QUASI-PERIODIC SOLUTIONS TO NONLINEAR BEAM EQUATION ON COMPACT LIE GROUPS WITH A MULTIPLICATIVE POTENTIAL

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ABSTRACT. The goal of this work is to study the existence of quasi-periodic solutions in time to nonlinear beam equations with a multiplicative potential. The nonlinearities are required to only finitely differentiable and the frequency is along a pre-assigned direction. The result holds on any compact Lie group or homogenous manifold with respect to a compact Lie group, which includes the standard torus $\mathbb{T}^d$, the special orthogonal group $SO(d)$, the special unitary group $SU(d)$, the spheres $S^d$ and the real and complex Grassmannians. The proof is based on a differentiable Nash-Moser iteration scheme.

1. INTRODUCTION

This paper concerns the existence of quasi-periodic solutions of the forced nonlinear beam equation

$$u_{tt} + \Delta^2 u + V(x)u = \epsilon f(\omega t, x, u), \quad x \in M,$$

where $M$ is any simply connected compact Lie group with dimension $d$ and rank $r$, $\epsilon > 0$, the frequency $\omega \in \mathbb{R}^\nu$, $V \in C^q(M; \mathbb{R})$ and $f \in C^q(\mathbb{T}^\nu \times M \times \mathbb{R}; \mathbb{R})$, where $q$ is large enough. Assume that the frequency vector $\omega$ satisfies

$$\omega = \lambda \omega_0, \quad \lambda \in \Lambda := [1/2, 3/2], \quad |\omega_0| \leq 1,$$

where $|\cdot|$ will be defined later in (2.4). For some $\gamma_0 > 0$, the following Diophantian condition holds:

$$|\omega_0 \cdot l| \geq 2\gamma_0 |l|^{-\nu}, \quad \forall l \in \mathbb{Z}^\nu \setminus \{0\}.$$  

Moreover we suppose

$$\Delta^2 + V(x) \geq \kappa_0 \mathbb{I} \quad \text{with} \quad \kappa_0 > 0.$$  

Equation (1.1) is interesting by itself. It is derived from the following Euler-Bernoulli beam equation

$$\frac{d^2}{dx^2} (EI \frac{d^2 u}{dx^2}) = g,$$

which describes the relationship between the applied load and the beam’s deflection, where the curve $u(x)$ describes the deflection of the beam at some position $x$ in the $z$ direction, $g$ is distributed load which may be a function of $x$, $u$ or other variables, $I$ is the second moment of area of the beam’s cross-section, $E$ is the elastic modulus, the product $EI$ is the flexural rigidity. Derivatives of the deflection $u$ have significant physical significance: $u_x$ is the slope of the beam, $-EIu_{xx}$ is the bending moment of the beam and $-(EIu_{xxx})_x$ is the shear force of the beam. The dynamic beam equation is the Euler-Lagrange equation

$$m \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 u}{\partial x^2} \right) = g,$$

where $m$ is the mass per unit length. If $E$ and $I$ are independent of $x$, then equation (1.5) can be reduced to

$$m u_{tt} + EI u_{xxxx} = g.$$
After a time rescaling $t \rightarrow ct$ with $c = \sqrt{EI/m}$, we obtain

$$u_{tt} + u_{xxxx} = \tilde{g} \quad \text{with} \quad \tilde{g} = \frac{m}{EI}g.$$ 

The search for periodic or quasi-periodic solutions to nonlinear PDEs has a long standing tradition. There are two main approaches: one is the infinite-dimensional KAM (Kolmogorov-Arnold-Moser) theory to Hamiltonian PDEs, refer to Kuksin [15], Wayne [22], Pöschel [19]. The main difficulty, namely the presence of arbitrarily small divisors in the expansion series of the solutions, is handled via KAM theory. Later, another more direct bifurcation approach was established by Craig and Wayne [10] and improved by Bourgain [5–7] based on Lyapunov-Schmidt procedure, to solved the small divisors problem, for periodic solutions, with an analytic Newton iterative scheme. This approach is often called as the Craig-Wayne-Bourgain method, which is different from KAM. The main advantage for this approach is to require only the so called first order Melnikov non-resonance conditions for solving the linearized equations at each step of the iteration.

Up to now, the existence of quasi-periodic solutions of the nonlinear beam equations using KAM theory and Nash-Moser iteration have received much attention by the mathematical communities. For the 1D beam equation, when $f(\omega, x, u) = O(u^3)$ is an analytic, odd function, in [11] they proved the existence of linearly stable small-amplitude quasi-periodic solution by an infinite KAM theorem [18]. In [9], Chang, Gao, and Li got quasi-periodic solutions for the 1-dimensional nonlinear beam equation with prescribed frequencies under Dirichlet boundary condition. Later, Wang and Si in [21] considered 1-dimensional beam equation with quasi-periodically forced perturbations $f(\omega, x, u) = \epsilon \phi(t)h(u)$. Also, in [16] they studied the existence of quasi-periodic solutions of the 1-dimensional completely resonant nonlinear beam equation. Some KAM-theorems for small-amplitude solutions of equations (1.1) on $\mathbb{T}^d$ with typical $V(x) = m$ were obtained in [13, 14]. Both works treat equations with a constant-coefficient analytical nonlinearity $f(\omega, x, u) = f(u)$. Subsequently, by a KAM type theorem, Mi and Cong in [17] proved the existence of quasi-periodic solutions of the nonlinear beam equation

$$u_{tt} + \Delta^2 u + V(x) * u + \epsilon \left( - \sum_{i,j=1}^d b_{ij}(u, \nabla u) \partial_i \partial_j u + g(u, \nabla u) \right) = 0, \quad x \in \mathbb{T}^d$$

and got the linear stability for corresponding solutions. In perturbations term, $\nabla u \equiv (\partial_{x_1} u, \partial_{x_2} u, \ldots, \partial_{x_d} u)$ the derivatives of $u$ with respect to the space variables, and $b_{ij}$, $g$ are real analytic function. In [11], Eliasson, Grébert, and Kuksin proved that for the nonlinear beam equation

$$u_{tt} + \Delta^2 u + mu + \partial_u f(x, u) = 0, \quad x \in \mathbb{T}^d,$$

where $f(x, u) = u^4 + O(u^5)$, has many linearly stable or unstable small-amplitude quasi-periodic solutions. All above proofs depend on some type of KAM techniques and are carried out in analytic nonlinearities cases. However, there is no existence result for nonlinear beam equation with perturbations having only finitely differentiable regularities. Recently, in [20], based on a Nash-Moser type implicit function theorem, Shi prove the existence of quasi-periodic solution of the following beam equation

$$u_{tt} + \Delta^2 u + mu = \epsilon f(\omega, x, u), \quad x \in \mathbb{T}^d,$$  \hspace{1cm} (1.6)

where $f$ is finitely differentiable, $\omega$ is the frequency, $m > 0$. Note that these previous results are confined to tori on existence of quasi-periodic solutions of the nonlinear beam equation. The reason why these results are confined to tori is that their proofs require specific properties of the eigenvalues, while the eigenfunctions must be the exponentials or, at least, strongly “localised close to exponentials”. According to the harmonic analysis on compact Lie groups and the theory of the highest weight which provides an accurate description of the eigenvalues of the Lapalce-Beltrami operator as well as the multiplication rules of its eigenfunctions. In [4] Berti and Procesi extended this result to the case of quasi-periodic solutions in time of the following NLW
Define an index set $\mathcal{N}$:

$$\{j \in \mathbb{R}^r : j = \sum_{k=1}^{r} j_k w_k, j_k \in \mathbb{N}\}.$$ 

Moreover, in [12], Berti and Bolle considered the NLW and NLS respectively, with a multiplicative potential:

$$u_{tt} - \Delta u + V(x)u = \epsilon f(\omega t, x, u), \quad \text{for} \quad x \in \mathbb{T}^d.$$ 

In the present paper, our goal is to prove the existence of quasi-periodic solutions in time of the nonlinear beam equation (1.1) with a multiplicative potential $V(x)$ and finite regularity nonlinearities on any compact Lie group or homogenous manifold with respect to a compact Lie group. There are three main difficulties in this work: (i) a multiplicative potential in higher dimensions. The eigenvalues of the operator $\Delta^2 + V(x)$ appear in clusters of unbounded sizes and the the eigenfunctions are (in general) not localized with respect to the exponentials. We will use the similar properties of the eigenvalues and the eigenfunctions of the operator $-\Delta + V(x)$ in [3]. (ii) the finite differentiable regularities of the nonlinearity. Clearly, a difficulty when working with functions having only Sobolev regularity is that the Green functions will exhibit only a polynomial decay off the diagonal, and not exponential (or subexponential). A key concept one must exploit is the interpolation/tame estimates. (iii) the nonlinear beam equation are defined not only on tori, but on any compact Lie group or homogenous manifold with respect to a compact Lie group, which includes the standard torus $\mathbb{T}^d$, the special orthogonal group $SO(d)$, the special unitary group $SU(d)$, the spheres $S^d$, the real and complex Grassmannians, and so on, recall [8].

The rest of the paper is organized as follows: we state the main result (see Theorem 2.2) and introduce several notations in subsection 2.1. In subsection 2.2, we define the strong $s$-norm of a matrix $M$ and introduce its properties. Section 3 is devoted to give the iterative theorem, see Theorem 3.27. In subsection 3.1, we give a multiscale analysis of the linearized operators $\mathcal{L}_\epsilon(x, u)$ (recall (2.13)) as [11], see Proposition 3.8. Our aim is to check that the assumption (A3) in Proposition 3.8 holds in subsection 3.2. Under that Proposition 3.8 and (3.17) hold, we have to remove some $\lambda$ in $\Lambda$, recall (1.2). In subsection 3.3, the measure of the excluded $\lambda$ satisfies (3.49) and (3.51) respectively. In subsection 3.4, we establish Theorem 3.27 and give the proof. At the end of the construction, we prove that the measure of the parameter $\lambda$ satisfying Theorem 2.2 is a large measure Cantor-like set in subsection 3.5. Finally, in section 4, we list the proof of some related results for the sake of completeness.

2. MAIN RESULTS

2.1. NOTATIONS. After a time rescaling $\varphi = \omega \cdot t$, we consider the existence of solutions $u(\varphi, x)$ of

$$(\lambda \omega \cdot \partial_\varphi)^2 u + \Delta^2 u + V(x)u = \epsilon f(\varphi, x, u), \quad x \in \mathcal{M}. \quad (2.1)$$

Define an index set $\mathcal{N}$ as

$$\mathcal{N} := \mathbb{Z}^r \times \Gamma_+(\mathcal{M}) \quad \text{with} \quad \Gamma_+(\mathcal{M}) := \left\{ j \in \mathbb{R}^r : j = \sum_{k=1}^{r} j_k w_k, j_k \in \mathbb{N} \right\},$$

where $\Gamma_+(\mathcal{M})$ is contained in an $r$-dimensional lattice (in general not orthogonal)

$$\Gamma := \left\{ j \in \mathbb{R}^r : j = \sum_{k=1}^{r} j_k w_k, \quad j_k \in \mathbb{Z} \right\}$$

generated by independent vectors $w_1, \cdots, w_r \in \mathbb{R}^r$. There exists an integer $\delta \in \mathbb{N}$ such that the fundamental weights satisfy

$$w_k \cdot w_{k'} \in \delta^{-1} \mathbb{Z}, \quad \forall k, k' = 1, \cdots, r. \quad (2.2)$$
Moreover $\Gamma_+(M)$ is required to satisfy a product structure, namely

$$j = \sum_{k=1}^{r} j_k w_k, \quad j' = \sum_{k=1}^{r} j'_k w_k$$

$$\Rightarrow j'' = \sum_{k=1}^{r} j''_k w_k \in \Gamma_+(M) \quad \text{if} \quad \min \{j_k, j'_k\} \leq j''_k \leq \max \{j_k, j'_k\}, \quad \forall \ k = 1, \ldots, r.$$  \quad (2.3)

Remark that (2.3) is used only in the proof of Lemma 3.11.

We briefly recall the relevant properties of harmonic analysis on compact Lie group, see [4]. The eigenvalues of the Laplace-Beltrami operator $\Delta$ on $M$ are

$$\lambda_j := -\|j + \rho\|^2 + \|\rho\|^2$$

with respect to the the eigenfunctions

$$e_{j,p}(x), \quad x \in M, \quad j \in \Gamma_+(M), \quad p = 1, \ldots, d_j,$$

where $\| \cdot \|$ stands for the Euclidean norm on $\mathbb{R}^d$, $\rho := \sum_{k=1}^{r} w_k$, $e_j(x)$ is the (unitary) matrix associated to an irreducible unitary representations $(R_{V_j}, \mathcal{V}_j)$ of $M$, namely

$$(e_{j}(x))_{p,p'} = \langle R_{V_j}(x) v_p, v_{p'} \rangle, \quad v_p, v_{p'} \in V_j,$$

where $(\mathcal{V}_p)_{p=1, \ldots, \dim \mathcal{V}_j}$ is an orthonormal basis of the finite dimensional euclidean space $V_j$ with scalar product $(\cdot, \cdot)$. Denote by $\mathcal{N}_j$ the eigenspace of $\Delta$ with respect to $\lambda_j$. The degeneracy of the eigenvalue $\lambda_j$ satisfies

$$d_j \leq \|j + \rho\|^{d-r}.$$  

Furthermore, by the Peter-Weyl theorem, we have the following orthogonal decomposition

$$L^2(M) = \bigoplus_{j \in \Gamma_+(M)} \mathcal{N}_j.$$  

Given $n = (l, j) \in \mathfrak{N}$ and $\mathfrak{N}_1, \mathfrak{N}_2 \subset \mathcal{E} \subset \mathfrak{N}$, define

$$|n| := \max \{\|l\|, |j|\}, \quad |l| := \max_{1 \leq k \leq r} |l_k|, \quad |j| := \max_{1 \leq l \leq r} |j_k|; \quad (2.4)$$

$$\text{diam}(\mathcal{E}) := \sup_{n, n' \in \mathcal{E}} |n - n'|, \quad d(\mathfrak{N}_1, \mathfrak{N}_2) := \inf_{n \in \mathfrak{N}_1, n' \in \mathfrak{N}_2} |n - n'|, \quad d(n, \mathfrak{N}) := \inf_{n' \in \mathfrak{N}} |n - n'|.$$  

Remark 2.1. We set $n - n' = 0$ if $n - n' \in \mathbb{Z}^r \times (\Gamma \setminus \Gamma_+(M))$.

For some constants $c_2 > c_1 > 0$, the following holds:

$$c_1 |n| \leq \sqrt{\|l\|^2 + \|j + \rho\|^2} \leq c_2 |n|, \quad \forall n = (l, j) \in \mathfrak{N}. \quad (2.5)$$

Decomposing

$$u(\varphi, x) = \sum_{n \in \mathfrak{N}} u_n e^{i\varphi \cdot x} e_j(x) = \sum_{(l,j) \in \mathfrak{N}} e^{i\varphi \cdot x} \sum_{p=1}^{d_j} u_{l,j,p} e_{j,p}(x),$$

the Sobolev space $H^s$ is defined by

$$H^s := H^s(\mathfrak{N}; \mathbb{R}) := \left\{ u = \sum_{n \in \mathfrak{N}} u_n e^{i\varphi \cdot x} e_j(x) : u_n \in \mathbb{C}^{d_j}, \|u\|^2 = \sum_{n \in \mathfrak{N}} \langle w_n \rangle^{2s} \|u_n\|^2 < +\infty \right\} \quad (2.6)$$

with $\langle w_n \rangle := \max\{c_1, 1, (\|l\|^2 + \|j + \rho\|^2)^{1/2}\}$, where $c_1$ is seen in (2.5), $\|u_n\|^2 := 2\pi \sum_{p=1}^{d_j} |u_{l,j,p}|^2$. There also exist $b_2 > b_1 > 0$ such that

$$b_1 |j| \leq \|j\| \leq b_2 |j|, \quad \forall j \in \Gamma_+(M). \quad (2.7)$$
For $s \geq s_0 > (\nu + d)/2$, the Sobolev space $H^s$ has the following properties:

1. $\|uv\|_s \leq C(s)\|u\|_s\|v\|_s$, $\forall u, v \in H^s$;
2. $\|u\|_{L^\infty} \leq C(s)\|u\|_s$, $\forall u \in H^s$;
3. $\|uv\|_s \leq C(s)(\|u\|_s\|v\|_{s_0} + \|u\|_{s_0}\|v\|_s)$, $\forall u, v \in H^s$.

The above properties (1), (2) and (3) are also seen in [4, Lemma 2.13].

Let $V(x) = m + \bar{V}(x)$, where $m$ is the average of $V(x)$ and $\bar{V}$ has zero average. Define the composition operator on Sobolev spaces

$$F : H^s \rightarrow H^s, \ u \mapsto f(\varphi, x, u),$$

where $f \in C^q(T^\nu \times M \times \mathbb{R}; \mathbb{R})$. The core of a Nash-Moser iteration is the invertibility of the following linearized operator

$$\mathcal{L}(\epsilon, \lambda, u) := L_\lambda - \epsilon(DF)(u) = L_\lambda - \epsilon(\partial_u f)(\varphi, x, u) = D_\lambda + \bar{V}(x) - \epsilon(\partial_u f)(\varphi, x, u),$$

where

$$L_\lambda := (\lambda\omega_0 - \partial_\varphi)^2 + \Delta^2 + V(x), \quad D_\lambda := (\lambda\omega_0 - \partial_\varphi)^2 + \Delta^2 + m.$$

In the Fourier basis $e^{it\cdot\varepsilon_j(x)}$, the operator $\mathcal{L}(\epsilon, \lambda, u)$ (see (2.9)) is represented by the infinite-dimensional self-adjoint matrix

$$A(\epsilon, \lambda, u) := D(\lambda) + T(\epsilon, u) = D(\lambda) + T' - \epsilon T''(u),$$

where $D(\lambda) := \text{diag}_{n \in \mathbb{N}}(\mu_n(\lambda)I_{b_2})$, with $\mu_n(\lambda) = - (\lambda\omega_0 \cdot l)^2 + \lambda_j^2 + m$, and

$$T'(a)_n^m := (\bar{V})_{j - j} \quad T''(a)_n^m := (a)_{m - n'} = a_{l - l'}(j - j').$$

with $a(\varphi, x) := (\partial_u f)(\varphi, x, u(\varphi, x))$. Similarly, we also define

$$\mathcal{L}(\epsilon, \lambda, u, \theta) := L_\lambda(\theta) - \epsilon(DF)(u) = L(\theta) - \epsilon(\partial_u f)(\varphi, x, u) = D(\theta) + \bar{V}(x) - \epsilon(\partial_u f)(\varphi, x, u),$$

where $F$ is seen in (2.8), and

$$L_\lambda(\theta) = (\lambda\omega_0 - \partial_\varphi + i0)^2 + \Delta^2 + V(x), \quad D_\lambda(\theta) = (\lambda\omega_0 - \partial_\varphi + i0)^2 + \Delta^2 + m.$$

For all $\theta \in \mathbb{R}$, the operator $\mathcal{L}(\epsilon, \lambda, u, \theta)$ (see (2.13)) is represented by the infinite-dimensional self-adjoint matrix depending on $\theta$

$$A(\epsilon, \lambda, u, \theta) := D(\lambda, \theta) + T(\epsilon, u) = D(\lambda, \theta) + T' - \epsilon T''(u),$$

where $D(\lambda, \theta) := \text{diag}_{n \in \mathbb{N}}(\mu_n(\lambda, \theta)I_{b_2})$, with $\mu_n(\lambda, \theta) = - (\lambda\omega_0 \cdot l + \theta)^2 + \lambda_j^2 + m$, and $T', T''$ are given in (2.12). In addition denote by $A_{N,l_0,j_0}(\epsilon, \lambda, u, \theta)$ the submatrices of $A(\epsilon, \lambda, u, \theta)$ centered at $(l_0, j_0)$, where

$$A_{N,l_0,j_0}(\epsilon, \lambda, u, \theta) := A_{|l - l_0| \leq N, |j - j_0| \leq N}(\epsilon, \lambda, u, \theta).$$

We use the simpler notations

$$A_{N,l_0,j_0}(\epsilon, \lambda, u, \theta) := A_{N,0,j_0}(\epsilon, \lambda, u, \theta) \quad \text{if} \quad l_0 = 0;$$
$$A_{N}(\epsilon, \lambda, u, \theta) := A_{N,0,0}(\epsilon, \lambda, u, \theta) \quad \text{if} \quad (l_0, j_0) = (0, 0);$$
$$A_{N,l_0,j_0}(\epsilon, \lambda, u) := A_{N,0,j_0}(\epsilon, \lambda, u, 0) \quad \text{if} \quad l_0 = 0, \theta = 0;$$
$$A_{N}(\epsilon, \lambda, u) := A_{N,0,0}(\epsilon, \lambda, u, 0) \quad \text{if} \quad l_0 = 0, j_0 = 0, \theta = 0.$$

Clearly, the following crucial covariance property holds:

$$A_{N,l_0,j_0}(\epsilon, \lambda, u, \theta) = A_{N,l_0,j_0}(\epsilon, \lambda, u, \theta + \lambda_0 \cdot l_0).$$

The main result of this paper is
Theorem 2.2. Let $M$ be any simply connected compact Lie group with dimension $d$ and rank $r$. Assume (1.3) holds, then there exist $s := s(\nu, d, r)$, $q := q(\nu, d, r) \in \mathbb{N}, \epsilon_0 > 0$, a map
$$u(\epsilon, \cdot) \in C^1(\Lambda; H^s) \quad \text{with} \quad u(0, \lambda) = 0,$$
and a Cantor-like set $\mathcal{D}_\epsilon \subset \Lambda$ of asymptotically full Lebesgue measure, namely
$$\text{meas}(\mathcal{D}_\epsilon) \to 1 \quad \text{as} \quad \epsilon \to 0,$$
such that, for all $V \in C^q$ satisfies (1.4), $f \in C^q$ and $\lambda \in \mathcal{D}_\epsilon$, $u(\epsilon, \lambda)$ is a solution of (2.1) with $\omega = \lambda \omega_0$.

Remark 2.3. If $V, f \in C^\infty$, then $u(\epsilon, \lambda) \in C^\infty(T^r \times M; \mathbb{R})$, which can be completed just by making small modifications in the proofs of lemmas 3.32 and 3.34, and formulae (3.112)-(3.113).

Remark 2.4. In fact, here $M$ may be a homogeneous manifold with respect to a compact Lie group, namely $M := G/G_0$, $G := G \times T^r$, where $G_0$ is a closed subgroup of $G$, $G$ is a simple connected compact Lie group, $T^r$ is a tori. The eigenvalues of the Laplace-Beltrami operator $\Delta$ on $M$ are
$$\lambda_j^2 := -\|\vec{j} + \vec{\rho}\|^2 + \|\vec{\rho}\|^2 = -\|j^{(1)} + \rho\|^2 + \|\rho\|^2 - \|j^{(2)}\|^2$$
with respect to the eigenfunctions
$$e_{j^{(1)},j^{(2)}}(x^{(1)},x^{(2)}), \quad \vec{x} = (x^{(1)},x^{(2)}) \in G \times T^r, \quad p = 1, \cdots, d_j,$$
where $\vec{\rho} = (\rho, 0)$, $\vec{j} = (j^{(1)}, j^{(2)})$ belongs to a subset of $\Gamma_+(G) \times \mathbb{Z}^r$, and $d_j \leq d_j(1)$ (recall [3 Theorem 2.9]).

2.2. Matrices with off-diagonal decay. For $\mathcal{B}, \mathcal{C} \subset \mathfrak{g}$, a bounded linear operator $\mathcal{L} : H^s_{\mathcal{B}} \to H^s_{\mathcal{C}}$ is represented by a matrix in
$$\mathcal{M}_{\mathcal{B} \mathcal{C}} = \{(M_{n''}^{n'})_{n' \in \mathcal{B}, n'' \in \mathcal{B}} \quad \text{with} \quad M_{n''}^{n'} \in \text{Mat}(\mathfrak{d}_{j'} \times \mathfrak{d}_{j''}, \mathcal{C})\},$$
where $H^s_{\mathcal{B}} := \left\{ u = \sum_{n \in \mathfrak{g}_\mathcal{B}} u_n e^{ir} \mathbf{e}_j(x) \in H^s : u_n = 0 \text{ if } n \notin \mathcal{B} \right\}$. Define the $L^2$-operator norm
$$\|M_{\mathcal{B}}\|_0 = \sup_{h \in H_{\mathcal{B}}} \frac{\|M_{\mathcal{B}} h\|_0}{\|h\|_0}.$$
Moreover we introduce the strong $s$-norm of a matrix $M \in \mathcal{M}_{\mathcal{B} \mathcal{C}}$ as follows:

Definition 2.5. The $s$-norm of any matrix $M \in \mathcal{M}_{\mathcal{B} \mathcal{C}}$ is defined by
$$\|M\|^2_s := K_0 \sum_{n \in \mathfrak{g}_\mathcal{B}} |M(n)|^2 \langle n \rangle^{2s} \quad (2.22)$$
where $K_0 > 4 \sum_{n \in \mathfrak{g}_\mathcal{B} \times \mathfrak{g}_\mathcal{B}} \langle n \rangle^{-2s_0}$, $\langle n \rangle = \max(1, |n|)$, and
$$\langle n \rangle := \begin{cases} \sup_{n' - n'' = n, n' \in \mathcal{B}, n'' \in \mathfrak{g}_\mathcal{B}} |M_{n''}^{n'}|_0 & \text{if} \quad n \in \mathcal{C} - \mathcal{B}, \\ 0 & \text{if} \quad n \notin \mathcal{C} - \mathcal{B}. \end{cases}$$

It is obvious that the $s$-norm in (2.22) satisfies that $| \cdot |_s \leq | \cdot |_{s'}$ for all $0 < s < s'$. The following properties (see lemmas 2.6 and 1.11) on the strong $s$-norm are given in [3].

