Stochastic Dilation of Symmetric Completely Positive Semigroups

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Abstract

This is a continuation of the study of the theory of quantum stochastic dilation of completely positive semigroups on a von Neumann or $C^*$ algebra, here with unbounded generators. The additional assumption of symmetry with respect to a semifinite trace allows the use of the Hilbert space techniques, while the covariance gives rise to better handle on domains. An Evans-Hudson flow is obtained, dilating the given semigroup.

Keywords: completely positive semigroups, quantum stochastic dilation.

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1 Introduction

In an earlier series of papers ([14], [15]), we had constructed a theory of stochastic dilation “naturally” associated with a given completely positive (CP) semigroup (heat semigroup) on a von Neumann or $C^*$ algebra with bounded generator. There the computations involved $C^*$ or von Neumann Hilbert modules, using the results of [7], map-valued quantum stochastic processes on modules and stochastic integration w.r.t. them ([17], [20]). It is then natural to consider the case of a CP semigroup with unbounded generator and ask the same questions about the associated stochastic dilations. As one would expect, the problem is too intractable in this generality and we impose some further structures on it, viz. we assume that the semigroup is symmetric w.r.t. a semifinite trace and covariant under the action of a Lie group on the algebra. This additional hypothesis enables us to control the domains of the various operator coefficients appearing in the quantum stochastic differential equations so that the Mohari-Sinha conditions ([18]) can be applied. The covariance is exploited again as in [1] along with the
assumption that the crossed-product von Neumann algebra $\mathcal{A} \rtimes G$ is isomorphic with the von Neumann algebra generated by $\mathcal{A}$ and the representation $u_g$ of $G$ in the GNS Hilbert space associated with the trace, to obtain the structure maps, and finally the Evans-Hudson (E-H) flow is constructed essentially along the lines of the proofs in [14]. As precursors of this work, we may mention those in [12] and [2]. While the first one deals with a general E-H flow with unbounded structure maps under some additional hypotheses, the second one treats the problem in a different spirit.

2 Preliminaries

Let $\mathcal{A}$ be a separable $C^\ast$-algebra and $\tau$ be a densely defined, semifinite, lower semicontinuous and faithful trace on $\mathcal{A}$. Let $\mathcal{A}_\tau \equiv \{ x : \tau(x^*x) < \infty \}$. Let $h = L^2(\tau)$, and $\mathcal{A}$ is naturally imbedded in $\mathcal{B}(h)$. We denote by $\hat{\mathcal{A}}$ the von Neumann closure of $\mathcal{A}$ with respect to the weak topology inherited from $\mathcal{B}(h)$. Clearly $\mathcal{A}_\tau$ is ultraweakly dense in $\hat{\mathcal{A}}$. Assume furthermore that $G$ is a second countable Lie group with $(\chi_i, i = 1, \ldots, N)$ a basis of its Lie algebra, $g \mapsto \alpha_g \in \text{Aut}(\mathcal{A})$ a strongly continuous representation. Suppose that $\alpha_g(\mathcal{A}_\tau) \subseteq \mathcal{A}_\tau$ and $\tau(\alpha_g(x^*y)) = \tau(x^*y)$ for $x \in \mathcal{A}_\tau, y \in \mathcal{A}, g \in G$ (by polarization this is equivalent to the assumption that $\tau(\alpha_g(x^*x)) = \tau(x^*x)$ for $x \in \mathcal{A}_\tau$). This allows one to extend $\alpha_g$ as a unitary linear operator (to be denoted by $u_g$) on $h$ and clearly $\alpha_g(x) = u_g xu_g^*$ for $x \in \mathcal{A}$. It is indeed easy to verify this relation on vectors in $\mathcal{A}_\tau$ and then it extends to the whole of $h$ by the fact that $h$ is the completion of $\mathcal{A}_\tau$. For $f \in C_c^\infty(G)$ (i.e. $f$ is smooth complex-valued function with compact support on $G$) and an element $x \in \mathcal{A}$, let us denote by $\alpha(f)(x)$ the norm-convergent integral $\int_G f(g)\alpha_g(x)dg$, where $dg$ denotes the left Haar measure on $G$.

**Lemma 2.1** $g \mapsto u_g$ is strongly continuous w.r.t the Hilbert-space topology of $h$.

**Proof :-**

Let $\mathcal{A}_1 \equiv \{ x \in \mathcal{A} | \tau(|x|) < \infty \}$. It is known that $\mathcal{A}_1$ is dense in $h$ in the topology of $h$. Furthermore, for $x \in \mathcal{A}_\tau$ and $y \in \mathcal{A}_1$, $|\tau((u_g(x) - x)^*y)| \leq \| (u_g(x) - x)^* \| \tau(|y|)$, which proves that $g \mapsto \tau((\alpha_g(x) - x)^*y)$ is continuous, by the strong continuity of $\alpha$ w.r.t. the norm topology of $\mathcal{A}$. But by the density of $\mathcal{A}_1$ and $\mathcal{A}_\tau$ in $h$ and the fact that $u_g$ is unitary, we conclude that
for fixed $\xi \in h$, $g \mapsto u_g \xi$ is continuous w.r.t. the weak topology of $h$, and hence is strongly continuous.

The above lemma allows us to define $\alpha(f)(\xi) = \int f(g)u_g(\xi)dg \in h$ for $f \in C^\infty_c(G), \xi \in h$. Furthermore, from the expression $\alpha g(x) = u_g xu^*_g$, it is possible to extend $\alpha_g$ to the whole of $B(h)$ as a normal automorphism group implemented by the unitary group $u_g$ on $h$ and we shall denote this extended automorphism group too by the same notation. Let $A_\infty \equiv \{ x \in A : g \mapsto \alpha_g(x) \}$ is infinitely differentiable w.r.t. the norm topology $\| \cdot \|$, i.e. $A_\infty$ is the intersection of the domains of $\partial_i \partial_{i_2} \ldots \partial_{i_k}; k \geq 1$, for all possible $i_1, i_2, \ldots \in \{ 1, 2, \ldots \}$, where $\partial_i$ denotes the closed $\ast$-derivation on $A$ given by the generator of the one-parameter automorphism group $\alpha_{exp(\xi)}$, where $exp$ denotes the usual exponential map for the Lie group $G$. The following result is essentially a consequence of the results obtained in [13], [19].

Proposition 2.2 (i) $A_\infty$ is dense $\ast$-subalgebra of $A$.

(ii) Similarly, we denote by $d_k$ the self-adjoint generator of the unitary group $u_{exp(\xi)}$ on $h$ such that $u_{exp(\xi)} = e^{itd_k}$, and let $h_\infty \equiv \cap_{i_1, i_2, \ldots} Dom(d_{i_1}d_{i_2} \ldots d_{i_k}; k = 1, 2, \ldots)$. Then $h_\infty$ is dense in $h$.

(iii) If we equip $A_\infty$ with a family of norms $\| \cdot \|_{\infty, n}; n = 0, 1, 2, \ldots$ given by:

$$\| x \|_{\infty, n} = \sum_{i_1, i_2, \ldots, i_k; k \leq n} \| \partial_1 \partial_{i_2} \ldots \partial_{i_k}(x) \|$$

for $n \geq 1$, and $\| x \|_{\infty, 0} = \| x \|$, and similarly define a family of Hilbertian norms $\| \cdot \|_{2, n}; n = 0, 1, 2, \ldots$ on $h_\infty$ by:

$$\| \xi \|_{2, n}^2 = \sum_{i_1, i_2, \ldots, i_k; k \leq n} \| d_{i_1}d_{i_2} \ldots d_{i_k}(\xi) \|^2$$

on $h_\infty$, then $A_\infty$ and $h_\infty$ are complete with respect to the locally convex topologies induced by the respective (countable) family of norms as defined above. In other words, $A_\infty$ and $h_\infty$ are Frechet spaces in the topologies (to be called “Frechet topologies” from now on) described above.

(iv) $\alpha g(A_\infty) \subseteq A_\infty$, $u_g(h_\infty) \subseteq h_\infty$ for all $g \in G$. Furthermore, $g \mapsto \alpha_g(x), g \mapsto u_g(\xi)$ are smooth ($C^\infty$) in the respective Frechet topologies for $x \in A_\infty, \xi \in h_\infty$.

(v) Let $A_{\infty, r} = A_\infty \cap h_\infty$. It is a $\ast$-closed two-sided ideal in $A_\infty$ and is dense in $A, A_\infty, h$ and $h_\infty$ w.r.t. the relevant topologies.

Proof:
The proof of (i) and (ii) will follow immediately from the references cited
before the statement of this proposition. The proof of (iii) is quite standard, which uses the fact that \( \partial_1, d_1 \)'s are closed maps in \( \mathcal{A} \) and \( h \) respectively.

Next we indicate briefly the proof of (iv) for \( \mathcal{A}_\infty \) only, since it is similar for \( h_\infty \). First of all, by the definition of \( \mathcal{A}_\infty \) and the fact that \( G \times G \ni (g_1, g_2) \mapsto g_1 g_2 \in G \) is \( C^\infty \) map, we observe that for \( x \in \mathcal{A}_\infty \), the map \( (g_1, g) \mapsto \alpha_{g_1}(\alpha_g(x)) = \alpha_{g_1 g}(x) \) is \( C^\infty \) on \( G \times G \), hence in particular for fixed \( g, G \ni g_1 \mapsto \alpha_{g_1}(\alpha_g(x)) \) is \( C^\infty \), i.e. \( \alpha_g(x) \in \mathcal{A}_\infty \). Similarly, for fixed \( x \in \mathcal{A}_\infty \) and any positive integer \( k \), the map \( F : R^k \times G \to \mathcal{A} \) given by \( F(t_1, ... t_k, g) = \alpha_{\exp(t_1 \chi_{i_1}) \cdots \exp(t_k \chi_{i_k}) g}(x) \) is \( C^\infty \). By differentiating \( F \) in its first \( k \) components at 0, we get that \( \partial_{i_1} \cdots \partial_{i_k}(\alpha_g(x)) \) is \( C^\infty \) in \( g \).

To prove (v), we need to note first that the elements of the form \( \alpha(f)(\xi) \), with \( f \in C^\infty_c(G) \) and \( \xi \in \mathcal{A}_\tau \) are clearly in \( \mathcal{A}_\infty, \tau \). Let us first consider the density in \( h \) and \( h_\infty \). Since the topology of \( h_\infty \) is stronger than that of \( h \) and since \( h_\infty \) is dense in \( h \) in the topology of \( h \), it suffices to prove that the set of elements of the above form is dense in \( h_\infty \) in the Frechet topology. For this, we take \( \xi \in h_\infty \), and choose a net \( x_\nu \) of elements from \( \mathcal{A}_\tau \) which converges in the topology of the Hilbert space \( h \) to \( \xi \), and then it is clear that \( \alpha(f)(x_\nu) \to \alpha(f)(\xi) \) \( \forall f \in C^\infty_c(G) \) w.r.t. the Frechet topology of \( h_\infty \), since \( d_{i_1} \cdots d_{i_k} \alpha(f)(x_\nu - \xi) = (-1)^k \alpha(X_{i_1} \cdots X_{i_k} f)(x_\nu - \xi) \). Thus, it is enough to show that \( \{ \alpha(f)(\xi), f \in C^\infty_c(G), \xi \in h_\infty \} \) is dense in \( h_\infty \) in the Frechet topology. For this, we choose a net \( f_\nu \in C^\infty_c(G) \) such that \( \int_G f_\nu dg = 1 \forall \nu \) and the support of \( f_\nu \) converges to the singleton set containing the identity element of the group \( G \), and then it is simple to see that \( \alpha(f_\nu)(\xi) \to \xi \) in the Frechet topology. Finally, the norm-density of \( \mathcal{A}_\infty, \tau \) in \( \mathcal{A} \) and the Frechet density in \( \mathcal{A}_\infty \) will follow by similar arguments.

\[ \square \]

**Remark 2.3** It may be noted that for \( x \in \mathcal{A}_\infty, \tau \), \( \delta_{i_1} \cdots \delta_{i_k}(x) = d_{i_1} \cdots d_{i_k}(x) \in \mathcal{A} \cap h \). This follows from the fact that if \( y_\nu \) is a net in \( \mathcal{A} \cap h \) which converges both in the norm topology of \( \mathcal{A} \) as well as in the Hilbert space topology of \( h \), then the norm-limit belongs to \( h \) and the two limits must coincide as vectors of \( h \).

Now we shall introduce some more useful notation and terminology and prove some preparatory results. If \( \mathcal{H} \) is any Hilbert space with a strongly continuous unitary representation of \( G \) given by \( U_g \), we denote by \( \mathcal{H}_\infty \) the intersection of the domains of the self-adjoint generators of different one-parameter subgroups, just as we did in case of \( h \). We denote the correspond-
ing family of “Sobolev-like” norms again by the same notation as in case of $h$ and consider $\mathcal{H}_\infty$ as a Frechet space as earlier. We call such a pair $(\mathcal{H}, U_g)$ a Sobolev-Hilbert space and for two such pairs $(\mathcal{H}, U_g)$ and $(\mathcal{K}, V_g)$, we denote by $\mathcal{B}(\mathcal{H}_\infty, \mathcal{K}_\infty)$ the space of all linear maps $S$ from $\mathcal{H}$ to $\mathcal{K}$ such that $S(\mathcal{H}_\infty) \subseteq \mathcal{K}_\infty$, and $S$ is continuous with respect to the Frechet topologies of the respective spaces. We call a linear map $L$ from $\mathcal{H}$ to $\mathcal{K}$ to be covariant if $\mathcal{H}_\infty \subseteq \text{Dom}(L)$ and $LU_g(\xi) = V_g L(\xi) \forall g \in G, \xi \in \mathcal{H}_\infty$.

