HARMONIC REDUCTION METHOD
FOR HOMOGENEOUS SECOND ORDER PDE’S

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Abstract. Under this method second order partial differential equations (PDE’s) can be reduced to first order PDE’s, simplifying the Initial value problem IVP or Border value Problem BVP for most cases of second-order differential equation, letting easiest proofs for existence and uniqueness of the PVI and BVP problems.

Algorithm for Harmonic Reduction Method for second order PDE

Theorem 0.1. Let \( \Omega \) a normed space, such that have a non trivial harmonic function \( \Gamma (\Delta \Gamma = 0) \).

Then all second order PDE of the form.

\[
\Delta u = A \cdot \nabla u + B(u)t + Cu
\]

Where \( \Omega \) is an \( n \)-th dimensional normed space.

Then, the PDE can be reduced to a first-order PDE.

Proof. Lets define \( F_\Omega : \Omega^* \rightarrow \Omega^* \), where \( \Omega^* \) is the dual space related to \( \Omega \). Such that \( F_\Omega(u_i) = A \cdot \nabla u + B(u)t + Cu - \Delta u = 0 \), describe in (0.1), such that the studied function is unvariant under this functional, then lets calculate \( F_\Omega(\Gamma u_i) \).

\[
F_\Omega(\Gamma u_i) = \Gamma (A_\Gamma \cdot \nabla u + B_\Gamma(u)t + C_\Gamma u) + u (A_\Gamma \cdot \nabla \Gamma + B_\Gamma \Gamma_t) - \Delta u - 2\nabla \Gamma \cdot \nabla u
\]

\[
= \Gamma ([A_\Gamma - A] \cdot \nabla u + [B_\Gamma - B] (u)t + [C_\Gamma - C] u) + u (A_\Gamma \cdot \nabla \Gamma + B_\Gamma \Gamma_t) - 2\nabla \Gamma \cdot \nabla u
\]

Therefore would be necessary to define the harmonic perturbator as follows.

\[
P(\Phi) = \Phi_\Gamma - \Phi
\]

With this definition, the last result gets reduce to the following after some algebraic steps.

\[
F_\Omega(\Gamma u) = \Gamma [P(A) \cdot \nabla u + P(B)(u)t + P(C)u] + u [A_\Gamma \cdot \nabla \Gamma + B_\Gamma \Gamma_t] - 2\nabla \Gamma \cdot \nabla u
\]

It’s important to notice, that \( F_\Omega(u \Gamma) = 0 \), therefore.

\[
2\nabla \Gamma \cdot \nabla u = \Gamma [P(A) \cdot \nabla u + P(B)(u)t + P(C)u] + u [A_\Gamma \cdot \nabla \Gamma + B_\Gamma \Gamma_t]
\]
Notice that under this arrangement equation (0.2) is first order PDE depends on the existence of the \( u \). Therefore the order of the PDE has been reduced as the theorem suggests as a consequence that harmonic function \( \Gamma \) is known and chosen.

This result is valid under the use of fixed point theorem for compact mapping funded on the locality of the harmonic function for infinitesimal balls centered at an arbitrary \( x_0 \).

From this proof would be necessary to understand the behavior of the harmonic perturbator, such that can be extended to a diversity of IVP/BVP linked to the PDE’s, such as are described in the **Theorem (0.1)**.

**Definition 0.2.** Harmonic disruptors are the coefficient function that are describe in equation (0.1) that can on the function \( u \).

To distinguish a harmonic disrupted function under the harmonic function \( \Gamma \) will be described by the subscript associated with the harmonic function before the direction index, for abstract analysis for this type of PDE.

**Definition 0.3.** Harmonic perturbation is defined as the difference between the disrupted function and the function without being disrupted by the function \( \Gamma \), define as \( P \).

\[
P(A_i) = A_{\Gamma_i} - A_i
\]

**Theorem 0.4.** If the PDE is linear, then, the harmonic perturbations are null.

*Proof.* Notice that if the PDE is linear then the harmonic disruptor does not depend on \( u \) or any of its derivatives therefore does holds that.

\[
A_{\Gamma_i} = A_i
\]

That leads to the conclusion \( P(A_i) = 0 \)

Notice that under this reduction method the nature of the PDE remains, this means if it’s a linear system remains as a linear system, if it’s nonlinear, remains as well.

