1}$ be an integer sequence such that $n - m_n \to \infty$. Then

$$\lim_{n \to \infty} \mathbb{E}\mathbf{P}^{(1:n)}(Z_{m,n,n} > 1|Z_n > 0) = 0$$

and

$$\mathbf{P}^{(n:1)}(Z_{m,n,n} > 1|Z_n > 0) \xrightarrow{n \to \infty} 0 \text{ a.s.}$$

If, furthermore, $\limsup_{n \to \infty} n^{-1}m_n < 1$

$$\mathbb{E}\log^2 A < \infty \quad \text{and} \quad \mathbb{E}\log^2 B < \infty,$$

then also

$$\mathbf{P}^{(1:n)}(Z_{m,n,n} > 1|Z_n > 0) \xrightarrow{n \to \infty} 0 \text{ a.s.}$$

holds true.

(b) For arbitrary $m \in \mathbb{N}$, let $(Z_k^*)_{0 \leq k \leq m}$ be a GWP in the random environment $e_{m:1}$ such that

$$\mathbf{P}^{(m:1)}(Z_0^* \in \cdot) = \text{Geom}_+\left(\frac{1 + R_m^{(-1)}}{1 + R_\infty^{(-1)}}\right).$$

Individuals in generation $k$ are supposed to have the same conditional offspring law as $(Z_{n-m+k,n})_{0 \leq k \leq m}$ under $\mathbf{P}^{(n:1)}(\cdot|Z_n > 0)$ for each $k \in \{0, \ldots, m-1\}$, thus...
Then
\[ \mathbb{P}^{(n:1)}((Z_{(k,n)})_{n-m \leq k \leq n} \in \cdot | Z_n > 0) - \mathbb{P}^{(m:1)}((Z_{(k)})_{0 \leq k \leq m} \in \cdot) \xrightarrow{n \to \infty} 0 \text{ a.s.} \quad (36) \]

Proof. (a) Put \( l_n := n - m_n \), \( \rho := \liminf_{n \to \infty} n^{-1} l_n \), and let \( \hat{\Pi}_k \) be a copy of \( \Pi_k \) independent of all other occurring random variables. Use (30), \( l_n \to \infty \) and \( R_\infty^{-1} \to R_\infty^{-1} \) a.s. to infer
\[ \mathbb{P}^{(n:1)}(Z_{m,n} > 1 | Z_n > 0) = \frac{R_n^{(-1)} - R_{l_n}^{(-1)}}{1 + R_n^{(-1)}} \xrightarrow{n \to \infty} 0 \text{ a.s.} \]

and thus (33). But this also implies (32) as
\[ \mathbb{P}^{(n:1)}(Z_{m,n} > 1 | Z_n > 0) \overset{d}{=} \mathbb{P}^{(1:n)}(Z_{m,n} > 1 | Z_n > 0) \]

for each \( n \) and by the dominated convergence theorem.

Turning to (35) under the stated extra moment conditions, observe that (33) entails
\[ \mathbb{P}^{(1:n)}(Z_{m,n} > 1 | Z_n > 0) = \frac{R_m^{(-1)}}{\Pi_n + R_m} \leq \frac{R_m^{(-1)}}{\Pi_n + R_m} \overset{d}{=} \frac{R^{(-1)}_{m_n}}{\hat{\Pi}_n + R^{(-1)}_{m_n}}. \]

Assuming (34) and putting \( \nu = \mathbb{E} \log A \), it follows by the Hsu-Robbins theorem (see \cite[Cor. 10.4.2]{12}) that
\[ \sum_{n \geq 1} \mathbb{P}(|S_n + n\nu| > \varepsilon n) < \infty \quad \text{for all } \varepsilon > 0, \]

and by \cite[Thm. 1.2]{4} that
\[ \mathbb{E} \log (1 + R^{(-1)}_\infty) < \infty. \]

In order to conclude (35), it suffices to show that
\[ \mathbb{P}\left( \mathbb{P}^{(1:n)}(Z_{m,n} > 1 | Z_n > 0) > \varepsilon \right) < \infty \]