**Lemma 2.4** If $L$ from $\mathcal{H}$ to $\mathcal{K}$ is bounded (in the usual Hilbert space sense) and covariant in the above sense, then $L \in \mathcal{B}(\mathcal{H}_\infty, \mathcal{K}_\infty)$.

**Proof:**

Let $d_i^H$ and $d_i^K$ be respectively the self-adjoint generator of the one parameter subgroup corresponding to $\chi_i$ in $\mathcal{H}$ and $\mathcal{K}$. From the relation $LU_g = V_g L$ it follows that (since $L$ is bounded) $L$ maps the domain of $d_i^H$ into the domain of $d_i^K$ and $Ld_i^H = d_i^K L$. By repeated application of this argument it follows that $Ld_{i_1}^H \ldots d_{i_k}^H(\xi) = d_{i_1}^K \ldots d_{i_k}^K L(\xi) \forall \xi \in \mathcal{H}_\infty$, and thus $\|L\xi\|_{2,n} \leq \|L\| \|\xi\|_{2,n}$. \(\square\)

We shall call an element of $\mathcal{B}(\mathcal{H}_\infty, \mathcal{K}_\infty)$ a “smooth” map, and if such a smooth map $L$ satisfies an estimate $\|L\xi\|_{2,n} \leq C\|\xi\|_{2,n+p}$ for all $n$ and for some integer $p$ and a constant $C$, then we say that $L$ is a smooth map of order $p$ with the bound $\leq C$. From the proof of the above lemma we observe that any bounded covariant map is smooth of order 0 with the bound $\leq \|L\|$. By a similar reasoning we can prove the following:

**Lemma 2.5** Suppose that $L$ is a closed (in the Hilbert space sense), covariant map from $\mathcal{H}$ to $\mathcal{K}$ and $\mathcal{H}_\infty$ is in the domain of $L$. Under these assumptions, $L$ is smooth of the order $p$ for some $p$.

**Proof:**

For simplicity of notation, we shall use the same symbol $d_i$ for both $d_i^H$ and $d_i^K$, and also we use the same symbols for the corresponding one parameter groups of unitaries acting on $\mathcal{H}$ and $\mathcal{K}$. Let $L$ be a map as above. Since $L$ is closed in the Hilbert space sense, and the Frechet topology in $\mathcal{H}_\infty$ is stronger than its Hilbert space topology, it follows that $L$ is closed as a map from the Frechet space $\mathcal{H}_\infty$ to the Hilbert space $\mathcal{K}$, and being defined on the entire $\mathcal{H}_\infty$, it is continuous w.r.t the above topologies. By the definition of Frechet space continuity, there exists some $C$ and $p$ such that $\|L(\xi)\|_{2,0} \leq C\|\xi\|_{2,p}$. Now, for any fixed $k$, let $u_t \equiv u_{\exp(t\chi_k)}$. Since $u_t$ maps $h_\infty$ into itself and $L$ is covariant,
we have that \( L(\frac{u(t) - \xi}{t}) = \frac{u(t\xi) - L\xi}{t} \). Now, since \( \frac{u(t\xi) - L\xi}{t} \to d_k(\xi) \) as \( t \to 0^+ \) in the Frechet topology, we have that \( L(\frac{u(t\xi) - L\xi}{t}) = \frac{u(t\xi) - L\xi}{t} \) converges to \( Ld_k\xi \) in the Hilbert space topology of \( \mathcal{K} \), and so by the closedness of \( d_k L\xi \) must belong to the domain of \( d_k \), with \( Ld_k\xi = d_k L\xi \). Repeated use of this argument proves that \( L(\mathcal{H}_\infty) \subseteq \mathcal{K}_\infty \) and \( L(d_{i_1} \ldots d_{i_k}\xi) = d_{i_1} \ldots d_{i_k}(L\xi) \forall \xi \in \mathcal{H}_\infty \). Now, a direct computation enables one to show that \( L \) is of order \( p \) with the bound \( \leq C \).

\[ \square \]

**Theorem 2.6** Let \( (\mathcal{H}, U_g), (\mathcal{K}, V_g) \) be two Sobolev-Hilbert spaces as in earlier discussion, and \( L \) be a closed (not as Frechet space map but as Hilbert space map) linear map from \( \mathcal{H} \) to \( \mathcal{K} \). Furthermore, assume that \( \mathcal{H}_\infty \) is in the domain of \( |L|^2 \) and is a core for \( |L|^2 \), and \( LU_g = V_g L \) on \( \mathcal{H}_\infty \). Then we have the following conclusions:

(i) \( L \) is a smooth covariant map with some order \( p \) and bound \( \leq C \) for some \( C \);

(ii) \( L^* \) (the densely defined adjoint in the Hilbert space sense ) will have \( \mathcal{K}_\infty \) in its domain;

(iii) \( L^* \) is also a smooth covariant map from \( \mathcal{K}_\infty \) to \( \mathcal{H}_\infty \); with order \( p \) and bound \( \leq C \) as in (i).

**Proof:**

Let the polar decomposition of \( L \) be given by \( L = W|L| \). We claim that both \( W \) and \( |L| \) are covariant maps. First we note that \( \mathcal{H}_\infty \) is also a core for \( L \) (being a core for \( |L|^2 \)) and since \( U_g \) is a unitary operator that maps \( \mathcal{H}_\infty \) into itself, clearly \( \mathcal{H}_\infty \) is a core for \( LU_g \) and also for \( V_g L \). Thus the relation \( LU_g = V_g L \) on \( \mathcal{H}_\infty \) implies that the operators \( LU_g \) and \( V_g L \) have the same domain and they are equal. Now, note that \( L \) being closed and \( V_g \) being bounded, we have that \( (V_g L)^* = L^* V_g^* = L^* V_g^{-1} \). Furthermore, since \( U_g^{-1} \) maps the core \( \mathcal{H}_\infty \) for \( L \) into itself, one can easily verify that \( (LU_g)^* = U_g^* L^* \). Thus, we get that \( U_g L^* = L^* V_g \forall g \). It then follows that \( U_g |L|^2 = |L|^2 U_g \) and hence by spectral theorem \( U_g \) and \( |L| \) will commute. By Lemma 2.3, we get that \( |L|(\mathcal{H}_\infty) \subseteq \mathcal{H}_\infty \), and \( |L| \) is a smooth covariant map of some order.

Now, if \( P \) denotes the projection onto the closure of the range of \( |L| \), then \( P \) clearly commutes with \( U_g \) for all \( g \), hence in particular \( U_g \text{Ran}(P)^\perp \subseteq \text{Ran}(P)^\perp \). Thus \( WU_g P^\perp = WP^\perp U_g = 0 = V_g WP^\perp \). On the other hand,
\[V_g WP = WU_g P, \text{ because } V_g W|L| = V_g L = LU_g = W|L|U_g = WU_g|L|.\]
Hence we have that \( W \) is a bounded covariant map, and thus by [2,4], it follows that \( W^* \) is covariant too, and in particular \( W^*(K_\infty) \subseteq H_\infty \subseteq Dom(|L|) \), so that \( K_\infty \subseteq Dom(L^*) = Dom(|L|W^*). \) Furthermore, from the fact that \( W \) and \( W^* \) are smooth maps of order 0 with bound \( \leq 1 \) (as \( \|W\| = \|W^*\| = 1 \)) and \( |L| \) is a smooth covariant map of some order \( p \) with bound \( \leq C \) for some \( C \), clearly both \( L = W|L| \) and \( L^* = |L|W^* \) are smooth covariant maps of order \( p \) and bound \( \leq C \), which completes the proof. \( \square \)

**Lemma 2.7** Let \((\mathcal{H}_i, U^i_g), i = 1, 2\) and \((\mathcal{K}_i, V^i_g), i = 1, 2\) be Sobolev Hilbert spaces and \( k \) be any Hilbert space. Then we can construct Sobolev Hilbert spaces \((\mathcal{H}_i \oplus \mathcal{K}_i, U^i_g \oplus V^i_g)\) and \((\mathcal{H}_i \otimes_k U^i_g \otimes I, \mathcal{K}_i \otimes_k V^i_g \otimes I)\) (with the symbols carrying their usual meanings) and if \( L \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2), M \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2) \), then we have the following:

(i) \( L \oplus M \) is a smooth map between appropriate spaces, and

(ii) \((\mathcal{H}_1 \otimes_k, \mathcal{K}_1 \otimes_k)_{\infty}\) is the completion of \( \mathcal{H}_1 \otimes_{alg} k \) under the respective Frechet topology and the map \( L \otimes_{alg} I \) on \( \mathcal{H}_1 \otimes_{alg} k \) extends as a smooth map on the respective Frechet space (we shall denote this smooth map by \( L \otimes I \) or sometimes \( L \)). Furthermore, if \( L \) is of order \( p \) with some constant \( C \), so will be \( L \).

**Proof:**

(i) is straightforward. To prove (ii), we fix any orthonormal basis \( \{e_l\} \) of \( k \) and let \( \xi = \sum \xi_l \otimes e_l \) be a vector in the domain of the self adjoint generator of the one parameter unitary group \( u_t \otimes I \), where \( u_t \) is as in the proof of Lemma 2.8 and the summation is over a countable set since \( \xi_l = 0 \) for all but countably many values of \( l \). So, without loss of generality we may assume that the set of \( l \)'s with \( \xi_l \) nonzero is indexed by \( 1, 2, ..., \) Since \( \sum \frac{u(t)\xi_l - \xi_l}{t} \otimes e_l \) is Cauchy (in the Hilbert space topology) suppose that \( \sum \frac{u(t)\xi_l - \xi_l}{t} \otimes e_l \rightarrow \sum \eta_l \otimes e_l. \) Clearly, for each \( l, \eta_l = \lim_{t \to 0} \left( \frac{u(t)\xi_l - \xi_l}{t} \right) \), which implies that \( \xi_l \in Dom(d_k) \) and \( d_k \xi_l = \eta_l \). Thus, if \( d_k \) denotes the self adjoint generator of the one parameter unitary group \( u_t \otimes I \), then we have proved that the domain of it consists of precisely the vectors \( \sum \xi_l \otimes e_l \) such that each \( \xi_l \in Dom(d_k) \) and \( \sum \|d_k(\xi_l)\|^2 < \infty \). Repeated use of this argument enables us to prove that \((\mathcal{H}_1 \otimes_k)_{\infty}\) consists of the vectors \( \xi = \sum \xi_l \otimes e_l \) with the property that \( \xi_l \in \mathcal{H}_1 \otimes_k = \forall l \) and for any \( n, \|\xi\|_n^2 = \sum \|\xi_l\|_n^2 < \infty \). From this, it is clear that \( \sum_{l=1}^m \xi_l \otimes e_l \) converges (as \( m \to \infty \)) to \( \xi \) in each of the \( \|\cdot\|_n \) norms, i.e.
in the Frechet topology. The rest of the proof follows by observing that for any $$\xi = \sum_{\text{finite}} \xi_l \otimes e_l \in \mathcal{H}_{1,\infty} \otimes \text{alg} k$$, $$\| \tilde{L}(\xi) \|_{2,n}^2 = \sum \| L\xi_l \|_{2,n}^2$$.

\begin{proof}
\end{proof}

3 Assumptions on the semigroup and its generator

Let $$T_t$$ be a q.d.s. on $$\mathcal{A}$$ which is $$\tau$$-symmetric (i.e. $$\tau(T_t(x)y) = \tau(xT_t(y))$$ for all positive $$x, y \in \mathcal{A}$$, and for all $$t \geq 0$$). We refer the reader to [8] for a detailed account of such semigroups from the point of view of Dirichlet forms. We shall need some of the results obtained in that reference. As it is mentioned in that reference, $$T_t$$ can be canonically extended to a normal $$\tau$$-symmetric q.d.s. on $$\bar{\mathcal{A}}$$ as well as to $$C_0$$-semigroup of positive contractions on the Hilbert space $$\mathcal{H}$$. We shall denote all these semigroups by the same symbol $$T_t$$ as long as no confusion can arise. Furthermore, we assume that $$T_t$$ on $$\bar{\mathcal{A}}$$ is conservative, i.e. $$T_t(1) = 1$$ for all $$t \geq 0$$.

Let us denote by $$\mathcal{L}$$ the $$C^*$$ generator of $$T_t$$ on $$\mathcal{A}$$, and by $$\mathcal{L}_2$$ the generator of $$T_t$$ on $$\mathcal{H}$$. Clearly, $$\mathcal{L}_2$$ is a negative self-adjoint map on $$\mathcal{H}$$. We also recall (from [8]) that there is a canonical Dirichlet form $$\eta$$ on $$\mathcal{H}$$ given by, $$\text{Dom}(\eta) = \text{Dom}((-\mathcal{L}_2)^\sharp)$$, $$\eta(a) = \|(-\mathcal{L}_2)^\sharp(a)\|_{2,0}^2, a \in \text{Dom}(\eta)$$. We recall from [8] that $$\mathcal{B} := \mathcal{A} \cap \text{Dom}(\eta)$$ is a $$\ast$$-algebra, called the Dirichlet algebra, which is norm-dense in $$\mathcal{A}$$.

We now make the following important assumptions:

**Assumptions:**

(A1) $$T_t$$ is covariant, i.e. $$T_t$$ commutes with $$\alpha_g$$ for all $$t \geq 0, g \in G$$.

(A2) $$\mathcal{L}$$ has $$\mathcal{A}_\infty$$ in its domain,

(A3) $$\mathcal{L}_2$$ has $$\mathcal{H}_\infty$$ in its domain.

