Decompositions of Hilbert spaces, stability analysis and convergence probabilities for discrete-time quantum dynamical semigroups

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Abstract
We investigate convergence properties of discrete-time semigroup quantum dynamics, including asymptotic stability, probability and speed of convergence to pure states and subspaces. These properties are of interest in both the analysis of uncontrolled evolutions and the engineering of controlled dynamics for quantum information processing. Our results include two Hilbert space decompositions that allow for deciding the stability of the subspace of interest and for estimating of the speed of convergence, as well as a formula to obtain the limit probability distribution for a set of orthogonal invariant subspaces.

Keywords: quantum channels, quantum dynamical semigroups, convergence, Perron–Frobenious theory

1. Introduction

Completely positive and trace-preserving (CPTP) maps on operator spaces have long been studied: they gained a central role in quantum open-system theory [1, 2], especially in the operator-algebra formulation [3], and more recently in quantum information theory and applications [4]. In essence, CPTP maps represent the most general evolution maps for the statistical description of quantum systems and thus are the main tool for describing the action of noise and the environment in quantum dynamical systems as well as quantum information processing protocols. In the control of quantum systems, open-system dynamics are needed...
whenever noise or measurements enter the model: CPTP evolutions are then associated to controlled open systems, in open or closed loop, and filtering equations [5].

In terms of the qualitative analysis of the dynamics, the main difference between CPTP maps and unitary (or coherent) evolutions, is that the former allow for contractive dynamics. Two types of dynamical effects associated with CPTP dynamics are typically studied for uncontrolled dynamics: decoherence and ergodicity. Decoherent dynamics describe the decay of the quantum correlations between orthogonal states of a preferred basis and have been one of the main focuses of research concerned the action of environmental noise on quantum systems [6]. However, if we are interested in the engineering of quantum systems for control and information processing protocols, it is typically more interesting to know if we have convergence towards a target unique state, often pure and possibly entangled, or to a subspace [7–20].

In this work, we develop analysis tools that address the issues emerging in problems of dynamical engineering. We consider an invariant subspace of the Hilbert space associated to the quantum system of interest, which we assume throughout the paper to be finite-dimensional. More precisely, we assume that the set of operators with support limited with that subspace is invariant. The class of dynamical models we focus on is associated to iteration of a given CPTP map, i.e. a discrete-time quantum dynamical semigroup (QDS) in the language of [12, 21]. We thus investigate the asymptotic behavior with respect to this subspace, i.e. whether the evolution converges towards it, how fast the convergence is and, the asymptotic probability distribution with respect to smaller invariant subspaces that are comprised in it. Clearly, convergence to a pure state, be it entangled or not, can be seen as a particular case of convergence to a (one-dimensional) subspace.

Some of these problems have been partially addressed for continuous-time CPTP semigroups, notably in the work of Baumgartner et al [22, 23] and Ticozzi et al [24], using linear-algebraic approaches. We here build on these ideas and develop the analysis directly in the discrete-time case. This is more general, as not all discrete-time semigroup can be seen as ‘sampled’ continuous-time semigroups [25], and has its own peculiarities.

We employ an approach based on Perron–Frobenious theory that allows to follow the same path taken in the study of the analogous classical Markov chain properties. We first propose two alternative methods, both based on suitable Hilbert space decompositions in ‘transient subspaces’, to analyse convergence. The first is based on the spectral structure of the linear CPTP map, while the second on the ‘dissipative links’ between the subspaces. They both provide constructive tests for the attractivity of an invariant subspace, and ways to estimate the asymptotic speed of convergence to it. Both the approaches build on existing ergodicity results and representations of CPTP dynamics [12, 26]. In the spirit of [24], we do not assume that the target subspace collects all the invariant sets of states, so the algorithms can be used as tests for the asymptotic stability of any sets.

In the case that the considered subspace is not asymptotically stable, they both allow to extend it to an attractive one. The dissipation-induced decomposition (DID), which is initially presented in Schrödinger’s picture, is further developed and clarified in the dual picture. Finally, we propose some explicit formulas to compute convergence to invariant, yet non attractive subspaces. These build on similar results for Markov chains, and suitable map decompositions.

Sections 2 and 3 presents the key definitions, notations and some basic results on CPTP maps and invariant structures. The two main decompositions and their properties are described in sections 4 and 5, respectively. Section 6 contains the derivations of the formulas for the asymptotic convergence probabilities, while an illustrative example is presented in section 7.
2. Preliminaries

2.1. Definitions and notations

Consider a finite-dimensional Hilbert space $\mathcal{H} \sim \mathbb{C}^d$. Let $\mathfrak{B}(\mathcal{H})$ be the (complex) space of linear operators on $\mathcal{H}$, $\mathfrak{H}(\mathcal{H})$ the (real) space of hermitian operators, and $\mathfrak{H}^+(\mathcal{H})$ the cone of positive semi-definite (PSD) operators. A linear map $T : \mathfrak{H}(\mathcal{H}) \to \mathfrak{H}(\mathcal{H})$ is said to be CPTP if it is:

1. **trace preserving** (TP):
   \[
   \forall A \in \mathfrak{H}(\mathcal{H}) : \quad \text{Tr}(T(A)) = \text{Tr}(A). \tag{1}
   \]

2. **Completely positive** (CP): $\forall n \in \mathbb{N}$, $T \otimes \text{id}_n$ is positive, where $\text{id}_n$ is the identity map on the operators of an Hilbert space of dimension $n$.

   All physically admissible evolutions are CPTP [4]. Consider the set of density operators on $\mathfrak{H}(\mathcal{H})$, or quantum states, $\mathfrak{D}(\mathcal{H})$, namely the set of self-adjoint, positive semi-definite operators with trace one. Clearly, if $T$ is a CPTP map, it leaves $\mathfrak{D}(\mathcal{H})$ invariant. Hence, $T$ can be seen as the generator of a (discrete) QDS in Schrödinger picture by considering iterated applications of the map:
   \[
   \rho(n + 1) = T(\rho(n)), \tag{2}
   \]
   for any $\rho(0) \in \mathfrak{D}(\mathcal{H})$.

   The Stinespring–Kraus representation theorem [2] ensures that a linear map is CP (not necessarily CPTP) if and only if it admits an operator-sum representation (OSR):
   \[
   T(\rho) = \sum_{k=1}^{K} M_k \rho M_k^\dagger, \tag{3}
   \]
   for some $M_k \in \mathfrak{B}(\mathcal{H})$, where $M^\dagger$ is the adjoint of $M$. For $T$ to be also CPTP it is easy to see, using the cyclic property of the trace, that it is necessary and sufficient to require that:
   \[
   \sum_{k=1}^{K} M_k^\dagger M_k = I, \tag{4}
   \]
   where $I$ is the identity operator on $\mathcal{H}$.

   As for any linear map $T$, it is possible to consider its Jordan decomposition:
   \[
   T = \sum_{\lambda_k \in \sigma(T)} \lambda_k P_k + N_k, \tag{5}
   \]
   where $P_k$ is the projection onto the generalized eigenspace in $\mathfrak{H}(\mathcal{H})$ relative to the eigenvalue $\lambda_k$, and $N_k$ is nilpotent with index $d_k$ ($N_k^{d_k} = 0$). The eigenvalues coincides with the roots of the characteristic polynomial of $T$:
   \[
   \Delta_T(\lambda) = \det(\lambda I - T). \tag{6}
   \]
   The nilpotent index $d_k$ is determined by the multiplicities of $\lambda_k$ as a root of the minimal polynomial of $T$, $\psi_T(\lambda)$, i.e. the one with minimal degree among the polynomials $p(\lambda)$ such that $p(T) = 0$. Moreover, the positivity of the map allows for a generalization of Perron–Frobenius theory [26], which guarantees that the set of eigenvalues contains its spectral radius.
The space $S(H)$ is a Hilbert space if endowed with the Hilbert–Schmidt inner product:

$$\langle A|B \rangle = \text{Tr} (AB), \quad A, B \in S(H).$$

(7)

Let $T^*$ be the dual of $T$ with respect to the above inner product. Given an OSR for $T$, it follows from (7) that $T^*$ acts on an element $A \in S(H)$ by:

$$T^* (A) = \sum_k M_k^* A M_k.$$

(8)

Thus, the dual of a CP map is still CP. However, it is immediate to show that the dual of a CPTP map does not need to be TP, but it must be unital, since (4) holds.

2.2. Block representations

To our aim, it is useful to introduce appropriate ‘block’ representations of maps and operators with respect to a decomposition of the underlying Hilbert space. Consider a decomposition of $H$ into two orthogonal subspaces:

$$H = H_S \oplus H_R.$$  

(9)

If we choose a basis:

$$\{ |\varphi_l \rangle \} = \{ |\phi_l^S \rangle \} \cup \{ |\phi_l^R \rangle \},$$

(10)

where $\{ |\phi_l^S \rangle \}$ is a basis for $H_S$ and $\{ |\phi_l^R \rangle \}$ for $H_R$, a block structure is induced on any matrix representing an element $X \in \mathcal{B}(H)$:

$$X = \begin{bmatrix} X_S & X_P \\ X_Q & X_R \end{bmatrix},$$

(11)

and in particular on the matrices $M_k$ appearing in an OSR of $T$.

