THE SO(3) MONOPOLE COBORDISM AND SUPERCONFORMAL SIMPLE TYPE

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Abstract. We show that the SO(3) monopole cobordism formula from [8] implies that all smooth, closed, oriented four-manifolds with \( b^1 = 0 \) and \( b^+ \geq 3 \) and odd with Seiberg-Witten simple type satisfy the superconformal simple type condition defined by Mariño, Moore, and Peradze, [29, 28]. This implies the lower bound, conjectured by Fintushel and Stern [19], on the number of Seiberg-Witten basic classes in terms of topological data.

1. Introduction

For a closed, four-manifold, \( X \), we will use the characteristic numbers,
\[
(1.1) \quad c_1^2(X) := 2e(X) + 3\sigma(X), \quad \chi_h(X) := (e(X) + \sigma(X))/4, \quad c(X) := \chi_h(X) - c_1^2(X),
\]
where \( e(X) \) and \( \sigma(X) \) are the Euler characteristic and signature of \( X \). We call a four-manifold standard if it is closed, connected, oriented, and smooth with \( b^+ \geq 3 \) and odd and \( b^1(X) = 0 \). We will write \( Q_X \) for the intersection form of \( X \) on both \( H_2(X; \mathbb{Z}) \) and \( H_2(X; \mathbb{Z}) \) as in [22, Definition 1.2.1].

For a standard four-manifold, \( X \), the Seiberg-Witten invariants define a function, \( SW_X : \text{Spin}^c(X) \to \mathbb{Z} \), on the set of spin\(^c\) structures on \( X \). The Seiberg-Witten basic classes of \( X \), \( B(X) \), are the image under \( c_1 : \text{Spin}^c(X) \to H^2(X; \mathbb{Z}) \) of the support of \( SW_X \). A manifold \( X \) has Seiberg-Witten simple type if \( K^2 = c_1^2(X) \) for all \( K \in B(X) \). All known standard four-manifolds have Seiberg-Witten simple type (see [27, Conjecture 1.6.2]).

Following [28, 29], one says that a standard four-manifold, \( X \), has superconformal simple type if \( c(X) \leq 3 \) or for \( w \in H^2(X; \mathbb{Z}) \) characteristic and \( c(X) \geq 4 \),
\[
(1.2) \quad SW_X^{w,i}(h) := \sum_{s \in \text{Spin}^c(X)} (-1)^{(w^2 + c_1(s) \cdot w)} SW_X(s) \langle c_1(s), h \rangle^i = 0, \quad \text{for } i \leq c(X) - 4,
\]
and all \( h \in H_2(X; \mathbb{R}) \). Mariño, Moore, and Peradze conjectured that all standard four-manifolds satisfy this condition [29, Conjecture 7.8.1].

In [6], we showed that if \( X \) was abundant in the sense that \( B(X)^\perp \) (the orthogonal complement with respect to \( Q_X \)) contained a hyperbolic summand, then \( X \) had superconformal simple type. In this article, we establish the following.

**Theorem 1.1.** Let \( X \) be a standard four-manifold of Seiberg-Witten simple type and assume Hypothesis [2, 5]. Then \( X \) has superconformal simple type.

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In [8], we proved the required SO(3)-monopole link-pairing formula, restated in this article as Theorem 2.6, assuming the validity of certain technical properties — comprising Hypothesis 2.5 and described in more detail in Remark 2.8 — of the local gluing maps for SO(3) monopoles constructed in [10]. A proof of the required local SO(3)-monopole gluing-map properties, which may be expected from known properties of local gluing maps for anti-self-dual SO(3) connections and Seiberg-Witten monopoles, is currently being developed by the authors [9].

One might draw a comparison between our use of the SO(3)-monopole link-pairing formula in our proof of Theorem 1.1 and Göttsche’s assumption of the validity of the Kotschick-Morgan Conjecture [25] in his proof [23] of the wall-crossing formula for Donaldson invariants. However, such a comparison overlooks the fact that our assumption of certain properties for local SO(3)-monopole gluing maps is narrower and more specific. Indeed, our monograph [8] effectively contains a proof of the Kotschick-Morgan Conjecture, modulo the assumption of certain technical properties for local gluing maps for anti-self-dual SO(3) connections which extend previous results of Taubes [39, 40, 41], Donaldson and Kronheimer [4], and Morgan and Mrowka [32, 33]. Our proof of Theorem 2.6 in [8] relies on our construction of a global gluing map for SO(3) monopoles and that in turn builds on properties of local gluing maps for SO(3) monopoles; the analogous comments apply to the proof of the Kotschick-Morgan Conjecture.

1.1. Background and applications. In [28, 29], Mariño, Moore, and Peradze originally defined the concept of superconformal simple type in the context of supersymmetric quantum field theory. With those methods, they argued that a four-manifold satisfying the superconformal simple type condition also satisfied the vanishing result for low degree terms of the Seiberg-Witten series given in (1.2). Because of the applications of (1.2) described here, we use (1.2) as the definition of superconformal simple type. Not only do all known examples of four-manifolds satisfy this definition, but the condition is preserved under the standard surgery operations (blow-up, torus sum, and rational blow-down) used to construct new examples (see [29, Section 7]). The article [6] establishes that abundant four-manifolds have superconformal simple type, but also provides an example of a non-abundant four-manifold which still has superconformal simple type. Hence, the results established here are strictly stronger than those of [6].

Mochizuki [30] proved a formula (see Theorem 4.1 in [24]) which expresses the Donaldson invariants of a complex projective surface in a form similar to that given by the SO(3)-monopole cobordism formula [8, Theorem 0.0.1], but with coefficients given as the residues of an explicit C*-equivariant integral over the product of Hilbert schemes of points on X. In [24], Göttsche, Nakajima, and Yoshioka showed how Witten’s Conjecture (given here as Conjecture 1.3) followed from Mochizuki’s formula. In addition, they conjectured [24, Conjecture 4.5] that Mochizuki’s formula (and hence their proof of Witten’s Conjecture) holds for all standard four-manifolds and not just complex projective surfaces. Their [24, Proposition 8.9] shows that all four-manifolds satisfying Mochizuki’s formula have superconformal simple type. The development in [24] relies on Mochizuki’s formula for the Donaldson invariant and that is conjectured in [24] to be equivalent to the version of the SO(3)-monopole cobordism formula given in [8, Theorem 0.0.1]. In contrast, this article uses a version of the SO(3)-monopole cobordism formula, Theorem 3.2, which does not involve
the Donaldson invariant and so the two proofs are quite different. Using Mochizuki’s techniques from \cite{30} to find an equation similar to that in Theorem \cite{5}, discovering what that equation would imply about its coefficients poses an interesting question for future research.

The superconformal simple type condition is not only relevant to physics and algebraic geometry. Using the vanishing condition \cite{12} as a definition, in \cite{29} Theorem 8.1.1], Mariño, Moore, and Peradze rigorously derived a lower bound on the number of basic classes for manifolds of superconformal simple type. Theorem \cite{11} and \cite{29} Theorem 8.1.1] therefore yield a proof of the following result, first conjectured by Fintushel and Stern \cite{19}.

**Corollary 1.2.** Let $X$ be a standard four-manifold of Seiberg-Witten simple type. If $B(X)$ is non-empty and $c(X) \geq 3$, then the SO$(3)$-monopole link-pairing formula (Theorem \cite{2.7}) implies that $|B(X)/\{\pm 1\}| \geq c(X)/2$.

Theorem \cite{11} also completes a proof of the derivation of Witten’s Conjecture relating Donaldson and Seiberg-Witten invariants from the SO$(3)$-monopole cobordism formula of \cite{8}. In \cite{43}, Witten conjectured the following relation between the Seiberg-Witten and Donaldson invariants (see \cite{12} Lemma 2.8] for this equivalent form of the conjecture).

**Conjecture 1.3** (Witten’s Conjecture) \cite{43} Let $X$ be a standard four-manifold with Seiberg-Witten simple type. Then for any $w \in H^2(X; \mathbb{Z})$, $h \in H_2(X; \mathbb{R})$, and positive generator $x \in H_0(X; \mathbb{Z})$, the Donaldson invariant, $D^w_X$, as defined in \cite{3, 26} satisfies

$$D^w_X(h^{\delta-2m} x^m) = 2^{2-c(X)} \sum_{i+2k = \delta-2m} \frac{(\delta-2m)!}{2^{k-m} k! i!} SW^w_X(h) Q_X(h)^k,$$

(1.3)

when $\delta$ is a non-negative integer obeying $\delta \equiv -w^2 - 3\chi_h(X) \pmod{4}$.

By definition, $D^w_X(h^{\delta-2m} x^m) = 0$ when $\delta$ is a non-negative integer that does not obey $\delta \equiv -w^2 - 3\chi_h(X) \pmod{4}$. In \cite{8}, using the moduli space of SO$(3)$ monopoles defined by Pidstrigatch and Tyurin \cite{37} for this purpose and assuming the technical properties for local SO$(3)$-monopole gluing maps described in Hypothesis \cite{2.5} we proved the SO$(3)$-monopole cobordism formula, which expresses the Donaldson polynomial $D^w_X$ of \cite{3, 26} as a polynomial in the Seiberg-Witten polynomials in \cite{12}, the intersection form, $Q_X$, and an additional cohomology class $\Lambda$ on $H_2(X; \mathbb{R})$,

$$D^w_X(h^{\delta-2m} x^m) = \sum_{i+j+2k = \delta-2m} a_{i,j,k} SW_X(K) (K, h)^i (\Lambda, h)^j Q_X(h)^k,$$

(1.4)

where the real coefficients, $a_{i,j,k}$, are unknown but depend only on homotopy invariants of the manifold. It became apparent in \cite{6, 12} that superconformal simple type functioned as an obstruction to determining these coefficients. That is, because the Seiberg-Witten polynomials $SW^w_X$ vanish when $i \leq c(X) - 3$ for all known examples, we could not use examples where Witten’s Conjecture held to determine the coefficients $a_{i,j,k}$ with $i < c(X) - 3$. However, in \cite{11}, we showed that while we could not determine the coefficients $a_{i,j,k}$ with $i < c(X) - 3$, we could show that they satisfied a difference equation and by combining the superconformal simple type condition with this difference equation, we could derive Witten’s Conjecture from \cite{1.3}. Thus, Theorem \cite{11} and the results of \cite{11} give the following
Corollary 1.4. Let $X$ be a standard four-manifold and assume Hypothesis 2.5. Then Witten’s Conjecture 1.3 holds.

Recall that Hypothesis 2.5 refers to certain expected properties for local gluing maps for SO(3) monopoles.

1.2. Outline. After reviewing definitions and basic properties of the Seiberg-Witten invariants in Section 2.1, we introduce the moduli space of SO(3) monopoles in Section 2.2 and review results from [13, 14, 15, 8] on the monopole cobordism formula. We consider a particular case of this formula in Section 3 to get, in Theorem 3.3, a form of the cobordism formula where the pairing with the link of the moduli space of anti-self-dual connections vanishes by a dimension-counting argument. This cobordism formula then states that a sum over $K \in B(X)$ of pairings with links of the Seiberg-Witten moduli space corresponding to $K$ vanishes, giving an equality of the form (see (3.6))

$$0 = \sum_{k=0}^{\ell} a_{c-2v+2k,0,\ell-k} SW_{X}^{w,c-2v+2k} \ell^{-k} X,$$

where we abbreviate $c = c(X)$. In Section 4 we show that the coefficient $a_{c-2v,0,\ell}$ appearing in (1.5) is non-zero by applying the methods used in [25] to the topological description of the link of the Seiberg-Witten moduli space given in [8]. We show that the coefficients $a_{c-2v+2k,0,\ell-k}$ in (1.5) vanish if $c - 2v + 2k \geq c - 3$ in Section 5. In Section 6 we combine this information on the coefficients and give an inductive argument proving Theorem 1.1.

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2. Preliminaries

2.1. Seiberg-Witten invariants. Detailed expositions of the Seiberg-Witten invariants, introduced by Witten in [43], are provided in [27, 31, 34]. These invariants define a map with finite support,

$$SW_{X} : \text{Spin}^{c}(X) \to \mathbb{Z},$$

from the set of spin$^{c}$ structures on $X$. A spin$^{c}$ structure $s = (W^{\pm}, \rho_{W})$ on $X$ consists of a pair of complex rank-two vector bundles, $W^{\pm} \to X$, and a Clifford multiplication map, $\rho_{W} : T^{*}X \to \text{Hom}_{\mathbb{C}}(W^{+},W^{-})$. If $s \in \text{Spin}^{c}(X)$, then $c_{1}(s) := c_{1}(W^{+}) \in H^{2}(X;\mathbb{Z})$ is characteristic.

One calls $c_{1}(s)$ a Seiberg-Witten basic class if $SW_{X}(s) \neq 0$. Define

$$B(X) := \{ c_{1}(s) : SW_{X}(s) \neq 0 \}.$$

If $H^{2}(X;\mathbb{Z})$ has 2-torsion, then $c_{1} : \text{Spin}^{c}(X) \to H^{2}(X;\mathbb{Z})$ is not injective. Because we will work with functions involving real homology and cohomology, we define

$$SW'_{X} : H^{2}(X;\mathbb{Z}) \to \mathbb{Z}, \quad K \mapsto \sum_{s \in c_{1}^{-1}(K)} SW_{X}(s).$$
Thus, we can rewrite the expression for $SW_X^{u,i}(h)$ in (1.2) as

\begin{equation}
SW_X^{u,i}(h) = \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w^2 + w - K)} SW^X_s(K, h)^i.
\end{equation}

A four-manifold, $X$, has Seiberg-Witten simple type if $SW_X(s) \neq 0$ implies that $c_1(s)^2 = c_1^2(X)$.