for all \( \varepsilon > 0 \). To this end, observe that
\[ \mathbb{P}\left( \mathbb{P}^{(1:n)}(Z_{m,n} > 1 | Z_n > 0) > \varepsilon \right) \leq \mathbb{P}\left( \frac{R^{(-1)}_{m_n}}{\hat{\Pi}_n + R^{(-1)}_{m_n}} > \varepsilon \right) \]
\[
\frac{\hat{R}^{-1}_{\infty}}{\hat{R}_{l_n} + \hat{R}^{-1}_{\infty}} > \varepsilon
\]
\[
\leq \mathbb{P} \left( \frac{R_{\infty}^{-1}}{R_{l_n} + R_{\infty}^{-1}} > \varepsilon \right)
\]
\[
\leq \mathbb{P} \left( R_{\infty}^{-1} > e^{\varepsilon n} \right) + \mathbb{P} \left( R_{\infty}^{-1} \leq e^{\varepsilon n}, \frac{e^{\varepsilon n}}{R_{l_n} + e^{\varepsilon n}} > \varepsilon \right)
\]
\[
\leq \mathbb{P} \left( \log R_{\infty}^{-1} > \varepsilon n \right) + \mathbb{P} \left( S_{l_n} > -\varepsilon n - \log(\varepsilon^{-1} - 1) \right)
\]
\[
\leq \mathbb{P} \left( \log R_{l_n}^{-1} > \varepsilon n \right) + \mathbb{P} \left( S_{l_n} + \nu l_n > (\nu - 2\rho \varepsilon) l_n \right)
\]
the last estimate being valid for sufficiently large \( n \) only. But for \( \varepsilon \) so small that \( \nu - 2\rho \varepsilon > 0 \) (recalling \( \nu > 0 \)), we now conclude the summability of the last line over \( n \geq 1 \) as required.

Since, by Lemma 3.3, \( Z_{k+1,n} = \sum_{i=1}^{Z_{k,n}} \zeta_{i}^{k-1,k} \) with \( \zeta_{1}^{k-1,k}, \zeta_{2}^{k-1,k}, \ldots \) as described there, it suffices to note that, by recalling (30) and (31),
\[
\mathbb{P}^{(n:1)}(Z_{n-m,n} \in \cdot | Z_n > 0) \xrightarrow{n \to \infty} \text{Geom}_+ \left( \frac{1 + R_{m}^{-1}}{1 + R_{\infty}^{-1}} \right)
\]
in total variation, and
\[
\mathbb{P}^{(n:1)}(S_{l_1}^{n-m-k,n-m+k+1} \in \cdot | Z_n > 0) = \text{Geom}_+ \left( \frac{1 + R_{m-k}^{-1}}{1 + R_{m-k}^{-1}} \right)
\]
for each \( k = 0, \ldots, m - 1 \).

Turning to annealed results under the common assumption
\[
\mathbb{E} f'(1) \log f'(1) = \mathbb{E}(1/A) \log(1/A) < \infty,
\]
three subregimes must be distinguished for which the annealed probability of survival \( \mathbb{P}(Z_n > 0) \) exhibits a different behavior as \( n \to \infty \), see [18, Sect. 2.4], [15] and also [13] which focusses on the linear fractional case. Let \( \psi(\theta) := \log \mathbb{E}^{\theta \log(1/A)} = \log \mathbb{E}^{(1/A)^{\theta}} \) denote the cumulant g.f. of \( \log(1/A) \) which, by [87], is finite for \( \theta \in [0, 1] \) with finite derivative
\[
\psi'(\theta) = e^{-\psi(\theta)} \mathbb{E} \log(1/A)(1/A)^{\theta}.
\]
A linear fractional GWP in i.i.d. random environment \((Z_n)_{n \geq 0}\) is called
- **strongly subcritical** if \( \psi'(0) < 0 \) and \( \psi'(1) < 0 \);
- **intermediately subcritical** if \( \psi'(0) < 0 \) and \( \psi'(1) = 0 \);
- **weakly subcritical** if \( \psi'(0) < 0 \) and \( \psi'(1) > 0 \).

Putting \( \kappa := \psi(1) = \log \mathbb{E}(1/A) \), we can define a new measure \( \hat{\mathbb{P}} \) with the help of the positive martingale \( (e^{-\kappa \Pi_{n}^{-1}})_{n \geq 0} \), namely
Linear fractional GWPRE and perpetuities

\[ \hat{P}(A) := \int_A e^{-\kappa n} \Pi^n dP = \int_A e^{S_n - \kappa n} dP \]

for \( A \in \sigma((A_k, B_k)_{0 \leq k \leq n}) \) and any \( n \in \mathbb{N}_0 \). Under \( \hat{P} \), the \( (A_k, B_k) \) are again i.i.d. and \( \hat{E} \log(1/A) = e^{\kappa} E_{1/A} \log(1/A) \). The convexity of \( \psi \) together with \( \psi'(0) < 0 \) entails that \( \kappa < 0 \) in the strongly and intermediately subcritical case, in fact

\[ e^{\kappa} = E(1/A) = \inf_{0 \leq \theta \leq 1} E(1/A) \theta. \]

