The SHAI property for the operators on $L^p$ *

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Abstract

A Banach space $X$ has the SHAI (surjective homomorphisms are injective) property provided that for every Banach space $Y$, every continuous surjective algebra homomorphism from the bounded linear operators on $X$ onto the bounded linear operators on $Y$ is injective. The main result gives a sufficient condition for $X$ to have the SHAI property. The condition is satisfied for $L^p(0,1)$ for $1 < p < \infty$, spaces with symmetric bases that have finite cotype, and the Schatten $p$-spaces for $1 < p < \infty$.

1 The main results

Following Horvath [9], we say that a Banach space $X$ has the SHAI (surjective homomorphisms are injective) property provided that for every Banach space $Y$, every surjective continuous algebra homomorphism from the space $L(X)$ of bounded linear operators on $X$ onto $L(Y)$ is injective, and hence by Eidelheit’s [6] classical theorem, $X$ is isomorphic as a Banach space to $Y$. The continuity assumption is redundant by an automatic continuity theorem of B. E. Johnson [5, Theorem 5.1.5]. The spaces $\ell^p$ for $1 \leq p \leq \infty$ are known to have the SHAI property [9, Proposition 1.2], as do some other classical spaces [9], [10], but there are many spaces that do not have the SHAI property [9]. Our research on the SHAI

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It is easy to see that the decomposition \((\ell^a)\) where \(a=0,1,2,\ldots\) is a Banach space that verifies the statement about the ranges of the strong limits of the nets. Finally, observe that the space \(L^\infty\) has the SHAI property because \(L^\infty\) is isomorphic as a Banach space to \(\ell^\infty\) [2, Theorem 4.3.10].

Before stating our theorems, we need to review the notion of an unconditional Schauder decomposition of a Banach space \(X\). A family \((E_\alpha)_{\alpha \in A}\) of closed subspaces of \(X\) is called an unconditional Schauder decomposition for \(X\) provided every vector \(x\) in \(X\) has a unique representation \(x = \sum_{\alpha \in A} x_\alpha\), where the convergence is unconditional and, for each \(\alpha \in A\), the vector \(x_\alpha\) is in \(E_\alpha\). Notice that by uniqueness of the representation, \(E_\alpha \cap E_\beta = \{0\}\) when \(\alpha \neq \beta\), and there are idempotents \(P_\alpha\) on \(X\) such that \(P_\alpha x = E_\alpha\) and \(P_\alpha P_\beta = 0\) for \(\alpha \neq \beta\). It is known that the \(P_\alpha\) are in \(L(X)\). Moreover, for any subset \(B\) of \(A\), the net \(\{\sum_{\alpha \in F} P_\alpha : F \subset B\) finite\} is bounded in \(L(X)\) and converges strongly to an idempotent \(P_B\) that has range \(\overline{\text{span}}_{\alpha \in B} E_\alpha\). The suppression constant of the decomposition is then defined to be \(\sup\{\|\sum_{\alpha \in F} P_\alpha\| : F \subset A\) finite\}. Note that \(\|P_B\|\) is bounded by this suppression constant for all subsets \(B\) of \(A\). In practice, this theorem is rarely used, since typically one constructs the idempotents \(P_\alpha\) and checks the uniform boundedness of the aforementioned nets and verifies the statement about the ranges of the strong limits of the nets. Finally, observe that a collection \((e_\alpha)_{\alpha \in A}\) forms an unconditional Schauder basis for \(X\) if and only if \((E_\alpha)_{\alpha \in A}\) is an unconditional Schauder decomposition of \(X\), where \(E_\alpha = \mathbb{K} e_\alpha\) (\(\mathbb{K}\) is the scalar field). In the sequel, we will most often use an unconditional Schauder decomposition \(E_\alpha\) where each \(E_\alpha\) is finite dimensional. Such a decomposition is called an unconditional FDD. FDD stands for finite dimensional decomposition. Schauder decompositions and FDDs are discussed in the monograph [13, Section 1.g]. Schauder bases, type/cotype theory, and other concepts from Banach space theory that are used in this paper are treated in the textbook [2].

