Exterior Isoclinism of Crossed Modules

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Abstract

In this paper, we define the notion of exterior isoclinism of crossed modules. Functions for computing with these structures have been written using the GAP computational discrete algebra programming language.

Keywords: exterior isoclinism, crossed module, GAP

MSC Classification: 18C40, 18G50.

1 Introduction

Isoclinism was introduced by [16] as a classification of prime power groups which is an equivalence relation and weaker than isomorphism. The idea based on isomorphism of central quotients and commutators rather than isomorphism of whole groups. The notion of that was studied in [15] with details. Additionally, the relation between isoclinism families and stem groups studied in [5]. For some related works, we refer [23, ?].

The notion of crossed module, generalizing the notion of a G-module, was introduced by Whitehead [25] in the course of his studies on the algebraic structure of the second relative homotopy group. Crossed modules are algebraic objects with rich structure and considered as “2-dimensional groups” [8]. They provide a simultaneous generalization of the concepts of normal subgroups and modules over a group. Furthermore we may regard any group as a crossed module. It is therefore of interest to seek generalizations of group theoretic concepts and structures to crossed modules.

A share package XMod [1], for the GAP [24], computational discrete algebra system was described by C.D. Wensley et al which contains functions for computing crossed modules of groups and cat1-groups and their morphisms. Later, the algebraic version of a GAP package XModAlg [2] was given by Arvasi and Odabas (see [3]). The notions of isoclinism and stem group generalized to crossed modules as 2-dimensional groups and computer implementations of these notions for GAP given in [21]. Also, n-isoclinism classes of crossed modules are defined in [22].

In considering the tensor and exterior products of (non-abelian) groups of Brown and Loday we see that their definitions have close connections with universal commutator relations in a group. In fact a special case of the exterior product of two groups was discovered

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by Miller [18] in her group theoretic interpretation of the second homology of a group with integer coefficients. The notion the exterior isoclinism of groups which yields a new classification on the class of all groups was introduced in [14].

The motivation of this paper lies in [10] and [21], where exterior square, exterior center and isoclinism of crossed modules are studied. In this paper we describe a new equivalence relation named “exterior isoclinism” for crossed modules similar to isoclinism using the non-abelian exterior product, exterior square and exterior center defined in [7, 10]. Additionally, we give computer implementations to determine exterior isoclinism families of small groups and crossed modules for GAP.

The work organized as follows:
In section 2, we will recall basic notions such as isoclinism of groups and crossed modules. In section 3, we will recall the notion of exterior isoclinism of groups. In section 4, we will introduce the notion of exterior isoclinism of crossed modules. In section 5, we will give functions for computing with these structures have been developed using the GAP computational discrete algebra programming language.

2 Preliminaries

In this section we will recall the notions of isoclinism with basic properties (see [16, 21]). In the sequel of the work, we assume all groups to be finite.

2.1 Isoclinism of groups

In general, the isoclinism is used for classification of finite groups, and there are many works concerning the enumeration of groups with finite order related to isoclinism [23, ?].

Definition 2.1. [16] Let $M$ and $N$ be groups; a pair $(\mu, \zeta)$ is termed an isoclinism from $M$ to $N$ if:

1. $\mu : M/Z(M) \rightarrow N/Z(N)$ is an isomorphism from to between central quotients;
2. $\zeta : [M, M] \rightarrow [N, N]$ is an isomorphism between derived subgroups;
3. the diagram

$$
\begin{array}{ccc}
M/Z(M) \times M/Z(M) & \xrightarrow{\zeta \times \mu} & [M, M] \\
\downarrow{\mu \times \mu} & & \downarrow{\zeta} \\
N/Z(N) \times N/Z(N) & \xrightarrow{c_N} & [N, N]
\end{array}
$$

commutative where $c_M, c_N$ are commutator maps.

