Conformal Theory of the Dimensions of Diffusion Limited Aggregates

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We employ the recently introduced conformal iterative construction of Diffusion Limited Aggregates (DLA) to study
the multifractal properties of the harmonic measure. The
support of the harmonic measure is obtained from a dynamical process which is complementary to the iterative cluster growth. We use this method to establish the existence of a series of random scaling functions that yield, via the thermodynamic formalism of multifractals, the generalized dimensions $D_q$ of DLA for $q \geq 1$. The scaling function is determined just by the last stages of the iterative growth process which are relevant to the complementary dynamics. Using the scaling relation $D_3 = D_0/2$ we estimate the fractal dimension of DLA to be $D_0 \approx 1.69 \pm 0.03$.

The diffusion limited aggregation (DLA) model was introduced in 1981 by T. Witten and L. Sander [1]. DLA has attracted enormous amount of research as an elegant example of the dynamical creation of a non-trivial fractal set, and as a model that underlies many pattern forming processes including dielectric breakdown [2], two-fluid flow [3], and electro-chemical deposition [4]. In spite of the significant amount of effort to understand the fractal and the multifractal properties of DLA, there exists to date no accepted calculation of these properties from first principles. In this Letter we present a theory that is based on an iterative conformal construction of DLA [5,6] that culminates with the construction of a series of (random) scaling functions which allow, via the thermodynamic formalism of multifractals [8,9,5], a convergent calculation of the multifractal properties of DLA.

Originally the DLA model was introduced as the outcome of N random walks. Fixing one particle at the center of coordinates in $d$-dimensions, and releasing random walkers from infinity one at a time, one allows them to walk around until they hit any particle belonging to the cluster. Upon hitting they are attached to the growing cluster. We are interested here in $N = 2$ for which numerical simulations indicated that for $N \to \infty$ the cluster retains a fractal dimension of about 1.71 [6]. The approach used here is different: we employ an iterative conformal construction of DLA that was recently proposed by Hastings and Levitov [3]. The basic idea is to follow the evolution of the conformal mapping $\Phi^{(n)}(w)$ which maps the exterior of the unit circle in the mathematical $w-$plane onto the complement of the cluster of $n$ particles in the physical $z-$plane. The unit circle is mapped to the boundary of the cluster which is parametrized by the arc length $s, z(s) = \Phi^{(n)}(e^{i\theta})$. This map $\Phi^{(n)}(w)$ is made from compositions of elementary maps $\phi_{\lambda,\theta}$,

$$\Phi^{(n)}(w) = \Phi^{(n-1)}(\phi_{\lambda,\theta}(w)),$$  

where the elementary map $\phi_{\lambda,\theta}$ transforms the unit circle to a circle with a "bump" of linear size $\sqrt{\lambda}$ around the point $w = e^{i\theta}$. Accordingly the map $\Phi^{(n)}(w)$ adds on a new bump to the image of the unit circle under $\Phi^{(n-1)}(w)$. The bumps in the physical $z-$plane simulate the accreted random walkers in the original formulation. The main idea in this construction is to choose the positions of the bumps $\theta_n$ and their sizes $\sqrt{\lambda}$ such as to achieve accretion of fixed linear size bumps on the boundary of the growing cluster according to the harmonic measure $P(s)ds$. The latter is the probability that a random walker would hit an infinitesimal arc $ds$ centered at the point $z(s)$. This is done as follows.

