OPTIMAL WEAK ESTIMATES FOR RIESZ POTENTIALS

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Abstract. In this note we prove a sharp reverse weak estimate for Riesz potentials

\[ \|I_s(f)\|_{L^{\frac{n}{n-s},\infty}} \geq \gamma_s \|f\|_{L^1} \quad \text{for } 0 < f \in L^1(\mathbb{R}^n), \]

where \( \gamma_s = 2^{-s} \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \). We also consider the behavior of the best constant \( C_{n,s} \) of weak type estimate for Riesz potentials, and we prove \( C_{n,s} = O(\gamma_s) \) as \( s \to 0 \).

1. Introduction

The Riesz potentials (fractional integral operators) \( I_s \), which play an important part in Analysis, are defined by

\[ I_s(f)(x) = \gamma_s \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-s}} dy, \]

where \( 0 < s < n \) and \( \gamma_s = 2^{-s} \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \). Such operators were first systematically investigated by M. Riesz [11]. The \((L^p, L^q)\)-boundedness of Riesz potentials were proved by G. Hardy and J. Littlewood [6] when \( n = 1 \) and by S. Sobolev [12] when \( n > 1 \). The \((L^1, L^{\frac{n}{n-s},\infty})\)-boundedness were obtained by A. Zygmund [16]. More precisely, they established the following theorem.

**Theorem A.** Let \( 0 < s < n \) and let \( p, q \) satisfy \( 1 \leq p < q < \infty \) and \( \frac{1}{p} - \frac{1}{q} = \frac{s}{n} \), then when \( p > 1 \),

\[ \|I_s(f)\|_{L^q(\mathbb{R}^n)} \leq C(n,p,s)\|f\|_{L^p(\mathbb{R}^n)}. \]

And when \( p = 1 \),

\[ \|I_s(f)\|_{L^{\frac{n}{n-s},\infty}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \{x \in \mathbb{R}^n : |I_s f| > \lambda\} \frac{n-s}{n} \leq C(n,s)\|f\|_{L^1(\mathbb{R}^n)}. \]

The best constant in the \((L^p, L^q)\) inequality when \( p = \frac{2n}{n+s}, q = \frac{2n}{n-s} \) was precisely calculated by E. Lieb [8] (see also [4]), and E. Lieb and M. Loss also offered an upper bound of \( C(n,p,s) \) (see chapter 4 in [9]).

Although the best constant of \((L^p, L^q)\) estimate for Riesz potentials has been studied for decades, to the best of the authors’ knowledge there is no result about the best constant of

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(\(L^1, L^\frac{n}{n-s}, \infty\)) estimate for Riesz potentials. In this paper, we will provide some estimates for the best constant of the weak type inequality.

In \([14]\) (see multilinear case in \([15]\)), the second author setted up the following limiting weak-type behavior for Riesz potentials,

\[
\lim_{\lambda \to 0} \lambda \left\{ x \in \mathbb{R}^n : |I_s f| > \lambda \right\} = \gamma^n_s v^n \| f \|_{L^1(\mathbb{R}^n)} \text{ for } 0 < f \in L^1(\mathbb{R}^n),
\]

which implies a reverse weak estimate

\[
\| I_s(f) \|_{L^\frac{n}{n-s}, \infty(\mathbb{R}^n)} \geq \gamma^n_s v^n \| f \|_{L^1(\mathbb{R}^n)} \text{ for } 0 < f \in L^1(\mathbb{R}^n),
\] (1.1)

where \(v^n\) is the volume of the unit ball in \(\mathbb{R}^n\). So a natural question that arises here is whether the constant \(\gamma^n_s v^n\) is sharp? In the paper, we will give an affirmative answer.