Building on this decomposition, we can see the space of hermitian operators $S(H)$ as a sum of 3 orthogonal operator subspaces:

$$S(H) = S_S \oplus S_{SR} \oplus S_R,$$

$$S_S = \left\{ \rho \in S(H) : \rho = \begin{bmatrix} \rho_S & 0 \\ 0 & 0 \end{bmatrix} \right\},$$

$$S_{SR} = \left\{ \rho \in S(H) : \rho = \begin{bmatrix} 0 & \rho_P \\ \rho_P^\dagger & 0 \end{bmatrix} \right\},$$

$$S_R = \left\{ \rho \in S(H) : \rho = \begin{bmatrix} 0 & 0 \\ 0 & \rho_R \end{bmatrix} \right\}.$$  

(12)

We shall use the notation $S_S^+$ for the PSD elements in $S_S$, and similarly for $S_R^+$. From (9) and (12) it is possible to construct three ‘reduced’ linear maps:

$$T_S : S_S \rightarrow S_S,$$

$$T_S (A_S) = \sum_k M_k A_S M_k^*,$$

(13)
\[ T_R : \mathcal{S}_R \to \mathcal{S}_R \]
\[ T_R(A_R) = \sum_k M_{k,R} A_R M_{k,R}^\dagger. \quad (14) \]

\[ T_{SR} : \mathcal{S}_R \to \mathcal{S}_S \]
\[ T_{SR}(A_R) = \sum_k M_{k,P} A_R M_{k,P}^\dagger. \quad (15) \]

Notice that these maps can be defined without explicitly referring to an OSR of \( T \), by considering e.g.
\[ T_{SR}(A_R) = \Pi_S T(A_R) \Pi_S, \quad A_R \in \mathcal{S}_R, \quad (16) \]
where \( \Pi_S \) is the orthogonal projection on \( \mathcal{H}_S \).

3. Invariant and asymptotically stable subspaces

3.1. Definitions

Given an \( X \in \mathcal{B}(\mathcal{H}) \), its support is defined as:
\[ \text{supp}(X) = (\ker(X))^\perp, \quad (17) \]
and coincides with its range if \( X \in \mathcal{S}(\mathcal{H}) \). Moreover, given a set of operators \( W \subset \mathcal{S}(\mathcal{H}) \), its support can be defined as the minimal subspace of \( \mathcal{H} \) that contains the supports of all the elements in \( W \):
\[ \text{supp}(W) = \bigvee_{\eta \in W} \text{supp}(\eta), \quad (18) \]
where \( \bigvee \) stands for the sum between subspaces.

Studying how the support of an operator evolves, along with a decomposition like (9), is instrumental to study the properties of invariant subspaces: \( \mathcal{H}_S \) is said invariant if any trajectory starting from an operator with support in it, has its support in \( \mathcal{H}_S \) for all times, or equivalently:
\[ \text{supp}(\rho) \subset \mathcal{H}_S \Rightarrow \text{supp}(T(\rho)) \subset \mathcal{H}_S. \quad (19) \]

A particular class of invariant subspaces, is that one of globally asymptotically stable (GAS). These subspaces are also globally attractive: an invariant \( \mathcal{H}_S \) is said to be GAS if:
\[ \lim_{n \to \infty} \| \Pi_S T^n(\rho) - \Pi_S T^n(\rho) \Pi_S \| = 0, \quad (20) \]
\[ \forall \rho \in \mathcal{D}(\mathcal{H}), \]
where \( \Pi_S \) is the projection onto the subspace. Namely, they are subspaces to which the evolution converges.

3.2. Invariant faces for CPTP maps

We begin by recalling some basic properties of invariant sets for CPTP maps acting on the cone \( \mathcal{S}(\mathcal{H})^+ \). In particular we shall focus on faces of the cone.
Given a CP map \( T \), a face \( F \) is invariant if
\[
\forall M \in F, \quad T(M) \in F.
\]
It is well-known that any face corresponds to the set of PSD operators with support on a subspace. Moreover, the correspondence between faces and subspaces is a lattice isomorphism [27]. Using this correspondence, it is easy to verify that a face is invariant if and only if the corresponding subspace is so. The next lemma shows that in fact to any invariant set corresponds an invariant face.

**Lemma 1.** Let \( T \) be a positive map with \( W \subset \mathcal{S}(\mathcal{H}) \) an invariant set. If \( X \in \mathcal{S}(\mathcal{H}) \) is such that \( \text{supp}(X) \subset \text{supp}(W) \) then \( \text{supp}(T(X)) \subset \text{supp}(W) \).

**Proof.** Let \( \tilde{W} \) be the convex hull of \( W \), so \( \text{supp}(\tilde{W}) = \text{supp}(W) \). \( \tilde{W} \) is invariant and contains an element \( A \) such that \( \text{supp}(\tilde{W}) = \text{supp}(A) \). Thus \( \text{supp}(T(A)) \subset \text{supp}(A) \) since \( T(A) \in \tilde{W} \).

By \( \text{supp}(X) \subset \text{supp}(W) \) there exists a constant \( c > 0 \) such that \( cA \geq X \) so \( T(cA) \geq T(X) \) which implies \( \text{supp}(T(X)) \subset \text{supp}(T(cA)) \).

The next property characterizes the invariance in terms of the OSR of a CP map \( T \), and slightly extends some results of [12].

**Proposition 1.** Suppose \( T \) is a CP map on \( \mathcal{S}(\mathcal{H}) \) and \( \mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R \) then

(i) \( \mathcal{H}_S \) is invariant if and only if in any OSR for \( T \) the matrices \( M_k \) have the block structure:
\[
M_k = \begin{bmatrix} M_{k,S} & M_{k,P} \\ 0 & M_{k,R} \end{bmatrix}.
\]

(ii) \( \mathcal{H}_S \) is invariant if and only if \( \mathcal{S}(\mathcal{H}) \oplus \mathcal{S}(\mathcal{R}) \) is invariant under the action of \( T \).

**Proof.**

(i) Fix an OSR for \( T \) and let \( A \in \mathcal{S}^+ \):
\[
A = \begin{bmatrix} A_S & 0 \\ 0 & 0 \end{bmatrix}.
\]

By applying \( T \) we obtain:
\[
T(A) = \begin{bmatrix} \sum_k M_{k,S}A_SM_{k,S}^* & \sum_k M_{k,S}A_SM_{k,Q}^* \\ \sum_k M_{k,Q}A_SM_{k,L}^* & \sum_k M_{k,Q}A_SM_{k,Q}^* \end{bmatrix}.
\]

If \( \mathcal{H}_S \) is invariant, we must have \( \sum_k M_{k,Q}A_SM_{k,Q}^* = 0 \) for any choice of \( A_S \). Since it is a sum of positive elements, all summands must be zero, irrespective of \( A_S \). This implies that \( M_{k,Q} = 0 \) for any \( k \). Conversely, if \( M_{k,Q} = 0 \) for any \( k \) then for any \( A \in \mathcal{S}^+ \), \( T(A) \in \mathcal{S}^+ \) since all the others blocks are zero as well, and \( \mathcal{H}_S \) is invariant.

(ii) If \( \mathcal{S}(\mathcal{H}) \oplus \mathcal{S}(\mathcal{R}) \) is invariant and \( T \) is positive, then the intersection \( \mathcal{S}^+ = \mathcal{S}(\mathcal{H}) \oplus \mathcal{S}(\mathcal{R}) \cap \mathcal{S}(\mathcal{H}) \) is invariant as well. On the other hand, if \( \mathcal{H}_S \) is invariant consider a generic \( A \in \mathcal{S}(\mathcal{H}) \oplus \mathcal{S}(\mathcal{R}) \),
If we now apply $T$ and use the OSR structure we derived in (21) we obtain the following structure:

$$T(A) = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}$$

and so $\mathcal{S}_S \oplus \mathcal{S}_R$ is invariant.

Using this result we can infer some properties for the reduced maps (13) and (14). Assuming that $\mathcal{H}_S$ is invariant, $T_S$ is just the restriction of $T$ to $\mathcal{S}_S$, while $T_R$ is its effect (pinching) on $\mathcal{H}_R$. Then, from the proposition it follows that that the product of the characteristic polynomials of $T_S$ and $T_R$ divide the characteristic polynomial of $T$, while their minimal polynomials divide that one of $T$. In particular the eigenvalue of $T_S$ and $T_R$ are eigenvalue of $T$ too and, moreover, the index of an eigenvalue of $T_S$ or $T_R$ is lesser or equal than the index of the same eigenvalue with respect to $T$.

4. Nested-face decomposition (NFD)

4.1. Preview of the key results

We are now ready to focus on the main objective of this paper: studying the convergence features of an iterated CPTP map $T$ by identifying ‘transient’ faces of the cone of PSD operators. The first approach proposed construct an increasing sequence of nested faces of PSD operators, each included in all the following ones. For this reason, we will name it NFD. Starting from an invariant subspace, such sequence is associated to a sequence of subspaces $\{\mathcal{H}_i\}$, and it is iteratively built relying on the positivity and spectral properties of the map.

The idea is the following: at each step, the part of the sequence already constructed determines a subspace $\mathcal{H}_S$, and the total space of the system is decomposed in the latter and its orthogonal complement $\mathcal{H}_R = \mathcal{H}_S^\perp$. The next face/subspace $\mathcal{H}_{S+i}$ is obtained as follows: we start by reducing the map $T$ to the orthogonal complement $\mathcal{H}_S^\perp$, and finding the operator eigenspace $D_i$ associate to its slowest eigenvalue (its spectral radius). By adding the support of $D_i$, call it $\mathcal{H}_T$, to $\mathcal{H}_S$, a new element of the nested sequence is obtained:

$$\mathcal{H}_{S+i} = \mathcal{H}_S + \mathcal{H}_T, \quad \mathcal{H}_S \subseteq \mathcal{H}_{S+i}.$$

At the end, the last face corresponds to the whole Hilbert space, which now can be decomposed as:

$$\mathcal{H} = \mathcal{H}_S \oplus \left( \bigoplus_i \mathcal{H}_T \right).$$

We next describe an extremely simple example that can be studied avoiding mathematical complications, and illustrates the basic principles of the decomposition. A more complete application of our results will be described in section 7.