A concept that is particularly relevant for us is that of bounded completeness. An unconditional Schauder decomposition \((E_\alpha)_{\alpha \in A}\) for \(X\) is said to be boundedly complete provided that whenever \(x_\alpha \in E_\alpha\) and \(\{\|\sum_{\alpha \in F} x_\alpha\| : F \subset A\) finite\} is bounded, then the formal sum \(\sum_{\alpha \in A} x_\alpha\) converges in \(X\), which is the same as saying that the net \(\{\sum_{\alpha \in F} x_\alpha : F \subset A\) finite\} converges. A convenient condition that obviously guarantees bounded completeness is that the decomposition has a disjoint lower \(p\) estimate for some \(p < \infty\). The decomposition \((E_\alpha)_{\alpha \in A}\) is said to have a disjoint lower; respectively, upper; \(p\) estimate provided that there is \(C < \infty\) so that whenever \(x_1, \ldots, x_n\) are finitely many vectors in \(X\) such that for every \(\alpha \in A\) there is at most one \(i\) with \(1 \leq i \leq n\) for which \(P_\alpha x_i \neq 0\), we have for \(x = \sum_{i=1}^n x_i\) the inequality

\[
\left\| \sum_{i=1}^n x_i \right\| \geq \frac{1}{C} \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p} \quad \text{respectively,} \quad \left\| \sum_{i=1}^n x_i \right\| \leq C \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p} .
\]

It is easy to see that the decomposition \((E_\alpha)_{\alpha \in A}\) has a disjoint lower \(p\) estimate with constant
Lemma 1.1 Suppose that \((E_\alpha)_{\alpha \in A}\) is an unconditional decomposition for \(X\) that has a disjoint lower \(p\) estimate with \(1 \leq p < \infty\), and let \(Y \supseteq X\). Then there is a constant \(C < \infty\) such that if \(A_1, \ldots, A_n\) are disjoint subsets of \(A\) and \(P_{A_j}\) is the basis projection onto \(E_{A_j} = \text{span} \{E_\alpha : \alpha \in A_j\}\) and \(T_1, \ldots, T_n\) are operators in \(L(Y)\), then

\[
\left\| \sum_{i=1}^{n} T_i P_1 \right\| \leq C \left( \sum_{i=1}^{n} \|T_i\|^q \right)^{1/q},
\]

where \(1/p + 1/q = 1\).
Proof: Suppose $x \in X$. Then
\[
\left\| \sum_{i=1}^{n} T_i P_i x \right\| \leq \sum_{i=1}^{n} \|T_i\| \|P_i x\| \leq \left( \sum_{i=1}^{n} \|T_i\|^q \right)^{1/q} \left( \sum_{i=1}^{n} \|P_i x\|^p \right)^{1/p}
\leq C \left( \sum_{i=1}^{n} \|T_i\|^q \right)^{1/q} \|x\|,
\]
where the constant $C$ is the disjoint lower $p$ constant of $(E_\alpha)_{\alpha \in A}$. ■

A family of sets is said to be almost disjoint provided the intersection of any two of them is finite.

**Definition 1.2** Suppose that $(E_n)_{n=1}^\infty$ is an unconditional FDD for a Banach space $X$. We say that $(E_n)$ has property (#) provided there is an almost disjoint continuum $\{N_\alpha: \alpha < c\}$ of infinite sets of natural numbers such that for each $\alpha < c$, $X$ is isomorphic to the closed linear span of the subspaces $E_n$ for $n \in N_\alpha$.

Subsymmetric bases are obvious examples of FDDs that have property (#). (A basis is subsymmetric if it is unconditional and every subsequence of the basis is equivalent to the basis. Symmetric bases are subsymmetric.) A second almost obvious example is the direct sum of two Banach spaces with subsymmetric bases. Such a space has an FDD with property (#) such that each space in the decomposition is two dimensional. In Corollary 1.6 we point out that the Haar basis for $L^p$ has property (#) when $1 < p < \infty$.

**Proposition 1.3** Let $(E_n)_{n=1}^\infty$ be an FDD for a Banach space $X$. Assume that $(E_n)$ has property (#), witnessed by an almost disjoint family $\{N_\alpha: \alpha < c\}$ of infinite subsets of the natural numbers. For $F \subseteq \mathbb{N}$, let $P_F$ be the basis projection from $X$ onto the closed linear span $E_F$ of the subspaces $E_n$ for $n \in F$. Suppose that $\Phi$ is a non zero, non injective continuous homomorphism from $L(X)$ onto a Banach algebra $\mathcal{A}$. Then for each $\alpha < c$, $\Phi(P_{N_\alpha})$ is a non zero idempotent in $\mathcal{A}$. Moreover, there is a constant $C < \infty$ such that if $F$ is any finite subset of $[\alpha < c]$, then $\left\| \sum_{\alpha \in F} \Phi(P_{N_\alpha}) \right\|_{\mathcal{A}} \leq C$. If $\mathcal{A}$ is a subalgebra of $L(Y)$ for some Banach space $Y$, then $(\Phi(P_{N_\alpha}))_{\alpha < c}$ is a family of commuting extensions to $Y$ of the projections associated with an unconditional Schauder decomposition for a subspace $Y_0$ of $Y$.

