FIELDS GENERATED BY POINTS ON SUPERELLIPTIC CURVES

LEA BENEISH AND CHRISTOPHER KEYES

Abstract. We give an asymptotic lower bound on the number of field extensions generated by algebraic points on superelliptic curves over \( \mathbb{Q} \) with fixed degree \( n \), discriminant bounded by \( X \), and Galois closure \( S_n \). For \( C \) a fixed curve given by an affine equation \( y^m = f(x) \) where \( m \geq 2 \) and \( \deg f(x) = d \geq m \), we find that for all degrees \( n \) divisible by \( \gcd(m, d) \) and sufficiently large, the number of such fields is asymptotically bounded below by \( X^{-n} \), where \( c_n \to 1/m^2 \) as \( n \to \infty \). This bound is determined explicitly by parameterizing \( x \) and \( y \) by rational functions, counting specializations, and accounting for multiplicity. We then give geometric heuristics suggesting that for \( n \) not divisible by \( \gcd(m, d) \), degree \( n \) points may be less abundant than those for which \( n \) is divisible by \( \gcd(m, d) \). Namely, we discuss the obvious geometric sources from which we expect to find points on \( C \) and discuss the relationship between these sources and our parametrization. When one a priori has a point on \( C \) of degree not divisible by \( \gcd(m, d) \), we argue that a similar counting argument applies. As a proof of concept we show in the case that \( C \) has a rational point that our methods can be extended to bound the number of fields generated by a degree \( n \) point of \( C \), regardless of divisibility of \( n \) by \( \gcd(m, d) \).

1. Introduction

Let \( K \) be a number field, and let \( C/K \) be a smooth curve of genus \( g \). Faltings [Fal83] proved that when \( g \geq 2 \), the set of \( K \)-rational points on \( C \), \( C(K) \), is finite, and in fact \( C(L) \) is finite for any finite extension \( L/K \). It is natural to ask if similar finiteness results hold for the higher degree points of \( C \). We say the degree of an algebraic point \( P \in C(\overline{K}) \) is the degree \( [K(P) : K] \), where \( K(P) \) is the minimal field of definition for \( P \). While in fact a curve of genus \( g \geq 2 \) may have infinitely many points of some degree \( n > 1 \), it is still an interesting problem to characterize when this occurs and prove finiteness results for “sporadic” points. There have been several recent works related to the study of higher degree points on families of hyperelliptic curves (see [BGW17, GM19]) and on various modular curves (see [BEL+19, Box21, BGRW20, BNI+20, DEvH+20, OS19]).

Instead of studying the points of \( C \), one can take the perspective of studying the set of field extensions \( K(P)/K \) generated by algebraic points \( P \in C(\overline{K}) \). This idea was suggested by Mazur and Rubin [MR18] in their program for Diophantine stability, where a variety over \( K \) is said to be Diophantine stable for \( L/K \) if its \( K \)-rational points and \( L \)-rational points coincide. A natural first question is to ask how many extensions generated by an algebraic point exist for a fixed degree when ordered by discriminant.

We define the following functions for counting extensions of number fields by discriminant. Let

\[
N_n(X) = \# \left\{ L/K : [L : K] = n, \ |\text{Disc } L/K| \leq X \right\},
\]

where \( X > 0 \) is a real number and \( n \geq 1 \) is any positive integer. For a fixed curve \( C/K \), we define the counting function for extensions generated by an algebraic point of \( C \) to be

\[
N_{n,C/K}(X) = \# \left\{ K(P)/K : P \in C(\overline{K}), \ [K(P) : K] = n, \ |\text{Disc } K(P)/K| \leq X \right\}.
\]

We further define

\[
N_{n,C/K}(X, G) = \# \left\{ K(P)/K : P \in C(\overline{K}), \ [K(P) : K] = n, \ |\text{Disc } K(P)/K| \leq X, \ \text{Gal}(\widetilde{K}(P)/K) \simeq G \right\}
\]

where \( G \) is a permutation subgroup of the symmetric group \( S_n \) and \( \widetilde{K}(P) \) denotes the Galois closure of \( K(P)/K \).
When $E$ is an elliptic curve over $\mathbb{Q}$, Lemke Oliver and Thorne [LT19] show $N_{n,E/\mathbb{Q}}(X, S_n) \gg X^{c_n-\epsilon}$ for a positive constant $c_n$ approaching $1/4$ from below as $n \to \infty$. Conditionally, this exponent can be improved to approach $1/4$ from above. In fact, they show something stronger, namely that $X^{c_n-\epsilon}$ is an asymptotic lower bound on degree $n$ extensions for which the Mordell–Weil ranks satisfy $\text{rk } E(K) > \text{rk } E(\mathbb{Q})$, with root number $\pm 1$. In [Key19] the second author proves that for a hyperelliptic curve $C/\mathbb{Q}$ of genus $g \geq 1$ and for $n$ sufficiently large relative to $C$, with $n$ even if the defining polynomial of $C$ has even degree, we have $N_{n,C/\mathbb{Q}}(X, S_n) \gg X^{c_n}$, where $c_n$ is again a constant depending on $g$ which tends to $1/4$ from below as $n \to \infty$.

We continue this program of studying the the set of fields generated by points on curves in the case of superelliptic curves. We fix $\mathbb{Q}$ to be the base field, and use $N_{n,C}(X, S_n)$ for $N_{n,C/\mathbb{Q}}(X, S_n)$ where it will not create confusion. For a positive integer $m \geq 2$, a superelliptic curve $C/\mathbb{Q}$ of exponent $m$ is given by an affine equation

$$C : y^m = f(x) = \sum_{i=0}^{d} c_i x^i,$$

where $f(x) \in \mathbb{Z}[x]$ is a squarefree polynomial of degree $d$. In this paper, we restrict further to the case where $m \leq d$. Such a curve possesses a degree $m$ map to the projective line $\mathbb{P}^1$ defined over $\mathbb{Q}$, sending a point $(x, y) \mapsto x$. When $\gcd(m, d) \mid n$ and $n$ is sufficiently large, we have the following asymptotic lower bound for $N_{n,C}(X, S_n)$.

**Theorem 1.** Let $C$ be a superelliptic curve with equation (1.1) and suppose $n$ is a multiple of $\gcd(m, d)$ satisfying

$$n \geq \max(d, \text{lcm}(m, d) - m - d + 1, 2m^2 - m).$$

Then we have

$$N_{n,C}(X, S_n) \gg X^{c_n},$$

where $c_n$ is a constant depending on $m, d$, and $n$ given explicitly in (5.8) and $c_n \to 1/m^2$ as $n \to \infty$. The implied constant in (1.2) depends only on $n$ and (the equation for) $C$.

Moreover, for all sufficiently large $n$ (relative to $m$ and $d$) with $\gcd(m, d) \mid n$, we can take

$$c_n = \frac{1}{m^2} \left(1 + \frac{(2m - 2dr + 1)n + d^2r^2 - mdr + mk - k^2}{2n(n-1)}\right),$$

where $1 \leq r < m$ and $0 \leq k < m$ are integers depending only on the residue classes of $n, d$ (mod $m$).

**Remark.** We make note of a few properties of the constant $c_n$ in Theorem 1.

- For any fixed choice of $m, d$, the constant $c_n$ in (1.2) satisfies $c_n - \frac{1}{m^2} \sim \frac{m^2-md-r}{m^2(n-1)}$ in the limit as $n \to \infty$, where $1 \leq r < m$ is an integer depending only on $n, d$ (mod $m$). In particular, $\frac{m^2-md-r}{m^2(n-1)}$ is negative, so we can say that in (1.2), $c_n$ approaches $\frac{1}{m^2}$ from below.
- In contrast, the improved exponent in (1.3) satisfies $c_n - \frac{1}{m^2} \sim \frac{m^2-md-r+1}{2m(n-1)}$, which is seen to be positive in the case $m = d$, so $c_n \to \frac{1}{m^2}$ from above as $n \to \infty$. If $m < d$, the improved $c_n$ will approach $\frac{1}{m^2}$ from below as in (1.2).
- The improved exponent in (1.3) takes effect when we have good enough asymptotic upper bounds for $N_{n,C}(X)$. The bound of Lemke Oliver–Thorne [LT20, Theorem 1.1] suffices when $n$ is taken to be large. We discuss how large $n$ must be for (1.3) to be known to hold in Section 5.4, see Figure 5.1.
- Theorem 1 agrees or improves upon known lower bounds for $N_{n,C}(X, S_n)$ in the cases where $C$ is an elliptic curve [LT19] or a hyperelliptic curve [Key19].
- We do not expect this lower bound to be sharp; in the case where $C$ is an elliptic curve, Lemke Oliver–Thorne [LT19] suggest a heuristic of $X^{3/4+o(1)}$ for the asymptotics of the number of fields $K/\mathbb{Q}$ for which $\text{rk } E(K) = \text{rk } E(\mathbb{Q}) + 2$. 
The strategy for proving Theorem 1, employed also in [LT19] and [Key19], is to use the equation for $C/Q$ to find an explicit parameterized family of polynomials generating degree $n$ extensions $Q(P)/Q$ with Galois closure $S_n$. Some effort is required to verify that the members of the family are in fact irreducible and have Galois group $S_n$. We then count the polynomials in this family and bound how often the number fields they generate are isomorphic.

A notable limitation of Theorem 1 is the condition that the count only applies for field extensions of degree $n$ where $n$ is such that gcd$(m, d) | n$. In the case where $C$ is a hyperelliptic curve, we have $m = 2$ and $d$ can be chosen to be odd if and only if $C$ has a rational Weierstrass point. In this case, gcd$(m, d) = 1$, and our parameterization produces infinite families of odd degree $n$ points for $n$ sufficiently large. In the general case however, we take $d = 2g + 2$, where $g$ is the genus of $C$, giving gcd$(m, d) = 2$, so this parameterization does not produce any odd degree points (cf. [Key19]).

This is consistent with a result of Bhargava–Gross–Wang [BGW17] that says a positive proportion of locally soluble hyperelliptic curves have no odd degree points. In Section 7 we speculate as to whether for superelliptic curves, points of degrees $n$ such that gcd$(m, d) | n$ are “more common” than points of degrees $n$ where gcd$(m, d) \nmid n$. This section contains a description of various geometric sources from which we expect to find infinitely many points on these curves. We also discuss the relationship of these sources to the points obtained by the parameterization strategy.

Having speculated that the parametrization on which Theorem 1 relies does not miss too many points on the curves, if one finds points missed by the parametrization, it is possible to use those points and apply a similar counting strategy. This is the case if for some $n$ with gcd$(m, d) \nmid n$ we have a point of degree $n$ on the curve. For an explicit example of this, we consider the case where $C$ has a (possibly non-Weierstrass) rational point, where it turns out that the asymptotic lower bound of Theorem 1 may be extended to all sufficiently large degrees $n$.

**Theorem 2.** Let $C$ be a superelliptic curve with equation (1.1) possessing a rational point. Then for all sufficiently large degrees $n$, there is a constant $c_n$, given explicitly in (6.5), for which

$$N_{n,C}(X, S_n) \gg X^{c_n},$$

satisfying $c_n \to \frac{1}{m^2}$ as $n \to \infty$. The implied constant depends only on $n$ and (the equation for) $C$.

This paper is organized as follows. In Section 2 we give an overview of the parameterization strategy used in the proof of Theorem 1. In Section 3 we give criteria for a transitive permutation group to be the symmetric group, based on containing cycles of certain lengths. We also recall the Newton polygon of a polynomial, and how it may be used to identify cycle types in its Galois group. Section 4 is devoted to proving that our parameterization strategy almost always produces irreducible polynomials with symmetric Galois group, so in Section 5 we can count polynomials produced by our parameterization and adjust for multiplicity to obtain a lower bound for $N_{n,C}(X, S_n)$. In Section 6 we extend the methods of the previous sections in the case where $C$ has a rational point to give a proof of Theorem 2. A discussion of the geometric sources for infinite collections of points on superelliptic curves, and their relevance to field counting problems of this flavor, is given in Section 7.

**Acknowledgments**

The authors are grateful to Henri Darmon, Hannah Larson, Robert Lemke Oliver, Jackson Morrow, Frank Thorne, Brooke Ullery, Isabel Vogt, and David Zureick-Brown for helpful conversations. The authors would further like to thank Abbey Bourdon, Hannah Larson, Robert Lemke Oliver, Jackson Morrow, Jeremy Rouse, and Isabel Vogt for their thoughtful comments on an earlier draft.
2. The parametrization strategy

To produce algebraic points on \( C \), our strategy is to parameterize the coordinates \( x \) and \( y \) as rational functions in an auxiliary variable \( t \). We set

\[
x(t) = \frac{\gamma(t)}{\eta(t)} \quad \text{and} \quad y(t) = \frac{g(t)}{h(t)}.
\]

Substituting into the equation for \( C \), given by (1.1), and clearing denominators, we obtain the polynomial equation

\[
(2.1) \quad F_{g,h,\gamma,\eta}(t) = h(t)^m \left( c_d \gamma(t)^d + c_{d-1} \gamma(t)^{d-1} \eta(t) + \cdots + c_1 \gamma(t) \eta(t) + c_0 \eta(t)^d \right) - g(t)^m \eta(t)^d = 0.
\]

Suppose \( g, h, \gamma, \eta \) are chosen in \( \mathbb{Z}[x] \) such that \( F_{g,h,\gamma,\eta}(t) \) is irreducible with some root \( \alpha \). Then

\[
P = (x(\alpha), y(\alpha)) = \left( \frac{\gamma(\alpha)}{\eta(\alpha)}, \frac{g(\alpha)}{h(\alpha)} \right)
\]

is a point on \( C \) defined over the field \( \mathbb{Q}(\alpha) \), and \( \mathbb{Q}(\alpha) \) is the field generated by \( P \). Given a degree \( n \), our approach is to count how many ways we can choose \( g, h, \gamma, \eta \) such that \( F_{g,h,\gamma,\eta} \) is degree \( n \), irreducible, and has Galois group \( S_n \).