The probability density $P(s)$ is given by the inverse of the derivative of the conformal map $P(s) = \left[\Phi^{(n)}(e^{i\theta})\right]^{-1}$. Using the obvious fact that $ds = |\Phi^{(n)}(e^{i\theta})|d\theta$ we see that $P(s)ds \equiv d\theta$, i.e. the harmonic measure is uniform on the unit circle. Thus choosing random positions $\theta_n$, and $\lambda_n$ in Eq. (1) according to

$$\lambda_n = \frac{\lambda_0}{|\Phi^{(n)}(e^{i\theta_n})|^2} \quad (2)$$

we accrete fixed size bumps in the physical plane according to the harmonic measure. Finally the elementary map $\phi_{\lambda,\theta}$ is chosen as

$$\phi_{\lambda,\theta}(w) = w^{1-a} \left\{ \frac{(1 + \lambda)}{2w} (1 + w) \right\}$$

$$\times \left[ 1 + w + w \left( 1 + \frac{1}{w^2 - \frac{1}{w^2 - 1}} \right)^{1/2} \right] - 1 \right\}^{\lambda} \quad (3)$$

$$\phi_{\lambda,\theta}(w) = e^{i\theta} \phi_{\lambda,\theta}(e^{-i\theta} w), \quad (4)$$

The parameter $a$ is confined in the range $0 < a < 1$, determining the shape of the bump. In this Letter we employ $a = 2/3$ which is consistent with semicircular bumps. The qualitative properties of this mapping that enter prominently our analysis are the following:

$$\phi_{\lambda,\theta}(w) \approx (1 + \lambda)^{a} w \quad \text{for} \quad \frac{|w| - 1}{\sqrt{\lambda}} \geq \text{const}, \quad (5)$$

where const here is a tolerance-dependent but $\lambda$-independent constant of the order of unity.

$$|\phi_{\lambda,\theta}(w)| \approx (1 + \sqrt{\lambda})|w| \quad \text{for} \quad \text{arg} w \in [\theta + \sqrt{\lambda}], \quad (6)$$

$$|\phi_{\lambda,\theta}(w)| \approx |w| \quad \text{for} \quad \text{arg} w \notin [\theta - \sqrt{\lambda}], \quad (7)$$

Eq. (1) means that points that do not belong to the bump are only reparametrized along the circle.
The final calculation of $D$ into balls of diameter $D$ follows [9]: consider a partition of the cluster boundary by dimensions of the harmonic measure. The latter are defined as a given history $\theta_1$, $\theta_2$, ..., $\theta_{n-1}$ according to the uniform measure on the unit circle, yielding a distribution of values of $\lambda_n$, which according to Eq. (2) is the distribution of the density of the harmonic measure. The moments of this distribution are connected to the generalized dimensions of the harmonic measure. The latter are determined as follows [3]: consider a partition of the cluster boundary into balls of diameter $\ell_i$ and measure $p_i$. The generalized dimensions $D_q$ are determined by the equation

$$\lim_{M \to \infty} \sum_{i=1}^{M} p_i^{q/D_q} = 1.$$  

The calculation of $D_q$ from the statistics of $\lambda_n$ was discussed in detail in Ref. [4] with the exact result

$$\lambda_n \sim n^{-2qD_{2q+1}/D}.$$  

It is well known [10] that the fractal dimension is $D_0$ in this language, the information dimension (known also as the dimension of the harmonic measure) is $D_1 = 1$ [1], and in general $D_q \geq D_{q'}$ for any $q' > q$. It was shown in [2] and in [3] that $D_1 = D_0/2$. This last result means of course that on the average $\lambda_n$ decreases like $n^{-1}$. The exact result found in [3] is

$$\lambda_n = \frac{1}{aD_n}.$$  

Consider the support of the harmonic measure. We use the uniformity of the measure on the unit circle to form, for a given cluster of $n$ bumps, an equi-measure partition of $M$ balls in the physical space, by selecting points $z_k$

$$z_k \equiv \Phi^{(n)}(e^{i\theta_k}) = \phi_{\lambda_1, \theta_1} \circ \phi_{\lambda_2, \theta_2} \circ \ldots \circ \phi_{\lambda_n, \theta_n} (e^{i\theta_k}),$$

where the points $\zeta_k$, $k = 1, 2, \ldots, M$ are uniformly spaced on the unit circle. We introduce the “complementary dynamics” to the cluster growth by

