Imperfect Best-Response Mechanisms

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joint work with
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Best-response mechanisms [Nisan et al., 2011]

- At each time step, a subset of agents is adversarially chosen
- The selected agents adopt their best-response
- Repeat until the equilibrium has been reached
- Agents utilities/costs are only evaluated at the equilibrium
Best-response mechanisms [Nisan et al., 2011]

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Examples

- BGP
- some TCP variants
- GSP auctions
- Interns-Hospital Matching (IHM)
Convergence & Incentive-Compatibility

Convergence

- The dynamics will eventually converge to a Nash equilibrium
Convergence & Incentive-Compatibility

Convergence

- The dynamics will eventually converge to a Nash equilibrium.

Incentive Compatibility

- If a player does not play the best response whenever is selected, the dynamics will reach a different equilibrium.
- The utility for this player at new equilibrium is lower than in the equilibrium reached by always playing the best response.
NBR-solvable games [Nisan et al., 2011]

NBR-solvable game

- NBR strategy: a strategy that can never be a best-response
- A game solvable by iterated elimination of NBR strategies
NBR-solvable games [Nisan et al., 2011]

NBR-solvable game

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Clear outcome

- A NBR solvable game has clear outcome if for each player \( i \)...
- ...there is a sequence of eliminations of NBR strategies...
- ...such that the equilibrium maximizes the utility of \( i \)...
- ...at the first time that \( i \) eliminate a strategy in this sequence
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BGP, TCP, GSP & IHM are NBR-solvable with clear outcomes
In this work...

Theorem (Nisan et al., 2011)

- If a game is NBR-solvable, then the best-response mechanism converges

- If the NBR-solvable game has a clear outcome, then the best-response mechanism is also incentive-compatible
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- If a game is NBR-solvable, then the best-response mechanism converges
- If the NBR-solvable game has a clear outcome, then the best-response mechanism is also incentive-compatible

Our contribution

- What happen if an agent can sometimes take a wrong action?
- How resistant are these results to small perturbations?
- Are convergence and incentive-compatibility robust?
Imperfect best-response mechanisms

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Imperfect best-response mechanisms

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\( p \)-imperfect best-response mechanism

- At each time step, a subset of agents is chosen by a non-adaptive adversary
- The selected agents adopt their best-response, except with probability \( p \)
- Repeat until the equilibrium has been reached
- Agents utilities/costs are only evaluated at the equilibrium
Does the convergence result hold?
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Obviously, if $p$ is small...
Does the convergence result holds?

Obviously, if $p$ is small...

WRONG!

- Even for $p$ exponentially small in the number of players...
- there is a schedule of players such that for any $t > 0$...
- the $p$-imperfect mechanism is in the equilibrium at time $t$...
- with probability at most $\varepsilon$
Convergence: a negative result

The game

- $n$ players with strategies $s_0$ and $s_1$
- player $i$ prefers strategy $s_1$ only if $1, \ldots, i - 1$ are playing $s_1$
Convergence: a negative result

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The $p$-imperfect mechanism

- if $1, \ldots, i-1$ play $s_1$, player $i$ gets wrong with probability $p$
- otherwise, she gets the wrong strategy with probability $q \ll p$
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Contribution 8
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  - Between two consecutive occurrence of $i$ always appears $j > i$
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  - The length of the sequence is $2^{n-1}$
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  - Between two consecutive occurrence of $i$ always appears $j > i$
  - The length of the sequence is $2^{n-1}$
  - $n$ appears only at the end of the sequence
- if $p = \Omega \left( \frac{1}{2^{n-1}} \right)$ and $q \to 0$, then $n$ always plays $s_0$ w.h.p.
Convergence: a positive result

Convergence is not robust

- For best-response mechanisms, convergence result holds regardless of the schedule
- For $p$-imperfect mechanism, convergence results must depend on the schedule
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- If $p$ is small enough and the game is NBR-solvable...
- then a $p$-imperfect mechanism converges...
Convergence: a positive result

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A positive result

- If $p$ is small enough and the game is NBR-solvable...
- then a $p$-imperfect mechanism converges...
- but the bound on $p$ depends on the schedule
Incentive-compatibility: a negative result

|      | left | right |
|------|------|-------|
| top  | 2,1  | 1,0   |
| bottom | 0,0  | 0,c   |

It is a NBR-solvable game with clear outcome.

If the row player gets wrong with prob. \( p \) and \( c = \Omega(1/p) \), then the column player prefers to play right.

We need a quantitative definition of clear outcome.
Incentive-compatibility: a negative result

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\begin{center}
\begin{tabular}{ccc}
 & left & right \\
\hline
\text{top} & 2,1 & 1,0 \\
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\end{tabular}
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Incentive-compatibility: a positive result

Theorem
A p-imperfect mechanism is incentive-compatible if for each $i$

$$u_i(NE) \geq \frac{1}{1-2\delta} \left( 2\delta \cdot u^*_i + u^k_i \right)$$

- $\delta = \delta(p) > 0$
- $u^k_i$: max utility player $i$ achieves at her first elimination
- $u^*_i$: max utility player $i$ achieves in the entire game

Proof idea.
- If the player follows the $p$-imperfect mechanism...
- ...then she gets $u_i(NE)$
- Otherwise she gets at most $u^*_i$ with prob. depending on $p$...
- ...and she gets at most $u^k_i$ with remaining probability
What happens for larger classes of games?

Different behavior for different schedules

|     | 0   | 1   |
|-----|-----|-----|
| 0   | 1,1 | 0,0 |
| 1   | 0,0 | 1,1 |
What happens for larger classes of games?

Different behavior for different schedules

\[
\begin{array}{cc}
0 & 1 \\
0 & 1,1 & 0,0 \\
1 & 0,0 & 1,1 \\
\end{array}
\]

Different behavior for different best-response mechanisms

\[
\begin{array}{cc}
0 & 1 \\
0 & 0,0 & 0,1 \\
1 & 0,1 & 1,0 \\
\end{array}
\]
Other results

- We try to describe how $p$-imperfect mechanism behave
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- We try to describe how $p$-imperfect mechanism behave
- ... with an application to PageRank games
Thank you!