The assumption that $$T_t$$ is covariant in particular implies that $$T_t$$ leaves $$\mathcal{A}_\infty$$ invariant, hence by Nelson’s theorem this domain is a core for $$\mathcal{L}$$, and clearly $$\alpha_g \mathcal{L} = \mathcal{L} \alpha_g$$ on $$\mathcal{A}_\infty$$. It follows that (by arguments similar to those in the proof of [2]), $$\mathcal{L}(\mathcal{A}_\infty) \subseteq \mathcal{A}_\infty$$. Similarly, $$\mathcal{L}_2(\mathcal{H}_\infty) \subseteq \mathcal{H}_\infty$$. Since the actions of $$\mathcal{L}$$ and $$\mathcal{L}_2$$ coincide on $$\mathcal{A}_{\infty,\tau}$$, one has that $$\mathcal{L}(\mathcal{A}_{\infty,\tau}) \subseteq \mathcal{A}_{\infty,\tau}$$. Furthermore, we have the following:
Lemma 3.1 $A_{\infty, \tau}$ is stable under the action of $T_t$ and hence is a core for both $L_2$ and $L$.

The proof of the lemma is straightforward and hence omitted.

By the Lemma 3.1, $h_\infty$ is also a core for $L_2$, as $A_{\infty, \tau} \subseteq h_\infty$. It is important to remark here that the assumption A3 is the only hypothesis on the generator of the semigroup which involves the generator at the $L^2$-level, not the norm generator. However, we shall later on see that in an important special case, where the group $G$ is compact and acts ergodically on the algebra $A$, the assumption A3 will follow automatically from the other hypotheses.

Modifying slightly the arguments of [8] and [22], we describe the structure of $L$.

Theorem 3.2 (i) There is a Hilbert space $K$ equipped with an $A$-$A$ bimodule structure. We denote the right action by $(a, \xi) \mapsto \xi a, \xi \in K, a \in A$ and the left action by $(a, \xi) \mapsto \pi(a)\xi, \xi \in K, a \in A$.

(ii) There is a densely defined closable linear map $\delta_0$ from $A$ into $K$ such that $A_{\infty, \tau} \subseteq B = \text{Dom}(\delta_0)$ (where $B$ is the Dirichlet algebra mentioned earlier), and $\delta_0$ is a bimodule derivation, i.e. $\delta_0(ab) = \delta_0(a)b + \pi(a)\delta_0(b)v a, b \in B$.

(iii) For $a, b \in A_{\infty, \tau}, \|\delta_0(a)b\|_K \leq C_{a}\|b\|_{2,0}$, where $\|\|_K$ denotes the Hilbert space norm of $K$, and $C_{a}$ is a constant depending only on $a$. Thus, for any fixed $a \in A_{\infty, \tau}$, the map $A_{\infty, \tau} \ni b \mapsto \sqrt{2}\delta_0(a)b \in K$ extends to a unique bounded linear map between the Hilbert spaces $h$ and $K$, and this bounded map will be denoted by $\delta(a)$.

(iv) $\partial L(a, b, c) \equiv \delta(a)^*\pi(b)\delta(c) = L(a^*bc) - L(a^*b)c - a^*L(bc) + a^*L(b)c$, for $a, b, c \in A_{\infty, \tau}$.

(v) $K$ is the closed linear span of the vectors of the form $\delta(a)b, a, b \in A_{\infty, \tau}$.

(vi) $\pi$ extends as a normal $*$-homomorphism on $A$.

Proof :-

We refer for the proof of (i) and (ii) to [22] and [8]. Now, we note that $A_{\infty, \tau}$ is contained in the “Dirichlet algebra” (c.f. [8]) and in fact is a form-core for the Dirichlet form $\eta$ mentioned earlier. Using the calculations made in the proof of Lemma 3.3 of [8, page 8], we see that for $a, b \in A_{\infty, \tau}$,

$$\|\delta_0(a)b\|_K^2 = \frac{1}{2}\tau(-b^*L(a)^*ab - b^*a^*L(a)b + b^*L(a^*a)b).$$
Here, we have also used the fact that \( a, a^*, a^*a \in \text{Dom}(\mathcal{L}) \). From the above expression (iii) immediately follows. We verify (iv) by direct and straightforward calculations, which we omit. To prove (v), we first recall from \([8]\) that \( \mathcal{K} \) can be taken to be the closed linear span of the vectors of the form \( \delta_0(a)b, a, b \in \mathcal{B} \). Now, by Lemma 3.3 of \([8]\), \( \|\delta_0(a)b\|_\mathcal{K}^2 \leq \|b\|_{\mathcal{K},0}^2 \eta(a, a) \). Since \( \mathcal{A}_{\infty, \tau} \) is on one hand norm-dense in \( \mathcal{A} \) and also form core for \( \eta \) on the other hand, (v) follows.

Let us now prove (vi). It is enough to prove that whenever we have a Cauchy net \( a_\mu \in \mathcal{A}_{\infty, \tau} \) in the weak topology, then \( \langle \xi, \pi(a_\mu)\xi \rangle \) is also Cauchy for any fixed \( \xi \) belonging to the dense subspace of \( \mathcal{K} \) spanned by the vectors of the form \( \delta(b)c, b, c \in \mathcal{A}_{\infty, \tau} \). But it is clear that for this, it suffices to show that \( a \mapsto \langle \delta(b)b', \pi(a)\delta(b)b' \rangle \) is weakly continuous. Now, by the symmetry of \( \mathcal{L} \) and the trace property of \( \tau \), we have that for \( a \in \mathcal{A}_{\infty, \tau} \),

\[
\langle \delta(b)b', \pi(a)\delta(b)b' \rangle = \langle b, ab\mathcal{L}(b'b'^*) \rangle = \langle b, a\mathcal{L}(bb'b'^*) \rangle = \langle \mathcal{L}(bb'b'^*), ab \rangle + \langle \mathcal{L}(bb'(bb')^*), a \rangle.
\]

The first three terms in the right hand side are clearly weakly continuous in \( a \), so we have to concentrate only on the last term, which is of the form \( \tau(\mathcal{L}(xx^*)a) \) for \( x \in \mathcal{A}_{\infty, \tau} \). Now, we have,

\[
\tau(\mathcal{L}(xx^*)a) = \tau(\mathcal{L}(x)x^*a) + \tau(x\mathcal{L}(x^*)a) + \tau(\delta(x^*)^*\delta(x^*)a),
\]

and since \( \mathcal{L}(\mathcal{A}_{\infty, \tau}) \subseteq \mathcal{A}_{\infty, \tau} \), the first two terms in the right hand side of the above expression are weakly continuous in \( a \), so we are left with the term \( \tau(\delta(x^*)^*\delta(x^*)a) \). Let us choose an approximate identity \( e_n \) of the \( C^* \)-algebra \( \mathcal{A} \) such that each \( e_n \) belongs to \( \mathcal{A}_r \) (this is clearly possible, since \( \mathcal{A}_r \) is a norm-dense *-ideal, and for \( z \in \mathcal{A}_r \), one has that \( |z| \in \mathcal{A}_r \)). By normality of \( \tau \), \( \tau(\delta(x^*)^*\delta(x^*)) = \sup_n \tau(e_n\delta(x^*)^*\delta(x^*)e_n) = 2\sup_n \|\delta_0(x^*)e_n\|_\mathcal{K}^2 \leq 2\|e_n\|_{\mathcal{K},0}^2 \eta(x^*, x^*) < \infty \), since \( \|e_n\|_{\mathcal{K},0} \leq 1 \) and \( x^* \in \mathcal{A}_{\infty, \tau} \subseteq \text{Dom}(\eta) \). Thus, \( \delta(x^*)^*\delta(x^*) = y^2 \) for some \( y \in \mathcal{A}_r \), hence \( \tau(\delta(x^*)^*\delta(x^*)a) = \tau(yay) \) with \( y \in \mathcal{A}_r \), which proves the required weak continuity.

\[ \square \]

Now we obtain the Christensen-Evans type form of the generator \( \mathcal{L} \).

**Theorem 3.3** Let \( R : h \to \mathcal{K} \) be defined as follows :

\[
\mathcal{D}(R) = \mathcal{A}_{\infty, \tau}, \quad Rx = \sqrt{2}\delta_0(x).
\]

Then \( R \) has a densely defined adjoint \( R^* \), whose domain contains the linear span of the vectors \( \delta(x)y, \ x, y \in \mathcal{A}_{\infty, \tau} \) and

\[
R^*(\delta(x)y) = x\mathcal{L}(y) - \mathcal{L}(x)y - \mathcal{L}(xy).
\]

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We denote the closure of $R$ by the same notation $R$. For $x, y \in A_{\infty, \tau}$,

$$(R^* \pi(x)R - \frac{1}{2}R^* Rx - \frac{1}{2} x R^* R)(y) = L(x)y.$$  

Furthermore,

$$\delta(x)y = (Rx - \pi(x)R)(y), x, y \in A_{\infty, \tau},$$

$$L_2 = -\frac{1}{2} R^* R.$$ 

Proof :-

For $x, y, z \in A_{\infty, \tau}$, we observe by using the symmetry of $L$ that

$$\langle \delta(x)y, Rz \rangle = 2 \langle \delta_0(x)y, \delta_0(z) \rangle$$

$$\tau(y^*L(x^*z) - y^*L(x^*)z - y^*x^*L(z))$$

$$\tau(L(y^*)x^*z - (L(x) y)^*z - L(xy)^*z)$$

$$\langle \{xL(y) - L(x)y - L(xy)\}, z \rangle.$$ 

This suffices for the proof of the statements regarding $R^*$. It can be verified by a straightforward computation that $(R^* \pi(x)R - \frac{1}{2}R^* Rx - \frac{1}{2} x R^* R)(y) = L(x)y$ holds for $x, y \in A_{\infty, \tau}$. The remaining statements are also verified in a straightforward manner. \hfill \Box

4 HP Dilation

We shall now prove the existence of a unitary HP dilation for $T_t$.

**Theorem 4.1** There exist a Hilbert space $k_1$ and a partial isometry $\Sigma : K \to h \otimes k_0$ (where $k_0 = L^2(G) \otimes k_1$) such that $\pi(x) = \Sigma^*(x \otimes I_{k_0})\Sigma$ and $\tilde{R} \equiv \Sigma R$ is covariant in the sense that $(u_g \otimes v_g)\tilde{R} = \tilde{R}u_g$ on $A_{\infty, \tau}$ where $v_g = L_g \otimes I_{k_1}$, $L_g$ denoting the left regular representation of $G$ in $L^2(G)$.

Proof : 

The proof is essentially by the ideas as those in [9], so we omit the details. First we construct a strongly continuous unitary representation $V_g$ of $G$ in $K$ (strong continuity will follow by covariance of $L$ on a dense set of vectors, and hence by unitarity for every vector) such that $\pi$ is covariant under this...
$G$-action in $\mathcal{K}$. This $V_g$ satisfies $V_g \delta(x) = \delta(\alpha_g(x))$ by the construction, which clearly implies that $V_g R = Ru_g$ on $\mathcal{A}_{\alpha,\tau}$. Thus, $\pi$ is a normal covariant $*$-representation of $\tilde{\mathcal{A}}$ in $\mathcal{K}$, hence extends to a normal $*$-representation, say $\tilde{\pi}$ of the crossed product von Neumann algebra $\tilde{\mathcal{A}} \rtimes_G$, which is the weak closure of the algebra generated by $(x \otimes I_{L^2(G)})$, $x \in \tilde{\mathcal{A}}$ and $u_g \otimes L_g$, $g \in G$ in $\mathcal{B}(h \otimes L^2(G))$. Thus there is $\Sigma : \mathcal{K} \to h \otimes L^2(G) \otimes k_1$ (for some $k_1$) such that $\Sigma^* (X \otimes I_{k_1}) \Sigma = \tilde{\pi}(X)$, for $X \in \tilde{\mathcal{A}} \rtimes_G$. So in particular $\Sigma^*(x \otimes I_{k_0}) \Sigma = \pi(x)$, and $\Sigma^*(u_g \otimes v_g) \Sigma = V_g$. The rest of the proof follows easily from the arguments similar to those in [11].

It is clear that for $x \in \mathcal{A}_{\alpha,\tau}$, $\mathcal{L}(x) = \tilde{R}^*(x \otimes 1_{k_0}) \tilde{R} - \frac{1}{2} \tilde{R}^* \tilde{R} x - \frac{1}{2} x \tilde{R}^* \tilde{R}$. This enables us to write down the candidate for the unitary dilation for the q.d.s. $T_t$.

Before stating and proving the main theorem concerning H-P dilation, we make a crucial observation. Let us consider the form-generator given by $\mathcal{B}(h) \ni x \mapsto (Ru_\tau, (x \otimes 1) \tilde{R} \nu) - \frac{1}{2} \langle xu_\tau, \tilde{R}^* \tilde{R} \nu \rangle - \frac{1}{2} \langle \tilde{R}^* \tilde{R} u_\tau, x \nu \rangle$, $u_\tau \in \mathcal{D}^1(\tilde{R}^* \tilde{R})$. By the construction of Davies ([11]), there exists a unique minimal q.d.s. on $\Sigma_k$. This enables us to write down the candidate for the unitary dilation for the q.d.s. $T_t$.