Proof: Since, for each $\alpha$, the range of $P_{N_\alpha}$ is isomorphic to $X$, and $\Phi$ is not zero, $\Phi(P_{N_\alpha})$ is a non zero idempotent in $\mathcal{A}$. Suppose that $F$ is a finite subset of $[\alpha < c]$. Take a finite set $S$ of natural numbers so that $N_\alpha \cap N_\beta \subseteq S$ for all distinct $\alpha, \beta$ in $F$. For $\alpha \in F$, let $Q_\alpha = P_{N_\alpha \setminus S}$ be the basis projection from $X$ onto $\overline{\text{span}} \{E_n: n \in N_\alpha \setminus S\}$. The kernel of $\Phi$ is
a non trivial ideal in $L(X)$ and hence contains the finite rank operators. Since $P_{N_\alpha} - Q_\alpha$ is a finite rank operator, $\Phi(P_{N_\alpha}) = \Phi(Q_\alpha)$ for each $\alpha \in F$. But the projections $Q_\alpha$, for $\alpha \in F$, are projections onto the closed spans of disjoint subsets of the FDD $(E_n)_{n=1}^\infty$, so

$$\left\| \sum_{\alpha \in F} \Phi(Q_\alpha) \right\|_A \leq \left\| \sum_{\alpha \in F} Q_\alpha \right\| \|\Phi\| \leq C \|\Phi\|,$$

where $C$ is the suppression constant of $(E_n)$. The last statement is now obvious. 

With the preliminaries out of the way, we state the main theorem in this article.

**Theorem 1.4** Let $(E_n)_{n=1}^\infty$ be an unconditional FDD for a Banach space $X$. Assume that $(E_n)_{n=1}^\infty$ has property $(\#)$ (Definition 1.2) and $(E_n)_{n=1}^\infty$ has a disjoint lower $p$ estimate for some $p < \infty$. Then $X$ has the SHAI property.

**Proof:** Suppose, for contradiction, that $\Phi$ is a non injective continuous homomorphism from $L(X)$ onto $L(Y)$ for some non zero Banach space $Y$. We continue with the set up in Proposition 1.3, where property $(\#)$ for $(E_n)$ is witnessed by an almost disjoint family $\{N_\alpha : \alpha < c\}$ of infinite subsets of the natural numbers, and for $F \subset \mathbb{N}$, the basis projection from $X$ onto the closed linear span $E_F$ of $\{E_n : n \in F\}$ is denoted by $P_F$.

We claim that to get a contradiction it is enough to prove that the subspace $Y_0$ is complemented in $Y$. Indeed, if $Y_0$ is complemented in $Y$, then $L(Y_0)$ is isomorphic as a Banach algebra to a subalgebra of $L(Y)$. However, defining $Y_\alpha = \Phi(P_{N_\alpha})Y$ for $\alpha < c$, we know that $(Y_\alpha)_{\alpha < c}$ is an unconditional Schauder decomposition for $Y_0$. But then for every subset $S$ of $\{\alpha : \alpha < c\}$ there is an idempotent $Q_S$ from $Y_0$ onto $\text{span}\{Y_\alpha : \alpha \in S\}$ with $Q_S$ zero on all $Y_\beta$ for which $\beta \notin S$. Thus if $S_1$ and $S_2$ are different subsets of $\{\alpha : \alpha < c\}$, then $\|Q_{S_1} - Q_{S_2}\| \geq 1$, and hence the density character of $L(Y_0)$, whence also of $L(Y)$, is at least $2^c$. However, since $X$ is separable, the density character of $L(X)$ is at most $c$ (actually, equal to $c$ since $X$ has an unconditional FDD), so $L(Y)$ cannot be a continuous image of $L(X)$. This completes the proof of the claim.

