Sextic double solids with Artin–Mumford obstructions to rationality

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Abstract

We study a double solid $X$ branched along a nodal sextic surface in a projective space and the 2-torsion subgroup in the third integer cohomology group of a resolution of singularities of $X$. This group can be considered as an obstruction to rationality of $X$. Studying this group we conclude that all sextic double solids admitting non-trivial obstructions to rationality are branched along determinantal surfaces of very specific type and we provide an explicit list of them.

1 Introduction

In this work we study rationally connected varieties having some obstructions to rationality over an algebraically closed field of characteristic zero. As a result we get descriptions of non-rational varieties which are however somewhat similar to rational. Rationally connected curves or surfaces are necessarily rational: this is an easy corollary of the Hurwitz theorem and the Castelnuovo criterion. For threefolds it is false, and M. Artin and D. Mumford [AM72] introduced one of the first examples of a rationally connected non-rational variety of dimension 3. They considered the so called quartic double solid; namely, the double cover $X$ of $\mathbb{P}^3$, branched along a quartic. They constructed a special nodal quartic surface $B$ (a quartic symmetroid; i.e., a zero locus of determinant of a matrix of linear forms over $\mathbb{P}^3$) and showed that $X$ branched along $B$ is rationally connected, but non-rational. In order to show non-rationality of the double solid, M. Artin and D. Mumford introduced a birational invariant of a smooth projective variety $M$; namely the torsion subgroup $T$ in $H^3(M, \mathbb{Z})$. In particular, they showed that if the group $T \neq 0$, then $M$ is non-rational; moreover, it is not stably rational.

Let us denote by $\tilde{\mathbb{P}}^3$ the blow up of $\mathbb{P}^3$ in all the singular points of $B$, by $\tilde{B}$ the proper transform of $B$ and by $\tilde{X}$ the double cover of $\tilde{\mathbb{P}}^3$ branched along $\tilde{B}$. Then we have the following diagram:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\pi}} & \tilde{\mathbb{P}}^3 \\
\downarrow{\tilde{\sigma}} & & \downarrow{\sigma} \\
X & \xrightarrow{\pi} & \mathbb{P}^3 \\
\end{array}
\]

Let us note that in the case of the double cover $M$ of the threefold with torsion-free integer cohomology groups all $p$-torsion subgroups in $H^3(M, \mathbb{Z})$ are trivial for all $p$ except $p = 2$. This can be deduced from the Mayer–Vietoris sequence for the standard partition of the double cover.

In [AM72] M. Artin and D. Mumford proved that for the nodal quartic symmetroid $B$ the group $H^3(\tilde{X}, \mathbb{Z})$ contains a non-trivial 2-torsion subgroup $T_2$, therefore $\tilde{X}$ is not rational.

Then S. Endrass [End99] studied all nodal quartic double solids and proved that the family of quartics constructed by M. Artin and D. Mumford is the only family of nodal quartics such that double solids branched along them have non-trivial group $T_2$.

Our goal is to study the next interesting class of rationally connected varieties: double solids branched along nodal sextic surfaces. Note that it is the last case when the question about rationality of a nodal double solid is not obvious. Indeed, nodal double solids branched along surfaces of degree greater than 6 are of non-negative Kodaira dimension; therefore, non-rational.

The case of sextic double solids contains more complicated examples of non-rational varieties than the case of quartic double solids. There are known examples of such varieties with non-trivial Artin–Mumford
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Z
going to provide an explicit classification of obstructed sextic double solids: we show that if a nodal sextic
double solid admits a non-trivial obstruction to rationality of Artin–Mumford type, then it belongs to one
of four concretely described families \( Z_{31}, Z_{32}, Z_{35} \) and \( Z_{40} \). Moreover, a general element of each of these
families actually admits a non-trivial obstruction.

Now let us give more details in order to describe our result. We are going to study the group \( T_2 \) using the
methods introduced by S. Endrass. He showed a connection between the 2-torsion subgroups \( T_2 \subset H^3(X, \mathbb{Z}) \)
and the theory of even sets of nodes of the surface \( B \). A set \( w \subset \text{Sing}(B) \) on a nodal surface \( B \subset \mathbb{P}^3 \) is called
a \( \delta/2 \)-even set of nodes for \( \delta = 0 \) or 1, if

\[
\delta H + \sum_{p \in w} E_p |_B \in 2 \cdot \text{Pic}(B).
\]

Here by \( E_p \) and \( H \) we denote the exceptional divisors of the blow up \( \overline{\mathbb{P}^3} \) and the inverse image of the class of
a plane on \( \mathbb{P}^3 \) respectively. For both \( \delta = 0 \) and 1 such sets are called even sets of nodes. These sets and their
properties were studied by F. Catanese [Cat81], W. Barth [Bar80], A. Beauville [Bea80], and D. B. Jaffe and
D. Ruberman [JR97]. Their main property is that even sets of nodes form a vector space over the field \( \mathbb{F}_2 \) of
two elements, i.e. the symmetric difference of two \( \delta/2 \)-even sets of nodes is an even set of nodes as well; we
will denote the space of all even sets of nodes on the surface \( B \) by \( \overline{\mathcal{C}}_B \). An important assertion, proved by
S. Endrass [End99] Lemma 1.2], claims that the group \( T_2 \) is a subspace of \( \overline{\mathcal{C}}_B \):

\[
\overline{\mathcal{C}}_B \cong T_2 \oplus \mathbb{F}_2^d.
\]

Here the number \( d \) is called the defect of the set \( \text{Sing}(B) \) and it has a geometric interpretation, for more
details see Section 4.

Another important fact about even sets of nodes is that any nodal surface containing such a set of nodes is
determinantal:

**Theorem 1.2** ([CC98], [Bar80] Lemma 4], cf. [CC97] Corollary 0.4]). If a nodal surface \( B \) in \( \mathbb{P}^3 \) contains a
non-empty \( \delta/2 \)-even set of nodes \( w \), then there exists a vector bundle \( \mathcal{E} \) over \( \mathbb{P}^3 \) and an injective morphism

\[
\Phi: \mathcal{E}'(-\deg(B) - \delta) \hookrightarrow \mathcal{E},
\]

such that \( \Phi \) is symmetric, i.e. induced by an element of \( H^0(\mathbb{P}^3, S^2 \mathcal{E}(\deg(B) + \delta)) \), and the following holds:

\[
B = B(\mathcal{E}) = \{ x \in \mathbb{P}^3 \mid \text{cork}(\Phi|_x) \geq 1 \};
\]

\[
w = w(\mathcal{E}) = \{ x \in \mathbb{P}^3 \mid \text{cork}(\Phi|_x) = 2 \}.
\]

Moreover, if the bundle \( S^2 \mathcal{E}(\deg(B) + \delta) \) is globally generated, then for a sufficiently general \( \Phi \), we have
that \( \text{Sing}(B) = w(\mathcal{E}) \).

**Definition 1.3.** If \( B \) is a nodal surface and \( w \) is an even set of nodes on it, we call the bundle \( \mathcal{E} \) from
Theorem 1.2 the Casnati–Catanese bundle of \( w \).

Actually, Theorem 1.2 gives an explicit construction of the vector bundle \( \mathcal{E} \) which helped to find all
possible constructions of 0-even sets of nodes on sextics:

**Theorem 1.4** ([CT07] Main Theorem A, Proposition 3.3]). If \( w \) is a 0-even set of nodes on a nodal sextic
surface \( B \), then its Casnati–Catanese bundle is one of the following:

\[
\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \text{ and } |w| = 24;
\]

\[
\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 3} \text{ and } |w| = 32;
\]

\[
\mathcal{E} = \Omega^1_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \text{ and } |w| = 40;
\]

\[
\mathcal{E} = \text{Ker}(\Omega^1_{\mathbb{P}^3}(-1)^{\oplus 3} \to \mathcal{O}(-2)^{\oplus 3}) \text{ and } |w| = 56.
\]
Remark 1.5. Let $B$ be a surface with a 0-even set of nodes $w$ of 56 nodes whose Casnati–Catanese bundle is the fourth vector bundle from the list in Theorem 1.4. Then there exists a 0-even set of 24 nodes $w' \subset w$ whose Casnati–Catanese bundle is the first one in the list. Similarly, the 0-even set of 32 nodes $w'' = w \setminus w'$ has Casnati–Catanese bundle, which is the second one in the list.

Our goal is to study the case of 1/2-even sets of nodes. We introduce an auxiliary notion of minimal even sets of nodes which allows us to avoid the situations described in Remark 1.5. Namely, we call $w \in C_B$ minimal if it does not contain a proper even subset of nodes. This notion also happens to be very useful because it provides some cohomological restrictions for Casnati–Catanese bundles. However, on each surface $B$ with non-trivial group $C_B$ we can find a minimal non-empty even set of nodes. So using the notion of minimality we get a precise description of arbitrary (not necessarily minimal) surfaces which have non-trivial group.

Theorem 1.6. Let $B$ be a nodal sextic with $C_B \neq 0$. Then there exists an even set of nodes $w$ on $B$ such that either $w$ is 0-even and its Casnati–Catanese bundle is one of the first three bundles from the list in Theorem 1.4 or $w$ is 1/2-even and its Casnati–Catanese bundle is

$$\mathcal{E} = \Omega^3_{P^3}(\Lambda) \oplus \bigoplus \mathcal{O}_{P^3}(-m_i)$$

for some non-negative integers $k$ and $m_i$.

We point out that Theorem 1.6 provides only necessary conditions for Casnati–Catanese bundles of minimal 1/2-even sets of nodes. It would be interesting to find a complete classification of these Casnati–Catanese bundles.