Let \(C_{n,s}\) be the best constant such that the \((L^1, L^\frac{n}{n-s}, \infty)\) estimate holds for Riesz potentials, i.e.

\[
C_{n,s} = \sup_{f \in L^1(\mathbb{R}^n)} \frac{\| I_s(f) \|_{L^\frac{n}{n-s}, \infty(\mathbb{R}^n)}}{\| f \|_{L^1(\mathbb{R}^n)}}.
\]

Then from (1.1), one can directly obtain a lower bound for \(C_{n,s}\),

\[
C_{n,s} \geq \gamma^n_s v^n.
\]

Our another goal in this paper is to provide upper and lower bounds of \(C_{n,s}\) to study the behavior of \(C_{n,s}\) as \(s \to 0\). Our approach depends on the weak \(L^\frac{n}{n-s}\) norm \(|| \cdot ||_{L^\frac{n}{n-s}, \infty}\) which is defined by

\[
|| f ||_{L^\frac{n}{n-s}, \infty(\mathbb{R}^n)} = \sup_{0 < |E| < \infty} |E|^{-\frac{1}{r} + \frac{n-s}{n}} \left( \int_E |f|^r \, dx \right)^{\frac{1}{r}}, \quad 0 < r < \frac{n}{n-s}.
\]

The norm \(|| \cdot ||_{L^\frac{n}{n-s}, \infty(\mathbb{R}^n)}\) is equivalent to \(|| \cdot ||_{L^\frac{n}{n-s}, \infty(\mathbb{R}^n)}\). In fact there holds (see Exercise 1.1.12 in \([3]\))

\[
|| f ||_{L^\frac{n}{n-s}, \infty(\mathbb{R}^n)} \leq || f ||_{L^\frac{n}{n-s}, \infty(\mathbb{R}^n)} \leq \left( \frac{n}{n-r(n-s)} \right)^{\frac{1}{r}} || f ||_{L^\frac{n}{n-s}, \infty(\mathbb{R}^n)}.
\] (1.2)

Closely related to the Riesz potentials is the centered fractional maximal function, which is defined by

\[
M_s f(x) = \sup_{r > 0} \frac{1}{|B(x,r)|^{1-s/n}} \int_{B(x,r)} |f(y)| \, dy, \quad 0 < s < n.
\]

\(M_s\) satisfy the same \((L^p, L^q)\) and \((L^1, L^\frac{n}{n-s}, \infty)\) inequality as \(I_s\) does, see \([1]\) and \([10]\). For any positive function \(f\) it is easy to see \(M_s(f) \leq 1/\gamma(s)v^n I_s(f)\). Although the reverse inequality does not hold in general, B.Muckenhoupt and R.Wheeden \([10]\) proved the two quantities are comparable in \(L^p\) norm \((1 < p < \infty)\) when \(f\) is nonnegative.

Now let us state our main results. First of all we consider the weak estimate of \(I_s(f)\) and \(M_s(f)\) under the norm \(|| \cdot ||_{L^\frac{n}{n-s}, \infty}\). Surprisingly identities for the weak type estimate of Riesz potentials and fractional maximal function can be established, which implies the two quantities are comparable in \(L^\frac{n}{n-s}, \infty\) (quasi)norm when \(f \in L^1(\mathbb{R}^n)\) is nonnegative.
Theorem 1.1. Let $0 < s < n$ and $f \in L^1(\mathbb{R}^n)$. When $1 \leq r < \frac{n}{n-s}$,
\[
\|I_s(f)\|_{L_{n-s}^\frac{n}{n-s}(\mathbb{R}^n)} \leq \gamma_s v_n^{\frac{n-s}{2}} \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \|f\|_{L^1(\mathbb{R}^n)},
\]
and
\[
\|M_s(f)\|_{L_{n-s}^\frac{n}{n-s}(\mathbb{R}^n)} = \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \|f\|_{L^1(\mathbb{R}^n)}.
\]
Moreover if $0 < f \in L^1(\mathbb{R}^n)$, then
\[
\|I_s(f)\|_{L_{n-s}^\frac{n}{n-s}(\mathbb{R}^n)} = \gamma_s v_n^{\frac{n-s}{2}} \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \|f\|_{L^1(\mathbb{R}^n)}.
\]

Remark 1. In fact, from the proof one can obtain the reverse weak estimate holds when $0 < r < \frac{n}{n-s}$. More precisely when $0 < r < \frac{n}{n-s}$,
\[
\|I_s(f)\|_{L_{n-s}^\frac{n}{n-s}(\mathbb{R}^n)} \geq \gamma_s v_n^{\frac{n-s}{2}} \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \|f\|_{L^1(\mathbb{R}^n)},
\]
and
\[
\|M_s(f)\|_{L_{n-s}^\frac{n}{n-s}(\mathbb{R}^n)} \geq \left(\frac{n}{n-(n-s)r}\right)^{\frac{1}{r}} \|f\|_{L^1(\mathbb{R}^n)},
\]
if $0 < f \in L^1(\mathbb{R}^n)$.

