Large-time behavior of the $H^{-m}$-gradient flow of length for closed plane curves

Kohei Nakamura
Saitama University, Japan
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Abstract

We consider the $H^{-m}$-gradient flow of length for closed plane curves. This flow is a generalization of curve diffusion flow. We investigate the large-time behavior assuming the global existence of the flow. Then we show that the evolving curve converges exponentially to a circle. To do this, we use interpolation inequalities between the deviation of curvature and the isoperimetric ratio, recently established by Nagasawa and the author.

Keywords: area-preserving flow, isoperimetric ratio, interpolation inequalities, higher order curvature flow, curve diffusion flow

1 Introduction

Let $f = (f_1, f_2) : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^2$ be a function such that $\text{Im} f$ is a closed plane curve with rotation number 1 and the variable of $f$ is the arc-length parameter. The unit tangent vector is $\tau = (f'_1, f'_2)$. Let $\nu = (-f'_2, f'_1)$ be the inward unit normal vector, and let $\kappa = f''$ be the curvature vector. The curvature $\kappa = \kappa \cdot \nu$ is positive when $\text{Im} f$ is convex. Since the curve has rotation number 1, the deviation of curvature is

$$\tilde{\kappa} = \kappa - \frac{1}{L} \int_0^L \kappa \, ds = \kappa - \frac{2\pi}{L}.$$
In this paper, we consider the flow

$$\frac{\partial_t f}{\partial_t} = (-1)^m (\partial_s^{2m} \tilde{\kappa}) \nu.$$  \hspace{1cm} (1.1)

This is the $H^{-m}$-gradient flow of length. Indeed, for any $\varphi \in H^{-m}$, we have

$$\frac{d}{d\epsilon} L(f + \epsilon \varphi) \bigg|_{\epsilon=0} = -\int_0^L \tilde{\kappa} \varphi \, ds = -\int_0^L \{(-1)^m \partial_s^{m} \tilde{\kappa}\} \{\partial_s^{-m} \varphi\} \, ds$$

$$= -\int_0^L \{\partial_s^{-m} \{(-1)^m \partial_s^{2m} \tilde{\kappa}\}\} \{\partial_s^{-m} \varphi\} \, ds$$

$$= -\langle (-1)^m \partial_s^{2m} \tilde{\kappa}, \varphi \rangle_{H^{-m}}.$$  

This flow has already been considered when $m = 0, 1$.

When $m = 0$, the flow \textbf{(1.1)} is

$$\frac{\partial_t f}{\partial_t} = \tilde{\kappa} \nu.$$  \hspace{1cm} (1.2)

This flow was first considered by Gage \cite{5}, who proved that a simple closed strictly convex initial curve remains so along the flow, and the evolving curve converges to a circle. However, we can not expect that a global solution exists if initial curve is not convex. Using numerical analysis, Mayer \cite{8} found an example of a curve which develops singularities in finite time under the flow. Very little appears to be known regarding sufficient conditions for global existence.

Nagasawa and the author \cite{9} considered large-time behavior assuming the global existence of the flow. We showed that the evolving curve converges exponentially to a circle assuming the global existence by using interpolation inequalities in section 2.

When $m = 1$, the flow \textbf{(1.1)} is

$$\frac{\partial_t f}{\partial_t} = -\partial_s^2 \tilde{\kappa} \nu.$$  \hspace{1cm} (1.3)

This flow was proposed by Mullins \cite{7} and we call it curve diffusion flow. The flow is a fourth-order parabolic partial differential equation. Hence we do not expect convexity to be preserved along the flow. Indeed, Giga and Ito \cite{6} showed the existence of a simple closed strictly convex plane curve that becomes non-convex in finite time under the flow. Also, Escher–Ito \cite{4} and Chou \cite{1} proved that evolving curves may develop singularities in finite time even when the initial curve is smooth.

On the other hand, there are some results for large-time behavior. Chou \cite{1} showed that the evolving curve converges exponentially to a circle assuming the global existence of the flow. Moreover Elliott–Garcke \cite{3} and Wheeler
showed the global existence and investigated the large-time behavior for initial data close to a circle.

In this paper, we would like to investigate the large-time behavior of (1.1) assuming the global existence of the flow. In order to do this, we introduce several inequalities in section 2. In section 3, by using these inequalities, we prove that the evolving curve converges exponentially to a circle.

