A MODEL OF MAGNETIC BRAKING OF SOLAR ROTATION THAT SATISFIES OBSERVATIONAL CONSTRAINTS

Pavel A. Denissenkov
Department of Physics & Astronomy, University of Victoria, P.O. Box 3055, Victoria, B.C., V8W 3P6, Canada; pavel.denisenkov@gmail.com
Received 2010 February 10; accepted 2010 June 8; published 2010 July 16

ABSTRACT

The model of magnetic braking of solar rotation considered by Charbonneau & MacGregor has been modified so that it is able to reproduce for the first time the rotational evolution of both the fastest and slowest rotators among solar-type stars in open clusters of different ages, without coming into conflict with other observational constraints, such as the time evolution of the atmospheric Li abundance in solar twins and the thinness of the solar tachocline. This new model assumes that rotation-driven turbulent diffusion, which is thought to amplify the viscosity and magnetic diffusivity in stellar radiative zones, is strongly anisotropic with the horizontal components of the transport coefficients strongly dominating over those in the vertical direction. Also taken into account is the poloidal field decay that helps to confine the width of the tachocline at the solar age. The model’s properties are investigated by numerically solving the azimuthal components of the coupled momentum and magnetic induction equations in two dimensions using a finite element method.

Key words: magnetohydrodynamics (MHD) – stars: evolution – stars: interiors – stars: magnetic field – Sun: rotation

Online-only material: color figures

1. INTRODUCTION

Helioseismology has revealed that the Sun’s radiative core rotates nearly uniformly, at least down to the radius $r \sim 0.2 R_\odot$, with a rate that is close to the mean rotation rate of its convective envelope (e.g., Tomczyk et al. 1995; Couvidat et al. 2003). It is also known that the Sun, like other solar-type stars, has been experiencing an effective braking of its surface rotation via a magnetized solar (stellar) wind (Schatzman 1962; Weber & Davis 1967; Kawaler 1988). Because the magnetic field lines that sling charged particles into space are rooted in the photosphere, it is the envelope rotation that should be decelerated by the wind torque, while the conservation of angular momentum in the radiative core should keep it rapidly rotating. This expectation appears to be in stark contrast with the measurements of internal solar rotation, which indicate that the difference in the angular velocity between the Sun’s radiative core and convective envelope must have been erased by some angular momentum redistribution mechanism. A similar conclusion can be reached from comparisons of the rotation period distributions for solar-type stars in open clusters of different ages. It turns out that the core and envelope rotation have to be coupled with a characteristic timescale of angular momentum transfer between them increasing from a few Myr to nearly a hundred Myr between the fastest and slowest rotators (Denissenkov et al. 2010; cf. Irwin et al. 2007). A simple model of rotational evolution of solar-type stars with angular momentum transport described as a purely diffusive process shows that this coupling time interval roughly corresponds to a diffusion coefficient decreasing from $10^6 \text{ cm}^2 \text{ s}^{-1}$ to $5 \times 10^4 \text{ cm}^2 \text{ s}^{-1}$. However, such large diffusion coefficients would lead to rates of surface Li destruction strongly exceeding the trend observed in the Sun and its twins (Section 2.4).

Therefore, we have to admit that the angular momentum redistribution in solar-type stars is unlikely to be accompanied by an equally rapid mass transfer. This restricts our search for possible mechanisms of angular momentum transport in solar-type stars to those employing waves and magnetic fields.

Presently, the most popular mechanisms of angular momentum redistribution in the Sun’s radiative core are the smoothing of its rotation profile by internal gravity waves (g-modes) generated by turbulent eddies in the convective envelope (Charbonnel & Talon 2005) and magnetic braking of differential rotation (Charbonneau & MacGregor 1992, 1993; Spruit 1999, 2002). Using a simplified analytical prescription for the spectrum of internal gravity waves and other approximations, Charbonnel & Talon (2005) have succeeded in obtaining a nearly flat rotation profile in the present-day Sun simultaneously with a time evolution of its internal rotation consistent with the observed solar Li abundance. However, because of the complexity of physical processes involved in the generation, propagation, and dissipation of internal gravity waves in the Sun, it is difficult to assess the validity of this solution. For instance, it would not be valid if a flatter spectrum, like that computed by Rogers & Glatzmaier (2005), were used. Nonlinear wave–wave interactions, as well as the interaction of gravity waves with a magnetic field, could also change the solution drastically (Rogers et al. 2008; Rogers & MacGregor 2010). Even in a case very similar to that considered by Charbonnel & Talon (2005) the question arises as to why gravity waves do not disturb the uniform solar rotation. Indeed, the power of gravity waves generated in the present-day Sun should be approximately the same as it was in the young Sun; consequently, given their peculiar anti-diffusive nature in shear flows (e.g., Plumb 1977; Ringot 1998), gravity waves should have forced the solar rotation profile away from the uniform one (Denissenkov et al. 2008). However, this has not occurred.

For a magnetic field configuration to influence differential rotation of the solar radiative core, it must have a non-vanishing azimuthal (toroidal) component. To achieve this, it is usually assumed that there is an axially symmetric poloidal magnetic field frozen into the plasma inside the radiative core which can be stretched around the rotation axis by the differential rotation itself to form the necessary toroidal field. As far as the origin of the poloidal field is concerned, it is widely believed that the field was either inherited from a protostellar cloud or
generated by a convective dynamo during the Sun’s pre-main-
sequence (pre-MS) evolution when its convective envelope
occupied a much larger volume. Spruit (1999) proposed
the original alternative explanation that the poloidal field might be
continuously replenished by radially stretching the toroidal field
in a dynamo-like cyclic process, the necessary radial plasma
displacements being caused by the non-axisymmetric \((m = 1)\)
kink instability of the toroidal field (the Tayler–Spruit dynamo).

Magnetic braking is implemented by the azimuthal compo-
nent of the magnetic tension term of the Lorentz force. Given that
its corresponding engagement time is of order several Alfvén
timescales for the poloidal field, magnetic breaking appears to
be very efficient even for a small field amplitude (Mestel &
Weiss 1987). The only problem associated with this mecha-
nism is that it leaves behind a torus-shaped “dead” zone inside
the radiative core with the angular velocity increasing toward
the center of the torus cross section. This is a direct conse-
quence of Ferraro’s law of isorotation (rotation that keeps the
angular velocity constant along each field line; Ferraro 1937)
and the presence of a constant dipole poloidal field. To solve
this problem, Charbonneau & MacGregor (1993, hereafter re-
ferred to as CHMG93) and other researchers (e.g., Ruediger &
Kitchatinov 1996) artificially amplified the viscous transport
with the justification that rotation and associated hydrodynamic
instabilities should lead to turbulence, and hence that the mi-
iceoscopic viscosity should be augmented by a much stronger
turbulent viscosity. However, the minimum value of the en-
hanced viscosity \(5 \times 10^{24} \text{ cm}^2 \text{ s}^{-1}\) with which it is possible to
get a nearly flat rotation profile in the present-day Sun results
in too much Li being transported below the bottom of convec-
tive envelope and thus destroyed, inconsistent with the observed
solar Li abundance. Therefore, the proposed isotropic viscosity
enhancement cannot be considered a satisfactory solution.

Only recently has the physics-based modeling of angular
momentum transport in solar-type stars begun to use the
extensive data sets of rotation periods for solar analogs in
open clusters as an observational constraint. The main goal of
previous studies was to demonstrate that a particular model
could (or could not) produce the solid-body rotation of the
Sun. In particular, Eggenberger et al. (2005) showed that the
Tayler–Spruit dynamo could account for the flat rotation profile
of the Sun. However, the correct model must also explain
the cause of the transition from the solid-body rotational
evolution of the fastest rotators to the evolution wherein a
radial differential rotation is sustained for a hundred Myr in
the slowest rotators. It turns out that the Tayler–Spruit dynamo
cannot pass this observational test because it always enforces
a nearly uniform rotation in solar-type stars, no matter how slowly
they rotate (Denissenkov et al. 2010).

Another important problem that the correct mechanism of
angular momentum transport in the Sun has to explain is the
thinness of the solar tachocline. The tachocline is a layer in
which the differential rotation (both radial and latitudinal) of
the convective envelope sharply changes to the nearly solid-
body rotation of the radiative core (Spiegel & Zahn 1992). Its
thickness, as measured by helioseismology, is only about 4% of
the solar radius (Basu & Antia 2003). It has been suggested that
no purely hydrodynamic mechanism can explain its existence
and that a large-scale magnetic field in the radiative core is
therefore needed to confine the tachocline structure (Gough &
McIntyre 1998; Ruediger & Kitchatinov 2007).

To summarize, because large-scale magnetic fields have
certainly been playing crucial roles during different phases of
the Sun’s rotational evolution—such as the synchronizing of
rotation of the young Sun and its surrounding protoplanetary
disk (the disk locking; e.g., Shu et al. 1994; Matt & Pudritz
2005), providing the leverage for the solar wind, confining the
tachocline, being generated through a convective dynamo and
then driving the solar activity—we believe that it is worthwhile
to develop a mechanism of magnetic breaking of differential
rotation of the Sun’s radiative core.