To show that $Y_0$ must be complemented in $Y$, we use the fact proved in Proposition 1.3 that there is a constant $C$ such that for every finite subset $F$ of $\{\alpha : \alpha < c\}$ we have $\|\sum_{\alpha \in F} \Phi(P_{N_\alpha})\|_{L(Y)} \leq C$. It was remarked in the introduction to this section that this condition guarantees that $Y_0$ is complemented in $Y$ when $(Y_\alpha)_{\alpha < c}$ is a boundedly complete decomposition. To see that $(Y_\alpha)_{\alpha < c}$ is boundedly complete, we use Lemma 1.1. We guarantee bounded completeness by proving that $(Y_\alpha)_{\alpha < c}$ has a disjoint lower $p$ estimate. That is, we just need to find a constant $C$ so that if $F_1, \ldots, F_m$ are disjoint finite subsets of $\{\alpha : \alpha < c\}$
and \( y \) is in \( Y \) (or even just in \( Y_0 \)), then
\[
\|y\| \geq \frac{1}{C} \left( \sum_{j=1}^{m} \left\| \sum_{\alpha \in F_j} \Phi(P_{N_\alpha})y \right\|^p \right)^{1/p}.
\] (1)

Just as in the proof Proposition 1.3, we can write \( \sum_{\alpha \in F_j} \Phi(P_{N_\alpha}) = \Phi(Q_j) \) with \( Q_j \), for \( 1 \leq j \leq m \), being the basis projections onto the closed spans of disjoint sets of FDD basis spaces \( (E_n) \). So (1) can be rewritten as
\[
\|y\| \geq \frac{1}{C} \left( \sum_{j=1}^{m} \|\Phi(Q_j)y\|^p \right)^{1/p}.
\] (2)

From Lemma 1.1 and the surjectivity of \( \Phi \), for any \( T_1, \ldots, T_m \) in \( L(Y) \) we have
\[
\left\| \sum_{j=1}^{m} T_j \Phi(Q_j) \right\| \leq C \left( \sum_{j=1}^{m} \|T_j\|^q \right)^{1/q},
\] (3)

where \( C \) depends only on \( p \) and on \( \|\Phi\| \cdot \|\Phi^*\|^{-1} \), and \( 1/p + 1/q = 1 \). Take any \( y \in Y \) and take \( \beta_j \geq 0 \) with
\[
\sum_{j=1}^{m} \beta_j = 1 \quad \text{and} \quad \sum_{j=1}^{m} \beta_j \|\Phi(Q_j)y\| = \left( \sum_{j=1}^{m} \|\Phi(Q_j)^* y\|^p \right)^{1/p}.
\]

Let \( y_0 \) be any unit vector in \( Y \) and let \( T_j \) be \( \Phi(Q_j) \) followed by a norm (at most) one projection onto the (at most) one dimensional space \( \mathbb{K} \Phi(Q_j)y \) followed by \( \Phi(Q_j)y \mapsto \beta_j \|\Phi(Q_j)y\|y_0 \). Then by (3),
\[
\left( \sum_{j=1}^{m} \|\Phi(Q_j)y\|^p \right)^{1/p} = \sum_{j=1}^{m} \beta_j \|\Phi(Q_j)y\| = \left\| \sum_{j=1}^{m} T_j \Phi(Q_j)y \right\| \leq C \left( \sum_{j=1}^{m} \|T_j\|^q \right)^{1/q} \|y\| = C\|y\|,
\]
which is (2).

Our first corollary of Theorem 1.4 is immediate. Its hypothesis are satisfied by many spaces that are used in analysis, including most Orlicz and Lorentz sequence spaces.

**Corollary 1.5** If \( X \) has a subsymmetric basis and has finite cotype, then \( X \) has SHAI.
The next corollary solves the problem that motivated our research into the SHAI property.

**Corollary 1.6** For $1 < p < \infty$, the space $L^p$ has the SHAI property.

**Proof:** In view of Theorem 1.4, it is enough to prove that the Haar basis for $L^p$ has property (♯). Let $\{N_\alpha : \alpha < c\}$ be a continuum of almost disjoint infinite subsets of the natural numbers $\mathbb{N}$. Define for $\alpha < c$

$$X_\alpha = \text{span}\{h_{n,i} : n \in N_\alpha \text{ and } 1 \leq i \leq 2^n\},$$

where $\{h_{n,i} : n = 0, 1, \ldots \text{ and } 1 \leq i \leq 2^n\}$ is the usual (unconditional) Haar basis for $L^p(0, 1)$, indexed in its usual way, so that $\{|h_{n,i}| : 1 \leq i \leq 2^n\}$ is the set of indicator functions of the dyadic subintervals of $(0, 1)$ that have length $2^{-n}$. By the Gamlen–Gaudet theorem [7], $X_\alpha$ is isomorphic to $L^p$ with the isomorphism constant depending only on $p$. \hfill \blacksquare