2 Several inequalities

In this section we introduce several inequalities which were established in [9]; for details of the proofs, see [9].

For a non-negative integer \( \ell \), we set

\[
I_\ell = L^{2\ell+1} \int_0^L |\tilde{\kappa}(\ell)|^2 ds,
\]

which is a scale invariant quantity (cf. [2]). It is important for the global analysis of evolving curves to estimate \( I_\ell \). We have the Gagliardo-Nirenberg inequalities

\[
I_\ell \leq C I_\ell^{\frac{\ell}{m}} I_0^{1-\frac{\ell}{m}},
\]

where \( 0 \leq \ell \leq m \) and \( C \) is a positive constant and independent of \( L \). Such inequalities are very useful but only these are not sufficient to estimate \( I_0 \) because of the way these inequalities make use of \( I_0 \). Hence we need a different type of inequality to estimate \( I_\ell \) for \( \ell \geq 0 \).

We introduce the quantity

\[
I_{-1} = 1 - \frac{4\pi A}{L^2},
\]

where the \( A \) is the (signed area) given by

\[
A = -\frac{1}{2} \int_0^L f \cdot \nu ds.
\]

\( I_{-1} \) is also scale invariant, and is non-negative by the isoperimetric inequality.

The following inequalities for \( I_0, I_{-1} \) were derived by Nagasawa and the author in [9].

**Theorem 2.1** We have

\[
8\pi^2 I_{-1} \leq I_0 \leq I_{-1}^{\frac{1}{2}} \left[ L^3 \int_0^L \{ \kappa^3 \tilde{\kappa} + (\kappa')^2 \} ds \right]^{\frac{1}{2}}.
\]

In both cases, equality holds only in the trivial case \( \tilde{\kappa} \equiv 0 \).
From this inequality, we have the following new interpolation inequalities.

**Theorem 2.2** Let \(0 \leq \ell \leq m\). There exists a positive constant \(C = C(\ell, m)\) independent of \(L\) such that

\[
I_\ell \leq C \left( I_{m-\ell}^{m} I_m + I_{m-\ell}^{m+1} I_m^{m+1} \right)
\]

holds.

### 3 Large-time bahavior

In this section, we investigate the large-time behavior of (1.1) assuming the global existence of the flow. Since (1.1) is a parabolic equation, \(f\) is smooth for \(t > 0\) as long as the solution exists. Hence by shifting the initial time, we may assume the initial data is smooth. Then we have the following theorem.

**Theorem 3.1** Assume that \(f\) is a global solution of (1.1) such that the initial rotation number is 1 and the initial (signed) area is positive. Then for each \(\ell \in \mathbb{N} \cup \{-1, 0\}\), there exist \(C_\ell > 0\) and \(\lambda_\ell > 0\) such that

\[
I_\ell(t) \leq C_\ell e^{-\lambda_\ell t}.
\]

**Proof.** We have

\[
\begin{align*}
\frac{dL}{dt} &= -\int_0^L \partial_t f \cdot \kappa \, ds = (-1)^{m+1} \int_0^L (\partial_s^2 \kappa) \, \kappa \, ds = -\frac{1}{L^{2m+1}} I_m, \\
\frac{dA}{dt} &= -\int_0^L \partial_t f \cdot \nu \, ds = (-1)^{m+1} \int_0^L (\partial_s^2 \kappa) \, ds = 0.
\end{align*}
\]

When \(\ell = -1\), we have

\[
\frac{d}{dt} I_{-1} = \frac{d}{dt} \left( -\frac{4\pi A}{L^2} \right) = \frac{8\pi A}{L^2} \frac{dL}{dt} = -\frac{8\pi A}{L^{2(m+2)}} I_m \leq -\lambda_{-1} I_{-1},
\]

where \(\lambda_{-1}\) is a positive constant. Hence, the exponential decay of \(I_{-1}\) follows.