Generally, the degree of \( F_{g,h,\gamma,\eta} \) is the maximum of \( m(\deg h) + d(\deg \gamma) \) and \( m(\deg g) + d(\deg \eta) \), both of which are multiples of \( \gcd(m, d) \). Since we will eventually count the number of such parameterizations, we want to choose \( g, h, \gamma, \eta \) so the sum of their degrees is as large as possible, giving us the most degrees of freedom to count. Recall that in this paper, we have assumed \( m \leq d \), so this sum of degrees will be maximized by letting \( \deg g \) and \( \deg h \) large, while keeping those of \( \gamma \) and \( \eta \) small. To that end, we simply take \( \eta = 1 \) and suppress the notation by writing \( F_{g,h,\gamma} \) for the remainder of this paper. However, in the general case, namely if \( m > d \), it would be useful to take \( \eta \) to be nonconstant.

We observe that when \( n \) is a sufficiently large multiple of \( \gcd(m, d) \), we can always choose the degrees of \( g, h, \gamma \) to make the polynomial (2.1) have degree \( n \) in general. This is done by using \( \deg \gamma \) to control the residue class of \( n \) modulo \( m \) if necessary, and letting \( \deg g, \deg h \) be as large as possible. It remains to determine how large \( n \) must be for such degrees to exist. It is clear that we must have at least \( n \geq d \) by looking at the minimum degree of \( F_{g,h,\gamma} \). To give a more precise answer we recall the classical definition of the Frobenius number, with a straightforward generalization to integers that are not coprime.

**Definition** (Frobenius number). Given natural numbers \( a, b \) with \( \gcd(a, b) = 1 \), the Frobenius number, denoted \( \text{frob}(a, b) \), is the largest natural number which is not a linear combination \( ax + by \) where \( x, y \geq 0 \).

When \( \gcd(a, b) \neq 1 \), we define a generalized Frobenius number, denoted by \( \text{Frob}(a, b) \), to be the largest multiple of \( \gcd(a, b) \) that is not a linear combination \( ax + by \) for \( x, y \geq 0 \).

We have the elementary result that for coprime integers \( a, b \), the Frobenius number is given by \( \text{frob}(a, b) = ab - a - b \). Recognizing that for any natural numbers \( a, b \) we have

\[
\text{Frob}(a, b)/\gcd(a, b) = \text{frob} \left( \frac{a}{\gcd(a, b)}, \frac{b}{\gcd(a, b)} \right),
\]

we find that the generalized Frobenius number satisfies \( \text{Frob}(a, b) = \text{lcm}(a, b) - a - b \).

For any \( n \geq \max(d, \text{Frob}(m, d) + 1) \) we can manipulate the degrees of \( g, h, \gamma \) such that \( \deg F_{g,h,\gamma} = n \) in (2.1). Moreover, this is sharp in the sense that (2.1) will not take degrees \( n < d \) or \( n = \text{Frob}(m, d) \). We conclude this section by summarizing our discussion in the following proposition.

**Proposition 2.1.** Let \( C \) be given by (1.1) with \( m \leq d \). For all degrees \( n \geq \max(d, \text{Frob}(m, d) + 1) \) such that \( \gcd(m, d) \mid n \), there exist \( g, h, \gamma, \eta \) such that \( F_{g,h,\gamma,\eta}(t) \) given in (2.1) has degree \( n \).
Explicitly, we can assume $\eta = 1$ and take $g, h, \gamma$ to have the degrees given below depending on whether or not $m$ divides $d$:

\begin{align*}
\deg g &= \frac{n}{m} \\
\deg h &= \lfloor \frac{(n - d)}{m} \rfloor & \text{when } m \mid n \\
\deg \gamma &= 1
\end{align*}

(2.2)

and

\begin{align*}
\deg g &= \lfloor \frac{n}{m} \rfloor \\
\deg h &= \frac{(n - rd)}{m} & \text{when } m \nmid n \\
\deg \gamma &= r
\end{align*}

(2.3)

where $r > 0$ is the minimal integer such that $n \equiv rd \pmod{m}$.

Notice that the choices above accomplish our goals of maximizing the total degrees of freedom by letting $g, h$ have the largest possible degree, while $\deg \gamma$ is kept small, with $1 \leq r < m$.

3. Group theory and Newton polygons

We now state some useful results on generating sets for the symmetric group and the theory of Newton polygons. These will be used later to show that the polynomials $F_{g, h, \tau, \eta}(t)$ given by Proposition 2.1 are generically irreducible and have Galois group $S_n$.

3.1. Generating sets for the symmetric group. We begin by stating some standard results from group theory on generating sets for the symmetric group.

**Lemma 3.1.** Let $G \subseteq S_n$ be a permutation subgroup acting on the set $\{1, \ldots, n\}$. Suppose the action of $G$ is transitive and that $G$ contains a transposition. Then if $G$ contains either an $(n - 1)$-cycle or a $p$-cycle for a prime $p > n/2$, we have $G = S_n$.

**Proof.** This is a standard group theory exercise. For a short proof see [Key19, Proposition 2.4].

Next, we have a useful lemma which shows that if a permutation group $G$ on $n$ elements contains a subgroup isomorphic to a sufficiently large symmetric group, then $G$ must be the full symmetric group $S_n$.

**Lemma 3.2** (Lifting transitive subgroups, [Key19, Lemma 2.3]). Let $G \subseteq S_n$ be a transitive permutation subgroup on the set $\{1, \ldots, n\}$. Assume $G$ contains a subgroup $H$ which is isomorphic as a permutation subgroup to $S_k$ for $k > n/2$. Then $G \simeq S_n$.

We can use Lemma 3.2 to give a criterion for a transitive permutation subgroup on $n$ elements to be isomorphic to the symmetric group. While these requirements are somewhat contrived, they turn out to be attainable using the Newton polygon approach on our parameterized polynomials in the coming subsection.

**Proposition 3.3.** Fix an integer $m \geq 2$. Suppose $n > m$ and $G \subseteq S_n$ is a permutation subgroup acting on the set $\{1, \ldots, n\}$, satisfying the following:

(i) $G$ is transitive,

(ii) $G$ contains a cycle $\sigma$ with $n - m \leq \text{length}(\sigma) < n$,

(iii) $G$ contains a transposition $\tau$,

(iv) $G$ contains a $q$-cycle $\theta$ for a prime $q > m$.

Then $G = S_n$.

**Proof.** When $m \geq \frac{n}{2}$, (i), (iii), and (iv) ensure that $G$ satisfies the hypotheses of Lemma 3.1, so $G = S_n$. Assume then that $n > 2m$. 

Assume $\sigma$ has length $(n - m)$, so after renumbering, we have $\sigma = (1 \ 2 \ \cdots \ n - m)$. The same argument works if $n - m < \text{length}(\sigma) < n$. Since $G$ is transitive, we may assume the transposition is of the form $\tau = (1 \ a)$ for $1 < a \leq n$ and that the $q$-cycle, $\theta$, acts nontrivially on $1$.

Suppose that $a > n - m$. Then consider the subgroup $H$ generated by $\sigma$ and $\tau$. This subgroup may be viewed as a permutation group acting on the set $\{1, \ldots, n-m, a\}$. This action is transitive, and hence $H \cong S_{n-m+1}$, by Lemma 3.1.

Suppose instead that $a \leq n-m$. Let $H'$ be the subgroup generated by $\tau$ and $\theta$. Again by 3.1, $H'$ is isomorphic to either $S_{q+1}$ or $S_q$ (depending on whether or not the $q$-cycle acts nontrivially on $a$). In either case, we have at least $m$ distinct transpositions $(1 \ a_i)$ for $1 < a_i \leq n$ contained in $G$.

Let $\rho$ be an element of $G$ which satisfies $\rho(1) = n - m + 1$; such an element exists by transitivity. Conjugating $(1 \ a_i)$ by $\rho$ for each $i$, we get (at least) $m$ distinct transpositions $(n - m + 1 \ \rho(a_i))$. However, there are only $m - 1$ integers between $n - m + 2$ and $n$, so by the pigeonhole principle, at least one of $\rho(a_i)$ is at most $n - m$. Possibly after renumbering again, we are in the previous case, where $a > n - m$.

\[ \Box \]

3.2. **Newton polygons.** We now introduce the Newton polygon, which associates to a polynomial over $\mathbb{Q}_p$ a diagram of line segments containing data about valuations of roots. We will use this to show our polynomials are irreducible and identify cycles in the Galois group. Let $p$ be a prime, $\mathbb{Q}_p$ the field of $p$-adic numbers, and $F(t) \in \mathbb{Q}_p[t]$ a polynomial.

**Definition** (Newton polygon). With the notation above, let $F(t)$ be given by $F(t) = \sum_{i=0}^{n} k_i t^i$. The $p$-adic Newton polygon of $F$ is the lower convex hull of the set

\[ \{ (i, v_p(k_i)) \in \mathbb{R}^2 \mid 0 \leq i \leq n \} , \]

where $v_p$ denotes the $p$-adic valuation, and we set $v_p(0) = \infty$ by convention. We will denote the Newton polygon of $F$ by $NP_{\mathbb{Q}_p}(F)$, or simply by $NP(F)$ when it will not create confusion.

The Newton polygon $NP(F)$ can be split up into segments of distinct slopes $s_j$, and if the $j$-th segment has length $l_j$, then $F(t)$ has $l_j$ roots of valuation $-s_j$. This key fact leads to the following lemmas, proven in [Key19].

**Lemma 3.4** (see [Key19, Lemma 2.7]). Suppose $NP_{\mathbb{Q}_p}(F)$ has a segment of length $l$ and slope $s$, and no other segments of this slope (i.e. the entire segment of slope $s$). Then $F$ factors as $F = F_0 F_1$ over $\mathbb{Q}_p$, where $\deg F_0 = l$ and $F_0$ has roots of valuation $-s$.

Moreover, if $s = r/l$ has reduced fraction form $r'/l'$ then all irreducible factors of $F_0$ over $\mathbb{Q}_p$ have degree divisible by $l'$. In particular, if $\gcd(r, l) = 1$ then the $F_0$ produced above is irreducible.

**Lemma 3.5** (see [Key19, Lemma 2.8]). Suppose $F(t) \in \mathbb{Q}[t]$ and $NP_{\mathbb{Q}_p}(F)$ has a segment of length $l$ and slope $r/l$ with $\gcd(r, l) = 1$. Assume further that $\gcd(n, p) = 1$. Then $\text{Gal}(F/\mathbb{Q})$ contains an $l$-cycle.

4. **Irreducibility and Galois groups**

Let $C$ be a superelliptic curve with exponent $m$ and defining polynomial $f(x)$, as in (1.1). As in Proposition 2.1, given any $n \geq n_0$ such that $\gcd(m, d) \mid n$, there exist choices of degrees (2.2) or (2.3) for $g, h, \gamma$ such that the polynomial $F_{g,h,\gamma}(t)$ given in (2.1) has degree $n$ in general. Writing

\[ g(t) = \sum_{i=1}^{\deg g} a_i t^i, \quad h(t) = \sum_{j=1}^{\deg h} b_j t^j, \quad \gamma(t) = \sum_{\ell=1}^{\deg \gamma} \alpha_{\ell} t^{\ell}, \]

we can view $F_{g,h,\gamma}(t)$ as a degree $n$ polynomial $F(a, b, \alpha, t) \in \mathbb{Q}(a, b, \alpha)[t]$. Here $\alpha$ indicates the tuple of indeterminates $(a_0, \ldots, a_{\deg g})$, and similarly for $b$ and $\alpha$. For simplicity, since we have fixed the curve $C$ and degree $n$, we will denote this polynomial family by $F \in \mathbb{Q}(a, b, \alpha)[t]$, and denote a rational specialization by $F_{a_0, b_0, \alpha_0} \in \mathbb{Q}[t]$, where $a_0 \in \mathbb{Q}^{\deg g + 1}$, $b_0 \in \mathbb{Q}^{\deg h + 1}$, $\alpha_0 \in \mathbb{Q}^{\deg \gamma + 1}$.
It is clear that since $F$ is degree $n$, almost all specializations $F_{a_0,b_0,c_0}$ have degree $n$. With Hilbert’s irreducibility theorem, we can say something stronger — that the irreducibility and Galois group structure of the polynomial family carry over to most specializations. We state this for a general polynomial $F(y,t) \in \mathbb{Q}(y)[t]$ where $y$ is some tuple of indeterminates.

**Lemma 4.1** (Hilbert’s irreducibility theorem). Let $F(y,t) \in \mathbb{Q}(y)[t]$ with Galois group $G$. Suppose $y_0$ is a rational specialization such that $F(y_0,t) \in \mathbb{Q}[t]$ is irreducible with Galois group $G_0$. Then $F(y,t)$ is irreducible and $G \simeq G_0$ for almost all $y_0$.

The following corollary is more specific, as it refers to the permutation representations of the Galois groups, may be found in [LT19, Theorem 4.2].

**Corollary 4.2.** Suppose $F(y,t) \in \mathbb{Q}(y)[t]$ is irreducible. If a permutation representation of $G_0$ contains a given cycle type for a positive proportion of integral specializations $y_0$, then $G$ contains an element of the same cycle type.

Using the Newton polygons from the previous section, our aim is to show that many integral specializations $F_{a_0,b_0,c_0}$ have certain cycle types in their Galois groups. Corollary 4.2 implies that $F$ must have those same cycles in its Galois group over $\mathbb{Q}(a,b,c)$. If there are cycle types corresponding to the hypotheses of Proposition 3.3, then we have $G = \text{Gal}(F/\mathbb{Q}(a,b,c)) \simeq S_n$. The remainder of this section is devoted to proving the following proposition, which states that $G \simeq S_n$ when $n$ is sufficiently large.

**Proposition 4.3.** Fix an integer $m \geq 2$, a squarefree integral polynomial $f(x)$ of degree $d \geq m$, and an integer $n > n_0 = \max(d, \text{Frob}(m,d) \mid n$. Then whenever $n \geq 2m^2 - m$ is satisfied, the degree $n$ polynomial family $F(t) \in \mathbb{Q}(a,b,c)[t]$, given in (2.1) with degrees (2.2) if $m \mid n$ or (2.3) if $m \nmid n$, is irreducible and has Galois group $S_n$.

Moreover, 100% of specializations of $F(t)$ to $\mathbb{Q}[t]$ are degree $n$, irreducible, and have Galois group $S_n$.

4.1. Finding short cycles. We will first prove an intermediate result. Namely, that sufficiently short cycles can be found in the Galois group of $F$, so long as $n$ is sufficiently large. To show this, we use the following lemma, which is an elementary consequence of the Chebotarev density theorem.

**Lemma 4.4.** Let $f(x) \in \mathbb{Z}[x]$ be a nonconstant polynomial. Then there are infinitely many primes dividing $f(x_0)$ for some integer $x_0$. Moreover, if $f(x)$ is squarefree, then there exist infinitely many primes dividing $f(x_0)$ exactly once for some integer $x_0$.