2.2. SO(3) monopoles. We now review the basic definitions and results on the moduli space of SO(3) monopoles. More detailed discussions of these results can be found in \cite{14,15}.

2.2.1. Spin\textsuperscript{u} structures. A spin\textsuperscript{u} structure $t = (V^\pm, \rho)$ on a four-manifold, $X$, is a pair of complex rank-four vector bundles, $V^\pm \rightarrow X$, with a Clifford module structure, $\rho : T^*X \rightarrow \text{Hom}_C(V^+, V^-)$. In more familiar terms, for a spin\textsuperscript{c} structure $\mathfrak{s} = (W^\pm, \rho_W)$ on $X$, a spin\textsuperscript{u} structure is given by $V^\pm = W^\pm \otimes E$, where $E \rightarrow X$ is a complex rank-two vector bundle and the Clifford multiplication map is given by $\rho = \rho_W \otimes \text{id}_E$. We define characteristic numbers of a spin\textsuperscript{u} structure $t = (W^\pm \otimes E, \rho)$ by

$$p_1(t) := p_1(\mathfrak{su}(E)), \quad c_1(t) := c_1(W^+) + c_1(E), \quad w_2(t) := c_1(E) \pmod{2}.$$

**Lemma 2.1.** Let $X$ be a standard four-manifold. Given $\phi \in H^4(X; \mathbb{Z})$, $\Lambda \in H^2(X; \mathbb{Z})$, and $\omega \in H^2(X; \mathbb{Z}/2\mathbb{Z})$, there is a spin\textsuperscript{u} structure $t$ on $X$ with $p_1(t) = \phi$, $c_1(t) = \Lambda$, and $w_2(t) = \omega$ if and only if:

1. There is a class $w \in H^2(X; \mathbb{Z})$ with $\omega = w \pmod{2}$,
2. $\Lambda \equiv w + w_2(X) \pmod{2}$,
3. $\phi \equiv w^2 \pmod{4}$.

**Proof.** Given $(\phi, \Lambda, \omega)$ and $w$ satisfying the three conditions above, we observe that $\Lambda - w$ is characteristic so there is a spin\textsuperscript{c} structure $\mathfrak{s} = (W^\pm, \rho_W)$ with $c_1(s) = \Lambda - w$. Let $E \rightarrow X$ be the rank-two complex vector bundle with $c_1(E) = w$ and $c_2(E) = (w^2 - \phi)/4$. Define $t$ by $V^\pm = W^\pm \otimes E$ and $\rho = \rho_W \otimes \text{id}_E$. Observe that $p_1(\mathfrak{su}(E)) = c_1(E)^2 - 4c_2(E) = w^2 - w^2 + \phi = \phi$ and $w_2(t) = w_2(\mathfrak{su}(E)) \equiv c_1(E) \equiv \omega \pmod{2}$, while $c_1(t) = c_1(E) + c_1(s) = \omega + \Lambda - w = \Lambda$, as required.

Given a spin\textsuperscript{u} structure $t$, these properties of its characteristic classes follow from easy computations.

2.2.2. The moduli space of SO(3) monopoles and fixed points of a circle action. For a spin\textsuperscript{u} structure $t = (W^\pm \otimes E, \rho)$ on $X$, the moduli space of SO(3) monopoles on $t$ is the space of solutions, modulo gauge equivalence, to the SO(3)-monopole equations (namely, \cite{13} Equation (1.1)) or \cite{14} Equation (2.32)) for a pair $(A, \Phi)$ where $A$ is a unitary connection on $E$ and $\Phi \in \Omega^0(V^+)$. We write this moduli space as $\mathcal{M}_t$.

Complex scalar multiplication on the section, $\Phi$, defines an $S^1$ action on $\mathcal{M}_t$ with stabilizer $\{\pm 1\}$ away from two families of fixed point sets: (1) zero-section points, $[A, 0]$, and (2) reducible points, $[A, \Phi]$, where $A$ is reducible.

By \cite{14} Section 3.2, the subspace of zero-section points is a manifold with a natural smooth structure diffeomorphic to the moduli space of anti-self-dual connections on $\mathfrak{su}(E)$ which we denote, following the notation of \cite{20}, by $M_\kappa^u$ where $\kappa = -p_1(t)/4$ and $w = c_1(E)$. 
By [14, Lemma 2.17], the subspace of reducible points where $A$ is reducible with respect to a splitting $V = (W \otimes L_1) \oplus (W \otimes L_2)$ is a manifold, $M_\kappa$, which is compactly cobordant to the moduli space of Seiberg-Witten monopoles associated with the spin$^c$ structure $s$ where $c_1(s) = c_1(W^+ \otimes L_1)$. By [15, Lemma 3.32], the possible splittings of $t$ are given by

$$\text{Red}(t) = \{ s \in \text{Spin}^c : (c_1(s) - c_1(t))^2 = p_1(t) \}.$$ 

Hence, the subspace of reducible points is

$$M_{t}^{\text{red}} = \bigcup_{s \in \text{Red}(t)} M_s.$$ 

We define

$$M_t^0 := M_t \setminus M_w^\kappa, \quad M_t^* := M_t \setminus M_t^{\text{red}}, \quad M_t^{*, 0} := M_t^0 \cap M_t^*.$$ 

We recall the

**Theorem 2.2.** [13, 12, 11] Let $t$ be a spin$^u$ structure on a standard four-manifold, $X$. For generic perturbations of the $\text{SO}(3)$ monopole equations, the moduli space, $M_t^{*, 0}$, is a smooth, orientable manifold of dimension

$$\dim M_t = 2d_a(t) + 2n_a(t),$$

where

$$d_a(t) := \frac{1}{2} \dim M_w^\kappa = -p_1(t) - 3\chi_h(X),$$

$$n_a(t) := \frac{1}{4} (p_1(t) + c_1(t)^2 - c_1^2(X) + 8\chi_h(X)),$$

with $\chi_h(X)$ and $c_1^2(X)$ as in (1.1).

**2.2.3. The compactification.** The moduli space $M_t$ is not compact but admits an Uhlenbeck compactification as follows (see [14, Section 2.2] or [13] for details). For $\ell \geq 0$, let $t(\ell)$ be the spin$^u$ structure on $X$ with

$$p_1(t(\ell)) = p_1(t) + 4\ell, \quad c_1(t(\ell)) = c_1(t), \quad w_2(t(\ell)) = w_2(t).$$

Let $\text{Sym}^\ell(X)$ be the $\ell$-th symmetric product of $X$ (that is, $X^\ell$ modulo the symmetric group on $\ell$ elements). For $\ell = 0$, we define $\text{Sym}^0(X)$ to be a point. The space of ideal $\text{SO}(3)$ monopoles is defined by

$$IM_t := \bigcup_{\ell=0}^N M_t(\ell) \times \text{Sym}^\ell(X).$$

We give $IM_t$ the topology defined by Uhlenbeck convergence (see [13, Definition 4.9]).

**Theorem 2.3.** [13] Let $X$ be a standard four-manifold with Riemannian metric, $g$. Let $\mathcal{M}_t \subset IM_t$ be the closure of $M_t$ with respect to the Uhlenbeck topology. Then there is a non-negative integer, $N$, depending only on $(X, g)$, $p_1(t)$, and $c_1(t)$ such that $\mathcal{M}_t$ is compact.

The $S^1$ action on $\mathcal{M}_t$ extends continuously over $IM_t$ and $\mathcal{M}_t$, in particular, but $\mathcal{M}_t$ contains fixed points of this $S^1$ action which are not contained in $\mathcal{M}_t$. The closure of $M^w_\kappa$ in
\( \tilde{\mathcal{M}}_4 \) is the Uhlenbeck compactification of the moduli space of anti-self-dual connections as defined in \cite{4}. There are additional reducible points in the lower strata of \( \mathcal{I} \tilde{\mathcal{M}}_4 \). Define
\[
\text{Red}(t) := \{ s \in \text{Spin}^c(X) : (c_1(s) - c_1(t))^2 \geq p_1(t) \}.
\]
If we define the level, \( \ell(t,s) \), in \( \tilde{\mathcal{M}}_4 \) of the spin\(^c\) structure \( s \) by
\[
(2.5) \quad \ell(t,s) := \frac{1}{4} ( (c_1(s) - c_1(t))^2 - p_1(t) ),
\]
then the strata of reducible points in \( \tilde{\mathcal{M}}_4 \) are given by
\[
(2.6) \quad \tilde{\mathcal{M}}^\text{red}_4 := \bigcup_{s \in \text{Red}(t)} M_s \times \text{Sym}^{\ell(t,s)}(X).
\]
Note that for \( s \in \text{Red}(t) \), we have \( \ell(t,s) \geq 0 \) by the definitions of \( \text{Red}(t) \) and \( \ell(t,s) \). By analogy with the corresponding definitions for \( \mathcal{M}_4 \), we write
\[
\tilde{\mathcal{M}}^0_4 := \tilde{\mathcal{M}}_4 \setminus \tilde{\mathcal{M}}^w_4, \quad \tilde{\mathcal{M}}^*_4 := \tilde{\mathcal{M}}_4 \setminus \tilde{\mathcal{M}}^\text{red}_4, \quad \tilde{\mathcal{M}}^{*,0}_4 := \tilde{\mathcal{M}}^0_4 \cap \tilde{\mathcal{M}}^*_4,
\]
and observe that the stabilizer of the \( S^1 \) action on \( \tilde{\mathcal{M}}^{*,0}_4 \) is \( \{ \pm 1 \} \).

### 2.2.4. Cohomology classes and geometric representatives

The cohomology classes used to define Donaldson invariants extend to \( \mathcal{M}^*_4 / S^1 \). For \( \beta \in H_1(X; \mathbb{R}) \), there is a cohomology class,
\[
\mu_p(\beta) \in H^{4-i}(\mathcal{M}^*_4 / S^1; \mathbb{R}),
\]
with geometric representative (in the sense of \cite{26} p. 588 or \cite{15} Definition 3.4),
\[
\mathbf{V}(\beta) \subset \mathcal{M}^*_4 / S^1.
\]
For \( h_i \in H_2(X; \mathbb{R}) \) and a generator \( x \in H_0(X; \mathbb{Z}) \), we define
\[
\mu_p(h_1 \ldots h_{\delta-2m}x^m) = \mu_p(h_1) \sim \cdots \sim \mu_p(h_{\delta-2m}) \sim \bar{\mu}_p(x)^m \in H^{2\delta}(\mathcal{M}^*_4 / S^1; \mathbb{R}),
\]
\[
\mathbf{V}(h_1 \ldots h_{\delta-2m}x^m) = \mathbf{V}(h_1) \cap \cdots \cap \mathbf{V}(h_{\delta-2m}) \cap \mathbf{V}(x) \cap \cdots \cap \mathbf{V}(x),
\]
and let \( \bar{\mathbf{V}}(h_1 \ldots h_{\delta-2m}x^m) \) be the closure of \( \mathbf{V}(h_1 \ldots h_{\delta-2m}x^m) \) in \( \mathcal{M}^*_4 / S^1 \).

Denote the first Chern class of the \( S^1 \) action on \( \mathcal{M}^{*,0}_4 \) with multiplicity two by
\[
\bar{\mu}_c \in H^2(\mathcal{M}^{*,0}_4 / S^1; \mathbb{Z}).
\]
This cohomology class has a geometric representative \( \bar{W} \).

### 2.2.5. The link of the moduli space of anti-self-dual connections

Let \( L^\text{asd}_4 \) be the link of \( \tilde{\mathcal{M}}^w_4 \subset \tilde{\mathcal{M}}_4 / S^1 \) (see \cite{14} Definition 3.7). The space \( L^\text{asd}_4 \) is stratified by smooth manifolds, with lower strata of codimension at least two. The top stratum of \( L^\text{asd}_4 \) is a smooth, codimension-one submanifold of \( \mathcal{M}^{*,0}_4 / S^1 \) and so has dimension twice
\[
(2.7) \quad \frac{1}{2} \dim L^\text{asd}_4 = d_a(t) + n_a(t) - 1.
\]
Just as an integral lift \( w \) of \( w_2(t) \) defines an orientation for \( \tilde{\mathcal{M}}^w_4 \) in \( \mathbb{Z} \), the choice of \( w \) defines a compatible orientation for the top stratum of \( L^\text{asd}_4 \) (see \cite{15} Lemma 3.27). The intersection of the geometric representatives in Section 2.2.4 with \( L^\text{asd}_4 \) can be used to compute Donaldson invariants \cite{3, 20} or spin polynomial invariants \cite{36}. We will need the
following vanishing result. We note that \( \mathbb{N} = \{0, 1, 2, \ldots\} \) denotes the set of non-negative integers here and throughout the remainder of our article.

**Proposition 2.4.** [15 Proposition 3.29] Let \( t \) be a spin\(^u\) structure on a standard four-manifold \( X \). For \( \delta, \eta_c, m \in \mathbb{N} \), if

\[
\begin{align*}
(2.8a) & \quad \delta - 2m \geq 0, \\
(2.8b) & \quad \delta + \eta_c = \frac{1}{2} \dim L^\text{asd}_t = d_a(t) + n_a(t) - 1, \\
(2.8c) & \quad \delta > \frac{1}{2} \dim M^w_\kappa = d_a(t) \geq 0,
\end{align*}
\]

then

\[
\# \left( \tilde{V}(h^{\delta-2m}x^m) \cap \tilde{W}^{\eta_c} \cap L^\text{asd}_t \right) = 0,
\]

where \# denotes the signed count of the points in the intersection.