Next we consider the behavior of \(I_0\). Since

\[
\partial_t \kappa = (-1)^m \partial_s^{2m+2} \kappa + (-1)^m \kappa^2 \partial_s^{2m} \kappa,
\]

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we have
\[
\frac{d}{dt} I_0 = \frac{dL}{dt} \int_0^L \kappa^2 \, ds + L \int_0^L 2\kappa \partial_t \kappa \, ds + L \int_0^L \kappa^2 \partial_t (ds)
\]
\[
= - \frac{I_m}{L^{2m+1}} \frac{I_0}{L} + 2L \int_0^L \kappa \left\{ (-1)^m \partial_s^{2m+2} \kappa + (-1)^m \kappa^2 \partial_s^{2m} \kappa + \frac{2\pi dL}{L^2 \, dt} \right\} \, ds
\]
\[
+ (-1)^{m+1} L \int_0^L \kappa^2 \partial_s^{2m} \kappa \, ds
\]
\[
= - \frac{I_0 I_m}{L^{2(m+1)}} - 2L \int_0^L \left( \partial_s^{2m+1} \kappa \right)^2 \, ds + (-1)^m L \int_0^L \kappa \kappa (2\kappa - \kappa) \partial_s^{2m} \kappa \, ds
\]
\[
= - \frac{I_0 I_m}{L^{2(m+1)}} - \frac{1}{L^{2(m+1)}} I_{m+1} + (-1)^m L \int_0^L \kappa \left( \kappa + \frac{2\pi}{L} \right) \left\{ 2 \left( \kappa + \frac{2\pi}{L} \right) - \kappa \right\} \partial_s^{2m} \kappa \, ds.
\]
Hence we find
\[
(3.1) \quad \frac{d}{dt} I_0 + \frac{I_0 I_m}{L^{2(m+1)}} + \frac{2I_{m+1}}{L^{2(m+1)}} = (-1)^m L \int_0^L \left( \kappa^3 + \frac{6\pi}{L} \kappa^2 + \frac{8\pi^2}{L^2} \kappa \right) \partial_s^{2m} \kappa \, ds.
\]
For \( k \in \mathbb{N} \cup \{0\} \) and \( m \in \mathbb{N} \), let \( P_m^k(\kappa) \) be any linear combination of the type
\[
P_m^k(\kappa) = \sum_{i_1 + \ldots + i_m = k} c_{i_1, \ldots, i_m} \partial_s^{i_1} \kappa \cdots \partial_s^{i_m} \kappa
\]
with universal, constant coefficients \( c_{i_1, \ldots, i_m} \). Similarly we define \( P_0^k \) as a universal constant. Then terms on the right-hand side of (3.1) are
\[
(-1)^m L \int_0^L \kappa^3 \partial_s^{2m} \kappa \, ds = L \int_0^L P_m^3(\kappa) \partial_s^{2m} \kappa \, ds,
\]
\[
(-1)^m 6\pi L \int_0^L \kappa^2 \partial_s^{2m} \kappa \, ds = 6\pi \int_0^L P_m^2(\kappa) \partial_s^{2m} \kappa \, ds,
\]
\[
(-1)^m \frac{8\pi^2}{L} \int_0^L \kappa \partial_s^{2m} \kappa \, ds = \frac{8\pi^2}{L} \int_0^L (\partial_s^{2m} \kappa)^2 \, ds = \frac{8\pi^2}{L^2(m+1)} I_m,
\]
by use of integration by parts. We set
\[
J_{k,p} = \left( L^{1+k} p^{-1} \int_0^L |\partial_s^{k+1} \kappa|^p \, ds \right)^{\frac{1}{p}}.
\]
Then we have
\[
J_{n,p} \leq C J_{m,q}^{1-\theta} J_{0,r}^{1-\theta}, \quad \theta = \frac{n - \frac{1}{p} + \frac{1}{r}}{m - \frac{1}{q} + \frac{1}{r}}.
\]
from the Gagliardo-Nirenberg inequalities. Hence we show, by using Young’s inequality,

\[
\left| \frac{L}{L^2(m+1)} \sum_{\ell=0}^{m} \sum_{k+j=\ell} J_{m,4} J_{k,4} J_{j,4} J_{m-k-j,4} \right|
\]

\[
\leq \frac{C}{L^2(m+1)} \sum_{\ell=0}^{m} \sum_{k+j=\ell} J_{m,4} J_{m+1,2} J_{0,2} J_{4(m+1),} J_{4(m+1),} J_{0,2} J_{m+1,2} J_{m+1,2} J_{0,2} J_{m+1,2} J_{0,2}
\]