Lemma 2.6. ([3 Corollary 3.3]) Let $\Sigma : u(\varphi, \mathbf{x}) \to g(\varphi, \mathbf{x}) u(\varphi, \mathbf{x})$ be a linear operator in $L^2$, self-adjoint. Then, $\forall s > (\nu + d)/2$, the following holds:
$$\|\Sigma\|_s \leq C(s) \|g\|_{s + \rho} \quad \text{with} \quad \rho = (2\nu + d + r + 1)/2. \quad (2.23)$$
Lemma 2.7. (Interpolation \[3, \text{Lemma 2.6}\]) There exists $C(s) \geq 1$, with $C(s_0) = 1$, such that for all subset $\mathcal{B}$, $\mathcal{C}, \mathcal{D} \subset \mathcal{K}$, one has:

$$|M_1 M_2|_s \leq \frac{1}{2} |M_1|_{s_0} |M_2|_s + C(s) \frac{1}{2} |M_1|_s |M_2|_{s_0}, \quad \forall s \geq s_0, \forall M_1 \in \mathcal{M}^\mathcal{B}_\mathcal{C}, M_2 \in \mathcal{M}^\mathcal{D}_\mathcal{C},$$

(2.24)
in particular,

$$|M_1 M_2|_s \leq C(s) |M_1|_s |M_2|_s, \quad \forall s \geq s_0, \forall M_1 \in \mathcal{M}^\mathcal{B}_\mathcal{C}, M_2 \in \mathcal{M}^\mathcal{D}_\mathcal{C}.\tag{2.25}$$

Lemma 2.8. (\[3, \text{Lemma 2.7}\]) For all subset $\mathcal{B}, \mathcal{C} \subset \mathcal{K}$, we have

$$\|M h\|_s \leq C(s)(\|M\|_{s_0} \|h\|_s + \|M\|_s \|h\|_{s_0}), \quad \forall M \in \mathcal{M}^\mathcal{B}_\mathcal{C}, \forall h \in H^s_\mathcal{B}.\tag{2.26}$$

Lemma 2.9. (Smoothing \[3, \text{Lemma 2.8}\]) For all subset $\mathcal{B}, \mathcal{C} \subset \mathcal{K}$ and all $s' \geq s \geq 0$, one has that, for $N \geq 2$,

1. If $M^{n'}_n = 0$ for all $|n' - n| \leq N$, then

$$|M|_s \leq N^{-(s'-s)} |M|_{s'}, \quad \forall M \in \mathcal{M}^\mathcal{B}_\mathcal{C}.\tag{2.27}$$

2. If $M^{n'}_n = 0$ for all $|n' - n| > N$, then

$$|M|_{s'} \leq N^{s'-s} |M|_s, \quad |M|_s \leq N^{s+\nu+r} |M|_0, \quad \forall M \in \mathcal{M}^\mathcal{B}_\mathcal{C}.\tag{2.28}$$

Lemma 2.10. (Decay along lines \[3, \text{Lemma 2.9-2.10}\]) For all subset $\mathcal{B}, \mathcal{C} \subset \mathcal{K}$ and all $s \geq 0$, one has that, for some constant $K_1 > 0$,

$$|M|_s \leq K_1 |M|_{s+\nu+r}, \quad \forall M \in \mathcal{M}^\mathcal{B}_\mathcal{C},\tag{2.29}$$

where $M_n, n \in \mathcal{C}$ denote its $n$-th line. Moreover

$$\|M\|_0 \leq |M|_{s_0}, \quad \forall M \in \mathcal{M}^\mathcal{B}_\mathcal{C}.\tag{2.30}$$

Denote by $[-1]M$ any left inverse of $M$.

Lemma 2.11. (Perturbation of left-invertible matrices \[3, \text{Lemma 2.12}\]) If $M \in \mathcal{M}^\mathcal{B}_\mathcal{C}$ has a left invertible matrix $[-1]M$, then, $\forall P \in \mathcal{M}^\mathcal{B}_\mathcal{C}$, with $[-1]M|_{s_0} P|_{s_0} \leq 1/2$, the matrix $M + P$ has a left inverse with

$$|[-1](M + P)|_{s_0} \leq 2 |[-1]M|_{s_0},\tag{2.31}$$

and, for all $s \geq s_0$,

$$|[-1](M + P)|_s \leq C(s)(|[-1]M|_s + |[-1]M|_0^2 |P|_s).\tag{2.32}$$

Moreover, if $||[-1]M||_0 \|P\|_0 \leq 1/2$, then there exists a left inverse $[-1](M + P)$ of $M + P$ with

$$||[-1](M + P)||_0 \leq 2 ||[-1]M||_0.\tag{2.33}$$

3. NASH-MOSER ITERATIVE SCHEME

Consider the orthogonal splitting $H^s = H_{N_n} \oplus H^s_{N_n}^\perp$, where $H^s$ is defined in (2.6) and

$$H_{N_n} := \left\{ u \in H^s : u = \sum_{|n| \leq N_n} u_n e^{i\nu \cdot \varphi} e_j(x) \right\}, \tag{3.1}$$

$$H^s_{N_n} := \left\{ u \in H^s : u = \sum_{|n| > N_n} u_n e^{i\nu \cdot \varphi} e_j(x) \right\},$$

with $u_n \in \mathbb{C}^{D_j}$ and

$$N_{n+1} := N_n^2, \quad \text{namely } N_{n+1} = N_0^{2n+1}.$$
Furthermore $P_{N_n}, P_{N_n}^\perp$ denote the orthogonal projectors onto $H_{N_n}$ and $H_{N_n}^\perp$ respectively, namely

$$P_{N_n} : H^s \to H_{N_n}, \quad P_{N_n}^\perp : H^s \to H_{N_n}^\perp. \tag{3.3}$$

Then, by means of (2.5), $\forall n \in \mathbb{N}, \forall s \geq 0, \forall \kappa \geq 0$, the following hold:

$$\|P_{N_n} u\|_{s+\kappa} \leq c_2 N_n^\kappa \|u\|_s, \quad \forall u \in H^s, \quad (3.4)$$

$$\|P_{N_n}^\perp u\|_s \leq c_1^{-\kappa} N_n^{-\kappa} \|u\|_{s+\kappa}, \quad \forall u \in H^{s+\kappa}. \quad (3.5)$$

In addition, for all $j_0 \in \Gamma^+(M)$, denote by $P_{N,j_0}$ the orthogonal projector from $H^s$ onto the subspace

$$H_{N,j_0} := \{ u \in H^s : u = \sum_{|l-j_0| \leq N} u_{l,j} e^{i\nu l} e_j(x), \ u_{l,j} \in C^\delta_j \}.$$  

This shows that $H_{N,0} = H_{N_n}$ (see (3.1)), $P_{N,0} = P_{N_n}$ (see (3.3)).

Moreover let $\tilde{P}_{N,j_0}$ denote the orthogonal projector from $H^{s_0}(M)$ onto the space

$$\tilde{H}_{N,j_0} := \{ u \in H^{s_0} : u = \sum_{|j-j_0| \leq N} u_j e_j(x), \ u_j \in C^\delta_j \}.$$  

Remark that the functions on $H^{s_0}(M)$ depend only on $x$.

For all $s \geq s_1$ with $s_1 \geq s_0 > \frac{\nu + d}{2}$, if $f \in C^q(T^\nu \times M \times \mathbb{R}, \mathbb{R})$ with

$$q \geq s_2 + q + 2, \tag{3.6}$$

then the composition operator $F$ (recall (2.3)) has the following standard properties (P1)-(P3) (see [1]):

(P1)(Regularity) $F \in C^2(H^s, H^s)$.

(P2)(Tame estimates) $\forall u, h \in H^s$ with $\|u\|_{s_1} \leq 1$,

$$\|F(u)\|_s \leq C(s)(1 + \|u\|_s), \tag{3.7}$$

$$\|(DF)(u)h\|_s \leq C(s)(\|h\|_s + \|u\|_s \|h\|_{s_1}), \tag{3.8}$$

$$\|D^2 F(u)[h,v]\|_s \leq C(s)(\|u\|_s \|h\|_{s_1} \|v\|_{s_1} + \|v\|_s \|h\|_{s_1} + \|v\|_{s_1} \|h\|_s). \tag{3.9}$$

(P3)(Taylor tame estimate) $\forall u, h \in H^s$ with $\|u\|_{s_1} \leq 1, \|h\|_{s_1} \leq 1$,

$$\|F(u + h) - F(u) - (DF)(u)h\|_{s_1} \leq C(s_1) \|h\|_s^2, \tag{3.10}$$

$$\|F(u + h) - F(u) - (DF)(u)h\|_s \leq C(s)(\|u\|_s \|h\|_{s_1}^2 + \|h\|_{s_1} \|h\|_s). \tag{3.11}$$

In addition the potential $V$ satisfies that, for some fixed constant $C$,

$$\|V\|_{C^q} \leq C, \tag{3.12}$$

where $q$ is defined in (3.6).

3.1. The multiscale analysis. Let

$$\mathcal{L}_N(\epsilon, \lambda, u) := P_N \mathcal{L}(\epsilon, \lambda, u)|_{H_{N_n}}, \tag{3.13}$$

where $\mathcal{L}(\epsilon, \lambda, u)$ given by (2.9). To guarantee the convergence of the iteration, we need sharper estimates on inversion of the linearized operators $\mathcal{L}_N(\epsilon, \lambda, u)$:

$$|\mathcal{L}^{-1}_N(\epsilon, \lambda, u)|_{s_1} = O(N^{\tau_2 + \delta s}), \quad \delta \in (0, 1), \tau_2 > 0, \forall s > 0,$$

Hence, for fixed $l \in \mathbb{Z}^\nu, \theta \in \mathbb{R}$, some extra properties on the following linear operator

$$(-D^2 + V(x))|_{\tilde{H}_{N,j_0}}$$

are required.
Lemma 3.1. For fixed \( l \in \mathbb{Z}^\nu, \theta \in \mathbb{R}, \) provided
\[
\|((-\lambda_0 l + \theta)^2)I_{s_j} + \hat{P}_{N,j_0}(\Delta^2 + V(x))_{H_{N,j_0}}\|_{L^2_x} \leq N^\tau,
\]
for \( N \geq \tilde{N}(s_2, V) \) large enough, one has:
\[
\|((-\lambda_0 l + \theta)^2)I_{s_j} + \hat{P}_{N,j_0}(\Delta^2 + V(x))_{H_{N,j_0}}\|_{L^2_x} \leq (1/2)N^{\tau_2 + \delta}, \quad \forall s \in [s_0, s_2].
\]

Proof. The proof is given in the Appendix. \( \square \)

Lemma 3.1 shows that there exists \( N_0 := N_0(s_2, k_0, V) \in \mathbb{N} \) (see the first step in the proof of Theorem 3.27) such that for fixed \( l \in [-N, N]^\nu \cap \mathbb{Z}^\nu, \theta \in \mathbb{R} \) with \( \tilde{N}(s_2, V) \leq N^{1/k_0} \leq N \leq N_0, \) if
\[
\|((-\lambda_0 l + \theta)^2)I_{s_j} + \hat{P}_{N,j_0}(\Delta^2 + V(x))_{H_{N,j_0}}\|_{L^2_x} \leq N^{\tau},
\]
holds, then we have
\[
\|((-\lambda_0 l + \theta)^2)I_{s_j} + \hat{P}_{N,j_0}(\Delta^2 + V(x))_{H_{N,j_0}}\|_{L^2_x} \leq (1/2)N^{(\tau_2 + \delta)}, \quad \forall s \in [s_0, s_2].
\]

In addition, in the Fourier basis \( e^{it\varphi}, \) definition (2.14) implies
\[
P_{N,j_0}(L_\theta)_{H_{N,j_0}} = \sum_{|\xi| \leq N}((-\lambda_0 l + \theta)^2)I_{s_j} + \hat{P}_{N,j_0}(\Delta^2 + V(x))_{H_{N,j_0}}e^{it\varphi},
\]
which gives rise to
\[
\|(P_{N,j_0}(L_\theta))_{H_{N,j_0}}\|_{L^2_x} \leq \frac{1}{2}N^{(\tau_2 + \delta)}, \quad \forall s \in [s_0, s_2].
\]

Then, from (2.23), (3.7), (3.16) and \( \|u\|_{s_1} \leq 1, \) we deduce
\[
\|(P_{N,j_0}(L_\theta))_{H_{N,j_0}}\|_{L^2_x} \leq \frac{e}{2}N^{\tau_2 + \delta(s_1 - \theta)}\|P_{N,j_0}(\Delta^2 + V(x))\|_{L^2_x} \leq 1/2
\]
for \( eN^{\tau_2 + \delta(s_1 - \theta)} \leq \tilde{c}(s_1) \) small enough. Hence it follows from (3.16) and Lemma 2.11 that
\[
\left|A^{-1}_{N,j_0}(\epsilon, \lambda, u, \theta)\right| \leq N^{\tau_2 + \delta}, \quad \forall s \in [s_0, s_1 - \theta].
\]

Based on the fact (3.17), we have the following definitions.

Definition 3.2 (N-good/N-bad matrix). The matrix \( A \in \mathbb{M}_0^\delta \) with \( \mathfrak{c} \subset \mathfrak{C} \) and \( \text{diam}(\mathfrak{c}) \leq 4N \) is N-good if \( A \) is invertible with
\[
\left|A^{-1}\right| \leq N^{\tau_2 + \delta}, \quad \forall s \in [s_0, s_1 - \theta].
\]

Otherwise \( A \) is N-bad.

Definition 3.3 (Regular/Singular sites). The index \( n = (l, j) \in \mathfrak{N} \) is regular for \( A \) if \( |\tilde{\mu}_n| \geq \tilde{\Theta}, \) where
\[
A^n_\theta := \tilde{\mu}_nI_{s_j} \quad \text{with} \quad \tilde{\mu}_\theta := \tilde{\mu}_n(\epsilon, \lambda, \theta) := -\lambda_0 l + \theta + 2N - \epsilon n.
\]

Note that \( \tilde{\mu} \) denotes the average of \( a(\varphi, x) \) on \( \mathbb{T}^\nu \times M, \) where \( a(\varphi, x) := (\partial_\alpha f)(\varphi, x, u(\varphi, x)) \), and that \( \tilde{\Theta} \) is given in Proposition 3.8.

Definition 3.4 ((A, N)-regular/(A, N)-singular site). For \( A \in \mathbb{M}_0^\delta, \) we say that \( n \in \mathfrak{A} \subset \mathfrak{N} \) is (\( A, N \))-regular if there exists \( \mathfrak{c} \subset \mathfrak{A} \) with \( \text{diam}(\mathfrak{c}) \leq 4N, \text{d}(n, \mathfrak{A} \setminus \mathfrak{c}) \geq N \) such that \( A^n_\theta \) is N-good.

Definition 3.5 ((A, N)-good/(A, N)-bad site). The index \( n = (l, j) \in \mathfrak{N} \) is (\( A, N \))-good if it is regular for \( A \) or (\( A, N \))-regular. Otherwise we say that \( n \) is (\( A, N \))-bad.

Define
\[
\mathbb{B}_N(j_0) := \{ \theta \in \mathbb{R} : A_{N,j_0}(\epsilon, \lambda, u, \theta) \text{ is N-bad} \}.
\]
\begin{definition}[\(N\)-good/\(N\)-bad parameters] A parameter \(\lambda \in \Lambda\) is \(N\)-good for \(\mathcal{A}\) if one has that, for all \(j_0 \in \Gamma_+(M)\),
\begin{equation}
\mathcal{B}_N(j_0) \subset \bigcup_{q=1}^{N^{\nu+d+r+5}} I_q, \text{ where } I_q = I_q(j_0) \text{ intervals with } \operatorname{meas}(I_q) \leq N^{-\tau}.
\end{equation}
Otherwise we say that \(\lambda\) is \(N\)-bad.
\end{definition}

In addition denote
\begin{equation}
\mathcal{G}_N(u) := \{ \lambda \in \Lambda : \lambda \text{ is } N\text{-good for } \mathcal{A} \}.
\end{equation}

As a result we have the following lemma:

\begin{lemma}
There exist \(N_0 := N_0(s_2, \kappa_0, V) \in \mathbb{N}\) and \(\tilde{c}(s_1) > 0\) such that if \(\epsilon N_0^{r_2+\delta(s_1-\theta)} \leq \tilde{c}(s_1)\), then \(\forall \kappa_0^{\frac{1}{s}} < N_0^{1/\chi} \leq N \leq N_0, \forall \|u\|_{s_1} \leq 1, \forall \epsilon \in [0, \epsilon_0],\) we have \(\mathcal{G}_N(u) = \Lambda\), where \(\mathcal{G}_N(u)\) is defined in (3.21).
\end{lemma}

\textbf{Proof.} Denote by \(\hat{\lambda}_{j,p}, p = 1, \ldots, \varrho_j\) the eigenvalues of \(\hat{P}_{N_0,j_0}(\Delta^2 + V(x))|_{\mathcal{B}_{N_0,j_0}}\). It follows from the fact of (3.17) deduced by (3.14) and Definition 3.2 that, \(\forall|l, j - j_0| \leq N, \forall 1 \leq p \leq \varrho_j,\)
\begin{equation}
| - (\lambda \omega_0 \cdot l + \theta)^2 + \hat{\lambda}_{j,p}| > N^{-\tau} \Rightarrow \mathcal{A}_{N,j_0}(\lambda, \epsilon, \theta) \text{ is } N\text{-good,}
\end{equation}
which carries out
\begin{equation}
\mathcal{B}_N(j_0) \subset \bigcup_{|l, j - j_0| \leq N, 1 \leq p \leq \varrho_j} \{ \theta \in \mathbb{R} : | - (\lambda \omega_0 \cdot l + \theta)^2 + \hat{\lambda}_{j,p}| \leq N^{-\tau} \}.
\end{equation}

Assumption (1.4) implies \(\hat{\lambda}_{j,p} \geq \kappa_0 > 0\). Hence, for all \(N > \kappa_0^{\frac{1}{s}}\), we get
\begin{equation}
\{ \theta \in \mathbb{R} : | - (\lambda \omega_0 \cdot l + \theta)^2 + \hat{\lambda}_{j,p}| \leq N^{-\tau} \} \subset \mathcal{J}_1 \cup \mathcal{J}_2,
\end{equation}
where
\begin{align*}
\mathcal{J}_1 &= \left\{ \theta \in \mathbb{R} : - \sqrt{\hat{\lambda}_{j,p} + N^{-\tau}} \leq \theta + \lambda \omega_0 \cdot l \leq - \sqrt{\hat{\lambda}_{j,p} - N^{-\tau}} \right\}, \\
\mathcal{J}_2 &= \left\{ \theta \in \mathbb{R} : \sqrt{\hat{\lambda}_{j,p} - N^{-\tau}} \leq \theta + \lambda \omega_0 \cdot l \leq \sqrt{\hat{\lambda}_{j,p} + N^{-\tau}} \right\}.
\end{align*}

It is easy that, for \(q = 1, 2,\)
\begin{align*}
\operatorname{meas}(\mathcal{J}_q) &= \sqrt{\hat{\lambda}_{j,p} + N^{-\tau}} - \sqrt{\hat{\lambda}_{j,p} - N^{-\tau}} = \frac{2N^{-\tau}}{\sqrt{\hat{\lambda}_{j,p} + N^{-\tau}} + \sqrt{\hat{\lambda}_{j,p} - N^{-\tau}}} \\
&\leq \frac{2N^{-\tau}}{\sqrt{\hat{\lambda}_{j,p}}} \leq \frac{2N^{-\tau}}{\sqrt{\kappa_0}}.
\end{align*}
Since \(|(l, j - j_0)| \leq N, 1 \leq p \leq \varrho_j\) with \(\varrho_j \leq |j + \rho|^{d-\tau},\) we obtain
\begin{equation}
\mathcal{B}_N(j_0) \subset \bigcup_{q=1}^{K_1CN^{\nu+d}} I_q, \text{ where } I_q \text{ intervals with } \operatorname{meas}(I_q) \leq N^{-\tau},
\end{equation}
where \(K_1 = [2/\sqrt{\kappa_0}] + 1.\) The symbol \([ \cdot ]\) denotes the integer part. \(\square\)

Our goal is to show that a matrix \(\mathcal{A}\) at the larger scale \(N',\) where
\begin{equation}
N' = N^\chi \text{ with } \chi > 1
\end{equation}
is \(N'\)-good under some conditions, see (3.23)-(3.25) and (A1)-(A3) in proposition 3.8.
Proposition 3.8. Assume
\[ \delta \in (0, 1/2), \quad \tau_2 > 2\tau + \nu + r + 1, \quad C_1 := C_1(\nu, d, r) \geq 2, \quad \chi \in [\chi_0, 2\chi_0], \quad (3.23) \]
\[ \chi_0(\tau_2 - 2\tau - \nu - r) > 3(\epsilon + C_1(s_0 + \nu + r)), \quad \chi_0\delta > C_1, \quad (3.24) \]
\[ 3\epsilon + 2\chi_0(\tau + \nu + r) + C_1s_0 < s_1 - \delta \leq s_2, \quad (3.25) \]
where \( \epsilon := \tau_2 + \nu + r + s_0. \) For all given \( \Upsilon > 0, \) there exist \( \tilde{\Theta} := \tilde{\Theta}(\Upsilon, s_1) > 0 \) large enough and \( N(\Upsilon, \tilde{\Theta}, s_2) \in \mathbb{N} \) such that: for all \( N \geq N(\Upsilon, \tilde{\Theta}, s_2) \) and \( \mathfrak{A} \in \mathfrak{M} \) with \( \text{diam}(\mathfrak{A}) \leq 4N', \) if \( \mathfrak{A} \in \mathcal{M}_\mathfrak{A}^3 \) satisfies
\[ (A1) \ |Q|_{s_1-\delta} \leq \Upsilon \text{ with } Q = \mathfrak{A} - \text{Diag}(\mathfrak{A}), \]
\[ (A2) \ |A^{-1}|_0 \leq (N')^r, \]
\[ (A3) \text{ The set of } (\mathfrak{A}, N)\text{-bad sites } \mathfrak{B} \text{ admits a partition } \bigcup \alpha \mathscr{O}_\alpha \text{ into disjoint clusters with} \]
\[ \text{diam}(\mathscr{O}_\alpha) \leq N^{C_1}, \quad \text{d}(\mathscr{O}_\alpha, \mathscr{O}_\beta) \geq N^2, \quad \forall \alpha \neq \beta, \quad (3.26) \]
then \( \mathfrak{A} \) is \( N'\)-good with
\[ |A^{-1}|_s \leq \frac{1}{4} (N')^r ((N')^r + |Q|_s), \quad \forall s \in [s_0, s_2]. \quad (3.27) \]

Proof. The proof of the proposition is shown in the Appendix. \( \square \)

3.2. Separation properties of bad sites. Let us check that the assumption \( (A3) \) in Proposition 3.8 holds.

Definition 3.9 ((\( \mathfrak{A}(u, \theta), N)\)-strongly-regular/(\( \mathfrak{A}(u, \theta), N)\)-weakly-singular site). The index \( n = (l, j) \in \mathfrak{N} \) is \( (\mathfrak{A}(u, \theta), N)\)-strongly-regular if \( \mathfrak{A}_{N,l,j}(\epsilon, \lambda, u, \theta) \) (see (2.16)) is \( N \)-good, where \( \mathfrak{A}(\epsilon, \lambda, u, \theta) \) is defined in (2.15). Otherwise \( n \) is \( (\mathfrak{A}(u, \theta), N)\)-weakly-singular.

Definition 3.10 ((\( \mathfrak{A}(u, \theta), N)\)-strongly-good/(\( \mathfrak{A}(u, \theta), N)\)-weakly-bad site). The index \( n = (l, j) \in \mathfrak{N} \) is \( (\mathfrak{A}(u, \theta), N)\)-strongly-good if it is regular for \( \mathfrak{A}(\epsilon, \lambda, u, \theta) \) or all the sites \( n' = (l', j') \) with \( \text{d}(n, n') \leq N \) are \( (\mathfrak{A}(u, \theta), N)\)-strongly-regular. Otherwise \( n \) is \( (\mathfrak{A}(u, \theta), N)\)-weakly-bad.

Lemma 3.11. For \( j_0 \in \Gamma_+^{\mathfrak{M}}, \chi \in [\chi_0, 2\chi_0], \) if the site \( n = (l, j) \in \mathfrak{N} \) with \( |l| \leq N', |j - j_0| \leq N' \) is \( (\mathfrak{A}(u, \theta), N)\)-strongly-good, then it is \( (\mathfrak{A}_{N,l,j_0}(\epsilon, \lambda, u, \theta), N)\)-good.

Proof. Let \( \mathfrak{R} = \mathfrak{M} \times \mathfrak{J} \) with
\[ \mathfrak{M} := [-N', N']^r \cap \mathbb{Z}^r, \quad \mathfrak{J} := \Bigl\{ j_0 + \left\{ \sum_{p=1}^r j_k w_k : j_k \in [-N', N'] \right\} \Bigr\} \cap \Gamma_+^{\mathfrak{M}}. \]

If \( n = (l, j) \in \mathfrak{N} \) with \( |l| \leq N', |j - j_0| \leq N' \) is regular for \( \mathfrak{A}(\epsilon, \lambda, u, \theta), \) then Definition 3.5 and (2.19) verify that it is \( (\mathfrak{A}_{N,l,j_0}(\epsilon, \lambda, u, \theta), N)\)-good.

If \( n = (l, j) \in \mathfrak{N} \) with \( |l| \leq N', |j - j_0| \leq N' \) is \( (\mathfrak{A}(u, \theta), N)\)-strongly-regular, and set \( l := (l_k)_{1 \leq k \leq r}, \quad \tau_k := (j_0)_k - N', \quad l_k := (j_0)_k + N', \quad \bar{\tau}_k := -N', \quad \bar{l}_k := N', \) then the \( N \)-ball of \( n \) is defined as \( \mathfrak{F}_N := \mathfrak{F}(n, N) := \mathfrak{M}_N \times \mathfrak{J}_N, \) where
\[ \mathfrak{M}_N := \left( \prod_{k=1}^r W_k \right) \cap \mathbb{Z}^r, \quad \mathfrak{J}_N := \left\{ \sum_{k=1}^r j_k w_k : j_k \in J_k \right\} \cap \Gamma_+^{\mathfrak{M}}, \quad \text{with} \]
\[ l_k - \bar{\tau}_k > N, \quad \bar{l}_k - l_k > N \Rightarrow W_k := [l_k - N, l_k + N], \]
\[ l_k - \bar{\tau}_k \leq N, \quad \bar{\tau}_k - l_k > N \Rightarrow W_k := [\bar{\tau}_k, \bar{\tau}_k + 2N], \]
\[ l_k - \bar{\tau}_k > N, \quad \bar{\tau}_k - l_k \leq N \Rightarrow W_k := [\bar{l}_k - 2N, \bar{\tau}_k]. \]
and
\[ j_k - r_k > N, \quad t_k - j_k > N \Rightarrow J_k := [j_k - N, j_k + N], \]
\[ j_k - r_k \leq N, \quad t_k - j_k > N \Rightarrow J_k := [r_k, t_k + 2N], \]
\[ j_k - r_k > N, \quad t_k - j_k \leq N \Rightarrow J_k := [t_k - 2N, t_k]. \]

It is obvious that \( d(n, s) > N \) and \( \text{diam}(\mathcal{N}_N) \leq 2N < 4N \). With the help of (3.3), there exists \( n' = (l', j') \in s \) with \( d(n, n') \leq N \) such that
\[ \mathcal{N}_N = \left( (l' + [-N, N]) \times \left( j' + \left\{ \sum_{k=1}^r l_k w_k : l_k \in [-N, N] \right\} \right) \right) \cap \mathcal{N}. \]

Since \( n \) is \((A(u, \theta), N)\)-strongly-regular, by Definition 3.10 then
\[ |(A_{\delta, N})^{-1}(\epsilon, \lambda, u, \theta)|_s = |(A_{\nu, j'})^{-1}(\epsilon, \lambda, u, \theta)|_s \leq N^{r_2 + \delta s}. \]

Lemma 3.11 establishes that if \( n_0 = (l_0, j_0) \in \mathcal{N} \) is \((A_{\nu, j_0}(\epsilon, \lambda, u, \theta), N)\)-bad, then it it \((A(u, \theta), N)\)-weakly-bad for \( A(\epsilon, \lambda, u, \theta) \) with \( |l - l_0| \leq N, \quad |j - j_0| \leq N \). Our goal is to get the upper bound of the number of \((A(u, \theta), N)\)-weakly-bad sites \((l_0, j_0)\) with \( |l_0| \leq N' \).

**Lemma 3.12.** Let \( \lambda \) be \( N \)-good for \( A(\epsilon, \lambda, u, \theta) \). For \( j_0 \in \Gamma_+(M), \chi \in [\chi_0, 2\chi_0], \) the number of \((A(u, \theta), N)\)-weakly-singular sites \((l_0, j_0)\) with \( |l_0| \leq 2N' \) does not exceed \( N^{\nu + d + r + 5} \).