**The Seiberg-Witten link.** For \( \ell(t, s) \geq 0 \), the link \( L_{t,s} \) of \( M_\delta \times \text{Sym}^\ell(X) \subset \tilde{M}_t / S^1 \) is defined in [8]. The space \( L_{t,s} \) is compact, stratified by smooth manifolds with corners, with lower strata of codimension at least two. The dimension of \( L_{t,s} \) equals that of \( L^\text{asd}_t \). As described in [8 Section 7.1.3], the top stratum of \( L_{t,s} \) is orientable with a natural choice of orientation.

**Hypothesis 2.5** (Properties of local SO(3)-monopole gluing maps). The local gluing map, constructed in [10], gives a continuous parametrization of a neighborhood of \( M_\delta \times \Sigma \) in \( \mathcal{M}_t \) for each smooth stratum \( \Sigma \subset \text{Sym}^\ell(X) \).

Hypothesis 2.5 is discussed in greater detail in [8 Section 6.7]. The question of how to assemble the local gluing maps for neighborhoods of \( M_\delta \times \Sigma \) in \( \mathcal{M}_t \), as \( \Sigma \) ranges over all smooth strata of \( \text{Sym}^\ell(X) \), into a global gluing map for a neighborhood of \( M_\delta \times \text{Sym}^\ell(X) \) in \( \mathcal{M}_t \) is itself difficult — involving the so-called ‘overlap problem’ described in [17] — but one which we do solve in [8]. See Remark 2.8 for a further discussion of this point.

**Theorem 2.6** (SO(3)-monopole link pairing formula). [8 Theorem 9.0.5] Let \( t \) be a spin\(^u\) structure on a standard four-manifold \( X \) of Seiberg-Witten simple type and assume Hypothesis 2.5. Denote \( \Lambda = c_1(t) \) and \( K = c_1(s) \) for \( s \in \text{Red}(t) \). Let \( \delta, \eta_c, m \in \mathbb{Z}_{\geq 0} \) satisfy \( \delta - 2m \geq 0 \) and

\[
\delta + \eta_c = \frac{1}{2} \dim L_{t,s} = d_a(t) + n_a(t) - 1.
\]

Let \( \ell = \ell(t, s) \) be as defined in (2.5). Then, for any integral lift \( w \in H^2(X; \mathbb{Z}) \) of \( w_2(t) \), and any \( h \in H_2(X; \mathbb{R}) \) and generator \( x \in H_0(X; \mathbb{Z}) \),

\[
\begin{align*}
\# \left( \tilde{V}(h^{\delta-2m}x^m) \cap \tilde{W}^{\eta_c} \cap L_{t,s} \right) \\
= SW_X(s) \sum_{\substack{i+j+2k = \delta-2m \\in \mathbb{Z}_{\geq 0}}} a_{i,j,k}(x_h, c_1^2, K \cdot \Lambda, \Lambda^2, m, \ell) \langle K, h \rangle^i \langle \Lambda, h \rangle^j Q_X(h)^k,
\end{align*}
\]

where \# denotes the signed count of points in the intersection and where for each triple of non-negative integers, \( i, j, k \in \mathbb{N} \), the coefficients,

\[
a_{i,j,k} : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R},
\]
are functions of the variables $\chi_h, c_1^2, c_1(s), \Lambda, \Lambda^2, m, \ell$ and vanish if $k > \ell(t, s)$.

**Remark 2.7.** In contrast to the version of this theorem presented in [12], the coefficients $a_{i,j,k}$ depend on the additional argument $\ell$ because we do not assume that $\delta = \frac{1}{2} \dim M^m_k$ in (2.9) as we do in [12].

**Remark 2.8.** The proof in [8] of Theorem 2.6 assumes the Hypothesis 2.5 (see [8, Section 6.7]) that the local gluing map for a neighborhood of $M_s \times \Sigma$ in $\bar{M}_t$ gives a continuous parametrization of a neighborhood of $M_s \times \Sigma$ in $\bar{M}_t$, for each smooth stratum $\Sigma \subset \text{Sym}^\ell(X)$. These local gluing maps are the analogues for SO(3) monopoles of the local gluing maps for anti-self-dual SO(3) connections constructed by Taubes in [39, 40, 41] and Donaldson and Kronheimer in [4, §7.2]; see also [32, 33]. We have established the existence of local gluing maps in [10] and expect that a proof of the continuity for the local gluing maps with respect to Uhlenbeck limits should be similar to our proof in [7] of this property for the local gluing maps for anti-self-dual SO(3) connections. The authors are currently developing a proof of the required properties for the local gluing maps for SO(3) monopoles [9]. Our proof will also yield the analogous properties for the local gluing maps for anti-self-dual SO(3) connections, as required to complete the proof of the Kotschick-Morgan Conjecture [25], based on our work in [8].

### 2.2.7. The cobordism formula

The compactification $\bar{M}_t^{s,0}/S^1$ defines a compact cobordism, stratified by smooth oriented manifolds, between

$$L_{t}^{\text{as}} \text{ and } \bigcup_{s \in \text{Red}(t)} L_{t,s}.$$  

For $\delta + \eta_c = \frac{1}{7} \dim L_{t}^{\text{as}}$, this cobordism gives the following equality [8, Equation (1.6.1)],

$$\# \left( \bar{V}(h^{\delta-2m,x^m}) \cap \bar{W}^{\eta_c} \cap L_{t}^{\text{as}} \right)$$

$$= - \sum_{s \in \text{Red}(t)} (-1)^{\frac{1}{8}(w^2-\sigma)+\frac{1}{8}(w^2+(w-c_1(t))c_1(s))} \# \left( \bar{V}(h^{\delta-2m,x^m}) \cap \bar{W}^{\eta_c} \cap L_{t,s} \right).$$ (2.10)

We note that the power of $-1$ in (2.10) is computed by comparing the different orientations of the links as described in [8, Lemma 7.1.8].

### 3. The cobordism with $c_1(t) = 0$

In this section, we will derive a formula (see (3.6)) relating the Seiberg-Witten polynomials $SW_{X}^{w,i}$ defined in [12] and the intersection form of $X$. We do so by applying the cobordism formula (2.10) in a case where Proposition 2.4 implies the left-hand-side of (2.10) vanishes. To extract a formula from the resulting vanishing sum that includes the Seiberg-Witten polynomials, $SW_{X}^{w,i}$, we apply Theorem 2.6 to the terms on the right-hand-side of (2.10). In the resulting sum over $\text{Red}$, the coefficients, $a_{i,j,k}$, appearing in equation (2.9)
in Theorem 2.6 depend on \( c_1(t) \cdot c_1(s) \). This dependence prevents the desired extraction of \( SW_{\text{w},i}^X \) (see Remark 3.4) from the cobordism sum. To ensure that \( c_1(t) \cdot c_1(s) \) is constant as \( c_1(s) \) varies in \( B(X) \) without further assumptions on \( B(X) \), such as the abundance condition mentioned in our Introduction, we assume \( c_1(t) = 0 \).

We begin by establishing the existence of a family of spin\( ^u \) structures with \( c_1(t) = 0 \).

**Lemma 3.1.** Let \( X \) be a standard four-manifold. For every \( n \in \mathbb{N} \) there is a spin\( ^u \) structure \( t_n \) on \( X \) satisfying

\[
(3.1) \quad c_1(t_n) = 0, \quad p_1(t_n) = 4n + c_1^2(X) - 8\chi_h(X), \quad w_2(t_n) = w_2(X),
\]

and such that \( n_a(t_n) = n \), where \( n_a(t) \) is the index defined in (2.4b).

**Proof.** By [1, p. 147] or [22, Exercise 1.2.23], \( w_2(X) \) admits an integral lift. Thus the existence of the spin\( ^u \) structure \( t_n \) with the characteristic classes in (3.1) follows from Lemma 2.7 and the observation that for \( c_1(t_n) = 0 \) and \( w_2(t_n) = w_2(X) \) we have \( w_2(t_n)^2 \equiv \sigma(X) \equiv c_1^2(X) - 8\chi_h(X) \) (mod 4) so the desired value of \( p_1(t_n) \) can be achieved for any \( n \in \mathbb{Z} \) with \( n \geq 0 \). The equality \( n_a(t_n) = n \) follows from (2.4b) and the value of \( p_1(t_n) \) in (3.1). \( \square \)

To apply Theorem 2.6 to the cobordism formula (2.10) for a spin\( ^u \) structure \( t_n \), we compute the level in \( \mathcal{M}_{t_n} \) of a spin\( ^c \) structure \( s \).

**Lemma 3.2.** Let \( X \) be a standard four-manifold of Seiberg-Witten simple type. For a non-negative integer, \( n \), let \( t_n \) be a spin\( ^u \) structure on \( X \) satisfying (3.1). For \( c_1(s) \in B(X) \), the level \( \ell = \ell(t_n, s) \) in \( \mathcal{M}_{t_n} \) of \( s \) is

\[
(3.2) \quad \ell(t_n, s) = 2\chi_h(X) - n.
\]

**Proof.** By the definition of Seiberg-Witten simple type, for any \( c_1(s) \in B(X) \),

\[
(3.3) \quad c_1(s)^2 = c_1^2(X).
\]

By (2.5), the level is given by

\[
\ell(t_n, s) = \frac{1}{4} \left( (c_1(s) - c_1(t_n))^2 - p_1(t_n) \right)
\]

\[
= \frac{1}{4} \left( c_1(s)^2 - 4n - c_1^2(X) + 8\chi_h(X) \right) \quad \text{(by 3.1)}
\]

\[
= 2\chi_h(X) - n \quad \text{(by 3.3)},
\]

as desired. \( \square \)

Combining (2.10) with Proposition 2.4 and Theorem 2.6 then gives the following

**Theorem 3.3.** Let \( X \) be a standard four-manifold of Seiberg-Witten simple type. Assume that \( m, n \in \mathbb{N} \) satisfy

\[
(3.4a) \quad n \leq 2\chi_h(X),
\]

\[
(3.4b) \quad 1 < n,
\]

\[
(3.4c) \quad 0 \leq c(X) - n - 2m - 1.
\]

We abbreviate the coefficients in equation (2.9) in Theorem 2.6 by

\[
(3.5) \quad a_{i,0,k} := a_{i,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, 2\chi_h(X) - n).
\]
Then, for \( A = c(X) - n - 2m - 1 \) and \( w \in H^2(X; \mathbb{Z}) \) characteristic,

\[
0 = \sum_{k=0}^{2\chi_h(X) - n} a_A + 2k, 2\chi_h(X) - n - k SW_X^{w, A + 2k}(h) Q_X(h) 2\chi_h(X) - n + k.
\]

Proof. Let \( t_n \) be a spin\(^u\) structure on \( X \) satisfying (3.1), where \( n \) is the non-negative integer in the statement of Theorem 3.3. The value of \( p_1(t_n) \) in (3.1) and the expression for \( d_a(t_n) \) given in (2.4a) and \( c(X) \) in (1.1) imply that

\[
\frac{1}{2} \dim M^w_\kappa = d_a(t_n) = c(X) + 4\chi_h(X) - 4n.
\]

The value of \( d_a(t_n) \) in (3.7), the equality \( n_a(t_n) = n \) given in Lemma 3.1 and the formula for half the dimension of \( L_{t_n}^{\text{asd}} \) given in (2.7) imply that

\[
\frac{1}{2} \dim L_{t_n}^{\text{asd}} = c(X) + 4\chi_h(X) - 3n - 1.
\]

We apply the cobordism formula (2.10) to the spin\(^u\) structure \( t_n \) with

\[
\delta := \frac{1}{2} \dim L_{t_n}^{\text{asd}} = c(X) + 4\chi_h(X) - 3n - 1 \quad \text{and} \quad \eta_c := 0,
\]

and claim that Proposition 2.4 implies that the left-hand-side of (2.10) vanishes. Assumption (3.4a) and (3.2) imply that for \( c_1(s) \in B(X) \),

\[
2\ell(t_n, s) = 4\chi_h(X) - 2n \geq 0.
\]

Assumption (3.4c), the definition of \( \delta \), and (3.10) imply that

\[
\delta - 2m \geq \delta - 2\ell - 2m = c(X) - n - 1 - 2m \geq 0.
\]

Thus, \( \delta - 2m \geq 0 \) so condition (2.8a) of Proposition 2.4 holds.

The choice of \( \delta \) and \( \eta_c \) imply that \( \delta + \eta_c = \frac{1}{2} \dim L_{t_n}^{\text{asd}} \), so condition (2.8b) of Proposition 2.4 holds.