\[
\leq \frac{C}{L^2(m+1)} \sum_{\ell=0}^{m} \sum_{k+j=\ell} J_{m,4} J_{m+1,2} J_{0,2} J_{4(m+1),} J_{4(m+1),} J_{0,2} J_{m+1,2} J_{m+1,2} J_{0,2}
\]

\[
= \frac{C}{L^2(m+1)} \left( \epsilon I_{m+1} + C_\epsilon I_0^{2m+3} \right)
\]

for any \( \epsilon > 0 \) and appropriate constant \( C_\epsilon \). Similarly, we also have

\[
\left| \frac{6\pi}{L^2(m+1)} \sum_{k=0}^{m} J_{k,3} J_{m-k,3} \right|
\]

\[
\leq \frac{C}{L^2(m+1)} \sum_{k=0}^{m} J_{m,3} J_{m+1,2} J_{0,2} J_{6(m+1),} J_{6(m+1),} J_{0,2} J_{m+1,2} J_{m+1,2} J_{0,2}
\]

\[
= \frac{C}{L^2(m+1)} \left( \epsilon I_{m+1} + C_\epsilon I_0^{2m+3} \right)
\]

Therefore, we have

\[
\frac{d}{dt} I_0 + \frac{I_0 I_m}{L^2(m+1)} + \frac{2 I_{m+1}}{L^2(m+1)} \leq \frac{C}{L^2(m+1)} \left( \epsilon I_{m+1} + C_\epsilon I_0^{2m+3} + C_\epsilon I_0^{2m+5} + I_m \right).
\]

By using Young’s inequality and Theorem 2.2, we obtain

\[
I_0^{\frac{2m+5}{3}} = I_0^{\frac{2m+3}{3}} I_0^2 \leq \epsilon I_0 + C_\epsilon I_0 \leq \epsilon I_0 + C_\epsilon \left( I_{m+1}^{\frac{m+1}{m+2}} I_{m+1}^{\frac{m+1}{m+2}} + I_{m+1}^{\frac{m+1}{m+2}} I_{m+1}^{\frac{m+1}{m+2}} \right)
\]

\[
\leq \epsilon I_0 + C_\epsilon \left( I_{m+1}^{\frac{m+1}{m+2}} + \epsilon' \right) I_{m+1} + C_\epsilon I_{m+1}.
\]
where \( \epsilon, \epsilon' > 0 \) and \( C_{\epsilon} \) and \( C_{\epsilon, \epsilon'} \) are appropriate constants. Similarly, for \( m \geq 1 \), we have

\[
I_m \leq C \left( I_{m+1}^{1/2} I_{m+1}^1 + I_{m+2}^{1/2} I_{m+2}^1 \right) \leq C \left( I_{m+1}^{1/2} + \epsilon \right) I_{m+1} + C\epsilon I_{m+1}.
\]

Taking \( \epsilon, \epsilon' \) sufficiently small, we have

\[
\frac{d}{dt} I_0 + \frac{I_0 I_m}{L^{2(m+1)}} + \frac{C_1 I_{m+1}}{L^{2(m+1)}} \leq \frac{C_2}{L^{2(m+1)}} I_0^{2m+3} + \frac{C_3}{L^{2(m+1)}} e^{-\lambda t}
\]

for sufficiently large \( t \). Since \( \frac{dL}{dt} + I_m L^{2(m+1)+1} = 0 \), we have

\[
\int_0^\infty I_m dt < \infty.
\]

From Wirtinger’s inequality, we obtain

\[
\int_0^\infty I_\ell dt < \infty
\]

for \( \ell \in \{0, \ldots, m\} \). From \((3.2)\) and \( \int_0^\infty I_0 dt < \infty \), we can show

\[
\frac{I_0 I_m}{L^{2(m+1)}} > \frac{C_2}{L^{2(m+1)}} I_0^{2m+3}
\]

for sufficiently large \( t \). Hence we have

\[
I_0 \leq C_0 e^{-\lambda_0 t}.
\]