**Lemma 4.2** $\tilde{T}_t$ is conservative.

**Proof**:

Let $\tilde{\mathcal{L}}$ denote the generator of $\tilde{T}_t$. We claim that $\mathcal{A}_{\alpha,\tau} \subseteq \mathcal{D}(\tilde{\mathcal{L}})$ and $\tilde{\mathcal{L}} = \mathcal{L}$ on $\mathcal{A}_{\alpha,\tau}$. Fix any $x \in \mathcal{A}_{\alpha,\tau}$. Let $\mathcal{D}_s$ be the linear span of operators of the form $(1 + \tilde{R}^* \tilde{R})^{-1} \sigma (1 + \tilde{R}^* \tilde{R})^{-1}$ for $\sigma \in \mathcal{B}_1(h)$. Clearly, for $\rho \in \mathcal{D}_s$, $tr(\tilde{\mathcal{L}}(x) \rho) = tr(x \tilde{\mathcal{L}}(x) \rho) = tr(\mathcal{L}(x) \rho)$ (using the explicit forms of $\mathcal{L}$ and $\tilde{\mathcal{L}}$), and since $\mathcal{D}_s$ is a core for $\tilde{\mathcal{L}}_s$ (see [11]), we have $tr(x \tilde{\mathcal{L}}(x) \rho) = tr(\mathcal{L}(x) \rho)$ for all $\rho \in \mathcal{D}(\tilde{\mathcal{L}})$. Now, for $\rho \in \mathcal{D}(\tilde{\mathcal{L}})$, $tr(\frac{t}{t} \tilde{T}_{s,*}(x) \rho ds) = tr(\mathcal{L}(x) t^{-1} \tilde{T}_{s,*}(x) \rho ds)$. So we extend this equality by continuity to all $\rho \in \mathcal{B}_1(h)$. Letting $t \to 0 +$, we get that $x \in \mathcal{D}(\tilde{\mathcal{L}})$ and $tr(\tilde{\mathcal{L}}(x) \rho) = tr(\mathcal{L}(x) \rho) \forall \rho \in \mathcal{B}_1(h)$, which implies that $\tilde{\mathcal{L}}(x) = \mathcal{L}(x)$. 12
From this, it follows by easy arguments using the fact that the resolvents of $L$ leaves $A^\infty_{\infty, \tau}$ invariant that $\tilde{T}_t(x) = T_t(x) \forall x \in A^\infty_{\infty, \tau}$, and hence by the ultraweak density of $A^\infty_{\infty, \tau}$ in $\tilde{A}$, $T_t$ and $\tilde{T}_t$ agree on $\tilde{A}$ (where we use the same notation for the $C^*$ semigroup $T_t$ and its canonical normal extension on $\tilde{A}$).

In particular $\tilde{T}_t(1) = 1$.

We note that since the set of smooth complex-valued functions on $G$ with compact supports is dense in $L^2(G)$ in the $L^2$-norm, it is clear that $k^0_{0\infty}$ is dense in the Hilbert space $k_0$, so let us choose and fix an orthonormal basis $\{e_i\}$ of $k_0$ from $k_{0, \infty}$. (note that $k_0$ can of course be chosen to be separable since $\bar{A}$ is $\sigma$-finite von Neumann algebra and $G$ is second countable.)

Theorem 4.3: The q.s.d.e.

\[ dU_t = U_t(a_R^\dagger(dt) - a_R(dt) - \frac{1}{2}\tilde{R}^\ast \tilde{R} dt); U_0 = I \]

on the space $h \otimes \Gamma(L^2(R^+) \otimes k_0)$ admits a unique unitary operator valued solution which implements a HP dilation for $T_t$.

For the meaning of such q.s.d.e. with unbounded coefficients we refer to [18] and [11] and for the notation $a_R, a_R^\dagger$ we refer to [14].

Proof :-

Since $\tilde{R}^\ast \tilde{R} = -2L_2$, and since $h_\infty \subseteq D(\mathcal{L}_2) \subseteq D(\tilde{R})$, the closed Hilbert space operator $\tilde{R}$ is also continuous as a map from $h_\infty$ to $h \otimes k_0$ w.r.t the Frechet topology and the Hilbert space topology of the domain and the range respectively. Thus the relation $\tilde{R}u_g = (u_g \otimes v_g)\tilde{R}$ on $A^\infty_{\infty, \tau}$ extends by continuity to $h_\infty$. That is, $\tilde{R}$ is covariant, and by the assumptions made on $\mathcal{L}_2$ at the beginning of this section it is easy to see that the conditions of the Theorem 2.6 are satisfied, so that there are $C, p$ such that $\|\tilde{R}\xi\|_{2,0} \leq C\|\xi\|_{2,p}$. Moreover, by 2.6, we obtain in particular that $\text{Dom}(\tilde{R}^\ast)$ (Hilbert space domain) contains $(h \otimes k_0)_\infty$. We recall from 14 the notation $\langle f, S \rangle$ (where $S$ is a linear map from $h$ to $h \otimes k_0$ and $f \in k_0$), which is defined to be a linear map from $h$ to itself with the same domain as that of $S$, and satisfying $\langle \langle f, S \rangle \beta, \gamma \rangle = \langle S\beta, \gamma \otimes f \rangle$. Now, for any vector $f \in k_{0\infty}$, it is clear that $h_\infty \subseteq \text{Dom}(\langle f, \tilde{R} \rangle \ast)$. So in particular, we have that $h_\infty \subseteq \text{Dom}(\tilde{R}_i), \text{Dom}(\tilde{R}_i^\ast)$, where $\tilde{R}_i = \langle e_i, \tilde{R} \rangle$. It is easy to note that $\tilde{R}_i, \tilde{R}_i^\ast$ keep $h_\infty$ invariant.
We shall now use the results of [18]. Let $Z_R$ denote the class of elements $L \equiv ((L^i_j))_{i,j \geq 0}, L^i_j \in \mathcal{B}(h)$ such that for each $j$ there is $c_j$ with $\sum_i \|L^i_j v\|_{2,0}^2 \leq c_j^2 \|v\|_{2,0}^2 \forall v \in h$. For $L \in Z_R$, we define $L^i_j \equiv L^i_j + (L^i_j)^* + \sum_{k \geq 1}(L^i_k)^* L^k_j$, $\mathcal{I}_R \equiv \{L \in Z_R : L^0_j = 0 \forall i, j\}$, and $Z^0_R \equiv \{L \in Z_R : \mathcal{L}_{2,0} \leq 0, \mathcal{S}' \subseteq N, \text{card} S < \infty\}$, where $\mathcal{L}_{S'} \equiv ((\mathcal{L}^i_j))_{i,j \in S'}$. For any family $L = ((L^i_j))_{i,j \geq 0}$ where $L^i_j$ are closed densely defined operators, we define $\tilde{\beta}$ and furthermore $\tilde{\mathcal{I}}_R$. We shall now use the results of [18]. Let $Z^0_R = Z \in \mathcal{Z}^{-}(\mathcal{D})$ the class of elements $Z \equiv ((Z^i_j))$ such that $Z^0_R$ is the generator of a strongly continuous contractive semigroup on $h$ with $\mathcal{D}$ as a core for $Z^0_R, \mathcal{D} \subseteq \text{Dom}(Z^0_j) \forall i, j$, and there is a sequence $Z(n) \in Z_R \cap \hat{Z}_R$ with $\lim_{n \rightarrow \infty} Z^i_j(n) v = Z^i_j v \forall v \in \mathcal{D}$. We define the bilinear forms $\mathcal{L}^i_j(X)$ for $X \in \mathcal{B}(h)$ by setting

$$\langle \beta, \mathcal{L}^i_j(X) \gamma \rangle = \langle \beta, X Z^i_j \gamma \rangle + \langle Z^i_j \beta, X \gamma \rangle + \sum_{k \geq 1} \langle Z^i_k \beta, X Z^j_k \gamma \rangle.$$ 

We also set $\mathcal{I} = \{Z \in \mathcal{Z}^{-}(\mathcal{D}) : \mathcal{L}^i_j(I) = 0 \forall i, j\}, \bar{\mathcal{I}} = \{Z \in \mathcal{Z}^{-}(\mathcal{D}) : \bar{Z} \in \mathcal{I}\}$, and furthermore $\beta_\lambda = \{X \geq 0, X \in \mathcal{B}(h) : \mathcal{L}^i_0(X) = \lambda X\}$ for $\lambda > 0$, and $\bar{\beta}_\lambda$ is defined by replacing $Z$ by $\bar{Z}$. With these notations, we have from [18] that the q.s.d.e. $dU_i = \sum_{j \geq 1} V^i_j d\tilde{\mathcal{L}^i_j}(t); V_0 = I$ has a unique unitary solution provided $Z \in \mathcal{I} \cap \bar{\mathcal{I}}$ and $\bar{\beta}_\lambda = \beta_\lambda = \{0\}$.

We now take $\mathcal{D} = h_\infty, Z^i_j = 0 \forall i, j \geq 1, Z^0 = \tilde{R}_i, Z^i_0 = \tilde{R}_i^*$ and $Z^0_2 = \mathcal{L}_2 = \frac{1}{2} R^* \tilde{R}$. Let $G_n = n(n - \mathcal{L}_2)^{-1}$. Let $Z^i_j(n) = G^i_j G^i_j G_n$. We shall show that this choice satisfies properties described above. We first note that $G_n$ is clearly a bounded (with $\|G_n\| \leq 1$) covariant map, hence smooth of order 0 with bound $\leq 1$, in particular maps $\mathcal{D}$ into itself. We have that for $\xi \in \mathcal{D}$,

$$\sum_i \|Z^0_i(n) \xi\|_{2,0}^2 \leq \|(-2\mathcal{L}_2)^{\frac{1}{2}} \xi\|_{2,0}^2$$

(as $G^0_i G^0_n R_i \leq R^*_i R_i$ and $G_n, \mathcal{L}_2$ commute).

Similarly we find that $\sum_i \|Z^i_i(n) \xi\|_{2,0}^2 < \infty$. For $j > 0$, we also have that $\|Z^i_j(n) \xi\|_{2,0}^2 = \|G^0_i \tilde{R}^j G^i_n \xi\|_{2,0}^2 \leq C^j_i \|\xi\|_{2,0}^2$, where we have used Theorem [2.10] to get constants $C^j_i, p_j$ such that $\|\tilde{R}^j_i \xi\|_{2,0} \leq C^j_i \|\xi\|_{2,0}$ for $\xi \in \mathcal{D}$. Finally we verify that $\lim_{n \rightarrow \infty} Z^i_j(n) v = Z^i_j v \forall v \in \mathcal{D}$. This follows from the following general fact:

If $L$ is a closed linear map from $h$ to $h$ with $h_\infty$ in its domain, so that $\|L \xi\|_{2,0} \leq M\|\xi\|_{2,r}$ for some $M, r$, then for $\xi \in h_\infty, G^i_n L G^i_n \xi \rightarrow L \xi$ as $n \rightarrow \infty$. To prove this fact, it suffices to observe that $G^i_n \xi$ clearly in $h_\infty$ and
\[ \|G_n \xi - \xi\|_2^2 = \sum_{i_1, i_2, \ldots, i_k, k \leq r} \| (G_n - I)(d_{i_1} d_{i_2} \ldots d_{i_k} \xi) \|_{2,0}^2 \] (as \( G_n \) is covariant), which goes to 0 as \( G_n \to I \) strongly. Thus we have \( \|G_n L(G_n \xi - \xi)\|_{2,0} \leq \|G_n L(G_n \xi - \xi)\|_{2,0} + \|(G_n - I) L \xi\|_{2,0} \leq M \|G_n \xi - \xi\|_{2,r} + \|(G_n - I) L \xi\|_{2,0} \), which completes the proof of the fact. Now, the existence, uniqueness and unitarity of \( U_t \) follows from \( [18] \), noting the facts that \( \tilde{Z} = Z \) in this case and the conservativeness of \( T_t \) proved earlier suffices for \( \beta_\lambda = \tilde{\beta}_\lambda = 0 \). \( \square \)

5 Evans Hudson type Dilation

We now study sufficient conditions for the existence of Evans-Hudson (E-H) dilation for the q.d.s \( T_t \) considered in the previous section. We shall make the following additional assumptions (either \( \text{A4,A5, A6,A7; or A4,A5,A6, A7'} \) on the algebra and the group action:

**A4:** \( \mathcal{A}_0 = \{ x \in \mathcal{A}_\infty : \exists C_x > 0 \text{s.t.} \|x\|_{\infty, n} \leq \|x\|_{\infty, 0} C_x \forall n \} \) is norm-dense in \( \mathcal{A} \);

**A5:** \( h_0 = \{ u \in h_\infty : \exists M_u > 0 \text{s.t.} \|u\|_{2, n} \leq \|u\|_{2, 0} M_u \forall n \} \) is \( L^2 \)-dense in \( h \);

**A6:** The canonical homomorphism of the crossed-product von Neumann algebra \( \bar{\mathcal{A}} \rtimes \mathcal{G} \) onto the weak closure of the \( \ast \)-algebra generated by \( \{ \bar{\mathcal{A}}, u_g ; g \in \mathcal{G} \} \) in \( \mathcal{B}(h) \) is an isomorphism;

and either

**A7:** Assume that the trace \( \tau \) is finite (hence \( \mathcal{A}_\infty \) coincides with \( \mathcal{A}_{\infty, \tau} \)) and we require the following.

Let \( \bar{\partial}_t \) denote the generator of the one-parameter group \( \bar{\mathcal{A}} \ni x \mapsto u_{g_t} x u_{g_t}^* \), w.r.t. the operator norm topology of \( \bar{\mathcal{A}} \), where \( g_t = \exp(t \chi_i) \), i.e. \( x \in \bar{\mathcal{A}} \) belongs to the domain of \( \bar{\partial}_t \) if and only if \( \frac{u_{g_t} x u_{g_t}^* - x}{t} \) converges as \( t \to 0^+ \) in the operator norm topology of \( \bar{\mathcal{A}} \) inherited from \( \mathcal{B}(h) \). Let \( \mathcal{A}_\infty \) denote the intersection of the domains of \( \bar{\partial}_{i_1} \ldots \bar{\partial}_{i_k} \) for all possible choices of \( i_1, \ldots i_k \). We assume that \( \mathcal{A}_\infty = \mathcal{A}_\infty \), i.e. any element of \( \bar{\mathcal{A}} \) which belongs to the intersection of all the domains ("smooth"), then it must belong to \( \mathcal{A} \).

or

**A7':** The trace \( \tau \) is not finite, and in this case we assume that \( \{ \bar{\mathcal{A}}, u_g ; g \in \mathcal{G} \}'' \) is a type I factor isomorphic with \( \mathcal{B}(h) \).