Assumption (3.4b) implies that \(-1 > -n\) so \(-3n - 1 > -4n\). This inequality, our choice of \( \delta \), and (3.7) imply that

\[
\delta = c(X) + 4\chi_h(X) - 3n - 1 > c(X) + 4\chi_h(X) - 4n = \frac{1}{2} \dim M^w_\kappa,
\]

so condition (2.8c) of Proposition 2.4 holds. Thus, all three conditions of Proposition 2.4 hold and the left-hand-side of (2.10) vanishes when applied with the given values of \( \delta \), \( \eta_c \) and the spin\(^u\) structure \( t_n \). Under these conditions, equation (2.10) becomes

\[
0 = - \sum_{s \in \text{Red}(t_n)} (-1)^{\frac{1}{2}(w^2 - \sigma) + \frac{1}{2}(w^2 + w \cdot c_1(s))} \# \left( \nabla^{h^{\delta - 2m}X^m} \cap L_{t_n, s} \right).
\]

For each \( s \in \text{Red}(t_n) \), equation (2.9) in Theorem 2.6 implies that each term in the sum on the right-hand-side of (3.12) contains a factor of \( SW_X(s) \). The terms in this sum given by
Because we have assumed $w \in H^2(X; \mathbb{Z})$ is characteristic, we have $w^2 \equiv \sigma(X) \pmod{8}$ by [22, Lemma 1.2.20], so

$$(-1)\frac{1}{2}(w^2 - \sigma(X)) = 1.$$ 

Our assumption that $\Lambda = c_1(t_n) = 0$ from (3.11) implies that all the terms in equation (2.9) with a factor of $(\Lambda, h)^2$ with $j > 0$ vanish. Thus, applying equation (2.9) in Theorem 2.6 to the terms in (3.13) and noting that $\ell = 2\chi_h(X) - n$ by (3.2) yields

$$0 = \sum_{s \in c_1^{-1}(K)} \sum_{K \in B(X)} (-1)\frac{1}{2}(w^2 + w\cdot c_1(s)) \# \left( \tilde{V}(h^{\delta - 2m}x^m) \cap L_{t_n,s} \right)$$

$$= \sum_{s \in c_1^{-1}(K)} \sum_{K \in B(X)} (-1)\frac{1}{2}(w^2 + w\cdot K) SW_X(s)$$

$$\times \sum_{i+2k = \delta - 2m} a_{i,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, 2\chi_h(X) - n)(K, h)^i Q_X(h)^k.$$ 

By the definition of $SW'_X(K)$ in (2.2), we can rewrite (3.14) as

$$0 = \sum_{K \in B(X)} (-1)\frac{1}{2}(w^2 + w\cdot K) SW'_X(K)$$

$$\times \sum_{i+2k = \delta - 2m} a_{i,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, 2\chi_h(X) - n)(K, h)^i Q_X(h)^k.$$ 

Because the coefficient $a_{i,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, 2\chi_h(X) - n)$ does not depend on $K \in B(X)$, using abbreviation $a_{i,0,k}$ of (3.5) we can rewrite (3.15) as

$$0 = \sum_{i+2k = \delta - 2m} a_{i,0,k} Q_X(h)^k \sum_{K \in B(X)} (-1)\frac{1}{2}(w^2 + w\cdot K) SW'_X(K)(K, h)^i$$

$$= \sum_{i+2k = \delta - 2m} a_{i,0,k} SW_{X,h}^{w,i}(h) Q_X(h)^k \quad \text{(by (2.3))}.$$ 

Because the coefficients $a_{i,0,k}$ in (3.16) vanish for $k > \ell = 2\chi_h(X) - n$ by Theorem 2.6 we can rewrite (3.16) as

$$0 = \sum_{k=0}^{\ell} a_{\delta - 2m - 2\ell + 2k,0,\ell - k} SW_{X,h}^{w,\delta - 2m - 2\ell + 2k}(h) Q_X(h)^{\ell - k}.$$
From (3.11) and the definition \( A = c(X) - n - 2m - 1 \) in the statement of the theorem, we have \( \delta - 2m - 2\ell + 2k = A + 2k \). Substituting that equality and \( \ell = 2\chi_h(X) - n \) into (3.17) completes the proof.

**Remark 3.4.** As discussed in the beginning of this section, we work with a spin\(^u\) structure \( t \) with \( c_1(t) = 0 \) in order to ensure that the coefficients \( a_{i,j,k} \) appearing in (2.9) do not depend on \( K \in B(X) \). Thus, after reversing the order of summation in (3.15) we can pull these coefficients out in front of the inner sum over \( K \in B(X) \) to get the expression (3.16) involving the Seiberg-Witten polynomials. Hence, the choice of spin\(^u\) structure with \( c_1(t) = 0 \) is a necessary step in the argument.

4. **The leading term computation**

To show equation (3.6) is non-trivial, we now demonstrate, in a computation similar to that of [25, Theorem 6.1.1], that the coefficient of the term in (3.6) including the highest power of \( Q_X \) is non-zero.

**Proposition 4.1.** Continue the notation and assumptions of Theorem 3.3. In addition, assume that there is \( K \in B(X) \) with \( K \neq 0 \). Let \( m \) and \( n \) be non-negative integers satisfying the conditions (3.4). Define \( A := c(X) - n - 2m - 1 \), and \( \delta := c(X) + 4\chi_h(X) - 3n - 1 \), and \( \ell = 2\chi_h(X) - n \). Then

\[
a_{A,0,\ell}(\chi_h(X), c_1(X), 0, 0, m, \ell) = (-1)^{m+\ell-\ell} \frac{(\delta - 2m)!}{\ell! A!}.
\]

**Remark 4.2.** Although the computation of the precise value of the coefficient in (4.1) is quite delicate, we are fortunate to require only the result that \( a_{A,0,\ell} \) is non-zero.

**Remark 4.3.** The methods in this section allow one to compute the coefficients \( a_{i,j,\ell} \) in greater generality (for example, without the assumption that \( c_1(t) = 0 \)). Because Theorem 1.1 does not require greater generality and indeed, as noted in Remark 3.4, requires the assumption that \( c_1(t) = 0 \), we omit the proof of the more general result in the interest of clarity.
is a continuous, $S^1$-equivariant map with $0 \in g^{-1}_t(0)$ and which is smooth away from the origin and vanishes transversely away from the origin. The dimensions satisfy

$$\tag{4.3} r_N - r_\Xi = \frac{1}{2} \ell M_t(\ell) = \frac{1}{2} M_1 - 3 \ell.$$  

We further note that because $\dim M_s = 0$ and $M_s$ is compact and oriented, $M_s$ is a finite set of points. If $1 \in H^0(M_s; \mathbb{Z})$ is a generator given by an orientation of $M_s$, then

$$\langle 1, [M_s] \rangle = SW_X(s),$$

as this pairing is just the count with sign of the points in the oriented moduli space $M_s$.

4.1.2. *The neighborhood of $M_s \times \{x\}$. In* [8] *Chapter 5*, we constructed a virtual neighborhood $\tilde{\mathcal{M}}^\text{vir}_{t,s}$ of $M_s \times \text{Sym}^\ell(X)$ in $\mathcal{M}_t$ admitting a continuous, surjective map (see [8] Lemma 5.8.2)),

$$\tag{4.4} \pi_X : \tilde{\mathcal{M}}^\text{vir}_{t,s} \rightarrow \text{Sym}^\ell(X).$$

The space, $\tilde{\mathcal{M}}^\text{vir}_{t,s}$, is stratified by smooth manifolds and contains a subspace, $\mathcal{O}_{t,s}$, which is homeomorphic to a neighborhood $\tilde{U}_{t,s}$ of $M_s \times \text{Sym}^\ell(X)$ in $\mathcal{M}_t$. Let $\tilde{\mathcal{O}}_{t,s}$ be the intersection of $\mathcal{O}_{t,s}$ with the top stratum of $\tilde{\mathcal{M}}^\text{vir}_{t,s}$. Then $\mathcal{O}_{t,s}$ is the zero locus of a transversely vanishing section of a vector bundle of rank $2r_\Xi + 2\ell$ over the top stratum of $\tilde{\mathcal{M}}^\text{vir}_{t,s}$ and $\mathcal{O}_{t,s}$ is diffeomorphic to the top stratum of $\tilde{U}_{t,s}$. The top stratum of $\tilde{\mathcal{M}}^\text{vir}_{t,s}$ has dimension determined by

$$\frac{1}{2} \dim \tilde{\mathcal{M}}^\text{vir}_{t,s} = \frac{1}{2} \dim M_1 + r_\Xi + \ell.$$  

There is an $S^1$ action on $\tilde{\mathcal{M}}^\text{vir}_{t,s}$ which restricts to the $S^1$ action on $\mathcal{M}_t$ discussed in Section 2.2.3. This $S^1$ action is free on the complement of its fixed point set, $M_s \times \text{Sym}^\ell(X) \subset \tilde{\mathcal{M}}^\text{vir}_{t,s}$.

Let $\Delta \subset \text{Sym}^\ell(X)$ be the ‘big diagonal’, given by points $\{x_1, \ldots, x_\ell\}$ where $x_i = x_j$ for some $i \neq j$. For $x \in \text{Sym}^\ell(X) \setminus \Delta$, let $U$ be an open set,

$$x \in U \Subset \text{Sym}^\ell(X) \setminus \Delta,$$

and let $\tilde{U} \subset X^\ell$ be the pre-image of $U$ under the branched cover $X^\ell \rightarrow \text{Sym}^\ell(X)$. Let $\text{CSO}(3)$ be the open cone on $\text{SO}(3)$. For $U$ sufficiently small, we define

$$\tag{4.5} N(t, s, U) := M_s \times \mathbb{C}^r \times \text{CSO}(3)^\ell \times \mathcal{G}_t \tilde{U},$$

where $\mathcal{G}_t$ is the symmetric group on $\ell$ elements, acting diagonally by permutation on the $\ell$ factors in $\text{CSO}(3)^\ell$ and $\tilde{U}$. Because $U$ is contained in the top stratum of $\text{Sym}^\ell(X)$, the construction of $\tilde{\mathcal{M}}^\text{vir}_{t,s}$ in [8] Section 5.1.5] and the map $\pi_X$ in [8] Lemma 5.8.2] imply that there is a commutative diagram,

$$\begin{array}{ccc}
N(t, s, U) & \xrightarrow{\gamma} & \tilde{\mathcal{M}}^\text{vir}_{t,s} \\
\downarrow & & \downarrow \pi_X \\
U & \longrightarrow & \text{Sym}^\ell(X)
\end{array}$$

where

1. The horizontal maps are open embeddings,
2. The vertical map on the left is projection onto the factor $U$,
(3) The image of $\gamma$ is a neighborhood of $M_s \times \{x\}$ in $\mathcal{M}_{t,s}^{\text{vir}}$.

(4) The embedding $\gamma$ is equivariant with respect to the diagonal $S^1$ action on the factors of $C$ and $SO(3)$ in (4.7) and the $S^1$ action on $\mathcal{M}_{t,s}^{\text{vir}}$.

Observe that because $U$ is in the top stratum of $\text{Sym}^\ell(X)$, $\mathcal{G}_\ell$ acts freely on $\tilde{U}$. Hence, for $x \in U$ the pre-image of $x$ under the left vertical arrow in the diagram (4.7) is

$$\mathcal{M}_s \times \mathbb{C}^r \times SO(3)^\ell \times \{x\}.$$

The commutativity of the diagram (4.7) implies that for $x \in U$, the embedding $\gamma$ defines a homeomorphism,

$$(4.8) \quad \mathcal{M}_s \times \mathbb{C}^r \times SO(3)^\ell \times \{x\} \to \pi_\ell^{-1}(x).$$

Note that for $x \in U$ represented by $\{x_1, \ldots, x_\ell\}$, each $x_\ell$ has multiplicity one, by definition of $\Delta$, for $1 \leq \ell \leq \ell$.

**Remark 4.4.** The virtual neighborhood $\mathcal{M}_{t,s}^{\text{vir}}$ is a union of the domains of the gluing maps defined in [10]. Therefore, the space (4.6) can be understood as follows. The factor $M_s \times \mathbb{C}^r$ represents pairs of ‘almost monopoles’ on the spinu structure $t(\ell)$. The factors $SO(3)$ represent centered, charge-one, framed instantons on $S^4$ which are spliced onto pairs defined by $M_s \times \mathbb{C}^r \times SO(3)^\ell$ at the points $\{x_1, \ldots, x_\ell\} \subset X$ defined by the factor $U \in \text{Sym}^\ell(X) \setminus \Delta$. This gluing construction is described in detail in [10], [16], [8] and is similar to that described for the moduli space of anti-self-dual connections in [3] Section 7.2], [20] Section 3.4], and [39] [40] [41].

For the cone point $c \in SO(3)$, define $c_\ell \in SO(3)^\ell$ by

$$c_\ell = \underbrace{\{c\} \times \{c\} \times \cdots \times \{c\}}_{\ell \text{ copies}} \in SO(3)^\ell.$$

Because $c_\ell$ is a fixed point of the $\mathcal{G}_\ell$ action on $SO(3)^\ell$,

$$(4.9) \quad \gamma^{-1} \left( M_s \times \text{Sym}^\ell(X) \right) = M_s \times \{0\} \times \{c_\ell\} \times U \subset M_s \times \mathbb{C}^r \times SO(3)^\ell \times \mathcal{G}_\ell \tilde{U},$$

where $\gamma$ is the embedding in (4.7).

4.1.3. The link and its branched cover. In [8] Proposition 8.0.4], we constructed a link $L_{t,s}^{\text{vir}} \subset \mathcal{M}_{t,s}^{\text{vir}}/S^1$ of $M_s \times \text{Sym}^\ell(X) \subset \mathcal{M}_{t,s}^{\text{vir}}/S^1$. We will need the following description of $\pi_\ell^{-1}(x) \cap L_{t,s}^{\text{vir}}$ and a branched cover of this space.