**Toy model: three-level system for NFD.** Consider a three-level system on a Hilbert space spanned by orthonormal vectors $\{|1\}, |2\}, |3\}$, and assume on each time interval of length $T$ it can undergo the following processes: populations on levels $|1\}, |2\}$ are swapped with probability $\gamma_1$, and left as they are with probability $\gamma_0$; level $|3\}$ decays to level $|1\}$ with probability $\gamma_2$, while leaving the other two levels untouched. Assume these processes can be represented, with respect to the reference basis, by the following OSR:
$$M_0 = \sqrt{\gamma_0} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad M_1 = \sqrt{\gamma_1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad M_2 = \sqrt{\gamma_2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (25)$$

Assuming $\sum \gamma_i = 1$ the map is TP. It is easy to see that the pure state $\rho = 1/2(|1\rangle + |2\rangle)(\langle 1| + \langle 2|)$ is invariant for the dynamics. For the purpose of illustration, we can thus try to use the ideas sketched above to analyze if it is GAS, and the convergence properties of the map. Let $H_{S_0} = \text{span} \{ \frac{1}{2}(|1\rangle + |2\rangle) \}$. The positive map reduced to its complement, $H_{R_0} = \text{span} \{ \frac{1}{2}(|1\rangle - |2\rangle), |3\rangle \}$, has spectral radius 1, since $\rho = \frac{1}{2}(|1\rangle - |2\rangle)(\langle 1| - \langle 2|)$ is also fixed. Then we can add $H_{R_1} = \text{span} \{ \frac{1}{2}(|1\rangle - |2\rangle) \}$ to $H_{S_0}$ and obtain $H_{S_1} = H_{S_0} \oplus H_{R_1}$. Now its complement $H_{R_2}$ has spectral radius $1 - \gamma_2 < 1$, and the reduced map is just the multiplication by this scalar. So we have $H_{S_2}$ is an enlargement of $H_{S_1}$ that makes it GAS.

In general, we shall prove that this construction assures that the reduced maps are characterized by strictly decreasing spectral radii, with corresponding eigenvectors having nontrivial support on the $H_{T_i}$. This allows to estimate the asymptotic speed with which the probability decays on the subspaces of the sequence, and thus the speed of convergence to the smaller faces. Finally, this allows us to decide if the starting subspace is GAS or not: it is necessary and sufficient that the spectral radius of the first reduced map is strictly less than one.

4.2. Construction

To start assume $H_{S_0} = H_S$, an initial invariant subspace, and $H_{R_0} = H_R = H \ominus H_S$. The general construction step of $H_{R,i+1}$ and $H_{S,i+1}$ from $H_{R,i}$ and $H_{S,i}$ can be carried out as follows:

1. define

$$D_i := \ker \left( T_{R,i} - \sigma (T_{R,i}) I \right)^{d_i} \cap \sigma_{T_{R,i}}^+, \quad (26)$$

$$H_{T_i} := \text{supp}(D_i), \quad (27)$$

where $d_i = \text{dim}(H_{R,i})$, and $\sigma(T_{R,i})$ is the spectral radius of $T_{R,i}$. Since $T_{R,i}$ is CP, $D_i$ contains elements different from zero [26]. Furthermore $D_i$ is the intersection of two invariant sets for $T_{R,i}$, so it is invariant. By lemma 1 $H_{T_i}$ is then invariant as well;

2. call $H_{S,i+1} = H_S \oplus H_{T_i}$. Notice that by construction this is an invariant subspace for $T$;

3. if $H_{T_i} = H_{R,i}$ the construction is ended, otherwise define $H_{R,i+1} = H_{S,i+1}^\perp$, and iterate.

Since at any step $H_{T_i}$ is not the zero space, $H_{S,i+1}$ has dimension strictly larger than the dimension of $H_{S,i}$, so in a finite number of steps its dimension must reach the dimension of $H$ and the construction stops. When this is the case, we have obtained a chain of subspaces with the following properties:

$$H_{S,i} = H_S, \quad (28)$$
Alternatively, by using the $\mathcal{H}_E$ we introduced in equation (27) of the iterative construction, this decomposition can be rewritten as

$$\mathcal{H} = \mathcal{H}_S \oplus \bigoplus_i \mathcal{H}_E \, .$$

(31)

It is then easy to see that, in a basis that reflect this structure, the matrices $M_k$ are in a block upper-triangular form, with the first diagonal block corresponding to $\mathcal{H}_S$ and each other diagonal block corresponding to one of the $\mathcal{H}_E$.

### 4.3. NFD Properties

**Proposition 2.** If the chain $\mathcal{H}_{(29)}$ is constructed as above, then the spectral radii of the reduced $T_R$ on $\mathcal{H}_R = \mathcal{H}_S \perp \mathcal{H}_S$ satisfy:

$$\sigma(T_R) > \sigma(T_{R_{i+1}}) \, .$$

(32)

**Proof.** See appendix.

This decomposition thus provides us with a nested sequence of faces to which the cone of PSD matrices asymptotically converge. In fact, if each of the $\mathcal{H}_E$ is characterized by a spectral radius strictly lesser than one, these subspaces tend to correspond to zero probability asymptotically. The next proposition shows how this is naturally related to attractivity.

**Proposition 3.** The invariant subspace $\mathcal{H}_S$ is GAS if and only if $\sigma(T_R) < 1$. If this is not the case, $\mathcal{H}_S = \mathcal{H}_S \oplus \mathcal{H}_R$ is then the minimal GAS subspace containing $\mathcal{H}_S$.

**Proof.** If $\sigma(T_R) < 1$ the support of any fixed points for $T$ is contained in $\mathcal{H}_S$ so by the results in [12] $\mathcal{H}_S$ is GAS. If $\sigma(T_R) = 1$ then $\sigma(T_{R_i}) < 1$ and $\mathcal{H}_S$ is GAS. Moreover there exist a density operator $\rho$ with $\text{supp}(\rho) = \mathcal{H}_R$ and $T_R(\rho) = \rho$, where the last is a consequence that for $T1$ has only simple eigenvectors [26] and so the same holds for $T_R$, by this any GAS subspace must contain $\mathcal{H}_R$.

We conclude the section with two remarks. First, all the construction is based on the maps $T_R$, which do not depend on a particular OSR, but only on the (whole) map $T$. Secondarily, if $\mathcal{H}_S$ is taken to be the support of the whole fixed points subspace, the above construction is similar in principle to the decomposition in ‘transient subspaces’ that has been proposed in [23] for continuous time evolutions. However, beside being developed for discrete-time semigroups, our results differs from the above in the following aspects: (i) we allow for the initial subspaces to be any invariant subspace, a generalization that is of interest in many control and quantum information protection tasks [9–11, 24, 28, 29]; (ii) we investigate the structure and the spectral properties of the emerging $\mathcal{H}_R$; (iii) we use the latter as tool for deciding asymptotic stability of the subspace.
5. DID decomposition

5.1. Preview of the key results

The second decomposition we study is the discrete-time version of a decomposition first proposed in [24] for continuous QDS. The starting point is again an invariant subspace $\mathcal{H}_S$, but in this case we construct a chain of subspaces that have to be ‘crossed’ by the state trajectory in order to reach the invariant subspace $\mathcal{H}_S$, also in the form:

$$\mathcal{H} = \mathcal{H}_S \oplus \left( \bigoplus_i \mathcal{H}_F \right).$$

The idea underlying the construction is the following: as in the case of the nested-face decomposition, at each step of the construction we start with the subspaces of the chain have already been obtained, namely $\mathcal{H}_S = \mathcal{H}_S \oplus \mathcal{H}_F \oplus \ldots \mathcal{H}_{E_i}$. The full space is then decomposed into $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$. The new element $\mathcal{H}_F$ is constructed as the part of $\mathcal{H}_R$ that is dynamically connected to the previous transient subspace $\mathcal{H}_{E_i}$, that is, the states with support on $\mathcal{H}_F$ will be mapped by $T$ into states that have support also on $\mathcal{H}_{E_i}$.

The iterative construction ensures that $\mathcal{H}_F$ can be connected only to the last $\mathcal{H}_{E_i}$, and allows for an approximate estimation of the rates at which the density operators with support on one of the $\mathcal{H}_F$ tend to flow towards $\mathcal{H}_S$, following the chain of subspaces. Finally, it is easy to use the construction to determine stability of the initial $\mathcal{H}_S$: if the last subspace is linked to the previous one, the whole chain is connected and we shall prove that this makes $\mathcal{H}_S$ GAS.

Let us briefly revisit the toy model we introduced in order to illustrate the NFD, and intuitively show how the DID procedure works and differs from the latter.

**Toy model: three-level system for DID.** Consider again the three-level system on a Hilbert space spanned by orthonormal vectors $\{|1\rangle, |2\rangle, |3\rangle\}$, and assume on each time interval of length $T$ undergoes a CPTP evolution associated to OSR $\{M_0, M_1, M_2\}$ as in (25).