If there is an isoclinism from $M$ to $N$, we shall say that $M$ and $N$ are isoclinic groups and denoted by $M \sim N$. It is well-known that isoclinism is an equivalence relation.
Examples 1.
(1) Isomorphic groups are also isoclinic.
(2) All abelian groups are isoclinic to each other. The pairs \((\mu, \zeta)\) consist of trivial homomorphisms.
(3) A stem group \(G\) is the group satisfying \(Z(G) \leq G'\). In every non-abelian isoclinism family there exist a stem group and is the lowest order group in the family. See [16] for details. Consider the isoclinism family of \(C_4 \ltimes C_8\), the stem group in the family is \(Q_8\) with order 8 which is the lowest order group in the family.

Gustafson [13] considered “what is the probability that two group elements commute?” The answer is given by what is known as the commutative degree of a group. Let \(G\) be a finite group, the commutative degree of \(G\) is denoted by \(d(G)\) and defined by

\[
d(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}
\]

Obviously, \(G\) is abelian if and only if \(d(G) = 1\); furthermore, the following results are given in [17]:

1. If \(G\) is abelian, then \(d(G) \leq \frac{5}{8}\)
2. If \(d(G) > \frac{1}{2}\), then \(G\) is nilpotent
3. If \(d(G) = \frac{1}{2}\) and \(G\) is not nilpotent, then the derived subgroup of \(G\) is isomorphic to cyclic group of order 3.

Moreover as an important result, Lescot proved that every isoclinic finite groups has same commutative degree (see [17] for details).

2.2 Isoclinism of crossed modules

A crossed module consists of a group homomorphism \(\partial : S \to R\), endowed with a left action \(R\) on \(S\) (written by \((r, s) \to rs\) for \(r \in R\) and \(s \in S\)) satisfying the following conditions:

\[
\begin{align*}
\partial(rs) &= rs(\partial s)^{-1}, & \forall s \in S, r \in R; \\
(\partial s_2)s_1 &= s_2s_1s_2^{-1}, & \forall s_1, s_2 \in S.
\end{align*}
\]

The first condition is called the pre-crossed module property and the second one the Peiffer identity. We will denote such a crossed module by \(\mathcal{X} = (\partial : S \to R)\). \(\mathcal{X}\) is called to be finite if the source and range groups are both finite. In this case, we define size of \(\mathcal{X}\) to be the ordered pair \((|S|, |R|)\).

A morphism of crossed modules \((\sigma, \rho) : \mathcal{X}_1 \to \mathcal{X}_2\), where \(\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)\) and \(\mathcal{X}_2 = (\partial_2 : S_2 \to R_2)\), consists of two group homomorphisms \(\sigma : S_1 \to S_2\) and \(\rho : R_1 \to R_2\) such that

\[
\partial_2 \circ \sigma = \rho \circ \partial_1, \quad \text{and} \quad \sigma(rs) = (\rho r)\sigma s \quad \forall s \in S, r \in R.
\]

Standard constructions for crossed modules include the following.
1. Any group $G$ can be regarded as a crossed module in two obvious ways. We can take $\xymatrix{1 \ar@{^{(}->}[r]^\text{inc} & R}$ with the inclusion map or $\xymatrix{R \ar[r]^\text{id} & R}$ with the identity map and action by conjugation.

2. A conjugation crossed module is an inclusion of a normal subgroup $N \trianglelefteq R$, where $R$ acts on $N$ by conjugation.

3. If $1 \rightarrow N \rightarrow S \xymatrix{\rightarrow & R \rightarrow 1}$ is a central extension of groups, then $S \xymatrix{\rightarrow & R}$ is a crossed module with the action of $R$ on $S$ via lifting and conjugation.

4. The direct product of $X_1$ and $X_2$ is $X_1 \times X_2 = (\partial_1 \times \partial_2 : S_1 \times S_2 \rightarrow R_1 \times R_2)$ is a crossed module with direct product action $(r_1,r_2)(s_1,s_2) = (r_1s_1, r_2s_2)$.