As a direct consequence of (22), we obtain

\[ P(Z_n > 0) = \mathbb{E}P^{(n:1)}(Z_n > 0) \quad (38) \]

\[ = \mathbb{E} \left( \frac{1}{H_n(1 + R_n^{-1})} \right) = e^{\kappa n} \hat{E} \left( \frac{1}{1 + R_n^{-1}} \right) \quad (39) \]

and also the inequality

\[ \frac{1}{1 + \hat{E}R_n^{-1}} \leq e^{-\kappa n} P(Z_n > 0) \leq 1 \quad (40) \]

for all \( n \in \mathbb{N} \). Here the upper bound is trivial while the lower one follows by Jensen’s inequality. We now see that the survival probability essentially behaves like \( e^{\kappa n} \) if \( \hat{E}R_n^{-1} \), the monotone limit of \( \hat{E}R_n^{-1} \), is finite which, as will be seen below, does only hold in the strongly subcritical case.

Regarding the annealed conditional law of \( Z_n \) given \( Z_n > 0 \), (22) implies that it is a mixture of geometric laws on \( \mathbb{N} \) with a mixing measure on the open unit interval \((0, 1)\) that again involves the finite perpetuity \( R_n^{-1} \) in terms of \((1 + R_n^{-1})^{-1}\). More precisely,

\[ P(Z_n \in \cdot | Z_n > 0) = \frac{1}{P(Z_n > 0)} \int P^{(n:1)}(Z_n \in \cdot | Z_n > 0) P^{(n:1)}(Z_n > 0) dP \]

\[ = \frac{1}{e^{-\kappa n} P(Z_n > 0)} \int Geom_+ \left( \frac{1}{1 + R_n^{-1}} \right) \frac{e^{-\kappa n}}{H_n(1 + R_n^{-1})} dP \]

\[ = \int_{(0,1)} \theta Geom_+(\theta) \Lambda_n(d\theta) \quad (41) \]

for each \( n \in \mathbb{N} \), where

\[ \Lambda_n(d\theta) = \theta \hat{P} \left( \frac{1}{1 + R_n^{-1}} \in d\theta \right) / \hat{E} \left( \frac{1}{1 + R_n^{-1}} \right) \]
As a particular consequence, using that $\text{Geom}_+(\theta)$ has mean $1/\theta$, we find the following annealed analog of (20):

$$E(Z_n|Z_n > 0) = \frac{1}{e^{-\kappa n}P(Z_n > 0)}. \quad (42)$$

In the strongly subcritical case, the previous identities quite directly lead to asymptotic results as $n \to \infty$, summarized in the subsequent theorem.

**Theorem 3.5** Let $(Z_n)_{n \geq 0}$ be strongly subcritical, that is $E(1/A) < \infty$ and $-\infty < E(1/A) \log(1/A) < 0$, and also $E \log B < \infty$. Then \( \hat{P}(R^{(-1)}_{\infty} < \infty) = 1 \) and

$$\left\| P(Z_n \in \cdot | Z_n > 0) - \int_{(0,1)} \theta \text{Geom}_+(\theta) \Lambda_{\infty}(d\theta) \right\| \xrightarrow{\text{n} \to \infty} 0, \quad (43)$$

with $\Lambda_{\infty}$ defined as $\Lambda_n$ above for $R^{(-1)}_{\infty}$. Furthermore,

$$e^{-\kappa n} P(Z_n > 0) = \frac{1}{E(Z_n|Z_n > 0)} \xrightarrow{n \to \infty} \hat{E}\left(\frac{1}{1 + R^{(-1)}_{\infty}}\right). \quad (44)$$

**Proof.** If $\hat{E}(1/A) = e^{-\kappa}E(1/A) \log(1/A) \in \mathbb{R}_-$ and $E \log B < \infty$, then Prop. 2.1 ensures that $R^{(-1)}_{\infty} < \infty \hat{P}$-a.s., which in turn implies the weak convergence of $\Lambda_n$ to $\Lambda_{\infty}$. Assertion (43) is now immediate because the involved distributions are living on $\mathbb{N}$. A combination of (38), (42) and the monotone convergence theorem further provides (44). \( \square \)