**Proof.** A prime $p$ divides $f(x_0)$ for some integer $x_0$ if and only if the reduction of $f$ modulo $p$ has a root. The set of such primes certainly contains the primes for which the reduction of $f$ modulo $p$ splits completely. These are precisely the primes which split completely in the splitting field of $f$. By the Chebotarev Density Theorem, this is an infinite set, proving the first claim.

If $f(x)$ is squarefree, suppose $p \mid \text{Disc } f$ and that $p \mid f(x_0)$ for some integer $x_0$, the existence of which is guaranteed by the first claim. Consider $f(x_0 + p)$,

$$f(x_0 + p) \equiv f(x_0) + f'(x_0)p \pmod{p^2}. $$

If $p^2 \mid f(x_0)$ and $p^2 \mid f(x_0 + p)$ then we must have $p \mid f'(x_0)$. However, this implies $x_0$ is a double root of $f(x) \pmod{p}$, contradicting $p \nmid \text{Disc } f$. Thus we conclude one of $f(x_0)$ and $f(x_0 + p)$ is divisible by $p$ exactly once. We have that the set of primes dividing some $f(x_0)$ is infinite, and all of these except the finitely many dividing $\text{Disc } f$ divide some $f(x_0)$ exactly once. 

**Lemma 4.5** (Short cycles). With notation as above, for all integers $\ell$ such that $\ell \leq n/m$, there exists a $\ell$-cycle in the Galois group of $F(t)$ over $\mathbb{Q}(a,b,c)$.

**Proof.** We claim $c_0$ is an $m$-th power residue for infinitely many primes. This is vacuous if $c_0 = 0$, and for $c_0 \neq 0$ we can apply Lemma 4.4 to the squarefree polynomial $x^m - c_0$ to find that there are
infnitely many such primes. Take $p > \max(c_0, m)$ to be such a prime, and let $a_0$ to be an integer such that $a_0^m \equiv c_0 \pmod{p}$ and $p^2 \nmid a_0^m - c_0$.

We write $g(t) = a_0 + a_1 t + \cdots + a_{\deg g} t^{\deg g}$, $h(t) = b_0 + b_1 t + \cdots + b_{\deg h} t^{\deg h}$, $\gamma(t) = a_0 + a_1 t + \cdots + \alpha_{\deg \gamma} t^{\deg \gamma}$, and $f(\gamma(t)) = c_0 + c_1 t + \cdots + c_{\deg \gamma} t^{\deg \gamma}$. Consider the case of $\ell = 2 \leq n/m$, which will illuminate the general approach. We begin by making the restrictions $b_0 \equiv 1 \pmod{p^2}$ and $b_i \equiv a_j \equiv 0 \pmod{p^2}$ for all $i > 0$ and $j \geq 0$. Modulo $p^2$, such a specialization $F_{a_0,b_0,a_0}$ becomes

$$F_{a_0,b_0,a_0}(t) = h(t)^m f(\gamma(t)) - g(t)^m \equiv b_0^m c_0 - g(t)^m \equiv c_0 - a_0^m - ma_0^{m-1} a_1 t - \left( ma_0^{m-1} a_2 + \binom{m}{2} a_0^m - a_1^2 \right) t^2 - \cdots.$$  

Recall that $a_0$ was chosen earlier so that the constant term, which is equivalent to $c_0 - a_0^m \pmod{p}$, is divisible by $p$ exactly once. We further impose that $a_1 \equiv 0 \pmod{p}$ and $a_2 \neq 0 \pmod{p}$. These ensure that the linear term is divisible by $p$ but the quadratic term is not.

This produces a segment of length 2 and slope $-1/2$ in $\NP(F)$ so by Lemma 3.5, any integral specialization $a_0, b_0, a_0$ satisfying these conditions implies $\Gal(F_{a_0,b_0,a_0}/\mathbb{Q})$ contains a transposition. Since we only imposed conditions on the residue classes of $a_0, b_0, a_0$ modulo powers of $p$, these are satisfied for a positive proportion of specializations. Hence Corollary 4.2 guarantees the existence of a transposition in $\Gal(F/\mathbb{Q}(a, b, \alpha))$.

For general $\ell$, recall that $\deg g = \lfloor n/m \rfloor$, so $\ell \leq \deg g$. Consider a specialization $a_0, b_0, a_0$ satisfying the following restrictions.

\begin{equation}
\begin{aligned}
a_0^m &\equiv c_0 \pmod{p} \text{ such that } p^2 \nmid c_0 - a_0^m \\
a_j &\equiv 0 \pmod{p} \text{ for } 0 < j < \ell \\
a_\ell &\neq 0 \pmod{p} \\
b_0 &\equiv 1 \pmod{p^2} \\
b_j &\equiv 0 \pmod{p^2} \text{ for all } j > 0 \\
a_j &\equiv 0 \pmod{p^2} \text{ for all } j \geq 0
\end{aligned}
\end{equation}

with no restrictions on $a_j$ for $j > \ell$. Again, we have that the constant term of $F_{a_0,b_0,a_0}(t)$ is equivalent modulo $p^2$ to $c_0 - a_0^m$, which is divisible by $p$ exactly once. For $0 < j < \ell$, the $t^j$ term of $F_{a_0,b_0,a_0}(t)$ is equivalent modulo $p^2$ to

\begin{equation}
- \sum_{0 \leq i_1, \ldots, i_m \leq \deg g \atop i_1 + \cdots + i_m = j} \left( \prod_{r=1}^m a_{i_r} \right)
\end{equation}

The restrictions in (4.1) ensure that $p$ divides every term in (4.2). The $t^\ell$ term of $F_{a_0,b_0,a_0}(t)$ is equivalent modulo $p^2$ to

\begin{equation}
- \sum_{0 \leq i_1, \ldots, i_m \leq \deg g \atop i_1 + \cdots + i_m = \ell} \left( \prod_{r=1}^m a_{i_r} \right) = - \left( ma_0^{m-1} a_\ell + \sum_{0 < i_1, \ldots, i_m < \ell \atop i_1 + \cdots + i_m = \ell} \left( \prod_{r=1}^m a_{i_r} \right) \right)
\end{equation}

The restrictions on $p$ and in (4.1) ensure that $p \nmid ma_0^{m-1} a_\ell$, but $p \mid \prod_{r=1}^m a_{i_r}$ when $i_1 + \cdots + i_m = \ell$ for all $i_r$ nonzero. Hence $p$ does not divide the coefficient of the $t^\ell$ term.

Thus the Newton polygon $\NP(F)$ has a segment of length $\ell$ and slope $-1/\ell$. Another application of Lemma 3.5 and Corollary 4.2 shows $\Gal(F/\mathbb{Q}(a, b, \alpha))$ contains an $\ell$-cycle, since the restrictions (4.1) are residue class conditions. \qed
Remark. The proof of Lemma 4.5 works the same for any parameterized family of the form (2.1). So long as \( \ell \leq \deg g \), one can specify similar residue class conditions, including appropriate ones for \( \beta_0 \), so that a segment of slope \(-1/\ell\) appears in \( \text{NP}(F_{a_0, b_0, \alpha_0, \beta_0})\).

4.2. Long cycles and the proof of Proposition 4.3. We are now ready to prove Proposition 4.3. The strategy is to use Newton polygons and Corollary 4.2 to satisfy the requirements (i) – (iv) of Proposition 3.3 for \( \text{Gal}(F/\mathbb{Q}(a, b, \alpha)) \).

Proof of Proposition 4.3. The hypothesis \( n \geq \max(\text{Frob}(m, d), d) \) ensures that the degrees given in (2.2) in the case of \( m \mid n \) or (2.3) in the case of \( m \nmid n \) produce a polynomial family \( F \in \mathbb{Q}(a, b, \alpha)[t] \) of degree \( n \).

We now argue that the hypothesis \( n \geq 2m^2 - m \) allows us to assume that \( G = \text{Gal}(F/\mathbb{Q}(a, b, \alpha)) \) contains a transposition and prime cycle of length \( q > m \). By Bertrand’s postulate, there exists a prime \( q \) such that \( m < q < 2m \). So long as \( 2 \leq q \leq n/m \), Lemma 4.5 ensures that the transposition and \( q \)-cycle will exist in \( G \). When \( n \geq 2m^2 - m \), we have \( n/m > 2m - 1 \geq q \), as desired. Thus the group \( G \) is a permutation subgroup of \( S_n \) and satisfies (iii) and (iv) from Proposition 3.3 when \( n \) is sufficiently large.

It remains to verify (i) and (ii), that \( G \) is transitive and contains a sufficiently long cycle. For this we consider the cases of \( \gcd(m, n) = 1 \), \( m \mid n \), and \( 1 < \gcd(m, n) < m \) separately.

Case (a): \( \gcd(m, n) = 1 \). In this case, the degrees of \( g(t), h(t) \), and \( \gamma(t) \) are given in (2.3). Let \( p \) be a prime such that \( p \nmid c_0, c_d \). We first consider a specialization \( a_0, b_0, \alpha_0 \) satisfying the following requirements.

\[
\begin{align*}
v_p(a_0) &= 0 \\
v_p(a_i) &\geq m \text{ for all } i > 0 \\
v_p(b_j) &\geq m \text{ for all } j < (n - rd)/m \\
v_p(b_{(n-rd)/m}) &= 1 \\
v_p(\alpha_\ell) &\geq 0 \text{ for all } \ell < r \\
v_p(\alpha_r) &= 0.
\end{align*}
\]

The choices in (4.4) ensure that all but the constant term of \( F_{a_0, b_0, \alpha_0} \) have \( p \)-adic valuation at least \( m \), while the leading term of is \( b_{(n-rd)/m}^m c_d \alpha_r^d \), which has \( p \)-adic valuation exactly \( m \). The constant term, \( b_0^m f(\alpha_0) - a_0^m \), has valuation 0. This produces the \( p \)-adic Newton polygon below in Figure 4.1.

**Figure 4.1.** \( \text{NP}_{\mathbb{Q}_p}(F_{a_0, b_0, \alpha_0}) \) with \( n \)-cycle.

Since the polygon has exactly one segment of slope \( m/n \) with \( \gcd(m, n) = 1 \), Lemma 3.4 implies that \( F_{a_0, b_0, \alpha_0} \) is irreducible. Hence \( F \) must be irreducible, and \( G \) is transitive, satisfying (i).

We can modify this procedure slightly to produce an \( (n - m) \)-cycle in \( G \). Keeping the same choice of prime \( p \), consider a specialization satisfying

\[
v_p(a_0), v_p(a_1) = 0
\]
\[ v_p(a_i) \geq m \text{ for all } i > 1 \]
\[ v_p(b_j) \geq m \text{ for all } j < (n - rd)/m \]
\[ v_p(b_{(n-rd)/m}) = 1 \]
\[ v_p(\alpha_\ell) \geq 0 \text{ for all } \ell < r \]
\[ v_p(\alpha_r) = 0. \]

Notice that (4.5) ensures that all terms of \( F_{a_0,b_0,\alpha_0} \) are divisible by \( p^m \) except those containing only \( a_0 \) and \( a_1 \). In particular, the \( t^m \) coefficient has valuation 0, owing to the presence of an \( a_m^{p^k} \). The leading coefficient again has valuation exactly \( m \). Thus the \( p \)-adic Newton polygon is shown in Figure 4.2.

**Figure 4.2.** NPQ\(_p\)(\( F_{a_0,b_0,\alpha_0} \)) with \((n - m)\)-cycle

The polygon in Figure 4.2 has a segment of slope \( m/(n - m) \), which is a reduced fraction since \( \gcd(m, n) = 1 \). Thus Lemma 3.5 implies that the Galois group of a specialization \( F_{a_0,b_0,\alpha_0} \) satisfying (4.5) contains an \((n - m)\)-cycle. Since the conditions (4.5) are satisfied for a positive proportion of specializations, we apply Corollary 4.2 to conclude that \( G \) contains an \((n - m)\)-cycle. Thus \( G \) satisfies (ii).

**Case (b):** \( m \mid n \). For this case, the degrees of \( g(t) \), \( h(t) \), and \( \gamma(t) \) are given by (2.2). Since \( f(x) \) is squarefree, we can use Lemma 4.4 to find a prime \( p \) not dividing \( c_d \) and an integer \( x_0 \) for which \( p \) divides \( f(x_0) \) exactly once. Consider a specialization of \( F \) satisfying

\[ v_p(a_i) \geq 1 \text{ for all } i < n/m \]
\[ v_p(a_{n/m}) = 0 \]
\[ v_p(b_j) = 0 \text{ for all } j \]
\[ \alpha_0 \equiv x_0 \pmod{p^2} \]
\[ v_p(\alpha_{\ell}) \geq 1 \text{ for all } \ell > 0. \]

The restriction on \( \alpha_0 \) ensures that the constant term of \( F_{a_0,b_0,\alpha_0} \), given by \( b_0^{p^m}f(\alpha_0) - a_0^{p^n} \) is divisible by \( p \) exactly once. All other terms are divisible by \( p \) at least once, except for the leading term, as ensured by the choice of \( a_0 \). This results in the \( p \)-adic Newton polygon shown in Figure 4.3, which has a single segment of slope \(-1/n\). Lemma 3.4 implies that \( F_{a_0,b_0,\alpha_0} \) and hence \( F \) is irreducible, so \( G \) is transitive and satisfies (i).
Keeping the same choice of $p$ and $x_0$, we can modify the restrictions (4.6) by requiring that $v_p(a_{n/m-1}) = 0$. This results in the Newton polygon shown below in Figure 4.4, which has a segment of length $n - m$ and slope $-1/(n - m)$. This, once again with Lemma 3.5 and Corollary 4.2, gives that $G$ contains an $(n - m)$-cycle and hence satisfies (ii).

**Case (c):** $1 < \gcd(m, n) < m$. For the final case, we need to combine the strategies of the previous cases. The degrees of $g(t), h(t), \gamma(t)$ are given by (2.3). Choosing some prime $p$ not dividing $c_0$ or $c_d$, the restrictions (4.4) produce the $p$-adic Newton polygon in Figure 4.1. However, since $\gcd(m, n) > 1$, we cannot conclude right away that $F_{a_0, b_0, \alpha_0}$ is irreducible. Since the segment has slope $m/n = m/\gcd(m, n)$, Lemma 3.4 gives that any irreducible factor of $F_{a_0, b_0, \alpha_0}$ has degree divisible by $n/\gcd(m, n)$. In particular, this means that any irreducible components of $F$ must also have degree divisible by $n/\gcd(m, n)$.