**Remark 1.7** Although our proof that $L^p$ has the SHAI property is simple enough, it is strange. The “natural” way of proving that a space $X$ has the SHAI property is to verify that for any non trivial closed ideal $\mathcal{I}$ in $L(X)$, the quotient algebra $L(X)/\mathcal{I}$ contains no minimal idempotents. (An idempotent $P$ is called minimal provided $P \neq 0$ and the only idempotents $Q$ for which $PQ = QP = Q$ are $P$ and 0. Rank one idempotents in $L(X)$ are minimal.) This suggests the following problem, which is related to the known problem whether every infinite dimensional complemented subspace of $L^p$ is isomorphic to its square.

**Problem 1.8** Is there a non trivial closed ideal $\mathcal{I}$ in $L(L^p)$ for which $L(L^p)/\mathcal{I}$ has a minimal idempotent?

If there is a positive answer to Problem 1.8, the witnessing ideal $\mathcal{I}$ cannot be contained in the ideal of strictly singular operators. This is because every infinite dimensional complemented subspace of $L^p$ contains a complemented subspace that is isomorphic either to $\ell^p$ or to $\ell^2$ [11], and the fact that idempotents in $L(X)/\mathcal{I}$ lift to idempotents in $L(X)$ when $\mathcal{I}$ is an ideal that is contained in $L(X)$ [4].

**Problem 1.9** Does $L^1$ have the SHAI property?

2 Examples and permanence properties

Here we present some more examples of spaces with property (♯) and with the SHAI property. We do not know whether every complemented subspace of $L^p$ has the SHAI property,
but we show that at least some of the known examples of such spaces do. Along the way we state and prove some permanence properties of (#).

The classical complemented subspaces of $L^p$ have the SHAI property when $1 < p < \infty$. This was known for $\ell^2$ and $\ell^p$ and proved above for $L^p$. The case of $\ell^p \oplus \ell^2$, the $\ell^p$ sum of $\ell^2$, has (#) and the SHAI property follows from Proposition 2.2 below. Before stating Proposition 2.2 we introduce a quantitative version of property (#).

**Definition 2.1** Suppose that $(E_n)_{n=1}^\infty$ is an unconditional FDD for a Banach space $X$ and $K$ is a positive constant. We say that $(E_n)_{n=1}^\infty$ has property (#) with constant $K$ provided there is an almost disjoint continuum $\{N_\alpha : \alpha < c\}$ of infinite sets of natural numbers such that for each $\alpha < c$, $X$ is $K$-isomorphic to the closed linear span of $\{E_n : n \in N_\alpha\}$.

Note that if $(E_n)_{n=1}^\infty$ has property (#) then it has property (#) for some positive constant $K$. Nevertheless, we need this quantitative notion for the full generality of Proposition 2.2.

Recall that if $(e_i)$ is an unconditional basis for some Banach space $Y$ and $X_i$, for $i = 1, 2, \ldots$, is a Banach space, $(\bigoplus_{i=1}^\infty X_i)_Y$ is the space of sequences $\bar{x} = (x_1, x_2, \ldots)$ whose norm, $\|\bar{x}\| = \left\| \sum_{i=1}^\infty \|x_i\| \cdot e_i \right\|_Y$, is finite. We denote the subspace of $(\bigoplus_{i=1}^\infty X_i)_Y$ of all sequences of the form $(0, \ldots, 0, x_i, 0, \ldots)$ by $X_i \otimes e_i$.  

**Proposition 2.2** For $i = 1, 2, \ldots$ let $(E_n^i)_{n=1}^\infty$ be an unconditional FDD for a Banach space $X_i$, all satisfying property (#) with a common $K$. Then for each subsymmetric basis $(e_i)$ of some Banach space $Y$, the unconditional FDD $(E_n^i \otimes e_i)_{n=1}^\infty$ of $(\bigoplus_{i=1}^\infty X_i)_Y$ satisfies (#). If, in addition, the decompositions $(E_n^i)_{n=1}^\infty$ have disjoint lower $p$ estimates with uniform constant and $(e_i)$ also has such an estimate, then $(\bigoplus_{i=1}^\infty X_i)_Y$ has the SHAI property.