Theorem 1.6 significantly simplifies the search of surface containing a 1/2-even set of nodes; moreover, we prove some useful bounds for the numbers $k$ and $m_i$, see Lemma 3.6 for details. However, the list of bundles $\mathcal{E}$ is not so short as in Theorem 1.4. Then we show the connection between global sections of the twist of the sheaf of ideals $H^0(P^3, I_{\text{Sing}(B)}(5))$ and the number $d$ from the equation (1.1). We estimate the dimension of the space $H^0(P^3, I_{\text{Sing}(B)}(5))$ using the dimensions of $H^0(P^3, I_w(5))$ for all even $w \in C_B$. Next, we compute spaces $H^0(P^3, I_w(5))$ using bundles $\mathcal{E}$ constructed in Theorem 1.6. This leads to the following result:

Theorem 1.7. If $B$ is a nodal sextic and $T_2(\tilde{X}) \neq 0$, then $B = B(\mathcal{E})$ is the zero locus of the determinant of the injective morphism (1.2), where $\mathcal{E}$ is one of the following bundles:

- $\mathcal{E} = \mathcal{O}_{P^3}(-2) \oplus \mathcal{O}_{P^3}(-2)$ with 1/2-even set $w(\mathcal{E})$ of 31 nodes;
- $\mathcal{E} = \mathcal{O}_{P^3}(-2) \oplus \mathcal{O}_{P^3}(-3)$ with 0-even set $w(\mathcal{E})$ of 32 nodes;
- $\mathcal{E} = \mathcal{O}_{P^3}(-2) \oplus \mathcal{O}_{P^3}(-3)$ with 1/2-even set $w(\mathcal{E})$ of 35 nodes;
- $\mathcal{E} = \Omega^3_{P^3}(-1) \oplus \mathcal{O}_{P^3}(-2)$ with 0-even set $w(\mathcal{E})$ of 40 nodes.

By construction we know that any surface $B$ described in Theorem 1.7 contains an even set of nodes $w$; however, similarly to Theorem 1.2, in some cases we can not be sure that $\text{Sing}(B) = w$. Moreover, such situation arises in case of the surface with an even set of 56 nodes described in Theorem 1.4 for more details see Section 7. Nevertheless, for general elements of families described in Theorem 1.7 this is true; therefore, these elements have non-trivial obstructions to rationality.

Proposition 1.8. If $\mathcal{E}$ is a bundle from the list in Theorem 1.7 then there exists a Zariski-open subset $U$ of $H^0(P^3, S^2 \mathcal{E}(6 + \delta))$ such that for any $\Phi \in U$ the singular locus of the sextic double solid $X$ branched over the surface $B = \{\det(\Phi) = 0\}$ coincides with $w(\mathcal{E})$ and it admits a non-trivial obstruction of Artin–Mumford type $T_2(\tilde{X}) \neq 0$.

In view of Theorem 1.7 and Proposition 1.8 we construct four families $Z_{31}, Z_{32}, Z_{35}$ and $Z_{40}$ of double solids branched along sextics such that a general element of each of them has exactly 31, 32, 35 or 40 nodes respectively and admits a non-trivial Artin–Mumford obstruction to rationality.

Examples of double solids branched over sextics with even sets of 35 or 31 nodes were already described in [KP14, Section 3.2] and [Bea16]. Actually, in [KP14, Section 3.2] the branch surface was described as the zero locus of the determinant of a different vector bundle than in Theorem 1.7 cf. [Cat81, Theorem 2.23].
It happens that for all examples of non-rational sextic double solids described in Proposition 1.8 their defects \(d\) vanish, so \([CP10, Corollary A, Corollary 4.2]\) gives another proof of non-rationality of these double solids. Nevertheless, using our method and \([AM72, Proposition 1]\) we prove stable non-rationality of all these varieties. So, Proposition 1.8 introduces two new families of rationally connected stably non-rational varieties. In Section 7 we discuss a possible example of nodal sextic double solid where non-rationality can be proved by an Artin–Mumford obstruction but cannot be deduced from \([CP10]\), and mention some other relevant open questions.

We point out that stable non-rationality of a very general (smooth) quartic double solid is known from \([HT16]\). However, the approach using the Artin–Mumford obstruction gives a more precise information about stable non-rationality for certain families of singular sextic double solids, cf. \([Voi15]\).

The paper is organized as follows: in Section 2 we recall the construction from \([CC97, Section 1]\) which helps us to associate with any even set of nodes \(w\) a special Casnati–Catanese bundle \(E\). In Section 3 we introduce useful properties of \(1/2\)-even sets of nodes on sextic surfaces and prove Theorem 1.6. Then, in Section 4 we study the notion of defect. In Section 5 we compute the defects of even sets of nodes with Casnati–Catanese bundles described in Theorem 1.6, which helps us to prove Theorem 1.7 in Section 6. Also, this section contains the proof of Proposition 1.8. In Section 7 we review an explicit construction of a non-general subfamily of sextic double solids in the family \(Z_{32}\) and discuss possible directions of further research.

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2 Casnati and Catanese construction

In this section we recall the main steps of the construction from \([CC97, Section 1]\). First of all, we associate a sheaf with a \(\delta/2\)-even set of nodes \(w\) on the surface \(B\). Consider the divisor \(\delta H + E_w\), which is divisible by 2 in \(\text{Pic}(\tilde{B})\). Denote by \(X_w\) the double cover of \(\tilde{B}\) branched along \(\delta H + E_w\):

\[
X_w \overset{\pi_w}{\longrightarrow} \tilde{B} \overset{\sigma}{\longrightarrow} B.
\]

The direct image of \(\mathcal{O}_{X_w}\) under \(\pi_w\) splits into a sum of line bundles

\[
\pi_w:\mathcal{O}_{X_w} = \mathcal{O}_{\tilde{B}} \oplus \mathcal{O}_{\tilde{B}}(-\delta H + E_w)/2).
\]

Denote by \(\mathcal{F}\) the direct image of the second summand under \(\sigma\):

\[
\mathcal{F} := \sigma_* \mathcal{O}_{\tilde{B}}(-\delta H + E_w)/2).
\]

We will consider \(\mathcal{F}\) as a sheaf on \(\mathbb{P}^3\). The sheaf \(\mathcal{F}\) is very interesting in view of Theorem 1.2 because it is the cokernel of the injective morphism \(\Phi\), so we have an exact sequence:

\[
0 \to \mathcal{E}'(-\deg(B) - \delta) \to \mathcal{E} \to \mathcal{F} \to 0.
\]

Now let us restrict to the case of a \(1/2\)-even set of nodes on some surface \(B\) and recall the construction \([CC97, Section 1]\) in detail. Denote by \(R\) the homogeneous coordinate ring of the projective space \(R = \mathbb{C}[x_0, x_1, x_2, x_3]\). By \(\Gamma_*\) we denote a functor associating with any coherent sheaf \(\mathcal{G}\) on \(\mathbb{P}^3\) an \(R\)-module of global section of all twists of \(\mathcal{G}\)

\[
\Gamma_*(\mathcal{G}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^0(\mathbb{P}^3, \mathcal{G}(i)).
\]

Let \(R\Gamma_*\) be its right derived functor in the derived category. Let \(\mathcal{C}\) is an object of the derived category of \(\mathbb{P}^3\) and \(W\) a graded submodule in \(\mathcal{H}^4(\mathcal{C})\). Then by \(\tau_{>\delta} \cap \tau_{\leq W}\) \(\mathcal{C}\) we denote a truncation of the object \(\mathcal{C}\) in the derived category of \(\mathbb{P}^3\). Denote by \(D'\) a full subcategory of \(D^b(R - \text{mod})\), whose objects are complexes \(\mathcal{C}\) with a property \(\mathcal{H}^i(\mathcal{C})\) is a module of finite length for \(0 < i < 3\), and \(\mathcal{H}^4(\mathcal{C})\) vanish for all other \(i\). Also we
say that the morphism between vector bundles is equivalent to zero if it factors through a direct sum of line bundles. If we denote by $\text{Bun}'_{\mathbb{P}^3} = \text{Bun}_{\mathbb{P}^3} / \sim$, then we have a functor between categories:

$$\tau_{>0}\tau_{<3}R\Gamma_* : \text{Bun}'_{\mathbb{P}^3} \to D'.$$

Then this functor is an equivalence.

**Theorem 2.3 ([Wal96, Proposition 2.10]).** There exists a functor $\text{Syz} : D' \to \text{Bun}'_{\mathbb{P}^3}$ which is inverse to the functor $\tau_{>0}\tau_{<3}R\Gamma_*$. 

Now we fix a 1/2-even $w$ on a sextic and fix the graded $R$-module $W$:

$$W = \bigoplus_{j \geq 2} H^1(\mathbb{P}^3, \mathcal{F}(j)).$$

Let us consider a truncation $\mathcal{C} = \tau_{>0}\tau_{<1,0}R\Gamma(\mathcal{F})$. By Theorem 2.3 there exists a vector bundle $\text{Syz}(\mathcal{C})$, such that we have isomorphisms of graded modules for $i = 1$ and 2:

$$\bigoplus_{j \in \mathbb{Z}} H^i(\mathbb{P}^3, \text{Syz}(\mathcal{C})(j)) \cong H^i(\mathcal{C}).$$

Then the canonical mapping $\mathcal{C} \to \bigoplus_{i,j} H^i(\mathbb{P}^3, \mathcal{F}(j))$ induces the morphism:

$$\epsilon : \text{Syz}(\mathcal{C}) \to \mathcal{F},$$

which can be non-surjective. Finally, the bundle $\mathcal{E}$ is defined as follows:

$$\mathcal{E} := \text{Syz}(\mathcal{C}) \oplus \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(-i)^{\oplus m_i},$$

where the bundle $\mathcal{E}/\text{Syz}(\mathcal{C}) = \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(-i)^{\oplus m_i}$ corresponds to the free graded $R$-module generated by

$$\bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^3, (\text{Coker}(\epsilon(j)),$$

where by $\epsilon(j)$ we denote the twisted map $\text{Syz}(\mathcal{C})(j) \to \mathcal{F}(j)$. By the definition $\mathcal{E}$ maps surjectively to $\mathcal{F}$ and by [CC97, Claim 2.1] the kernel of this surjective morphism equals $\mathcal{E}^\vee(-\deg(B) - \delta)$.

So by [CC97, Corollary 0.4] for any even set of nodes $w$ we can associate a unique up to an isomorphism Casnati-Catanese vector bundle $\mathcal{E}$ of $w$. From now on we consider only Casnati–Catanese bundles obtained from this construction.

### 3 Minimal 1/2-even sets of nodes

Here we are going to describe precisely sextic surfaces with a minimal 1/2-even set of nodes and prove Theorem 1.6. The first lemma is useful for surfaces of any degree and any minimal even set of nodes. Here by $\mathcal{F}$ we also denote the direct image of the sheaf defined in (2.2) to the projective space.