Let $\mathcal{X} = (\partial : S \rightarrow R)$ be a crossed module. We get following subgroups of $S$ and $R$ respectively

\[ S^R = \{ s \in S : r = s, \forall r \in R \} \]
\[ \text{St}_R(S) = \{ r \in R : s = s, \forall s \in S \} . \]

Clearly $S^R$ is in fact a subgroup of $Z(S)$ (center of $S$), called fixed point subgroup of $S$. Note that $\text{st}_R(S)$ is kernel of the homomorphism from $R$ to $Aut(S)$ which defines the action of $\mathcal{X}$. $\text{st}_R(S)$ is called stabilizer subgroup of $R$. If the action of $R$ on $S$ is faithful, that is if $\text{st}_R(S) = 1$ then $\mathcal{X}$ called is faithful crossed module. Also $D_R(S) = \langle \{ r^ss^{-1} : s \in S, r \in R \} \rangle$ is the subgroup of $S$ and $D_R(S)$ is called the displacement subgroup.

**Definition 2.2.** Let $\mathcal{X} = (\partial : S \rightarrow R)$ be a crossed module. Then

\[ Z(\mathcal{X}) : S^R \xymatrix{\rightarrow & Z(R) \cap \text{st}_R(S)} \]

is a crossed module. $Z(\mathcal{X})$ is called center of the crossed module $\mathcal{X}$.

$\mathcal{X} = (\partial : S \rightarrow R)$ is abelian if and only if $R$ is abelian and acts trivially on $S$, which implies that $S$ is also abelian. With this in mind,

\[ \mathcal{X}' = D_R(S) \xymatrix{\rightarrow & R'} \]

is commutator subcrossed module of $\mathcal{X}$ where $R' = [R, R]$ is the commutator subgroup of $R$. (see [20])

**Definition 2.3.** Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be finite crossed modules; two pair $(\mu_1, \mu_0)$ and $(\zeta_1, \zeta_0)$ are termed an isoclinism of crossed modules from $\mathcal{X}_1$ to $\mathcal{X}_2$ if:

1. $(\mu_1, \mu_0)$ is an isomorphism from $\mathcal{X}_1/(Z(\mathcal{X}_1))$ to $\mathcal{X}_2/(Z(\mathcal{X}_2))$;
2. $(\zeta_1, \zeta_0)$ is an isomorphism from $\mathcal{X}_1'$ to $\mathcal{X}_2'$;
3. the diagram

\[ \begin{array}{ccc}
\mathcal{X}_1/(Z(\mathcal{X}_1)) \times \mathcal{X}_1/(Z(\mathcal{X}_1)) & \xymatrix{\rightarrow & \mathcal{X}_1'} \\
\mathcal{X}_2/(Z(\mathcal{X}_2)) \times \mathcal{X}_2/(Z(\mathcal{X}_2)) & \xymatrix{\rightarrow & \mathcal{X}_2'}
\end{array} \]

\[ \xymatrix{\xymatrix{\ar[r]^{(\mu_1,\mu_0) \times (\mu_1,\mu_0)} & \ar[r]^{(\zeta_1,\zeta_0)} & \} \}

is commutative.
If there is an isoclinism of crossed modules from \( X_1 \) to \( X_2 \), we shall say that \( X_1 \) and \( X_2 \) are isoclinic crossed modules.

**Examples 2.**

1. All abelian crossed modules are isoclinic. The pairs \((\mu_1, \mu_0), (\zeta_1, \zeta_0)\) consist of trivial homomorphisms.
2. Let \((\mu, \zeta)\) be an isoclinism from \( M \) to \( N \). Then \( X_1 = (\text{id}_M : M \to M) \) is isoclinic to \( X_2 = (\text{id}_N : N \to N) \) where \((\mu_1, \mu_0) = (\mu, \mu)\) and \((\zeta_1, \zeta_0) = (\zeta, \zeta)\).
3. Let \( M \) be a group and let \( N \) be a normal subgroup of \( M \) with \( N\Z(M) = M \). Then \( N \hookrightarrow M \) is isoclinic to \( M \xrightarrow{id} M \). Here \((\mu_1, \mu_0)\) and \((\zeta_1, \zeta_0)\) are defined by \((\text{inc}, \text{inc}), (\text{id}_{G_1}, \text{id}_{G_0})\), respectively.