**Proof.** Definition 3.9 implies that \( A_{\nu, j_0}(\epsilon, \lambda, u, \theta) \) is \( N \)-bad if \( (l_0, j_0) \) is \((A(u, \theta), N)\)-weakly-singular. Using the covariance property (2.21), we obtain that \( A_{\nu, j_0}(\epsilon, \lambda, u, \theta + \lambda \omega_0 \cdot l_0) \) is \( N \)-bad, which leads to \( \theta + \lambda \omega_0 \cdot l_0 \in \mathcal{N} \) (recall (3.19)). By assumption that \( \lambda \) is \( N \)-good, we have that (3.20) holds. For \( \tau > 2\chi_0 \nu \), we claim that there exists at most one element \( \theta + \lambda \omega_0 \cdot l_0 \) with \( |l_0| \leq 2N' \) in each interval \( I_q \), which implies the conclusion of the lemma by (3.20).

Let us check the claim. Suppose that there exists \( l_0 \neq l'_0 \) with \( |l_0|, |l'_0| \leq 2N' \) such that \( \theta + \lambda \omega_0 \cdot l_0, \theta + \lambda \omega_0 \cdot l'_0 \in I_q \). Then
\[ |\lambda \omega_0 \cdot (l_0 - l'_0)| = |(\theta + \lambda \omega_0 \cdot l_0) - (\theta + \lambda \omega_0 \cdot l'_0)| \leq \text{meas}(I_q) \leq N^{-\tau}. \tag{3.28} \]

Moreover (1.3) gives that, for \( \lambda \in \Lambda = [1/2, 3/2], \)
\[ |\lambda \omega_0 \cdot l| \geq \gamma_0 |l|^{-\nu}, \quad \forall l \in Z^\nu \setminus \{0\}, \]
which carries out
\[ |\lambda \omega_0 \cdot (l_0 - l'_0)| \geq \frac{\gamma_0}{|l_0 - l'_0|^\nu} \geq \frac{\gamma_0}{(4N')^\nu} \frac{\gamma_0}{4^\nu} N^{-\chi \nu}. \]

If \( \tau > 2\chi_0 \nu \), then this leads to a contradiction to (3.28) for \( N \geq \tilde{N}(\gamma_0, \nu) \) large enough.

**Corollary 3.13.** Let \( \lambda \) be \( N \)-good for \( A(\epsilon, \lambda, u, \theta) \). For \( j_0 \in \Gamma_+(M), \) the number of \((A(u, \theta), N)\)-weakly-bad sites \((l_0, j_0)\) with \( |l_0| \leq N \) does not exceed \( N^{2\nu + d + 2r + 6} \).

**Proof.** Since \( |l - l_0| \leq N \Rightarrow |l| \leq N' + N \), Lemma 3.12 establishes that the number of \((A(u, \theta), N)\)-weakly-singular sites \((l, j)\) with \( |l - l_0| \leq N, |j - j_0| \leq N \) is bounded from above by \( N^{\nu + d + r + 5} \times 4^r N^r \). By Definition 3.10 each \((l_0, j_0)\), which is \((A(u, \theta), N)\)-weakly-bad, is included in some \( N \)-ball centered at an \((A(u, \theta), N)\)-weakly-singular site. Moreover each of these balls contain at most \( 4^r N^{\nu} \) sites of the form \((l, j_0)\). Hence the number of \((A(u, \theta), N)\)-weakly-bad sites is at most \( 4^r N^{\nu + d + 2r + 5} \times N^r \).
**Definition 3.14.** Denote by \( \{n_k, k \in [0, L] \cap N\} \) a sequence of site with \( n_k \neq n_{k'}, \forall k \neq k' \). For \( \bar{B} \geq 2 \), we call \( \{n_k, k \in [0, \bar{L}] \cap N\} \) a \( \bar{B} \)-chain of length \( \bar{L} \) with \( |n_{k+1} - n_k| \leq \bar{B}, \forall k = 0, \ldots, \bar{L} - 1 \).

**Lemma 3.15.** For \( \epsilon \) small enough, there exists \( C(\nu, d, r) > 0 \) such that, \( \forall \theta \in \mathbb{R} \) and \( \forall N \in \mathbb{N}^+ \), any \( \bar{B} \)-chain of the \( (A(u, \theta), N) \)-weakly-bad sites for \( A(\epsilon, \lambda, u, \theta) \) with \( |(l, j - j_0)| \leq N' \) has length \( \bar{L} \leq (\bar{B}N)^{C(\nu, d, r)} \).

**Proof.** Here we exploit that \( n \in \mathfrak{N} \) is singular if it is \((A(u, \theta), N)\)-weakly-bad. Denote by \( \{n_k, k \in [0, \bar{L}] \cap N\} \) a \( \bar{B} \)-chain of singular sites. Then

\[
\max \{|l_{k+1} - l_k|, |j_{k+1} - j_k|\} \leq \bar{B}, \quad \forall k = 0, \ldots, \bar{L} - 1.
\]

(3.29)

It follows from Definition 3.3 and the definition of \( \lambda \) that

\[
|-(\lambda \omega_0 \cdot l_k + \theta)^2 + (\|j_k + \rho\|^2 - \|\rho\|^2)^2 + m| < \bar{\Theta} + 1
\]

(3.30)

In fact, Definition 3.3 shows that

\[
|-(\lambda \omega_0 \cdot l + \theta)^2 + \lambda^2 - m| < \bar{\Theta}.
\]

Clearly, we obtain that \( \epsilon \bar{m} \bar{n} \geq 1 \) if \( \epsilon \) is small enough, which lead to (3.30). With the help of (3.30), we deduce

\[
\Rightarrow |-(\lambda \omega_0 \cdot l_k + \theta)^2 + (\|j_k + \rho\|^2 - \|\rho\|^2)^2| < \sqrt{\bar{\Theta} + 1 + m}
\]

or

\[
|\lambda \omega_0 \cdot l_k + \theta + \|j_k + \rho\|^2 - \|\rho\|^2| < \sqrt{\bar{\Theta} + 1 + m}.
\]

Then one of the following \( \theta \)-independent inequalities holds:

\[
|\pm(\lambda \omega_0 \cdot (l_{k+1} - l_k)) + (\|j_{k+1} + \rho\|^2 \pm \|j_k + \rho\|^2)| < 2 \left( \sqrt{\bar{\Theta} + 1 + m} + \|\rho\|^2 \right)
\]

which leads to

\[
\|j_{k+1} + \rho\|^2 \pm \|j_k + \rho\|^2 < 2 \left( \sqrt{\bar{\Theta} + 1 + m} + \|\rho\|^2 + \bar{B} \right).
\]

Combining this with the inequality

\[
\|j_{k+1} + \rho\|^2 \pm \|j_k + \rho\|^2 \leq \|j_{k+1} + \rho\|^2 \pm \|j_k + \rho\|^2
\]

yields

\[
\|j_k + \rho\|^2 - \|j_{k_0} + \rho\|^2 \leq 2 \left( \sqrt{\bar{\Theta} + 1 + m} + \|\rho\|^2 + \bar{B} \right) |k - k_0|.
\]

(3.31)

Due to (2.7), (3.29), (3.31) and the equality

\[
(j_k + \rho) \cdot (j_k - j_{k_0}) = \frac{1}{2} \left( \|j_k + \rho\|^2 - \|j_{k_0} + \rho\|^2 - \|j_k - j_{k_0}\|^2 \right),
\]

we obtain

\[
|(j_k + \rho) \cdot (j_k - j_{k_0})| \leq \left( \sqrt{\bar{\Theta} + 1 + m} + \|\rho\|^2 + \bar{B} \right) |k - k_0| + (b_2^2/2)|k - k_0|^2 \bar{B}^2
\]

\[
\leq \left( \sqrt{\bar{\Theta} + 1 + m} + \|\rho\|^2 + b_2^2 + 1 \right) |k - k_0|^2 \bar{B}^2.
\]

(3.32)

Define the following subspace of \( \mathbb{R}^t \) by

\[
\mathcal{F} := \text{span}_{\mathbb{R}} \{j_k - j_{k'} : k, k' = 0, \ldots, \bar{L}\} = \text{span}_{\mathbb{R}} \{j_k - j_{k_0} : k = 0, \ldots, \bar{L}\}.
\]

Let \( t \) be the dimension of \( \mathcal{F} \). Denote by \( \zeta_1, \ldots, \zeta_t \) a basis of \( \mathcal{F} \). It is clear that \( t \leq r \).

**Case1.** For all \( k_0 \), \( \zeta_1, \ldots, \zeta_t \) is a basis of \( \mathcal{F} \) and \( \zeta_{k_0} = \text{span}_{\mathbb{R}} \{j_k - j_{k_0} : |k - k_0| \leq \bar{L}, k = 0, \ldots, \bar{L}\} = \mathcal{F} \).

**Formula (3.29)** indicates that

\[
|\zeta_p| = |j_p - j_{k_0}| \leq |p - k_0| \bar{B} \leq \bar{L}^v \bar{B}, \quad p = 0, \ldots, \bar{L}.
\]

(3.33)
Let \( \Pi_\mathcal{F} \) denote the orthogonal projection on \( \mathcal{F} \). Then

\[
\Pi_\mathcal{F}(j_{k_0} + \rho) = \sum_{p=1}^{t} z_p \zeta_p
\]  

(3.34)

for some \( z_p \in \mathbb{R}, p = 1, \cdots, t \). Hence we get

\[
\Pi_\mathcal{F}(j_{k_0} + \rho) \cdot \zeta_{p'} = \sum_{p=1}^{t} z_p \zeta_p \cdot \zeta_{p'}.
\]

Based on above fact, we consider the linear system \( Qz = y \), where

\[
Q = (Q_{pp'})_{p,p'=1,\cdots,t} \quad \text{with} \quad Q_{pp'} = \zeta_p \cdot \zeta_{p'}, \quad y_{p'} = \Pi_\mathcal{F}(j_{k_0} + \rho) \cdot \zeta_{p'} = (j_{k_0} + \rho) \cdot \zeta_{p'}.
\]

It follows from (3.32)-(3.33) that

\[
|y_{p'}| \leq \left( \sqrt{\Theta + 1 + m + \|\rho\|^2 + b_2^2 + 1} \right) (\tilde{L}^\nu \tilde{B})^2, \quad |Q_{pp'}| \leq (\tilde{L}^\nu \tilde{B})^2.
\]

(3.35)

In addition, by formula (2.2), we verify

\[
3^t \det(Q) \in \mathbb{Z}, \quad \text{namely} \quad 3^t \det(Q) \geq 1.
\]

(3.36)

Let \( Q^* \) be the adjoint matrix of \( Q \). It follows from Hadamard inequality that

\[
|\langle Q^* \rangle_{pp'}| \leq \prod_{p \neq p', 1 \leq p \leq t} \left( \sum_{p \neq p', 1 \leq p' \leq t} |Q_{pp'}|^2 \right)^{1/2},
\]

which leads to

\[
|\langle Q^* \rangle_{pp'}| \leq (t - 1) \frac{3^t}{t} (\tilde{L}^\nu \tilde{B})^{2(t-1)}.
\]

Based on above inequality, (3.35)-(3.36) and Cramer’s rule, we can obtain

\[
|z_p| \leq \sum_{p'=1}^{t} |Q^{-1}_{pp'} y_{p'}| \leq 3^t t \left( \sqrt{\Theta + 1 + m + \|\rho\|^2 + b_2^2 + 1} \right) (\tilde{L}^\nu \tilde{B})^2 t.
\]

Combining this with formulae (3.33)-(3.34) derives

\[
\Pi_\mathcal{F}(j_{k_0} + \rho) \leq \sqrt{t} |z_p| |\zeta_p| \leq 3^t t^{1+1} \left( \sqrt{\Theta + 1 + m + \|\rho\|^2 + b_2^2 + 1} \right) (\tilde{L}^\nu \tilde{B})^{2t+1}.
\]

As a consequence

\[
|j_{k_1} - j_{k_2}| = |(j_{k_1} - j_{k_0}) - (j_{k_2} - j_{k_0})| = |\Pi_\mathcal{F}(j_{k_1} + \rho) - \Pi_\mathcal{F}(j_{k_2} + \rho)|
\]

\[
\leq 2^r \cdot 3^t \cdot t^{r+1} \left( \sqrt{\Theta + 1 + m + \|\rho\|^2 + b_2^2 + 1} \right) (\tilde{L}^\nu \tilde{B})^{2r+1}.
\]

Counted without multiplicity, the number of \( j_k \in \Gamma^+(M) \) is bounded from above by

\[
4^r \left( 2 \cdot 3^t \cdot t^{r+1} \left( \sqrt{\Theta + 1 + m + \|\rho\|^2 + b_2^2 + 1} \right) (\tilde{L}^\nu \tilde{B})^{2r+1} \right)^r,
\]

namely

\[
2 \left\{ j_k : 0 \leq k \leq \tilde{L} \right\} \leq 2^{3^t} \cdot 3^t \cdot t^{r+1} \left( \sqrt{\Theta + 1 + m + \|\rho\|^2 + b_2^2 + 1} \right) (\tilde{L}^\nu \tilde{B})^{2r+1}.
\]

(3.37)

For each \( k_0 \in [0, \tilde{L}] \), Corollary 3.13 shows that the number of \( k \in [0, \tilde{L}] \) such that \( j_k = j_{k_0} \) does not exceed \( N^{2r+d+2r+6} \). Hence, in view of (3.37), the following holds:

\[
\tilde{L} \leq 2^{3^t} \cdot 3^t \cdot t^{r+1} \left( \sqrt{\Theta + 1 + m + \|\rho\|^2 + b_2^2 + 1} \right) (\tilde{L}^\nu \tilde{B})^{2r+1} N^{2r+d+2r+6}.
\]

(3.38)
If \( v < \frac{1}{2(2r+1)} \), then (3.38) derives

\[
\tilde{L} \leq 2^{3r} \frac{2^r}{3} r^{2r(1+\frac{1}{2})} (\sqrt{\tilde{\Theta} + 1 + m + \|\rho\|^2 + b_2}^r) \tilde{B}^{r(2r+1)} N^{2(\nu + d + 2r) + 6} \]

\[
\Rightarrow \tilde{L} \leq 2^{6r} \frac{2^r}{3} r^{2r(1+\frac{1}{2})} (\sqrt{\tilde{\Theta} + 1 + m + \|\rho\|^2 + b_2}^r) \tilde{B}^{2r(2r+1)} N^{2(\nu + d + 2r) + 6}.
\]

Case 2. If there exist some \( k_0 \in [0, \tilde{L}] \cap \mathbb{N} \) such that \( \dim \mathcal{F}_{k_0} \leq r - 1 \), for \( k_0 \in \tilde{F} \), then we consider

\[
\mathcal{F}_{k_0} := \text{span}_{\mathbb{R}} \{ j_k - j_{k_0} : |k - k_0| < \tilde{L}^v, k \in \tilde{F} \} = \text{span}_{\mathbb{R}} \{ j_k - j_{k_0} : k \in \tilde{F} \}
\]

where

\[
\tilde{L}_1 = \tilde{L}^v, \quad \tilde{F} := \{ k : |k - k_0| < \tilde{L}^v \} \cap ([0, \tilde{L}] \cap \mathbb{N}).
\]

The upper bound of \( \tilde{L}_1 \) can be proved by the same method as employed on \( \tilde{L} \), namely

\[
\tilde{L}_1 = \tilde{L}^v \leq 2^{6r} \frac{2^r}{3} r^{2r(1+\frac{1}{2})} (\sqrt{\tilde{\Theta} + 1 + m + \|\rho\|^2 + b_2}^r) \tilde{B}^{2r(2r+1)} N^{2(\nu + d + 2r) + 6}.
\]

In addition the iteration is carried out at most \( r \) steps owing to the fact \( t \leq r \). Hence

\[
\tilde{L}_r = \tilde{L}^{rv} \leq 2^{6r} \frac{2^r}{3} r^{2r(1+\frac{1}{2})} (\sqrt{\tilde{\Theta} + m + \|\rho\|^2 + b_2}^r) \tilde{B}^{2r(2r+1)} N^{2(\nu + d + 2r) + 6}.
\]

Hence there exists some constant \( C(\nu, d, r) > 0 \) such that \( \tilde{L} \leq (\tilde{B} N)^{C(\nu, d, r)} \). \( \square \)

In addition, the following equivalence relation is defined.

**Definition 3.16.** We say that \( \tilde{\xi} \equiv \tilde{\eta} \) if there is a \( N^2 \)-chain \( \{ n_k, k \in [0, \tilde{L}] \cap \mathbb{N} \} \) connecting \( \tilde{\xi} \) to \( \tilde{\eta} \), namely

\[
n_0 = \tilde{\xi}, n_{\tilde{L}} = \tilde{\eta}.
\]

Let us state the following proposition.

**Proposition 3.17.** If we suppose

\[
(1) \ \lambda \text{ is } N\text{-good for } A, \quad (2) \ \tau > 2\chi_0 \nu,
\]

then there exist \( C_1 := C_1(\nu, d, r) \geq 2 \) and \( \hat{N} := \hat{N}(\nu, d, r, \gamma_0, m, \tilde{\Theta}, \rho, b_2) \) such that, \( \forall \nu, \tau \in \mathbb{R}, \) the \( (A(\nu, \theta), N) \)-weakly-bad sites for \( A(\epsilon, \lambda, u, \theta) \) with \( |l| \leq \hat{N}', |j - j_0| \leq \hat{N}' \) admits a partition \( \cup_{\alpha} \mathcal{D}_{\alpha} \), where

\[
\text{diam}(\mathcal{D}_{\alpha}) \leq N^{C_1}, \quad \text{d}(\mathcal{D}_{\alpha}, \mathcal{D}_{\beta}) > N^2, \quad \forall \alpha \neq \beta.
\]

**Proof.** Let \( \tilde{B} = N^2 \). By Definition 3.16 the equivalence relation induces that a partition of the \( (A(\nu, \theta), N) \)-weakly-bad sites for \( A(\epsilon, \lambda, u, \theta) \) with \( |l| \leq \hat{N}', |j - j_0| \leq \hat{N}' \) satisfies

\[
\text{diam}(\mathcal{D}_{\alpha}) \leq \tilde{L} \tilde{B} \leq N^{C_1}, \quad \text{d}(\mathcal{D}_{\alpha}, \mathcal{D}_{\beta}) > N^2, \quad \forall \alpha \neq \beta,
\]

where \( C_1 := C_1(\nu, d, r) = 3C(\nu, d, r) + 2 \). \( \square \)

Thus the assumption (A3) in Proposition 3.8 holds by Proposition 3.17 for \( j_0 = 0, \theta = 0 \).

### 3.3. Measure and “complexity” estimates.

We define

\[
\mathcal{B}^0_N(j_0) := \mathcal{B}^0_N(j_0, \epsilon, \lambda, u) := \left\{ \theta \in \mathbb{R} : \| A_{N,j_0}^{-1}(\epsilon, \lambda, u, \theta) \|_0 > N^{\tau} \right\}
\]

\[
= \left\{ \theta \in \mathbb{R} : \exists \text{ an eigenvalue of } A_{N,j_0}(\epsilon, \lambda, u, \theta) \text{ with modulus less than } N^{-\tau} \right\}.
\]
where \( \| \cdot \|_0 \) is the operator \( L^2 \)-norm. Moreover we also define

\[
\mathcal{B}_N^0(u) := \left\{ \lambda \in A : \forall j_0 \in \Gamma_+(M), \mathcal{B}_N^0(j_0) \subset \bigcup_{q=1}^{N^{\nu+d+r+5}} I_q, \text{ where } I_q = I_q(j_0) \right\}.
\]

are disjoint intervals with measure \( \text{meas}(I_q) \leq N^{-\tau} \).

(3.40)

It follows from (2.13), (2.15), (2.17), (3.7), (3.12), \( \| \lambda \| \) by (2.7) and the definition of \( \eta \) denotes the cardinality of the set \( E \).}

Lemma 3.18. Let \( \hat{\mu}_k(M_1), \hat{\mu}_k(M_2) \) be eigenvalues of \( M_1, M_2 \) respectively, where \( M_p, p = 1, 2 \) are self-adjoint matrices of the same dimension, then \( \hat{\mu}_k(M_p), p = 1, 2 \) are ranked in nondecreasing order with

\[
|\hat{\mu}_k(M_1) - \hat{\mu}_k(M_2)| \leq \|M_1 - M_2\|_0.
\]

(3.41)

Lemma 3.19. Let \( M(\eta) \) be a family of self-adjoint matrices in \( M^\mathcal{E}_\mathfrak{g} \) with \( C^1 \) depending on the parameter \( \eta \in \mathcal{H} \subset \mathbb{R} \). Assume \( \partial_\eta M(\eta) \geq b \tau \) for some \( b > 0 \), then, for all \( a > 0 \), the Lebesgue measure

\[
\text{meas} \left( \{ \eta \in \mathcal{H} : \|M^{-1}(\eta)\|_0 \geq a^{-1} \} \right) \leq 2^\mathcal{E}ab^{-1},
\]

where \( \mathcal{E} \) denotes the cardinality of the set \( \mathcal{E} \). Furthermore one has

\[
\{ \eta \in \mathcal{H} : \|M^{-1}(\eta)\|_0 \geq a^{-1} \} \subset \bigcup_{1 \leq q \leq \mathcal{E}} I_q, \text{ with } \text{meas}(I_q) \leq 2ab^{-1}.
\]

\( \|M^{-1}(\eta)\|_0 := \infty \) if \( M(\eta) \) is not invertible.

Proof. The proof is given by Lemma 5.1 of [1].

Letting \( v \geq 1 \) and

\[
N \geq 2\|\rho\|,
\]

(3.42)

by (2.7) and the definition of \( \lambda_j \), simple calculation yields

\[
\lambda_j^2 > v^2(v - 1)^2N^4 \quad \text{if} \quad |j| > vb_1^{-1}N,
\]

(3.43)

\[
\lambda_j^2 \leq 4v^2(v + 1)^2(b_2/b_1)^4N^4 \quad \text{if} \quad |j| \leq vb_1^{-1}N,
\]

(3.44)

where \( b_1, b_2 \) are given in (2.7). In addition, we assume that

\[
N \geq \bar{N}(V, \nu, d, \rho) > 0 \text{ large enough, and } \varepsilon \kappa_0^{-1}(\|T''\|_0 + \|\partial_\lambda T''\|_0) \leq c
\]

(3.45)

for some constant \( c > 0 \).

Lemma 3.20. \( \forall j_0 \in \Gamma_+(M), \) with \( |j_0| > \frac{b_1+3}{b_1}N, \forall \lambda \in A, \) we have

\[
\mathcal{B}_N^0(j_0) \subset \bigcup_{q=1}^{N^{\nu+d+2}} I_q, \text{ where } I_q = I_q(j_0) \text{ are intervals with } \text{meas}(I_q) \leq N^{-\tau}.
\]
Lemma 3.21. With the help of the upper bound of \( \frac{3}{2} N \), which leads to \( \lambda_j^2 > 36N^4 \) owing to (3.43). Combining this with \( |\lambda \omega_0| \leq \frac{3}{2} \) and \( |l| \leq N \) yields
\[
-(\lambda \omega_0 \cdot l + \theta)^2 + \lambda_j^2 > -(\frac{3}{2} + |\theta|)^2 + 36N^4 > 15N^2, \quad \forall |\theta| \leq 3N.
\]
Therefore, by means of (3.45) and (3.12), we deduce \( \hat{\mu}_{l,j,p}(\theta) \geq N^2 \), which implies
\[
B_{N,j}(j_0) \cap [-3N, 3N] = \emptyset.
\]
Based on above fact, denote by
\[
B_{N}^+(j_0) = B_{N}(j_0) \cap (3N, +\infty), \quad B_{N}^-(j_0) = B_{N}(j_0) \cap (-\infty, -3N).
\]
We restrict our attention for \( \theta > 3N \). Define
\[
E := \{(l, j, p) \in \mathbb{N} \times \mathbb{N} : |l| \leq N, |j-j_0| \leq N, p \leq \mathcal{O}_j \}.
\]
It follows from the inequality \( \mathcal{O}_j \leq \|j + \rho\|^{d-r} \) that \( \|E\| \leq CN^u+d. \) In addition,
\[
\mathcal{O}_j(-A_{N,j_0}(\epsilon, \lambda, \theta)) = \text{diag} \max_{|l| \leq N, |i-j| \leq N} 2(\lambda \omega_0 \cdot l + \theta) I \geq (6N-3N) I = 3NI.
\]
Applying Lemma 3.19 for \( a = N^{-\tau}, b = 3N \) and \( \mathcal{O} \leq CN^{u+d}, \) we have
\[
B_{N}^+(j_0) \subset \bigcup_{q=1}^{N_d+1} I_q,
\]
where the intervals \( I_q = I_q(j_0) \) satisfy \( \text{meas}(I_q) \leq 2N^{-\tau}(3N)^{1} \leq N^{-\tau} \).

The proof on \( B_{N}^+ \) can apply the similar step as above. For the sake of convenience, we omit the process. Therefore, we have
\[
B_{N}^+(j_0) \subset \bigcup_{q=1}^{N_d+2} I_q, \quad \text{where } I_q = I_q(j_0) \text{ are intervals with } \text{meas}(I_q) \leq N^{-\tau}.
\]

Now consider the case \( j_0 \leq \frac{b_1+3}{b_1} N \). We have to study the measure of the set
\[
B_{2,N}^+(j_0) := B_{2,N}^+(j_0; \epsilon, \lambda, u) := \left\{ \theta \in \mathbb{R} : \|A_{N,j_0}^{-1}(\lambda, \epsilon, u, \theta)\|_0 > N^{\tau/2} \right\}.
\]
With the help of the upper bound of \( \text{meas}(B_{2,N}^+(j_0)) \), a complexity estimate for \( B_{N}^+(j_0) \) can be obtained.

Lemma 3.21. \( \forall j_0 \in \Gamma_+(M) \), with \( |j_0| \leq \frac{b_1+3}{b_1} N \), \( \forall \lambda \in \Lambda \), we have
\[
B_{2,N}^+(j_0) \subset [-sN^2, sN^2] \quad \text{with} \quad s = 2((2b_1 + 4)^2 + (b_2/b_1)^2).
\]

Proof. If \( \|\theta\| > sN^2 \), then one has
\[
|\lambda \omega_0 \cdot l + \theta| \geq |\theta| - |\lambda \omega_0 \cdot l| > sN^2 - \frac{3}{2}N > 2(b_1 + 4)^2(b_2/b_1)^2N^2.
\]
Since \( |j-j_0| \leq N \Rightarrow |j| \leq \frac{2b_1+3}{b_1} N \), by (3.44), it gives
\[
\lambda_j^2 \leq (2(b_1 + 3)(b_1 + 4)) (b_2/b_1)^4 N^4.
\]
Then, due to (3.12) and (3.45), all the eigenvalues \( \hat{\mu}_{l,j,p}(\theta), p = 1, \cdots, \mathcal{O}_j \) of \( A_{N,j_0}(\epsilon, \lambda, u, \theta) \) satisfy that, for all \( |\theta| > sN^2 \),
\[
\hat{\mu}_{l,j,p}(\theta) = -(\lambda \omega_0 \cdot l + \theta)^2 + \lambda_j^2 + O(\|V\|_0 + \epsilon\|T\|_0) \leq -(b_2/b_1)^4 N^4.
\]
which implies the conclusion of the lemma. \( \square \)

**Lemma 3.22.** Let \( \mathfrak{h} = \text{meas}(\mathfrak{B}_{2,N}(j_0)) \). \( \forall j_0 \in \Gamma_+(\mathbf{M}) \), with \( |j_0| \leq \frac{b_0 + \beta_0}{b_0} N \), \( \forall \lambda \in \Lambda \), the following holds

\[
\mathfrak{B}_N^0(j_0) \subset \bigcup_{q=1}^{\infty} I_q,
\]

where \( I_q = I_q(j_0) \) are intervals with \( \text{meas}(I_q) \leq N^{-\tau} \).