Then we prove the following sharp reverse weak estimates for Riesz potentials.

Theorem 1.2. Let $0 < f \in L^1(\mathbb{R}^n)$, then
\[
\|I_s(f)\|_{L_{n-s}^\frac{n}{n-s}(\mathbb{R}^n)} \geq \gamma_s v_n^{\frac{n-s}{2}} \|f\|_{L^1(\mathbb{R}^n)}.
\]
And the equality holds when $f = (\frac{a}{b+|x-x_0|^2})^{\frac{n}{n-s}}$, where $a, b > 0$ and $x_0 \in \mathbb{R}^n$.

As a corollary of Theorem 1.1 and Theorem 1.2 we can obtain the following sharp reverse inequality.

Corollary 1.1. Let $f \in L^1(\mathbb{R}^n)$, then
\[
\|M_s f\|_{L_{n-s}^\frac{n}{n-s}(\mathbb{R}^n)} \geq \|f\|_{L^1}.
\]
And the equality holds when $f(x) = h(|x-x_0|)$ where $h$ is a radial decreasing function.

At last we offer an upper and a lower bound for $C_{n,s}$, which implies the behavior of the best constant $C_{n,s}$ for small $s$ is optimal, i.e. $C_{n,s} = O(\frac{n}{s}) = O(1)$ as $s \to 0$.

Theorem 1.3. When $n > 2$ and $0 < s < \frac{n-2}{n}$,
\[
\gamma_s v_n^{\frac{n-s}{s}} \frac{n-2-4s}{2s(n-2-s)} \leq C_{n,s} \leq \gamma_s v_n^{\frac{n-s}{s}} \frac{n}{s}.
\]

Remark 2. Besides using the rearrangement inequality to obtain an upper bound $\gamma_s v_n^{\frac{n-s}{s}} \frac{n}{s}$, we can take the heat-diffusion semi-group as a tool (see the Appendix), which was used by E. Stein and J. Strömberg in [13] to study the $(L^1, L^{1,\infty})$ bound for centered maximal function, to obtain another upper bound which is equal to $O(\gamma_s v_n^{\frac{n-s}{s}} \frac{n}{s}) = O(1)$ as $(s, n) \to (0, \infty)$.
2. The identity for $I_s(f)$ and $M_s(f)$ in $\| \cdot \|_{L_{\frac{n-s}{n-s}}}^\infty$

In this section, we will prove Theorem 1.1. Without loss of generality let us assume $\|f\|_{L^1(\mathbb{R}^n)} = 1$. Since $I_s(f) \leq I_r(|f|)$ and $r \geq 1$, using Minkowski inequality one have for any measurable set $E$ with $|E| < \infty$,

$$|E|^{-\frac{1}{r} + \frac{n-s}{n}} \left[ \int_E |I_s f(x)|^r \, dx \right]^\frac{1}{r} \leq \gamma_s |E|^{-\frac{1}{r} + \frac{n-s}{n}} \int \left[ \int_E \frac{dx}{|x-y|^{(n-s)r}} \right]^\frac{1}{r} |f(y)| \, dy. \quad (2.1)$$

Then by Hardy Littlewood rearrangement inequality, there holds

$$\int_E \frac{dx}{|x-y|^{(n-s)r}} \leq \int_{E^*} \frac{dx}{|x|^{(n-s)r}} = \frac{\nu^n_{n-r}}{n - (n-s)r} |E|^{1 - \frac{n-s}{n-r}}, \quad (2.2)$$

where $E^*$ is the symmetric rearrangement of the set $E$, i.e. $E^*$ is an open ball centered at the origin whose volume is $|E|$. Therefore by (2.1) and (2.2) one can obtain

$$\| I_s(f) \|_{L_{\frac{n}{n-s}}}^\infty \leq \gamma_s v_n^n \left( \frac{n}{n - (n-s)r} \right)^\frac{1}{r} \| f \|_{L^1}.$$

Next, let us prove when $0 \leq f \in L^1(\mathbb{R}^n)$ and $0 < r < \frac{n}{n-s}$,

$$\| I_s(f) \|_{L_{\frac{n}{n-s}}}^\infty \geq \gamma_s v_n^n \left( \frac{n}{n - (n-s)r} \right)^\frac{1}{r} \| f \|_{L^1}. \quad (2.3)$$