We note that \( \text{A4 and A5} \) hold for \( G \) compact or \( G = \mathbb{R}^n \). For compact groups, the verification of \( \text{A4,A5} \) can be done by the Peter-Weyl decomposition, and for \( \mathbb{R}^n \), the space \( \mathcal{S}(\mathbb{R}^n) \) of Schwarz functions can be taken as a candidate for both \( \mathcal{A}_0 \) as well as \( h_0 \). Similar thing will be true for many
Lemma 5.1 Under the assumption (A6), it follows that \( \mathcal{A}' \triangleright \triangleleft G \) is isomorphic with \( \{ \mathcal{A}', \mathcal{g}; g \in G \}'' \).

Proof:-
The proof is an easy consequence of the Tomita-Takesaki modular theory. Let \( J \) denote the closed extension of the canonical anti-linear isometric involution of that theory which sends \( x \in \mathcal{A}_\infty \to x^* \). Since \( \mathcal{g} = \alpha_g(x) \) for \( x \in \mathcal{A}_\infty \), and \( \alpha_g \) is *-preserving, it is clear that \( \mathcal{g} = J_\mathcal{g}J \) and thus (since we also have \( J^2 = \text{id} \)) \( J_\mathcal{g}J = \mathcal{g} \). But we know from the Tomita-Takesaki theory that \( J \mathcal{A} J = \mathcal{A}' \), hence the von Neumann algebra \( \mathcal{B} \equiv \{ \mathcal{A}, \mathcal{g}; g \in G \}'' \) is anti-isomorphic under the map \( \mathcal{J}.\mathcal{J} \) with \( \mathcal{C} \equiv \{ \mathcal{A}', \mathcal{g}; g \in G \}'' \). But by assumption A6, \( \mathcal{B} \) is isomorphic with \( \mathcal{A} \triangleright \triangleleft G \), which is nothing but the von Neumann algebra generated by \( \mathcal{A} \otimes 1 \) and \( \mathcal{g} \otimes L_g \) in \( \mathcal{B}(h \otimes L^2(G)) \), where \( L_g \) is the left regular representation. Clearly \( \mathcal{A} \triangleright \triangleleft G \) is anti-isomorphic with \( \mathcal{A}' \triangleright \triangleleft G \), under the obvious anti-isomorphism \( (\mathcal{J} \mathcal{J}) \otimes \text{id} \). Thus \( \mathcal{A}' \triangleright \triangleleft G \) is isomorphic with \( \mathcal{C} \), since the composition of two anti-isomorphisms is an isomorphism.

Theorem 5.2 Under the assumption A6 above, there exist a Hilbert space \( k_0 \), a partial isometry \( \Sigma : h \otimes k_0 \to h \otimes k_0 \) and a closed linear map \( \tilde{S} \) from \( h \) into \( h \otimes k_0 \) with \( h_\infty \) in its domain, such that the followings hold:

(i) \( \tilde{S} \) is covariant, i.e. \( \Sigma(\mathcal{g} \otimes \text{id}) = (\mathcal{g} \otimes \text{id})\Sigma; \)
(ii) \( \tilde{\pi}(x) \equiv \Sigma(x \otimes 1_{k_0})\Sigma^* \) is a covariant normal *-homomorphism of \( \mathcal{A} \) into \( \mathcal{B}(h \otimes k_0) \), which is structural in the sense of (A6), i.e. \( \tilde{\pi}(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{B}(k_0) \) and we also have \( \tilde{\pi}(\alpha_g(x))(\mathcal{g} \otimes \text{id}) = (\mathcal{g} \otimes \text{id})\tilde{\pi}(x); \)
(iii) \( \tilde{S} \) from \( h \) to \( h \otimes k_0 \) is smooth covariant map (w.r.t. the same \( G \)-action as in (i)), i.e. \( \tilde{S} \mathcal{g} = (\mathcal{g} \otimes \text{id})\tilde{S} \); and \( \tilde{\pi}(x) \equiv \tilde{\pi}(x)\tilde{S} \) for \( x \in \mathcal{A}_{\infty},r \) extends as a bounded map from \( h \) to \( h \otimes k_0 \) which is also structural, i.e. \( \tilde{\delta}(x) \in \mathcal{A} \otimes k_0, x \in \mathcal{A}_{\infty},r \), and covariant in the sense that \( \tilde{\delta}(\alpha_g(x)) = \gamma_g(\delta(x)), \)
where \( \gamma_g : \mathcal{A} \otimes k_0 \to \mathcal{A} \otimes k_0 \) by \( \gamma_g(.) = (\mathcal{g} \otimes \text{id})(u_g^* \otimes \text{id}); \)
(iv) \( \mathcal{L}(x) \equiv \tilde{S}^*\tilde{\pi}(x)\tilde{S} - \frac{1}{2}\tilde{S}^*\tilde{\pi}(x)\tilde{S} \) for \( x \in \mathcal{A}_{\infty},r \), in the sense that the LHS is a bounded extension of the RHS, which is defined on its natural domain.
(v) Furthermore, if we also make the assumption A7, then we have a stronger structurality in the following sense:

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For $x \in \mathcal{A}_\infty$ and $\xi, \eta \in k_0$, $< \xi, \tilde{\delta}(x) > \in \mathcal{A}_\infty$ and $< \xi, \tilde{\pi}(x)\eta >\equiv <. \otimes \xi, \tilde{\pi}(x)(. \otimes \eta)> \in \mathcal{A}_\infty$.

Proof :-

First of all we proceed in the line of the proof of Theorem 4.1, and consider the covariant normal *-homomorphism $\pi$ as in that theorem and lift it to a normal *-homomorphism $\tilde{\pi}$ of the crossed product, which is isomorphic by the assumption with $\{\tilde{\mathcal{A}}, u_g; g \in G\}''$, and thus we take $\tilde{\pi}$ to be a normal *-homomorphism on $\{\tilde{\mathcal{A}}, u_g; g \in G\}''$ satisfying $\tilde{\pi}(x) = \pi(x)$ for $x \in \tilde{\mathcal{A}}$ and $\tilde{\pi}(u_g) = V_g$. Then the construction as in [1] ensures that there exist some Hilbert space $k_1$ and isometry $\Sigma$ from $\mathcal{K}$ to $h \otimes k_1$ (where $\mathcal{K}$ is as in the proof of [1]) such that $\Sigma^*(x \otimes 1_{k_1})\Sigma = \tilde{\pi}(x), \Sigma^*(u_g \otimes 1_{k_1})\Sigma = V_g$. Let $\mathcal{K}_1 \equiv \Sigma(\mathcal{K}) \subseteq h \otimes k_1$, and let $P_1 = \Sigma \Sigma^*$ and $\delta_1(x) = \Sigma \delta(x), x \in \mathcal{A}_\infty$. Clearly $\{\delta_1(x)v, x \in \mathcal{A}_\infty, v \in h\}$ is dense in $\mathcal{K}_1$. As in [8], we now construct a normal *-homomorphism $\pi'$ of $\mathcal{A}'$ in $\mathcal{B}(\mathcal{K}_1)$ by setting $\pi'(a)\delta_1(x)v = \delta_1(x)av$, and extending by linearity and continuity (details can be found in [8] or [14]). We extend this $\pi'(a)$ on the whole of $h \otimes k_1$ by putting it equal to 0 on $P_1^\perp(h \otimes k_1)$, and we denote this trivial extension also by $\pi'(a)$. Clearly, $\pi'$ is covariant w.r.t. $g \mapsto (u_g \otimes 1)$, and thus we can extend $\pi'$ to a normal *-homomorphism of $\{\mathcal{A}', u_g; g \in G\}''$ (which is isomorphic with the crossed product von Neumann algebra $\mathcal{A}' \otimes_\pi G$), say $\pi''$, satisfying $\pi''(u_g) = u_g \otimes 1_{k_1}$. Hence there exist a Hilbert space $k_2$ and a partial isometry $\Sigma_1 : h \otimes k_1 \to h \otimes k_2$ such that $\Sigma_1^*(a \otimes 1_{k_2})\Sigma_1 = P_1(a \otimes 1_{k_1})$ and $\Sigma_1^*(u_g \otimes 1_{k_2})\Sigma_1 = P_1(u_g \otimes 1_{k_1})$. We take $k_0 = k_1 \oplus k_2$, $\tilde{\Sigma} = \Sigma_1 \oplus 0_{h \otimes k_2} : h \otimes k_0 \equiv (h \otimes k_1) \oplus (h \otimes k_2) \to h \otimes k_0$, and $\tilde{S}v := \Sigma(\Sigma Rv \oplus 0)$. The remaining is verified as in [8], and hence the details are omitted.

Finally, to prove (v), we first note that since $\tilde{\Sigma}$ is covariant and bounded (in Hilbert space sense), $u_{g_t} < \xi, \tilde{\pi}(x)\eta > = u_{g_t}^* < \xi, \tilde{\pi}(\alpha_{g_t}(x))\eta >$, (where $g_t = \exp(t\chi_i)$, for some fixed $i$ ) and thus we have that for $x \in \mathcal{A}_\infty$, $u_{g_t}^* < \xi, \tilde{\pi}(x)\eta > = u_{g_t}^* < \xi, \tilde{\pi}(\alpha_{g_t}(x))\eta > = \int_0^1 \xi, \tilde{\pi}(\partial_t(x))\eta = \int_0^1 < \xi, \tilde{\pi}(\partial_t(x))\eta > ds$; which goes to 0 in norm as $\tilde{\pi}$ is a norm-contractive map. This shows that $< \xi, \tilde{\pi}(x)\eta >$ belongs to the domain of $\partial_t$ for any $i$, and repetition of similar arguments proves that it belongs to $\mathcal{A}_\infty = \mathcal{A}_\infty$. We can prove similar fact about $\tilde{\delta}$ by using the covariance of $\tilde{\delta}$. We first note that $\mathcal{L}$, being covariant, norm-closed and having $\mathcal{A}_\infty$ in its domain, is continuous in the Frechet topology. Using the Frechet continuity of $\mathcal{L}$ and the cocycle identity $\tilde{\delta}(x)^* \tilde{\delta}(x) = \mathcal{L}(x^* x) - \mathcal{L}(x)^* x - x^* \mathcal{L}(x)$ for $x \in \mathcal{A}_\infty$, (by Assumption $A7$ $\mathcal{A}_{\infty,\tau} = \mathcal{A}_\infty$) we conclude that $\tilde{\delta} : \mathcal{A}_\infty \to \mathcal{B}(h, h \otimes k_0)$ is continuous w.r.t. the
Lemma 5.3 There exist a Hilbert space $B$ and the “structural” maps $\Theta$, as in [14], which combines all the section and proceed to prove the existence of an Evans-Hudson dilation. Our first step is to introduce a map $\Theta$, as in [14], which combines all the “structural” maps $L, \tilde{\delta}, \tilde{\pi}$ into a single map. Let $k_0 \equiv C \oplus k_0$ and $\Theta : A_{\infty} \rightarrow B(h \otimes \hat{k}_0)$ given by

$$\Theta(x) = \left( L(x), \tilde{\delta}^*(x), \tilde{\sigma}(x) \right),$$

where $\tilde{\delta}^*(x) = \tilde{\delta}(x^*)^* \text{ and } \tilde{\sigma}(x) = \tilde{\pi}(x) - (x \otimes 1_{k_0}).$

Lemma 5.3 There exist a Hilbert space $k'_0$ and two covariant smooth maps $B : h \otimes \hat{k}_0 \rightarrow h \otimes k'_0$, $A : h \otimes k'_0 \rightarrow h \otimes k_0$ such that $\Theta(x) = A(x \otimes 1_{k'_0})B$, where covariance is with respect to the representation $g \mapsto u_g \otimes 1$ in all cases.

Proof :-

We take $k'_0 = \hat{k}_0 \otimes C^3 \equiv \hat{k}_0 \oplus \hat{k}_0 \oplus \hat{k}_0$. Let $A = (A_1, A_2, I), B = \begin{pmatrix} B_1 \\ I \\ B_2 \end{pmatrix}$, where $A_i, B_j$ s are covariant smooth maps from $h \otimes \hat{k}_0$ to itself given by (w.r.t. the direct sum decomposition $h \otimes \hat{k}_0 = h \oplus (h \otimes k_0)$):

$A_1 = \begin{pmatrix} 0 & S^* \Sigma \\ 0 & -\Sigma \end{pmatrix}, \quad A_2 = \begin{pmatrix} -\frac{1}{2} \tilde{S}^* \tilde{S} & 0 \\ \tilde{S} & -\frac{1}{2} 1_{h \otimes k_0} \end{pmatrix}, \quad B_1 = A_1^*, B_2 = A_2^*.$

We note that all the maps above are defined with their usual domains and from the results of the previous sections (since $\tilde{\Sigma}$ is a bounded covariant map and $\tilde{S}$ is smooth covariant, and satisfies the condition of (2.1), so that its adjoint is smooth covariant too, and furthermore composition of smooth covariant maps is again smooth covariant) it follows that $A$ and $B$ are indeed smooth covariant. That $\Theta(x) = A(x \otimes 1_{k'_0})B$ can be verified by direct and easy computation. $\square$
We now extend the definition of the map $\Theta$, taking advantage of the fact that $(A \otimes I)$ and $(B \otimes I)$ are also smooth covariant maps with the same order and bounds as $A$ and $B$ respectively where $I$ denotes identity on any separable Hilbert space with trivial $G$-action (see Lemma 2.7).