**Proof.** For brevity, we write \( A_{N,j_0}(\theta) := A_{N,j_0}(\epsilon, \lambda, u, \theta) \). If \( \theta \in \mathfrak{B}_N^0(j_0) \), then there exists an eigenvalue \( \hat{\mu}_{1,j_1,j_0}(A_{N,j_0}(\theta)) \) of \( A_{N,j_0}(\theta) \) with

\[
|\hat{\mu}_{1,j_1,j_0}(A_{N,j_0}(\theta))| \leq N^{-\tau}.
\]

Consequently, we have that, for \( |\Delta \theta| \leq 1 \),

\[
\|A_{N,j_0}(\theta + \Delta \theta) - A_{N,j_0}(\theta)\|_0 = \|\text{Diag}([|\theta| \leq N, |j-j_0| \leq N](\lambda \omega_0 \cdot l + \theta + \Delta \theta)^2 - (\lambda \omega_0 \cdot l + \theta + \Delta \theta)^2)I_{j_0}\|_0
\]

\[
= |(2 \lambda \omega_0 + 2 \theta + \Delta \theta)(\Delta \theta)| \leq (4N + 2|\theta| + 1)|\Delta \theta|.
\]

If \( (4N + 2|\theta| + 1)|\Delta \theta| \leq N^{-\tau} \), then it follows from (3.41) that

\[
|\hat{\mu}_{1,j_1,j_0}(A_{N,j_0}(\theta + \Delta \theta)) - \hat{\mu}_{1,j_1,j_0}(A_{N,j_0}(\theta))| \leq N^{-\tau},
\]

which gives rise to

\[
\hat{\mu}_{1,j_1,j_0}(A_{N,j_0}(\theta + \Delta \theta)) \leq 2N^{-\tau}.
\]

Hence \( \theta + \Delta \theta \in \mathfrak{B}_{2,N}(j_0) \), that is

\[
(4N + 2|\theta| + 1)|\Delta \theta| \leq N^{-\tau} \Rightarrow \theta + \Delta \theta \in \mathfrak{B}_{2,N}(j_0).
\]

By Lemma 3.21 we have to guarantee

\[
(4N + 2\alpha N + 1)|\Delta \theta| \leq N^{-\tau} \quad \text{with} \quad \alpha = 2((2b_1 + 4)^2 + 1)(b_2/b_1)^2.
\]

This indicates \( |\Delta \theta| \leq c(\alpha) N^{-(\tau + 1)} \), which carries out

\[
[\theta - c(\alpha) N^{-(\tau + 1)}, \theta + c(\alpha) N^{-(\tau + 1)}] \subset \mathfrak{B}_{2,N}(j_0).
\]

Therefore \( \mathfrak{B}_N^0(j_0) \) is included in an union of intervals \( \mathcal{I}_p \) with disjoint interiors, namely

\[
\mathfrak{B}_N^0(j_0) \subset \bigcup_p \mathcal{I}_p \subset \mathfrak{B}_{2,N}(j_0) \quad \text{with} \quad \text{meas}(\mathcal{I}_p) \geq 2c(\alpha) N^{-(\tau + 1)}.
\]

Now we decompose each \( \mathcal{I}_p \) as an union of non-overlapping intervals with

\[
\frac{1}{2} c(\alpha) N^{-(\tau + 1)} \leq \text{meas}(I_q) \leq c(\alpha) N^{-(\tau + 1)}.
\]

Thus we get

\[
\mathfrak{B}_N^0(j_0) \subset \bigcup_{q=1}^{\infty} I_q \subset \mathfrak{B}_{2,N}(j_0),
\]

where \( I_q \) satisfies (3.46). Since \( I_q \) does not overlap, we deduce

\[
\frac{1}{2} c(\alpha) N^{-(\tau + 1)} q_0 \leq \sum_{q=1}^{q_0} \text{meas}(I_q) \leq \text{meas}(\mathfrak{B}_{2,N}(j_0)) =: \mathfrak{h},
\]

which gives \( q_0 \leq \mathfrak{C} h N^{\tau + 1} \). \( \square \)

Define

\[
\mathfrak{B}_{2,N}(j_0) := \mathfrak{B}_{2,N}(j_0 ; \epsilon, u) := \left\{ (\lambda, \theta) \in \Lambda \times \mathbf{R} : \|A_{N,j_0}^{-1}(\epsilon, \lambda, u, \theta)\|_0 > N^\tau / 2 \right\}.
\]
Lemma 3.23. \( \forall j_0 \in \Gamma_+(M) \), with \(|j_0| \leq \frac{b_1+3}{b_1} N \), \( \forall \lambda \in \Lambda \), there exists some constant \( \mathcal{C}_0 > 0 \) such that
\[
\text{meas}(\mathcal{B}^0_{2,N}(j_0)) \leq \mathcal{C}_0 N^{-\tau + \nu + d + 2}.
\] (3.47)

Proof. Let
\[
\xi = \frac{1}{\lambda^2}, \quad \zeta = \frac{\theta}{\lambda}, \quad (\xi, \zeta) \in [4/9, 4] \times [-2sN^2, 2sN^2]
\]
with \( s = 2((2b_1 + 4)^2 + 1)(b_2/b_1)^2 \). Thus we consider the following self-adjoint matrices
\[
L(\xi, \zeta) := \lambda^{-2} \mathcal{A}_{N,j_0}(\epsilon, \lambda, u, \theta) = \text{diag}_{[i \leq N, |j-j_0| \leq N} \left( (-\omega_0 \cdot l + \zeta)^2 I_0 \right)
+ \xi \tilde{P}_{N,j_0}(\Delta^2 + V(x))|_{\tilde{H}_{N,j_0}} - \epsilon \zeta T''(\epsilon, 1/\sqrt{\xi}).
\]
Formula (1.4) implies \( \tilde{P}_{N,j_0}(\Delta^2 + V(x))|_{\tilde{H}_{N,j_0}} \geq \kappa_0 I \), which leads to
\[
\partial_\xi L(\xi, \zeta) = \tilde{P}_{N,j_0}(\Delta^2 + V(x))|_{\tilde{H}_{N,j_0}} - \epsilon T'' - \frac{\epsilon}{2} \xi^{-1/2} \partial_\xi T'' \geq \kappa_0/2.\] (3.45)
Combining this with Lemma 3.19 for \( a = 8N^{-\tau}, b = \kappa_0/2, x \in C \), \( b \leq C N^{\nu + d} \), for each \( \zeta \in [-2sN^2, 2sN^2] \), we obtain
\[
\text{meas}(\{ \xi \in [4/9, 4] : ||L^{-1}(\xi, \zeta)||_0 > N^{\tau/8} \}) \leq 2CN^{\nu + d} 8N^{-\tau}/2 \kappa_0 \leq C' N^{-\tau + \nu + d}.
\]
Integrating in \( \zeta \in [-2sN^2, 2sN^2] \) yields \( ||L^{-1}(\xi, \zeta)||_0 \leq N^{\tau/8} \) except for \( \xi \) in a set of measure \( O(N^{-\tau + \nu + d + 2}) \).

In addition
\[
\left| \text{det} \left( \frac{\partial (\xi, \zeta)}{\partial (\lambda, \theta)} \right) \right| = \frac{1}{\lambda^4} \geq \frac{1}{6},
\]
Combining this with \( \mathcal{A}_{N,j_0}(\epsilon, \lambda, u, \theta) = \lambda^2 L(\xi, \eta) \) yields
\[
||\mathcal{A}^{-1}_{N,j_0}(\lambda, \epsilon, u, \theta)||_0 \leq \lambda^{-2} ||L^{-1}(\xi, \zeta)||_0 \leq \lambda^{-2} N^{\tau/8} \leq N^{\tau/2},
\]
for all \( (\lambda, \theta) \in \Lambda \times \mathbb{R} \) except for \( \lambda \) in a set of measure \( O(N^{-\tau + \nu + d + 2}) \). \( \square \)

Define
\[
\mathcal{W}_N(u) := \{ ||A^{-1}_N(\epsilon, \lambda, u)||_0 \leq N^{\tau} \},
\] (3.48)
where \( A^{-1}(\epsilon, \lambda, u), A^{-1}_N(\epsilon, \lambda, u) \) are defined in (2.11), (2.20) respectively. A similar computation as the proof of Lemma 3.23 verifies
\[
\text{meas}(\Lambda \setminus \mathcal{W}_N) \leq N^{-\tau + \nu + d + 3} \leq N^{-1} \quad (3.49)
\]
for \( \tau \geq \nu + d + 4 \). Moreover define the following set
\[
\mathcal{E}_N(j_0) = \left\{ \lambda \in \Lambda : \text{meas}(\mathcal{B}^0_{2,N}(j_0)) \geq \mathcal{C}^{-1} N^{-\tau + \nu + d + \tau + 3} \right\},
\] (3.50)
where \( \mathcal{C} \) is given in Lemma 3.22.

Lemma 3.24. \( \forall j_0 \in \Gamma_+(M) \), with \(|j_0| \leq \frac{b_1+3}{b_1} N \), \( \forall \lambda \in \Lambda \), one has \( \text{meas}(\mathcal{E}_N(j_0)) = O(N^{-\tau + 1}) \).

Proof. Fubini Theorem yeilds that, for \( r = \mathcal{C}^{-1} N^{-\tau + \nu + d + r + 3} \),
\[
\text{meas}(\mathcal{B}^0_{2,N}(j_0)) = \int_{\frac{1}{r}}^\frac{1}{r} \text{d}\lambda \text{meas}(\mathcal{B}^0_{2,N}(j_0)) \geq r \text{meas}\{ \lambda \in \Lambda : \text{meas}(\mathcal{B}^0_{2,N}(j_0)) \geq r \},
\]
which leads to
\[
\text{meas}\{ \lambda \in \Lambda : \text{meas}(\mathcal{B}^0_{2,N}(j_0)) \geq r \} \leq \frac{1}{r} \text{meas}(\mathcal{B}^0_{2,N}(j_0)) \leq \frac{1}{r} \mathcal{C}_0 N^{-\tau + \nu + d + 2} = \mathcal{C}_0 \mathcal{E}N^{-\tau - 1}.
\] (3.47) \( \square \)
**Proposition 3.25.** Assume \((3.45)\) holds. There exists some constant \(C_1 > 0\) such that the set \(\Lambda \setminus \mathcal{G}_N^0\) has measure
\[
\text{meas}(\Lambda \setminus \mathcal{G}_N^0(u)) \leq C_1 N^{-1},
\tag{3.51}
\]
where \(\Lambda, \mathcal{G}_N^0(u)\) are defined in (1.2), (3.40) respectively.

**Proof.** By definition \((3.50)\), \(\forall \lambda \notin \mathcal{C}_N(j_0), \) we have
\[
\text{meas}(\mathcal{B}_2^0(j_0)) \leq C^{-1} N^{-\tau+\nu+d+r+3}.
\]
Combining this with Lemma \((3.22)\) deduces that, \(\forall \lambda \notin \mathcal{C}_N(j_0), \forall j_0 \in \Lambda_+(M)\) with \(|j_0| \leq \frac{b_1+3}{b_1} N,\)
\[
\mathcal{B}_N^0(j_0) \subset \bigcup_{q=1}^{\frac{b_1}{N}} I_q \subset \bigcup_{q=1}^{\frac{b_1}{N}} I_q \text{ with } \text{meas}(I_q) \leq N^{-\tau}.
\tag{3.52}
\]
It follows from \((3.52)\) and Lemma \((3.20)\) that, \(\forall \lambda \notin \mathcal{C}_N(j_0), \forall j_0 \in \Gamma_+(M),\)
\[
\mathcal{B}_N^0(j_0) \subset \bigcup_{q=1}^{\frac{b_1}{N}} I_q, \text{ where } I_q = I_q(j_0) \text{ are intervals with } \text{meas}(I_q) \leq N^{-\tau}.
\]
Hence, for all \(j_0 \in \Gamma_+(M)\) with \(|j_0| \leq \frac{b_1+3}{b_1} N,\) we have
\[
\Lambda \setminus \mathcal{G}_N^0(u) \subset \bigcup_{|j_0| \leq (b_1+3)b_1^{-1} N} \mathcal{C}_N(j_0).
\]
Applying Lemma \((3.24)\) yields
\[
\text{meas}(\Lambda \setminus \mathcal{G}_N^0(u)) \leq \sum_{|j_0| \leq (b_1+3)b_1^{-1} N} \text{meas}(\mathcal{C}_N(j_0)) = O(N^{-1}).
\]
\[\square\]

### 3.4. Nash-Moser iteration

We first give the following iterative theorem. From now on, we fix
\[
\delta := 1/4, \quad \tau_1 := 3\nu + d + 1, \quad \lambda_0 := 3C_1 + 9, \quad (3.53)
\]
\[
\tau := \max\{\tau_1 + 3, 2\chi_0\nu + 1\} = \max\{3\nu + d + 4, 2\chi_0\nu + 1\}, \quad \tau_2 := 3\tau + 2(\nu + r) + (\nu + d), \tag{3.54}
\]
\[
s_0 := \nu + d, \quad s_1 := 12\chi_0(\tau + (\nu + r) + (\nu + d)), \quad s_2 := 12\tau_2 + 8s_1 + 12, \quad (3.55)
\]
and \(\sigma = \tau_2 + 3\delta s_1 + 3.\) \((3.56)\)

**Remark 3.26.** Formulae \((3.53), (3.55)\) satisfy \(\tau \geq \max\{\tau_1 + 1, \nu + d + 4\}\) (see \((3.49)\)), assumptions \((3.23), (3.25)\) and \((2)\) in Propositions \((3.8)\) and \((3.17)\) respectively. The choice of \(\tau_1\) is seen in Lemma \((3.37)\). Moreover the choices of \(\delta, s_1\) satisfy \(\delta s_1 \geq \varrho/2,\) where \(\varrho\) is given in Lemma \((2.6)\)

Setting \(\gamma > 0\), we restrict \(\lambda\) to the set
\[
\mathcal{H} := \left\{ \lambda \in \Lambda : \|((-\lambda \omega_0 \cdot l)^2) I_{b}, \hat{P}_{N_0,0}(\Delta^2 + V(x)) I_{H_{N_0,0}} \right\}^{-1} L_2^\frac{1}{2} \leq \gamma^{-1} N_{0}^{\tau_1}, \quad \forall |l| \leq N_0 \right\}
\tag{3.57}
\]
where \(\hat{\lambda}_{j,p}, p = 1, \cdots, \delta_j\) are eigenvalues of \(\hat{P}_{N_0,0}(\Delta^2 + V(x)) I_{H_{N_0,0}}.\)

**Theorem 3.27.** There exist \(\bar{c}, \bar{\gamma}\) (depending on \(\nu, d, r, V, \gamma_0, \kappa_0\)) such that if
\[
N_0 \geq 16\gamma^{-1}, \quad \gamma \in (0, \bar{\gamma}), \quad \epsilon_{0} N_{0}^{\alpha} \leq \bar{c},
\tag{3.58}
\]
then there is a sequence \((u_n)_{n \geq 0}\) of \(C^1\) maps \(u_n : [0, \epsilon_0] \times \Lambda \rightarrow H^{s_1}\) satisfying
\[
(F1)_{n \geq 0} u_n \in H_{N_0}, u_n(0, \lambda) = 0, \quad \|u_n\|_{s_1} \leq 1, \quad \|\partial \lambda u_n\|_{s_1} \leq C N_{0}^{\tau_1 + s_1 + 1} \gamma^{-1}.
\]
(F2) \( n \geq 1 \) \( \| u_k - u_{k-1} \| s_1 \leq N_k^{\sigma - 1}, \| \partial_\lambda (u_k - u_{k-1}) \| s_1 \leq N_k^{\frac{1}{2}}, \forall 1 \leq k \leq n \).
(F3) \( n \geq 1 \) one has
\[
\| u - u_{n-1} \| s_1 \leq N_n^{-\sigma} \Rightarrow \bigcap_{k=1}^{n} \mathcal{G}_n^0 (u_{k-1}) \subset \mathcal{G}_n (u),
\]
where \( \mathcal{G}_n^0 (u), \mathcal{G}_n (u) \) are defined in (3.40), (3.21) respectively.

(F4) \( n \geq 0 \) Set
\[
\mathcal{D}_n := \bigcap_{k=1}^{n} \mathcal{U}_n (u_{k-1}) \cap \bigcap_{k=1}^{n} \mathcal{G}_n^0 (u_{k-1}) \cap \mathcal{W},
\]
where \( \mathcal{U}_n (u), \mathcal{W} \) are defined in (3.48), (3.57) respectively. If \( \lambda \in \mathcal{B} (\mathcal{D}_n, N_n^{-\sigma}) \), then \( u_n (\epsilon, \lambda) \) solves the equation
\[
(P_n)_{\epsilon} \quad u_n (L_\lambda u - \epsilon F(u)) = 0.
\]

(F5) \( n \geq 0 \) Letting \( \mathcal{A}_n := \| u_n \| s_2 \) and \( \mathcal{A}_n' := \| \partial_\lambda u_n \| s_2 \), we have
\[
(1) \quad \mathcal{A}_n \leq N_n^{2^2+2s_1+2}, \quad \mathcal{A}_n' \leq N_n^{4^2+2s_1+6}.
\]
The sequence \( (u_n)_{n \geq 0} \) converges in \( s_1 \)-norm to a map
\[
u(\epsilon, \cdot) \in C^1 (\Lambda; H^{s_1}) \quad \text{with} \quad u(0, \lambda) = 0.
\]
Moreover
\[
\mathcal{D}_\epsilon := \bigcap_{n \geq 0} \mathcal{D}_n
\]
is the Cantor-like set, and for all \( \lambda \in \mathcal{D}_\epsilon \), \( u(\epsilon, \lambda) \) is a solution of (2.1) with \( \omega = \lambda \omega_0 \).

Remark 3.28. The case \( (F2)_{n=0} \) for \( u_{-1} := 0 \) is seen as (F1)0.

Step 1: Initialization. Let us check that (F1)0, (F4)0, (F5)0 hold. In the first step of iteration, the equation \( (P_{N_0}) \) is written as the following form
\[
\mathcal{L}_{N_0} u = \epsilon P_{N_0} F(u) \quad \text{with} \quad \mathcal{L}_{N_0} := P_{N_0} (L_\lambda |_{H_{N_0}}), \quad \forall u \in H_{N_0},
\]
where \( L_\lambda \) is given by (2.10).

Lemma 3.29. For all \( \lambda \in \mathcal{B} (\mathcal{W}, 2N_0^{-\sigma}) \), the operator \( \mathcal{L}_{N_0} \) is invertible with
\[
\| \mathcal{L}_{N_0}^{-1} \| s_1 \leq 2 \epsilon_2 \gamma^{-1} N_0^{\tau_1 + s_1}.
\]

Proof. For all \( \lambda \in \mathcal{B} (\mathcal{W}, 2N_0^{-\sigma}) \), there exists \( \lambda_1 \in \mathcal{W} \) such that \( |\lambda_1 - \lambda| \leq 2N_0^{-\sigma} \). If \( N_0 \geq 16 \gamma^{-1} \geq 2 \), then it follows (1.2) and (3.57) that, for all \( \sigma \geq \tau_1 + 3,
\]
\[
|-(\lambda \omega_0 \cdot l)^2 + \hat{\lambda}_{j,k}| \geq \gamma N_0^{-\tau_1} - |(\lambda_1 \omega_0 \cdot l)^2 - (\lambda \omega_0 \cdot l)^2|
\]
\[
\geq \gamma N_0^{-\tau_1} - |\lambda_1 - \lambda| \| \lambda_1 + \lambda \| \omega_0 \| l \| l \| \omega_0 |l |^2 \geq \gamma N_0^{-\tau_1} - 2N_0^{-\sigma} 4 |\omega_0|^2 N_0^2
\]
\[
\geq \gamma \frac{2}{2} N_0^{-\tau_1} - \sigma.
\]
This gives that \( \| \mathcal{L}_{N_0}^{-1} \| \leq 2 \gamma^{-1} N_0^{\tau_1} \). Combining this with formula (3.4), we get the conclusion of the lemma.

Then solving equation (3.61) is reduced to the fixed point problem \( u = U_0 (u) \), where
\[
U_0 : H_{N_0} \to H_{N_0}, \quad u \mapsto \epsilon \mathcal{L}_{N_0}^{-1} P_{N_0} F(u).
\]

Lemma 3.30. If \( \epsilon \gamma^{-1} N_0^{\tau_1 + s_1 + \sigma} \leq \epsilon_0 (s_1) \) is small enough, \( \forall \lambda \in \mathcal{B} (\mathcal{W}, 2N_0^{-\sigma}) \), the map \( U_0 \) is a contraction in \( \mathcal{B} (0, \rho_0) := \{ u \in H_{N_0} : \| u \| s_1 \leq \rho_0 := N_0^{-\sigma} \} \).
Proof. If \( \epsilon \gamma^{-1}N^{\tau_1 + s_1 + \sigma} \leq c_0(s_1) \) is small enough, owing to (3.7), (3.62) and \( \|u\|_{s_1} \leq 1 \), then it shows

\[
\|U_0(u)\|_{s_1} \leq 2\epsilon c_2^1 \gamma^{-1}N^{\tau_1 + s_1} \|F(u)\|_{s_1} \leq 2\epsilon c_2^1 \gamma^{-1}N^{\tau_1 + s_1} C(s_1)(1 + \|u\|_{s_1}) \leq N_0^{-\sigma}.
\]

In addition, by (3.7), (3.62) and \( \|u\|_{s_1} \leq 1 \), we have that, for \( \epsilon \gamma^{-1}N^{\tau_1 + s_1 + \sigma} \leq c_0(s_1) \) small enough,

\[
\|(DU_0)(u)\|_{s_1} = \epsilon \|\mathcal{L}^{-1}_N P_N((DF)(u))_{H_{\lambda}}\|_{s_1} \leq 2\epsilon c_2^1 \gamma^{-1}N^{\tau_1 + s_1} \|P_N((DF)(u))_{H_{\lambda}}\|_{s_1} \leq 2\epsilon c_2^1 \gamma^{-1}N^{\tau_1 + s_1} C(s_1)(1 + \|u\|_{s_1}) \leq 1/2.
\]

Thus the map \( U_0 \) is a contraction in \( B(0, \rho_0) \).

Denote by \( \tilde{u}_0 \) the unique solution of equation (3.61) in \( B(0, \rho_0) \). The map \( U_0 \) (see (3.64)) has \( u_0 \) as a fixed point for \( \epsilon = 0 \). By uniqueness we deduce \( \tilde{u}_0(0, \lambda) = 0 \). The implicit function theorem implies that \( \tilde{u}_0(\cdot, \cdot) \in C^1(B(\mathcal{W}, 2N_0^{-\sigma}); H_{\lambda}) \) with

\[
\partial_\lambda \tilde{u}_0 = -\mathcal{L}^{-1}_{N_0}(\epsilon, \lambda, \tilde{u}_0)(\partial_\lambda \mathcal{L}_{N_0})\tilde{u}_0,
\]

where

\[
\mathcal{L}_{N_0}(\epsilon, \lambda, \tilde{u}_0) := \mathcal{L}_{N_0} - \epsilon P_N((DF)(\tilde{u}_0))_{H_{\lambda}}, \quad \partial_\lambda \mathcal{L}_{N_0} = P_N(2\lambda(\omega_0 \cdot \varphi)^2)_{H_{\lambda}}.
\]

Formulae (3.65) and (3.62) give that \( \mathcal{L}_{N_0}(\epsilon, \lambda, \tilde{u}_0) \) is invertible with

\[
\|\mathcal{L}^{-1}_{N_0}(\epsilon, \lambda, \tilde{u}_0)\|_{s_1} = \|(I - \epsilon \mathcal{L}^{-1}_{N_0} P_N((DF)(\tilde{u}_0))_{H_{\lambda}})^{-1} \mathcal{L}^{-1}_{N_0}\|_{s_1} \leq 2\|\mathcal{L}^{-1}_{N_0}\|_{s_1} \leq 4c_2^1 \gamma^{-1}N^{\tau_1 + s_1}.
\]

Consequently, applying \( \|\tilde{u}_0\|_{s_1+2} \leq c_2^2 N_0^{-\sigma} \|\tilde{u}_0\|_{s_1} \leq c_2^2 N_0^{-\sigma} \), we derive that, for all \( \sigma \geq 2 \),

\[
\|\partial_\lambda \tilde{u}_0\|_{s_1} \leq 4c_2^1 \gamma^{-1}N^{\tau_1 + s_1} \|\partial_\lambda \mathcal{L}_{N_0}\|_{s_1} \leq 12c_2^1 \|\omega_0\|^2 \gamma^{-1}N^{\tau_1 + s_1} \|\tilde{u}_0\|_{s_1+2} \leq 12c_2^1 + 2 \gamma^{-1}N^{\tau_1 + s_1} \leq 12c_2^1 + 2 \gamma^{-1}N^{\tau_1 + s_1}.
\]

Define a \( C^\infty \) cut-off function \( \psi_0 : \Lambda \to [0, 1] \) as

\[
\psi_0 := \begin{cases} 1 & \text{if } \lambda \in \mathcal{B}(\mathcal{W}, N_0^{-\sigma}), \\ 0 & \text{if } \lambda \in \mathcal{B}(\mathcal{W}, 2N_0^{-\sigma}) \end{cases} \quad \text{with} \quad |\partial_\lambda \psi_0| \leq 12c_2^1 + 2 N_0^{-\sigma}.
\]

Then \( u_0 := \psi_0 \tilde{u}_0 \) in \( C^1(\Lambda; H_{\lambda}) \). It follows from \( \|\tilde{u}_0\|_{s_1} \leq \rho_0 \) (see Lemma 3.30, 3.68-3.69) that, for all \( \sigma \geq 2 \),

\[
\|u_0\|_{s_1} \leq \|\tilde{u}_0\|_{s_1} \leq N_0^{-\sigma}, \quad \|\partial_\lambda u_0\|_{s_1} \leq |\partial_\lambda \psi_0|\|\tilde{u}_0\|_{s_1} + |\psi_0|\|\partial_\lambda \tilde{u}_0\|_{s_1} \leq 12c_2^1 + 2N_0^{-\sigma} + 12c_2^1 + 2 \gamma^{-1}N^{\tau_1 + s_1} \leq 12c_2^1 + 2 \gamma^{-1}N^{\tau_1 + s_1 + 1}.
\]

Formulae (3.70)-(3.71) show that \( (F1)_0 \) is satisfied. Next, we verify the property \( (F4)_0 \). Since \( \mathcal{D} = \mathcal{W} \), by the definition of \( \psi_0 \) (see (3.69)), the \( u_0(\epsilon, \lambda) \) solves the equation \( (P_0) \) for all \( \lambda \in \mathcal{B}(\mathcal{D}, 2N_0^{-\sigma}) \).