For any $\epsilon > 0$, choose $R$ large enough such that $\int_{B_R(0)} f(y) \, dy = 1 - \epsilon$. Let $E = B_{lR}(0)$. Since

$$\int_{B_R(0)} \frac{f(y)}{|x-y|^{n-s}} \, dy \geq \int_{B_R(0)} \frac{f(y)}{(|x| + R)^{n-s}} \, dy = (1 - \epsilon)(|x| + R)^{s-n},$$

then

$$\| I_s(f) \|_{L_{\frac{n}{n-s}}}^\infty \geq \gamma_s |E|^{-\frac{1}{r} + \frac{n-s}{n}} \left[ \int_E \left( \int_{B_R(0)} \frac{f(y)}{|x-y|^{n-s}} \, dy \right)^r \, dx \right]^\frac{1}{r}$$

$$\geq \gamma_s |E|^{-\frac{1}{r} + \frac{n-s}{n}} \left( 1 - \epsilon \right) \left[ \int_E \frac{dx}{(|x| + R)^{(n-s)r}} \right]^\frac{1}{r}$$

$$= \gamma_s v_n^n \frac{n-s}{n} \left[ \int_0^l \frac{t^{n-1}}{(t+1)^{(n-s)r}} \, dt \right]^\frac{1}{r}.$$

By the fact that this inequality holds for any $l > 0$, then letting $l \to \infty$, one obtain

$$\| I_s(f) \|_{L_{\frac{n}{n-s}}}^\infty \geq \gamma_s v_n^n \left( 1 - \epsilon \right) \left( \frac{n}{n - (n-s)r} \right)^\frac{1}{r},$$

which implies (2.3). And we finish the proof of the identity for Riesz potential.
For fractional maximum function $M_s$, since
\[
M_s(f)(x) \geq \frac{1}{v_n^{n-s}(|x| + R)^{n-s}} \int_{|y-x| \leq R} |f(y)|dy
\geq \frac{1}{v_n^{n-s}(|x| + R)^{n-s}} \int_{|y| \leq R} |f(y)|dy = \frac{1 - \epsilon}{v_n^{n-s}(|x| + R)^{n-s}},
\]
then one can use the same method to get
\[
\|M_s(f)\|_{L^{\frac{n}{n-s}, \infty}} \geq \left( \frac{n}{n - (n-s)r} \right)^{\frac{1}{r}} \text{ when } 0 < r < \frac{n}{n-s}.
\]
On the other hand,
\[
\|M_s(f)\|_{L^{\frac{n}{n-s}, \infty}} \leq \|1/\gamma(s)v_n^{n-s}I_s(|f|)\|_{L^{\frac{n}{n-s}, \infty}} = \left( \frac{n}{n - (n-s)r} \right)^{\frac{1}{r}}.
\]
Thus one can obtain the desired identity for $M_s$.

3. The Sharp Reverse Weak Estimate for $I_s$ and $M_s$

In this section, first we prove the sharp reverse weak estimate for Riesz potentials $I_s$. By (1.2) and Theorem 1.1 there holds
\[
\|I_s(f)\|_{L^{\frac{n}{n-s}, \infty}} \geq \gamma_s v_n^{n-s} \|f\|_{L^1}, \ 0 < f \in L^1(\mathbb{R}^n).
\]
Next, we will prove that the equality can be attained by the function $g(x) = \left( \frac{a}{p + |x - x_0|^2} \right)^{\frac{n+s}{2}}$, where $a, b > 0$ and $x_0 \in \mathbb{R}^n$. Since the translation and dilation of $g$ do not change the ratio $\|I_s(g)\|_{L^{\frac{n}{n-s}, \infty}}/\|g\|_{L^1}$, we only need to consider $g(x) = \left( \frac{2}{1 + |x|^2} \right)^{\frac{n+s}{2}}$. In our calculus we will use the stereographic projection, so we will introduce some notations about the stereographic projection here.