Choose another prime $p$ such that $p \mid f(x_0)$ exactly once, as exists by Lemma 4.4. Take the restrictions (4.6), noting that we must set $v_p(a_{n/m}) = 0$, since $m \nmid n$. This produces a Newton polygon much like that in Figure 4.4, but with a segment of length $\ell = \lfloor n/m \rfloor$ and slope $-1/\ell$. By Lemma 3.5 and Corollary 4.2, we have that $G$ contains an $\ell$-cycle, and $n - m < \ell < n$, so (ii) is satisfied.

Lemma 3.4 implies that with these restrictions, $F_{a_0, b_0, \alpha_0}$ has an irreducible factor of degree $\ell$. Hence $F$ must have an irreducible factor of degree at least $\ell$. Recall that the irreducible components of $F$ have degree divisible by $n/\gcd(m, n)$. We have by our hypothesis $n \geq 2m^2 - m$ that

$$\frac{n}{\gcd(m, n)} \geq \frac{2m^2 - m}{\gcd(m, n)} = \frac{2m^2 - m}{m} > m > n - \ell.$$ 

Thus if $F$ is reducible, it has a factor of degree at least $\ell$, with the remaining factor(s) having degree(s) at most $n - \ell$. This cannot be divisible by $n/\gcd(m, n)$, producing a contradiction. Hence it must be that $F$ is irreducible and $G$ is transitive.
In all three cases (a), (b), and (c), we have shown that when \( n \) is sufficiently large, \( G = \text{Gal}(F/\mathbb{Q}(a, b, \alpha)) \) satisfies hypotheses (i) – (iv) of Proposition 3.3, so we have \( G \cong S_n \). Theorem 4.1 then implies that for almost all specializations \( a_0, b_0, \alpha_0 \), the polynomial \( F_{a_0, b_0, \alpha_0} \) is irreducible with Galois group \( S_n \).

We conclude this section by noting that when \( F \) is given by (2.1) and known to be irreducible with Galois group \( S_n \), we can partially specialize \( F \) and obtain that almost all of the partial specializations remain irreducible and have the same Galois group. This will be useful in the following section, when we wish to count extensions by counting only the possible choices of \( g(t) \) and \( h(t) \), and have \( \gamma(t) \) a fixed polynomial, depending on \( f(x) \) and \( n \). We summarize this in the following corollary.

**Corollary 4.6.** Fix an integer \( m \geq 2 \), a squarefree polynomial \( f(x) \) of degree \( d \geq m \), and an integer \( n \), satisfying all the hypotheses of Proposition 4.3. There exists a choice of \( \gamma_0(t) \in \mathbb{Z}[x] \), i.e. a vector \( \alpha_0 \in \mathbb{Z}^{k+1} \) for which the partial specialization \( F_{\alpha_0} \in \mathbb{Q}(a, b)[t] \) is irreducible, with Galois group \( S_n \) over \( \mathbb{Q}(a, b) \), and such that \( f(\gamma_0(t)) \) is squarefree.

Moreover, the absolute value of the coefficients \( \alpha_0 \) may be chosen to bounded above by some constant depending only on \( m, n, \) and \( f(x) \).

**Proof.** By Proposition 4.3, we have that \( F \) is irreducible with Galois group \( S_n \) over \( \mathbb{Q}(a, b, \alpha) \). Hilbert’s irreducibility theorem (Theorem 4.1 but for arbitrary base field) implies that for almost all choices of \( \alpha_0 \), \( F_{\alpha_0} \) is irreducible with Galois group \( S_n \) over \( \mathbb{Q}(a, b) \). Therefore, for each \( m, n, \) and \( f(x) \), we may choose some \( \alpha_0 \) for which this is true.

So long as \( \gamma_0(t) \) doesn’t have a multiple root at a root of \( f(x) \), the composition \( f(\gamma_0(t)) \) will be squarefree. Since the space of \( \alpha_0 \) giving rise to squarefree \( \gamma_0 \) is dense (it is Zariski open), some \( \gamma_0 \) from the previous paragraph also satisfies \( f(\gamma_0(t)) \) is squarefree. \( \Box \)

5. Lower bounds for \( N_{n, C}(X, S_n) \)

In this section, we describe how to obtain the asymptotic lower bounds in Theorem 1.

5.1. **Coefficient bounds.** In this section, we construct a family of polynomials \( P_{f,n}(Y) \) arising from certain specializations of (2.1) in Section 2. We will do this by imposing bounds on the coefficients of \( g(t) \) and \( h(t) \) in \( F(t) = g(t)^m - h(t)^m f(\gamma(t)) \). These bounds will be useful for counting multiplicities of fields generated by this family of polynomials because of the following lemma that relates the absolute value of a polynomial’s coefficients to the absolute values of its roots.

**Lemma 5.1.** Let \( f(x) = \sum_{i=0}^{n} c_i x^i \in \mathbb{C}[x] \) be monic and have degree \( n \). There exist positive constants \( A_i \) such that for any \( Y > 0 \), if \( |c_i| \leq A_i Y^{n-i} \) for \( 0 \leq i \leq n \) then \( |\alpha| \leq Y \) for all roots \( \alpha \) of \( f(x) \).

**Proof.** The result follows from a bound of Fujiwara [Fuj16], see [Key19, Lemma 4.1]. \( \Box \)

Let \( C \) be a nonsingular superelliptic curve over \( \mathbb{Q} \), as in (1.1). Let \( n \) be a positive integer satisfying the hypotheses of Proposition 4.3, and fix some \( \gamma_0(t) \in \mathbb{Z}[t] \) according to Corollary 4.6. We now take \( Y \) to be a a positive real number. Let \( P_{f,n}(Y) \) be the set of polynomials of the form

\[
F(t) = h(t)^m f(\gamma(t)) - g(t)^m
\]

that arise from certain specializations of (2.1) for which we will give certain constraints on \( g(t) \) and \( h(t) \) below. That is, \( P_{f,n}(Y) \) is a set of integral specializations \( F_{a_0, b_0, \alpha_0} \) where the choice of \( Y \) imposes constraints on \( a_0, b_0 \) and \( \alpha_0 \) is precisely the coefficients of \( \gamma_0 \). We write the coefficients of \( F(t) \) as follows:

\[
F(t) = d_n t^n + d_{n-1} t^{n-1} + \cdots + d_0.
\]
In order to apply Lemma 5.1, we need bounds on the coefficients $d_i$ in terms of $Y$. To achieve this, we impose restrictions on the coefficients of $g$ and $h$. In the case where $m \mid n$ we take
\[
g(t) = a_{n/m} t^{n/m} + a_{n/m-1} t^{n/m-1} + \cdots + a_0,
\]
\[
h(x) = b_{(n-d-k_1)/m} t^{(n-d-k_1)/m} + b_{(n-d-k_1)/m-1} t^{(n-d-k_1)/m-1} + \cdots + b_0,
\]
\[
f(g(t)) = (c_d x^d + \cdots + c_1 x + c_0).
\]

Here $k_1$ is the minimal nonnegative integer such that $(n-d-k_1)/m$ is an integer. This realizes the degrees in (2.2). Fix $a_{n/m}$ to be an integer so that the partial specialization $F_{a_{n/m},a_0}$ is irreducible and $\text{Gal}(F_{a_{n/m},a_0}, \mathbb{Q}(a,b)) \simeq S_n$. Such $a_{n/m}$ exists by Theorem 4.1. If $k_1 = 1$, we similarly take $b_{(n-d)/m}$ such that $F_{a_{n/m},b_{(n-d)/m},a_0}$ is irreducible and $\text{Gal}(F_{a_{n/m},b_{(n-d)/m},a_0}/\mathbb{Q}(a,b)) \simeq S_n$. We then impose the restrictions that $a_i, b_j$ are integers satisfying $|a_{n/m-i}| \leq Y^i$ for $i > 0$, and $|b_{(n-d-k_1)/m-j}| \leq Y^{k_1/m+j}$ for $j > 0$, with $|b_{(n-d-k_1)/m-j}| \leq Y^{k/m}$ if $k_1 \neq 0$.

In the case where $m \nmid n$, we choose $r$, the degree of $g(t)$, to be the minimal positive integer for which $n \equiv dr \pmod{m}$. As above, we have
\[
g(t) = a_{(n-k_2)/m} t^{(n-k_2)/m} + a_{(n-k_2)/m-1} t^{(n-k_2)/m-1} + \cdots + a_0,
\]
\[
h(t) = b_{(n-dr)/m} t^{(n-dr)/m} + b_{(n-dr)/m-1} t^{(n-dr)/m-1} + \cdots + b_0,
\]
\[
f(g(t)) = (c_d x^d + \cdots + c_1 x + c_0).
\]

Here $k_2$ is the minimal positive integer such that $(n-k_2)/m$ is an integer so this realizes the degrees in (2.3). This time, we use Theorem 4.1 to find an integer $b_{(n-dr)/m}$ such that the partial specialization $F_{b_{(n-dr)/m},a_0}$ is irreducible with $\text{Gal}(F_{b_{(n-dr)/m},a_0}/\mathbb{Q}(a,b)) \simeq S_n$. We then impose the restrictions that $a_i, b_j$ are integers satisfying $|a_{(n-k_2)/m-i}| \leq Y^{k_2/m+i}$ for $i > 0$, and $|b_{(n-dr)/m-j}| \leq Y^j$ for $j > 0$.

We note that these polynomials $F \in P_{f,n}(Y)$ have degree $n$, and these restrictions on the coefficients imply that $|d_i| \ll A_i Y^{n-i}$. Applying Lemma 5.1 and accounting for the implied constant, we see that for all $F \in P_{f,n}(Y)$, we have that all roots $\alpha$ of $F$ satisfy $|\alpha| \ll n^f Y$ and thus we also have $|\text{Disc}(F)| \leq BY^{n(n-1)}$ for a constant $B$ depending on $f$ and $n$.

5.2. Bounding multiplicities. We count the number of fields arising from specializations in (2.1) by counting the number of polynomials in $P_{f,n}(Y)$ and adjusting for two possible sources of multiplicity. The first potential source of multiplicity is the case where two different $g(t), h(t)$ give rise to the same element $F(t)$ in $P_{f,n}(Y)$. The second potential source of multiplicity is that multiplicity $F(t)$ in $P_{f,n}(Y)$ produce isomorphic number fields. The first potential source of multiplicity is dealt with by the following lemma:

**Lemma 5.2.** Let $F(t) \in \mathbb{Z}[t]$ be a polynomial of degree $n$. The number of ways to choose $g(t), h(t) \in \mathbb{Z}[t]$ where one of the leading coefficients of $g$ or $h$ is fixed, such that $F(t) = g(t)^m - f(t)h(t)^m$ is $O_n(1)$.

**Proof.** Note that we assumed $f(x)$ is squarefree in our definition of a superelliptic curve in (1.1). We then chose $\gamma_0(t)$ in Corollary 4.6 such that $f(\gamma_0(t))$ is squarefree. Thus the coordinate ring $\mathbb{C}[t,y]/(y^m - f(\gamma_0(t)))$ is a Dedekind domain by Theorem II.5.10 of [Lor96]. With this, the justification is the same as in [LT19, Lemma 7.4].

In other words, this lemma gives us that each choice of $g(t), h(t)$ coincides with at most a constant number of other choices. Thus we can give a count for the number of $F(t)$ in $P_{f,n}(Y)$ based on the number of choices for $g(t)$ and $h(t)$. More precisely, $\#P_{f,n}(Y) \asymp Y^c$ where we define $c$ as follows:

In the case where $m \mid n$, we have
\[
c = \sum_{i=1}^{n/m} i + \sum_{j=0}^{(n-d-k_1)/m} \left( j + \frac{k_1}{m} \right) = \frac{1}{m^2} \left( n^2 + n(m-d) + d^2 + (k_1-d)m - k_1^2 \right).
\]
In the case where \( m \nmid n \), we have

\[
(5.3) \quad c = \sum_{i=0}^{(n-k_2)/m} \left( \frac{k_2}{m} + i \right) + \sum_{j=1}^{(n-r_2)/m} \frac{1}{m^2} \left( n^2 + n(m - d) + \frac{d^2 r^2 + (k_2 - dr)m - k^2}{2} \right).
\]

Because the polynomials \( F(t) \) that we are counting are specializations of the family (2.1), with appropriately chosen \( a_{n/m} \) or \( b_{(n-dr)/m} \), Corollary 4.6 implies that \( P_{f,n}(Y, S_n) \asymp Y^c \). Where \( P_{f,n}(Y, S_n) \) is the subset of \( P_{f,n}(Y) \) for which the elements \( F(t) \) are irreducible and have Galois group \( S_n \).

To address the second source of potential multiplicity (that there may be multiple elements of \( F(t) \) that produce isomorphic number fields), we use results of Ellenberg and Venkatesh [EV06] for counting number fields, and the multiplicity counts of Lemke Oliver and Thorne [LT19]. See also [Key19] for a more detailed discussion.

As mentioned previously our assumptions on the sizes of \( |a_i|, |b_j| \), ensure that the coefficients of (5.1) are bounded by \( |d_{n-i}| \leq AY^t \) for some constant \( A \). In particular the leading terms are bounded, and hence we may divide some constant integer \( w \). We define the set

\[
S(Y) := \left\{ F = t^n + d'_{n-1}t^{n-1} + \ldots + d'_0 \in \mathbb{Z} \left( \frac{1}{w} \right) [t] : |d'_{n-1}| \ll n_f Y^t \right\}
\]

with the additional condition that \( F(t) \) is irreducible. Note that by this construction elements of \( P_{f,n}(Y, S_n) \) are in bijection with a subset of \( S(Y) \), provided we choose the implied constant appropriately.

We define the multiplicity of a number field \( K \) of degree \( n \) in \( S(Y) \) to be as follows

\[
M_K(Y) := \# \{ F \in S(Y) \mid \mathbb{Q}[t]/F(t) \simeq K \}.
\]

We state here several bounds related to this multiplicity \( M_K(Y) \) that we will use to compute bounds on \( N_{n,C}(X, S_n) \). The following is a bound of Lemke Oliver and Thorne on \( M_K(Y) \).