Let us show that the upper bounds of \( u_0, \partial_\lambda u_0 \) on \( s_2 \)-norm. Applying the equality \( u_0 = U_0(u_0) \), (3.4), (3.64), (3.62), (3.7), (3.58), (3.54) and the inequality \( \|u_0\|_{s_1} \leq 1 \), we derive

\[
\|u_0\|_{s_2} = \|U_0(u_0)\|_{s_2} \leq c_2^{s_2 - s_1} N_0^{s_2 - s_1} \|U_0(u_0)\|_{s_1} \leq \epsilon c_2^{s_2} N_0^{s_2 - s_1} 2\gamma^{-1}N^{\tau_1 + s_1} C(s_1)(1 + \|u_0\|_{s_1}) \leq N_0^{\tau_1 + 2} \leq N_0^{2\tau_2 + 2\delta s_1 + 2}.
\]

For all \( \lambda \in \mathcal{B}(\mathcal{D}, 2N_0^{-\sigma}) \), if \( \sigma \geq \tau_1 + 1 \) \( N_0 \geq 2\gamma^{-1} \), then (3.63) infers

\[
| - (\lambda \omega_0 \cdot l)^2 + \hat{\lambda}_{j,p} | \geq \frac{\gamma}{2} N_0^{-\tau_1} \geq N_0^{-\sigma},
\]

which indicates that, for \( N = N_0, \theta = 0, j_0 = 0 \),

\[
|\mathcal{L}^{-1}_{N_0} s| = |(P_N(\mathcal{L}_\lambda)_{H_{\lambda}})^{-1}_{s}| \leq \frac{1}{2} N_0^{-\tau_2 + \delta s}, \quad \forall s \in [s_0, s_2].
\]
Consequently, for all $s_1 \geq s_0 + \varrho$ and $\|u_0\|_{s_1} \leq 1$, we verify
\[
|\varSigma_{N_0}^{-1}(\epsilon, \lambda, u_0)|_s \leq C(s) \left( |\varSigma_{N_0}^{-1}|_s + |\varSigma_{N_0}^{-1}|^2 \right) |\epsilon P_{N_0}((DF)(u_0))|_{H_{N_0}} |s| \leq \frac{1}{2} N_0^2 + \delta_0 C(s)(1 + \|u_0\|_{s_0 + \varrho}) \leq \frac{1}{2}.
\]
Hence, for all $s_1 \geq s_0 + \varrho$ and $\varrho/2 \leq \delta s_1$, it follows from Lemma 2.11 (2.23), (3.4), (3.73), (3.7), (3.55), (3.72), (3.58), that, for all $s \in [s_1, s_2]$,
\[
|\varSigma_{N_0}^{-1}(\epsilon, \lambda, u_0)|_s \leq C(s) \left( |\varSigma_{N_0}^{-1}|_s + |\varSigma_{N_0}^{-1}|^2 \right) |\epsilon P_{N_0}((DF)(u_0))|_{H_{N_0}} |s| \leq \frac{1}{2} N_0^2 + \delta_0 C(s)(1 + \|u_0\|_{s_2}) \leq \frac{1}{2} N_0^2 + \delta s.
\]
Combining this with (3.66), (2.26), (3.67), (3.4), (3.72), (3.55), $\delta = \frac{1}{4}$ and $\|u_0\|_{s_1} \leq 1$ yields
\[
\|\partial_\lambda u_0\|_{s_2} \leq C(s_2) \left( |\varSigma_{N_0}^{-1}(\epsilon, \lambda, u_0)|_{s_2} \right) |\varSigma_{N_0}^{-1}(\epsilon, \lambda, u_0)|_{s_2} \leq 3C(s_2) C(s_2) N_0^2 \|u_0\|_{s_2} + N_0^2 + \delta s N_0^2 \|u_0\|_{s_1} \leq N_0^{4\tau_2 + 2s_1 + 6}.
\]
Formul\ae\ (3.72) and (3.74) give that $(F5)_{0}$ holds.

Step 2: assumption. Assume that we have get a solution $u_n \in C^1(\Lambda; H_{N_n})$ of $(P_{N_n})$ and that properties $(F1)_k - (F5)_k$ hold for all $k \leq n$.

Step 3: iteration. Our goal is to find a solution $u_{n+1} \in C^1(\Lambda; H_{N_{n+1}})$ of $(P_{N_{n+1}})$ and to prove the statements $(F1)_{n+1} - (F5)_{n+1}$. Denote by
\[
\bar{u}_{n+1} = u_n + h \quad \text{with} \quad h \in H_{N_{n+1}}
\]
a solution of $(P_{N_{n+1}})$. In addition it follows from (3.2), $\sigma \geq 2$ (see (3.56)) and $N_0 \geq 2$ that
\[
\mathcal{B}(\varPhi_{n+1}, 2N_{n+1}^{-\sigma}) \subset \mathcal{B}(\varPhi, 2N_n^{-\sigma}),
\]
which derives $P_{N_n}(L\lambda u_n - \epsilon F(u_n)) = 0$ by (F4). This implies
\[
P_{N_{n+1}}(L\lambda \bar{u}_n + h - \epsilon F(u_n + h)) = P_{N_{n+1}}(L\lambda u_n - \epsilon F(u_n)) + P_{N_{n+1}}(L\lambda \bar{h} - \epsilon (F(u_n + h) - F(u_n)))
\]
\[
= \varSigma_{N_{n+1}}(\epsilon, \lambda, u_n) h + \varrho_n(h) + \varrho_{n},
\]
where $\varSigma_{N_{n+1}}(\epsilon, \lambda, u_n)$ is defined in (3.13) and
\[
\varrho_n(h) := -\epsilon P_{N_{n+1}}(F(u_n) + h - \epsilon F(u_n)) - (DF)(u_n) h, \quad \varrho_n := P_{N_{n+1}+1}(P_{N_n} L\lambda u_n - \epsilon F(u_n)) = P_{N_{n+1}+1} P_{N_n} (\bar{V} u_n - \epsilon F(u_n)).
\]
Remark that $P_{N_{n+1}} P_{N_n}^{-1}((D\lambda) u_n) = 0$ according to (2.10). Our aim is to prove the linearized operators $\varSigma_{N_{n+1}}(\epsilon, \lambda, u_n)$ (recall (3.13)) is invertible and to give the tame estimates of its inverse using Proposition 3.8. In addition formul\ae\ (2.9), (2.11) and (2.20) deduce that $\varSigma_{N_{n+1}}(\epsilon, \lambda, u_n)$ may be represented by the matrix $\varLambda_{N_{n+1}}(\epsilon, \lambda, u_n)$. We distinguish two cases.

If $2^{n+1} \leq \chi_0$, then there exists $\chi \in [\chi_0, 2\chi_0]$ such that
\[
N_{n+1} = \bar{N}, \quad \bar{N} := [N_{n+1}^{1/\chi_0}] \in (N_0^{1/\chi}, N_0).
\]
If $2^{n+1} > \chi_0$, then there exists a unique $p \in [0, n]$ such that
\[
N_{n+1} = N_p, \quad \chi = 2^{n+1-p} \in [\chi_0, 2\chi_0].
\]
Lemma 3.31. Let \( A(\epsilon, \lambda, u, \theta) \) be defined in (2.15). \( \forall \theta \in \mathbb{R}, \forall j_0 \in \Gamma_+(M), \forall \lambda \in \bigcap_{k=1}^{n+1} \mathcal{G}_0^{\lambda_{ij}}(u_{k-1}) \), the assumption (A3) of Proposition 3.8 may be applied to \( \mathcal{A}_{N+1, j_0}(\epsilon, \lambda, u, \theta) \).

Proof. Define \( \mathfrak{N}_{N+1} := \mathfrak{M}_{N+1} \times \mathfrak{J}_{N+1} \) with

\[
\mathfrak{M}_{N+1} := [-N_{n+1}, N_{n+1}]^\nu \cap \mathbb{Z}^\nu, \quad \mathfrak{J}_{N+1} := \{ j_0 + \sum_{k=1}^r j_k w_k : j_k \in [-N_{n+1}, N_{n+1}] \} \cap \Gamma_+(M).
\]

Let us consider the case \( 2^{n+1} \leq \chi_0 \). By Lemma 3.11 if the site \( n = (l, j) \) is in \( \mathfrak{M}_{N+1} \), then it is \( \mathcal{A}(\epsilon, \lambda, u, \theta), \tilde{N} \)-strongly-good for \( \mathcal{A}(\epsilon, \lambda, u, \theta) \) with \( |(l, j - j_0)| \leq N_{n+1} \), then it is \( \mathcal{A}_{N+1, j_0}(\epsilon, \lambda, u, \theta), \tilde{N} \)-good. This implies

\[
\mathcal{A}(\epsilon, \lambda, u, \theta), \tilde{N} \)-good sites \( \subset \mathcal{A}(\epsilon, \lambda, u, \theta) \) with \( |(l, j - j_0)| \leq N_{n+1} \).
\]

It follows from Lemma 3.17 and (F1) that \( \mathcal{G}_N(u_n) = \Lambda \), which shows that \( \lambda \in \bigcap_{k=1}^{n+1} \mathcal{G}_N^0(u_{k-1}) \subset \mathcal{G}_N(u_n) \) is \( \tilde{N} \)-good for \( \mathcal{A}_{N+1, j_0}(\epsilon, \lambda, u, \theta) \). Combining this with (3.54), (3.79) and Proposition 3.17 gives that the assumption (A3) of Proposition 3.8 applies to \( \mathcal{A}_{N+1, j_0}(\epsilon, \lambda, u, \theta) \).

If \( 2^{n+1} > \chi_0 \), then a simple discussion as above yields

\[
\mathcal{A}(\epsilon, \lambda, u, \theta), N_p \)-good sites \( \subset \mathcal{A}(\epsilon, \lambda, u, \theta) \) with \( |(l, j - j_0)| \leq N_{n+1} \).
\]

If the following

\[
\bigcap_{k=1}^{n+1} \mathcal{G}_N^0(u_{k-1}) \subset \mathcal{G}_N(u_n) \quad (3.81)
\]

holds, by (3.54), (3.80) and Proposition 3.17 we have that the assumption (A3) of Proposition 3.8 applies to \( \mathcal{A}_{N+1, j_0}(\epsilon, \lambda, u, \theta) \).

Let us verify formula (3.81). In fact, for \( p = 0 \), it follows from Lemma 3.17 and (F1) that \( \mathcal{G}_N(u_n) = \Lambda \). Hence it is clear that (3.81) holds for \( p = 0 \). For \( p \geq 1 \), one has

\[
\|u_n - u_{p-1}\|_{s_1} \leq \sum_{k=p}^{n} \|u_k - u_{k-1}\|_{s_1} \leq \sum_{k=p}^{n} N_k^{-\sigma - 1} \leq N_p^{-\sigma} \sum_{k=p}^{n} N_k^{-1} \leq N_p^{-\sigma},
\]

which leads to

\[
\bigcap_{k=1}^{n+1} \mathcal{G}_N^0(u_{k-1}) \subset \bigcap_{k=1}^{p} \mathcal{G}_N^0(u_{k-1}) \subset \mathcal{G}_N(u_n).
\]

Lemma 3.32. For all \( \lambda \in \mathcal{B}(\mathcal{D}_{n+1}, 2N_{n+1}^{-\sigma}) \), the operator \( \mathfrak{L}_{N+1}(\epsilon, \lambda, u_n) \) is invertible with

\[
|\mathfrak{L}_{N+1}^0(\epsilon, \lambda, u_n)|_{s_1} \leq N_{n+1}^{\tau_2 + \delta s_1}, \quad |\mathfrak{L}_{N+1}^{-1}(\epsilon, \lambda, u_n)|_{s_2} \leq C'(s_2)N_{n+1}^{\tau_2 + \delta s_2}.
\]

Proof. The operator \( \mathfrak{L}_{N+1}(\epsilon, \lambda, u_n) \) is represented by the matrix \( \mathcal{A}_{N+1}(\epsilon, \lambda, u_n) \). Let \( \lambda \in \mathcal{D}_{n+1} \) (recall (3.59)), which leads to \( \lambda \in \mathcal{D}_{N+1}(u_n) \). Definition (3.48) gives that \( \mathcal{A}_{N+1}(\epsilon, \lambda, u_n) \) is invertible with

\[
|\mathcal{A}_{N+1}^{-1}(\epsilon, \lambda, u_n)|_{s_1} \leq C(s_1)(\|V\|_{s_1} + \epsilon \|DF(u_n)\|_{s_1}) \leq C'(V).
\]

In addition formulae (2.23), (3.7), (3.12) and (F1) deduce

\[
|\mathcal{A}_{N+1}(\epsilon, \lambda, u_n) - \text{Diag}(\mathcal{A}_{N+1}(\epsilon, \lambda, u_n))|_{s_1 - \theta} \leq C(s_1)(\|V\|_{s_1} + \epsilon \|DF(u_n)\|_{s_1}) \leq C'(V).
\]
Under (3.83)-(3.84) and Lemma 3.31 for \( \theta = 0, j_0 = 0 \), the assumptions (A1)-(A3) in Proposition 3.8 are satisfied. If \( 2^{n+1} \leq \chi_0 \) (resp. \( 2^{n+1} > \chi_0 \)), combining this with Remark 3.26 yields that Proposition 3.8 is applied to \( \mathcal{A} := \mathcal{A}_{N_{n+1}}(\epsilon, \lambda, u_n) \) with

\[
\mathcal{A} := \left[-N_{n+1}, N_{n+1}\right] \times \left( \{j_0 + \sum_{p=1}^{r} j_k w_k : j_k \in \left[-N_{n+1}, N_{n+1}\right]\} \cap \Gamma_+(M) \right),
\]

\[
N' := N_{n+1}, \quad N := \tilde{N} \quad \text{(resp.} \quad N := N_p).\]

Hence, for \( s = s_1, \delta s_1 \geq \varrho/2 \), it follows from (3.27), (2.23), (3.4), (3.12), (3.7), (3.2), and (F1), that, for all \( \lambda \in \mathcal{D}_{n+1}, \)

\[
|A_{N_{n+1}}^{-1}(\epsilon, \lambda, u_n)|_{s_1} \leq \frac{1}{4} N_{n+1}^{-2} (N_{n+1}^{\delta s_1} + C(s_1)(\|V\|_{s_1+\varrho} + \|DF(u_n)\|_{s_1+\varrho}))
\]

\[
\leq \frac{1}{4} N_{n+1}^{-2} \left( N_{n+1}^{\delta s_1} + C' + \epsilon C'(s_1)(1 + \epsilon c_0^2 N_{n+1}^{\delta s_1} \|u_n\|_{s_1}) \right)
\]

\[
\leq \frac{1}{2} N_{n+1}^{-2+\delta s_1}. \quad (3.85)
\]

Moreover for \( s = s_2, \delta s_1 \geq \varrho/2 \), by (3.27), (2.23), (3.4), (3.12), (3.7), (3.2), \( \delta = 1/4, (3.55) \) and (F5), we have that, for all \( \lambda \in \mathcal{D}_{n+1}, \)

\[
|A_{N_{n+1}}^{-1}(\epsilon, \lambda, u_n)|_{s_2} \leq \frac{1}{4} N_{n+1}^{-2} (N_{n+1}^{\delta s_2} + C(s_2)(\|V\|_{s_2+\varrho} + \|DF(u_n)\|_{s_2+\varrho}))
\]

\[
\leq \frac{1}{4} N_{n+1}^{-2} \left( N_{n+1}^{\delta s_2} + C' + \epsilon C'(s_2)(1 + \epsilon c_0^2 N_{n+1}^{\delta s_2} \|u_n\|_{s_2}) \right)
\]

\[
\leq \frac{1}{2} N_{n+1}^{-2+\delta s_2}. \quad (3.86)
\]

Formulae (3.85)-(3.86) give that, for all \( \lambda \in \mathcal{D}_{n+1}, \)

\[
|\mathcal{L}_{N_{n+1}}^{-1}(\epsilon, \lambda, u_n)|_s \leq \frac{1}{2} N_{n+1}^{2+\delta s}, \quad s = s_1, s_2. \quad (3.87)
\]

For all \( \lambda' \in \mathcal{B} \left( \mathcal{D}_{n+1}, 2N_{n+1}^{-\sigma} \right), \) there exists some \( \lambda \in \mathcal{D}_{n+1} \) such that \( \lambda' - \lambda < 2N_{n+1}^{-\sigma} \). It is obvious that

\[
\mathcal{L}_{N_{n+1}}(\epsilon, \lambda', u_n(\epsilon, \lambda')) = \mathcal{L}_{N_{n+1}}(\epsilon, \lambda, u_n(\epsilon, \lambda)) + \mathcal{R},
\]

where

\[
\mathcal{R} = \mathcal{L}_{N_{n+1}}(\epsilon, \lambda', u_n(\epsilon, \lambda')) - \mathcal{L}_{N_{n+1}}(\epsilon, \lambda, u_n(\epsilon, \lambda))
\]

\[
= P_{N_{n+1}}((\lambda' + \lambda)(\lambda' - \lambda)(\omega_0 \cdot \partial^2))_{H_{N_{n+1}}}
\]

\[
- \epsilon P_{N_{n+1}}((DF)(u_n(\epsilon, \lambda')) - (DF)(u_n(\epsilon, \lambda)))_{H_{N_{n+1}}}. \]

It follows from (122), (2.28), (3.2), \( \sigma \geq 2 \) (see (3.56)) and \( N_0 \geq 2 \) that

\[
|P_{N_{n+1}}((\lambda' + \lambda)(\lambda' - \lambda)(\omega_0 \cdot \partial^2))_{H_{N_{n+1}}}|_{s} \leq 8c_0^2 N_{n+1}^{\sigma+2} \quad \text{for} \quad s = s_1, s_2. \quad (3.88)
\]

In addition, applying (2.23), (3.4), (3.8), (3.55), (3.58), (F1), \( \delta s_1 \geq \varrho/2 \), we deduce

\[
|\epsilon P_{N_{n+1}}((DF)(u_n(\epsilon, \lambda')) - (DF)(u_n(\epsilon, \lambda)))_{H_{N_{n+1}}}|_{s_1} \leq C(s_1)N_{n+1}^{-\sigma+\delta s_1}, \quad (3.89)
\]

\[
|\epsilon P_{N_{n+1}}((DF)(u_n(\epsilon, \lambda')) - (DF)(u_n(\epsilon, \lambda)))_{H_{N_{n+1}}}|_{s_2} \leq C(s_2)N_{n+1}^{-\sigma+2\sigma_1}N_{n+1}^{-\sigma+\delta s_1}. \quad (3.90)
\]

As a consequence

\[
|\mathcal{L}_{N_{n+1}}^{-1}(\epsilon, \lambda, u_n(\epsilon, \lambda))|_{s_1} \leq C'(s_2)N_{n+1}^{-\sigma+2\sigma_1+\delta s_1+2} \leq 1/2. \quad (3.56)
\]
Hence Lemma 3.33 gives that $\Sigma_{N_{n+1}}(\epsilon, \lambda', u_n(\epsilon, \lambda'))$ is invertible with
\[
|\Sigma_{N_{n+1}}^{-1}(\epsilon, \lambda', u_n(\epsilon, \lambda'))|_{s_1} \leq 2|\Sigma_{N_{n+1}}^{-1}(\epsilon, \lambda, u_n(\epsilon, \lambda))|_{s_1} \leq N_{n+1}^{\gamma_2+\delta_1},
\]
\[
|\Sigma_{N_{n+1}}^{-1}(\epsilon, \lambda', u_n(\epsilon, \lambda'))|_{s_2} \leq C(s_2)(|\Sigma_{N_{n+1}}^{-1}(\epsilon, \lambda, u_n(\epsilon, \lambda))|_{s_2} + |\Sigma_{N_{n+1}}^{-1}(\epsilon, \lambda, u_n(\epsilon, \lambda))|_{s_1}^2 |{\mathcal{A}}|_{s_2}) \leq C(s_2)(\frac{1}{2}N_{n+1}^{\gamma_2+\delta_1} + \frac{1}{4}N_{n+1}^{2(\gamma_2+\delta_1)}C'(s_2)N_n^{4\gamma_2+8\delta_1+6}N_{n+1}^{-\sigma+\delta_1}) \leq C''(s_2)N_{n+1}^{\gamma_2+\delta_2}.
\]

The proof of the lemma is completed. □

Then, owing to Lemma 3.32 solving the equation $(P_{N_{n+1}})$ (see also (3.76)) is reduced to the fixed point problem $h = U_{n+1}(h)$ with
\[
U_{n+1} : H_{N_{n+1}} \to H_{N_{n+1}}, \quad h \mapsto -\Sigma_{N_{n+1}}^{-1}(\epsilon, \lambda, u_n)(R_n(h) + r_n),
\]
where $R_n(h), r_n$ are defined in (3.77)-(3.78).

**Lemma 3.33.** For all $\lambda \in \mathcal{B}(\mathcal{D}_{n+1}, 2N_{n+1}^{-\sigma})$, the map $U_{n+1}$ is a contraction in
\[
\mathcal{B}(0, \rho_{n+1}) := \{h \in H_{N_{n+1}} : \|h\|_{s_1} \leq \rho_{n+1} := N_{n+1}^{-\sigma-1}\},
\]
Moreover the unique fixed point $\tilde{h}_{n+1}(\epsilon, \lambda)$ of $U_{n+1}$ satisfies
\[
\|\tilde{h}_{n+1}\|_{s_1} \leq 2K(s_2)N_{n+1}^{\gamma_2+\delta_1}N_n^{-(s_2-s_1)}\mathcal{A}_n,
\]
where $\mathcal{A}_n$ is seen in (F5)$_n$.

**Proof.** For all $\lambda \in \mathcal{B}(\mathcal{D}_{n+1}, 2N_{n+1}^{-\sigma})$, it follows from (3.77)-(3.78), (3.5), (3.7), (3.10)-(3.12) and (F5)$_n$ that
\[
\|r_n\|_{s_1} + \|R_n(h)\|_{s_1} \leq C\|\tilde{V}_{n+1}\|_{s_2} + \epsilon\|F(u_n)\|_{s_2} + \epsilon C(s_1)\|h\|_{s_1}^2 \leq C(s_2)C_1^{-1}N_{n+1}^{-(s_2-s_1)}(1 + \mathcal{A}_n) + \epsilon C(s_1)\|h\|_{s_1}^2 \leq 2C(s_2)C_1^{-1}N_{n+1}^{-(s_2-s_1)}+2\epsilon\|h\|_{s_1}^2.
\]
Letting $\|h\|_{s_1} \leq \rho_{n+1}$, by (3.91), (2.26), (3.82), (3.94) and (3.2), we check that, for some constants $K(s_1), K(s_2) > 0$,
\[
\|U_{n+1}(h)\|_{s_1} \leq K(s_2)N_{n+1}^{\gamma_2+\delta_1}N_n^{-(s_2-s_1)} + \epsilon K(s_1)N_{n+1}^{\gamma_2+\delta_1}\rho_{n+1}^2.
\]
Formulæ (3.55)-(3.56) imply that, for $N \geq N_0(s_2)$ large enough,
\[
K(s_2)N_{n+1}^{\gamma_2+\delta_1}N_n^{-(s_2-s_1)} \leq \rho_{n+1}/2, \quad K(s_1)N_{n+1}^{\gamma_2+\delta_1}\rho_{n+1} \leq 1/2,
\]
which leads to $\|U_{n+1}(h)\|_{s_1} \leq \rho_{n+1}$. Moreover differentiating (3.91) with respect to $h$ yields
\[
DU_{n+1}(h)[v] = \Sigma_{N_{n+1}}^{-1}(\epsilon, \lambda, u_n)P_{N_{n+1}}((DF)(u_n + h)[v] - (DF)(u_n)[v]).
\]
Using (2.26), (3.82), (3.9), (F1)$_n$ and (3.95), we deduce
\[
\|DU_{n+1}(h)[v]\|_{s_1} \leq \epsilon C(s_1)N_{n+1}^{\gamma_2+\delta_1}\|P_{N_{n+1}}((DF)(u_n + h)[v] - (DF)(u_n)[v])\|_{s_1} \leq \epsilon K(s_1)N_{n+1}^{\gamma_2+\delta_1}\rho_{n+1}\|v\|_{s_1} \leq \frac{1}{2}\|v\|_{s_1}.
\]
Hence $U_{n+1}$ is a contraction in $\mathcal{B}(0, \rho_{n+1})$. Let $\tilde{h}_{n+1}(\epsilon, \lambda)$ denote the unique fixed point of $U_{n+1}$. In addition, by means of (2.26), (3.82), (3.91), (3.93), (3.2) and $\|\tilde{h}_{n+1}\|_{s_1} \leq \rho_{n+1}$, we obtain
\[
\|\tilde{h}_{n+1}\|_{s_1} \leq K(s_2)N_{n+1}^{\gamma_2+\delta_1}N_n^{-(s_2-s_1)}\mathcal{A}_n + \epsilon K(s_1)N_{n+1}^{\gamma_2+\delta_1}\rho_{n+1}\|\tilde{h}_{n+1}\|_{s_1}.
\]
Combining this with (3.95) gives that (3.92) holds. □
Consequently, by (3.98)-(3.99), (3.2), (3.55)-(3.56) and the definition of \( u \sim \), let us verify the upper bound of \( h_{n+1} \).

**Proof.** Applying (3.91), (2.26) and (3.82) yields

\[
\|h_{n+1}\|_{s_2} = \|Q_{N+1}^{-1}(\epsilon, \lambda, u_n)(R_n(h_{n+1}) + r_n)\|_{s_2} \\
\leq C(s_2)(N_{n+1}^{r_2 + \delta s_1} + N_{n+1}^{r_2 + \delta s_2} + C''(s_2)N_{n+1}^{r_2 + \delta s_2}(\|R_n(h_{n+1})\|_{s_1} + \|r_n\|_{s_1})).
\]

Let us check the upper bounds of \( R_n(h_{n+1}) \) for \( s = s_1, s_2 \). It follows from (3.77), (3.10), (3.92), \( \|R_n(h_{n+1})\|_{s_1} \leq \rho_{n+1}, \) \((F5)_n\), (3.11) and (3.95) that

\[
\|R_n(h_{n+1})\|_{s_1} \leq C(s_2)N_{n+1}^{-r_2 - s_1} A_{n}, \quad \|R_n(h_{n+1})\|_{s_2} \leq C(s_2)(\epsilon \rho_{n+1}^2 A_{n} + \epsilon \rho_{n+1}\|\tilde{h}_{n+1}\|_{s_2}).
\]

Moreover, using (3.78), (3.5), (3.7), (3.12) and \((F5)_n\), we deduce

\[
\|r_n\|_{s_1} \leq C(s_2)N_{n+1}^{-r_2 - s_1} A_{n}, \quad \|r_n\|_{s_2} \leq C(s_2)A_{n}.
\]

Consequently, by (3.98)-(3.99), (3.2), (3.55)-(3.56) and the definition of \( \rho_{n+1} \), we get that, for \( \epsilon \) small enough,

\[
\|h_{n+1}\|_{s_2} \leq C(s_2)(N_{n+1}^{r_2 + \delta s_1} + C(s_2)(2A_{n} + \epsilon \rho_{n+1}\|\tilde{h}_{n+1}\|_{s_2} + C''(s_2)N_{n+1}^{r_2 + \delta s_2}2C(s_2)N_{n+1}^{-r_2 - s_1} A_{n})) \\
\leq C''(s_2)N_{n+1}^{r_2 + \delta s_1} A_{n} + C(s_2)N_{n+1}^{r_2 + \delta s_1 - 1}\|\tilde{h}_{n+1}\|_{s_2} \\
\leq C''(s_2)N_{n+1}^{r_2 + \delta s_1} A_{n} + \frac{1}{2}\|\tilde{h}_{n+1}\|_{s_2},
\]

which leads to \( \|h_{n+1}\|_{s_2} \leq 2C''(s_2)N_{n+1}^{r_2 + \delta s_1} A_{n} \).

Next, let us estimate the derivatives of \( h_{n+1} \) with respect to \( \lambda \).