Notice that coordinate functions are Harmonic functions, therefore using this method will imply in the partial derivatives along each coordinate, also notice that if this coordinate function is multiplied with a temporal variable, still a harmonic function, being the next step under this method.

**Theorem 0.5.** For Harmonic functions such as the following.

\[
\Gamma = f(t)x_i
\]

under the harmonic reduction method, the behavior of the total derivative is determined for a linear PDE.

*Proof.* Recaling from **Theorem 0.1** equation (0.2), does hold for the harmonic define for this theorem the following.

\[
2f(t)\frac{\partial u}{\partial x_i} = f(t)x_i [P(A) \cdot \nabla u + P(B)(u) + P(C)u] + u [f(t)A_{\Gamma} \cdot \hat{e}_i + B_{\Gamma} \Gamma_i]
\]
Now as a consequence that the PDE by hypothesis is linear, does imply that the harmonic perturbators are null, therefore implying.

\[ 2f(t) \frac{\partial u}{\partial x_i} = u \left[ f(t)A \cdot \hat{e}_i + Bx_i \frac{\partial f(t)}{\partial t} \right] \]

Equivalent to the following.

\[ \frac{\partial u}{\partial x_i} = u \left[ \frac{A}{2} + \frac{B}{2f(t)} \frac{\partial f(t)}{\partial t} \right] \]

Notice that this holds for any coordinate, therefore by changing the coordinate does imply information for the gradient.

\[ \nabla u = u \left[ \frac{A}{2} + \frac{B}{2f(t)} \frac{\partial f(t)}{\partial t} \right] \]

Now by calculating the divergence of this gradient does imply.

\[ \Delta u = u \left[ \frac{B}{2f(t)} \frac{\partial f(t)}{\partial t} \right]^2 + u \left[ \frac{A}{2} + \frac{B}{2f(t)} \frac{\partial f(t)}{\partial t} \right] \]

With the information related to equations (0.3) and (0.4) on the PDE does imply.

\[ \frac{\partial u}{\partial t} = u \left[ \frac{nB}{2f(t)} \frac{\partial f(t)}{\partial t} + \nabla \cdot \frac{A}{2} + \frac{B}{2f(t)} \frac{\partial f(t)}{\partial t} \cdot \frac{r}{r} \right] \]

(0.5)

Now with all this information using equations (0.3) and (0.5)

\[ \frac{1}{u} \frac{du}{dt} = u \left[ \frac{n}{2f(t)} + \frac{B}{2f(t)} \frac{\partial f(t)}{\partial t} + \frac{A}{2} + \frac{B}{2f(t)} \frac{\partial f(t)}{\partial t} \cdot \frac{r}{r} \right] \]

to finish the proof would be necessary to determine the behavior of \( f \), for this reason, it will be done the time integral as suggest the total differential.

\[ \ln \left| \frac{u(t, \vec{r})}{u_0(\vec{r})} \right| = \frac{n}{2} \ln \left| \frac{\partial f(t)}{f(0)} \right| + \int_0^t \left[ B \left( \frac{1}{f(s)} \frac{\partial f(s)}{\partial s} \right)^2 - \frac{\partial}{\partial s} \left( \frac{B}{f(s)} \frac{\partial f(s)}{\partial s} \right) \right] \frac{r^2}{4} ds \]

\[ + \frac{B}{4f(s)} \frac{\partial f(s)}{\partial s} r^2 \bigg|_{s=0}^{s=t} + \int_0^t \frac{A}{2} \frac{dr}{ds} - \frac{1}{B} \left( C + \nabla \cdot \frac{A}{2} f(s) \right) \frac{ds}{ds} \]

Therefore by setting \( f \), such that solves the following ODE.

\[ \frac{\partial}{\partial s} \left[ \frac{B}{f(s)} \frac{\partial f(s)}{\partial s} \right] - B \left( \frac{1}{f(s)} \frac{\partial f(s)}{\partial s} \right)^2 = 0 \]

With initial condition \( f(0) = 1 \) and \( (\partial f/\partial s)(0) = K \), does imply for the integral result of the total differential becomes the following implying a solution for \( u \).
Therefore as a consequence that the method does give an explicit solution for the PDE, the total differential is fully determined by differentiating over each coordinate, finishing the proof.