**Definition 5.4** For any two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and a smooth (not necessarily covariant) map $X$ from the Sobolev-Hilbert space $(h \otimes \mathcal{H}_1, u_g \otimes 1)$ to $(h \otimes \mathcal{H}_2, u_g \otimes 1)$ we define $\Theta(X)$ to be a smooth map from $(h \otimes \hat{k}_0 \otimes \mathcal{H}_1, u_g \otimes 1 \otimes 1)$ to $(h \otimes \hat{k}_0 \otimes \mathcal{H}_2, u_g \otimes 1 \otimes 1)$, given by,

$$\Theta(X) = (A \otimes 1_{\mathcal{H}_2})P_{23}'(X \otimes 1_{\hat{k}_0'})P_{23}(B \otimes 1_{\mathcal{H}_1}),$$

where $P_{23} : h \otimes \hat{k}_0' \otimes \mathcal{H}_1 \rightarrow h \otimes \mathcal{H}_1 \otimes \hat{k}_0'$ denotes the operator which interchanges 2nd and 3rd tensor components, and a similar definition is given for $P_{23}'$.

It is clear that for $\mathcal{H}_1 = \mathcal{H}_2 = C$, we indeed recover the definition of $\Theta(x)$ for $x \in \mathcal{A}_\infty, \tau$, and furthermore we make sense of the same symbol even for which are not necessarily bounded operator on $h$, but are smooth map on $h$ w.r.t. the $G$-action given by $u_g$. The above extended definition enables us to compose $\Theta$ with itself, i.e. we can make sense of $\Theta(\Theta(X))$ and so on. We shall denote $\Theta(...\Theta(X))$ $(n$-times composition) by $\Theta^n(X)$ for $X$ as in the definition 5.4. The following estimate will be a fundamental tool for proving the homomorphism property of the E-H dilation which we are going to construct.

**Theorem 5.5** For $x \in \mathcal{A}_\infty, \xi \in (h \otimes \hat{k}_0^n)_{\infty}$, (where $\hat{k}_0^n$ denotes $n$-fold tensor product of $\hat{k}_0$ with itself, and $G$-action is taken to be $u_g \otimes 1_{\hat{k}_0^n}$) we have that

$$\|\Theta^n(x)\xi\|_{2,m} \leq C_1 C_2^n \|x\|_{\infty, np_1+m} \|\xi\|_{2,n(p_1+p_2)+m};$$

where $N$ is the dimension of the Lie algebra of $G$, $C_1 = \sqrt{\frac{2^2}{4N-1}(2\sqrt{N})^{m+1}},$ $C_2 = c_1 c_2 (2\sqrt{N})^{p_1}$, $c_1, c_2, p_1, p_2$ are such that $A$ is a smooth map of order $p_1$ and bound $\leq c_1$, and $B$ is a smooth map of order $p_2$ and bound $\leq c_2$. In particular, let $x \in \mathcal{A}_0, \xi_n = v \otimes w_1 \otimes \ldots \otimes w_n$ with $v \in h_0, w_i \in \hat{k}_0$, $\|w_i\|_{2,0} \leq K$ for some $K$, and let $V$ be a smooth covariant map of order 0 and bound $\leq 1$ on $(h \otimes \hat{k}_0^n \otimes \mathcal{K}')_\infty$ (where $\mathcal{K}'$ is some separable Hilbert space, with trivial $G$-action) and $\eta \in \mathcal{K}'$. Then we have that $\|(\Theta^n(x) \otimes 1)V(\xi_n \otimes \eta)\|_{2,0} \leq \|\eta\|_{2,0} K_1 K_2^n$ for some constants $K_1, K_2$ depending on $x, v, K$ only.

Furthermore, for $x, y \in \mathcal{A}_0$ and $v, V, \eta$ as above, we get constants $K_1', K_2'$ depending on $x, y, v$ only such that $\|(\Theta^n(x) \Theta^n(y)) \otimes 1)V(v \otimes \eta)\|_{2,0} \leq \|\eta\|_{2,0} K_1' K_2'^n.$
Proof :-
First we note that for \( x \in A_\infty \) and \( v \in h_\infty \), we have that

\[
\|xv\|_{2,m} \leq \sqrt{\frac{2}{4N-1}(2\sqrt{N})^{m+1}\|x\|_{\infty,m}\|v\|_{2,m}}.
\]  

(2)

To prove this estimate we note that for any fixed \( k \)-tuple \( i_1,...i_k \) with \( k \leq m \), where each \( i_j \in \{1,...,N \} \), we have that \( \|d_{i_1}...d_{i_k}\|_{2,0}^2 = \| \sum_{J \subseteq \{1,...,k \}} \partial_J(x)d_J(v)\|_{2,0}^2 \), where for any subset \( J \) of \( \{1,...,k \} \), we denote by \( \partial_J \) the map \( \partial_{i_1}...\partial_{i_k} \), with \( p_1 < p_2 < ...p_l \) being the arrangement of the elements of \( J \) (\( |J| = t \)) in the increasing order; and a similar definition is given for \( d_J, J^c \) being the complement of \( J \). Clearly, for any \( l \) nonnegative numbers \( a_1,...a_l \), one has that \( (a_1 + ...a_l)^2 \leq 2l(a_1^2 + ...a_l^2) \) and using this we see that \( \| \sum_{J \subseteq \{1,...,k \}} \partial_J(x)d_J(v)\|_{2,0}^2 \leq 2^{k+1} \sum_J \| \partial_J(x)\|_{2,0}^2 \leq 2^{k+1} \sum_J \|x\|_{\infty,m}^2\|v\|_{2,m}^2 = 2^{2k+1}\|x\|_{\infty,m}\|v\|_{2,m}^2 \). The proof of the estimate then follows by noting that the number of possible \( k \)-tuples as above is \( N^k \), and \( \sum_{k=0}^{m}2^{2k+1}N^k = \frac{2}{4N-1}((4N)^{m+1} - 1) \).

Now we come to the proof of the main estimate in the present theorem. It is easy to see that the estimate (2) holds even when \( x \in A_\infty \) is replaced by \( (x \otimes 1_{\mathcal{H}}) \) for any separable Hilbert space \( \mathcal{H} \) and \( v \in h_\infty \) is replaced by \( v \in (h \otimes \mathcal{H})_\infty \) where \( \mathcal{H} \) carries the trivial \( G \)-representation. Hence \( \|\Theta^n(x)\xi\|_{2,m} \leq C_1^n\|(x \otimes 1)P_{23}(B \otimes 1)...P_{23}(B \otimes 1)\xi\|_{2,m+np_1} \leq C_1^n\sqrt{\frac{2}{4N-1}(2\sqrt{N})^{m+np_1+1}\|x\|_{\infty,m+np_1}\|B \otimes 1\|...P_{23}(B \otimes 1)\xi\|_{2,m+np_1} \), from which the desired estimate follows. (Note that \( P_{23}, P'_{23} \) are trivially smooth covariant maps of order 0 and bound \( \leq 1 \).)

The last two assertions of the result follows by noting that for \( x \in A_0, \), \( v \in h_0 \), we have that \( \|x\|_{\infty,mp_1+m} \leq \|x\|_{\infty,0}M_x^{mp_1+m} \) and a similar estimate is valid for \( v \), and furthermore we have that \( \|V(\xi_n \otimes \eta)\|_{2,(n(p_1+p_2)+m)K^{n}\|\eta\|^2} \). \[ \square \]

We now prove an important algebraic property of \( \Theta \).

Lemma 5.6 For \( x, y \) of the form \( < \xi, \Theta^n(a)\eta > \) for some \( a \in A_\infty, \xi, \eta \in (h \otimes k_0 \otimes^a_{\infty})_\infty \), we have that

\[
\Theta(xy) = \Theta(x)(y \otimes 1) + (x \otimes 1)\Theta(y) + \Theta(x)Q\Theta(y),
\]

where \( Q = \left( \begin{array}{cc} 0, 0 \\ 0, 1_{h \otimes k_0} \end{array} \right) \) as in [4].

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Proof :-
We extend the definition of $\tilde{\pi}$ in the obvious manner to make sense of $\tilde{\pi}(x)$ for any $x \in \mathcal{B}(h_\infty, h_\infty)$ so that $\tilde{\pi}(x)$ is also a smooth map. However, this extended $\tilde{\pi}$ need not be a homomorphism on whole of the $\mathcal{B}(h_\infty, h_\infty)$. For proving the lemma, it suffices to verify that $\tilde{\pi}(xy) = \tilde{\pi}(x)\tilde{\pi}(y)$ for $x, y$ of the form $< \xi, \Theta^n(a)\eta >$ for some $a \in \mathcal{A}_\infty$, $\xi, \eta \in (h \otimes k_0 ^* \otimes ^n)_{\infty}$. We now consider the cases corresponding to the assumptions A7 and A7' separately. In the first case, we have already shown that $< \xi, \Theta^n(a)\eta > \in \mathcal{A}_\infty$ for $a, \xi, \eta$ as in the statement of the lemma, and then it is trivial to see that the above algebraic relation holds, since $\tilde{\pi}$ is indeed a homomorphism on $\mathcal{A}_\infty$. Now, we consider the case when A7' is assumed. In this situation, the proof of the theorem 5.2 enables us to see that $k_1, k_2, k_0$ (in the notation of the theorem mentioned above) can be chosen in such a way that $\tilde{\pi}(x) = \tilde{\Sigma}(x \otimes I_{h})^{\otimes 1}$ for $x \in \mathcal{B}(h_\infty, h_\infty)$, $\tilde{\Sigma}$ is a homomorphism. This completes the proof of the required algebraic property of $\Theta$ in the present case.

Theorem 5.7 Let $\tilde{\Sigma}, \tilde{S}, k_0$ be as in the statement of theorem 5.2. Then the following operator q.s.d.e. in $h \otimes \Gamma(L^2(R_+, k_0))$ admits a contraction-valued solution.

$$dV_t = V_t \left( a^{\dagger}_{\tilde{S}}(dt) + \Lambda_{\tilde{\Sigma}-I}(dt) - a_{\tilde{\Sigma}, \tilde{S}}(dt) - \frac{1}{2} \tilde{S}^* \tilde{S}(dt) \right); V_0 = I. \quad (3)$$

Furthermore, $V_t$ is covariant w.r.t. the $G$-action $u_g \otimes 1$ on $h \otimes \Gamma(L^2(R_+, k_0))$.

Proof :-
The existence and uniqueness of the solution $V_t$ can be obtained essentially as in the Theorem 4.3 (see also [18] and [11]). Covariance of $V_t$ is straightforward to show. It is important to note that $V_t$ is not unitary since $\tilde{\Sigma}$ is a strict partial isometry.

Now, let us recall the map-valued quantum stochastic calculus developed in [14]. We want to define integrals of the form $\int_0^t Y_s \circ M(ds)$ where $M$ stands for $(a_{\tilde{\sigma}}(ds) + a^{\dagger}_{\tilde{\sigma}}(ds) + \Lambda_{\tilde{\sigma}}(ds) + \mathcal{I}_C(ds))$ (see notations in [14]), where $\tilde{\sigma}(x) = \tilde{\pi}(x) - (x \otimes 1_{k_0})$. The ideas and construction will be almost the same as those in [14], and hence we only briefly sketch the steps, omitting details.

We shall adopt the following convention throughout the rest of the paper : unless otherwise stated, $G$-action on a Hilbert space of the form $h \otimes \mathcal{H}'$ will be taken to be $u_g \otimes 1$.
In [14], the integrator $\Theta$ was a norm-bounded map, and thus the above integral could have been defined on the whole of $\mathcal{A} \otimes \Gamma$ (with $\Gamma = \Gamma(L^2(R_+, k_0))$). But here we are dealing with unbounded $\Theta$, and thus we shall make sense of the above integral only on a restricted domain. Let $\mathcal{D}$ denote the algebraic linear span of the elements of the form $x \otimes e(f)$, with $x \in \mathcal{B}(h_\infty, h_\infty)$, i.e. $x$ is a smooth map from $h_\infty$ to itself, $f$ is a bounded continuous $k_0$-valued function on $[0, \infty)$, and let $\mathcal{S}$ denote the space $\mathcal{B}(h_\infty, (h \otimes \Gamma)_\infty)$. Let $(Y_t)_{t \geq 0}$ be a family of maps from $\mathcal{D}$ to $\mathcal{S}$, with the adaptedness in the obvious sense as in [14], and also satisfying the following condition (an analogue of 3.12 of [14]):

$$\sup_{0 \leq s \leq t} \|Y_s(x \otimes e(f))v\|_{2,0} \leq \|r_1(x \otimes 1_{\mathcal{H}'})r_2v\|_{2,0}$$

for some Hilbert space $\mathcal{H}'$, where $r_1, r_2$ are smooth maps between appropriate spaces, in contrast to their being bounded in [14]. We call such a process $Y_t$ regular, and we define $Z_t = \int_0^t Y_s \circ M(ds)$ exactly in the same way as in [14]. Indeed, $\Theta_{f(s)}(x)$ (with the already given definition of $\Theta$, and notations as in [14]) is a densely defined closable operator on $h$, with $h_\infty$ in its domain, and also $< f(s), \Theta(x)g(s) >$ clearly is an element of $\mathcal{B}(h_\infty, h_\infty)$ (i.e. smooth map), hence the conditions required in [14] for defining the maps $S(s), T(s)$ by $S(s)(ve(f_s)) = \tilde{Y}_s(\tilde{\delta}(x) \otimes e(f_s))v$ and $T(s)(ve(g_s) \otimes f(s)) = \tilde{Y}_s(\tilde{\sigma}(x)f(s) \otimes e(g_s))v$ as in [14] (with similar notation) are satisfied. Then an analogue of Prop 3.3.5 of [14] can be applied to $Z_t$ (see Cor. 2.2.4 (ii) of [14]), and we conclude that $Z_t$ is well-defined and regular. However, in the present situation $Z_t(x \otimes e(f))$ need not be a bounded operator from $h$ to $h \otimes \Gamma$, and need not belong to $\tilde{\mathcal{A}} \otimes \Gamma$. Nevertheless, at least in case A7 is assumed, we shall show that the EH flow $J_t$ to be constructed by us will map $\tilde{\mathcal{A}} \otimes \Gamma$ to itself.