**Lemma 4.5.** For $x \in \text{Sym}^\ell(X) \setminus \Delta$, the space $\pi_\ell^{-1}(x) \cap L_{t,s}^{\text{vir}}$ is homeomorphic to the link of

$$M_s \times \{0\} \times \{c_\ell\} \times \{x\} \quad \text{in} \quad M_s \times \mathbb{C}^r \times S^1 \times SO(3)^\ell \times \{x\}.$$

**Proof.** From the description in (4.9) of the intersection of the image $\gamma$ with $M_s \times \text{Sym}^\ell$ and by the $S^1$ equivariance of this embedding, we see that pre-image of $M_s \times \text{Sym}^\ell(X)$ under the homeomorphism (4.8) is

$$M_s \times \{0\} \times \{c_\ell\} \times \{x\}.$$

Because the homeomorphism (4.8) is $S^1$ equivariant, it identifies the link of the preceding space in the $S^1$ quotient of the domain of (4.8) with $\pi_\ell^{-1}(x) \cap L_{t,s}^{\text{vir}}$, as asserted. □
The computations in our proof of Proposition 4.1 require the following branched cover of this link.

**Lemma 4.6.** There is a degree \((-1)^{\ell}2^{r_N+\ell-1}\) branched cover

\[
\tilde{f} : M_g \times \mathbb{CP}^{r_N+2\ell-1} \to \pi^{-1}_X(x) \cap L^{vir}_{L_g}.
\]

If \(\nu\) is the first Chern class of the \(S^1\) action on \(\mathcal{M}^{vir}_{L_g}\), then

\[
(4.10) \quad \tilde{f}^* \nu = 1 \times 2\tilde{\nu},
\]

where \(1 \in H^0(M_g; \mathbb{Z})\) satisfies \((4.3)\) and \(\tilde{\nu} \in H^2(\mathbb{CP}^{r_N+2\ell-1}; \mathbb{Z})\) satisfies

\[
(4.11) \quad (\tilde{\nu}^{r_N+2\ell-1}, [\mathbb{CP}^{r_N+2\ell-1}]) = (-1)^{r_N+2\ell-1}.
\]

**Proof.** The product of the degree \((-2)\) branched cover (see [35] for an explanation of the sign) \(\mathbb{CP}^2 \to \text{CSO}(3)\) with the map \(z \to z^2\) on the factors of \(\mathbb{C}\) gives a degree \((-1)^{\ell}2^{r_N+\ell}\) branched cover,

\[
M_g \times \mathbb{CP}^{r_N+2\ell} \to M_g \times \mathbb{CP}^{r_N} \times \text{CSO}(3)^\ell,
\]

mapping \(M_g \times \{0\}\) to \(M_g \times \{0\} \times \{e_\ell\}\) and which is \(S^1\) equivariant if \(S^1\) acts with weight two on the image. Consequently, this map takes the link of \(M_g \times \{0\}\) in its domain to the link of \(M_g \times \{0\} \times \{e_\ell\}\) in its image. By Lemma 4.5 the link of \(M_g \times \{0\} \times \{e_\ell\}\) in the \(S^1\) quotient of \(M_g \times \mathbb{CP}^{r_N} \times \text{CSO}(3)^\ell\) is homeomorphic to \(\pi^{-1}_X(x) \cap L^{vir}_{L_g}\). Hence, there is a degree \((-1)^{\ell}2^{r_N+\ell-1}\) branched cover

\[
\tilde{f} : M_g \times \mathbb{CP}^{r_N+2\ell-1} \to \pi^{-1}_X(x) \cap L^{vir}_{L_g}.
\]

Because this map doubles the weight of the \(S^1\) action, \(\tilde{f}^* \nu\) is twice the first Chern class of the \(S^1\) bundle,

\[
M_g \times \left(\mathbb{CP}^{r_N+2\ell} \setminus \{0\}\right) < S^1 \mathbb{C} \to M_g \times \mathbb{CP}^{r_N+2\ell-1},
\]

whose first Chern class is \(1 \times \tilde{\nu}\) where \(\tilde{\nu}\) is the negative of the hyperplane class. \(\Box\)

4.2. **Multilinear algebra.** The proof of Proposition 4.1 requires us to consider the intersection number with \(L_{L_g}\) in (2.9) as a symmetric multilinear map on \(H_2(X; \mathbb{R})\) rather than a polynomial. We thus introduce some notation to translate between the two concepts.

For a finite-dimensional, real vector space \(V\), let \(S_d(V)\) be the vector space of \(d\)-linear, symmetric maps, \(M : V^{\otimes d} \to \mathbb{R}\), and let \(P_d(V)\) be the vector space of degree \(d\) homogeneous polynomials on \(V\). The map \(\Phi : S_d(V) \to P_d(V)\) defined by \(\Phi(M)(v) = M(v, \ldots, v)\) is an isomorphism of vector spaces (see [20] Section 6.1.1]). For \(M_i \in S^d_i(V)\), we define a product on \(S_* (V) = \oplus_{d \geq 0} S_d(V)\) by

\[
(4.12) \quad (M_1 M_2)(h_1, \ldots, h_{d_1+d_2}) := \frac{1}{(d_1+d_2)!} \sum_{\sigma \in \mathcal{G}_{d_1+d_2}} M_1(h_{\sigma(1)}, \ldots, h_{\sigma(d_1)}) M_2(h_{\sigma(d_1+1)}, \ldots, h_{\sigma(d_1+d_2)}),
\]

where \(\mathcal{G}_d\) is the symmetric group on \(d\) elements. When \(S_*(V)\) has this product and \(P_*(V)\) has its usual product, \(\Phi\) is an algebra isomorphism.
Lemma 4.7. Continue the assumptions and notation of Proposition 4.1. For \( n \in \mathbb{N} \) as in Proposition 4.1 let \( t_n \) be a spin\(^s\) structure satisfying (3.1). Then,

\[
\# \left( \mathcal{V}(h_1 \ldots h_{\delta-2m}x^m) \cap L_{t_n,s} \right) = \frac{SW_X(s)}{(\delta - 2m)!} \sum_{i \geq 2k} a_{i,0,k}(\chi(X), c_2^2(X), 0, 0, m, \ell) \sum_{\sigma \in \mathfrak{S}_{\delta - 2m}} \prod_{u=1}^{i} (K, h_{\sigma(u)}) \prod_{u=1}^{(\delta - 2m - i)/2} Q_X(h_{\sigma(i+2u-1)}, h_{\sigma(i+2u)})
\]

(4.13)

where \( \mathfrak{S}_{\delta - 2m} \) is the symmetric group on \( (\delta - 2m) \) elements and \( K = c_1(s) \).

**Proof.** Because we are assuming \( \Lambda = 0 \), all terms on the right-hand-side of (2.9) containing a factor of \( \langle \Lambda, h \rangle^j \) with \( j > 0 \) vanish. Applying \( \Phi \) to both sides of (4.13) then yields (2.9) in Theorem 2.6. Because \( \Phi \) is an isomorphism, the result follows. \( \square \)

The following shows the computation which will yield the coefficient appearing in (4.1).

Corollary 4.8. Continue the notation and hypotheses of Lemma 4.7 and abbreviate \( A = \delta - 2m - 2\ell \). There is a class \( h \in \text{Ker} \, K \subset H_2(X; \mathbb{R}) \) with \( Q_X(h) = 1 \) and if

\[
h_u = h \in \text{Ker} \, K \subset H_2(X; \mathbb{R}) \quad \text{for } A + 1 \leq u \leq \delta - 2m,
\]

then

\[
\# \left( \mathcal{V}(h_1 \ldots h_{\delta-2m}x^m) \cap L_{t_n,s} \right) = \frac{SW_X(s)A!(2\ell)!}{(\delta - 2m)!} a_{A,0,\ell}(\chi(X), c_2^2(X), 0, 0, m, \ell) \prod_{u=1}^{A} (K, h_u).
\]

(4.14)

**Proof.** Because \( b^+(X) \geq 3 \), \( Q_X \) is positive on a three-dimensional subspace \( P \subset H_2(X; \mathbb{R}) \). The codimension of \( \text{Ker} \, K \subset H_2(X; \mathbb{R}) \) is at most one so \( P \cap \text{Ker} \, K \) has dimension at least two. Hence, there is a class \( h' \in \text{Ker} \, K \) with \( Q_X(h') > 0 \). Then \( h = h'/Q_X(h')^{1/2} \in \text{Ker} \, K \) satisfies \( Q_X(h) = 1 \), as required.

The assumption (4.14) implies that only \( A \) elements of \( \{h_1, \ldots, h_{\delta-2m}\} \) are not in \( \text{Ker} \, K \). Therefore, in all terms in the sum in (4.13) with \( k < \ell \), we have \( i = \delta - 2m - 2k > \delta - 2m - 2\ell = A \) and thus in such a term, the product of the \( i > A \) factors,

\[
\prod_{u=1}^{i} (K, h_{\sigma(u)}),
\]

must vanish. Hence, all terms with \( k < \ell \) in the sum in (4.13) vanish. We know the terms with \( k > \ell \) vanish because the coefficients \( a_{i,0,k} \) with \( k > \ell \) vanish by Theorem 2.6 so only
the terms with \( k = \ell \) are non-zero. Thus, (4.13) and the equality \((\delta - 2m - A)/2 = \ell \) imply

\[
\# \left( \mathcal{V}(h_1 \ldots h_{\delta - 2m}x^m) \cap \mathcal{W}^m \cap L_{4,\delta} \right) \\
= \frac{SW_X(\delta)}{(\delta - 2m)!} \sum_{\sigma \in \mathfrak{S}_{\delta - 2m}} a_{A,0,\ell}(\chi h(X), c_1^2(X), 0, 0, m, \ell) \\
\times \prod_{u=1}^A \langle K, h_{\sigma(u)} \rangle \prod_{u=1}^\ell Q_X(h_{\sigma(A+2u-1)}, h_{\sigma(A+2u)}).
\]

(4.16)

If \( \sigma_0 \in \mathfrak{S}_{\delta - 2m} \) and for \( u \leq A = \ell \), we have \( \sigma_0(u) > A \), then the term given by that \( \sigma_0 \) in the sum on the right-hand side of (4.16) contains the factor

\[
\prod_{u=1}^A \langle K, h_{\sigma_0(u)} \rangle \prod_{u=1}^\ell Q_X(h_{\sigma_0(A+2u-1)}, h_{\sigma_0(A+2u)}),
\]

which vanishes by the assumption (4.14) that \( h_u \in \text{Ker} K \) for \( u > A \). Thus, for \( h_u \) as given in (4.14), the sum over \( \mathfrak{S}_{\delta - 2m} \) in (4.16) reduces to a sum over the subset

\[
\mathfrak{S}_{\delta - 2m}(A) := \{ \sigma \in \mathfrak{S}_{\delta - 2m} : \sigma(u) \leq A \text{ for all } u \leq A \}.
\]

The pigeonhole principle then implies that elements of \( \mathfrak{S}_{\delta - 2m}(A) \) preserve the subsets \( \{1, \ldots, A\} \) and \( \{A + 1, \ldots, \delta - 2m\} \). If we identify \( \mathfrak{S}_A \) and \( \mathfrak{S}_{2\ell} \) (using \( 2\ell = \delta - 2m - A \)) with the subgroups of \( \mathfrak{S}_{\delta - 2m} \) which are the identity on \( \{A + 1, \ldots, \delta - 2m\} \) and \( \{1, \ldots, A\} \) respectively, then there is an isomorphism

\[
S : \mathfrak{S}_A \times \mathfrak{S}_{2\ell} \to \mathfrak{S}_{\delta - 2m}(A), \quad S(\sigma_1, \sigma_2) = \sigma_1 \sigma_2.
\]

This isomorphism, the identity (4.16), and the equality \((\delta - 2m - A)/2 = \ell \) imply that

\[
\# \left( \mathcal{V}(h_1 \ldots h_{\delta - 2m}x^m) \cap \mathcal{W}^m \cap L_{4,\delta} \right) \\
= \frac{SW_X(\delta)}{(\delta - 2m)!} \sum_{\sigma_1 \in \mathfrak{S}_A} \sum_{\sigma_2 \in \mathfrak{S}_{2\ell}} a_{A,0,\ell}(\chi h(X), c_1^2(X), 0, 0, m, \ell) \\
\times \prod_{u=1}^A \langle K, h_{\sigma_1(u)} \rangle \prod_{u=1}^\ell Q_X(h_{\sigma_2(A+2u-1)}, h_{\sigma_2(A+2u)}).
\]

(4.18)

Observe that for all \( \sigma_1 \in \mathfrak{S}_A \),

\[
\prod_{u=1}^A \langle K, h_{\sigma_1(u)} \rangle = \prod_{u=1}^A \langle K, h_u \rangle
\]

while for all \( \sigma_2 \in \mathfrak{S}_{2\ell} \),

\[
\prod_{u=1}^\ell Q_X(h_{\sigma_2(A+2u-1)}, h_{\sigma_2(A+2u)}) = Q_X(h)^\ell = 1.
\]
Thus, all the \(|\mathcal{G}_A|\mathcal{G}_{2\ell} = A!(2\ell)!\) terms in the double sum in (4.18) are equal and we can rewrite (4.18) as

\[
\# \left( \widetilde{\mathcal{V}}(h_1 \ldots h_{\delta-2m}x^m) \cap \mathcal{V}_{MC} \cap L_{i,s} \right)
= \frac{SW_X(x)A!(2\ell)!}{(\delta - 2m)!} \cdot a_{A,0,\ell}(\chi_h(X), c_1^2(X), 0, 0, m, \ell) \prod_{u=1}^{A} \langle K, h_u \rangle.
\]

This completes the proof of the corollary.  

\[\square\]