## 3 Exterior isoclinism of groups

Hakima and Jafari [14] introduced the exterior isoclinism of finite groups. We now recall that the non-abelian exterior product of groups from [10].

Let \( M, N \) be normal subgroups of a group \( G \). The non-abelian exterior product of \( M \) and \( N \) is the group \( M \wedge N \) generated by the elements \( m \wedge n \) with \((m, n) \in M \times N\), subject to relations
\[
mm' \wedge n =^m (m' \wedge n)(m \wedge n), \\
m \wedge nn' = (m \wedge n)^n(m \wedge n'), \\
m \wedge n = 1 \text{ whenever } m = n
\]
where by definition \( x(y \wedge z) = (xy^x) \wedge (zx^{x^{-1}}) \) and conjugation \( xy = yx^{-1} \) is taken in the group \( G \) (see [10]). A more general construction \( M \wedge N \) is given in [6] for arbitrary crossed \( G \)-modules \( M, N \). The exterior square of a group \( G \) (sometimes also called the non-abelian exterior square), denoted by \( G \wedge G \).

The exterior product \( M \wedge N \) can also be defined by its universal property. Given a group \( H \) and a function \( h : M \times N \to H \), we say that \( h \) is an exterior pairing if for all \( m, m' \in M, n, n' \in N \),
\[
h(mm', n) = h(mm', n)h(m, n), \\
h(m, mm') = h(m, n)h(m', n'), \\
h(m, n) = 1 \text{ whenever } m = n
\]
then \( h \) is an exterior pairing from \( M \times N \) into \( H \). The exterior center of a group defined by [11] as:
\[
Z \wedge (G) = \{ g \in G : 1 = g \wedge x \in G \wedge G \text{ for all } x \in G \}
\]

**Definition 3.1.** [14] Let \( M \) and \( N \) be groups; a pair \((\mu, \zeta)\) is termed an exterior isoclinism from \( M \) to \( N \) if:

1. \( \mu \) is an isomorphism from \( \mu : M/\Z \wedge (M) \) to \( N/\Z \wedge (N) \);
2. \( \zeta : M \wedge M \to N \wedge N \) is an isomorphism between exterior squares;
3. the diagram

\[
\begin{array}{ccc}
M/Z \wedge (M) \times M/Z \wedge (M) & \xrightarrow{h_M} & M \wedge M \\
\downarrow \mu \times \mu & & \downarrow \zeta \\
N/Z \wedge (N) \times N/Z \wedge (N) & \xrightarrow{h_N} & N \wedge N
\end{array}
\]

commutative where \(h_M, h_N\) are universal exterior pairings.

If there is an exterior isoclinism from \(M\) to \(N\), we shall say that \(M\) and \(N\) are exterior isoclinic groups and denoted by \(M \approx N\).

**Proposition 3.2.** Exterior isoclinism is an equivalence relation.

**Examples 3.**
1. Isomorphic groups are also exterior isoclinic.
2. All cyclic groups are exterior isoclinic to each other.
3. \(S_3 \approx C_3 \rtimes C_4\)

**Corollary 3.3.** There are no relation between isoclinism and exterior isoclinism. For example \(D_{16} \sim Q_{16}\) but they are not exterior isoclinic. Conversely, \(C_4 \rtimes C_4 \approx Q_{16}\) but they are not isoclinic. (please see Table I and Table II) But if they are cyclic then every two groups are both isoclinic and exterior isoclinic to each other.