One can also state annealed versions of Theorem 3.2 and Theorem 3.4, for the latter see also [13, Thm. 2], but we refrain from doing so here. The neat representation of the asymptotic Yaglom law as a mixture of geometric laws does no longer hold in the intermediately subcritical case, nor in the weakly subcritical case because the a.s. monotone limit $R^{(-1)}_{\infty}$ of the $R^{(-1)}_{1n}$ is no longer finite under the necessary change of measure, at least when ruling out the case that $Ax + B = x$ a.s. for some $x \in \mathbb{R}$. Regarding the intermediately subcritical case, this measure change is the same as in the strongly subcritical case, but since

$$\hat{E}(1/A) = e^{-\kappa}E(1/A) \log(1/A) = 0,$$

it follows that

$$\limsup_{n \to \infty} \Pi_{n}^{-1} = \limsup_{n \to \infty} e^{S_{n}} = \infty \hat{P}$$.a.s. by the classical Chung-Fuchs theorem for centered random walks and then $R^{(-1)}_{\infty} = \infty \hat{P}$-a.s. by another appeal to Prop. 2.1. On the other hand, the existence of an asymptotic Yaglom law can also be shown in these two subregimes (providing some extra conditions), and we refer to [15] as well as the monograph [18] for more details and an account of further relevant literature.
4 The supercritical case

Suppose now that \( (Z_n)_{n \geq 0} \) is supercritical, thus \( R_\infty < \infty = R_\infty^{(-1)} \) and also \( \Pi_n \to 0 \) a.s. Recall that \( q(e) \) denotes the extinction probability given \( e \) and also \( e_{\geq n} = (A_k, B_k)_{k \geq n} \) for \( n \in \mathbb{N} \), thus \( e_{\geq 2} = e \). Then it is well-known [7] that

\[
q(e_{\geq 1}) = f_1(q(e_{\geq 2})) \quad \text{a.s.} 
\]

and that \( \{q(e_{\geq 1}) = 1\} \) is a.s. shift-invariant, i.e.

\[
\{q(e_{\geq 1}) = 1\} = \{q(e_{\geq 2}) = 1\} \quad \text{a.s.}
\]

Consequently, by ergodicity of the environment,

\[
P(q(e_{\geq 1}) = 1) \in \{0, 1\}. 
\]

In the case when \( q(e_{\geq 1}) < 1 \) a.s., we can restate (45) as

\[
\frac{1}{1 - q(e_{\geq 1})} = \frac{A_1}{1 - q(e_{\geq 2})} + B_1, 
\]

thus \( q(e_{\geq 1}) = \varphi^{-1} \circ g_1 \circ \varphi(q(e_{\geq 2})) \). The next theorem is now immediate.

**Theorem 4.1** Let \( (Z_n)_{n \geq 0} \) be supercritical. Then

\[
\frac{1}{1 - q(e)} = R_\infty \in [1, \infty) \quad \text{a.s.,} \quad (48)
\]

in particular \( q(e) < 1 \) a.s. Furthermore, recalling \( P = P(\cdot|e) \),

\[
\lim_{n \to \infty} \frac{1}{\Pi_n} \left( \frac{1}{R_n} - P(Z_n > 0) \right) = \frac{1}{R_\infty^2} \quad \text{a.s.} 
\]

(49)

Observe that (49) may also be stated as

\[
P(Z_n > 0) = \frac{1}{R_n} - \frac{\Pi_n}{R_\infty^2} + \Gamma_n 
\]

as \( n \to \infty \), where \( \Gamma_n \) satisfies \( \Pi_n^{-1} \Gamma_n \to 0 \) a.s.

**Proof.** The a.s. finiteness of \( R_\infty \) follows by Prop. 2.1. By iteration of (47) (or taking the limit in (10)), we then infer

\[
1 \leq \frac{1}{1 - q(e_{\geq 1})} = \frac{\Pi_n}{1 - q(e_{\geq n})} + R_n \overset{n \to \infty}{\to} R_\infty \quad \text{a.s.}
\]

and therefore \( q(e) < 1 \) a.s. In order to get (49), observe that, by another use of (10),
\[
\frac{1}{R_n} \left( \frac{1}{R_n} - P(Z_n > 0) \right) = \frac{1}{R_n(R_n + P(Z_n > 0))} \xrightarrow{n \to \infty} \frac{1}{R_\infty^2} \text{ a.s.}
\]

This completes the proof. \(\square\)

The following result describes the global limit behaviour of a supercritical process conditioned on non-extinction.

**Theorem 4.2** Let \((Z_n)_{n \geq 0}\) be supercritical. Then \(W_n := \prod_{n} Z_n\) for \(n \geq 0\) forms a.s. a mean one nonnegative martingale under the quenched probability measure \(P\) and thus converges a.s. to a random variable \(W_\infty\) having conditional law

\[
P(W_\infty \in \cdot) = \left( 1 - \frac{1}{R_\infty} \right) \delta_0 + \frac{1}{R_\infty} \text{Exp} \left( \frac{1}{R_\infty} \right)
\]

and particularly also mean one.