**Proof:** For each $i$, let $\{N^i_\alpha : \alpha < c\}$ be an almost disjoint continuum of infinite sets of natural numbers such that for every $\alpha < c$, $X_\alpha$ is $K$-isomorphic to the closed linear span of the subspaces $E_n^i$ for $n \in N_\alpha$. Also, let $\{N_\alpha : \alpha < c\}$ be an almost disjoint continuum of infinite sets of natural numbers. Then

$$\{ (i, n) : i \in N_\alpha \text{ and } n \in N^i_\alpha \}$$

is a continuum of almost disjoint subsets of $\mathbb{N} \times \mathbb{N}$. It is easy to see that this continuum satisfies what is required of the unconditional FDD $(E_n^i \otimes e_i)_{i,n=1}^\infty$ to satisfy (#). If the decompositions $(E_n^i)_{n=1}^\infty$ have disjoint lower $p$ estimates with uniform constant and $(e_i)$ also has such an estimate, then the FDD $(E_n^i \otimes e_i)_{i,n=1}^\infty$ clearly has a disjoint lower $p$ estimate as well, so the SHAI property follows from Theorem 1.4. 

\[ \blacksquare \]
Remark 2.3 Note that the proof above works with only notational differences if we deal with only finitely many \( X_i \) (and here we do not need to assume the uniformity of the (\( \# \)) property). In particular, if each of \( X \) and \( Y \) has an unconditional FDD with (\( \# \)), then so does \( X \oplus Y \).

As we said above, this takes care of the space \( \ell^p(\ell^2) \). The first non classical complemented subspace of \( L^p \) is the space \( X_p \) of Rosenthal [15]. We recall its definition. Let \( p > 2 \) and let \( \bar{w} = (w_i)_{i=1}^\infty \) be a bounded sequence of positive real numbers. Let \( (e_i)_{i=1}^\infty \) and \( (f_i)_{i=1}^\infty \) be the unit vector bases of \( \ell^p \) and \( \ell^2 \). Let \( X_{p,\bar{w}} \) be the closed span of \( (e_i \oplus w_i f_i)_{i=1}^\infty \) in \( \ell^p \oplus \ell^2 \). If the \( w_i \) are bounded away from zero, then \( X_{p,\bar{w}} \) is isomorphic to \( \ell^2 \). If \( \sum_{i=1}^\infty w_i^{2p/(p-2)} < \infty \), then \( X_{p,\bar{w}} \) is isomorphic to \( \ell^p \). If one can split the sequence \( \bar{w} \) into two subsequences, one bounded away from zero and the other such that the sum of the \( 2p/(p-2) \) powers of its elements converges, then \( X_{p,\bar{w}} \) is isomorphic to \( \ell^p \oplus \ell^2 \). Rosenthal proved that in all other situations one gets a new space, isomorphically unique (i.e., any, two spaces corresponding to two choices of \( \bar{w} \) with this condition are isomorphic). Moreover, \( X_{p,\bar{w}} \) is isomorphic to a complemented subspace of \( L^p \). The constants involved (isomorphisms and complementations) are bounded by a constant depending only on \( p \). This common (class of) space(s) is denoted by \( X_p \). For \( 1 < p < 2 \), \( X_p \) is defined to be \( X_{p/(p-1)}^* \).

**Proposition 2.4** Let \( p \in (1,\infty) \setminus \{2\} \). Then \( X_p \) has (\( \# \)) and has the SHAI property.

**Proof:** Let \( p > 2 \). Write \( \mathbb{N} \) as a disjoint union of finite subsets \( \sigma_j \) for \( j = 1,2,\ldots \), with \( |\sigma_j| \to \infty \). For \( i \in \sigma_j \) put \( w_i = |\sigma_j|^{1/(2p)} \), so \( w_i \to 0 \) and for each \( j \), \( \sum_{i \in \sigma_j} w_i^{2p/(p-2)} = 1 \). Set \( E_j = \text{span} (e_i \oplus w_i f_i)_{i \in \sigma_j} \). It follows that for any infinite subsequence of the unconditional FDD \( (E_j) \), the closed span of this subsequence is isomorphic to \( X_p \). The FDD is unconditional and, as it lives in \( L^p \), has a lower \( p \) estimate. So the result in this case follows from Theorem 1.4. The case \( 1 < p < 2 \) follows by looking at the dual FDD.