Next we consider the behavior of \( I_\ell \) for \( \ell \geq 1 \). By direct calculations, we have

\[
\frac{d}{dt} I_\ell = (2\ell + 1) L^{2\ell} \frac{dL}{dt} \int_0^L \left( \partial_s^\ell \tilde{\kappa} \right)^2 ds + 2L^{2\ell+1} \int_0^L \left( \partial_s^\ell \tilde{\kappa} \right)^2 \partial_s (ds) + 2L^{2\ell+1} \int_0^L \left( \partial_s^\ell \tilde{\kappa} \right)^2 \partial_s (ds)
\]

and

\[
(2\ell + 1) L^{2\ell} \frac{dL}{dt} \int_0^L \left( \partial_s^\ell \tilde{\kappa} \right)^2 ds = - (2\ell + 1) L^{2\ell} \int_0^L \left( \partial_s^\ell \tilde{\kappa} \right)^2 ds \int_0^L \left( \partial_s^\ell \tilde{\kappa} \right)^2 ds
\]

\[
= - \frac{2\ell + 1}{L^{2(m+1)}} I_m I_\ell,
\]

\[
L^{2\ell+1} \int_0^L \left( \partial_s^\ell \tilde{\kappa} \right)^2 \partial_s (ds) = (-1)^{m+1} L^{2\ell+1} \int_0^L \left( \partial_s^m \tilde{\kappa} \right)^2 \partial_s \tilde{\kappa} ds
\]

\[
= (-1)^m L^{2\ell+1} \int_0^L \left( \partial_s^\ell \tilde{\kappa} \right) \sum_{n=0}^1 L^{-(1-n)} P_{n+2}^{2m+\ell} (\tilde{\kappa}) ds.
\]


We can show

\[(3.3) \quad \partial_t \partial_s^\ell \kappa = (-1)^m \partial_s^{2m+\ell+2} \kappa + (-1)^m \sum_{n=0}^{2} L^{-(2-n)} P_{n+1}^{2m+\ell}(\kappa)\]

by induction on \(\ell\). Indeed, since

\[\partial_t \kappa = (-1)^m \partial_s^{2m+2} \kappa + (-1)^m \kappa^2 \partial_s^{2m} \kappa,\]

we have

\[\partial_t \partial_s \kappa = \partial_t \partial_s \kappa + (\partial_t f \cdot \kappa) \partial_s \kappa\]

\[= \partial_s \{(-1)^m \partial_s^{2m+2} \kappa + (-1)^m \kappa^2 \partial_s^{2m} \kappa\} + (-1)^m \kappa (\partial_s^{2m} \kappa) \partial_s \kappa\]

\[= (-1)^m \partial_s^{2m+3} \kappa + (-1)^m 2 \kappa \partial_s (\kappa) (\partial_s^{2m} \kappa) + (-1)^m \kappa^2 \partial_s^{2m+1} \kappa + (-1)^m \kappa (\partial_s^{2m} \kappa) \partial_s \kappa\]

\[= (-1)^m \partial_s^{2m+3} \kappa + (-1)^m \sum_{n=0}^{2} L^{-n} P_{n+1}^{2m+1}(\kappa).\]

Hence we obtain (3.3) when \(\ell = 1\). If (3.3) holds for \(\ell \geq 1\), since

\[\partial_t \partial_s^{\ell+1} \kappa = \partial_t \partial_s \partial_s^{\ell} \kappa + (\partial_t f \cdot \kappa) \partial_s^{\ell} \kappa\]

\[= \partial_s \{(-1)^m \partial_s^{2m+\ell+2} \kappa + (-1)^m \sum_{n=0}^{2} L^{-(2-n)} P_{n+1}^{2m+\ell}(\kappa)\} + (-1)^m \kappa (\partial_s^{2m} \kappa) \partial_s (\partial_s^{\ell} \kappa)\]

\[= (-1)^m \partial_s^{2m+\ell+3} \kappa + (-1)^m \sum_{n=0}^{2} L^{-n} P_{n+1}^{2m+\ell+1}(\kappa),\]

we show (3.3) for \(\ell + 1\). Hence we have

\[2L^{2\ell+1} \int_0^L \partial_s \partial_s^{\ell} \kappa \ (\partial_t \partial_s^{\ell} \kappa) \ ds\]

\[= 2L^{2\ell+1} \int_0^L (\partial_s^{\ell+m+1} \kappa)^2 \ ds + (-1)^m 2L^{2\ell+1} \int_0^L (\partial_s^{\ell} \kappa) \sum_{n=0}^{2} L^{-n} P_{n+1}^{2m+\ell}(\kappa) \ ds\]

\[= -\frac{2}{L^{2(m+1)}} I_{m+\ell+1} + (-1)^m 2L^{2\ell+1} \int_0^L (\partial_s^{\ell} \kappa) \sum_{n=0}^{2} L^{-n} P_{n+1}^{2m+\ell}(\kappa) \ ds.\]