In this paper, we elaborate upon and extend the magnetic brak-
ing mechanism that was originally proposed by Mestel & Weiss
(1987) and then studied in detail numerically by CHMG93 fol-
lowed by Ruediger & Kitchatinov (1996). Our only modification
of the mechanism is to assume that the turbulence induced by
rotation-driven hydrodynamic instabilities in the Sun’s radia-
tive core is highly anisotropic with its corresponding horizontal
temperature of turbulent viscosity \(\nu_h\) strongly dominating over
that in the vertical direction \(\nu_v\). The hypothesis of anisotropic
turbulent diffusion in stellar radiative zones was first advanced
by Zahn (1992) and it had since been productively used by
many other researchers to study rotational mixing in both MS
and red giant branch stars (Talon & Zahn 1997; Talon et al.
1997; Maeder 2003; Palacios et al. 2003, 2006). Initially, it was
ever even considered to explain the thinness of the solar tachocline by
Spiegel & Zahn (1992). We will show that this hypothesis alone
permits a solution of almost all of the aforementioned problems
associated with this particular mechanism of magnetic braking,
such as the Li problem (a sufficiently small value of the diffusion
coefficient \(\nu_v\) can be chosen so that it does not lead to excessive
Li destruction), the dead-zone problem, and the difference in
spin-down of the fastest and slowest rotators among solar-type
stars in open clusters.

The paper is organized as follows. In Section 2, we briefly
discuss simple models of rotational evolution of solar-type stars
and review the main results that were obtained by comparison of
the model predictions with the observed period distributions for
solar counterparts in open clusters. In Section 3.1, we reproduce
the two-dimensional MHD computations of CHMG93 and show
that they cannot account for the spin-down of the fastest rotators
in open clusters, especially when one reduces the vertical
viscosity to a value more or less compatible with the surface
Li abundance evolution in solar twins. In Section 3.2, we
present our new dynamic model of magnetic braking with
the anisotropic turbulent diffusion \(\nu_{h} \gg \nu_{v}\) and discuss its advantages compared to the original model. Section 3.3
describes a number of stationary solutions that address the
problem of the penetration of the latitudinal differential rotation
from the bottom of convective envelope into the radiative interior
in the present-day Sun. Finally, we summarize our conclusions
in Section 4.

2. SIMPLE MODELS OF ROTATIONAL EVOLUTION OF
SOLAR-TYPE STARS

The simple models of rotational evolution of solar-type
stars—one, the double-zone model and the purely diffusive
one—have been discussed in detail and compared with each other
by Denissenkov et al. (2010). Here, we will only briefly
describe the models and summarize the main results obtained
by comparing their predictions with distributions of angular
velocity of the (assumed) rigidly rotating convective envelope
for solar analogs in open clusters, \(\Omega e = 2\pi/P\), which are
derived from the corresponding observed rotation period \(P\)
distributions. We will also show that the purely diffusive
model fails to simultaneously comply with the observational
constraints imposed by the spin-down of solar analogs in open clusters and the time evolution of the surface Li abundance in solar twins. Although the simple models do not identify the physics behind the internal transport of angular momentum in solar-type stars, they nevertheless provide us with useful estimates of characteristic timescales of relevant processes and their dependence on the rotation rate. Therefore, they can serve as a good starting point for our analysis.

2.1. The Pre-main-sequence Disk Locking

In Figure 1, crosses with vertical error bars represent the upper 90th and lower 10th percentiles of the $\Omega_e$ distributions for stars having masses in the interval $0.9 \lesssim M/M_\odot \lesssim 1.1$ sampled from open clusters with ages between 2 Myr and 600 Myr (for details, see Denissenkov et al. 2010). The data for the youngest clusters set up the initial $\Omega_e$ distributions for rotational evolution computations. Of course, these are not the true initial distributions inherited by the stars during their formation epoch because they have already been modified (reduced) by a disk-locking process compared to what they would have been if the stars’ pre-MS evolution had taken place in complete isolation. The fact is that after its birth a protostar continues to contract before it settles down on the zero-age MS (ZAMS) at an age of about 40 Myr (for $M \approx 1 M_\odot$). During this period of time, the star’s radius $R$ and total moment of inertia $I$ are decreasing, hence its $\Omega_e$ should be growing as a consequence of angular momentum conservation. Contrary to this, it is the surface angular velocity rather than the angular momentum that is observed to be nearly preserved in contracting pre-MS stars (e.g., Rebull et al. 2004). This is usually explained by a complex magnetic interaction that pumps out angular momentum from the protostar to its surrounding protoplanetary disk (e.g., Shu et al. 1994; Matt & Pudritz 2005). As a result, the magnetic disk locking keeps $\Omega_e$ nearly constant (this is modeled by horizontal fragments of dotted curves in Figure 1) for a period of time up to 10–20 Myr at most before the disk disappears. After that, the star’s residual contraction and conservation of angular momentum cause an increase of its $\Omega_e$ that continues until the star reaches the ZAMS.

2.2. The Surface Loss of Angular Momentum

From the moment when the young Sun arrives at the ZAMS ($t_{\text{ZAMS}} \approx 40$ Myr) until the Sun’s present-day age of $t_\odot = 4.57$ Gyr, its internal structure does not change significantly. In particular, the dimensionless moment of inertia of the Sun’s convective envelope $i_e \equiv I_e/(R_\odot^2 M_\odot) \approx 0.0105$ remains constant to within 6%. Note that $i_e$ amounts only to 14% of the Sun’s total moment of inertia $i \equiv I/(R_\odot^2 M_\odot) \approx 0.074$. If there were no angular momentum transfer from the Sun’s radiative core to its convective envelope then the only process involved in the Sun’s rotational evolution during its life on the MS ($t_{\text{ZAMS}} \leq t \leq t_\odot$) would be the braking of envelope rotation by the magnetized solar wind. The corresponding rate of angular momentum loss is often approximated by the following equation:

$$\dot{j} = \frac{d}{dt}(I_e \Omega_e) \approx I_e \frac{d\Omega_e}{dt} \approx -K_w \left(\frac{R/R_\odot}{M/M_\odot}\right) \min(\Omega_e\Omega_{\text{sat}}, \Omega_e^3),$$

where $K_w$ is a constant calibrated to give $\Omega_e(t_{\odot}) = \Omega_{\odot} = 2.86 \times 10^{-6}$ rad s$^{-1}$ (for $P_\odot = 25.4$ days), and $\Omega_{\text{sat}}$ is a mass-dependent velocity above which the wind gets saturated (Chaboyer et al. 1995; Krishnamurthi et al. 1997; Andronov et al. 2003). The value of $\Omega_{\text{sat}} = 8 \Omega_{\odot}$ has been adjusted so that the upper solid curve in Figure 1 simulating the solid-body rotational evolution of rapidly rotating open-cluster solar-type stars could fit the upper 90th percentiles of their $\Omega_e$ distributions as closely as possible (Denissenkov et al. 2010).

The differential Equation (1) can easily be solved. If the initial angular velocity $\Omega_{e,\text{ZAMS}} \equiv \Omega_e(t_{\text{ZAMS}})$ exceeds the saturation threshold $\Omega_{\text{sat}}$, then the solution consists of two different parts. The first one, which is valid for $t_{\text{ZAMS}} \leq t \leq t_{\text{sat}}$, is

$$\Omega_e = \Omega_{e,\text{ZAMS}} \exp \left[-(t - t_{\text{ZAMS}})/t_w\right].$$

where the wind braking timescale is $t_w = (t_{\text{sat}} - t_{\text{ZAMS}})/(A/2 + B)$, while $t_{\text{sat}} = t_{\text{ZAMS}} + Bt_w$ is the time when $\Omega_e$ has decreased to a value of $\Omega_{\text{sat}}$. To shorten the equations, we have used the notations $A \equiv (\Omega_{\text{sat}}/\Omega_{\odot})^2 - 1$ and $B \equiv \ln(\Omega_{e,\text{ZAMS}}/\Omega_{\text{sat}})$. The exponential decay of $\Omega_e$ is replaced by the Skumanich relation (Skumanich 1972)

$$\Omega_e = \frac{\Omega_{\odot}}{\sqrt{1 + 2[(A/2 + B)/(1 + A)](t - t_{\odot})/(t_{\text{sat}} - t_{\text{ZAMS}})}}$$

for the time interval $t_{\text{sat}} < t \leq t_{\odot}$. Finally, the solar-calibrated wind constant has to be equal to

$$K_w = \frac{R_\odot^2 M_\odot (A/2 + B)}{\Omega_{\odot}^2} \frac{i_e}{(1 + A) (t_{\text{sat}} - t_{\text{ZAMS}})}.$$

Alternatively, if the star arrives at the ZAMS with $\Omega_{e,\text{ZAMS}} \leq \Omega_{\text{sat}}$ then its surface angular velocity will be declining according to the Skumanich law from the very beginning. The above equations can still be used provided that one substitutes $\Omega_{\text{sat}} = \Omega_{e,\text{ZAMS}}$ in all of them.
Equations (2) and (3) suffice to describe the Sun’s spin-down for the two limiting cases in which its core and envelope rotation are either completely decoupled or, alternatively, rigidly coupled. For the first case, which is only of academic interest because it is in conflict with uniform rotation in the solar interior, Equation (4) gives \( K_w = 4.60 \times 10^{46} \text{ cm}^2 \text{ g s} \) for \( \Omega_{ZAMS} = 100 \Omega_\odot \), and \( K_w = 4.13 \times 10^{46} \text{ cm}^2 \text{ g s} \) for \( \Omega_{ZAMS} = 4.7 \Omega_\odot \) (the upper and lower solid curves in Figure 1). In the second case, which appears to adequately describe the fastest rotators, we have to substitute the total moment of inertia \( I \) instead of \( i \) into Equation (4), resulting in \( K_w = 3.24 \times 10^{47} \text{ cm}^2 \text{ g s} \) and \( K_w = 2.91 \times 10^{47} \text{ cm}^2 \text{ g s} \) for the same initial angular velocities. It is interesting that the solutions (2) and (3) do not depend on the moment of inertia; therefore, the curves representing them for the two limiting cases completely coincide in Figure 1. The entire difference between the cases is contained in the values of the wind constant \( K_w \). These values have to be larger, in proportion to the ratio \( i/i_e \) of \( 0.074/0.0105 \approx 7.05 \). For the solid-body rotational evolution because in this case the wind has to be faster to remove the correspondingly increased amount of angular momentum, so that finally we could get \( \Omega_e(t_0) = \Omega_\odot \). Note that the last constraint comes directly from the observed rotation period distributions for solar-type stars in open clusters that show a convergence of \( P \) to \( P_c \) by the solar age. It is also clear that the correct mechanism of angular momentum redistribution in solar-type stars should not tolerate a large variation of the wind constant \( K_w \) between its applications to the fastest and slowest rotators as long as we believe that the dependence of \( J \) on \( \Omega_e \) has correctly been factored out in Equation (1).