**Lemma 5.3** (Lemke Oliver – Thorne [LT19, Proposition 7.5]). We have

\[
M_K(Y) \ll \max \left( Y^n |\text{Disc}(K)|^{-1/2}, Y^{n/2} \right).
\]

The proof of this lemma uses the geometry of numbers, building on the strategy suggested in [EV06].

This bound of Lemke Oliver and Thorne for \( M_K(Y) \) together with the following theorem of Schmidt on general number field counts with bounded discriminant are used in [Key19] to give a bound for the sum of multiplicities of fields with discriminant bounded by \( T \).

**Theorem 5.4** (Schmidt, [Sch95]). For \( n \geq 3 \), we have

\[
(5.4) \quad N_n(X) \ll X^{2n/3}.
\]

**Lemma 5.5** (Keyes, [Key19, Lemma 5.4]). Let \( T \leq Y^n \). Then

\[
\sum_{|\text{Disc}(K)| \leq T} M_K(Y) \ll Y^n T^{n/4},
\]

where the sum runs over all degree \( n \) number fields \( K \) such that \( |\text{Disc}(K)| \leq T \).

**5.3. Bounding** \( N_{n,C}(X, S_n) \). We now have all the tools to compute the bound on \( N_{n,C}(X, S_n) \), completing the proof of (1.2) in Theorem 1.

**Proof of (1.2).** By our construction, for any \( F \in P_{f,n}(Y, S_n) \) and any root \( \alpha \) of \( F \), we have \((\alpha, \frac{w(a_{i})}{a_{i}^{m}}) \in C(K) \) where \( K = \mathbb{Q}(\alpha) \) is a field of degree \( n \) with \( \text{Gal}(\overline{K}/\mathbb{Q}) \simeq S_n \). Recall also that we have \( |\text{Disc}(K)| \leq BY^{n(n-1)} \) for a constant \( B \). Roughly speaking, we are taking our count for the polynomials and dividing by a bound for the multiplicity (i.e. the number of polynomials per field) to get the number of fields.

First we will show that fields of low discriminant here are negligible in their contributions to \( N_{n,C}(X, S_n) \). Using Lemma 5.5, we choose \( T = \kappa Y^{-\mu}t\), \( \mu = (4n-4(dr+(m-1)m)+(2(dr-k)(dr+k-m))/n) \) so that
and we recall that
\begin{equation}
\#P_{f,n}(Y, S_n) \asymp Y^c
\end{equation}
where \(c\) is given either by (5.2) or (5.3). We choose \(\kappa\) to be sufficiently small so that the quantity in 5.5 is at most \(\#P_{f,n}(Y, S_n)/2\). Thus our parameterization produces negligibly many fields of discriminant at most \(T\). Since the bound in Lemma 5.3 is decreasing with respect to \(|\text{Disc}(K)|\), thus we have \(M_K(Y) \ll T^{-1/2}Y^n\) for all \(K\) of discriminant \(T < |\text{Disc}(K)| \leq BY^n(n-1)\). We obtain an asymptotic lower bound for \(N_{n,C}(BY^n(n-1), S_n)\) by dividing \(\#P_{f,n}(Y, S_n)\) by this worst case multiplicity.

\begin{equation}
N_{n,C}(BY^n(n-1), S_n) \gg Y^{c-n}T^{1/2}
\end{equation}

To obtain the exponent \(c_n\) in (1.2), we replace \(Y\) in (5.7) by \((X/B)^{1/n(n-1)}\). This produces
\begin{equation}
c_n = \frac{1}{m^2} + \frac{2n^2(m - m^2 - 3) + n(km - k^2 + 4(m - m^2 - dr) + 2(km - m^2 + d^2 - km - dr))}{2m^2(n - 1)^2}
\end{equation}
and thus \(N_{n,C}(X, S_n) \gg X^{c_n}\), which is the first statement of Theorem 1. Note that \(r\) and \(k\) are natural numbers depending on \(m, n, d\). If we are in the case that \(m \mid n\), then we have \(r = 1\) and \(k\) is defined as follows
\[k := \begin{cases} k_1 & \text{if } m \mid n, \\ k_2 & \text{if } m \nmid n. \end{cases}\]
In all cases, \(1 \leq r < m\) and \(0 \leq k < m\). \(\Box\)

5.4. Improvements for \(n\) sufficiently large. As in [Key19, Section 5.4], we can improve on our lower bound when \(n\) is sufficiently large by employing better known upper bounds for \(N_n(X)\). The idea is to show that if the upper bound for \(N_n(X)\) is good enough, then the best case scenario of Lemma 5.3 applies, and we can assume \(M_K(Y) \ll Y^{n/2}\). Thus
\[N_n(Y^{n(n-1)}) \gg Y^{c - \frac{n}{2}}\]
where \(c\) is given in (5.2) or (5.3), as appropriate. It remains to compute this exponent and determine when the improved upper bounds for \(N_n(X)\) take effect.

Assume we have an upper bound of the form
\begin{equation}
(*) \quad N_n(X) \ll X^{\alpha(n,m,d)},
\end{equation}
where \(\alpha(n, m, d) \geq 1\) is a constant depending on \(n\) and the \(m, d\) values for our curve \(C\). We will use a modification of (the proof of) Lemma 5.5 which is somewhat more flexible.

**Lemma 5.6.** Let \(T \leq Y^n\). Assume \((*)\) for some constant \(\alpha(n,m,d)\). Then
\[\sum_{|\text{Disc} K| \leq T} M_K(Y) \ll Y^nT^{\alpha(n,m,d) - 1/2} + \frac{Y^nT^{\alpha(n,m,d) - 1}}{2\alpha(n, m, d) - 1}.
\]
In particular, when we take \(T = Y^n\) we have
\[\sum_{|\text{Disc} K| \leq Y^n} M_K(Y) \ll Y^{\frac{n}{2} + \alpha(n,m,d)}.
\]
Proof. We set up a Riemann-Stieljes integral as in [Key19, Lemma 5.4].
\[ \sum_{|\text{Disc } K| \leq T} M_K(Y) = \int_{1^-}^{T} M_K(Y) dN_n(t) \]
\[ \ll \int_{1^-}^{T} Y^n t^{-\frac{3}{2}} dN_n(t) \]
\[ = Y^n T^{-\frac{3}{4}} N_n(T) + \frac{Y^n}{2} \int_{1^-}^{T} t^{-\frac{3}{4}} N_n(t) dt. \]
Substituting (*) into the last line above gives the first statement of the lemma. \[\square\]

Note that Lemma 5.5 follows from this by taking \(\alpha(n, m, d) = \frac{n+2}{4}\), the Schmidt bound [Sch95]. However, this isn’t good enough for \(Y^{\frac{n}{2} + n\alpha(n, m, d)}\) to be \(o(Y^c)\). For this we need
\[ (**) \quad \alpha(n, m, d) < \frac{c}{n} - \frac{1}{2}. \]
Using the best known upper bounds we can find when (**) is satisfied for a given \(C\) and \(n\).

**Theorem 3** (Lemke Oliver – Thorne, [LT20, Theorem 1.1]). For \(n \geq 6\) we have
\[ N_n(X) \ll X^{1.564(\log n)^2}. \]

This is sufficient to give the proof of (1.3) in Theorem 1, which we state as a corollary.

**Corollary 5.7.** For any given curve \(C\) (i.e. a choice of \(m, d\)), if \(n \gg 0\) then we have
\[ N_n(X, S_n) \gg X^{c_n}, \]
where
\[ c_n = \frac{1}{m^2} \left( 1 + \frac{(2m - 2dr + 1)n + d^2r^2 - mdr + mk - k^2}{2n(n-1)} \right). \]
This is precisely (1.3) in Theorem 1.

**Proof.** Fix a choice of \(C\), so \(m\) and \(d\) are fixed. Assume \(n \geq 6\) and set \(\alpha(n, m, d) = 1.564(\log n)^2\), so Theorem 3 ensures (*) is satisfied. Recalling \(c\) from (5.2) or (5.3) we see that in either case, \(\frac{c}{n} - \frac{1}{2}\) grows linearly with \(n\), as \(k_1, k_2, \) and/or \(r\) are bounded, depending on \(m, d\). Clearly \((\log n)^2\) grows more slowly with \(n\), so for \(n\) sufficiently large (**) is satisfied.

As noted above, Lemma 5.6 together with (**) implies that
\[ \sum_{|\text{Disc } K| \leq Y^n} M_K(Y) = o(Y^c). \]
Thus the contribution of fields with discriminant up to \(Y^n\) to \(\#P_{f,n}(Y, S_n)\) is negligible. For fields \(K\) with \(Y^n < |\text{Disc } K| \leq Y^{n(n-1)}\) we have \(M_K(Y) \ll Y^{n/2}\) by Lemma 5.3. Hence, we have
\[ N_{n,C}(Y^{n(n-1)}, S_n) \gg \#P_{f,n}(Y, S_n) Y^{-\frac{2}{2}} \gg Y^{c - \frac{n}{2}}. \]
To get \(c_n\) we set \(Y = X^{\frac{n}{n(n-1)}}\) and take \(c_n = \frac{c - n/2}{n(n-1)}\), which we can compute explicitly to obtain the stated value. \[\square\]

The question remains to find when the improved asymptotic lower bound above takes effect. That is, to determine when (*) and (**) are both satisfied. To do this, we make use of a more flexible version upper bound of Lemke Oliver and Thorne, which we state below with some variables changed to avoid confusion with our notation.

**Theorem 4** (Lemke Oliver – Thorne, [LT20, Theorem 1.2]). Let \(n \geq 2\).

1. Let \(a\) be the least integer for which \(\left(\frac{a+2}{2}\right) \geq 2n + 1\). Then
\[ N_n(X) \ll X^{2a - \frac{n(a - 1)(a + 4)}{6(n-1)}}. \]
For a fixed superelliptic curve, i.e. choice of $m$ and $d$, we aim to find an integer $N$ such that for all $n \geq N$ satisfying $\gcd(m, d) \mid n$, Corollary 5.7 is true. Below we summarize this procedure.

1. Set $\alpha(n, m, d) = 1.564(\log n)^2$ and find $N_0$ such that $(\ast \ast)$ is satisfied for all $n \geq N_0$. ($(\ast)$ satisfied by Theorem 3).

2. For $n_0 = \max(d, \lcm(m, d) - m - d, 2m^2 - m) \leq N < N_0$, use Theorem 4 search for $a, b$ values to find $\alpha(n, m, d)$ satisfying both $(\ast)$ and $(\ast \ast)$.

For several small values of $m$ and $d$, we compute $N$ with this procedure, displayed below in Figure 5.1.

| $m$ | 2 | 3 | 4 | 5 | 6 | 7 | 10 |
|-----|---|---|---|---|---|---|----|
| $d$ | $n_0$ | $N$ | $n_0$ | $N$ | $n_0$ | $N$ | $n_0$ |
| 3   | 6  | 106 | 15  | 552|    |    |    |
| 4   | 6  | 108 | 15  | 553| 28 | 1164|    |
| 5   | 6  | 110 | 15  | 555| 28 | 1161| 45 | 2015|
| 6   | 6  | 112 | 15  | 558| 28 | 1162| 45 | 2014| 66 | 3192|
| 7   | 7  | 114 | 15  | 559| 28 | 1163| 45 | 2015| 66 | 3187| 91 | 4438|
| 10  | 10 | 120 | 17  | 565| 28 | 1166| 45 | 2020| 66 | 3190| 91 | 4438|
| 100 | 100| 234 | 197 | 662| 100| 2146| 100| 2110| 194| 3278| 593| 5525| 190| 10940|
| 1000| 1000|1000|1997|1997|1000|2040|1000|3045|1994|4139|5993|5993|1000|11800|

Figure 5.1. When is Corollary 5.7 taking effect?

6. THE CASE WHERE $C$ HAS A RATIONAL POINT

Over the course of this section we will prove Theorem 2. Suppose $C$ has a rational point. We claim that after an appropriate change of variables, we can take the defining polynomial $f(x)$ to be monic. We observe that, after composing with an automorphism of $\mathbb{P}^1$, we may assume this rational point lies in the fiber over infinity. Considering the homogeneous equation for $C$ in the weighted projective space $\mathbb{P} \left( \frac{\lcm(m, d)}{d}, \frac{\lcm(m, d)}{m}, \frac{\lcm(m, d)}{d} \right)$,

$$C: y^m = c_dx^d + c_{d-1}x^{d-1}z + \cdots + c_1xz^{d-1} + c_0z^d,$$

this equates to a rational point of the form $[x_0 : y_0 : 0]$, i.e. a rational solution to $y_0^m = c_dx_0^d$.

Writing $c_d = \frac{y_0^m}{x_0^d}$, and noting that $m$ and $d$ are both multiples of $\gcd(m, d)$ we have that $c_d$ is a $\gcd(m, d)$-th power. Since $\frac{m}{\gcd(m, d)}$ and $\frac{d}{\gcd(m, d)}$ are relatively prime, there exist positive integers $k, \ell$ such that $\frac{km}{\gcd(m, d)} = \frac{\ell d}{\gcd(m, d)} - 1$. Multiplying the equation for $C$ by $c_d^{\frac{km}{\gcd(m, d)}}$, we get

$$c_d^{\frac{km}{\gcd(m, d)}}y^m = c_d^{\frac{\ell d}{\gcd(m, d)}}f(x)$$

$$= c_d^{\frac{\ell d}{\gcd(m, d)}}x^d + c_d^{\frac{km}{\gcd(m, d)}}c_d^{-1}x^{d-1} + \cdots + c_0$$

$$\implies \left( c_d^{\frac{k}{\gcd(m, d)}} \right)^m \left( c_d^{\frac{\ell d}{\gcd(m, d)}} \right)^x + c_d^{\frac{\ell d}{\gcd(m, d)}}c_d^{-1} \left( c_d^{\frac{\ell d}{\gcd(m, d)}} \right)^x + \cdots + c_d^{\frac{km}{\gcd(m, d)}}c_0.$$

This shows that after a change of coordinates we may assume $f$ is monic, and since $c_d$ is a $\gcd(m, d)$-th power, this change of coordinates is defined over the rationals. For the remainder of this section, we assume $f(x)$ is monic, i.e. $c_d = 1$, which will make the following parameterization strategy more convenient.
6.1. **Parameterization.** Let \( n \) be a positive integer. If \( n \) is a multiple of \( \gcd(m, d) \), we fall into the case of Theorem 1. Suppose instead that \( n \) is not a multiple of \( \gcd(m, d) \). In this case, our parameterization strategy is essentially the same as described in Section 2, in that we parameterize the coordinates by rational functions \( x(t) = \gamma(t) \) and \( y(t) = \frac{g(t)}{h(t)} \); however this time we need to add extra conditions to ensure that \( F(t) \) as given in (2.1) has degree \( n \). The key is to use the monicity of \( f \) and request dependence relations among the coefficients of \( g, h, \) and \( \gamma \) such that the leading terms of \( F(t) \) vanish. We summarize this in the following proposition, analogous to Proposition 2.1.