Therefore we have

\[\frac{d}{dt} I_{\ell+1} + \frac{2\ell + 1}{L^{2(m+1)}} I_{m+\ell+1} + \frac{2}{L^{2(m+1)}} I_{m+\ell+1} = (-1)^m 2L^{2\ell+1} \int_0^L (\partial_s^{\ell} \kappa) \sum_{n=0}^{2} L^{-n} P_{n+1}^{2m+\ell}(\kappa) \ ds.\]
When \( n = 0 \), after integration by parts \( m \) times, using Theorem 2.2 and Young’s inequality, we have

\[
(-1)^m 2L^{2\ell+1} \int_0^L \left( \partial_s^m \tilde{\kappa} \right) L^{-2} P_{1}^{2m+\ell} (\tilde{\kappa}) \, ds
\]

\[
= cL^{2\ell-1} \int_0^L \left( \partial_s^{m+\ell} \tilde{\kappa} \right)^2 \, ds = \frac{c}{L^{2(m+1)}} I_{m+\ell}
\]

\[
\leq \frac{C}{L^{2(m+1)}} \left( I_{-1}^{\frac{1}{2}} I_{m+\ell+1} + I_{-1}^{\frac{1}{m+\ell+1}} I_{m+\ell+1}^{\frac{m+\ell+1}{m+\ell+2}} \right)
\]

\[
\leq \frac{C}{L^{2(m+1)}} \left\{ \left( I_{-1}^{\frac{1}{2}} + \epsilon \right) I_{m+\ell+1} + C \epsilon I_{-1} \right\}.
\]

When \( n = 1 \), we have

\[
(-1)^m 2L^{2\ell+1} \int_0^L \left( \partial_s^1 \tilde{\kappa} \right) L^{-1} P_{2}^{2m+\ell} (\tilde{\kappa}) \, ds = (-1)^m 2L^{2\ell} \int_0^L \sum_{k=0}^{2m+\ell} c_k \left( \partial_s^k \tilde{\kappa} \right) \left( \partial_s^{k} \tilde{\kappa} \right) \left( \partial_s^{2m+\ell-k} \tilde{\kappa} \right) \, ds.
\]

We set

\[
K_1 = \{ k \in \{0, \ldots, 2m + \ell \} \mid \text{max}\{k, 2m + \ell - k\} > m + \ell \},
\]

\[
K_2 = \{ k \in \{0, \ldots, 2m + \ell \} \mid \text{max}\{k, 2m + \ell - k\} \leq m + \ell \}.
\]

If \( \text{max}\{k, 2m + \ell - k\} > m + \ell \), then \( \text{min}\{k, 2m + \ell - k\} < m + \ell \). When
$k \in K_1$, from integration by parts $\max\{k, 2m + \ell - k\} - m - \ell$ times, we have

\[
(-1)^m 2L^{2\ell} \int_0^L \sum_{k=0}^{2m+\ell} c_k (\partial_s^k \tilde{\kappa}) (\partial_s^{m+\ell-k} \tilde{\kappa}) \, ds
\]

\[
= 2L^{2\ell} \int_0^L \sum_{k \in K_1} (-1)^{\max\{k, 2m+\ell-k\}-\ell} c_k (\partial_s^{m+\ell} \tilde{\kappa}) P_{2m+\ell}^m(\tilde{\kappa}) \, ds
\]

\[
+ (-1)^m 2L^{2\ell} \int_0^L \sum_{k \in K_2} c_k (\partial_s^k \tilde{\kappa}) (\partial_s^{m+\ell} \tilde{\kappa}) (\partial_s^{2m+\ell-k} \tilde{\kappa}) \, ds
\]