The illustrated degeneracy of the solutions (2) and (3) with respect to the two limiting cases of coupling between the core and envelope rotation is lifted when we begin to consider more realistic cases in which the coupling is allowed to increase progressively in time. To demonstrate this via a simple example, let us assume that the moment of inertia of the star’s rigidly rotating envelope that is decoupled from the rest of it obeys the following rule:

\[
I(t) = I_e \left[ \frac{\Omega_{e,ZAMS}}{\Omega_e(t)} \right]^p,
\]

where \( p \equiv \ln(I/I_e)/\ln(\Omega_{e,ZAMS}/\Omega_\odot) \). Although this case does not have a physical justification, it admits an analytical solution of Equation (1) and it also guarantees that \( I(t_{ZAMS}) = I_e \) (no core/envelope coupling at the beginning) and \( I(t_{0}) = I \) (full coupling at the end). For \( \Omega_{e,ZAMS} = 100 \Omega_\odot \), the solution of Equation (1) for the dependence (Equation (5)) is plotted in Figure 1 with a dot-dashed curve. It needs \( K_w = 2.56 \times 10^{47} \text{ cm}^2 \text{ g s} \). Thus, in spite of the fact that the evolution begins with angular momentum being lost only from the convective envelope, the wind constant has a value much closer to the previously considered case of solid-body rotational evolution. This is explained by the fact that the star will later have to get rid of more angular momentum than in the old decoupled case because the moment of inertia of its decoupled and uniformly rotating envelope (not just the convective envelope but also an outer part of the radiative core adjoining it) increases well above \( I_e \). Therefore, it is necessary to lose additional angular momentum from the surface of the convective envelope in the future, which will be supplied by the radiative core as its rotation gets more and more coupled to that of the envelope, demanding a larger wind constant which, in turn, leads to a steep initial decline of \( \Omega_e \).

The double-zone model (MacGregor 1991) assumes that the Sun (or a solar-type star) consists of two uniformly rotating zones, the core and envelope, and that an excess of angular momentum in the core, compared to the case in which the whole star rotates rigidly, is transferred to the envelope on a specified constant timescale \( \tau_c \). It is evident that, for this core–envelope rotational coupling to replenish the angular momentum content of the envelope faster than it is drained by the wind, one needs to have \( \tau_c < \tau_w = 133 \text{ Myr} \) (for \( \Omega_{ZAMS} = 100 \)), and vice versa. Computations show (e.g., Denissenkov et al. 2010) that the solid-body rotational evolution of a solar-type star, in the case that its core and envelope rotate nearly synchronously all the time, can only be achieved with \( \tau_c \leq 1 \text{ Myr} \).

It turns out that the upper 90th percentiles of \( \Omega_e \) distributions (the fastest rotators) for solar analogs in open clusters are very well reproduced by the double-zone model with \( \tau_c = 1 \text{ Myr} \) (upper dotted curves in panels (A) and (B) in Figure 2). In contrast, the slowest rotators (the lower 10th percentiles) at ages of order 100 Myr are found to be located below the curves representing the solid-body rotational evolution (\( \tau_c = 1 \text{ Myr} \)) that starts from the 10th percentiles of \( \Omega_e \) distributions for the youngest clusters, even when the safe upper limit of \( \tau_c = 20 \text{ Myr} \) is chosen for the disk-locking time (the lower dotted curve in Figure 2(A)). A rigorous statistical analysis that uses Monte Carlo simulations and compares the full \( \Omega_e \) distributions (modeled versus observed ones) rather than just their percentiles confirms this conclusion. It also gives the following estimates of the coupling time for the slowest rotators: \( \tau_c \approx 55 \pm 25 \text{ Myr} \) for stars with \( 0.9 \lesssim M/M_\odot \lesssim 1.1 \) and \( \tau_c \approx 175 \pm 25 \text{ Myr} \) for stars with \( 0.7 \lesssim M/M_\odot < 0.9 \) (Denissenkov et al. 2010).

2.4. The Purely Diffusive Model

The purely diffusive model uses appropriate initial and boundary conditions, including Equation (1), to solve the following equation:

\[
\rho r^4 \frac{\partial \Omega}{\partial t} = \frac{\partial}{\partial r} \left( \rho r^4 \nu \frac{\partial \Omega}{\partial r} \right),
\]

where \( \nu \) is a constant viscosity whose physical nature is not specified. The time-dependent density distribution \( \rho(r, t) \) in the radiative core and other necessary stellar structure parameters are taken from full stellar evolution computations. Denissenkov et al. (2010) have demonstrated that the solid-body rotational evolution of solar-type stars can only be simulated with \( \nu \gtrsim 10^5 \text{ cm}^2 \text{ s}^{-1} \) (solid curves in Figure 2). They have also established an approximate correspondence between the diffusion coefficient \( \nu \) and the coupling time \( \tau_c \) from the double-zone model that produce a similar rotational evolution. In particular, this relation says that the spin-down of a slowly rotating star computed using the double-zone model with \( \tau_c \approx 40 \text{ Myr} \) looks almost identical to that obtained with the purely diffusive angular momentum redistribution for \( \nu \approx 5 \times 10^4 \text{ cm}^2 \text{ s}^{-1} \) (dot-dashed curves in Figure 2). This solution gives an example similar to Equation (5) when the moment of inertia of the rigidly rotating envelope effectively increases with time as the diffusive transport of angular momentum gradually penetrates into the radiative core. It should be noted that even for a coupling time as long as 90 Myr the corresponding diffusion coefficient still remains as large as \( \nu \approx 2.5 \times 10^4 \text{ cm}^2 \text{ s}^{-1} \).
The angular momentum redistribution in solar-type stars is known to be accompanied by a chemical element transport, the most pronounced manifestation of which is the strongly reduced abundance of Li in the solar atmosphere. Meléndez et al. (2009) have recently reported that the Sun does not seem to be unique in this respect. It turns out that solar twins (stars with mass and metallicity not different from the solar values to within a few percent) show a dependence of the surface Li abundance on age to which the Sun also belongs (Figure 3). A natural interpretation of this relation is that a rate of Li destruction in the atmospheres of solar twins is approximately the same, the Sun simply being a relatively old star.

We have added diffusion terms to a network of nuclear kinetics equations relevant for the Sun’s evolution (the pp chains and CNO cycle reactions). This network has been solved in a post-processing way using, as a background, previously stored files with all necessary information about the Sun’s internal structure from its fully convective pre-MS evolution. If this is true then the Li mixing may have nothing to do with the Sun’s nearly uniform rotation nor the spin-down of the fastest rotators in open clusters (dashed curves in Figure 2).

Finally, it is important to note that Be, another fragile element, does not seem to share the fate of Li in evolved MS stars with close to solar effective temperatures, including the Sun (Randich 2008). This means that the chemical mixing that causes the Li destruction does not penetrate below $r \approx r_e - 0.16 R_\odot \approx 0.55 R_\odot$ in the Sun (e.g., see Figure 1 of Barnes et al. 1999), $r_e \approx 0.713 R_\odot$ being the radius of its core–envelope interface, where the temperature is high enough for protons to begin destroying Be. Alternatively, it is possible that the diffusion coefficient rapidly declines with depth (e.g., Richer et al. 2000). If this is true then the Li mixing may have nothing to do with the global angular momentum redistribution in the Sun. On the other hand, to be in accord with observations the latter process cannot be assisted or accompanied by a mass transfer whose rate, when described in terms of a diffusion coefficient, greatly exceeds $v \approx 4 \times 10^3 \text{ cm}^2 \text{s}^{-1}$.

3. DYNAMIC AND STATIONARY MODELS OF SOLAR ROTATION WITH LARGE-SCALE MAGNETIC FIELDS

3.1. Magnetic Braking of Solar Rotation: The Old Solution

Because our main goal is to modify the well-known model of magnetic braking of solar internal differential rotation to enable

![Figure 2. Spin-down of the surface solar rotation for the case in which the angular momentum redistribution in the Sun’s interior is modeled as a purely diffusive process (Equation (6)) with a diffusion coefficient $\nu$ (displayed in panels (A) and (B) for the fastest and slowest rotators, respectively). The corresponding internal rotation profiles at the solar age are shown in panels (C) and (D) with crosses representing the observational data from helioseismology measurements by Couvidat et al. (2003). Horizontal line segments at $r/R \geq 0.713$ in the same panels show the latitudinal differential rotation of the (bottom of) the Sun’s convective envelope at $\theta = 30^\circ, 45^\circ, 60^\circ,$ and $90^\circ$ (from the lower to upper).]
it to explain the recent observational data on the $\nu$ distributions for solar-type stars in open clusters, without causing a conflict with the Li abundance data for solar twins, we will only briefly describe the model employed by CHMG93 that we want to develop. For more details about the model, the interested reader is referred to the original paper for an excellent analysis of numerical results obtained with it and also a discussion of its shortcomings. To save space, we introduce, at the outset, the general forms of relevant equations that encompass both the original case considered by CHMG93 and our modified case with anisotropic turbulent diffusion. In fact, we have chosen an equivalent system of equations similar to that used in the follow-up study by Ruediger & Kitchatinov (1996), which allows the poloidal field potential to fade with time.

