RECOVERING SURFACE PROFILES OF SOLITARY WAVES ON A UNIFORM STREAM FROM PRESSURE MEASUREMENTS

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Abstract. In this paper, we derive an explicit formula that permits to recover the free surface wave profile of an irrotational solitary wave with a uniform underlying current from pressure data measured at the flat bed of the fluid. The formula is valid for the governing equations and applies to waves of small and large amplitude.

1. Introduction. There are several different ways to measure surface wave elevation. A convenient way is through the use of underwater pressure transducers. In fact, the pressure plays an important role in qualitative studies of travelling water waves in irrotational flow and is essential in the description of the particle trajectories beneath the waves (see [6],[8],[11]). The pressure is also important in quantitative studies because the elevation of a surface water waves is often determined in field investigation from pressure data obtained at the sea bed, especially used in the tsunami detection made by the Pacific Tsunami Warning Center (see [2],[17],[19]).

The standard approach to reconstruct the free surface from pressure measurements consists of assuming that the hydrostatic assumption is sufficiently accurate (see [19]). However, one can not expect that this hydrostatic assumption can capture the nonlinear effects and the discussion in Bishop and Donelan [3] shows that even for waves of moderate amplitude predictions errors are significant. Within the linear regime of water waves of small amplitude in finite water depth, one can derive a better approximation (see [14]), but it is limited for waves of moderate amplitude compared with the experimental data by Tsai et al. [21].

These considerations motivate the quest for a reconstruction formula that accounts for nonlinear effects and that is thus applicable to waves of moderate and large amplitude. Recently, nonlinear nonlocal equations relating the dynamic pressure on the bed and the wave profiles were obtained without approximation from the governing equations (see [9],[13],[19]). Clamond and Constantin [4] and Clamond [5] obtained exact tractable relations and straightforward numerical procedures can be used to derive the free surface from the pressure at the bed.

The above discussions are focused on irrotational travelling water waves. It is well known that wave-current interaction is important for sediment transport and

2010 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.
Key words and phrases. Pressure, solitary wave, current.
pollution dispersion in the nearshore zone. It is therefore highly desirable to extend
the theoretical investigations to the irrotational solitary waves with uniform current.
In this paper, we follow the method of Constantin [9] to derive an explicit formula
providing a parametric representation of the wave profile in the irrotational solitary
waves with uniform current in terms of the pressure on the flat bed. It is noted
that the case of non-uniform currents is quite open [16].

2. Preliminaries. The governing equations for a two-dimensional travelling waves
\( y = \eta(X - ct) \) at the surface of water in irrotational flow over the flat bed \( Y = -d \)
are the equation of mass conservation
\[
  u_X + v_Y = 0
\]  
 coupled with Euler's equation
\[
  (u - c)u_X + v u_Y = -P_X
\]
\[
  (u - c)v_X + v v_Y = -P_Y - g,
\]
Here the velocity filed is represented by \( (u = u(X - ct), v = v(X - ct)) \), where \( c > 0 \)
is the wave speed, \( P = P(X - ct, Y) \) denotes the pressure and \( g \) is the gravitational
constant of acceleration. The absence of vorticity is expressed by the irrotational
condition
\[
  u_Y = v_X,
\]
throughout the flow. The boundary conditions are
\[
  v = (u - c)\eta_X \text{ on } Y = \eta(X - ct)
\]
\[
  v = 0 \text{ on } Y = -d
\]
\[
  P = P_{atm} \text{ on } Y = \eta(X - ct)
\]
where \( P_{atm} \) is the constant atmospheric pressure. The flow is uniform for \( X \to \pm \infty \),
with the free surface approaching the height \( d > 0 \) above the flat bed, and
\[
  c > \sqrt{gd}
\]
must hold for non-trivial solutions (see [1]). Moreover, all solitary waves are waves
of elevation above their asymptotic flat limit, symmetric about a single crest and
with a strictly monotonic wave profile on either side of this crest (see [12]). Hence,
for solitary waves we additionally impose the far-field conditions
\[
  \lim_{X \to \infty} \eta(X) \to 0,
\]
\[
  \lim_{|X| \to \infty} v(X - ct, Y) \to 0 \text{ uniformly in } Y,
\]
Throughout the paper we are only concerned with smooth solitary waves. For these
waves it is known that the wave speed exceeds the horizontal velocity component:
\[
  u < c \text{ throughout the flow.}
\]
In this paper we consider that the ambient flow is a uniform current \( U \), so that we
can write the solution as follows
\[
  u = U + \tilde{u}, \quad v = \tilde{v}, \quad P = \tilde{P},
\]
Substituting (12) into (1)-(7), we can obtain the problem as
\[
  \tilde{u}_X + \tilde{v}_Y = 0,
\]
where \( \tilde{c} = c - U \).