The inverse stereographic projection $S : \mathbb{R}^n \to \mathbb{S}^n \setminus \{S\}$, where $S = -e_{n+1}$ denotes the southpole, is given by
\[
(S(x))_i = \frac{2x_i}{1 + |x|^2}, \quad i = 1, \ldots, n, \quad (S(x))_{n+1} = \frac{1 - |x|^2}{1 + |x|^2}.
\]
Correspondingly, the stereographic projection is given by $S^{-1} : \mathbb{S}^n \setminus \{S\} \to \mathbb{R}^n$,
\[
(S^{-1}(\xi))_i = \frac{\xi_i}{1 + \xi_{n+1}}, \quad i = 1, \ldots, n.
\]
And the Jacobian of the (inverse) stereographic projection are
\[
J_S(x) = \left( \frac{2}{1 + |x|^2} \right)^n \text{ and } J_{S^{-1}}(\xi) = (1 + \xi_{n+1})^{-n}.
\]
By changing of variables,
\[\|g\|_{L^1} = \int_{\mathbb{R}^n} \left(\frac{2}{1 + |x|^2}\right)^{\frac{n+s}{2}} dx = \int_{\mathbb{S}^n} \left(\frac{2}{1 + |S^{-1}(\xi)|^2}\right)^{\frac{s+n}{2}} d\xi\]
\[= \int_{\mathbb{S}^n} (1 + \xi_{n+1})^{\frac{s-n}{2}} d\xi = |S^{n-1}| \int_{-1}^1 (1 + t)^{\frac{s-2}{2}} (1 - t)^{\frac{2-n}{2}} dt\]
\[= \pi^{n/2} 2^{\frac{s+n}{2}} \frac{\Gamma(s/2)}{\Gamma(\frac{s+n}{2})}. \quad (3.1)\]

Denote
\[c_{n,s} = \pi^{n/2} 2^{\frac{s+n}{2}} \frac{\Gamma(s/2)}{\Gamma(\frac{s+n}{2})}.\]

Since
\[|S^{-1}(\xi) - S^{-1}(\eta)|^2 = J_{S^{-1}}(\xi)^{\frac{1}{n}}|\xi - \eta|^2 J_{S^{-1}}(\eta)^{\frac{1}{n}}, \text{ for any } \xi, \eta \in \mathbb{S}^n,\]
and
\[\int_{\mathbb{S}^n} \frac{d\eta}{|\xi - \eta|^{n-s}} = \frac{2\pi^{n/2} \Gamma(s/2)}{\Gamma(\frac{n+s}{2})} = \frac{c_{n,s}}{2^{\frac{n-s}{2}}} \text{ for any } \eta \in \mathbb{S}^n \text{ (see D.4 in [5]),}\]
once can obtain
\[I_s(g)(x) = \gamma(s) \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-s}} \left(\frac{2}{1 + |y|^2}\right)^{\frac{s+n}{2}} dy\]
\[= \gamma(s) \int_{\mathbb{R}^n} \frac{1}{|S^{-1}(\xi) - S^{-1}(\eta)|^{n-s}} \left(\frac{2}{1 + |S^{-1}(\eta)|^2}\right)^{\frac{s+n}{2}} d\eta\]
\[= \gamma(s) \int_{\mathbb{S}^n} \frac{1}{|\xi - \eta|^{n-s} |J_{S^{-1}}(\xi)|^{\frac{n-s}{2n}} |J_{S^{-1}}(\eta)|^{\frac{n-s}{2n}} (1 + |S^{-1}(\eta)|^2)^{\frac{s-n}{2}}} d\eta\]
\[= \gamma(s) \int_{\mathbb{S}^n} \frac{d\eta}{|\xi - \eta|^{n-s}} (1 + \xi_{n+1})^{\frac{s-n}{2}} = \gamma(s) \frac{c_{n,s}}{(1 + |x|^{\frac{n-s}{2}})}.\]

Thus for any \(\lambda > 0,\)
\[\left|\{I_s(g) > \lambda\}\right| = v_n \left(\frac{\gamma(s)c_{n,s}}{\lambda} \frac{2}{n-s} - 1\right)^{\frac{2}{2}}. \quad (3.2)\]

Therefore combining (3.1) and (3.2) one have
\[\frac{\|I_s(g)\|_{L^{\frac{n-s}{2}}} \infty}{\|g\|_{L^1}} = v_n^{\frac{n-s}{2}} \sup_{\lambda > 0} \left(\frac{\gamma(s)}{\lambda} \frac{2}{n-s} - \left(\frac{\lambda}{c_{n,s}}\right) \frac{2}{n-s}\right)^{\frac{n-s}{2}} = \gamma(s) v_n^{\frac{n-s}{2}}.\]