In the spherical polar coordinates $\{r, \theta, \phi\}$, the magnetic breaking of solar rotation in the presence of anisotropic turbulence in the radiative core is described by the following equations:

\[
\begin{align*}
\rho r \sin \theta \frac{\partial \Omega}{\partial t} &= \sin \theta \frac{\partial}{\partial r} \left( \rho r^4 v_c \frac{\partial \Omega}{\partial r} \right) + \frac{\rho}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin^3 \theta v_h \frac{\partial \Omega}{\partial \theta} \right) + \frac{1}{4\pi r^2} \frac{\partial}{\partial \theta} \left[ \frac{1}{r} \frac{\partial A}{\partial \theta} \frac{\partial (Br)}{\partial r} \right] \\
&\quad - \frac{1}{\sin \theta} \frac{\partial A}{\partial r} \frac{\partial (B \sin \theta)}{\partial \theta} + (\rho r \sin \theta) S_f \Omega,
\end{align*}
\]

(7)

\[
\begin{align*}
\frac{\partial B}{\partial t} &= \frac{1}{r} \frac{\partial}{\partial r} \left[ \eta_r \frac{\partial (Br)}{\partial r} \right] + \frac{\eta_h}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{1}{\sin \theta} \frac{\partial (B \sin \theta)}{\partial \theta} \right] \\
&\quad + \frac{1}{r} \left( \frac{\partial \Omega}{\partial \theta} \frac{\partial A}{\partial \theta} - \frac{\partial \Omega}{\partial \theta} \frac{\partial A}{\partial \theta} \right),
\end{align*}
\]

(8)

\[
\begin{align*}
\frac{\partial A}{\partial t} &= \eta_r \frac{\partial^2 A}{\partial r^2} + \eta_h \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial A}{\partial \theta} \right),
\end{align*}
\]

(9)

where $S_f = J/(I_c \Omega_c)$ is a normalized angular momentum loss rate, with $I_c$ and $\Omega_c$ being, as before, the moment of inertia and angular velocity of the convective envelope. For the rate of angular momentum loss from the surface $J$, we continue to use Equation (1), whereas CHMG93 used the formulation of Weber & Davis (1967).

Equations (7) and (8) represent the $\varphi$—(azimuthal) components of the momentum and induction equations. Following CHMG93, we neglect any meridional motion of either rotational or magnetic origin. We also omit the Coriolis force, while solving the problem in an inertial (non-rotating) frame of reference in which the polar axis is directed along the rotation axis. The right-hand side of the momentum equation (7) contains the viscous term, which has been split into its vertical and horizontal components, the magnetic tension part of the Lorentz force $(1/4\pi)(B, \nabla)B_e$, where the total (poloidal plus toroidal) field is $B \equiv B_p + B_e$, in which the poloidal field has been expressed via its potential $A$ as follows:

\[
B_e = \nabla \times \left( \frac{A e_p}{r \sin \theta} \right) = \left\{ \frac{1}{r^2 \sin \theta} \frac{\partial A}{\partial \theta}, -\frac{1}{r \sin \theta} \frac{\partial A}{\partial r}, 0 \right\},
\]

(10)

and, finally, the term simulating the surface angular momentum loss by smearing $J$ over the entire convective envelope, which therefore must vanish in the radiative core.

Equation (9) is decoupled from the first two equations, therefore it can be solved separately. Following Ruediger & Kitchatinov (1996), we expand the poloidal field potential into a series of the associated Legendre polynomials and keep only the longest living dipole term

\[
A(r, \theta, t) \approx -a(r) \sin^2 \theta \exp(-t/\tau),
\]

(11)

which we substitute into Equation (9). As a result, we have to solve the eigenvalue problem

\[
\eta_r \frac{d^2 a}{dr^2} - 2 \frac{\eta_h}{r^2} \frac{d a}{dr} = -\frac{a}{\tau}
\]

(12)
for the boundary conditions \( a(0) = a(r_b) = 0 \), where \( 0 \leq r_b \leq R_{\odot} \), and the potential \( A \) will thus be determined.

CHMG93 considered the isotropic case in which \( \nu \equiv \nu_p \equiv \nu_H \) and \( \eta \equiv \eta_p \equiv \eta_h \). They artificially amplified the microscopic kinematic viscosity \( \nu \) and magnetic diffusivity \( \eta \) by factors of order \( 10^3 \) and \( 10^2 \), respectively, making them as large as \( \nu = 5 \times 10^5 \text{ cm}^2 \text{ s}^{-1} \) and \( \eta = 5 \times 10^5 \text{ cm}^2 \text{ s}^{-1} \) and keeping their values constant throughout the radiative core. It is convenient to transform Equation (12) into its dimensionless form

\[
d\gamma a / dt^2 - 2 \gamma a = -\lambda a,
\]

where \( \gamma \equiv r/r_b, \gamma \equiv \eta_h/\eta_v, \) and \( \lambda \equiv (r_b^2/\tau \eta_v) \). Taking the values of \( \gamma = 1 \) and \( \eta = 5 \times 10^5 \text{ cm}^2 \text{ s}^{-1} \) used by CHMG93, we solve the eigenvalue problem (Equations (12) and (13)) and find that, in this case, the poloidal field potential can be approximated by the following relation:

\[
A(r, \theta, t) \approx B_0 \frac{r^2}{2} \left( 1 - \frac{r}{r_b} \right)^2 \sin^2 \theta \times \exp(-t/\tau),
\]

where \( \tau = 35 \text{ Myr} \) for \( r_b = r_e = 0.713 \ R_{\odot} \). The same, or very similar, form of the potential was used by both CHMG93 and Ruediger & Kitchatinov (1997). However, in both of those investigations the time dependence of \( A \) was ignored, which might not be a good assumption, given the short decay time for \( A \) obtained with that for the enhanced magnetic diffusivity.

The decay time can be as high as \( 5.4 \text{ Gyr} \), which is comparable to the solar age, only when the maximum value of the microscopic magnetic diffusivity in the Sun’s radiative core \( \eta_{\text{max}} = \eta(r_e) \approx 3 \times 10^3 \text{ cm}^2 \text{ s}^{-1} \) is substituted into the above expression for \( \lambda \) instead of \( \eta_v \) (cf. Ruediger & Kitchatinov 1996). But, in this case, magnetic braking will hardly work, especially, after our having correspondingly decreased the amplified microscopic viscosity either to the maximum microscopic value of \( \nu_{\text{max}} = \nu(r_e) \approx 20 \text{ cm}^2 \text{ s}^{-1} \) or to its minimum turbulent value of \( \nu \approx \eta \approx \eta_{\text{max}} \) (assuming that \( \eta \) is still of turbulent origin but has also reached its minimum possible value). The latter substitution brings the turbulent magnetic Prandtl number \( P_{\text{m}} \equiv \eta/\nu \) close to its most plausible value of unity.

Keeping this inconsistency in mind, we proceed with our revision of the original model of CHMG93. Like them, we apply Equations (7), (8), and (14) to the whole star assuming that \( r \equiv r_b = r_e \equiv 0.713 \ R_{\odot} \). The same, or very similar, form of the potential was used by both CHMG93 and Ruediger & Kitchatinov (1997). However, in both of those investigations the time dependence of \( A \) was ignored, which might not be a good assumption, given the short decay time for \( A \) obtained with that for the enhanced magnetic diffusivity.

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in which $B_F \propto B_0$ and $B_0 \propto B_0$ because the total magnetic field in the radiative core is $B = B_F + B_\psi$.

CHMG93 have distinguished the following three important epochs in the magnetic braking of solar rotation that differ from each other by the relative strength of the magnetic and wind torques. During the first epoch, the magnetic torque is much weaker than the wind torque and, as a consequence, the rotational shear produced by the spin-down of the convective envelope at its interface with the radiative core is spread by the viscosity almost without hindrance to the core surface. So, the shear in the outer core grows almost linearly with time and, as a result, the toroidal field is amplified there almost quadratically in time. This short epoch, lasting between a few thousand to a few million years, is followed by the second epoch during which the differential rotation profile begins to interact with the growing toroidal field. For the poloidal field configurations in which the core and envelope are magnetically coupled (e.g., the D2 case), this epoch is characterized by vigorous phase mixing throughout the radiative core. This process is initiated by large-scale toroidal field oscillations that originate close to the bottom of the convective envelope and propagate through the radiative core along the poloidal field lines using them as elastic rubber strings. Because the strings are rooted in the convective envelope (for the D2 configuration) at different colatitudes and therefore have different lengths, the Alfvén waves carrying the perturbations back and forward along the strings get out of phase as time progresses. Eventually, this causes large gradients of toroidal field to build up locally at different places and, as a result, the large-scale toroidal field oscillations are diffused away either via ohmic dissipation or via turbulent magnetic reconnection, depending on the nature of $\eta$ (e.g., Spruit 1999).

Finally, during the third epoch, a state of dynamical balance that is achieved at the end of the second epoch between the magnetic torque in the radiative core and the total stresses (magnetic plus viscous) at the bottom of convective envelope is maintained as the surface wind torque decreases with time.

First, we have repeated the computations of CHMG93 for one of their favorite models, namely, the one that starts with $\Omega_{\text{ZAMS}} = 50 \Omega_{\odot}$ and has the D3 configuration of time-independent poloidal field with the amplitude $B_0 = 1$ G. Our results are presented in Figure 4 (thick solid curves in panels (A) and (C) as well as panels (B) and (D)). The geometry and magnitude of the residual toroidal field in the present-day Sun (panel (D)) look very similar to those obtained by CHMG93 (upper quadrant of the lower right panel in their Figure 6). As they reported, the model has a nearly flat rotation profile by the solar age (panels (B) and (C)). However, it seems unlikely that it will be able to fit the upper 90th percentiles of the $\Omega_e$ distributions for open-cluster solar analogs after its initial rotation rate is increased to a value of $\Omega_e = 100 \Omega_{\odot}$ corresponding to the observed maximum (the peak on the dotted curve representing the solid-body rotational evolution...
simulated using the double-zone model). This supposition is confirmed by our additional test computations that started with \( \Omega_{\text{ZAMS}} = 100 \Omega_\odot \) (thin solid curve in panel (A)).