**Lemma 3.35.** For all \( \lambda \in B(\mathcal{P}_{n+1}, 2N_{n+1}^{-r_2}) \), one has \( h_{n+1}(\epsilon, \lambda) \in B(\mathcal{P}_{n+1}, 2N_{n+1}^{-r_2}) \) with

\[
\|\partial_\lambda h_{n+1}\|_{s_1} \leq N_{n+1}^{-1}, \quad \|\partial_\lambda h_{n+1}\|_{s_2} \leq N_{n+1}^{r_2 + \delta s_1 + 1}(N_{n+1}^{r_2 + \delta s_1 + 2} A_{n} + A'_{n}).
\]

**Proof.** Lemma 3.33 shows that for all \( \lambda \in B(\mathcal{P}_{n+1}, 2N_{n+1}^{-r_2}) \), \( h_{n+1} \) is a solution of equation (3.96). Applying (3.96) and (3.13) yields

\[
D_{\lambda}h_{n+1}(\epsilon, \lambda, \tilde{h}_{n+1}) = \mathcal{F}_{n+1}(\epsilon, \lambda, u_n + \tilde{h}_{n+1}) = \mathcal{F}_{n+1}(\epsilon, \lambda, u_n) + \mathcal{R}',
\]

where \( \mathcal{R}' = -\epsilon P_{n+1}(DF)(u_n + \tilde{h}_{n+1}) - (DF)(u_n)|_{H_{N_{n+1}}} \). It follows from (2.23), (3.8), (3.4), \( \delta s_1 \geq \delta /2 \), \((P1)_n\), \((F1)_n\), \((F5)_n\), \( \|\tilde{h}_{n+1}\| \leq \rho_{n+1} < 1 \) and (3.97) that

\[
\|\mathcal{R}'\|_{s_1} \leq C(s_1)N_{n+1}^{2\delta s_1} A_{n+1}, \quad \|\mathcal{R}'\|_{s_2} \leq C(s_2)N_{n+1}^{r_2 + 3\delta s_1} A_{n}.
\]

Using (3.82), (3.102) together with (3.56), we deduce

\[
\|\mathcal{F}_{n+1}^{-1}(\epsilon, \lambda, u_n)\|_{s_1}, |\mathcal{R}'|_{s_1} \leq 1/2.
\]
Combining this with Lemma 2.11 yields
\begin{equation}
|\Sigma_{N_{n+1}^{-1}}(\epsilon, \lambda, u_n + \tilde{h}_{n+1})|_{s_1} \leq 2|\Sigma_{N_{n+1}^{-1}}(\epsilon, \lambda, u_n)|_{s_1} \leq 2N_{n+1}^{\tau_2 + \delta s_1}, \quad (3.103)
\end{equation}
\begin{equation}
|\Sigma_{N_{n+1}^{-1}}(\epsilon, \lambda, u_n + \tilde{h}_{n+1})|_{s_1} \leq C(s_2)(|\Sigma_{N_{n+1}^{-1}}(\epsilon, \lambda, u_n)|_{s_2} + |\Sigma_{N_{n+1}^{-1}}(\epsilon, \lambda, u_n)|_{s_1} |\mathcal{A}'|_{s_2}) \leq C(s_2)N_{n+1}^{\tau_2 + \delta s_2}. \quad (3.104)
\end{equation}

Therefore the implicit function theorem establishes \(\tilde{h}_{n+1} \in C^1(\mathcal{D}_{n+1}, 2N_{n}^{-\sigma}); H_{N_{n+1}}\), which then infers
\begin{equation}
\partial\lambda \mathcal{D}_{n+1}(\epsilon, \lambda, \tilde{h}_{n+1}) + D_h \mathcal{D}_{n+1}(\epsilon, \lambda, \tilde{h}_{n+1})\partial h_{n+1} = 0.
\end{equation}

Consequently, using the fact that \(u_n\) solves \((P_{N_n})\) deduced by (3.75) and (F4)n, we deduce
\begin{equation}
\partial\lambda \tilde{h}_{n+1} = -\Sigma_{N_{n+1}^{-1}}(\epsilon, \lambda, u_n + \tilde{h}_{n+1})\partial h_{n+1}(\epsilon, \lambda, \tilde{h}_{n+1}),
\end{equation}
where
\begin{equation}
\partial\lambda \mathcal{D}_{n+1}(\epsilon, \lambda, \tilde{h}_{n+1}) = P_{N_{n+1}}((\partial\lambda L_{\lambda}) \tilde{h}_{n+1}) + P_{N_{n+1}}(\hat{V} \partial\lambda u_n) - \epsilon P_{N_{n+1}}P_{N_{n+1}}(\mathcal{D}(\hat{F})(u_n)\partial\lambda u_n)
- \epsilon P_{N_{n+1}}(\mathcal{D}(\hat{F})(u_n + \tilde{h}_{n+1})\partial h_{n+1} - (\mathcal{D})(\hat{F})(u_n)\partial h_{n+1}).
\end{equation}

Remark that \(P_{N_{n+1}}, P_{N_{n+1}}(\hat{F}_n) = 0\) according to (2.10). To establish (3.100), we have to verify the upper bounds of \(\partial\lambda \mathcal{D}_{n+1}(\epsilon, \lambda, \tilde{h}_{n+1})\) in \(s_1, s_2\)-norms. It follows from (2.10), (3.4)-3.5, (3.9), (3.12), (F1)n, (F5)n, \(|\tilde{h}_{n+1}||_{s_1} \leq P_{n+1} < 1\), (3.8), (3.9), (3.58) and (3.97) that
\begin{equation}
|\partial\lambda \mathcal{D}_{n+1}(\epsilon, \lambda, \tilde{h}_{n+1})|_{s_1} \leq C(s_2)(N_{n+1}^{\tau_2 + \delta s_1} + 2N_{n}^{-\sigma(s_2-s_1)} \mathcal{A}_n + N_{n}^{-\sigma(s_2-s_1)} \mathcal{A}'_n),
\end{equation}
\begin{equation}
|\partial\lambda \mathcal{D}_{n+1}(\epsilon, \lambda, \tilde{h}_{n+1})|_{s_2} \leq C(s_2)(N_{n+1}^{\tau_2 + \delta s_1} \mathcal{A}_n + \mathcal{A}'_n),
\end{equation}
which give rise to
\begin{equation}
|\partial\lambda \tilde{h}_{n+1}|_{s_1} \leq C(s_2)N_{n+1}^{\tau_2 + \delta s_1} (N_{n+1}^{\tau_2 + \delta s_1} + 2N_{n}^{-\sigma(s_2-s_1)} \mathcal{A}_n + N_{n}^{-\sigma(s_2-s_1)} \mathcal{A}'_n),
\end{equation}
\begin{equation}
|\partial\lambda \tilde{h}_{n+1}|_{s_2} \leq C(s_2)(N_{n+1}^{\tau_2 + \delta s_1} \mathcal{A}_n + \mathcal{A}'_n).
\end{equation}

Define a \(C^\infty\) cut-off function \(\psi_{n+1} : \Lambda \rightarrow [0, 1]\) as
\begin{equation}
\psi_{n+1} := \begin{cases} 1 & \text{if } \lambda \in \mathcal{B}(\mathcal{D}_{n+1}, N_{n}^{-\sigma}), \\ 0 & \text{if } \lambda \in \mathcal{B}(\mathcal{D}_{n+1}, 2N_{n+1}^{-\sigma}), \end{cases}
\end{equation}
with \(|\partial\lambda \psi_{n+1}| \leq CN_{n+1}^{\sigma}. \quad (3.105)

Then we define \(h_{n+1}\) as
\begin{equation}
h_{n+1}(\epsilon, \lambda) := \begin{cases} \psi_{n+1} \tilde{h}_{n+1}(\epsilon, \lambda) & \text{if } \lambda \in \mathcal{B}(\mathcal{D}_{n+1}, 2N_{n+1}^{-\sigma}), \\ 0 & \text{if } \lambda \notin \mathcal{B}(\mathcal{D}_{n+1}, 2N_{n+1}^{-\sigma}). \end{cases}
\end{equation}

Hence, by (3.105) (3.106), Lemma 3.33 (3.97), (3.100), (F5)n, (3.56) and \(\delta = 1/4\), one has that \(h_{n+1} \in C^1(\Lambda; H_{N_{n+1}})\) with
\begin{equation}
|h_{n+1}|_{s_1} \leq |\psi_{n+1}||\tilde{h}_{n+1}|_{s_1} \leq |\tilde{h}_{n+1}|_{s_1} \leq N_{n+1}^{-\sigma_1},
\end{equation}
\begin{equation}
|h_{n+1}|_{s_2} \leq |\psi_{n+1}||\tilde{h}_{n+1}|_{s_2} \leq |\tilde{h}_{n+1}|_{s_2} \leq K(s_2)N_{n+1}^{\tau_2 + \delta s_1 + 1},
\end{equation}
\begin{equation}
|\partial\lambda h_{n+1}|_{s_1} \leq |\lambda^{\psi_{n+1}}||\tilde{h}_{n+1}|_{s_1} + |\psi_{n+1}||\partial\lambda \tilde{h}_{n+1}|_{s_1} \leq N_{n+1}^{-1/2},
\end{equation}
\begin{equation}
|\partial\lambda h_{n+1}|_{s_2} \leq |\lambda^{\psi_{n+1}}||\tilde{h}_{n+1}|_{s_2} + |\psi_{n+1}||\partial\lambda \tilde{h}_{n+1}|_{s_2} \leq K'(s_2)N_{n+1}^{3\tau_2 + 2\delta s_1 + 4}.
\end{equation}
Moreover it is clear that \( h_{n+1}(0, \lambda) = 0 \). As a consequence, we define \( u_{n+1}(\epsilon, \cdot) \in C^1(\Lambda; H_{N_{n+1}}) \) as

\[
 u_{n+1} := u_n + h_{n+1},
\]

where \( h_{n+1} \) is given by (3.106). Let us check that properties \((F1)_{n+1}-\)\((F5)_{n+1}\) hold. It follows from (3.70)-(3.71), \((F2)_{n+1}\), (3.107), (3.109) and (3.111) that

\[
\|u_{n+1}\|_{s_1} \leq \|u_0\|_{s_1} + \sum_{k=1}^{n+1} \|h_k\|_{s_1} \leq N_0^{-\sigma} + \sum_{k=1}^{n+1} N_k^{-\sigma - 1} \leq \frac{1}{2} + N_1^{-\sigma} \sum_{k=1}^{n+1} N_k^{-1} \leq 1,
\]

(3.112)

\[
\|\partial_\lambda u_{n+1}\|_{s_1} \leq \|\partial_\lambda u_0\|_{s_1} + \sum_{k=1}^{n+1} \|\partial_\lambda h_k\|_{s_1} \leq 12c_2^{s_1+2} N_0^{-s_1+1} + \sum_{k=1}^{n+1} N_k^{-1/2} \leq C\gamma^{-1} N_0^{-s_1+1}.
\]

(3.113)

Therefore property \((F1)_{n+1}\) holds. It is straightforward that property \((F2)_{n+1}\) holds according to (3.107), (3.109) and (3.111). By the definitions of \( \psi_{n+1}, h_{n+1} \) (see (3.105)-(3.106)), we have that \( h_{n+1} = h_{n+1} \) on \( B(\mathcal{D}_{n+1}, N_{n+1}^{\sigma}) \). Moreover \( h_{n+1} \) solves equation \(\mathcal{D}_{n+1}(\epsilon, \lambda, h)\) (see (3.96)), which leads to that \( u_{n+1} \) solves equation \((F_{n+1})\), namely property \((F4)_{n+1}\). Let us check that property \((F5)_{n+1}\) holds. Indeed, the definitions of \( G_{n+1} \) and \( G'_{n+1} \) establish that

\[
 G_{n+1} \leq G_n + \|\partial_\lambda h_{n+1}\|_{s_2} \leq \sum_{k=1}^{n+1} \|\partial_\lambda h_k\|_{s_2} \leq N_{n+1}^{2s_2+2s_1+2} + K(s_2) N_{n+1}^{2s_2+2s_1+2} \leq N_{n+1}^{2s_2+2s_1+2}.
\]

(3.110)

\[
 G'_{n+1} \leq G'_n + \|\partial_\lambda h_{n+1}\|_{s_2} \leq N_{n+1}^{4s_2+3s_1+4} + K'(s_2) N_{n+1}^{4s_2+3s_1+4} \leq N_{n+1}^{4s_2+3s_1+4}.
\]

(3.111)

Now, we are denoted to prove property \((F3)_{n+1}\).

**Lemma 3.36.** Property \((F3)_{n+1}\) holds.

**Proof.** Firstly, if \( \|u - u_n\|_{s_1} \leq N_{n+1}^{-\sigma} \), then we claim that

\[
\|A_{N_{n+1}, j_0}^{-1}(\epsilon, \lambda, u_n, \theta)\|_0 \leq N_{n+1}^{s_1} \Rightarrow A_{N_{n+1}, j_0}(\epsilon, \lambda, u, \theta) \text{ is } N_{n+1}-\text{good,}
\]

(3.114)

where \( A(\epsilon, \lambda, u, \theta) \) is defined in (2.15). This implies that

\[
\mathcal{B}_{N_{n+1}}(j_0; \epsilon, \lambda, u) \subset \mathcal{B}_0^{N_{n+1}}(j_0; \epsilon, \lambda, u_n), \quad \forall j_0 \in \Gamma_+(M), \forall \theta \in R
\]

by definitions (3.19) and (3.39). Hence, using (3.21), (3.40), we establish

\[
\lambda \in \mathcal{B}_0^{N_{n+1}}(u_n) \Rightarrow \lambda \in \mathcal{B}_{N_{n+1}}(u), \quad \text{that is}
\]

\[
\|u - u_n\|_{s_1} \leq N_{n+1}^{-\sigma} \text{ and } \lambda \in \bigcap_{k=1}^{n+1} \mathcal{B}_0^{N_{n+1}}(u_{k-1}) \Rightarrow \lambda \in \mathcal{B}_{N_{n+1}}(u).
\]

Let us prove the claim (3.114). It follows from (2.23), (3.7), (3.12) and (F1)\(_n\) that

\[
|A_{N_{n+1}, j_0}^{-1}(\epsilon, \lambda, u_n, \theta) - \text{Diag}(A_{N_{n+1}, j_0}(\epsilon, \lambda, u_n, \theta))|_{s_1-\rho} \leq C(s_1) (\|V\|_{s_1} + \|v\|_{(DF)(u_n)}|_{s_1}) \leq C(V) =: \Upsilon.
\]

(3.115)

Combining this with the fact \( \|A_{N_{n+1}, j_0}^{-1}(\epsilon, \lambda, u_n, \theta)\|_0 \leq N_{n+1}^\tau \) and Lemma 3.31 yields that the assumptions (A1)-(A3) in Proposition 3.8 are satisfied. If \( 2^{n+1} \leq \chi_0 \) (resp. \( 2^{n+1} > \chi_0 \)), then we apply Proposition 3.8 to \( A := A_{N_{n+1}}(\epsilon, \lambda, u_n, \theta) \) with

\[
\mathfrak{A} := [-N_{n+1}, N_{n+1}]^r \times \left\{ j_0 + \sum_{p=1}^{r} j_p w_k : j_k \in [-N_{n+1}, N_{n+1}] \right\} \cap \Gamma_+(M),
\]

\[
N' := N_{n+1} \quad N := N_p \text{ (resp. } N := N_p).
\]
Hence, for all \( s \in [s_0, s_1 - \delta] \), it follows from (3.27), (2.23), (3.7), (3.12), and (F1)\(_n\) that
\[
|A_{N_{n+1,j_0}}^{-1}(\epsilon, \lambda, u_n, \theta)|_s \leq \frac{1}{4} N_{n+1}^{\tau_2} (\Lambda_{n+1}^{\delta s} + C(s_1) (\|V\|_{s_1} + \epsilon \|(DF)(u_n)\|_{s_1})) \leq \frac{1}{2} N_{n+1}^{\tau_2 + \delta s}. \tag{3.116}
\]
Moreover
\[
A_{N_{n+1,j_0}}(\epsilon, \lambda, u, \theta) := A_{N_{n+1,j_0}}(\epsilon, \lambda, u_n, \theta) + \mathcal{R}
\]
with \( \mathcal{R} = A_{N_{n+1,j_0}}(\epsilon, \lambda, u, \theta) - A_{N_{n+1,j_0}}(\epsilon, \lambda, u_n, \theta) \). It is clear that
\[
|\mathcal{R}|_{s_1 - \delta} \lesssim C(s_1) \|(DF)(u) - (DF)(u_n)\|_{s_1} \lesssim C'(s_1) \|u - u_n\|_{s_1} \leq C'(s_1) N_{n+1}^{-\sigma},
\]
which carries out
\[
|A_{N_{n+1,j_0}}^{-1}(\epsilon, \lambda, u_n, \theta)|_{s_1 - \delta} |\mathcal{R}|_{s_1 - \delta} \leq \frac{1}{2} N_{n+1}^{\tau_2 + \delta s}.
\]
By Lemma 2.11 we obtain that, for all \( s \in [s_0, s_1 - \delta] \),
\[
|A_{N_{n+1,j_0}}^{-1}(\epsilon, \lambda, u_n, \theta)|_s \leq N_{n+1}^{\tau_2 + \delta s},
\]
which leads to that \( A_{N_{n+1,j_0}}(\epsilon, \lambda, u, \theta) \) is \( N_{n+1} \)-good.

3.5. **Measure estimate.**

**Lemma 3.37.** The complementary of the set \( \mathcal{U} \) defined in (3.57) has that, for some constant \( \mathcal{C}_2 > 0 \),
\[
\operatorname{meas}(\Lambda \setminus \mathcal{U}) \leq \mathcal{C}_2 \gamma. \tag{3.117}
\]

**Proof.** Definition (3.57) gives
\[
\Lambda \setminus \mathcal{U} \subset \bigcup_{|l|, |j| \leq N_0, 1 \leq p \leq \delta_j} \mathcal{U}_{l,j,p} \quad \text{with} \quad \mathcal{U}_{l,j,p} := \left\{ \lambda \in \Lambda : |-(\lambda \omega_0 \cdot l)^2 + \hat{\lambda}_{j,p}| \leq \gamma N_0^{-\tau_1} \right\},
\]
where \( \hat{\lambda}_{j,p} \) is eigenvalues of \( \tilde{P}_{N_0,0}(\Delta^2 + V(x))|_{H_{N_0,0}} \). Formula (1.4) implies \( \hat{\lambda}_{j,p} \geq \kappa_0 \). If \( \gamma N_0^{-\tau_1} < \kappa_0 \), then \( \mathcal{U}_{0,0,p} = \emptyset \). For \( l \neq 0 \), letting \( \xi = \lambda^2 \), we have
\[
\hat{f}_{l,j,p}(\xi) = (\omega_0 \cdot l)^2 - \hat{\lambda}_{j,p}.
\]
Using (1.3) and \( |l| \leq N_0 \), we deduce
\[
\partial_{\xi} \hat{f}_{l,j,p}(\xi) = (\omega_0 \cdot l)^2 \geq 4 \gamma_{N_0}^{-2} N_0^{-2\nu}.
\]
Thus one has
\[
|\xi_1 - \xi_2| \leq \frac{2 \gamma N_0^{-\tau_1}}{4 \gamma_{N_0}^{-2} N_0^{-2\nu}} = \frac{\gamma}{2 \gamma} N_0^{-\tau_1 + 2\nu},
\]
which carries out
\[
\operatorname{meas}(\mathcal{U}_{l,j,p}) \leq \frac{\gamma}{2 \gamma} N_0^{-\tau_1 + 2\nu}.
\]
Thus, if \( \tau_1 - 3\nu - d \geq 1 \), then the following holds:
\[
\operatorname{meas}(\Lambda \setminus \mathcal{U}) \leq \sum_{|l|, |j| \leq N_0, p \leq \delta_j} \operatorname{meas}(\mathcal{U}_{l,j,p}) \leq \frac{\gamma}{2 \gamma} N_0^{-\tau_1 + 2\nu} C N_0^{\nu + d} = \mathcal{O}(\frac{\gamma}{2 \gamma} N_0^{-1}).
\]
\[ \square \]
Finally, for $\epsilon_0$ small enough, we choose
\[
\gamma = \frac{1}{\epsilon_0^{2/3}}, \quad N_0 = 32\gamma^{-1}
\]  
(3.118)
to guarantee that (3.58) is satisfied, and that $16\gamma^{-1} \geq \max\lbrace \kappa_0^{-1}, 1, 2\|\rho\|\}$ (recall Lemma 3.7 and (3.42)). To apply Proposition 3.25 to $\mathcal{D}_N^0(k-1), k \geq 1$, we have to check
\[
\epsilon\kappa_0^{-1}(\|\mathcal{T}^0_N (u_{k-1})\|_0 + \|\partial_\lambda \mathcal{T}^0_N (u_{k-1})\|_0) \leq c \text{ (recall (3.45))},
\]  
(3.119)
where $\mathcal{T}^\nu(u)$ is defined in (2.12). It is easy that
\[
\|\mathcal{T}^0_N (u_{k-1})\|_0 \leq |\mathcal{T}^0_N (u_{k-1})|_{s_0} \leq C(s_0)(1 + \|u_{k-1}\|_{s_0 + \rho}) \leq C'(s_0),
\]
where $\rho = (2\nu + d + r + 1)/2$ (recall Lemma (2.6)). Moreover
\[
\|\partial_\lambda \mathcal{T}^0_N (u_{k-1})\|_0 \leq C(s_0)(1 + \|\partial_\lambda u_{k-1}\|_{s_0 + \rho}) \leq C'(s_0)N^{\gamma_2 + s_1 + 1}\gamma^{-1}.
\]
Hence, by (3.58), we get (3.119). Consequently, the complement of $\mathcal{D}_p$ in $\Lambda$ has measure
\[
\text{meas} (\mathcal{D}_p) \leq \text{meas} \left( \bigcup_{k=1}^{n} (\mathcal{U}_{N(k)}(u_{k-1}))^c \cup \bigcup_{k=1}^{n} (\mathcal{G}_{N(k)}^0(u_{k-1}))^c \cup \mathcal{U}^c \right)
\]
\[
\leq \sum_{k \geq 1} \text{meas} (\mathcal{U}_{N(k)}(u_{k-1}))^c + \sum_{k \geq 1} \text{meas} (\mathcal{G}_{N(k)}^0(u_{k-1}))^c + \text{meas} (\mathcal{U}^c)
\]
\[
\leq \sum_{k \geq 1} N_k^{-1} + c_1 \sum_{k \geq 1} N_k^{-1} + c_2 \gamma = \text{O}(\gamma).
\]

4. APPENDIX

4.1. Proof of lemma 3.1

Before proving the lemma, we have to give some definitions. We call that the site $j \in E$ is regular if $|-(\lambda\omega \cdot l + \theta)^2 + \lambda_j^2 + m| \geq \Theta$, where $A_j = \Theta = -(\lambda\omega \cdot l + \theta)^2 + \lambda_j^2 + m)I_{3j}$. Otherwise $j$ is singular. Let $R, S$ denote the following sets
\[
R := \{ j \in E \mid j \text{ is regular} \}, \quad S := \{ j \in E \mid j \text{ is singular} \}.
\]
It is straightforward that $E = R + S$. For fixed $l \in \mathbb{Z}^\nu, \theta \in \mathbb{R}$, let $A$ represent the following linear operator:
\[
-(\lambda\omega \cdot l + \theta)^2)I_{3j} + \tilde{P}_{N_0, \delta_{0}}(\Delta^2 + V(x))|_{\mathbb{R}_{N_0, \delta_{0}}},
\]
Abusing the notations, we write $A_E^E := A, u_E := u, h_E := h$, where $A_E^R \in \mathcal{M}_R^E, u_E, h_E \in H_E^R$. Consider the Cramer system
\[
A_E^E u_E = h_E.
\]  
(4.1)

Remark 4.1. We choose $\Theta$ satisfying $\Theta \leq \text{O}(\gamma)$ for some constant $c_1(s_0) > 0$.

Proof. Denote $E := \{ j = j_0 + \sum_{p=1}^{n} j_p w_p \mid j_p \in [-N, N] \} \cap \Gamma_+(M)$.

The first reduction: There exist $M_{R}^E, N_R^E \in \mathcal{M}_R^E$ such that
\[
A_{E}^E u_E = h_E \Rightarrow u_R + M_{R}^E u_E = N_{R}^E h_E.
\]
Moreover, $M_{R}^E, N_{R}^E$ satisfy (4.5) - (4.7).

In fact, for $j \in E$ is regular, we obtain
\[
A_j^E u_j + A_j^E(\{j\})_E u_{E \setminus \{j\}} = h_j \Rightarrow u_j + (A_j^E)^{-1} A_j^E(\{j\})_E u_{E \setminus \{j\}} = (A_j^E)^{-1} h_j.
\]
From (2.24)-(2.25), (2.23), it yields that
\[ \begin{align*}
|((A^j_j)^{-1}A^E\{j\})|_{s_0+\nu+r} & \leq C'((A^j_j)^{-1}|_{s_0+\nu+r}A^E\{j\})|_{s_0+\nu+r} \leq C((A^j_j)^{-1}|_{s_0+\nu+r}V||_{s_0+\nu+r+\varrho} \\
& \leq C\Theta^{-1}||V||_{s_0+\nu+r+\varrho}, \\
|((A^j_j)^{-1}A^E\{j\})|_{s+\nu+r} & \leq \frac{1}{2}|(A^j_j)^{-1}|_{s+\nu+r}|A^E\{j\}|_{s_0} + \frac{C(s)}{2}((A^j_j)^{-1}|_{s_0}A^E\{j\})|_{s+\nu+r} \\
& \leq C(s)(\Theta^{-1}||V||_{s_0+\nu+r+\varrho} + \Theta^{-1}||V||_{s+\nu+r+\varrho}).
\end{align*} \] (4.2)

(4.3)

Define
\[ M^j_j := \left\{ \begin{array}{ll}
((A^j_j)^{-1}A^E\{j\})^j, & j' \in E\{j\}, \\
0, & j' = j,
\end{array} \right. \quad \text{and} \quad N^j_j := \left\{ \begin{array}{ll}
((A^j_j)^{-1})^j, & j' = j, \\
0, & j' \in E\{j\}.
\end{array} \right. \] (4.4)

Then (4.1) becomes
\[ u_j + M^E_j u_E = N^E_j h_E. \]

In addition, it follows from (2.29), (4.2)-(4.4) and the definition of the set \( R \) that
\[ \begin{align*}
|M^E_R|_{s_0} & \leq K_1|M^E_j|_{s_0+\nu+r} \leq C'\Theta^{-1}||V||_{s_0+\nu+r+\varrho}, \\
|M^E_R|_s & \leq K_1|M^E_j|_{s+\nu+r} \leq C'(s)(\Theta^{-1}||V||_{s_0+\nu+r+\varrho} + \Theta^{-1}||V||_{s+\nu+r+\varrho}), \\
|N^E_R|_{s_0} & \leq K_1|N^E_j|_{s_0+\nu+r} \leq K_1\Theta^{-1}, \quad |N^E_R|_s \leq K_1|N^E_j|_{s+\nu+r} \leq K_1\Theta^{-1}.
\end{align*} \] (4.5)

The second reduction: For \( \Theta^{-1}||V||_{s_0+\nu+r+\varrho} \leq \epsilon_1(s_0) \) small enough, there exist \( \hat{M}^S_R \in M^S_R, \hat{N}^E_R \in M^E_R \)
satisfying (4.12)-(4.14) such that
\[ A^E_R u_E = h_E \Rightarrow u_R = \hat{M}^S_R u_S + \hat{N}^E_R h_E. \] (4.8)

In fact, since \( E = R + S \), then
\[ u_R + M^E_R u_E = N^E_R h_E \Rightarrow u_R + M^R_R u_R + M^S_R u_S = N^E_R h_E, \]

namely,
\[ (I^R_R + M^R_R)u_R + M^S_R u_S = N^E_R h_E. \] (4.9)

For \( \Theta^{-1}||V||_{s_0+\nu+r+\varrho} \leq \epsilon_1(s_0) \) small enough, formula (4.5) shows
\[ |(I^R_R)^{-1}|_{s_0}|M^R_R|_{s_0} \leq C'\Theta^{-1}||V||_{s_0+\nu+r+\varrho} \leq 1/2. \]

Then it follows from Lemma 2.11 that \( (I^R_R + M^R_R) \) is invertible with
\[ |(I^R_R + M^R_R)^{-1}|_{s_0} \leq 2, \] (4.10)
\[ |(I^R_R + M^R_R)^{-1}|_s \leq C(s)(|I^R_R|^{-1}|_s + |I^R_R|^{-1}|_{s_0}|M^R_R|_s) \leq C''(s)(1 + \Theta^{-1}||V||_{s+\nu+r+\varrho}). \] (4.11)

As a consequence equation (4.9) is reduced to
\[ u_R = \hat{M}^S_R u_S + \hat{N}^E_R h_E \]

where
\[ \hat{M}^S_R = -(I^R_R + M^R_R)^{-1}M^S_R, \quad \hat{N}^E_R = (I^R_R + M^R_R)^{-1}N^E_R. \]
For $\Theta^{-1}\|V\|_{s_0+\nu+r+\rho} \leq c_1(s_0)$ small enough, applying (2.24)-(2.25), (4.5)-(4.7) and (4.10)-(4.11) yields that

$$|\tilde{M}_E^S|_{s_0} \leq 2C|M_R^S|_{s_0} \leq C, \quad (4.12)$$

$$|\tilde{M}_E^S|_{s} \leq \frac{1}{2}([\Omega^R_R + M_R^R]^{-1}_{s_0})|M_R^S|_{s} + \frac{C(s)}{2}([\Omega^R_R + M_R^R]^{-1}_{s})|M_R^S|_{s_0} \leq C''(s)(1 + \Theta^{-1}\|V\|_{s_0+\nu+r+\rho}), \quad (4.13)$$

$$|\tilde{N}_E^E|_{s_0} \leq C'\Theta^{-1}, \quad |\tilde{N}_E^E|_{s} \leq C''(s)\Theta^{-1}(1 + \Theta^{-1}\|V\|_{s_0+\nu+r+\rho}). \quad (4.14)$$

The third reduction: There exist $\tilde{M}_E^E \in M_E^E, \tilde{N}_E^E \in M_E^E$ satisfying (4.19)-(4.22) such that

$$A_E^E u_E = h_E \Rightarrow \tilde{M}_E^E u_S = \tilde{N}_E^E h_E.$$ 

Furthermore, $((A_E^E)^{-1})_E^S$ is a left inverse of $\tilde{M}_E^S$.