**Lemma 3.1.** If $w$ is a minimal $\delta/2$-even set of nodes on a nodal surface $B$, then $h^1(\mathbb{P}^3, \mathcal{F}(n))$ vanishes for any $n \leq 0$ and $n \geq \deg(B) - 4 + \delta$.

**Proof.** Let us denote by $2^w$ the $\mathbb{F}_2$-vector space generated by all elements of any even set of nodes $w$. It follows from [Bea80, Lemma 2] that we have the following inequality (cf. also [JR97, Theorem 4.5]):

$$\dim(\mathcal{O}_B \cap 2^w) \geq 1 + h^1(X_w, \mathbb{Z}) = 1 + 2h^1(X_w, \mathcal{O}_{X_w}).$$

So by the definition of minimality we conclude that $h^1(X_w, \mathcal{O}_{X_w}) = 0$. Moreover, since $h^1(B, \mathcal{O}_B)$ vanishes for any surface in $\mathbb{P}^3$, we have that $h^1(\mathbb{P}^3, \mathcal{F}) = h^1(X_w, \mathcal{O}_{X_w}) = 0$.
To show the same for all negative twists of $\mathcal{F}$, we choose a general hyperplane section $H$ of $B$. Then the intersection $H \cap B$ is a smooth curve. Since the inverse of a line bundle $\mathcal{F}(n)|_H$ admits a global section for all $n \leq 0$, by looking at the exact sequence

$$0 \rightarrow \mathcal{F}(n-1) \rightarrow \mathcal{F}(n) \rightarrow \mathcal{F}(n)|_H \rightarrow 0,$$

we see that $h^1(\mathbb{P}^3, \mathcal{F}(n)) = 0$ for all negative $n$.

Now let us use the Serre duality on the smooth surface $\bar{B}$. The canonical class of this surface equals to $\sigma^*(\mathcal{O}_{\mathbb{P}^3}(\deg(B) - 4)|_B)$. Then by \cite{Cat81} Remark 2.15 we have

$$H^i(\mathbb{P}^3, \mathcal{F}(n)) = H^i\left(\bar{B}, \mathcal{O}_{\bar{B}}\left(\frac{2n - \delta}{2}H - E_w\right)\right) \cong H^{2-i}\left(\bar{B}, \mathcal{O}_{\bar{B}}\left(\frac{\delta + (2\deg(B) - 4 - n)H + E_w}{2}\right)^{\vee}\right),$$

$$\cong H^{2-i}\left(\bar{B}, \mathcal{O}_{\bar{B}}\left(\frac{-\delta + (2\deg(B) - 3 - n)H - E_w}{2}\right)^{\vee}\right) = H^{2-i}(\mathbb{P}^3, \mathcal{F}(\deg(B) - 3 - n))$$

(3.2)

Applying this isomorphism we prove the assertion of the lemma.

Now let us restrict to the case when $B$ is a sextic and $w$ is 1/2-even set of nodes. We will say that $w$ is cut out by a plane if there exists a plane $\Pi$ intersecting $B$ along a curve $C$ with multiplicity 2, so $\Pi \cdot B = 2C$ and $w = \text{Sing}(B) \cap C$. Even sets of nodes which are cut out by a plane were studied in \cite[Proposition 2.4]{End99}, see also Proposition \ref{Proposition 3.1} so we are going to discuss all other even sets of nodes. In this case we state that the $R$-module $M = \bigoplus_n H^0(\mathbb{P}^3, \mathcal{F}(n))$ has the following set of generators:

\textbf{Lemma 3.3.} If $w \in \mathcal{G}_B$ is not cut out by a plane, then the graded module $M$ is generated by its graded components $H^0(\mathbb{P}^3, \mathcal{F}(2))$, $H^0(\mathbb{P}^3, \mathcal{F}(3))$, and $H^0(\mathbb{P}^3, \mathcal{F}(4))$.

\textbf{Proof.} By Lemma \ref{Lemma 3.1} we have $h^1(\mathbb{P}^3, \mathcal{F}(n)) = 0$ for $n \neq 1, 2$. Also, since the global sections of $\mathcal{F}(1)$ are precisely the planes cutting out $w$, we have $h^0(\mathbb{P}^3, \mathcal{F}(1)) = 0$; and therefore, $h^0(\mathbb{P}^3, \mathcal{F}(n)) = 0$ for all $n \leq 1$. Then by Serre duality we get:

$$h^2(\mathbb{P}^3, \mathcal{F}(n)) = 0, \text{ for } n \geq 2.$$

Therefore, $\mathcal{F}(4)$ is 0-regular in the sense of \cite[Lecture 14]{Mum66}. Then by loc. cit. we get the assertion.

All 1/2-even sets of nodes are studied in details in \cite[Proposition 2.4]{End99}. Now using Lemma \ref{Lemma 3.3} we can describe the Casnati–Catanese vector bundles $\mathcal{E}$ of all the remaining 1/2-even minimal sets.

\textbf{Proposition 3.4.} If $w \in \mathcal{G}_B$ is a minimal 1/2-even set of nodes, then its Casnati–Catanese bundle are isomorphic to

$$\mathcal{E} = \Omega^1_{\mathbb{P}^3}(-2)^{\oplus k} \oplus \bigoplus \mathcal{O}_{\mathbb{P}^3}(-i)^{\oplus m_i},$$

for some non-negative integers $k$ and $m_i$. Moreover, if $w$ is not cut out by a plane, then

$$\mathcal{E} = (\Omega^1_{\mathbb{P}^3}(-2) \otimes W) \oplus (\mathcal{O}_{\mathbb{P}^3}(-2) \otimes U_2) \oplus (\mathcal{O}_{\mathbb{P}^3}(-3) \otimes U_3) \oplus (\mathcal{O}_{\mathbb{P}^3}(-4) \otimes U_4).$$

(3.5)

where $W$ and $U_i$ are respectively $k = k(\mathcal{E})$-dimensional and $m_i = m_i(\mathcal{E})$-dimensional vector spaces.

\textbf{Proof.} Let us follow the construction in Section 2 first of all, we should find the vector bundle $\text{Syz}(\mathcal{E})$. Since by Lemma \ref{Lemma 3.1} in the minimal case the graded $R$-module $W$ is just a vector space concentrated in a single degree.

The vector bundle $\Omega^1_{\mathbb{P}^3}(-2) \otimes W$ has the similar module structure on intermediate cohomology groups. By \cite[Corollary 2.9]{Wal96} any vector bundle $\mathcal{G}$ on $\mathbb{P}^n$ is a direct sum of several line bundles and the bundle $\text{Syz}(r \geq 0, r < 3, R\mathcal{G})$. Since $\Omega^1_{\mathbb{P}^3}(-2) \otimes W$ is semistable which slope is not integer, then it does not splits into a sum of a line bundle and some other vector bundle. Therefore, we have

$$\text{Syz}(\mathcal{E}) = \Omega^1_{\mathbb{P}^3}(-2) \otimes W.$$

Moreover, from the construction of linear summands in the Casnati–Catanese bundle $\mathcal{E}$ and by Lemma \ref{Lemma 3.3} we get that only the numbers $m_2$, $m_3$, and $m_4$ in the equation (3.5) can be non-zero; this finishes the proof.
Proposition 3.4 fixes the precise form of Casnati–Catanese bundles of minimal 1/2-even sets of nodes on sextics. In particular, Theorem 1.6 follows from it. Moreover, we see that the Casnati–Catanese bundles of minimal 1/2-even sets of nodes which are not cut out by planes depend only on the four numbers \( k = k(\mathcal{E}) \) and \( m_i = m_i(\mathcal{E}) \). Let us introduce some additional restrictions them.

**Lemma 3.6.** Suppose that \( B \) is a sextic surface, \( w \in \overline{\mathcal{B}} \) is minimal 1/2-even, and \( w \) is not cut out by a plane. Let \( \mathcal{E} \) be the Casnati–Catanese bundle of \( w \). Then

1. \( m_2 = k + \frac{35 - |w|}{4} \);
2. \( m_3 \leq 6 - k \);
3. \( k + 3m_2 + m_3 - m_4 = 6 \).

**Proof.** The first assertion is due to the Riemann–Roch theorem for the sheaf \( \mathcal{F}(2) \) (see [JR97] Lemma 4.1). The cohomology group \( H^0(\mathbb{P}^3, \mathcal{F}(3)) \) is generated by the image of \( H^0(\mathbb{P}^3, \mathcal{F}(2)) \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \) in it and some additional generators. The number of these generators is less than or equal to \( h^0(\mathbb{P}^3, \mathcal{F}(3)) - m_2 \), so we get the second assertion. Finally, since the degree of the support surface of \( \text{Coker}(\Phi) \) equals 6. Than the difference of the degrees of \( \mathcal{E} \) and \( \mathcal{E}^\vee (\deg(B) - 3) \) equals 6. Computing this difference, we get the last assertion.

Moreover, we have one more condition on the numbers \( m_i \). In order to do it, let us consider the Casnati–Catanese bundle \( \mathcal{E} \) as a direct sum of four bundles

\[
\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3 \oplus \mathcal{E}_4,
\]

where \( \mathcal{E}_1 = W \otimes \Omega^1_{\mathbb{P}^3}(-2) \) and \( \mathcal{E}_i = U_i \otimes \mathcal{O}_{\mathbb{P}^3}(-i) \) for \( i = 2, 3, 4 \). This decomposition induces the decomposition of the morphism \( \Phi = (\Phi_{ij}) \). Thus \( \Phi \) is a symmetric \( 4 \times 4 \) matrix, \( \Phi_{ij} \) corresponds by an element in \( H^0(\mathbb{P}^3, \mathcal{E}_i \otimes \mathcal{E}_j(7)) \) if \( i \neq j \), or in \( H^0(\mathbb{P}^3, S^2 \mathcal{E}_i \otimes \mathcal{O}_{\mathbb{P}^3}(7)) \), if \( i = j \) and it induces the following morphism:

\[
\Phi_{ij} : E_i \rightarrow E_j.
\]

Moreover \( \Phi_{ij} = \Phi^t_{ji} \). Now we are ready to formulate the assertion.