Being the state $\rho_S = \frac{1}{2}(|1\rangle + |2\rangle)(|1\rangle + |2\rangle)$ invariant, we can start by the invariant subspace $\mathcal{H}_S = \text{span}\{\frac{1}{2}(1 + |2\rangle)\}$. The only part of the Hilbert space dynamically connected to $\mathcal{H}_S$ is $\mathcal{H}_F = \text{span}\{|3\rangle\}$, via the decay process associated to $M_2$. So we can define $\mathcal{H}_S = \mathcal{H}_S \oplus \mathcal{H}_F$ and see how the remaining $\mathcal{H}_R = \text{span}\{\frac{1}{2}(|1\rangle - |2\rangle)\}$ is connected to the preceding. It is clear that while there is decay from $\mathcal{H}_F$ to $\mathcal{H}_R$, the viceversa is not true and $\mathcal{H}_R$ remains invariant. Thus, $\mathcal{H}_S$ cannot be GAS.

So while $\mathcal{H}_F$ for the NFD was span $\{\frac{1}{2}(|1\rangle - |2\rangle)\}$, being the support of the spectral radius eigenoperator of the reduced map, in this case it is span $\{|2\rangle\}$, collecting the part of the Hilbert space that is dynamically connected to the initial invariant subspace. □

5.2. Construction

Fix an OSR $\{M_t\}$ for $T$. To start, let $\mathcal{H}_S = \mathcal{H}_S$, an initial invariant subspace, and $\mathcal{H}_R = \mathcal{H}_R = \mathcal{H}_S^2$. We proceed iteratively: at each step we start from a decomposition of the form

$$\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R,$$  \hspace{1cm} (33)
where
\[ H_S = H_S \oplus \bigoplus_{j=1}^{i-1} H_T \] (34)

Let \( M_{k,P} \) and \( M_{k,R} \) be the \( P \)- and \( R \)-blocks in (21), with respect to the decomposition (33). First \( H_{R_{i+1}} \) is defined as follows
\[ H_{R_{i+1}} = \bigcap_k \ker(M_{k,P}). \] (35)

Three cases are possible:

1. \( M_{k,P} = 0 \) for all \( k \), i.e. \( H_{R_{i+1}} = H_R \). In this case \( H_T = H_R \), so that \( H_{S_{i+1}} = H \), and the construction is terminated.
2. \( H_{R_{i+1}} = \{0\} \): in this case the construction is concluded. Again \( H_T = H_R \).
3. If none of the above cases applies, we choose as \( H_T \) the orthogonal complement of \( H_{R_{i+1}} \) in \( H_R \)
\[ H_T = H_T \oplus H_{R_{i+1}}. \] (36)

Since degenerate cases have been dealt separately, both \( H_T \) and \( H_{R_{i+1}} \) have positive dimension and then \( H_{R_{i+1}} \) has dimension strictly less than \( H_R \). Define \( H_{S_{i+1}} = H_S \oplus H_T \) and iterate.

Notice that at any step the dimension of \( H_R \) strictly decreases, or the procedure is halted. Thus, in a finite number of steps the procedure terminates. When the algorithm stops the subspace is decomposed in a direct sum of the form
\[ H = H_S \oplus \bigoplus_{j=1}^{N} H_T \] (37)

We shall call this the discrete-time DID. In addition, we will call DID constructions concluded by case 2 above successful, while those terminated by case 1 unsuccessful. The reason will be clarified in proposition 4, where we show that case 2 corresponds to a GAS initial subspace.

When we choose a basis according to the DID, the matrices \( M_k \) of the OSR acquire a specific block structure, in which one block depends on which case has stopped the construction. If the DID terminates successfully, as in case 2, then we get:

\[ M_k = \begin{bmatrix}
    M_{k,S} & M_{k,R} & 0 & \ldots & 0 \\
    0 & M_{k,T_1} & M_{k,P_1} & 0 & \ldots \\
    0 & M_{k,Q_1} & M_{k,T_2} & \ddots & \vdots \\
    \vdots & \vdots & \ddots & \ddots & \ddots \\
    0 & M_{k,Q_{N-1,1}} & M_{k,Q_{N-1,2}} & \ldots & M_{k,T_{N-1}} & M_{k,P_N} \\
    0 & M_{k,Q_N,1} & M_{k,Q_N,2} & \ldots & M_{k,Q_{N,N-1}} & M_{k,T_N}
\end{bmatrix} \]

\[ k = 1, \ldots K, \] (38)

with \( \cap_k \ker(M_{k,P}) = 0 \) for \( i = 1, \ldots N \). The zero blocks in the first column indicate that \( H_S \) is invariant, while those over the first upper diagonal show that a state with support on one of the \( H_T \) can only transition to the preceding subspace in the chain \( H_{T_{i-1}} \) and cannot e.g. ‘jump’ directly to \( H_S \).
If instead the decomposition is terminated due to case 1, we obtain:

\[
\begin{bmatrix}
M_{k,S} & M_{k,T} & 0 & \ldots & 0 \\
0 & M_{k,T_1} & M_{k,P} & 0 & \ldots & 0 \\
0 & M_{k,Q_{1,1}} & M_{k,T_2} & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & M_{k,Q_{N-1,1}} & M_{k,Q_{N-1,2}} & \ldots & M_{k,T_{N-1}} & 0 \\
0 & M_{k,Q_{N,1}} & M_{k,Q_{N,2}} & \ldots & M_{k,Q_{N,N-1}} & M_{k,T_N}
\end{bmatrix}
\]

\[k = 1, \ldots K.\]  

(39)

In this case, the first \(N - 1\) zero blocks in the last column, on top of \(M_{k,T_i}\), for all \(k\), imply the invariance of the last subspace, \(H_{T_N}\).

5.3. DID properties

As in the previous case, the obtained decomposition allows us to decide the asymptotic stability of the face.

**Proposition 4.** \(H_S\) is GAS if and only if the DID is terminated successfully.

**Proof.** If the DID is terminated due to case 1 the matrices have the structure (39). By a reordering of the basis, and hence of the blocks, and by using proposition 1, is easily seen that \(H_{T_N}\) is invariant. Hence \(H_S\) cannot be GAS.

If instead the DID is terminated successfully it is possible to verify that no fixed point has support in \(H_R\), and by the results of [12] this is equivalent to show that the target subspace is attractive. Suppose by contradiction that \(\rho\) is a density operator with support in \(H_R\) and a fixed point for the map, i.e.:

\[
\text{supp}(\rho) \subset \bigoplus_{i=2}^{N} H_{T_i}.
\]

(40)

Then there exists a maximal \(j\) between 2 and \(N\) (note that the next subspace is minimal) such that

\[
\text{supp}(\rho) \subset \bigoplus_{i=j}^{N} H_{T_i}.
\]

(41)

Then \(\rho_{T_j}\) (the block of \(\rho\) corresponding to \(H_{T_j}\), when it is decomposed accordingly to the DID) is not zero, otherwise, due to positiveness of \(\rho\), the corresponding columns and rows should be zero and (41) would hold with \(j + 1\). Using the block structure (38) and the fact that \(\rho\) is a fixed point, we must have:

\[
T(\rho)_{T_{j-1}} = \sum_{k} M_{k,T_j} \rho_{T_j} M_{k,T_j}^\dagger = 0,
\]

(42)

this is a sum of PSD matrices so any terms must be zero.

If \(\rho_{T_j} = C C^\dagger\) with \(C \in \mathcal{B}(H_T)\) using the last

\[
M_{k,T_j} C = 0,
\]

(43)

for any \(k\), then the columns of \(C\) should be in \(\bigcap_k \ker(M_{k,T})\) which is impossible by construction, since at each step we choose \(\bigcap_k \ker(M_{k,T}) = \{0\}. \]

\[\square\]
Suppose now that the DID is completed successfully, and that we are interested in estimating the convergence speed to the target subspace. Fix a state \( \rho \), with support in \( \mathcal{H}_T \), i.e.
that verify:
\[
\Pi_T \rho \Pi_T = \rho,
\]
if \( \Pi_T \) is the orthogonal projection onto \( \mathcal{H}_T \). Consider the probability of finding the system in the preceding subspace \( \mathcal{H}_{T_{-1}} \) after one application of \( T \):
\[
P_T(\rho) \left( \Pi_{T_{-1}} \right) = \text{Tr} \left( \Pi_{T_{-1}} T(\rho) \right),
\]
where \( \Pi_{T_{-1}} \) is the orthogonal projection onto \( \mathcal{H}_{T_{-1}} \). It is possible to obtain bounds on the growth of this probability (recall it was zero before the action of the map):
\[
P_T(\rho) \left( \Pi_{T_{-1}} \right) = \text{Tr} \left( \Pi_{T_{-1}} T(\rho) \right)
= \text{Tr} \left( \sum_k M_{k,R} \rho_T M_{k,R}^\dagger \right)
= \text{Tr} \left( \sum_k M_{k,R}^\dagger M_{k,R} \rho_T \right).
\]
In fact, the increment in probability is at least
\[
\gamma_i \text{Tr} \left( \rho_T \right),
\]
if \( \gamma_i \) is the least eigenvalue of \( \sum_k M_{k,R}^\dagger M_{k,R} \), as is seen putting \( \sum_k M_{k,R}^\dagger M_{k,R} \) in its diagonal form. In the same way it is possible to give an upper bound for (45) by the maximum eigenvalue of \( \sum_k M_{k,R}^\dagger M_{k,R} \). Note that these bounds are always meaningful: \( \sum_k M_{k,R}^\dagger M_{k,R} \) is positive definite by construction, so its lowest eigenvalue cannot be zero. Moreover, exists a density operator (e.g. the projection on the subspace generated by an eigenvector corresponding to \( \gamma_i \)) for which the lower bound is reached, and the same is true for the upper bound (this shows also that the maximum eigenvalue is less than or equal to one).