**Proposition 6.1.** Let \( C \) be given by (1.1) with \( f \) monic and \( m \leq d \). For all degrees \( n \) not divisible by \( \gcd(m, d) \) and sufficiently large, there exist \( g, h, \gamma, \eta \) such that \( F_{g,h,\gamma,\eta}(t) \) given in (2.1) has degree \( n \) in the ring \( \mathbb{Q}(a, b, \alpha)[t] \).

Explicitly, if \( (s-1)m < n < sm \), we take \( \eta = 1 \) and \( g, h, \gamma \) with degrees

\[
\begin{align*}
\deg g &= s \\
\deg h &= s - \frac{rd}{m} \\
\deg \gamma &= r,
\end{align*}
\]

where \( r > 0 \) is the minimal integer such that \( m \mid rd \), and impose additional relations given in (6.2) and (6.3) on the highest \( sm - n \) coefficients of \( g(t) \).

**Proof.** A priori, before imposing any relations on the coefficients, \( F(t) \) is seen to have degree \( sm \). Explicitly, write

\[
\begin{align*}
g(t) &= a_s t^s + \cdots + a_0 \\
h(t) &= b_{s-rd/m} t^{s-rd/m} + \cdots + b_0 \\
\gamma(t) &= a_r \cdot \alpha_r^{\lcm(m, d)} t^r + a_{r-1} t^{r-1} + \cdots + a_0,
\end{align*}
\]

noting that the leading coefficient of \( \gamma(t) \) is a \( \lcm(m, d) \)-th power, to be used momentarily. As we can see, this only works for \( s \geq rd/m \), which is taken care of by the hypothesis that \( n \) is sufficiently large.

The leading coefficient of \( F(t) \) is given by

\[
\left( a_s^m - b_{s-rd/m}^{\lcm(m, d)} \right) t^{sm}.
\]

We can force this to be zero by setting

\[
a_s = b_{s-rd/m}^{\lcm(m, d)}/m.
\]

Write \( n = (s-1)m + k \), where \( 0 < k < m \). By imposing relations on more coefficients, we can ensure \( \deg F = n \). For \( 0 < u < m - k \), the \( t^{sm-u} \) coefficient of \( F(t) \) is of the form

\[
\sum_{i_1 + \cdots + i_m = sm-u} (a_{i_1} \cdots a_{i_m}) \prod_{0 \leq w \leq sm-u} c_w \left( \sum_{w+\ell+\cdots+\ell = sm-u-v} \prod (\alpha_{i_1} \cdots \alpha_{i_m}) \right)
\]

Thus by taking

\[
a_{s-u} = \frac{1}{bd_m} \left[ - \sum_{i_1 + \cdots + i_m = sm-u} (a_{i_1} \cdots a_{i_m}) \prod_{0 \leq w \leq sm-u} c_w \left( \sum_{w+\ell+\cdots+\ell = sm-u-v} \prod (\alpha_{i_1} \cdots \alpha_{i_m}) \right) \right],
\]

we ensure the \( t^{sm-u} \) coefficient is zero. Again, we are using the hypothesis that \( n \) is sufficiently large here to ensure that there are enough coefficients to complete the process. Doing this for all \( 0 < u < m - k \), we have that \( F(t) \) has degree \( n \) in \( \mathbb{Q}(a, b, \alpha)[t] \). \( \square \)
6.2. Irreducibility and the Galois group of $F$. Following a similar strategy as in Proposition 4.3 — using Newton polygons to show specializations of $F(t)$ have certain cycles in their Galois groups — we can show that the polynomial family described above is irreducible and has Galois group $S_n$.

Proposition 6.2. Fix an integer $m \geq 2$ and a monic squarefree integral polynomial $f(x)$ of degree $d \geq m$. Then for all sufficiently large integers $n$ such that $\gcd(m, d) \nmid n$, the polynomial family $F(t) \in \mathbb{Q}(\alpha, \beta, \gamma)[t]$ described in Proposition 6.1 is irreducible and has Galois group $S_n$.

Proof. Fix a prime $p \nmid \text{Disc} f$ such that $p$ divides $f(x_0)$ exactly once for some integer $x_0$, which exists by Lemma 4.4. Writing $n = (s-1)m + k$ as in the proof of Proposition 6.1, consider a specialization of $F$ satisfying

\begin{equation}
\begin{align*}
v_p(a_i) &\geq 1 \text{ for all } i < s - (m - k) \\
v_p(a_{s-(m-k)}) &= 0 \\
v_p(b_0) &= 0 \\
v_p(b_j) &\geq 1 \text{ for all } 0 < j \leq s - rd/m \\
\alpha_0 &\equiv x_0 \pmod{p^2} \\
v_p(\alpha_r) &\geq 1 \text{ for all } 0 < j \leq r.
\end{align*}
\end{equation}

These restrictions ensure that the leading term, $a^m_0 - b^m_0 f(\alpha_0)$, is divisible by $p$ exactly once. All other terms of $F(t)$ are divisible by $p$ at least once, until the $t^{(m-(m-k))}$ term, thanks to $a^m_{s-(m-k)}$ not being divisible by $p$. The result is that the Newton polygon of $F_{\alpha_0}[F_{\alpha, \beta, \gamma}]$ has a segment of length $ms - m(m-k) = n - (m-1)(m-k)$ and slope $\frac{1}{n-(m-1)(m-k)}$, so by Lemma 3.5 and Corollary 4.2, $G = \text{Gal}(F(t)/\mathbb{Q}(\alpha, \beta, \gamma))$ contains an $(n - (m-1)(m-k))$-cycle.

By Bertrand’s postulate, there exists a prime $q$ such that $(m-1)(m-k) < q < 2(m-1)(m-k)$. Using the technique in the proof of Lemma 4.5, we see that whenever $s-(m-k) \geq 2(m-1)(m-k)$, we can produce a $q$ cycle in $G$. This condition is satisfied by the hypothesis that $n$ is sufficiently large, and allows us to also produce a transposition and 3-cycle in $G$.

Finally, we argue that $G$ is transitive. The specialization in (6.4) has an irreducible factor of degree $n - (m-1)(m-k)$ by Lemma 3.4. Similarly, in showing there exists a $q$-cycle in $G$ we have produced specialization whose Newton polygon has a segment of length $q$ and slope $-1/q$, hence an irreducible factor of degree $q$. Since $q > (m-1)(m-k)$, it is not possible for $F(t)$ to have irreducible factors of both degrees $n - (m-1)(m-k)$ and $q$ over $\mathbb{Q}(\alpha, \beta, \gamma)$, so we conclude $F(t)$ is irreducible and $G$ is transitive. We appeal now to Proposition 3.3, with $(m-1)(m-k)$ playing the role of $m$ in the statement of the result, to conclude $G \simeq S_n$. \hfill $\Box$

We conclude by noting that Hilbert’s irreducibility theorem allows us to find a partial specialization $\gamma_0 \in \mathbb{Z}[t]$ so that $F_{\alpha_0} \in \mathbb{Q}(\alpha, \beta, \gamma)[t]$ is irreducible, with Galois group $S_n$. The justification is identical to that of Corollary 4.6.

Corollary 6.3. Fix an integer $m \geq 2$ and a monic squarefree integral polynomial $f(x)$ of degree $d \geq m$. Then for all sufficiently large integers $n$ such that $\gcd(m, d) \nmid n$, there exists $\gamma_0 \in \mathbb{Z}[t]$ such that the partial specialization $F_{\alpha_0} \in \mathbb{Q}(\alpha, \beta, \gamma)[t]$ of the $F$ described in Proposition 6.1 is irreducible, has Galois group $S_n$, and satisfies $f(\gamma_0(t))$ is squarefree.

6.3. Counting. Fix $\gamma_0 \in \mathbb{Z}[t]$ as in Corollary 6.3 and take $Y$ to be a positive real number. Let $P_f, n(Y)$ be the subset of polynomials $F = h^n f(\gamma_0) - g^m$, where

\begin{align*}
g(t) &= a_0^m + a_{m-1} t^{m-1} + \cdots + a_0, \\
h(t) &= b_s - rd/m t^{s-rd/m} + b_{s-rd/m-1} t^{s-rd/m-1} + \cdots + b_0, \\
f(\gamma_0(t)) &= c_0^m + c_1^m t + \cdots + c_n^m t^n.
\end{align*}
satisfy the conditions in Proposition 6.1. Moreover, fix the integers $a_{s-i}$ for $0 \leq i \leq m - k$ and $b_{s-rd/m-j}$ for $0 \leq j \leq m - k$ satisfying the restrictions of Proposition 6.1 such that the partial specialization of $F_{a_0}$ is irreducible and has Galois group isomorphic to $S_n$ over $\mathbb{Q}(a, b)$. This is possible by Corollary 6.3 and Theorem 4.1. To complete the definition of $P_{f,n}(Y)$, we require that

$$\begin{align*}
|a_{s-(m-k)-i}| &\leq Y^i \text{ for all } 0 < i \leq s - (m-k), \\
|b_{s-rd/m-(m-k)-j}| &\leq Y^j \text{ for all } 0 < j \leq s - rd/m - (m-k),
\end{align*}$$

so that the $i$-th coefficient of $F \in P_{f,n}(Y)$ is at most a constant multiple of $A_s Y^{s-i}$. Applying Lemma 5.1 and accounting for the implied constant, we see that the root $\alpha$ of such $F$ satisfies $|\alpha| \ll Y$, so we have $|\text{Disc}(F)| \leq BY^m(n-1)$ for a constant $B$.

Counting as in (5.2) and (5.3), we have $\#P_{f,n}(Y) \gg Y^c$ where

$$c = \frac{1}{m^2} \left( n^2 + n (-2m^2 + (2+2k) m - 2k - rd) + m^4 - (2k + 3)m^3 + (k^2 + 5k + dr + 2)m^2 \
- (2k^2 + dr + \frac{3dr}{2} + 3k)m + \frac{d^2r^2}{2} + drk + k^2 \right).$$

Note that while $k, r$ depend on $n$, we always have $0 < k < m$ and $0 < r < m$. By Corollary 6.3 and our choices in the previous paragraph for $a_{s-i}$ for $0 \leq i \leq m - k$ and $b_{s-rd/m-j}$ for $0 \leq j \leq m - k$, we have that $P_{f,n}(Y, S_n) \sim Y^c$.

Finally, when $n$ is sufficiently large, we can use the methods of Section 5.4, namely the proof of Corollary 5.7, to bound $N_{n,C}(X, S_n)$. This shows that when $n$ is sufficiently large we have $N_{n,C}(Y^{n-1}, S_n) \gg Y^{ce/2}$, hence after replacing $Y$ with $X^{\frac{n}{m(n-1)}}$ we have

$$N_{n,C}(X, S_n) \gg X^{\frac{c-n/2}{n(n-1)}} = X^{\frac{1}{m^2} \left( 1 + \frac{2m^2 + (4 + 4k)m - 4k - 2rd + 2}{2m(n-1)} \right)}.$$

That is, $N_{n,C}(X, S_n) \gg X^{c_n}$ where

$$c_n = \frac{1}{m^2} \left( 1 + \frac{-5m^2 + (4 + 4k)m - 4k - 2rd + 2}{2m(n-1)} \right),$$

from which we see that $c_n \to \frac{1}{m^2}$ as $n \to \infty$. This completes the proof of Theorem 2.

7. Geometric sources of higher degree points

Let $C$ be a superelliptic curve given by affine equation of the form $y^m = f(x)$ where $f(x)$ has degree $d$. The parametrization strategy in (2.1) produces points on superelliptic curves that generate degree $n$ field extensions. The strategy fails to produce degree $n$ extensions when $\gcd(m, d) \nmid n$ in general. In this section, we attempt to provide some heuristics for why one should expect degree $n$ points on superelliptic curves with $\gcd(m, d) \nmid n$ to appear less often compared to degree $n$ points with $\gcd(m, d) \mid n$.

In the case of hyperelliptic curves, $m = 2$ and $\gcd(2, d) = 2$, this parametrization does not produce any odd degree points (cf. [Key19]). This is consistent with a result of Bhargava–Gross–Wang [BGW17] which says that a positive proportion of locally soluble hyperelliptic curves have no odd degree points (and thus that a positive proportion of all hyperelliptic curves have no odd degree points).

While we are far from proving an analogous result to [BGW17] for degree $n$ points with $\gcd(m, d) \nmid n$ on superelliptic curves, we attempt to give some heuristics and examples suggesting that points of degree $n$ with $\gcd(m, d) \mid n$ appear more often than those with $\gcd(m, d) \nmid n$ and we ask the following:
Question. What, if anything, can be said about the sparcity or abundance of various degrees $n$ of points on superelliptic curves given by affine equation of the form $C : y^m = f(x)$ where $f(x)$ has degree $d$? In particular, can something be said in terms of the relationship of $n$ to the quantities $m$, $d$, and $\gcd(m, d)$?

Another way to phrase this question is in terms of the index of the curve $C/K$. The index of a curve $C$, denoted $I(C)$, is the greatest common divisor of degrees $[L : K]$, where $L/K$ ranges over algebraic extensions such that $C(L) \neq \emptyset$. See [GLLI13, Sha18] for more on the index of a curve. In terms of the index, we can ask whether the index of our superelliptic curve $C/\mathbb{Q}$ is related to $\gcd(m, d)$. It is already clear for instance that $I(C)\mid gcd(m, d)$ but we ask if more is true.

7.1. Arithmetic from geometry. A geometric source from which we can expect to find infinitely many points on $C$ are maps to $\mathbb{P}^1$. The most apparent of these are the natural maps of degree $m$ and $d$ from our curve $C$ to $\mathbb{P}^1$. That is, we can get infinitely many points by pulling back along the degree $m$ and degree $d$ maps to $\mathbb{P}^1$. Thus we know there are infinitely many degree $n$ points that are either multiples of $d$ or multiples of $m$. For other discussions on sources of infinitely many points on different types of curves, or more general curves, see [AH91, BEL+19, DF93, HS91, SV20].