It is also not clear what could possibly force the rotational evolution of the CHMG93 model to change its morphology in the \( \Omega_e \) versus age plane from the convex one for the fastest rotators to the concave one for the slowest rotators, as is indicated by both direct observations (e.g., compare the solid and dot-dashed curves in panels (A) and (B) in Figure 2, respectively) and their rigorous statistical analysis (Denissenkov et al. 2010). Physically, the change implies a transition from solid-body rotational evolution to evolution with a differential rotation sustained for a few tens to a hundred Myr. As has been mentioned, this cannot be produced by a variation of \( B_0 \). We can see now that an increase of the initial rotation rate does not make the rotational evolution of the CHMG93 model approach the dotted curve either. In other words, this model does not have enough internal degrees of freedom to explain the whole spectrum of observed rotation rates. Besides, its high viscosity (\( v = 5 \times 10^5 \text{ cm}^2 \text{ s}^{-1} \)) causes the model to conflict with the observed Li abundances in solar twins (Section 2.4). When we reduce the viscosity to the value of \( v = 5 \times 10^3 \text{ cm}^2 \text{ s}^{-1} \), which is more or less compatible with the Li data, then the model’s rotational evolution becomes absolutely incapable of reproducing the upper 90th percentiles of the \( \Omega_e \) distributions (dashed curve in Figure 4(A)).

As far as the importance of taking into account the exponential decay of the poloidal field is concerned, it depends on how big the values of \( \gamma = \eta_\text{h}/\eta_v \) and \( \eta_v \) are in the model in question. For the above considered case of \( \Omega_{\text{ZAMS}} = 100 \Omega_\odot \), \( \gamma = 1 \), and \( \eta_v = 5 \times 10^5 \text{ cm}^2 \text{ s}^{-1} \), a more consistent solution with \( A \) decaying on the timescale of \( \tau = 35 \text{ Myr} \) (Equation (14)) is plotted in Figure 4(A) as a dot-dashed curve. Although it does not appear to strongly deviate from the old solution (the thin solid curve), the difference between the two solutions will probably be significant for Monte Carlo simulations and a rigorous statistical analysis of rotation period distributions for solar analogs in open clusters like those performed by Denissenkov et al. (2010).

Finally, it is worth noting that the purely diffusive model with the same viscosity \( v = 5 \times 10^5 \text{ cm}^2 \text{ s}^{-1} \) as the one used by CHMG93 seems to produce both the Sun’s rotational evolution and its final internal rotation profile (the dot-dashed curves in panels (A) and (C) in Figure 2) that are not very different from those computed with the model of CHMG93, especially when the poloidal field potential is allowed to decrease with time (the dot-dashed curves in panels (A) and (C) in Figure 4). The secondary role of magnetic braking compared to the viscous transport of angular momentum is emphasized when we take a much smaller value of \( v = 5 \times 10^3 \text{ cm}^2 \text{ s}^{-1} \), consistent with the solar Li data (compare dashed curves in the same four panels).

In this case, the only noticeable effect produced by the presence of magnetic fields is a slightly slower rotation of the radiative core at the solar age, especially close to its center, which still strongly deviates from the helioseismology data.

### 3.2. Magnetic Braking with Anisotropic Turbulent Diffusion

We start with the aforementioned failed D3 model of magnetic braking of solar rotation that has an isotropic viscosity \( v = 5 \times 10^3 \text{ cm}^2 \text{ s}^{-1} \) and magnetic diffusivity \( \eta = 5 \times 10^4 \text{ cm}^2 \text{ s}^{-1} \) and which begins its rotational MS evolution with \( \Omega_{\text{ZAMS}} = 100 \Omega_\odot \) (thin dashed curves in Figure 6). Note that its viscosity is already increased by nearly 2 orders of magnitude compared to the maximum value of the microscopic (molecular for the Sun) viscosity in the radiative core. This is not a problem, though, given that vertical diffusion with a coefficient of a similar or slightly stronger magnitude is indeed predicted to occur in radiative zones of solar-metallicity MS stars with \( M = 1.5 \, M_\odot \), especially close to the ZAMS (Palacios et al. 2003), where internal gradients of the mean molecular weight are too weak to effectively hinder it. This diffusion is associated with either rotation-driven meridional circulation or small-scale turbulence caused by rotational shear instabilities. It has been successfully used to explain chemical composition anomalies in the atmospheres of massive MS stars and their descendants (e.g., Meynet & Maeder 2000; Denissenkov 2005).

Whereas the model of magnetic angular momentum transport based on the hypothetical Tayler–Spruit dynamo has a problem reproducing differential rotation in the slowest rotators (Denissenkov et al. 2010), the D3 model of CHMG93, in contrast, cannot provide a rapidly rotating solar-type star with a means to spin-down its radiative core fast enough for the star’s rotational evolution to fit the upper 90th percentiles of the \( \Omega_e \) distributions for solar analogs in open clusters. The question is whether we can make a reasonable modification of the CHMG93 model such that it will become capable of reproducing the rotational evolution of both the fastest and slowest rotators while being consistent with the other observational constraints. We have shown that a variation of the isotropic viscosity does not help to solve the problem self-consistently with or without a magnetic field. The major obstacles are the observational limit of \( v \lesssim 5 \times 10^5 \text{ cm}^2 \text{ s}^{-1} \) imposed by the rate of the evolutionary Li depletion in solar twins (Figure 3) and the rapid decay of the poloidal field caused by high viscosity and magnetic diffusivity. On the theoretical side, stellar hydrodynamics does not predict a build-up of vertical turbulent viscosities as large as \( v \sim 10^6 \text{ cm}^2 \text{ s}^{-1} \) in the radiative cores of rapidly spinning MS solar-type stars, as required for the stars to evolve like solid-body rotators. Besides, such large diffusion coefficients would solve the problem without being assisted by magnetic fields.

It should be noted that the poloidal magnetic field does help the viscous transport to reduce, even a little, the degree of differential rotation in the Sun’s radiative core, as claimed by CHMG93. Indeed, panels (C) in our Figures 8 and 5 indicate that, whereas the contours of constant angular velocity in the present-day Sun’s D0 model are concentric spheres, they become consecutive layers in meridian cross sections of an axisymmetric torus in the D3 model. This torus, called the “dead zone” by Mestel & Weiss (1987), is shaped by the poloidal field, whose lines form a similar toroidal structure (Figure 5(A)), according to Ferraro’s law \( (B_p, \nabla \Omega) = 0 \). Thus, in the D3 configuration, the viscous transport has to redistribute angular momentum throughout a smaller volume than in the absence of magnetic fields (the D0 case). However, when \( v = 5 \times 10^3 \text{ cm}^2 \text{ s}^{-1} \), it does not have enough time to finish its work by the solar age. As CHMG93 reported, the situation is slightly improved when the poloidal field configuration is switched to D2. In this case, there is no waiting time before the spin-down of the convective envelope gets communicated to the radiative core because poloidal field lines are now rooted in the envelope, hence they can start transferring the wind torque to the core from the very beginning. Besides, because not every poloidal field line is now parallel to the bottom of the convective envelope (some of them emanate from it, being directed inside the core), a larger cylindrical region around the rotation axis (not just the axis and its immediate adjacent layers as it was
in the D3 case) acquires a rotation rate close to that of the envelope. As a result, the dead zone becomes thinner (Figure 5(D)). However, even in this case the rotational evolution is still far from being close to that of a solid-body rotator (thick dashed curves in Figure 6).

When analyzing the above results, especially contour plots (C) and (D) in Figure 5, we have noticed that a strong horizontal turbulence, a concept with different approximate descriptions that have already been used in other stellar evolution computations, might solve the problem of magnetic braking of solar rotation as well. In fact, the same hydrodynamic models that give an estimate of \( \nu_v \sim 10^2 - 10^4 \text{ cm}^2 \text{ s}^{-1} \) for the vertical component of turbulent diffusion also predict a much stronger value, of order \( 10^4 - 10^6 \text{ cm}^2 \text{ s}^{-1} \), for its horizontal component in the radiative core of the solar-metallicity MS star with \( M = 1.5 M_\odot \) and \( \Omega_{e,ZAMS} \approx 50 \Omega_\odot \) (Palacios et al. 2003). In a solar-mass star, the corresponding turbulent viscosities are expected to be lower by a factor of 10, which is roughly proportional to the ratio of luminosities for MS stars with \( M = 1.5 M_\odot \) and \( M = M_\odot \), because the baroclinic and secular Kelvin–Helmholtz instabilities driving this anisotropic turbulence have growth rates inversely proportional to the thermal timescale. On the other hand, the coefficient of horizontal turbulent diffusion has been suggested to take on much greater values (by more than 4 orders of magnitude) than previously thought (cf. Zahn 1992; Maeder 2003; Mathis et al. 2007).

Zahn (1992) was the first to bring anisotropic turbulent diffusion to the attention of the stellar astrophysics community.
Anisotropic turbulence has both a quite natural physical justification and an experimental basis in geophysics and oceanography. Tassoul (2000) notes that “in the Earth's lower atmosphere one has $v_{h}/v_{v} \lesssim 10^{2}$, whereas this ratio may be as large as $10^{5}$ in the surface layer of the ocean where large-scale currents are observed.” It is even more likely to occur in stellar radiative zones with their extremely large Reynolds numbers and strong thermal stratifications in the vertical direction.