**Proof.** That \((W_n)_{n \geq 0}\) forms a nonnegative mean one \(P\)-martingale for almost all realizations of \(e\) is a well-known fact. Its a.s. convergence to some \(W_\infty\) then follows by the martingale convergence theorem. Moreover, by computing the conditional Laplace transform of \(W_n\) given \(e\), which a.s. converges to the conditional Laplace transform of \(W_\infty\) given \(e\), we find

\[
E e^{-uW_n} = f_{1,n}(e^{-uH_n}) = 1 - (1 - f_{1,n}(0)) \frac{1 - f_{1,n}(e^{-uH_n})}{1 - f_{1,n}(0)}
\]

\[
= 1 - \frac{1}{R_n + P(Z_n > 0)} \cdot \frac{R_n + P(Z_n > 0)}{R_n - P(Z_n > 0)}
\]

\[
xrightarrow{n \to \infty} 1 - \frac{1}{R_\infty} \cdot \frac{uR_\infty}{1 + uR_\infty} = \left( 1 - \frac{1}{R_\infty} \right) + \frac{1}{R_\infty} \cdot \frac{R_\infty^{-1}}{R_\infty^{-1} + u} \text{ a.s.}
\]

and this shows (50). \(\square\)

It is well-known, see [8, p. 47ff], that a supercritical GWP \(Z = (Z_n)_{n \geq 0}\) with one ancestor and extinction probability \(0 < q < 1\) can be decomposed into two nontrivial parts, say \(Z_1 = (Z_{1,n})_{n \geq 0}\) and \(Z_2 = (Z_{2,n})_{n \geq 0}\), by dividing each generation into their individuals with a finite line of descent and those with an infinite line of descent, respectively. Then \(Z_1\) and \(Z_2\) are both again nontrivial GWP’s, though for the last one the underlying probability measure must be chosen as \(\hat{P} := P(\cdot \mid Z_n \to \infty)\) (conditioning upon survival). If \(f\) denotes the g.f. of the offspring distribution of \(Z\), then the offspring distributions of \(Z_1\) and \(Z_2\) have g.f. \(g(s) = q^{-1}f(qs)\) and \(h(s) = (1 - q)^{-1}f(q + (1 - q)s)\), respectively. With \(P\) denoting the transition kernel of the Markov chain \(Z\), the law of \(Z_1\), known as the Harris-Sevastyanov transform, is actually nothing but the Doob \(h\)-transform of \(Z\) under the positive \(P\)-harmonic function \(N_0 \ni\)
It is also stated there that, if $f$ is linear fractional, then $g$ is linear fractional, see [20, Prop. 3.1], and it should not be surprising that the same holds true for $h$. More precisely, if the law associated with $f$ is $\text{LF}(a,b)$, then the laws associated with $g$ and $h$ are

$$\text{LF}\left(\frac{1}{a}, \frac{a+b-1}{a}\right) \quad \text{and} \quad \text{LF}(a, 1-a) = \text{Geom}_+(a),$$

respectively, where $q = b^{-1}(a + b - 1)$ should be recalled.

After these preliminary remarks about the fixed environment case, we return to the situation of i.i.d. random linear fractional offspring laws. Plainly, the decomposition into individuals with finite and infinite line of descent works and the following result shows that the obtained processes $Z_1, Z_2$ are again GWPRE’s with random linear fractional reproduction. On the other hand, the environment is no longer i.i.d.

**Theorem 4.3** Let $Z = (Z_n)_{n \geq 0}$ be supercritical with $0 < q(e) < 1$ a.s. Put also $C_n := A_n + B_n$ and $R_{n,\infty} = \Pi_n^{-1}(R_{\infty} - R_n)$. Then the following assertions hold in the given notation and for $Z_1$ and $Z_2$ as introduced before:

(a) $Z_1$ is a subcritical GWP in the ergodic stationary environment $(e_{\geq n})_{n \geq 1}$ with quenched random linear fractional offspring law

$$\frac{(C_n - 1)R_{n,\infty}}{C_n(R_{n,\infty} - 1)} \delta_0 + \frac{R_{n,\infty} - C_n}{C_n(R_{n,\infty} - 1)} \text{Geom}_+ \left( \frac{R_{n,\infty}}{C_n R_{n+1,\infty}} \right)$$

and associated g.f.

$$g_n(s) = \frac{f_n(q(e_{\geq n+1})s)}{q(e_{\geq n})} = \frac{R_{n,\infty}}{R_{n,\infty} - 1} f_n \left( \frac{(R_{n+1,\infty} - 1)s}{R_{n+1,\infty}} \right)$$

for each $n \geq 1$.