In the light of these observations, the \( \gamma_i \) are indications of the (maximal and minimal) probabilities that a transition from \( \mathcal{H}_T \) to \( \mathcal{H}_{T_{-1}} \) occurs. Knowing all these transition rates we can use the minimal ones to find the convergence bottlenecks, and estimate the worst-case time needed to reach \( \mathcal{H}_S \) starting from any state.

It should be noted that the bounds derived are significant only when (44) holds, due to the possible presence of blocks which connect the subspaces \( \mathcal{H}_T \) in the opposite way, namely making probability flow down the chain. However, since we assume \( \mathcal{H}_S \) to be GAS, the transitions towards it ‘dominates’ the dynamics, so the \( \gamma_i \) can be effectively used to estimate the convergence speed.

5.4. Dual characterization

The DID can be also studied, and in fact characterized, in the Heisenberg picture. In this dual framework some properties become more explicit, e.g. its independence from the chosen OSR. The following characterization of invariant subspaces which refers to the dual map will be needed later.

**Proposition 5.** Let \( T \) be a CPTP map, and \( T^\dagger \) its dual. A subspace \( \mathcal{H}_S \) is invariant for \( T \) if and only if for any \( n \)
Proof. By unitality of the dual map we have
\[
\Pi_S + \Pi_R = I = T^*(I) = T^*(\Pi_S + \Pi_R) = T^*(\Pi_S) + T^*(\Pi_R).
\] (49)

Given proposition 1, it is easy to see that if \( H_S \) is invariant for \( T \) then \( H_R \) is invariant for \( T^* \). Hence we have that \( T^*(\Pi_R) = T_R^*(\Pi_R) \), which has support only on \( H_R \). Hence, for (49) to be true, it must be
\[
T^*(\Pi_S) = T_S^*(\Pi_S) + T_{SR}^*(\Pi_S) = \Pi_S + T_{SR}^*(\Pi_S).
\] (50)

Applying \( n \)-times \( T^* \) and using the invariance of \( H_R \) for the dual map, we thus have:
\[
T^{*n}(\Pi_S) = \Pi_S + \sum_{i=1}^{n-1} T_{SR}^*(\Pi_S).
\] (51)

Since \( T_{SR}^*(\Pi_S) \) is non decreasing. Suppose now
\[
T^*(\Pi_S) \geq \Pi_S.
\] (52)

Let \( H_R \) be the orthogonal complement of \( H_S \) so that
\[
\Pi_S + \Pi_R = I,
\] (53)

and then
\[
T^*(\Pi_S) + T^*(\Pi_R) = I.
\] (54)

Rewriting these two terms in their block form
\[
T^*(\Pi_S) = \sum_k \begin{bmatrix}
M_{k,S}^+ M_{k,S} & M_{k,S}^+ M_{k,P} \\
M_{k,P} M_{k,S} & M_{k,P} M_{k,P}
\end{bmatrix},
\] (55)
\[
T^*(\Pi_R) = \sum_k \begin{bmatrix}
M_{k,Q}^+ M_{k,Q} & M_{k,Q}^+ M_{k,R} \\
M_{k,R} M_{k,Q} & M_{k,R} M_{k,R}
\end{bmatrix},
\] (56)

now by unitality
\[
\sum_k M_{k,S}^+ M_{k,S} \leq I,
\] (57)

and from (52)
\[
\sum_k M_{k,S}^+ M_{k,S} \geq I,
\] (58)

so
\[
\sum_k M_{k,S}^+ M_{k,S} = I.
\] (59)
But now (54), together with (56), implies
\[ M_{k,O}^LM_{k,O} = 0, \]  
(60)
for any \( k \), and by proposition 1 invariance of \( \mathcal{H}_S \) is proved. \( \square \)

So a subspace is invariant if the sequence generated by the corresponding orthogonal projection is monotonically non decreasing. From this result immediately follows:

**Corollary 1.** Let \( T \) be a CPTP map and \( \mathcal{H}_S \) a subspace:

(i) \( \mathcal{H}_S \) is invariant if and only if \( T^n(\Pi_S) \supseteq \Pi_S \).

(ii) If \( \mathcal{H}_S \) is invariant then for any \( n \)
\[ \text{supp}(T^n(\Pi_S)) \subseteq \text{supp}(T^{n+1}(\Pi_S)). \]  
(61)

The first point is already known, and projections satisfying this property are called sub-harmonic [26]. Using these results, we can give a dual characterization of the DID.

**Proposition 6.** Let \( T \) be a CPTP map, \( \mathcal{H}_S \) a GAS subspace and consider the decomposition induced by the DID. Then for \( n \leq N \)
\[ \text{supp}(T^n(\Pi_S)) = \mathcal{H}_S \oplus \bigoplus_{i=1}^n \mathcal{H}_{T^i} \]  
(62)

and for \( n > N \) the support is the whole \( \mathcal{H} \).

**Proof.** We shall prove that (62) holds by induction on \( n \). First consider the case \( n = 1 \). By using the matrix block-decomposition provided in (38), it is easy to show that:
\[ T^*(\Pi_S) = \sum_k M_k^* \Pi_S M_k \]

\[ = \sum_k \begin{bmatrix} M_{k,S}^* M_{k,S} & M_{k,R}^* M_{k,R} & 0 & \cdots & 0 \\ M_{k,S}^* M_{k,S} & M_{k,R}^* M_{k,R} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \]  
(63)

Recall that by (61)
\[ \mathcal{H}_S \subseteq \text{supp}(T^*(\Pi_S)). \]  
(64)

so, by (63), it suffices to show that \( \mathcal{H}_{T^1} \) is contained in \( \text{supp}(T^*(\Pi_S)) \). Choosing a set of vector \( |\varphi_{1,h}\rangle \) in \( \mathcal{H}_{T^1} \), applying \( T^*(\Pi_S) \) results in:
By a proper choice of $|\phi_{l,h}\rangle$, it is possible to obtain that the $|\phi_{l,h}\rangle$ are a basis for $\mathcal{H}_T$, since by construction the range of $\sum_k M_{L,R}^k M_{L,R}^\dagger$ is $\mathcal{H}_T$. Hence, the case $n = 1$ is completed.

Now assume (62) true for $n < l$, and call $\Pi_l$ the projection on $\bigoplus_{j=1}^l \mathcal{H}_j$. So that

$$T^*(\Pi_l) |\psi_{l,h}\rangle = \begin{bmatrix}
M_{L,R}^l M_{L,R}^\dagger & |\phi_{l,h}\rangle \\
M_{L,R}^l M_{L,R}^\dagger & |\phi_{l,h}\rangle \\
0 & |\phi_{l,h}\rangle \\
\vdots & \vdots \\
0 & |\phi_{l,h}\rangle \\
0 & |\phi_{l,h}\rangle \\
\end{bmatrix} |\phi_{l,h}\rangle = |\psi_{l,h}\rangle.$$  

(65)

and for some real constant $c > 0$

$$c\Pi_{l-1} \leq T^{n-l}(\Pi_l) \leq C\Pi_{l-1}.$$  

(67)

This implies that the support of $T^*(\Pi_{l-1})$ and $T^{n-l}(\Pi_l)$ is the same and the statement is proved if $\text{supp}(T^*(\Pi_{l-1}))$ coincides with $\mathcal{H}_S \bigoplus \left( \bigoplus_{j=1}^l \mathcal{H}_j \right)$, which can be verified proceeding as above. By (61) in corollary 1, we have:

$$\mathcal{H}_S \bigoplus \left( \bigoplus_{j=1}^{l-1} \mathcal{H}_j \right) \subset \text{supp}(T^*(\Pi_l)) = \text{supp}(T^*(\Pi_{l-1})).$$  

(68)

Applying $T^*(\Pi_{l-1})$ to a set of vectors $|\phi_{l,h}\rangle$ in $\mathcal{H}_T$ results in:

$$T^*(\Pi_{l-1}) |\phi_{l,h}\rangle = \sum_k M_{L,R}^k M_{L,R}^\dagger |\phi_{l,h}\rangle = |\psi_{l,h}\rangle.$$  

(69)

where $|\phi_{l,h}\rangle = \sum_k M_{L,R}^k M_{L,R}^\dagger |\phi_{l,h}\rangle$ and $|\psi_{l,h}\rangle$ accounts for the first blocks of elements. By the (69) $|\psi_{l,h}\rangle \oplus |\phi_{l,h}\rangle$ are in $\text{supp}(T^*(\Pi_{l-1}))$. Again choosing properly $|\phi_{l,h}\rangle$, and noting that
\[ \sum \alpha_{i} M_{\alpha} M_{\alpha} \] is strictly positive, we prove that the desired property (62) holds for \( l \) as well. Finally, the last statement follows directly from the first and the hypothesis of \( \mathcal{H}_S \) being GAS, which in turn implies that the DID algorithm runs to completion and \( \mathcal{H}_S \left( \bigoplus_{i=1}^{N} \mathcal{H}_{T_i} \right) = \mathcal{H} \).

So the DID is determined by, and is in fact equivalent to, the sequence of supports (62). Since it depends only from the form of \( T^* \) we readily obtain that the DID is independent from the chosen OSR. From the last result another useful property is also obtained.

**Corollary 2.** Suppose \( \mathcal{H}_S \) is an invariant subspace for the CPTP map \( T \). Then it is GAS if and only if the sequence \( \text{supp}(T^{(n)}(\mathcal{H}_S)) \) is strictly increasing until it covers the whole space.