$$z_{j,k} \equiv \phi_{\lambda_j, \theta_j} \circ \phi_{\lambda_{j+1}, \theta_{j+1}} \circ \ldots \circ \phi_{\lambda_n, \theta_n} (e^{i\zeta_k}),$$

and $z_{1,k} = z_k$. The complementary dynamics creates the points $z_k$ from the seeds $e^{i\zeta_k}$. In Fig.1 we display the typical evolution of the set $\{z_{j,k}\}$ for different values of $j$. The striking observation is that the equi-measure partition is fixed in shape very early (large $j$) in the complementary dynamics. Later stages only serve to inflate the fixed shape, with the last steps being most prominent in determining the final radius of this set which of course is the radius of the cluster.

![Fig. 1. The evolution of the set \(\{z_{j,k}\}\). (a) \(j=11\) and 1, (b) \(j = 10^5\), 6000 and 2000. Note that the straight lines are not parts of the set, they simply connect points on the set.](image)

This observation is easy to understand. As noted in Eqs. (14) the elementary map $\phi_{\lambda, \theta}(w)$ distorts circles of radius close to unity, but acts as a uniform multiplication for points with larger absolute values. For $j = n$ we always start the complementary dynamics on the unit circle, and (15) are applicable. We estimate now how many iterations of the complementary dynamics are necessary before (15) becomes applicable. Consider an arbitrary value of $\zeta_k$. The first iteration of the complementary dynamics $z_{\zeta,k}$ have $|z_{\zeta,k}| \approx 1 + \sqrt{\lambda_n}$ if $\zeta \in [\theta_n, \pm \sqrt{\lambda_n}]$, an event of probability $\kappa_n = \sqrt{\lambda_n}/2\pi$. Otherwise $|z_{\zeta,k}| = 1$. If the dynamics failed to increase $|z_{\zeta,k}|$, it can do it with $|z_{\zeta-1,k}|$ after two steps, with probability $(1 - \kappa_n)\kappa_n - 1$. The average number of steps $A$ needed to grow out with certainty is therefore

$$A = \sum_{k=0}^{\infty} k(1 - \kappa_n)(1 - \kappa_{n-1}) \ldots (1 - \kappa_{n-k+2})\kappa_{n-k+1}$$

(14)

Averaging over the history $\lambda_1, \ldots, \lambda_n$, approximating $\langle\lambda_{n-k}\rangle \sim \langle\lambda_{n-k}\rangle$ (which was justified for $k \neq k', k, k' \ll n$ in (14)), we find

$$\langle A \rangle = \sum_{k=0}^{\infty} k(1 - \kappa_n)(1 - \kappa_{n-k+2})\kappa_{n-k+1}$$

(15)

We assume and show self-consistently that the sum is dominated by $k \ll n$. In that case

$$\langle A \rangle \approx \sum_{k=0}^{\infty} k(1 - \kappa_n)^{k-1}\kappa_n = \frac{1}{\kappa_n} n D_2/D$$

(16)
where (10) has been used. The variance is estimated similarly, and is of the same order. Since 0.5 = D_2/D \leq D_3/D \leq D_1/D = 0.58 we see that for large n we need relatively few iterations of the complementary dynamics to increase the radius of the set \{z_{j,k}\} to a value after which the overwhelming majority of the iterations serve simply to inflate the radius by factors of \((1+\lambda_j)^{\ell}\). In light of Eq.(12) we understand the phenomenon exhibited in Fig.1, that the last steps of the complementary dynamics have the largest effect on the radius of the set \{z_{j,k}\}. We note in passing that these comments also explain the finding of (7) that the first Laurent coefficient of \(\Phi^{(m)}(w)\), denoted there as \(F_1^{(m)}\) is a measure of the radius of the cluster. The exact result is that \(F_1^{(m)} = \prod_{i=1}^{n} (1+\lambda_i)^{\sigma}\). Finally note that the set \{z_{j,k}\} for \(k \to \infty\) generates the support of the harmonic measure. It will reveal the generalized dimensions \(D_q\) for \(q \geq 1\) only.