In fact, since $E = R + S$, for $j \in E$ is regular, this holds:

$$(A_E^E u_E)_j = h_j \Rightarrow (A_R^E u_R + A_S^E u_S)_j = h_j \Rightarrow (A_E^R \tilde{M}_R^S u_S + \tilde{N}_E^E h_E) + A_S^E u_S)_j = h_j \Rightarrow (\tilde{M}_E^E u_S)_j = (\tilde{N}_E^E h_E)_j, \quad (4.15)$$

where

$$\tilde{M}_E^S = A_E^R \tilde{M}_R^S + A_S^E, \quad \tilde{N}_E^E = I_E^E - A_E^R \tilde{N}_R^E.$$ 

Since $j \in R$, formula (4.15) infers that $\tilde{M}_E^S = 0$, which then gives that $\tilde{N}_E^E = 0$. Therefore

$$|A_E^E|_{s_0} = |A_E^E|_{s_0} \leq |\text{Diag}(A_E^S)|_{s_0} + |A_E^E - \text{Diag}(A_E^S)|_{s_0} \leq \Theta + C + C'|\|V\|_{s_0+\rho}, \quad (4.17)$$

$$|A_E^E|_{s} = |A_E^S|_{s} \leq |\text{Diag}(A_E^S)|_{s} + |A_E^E - \text{Diag}(A_E^S)|_{s} \leq \Theta + C(s)|\|V\|_{s_0+\rho}. \quad (4.18)$$

It follows from (2.24)-(2.25), (4.12)-(4.14) and (4.16)-(4.18) that

$$|\tilde{M}_E^S|_{s_0} = |\tilde{M}_E^S|_{s_0} \leq C|A_R^S|_{s_0}|\tilde{M}_R^S|_{s_0} + |A_S^S|_{s_0} \leq C'(|\|V\|_{s_0+\rho}), \quad (4.19)$$

$$|\tilde{M}_E^S|_{s} = |\tilde{M}_E^S|_{s} \leq \frac{1}{2}|A_R^S|_{s_0}|\tilde{M}_R^S|_{s} + \frac{C(s)}{2}|A_R^S|_{s}|\tilde{M}_R^S|_{s_0} + |A_S^S|_{s} \leq C(s, \Theta)(1 + \|V\|_{s_0+\rho}), \quad (4.20)$$

$$|\tilde{N}_E^E|_{s_0} = |\tilde{N}_E^E|_{s_0} \leq |I_E^E|_{s_0} + C|A_S^E|_{s_0}|\tilde{N}_R^E|_{s_0} \leq C', \quad (4.21)$$

$$|\tilde{N}_E^E|_{s} \leq |I_E^E|_{s_0} + \frac{1}{2}|A_R^S|_{s_0}|\tilde{N}_R^E|_{s} + \frac{C(s)}{2}|A_S^S|_{s}|\tilde{N}_R^E|_{s_0} \leq C(s, \Theta)(1 + \|V\|_{s_0+\rho}). \quad (4.22)$$

In addition, by (4.16), it is obvious that

$$((A_E^E)^{-1})_E^S \tilde{M}_E^E = ((A_E^E)^{-1})_E^S \tilde{M}_E^S = ((A_E^E)^{-1})_E^S (A_R^E \tilde{M}_R^S + A_S^S) = I_E^S.$$ 

This shows that $((A_E^E)^{-1})_E^S$ is a left inverse of $\tilde{M}_E^S$.

If $\delta < 1$, then there exists some constant $C(r) > 0$ such that $S$ (the set of singular sites) admits a partition with

$$\text{diam}(\Omega_\alpha) \leq LB \leq B^{1+C(r)} = N^\delta, \quad \text{d}(\Omega_\alpha, \Omega_\beta) > N^{\frac{\delta}{2+C(r)}}, \quad \forall \alpha \neq \beta. \quad (4.23)$$

Therefore we have the following result:

The final reduction: Define $X_E^S \in M_E^S$ by

$$X^j_E := \begin{cases} 
\tilde{M}_j^E & \text{if} \quad (j, j') \in \bigcup_\alpha (\Omega_\alpha \times \hat{\Omega}_\alpha), \\
0 & \text{if} \quad (j, j') \notin \bigcup_\alpha (\Omega_\alpha \times \hat{\Omega}_\alpha), 
\end{cases} \quad (4.24)$$

where $\hat{\Omega}_\alpha := \left\{ j \in E : \text{d}(j, \Omega_\alpha) \leq N^{\frac{\delta}{2+C(r)}} \right\}$. The definition of $\hat{\Omega}_\alpha$ together with (4.23) may indicate $\hat{\Omega}_\alpha \cap \hat{\Omega}_\beta = \emptyset, \forall \alpha \neq \beta.$
Step1: Let us claim that \( X^S_E \in \mathcal{M}^S_E \) has a left inverse \( Y^F_S \in \mathcal{M}^F_S \) with
\[
\| Y^F_S \|_0 \leq 2N^\tau. \tag{4.25}
\]
Define \( Z^S_E := M^S_E - X^S_E \), which then gives that \( X^S_E = M^S_E - Z^S_E \). Definition (4.24) implies that
\[
Z^j_E = 0 \quad \text{if} \quad d(j, j') \leq N^{4\delta(1+\sigma(r))}/4.
\]
Combining this with (2.27), (4.20), (3.12), for all \( s_1 \geq s_0 + \nu + r + \varrho \), we obtain
\[
|Z^S_E|_{s_0} \leq (N^{4\delta(1+\sigma(r))}/4)^{(s_1-\nu-r-\varrho-s_0)}|Z^S_E|_{s_1-\nu-r-\varrho} \leq 4^{s_1}(N^{4\delta(1+\sigma(r))})^{-(s_1-\nu-r-\varrho-s_0)}|M^S_E|_{s_1-\nu-r-\varrho}
\leq C'(s_1, \Theta)(N^{4\delta(1+\sigma(r))})^{-(s_1-\nu-r-\varrho-s_0)}, \tag{4.26}
\]
\[
|Z^S_E|_s \leq |M^S_E|_s \leq C(s, \Theta)(1 + \|V\|_{s+s_0+r+\varrho}). \tag{4.27}
\]
In addition, for \( N \geq \tilde{N}(s_1, V) \) large enough and \( s_1 > \frac{2(1+C(r))}{\delta} \tau + (s_0 + \nu + r + \varrho) \), formulae (2.30) and (4.27) establish
\[
\|((A^E)^{-1})^F_S\|_0 \|Z^S_E\|_0 \leq \|((A^E)^{-1})^F_S\|_0 \|Z^S_E|_{s_0} \leq C(s, \Theta)N^\tau(N^{4\delta(1+\sigma(r))})^{-(s_1-\nu-r-s_0)} \leq 1/2.
\]
Lemma 2.11 verifies that \( X^S_E \) has a left inverse \( Y^F_S \in \mathcal{M}^F_S \) with
\[
\| Y^F_S \|_0 \leq 2\|((A^E)^{-1})^F_S\|_0 \leq 2N^\tau. \tag{4.33}
\]

Step2: Define \( \tilde{Y}^F_S \) by
\[
\tilde{Y}^j_E := \begin{cases} 
Y^j_E & \text{if} \quad (j', j) \in \bigcup_{\alpha}(\Omega_\alpha \times \hat{\Omega}_\alpha) \\
0 & \text{if} \quad (j', j) \notin \bigcup_{\alpha}(\Omega_\alpha \times \hat{\Omega}_\alpha).
\end{cases} \tag{4.29}
\]
If the fact \( (Y^F_S - \tilde{Y}^F_S)X^S_E = 0 \) holds, then \( \tilde{Y}^F_S \) is a left inverse of \( X^S_E \) with
\[
|\tilde{Y}^F_S|_s \leq C(s)N^{4\delta(\nu+r)+\tau}. \tag{4.30}
\]
Let us prove above fact. For \( j \in S = \bigcup_{\alpha}(\Omega_\alpha \times \hat{\Omega}_\alpha), \) there is \( \alpha \) such that \( j \in \Omega_\alpha \). Moreover, since \( (Y^F_S - \tilde{Y}^F_S)_{j''} = Y^{j''} - \tilde{Y}^{j''} = 0 \) for \( j'' \in \hat{\Omega}_\alpha \), we get
\[
((Y^F_S - \tilde{Y}^F_S)X^S_E)_{j''} = \sum_{j'' \notin \Omega_\alpha} (Y^F_S - \tilde{Y}^F_S)_{j''} X^S_E_{j''}.
\]
If \( j \in \Omega_\alpha \), then definition (4.24) implies that \( X^S_E_{j''} = 0 \), which shows that \( ((Y^F_S - \tilde{Y}^F_S)X^S_E)_{j''} = 0 \).
If \( j \notin \Omega_\beta \) with \( \alpha \neq \beta \), for \( j'' \notin \hat{\Omega}_\beta \), then \( X^S_E_{j''} = 0 \) owing to (4.24). As a consequence
\[
((Y^F_S - \tilde{Y}^F_S)X^S_E)_{j''} = \sum_{j'' \notin \Omega_\alpha} (Y^F_S - \tilde{Y}^F_S)_{j''} X^S_E_{j''} = \sum_{j'' \notin \Omega_\alpha} Y^{j''} X^{j''} = \sum_{j'' \in E} Y^{j''} X^{j''} = 0.
\]
Since \( \text{diam}(\hat{\Omega}_\alpha) < 2N^\delta \) (see 4.23), by definition (4.29), we obtain that \( \tilde{Y}^j_E = 0 \) for all \( |j - j'| \geq 2N^\delta \). Then (2.28) infers
\[
|\tilde{Y}^F_S|_s \leq C(s)N^{4\delta(\nu+r)+\tau} \leq C(s)N^{4\delta(\nu+r)+\tau}.
\]
Step 3: For $N \geq \tilde{N}(s_1, V)$ large enough and $s_1 > (1 + C(r))(s_0 + \nu + r) + \frac{2(1 + C(r))}{\delta} r + (s_0 + \nu + r + \varrho)$, it follows from (4.27) and (4.30) that
\[
|\tilde{\mathcal{Y}}^E|_{s_0}|Z^S_E|_{s_0} \leq CN^\frac{\delta}{2}(s_0 + \nu + r) + C(s_1, \Theta)(N^\frac{\delta}{2} + r)^{-\nu - r - \varrho - s_0} \leq 1/2.
\]
Combining this with the equality $\tilde{M}^S_E = X^S_E + Z^S_E$ and Lemma 2.11 establishes that $\tilde{M}^S_E$ has a left inverse $[-1](\tilde{M}^S_E)$ with
\[
\begin{align*}
|[-1](\tilde{M}^S_E)|_{s_0} & \leq 2|\tilde{\mathcal{Y}}^E|_{s_0} \leq 2CN^\frac{\delta}{2}(s_0 + \nu + r) + r, \\
|[-1](\tilde{M}^S_E)|_s & \leq C(s)(|\tilde{\mathcal{Y}}^E|_s + |\tilde{\mathcal{Y}}^E|_{s_0}^2|Z^S_E|_s) \\
& \leq C\delta(s, \Theta)N^{\delta(s_0 + \nu + r)} + 2\tau(N^{\delta s} + \|V\|_{s + \nu + r + \varrho}).
\end{align*}
\]
Thus the system (4.1) is equivalent to
\[
\begin{cases}
u_R = \tilde{M}^S_E([-1](\tilde{M}^S_E))^\dagger \tilde{\mathcal{Y}}^E h_E + \tilde{N}^E h_E, \\
u_S = (-1)(\tilde{M}^S_E)^\dagger \tilde{N}^E h_E.
\end{cases}
\]
This implies that
\[
((A^E_E)^{-1})^E_R = \tilde{M}^S_E([-1](\tilde{M}^S_E))^\dagger \tilde{N}^E h_E + \tilde{N}^E_R, \quad ((A^E_E)^{-1})_S^E = (-1)(\tilde{M}^S_E)^\dagger \tilde{N}^E E.
\]
From (2.24), (4.12)-(4.14), (4.21)-(4.22) and (4.31)-(4.32), it yields that
\[
\begin{align*}
|((A^E_E)^{-1})^E_R|_s & \leq C''(s, \Theta)N^{\delta(s_0 + \nu + r) + 2\tau}(N^{\delta s} + \|V\|_{s + \nu + r + \varrho}), \\
|((A^E_E)^{-1})_S^E|_s & \leq C''(s, \Theta)N^{\delta(s_0 + \nu + r) + 2\tau}(N^{\delta s} + \|V\|_{s + \nu + r + \varrho}).
\end{align*}
\]
The definition of $E$ and (3.4) give $\|V\|_{s + \nu + r + \varrho} \leq \nu^{\nu + r + \varrho} N^{\nu + r + \varrho} \|V\|_s$. Consequently, for $N \geq \tilde{N}(s_2, V)$ large enough, combining this with (3.12) yields
\[
|((A^E_E)^{-1})^E_R|_s \leq |((A^E_E)^{-1})^E_R|_s + |((A^E_E)^{-1})_S^E|_s \leq C''(s, \Theta)N^{\delta(s_0 + \nu + r) + 2\tau + \nu + r + \varrho + 1} + 2\tau + \frac{\varrho}{\delta}(\nu + \varrho) + \frac{\varrho}{\delta}(\nu + \varrho) + \frac{\varrho}{\delta}(\nu + \varrho) + \frac{\varrho}{2}.
\]
where $\tau_2 > \delta(s_0 + \nu + r) + 2\tau + \nu + r + \varrho + 1 = 2\tau + \frac{\varrho}{\delta}(\nu + \varrho) + \frac{\varrho}{\delta}(\nu + \varrho) + \frac{\varrho}{2}$. \hfill \Box

Let us verify that the fact (4.23) holds.

**Definition 4.2.** Denote by $\{j_k, k \in [0, L] \cap \mathbb{N}\}$ a sequence of sites with $j_k \neq j_{k'}, \forall k \neq k'$. For $B \geq 2$, we call $\{j_k, k \in [0, L] \cap \mathbb{N}\}$ a $B$-chain of length $L$ with $|j_{k+1} - j_k| \leq B, \forall k = 0, \cdots, L - 1$.

**Lemma 4.3.** There exists $C(r) > 0$ such that, for fixed $l \in \mathbb{Z}'$, $\theta \in \mathbb{R}$, any $B$-chain of singular sites has length $L \leq B^C(r)$.

**Proof.** Denote by $\{j_k, k \in [0, L] \cap \mathbb{N}\}$ a $B$-chain of singular sites. Then
\[
|j_{k+1} - j_k| \leq B, \quad \forall k = 0, \cdots, L - 1.
\]
(4.33)

Letting $\vartheta := \lambda \omega_0 \cdot l + \theta$, by the definitions of the singular site and $\lambda_j$, we give
\[
| - \vartheta^2 + (|j_k + \rho|^2 - ||\rho||^2)^2 + m| < \Theta \Rightarrow | - \vartheta^2 + (|j_k + \rho|^2 - ||\rho||^2)^2 | < \Theta + m
\]
\[
\Rightarrow | - \vartheta + |j_k + \rho|^2 - ||\rho||^2 | < \sqrt{\Theta + m} \quad \text{or} \quad | \vartheta + |j_k + \rho|^2 - ||\rho||^2 | < \sqrt{\Theta + m}
\]
\[
\Rightarrow \left| \begin{array}{l}
|j_{k+1} + \rho|^2 + |j_k + \rho|^2 < 2(\sqrt{\Theta + m} + ||\rho||^2) \quad \text{or} \\
|j_{k+1} + \rho|^2 - |j_k + \rho|^2 | < 2(\sqrt{\Theta + m} + ||\rho||^2),
\end{array} \right.
\]
which leads to
\[ \|j_{k+1} + \rho\|^2 - \|j_k + \rho\|^2 < 2(\sqrt{\Theta} + m + \|\rho\|^2). \]
This implies
\[ \|j_k + \rho\|^2 - \|j_{k_0} + \rho\|^2 \leq 2(\sqrt{\Theta} + m + \|\rho\|^2)|k - k_0|. \]
Combining this with (2.7), (4.33) and the equality
\[ (j_{k_0} + \rho) \cdot (j_k - j_{k_0}) = \frac{1}{2} (\|j_k + \rho\|^2 - \|j_{k_0} + \rho\|^2 - \|j_k - j_{k_0}\|^2) \]
yields
\[ |(j_{k_0} + \rho) \cdot (j_k - j_{k_0})| \leq (\sqrt{\Theta} + m + \|\rho\|^2)|k - k_0| + (b_2^2/2)|k - k_0|^2B^2 \leq (\sqrt{\Theta} + m + \|\rho\|^2 + b_2^2)|k - k_0|^2B^2. \]

(4.34)
Define the following subspace of \( \mathbb{R}^r \) by
\[ \mathcal{E} := \text{span}_\mathbb{R}\{j_k - j_{k'} : k, k' = 0, \cdots, L\} = \text{span}_\mathbb{R}\{j_k - j_{k_0} : k = 0, \cdots, L\}. \]
Let \( r_0 \) be the dimension of \( \mathcal{E} \). Denote by \( \xi_1, \cdots, \xi_{r_0} \) a basis of \( \mathcal{E} \). It is straightforward that \( r_0 \leq r \).

Case 1. For all \( k_0 \in [0, L] \cap \mathbb{N} \), we have
\[ \mathcal{E}_{k_0} := \text{span}_\mathbb{R}\{j_k - j_{k_0} : |k - k_0| \leq L' \}, k = 0, \cdots, L \} = \mathcal{E}. \]
Formula (4.33) indicates that
\[ |\xi_p| = |j_p - j_{k_0}| \leq |p - k_0|B \leq L'B, \quad p = 1, \cdots, r_0. \]
Let \( \Pi_{\mathcal{E}} \) denote the orthogonal projection on \( \mathcal{E} \). Then
\[ \Pi_{\mathcal{E}}(j_{k_0} + \rho) = \sum_{p=1}^{r_0} z_p \xi_p \]
for some \( z_p \in \mathbb{R}, p = 1, \cdots, r_0 \). Hence we get
\[ \Pi_{\mathcal{E}}(j_{k_0} + \rho) \cdot \xi_{p'} = \sum_{p=1}^{r_0} z_p \xi_p \cdot \xi_{p'}. \]
Based on above fact, we consider the linear system
\[ Qz = y, \quad \text{where} \quad Q = (Q_{pp'})_{p,p'=1,\cdots,r_0}. \]
It follows from (4.34)-(4.35) that
\[ |y_{p'}| \leq (\sqrt{\Theta} + m + \|\rho\|^2 + b_2^2)(L'B)^2, \quad |Q_{pp'}| \leq (L'B)^2. \]
Moreover formula (2.2) verifies
\[ \delta^{r_0} \det(Q) \in \mathbb{Z}, \quad \text{namely} \quad \delta^{r_0} |\det(Q)| \geq 1. \]
Hadamard inequality gives
\[ |(Q^*)_1| \leq \prod_{p \neq 1, 1 \leq p \leq r_0} \left( \sum_{p' \neq 1, 1 \leq p' \leq r_0} |Q_{pp'}|^2 \right)^{1/2}, \]
where \( Q^* \) is the adjoint matrix of \( Q \). This establishes that
\[ |(Q^*)_1| \leq (r_0 - 1)^{1/2}(L'B)^2(r_0 - 1). \]
Based on this and Cramer’s rule, (4.37)-(4.38), we obtain that
\[ |z_p| \leq \sum_{p'=1}^{r_0} |Q_{pp'}^{-1}y_{p'}| \leq \delta^{r_0}r_0^{1/2}(\sqrt{\Theta} + m + \|\rho\|^2 + b_2^2)(L'B)^2r_0. \]
Combining this with formulae (4.35)–(4.36), we derive
\[ |\Pi_\varepsilon(j_{k_0} + \rho)| \leq r_0|z_0| |\xi_0| \leq 2^v r_0^{v_0+1} (\sqrt{\Theta} + m + \|\rho\|^2 + b_2^2) (L^v B)^{2r+1}. \]
As a consequence
\[ |j_{k_1} - j_{k_2}| = |(j_{k_1} - j_{k_0}) - (j_{k_2} - j_{k_0})| = |\Pi_\varepsilon(j_{k_1} + \rho) - \Pi_\varepsilon(j_{k_2} + \rho)| \]
\[ \leq 2^v r^{v_0+1} (\sqrt{\Theta} + m + \|\rho\|^2 + b_2^2) (L^v B)^{2r+1}, \]
which then implies
\[ L \leq 4^v (2^v r^{v_0+1} (\sqrt{\Theta} + m + \|\rho\|^2 + b_2^2) (L^v B)^{2r+1})^r. \tag{4.39} \]
If \( v < \frac{1}{2r(2r+1)} \), then (4.39) yields that
\[ L^\frac{1}{2r} \leq 2^{2r^2} r^{r(r+1)} (\sqrt{\Theta} + m + \|\rho\|^2 + b_2^2)^r B^{r(2r+1)} \]
\[ \Rightarrow L \leq 2^{2r^2} r^{r(r+1)} (\sqrt{\Theta} + m + \|\rho\|^2 + b_2^2)^r B^{2r(2r+1)}. \]

Case 2. If there exists some \( k_0' \in [0, L] \cap \mathbb{N} \) such that \( \dim \mathcal{E}_{k_0} \leq r - 1 \), for \( k_0 \in \mathcal{J} \), then we consider
\[ \mathcal{E}_{k_0} := \text{span}_{\mathbb{R}} \{ j_k - j_{k_0} : |k - k_0| < L_1, k \in \mathcal{J} \} = \text{span}_{\mathbb{R}} \{ j_k - j_{k_0} : k \in \mathcal{J} \}, \]
where
\[ L_1 = L^v, \quad \mathcal{J} := \{ k : |k - k_0'| < L^v, k = 0, \ldots, L \} \cap ([0, L] \cap \mathbb{N}). \]
The upper bound of \( L_1 \) can be proved by the same method as employed on \( L \), namely
\[ L_1 = L^v \leq 2^{2v^2} r^{v^2} r^{2r(r+1)} (\sqrt{\Theta} + m + \|\rho\|^2 + b_2^2) B^{2r(2r+1)}. \]
The fact \( r_0 \leq r \) leads to that the iteration is carried out at most \( r \) steps. Thus
\[ L_r = L^{rv} \leq 2^{2r^2} r^{2r(r+1)} (\sqrt{\Theta} + m + \|\rho\|^2 + b_2^2) r^{2r(2r+1)} \Rightarrow L \leq C(r) \]
for some constant \( C(r) > 0 \). Let \( B = N^{\frac{1}{2}(1+C(r))} \).

**Definition 4.4.** We say that \( \varepsilon \equiv \eta \) if there is a \( N^{\frac{1}{2}(1+C(r))} \)-chain \( \{ j_k, k \in [0, L] \cap \mathbb{N} \} \) connecting \( \varepsilon \) to \( \eta \), namely, \( j_0 = \varepsilon, j_L = \eta \).

The equivalence relation induces that a partition of \( S \) satisfies
\[ \text{diam}(\Omega_\alpha) \leq LB \leq B^{1+C(r)} = N^\frac{1}{2}, \quad \text{d}(\Omega_\alpha, \Omega_\beta) > N^{\frac{1}{2}} \], \( \forall \alpha \neq \beta \).