Also Notice that under Minkowsky metric over a complex vector field, this method does not work for the d’Alambertian operator as consequence of the reduction under the four-vector method Being solvable with a planar wave anzat, implying the existence of the temporal component is essential on building this method to finding solutions for dissipative PDE’s. Also because of the locality of the harmonic this method can generate a contractive mapping upon the domain space for the homogeneous case of the PDE.

As you the reader can imagine for nonlinear PDE’s would be more complex to analyze as a consequence the Harmonic Perturbators influences that should count on considerations.

1. Examples

Let’s do some examples, to see how this method works.

**Example 1.1.** Let’s find the general solution of the Heat Equation, known as well as the diffusion equation.

\[ \Delta u = \alpha \frac{\partial u}{\partial t} \]

With initial and border conditions as follows.

\[ u(\vec{r}, 0) = u_0(\vec{r}) \]
\[ u(\vec{r}, t) = 0 \quad \forall \vec{r} \in \partial \mathbb{R}^n \]

By applying the Harmonic reduction method, the solution for the PDE is the following as a consequence of **Theorem 0.5**, letting \( A = 0, B = \alpha, \) and \( C = 0. \)

\[ u = u_0 \sqrt{f(t)} \exp \left\{ \frac{B}{4f(s)} \frac{\partial f(s)}{\partial s} \bigg|_{s=0}^{s=t} + \int_0^t A \frac{d\vec{r}}{ds} - \frac{1}{B} \left( C + \nabla \cdot \frac{A}{2} \right) ds \right\} \]

Where \( f \) is the solution for the following ODE.

\[ \frac{\partial}{\partial t} \left[ \frac{1}{f(t)} \frac{\partial f(t)}{\partial t} \right] = \left( \frac{1}{f(t)} \frac{\partial f(t)}{\partial t} \right)^2 \]

Where the solution of the ODE is the following.

\[ f(t) = \frac{1}{1 - Kt} \]

For \( K < 0, \) this way the function \( f \) is fully determined for all \( t > 0, \) this implies that the solution becomes.
The only thing left to do is adjust $K$ such the PDE does hold.

**Example 1.2.** Let’s take the following PDE.

$$\Delta u = \vec{r} \cdot \nabla u - (t + 1) \frac{\partial u}{\partial t}$$

This PDE, still a linear PDE but allows to see properly the coefficient $A = r$, $B = -(t + 1)$ and $C = 0$, therefore as consequence of Theorem (0.5), does imply.

$$u = u_0 \left( \frac{f(t)}{t + 1} \right)^n \exp \left\{ \frac{r^2}{4} - t + 1 \frac{\partial f(s)}{f(s)} \bigg|_{s=0} \right\}$$

Also is known that the ODE related to $f$ is the following.

$$\frac{\partial}{\partial s} \left[ \frac{s + 1}{f(s)} \frac{\partial f(s)}{\partial s} \right] - (s + 1) \left( \frac{1}{f(s)} \frac{\partial f(s)}{\partial s} \right)^2 = 0$$

equivalent to.

$$\frac{\partial}{\partial s} \left[ \frac{1}{f(s)} \frac{\partial f(s)}{\partial s} \right] = \left[ \frac{1}{f(s)} \frac{\partial f(s)}{\partial s} \right]^2 - \frac{1}{s + 1} \left[ \frac{1}{f(s)} \frac{\partial f(s)}{\partial s} \right]$$

To visualize the ODE will be using the change of variable $\phi = f^{-1} \frac{\partial f}{\partial t}$, therefore.

$$\frac{\partial \phi}{\partial s} = \phi^2 - \frac{\phi}{s + 1}$$

Notice that this equation can be solved by the change of variable $z = \phi^{-1}$, this does imply.

$$\frac{\partial z}{\partial s} \frac{z}{s + 1} = -1$$

this imply that the $f$ solution for the ODE use in Theorem (0.5) is the following.

$$f(t) = \frac{K}{K - \ln |t + 1|}$$

Therefore the solution for the PDE becomes.

