Generalized Born Infeld gravitational action

Denis Comelli
INFN - Sezione di Ferrara, via Saragat 1, I-35131 Ferrara, Italy
E-mail: comelli@fe.infn.it

Abstract. Non linear gravitational actions obtained generalizing the tensor scalar density \( \sqrt{-g} \det |g_{\alpha\beta}| \) along the line of the Born infeld actions are studied. The requirements that the theory reduces to Einstein-Hilbert action for small curvature, be free of spin two ghosts and free of Coulomb like Schwarschild singularity, selects one effective lagrangian whose dynamics is dictated by the tensors \( g_{\mu\nu} \) and \( R_{\mu\nu\rho\sigma} \) (not \( R_{\mu\nu} \) or the scalar \( R \)).

1. Introduction
There are numerous suggestions in the literature for modification of the classical Einstein Hilbert (EH) action of general relativity. Many modification attempts (at least in 4-dimensional space-time) to add to the Einstein term some scalar functions of the curvature \( R \) and/or of combinations of the Ricci, \( R_{\mu\nu} \), and/or Riemann, \( R_{\mu\nu\alpha\beta} \) tensors

\[
S_{EH} = \frac{m_{Pl}^2}{16\pi} \int d^4x \sqrt{-g} (R - 2\Lambda) \rightarrow \int d^4x \sqrt{-g} \mathcal{L}(R, R_{\mu\nu}, R_{\mu\nu\alpha\beta})
\]

(1)

where \( g = \det |g_{\mu\nu}| \) is the determinant of the metric tensor, \( \Lambda \) the cosmological constant and \( m_{Pl} \) is the Planck mass. Such a form of the Lagrangian density, i.e. a scalar function multiplied by the determinant of metric tensor, is dictated by the demand of invariance of the action with respect to general coordinate transformation. Here we intend to show a generalization of the above procedure that concentrate on the modification of the first density part of the \( S_{EH} \) action, that means the \( \sqrt{-g} \) part:

\[
S_{EH} \rightarrow \int d^4x \sqrt{-g} \det |G_{\mu\nu}(g_{\alpha\beta}, R, R_{\alpha\beta}, R_{\alpha\beta\gamma})|\]

(2)

that can be further generalized extending the concept of determinant to non quadratic matrices\(^1\).

This generalization is obtained using the properties of the Levi Civita tensor \( \epsilon_{\mu\nu\rho\sigma} = \sqrt{-g} e_{\mu\nu\rho\sigma} \), where \( e_{\mu\nu\rho\sigma} \) is the Levi Civita pseudo tensor (or permutation operator, \( e_{0123} = 1 \)), a tensorial density of weight \( w = -1 \) (note that \( e^{\mu\nu\rho\sigma} \) is a tensorial density of weight \( w = 1 \))\(^2\). In this spirit, Born and Infeld (BI) proposed for the electromagnetism the action \([1, 2]\]

\[
\int d^4x \sqrt{-g} \det |g_{\mu\nu} + \lambda F_{\mu\nu}|\]

(3)

\(^1\) The usual determinant definition is: \( \det |T_{\alpha\beta}| = \frac{1}{n!} e^{\mu_1\nu_1\rho_1\sigma_1} e^{\mu_2\nu_2\rho_2\sigma_2} T_{\mu_1\nu_1} T_{\rho_1\sigma_1} T_{\rho_2\sigma_2} T_{\nu_2\mu_2} \).

\(^2\) The weight \( w \) of a tensor density \( T^\nu_\mu \) is defined by the transformation law: \( T^\nu_\mu = |\frac{\partial x^\nu}{\partial x'^\nu} | \frac{\partial x'^\mu}{\partial x^\mu} T^\beta_\beta \).
while, Eddington [3] indicated, as purely affine gravitational action, the term

$$\int d^4x \sqrt{\det|T_{\mu\nu}(\Gamma)|}$$  \hspace{1cm} (4)$$

whose possible generalizations are analyzed in ref.[4].