**4.2. Proof of Proposition 3.8.** For \( A \in \mathcal{M}_3^\mathfrak{A} \), define
\[ \text{Diag}(A) := (\delta_{nn} A^n_{n,n'})_{n,n' \in \mathfrak{A}}. \]
Denote by \( \mathfrak{G}, \mathfrak{B} \) the following sets
\[ \mathfrak{G} := \{ j \in \mathfrak{A} \mid j \text{ is } (A, N)-\text{good} \}, \quad \mathfrak{B} := \{ j \in \mathfrak{A} \mid j \text{ is } (A, N)-\text{bad} \}. \]
It is clear that \( \mathfrak{A} = \mathfrak{G} \cup \mathfrak{B} \). Moreover \( \mathfrak{G} = \mathfrak{R} \cup \mathfrak{R}, \) where
\[ \mathfrak{R} := \{ j \in \mathfrak{G} \mid j \text{ is } (A, N)-\text{regular} \}, \quad \mathfrak{R} := \{ j \in \mathfrak{G} \mid j \text{ is regular} \}. \]

**Proof.** Abusing the notations, we write \( \mathcal{A}_\mathfrak{A} := A, \omega_\mathfrak{A} := u, h_\mathfrak{A} := h, \) where \( \mathcal{A}_\mathfrak{A} \in \mathcal{M}_3^\mathfrak{A}, \omega_\mathfrak{A}, h_\mathfrak{A} \in H^s_{\mathfrak{R}}. \)

Consider the following Cramer system
\[ \mathcal{A}_\mathfrak{A} \omega_\mathfrak{A} = h_\mathfrak{A}. \tag{4.40} \]

**The first reduction:** For \( N \geq \tilde{N}(\tilde{\Theta}, \Upsilon, s_1) \) large enough, there exist \( P_{\mathfrak{A}}^\mathfrak{A}, S_{\mathfrak{A}}^\mathfrak{A} \in \mathcal{M}_3^\mathfrak{A} \) with
\[ |P_{\mathfrak{A}}^\mathfrak{A}|_{s_0} \leq C(s_1) \tilde{\Theta}^{-1} \Upsilon, \quad |S_{\mathfrak{A}}^\mathfrak{A}|_{s_0} \leq N^\varepsilon. \tag{4.41} \]
and, for all \( s \geq s_0 \),
\[
|P_{\alpha}^n|_s \leq C(s)N^{r}(N^{s-s_0} + N^{-(\nu+r)}|Q|_{s+\nu+r}), \quad |S_{\alpha}^n|_s \leq C(s)N^{r+s-s_0} \tag{4.42}
\]
such that
\[
A_{\alpha}^3 u_{\alpha} = h_{\alpha} \Rightarrow u_{\phi} + P_{\alpha}^3 u_{\alpha} = S_{\alpha}^3 h_{\alpha}. \tag{4.43}
\]

In fact, since \( n \in \mathcal{A} \) is \((A, N)\)-regular, there exist \( \mathcal{F} \subset \mathcal{A} \) with \( \text{diam} (\mathcal{F}) \leq 4N \), \( d(n, \mathcal{A}\setminus \mathcal{F}) \geq N \) such that \( A_{\mathcal{F}}^3 \) is \( N \)-good. Then we have
\[
A_{\mathcal{F}}^3 u_{\mathcal{F}} = h_{\mathcal{F}} \Rightarrow A_{\mathcal{F}}^3 u_{\mathcal{F}} + A_{\mathcal{F}}^3 \bigcap u_{\mathcal{A}\setminus \mathcal{F}} = h_{\mathcal{F}} \Rightarrow u_{\mathcal{F}} + (A_{\mathcal{F}}^3)^{-1}A_{\mathcal{F}}^3 u_{\mathcal{A}\setminus \mathcal{F}} = (A_{\mathcal{F}}^3)^{-1}h_{\mathcal{F}}.
\]
From (2.25), (3.18) and (A1), it yields that
\[
|(A_{\mathcal{F}}^3)^{-1}A_{\mathcal{F}}^3|_{s_1-\partial} \leq C(s_1)\big((A_{\mathcal{F}}^3)^{-1}|_{s_1-\partial}Q|_{s_1-\partial} \leq C(s_1)N^{\tau_2+\delta(s_1-\partial)}Y. \tag{4.44}
\]
The fact \( \text{diam} (\mathcal{F}) \leq 4N \) shows that \(((A_{\mathcal{F}}^3)^{-1})_{n} = 0 \) for all \(|n' - n| > 4N \). Combining this with (2.24), (2.28), (3.18) and (A1) verifies
\[
|((A_{\mathcal{F}}^3)^{-1} A_{\mathcal{F}}^3)^n|_{s+\nu+r} \leq \frac{1}{2}(|(A_{\mathcal{F}}^3)^{-1}|_{s_0} Q|_{s+\nu+r} + \frac{C(s)}{2} |(A_{\mathcal{F}}^3)^{-1}|_{s+\nu+r} Q|_{s_0}) \leq \frac{1}{2}(|(A_{\mathcal{F}}^3)^{-1}|_{s_0} Q|_{s+\nu+r} + \frac{C(s)}{2} |(A_{\mathcal{F}}^3)^{-1} N^{s+\nu+r-s_0} Q|_{s_0}) \leq \frac{1}{2} N^{\tau_2+\delta s_0} |Q|_{s+\nu+r} + \frac{C(s)}{2} N^{s+\nu+r-s_0} N^{\tau_2+\delta s_0} Y \leq C'(s) N^{(\delta-1)s_0} (N^{s+\nu+r+s_2} Y + N^{\tau_2+s_0} Q|_{s+\nu+r}). \tag{4.45}
\]
Define
\[
P'_{n} := \begin{cases} \left( (A_{\mathcal{F}}^3)^{-1} A_{\mathcal{F}}^3 \right)_{n}', & \text{if } n' \in \mathcal{A}\setminus \mathcal{F}, \\ 0, & \text{if } n' \in \mathcal{F}, \end{cases} \quad \text{and} \quad S'_{n} := \begin{cases} (A_{\mathcal{F}}^3)^{-1} n', & \text{if } n' \in \mathcal{F}, \\ 0, & \text{if } n' \in \mathcal{A}\setminus \mathcal{F}. \end{cases} \tag{4.46}
\]
Then (4.40) becomes
\[
u_{n} + \sum_{n' \in \mathcal{A}} P'_{n} n' = \sum_{n' \in \mathcal{A}} S'_{n} n' h_{n'}.
\]
Since \( d(n, \mathcal{A}\setminus \mathcal{F}) \geq N \), we have that \( P'_{n} = 0 \) for \(|n-n'| \leq N \). Hence, by means of (2.27), (4.44), (4.46), for \( s_1 > 1 \), \( (2+\nu+r+s_0) + \), we get that, for \( N \geq N(\Theta, s_1) \) large enough,
\[
|P_{\mathcal{F}n}|_{s+\nu+r} \leq N^{-(s_1-s_0-\nu-r-\phi)}|P_{\mathcal{F}n}|_{s_1-\phi} \leq C(s_1) Y N^{(\delta-1)s_1+s_2+\nu+r+s_0+(1-\delta)} \leq C(s_1) \Theta^{-1} Y,
\]
which leads to
\[
|P_{\mathcal{F}n}|_{s_0} \leq K_1 |P_{\mathcal{F}n}|_{s+\nu+r} \leq C'(s_1) \Theta^{-1} Y. \tag{4.49}
\]
Letting \( \epsilon := \tau_2+\nu+r+s_0 \), for \( N \geq N(Y, s_1) \) large enough, we get
\[
|P_{\mathcal{F}n}|_{s+\nu+r} \leq |(A_{\mathcal{F}}^3)^{-1} A_{\mathcal{F}}^3|_{s+\nu+r} \leq C'(s) N^{(\delta-1)s_0} (N^{s+\nu+r+s_2} Y + N^{\tau_2+s_0} Q|_{s+\nu+r}) \leq C'(s) N^{(\delta-1)s_0} + N^{-(\nu+r)} Q|_{s+\nu+r}), \tag{4.45}
\]
which carries out
\[
|P_{\mathcal{F}n}|_{s} \leq K_1 |P_{\mathcal{F}n}|_{s+\nu+r} \leq C''(s) N^\epsilon (N^{s-s_0} + N^{-(\nu+r)} Q|_{s+\nu+r}). \tag{4.46}
\]
In addition, definition (4.46) gives that \( S'_{n} = 0 \) for \(|n-n'| \geq 4N \). As a consequence
\[
|S'_{n}|_{s_0} \leq K_1 |S'_{n}|_{s+\nu+r} \leq K_1 (A_{\mathcal{F}}^3)^{-1} |_{s_0+\nu+r} \leq K_1 N^{\tau_2+\delta(s_0+\nu+r)} \leq N^\epsilon,
\]
which gives
\[ |S^3_{R^1}| s \leq K_1 |S^3_{n^1}|_{s+\nu+r} \leq K_1(4N)^{s-s_0}|S^3_{n^1}|_{s_0+\nu+r} \leq C(s)N^{\nu+s-s_0}. \]

If \( n \in \mathfrak{A} \) is regular, then a similar argument as the first reduction shown in the proof of Lemma 3.1 yields
\[ |P^3_{R^1}| s \leq K_1 |P^3_{n^1}|_{s+\nu+r} \leq C'(s)\tilde{\Theta}^{-1} |Q|_{s+\nu+r} \leq C'(s)\tilde{\Theta}^{-1} \Upsilon, \]
\[ |P^3_{R^1}| s \leq K_1 |P^3_{n^1}|_{s+\nu+r} \leq C'(s)\tilde{\Theta}^{-1} (|Q|_{s_0+\nu+r} + |Q|_{s+\nu+r}), \]
\[ |C^3_{R^1}| s \leq K_1 |C^3_{n^1}|_{s+\nu+r} \leq K_1\tilde{\Theta}^{-1}, \quad |S^3_{R^1}| s \leq K_1 |S^3_{n^1}|_{s+\nu+r} \leq K_1\tilde{\Theta}^{-1}. \]

Thus formulae (4.41)-(4.42) hold.

**The second reduction:** If \( \tilde{\Theta} \) is large enough subject to \( \Upsilon \), then there exist \( \tilde{P}^3_\Theta \in M^3_\Theta, \tilde{S}^3_\Theta \in M^3_\Theta \) with
\[ |\tilde{P}^3_\Theta| s \leq C(s_1)\Upsilon^{-1}, \quad |\tilde{S}^3_\Theta| s \leq C(s_1)N^\nu, \]
and, for all \( s \geq s_0 \),
\[ |\tilde{P}^3_\Theta| s \leq C(s)N^\nu(N^{s-s_0} + N^{-(\nu+r)}|Q|_{s+\nu+r}), \quad |\tilde{S}^3_\Theta| s \leq C(s)N^{2\nu}(N^{s-s_0} + N^{-(\nu+r)}|Q|_{s+\nu+r}), \]
such that
\[ A^3_\Theta u_\Theta = h_\Theta \Rightarrow u_\Theta = \tilde{P}^3_\Theta u_\Theta + \tilde{S}^3_\Theta h_\Theta. \]

In fact, by means of the fact \( \mathfrak{A} = \mathfrak{G} + \mathfrak{B} \) and formula (4.44), we infer
\[ u_\Theta + P^3_\Theta u_\Theta = S^3_\Theta h_\Theta \Rightarrow u_\Theta + \tilde{P}^3_\Theta u_\Theta + P^3_\Theta u_\Theta = S^3_\Theta h_\Theta, \]
namely,
\[ (I^\Theta + P^3_\Theta)u_\Theta + P^3_\Theta u_\Theta = S^3_\Theta h_\Theta. \]

If \( \tilde{\Theta} \) is large enough subject to \( \Upsilon \), then we have
\[ |(I^\Theta + P^3_\Theta)| s_0 \leq 1/2. \]

Hence Lemma 2.17 gives that \( I^\Theta + P^3_\Theta \) is invertible with
\[ |(I^\Theta + P^3_\Theta)^{-1}| s_0 \leq 2, \]

\[ |(I^\Theta + P^3_\Theta)^{-1}| s \leq C(s)(1 + |P^3_\Theta| s) \leq C'(s)N^\nu(N^{s-s_0} + N^{-(\nu+r)}|Q|_{s+\nu+r}). \]

As a consequence equation (4.50) is reduced to
\[ u_\Theta = \tilde{P}^3_\Theta u_\Theta + \tilde{S}^3_\Theta h_\Theta \]
where
\[ \tilde{P}^3_\Theta = -(I^\Theta + P^3_\Theta)^{-1}P^3_\Theta, \quad \tilde{S}^3_\Theta = (I^\Theta + P^3_\Theta)^{-1}S^3_\Theta. \]

Hence, due to (2.23)-(2.25), (4.41)-(4.42) and (4.51)-(4.52), we get that (4.47)-(4.48) hold.

**The third reduction:** There exist \( \tilde{P}^3_\Theta \in M^3_\Theta, \tilde{S}^3_\Theta \in M^3_\Theta \) with
\[ |\tilde{P}^3_\Theta| s_0 \leq C(s_1, \tilde{\Theta}), \quad |\tilde{S}^3_\Theta| s_0 \leq C(s_1)N^\nu, \]
and, for all \( s \geq s_0 \),
\[ |\tilde{P}^3_\Theta| s \leq C(s, \tilde{\Theta})N^\nu(N^{s-s_0} + N^{-(\nu+r)}|Q|_{s+\nu+r}), \]
\[ |\tilde{S}^3_\Theta| s \leq C(s, \tilde{\Theta})N^{2\nu}(N^{s-s_0} + N^{-(\nu+r)}|Q|_{s+\nu+r}), \]
such that
\[ A^3_\Theta u_\Theta = h_\Theta \Rightarrow \tilde{P}^3_\Theta u_\Theta = \tilde{S}^3_\Theta h_\Theta. \]

Furthermore \((A^3_\Theta)^{-1}S^3_\Theta \) is a left inverse of \( \tilde{P}^3_\Theta \).
In fact, with the help of the quality $\mathfrak{A} = \mathfrak{G} + \mathfrak{B}$, this holds:

$$\mathcal{A}^\mathfrak{G}_\mathfrak{A}^\mathfrak{G} u_{\mathfrak{A}} = h_{\mathfrak{A}} \Rightarrow \mathcal{A}^\mathfrak{G}_\mathfrak{A}^\mathfrak{G} u_{\mathfrak{G}} + \mathcal{A}^\mathfrak{G}_\mathfrak{A}^\mathfrak{B} u_{\mathfrak{B}} = h_{\mathfrak{A}}.$$  

Combining this with (4.49) leads to

$$\mathcal{A}^\mathfrak{G}_\mathfrak{A} (\mathcal{P}^\mathfrak{B}_\mathfrak{G} u_{\mathfrak{B}} + \mathcal{S}^\mathfrak{B}_\mathfrak{G} h_{\mathfrak{A}}) + \mathcal{A}^\mathfrak{G}_\mathfrak{A}^\mathfrak{B} u_{\mathfrak{B}} = h_{\mathfrak{A}}, \text{ namely } \hat{\mathcal{P}}^\mathfrak{B}_\mathfrak{A} u_{\mathfrak{B}} = \hat{\mathcal{S}}^\mathfrak{B}_\mathfrak{A} h_{\mathfrak{A}},$$

where

$$\hat{\mathcal{P}}^\mathfrak{B}_\mathfrak{A} = \mathcal{A}^\mathfrak{G}_\mathfrak{A} \mathcal{P}^\mathfrak{B}_\mathfrak{G} + \mathcal{A}^\mathfrak{G}_\mathfrak{A}^\mathfrak{B}, \quad \hat{\mathcal{S}}^\mathfrak{B}_\mathfrak{A} = \mathcal{I}^\mathfrak{G} - \mathcal{A}^\mathfrak{G}_\mathfrak{A} \mathcal{S}^\mathfrak{B}_\mathfrak{G}.$$

Since formulae (4.53)-(4.54) are proved in the similar way as shown in the proof of Lemma 3.1 (see the third reduction), the detail is omitted. Moreover if there exists some constant $C_1 := C_1(\nu, d, \tau) \geq 2$ such that $\mathfrak{B}$ (the set of $(\mathcal{A}, N)$-bad sites) admits a partition with

$$\text{diam}(\mathcal{D}_\alpha) \leq N^{C_1}, \quad d(\mathcal{D}_\alpha, \mathcal{D}_\beta) > N^2, \quad \forall \alpha \neq \beta,$$

then we have the following result:

**The final reduction:** Define $\mathcal{X}^\mathfrak{G}_\mathfrak{A} \in \mathcal{M}^\mathfrak{G}_\mathfrak{A}$ by

$$\mathcal{X}^\mathfrak{G}_\mathfrak{A} := \begin{cases} \mathcal{P}^\mathfrak{B}_{n'} & \text{if } (n, n') \in \bigcup_\alpha (\mathcal{D}_\alpha \times \hat{\mathcal{D}}_\alpha), \\ 0 & \text{if } (n, n') \notin \bigcup_\alpha (\mathcal{D}_\alpha \times \hat{\mathcal{D}}_\alpha), \end{cases}$$

where $\hat{\mathcal{D}}_\alpha := \{ n \in \mathfrak{A} : d(n, \mathcal{D}_\alpha) \leq N^2/4 \}$. The definition of $\hat{\mathcal{D}}_\alpha$ together with (3.26) may indicate $\hat{\mathcal{D}}_\alpha \cap \hat{\mathcal{D}}_\beta = \emptyset$, $\forall \alpha \neq \beta$.

**Step 1:** Let us claim that $\mathcal{X}^\mathfrak{G}_\mathfrak{A} \in \mathcal{M}^\mathfrak{G}_\mathfrak{A}$ has a left inverse $\mathcal{Y}^\mathfrak{G}_\mathfrak{A} \in \mathcal{M}^\mathfrak{G}_\mathfrak{A}$ with

$$\|\mathcal{Y}^\mathfrak{G}_\mathfrak{A}\|_0 \leq 2N^\tau.$$  

Define $\mathcal{Z}^\mathfrak{G}_\mathfrak{A} := \mathcal{P}^\mathfrak{B}_\mathfrak{A} - \mathcal{X}^\mathfrak{G}_\mathfrak{A}$, which then gives that $\mathcal{X}^\mathfrak{G}_\mathfrak{A} = \mathcal{P}^\mathfrak{B}_\mathfrak{A} - \mathcal{Z}^\mathfrak{G}_\mathfrak{A}$. Definition (4.57) implies that

$$\mathcal{Z}^\mathfrak{G} = 0 \quad \text{if } d(n, n') \leq N^2/4.$$  

Combining this with (2.27), (4.54) and (A1), for all $s_1 \geq s_0 + \nu + r + \rho$, we obtain

$$|\mathcal{Z}^\mathfrak{G}|_s \leq (N^2/4)^{(s_1 - \nu - r - \rho - s_0)} |\mathcal{Z}^\mathfrak{G}|_{s_1 - \nu - r - \rho} \leq 4^{s_1} N^{-2(s_1 - \nu - r - \rho - s_0)} |\mathcal{P}^\mathfrak{B}_\mathfrak{A}|_{s_1 - \nu - r - \rho}$$

$$\leq C(s_1, \tilde{\Theta}) 4^{s_1} N^{-2(s_1 - \nu - r - \rho - s_0)} N^\nu (N^{s_1 - \nu - r - s_0} + N^{\nu + r}) |\mathcal{Q}|_{s_1 - \rho}$$

$$\leq C'(s_1, \tilde{\Theta}) N^{2\xi - (s_1 - \rho)}, \quad (4.58)$$

$$|\mathcal{Z}^\mathfrak{G}|_s \leq |\mathcal{P}^\mathfrak{B}_\mathfrak{A}|_s \leq C(s, \tilde{\Theta}) N^\nu (N^{s - s_0} + N^{\nu + r}) |\mathcal{Q}|_{s + \nu + r}. \quad (4.59)$$

In addition, for $N \geq N(\tilde{\Theta}, s_1)$ large enough and $s_1 > 2\xi + \chi + \rho$, formulae (2.30) and (4.58) give

$$||((\mathcal{A}^\mathfrak{G}_\mathfrak{A})^{-1})^\mathfrak{G}_\mathfrak{A}_s||_0 \leq ||((\mathcal{A}^\mathfrak{G}_\mathfrak{A})^{-1})^\mathfrak{G}_\mathfrak{A}_s||_0 |\mathcal{Z}^\mathfrak{G}|_s \leq C(s_1, \tilde{\Theta}) N^{2\xi - (s_1 - \rho)} (N')^\tau$$

$$\leq C'(s_1, \tilde{\Theta}) N^{2\xi - (s_1 - \rho) + \chi\tau} \leq 1/2. \quad (4.60)$$

Lemma 2.11 verifies that $\mathcal{X}^\mathfrak{G}_\mathfrak{A}$ has a left inverse $\mathcal{Y}^\mathfrak{G}_\mathfrak{A} \in \mathcal{M}^\mathfrak{G}_\mathfrak{A}$ with

$$||\mathcal{Y}^\mathfrak{G}_\mathfrak{A}\|_0 \leq 2||((\mathcal{A}^\mathfrak{G}_\mathfrak{A})^{-1})^\mathfrak{G}_\mathfrak{A}_s||_0 \leq 2(N')^\tau.$$  

**Step 2:** Define $\mathcal{Y}^\mathfrak{G}_\mathfrak{A} \in \mathcal{M}^\mathfrak{G}_\mathfrak{A}$ by

$$\mathcal{Y}^\mathfrak{G}_\mathfrak{A} := \begin{cases} \mathcal{Y}^\mathfrak{G}_\mathfrak{A} & \text{if } (n, n') \in \bigcup_\alpha (\mathcal{D}_\alpha \times \hat{\mathcal{D}}_\alpha), \\ 0 & \text{if } (n, n') \notin \bigcup_\alpha (\mathcal{D}_\alpha \times \hat{\mathcal{D}}_\alpha). \end{cases}$$

If the fact

$$(\mathcal{Y}^\mathfrak{G}_\mathfrak{A} - \mathcal{X}^\mathfrak{G}_\mathfrak{A}) \mathcal{X}^\mathfrak{G}_\mathfrak{A} = 0 \quad (4.61)$$
holds, then \( \tilde{Y}^{B}_{\beta} \) is a left inverse of \( \mathcal{X}^{B}_{\alpha} \) with

\[
|\tilde{Y}^{B}_{\beta}| \leq C(s)N^{C_{1}(s+\nu+r)+\chi r}.
\]  

(4.62)

Let us prove formula (4.61). For any \( n \in B = \bigcup_{\alpha} \Omega_{\alpha} \), there is \( \alpha \) such that \( n \in \Omega_{\alpha} \) and

\[
((Y^{B}_{\beta} - \tilde{Y}^{B}_{\beta})\mathcal{X}^{B}_{\alpha})n' = \sum_{n'' \in \Omega_{\beta}} (Y^{B}_{\beta} - \tilde{Y}^{B}_{\beta})n'' \mathcal{X}^{(B)}_{\alpha}n''.
\]

Remark that \((Y^{B}_{\beta} - \tilde{Y}^{B}_{\beta})n'' = Y^{B}_{n} - \tilde{Y}^{B}_{n} = 0 \) if \( n'' \in \hat{\Omega}_{\alpha} \).

If \( n' \in \Omega_{\alpha} \), then definition (4.56) implies that \( \mathcal{X}^{(B)}_{\alpha}n'' = 0 \), which shows \((Y^{B}_{\beta} - \tilde{Y}^{B}_{\beta})\mathcal{X}^{B}_{\alpha}n' = 0 \). As a consequence

\[
\begin{align*}
((Y^{B}_{\beta} - \tilde{Y}^{B}_{\beta})\mathcal{X}^{B}_{\alpha})n' &= \sum_{n'' \in \Omega_{\beta}} (Y^{B}_{\beta} - \tilde{Y}^{B}_{\beta})n'' \mathcal{X}^{(B)}_{\alpha}n'' \\
&= (Y^{B}_{\beta} - \tilde{Y}^{B}_{\beta})\mathcal{X}^{B}_{\alpha}n' = (I^{B}_{\beta}n') = 0.
\end{align*}
\]

It is obvious that \( \text{diam}(\hat{\Omega}_{\alpha}) < 2N^{C_{1}} \) due to (3.26). Based on this and definition (4.60), we obtain that \( \tilde{Y}^{B}_{n} = 0 \) for all \( n - n' \geq 2N^{C_{1}} \). Thus it follows from (2.28), (3.22), (4.57) and (4.60) that

\[
|\tilde{Y}^{B}_{\beta}|_{s} \leq C(s)N^{C_{1}(s+\nu+r)+\chi r} ||\hat{Y}^{B}_{\beta}||_{0} \leq C(s)N^{C_{1}(s+\nu+r)+\chi r}.
\]

(4.63)

\[
|\tilde{Y}^{B}_{\beta}|_{s} \leq C(s)(|\hat{Y}^{B}_{\beta}|_{s} + |Z^{B}_{\beta}|_{s}) \leq C'(s, \Theta)N^{2\chi r + 2C_{1}(s+\nu+r)}(N^{C_{1}s} + |Q|_{s+\nu+r}).
\]

(4.64)

Step 3: For \( N \geq \hat{N}(\hat{\gamma}, \tilde{\Theta}, s_{1}) \) large enough and \( s_{1} > C_{1}(s_{0} + \nu + r) + \chi r + 2\epsilon + \varphi \), it follows from (4.58) and (4.62) that

\[
|\tilde{Y}^{B}_{\beta}|_{s_{0}} \leq C N^{C_{1}(s_{0} + \nu + r) + \chi r} C(s, \Theta)N^{2\chi r + 2C_{1}(s_{0} + \nu + r)}(N^{C_{1}s} + |Q|_{s+\nu+r}).
\]

Thus system (4.40) is equivalent to

\[
\begin{cases}
\hat{u}_{\beta} = \tilde{P}^{B}_{\beta}(-1)(\tilde{P}^{B}_{\beta})\hat{S}^{B}_{\alpha}h_{\alpha} + \hat{S}^{B}_{\alpha}h_{\alpha}, \\
\hat{u}_{\beta} = (-1)(\tilde{P}^{B}_{\beta})\hat{S}^{B}_{\alpha}h_{\alpha}.
\end{cases}
\]

This implies that

\[
((A^{B}_{\beta})^{-1})^{\beta}_{\alpha} = \tilde{P}^{B}_{\beta}(-1)(\tilde{P}^{B}_{\beta})\hat{S}^{B}_{\alpha} + \hat{S}^{B}_{\alpha}, \quad (A^{B}_{\beta})^{-1}^{\beta}_{\alpha} = (-1)(\tilde{P}^{B}_{\beta})\hat{S}^{B}_{\alpha}.
\]

The fact \( \mathfrak{A} \in \mathfrak{A} \) with \( \text{diam}(\mathfrak{A}) \leq 4N' \) shows

\[
Q^{n}_{n'} = 0 \quad \text{if} \quad d(n, n') > 8N',
\]

which leads to \( |Q|_{s+\nu+r} \leq C(N')^{\nu+r}|Q|_{s} = C(N^{\nu+r})|Q|_{s} \), due to (2.28) and (3.22). Then it follows from (2.24), (4.53)-(4.54) and (4.63) that

\[
((A^{(B)}_{\beta})^{-1})^{\beta}_{\alpha} \leq C''(s, \tilde{\Theta})N^{2\chi r + 2C_{1}(s_0 + \nu + r)}(N^{C_{1}s} + |Q|_{s+\nu+r}) \leq C''(s_1)N^{1+\chi r + C_1(s_0 + \nu + r)}.
\]

(4.65)

In addition

\[
((A^{(B)}_{\beta})^{-1})^{\beta}_{\alpha} \leq C|\tilde{P}^{B}_{\beta}|_{s_{0}}|\hat{S}^{B}_{\alpha}|_{s_{0}} \leq C'(s_1)N^{1+\chi r + C_1(s_0 + \nu + r)}.
\]
Combining this with (2.24), (4.47)-(4.48), (4.65) gives that
\[
((A_2^3)^{-1})_{|s} \leq C_4(s, \Theta) N^{2\varepsilon+2\chi}\tau+2C_1(s_0+\nu+r)+\chi(\nu+r) (N^C_1 s + |Q|_s).
\]
Consequently, for \( N \geq \bar{N}(\Theta, s_1) \) large enough, we have
\[
|((A_2^3)^{-1})_{|s} | \leq |((A_2^3)^{-1})_{|s} | + |((A_2^3)^{-1})_{|s} | \\
\leq C_5(s, \Theta) N^{2\varepsilon+2\chi}\tau+2C_1(s_0+\nu+r)+\chi(\nu+r) (N^C_1 s + |Q|_s) \\
\leq \frac{1}{4} (N')^{\gamma_2} ((N')^{\delta s} + |Q|_s),
\]
if \( \chi^{-1} C_1 < \delta, \chi^{-1} (2\varepsilon + 2\chi\tau + 2C_1(s_0 + \nu + r) + \chi(\nu + r)) < \gamma_2. \)

\[\Box\]

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