**Proof.** One implication is just a restatement of the last proposition. For the converse, it suffices to follow the proof, taking into account that the failure of the DID returns the block structure of the form (39), from which the sequence \( \text{supp}(T^{(n)}(\mathcal{H}_S)) \) cannot cover the whole space.

### 6. Asymptotic probabilities

The two decompositions we introduced in the previous sections essentially study the transient structure and dynamical behavior of a subspace \( \mathcal{H}_R \), complementary to \( \mathcal{H}_S \), when the last one is invariant. When \( \mathcal{H}_S \) is GAS, we can further investigate its ‘internal’ asymptotic behavior, and understand where the system will converge to, and with which probability.

If a subspace \( \mathcal{H}_S \) is GAS, it must contain the supports of all fixed points. It is well known (see e.g. [26, 30]), that for a CPTP map \( T \) the subspace of fixed points has a structure of the form:

\[
\mathcal{T}_T = U^\dagger \left( \bigoplus_{k=1}^{K} \mathcal{M}_{d_k} \otimes \rho_k \bigoplus 0 \right) U,
\]

where \( U \) is a unitary operator, \( \mathcal{M}_{d_k} \) stands for the full algebra of complex matrices on \( \mathcal{C}^{d_k} \) and \( \rho_k \) are positive definite density matrices. When the action of \( T \) is restricted to the convex, invariant set of density operators, to find the invariant sets it is enough to substitute the full algebras with the sets of density operators contained in the \( \mathcal{M}_{d_k} \). This highlights a direct-sum structure for the minimal GAS subspace (or collecting subspace [22]):

\[
\mathcal{H}_S = \bigoplus_{k=1}^{K} \mathcal{H}_S_k,
\]

with all the \( \mathcal{H}_S_k \) invariant, being supports of invariant states. This decompostion of the minimal GAS subspace is of particular interest in quantum information applications, since to each component \( \mathcal{H}_S_k \) remains associated an information preserving structure [31]: namely, the subspace \( \mathcal{H}_S_k \) contains a perfectly noiseless subsystem of dimension \( d_k \) that allows to store and preserve quantum information [29].

An interesting question is whether the dynamics will drive the state into a protected subsystem of interest, and with what probability this will happen, depending on the initial state of the dynamics. The first part of the question can be addressed by checking if the support of the noiseless subsystem is GAS. If this is not case, it is possible to derive explicit formulas for the asymptotic probabilities to find the state in one of the \( \mathcal{H}_S_k \), i.e. to evaluate
given the initial state $\rho$, where $\Pi_S$ is the orthogonal projection on $\mathcal{H}_S$. A particular case of interest emerges when all the subspaces are one-dimensional, and the evolution induces decoherence with respect to this (partial) orthogonal preferred basis. Given an initial state, what is the probability of finding it in one of the orthogonal and pure pointer states?

In deriving a suitable tool to answer these questions, a key preliminary result is represented by the following lemma.

**Lemma 2.** Let $T$ be a CPTP map, $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$, where $\mathcal{H}_S = \bigoplus_{k=1}^K \mathcal{H}_{S_k}$ with each of the $\mathcal{H}_{S_k}$ being invariant. If $\Pi_{S_k}$ is the orthogonal projection on $\mathcal{H}_{S_k}$ then for any $i$

$$T^*(\Pi_{S_i}) = \Pi_{S_i} + T_{S\bar{S}}(\Pi_{S_i}).$$

**Proof.** We will first explicitly prove the statement for $K = 2$. Given that the $\mathcal{H}_{S_1,2}$ are invariant, with respect to the orthogonal sum $\mathcal{H}_S = \bigoplus_{k=1}^K \mathcal{H}_{S_k}$, the matrices $M_k$ have the block-structure:

$$
\begin{bmatrix}
M_{k,S_1} & 0 & M_{k,R} \\
0 & M_{k,S_2} & M_{k,P} \\
0 & 0 & M_{k,R}
\end{bmatrix}.
$$

Taking into account the unitality condition, by the block form we derive the relations:

$$
\sum_k M_{k,S_i}^\dagger M_{k,S_i} = I, \quad \text{(75)}
$$

$$
\sum_k M_{k,R}^\dagger M_{k,S_i} = 0, \quad \text{(76)}
$$

$$
\sum_k M_{k,S_2}^\dagger M_{k,S_i} = I, \quad \text{(77)}
$$

$$
\sum_k M_{k,P}^\dagger M_{k,S_i} = 0. \quad \text{(78)}
$$

Let us focus on $\mathcal{H}_{S_1}$, as the same reasoning applies to $\mathcal{H}_{S_2}$ up to a relabeling. In the same block-representation, the projection of interest is

$$
\Pi_{S_1} = \begin{bmatrix}
I & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

\text{(79)}
we thus have:

\[ T^*(\Pi_i) = \sum_k \begin{bmatrix} M_{k,S_i}^+ M_{k,S_i} & 0 & M_{k,S_i}^+ M_{k,R} \\ 0 & 0 & 0 \\ M_{k,R}^+ M_{k,S_i} & 0 & M_{k,R}^+ M_{k,R} \end{bmatrix} \]

\[ = \Pi_{S_i} + T_{SR} \left( \Pi_{S_i} \right). \quad (80) \]

In the general case \( \mathcal{H}_S = \bigoplus_j \mathcal{H}_{S_j} \), for any \( j = 1, \ldots, K \) we can consider the decomposition \( \mathcal{H}_S = \mathcal{H}_{S_j} \oplus \bigoplus_{i \neq j} \mathcal{H}_{S_i} \). These two orthogonal subspaces in the sum are both invariant so by the reasoning above the evolution of \( \Pi_{S_j} \) has the desired form. □

Using the above lemma, we can then provide a formula to compute the asymptotic probability analytically, depending on the initial state.

**Proposition 7.** Under the hypothesis of the preceding lemma, assume also that \( \mathcal{H}_S \) is GAS, then

\[ \lim_{n \to \infty} \text{Tr} \left( \Pi_{S_i} T^n(\rho) \right) = \text{Tr} \left( \Pi_{S_i} \rho_{S_i} \right) + \text{Tr} \left( \Pi_{S_i} T_{SR} \left( (I - T_R)^{-1}(\rho_R) \right) \right). \quad (81) \]

**Proof.** The limit of \( \Pi_{S_i} \) under the action of \( T^* \) is easily computed:

\[ T^* \left( \Pi_{S_i} \right) = \Pi_{S_i} + T_{SR} \left( \Pi_{S_i} \right), \]

\[ T^{*2} \left( \Pi_{S_i} \right) = \Pi_{S_i} + \left( T_R^* \left( T_{SR} \left( \Pi_{S_i} \right) \right) + T_{SR} \left( \Pi_{S_i} \right) \right), \]

\[ \vdots \]

\[ T^{*n} \left( \Pi_{S_i} \right) = \Pi_{S_i} + \sum_{k=0}^{n-1} T_R^k \left( T_{SR} \left( \Pi_{S_i} \right) \right). \quad (82) \]

Letting \( n \) go to infinity

\[ \lim_{n \to \infty} T^{*n} \left( \Pi_{S_i} \right) = \lim_{n \to \infty} \Pi_{S_i} + \left( \sum_{k=0}^{n-1} T_R^k \left( T_{SR} \left( \Pi_{S_i} \right) \right) \right) \]

\[ = \Pi_{S_i} + \left( (I - T_R)^{-1}(T_{SR} \left( \Pi_{S_i} \right)) \right). \quad (83) \]

where the last equality follows from the fact that since \( \sigma(T_R) < 1 \)

\[ \sum_{k=0}^{\infty} T_R^k = (I - T_R)^{-1}. \quad (84) \]

By using the relation:

\[ \lim_{n \to \infty} \text{Tr} \left( \Pi_{S_i} T^n(\rho) \right) = \lim_{n \to \infty} \text{Tr} \left( T^{*n} \left( \Pi_{S_i} \right) \rho \right). \quad (85) \]

we get the statement. □
In conclusion, the asymptotic probability of converging to an invariant subspace inside the minimal GAS subspace is given by the sum of two terms: the initial probability of finding the state there (the term $\text{Tr} \left( \Pi S \right)$), plus a (linear) term that can be computed explicitly knowing the map decomposition as in (14)--(15).

7. An illustrative example

In this section we put our results at work, showing how they can be employed to study the dynamical behavior of different faces of the positive cone and their asymptotic probabilities.

7.1. Description of the dynamics

Consider a 7 level quantum system associated to the Hilbert space $\mathcal{H} = \text{span} \{ | j \rangle \}_{j=1}^7$, on which, within each fixed time step, one of the following ‘noise actions’ may occur:

(i) with probability $\gamma_1 < 1$, level 1, 3 and 2, 4 are swapped,
(ii) with probability $\gamma_2 < 1$, level 3 decays to 1 and 4 to 2,
(iii) with probability $\gamma_3 < 1$, level 5 decays to level 4 and 3 in the same proportion,
(iv) with probability $\gamma_4 < 1$, level 6 decays to 5,
(v) with probability $\gamma_5 < 1$, level 7 decays to 5;

where $\sum_i \gamma_i = 1$, $\gamma_i > 0$ for any $i$ and $\gamma_3 < \gamma_4 < \gamma_5$. An OSR for the map $T$ jointly describing these processes can be obtained by the following matrices, associated to each of the processes in the ordered basis for $\mathcal{H}$ given above (see e.g. [4], chapter 8 for details on phenomenological description of noise actions):

(i)

$$
N_1 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

(ii)

$$
N_2 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/\sqrt{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/\sqrt{2}
\end{bmatrix}.
$$
\[
N_3 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}
\end{bmatrix}
\]

(iii)

\[
N_4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(iv)

\[
N_5 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
N_6 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
N_7 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Defining the probability-weighed matrices $N_1 = \sqrt[n]{\gamma_1} N_1$, $M_2 = \sqrt[n]{\gamma_2} N_2$, $M_3 = \sqrt[n]{\gamma_3} N_3$, $M_4 = \sqrt[n]{\gamma_4} N_4$, $M_5 = \sqrt[n]{\gamma_5} N_5$, $M_6 = \sqrt[n]{\gamma_6} N_6$, $M_7 = \sqrt[n]{\gamma_7} N_7$, $M_8 = \sqrt[n]{\gamma_8} N_8$ and $M_9 = \sqrt[n]{\gamma_9} N_9$ we obtain a representation for the whole process $T$.