**Theorem 3.4.** [14] Let \(G\) be any finite group and \(H \leq G, N \trianglelefteq G\).

(i) If \(G = HZ \wedge (G)\), then \(G \approx H\). The converse of this condition is true when \(H \cap Z \wedge (G) = 1\).

(ii) \(G/N \approx G\) if and only if \(N \leq Z \wedge (G)\).

### 3.1 Exterior Degrees of Finite Groups

Niroomand and Rezaeiv [19] introduced the exterior degree of a finite group \(G\) to be the probability for two elements \(x\) and \(y\) in \(G\) such that \(x \wedge y = 1\) and wrote a simple algorithm to compute the exterior degree for some small groups. Let \(G\) be a finite group, the exterior degree of \(G\) is denoted by \(d \wedge (G)\) and defined by

\[
d \wedge (G) = \frac{|\{(x, y) \in G \times G : x \wedge y = 1\}|}{|G|^2}
\]

**Proposition 3.5.** Every exterior isoclinic finite groups has same exterior degree.

*Proof.* If \(M\) and \(N\) are two exterior isoclinic finite groups, then we will show that \(d \wedge (M) = d \wedge (N)\).
Let \((\mu, \zeta)\) be an exterior isoclinism from \(M\) to \(N\); one gets
\[
|M/Z \wedge (M)|^2 d \wedge (M) = \frac{1}{|Z \wedge (M)|^2} |M|^2 d \wedge (M)
= \frac{1}{|Z \wedge (M)|^2} |\{(x, y) \in M \times M : x \wedge y = 1\}|
= \frac{1}{|Z \wedge (M)|^2} |\{(x, y) \in M \times M : h_M(xZ \wedge (M), yZ \wedge (M)) = 1\}|
= |\{(\alpha, \beta) \in M/Z \wedge (M) \times M/Z \wedge (M) : \zeta(h_M(\alpha, \beta)) = 1\}|
= |\{(\alpha, \beta) \in M/Z \wedge (M) \times M/Z \wedge (M) : h_N(\mu(\alpha), \mu(\beta)) = 1\}|
= |\{(\gamma, \delta) \in N/Z \wedge (N) \times N/Z \wedge (N) : h_N(\gamma, \delta) = 1\}|
= |N/Z \wedge (N)|^2 d \wedge (N)
\]
thus we have \(d \wedge (M) = d \wedge (N)\). \qed

**Definition 3.6.** An exterior stem group is a group whose exterior center is contained in its exterior square. In other words a group \(G\) is an exterior stem group if \(M/Z \wedge (M) \leq M \wedge M\).

Every group is exterior isoclinic to an exterior stem group, but distinct exterior stem groups may be exterior isoclinic.

### 4 Exterior isoclinism of crossed modules

In this section we define exterior center and exterior commutator sub-crossed modules to introduce exterior isoclinism of crossed modules.

**Definition 4.1.** Let \(\mathcal{X} = (\partial : S \to R)\) be a crossed module. Then
\[
Z \wedge (\mathcal{X}) = (\partial : S^R \to St_R(S) \cap Z \wedge (R))
\]
is a crossed module. \(Z(\mathcal{X})\) is called exterior center of the crossed module \(\mathcal{X}\). Moreover,
\[
\mathcal{X} \wedge \mathcal{X} = (\partial : D_R(S) \to R \wedge R)
\]
is exterior derived sub-crossed module of the crossed module \(\mathcal{X}\).