Next let us prove the sharp reverse weak estimate for \(M_s.\) By the identity in Theorem 1.1 for \(M_s\) and (1.2) one can find for any \(f \in L^1,\)
\[\|M_s(f)\|_{L^{\frac{n-s}{n}}} \infty \geq \|f\|_{L^1}. \quad (3.3)\]

On the other hand, since \(M_s(f) \leq 1/\gamma(s) v_n^{\frac{n-s}{2}} I_s(f)\) and we already proved that the function \(g = \left(\frac{\alpha}{n+s|x|^{\frac{n-s}{2}} x}\right)^{\frac{s+n}{2}}\) satisfies \(\|I_s(g)\|_{L^{\frac{n-s}{2}}} \infty = \gamma(s) v_n^{\frac{n-s}{2}} \|g\|_{L^1},\) then by (3.3) the following equality holds
\[\|M_s(g)\|_{L^{\frac{n-s}{n}}} \infty = \|g\|_{L^1}. \quad (3.4)\]
In fact, one can prove (3.4) holds for any $L^1$ function $f(x) = h(|x - x_0|)$, where $h$ is a radial decreasing function, by using an approach from [2]. First assume $\|f\|_{L^1} = 1$. Let $\delta_{x_0}$ denote the Dirac delta mass placed at the $x_0$. It is easy to check that

$$M(\delta_{x_0})(x) = \frac{1}{|B(x, |x|)|},$$

where $M$ is the centered Hardy-Littlewood maximum function. Hence, for every $\lambda > 0$, there holds

$$\lambda|\{x : M(\delta_{x_0})(x) > \lambda\}|^{\frac{1}{n-\alpha}} = 1.$$

Since $h$ is a radial decreasing function with $\|h\|_{L^1} = 1$, then by the Lemma 2.1 in [2], one have

$$M(f)(x) \leq M(\delta_{x_0})(x)$$

for every $x \in \mathbb{R}^n$.

Then for any $r > 0$ and $x \in \mathbb{R}^n$,

$$\frac{1}{|B(x, r)|^{1-\alpha}} \int_{B(x, r)} f(y)dy \leq (\frac{1}{|B(x, r)|} \int_{B(x, r)} f(y)dy\|f\|_{L^1}^{\frac{n-\alpha}{n}} \leq (M(\delta_{x_0})(x))^{\frac{n-\alpha}{n}},$$

which implies that

$$\|M_s f\|_{L^{n-\alpha, \infty}} \leq 1 = \|f\|_{L^1}.$$  \hspace{1cm} (3.5)

Combining this inequality with (3.3), one can obtain the desired result for $M_s$.

What is noteworthy at the end of the section is that this result is also true for centered Hardy-Littlewood maximal function. That is because using the same method one can prove (3.5) when $s = 0$, i.e. (3.5) is true for centered Hardy-Littlewood maximal function. On the other hand, using the limiting weak type behavior for maximum function in [7], (3.3) is also true for centered Hardy-Littlewood maximal function.

4. The upper and lower bound of $C_{n,s}$

In this section, we will provide an upper and a lower bound for $C_{n,s}$. Using Theorem 1.1 and (1.2), we can get an upper bound

$$C_{n,s} \leq \gamma_s \frac{n}{s} v_n^{\frac{n-s}{n}}.$$

To obtain the lower bound, we will use the following formula(see section 5.10 in [9]). Let $0 < \alpha < n$, $0 < s < n$ and $\alpha + s < n$, then

$$\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-s}} \frac{1}{|y|^{n-\alpha}} dy = C_{n,\alpha,s} \frac{1}{|x|^{n-s-\alpha}}$$

with

$$C_{n,\alpha,s} = \pi^\frac{n}{2} \frac{\Gamma(\frac{\alpha}{2}) \Gamma(\frac{s}{2}) \Gamma(\frac{n-s-\alpha}{2})}{\Gamma(\frac{n-s}{2}) \Gamma(\frac{n-\alpha}{2}) \Gamma(\frac{2+\alpha}{2})}.$$