Let us introduce in the moving frame
\[
x = X - ct, \quad y = Y,
\]
the velocity potential \( \phi \) by
\[
\phi_x = \tilde{u} - \tilde{c}, \quad \phi_y = \tilde{v}, \quad \phi(0, -d) = 0,
\]
that is, using (16)
\[
\phi(x, y) = \int_0^x [\tilde{u}(l, -d) - \tilde{c}] dl + \int_{-d}^y \tilde{v}(x, s) ds,
\]
Using (13), we can also define the stream function \( \psi \) by
\[
\psi_x = -\tilde{v}, \quad \psi_y = \tilde{u} - \tilde{c}, \quad \psi(0, \eta(0)) = 0,
\]
that is
\[
\psi(x, y) = m + \int_{-d}^y [\tilde{u}(x, s) - \tilde{c}] ds, \quad (x, y) \in \bar{\Omega}
\]
for a suitable constant \( m \in \mathbb{R} \). It follows from the normalization in (23) and from (17)-(18) that \( \psi \) vanishes on the free surface \( y = \eta(x) \) and that \( \psi = m \) on the flat bed \( y = -d \). In particular, we have
\[
m = \psi(x, -d) = -\int_{-d}^{\eta(x)} \psi_y(x, s) ds = \int_{-d}^{\eta(x)} [\tilde{c} - \tilde{u}(x, s)] ds, \quad (x, y) \in \bar{\Omega}
\]
so that \( m = \tilde{c} d \) as \( \eta(x) \to 0 \) and \( \tilde{u}(x, s) \to 0 \) for \( x \to \infty \). Furthermore, notice that the Euler equations (14)-(15) can be recast as stating that the expression \( \frac{|\nabla \psi(x, y)|^2}{2} + gy + \tilde{P} \) is constant throughout the flow. The flow being uniform at infinity, we can evaluate this constant on the free surface where we can use (19) to obtain that
\[
\frac{|\nabla \psi(x, y)|^2}{2} + gy + \tilde{P} = \frac{c^2}{2} + P_{atm},
\]
throughout the flow (that is Bernoulli’s law). Summarizing, the governing equations for irrotational two-dimensional solitary waves on a uniform current above a flat bed are equivalent to the study of the following elliptic free boundary problem
\[
\Delta \psi = 0, \quad x \in \mathbb{R}, \quad -d \leq y \leq \eta(x),
\]
\[ |\nabla \psi(x,y)|^2 + 2g\eta(x) = \bar{c}^2, \ x \in \mathbb{R}, \]  
\[ \psi(x,\eta(x)) = 0, \ x \in \mathbb{R}, \]  
\[ \psi(x,-d) = \bar{c}d, \ x \in \mathbb{R}, \]  

coupled with the asymptotic limits,

\[
\lim_{|x| \to \infty} \eta(x) = 0 \\
\lim_{|x| \to \infty} \nabla \psi(x,y) = (-\bar{c},0) \text{ uniformly for } -d \leq y \leq \eta(x) \tag{31}
\]

The Cauchy-Riemann equations and (27)-(29) imply that the complex functions

\[(x + iy) \mapsto \phi(x,y) + i\psi(x,y), \]  

is holomorphic, so that the smooth functions \(\phi\) and \(\psi\) are harmonic conjugated.