4.3. **Cohomology classes and duality.** By [8, Proposition 8.0.4], there are a topological space, \(L_{i,s}^{vir} \subset \mathcal{M}_{i,s}^{vir} / S^1\), with fundamental class \([L_{i,s}^{vir}]\) and cohomology classes

\[
\bar{\mu}_p(h_i), \bar{\mu}_c, \bar{e}_I, \bar{e}_s \in H^\bullet \left( \mathcal{M}_{i,s}^{vir} / S^1 \setminus \left( M_s \times \text{Sym}^\ell(X) \right) ; \mathbb{R} \right),
\]

such that

\[
\# \left( \bar{\mathcal{V}}(h_1 \ldots h_{\delta-2m}x^m) \cap L_{i,s} \right)
= \langle \bar{\mu}_p(h_1) \sim \cdots \sim \bar{\mu}_p(h_{\delta-2m}) \sim \bar{\mu}_p(x)^m \sim \bar{e}_I \sim \bar{e}_s, [L_{i,s}^{vir}] \rangle.
\]

We note that the cohomology classes \(\bar{\mu}_p(h_i)\) and \(\bar{\mu}_p(x)\) are extensions of the classes \(\mu_p(h_i)\) and \(\mu_p(x)\) defined in Section 2.2.4. For \(\beta \in H_\bullet(X; \mathbb{R})\), the cohomology class \(S^\ell(\beta) \in H^{4-\bullet}(\text{Sym}^\ell(X); \mathbb{R})\) is defined by the property that, for \(\bar{\pi} : X^\ell \to \text{Sym}^\ell(X)\) denoting the degree-\(\ell!\) branched covering map,

\[
\bar{\pi}^* S^\ell(\beta) = \sum_{i=1}^n \pi_i^* \text{PD}[\beta],
\]

where \(\pi_i : X^\ell \to X\) is projection onto the \(i\)-th factor. Thus (compare [25, p. 454]),

\[
\langle S^\ell(h_1) \sim \cdots \sim S^\ell(h_{2\ell+k}), [\text{Sym}^\ell(X)] \rangle = \left\{ \begin{array}{ll}
\frac{(2\ell)!}{\ell!2^\ell} Q_X(h_1, \ldots, h_{2\ell}) \text{PD}[x] & \text{if } k = 0, \\
0 & \text{if } k > 0,
\end{array} \right.
\]

where \(x \in \text{Sym}^\ell(X) \setminus \Delta\) is a point in the top stratum. Note that if \(h_u = h\) for \(u = 1, \ldots, 2\ell\), then by the definition of the product in (4.12),

\[
Q_X(h_1, \ldots, h_{2\ell}) = Q_X(h)\ell.
\]

From [8, Equations (8.3.21), (8.3.24), and (8.3.25)] and [8, Lemma 8.4.1] we have, denoting \(K = c_1(s), \Lambda = c_1(t), h \in H_2(X; \mathbb{R})\), and a generator \(x \in H_0(X; \mathbb{Z})\),

\[
\bar{\mu}_p(h) = \frac{1}{2} \langle \Lambda - K, h \rangle \nu + \pi^*_X S^\ell(h),
\]

\[
\bar{\mu}_p(x) = -\frac{1}{4} \nu^2 + \pi^*_X S^\ell(x),
\]

\[
\bar{e}_s = (-\nu)^{\pi_e},
\]

where \(\nu\) is the first Chern class of the \(S^1\) action on \(\mathcal{M}_{i,s}^{vir}\) and \(\pi_X\) is defined in (4.3).
4.4. The computation. We can now give the

Proof of Proposition 4.1. For \( n \in \mathbb{N} \), as appearing in Proposition 4.1, let \( t_n \) be a spin\( ^u \) structure satisfying (3.1). We will apply Corollary 4.8 to verify the expression (4.1) for the coefficient \( a_{A,0,\ell} \). From the definitions of \( \delta, A, \) and \( \ell \) in the statement of Proposition 4.1 and the expression for \( \delta \) in (3.9),

\[
A + 2\ell + 2m = \delta = \frac{1}{2} \dim \mathcal{L}_{t_n}^\text{and} = \frac{1}{2} \dim \mathcal{M}_{t_n} - 1.
\]

By hypothesis in Proposition 4.1, there is a class \( K \in B(X) \) with \( K \not= 0 \). Let \( s \in \text{Spin}^c(X) \) satisfy \( c_1(s) = K \). As in the proof of Corollary 4.8, there are \( h_0, h'_0 \in H_2(X; \mathbb{R}) \) which satisfy

\[
\langle K, h_0 \rangle = 0, \quad Q_X(h_0) = 1, \quad \langle K, h'_0 \rangle = -1.
\]

Define

\[
h_u := \begin{cases} 
h'_0 & \text{for } 1 \leq u \leq A, \\
h_0 & \text{for } A + 1 \leq u \leq \delta - 2m.
\end{cases}
\]

Corollary 4.8, the identity (4.25), and the definition (4.26) imply that

\[
\# \left( \bar{\mathcal{V}}(h_1 \ldots h_{\delta-2m}x^m) \cap L_{t_n,s} \right)
\]

\[
= (-1)^A \frac{SW_X(s)A!(2\ell)!}{(\delta - 2m)!} a_{A,0,\ell}(\chi_{h}(X), c_1^2(X), 0, 0, m, \ell).
\]

We now use the work of the previous sections to compute the left-hand-side of (4.27). Applying (4.20) with \( t = t_n \) gives

\[
\# \left( \bar{\mathcal{V}}(h_1 \ldots h_{\delta-2m}x^m) \cap L_{t_n,s} \right)
\]

\[
= \langle \tilde{\mu}_p(h_1) \sim \ldots \sim \tilde{\mu}_p(h_{\delta-2m}) \sim \tilde{\mu}_p(x)^m \sim e_I \sim e_s, [L_{t_n,s}] \rangle.
\]

By (4.23) (with \( \Lambda = c_1(t_n) = 0 \), (4.25), and (4.26),

\[
\tilde{\mu}_p(h_u) = \begin{cases} \frac{1}{2} \nu + \pi_X^* S^\ell(h'_0) & \text{for } 1 \leq u \leq A \\
\pi_X^* S^\ell(h_0) & \text{for } A + 1 \leq u \leq \delta - 2m.
\end{cases}
\]

Substituting the preceding expressions for \( \tilde{\mu}_p(h_u) \) and the expressions for \( \tilde{\mu}_p(x) \) and \( e_s \) from (4.23) into (4.28) and using the equality \( \delta - 2m - A = 2\ell \) in (4.24) gives

\[
\# \left( \bar{\mathcal{V}}(h_1 \ldots h_{\delta-2m}x^m) \cap L_{t_n,s} \right)
\]

\[
= \left\langle \left( \frac{1}{2} \nu + \pi_X^* S^\ell(h'_0) \right)^A \sim \left( \pi_X^* S^\ell(h_0) \right)^{2\ell} \sim \left( -\frac{1}{4} \nu^2 + \pi_X^* S^\ell(x) \right)^m \sim (-\nu)^{\pi_X^*} \sim e_I, [L_{t_n,s}] \right\rangle.
\]
Applying the computations (4.21) and (4.22) and our assumption in (4.25) that \( Q_X(h_0) = 1 \) to (4.29) yields

\[
\# (\mathcal{V}(h_1 \ldots h_{\delta-2m}x^m) \cap L_{t_0,s})
\]

\[
= \frac{(2\ell)!}{\ell!2^\ell} \left( \frac{1}{2} \nu + \pi^*_X S^\ell(h'_0) \right)^\ell - \left( -\frac{1}{4} \nu^2 + \pi^*_X S^\ell(x) \right)^m
\]

\[
\sim (-\nu)^{\epsilon \ell} \sim \bar{e}_I \pi_X^* \text{PD}[x] \cap [L_{t_0,s}]^\ell.
\]

Because

\[ S^\ell(h'_0) \sim \text{PD}[x] = 0 \sim S^\ell(x) \sim \text{PD}[x], \]

by dimension-counting on Sym^\ell(X), the identity (4.30) simplifies to

\[
\# (\mathcal{V}(h_1 \ldots h_{\delta-2m}x^m) \cap L_{t_0,s})
\]

\[
= \frac{(2\ell)!}{\ell!2^\ell} (-1)^{m+r_\Xi} 2^{-A-2m-\ell} (\nu^{A+2m+r_\Xi} \sim \bar{e}_I, [\pi^{-1}_X(x) \cap L_{t_0,s}^\text{vir}]).
\]

Finally, we apply the computation from [16, Lemma 4.12], where it is computed that the restriction of \( \bar{e}_I \) to \( \pi^{-1}_X(x) \cap L_{t_0,s}^\text{vir} \) equals \( (-2)^{-\ell} \nu^\ell \) to rewrite (4.31) as

\[
\# (\mathcal{V}(h_1 \ldots h_{\delta-2m}x^m) \cap L_{t_0,s})
\]

\[
= \frac{(2\ell)!}{\ell!2^\ell} (-1)^{m+r_\Xi+\ell} 2^{-A-2m-\ell} (\nu^{A+2m+r_\Xi+\ell}, [\pi^{-1}_X(x) \cap L_{t_0,s}^\text{vir}]).
\]

Now observe that, because (4.3) gives

\[ r_N - r_\Xi = \frac{1}{2} \dim \mathcal{M}_{t_0} - 3\ell = \delta + 1 - 3\ell, \]

the equality (4.24) implies that we have

\[
A + 2m + r_\Xi + \ell = \delta - \ell + r_\Xi = r_N + 2\ell - 1.
\]

Hence, using the branched cover \( \tilde{f} \) of degree \((-1)^{\ell}2^{r_N+\ell-1} \) in Lemma (4.6) we can write

\[
\langle \nu^{A+2m+r_\Xi+\ell}, [\pi^{-1}_X(x) \cap L_{t_0,s}^\text{vir}] \rangle
\]

\[
= (-1)\ell 2^{-r_N-\ell+1} \langle \nu^{r_N+2\ell-1}, \tilde{f}_* [M_6 \times \mathbb{C}P^{r_N+2\ell-1}] \rangle.
\]

\[
= (-2)^\ell (1 \times \tilde{\nu})^{r_N+2\ell-1, \mathbb{C}P^{r_N+2\ell-1} - \ell} \text{ (by (4.10))}
\]

\[
= (-2)^\ell (1, [M_6]) \times \tilde{\nu}^{r_N+2\ell-1, \mathbb{C}P^{r_N+2\ell-1} - \ell} \text{ (by [38] Theorem 5.6.13)),}
\]

and thus, applying (4.3) and (4.11) to the preceding expression yields,

\[
\langle \nu^{A+2m+r_\Xi+\ell}, [\pi^{-1}_X(x) \cap L_{t_0,s}^\text{vir}] \rangle = (-1)^{r_N+1+\ell} 2^\ell SW_X(s).
\]

Combining (4.32) and (4.31) implies that under the assumptions (4.25) on \( h_u \),

\[
\# (\mathcal{V}(h_1 \ldots h_{\delta-2m}x^m) \cap L_{t_0,s})
\]

\[
= \frac{(2\ell)!}{\ell!2^\ell} (-1)^{m+r_\Xi+r_N+1} 2^{-A-2m} SW_X(s).
\]
Comparing (4.27) and (4.35) gives
\[
(-1)^{\frac{3}{2}A}SW_X(s)A!(2\ell)!^{-} a_{A,0,\ell}(\chi_h(X), c_2^j(X), 0, 0, m, \ell) \\
= \frac{(2\ell)!}{\ell!2^\ell}(-1)^{m+r_{\delta}+r_{N}+1}2^{-A-2m}SW_X(s),
\]
which we solve to get
\[
(4.36) \quad a_{A,0,\ell}(\chi_h(X), c_2^j(X), 0, 0, m, \ell) = \frac{(-1)^{A+m+r_{\delta}+r_{N}+1}2^{-A-2m-\ell}}{\ell!A!}.
\]
Equation (4.24) implies that \(-A-2m-\ell = \ell - \delta\) while (4.33) implies that
\[
A + m + r_{\delta} + r_{N} + 1 = \ell + m \pmod{2}.
\]
Hence, (4.36) implies the desired equality (4.1), completing the proof of Proposition 4.1. □

5. Vanishing coefficients

We now determine the coefficients \(a_{i,0,k}\) with \(i \geq c(X) - 3\) appearing in (3.6). Although, as pointed out in Remark 2.7, the coefficients in (3.6) are not those determined in [12 Proposition 4.8], the techniques used in the proof of [12 Proposition 4.8] also determine the coefficients \(a_{i,0,k}\) with \(i \geq c(X) - 3\) appearing in (3.6).

Proposition 5.1. Continue the hypothesis and notation of Theorem 3.3. In addition, assume \(c(X) \geq 3\) and
\[
(5.1) \quad n \equiv 1 \pmod{2}.
\]
Then for \(p \geq c(X) - 3\) and \(k \geq 0\) an integer such that \(p+2k = c(X) + 4\chi_h(X) - 3n - 1 - 2m\),
\[
a_{p,0,k}(\chi_h(X), c_2^j(X), 0, 0, m, 2\chi_h - n) = 0.
\]

We prove Proposition 5.1 by showing that on certain standard four-manifolds, the vanishing result (3.15) forces each of the coefficients in the sum to be zero by using the following generalization of [20 Lemma VI.2.4].