### 7.2. Checking GAS

By looking at the structure of the matrices, it is easy to note that the subspace $\mathcal{H}_S = \text{span}([\{1\}, [3] \})$ is invariant. This allows us to employ the results of section 5.4 in order to check if it is also GAS. We must look at the sequence of supports $T^*([1] \langle 1 \rangle + [3] \langle 3 \rangle)$. We obtain:

$$\text{supp}(T^*([1] \langle 1 \rangle + [3] \langle 3 \rangle)) = \text{span}([1], [3], [5]),$$

$$\text{supp}(T^{*2}([1] \langle 1 \rangle + [3] \langle 3 \rangle)) = \text{span}([1], [3], [5], [6], [7]),$$

$$\text{supp}(T^{*3}([1] \langle 1 \rangle + [3] \langle 3 \rangle)) = \text{span}([1], [3], [5], [6], [7]).$$

Since this sequence stops before covering the whole $\mathcal{H}$, by corollary 2 $\mathcal{H}_S$ is not GAS.

### 7.3. NFD and DID

It is interesting find out what is the minimal subspace that contains $\mathcal{H}_S$ and is GAS. This can be done using the nested faces construction, thanks to the results in section 4.3. Decomposition (31) returns in this case to the following subspaces, each characterized by the spectral radius of the corresponding $T_{R_i}$

$$\mathcal{H}_R = \text{span}([\{2\}, [4] \}), \quad \sigma(T_{R_1}) = 1,$$

$$\mathcal{H}_T = \text{span}([\{5\} \}), \quad \sigma(T_{R_1}) = 1 - \gamma_3,$$

$$\mathcal{H}_R = \text{span}([\{6\} \}), \quad \sigma(T_{R_1}) = 1 - \gamma_4,$$

$$\mathcal{H}_T = \text{span}([\{7\} \]), \quad \sigma(T_{R_1}) = 1 - \gamma_5.$$
As expected, given proposition 3, $\sigma(T_R) = 1$; moreover, the same proposition permits to obtain the minimal GAS subspace, which is $H_S = H_S \oplus H_S$. In our case where $H_S = H_S = \text{span}(\{2\}, \{4\})$, we obtain:

$$H_S = \text{span}(\{1\}, \{3\}, \{2\}, \{4\})$$

The subspace $H_S$ can be used as the starting point for the DID; doing so, decomposition (37) is given by:

$$H_{T_1} = \text{span}(\{5\})$$
$$H_{T_2} = \text{span}(\{6\}, \{7\})$$

For any of these subspaces there is a minimal and a maximal transition rate, as explained in section 5.3, the least of which has value $\gamma_3$ (in this case it can be read out directly from the form of the dynamics, and in particular $M_4$). A comparison with the maximal spectral radius of the nested faces decomposition shows that both the constructions give the same estimation for the convergence speed towards $H_S$.

### 7.4. Asymptotic probabilities

Knowing that $H_S$ is GAS, it is possible to use the results in section 6, to evaluate the asymptotic probabilities of finding the state in one of the two subspaces $H_S$ and $H_S$. In order to make the structure of the fixed-point set explicit, it is useful to note that by representing the dynamics restricted to $H_S$ in the basis $\{|1\}, \{3\}, \{2\}, \{4\}$, one directly obtains a tensor structure. In fact, by relabeling these four states as $|1\> = |0\>_N \otimes |0\>_F$, $|2\> = |1\>_N \otimes |0\>_F$, $|3\> = |0\>_N \otimes |1\>_F$ and $|4\> = |1\>_N \otimes |1\>_F$, we define a decomposition of $H_S$ in two ‘virtual’ subsystem of dimension-two: $H_S = H_N \otimes H_F$. With respect to this decomposition the matrices that generates the dynamics inside $H_S$ can be written as:

$$B_1 = \sqrt{\gamma_1} I_2 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$B_2 = \sqrt{\gamma_2} I_2 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$B_3 = \sqrt{\gamma_3} I_2 \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$B_4 = \sqrt{1 - \gamma_1 - \gamma_2} I_2 \otimes I_2.$$

Any of the $B_i$ factorizes in an operator proportional to the identity on $H_N$ times another on $H_F$, making $H_N$ a noiseless subsystem [28, 29]. Moreover, projecting onto the the subspaces $H_S$ and $H_S$, as defined above, corresponds to projecting onto the subsystem states $|0\>_N$ and $|1\>_N$. Thus evaluating the trace of the state projected onto one of them returns the probability of having prepared the corresponding state in $H_N$. To do this in the asymptotic limit we can use the results of section 6, since both subspaces are invariant.

Turning to the asymptotic probabilities, it is convenient to evaluate the limits of the projections, as is done in the proof of proposition 7, and then apply them to the initial state:

$$\lim_{n \to \infty} T^{na}(\Pi_{S_1}) = |1\>\langle 1| + |3\>\langle 3| + \frac{1}{2}(|5\>\langle 5| + |6\>\langle 6| + |7\>\langle 7|),$$
$$\lim_{n \to \infty} T^{na}(\Pi_{S_2}) = |2\>\langle 2| + |4\>\langle 4| + \frac{1}{2}(|5\>\langle 5| + |6\>\langle 6| + |7\>\langle 7|).$$
By these, if the initial state is \( \rho_0 = \frac{1}{2} \mathbb{I} \), we obtain:

\[
\lim_{n \to \infty} \text{Tr} \left( \Pi S_t T^n (\rho_0) \right) = \frac{1}{2},
\]

\[
\lim_{n \to \infty} \text{Tr} \left( \Pi S_2 T^n (\rho_0) \right) = \frac{1}{2}.
\]

If instead the initial state is \( \rho_0 = \frac{1}{2} (|1\rangle \langle 1| + |7\rangle \langle 7|) \), then we have

\[
\lim_{n \to \infty} \text{Tr} \left( \Pi S_5 T^n (\rho_0) \right) = \frac{3}{4},
\]

\[
\lim_{n \to \infty} \text{Tr} \left( \Pi S_2 T^n (\rho_0) \right) = \frac{1}{4}.
\]

8. Discussion and conclusions

A thorough understanding of open system dynamics, and specifically iterated CPTP maps, plays a central role in the development of quantum information and control methods. Their asymptotic behavior reveals the effectiveness of error protection and correction strategies [28, 30] and protocols for state preparation [12].

In this work we presented a rich set of linear-algebraic tools for analyzing the convergence features of a discrete-time QDS generated by a given CPTP map. The work complements known results on the decomposition of the peripheral eigen-operators of a CPTP map, and the emerging Hilbert-space structure [26, 31], by introducing decompositions of the ‘decaying’ part of the Hilbert space that highlight the mechanism and speed of convergence. Similarly to those results, our results build on a generalization of Perron–Frobenius theory. With respect to the existing results in continuous-time, the discrete-time is more general, and has direct applications in the study of engineered dynamics in quantum ‘digital’ simulators [19].

Our tools include two Hilbert space decompositions in ‘transient’ subspaces. We believe that both have their place and potential advantages for certain tasks, which we next briefly discuss and compare.

The NFD is based on the positivity and spectral properties of the evolution, and its constructions is directly derived from the Perron–Frobenius analysis. Its potential advantages include: (i) the ability of deciding GAS for the initial subspace at the first iteration—in fact, if the first reduced spectral ratio is strictly lesser than one, convergence is guaranteed; (ii) the spectral radii correspond to exact asymptotic convergence speed; (iii) the nested subspaces \( \mathcal{H}_S \) are all invariant subspaces, allowing for reduced description of the dynamics if needed. However, in the construction the initial representation of the dynamics may be soon lost, and is in general difficult to assess the role of a certain physical variable on the qualitative behavior.

On the other hand, the DID is based on the directed dynamical links between subspaces. Its construction decomposes the OSR matrices, and in practical examples where the maps are associated to physical decay processes (e.g. spontaneous or stimulated emissions), it often only requires a re-arranging of the natural basis. This can help in identifying the physical parameters leading to, or hindering, convergence. However, (i) to decide GAS one has to run the algorithm to its end; (ii) the transition rates are not exact asymptotic convergence speeds; (iii) its \( \mathcal{H}_S \) are not invariant, with exception of the first one.
Summing up, the first decomposition is more natural from a mathematical perspective, however the second is more likely to highlight the physical mechanisms leading to convergence [24].

Nonetheless, both decompositions associate subspaces with different decay speeds: eigenvalues of the map in the NFD, or transition rates in the DID. These speeds can help identifying bottlenecks for the convergence, typically associated to a particular physical process, as we illustrated in the example.