Now assume $n - 2 > 4s$. Choose $f(y) = \frac{1}{|y|^{n-2}} \chi(|y| \leq 1)$ and let us prove

$$\|I_{s} f\|_{L^{\frac{n}{n-s}, \infty}} \geq \gamma_s \frac{v_n^{\frac{n-s}{n}}}{s} \frac{n-2-4s}{2(n-2-s)} \|f\|_{L^1}.$$
Since \( |x| \leq \frac{1}{2}, |y| > 1 \) implies \( |y - x| \geq \frac{|y|}{2} \), using (4.1) with \( \alpha = 2 \) one have
\[
\frac{1}{\gamma_s} I_s(f)(x) = \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-s}} f(y) dy = \int_{|y| \leq 1} \frac{1}{|x - y|^{n-s}} \frac{1}{|y|^{n-2}} dy
\]
\[
= \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-s}} \frac{1}{|y|^{n-2}} dy - \int_{|y| > 1} \frac{1}{|x - y|^{n-s}} \frac{1}{|y|^{n-2}} dy
\]
\[
\geq \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-s}} \frac{1}{|y|^{n-2}} dy - \int_{|y| > 1} \frac{2^{n-s}}{|y|^{2n-2s}} dy
\]
\[
= \frac{c}{|x|^{n-s-2}} - d, \tag{4.2}
\]
where
\[
c = \frac{4\pi^{n/2}}{(n-s-2)\Gamma(n/2-1)s} \quad \text{and} \quad d = \frac{2^{n-s+1}\pi^{n/2}}{(n-s-2)\Gamma(n/2)s}.
\]
Choose \( \lambda_0 = \gamma_s(2^{n-s-2}c - d) \), since \( \frac{c}{d} = \frac{n-2}{2n-s} > \frac{1}{2^{n-s-2}} \), then \( \lambda_0 \) is positive. Thus by (4.2), there holds
\[
|\{I_s f > \lambda_0\}| \geq \frac{1}{\gamma_s} \left\{ |x| \leq \frac{1}{2}, \frac{c}{|x|^{n-s-\alpha}} - d > \frac{\lambda_0}{\gamma_s} \right\} = v_n \left( \frac{1}{2} \right)^n. \tag{4.3}
\]
Using the fact \( \|f\|_{L^1(\mathbb{R}^n)} = \omega_{n-1}^{s-1} \) and (4.3) one can obtain
\[
\frac{\|I_s f\|_{L^1(\mathbb{R}^n)}}{\|f\|_{L^1}} \geq \lambda_0 \frac{\|I_s f > \lambda_0\|_{L^1}}{\|f\|_{L^1}} = \lambda_0 v_n^{n-s} \frac{\Gamma(n/2)}{2^{n-s}\pi^{n/2}} = \gamma_s v_n^{n-s} \frac{\Gamma(n/2)}{2(n-2-s)}.
\]
So we complete the proof of Theorem 1.3.

**Appendix**

In this Appendix, we give an alternative approach to prove the \( (L^1, L^{n-s}_{\infty}) \) estimate for Riesz potentials, and at the same time this approach also provide an upper bound for \( C_n \), which have the same behavior with \( \gamma_s v_n^{(n-s)/n} n/s \) as \( (s, n) \rightarrow (0, \infty) \). First, we state a lemma (see section 3 in [13], also see the Hopf abstract maximal ergodic theorem in [3]) about the weak estimate of the average of the heat-diffusion semi-group \( T^t(f) = P_t * f \), where
\[
P_t = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.
\]

**Lemma 4.1.** For any \( f \in L^1(\mathbb{R}^n) \), there holds
\[
|\{ x \in \mathbb{R}^n : \sup_{s > 0} \frac{1}{s} \int_0^s P_t f(x) dt > \lambda \}| \leq \frac{1}{\lambda} ||f||_{L^1(\mathbb{R}^n)}, \lambda > 0.
\]
Now let prove the \( (L^1, L^{n-s}_{\infty}) \) estimate for Riesz potentials \( I_s(f) \), which also can be presented by the following formula related to \( T^t(f) \),
\[
I_s(f)(x) = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s-1} P_t * f(x) dt.
\]
We divide the integral into two parts

\[
\int_0^\infty t^{\frac{s}{2}-1} P_t * f(x) dt = J_1(f)(x) + J_2(f)(x),
\]

where

\[
J_1(f)(x) = \int_0^R t^{\frac{s}{2}-1} P_t * f(x) dt,
\]

\[
J_2(f)(x) = \int_R^\infty t^{\frac{s}{2}-1} P_t * f(x) dt,
\]

for some \( R \) to be determined later.