Building on \( X_p \) and the classical complemented subspaces of \( L^p \), Rosenthal [15] lists a few more isomorphically distinct spaces that are isomorphic to complemented subspaces of \( L^p \) when \( p \in (1,\infty) \setminus \{2\} \). Using the discussion above one can easily show that they all have (\( \# \)) and the SHAI property. Here we just comment on one of them for which the full power of Proposition 2.2 is needed. This is the space denoted in [15] by \( B_p \). It is the \( \ell^p \) sum of spaces \( X_i \) each having a 1-symmetric basis, and thus having (\( \# \)) with uniform constant. Each \( X_i \) is isomorphic to \( \ell^2 \), but the isomorphism constant tends to infinity as \( i \to \infty \). By Proposition 2.2, \( B_p \) has (\( \# \)) and the SHAI property.

The first infinite collection of mutually non isomorphic complemented subspaces of \( L^p \) for \( p \in (1,\infty) \setminus \{2\} \) was constructed in [16]. We recall the simple construction. Given two
subspaces $X$ and $Y$ of $L^p(\Omega)$ with $1 \leq p \leq \infty$, $X \otimes_p Y$ denotes the subspace of $L^p(\Omega^2)$ that is the closed span of all functions of the form $h(s,t) = f(s)g(t)$ with $f \in X$ and $g \in Y$. It is easy to see (and was done in [16]) that the isomorphism class of $X \otimes_p Y$ depends only on the isomorphism classes of $X$ and $Y$ and that, if $X$ and $Y$ are complemented in $L^p(\Omega)$, then $X \otimes_p Y$ is complemented in $L^p(\Omega^2)$. More generally, if $X_1, X_2, Y_1, Y_2$ are subspaces of $L^p(\Omega)$ and $T_i \in L(X_i, Y_i)$, then $T_1 \otimes_p T_2 \in L(X_1 \otimes_p X_2, Y_1 \otimes_p Y_2)$. Note also that if $(E_n^i)_{n=1}^\infty$ is an unconditional FDD for $X_i$ for $i = 1, 2$, then $(E_n^1 \otimes_p E_m^2)_{n,m=1}^\infty$ is an unconditional FDD for $X_1 \otimes_p X_2$. This follows from iterating Khinchine’s inequality.

With a little abuse of notation we denote by $X_p$ some isomorph of $X_p$ that is complemented in $L^p[0,1]$. Set $Y_1 = X_p$, and for $n = 2, 3, \ldots$, let $Y_n = Y_{n-1} \otimes_p X_p$. From the above it is clear that the spaces $Y_n$ are complemented (alas, with norm of projection depending on $n$) in some $L^p$ space isometric to $L^p[0,1]$. The main point in [16] was to prove that the spaces $Y_n$ are isomorphically different. That all the spaces $Y_n$ have (#) follows now from the following general proposition, because it is clear that $\otimes_p$ satisfies Conditions (1) and (2) in Proposition 2.5 for the class of all $m$ tuples of subspaces of $L^p(\mu)$ spaces.

**Proposition 2.5** Assume that $X_1, \ldots, X_m$ are Banach spaces, each of which has an unconditional FDD satisfying (#). Let $Y_1 \otimes \cdots \otimes Y_m$ denote an $m$ fold tensor product endowed with norm defined on some class of $m$ tuples of Banach spaces with the following two properties:

1. If $T_i \in L(Y_i, Z_i)$ for $i = 1, \ldots, m$, then
   $$T_1 \otimes \cdots \otimes T_m : Y_1 \otimes \cdots \otimes Y_m \to Z_1 \otimes \cdots \otimes Z_m$$
   is bounded.

2. If $Y_i$ has an unconditional FDD $(F^i_n)_{n=1}^\infty$ for each $i$, then $(F^1_{n_1} \otimes \cdots \otimes F^m_{n_m})_{n_1,\ldots,n_m=1}^\infty$ is an unconditional FDD for the completion of $Y_1 \otimes \cdots \otimes Y_m$.

Then, if we assume in addition that $(X_1, \ldots, X_m)$ is in this class, the completion of $X_1 \otimes \cdots \otimes X_m$ has an unconditional FDD with (#).

**Proof:** For each $i = 1, \ldots, m$, let $(E_n^i)_{n=1}^\infty$ be an unconditional FDD for a Banach space $X_i$ such that there is an almost disjoint continuum $\{N^i_\alpha : \alpha < c\}$ of infinite sets of $\mathbb{N}$ such that for each $\alpha < c$, $X_i$ is isomorphic to the closed linear span of the spaces $E_n^i$ for $n \in N^i_\alpha$.