(b) Conditioned upon $e$ and survival of $Z$, i.e. under $\hat{P} := P(\cdot | Z_n \to \infty)$, $Z_2$ is a nonextinctive GWP in varying environment with positive geometric offspring law

$$\text{Geom}_+ \left( 1 - \frac{B_n}{R_{n,\infty}} \right)$$

and associated g.f.

$$h_n(s) = \frac{f_n(q(e_{\geq n+1}) + (1 - q(e_{\geq n+1}))s) - q(e_{\geq n})}{1 - q(e_{\geq n})}$$

$$= \frac{R_{n,\infty} - B_n}{R_{n,\infty} - B_n}$$

for each $n \geq 1$. 
We note that \( q(e_{\geq n}) \) in the above formulae may also be expressed in terms of \( \Pi_n \) and \( R_n \) for each \( n \). Namely,

\[
R_\infty = \frac{1}{1 - q(e_{\geq 1})} = \frac{\Pi_n}{1 - q(e_{\geq n})} + R_n
\]

implies

\[
\frac{1}{1 - q(e_{\geq n})} = \frac{R_\infty - R_n}{\Pi_n} = R_{n, \infty} \quad \text{and thus} \quad q(e_{\geq n}) = \frac{R_{n, \infty} - 1}{R_{n, \infty}}.
\]

**Proof.** (a) Consider \( Z \) under the quenched probability measure \( P \). Then an individual \( \upsilon \), say, in generation \( n - 1 \) for arbitrary \( n \in \mathbb{N} \) produces a random number of offspring with law \( LF(A_n, B_n) \) and g.f. \( f_n \), and each of these children has a finite line of descent with probability \( q(e_{\geq n}) \), independent of the other children. Therefore, by exactly the same argument as in the ordinary Galton-Watson case and recalling from (45) that \( f_n(q(e_{\geq n+1})) = q(e_{\geq n}) \), the law of the number of children of \( \upsilon \) whose families eventually die out is again linear fractional with the asserted g.f. \( g_n \). In order to get (51), we argue as follows: It follows by (4) that

\[
LF(A_n, B_n) = P_{n, 0} \delta_0 + (1 - P_{n, 0}) Geom_+(P)
\]

with \( P = A_n/C_n \) and \( P_{n, 0} = (C_n - 1)/C_n \). Let \( \hat{P}_{n, 0} \) and \( \hat{P} \) denote the corresponding parameters of the law associated with \( g_n \). Then one can directly see from the relation between \( f_n \) and \( g_n \) that \( 1 - \hat{P} = q(e_{\geq n+1})(1 - P) \) and

\[
\hat{P}_{n, 0} = g_n(0) = \frac{f_n(0)}{q(e_{\geq n})} = \frac{P_{n, 0}}{q(e_{\geq n})}.
\]

Finally use (3) and again (4) to obtain (51) after a little algebra.

(b) As for (54), the argument is again the same as in the ordinary Galton-Watson case after conditioning with respect to \( e \) and \( Z \to \infty \) and fixing an arbitrary individual \( \upsilon \). By regarding its offspring with infinite line of descent, we arrive at a number the (quenched) law of which has indeed the asserted g.f. \( h_n \). In contrast to (a), a look at \( 1/(1 - h_n) \) then also easily provides (53). Further details are therefore omitted. \( \square \)

The following example shows that it is possible to have \( q(e_{\geq n}) = q \) a.s. for all \( n \in \mathbb{N} \) and some \( q \in (0, 1) \) in a truly varying linear fractional environment.

**Example 4.4** Fix an arbitrary \( q \in (0, 1) \) and then a sequence \( e = (A_n, B_n)_{n \geq 1} \) of i.i.d. nonconstant random vectors with generic copy \( (A, B) \) and taking values in \( \mathbb{R}_+ \times \mathbb{R}_+ \) satisfying