In what follows, for $n$ the degree of the points and $g$ the genus of the curve, we discuss maps from $C$ to $\mathbb{P}^1$ in the case $n < g$ and in the case $n \geq 2g$.

7.1.1. The case of $n < g$. We first wish to characterize potential sources of infinitely many points on $C$ of degree $n < g$. Suppose further that the exponent $m$ is prime (we remark about the composite case below), so we have either $\gcd(m, d) = 1$ or $\gcd(m, d) = m$. In the former case, the normalization of $C$ has a ramified rational point at infinity, so by Theorem 2 there exist infinitely many points of sufficiently large degree $n$. If $\gcd(m, d) = m$, then we are only guaranteed the existence of points of degree $n$ a multiple of $m$.

Let $n < g$, and define the $n$th symmetric product of $C$ as usual by $\text{Sym}^n(C) := C^n/S_n$. The points of $\text{Sym}^n(C)$ correspond to effective degree $n$ divisors on $C$. We have a natural map

$$\alpha : \text{Sym}^n(C) \to \text{Pic}^n(C),$$

defined by taking $D \mapsto [D]$. $\text{Pic}^n(C)$ is a $g$-dimensional variety (it is a torsor of the Jacobian of $C$), and the image $\alpha(\text{Sym}^n(C))$, often denoted by $W_n$, is a proper closed subvariety of $\text{Pic}^n(C)$.

Suppose there exists a degree $n$ divisor class $[D_0]$, defined over $\mathbb{Q}$. Then $\text{Pic}^n(C)$ is isomorphic to the Jacobian of $C$, denoted $J_C$, by the map $[D] \mapsto [D] - [D_0]$, and we extend the map $\alpha$ above to $J_C$ by composition with the isomorphism. In the case where $m$ is prime, by a result of Zarhin [Zar18, Theorem 1.2] we have that for a generic $C$, $J_C$ is geometrically simple. That is, generically $J_C$ does not contain a translated proper abelian subvariety and therefore $\alpha(\text{Sym}^n(C))$ does not contain an abelian subvariety.

By a theorem of Faltings [Fal94], this implies there are only finitely many points of $\alpha(\text{Sym}^n(C))$ and therefore only finitely many points of $\text{Sym}^n(C)$ that do not come from a $g_d^m$ on $C$.

Theorem 5 (Faltings, [Fal94]). Let $X$ be a closed subvariety of an abelian variety $A$, with both defined over a number field $K$. Then the set $X(K)$ equals a finite union $\cup B_i(K)$, where each $B_i$ is a translated abelian subvariety of $A$ contained in $X$.

In other words, there are only finitely many points of $\text{Sym}^n(C)$ apart from those coming from the positive dimensional fibers of $\alpha$. We know that for some $n$ (namely, $n = m$ or a multiple of $m$) the map $\alpha$ must have positive dimensional fibers, because in particular the points of $\text{Sym}^n(C)$ that are the result of pulling back points from maps to $C$ to $\mathbb{P}^1$ (e.g. a $g_d^m$) map to a point of $J_C$. This is because the Jacobian of $\mathbb{P}^1$ is trivial. However, the lack of a complete characterization of the positive dimensional fibers prevents us from concluding anything about finiteness of $C(K)$ in certain degrees.

For hyperelliptic curves, there is a complete characterization of the positive dimensional fibers (see e.g., Arbarello–Cornalba–Griffiths–Harris, [ACGH85] page 13). Any effective degree $n$ divisor
D having positive rank on a hyperelliptic curve \( H \) must contain a sub-divisor of the form \( P + \iota(P) \) where \( P \) is some point on \( H \) and \( \iota \) is the hyperelliptic involution. In other words, the only positive dimensional fibers of the map \( \alpha \) when \( C \) is a hyperelliptic curve are multiples of the \( g_2^1 \) (i.e. the only source of infinitely many points is pulling back along the degree 2 map to \( \mathbb{P}^1 \)). Gunter–Morrow in [GM19, Proposition 2.6] use this and argue as above to show that for 100% of hyperelliptic curves \( C \) (asymptotically as \( g \to \infty \)), \( C \) has finitely many degree \( n < g \) points that do not arise from pulling back a degree \( n/2 \) point of \( \mathbb{P}^1 \).

**Remark.** In the case of \( m \) composite, we no longer have that \( J_C \) is geometrically simple, however work of Occhipinti–Ulmer [OU15] provides a useful understanding of the abelian subvarieties that appear in the Jacobian. More precisely, for a fixed polynomial \( f(x) \) with \( m = p_1^{a_1} \cdots p_i^{a_i} \) (composite), the curve \( C_m; y^m = f(x) \) has maps to other curves of the form \( C_{m'; y^{m'/p_i^{b_i}}} = f(x) \) where \( 1 \leq b_i \leq a_i \) and \( m' := m/p_i^{b_i} \). These maps between curves induce homomorphisms from the Jacobian \( J_{C_m} \) to \( J_{C_{m'}} \). They define \( J_m^{\text{new}} \) to be the quotient of \( J_{C_m} \) by the sum of the images of these morphisms for all proper divisors \( m' \) of \( m \). \( J_m^{\text{new}} \) is isogenous to the product of \( J_{m'}^{\text{new}} \) with \( m' \) ranging over all divisors of \( m \). They show that for some sufficiently large \( M \), \( J_m^{\text{new}} \) does not contain any abelian subvarieties of dimension less than or equal to the genus of \( C \).

### 7.1.2. The case of \( n \geq 2g \)

For a fixed curve \( y^m = f(x) \) where \( f(x) \) has degree \( d \), the parametrization in Proposition 2.1 produces infinitely many points of sufficiently large degrees \( n \) divisible by \( \gcd(m, d) \). Choose finitely many such points \( P_1 \cdots P_w \) of degrees \( n_1 \cdots n_w \) on \( C \).

We now illustrate how one can use such points to produce a degree \( n = \sum_{i=1}^w n_i \) map to \( \mathbb{P}^1 \), that is, another source of infinitely many points of degree \( n \). In this case \( n \) will (by construction) be a multiple of \( \gcd(m, d) \).

To each point \( P_i \), one can associate an element of \( \text{Sym}^{n_i}(C) \) i.e., the effective degree \( n_i \) divisors \( D_i \) defined over \( \mathbb{Q} \) corresponding to the Galois conjugates of \( P_i \). Take \( D := D_1 + \cdots + D_w \). Let \( w \) be a positive integer large enough such that \( n \geq 2g \). Using that \( C \) is smooth and integral, we may identify Weil divisors with line bundles (see e.g., [Har77], II.6.16), and hence consider the line bundle \( L(D) \), which is defined over \( \mathbb{Q} \). By Riemann–Roch (see e.g., [Har77], IV.1.3), the line bundle \( L(D) \) is basepoint free and has

\[
h^0(C, L(D)) = h^1(C, L(D)) + n + 1 - g \geq g + 1 \geq 2,
\]

and so the sections of \( L(D) \) define a map to \( \mathbb{P}^1 \). We may assume that the sections of \( L(D) \) define a degree \( n \) map to \( \mathbb{P}^1 \). If \( h^0(C, L(D)) \) is greater than 2, we may instead take a sub-linear series. Using a geometric version of the Hilbert Irreducibility Theorem (see e.g., [Ser97] §9.2, Proposition 1), the fibers over all but a thin set of the rational points on \( \mathbb{P}^1 \) give us degree \( n \) points on \( C \). Note that if for our given curve, \( \gcd(m, d) = 1 \), then this produces a degree \( n \) map to \( \mathbb{P}^1 \) giving us infinitely many points on \( C \) for all \( n \) sufficiently large.

**Remark.** The above construction of a degree \( n \) map to \( \mathbb{P}^1 \) began with points \( P_1 \cdots P_w \) coming from parametrization (2.1) that each had degrees that were multiples of \( \gcd(m, d) \). The same construction could be carried out with \( P_1 \cdots P_{w+1} \) if one found a point \( P_{w+1} \) on the curve not coming from the parametrization, but instead having some degree \( n_{w+1} \) that is not a multiple of \( \gcd(m, d) \). The result of this would be that for \( n \) sufficiently large, there is an infinite source of points that have degree \( n \) (i.e. a degree \( n \) map to \( \mathbb{P}^1 \)) where \( n \) is not a multiple of \( \gcd(m, n) \).

**Remark.** If \( g + 1 \leq n < 2g \) and \( L(D) \) is not basepoint free, we can still obtain a degree \( n \) map to \( \mathbb{P}^1 \) from the curve minus the base point locus. By the “curve to projective” extension theorem, such a map extends to a map to \( \mathbb{P}^1 \) from the curve but the degree can be smaller by the degree of the base locus divisor. The degree of the base locus divisor must be divisible by the index of the curve.

### 7.2. On the degree \( n \) points from the parametrization

Recall that points coming from the parametrization (2.1) generate fields of degree \( n \) and we show in Proposition 4.3 that 100% of these
fields have Galois closure $S_n$. As a special case of the points produced by the parameterization, when $n = m$ our parametrization gives rise to points that come from pulling back along the $n$-to-one map to $\mathbb{P}^1$. This case is special because the fields generated by these points have Galois closure $C_n$ not $S_n$ and so on by Proposition 4.3 they make up 0% of the points produced by our parametrization.

To see this, consider for example, the curve $y^m = f(x)$, and set $\gamma(t) = \alpha_0$, $g(t) = t$, and $\eta(t) = h(t) = 1$, this gives us the polynomial $F(t) = t^m - f(\alpha_0)$. We view this as a map from $\mathbb{P}^1$ to $\text{Sym}^m(C)$ sending the point $[\alpha_0 : 1]$ on $\mathbb{P}^1$ to the degree $n$ divisor consisting of the conjugates of the point $(\alpha_0, f(\alpha_0))^{1/m}$ on $C$. Thus the parametrization given in (2.1) recovers degree $m$ points that come from pulling back along the $m$-to-one map to $\mathbb{P}^1$.

A description as above of the points coming from parametrization (1.1) as some copy of $\mathbb{P}^r$ (for some $r$) does not hold in general, so we ask the following question:

**Question.** Is there a nice geometric characterization of the points that arise from the parametrization given in (2.1)?

### 7.3. Heuristics for a special case using a result of Bhargava–Gross–Wang

Suppose we have a curve $C$ given by an affine equation $y^m = f(x)$ where $f(x)$ has degree $d > 4$. Suppose further that $m$ and $d$ satisfy $2^i | \gcd(m, d)$ where $i \geq 2$. Let $N = 2k$ for $k$ an odd prime. In particular, for this case we have that $\gcd(m, d) \nmid N$. In what follows we suggest that for $k$ sufficiently large, one should not expect to find many points of degree $N$.

Let $C$ be the superelliptic curve given by $y^m = f(x)$ with $f(x)$ of degree $d$ and let $H$ be the hyperelliptic curve given by $y^2 = f(x)$ (note that this is the same $f(x)$ as in the equation of $C$). We made the assumption that $d > 4$, so $H$ has genus at least 2. We have a natural map $\phi$ from $C$ to $H$, given by sending points $\{(x, \sqrt{f(x)})\}$ to $\{(x, \sqrt[2]{f(x)})\}$. If $P$ is a point of degree $N$ on $C$, we can map it to a point $P'$ on $H$ as below. Let $\mathbb{Q}(P)$ and $\mathbb{Q}(P')$ be the extensions generated by a point $P$ on $C$ and by a point $P'$ on $H$, respectively.

$$
\begin{array}{ccc}
\mathbb{Q}(P) & C & \{(x_0, y_0)\} \\
N \downarrow & \phi & \downarrow \\
\mathbb{Q}(P') & H & \{(x_0, y_0^{m/2})\} \\
\downarrow & \psi & \downarrow \\
\mathbb{Q} & \mathbb{P}^1 & \{(x_0 : 1)\}
\end{array}
$$

By assumption, the degree of $[\mathbb{Q}(P) : \mathbb{Q}]$ is $N$. This means that the possibilities for $d_\psi$ and $d_\phi$ are as follows:

|   | Case 1 | Case 2 | Case 3 | Case 4 |
|---|--------|--------|--------|--------|
| $d_\psi$ | 1      | $N$    | $k$    | 2      |
| $d_\phi$ | $N$    | 1      | 2      | $k$    |

**Case 1:** One should expect this to happen rarely as this would imply $\mathbb{Q}(P) = \mathbb{Q}(P')$, or equivalently $\mathbb{Q}\left(x_0, \sqrt[2]{f(x_0)}\right) = \mathbb{Q}\left(x_0, \sqrt{f(x_0)}\right)$, is an equality of degree $N$ number fields.

**Case 2:** In this case $H$ has a rational point. Since we assumed $g(H) \geq 2$, Faltings’ theorem [Fal83] implies that the set $H(\mathbb{Q})$ is finite. In fact, Shankar–Wang [SW18] show that for even, monic hyperelliptic curves $H$ of genus $g(H) \geq 9$ with a marked rational non-Weierstrass point $\infty$, a positive proportion (tending to 100% as $g(H) \to \infty$) have exactly two rational points, namely $\infty$ and $-\infty$, the conjugate of $\infty$ under the hyperelliptic involution. By assumption we have that $H$ is even, but even if $H$ is not monic, we may still be able to bound the number of rational points. Under certain technical assumptions (when $r \leq g(H) - 3$, for $r$ the rank of $J_H$), Stoll [Sto19] gives an explicit
uniform bound for \( \#H(\mathbb{Q}) \) depending only on the genus of the curve and the rank of its Jacobian using the Chabauty–Coleman method [Cha41, Col85] (see also [MP12]). Therefore we may say that this does not happen often.

Case 3: We have that \( d_\phi \) is bounded above by the degree of \( \phi \), so the Riemann–Hurwitz formula gives an upper bound

\[
d_\phi \leq \deg(\phi) \leq \frac{g(C) - 1}{g(H) - 1}.
\]

Thus for \( N \) sufficiently large (i.e., \( k \) sufficiently large), Case 3 is excluded entirely.

Case 4: Here \( P' \) is an odd degree point of \( H \). However, Bhargava–Gross–Wang [BGW17] show that a positive proportion of hyperelliptic curves \( H \) have no odd degree points, excluding this case. Note that for this positive proportion of curves, Case 2 also does not occur.

We conclude with an illustrative special case, in which we show that for many curves \( C \) satisfying some conditions on \( m, d, k \), we have at most finitely many points of degree \( N \).