Unfortunately, neither $v_{h}$ nor $v_{v}$ can be derived from first principles yet. Therefore, crude estimates are usually adopted and then tuned up by making an order of magnitude evaluation of the relevant energy balance, using appropriate results from laboratory or numerical experiments, and, last but not least, trying to comply with available observational constraints concerning mixing in stars. Given this uncertainty, while pursuing our modest goal of producing a modified CHMG93 model that should be as simple as the original one, we have chosen the following prescriptions for the anisotropic diffusion coefficients in solar-type MS stars:

$$v_{v} = 5 \times 10^{3} \text{ cm}^{2} \text{s}^{-1}, \quad \text{and} \quad v_{h} = 10^{6} \left( \frac{\Omega_{c,ZAMS}}{100 \Omega_{\odot}} \right) \text{ cm}^{2} \text{s}^{-1}.$$  \hspace{1cm} (15)

The first expression represents the approximate upper limit for the diffusion rate associated with the radial mass transport that is more or less consistent with Li data in solar twins. The more realistic value would be closer to $2 \times 10^{3} \text{ cm}^{2} \text{s}^{-1}$ (dotted curve in Figure 3). However, the observational data do not exclude the possibility that $v_{h}$ could be as large as $3 \times 10^{4} \text{ cm}^{2} \text{s}^{-1}$ during the first 300 Myr and then reduced to $1.75 \times 10^{3} \text{ cm}^{2} \text{s}^{-1}$ (dotted curve in Figure 3; however, our rotation evolution computations show that this variable viscosity does not solve the problem, as is illustrated by Figure 7). In other words, our assumed constant value for $v_{v}$ should instead be considered a time average, therefore it can be a bit larger, say like $v_{h} = 4 \times 10^{3} \text{ cm}^{2} \text{s}^{-1}$ (short-dashed curve in Figure 3). But it definitely cannot be as large as $v_{v} = 5 \times 10^{4} \text{ cm}^{2} \text{s}^{-1}$ (solid curve in Figure 3) and remain constant, as CHMG93 assumed. As for our choice of $v_{h}$, its amplitude seems to have a correct order of magnitude (in agreement with most other recent applications) for a ZAMS solar model rotating with the angular velocity $\Omega_{c,ZAMS} = 100 \Omega_{\odot}$, taking into account that Mathis et al. (2007) have estimated $v_{h} \approx 10^{7} - 10^{8} \text{ cm}^{2} \text{s}^{-1}$ for a young (252 Myr) solar-metallicity 1.5 $M_{\odot}$ model star that started its MS evolution with approximately one half the rotation rate. On the other hand, the assumed dependence of $v_{h}$ on $\Omega_{c,ZAMS}$ does not seem unnatural and it provides us with the
necessary additional degree of freedom, such that a variation of its value can probably be used to model the transition from the fastest to slowest rotators, without causing a conflict with other observational constraints.

Thick solid curves in Figure 6 have been computed using the anisotropic diffusion coefficients (Equation (15)) for the D2 poloidal field configuration. We have assumed that $\eta_v = 2.5 \times 10^3 \text{ cm}^2 \text{s}^{-1}$ and $\eta_h = \nu_v$. The poloidal field decay has been taken into account (Equation (14)) with an appropriate value of the decay time obtained as a solution of Equation (13). We can see that now the surface rotation evolution curve in panel (A) fits the upper 90th percentiles of the $\Omega_e$ distributions for solar analogs in open clusters, while its corresponding internal rotation profile at the solar age appears to be nearly uniform (panel (B)). A comparison of four similar models in panel (A) demonstrates that it is the combination of the D2 poloidal field geometry and strong horizontal turbulence (the D2H model) that results in the rotational evolution resembling that of the fastest rotators because neither the D2 nor the D3H models are able to produce a convex $\Omega_e$ evolutionary curve. The contours of constant angular velocity and toroidal magnetic field for the D2H model of the solar age are shown in panels (B) and (D) in Figure 8. It is important to note that, in the absence of internal large-scale magnetic fields, strong horizontal turbulent diffusion could not erase the differential rotation in the radiative core because it is the dipole poloidal field and its interaction with the rotational shear that change the geometry of differential rotation from the spherically symmetric one (Figure 8(C)) to the toroidal one (Figure 5(D)). On the other hand, the assumption of anisotropic turbulence with $\nu_h \gg \nu_v$ allows us to choose a sufficiently small value of $\nu_v$, thus avoiding a conflict with the observed surface Li abundances in solar twins.

Finally, when we apply the D2H configuration to model the rotational evolution of solar-type stars with initial angular velocities ranging from $\Omega_{\text{ZAMS}} = 100 \Omega_\odot$ to $\Omega_{\text{ZAMS}} = 5 \Omega_\odot$, we indeed obtain a nice transition from the convex evolutionary curves for the fastest rotators to the concave ones for the slowest rotators (Figure 9), as hinted by observations. Note that all of these models end up with nearly flat internal rotation profiles, and their solar-calibrated wind constants do not differ by very much: whereas the fastest rotating model has $K_w = 3.64 \times 10^{17} \text{ cm}^2 \text{ g s}$, the slowest rotating model needs $K_w = 4.44 \times 10^{17} \text{ cm}^2 \text{ g s}$.

### 3.3. Stationary Solutions: The Width of the Solar Tachocline

The conclusion reached by MacGregor & Charbonneau (1999) that it is the D3 configuration, rather than the D2 one, that seems to be the most appropriate one for the model of magnetic braking of solar rotation is based on their consideration of the thinness of the solar tachocline. Because of the direct magnetic coupling of the convective envelope with the radiative core through the D2 poloidal field, the latitudinal differential rotation at the core–envelope interface $\Omega(r_c, \theta, t_d)$ penetrates into the bulk of the radiative core more easily in the D2 case than in the D3 case. The differential rotation of the convective envelope in the present-day Sun is maintained by the inhomogeneous
ν

ture is isotropic and has its near maximum microscopic value
velocity) for one such model in which the kinematic viscos-
shows a stationary solution (contours of constant angular ve-
using Equation (13). The solar-calibrated wind constant varies between
initially and is decaying exponentially with an e-folding time that is calculated
using Equation (13). The solar-calibrated wind constant varies between
K_w = 5.64 x 10^{37} cm^2 g s and K_w = 4.44 x 10^{37} cm^2 g s from the fastest to slowest
rotator.

Figure 9. Solid curves show the rotational evolution of solar-type stars computed
with our D2H model using the anisotropic turbulent viscosity (Equation (15)). Other
model parameters are \eta_0 = 2.5 x 10^{13} cm^2 s^{-1} (except for the slowest
rotating model that has \eta_0 = 5 x 10^{13} cm^2 s^{-1}), \eta_0 = v_h, B_0 = 0.1 G
initially and is decaying exponentially with an e-folding time that is calculated
using Equation (13). The solar-calibrated wind constant varies between
K_w = 5.64 x 10^{37} cm^2 g s and K_w = 4.44 x 10^{37} cm^2 g s from the fastest to slowest
rotator.

turbulence of a rotating fluid (e.g., Kitchatinov & Ruediger
1993; Ruediger & Kitchatinov 1996). Following Charbonneau
et al. (1999) and Ruediger & Kitchatinov (2007), we use the
simple approximate relation
\[ \Omega(r_e, \theta, t_0) = \Omega(1 - 0.15 \cos^2 \theta) \]
Its values at \theta = 30^\circ, 45^\circ, 60^\circ, and 90^\circ are plotted in panels (C)
and (D) in Figure 2 as horizontal line segments at \( r/R \gtrsim 0.713 \). It is seen that the variation of the interface angular velocity with
the colatitude in the present-day Sun is much smaller than the
decrease of \Omega, during the Sun’s possible previous MS evolution
(Figure 2(A)). Therefore, we have neglected this variation in our
rotational evolution computations. However, now, when we
consider the final model of the solar age, we have to be sure
that our assumptions and, in particular, our preference for the
D2 poloidal field configuration, do not lead to the formation of
a thick tachocline.

A stationary solution that has to be tested for the spread
of the interfacial differential rotation can easily be obtained
with the same COMSOL Multiphysics code that we have used
to compute the rotational evolution of solar-type stars. We
have only to set to zero the left-hand sides of Equations (7)
and (8), omit the term proportional to \( S_z \), use the relation (16) as
a new outer boundary condition, and shrink the computational
space domain from \( r = R_0 \) to \( r = r_e = 0.713 R_0 \). We have
used this approach to obtain a number of stationary solutions,
the results of which are discussed below.

Let us start with non-magnetic solar models. Figure 10(A)
shows a stationary solution (contours of constant angular ve-
locity) for one such model in which the kinematic viscosity
is isotropic and has its near maximum microscopic value of \( v = 20 \text{ cm}^2 \text{ s}^{-1} \) that is assumed (as before) to be con-
stant throughout the radiative core. This solution is evidently
in conflict with helioseismology data indicating that both the
latitudinal and the radial differential rotation of the solar con-
vective envelope are smoothed out when one crosses the thin
(\( \Delta r \lesssim 0.04 R_0 \)) tachocline, so that the radiative core ro-
tates like a solid body at least down to \( r \approx 0.2 R_0 \) (e.g.,
Ruediger & Kitchatinov 2007, and references therein). This
result is not surprising because the isotropic viscosity is known
to spread a horizontal velocity inhomogeneity of a given length
scale to a radial velocity inhomogeneity of the same length
scale (Ruediger & Kitchatinov 1997). Figure 10(B) presents
a stationary solution for the same non-magnetic model but
with the anisotropic viscosity whose horizontal component is
\( \eta_h = 10^4 \text{ cm}^2 \text{ s}^{-1} \gg v_h = 20 \text{ cm}^2 \text{ s}^{-1} \). In this case, the strong
horizontal turbulence smooths out the interfacial differential rotation in a sufficiently thin layer located immediately beneath
the core–envelope interface (cf. Spiegel & Zahn 1992).