\[
\mathbb{P}(B > 0) < 1 \quad \text{and} \quad \frac{A + B - 1}{B} = q \quad \text{a.s.}
\]
It follows that \( A + B = 1 + Bq \geq 1 \) a.s. and \( A = 1 - B(1 - q) \in (0, 1] \) a.s. (Case (C1.2) from Prop. 2.3), thus our standing assumption \( 0 \) is fulfilled. Moreover,
\[
\frac{1}{1 - q} A + B = \frac{1}{1 - q} \quad \text{a.s.}
\]
The last degeneracy property ensures, as it must, that the perpetuity \( R_\infty \) is a.s. constant, namely (see (48))
\[
R_\infty = \sum_{n \geq 1} \Pi_{n-1} B_n = \frac{1}{1 - q} \sum_{n \geq 1} \Pi_{n-1} (1 - A_n) = \frac{1}{1 - q} \quad \text{a.s.}
\]
Writing \( LF(A, B) \) as
\[
LF(A, B) = P_0 \delta_0 + (1 - P_0) Geom_+(P),
\]
we further have
\[
P_0 = \frac{Bq}{1 + Bq} \quad \text{and} \quad P = \frac{B}{1 + Bq} \quad \text{a.s.}
\]
This indicates that the random environment, by means of the random parameter \( B \in [0, 1/(1-q)] \), modulates both the probability for having no offspring \( P_0 \) as well as the tail index \( 1 - P \) of the offspring law, but keeps the extinction probability constant. As a consequence, the g.f.'s \( g_n \) and \( h_n \) in (52) and (54), respectively, of the previous decomposition result also take the much simpler form \( g_n(s) = q^{-1} f_n(qs) \) and \( h_n(s) = (1 - q)^{-1} (f_n(q + (1-q)s) - q) \) for each \( n \geq 1 \) and are thus of the same form as in the ordinary Galton-Watson case. As a further complete analogy, we finally mention that the a.s. limit \( W_\infty \) of the normalized martingale \( W_n = Z_n/\Pi_n, n \geq 0 \), has quenched law (see (50))
\[
q \delta_0 + (1 - q) Exp(1 - q)
\]
which does therefore not depend on the environment \( e \). On the other hand, the latter enters in the normalization of \( Z_n \) as shown.

5 The critical case

Finally, we take a quick look at the critical case when \( R_\infty = R_\infty^{-1} = \infty \) and note first that (19) and (20) are still valid. But unlike the subcritical case, the Markov chain and autoregressive sequence \( M_n := R_n/\Pi_n, n \geq 0 \), which satisfies the recursion
\[
M_n = \frac{1}{A_n} M_{n-1} + \frac{B_n}{A_n}
\]
and figures in the parameters (see (51) and (60) below), is no longer positive recurrent but convergent to $\infty$ in probability $(M_n \overset{d}{=} R_n^{(-1)} \uparrow R_{\infty}^{(-1)} = \infty$ a.s.). Nonetheless the chain may still exhibit two different kinds of behavior depending on whether it be null recurrent or transient. The following theorem reflects this dichotomy as for its consequences for the quenched survival probability and the quenched conditional law of $Z_n$ and its mean given survival. Note that (20) directly implies that $R_n P(Z_n > 0) = M_n / E(Z_n | Z_n > 0)$ for each $n \geq 1$.

**Theorem 5.1** Let $(Z_n)_{n \geq 0}$ be critical and thus $R_{\infty}^{(-1)} = \infty = R_{\infty}$ a.s. Denote by $L = L(\cdot | e)$ the random set of accumulation points of the sequence $(R_n P(Z_n > 0))_{n \geq 1} = (M_n / E(Z_n | Z_n > 0))_{n \geq 1}$, and by $D = D(\cdot | e)$ the random set of accumulation points of $(P(Z_n \in \cdot | Z_n > 0))_{n \geq 1}$ with respect to total variation distance. Then

$$R_n P(Z_n > 0) = \frac{M_n}{E(Z_n | Z_n > 0)} \overset{n \to \infty}{\longrightarrow} 1 \text{ a.s.},$$

(55)

$$P\left(\frac{Z_n}{M_n} \in \cdot | Z_n > 0\right) \overset{w}{\longrightarrow} \text{Exp}(1) \text{ a.s.}$$

(56)

if $(M_n)_{n \geq 0}$ is transient, whereas

$$L = \left\{ \frac{x}{1 + x} : x \in \mathbb{R} \text{ recurrence point of } (M_n)_{n \geq 0} \right\} \text{ a.s.},$$

(57)

$$D = \left\{ \text{Geom}_+\left(\frac{1}{1 + x}\right) : x \in \mathbb{R} \text{ recurrence point of } (M_n)_{n \geq 0} \right\} \text{ a.s.}$$

(58)

if $(M_n)_{n \geq 0}$ is recurrent.