$$u = u_0 \left( \frac{K}{(t + 1)(K - \ln |t + 1|)} \right)^n \exp \left\{ \frac{r^2}{4} \left( 1 + \frac{1}{\ln |t + 1| - K} \right) \bigg|_{s=0} \right\}$$

Notice that even with all this information PDE’s of this type also require the behavior of the position $r$ over the time variable, therefore for this PDE is needed
more information to allow a complete solution, regardless of this still gives full view upon the behavior of the function $u$ in terms of the coordinate system, time and a fixed value $K < 0$. But for static $r$ does hold.

$$u = u_0 \sqrt{\left( \frac{K}{(t + 1)(K - \ln |t + 1|)} \right)^n} \exp \left\{ \frac{\ln |t + 1|}{\ln |t + 1| - K} \frac{r^2}{4K} \right\}$$

2. Some aditional considerations

If the function $f$ that describe temporal behavior does not exist, does not imply that the function $u$ does not exist, therefore more information is needed.

3. How does this method was inspired.

This method was inspired by many humanities disciplines as art (realism [selection of colors]) and music (instrument tuning), therefore was necessary to find functions that have special properties such as compact mapping inside an infinitesimal ball, also under this process has to give a reference about the local behavior of the PDE in lower order PDE’s, but most important was needed the existence of invariant functionals where the PDE does hold, that was the main reason for the use for harmonic functions.

4. Applications

This method has applications to do proofs on existence and uniqueness for second order PDE’s of the form described in Theorem (0.1), doing reference to diffusive systems, giving the information about $u$ in terms of the harmonic function, that has a link among coordinate systems, working from the gradient and the total derivative.

5. Summary of Harmonic reduction method

The method consists of 3 main steps.

I Define $F$ invariant in terms of the PDE.

II Evaluate $u\Gamma$ on the function $F$.

III Find the Behavior of the partial differentials in terms of the harmonic $\Gamma$, going in the following order:

(a) Spatial first order partial derivatives of the function $u$.

(b) Laplacian of the function $u$

(c) temporal partial derivative of the function $u$

After following this 3 main step would be enough information to determine the total derivative of the function $u$ implying an explicit solution for the PDE. This explicit solution would have a function $f$ defining the temporal behavior of the main PDE, under this assumption the PDE problem becomes an ODE problem by working with equation (0.6).
6. Why does work?

Does work as a consequence of the Green Identities, in particular as a consequence of the first identity.

\[ \int_S u \nabla \Gamma \cdot d\vec{S} = \int_V \nabla u \cdot \nabla \Gamma dV \]
\[ \int_S \Gamma \nabla u \cdot d\vec{S} = \int_V \nabla u \cdot \nabla \Gamma dV + \int_V \Gamma \Delta u dV \]

By adding up this equation.

\[ \int_S \nabla (u \Gamma) \cdot d\vec{S} = \int_V \Delta (u \Gamma) dV \]

And by the definition of the function, \( F \) does hold.

\[ \int_S (\Gamma \nabla u + u \nabla \Gamma) \cdot d\vec{S} = \int_V \left( A \Gamma \cdot \nabla (\Gamma u) + B \Gamma \frac{\partial (\Gamma u)}{\partial t} + C \Gamma u \Gamma \right) dV \]

And by comparing this result upon the second Green Identity.

\[ \int_S (\Gamma \nabla u - u \nabla \Gamma) \cdot d\vec{S} = \int_V \Gamma \left( A \cdot \nabla u + B \frac{\partial u}{\partial t} + C u \right) dV \]

Notice that the difference between these two implies.

\[ 0 = \int_V \left( \frac{P(A) \cdot \nabla u}{u} + \frac{P(B)}{u} \frac{\partial u}{\partial t} + P(C) + \frac{A \Gamma \cdot \nabla \Gamma}{\Gamma} + \frac{B \Gamma \partial \Gamma}{\Gamma \partial t} - 2 \frac{\nabla u \cdot \nabla \Gamma}{u \Gamma} \right) dV \]

Which is equivalent to the expression shown in equation 0.2, but differentials are much appealing to the sight of the reader for mistake checking, instead of the integral approach of Theorem (0.1).

References

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