**Proposition 4.2.** Given a normal sub-crossed module \(\mathcal{X}_1 = (\partial : S_1 \to R_1)\) of a crossed module \(\mathcal{X} = (\partial : S \to R)\), there is a quotient crossed module
\[
\mathcal{X}/\mathcal{X}_1 = (\bar{\partial} : S/S_1 \to R/R_1)
\]
where the action is defined by
\[
r_{R_1}(sS_1) := (\bar{s})S_1
\]
and the boundary map is given by
\[
\bar{\partial}(sS_1) := (\partial s)R_1.
\]
Therefore
\[
\mathcal{X}/Z \wedge (\mathcal{X}) = (\bar{\partial} : S/S^R \to R/(St_R(S) \cap Z \wedge (R)))
\]
is exterior central quotient crossed module.
**Proposition 4.3.** Let $\mathcal{X} = (\partial : S \to R)$ be a crossed module. Define the maps
\[
\sigma : S/S^R \times R/(St_R(S) \cap Z \wedge (R)) \to D_R(S) \\
(ss^{R}, r(St_R(S) \cap Z \wedge (R))) \mapsto \theta_{SS^{-1}}
\]
and
\[
\omega : R/(St_R(S) \cap Z \wedge (R)) \times R/(St_R(S) \cap Z \wedge (R)) \to R \wedge R \\
(r(St_R(S) \cap Z \wedge (R)), r'(St_R(S) \cap Z \wedge (R))) \mapsto r \wedge r',
\]
for all $s \in S$, $r, r' \in R$. Then the maps $\sigma$ and $\omega$ are well-defined.

**Proof.** It can be easily checked by a similar way of Proposition 1 in [21].

**Definition 4.4.** Let $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ and $\mathcal{X}_2 = (\partial_2 : S_2 \to R_2)$ be crossed modules; two pair $(\mu_1, \mu_0)$ and $(\zeta_1, \zeta_0)$ are termed an exterior isoclinism of crossed modules from $\mathcal{X}_1$ to $\mathcal{X}_2$ if:

1. $(\mu_1, \mu_0)$ is an isomorphism from $(\bar{\partial}_1 : S_1/S_{1}^{R_1} \to R_1/(St_{R_1}(S_1) \cap Z \wedge (R_1)))$ to $(\bar{\partial}_2 : S_2/S_{2}^{R_2} \to R_2/(St_{R_2}(S_2) \cap Z \wedge (R_2)))$;
2. $(\zeta_1, \zeta_0)$ is an isomorphism from $(\partial_1 : D_{R_1}(S_1) \to R_1 \wedge R_1)$ to $(\partial_2 : D_{R_2}(S_2) \to R_2 \wedge R_2)$;
3. the diagrams

\[
\begin{array}{ccc}
S_1/S_1^{R_1} \times R_1/(St_{R_1}(S_1) \cap Z \wedge (R_1)) & \xrightarrow{\sigma_1} & D_{R_1}(S_1) \\
\downarrow{\mu_1 \times \mu_0} & & \downarrow{\zeta_1} \\
S_2/S_2^{R_2} \times R_2/(St_{R_2}(S_2) \cap Z \wedge (R_2)) & \xrightarrow{\sigma_2} & D_{R_2}(S_2)
\end{array}
\tag{1}
\]

and

\[
\begin{array}{ccc}
R_1/(St_{R_1}(S_1) \cap Z \wedge (R_1)) \times R_1/(St_{R_1}(S_1) \cap Z \wedge (R_1)) & \xrightarrow{\omega_1} & R_1 \wedge R_1 \\
\downarrow{\mu_0 \times \mu_0} & & \downarrow{\zeta_0} \\
R_2/(St_{R_2}(S_2) \cap Z \wedge (R_2)) \times R_2/(St_{R_2}(S_2) \cap Z \wedge (R_2)) & \xrightarrow{\omega_2} & R_2 \wedge R_2
\end{array}
\tag{2}
\]

are commutative.

If there is an exterior isoclinism of crossed modules from $\mathcal{X}_1$ to $\mathcal{X}_2$, we shall say that $\mathcal{X}_1$ and $\mathcal{X}_2$ are exterior isoclinic crossed modules.