Taking into account only a purely gravitational theory without any matter content and considering a purely metric approach, in ref.[5], Deser and Gibbons proposed the general covariant action:

$$S = \int d^4x \sqrt{\det|g_{\mu\nu} + \lambda R_{\mu\nu} + X_{\mu\nu}|}$$ \hspace{1cm} (5)$$

where $X_{\mu\nu}$ contains terms of second or higher order in the curvature and formulated the minimum physical request that such a theory should satisfy:

1) Reduction to EH action for small curvature; 2) Ghost freedom; 3) Regularization of some singularities (for example the Coulombian like in the Schwarzschild case); 4) Supersymmetrizability 3.

In ref.[7] there is an extensive analysis of the cosmological behaviors of actions like (5) in Friedmann Robertson Walker (FRW) background.

A straightforward generalization of action (5) can be written in the general form of “determinant-action”:

$$S_{\text{det}} = \int d^4x \sqrt{\det|G_{\mu\nu}(g_{\alpha\beta}, R, R_{\alpha\beta}, R_{\alpha\beta\delta\gamma})|}$$  \hspace{1cm} (6)$$

where $G_{\mu\nu}$ is a two index covariant tensor, combination of $g_{\alpha\beta}$, $R_{\alpha\beta}$ and $R_{\alpha\beta\delta\gamma}$.

Using the properties of the Levi Civita pseudo tensor $e$, we can rewrite the determinant-action (6) as:

$$\int d^4x \sqrt{1/4! e\cdots e G_{\mu_1\nu_1} G_{\mu_2\nu_2} G_{\mu_3\nu_3} G_{\mu_4\nu_4}}$$ \hspace{1cm} (7)$$

where $e\equiv e^{\mu_1\mu_2\mu_3\mu_4} e_{\nu_1\nu_2\nu_3\nu_4}$. This form will be our guideline for the new generalizations of BI type gravity in the next chapter.

2. Generalized Born-Infeld Gravity for more detail see ref.[6]

The first generalization of the action (7) is obtained inserting an arbitrary number $(n)$ of Levi Civita tensors:

$$\int d^4x (e\cdots e G_{\cdots 2n} G_{\cdots 4})^{1/2} \rightarrow \int d^4x (e\cdots e G_{\cdots 2n} G_{\cdots 4})^{1/n}$$ \hspace{1cm} (8)$$

then, also the $G$ tensors, can be taken independents at each insertion:

$$\int d^4x (e\cdots e G_{\cdots 2n} G_{\cdots 4})^{1/n} \rightarrow \int d^4x (e\cdots e G_{\cdots 2n} G_{\cdots 4})^{1/n}$$ \hspace{1cm} (9)$$

We stress that $e^{\mu\nu\rho\sigma}$, being a tensor density of weight $w = -1$ generates scalar densities that need to be corrected taking the appropriate power $1/n$ for the full expression. This allow us also

3 This requirement is quite stringent and is probably implemented if gravity descends from String/M-Theory [8, 10, 2].
to introduce directly into the lagrangian density higher index tensors, like $R_{\mu\nu\rho\sigma}$, generalizing the determinant operation to tensors with more than two indices.

A generic term in $d$-dimensions, having $n$ times $\epsilon^{\mu\nu\sigma}$, $r$ times $g_{\mu\nu}$, $s$ times $R_{\mu\nu}$, and $t$ times $R_{\mu\nu\rho\sigma}$ reads $^4$

$$\int d^4x M^{d-2(n+s)+2} \left( \epsilon^{\mu\nu\rho\sigma} g_{\mu\nu} g_{\nu\rho} R_{\rho\sigma} \right)^{1/n}$$

(10)

where the “conservation of the number of indices” requires $d n = 2 r + 2 s + 4 t$ and $M$ is a mass scale. The range of variations of $t$ is $0 \leq t \leq d n/4$, (when $s + r = d n/2$ and $s = r = 0$ respectively) and the mass scale coefficient varies from $M^{d-2s}$ to $M^2$.