**Conjecture:** For \(1 < M_2 < n\) the binary scaling function (17) converges in distribution to a universal scaling function independent of \(M_2\) and the history \(\theta_1, \ldots, \theta_n\). We denote the distribution as \(P_2(\sigma)\).

In Fig.2 we show the numerical evidence for this conjecture. We employ clusters with \(n = 10^4\) and \(n = 10^5\), and various values of \(M_2\). Note that \(M_2 < n\); increasing \(M_2\) beyond, say, 128 for \(n = 10^5\) results in exposing the ultra-violet cutoff of the smooth bumps, yielding a spurious peak at \(\sigma = 1/2\). The harmonic measure is extremely concentrated near the tips of the cluster, and any attempt to resolve the fjords leads to oversampling of the smooth bumps on the tips. Note that the scaling function is defined as a ratio, and the discussion above implies that it is sensitive to only \(n D_2/D\) last growth steps. The majority of the iterations of the fundamental map \(\phi_{\lambda,\theta}\) are irrelevant, as they cancel in the ratio.

Ordering the \(2^n\) diameters \(z_j - \tilde{z}_j\) in a binary basis \(\ell(\epsilon_m \ldots \epsilon_1)\), redefining the ratios \(\sigma(\epsilon_m \ldots \epsilon_1)\) accordingly, and using in Eq.(13) the fact that \(p_i = 2^{-m}\) for the \(M_1\) balls, we derive the equation (18)

\[
\sum_{\epsilon_{m+1} \ldots \epsilon_1} \sigma^{-\tau(q)}(\epsilon_{m+1} \ldots \epsilon_1) \ell^{-\tau(q)}(\epsilon_{m} \ldots \epsilon_1) = 2^q \sum_{\epsilon_{m} \ldots \epsilon_1} \ell^{-\tau(q)}(\epsilon_{m} \ldots \epsilon_1),
\]

where \(\tau(q) \equiv (q-1)D_q\). Iterating, we find

\[
S_{m+1} = \sum_{\epsilon_{m+1} \ldots \epsilon_1} \sigma^{-\tau(q)}(\epsilon_{m+1} \ldots \epsilon_1) \sigma^{-\tau(q)}(\epsilon_{m} \ldots \epsilon_1) \ldots \sigma^{-\tau(q)}(\epsilon_1) = 2^{(m+1)q}.
\]

To compute \(D_1\) we notice that \(\tau(1) = 0\). For \(\tau(q) \to 0^+\) all the realizations of the products of random numbers \(\sigma^{-\tau}\) are comparable. Since the most probable product for \(m \to \infty\) is \(\lambda, \theta\), we can estimate the sum in (13) as \(2^{(m+1)} \log[\log \sigma]\). Substituting in (19) and taking the \(m\)th root and the log to base 2 gives

\[
\lim_{q \to 1^+} \tau(q) = \frac{1 - q}{\log_2 \sigma}.
\]

Using the scaling function \(P_2(\sigma)\) we compute \(\log_2 \sigma \approx -1 \pm 0.01\), yielding the expected result \(D_1 = 1\). This is the first nontrivial calculation of a generalized dimension in this approach.

For \(q > 1\) the approach using the most probable product of random numbers is not applicable. We have \(2^m\) realizations of products of \(m\) random numbers, and rare events are relevant. Moreover, one should notice that even though every factor of \(\sigma^{-\tau}\) is independent, the products in the sum are not, as they have common factors. One can use the fact that the random products are organized on a binary tree to write an exact recursion relation

\[
S_{m+1} = \sigma(0)^{-\tau(q)}(0)S_m^{(0)} + \sigma(1)^{-\tau(q)}S_m^{(1)},
\]

FIG. 2. (a) The random scaling function \(P_2(\sigma)\) obtained from two consecutive partitions of 16 and 32 equi-measure balls. 550 clusters of 10000 bumps were used. (b) \(P_2(\sigma)\) computed from 32 and 64 balls, with 550 clusters of 10000 bumps. (c) \(P_2(\sigma)\) computed from 32 and 64 balls, with 44 clusters of 100 000 bumps.