Let \( f(x) \) be a squarefree polynomial of even degree \( d = 2g + 2 \). This gives a hyperelliptic curve with affine equation

\[
y^2 = f(x) = c_{2g+2}x^{2g+2} + c_{2g+1}x^{2g+1} + \cdots + c_0
\]

with coefficients \( c_i \in \mathbb{Z} \). We define the height of the polynomial \( f(x) \) to be

\[
\text{ht}(f) := \max\{|c_i|\},
\]

where \( c_i \) are as above.

We remark that Propositions 2.5 and 2.6(2) of [GM19] hold for even degree hyperelliptic curves as in (7.1). The results of Gunther–Morrow are stated for (odd) hyperelliptic curves with a rational Weierstrass point and their hyperelliptic curves are ordered by a slightly different height. We phrase the result in terms of densities of polynomials \( f(x) \) so that our height is compatible with the height used in [BGW17]. We record the minor differences in the proofs in the following Lemma:

**Lemma 7.1.** For \( n \) an even positive integer and \( g > n \), for 100\% of squarefree polynomials \( f(x) \) ordered by height, the corresponding hyperelliptic curve \( H \) given in (7.1) of genus \( g \) over \( \mathbb{Q} \) have finitely many degree \( n \) points not obtained by pulling back degree \( \frac{g}{2} \) points of \( \mathbb{P}^1 \).

**Proof.** First we show, as in Proposition 2.5 of [GM19] that for 100\% of squarefree polynomials \( f(x) \) the corresponding genus \( g \) hyperelliptic curves with affine equation \( y^2 = f(x) \) have geometrically simple Jacobian.

To see this, let \( t_0, \ldots, t_{2g+2} \), so that we may show that polynomial \( F(x, t_0, \ldots, t_n) = t_{2g+2}x^{2g+2} + \cdots + t_0 \) has Galois group \( S_{2g+2} \) over \( \mathbb{Q}(t_0, \ldots, t_{2g+2}) \). Take the specialization with \( t_2 = \cdots = t_{2g} = 0 \), \( t_0 = t_1 = -1 \), and \( t_{2g+2} = 1 \). This gives us the polynomial \( x^{2g+2} - x - 1 \), which is irreducible and has Galois group \( S_{2g+2} \) by Corollary 3 of [Osa87]. This implies that the curve \( H \) given by affine equation \( H: y^2 = f(x) \) has geometrically simple Jacobian by a result of Zarhin [Zar10]. By the Hilbert Irreducibility Theorem 4.1 we see that 100\% of specializations of \( F(x, t_1, \ldots, t_n) \) have Galois group \( S_{2g+2} \) and thus for 100\% of squarefree polynomials \( f(x) \) the corresponding genus \( g \) hyperelliptic curve \( H \) with affine equation \( y^2 = f(x) \) has geometrically simple Jacobian.

The rest follows from exactly the same proof as [GM19, Proposition 2.6], outlined in Section 7.1, except that since we do not assume there is a rational Weierstrass point, one defines an Abel Jacobi map \( \text{Sym}^n(H) \to J_H \) using a fixed degree 2 divisor \( D_0 \) on the curve and sending \( D \mapsto 2D - nD_0 \) (we know such a divisor exists because of the map to \( \mathbb{P}^1 \)).

We also note that the above height on the polynomials \( \text{ht}(f) := \max\{|c_i|\} \) agrees with the height defined by Bhargava–Gross–Wang in [BGW17] when \( H \) is embedded in the weighted projective space...
\[ \mathbb{P}(1,g + 1, 1) \] and expressed by the following equation:

(7.2) \quad y^2 = f(x, z) = c_{2g+2}x^{2g+2} + c_{2g+1}x^{2g+1}z + \cdots + c_0z^{2g+2}.

Bhargava–Gross–Wang define the height of such a curve \( C \) to be \( ht'(H) := \max\{|c_i|\} \). This height on curves in weighted projective space corresponds exactly to the height \( ht(f) \) on the defining polynomial \( f(x) \) when we dehomogenize by taking \( z = 1 \). Thus by [BGW17, Theorem 1] when ordered by height, a positive proportion of squarefree polynomials \( f(x) \) have no odd degree points.

**Proposition 7.2.** Suppose \( m \) and \( d \) are positive even integers and \( k \) is an odd prime satisfying

- \( 4 \mid m \mid d \),
- \( \frac{m}{2} < k \),
- \( N = 2k < \frac{d}{2} - 1 \).

Then for a positive proportion of squarefree degree \( d \) polynomials \( f(x) \) ordered by height, the superelliptic curve given by \( C: y^m = f(x) \) has finitely many points of degree \( N \).

Moreover, for such a curve \( C \) and point \( P \in C \) of degree \( N \), the image \( \phi(P) \in H \) as defined above is of degree \( N \) and is not the pullback of a degree \( k \) point on \( \mathbb{P}^1 \).

**Proof.** Let \( H \) be the corresponding hyperelliptic curve with equation \( H: y^2 = f(x) \). By methods of [GM19] (see Lemma 7.1), we have that for 100\% of polynomials \( f(x) \) of degree \( d \), the corresponding hyperelliptic curve has only finitely many points of degree \( n < g(H) \) that are not the pullback of a degree \( \frac{d}{2} \) point on \( \mathbb{P}^1 \). We also know that a positive proportion of such hyperelliptic curves do not have any odd degree points by [BGW17, Theorem 1]. Thus for a positive proportion of polynomials \( f(x) \), the hyperelliptic curve \( H \) has both of these properties. For such \( H \), let \( C: y^m = f(x) \) be the superelliptic curve with map to \( H \) given by \( \phi: (x_0, y_0) \mapsto (x_0, y_0^{m/2}) \) as above. Take \( P \) to be a point on \( C \) of degree \( N \) with \( \phi(P) = P' \) its image in \( H \). By considering Cases 1 — 4 above, we show there are only finitely many such \( P \).

Cases 2 and 4 are excluded by the fact that \( H \) has no odd degree points. To see that Case 3 is impossible, we recall \( k = d_\phi \leq \deg(\phi) = \frac{m}{2} \). This contradicts the hypothesis, so \( d_\phi \) cannot be equal to \( k \).

All that remains is Case 1, in which both \( P \) and its image \( P' \) are degree \( N \) points. Suppose \( P = (x_0, y_0) \), so \( P' = (x_0, y_0^{m/2}) \), and the image of \( P, P' \) in \( \mathbb{P}^1 \) is \( x_0 \). If \( P' \) is the pullback of a degree \( k = N/2 \) point of \( \mathbb{P}^1 \), then \( \mathbb{Q}(x_0) : \mathbb{Q} = k \) and \( f(x_0) \) is not a square in \( \mathbb{Q}(x_0) \). However, this implies that the degree of \( \sqrt{f(x_0)} \) over \( \mathbb{Q}(x_0) \) is greater than 2, which contradicts that the degree of \( P \) is \( N \).

Thus we see that \( P' \) cannot be the pullback of a degree \( k \) point on \( \mathbb{P}^1 \). Since \( N < \frac{d}{2} - 1 = g(H) \) by our assumption, we have that only finitely many such \( P' \) can exist. Hence at most finitely many points \( P \) on \( C \) of degree \( N \) can exist. \( \Box \)

The argument for Case 1 in the proof of Proposition 7.2 can be refined to give a slightly more general result, at the expense of the description of the image of \( P \) in \( H \). For this, we do not use [BGW17, Theorem 1], allowing us to obtain a proportion approaching 100\%.

**Proposition 7.3.** Suppose \( m, d \) are positive even integers such that \( d > 4 \). Let \( N < \frac{d}{2} - 1 \) have 2-adic valuation strictly less than that of \( m \), i.e. \( v_2(N) < v_2(m) \). Then for a positive proportion approaching 100\% of squarefree degree \( d \) polynomials \( f(x) \), ordered by height, the superelliptic curve \( C: y^m = f(x) \) has finitely many points of degree \( N \).

**References**

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves*. Vol. I, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[AH91] Dan Abramovich and Joe Harris. Abelian varieties and curves in \( W_d(C) \). *Compositio Math.*, 78(2):227–238, 1991.
[BEL+19] Abbey Bourdon, Özlem Ejder, Yuan Liu, Frances Odumodu, and Bianca Viray. On the level of modular curves that give rise to isolated $j$-invariants. *Adv. Math.*, 357:106824, 33, 2019.

[BGRW20] Abbey Bourdon, David Gill, Jeremy Rouse, and Lori Watson. Odd degree isolated points on $X_1(N)$ with rational $j$-invariant. https://arxiv.org/abs/2006.14966, 2020.

[BGW17] Manjul Bhargava, Benedict H. Gross, and Xiaoheng Wang. A positive proportion of locally soluble hyperelliptic curves over $Q$ have no point over any odd degree extension. *J. Amer. Math. Soc.*, 30(2):451–493, 2017. With an appendix by Tim Dokchitser and Vladimir Dokchitser.

[BN15] Peter Bruin and Filip Najman. Hyperelliptic modular curves $X_0(n)$ and isogenies of elliptic curves over quadratic fields. *LMS J. Comput. Math.*, 18(1):578–602, 2015.

[Box21] Josha Box. Quadratic points on modular curves with infinite Mordell-Weil group. *Math. Comp.*, 90(327):321–343, 2021.

[Cha41] Claude Chabauty. Sur les points rationnels des courbes algébriques de genre supérieur à l’unité. *C. R. Acad. Sci. Paris*, 212:882–885, 1941.

[Co85] Robert F. Coleman. Effective Chabauty. *Duke Math. J.*, 52(3):765–770, 1985.

[DEv+20] Maarten Derickx, Anastassia Etropolski, Mark van Hoeij, Jackson S. Morrow, and David Zureick-Brown. Sporadic cubic torsion. Accepted in *Algebra and Number Theory*, see https://arxiv.org/abs/2007.13929, 2020.

[DF93] Olivier Debarre and Rachid Fahlaoui. Abelian varieties in $W_2^j(C)$ and points of bounded degree on algebraic curves. *Compositio Math.*, 88(3):235–249, 1993.

[EV06] Jordan S. Ellenberg and Akshay Venkatesh. The number of extensions of a number field with fixed degree and bounded discriminant. *Ann. of Math. (2)*, 163(2):723–741, 2006.

[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varitäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983.

[Fal94] Gerd Faltings. The general case of S. Lang’s conjecture. In *Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991)*, volume 15 of *Perspect. Math.*, pages 175–182. Academic Press, San Diego, CA, 1994.

[Fuj16] Matsusaburô Fujiwara. Über die obere schranke des absoluten betrages der wurzeln einer algebraischen gelichung. *Tôhoku Math. J.*, 10:167–171, 1916.

[GM91] Ofer Gabber, Qing Liu, and Dino Lorenzini. The index of an algebraic variety. *Invent. Math.*, 192(3):567–626, 2013.

[GM19] Joseph Gunther and Jackson S. Morrow. Irrational points on hyperelliptic curves. https://arxiv.org/abs/1805.11943, 2019.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HS91] Joe Harris and Joe Silverman. Bielliptic curves and symmetric products. *Proc. Amer. Math. Soc.*, 112(2):347–356, 1991.

[Key19] Christopher Keyes. Growth of points on hyperelliptic curves. https://arxiv.org/abs/1909.04098, 2019.

[Har96] Dino Lorenzini. *An invitation to arithmetic geometry*, volume 9 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.

[LT19] Robert J. Lemke Oliver and Frank Thorne. Rank growth of elliptic curves in nonabelian extensions. *Int. Math. Res. Not. IMRN*, Dec. 2019.

[LT20] Robert J. Lemke Oliver and Frank Thorne. Upper bounds on number fields of given degree and bounded discriminant, 2020.

[MP12] William McCallum and Bjorn Poonen. The method of Chabauty and Coleman. In *Explicit methods in number theory*, volume 36 of *Panor. Synthéses*, pages 99–117. Soc. Math. France, Paris, 2012.

[MR18] Barry Mazur and Karl Rubin. Diophantine stability. *Amer. J. Math.*, 140(3):571–616, 2018. With an appendix by Michael Larsen.

[OS19] Ekin Ozman and Samir Siksek. Quadratic points on modular curves. *Math. Comp.*, 88(319):2461–2484, 2019.

[Osa87] Hiroyuki Osada. The Galois groups of the polynomials $X^n + aX^l + b$. *J. Number Theory*, 25(2):230–238, 1987.

[OU15] Thomas Occhipinti and Douglas Ulmer. Low-dimensional factors of superelliptic Jacobians. *Eur. J. Math.*, 1(2):279–285, 2015.

[Sch95] Wolfgang M. Schmidt. Number fields of given degree and bounded discriminant. *Astérisque*, (228):4, 189–195, 1995. Columbia University Number Theory Seminar (New York, 1992).

[Ser97] Jean-Pierre Serre. *Lectures on the Mordell-Weil theorem*. Aspects of Mathematics. Friedr. Vieweg & Sohn, Braunschweig, third edition, 1997. Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt, With a foreword by Brown and Serre.

[Sha18] Shahed Sharif. Period and index for higher genus curves. *J. Number Theory*, 186:259–268, 2018.

[Sto19] Michael Stoll. Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell-Weil rank. *J. Eur. Math. Soc. (JEMS)*, 21(3):923–956, 2019.
[SV20] Geoffrey Smith and Isabel Vogt. Low Degree Points on Curves. *International Mathematics Research Notices*, 06 2020. rnaa137.

[SW18] Arul Shankar and Xiaoheng Wang. Rational points on hyperelliptic curves having a marked non-Weierstrass point. *Compos. Math.*, 154(1):188–222, 2018.

[Zar10] Yuri G. Zarhin. Families of absolutely simple hyperelliptic Jacobians. *Proc. Lond. Math. Soc. (3)*, 100(1):24–54, 2010.

[Zar18] Yuri G. Zarhin. Endomorphism algebras of abelian varieties with special reference to superelliptic Jacobians. In *Geometry, algebra, number theory, and their information technology applications*, volume 251 of *Springer Proc. Math. Stat.*, pages 477–528. Springer, Cham, 2018.

Lea Beneish, Department of Mathematics and Statistics, McGill University, Montréal, QC, H3A 0B9, Canada

Email address: lea.beneish@mail.mcgill.ca

Christopher Keyes, Department of Mathematics, Emory University, Atlanta, GA 30322, United States

Email address: christopher.keyes@emory.edu