For the rest, we will reduce our analysis to the simplest case: $d = 4$ and $n = 2$ so that the counting rule of eq.(10) fix the structure

$$\int d^4x M^{4-t-s} \left( \epsilon^{\mu\nu\rho\sigma} g_{\mu\nu} g_{\nu\rho} R_{\rho\sigma} \right)^{1/2}$$

(11)

from which we can generate the following operators $^6$:

$$\epsilon\epsilon g_{\mu_1\nu_1} g_{\mu_2\nu_2} g_{\mu_3\nu_3} g_{\mu_4\nu_4} = 4! g; \quad \epsilon\epsilon g_{\mu_1\nu_1} g_{\mu_2\nu_2} R_{\mu_3\mu_4\nu_3\nu_4} = 4 g R;$$

$$\epsilon\epsilon g_{\mu_1\nu_1} g_{\mu_2\nu_2} g_{\mu_3\nu_3} R_{\mu_4\nu_4} = 6 g R; \quad \epsilon\epsilon g_{\mu_1\nu_1} R_{\mu_2\nu_2} R_{\mu_3\mu_4\nu_3\nu_4} = 2 g (R^2 - 2[R^2]);$$

$$\epsilon\epsilon R_{\mu_1\nu_1} R_{\mu_2\nu_2} R_{\mu_3\mu_4\nu_3\nu_4} = \frac{9}{4} (R^2 - 4[R^2] + R^2); \quad \epsilon\epsilon g_{\mu_1\nu_1} g_{\mu_2\nu_2} R_{\mu_3\mu_4\nu_3\nu_4} = 2 g (R^2 - [R^2]);$$

$$\epsilon\epsilon R_{\mu_1\nu_1} R_{\mu_2\nu_2} R_{\mu_3\mu_4\nu_3\nu_4} = 4 g ([R R R] + \frac{1}{2} R^3 - \frac{5}{2} R [R^2] + 2 [R^3]);$$

$$\epsilon\epsilon R_{\mu_1\nu_1} R_{\mu_2\nu_2} R_{\mu_3\mu_4} R_{\nu_3\nu_4} = 4 g (R^3 - 3 R [R^2] + 2 [R^3]);$$

$$\epsilon\epsilon R_{\mu_1\nu_1} R_{\mu_2\nu_2} R_{\mu_3\mu_4\nu_3\nu_4} = 4! \det ||R_{\alpha\beta}||$$

where $[R^2] \equiv R^{\mu\nu} R_{\mu\nu}, \ [R R R] \equiv R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} R^{\rho\sigma}$ and the combination $(R^2 - 4[R^2] + R^2) \equiv \mathcal{E}$ is the Gauss-Bonnet term.

In order to get a phenomenological acceptable model we will follow the usual attitude to cancel “by hand” the cosmological constant with the following trick

$$\int d^4x \sqrt{-g} M^4 \left( \sqrt{1 + \frac{\alpha}{M^2} R + \frac{1}{M^4} \ldots \ldots} - 1 \right)$$

(13)

with no justification from the symmetry point of view.

The existence of some guiding symmetry principle to build our action will strongly improve the prediction of our approach. We know for example that the effective gravity lagrangians derived by String Theory to the fourth order in curvature expansion are ghost free [14]. This it means that in second curvature order we have the Gauss Bonnet combination $\mathcal{E} = R^2 - 4[R^2] + R^2$. In higher order we find cubic corrections for bosonic string theories while the supersymmetric extensions predict zero cubic and non zero quartic corrections [15]. Because string inspired effective lagrangians are computed evaluating on shell graviton amplitudes ($R_{\mu\nu} = 0$), the terms which contains at least twice the Ricci tensor or the Ricci scalar are non properly included.