Now, we return to the isotropic viscosity but add a poloidal magnetic field. Figure 10(C) shows a stationary solution for
the case similar to that considered by CHMG93, namely, for
the D3 poloidal field configuration of amplitude \( B_0 = 0.1 \text{ G} \)
interacting with the rotational shear via the (turbulent) viscosity
\( v = 5 \times 10^4 \text{ cm}^2 \text{ s}^{-1} \), the induced toroidal field being diffused
away at the rate \( \eta = 5 \times 10^5 \text{ cm}^2 \text{ s}^{-1} \). In this case, our contours
of constant angular velocity indeed resemble very closely those
plotted in the lower right panel in Figure 1 by MacGregor &
Charbonneau (1999). After returning to the isotropic viscosity,
we would expect the interfacial differential rotation to again be
spread radially deep into the core but this has not happened.
The reason is that now it is the latitudinal transport of angular
momentum by the magnetic stress, proportional to the product
\( B_0 B_z \), that suppresses the differential rotation and restricts its
penetration into the radiative interior (Ruediger & Kitchatinov
1997). However, the original CHMG93 model did not take into
account the decay of the poloidal field in spite of the fact that
its assumed extremely high value of the magnetic diffusivity
implies the very short decay time of 35 Myr. With this
e-folding time, the poloidal field will already have the amplitude of merely
\( 10^{-10} \text{ G} \) by the age of 725 Myr. Figure 10(D) shows contours of
constant angular velocity in the D3 model of the solar age with
this weak field. Because there is actually no poloidal field left,
the solution looks almost identical to that for the pure D0 case.

We have chosen the D2 configuration because it more effec-
tively transfers the wind torque and rotational shear into the
radiative core than the D3 configuration. However, this same prop-
eity plays a negative role when we try to prevent the interfacial
differential rotation in the present-day Sun from penetrating the
radiative interior. Fortunately, our D2H model combines the D2
geometry with the assumption of strong horizontal turbulent dif-
fusion which, as we have seen in Figure 10(B), can potentially
counteract the negative effect produced by the non-vanishing
radial component of the D2 poloidal field at the core–envelope
interface. To test this possibility, we have computed four station-
ary solutions for our favorite D2H model. Figure 11(A) shows
contours of constant angular velocity at the solar age for a D2H
model in which neither the anisotropic diffusion coefficients nor
the poloidal field amplitude were allowed to change with time.
This is the worst possible solution in terms of the tachocline
width. However, like the original CHMG93 solution, our first
D2H test solution is not fully consistent because it does not take
into account the poloidal field decay. For our assumed values
of \( \eta_0 = 2.5 \times 10^3 \text{ cm}^2 \text{ s}^{-1} \) and \( \eta_0 = v_h = 10^6 \text{ cm}^2 \text{ s}^{-1} \)
(for the fastest solar-type rotator with \( \Omega_{ZAMS} = 100 \Omega_0 \))
Equation (13) gives \( \tau = 1.1 \text{ Gyr} \). Hence, by the solar age,
the initial amplitude \( B_0 = 0.1 \text{ G} \) will be reduced to \( 2 \times 10^{-3} \text{ G} \).
Figure 11(B) demonstrates that for a D2 poloidal field of such
an order of magnitude the tachocline can be very thin, due to
the continuing strong horizontal turbulent diffusion. Now, given our assumption about the dependence of $v_h$ on the angular velocity (Equation (15)), it would be fair to construct a solution for the case of $v_h$ having been reduced to $10^4$ cm$^2$ s$^{-1}$, proportionally to the decrease of $\Omega_e$ from $100 \Omega_\odot$ to $\Omega_\odot$. Its respective contours of constant $\Omega(r, \theta, t)$ are presented in Figure 11(C). We see that even in this case the obtained tachocline width is not more discrepant than in the original CHMG93 solution (Figure 10(C)). Finally, Figure 11(D) illustrates that the previous solution is greatly improved when the amplitude of the residual poloidal field is reduced by another order of magnitude. So, the anisotropic turbulence with the horizontal components of turbulent viscosity strongly dominating over those in the vertical direction plus the account of the poloidal field decay incorporated into our D2H model make it possible to obtain a sufficiently thin tachocline in the present-day Sun’s model even for the D2 poloidal field configuration.

4. CONCLUSION

The main goal of this work has been to modify the model of magnetic braking of solar rotation presented and discussed in detail by CHMG93 so that it could reproduce the most recent observational data concerning the rotation of solar analogs in open clusters of different ages, without coming into conflict with other available observational constraints. Although we have used a different prescription for the surface angular momentum loss rate (Equation (1)), its corresponding dependence of the envelope braking rate $-d(\Omega_e/\Omega_\odot)/dt$ (yr$^{-1}$) on the rotation rate $\Omega_e/\Omega_\odot$ does not deviate significantly from the dependence predicted by the Weber–Davis MHD wind model that was employed by CHMG93 (for comparison, we used Figure 1 from Charbonneau 1992). It turns out that even in its original formulation the CHMG93 model experiences difficulties in fitting the upper 90th percentiles of the $\Omega_e$ distributions for cluster solar-type stars. The discrepancy with the observationally constrained rotational evolution of the fastest rotators becomes much worse when we reduce the kinematic viscosity by 1 order of magnitude and/or take into account the poloidal field decay caused by the high magnetic diffusivity in the CHMG93 model. The viscosity reduction makes the model more consistent with the data for the atmospheric Li depletion in solar twins. The value of $v = 5 \times 10^4$ cm$^2$ s$^{-1}$ used by CHMG93 leads to excessive Li destruction unless it describes, in terms of an effective diffusion coefficient, another angular momentum transport mechanism implemented as a process without a radial mass transfer, e.g., angular momentum redistribution by internal gravity waves. However, in this case the question would arise as to why that additional process could not do the entire work alone. Poloidal field decay also seems to be necessary, given that the model already follows the decay of the induced toroidal field. After these reasonable modifications are made, the CHMG93 model produces a rotational evolution of a rapidly rotating solar-type star that differs greatly from that of the fastest rotators among solar analogs in open clusters. Besides, at the solar age, the model still possesses strong differential rotation in the
radiative core, in conflict with helioseismology measurements. This differential rotation has such a geometry that its corresponding contours of constant angular velocity form consecutive layers in cross sections of an axisymmetric torus, a region called the “dead zone” by Mestel & Weiss (1987). Its shape is a manifestation of Ferraro’s law of isorotation, \((\mathbf{B}_p, \nabla)\Omega = 0\), with the poloidal field \(\mathbf{B}_p\) having a dipole structure.

To save the CHMG93 model, we have assumed that the rotation-driven turbulence that is thought to amplify the viscosity and magnetic diffusivity compared to their microscopic values in stellar radiative zones is actually anisotropic with the horizontal components of the transport coefficients strongly dominating over those in the vertical direction. This is not a new hypothesis. It was first introduced to the stellar astrophysics community by Zahn (1992) and, since then, it has widely been used to model both chemical mixing and angular momentum transport in stars. The strong horizontal turbulent diffusion serves a two-fold goal in our modified model: it erases the dead zone along isobaric surfaces (a horizontal erosion of latitudinal differential rotation), while allowing us to choose a sufficiently small value for the coefficient of vertical diffusion that does not lead to a conflict with the Li data in solar twins. We have also found it necessary to switch from the D3 poloidal field configuration, that was considered to be the most appropriate one in the original CHMG93 model, to the D2 configuration. In the D2 geometry, the dipole poloidal field partially penetrates the convective envelope, which allows it to more effectively transfer the surface wind torque to the radiative interior. As a result, a thinner dead zone develops, which makes it easier for the horizontal turbulence to erase it. The horizontal component of the turbulent viscosity gives us an additional degree of freedom whose assumed linear dependence on the initial (ZAMS) angular velocity enables the models to properly reproduce the transition from the convex morphology of the surface rotation evolution of the fastest rotators to the concave \(\Omega_e(r)\) curves for the slowest rotators, as suggested by observations.

MacGregor & Charbonneau (1999) have chosen the D3 geometry as the most suitable one for the modeling of magnetic breaking of solar rotation because its corresponding poloidal field configuration does not lead to a penetration of the differential rotation of the Sun’s convective envelope into its radiative interior, consistent with the observations. The helioseismology measurements show that the envelope differential rotation gets smoothed out in a thin (\(\Delta r \lesssim 0.04 \, R_\odot\)) transition layer (the solar tachocline) immediately beneath the bottom of the Sun’s convective envelope and that the radiative core below the tachocline rotates like a solid body at least down to \(r \approx 0.2 \, R_\odot\). Contrary to this, it has been shown that the non-vanishing radial component of the D2 poloidal field at the core–envelope interface causes the interfacial latitudinal differential rotation to move into the bulk of the radiative core. However, after our modifications of the original CHMG93 model, the new D2H model appears to be free of this problem. Both the strong horizontal turbulent

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**Figure 11.** Same as in Figure 10 but for our magnetic braking model (the D2 poloidal field with the anisotropic turbulent diffusion). D2HSTAT, \(B_0 = 0.1\) (G): this case assumes that all the model parameters remain constant, including the value of \(v_h = 10^6\) cm s\(^{-1}\) (for \(\Omega_{ZAMS} = 100\Omega_\odot\) in Equation (15)). D2HSTAT, \(B_0 = 0.001\) (G): same as in the previous panel but with the decayed poloidal field. D2H, \(B_0 = 0.001\) (G) and \(B_0 = 0.0001\) (G): here, it is additionally assumed that \(v_h\) has been reduced to \(10^6\) cm s\(^{-1}\), proportionally to the decrease of \(\Omega_e\) from 100\(\Omega_\odot\) to \(\Omega_\odot\).