Consider the continuum
$$\{N^1_\alpha \times \cdots \times N^m_\alpha : \alpha < c\}$$

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of subsets of \( \mathbb{N}^m \). This is an almost disjoint family whose cardinality is the continuum. Property (2) of the tensor norms we consider guarantees that \( (E^1_{n_1} \otimes \cdots \otimes E^m_{n_m})_{n_1,\ldots,n_m=1}^\infty \) is an unconditional FDD for the completion of \( X_1 \otimes \cdots \otimes X_m \). Property (1) implies that for each \( \alpha < c \), the closed linear span of \( (E^1_{n_1} \otimes \cdots \otimes E^m_{n_m})_{n_1,\ldots,n_m} \in N_1^\alpha \times \cdots \times N_m^\alpha \) is isomorphic to the completion of \( X_1 \otimes \cdots \otimes X_m \).

**Remark 2.6** Note that in general Property (1) does not imply Property (2). The Schatten classes \( C_p \) for \( p \neq 2 \) are examples of tensor norms that satisfy (1) but not (2).

We note that it is clear from Proposition 2.5 that if \( X_1,\ldots,X_m \) are subspaces of \( L^p \) for \( 1 \leq p < 2 \) that have (sub)symmetric bases, then \( X_1 \otimes \cdots \otimes \otimes p X_m \) has (#) and the SHAI property. The class of subspaces of \( L^p \) for \( 1 \leq p < 2 \) that have a symmetric basis (i.e., the norm of a vector is invariant, up to a constant, under all permutations and changes of signs of its coefficients) is a rich family. (For \( p > 2 \), up to isomorphism it includes only \( \ell^1 \) and \( \ell^2 \).) Thus the class of tensor products above includes, for example, \( \ell^p_1 (\ell^p_2 (\cdots (\ell^p_m) \cdots)) \) whenever \( p \leq p_1 < p_2 < \cdots < p_m \leq 2 \).

**Problem 2.7** Suppose \( p \in (1, \infty) \setminus \{2\} \) and let \( X \) be a complemented subspace of \( L^p \). Does \( X \) have the SHAI property? What if, in addition, \( X \) has an unconditional basis? What if, in addition, \( X \) is one of the \( \aleph_1 \) spaces constructed in [1]?
norm one complemented in $T_p$ and, similarly, $T_p$ is isometric to a norm one complemented subspace of $T_p(M)$. The space $T_p$ is isomorphic to $\ell^p(T_p)$ [3, p. 85], so the decomposition method [2, Theorem 2.2.3] shows that $T_p$ is isomorphic to $T_p(M)$. Thus every almost disjoint family of infinite subsets of $\mathbb{N}$ witnesses that $T_p$ has property (♯). Now for $1 < p < \infty$, $T_p$ is complemented in $C_p$ via the projection that zeroes out the entries that lie above the diagonal [14], [8], from which it follows easily [3] that $T_p$ is isomorphic to $C_p$. We record these observations in Proposition 2.8.

**Proposition 2.8** For $1 \leq p \leq \infty$, the space $T_p$ has property (♯). Moreover, for $1 < p < \infty$, the space $C_p$ has property (♯).

As we mentioned above, it can be proved that $C_1$ and $C_\infty$ have the SHAI property even though neither has an unconditional FDD. However, the $C_p$ norms for $1 \leq p \leq \infty$ are what Kwapień and Pełczyński [12] call unconditional matrix norms; i.e., the norm $\left\| \sum_{i,j} a_{i,j} e_{i,j} \right\|$ of a linear combination $\sum_{i,j} a_{i,j} e_{i,j}$ of the natural basis elements $(e_{i,j})_{i,j=1}^\infty$, is equivalent (in our case even equal) to the norm of $\sum_{i,j} \varepsilon_i \delta_j a_{i,j} e_{i,j}$ for all sequences of signs $(\varepsilon_i)_{i=1}^\infty$ and $(\delta_j)_{j=1}^\infty$.

One can define a variation of property (♯) for bases with this unconditionality property, check that the natural bases for $C_p$, for $1 \leq p \leq \infty$, satisfy this property, and prove a version of Theorem 1.4. This shows that $C_1$ has the SHAI property (and gives an alternative proof also for $C_p$ for $1 < p < \infty$). This variation of Theorem 1.4 does not apply to $C_\infty$, which does not have finite cotype, and we do not know whether $C_\infty$ has the SHAI property. Since our focus in this paper is on spaces that are more closely related to $L^p$ than are the $C_p$ spaces, we do not go into more detail. Our main reason for bringing up $C_p$ is to point out why the definition of property (♯) is made for unconditional FDDs rather than just for unconditional bases.

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