**Example 4.** Let $(\mu, \zeta)$ be an exterior isoclinism from $M$ to $N$. Then $\mathcal{X}_1 = (id_M : M \to M)$ is exterior isoclinic to $\mathcal{X}_2 = (id_N : N \to N)$ where $(\mu_1, \mu_0) = (\mu, \mu)$ and $(\zeta_1, \zeta_0) = (\zeta, \zeta)$. 

8
Remark 4.5. If the crossed modules $X_1$ and $X_2$ are simply connected or finite, then the commutativity of diagrams (1) with (2) in Definition 4.4 are equivalent to the commutativity of following diagram.

\[
\begin{array}{ccc}
X_1/(Z \wedge (X_1)) \times X_1/(Z \wedge (X_1)) & \longrightarrow & X_1 \wedge X_1 \\
\downarrow_{(\mu_1,\mu_0) \times (\mu_1,\mu_0)} & & \downarrow_{(\xi_1,\xi_0)} \\
X_2/(Z \wedge (X_2)) \times X_2/(Z \wedge (X_2)) & \longrightarrow & X_2 \wedge X_2
\end{array}
\]

A crossed module $\mathcal{X}$ is a exterior stem crossed module if $Z \wedge (\mathcal{X}) \leq \mathcal{X} \wedge \mathcal{X}$. Every crossed module is exterior isoclinic to an exterior stem crossed module, but distinct exterior stem crossed modules may be exterior isoclinic.

5 Computer Implementation

GAP (Groups, Algorithms, Programming [24]) is the leading symbolic computation system for solving computational discrete algebra problems. Symbolic computation has underpinned several key advances in Mathematics and Computer Science, for example, in number theory and coding theory (see [4]). The Small Groups library has been used in such landmark computations as the "Millennium Project" to classify all finite groups of order up to 2000 by Besche, Eick and O'Brien in [9]. The HAP [12] package has some functions to calculate about non-abelian exterior product, exterior square and exterior center. Moreover, The XMod package has many functions for isoclinism classes of groups and crossed modules and some family invariants.

We have developed new functions for GAP which constructs exterior isoclinism of groups and exterior stem groups. The function AreExteriorIsoclinicGroups is used for checking whether or not two groups are exterior isoclinic, while the function ExteriorIsoclinicGroup returns a group which is exterior isoclinic to given group. Two groups of different order can be exterior isoclinic. So, the function ExteriorIsoclinicFamily may be called in two ways: as ExteriorIsoclinicFamily($G$) to construct a list of the positions in the given order of groups that are exterior isoclinic to $G$; or as ExteriorIsoclinicFamily($G$,list) to construct a list of the positions in given list partitioned according to exterior isoclinism of $G$.

In the following GAP session, we compute all isoclinism families and some external isoclinism families of groups of order 16 using these functions.

```
gap> D16 := SmallGroup(16,7);; StructureDescription(D16);
"D16"

gap> Q16 := SmallGroup(16,9);; StructureDescription(Q16);
"Q16"

gap> AreIsoclinicGroups(D16,Q16)
true

gap> AreExteriorIsoclinicGroups(D16,Q16)
false

gap> C4xC4 := SmallGroup(16,4);; StructureDescription(C4xC4);
```
On the other hand, we implement the functions `CommutativeDegreeOfGroup` and `ExteriorDegreeOfGroup` which used to compute commutative degree and exterior degree of a group.

In the following GAP session, we obtain commutative degrees and exterior degrees of several groups.

```
gap> CommutativeDegreeOfGroup(AlternatingGroup(4));
1/3
gap> ExteriorDegreeOfGroup(AlternatingGroup(4));
7/24
gap> CommutativeDegreeOfGroup(QuaternionGroup(40));
13/40
gap> ExteriorDegreeOfGroup(QuaternionGroup(40));
13/40
```

For the groups of order 16, there are 14 isomorphism classes and 3 isoclinism families with representatives $C_{16}, (C_4 \times C_2) \ltimes C_2$ and $D_{16}$. The commutative degree of the groups in the same isoclinism family are equal, as expected. See following table for details.