\[ y = \eta(x) \]

\[ p = \frac{-\bar{c}d}{c} = -\bar{d} \]

\[ q = -\frac{\phi}{c} \]

\[ p = -\frac{\psi}{c} \]

**Figure 1.** The hodograph transform.

3. **A conformal hodograph transform.** In the fluid domain \(\Omega = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}, -d \leq y \leq \eta(x)\}\) in the moving fame, we introduce (see figure 1) the hodograph transform \(H\) induced by the pair \((\phi, \psi)\) as

\[ H : \Omega \to \mathbb{R}^2, \ H(x,y) := -\frac{1}{c}(\phi(x,y), \psi(x,y)). \]  

\(H\) is an analytical bijection from \(\Omega\) on to \(\mathbb{R} \times [-\bar{d},0]\), where \(\bar{d} = \bar{c}d/c\) (see \([10,16]\)). It is convenient to denote the coordinates in \(H(\Omega) = \mathbb{R}^2 \times [-\bar{d},0]\) by \((q, p)\), that is,

\[ q = -\frac{1}{c}\phi(x,y), \ p = -\frac{1}{c}\psi(x,y), \]  

Introducing the height function


\[ h(q, p) = y + \tilde{d} \text{ for } (q, p) \in \mathbb{R} \times [-\tilde{d}, 0], \]  

\( (35) \)

notice that

\[
\begin{aligned}
\partial_q &= h_p \partial_x + h_q \partial_y, \\
\partial_p &= -h_q \partial_x + h_p \partial_y,
\end{aligned}
\]

\( (36) \)

and

\[
\begin{aligned}
h_q &= -c \frac{\tilde{v}}{(\tilde{c} - \tilde{u})^2 + \tilde{v}^2} = -\frac{\partial_x}{\partial p} = \frac{\partial y}{\partial q}, \\
h_p &= c \frac{\tilde{c} - \tilde{u}}{(\tilde{c} - \tilde{u})^2 + \tilde{v}^2} = \frac{\partial x}{\partial q} = \frac{\partial y}{\partial p},
\end{aligned}
\]

\( (37) \)

Thus (16) is equivalent to the following nonlinear boundary value problems:

\[
\begin{aligned}
\Delta_{p,q} h &= 0, \quad q \in \mathbb{R}, \quad -\tilde{d} < p < 0, \\
h(q, -\tilde{d}) &= 0, \quad q \in \mathbb{R}, \\
(h^2_q(q, 0) + h^2_p(q, 0))(\tilde{c}^2 - 2gh(q, 0) + 2g\tilde{d}) &= c^2, \quad q \in \mathbb{R},
\end{aligned}
\]

\( (38) \)

with the asymptotic limits

\[
\begin{aligned}
\lim_{|q| \to \infty} h(q, 0) &= \tilde{d}, \\
\lim_{|q| \to \infty} \nabla_{q,p} h(q, p) &= (0, c), \quad \text{uniformly for } -\tilde{d} \leq p \leq 0.
\end{aligned}
\]

\( (39) \)

4. **Main result.** The above considerations will be developed by means of Fourier analysis. In using the Fourier transform the growth at infinity is crucial. Following Constantin [9], we can provide an approach that just relies upon classical distribution theory. Indeed, from (37), we have

\[
(c - \tilde{u})^2 + \tilde{v}^2 = \frac{c^2}{h^2_q + h^2_p}
\]

so that Bernoulli’s law (26) becomes

\[
\frac{c^2}{2(h^2_q + h^2_p)} + gh + \bar{P} - P_{atm} - g\tilde{d} = \frac{c^2}{2}, \quad q \in \mathbb{R}, \quad -\tilde{d} \leq p \leq 0.
\]

\( (41) \)

Evaluating the above relation on \( p = -\tilde{d} \), we infer from (38) that

\[
\frac{c^2}{h^2_p(q, -\tilde{d})} = c^2 - 2f(q), \quad q \in \mathbb{R},
\]