\[
\leq \frac{C}{L^{2(m+1)}} \sum_{k \in K_1} \sum_{k' = 0}^{m+\ell} J_{k'-3} J_{m+\ell-k',3} J_{m+\ell,3} + \frac{C}{L^{2(m+1)}} \sum_{k \in K_2} J_{\ell,3} J_{k,3} J_{2m+\ell-k,3}
\]

\[
\leq \frac{C}{L^{2(m+1)}} \sum_{k \in K_1} \sum_{k' = 0}^{m+\ell} \frac{6^{k'+1}}{m+\ell+12} \frac{6^{(m+\ell-k')^2+5}}{J_{m+\ell,0,2}^6} \frac{6^{(m+\ell-k')^1+1}}{J_{m+\ell,0,2}^6} \frac{6^{(m+\ell-k')^0}}{J_{m+\ell,0,2}^6}
\]

\[
+ \frac{C}{L^{2(m+1)}} \sum_{k \in K_2} \frac{6^{k+1}}{m+\ell+12} \frac{6^{m+5}}{J_{m+\ell,0,2}^6} \frac{6^{m+1}}{J_{m+\ell,0,2}^6} \frac{6^{(m+\ell-k)^+5}}{J_{m+\ell,0,2}^6} \frac{6^{(2m+\ell-k)^+1}}{J_{m+\ell,0,2}^6}
\]

\[
\leq \frac{C}{L^{2(m+1)}} \left( \epsilon I_{m+\ell+1} + C_\ell I_0^{2(m+\ell)+5} \right).
\]

When $n = 2$, we have

\[
P_{3}^{2m+\ell}(\tilde{\kappa}) = \sum_{\alpha=0}^{2m+\ell} \sum_{k+j=\alpha} c_{k,j} (\partial_s^k \tilde{\kappa}) (\partial_s^j \tilde{\kappa}) (\partial_s^{2m+\ell-k-j} \tilde{\kappa}).
\]

We set

\[
K_{\alpha,1} = \{(k,j) \mid k + j = \alpha, \max\{k, j, 2m + \ell - k - j\} > m + \ell\},
\]

\[
K_{\alpha,2} = \{(k,j) \mid k + j = \alpha, \max\{k, j, 2m + \ell - k - j\} \leq m + \ell\}.
\]

If $\max\{k, j, 2m + \ell - k - j\} > m + \ell$, the other terms are less than $m + \ell$. When $k \in K_{\alpha,1}$, from integration by parts $\max\{k, j, 2m + \ell - k - j\} - m - \ell$.
times, we have

\[-1)^m 2L^{2\ell+1} \int_0^L \sum_{\alpha=0}^{2m+\ell} \sum_{k+j=\alpha} c_{kj} \left( \partial_s^{k} \kappa \right) \left( \partial_s^{j} \kappa \right) \left( \partial_s^{2m+\ell-k-j} \kappa \right) ds \]

\[= 2L^{2\ell+1} \int_0^L \sum_{\alpha=0}^{2m+\ell} \sum_{(k,j) \in K_{n,1}} (-1)^{\max\{k,j,2m+\ell-k-j\}} c_{kj} \left( \partial_s^{m+\ell} \kappa \right) P_{m+\ell}^{(j,0)} ds \]

\[+ (-1)^m 2L^{2\ell+1} \int_0^L \sum_{\alpha=0}^{2m+\ell} \sum_{(k,j) \in K_{n,2}} (\partial_s^{k} \kappa) \left( \partial_s^{j} \kappa \right) \left( \partial_s^{2m+\ell-k-j} \kappa \right) ds \]

\[\leq \frac{C}{L^{2(m+1)}} \sum_{\alpha=0}^{2m+\ell} \sum_{(k,j) \in K_{n,1}} \sum_{j'=\beta=0}^{m+\ell} J_k J_{k} J_{j} J_{m+\ell-\kappa-j,4} J_{m+\ell,4} \]

\[+ \frac{C}{L^{2(m+1)}} \sum_{\alpha=0}^{2m+\ell} \sum_{(k,j) \in K_{n,2}} \sum_{j'=\beta=0}^{m+\ell} J_k J_{k} J_{j} J_{m+\ell-\kappa-j,4} J_{m+\ell,4} \]

\[= \frac{C}{L^{2(m+1)}} \int_{m+\ell+1}^{2(m+\ell)+1} \int_{m+\ell+1}^{2(m+\ell)+1} \int_{m+\ell+1}^{2(m+\ell)+1} \int_{m+\ell+1}^{2(m+\ell)+1} \int_{m+\ell+1}^{2(m+\ell)+1} \]

\[\leq \frac{C}{L^{2(m+1)}} \left( \epsilon I_{m+\ell+1} + C_4 I_0^{(m+\ell)+3} \right). \]