**Lemma 3.8.** Suppose \( B \) is an irreducible sextic surface, \( w \in \overline{\mathcal{B}} \) is minimal 1/2-even, and \( \mathcal{E} \) is its Casnati–Catanese bundle. If \( m_2 = 1 \), then \( m_3 \geq m_4 \).

**Proof.** Assume that \( m_2 = 1 \) and \( m_4 > m_3 \). In this situation \( \Phi \) is of the following form (for simplicity we write \( \mathcal{O} \) instead of \( \mathcal{O}_{\mathbb{P}^3} \):

\[
T_{\mathbb{P}^3}(-5)^{\oplus k} \oplus \mathcal{O}(-5) \oplus \mathcal{O}(-4)^{\oplus m_3} \oplus \mathcal{O}(-3)^{\oplus m_4} \overset{\Phi}{\longrightarrow} \Omega^1_{\mathbb{P}^3}(-2)^{\oplus k} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-3)^{\oplus m_3} \oplus \mathcal{O}(-4)^{\oplus m_4}.
\]

Since \( \Phi \) is an injection, its restriction to \( \mathcal{O}(-3)^{\oplus m_3} \) maps it injectively to \( \mathcal{O}(-2) \oplus \mathcal{O}(-3)^{\oplus m_3} \). Thus, if \( m_4 > 1 + m_3 \), then \( \Phi \) is not injection, which contradicts to the definition of the Casnati–Catanese bundle. Therefore, we can suppose that \( m_4 = 1 + m_3 \). Then \( \Phi \) is of the following form:

\[
\Phi = \begin{pmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & 0 \\
\Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24} \\
\Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} \\
0 & \Phi_{42} & \Phi_{43} & 0
\end{pmatrix}
\]

The block \( \Phi_{42} \) is just a column. Moreover, we can choose the basis in which the squared matrix \((\Phi_{42}, \Phi_{43})\) is of the following form:

\[
(\Phi_{42}, \Phi_{43}) = \begin{pmatrix}
l & l_1 & \cdots & l_{m_3} \\
0 & \lambda_1 & \cdots & \lambda_{m_3} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \lambda_{m_31} & \cdots & \lambda_{m_3 m_3}
\end{pmatrix}.
\]

Here \( l, l_1, \ldots, l_{m_3} \) are some linear forms on \( \mathbb{P}^3 \), and the matrix \( (\lambda_{ij}) \) is the map from \( \mathcal{O}(-3)^{\oplus m_4} \) to \( \mathcal{O}(-3)^{\oplus m_3} \). Since our field is algebraically closed, in some basis this matrix is upper-triangular. In these notations if we compute the determinant of \( \Phi \), we see that \( \det(\Phi) \) is divided by \( l \). However, it is impossible since \( B \) is irreducible. Then the assertion is proved. \( \square \)
We will also need the following assertion:

**Lemma 3.9.** If $w$ is a $\delta/2$-even minimal set of nodes on a sextic $B$, then 

$$h^0(\mathbb{P}^3, \mathcal{F}(4)) = h^0(\mathbb{P}^3, \mathcal{F}(3)) + 12$$

**Proof.** By the Riemann–Roch theorem we have the following (see [JR97, Lemma 4.1]):

\[
\chi(\mathcal{F}(3)) = \frac{59 - |w|}{4}; \\
\chi(\mathcal{F}(4)) = \frac{107 - |w|}{4}.
\]

By Lemma 3.1 we have that $\chi(\mathcal{F}(i)) = h^0(\mathbb{P}^3, \mathcal{F}(i))$ for any $i > 2$; therefore, we get the result. \qed

## 4 Defect

In this section we give a geometric description of the number $d$ in the formula (1.1) and study its properties. We begin with a definition:

**Definition 4.1.** For any finite set of points $S = \{p_1, \ldots, p_k\}$ in $\mathbb{P}^3$ and any integer $N$ the $N$-defect of this set the difference between the dimension of the space of surfaces of degree $N$ which contain $\{p_1, \ldots, p_k\}$ and the dimension of the space of surfaces containing $k$ general points in $\mathbb{P}^3$:

$$d_N(S) := \dim \left( H^0(\mathbb{P}^3, \mathcal{I}_S(N)) - \dim \left( H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(N)) \right) - k \right),$$

where $\mathcal{I}_S$ is a sheaf of ideals of the set $S$ in $\mathbb{P}^3$.

The notion of the $N$-defect is important for us in view of the following theorem.

**Theorem 4.2 ([Cle83, Section 3]).** Let $B$ be a nodal surface in $\mathbb{P}^3$ and $N = 3 \deg(B)/2 - 4$. Then in the notation of formula (1.1) we have $d = d_N(\text{Sing}(B))$.

To simplify the notation, from now on we fix some number $N$ and write $d(S)$ and use the word “defect” instead of writing “$N$-defect” $d_N(S)$. Our goal is to show that if the defect of all special subsets (such as even sets of nodes in the set of all nodes) is large enough, then the defect of the whole set is also large. Let $S$ be a finite set of points in $\mathbb{P}^3$. Denote by $A_S$ the space of all surfaces $Z$ of degree $N$ in $\mathbb{P}^3$ which contain $S$.

**Lemma 4.3.** Assume that the dimension of the space $A_S$ equals $M$. Then $\dim(A_S \setminus \{p\})$ equals either $M$ or $M + 1$, depending on $p \in S$.

**Proof.** Since $A_S \subset A_{S \setminus \{p\}}$, it is enough to prove that $\dim(A_{S \setminus \{p\}}) \leq M + 1$. However, $A_S$ and $A_{S \setminus \{p\}}$ are hyperplanes in the space of surfaces of degree $M$ in $\mathbb{P}^3$, and $A_S$ in $A_{S \setminus \{p\}}$ is the zero locus of an linear function of evaluation in the point $p$. Therefore, the codimension of $A_S$ in $A_{S \setminus \{p\}}$ is less than or equal to 1. \qed

Now we need the definition:

**Definition 4.4.** We say that a subset $S'$ of $S$ generates $S$ if any surface $Z$ of degree $N$ containing $S'$ also contains $S$.

In view of Definition 4.1 of a set of points has the following precise description:

**Lemma 4.5.** The defect $d(S)$ equals the difference $|S| - |S'|$, where $S'$ is a generating subset in $S$ of the minimal cardinality.
Proof. We prove the assertion by induction on \( n = |S| \). If \( n = 1 \), then \( S \) contains just one point, so \( d(S) \) vanishes. In this case the smallest generating subset \( S' \) coincides with \( S \), so the assertion is true.

Now suppose that we have the assertion for all \( k < n \) and \( S \) consists of \( n \) points. Let us choose a point \( p \in S \) and consider the set \( T = S \setminus \{p\} \). By the inductive assumption, we can choose a minimal subset \( T' \) generating \( T \) and the following holds:

\[
d(T) = |T| - |T'|.
\]

However, since every surface of degree \( N \) containing \( S \) should contain \( T \), we have an inequality \( d(T) \leq d(S) \). Now let us consider the possibilities case by case.

If \( d(T) = d(S) \), then there exists a surface \( Z \) of degree \( N \) such that \( T \subset Z \) and \( p \notin Z \). Then the set \( S' = T' \cup \{p\} \) generates \( S \). Let us prove that \( S' \) is minimal.

Assume the contrary: there exists a generating subset \( S'' \) of \( S \) of cardinality less than \( |S| \). Then \( S'' \) should contain \( p \) since we assume the existence of \( Z \). As we assume, surfaces of degree \( N \) passing through \( S \) and therefore through \( S'' \) form an \( M \)-dimensional space where

\[
M = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(N)) - |S| - d(S).
\]

Also by our assumption on the defect of \( T \), the space of surfaces of degree \( N \) passing through \( T \) is of dimension \((M + 1)\).

All surfaces of degree \( N \) that contain \( T \) should pass through \( S'' \cap T \). Then there exist at least \((M + 1)\)-dimensional space of surfaces of degree \( M \) passing \( S'' \cap T \). However, by Lemma 4.3 the dimension of the space of surfaces of degree \( N \) passing through \( S'' \setminus \{p\} \) can not be more than \((M + 1)\), so \( S'' \cap T \) generates \( T \). Therefore, \( S'' \) contains \( p \) and the generating set of \( T \), so it can not have more elements than \( S' \).

If \( d(T) \leq d(S) - 1 \), then any surface \( Z \) of degree \( N \) passing through \( T \) should pass through \( S \). Therefore, \( d(T) = d(S) - 1 \) and \( T \) generates \( S \). By induction, we can construct a minimal generating subset \( T' \) in \( S \) satisfying the condition in the assertion. Then \( S' = T' \) is obviously a minimal generating subset in \( S \) and it also satisfies the condition in assertion.

As an easy corollary of Lemma 4.5, we can show that the defect of the subset can not be greater than the defect of the whole set.

Lemma 4.6. If \( T \subset S \), then \( d(T) \leq d(S) \).

Proof. Let us denote by \( T' \) some generating subset of \( T \). Then \( S \) can be generated by \( S' := (S \setminus T) \cup T' \). Therefore, we have

\[
d(S) \geq |S| - |S'| = |S| - |S\setminus T| - |T'| = |T| - |T'| = d(T).
\]

Lemma 4.6 connects the notion of the defect with some special subsets which generate the whole set. In the next lemma we show that we can construct a lot of different minimal generating subsets.

Lemma 4.8. If \( T \subset S \) is a subset and \( d(T) = 0 \), then we can construct a minimal generating subset \( S' \) of \( S \) such that \( S' \) contains \( T \).

Proof. We prove the assertion by induction on \( n = |S| - |T| \). If \( n = 0 \), then \( d(S) = d(T) = 0 \), so \( S' = T \) is a minimal generating set. Now, assume that the assertion is true for all \( k < n \), and assume that we have sets \( S \) and \( T \) such that \( |S| - |T| = n \). In this situation we have two possibilities: either there exists a point \( p \in S \setminus T \) such that \( d(S \setminus \{p\}) = d(S) - 1 \) or not.

In the first case it follows that \( S \setminus \{p\} \) generates \( S \). By induction, we can construct a minimal generating subset \( S' \subset S \setminus \{p\} \) containing \( T \). Thus \( S' \) will be the required minimal generating subset.