\( (42) \)

with \( f(q) = \bar{P}(q, -\tilde{d}) - P_{atm} - g\tilde{d} \). Taking into account (11) and (37), this yields

\[
h_p(q, -\tilde{d}) = \frac{c}{\sqrt{c^2 - 2f(q)}}, \quad q \in \mathbb{R}.
\]

\( (43) \)

The function \( h \) is bounded so that it represents a tempered distribution, cf. Friedlander [15]. We perform Fourier transforms in this distributional sense throughout this paper all the references to (tempered) distributions are with respect to the
q-variable, and the p-variable is treated as a parameter. Taking the Fourier transform \( \hat{h} \) in the q-variable of the partial differential equation in (38), we obtain the differential equation

\[
\hat{h}_{pp}(k, p) - k^2 \hat{h}(k, p) = 0
\]

(44)

where \( \hat{h}(k, p) = \mathcal{F}\{h\}(k) = \int_{\mathbb{R}} h(k, p)e^{-ikq}dq, \ k \in \mathbb{R}, \) stands for the Fourier transform of the function \( q \mapsto h(q, p) \).

Equation (44) has the general solution

\[
\hat{h}_p(k, p) = A(k)e^{kp} + B(k)e^{-kp}, \quad -\tilde{d} \leq p \leq 0.
\]

(45)

The left-hand side is a tempered distribution (being the Fourier transform of a tempered distribution), while on the right-hand side \( A \) and \( B \) are merely distributions so that their multiplication with smooth functions is well-defined. Since \( \hat{h}(k, -\tilde{d}) = 0 \) by (38), we have \( B(k) = -A(k)e^{-2k\tilde{d}} \) so that

\[
\hat{h}_p(k, -\tilde{d}) = 2A(k)e^{-k\tilde{d}}
\]

(46)

for some distribution \( A(k) \). Now both sides of (43) are bounded functions and therefore represent tempered distributions. Taking the Fourier transform of (43) we obtain that

\[
\hat{h}_p(k, -\tilde{d}) = \mathcal{F}\left\{ \frac{c}{\sqrt{c^2 - 2f(q)}} \right\}(k)
\]

(47)

in the sense of tempered distributions. On the other hand, differentiating (46) with respect to \( p \) and subsequently setting \( p = -\tilde{d} \), we get

\[
\hat{h}_p(k, -\tilde{d}) = 2A(k)e^{-k\tilde{d}}k
\]

(48)

in the sense of tempered distributions. Comparing this with (47) enables us to infer that

\[
2A(k)e^{-k\tilde{d}}k = \mathcal{F}\left\{ \frac{c}{\sqrt{c^2 - 2f(q)}} \right\}(k)
\]

(49)

as distributions. Thus, in view of (46), we obtain

\[
\hat{h}(k, p) = \frac{\sinh(k(p + \tilde{d}))}{k} \mathcal{F}\left\{ \frac{c}{\sqrt{c^2 - 2f(q)}} \right\}(k), \quad -\tilde{d} \leq p \leq 0,
\]

(50)

in the sense of distributions. Notice that in view of equation (35), the inverse of \( H(\Omega) : \mathbb{R} \times [-p, 0] \to \Omega \) is given by

\[
x = \theta(q, p), \ y = h(q, p) - \tilde{d},
\]

(51)

for some smooth function \( \theta \), so that

\[
(x, \eta(x)) = (\theta(q, 0), h(q, 0) - \tilde{d}).
\]

(52)

Since \( h_p = \theta_q \) by (37), setting

\[
\theta_0(q) = \theta(q, 0), \ h_0 = h(q, 0) - \tilde{d}.
\]