**Theorem 5.8** We set $j_t : \tilde{\mathcal{A}} \rightarrow \mathcal{B}(h \otimes \Gamma)$ by $j_t(x) = V_t(x \otimes 1_\Gamma)\tilde{V}_t^*$ (where $V_t$ as in theorem 7.7) and also we define $J_t : \tilde{\mathcal{A}}_{\text{alg}} \otimes \Gamma \rightarrow \mathcal{B}(h, h \otimes \Gamma)$ by $J_t(x \otimes e(f))v = j_t(x)(ve(f))$ and extending by linearity. Furthermore, the above definition of $j_t(x)$ can be extended to $x \in \mathcal{B}(h_\infty, h_\infty)$ (i.e. for $x$ which are possibly unbounded as Hilbert space map) and $J_t : \mathcal{D} \rightarrow \mathcal{S}$ also similarly. Then we have the following:

(i) $J_t$ is a regular process and satisfies the q.s.d.e. $dJ_t = J_t \circ M(dt)$; $J_0 = id$;
(ii) $j_t : \tilde{\mathcal{A}} \rightarrow \mathcal{B}(h \otimes \Gamma)$ is a normal covariant (w.r.t. $\alpha_g \otimes id$) $\ast$-homomorphism which dilates $T_t$ in the sense of [4], [14];
and 

(iii) If \( A7 \) (and not \( A7' \)) is assumed, then \( j_t(\hat{A}) \subseteq \hat{A} \otimes \mathcal{B}(\Gamma) \), and 
\[
<e(f), j_t(x)e(f')> \in \mathcal{A}_\infty \quad \text{for} \quad x \in \mathcal{A}_\infty, f, f' \in L^2(R_+, k_0).
\]

Proof :-

Since \( V_t \) and \( V_t^* \) are smooth covariant of order 0, the regularity (even in the sense of [14]) of \( J_t \) easily follows, and thus \( J_t^t J_s \circ M(ds) \) makes sense. Furthermore, by Ito formula and the q.s.d.e. satisfied by \( V_t \), it is simple to verify that \( J_t \) satisfies the required map-valued q.s.d.e. This proves (i).

For proving (ii), our strategy is as in [14]. We define a family of maps \( \Phi_t \) as follows. First we extend the definition of \( J_t \) to make sense of \( J_t(X \otimes e(f)) \) for \( X \in \mathcal{B}(h_\infty, (h \otimes \Gamma_{fr})_\infty) \) where \( \Gamma_{fr} \) is the free Fock space over \( \hat{k}_0 \) (see [14]), by setting \( J_t(X \otimes e(f))v = (V_t \otimes 1_{\Gamma_{fr}})P_{23}(X \otimes 1_{\Gamma})V_t^*(ve(f)) \) and then we define for fixed \( u, v \in h_0 \) and \( f, f' \) being bounded continuous \( k_0 \)-valued functions on \( R_+ \), \( \Phi_t(X, Y) = < J_t(X \otimes e(f))u, J_t(Y \otimes e(f'))v > - < u e(f), J_t(X^*Y \otimes e(f'))v > \), for all \( X, Y \in \mathcal{B}(h_\infty, (h \otimes \Gamma_{fr})_\infty) \) satisfying that \( X^* \in \mathcal{B}(h \otimes \Gamma_{fr})_\infty, h_\infty \) (and thus \( X^*Y \in \mathcal{B}(h_\infty, \infty) \)), and in case it is \( A7 \), not \( A7' \), the assumption taken by us, we also require that for any vector \( \beta \) in \( \Gamma_{fr} \), \( \phi^{\beta} \), \( \beta, X >, < \beta, Y > \in \mathcal{A}_\infty \). With this definition, we now fix \( x, y \in \mathcal{A}_0 \) too. We identify \( \hat{k}_0^n \) as canonically embedded subspace of \( \Gamma_{fr} \), and it is clear that for \( w \in \hat{k}_0^n \), \( \Theta^n(x)w \) is a smooth map with its adjoint being smooth too (which follows from the explicit structure of \( \Theta(,) = A(, \otimes 1)B \), where \( A, B \) are smooth covariant, hence have smooth adjoints). Thus, for any \( n \), and \( w, w' \in \hat{k}_0^n \), \( \Phi_t(\Theta^n(x)w, \Theta^n(y)w') \) is well-defined and furthermore one can easily verify relation (3.18) of [14] with \( X = \Theta^n(x)w, Y = \Theta^n(y)w' \) by using Lemma 5.6. We can now iterate this relation arbitrarily many times, and by noting the estimate (1) of Theorem 5.5, (and also by using the fact that \( \|\Phi_t(X, Y)\| \leq \|((X \otimes 1) V_t^*(ve(f)))\|_2.0 + \|ve(f)\|_2.0 \|((X^*Y \otimes 1)(V_t^*(ue(f')))\|_2.0 \), and \( V_t^* \) is a smooth covariant map of order 0, bound \( \leq 1 \) conclude that \( \Phi_t(x, y) = 0 \) for \( x, y \in \mathcal{A}_0 \). This proves the weak homomorphism property of \( j_t \). Since \( j_t(x) \) is by the very definition in terms of \( V_t \) is a bounded map for all \( x \in \hat{A} \) and \( \|j_t(x)\| \leq \|x\|_{\infty, 0} \), the strong homomorphism property follows. The covariance and other properties of \( j_t \) as in (ii) of the statement of the present theorem are straightforward to see.

Finally to prove (iii), first of all we show that \( j_t(\hat{A}) \subseteq \hat{A} \otimes \mathcal{B}(\Gamma) \). For this, we construct iteratively \( J_t^{(n)} \), by setting \( J_t^{(0)}(x \otimes e(f))v = (x \otimes e(f))v \), and \( J_t^{(n+1)} = \int_0^t J_s^{(n)} \circ M_\Theta(ds) \) for \( n \geq 0 \), and furthermore define \( J_t(x \otimes e(f))v =
\]
\[ \sum_n J_t^{(n)} (x \otimes e(f))v, \text{ for all } x, v \text{ such that the above sum is convergent in the } \text{Hilbert space sense.} \] One can verify that for \( x \in \mathcal{A}_0, v \in h_0, J_t(x \otimes e(f))v \) exists and satisfies the same q.s.d.e. as \( J_t \) with the same initial condition, hence by the standard iteration argument used to prove the uniqueness of solution of q.s.d.e., it follows that \( J_t(x \otimes e(f))v = J_t((x \otimes e(f))av) \) for \( x \in \mathcal{A}_0, v \in h_0 \) and \( a \in \mathcal{A}' \) of the form \( JyJ \) for some \( y \in \mathcal{A}_0 \), where \( J \) is the anti-unitary operator of Tomita-Takesaki theory mentioned before. Clearly such \( a \) maps \( h_\infty \) into \( h_\infty \) and also \( JA_0J \) is strongly dense in \( \mathcal{A}' \). From this we obtain that \( (a \otimes 1)J_t(x \otimes e(f))v = J_t(x \otimes e(f))av \) for \( x, a, v \) as before. Note that we needed \( x \in \mathcal{A}_0, v \in h_0 \) for showing the summability of \( J_t^{(n)}(x \otimes e(f))v \). Thus \( j_t(x) \) commutes with all \( (a \otimes 1), a \in JA_0J \); and since \( j_t(x) \) is bounded operator, the same holds for all \( a \in \mathcal{A}' \). This proves that \( j_t(A_0) \subseteq \bar{A} \otimes B(\Gamma) \), and then due to the normality and boundedness of the map \( x \mapsto j_t(x) \), the same thing will follow for all \( x \). Now, take \( x \in \mathcal{A}_\infty \). Since \( V_t \) is covariant contractive map, we can easily verify that \( \langle e(f), j_t(x)e(f') \rangle \in \bar{A}_\infty \) for \( x \in \mathcal{A}_\infty, f, f' \in L^2(k_0)_\infty \), and hence by the assumption \( \text{A7,} \langle e(f), j_t(x)e(f') \rangle \in \mathcal{A}_\infty \), which completes the proof. 

\[ \square \]

6 Applications and examples

In this section we shall show that it is indeed possible to accommodate many interesting classical and noncommutative semigroups in our framework.

6.1 Classical (commutative) examples

First of all we prove that the assumption \( \text{A6} \) regarding crossed product is valid in a typical classical situation.

**Theorem 6.1** Let \( X \) be a locally compact separable Hausdorff space with a regular Borel measure \( \mu \) on it, and let a locally compact group \( G \) act on \( X \) freely and transitively, and the measure \( \mu \) is \( G \)-invariant. Then the von Neumann algebra \( L^\infty(X, \mu) \) satisfies the assumption \( \text{A6}, \) i.e. the crossed product von Neumann algebra \( L^\infty(X, \mu) \rtimes G \) is isomorphic with the weak closure of the \( \ast \)-algebra generated by \( L^\infty(X, \mu) \) and \( u_g \) in \( \mathcal{B}(L^2(X, \mu)) \), which is in fact
the whole of $\mathcal{B}(L^2(X, \mu))$, where $u_g$ denotes the unitary representation of $G$ in $L^2(X, \mu)$ induced by the $G$-action on $X$.

Proof:-
Let $x_0$ be any point of $X$. It is clear that the bijective continuous map $G \ni g \mapsto gx_0 \in X$ is a homeomorphism (because $G$ is locally compact and $X$ is Hausdorff, so that any continuous bijection from $G$ to $X$ is automatically a homeomorphism). Let us denote $L^\infty(X, \mu)$ and $L^2(X, \mu)$ by $\mathcal{A}$ and $h$ respectively. We recall that $\mathcal{A} \bowtie G$ can be defined to be the von Neumann algebra generated by $f \otimes 1, f \in \mathcal{A}$ and $u_g \otimes L_g, g \in G$ in $\mathcal{B}(h \otimes L^2(G))$ (where $L_g$ is the left regular representation). So, the commutant of this von Neumann algebra is Hausdorff, so that any continuous bijection from $G$ to $X$ is automatically a homeomorphism. But $\mathcal{A} \bowtie \mathcal{B}(L^2(G))$ and $\{u_g \otimes L_g, g \in G\}'$ (since $\mathcal{A}$ is maximal abelian). But $\mathcal{A} \bowtie \mathcal{B}(L^2(G))$ can be identified with the direct integral of copies of $\mathcal{B}(L^2(G))$ over $(X, \mu)$. In this direct integral picture, we can view any element $B$ of $\mathcal{A} \bowtie \mathcal{B}(L^2(G))$ as a measurable map $B : x \mapsto B(x) \in \mathcal{B}(L^2(G))$; and then it is easy to see that $B$ also commutes with all $u_g \otimes L_g$ if and only if $B(gx) = L_g^*B(x)L_g \forall x, g$. Thus, the map $B(.)$ is determined by the value of $B(.)$ at any one point of $X$ (since the action is free and transitive). It is now easily seen that $(\mathcal{A} \bowtie G)'$ is isomorphic with $1 \otimes \mathcal{B}(L^2(G))$. To verify this, we denote the inverse of the map $g \mapsto gx_0$ (which is a homeomorphism as noted earlier) by $\Psi$ and consider the unitary $U$ on $h \otimes L^2(G)$ given by $(U\phi)(x, g) = \phi(x, \Psi(x)g)$, for $\phi \in h \otimes L^2(G) \cong L^2(X \times G)$. Clearly, for any $B \in (\mathcal{A} \bowtie G)'$, i.e. $B(x) = L_g^*B(x_0)L_g$, with $g = \Psi(x)$, we have that $UBU^* = 1 \otimes B(x_0)$, and since $B(x_0)$ can be allowed to be an arbitrary element of $\mathcal{B}(L^2(G))$, we have shown that $U(\mathcal{A} \bowtie G)U^* = (1 \otimes \mathcal{B}(L^2(G)))$, hence $U(\mathcal{A} \bowtie G)U^* = \mathcal{B}(h) \otimes 1$. Now it is enough to prove that the von Neumann algebra generated by $\mathcal{A}$ and $u_g, g \in G$ in $\mathcal{B}(h)$ is the whole of $\mathcal{B}(h)$, which is a direct consequence of the imprimitivity theorem. \hfill \Box

Remark 6.2 Remark :-
1. The assumption of transitivity is not so crucial and can be weakened to the assumption that the bundle $(X, X/G, \pi)$ (where $\pi : X \rightarrow X/G$ is the canonical quotient map) is trivial (which will be true in particular whenever $X/G$ is contractible).

Now, we proceed to give classical examples where our theory works. Let $G$ be a separable Lie group, equipped with an invariant (w.r.t. the action of $G$ on itself) Riemannian metric, $\mathcal{A}$ be the $C^*$-algebra of continuous functions.
on $G$ vanishing at “$\infty$”, and let $G$ act on itself by the left regular action, which is trivially free and transitive, so that $A_6$ will be valid. In case $G$ is compact, we check $A_7$ by using the fact that any element of $L^2(G)$ which is almost everywhere differentiable and all the partial derivatives are again in $L^2$ obviously belongs to the $C^\infty$-class.

For noncompact $G$, $A_7'$ follows by Theorem 4.1. The Laplace-Beltrami operator on $G$ commutes with the isometry group associated with the $G$-invariant Riemannian metric, and hence the heat semigroup generated by it satisfies the covariance and symmetry (w.r.t. the left Haar measure) conditions, and also the other assumptions needed for HP dilation theory are satisfied.

As far as the assumptions $A_4$, $A_5$, $A_6$, $A_7$ (or $A_7'$) needed for the EH theory are concerned, we have already verified $A_6, A_7$ (/ $A_7'$). However, one has to verify $A_4$ and $A_5$ case by case, and in many cases (e.g. compact groups, $R^n$ etc.) these can indeed be verified.