Otherwise, let us fix some point \( p \in S \setminus T \). By assumption, the set \( S \setminus \{p\} \) does not generates \( S \). Then we denote by \( S' \) the union of \( \{p\} \) and a minimal generating subset of \( S \setminus \{p\} \) containing \( T \). By the similar reasons as in Lemma 4.5 the subset \( S' \) is minimal and obviously it generates \( S \).

Now let us move closer to the situation of even sets of nodes. From now on we fix a subspace \( C \subset \mathbb{P}^2 \) in the \( \mathbb{P}_2 \)-space spanned by the set \( S = \{p_1, \ldots, p_n\} \) of points in \( \mathbb{P}^3 \). Let us assume that subsets \( S_1, \ldots, S_n \) of \( S \) form a basis in \( C \). In the next lemma we are going to show how to construct a basis with useful properties by the fixed one. To formulate it we need a new notation:
Definition 4.9. Denote by $P_{i_1,\ldots,i_k}^n$ the following subset of $S$:

$$P_{i_1,\ldots,i_k}^n := (S_{i_1} \cap \cdots \cap S_{i_k}) \setminus \bigcup_{j \notin \{i_1,\ldots,i_k\}} S_j.$$  

Lemma 4.10. For any set of indices $I = \{i_1,\ldots,i_k\}$ with $i_1 = 1$ there exist subsets $T_1,\ldots,T_n$ of $S$ such that they form a basis of $C$, $P_I^n \subset T_1 = S_1$, and $P_I^n \cap T_j$ is empty for all $j > 1$.

Proof. Let us reformulate our assertion in terms of linear algebra. As well as all disjoint subsets $P_I^n$ of $S$, any element $S_I = \sum_{i \in I} S_i$ of $C$ corresponds to subset of indices $I$. These subsets can be considered as elements of the vector space $\mathbb{F}_2^n$. We endow this space with the standard scalar product $\langle \cdot, \cdot \rangle$. Then we have the following:

$$S_J \cap P_I^n = \begin{cases} P_I^n, & \text{if } \langle I, J \rangle = 1; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Now let us consider a linear function $L_I$ on $C$, such that $L_I(S_J) = \langle I, J \rangle$. Then all we need in the assertion is just to construct the basis $T_{J_1},\ldots,T_{J_n}$ in $C$ such that $L_I$ is non-trivial on the first element of the basis and vanishes on all others. This finishes the proof. \hfill \square

In the next assertion we apply previous results in order to connect the defect of a set with the defects of its subsets.

Proposition 4.11. If $d(T) \geq 1$ for any element $T \in C$, then $d(S) \geq \dim(C)$.

Proof. Let us prove this assertion by induction on $n = \dim(C)$. For $n = 1$ it is trivial. Let us assume that it holds for any $k < n$ and consider the $n$-dimensional space of sets $C$. Fix a basis $S_1,\ldots,S_n$ some basis in $C$. Then the defect of $S_1$ is greater or equal than 1 by assumption. Let us choose a generating subset $S'_1$ of $S_1$. By Lemma 4.5, the difference $S_1 \setminus S'_1$ is not empty and contains a point $p_I$. By Definition 4.10 we have

$$\prod_{I \subset \{p_1,\ldots,p_n\}} P_I^n = S.$$ 

Therefore, there exists $I$, such that $p_I \in P_I^n$ and $p \notin S_J^n$ for any $J \neq I$. By Lemma 4.10 we can construct a basis $T_1,\ldots,T_n$ of $C$ such that $S_J^n \subset T_1 = S_1$ and $S_J^n \cap T_j = \varnothing$ for $j > 1$.

Let us denote $T = \bigcup_{j=2}^n T_j$. Since by construction the set $S'_1$ is the generating subset for $T_1$ and $d(S'_1) = 0$, we have

$$d(S'_1 \cap T) = 0.$$ 

Then by Lemma 4.3 we can complete the intersection $S'_1 \cap T$ to a generating subset $T'$ of $T$. However, by the inductive assumption applied to the subspace $\langle T_2,\ldots,T_n \rangle$ we can suppose that $d(T) \geq n - 1$. Thus $|T \setminus T'| \geq n - 1$; moreover, by choice of the basis $p \notin T'$.

Therefore, we construct the generating set $T' \cup S'_1$ of $S$ that consists of less than or equal to $\mu - n$ points. Using Lemma 4.5 we prove the assertion. \hfill \square

Finally, we are ready to state the main property of defects of even sets of nodes.

Corollary 4.12. If $B$ is a nodal surface and for any even set of nodes $w \in \overline{C_B}$ one has $d(w) \geq 1$ then $d(\text{Sing}(B)) \geq \dim(\overline{C_B})$.

Proof. Let us consider the $\mathbb{F}_2$-space $\overline{C_B}$. Its elements are subsets of points in $\mathbb{F}_2^3$ and by assumption the defect of any of these subsets is greater than zero. Thus, the conditions of Proposition 4.11 hold. Therefore, we get the result. \hfill \square
5 Defect of minimal 1/2-even sets of nodes

In this section we are going to estimate the defects of the even sets of nodes which appear in Proposition 3.4.

From now on by defect we set $N = 5$; thus, the defect is the 5-defect. Note that we have a useful informations on the defects of the even sets of nodes cut out by planes.

**Proposition 5.1** ([End99 Page 6]). If $w$ is cut out by a plane, then its defect is greater than zero.

We are looking for 1/2-even sets of nodes with zero defect; thus, we assume that they are not cut out by planes. Our goal is to prove the following assertion:

**Proposition 5.2.** Assume that $B$ is a nodal sextic, $w$ is a minimal 1/2-even set of nodes on $B$ and $d(w) = 0$. Then, in the notation of Proposition 3.4, we have that either

1. $m_2 = 0$ and $k \geq 3$ or $k = 0$;
2. $m_2 = 1$ and $k = 0$.

Here will be very useful the following interpretation of defect of set of some set of points:

**Lemma 5.3.** For any set of points $w \subset \mathbb{P}^3$ we have

$$d(w) = h^1(\mathbb{P}^3, \mathcal{I}_w(5)) = h^0(\mathbb{P}^3, \mathcal{I}_w(5)) - 56 + |w|.$$

**Proof.** Let us consider the following exact sequence:

$$0 \to \mathcal{I}_w(5) \to \mathcal{O}_{\mathbb{P}^3}(5) \to \mathcal{O}_w \to 0.$$  \hspace{1cm} (5.5)

Since $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5)) = 56$ and $h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5)) = 0$, the long exact sequence of cohomology gives us the result.

We will use the following free resolution of the sheaf of ideals $F_\bullet \to \mathcal{I}_w(5)$:

**Lemma 5.4.** If $B$ is a nodal sextic surface with a $\delta/2$-even set of nodes $w$ and $\mathcal{E}$ its Casnati-Catanese bundle, then we have the following exact sequence:

$$0 \to \Lambda^2 \mathcal{E}^\vee(n_3) \to sl(\mathcal{E})(n_2) \xrightarrow{\phi} S^2 \mathcal{E}(n_1) \to \mathcal{I}_w(5) \to 0,$$

where $(n_1, n_2, n_3) = (5, -1, -7)$ or $(6, -1, -8)$ for $\delta = 0$ and $1$ respectively. By $sl(\mathcal{E})$ we denote the bundle of traceless endomorphisms of $\mathcal{E}$. The morphism $\phi$ is the composition:

$$\phi : sl(\mathcal{E})(-1) \hookrightarrow \mathcal{E} \otimes \mathcal{E}^\vee(-1) \xrightarrow{\Id_{\mathcal{E}} \otimes \phi(6)} \mathcal{E} \otimes \mathcal{E}(6) \to S^2 \mathcal{E}(6).$$

**Proof.** The result follows from [Joz78 Theorem 3.1]. Actually, there we have such a situation: we consider a commutative Noetherian $\mathbb{Q}$-algebra $R$ and denote by $\text{Mat}_n(R)$, $\text{Alt}_n(R)$, and $\text{Sym}_n(R)$ the space of $n \times n$ matrices over $R$ and its subspaces of alternating and symmetric matrices respectively. If $M \in \text{Sym}_n(R)$ is some symmetric matrix over $R$, we have the following exact sequence:

$$0 \to \text{Alt}_n(R) \to \text{Ker} \left( \text{Mat}_n(R) \xrightarrow{\text{Tr}} R \right) \to \text{Sym}_n(R) \to I_{n-1} \to 0,$$

where by $I_{n-1}$ we denote the ideal of $(n-1)$-minors of the matrix $M$ and $\text{Tr}$ is the trace map. The differentials in this sequence are described in terms of $M$, for more details see [Joz78 Section 3]. Note that the kernel $\text{Ker} \left( \text{Mat}_n(R) \xrightarrow{\text{Tr}} R \right)$ is isomorphic to the quotient of the space of matrices by the subgroup of scalar matrices:

$$\text{Ker} \left( \text{Mat}_n(R) \xrightarrow{\text{Tr}} R \right) \cong \text{Mat}_n(R)/R.$$

Now let us return to the exact sequence in the assertion of the lemma. Let us prove its existence locally: we restrict to some affine open subset of $\mathbb{P}^3$ and consider the restriction of the operator $\Phi$ as a symmetric matrix $M$. Then, in view of the discussion above, this sequence can be constructed since all morphisms agree on the intersections of the affine charts. Moreover, we know, that

$$n_1 - n_2 = n_2 - n_3 = \deg(\Phi) = 6 + \delta.$$

Since $\deg(\mathcal{I}_w(5)) = 5$, we can compute all the numbers $n_1$, $n_2$ and $n_3$. □
For simplicity let us denote the bundles in the resolution as follows

\[ F_0 := S^2 E(6); \]
\[ F_1 := sl(E)(-1); \]
\[ F_2 := \Lambda^2 E'(8). \]

Denote by \( V \) the vector space of dimension 4 such that \( \mathbb{P}^3 = \mathbb{P}(V) \). We can describe cohomologies of the bundles \( F_i \) in terms of the spaces of the bundles \( W \) and \( U_i \):

**Lemma 5.6.** The bundles \( F_0 \) and \( F_1 \) have the following cohomology groups:

\[
\begin{align*}
H^0(\mathbb{P}^3, S^2 E(6)) &= (W \otimes U_2 \otimes \Lambda^2 V') \oplus (S^2 U_2 \otimes S^2 V') \oplus (U_2 \otimes U_3 \otimes V') \oplus (S^2 U_3 \otimes U_2 \otimes U_4); \\
H^1(\mathbb{P}^3, S^2 E(6)) &= (S^2 W \otimes \Lambda^2 V') \oplus (W \otimes U_4); \\
H^0(\mathbb{P}^3, sl(E)) &= (U_2 \otimes W' \otimes V) \oplus (U_2 \otimes U_3) \oplus (U_3 \otimes U_4) \oplus (U_2 \otimes U_4 \otimes V'); \\
H^0(\mathbb{P}^3, sl(E)) &= (W \otimes W' \otimes V) \oplus (W \otimes U_3); \\
H^0(\mathbb{P}^3, \Lambda^2 E'(8)) &= \Lambda^2 U'_4; \\
H^1(\mathbb{P}^3, \Lambda^2 E'(8)) &= S^2 W'; \\
H^2(\mathbb{P}^3, \Lambda^2 E'(8)) &= U'_4 \otimes W'; \\
H^3(\mathbb{P}^3, \Lambda^2 E'(8)) &= \Lambda^2 U'_4.
\end{align*}
\]

And all other cohomology groups vanish.