$^4$ We assume a sort of minimal dimensional analysis not considering operators that saturate indices between them self, as $R_{\mu\nu\rho\sigma} R^{\rho\sigma}_{\mu\nu}$ and many others. Each single tensor will saturate the respective indices only with the Levi Civita pseudo tensors.
Supersymmetric BI type generalizations of Weyl supergravity action are given in [10]. Since there is no decisive hint about the correct string model, we will attempt a phenomenological approach guessing some ad hoc principles and testing the possible implications.

Here we will introduce a sort of “selection rule” such that only some tensors generate the gravitational dynamics.

If for example, only the tensor $\mathcal{R}_{\mu\nu\rho\sigma}$ ($t = 2$) is present, the lagrangian results:

$$\int d^4x \sqrt{-g} M^2 \sqrt{R^2 - 4|R|^2 + R^2}$$

(14)

In the case with only the tensors $g_{\mu\nu}$ and $\mathcal{R}_{\mu\nu\rho\sigma}$ we get:

$$\int d^4x \sqrt{-g} M^4 \sqrt{1 + \frac{\alpha''}{M^2} R + \frac{\beta''}{M^4} E}$$

(15)

and it fits exactly the constraints for ghost freedom and the EH asymptotic (see later).

While in the opposite case where no $\mathcal{R}_{\mu\nu\rho\sigma}$ can enter we have the lagrangian:

$$\int d^4x \sqrt{-g} M^4 \sqrt{1 + \frac{\alpha}{M^2} R + \frac{\beta}{M^4} (R^2 - 3R|R|^2 + 2|R|^3) + \frac{\gamma}{M^8} \frac{\det |R_{\mu\nu}|}{g}}$$

where ghosts show up. When only $\mathcal{R}_{\mu\nu\rho\sigma}$ and $R_{\mu\nu}$ tensors are present :

$$\int d^4x \sqrt{-g} \sqrt{M^4 (E) + M^2 (...) + \gamma \frac{\det |R_{\mu\nu}|}{g}}$$

(16)

Finally we have only one operator with $R_{\mu\nu}$ tensors:

$$\int d^4x \sqrt{\det |R_{\mu\nu}|}$$

(17)

with no EH matching.

The only phenomenologically interesting lagrangian coming from the above selection rules is certainly [6]:

$$\int d^4x \sqrt{-g} M^4 \left(1 - \sqrt{1 - \frac{\alpha R}{M^2} + \frac{\beta}{M^4} (R^2 - 4|R|^2 + R^2)}\right)$$

(18)

where we have applied the cancellation mechanism of cosmological constant.

It is clear that, dis-carting an extremely large $\alpha$ coefficient, the scale of the Cosmological constant is automatically $m_{pl}$ and the only way to enter in a small curvature regime, without falling directly in a de sitter planckian era, is to invoke a cancellation mechanism which, in principle, can be isolated in the gravity-matter interaction side. In any case, at our presentation level, we choose such a simplified cancellation procedure in order to have Minkowski space as asymptotic solution without taking into account the cosmological measurements which give the present period in an accelerated phase. Generation of such a behaviors can be related in various ways always to gravity-matter interactions.

At this point, we will reanalyze more carefully the above lagrangian to the light of the physical criteria suggested by Deser and Gibbons in [5].