In the context of $R^n$, we now discuss two interesting classes of q.d.s. First, we consider the expectation semigroup of a diffusion process, such that the generator $L$ is of the form $L(f) = -\sum_{ij} a_{ij} \partial_i \partial_j f$ for smooth $f$ with compact support, where $\partial_i$ denotes the $i$-th partial derivative, and $a_{ij}$ are smooth functions, with the matrix $((a_{ij}))$ being pointwise nonsingular and positive definite, and $x \mapsto ((a_{ij}(x)))^{-\frac{1}{2}}$ is a smooth bounded function. Although if $a_{ij}$ are non-constant functions, then the above $L$ is not covariant w.r.t. the action of translation group, we show how to choose a different group acting on $R^n$ such that $L$ can be written as $L_0 + \delta_0$ for some covariant 2nd order operator $L_0$ and a derivation $\delta_0$. To achieve this, we change the canonical Riemannian metric of $R^n$ and equip it with a different metric given by $<\partial_i, \partial_j> |_x = b_{ij}(x)$, where $((b_{ij}(x))) := ((a_{ij}(x)))^{-1}$. Let us denote by $X$ the Riemannian manifold $R^n$ with this new metric and let $G$ be the group of Riemannian isometries of $X$. It is easy to verify that if we choose $L_0$ to be the generator of the heat semigroup on $X$, then $L_0$ is $G$-covariant and symmetric w.r.t. the Riemannian volume measure, and moreover $L$ is indeed the same as $L_0$ up to some first order operators, which can be written as $\delta_0(.)$ for some suitable closed derivation $\delta_0$ which generates a 1-parameter automorphism group of the underlying function algebra (we omit some technical conditions that may be required to ensure the existence of such an automorphism group, but at least for nice enough $a_{ij}$ this will be possible). We can now apply our theory on $L_0$ to construct dilations, and then it is trivial to obtain dilations of $e^{tL}$ from the dilation of $e^{tL_0}$, using some standard perturbation techniques.
However, we must point out here that one needs to verify \( \text{A1-A7} \) case by case. A sufficient condition for verifying these assumptions is that there is a nice and large enough Lie subgroup of \( G \) which acts freely and transitively on \( X \), (which in particular will imply that \( X \) is a Riemannian homogeneous space) and such that \( \mathcal{A}_\infty \) and \( h_\infty \) will coincide as sets with those as in the case of \( \mathbb{R}^n \) with the action of itself. For the simple case when \( n = 1 \), we can show the existence of such a subgroup by direct computation which gives us an explicit description of the group of isometries of \( X \).

### 6.2 Noncommutative examples

We shall give a class of examples which are closely connected with noncommutative geometry.

**Proposition 6.3** Let \( \mathcal{A} \) be a unital C\(^*\) algebra and let \( G \) be a compact Lie group acting ergodically on \( \mathcal{A} \). Then the assumptions \( \text{A4-A7} \) are valid for \( \mathcal{A} \) with the above group action and the unique \( G \)-invariant normalized trace described in [3], [16].

**Proof :-**

Since \( G \) is compact, \( \text{A4,A5} \) are easy to verify. We shall now prove that \( \text{A6 and A7} \) are also valid. By combining the results of [23], [3] and [16] there is a set of elements \( t_{\pi ij} \), \( \pi \in \hat{G} \), \( i = 1, \ldots, d_\pi \), \( j = 1, \ldots, m_\pi \) of \( \mathcal{A} \), where \( \hat{G} \) is the set of irreducible representations of \( G \), \( d_\pi \) is the dimension of the irreducible representation space denoted by \( \pi \), \( m_\pi \leq d_\pi \) is a natural number, such that the followings hold:

(i) The linear span of \( \{ t_{\pi ij} \} \) is norm-dense in \( \mathcal{A} \),

(ii) \( \{ t_{\pi ij} \} \) is an orthonormal basis of \( h = L^2(\mathcal{A}, \tau) \),

(iii) The action of \( u_g \) coincides with the \( \pi \)-th irreducible representation of \( G \) on the vector space spanned by \( t_{\pi ij} \), \( i = 1, \ldots, d_\pi \) for each fixed \( j \) and \( \pi \),

(iv) \( \sum_{i=1}^{d_\pi} (t_{\pi ij})^*t_{\pi ik} = \delta_{jk}d_\pi 1 \), where \( \delta_{jk} \) denotes the Kronecker delta symbol. Thus, in particular, \( \| t_{\pi ij} \|_\infty,0 \leq \sqrt{d_\pi} \forall \pi, i, j \).

Now, we first prove \( \text{A6} \). We recall that the crossed product von Neumann algebra \( \mathcal{C} := \mathcal{A} \bowtie G \) is by definition the von Neumann algebra generated by \( \{ (t_{\pi ij} \otimes 1), \pi, i, j; (u_g \otimes L_g), g \in G \} \) in \( L^2(\mathcal{A}) \otimes L^2(G) \), where \( L_g \) is the regular representation of \( G \) in \( L^2(G) \). Let \( \rho \) be the normal *-homomorphism from \( \mathcal{C} \) onto \( \{ \mathcal{A}, u_g, g \in G \}'' \subseteq \mathcal{B}(L^2(\tau)) \) which satisfies \( \rho(t_{\pi ij} \otimes 1) = t_{\pi ij} \) and
\[ \rho(u_g \otimes L_g) = u_g. \] We have to show that this is an isomorphism, i.e. the kernel of \( \rho \) is trivial. Clearly, the set of elements of the form \( \sum c_{\pi ij} t_{ij}^{p} u_{g_{nij}} \) (finitely many terms), with \( c_{\pi ij} \in \mathbb{C}; g_{nij} \in G \) is dense w.r.t. the strong-operator topology in \( \{ A, u_{g}, g \in G \}^{\prime}. \) Similarly, the set of elements of the form \( \sum c_{\pi ij} (t_{ij}^{p} \otimes 1) (u_{g_{nij}} \otimes Lg_{nij}^{(p)}) \) (finitely many terms) will be strongly dense in \( \mathcal{C}. \) Now, let \( \mathcal{I} \equiv \{ X \in \mathcal{C} : \rho(X) = 0 \}. \) We need to show that \( \mathcal{I} = \{0\}. \)

Let \( X \in \mathcal{I} \) and let \( X_n = \sum c_{\pi ij}^{(p)} (t_{ij}^{p} \otimes 1) (u_{g_{nij}}^{(p)} \otimes Lg_{nij}^{(p)}) \) be a net (indexed by \( p \)) of elements from the above dense algebra such that \( X_n \) converges strongly to \( X. \) Hence we have, \( \sum |c_{\pi ij}^{(p)}|^2 = \| \rho(X_p) (1) \|^{2}_{0} \to 0. \) This implies that for any \( \phi \in L^2(G), \) \( \| X_p (1 \otimes \phi) \|^2 = \sum |c_{\pi ij}^{(p)}|^2 \| Lg_{nij}^{(p)} \phi \|^2 \leq \| \phi \|^2 \sum |c_{\pi ij}^{(p)}|^2 \to 0, \) which proves that \( X (1 \otimes \phi) = 0 \) for every \( X \in \mathcal{I}. \) But since \( \mathcal{I} \) is an ideal in \( \mathcal{C}, \) this shows that for \( a \in A, X (a \otimes \phi) = (X (a \otimes 1)) (1 \otimes \phi) = 0, \) and by the fact that \( \{ a \otimes \phi, a \in A, \phi \in L^2(G) \} \) is total in \( h \otimes L^2(G) \) we conclude that \( \mathcal{I} = \{0\}. \)

We are now left with the proof of \( A7. \) Since in this case the trace \( \tau \) is finite, we have that \( \mathcal{A}_\infty \subseteq \mathcal{A}_\infty \subseteq \mathcal{h}_\infty. \) Hence it suffices to prove \( h_\infty = \mathcal{A}_\infty. \)

Let \( \Delta_G \) be the Laplacian on the compact Lie group \( G, \) and \( \lambda_{\pi}, \pi \in \mathbb{G} \) be its set of eigenvalues, where \( \lambda_{\pi} \) occurs with \( d_{\pi} \) multiplicity. Let \( v \in \mathcal{h}_\infty \) be given by an \( L^2 \)-convergent series \( v = \sum c_{\pi ij} t_{ij}^{\pi}. \) The assumption that \( v \in \mathcal{h}_\infty \) implies that \( \sum |\lambda_{\pi}|^{2n} |c_{\pi ij}|^2 < \infty \) for every positive integer \( n. \) Now, using the well-known Weyl asymptotics for the Laplacian on a compact manifold and the fact that \( m_{\pi} \leq d_{\pi} \) in our case, it is easy to see that for any large enough \( n, \) \( \sum_{i,j} \frac{d_{\pi}}{\lambda_{\pi}^{2n}} < \infty. \) Now, since \( \| t_{ij}^{\pi} \|_{\infty,0} \leq \sqrt{d_{\pi}}, \) we have, \( \sum_{i,j} |c_{\pi ij}| \| t_{ij}^{\pi} \|_{\infty,0} \leq \sum |c_{\pi ij}| \sqrt{d_{\pi}} \leq (\sum |c_{\pi ij}|^2 |\lambda_{\pi}|^{2n})^{\frac{1}{2}} \left( \sum \frac{d_{\pi}}{\lambda_{\pi}^{2n}} \right)^{\frac{1}{2}} < \infty \) for all large enough \( n. \) This proves that the series \( \sum c_{\pi ij} t_{ij}^{\pi} \) converges in the norm of \( \mathcal{A}, \) and hence \( v \in \mathcal{A}. \) Similar arguments will enable us to prove that indeed \( v \in \mathcal{A}_\infty, \) thereby proving that \( \mathcal{h}_\infty \subseteq \mathcal{A}_\infty, \) hence \( \mathcal{h}_\infty = \mathcal{A}_\infty. \)

We now note that since we have shown in the above proof that \( \mathcal{h}_\infty = \mathcal{A}_\infty, \) and since in the present case the finiteness of the trace implies that the Frechet topology of \( \mathcal{A}_\infty \) is stronger than that of \( \mathcal{h}_\infty, \) it is clear that the assumption \( A2 \) implies \( A3. \) Thus, we have the following nice sufficient condition to have EH dilation in this case:

**Proposition 6.4** With the set-up of the Proposition [6.3], any covariant, symmetric, conservative q.d.s. \( T_t \) on \( \mathcal{A} \) such that the domain of its norm-

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6.3 Connection with Arveson-Powers Index Theory

In this final subsection, we would like to apply our results to obtain the numerical index as defined by Arveson and Powers \[21\],\[4\] and studied by many other authors. Let us use the notation of the section on HP dilation theory. In the notation of the lemma \[4.2\], \( \tilde{T}_t \) is a covariant q.d.s. on \( \mathcal{B}(h) \) which extends \( T_t \). Furthermore, we can choose \( k_0 \) in a “minimal” way, in the sense that there does not exist any nonzero vector \( f \in k_0 \) satisfying \( \langle f, \tilde{R} \rangle = 0 \). To show this, let us denote by \( k' \) the set of all \( f \in k_0 \) such that \( \langle f, \tilde{R} \rangle = 0 \). It is easy to check that this set is a closed subspace of \( k_0 \), and \( (x \otimes 1) \tilde{R} \xi \in (h \otimes k')^\perp \) for all \( x \in \mathcal{B}(h) , \xi \in h \). Thus we can replace \( k_0 \) by \( k_0 \ominus k' \) to achieve the required minimality. For simplicity, let us assume that the trace \( \tau \) is finite, so that \( 1 \in h = L^2(\tau) \). Assume w.l.g. that \( k_0 \) has been chosen in a minimal way as described above, and \( \{e_i\} \) be an o.n.b. of \( k_0 \) consisting of “smooth” vectors as in the section on HP dilation. Let \( \tilde{\mathcal{R}}_i = \langle e_i, \tilde{R} \rangle \). Note that by construction, \( \tilde{\mathcal{R}}(1) = 0 \). We now prove that the result by Bhat \[5\] on the minimality of dilation can be extended to the present situation.

**Theorem 6.5** The dilation \( \tilde{\mathcal{R}}_i \) of \( \tilde{T}_t \), given by \( \tilde{j}_i(x) = U_t(x \otimes 1\Gamma)U_t^* \), (where \( U_t \) is the unitary process constructed in the section on HP dilation) is minimal in the sense of Bhat and Parthasarathy \[2\], i.e. \( \{\tilde{j}_i(1_1)\ldots\tilde{j}_i(x_n)\xi; x_1,\ldots,x_n \in \mathcal{B}(h), \xi \in h; t_1 \geq t_2 \geq \ldots t_n \geq 0\} \) is total in \( h \otimes \Gamma \). Thus, the numerical index of \( \tilde{T}_t \) in the sense of Arveson is the dimension of \( k_0 \).

**Proof :-**

We first verify the following condition which is sort of \( L^2 \)-independence of the family of operators \( \{I, \tilde{R}_i; i = 1,2,\ldots\} \) :

If \( c_0, c_1, \ldots \) is an \( L^2 \)-sequence of complex numbers such that \( (c_0 I + \sum_i c_i \tilde{R}_i)\xi = 0 \forall \xi \in h_{\infty} \), then \( c_i = 0 \forall i = 0,1,2,\ldots \) clearly \( c_0 \) must be 0. Now, consider the vector \( f \in k_0 \) given by \( f = \sum_{i \geq 1} c_i e_i \) (which is well-defined as \( c_i \) is \( L^2 \)), and note that the condition \( \sum c_i \tilde{R}_i = 0 \) on \( h_{\infty} \) implies that \( \langle f, \tilde{R} \rangle = 0 \) on \( h_{\infty} \), and since \( h_{\infty} \) is a core for \( \tilde{R} \), we have that \( \langle f, \tilde{R} \rangle = 0 \), which implies that \( f = 0 \) by our assumption on \( k_0 \).

The rest of the proof will be exactly the same as that of \[3\] by Bhat. We note that the boundedness of the coefficients (which was assumed by Bhat)
is not really used in the proof, since \( h_\infty \)-vectors are used.

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