**Proof.** Bundles \( F_0, F_1 \) and \( F_2 \) split into a direct sum of tensor degrees of tangent and line bundles of \( \mathbb{P}^3 \):

\[
S^2 E(6) = (S^2 W \otimes S^2 \Omega^1(2)) \oplus (\Lambda^2 W \otimes \Lambda^2 \Omega^1(2)) \oplus (W \otimes U_2 \otimes \Omega^1(2)) \oplus (W \otimes U_3 \otimes \Omega^1(1)) \\
\oplus (W \otimes U_4 \otimes \Omega^1) \oplus (S^2 U_2 \otimes O(2)) \oplus (U_2 \otimes U_3 \otimes O(1)) \oplus (S^2 U_3 \otimes O) \oplus (U_2 \otimes U_4 \otimes O) \\
\oplus (U_3 \otimes U_4 \otimes O(-1)) \oplus (S^2 U_4 \otimes O(-2)); \\
sl(E)(-1) = (W' \otimes W \otimes \Omega^1 \otimes T(-1)) \oplus (W' \otimes U_2 \otimes T(-1)) \oplus (W' \otimes U_3 \otimes T(-2)) \\
\oplus (W' \otimes U_4 \otimes T(-3)) \oplus (U'_3 \otimes W \otimes \Omega^1(-1)) \oplus (U'_3 \otimes U_2 \otimes O(-1)) \oplus (U'_3 \otimes U_3 \otimes O(-2)) \\
\oplus (U'_3 \otimes U_4 \otimes O(-3)) \oplus (U'_3 \otimes W \otimes \Omega^1) \oplus (U'_3 \otimes U_2 \otimes O) \oplus (U'_3 \otimes U_3 \otimes O(-1)) \\
\oplus (U'_3 \otimes U_4 \otimes O(-2)) \oplus (U'_3 \otimes W \otimes \Omega^1(1)) \oplus (U'_4 \otimes U_2 \otimes O(1)) \oplus (U'_4 \otimes U_3 \otimes O) \\
\oplus (U'_4 \otimes U_4 \otimes O(-1)); \\
\Lambda^2 E'(8) = (S^2 W' \otimes \Lambda^2 T(-4)) \oplus (\Lambda^2 W' \otimes S^2 T(-4)) \oplus (W' \otimes U'_2 \otimes T(-4)) \oplus (W' \otimes U'_3 \otimes T(-3)) \\
\oplus (W' \otimes U'_4 \otimes T(-2)) \oplus (\Lambda^2 U'_2 \otimes O(-4)) \oplus (U'_2 \otimes U'_3 \otimes O(-3)) \oplus (U'_2 \otimes U'_4 \otimes O(-2)) \\
\oplus (\Lambda^2 U'_3 \otimes O(-2)) \oplus (U'_3 \otimes U'_4 \otimes O(-1)) \oplus (\Lambda^2 U'_4 \otimes O)
\]

Computing cohomology groups of standard bundles on \( \mathbb{P}^3 \), we get the result. \( \square \)

The decomposition \([3.7]\) induces the decomposition of the morphism \( \phi \). Actually, since the bundles \( F_0 \) and \( F_1 \) can be included in the tensor degrees of \( E \), their decomposition induces \( \phi = (\phi_{i,j}, \phi'_{i,j}) \). Thus this decomposition allows us to describe morphisms between cohomologies of \( F_i \):

**Lemma 5.7.** Assume \( D \): \( H^1(\mathbb{P}^3, F_1) \rightarrow H^1(\mathbb{P}^3, F_0) \) is induced by \( \phi \).

1. If \( k = 1 \), then the codimension of \( \text{Coker}(D) \) greater of equal than \( 6 + m_4 - m_3 \);
2. If \( k = 2, m_4 = 0 \) and \( m_3 = 4 \), then \( \text{Coker}(D) \) is non-zero.
3. If \( k > 0 \) and \( m_2 = 1 \) then \( \text{Coker}(D) \) is at least 4-dimensional.
Proof. By Lemma 5.6 and the decomposition of $\phi$ we see that $D$ is the linear map:

$$D = \left( \begin{array}{cc} \phi_{11,11}^{11,11} & \phi_{11,14}^{11,11} \\ \phi_{14,11}^{11,11} & \phi_{13,14}^{11,11} \end{array} \right) : (W \otimes W^\vee \otimes V) \oplus (W \otimes U_3^\vee) \to (S^2W \otimes \Lambda^2V) \oplus (W \otimes U_4).$$

The map $\phi_{11,14}^{11,11}$ is induced by the component $\Phi_{14}$ of $\Phi$. Since by construction

$$\Phi_{14} \in H^0(\mathbb{P}^3, \mathcal{E}_1 \otimes \mathcal{E}_4(7)) = W \otimes U_4 \otimes H^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(1)) = 0,$$

we get that $\phi_{11,11}^{11,11} = 0$. The map $\phi_{11,11}^{11,11}$ is induced by $\Phi_{11}$, which lies in the space

$$\Phi_{11} \in H^0(\mathbb{P}^3, S^2 \mathcal{E}_1 \otimes \mathcal{O}_{\mathbb{P}^3}(7)) = H^0(\mathbb{P}^3, S^2(\Omega^1(-2) \otimes W) \otimes \mathcal{O}_{\mathbb{P}^3}(7)) = \Lambda^2W \otimes V.$$

If $k = 1$, then $\Lambda^2W = 0$. Thus, $\Phi_{11}$; and therefore, $\phi_{11,11}^{11,11}$ vanish. Then we have $\phi_{11,11}^{11,11} = \phi_{11,14}^{11,14} = 0$, and the restriction $D|_{W \otimes W^\vee \otimes V}$ equals zero. Therefore, $D(H^1(\mathbb{P}^3, F_1)) = D(W \otimes U_3^\vee)$ is of dimension less or equal than $m_3$. Since $\dim(H^1(\mathbb{P}^3, F_0)) = 6 + m_4$, we prove the first assertion.

If $k = 2$, then $\phi_{11,14}^{11,14}$ still vanishes, and $\phi_{11,11}^{11,11}$ is induced by $\Phi_{11} \in \Lambda^2W \otimes V$. Since by our assumption $\dim(\Lambda^2W) = 1$, we can suppose that $\Phi_{11} = (w_1 \wedge w_2) \otimes v_{12}$ for some basis $w_1, w_2$ of $W$ and some $v_{12}$ in $V$. Then, if we apply $\phi_{11,11}^{11,11}$ to some element $\xi \otimes v$ in $\text{Hom}(W, W) \otimes V = W \otimes W^\vee \otimes V$, we get the following:

$$\phi_{11,11}^{11,11}(\xi \otimes v) = (\xi(w_1) \cdot w_2 - \xi(w_2) \cdot w_1) \otimes (v \wedge v_{12}). \quad (5.8)$$

Thus, for any $\xi \in \text{Hom}(W, W)$ and any $v \in V$ the image of $\phi_{11,11}^{11,11}$ is a multiple of $v_{12}$. Therefore, we have

$$D(W \otimes W^\vee \otimes V) \subset S^2W \otimes (V \wedge v_{12}).$$

The space $S^2W \otimes (V \wedge v_{12})$ is of codimension 9 in $S^2W \otimes \Lambda^2V$. Since $\dim(W \otimes U_3^\vee) = km_2 = 8$, we prove the second assertion of the lemma.

Now, if $m_2 = 1$, then by Lemma 5.6 the dimensions of $H^1(\mathbb{P}^3, F_1)$ and $H^1(\mathbb{P}^3, F_0)$ are equal. Then we have that $\dim(\ker(D)) = \dim(\text{coker}(D))$. Since $\phi_{11,14}^{11,14}$ vanishes on $W \otimes W^\vee \otimes V$ and $\phi_{11,11}^{11,11}$ maps any element of the subspace $\text{Id}_W \otimes V$ to zero by (5.8), the kernel of $D$ is at least 4-dimensional. This proves the third assertion.

Now we are ready to prove Proposition 5.2.

Proof of Proposition 5.2. Let us consider the spectral sequence $E_1^{pq} = H^p(\mathbb{P}^3, F_q) \Rightarrow H^{p-q}(\mathbb{P}^3, \mathcal{I}_w(5))$:

$$\begin{array}{c}
\Lambda^2U_2^\vee \\
W^\vee \otimes U_2^\vee \\
S^2W^\vee \\
H^0(\mathbb{P}^3, F_2) \\
\end{array} \longrightarrow \begin{array}{c}
0 \\
0 \\
(V \otimes W \otimes W^\vee) \oplus (W \otimes U_3^\vee) \\
H^0(\mathbb{P}^3, F_1) \\
\end{array} \longrightarrow \begin{array}{c}
0 \\
(D) \\
(H^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(1))) \\
H^0(\mathbb{P}^3, F_0) \\
\end{array}$$

This spectral sequence converges to the cohomology groups of $\mathcal{I}_w(5)$. Since $d(w) = 0$, we have $h^1(\mathbb{P}^3, F_0) = 0$ by Lemma 5.6. Then on the infinity page of the spectral sequence we have $E_\infty^{2,0} = 0$. Therefore, we get that $E_1^{3,3} = 0$ and $E_\infty^{3,1} = 0$. Then, since $0 = E_1^{3,3} = \Lambda^2U_2^\vee$, we get $m_2 = 0$ or 1.