| No | Number | Representator | Members | $d(G)$ |
|----|--------|---------------|---------|--------|
| 1  | 5      | [16,1]        | $C_{16}, C_4 \times C_4, C_8 \times C_2, C_4 \times C_2^2, C_2^4$ | 1      |
| 2  | 6      | [16,3]        | $(C_4 \times C_2) \times C_2, C_4 \times C_4, C_8 \times C_2, C_2 \times D_8, C_2 \times Q_8, (C_4 \times C_2) \times C_2$ | 5/8    |
| 3  | 3      | [16,7]        | $D_{16}, QD_{16}, Q_{16}$ | 7/16   |

Moreover, there are 9 exterior isoclinism families for the groups of order 16. The exte-
rior degree of the groups in the same exterior isoclinism family are equal, as expected. See following table for details.

### Table II

| No | Number | Representator | Members | \( d \wedge (G) \) |
|----|--------|---------------|---------|-------------------|
| 1  | 1      | [16,1]        | \( C_{16} \) | 1                 |
| 2  | 1      | [16,2]        | \( C_4 \times C_4 \) | 11/32             |
| 3  | 1      | [16,3]        | \((C_4 \times C_2) \times C_2\) | 19/64             |
| 4  | 3      | [16,4]        | \( C_4 \times C_4, QD_{16}, Q_{16}\) | 7/16              |
| 5  | 2      | [16,5]        | \( C_8 \times C_2, C_8 \times C_2\) | 5/8               |
| 6  | 1      | [16,7]        | \( D_{16}\) | 11/32             |
| 7  | 3      | [16,10]       | \( C_4 \times C_2^2, C_2 \times Q_8, (C_4 \times C_2) \times C_2\) | 11/32             |
| 8  | 1      | [16,11]       | \( C_2 \times D_8\) | 1/4               |
| 9  | 1      | [16,14]       | \( C_2^4\) | 23/128            |

The function IsExteriorStemGroup is used to verify that the condition in definition 3.6 is satisfied, while the function ExteriorIsoclinicExteriorStemGroup returns a exterior stem group that it is exterior isoclinic to given group. The function AllExteriorStemGroupIds returns the IdGroup list of the exterior stem groups of a specified size.

The following GAP session illustrates the use of these functions.

```gap
gap> c3c3 := Group([ (1,2,3), (4,5,6) ]);;
gap> IsStemGroup(c3c3);
false
gap> IsExteriorStemGroup(c3c3);
true
gap> AllExteriorStemGroupIds(16);
[ [ 16, 2 ], [ 16, 3 ], [ 16, 7 ], [ 16, 11 ], [ 16, 14 ] ]
gap> StructureDescription(ExteriorIsoclinicExteriorStemGroup(c4c2));
"C2 x C2"
```

We also have developed new functions for GAP which constructs exterior isoclinism of crossed modules and exterior stem crossed modules. By the definition of exterior center of a crossed module and exterior derived sub-crossed module given in definition 4.1; we added the function ExteriorCentreXMod and ExteriorDerivedSubXMod. The function AreExteriorIsoclinicXMods is used for checking whether or not two crossed modules are exterior isoclinic, while ExteriorIsoclinicXModFamily to construct a list of the positions in given list partitioned according to exterior isoclinism of the given crossed module.

Following GAP session shows that two crossed modules of different orders would be exterior isoclinic.

```gap
gap> all4 := AllXMods(4,4);;
gap> iso_all4 := AllXModsUpToIsomorphism(all4);;
```

11
The function \texttt{IsExteriorStemXMod} is used to check the given crossed modules is a stem crossed module or not.

Following \texttt{GAP} session shows that the numbers of all crossed modules of order \([4,4]\) is 60, which give rise 18 isomorphism classes, 2 isoclinism families and 5 exterior isoclinism families.


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**Competing Interest Statement:** The authors declare they have no competing interests.

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