Denote \( \mathcal{M}^0 f(x) = \sup_{r>0} \frac{1}{r} \int_0^r P_t * f(x) dt \), then we have

\[
J_1(f)(x) = \sum_{i=1}^{\infty} \int_{2^{-i}R}^{2^{i+1}R} t^{\frac{s}{2}-1} P_t * f(x) dt \]
\[
\leq \sum_{i=1}^{\infty} (2^{-i}R)^{\frac{s}{2}-1} 2^{-i+1} R \left( \frac{1}{2^{-i+1} R} \int_0^{2^{-i+1} R} P_t * f(x) dt \right) \]
\[
\leq 2R^{\frac{s}{2}} \frac{2^{-\frac{s}{2}}}{1 - 2^{-\frac{s}{2}}} \mathcal{M}^0 f(x). \tag{4.4}
\]

On the other hand, by direct computation, we obtain that

\[
J_2(f)(x) \leq \int_R^\infty t^{\frac{s}{2}-1} \|P_t\|_{L^\infty} \|f\|_{L^1} dt \]
\[
\leq \frac{2}{n-s} (4\pi)^{-\frac{s}{2}} R^{\frac{s}{2}-\frac{n}{2}} \|f\|_{L^1}. \tag{4.5}
\]

Combining (4.4) and (4.5), we obtain that

\[
I_s(f)(x) \leq \frac{1}{\Gamma(s/2)} \left( 2R^{\frac{s}{2}} \frac{2^{-\frac{s}{2}}}{1 - 2^{-\frac{s}{2}}} \mathcal{M}^0 f(x) + \frac{2}{n-s} (4\pi)^{-\frac{s}{2}} R^{\frac{s}{2}-\frac{n}{2}} \|f\|_{L^1} \right). \tag{4.6}
\]

for all \( R > 0 \). The choice of

\[
R = \left( \frac{(4\pi)^{-\frac{s}{2}} \|f\|_{L^1}}{\mathcal{M}^0 f(x)} \right)^{\frac{2}{n}}
\]

minimizes the right side of the expression in (4.6). Thus

\[
I_s(f)(x) \leq \tau_s(\mathcal{M}^0 f(x))^{-\frac{n}{n-s}} \|f\|_{L^1}^{\frac{s}{n}}, \tag{4.7}
\]

where

\[
\tau_s = 2(4\pi)^{-\frac{s}{2}} (2^{\frac{s}{2}} - 1) \frac{s-n}{n-s} \frac{1}{\Gamma(s/2)}.
\]
Now using lemma 4.1 one can see that
\[
\lambda \{ I_s f > \lambda \}^{\frac{n-s}{n}} \leq \lambda \{ \tau_s (M^0 f(x))^{\frac{n-s}{n}} \| f \|_{L^1}^\frac{1}{n} > \lambda \}^{\frac{n-s}{n}}
\]
\[
\leq \lambda \left( \frac{\tau_s \| f \|_{L^1}^\frac{1}{n}}{\lambda} \right)^{\frac{n-s}{n}} \| f \|_{L^1}^{\frac{n-s}{n}}
\]
\[
\leq \tau_s \| f \|_{L^1}.
\]
Notice that
\[
2^{\frac{s}{n}} - 1 > \frac{\ln 2}{2} s \text{ for } s > 0,
\]
thus,
\[
\tau_s \leq \frac{2}{\ln 2} \left( \frac{1}{4\pi} \right)^{\frac{1}{n}} \frac{1}{\Gamma(\frac{s}{2} + 1)} \frac{n}{n-s}.
\]
So by this approach, one can obtain that
\[
C_{n,s} \leq \frac{2}{\ln 2} \left( \frac{1}{4\pi} \right)^{\frac{1}{n}} \frac{1}{\Gamma(\frac{s}{2} + 1)} \frac{n}{n-s}
\]
and it is easy to check that when \((s, n) \to (0, \infty)\),
\[
\frac{2}{\ln 2} \left( \frac{1}{4\pi} \right)^{\frac{1}{n}} \frac{1}{\Gamma(\frac{s}{2} + 1)} \frac{n}{n-s} = O(\gamma_s v_{n}^{\frac{n-s}{s}})
\]

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