First assume that $m_2 = 0$ and $k = 1$. By the third assertion of Lemma 5.6 and Lemma 5.7 we have:

$$\dim(E_\infty^{1,1}) \geq \dim(\text{coker}(D_1)) - \dim(E_1^{2,3}) = 6 + m_4 - m_3 - m_2 = k \geq 1,$$

and this contradicts to assumption $E_\infty^{1,1} = 0$. 

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Now assume, \( m_2 = 0 \) and \( k = 2 \). By second and third assertions of Lemma \([5,3]\) we have that \( m_4 = 0 \) and \( m_3 = 4 \). By Lemma \([5,7]\) we get that \( \dim(\text{Coker}(D)) \neq 0 \). Since \( \dim(E^{2,3}) = km_2 = 0 \), we get that \( E^{1,1}_k \neq 0 \).

Therefore, we conclude that if \( m_2 = d(w) = 0 \), then either \( k = 0 \) or \( k \geq 3 \) and this is the first option of Proposition \([5,2]\). Now let us consider the case \( m_2 = 1 \) and \( 0 < k < 4 \). Then by the third assertion of Lemma \([5,7]\) we have that \( \dim(\text{Coker}(D)) \) is greater than or equal to 4. Therefore, since \( \dim(E^{2,3}) = k < 4 \), we have that \( E^{1,1}_k \neq 0 \) and this contradicts our assumptions.

Finally, assume that \( m_2 = 1 \) and \( k \geq 4 \). By the third assertion of Lemma \([3,6]\) we have that in this case

\[
m_4 = m_3 + k - 3 > m_3.
\]

In view of Lemma \([3,8]\) this contradicts the irreducibility of \( B \), so this case is impossible.

Therefore, we conclude that if \( m_2 = 1 \) and the defect of \( w \) vanishes, then \( k = 0 \) and this is the second option of Proposition \([5,2]\). \( \square \)

Now let us provide here also some useful computations:

**Lemma 5.9.** Assume \( w \) is a \( \delta/2 \)-even set of nodes on a nodal sextic \( B \), and \( \mathcal{E} \) is its Casnati-Catanese bundle.

1. If \( \delta = 0 \) and \( \mathcal{E} = \mathcal{O}_{P^3}(-2)^{\oplus 3} \) or \( \Omega^1_{P^3}(-1) \oplus \mathcal{O}_{P^3}(-2) \), then \( d(w) = 0 \);
2. If \( \delta = 1 \) and \( \mathcal{E} = \mathcal{O}_{P^3}(-3)^{\oplus 3} \oplus \mathcal{O}_{P^3}(-2) \) or \( \mathcal{O}_{P^3}(-3)^{\oplus 6} \), then \( d(w) = 0 \).

**Proof.** Let us start with a \( 0 \)-even \( w \) with the Casnati–Catanese bundle \( \mathcal{O}_{P^3}(-2)^{\oplus 3} \). Substituting it into the exact sequence \([5,3]\), we get the following:

\[
0 \to \mathcal{O}_{P^3}(-3)^{\oplus 3} \to \mathcal{O}_{P^3}(-1)^{\oplus 8} \to \mathcal{O}_{P^3}(1)^{\oplus 6} \to \mathcal{I}_w(5) \to 0.
\]

Therefore, \( h^1(P^3, \mathcal{I}_w(5)) = 0 \) and so by Lemma \([5,3]\) we have \( d(w) = 0 \).

Now if \( \delta = 0 \) and \( \mathcal{E} = \Omega^1_{P^3}(-1) \oplus \mathcal{O}_{P^3}(-2) \), the resolution of \( \mathcal{I}_w(5) \) is as follows by Lemma \([5,3]\):

\[
0 \to \Lambda^2 T_{P^3}(-5) \oplus T_{P^3}(-4) \to T_{P^3} \oplus \Omega^1_{P^3}(-1) \oplus T_{P^3}(-2) \oplus \Omega^1_{P^3} \to \mathcal{S}^2 \Omega^1_{P^3}(-1) \oplus \Omega^1_{P^3}(2) \oplus \mathcal{O}_{P^3}(1) \to \mathcal{I}_w(5) \to 0.
\]

Then in view of the following table of cohomologies we get that \( h^1(P^3, \mathcal{I}_w(5)) = 0 \) and \( d(w) \) vanishes by Lemma \([5,3]\).

| \( F \) | \( H^0(P^3, F) \) | \( H^1(P^3, F) \) | \( H^2(P^3, F) \) | \( H^3(P^3, F) \) |
|---|---|---|---|---|
| \( \Lambda^2 T_{P^3}(-5) \) | 0 | 0 | \( \text{det}(V) \) | 0 |
| \( T_{P^3}(-4) \) | 0 | 0 | \( \text{det}(V) \) | 0 |
| \( T_{P^3} \oplus \Omega^1_{P^3}(-1) \) | 0 | \( V \) | 0 | 0 |
| \( T_{P^3}(-2) \) | 0 | 0 | 0 | 0 |
| \( \Omega^1_{P^3} \) | 0 | \( \mathbb{C} \) | 0 | 0 |
| \( S^2 \Omega^1_{P^3}(-1) \) | 0 | 0 | 0 | 0 |
| \( \Omega^1_{P^3}(2) \) | \( \Lambda^2 V^{\ast} \) | 0 | 0 | 0 |
| \( \mathcal{O}_{P^3}(1) \) | \( V^{\ast} \) | 0 | 0 | 0 |

Finally, let us consider an \( 1/2 \)-even \( w \) with \( \mathcal{E} = \mathcal{O}_{P^3}(-3)^{\oplus 3} \oplus \mathcal{O}_{P^3}(-2) \) or \( \mathcal{O}_{P^3}(-3)^{\oplus 6} \). Then in view of Lemma \([5,6]\) we see that \( h^1(P^3, \mathcal{I}_w(5)) = 0 \). So \( d(w) = 0 \) by Lemma \([5,3]\). \( \square \)

Also we will need the following assertion:

**Lemma 5.10.** If \( B \) is a nodal sextic, \( w \) is a minimal \( 1/2 \)-even set of nodes on \( B \) which is not cut out by a plane and \( m_2 = 1 \), then the canonical morphism

\[
\gamma : H^0(P^3, F(2)) \otimes V^{\ast} \to H^0(P^3, F(3))
\]

is an embedding.
Proof. If $E$ is a Casnati–Catanese bundle of $w$, then $\gamma$ extends to a morphism between exact sequence of cohomologies of twists of the exact sequence (1.2):

$$
\begin{array}{cccc}
0 & H^0(\mathbb{P}^3, \mathcal{E}(-5)) \otimes V^\vee & H^0(\mathbb{P}^3, \mathcal{E}(2)) \otimes V^\vee & H^1(\mathbb{P}^3, \mathcal{E}(-5)) \otimes V^\vee \\
\downarrow{\alpha} & \downarrow{\beta} & \downarrow{\gamma} & \\
0 & H^0(\mathbb{P}^3, \mathcal{E}(-4)) & H^0(\mathbb{P}^3, \mathcal{E}(3)) & H^1(\mathbb{P}^3, \mathcal{E}(-4))
\end{array}
$$

By Proposition 3.3 groups $H^2(\mathbb{P}^3, \mathcal{E}(-5))$ and $H^1(\mathbb{P}^3, \mathcal{E}(-4))$ vanish. In view of the above notation this diagram is as follows:

$$
\begin{array}{cccc}
0 & U_2 \otimes V^\vee & H^0(\mathbb{P}^3, \mathcal{F}(2)) \otimes V^\vee & 0 \\
\downarrow{\alpha=0} & \downarrow{\beta} & \downarrow{\gamma} & \\
0 & U_2 \otimes V^\vee & U_3 & H^0(\mathbb{P}^3, \mathcal{F}(3))
\end{array}
$$

The morphism $\beta$ is an embedding of the first direct summand. The map $\varphi$ is induced by the twist of the morphism $\Phi$, so we have $\varphi = \Phi_{24} + \Phi_{34}$. The map $\gamma$ is an embedding only if the component $\Phi_{34}$ is injective.

But if we assume that $\Phi_{34}(\varphi) = 0$ for some $\varphi \in U_4^\vee$, then $\det(\Phi)$ is divisible by the linear form $\Phi_{24}(\varphi)$ and $B$ is not irreducible. Then $\gamma$ is an embedding.

6 Proof of Theorem 1.7

In this section we finally prove Theorem 1.7 and specify surfaces such that double covers of $\mathbb{P}^3$ branched in them have a non-trivial obstructions to rationality of Artin–Mumford type. First, we list all 1/2-even minimal sets of nodes on a sextic surface with zero defect.

Proposition 6.1. If $B$ is a nodal sextic, $w$ is a minimal 1/2-even set of nodes on $B$ which is not cut out by a plane with $d(w) = 0$, then either $(k, m_2, m_3, m_4) = (0, 0, 6, 0)$ or $(0, 1, 3, 0)$.

Proof. By Proposition 5.2 we have either $m_2 = 0$ or 1. Let us start with the case $m_2 = 0$. Then by the second and the third assertions of Lemma 3.6 we have that $m_4 = 0$. Then $H^0(\mathbb{P}^3, \mathcal{F}(2))$ and $H^0(\mathbb{P}^3, \mathcal{F}(3))$ generates the graded $\mathbb{R}$-module $M$; thus, we have a surjection

$$
\varepsilon : H^0(\mathbb{P}^3, \mathcal{F}(3)) \otimes V^\vee \rightarrow H^0(\mathbb{P}^3, \mathcal{F}(4)).
$$

This implies that $4h^0(\mathbb{P}^3, \mathcal{F}(4)) \geq h^0(\mathbb{P}^3, \mathcal{F}(4))$; then since $m_4 = h^0(\mathbb{P}^3, \mathcal{F}(3))$ in this case by Lemma 3.3 we have that

$$
4m_4 = 4h^0(\mathbb{P}^3, \mathcal{F}(4)) \geq h^0(\mathbb{P}^3, \mathcal{F}(4)) = h^0(\mathbb{P}^3, \mathcal{F}(3)) + 12 = m_3 + 12.
$$

Thus $m_3 \geq 4$; by the second assertion of Lemma 3.6 we get $k \leq 2$; and therefore, by Proposition 5.2 the assumption $d(w) = 0$ implies that $k = 0$. So the only possible collection of numbers $(k, m_2, m_3, m_4)$ is $(0, 0, 6, 0)$. By Lemma 5.9 in this case $d(w)$ is actually zero.