(A color version of this figure is available in the online journal.)
diffusion and the poloidal field decay work toward confining the width of the solar tachocline.

It is important to note that our D2H model is very simplistic and it is based on a number of assumptions whose validity has not been proven. For instance, it is not clear how the rotation-driven turbulence should interact with the strong large-scale magnetic fields. Also, the formation of the solar tachocline is a much more complex process than described by our model. In particular, it must take into account a penetration of the meridional circulation from the convective zone into the radiative core (e.g., Gough & McIntyre 1998; Ruediger & Kitchatinov 2007; Gough 2010), which our model completely ignores. We are aware of these shortcomings. Therefore, we consider our model only as a legitimate extension of the CHMG93 model in the sense that we have not made any modifications of the latter that are very different from the original assumptions and approximations used by CHMG93. On the other hand, we have shown that just one additional assumption, that of strong horizontal viscosity, helps to solve several problems with the old model.

Finally, we address a problem of the so-called $\Lambda$-terms that enable angular momentum transport even in the state of solid-body rotation in the case of anisotropic turbulence. It is these extra terms that are thought to maintain differential rotation of the Sun’s convective envelope (Kitchatinov & Ruediger 1993; Ruediger & Kitchatinov 1996). To take the $\Lambda$-effect into account, we have to multiply the vertical and horizontal viscosities in the momentum Equation (7) by the following factors:

$$f_v = 1 - \frac{V}{\partial \ln \Omega_c / \partial y}, \quad \text{and} \quad f_h = 1 - \frac{H}{\sin^2 \theta \partial \ln (\cos \theta)}.$$  

(17)

where $V$ and $H$ are normalized radial and latitudinal angular momentum fluxes. The quasilinear theory of the angular momentum transport by rotating turbulence in density-stratified fluids (Kitchatinov & Ruediger 2005) results in the following expressions:

$$V = V^{(0)} - H^{(1)} \cos^2 \theta, \quad H = H^{(1)} \sin^2 \theta,$$  

(18)

where $V^{(0)}$ and $H^{(1)}$ are simple functions of the Coriolis number $\Omega^* = 2 \Omega \Omega_c$, a dimensionless anisotropy parameter, and the square of the ratio $l_{corr}/H$, with $H$ being the density scale height. In the last relations, $\Omega_c$, $l_{corr}$ and $H$ are estimates of the turbulent eddy turnover time and length scale.

A straightforward analysis shows that, unlike the convective envelope, the $\Lambda$-effect turns out to be unimportant in the radiative core of a solar-type star. Indeed, first of all, the Coriolis number $\Omega^* \gg 1$, even for the present-day solar rotation (e.g., see Figure 2 of Kitchatinov & Ruediger 2005). In this limit, the expressions (18) are simplified to

$$V \approx -H^{(1)} \cos^2 \theta, \quad H \approx H^{(1)} \sin^2 \theta, \quad H^{(1)} \approx \left(\frac{l_{corr}}{H}\right)^2 \left(\frac{\pi}{4 \Omega^*}\right).$$  

(19)

Second, given that the turbulent viscosity (vertical or horizontal) can be estimated as $v_t \sim l_{corr}^2 \tau_{corr}$, the last relationship in Equation (19) is transformed to

$$H^{(1)} \sim \frac{\pi v_t}{8 \Omega H^2}.$$  

It becomes extremely small, $H^{(1)} \lesssim v_t/(10^{14} \text{ cm}^2 \text{s}^{-1}) \lesssim 10^{-8}$ (for $H \sim 10^{10} \text{ cm}$, $\Omega \gtrsim 10^{-6} \text{ rad s}^{-1}$, and $v_t \sim 10^8 \text{ cm}^2 \text{s}^{-1}$), in the radiative core and, therefore, the $\Lambda$-terms can be neglected there. The main reason behind this result is the smallness of the ratio $l_{corr}/H$ in the core. In contrast, we have $l_{corr}/H_p \approx 2$ in the convective envelope, where the Coriolis number is also comparable to or even much less than one. Consequently, the $\Lambda$-effect plays a very important role in shaping the differential rotation of the Sun’s convective envelope. Note that in our numerical analysis of the thickness of the solar tachocline we have chosen to use the empirical relation (16) for $\Omega r_c, \theta$ that substitutes for a stationary solution of the momentum equation with the $\Lambda$-terms for the bottom of the Sun’s convective envelope.

The author is grateful to Don VandenBerg who has supported this work through his Discovery Grant from Natural Sciences and Engineering Research Council of Canada. The author also appreciates discussions with Marc Pinsonneault, Don Terndrup, and Keith MacGregor that have greatly stimulated this work.

REFERENCES

Andronov, N., Pinsonneault, M. H., & Sills, A. 2003, ApJ, 582, 358
Barnes, G., Charbonneau, P., & MacGregor, K. B. 1999, ApJ, 511, 466
Basu, A., & Antia, H. M. 2003, ApJ, 585, 553
Chaboyer, B., Demarque, P., & Pinsonneault, M. H. 1995, ApJ, 441, 865
Charbonne P. 1992, in ASP Conf. Ser. 26, Cool Stars, Stellar Systems, and the Sun, ed. M. S. Giampapa & J. A. Bookbinder (San Francisco, CA: ASP), 416
Charbonne P., DiPati, M., & Gilman, P. A. 1999, ApJ, 526, 523
Charbonne P., & MacGregor, K. B. 1992, ApJ, 387, 639
Charbonne P., & MacGregor, K. B. 1993, ApJ, 417, 762
Charbonnel, C., & Talon, S. 2005, Science, 309, 2189
Covidat, S., Garcia, R. A., Turck-Chièze, Corbath, T., Henney, C. J., & Jimenez-Reyes, S. 2003, ApJ, 597, L27
Denissenkov, P. A. 2005, ApJ, 622, 1058
Denissenkov, P. A., Pinsonneault, M., & MacGregor, K. B. 2008, ApJ, 684, 757
Denissenkov, P. A., Pinsonneault, M., Terndrup, D. M., & Newgum, S. 2010, ApJ, 716, 1269
Eggenberger, P., Maeder, A., & Meynet, G. 2005, A&A, 440, L9
Ferraro, V. C. A. 1937, MNRAS, 97, 458
Gough, D. O. 2010, in Magnetic Coupling Between the Interior and Atmosphere of the Sun, ed. S. S. Hasan & R. J. Ruten (Berlin: Springer), 68
Gough, D. O., & McIntyre, M. E. 1998, Nature, 394, 755
Irwin, J., Hodgkin, S., Aigrain, S., Hebb, L., Bouvier, J., Clarke, C., Moraux, E., & Bramich, D. M. 2007, MNRAS, 377, 741
Kawaler, S. D. 1988, ApJ, 333, 236
Kitchatinov, L. L., & Ruediger, G. 1993, A&A, 276, 96
Kitchatinov, L. L., & Ruediger, G. 2005, Astron. Nachr., 326, 379
Krishnamurthi, A., Pinsonneault, M. H., Barnes, S., & Sofia, K. 1997, ApJ, 480, 303
MacGregor, K. B. 1991, in Angular Momentum Evolution of Young Stars, ed. S. Catalano & J. R. Stauffer (Dordrecht: Kluwer), 315
MacGregor, K. B., & Charbonneau, P. 1999, ApJ, 519, 911
Maeder, A. 2003, A&A, 399, 263
Mathis, S., Palacios, A., & Zahn, J.-P. 2007, A&A, 462, 1063
Meynet, G., & Maeder, A. 2000, A&A, 361, 101
 Metric, L., & Weiss, N. O. 1987, MNRAS, 226, 123
Mestel, L., & Weiss, N. O. 1987, MNRAS, 226, 123
Mestel, L., & Weiss, N. O. 1987, MNRAS, 226, 123
Mestel, L., & Weiss, N. O. 1987, MNRAS, 226, 123
Meynet, G., & Maeder, A. 2000, A&A, 361, 101
Plumb, R. A. 1977, J. Atmos. Sci., 34, 1847
Rogers, T. M., & Glatzmaier, G. A. 2005, MNRAS, 364, 1135
Rebull, L. M., Wolff, S. C., & Strom, S. E. 2004, AJ, 127, 1029
S. Catalano & J. R. Stauffer (Dordrecht: Kluwer), 315
S. Catalano & J. R. Stauffer (Dordrecht: Kluwer), 315
Ruediger, G., & Kitchatinov, L. L. 2007, New J. Phys., 9, 302
Schatzman, E. 1962, Ann. Astrophys., 25, 18
Shu, F., Najita, J., Ostriker, E., Wilkin, F., Ruden, S., & Lizano, S. 1994, ApJ, 429, 781
Skumanich, A. 1972, ApJ, 171, 565
Spiegel, E. A., & Zahn, J.-P. 1992, A&A, 265, 106
Spruit, H. C. 1999, A&A, 349, 189
Spruit, H. C. 2002, A&A, 381, 923
Talon, S., & Zahn, J.-P. 1997, A&A, 317, 749
Talon, S., Zahn, J.-P., Maeder, A., & Meynet, G. 1997, A&A, 322, 209
Tassoul, J.-L. 2000, Stellar Rotation/Jeans-Louis Tassoul Cambridge (Cambridge Astrophys. Ser. 36; Cambridge: Univ. Press)
Tomczyk, S., Schou, J., & Thompson, M. J. 1995, ApJ, 448, L57
Weber, E. J., & Davis, L., Jr. 1967, ApJ, 148, 217
Zahn, J.-P. 1992, A&A, 256, 115