(53)
we obtain
\[ F\{\theta'_0\}(k) = \cosh(k\tilde{d})F\left(\frac{c}{\sqrt{c^2 - 2f(q)}}\right)(k), \tag{54} \]
if we recall (50). On the other hand, (50) also yields
\[ F\{\tilde{h}_0\}(k) = \frac{\sinh(k\tilde{d})}{k}F\left(\frac{c}{\sqrt{c^2 - 2f(q)}}\right)(k) - 2\pi\delta(k). \tag{55} \]
where \( \delta \) is the Dirac mass, since \( F\{1\}(k) = 2\pi\delta(k) \). Note that
\[ 2\pi\delta(k) = \frac{\sinh(k\tilde{d})}{k}2\pi\delta(k) = \frac{\sinh(k\tilde{d})}{k}F\{1\}(k) \]
\( F\{1\}(k) = \cosh(k\tilde{d})F\{1\}(k). \)
Therefore (54) and (55) can be written as
\[ F\left\{\theta'_0 - \frac{c}{\tilde{c}}\right\}(k) = \cosh(k\tilde{d})F\left(\frac{c}{\sqrt{c^2 - 2f(q)}} - \frac{c}{\tilde{c}}\right)(k), \tag{57} \]
and
\[ F\{\tilde{h}_0\}(k) = \frac{\sinh(k\tilde{d})}{k}F\left(\frac{c}{\sqrt{c^2 - 2f(q)}} - \frac{c}{\tilde{c}}\right)(k). \tag{58} \]
respectively, holding in the sense of distributions. Moreover, from (43) we get
\[ |\tilde{c} - \tilde{u}(x, -\tilde{d})|^2 = \tilde{c}^2 - 2f(q) \tag{59} \]
The decay properties of \( \tilde{u}(x, -\tilde{d}) \) as \( |x| \to \infty \), established in Craig and Sternberg [12] and in Mcleod [18], together with (36) ensure that \( f \) and its derivative are square-integrable. Consequently \( \lim_{|q|\to\infty} f(q) = 0 \), cf. Strichartz [20]. The relation
\[ \frac{c}{\sqrt{c^2 - 2f(q)}} - \frac{c}{\tilde{c}} = \frac{2c}{\tilde{c}\sqrt{c^2 - 2f(q)}(\tilde{c} + \sqrt{c^2 - 2f(q)})} \tag{60} \]
guarantees that the expression on the left-hand side is a square-integrable function. This will be also the case for its Fourier transform, cf. Strichartz [17]. Thus (57)-(58) are not just equalities in the sense of distributions but are equalities for functions. Finally, since (37) ensures \( \theta'_0(q) \to 1 \) for \( q \to \infty \), a glance at (52), (57) and (58) yields (61) with \( x = \theta_0 \) and \( \eta = h_0 \),
\[ x(q) = \frac{c}{\tilde{c}}q + \int_{-\infty}^{q} F^{-1}\left\{\cosh(k\tilde{d})F\left[\frac{c}{\sqrt{c^2 - 2f(q)}} - \frac{c}{\tilde{c}}\right](k)\right\}(s)ds, \tag{61} \]
\[ \eta(q) = F^{-1}\left\{\frac{\sinh(k\tilde{d})}{k}F\left[\frac{c}{\sqrt{c^2 - 2f(q)}} - \frac{c}{\tilde{c}}\right](k)\right\}(q) \]
Finally, the exact recovery formula for the wave profile of a solitary wave on a uniform current is obtained. This parametric representation of the surface wave profile is valid for smooth solitary wave with solutions of the governing equations, up to the wave of greatest height. The usefulness (61) holds without the restriction to small-amplitude waves and is explicit. The usefulness solution of (61) lies in
showing that the surface elevation of a solitary water wave on a uniform stream can always be recovered from pressure data at the flat bed of the fluid domain. This is important since a direct measurement of the elevation is difficult, and often such data are inferred from pressure measurements. It is interesting to pursue a numerical or experimental study to investigate the practical use of (61). This is work in progress.

Acknowledgments. The author is visiting the Department of Mathematics, King’s College of London during the preparation of this manuscript, and acknowledges the support of the National Science Council of Taiwan through the Research Cooperation Initiative between Top UK and Taiwan Universities and International Wave Dynamics Research Center(NSC 103-2911-I-006-302). The author is grateful to Prof. Adrian Constantin for many constructive and stimulating discussions. The author would like to acknowledge the insightful critiquing of the referees.

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Received April 2013; revised September 2013.

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