Taking \( \epsilon > 0 \) sufficiently small, we obtain

\[\frac{d}{dt} I_{\ell} + 2\ell + 1 \int_{2(m+\ell)+1} \int_{m+\ell+1} \int_{m+\ell+1} \int_{m+\ell+1} \int_{m+\ell+1} \leq \frac{C_2}{L^{2(m+1)}} \left( I_{m+\ell+1} + I_0^{2(m+\ell)+5} + \epsilon^{2(m+\ell)+3} \right) \]

for sufficiently large \( t > 0 \). Hence we obtain

\[I_{\ell} \leq C_4 e^{-\lambda t}. \]
Moreover we obtain the next theorem.

**Theorem 3.2** Let $f$ be as in Theorem 3.1, and let $f(s, t) = \sum_{k \in \mathbb{Z}} \hat{f}(k)(t) \varphi_k(s)$ be the Fourier expansion for any fixed $t > 0$. Set

$$c(t) = \frac{1}{\sqrt{L(t)}}(\Re \hat{f}(0)(t), \Im \hat{f}(0)(t)),$$

and define $r(t) \geq 0$ and $\sigma(t) \in \mathbb{R}/2\pi\mathbb{Z}$ by

$$\hat{f}(1)(t) = \sqrt{L(t)}r(t)\exp\left(\frac{2\pi\sigma(t)}{L(t)}\right).$$

Furthermore we set

$$\tilde{f}(\theta, t) = f(L(t)\theta - \sigma(t), t), \quad \text{for } (\theta, t) \in \mathbb{R}/\mathbb{Z} \times [0, \infty).$$

Then the following claims hold.

1. There exists $c_{\infty} \in \mathbb{R}^2$ such that

$$\|c(t) - c_{\infty}\| \leq Ce^{-\gamma t}.$$

2. The function $r(t)$ converges exponentially to the constant $\frac{L_{\infty}}{2\pi}$ as $t \to \infty$:

$$\left|r(t) - \frac{L_{\infty}}{2\pi}\right| \leq Ce^{-\gamma t}.$$

3. There exists $\sigma_{\infty} \in \mathbb{R}/2\pi\mathbb{Z}$ such that

$$|\sigma(t) - \sigma_{\infty}| \leq Ce^{-\gamma t}.$$

4. For any $k \in \mathbb{N} \cup \{0\}$ there exist $C_k > 0$ and $\gamma_k > 0$ such that

$$\|\tilde{f}(\cdot, t) - \tilde{f}_{\infty}\|_{C^k(\mathbb{R}/\mathbb{Z})} \leq C_ke^{-\gamma_k t},$$

where

$$\tilde{f}_{\infty}(\theta) = c_{\infty} + \frac{L_{\infty}}{2\pi}(\cos 2\pi\theta, \sin 2\pi\theta).$$

5. For sufficiently large $t$, $\text{Im} \tilde{f}(\cdot, t)$ is the boundary of a bounded domain $\Omega(t)$. Furthermore, there exists $T_* \geq 0$ such that $\Omega(t)$ is strictly convex for $t \geq T_*$. 

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(6) Let $D_{r_\infty}(c_\infty)$ be the closed disk with center $c_\infty$ and radius $r_\infty$. Then we have
\[ d_H(\Omega(t), D_{r_\infty}(c_\infty)) \leq C e^{-\gamma t}, \]
where $d_H$ is the Hausdorff distance.

(7) Let $b(t) = \frac{1}{A(t)} \int_{\Omega(t)} x \, dx$ be the barycenter of $\Omega(t)$. Then we have
\[ ||A(t)(b(t) - c(t))|| \leq C e^{-\gamma t}. \]

Proof. From Theorem 3.1 we have
\[ \|\tilde{\kappa}\|_{C^\infty} \leq C e^{-\lambda t}. \]
Hence we can show each of the above assertions in the same way as in [9, Theorem 4.3].

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