Lemma 5.2. [12 Lemma 4.1] Let \(V\) be a finite-dimensional real vector space. Let \(T_1, \ldots, T_n\) be linearly independent elements of the dual space \(V^*\). Let \(Q\) be a quadratic form on \(V\) which is non-zero on \(\cap_{i=1}^n \text{Ker} \ T_i\). Then \(T_1, \ldots, T_n, Q\) are algebraically independent in the sense that if \(F(z_0, \ldots, z_n) \in \mathbb{R}[z_0, \ldots, z_n]\) and \(F(Q,T_1,\ldots,T_n) : V \to \mathbb{R}\) is the zero map, then \(F(z_0, \ldots, z_n)\) is the zero element of \(\mathbb{R}[z_0, \ldots, z_n]\).

In [12 Section 4.2], we used the manifolds constructed by Fintushel, Park and Stern in [18] to give the following family of standard four-manifolds.

Lemma 5.3. For every integer \(q \geq 2\), there is a standard four-manifold \(X_q\) of Seiberg-Witten simple type satisfying
\[
(5.2a) \quad \chi(X_q) = q \quad \text{and} \quad c(X_q) = 3,
\]
\[
(5.2b) \quad B(X_q) = \{ \pm K \} \quad \text{and} \quad K \neq 0,
\]
\[
(5.2c) \quad \text{The restriction of} \ Q_{X_q} \ \text{to} \ Ker \ K \subset H_2(X_q; \mathbb{R}) \ \text{is non-zero}.
\]
We write the blow-up of $X_q$ at $r$ points as $X_q(r)$, so
\[ \chi_h(X_q(r)) = \chi_h(X_q) = q, \]
\[ c_1^2(X_q(r)) = c_1^2(X_q) - r, \]
\[ c(X_q(r)) = c(X_q) + r = r + 3, \]
where we recall from (1.1) that $c(X) := \chi_h(X) - c_1^2(X)$. We consider both the homology and cohomology of $X_q$ as subspaces of the homology and cohomology of $X_q(r)$, respectively. Let $e_u^* \in H^2(X_q(r); \mathbb{Z})$ be the Poincaré dual of the $u$-th exceptional class. Let $\pi_u : (\mathbb{Z}/2\mathbb{Z})^r \to \mathbb{Z}/2\mathbb{Z}$ be projection onto the $u$-th factor. For $\varphi \in (\mathbb{Z}/2\mathbb{Z})^r$ and $K \in B(X_q)$, we define
\[ K_\varphi := K + \sum_{u=1}^r (-1)^{\pi_u(\varphi)} e_u^* \quad \text{and} \quad K_0 := K + \sum_{u=1}^r e_u^*. \]
Then, by the blow-up formula for Seiberg-Witten invariants [21 Theorem 14.1.1],
\[ B'(X_q(r)) = \{ K_\varphi : \varphi \in (\mathbb{Z}/2\mathbb{Z})^r \}, \]
\[ \text{SW}_{X_q(r)}(K_\varphi) = \text{SW}_{X_q}(K). \]
In preparation for our application of Lemma 5.2, we have the

**Lemma 5.4.** Let $q \geq 2$ and $r \geq 0$ be integers. Let $X_q(r)$ be the blow-up of the four-manifold $X_q$ given in Lemma 5.3 at $r$ points. Then the set
\[ \{ K, e_1^*, \ldots, e_r^*, Q_{X_q(r)} \} \]
is algebraically independent in the sense of Lemma 5.2 for the vector space $H_2(X_q(r); \mathbb{R})$.

**Proof.** The cohomology classes $K, e_1^*, \ldots, e_r^*$ are linearly independent in $H^2(X_q(r); \mathbb{R})$. The restriction of $Q_{X_q(r)}$ to the intersection of the kernel of these cohomology classes equals the restriction of $Q_{X_q}$ to the kernel of $K$ in $H_2(X_q; \mathbb{R})$, which is non-zero by [5.2c]. Hence, Lemma 5.2 implies that $\{ K, e_1^*, \ldots, e_r^*, Q_{X_q(r)} \}$ is algebraically independent. \( \square \)

**Proof of Proposition 5.1.** Because $c(X) \geq 3$, if $q = \chi_h(X)$ and $r = c(X) - 3 \geq 0$, then
\[ \chi_h(X) = \chi_h(X_q(r)) \quad \text{and} \quad c_1^2(X) = c_1^2(X_q(r)) \]
by Lemma 5.3 and so
\[ a_{i,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, \ell) = a_{i,0,k}(\chi_h(X_q(r), c_1^2(X_q(r)), 0, 0, m, \ell). \]
As in the proof of Theorem 3.3, the assumptions on $m$ and $n$ allow us to apply the cobordism formula (2.10) with a spin$^u$ structure $t_u$ on $X_q(r)$ satisfying (3.1), $\bar{w} \in H^2(X_q(r); \mathbb{Z})$ characteristic,
\[ \delta := c(X) + 4\chi_h(X) - 3n - 1 \quad (\text{from (3.9)}) \]
\[ = c(X_q(r)) + 4\chi_h(X_q(r)) - 3n - 1 \quad (\text{by (5.5)}), \]
and \( \ell(t_n,s) = 2\chi_h(X_q(r)) - n \) from (3.2) to get (see (3.15))

\[
0 = \sum_{K \in B(X_q(r))} (-1)^{\frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot K)}SW'_{X_q(r)}(K) \\
\times \sum_{i+2k \equiv \delta-2m} a_{i,0,k}(\chi_h(X_q(r)), c_1^2(X_q(r)), 0, 0, m, \ell)\langle K, h \rangle^i Q_{X_q(r)}(h)^k. \tag{5.7}
\]

Because \( SW_X(-K) = (-1)^{\chi_h(X)}SW_X(K) \) by [31 Corollary 6.8.4], the set \( B(X_q(r)) \) is closed under the action of \( \{ \pm 1 \} \). Let \( B'(X_q(r)) \) be a fundamental domain for the action of \( \{ \pm 1 \} \) on \( B(X_q(r)) \). We will rewrite (5.7) as a sum over \( B'(X_q(r)) \) by combining the terms given by \( K \) and \( -K \). First observe that

\[
\frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot (-K)) \equiv \frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot K) + \bar{\omega} \cdot K \pmod{2}
\]

\[
\equiv \frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot K) + K^2 \pmod{2} \quad \text{(because \( \bar{\omega} \) is characteristic)}
\]

\[
\equiv \frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot K) + c_1^2(X) \pmod{2} \quad \text{(by (3.3)).}
\]

Combining this equality with \( SW_X(-K) = (-1)^{\chi_h(X)}SW_X(K) \) yields

\[
(-1)^{\frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot K)}SW'_{X_q(r)}(K)\langle K, h \rangle^i + (-1)^{\frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot (-K))}SW'_{X_q(r)}(-K)\langle -K, h \rangle^i
\]

\[
= (-1)^{\frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot K)}SW'_{X_q(r)}(K)\langle K, h \rangle^i \left( 1 + (-1)^{c_1^2(X)+\chi_h(X)+i} \right)
\]

\[
= (-1)^{\frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot K)}SW'_{X_q(r)}(K)\langle K, h \rangle^i \left( 1 + (-1)^{c(X)+i} \right).
\]

Because \( n \equiv 1 \pmod{2} \) by our assumption (5.1), we have \( \delta = c(X) + 4\chi_h(X) - 3n -1 \equiv c(X) \pmod{2} \) so \( \delta = i + 2k \equiv c(X) \pmod{2} \) implies \( c(X) + i \equiv 0 \pmod{2} \). Hence, the preceding identity simplifies to give

\[
(5.8)
\]

\[
(-1)^{\frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot K)}SW'_{X_q(r)}(K)\langle K, h \rangle^i + (-1)^{\frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot (-K))}SW'_{X_q(r)}(-K)\langle -K, h \rangle^i
\]

\[
= (-1)^{\frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot K)}2 SW'_{X_q(r)}(K)\langle K, h \rangle^i.
\]

Equation (5.8) allows us to rewrite (5.7) as

\[
0 = \sum_{K \in B'(X_q(r))} (-1)^{\frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot K)}SW'_{X_q(r)}(K) \\
\times \sum_{i+2k \equiv \delta-2m} 2a_{i,0,k}(\chi_h(X_q(r)), c_1^2(X_q(r)), 0, 0, m, \ell)\langle K, h \rangle^i Q_{X_q(r)}(h)^k. \tag{5.9}
\]

If we abbreviate \( a_{i,0,k} = a_{i,0,k}(\chi_h(X_q(r)), c_1^2(X_q(r)), 0, 0, m, \ell) \) and

\[
\varepsilon(\bar{\omega}, K) = \frac{1}{2}(\bar{\omega}^2 + \bar{\omega} \cdot K),
\]
and use the description of $B'(X_q(r))$ in (5.4), then (5.9) yields

\[ 0 = \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^r} \sum_{i=0}^{2r} (-1)^{\varepsilon(\tilde{w}, K_\varphi)} SW_{X_q(r)}' (K_\varphi) 2a_{i,0,k} (K_\varphi, h)^{i} Q_{X_q(r)}(h)^k. \] (5.10)

To apply Lemma 5.2 to (5.10) and get information about the coefficients $a_{i,0,k}$, we will replace $B'(X_q(r))$ with the set $\{K, e_1^*, \ldots, e_r^*\}$ appearing in Lemma 5.4.

Because $\bar{w} \in H^2(X_q(r); \mathbb{Z})$ is characteristic, $\bar{w} \cdot e_u^* \equiv (e_u^*)^2 \equiv 1 \pmod{2}$. Hence,

\[ \varepsilon(\bar{w}, K_\varphi) \equiv \varepsilon(\bar{w}, K_0) + \sum_{u=1}^{r} \frac{1}{2} \left( (-1)^{\pi_u(\varphi)} - 1 \right) \bar{w} \cdot e_u^* \pmod{2}, \] (5.11)

simplifies to give,

\[ \varepsilon(\bar{w}, K_\varphi) \equiv \varepsilon(\bar{w}, K_0) + \sum_{u=1}^{r} \pi_u(\varphi) \pmod{2}. \] (5.12)

Using the definition (5.3) of $K_\varphi$, we expand the factor $\langle K_\varphi, h \rangle^i$ in (5.10) as

\[ \langle K_\varphi, h \rangle^i = \sum_{i_0 + \cdots + i_r = i} (-1)^{\sum_{u=1}^{r} \pi_u(\varphi) i_u} (i_0 \cdots i_r) \langle K, h \rangle^{i_0} \prod_{u=1}^{r} (e_u^*, h)^{i_u}. \] (5.13)

Substituting (5.11) and (5.12) into (5.10), yields

\[ 0 = (-1)^{\varepsilon(\bar{w}, K_0)} SW_{X_q'(r)} (K) \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^r} \sum_{i_0 + \cdots + i_r = i} \left( \frac{i_0 + \cdots + i_r}{i_0 \cdots i_r} \right) (-1)^{\sum_{u=1}^{r} (1+i_u)} \prod_{u=1}^{r} (e_u^*, h)^{i_u} \]

By Lemma 5.4, the set $\{K, e_1^*, \ldots, e_r^*, Q_{X_q(r)}\}$ is algebraically independent and so the monomials,

\[ K^{i_0} \left( \prod_{u=1}^{r} (e_u^*)^{i_u} \right) Q_h^k, \]

are linearly independent. For the integer $p$ appearing in the statement of Proposition 5.1, we have $p \geq c(X) - 3$ by assumption, so $p \geq r$ by the equality $r = c(X) - 3$ preceding (5.5). Hence, equation (5.13) and Lemma 5.2 imply that the coefficient of the term

\[ \langle K, h \rangle^{p-r} \prod_{u=1}^{r} (e_u^*, h) Q_{X_q(r)}(h)^k \]
in (5.13) must vanish. Because \( i_u = 1 \) for \( u = 1, \ldots, r \) in this term and \( p = i_0 + \cdots + i_r \), we can write this coefficient as

\[
(-1)^{\varepsilon(\tilde{w}, K_0)} SW_{X_q}^\ell(K) \frac{p!}{(p-r)!} \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^r} (-1)^\sum_{u=1}^r 2\pi u(\varphi) 2a_{p,0,k}
\]

\[
= (-1)^{\varepsilon(\tilde{w}, K_0)} SW_{X_q}^\ell(K) \frac{p!}{(p-r)!} \sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^r} 2a_{p,0,k}
\]

\[
= (-1)^{\varepsilon(\tilde{w}, K_0)} SW_{X_q}^\ell(K) \frac{p!}{(p-r)!} a_{p,0,k}.
\]

Hence, the coefficient \( a_{p,0,k} \) must vanish, as asserted, and this concludes the proof of Proposition 5.1.

6. Proof of the main result

We will prove Theorem 1.1 by applying the computations of the coefficients in Proposition 4.1 and Proposition 5.1 to the vanishing sum (3.6).

To apply Proposition 4.1 we need to assume that there is a class \( K \in B(X) \) with \( K \neq 0 \). We can make this assumption if we can replace \( X \) with its blow-up \( \tilde{X} \). In the following, we show that the superconformal simple type condition is invariant under blow-up, allowing the desired replacement of \( X \) with \( \tilde{X} \) in the proof of Theorem 1.1.

Lemma 6.1. Let \( X \) be a standard four-manifold of Seiberg-Witten simple type with \( c(X) \geq 3 \). Then \( X \) has superconformal simple type if and only if its blow-up \( \tilde{X} \) does.