\footnote{Note that the expression under square root corresponds to the Lovelock lagrangian in four dimensions [12].}
The small curvature limit of eq.(18) results:
\[
\int d^4 x \sqrt{-g} \left( \frac{\alpha M^2}{2} R + \frac{\alpha^2}{8} R^2 - \frac{\beta}{2} (R^2 - 4 |R|^2 + R^2) + \frac{\alpha R}{16M^2} \left( (\alpha^2 - 4\beta)R^2 - 4\beta(R^2 - 4 |R|^2) \right) + \ldots \right) = \int d^4 x \sqrt{-g} \frac{m^2_{Pl}}{16\pi} \left( R + \frac{1}{6 m_0^2} R^2 + \ldots \right)
\]
\[(19)\]

So, the requirement of a correct EH leading gravitational operator fix \( \alpha M^2 = \frac{m^2_{Pl}}{16\pi} \) while the coefficient of the \( R^2 \) operator results \( \frac{\alpha^2}{8} = \frac{m^4_{Pl}}{512\pi^2 M^2} \) generating an extra scalar degree of freedom with mass \( m_0 = \sqrt{\frac{16\pi^3}{3 m^2_{Pl}} M^2} \). The exchange of such a scalar between two test particles changes the \( 1/r \) static gravitation potential slope to \( \frac{1}{r} \left( 1 + \frac{1}{3} e^{-m_0 r} \right) \). Using the results of ref.[11] with the strength parameter equal to \( 1/3 \) we can obtain a lower bound on \( m_0 \) of \( \sim 2 \times 10^{-2} eV \) corresponding to a value for the mass parameter \( M \geq 250 \) GeV.

The general analysis of the particle content in higher derivative lagrangians of the form
\[
\int d^4 x \sqrt{-g} \ F[R, |R|^2, R^2]
\]
shows the existence of massless gravitons plus new degrees of freedom. In general there is a massive spin zero field \( (m_0 \) mass) and a massive spin two field \( (m_2 \) mass) with a wrong sign of kinetic term: a ghost fields. Due to the fact that the mass of such particles and their potential ghostlike may be very different around different vacuum states we give the full set of eqs that fix such a parameters around solutions characterized by a constant curvature \( R = R_0 \) in a maximally symmetric background [13], \( R_{\mu\nu\rho\sigma} = \frac{R_0}{12} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \).

The equations of motion that fix \( R_0 \) are:
\[
F - \frac{R}{2} F_R - \frac{R^2}{4} F_P - \frac{R^2}{6} F_Q \bigg|_{R_0} = 0
\]
\[(21)\]

where \( P \equiv |R|^2 \) and \( Q \equiv R^2 \) and \( F_R = \frac{\partial F}{\partial R}, F_P = \frac{\partial F}{\partial P}, F_Q = \frac{\partial F}{\partial Q}, \) \( F_{PP} = \frac{\partial^2 F}{\partial P^2} \) and so on.

To each solution of (21) it corresponds two extra degrees of freedom scalar and tensorial with mass \( m_0 \) and \( m_2 \) given by:
\[
\frac{m^2_{Pl}}{96\pi m_0^2} = \frac{1}{2} F_{RR} + \frac{1}{3} F_P + F_Q + \frac{R}{6} (3F_{RP} + 2F_{RQ}) + R^2 \left( \frac{1}{8} F_{PP} + \frac{1}{6} F_{PQ} + \frac{1}{18} F_{QQ} \right) \bigg|_{R_0}
\]
\[(22)\]

\[
-\frac{m^2_{Pl}}{32\pi m_2^2} = \frac{1}{2} F_P + 2F_Q \bigg|_{R_0}
\]
\[(23)\]

where we always take \( R = R_0 \) and \( P = \frac{R_0^2}{4}, \) \( Q = \frac{R_0^3}{6}. \)

A ghost free spectrum (as required for example from string theory [14]) is realized when \( m_2 \to \infty \) and this request is automatically satisfied for lagrangians of the form
\[
F[R, R^2 - 4 |R|^2] = f[R, E]
\]
\[(24)\]

that fit the structure of eq.(18).
For this particular lagrangian (18) we have two possible background solutions: one flat background \( R_0 = 0 \) with zero cosmological constant, that corresponds to the small curvature EH limit previously described; one with \( R_0 \neq 0 \) with an extra scalar degree of mass \( m_0^2 \sim \frac{25\pi m^4}{3m_{Pl}^4} + O(\alpha) \).