Now assume $m_2 = 1$. Then by Proposition 5.2 we get that $k = 0$. Now let us consider the exact sequence on $\mathbb{P}^3$:

$$
0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}(1) \otimes V \rightarrow \mathcal{F}(2) \otimes \Lambda^2 V^\vee \rightarrow \mathcal{F}(3) \otimes V^\vee \rightarrow \mathcal{F}(4) \rightarrow 0.
$$

The spectral sequence of cohomology groups $E_1^{p,q} = H^p(\mathbb{P}^3, \mathcal{F}(q)) \otimes \Lambda^q V$ is as follows:

$$
\begin{array}{cccc}
\cdots & H^2(\mathcal{F}) & \overset{\rho}{\rightarrow} H^1(\mathcal{F}(1)) \otimes V & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & H^0(\mathcal{F}(2)) \otimes \Lambda^2 V^\vee & H^0(\mathcal{F}(3)) \otimes V^\vee & 0 & 0 & 0 \\
\end{array}
$$

$$
\delta' : H^0(\mathbb{P}^3, \mathcal{F}(4)) \rightarrow H^0(\mathbb{P}^3, \mathcal{F}(3)) \otimes V^\vee \rightarrow H^0(\mathbb{P}^3, \mathcal{F}(2)) \otimes \Lambda^2 V^\vee
$$

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By (3.2) the map $\rho$ is Serre dual to the canonical morphism $\gamma$ in Lemma 5.10. Since $\gamma$ is an embedding, $\rho$ is surjective, so $E_2^{2,1} = 0$. Then $D'$ is an only non-zero differentials on next pages. Since we started with an exact sequence, the spectral sequence converges to zero, so $0 = E_2^{0,4} = \text{Coker}(\varepsilon)$. So we get that the graded $R$-module $M$ is generated by its components $H^0(\mathbb{P}^3, F(2))$ and $H^0(\mathbb{P}^3, F(3))$. This implies $m_4 = 0$.

Finally, by Lemma 5.9 in case $(k, m_2, m_3, m_4) = (0, 1, 3, 0)$ we get that $d(w) = 0$. □

Now we specify all 0-even sets of nodes with zero defect.

**Proposition 6.2.** If $w$ is a 0-even set of nodes on a nodal sextic $B$ and $d(w) = 0$, then $|w| = 32$ or $40$, and Casnati–Catanese bundle of $w$ is isomorphic to $\mathcal{O}_{\mathbb{P}^3}(-2)\oplus 3$ or $\Omega^1_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2)$ respectively.

**Proof.** By Theorem 1.4 we have an explicit description of 0-even sets of nodes on nodal sextics. In this case there can appear only four different Casnati–Catanese bundles, which are listed in Theorem 1.3. The case with an even set of 56 nodes obviously has non-zero defect since $h^0(\mathcal{O}_{\mathbb{P}^3}(5)) = 56$ and $w$ is contained in the zero loci of the partial derivatives of the equation of $B$, which are quintics. By [End99, Lemma 3.1] the even set of 24 nodes has non-zero defect. Finally, by Lemma 5.9 in case $|w| = 32$ or $40$ we get that $d(w) = 0$. □

Finally, we are ready to describe sextic surfaces providing obstructions to rationality of double solids.

**Proof of Theorem 1.7.** Assume that $B$ is a nodal sextic surface such that $T_2(\tilde{X}) \neq 0$. By Theorem 1.4 it means that $\dim(C_B) > d(\text{Sing}(B))$ and so by Corollary 1.12 there exists a non-trivial even set of nodes $w \in C_B$ such that $d(w) = 0$.

If $w$ is a 0-even set of nodes, then by Proposition 6.2 its Casnati–Catanese bundle is isomorphic to either $\mathcal{O}_{\mathbb{P}^3}(-2)\oplus 3$ or $\Omega^1_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2)$.

If $w$ is a 1/2-even set of nodes, then we may assume that it is minimal. Otherwise, we can replace it by its proper 1/2-even subset and its defect should also vanish by Lemma 1.0. Moreover, by Proposition 5.1 we get that $w$ is not cut out by a plane. Then by Proposition 6.1 we get the result.

Theorem 1.7 gives us an explicit description of surfaces which can arise as branch surfaces of double solids with non-vanishing Artin–Mumford obstruction to rationality. However, we can not ensure that any surface of this type actually admits this obstruction. Nevertheless, Proposition 1.8 claims that a general surface of this type indeed admits it.

**Proof of Proposition 1.8.** Any vector bundle $\mathcal{E}$ from the list in Theorem 1.7 except the last one is a sum of line bundles over $\mathbb{P}^3$, so $S^2 \mathcal{E}(6 + \delta)$ is also a sum of line bundles, and we can see that all these bundles have positive degrees. So the bundle $S^2 \mathcal{E}(6 + \delta)$ is generated by its global sections.

Also, if we have $\mathcal{E} = \Omega^1_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2)$, then $S^2 \mathcal{E}(6) = S^2 \Omega^1_{\mathbb{P}^3}(4) \oplus \Omega^1_{\mathbb{P}^3}(3) \oplus \mathcal{O}_{\mathbb{P}^3}(2)$. Since each summand is generated by its global sections, the same holds for the bundle $S^2 \mathcal{E}(6)$.

Therefore, by Theorem 1.2 singularities of a general surface $B$ consist just of the even set of nodes $w(\mathcal{E})$. The defect of this even set of nodes vanishes by Lemma 5.9 so by formula (1.1) we get that $T_2(\tilde{X}) \neq 0$. □

### 7 Discussion

**Examples with positive defect**

By Theorem 1.7 we showed that all sextic double solids admitting a non-trivial Artin–Mumford obstruction to rationality are necessarily branched in symmetric surfaces of four exactly defined types. Moreover, by Proposition 1.8 we get that general surfaces of this types gives actual examples of double solids with a non-trivial Artin–Mumford obstruction. Remarkably, for any general surface $B$ of each type the space $\overline{C}_B$ is one-dimensional by Theorem 1.2. So we do not know any examples of a surface $B$ such that $\dim(\overline{C}_B) > 1$ and $T_2(\tilde{X}) \neq 0$.

However, in [CT07, Proposition 3.3] is given a bundle $\mathcal{E}$ which defines a surface $B$ with an even set $w$ of 56 nodes which consists of all singularities of $B$. Another construction of this variety is given in [vGZ18].
By [JR97, Theorem 4.5] in the notation of Section 3 we have \( \dim(C_B) = 1 + 2h^1(\mathbb{P}^3, \mathcal{F}) \). However, by the description [CT07, Proposition 3.3], we can conclude that
\[
\dim(C_B) = 7.
\]
Also the defect \( d(\text{Sing}(B)) > 0 \) in this case, because all 56 points of \( \text{Sing}(B) \) are contained in four quintic surfaces which are zero loci of partial derivative of the equation of \( B \). So if \( d(\text{Sing}(B)) < 7 \) we would get the new family of examples of non-rational sextic double solid and this non-rationality would not be a corollary of [CP10, Corollary A].

The group \( C_B \) is generated by some set of minimal even sets of nodes, moreover, if \( w' \in C_B \), then we have \( w \setminus w' \in C_B \). So \( C_B \) is generated by even sets of 15, 24, 32 and 39 nodes. Moreover, since the sum of any two even sets of nodes is also even, we can conclude, that \( C_B \) is generated just by even sets of 24 and 32 nodes.

Then we have at least two possible ways to estimate \( d(\text{Sing}(B)) \). The first one is to make a straightforward computation similar to the one in Section 5 using the vector bundle \( E \) constructed in [CT07, Proposition 3.3]. The second way to estimate the defect is to use the structure of the group \( C_B \) and to connect somehow the defect \( d(\text{Sing}(B)) \) with defects of all \( w \neq w' \in C_B \) which defects are less than or equal to 1. However, we cannot find any reasonable bound even for the defect of the disjoint union of two sets of points with known defect. The only very rough idea is that it is less than the cardinality of the smaller set, but this is insufficient for our goals.

### Very general sextic double solid

By the degeneration method introduced in [Vol15], the existence of sextic double solids with Artin–Mumford obstructions to stable rationality implies that a very general nodal sextic double solid from a family containing such an example is not stably rational. Therefore, it would be interesting to find out have the examples provided by Theorem 1.7 fit into the parameter space of nodal sextic double solids, similarly to what was done in [Vol15, Section 2]. Note that the existence of any of these examples implies that a very general smooth sextic double solid is not stably rational, but this also follows from the more general result of [HT16].

### Other double covers

It would be interesting to obtain a classification similar to [End99] and Theorem 1.7 for other Fano threefolds that have a double cover structure. It seems that except quartic and sextic double solids three-dimensional Artin–Mumford-type examples are known only for a particular family of double covers of quartics over an intersection with a quartics which have 20 nodes, see [PS16, Theorem 1.2]. In particular, it would be interesting to know if such examples exist for Fano threefolds of type \( X_{10} \) which are double covers of the del Pezzo threefold \( V_5 \), for del Pezzo threefolds of type \( V_1 \), for complete intersection of quadrics and cubics \( X_6 \) that are double covers of cubic hypersurfaces and for complete intersection of three quadrics \( X_8 \) that are double covers of intersection of two quartics.

### Casnati–Catanese bundles

Theorem 1.6 gives necessary conditions for Casnati–Catanese bundles of minimal 1/2-even sets of nodes. It would be interesting to find a complete classification of such Casnati–Catanese bundles, and furthermore, of Casnati–Catanese bundles of arbitrary 1/2-even sets of nodes similar to that of Theorem 1.4.

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