The asymptotic probability formula given in proposition 7 allows for computing the asymptotic probability distribution of converging to a set of orthogonal subspaces. This type of problems are relevant e.g. in verifying the efficiency of an initialization procedure for a quantum noiseless code, the probability of decoherence driving the state to one of a set of orthogonal pointer states, or in general to be able to assess the asymptotic properties when the evolution is not mixing, i.e. it does not have a unique GAS state.

In conclusion, we have presented a set of analysis tools that can aid in the design of evolution for quantum control and quantum information processing. In particular, they should provide suitable means for analyzing the convergence speed of quantum information protocols based on dissipation, including entanglement preparation and computation [17, 18, 32, 33].

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Appendix. Proof of proposition 2

The proof is based on two lemmas. The first one shows that an eigenvector for a reduced map on the complement of an invariant subspace can always be extended to a generalized eigenvector of the whole map, corresponding to the same eigenvalue.

**Lemma 3.** Let \( T \) be a linear map on a vector space \( \mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \) of dimension \( d \), with \( \mathcal{V}_1 \) invariant, so that

\[
T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}
\]  
(A.1)

If \( \eta \in \mathcal{V}_2 \), \( \eta = \begin{bmatrix} 0 \\ \eta_2 \end{bmatrix} \) and \( T_3 \eta_2 = \sigma \eta_2 \), then exists \( \xi \in \mathcal{V}_1 \) such that

\[
\xi + \eta \in \ker\left( (T - \sigma I)^d \right).
\]  
(A.2)

**Proof.** If \( \xi \in \mathcal{V}_1 \) then

\[
\xi = \begin{bmatrix} \xi_1 \\ 0 \end{bmatrix}
\]
and the lemma is proved if the equation in $\xi$

$$(T - \sigma I)^d(\xi + \eta) = \begin{pmatrix} (T_1 - \sigma I)^d \xi_1 + T_2 \eta_2 \\ (T_3 - \sigma I)^d \eta_2 \end{pmatrix} = 0 \quad (A.3)$$

has solution.

Since $\eta_2$ is an eigenvector of $T_3$, $(T_3 - \sigma I)^d \eta_2 = 0$ and (A.3) reduces to

$$(T_1 - \sigma I)^d \xi_1 = -T_2 \eta_2. \quad (A.4)$$

If $\sigma$ is not an eigenvalue of $T_1$, this system is clearly solvable in $\xi_1$. In the other case it is solvable only if $T_2 \eta_2$ belongs to the image of $(T_1 - \sigma I)^d$ i.e. if $T_2 \eta_2$ has no component with respect to the generalized eigenspace relative to $\sigma$. Since generalized eigenspaces are in direct sum, $T_2 \eta_2$ admits a unique decomposition as sum of generalized eigenvectors of $T_1$:

$$T_2 \eta_2 = \sum_k v_k, \quad (A.5)$$

where any $v_k$ is relative to a different eigenvalue $\lambda_k$, with $\lambda_j \neq \lambda_i$ if $j \neq i$. Also, due to the invariance of $\mathcal{V}_1$, any $v_k$ can be (trivially) extended to a generalized eigenvector $w_k$ for $T$, also relative to $\lambda_k$. We can thus write:

$$(T - \sigma I)^d \eta = \begin{pmatrix} T_2 \eta_2 \\ 0 \end{pmatrix} = \sum_k \begin{pmatrix} v_k \\ 0 \end{pmatrix} = \sum_k w_k. \quad (A.6)$$

Notice that if $\sigma$ is an eigenvalue of $T_1$, as it is in the case we are discussing, then it is also an eigenvalue of $T$ since $\mathcal{V}_1$ is invariant. It is thus apparent that none of the $w_k$ can be a generalized eigenvalue corresponding to $\sigma$; in fact, due to the Jordan structure of $T$, $T - \sigma I$ restricted to the generalized $\sigma$-eigenspace is a nilpotent matrix of order at most $d$, and hence any generalized $\sigma$-eigenvector for $T$ is mapped to zero after $d$ applications of $T - \sigma I$. This allows us to conclude that none of the $v_k$ is relative to $\sigma$ and the system (A.3) is always solvable. \hfill \square

**Lemma 4.** Let $\mathcal{H}$ be a finite dimensional Hilbert space, and $\mathcal{H}_S \oplus \mathcal{H}_R$ an orthogonal decomposition. If $Z \in \mathfrak{S}(\mathcal{H})$ has block form

$$Z = \begin{bmatrix} Z_S & Z_P \\ Z_P^* & Z_R \end{bmatrix}, \quad (A.7)$$

with $Z_R > 0$ and $X \in \mathfrak{S}(\mathcal{H})$ is such that

$$X = \begin{bmatrix} X_S & 0 \\ 0 & 0 \end{bmatrix}, \quad (A.8)$$

with $X_S > 0$, then there exists a scalar $c > 0$ such that $Z + cX > 0$.

**Proof.** Considering the block form induced by the decomposition of $\mathcal{H}$, if $|\varphi\rangle = |\varphi_S\rangle \oplus |\varphi_R\rangle$ then...
Since \( X_S > 0 \) exists a \( c_1 \) such that \( c_1 X_S + Z_S > 0 \), so that redefining \( c = c' + c_1 \): 

\[
\langle \varphi_S | c X_S + Z_S | \varphi_S \rangle \geq \langle \varphi_S | c' X_S | \varphi_S \rangle
\]  

(A.10)

for any \(| \varphi_S \rangle \).

The set of vectors \(| \varphi_S \rangle \oplus | \varphi_R \rangle \), under the condition \( \langle \varphi_S | \varphi_S \rangle = \langle \varphi_R | \varphi_R \rangle = 1 \), is compact and \( \langle \varphi_R | Z_P^+ | \varphi_S \rangle + \langle \varphi_S | Z_P | \varphi_R \rangle \) is a real continuous function, then exists \( m > 0 \) such that

\[
\langle \varphi_R | Z_P^+ | \varphi_S \rangle + \langle \varphi_S | Z_P | \varphi_R \rangle \geq -2m.
\]  

(A.11)

if \(| \varphi_S \rangle = | \varphi_R \rangle = 1 \). So for \(| \varphi_S \rangle \) and \(| \varphi_R \rangle \) with \(| \varphi_S \rangle \neq 0 \) and \(| \varphi_R \rangle \neq 0 \)

\[
\langle \varphi_R | Z_P^+ | \varphi_S \rangle + \langle \varphi_S | Z_P | \varphi_R \rangle \geq -2m | \varphi_R \rangle | \varphi_S \rangle
\]  

(A.12)

By positiveness of \( X_S \) and \( Z_R \)

\[
\langle \varphi_S | X_S | \varphi_S \rangle \geq a_1^2 | \varphi_S \rangle^2
\]

\[
\langle \varphi_R | Z_R | \varphi_R \rangle \geq a_2^2 | \varphi_R \rangle^2
\]  

(A.13)

for some \( a_1, a_2 > 0 \) (for example the roots of their minimum eigenvalues).

Suppose \(| \varphi_S \rangle \neq 0 \) and \(| \varphi_R \rangle \neq 0 \) and put (A.10), (A.12) and (A.13) in (A.9)

\[
\langle \varphi_S \rangle \oplus \langle \varphi_R | c X + Z | \varphi_S \rangle \oplus | \varphi_R \rangle \rangle 
\]

\[
\geq a_1^2 | \varphi_S \rangle^2 c' + a_2^2 | \varphi_R \rangle^2 - 2m | \varphi_S \rangle | \varphi_R \rangle
\]  

(A.14)

choose \( c' > \left( \frac{m}{a_2} \right)^2 \)

\[
\frac{m^2}{a_2^2} | \varphi_S \rangle^2 + a_2^2 | \varphi_R \rangle^2 - 2m | \varphi_S \rangle | \varphi_R \rangle \rangle 
\]

\[
= \left( \frac{m}{a_2} | \varphi_S \rangle - a_2 | \varphi_R \rangle \right)^2 > 0.
\]  

(A.15)

so \( c X + Z > 0 \).

It is now possible to prove proposition 2.

**Proof of proposition 2.** Suppose by contradiction that \( \sigma = \sigma(T_{R_i}) = \sigma(T_{R_{i+1}}) \). It would be then possible to find \( A \geq 0 \) such that \( T_{R_{i+1}}(A) = \alpha A. H_T = \text{supp}(A) \) is invariant for \( T_{R_{i+1}} \), so
that $H_{R} = H_{T} \oplus H_{F}$ is invariant for $T_{R}$. Consider $T_{R}$ the restriction of $T_{R}$ to $H_{R}$. For this map $\mathfrak{S}(H_{T}) \oplus \mathfrak{S}(H_{F})$ is invariant and, being $A \in \mathfrak{S}(H_{T})$ an eigenoperator for $T_{R}$, it is thus possible to apply lemma 3 to extend $A$ to a generalized eigenvector of $T_{R}$, of the form

$$A' = \begin{bmatrix} A_1 & A_2 \\ A_2^T & A \end{bmatrix}$$

(A.16)

with $A > 0$. By the definition of $H_{T}$ there exists, for $T_{R}$, a generalized eigenvector $X \geq 0$ relative to $\sigma$, such that supp$(X) = H_{T}$. Since $H_{R} \subset H_{T}$, $X$ is an eigenvalue of $T_{R}$ as well. Now by lemma 4 it is possible to find a constant $c > 0$ such that $B = cX + A > 0$. Since $B$ is a generalized eigenvector of $T_{R}$ relative to $\sigma$, the same holds for $T_{R}$. However, this is not possible since $H_{T}$ is strictly contained in supp$(B)$, in contradiction with its definition. \hfill \quad $\Box$

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