Many generalizations of EH action induce corrections to Schwarzschild metric which could have interesting consequences. In the class of Born-Infeld action there are some studies about spherically symmetric Schwarzschild solutions (see [8, 9]). Here we are not interested in a full analysis of new solutions because we can reduce our action to the one studied in [9] in some portion of parameter space.

Following the lines of [9], we can neglect, in the action, terms proportional to \( R \) and \( R_{\mu\nu} \), due to that fact that we are looking for solutions similar to the Schwarzschild one (\( R_{\mu\nu} \sim 0 \)). Only the presence of terms proportional to the Weyl tensor can, in principle, remove the black hole singularity at the origin. This observation reduce our action to

\[
\int d^4x \sqrt{-g} M^4 \left( 1 - \sqrt{1 + \frac{\beta}{M^4} R^2} \right)
\]

(25)

and this exact form is studied in chapter four of [9] (see there for details). The main results in [9] are the existence of solutions which behave asymptotically as black holes and becomes spaces of constant \( R^2 \) at small radii. In some portion of parameters space there is not even an event horizon with the presence of a bare mass instead of a black hole (bare in the sense that is not hidden behind an event horizon) without a naked singularity.

References
[1] M. Born and L. Infeld, Proc. Roy. Soc. Lond. A 144 (1934) 425.
[2] J. H. Schwarz, Preprint arXiv:hep-th/0103165. S. V. Ketov, Preprint arXiv:hep-th/0108189; G. A. Silva, Preprint arXiv:hep-th/0012267.
[3] A.S. Eddington, The Mathematical Theory of Gravity (CUP 1924).
[4] D. N. Vollick, Phys. Rev. D 69 (2004) 064030.
[5] S. Deser and G. W. Gibbons, Class. Quant. Grav. 15 (1998) L35.
[6] D. Comelli, Phys. Rev. D 72 (2005) 064018.
[7] D. Comelli and A. Dolgov, JHEP 0411 (2004) 062.
[8] M. N. R. Wohlfarth, Class. Quant. Grav. 21 (2004) 19.27 [Erratum-ibid. 21 (2004) 5297].
[9] J. A. Feigenbaum, P. G. O. Freund and M. Pigli, Phys. Rev. D 57 (1998) 4738; J. A. Feigenbaum, Phys. Rev. D 58 (1998) 124023.
[10] S. J. J. Gates and S. V. Ketov, Class. Quant. Grav. 18 (2001) 3561; S. M. Kuzenko and S. A. McCarthy, JHEP 0302 (2003) 038.
[11] C. D. Hoyle, D. J. Kapner, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, U. Schmidt and H. E. Swanson, Phys. Rev. D 70 (2004) 042004.
[12] D. Lovelock, J. Math. Phys. 12 (1971) 498.
[13] D. Wands, Class. Quant. Grav. 11 (1994) 269; A. Hindawi, B. A. Ovrut and D. Waldram, Phys. Rev. D 53 (1996) 5597; A. Nunez and S. Solganik, Phys. Lett. B 608 (2005) 189; T. Chiba, JCAP 0503 (2005) 008.
[14] B. Zwiebach, Phys. Lett. B 156 (1985) 315; D. G. Boulware and S. Deser, Phys. Rev. Lett. 55 (1985) 2656; S. Deser and A. N. Redlich, Phys. Lett. B 176 (1986) 350; I. Jack, D. R. T. Jones and A. M. Lawrence, Phys. Lett. B 203 (1988) 378;
[15] D. J. Gross and E. Witten, Nucl. Phys. B 277 (1986) 1; R. R. Metsaev and A. A. Tseytlin, Phys. Lett. B 185 (1987) 52; E. A. Bergshoeff and M. de Roo, Nucl. Phys. B 328 (1989) 439; M. de Roo, H. Suelmann and A. Wiedemann, Phys. Lett. B 280 (1992) 39.