**Proof.** By (19) and (20), we have

$$P(Z_n \in \cdot | Z_n > 0) = \text{Geom}_+\left(\frac{1}{1 + M_n}\right) \text{ a.s.}$$

(59)

and

$$R_n P(Z_n > 0) = \left(1 + \frac{\Pi_n}{R_n}\right)^{-1} \left(1 + \frac{1}{M_n}\right)^{-1} \text{ a.s.}$$

(60)

Further noting that, if $Y(\theta) \overset{d}{=} \text{Geom}_+(\theta)$, then $\theta Y(\theta) \overset{d}{\to} \text{Exp}(1)$ as $\theta \downarrow 0$, all assertions are easily verified.

**Remark 5.2** As for the autoregressive Markov chain $(M_n)_{n \geq 0}$, it must be acknowledged that, unlike positive recurrence, there seems to be no complete classification of null recurrence and transience of that chain in terms of the random parameter $(A, B)$; for the contractive case when $H_n \to 0$ a.s. we mention the work by Buraczewski and Iksanov [3] and by Zerner [27], and for the critical case considered in this section the classical
work by Babillot, Bougerol and Elie [9] and the very recent article by the author with Iksanov [5]. A look at the latter one gives rise to the conjecture that necessary and sufficient conditions are difficult to come by.

**Remark 5.3** Since $\mathbf{P}^{(n:1)}(Z_n \in \cdot | Z_n > 0) = \text{Geom} + ((1 + R_n^{(-1)})^{-1})$ for any $n$ and $R_n^{(-1)} \to \infty$ a.s., the dichotomy encountered in the above theorem disappears under reversal of the environment. Namely,

$$
\mathbf{P}^{(n:1)}\left(\frac{Z_n}{R_n^{(-1)}} \in \cdot \mid Z_n > 0\right) \underset{d}{\to} \text{Exp}(1) \text{ a.s.}
$$

Let us finally touch very briefly on annealed results by taking the survival probability $\mathbb{P}(Z_n > 0)$ as an example. Eqs. (60) and (20) provide

$$
\mathbb{P}(Z_n > 0) = \mathbb{E}\left(\frac{1}{R_n} \left(1 + \frac{1}{M_n}\right)^{-1}\right) = \mathbb{E}\left(\frac{1}{\Pi_n + R_n}\right),
$$

and Kozlov [21] embarked on the last expression, written in the form

$$
\mathbb{E}\left(e^{-S_n} + \sum_{k=1}^{n} e^{-S_{k-1}} B_k\right)^{-1}
$$

(see his Eq. (10)), to show that

$$
n^{1/2} \mathbb{P}(Z_n > 0) \to \beta
$$

for some positive constant $\beta$ under the additional moment conditions

$$
0 < \mathbb{E}\log^2 A < \infty, \quad \mathbb{E}B < \infty \quad \text{and} \quad \mathbb{E}B|\log A| < \infty.
$$

This result was later extended by Geiger and Kersting [14] to general critical GWPRE under corresponding moment conditions. Regarding the behavior of the annealed law of $Z_n$ given $Z_n > 0$ in the linear fractional case, we finally mention that Afanasyev [1, 2] showed the weak convergence of the process $(\Pi_{[nt]}Z_{[nt]})_{0 \leq t \leq 1}$ in the Skorohod space $D([0,1])$ for each $u \in (0,1)$. In view of Theorem 5.1, notably (50), this shows that the normings in quenched and annealed regime are different, namely $M_n = R_n/\Pi_n$ versus $1/\Pi_n = o(M_n)$, when the chain $(M_n)_{n \geq 0}$ is transient. For extensions and further relevant literature, we refer again to [13] Section 5.8.

### 6 Concluding remarks

Being aware that our selection of – essentially known – results might be seen as somewhat arbitrary and therefore cause reservations of readers especially
in the branching process community, we would like to stress once again that we have aimed at offering a different vantage point than earlier publications by adopting a perspective (with respective notation) that is more familiar in the study of random difference equations and their asymptotic properties. By thus putting the focus on the connections of linear fractional GWPRE with these equations, one can observe in a very explicit way how a “breathing” or “fluctuating” environment impacts on a process evolving in it. Needless to say that we could have discussed many more results, and that there is also plenty of room for extensions, like to the multitype setting or even to branching models with interaction. In another direction, Lindo and Sagitov [23], based on the dissertation of the second author [22], have introduced a special class of Galton-Watson processes with explosions, called theta-branching processes, that have similar closure properties as linear fractional branching processes regarding the laws of their marginals. They could therefore be studied in a random environment setting with a similar focus. We refer to future work.

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