Proof. If \( X \) has superconformal simple type, then \( \tilde{X} \) does by [29, Theorem 7.3.1]. We prove the converse. If \( c(X) \leq 3 \), the result is trivial; we will show that if \( c(X) \geq 4 \) and \( \tilde{X} \) has superconformal simple type, then \( X \) satisfies (1.2). Note that \( c(X) \geq 4 \) implies that \( c(\tilde{X}) \geq 5 \) so \( \tilde{X} \) having superconformal simple type implies that \( \tilde{X} \) satisfies (1.2).

Let \( e^* \in H^2(\tilde{X}; \mathbb{Z}) \) be the Poincaré dual of the exceptional curve. Let \( w \in H^2(X; \mathbb{Z}) \) be characteristic, so \( \tilde{w} := w - e^* \in H^2(\tilde{X}; \mathbb{Z}) \) is also characteristic. By [21, Theorem 14.1.1],

\[
B(\tilde{X}) = \{ K \pm e^* : K \in B(X) \} \quad \text{and} \quad SW_{\tilde{X}}^\ell(K \pm e^*) = SW_X^\ell(K).
\]

For \( K \in B(X) \),

\[
\frac{1}{2}(\tilde{w}^2 + \tilde{w} \cdot (K + e)) \equiv \frac{1}{2}(w^2 + w \cdot K) \quad (\text{mod } 2),
\]

\[
\frac{1}{2}(\tilde{w}^2 + \tilde{w} \cdot (K - e)) \equiv \frac{1}{2}(w^2 + w \cdot K) + 1 \quad (\text{mod } 2).
\]
Then, by the expression for $SW_{X}^{w,i}$ in (2.3) and applying the sign identities just noted,

$$
SW_{X}^{w,i}(h) = \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w^2 + \bar{w} - (K+e))} SW_{X}'(K + e^*)(K + e,h)^i + (-1)^{\frac{1}{2}(w^2 + \bar{w} - (K-e))} SW_{X}'(K - e^*)(K - e,h)^i
$$

$$
= \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w^2 + w \cdot K)} SW_{X}'(K) \sum_{u=0}^{i} \left( \binom{i}{u} \right) (1 - (-1)^u) \langle K, h \rangle^i - u \langle e^*, h \rangle^u,
$$

and thus, for all $h \in H_2(\widetilde{X}; \mathbb{R}),$

$$
(6.1) \quad SW_{X}^{\bar{w},i}(h) = \sum_{u=0}^{i} \left( \binom{i}{u} \right) SW_{X}^{w,i-u}(h) (1 - (-1)^u) \langle e^*, h \rangle^u.
$$

As noted at the beginning of this proof, we can assume that $\widetilde{X}$ satisfies (1.2) so $SW_{X}^{\bar{w},i}$ vanishes for $i \leq c(\widetilde{X}) - 4$ by (1.2). For $h_0 \in H_2(X; \mathbb{R})$ and $e \in H_2(\widetilde{X}; \mathbb{R}),$ the homology class of the exceptional curve, and $s, t \in \mathbb{R},$ equation (6.1) implies that

$$
SW_{X}^{\bar{w},i}(sh_0 + te) = \sum_{u=0}^{i} \left( \binom{i}{u} \right) SW_{X}^{w,i-u}(sh_0) (1 - (-1)^u) \langle e^*, te \rangle^u
$$

$$
= \sum_{u=0}^{i} \left( \binom{i}{u} \right) SW_{X}^{w,i-u}(h_0) (1 - (-1)^u) (-1)^u s^{i-u} t^u.
$$

Assume $i \geq 1.$ Because $SW_{X}^{\bar{w},i}$ vanishes for $i \leq c(\widetilde{X}) - 4,$ we have

$$
0 = \left. \left( \frac{\partial^i}{\partial s^{i-1} \partial t} SW_{X}^{\bar{w}}(sh_0 + te) \right) \right|_{s=t=0}
$$

$$
= -2iSW_{X}^{w,i-1}(h_0).
$$

Thus, $SW_{X}^{w,i-1}$ vanishes for $1 \leq i \leq c(\widetilde{X}) - 4$ or $0 \leq i - 1 \leq c(X) - 4$ as required. \hfill \Box

Half of the polynomials $SW_{X}^{w,i}$ vanish for the following trivial reasons. (This result appears in the remarks following [29 Proposition 6.1.3]; we include it here for completeness.)

**Lemma 6.2.** If a standard four-manifold $X$ has Seiberg-Witten simple type, $i \geq 0$ is any integer obeying $c(X) + i \equiv 1 \pmod{2},$ and $w \in H^2(X; \mathbb{Z})$ is characteristic, then $SW_{X}^{w,i}$ vanishes.

**Proof.** Because $SW_{X}'(K) = (-1)^{\chi_{0}(X)} SW_{X}'(-K)$ by [31 Corollary 6.8.4], the terms in (2.2) corresponding to $K$ and $-K$ in $SW_{X}^{w,i},$ namely

$$
(-1)^{\frac{1}{2}(w^2 + w \cdot K)} SW_{X}'(K, h)^i \quad \text{and} \quad (-1)^{\frac{1}{2}(w^2 - w \cdot K)} SW_{X}'(-K, -K, h)^i
$$

differ by the factor

$$
(-1)^{\chi_{0}(X) + w \cdot K + i}.
$$
Because $w$ is characteristic and because $X$ has Seiberg-Witten simple type, we have $w \cdot K \equiv K^2 \equiv c_1^2(X) \pmod{2}$. Hence,

$$
\chi_h(X) + w \cdot K + i \equiv \chi_h(X) + K^2 + i \equiv \chi_h(X) + c_1^2(X) + i \equiv c(X) + i \pmod{2}.
$$

Thus, if $c(X) + i \equiv 1 \pmod{2}$, then the terms for $K$ and $-K$ in $SW_{X,i}^{w,j}$ cancel and the function $SW_{X,i}^{w,j}$ vanishes.

The vanishing of the sum (3.6) will give information about the Seiberg-Witten polynomial $SW_{X,A}^{w,i}$ of degree $A = c(X) - n - 2m - 1$ which appears in this sum with a non-zero coefficient. We write $A = c(X) - 2v$ where $v$ is a non-negative integer such that $2v = n + 2m + 1$ as in the statement of Theorem 3.3 and note some of the values for this degree to which we can apply Theorem 3.3 and Proposition 5.1. Observe that if $n = 3$, then the equality $2v = n + 2m + 1$ implies $m = v - 2$.

**Lemma 6.3.** Let $X$ be a standard four-manifold with $c(X) \geq 3$. For any $v \in \mathbb{N}$ with $4 \leq 2v \leq c(X)$, the natural numbers $n = 3$ and $m = v - 2$ satisfy the conditions (3.4) in Theorem 3.3 and the parity condition (5.1) in Proposition 5.1.

**Proof.** Because $\chi_h(X) \geq 2$ for a standard four-manifold, $n = 3$ will satisfy the conditions (3.4a), (3.4b), and (5.1). The hypothesis $4 \leq 2v$ implies that $2m = 2v - 4 \geq 0$ while the hypothesis $2v \leq c(X)$ implies $2m = 2v - 4 \leq c(X) - 4 = c(X) - n - 1$, which is the condition (3.4c). \qed

**Remark 6.4.** We note that the requirement $0 < n - 1$ in (3.4b) implies that $n \geq 2$ and so $2v = n + 1 + 2m \geq 3$. Hence, the methods of this article do not imply that $SW_{X,i}^{w,j}$ vanishes when $i > c(X) - 3$, which does not hold in general.

**Proof of Theorem 1.1.** By Lemma 6.1 it suffices to prove that the blow-up of $X$ has superconformal simple type. Because $c_1^2(\tilde{X}) = c_1^2(X) - 1$, we can assume $c_1^2(X) \neq 0$ by replacing $X$ with its blow up if necessary. If we assume $c_1^2(X) \neq 0$ and $K \in B(X)$, then $K^2 = c_1^2(X) \neq 0$ by (3.3), so $K \neq 0$. Thus, we can assume $0 \notin B(X)$ by replacing $X$ with its blow-up if needed.

We now abbreviate $c = c(X)$ and $\chi_h = \chi_h(X)$. If $w \in H^2(X;\mathbb{Z})$ is characteristic, then $SW_{X,i}^{w,j}$ vanishes unless $i \equiv c \pmod{2}$ by Lemma 6.2. Thus, it suffices to prove that $SW_{X,i}^{w,c-2v} = 0$ for $4 \leq 2v \leq c$, which we will do by induction on $v$.

By Lemma 6.3 the values $n = 3$ and $m = v - 2$ satisfy the conditions (3.4) in Theorem 3.3. Substituting these values into (3.6) (noting that $A = c - n - 2m - 1 = c - 2v$), yields

$$
(6.2) \quad 0 = \sum_{k=0}^{2\chi_h-3} a_{c-2v+2k,0,2\chi_h-3-k} SW_{X,i}^{w,c-2v+2k}(h) Q_X(h^{2\chi_h-3+k}),
$$

where the coefficients $a_{i,0,k}$ are defined in (3.5). Because $n = 3$ satisfies the assumption (5.1), Proposition 5.1 implies that

$$
(6.3) \quad a_{c-2v+2k,0,2\chi_h-3-k} = 0 \quad \text{for} \quad 2k - 2v \geq -3.
$$

Because of our assumption that $0 \notin B(X)$, an application of Proposition 4.1 with $n = 3$ gives

$$
(6.4) \quad a_{c-2v,0,2\chi_h-3} \neq 0.
$$
We now begin the induction on \( v \). If \( 2v = 4 \), the identity (6.2) becomes

\[
0 = \sum_{k=0}^{2\chi h-3} a_{c-4+2k,0,2\chi h-3-k} SW^{w,c-4+2k}_X(h)Q_X(h)^{2\chi h-3-k}
= a_{c-4,0,2\chi h-3} SW^{w,c-4}_X(h)Q_X(h)^{2\chi h-3} \quad \text{(by (6.3))},
\]

that is,

\[
0 = a_{c-4,0,2\chi h-3} SW^{w,c-4}_X(h)Q_X(h)^{2\chi h-3}.
\]

Because \( 2v = 4 \), equations (6.4) and (6.5) imply that \( SW^{w,c-4}_X(h)Q_X(h)^{2\chi h-3} = 0 \) for all \( h \in H_2(X;\mathbb{R}) \).

If \( Z \subset H_2(X;\mathbb{R}) \) is the (codimension-one) zero locus of \( Q_X \), the preceding equality implies that the polynomial \( SW^{w,c-4}_X \) vanishes on the open, dense subset \( H_2(X;\mathbb{R}) \) \( \setminus \) \( Z \) of \( H_2(X;\mathbb{R}) \) and hence \( SW^{w,c-4}_X \) vanishes on \( H_2(X;\mathbb{R}) \), completing the proof of the initial case of the induction on \( v \).

For our induction hypothesis, we assume that \( SW^{w,c-2v'}_X = 0 \) for all \( v' \) with \( 4 \leq 2v' < 2v \leq c \). We split the sum in (6.2) into three terms:

\[
0 = a_{c-2v,0,2\chi h-3} SW^{w,c-2v}_X(h)Q_X(h)^{2\chi h-3}
+ \sum_{k=1}^{v-2} a_{c-2v+2k,0,2\chi h-3-k} SW^{w,c-2v+2k}_X(h)Q_X(h)^{2\chi h-3-k}
+ \sum_{k=v-1}^{2\chi h-3} a_{c-2v+2k,0,2\chi h-3-k} SW^{w,c-2v+2k}_X(h)Q_X(h)^{2\chi h-3-k}.
\]

If either of the two sums in (6.6) are sums over empty indexing sets, then the notation is meant to indicate that those sums vanish. We now show the two sums will vanish even if their indexing sets are non-empty. If we write \( c-2v+2k = c-2(v-k) \) and define \( v' = v-k \), then for \( 1 \leq k \leq v-2 \), we have \( v-1 \geq v' \geq 2 \). Hence, by our induction hypothesis, we see that

\[
SW^{w,c-2v+2k}_X = SW^{w,c-2v'}_X = 0 \quad \text{for } 1 \leq k \leq v-2.
\]

If \( v-1 \leq k \), then \( 2k-2v \geq -3 \) and so (6.3) implies that

\[
a_{c-2v+2k,0,2\chi h-3-k} = 0 \quad \text{for } v-1 \leq k \leq 2\chi h - 3.
\]

The vanishing results (6.7) and (6.8) imply that the two sums in (6.6) vanish while (6.4) implies that the coefficient of the first term on the right-hand-side of (6.6) is non-zero. Therefore, the identity (6.6) reduces to

\[
0 = SW^{w,c-2v}_X(h)Q_X(h)^{2\chi h-3}.
\]

If \( Z \) is the zero locus of \( Q_X \), then (6.9) implies that the polynomial \( SW^{w,c-2v}_X \) vanishes on the open dense subset \( H_2(X;\mathbb{R}) \) \( \setminus Z \) of \( H_2(X;\mathbb{R}) \) and hence \( SW^{w,c-2v}_X \) vanishes identically, completing the induction and the proof of Theorem 1.3. \( \square \)
Proof of Corollary 1.2. The lower bound on the number of basic classes on $X$ is true for manifolds of superconformal simple type by [29, Theorem 8.1.1]. Hence the Corollary follows immediately from Theorem 1.1. □

Proof of Corollary 1.4. Witten’s Conjecture follows from the SO(3)-monopole cobordism formula, [8, Theorem 0.0.1] for standard four-manifolds of superconformal simple type by [11, Theorem 1]. Thus Corollary 1.4 follows from [11, Theorem 1] and Theorem 1.1. □

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