At this point we can introduce a “binary” scaling function. Consider two values of \(M\), \(M_1 = 2^m\) and \(M_2 = 2^{m+1}\). Consider the two associated sets \(\{\tilde{z}_i\}_{i=1}^{M_1}, \{z_k\}_{k=1}^{M_2}\) such that \(\tilde{z}_i = z_{2i}\). For every difference \(\tilde{z}_i - \tilde{z}_{i-1}\) in the coarser resolution \(M_1\) (denoted as the \(i\)th “mother”), there are two differences \(z_{2i} - z_{2i-1}\) and \(z_{2i-1} - z_{2i-2}\) in the finer resolution, which are denoted as “daughters”. The binary scaling function at every resolution has \(M_2\) values which are obtained as the ratio of daughters to mothers,

\[
\sigma_{2i} = (|z_{2i} - z_{2i-1}|/|\tilde{z}_i - \tilde{z}_{i-1}|),
\]

\[
\sigma_{2i-1} = (|z_{2i-1} - z_{2i-2}|/|\tilde{z}_i - \tilde{z}_{i-1}|). \tag{17}
\]
where $S_{m}^{(\epsilon)}$ are two independent realizations of the sum of $2^{m}$ products, each of $m$ random variables. We failed to find the exact asymptotics of $S_{m}$, which is necessary for computing $\tau(q)$. We recognize however that our binary partition is in fact arbitrary, and instead we can perform refinements into $k$ daughters at each step. Accordingly we will have $k$ scaling ratios $\sigma(\epsilon)$ for each mother, with $\epsilon$ now taking on $k$ values $k=0,1,\ldots,k-1$. The random scaling function is now denoted as $P_{k}(\sigma)$; its existence as a universal function for $k=2^{m}$ emanates in an obvious fashion from the existence of $P_{2}(\sigma)$, and its existence for any $k$ can be demonstrated independently as done above for $P_{2}(\sigma)$. The important point is that the asymptotics of Eq. (19) is computable in the limit $k \to \infty$:

$$S_{m+1} = \sum_{\epsilon_{1}=0}^{k-1} \sigma(\epsilon_{1})^{-\tau(q)} S_{m}^{(\epsilon_{1})}. \quad (22)$$

In the limit of $k \to \infty$ this equation reads

$$S_{m+1} \to k\sigma(\epsilon_{1})^{-\tau(q)} S_{m}^{(\epsilon_{1})} = k\sigma^{-\tau(q)} S_{m} \quad (23)$$

where we have used the fact that $\sigma(\epsilon_{1})^{-\tau(q)}$ is random, independent of the consecutive factors $\sigma(\epsilon_{1},\epsilon_{2})^{-\tau(q)} \ldots$ which consist $S_{m}^{(\epsilon_{1})}$. Asymptotically $S_{m+1} \to [k\sigma^{-\tau(q)}]^{m+1}$, and substituting in (23) we compute $\tau(q)$ from

$$\int P_{k}(\sigma)\sigma^{-\hat{\tau}(q)} d\sigma = k^{q-1}, \quad \lim_{k \to \infty} \hat{\tau}(q) = \tau(q). \quad (24)$$

The best fit predicts $2D_{3} = 2\tilde{D}_{3}(k \to \infty) = 1.69$. Examining different partitions of the unit circle and different integration schemes we concluded that we can bound the errors around $D_{0} = 2D_{3} = 1.69 \pm 0.03$.

The excellent result of this calculation leaves for future research the analytic determination of the random scaling functions $P_{k}(\sigma)$. If these could be written down from first principles the problem of DLA will be settled. In particular only analytic $P_{k}(\sigma)$ will allow reliable calculations of $D_{q}$ for $q \to \infty$ due to the sensitivity to the small $\sigma$ tail where the statistics is low.

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